PREFIX CODES: EQUIPROBABLE WORDS, UNEQUAL LETTER COSTS

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Abstract. We consider the following variant of Huffman coding in which the costs of the letters, rather than the probabilities of the words, are non-uniform: “Given an alphabet of \( r \) letters of non-uniform length, find a minimum-average-length prefix-free set of \( n \) codewords over the alphabet;” equivalently, “Find an optimal \( r \)-ary search tree with \( n \) leaves, where each leaf is accessed with equal probability but the cost to descend from a parent to its \( i \)th child depends on \( i \).” We show new structural properties of such codes, leading to an \( O(n \log^2 r) \)-time algorithm for finding them. This new algorithm is simpler and faster than the best previously known \( O(nr \min\{\log n, r\}) \)-time algorithm due to Perl, Garey, and Even [7].

Key words. Algorithms, Huffman Codes, Prefix Codes, Trees.

AMS subject classification. Analysis of Algorithms.

1. Introduction. The well-known Huffman coding problem is the following: given a sequence of access probabilities \( \langle p_1, p_2, \ldots, p_n \rangle \), construct a binary prefix code \( \langle w_1, w_2, \ldots, w_n \rangle \) minimizing the expected length \( \sum_i p_i \cdot \text{length}(w_i) \). A binary prefix code is a set of binary strings, none of which is a prefix of another.

A natural generalization of the problem is to allow the words of the code to be strings over an arbitrary alphabet of \( r \geq 2 \) letters and to allow each letter to have an arbitrary non-negative length. The length of a codeword is then the sum of the lengths of its letters. For instance, the “dots and dashes” of Morse code are a variable-length alphabet with length corresponding to transmission time. (See Figure 2.1.) This generalization of Huffman coding to a variable-length alphabet has been considered by many authors, including Altenkamp and Mehlhorn, and Karp. Apparently no polynomial-time algorithm for it is known, nor is it known to be NP-hard.

A prefix code in which the codewords \( \langle w_1, w_2, \ldots, w_n \rangle \) are in alphabetical order is called alphabetic. In this case the underlying tree represents an \( r \)-ary search tree. The length of the \( i \)th letter corresponds to the time required to descend from a node into its \( i \)th subtree. This time is often a function of \( i \) in search-tree algorithms, for instance, when the subtree to descend into is chosen by sequential search. An optimal alphabetic code thus corresponds to a minimum expected-cost search tree.

In this paper we consider the special case in which the codewords occur with equal probability, i.e., each \( p_i \) equals \( 1/n \). With this restriction, the alphabetic and non-alphabetic problems are equivalent. The problem may be viewed as a variant of Huffman coding in which the lengths of the letters, rather than the codeword probabilities, are non-uniform. Alternatively, it may viewed as the problem of finding an optimal \( r \)-ary search tree, where the search queries are uniformly distributed but the time to descend from a parent to its \( i \)th child depends on \( i \). For the complexity results stated in this paper, the algorithms return a tree representing an optimal code.

In 1989, Kapoor and Reingold described a simple \( O(n) \)-time algorithm for the binary case \( r = 2 \). In 1975, Perl, Garey, and Even gave an \( O(rn \min\{r, \log n\}) \)-time algorithm.

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algorithm. (Although due a typographical error their abstract incorrectly claims an $O(n)$-time algorithm.) In the same year Cot [2] described an $O(r^2n)$-time algorithm.

In 1971, Varn [8] gave an algorithm without analyzing its complexity. It appears Varn’s algorithm requires $\Omega(rn)$ time.

In this paper we describe an $O(n \log r)$-time algorithm based on new insights into the structure of optimal trees. In Section 2 we define shallow and proper trees and prove that some proper shallow tree is optimal. In Section 3 we develop the algorithm, which efficiently constructs all proper shallow trees and returns one representing an optimal prefix code.

2. Shallow Trees. Fix an instance of the problem, given by the respective lengths $(c_1 \leq c_2 \leq \cdots \leq c_r)$ of the $r$ letters in the alphabet and the number $n$ of (equiprobable and prefix-free) codewords required. We assume the standard tree representation of prefix codes, as described in the following definition.

Definition 2.1. The infinite $r$-ary tree is the infinite, rooted, $r$-ary tree. Each tree edge has a length and a label — an edge going from a node to its $i$th child has length $c_i$ and is labeled with the $i$th letter in the alphabet.

