Variable-Length Lossy Compression Allowing Positive Overflow and Excess Distortion Probabilities

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Abstract—This paper investigates the problem of variable-length lossy source coding. We deal with the case where both the excess distortion probability and the overflow probability of codeword length are less than or equal to positive constants. The infimum of the thresholds on the overflow probability is characterized by a smooth max entropy-based quantity. Both non-asymptotic and asymptotic cases are analyzed. To show the achievability results, we do not utilize the random coding argument but give an explicit code construction.

I. INTRODUCTION

The problem of variable-length lossy source coding is one of the fundamental research topics in Shannon theory. For this problem, several studies have adopted the excess distortion probability as a distortion criterion (e.g., [4], [5], [13], [12]). The excess distortion probability is defined as the probability that the distortion between a source sequence and its reproduction is greater than a certain threshold.

For the problem of variable-length lossy source coding under the excess distortion probability, there are mainly two criteria on codeword length: the mean codeword length and the overflow probability. Kostina et al. [4] have considered the mean codeword length and shown the non-asymptotic characterization on the optimal mean codeword length. They also have performed the asymptotic analysis on the optimal mean codeword length for i.i.d. sources. On the other hand, Yagi and Nomura [13] and Nomura and Yagi [5] have considered the overflow probability. In [13], they have treated the case where either the overflow probability or the excess distortion probability is less than or equal to a positive constant asymptotically for general sources. In [5], they have dealt with the case where the probability of union of events that the overflow occurs and the excess distortion occurs is less than or equal to a positive constant asymptotically for general sources.

This paper considers the excess distortion probability and the overflow probability as in [5] and [13]. However, the primary differences are 1) we address the case where both the excess distortion probability and the overflow probability may be positive and 2) we analyze both non-asymptotic and asymptotic cases.

The contribution of this paper is the non-asymptotic (one-shot) and asymptotic characterizations on the minimum threshold of the overflow probability by using a new smooth max entropy-based quantity. In the non-asymptotic regime, coding theorems are shown for both stochastic and deterministic encoders. To show the achievability results, we give an explicit code construction instead of using the random coding argument. It turns out that the constructed code satisfies the properties of the optimal code shown in [4]. Further, using the results obtained in the non-asymptotic regime, we establish asymptotic coding theorem for general sources.

II. ONE-SHOT CODING THEOREM

A. Problem Formulation

Let $\mathcal{X}$ be a source alphabet and $\mathcal{Y}$ be a reproduction alphabet, where both are finite sets. Let $X$ be a random variable taking a value in $\mathcal{X}$ and $x$ be a realization of $X$. The probability distribution of $X$ is denoted as $P_X$. A distortion measure $d$ is defined as $d: \mathcal{X} \times \mathcal{Y} \to [0, +\infty)$.

The pair of an encoder and a decoder $(f, g)$ is defined as follows. An encoder $f$ is defined as $f: \mathcal{X} \to \{0, 1\}^*$, where $\{0, 1\}^*$ denotes the set of all binary strings and the empty string $\lambda$, i.e., $\{0, 1\}^* = \{\lambda, 0, 1, 00, \ldots\}$. An encoder $f$ is possibly stochastic and produces a non-prefix code. For $x \in \mathcal{X}$, the codeword length of $f(x)$ is denoted as $\ell(f(x))$. A deterministic decoder $g$ is defined as $g: \{0, 1\}^* \to \mathcal{Y}$.

Variable-length source coding without the prefix condition is discussed as in, for example, [3] and [2].

The performance criteria considered in this paper are the excess distortion and the overflow probabilities.

Definition 1: Given $D \geq 0$, the excess distortion probability for a code $(f, g)$ is defined as $\Pr\{d(X, g(f(X))) > D\}$.

Definition 2: Given $R \geq 0$, the overflow probability for a code $(f, g)$ is defined as $\Pr\{\ell(f(X)) > R\}$.

Using these criteria, we define a $(D, R, \epsilon, \delta)$ code.

Definition 3: Given $D, R \geq 0$ and $\epsilon, \delta \in [0, 1)$, a code $(f, g)$ satisfying

$$\Pr\{d(X, g(f(X))) > D\} \leq \epsilon,$$

$$\Pr\{\ell(f(X)) > R\} \leq \delta$$

is called a $(D, R, \epsilon, \delta)$ code.

1 The smooth max entropy has first introduced by Renner and Wolf [6]. Recently, several studies have characterized the optimal rate by the smooth max entropy (e.g., [7], [9], [10], [11]).
The fundamental limits are the minimum thresholds $R^*(D, \epsilon, \delta)$ and $R(D, \epsilon, \delta)$ for given $D$, $\epsilon$, and $\delta$.

