CHOW QUOTIENTS AND PROJECTIVE BUNDLE FORMULAS FOR 
EULER-CHOW SERIES

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Abstract. Given a projective algebraic variety $X$, let $\Pi_p(X)$ denote the monoid of effective algebraic equivalence classes of effective algebraic cycles on $X$. The $p$-th Euler-Chow series of $X$ is an element in the formal monoid-ring $\mathbb{Z}[\Pi_p(X)]$ defined in terms of Euler characteristics of the Chow varieties $C_p,\alpha(X)$ of $X$, with $\alpha \in \Pi_p(X)$. We provide a systematic treatment of such series, and give projective bundle formulas which generalize previous results by [LY87] and [Eli94]. The techniques used involve the Chow quotients introduced in [KSZ91], and this allows the computation of various examples including some Grassmannians and flag varieties. There are relations between these examples and representation theory, and further results point to interesting connections between Euler-Chow series for certain varieties and the topology of the moduli spaces $\overline{M}_{0,n+1}$.

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The use of topological invariants on moduli spaces has played a vital role in various branches of mathematics and mathematical physics in the last two decades. A light sampling under this vast umbrella includes works in gauge theory, the theory of instantons, various moduli spaces of vector bundles, moduli spaces of curves and their compactifications, Chow varieties and Hilbert schemes.

In this work we study a class of invariants for projective varieties arising from the Euler characteristics of their Chow varieties. These invariants first appeared in the work of H. B. Lawson and Steve S. T. Yau [LY87], whose techniques play an important role in this paper, and they present, in various instances, a quite nice and elegant behavior which can often be codified in simple generating functions.

As a motivation, we start with some particular cases, which are well studied in the literature. Let \( X \) be a connected projective variety and let \( SP(X) \) denote the disjoint union \( \bigsqcup_{d \geq 0} SP_d(X) \) of all symmetric products of \( X \), with the disjoint union topology, where \( SP_0(X) \) is a single point. One can define a function \( E_0(X) : \mathbb{Z}_+ = \pi_0(SP(X)) \rightarrow \mathbb{Z} \) which sends \( d \) to the Euler characteristic \( \chi(SP_d(X)) \) of the \( d \)-fold symmetric product of \( X \). This is what we call the \( 0 \)-th Euler-Chow function of \( X \). The same information can be codified as a formal power series \( E_0(X) = \sum_{d \geq 0} \chi(SP_d(X)) t^d \), and a result of Macdonald [Mac62] shows that \( E_0(X) \) is given by the rational function \( E_0(X) = (1/(1 - t))^\chi(X) \).

Another familiar instance arises in the case of divisors. Given an \( n \)-dimensional projective variety \( X \), let \( \text{Div}_+(X) \) denote the space of effective divisors on \( X \) and assume that \( \text{Pic}_0(X) = \{0\} \). Consider the function \( E : \text{Pic}(X) \rightarrow \mathbb{Z} \) which sends \( L \in \text{Pic}(X) \) to \( \dim H^0(X; \mathcal{O}(L)) \). Observe that

1. Given \( L \in \text{Pic}(X) \), then \( E(L) \neq 0 \) if and only if \( L = \mathcal{O}(D) \) for some effective divisor \( D \);
2. Under the given hypothesis, algebraic and linear equivalence coincide, and two effective divisors \( D \) and \( D' \) are algebraically equivalent if and only if they are in the same linear system.

The last observation implies that \( \text{Div}_+(X) \) can be written as \( \text{Div}_+(X) = \bigsqcup_{\alpha \in A^\geq_{n-1}(X)} \text{Div}_+(X)_\alpha \), where \( A^\geq_{n-1}(X) \) is the monoid of algebraic equivalence classes of effective divisors (cf. Fulton [Ful84, §12]), and \( \text{Div}_+(X)_\alpha \) is the linear system associated to \( \alpha \in A^\geq_{n-1}(X) \). The first observation shows that the only relevant data to \( E \) is given by \( A^\geq_{n-1}(X) \subset \text{Pic}(X) \). Therefore, we might as well restrict \( E \) and define the \((n - 1)\)-st Euler-Chow function of \( X \) as the function \( E_{n-1}(X) : A^\geq_{n-1}(X) \rightarrow \mathbb{Z}_+ \) which sends \( \alpha \in A^\geq_{n-1}(X) \) to the Euler characteristic \( \chi(\text{Div}_+(X)_\alpha) = \dim H^0(X; \mathcal{O}(L_\alpha)) \), where \( L_\alpha \) is the line bundle associated to \( \alpha \).

**Example 1.1.** An even more restrictive case arises when \( \text{Pic}(X) \cong \mathbb{Z} \), and \( A^\geq_{n-1}(X) \cong \mathbb{Z}_+ \) is generated by the class of a very ample line bundle \( L \). Then the \((n - 1)\)-st Euler-Chow function \( E_{n-1}(X) = \sum_{d \geq 0} \dim H^0(X; \mathcal{O}(L^\otimes d)) t^d \) is just the Hilbert function associated to the projective embedding of \( X \) induced by \( L \). This is once again a rational function.
In general, the situation is not so simple, and we need to introduce additional notions in order to approach cycles of arbitrary dimension. We start with the Chow monoid \( \mathcal{C}_p(X) \) of effective \( p \)-cycles on \( X \), which can be written as a disjoint union \( \coprod_{\alpha \in \mathcal{P}_p(X)} \mathcal{C}_{p,\alpha}(X) \) of connected projective (Chow) varieties \( \mathcal{C}_{p,\alpha}(X) \); cf. Section 3. Here, \( \Pi_p(X) = \pi_0(\mathcal{C}_p(X)) \) denotes the monoid of effective algebraic equivalence classes of effective \( p \)-cycles. This monoid should be contrasted with \( A^E_p(X) \), the monoid of algebraic equivalence classes of effective \( p \)-cycles; cf. [Ful84, §12]. In fact, there is a finite surjective monoid morphism \( \Pi_p(X) \to A^E_p(X) \), and the Grothendieck group associated to both monoids is \( A_p(X) \), the group of algebraic equivalence classes of \( p \)-cycles on \( X \); cf. Friedlander [Fri91]. The properties and relations among these monoids is discussed in Section 3.

The \( p \)-th Euler-Chow function of the projective variety \( X \) is then defined as the function
\[
E_p(X) : \Pi_p(X) \to \mathbb{Z}
\]
\[
\alpha \mapsto \chi(\mathcal{C}_{p,\alpha}(X))
\]
which can be completely encoded as a formal power series \( E_p(X) = \sum_{\alpha \in \Pi_p(X)} \chi(\mathcal{C}_{p,\alpha}(X)) t^\alpha \), on variables \( t^\alpha \) associated to the elements of \( \Pi_p(X) \), and satisfying the relations \( t^\alpha t^\beta = t^{\alpha+\beta} \). For this reason, we also call \( E_p(X) \) the \( p \)-th Euler-Chow series of \( X \).

Previous works in which such functions have been explicitly computed include the cases of products of two projective spaces in Lawson and Yau [LY87], simplicial projective toric varieties in [Eli94], and some cases of principal bundles whose structure group is an abelian variety in Elizondo and Hain [EH96].

The two main results of this paper, Theorems 4.1 and 5.5 consider instances where one can reduce the calculation of Euler-Chow functions to simpler situations. In the first case we consider the projectivization \( \mathbb{P}(E) \) of an algebraic vector bundle \( E \) over a projective variety \( W \), which splits as a direct sum of bundles \( E_1 \oplus E_2 \). We define a “trace map” \( t_{p-1} : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \mathcal{C}_p(\mathbb{P}(E)) \) which, when combined with the inclusions \( i_1 : \mathbb{P}(E_1) \to \mathbb{P}(E) \) and \( i_2 : \mathbb{P}(E_2) \to \mathbb{P}(E) \) produces a monoid morphism \( \Psi_p : \Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \times \Pi_p(\mathbb{P}(E_1)) \times \Pi_p(\mathbb{P}(E_2)) \to \Pi_p(\mathbb{P}(E)) \). This morphism is the key ingredient for the following “split bundle” formula.

**Theorem 4.1.** Let \( E_1 \) and \( E_2 \) be algebraic vector bundles over a connected projective variety \( W \), of ranks \( e_1 \) and \( e_2 \), respectively, and let \( 0 \leq p \leq e_1 + e_2 - 1 \). Then the \( p \)-th Euler-Chow function of \( \mathbb{P}(E_1 \oplus E_2) \) is given by
\[
E_p(\mathbb{P}(E_1 \oplus E_2)) = \Psi_{p_2} \left( E_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \odot E_p(\mathbb{P}(E_1)) \odot E_p(\mathbb{P}(E_2)) \right).
\]

In the formula, \( \Psi_{p_2} \) denotes the push-forward induced by \( \Psi_p \), introduced in Definition 2.3, and \( \odot \) denotes the “exterior product” of Euler-Chow functions, cf. Definition 2.3.

This formula applies to several situations, where the main technical difficulty is reduced to the computation of \( \Psi_p \). For example, in the case where \( E_1 = L \) is a line bundle generated by its global
sections and $E_2$ is the trivial line bundle, then $\Psi_p$ is completely determined in terms of the first Chern class of $L$, as one expects; cf. Corollary 4.3. In order to illustrate such a case, let $L$ be $O_{P^n}(d)$, $d \geq 0$. Then we have $\Pi_p(\mathcal{P}(O_{P^n}(d) \oplus 1)) \cong \mathbb{Z}_+ \oplus \mathbb{Z}_+$. If one writes $t^{(r,s)} = x^r y^s$, then the $p$-th Euler-Chow function of $\mathcal{P}(O_{P^n}(d) \oplus 1)$ is determined by the following generating function

$$E_p(\mathcal{P}(O_{P^n}(d) \oplus 1)) = \left( \frac{1}{1-x} \right)^{(n+1)/p} \left( \frac{1}{1-y} \right)^{(n+1)/(p+1)} \left( \frac{1}{1-x^d y} \right)^{(n+1)/(p+1)}$$



Remark 1.2. The case $d = 0$, i.e., $\mathbb{P}^n \times \mathbb{P}^1$ was computed in [LY87]. The Euler-Chow series of the blow-up $\mathbb{P}^{n+1} \cong \mathcal{P}(O_{P^n}(1) \oplus 1)$ of $\mathbb{P}^{n+1}$ at a point, and of the Hirzebruch surfaces $F_d \cong \mathcal{P}(O_{P^1}(d) \oplus 1)$ were computed in Elizondo [EL94], using general techniques for toric varieties.

The next class of results applies to the case where $X$ is a smooth projective variety on which $\mathbb{C}^*$ acts via automorphisms, and such that the fixed point set $X^{\mathbb{C}^*}$ has only two connected components $X_1$ and $X_2$. In this case, following Kapranov, Sturmfels and Zelevinski [KSZ91], we introduce the Chow quotient $X/\mathbb{C}^*$ of $X$ by $\mathbb{C}^*$, which comes equipped once again with a “trace map” $t_{p-1} : \mathcal{C}_{p-1}(X/\mathbb{C}^*) \to \mathcal{C}_p(X)$, defined using techniques from Friedlander and Lawson [FL92].

In a similar fashion to the previous theorem we introduce a monoid morphism $\Psi_p : \Pi_{p-1}(X/\mathbb{C}^*) \times \Pi_p(X_1) \times \Pi_p(X_2) \to \Pi_p(X)$ which produces the following result.

**Theorem 5.5.** Let $X$ be an smooth projective variety on which $\mathbb{C}^*$ acts algebraically. If $X^{\mathbb{C}^*}$ is the union of two connected components $X_1$ and $X_2$, then for each $0 \leq p \leq \dim X$ one has

$$E_p(X) = \Psi_{p_2}(E_{p-1}(X/\mathbb{C}^*) \odot E_p(X_1) \odot E_p(X_2)).$$

**Remark 1.3.** In the case were $W$ is smooth, then Theorem 4.1 can be shown to be a particular case of Theorem 5.5.

Examples of such Chow quotients and resulting trace maps $t_p$’s and homomorphisms $\Psi_p$’s, are computed in Section 5.1. There we consider the case where $\mathbb{C}^*$ acts linearly on the last coordinate of $\mathbb{C}^{n+1}$, inducing an action on all (partial) flag varieties $F(d_1, \ldots, d_r; \mathbb{P}^n)$, with $0 \leq d_1 < \cdots < d_r \leq n$. Let $G_d(\mathbb{P}^n)$ be the space of all $d$-planes in $\mathbb{P}^n$, then we show that $G_d(\mathbb{P}^n)/\mathbb{C}^* \cong F(d - 1, 1; \mathbb{P}^{n-1})$, which implies the following “almost” recursive formula

$$E_p(G_d(\mathbb{P}^n)) = \Psi_p \left( E_{p-1}(F(d - 1, 1; \mathbb{P}^{n-1})) \odot E_p(G_d(\mathbb{P}^{n-1})) \odot E_p(G_{p-1}(\mathbb{P}^{n-1})) \right).$$

As another example, we show that $F(0, 1; \mathbb{P}^n)/\mathbb{C}^*$ is the blow-up of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ along the diagonal. In this context, this variety is naturally identified with $\mathbb{P}^{n-1}[2]$, the compactification of the configuration space of two distinct points in $\mathbb{P}^{n-1}$, introduced by Fulton and MacPherson in [FM94]. The main formula in Theorem 5.5 does not apply in this case, since the fixed point set of the $\mathbb{C}^*$ action on $F(0, 1; \mathbb{P}^n)$ has three connected components. Nevertheless, this Chow quotient can
still be used to obtain explicit computations in some cases; cf. Theorem 5.16, Corollary 5.19 and Proposition 5.20. In the tables below we exhibit the Euler-Chow series of \( F(0, 1; \mathbb{P}^2) \) and \( G_1(\mathbb{P}^3) \). We use the fact that for these spaces the monoids \( \Pi_\ast(-) \) are freely generated by the classes of Schubert cycles, and indicate in the table the appropriate association \{ variables \} ↔ \{ Schubert classes \}.

|                   | \( F(0, 1; \mathbb{P}^2) \) | Euler-Chow series |
|-------------------|-----------------------------|-------------------|
| \( t \leftrightarrow \omega_{0,0.1} \) | \( E_0 = \frac{1}{(1-t)s} \) |
| \( r \leftrightarrow \omega_{0,0.2} \); \( s \leftrightarrow \omega_{1,1.2} \) | \( E_1 = \frac{1}{(1-r)^4(1-s)^4(1-rs)^4} \) |
| \( x \leftrightarrow \omega_{1,1.2} \); \( y \leftrightarrow \omega_{0,0.2} \) | \( E_2 = \frac{1}{(1-x)^4(1-y)^4} \) |

|                   | \( G_1(\mathbb{P}^3) \) | Euler-Chow series |
|-------------------|-----------------------------|-------------------|
| \( t \leftrightarrow \omega_{0,1} \) | \( E_0 = \frac{1}{(1-t)s} \) |
| \( s \leftrightarrow \omega_{0,2} \) | \( E_1 = \frac{1}{(1-s)4^4} \) |
| \( x \leftrightarrow \omega_{0,3} \); \( y \leftrightarrow \omega_{1,2} \) | \( E_2 = \frac{1}{(1-x)^4(1-y)^4(1-xy)^2} \) |
| \( z \leftrightarrow \omega_{1,3} \) | \( E_3 = \frac{1}{(1-z)^4} \) |

There are several open questions suggested by the results we have computed so far. For example, what is the precise relation between the Euler-Chow series of generalized flag varieties and the combinatorics of such a variety? Even in apparently simple cases, such as \( G_1(\mathbb{P}^n) \) (not fully computed yet), one finds a direct relation between their Euler-Chow series and the ones for the compactified moduli space \( \overline{M}_{0,n+1} \) of stable \((n+1)\)-punctured curves of genus 0. The latter space arises as the Chow quotient \( G_1(\mathbb{P}^n) // (\mathbb{C}^*)^n \); cf. [Kap93]. Other questions, such as general blow-up formulas and rationality of Euler-Chow series (not expected in general) are also quite challenging.

