Decoherence of number states in phase-sensitive reservoirs

Alessio Serafini, Fabrizio Illuminati and Silvio De Siena

Dipartimento di Fisica “E. R. Caianiello”, Università di Salerno, INFM UdR Salerno, INFN Sez. Napoli, Gruppo Collegato di Salerno, Via S. Allende, 84081 Baronissi (SA), Italy

The non-unitary evolution of initial number states in general Gaussian environments is solved analytically. Decoherence in the channels is quantified by determining explicitly the purity of the state at any time. The influence of the squeezing of the bath on decoherence is discussed. The behavior of coherent superpositions of number states is addressed as well.

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1 Introduction

Recent developments in experimental cavity Quantum Electrodynamics and controlled atom-photon interactions allow for a direct investigation of deeply quantum mechanical configurations of the light field. In particular, the deterministic production of low-order number states seem to be at hand, by means of micromaser techniques in high-Q cavities [1] and of strong coupling with trapped atoms [2]. The probabilistic generation of number states via conditional measurements and post-selection [3] has been demonstrated as well, by exploiting parametric down conversion and low multiplication noise detectors [4]. A further possibility to generate number states with high fidelities by atom-field interactions in high-Q cavities has been recently suggested [5]. We also mention that, as for motional degrees of freedom, effective techniques to create number states have been developed and mastered [6]. Moreover, the upcoming VLPC (‘visible light photon counter’) technology holds promising perspectives about the actual possibility of selectively detect low-order number states [7].

Besides being probes of fundamental quantum mechanical features, Fock states of the electromagnetic field are needed in several quantum information applications whenever, for instance, reliable single–photon pulses are required [8, 9]. These possibilities bring to the attention the problem of preserving the quantum mechanical properties of number states, which are unavoidably corrupted by environmental decoherence. Indeed, their inherent non–classical nature makes such states especially fragile and difficult to maintain. More specifically, the numerical analysis strongly suggests that the very possibility of generating pure number states is seriously hampered by environmental decoherence, even in high-Q cavity settings [10]. In this paper we study the rate of decoherence of initial number states in general Gaussian noisy channels, i.e. in presence of dissipation in general Gaussian environments. The dynamical behavior of the system is described by the quantum optical master equation, allowing for arbitrary phase–sensitive (‘squeezed’) baths. The dynamics will be solved in terms of the symmetrically ordered characteristic function, while decoherence will be quantified by computing the purity $\mu = \text{Tr} \varrho^2$ during the evolution of the state.

2 Solving the master equation

Let us consider a denumerable, infinite-dimensional Hilbert space $\mathcal{H}$, spanned by a Fock basis $\{|n\>$, with $n \in \mathbb{N}$, of eigenstates of the hermitian operator $\hat{n} = a^\dagger a$. The annihilation and creation operators $a$ and $a^\dagger$ satisfy the canonical commutation relation $[a, a^\dagger] = 1$. We define the quadrature operators $\hat{x} = (a + a^\dagger)/\sqrt{2}$ and $\hat{p} = -i(a - a^\dagger)/\sqrt{2}$ describing, for instance, amplitude and phase quadratures of a single mode of the electromagnetic field, or position and momentum operators of a material harmonic oscillator.

Any quantum state of this system can be described either by its density matrix $\varrho$ or by its symmetrically ordered characteristic function $\chi(\alpha)$ [11], defined as

$$\chi(\alpha) = \text{Tr}(\varrho D_\alpha) ,$$

where $D_\alpha = \exp(\alpha a^\dagger - \alpha^* a)$ is the unitary displacement operator. In the following we will make use of phase space variables $x$ and $p$, defined by $\alpha = (x + ip)/\sqrt{2}$. Moreover, it is useful to define the covariance matrix $\sigma$, associated to a state $\varrho$ by

$$\sigma_{ij} = \frac{1}{2} \langle \hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i \rangle - \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle ,$$

with $\hat{x}_1 = \hat{x}, \hat{x}_2 = \hat{p}$ and $\langle O \rangle = \text{Tr}(\varrho O)$ for the operator $O$. The dynamics we will study can be modeled by the coupling with a continuum of oscillators, described by the
following interaction Hamiltonian

\[ H_{\text{int}} = \hbar \int [W(\omega) a^\dagger b(\omega) + W^*(\omega) a b^\dagger] \, d\omega , \]

