Decoupled reference governors: a constraint management technique for MIMO systems

Yudan Liu, Joyce Osorio and Hamid R. Ossareh

Abstract

This paper presents a computationally efficient solution for constraint management of multi-input and multi-output (MIMO) systems. The solution, referred to as the Decoupled Reference Governor (DRG), maintains the highly-attractive computational features of Scalar Reference Governors (SRG) while having performance comparable to Vector Reference Governors (VRG). DRG is based on decoupling the input–output dynamics of the system, followed by the deployment of a bank of SRGs for each decoupled channel. We present two formulations of DRG: DRG-tf, which is based on system decoupling using transfer functions, and DRG-s, which is built on state feedback decoupling. A detailed set-theoretic analysis of DRG, which highlights its main characteristics, is presented. We also show a quantitative comparison between DRG and the VRG to illustrate the computational advantages of DRG. The robustness of this approach to disturbances and uncertainties is also investigated.

1. Introduction

Control and constraint management of systems with multiple inputs and multiple outputs (i.e. MIMO systems) have been studied in the field of controls for many decades. The control of MIMO systems has been the focus of many works in the literature, for example, the Linear Quadratic Regulator (LQR), state feedback control methods, sliding mode control, $H_2$ and $H_{\infty}$ control, and decentralised and centralised control methods, please see Zhou et al. (1996), Ge and Li (2014), Garelli et al. (2006), Burl (1998) and Skogestad and Postlethwaite (2007) and the references therein. The problem of constraint management of MIMO systems has been explored as well. One route is to first find a suitable compensator to decouple the input–output dynamics, see Skogestad and Postlethwaite (2007), MacFarlane (1970) and MacFarlane and Hung (1983). Afterwards, a diagonal controller for the newly decoupled plant is designed. The constraint management part is handled by nonlinear functions (e.g. saturation functions) that maintain the constrained signal within the desired bounds (Åström & Hägglund, 1995). However, this approach can compromise the closed-loop stability and may not enforce state constraints. Another approach is Model Predictive Control (MPC), see Shah and Engell (2011) and Bemporad et al. (2002), which addresses both tracking and constraint management at the same time. This approach for constraint management in MIMO systems is explored in works like (Elliott & Rasmussen, 2013), where decentralised MPC strategies are proposed. Other MPC solutions are centralised (Wang et al., 2003), distributed (Camponogara et al., 2002), and cascade or hierarchical strategies (Scattolini, 2009). However, MPC tends to be computationally demanding, which has limited its applicability, especially for systems with fast dynamics and/or high order. Theoretical guarantees such as stability are also difficult to obtain in practice. Other approaches to solve constraint management are $l_1$-optimal control, see McDonald and Pearson (1991), barrier Lyapunov function, see Tee et al. (2009) and Sun et al. (2020), and constrained LQR, see Scokaert and Rawlings (1998).

Another constraint management technique, which can be designed independently of the tracking controller and alleviates the above shortcomings of MPC, is the Reference Governor (Kolmanovsky et al., 2014) and Garone et al. (2017), also referred to as the Scalar Reference Governor (SRG). It is an add-on scheme for enforcing pointwise-in-time state and control constraints by modifying, whenever required, the reference to a well-designed stable closed-loop system. A block diagram of SRG is shown in Figure 1, where $y(t)$ is the constrained output, $r(t)$ is the reference, $v(t)$ is the governed reference, and $x(t)$ is the system state (measured or estimated). To compute $v(t)$, SRG employs the so-called Maximal Admissible Set (MAS) (Gilbert & Tan, 1991), which is defined as the set of all constant inputs and initial conditions that are constraint-admissible. By solving a simple linear program (LP) over this set, SRG selects a $v(t)$ that is as close as possible to $r(t)$ such that the constraints are satisfied for all time.

Standard SRG uses a single decision variable in the LP to simultaneously govern all the channels of an MIMO system. As a result, it tends to have a conservative response. A modification of the SRG, which performs well in MIMO systems, is the so-called Vector Reference Governor (VRG), see Garone...
et al. (2017). This technique handles constraint management by solving a quadratic program (QP) with multiple decision variables (one for each reference input). Even though VRG shares some properties with SRG, its implementation demands a higher computational load in comparison with SRG. This is because of the QP with multiple decision variables that must be solved at each time step, either by implicit methods or multi-parametric explicit methods. In this paper, we present a new reference governor solution for MIMO systems that maintains the computational simplicity of the SRG, but with performance similar to VRG. The solution, referred to as the Decoupled Reference Governor (DRG), is based on decoupling the input–output dynamics of the system, followed by the deployment of a bank of SRGs for each decoupled channel. Since the decoupling operation can be performed in both transfer function and state-space domains, we investigate two DRG formulations: DRG-tf, and DRG-ss, as summarised next.

The block diagram of the DRG-tf method is shown in Figure 2, where $G(z)$ is the closed-loop system with inputs $u_i$ and constrained outputs $y_i$. In this block diagram, we have assumed that the system is square, i.e. it has $m$ inputs and $m$ outputs, because the decoupling operation can only be applied to square systems. We will extend the theory to non-square systems in Section 7, but for the ease of illustration, assume for now that $G(z)$ is a square system. Over the output, the constraints are imposed: $y_i(t) \in Y_i, \forall t$, where $Y_i$ are specified sets. Given the set-points $r_i$, the goal is to select each $u_i$ as close as possible to $r_i$ (to ensure that the tracking outputs, which are not shown in the figure, follow $r_i(t)$ as closely as possible) while ensuring that the output constraints are satisfied, i.e. $y_i(t) \in Y_i, \forall t$. The DRG-tf method achieves these goals as follows: first, system $G(z)$ is decoupled by finding a suitable filter, $F(z)$, that eliminates the coupling dynamics of $G(z)$. The resulting decoupled system is $W(z) := G(z)F(z)$, which is diagonal; that is, each output $y_i$ depends only on the new input $v_i$. Second, we introduce a bank of $m$ decoupled SRGs, where the goal of the $i$th SRG is to select $v_i$ as close as possible to $r_i$ while ensuring $y_i \in Y_i$. Each SRG (see Figure 2) uses only the states of the $i$th decoupled sub-system. Finally, since we would like to ensure that $u_i = r_i$ when $r_i$ is constraint-admissible, we introduce the inverse of the filter, $F^{-1}(z)$, to cancel the effects of $F(z)$. Note that $F^{-1}(z)$ also ensures that $u_i$ and $r_i$ are close if $r_i$ is not constraint-admissible.

Similar to DRG-tf, DRG-ss is based on decoupling the input–output dynamics as shown in Figure 3. The difference is that the system $G$ is decoupled by using state feedback, where the feedback matrices $\Phi$ and $\Gamma$ are properly chosen as will be discussed later in this paper. Second step is introducing $m$ decoupled SRGs, whose goal is the same as the SRGs in DRG-tf. Finally, to make sure that $u_i = r_i$ when $r_i$ is constraint-admissible, $x$ is fed back through $\Gamma^{-1}(r - \Phi x)$.

Finally, we handle non-square systems by transforming them into square ones and applying the DRG theory explained above to the resulting square system. Detailed information will be provided in Section 7.

Because of the decoupling process, DRG-ss differs from DRG-tf in its analysis, implementation, and observer design. Furthermore, DRG-ss contains an additional feedback loop, which may compromise closed-loop stability. Thus, in this paper, we present a detailed analysis of both methods, including stability, transient and steady-state properties, and observer design considerations. We also study the class of systems for which DRG performs well, and present an analysis of the robustness of DRG to unmeasured disturbances and parametric uncertainties. Note that a preliminary exposition of DRG-tf was presented in a conference version of this paper in Liu et al. (2018). The current paper improves on Liu et al. (2018) by presenting a complete analysis of the transient and steady-state characteristics of DRG-tf, introducing and studying DRG-ss, discussing observer design considerations, presenting the robustness analysis mentioned above, and introducing the extension of DRG-tf and DRG-ss to non-square systems.

The main contributions of this research are as follows:

- A computationally efficient constraint management technique for square MIMO systems (i.e. the DRG), which is a novel extension of the SRG. Two formulations of DRG (i.e. DRG-ss and DRG-tf), and their advantages and disadvantages, are studied.

- Analysis of stability and performance of DRG in comparison with VRG. We show that the proposed approach is most suitable for a specific class of systems and illustrate this by examples.

- Quantitative comparison of explicit and implicit optimisation techniques for VRG and DRG, where we show that DRG algorithm can run two orders of magnitude faster than VRG at every time step.

**Figure 1.** Scalar reference governor block diagram.

**Figure 2.** DRG-tf block diagram.

**Figure 3.** DRG-ss block diagram. $r, r', v, u, y$ represent $[r_1, r_2, \ldots, r_m]'$, $[r'_1, r'_2, \ldots, r'_m]'$, $[v_1, v_2, \ldots, v_m]'$, $[u_1, u_2, \ldots, u_m]'$, and $[y_1, y_2, \ldots, y_m]'$, respectively.
A novel extension of DRG to systems that are affected by unknown additive disturbances and parametric uncertainties.

An extension of DRG to non-square MIMO systems, which enhances the applicability of DRG.

2. Preliminaries

In this section, we introduce the notations and norms that are used in this paper. Then, we review the decoupling methods and reference governor schemes.

The following notations are used in this paper. \( \mathbb{Z}_+ \) denotes the set of all non-negative integers. Let \( V, U \subset \mathbb{R}^n \). Then, \( V \sim U := \{z \in \mathbb{R}^n : z + u \in V, \forall u \in U\} \) is the Pontryagin-subtraction (P-subtraction) (Kolmanovsky & Gilbert, 1998). The identity matrix with dimension \( i \times i \) is denoted by \( I_i \). Given a discrete-time signal \( u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \), the \( L_2 \) norm is defined as: \( \|u\|_2 = \sum_{j=1}^{m} u_j^2 \), and its \( L_\infty \) norm is represented as: \( \|u(t)\|_\infty = \sup_j |u_j(t)| \). For a system with transfer function \( F(z) \) and impulse response \( f(t) \), the \( H_\infty \) norm is defined as: \( \|F\|_{H_\infty} = \max_{\delta}(\|F(\delta\omega)\|) \), where \( \delta \) represents the maximum singular value, and the \( L_1 \) norm is defined as: \( \|f(t)\|_1 = \max_i \sum_{j=1}^{m} |f_{ij}(t)| \), where \( f_{ij} \) is the \( ij \)th element of \( f \) and \( m \) is the number of columns of \( f \). We denote the condition number of a matrix (defined by the ratio of the maximum to the minimum singular values) by \( \gamma \). A zero matrix with dimension \( i \times j \) is denoted as \( 0_{ij} \).

2.1 Review of decoupling methods

In this section, we review two decoupling methods, one based on transfer functions (Skogestad & Postlethwaite, 2007) and the other based on state space (Falb & Wolovich, 1967).

2.1.1 Decoupling method based on transfer functions

Consider the square coupled system \( G(z) \) shown in Figure 2 and defined as:

\[
\begin{bmatrix}
Y_1(z) \\
\vdots \\
Y_m(z)
\end{bmatrix} =
\begin{bmatrix}
G_{11}(z) & \cdots & G_{1m}(z) \\
\vdots & \ddots & \vdots \\
G_{m1}(z) & \cdots & G_{mm}(z)
\end{bmatrix}
\begin{bmatrix}
U_1(z) \\
\vdots \\
U_m(z)
\end{bmatrix} = G(z)
\]

(1)

where \( Y_i \) and \( U_i \) are the \( Z \)-transforms of \( y_i \) and \( u_i \), respectively. The system \( G(z) \) consists of diagonal subsystems with dynamics \( G_{ii}(z) \) and off-diagonal (interaction) subsystems with dynamics \( G_{ij}(z), i \neq j \). A decoupled system is perfectly diagonal (i.e. each output depends on only one input). As shown in Figure 2, we decouple the system by adding a filter, \( F(z) \), before \( G(z) \), so that the product \( G(z)F(z) \) yields a diagonal transfer function matrix \( W(z) := G(z)F(z) \) (Skogestad & Postlethwaite, 2007). By doing so, each output \( Y_i \) depends only on the new input \( V_i \) through: \( Y_i(z) = W_{ii}(z)V_i(z) \), where \( W_{ii}(z) \) is the \( i \)th diagonal elements of \( W(z) \) and \( V_i(z) \) is the \( Z \)-transform of \( v_i \).

