An experimental study of stochastic resonance in a bistable mechanical system

Honggang Hu¹, Kimihiko Nakano², Matthew P. Cartmell³, Rencheng Zheng⁴ and Masanori Ohori⁵

¹, ², ⁴, ⁵ Institute of Industrial Science, University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo, 153-8505, Japan
³ School of Engineering, University of Glasgow, James Watt Building, Glasgow, G12 8QQ, UK

E-mail: huhg@iis.u-tokyo.ac.jp

Abstract. Potential applications for stochastic resonance have developed strongly in recent years. This paper presents a study of an application of stochastic resonance in a mechanical system. Since a linear system cannot normally exhibit stochastic resonance, a cantilever beam with an end magnet was used to constitute a bistable nonlinear oscillator. Excited by ambient random vibration, the elastic beam undergoes a modulation of the potential well by means of a periodic excitation and flips between bistable states as a result of this. By adjusting the distance between the end magnet and a fixed magnet it is possible to drive the system controllably between bistable states. An electromagnet was used to provide the periodical parametric excitation which can result in stochastic resonance. The conditions for the occurrence of stochastic resonance are also discussed in the paper. Furthermore, simulations and experimental studies have been implemented to illustrate the application. The experimental results prove that stochastic resonance can occur, and that it can be usefully applied in such a mechanical system under specific conditions.

1. Introduction

Stochastic resonance is a physical phenomenon that can occur in nonlinear systems, whereby generally feeble noise signals can be amplified and optimised by the assistance of a parametric excitation. The effect requires three basic ingredients: (a) an energetic activation barrier, (b) a weak coherent input such as a periodic signal, and (c) a source of noise that is inherent to the system [1]. Recently, stochastic resonance has been considered in several fields, such as signal processing [2], circuit experiments [3], and image visualisation [4]. However, experimental research on the application of stochastic resonance in real mechanical vibration systems is relatively rare. McInnes et al showed that stochastic resonance can be considered to be an effective method for enhancing vibration energy harvesting and confirmed that stochastic resonance can occur in certain mechanical vibration systems [5].

For a nonlinear mechanical system we can consider the ambient vibration as the source of noise which can excite the stochastically resonant system into a bistable nonlinear response. The presence of bistability makes the system capable of rapidly switching between the stable states under an external nonlinear force. This generates a double well potential. The probability of a transition between the two potential wells is determined by the Kramers rate, and will be small if the potential barrier between the
two wells is very large. If we apply a weak parametric excitation to the system the double well potential periodically raises and lowers the potential barrier, and noise-induced hopping between the potential wells can become synchronised with the parametric excitation, leading to stochastic resonance [1]. On that basis an experimental study on stochastic resonance in a vibrating mechanical system is discussed in this paper.

2. Methodology

2.1. Modelling of a nonlinear bistable vibrating system

Based on the theory of stochastic resonance a bistable nonlinear system is conjectured, as in Figure 1 where there is a cantilever beam with an end magnet. This system is excited by ambient vibration, \( Y(t) \), and there is an interaction between the elastic force of the beam and the magnetic force, generating a nonlinear response. By adjusting the distance \( d \) between the two magnets, the system can show bistability. A non-contact actuator is used to provide the source of the periodic parametric excitation.

![Figure 1. A bistable vibrating system](image)

A repulsive force \( F_M \) acts between the two magnets. The vertical component, \( F_V \), can be written in terms of \( F_M \) and expanded into a Taylor series, computed around \( y = 0 \) and truncated as follows,

\[
F_V = F_M \sin \theta = \frac{F_M}{d} y \left[ 1 + \frac{y^2}{d^2} \right]^{1/2} \approx \frac{F_M}{d} y - \frac{F_M}{2d^3} y^3
\]

It can be seen from this that \( F_V \) is a nonlinear force. For an elastic beam the stiffness can be approximated by a linear spring of stiffness \( k \), defined by,

\[
k = \frac{3EI}{l^3}
\]

where \( E \) is Young's modulus, \( I \) is the area moment of inertia of the beam, and \( l \) is the length of the beam. The system in Figure 1 comprises an undamped nonlinear system containing a mass \( m \) and spring \( k \), with the addition of the nonlinear source of force \( F_V \), noting that in practice low inherent viscous damping will also be evident. The free movement of the mass through displacement \( y \) is given by the following differential equation,

\[
m \ddot{y} + ky - F_V = -m \ddot{Y}
\]

