An invariance principle for maps with polynomial decay of correlations

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Abstract

We give a general method of deriving statistical limit theorems, such as the central limit theorem and its functional version, in the setting of ergodic measure preserving transformations. This method is applicable in situations where the iterates of discrete time maps display a polynomial decay of correlations.

1 Introduction

The decay of correlations in dynamical systems, or, more generally, the rate of approach of a given initial distribution to an invariant one, is an area of long standing interest and research. These rates are usually described in terms of the speed at which the iterates of a corresponding Frobenius-Perron operator, acting on a subspace of a functional space, decay to zero. Quasi-compactness of this operator on the space of function of bounded variation led to an exponential decay of correlations in the case of uniformly expanding maps on the interval. Recently, a significant body of work has been directed at an examination of sub-exponential decay for specific families of maps. The simplest example is the Manneville-Pomeau map (for fixed $\gamma > 0$ let $T_\gamma : [0, 1] \to [0, 1]$ be given by $T_\gamma(y) = y + y^{1+\gamma} \pmod{1}$) for which polynomial decay was demonstrated for Hölder continuous functions.

Throughout this paper, $(Y, B, \nu)$ denotes a probability measure space (a measure space with $\nu(Y) = 1$) and $T : Y \to Y$ a (non-invertible) measure preserving transformation. Thus $\nu$ is invariant for $T$, i.e. $\nu(T^{-1}(A)) = \nu(A)$ for all $A \in B$. Recall that $T$ is ergodic (with respect to $\nu$) if for each $A \in B$ with $T^{-1}(A) = A$ we have $\nu(A) \in \{0, 1\}$ and $T$ is mixing (with respect to $\nu$) if and only if

$$\nu(A \cap T^{-n}(B)) \to \nu(A)\nu(B) \quad \text{for every} \quad A, B \in B.$$
In terms of the correlation function
\[
\text{Cor}(f, g \circ T^n) := \int f(y)g(T^n(y))\nu(dy) - \int f(y)\nu(dy) \int g(y)\nu(dy)
\]
mixing is equivalent to \(\text{Cor}(f, g \circ T^n) \to 0\) for all \(f \in L^1(Y, \mathcal{B}, \nu)\) and \(g \in L^\infty(Y, \mathcal{B}, \nu)\). The transfer operator \(P_{T,\nu}^n : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu)\), by definition, satisfies
\[
\int P_{T,\nu}^n f(y)g(y)\nu(dy) = \int f(y)g(T^n(y))\nu(dy),
\]
which leads to
\[
|\text{Cor}(f, g \circ T^n)| \leq ||g||_\infty ||P_{T,\nu}^n f - \int f(y)\nu(dy)||_1,
\]
valid for all \(f \in L^1(Y, \mathcal{B}, \nu)\) and \(g \in L^\infty(Y, \mathcal{B}, \nu)\), so if one is able to estimate \(||P_{T,\nu}^n f - \int f(y)\nu(dy)||_L\) for some norm \(|| \cdot ||_L \geq || \cdot ||_1\), then one obtains an upper bound on \(|\text{Cor}(f, g \circ T^n)|\) for \(g \in L^\infty\) and \(f \in L^1\). This line of approach to the decay of correlations was taken in the work cited above. A general method of obtaining polynomial decay of the \(L^1\) norm is presented in [32].

In this paper we address the question of the range of validity of the central limit theorem and its functional counterpart, and generalize results of Gordin [15], Keller [20], Liverani [22], and Viana [30]. For measurable \(h : Y \to \mathbb{R}\) with \(\int h(y)\nu(dy) = 0\), we say that the Central Limit Theorem (CLT) holds for \(h\) if the distributions of the random variables \(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h \circ T^j\) converge weakly to a normal distribution \(N(0, \sigma^2)\)
\[
\lim_{n \to \infty} \nu\{y : \sum_{j=0}^{n-1} h(T^j(y)) < \sqrt{n}t\} = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^t e^{-x^2/2\sigma^2} dx, \quad t \in \mathbb{R}.
\]
This will be denoted by
\[
\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h \circ T^j \to^d \sigma N(0, 1).
\]
We introduce this notation because, for \(\sigma > 0\), we have \(N(0, \sigma^2) = \sigma N(0, 1)\), while \(\sigma N(0, 1)\) is the point measure \(\delta_0\) for \(\sigma = 0\). This allows us to state our results in a unified way. There will be always a separate issue of determining whether \(\sigma\) is positive or zero.