A node is a node of the infinite $r$-ary tree. The finite words over the alphabet of $r$ letters correspond to the nodes. The labels along the path from the root to any node spell the corresponding word and the length of the path is the length of this word. A prefix code corresponds to a set of nodes none of which is a descendant of another. (See Figure 2.1.)

Definitions 2.2. A tree is any subtree $T$ of the infinite $r$-ary tree containing the root. In any tree, $n$ of the leaves will be identified as terminals; their corresponding words form a prefix code. The remaining nodes in the tree are referred to as non-terminals.

Give a node $u$, the notation $\text{child}_i(u)$ denotes $u$’s $i$th child; $\text{depth}(u)$ denotes the

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Fig. 2.1. Two trees for the 6 symbols $a, b, c, d, e, f$, each occurring with probability $1/6$. The tree on the left is the optimal tree that uses the alphabet $\{0, 1\}$, length($0$) = length($1$) = 1, while the tree on the right is for the alphabet $\{\_\_\_\_, \_\_\_\_\_\_\_\_\_\_\}$ with length(\_\_\_\_) = 1 and length(\_\_\_\_\_\_\_\_\_) = 2. The corresponding sets of codewords are

\[
\begin{align*}
& a = 000, \quad b = 001, \quad c = 011, \quad d = 011, \quad e = 10, \quad f = 11 \\
& a = \_\_\_\_, \quad b = \_\_\_\_, \quad c = \_\_\_\_, \quad d = \_\_\_\_, \quad e = \_\_\_, \quad f = \_\_\_\_\_\_.
\end{align*}
\]
depth (the length of the corresponding codeword); parent(u) denotes the parent.

The cost \( c(T) \) of such a tree is the sum of the depths of the terminals — also
called the external weighted path length of the tree.

A proper tree is a tree in which every non-terminal has at least two children. The
goal is to find an optimal tree with \( n \) terminals. It is easy to see that some optimal
tree is proper; thus, we restrict our attention to proper trees.

Our basic tool for understanding the structure of optimal trees is a swapping
argument. For example, in any proper optimal tree, no non-terminal is deeper than
any terminal. Otherwise, the terminal and the subtree rooted at the non-terminal
could be swapped, decreasing the average depth of the terminals.

We use a swapping argument to prove that an optimal proper tree has the fol-
lowing form for some \( m \). The non-terminals are the \( m \) shallowest (i.e., least-depth)
nodes of the infinite tree, while the terminals are the \( n \) shallowest available children
of these nodes in the infinite tree. We call such a tree shallow; here is the precise
definition:

**Definition 2.3.** A tree \( T \) is shallow provided that

(i) for any non-terminal \( u \in T \) and any node \( w \) (not necessarily in \( T \)) that is
not a non-terminal, depth(u) \( \leq \) depth(w) and

(ii) for any terminal \( u \in T \) and any node \( w \) that is not in \( T \) but is a child of a
non-terminal, depth(u) \( \leq \) depth(w).

Note that a non-terminal of an (improper) shallow tree might have no children
in the tree. This is why we refer to “terminal” and “non-terminal” nodes in place of
the more common “internal nodes” and “leaves”.

As a simple example consider the basic binary tree; \( r = 2 \), \( c_1 = c_2 = 1 \). A
proper binary tree \( T \) will be shallow if and only if there is some depth \( l \) such that
(a) every node \( u \) in the infinite tree with depth(\( u \)) < \( l \) is a non-terminal in \( T \) and (b)
all terminals of \( T \) are on levels \( l \) and \( l + 1 \). Conditions (a) and (b) are necessary and
sufficient conditions for \( T \) to have minimum external path length among all binary
trees with the same number of leaves, see e.g., [6, §5.3.1]. So, a binary tree has
minimum external path length for its number of leaves if and only if it is shallow. For
example, the binary tree on the left of Figure 2.1 has minimum external path length
among all trees with 6 leaves because it fulfills conditions (a) and (b) with \( l = 2 \). As
we will see later, though, for most values of \( r \) and \( c_i \), shallowness alone does not imply
optimality. However, if a shallow tree has the right number of non-terminals, then it
is optimal:

**Lemma 2.4.** Let \( m^* \) be the minimum number of non-terminals in any optimal
tree. Then any shallow tree with \( m^* \) non-terminals is optimal and proper.

