Small Knudsen Rate of Convergence to Rarefaction Wave for the Landau Equation

RENJUN DUAN, DONGCHENG YANG & HONGJUN YU

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Abstract

In this paper, we are concerned with the hydrodynamic limit to rarefaction waves of the compressible Euler system for the Landau equation with Coulomb potentials as the Knudsen number $\varepsilon > 0$ is vanishing. Precisely, whenever $\varepsilon > 0$ is small, for the Cauchy problem on the Landau equation with suitable initial data involving a scaling parameter $a \in \left[ \frac{2}{3}, 1 \right]$, we construct the unique global-in-time uniform-in-$\varepsilon$ solution around a local Maxwellian whose fluid quantities are the rarefaction wave of the corresponding Euler system. In the meantime, we establish the convergence of solutions to the Riemann rarefaction wave uniformly away from $t = 0$ at a rate $\varepsilon^{\frac{3}{5} - \frac{2}{5} a | \ln \varepsilon |}$ as $\varepsilon \to 0$. The proof is based on the refined energy approach combining Guo (Commun Math Phys 231:391–434, 2002) and Liu et al. (Physica D 188:178–192, 2004) under the scaling transformation $(t, x) \to (\varepsilon^{-a} t, \varepsilon^{-a} x)$.

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1. Introduction

The Landau equation is one of the most fundamental equations in plasma physics. A lot of great contributions in the mathematical study of the spatially inhomogeneous Landau equation have been made by many people, including Lions [31],
Villani [44], Alexander–Villani [2], Degond–Lemou [10] and Guo [21]. In particular, Guo [21] gave the first proof for constructing the global classical solutions close to a constant equilibrium state in a periodic box, and later Strain and Guo [39,40] established the large time asymptotic behavior of those global solutions. Since then, the spatially inhomogeneous perturbation theory of the Landau equation around global Maxwellians was further developed in different settings; see Yu [48], Carrapatoso–Tristani–Wu [7], Carrapatoso–Mischler [6], Guo–Hwang–Jang–Ouyang [22], and Duan–Liu–Sakamoto–Strain [11]. In the case of the whole space, the global classical solution near vacuum was also constructed by Luk [36]. Recently, lots of research has been done investigating the regularity of solutions to the spatially inhomogeneous Landau equation for general initial data under certain conditions; see Golse–Imbert–Mouhot–Vasseur [16] and Henderson–Snelson [23], for instance. In this paper, we consider another interesting topic on the hydrodynamic limit of the Landau equation for which quite few results are known, although it has been extensively studied in the Boltzmann theory, cf. Grad [18], Golse [15] and Saint-Raymond [38].

1.1. Problem

We consider the one-dimensional Landau equation

$$\partial_t F + v_1 \partial_x F = \frac{1}{\varepsilon} Q(F, F),$$

(1.1)

where the unknown $F = F^\varepsilon(t, x, v) \geq 0$ stands for the density distribution function for the gas particles with space position $x \in \mathbb{R}$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t > 0$. On the right hand side of (1.1), the parameter $\varepsilon > 0$ is the Knudsen number which is proportional to the mean free path, and the Landau collision operator $Q(\cdot, \cdot)$ is a bilinear integro-differential operator acting only on velocity variables, taking the form of

$$Q(F_1, F_2)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v - v_*) \left\{ F_1(v_*) \nabla_v F_2(v) - \nabla_{v_*} F_1(v_*) F_2(v) \right\} \, dv.$$  

(1.2)

The non-negative matrix $\phi$ in the integral above is given by

$$\phi(v) = \left( I - \frac{v \otimes v}{|v|^2} \right) |v|^\gamma + 2, \quad \gamma \geq -3,$$

(1.3)

where $I$ is the $3 \times 3$ identity matrix and $v \otimes v$ is the tensor product. Note that (1.2) in the case $\gamma = -3$ corresponds to the original (Fokker–Planck–Landau collision operator for Coulomb potentials, see [2,10,21]. Throughout the paper, we are focused on the very soft potentials case $-3 \leq \gamma < -2$, since this is similar to treating the other cases $\gamma \geq -2$ in an easier way for which the linearized Landau operator has the spectral gap. Some further comments on the main differences of the proof between the very soft potentials case and the other cases will be given later on.
Formally, when the Knudsen number $\varepsilon$ tends to zero, the limit of the Landau equation (1.1) gives rise to the one-dimensional compressible Euler system

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_1)}{\partial x} &= 0, \\
\frac{\partial (\rho u_1)}{\partial t} + \frac{\partial (\rho u_1^2) + p}{\partial x} &= 0, \\
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x} &= 0, \quad i = 2, 3, \\
\left\{ \rho (e + \frac{|u|^2}{2}) \right\}_t + \left\{ \rho u_1 \left( e + \frac{|u|^2}{2} \right) + p u_1 \right\}_x &= 0,
\end{align*} \tag{1.4}$$

where

$$\begin{align*}
\rho(t, x) &= \int_{\mathbb{R}^3} \psi_0(v) F \, dv, \\
\rho u_i(t, x) &= \int_{\mathbb{R}^3} \psi_i(v) F \, dv, \quad \text{for } i = 1, 2, 3, \\
\rho(e + \frac{1}{2}|u|^2)(t, x) &= \int_{\mathbb{R}^3} \psi_4(v) F \, dv.
\end{align*} \tag{1.5}$$

Here $\rho = \rho(t, x) > 0$ is the mass density, $u = u(t, x) = (u_1, u_2, u_3)$ is the fluid velocity, $e = e(t, x)$ is the internal energy, and $p = R \rho \theta$ is the pressure, where $R$ is the gas constant that we will set to be $\frac{2}{3}$ throughout the paper for convenience and $\theta = \theta(t, x) > 0$ is the temperature related to the internal energy $e$ by $e = \frac{3}{2} R \theta = \theta$. Moreover, the five collision invariants $\psi_i(v)$ ($i = 0, 1, 2, 3, 4$) are given by

$$\psi_0(v) = 1, \quad \psi_i(v) = v_i \ (i = 1, 2, 3), \quad \psi_4(v) = \frac{1}{2} |v|^2,$$

satisfying

$$\int_{\mathbb{R}^3} \psi_i(v) Q(F, F) \, dv = 0, \quad \text{for } i = 0, 1, 2, 3, 4. \tag{1.6}$$

The rigorous mathematical justification of establishing the hydrodynamic limit to the Euler system (1.4) for the Landau equation (1.1) in a general setting is an outstanding open problem in kinetic theory, which is similar to the case of the Boltzmann equation with or without angular cutoff, cf. [15,18,38]. Regarding solutions with basic wave patterns (cf. [28,42]), there have been extensive studies of the global existence and large time asymptotic behavior of solutions (cf. [5,25,33,35,50]) and small Knudsen rate of convergence (cf. [26,27,30,47,49]) in the context of the cutoff Boltzmann equation; some relevant literature will be reviewed in detail later on. However, to the best of our knowledge, few results on this topic are known for either the non-cutoff Boltzmann or Landau equation, essentially due to the effect of grazing singularity of both collision operators on non-trivial profiles with even small space variations connecting two distinct global Maxwellians, that makes it necessary to develop new perturbation approaches beyond the situation where solutions are close to a constant equilibrium (cf. [1,17,21]). Recently, the first and third authors of this paper studied in [12] the nonlinear stability as well as the large-time asymptotic of rarefaction waves for the Landau equation (1.1) with Coulomb potentials. In the present work, we expect to further study the hydrodynamic limit with rarefaction waves of the one-dimensional Landau equation (1.1) as Knudsen number $\varepsilon > 0$ is sufficiently small.
1.2. Macro–Micro Decomposition

For our purpose above, as in [33,34], we define the local Maxwellian $M$ associated with the solution $F$ to the equation (1.1) in terms of the fluid quantities of $F$ as in (1.5) by

$$M = M_{[\rho, u, \theta]}(t,x)(v) = \frac{\rho(t, x)}{(2\pi R \theta(t, x))^{3/2}} \exp\left(-\frac{|v - u(t, x)|^2}{2R \theta(t, x)}\right). \quad (1.7)$$

We denote an $L^2_ν(\mathbb{R}^3)$ inner product as $\langle h, g \rangle = \int_{\mathbb{R}^3} h(v) g(v) \, dv$. Then, considering the linearized Landau operator around the local Maxwellian $M$ of the form

$$L_M h = Q(h, M) + Q(M, h), \quad (1.8)$$

the macroscopic kernel space is spanned by the following five pairwise-orthogonal base:

$$\begin{align*}
\chi_0(v) &= \frac{1}{\sqrt{\rho}} M, \\
\chi_i(v) &= \frac{v_i - u_i}{\sqrt{\rho \theta}} M, \quad \text{for } i = 1, 2, 3, \\
\chi_4(v) &= \frac{1}{\sqrt{6\rho}} \left(\frac{|v - u|^2}{R \theta} - 3\right) M, \\
\langle \chi_i, \frac{\chi_j}{M} \rangle &= \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4.
\end{align*}$$

In terms of these five orthonormal functions, we define the macroscopic projection $P_0$ and the microscopic projection $P_1$ as follows:

$$P_0 h = \sum_{i=0}^{4} \langle h, \frac{\chi_i}{M} \rangle \chi_i, \quad P_1 h = h - P_0 h. \quad (1.9)$$

A function $h(v)$ is called microscopic or non-fluid if

$$\int_{\mathbb{R}^3} h(v) \psi_i(v) \, dv = 0, \quad \text{for } i = 0, 1, 2, 3, 4. \quad (1.10)$$

Initiated by Liu–Yu [33] and developed by Liu–Yang–Yu [34], for a non-trivial solution profile connecting two different global Maxwellians at $x = \pm \infty$, we decompose the equation (1.1) and its solution with respect to the local Maxwellian (1.7) as

$$F = M + G, \quad P_0 F = M, \quad P_1 F = G, \quad (1.11)$$

where the local Maxwellian $M$ as (1.7) and $G = G(t, x, v)$ represent the macroscopic and microscopic component in the solution respectively. Then the equation (1.1) becomes

$$(M + G)_t + v_1(M + G)_x = \frac{1}{\varepsilon} Q(G, M) + \frac{1}{\varepsilon} Q(M, G) + \frac{1}{\varepsilon} Q(G, G), \quad (1.12)$$
due to \( Q(M, M) = 0 \). Multiplying (1.12) by the collision invariants \( \psi_i(v) \) \((i = 0, 1, 2, 3, 4)\) and integrating the resulting equations with respect to \( v \) over \( \mathbb{R}^3 \), one gets the following macroscopic system:

\[
\begin{align*}
\rho_t + (\rho u_1)_x &= 0, \\
(\rho u_i)_t + (\rho u_i^2)_x + p_x &= -\int_{\mathbb{R}^3} v_i^2 G_x \, dv, \\
(\rho u_i)_t + (\rho u_1 u_i)_x &= -\int_{\mathbb{R}^3} v_i v_j G_x \, dv, \quad i = 2, 3,
\end{align*}
\]

Here we have used (1.5), (1.6) and the fact that \( G_t \) is microscopic by (1.10).

Applying the projection operator \( P_1 \) to (1.12) and using (1.11), we obtain the following microscopic system:

\[ G_t + P_1(v_1 G_x) + P_1(\mu_1 M_x) = \frac{1}{\epsilon} L_M G + \frac{1}{\epsilon} Q(G, G). \tag{1.14} \]

Here the linearized operator \( L_M \) is defined in (1.8) and the null space \( \mathcal{N} \) of \( L_M \) is spanned by \( \chi_i \) \((i = 0, 1, 2, 3, 4)\). It follows by (1.14) that

\[ G = \epsilon L_M^{-1}[P_1(v_1 M_x)] + L_M^{-1} \Theta, \quad \Theta := \epsilon G_t + \epsilon P_1(v_1 G_x) - Q(G, G). \tag{1.15} \]

Substituting the expression of \( G \) in (1.15) into (1.13), we further obtain the following fluid-type system

\[
\begin{align*}
\rho_t + (\rho u_1)_x &= 0, \\
(\rho u_1)_t + (\rho u_1^2)_x + p_x &= \frac{4}{3} \epsilon (\mu(\theta) u_{1x})_x - (\int_{\mathbb{R}^3} v_i^2 L_M^{-1} \Theta \, dv)_x, \\
(\rho u_i)_t + (\rho u_1 u_i)_x &= \epsilon (\mu(\theta) u_{ix})_x - (\int_{\mathbb{R}^3} v_i v_j L_M^{-1} \Theta \, dv)_x, \quad i = 2, 3,
\end{align*}
\]

Here the viscosity coefficient \( \mu(\theta) > 0 \) and the heat conductivity coefficient \( \kappa(\theta) > 0 \), both are smooth functions depending only on \( \theta \). The explicit formulas of \( \mu(\theta) \) and \( \kappa(\theta) \) are defined by (5.3).

### 1.3. Rarefaction Wave and Its Smooth Approximation

Now we turn to define the rarefaction wave profile to the system (1.1) as in [32, 35, 37]. Consider the Euler system (1.4) with the state equation \( p = \frac{\gamma}{\gamma-1} \rho^\gamma \), where \( k_0 = \frac{1}{2\pi_0} \) and \( S \) is the macroscopic entropy, supplemented with the following Riemann initial data

\[ (\rho, u, \theta)(t, x)|_{t=0} = \begin{cases} 
(\rho_0^R, u_0^R, \theta_0^R)(x) = \left\{ \begin{array}{ll} 
(\rho_+, u_+, \theta_+), & x > 0, \\
(\rho_-, u_-, \theta_-), & x < 0.
\end{array} \right. \tag{1.17}
\end{cases} \]

Here \( \rho_0 > 0, u_0 = (u_1, 0, 0) \) and \( \theta_0 > 0 \) are assumed to be constant.
The rarefaction wave \((ρ^R, u^R, θ^R)(x/t)\) with \(u^R(x/t) = (u_1^R(x/t), 0, 0)\) to the Riemann problem (1.4) and (1.17) can be constructed as follows. We can rewrite the system (1.4) in terms of \((ρ, u, S)\) with \(u = [u_1, 0, 0]\) as

\[
\begin{align*}
ρ_t + (ρu_1)_x &= 0, \\
u_{1t} + u_1u_{1x} + \frac{1}{ρ}p_x &= 0, \\
S_t + u_1S_x &= 0.
\end{align*}
\]

It is well known that the Euler system (1.18) has three distinct eigenvalues:

\[
λ_i(ρ, u_1, S) = u_1 + (-1)^{i+1}\sqrt{p_ρ(ρ, S)}, \quad i = 1, 3, \quad λ_2(ρ, u_1, S) = u_1.
\]

Here \(p_ρ(ρ, S) = \frac{5}{3}k_0ρ^{\frac{3}{2}}e^S > 0\). In terms of the two Riemann invariants of the third eigenvalue \(λ_3(ρ, u_1, S)\), we define the 3-rarefaction wave curve for the given left constant state \((ρ_-, u_{1-}, θ_-)\), with \(ρ_- > 0\) and \(θ_- > 0\) as follows (cf. [28, 42]):

\[
R_3(ρ_-, u_{1-}, θ_-) = \{(ρ, u_1, θ) ∈ \mathbb{R}_+ × \mathbb{R} × \mathbb{R}_+ | S = S_\ast, u_1 - \sqrt{15k_0e^{\frac{5}{2}}ρ_{\ast}} = u_{1-} - \sqrt{15k_0e^{\frac{5}{2}}ρ_{\ast}}, \quad ρ > ρ_-, \quad u_1 > u_{1-}\}.
\]

Here and to the end, \(S_\ast := S_{\ast -} = -\frac{2}{3}ln(ρ_+ + ln(\frac{4}{7}πθ_-) + 1\) is a constant.