**Definition 4:** Given $D \geq 0$ and $\epsilon, \delta \in [0, 1)$, we define
\[
R^*(D, \epsilon, \delta) := \inf \{ R : \exists a (D, R, \epsilon, \delta) \text{ code} \},
\]
\[
\tilde{R}(D, \epsilon, \delta) := \inf \{ R : \exists \text{ a deterministic } (D, R, \epsilon, \delta) \text{ code} \}.
\]

**Remark 1:** Consider the special case $\delta = 0$. From a $(D, R, \epsilon, 0)$ code, we can construct a fixed-length code achieving rate $R$ and the excess distortion probability $\leq \epsilon$. Thus, $R^*(D, \epsilon, 0)$ or $\tilde{R}(D, \epsilon, 0)$ represents the minimum rate in fixed-length lossy source coding under the excess distortion criterion.

**B. Smooth Max Entropy-Based Quantity**

The smooth max entropy, which is also called the smooth Rényi entropy of order zero, has first introduced by Renner and Wolf [6]. Later, Uyematsu [10] has shown that the smooth max entropy can be defined in the following form.

**Definition 5:** Given $\delta \in [0, 1)$, the smooth max entropy $H^\delta(X)$ is defined as
\[
H^\delta(X) := \min_{A : |X|} \log |A|,
\]
where $|\cdot|$ represents the cardinality of the set.

One of the useful properties of the smooth max entropy, which is used in the proof of the achievable result in our main theorem, is Schur convexity. To state the definition of a Schur concave function, we first review the notion of majorization.

**Definition 6:** Let $\mathbb{R}_+$ be the set of non-negative real numbers and $\mathbb{R}_+^m$ be the $m$-th Cartesian product of $\mathbb{R}_+$, where $m$ is a positive integer. Suppose that $x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}_+^m$ satisfy
\[x_i \geq x_{i+1}, \quad y_i \geq y_{i+1} \quad (i = 1, 2, \ldots, m-1).
\]
If $x \in \mathbb{R}_+^m$ and $y \in \mathbb{R}_+^m$ satisfy, for $k = 1, \ldots, m-1$,
\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \text{and} \quad \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i,
\]
then we say that $y$ majorizes $x$ (it is denoted as $x \prec y$).

Schur concave functions are defined as follows.

**Definition 7:** We say that a function $h(\cdot) : \mathbb{R}_+^m \to \mathbb{R}$ is a Schur concave function if $h(y) \leq h(x)$ for any $x, y \in \mathbb{R}_+^m$ satisfying $x \prec y$.

From the definition of the smooth max entropy and Schur concave functions, it is easy to see that the smooth max entropy is a Schur concave function.

Next, using the smooth max entropy, we introduce a new quantity, which plays an important role in producing our main results.

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1. All logarithms are of base 2 throughout this paper.
2. In [10], by using the notion of majorization, it is shown that the smooth Rényi entropy of order $\alpha$ is a Schur concave function for $0 \leq \alpha < 1$ and a Schur convex function for $\alpha > 1$.

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**Definition 8:** Given $D \geq 0$ and $\epsilon, \delta \in [0, 1)$, $G^{\delta}_{D,\epsilon}(X)$ is defined as
\[
G^{\delta}_{D,\epsilon}(X) := \min_{\Pr[d(X,Y) > D] \leq \epsilon} H^{\delta}(Y) \tag{4}
\]
\[
= \min_{\Pr[d(X,Y) > D] \leq \epsilon} \min_{B \in \mathcal{Y}} \log |B|,
\]
where $P_{Y|X}$ denotes a conditional probability distribution of $Y$ given $X$.

**Remark 2:** For a given $D \geq 0$ and $\epsilon \in [0, 1)$, suppose that
\[
\Pr\left\{ y \in Y : d(X, y) > D \right\} > \epsilon.
\]
Then, there are no codes whose excess distortion probability is less than or equal to $\epsilon$. Conversely, if such codes do not exist for given $D$ and $\epsilon$, [5] holds. In this case, we define $R^*(D, \epsilon, \delta) = +\infty$ and $\tilde{R}(D, \epsilon, \delta) = +\infty$. Further, if [5] holds, we also define $G^{\delta}_{D,\epsilon}(X) = +\infty$ because there is no conditional probability distribution $P_{Y|X}$ on $\mathcal{Y}$ satisfying $\Pr\{d(X,Y) > D\} \leq \epsilon$.

**C. One-Shot Coding Theorem for Stochastic Codes**

The next lemma shows the achievable result on $R$ of a $(D, R, \epsilon, \delta)$ code.