A stronger result than the CLT is the Weak Invariance Principle, also called a Functional Central Limit Theorem (FCLT). Let \(\sigma > 0\) and define the process \(\{\psi_n(t), t \in [0, 1]\}\) by
\[
\psi_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T^j \quad \text{for} \quad t \in [0, 1], \ n \geq 1
\]
One of our main results is the following

**Theorem 1** Let $T : Y \to Y$ be ergodic with respect to the invariant measure $\nu$ and let $h \in L^2(Y, B, \nu)$ be such that $\int h(y)\nu(dy) = 0$. If there is $\beta > \frac{1}{2}$ such that

$$\lim_{n \to \infty} n^\beta \|P^n_{T,\nu}h\|_2 < \infty,$$

then the CLT and FCLT hold for $h$ provided that

$$\sigma = \lim_{n \to \infty} \frac{\|\sum_{j=0}^{n-1} h \circ T^j\|}{\sqrt{n}} > 0.$$

Many CLT results and invariance principles for maps have been proven, cf. the survey [8] which, in particular, reviews the case of uniformly expanding maps on the interval; for mixing maps the $L^1$ norm of $P^n h$ decay exponentially for functions of bounded variation thus Theorem 1 applies. Observe that

$$\|P^n_{T,\nu}h\|_1 \leq \|P^n_{T,\nu}h\|_2 \leq \|P^n_{T,\nu}h\|_\infty$$

for every $h \in L^\infty(Y, B, \nu)$. On the other hand, if $T$ is ergodic, then $P_{T,\nu}$ is a contraction in every space $L^p(Y, B, \nu)$, $1 \leq p \leq \infty$. Therefore

$$\|P^n_{T,\nu}h\|_2 \leq \|h\|_\infty^{1/2}\|P^n_{T,\nu}h\|_1^{1/2}$$

for $h \in L^\infty(Y, B, \nu)$. Thus Theorem 1 is applicable when $h \in L^\infty(Y, B, \nu)$ and the $L_1$ norm of $P^n h$ decays polynomially as $n^{-\alpha}$ with $\alpha > 1$. Although the CLT for such decay can be deduced from the result of Liverani [22], Theorem 1 gives both the CLT and FCLT. To prove only the CLT a weaker condition than Condition 1 is sufficient (cf. Theorem 3) while the polynomial rate is needed in the proof of the FCLT.

Only recently the FCLT was established by Pollicott and Sharp [29] for Hölder continuous functions $h$ with $\int h(y)\nu(dy) = 0$ and for maps $T_\gamma$ such as the Manneville-Pomeau map under the hypothesis that $0 < \gamma < \frac{1}{3}$. The CLT was proved by Young [32] by establishing that the $L_1$ norm of $P^n h$ decays polynomially as $n^{-\alpha}$ with $\alpha = \frac{1}{\gamma} - 1$ which is greater than 1 exactly when $0 < \gamma < \frac{1}{2}$. Thus our Theorem 1 gives both the CLT and FCLT when $0 < \gamma < \frac{1}{2}$ for the Manneville-Pomeau map.

The structure of the paper is as follows. Section 2 briefly summarizes the required background and notation. In Section 3 we state and prove, using ideas of [26, 8], our main results (Theorem 1 and Theorem 2) from which Theorem 1 directly follows. We also discuss the case of $\sigma = 0$. The last section contains examples of the applicability of our abstract theorems. As our aim was to go beyond the exponential decay of correlations, we give some examples of transformations for which polynomial decay of correlations has been proved.
2 Preliminaries

The definition of the Frobenius-Perron (transfer) operator for \( T \) depends on a given \( \sigma \)-finite measure \( \mu \) on the measure space \( (Y, B) \) with respect to which \( T \) is nonsingular, i.e. \( \mu(T^{-1}(A)) = 0 \) for all \( A \in B \) with \( \mu(A) = 0 \). This in turn gives rise to different operators for different underlying measures on \( B \).

Thus if \( \nu \) is invariant for \( T \), then \( T \) is nonsingular and the transfer operator \( \mathcal{P}_{T,\nu} : L^1(Y, B, \nu) \to L^1(Y, B, \nu) \) is defined as follows. For any \( f \in L^1(Y, B, \nu) \), there is a unique element \( \mathcal{P}_{T,\nu}f \) in \( L^1(Y, B, \nu) \) such that

\[
\int_A \mathcal{P}_{T,\nu}f(y)\nu(dy) = \int_{T^{-1}(A)} f(y)\nu(dy) \quad \text{for } A \in B. \tag{3}
\]

We are writing here \( \mathcal{P}_{T,\nu} \) to underline the dependence on \( T \) and \( \nu \). The Koopman operator is defined by

\[
U_T f = f \circ T
\]

for every measurable \( f : Y \to \mathbb{R} \). In particular, \( U_T \) is also well defined for \( f \in L^1(Y, B, \nu) \) and is an isometry of \( L^1(Y, B, \nu) \) into \( L^1(Y, B, \nu) \), i.e. \( ||U_T f||_1 = ||f||_1 \) for all \( f \in L^1(Y, B, \nu) \). The following relation holds between the operators \( U_T, \mathcal{P}_{T,\nu} : L^1(Y, B, \nu) \to L^1(Y, B, \nu) \)

\[
\mathcal{P}_{T,\nu}U_T f = f \quad \text{and} \quad U_T \mathcal{P}_{T,\nu} f = E(f(T^{-1}(B))) \tag{4}
\]

for \( f \in L^1(Y, B, \nu) \), where \( E(|T^{-1}(B)|) : L^1(Y, B, \nu) \to L^1(Y, T^{-1}(B), \nu) \) denotes the operator of conditional expectation. Since the measure \( \nu \) is finite, we have \( L^p(Y, B, \nu) \subset L^1(Y, B, \nu) \) for \( p \geq 1 \). The operator \( U_T : L^p(Y, B, \nu) \to L^p(Y, B, \nu) \) is also an isometry on this space. Note that if the conditional expectation operator \( E(|T^{-1}(B)|) : L^1(Y, B, \nu) \to L^1(Y, B, \nu) \) is restricted to \( L^2(Y, B, \nu) \), then this is the orthogonal projection of \( L^2(Y, B, \nu) \) onto \( L^2(Y, T^{-1}(B), \nu) \).