In this paper, we study two structures for \( W(z) \), which lead to the following two decoupling methods:

- Diagonal Method: We find \( F(z) \) such that \( W(z) = \text{diag}(G_{11}, G_{22}, \ldots, G_{mm}) \). The filter and the inverse filter are defined as:

\[
F(z) = G^{-1}(z)W(z), \quad F^{-1}(z) = W^{-1}(z)G(z)
\]

(2)

- Identity Method: We find \( F(z) \) such that \( W(z) \) equals the identity matrix. The filter and the inverse filter are defined as:

\[
F(z) = G^{-1}(z), \quad F^{-1}(z) = G(z)
\]

(3)

Notice that in both methods, the elements of either \( F(z) \) or \( F^{-1}(z) \) (or both) may be improper transfer functions because of \( G^{-1}(z) \) and \( W^{-1}(z) \). If this is the case, they cannot be implemented in the DRG scheme of Figure 2. In order to make them proper, we multiply \( F(z) \) and \( F^{-1}(z) \) by time-delays of the form \( 1/z^\beta \), where \( \beta \) is the smallest integer required to make these transfer functions proper. Note that if delays are added to either \( F \) or \( F^{-1} \), the system response will be delayed under the DRG scheme, even if no constraint violation is likely. This is a caveat of the DRG approach; however, if the sample time is small enough, the introduced delay would be negligible as compared with the system dynamics. Also note that \( G^{-1}(z) \) might introduce unstable poles to \( F(z) \) or \( F^{-1}(z) \), which will cause the system to become unstable. Further assumptions are introduced later in the paper to avoid such situations.

Remark 2.1: In the above discussion, the matrix \( W(z) \) is assumed to be diagonal, which means that every \( y_i \) depends only on \( v_i \). This, however, is only one possible structure for \( W(z) \). It is also possible to decouple the system by having each \( y_i \) depend on only one \( v_j \). In this case, the structure of \( W(z) \) will be such that every row will have only one non-zero element. Similarly, each column will have only one non-zero element. The DRG scheme presented in this paper can be used with this structure of \( W \). However, for the sake of simplicity, in the rest of this paper, we will assume that \( W \) is diagonal.

2.1.2 Decoupling method based on state feedback

In this section, we describe input/output decoupling via state-feedback, as presented in Falb and Wolovich (1967) and Lloyd (1970). Consider a discrete-time coupled system, \( G \) (see Figure 3), given in state-space form by:

\[
x(t + 1) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

(4)

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the input, and \( y \in \mathbb{R}^m \) is the output vector. Note that the number of inputs is equal to the number of outputs.

In the remainder of this discussion, we assume no direct feed through between \( u \) and \( y \) (i.e. \( D = 0 \)) as required by Falb and Wolovich (1967) and Lloyd (1970). Note that the case where \( D \neq 0 \) can be handled as well (e.g. see Silverman, 1970), but for the sake of simplicity, here we will only present the case where \( D = 0 \).
The substitution of \( u = \Phi x + \Gamma y \), where \( \Phi \) is an \( m \times n \) matrix and \( \Gamma \) is an \( m \times m \) matrix, into (4) results in:

\[
x(t + 1) = \begin{pmatrix} A + B\Phi \end{pmatrix}x(t) + B\Gamma y(t), \quad y(t) = Cx(t)
\]

Let \( d_1, d_2, \ldots, d_m \) be defined by:

\[ d_i = \min\{j : C_iA^jB \neq 0, j = 0, 1, \ldots, n - 1\} \]

where \( C_i \) denotes the \( i \)th row of \( C \). If \( C_iA^jB = 0 \) for all \( j = 0, 1, \ldots, n - 1 \), then we set \( d_i = n - 1 \). Let \( A^* \in \mathbb{R}^{m \times n} \) and \( B^* \in \mathbb{R}^{m \times m} \) be defined by:

\[
A^* = \begin{bmatrix} C_1A^{d_1+1} & \ldots & C_mA^{d_m+1} \end{bmatrix}, \quad B^* = \begin{bmatrix} C_1A^{d_1} & \ldots & C_mA^{d_m} \end{bmatrix}
\]

It is shown in Falb and Wolovich (1967) that there exist a pair of matrices \( \Phi \) and \( \Gamma \) that decouple the system from \( v \) to \( y \) if and only if \( B^* \) is nonsingular.

Below, we study two structures for \( \Phi \) and \( \Gamma \), which lead to the following two decoupling methods:

- **Identity method**: The pair
  \[
  \Phi = -B^*^{-1}A^*, \quad \Gamma = B^*^{-1}
  \]
  leads to \( y_i(t + d_i + 1) = y_i(t) \), which means that the \( i \)th output depends only on the \( i \)th input with one or more time delays.

- **Pole-assignment method**: We can decouple the system while simultaneously assigning the poles of the decoupled system by using the following choice of \( \Gamma \) and \( \Phi \):
  \[
  \Phi = B^*^{-1}\left[ \sum_{k=0}^{\delta} M_kCA^k - A^* \right], \quad \Gamma = B^*^{-1}
  \]
  where \( \delta = \max d_i \) and \( M_k \) are \( m \times m \) diagonal matrices that are designed to assign the poles at specific locations. For more details, please see Falb and Wolovich (1967). Note that not all of the eigenvalues of \( \hat{A} \) can be arbitrarily assigned. However, it is shown in Falb and Wolovich (1967) that if \( m + \sum_{i=1}^{m} d_i = n \), all the poles of the decoupled system can be assigned.

### 2.2 Review of reference governors

This section reviews the SRG and VRG as presented in Kolmanovsky et al. (2014) and Garone et al. (2017). Consider a discrete-time square linear system described by the state-space model in (4). Suppose \( y(t) \in \mathbb{R}^m \) is the constrained output vector, over which the following constraints are imposed:

\[ y_i(t) \in Y_i, i = 1, \ldots, m, \forall t \in \mathbb{Z}_+, \] where \( Y_i \subset \mathbb{R} \) are specified constraint sets. The constraints can also be expressed in vector form as: \( y(t) \in Y \), where \( Y \) is given by the Cartesian product \( Y = Y_1 \times \cdots \times Y_m \). A review of SRG and VRG is provided next.

#### 2.2.1 Scalar reference governor (SRG)

SRG computes the input \( u(t) \) in (4) as a convex combination of the previous input \( u(t - 1) \) and the current reference \( r(t) \), i.e.

\[
u(t) = u(t - 1) + \kappa(r(t) - u(t - 1))
\]

where \( \kappa \) is the solution of the following linear program:

\[
\begin{align*}
\text{maximize} & \quad \kappa \\
\text{s.t.} & \quad u(t) = u(t - 1) + \kappa(r(t) - u(t - 1)) \\
& \quad (x(t), u(t)) \in O_{\infty}
\end{align*}
\]

where \( O_{\infty} \) is the Maximal Admissible Set (MAS) discussed below. In the above optimisation problem, \( x(t), r(t), \) and \( u(t - 1) \) are known parameters, and \( \kappa \) is the optimisation variable. Note that \( \kappa = 0 \) means that in order to keep the system safe, \( u(t) = u(t - 1) \), and \( \kappa = 1 \) means that no violation is predicted and, therefore, \( u(t) = r(t) \). This SRG formulation ensures closed-loop stability and recursive feasibility. Note that if constraint violation is predicted for any output, all inputs will be affected equally because a single \( \kappa \) is used. Thus, the response of SRG may be conservative for MIMO systems.

MAS is the set of all safe initial conditions and inputs, defined as:

\[
O_{\infty} := \{ (x_0, u_0) : x(0) = x_0, u(t) = u_0, y(t) \in Y, \forall t \geq 0 \}
\]

where it is assumed that \( u(t) = u_0 \) is held constant for all time. Computation of MAS is possible, as \( y(t) \) can be expressed explicitly as a function of \( x(0) = x_0 \) and \( u_0 \). \( y(t) = CA^tx_0 + (C(I - A)^{-1}(I - A^t)B + D)u_0 \). MAS can be computed using the above and can be shown to be a polytope of the form:

\[
O_{\infty} = \{ (x_0, u_0) : H_kx_0 + H_uy_0 \leq h \}
\]

Conditions for \( O_{\infty} \) to be finitely determined (i.e. matrices \( H_k, H_u, h \) to be finite dimensional) are discussed in Kolmanovsky and Gilbert (1995) and Gilbert and Kolmanovsky (1995). Basically, to ensure that \( O_{\infty} \) is finitely determined, the steady-state constraint is first tightened, resulting in the steady-state admissible set, \( P_{ss} \):

\[
P_{ss} := \{ (x_0, u_0) : G_0x_0 \in Y \}
\]

where \( G_0 \) is the DC gain of system (4), and \( Y := (1 - \epsilon)Y \) for some small positive \( \epsilon \). The intersection of \( P_{ss} \) with \( O_{\infty} \) (i.e. adding the inequality in (12) to (11)) leads to a finitely determined inner approximation of \( O_{\infty} \). In the sequel, with some abuse of notation, we assume that \( O_{\infty} \) includes the tightened steady-state constraint and is, hence, finitely determined.

#### 2.2.2 Vector reference governor (VRG)

VRG extends the capabilities of SRG and uses diagonal matrix \( K \) instead of scalar \( \kappa \). More specifically, (9) is reformulated as:

\[
u(t) = u(t - 1) + K(r(t) - u(t - 1))
\]
where \( K = \text{diag}(\kappa_i) \). The values of \( \kappa_i, i = 1, \ldots, m \), are chosen by solving a Quadratic Program (QP):

\[
\begin{align*}
\text{minimize} & \quad \|u(t) - r(t)\| \\
\text{s.t.} & \quad u(t) = u(t-1) + K(r(t) - u(t-1)) \\
& \quad (x(t), u(t)) \in O_\infty
\end{align*}
\]

Note that for VRG, \( O_\infty \subset \mathbb{R}^{n+m} \) can be computed in the same way as explained in Section 2.2.1. Because of the increased number of optimisation variables and the QP formulation, VRG is more computationally demanding than SRG.

### 2.2.3 Maximal admissible sets (MAS) for systems with disturbances

In this section, we review the concept of robust MAS for systems affected by additive disturbances:

\[
x(t + 1) = Ax(t) + Bu(t) + B_w w(t) \\
y(t) = Cx(t) + Du(t) + D_w w(t) \in \mathbb{Y}
\]

where \( \mathbb{Y} \), as before, is the constraint set. The disturbance input satisfies \( w \in \mathbb{W} \), where \( \mathbb{W} \subset \mathbb{R}^d \) is a compact polytope with the origin in its interior. Works that have explored unknown disturbances for RG schemes can be found in Osorio and Ossareh (2018), Kolmanovsky and Gilbert (1998), Gilbert and Kolmanovsky (1999) and Osorio et al. (2019).

In order to define the robust MAS for system (14), we write \( y(t) \) as a function of the initial state, \( x_0 \), the constant input, \( u(t) = u_0 \), and the disturbances:

\[
y(t) = CA^t x_0 + (C(I - A)^{-1}(I - A^t)B + D)u_0 \\
\quad + C \sum_{j=0}^{t-1} A^{t-j-1}B_w w(j) + D_w w(t)
\]

We now define the sets \( \mathbb{Y}_t \) using the following recursion:

\[
\mathbb{Y}_0 = \mathbb{Y} \sim D_w \mathbb{W}, \quad \mathbb{Y}_{t+1} = \mathbb{Y}_t \sim CA^t B_w \mathbb{W}
\]

P-subtraction allows us to rewrite the requirement \( y(t) \in \mathbb{Y} \), \( \forall w(j) \in \mathbb{W}, j = 0, \ldots, t \) as:

\[
CA^t x_0 + (C(I - A)^{-1}(I - A^t)B + D)u_0 \in \mathbb{Y}_t
\]

Finally, we define the robust MAS as:

\[
O_\infty := \{(x_0, u_0) \in \mathbb{R}^{n+m} : G_0 u_0 \in \tilde{\mathbb{Y}}, \\
CA^t x_0 + (C(I - A)^{-1}(I - A^t)B + D)u_0 \in \mathbb{Y}_t\}
\]

where \( G_0 \) is the DC gain of (14) from \( u \) to \( y \), and \( \tilde{\mathbb{Y}} := (1 - \epsilon)\mathbb{Y} \), for some \( 0 < \epsilon \ll 1 \) and large \( t \). Note that \( \tilde{\mathbb{Y}} \) is introduced to ensure finite-determinism of \( O_\infty \) (similar to Section 2.2.1).

### 3. Decoupled reference governor based on transfer function decoupling: DRG-tf

As mentioned in the Introduction, DRG-tf is based on decoupling the system using the method described in Section 2.1.1 to obtain a completely diagonal system \( W(z) \), where \( W(z) := \text{diag}(W_{11}(z), \ldots , W_{nn}(z)) \), followed by implementing \( m \) independent SRGs for the resulting decoupled subsystems, and coupling the dynamics using \( F^{-1}(z) \) to cancel the effects of \( F(z) \).

Because the SRGs are inherently nonlinear elements, one challenge with DRG-tf is quantifying the tracking performance of the system. State estimation is another challenge. More specifically, how can the states of the decoupled subsystems be obtained and fed back to the SRGs? In this section, we elaborate on DRG-tf with a special focus on the above challenges.