Without loss of generality, we consider only the simple 3-rarefaction wave in this paper, and the case for 1-rarefaction wave can be treated similarly. The 3-rarefaction wave to the Euler system (1.4) with (1.17) can be expressed explicitly by the Riemann solution to the inviscid Burgers equation

\[
\begin{align*}
ω_t + ωω_x &= 0, \\
ω(0, x) &= \begin{cases} ω_-, & x < 0, \\
ω_+, & x > 0. \end{cases}
\end{align*}
\]

If two constants \(ω_- < ω_+\) are chosen, then (1.20) admits a centered rarefaction wave solution \(ω^R(x, t) = ω^R(\frac{x}{t})\) connecting \(ω_-\) and \(ω_+\) (cf. [37]) in the form of

\[
ω^R(\frac{x}{t}) = \begin{cases} ω_-, & \frac{x}{t} ≤ ω_- \\
\frac{x}{t}, & ω_- < \frac{x}{t} ≤ ω_+ \\
ω_+, & \frac{x}{t} > ω_+. \end{cases}
\]

For \((ρ_\ast, u_{1\ast}, θ_\ast) ∈ R_3(ρ_-, u_{1-}, θ_-)\), the 3-rarefaction wave \((ρ^R, u^R, θ^R)(\frac{x}{t})\) with \(u^R(\frac{x}{t}) = (u_1^R, u_2^R, u_3^R)(\frac{x}{t})\) to the Riemann problem (1.4) with (1.17) can be defined explicitly by

\[
\begin{align*}
λ_3(ρ^R(\frac{x}{t}), u_1^R(\frac{x}{t}), S_\ast) &= \begin{cases} λ_3(ρ_-, u_{1-}, S_\ast), & \frac{x}{t} ≤ λ_3(ρ_-, u_{1-}, S_\ast), \\
\frac{x}{t}, & λ_3(ρ_-, u_{1-}, S_\ast) < \frac{x}{t} ≤ λ_3(ρ_+, u_{1+}, S_\ast), \\
λ_3(ρ_+, u_{1+}, S_\ast), & \frac{x}{t} > λ_3(ρ_+, u_{1+}, S_\ast), \end{cases} \\
u_1^R(\frac{x}{t}) - \sqrt{15k_0e^{\frac{5}{2}}(ρ^R(\frac{x}{t})}_1^\frac{1}{2} = u_{1-} - \sqrt{15k_0e^{\frac{5}{2}}ρ_{\ast}}, & u_2^R(\frac{x}{t}) = u_3^R(\frac{x}{t}) = 0,
\end{align*}
\]

\[
\theta^R(\frac{x}{t}) = \frac{3}{2}k_0e^{S_\ast}(ρ^R(\frac{x}{t})_3^\frac{1}{2} \]
Since the above 3-rarefaction wave is only Lipschitz continuous, we shall construct an approximate smooth rarefaction wave to the 3-rarefaction wave defined in (1.21). Motivated by [37,45], the approximate smooth rarefaction wave can be constructed by the Burgers equation

\[
\begin{cases}
\bar{\omega}_t + \bar{\omega}\bar{\alpha}_x = 0, \\
\bar{\omega}(0, x) = \bar{\omega}_\delta(x) = \bar{\omega}(\frac{x}{\delta}) = \frac{\omega_{\alpha} + \omega_{\beta}}{2} + \frac{\omega_{\alpha} - \omega_{\beta}}{2} \tanh(\frac{x}{\delta}),
\end{cases}
\tag{1.22}
\]

where \( \delta > 0 \) is a small constant depending on the Knudsen number \( \varepsilon \) and \( \tanh(\cdot) \) is the usual hyperbolic tangent. In fact, as given in (3.3) later on, we will choose \( \delta = \frac{1}{k} \varepsilon^{\frac{3}{2} - \frac{7}{2}a} \) for a suitably small constant \( k > 0 \) independent of \( \varepsilon \). By the method of characteristic curves, the solution \( \bar{\omega}_\delta(t, x) \) to the problem (1.22) can be given by

\[
\bar{\omega}_\delta(t, x) = \bar{\omega}_\delta(x_0(t, x)), \quad x = x_0(t, x) + \bar{\omega}_\delta(x_0(t, x))t.
\]

The properties of \( \bar{\omega}_\delta(t, x) \) are given by Lemma 5.1 in Section 5.

Correspondingly, the approximate smooth 3-rarefaction wave \( (\bar{\rho}_\delta, \bar{u}_\delta, \bar{\theta}_\delta)(t, x) \) to (1.21) for the Euler system (1.4) and (1.17) can be defined by

\[
\begin{cases}
\bar{\omega}_\delta(t, x) = \lambda_3(\bar{\rho}_\delta(t, x), \bar{u}_\delta(t, x), S_\delta), \quad \omega_\pm = \lambda_3(\rho_\pm, u_{1\pm}, S_\delta), \\
\bar{u}_{1\delta}(t, x) - \sqrt{15k_0 \varepsilon^2 \rho^\frac{1}{2}_\delta} (t, x) = u_{1\pm} - \sqrt{15k_0 \varepsilon^2 \rho^\frac{1}{2}_\delta}, \\
\bar{u}_3(t, x) = \bar{u}_{3\delta}(t, x) = 0, \quad \bar{\theta}_\delta(t, x) = \frac{3}{2}k_0 \varepsilon \rho^\frac{1}{2}_\delta(t, x), \\
\lim_{x \to \pm \infty} (\bar{\rho}_\delta, \bar{u}_{1\delta}, \bar{\theta}_\delta)(t, x) = (\rho_\pm, u_{1\pm}, \theta_\pm), \quad (\rho_+, u_+, \theta_+) \in R_3(\rho_-, u_-, \theta_-),
\end{cases}
\tag{1.23}
\]

where \( \bar{\omega}_\delta(t, x) \) is the solution of Burger equation (1.22). From now on, we shall omit the explicit dependence of \( (\bar{\rho}_\delta, \bar{u}_\delta, \bar{\theta}_\delta)(t, x) \) on \( \delta \) and denote it by \( (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) \) for simplicity. Then the approximate smooth 3-rarefaction wave \( (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) \) satisfies the Euler system

\[
\begin{cases}
\bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\
(\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2)_x + \bar{p}_x = 0, \\
(\bar{\rho}\bar{u}_i)_t + (\bar{\rho}\bar{u}_i\bar{u})_x = 0, \quad i = 2, 3, \\
(\bar{\rho}\bar{\theta})_t + (\bar{\rho}\bar{\theta} \bar{u})_x + \bar{p}\bar{u}_{1x} = 0,
\end{cases}
\tag{1.24}
\]

where \( \bar{p} = R\bar{\rho}\bar{\theta} \). Note that the initial function \( \bar{\omega}_\delta \) in (1.22) connecting two distinct constant states is strictly increasing and smooth, so \( \bar{\omega}(t, x) \) and \( (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) \) are smooth in \( t \) and \( x \) up to any order by [45] and [24]. Without any proof we also list the main properties of \( (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) \) in Lemma 5.2 in Section 5. In terms of the approximate rarefaction wave \( (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) \), we denote

\[
\bar{M} = M_{[-\bar{\rho}, \bar{u}, \bar{\theta}]}(t, x) = \frac{\bar{\rho}(t, x)}{(2\pi R\bar{\theta}(t, x))^{3/2}} \exp\left(-\frac{|v - \bar{u}(t, x)|^2}{2R\bar{\theta}(t, x)}\right).
\]
For the technical reason as in [47], we choose the far-field data \((\rho_+, u_+, \theta_+)\) and \((\rho_-, u_-, \theta_-)\) in (1.17) to be close enough to the constant state \((1, 0, \frac{3}{2})\) such that the approximate smooth rarefaction wave further satisfies that

\[
\begin{align*}
\eta_0 &:= \sup_{t \geq 0, x \in \mathbb{R}} \{|\tilde{\rho}(t, x) - 1| + |\tilde{u}(t, x)| + |\tilde{\theta}(t, x) - \frac{3}{2}|\} \text{ is small,} \\
\frac{1}{2} \sup_{t \geq 0, x \in \mathbb{R}} \tilde{\theta}(t, x) &< \frac{3}{2} < \inf_{t \geq 0, x \in \mathbb{R}} \tilde{\theta}(t, x).
\end{align*}
\] (1.25)

As in [35], associated with the constant state \((1, 0, \frac{3}{2})\), we will use throughout the paper a global Maxwellian

\[
\mu = M_{[1,0,\frac{3}{2}]}(v) = (2\pi)^{-\frac{3}{2}} \exp\{-|v|^2/2\}.
\] (1.26)

We remark that the main reason for assuming (1.25) and introducing the global Maxwellian \(\mu\) is that the fluid quantities \((\rho, u, \theta)(t, x)\) associated with any solution \(F(t, x, v)\) connecting two distinct global Maxwellsians \(M_{\pm} := M_{[\rho_\pm, u_\pm, \theta_\pm]}\) should also satisfy (1.25), (see (3.5) for the proof) and hence one can rewrite the microscopic equation of \(G\) with the linearized Landau operator \(L_M\) around the local Maxwellian \(M = M_{[\rho, u, \theta]}\) as the microscopic equation of \(f\) with the linearized Landau operator \(\mathcal{L}\) around the global Maxwellian \(\mu\) by setting \(G = \overline{G} + \sqrt{\mu} f\), where \(\overline{G}\) is a microscopic correction term, see (2.3), (2.9) and (2.10) later on. Therefore, it is more convenient to make use of the method in [20, 21] to carry out the higher order energy estimates on \(f\) than on \(G\).

### 1.4. Main Results

To present the results, we need to first introduce the solution space. Recall that associated with any function \(F = F(t, x, v)\), one has the corresponding functions \(M = M_{[\rho, u, \theta](t, x)}(v), G = G(t, x, v), \overline{M} = M_{[\overline{\rho}, \overline{u}, \overline{\theta}](t, x)}(v)\) and \(\overline{G} = G(t, x, v)\) in terms of (1.11), (1.25), (2.2) and (2.3). Then, for any \(0 < T \leq +\infty\), we define

\[
X(0, T) = \left\{ F = F(t, x, v) \mid w^{|\beta|} \partial^\alpha \rho \left\{ \frac{F - \overline{M}}{\sqrt{\mu}} \right\} \in C^0([0, T]; L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)), \right. \\
\left. \partial^\alpha (\rho - \overline{\rho}, u - \overline{u}, \theta - \overline{\theta}) \in L^2(0, T; L^2(\mathbb{R}^3_x)), \right. \\
\left. w^{|\beta|} \langle v \rangle^{\gamma} \nabla_v \partial^\alpha \left\{ \frac{G - \overline{G}}{\sqrt{\mu}} \right\} \times \frac{v}{|v|}, \right. \\
\left. w^{|\beta|} \langle v \rangle^{\gamma+2} \partial^\alpha \left\{ \frac{G - \overline{G}}{\sqrt{\mu}} \right\} \in L^2(0, T; L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)), \right. \\
\left. |\alpha| + |\beta| \leq 2 \bigg\},
\] (1.27)

where \(w = \langle v \rangle^{\gamma+2}\) and we may refer to Section 2.2 for all other notations.

The main results of the paper can be stated as follows:
Theorem 1.1. Let $-3 \leq \gamma < -2$ in (1.3). Assume that the far-field data $(\rho_{\pm}, u_{\pm}, \theta_{\pm})$ satisfy $u_{2\pm} = u_{3\pm} = 0$ and $(\rho_{\pm}, u_{1\pm}, \theta_{\pm}) \in R_3(\rho_{\pm}, u_{1\pm}, \theta_{\pm})$ in (1.19), and $\delta_r := |\rho_r - \rho_-| + |u_r - u_-| + |\theta_r - \theta_-|$ is the wave strength. Let $(\rho^R, u^R, \theta^R)(t, x)$ be the Riemann solution (1.21) of the Euler system (1.4) and (1.17), and $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x)$ be the corresponding approximate smooth profile satisfying (1.23), (1.24) and (1.25) induced by the Burgers equation (1.22) with $\delta = \frac{1}{r} \epsilon^{\frac{5}{3} - \frac{3}{5} a}$ for $\frac{2}{3} \leq a \leq 1$ and $k > 0$. Then, there are small constants $\epsilon_0 > 0$, $\eta_0 > 0$ and $k > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and any $\delta_r > 0$ and any $\eta_0 > 0$ with $\delta_r + \eta_0 < \eta_0$, the Cauchy problem on the Landau equation (1.1) with non-negative initial data

$$F(0, x, v) = F_0(x, v) \rightarrow M_{[\rho_{\pm}, u_{\pm}, \theta_{\pm}]}(v), \quad x \rightarrow \pm \infty,$$  \hspace{1cm} (1.28)

satisfying (3.2), admits a unique global-in-time solution $F(t, x, v) \geq 0$ with $F(t, x, v) \in X(0, +\infty)$ such that the following things hold true:

(a) Under the scaling transformation $(\tau, y) = (\epsilon^{-a} t, \epsilon^{-a} x)$ as in (2.1), there are an energy functional $E_2(\tau)$ and a corresponding energy dissipation functional $D_2(\tau)$, given by (2.15) and (2.16) in terms of $(\tau, y)$ coordinates, respectively, such that

$$\sup_{\tau \geq 0} E_2(\tau) + \int_0^{+\infty} D_2(\tau) \, d\tau \leq Ck\epsilon^{\frac{5}{3} - \frac{3}{5} a}.$$  \hspace{1cm} (1.29)

(b) For any $l > 0$, there is a constant $C_{l, k} > 0$, independent of $\epsilon$, such that

$$\sup_{l \geq 1} \left\| F(t, x, v) - M_{[\rho^R, u^R, \theta^R]}(v) \right\|_{L^\infty_x L^2_v} \leq C_{l, k, \epsilon^{\frac{3}{5} - \frac{3}{5} a}} \ln \epsilon.$$  \hspace{1cm} (1.30)

Remark 1.2. To the best of our knowledge, Theorem 1.1 seems to provide the first result regarding the hydrodynamic limit with rarefaction waves for the Landau equation. It remains open to obtain similar results in case of other kinds of basic wave patterns such as shock wave and contact discontinuity. Moreover, we expect that the current work may shed a little light on the study of the same topic on the wave patterns such as shock wave and contact discontinuity. Moreover, we expect that the current work may shed a little light on the study of the same topic on the wave patterns such as shock wave and contact discontinuity. Moreover, we expect that the current work may shed a little light on the study of the same topic on the wave patterns such as shock wave and contact discontinuity. Moreover, we expect that the current work may shed a little light on the study of the same topic on the wave patterns such as shock wave and contact discontinuity.

Remark 1.3. Estimate (1.30) shows that under the condition (1.28) on initial data, the uniform convergence rate in small Knudsen number $\epsilon > 0$ can be variable with respect to the scaling parameter $a \in \left[\frac{7}{3}, 1\right]$. In particular, choosing $a = \frac{7}{3}$ can give the fastest convergence rate $\epsilon^{\frac{1}{3}} \ln \epsilon$.

Remark 1.4. It should be pointed out that the scaling argument was first used in Xin [45] to study under the transformation $(\tau, y) = (\epsilon^{-a} t, \epsilon^{-a} x)$ the vanishing viscosity limit to rarefaction waves for the one-dimensional compressible Navier–Stokes system, where the convergence rate is $\epsilon^{\frac{1}{4}} \ln \epsilon$. Later, Xin–Zeng [47] justified the hydrodynamic limit with rarefaction waves of the Boltzmann equation for the hard sphere model with the convergence rate $\epsilon^{\frac{1}{3}} \ln \epsilon$ through the scaling transformation $(\tau, y) = (\epsilon^{-1} t, \epsilon^{-1} x)$; this convergence rate was later improved by Li [30] to be $\epsilon^{\frac{1}{3}} \ln \epsilon^2$ under the scaling $(\tau, y) = (\epsilon^{-\frac{2}{3}} t, \epsilon^{-\frac{2}{3}} x)$.\
1.5. Relevant Literature

We now review some works related to the study in this paper. Mathematically it is an important and challenging problem to rigorously justify the hydrodynamic limit of kinetic equations in a general setting. Great contributions have been made into different topics of the Boltzmann equation with cutoff. We only refer readers to [15,18,38] mentioned before, as well as two recent progresses [13,14] and reference therein, and also refer to [43] for numerical investigations. Thus, we mainly focus on those known results on the limit of the Boltzmann equation to the compressible Euler system admitting solutions of basic wave patterns, such as rarefaction waves, contact discontinuities and shock waves. In particular, Yu [49] first established the validity of hydrodynamic limit of the Boltzmann equation for the hard-sphere model when the solution of the Euler system contains only the non-interacting shocks. Move precisely, he showed that the Boltzmann solution converges to a local Maxwellian defined by the solution of the Euler system uniformly away from the shock in any fixed time interval. Later, Huang–Wang–Yang [26] proved the hydrodynamic limit to a single contact discontinuity wave, and Xin–Zeng [47] showed the hydrodynamic limit to the Euler system with non-interacting rarefaction waves. As mentioned before, the convergence rate in [47] was improved by Li [30] through a different scaling transformation. Furthermore, Huang–Wang–Wang–Yang [27] obtained the hydrodynamic limit in the general setting of Riemann solutions that contains the superposition of shock, rarefaction wave and contact discontinuity.

Although the hydrodynamic limit from the Boltzmann equation for the hard-sphere model to the Euler system with basic wave patterns has been greatly studied as mentioned above, to the best of our knowledge there are few results on the same topic for the Landau equation or the non-cutoff Boltzmann equation when grazing collisions of particles are dominated. Notice that the cutoff Boltzmann operator is an integral one without angular singularity while the Landau operator or the non-cutoff Boltzmann operator features the velocity diffusion, so it is formally much harder to treat the latter case for the hydrodynamic limit to the non-trivial profiles with space variations.

In this paper, we prove the existence of global-in-time solutions to the one-dimensional Landau equation with suitable initial data as Knudsen number $\varepsilon > 0$ is sufficiently small. And the solution of the Landau equation converges to the local Maxwellian defined by the rarefaction wave of the Euler system uniformly away from $t = 0$ as $\varepsilon \to 0$. Moreover, we obtain the uniform convergence rate $\varepsilon^{\frac{3}{5} - \frac{2}{5}a|\ln \varepsilon|}$ with $a \in [\frac{2}{3}, 1]$ by using the scaling transformation $y = \varepsilon^{-a}x$ and $\tau = \varepsilon^{-a}t$. It should be pointed out that the energy estimates in the current work are performed in the Eulerian coordinates instead of the Lagrangian coordinates as used in [26,27,30,47]. In general, the Eulerian coordinates is preferable from the point of view of both mathematics and physics as mentioned in [9, page 13]. We also remark that the equations in Eulerian coordinates are more suitable than those in Lagrangian coordinates when vacuum is involved.
1.6. Main Strategy of the Proof

In what follows we present a few key points on the proof of the main result Theorem 1.1. In fact, the strategy is based on the scaling transformation (2.1) of the independent variables \((t, x)\), inspired by [45], and further the combined application of the energy methods in [19,21] and [33,34]. We shall point out main difficulties to be overcome as well as main differences of the proof compared to the previous results [21,47].