The significance of using the transfer operator \( \mathcal{P}_{T,\nu} \) is that it allows a unified approach to the study of statistical properties of the transformation \( T \). Extending the approach of Gordin [15], Keller [20], Liverani [22], and Viana [30], we have the following

**Theorem 2** Let \( (Y, B, \nu) \) be a probability measure space and \( T : Y \to Y \) be ergodic with respect to \( \nu \). Suppose that \( h \in L^2(Y, B, \nu) \) is such that \( \mathcal{P}_{T,\nu} h = 0 \). Then the CLT and FCLT hold for \( h \) provided that \( ||h||_2 > 0 \).

Moreover, for each \( n \geq 1 \) we have \( \text{Cor}(h, g \circ T^n) = 0 \) for all \( g \in L^2(Y, B, \nu) \) and

\[
||\sum_{j=0}^{n-1} h \circ T^j||_2 = \sqrt{n}||h||_2.
\]

For a direct proof of this result see [24], where the proof relies on the fact that the family

\[
\{T^{-n+j}(B) : 1 \leq j \leq n, n \geq 1\}
\]
is a martingale difference array for which the central limit theorem may be proved by using the Martingale Central Limit Theorem (cf. [2, Theorem 35.12]) and the Birkhoff Ergodic Theorem. If the assumption of ergodicity appearing in Theorem 2 is omitted, then we obtain weak convergence to mixtures of normal distributions, that is the distributions of the random variables \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} h \circ T^j \) converge weakly to a distribution with a characteristic function of the form \( \varphi(r) = E(\exp(-\frac{1}{2}r^2\eta)) \) where \( \eta \) is such that \( \eta \circ T = \eta \) and \( \int \eta(y)\nu(dy) = \int h^2(y)\nu(dy) \). This again is a consequence of the Birkhoff Ergodic Theorem and another version of the Martingale Central Limit Theorem due to Eagleson [12, Corollary p. 561].

In general, for a given \( h \) the equation \( P_{T,\nu}h = 0 \) might not be satisfied. Then the idea is to write \( h \) as a sum of two functions in which one satisfies the assumptions of Theorem 2 while the other is irrelevant for the CLT or FCLT to hold. This is strongly connected with the property of weak convergence which says that if two sequences differ by a sequence converging in probability to zero and one of them is weakly convergent then the other is weakly convergent to the same limit \([2, \text{Theorem } 4.1]\). In particular, in our setting for the CLT for \( h \) to hold it is enough to show that there is a \( \tilde{h} \) satisfying the assumptions of Theorem 2 such that the sequence \( \left( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \right)_{n \geq 1} \) is convergent in \( \nu \)-measure to zero. If, additionally, the sequence \( \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} | \sum_{j=0}^{k-1} (h - \tilde{h}) \circ T^j | \right)_{n \geq 1} \) is convergent to zero in \( \nu \)-measure, then the FCLT also holds for \( h \).

Finally, we illustrate Theorem 2 with an example. The Chebyshev maps [1] on \([-1,1]\) are given by

\[
S_N(y) = \cos(N \arccos y), \quad N = 0, 1, \ldots
\]

with \( S_0(y) = 1 \) and \( S_1(y) = y \).

For \( N \geq 2 \) they are ergodic (and in fact mixing) with respect to the measure \( \nu \) with the density

\[
g_\nu(y) = \frac{1}{\pi \sqrt{1 - y^2}}.
\]

For instance, for \( N = 2 \) the transfer operator on \( L^1([-1,1],\mathcal{B}([-1,1]),\nu) \) is given by

\[
P_{S_2,\nu}f(y) = \frac{1}{2} \left[ f \left( \sqrt{\frac{1}{2}y + \frac{1}{2}} \right) + f \left( -\sqrt{\frac{1}{2}y + \frac{1}{2}} \right) \right].
\]

For even \( N \geq 2 \) and any odd function \( h : [-1,1] \to \mathbb{R} \) which is square integrable with respect to \( \nu \), we have \( P_{S_N,\nu}h = 0 \). We also have \( P_{S_N,\nu}h = 0 \) for the function \( h(y) = y \) and all \( N \) (either even or odd). By Theorem 2 the CLT and FCLT hold for \( h \). This gives a theoretical basis for the numerical observations of Hilgers and Beck [17].
3 The main results

In this section we state and prove our main results. We start with the following abstract theorem which gives the CLT under less restrictive and easily verifiable assumptions when compared with the theorem of [15]. We adapt here the ideas of [26].