The following assumptions are made in this section:

**A. 1:** System \( G(z) \) in Figure 2 reflects the combined closed-loop dynamics of the plant with a stabilising controller. Consequently, \( G(z) \) is asymptotically stable (i.e., \( |\lambda_i(A)| < 1, i = 1, \ldots, n \)). Furthermore, we assume that all diagonal subsystems of the decoupled system \( W(z) \) are also asymptotically stable. \( \blacksquare \)

**A. 2:** \( G(z) \) in Figure 2 is invertible and has a stable inverse. \( \blacksquare \)

**A. 3:** The constraint sets \( \mathbb{Y}_i \) are closed intervals of the real line containing the origin in their interiors. This is in agreement with the assumptions commonly made in the literature of reference governors. We thus assume that \( \mathbb{Y}_i = \{y_i : \bar{\epsilon} \leq y_i \leq \bar{\epsilon}\} \). \( \blacksquare \)

Consider the system in Figure 2 with \( G(z) \) given in (1). To design the SRGs, we compute the MAS for each \( W_{ii} \), denoted by \( \mathbb{O}_\infty^{W_{ii}} \). To obtain these sets, we find a minimal state-space realisation of each subsystem \( W_{ii} \), and compute its MAS as:

\[
\mathbb{O}_\infty^{W_{ii}} := \{(x_i, v_i) \in \mathbb{R}^{n+1} : x_i(0) = x_{i0}, \\
v_i(t) = v_{i0}, y_i(t) \in \mathbb{Y}_i, \forall t \in \mathbb{Z}_+\}
\]

where \( x_i \) and \( n_i \) are the state and the order of \( W_{ii} \), respectively. If the states of \( G \) are unknown, an observer can be designed, which will be explained later.

The DRG-tf formulation is based on the sets \( \mathbb{O}_\infty^{W_{ii}} \). Specifically, the inputs \( v_i \) are defined, similar to (9), by:

\[
v_i(t) = v_i(t-1) + \kappa_i(r'_i(t) - v_i(t-1))
\]

where \( \kappa_i \) are computed by \( m \) independent linear programs:

\[
\begin{align*}
\text{maximize} & \quad \kappa_i \\
\text{s.t.} & \quad v_i(t) = v_i(t-1) + \kappa_i(r'_i(t) - v_i(t-1)) \\
& \quad (x_i(t), v_i(t)) \in \mathbb{O}_\infty^{W_{ii}}
\end{align*}
\]

**Remark 3.1:** Note that, since \( F(z) \) and \( F^{-1}(z) \) are both assumed to be stable, the DRG formulation above inherits the stability and recursive feasibility properties of SRG theory. Specifically, for a constant signal \( r(t) = r, r'(t) \) converges (because of stability of \( F^{-1} \)), which implies that \( v(t) \) converges (because of stability of SRGs). Thus, the system of Figure 2 is guaranteed to be stable.

Below, we specialise the DRG-tf formulation to the two decoupling methods presented in Section 2.1.1, namely, the diagonal and the identity methods. We address the subtleties
associated with both methods, including observer design and the structure of the maximal admissible sets. We also illustrate, using examples, that DRG-tf is most effective for systems with small condition numbers and singular values. We will provide a theoretical basis for this observation in Section 3.3. Finally, we show, using examples and analysis, that the DRG-tf with identity method may not perform well for certain systems even if the plant model is known precisely.

3.1 Diagonal method

Recall that decoupling using the diagonal method leads to the decoupled system

\[ W(z) := \text{diag}(G_{11}, \ldots, G_{mm}), \]

where \( F(z) \) and \( F^{-1}(z) \) are chosen as \( G^{-1}(z) W(z) \) and \( W^{-1}(z) G(z) \), respectively. We assume that \( O_{\infty}^{W_{ii}} \) are created using minimal realizations of \( G_{ii} \). In this section, we first present an example to highlight the key attributes of the DRG-tf with diagonal method. For this example, we assume that the states of all \( G_{ii} \) are measured and are available for feedback to the SRGs. This assumption will be relaxed in the next subsection, which discusses the issue of observer design.

3.1.1 Motivating example

Consider the system \( G(z) \) in (1) given by:

\[
G(z) = \begin{bmatrix}
0.9 \\
(z - 0.2)^2 \\
\frac{q}{3z + 1} \\
0.4 \\
(2z - 1)^2 \\
\frac{1}{z - 0.6}
\end{bmatrix}
\]  

and the constraints defined by \(-1.2 \leq y_1 \leq 1.2\) and \(-3.9 \leq y_2 \leq 3.9\). The parameter \( q \) will be selected later.

Next, we use (2) to find \( F(z) \). Noticing that in this example, we encounter the situation that both \( F(z) \) and \( F^{-1}(z) \) are not proper, we multiply them by \( \frac{1}{z} \). Finally, we obtain the decoupled system:

\[ W(z) = \frac{1}{z} \text{diag} \left( \frac{0.9}{(z - 0.2)^2}, \frac{0.4}{(z - 0.6)} \right). \]

In the discussion below, we denote the DC-gain matrix of \( F(z) \) by \( F_0 \).

As mentioned in the Introduction, a requirement for DRG is that the signals \( u(t) \) and \( r(t) \) should be as close as possible to ensure that the tracking performance of the system does not degrade significantly as compared with VRG. As it turns out, this will be the case if the maximum singular value of \( F_0 \), i.e. \( \bar{\sigma} \), is small. On the other hand, this will not be the case if the minimum singular value of \( F_0 \), \( \sigma \), is large. We will analytically prove these statements in Section 3.3, but here we illustrate them via our example. For this, we consider two different \( q \)'s in (21): \( q = 0.5 \) and \( q = 0.05 \). If \( q = 0.5 \), then the maximum and minimum singular values of \( F_0 \) are \( \bar{\sigma} = 4.51 \) and \( \sigma = 0.30 \), and its condition number is \( \gamma = 14.94 \). If \( q = 0.05 \), then \( \bar{\sigma} = 3.39 \), \( \sigma = 0.30 \), and \( \gamma = 11.21 \). The second case has smaller condition number and maximum singular value compared to the first one.

We proceed to design the DRG-tf based on \( W(z) \). In this case, we obtain \( O_{\infty}^{W_{11}} \) and \( O_{\infty}^{W_{22}} \) based on (18). We simulate the response of this system to a step of size 1 in both \( r_1 \) and \( r_2 \). The simulation results for both \( q = 0.5 \) and \( q = 0.05 \) are depicted in Figure 4.

As can be seen, the outputs in both cases satisfy the constraints, as required. However, \( u(t) \) is closer to \( r(t) \) for the

![Figure 4](image-url). Comparison of DRG-tf for systems with small and large condition numbers (\( \gamma \)). Top plot is the output (constraints shown by dashed lines) and the bottom plot is the reference \( r(t) \) and the plant input \( u(t) \).
system with smaller condition number and singular value (i.e. for $q = 0.05$), which indicates better tracking performance.

Furthermore, Figure 5 shows the performance of the SRGs in the DRG scheme (recall Figure 2). Here, two observations can be made. First, $v(t)$ is below $r'(t)$ for all time, which is expected since in SRG theory, the output of SRG is always bounded above (or below) by its input (in this case, $r'$). However, note from Figure 4 that $u(t)$ may be above $r(t)$, which is a situation that does not arise in SRG or VRG applications. The reason can be explained as follows: at steady state, $u$ converges to $F_0v$ and $r'$ converges to $F_0^{-1}r$ (see Figure 2). Thus, $r - u$ converges to $F_0(r' - v)$, which indicates that, even if $r'_i > v_i$, $u_i$ may be above or below $r_i$ depending on $F_0$. Note that $u_i$ above $r_i$ may or may not be acceptable depending on the specific application. An example where this situation is acceptable is a distillation process (Skogestad & Postlethwaite, 2007), because the constrained outputs (i.e. product compositions) determine the efficiency of the process. On the other hand, an example where this situation is not acceptable is in aerospace applications like controlling a drone, since the roll, pitch, and yaw angles cannot significantly exceed their commands. The second observation is that $v(t)$ is close to $r'(t)$, which, as mentioned in Section 2.2, is one of the features of SRG.

Note from Figures 4 and 5 that, while $u_i$ for the two systems (i.e. one with smaller condition number and one with larger condition number) are visibly different, $r'_i$ and $v_i$ are similar. We highlight that this is a feature of the specific system chosen in this example and is not true in general. The reason for this behaviour is that the inverse filters ($F^{-1}(z)$) for the two systems are almost the same, which means that $r'$ and, in turn, $v$, will be almost the same.

Figure 6 shows a comparison between VRG and DRG-tf for $q = 0.05$. There is a time delay in the response of DRG-tf that is caused by the delay added to $F$ and $F^{-1}$ to make them proper. Note that $u(t)$ is below $r(t)$ for the VRG but not for DRG-tf, as explained above. More interestingly, the rise time for DRG-tf is much faster than that of VRG. This is because, for this example, the interacting dynamics are slow and dominant, which causes the VRG to generate slow inputs. The DRG-tf, on the other hand, operates on the decoupled system where these slow dynamics have been cancelled. This shows that, in addition to computational advantages, the DRG-tf may also have performance advantages compared to VRG.

To investigate the above observation more thoroughly, a comparison between the volumes for the MAS’s of DRG-tf (i.e. volumes of $O^{W_{11}}_\infty$ and $O^{W_{22}}_\infty$) and VRG (i.e. volume of $O_\infty$) is as follows for $q = 0.05$: $\text{vol}(O^{W_{11}}_\infty) = 1.698$, $\text{vol}(O^{W_{22}}_\infty) = 7.761$, and $\text{vol}(O_\infty) = 7.761$ (volumes are computed using the MPT toolbox that is introduced in Herceg et al., 2013). Clearly, the sum of the volumes for DRG-tf exceeds the volume of the MAS for VRG, which is in agreement with the observations of Figure 6 regarding a less conservative response from DRG-tf in comparison to VRG. A deeper analysis of the geometric properties of the MAS’s is outside of the scope of this paper; however, it may be considered as an interesting topic for future work.

Figure 5. Comparison of $r'(t)$ and $v(t)$ in DRG-tf with large (top plot) and small (bottom plot) condition number systems.
by computing the state estimate recursively:

$$\hat{x}_i(t+1) = A_{ii}\hat{x}_i(t) + B_{ii}y_i(t)$$  \hspace{1cm} \text{(22)}$$

where $\hat{x}_i$ is the estimate of the state $x_i$. In real-time, the SRGs in the DRG-tf formulation use $\hat{x}_i$ instead of $x_i$. Note that the open loop observer works well only when the system model and the initial conditions are both accurately known, which is not always the case.

To improve upon the open loop observer, feedback can be implemented from the measured output, as is done in standard observer design. We consider two observer design strategies below. The first assumes that all $y_i$ are measured, which leads to $m$ decoupled observers, and the second assumes that some $y_i$ are not measured, necessitating a centralised observer. Both strategies lead to subtleties for DRG-tf that we highlight in this section.

**Decoupled observers:** First suppose that all $y_i$ are available for measurement. In this case, we can design $m$ decoupled Luenberger observers as follows:

$$\hat{x}_i(t + 1) = A_{ii}\hat{x}_i(t) + B_{ii}y_i(t) + L_i(y_i(t) - C_{ii}\hat{x}_i(t) - D_{ii}v(t))$$  \hspace{1cm} \text{(23)}$$

where $L_i$ is designed to assign the eigenvalues of $A_{ii} - L_iC_{ii}$ in the unit circle. Note that for the DRG-tf implementation, the state that feeds back to SRGI is $\hat{x}_i$.

A challenge with the above observer is that of selecting the initial conditions for each $\hat{x}_i$. Indeed, if the observers are not initialised properly, the DRG-tf scheme may not be able to enforce the constraints. We provide a solution to this problem below, for the case where the initial condition of $G(z)$, denoted by $x_0$, is known precisely. We will treat the case of unknown $x_0$ later.

Our solution is to modify the input to $G(z)$ in Figure 2 to explicitly cancel the effects of $x_0$. To see how this can be done, note that the output of $G(z)$ with initial condition $x_0$ can be written as:

$$y(t) = CA^t x_0 + (C(I - A^t)(I - A)^{-1}B + D)u_0$$

where $A$, $B$, $C$, and $D$ are the state space matrices of $G$. Denote by $M(z)$ the $Z$-transform of $CA^t$ for the sake of simplicity of notation. Note that $M(z)$ represents the initial condition response of the system. In order to get $Y(z) = W(z)V(z)$, where $W$ is a desired diagonal matrix as before, we define $U(z)$ as:

$$U(z) = F(z)V(z) - G^{-1}(z)M(z)x_0$$  \hspace{1cm} \text{(24)}$$

where $F(z) = G(z)^{-1}W(z)$ as before (compare (24) with $U(z) = F(z)V(z)$ in Figure 2). This will effectively cancel the initial conditions and result in a completely decoupled system.