This paper is organized as follows. In Section 2 we provide the necessary algebraic terminology and background. This material is complemented with Appendix A, where we describe a more general approach to Euler-Chow series using the notion of monoid-graded algebras and their invariants. In Section 3 we introduce the Chow varieties and Chow monoids of projective varieties, along with various properties. The monoids \( \Pi_p(X) \) and \( A_{\geq p}(X) \) are introduced and compared in this section, and relations between them and their common Grothendieck group are presented. In this section we also introduce the Euler-Chow functions. In Section 4 we study the case of projective bundles and prove Theorem 4.1. We also study the projective closure of line bundles and exhibit various explicit examples. In Section 5 we introduce the Chow quotient \( X // (\mathbb{C}^*)^n \) of a projective variety \( X \) under an algebraic action of \((\mathbb{C}^*)^n\), following [KSZ91]. We combine the notions of Chow quotients and the trace maps of [FL92] to prove Theorem 5.5. Various examples of Chow quotients, resulting trace maps, and Euler-Chow series are exhibited. In the Appendix A we present an algebraic framework which places the Euler-Chow series \( E_p(X) \) in a broader context, as an invariant of the Pontrjagin ring of the Chow monoid \( \mathcal{E}_p(X) \). Functoriality and “change of monoid” functors are studied in the more general context of monoids with proper multiplication.

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2. Preliminaries

Let us start with an abelian monoid $M$, whose multiplication we denote by $* : M \times M \rightarrow M$. When no confusion is likely to arise we use an additive notation $+: M \times M \rightarrow M$ with no subscripts attached. We say that $M$ has finite multiplication if $*$ has finite fibers. Typical examples are the freely generated monoids, such as the non-negative integers $\mathbb{Z}_+$ under addition.

Definition 2.1. Given a monoid with finite multiplication $M$, and a commutative ring $S$, denote by $S^M$ the set of all functions from $M$ to $S$. If $f$ and $f'$ are elements in $S^M$, let $f + f' \in S^M$ be defined by pointwise addition, i.e. $(f + f')(m) = f(m) + f'(m)$. Define the product $f * f' \in S^M$ as the “convolution”

$$(f * f')(m) = \sum_{a * b = m} f(a)f'(b).$$

It is easy to see that $S^M$ then becomes a commutative ring with unity, under these operations.

Remark 2.2. The ring $S^M$ can be identified with the completion $S[M]$ of the monoid algebra $S[M]$ at its augmentation ideal. Therefore, the elements of $S^M$ can be written as a formal power series $f = \sum_{m \in M} s_m \cdot t^m$, on variables $t^m$ and coefficients in $S$. In this form the multiplication is given by the relation $t^m t^{m'} = t^{m+m'}$ for elements $m, m' \in M$.

Definition 2.3. Given a monoid morphism $\Psi : M \rightarrow N$, $f \in S^M$ and $g \in S^N$, define $\Psi^* g \in S^M$ and $\Psi_* f \in S^N$ by

$$(\Psi^* g)(m) = g(\Psi(m))$$

and

$$(\Psi_* f)(n) = \sum_{m \in \Psi^{-1}(n)} f(m)$$

if $\Psi$ has finite fibers.

Proposition 2.4. Let $M$ and $N$ be monoids with finite multiplication, and let $\Psi : M \rightarrow N$ be a monoid morphism. Then

1. The pull-back map $\Psi^* : S^N \rightarrow S^M$ is an $S$-module homomorphism.
2. If $\Psi$ has finite fibers then the push-forward map $\Psi_* : S^M \rightarrow S^N$ is a morphism of $S$-algebras.
3. Any ring homomorphism $\Psi : S \rightarrow S'$ induces a ring homomorphism $\Psi_* : S^M \rightarrow S'^M$. 
Proof. 1. Given \( f, g \in S^M' \) and \( m \in M, s \in S \), one has:
\[
\Psi^x(f + sg)(m) = (f + sg)(\Psi(m)) = f(\Psi(m)) + sg(\Psi(m)) = (\Psi^x f)(m) + s(\Psi^x g)(m)
\]

2. The proof that \( \Psi^x \) is an \( S \)-module morphism follows the same pattern as the previous one. As to the multiplicative structure one has for \( f, g \in S^M \) and \( n \in N \):
\[
\Psi^x(f \ast g)(n) = \sum_{m \in \Psi^{-1}(n)} (f \ast g)(m) = \sum_{m \in \Psi^{-1}(n)} \sum_{a, b \in M \atop a \ast N b = m} f(a)g(b)
\]
\[
= \sum_{a, \beta \in N \atop \alpha \ast N \beta = n} \sum_{a \in \Psi^{-1}(\alpha)} f(a)g(b) = \sum_{a, \beta \in N \atop \alpha \ast N \beta = n} (\Psi^x f)(a) \cdot (\Psi^x g)(b)
\]
\[
= (\Psi^x f) \ast (\Psi^x g)(n)
\]

3. The proof is evident. \( \square \)

The last operation we need to introduce is the following exterior product.

**Definition 2.5.** Given monoids \( M \) and \( N \), and a commutative ring \( S \), one can define a map \( \odot : S^M \otimes_S S^N \to S^{M \times N} \). This map sends \( f \otimes g \) to the function \( f \odot g \in S^{M \times N} \) which assigns to \( (m, n) \) the element \( f(m)g(n) \in S \).

**Proposition 2.6.** The operation \( \odot \) is bilinear and associative. In other words, the following diagram commutes:

\[
\begin{array}{ccc}
(S^M \otimes_S S^N) \otimes_S S^P & \cong & S^M \otimes_S (S^N \otimes_S S^P) \\
\odot \otimes 1 & & 1 \otimes \odot \\
S^{M \times N} \otimes_S S^P & \Rightarrow & S^{M \times N} \times_S S^{P \times N}
\end{array}
\]

3. The Euler-Chow series of projective varieties

Let \( X \) be a projective algebraic variety over \( \mathbb{C} \), and let \( p \) be a non-negative integer such that \( 0 \leq p \leq \dim X \). The **Chow monoid** \( \mathcal{C}_p(X) \) of effective \( p \)-cycles on \( X \) is the free monoid generated by the irreducible \( p \)-dimensional subvarieties of \( X \). It is well-known that \( \mathcal{C}_p(X) \) can be written as a countable disjoint union of projective algebraic varieties \( \mathcal{C}_{p, \alpha}(X) \), the so-called **Chow varieties**.

We summarize, in the following statements, a few basic properties of the Chow monoids and varieties which are found in Hoyt [Hoy66], Friedlander [Fri91], and Friedlander and Mazur [FM94a]. For a recent survey and extensive bibliography on the subject, we refer the reader to Lawson [Law95].
Properties 3.1. Let $X$ be a projective variety, and fix $0 \leq p \leq \dim X$.

1. The disjoint union topology on $\mathcal{C}_p(X)$ induced by the classical topology on the Chow varieties, is independent of the projective embedding of $X$; cf. [Hoy66].

2. The restriction of the monoid addition to products of Chow varieties is an algebraic map; [Fri91].

3. An algebraic map $f : X \to Y$ between projective varieties (hence a proper map), induces a natural monoid morphism $f^* : \mathcal{C}_p(Y) \to \mathcal{C}_p(X)$ which is an algebraic continuous map when restricted to a Chow variety; cf. [Fri91]. This is the proper push-forward map.

4. A flat map $f : X \to Y$ of relative dimension $k$, induces a natural monoid morphism $f^* : \mathcal{C}_p(Y) \to \mathcal{C}_{p+k}(X)$ which is an algebraic continuous map when restricted to a Chow variety; cf. [Fri91]. This is the flat pull-back map.

Remark 3.2. Since we work over $\mathbb{C}$, we can define an algebraic continuous map as a continuous map $f : X \to Y$ between varieties which induces an algebraic map $f^\nu : X^\nu \to Y^\nu$ between their weak normalizations; cf. [FM94]. One could alternatively define Chow varieties as the weak normalization of those we consider here, as in Kollár [Kol96]. This does not alter their topology, but transforms Chow varieties into a functor in the category of projective varieties and algebraic morphisms. Either approach can be used in this paper, without altering the results.

Definition 3.3.

1. We denote by $\Pi_p(X)$, the monoid $\pi_0(\mathcal{C}_p(X))$ of path-components of $\mathcal{C}_p(X)$. This is the monoid of “effective algebraic equivalence classes” of effective $p$-cycles on $X$. We use the notation $a \sim_{\text{alg}} b$ to express that two effective cycles $a,b$ are effectively algebraically equivalent.

2. The group of all algebraic $p$-cycles on $X$ modulo algebraic equivalence is denoted $A_p(X)$, and the submonoid of $A_p(X)$ generated by the classes of cycles with non-negative coefficients is denoted by $A^\geq_p(X)$; cf. Fulton [Ful84, §12]. We use the notation $a \sim_{\text{alg}} b$ to express that two cycles $a,b$ are algebraically equivalent.

3. Let $c : \mathcal{C}_p(X) \to H_{2p}(X,\mathbb{Z})$ be the cycle map into singular homology; cf. [Ful84, §19]. The image of $c$ is denoted by $M_p(X)$.

The following result explains the relation between the monoids above.

Proposition 3.4.

1. The Grothendieck group associated to the monoid $\Pi_p(X)$ is $A_p(X)$. In particular, there is a natural monoid morphism $\iota_p : \Pi_p(X) \to A_p(X)$ which satisfies the universal property that any monoid morphism $f : \Pi_p(X) \to G$, from $\Pi_p(X)$ into a group $G$, factors through $A_p(X)$.

2. The image of $\iota_p$ is $A^\geq_p(X)$, and the image of $A^\geq_p(X)$ under the cycle map is $M_p(X)$.
3. The monoid surjection $\tau_p : \Pi_p(X) \to A^\leq_p(X)$ induced by $\iota_p$ is an isomorphism if and only if $\Pi_p(X)$ has cancellation law.

4. Both $\tau_p : \Pi_p(X) \to A^\leq_p(X)$ and $c_p : A^\leq_p(X) \to M_p(X)$ are finite monoid morphisms.

Proof. The first assertion is proven in [Fri91], and follows from standard arguments, e.g. in Samuel [Sam71]. The second assertion follows from the definitions and the universal property just described. The third assertion follows from the elementary fact that the universal map from an abelian monoid into its group completion is injective if and only if the monoid has cancellation law. To prove the last assertion, consider a projective embedding of $X$. The cycles supported in $X$ with a fixed degree $d$ in the ambient projective space form a projective variety, which is then a finite union of components of $\mathcal{C}_p(X)$. The assertion now follows easily from these observations.

The following result is found in Elizondo [Eli94].

**Proposition 3.5.** Given a complex projective algebraic variety $X$ and $0 \leq p \leq \dim X$, the monoids $\mathcal{C}_p(X)$, $\Pi_p(X)$, $A^\leq_p(X)$ and $M_p(X)$ all have finite multiplication. In particular, they all belong to $\mathfrak{Atm}_p$; cf. Appendix A.

Proof. One has surjective monoid morphisms:

\[
\mathcal{C}_p(X) \xrightarrow{\pi_p} \Pi_p(X) \xrightarrow{\iota_p} A^\leq_p(X) \xrightarrow{c_p} M_p(X),
\]

so that when a projective embedding $j : X \hookrightarrow \mathbb{P}^n$ is chosen one obtains a commutative diagram

$$
\begin{array}{cccccc}
\mathcal{C}_p(X) & \xrightarrow{j^*} & \Pi_p(X) & \xrightarrow{j^*} & A^\leq_p(X) & \xrightarrow{j^*} & M_p(X) \\
\mathcal{C}_p(\mathbb{P}^n) & \xrightarrow{\cong} & \Pi_p(\mathbb{P}^n) & \xrightarrow{\cong} & A^\leq_p(\mathbb{P}^n) & \xrightarrow{\cong} & M_p(\mathbb{P}^n) \\
\mathbb{Z}_+ & \xrightarrow{\cong} & \mathbb{Z}_+ & \xrightarrow{\cong} & \mathbb{Z}_+ & \xrightarrow{\cong} & \mathbb{Z}_+
\end{array}
$$

where the leftmost vertical arrow is a closed inclusion. Recall that $\mathcal{C}_p(\mathbb{P}^n) = \bigsqcup_{d \in \mathbb{Z}_+} \mathcal{C}_{p,d}(\mathbb{P}^n)$ where $\mathcal{C}_{p,d}(\mathbb{P}^n)$ is a projective connected algebraic (Chow) variety. Furthermore, $j(\mathcal{C}_p(X)) \cap \mathcal{C}_{p,d}(\mathbb{P}^n)$ is a subvariety for all $d$. It follows that $\Pi_p(X)$, $A^\leq_p(X)$ and $M_p(X)$ are all discrete, and that $\mathcal{C}_p(X)$ has finite multiplication, since it is free. Proposition A.3 now shows that $\Pi_p(X)$, $A^\leq_p(X)$ and $M_p(X)$ also have finite multiplication.