**Theorem 3** Let $T$ be a measure-preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$ and let $h \in L^2(Y, \mathcal{B}, \nu)$ be such that $\int h(y)\nu(dy) = 0$. Suppose that

$$\sum_{n=1}^{\infty} n^{-2} \left\| \sum_{k=0}^{n-1} P_{T, \nu}^k h \right\|_2 < \infty. \quad (5)$$

Then there exists $\tilde{h} \in L^2(Y, \mathcal{B}, \nu)$ such that $P_{T, \nu} \tilde{h} = 0$ and $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j$ converges to zero in $L^2(Y, \mathcal{B}, \nu)$ as $n \to \infty$.

In particular, if $T$ is ergodic, then the CLT for $h$ provided that $\|\tilde{h}\|_2 > 0$.

**Proof 1** For $\epsilon > 0$ define $f_\epsilon = \sum_{k=1}^{\infty} \frac{P_{T, \nu}^{k-1} h}{(1+\epsilon)^k}$. Observe that

$$f_\epsilon = \epsilon \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{P_{T, \nu}^k \tilde{h}}{(1+\epsilon)^n} \quad (6)$$

Since $P_{T, \nu}$ is a contraction in $L^2(Y, \mathcal{B}, \nu)$, we have $f_\epsilon \in L^2(Y, \mathcal{B}, \nu)$ and $h = (1+\epsilon)f_\epsilon - P_{T, \nu}f_\epsilon$. Let us put

$$h_\epsilon = f_\epsilon - U_T P_{T, \nu}f_\epsilon.$$

Then $P_{T, \nu}h_\epsilon = 0$ and

$$h = h_\epsilon + \epsilon f_\epsilon + U_T P_{T, \nu}f_\epsilon - P_{T, \nu}f_\epsilon \quad (7)$$

Now the arguments of [26] apply. Using

$$\int h_\epsilon(y)h_\delta(y)\nu(dy) = \int f_\epsilon(y)f_\delta(y)\nu(dy) - \int P_{T, \nu}f_\epsilon(y)P_{T, \nu}f_\delta(y)\nu(dy)$$

and $P_{T, \nu}f_\epsilon = (1+\epsilon)f_\epsilon - h$ for any $\epsilon, \delta > 0$ we obtain

$$\|h_\epsilon - h_\delta\|_2^2 \leq (\epsilon + \delta)(\|f_\epsilon\|_2^2 + \|f_\delta\|_2^2). \quad (8)$$

Condition [2] and Equation [7] imply that $\sqrt{\epsilon}\|f_\epsilon\|_2 \to 0$ as $\epsilon \to 0$ and

$$\sum_{k=1}^{\infty} \sqrt{\delta_k} \sup_{\delta_k \leq \epsilon \leq \delta_{k-1}} \|f_\epsilon\|_2 < \infty,$$

where $\delta_k = 2^{-k}$ for $k \geq 0$ ([26, Lemma 1]). Consequently, $\tilde{h} = \lim_{\epsilon \to 0} h_\epsilon$ exists in $L^2(Y, \mathcal{B}, \nu)$ and $P_{T, \nu}h = 0$. Let $\epsilon_n = 2^{-j_n}$ for $n \geq 1$ where $j_n$ is the unique integer $j$ for which $2^{j-1} \leq n < 2^j$. Then

$$\sum_{k=0}^{n-1} (h - \tilde{h}) \circ T^k = \sum_{k=0}^{n-1} (h_{\epsilon_n} - \tilde{h}) \circ T^k + \epsilon_n \sum_{k=0}^{n-1} f_{\epsilon_n} \circ T^k + U_T^n P_{T, \nu}f_{\epsilon_n} - P_{T, \nu}f_{\epsilon_n} \quad (9)$$

+
by Equation 4. Since $P_{T,\nu}(h_\epsilon_n - \tilde{h}) = 0$, we have
\[
\frac{\|\sum_{k=0}^{n-1} (h - \tilde{h}) \circ T^k\|_2}{\sqrt{n}} \leq \|h_\epsilon_n - \tilde{h}\|_2 + (\epsilon_n \sqrt{n} + \frac{2}{\sqrt{n}})\|f_\epsilon_n\|_2
\leq \|h_\epsilon_n - \tilde{h}\|_2 + 6\epsilon_n\|f_\epsilon_n\|_2,
\]
but the right-hand side of this inequality converges to 0 as $n \to \infty$, which completes the proof. \hfill \Box

One situation in which all of the assumptions of the preceding theorem are met is described in the following

**Corollary 1** Let $T$ be a measure-preserving transformation on the probability space $(Y, B, \nu)$ and let $h \in L^2(Y, B, \nu)$ be such that $\int h(y)\nu(dy) = 0$. Suppose that
\[
\sum_{n=1}^{\infty} \frac{\|P_{T,\nu}^n h\|_2}{\sqrt{n}} < \infty.
\]
Then
\[
\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h \circ T^k \to^d \sigma N(0, 1)
\]
where
\[
\sigma = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \|\sum_{k=0}^{n-1} h \circ T^k\|_2.
\]

By imposing stronger assumptions on the growth of the norm in Condition 5 we can deduce a stronger version of the central limit theorem. Here we adapt the ideas of [9, 11]. We use the standard notation $b(n) = O(a(n))$ if $\lim \sup_{n \to \infty} b(n)/a(n) < \infty$.