First of all, since we consider the hydrodynamic limit to the rarefaction wave for the Landau equation, the solution is constructed by connecting two distinct global Maxwellians through the rarefaction wave solution of the Euler system, and hence the perturbation has to be made around the time-asymptotic local Maxwellian. Under this formulation, we turn to the macro-micro decomposition of the solution for the Landau equation with respect to the local Maxwellian that was initiated by Liu–Yu [33] and developed by Liu–Yang–Yu [34] in the Boltzmann theory. We thus can rewrite the Landau equation as the form of the compressible Navier–Stokes-type system so that the analysis in the context of the viscous conservation laws can be applied to capture the dissipation of the fluid part around wave patterns, in particular the dissipation term

\[
\int_0^T \int_{\mathbb{R}} \tilde{u}_{1y} |(\tilde{\rho}, \tilde{u}_1, \tilde{\theta})|^2 \, dy \, ds;
\]

see Lemma 3.1, where \(\tilde{u}_{1y} > 0\) holds, by Lemma 5.2.

Moreover, to make the energy estimates on the fluid-type system (1.16) around the rarefaction wave, one has to treat those integral terms involving the inverse of the linearized operator \(L_M\) for the local Maxwellian \(M\), such as

\[
\int_{\mathbb{R}^3} v_i v_j L_M^{-1} \Theta \, dv, \quad \int_{\mathbb{R}^3} v_i |v|^2 L_M^{-1} \Theta \, dv,
\]

where \(\Theta\) is defined in (1.15). Like the Boltzmann equation for the hard-sphere model considered in [35,47], using the coercivity dissipation estimate of \(-L_M\), one may similarly derive in the Landau case that

\[
\int_{\mathbb{R}^3} \frac{v |L_M^{-1} h|^2}{M} \, dv \leq C \int_{\mathbb{R}^3} \frac{v^{-1} |h|^2}{M} \, dv
\]

(1.32)

for \(h \in (\ker L_M)^\perp\), where \(v = v(v) = (1 + |v|)^\gamma + 2\). The above inequality is applicable to estimate (1.31) in the dissipative case \(\gamma + 2 \geq 0\) but fails in the very soft case \(\gamma + 2 < 0\), since for the latter case \(v^{-1}\) on the right-hand side of (1.32) gives rise to the extra velocity growth terms that can not be controlled when the energy dissipation is degenerate in large velocity. We remark that for the dissipative case \(\gamma + 2 \geq 0\) we may not need to include in the energy functional the polynomial velocity weight function depending also on the order of velocity derivatives, which would make it much easier to derive the energy estimates than in case of \(\gamma + 2 < 0\).

To overcome the difficulty above, inspired by [46] and [12], we make use of the Burnett functions \(\hat{A}_i\) and \(\hat{B}_{ij}\) as in (5.1), see Section 5 for the basic properties of
the Burnett functions. Indeed, in terms of the Burnett functions, the integral terms in (1.31) can be represented as the inner products of $A_i$ and $B_{ij}$ with $\Theta$, where $A_i$ and $B_{ij}$ defined in (5.2) are the inverse of $\hat{A}_i$ and $\hat{B}_{ij}$ under the linear operator $L_M$, respectively; see the identities (3.14), (3.15) and (3.16) for details. Notice that $A_i$ and $B_{ij}$ enjoy the fast velocity decay so as to bound any polynomial velocity growth in $\Theta$, see (3.17) and its applications in the proof of Lemma 3.1.

For the energy estimate on the microscopic component $G = F - M$, instead of using the perturbation approach in [33,34] around local Maxwellians, we would follow the strategy of the proof in [21] around global Maxwellians. In such way, the computations can be much simplified, in particular when treating iteratively the velocity derivative estimates with the negative-power polynomial velocity weight. Specifically, to transfer the microscopic equation to the one around the reference global Maxwellian $\mu$ we have chosen in (1.26), similar for showing the large-time asymptotic stability in [12], we need to subtract $G(t, x, v)$ from $G(t, x, v)$ to remove the slow time-decay inhomogeneous terms, because those terms induce the energy term $\| [\bar{u}_x, \bar{\theta}_x] \|_2$ in the lower order estimates that is not integrable in time. Notice that the term $G(t, x, v)$ can be represented precisely by using the Burnett functions so that its estimates can be made by using the properties of Burnett functions, cf. (5.22). Therefore, we set

$$f = \frac{G - \overline{G}}{\sqrt{\mu}} = \frac{F - M - \overline{G}}{\sqrt{\mu}} \quad (1.33)$$

and deduce that

$$f_t + v_1 f_y - \varepsilon^{a-1} \mathcal{L} f = \varepsilon^{a-1} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) + \varepsilon^{a-1} \left\{ \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right) \\
+ \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right) \right\} + \cdots \quad (1.34)$$

in $(\tau, y)$-coordinates under the scaling, where $\mathcal{L}$ and $\Gamma$ are the linearized and nonlinear Landau operator around global Maxwellians, respectively. Then many known estimates on $\mathcal{L}$ and $\Gamma$ in [21] can be directly employed; see, for instance Lemma 5.6 and Lemma 5.7, which further lead to Lemma 5.8 and Lemma 5.9, for treating the estimates on the source terms on the right-hand side of (1.34).

It should be pointed out that the unknown $f$ in (1.33) and (1.34) is purely microscopic. It is also the case even if the rarefaction wave is reduced to a constant state and thus one has $\overline{G} \equiv 0$ and $f = (F - M)/\sqrt{\mu}$, where $M$ is still the local Maxwellian determined by the Landau solution as in (1.7). Therefore, the reformulated equation (1.34) is essentially different from the one in [21] even in the perturbation framework around global Maxwellians. One advantage of using (1.34) is to obtain the trilinear estimate in the one-dimensional setting. In fact, induced by the first term on the right-hand side of (1.34), cf. (5.20), it holds that

$$\langle \Gamma(f, f), f \rangle \leq C \mathcal{E}^{\frac{3}{2}} \mathcal{D} \quad (1.35)$$

for the energy functional $\mathcal{E}$ and dissipation rate $\mathcal{D}$, since $f$ itself is microscopic and hence dissipative. However, whenever $f$ involves any macroscopic part, it seems
impossible to use the energy method in [19] to obtain the trilinear estimate of the above form (1.35) in case of the one-dimensional whole space $\mathbb{R}$, although it is true in the three-dimensional whole space $\mathbb{R}^3$. In fact, let $f = Pf + (I - P)f$, where $P$ is the projection on the kernel space of $L$ introduced in [21], then $(\Gamma(f, f), f)$ contains the lower order integral term $I = (\Gamma(Pf, Pf), (I - P)f)$. In case of three space dimensions, an estimate similar to (1.35) still holds true by using the $L^3-L^6-L^2$ Hölder inequality and the Sobolev inequality $\|f\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}$ as in [19], whereas for one space dimension, using the Sobolev inequality $\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}^{1/2}\|f'\|_{L^2(\mathbb{R})}^{1/2}$, one may only have

$$I \leq C\|Pf\|^{3/2}\|(Pf)_x\|^{1/2}\|(I - P)f\|_{\sigma},$$

which is out of control by $C\mathcal{E}^{1/2}\mathcal{D}$ as in (1.35), since $\|Pf\|^2$ is no longer in the dissipation rate. Hence, the global solution to the Landau equation near global Maxwellians in case of the one-dimensional whole space $\mathbb{R}$ can be constructed in terms of (1.34) and (1.35), that is different from the situation in [21] for three dimensions. In the current work, we construct the global solution around the one-dimensional non-constant rarefaction wave profile that seems more difficult to treat.

The new scaling transformation of the independent variables takes the form of $y = \varepsilon^{-a}x$, $\tau = \varepsilon^{-a}t$ involving with a free parameter $a \in [\frac{2}{3}, 1]$. The result in Theorem 1.1 shows that the solution converges to the local Maxwellian defined by the rarefaction wave of the Euler system at a rate $\varepsilon^{\frac{3}{2}-\frac{2}{3}a}\ln\varepsilon$. In particular, we can obtain the fastest convergence rate $\varepsilon^{\frac{1}{4}}\ln\varepsilon$ if one takes $a = \frac{2}{3}$. As mentioned before, the scaling argument through the change of variables $y = \varepsilon^{-\frac{3}{2}}x$ and $\tau = \varepsilon^{-\frac{3}{2}}t$ was first used in [45] to study the zero dissipation limit to the rarefaction wave for the one-dimensional compressible Navier–Stokes system, where the obtained convergence rate is $\varepsilon^{\frac{1}{2}}\ln\varepsilon$. Note that if one takes $a = \frac{3}{4}$ in terms of Theorem 1.1, then we obtain the convergence rate $\varepsilon^{\frac{3}{4}}\ln\varepsilon$ which is still a little sharper than the one in [45]. This is basically due to the detailed energy analysis such that both the lower and higher order estimates possess the same convergence rate of the Knudsen number $\varepsilon$. We also notice that under the scaling transformations $y = \varepsilon^{-1}x$, $\tau = \varepsilon^{-1}t$ in [47] and $y = \varepsilon^{-\frac{2}{3}}x$, $\tau = \varepsilon^{-\frac{2}{3}}t$ in [30] for the Boltzmann equation with hard sphere model, the convergence rates $\varepsilon^{\frac{1}{2}}\ln\varepsilon$ and $\varepsilon^{\frac{1}{4}}\ln\varepsilon$ are obtained respectively. Therefore, Theorem 1.1 also implies that the results of [30, 45, 47] can be improved to give rise to the faster rate $\varepsilon^{\frac{1}{4}}\ln\varepsilon$ by choosing $a = \frac{2}{3}$. At this moment, we would remark that we expect that such scaling argument also could be applied to study convergence to basic wave patterns (i.e., rarefaction waves, contact waves, and shock waves) for the Boltzmann equation, Navier–Stokes system, radiative hydrodynamic equations and many other related models. Once we use the scaling transformation (2.1) for an arbitrary parameter $a \in [\frac{2}{3}, 1]$, we have to deal with some difficulties caused by the higher order derivatives estimates such as (3.69). For this purpose, we design the energy functional $\mathcal{E}_2(\tau)$ and the corresponding dissipation functional $\mathcal{D}_2(\tau)$ involving the Knudsen number $\varepsilon$, see (2.15) and (2.16). The desired goal is to obtain the uniform a priori estimate (1.29) and then derive the convergence rate (1.30).
1.7. Organization of the Paper

The rest of this paper is arranged as follows: in Section 2, we will reformulate the system (1.1) and introduce a scaling for the new independent variable and the perturbation. In Section 3, we will establish the a priori estimates including low order energy estimates, high order energy estimates and the weighted energy estimates, see Lemmas 3.1, 3.2 and 3.3, respectively. In Section 4, we will establish the existence of global-in-time solutions as well as the convergence to the local Maxwellian defined by the rarefaction wave of the Euler system uniformly away from $t = 0$ as $\varepsilon \to 0$. In the Section (the “Appendix”), we will give some basic estimates frequently used in the previous sections.

2. Reformulation of the Problem

2.1. Scaling and Reformulation

In this section, we will reformulate the system and introduce a scaling for the independent variable and the perturbation. Firstly, we define the scaled independent variables

$$y = \frac{x}{\varepsilon^a}, \quad \tau = \frac{t}{\varepsilon^a}, \quad \text{for} \quad a \in \left[\frac{2}{3}, 1\right]. \quad (2.1)$$

Here the range of the parameter $a$ will be determined by the requirements (3.74) and (4.2) in the later proof under the conditions (3.3) and (3.4). This scaling transformation is crucial in the energy estimates. In fact, the hydrodynamic limit problem, in particular looking for convergence rates in $\varepsilon$, is transferred to showing the global-in-time dynamical stability of the rarefaction waves to the Landau equation under the scaling. From the proof, the obtained convergence rate in (1.30) seems optimal; see the detailed explanations of this point at the beginning of Section 3.

Correspondingly, we set the scaled perturbation $(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tau, y)$ and $\tilde{G} = \tilde{G}(\tau, y, v)$ as

$$\begin{cases}
    \tilde{\rho} = \rho(t, x) - \bar{\rho}(t, x), \\
    \tilde{u} = u(t, x) - \bar{u}(t, x), \\
    \tilde{\theta} = \theta(t, x) - \bar{\theta}(t, x), \\
    \tilde{G} = G(t, x, v) - \bar{G}(t, x, v), \quad \tilde{G} = \sqrt{\mu} f(\tau, y, v),
\end{cases} \quad (2.2)$$

Here the term $\bar{G} = \bar{G}(t, x, v) = \varepsilon L_M^{-1} P_1 v_1 M \left\{ \frac{|v-u|^2 \tilde{\theta}_y}{2 R \theta^2} + \frac{(v-u) \tilde{u}_y}{R \theta} \right\}$, that is

$$\bar{G} = \varepsilon^{1-a} L_M^{-1} P_1 v_1 M \left\{ \frac{|v-u|^2 \tilde{\theta}_y}{2 R \theta^2} + \frac{(v-u) \tilde{u}_y}{R \theta} \right\}, \quad (2.3)$$

which corresponds to the first-order correction term in the Chapman-Enskog expansion, see also the first term on the right-hand side of (1.15). The reason for introducing $\tilde{G}$ as the subtraction in (2.2) is that the term $P_1(v_1 M_x)$ in (1.14) contains $[\tilde{u}_y, \tilde{\theta}_y](\tau)$ under the scaling transform (2.1) and the time decay of $\| [\tilde{u}_y, \tilde{\theta}_y](\tau) \|^2$
is $ε^a(δ + ε^aτ)^{-1}$ by Lemma 5.3, which is not integrable in $τ$ over $(0, +∞)$. It should be pointed out that the term $G$ can be represented precisely by using the Burnett functions so that the estimates on those terms involving $G$ can be computed by using the properties of Burnett functions, see (5.22).

Subtracting (1.24) from system (1.13) and using the scaling (2.1), we can obtain

$$
\begin{align*}
\tilde{ρ}_τ + \tilde{ρ}u_{1y} + \hat{ρ}_y u_1 &= -J_1, \\
\tilde{u}_{1τ} + \tilde{u}_1 u_{1y} + \frac{2}{3} \tilde{ρ}_y &= -J_2 + \frac{1}{ρ} \int_{\mathbb{R}^3} v_1^2 G_y \, dv,
\end{align*}
\tag{2.4}
$$

where

$$
\begin{align*}
J_1 &= (\tilde{ρ}u_1)_y + \tilde{u}_1 \hat{ρ}_y + \tilde{u}_1 \hat{ρ}, \\
J_2 &= \tilde{u}_1 \tilde{u}_1 u_{1y} + \frac{2}{3} \tilde{ρ}_y \tilde{ρ} - \tilde{ρ} \tilde{ρ}, \\
J_3 &= \frac{2}{3} (\tilde{θ} \tilde{u}_1 + \tilde{θ} \tilde{u}_1) + (\tilde{θ}_y \tilde{u}_1 + \tilde{θ}_y \tilde{u}_1).
\end{align*}
\tag{2.5}
$$

Moreover, we also get, from (1.16), (1.24) and (2.1), that

$$
\begin{align*}
\tilde{ρ}_τ + \tilde{ρ}u_{1y} + \hat{ρ}_y u_1 &= -J_1, \\
\tilde{u}_{1τ} + \tilde{u}_1 u_{1y} + \frac{2}{3} \tilde{ρ}_y &= -J_2 + ε^{1-a} \frac{4}{3ρ} (μ(θ)u_{1y})_y - \frac{1}{ρ} (\int_{\mathbb{R}^3} v_1^2 L^{-1}_M(θ) \, dv)_y, \\
\tilde{θ}_y + \frac{2}{3} \tilde{θ}_y &= -J_3 + ε^{1-a} \frac{1}{ρ} (μ(θ))_{1y} + ε^{1-a} \frac{4}{3ρ} μ(θ) u_{1y}^2 \\
&+ ε^{1-a} \frac{1}{ρ} (μ(θ))/(2 u_{1y} + u_{3y})^2 - \frac{1}{ρ} (\int_{\mathbb{R}^3} v_1 |v|^2 L^{-1}_M(θ) \, dv)_y + \frac{1}{ρ} u \cdot (\int_{\mathbb{R}^3} v_1 v L^{-1}_M(θ) \, dv)_y.
\end{align*}
\tag{2.6}
$$

Here $J_1$, $J_2$ and $J_3$ are defined in (2.5) and $θ$ in (1.15) can be rewritten as

$$
θ = ε^{1-a} G_T + ε^{1-a} P_1(v_1 G_y) - Q(G, G).
$$

On the other hand, we need to derive the equation of the microscopic component $f$ as in (2.2). For this, we first denote

$$
Γ(h, g) := \frac{1}{\sqrt{μ}} Q(√μh, √μg), \quad ℋh := Γ(h, √μ) + Γ(√μ, h).
\tag{2.7}
$$

This, together with the definition of $L_M$ in (1.14), implies that

$$
\frac{1}{\sqrt{μ}} L_M(√μf) = \frac{1}{\sqrt{μ}} \{Q(M, √μf) + Q(√μf, M)\} = ℋf + Γ\left( f, \frac{M - μ}{√μ} \right) + Γ\left( \frac{M - μ}{√μ}, f \right).
\tag{2.8}
$$

From (1.14) and (2.1), one can easily see that

$$
G_T + P_1(v_1 G_y) + P_1(v_1 M_y) = ε^{a-1} L_M G + ε^{a-1} Q(G, G).
\tag{2.9}
$$
By using (2.8), $P_1 = I - P_0$ and $G = \sqrt{\mu} f$, we can rewrite equation (2.9) as

$$f_\tau + v_1 f_y - \varepsilon^{a-1} L f = \varepsilon^{a-1} \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right) + \varepsilon^{a-1} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) + \frac{P_0 (v_1 \sqrt{\mu} f_y)}{\sqrt{\mu}} - \frac{1}{\sqrt{\mu}} P_1 v_1 M \left\{ \frac{|v - u|^2}{2R\theta^2} + \frac{(v - u) \cdot \tilde{u}_y}{R\theta} \right\}$$

$$= \frac{P_1 (v_1 \sqrt{\mu} G_y)}{\sqrt{\mu}} - \frac{G}{\sqrt{\mu}} \tau - \frac{P_1 G}{\sqrt{\mu}}.$$