**Proof.** Fix a shallow tree \( T \) with \( m^* \) non-terminals. We will show the existence
of an optimal tree with the same non-terminals as \( T \). Since \( T \) is shallow, by property
(ii), this will imply \( T \) is optimal. By the choice of \( m^* \), \( T \) is also proper (otherwise
there would be an optimal proper tree with fewer non-terminals).

It remains to show the existence of an optimal tree with the same non-terminals
as \( T \). Let \( T^* \) be an optimal (and therefore proper) tree with \( m^* \) non-terminals. Let
\( N \) and \( N^* \) be the sets of non-terminals of \( T \) and \( T^* \), respectively. If \( N = N^* \) we are
done. Otherwise, let \( u \) be a minimum-depth node in \( N - N^* \), so that \( u \)'s parent is in
\( N^* \). Let \( u^* \) be a node in \( N^* - N \). Note that, since \( T \) is shallow, depth(\( u^* \)) \( \geq \) depth(\( u \)),
but that, in \( T^* \), \( u^* \) is a non-terminal (with at least two terminal descendants) while
\( u \) is either a terminal or not present.

In \( T^* \), swap the subtrees rooted at \( u \) and \( u^* \). Specifically, make \( u \) a non-terminal
and, for each descendant $v^*$ of $u^*$, delete it and add the corresponding descendant $v$ of $u$. If $v^*$ was a terminal, make $v$ a terminal, otherwise make $v$ a non-terminal. If $u$ was a terminal, make $u^*$ a terminal, otherwise delete $u^*$. Call the resulting tree $T'$.

From $\text{depth}(u^*) \geq \text{depth}(u)$ it follows that $c(T') \leq c(T^*)$. Thus, $T'$ is also optimal. Note that $T'$ shares one more non-terminal with $T$ than does $T^*$. Thus, repeated swapping produces an optimal tree with the same non-terminals as $T$.

Note that $m^* \geq (n-1)/(r-1)$, since each node has degree at most $r$.

**Corollary 2.5.** Let $m_{\text{min}} = \lceil (n-1)/(r-1) \rceil$. Let $(T_{m_{\text{min}}}, T_{m_{\text{min}}+1}, T_{m_{\text{min}}+2}, \ldots)$ be any sequence of shallow trees such that for each $m$, $T_m$ has $m$ non-terminals. Then one of the $T_m$ is proper and optimal.

The algorithm generates a sequence of shallow trees as above and returns the one which has minimum cost. The lemma guarantees that this tree will be optimal. The rest of the paper is devoted to examining the properties of shallow trees which enable the enumeration of the proper shallow trees in $O(n \log^2 r)$ time.

**2.1. Defining the Trees.**
Ordering the nodes. Label the nodes of the infinite tree as 1, 2, 3, ..., in order of increasing depth. Break ties arbitrarily, except that if two nodes u and w are of equal depth, and both are ith children of their respective parents, and parent(u) < parent(w), then let u < w (this is needed for Lemma 2.7). For the sake of notation, identify each node with its label, so that 1 is the root, 2 is a minimum-depth child of the root, etc. Figure 2.3 illustrates the top section of such a labeling for r = 3, c1 = 2, c2 = 2, and c3 = 5. These values of r and c are the ones we use in all later examples.

Definition 2.6. For each m ≥ mmin define Tm to be the tree whose non-terminals are {1, ..., m} and whose terminals are the minimum n nodes among the children of {1, ..., m} in {m + 1, m + 2, ...}. Thus, Tm is the “shallowest” tree with m non-terminals with respect to the ordering of the nodes. Since the ordering of the nodes respects depth, each Tm is shallow. Figure 2.3 presents T5, T6, T7, and T8 for n = 10 using the labeling of Figure 2.2.

2.2. Relation of Successive Trees. Next we turn our attention to the relation of Tm+1 to Tm.

Lemma 2.7. For m ≥ mmin, the new non-terminal (node m + 1) in Tm+1 is the minimum terminal of Tm.

Proof. The parent of m + 1 is in {1, ..., m}, so m + 1 is the minimum child of {1, ..., m} in {m + 1, m + 2, ...}. The result follows from the definition of Tm.