**Lemma 1:** Assume that $G^{\delta}_{D,\epsilon}(X) < +\infty$. For any $D \geq 0$ and $\epsilon, \delta \in [0, 1)$, there exists a $(D, R, \epsilon, \delta)$ code such that $R = G^{\delta}_{D,\epsilon}(X)$.

**Proof:** See Section IV-A.

**Remark 3:** To prove the achievable result, we do not use the random coding argument but give an explicit code construction. The constructed code satisfies the property of the optimal code shown in [4].

The next lemma shows the converse bound on $R$ of a $(D, R, \epsilon, \delta)$ code.

**Lemma 2:** For any $D \geq 0$ and $\epsilon, \delta \in [0, 1)$, any $(D, R, \epsilon, \delta)$ code satisfies
\[
R > G^{\delta}_{D,\epsilon}(X) - 1.
\]

**Proof:** See Section IV-B.

Combining Lemmas 1 and 2, we can immediately obtain the following result on $R^*(D, \epsilon, \delta)$.

**Theorem 1:** For any $D \geq 0$ and $\epsilon, \delta \in [0, 1)$, it holds that
\[
G^{\delta}_{D,\epsilon}(X) - 1 < R^*(D, \epsilon, \delta) \leq G^{\delta}_{D,\epsilon}(X) \tag{8}
\]
By Theorem 1, the minimum threshold $R^*(D, \epsilon, \delta)$ can be specified within one bit in the interval not greater than $G^{\delta}_{D,\epsilon}(X)$, regardless of the values $D$, $\epsilon$, and $\delta$. This result is mainly due to an explicit construction of good codes, rather than the random coding argument, given in Section IV-A.

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3. Under the constraint of the excess distortion probability, Kostina et al. [4] have discussed the optimal variable-length code which achieves the minimum mean codeword length. In [4], several properties of the optimal code are pointed out.
D. One-Shot Coding Theorem for Deterministic Codes

The next theorem shows the achievability result on \( R \) of a deterministic \((D, R, \epsilon, \delta)\) code.

Lemma 3: Assume that \( G_{D,\epsilon}^\delta(X) < +\infty \). For any \( D \geq 0 \) and \( \epsilon, \delta \in [0, 1) \), there exists a deterministic \((D, R, \epsilon, \delta)\) code such that

\[
R = \left[ G_{D,\epsilon}^\delta(X) + \frac{2 \log e}{2D_{D,\epsilon}^\delta(X)} \right].
\]  

Proof: See Section IV-C.

From Lemma 3 and the fact that \( R^*(D, \epsilon, \delta) \leq \tilde{R}(D, \epsilon, \delta) \), the following result on \( R^*(D, \epsilon, \delta) \) is obtained.

Theorem 2: For any \( D \geq 0 \) and \( \epsilon, \delta \in [0, 1) \), it holds that

\[
G_{D,\epsilon}^\delta(X) - 1 < \tilde{R}(D, \epsilon, \delta) \leq \left[ G_{D,\epsilon}^\delta(X) + \frac{2 \log e}{2D_{D,\epsilon}^\delta(X)} \right].
\]

By Theorem 2, \( \tilde{R}(D, \epsilon, \delta) \) can be specified in the interval within four bits, which is slightly weaker than the result for stochastic codes.

III. Asymptotic Coding Theorem

A. Problem Formulation

Let \( X^n \) and \( Y^n \) be the \( n \)-th Cartesian product of \( X \) and \( Y \), respectively. Let \( X^n \) be a random variable taking a value in \( X^n \) and \( x^n \) be a realization of \( X^n \). The probability distribution of \( X^n \) is denoted as \( P_{X^n} \). In this section, coding problem for general sources \( X = \{X_n\}_{n=1}^{\infty} \) is considered. A distortion measure \( d_n \) is defined as \( d_n : X^n \times Y^n \rightarrow [0, +\infty) \).

An encoder \( f_n : X^n \rightarrow \{0, 1\}^n \) is possibly stochastic and produces a non-prefix code. A decoder \( g_n : \{0, 1\}^n \rightarrow Y^n \) is deterministic.

We define an \((n, D, R, \epsilon, \delta)\) code as follows.

Definition 9: Given \( D, R, \epsilon, \delta \in [0, 1) \), a code \((f_n, g_n)\) satisfying

\[
\Pr\{d_n(X^n, g_n(f_n(X^n))) > nD\} \leq \epsilon, \\
\Pr\{\ell(f_n(X^n)) > nR\} \leq \delta
\]

for all \( n \geq n_0 \) with some \( n_0 > 0 \) is called an \((n, D, R, \epsilon, \delta)\) code.

The fundamental limit is the following minimum threshold.