(2)

where \( b(\omega) \) stands for the annihilation operator of the bath mode labeled by the variable \( \omega \), whereas \( W(\omega) \) represents the coupling. The state of the bath has been assumed to be stationary. Under the Markovian approximation, such a coupling gives rise to a time evolution ruled by the following master equation (in the interaction picture) \[12\]

\[ \dot{\rho} = \gamma \left( N L[a^\dagger] \rho + (N + 1) L[a] \rho - M^* D[a^\dagger] \rho + M D[a^\dagger] \rho \right) , \]

(3)

where the dot stands for time–derivative, the Lindblad superoperators are defined as \( L[a] \rho = 2O a^\dagger O \rho - \rho O^\dagger O \) and \( D[a] \rho = 2O \rho O - OO^\dagger \rho - \rho OO^\dagger \), the coupling is \( \gamma = 2\pi W^2(0) \), while the coefficients \( N \) and \( M \) are defined in terms of the correlation functions \( \langle b^\dagger(0)b(\omega) \rangle = N \delta(\omega) \)

and \( \langle b(0)b(\omega) \rangle = M \delta(\omega) \), where averages are computed over the state of the bath. The requirement of positivity of the density matrix imposes the constraint \( |M|^2 \leq N(N + 1) \). At thermal equilibrium, i.e. for \( M = 0 \), \( N \) coincides with the average number of thermal photons in the bath. If \( M \neq 0 \) then the bath is said to be ‘squeezed’, or phase-sensitive, entailing reduced fluctuations in one field quadrature. A squeezed reservoir may be modeled as the interaction with a bath of oscillators excited in squeezed quadrature. A squeezed reservoir may be modeled as the interaction with a bath of oscillators excited in squeezed quadrature. A squeezed reservoir may be modeled as the interaction with a bath of oscillators excited in squeezed quadrature.

Decoherence of number states

Decoherence of the initial pure state in the channel will be quantified by following the evolution of the purity \( \mu = \text{Tr} \rho^2 \). Such a quantity properly describes the degree of mixedness of a quantum state \( \rho \). For continuous-variable (CV) systems it takes the value 1 for pure states (represented by normalized projectors) and the value zero for maximally mixed states. The conjugate of \( \mu \) is referred to as the ‘linear entropy’ \( S_L \) in information theory: \( S_L = 1 - \mu \).

The purity of a quantum state of a single-mode CV system is easily computed as an integral over the whole phase space \[13\]

\[ \mu = \frac{1}{2\pi} \int_R \int_R |\chi|^2 \, dx \, dp . \]

(14)
The generalization of Eq. (12) to multi-mode systems is straightforward, and allows to track the dynamics in noisy channels of entangled two-mode Gaussian states [16] and of Schrödinger cat-like states [17]. Moreover, for CV systems, it has been recently proved that knowledge of the global and marginal purities provides a strong and experimentally reliable characterization of the entanglement of arbitrary two-mode Gaussian mixed states [18]. In the present instance, Eqs. (12) and (13) allow to compute the purity \( \mu_n(t) \) of an initially pure number state of order \( n \) evolving in the Gaussian channel.

### 3.1 Thermal bath

The instance of a reservoir at thermal equilibrium corresponds to the choice \( M = 0 \). In this case, both the environmental Gaussian state and the initial condition are rotationally invariant in phase space, see Eqs. (4) and (10), so that polar coordinates constitutes a suitable and convenient choice. Employing the variable \( s = e^{-\gamma t}|\alpha|^2 \), one gets

\[
\mu_n(t) = e^{\gamma t} \int_0^\infty L_n^2(s) e^{-\left(e^{\gamma t}(2N+1) - 2N\right)s} \, ds. \tag{15}
\]

Which can be analytically solved, using the relation [19]

\[
\int_0^\infty e^{-ax} L_n^2(x) \, dx = (a - 2)^n P_n \left(1 + \frac{2}{a^2 - 2a}\right),
\]

where \( P_n \) is the Legendre polynomial of order \( n \)