Lemma 2.8. For m ≥ mmin, provided the new non-terminal (node m + 1) in Tm+1 has at least one child, each terminal of Tm+1 is either a child of m + 1 or a terminal of Tm.

Proof. Let node m + 1 have d children in Tm+1. Let C denote the set of children of nodes {1, ..., m} in {m + 1, m + 2, ...}. The terminals of tree Tm+1 consist of the minimum d children of node m + 1 together with the minimum n − d nodes in C − {m + 1}. These n − d nodes, together with node m + 1 (the minimum node in C), are the n − d + 1 minimum nodes in C. If d ≥ 1, then by the definition of Tm, each such node is a terminal in Tm.

The main significance of Lemmas 2.7 and 2.8 is that they will allow an efficient construction of Tm+1. Moreover, they imply that, if Tm is not proper, neither is any subsequent tree.

Lemma 2.9. One of the trees {Tmin, Tmin+1, ..., Tmax} is optimal and proper, where mmax = min{m : Tm + 1 is improper}.

Proof. By Lemma 2.8 if Tm is improper, then so is Tm+1 — either node m + 1 has no children in Tm+1 or the non-terminal in Tm that had less than two children also has less than two children in Tm+1. Hence, for each m > mmax, tree Tm is improper. Thus Corollary 2.2 implies that one of the trees {Tmin, Tmin+1, ..., Tmax} is proper and optimal.

For n = 10, mmin = ⌈4n + 1⌉ = 5 and (as shown in Figure 2.3) T8 is improper. The lemma then implies that one of T5, T6, or T7 must have minimum external path length. Calculation shows that T6 with c(T6) = 59 is the optimal one.

3. Computing the Trees. The algorithm uses the following two operations to compute the trees.

To SPROUT a tree is to make its minimum terminal a non-terminal and to add the minimum child of this non-terminal as a terminal.

To LEVEL a tree is to add c children of the maximum non-terminal to the tree as terminals and to remove the c largest terminals in the tree. The c children are the minimum c children not yet in the tree, where c is maximum such that all children added are less than all terminals deleted.
Fig. 2.3. The trees $T_5$, $T_6$, $T_7$, and $T_8$ for $r = 3$, $c_1 = 2$, $c_2 = 2$, $c_3 = 5$ and $n = 10$. The node numbering is that of the previous figure. Calculating the external path lengths we find that $c(T_5) = 60$, $c(T_6) = 59$, $c(T_7) = 60$, and $c(T_8) = 62$.

The algorithm computes the initial tree $T_{m_{\text{min}}}$, then repeatedly sprouts and levels to obtain successive trees until the tree so obtained is not proper. Lemmas 2.7 and 2.8 imply that, as long as node $m + 1$ has at least one child in $T_{m+1}$ (it will if $T_{m+1}$ is proper), sprouting and leveling $T_m$ yields $T_{m+1}$. Figure 3.1 illustrates this operation.

**Observation 3.1.** Let $m = m_{\text{max}}$. If node $m + 1$ has at least one child in $T_{m+1}$ then sprouting and leveling $T_m$ yields tree $T_{m+1}$. If node $m + 1$ has no children in $T_{m+1}$, then the maximum terminal in $T_m$ is less than the minimum child of node $m + 1$ and sprouting and leveling $T_m$ yields a tree in which non-terminal $m + 1$ has one child. Hence, the algorithm always correctly identifies $T_{m_{\text{max}}}$ and terminates correctly, having considered all relevant trees.
To **Sprout** requires identification and conversion of the minimum terminal of the current tree, whereas to **Level** requires identification and replacement of (no more than \(r\)) maximum terminals by children of the new non-terminal. One could identify the maximum and minimum terminals in \(O(\log n)\) time by storing all terminals in two standard priority queues (one to detect the minimum, the other to detect the maximum). At most \(r\) terminals would be replaced in computing each tree and, because \(m_{\text{max}} \leq n - 1\), only \(O(n)\) trees would be computed. This approach yields an \(O(rn \log n)\)-time algorithm.

By a more careful use of the structure of the trees, we improve this in two ways. First, we give an amortized analysis showing that in total, only \(O(n \log r)\), rather than \(O(rn)\), terminals are replaced. Second, we show how to reduce the number of non-terminals in each priority queue to at most \(r\). This yields an \(O(n \log^2 r)\)-time algorithm.