The observers given in (23) and the SRGs can now be applied as before. Note that the inverse filter $F^{-1}(z)$ in Figure 2 need not be altered.

**Centralized observer:** Now consider the more interesting case, where either some $y_i$ are not measured, or outputs other

![Figure 6. Comparison of VRG and DRG-tf for the system shown in (21) with $q = 0.05$. Top plot is the output. The dashed lines are the output constraints. Bottom plot is $u$ compared with $r$. The dashed yellow line is the reference.](image-url)
than $y_1$ are measured. Since the dynamics from $v$ to $y$ are still required to be decoupled, $m$ decoupled SRGs can still be used in the DRG-tf formulation. However, we cannot design $m$ decoupled observers for each $W_i$, as we did before (since independent measurements are not available), and must instead design one centralised observer for $W$. This, in turn, implies that the SRGs must use an MAS different from (18). To elaborate on these ideas, let $y_i(t)$, as before, denote the constrained output vector, and let $y_m$ denote the measured output vector. Let $(A, B, C, D)$, $(A_f, B_f, C_f, D_f)$ be realizations of $G(z)$ from $u$ to $y$, $G(z)$ from $u$ to $y_m$, and $F(z)$, respectively. The states of $F(z)$, $x_f$, are known at the time of implementation so they do not need to be estimated. To estimate the states of $G(z)$, $x$, an observer is designed using feedback on the measurements $y_m$:

$$
\hat{x}(t + 1) = A\hat{x}(t) + Bu(t) + L(y_m(t) - C_m\hat{x}(t) - D_mu(t))
$$

(25)

Using the above, the states of the entire system, i.e. $x_w = (x_f, x)$, can be estimated by $\hat{x}_w = (\hat{x}_f, \hat{x})$. Note that initialisation of this observer is simple if the initial condition of $G(z)$, i.e. $x_0$ is known: in this case, the initial condition of the observer is set to $(0, x_0)$.

Recall that to construct $O_{\infty}^W$, the state-space model of the $i$th diagonal subsystem of $W$, $W_{ii}$, is required. However, the states of each individual $W_{ii}$ is not directly available, which is why the SRGs can no longer use the $O_{\infty}^W$ sets as described in (18). To remedy this, we use the following realisation of $W$, which is the augmented dynamics of $F(z)$ and $G(z)$:

$$
x_w(t + 1) = \begin{bmatrix}
A_f & 0 \\
BC_f & A
\end{bmatrix} x_w(t) + \begin{bmatrix}
B_f \\
BD_f
\end{bmatrix} v(t)
$$

$$
y(t) = \begin{bmatrix}
D_f \\
C_w
\end{bmatrix} x_w(t) + DD_f v(t)
$$

(26)

Using (26), the state-space model of $W_{ii}$ is given by: $(A_{w_i}, B_{w_i}(, i), C_{w_i}(, i), D_{w_i}(, i))$, where $B_{w_i}(, i)$ is the $i$th column of $B_w$, $C_{w_i}(, i)$ is the $i$th row of $C_w$, and $D_{w_i}(, i)$ is the $(i, i)$th element of $D_w$. Thus, we construct $O_{\infty}^W$ based on the state-space realisation $(A_{w_i}, B_{w_i}(, i), C_{w_i}(, i), D_{w_i}(, i))$ and, for real-time implementation, each SRG uses the state of the entire system (i.e. $\hat{x}_w = (x_f, \hat{x})$) as feedback.

Finally, for the case where the initial conditions are not known, either observer ((23) or (25)) can be used to estimate the states; however, during the transient phase of the observer, the states may be incorrect, which may lead to constraint violation. To remedy this issue, one can ‘robustify’ $O_{\infty}^W$ as discussed in Section 6, or alternatively, one could allow the transients to subside before running the system with the reference governor.

### 3.2 Identity method

As previously mentioned, for the identity method, $W(z)$ is either the identity matrix (if $G^{-1}(z)$ is proper) or the identity matrix with one or more time delays (if $G^{-1}(z)$ is not proper). In other words, the input–output behaviour of the $i$th channel is given by $y_i(t) = v_i(t - \beta)$, where $\beta \in \mathbb{Z}_+$ is the delay added to make $G^{-1}(z)$ proper. An interesting observation can be made: the MAS for a pure delay system is independent of the state and is given by:

$$
O_{\infty}^W = \{(x_0, v_0) : v_0 \in \mathbb{V}_i\}.
$$

(27)

The above follows directly from the definition of $O_{\infty}$ in Section 2.2.1 and by noting that the initial states (i.e. previous outputs) of the time-delay system can be chosen as 0, which is automatically admissible. Note also that, MAS for this case is finitely determined, without the need to tighten the steady-state constraint.

The DRG formulation for the case of identity method is the same as (19), (20). However, the implementation is greatly simplified due to the structure of $O_{\infty}^W$. To see this, let that the structure of (27) implies that $\kappa_1$ in (20) is chosen so that $v_i(t) \in \mathbb{V}_i$. Since $\mathbb{V}_i$ is an interval, this implies that $\kappa_1$ is selected so that $v_i(t)$ is simply clipped (i.e. saturated) at the constraint. Thus, the overall DRG can be implemented as a bank of $m$ decoupled saturation functions, which greatly simplifies real-time implementation.

Similar to the diagonal method, if $G(z)$ has a small condition number or maximal singular value, the inputs to system $G(z)$ would be far away from the references and, hence, tracking performance may suffer. Since this is the same phenomenon as the diagonal case, we will not provide numerical examples.

While the identity method is simpler and computationally superior to the diagonal method, it has a drawback. If system $G(z)$ has under-damped dynamics, then this method would cause large oscillation in the output, even if the plant model is known precisely. To illustrate, we select $q = 0.05$ in the example of Section 3.1 and change $G_{11}(z)$ in $G(z)$ to a underdamped system: $G_{11}(z) = \frac{0.54z - 0.49}{z^2 - 1.85z + 0.97}$. A comparison between the outputs of this system after applying DRG with the diagonal and identity methods is shown in Figure 7. It can be seen that the constraints are satisfied for both outputs. However, unlike the diagonal method, the output using the identity method has large oscillations.

The reason for this behaviour can be explained as follows. Because $G(z)$ has slow under-damped dynamics, and since $F^{-1}(z) = G(z)$ for the identity method, applying a step to $r(t)$ causes oscillatory response in $r'(t)$. Viewing DRG as saturations in this case, $v(t)$ is computed as $r'(t)$ clipped at the constraints. Finally, since $W(z)$ is an identity matrix or identity matrix with some time delays, these oscillations will directly show up at the output $y(t)$.

Because of the above shortcoming, it is recommended, before selecting a specific decoupling method, to perform an analysis of the system dynamics similar to the above.

### 3.3 Analysis of DRG-tf

In this section, we present an analysis of DRG-tf, both in steady-state and transient.

#### 3.3.1 Steady-state analysis

Recall the steady-state constraint in (12) for a generic system. The steady-state constraint for $O_{\infty}^W$ can be defined similarly. In order to study the steady-state admissible inputs, we consider the projection of the steady-state constraint onto the $v_i$
coordinate, which results in:

\[ V_{ss}^{W} := \{ v_i \in \mathbb{R} : W_{i0} v_i \in \mathcal{Y}_{ss} \} \]  \tag{28}

where \( W_{i0} \in \mathbb{R} \) is the DC gain of subsystem \( W_i \) and \( \mathcal{Y}_{ss} = (1 - \epsilon) \mathcal{Y}_i \) (recall that \( \mathcal{Y}_i \) is the constraint set for \( y_i \)). Since \( W \) is diagonal, it follows that the steady-state constraint-admissible input set for \( W \) is:

\[ V_{ss}^{W} := \{ v_{ss}^{W1} \times v_{ss}^{W2} \times \cdots \times v_{ss}^{Wm,n} \} \]  \tag{29}

We now compare the above set with the steady-state constraint-admissible set of system \( G \) (projected onto the \( u \) coordinate), which arises in VRG applications. This set, noted by \( U_{ss} \), is defined by:

\[ U_{ss} := \{ u \in \mathbb{R}^m : G_0 u \in \mathcal{Y}_{ss} \}. \]  \tag{30}

From the above, the following theorems emerge. Note that Theorems 3.1 and 3.3 below are also presented in the conference version of this paper (see Liu et al., 2018). Therefore, we will not present the proofs for brevity but will present the theorem statements for the sake of completeness.

**Theorem 3.1:** For the system of Figure 2, and \( U_{ss} \) and \( V_{ss}^{W} \) defined in (29) and (30), the following relation holds

\[ V_{ss}^{W} = F_0^{-1} \times U_{ss}, \]  \tag{31}

where \( F_0 \) is the DC gain of \( F(z) \) and the operation \( F_0^{-1} \times U_{ss} \) is the point-by-point mapping of the set \( U_{ss} \) through \( F_0^{-1} \).

An important implication of this theorem is as follows. If \( r \) is not admissible with respect to system \( G \) (i.e. \( r \notin U_{ss} \)), then \( r' \) (see Figure 2) must also not be admissible with respect to the system \( W \) (i.e. \( r' \notin V_{ss}^{W} \)).

The sets (29) and (30) describe the steady-state operations of DRG-tf and VRG, respectively. Note that VRG solves a QP, whereas DRG-tf solves an LP. This implies that, for non-admissible references, DRG-tf finds a solution on a vertex of \( V_{ss}^{W} \), or from Theorem 3.1, a vertex of \( U_{ss} \). On the other hand, VRG finds a solution that may or may not be at a vertex of \( U_{ss} \). Therefore, DRG-tf leads to a suboptimal solution with respect to the objective function of VRG. In the following theorem, we show this more clearly by finding the explicit expression of \( v \) computed by DRG-tf at steady-state. For this theorem, recall that the constraint on \( y_i(t) \) has the form \( \xi \leq y_i \leq \tilde{\xi} \).

**Theorem 3.2:** For the system of Figure 2, at steady state:

\[ v_i = \text{sat}(F_0^{-1} r) \]  \tag{32}

where \( \text{sat} \) refers to the saturation operator with bounds \( W_{i0}^{-1} \xi \) and \( W_{i0}^{-1} \tilde{\xi} \) and \( F_0^{-1} \) represents the DC gain of the \( i \)th row of \( F^{-1} \).

**Proof:** At steady state, \( \xi \leq W_{i0} v_i \leq \tilde{\xi} \) since \( y_i = W_{i0} v_i \). If \( r_i' \) is constraint admissible, i.e. \( \xi \leq W_{i0} r_i' \leq \tilde{\xi} \), then from (20), \( v_i = r_i' \). Otherwise, \( v_i \) would either be equal to \( W_{i0}^{-1} \xi \) or equal to \( W_{i0}^{-1} \tilde{\xi} \). Combining the fact that \( r_i' = F_0^{-1} r \) at steady state, the result follows.

This theorem shows that if \( r_i' \) is in the steady-state admissible set for \( v_i \) (i.e. \( V_{ss}^{W_i} \)), then \( v_i = r_i' \). If this holds for all \( i \), then \( u = r \) since \( F_0 F_0^{-1} = I_m \). If \( r_i' \) is not in \( V_{ss}^{W_i} \), then \( v_i \) can be calculated explicitly as shown in (32), which means that, from \( u = F_0 v \), \( u \) also can be computed explicitly at steady-state.

We now use an example to show the geometric interpretation of this theorem. Consider the same example as shown in (21) with \( q = 0.05 \). As before, the constraints are defined by \(-1.2 \leq y_1 \leq 1.2 \) and \[-3.9 \leq y_2 \leq 3.9 \). The steady-state admissible set for \( v_1 \) and \( v_2 \) is shown by the orange region in Figure 8. If \( r_i' = (r_1', r_2') = (1, 4.2) \) (blue dot in Figure 8), which is outside of the admissible set, then \( v_i \) is given by the closest point along the \( v_{ss} \) axis to \( r_i' \) (green star in Figure 8). Similarly, \( v_2 \) is given by the closest point along \( v_{ss} \) axis to \( r_2' \) (red star in Figure 8). If \( r' \) is in the admissible set (purple dot in Figure 8), then \( v = r' \) and \( u = r \) at steady-state.

As previously mentioned, a requirement for DRG is that the plant input, \( u \), and the setpoint, \( r \), should be equal if no constraint violation is predicted, and that they should be as close as possible if constraint violation is predicted. This is to ensure that the degradation of tracking performance is minimal. We note that each SRG in Figure 2 ensures that \( v_i \) and \( r_i' \) are close; however, \( u \) and \( r \) may be far. In the following theorem, we show
that, at steady-state, the closeness of $u$ and $r$ and, hence, the performance of DRG-tf, depends on the decoupling filter, $F(z)$.