**Definition 3.6.** The (algebraic) $p$-th Euler-Chow function of $X$ is the function

\[
E_p(X) : \Pi_p(X) \to \mathbb{Z} \\
\alpha \mapsto \chi(\mathcal{C}_{p,\alpha}(X)),
\]

which sends $\alpha \in \Pi_p(X)$ to the topological Euler characteristic of $\mathcal{C}_{p,\alpha}(X)$ (in the classical topology).
Following Remark 2.2 we associate a variable $t^\alpha$ to each $\alpha \in \Pi_p(X)$ and express the $p$-th Euler-Chow function as a formal power series

$$E_p(X) = \sum_{\alpha \in \Pi_p(X)} \chi(\mathcal{C}_p,\alpha(X)) \ t^\alpha.$$  

(4)

Remark 3.7.

1. The Pontrjagin ring $H_*(\mathcal{C}_p(X), \mathbb{Z})$ comes equipped with lots of additional structure, which can be better expressed using the terminology of Appendix A. It follows from Example A.6 that it becomes a finite $\Pi_p(X)$-graded ring. In other words, $H_*(\mathcal{C}_p(X), \mathbb{Z}) \in \mathfrak{A}_R^{fin}(\Pi_p(X))$. Under this framework, given $X$ and $p$ as above, one could define the (algebraic) Hilbert-Chow function of $X$ as the function $P^X_t \in \mathbb{Z}[t]^{\Pi_p(X)}$ obtained as the Hilbert $\Pi_p(X)$-series of the Pontrjagin ring $H_*(\mathcal{C}_p(X), \mathbb{Z})$ in the sense of Definition A.10. In particular, the algebraic Euler-Chow function $E_p(X) \in \mathbb{Z}[t]^{\Pi_p(X)}$ is the Euler $\Pi_p(X)$-series of $H_*(\mathcal{C}_p(X), \mathbb{Z})$; cf. Definition A.10.

2. One could in a similar fashion define the $p$-th Euler-Chow function mapping either $A_{\geq p}(X)$ or $M_p(X)$ to $\mathbb{Z}$. These would simply be the functions $\tau(E_p(X))$ and $(c_p \circ \tau_p)(E_p(X))$; cf. Definition 2.3.

Example 3.8.

1. If $X$ is a connected variety, then $\mathcal{C}_0(X) = \bigsqcup_{d \in \mathbb{Z}_+} SP_d(X)$, where $SP_d(X)$ is the $d$-fold symmetric product of $X$. Therefore, the 0-th Euler-Chow function is given by

$$E_0(X) = \sum_{d \geq 0} \chi(SP_d(X)) \ t^d = \left( \frac{1}{1-t} \right)^{\chi(X)},$$

according to McDonald’s formula [Mac62].

2. For $X = \mathbb{P}^n$, one has $\Pi_p(\mathbb{P}^n) \cong \mathbb{Z}_+$, with the isomorphism given by the degree of the cycles. In this case, the $p$-th Euler-Chow function was computed in [LY87]:

$$E_p(\mathbb{P}^n) = \sum_{d \geq 0} \chi(\mathcal{C}_{p,d}(\mathbb{P}^n)) \ t^d = \left( \frac{1}{1-t} \right)\binom{n+1}{p+1}.$$

3. Suppose that $X$ is an $n$-dimensional variety such that $\text{Pic}(X) \cong \mathbb{Z}$, generated by a very ample line bundle $L$. Then, we have seen in Example 1.1 that $\Pi_{n-1}(X) \cong \mathbb{Z}_+$ and that $E_{n-1}(X)$ is precisely the Hilbert function for the projective embedding of $X$ induced by $L$.

4. Projective bundle formulas

In this section we exhibit a formula for the Euler-Chow function of certain projective bundles over a variety $W$, and compute several examples. The basic setup is the following. Consider two
algebraic vector bundles $E_1 \to W$ and $E_2 \to W$ over a complex projective variety $W$. The various maps involved in our discussion are displayed in the commutative diagram below:

\[ \begin{array}{ccc}
\mathbb{P}(E_1) & \xLeftarrow{i_1} & \mathbb{P}(E_1 \oplus E_2) & \xLeftarrow{i_2} & \mathbb{P}(E_2) \\
p_1 \downarrow & & q \downarrow & & p_2 \\
W & & W & & W
\end{array} \]

where $p_1$, $p_2$ and $q$ are projections from the indicated projective bundles, and $i_1$, $i_2$ are the canonical inclusions.

We will introduce a monoid monomorphism $t_p : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))$ in Definition 4.3, which is a closed inclusion. In this way we become equipped with three morphisms of monoids with finite multiplication:

\begin{align*}
(5) & \quad i_{1p} : \Pi_p(\mathbb{P}(E_1)) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2)) \\
(6) & \quad i_{2p} : \Pi_p(E_2) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2)) \\
(7) & \quad \varphi_p : \Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2))
\end{align*}

induced by $i_1$, $i_2$, and $\varphi_p$, respectively, and let $\psi_p(a,b,c) = \varphi_p(a) + i_{1p}(b) + i_{2p}(c)$.

The main result in this section is the following.

**Theorem 4.1.** Let $E_1$ and $E_2$ be algebraic vector bundles over a connected projective variety $W$, of ranks $e_1$ and $e_2$, respectively, and let $0 \leq p \leq e_1 + e_2 - 1$. Then the $p$-th Euler-Chow function of $\mathbb{P}(E_1 \oplus E_2)$ is given by

\[ E_p(\mathbb{P}(E_1 \oplus E_2)) = \Psi_p(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \circ E_p(\mathbb{P}(E_1)) \circ E_p(\mathbb{P}(E_2)) \]
is precisely the blow-up of $\mathbb{P}(E_1 \oplus E_2)$ along $\mathbb{P}(E_1) \amalg \mathbb{P}(E_2)$, which we denote by $Q$, for short; see Lascou and Scott [LS75] for details. Let $b : Q \to \mathbb{P}(E_1 \oplus E_2)$ denote the blow-up map.

Since $\pi$ is a flat map of relative dimension 1, and $b$ is a proper map, one has two algebraic continuous homomorphisms (cf. 3.1), given by the flat pull-back
\begin{equation}
\pi^* : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times W \mathbb{P}(E_2)) \to \mathcal{C}_p(Q) \tag{11}
\end{equation}
and the proper push-forward
\begin{equation}
b_* : \mathcal{C}_p(Q) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2)). \tag{12}
\end{equation}

**Definition 4.3.** The map $t_p : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times W \mathbb{P}(E_2)) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))$ is defined as the composition $t_p = b_* \circ \pi^*$.

The following lemma will be used as a reference in a few places throughout the paper.

**Lemma 4.4.** Let $X = \coprod_{i \in \mathbb{N}} X_i$ and $Y = \coprod_{j \in \mathbb{N}} Y_j$ be spaces which are a countable disjoint union of connected projective varieties, and let $f : X \to Y$ be a continuous map such that the restriction $f|_{X_i}$ is an algebraic continuous map from $X_i$ into some $Y_j$. If $f$ is a bijection, then it is a homeomorphism in the classical topology.

**Proof.** Given $j \in \mathbb{N}$, one can write $f^{-1}(Y_j)$ as a disjoint union $\coprod_{k \in \Lambda} X_{i_k}$, for a certain collection of connected components of $X$. On the other hand, since $f$ is an algebraic continuous map, then $f(X_{i_k})$ is a Zariski closed subset of $Y_j$, and since $f$ is one-to-one, then $Y_j$ is written as a union of a (at most countable) family of disjoint closed algebraic subsets. This contradicts the connectedness of $Y_j$, unless there is only one $j(i)$ such that $f^{-1}(Y_j) = X_{j(i)}$, in which case $f$ sends $X_{j(i)}$ homeomorphically onto $Y_j$. \qed

We now prove Theorem [4.4].

**Proof of theorem [4.4].** Consider the action of $\mathbb{C}^*$ on $\mathbb{P}(E_1 \oplus E_2)$ given by scalar multiplication on one of the factors of $E_1 \oplus E_2$, whose fixed point set $\mathbb{P}(E_1 \oplus E_2)^{\mathbb{C}^*}$ consists of $\mathbb{P}(E_1) \amalg \mathbb{P}(E_2)$. It is easy to see that this induces an algebraic continuous action on $\mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))$ via monoid automorphisms, and our next step is to identify its fixed point set.

Consider the map
\begin{equation}
\phi_p : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times W \mathbb{P}(E_2)) \times \mathcal{C}_p(\mathbb{P}(E_1)) \times \mathcal{C}_p(\mathbb{P}(E_2)) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2)), \tag{13}
\end{equation}
defined as $\phi_p(a, b, c) = t_p(a) + i_1(b) + i_2(c)$. We claim that $\phi_p$ is a homeomorphism onto the fixed point set $\mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))^{\mathbb{C}^*}$.

If an element $\sigma = \sum_i n_i V_i \in \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))$ is fixed under the action, then each of its irreducible components must clearly be invariant under the action, as well. On the other hand, an invariant irreducible subvariety $V \subset \mathbb{P}(E_1 \oplus E_2)$ may be of two types:
i: those whose general points are fixed under the action (and hence all points of $V$ are fixed);

ii: and those whose general points have non-trivial orbits (of dimension 1).

Recall that the Chow monoids $\mathcal{C}_p(X)$ of any variety $X$ are freely generated by the irreducible $p$-dimensional subvarieties of $X$. Given cycles $\sigma_k \in \mathcal{C}_p(\mathbb{P}(E_k))$, $k = 1, 2$, the support of $i_{k*}(\sigma_k)$ is contained in $\mathbb{P}(E_i)$, and hence the images of $\mathcal{C}_p(\mathbb{P}(E_1))$ and $\mathcal{C}_p(\mathbb{P}(E_2))$ under $i_1$ and $i_2$ are freely generated by disjoint subsets of the generating set of $\mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))$, consisting of varieties of the first type described above.

On the other hand, given a $(p - 1)$-dimensional subvariety $Z$ of $\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)$, its inverse image $\pi^{-1}(Z)$ is a $p$-dimensional subvariety of $Q$ whose points outside the exceptional divisor of the blow-up map $b$, have orbits of dimension 1. Since $b$ is a $C^*$-equivariant birational map, the image $b(\pi^{-1}(Z))$ is an invariant irreducible subvariety of $\mathbb{P}(E_1 \oplus E_2)$ of the second type. It follows that the images of $i_{1*}, i_{2*}$ and $t_p$ are respectively freely generated by disjoint subsets, and this proves that the map $\phi_p$ is injective.

In order to show surjectivity, one just needs to show that every invariant, irreducible subvariety $V \subset \mathbb{P}(E_1 \oplus E_2)$ of the second type, is of the form $b(\pi^{-1}(Z))$ for some $(p - 1)$-dimensional subvariety of $\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)$, since those of the first type are, by definition, in the image of $i_{1*}$ and $i_{2*}$. Indeed, let $\tilde{V} \subset Q$ be the proper transform of $V$ under $b$, and let $Z = \pi(\tilde{V}) \subset \mathbb{P}(E_1) \times_W \mathbb{P}(E_2)$. Since the general points of $V$ have orbits of dimension 1, so do the general points of $\tilde{V}$, and this shows that the general fiber of $\pi|_{\tilde{V}} : \tilde{V} \to Z$ has dimension 1. In particular, $\tilde{V} = \pi^{-1}(Z)$ and, since $\pi|_{\tilde{V}}$ is a bundle projection and $b$ sends $\tilde{V}$ birationally onto $V$, one concludes that $t_p(Z) = b_* \circ \pi^* (Z) = V$.

The arguments above show that $\phi_p$ provides an algebraic continuous bijection from the product $\mathcal{C}_{p-1}((\mathbb{P}(E_1) \times_W \mathbb{P}(E_2))) \times \mathcal{C}_p(\mathbb{P}(E_1)) \times \mathcal{C}_p(\mathbb{P}(E_2))$ onto the fixed point set $\mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2))^{C^*}$. Notice that for each $\alpha \in \Pi_p(\mathbb{P}(E_1 \oplus E_2))$ the fixed point set $\mathcal{C}_{p,\alpha}(\mathbb{P}(E_1 \oplus E_2))^{C^*}$ is an algebraic subset of $\mathcal{C}_{p,\alpha}(\mathbb{P}(E_1 \oplus E_2))$ (not necessarily connected), and hence it can also be written as a countable disjoint union of projective varieties. One then applies Lemma 4.4 to conclude that $\phi_p$ is a homeomorphism onto its image.

It is a general fact that the Euler characteristic of a variety with an algebraic torus action equals that of its fixed point set; see Lawson and Yau [LY87] or Elizondo and Hain [EH96]. Therefore, given $\alpha \in \Pi_p(\mathbb{P}(E_1 \oplus E_2))$, it follows that

\begin{equation}
\chi(\mathcal{C}_{p,\alpha}(\mathbb{P}(E_1 \oplus E_2))) = \chi(\mathcal{C}_{p,\alpha}(\mathbb{P}(E_1 \oplus E_2))^{C^*}).
\end{equation}

On the other hand, if $\Psi_p$ is the morphism induced by $\phi_p$ between the monoids of connected components then

\begin{equation}
\mathcal{C}_{p,\alpha}(\mathbb{P}(E_1 \oplus E_2))^{C^*} = \prod_{(a,b,c) \in \Psi_p^{-1}(\alpha)} \mathcal{C}_{p-1,a}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \times \mathcal{C}_{p,b}(\mathbb{P}(E_1)) \times \mathcal{C}_{p,c}(\mathbb{P}(E_2)),
\end{equation}
since \( \phi_p \) is a homeomorphism onto its image. Therefore (14) implies that

\[
\chi (\mathcal{C}_{p,a} (\mathbb{P}(E_1 \oplus E_2))) = \sum_{(a,b,c) \in \mathbb{P}_p^{-1}(\alpha)} \chi(\mathcal{C}_{p-1,a} (\mathbb{P}(E_1) \times W \mathbb{P}(E_2))) \cdot \chi(\mathcal{C}_{p,b} (\mathbb{P}(E_1))) \cdot \chi(\mathcal{C}_{p,c} (\mathbb{P}(E_2))).
\]

It follows from Definitions 4.3 and 2.3 that (10) holds. \( \square \)

**Remark 4.5.** The preceding proof also shows that the map \( t_p \), introduced in Definition 4.3, is a closed inclusion, as claimed.