(2.10)

Here we have used the fact that

$$P_1 (v_1 M_y) = P_1 v_1 M \left\{ \frac{|v - u|^2}{2R\theta^2} + \frac{(v - u) \cdot \tilde{u}_y}{R\theta} \right\} + \varepsilon^{a-1} L_M \overline{G}.$$

Finally, we obtain by (1.1) and the scaling (2.1), that

$$F_\tau + v_1 F_y = \varepsilon^{a-1} Q(F, F).$$

(2.11)

2.2. Notations and Norms

The following notations are needed in the energy analysis for convenience of presentation: we shall use $\langle \cdot, \cdot \rangle$ to denote the standard $L^2$ inner product in $\mathbb{R}^3$ with its corresponding $L^2$ norm $|\cdot|_2$. We also use $(\cdot, \cdot)$ to denote $L^2$ inner product in $\mathbb{R}^y$ or $\mathbb{R}^y \times \mathbb{R}^3$ with its corresponding $L^2$ norm $||\cdot||$. Let $\alpha$ and $\beta$ be multi indices $\alpha = [\alpha_1, \alpha_2]$ and $\beta = [\beta_1, \beta_2, \beta_3]$, respectively. Denote a high order derivative

$$\partial_\beta^\alpha = \partial_\tau^{\alpha_1} \partial_y^{\alpha_2} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$

If each component of $\beta$ is not greater than the corresponding one of $\overline{\beta}$, we use the standard notation $\beta \leq \overline{\beta}$. And $\beta < \overline{\beta}$ means that $\beta \leq \overline{\beta}$ and $|\beta| < |\overline{\beta}|$. $C_\overline{\beta}$ is the usual binomial coefficient. Throughout the paper, generic positive constants are denoted by $C$ (generally large) and $c$ (generally small) which are independent of time $\tau$ and $\varepsilon$ unless otherwise stated. The notation $A \approx B$ is used to denote that there exists $c_0 > 1$ such that $c_0^{-1} B \leq A \leq c_0 B$. Motivated by [21], we introduce the following velocity weight function:

$$w = w(v) \equiv \langle v \rangle^{\gamma+2}, \quad \langle v \rangle = \sqrt{1 + |v|^2}.$$

(2.12)

Denote weighted $L^2$ norms as

$$|g|_{2, \ell}^2 \equiv \int_{\mathbb{R}^3} w^{2\ell} |g|^2 \, dv, \quad \|g\|_{2, \ell}^2 \equiv \int_{\mathbb{R}} |g|_{2, \ell}^2 \, dy.$$

The Landau collision frequency is

$$\sigma^{ij}(v) = \phi^{ij} * \mu = \int_{\mathbb{R}^3} \phi^{ij}(v - v_*) \mu(v_*) \, dv_*, \quad 1 \leq i, j \leq 3.$$

(2.13)
where $\phi^{ij}$ is given in (1.3). We remark that $[\sigma^{ij}(v)]_{1 \leq i, j \leq 3}$ is a positive-definite self-adjoint matrix. With (2.13), we define the weighted dissipation norms as

$$
|g|_{\sigma, \ell}^2 \equiv \sum_{i, j=1}^{3} \int_{\mathbb{R}^3} u^{2\ell} \left\{ \sigma^{ij} \partial_i g \partial_j g + \sigma^{ij} \frac{v_i v_j}{2} |g|^2 \right\} \, dv, \quad \|g\|_{\sigma, \ell}^2 \equiv \int_{\mathbb{R}} |g|_{\sigma, \ell}^2 \, dy,
$$

and let $|g|_{\sigma} = |g|_{\sigma, 0}$ and $\|g\|_{\sigma} = \|g\|_{\sigma, 0}$. From [40, Lemma 5, p.315], one has

$$
|g|_{\sigma} \approx \left| \langle v \rangle \frac{v^2}{\nu} g \right|_2 + \left| \langle v \rangle \frac{v}{|v|} \nabla v g \cdot \frac{v}{|v|} \right|_2 + \left| \langle v \rangle \frac{v^2}{\nu} \nabla v g \times \frac{v}{|v|} \right|_2. \quad (3.1)
$$

Now, we define the instant energy functional $E_2(\tau)$ by

$$
E_2(\tau) = \sum_{|\alpha| \leq 1} \| \partial^\alpha (\rho, \vec{u}, \vec{\theta})(\tau) \|^2 + \varepsilon^{2-2a} \sum_{|\alpha| = 2} \left\{ \| \partial^\alpha (\rho, \vec{u}, \vec{\theta})(\tau) \|^2 + \| \partial^\alpha f(\tau) \|^2 \right\} + \sum_{|\alpha| \leq 1} \| \partial^\alpha f(\tau) \|^2 + \sum_{|\alpha| + |\beta| \leq 2, |\beta| \geq 1} \| \partial^\alpha \rho f(\tau) \|^2_{2,|\beta|}. \quad (3.2)
$$

As usual, the instant energy functional $E_2(\tau)$ is assumed to be small enough a priori. And this will be closed by the energy estimates in the end. The corresponding dissipation rate $D_2(\tau)$ is given by

$$
D_2(\tau) = \varepsilon^{1-a} \sum_{1 \leq |\alpha| \leq 2} \| \partial^\alpha (\rho, \vec{u}, \vec{\theta})(\tau) \|^2 + \varepsilon^{1-a} \sum_{|\alpha| = 2} \| \partial^\alpha f(\tau) \|^2_{\sigma} + \varepsilon^{a-1} \sum_{|\alpha| \leq 1} \| \partial^\alpha f(\tau) \|^2_{\sigma} + \varepsilon^{a-1} \sum_{|\alpha| + |\beta| \leq 2, |\beta| \geq 1} \| \partial^\alpha \rho f(\tau) \|^2_{\sigma,|\beta|}. \quad (3.3)
$$

### 3. A Priori Estimates

This section is devoted to deducing the a priori estimates for the equation (2.11) around the smooth rarefaction wave. To this end, we first choose the initial datum of the equation (2.11) as

$$
F(0, y, v) \equiv F(0, x, v) = F_0(e^a y, v) \to M_{(\rho_\pm, u_\pm, \theta_\pm)}(v), \quad \text{as} \quad y \to \pm \infty, \quad (3.1)
$$

in terms of (1.28), such that it holds true that

$$
E_2(\tau) \big|_{\tau=0} \leq k^1 \varepsilon^{2} \quad (3.2)
$$

for a small constant $k > 0$ independent of $\varepsilon$. Here $E_2(\tau)$ is defined by (2.15) and $(\rho_\pm, u_\pm, \theta_\pm)$ is given by (1.17).

Since the local existence of the solutions to the Landau equation and Boltzmann equation near a global Maxwellian is well known in the torus or the whole space, cf. [19,21]. By the estimates of $(\vec{\rho}, \vec{u}, \vec{\theta})$ and a straightforward modification of the arguments there, the local existence of the solutions to the Landau equation (2.11)
and (3.1) can be obtained under the assumptions in Theorem 1.1. To obtain the global-in-time existence of solution, we mainly focus on deducing the uniform a priori estimates of solutions.

Throughout this section, we need to find out small positive constants $k$, $\delta_0$ and $\varepsilon_0$ with $0 < \varepsilon_0 \ll k \ll 1$ and $0 < \delta_0 \ll 1$, independent of $\varepsilon$, $\delta$ and $\tau$, then we choose

$$\delta = \frac{1}{k} \varepsilon^\frac{3}{5} - \frac{2}{5} a,$$

where $\delta$ is given in (1.22), and we further let $\varepsilon$ be arbitrarily chosen such that $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \delta_0$. Suppose that the Cauchy problem (2.11) and (3.1) has a non-negative solution $F(\tau, y, v) \in \hat{X}(0, \tau_1)$ for an arbitrary $\tau_1 \in (0, +\infty)$, where the space $\hat{X}(0, \tau_1)$ corresponds to (1.27) under the scaling transform (2.1). We make the a priori assumption

$$\sup_{0 \leq \tau \leq \tau_1} E_2(\tau) \leq \frac{1}{k^6} \varepsilon^{\frac{6}{5} - \frac{4}{5}} a,$$

where $E_2(\tau)$ is defined by (2.15). In what follows we formally explain why one has to choose such $\delta$ as (3.3) and the a priori assumption as (3.4). Indeed, if one assumes that $\sup_{0 \leq \tau \leq \tau_1} E_2(\tau) \leq O(1) \varepsilon^q$, with a constant $q > 0$, then it follows from this assumption and Lemma 5.2 that

$$\| (\rho, u, \theta)(t, x) - (\rho^R, u^R, \theta^R) \left( \frac{X}{t} \right) \|_{L^\infty} \leq \| (\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tau, y) \|_{L^\infty} + \| (\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x) - (\rho^R, u^R, \theta^R) \left( \frac{X}{t} \right) \|_{L^\infty} \leq C \varepsilon^\frac{q}{2} + C t^{-1} \delta (\ln(1 + t) + |\ln \delta|)$$

for any $t > 0$. Hence the above estimate in vanishing Knudsen number $\varepsilon > 0$ is optimal by taking $\delta = O(1) \varepsilon^\frac{q}{2}$. On the other hand, we have to deal with the slow time decay of the term in (3.13) in the way that

$$\varepsilon^{1-a} \int_0^{\infty} \| \tilde{\theta}^{\frac{3}{2}} \|_{L^1} \frac{4}{5} d\tau \leq C \varepsilon^{1-a} \int_0^{\infty} \varepsilon^\frac{q}{2} \varepsilon^{\frac{3}{5} - \frac{4}{5} a} (\delta + \varepsilon^a \tau)^{-\frac{2}{5}} d\tau \leq C \varepsilon^{1-a+\frac{4}{5}} a + \frac{4}{5} a = O(1) \varepsilon^{1-\frac{2}{5} a + \frac{4}{5} a},$$

where we have replaced $\delta = O(1) \varepsilon^\frac{q}{2}$ in the last identity. To close the a priori assumption, we need to require that

$$\varepsilon^{1-\frac{2}{5} a + \frac{4}{5} a} \leq \varepsilon^q,$$

that is $q \leq \frac{6}{5} - \frac{4}{5} a$. Notice that the convergence rate is the fastest by choosing $q = \frac{6}{5} - \frac{4}{5} a$. Hence, we can obtain the sharp convergence rate under the condition of (3.3) and (3.4).
By the a priori assumption (3.4) and the one-dimensional Sobolev imbedding inequality
\[ \|g\|_{L^\infty} \leq \sqrt{2}\|g\|_\frac{1}{2}\|g'\|_\frac{1}{2}, \] for any \( g = g(y) \in H^1(\mathbb{R}) \),
one can see that
\[ \sup_{\tau \geq 0, y \in \mathbb{R}} |(\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tau, y)| \leq 2k^{\frac{1}{12}}\varepsilon^{\frac{5}{3}} - \frac{5}{2}a. \]
Using this, (2.2) and (1.25) implies that
\[ |\rho(t, x) - 1| \leq |\rho(t, x) - \tilde{\rho}(t, x)| + |\tilde{\rho}(t, x) - 1| \leq 2k^{\frac{1}{12}}\varepsilon^{\frac{5}{3}} - \frac{5}{2}a + \eta_0. \]
Similar estimates also hold for \( u(t, x) \) and \( \theta(t, x) \). Therefore, one has
\[ \left\{ \begin{array}{ll}
\sup_{t \geq 0, x \in \mathbb{R}} \{ |\rho(t, x) - 1| + |u(t, x)| + |\theta(t, x) - \frac{3}{2} |\} \leq 6(k^{\frac{1}{12}}\varepsilon^{\frac{5}{3}} - \frac{5}{2}a + \eta_0),
\frac{1}{2} \sup_{t \geq 0, x \in \mathbb{R}} \theta(t, x) < \frac{3}{2} < \inf_{t \geq 0, x \in \mathbb{R}} \theta(t, x)
\end{array} \right. \] (3.5)
for some suitably small constants \( \varepsilon, k \) and \( \eta_0 \). We point out that (3.5) will be frequently used in the later energy estimates.

From now on, we will focus on the reformulated system (2.4), (2.6), (2.10) and (2.11) with initial data (3.1), and assume that the solution exists up to an arbitrary time \( \tau_1 \in (0, +\infty) \). We will first derive the lower order energy estimates for the macroscopic component \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) and the microscopic component \( f \) in Section 3.1. Then, Section 3.2 is devoted to obtaining the high order energy estimates of \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) and \( f \). The weighted energy estimates of \( f \) will be given in Section 3.3. We would emphasize that in all estimates below, the generic constant \( C \) at different places is independent of \( \tau_1 \) as well as all small parameters \( \varepsilon, k \) and \( \delta_0 \).

\section{Lower Order Energy Estimates}

In this subsection, we will consider the lower order energy estimates.

\textbf{Lemma 3.1.} Let (3.3), (3.4) and (3.5) hold. Then one has
\[ \|\tilde{\rho}, \tilde{u}, \tilde{\theta}\|^{2} + \|f\|^{2} + \int_{0}^{\tau} \|\tilde{\rho}, \tilde{u}, \tilde{\theta}\|^{2} \, ds \]
\[ + \varepsilon^{1-a} \sum_{|\alpha|=1} \int_{0}^{\tau} \|\partial^{\alpha}(\tilde{\rho}, \tilde{u}, \tilde{\theta})(s)\|^{2} \, ds + \varepsilon^{a-1} \int_{0}^{\tau} \|f(s)\|_{\sigma}^{2} \, ds \leq C_{\varepsilon}^{2(1-a)} \|\tilde{\rho}, \tilde{\rho}\|^{2} \]
\[ + Ck^{\frac{1}{12}}\varepsilon^{\frac{5}{3}} - \frac{5}{2}a + C\varepsilon^{1-a} \sum_{|\alpha|=1} \int_{0}^{\tau} \|\partial^{\alpha}f(s)\|_{\sigma}^{2} \, ds + Ck^{\frac{1}{12}}\varepsilon^{\frac{5}{3}} - \frac{5}{2}a \int_{0}^{\tau} D_{2}(s) \, ds \] (3.6)
for any \( \tau \in [0, \tau_1] \).
Proof. We start from the lower order estimates for \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) by the entropy and entropy flux. As in [34,35], the following macroscopic entropy \(S\) will be estimated for the lower order energy estimates: set
\[
-\frac{3}{2} \rho S = \int_{\mathbb{R}^3} M \ln M \, dv.
\]

Multiplying (2.11) by \(\ln M\) and integrating over \(v\), we have from this, (1.7), (1.10) and a direct calculation that
\[
\left( -\frac{3}{2} \rho S \right)_\tau + \left( -\frac{3}{2} \rho u_1 S \right)_y + \left( \int_{\mathbb{R}^3} v_1 G \ln M \, dv \right)_y - \int_{\mathbb{R}^3} v_1 G(\ln M)_y \, dv = 0.
\]

By a direct calculation, we also have that
\[
S = -\frac{2}{3} \ln \rho + \ln \left( \frac{4\pi}{3} \theta \right) + 1, \quad p = \frac{2}{3} \rho \theta = \frac{1}{2\pi e} \rho^\frac{5}{3} \exp(S). \tag{3.7}
\]

In terms of the scaling transformation (2.1), we can rewrite the conservation laws (1.16) as \(X_\tau + Y_y =\)
\[
\begin{pmatrix}
\begin{array}{c}
0
\end{array}
\end{pmatrix}
\begin{pmatrix}
\frac{4}{3} e^{1-\alpha} (\mu(\theta) u_{1y})_y - (\int v_1^2 L^{-1}_M \Theta \, dv)_y
\end{pmatrix}
\begin{pmatrix}
\varepsilon^{1-\alpha} (\mu(\theta) u_{2y})_y - (\int v_1 v_2 L^{-1}_M \Theta \, dv)_y
\end{pmatrix}
\begin{pmatrix}
\varepsilon^{1-\alpha} (\mu(\theta) u_{3y})_y - (\int v_1 v_3 L^{-1}_M \Theta \, dv)_y
\end{pmatrix}
\begin{pmatrix}
\kappa(\theta) \theta_y + \frac{3}{2} (\mu(\theta) u_1 u_{1y})_y + \sum_{i=2}^{3} (\mu(\theta) u_i u_{iy})_y - \frac{1}{2} (\int v_1 |v|^2 L^{-1}_M \Theta \, dv)_y
\end{pmatrix}
\end{pmatrix}
\]

Here
\[
X = (X_0, X_1, X_2, X_3, X_4)_\tau = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho(\theta + \frac{|u|^2}{2}))_\tau,
\]
\[
Y = (Y_0, Y_1, Y_2, Y_3, Y_4)_\tau = (\rho u_1, \rho u_1^2 + p, \rho u_1 u_2, \rho u_1 u_3, \rho u_1(\theta + \frac{|u|^2}{2}) + pu_1)_\tau,
\]

where \((\cdot, \cdot, \cdot)_\tau\) is the transpose of the vector \((\cdot, \cdot, \cdot)\). We define an entropy-entropy flux pair \(\eta, q)(\tau, y)\) around a Maxwellian \(\bar{M} = M[\bar{\rho}, \bar{u}, \bar{\theta}]\) \((\bar{u}_2 = \bar{u}_3 = 0)\) as
\[
\begin{pmatrix}
\eta(\tau, y) = \bar{\theta} \left\{ -\frac{3}{2} \rho \theta + \frac{3}{2} \bar{\rho} \bar{\theta} + \frac{3}{2} \nabla_X (\rho S)|_{X = \bar{X}} \cdot (X - \bar{X}) \right\},
q(\tau, y) = \bar{\theta} \left\{ -\frac{3}{2} \rho u_1 \bar{\theta} + \frac{3}{2} \bar{\rho} u_1 \bar{\theta} + \frac{3}{2} \nabla_X (\rho S)|_{X = \bar{X}} \cdot (Y - \bar{Y}) \right\}
\end{pmatrix}. \tag{3.8}
\]