**Theorem 3.3:** Given the system of Figure 2, at steady-state, we have that:

$$
\|F_0^{-1}\|^{-1}\|v - r'\| \leq \|u - r\| \leq \|F_0\|\|v - r'\|
$$

where $\|\cdot\|$ refers to any vector norm and its associated induced matrix norm.

This theorem shows that $\|u - r\|$ is bounded above and below by $\|v - r'\|$ scaled by the induced norms of $F_0$ and $F_0^{-1}$, which are known a-priori. More specifically, if $\|F_0\|_1$ is small, then small $\|v - r'\|$ implies small $\|u - r\|$, which is desirable. Also, if $\|F_0^{-1}\|_1$ is large, then small $\|v - r'\|$ implies large $\|u - r\|$, which is undesirable. In the case of large $\|F_0\|$ or small $\|F_0^{-1}\|_1$, no conclusion can be made.

Note that if 2-norm is chosen, then $\|F_0\|_2 = \tilde{\sigma}(F_0)$, where $\tilde{\sigma}(F_0)$ is the largest singular value of $F_0$. Similarly, $\|F_0^{-1}\|_1 = \overline{\sigma}(F_0)$, where $\overline{\sigma}(F_0)$ is the smallest singular value of $F_0$. Therefore,

$$
\overline{\sigma}(F_0)\|v - r'\|_2 \leq \|u - r\|_2 \leq \tilde{\sigma}(F_0)\|v - r'\|_2.
$$

Since $\|u - r\|_2$ is exactly the objective function in VRG optimisation, the above shows that the performance of DRG-tf and VRG will be close if $F_0$ has small singular values. Note that the quantity $\|v - r'\|$ depends on the value of $r$ and can be computed from Theorem 3.2.

Finally, note that if the identity decoupling method is implemented, then $F = G^{-1}$. Hence, using Theorem 3.3, the following relation follows:

$$
\|G_0\|^{-1}\|v - r'\| \leq \|u - r\| \leq \|G_0^{-1}\|\|v - r'\|,
$$

which allows us to study closeness of $u$ and $r$ using the original system $G(z)$ instead of filter $F(z)$.

### 3.3.2 Transient analysis

Here, we extend the steady-state results of the previous section and study the transient performance of DRG-tf. The analysis of this section relies on the $H_\infty$ and $L_1$ norm of $F(z)$. Because of the delays introduced in $F(z)$ and/or $F^{-1}(z)$ to make them proper, care must be taken in interpreting the results, as we show below.

**Theorem 3.4:** For the system of Figure 2, the following relationship holds:

$$
\|u(t + \beta_1) - r(t - \beta_2)\|_{L_\infty} \leq \|F\|_{H_\infty}\|v - r'\|_{L_2}
$$

where $\beta_1$ and $\beta_2$ are the number of delays added to make $F$ and $F^{-1}$ proper, respectively.

**Proof:** By Parseval’s theorem, $\|u - r\|_{L_2} = \|U - R\|_{H_2}$ and $\|v - r'\|_{L_2} = \|V - R'\|_{H_2}$, where $R'$, $R$, $U$, and $V$ are the $Z$-transforms of $r'$, $r$, $u$, and $v$, respectively. From Figure 2 the following equations hold:

$$
U(z) = \frac{1}{z^{\beta_1}}F(z)V(z), \quad R'(z) = \frac{1}{z^{\beta_2}}F(z)^{-1}R(z)
$$

Then,

$$
\|z^{\beta_1}U(z) - z^{-\beta_2}R(z)\|^2_{H_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|F(e^{jw})(V(e^{jw}) - R'(e^{jw}))\|^2_2 dw
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|F(e^{jw})\|^2_2 \|V(e^{jw}) - R'(e^{jw})\|^2_2 dw
$$

where $\|\cdot\|_2$ refers the Euclidean norm. Since $\|F\|_{H_\infty} = \max_w \tilde{\sigma}(F(e^{jw}))$, we have that:

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \|F(e^{jw})\|^2_2 \|V(e^{jw}) - R'(e^{jw})\|^2_2 dw \leq \|F\|^2_{H_\infty}\|V - R'\|^2_{L_2}
$$

By Parseval’s theorem, the result follows.

Note that (34) can be rewritten as:

$$
\|u(t) - r(t - \beta_2 - \beta_1)\|_{L_2} \leq \|F\|_{H_\infty}\|v - r'\|_{L_2}
$$

This equation shows that the average distance between $u$ and the delayed version of $r$ is bounded by the average distance between $v$ and $r'$ scaled by $\|F\|_{H_\infty}$. Thus, if $\|F\|_{H_\infty}$ is small, the DRG-tf and VRG will perform similarly in transient (although, DRG-tf will exhibit delays).

Since Theorem 3.4 only discusses time averages, below we provide another theorem to show that the peak of the distance between $u$ and $r$ is related to $\|f\|_{L_1}$, where $f$ is the impulse response matrix of $F(z)$ and $\|\cdot\|_{L_1}$ refers to the $L_1$ norm of $f$.

**Theorem 3.5:** For the system of Figure 2, the following relationship holds with respect to the $L_1$ norm:

$$
\|u(t + \beta_1) - r(t - \beta_2)\|_{L_\infty} \leq \|f\|_{L_1}\|v - r'\|_{L_\infty}
$$

**Proof:** Based on the inverse $Z$-transform of (35), we have:

$$
|u_i(t + \beta_1) - r_i(t - \beta_2)|
$$
where \( f_i \) refers to the \( j \)th element of \( f \), \( * \) denotes the convolution operator, and in the last inequality, we have used the fact that \( \| v - r' \|_{L_\infty} \) is the maximal value of \( |v_j(t) - r_j(t)| \) over \( j \) and over \( t \). Taking the maximum of both sides of the above with respect to \( i \), we get:

\[
\max_i |u_i(t + \beta_1) - r_i(t) - \beta_2| \leq \| f \|_{L_1} \| v - r' \|_{L_\infty}
\]  

and the result follows.

This theorem implies that if \( \| f \|_{L_1} \) is small, then, the DRG-tf will perform similarly to VRG in transient. If, however, \( \| f \|_{L_1} \) is large, no conclusion can be drawn.

### 4. Decoupled Reference Governor based on State Feedback Decoupling: DRG-ss

In this section, we will introduce DRG-ss and its corresponding steady-state and transient analyses. Because DRG-ss uses state feedback decoupling, we assume that all the states are known or measured. If this is not the case, a standard observer can be designed, which we will not discuss in this paper for the sake of brevity. The following assumptions are made for the development of the theory presented in this section:

**A. 4:** Similar to A. 1, system \( G(z) \) in Figure 3 is asymptotically stable.

**A. 5:** \( B^* \) matrix in (6) is nonsingular.

In addition, we assume that the sets \( \mathbb{Y}_i \) satisfy Assumption A. 3 in Section 3.

Consider the system in Figure 3, where we have applied the state feedback decoupling method to get a diagonal system, \( W \), which has state space form \((\bar{A}, \bar{B}, C, 0)\) given by (5). Note that the feedthrough matrix \( D \) is taken to be 0 as discussed in Section 2.1.2, but this assumption can be relaxed. A state-space realisation for each decoupled subsystem, \( W_{ij} \), is given by: \((\bar{A}, \bar{B}(i,:), C(i,:), 0)\), where \( \bar{B}(i,:) \) is the \( i \)th column of \( \bar{B} \), and \( C(i,:) \) is the \( i \)th row of \( C \). Next, for each decoupled subsystem, we compute the MAS, denoted by \( O_{W_{ij}}^{W_{ij}} \), as:

\[
O_{W_{ij}}^{W_{ij}} := \{ (x_{w_{ij}}, v_{ij}) \in \mathbb{R}^{n+1} : x_{w_{ij}} = x_{w}(0), \quad v_{ij}(t) = v_{ij}, \quad t \in \mathbb{Z}_+ \}
\]  

where \( x_{w} \) represents the state of \( W \). Note that in comparison with DRG-tf, which, depending on the observer design method, may use the states of \( W_{ij} \) or \( W \) to create \( O_{W_{ij}}^{W_{ij}} \), DRG-ss uses the states of \( W \) to create \( O_{W_{ij}}^{W_{ij}} \).

As for implementation, the SRGs within DRG-ss compute the inputs, \( v_{ij} \), to the decoupled system the same as (19) and \( \kappa_i \) is computed by the same linear program as (20). Note that, for the identity decoupling method, the construction of MAS is similar to that of DRG-tf with identity method (see (27)); that is, the SRGs can be replaced by a bank of decoupled saturation functions.

Because of the additional feedback loop (i.e. \(-\Phi x\) shown in Figure 3), the stability of DRG-ss is not guaranteed (unlike DRG-tf). Below, we provide a sufficient condition for stability of the DRG-ss scheme.

The block diagram of DRG-ss (Figure 3) can be rearranged as shown in Figure 9, where

\[
Q(z) = G\Phi(z) - G_{\infty}(z)J
\]

and \( G_{\infty}(z) = (I - A)^{-1}B \). From Small Gain Theorem (Khalil, 2002), if there exist four constants \( J_1, J_2, K_1, \) and \( K_2, \) with \( J_1J_2 < 1 \), such that:

\[
\|v\| \leq K_1 + J_1\|r'\|, \quad \|Q(z)v\| \leq K_2 + J_2\|v\|
\]

then, the system is bounded input bounded output stable (i.e. BIBO). While \( \| \cdot \| \) can be chosen to be any signal norm, we use the \( \infty \)-norm in the discussion that follows. Recall that in the SRG optimisation (20), \( \kappa_j \) satisfies 0 \( < \kappa_j \leq 1 \), which implies that:

\[
\|v(t)\| = \|v(t - 1) + K r'(t) - v(t - 1)\|_{\infty} \
\leq \| (I - K)v(t - 1) \|_{\infty} + \| K r'(t) \|_{\infty} \
\leq \| v(t - 1) \|_{\infty} + \| r'(t) \|_{\infty}
\]

where \( K \) is diagonal matrix with \( \kappa_j \) as its main-diagonal elements. Since \( v \) is bounded (because \( O_{W_{ij}}^{W_{ij}} \) is compact, see Gilbert & Tan, 1991), we have that \( \|v(t - 1)\|_{\infty} \leq M \) for some \( M > 0 \). Thus, \( \|v(t)\|_{\infty} \leq M + \|r'(t)\|_{\infty} \) (i.e. \( J_1 = 1, J_2 = M \)). Then, from small gain theorem, the system is BIBO stable if there exist a \( K_2 \) and \( J_2 < 1 \), such that:

\[
\|Q(z)v\|_{\infty} \leq K_2 + J_2\|v\|_{\infty}. \quad \text{Recall that the induced system norm } \|q\|_{L_1} \text{, where } q \text{ is the impulse response matrix of } Q(z), \text{ is defined as: } \|q\|_{L_1} = \sup \|Qv\|_{\infty} \|v\|_{\infty}. \text{ Then, for } J_2 \text{ to exist, the following inequality needs to be satisfied:}
\]

\[
\|q\|_{L_1} < 1
\]

In summary, the DRG-ss scheme is BIBO stable if \( \|q\|_{L_1} < 1 \). It is important to note that \( Q(z) \) depends on \( \Gamma \) and \( \Phi \). Thus, stability needs to be checked after \( \Gamma \) and \( \Phi \) have been designed, which means that iterations might be needed if the stability condition above is not satisfied. Finally, asymptotic stability can also be proved by applying the results from absolute stability (Harris & Valena, 1983) to the system of Figure 9 and using the fact that \( 0 < \kappa_j \leq 1 \).

Next, we provide an example for DRG-ss, where the two decoupling methods in Section 2.1.2 are applied to decouple the
system. Consider the system \( G \) given by:

\[
A = \begin{bmatrix}
0.1 & 1 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0.1
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

We use (7) and (8) to find \( \Phi_1 \) and \( \Gamma_1 \), and proceed to compute \( O_{11}^\infty \) and \( O_{22}^\infty \) based on (27) and (40) (for the identity and pole assignment methods, respectively). Note that for pole assignment method, we choose \( M_k = \text{diag}(0.9, 0.9) \) to locate two of the poles of \( W \) at 0.9. The constraint set is defined as \( Y := \{ (y_1, y_2) : y_1 \leq 2.1, y_2 \leq 1.1 \} \). We simulate the response of this system to a step of size 1 in both \( r_1 \) and \( r_2 \). The simulation results are depicted in Figure 10.

Figure 10 (top) shows that the outputs are within the constraints for both identity and pole assignment methods. Note, from the bottom plots of Figure 10, that there is a gap between \( u \) and \( r \). Later, we will investigate this gap.

As a final remark, similar to the identity method for DRG-tf, while the identity method for DRG-ss is simpler and computationally superior to the pole assignment method, it has a drawback: it may lead to large oscillations for underdamped systems.