### 4.1. Projective closure of line bundles

Here we consider the case where \( E_1 = E \) is a line bundle over a projective variety \( W \), and \( E_2 = 1_W \) is the trivial line bundle. In this case,

\[
\mathbb{P}(E_1) = \mathbb{P}(E_2) = \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) = W,
\]

and the inclusions \( i_k : \mathbb{P}(E_k) \to \mathbb{P}(E_1 \oplus E_2) \), \( k = 1, 2 \) (cf. Diagram 3), become sections \( i : W \to \mathbb{P}(E \oplus 1) \) and \( j : W \to \mathbb{P}(E \oplus 1) \) of the projective bundle \( \mathbb{P}(E \oplus 1) \) over \( W \).

If \( \xi = c_1 (\mathcal{O}_{E \oplus 1}(1)) \) denotes the first Chern class of the canonical bundle over \( \mathbb{P}(E \oplus 1) \), then the map

\[
T : \mathcal{A}_{p-1}(W) \oplus \mathcal{A}_p(W) \to \mathcal{A}_p(\mathbb{P}(E \oplus 1))
\]

\[
(\alpha, \beta) \mapsto q^* \alpha + \xi \cap q^* \beta
\]

is an isomorphism. Note that, since \( \xi \cap q^* \beta = i_* \beta \), the isomorphism above restricts to an injection

\[
T^\geq : \mathcal{A}_{p-1}^\geq (W) \oplus \mathcal{A}_p^\geq (W) \to \mathcal{A}_p^\geq (\mathbb{P}(E \oplus 1)).
\]

**Lemma 4.6.** Let \( E \) be a line bundle over \( W \) which is generated by its global sections. Then:

- **a:** The injection \( T^\geq \) is an isomorphism;
- **b:** If \( \Pi_*(W) \) are monoids with cancellation law for every *, then so are \( \Pi_*(\mathbb{P}(E \oplus 1)) \). Equivalently, if the natural surjections \( \Pi_p(W) \to \mathcal{A}_p^\geq(W) \) are isomorphisms for all \( p \), then so are the surjections \( \Pi_p(\mathbb{P}(E \oplus 1)) \to \mathcal{A}_p^\geq(\mathbb{P}(E \oplus 1)) \).

**Proof.** **a.** It follows from the proof of Theorem 4.1 that every effective \( p \)-cycle \( a \) in \( \mathbb{P}(E \oplus 1) \) is effectively algebraically equivalent to a cycle of the form \( i_* b + j_* c + q^* d \), where \( b, c \in \mathcal{C}_p(W) \) and \( d \in \mathcal{C}_{p-1}(W) \). Since \( E \) is generated by its sections, one can find a section \( s : W \to E \) of \( E \) whose zero locus \( Z \subset W \) intersects \( c \) properly. Let \( \tilde{s} : X \to \mathbb{P}(E \oplus 1) \) denote the composition \( \iota \circ s \), where \( \iota : E \hookrightarrow \mathbb{P}(E \oplus 1) \) is the open inclusion. Then, the closure of the orbit of \( \tilde{s}_* c \) under the \( \mathbb{C}^* \) action on \( \mathcal{C}_p(\mathbb{P}(E \oplus 1)) \) contains two fixed points: \( j_* c \) and \( i_* c + q^*(Z \cap c) \). It follows that \( a \sim_{\text{alg}} i_* (b + c) + q^*(d + Z \cap c) \), and this shows that \( T^\geq \) is surjective.
b. We need to show that if \( a \sim_{\text{alg}} a' \) then \( a \sim_{\text{alg}^+} a' \), where \( a, a' \in \mathcal{C}_p(\mathbb{P}(E \oplus 1)) \). The arguments described in the first half of the proof show that one can find cycles \( b, b' \in \mathcal{C}_p(W) \) and \( c, c' \in \mathcal{C}_{p-1}(W) \) such that \( a \sim_{\text{alg}^+} i_* b + q^* c \) and \( a' \sim_{\text{alg}^+} i_* b' + q^* c' \). Since \( a \sim_{\text{alg}} a' \), one concludes that \( b = q_* a \sim_{\text{alg}} q_* a' = b' \), and hence the hypothesis implies that \( b \sim_{\text{alg}^+} b' \). Also, \( j_* W \) intersects properly both \( i_* b + q^* c \) and \( i_* b' + q^* c' \), and hence \( j_* c = j_* W \cdot (i_* b + q^* c) \sim_{\text{alg}} j_* W \cdot (i_* b' + q^* c') = j_* c' \). Applying \( q_* \) shows that \( c \sim_{\text{alg}} c' \) and hence \( c \sim_{\text{alg}^+} c' \), and this concludes the proof. The equivalence of both statements follows from Proposition 3.4.

\[ \square \]

**Corollary 4.7.** Under the same hypothesis, the homomorphism

\[
\Psi_p : \Pi_{p-1}(W) \oplus \Pi_p(W) \oplus \Pi_p(W) \to \Pi_p(\mathbb{P}(E \oplus 1)) \cong \Pi_{p-1}(W) \oplus \Pi_p(W)
\]

sends \((\alpha, \beta, \gamma)\) to \((\alpha + \beta, \xi \cap \gamma + \alpha)\).

**Proof.** This is clear. \( \square \)

**Example 4.8.** Let \( W = \mathbb{P}^n \) and \( E = \mathcal{O}_{\mathbb{P}^n}(d) \), with \( d \geq 0 \). We will compute \( \Pi_p(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1)) \) for \( 1 \leq p \leq n \). First observe that \( W \) and \( E \) satisfy the hypothesis of Lemma 4.6, and that

\[
\Pi_{p-1}(\mathcal{O}(E)) \cong \Pi_p(\mathbb{P}^n) = \mathbb{Z}_+ : [p^{p-1}] \cong \mathbb{Z}_+ \quad \text{and} \quad \Pi_p(\mathbb{P}(E)) \cong \Pi_p(\mathbb{P}^n) \cong \mathbb{Z}_+ \cdot [p^p] \cong \mathbb{Z}_+,
\]

where \([p^p]\) denotes the class of a \( p \)-dimensional linear subspace of \( \mathbb{P}^n \). By definition, the map \( \Psi_p \) fits into a commutative diagram

\[
\begin{array}{ccc}
\Pi_{p-1}(\mathcal{O}_{\mathbb{P}^n}(d)) \times \Pi_p(\mathcal{O}_{\mathbb{P}^n}(d)) \times \Pi_p(\mathbb{P}^n) & \xrightarrow{\Psi_p} & \Pi_{p-1}(\mathbb{P}^n) \times \Pi_p(\mathbb{P}^n)
\end{array}
\]

and sends \( \Psi_p : (a[p^{p-1}], b[p^p], c[p^p]) \mapsto (a[p^{p-1}] + c \cdot c_1(\mathcal{O}_{\mathbb{P}^n}(d)) \cap [p^p], (b + c)[p^p]) \). Since \( c_1(\mathcal{O}_{\mathbb{P}^n}(d)) \cap [p^p] = d[p^{p-1}] \) we then identify \( \Psi_p \) with \((a, b, c) \mapsto (a + c \cdot d, b + c)\).

One can associate variables \( t_0, t_1 \) to the generators of \( \Pi_p(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1)) \cong \mathbb{Z}_+ \oplus \mathbb{Z}_+ \), and identify \((\alpha_0, \alpha_1) \in \Pi_p(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1)) \) with \( t_0^{\alpha_0} t_1^{\alpha_1} \). It follows from Theorem 4.1 and Remark 2.2...
that $E_p(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1)$ can be written as

$$E_p(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1) =$$

$$= \sum_{\alpha_0, \alpha_1} E_p(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1))(\alpha_0, \alpha_1) t_0^{\alpha_0} t_1^{\alpha_1}$$

$$= \sum_{\alpha_0, \alpha_1} \Psi_{p, \#}(E_{p-1}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d))) \odot E_p(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d))) \odot E_p(\mathbb{P}^n))(\alpha_0, \alpha_1) t_0^{\alpha_0} t_1^{\alpha_1}$$

$$= \sum_{\alpha_0, \alpha_1} \left( \sum_{(a,b,c) \in \Psi_p^{-1}(\alpha_0, \alpha_1)} E_{p-1}(\mathbb{P}^n)(a) \cdot E_p(\mathbb{P}^n)(b) \cdot E_p(\mathbb{P}^n)(c) \right) t_0^{a} t_1^{b}$$

$$= \left( \sum_{a \geq 0} E_{p-1}(\mathbb{P}^n)(a) \cdot t_0^a \right) \left( \sum_{b \geq 0} E_p(\mathbb{P}^n)(b) \cdot t_1^b \right) \left( \sum_{c \geq 0} E_p(\mathbb{P}^n)(c) \cdot (t_0^c t_1^c) \right).$$

In Lawson-Yau [LY87] it was shown that

$$\sum_{k \geq 0} E_p(\mathbb{P}^n)(k) t^k = \left( \frac{1}{1-t} \right)^{n+1}$$

and hence $E_p(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1))$ is then written as:

$$E_p(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1)) = \left( \frac{1}{1-t_0} \right)^{n+1} \left( \frac{1}{1-t_1} \right)^{n+1} \left( \frac{1}{1-t_0^d t_1} \right)^{n+1}$$

**Subexamples 4.9.**

1. When $d = 0$ one has $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1) = \mathbb{P}^n \times \mathbb{P}^1$ and our computations recover the formula

$$E_p(\mathbb{P}^n \times \mathbb{P}^1) = \left( \frac{1}{1-t_0} \right)^{n+1} \left( \frac{1}{1-t_1} \right)^{n+1} \left( \frac{1}{1-t_0^d t_1} \right)^{n+1}$$

obtained in [LY87]. Furthermore, our general formula in Theorem 4.1 gives an inductive process to compute $E_p(\mathbb{P}^n \times \mathbb{P}^m)$ in general.

2. When $n = 1$, the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d) \oplus 1)$ over $\mathbb{P}^1$ is the Hirzebruch surface $F_d$, and the formula

$$E_1(F_d) = \left( \frac{1}{1-t_0} \right)^2 \left( \frac{1}{1-t_1} \right) \left( \frac{1}{1-t_0^d t_1} \right)$$

recovers the one obtained in Elizondo [Eli94].
3. When \( d = 1 \), the \( \mathbb{P}^1 \)-bundle \( \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^n}(1) \oplus 1 \right) \) over \( \mathbb{P}^n \) is just the blow-up \( \widetilde{\mathbb{P}}^{n+1} \) of \( \mathbb{P}^{n+1} \) at a point, and the expression

\[
E_p(\widetilde{\mathbb{P}}^{n+1}) = \left( \frac{1}{1 - t_0} \right)^{n+1} \left( \frac{1}{1 - t_1} \right)^{n+1} \left( \frac{1}{1 - t_0 t_1} \right)^{n+1}
\]

recovers the formula obtained in [Eli94].

5. Chow quotients and Euler-Chow series

In this section we consider a projective algebraic variety \( X \) equipped with an algebraic action of \( \mathbb{C}^* \). In general, an algebraic action of a torus \( T = (\mathbb{C}^*)^k \) provides stratifications on \( X \), and here we introduce the following one, much in the spirit of Kapranov [Kap93].

**Definition 5.1.** Let \( T = (\mathbb{C}^*)^k \) act algebraically on \( X \) and let \( \alpha \in \Pi_p(X) \) be a fixed element with \( 0 \leq p \leq k \). We say that a \( p \)-dimensional orbit \( T \cdot x \) is of type \( \alpha \) if its closure \( \overline{T \cdot x} \) lies in the component \( C_{p,\alpha}(X) \), when viewed as a \( p \)-dimensional effective cycle. One can stratify \( X \) according to the orbit type of its elements, introducing the *Chow stratification* of \( X \).

Assume that \( T \) has orbits of maximal dimension \( k \). It is easy to see that when \( X \) is irreducible then there is a unique maximal open stratum \( X^o \) consisting of points whose orbits have maximal dimension \( k \) and are of the same type \( \alpha_0 \), for some \( \alpha_0 \in \Pi_k(X) \). In particular, there is an embedding \( X^o/T \hookrightarrow C_{k,\alpha_0}(X) \) and following [KSZ91] we introduce the following notion.

**Definition 5.2.** The *Chow quotient* \( X/\!\!/T \) is the weak normalization of the closure of \( X^o/T \) in \( C_{k,\alpha_0}(X) \).

**Remark 5.3.** When \( X^o/T \) is proper, the Chow quotient gives a closed embedding of \( X^o/T \) into the appropriate Chow variety.

We combine the use of Chow quotients with the notion of *trace maps* introduced in Friedlander and Lawson [FL92 §7.1], which is described as follows. A Chow variety \( C_{k,\alpha}(X) \), being projective, has its own Chow monoid \( C_p(C_{k,\alpha}(X)) \). The trace map is a continuous monoid morphism

\[
tr : C_p(C_{k,\alpha}(X)) \to C_{p+k}(X)
\]

that, roughly speaking, associates to an irreducible \( p \)-dimensional subvariety \( W \subset C_{k,\alpha}(X) \) its “total fundamental cycle”, which is a \((p + k)\)-cycle in \( X \).

Combining the two constructions above, one associates to such a \( T \)-action on an irreducible variety \( X \), a trace map

\[
t_p : C_p(X/\!\!/T) \to C_{p+k}(X).
\]
This map, in turn, induces a monoid morphism

\[ \varphi_p : \Pi_p(X/T) \to \Pi_{p+k}(X), \]

in the level of \( \pi_0 \). In other words, \( t_p(\mathcal{C}_{p,a}(X/T)) \subset \mathcal{C}_{p+k, \varphi_p(\alpha)}(X) \). Explicit examples of such monoid morphisms are given in Proposition 5.11 and Corollary 5.12 below.

**Proposition 5.4.** Let \( \mathbb{T} \cong (\mathbb{C}^*)^k \) act on an irreducible projective variety \( X \) with generic orbits of maximal dimension \( k \). If \( \dim(X/T - X^o/T) < p \leq \dim X - k \) then the trace map \( t_p : \mathcal{C}_p(X/T) \to \mathcal{C}_{p+k}(X) \) is injective.

**Proof.** Let \( W \subset X/T \) be an irreducible variety of dimension \( p \). The hypothesis on dimensions implies that \( W^o := W \cap X^o/T \) is an open dense subvariety of \( W \). In order to define the trace \( t_p(W) \) one considers the cycle \( Z_W \subset W \times X \), defined as the “closure of the cycle”

\[ Z_W^o = \{(w, c_w) \mid w \in W^o \text{ and } c_w = \text{ cycle whose Chow point is } w\}; \]

cf. [PL92, §7.1]. By definition, \( t_p(W) = pr_{2*}(Z_W) \), where \( pr_2 \) is projection onto the second factor of \( W \times X \).