Thanks to
\[
\theta = \frac{X_4}{X_0} - \frac{X_1^2 + X_2^2 + X_3^2}{2X_0^2}, \quad |u|^2 = \frac{X_1^2 + X_2^2 + X_3^2}{X_0^2},
\]

By this and (3.7), we have from a direct computation that
\[
(\rho S)_0 = S + \frac{|u|^2}{2\theta} - \frac{5}{3}, \quad (\rho S)_i = -\frac{u_i}{\theta}, \quad i = 1, 2, 3, \quad (\rho S)_4 = \frac{1}{\theta}.
\]
Hence, by using this and (3.8), we further obtain
\[
\begin{aligned}
\eta(\tau, y) &= \frac{3}{2}(\rho \theta - \tilde{\theta}) + \rho[(\tilde{S} - \frac{3}{2})\tilde{\theta} + \frac{|u - \tilde{u}|^2}{2}] + \frac{2}{3} \tilde{\rho} \tilde{\theta} \\
&= \tilde{\rho} \tilde{\Phi}'(\tilde{\rho}) + \frac{3}{2} \rho \tilde{\Phi}(\rho) + \frac{3}{4} \rho |u - \tilde{u}|^2, \\
q(\tau, y) &= u_1 \eta(\tau, y) + (u_1 - \tilde{u}_1)(\rho \theta - \tilde{\rho})
\end{aligned}
\]
where the convex function \( \Phi(s) \) is defined as \( \Phi(s) = s - \ln s - 1 \). By this, we know that there exists a constant \( c_1 > 1 \) such that
\[
c_1^{-1} \| (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 \leq \eta(\tau, y) \leq c_1 \| (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2.
\] (3.9)

In view of the definition of (3.8), we have by a direct computation that
\[
\eta_x(\tau, y) + q_y(\tau, y) - \nabla_{[\tilde{\rho}, \tilde{u}, \tilde{S}]} \eta(\tau, y) \cdot (\tilde{\rho}, \tilde{u}, \tilde{S})_x - \nabla_{[\tilde{\rho}, \tilde{u}, \tilde{S}]} q(\tau, y) \cdot (\tilde{\rho}, \tilde{u}, \tilde{S})_y
\]
\[
= \tilde{\theta} \{-\frac{3}{2} \rho \tilde{S}_x + (-\frac{3}{2} \rho u_1 S)_y\} + \frac{3}{2} \tilde{\theta} [\nabla S(\rho S)|_{X = \tilde{X}}(X_\tau + Y_y)\}.
\]
Owing to these facts, a direct but tedious computation shows that
\[
\begin{aligned}
\eta_x(\tau, y) + q_y(\tau, y) &= \varepsilon^{1-a} 2 \tilde{\theta} \mu(\theta) \tilde{u}_1^2 \\
&+ \varepsilon^{1-a} \frac{3 \tilde{\theta}}{2 \theta} \sum_{i=2}^3 \mu(\theta) \tilde{u}_i^2 + \varepsilon^{1-a} \frac{3 \tilde{\theta}}{2 \theta^2} \kappa(\theta) \tilde{\theta}_y^2 \\
&- \{\nabla_{[\tilde{\rho}, \tilde{u}, \tilde{S}]} \eta(\tau, y) \cdot (\tilde{\rho}, \tilde{u}, \tilde{S})_x + \nabla_{[\tilde{\rho}, \tilde{u}, \tilde{S}]} q(\tau, y) \cdot (\tilde{\rho}, \tilde{u}, \tilde{S})_y\}
\end{aligned}
\]
\[
q(\tau, y) \cdot (\tilde{\rho}, \tilde{u}, \tilde{S})_x - (\cdots)_y = \sum_{i=1}^4 H_i,
\] (3.10)
where we have denoted
\[
\sum_{i=1}^4 H_i = \varepsilon^{1-a} \left\{ \frac{3 \kappa(\theta)}{2 \theta^2} (\tilde{\theta}_y + \tilde{\theta}_y) \tilde{\theta}_y \tilde{\theta} - \frac{3 \tilde{\theta}}{2 \theta^2} \kappa(\theta) \tilde{\theta}_y \tilde{\theta}_y \right\}
\]
\[
+ \varepsilon^{1-a} \left\{ \frac{2 \mu(\theta)}{\theta} (\tilde{u}_1 y + \tilde{u}_1 y) \tilde{u}_1 y \tilde{\theta} - \frac{2 \tilde{\theta}}{\theta} \mu(\theta) \tilde{u}_1 y \tilde{u}_1 y \right\}
\]
\[
+ \left\{ \frac{3}{2} \tilde{\theta} (\tilde{u}_y) \int_{\mathbb{R}^3} \left( \frac{1}{2} v_1 |v|^2 - u \cdot v v_1 \right) L_M^{-1} \Theta \, dv \right\}
\]
\[
+ \left\{ \frac{3}{2} \tilde{u}_y - \frac{3}{2 \theta} u_v \right\} \cdot \int_{\mathbb{R}^3} v v_1 L_M^{-1} \Theta \, dv \right\}.
\]
Here the notation \((\cdots)_y\) represents the term in the conservative form so that it vanishes after integration. In the following energy analysis, we will frequently use the facts that \((\rho, u, \theta)\) and \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) are close enough to the state \((1, 0, \frac{1}{2})\) by (3.5) and (1.25).
By the similar arguments as [29,35], there exists $c_2 > 0$ such that

$$
- \{ \nabla_{[\hat{\rho},\hat{u},\tilde{S}]} \eta(\tau, y) \cdot (\hat{\rho}, \hat{u}, \tilde{S})_\tau + \nabla_{[\hat{\rho},\hat{u},\tilde{S}]} q(\tau, y) \cdot (\hat{\rho}, \hat{u}, \tilde{S})_y \}
= \frac{3}{2} \rho \tilde{u}_1 y (u_1 - \tilde{u}_1)^2 + \frac{2}{3} \rho \tilde{u}_1 y \Phi \left( \frac{\theta}{\rho} \right)
+ \rho \tilde{u}_1 y \Phi \left( \frac{\theta}{\theta} \right) + \frac{3}{2} \rho \tilde{u}_1 y (u_1 - \tilde{u}_1) \left( \frac{2}{3} \ln \frac{\tilde{\rho}}{\rho} + \ln \frac{\theta}{\theta} \right)
\geq c_2 \rho \tilde{u}_1 y (\rho^2 + \tilde{u}_1^2 + \tilde{\theta}^2).
$$

(3.11)

Since both $\mu(\theta)$ and $\kappa(\theta)$ are smooth functions of $\theta$, there exists a constant $c_3 > 1$ such that $\mu(\theta), \kappa(\theta) \in [c_3^{-1}, c_3]$. Plugging (3.11) into (3.10) and integrating the resulting equation with respect to $y$, we obtain

$$
\frac{d}{d\tau} \int_{\mathbb{R}} \eta(\tau, y) \, dy + c_\epsilon 1 - a \| [\tilde{u}_y, \tilde{\theta}_y] \|^2
+ c_2 \| \sqrt{\tilde{u}_1 y} (\tilde{\rho}, \tilde{u}_1, \tilde{\theta}) \|^2 \leq \sum_{i=1}^{4} \int_{\mathbb{R}} H_i \, dy.
$$

(3.12)

We are now ready to estimate the terms of (3.12) involving $H_i$. By the integration by parts and the Cauchy–Schwarz inequality, one gets that

$$
\int_{\mathbb{R}} H_1 \, dy = \epsilon^{1-a} \int_{\mathbb{R}} \left\{ \frac{3 \kappa(\theta)}{2 \theta^2} (\tilde{\theta}_y + \tilde{\theta}_y) \tilde{\theta}_y \tilde{\theta} - \frac{3 \tilde{\theta}}{2 \theta^2} \kappa(\theta) \tilde{\theta}_y \tilde{\theta}_y \right\} \, dy
\leq C \epsilon^{1-a} \int_{\mathbb{R}} \left| \tilde{\theta} \| [\tilde{\theta}_y]_{\tilde{\theta}_y} + |\tilde{\theta}_y|^2 + |\tilde{\theta}_y| |\tilde{\theta}_y| + |\tilde{\theta}_y| \right| \, dy
\leq C \epsilon^{1-a} \| \tilde{\theta} \|_{L^\infty} \{ \| \tilde{\theta}_y \|_{L^1} + \| \tilde{\theta}_y \|^2 + \| \tilde{\theta}_y \|^2 \},
$$

which further implies that

$$
\int_{\mathbb{R}} H_1 \, dy \leq \epsilon^{1-a} \left\{ \eta \| \tilde{\theta}_y \|^2 + C_\eta \| \tilde{\theta}_y \|^2 \| \tilde{\theta}_y \| \frac{4}{L^1}
+ C_\eta \| \tilde{\theta}_y \|^2 \| \tilde{\theta}_y \| \frac{8}{8} + C \| \tilde{\theta}_y \| \frac{1}{2} \| \tilde{\theta}_y \| \frac{1}{2} \| \tilde{\theta}_y \| \frac{2}{2} \right\}
\leq \eta \epsilon^{1-a} \| \tilde{\theta}_y \|^2 + C_\eta \epsilon^{1-a} (k \frac{1}{\epsilon} \frac{5}{4} a + \frac{1}{\epsilon} \frac{4}{a})
(\delta + \epsilon^a \tau)^{-\frac{4}{4}} + C \sqrt{E_2(\tau)} D_2(\tau)
\leq \eta \epsilon^{1-a} \| \tilde{\theta}_y \|^2 + C_\eta \epsilon^\frac{3}{4} + \frac{\epsilon^a}{a} (\delta + \epsilon^a \tau)^{-\frac{4}{4}}
+ C \epsilon^{\frac{3}{4} - \frac{3}{2} a} D_2(\tau),
$$

(3.13)

by the Sobolev imbedding inequality, Lemma 5.3, (3.4), (2.16) and the smallness of $k$. 


Following the same method used as (3.13), it holds that

\[
\int_{\mathbb{R}} H_2 \, dy = \varepsilon^{1-a} \int_{\mathbb{R}} \left\{ \frac{2\mu(\theta)}{\theta} (\bar{u}_{1y} + \bar{v}_{1y}) \bar{u}_{1y} - \frac{2\bar{\theta}}{\theta} \mu(\theta) \bar{u}_{1y} \bar{u}_{1y} \right\} dy \\
\leq \eta \varepsilon^{1-a} \left\Vert [\bar{u}_{1y}, \bar{\theta}_y] \right\Vert^2 + C_n \varepsilon^{\frac{7}{2} + \frac{1}{2} \alpha}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_k \varepsilon^{\frac{3}{2} - \frac{3}{2} \alpha} D_2(\tau).
\]

By using the self-adjoint property of \( L_M^{-1}, (1.9), (5.1) \) and (5.2), one can show that

\[
\int_{\mathbb{R}^3} \left( \frac{1}{2} v_1 |v|^2 - v_1 u \cdot v \right) L_M^{-1} \Theta \, dv = \int_{\mathbb{R}^3} L_M^{-1} \left\{ P_1 \left( \frac{1}{2} v_1 |v|^2 - v_1 u \cdot v \right) \right\} \Theta \, dv \\
= \int_{\mathbb{R}^3} L_M^{-1} \left\{ (R\theta)^{\frac{3}{2}} A_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) M \right\} \Theta \, dv = (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \Theta \, dv. \tag{3.14}
\]

and

\[
\int_{\mathbb{R}^3} v_1 v_i L_M^{-1} \Theta \, dv = \int_{\mathbb{R}^3} L_M^{-1} \{ P_1 (v_1 v_i M) \} \Theta \, dv \\
= \int_{\mathbb{R}^3} L_M^{-1} \left\{ R\theta \hat{B}_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) M \right\} \Theta \, dv = R\theta \int_{\mathbb{R}^3} B_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \Theta \, dv. \tag{3.15}
\]

Both (3.14) and the expression of \( H_3 \) in (3.10) imply

\[
\int_{\mathbb{R}} H_3 \, dy = \int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\bar{\theta}}{\bar{\theta}} \right) y \int_{\mathbb{R}^3} \left( \frac{1}{2} v_1 |v|^2 - u \cdot v v_1 \right) L_M^{-1} \Theta \, dv \right\} dy \\
= \int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\bar{\theta}}{\bar{\theta}} \right) (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \Theta \, dv \right\} dy. \tag{3.16}
\]

In order to compute (3.16), for any multi-index \( \beta \) and \( m \geq 0 \), we have

\[
\int_{\mathbb{R}^3} \left| (v)^m \sqrt{\mu} \partial_\beta A_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \right|^2 \, dv + \int_{\mathbb{R}^3} \left| (v)^m \sqrt{\mu} \partial_\beta B_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \right|^2 \, dv \leq C, \tag{3.17}
\]

by using the fast decay of the Burnett functions (5.4) and (3.5). Recalling that

\[
\Theta = \varepsilon^{1-a} G_\tau + \varepsilon^{1-a} P_1 (v_1 G_y) - Q(G, G). \tag{3.18}
\]

For the first term on the right-hand side of (3.18). Recalling that \( G = \bar{G} + \sqrt{\mu} f \), applying (3.17), (5.25), the Cauchy–Schwarz inequality and Lemma 5.3 and using \( \mathcal{E}_2(\tau) \leq k \varepsilon^{\frac{3}{2} - \frac{3}{2} \alpha} \) and \( \varepsilon^a \delta^{-1} \leq k \varepsilon^{\frac{3}{2} - \frac{3}{2} \alpha} \) by (3.4) and (15.1), one has

\[
\int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\bar{\theta}}{\bar{\theta}} \right) (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \Theta \, dv \right\} dy \leq C \varepsilon^{1-a} (||\bar{\theta}_y|| + ||\bar{\theta}_0||) \times \left\| \frac{G_\tau}{\sqrt{\mu}} \right\| \\
\leq C \varepsilon^{2(1-a)} (||\bar{\theta}_y|| + ||\theta_0||) \times (||[\bar{u}_{1y}, \bar{\theta}_y]|| + ||[\bar{u}_{1y}, \bar{\theta}_y] \cdot [u_\tau, \theta_\tau]||) \\
\leq C \eta \varepsilon^{1-a} ||\bar{\theta}_y||^2 + C_n \varepsilon^{\frac{7}{2} + \frac{1}{2} \alpha}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_n \varepsilon^a \delta^{-1} + \mathcal{E}_2(\tau) D_2(\tau) \\
\leq C \eta \varepsilon^{1-a} ||\bar{\theta}_y||^2 + C_n \varepsilon^{\frac{7}{2} + \frac{1}{2} \alpha}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_n k \varepsilon^{\frac{3}{2} - \frac{3}{2} \alpha} D_2(\tau). \tag{3.19}
\]
Similarly, it holds that

\[
\int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\tilde{\theta}}{\theta} \right)_y (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v-u}{\sqrt{R\theta}} \right) \frac{e^{1-a} \sqrt{\mu} f_\tau}{M} \, dv \right\} \, dy \\
\leq C \eta e^{(1-a)}(\|\tilde{\theta}_y\| + \|\tilde{\theta}_\theta\|) \times \|v\|^{-\frac{1}{2}} f_\tau \\
\leq C \eta e^{1-a} \|\tilde{\theta}_y\|^2 + C \eta e^{\frac{3}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C k \frac{1}{12} e^{\frac{3}{2} - \frac{7}{2} a} D_2(\tau).
\]  
(3.20)

It follows from (3.19) and (3.20) that

\[
\int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\tilde{\theta}}{\theta} \right)_y (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v-u}{\sqrt{R\theta}} \right) \frac{e^{1-a} G}{M} \, dv \right\} \, dy \\
\leq C \eta e^{1-a} \|\tilde{\theta}_y\|^2 + C \eta e^{1-a} f_\tau \|v\|_\sigma^2 + C \eta e^{\frac{3}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{12} e^{\frac{3}{2} - \frac{7}{2} a} D_2(\tau).
\]  
(3.21)

For the second term on the right-hand side of (3.18). Similar arguments as (3.21) imply

\[
\int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\tilde{\theta}}{\theta} \right)_y (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v-u}{\sqrt{R\theta}} \right) \frac{e^{1-a} P_1(v_1 G_y)}{M} \, dv \right\} \, dy \\
\leq C \eta e^{1-a} \|\tilde{\theta}_y\|^2 + C \eta e^{1-a} f_\tau \|v\|_\sigma^2 + C \eta e^{\frac{3}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{12} e^{\frac{3}{2} - \frac{7}{2} a} D_2(\tau).
\]