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,
\]

to find

\[
\mu_n(t) = e^{\gamma t} \frac{\xi - 2}{\xi + 1} P_n \left(1 + \frac{2}{\xi^2 - 2\xi}\right), \tag{16}
\]

with \( \xi = e^{\gamma t}(2N + 1) - 2N. \tag{17} \)

Eq. (16) provides the exact evolution of the purity of an initial number state \( |n\rangle \) in a thermal channel, fully determined by its mean photon number \( N \). Quite clearly, the purity in the channel is a decreasing function of \( N \). It is a decreasing function of \( n \) as well: higher order number states are more fragile and decohere faster. Moreover, number states with \( n > 0 \) can show a local minimum of the purity and a partial revival up to the asymptotic purity \( \mu_\infty = 1/(2N + 1) \).

### 3.2 Squeezed bath

We will now deal with the general instance \( M \neq 0 \). Due to the rotational symmetry of the characteristic functions of number states, we are free to choose any direction of squeezing of the environmental state of the channel. Indeed, the rate of decoherence can only depend on the module \( |M| \) of the parameter \( M \), which can therefore be chosen real and positive (corresponding to the choice \( \varphi = 0 \)), without loss of generality. Making such a choice and exploiting [19]

\[
\int_0^{2\pi} e^{p \cos \varphi} \, d \varphi = 2\pi I_0(|p|)
\]

(where \( I_0(x) = J_0(ix) = \sum_{k=0}^\infty \frac{x^{2k}}{(2k)!} \) is the zero order modified Bessel function of the first kind), one eventually finds

\[
\mu_n(t) = e^{\gamma t} \int_0^\infty e^{-\xi L_n(s)} I_0 \left(|M| (e^{\gamma t} - 1)s\right) \, ds. \tag{18}
\]

Such an integral cannot be further simplified, but can be numerically estimated to analyze the effect of the squeezing of the bath on the decoherence of number states. Eq. (18) obviously reduces to Eq. (16) for \( M = 0 \). Besides, the asymptotic expressions can be analytically integrated to yield

\[
\lim_{t \to \infty} \mu_n(t) = \mu_\infty = \frac{1}{\sqrt{(sN + 1)^2 - 4|M|^2}},
\]

which, according to Eq. (5), is the asymptotic purity in the channel, fixed by the reservoir state, irrespective of the chosen initial condition.

Eq. (18) shows that \( \mu_n \) is an increasing function of \( |M| \). However, the dependence on squeezing has to be analyzed by considering the parameters \( \mu_\infty \) and \( r \) instead of \( N \) and \( M \), because they permit to study the effect of squeezing (quantified by \( r \)) at given asymptotic purity \( \mu_\infty \). Such a dependence can be reconstructed inserting Eqs. (4) (6) into

![Figure 1: Evolution of purity for initial number states in Gaussian channels with \( \mu_\infty = 0.5 \) (purity revivals are evident in the plot). The solid line refers to state \( |1\rangle \) in a non squeezed bath; the dotted line refers to state \( |2\rangle \) in a non squeezed bath; the dot–dashed line refers to state \( |1\rangle \) in a bath with \( r = 1 \); the dashed line refers to state \( |2\rangle \) in a bath with \( r = 1 \).](image-url)
Eq. (18) and turns out to be quite involved. Anyway, a numerical analysis has been carried out, and is summarized in Fig. 1 where the evolution at short times \((t \leq \gamma)\) is considered. Note that this is the interesting time range, in which decoherence takes place before the system is driven towards the squeezed thermal state of the environment. Such an analysis definitely shows that squeezing the bath does not slow down the decoherence rate of number states in Gaussian channels. At a given asymptotic purity \(\mu_{\infty}\) the highest purity is maintained for \(r = 0\).