Definition 10: Given \( D \geq 0 \) and \( \epsilon, \delta \in [0, 1) \),

\[
R(D, \epsilon, \delta) := \inf\{R : \exists \text{ an } (n, D, R, \epsilon, \delta) \text{ code}\}.
\]

B. Coding Theorem: General Formula

The next theorem characterizes \( R(D, \epsilon, \delta) \) by the smooth max entropy-based quantity \( G_{D,\epsilon}^\delta(X^n) \).

Theorem 3: For any \( D \geq 0 \) and \( \epsilon, \delta \in [0, 1) \),

\[
R(D, \epsilon, \delta) = \limsup_{n \rightarrow \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n),
\]

where \( G_{D,\epsilon}^\delta(X^n) \) is defined as

\[
G_{D,\epsilon}(X^n) := \min_{P_{X^n}, n \geq n: \Pr(d_n(X^n, Y^n) > nD) \leq \epsilon} H^\delta(Y^n).
\]

Proof: See Section IV-D.

As shown in Theorem 3, we characterize the minimum threshold on the overflow probability by the quantity related to the entropy, whereas previous studies such as [5] and [13] have characterized it by the quantity related to the mutual information.

Remark 4: In the non-asymptotic (one-shot) regime, the results on the minimum threshold are different for stochastic encoders and deterministic encoders as shown in Theorems 1 and 2. In the asymptotic regime, however, the restriction to only deterministic encoders does not change the minimum threshold.

Remark 5: Instead of (11) and (12), as a generalization of the problem in [13], we can consider the conditions

\[
\limsup_{n \rightarrow \infty} \Pr\{d_n(X^n, g_n(f_n(X^n))) > nD\} \leq \epsilon, \\
\limsup_{n \rightarrow \infty} \Pr\{\ell(f_n(X^n)) > nR\} \leq \delta
\]

and define \( \hat{R}(D, \epsilon, \delta) \) as the minimum threshold under the conditions [13] and [13]. Then, by almost the same proof of Theorem 3, we have

\[
\hat{R}(D, \epsilon, \delta) = \limsup_{n \rightarrow \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n)
\]

for any \( D \geq 0 \) and \( \epsilon, \delta \in [0, 1) \).

IV. Proofs of Main Results

A. Proof of Lemma 1

First, some notations are defined before the construction of the encoder and decoder is described.

- For \( y \in Y \) and \( D \geq 0 \), \( B_D(y) \) is defined as

\[
B_D(y) := \{x \in X : d(x, y) \leq D\}.
\]

- We define \( y_i \) (\( i = 1, 2, \ldots \)) by the following procedure.

Let \( y_1 \) be defined as

\[
y_1 := \arg\max_{y \in Y} \Pr\{X \in B_D(y)\},
\]

and for \( i = 2, 3, \ldots \), let \( y_i \) be defined as

\[
y_i := \arg\max_{y \in Y} \Pr\left\{ X \in B_D(y) \setminus \bigcup_{j=1}^{i-1} B_D(y_j) \right\}.
\]

- For \( i = 1, 2, \ldots \), \( A_D(y_i) := B_D(y_i) \setminus \bigcup_{j=1}^{i-1} B_D(y_j) \).

From the definition, we have

\[
\bigcup_{j=1}^{i} A_D(y_j) = \bigcup_{j=1}^{i} B_D(y_j) \quad (i \geq 1),
\]

\[
A_D(y_i) \cap A_D(y_j) = \emptyset \quad (\forall i \neq j),
\]

\[
\Pr\{X \in A_D(y_1)\} \geq \Pr\{X \in A_D(y_2)\} \geq \cdots
\]

\footnote{In this paper, we assume that \( X \) and \( Y \) are finite sets. However, we can assume countably infinite \( X \) and \( Y \) if this operation is admitted for countably infinite \( X \) and \( Y \).}
• If 0 < \epsilon + \delta < 1, let \( i^* \geq 1 \) be the integer satisfying

\[
\sum_{i=1}^{i^*-1} \Pr\{X \in A_D(y_i)\} < 1 - \epsilon - \delta, \tag{21}
\]

\[
\sum_{i=1}^{i^*} \Pr\{X \in A_D(y_i)\} \geq 1 - \epsilon - \delta. \tag{22}
\]

If \( \epsilon + \delta \geq 1 \), we define \( i^* = 1 \).

• Let \( k^* \geq 1 \) be the integer satisfying

\[
\sum_{i=1}^{k^*-1} \Pr\{X \in A_D(y_i)\} < 1 - \epsilon, \tag{23}
\]

\[
\sum_{i=1}^{k^*} \Pr\{X \in A_D(y_i)\} \geq 1 - \epsilon. \tag{24}
\]

From this definition, it holds that \( k^* \geq i^* \).