**Theorem 4** Let $T$ be a measure-preserving transformation on the probability space $(Y, B, \nu)$ and let $h \in L^2(Y, B, \nu)$ be such that $\int h(y)\nu(dy) = 0$. Suppose that
\[
\|\sum_{k=0}^{n-1} P_{T,\nu}^k h\|_2 = O(n^\alpha) \quad \text{with} \quad \alpha < \frac{1}{2}.
\]
Then $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j$ converges to zero $\nu$–a.e and in $L^2(Y, B, \nu)$ as $n \to \infty$.

In particular, if $T$ is ergodic, then the CLT and FCLT hold for $h$ provided that $\|\tilde{h}\|_2 > 0$.

**Proof 2** Condition 11 and Equation 6 imply that $\|f_\epsilon\|_2 = O(\epsilon^{-\alpha})$ as $\epsilon \to 0$.

We are going to show that
\[
\|\sum_{k=0}^{n-1} (h - \tilde{h}) \circ T^k\|_2 = O(n^\alpha).
\]
Since

\[ ||h_{\epsilon_n} - \tilde{h}||_2 \leq \sum_{k=j_n+1}^{\infty} ||h_{\delta_k} - h_{\delta_{k-1}}||_2, \]

we obtain the estimate \( ||h_{\epsilon_n} - \tilde{h}||_2 = O(n^{\alpha-1/2}) \) using inequality (8) and the definition of \( \epsilon_n \). We also have \( \sqrt{n}||f_{\epsilon_n}||_2 = O(n^{\alpha-1/2}) \) and the desired assertion follows from Equation (8). Now the arguments of [11] apply. By Theorem 2.17 of [11], the estimate (7) implies that \( (\hat{h} - \tilde{h}) \in (\mathcal{U}_T)^\beta(L^2(Y, \mathcal{B}, \nu)) \) for \( \frac{1}{2} < \beta < 1 - \alpha \). Hence by Theorem 3.2(i) of [11], with \( p = \frac{1}{2} \), we obtain

\[ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\hat{h} - \tilde{h}) \circ T_k = 0 \quad \nu - a.e. \]

and this in turn implies that

\[ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \max_{0 \leq n-1} \left| \sum_{j=0}^{k} (\hat{h} - \tilde{h}) \circ T^j \right| = 0 \quad \nu - a.e., \]

which completes the proof. \( \square \)

**Corollary 2** Let \((Y, \mathcal{B}, \nu)\) be a probability measure space and \( T : Y \to Y \) be ergodic with respect to \( \nu \). Let \( h \in L^2(Y, \mathcal{B}, \nu) \) be such that \( \int h(y) \nu(dy) = 0 \). Then

\[ ||\sum_{k=0}^{n-1} \mathcal{P}_T^k h||_2 = O(1) \]  \hspace{1cm} (13)

if and only if there exist \( \tilde{h}, f \in L^2(Y, \mathcal{B}, \nu) \) such that \( \mathcal{P}_T^\infty \tilde{h} = 0, h = \tilde{h} + f \circ T - f \).

In particular, under Condition (13) the CLT and FCLT hold for \( h \) provided that \( h \neq f \circ T - f \) for any \( f \).

**Proof 3** Since \( L^2(Y, \mathcal{B}, \nu) \) is a reflexive Banach space, Condition (13) is equivalent to \( h = g - \mathcal{P}_{T,\nu} g \) with some \( g \in L^2(Y, \mathcal{B}, \nu) \) (Butzer and Westphal [2], Proposition 1). First assume that \( h = \tilde{h} + f \circ T - f \) with \( \mathcal{P}_{T,\nu} \tilde{h} = 0 \). By taking \( g = \tilde{h} + f \circ T \) and noting that \( \mathcal{P}_{T,\nu} g = \mathcal{P}_{T,\nu} U_T f = f \) we arrive at \( g = h + \mathcal{P}_{T,\nu} g \) which implies Condition (13).

Now assume that Condition (13) holds. Let \( g \) be such that \( g = g - \mathcal{P}_{T,\nu} g \). Taking \( h_1 = g - U_T \mathcal{P}_{T,\nu} g \) and observing that \( \mathcal{P}_{T,\nu} h_1 = 0 \), we arrive at the decomposition

\[ h = h_1 + f \circ T - f \]

where \( f = \mathcal{P}_{T,\nu} g \). By Theorem 4 there is \( \tilde{h} \) such that \( \mathcal{P}_{T,\nu} \tilde{h} = 0 \) and

\[ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\hat{h} - \tilde{h}) \circ T^j \to 0. \]

Since \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h_1 - \tilde{h}) \circ T^j \to 0 \) and \( \mathcal{P}_{T,\nu} (h_1 - \tilde{h}) = 0 \) implies \( ||\sum_{j=0}^{n-1} (h_1 - \tilde{h}) \circ T^j ||_2 = \sqrt{n} ||h_1 - \tilde{h}||_2 \), which completes the proof. \( \square \)
Now we give a simple result that derives a CLT and FCLT from a decay of correlations. Although the CLT in this case is due to [22], we also obtain the functional version.