### 4.1 Analysis of DRG-ss

In this section, we present steady-state and transient analyses of DRG-ss.

#### 4.1.1 Steady-state analysis

We begin by noting that the definitions of the steady-state half-space for DRG-ss, i.e. \( V_{ss}^W \), and VRG, i.e. \( U_{ss} \), are the same as (29) and (30). Below, we present a theorem to relate \( U_{ss} \) and \( V_{ss}^W \), which parallels Theorem 3.1 for DRG-tf.

**Theorem 4.1:** For the system of Figure 3, and \( U_{ss} \) and \( V_{ss}^W \) defined in (30) and (29), the following relation holds

\[
V_{ss}^W = C(I - \bar{A})^{-1} \bar{B}(C(I - A)^{-1}B)^{-1} \times U_{ss}
\]

where \( \bar{A} = A + B\Phi \) and \( \bar{B} = BG \).

**Proof:** Given the state-space realisation \((A, B, C, 0)\) for \( G(z) \), the DC-gain of \( G \) is given by \( G_0 = C(I - A)^{-1}B \). Similarly, the DC-gain of \( W \) is given by \( W_0 = C(I - (A + B\Phi))^{-1}B\Gamma \). Therefore, the relationship between \( W_0 \) and \( G_0 \) is as follows:

\[
W_0 = C(I - (A + B\Phi))^{-1}B\Gamma(C(I - A)^{-1}B)^{-1} \times G_0
\]

The proof follows from the definitions of \( U_{ss} \) and \( V_{ss}^W \).

This theorem shows that if \( r \) is not admissible with respect to system \( G \) (i.e. \( r \notin U_{ss} \)), then, after feeding through \( \Gamma^{-1} \), \( r' \) must also not be admissible with respect to the system \( W \) (i.e. \( r' \notin V_{ss}^W \)).

Before, we mentioned one requirement for DRG, which was \( u \) and \( r \) should be as close as possible. From Figure 3, we see that
\(v\) and \(r'\) are as close as possible, but \(u\) and \(r\) may not be close. Below, we provide a theorem to quantify the closeness of \(u\) and \(r\) in steady state.

**Theorem 4.2:** For the system of Figure 3, the following relation holds at steady state:

\[
\|\Gamma^{-1}\|_2 \|v - r'\| \leq \|u - r\| \leq \|\Gamma\|_2 \|v - r'\|
\]

where \(\|\cdot\|\) refers to any vector norm and its associated induced matrix norm.

**Proof:** At steady state, we have that \(u = \Gamma v + \Phi x\) and \(r = \Gamma r' + \Phi x\). Therefore:

\[
\|u - r\| = \|\Gamma v - \Gamma r'\| = \|\Gamma(v - r')\| \leq \|\Gamma\|_2 \|v - r'\|.\]

This proves the right hand inequality. To show the left hand inequality, write \(\|v - r'\| = \|\Gamma^{-1}u - \Gamma^{-1}r\| = \|\Gamma^{-1}(u - r)\| \leq \|\Gamma^{-1}\| \|u - r\|\). This can be re-written as \(\|\Gamma^{-1}\|_2 \|v - r'\| \leq \|u - r\|\), which concludes the proof.

The interpretation of this theorem and the steady-state analysis of \(v\) is similar to those in DRG-tf (see Theorem 3.3 and Theorem 3.2), except that instead of having \(\|F_0\|\) and \(F_0^{-1}r\) in DRG-tf, we have \(\|\Gamma\|\) and \(r = \Gamma^{-1}(r - \Phi x)\) in DRG-ss. For the sake of brevity, we will not provide the detailed analysis in this section.

**Remark 4.1:** Similar to DRG-tf, DRG-ss may compute \(u_t\) to be larger or smaller than \(r_i\) depending on the matrix \(\Gamma^{-1}\). Note that \(u_i > r_i\) may or may not be desirable, as we discussed in Section 3.

### 4.1.2 Transient analysis

Recall from Figure 3 that the following relationship holds:

\[
r' = \Gamma^{-1}(r - \Phi x), \quad u = \Gamma v + \Phi x \quad (45)
\]

From these equations, the following theorem emerges, which discusses the transient performance of DRG-ss:

**Theorem 4.3:** For the system in Figure 3, the following inequalities hold:

\[
\|u - r\|_{L_2} \leq \sqrt{\sum_{ij} \Gamma_{ij}^2} \times \|v - r'\|_{L_2} \quad (46)
\]

\[
\|u - r\|_{L_\infty} \leq \max_{ij} \|\Gamma_{ij}\| \times \|v - r'\|_{L_\infty} \quad (47)
\]

where \(\Gamma_{ij}\) is the \(ij\)th element of \(\Gamma\).

**Proof:** From (45), the following equation holds: \(u - r = \Gamma (v - r')\). Then,

\[
\|u - r\|_{L_2} = \|\Gamma (v - r')\|_{L_2} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{m} (\Gamma_{ij}(v(t) - r'(t)))^2\right)^{1/2} \quad (47)
\]

where \(\Gamma_i\) refers to the \(i\)th row of \(\Gamma\). By Cauchy-Schwarz inequality, we have:

\[
\sum_{i=0}^{\infty} \sum_{j=1}^{m} (\Gamma_{ij}(v(t) - r'(t)))^2 \leq \sum_{i=0}^{\infty} \sum_{j=1}^{m} (\Gamma_{ij}(v(t) - r'(t)))^2 \quad (47)
\]

Taking the square root of both sides proves (46). Next, we will show the proof of (47). We have that:

\[
\|u - r\|_{L_\infty} = \|\Gamma (v - r')\|_{L_\infty} \leq \max_{ij} \|\Gamma_{ij}\| \times \|v - r'\|_{L_\infty}
\]

Then, (47) follows.

Theorem 4.2 presents the relationship between \(v - r'\) and \(u - r\) and shows that if the elements of \(\Gamma\) are small, then the distance between \(u\) and \(r\) would also be small. This implies that tracking will not be significantly deteriorated as compared with VRG.

### 5. Computational considerations

In this section, we discuss the computational aspects of DRG and compare the run-time of DRG with VRG. Since the filters \(F(z)\) and \(F^{-1}(z)\) in DRG-tf and the matrix multiplications in DRG-ss can be implemented easily, we will not focus on them. The focus of this section will instead be on the implementation of the SRGs that are used in the DRG formulation. Note that the SRGs in DRG-tf and DRG-ss are the same, so we will only consider DRG-tf in this section.

Recall that the implementation of the DRG on an \(m\)-input \(m\)-output system involves solving \(m\) linear programs (LP), described by (20). These LPs can be solved implicitly via LP solvers, or explicitly as explained below. VRG, on the other hand, requires the solution to a Quadratic Program (QP), which can be solved implicitly via online optimisation or explicitly via multi-parametric programming. In this work, we use the MPT Toolbox in Matlab to implement implicit QP and implicit LP (MPT was the fastest among other solvers such as Gurobi). Also, we use the algorithm that is introduced in Tøndel et al. (2003) to implement explicit QP.

For the explicit DRG mentioned above, we implement Algorithm 1, which provides an algorithm to compute \(e^\cdot\) in (20) for each SRG. For this algorithm, we have assumed that \(O_{\infty}^{ui}\) is given by polytopes of the form (11), and that \(j^u\) denotes the number of rows of \(H_v, H_u, h\). Note that we have used the notation \(H_v\) instead of \(H_{vi}\) because the output of the SRGs in DRG are \(v_i\) and not \(u_i\). In this algorithm, with some abuse of notation, we use \(x\) to refer to the state that is fed back to the \(i\)th SRG (i.e., either the state of the \(i\)th subsystem or the state of the entire system as explain in Section 3.1.2).

To compare the performance of DRG with VRG, we use an example of a distillation process, which is a two-input and two-output coupled system presented in Skogestad
and Postlethwaite (2007). The DRG formulation for this system requires the solution to two LPs, whereas the VRG formulation requires the solution to a single LP. All simulations were performed in Matlab R2017b. The simulation device is a Macbook with 1.1 GHz Intel Core m3 processor and 8 GB memory.

We simulate the distillation process using 4 different governor/solver combinations: explicit DRG (i.e. Algorithm 1), implicit DRG (i.e. implicit LP), explicit VRG (i.e. explicit QP), and implicit VRG (i.e. implicit QP). The simulation length is 10,000 time steps in all cases with a sample time of 0.01s. Upon simulating the system, we compute the average and maximum computation times of the solvers. In order to eliminate the effects of background processes running on the computer, each of the above experiments are run 5 times and the averages are computed. The results are shown in Tables 1 and 2. As can be seen, the average time indicates that the Explicit RG is two orders of magnitude faster than explicit VRG and explicit VRG runs three orders of magnitude faster than the rest of the governors, which means that DRG computation terminates faster than VRG.

6. Robust DRG

In Section 2.2.3, we provided a brief explanation of how SRG can be modified to handle systems affected by unknown disturbances and sensor noise. Essentially, MAS is ‘robustified’ (i.e. shrunk) to account for the worst-case realisation of the disturbances. In this section, we extend these ideas to DRG-tf and DRG-ss, where we show that an initial pre-processing is required to have the system in the form (14). Secondly, we consider the case where the system model is uncertain, where we present an innovative solution for handling these systems.

### 6.1 DRG for systems with unknown disturbances

#### 6.1.1 DRG-tf for systems with unknown disturbances

Suppose system (1) is now affected by an unknown disturbance \( d(t) \in \mathbb{R}^d \):

\[
Y(z) = G(z)U(z) + G_w(z)D(z)
\]

where \( D(z) \) is the \( Z \)-transform of \( d(t) \). Consistent with the literature of SRG, it is assumed that \( d \in \mathbb{D} \), where \( \mathbb{D} \) is a compact polytopic set.

In this section, we consider DRG-tf with the diagonal decoupling method explained in Section 3 (the identity decoupling method can be applied similarly). Under Assumption A.2, we compute the filter \( P(z) \) defined in (2). This leads to each \( y_i \) described by:

\[
Y_i(z) = G_{ii}(z) V_i(z) + \sum_{j=1}^d G_{wj}(z) D_j(z),
\]

which is decoupled from \( v \) to \( y_i \) but not from \( d \) to \( y \). To address this, we convert the dynamics of each \( y_i \) to state-space form:

\[
x_i(t+1) = A_i x_i(t) + B_i v_i(t) + B_{wi} d(t) \\
y_i(t) = C_i x_i(t) + D_{wi} d(t) \in \mathbb{Y}_i
\]

For each subsystem (49), we now proceed to compute the corresponding robust MAS using the procedure described in Section 2.2.3. The implementation of DRG-tf is otherwise unchanged.

#### 6.1.2 DRG-ss for systems with unknown disturbances

In order to decouple system (14) from the inputs \( u \) to the outputs \( y \), we apply the pole assignment decoupling method explained in Section 4; similar results can be obtained for the identity decoupling method. The decoupled system to consider is:

\[
x(t+1) = (A + B\Phi) x(t) + B\Gamma v(t) + B_w d(t) \\
y(t) = C x(t) + D_w d(t) \in \mathbb{Y}
\]

where \( \Phi \) and \( \Gamma \) are computed based on (8), and \( v \) is the input obtained from the SRGs (see Figure 3). The \( i \)th decoupled subsystem can then be written as:

\[
x(t+1) = \tilde{A} x(t) + B_i v_i(t) + B_{wi} d(t) \\
y_i(t) = C_i x(t) + D_{wi} d(t) \in \mathbb{Y}_i
\]

where \( \tilde{A} = A + B\Phi, B_i \) is the \( i \)th column of \( B\Gamma \), \( C_i \) is the \( i \)th row of \( C \), and \( D_w \) is the \( i \)th row of \( D_w \). Based on (51) we create the corresponding robust MAS for the \( i \)th subsystem. The DRG-ss implementation is otherwise unchanged.

Next, we will illustrate the above ideas with two examples, one for DRG-tf and another for DRG-ss. Both examples are necessary in order to highlight the subtleties of the two approaches.

For DRG-tf, we consider the system (48) with \( q = 0.05 \) and

\[
G_w(z) = \begin{bmatrix}
0.2 \\
\frac{(z-0.5)^2(3z+1)}{(2z+1)(z-0.7)^2} \\
0.3
\end{bmatrix}
\]

and the constraints: \(-1.2 \leq y_1 \leq 1.2, -3.9 \leq y_2 \leq 3.9\). We implement DRG-tf for this system assuming the disturbance satisfies \( d \in \mathbb{D} := [-0.1, 0.1] \). For DRG-ss, we consider again system (43) used in Section 4 with the output constraints: \( y_1 \leq
2.1, \( y_2 \leq 1.1 \). Assume \( D_w \) is zero, \( B_w = [1.3, 0.3, 2.51] \top \), and that the disturbance also satisfies \( d(t) \in \mathbb{D} := [-0.1, 0.1] \). We decouple the system using the pole assignment method, placing the closed-loop poles at 0.1. For the purpose of simulations, the disturbance in both cases is generated randomly and uniformly from the interval \([-0.1, 0.1]\).