Both improvements follow from the tie-breaking condition on the ordering of the nodes, which guarantees that \(T_m\) must have the following structure.

**Lemma 3.2.** In any \(T_m\), if \(u\) and \(w\) are non-terminals with \(u < w\), and the \(i\)th child of \(w\) is in the tree, then so is the \(i\)th child of \(u\). If the \(i\)th child of \(w\) is a non-terminal, then so is the \(i\)th child of \(u\).
The result follows from average (while obtaining $T \sum$ then maintaining these in $T$ \{number of children of the $m$ children (i.e., queue. This reduces the cost of finding the minimum from $O$ between these terminals is of the form $O$ \{} among these terminals is child $i$ and $w_i$. The minimum among terminals that are child $i$ (the $i$th child of $u_i$). The maximum among these terminals is child$_i(w_i)$. Proof. A straightforward consequence of Lemma 3.2. Figure 2.3 presents $u_i$ and $w_i$ for the trees $T_5$, $T_6$, $T_7$, and $T_8$ when $n = 10$. This lemma implies that the minimum terminal in $T_m$ is the minimum among \{child$_i(u_i) : i = 1, \ldots, r\}$. Our algorithm finds the minimum terminal in $T$ by maintaining these $r$ particular children (rather than all $n$ terminals) in a priority queue. This reduces the cost of finding the minimum from $O(\log n)$ to $O(\log r)$. Similarly the algorithm finds the maximum terminal in $O(\log r)$ time by maintaining \{child$_i(w_i) : i = 1, \ldots, r\}$ in an additional priority queue.

Observation 3.6. As an aside, one can prove using Lemma 3.2 that, for any $m$ such that $m_{\text{min}} < m < m_{\text{max}}$, $c(T_{m+1}) - c(T_m) \geq c(T_m) - c(T_{m-1})$. That is, the sequence of tree costs is unimodal. To prove this, consider building $T_{m+1}$ from $T_m$. Sprouting increases the cost by $c_1$; leveling decreases the cost with each swap. For each swap in building $T_{m+1}$ from $T_m$, one can show there was a corresponding swap in building $T_m$ from $T_{m-1}$ and that the decrease in cost (from $T_m$ to $T_{m+1}$) due to the former is bounded by the decrease in cost (from $T_{m-1}$ to $T_m$) due to the latter. Thus, in practice the algorithm could be modified to stop and return $T_{m-1}$ when $c(T_m) \geq c(T_{m-1})$.

Corollary 3.3. Node $m$ has a minimum number of children among all non-terminals in $T_m$.

3.1. Only $O(n \log r)$ Replacements Total. The number of terminals replaced while obtaining $T_m$ from $T_{m-1}$ is at most the number of children of non-terminal $m$ in $T_m$. Although this might be $r$ for many $m$, the sum of the numbers of children is $O(n \log r)$:

Lemma 3.4. Let $d_m$ be the number of children of non-terminal $m$ in tree $T_m$. Then $\sum m d_m$ is $O(n \log r)$.

Proof. By Corollary 3.3, within $T_m$, node $m$ has the fewest children. The total number of children of the $m$ non-terminals is $m + n - 1$. Thus, $d_m$ is at most the average $(m + (n - 1)/m = 1 + (n - 1)/(1/m)$.

$$\sum_{m=m_{\text{min}}}^{m_{\text{max}}} d_m \leq (m_{\text{max}} - m_{\text{min}} + 1) + (n - 1) \sum_{m=m_{\text{min}}}^{m_{\text{max}}} 1/m = O(m_{\text{max}} - m_{\text{min}} + n \log(m_{\text{max}}/m_{\text{min}})).$$

The result follows from $m_{\text{min}} = \lceil \frac{n-1}{r-1} \rceil$ and $m_{\text{max}} \leq n - 1$.

3.2. Limiting the Relevant Terminals. To reduce the number of terminals that must be considered in finding the minimum and maximum terminals, we partition the terminals into $r$ groups. The $i$th group consists of the terminals that are $i$th children ($i = 1, \ldots, r$).