This can be re-written as,

\[
m \ddot{y} + \left( k - \frac{F_M}{d} \right) y + \frac{F_M}{2d^3} y^3 = -m \ddot{Y}
\]

The potential energy \( U(y) \) of the system can therefore be stated as,

\[
U(y) = \frac{1}{2} \left( k - \frac{F_M}{d} \right) y^2 + \frac{1}{4} \frac{F_M}{2d^3} y^4
\]
In equation (5) the distance \( d \) between the two magnets can be adjusted and therefore the value of \( k - \frac{F_m}{d} \) can become negative. We symbolise \( k - \frac{F_m}{d} \) as \( k - a \), and \( \frac{F_m}{2d^3} \) as \( b \), and the potential energy is presented in Figure 2 for different \( k - a \) and \( b \) values. It is clear from the Figure that the double potential wells appear when \( k - a \) is negative, leading to the possibility of bistability.

\[
\text{Figure 2. Potential energies } U(y) \text{ for different values of } k-a \text{ and } b
\]

2.2. Forced vibration

In order to achieve stochastic resonance within the system of Figure 1 an electromagnet is used as a non-contact actuator. A small magnet is fitted to the beam and positioned above the electromagnet so that when a periodic voltage is applied to the electromagnet coil a periodic excitation excites the beam and this force correspondingly adjusts the height of the potential well to drive the system into stochastic resonance. We can consider \( h \) as the distance from the electromagnet to the static position of the beam, and \( y_0 \) as the vibration displacement of the magnet attachment point on the beam. Therefore, \( (h+y_0) \) defines the distance between the electromagnet and the beam when the system vibrates, and we define \( F_E \) as the force between the electromagnet and the magnet, noting that this acts as the source of the parametric excitation.

When the distance between the electromagnet and magnet increases \( F_E \) will reduce, and it is clear that there is a nonlinear relationship between the force and the distance. In order to determine this relationship a calibration experiment for \( F_E \) was carried out. The result shows that \( F_E \) is in inverse proportion to the square of the distance, and so this relationship can be represented by

\[
F_E = \frac{\lambda}{(h+y_0)^2} \cos(\omega t) = \frac{\lambda}{h^2} \frac{1}{(1 + y_0/h)^2} \cos(\omega t) = \frac{\lambda}{h^2} \frac{(1 - y_0/h)^2}{(1 + y_0/h)^2(1 - y_0/h)^2} \cos(\omega t)
\]

where \( \lambda = 28 \) for this physical system. For a different electromagnet and voltage the parameter \( \lambda \) will assume a different and unique value. Therefore, when a harmonically oscillating voltage at angular frequency \( \omega \) is applied to the electromagnet the periodic force that is generated can be defined by the following,

\[
F_E = \frac{\lambda}{(h+y_0)^2} \cos(\omega t) = \frac{\lambda}{h^2} \frac{1}{(1 + y_0/h)^2} \cos(\omega t) = \frac{\lambda}{h^2} \frac{(1 - y_0/h)^2}{(1 + y_0/h)^2(1 - y_0/h)^2} \cos(\omega t)
\]

where \( y_0 \) is the peak vibration displacement of the magnet’s attachment point on the beam. This has to be related to the displacement \( y \) at the location of the end mass. The static deflection formula for a cantilever can readily be used to define this relationship since this is physically similar in shape to the first bending mode. This is given by,

\[
y_0(x,t) = \left( \frac{3x^2}{2I^2} - \frac{x^3}{2I} \right) y(l,t)
\]
In equation (7) \( x \) is the horizontal location along the axis of the beam and \( l \) is length of the beam. For the practical experimental system this relationship is found to be \( y_0 = 0.25y \). In addition, when the beam vibrates \( y_0 \) should really be a very small value compared with \( h \) which means that we can consider \( y_0 \ll h \). So, if we neglect the nonlinear term \((y_0/h)^2\) and then apply the relationship \( y_0 = 0.25y \), equation (6) can now be written as follows,

\[
F_x = \frac{\lambda}{h^2} (1 - 2 \frac{y_0}{h}) \cos(\omega t) = (\frac{\lambda}{h^2} - \frac{\lambda y}{2h^3}) \cos(\omega t)
\]  