**Corollary 3**

Let \((Y, \mathcal{B}, \nu)\) be a probability measure space, \(T : Y \to Y\) be ergodic with respect to \(\nu\), and let \(h \in L^\infty(Y, \mathcal{B}, \nu)\) be such that \(\int h(y)\nu(dy) = 0\). Suppose that there are \(\beta > 1\) and \(c > 0\) such that

\[
\left| \int h(y)g(T^n(y))\nu(dy) \right| \leq c n^{-\beta} ||g||_{\infty}
\]

for all \(g \in L^\infty(Y, \mathcal{B}, \nu)\) and sufficiently large \(n\). Then \(\sigma \geq 0\) given by

\[
\sigma^2 = \int h^2(y)\nu(dy) + 2 \sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy)
\]

is finite and if \(\sigma > 0\) the CLT and FCLT hold for \(h\).

Moreover, \(\sigma = 0\) if and only if \(h = f \circ T - f\) for some \(f \in L^1(Y, \mathcal{B}, \nu)\).

**Proof 4**

Condition 14 implies that

\[
||P^n_{T,\nu}h||_2 \leq ||h||^{1/2}_{\infty} ||P^n_{T,\nu}h||_1^{1/2} \quad \text{and} \quad ||P^n_{T,\nu}h||_1 \leq \frac{c}{n^{\beta}}. 
\]

(cf. [27], Proposition 1). Since all assumptions of Theorem 4 are met and the series \(\sum_{n=1}^{\infty} \int h(y)h(T^n(y))\nu(dy)\) is convergent, the assertions follow. It remains to discuss the case of \(\sigma = 0\). As in the proof of Theorem 4 let \(f_\epsilon = \sum_{k=1}^{\infty} \frac{P^{k-1}_{T,\nu}h}{(1+\epsilon)^k}\). Then the estimate of the norm \(||P^n_{T,\nu}h||_1\) allows us to conclude that \(f_\epsilon\) converges as \(\epsilon \to 0\) to \(\bar{f} = \sum_{k=0}^{\infty} P^k_{T,\nu}h\) and \(\bar{f} \in L^1(Y, \mathcal{B}, \nu)\). From Equation 7 it then follows that \(h = U_T f - f\) where \(f = P_{T,\nu} \bar{f}\), which completes the proof. \(\Box\)

4 Some examples

4.1 Maps with a neutral fixed point

Let \(Y = [0, 1]\) and \(\mathcal{B} = \mathcal{B}([0, 1])\) be the \(\sigma\)-algebra of Borel subsets of \([0, 1]\). For fixed \(\gamma > 0\) let us consider the map \(T_\gamma : [0, 1] \to [0, 1]\) given by

\[
T_\gamma(y) = \begin{cases} 
  y(1 + 2^\gamma y) & 0 \leq y \leq \frac{1}{2} \\
  2y - 1 & \frac{1}{2} < y \leq 1
\end{cases}
\]

which was introduced by Liverani et al. [23] to illustrate a probabilistic approach to prove polynomial decay of correlations. The transformation \(T_\gamma\) is a simple model of maps with a neutral (indifferent) fixed point at \(p = 0\), i.e. \(T_\gamma(p) = p\) and \(|T_\gamma'(p)| = 1\). As shown in [23] the transformation \(T = T_\gamma\) has a unique absolutely continuous invariant probability measure \(\nu = \nu_\gamma\), whose density is
Lipschitz continuous on any interval \((c, 1]\) and for each \(h \in C^1([0, 1])\) there exists a constant \(C = C(h)\) such that for all \(g \in L^\infty([0, 1], \mathcal{B}([0, 1]), \nu)\) and \(n \geq 1\)

\[
\left| \int h(y)g(T^n(y))\nu(dy) - \int h(y)\nu(dy) \int g(y)\nu(dy) \right| \leq C\rho_n||g||_\infty \tag{17}
\]

where \(\rho_n = n^{1-\frac{1}{\tau}}(\log n)^{\frac{1}{\tau}}\).

Let \(0 < \gamma < \frac{1}{\tau}\). Then there is \(\beta \in (1, \frac{1}{\gamma} - 1)\) such that \(\rho_n \leq C \gamma^n\) for sufficiently large \(n\). Thus by Corollary 3 the CLT and FCLT hold for \(h \in C^1([0, 1])\) with \(\int h(y)\nu(dy) = 0\) provided that \(h \neq f \circ T - f\) for any \(f\).

Young [32] uses an abstract coupling approach to obtain sub-exponential decay of correlations through the tail behaviour of a return time function, applies her method to more general one-dimensional maps with an indifferent fixed point, where in particular a finite number of expanding branches are allowed and it is assumed that \(yT^n(y) \approx y^\gamma\) near the indifferent fixed point, and shows that for Hölder continuous functions \(h\) on \([0, 1]\) we have \(\rho_n = n^{1-\frac{1}{\gamma}}\) in Equation 17. This family of maps contains the interval maps with an indifferent fixed point studied by Pollicott and Sharp [29] and, in particular, the Manneville-Pomeau map. Consequently, our Corollary 3 extends Theorem 1 of [29] to all \(\gamma \in (0, \frac{1}{2})\).