Lemma 3.5. In any $T_m$, for any $i$, the set of non-terminals whose $i$th children are terminals is of the form $\{u_i, u_i + 1, \ldots, w_i\}$ for some $u_i$ and $w_i$. The minimum among terminals that are $i$th children is child$_i(u_i)$ (the $i$th child of $u_i$). The maximum among these terminals is child$_i(w_i)$.

Proof. A straightforward consequence of Lemma 3.2.

As an aside, one can prove using Lemma 3.2 that, for any $m$ such that $m_{\text{min}} < m < m_{\text{max}}$, $c(T_{m+1}) - c(T_m) \geq c(T_m) - c(T_{m-1})$. That is, the sequence of tree costs is unimodal. To prove this, consider building $T_{m+1}$ from $T_m$. Sprouting increases the cost by $c_1$; leveling decreases the cost with each swap. For each swap in building $T_{m+1}$ from $T_m$, one can show there was a corresponding swap in building $T_m$ from $T_{m-1}$ and that the decrease in cost (from $T_m$ to $T_{m+1}$) due to the former is bounded by the decrease in cost (from $T_{m-1}$ to $T_m$) due to the latter. Thus, in practice the algorithm could be modified to stop and return $T_{m-1}$ when $c(T_m) \geq c(T_{m-1})$.

\footnote{This observation is due to R. Fleischer.}
3.3. The Algorithm in Detail. The full algorithm has two distinct phases. The first phase constructs the base tree $T_{m_{\min}}$. The second phase starts with $T_{m_{\min}}$ and, by Sprouting and Leveling, iteratively constructs the sequence of shallow trees

$$\langle T_{m_{\min}}, T_{m_{\min}+1}, T_{m_{\min}+2}, \ldots, T_{m_{\max}} \rangle$$

and returns one which has smallest external path length. $T_{m_{\max}}$ is the last proper tree in the sequence, i.e., $T_{m_{\max}+1}$ is improper. Lemma 2.9 guarantees that the algorithm returns an optimal tree. We now describe how to implement the first part of the algorithm in $O(n \log r)$ time and the second in $O(n \log^2 r)$ time; the full algorithm therefore runs in $O(n \log^2 r)$ time.

The skeleton of the final algorithm is shown in Fig. 3.2. Procedure Create-$T_{m_{\min}}$ creates tree $T_{m_{\min}}$, the variable $C$ contains the external path length of current tree $T_m$ and $mDeg$ contains the number of children of node $m$ in tree $T_m$. As presented, the algorithm computes only the cost of an optimal tree. It can easily be modified to compute the actual tree. Note that to check that the current tree $T_m$ is proper, by Observation 3.3 and Corollary 3.5, it suffices to check that non-terminal $m$ has at least two children.

**Algorithm to find an optimal variable-length prefix code**

The routines Sprout and Level are shown in Figure 3.3. Recall that the nodes of the infinite tree are labeled in order of increasing depth with ties broken arbitrarily except for the requirement that if $u$ and $v$ are both of equal depth and both are $i$th children of their respective parents, then $u < v$ if parent$(u) <$ parent$(v)$. Depending upon $c_1, c_2, \ldots, c_r$, there may be many such labelings. The algorithm we present breaks ties lexicographically — suppose $u$ and $v$ have the same depth and let $u = \text{child}_i(u')$ and $v = \text{child}_j(v')$; then $u < v$ if $u' < v'$ (or $u' = v'$ and $i < j$). Figure 2.2 illustrates this labeling for $r = 3$, $c_1 = 2$, $c_2 = 2$, and $c_3 = 5$. The sequence of shallow trees is fully determined by this labelling. Figure 2.3 illustrates the shallow trees with 10 non-terminals for these $r$ and $c$ values.

The algorithm represents the current tree $T_m$ with the following data structures:

- **N** — The number of terminals.
- **m** — The number of non-terminals. Also the rank of the maximum non-terminal.
- **C** — The sum of the depths of the terminals.
- **mDeg** — The number of children of non-terminal $m$.
- **D[u]** — The depth of each non-terminal $u$.
- **u[i]** — The rank of the minimum non-terminal (if any) whose $i$th child is a terminal $(1 \leq i \leq r)$.
- **w[i]** — The rank of the maximum non-terminal (if any) whose $i$th child is a terminal $(1 \leq i \leq r)$. If no non-terminal has a terminal $i$th child, then $u[i] > w[i]$. 