Note that, since the fibers of the projection \( Z_W^o \to W^o \) are irreducible and \( W^o \) is irreducible, then \( Z_W \) is irreducible of dimension \( p + k \leq \dim X \). Furthermore, if \( (a, b), (a', b) \in Z_W^o \) then \( a = a' \), for if \( a, a' \in W^o \subset X^o/T \) then the orbits in \( X \) represented by \( a \) and \( a' \) have a common point \( b \), hence \( a = a' \). It follows that \( pr_2 \) maps \( Z_W \) birationally onto its image, and thus \( t_p(W) := pr_{2*}(Z_W) \) is an irreducible cycle with multiplicity 1.

Suppose that \( t_p(W) = t_p(W') \), where both \( W \) and \( W' \) are irreducible. An element \( w \in W^o \) corresponds to an orbit \( \mathbb{T} \cdot x \subset pr_2(Z_W) = pr_2(Z_{W'}) \), with \( x \in X^o \). Therefore \( w \in W'^o \), and hence \( W = W' \). Since that Chow monoids are freely generated by the irreducible subvarieties, one concludes that the trace map \( t_p \) is injective. \( \Box \)

We now consider the situation where \( X \) is an irreducible projective variety, on which \( \mathbb{C}^* \) acts in such a fashion that the fixed point set \( X^{\mathbb{C}^*} \) has only two connected components \( X_1 \) and \( X_2 \). Let \( i_1 : X_1 \hookrightarrow X \) and \( i_2 : X_2 \hookrightarrow X \) denote the inclusion maps, and let \( t_{p-1} : \mathcal{C}_{p-1}(X/\mathbb{C}^*) \to \mathcal{C}_p(X) \) be the trace morphism (22). These maps induce a monoid morphism

\[ \Psi_p : \Pi_{p-1}(X/\mathbb{C}^*) \times \Pi_p(X_1) \times \Pi_p(X_2) \to \Pi_p(X) \]

\[ (\alpha, \beta, \gamma) \mapsto \varphi_{p-1}(\alpha) + i_{1*}\beta + i_{2*}\gamma, \]

which yields our next result.

**Theorem 5.5.** Let \( X \) be an smooth projective variety on which \( \mathbb{C}^* \) acts algebraically. If \( X^{\mathbb{C}^*} \) is the union of two connected components \( X_1 \) and \( X_2 \), then for each \( 0 \leq p \leq \dim X \) one has

\[ E_p(X) = \Psi_p \left( E_{p-1}(X/\mathbb{C}^*) \odot E_p(X_1) \odot E_p(X_2) \right). \]
Proof. The constructions above give a continuous monoid morphism

\[ \phi_p : \mathcal{C}_{p-1}(X//\mathbb{C}^*) \times \mathcal{C}_p(X_1) \times \mathcal{C}_p(X_2) \to \mathcal{C}_p(X) \]

\[ (a, b, c) \mapsto t_{p-1}(a) + i_1 b + i_2 c, \]

which induces \( \Psi_p \) in the level of \( \pi_0 \); cf. (23).

It is clear that the image of \( \phi_p \) lies in the fixed point set \( \mathcal{C}_p(X)^{\mathbb{C}^*} \). We show that \( \phi_p \) is injective and that it surjects onto \( \mathcal{C}_p(X)^{\mathbb{C}^*} \).

Suppose that \( \phi_p(a, b, c) = \phi_p(a', b', c') \). By “disjointness of support” one sees immediately that \( b = b', c = c' \) and \( t_{p-1}(a) = t_{p-1}(b) \).

It follows from the decomposition of \( X \) described in Bialynicki-Birula [Bia73, §4] that \( X^o = X - X^{\mathbb{C}^*} \), and that \( X^o/\mathbb{C}^* \) is proper. Therefore \( X//\mathbb{C}^* = X^o/\mathbb{C}^* \), and Proposition 5.4 implies that the trace map \( t_{p-1} \) is injective, which together with (26) shows that \( \phi_p \) is injective.

Given an irreducible, \( p \)-dimensional, and \( \mathbb{C}^* \)-invariant subvariety \( V \subset X \), then either \( V \subset X^{\mathbb{C}^*} \), in which case it clearly lies in the image of \( \phi_p \), or \( V^o = V \cap X^o \neq \emptyset \). In the latter case, define \( W = V^o/\mathbb{C}^* \subset X//\mathbb{C}^* \). A simple inspection shows that \( t_p(W) = V \). This suffices to show that \( \phi_p \) surjects onto \( \mathcal{C}_p(X)^{\mathbb{C}^*} \) and, using Lemma 4.4, one concludes that \( \phi_p \) is a homeomorphism onto its image.

The rest of the proof uses the same arguments as in the last paragraph in the proof of Theorem 4.1. \( \square \)

Remark 5.6. One can directly prove that the Chow quotient \( \mathbb{P}(E_1 \oplus E_2)//\mathbb{C}^* \) is isomorphic to \( \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \), whenever \( E_1 \) and \( E_2 \) are algebraic vector bundles over a variety \( W \). In this case, Theorem 4.1 becomes a consequence of the result above whenever \( W \) is smooth.

Next, we describe some examples of Chow quotients, trace maps and resulting computations of Euler-Chow series.

5.1. Examples.

The linear action of \( \mathbb{C}^* \) on the last coordinate of \( \mathbb{C}^{n+1} \) induces an algebraic \( \mathbb{C}^* \) action on the Grassmannian \( G_d(\mathbb{P}^n) \) of \( d \)-planes in \( \mathbb{P}^n \). More generally, one obtains an algebraic action on all partial flag varieties \( F(d_1, \ldots, d_r; \mathbb{P}^n) \) of nested linear subspaces \( D_1 \subset \cdots \subset D_r \) of \( \mathbb{P}^n \), satisfying \( \dim D_i = d_i \).

We first describe the orbit structure and the Chow quotient of both \( G_d(\mathbb{P}^n) \) and \( F(0, 1; \mathbb{P}^n) \), under this \( \mathbb{C}^* \) action.
5.1.1. The Grassmannian case. We adopt the convention that $G_d(\mathbb{P}^n) = \emptyset$, whenever $d < 0$, and that $\chi(0) = 0$. Fix $p_\infty = [0 : \cdots : 0 : 1] \in \mathbb{P}^n$, and let $L \in G_d(\mathbb{P}^n)$ be fixed under the above action. Then the corresponding $d$-dimensional subspace $L \subset \mathbb{P}^n$ can be of two types:

1. Either $L$ is contained in $\mathbb{P}^{n-1} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_n = 0\}$;
2. Or $L$ has the form $p_\infty \# \ell$, where $\ell$ is a $(d-1)$-dimensional subspace of $\mathbb{P}^{n-1}$, and $p_\infty \# \ell$ denotes the ruled join of $p_\infty$ and $\ell$ in $\mathbb{P}^n$.

In other words, the fixed point set $G_d(\mathbb{P}^n)^{\mathbb{C}^*}$ has two connected components which are naturally isomorphic to $G_d(\mathbb{P}^{n-1})$ and $G_{d-1}(\mathbb{P}^{n-1})$, and whose inclusions in $G_d(\mathbb{P}^n)$ are denoted by $i : G_d(\mathbb{P}^{n-1}) \hookrightarrow G_d(\mathbb{P}^n)$ and $j : G_{d-1}(\mathbb{P}^{n-1}) \hookrightarrow G_d(\mathbb{P}^n)$, respectively. In particular, this $\mathbb{C}^*$ action on $G_d(\mathbb{P}^n)$ satisfies the hypothesis of Theorem 5.5. Furthermore, all points in the generic locus $G_d(\mathbb{P}^n)^0 = G_d(\mathbb{P}^n) - G_d(\mathbb{P}^n)^{\mathbb{C}^*}$ have the same orbit type.

**Proposition 5.7.** The Chow quotient $G_d(\mathbb{P}^n)//\mathbb{C}^*$ is naturally isomorphic to the flag variety $F(d-1, d; \mathbb{P}^{n-1})$.

**Proof.** Let $\pi : \mathbb{P}^n - \{p_\infty\} \rightarrow \mathbb{P}^{n-1}$ denote the projection onto the hyperplane $\mathbb{P}^{n-1}$. Standard arguments show that

\[
q : G_d(\mathbb{P}^n)^0 \rightarrow F(d-1, d; \mathbb{P}^{n-1})
\]

\[
L \mapsto (L \cap \mathbb{P}^{n-1}, \pi(L))
\]

is a regular, surjective map. This map descends to the quotient $G_d(\mathbb{P}^n)^0/\mathbb{C}^*$ and produces a closed inclusion $G_d(\mathbb{P}^n)/\mathbb{C}^* \equiv G_d(\mathbb{P}^n)^0/\mathbb{C}^* \hookrightarrow F(d-1, d; \mathbb{P}^{n-1})$; cf. Proposition 5.4 and Remark 5.3. This is then easily seen to be an isomorphism. \(\square\)

**Corollary 5.8.** The $p$th Euler-Chow series of the Grassmannian $G_d(\mathbb{P}^n)$ is given by

\[
E_p(G_d(\mathbb{P}^n)) = \Psi_p\big(E_{p-1}(F(d-1, d; \mathbb{P}^{n-1})) \odot E_p(G_d(\mathbb{P}^{n-1})) \odot E_p(G_{d-1}(\mathbb{P}^{n-1}))\big),
\]

where $\Psi_p$ is given by (23).

We proceed to explicitly describe $\Psi_p$ in this example. We follow closely the projective notation for the Schubert cycles in Grassmannians and flag varieties, as described in Fulton [Ful84, §14.7]. However, since we are dealing with various projective spaces, we add an upperscript in the notation to denote the dimension of the ambient projective space.

First, fix a complete flag $L_0 \subset L_1 \subset \cdots \subset L_n = \mathbb{P}^n$ of linear subspaces, and associate to a sequence of length $d$ $a : 0 \leq a_0 < a_1 < \cdots < a_d \leq n$, the Schubert variety $\Omega_a^n = \Omega_{a_0, \ldots, a_d}^n \subset G_d(\mathbb{P}^n)$ defined by

\[
\Omega_a^n = \{l \in G_d(\mathbb{P}^n) \mid \dim (l \cap L_i) \geq a_i, i = 0, \ldots, d\}.
\]
Proposition 5.11. Define the Schubert variety $\Omega_{a^1, \ldots, a^r}^n \subset F(d_1, \ldots, d_r; \mathbb{P}^n)$ by

$$\Omega_{a^1, \ldots, a^r}^n = \{(l_1, \ldots, l_r) \in F(d_1, \ldots, d_r; \mathbb{P}^n) \mid l_i \in \Omega_{a^i}^n\}.$$ 

Facts 5.9.

1. The Schubert variety $\Omega_{a^1, \ldots, a^r}^n$ is an irreducible subvariety of $F(d_1, \ldots, d_r; \mathbb{P}^n)$ of dimension $d(a^1; \ldots; a^n) = \sum_{i=1}^r \sum_{j=0}^{d_i} (a^i_j - j)$. In this sum, any term $(a^i_j - j)$ is omitted if $a^i_j$ appears in the previous $a^{i-1}$.

2. The surjection $\Pi_p(F(d_1, \ldots, d_r; \mathbb{P}^n)) \to \mathbb{A}_F^d(F(d_1, \ldots, d_r; \mathbb{P}^n))$ is an isomorphism. Therefore, if $\omega_{a^1, \ldots, a^r}^n$ denotes the class of $\Omega_{a^1, \ldots, a^r}^n$ in $\Pi_*(F(d_1, \ldots, d_r; \mathbb{P}^n))$, then the monoid $\Pi_p(F(d_1, \ldots, d_r; \mathbb{P}^n))$ is freely generated by the collection

$$\{(\omega_{a^1, \ldots, a^r}^n \mid d(a^1; \ldots; a^r) = p)\}.$$

Remark 5.10. Let $a$ be the sequence $0 < 1 < \cdots < d - 1 < d + 1$. Then, it is easy to see that the class of a 1-dimensional orbit $[\mathbb{P} \cdot x] \in \Pi_1(G_d(\mathbb{P}^n))$, is precisely $\omega_{a}^n$.

Consider the universal bundles $U_d$ and $U_{d+1}$ over the flag variety $F(d-1, d; \mathbb{P}^{n-1})$, of ranks $d$ and $d+1$, respectively. Then, denote the quotient line bundle $U_{d+1}/U_d$ by $L$, and let $\rho : \mathbb{P}(L \oplus 1) \to F(d-1, d; \mathbb{P}^{n-1})$ be the bundle projection. The following result describes the trace map $t_p : \mathbb{C}_p(F(d-1, d; \mathbb{P}^{n-1})) \to \mathbb{C}_{p+1}(G_d(\mathbb{P}^n))$, in a very explicit and geometric manner.

Proposition 5.11.

1. The blow-up of $G_d(\mathbb{P}^n)$ along the fixed point set $G_d(\mathbb{P}^n)^C = G_{d-1}(\mathbb{P}^{n-1}) \Pi G_d(\mathbb{P}^{n-1})$ is the variety $\mathbb{P}(L \oplus 1)$.

2. Let $b : \mathbb{P}(L \oplus 1) \to G_d(\mathbb{P}^n)$ denote the blow-up map. Then $\phi_p : \mathbb{C}_p(F(d-1, d; \mathbb{P}^{n-1})) \to \mathbb{C}_{p+1}(G_d(\mathbb{P}^n))$ is the composition $b_* \circ \rho^*$, where $b_*$ denote the proper push-forward under $b$, and $\rho^*$ is the flat pull-back.

Proof. Left to the reader. 

The induced map $\varphi_p : \Pi_p(F(d-1, d; \mathbb{P}^{n-1})) \to \Pi_{p+1}(G_d(\mathbb{P}^n))$ can be computed either from the proposition or by a direct argument.

Corollary 5.12. Given a sequence $0 \leq a_1 < \cdots < a_d \leq n - 1$ and $0 \leq j \leq d$, one has:

$$\varphi_p(\omega_{a_0, \ldots, a_j, \ldots, a_d}^{n-1}) = \omega_{a_0, \ldots, a_{j-1}, a_j + 1, \ldots, a_d + 1}^n.$$

Proof. Exercise for the reader.
5.1.2. The flag varieties $F(0, 1; \mathbb{P}^n)$. The orbit structure of the $\mathbb{C}^*$ action on $F(0, 1; \mathbb{P}^n)$ is described in the following statement. Here we denote by $Q_{d,n}$ and $S_{d,n}$ the universal quotient bundle and the tautological bundle over $G_d(\mathbb{P}^n)$, respectively.