For the last term of (3.18), by using (2.7), (3.17) and the similar arguments as (5.19), we get

\[
\left\| \int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\tilde{\theta}}{\theta} \right)_y (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_1 \left( \frac{v-u}{\sqrt{R\theta}} \right) \frac{Q(G, G)}{M} \, dv \right\} \, dy \right\|
= \left\| \int_{\mathbb{R}} \left\{ \frac{3}{2} \left( \frac{\tilde{\theta}}{\theta} \right)_y (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} \frac{\sqrt{\mu} A_1 \left( \frac{v-u}{\sqrt{R\theta}} \right)}{M} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) \, dv \right\} \, dy \right\|
\leq C \eta e^{1-a} \|\tilde{\theta}_y\|^2 + C \eta e^{\frac{3}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{12} e^{\frac{3}{2} - \frac{7}{2} a} D_2(\tau).
\]  
(3.22)

By the estimates from (3.21) to (3.22), we have from (3.16) that

\[
\int_{\mathbb{R}} H_3 \, dy \leq C \eta e^{1-a} \|\tilde{\theta}_y\|^2 + C \eta e^{1-a} \sum_{|\alpha|=1} \|\partial^\alpha f\|_\sigma^2 \\
+ C \eta e^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{12} e^{\frac{3}{2} - \frac{7}{2} a} D_2(\tau).
\]  
(3.23)
By using (3.15) and following the same strategies used in the estimates of (3.16), then similar arguments as to those of (3.23) imply

$$
\int_{\mathbb{R}} H_4 \, dy = \int_{\mathbb{R}} \left\{ \left( \frac{3}{2} \tilde{u}_y - \frac{3}{2} \tilde{\vartheta} \right) \cdot \int_{\mathbb{R}^3} v_1 v L_M^{-1} \Theta \, dv \right\} \, dy
$$

$$
= \int_{\mathbb{R}} \left\{ \left( \frac{3}{2} \sum_{i=1}^{3} \tilde{u}_{iy} - \frac{3}{2} \tilde{\vartheta} \sum_{i=1}^{3} u_{iy} \right) R\vartheta \int_{\mathbb{R}^3} B_{1i} \left( \frac{v - u}{\sqrt{R\vartheta}} \right) \Theta \, dv \right\} \, dy
$$

$$
\leq C \eta \varepsilon^{1-a} \| \tilde{u}_y \|^2 + C \eta \varepsilon^{1-a} \sum_{|\alpha|=1} \| \partial^\alpha f \|^2 \sigma
$$

$$
+ C \eta \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau).
$$

Substituting the estimates of $H_1 - H_4$ into (3.12) and taking $\eta > 0$ small enough, one has

$$
\frac{d}{d\tau} \int_{\mathbb{R}} \eta(\tau, y) \, dy + c_2 \| \sqrt{u_{1y}(\rho, \tilde{u}, \tilde{\vartheta})} \|^2 + c_\varepsilon \| \tilde{u}_y \|^2 
\leq C \varepsilon^{1-a} \sum_{|\alpha|=1} \| \partial^\alpha f \|^2 \sigma
$$

$$
+ C \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C k \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau).
$$

Since there is no dissipation for density function and the temporal derivatives for $(\rho, \tilde{u}, \tilde{\vartheta})$ in (3.24), to get the estimation of $\| \tilde{\rho}_y \|^2$ and $\| (\rho_{\tau}, \tilde{u}_{\tau}, \tilde{\vartheta}_{\tau}) \|^2$, we first take the inner product of (2.4) with $\tilde{\rho}_y$ over $\mathbb{R}$ to get

$$
\varepsilon^{1-a} \left( \frac{2\tilde{\vartheta}}{3\tilde{\rho}} \tilde{\rho}_y, \tilde{\rho}_y \right) = \varepsilon^{1-a} \left( -\tilde{u}_{1\tau} - \tilde{u}_1 \tilde{u}_{1y} - \frac{2}{3} \tilde{\vartheta}_y - J_2 - \frac{1}{\rho} \int_{\mathbb{R}^3} v_1^2 G_y \, dv, \tilde{\rho}_y \right).
$$

By using (2.4), the integration by parts, the Cauchy inequality and Lemma 5.3, one has

$$
-\varepsilon^{1-a} (\tilde{u}_{1\tau}, \tilde{\rho}_y) = -\varepsilon^{1-a} (\tilde{u}_1, \tilde{\rho}_y)_\tau - \varepsilon^{1-a} (\tilde{u}_{1y}, \tilde{\rho}_y)
$$

$$
= -\varepsilon^{1-a} (\tilde{u}_1, \tilde{\rho}_y)_\tau + \varepsilon^{1-a} (\tilde{u}_{1y}, \rho \tilde{u}_{1y} + \tilde{\rho} \tilde{u}_1)
$$

$$
+ (\tilde{\vartheta} \tilde{u}_1)_y + \tilde{u}_1 \tilde{\vartheta}_y + \tilde{u}_{1y} \tilde{\vartheta}
$$

$$
\leq -\varepsilon^{1-a} (\tilde{u}_1, \tilde{\rho}_y)_\tau + C \eta \varepsilon^{1-a} \tilde{\rho}_y \| \tilde{u}_{1y} \|^2 + C \eta \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2
$$

$$
+ C \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C k \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau),
$$

where in the last inequality, we have dealt with the typical terms as follows:

$$
\varepsilon^{1-a} (\tilde{u}_{1y}, \rho \tilde{u}_{1y}) \leq C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 \leq C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 \leq C k \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau),
$$

and

$$
\varepsilon^{1-a} (\tilde{u}_{1y}, \tilde{\rho} \tilde{u}) \leq C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 + C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 \| \tilde{\rho} \|^2
$$

$$
\leq C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 + C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 \| \tilde{\rho} \|^2
$$

$$
\leq C \varepsilon^{1-a} \| \tilde{u}_{1y} \|^2 + C \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}}.
$$
By (2.5) and the similar above estimates, the term involving \(J_2\) can be controlled by
\[
\varepsilon^{1-a} |(J_2, \tilde{\rho}_y)| = \varepsilon^{1-a} \left| \left( \tilde{u}_1 \tilde{u}_1 y + \tilde{u}_1 \tilde{u}_1 y + \frac{2}{3} \rho_y \tilde{\rho} \tilde{\rho} - \tilde{\rho} \tilde{\rho}, \tilde{\rho}_y \right) \right| \\
\leq \eta \varepsilon^{1-a} \| \tilde{\rho}_y \|^2 + C \eta \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + Ck \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau).
\]

The Cauchy inequality implies
\[
\varepsilon^{1-a} \left| \left( \tilde{u}_1 \tilde{u}_1 y + \frac{2}{3} \tilde{\theta}_y, \tilde{\rho}_y \right) \right| \leq \eta \varepsilon^{1-a} \| \tilde{\rho}_y \|^2 + C \eta \varepsilon^{1-a} \| [\tilde{u}_1 y, \tilde{\theta}_y] \|^2.
\]

By \(G = \overline{G} + \sqrt{\mu} f\), we have from the similar arguments as (3.19) and (3.20) that
\[
\varepsilon^{1-a} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{v^2 G_y \tilde{\rho}_y}{\rho} \, dv \, dy \right| \leq \varepsilon^{1-a} \| \tilde{\rho}_y \| \left( \| \langle v \rangle \|^{\frac{1}{2}} f_y + \| \frac{G_y}{\sqrt{\mu}} \| \right) \\
\leq \eta \varepsilon^{1-a} \| \tilde{\rho}_y \|^2 + C \eta \varepsilon^{1-a} \| f_y \|_{\sigma}^2 + C \eta \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + Ck \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau).
\]

Hence, plugging the above related estimates into (3.25), we have by choosing \(\eta > 0\) small enough and \(C_0 > 0\) that
\[
\varepsilon^{1-a} \| \tilde{\rho}_y \|^2 \leq -C_0 \varepsilon^{1-a} (\tilde{u}_1, \tilde{\rho}_y) + C \varepsilon^{1-a} \| [\tilde{u}_1 y, \tilde{\theta}_y] \|^2 + C \varepsilon^{1-a} \| f_y \|_{\sigma}^2 \\
+ C \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + Ck \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau). \tag{3.26}
\]

On the other hand, by using the system (2.4) again, we can arrive at
\[
\varepsilon^{1-a} \| (\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x) \|^2 \leq C \varepsilon^{1-a} \| (\tilde{\rho}_y, \tilde{\theta}_y, \tilde{\theta}_y) \|^2 + \| f_y \|_{\sigma}^2 + C \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + Ck \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau). \tag{3.27}
\]

For some suitably large constant \(\tilde{C}_0 \gg C_0 > 0\), a suitable linear combination of (3.27), (3.26) and (3.24) yields
\[
\frac{d}{d\tau} \left( \tilde{C}_0 \int_{\mathbb{R}} \eta(\tau, y) \, dy + C_0 \varepsilon^{1-a} \int_{\mathbb{R}} \tilde{u}_1 \tilde{\rho}_y \, dy \right) \\
+ c \| \sqrt{\tilde{u}_1 y}(\tilde{\rho}, \tilde{u}_1, \tilde{\theta}) \|^2 + C \varepsilon^{1-a} \sum_{|\alpha|=1} \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 \\
\leq C \varepsilon^{1-a} \sum_{|\alpha|=1} \| \partial^\alpha f \|_{\sigma}^2 + C \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + Ck \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} D_2(\tau). \tag{3.28}
\]

This completes the proof of lower order energy estimates for the macroscopic component \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\).
Next, we turn to prove lower order energy estimates for the microscopic component $f$. Taking the inner product of (2.10) with $f$ over $\mathbb{R}_y \times \mathbb{R}^3_v$ gives

\[
(f_\tau + v_1 f_y - \varepsilon^{a-1} \mathcal{L} f, f) = \varepsilon^{a-1} \left( \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right) + \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}, f \right) + \left( \frac{P_0(v_1 \sqrt{\mu} f_y)}{\sqrt{\mu}} - \frac{1}{\sqrt{\mu}} P_1 v_1 M \left\{ |v - u|^2 \tilde{T}_y \right\} + \frac{(v - u) \cdot \tilde{u}_y}{\sqrt{\theta}} \right) - \frac{P_1(v_1 \tilde{G}_y)}{\sqrt{\mu}} - \frac{G_\tau}{\sqrt{\mu}}, f \right),
\]

(3.29)

We will estimate each term for (3.29). First of all, we have from the integration by parts and (5.5) that

\[
(f_\tau + v_1 f_y - \varepsilon^{a-1} \mathcal{L} f, f) \geq \frac{1}{2} \frac{d}{d\tau} \| f \|^2 + \sigma_1 \varepsilon^{a-1} \| f \|^2_\sigma.
\]

By using (5.7), (5.12) and (2.14), we can obtain

\[
\varepsilon^{a-1} \left| \left( \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), f \right) + \left( \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right) \right) \right| \leq C(n_0 + k_1 \varepsilon^{\frac{3}{2} - \frac{2}{a}}) \varepsilon^{a-1} \| f \|^2_\sigma.
\]

From (5.19), it is easily known that

\[
\varepsilon^{a-1} \left| \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}, f \right) \right| \leq C \eta \varepsilon^{a-1} \| f \|^2_\sigma + C \eta \varepsilon^{\frac{7}{4} - \frac{2}{a}} \varepsilon^{a-1} \| f \|^2_\sigma.
\]

In view of the properties of $P_0$ in (1.9) and (2.14), one can show that

\[
\left| \left( \frac{P_0(v_1 \sqrt{\mu} f_y)}{\sqrt{\mu}}, f \right) \right| = \left| \sum_{i=0}^{4} \frac{1}{\sqrt{\mu}} \left( v_1 \sqrt{\mu} f_y, \frac{x_i}{M} \right) x_i, f \right| \leq C \| \langle v \rangle^{-\frac{1}{4}} f \| \| \langle v \rangle^{-\frac{1}{4}} f_y \| \leq C \eta \varepsilon^{a-1} \| f \|^2_\sigma + C \eta \varepsilon^{1-a} \| f_y \|^2_\sigma.
\]

By using (5.1), a direct computation shows that

\[
P_1 v_1 M \left\{ \frac{|v - u|^2 \tilde{T}_y}{2R\theta^2} + \frac{(v - u) \cdot \tilde{u}_y}{\sqrt{\theta}} \right\} = \sqrt{\frac{K}{\theta}} \tilde{T}_y \hat{A}_1 \left\{ \frac{v - u}{\sqrt{R\theta}} \right\} M + \sum_{j=1}^{3} \tilde{u}_j \hat{B}_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) M,
\]
which implies that
\[
\left| \left( \frac{1}{\sqrt{\mu}} P_1 v_1 M \left\{ \frac{|v-u|^2 \partial_y}{2R\theta^2} + \frac{(v-u) \cdot \tilde{\theta}_y}{R\theta} \right\}, f \right) \right| 
\leq C \eta \epsilon^{a-1} \| f \|_\sigma^2 + C \eta \epsilon^{1-a} \| [\tilde{u}_y, \tilde{\theta}_y] \|^2.
\]

Here we have used the fact that \(|(v)^m \mu^{-\frac{1}{2}} M| \leq C\) by (3.5) for any \(m \geq 0\).

In addition, we use (1.9), (5.25), (3.3), (3.4), (2.14), the Sobolev imbedding inequality and Lemma 5.3 to obtain
\[
\left| \left( \frac{P_1(v_1 \tilde{G}_y)}{\sqrt{\mu}} + \tilde{G}_\tau \frac{f}{\sqrt{\mu}} \right) \right| = \left| \left( v_1 \tilde{G}_y \frac{1}{\sqrt{\mu}} - \frac{P_0(v_1 \tilde{G}_y)}{\sqrt{\mu}} + \tilde{G}_\tau \frac{f}{\sqrt{\mu}} \right) \right|
\leq C \epsilon^{1-a} (\| [\tilde{u}_{1y}, \tilde{\theta}_{yy}] \| + \| [\tilde{u}_{1y}, \tilde{\theta}_y] \cdot [u_y, \theta_y] \|
+ \| [\tilde{u}_{1\tau}, \tilde{\theta}_{\tau}] \| + \| [\tilde{u}_{1\tau}, \tilde{\theta}_\tau] \cdot [u_\tau, \theta_\tau] \|) \| (v)^{\frac{1}{2}} f \|
\leq C \eta \epsilon^{a-1} \| f \|_\sigma^2 + C \eta \epsilon^{\frac{7}{2} + \frac{1}{15} a} (\delta + \epsilon^a \tau)^{-\frac{3}{2}} + C \eta k \epsilon^{\frac{7}{2} - \frac{2}{3} a} D_2(\tau).
\]

Plugging the above related estimates into (3.29), we get
\[
\frac{1}{2} \frac{d}{d\tau} \| f \|^2 + C \epsilon^{a-1} \| f \|_\sigma^2 \leq C \epsilon^{1-a} (\| [\tilde{u}_y, \tilde{\theta}_y] \|^2 + \| f_y \|_\sigma^2)
+ C \epsilon^{\frac{7}{2} + \frac{1}{15} a} (\delta + \epsilon^a \tau)^{-\frac{3}{2}} + C k \epsilon^{\frac{7}{2} - \frac{2}{3} a} D_2(\tau),
\]
(3.30)

by choosing suitably small \(\eta\) and using the smallness of \(\eta_0\).

In summary, for some suitably large constant \(\tilde{C}_1 > 1\), adding (3.28) \(\times \tilde{C}_1\) to (3.30) gives
\[
\frac{d}{d\tau} \left\{ \tilde{C}_1 \int_{\mathbb{R}} \eta(\tau, y) \, dy + C_0 \epsilon^{1-a} \int_{\mathbb{R}} \tilde{u}_1 \tilde{\rho}_y \, dy + \frac{1}{2} \| f \|^2 \right\}
+ c \| \tilde{u}_{1y}(\tilde{\rho}, \tilde{u}_1, \tilde{\theta}) \|^2 + C \epsilon^{1-a} \sum_{|\alpha| = 1} \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 + C \epsilon^{a-1} \| f \|_\sigma^2
\leq C \epsilon^{1-a} \sum_{|\alpha| = 1} \| \partial^\alpha f \|_\sigma^2 + C \epsilon^{\frac{7}{2} + \frac{1}{15} a} (\delta + \epsilon^a \tau)^{-\frac{3}{2}} + C k \epsilon^{\frac{7}{2} - \frac{2}{3} a} D_2(\tau).
\]
(3.31)

Integrating (3.31) with respect to \(\tau\), we can obtain (3.6) by using (3.2) and (3.3) as well as the fact \(\eta(\tau, y) \approx \| (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2\) due to (3.9). We consequently finish the proof of Lemma 3.1.

\[
\square
\]

3.2. High Order Energy Estimates

In this subsection, we will derive high order energy estimates on time-spatial derivatives for \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) and \(f\).
Lemma 3.2. Letting (3.3), (3.4) and (3.5) hold, one has

\[
\sum_{|\alpha|=1} \left\{ \| \partial^{\alpha} (\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tau) \|^2 + \| \partial^{\alpha} f(\tau) \|^2 \right\} \\
+ \varepsilon^{2(1-a)} \sum_{|\alpha|=2} \left\{ \| \partial^{\alpha} (\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tau) \|^2 + \| \partial^{\alpha} f(\tau) \|^2 \right\} \\
+ \varepsilon^{1-a} \sum_{|\alpha|=2} \int_0^\tau \left\{ \| \partial^{\alpha} (\tilde{\rho}, \tilde{u}, \tilde{\theta})(s) \|^2 + \| \partial^{\alpha} f(s) \|^2_{\sigma} \right\} \, ds \\
+ \varepsilon^{a-1} \sum_{|\alpha|=1} \int_0^\tau \| \partial^{\alpha} f(s) \|^2_{\sigma} \, ds \\
\leq C k^{1\frac{1}{2}} \varepsilon^{3\frac{1}{2}-\frac{3}{2}a} + C \left( \eta_0 + k^{1\frac{1}{2}} \varepsilon^{3\frac{1}{2}-\frac{3}{2}a} + k^{1\frac{1}{2}} \varepsilon^{3\frac{1}{2}-\frac{3}{2}a} \right) \int_0^\tau D_2(s) \, ds 
\]

(3.32)

for any \( \tau \in [0, \tau_1] \).