![Algorithm to find an optimal variable-length prefix code](image-url)
Sprout ($T$) — Make the minimum terminal a non-terminal —
1. $m \leftarrow m + 1$;
2. Let child$_i(u[i])$ be the minimum terminal in low-queue.
3. $D[m] \leftarrow D[u[i]] + c_i$; $u[i] \leftarrow u[i] + 1$; update Qs($T, i$)
4. $C \leftarrow C - D[m]$; mDeg $\leftarrow 0$;
   — Add smallest child as a terminal —
5. Add-Terminal($T$)

Level($T$)
1. WHILE ($m$Deg < $r$ and child$_{mDeg+1}(m)$ is less than the max. terminal child$_i(w[i])$ in high-queue) DO
2. Add-Terminal($T$)
   — Delete the maximum terminal —
3. $C \leftarrow C - (D[w[i]] + c_i)$
4. $w[i] \leftarrow w[i] - 1$; update Qs($T, i$)

Add-Terminal($T$)
1. $m$Deg $\leftarrow m$Deg + 1; $C \leftarrow C + D[m] + c_{mDeg}$;
2. $w[m$Deg$] \leftarrow m$; update Qs($T, m$Deg$)$

**Fig. 3.3. The Operations Sprout and Level.**

**low-queue** — A priority queue for finding the minimum terminal.
Contains \{child$_i(u[i]) : u[i] \leq w[i]\}.

**high-queue** — A priority queue for finding the maximum terminal.
Contains \{child$_i(w[i]) : u[i] \leq w[i]\}.

For an example refer back to Figure 2.3. Tree $T_6$ has

$N = 10$, \hspace{1cm} $C = 59$, \hspace{1cm} mDeg = 2,

$D[1] = 0$, \hspace{1cm} $D[2] = 2$, \hspace{1cm} $D[3] = 3$, \hspace{1cm} $D[4] = 4$, \hspace{1cm} $D[5] = 4$, \hspace{1cm} $D[6] = 4$,

$u[1] = 4$, \hspace{1cm} $u[2] = 3$, \hspace{1cm} $u[3] = 1$, \hspace{1cm} $w[1] = 6$, \hspace{1cm} $w[2] = 6$, \hspace{1cm} $w[3] = 3$

low-queue = \{child$_1(4), \text{child}_2(3), \text{child}_3(1)\}$,

high-queue = \{child$_1(6), \text{child}_2(6), \text{child}_3(3)\}$.

The priority queues are maintained as follows. In general, a terminal in $T_m$ can have rank (label) arbitrarily larger than $m$. The algorithm explicitly maintains the ranks and depths of the $m$ non-terminals in the current tree; the algorithm compares the ranks of terminals in the priority queues via the ranks and depths of their (non-terminal) parents. When $u[i]$ or $w[i]$ changes to reflect a new current tree, the queues are updated by the following routine:

**Update Qs($T, i$)**
1. IF ($u[i] \leq w[i]$) THEN
2. Update child$_i(u[i])$ in low-queue and child$_i(w[i])$ in high-queue to maintain the queues’ invariants.
3. ELSE Delete both nodes from their respective queues.

Line 2 replaces the old child$_i(u[i])$ in low-queue (child$_i(w[i])$ in high-queue) by
the new one when \( u[i] \) (\( w[i] \)) changes. Line 3 will only be executed if \( \text{child}_i(u[i]) > \text{child}_i(w[i]) \), which will only happen if the tree no longer contains any \( i \)th child as a terminal. Note that Lemmas 2.7 and 3.2 imply that, for some \( i \) and \( T_m \), no non-terminal has an \( i \)th child in \( T_m \), then no non-terminal has an \( i \)th child in \( T_{m+1} \).

Construction of the First Tree.. Tree \( T_{m_{\text{min}}} \) has a simple structure. Its non-terminals are the nodes \( (1, 2, \ldots, m_{\text{min}}) \). Its terminals are the \( n \) shallowest children of nodes \( (1, 2, \ldots, m_{\text{min}}) \).