The results of DRG with disturbance are shown in Figures 11 and 12. In the top subplots of these figures, ‘\( y_1 \) coupled’ and ‘\( y_2 \) Coupled’ refer to the response of the system without DRG (i.e. \( r \) applied to \( G \) directly), which shows that, without a DRG, the constraints are violated. These results confirm that DRG is able to satisfy the constraints in the presence of disturbances. As can be seen from the plots, the disturbance affects both outputs (the outputs appear noisy). Interestingly, the disturbance does not affect \( u \) for DRG-tf (see Figure 11), but it affects \( u \) for DRG-ss (see \( u \) in Figure 12). The reason for this behaviour can be explained as follows: it can be seen from Figure 3 that the outer feedback in DRG-ss may transmit the effects of disturbances and sensor noise to \( r' \). As a result of this, the effect of the disturbance on the output may be higher in DRG-ss than in DRG-tf. This may be a decisive argument to select between DRG-ss and DRG-tf, since the latter does not show this type of behaviour.

**Remark 6.1:** For a system in which the states are not measured, a standard observer may not provide accurate estimation of the state if unknown disturbances affect the system. In such a case, we refer to the work developed in Kalabic (2015), where
an observer which considers the error introduced by unknown disturbances is implemented.

6.2 DRG with parametric uncertainty

In this section, we briefly sketch the approach that can be used for cases when system $G(z)$ in Figures 2 and 3 has parametric uncertainty, that is, matrices $A$ and $B$ are uncertain or vary in time. For simplicity, we assume matrices $C$ is known and $D = 0$. The approach we take is similar to Kerrigan (2001). Note that we consider parametric uncertainties in the state-space matrices, because the RG approach is a time-domain approach. Therefore, frequency domain uncertainties are not investigated. We assume that the uncertainty, that is, matrices $\bar{G}(z)$ is asymptotically stable. Therefore, stability is still not a concern in DRG-tf, but additional analysis must be carried out to ensure stability of DRG-ss. This is similar to our prior discussion in Section 4 so we will not dwell on the issue of stability.

For this discussion, reconsider system $G(z)$, but now with parametric uncertainty on the $A$ and $B$ matrices, which leads to the square linear system given by:

$$x(t + 1) = A(t)x(t) + B(t)u(t)$$
$$y(t) = Cx(t) \in \mathcal{Y}$$  \hspace{1cm} (52)

In Kerrigan (2001), in order to compute the robust MAS for this type of systems, it is assumed that the pair $(A(t), B(t))$ belongs to a given uncertainty polytope defined by the convex hull of the matrices $(A_i, B_i)$, that is

$$(A(t), B(t)) \in \text{conv}\{A_1, B_1, \ldots, A_n, B_n\},$$

where $n$ is the number of vertices in the uncertainty polytope (Pluymers et al., 2005). Applying this idea directly to DRG, however, may not guarantee constraint satisfaction because the parametric uncertainties will prevent us from perfectly decoupling the system. To explain, suppose we select a nominal pair of $A$ and $B$ matrices from the convex hull, and decouple this nominal system by computing the matrices $\Phi$ and $\Gamma$ using (7) or (8). Since the matrices of the actual system will be different from the nominal ones, this decoupling process results in:

$$x(t + 1) = \tilde{A}(t)x(t) + \tilde{B}(t)v(t), \quad y(t) = Cx(t)$$ \hspace{1cm} (53)

where the pair $(\tilde{A}(t), \tilde{B}(t))$ satisfies:

$$\tilde{A}(t), \tilde{B}(t) \in \text{conv}\{A_1, B_1, \ldots, A_n, B_n\}, \hspace{1cm} (54)$$

where $A_1 = A(1), B_1 = B(1)$, $\tilde{A}(t), \tilde{B}(t)$ are not decoupled for all matrices in the uncertainty polytope. This implies that DRG implemented on (53) may not achieve perfect decoupling and thus may not enforce the constraints.

To address the above problem, we introduce a novel margin in each $W_i^\infty$ to robustify each channel against these coupling dynamics. To explain, consider the dynamics of the $i$th output of (53):

$$x(t + 1) = \tilde{A}(t)x(t) + \tilde{B}(t)v_i(t) + B_w(t)v(t)$$
$$y_i(t) = C_i x(t)$$ \hspace{1cm} (55)

where $C_i$ is the $i$th row of $C$, $\tilde{B}(t)$ corresponds to the $i$th column of $\tilde{B}(t)$, $B_w(t)$ gathers all columns of $\tilde{B}(t)$ except the $i$th one, and $\bar{v}(t)$ represents the vector containing all inputs except the $i$th one, i.e. vector of all $v_k$, $k \neq i$. Our solution below treats $\bar{v}$ as an unknown bounded disturbance. To accomplish this, we quantify a lower and an upper bound on $\bar{v}$ and robustify $G_i^\infty$ using results similar to Section 6.1. Specifically, to find the bounds, we leverage the fact that each element of $\bar{v}(t)$, $\bar{v}_k$, is the output of an SRG, whose goal is to enforce the constraints on the $k$th output (i.e. $y_k(t) \in \mathcal{Y}_k$). Thus, we can define upper and lower bounds on each element of $\bar{v}$ using the steady-state constraints (28):

$$\bar{v}_k^\text{max} = \max\{\bar{v}_k : W_{kk}^{ij} \bar{v}_k \in (1 - \epsilon)\mathcal{Y}_k, j = 1, \ldots, N\}$$
$$\bar{v}_k^\text{min} = \min\{\bar{v}_k : W_{kk}^{ij} \bar{v}_k \in (1 - \epsilon)\mathcal{Y}_k, j = 1, \ldots, N\}$$ \hspace{1cm} (56)

where $W_{kk}^{ij}$ represents the DC gain of the system from the $k$th input to the $k$th output given the pair $(\tilde{A}(t), \tilde{B}(t))$. Since we have that each $\bar{v}_k(t) \in [\bar{v}_k^\text{min}, \bar{v}_k^\text{max}]$, we can now treat $\bar{v}(t)$ in (55) as an unknown bounded disturbance to create a robust MAS set for the $i$th channel, which can be accomplished using the ideas from Section 6.1 (for unknown disturbances) and references Kerrigan (2001) and Pluymers et al. (2005) (for polytopic uncertainties). Implementation of DRG using these MAS’s will ensure that the system is robust to the plant/model mismatch and, thus, the constraints will be satisfied. It is important to mention that this approach may lead to conservative results depending on how much the MAS is shrunk. However, if the system is ‘almost’ decoupled (i.e. the nominal system is close to the actual one), then the shrinkage will be negligible. For the sake of brevity, numerical examples and further analysis on this topic will appear in our future work.

7. Extension of DRG to non-square MIMO systems

In this section, we will briefly introduce the extension of DRG to non-square MIMO systems, i.e. systems where the number of inputs is either larger or smaller than the number of outputs. We will treat these cases separately in the following subsections. Generally speaking, we achieve this by either introducing fictitious outputs to transform the system into a square one (see Figure 13), or only decoupling a square subsystem of it (see Figure 14). For the sake of clarity, we will only focus on the extension of DRG-tf with the diagonal method; the same process can be applied to DRG-tf with identity method and DRG-ss.

7.1 Systems with larger number of inputs

Assume that system $G$ in Figure 13 has $m$ inputs and $p$ outputs, with $m > p$:

$$\begin{bmatrix}
Y_1(z) \\
\vdots \\
Y_p(z)
\end{bmatrix} = \begin{bmatrix}
G_{11}(z) & \ldots & G_{1m}(z) \\
\vdots & \ddots & \vdots \\
G_{p1}(z) & \ldots & G_{pm}(z)
\end{bmatrix} \begin{bmatrix}
U_1(z) \\
\vdots \\
U_m(z)
\end{bmatrix}$$ \hspace{1cm} (57)

We transform $G(z)$ into a square system as follows. We manually introduce $m-p$ outputs, $\bar{Y}_{p+1}, \ldots, \bar{Y}_m$, leading to the square
system \( \tilde{G} \), described below:

\[
\begin{bmatrix}
Y_1(z) \\
\vdots \\
Y_p(z) \\
\hat{Y}_{p+1}(z) \\
\vdots \\
\hat{Y}_m(z)
\end{bmatrix}
= \begin{bmatrix} G_{c_1p} & G_{c_p+1,m} \\ G_{1m} \end{bmatrix} \begin{bmatrix} U_1(z) \\
\vdots \\
U_p(z) \\
\hat{U}_{m+1}(z) \\
\vdots \\
\hat{U}_m(z)
\end{bmatrix}
\]

where \( \tilde{G} \) is an \((m - p) \times (m - p)\) transfer matrix representing the fictitious outputs, and \( G_{c_1p} \) and \( G_{c_p+1,m} \) denote the first \( p \) columns of \( G \) and the last \((m - p)\) columns of \( G \), respectively.

Note that the choice of the fake dynamics (i.e. \([0_{m-p}\tilde{G}]\)) in (58) is not unique. The reason we use this structure of \([0_{m-p}\tilde{G}]\) is that \( \tilde{G}^{-1} \) and \( F \) can be easily obtained through block matrix inversion (Lu & Shiu, 2002), and the structure of \( F \) is easy to study, as will be explained below. For the diagonal method in DRG-tf, the decoupled system \( W \) (see Figure 13) is constructed as:

\[
W = \begin{bmatrix} G_{11}(z) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0(z) & \cdots & G_{pp}(z) \\
W_p & \cdots & \cdots \end{bmatrix}
\]

where \( \tilde{G}_w \) is a \((m - p) \times (m - p)\) transfer function matrix that is chosen such that it has a stable inverse, so that \( F^{-1} \) can be computed (see (2)). Recall that the true outputs of the system are \( Y_1, \ldots, Y_p \) and the constraints are on these outputs. So, as Figure 13 shows, only \( p \) different SRGs are needed to ensure these outputs satisfy the constraints and there is no need to design SRGs for \( \hat{G}_w \). Finally, \( F^{-1} \) is introduced to ensure that \( u \) is close to \( r \), as before. By choosing \( \tilde{G} \) and \( W \) as shown in (58) and (59), \( F \) can be written as:

\[
F = \begin{bmatrix} \tilde{G}_{c_1p}^{-1} W_p \\
0_{m-p,p} \end{bmatrix}
\]

Note that if we choose \( \tilde{G} \) to be equal to \( \tilde{G}_w \), then \( \tilde{G}^{-1} \tilde{G}_w \) in (60) will become an identity matrix, which means that \( F \) is unrelated to the choice of \( \tilde{G} \). Of course, for this to hold, \( \tilde{G} \) needs to be invertible to ensure that (60) exists.

Remark 7.1: As can be seen from (60), if \( \tilde{G} \neq \tilde{G}_w \), then \( F \) is related to both \( \tilde{G} \) and \( \tilde{G}_w \). This implies that a proper set of \( \tilde{G} \) and \( \tilde{G}_w \) can be chosen such that the norm of \( F \) is small, which as discussed in Section 3.3, will lead to a small distance between \( u \) and \( r \) (see Figure 13) and, hence, good tracking performance.

Since \( G(z) \) has been transformed into a square system, the same analysis presented in Section 3.3 can be applied to study the steady-state and transient performance of DRG-tf for non-square systems. Hence, we will not repeat this analysis.

7.2 Systems with larger number of outputs

Assume system \( G(z) \) in Figure 14 has \( m \) inputs and \( p \) outputs, with \( p > m \). Instead of decoupling the entire \( G(z) \) as done in Section 3, only a square subsystem of \( G \) is decoupled. Without loss of generality, we assume that the square subsystem corresponds to the first \( m \) outputs of \( G \), but the method can be applied to other square subsystems as well. Let us denote the \( m \times m \) square subsystem of \( G \) as \( G_m \). Same as DRG-tf for square systems (see Section 3.1), \( F \) is designed to decouple \( G_m \), resulting in the diagonal subsystem, \( W_m \), shown below:

\[
W_m = \begin{bmatrix} G_{m11}(z) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0(z) & \cdots & G_{mmn}(z)
\end{bmatrix}
\]
Then, the whole system $W$ (i.e. $GF$) can be described by:

$$
W = \begin{bmatrix}
W_m \\
FG_{m+1,p} \\
W_p
\end{bmatrix}
$$

where $G_{m+1,p}$ represents the last $(p - m)$ rows of $G$.