When \(\gamma \in (\frac{1}{2}, 1)\) and \(h\) is Hölder continuous with \(h(0) \neq 0\) then the CLT does not hold as shown in [14].

### 4.2 One-dimensional maps with critical points

Consider the system studied by Bruin et al. [5]. Let \(T : I \rightarrow I\) be a \(C^3\) interval or circle map with a finite set \(C\) of critical points \((c \in C \text{ if } T'(c) = 0)\) and no stable or neutral periodic orbit. \(T\) is unimodal if it has only one critical point, and multimodal if it has more than one. All critical points are assumed to have the same finite critical order \(l \in (1, \infty)\), i.e. for \(c \in C\) there exists a diffeomorphism \(r : \mathbb{R} \rightarrow \mathbb{R}\) with \(r(0) = 0\) such that for \(y\) close to \(c\)

\[T(y) = \pm |r(y-c)|^l + T(c)\]

where the \(\pm\) may depend on the sign of \(y-c\). For a critical point \(c\), let \(D_n(c) = |(T^n)'(T(c))|\). For simplicity consider the case of unimodal maps. In [5] the method of Young [32] is adapted and the rate \(\rho_n\) in Equation 17 is related to the growth of \(D_n(c)\). In particular, if there exists \(C > 0\), \(\tau > 2l - 1\) such that \(D_n(c) \geq Cn^\tau\), for all \(n \geq 1\), then the map \(T\) has an absolutely continuous invariant probability measure, the measure is ergodic, and for any \(\hat{\tau} < \frac{2l - 1}{\tau - 1}\), we have \(\rho_n = n^{-\hat{\tau}}\). Consequently, our Corollary 3 implies both the CLT and FCLT for any Hölder continuous function \(h\).

In the study of asymptotic laws of return times in [5] the CLT for \(h = \log |T'| - \int \log |T'(y)|\nu(dy)\) is proved. It is shown that \(h \in L^2(Y, \mathcal{B}, \nu)\) and that the \(L^2\) norm of \(\mathcal{P}_n^T, h\) constitute a convergent series provided that \(D_n(c) \geq Cn^\tau\)
with $\tau > 4l - 3$ and $C > 0$. Then Gordin’s theorem as stated in [30] is used. Our Corollary 1 gives a more refined result in this case as it can be used for $h \in L^2(Y, \mathcal{B}, \nu)$. Note that Theorem 1.1 of [22] requires $h \in L^\infty(Y, \mathcal{B}, \nu)$.

4.3 Transformations on metric spaces

Let $Y = X$ be a metric space with some metric $d$ and $\mathcal{B} = \mathcal{B}(X)$ be the $\sigma$-algebra of Borel subsets of $X$. Consider a transformation $T : X \to X$ such that $T^{-1}(x)$ is countable or finite for each $x \in X$ and a strictly positive measurable function $\psi : X \to \mathbb{R}$, called a potential, such that for each $x \in X$ the sum $\sum_{y \in T^{-1}(x)} \psi(y)$ is convergent. The Ruelle-Perron-Frobenius operator is defined formally on bounded measurable functions $\phi : X \to \mathbb{R}$ by

$$(L_\psi \phi)(x) = \sum_{T(y) = x} \psi(y)\phi(y).$$

For a thorough and up to date presentation of the concept of Ruelle-Perron-Frobenius in studying decay of correlations we refer to [2].

Recently Pollicott [28], in the context of subshifts of finite type, gave an estimate of the convergence speed of $L^1$ norm of iterates $L^n \phi$, $n \geq 1$, when $\psi$ has a summable variation. Later on, Fan and Jiang [14] extended it to locally expansive Dini dynamical system and gave an estimate in the supremum norm of $C(X, \mathbb{R})$.

Let us recall the setting and notations of [14]. Let $X$ be compact, $T$ be a continuous transformation and $\psi$ be a continuous function. $T$ is said to be locally expanding if there are constants $\lambda > 1$ and $b > 0$ such that $d(T(x), T(y)) \geq \lambda$ if $d(x, y) \leq b$. This implies that $T$ is a local homeomorphism and the operator $L_\psi$ acts on the Banach space $C(X, \mathbb{R})$ of real valued continuous functions equipped with the supremum norm $\|\psi\|_\infty = \max_{x \in X} |\psi(x)|$. Recall that a right continuous and increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$ is called a modulus of continuity. Denote by $\mathcal{H}_\omega$ the space of all functions $\phi \in C(X, \mathbb{R})$ for which

$$[\phi]_\omega = \sup_{0 < d(x, y) \leq a} \frac{|\phi(x) - \phi(y)|}{\omega(d(x, y))} < \infty,$$

where $0 < a \leq b$ is a constant for which $T^{-1}(y) = \{x_1, \ldots, x_n\}$ and $T$ has local inverses $S_1, \ldots, S_n$ defined on the pairwise disjoint sets $S_j(B(y, a))$. Finally, $\omega$ is said to satisfy the Dini condition if