To construct \( T_{m_{\text{min}}} \) we assume that \( n > r \), otherwise \( T_{m_{\text{min}}} \) is simply the root and its first \( n \) children. For \( 1 \leq m < m_{\text{min}} \), define \( T_m \) to be the tree with non-terminals \( \{1, \ldots, m \} \) and all of the \( (r - 1)m + 1 \) children of \( \{1, \ldots, m \} \) as terminals. The proof of Lemma 2.7 generalizes easily to these trees; node \( m + 1 \) is the minimum terminal of \( T_m \).

**CREATE-T_{m_{\text{min}}}(T)**

— Create \( T_1 \) —

1. \( m_{\text{min}} = \lceil \frac{n}{r} \rceil \); \( D[1] ← 0 \); \( C = \sum_{i=1}^{\min\{r, n\}} c_i \);
2. CREATE low-queue; CREATE high-queue;
3. FOR \( i = 1 \) to \( \min\{r, n\} \) DO
   4. \( u[i] ← w[i] ← 1 \); UPDATE-Qs(T, i);

— Create \( \langle T_2, T_3, \ldots, T_{m_{\text{min}}-1} \rangle \) —

5. FOR \( m = 2 \) to \( (m_{\text{min}} - 1) \) DO
6.   Let \( \text{child}_i(u[i]) \) be the minimum terminal in low-queue.
7.   \( D[m] ← D[u[i]] + c_i \); \( u[i] ← u[i] + 1 \); UPDATE-Qs(T, i);
8.   FOR \( j = 1 \) to \( r \) DO
9.     \( w[j] ← m \); UPDATE-Qs(T, j);
10. \( C ← C - D[m] + \sum_{j=1}^{r} (D[m] + c_j) \);

— Create \( T_{m_{\text{min}}} \) —

11. \( m = m_{\text{min}} \); \( \Delta = n - (r - 1)(m_{\text{min}} - 1) \);
12. Let \( \text{child}_i(u[i]) \) be the minimum terminal in low-queue.
13. \( D[m] ← D[u[i]] + c_i \); \( u[i] ← u[i] + 1 \); UPDATE-Qs(T, i)
14. FOR \( j = 1 \) to \( \Delta \) DO
15.     \( w[j] ← m \); UPDATE-Qs(T, j);
16. \( C ← C - D[m] + \sum_{j=1}^{\Delta} (D[m] + c_j) \);
17. \( m_{\text{Deg}} = \Delta \);
18. LEVEL(T);

**Fig. 3.4. Operation CREATE-T_{m_{\text{min}}}**.

The tree \( T_1 \) is easy to construct. It is the tree with 1 root and \( r \) children. Inductively construct the tree \( T_m \) from the tree \( T_{m-1} \), \( m < m_{\text{min}} \) as follows: find the minimum terminal in \( T_m \) by taking the minimum terminal out of low-queue. Label this node \( m \), make it a non-terminal, and add all of its children to \( T_m \) as terminals. The details are shown in Fig. 3.4.

Finally, construct \( T_{m_{\text{min}}} \) from \( T_{m_{\text{min}}-1} \) by making the lowest terminal of \( T_{m_{\text{min}}-1} \) into node \( m_{\text{min}} \). Add the \( n - (r - 1)(m_{\text{min}} - 1) \) minimum children of node \( m_{\text{min}} \) as terminals bringing the total number of terminals in the current tree to \( n \). Level the resulting tree.

Since only \( O(n/r) \) trees are constructed while computing \( T_{m_{\text{min}}} \) and each tree
can be constructed from the previous tree in $O(r \log r)$ time, the time required to compute $T_{m_{\text{min}}}$ is $O(n \log r)$. (If desired, the time for each tree $T_m$ with $m < m_{\text{min}}$ can be reduced to $O(\log r)$, because maximum terminals are not replaced in constructing such a tree.)

**Construction of the Remaining Trees.** The algorithm constructs the sequence of trees

$$\langle T_{m_{\text{min}}}, T_{m_{\text{min}}+1}, T_{m_{\text{min}}+2}, \ldots, T_{m_{\text{max}}} \rangle$$

as described previously. Tree $T_m$ is found by **Sprouting** and then **Leveling** its predecessor $T_{m-1}$. The cost is $O(d_m \log r)$ time, where $d_m$ is the number of children of the new non-terminal $m$ in $T_m$. By Lemma 3.4 this part of the algorithm runs in $O((\sum_m d_m) \log r) = O(n \log^2 r)$ time.

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