(8)

The dynamics of the mechanism can now be defined by the following equation of motion,

\[
m\ddot{y} + (k - \frac{F_M}{d})y + \frac{F_M}{2d^3} \dot{y}^3 = -m\ddot{Y} + (\frac{\lambda}{h^2} - \frac{\lambda y}{2h^3}) \cos(\omega t)
\]  

(9)

A nondimensional position coordinate \( \delta = (F_M/2d^3)^{1/2}y \), and a free parameter \( \mu = (k - F_M/d) \) can now be usefully defined. Equation (9) is therefore simplified and can be stated as,

\[
\ddot{\delta} + \mu(1 + \eta \cos(\omega t))\delta + \delta^3 = N(t)
\]  

(10)

where \( N(t) = -m(\ddot{Y} - \frac{\lambda}{h^2} \cos(\omega t))(\frac{F_M}{2d^3})^{1/2} \) and is considered to be the ambient excitation function, containing the noise term, and \( \eta = (\frac{F_M}{8d^3})^{1/2} \frac{\lambda}{\mu h^2} \) represents the magnitude of the parametric excitation. From this the potential function can be defined as,

\[
U(\delta) = \frac{1}{2} \mu(1 - \eta \cos(\omega t))\delta^2 + \frac{1}{4} \delta^4
\]  

(11)

2.3. Analysis of stochastic resonance

It is seen from equation (11) that the periodic force can adjust the height of the potential well to push the system into stochastic resonance. The probability of this is determined by the Kramers rate, which is the characteristic escape rate from a stable state of a potential in the absence of a signal. For the system in this paper it can be defined by \( r \) as,

\[
r = \frac{|\mu|}{\sqrt{2\pi}} \exp(-\frac{\mu^2}{4D})
\]  

(12)

where \( D \) is the intensity of the ambient vibration. For example when the ambient vibration is white noise it can be defined as \( N(t) = \sqrt{2D}g(t) \), where \( g(t) \) is the Gauss function. It should be noted that by definition when the excitation frequency is half the maximum value of the Kramers rate, stochastic resonance will occur [6]. Contrary to this when the excitation frequency in Hz \( f \) exceeds that value then stochastic resonance cannot occur, where \( f = \omega/2\pi \). Therefore the value of excitation frequency in Hz for stochastic resonance can be estimated by the following,

\[
\frac{1}{2} r_{\text{lim}} = \frac{1}{2} \frac{|\mu|}{\sqrt{2\pi}} \Rightarrow \frac{|\mu|}{\sqrt{8\pi}}
\]  

(13)

3. Experimental study

3.1. System design

The experimental system is shown in Figure 3 and mainly includes an aluminium alloy beam, two magnets, and an electromagnet to provide a periodic force. When a harmonic voltage is applied to the electromagnet coil a corresponding force is generated. The linear stiffness \( k \) of the beam is calculated to be 46.15 N/m using equation (2). Parameter \( \mu \) from equation (10) changes while the beam vibrates because of the variation in \( F_M \). A preliminary experiment showed that for \( d = 4 \text{ cm} \) bistability ensues and the maximum peak displacement of the beam is then about 25 mm. The corresponding minimum
value for \( F_M \) is then \( F_{M(y=25mm)} = 2.83 \text{ N} \), therefore this gives a maximum numerical value for \( \mu \) of \( \mu_{\text{max}} = (k - F_M/\varepsilon) = -24.5 \). For this value equation (13) predicts that the parametric excitation frequency is 2.76 Hz for stochastic resonance. When the system vibrates \( F_M \) will be the same as, or a little smaller than, \( F_{M(y=25mm)} \), therefore the central value for the parametric excitation frequency should be around about, or a little bigger, than 2.76 Hz for stochastic resonance. Also, when the predominant frequency of the ambient vibration is near the resonance frequency of the beam, the vibration will also be enhanced. In order to distinguish between the effects of stochastic resonance and normal resonance it is necessary to ensure at the design stage that the parametric excitation frequency range for stochastic resonance is not coincident with the resonance frequency of the beam. The damped natural frequency of this beam was also measured by experiment and was found to be 5.87 Hz, which is well away from the value of 2.76 Hz that was calculated for stochastic resonance.