• Let \( k^* \) be defined as \( \alpha : = \sum_{i=1}^{i^*-1} \Pr\{X \in A_D(y_i)\} \) and \( \beta : = 1 - \epsilon - \alpha \).

• Let \( w_i \) be the \( i \)-th binary string in \( \{0, 1\}^* \) in the increasing order of the length and ties are arbitrarily broken. For example, \( w_1 = \lambda, w_2 = 0, w_3 = 1, w_4 = 00, w_5 = 01 \), etc.

We construct the following encoder \( f : \mathcal{X} \to \{0, 1\}^* \) and decoder \( g : \{0, 1\}^* \to \mathcal{Y} \).

[Encoder]

1) For \( x \in A_D(y_i) \) (\( i = 1, \ldots, k^* - 1 \)), set \( f(x) = w_i \).

2) For \( x \in A_D(y_{k^*}) \), set

\[
f(x) = \begin{cases} w_{k^*} & \text{with prob.} \frac{\beta}{\Pr\{X \in A_D(y_{k^*})\}}, \\ w_1 & \text{with prob.} \frac{1 - \beta}{\Pr\{X \in A_D(y_{k^*})\}}. 
\end{cases}
\]

\[
\tag{25}
\]

3) For \( x \notin \bigcup_{i=1}^{k^*} A_D(y_i) \), set \( f(x) = w_1 \).

[Decoder] Set \( g(w_i) = y_i \) (\( i = 1, \ldots, k^* \)).

Now, we evaluate the excess distortion probability. We have \( d(x, g(f(x))) \leq D \) for \( x \in A_D(y_i) \) (\( i = 1, \ldots, k^* - 1 \)) since \( g(f(x)) = y_i \). Furthermore, we have \( d(x, g(f(x))) \leq D \) with probability \( \beta / \Pr\{X \in A_D(y_{k^*})\} \) for \( x \in A_D(y_{k^*}) \). Thus,

\[
\Pr\{d(X, g(f(X))) \leq D\} = \sum_{i=1}^{k^*-1} \Pr\{X \in A_D(y_i)\} + \Pr\{X \in A_D(y_{k^*})\} = \alpha + \beta \geq 1 - \epsilon.
\]

Therefore, we have

\[
\Pr\{d(X, g(f(X))) > D\} = \epsilon. \tag{26}
\]

Note that we have \( \Pr\{X \in A_D(y_{k^*})\} \geq \beta \) from \( \text{(25)} \).

Next, we evaluate the overflow probability. From the construction of the encoder, it is easily verified that \( \ell(w_i) = [\log i] \) (\( i = 1, \ldots, k^* \)). Hence, setting \( R = [\log i^*] \), we have

\[
\Pr\{\ell(f(X)) > R\} \leq \sum_{i=i^*+1}^{k^*} \Pr\{f(X) = w_i\}
\]

\[
= \sum_{i=i^*+1}^{k^*} \Pr\{X \in A_D(y_i)\} + \Pr\{X \in A_D(y_{k^*})\}
\]

\[
= \sum_{i=1}^{k^*} \Pr\{X \in A_D(y_i)\} - \sum_{i=1}^{i^*} \Pr\{X \in A_D(y_i)\} + \beta
\]

\[
\leq \alpha - (1 - \epsilon - \delta) + \beta = \delta,
\]

where the last inequality is due to the definition of \( \alpha \) and \( \beta \) and the last equality is due to the definition of \( \beta \).

Therefore, the code \((f, g)\) is a \((D, R, \epsilon, \delta)\) code with \( R = [\log i^*] \). To complete the proof of the theorem, we shall show \( \log i^* = G_{D, \epsilon}^\delta(X) \). We define \( Y : = g(f(X)) \). Notice that

\[
P_Y(y_i) \overset{(a)}{=} \Pr\{X \in A_D(y_i)\} + \Pr\{X \in \bigcup_{i=k^*+1}^{i^*} A_D(y_i)\}
\]

\[
+ \Pr\{f(X) = w_{i^*}, X \in A_D(y_{i^*})\}
\]

\[
\overset{(b)}{=} \Pr\{X \in A_D(y_i)\} + \Pr\{d(X, g(f(X))) > D\}
\]

\[
\overset{(c)}{=} \Pr\{X \in A_D(y_i)\} + \epsilon, \tag{27}
\]

\[
P_Y(y_i) = \Pr\{X \in A_D(y_i)\} \quad (i = 2, \ldots, k^* - 1), \tag{28}
\]

where (a) and (b) follow from the definition of the encoder and decoder and (c) is due to \( \text{(26)} \). Then

\[
\sum_{i=1}^{i^*-1} \Pr\{X \in A_D(y_i)\} + \epsilon < 1 - \delta,
\]

\[
\sum_{i=1}^{i^*} \Pr\{X \in A_D(y_i)\} + \epsilon \geq 1 - \delta, \tag{29}
\]

\[
P_Y(y_1) \geq P_Y(y_2) \geq \cdots \geq P_Y(y_{k^*}),
\]

which implies that \( \log i^* = G_{D, \epsilon}^\delta(Y) \). Hence, if

\[
H^\delta(Y) = G_{D, \epsilon}^\delta(X) \tag{30}
\]

is shown, the desired equation \( \log i^* = G_{D, \epsilon}^\delta(X) \) is obtained.