## 4 Coherent superpositions

We now consider the behavior of initial coherent superpositions of number states evolving in a general Gaussian noisy channel. To properly exemplify the decoherence of such states, we focus on the simplest coherent normalized superposition \(|\psi_{01}\rangle = ([0] + e^{i\varphi}[1])/\sqrt{2}\) (which constitutes a ‘microscopic Schrödinger cat’). The characteristic function \(\chi_{01}\) of this state is simply found [11]

\[
\chi_{01}(\alpha) = \frac{e^{-|\alpha|^2}}{2} \left[ 2 - e^{-2\gamma t} |\alpha|^2 - e^{-2\gamma t} (\alpha^* e^{-i\varphi} - \alpha e^{i\varphi}) \right]
\]  

(19)

Inserting \(\chi_{01}\) as the initial condition in Eq. (12) and performing the integration of Eq. (14) yields, for the purity of the initial cat-like state evolving in the channel

\[
\mu_{01}(t, r) = 4\nu - e^{-2\gamma t} \frac{\mu_{\infty}^2}{2\mu_{\infty}} \left( \mu_{\infty} + (e^{\gamma t} - 1)(\cosh(2r) \right.
\]

\[
+ \cos(2\varphi - 2\varphi_0) \sinh(2r)) \right)
\]

\[
+ e^{-4\gamma t} \frac{\sqrt{5}}{2\mu_{\infty}^2} \left( 4\mu_{\infty}^2 + 8(e^{\gamma t} - 1)\mu_{\infty} \cosh(2r) \right)
\]

\[
+ (e^{\gamma t} - 1)^2 \left( 3 \cosh(4r) + 1 \right)
\]

(20)

where

\[
\nu = \left[ \frac{1}{\mu_{\infty}^2} (1 - e^{-\gamma t})^2 + e^{-2\gamma t} + 2 \frac{1}{\mu_{\infty}} \cosh(2r) \right]^{-1/2}
\]

(21)

is the purity of an initial vacuum in the channel [15]. Eq. (20) shows that, quite interestingly, the evolution of the coherent superposition is sensitive to the phase \(\varphi\) of the bath. It is straightforward to see that the optimal choice maximizing purity at any given time is provided by \(\varphi = \varphi + \pi/2\). Fixing such a choice, we have numerically analyzed the dependence of \(\mu_{01}\) on \(r\) for small squeezing parameters, the purity \(\mu_{01}\) does increase with \(r\). The optimal choice for \(r\) depends on time, for \(\gamma t = 0.5\) it turns out to be \(r \approx 0.28\). The relative increase in purity is plotted in Fig. 2 as a function of time, for various choices of the squeezing parameter \(r\).

![Figure 2: The relative increase in purity, defined by \(\Delta\mu/\mu = (\mu_{01}(t, r) - \mu_{01}(t, 0))/\mu_{01}(t, 0)\), as a function of time during the evolution of the superposition |\psi_{01}\rangle in Gaussian channels. The optimal condition \(\varphi = \varphi + \pi/2\) is always assumed, while \(\mu_{\infty} = 0.5\). The solid line refers to a bath with \(r = 0.28\), close to the optimal value; the dotted line refers to a bath with \(r = 0.4\) and the dot-dashed line refers to a bath with \(r = 0.1\).](image)

## 5 Comments and Conclusions

We have analytically solved the non-unitary evolution of number states in Gaussian noisy channels, and quantitatively estimated their decoherence. The dissipative model we have considered covers a variety of physical situations, from thermal dissipation of stationary modes in optical cavities to corruption of traveling waves in lossy fibers. In particular, we have straightforwardly shown that Fock states of higher order decohere faster and that squeezed baths do not help to preserve the quantum coherence of initially pure number states. On the other hand, when considering coherent superpositions of number states, we have shown that squeezed reservoirs can help to slow down decoherence, provided that the phase of the bath is optimally locked to the coherent phase of the superposition, and that the intensity of the squeezing is properly chosen. This suggests, looking towards practical implementations, that feedback schemes (simulating squeezed reservoirs by quantum non-demolition measurements) could indeed be helpful in prolonging the lifetime of coherent superpositions of Fock states, i.e. number cat-like states. Moreover, it is worth noticing that the same effect on the preservation of the purity of the superposition, warranted by squeezing the bath, can be obtained by an opposite squeezing of the initial state. In thermal baths, a superposition of squeezed number states proves to be more robust against decoherence than a superposition of non-squeezed number states.

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