![Figure 3. The experimental system](image)

### 3.2. Experimental results

In the experiment, the ambient vibration was provided by a large-scale vibration generator. This generates a gaussian white noise defined by \( n(t) = \sqrt{2D_0(t)} \), with \( D = 6 \text{ mm} \) and a bandwidth of 0 ~ 10 Hz. An amplified signal generator was used to provide a variable frequency voltage \( V \) of \( 3.5 \cos(2\pi ft) \) to the electromagnet to generate the periodic force required for the parametric excitation, where \( f \) is the frequency in Hz. A laser vibrometer was used to measure the displacement of the beam.

Figure 4 (a) shows the displacement response of the end mass just under ambient vibration excitation and it can be seen that the beam is unable to jump between the two potential wells. Conversely, Figure 4 (b) presents the displacement response for the parametric excitation on its own, operating at 3.0 Hz. This excitation alone is unable to excite the beam into bistable vibration.

![Figure 4. Displacement response under: (a) ambient vibration excitation, (b) parametric excitation](image)

In order to observe stochastic resonance when the beam is being excited by ambient vibration, parametric excitations at different frequencies were tested, at 1.0 Hz, 2.0 Hz, 3.0 Hz, 4.0 Hz, 6.0 Hz, and 8.0 Hz, respectively. Clearly 3.0 Hz is near to the theoretically predicted parametric excitation frequency for stochastic resonance and 6.0 Hz is near to the measured damped natural frequency of the beam. Figure 5 shows the displacements of the beam response under different parametric excitation frequencies.
Figure 5. Beam responses under ambient vibration and parametric excitations at different frequencies: (a) 1.0 Hz, (b) 2.0 Hz, (c) 3.0 Hz, (d) 4.0 Hz, (e) 6.0 Hz, and (f) 8.0 Hz.

Figure 5 shows that when the parametric excitation frequency is 3.0 Hz the beam response is enhanced. Then, when the parametric excitation frequency is shifted to 4.0 Hz, the response amplitude decreases. A parametric excitation frequency of 6.0 Hz is near to the natural frequency of damped vibration of the beam so the response is more intense than at 4.0 Hz but weaker than at 3.0 Hz. It becomes weaker again when the parametric excitation frequency exceeds 6 Hz. Therefore it has been shown here that when the parametric excitation frequency is 3.0 Hz the response is considerably enhanced by stochastic resonance.

4. Conclusions
An experimental study was undertaken to investigate the possibility of generating stochastic resonance in real mechanical systems and it has been shown that this is indeed possible under specific conditions. The frequency of the parametric excitation is confirmed as being one of the most important parameters for achieving this, and that when the parametric excitation frequency is half the maximum value of the predicted Kramers rate then stochastic resonance can be realised.

Reference
[1] L. Gammaitoni, P. Hanggi, P. Jung, F. Marchesoni, Stochastic resonance, Reviews of Modern Physics 70 (1) (1998) 223-287.
[2] B.E. Klamecki, Use of stochastic resonance for enhancement of low-level vibration signal components, Mechanical Systems and Signal Processing 19 (2005) 223-237.
[3] W. Korneta, I. Gomes, C.R. Mirasso, R. Toral, Experimental study of stochastic resonance in a Chua’s circuit operating in a chaotic regime, Physica D 219 (2006) 93-100.
[4] V.P. Rallabandi, Prasun Kumar Roy, Magnetic resonance image enhancement using stochastic resonance in Fourier domain, Magnetic Resonance Imaging 28 (2010) 1361-1373.
[5] C.R. McInnes, D.G. Gorman, M.P. Cartmell, Enhanced vibrational energy harvesting using nonlinear stochastic resonance, Journal of Sound and Vibration 318 (2008) 655-662.
[6] Y. Leng, Mechanism of parameter-adjusted stochastic resonance based on Kramer’s rate, Physica Sinica 58 (8) (2009) 5916-5200.