To show \( \text{(30)} \), the following lemma is useful.

[Lemma 4]: If \( P_Y \cdot \), which is induced by \( P_{Y_{[-X]}} \) satisfying \( \Pr\{d(X, Y^*) > D\} \leq \epsilon \) majorizes any \( P_Y \), which is induced by \( P_{Y_{[X]}} \) satisfying \( \Pr\{d(X, Y^*) > D\} \leq \epsilon \), then it holds that \( H^\delta(Y^*) = G_{D, \epsilon}^\delta(X) \).

Proof: The lemma follows from the fact that the smooth max entropy is a Schur concave function and the definition of \( G_{D, \epsilon}^\delta(X) \).

\]
In view of Lemma 4, we shall show that $P_Y$ majorizes any $P_Y$ induced by $P_{Y|X}$ satisfying $\Pr\{d(X,Y) > D\} \leq \epsilon$. To show this fact, suppose the following condition:

(iii) There exists a $P_Y$ satisfying $\Pr\{d(X,Y) > D\} \leq \epsilon$ but not being majorized by $P_Y$.

Assuming (iii), we shall show a contradiction.

Let $y_{\tau(1)}$ give the largest $P_Y(y)$ in $\mathcal{Y}$, $y_{\tau(2)}$ give the largest $P_Y(y)$ in $\mathcal{Y} \setminus \{y_{\tau(1)}\}$, $y_{\tau(3)}$ give the largest $P_Y(y)$ in $\mathcal{Y} \setminus \{y_{\tau(1)}, y_{\tau(2)}\}$, etc. That is, $P_Y(y_{\tau(1)}) \geq P_Y(y_{\tau(2)}) \geq \cdots \geq P_Y(y_{\tau(k^*)})$ and $P_Y(y_{\tau(k^*)}) \geq P_Y(y_{\tau(i)})$ for all $i = k^* + 1, k^* + 2, \ldots$. Considering the fact that the support of $P_Y$ is $\{1, 2, \ldots, k^*\}$ and the assumption (iii), we can say that there exists a $1 \leq j_0 \leq k^* - 1$ satisfying

$$\sum_{i=1}^{j_0} (P_Y(y_{\tau(i)}) - P_Y(y_i)) > 0. \quad (31)$$

On the other hand, the excess distortion probability under $P_X P_{Y|X}$ is evaluated as

$$\Pr\{d(X,Y) > D\} \geq \sum_{x \in \mathcal{X}} \sum_{i=1}^{j_0} P_X(x) P_{Y|X}(y_{\tau(i)}|x) I\{d(x,y_{\tau(i)}) > D\}$$

$$= \sum_{x \in \mathcal{X}} \sum_{i=1}^{j_0} P_X(x) P_{Y|X}(y_{\tau(i)}|x)$$

$$- \sum_{x \in \mathcal{X}} \sum_{i=1}^{j_0} P_X(x) P_{Y|X}(y_{\tau(i)}|x) I\{x \in B_D(y_{\tau(i)})\}$$

$$= \sum_{i=1}^{j_0} P_Y(y_{\tau(i)})$$

$$- \sum_{x \in \mathcal{X}} \sum_{i=1}^{j_0} P_X(x) P_{Y|X}(y_{\tau(i)}|x) I\{x \in B_D(y_{\tau(i)})\}$$

$$\geq \sum_{i=1}^{j_0} P_Y(y_{\tau(i)}) - \Pr\left\{X \in \bigcup_{i=1}^{j_0} B_D(y_{\tau(i)})\right\}, \quad (32)$$

where $I(\cdot)$ is the indicator function and the last inequality is due to $\sum_{i=1}^{j_0} P_{Y|X}(y_{\tau(i)}|x) I\{x \in B_D(y_{\tau(i)})\} \leq I\{x \in \bigcup_{i=1}^{j_0} B_D(y_{\tau(i)})\}$ for all $x \in \mathcal{X}$. For the second term in (32), it holds that