**Proposition 5.13.**

1. The fixed point set $F(0, 1; \mathbb{P}^n)^{\mathbb{C}^*}$ has three connected components $F_1, F_2$ and $F_3$, respectively isomorphic to $F(0, 1; \mathbb{P}^{n-1})$, $\mathbb{P}^{n-1}$ and $\mathbb{P}^n$.
2. There are two $\mathbb{C}^*$-invariant subvarieties $W_1$ and $W_2$ of $F(0, 1; \mathbb{P}^n)$ which are equivariantly isomorphic to the projective bundles $\mathbb{P}(Q_{0,n-1} \oplus 1)$ and $\mathbb{P}(S_{0,n-1} \oplus 1)$ over $\mathbb{P}^{n-1}$, respectively. Furthermore, $W_1$ and $W_2$ are Schubert cycles for an appropriate choice of coordinates, such that their Schubert classes are given by $[W_1] = \omega^n_{n-1; n-1, n}$ and $[W_2] = \omega^n_{0,0,n}$. 
3. The component $F_1$ is precisely $\mathbb{P}(Q_{0,n-1}) \subset \mathbb{P}(Q_{0,n-1} \oplus 1)$. The strata $W_1$ and $W_2$ intersect at $F_2$, where $F_2 \equiv \mathbb{P}(1) \subset \mathbb{P}(Q_{0,n-1} \oplus 1)$ and $F_2 \equiv \mathbb{P}(S_{0,n-1}) \subset \mathbb{P}(S_{0,n-1} \oplus 1)$. The component $F_3$ is identified with $\mathbb{P}(1) \subset \mathbb{P}(S_{0,n-1} \oplus 1)$.

**Proof.** The first assertion is clear.

To prove the second assertion, one needs only to consider the $\mathbb{C}^*$-equivariant projections $\pi_1 : F(0, 1; \mathbb{P}^n) = \mathbb{P}(Q_{0,n}) \rightarrow G_0(\mathbb{P}^n) \equiv \mathbb{P}^n$ and $\pi_2 : F(0, 1; \mathbb{P}^n) = \mathbb{P}(\tilde{S}_{1,n}) \rightarrow G_1(\mathbb{P}^n)$. Then $W_1$ is defined to be $\pi_1^{-1}(\mathbb{P}^{n-1}) \subset F(0, 1; \mathbb{P}^n)$ and $W_2 = \pi_2^{-1}(G_0(\mathbb{P}^{n-1})) \subset F(0, 1; \mathbb{P}^n)$, where $G_0(\mathbb{P}^{n-1})$ is one of the components of $G_1(\mathbb{P}^n)^{\mathbb{C}^*}$ and $\mathbb{P}^{n-1} = \{x_n = 0\}$. The rest of the proof is an easy consequence of this description

Applying [Bia73, Lemma 4.1] to our situation, one concludes that the closure of a 1-dimensional orbit $\mathbb{C}^*:x$ in $F(0, 1; \mathbb{P}^n)$ intersects precisely 2 connected components of the fixed point set. Furthermore, an application of [Bia73, Theorems 4.1 and 4.3] implies that these two components completely determine the type of the orbit; cf. Definition 5.1.

There are 3 types of 1-dimensional orbits:

1. **Type 1:** The 1-dimensional orbits contained in the stratum $W_1$. These are the orbits of the points in $W_1 - \{F_1 \cup F_2\}$.
2. **Type 2:** The 1-dimensional orbits contained in the stratum $W_2$. These are the orbits of the points in $W_2 - \{F_2 \cup F_3\}$.
3. **Generic type:** These are the orbits of points in the generic locus $F(0, 1; \mathbb{P}^n)^o = F(0, 1; \mathbb{P}^n) - \{W_1 \cup W_2\}$ of the action.

**Proposition 5.14.**

1. The Chow quotient $W_1//\mathbb{C}^*$ is naturally isomorphic to $F_1$.
2. The Chow quotient $W_2//\mathbb{C}^*$ is naturally isomorphic to $F_3$.
3. The Chow quotient $F(0, 1; \mathbb{P}^n)//\mathbb{C}^*$ is naturally isomorphic to $\mathbb{P}^{n-1}[2]$, the blow-up of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ along the diagonal.
Proof. The two first assertions are simple and follow from Remark 5.4. In order to prove the last assertion, we first identify \( \mathbb{P}^{n-1}[2] \) with the fibered product

\[
F(0, 1; \mathbb{P}^{n-1}) \times G_1(\mathbb{P}^{n-1}) \rightarrow F(0, 1; \mathbb{P}^{n-1}) = \{(l, l', L) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times G_1(\mathbb{P}^{n-1}) \mid l, l' \subset L\}.
\]

Using the notation of Proposition 5.13 we denote \( F(0, 1; \mathbb{P}^{n}) = F(0, 1; \mathbb{P}^{n}) \setminus \{W_1 \cup W_2\} \) and define an algebraic map

\[
(29) \quad \tau : F(0, 1; \mathbb{P}^{n})^o \rightarrow \mathbb{P}^{n-1}[2]
\]

\[
(l, L) \mapsto (l \cap \mathbb{P}^{n-1}, \pi(l), \pi(L)).
\]

Notice that this map factors through \( F(0, 1; \mathbb{P}^{n})^o / \mathbb{C}^* \), defining \( \tau' : F(0, 1; \mathbb{P}^{n})^o / \mathbb{C}^* \rightarrow \mathbb{P}^{n-1}[2] \), and that it actually defines an isomorphism \( \tau' : F(0, 1; \mathbb{P}^{n})^o / \mathbb{C}^* \rightarrow \mathbb{P}^{n-1}[2] - D \), where \( D = \{(l, l, L) \mid l \in L\} \) is the exceptional divisor of \( \mathbb{P}^{n-1}[2] \). Let \( \delta' : \mathbb{P}^{n-1}[2] - D \rightarrow F(0, 1; \mathbb{P}^{n})^o / \mathbb{C}^* \) denote the inverse of this isomorphism. Since \( \mathbb{P}^{n-1}[2] \) is smooth and the Chow quotient \( F(0, 1; \mathbb{P}^{n}) / \mathbb{C}^* \) is irreducible and weakly normal, one concludes that \( \delta' \) has a unique extension to a surjective map

\[
(30) \quad \delta : \mathbb{P}^{n-1}[2] \rightarrow F(0, 1; \mathbb{P}^{n}) / \mathbb{C}^*.
\]

In order to prove that \( \delta \) is an isomorphism, one just needs to prove that it is injective, due to the properties of birational maps between weakly normal spaces; cf. Kollár [Kol96]. The injectivity will follow from the following description of \( \delta \).

Let \( \Gamma \subset \mathbb{P}^{n-1}[2] \times F(0, 1; \mathbb{P}^{n}) \) be defined as the closure of

\[
(31) \quad \Gamma^o = \{(a, b, C) \times (l, L) \mid (a, b, C) \in \mathbb{P}^{n-1}[2] - D, \ (l, L) \in F(0, 1; \mathbb{P}^{n}), \ L \cap \mathbb{P}^{n-1} = a, \ \pi(l) = b\}.
\]

We observe that the projection \( q : \Gamma \rightarrow \mathbb{P}^{n-1}[2] \) is equidimensional. Indeed, it is easy to see that if \( (l, l', L) \in \mathbb{P}^{n-1}[2] - D \) then \( q^{-1}(l, l', L) = \text{supp} \delta(l, l', L) \). In other words, \( q^{-1}(l, l', L) \) is the closure of the orbit of \( (a, l \# a) \), where \( a \) is any point in \( l' \# p_{\infty} \setminus \{l', p_{\infty}\} \). Now, given \( (l, l, L) \in D \), one can think of \( (l, L) \) as an element in \( F_1 \subset F(0, 1; \mathbb{P}^{n}) \), and of \( l \) as an element in \( F_3 \subset F(0, 1; \mathbb{P}^{n}) \). It follows from the first two assertions of this proposition that there are 1-dimensional orbits \( \psi_1(l, L) \subset W_1 \) and \( \psi_2(l) \subset W_2 \) uniquely determined by \( (l, l, L) \). It is easy to see that \( q^{-1}(l, l, L) = \psi_1(l, L) \cup \psi_2(l) \).

It follows from [FL92] that the equidimensional projection \( q : \Gamma \rightarrow \mathbb{P}^{n-1}[2] \) induces an algebraic map \( q_\Gamma : \mathbb{P}^{n-1}[2] \rightarrow \mathcal{C}_{1, \alpha}(F(0, 1; \mathbb{P}^{n})) \). This map coincides with \( \delta \) on \( \mathbb{P}^{n-1}[2] - D \) and hence, by uniqueness, one concludes that \( \delta = q_\Gamma \). In this case, \( \delta(l, l, L) = \psi_1(l, L) + \psi_2(l) \), and hence \( \delta \) is injective. This concludes the proof. \( \square \)

We now compute the Euler-Chow series for the flag variety \( F(0, 1; \mathbb{P}^{2}) \), using the above information. In this case one has

\[
F_1 \cong W_1 / \mathbb{C}^* \cong F_2 \cong F_3 \cong W_2 / \mathbb{C}^* \cong \mathbb{P}^1,
\]
and
\[ F(0, 1; \mathbb{P}^2) \cap C^* \cong \mathbb{P}^1[2] = \mathbb{P}^1 \times \mathbb{P}^1. \]

It follows from the isomorphisms \( \Pi_1(F(0, 1; \mathbb{P}^2)) \cong \mathbb{Z}_+^2 \otimes \mathbb{Z}_+^2 \cong \mathbb{Z}_+^2 \), that we may denote the connected components of the Chow monoid \( C_1(F(0, 1; \mathbb{P}^2)) \) by \( C_{1,(r,s)}(F(0, 1; \mathbb{P}^2)) \), where \((r,s) \in \mathbb{Z}_+^2 \).

The Chow quotients described above induce inclusions:

1. \( \mathbb{P}^1 \cong W_1/\!\!/C^* \subset C_1(F(0, 1; \mathbb{P}^2)) \),
2. \( \mathbb{P}^1 \cong W_2/\!\!/C^* \subset C_1(F(0, 1; \mathbb{P}^2)) \), and
3. \( \mathbb{P}^1 \times \mathbb{P}^1 \cong F(0, 1; \mathbb{P}^2) \cap C^* \subset C_1(F(0, 1; \mathbb{P}^2)) \),

whose associated trace maps are given as follows.

**Lemma 5.15.**

1. \( \psi_1 : C_{1,d}(\mathbb{P}^1) \equiv C_{1,d}(W_1/\!\!/C^*) \to C_{2,(d,0)}(F(0, 1; \mathbb{P}^2)) \), sends \( d \cdot \mathbb{P}^1 \) to \( d \cdot W_1 \);
2. \( \psi_2 : C_{1,d}(\mathbb{P}^1) \equiv C_{1,d}(W_2/\!\!/C^*) \to C_{1,(0,d)}(F(0, 1; \mathbb{P}^2)) \), sends \( d \cdot \mathbb{P}^1 \) to \( d \cdot W_2 \);
3. Let \( t_1 : C_{1,(r,s)}(\mathbb{P}^1 \times \mathbb{P}^1) \to C_{2,(r,s)}(F(0, 1; \mathbb{P}^2)) \) be the trace map induced by the Chow quotient \( \mathbb{P}^1 \times \mathbb{P}^1 = F(0, 1; \mathbb{P}^2) \cap C^* \), and let \( Z \) be an irreducible subvariety of \( \mathbb{P}^1 \times \mathbb{P}^1 \). If the 2-cycle \( t_1(Z) \) contains either \( W_1 \) or \( W_2 \) as an irreducible component, then \( Z = \Delta = \) the diagonal of \( \mathbb{P}^1 \times \mathbb{P}^1 \), in which case \( t_1(\Delta) = W_1 + W_2 \).

**Proof.** The first two assertions are obvious from the definitions. In order to prove the last assertion, consider an irreducible 1-dimensional subvariety \( Z \subset \mathbb{P}^1 \times \mathbb{P}^1 \), distinct from the diagonal \( \Delta \), and assume that \( W_1 \subset \text{supp} \, t_1(Z) \). Since \( \text{supp} \, t_1(Z) = \bigcup_{w \in Z} \text{supp} \, \delta(w) \) and \( Z \cap \Delta \) is finite, one concludes that \( W_1 \cap (\bigcup_{w \in Z - \Delta} \text{supp} \, \delta(w)) \) is open and dense in \( W_1 \). On the other hand, if \( w \in Z - \Delta \) then \( \text{supp} \, \delta(w) \) is the closure of a generic orbit, and hence it only intersects \( W_1 \) at \( F_1 \). This implies that \( W_1 \cap (\bigcup_{w \in Z - \Delta} \delta(w)) \subset F_1 \) is open in \( W_1 \), which is a contradiction. Similar arguments are used in the case \( W_2 \subset \text{supp} \, t_1(Z) \). One then concludes that \( Z = \Delta \), and the rest of the proof follows easily. \( \square \)

With the above data, we can prove the following:

**Theorem 5.16.** Let \( x \) and \( y \) be variables associated to the Schubert classes \( \omega_{1,1}^2 \) and \( \omega_{2,0}^2 \), respectively. Then the 2nd Euler-Chow series of \( F(0, 1; \mathbb{P}^2) \) is given by the generating function

\[ E_2(F(0, 1; \mathbb{P}^2)) = \frac{1 - xy}{(1 - x)^3(1 - y)^3}. \]

In other words,

\[ E_2(F(0, 1; \mathbb{P}^2))(r, s) := \chi \left(C_{2,(r,s)}(F(0, 1; \mathbb{P}^2))\right) = \frac{1}{2}(r + 1)(s + 1)(r + s + 2). \]
Proof. Let \( \nu_{(r,s)} : F(0,1;\mathbb{P}^2) \to \mathbb{P}(\text{Sym}^r(\Lambda^1\mathbb{C}^3) \otimes \text{Sym}^s(\Lambda^2\mathbb{C}^3)) \), denote the composition
\[
F(0,1;\mathbb{P}^2) \hookrightarrow G_0(\mathbb{P}^2) \times G_1(\mathbb{P}^2) \xrightarrow{\nu_1 \times \nu_2} \mathbb{P}(\Lambda^1\mathbb{C}^3) \times \mathbb{P}(\Lambda^2\mathbb{C}^3)
\]
where the first map is the canonical inclusion, the second one is a product of Plücker embeddings, the third map is the product of the appropriate Veronese maps (embeddings when \( r > 0 \) and \( s > 0 \)) and the last one is a Segre embedding. Let \( \mathcal{O}_F(r,s) \) denote the pull-back \( \nu_{(r,s)}^*(\mathcal{O}(1)) \) of the hyperplane bundle. Then, it is easy to see that the Chow variety \( \mathcal{C}_{2,(r,s)}(F(0,1;\mathbb{P}^2)) \) is precisely the linear system \( \mathbb{P}(H^0(F(0,1;\mathbb{P}^2),\mathcal{O}_F(r,s))) \). It follows from the Borel-Weil theorem that if \( \mathbb{C}^3 \) denotes the canonical representation of \( \text{GL}(3,\mathbb{C}) \), then \( H^0(F(0,1;\mathbb{P}^2),\mathcal{O}_F(r,s)) \) is the irreducible \( \text{GL}(3,\mathbb{C}) \)-module \( S_{\lambda}(\mathbb{C}^3) \) of highest weight \( \lambda = (r+s,s,0) \). See [Ful97, §9.3] for complete details. Therefore, well-known formulas [FH91, Theorem 6.3] for the dimension of the Schur module \( S_{\lambda}(\mathbb{C}^3) \) gives
\[
\chi \left( \mathcal{C}_{2,(r,s)}(F(0,1;\mathbb{P}^2)) \right) = \dim S_{\lambda}(\mathbb{C}^3) = \frac{1}{2} (r+1)(s+1)(r+s+2).
\]
This concludes the proof. \( \square \)

Remark 5.17. We thank the referee for suggesting this proof, and for the observation that all divisorial Euler-Chow functions for arbitrary flag varieties should be given in terms of dimensions of irreducible representations of \( \text{GL}(n,\mathbb{C}) \). The various resulting generating functions then should be given by well-known formulas; cf. [Mac79].

