We are considering the integration of functions as in the subject of classical integration theory.

The modern interpretation of the Riemann integral is a subject that continues to arouse interest in diverse areas of analysis and its applications to such areas as optimisation and mathematical economics, as well as in the subject of classical integration theory.

We are considering the integration of functions $f$ defined on a bounded interval of the real line which for simplicity we take to be $[0, 1]$. A key concept is a tagged partition $T = \{\Delta_j, t_j\}_{j=1}^n$, $n = 1, 2, \ldots$, of $[0, 1]$ with

$$0 = a_1 < b_1 < a_2 < b_2 < \cdots < a_j < b_j = \cdots = a_n < b_n = 1,$$

$\Delta_j = [a_j, b_j]$, $|\Delta_j| = b_j - a_j$ and $t_j \in [a_j, b_j]$ for $j = 1, \ldots, n$. The uniform convergence of the associated Riemann sums $S(f, T)$ as $|\Delta_j| \to 0$ ensures that $f$ is Riemann integrable. Convergence over gauges is associated with the Henstock-Kurzweil integral.

The present paper deals with the Riemann integral of a multifunction $F : [0, 1] \to 2^X \setminus \emptyset$ taking values in the nonempty subsets of a real Banach space $X$. It is assumed that $F([0, 1]) \subseteq B(X)$, the collection of nonempty bounded subsets of $X$ equipped with the Hausdorff distance. In Definition 1.3 of the paper, the Riemann integral $\int_0^1 F(t) \, dt$ is a closed bounded set defined analogously to the Riemann integral of a scalar valued function.

It is not difficult to show that the Riemann integral $\int_0^1 F(t) \, dt$ of the multifunction $F$ is convex and its convex hull $\text{conv} F : t \mapsto \text{conv}(F(t))$, $t \in [0, 1]$, is also Riemann integrable and

$$\int_0^1 \text{conv} F(t) \, dt = \int_0^1 F(t) \, dt.$$ 

On the other hand, the assertion that $\text{conv} F$ is integrable necessarily implies that $F$ is integrable depends on the Banach space geometry of $X$ studied by G. Pisier in Seminaire Maurey-Schwartz 1973-1974. The required property is that there exists $C > 0$ and $p > 1$ such that

$$\min_{\alpha_j = \pm 1} \left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq C \left( \sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}$$

for all $x_j \in X$, $j = 1, \ldots, n$ and $n = 1, 2, \ldots$, in which case $X$ is said to have infratype $p$. The paper has a clear discussion of infratype and its consequences for the local theory of Banach spaces.

If $\text{conv} F$ is integrable and in addition $\int_0^1 \text{conv} F(t) \, dt$ is compact, then $F$ is Riemann integrable without any further assumption on the Banach space $X$. The proof is achieved by a clever appeal to the approximation property for $C(K)$ with $K$ compact.

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