As can be seen from Figure 14, we design one DRG (which contains $m$ decoupled SRGs) for $W_m$ to ensure that the outputs $y_1, \ldots, y_m$ satisfy the constraints. Then, we design a single SRG for $W_p$ to make sure that the outputs $y_{m+1}, \ldots, y_p$ satisfy the constraints. The challenge is that two sets of $v$’s are computed: one by the DRG and one by the SRG (as shown in Figure 14). Thus, the question is, how can the two sets of $v$’s be ‘fused’ together while satisfying the constraints on all outputs. There are several ways to accomplish this task. The easiest solution is to select the smallest $\kappa$ among the $m+1$ different $\kappa$’s ($\kappa$ is calculated based on (10)), denoted as $\tilde{\kappa}$, that is:

$$
\tilde{\kappa} = \min(\kappa_1, \ldots, \kappa_{m+1})
$$

and the update law for $v$ becomes:

$$
v(t + 1) = v(t) + \tilde{\kappa}(r'(t) - v(t)).
$$

With the above $\tilde{\kappa}$, the convexity of the maximal admissible sets (MAS) guarantees that the constraints for all outputs are satisfied and the solutions from the DRG and SRG are unified. However, the response of this approach may be conservative compared to the identity method. An alternative way to fuse the $v$’s is as follows. First, denote the set of $v$’s given by the SRG (see Figure 14) as $v_G$ and the set of $v$’s given by the DRG as $v_D$. We solve an RG-like LP (see (10)) to find the point in $O_{\infty}^{W_m}$ that is closest to $v_G$ (recall that $O_{\infty}^{W_m}$ refers to the MAS for $W_m$), denoted as $v_1$. Similarly, we solve another LP to find the closest point to $v_D$ in $O_{\infty}^{W_p}$ where $O_{\infty}^{W_p}$ represents the MAS for $W_p$, denoted as $v_2$. Note that $v_1$ and $v_2$ are both constraint-admissible for all outputs since they are in $O_{\infty}^{W_p}$ and $O_{\infty}^{W_m}$ at the same time.

Finally, we choose the actual set of $v$’s that is applied to $F(z)$ as:

$$
v = \begin{cases}
  v_1 & \text{if } \|r' - v_1\| \leq \|r' - v_2\| \\
  v_2 & \text{otherwise}
\end{cases}
$$

and introduce $F^{-1}$ to ensure that $u$ is close to $r$, as before. If choosing $v$ above, the constraints for all outputs are guaranteed to be satisfied. However, computational burden of this approach is higher than that for the first approach (i.e. select the smallest $\kappa$ among all $\kappa$’s) since two more LPs are required.

8. Conclusion

In this work, a method for constraint management of coupled MIMO systems was studied. The method is referred to as the Decoupled Reference Governor (DRG) and is based on decoupling the input-output dynamics, followed by the application of Scalar Reference Governors to each decoupled channel. We presented the DRG formulation with two different decoupling techniques based on transfer functions and state-space (i.e. DRG-tf and DRG-ss) and demonstrated the applicability of the method as a function of the singular values of the system and the decoupling matrix. We presented steady-state and transient analyses of the DRG and compared the computation time of DRG with Vector Reference Governors (VRG). It was shown that DRG can run faster than VRG by two orders of magnitude. Unknown disturbances and parametric uncertainties were also addressed. Finally, extensions to non-square MIMO systems were presented.

A notable contribution of our paper is the study of the similarities and differences between DRG-tf and DRG-ss, which we now summarise. The similarities are as follows: first, both methods guarantee constraint satisfaction for all time and may have performance comparable to VRG. Second, both may lead to values of $u(t)$ that are larger than $r(t)$, which may or may not be acceptable depending on the specific application. Third, both may lead to oscillations if the identity decoupling method is implemented, although the identity method is computationally superior. The differences are as follows: first, the outer feedback in DRG-ss may transmit the effects of disturbances and sensor noise to $r'$. This is not the case with DRG-tf. Second, DRG-tf is guaranteed to be stable, whereas stability of DRG-ss must be investigated a posteriori. Third, for DRG-ss, the tracking performance is affected by the decoupling feedback gain, which is not the case with DRG-tf. In sum, depending on the application and the design requirements, an appropriate DRG variant can be chosen by considering the above points.

Future work will explore modifications of DRG to ensure that the inputs to the closed-loop system (i.e. $u$ in Figure 2) remain below the references (i.e. $r$). We will also explore DRG formulations that have the ability to recover from constraint violations, should unknown disturbances or observer errors push the system outside of the maximal admissible sets. Finally, we will study the extension of DRG to nonlinear systems.

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

Yudan Liu http://orcid.org/0000-0002-7496-7033

Hamid R. Ossareh http://orcid.org/0000-0002-4964-569X

References

Áström, K. J., & Hägglund, T. (1995). *PID controllers: Theory, design, and tuning* (Vol. 2). Instrument Society of America Research Triangle Park.

Bemporad, A., Borrelli, F., & Morari, M. (2002). Model predictive control based on linear programming - the explicit solution. *IEEE Transactions on Automatic Control*, 47(12), 1974–1985. https://doi.org/10.1109/TAC.2002.805688

Burl, J. B. (1998). *Linear optimal control: H2 and H∞ methods Linear optimal control: H2 and H∞ methods*. Addison-Wesley Longman Publishing Co., Inc.

Camponogara, E., Jia, D., Krog, B. H., & Talukdar, S. (2002). Distributed model predictive control. *IEEE Control Systems*, 22(1), 44–52. https://doi.org/10.1109/79.980246

Elliott, M. S., & Rasmussen, B. P. (2013). Decentralized model predictive control of a multi-evaporator air conditioning system. *Control Engineering Practice*, 21(12), 1665–1677. https://doi.org/10.1016/j.conengprac.2013.08.010
Falb, P. L., & Wolovich, W. A. (1967). Decoupling in the design and synthesis of multivariable control systems. IEEE Transactions on Automatic Control, 12(6), 651–659. https://doi.org/10.1109/TAC.1967.1098737

Garelli, F., Manz, R., & De Battista, H. (2006). Limiting interactions in decentralized control of MIMO systems. Journal of Process Control, 16(5), 473–483. https://doi.org/10.1016/j.jprocont.2005.09.001

Garone, E., Di Cairano, S., & Kolmanovsky, I. (2017). Reference and command governors for systems with constraints: A survey on theory and applications. Automatica, 75(9), 306–328. https://doi.org/10.1016/j.automatica.2016.08.013

Ge, S. S., & Li, Z. (2014). Robust adaptive control for a class of MIMO nonlinear systems by state and output feedback. IEEE Transactions on Automatic Control, 59(6), 1624–1629. https://doi.org/10.1109/TAC.2013.2294826

Gilbert, E. G., & Kolmanovsky, I. (1998). Theory and computation of disturbance invariant sets. In Proceedings of the 1998 American Control Conference - ACC'98 (Vol. 3, pp. 1995–1999). IEEE.

Herceg, M., Kvasnica, M., Jones, C., & Morari, M. (2013). Multi-parametric toolbox 3.0. In Proc. of the European control conference. IEEE.

Kolmanovsky, I., & Gilbert, E. G. (1995). Fast reference governors for systems with state and control constraints and disturbance inputs. International Journal of Robust and Nonlinear Control, 9(15), 1117–1141. https://doi.org/10.1002/(SICI)1099-1239(19991230)9:15<1117::AID-RNC447>3.0.CO;2-1

Kolmanovsky, I., & Gilbert, E. G. (1999). Fast reference governors for systems with state and control constraints and disturbance inputs. International Journal of Robust and Nonlinear Control, 9(15), 1117–1141. https://doi.org/10.1002/(SICI)1099-1239(19991230)9:15<1117::AID-RNC447>3.0.CO;2-1

Kolmanovsky, I., & Gilbert, E. G. (1995). Maximal output admissible sets. IEEE Transactions on Automatic Control, 40(3), 489–497. https://doi.org/10.1109/9.383532

Harris, C. J., & Valença, J. E. (1983). The stability of input-output dynamical systems. Mathematics in science and engineering (Vol. 168). Academic Press.

Herceg, M., Kvasnica, M., Jones, C., & Morari, M. (2013). Multi-parametric toolbox 3.0. In Proc. of the European control conference. IEEE.

Kalabic, U. (2015). Reference governors: Theoretical extensions and practical applications.

Kerrigan, E. C. (2001). Robust constraint satisfaction: Invariant sets and predictive control [Unpublished Doctoral Dissertation]. University of Cambridge.

Khalil, H. K. (2002). Nonlinear systems. Prentice Hall.

Kolmanovsky, I., Garone, E., & Di Cairano, S. (2014). Reference and command governors: A tutorial on their theory and automotive applications. In American Control Conference (ACC) (pp. 226–241). IEEE.

Kolmanovsky, I., & Gilbert, E. G. (1995). Maximal output admissible sets for discrete-time systems with disturbance inputs. In Proceedings of 1995 American Control Conference - ACC'95 (Vol. 3, pp. 1995–1999). IEEE.

Kolmanovsky, I., & Gilbert, E. G. (1998). Theory and computation of disturbance invariant sets for discrete-time linear systems. Mathematical Problems in Engineering, 4(4), 317–367. https://doi.org/10.1155/S1024123X9800086X

Liu, Y., Osorio, J., & Ossareh, H. (2018). Decoupled reference governors for multi-input multi-output systems. In 2018 IEEE Conference on Decision and Control (CDC) (pp. 1839–1846). IEEE.

Lloyd, S. (1970). Decoupling a multivariable discrete-time system. Electronics Letters, 6(26), 831. https://doi.org/10.1049/el:19700572

Lu, T. T., & Shiu, S. H. (2002). Inverses of 2x2 block matrices. Computers and Mathematics with Applications, 43(1–2), 119–129. https://doi.org/10.1016/S0898-1221(01)00278-4

MacFarlane, A. G. J. (1970). Commutative controller: A new technique for the design of multivariable control systems. Electronics Letters, 6(5), 121–123. https://doi.org/10.1049/el:19700083

MacFarlane, A., & Hung, Y. (1983). A quasi-classical approach to multivariable feedback systems design. In Computer Aided Design of Multivariable Technological Systems (pp. 43–52). Elsevier.

McDonald, J., & Pearson, J. (1991). II-optimal control of multivariable systems with output norm constraints. Automatica, 27(2), 317–329. https://doi.org/10.1016/0005-1098(91)90080-L

Osorio, J., & Ossareh, H. R. (2018). A stochastic approach to maximal output admissible sets and reference governors. In 2018 IEEE Conference on Control Technology and Applications (CCTA) (pp. 704–709). IEEE.

Osorio, J., Santillo, M., Seeds, J. B., Jankovic, M., & Ossareh, H. R. (2019). A reference governor approach towards recovery from constraint violation. In 2019 American Control Conference (ACC) (pp. 1779–1785). IEEE.

Phybers, R., Rossiter, J., Suykens, J., & De Moor, B. (2005). The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In Proceedings of the 2005, American Control Conference (pp. 804–809). IEEE.

Scattolini, R. (2009). Architectures for distributed and hierarchical model predictive control—A review. Journal of Process Control, 19(5), 723–731. https://doi.org/10.1016/j.jprocont.2009.02.003

Scokaert, P. O., & Rawlings, J. B. (1998). Constrained linear quadratic regulation. IEEE Transactions on Automatic Control, 43(8), 1163–1169. https://doi.org/10.1109/9.704994

Shah, G., & Engell, S. (2011). Tuning MPC for desired closed-loop performance for MIMO systems. In Proceedings of the 2011 American Control Conference (pp. 4404–4409). IEEE.

Silverman, L. (1970). Decoupling with state feedback and precompensation. IEEE Transactions on Automatic Control, 15(4), 487–489. https://doi.org/10.1109/TAC.1970.1099504

Skogestad, S., & Postlethwaite, I. (2007). Multivariable feedback control: Analysis and design (Vol. 2). Wiley New York.

Sun, Z. Y., Peng, Y., Wen, C., & Chen, C. C. (2020). Fast finite-time adaptive stabilization of high-order uncertain nonlinear system with an asymmetric output constraint. Automatica, 121, 109170. https://doi.org/10.1016/j.automatica.2020.109170

Tee, K. P., Ge, S. S., & Tay, E. H. (2009). Barrier Lyapunov functions for the control of output-constrained nonlinear systems. Automatica, 45(4), 918–927. https://doi.org/10.1016/j.automatica.2008.11.017

Tondel, P., Johansen, T. Å., & Bemporad, A. (2003). An algorithm for multi-parametric quadratic programming and explicit MPC solutions. Automatica, 39(3), 489–497. https://doi.org/10.1016/S0005-1098(02)00250-9

Wang, W., Rivera, D. E., & Kempf, K. G. (2003). Centralized model predictive control strategies for inventory management in semiconductor manufacturing supply chains. In Proceedings of the 2003 American Control Conference (Vol. 1, pp. 585–590). IEEE.

Zhou, K., Doyle, J. C., & Glover, K. (1996). Robust and optimal control (Vol. 40). Prentice Hall.