$$\int_0^1 \frac{\omega(t)}{t} \, dt < \infty.$$

Suppose that $T$ is locally expanding and (topologically) mixing, the modulus of continuity $\omega$ satisfies the Dini condition, and $\psi \in \mathcal{H}_\omega$. From the Ruelle theorem proved in [13] it follows that there exists a strictly positive number $\rho$ and a strictly positive continuous function $\phi_*$ such that $L_\psi \phi_* = \rho \phi_*$, and a
unique probability measure $\mu_\psi$ such that
\[
\int L\psi \phi(x) \mu_\psi(dx) = \int \phi(x) \mu_\psi(dx).
\]
If we take $\phi_*$ to be normalized so $\int \phi_*(x) \mu_\psi(dx) = 1$, then for any $\phi \in C(X, \mathbb{R})$
\[
\|\phi - \phi_* \|_{\infty} \rightarrow 0.
\]
The measure $\mu_\psi$ has the so-called Gibbs property and we call the measure
\[
\nu = \phi_* \mu_\psi
\]
the Gibbs measure for $T$. It is an invariant probability measure for $T$.

Instead of working with the operator $L\psi$ let us consider its normalization $\tilde{L}$, which is defined as follows. Let
\[
\tilde{\psi} = \frac{\psi \phi_*}{\rho \phi_* \circ T}
\]
and define
\[
\tilde{L} = L\tilde{\psi}.
\]
The important feature for $\tilde{L}$ is that $\tilde{L}1 = 1$ and the transfer operator $\mathcal{P}_{T,\nu}$ on $L^1(X, \mathcal{B}, \nu)$ and the operator $\tilde{L}$ are related by
\[
\mathcal{P}_{T,\nu} \phi = \tilde{L} \phi - \nu \text{ a.e., } \phi \in C(X, \mathbb{R}).
\]
This yields
\[
\|\mathcal{P}_{T,\nu}^n \phi\|_2 \leq \|\tilde{L}^n \phi\|_{\infty} \quad \text{for } \phi \in C(X, \mathbb{R})
\]
and Theorem 4 of [14] can be applied directly to obtain an estimate on $\mathcal{P}_{T,\nu}^n \phi$, $n \geq 1$ through the rate of decay to zero of $\|\tilde{L}^n \phi\|_{\infty}$ which depends on the modulus of continuity of $\phi$ and the choice of $\omega$, so that we limit ourselves to recall two consequences of the estimates in [14]:

1. Let $\omega(t) = Ct^\theta$ for some constants $C > 0$ and $0 < \theta \leq 1$. Then $\mathcal{H}_\omega = C^\theta$ is the space of $\theta$-Hölder continuous functions. Thus $\psi \in C^\theta$ and it is known that the convergence speed is exponential, so that there are constants $C > 0$ and $\theta > 0$ such that for any $\phi \in C^\theta$ with $\int \phi(x) \nu(dx) = 0$
\[
\|\mathcal{L}_\psi^n \phi\|_\infty \leq Ce^{-\theta n}, \quad n \geq 1.
\]

2. Let $\omega(t) = \frac{1}{|\log t|^{1+\varepsilon}}$ and $\omega_0(t) = \frac{1}{|\log t|^{1+\varepsilon}}$ with $\varepsilon > 0$. If the potential $\psi \in \mathcal{H}_\omega$ and $\phi \in \mathcal{H}_{\omega_0}$ with $\int \phi(x) \nu(dx) = 0$, then there exists a constant $C > 0$ such that
\[
\|\mathcal{L}_\psi^n \phi\|_\infty \leq C\frac{(\log n)^{\frac{1}{2}+\varepsilon}}{n^{\frac{1}{2}+\varepsilon}}, \quad n \geq 1.
\]

From Theorem 11 it follows that the CLT and FCLT hold for $\phi$ in both cases. In the case when $\psi \in \mathcal{H}_{\omega_1}$ with $\omega_1(t) = \frac{1}{|\log t|^{1+\varepsilon}}$, it was proved in [14, Theorem 6.] that the CLT holds for $\phi \in \mathcal{H}_{\omega_0}$ as in [2]. Thus Theorem 11 generalizes the result of [14].
5 Conclusions

Here we have reviewed and extended central and functional central limit theorems as established by particular types of temporal decay of correlations. In particular, for the first time, we have established criteria for CLT and FCLT validity based on polynomial decay of correlations. Three concrete examples demonstrate the utility of these results, and show that they are applicable directly after establishing the decay through, for example, the coupling method of Young [31, 32] which is very flexible or through functional-analytic method using Ruelle’s operator. Another method has been introduced in [21] to deal with maps with discontinuities and to obtain exponential decay. It involves a direct study of the Ruelle-Perron-Frobenius operator but using the so-called Birkhoff metrics and the notion of invariant cones. Moreover it has been adapted in [25] to deal with systems with sub-exponential decay. See the excellent texts [2, 30] for detailed discussions of the functional-analytic methods.

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