$$\Pr\left\{X \in \bigcup_{i=1}^{j_0} B_D(y_{\tau(i)})\right\} \leq \Pr\left\{X \in \bigcup_{i=1}^{j_0} B_D(y_i)\right\}$$

$$= \frac{j_0}{c} \sum_{i=1}^{j_0} \Pr\{X \in A_D(y_i)\} - \epsilon, \quad (33)$$

where (a) follows from the definition of $y_i$, (b) follows from (18) and (19), and (c) follows from (27) and (28). Plugging (33) into (32) gives

$$\Pr\{d(X,Y) > D\} \geq \sum_{i=1}^{j_0} (P_Y(y_{\tau(i)}) - P_Y(y_i)) + \epsilon > \epsilon,$$

where the last inequality is due to (31). This is a contradiction to the fact that $\Pr\{d(X,Y) > D\} \leq \epsilon$.

B. Proof of Lemma 2

For any $(D,R,\epsilon,\delta)$ code $(f,g)$, set $Y := g(f(X))$. The definition of a $(D,R,\epsilon,\delta)$ code gives

$$\Pr\{\ell(f(x)) > R\} \leq \delta, \quad \Pr\{d(X,Y) > D\} \leq \epsilon. \quad (34)$$

Let $T := \{g(f(x)) \in \mathcal{Y} : x \text{ satisfies } \ell(f(x)) > R\}$. Then, the first inequality in (34) is rewritten as $\Pr\{Y \in T\} \leq \delta$.

Hence,

$$\Pr\{Y \in T^c\} \geq 1 - \delta, \quad (35)$$

where the superscript “$c$” represents the complement. From (35) and the definition of the smooth max entropy,

$$H^\delta(Y) \leq \log |T^c|. \quad (36)$$

On the other hand, since $\ell(g^{-1}(y)) \leq |R|$ for $y \in T^c$,

$$|T^c| \leq 1 + 2 + \cdots + 2^{|R|} = 2^{|R|+1} - 1 < 2^{|R|+1}. \quad (37)$$

Combining (36) and (37) yields $H^\delta(Y) < R + 1$. Thus, from the second inequality in (34), we have $G_{D,c}(X) < R + 1$.

C. Proof of Lemma 3

First, some notations are defined.

- Let $k^* \geq 1$ be the integer satisfying (24) and (23).
- Define $\gamma$ as $\gamma := 1 - \sum_{i=1}^{k^*} \Pr\{X \in A_D(y_i)\}$. Then, it holds that $\gamma \leq \epsilon$.
- Let $j^* \geq 1$ be the integer satisfying

$$\sum_{i=1}^{j^*-1} \Pr\{X \in A_D(y_i)\} < 1 - \gamma - \delta, \quad (38)$$

$$\sum_{i=1}^{j^*} \Pr\{X \in A_D(y_i)\} \geq 1 - \gamma - \delta. \quad (39)$$

We construct the following deterministic encoder $f : \mathcal{X} \rightarrow \{0,1\}^*$ and decoder $g : \{0,1\}^* \rightarrow \mathcal{Y}$.

[Encoder]

1) For $x \in A_D(y_i)$ ($i = 1, \ldots, k^*$), set $f(x) = w_i$.
2) For $x \not\in \bigcup_{i=1}^{k^*} A_D(y_i)$, set $f(x) = w_1$.

[Decoder] Set $g(w_i) = y_i$ ($i = 1, \ldots, k^*$).

Now, we evaluate the excess distortion probability. From the definition of the encoder and decoder,

$$\Pr\{d(X,g(f(X))) \leq D\} = \sum_{i=1}^{k^*} \Pr\{X \in A_D(y_i)\} \geq 1 - \epsilon.$$

Therefore, we have $\Pr\{d(X,g(f(X))) > D\} \leq \epsilon$.

Next, we evaluate the overflow probability. From the definition of the encoder, we have

$$\Pr\{f(X) = w_1\} = \Pr\{X \in A_D(y_1)\} + \gamma,$n
$$\Pr\{f(X) = w_i\} = \Pr\{X \in A_D(y_i)\} \quad (i = 2, \ldots, k^*).$$


Setting $R = \lfloor \log \min(j^*,k^*) \rfloor$, it holds that\footnote{If $R = \lfloor \log k^* \rfloor$, then $\Pr\{\ell(f(X)) > R\} = 0$.}

\[
\Pr\{\ell(f(X)) > R\} \leq 1 - \sum_{i=1}^{j^*} \Pr\{f(X) = w_i\}
\]

\[
= 1 - \left( \sum_{i=1}^{j^*} \Pr\{X \in A_D(y_i)\} + \gamma \right)
\]

\[
\leq 1 - \left( (1 - \gamma - \delta) + \gamma \right) = \delta,
\]

where the last inequality is due to (39).