Proof. We first consider the fluid variables \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\). Differentiating (2.6)_1 with respect to \( y \), we then multiply the resulting equation by \( \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \) and integrate with respect to \( y \) to obtain

\[
\left( \tilde{\rho}_y + \tilde{\rho}_1 \bar{u}_{1yy} + 2\tilde{\rho}_y \bar{u}_{1y} + \tilde{\rho}_{yy} \bar{u}_1, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) = - \left( J_{1y}, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right). 
\]

(3.33)

This, together with the integration by parts, leads to

\[
\frac{1}{2} \frac{d}{d\tau} \left\| \left( \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \right)^{1/2} \tilde{\rho}_y \right\|^2 + \left( \bar{u}_{1yy}, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) \\
= \left( \frac{\tilde{\rho}_y^2}{2}, \left( \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \right) \right)_\tau - \left( 2\tilde{\rho}_y \bar{u}_{1y} + \tilde{\rho}_{yy} \bar{u}_1 + J_{1y}, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right). 
\]

(3.34)

We are going to estimate the terms on the right-hand side of (3.34). By the Sobolev imbedding inequality and Lemma 5.3, we get

\[
\left| \left( \frac{\tilde{\rho}_y^2}{2}, \left( \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \right) \right)_\tau \right| + \left| (2\tilde{\rho}_y \bar{u}_{1y}, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y) \right| \\
\leq C \| [\tilde{\rho}_r, \tilde{\theta}_r] \|_{L^\infty} \| \tilde{\rho}_y \|^2 + C \| \tilde{\rho}_y \|_{L^\infty} \| \bar{u}_{1y} \| \| \tilde{\rho}_y \| \\
\leq C \varepsilon^a \delta^{-1} \| \tilde{\rho}_y \|^2 + C \varepsilon^a \delta^{-1} \| \bar{u}_{1y} \| \| \tilde{\rho}_y \| \\
\leq C \varepsilon^a \delta^{-1} \varepsilon^{a-1} \delta^{1-a} (\| \tilde{\rho}_y \|^2 + \| \bar{u}_{1y} \|^2) \\
\leq C \varepsilon^{2a-1} \delta^{-(\frac{3}{2}-\frac{3}{2}a)} D_2(\tau) = Ck \varepsilon^{\frac{12}{3}a - \frac{8}{3}} D_2(\tau),
\]

(3.35)
where in the last line, we have used (3.3) and (2.16). It also holds, by using the Sobolev imbedding inequality, Lemma 5.3, (2.16), (3.4) and (3.3), that

\[
\left| \left( \tilde{\rho}_{yy} \tilde{u}_1, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) \right| \leq C \| \tilde{u}_1 \|_{L^\infty}(\| \tilde{\rho}_{yy} \|^2 + \| \tilde{\rho}_y \|^2) \\
\leq C k^{\frac{1}{11}} \| \tilde{\rho}_y \|^\frac{3}{\alpha} \left\{ \varepsilon^{3a} \delta^{-1}(\delta + \varepsilon^a \tau)^{-2} + \varepsilon^{a-1} D_2(\tau) \right\} \\
\leq C \varepsilon^{\frac{7}{3} + \frac{1}{11}}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C k^{\frac{1}{11}} \varepsilon^\frac{3a-2}{3} D_2(\tau). \tag{3.36}
\]

Based on the estimates in (3.35) and (3.36), we are now ready to compute the term with $J_{1y}$. Recalling $J_1 = \tilde{\rho}_y \tilde{u}_1 + \tilde{u}_1 \tilde{\rho} + (\tilde{\rho} \tilde{u}_1)_y$ by (2.5), performing calculations similar to (3.35) and (3.36), we can arrive at

\[
\left| \left( \tilde{\rho}_y \tilde{u}_1 \right)_y + (\tilde{u}_1 \tilde{\rho})_y, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right| \\
\leq \left| \left( \left( \frac{2\tilde{\theta} \tilde{u}_1}{3\tilde{\rho}^2} \right)_y, \frac{\tilde{\rho}_y}{\tilde{\rho}} \right) \right| + \left| \left( \tilde{u}_1 \tilde{\rho}_y + \tilde{u}_1 \tilde{\rho}_y + \tilde{u}_1 \tilde{\rho}_y, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) \right| \\
\leq C \varepsilon^{\frac{7}{3} + \frac{1}{11}}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C(k \epsilon^{\frac{12}{3}a - \frac{8}{3}} + k^{\frac{1}{11}} \varepsilon^{\frac{3a-2}{3}}) D_2(\tau).
\]

On the other hand, we have from the Sobolev imbedding inequality and (3.4) that

\[
\left| \left( (\tilde{\rho} \tilde{u}_1)_{yy}, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) \right| = \left| \left( \tilde{\rho}_{yy} \tilde{u}_1 + \tilde{\rho} \tilde{u}_{1yy} + 2\tilde{\rho}_y \tilde{u}_1 \tilde{\rho}_y, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) \right| \\
\leq C \varepsilon^{a-1} \sqrt{E_2(\tau) D_2(\tau)} \leq C k^{\frac{1}{11}} \varepsilon^{\frac{3a-2}{3}} D_2(\tau).
\]

With the help of the above two estimates, we get

\[
\left| \left( J_{1y}, \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \tilde{\rho}_y \right) \right| \leq C \varepsilon^{\frac{7}{3} + \frac{1}{11}}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C(k \epsilon^{\frac{12}{3}a - \frac{8}{3}} + k^{\frac{1}{11}} \varepsilon^{\frac{3a-2}{3}}) D_2(\tau).	ag{3.37}
\]

Hence, substituting the estimates (3.35), (3.36) and (3.37) into (3.34), we obtain

\[
\frac{1}{2} \frac{d}{d\tau} \left\| \left( \frac{2\tilde{\theta}}{3\tilde{\rho}^2} \right)^{1/2} \tilde{\rho}_y \right\|^{2} + \left( \tilde{u}_{1yy}, \frac{2\tilde{\theta}}{3\tilde{\rho}} \tilde{\rho}_y \right) \leq C \varepsilon^{\frac{7}{3} + \frac{1}{11}}(\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C k^{\frac{1}{11}} \varepsilon^{\frac{3a-2}{3}} D_2(\tau).	ag{3.38}
\]

Here we have used the smallness of $k$ and the assumption of $\frac{2}{3} \leq a \leq 1$ such that

\[
k \epsilon^{\frac{12}{3}a - \frac{8}{3}} \leq k^{\frac{1}{11}} \epsilon^{\frac{3a-2}{3}}.	ag{3.39}
\]
Similar for deducing (3.33), by differentiating the equation (2.6)\textsubscript{2} with respect to \( y \), we then take the inner product of the resulting equation with \( \tilde{u}_{1y} \) to obtain

\[
\frac{1}{2} \frac{d}{d\tau} \| \tilde{u}_{1y} \|^2 + \langle (\tilde{u}_1 \tilde{u}_{1y})_y, \tilde{u}_{1y} \rangle + \left( \frac{2}{3} \tilde{\theta}_{yy}, \tilde{u}_{1y} \right) + \left( \frac{2 \tilde{\theta}}{\tilde{\rho} \tilde{\rho}} \tilde{\rho} \right)_y, \tilde{u}_{1y} \rangle = -(J_{2y}, \tilde{u}_{1y}) + \varepsilon^{1-a} \left( \frac{4}{3\rho} (\mu(\theta)u_{1y})_y \right)_y, \tilde{u}_{1y} \rangle \\
- \left( \frac{1}{\rho} \left( \int_{\mathbb{R}^3} v^2 L^{-1} \Theta \, dv \right) \right)_y, \tilde{u}_{1y} \rangle.
\]

(3.40)

By the integration by parts and using the arguments similar to those of (3.35) and (3.39), we get

\[
|\langle (\tilde{u}_1 \tilde{u}_{1y})_y, \tilde{u}_{1y} \rangle | = |\langle \tilde{u}_1 \tilde{u}_{1y}, \tilde{u}_{1yy} \rangle | = \frac{1}{2} |\langle \tilde{u}_{1y}, \tilde{u}_{1y} \rangle |
\leq C \| \tilde{u}_{1y} \|_{L^2} \| \tilde{u}_{1y} \|^2 \leq C_5 \varepsilon^{\frac{12}{5}} a^{-\frac{8}{5}} D_2(\tau) \leq C k \varepsilon^{\frac{12}{5}} a^{-\frac{8}{5}} D_2(\tau).
\]

We will compute the right-hand side of (3.40) term by term. Recalling \( J_2 \) by (2.5) and performing the similar calculations as (3.37), we thereby obtain

\[
|\langle J_{2y}, \tilde{u}_{1y} \rangle | = \left| \left[ \tilde{u}_1 \tilde{u}_{1y} + \tilde{u}_1 \tilde{u}_{1y} + \frac{2}{3} \rho \tilde{\theta} - \tilde{\rho} \tilde{\theta} \right]_y , \tilde{u}_{1y} \right| \\
\leq C \varepsilon^{\frac{7}{5} + \frac{1}{5} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{5}} + C k \varepsilon^{\frac{1}{5}} a^{-\frac{2}{5}} D_2(\tau).
\]

For the second term on the right-hand side of (3.40), we first use an integration by parts about \( y \) to obtain

\[
\left( \frac{4}{3\rho} (\mu(\theta)u_{1y})_y \right)_y, \tilde{u}_{1y} \rangle = - \left( \frac{4}{3\rho} (\mu(\theta)\tilde{u}_{1y})_y, \tilde{u}_{1yy} \right) - \left( \frac{4}{3\rho} (\mu(\theta)\tilde{u}_{1y})_y, \tilde{u}_{1yy} \right).
\]

In view of the Sobolev imbedding inequality, Lemma 5.3, (3.3) and (3.4), one can show that

\[
- \varepsilon^{1-a} \left( \frac{4}{3\rho} (\mu(\theta)\tilde{u}_{1y})_y, \tilde{u}_{1yy} \right)
= -\varepsilon^{1-a} \left( \frac{4}{3\rho} (\mu(\theta)\tilde{u}_{1yy}, \tilde{u}_{1yy} \right) - \varepsilon^{1-a} \left( \frac{4}{3\rho} \mu'(\theta) \tilde{u}_{1yy}, \tilde{u}_{1yy} \right)
\leq -c_5 \varepsilon^{1-a} \| \tilde{u}_{1yy} \|^2 + C \varepsilon^{1-a} \| \theta_y \|_{L^\infty} \| \tilde{u}_{1y} \| \| \tilde{u}_{1yy} \|
\leq -c_5 \varepsilon^{1-a} \| \tilde{u}_{1yy} \|^2 + C k \varepsilon^{\frac{12}{5}} a^{-\frac{8}{5}} D_2(\tau)
\]

for some constant \( c_5 > 0 \). Similarly, it holds that

\[
\varepsilon^{1-a} \left| \left( \frac{4}{3\rho} (\mu(\theta)\tilde{u}_{1y})_y, \tilde{u}_{1yy} \right) \right| \leq C \eta \varepsilon^{1-a} \| \tilde{u}_{1yy} \|^2
+ C \eta \varepsilon^{\frac{7}{5} + \frac{1}{5} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{5}} + C \eta k \varepsilon^{\frac{12}{5}} a^{-\frac{8}{5}} D_2(\tau).
\]
Hence, by taking $\eta > 0$ small enough, there exists a constant $c_6 > 0$ such that
\[
\varepsilon^{1-a} \left( \left[ \frac{4}{3} \mu(\theta) u_{1y} \right]_{y}, \tilde{u}_{1y} \right) \leq -c_6 \varepsilon^{1-a} \| \tilde{u}_{1yy} \|^2 \\
+ C \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C k \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3}} D_2(\tau).
\] (3.41)

The estimations for the last term of (3.40) is more complicated. The integration by parts and (3.15) give that
\[
\left( \left[ \frac{1}{\rho} \int_{\mathbb{R}^3} v^2 L^{-1}_M \Theta \, dv \right]_{y}, \tilde{u}_{1y} \right) = \left( \frac{1}{\rho} \int_{\mathbb{R}^3} R \theta B_{11} \left( \frac{v - u}{\sqrt{R \theta}} \right) \left( \Theta_y M \right) \, dv, \tilde{u}_{1yy} \right) \\
= \left( \frac{1}{\rho} \int_{\mathbb{R}^3} R \theta B_{11} \left( \frac{v - u}{\sqrt{R \theta}} \right) \left( \frac{1}{M} \right) \Theta_y \, dv, \tilde{u}_{1yy} \right) + \left( \frac{1}{\rho} \int_{\mathbb{R}^3} R \theta B_{11} \left( \frac{v - u}{\sqrt{R \theta}} \right) \Theta_y \, dv, \tilde{u}_{1yy} \right).
\] (3.42)

Notice that the first term on the right hand side of (3.42) is the higher nonlinear term compared with (3.16) and is easier to estimate. Therefore, we can follow a method similar to that used in (3.17)–(3.22) to deal with this term. Then we can arrive at
\[
\left| \left( \frac{1}{\rho} \int_{\mathbb{R}^3} R \theta B_{11} \left( \frac{v - u}{\sqrt{R \theta}} \right) \left( \frac{1}{M} \right) \Theta_y \, dv, \tilde{u}_{1yy} \right) \right| \\
\leq \eta \varepsilon^{1-a} \| \tilde{u}_{1yy} \|^2 + C \eta \varepsilon^{\frac{7}{3} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{12} \varepsilon^{\frac{3}{2} - \frac{2}{3}} D_2(\tau).
\] (3.43)
Substituting the above related estimates into (3.40) and taking $\eta > 0$ small enough, we get

$$\frac{1}{2} \frac{d}{d \tau} \| \tilde{u}_{1y} \|^2 + \left( \frac{2}{3} \hat{\theta}_{yy}, \tilde{u}_{1y} \right) + \left( \frac{2 \hat{\theta}}{3 \bar{\rho}} \hat{\rho}_{y}, \tilde{u}_{1y} \right) + c \varepsilon^{1-a} \| \bar{u}_{1yy} \|^2 \leq C \varepsilon^{1-a} \sum_{|\alpha| = 2} \| \partial^\alpha f \|_\sigma^2 + C \varepsilon^{\frac{7}{5} + \frac{1}{3}a} (\delta + \varepsilon \alpha \tau)^{-\frac{4}{5}} + C \left( k \frac{1}{12} \varepsilon^{\frac{3}{5} - \frac{2}{5}a} + k \frac{1}{12} \varepsilon^{\frac{3}{5}a - \frac{2}{5}} \right) D_2(\tau).$$

(3.44)

Similar to (3.40), by differentiating (2.6)_3 with respect to $y$ and taking the inner product of the resulting equation with $\tilde{u}_{iyy}$ ($i = 2, 3$), similar arguments as to those (3.44) imply that

$$\frac{1}{2} \frac{d}{d \tau} \| \tilde{u}_{iyy} \|^2 + c \varepsilon^{1-a} \| \tilde{u}_{iyy} \|^2 \leq C \varepsilon^{1-a} \sum_{|\alpha| = 2} \| \partial^\alpha f \|_\sigma^2 + C \varepsilon^{\frac{7}{5} + \frac{1}{3}a} (\delta + \varepsilon \alpha \tau)^{-\frac{4}{5}} + C \left( k \frac{1}{12} \varepsilon^{\frac{3}{5} - \frac{2}{5}a} + k \frac{1}{12} \varepsilon^{\frac{3}{5}a - \frac{2}{5}} \right) D_2(\tau).$$

(3.45)

Finally, we still deal with (2.6)_4. Differentiating (2.6)_4 with respect to $y$, we then take the inner product of the resulting equation with $\frac{1}{\bar{\theta}} \hat{\theta}_y$ to get

$$\left( \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) + \left( \frac{2}{3} \hat{\theta}_{yyy}, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) + \left( \frac{2}{3} \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) + \left( \frac{1}{\bar{\theta}} \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) = - \left( J_{3y}, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) + \varepsilon^{1-a} \left( \left[ \frac{1}{\rho} (\kappa(\theta) \hat{\theta}_y)_y \right], \frac{1}{\bar{\theta}} \hat{\theta}_y \right) + \varepsilon^{1-a} \left( \left[ \frac{4}{3 \rho} \mu(\theta) \hat{u}^2 \right]_y \right) + \varepsilon^{1-a} \left( \left[ \frac{1}{\rho} \left( \int_{\mathbb{R}^3} |v| \frac{1}{2} L^{-1}_M (\Theta) \, dv \right)_y \right], \frac{1}{\bar{\theta}} \hat{\theta}_y \right) - \left( \left[ \frac{1}{\rho} \left( \int_{\mathbb{R}^3} \frac{|v|^2}{2} L^{-1}_M (\Theta) \, dv \right)_y \right], \frac{1}{\bar{\theta}} \hat{\theta}_y \right).$$

(3.46)

We will estimate (3.46) term by term. First of all, one has, from arguments similar to those of (3.35) and (3.39), that

$$\left| \left( \frac{2}{3} \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) \right| \leq C \| \hat{\theta}_y \|_{L^\infty} \| \tilde{u}_{1y} \| \| \tilde{\theta}_y \| \leq C k \frac{1}{12} \varepsilon^\frac{3}{5} a - \frac{2}{5} D_2(\tau).$$