In order to illustrate a more topological approach, using techniques which can extended to more general situations, we present our original proof below.

Proof. We use the notation of 5.13. Denote \( F^0_{r,s} = \mathcal{C}_{2,(r,s)}(F(0,1;\mathbb{P}^2)) \), and let \( F^1_{r,s} \subset F^0_{r,s} \) be the (Zariski) closed subset consisting of those effective divisors which contain either \( W_1 \) or \( W_2 \) in their support. Note that
\[
F^1_{r,s} = (W_1 + F^0_{r-1,s}) \cup (W_2 + F^0_{r,s-1})
\]
and that
\[
(W_1 + F^0_{r-1,s}) \cap (W_2 + F^0_{r,s-1}) = (W_1 + W_2) + F^0_{r-1,s-1}.
\]

It follows from Lemma 5.14(3) that the commutative diagram
\[
\begin{array}{ccc}
\Delta + \mathcal{C}_{1,(r-1,s-1)}(\mathbb{P}^1 \times \mathbb{P}^1) & \longrightarrow & \mathcal{C}_{1,(r,s)}(\mathbb{P}^1 \times \mathbb{P}^1) \\
F^1_{r,s} & \longrightarrow & F^0_{r,s}
\end{array}
\]
is a fiber square whose horizontal arrows are closed inclusions, and such that the map of pairs
\[
(\mathcal{C}_{1,(r,s)}(\mathbb{P}^1 \times \mathbb{P}^1), \Delta + \mathcal{C}_{1,(r-1,s-2)}(\mathbb{P}^1 \times \mathbb{P}^1)) \longrightarrow (F^0_{r,s}, F^1_{r,s})
\]
Proof. Let $L_i = \{ z_i = 0 \}, \, i = 0, 1, 2$, be the coordinate lines in $\mathbb{P}^2$ and let $p_i, \, i = 0, 1, 2$ denote the fixed points of the action in $\mathbb{P}^2$, labeled such that $p_i \notin L_i, \, i = 0, 1, 2$. The orbits of the first type

$$a_{r,s}^0 = \chi(F_{r,s}^0), \quad a_{r,s}^1 = \chi(F_{r,s}^1) \quad \text{and} \quad b_{r,s} = \chi(C_{1,r,s}(\mathbb{P}^1 \times \mathbb{P}^1)).$$

It follows from (33) and (36), and the additivity properties of the Euler characteristic, that

$$a_{r,s}^0 = a_{r-1,s}^1 + b_{r,s} - b_{r-1,s-1}.$$  

Furthermore, (33) and (34) imply that

$$a_{r,s}^1 = a_{r-1,s}^0 + a_{r,s-1}^0 - a_{r-1,s-1}^0.$$  

With $x$ and $y$ as in the statement of the theorem, one has

$$E = E_2(F(0,1; \mathbb{P}^2)) = \sum_{r \geq 0, \, s \geq 0} a_{r,s}^0 x^r y^s$$

$$= \sum_{r \geq 0, \, s \geq 0} (a_{r,s}^1 + b_{r,s} - b_{r-1,s-1}) x^r y^s$$

$$= \sum_{r \geq 0, \, s \geq 0} (a_{r-1,s}^0 + a_{r,s-1}^0 - a_{r-1,s-1}^0 + b_{r,s} - b_{r-1,s-1}) x^r y^s$$

$$= \sum_{r \geq 0, \, s \geq 0} a_{r-1,s}^0 x^r y^s + \sum_{r \geq 0, \, s \geq 0} a_{r,s-1}^0 x^r y^s - \sum_{r \geq 0, \, s \geq 0} a_{r-1,s-1}^0 x^r y^s$$

$$+ \sum_{r \geq 0, \, s \geq 0} b_{r,s} x^r y^s - \sum_{r \geq 0, \, s \geq 0} b_{r-1,s-1} x^r y^s$$

$$= \sum_{r \geq 0, \, s \geq 0} a_{r-1,s}^0 x^r y^s + \left( \sum_{r \geq 0, \, s \geq 0} a_{r,s-1}^0 x^r y^{s-1} \right) y - \left( \sum_{r \geq 0, \, s \geq 0} a_{r-1,s-1}^0 x^{r-1} y^s \right) x$$

$$+ \sum_{r \geq 0, \, s \geq 0} b_{r,s} x^r y^s - \left( \sum_{r \geq 0, \, s \geq 0} b_{r-1,s-1} x^{r-1} y^{s-1} \right) xy,$$

where the third identity follows from (33) and the fourth one follows from (39). If $F = \sum_{r \geq 0, \, s \geq 0} b_{r,s} x^r y^s$ then $E = xE + yE - xyE + F - xyF$ and hence $(1 - x - y - xy)E = (1 - xy)F$. It follows from Subexample 1.9(1) that $F = \frac{1}{(1-x)(1-y)}$, therefore $E = \frac{1-xy}{(1-x)(1-y)}$. 

The shortest path to compute the other Euler-Chow series is using the full $\mathbb{T} = (\mathbb{C}^*)^2$-action on $F(0,1; \mathbb{P}^2)$, for it has finitely many fixed points and orbits of dimension 1, described in the following lemma.

Lemma 5.18. 1. The $\mathbb{T}$ action on $F(0,1; \mathbb{P}^2)$ has 6 fixed points.

2. The $\mathbb{T}$ action has 9 orbits of dimension 1, divided according to types as follows:

   a: There are 3 orbits of type $\omega_{2,0,1}^2$;

   b: There are 3 orbits of type $\omega_{0,2}^2$;

   c: There are 3 orbits of type $\omega_{2,0,1}^2 + \omega_{0,0,2}^2$.

Proof. Let $L_i = \{ z_i = 0 \}, \, i = 0, 1, 2$, be the coordinate lines in $\mathbb{P}^2$ and let $p_i, \, i = 0, 1, 2$ denote the fixed points of the action in $\mathbb{P}^2$, labeled such that $p_i \notin L_i, \, i = 0, 1, 2$. The orbits of the first type
 consist of elements of the form \((l, L_i), i = 0, 1, 2, \) with \(l \in L_i\). The orbits of second type consist of elements of the form \((p_i, L), p_i \in L\). The orbits of third type consist of elements of the form \((p, p_i), p \in L\) and \(p_i, p\) denotes the line determined by \(p_i\) and \(p\).

**Corollary 5.19.** The 0th and 1st Euler-Chow series of \(F(0, 1; \mathbb{P}^2)\) are given by the following generating functions:

\[
E_0(F(0, 1; \mathbb{P}^2)) = \frac{1}{(1 - t)^6}
\]

and

\[
E_1(F(0, 1; \mathbb{P}^2)) = \frac{1}{(1 - r)^3(1 - s)^3(1 - rs)^3},
\]

where \(t\) is a variable associated to \(\omega^2_{0,0,1}\) and \(r,s\) are associated to \(\omega^2_{0,0,2}\) and \(\omega^2_{1,1,2}\), respectively.

One now can use the above information to compute the \(p\)-th Euler-Chow series of the Grassmannian \(G_1(\mathbb{P}^3)\), using the prescription of Corollary 5.8.

In the next result we use the following association \{ variable \} ↔ \{ Schubert class \}:

\[
t \leftrightarrow \omega^2_{0,1}; \quad s \leftrightarrow \omega^2_{0,2}; \quad x \leftrightarrow \omega^2_{0,3}; \quad y \leftrightarrow \omega^2_{1,2}; \quad z \leftrightarrow \omega^2_{1,3}.
\]

**Proposition 5.20.** The Euler-Chow series of the Grassmannian \(G_1(\mathbb{P}^3)\) are given by the following generating functions:

\[
E_0(G_1(\mathbb{P}^3)) = \frac{1}{(1 - t)^6}, \quad E_1(G_1(\mathbb{P}^3)) = \frac{1}{(1 - s)^{12}},
\]

\[
E_2(G_1(\mathbb{P}^3)) = \frac{1}{(1 - x)^4(1 - y)^4(1 - xy)^3}, \quad E_3(G_1(\mathbb{P}^3)) = \frac{1 + z}{(1 - z)^5},
\]

where the latter coincides with the Hilbert series, as explained in Example 1.1.

**Proof.** It follows directly from Corollaries 5.8 and 5.12 and the computations of the Euler-Chow functions of the flag varieties \(F(0, 1; \mathbb{P}^2)\). \(\square\)

**Appendix A. Algebraic Constructions**

**A.1. Monoids with proper multiplication.** Throughout this discussion, all topological spaces considered are “well behaved” such as finite or countable CW-complexes.

Here we deal with abelian topological monoids, although we place special emphasis on discrete ones. However, the topological approach yields a more unified perspective while retaining the origins of the subject. The category of all abelian topological monoids and monoid morphisms will be denoted by \(\mathfrak{Atm}\).

**Definition A.1.** We say that \(M \in \mathfrak{Atm}\) is a monoid with proper multiplication if the multiplication map is a proper map; cf. Bourbaki [Bou89]. We denote by \(\mathfrak{Atm}_p\) the full subcategory of \(\mathfrak{Atm}\) consisting of all abelian topological monoids with proper multiplication.
Example A.2. Any monoid $M \in \mathcal{Atm}$ with finite multiplication, cf. Section 2, is an element of $\mathcal{Atm}_p$. In particular, so are all free monoids.

We now discuss some functorial properties of the category $\mathcal{Atm}_p$.

Proposition A.3.

1. If $\Psi : M \to N$ is a proper and surjective monoid morphism between abelian topological monoids, and $M \in \mathcal{Atm}_p$, then so does $N$.

2. If $\Psi : M \to N$ is a proper monoid morphism and $N \in \mathcal{Atm}_p$, then so does $M$. In particular, the category $\mathcal{Atm}_p$ is closed under pull-backs over proper morphisms and under closed inclusions.

Proof. 1. Consider the following commutative diagram

$$
\begin{array}{ccc}
M \times M & \xrightarrow{\ast_M} & M \\
\Psi \times \Psi & \downarrow & \downarrow \Psi \\
N \times N & \xrightarrow{\ast_N} & N
\end{array}
$$

and let $C \subset N$ be compact. Since $\Psi$ is surjective, one has $\ast_N^{-1}(C) = \Psi \times \Psi (\ast_M^{-1}(\Psi^{-1}(C)))$ which is then compact, by the properness of $\Psi$ and $\ast_M$.

2. We denote by $Q$ the pull-back in the following commutative diagram

$$
\begin{array}{ccc}
M \times M & \xrightarrow{\ast_M} & Q \\
\downarrow \pi_2 & & \downarrow \Psi \times \Psi \\
N \times N & \xrightarrow{\ast_N} & N
\end{array}
$$

Since $\Psi$ and $\ast_N$ are both proper maps, it follows that $\pi_1$ is also a proper map. The map $j : M \times M \to Q$ gives the homomorphism of $M \times M$ onto the graph of $\ast_M$ contained in $Q \subset M \times M \times M$. In other words $j(m, m') = (m \ast_M m', m, m') \in Q$, and $j$ is a closed inclusion. The properness of $\ast_M$ then follows from that of $\pi_1$ and the identity $\pi_1 \circ j = \ast_M$.

Remark A.4. It follows from the previous proposition that if a monoid $M \in \mathcal{Atm}$ has a proper augmentation $\Psi : M \to \mathbb{Z}_+$, then $M$ also belongs to $\mathcal{Atm}_p$. This applies, in particular, to the Chow monoids introduced in Section 3.
Now let $R$ be an arbitrary commutative ring and let $M$ be an abelian topological monoid, which may be assumed discrete.

**Definition A.5.**

1. An $M$-graded algebra over $R$ is an $R$-algebra $A$ which can be written as $A = \bigoplus_{m \in M} A_m$, where each $A_m$ is a $\mathbb{Z}_+$-graded $R$-module and the multiplication $\cdot : A \times A \to A$ induces a pairing of $\mathbb{Z}_+$-graded $R$-modules $\cdot : A_{m,r} \times A_{m',r'} \to A_{m \cdot m', r+r'}$.

Here $A_m = \bigoplus_{r \in \mathbb{Z}_+} A_{m,r}$ denotes the $\mathbb{Z}_+$-grading. Let $\mathfrak{A}_R(M)$ be the category of all $M$-graded algebras over $R$ and grading preserving $R$-algebra homomorphism.

2. One says that an $R$-algebra $A \in \mathfrak{A}_R(M)$ is $M$-finite or a finite $M$-graded algebra if each $A_m$ is an $R$-module of finite type. The full subcategory of $\mathfrak{A}_R(M)$ consisting of those algebras which are $M$-finite is denoted by $\mathfrak{A}_R^{\text{fin}}(M)$.