Therefore, the code $(f,g)$ is a deterministic $(D,R,\epsilon,\delta)$ code with $R = \lfloor \log \min(j^*,k^*) \rfloor$.

Let $i^*$ be the integer satisfying (22) and (21). Then, from the proof of Lemma 1, it holds that $\log i^* = G_{D,\epsilon}^\delta(X)$. Since $\gamma \leq \epsilon$, it is easily verified that $i^* \leq j^*$ and $i^* \leq k^*$, meaning that $i^* \leq \min(j^*,k^*)$. If $i^* = \min(j^*,k^*)$, $\min(j^*,k^*) \leq i^* + 2$ obviously holds. Then, assuming that $i^* < \min(j^*,k^*)$, we shall show that

\[
\min(j^*,k^*) \leq i^* + 2.
\]

This leads to

\[
\log \min(j^*,k^*) \leq \log(i^* + 2) \leq \log i^* + \frac{2 \log \epsilon}{i^*},
\]

where the rightmost inequality is due to Taylor’s expansion, and we obtain (9).

The first step to show $\min(j^*,k^*) \leq i^* + 2$ is the following inequality:

\[
\begin{aligned}
\sum_{i=1}^{j^*} \Pr\{X \in A_D(y_i)\} - \sum_{i=1}^{i^*} \Pr\{X \in A_D(y_i)\} & \\
& \leq 1 - \gamma - \delta + \Pr\{X \in A_D(y_{j^*})\} - (1 - \epsilon - \delta) \\
& = \Pr\{X \in A_D(y_{j^*})\} + \epsilon - \gamma \\
& \leq \Pr\{X \in A_D(y_{i^*})\} + \Pr\{X \in A_D(y_{i^*+1})\},
\end{aligned}
\]

(40)

where (a) follows from (22) and (38) and (b) follows from

\[
\epsilon - \gamma \leq \left( 1 - \sum_{i=1}^{k^*} \Pr\{X \in A_D(y_i)\} \right)
\]

\[
- \left( 1 - \sum_{i=1}^{k^*} \Pr\{X \in A_D(y_i)\} \right)
\]

\[
= \Pr\{X \in A_D(y_{i^*})\} \leq \Pr\{X \in A_D(y_{i^*+1})\}.
\]

Inequality (40) is equivalent to $\sum_{i=1}^{j^*+1} \Pr\{X \in A_D(y_i)\} \leq \sum_{i=1}^{i^*+1} \Pr\{X \in A_D(y_i)\}$. Thus, we obtain $j^* - 1 \leq i^* + 1$, implying that $\min(j^*,k^*) \leq i^* + 2$.

D. Proof of Theorem 3

(Direct part) From Lemma 3, there exists a code $(f_n,g_n)$ satisfying

\[
\Pr\{d_n(X^n,g_n(f_n(X^n))) > nD\} \leq \epsilon,
\]

\[
\Pr\left\{ \frac{1}{n} \ell(f_n(X^n)) > \frac{1}{n} G_{D,\epsilon}^\delta(X^n) + \frac{2}{n2^{G_{D,\epsilon}^\delta(X^n)}} \right\} \leq \delta.
\]

Fix $\gamma > 0$ arbitrarily. Then, it holds that

\[
\frac{1}{n} G_{D,\epsilon}^\delta(X^n) \leq \limsup_{n \to \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n) + \gamma
\]

and

\[
\frac{2}{n2^{G_{D,\epsilon}^\delta(X^n)}} \leq \gamma
\]

for all $n \geq n_0$ with some $n_0 > 0$. Then, from (42), we have

\[
\Pr\left\{ \frac{1}{n} \ell(f_n(X^n)) > \limsup_{n \to \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n) + 2\gamma \right\} \leq \delta
\]

for all $n \geq n_0$. Thus, from (41) and (43), $(f_n,g_n)$ is an $(n,D,R,\epsilon,\delta)$ code with

\[
R = \limsup_{n \to \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n) + 2\gamma
\]

for all $n \geq n_0$. Since $\gamma > 0$ is arbitrary, this indicates that

\[
R(D,\epsilon,\delta|X) \leq \limsup_{n \to \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n).
\]

(Converse part) For any $(n,D,R,\epsilon,\delta)$ code, Lemma 2 gives $nR > G_{D,\epsilon}^\delta(X^n) - 1$. Therefore, it holds that

\[
R \geq \limsup_{n \to \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n)
\]

for any $(n,D,R,\epsilon,\delta)$ code. Hence, we have

\[
R(D,\epsilon,\delta|X) \geq \limsup_{n \to \infty} \frac{1}{n} G_{D,\epsilon}^\delta(X^n).
\]

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