The following example plays a major role in this paper, since it includes the Chow monoids as a particular case.

**Example A.6.** Let $M$ be a monoid with proper multiplication with the property that all of its (path) components are compact. It follows from Proposition A.3 that the (discrete) monoid $N = \pi_0(M)$ is also in $\mathfrak{At}_p$, and hence its multiplication $*_N$ (induced by $*_M$) has finite fibers, cf. A.2. Now, given a commutative ring $R$, the singular homology $A = H_*(M, R)$ of $M$ with coefficients in $R$ together with its Pontriagin ring structure becomes an $N$-graded algebra over $R$, since $A = \bigoplus_{n \in N} H_*(M_n, R)$, where $M_n$ denotes the connected component associated to $n \in N = \pi_0(M)$. Furthermore, since each $M_n$ is a compact CW-complex, each $H_*(M_n, R) \equiv A_n$ is a finite module over $R$. In other words, $A = H_*(M, R)$ is an element of $\mathfrak{A}_R^{\text{fin}}(N) = \mathfrak{A}_R(\pi_0(M))$; cf. Definition A.3.

**Remark A.7.** A typical example occurs when $M$ is an abelian topological monoid which comes with a proper augmentation $\Psi : M \to \mathbb{Z}_+$, since $\Psi$ factors through $\pi_0(M)$. See Proposition A.3.

We now discuss the “change of grading” behavior of our algebras under monoid morphisms $\Psi : M \to N$.

**Definition A.8.** Let $A$ be an $M$-graded algebra over $R$ and $B$ be an $N$-graded algebra over $R$. Given a monoid morphism $\Psi : M \to N$, define $\Psi^*B$ by

\begin{equation}
\Psi^*B = \bigoplus_{m \in M} (\Psi^*B)_m
\end{equation}
where \((\Psi^* B)_m = B_{\Psi(m)}\), and \(\Psi_* A\) by

\[
\Psi_* A = \bigoplus_{n \in \mathbb{N}} (\Psi_* A)_n
\]

where \((\Psi_* A)_n = \bigoplus_{m \in \Psi^{-1}(n)} A_m\), and \((\Psi_* A)_n = 0\) if \(\Psi^{-1}(n) = \emptyset\). Given homomorphisms \(\varphi : A \to A'\) in \(\mathfrak{A}_R(M)\) and \(\eta : B \to B'\) in \(\mathfrak{A}_R(N)\), define:

\[
\Psi^*(\eta) : \Psi^* B \to \Psi^* B' \quad \text{by} \quad \Psi^*(\eta)(b) = \eta(b)
\]

and

\[
\Psi_*(\varphi) : \Psi_* A \to \Psi_* A' \quad \text{by} \quad \Psi_* (\varphi)(a) = \varphi(a).
\]

We have the following

**Proposition A.9.** Let \(\Psi : M \to N\) be a homomorphism of abelian topological monoids.

1. The assignment of \(\Psi^* B \in \mathfrak{A}_R(M)\) to \(B \in \mathfrak{A}_R(N)\), and of \(\Psi_* B \to \Psi^* B'\) to \(\eta \in \text{Mor}_{\mathfrak{A}_R(N)}(B, B')\) defines a covariant functor \(\Psi^* : \mathfrak{A}_R(N) \to \mathfrak{A}_R(M)\). Furthermore, \(\Psi^*\) preserves the algebras of finite type, in other words, \(\Psi^*\) takes \(\mathfrak{A}_R^{\text{fin}}(N)\) to \(\mathfrak{A}_R^{\text{fin}}(M)\).

2. Similarly, \(\Psi_*\) induces a covariant functor \(\Psi_* : \mathfrak{A}_R(M) \to \mathfrak{A}_R(N)\). If \(\Psi : M \to N\) is a finite monoid morphism (i.e. it has finite fibers), then \(\Psi_*\) takes \(\mathfrak{A}_R^{\text{fin}}(M)\) into \(\mathfrak{A}_R^{\text{fin}}(N)\).

**Proof.** 1. We need to show the following: Given \(B, B' \in \mathfrak{A}_R(N)\) and a homomorphism \(\varphi : B \to B'\) in \(\mathfrak{A}_R(N)\), one has:

   a: \(\Psi^* B \in \mathfrak{A}_R(M)\).

   b: \(\Psi^*(\varphi) : \Psi^* B \to \Psi^* B'\) is a morphism of M-graded algebras over \(R\) and \(\Psi^*(1_B) = 1_{\Psi^* B}\).

   c: Given \(\varphi : B \to B'\) and \(\varphi' : B' \to B''\) then one has \(\Psi^*(\varphi' \circ \varphi) = \Psi^*(\varphi') \circ \Psi^*(\varphi)\).

   By definition we have \((\Psi^* B)_m = B_{\Psi(m)}\) and the pairing \(B_{\Psi(m)} \times B_{\Psi(m')} = B_{\Psi(m+m')}\) of \(R\)-modules, thus one obtains a pairing \((\Psi^* B)_m \times (\Psi^* B)_{m'} = (\Psi^* B)_{m+m'}\) as required. Furthermore, if each \(B_n\) is a finite \(R\)-module, for all \(n \in \mathbb{N}\), then so is \((\Psi^* B)_m\) for all \(m \in M\). Now, given \(\varphi : B \to B'\) an \(R\)-algebra homomorphism such that \(\varphi(B_n) \subset B'_n\) then if \(b \in (\Psi^* B)_m = B_{\Psi(m)}\) and \(b' \in (\Psi^* B)_{m'} = B_{\Psi(m')}\) one has that if \(bb' \in B_{\Psi(m+m')}\) then \(\Psi^* \varphi(bb') = \varphi(bb') = \varphi(b)\varphi(b') = \Psi^* \varphi(b) \cdot \Psi^* \varphi(b')\), and also \(\Psi^* \varphi(b) = \varphi(b) \in B'_{\Psi(m)} = (\Psi^* B')_m\).

   Therefore \(\Psi^* \varphi\) is a homomorphism of \(M\)-graded \(R\)-algebras. The remaining properties are proven in a similarly trivial fashion.

2. We need to show corresponding assertions for \(\Psi_* : \mathfrak{A}_R(M) \to \mathfrak{A}_R(N)\). By definition, given \(A \in \mathfrak{A}_R(M)\) one has \((\Psi_* A)_n = \bigoplus_{m \in \Psi^{-1}(n)} A_m\). Therefore, given elements \(\alpha \in (\Psi_* A)_n\) and \(\beta \in (\Psi_* A)_{n'}\) one may assume that there are \(m \in \Psi^{-1}(n)\) and \(m' \in \Psi^{-1}(n')\) such that \(\alpha \in A_m\) and \(\beta \in A_{m'}\). In this case \(\alpha \cdot \beta \in A_m \cdot A_{m'} \subset A_{m+m'}\), and hence \(\alpha \beta \in (\Psi_* A)_{n+n'}\), since \(\Psi(m+m') = \Psi(m) \ast_N \Psi(m') = n \ast_N n'\). In other words, \(\Psi_* A\) is an \(N\)-graded \(R\)-algebra. Now, given \(\varphi : A \to A'\) and \(\varphi' : A' \to A''\) morphisms in \(\mathfrak{A}_R(M)\) one has \(\Psi_* (\varphi' \circ \varphi) = \Psi_* (\varphi') \circ \Psi_* (\varphi)\).
by definition, and if \( \alpha \in (\Psi A)_n \) then \( \alpha \in A_m \) for some \( m \in \Psi^{-1}(n) \) (otherwise \( \alpha = 0 \), by definition). Therefore \( \varphi(\alpha) \in A'_m \subset (\Psi A')_n \), and hence \( \Psi(\varphi) \) is a morphism in \( \mathfrak{A}_R(N) \). Finally, if \( \Psi \) has finite fibers, then \( (\Psi A)_n = \bigoplus_{m \in \Psi^{-1}(n)} A_m \) is a sum over finitely many indices, showing that \( \Psi \) then sends \( \mathfrak{A}^{fin}_R(M) \) into \( \mathfrak{A}^{fin}_R(N) \).

\[ \square \]

### A.2. Invariants for \( M \)-graded algebras.

We now restrict our attention to elements in \( \mathfrak{A}_{\text{fin}} M \) which have finite multiplication, and refer the reader to Section 2 for the notation used here. Our goal is to introduce the following invariants of \( M \)-graded algebras over \( R \).

**Definition A.10.** Let \( M \in \mathfrak{A}_{\text{fin}} M \) be a monoid with finite multiplication and let \( A \in \mathfrak{A}_{\text{fin}} R \langle M \rangle \) be a finite \( M \)-graded algebra over \( R \).

1. If \( R \) is a principal ideal domain (PID) define the **Hilbert \( M \)-function** (or Hilbert \( M \)-series) \( P_A(t) \in \mathbb{Z}[t]^M \) of \( A \) to be the function \( P_A(t) : M \rightarrow \mathbb{Z}[t] \) which sends \( m \in M \) to
\[
P_A(t)(m) = \sum_{k \in \mathbb{Z}_+} (rk_R A_{m,k}) t^k.
\]

2. Under the same hypothesis, define the **Euler \( M \)-function** (or Euler \( M \)-series) \( E_A \in \mathbb{Z}^M \) of \( A \) to be the image of \( P_A(t) \) in \( \mathbb{Z}^M \) under the “evaluation homomorphism” at \( -1 \): \( (e_{-1})_* : \mathbb{Z}[t]^M \rightarrow \mathbb{Z}^M \) cf. Proposition 2.4 (3). In other words, given \( m \in M \),
\[
E_A(m) = P_A(-1)(m) = \sum_{k \in \mathbb{Z}_+} (-1)^k \operatorname{rk}(A_{m,k}).
\]

We have the following

**Proposition A.11.** Let \( \Psi : M \rightarrow N \) be a finite morphism of monoids with finite multiplication, and let \( A \in \mathfrak{A}_{\text{fin}}^R(M) \) be a finite \( M \)-graded algebra over \( R \). Then
\[
P_{\Psi A}(t) = \Psi^*(P_A(t)).
\]

Similarly, if \( B \in \mathfrak{A}_{\text{fin}}^R(N) \) then
\[
P_{\Psi B}(t) = \Psi^*(P_B(t)).
\]

**Proof.** Given \( n \in N \), by definition one has
\[
P_{\Psi A}(t)(n) = \sum_{k \in \mathbb{Z}_+} \operatorname{rk}_R \left( (\Psi_A)_{n,k} \right) t^k = \sum_{k \in \mathbb{Z}_+} \operatorname{rk}_R \left( \sum_{m \in \Psi^{-1}(n)} A_{m,k} \right) t^k
\]
\[
= \sum_{m \in \Psi^{-1}(n)} \sum_{k \in \mathbb{Z}_+} (\operatorname{rk}_R A_{m,k}) t^k = \sum_{m \in \Psi^{-1}(n)} P_A(t)(m) = (\Psi^*(P_A(t)))(n).
\]
Similarly,
\[
\left(\Psi^\sharp \left(P_B(t)\right)\right)(m) = P_B(t)\left(\Psi(m)\right) = \sum_{k \in \mathbb{Z}_+} \text{rk}_R \left(B\Psi_{(m),k}\right) t^k
\]
\[
= \sum_{k \in \mathbb{Z}_+} \text{rk}_R \left(B\Psi_{(m),k}\right) t^k = P_{\Psi^\sharp B}(t).
\]

Remark A.12. Note that if \( M \) is the Chow monoid \( C_p(X) \) of a projective variety \( X \), then its Pontrjagin ring \( H_*(M; \mathbb{Z}) \) is a finite \( \Pi_p(X) \)-graded algebra whose Euler \( \Pi_p(X) \)-function is precisely the \( p \)-th Euler-Chow function \( \mathcal{E}_p(X) \) of \( X \).

References

[Bia73] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) 98 (1973), 480–497.

[Bou89] N. Bourbaki, *General topology*, Springer-Verlag, New York, 1989.

[Eli94] E. J. Elizondo, *The Euler series of restricted Chow varieties*, Compositio Math. 94 (1994), no. 3, 297–310.

[EH96] E. J. Elizondo and R. Hain, *Chow varieties of Abelian varieties*, Bol. Soc. Mat. Mex. 2 (1996), no. 3, 95–99.

[Fri91] E. Friedlander, *Algebraic cycles, Chow varieties and Lawson homology*, Compositio Math. 77 (1991), 55–93.

[FL92] E. Friedlander and H. B. Lawson, Jr., *A theory of algebraic cocycles*, Ann. of Math. (2) 136 (1992), 361–428.

[FM94a] E. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties*, Mem. Amer. Math. Soc. 110 (1994), no. 529.

[Ful84] W. Fulton, *Intersection theory*, 1st ed., Springer-Verlag, Heidelberg, 1984.

[FH91] W. Fulton and J. Harris, *Representation theory, a first course*, 1st ed., Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.

[FM94b] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) 139 (1994), 183–225.

[Ful97] W. Fulton, *Young tableaux*, Student Texts, vol. 35, London Mathematical Society, 1997.

[Hoy66] W. Hoyt, *On the Chow bunches of different projective embeddings of a projective variety*, Amer. J. Math. 88 (1966), 273–278.

[Kap93] M. M. Kapranov, *Chow quotients of Grassmannians I*, Adv. Soviet Math. 16 (1993), no. 2, 29–110, I. M. Gelfand Seminar.

[KSZ91] *Quotients of toric varieties*, Mathematische Annalen 290 (1991), 643–655.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Sprin-Verlag, Berlin, 1996.

[LS75] A. T. Lascu and D. B. Scott, *An algebraic correspondence with applications to blowing up Chern classes*, Ann. di Mat. 102 (1975), 1–36.

[LY87] H. B. Lawson, Jr. and Steve S. T. Yau, *Holomorphic symmetries*, Ann. Sci. École Norm. Sup. (4) 20 (1987), 557–577.

[Law95] H. B. Lawson, Jr., *Spaces of Algebraic Cycles*, Surveys in Differential Geometry, vol. 2, 137–213, Surveys in Differential Geometry, International Press, 1995, pp. 137–213.

[Mac62] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Math. Proc. Cambridge Philos. Soc. 58 (1962), 563–568.

[Mac79] I. G. Macdonald, *Symmetric functions and hall polynomials*, Clarendon Press, Oxford, 1979.
[Sam71] P. Samuel, *Séminaire sur l’équivalence rationelle*, Paris-Orsay, 1971.

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