Similarly, it holds that

$$\left| \left( \frac{1}{\bar{\theta}} \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) \right| \leq \left| \left( \hat{u}_{1y} \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) \right| + \left| \left( \tilde{u}_{1y} \hat{\theta}_y, \frac{1}{\bar{\theta}} \hat{\theta}_y \right) \right| \leq C k \frac{1}{12} \varepsilon^\frac{3}{5} a - \frac{2}{5} D_2(\tau).$$
Recalling $J_3$ defined by (2.5) and following similar to those of arguments used in (3.37), as well as (3.39), we have
\[
\left| \left( J_{3y}, \frac{1}{\theta} \tilde{\theta}_y \right) \right| = \left| \left[ \frac{2}{3} (\tilde{\theta} \tilde{u}_{1y} + \tilde{\theta} \tilde{u}_{1y}) + (\tilde{\theta}_y \tilde{u}_1 + \tilde{\theta}_y \tilde{u}_1) \right] \frac{1}{\theta} \tilde{\theta}_y \right| 
\leq C \varepsilon \frac{2}{3} + \varepsilon \frac{a}{2} (\delta + \varepsilon \tau)^{-\frac{4}{3}} + C k \frac{1}{3} \varepsilon \frac{3}{2} a^{-\frac{2}{3}} D_2(\tau).
\]

Similar arguments as to those of (3.41) imply that
\[
e^{1-a} \left( \left[ \frac{1}{\rho} (\kappa(\theta) \theta_y)_y \right] \frac{1}{\theta} \tilde{\theta}_y \right) + e^{1-a} \left( \left[ \frac{4}{3 \rho} \mu(\theta) u^2_y \right] \frac{1}{\theta} \tilde{\theta}_y \right) \leq -C \varepsilon \frac{1}{a} \left\| \tilde{\theta}_y \right\|^2 + C \varepsilon \frac{2}{3} + \varepsilon \frac{a}{2} (\delta + \varepsilon \tau)^{-\frac{4}{3}} + C k \frac{1}{3} \varepsilon \frac{3}{2} a D_2(\tau).
\]

On the other hand, we use the integration by parts, (3.14) and (3.15) to obtain
\[
\left( \frac{1}{\rho} u \cdot (\int_{\mathbb{R}^3} v_1 v L^{-1}_M \Theta \, dv)_y \right) \frac{1}{\theta} \tilde{\theta}_y = \left( \frac{1}{\rho} \left( \int_{\mathbb{R}^3} \frac{|v|^2}{2} L^{-1}_M \Theta \, dv \right)_y \right) \frac{1}{\theta} \tilde{\theta}_y 
= \left( \int_{\mathbb{R}^3} \frac{|v|^2}{2} L^{-1}_M \Theta \, dv \right) \frac{1}{\theta} \tilde{\theta}_y + \left( \int_{\mathbb{R}^3} v_1 v L^{-1}_M \Theta \, dv \right) \frac{1}{\theta} \tilde{\theta}_y 
= \left( \int_{\mathbb{R}^3} A_1 \left( \frac{v - u}{\sqrt{R\theta}} \right) \frac{\Theta}{M} \, dv \right) \frac{1}{\theta} \tilde{\theta}_y 
+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} B_{i1} \left( \frac{v - u}{\sqrt{R\theta}} \right) \frac{\Theta}{M} \, dv \frac{1}{\theta} \tilde{\theta}_y 
:= H_5.
\]

Notice that the above terms are similar to (3.42), and then arguments similar to those of (3.43) imply that
\[
H_5 \leq \eta \varepsilon^{1-a} \left\| \tilde{\theta}_y \right\|^2 + C \eta \varepsilon^{1-a} \sum_{|\alpha| = 2} \left\| \partial^\alpha f \right\|^2_\sigma + C \varepsilon \frac{2}{3} + \varepsilon \frac{a}{2} (\delta + \varepsilon \tau)^{-\frac{4}{3}} 
+ C \eta k \frac{1}{3} \varepsilon \frac{3}{2} a^{-\frac{2}{3}} D_2(\tau).
\]

Substituting the above related estimates into (3.46) and choosing $\eta > 0$ small enough, we can arrive at
\[
\frac{1}{2} \frac{d}{d \tau} \left\| \left( \frac{1}{\theta} \right)^{1/2} \tilde{\theta}_y \right\|^2 + \left( \frac{2}{3} \tilde{u}_{1yy}, \tilde{\theta}_y \right) + C \varepsilon^{1-a} \left\| \tilde{\theta}_y \right\|^2 
\leq C \varepsilon^{1-a} \sum_{|\alpha| = 2} \left\| \partial^\alpha f \right\|^2_\sigma + C \varepsilon^{2} + \varepsilon \frac{a}{2} (\delta + \varepsilon \tau)^{-\frac{4}{3}} + C \left( k \frac{1}{3} \varepsilon \frac{3}{2} a^{-\frac{2}{3}} + k \frac{1}{3} \varepsilon \frac{3}{2} a^{-\frac{2}{3}} \right) D_2(\tau).
\]

(3.47)
As a consequence, the sum of (3.38), (3.44), (3.45) and (3.47) gives

\[
\frac{\partial}{\partial \tau} \left\| \left( \frac{2\tilde{\theta}}{3\rho^2} \right)^{1/2} \tilde{\rho}_y \right\|^2 + \left\| \tilde{u}_y \right\|^2 + \left\| \left( \frac{1}{\tilde{\rho}} \right)^{1/2} \tilde{\theta}_y \right\|^2 \] + C\epsilon^{1-a} \left\| \left[ \tilde{u}_{yy}, \tilde{\theta}_{yy} \right] \right\|^2 
\leq C\epsilon^{1-a} \sum_{|\alpha|=2} \left\| \partial^\alpha f \right\|_\sigma^2 + C\epsilon^{\frac{7}{\sigma} + \frac{1}{\sigma} a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} 
+ C \left( k \frac{1}{\tau^2} \epsilon^{\frac{3}{2} - \frac{2}{\sigma} a} + k \frac{1}{\tau^2} \epsilon^{\frac{3}{2} a - \frac{2}{\sigma}} \right) D_2(\tau). \tag{3.48}
\]

Differentiating the equations (2.6) with respect to \( \tau \) and multiplying the resulting equations by \( \frac{2\tilde{\theta}}{3\rho^2} \tilde{\rho}_t, \tilde{u}_t, \tilde{\theta}_t \) with \( i = 2, 3 \) and \( \frac{1}{\tilde{\rho}} \tilde{\theta}_t \) respectively, then adding them together and integrating with respect to \( y \) over \( \mathbb{R} \), we have, by arguments similar to those of (3.48) such, that

\[
\frac{\partial}{\partial \tau} \left\| \tilde{\rho}_t \right\|^2 + \left\| \tilde{u}_t \right\|^2 + \left\| \left( \frac{1}{\tilde{\rho}} \right)^{1/2} \tilde{\theta}_t \right\|^2 \] + C\epsilon^{1-a} \left\| \left[ \tilde{u}_{ty}, \tilde{\theta}_{ty} \right] \right\|^2 
\leq C\epsilon^{1-a} \sum_{|\alpha|=2} \left\| \partial^\alpha f \right\|_\sigma^2 + C\epsilon^{\frac{7}{\sigma} + \frac{1}{\sigma} a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} 
+ C \left( k \frac{1}{\tau^2} \epsilon^{\frac{3}{2} - \frac{2}{\sigma} a} + k \frac{1}{\tau^2} \epsilon^{\frac{3}{2} a - \frac{2}{\sigma}} \right) D_2(\tau). \tag{3.49}
\]

The dissipative terms \( \|\tilde{\rho}_{yy}\|^2 \) and \( \|\tilde{\theta}_{yy}\|^2 \) are not included in the dissipation of (3.48) and (3.49). For this, we use the system (2.4). Differentiating the first and second system of (2.4) with respect to \( y \) yields

\[
\begin{aligned}
\tilde{\rho}_{ty} + (\tilde{\rho}_1 \tilde{u}_1)_y + (\tilde{\rho}_y \tilde{u}_1)_y &= -J_{1y}, \\
\tilde{u}_{1ty} + (\tilde{u}_1 \tilde{u}_1)_y + \frac{2\tilde{\theta}}{3\rho^2} \tilde{\rho}_{yy} + \frac{2\tilde{\rho}}{3\rho^2} \tilde{\rho}_{yy} &= - \left( \frac{1}{\tilde{\rho}} \int_{\mathbb{R}} v^2 G_y dv \right)_y - J_{2y}.
\end{aligned}
\tag{3.50}
\]

Taking the inner product of (3.50) with \( \tilde{\rho}_{yy} \) and performing similar calculations as to those of (3.25), we arrive at

\[
\epsilon^{1-a} \|\tilde{\rho}_{yy}\|^2 \leq - C_1 \epsilon^{1-a} \left( \tilde{u}_1, \tilde{\rho}_1, \tilde{\rho}_{yy} \right)_\tau + C\epsilon^{1-a} \left\| \left[ \tilde{u}_{yy}, \tilde{\theta}_{yy} \right] \right\|^2 + C\epsilon^{\frac{7}{\alpha} + \frac{1}{\alpha} a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C k \frac{1}{\tau^2} \epsilon^{\frac{3}{2} - \frac{2}{\alpha} a} D_2(\tau). \tag{3.51}
\]

Similarly, the following estimate holds:

\[
\epsilon^{1-a} \|\tilde{\theta}_{yy}\|^2 \leq C\epsilon^{1-a} \left\| \left[ \tilde{\rho}_{yy}, \tilde{u}_{yy} \right] \right\|^2 + C\epsilon^{\frac{7}{\alpha} + \frac{1}{\alpha} a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C k \frac{1}{\tau^2} \epsilon^{\frac{3}{2} - \frac{2}{\alpha} a} D_2(\tau). \tag{3.52}
\]

It remains to get the dissipative term \( \|\left( \tilde{\rho}_{\tau \tau}, \tilde{u}_{1\tau \tau}, \tilde{\theta}_{\tau \tau} \right)\|^2 \). Differentiating the equations (2.4) with respect to \( \tau \) and multiplying the resulting equations by \( \tilde{\rho}_{\tau \tau}, \tilde{u}_{1\tau \tau}, \tilde{\theta}_{\tau \tau} \),
\( \vec{u}_{i\tau} \) with \( i = 2, 3 \) and \( \vec{\theta}_{\tau} \) respectively, then adding them together and integrating with respect to \( y \) over \( \mathbb{R} \), one can arrive at

\[
\varepsilon^{1-a} \| (\vec{\rho}_{\tau}, \vec{u}_{\tau}, \vec{\theta}_{\tau}) \|^2 \leq C \varepsilon^{1-a} \| (\vec{\rho}_{\tau y}, \vec{u}_{\tau y}, \vec{\theta}_{\tau y}) \|^2 + C \varepsilon^{1-a} \sum_{|\alpha|=2} \| \partial^{\alpha} f \|^2 \sigma \\
+ C \varepsilon^{\frac{7}{5} + \frac{1}{15} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{5}} + C \left( k^{\frac{1}{12}} \varepsilon^{\frac{3}{5}} - \frac{3}{2} a + k^{\frac{1}{12}} \varepsilon^{\frac{3}{5}} a - \frac{5}{2} \right) D_2(\tau). 
\]

(3.53)

In summary, for some suitably large constants \( \bar{c}_1 \gg C_1 > 0 \), we have from a suitable linear combination of (3.48), (3.49), (3.51), (3.52) and (3.53) that

\[
\frac{d}{d\tau} \sum_{|\alpha|=1} \{ \mathcal{C}_1 \left( \left( \frac{2\varepsilon}{\sqrt[4]{5} \rho} \right)^{1/2} \partial^{\alpha} \rho \right) \|^2 + \| \partial^{\alpha} \vec{u} \|^2 + \left\| \left( \frac{1}{\rho} \right)^{1/2} \partial^{\alpha} \theta \right\|^2 \} + C_1 \varepsilon^{1-a} (\vec{u}_{1y}, \vec{\rho}_{yy})
\]

\[
+ C \varepsilon^{1-a} \sum_{|\alpha|=2} \| \partial^{\alpha} f \|^2 \sigma 
+ C \varepsilon^{\frac{7}{5} + \frac{1}{15} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{5}} + C \left( k^{\frac{1}{12}} \varepsilon^{\frac{3}{5}} - \frac{3}{2} a + k^{\frac{1}{12}} \varepsilon^{\frac{3}{5}} a - \frac{5}{2} \right) D_2(\tau). 
\]

(3.54)

Integrating (3.54) with respect to \( \tau \) and using (3.2) with (3.3), we can obtain

\[
\sum_{|\alpha|=1} \| \partial^{\alpha} (\vec{\rho}, \vec{u}, \vec{\theta}) \|^2 + C \varepsilon^{1-a} \sum_{|\alpha|=2} \int_0^\tau \| \partial^{\alpha} (\vec{\rho}, \vec{u}, \vec{\theta}) \|^2 \, ds 
\leq C \varepsilon^{2(1-a)} \| \vec{\rho}_{yy} \|^2 + C k^{\frac{1}{12}} \varepsilon^{\frac{6}{5} - \frac{4}{5} a} + C \varepsilon^{1-a} \sum_{|\alpha|=2} \int_0^\tau \| \partial^{\alpha} f \|^2 \, ds 
+ C \left( k^{\frac{1}{12}} \varepsilon^{\frac{3}{5} - \frac{5}{2} a} + k^{\frac{1}{12}} \varepsilon^{\frac{3}{5} a - \frac{2}{5}} \right) \int_0^\tau D_2(s) \, ds. 
\]

(3.55)

Next, we will derive the first order derivative estimates for the microscopic component \( f \). Taking the derivative \( \partial^{\alpha} \) of (2.10) with \( |\alpha| = 1 \) and taking the inner product with \( \partial^{\alpha} f \) over \( \mathbb{R}_y \times \mathbb{R}_y^3 \), we can obtain

\[
\left( \partial^{\alpha} f_{\tau} + v_1 \partial^{\alpha} f_y - \varepsilon^{a-1} L \partial^{\alpha} f, \partial^{\alpha} f \right) - \varepsilon^{a-1} \left( \partial^{\alpha} \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), \partial^{\alpha} f \right) 
= \varepsilon^{a-1} \left( \partial^{\alpha} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), \partial^{\alpha} f \right) + \left( \frac{\partial^{\alpha} P_0 (v_1 \sqrt{\mu} f_y)}{\sqrt{\mu}}, \partial^{\alpha} f \right) - \left( \frac{\partial^{\alpha} P_1 (v_1 \vec{G}_y)}{\sqrt{\mu}}, \partial^{\alpha} f \right) 
- \left( \frac{1}{\sqrt{\mu}} \partial^{\alpha} P_1 v_1 M \left\{ \left| v - u \right|^2 \vec{\gamma}_y \left( \frac{1}{2R \theta^2} + \frac{(v - u) \cdot \vec{u}_y}{R \theta} \right), \partial^{\alpha} f \right) - \left( \frac{\partial^{\alpha} \vec{G} \left( \vec{e}_\tau \right)}{\sqrt{\mu}}, \partial^{\alpha} f \right). 
\]

(3.56)

We will compute each term for (3.56). First of all, we have from the integration by parts and (5.5) that

\[
\left( \partial^{\alpha} f_{\tau} + v_1 \partial^{\alpha} f_y - \varepsilon^{a-1} L \partial^{\alpha} f, \partial^{\alpha} f \right) \geq \frac{1}{2} \frac{d}{d\tau} \| \partial^{\alpha} f \|^2 + \sigma_1 \varepsilon^{a-1} \| \partial^{\alpha} f \|^2 \sigma. 
\]
From (5.10) and (5.19), we immediately know that
\[ \varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right) + \partial^a \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f, \partial^a f \right) \right) \right| \]
\[ \leq C \eta \varepsilon^{a-1} \| \partial^a f \|_\sigma^2 + C \eta \left( \eta_0 + k \frac{4}{12} \varepsilon^{\frac{3}{2} - \frac{2}{a}} \right) D_2(\tau), \]
and
\[ \varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}, \partial^a f \right) \right) \right| \leq C \eta \varepsilon^{a-1} \| \partial^a f \|_\sigma^2 \]
\[ + C \eta \varepsilon^{\frac{7}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{4}{12} \varepsilon^{\frac{3}{2} - \frac{2}{a}} D_2(\tau). \]

By (1.9), (2.14), the Sobolev imbedding inequality, Lemma 5.3, (3.3) and (3.4), one has
\[ \left| \left( \frac{\partial^a P_0(v_1 \sqrt{\mu} f_y)}{\sqrt{\mu}}, \partial^a f \right) \right| \leq \sum_{i=0}^4 \left\| \langle v \rangle \frac{1}{2} \mu^{-\frac{1}{2}} \partial^a \left( \langle v_1 \sqrt{\mu} f_y, \chi_i \rangle \chi_i \right) \right\| \| \langle v \rangle \|^{-\frac{1}{2}} \partial^a f \| \]
\[ \leq \eta \varepsilon^{a-1} \| \partial^a f \|_\sigma^2 + C \eta \varepsilon^{1-a} \| \partial^a f_{y} \|_\sigma^2 + C \eta \varepsilon^{\frac{7}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} \]
\[ + C \eta k \frac{4}{12} \varepsilon^{\frac{3}{2} - \frac{2}{a}} D_2(\tau). \]

Here we have used the fact that \| \langle v \rangle^m \mu^{-\frac{1}{2}} M \|_2 \leq C (3.5) for any \( m \geq 0 \). The terms involving \( \tilde{G} \) are dominated by
\[ \left| \left( \frac{\partial^a P_1(v_1 \tilde{G}_y)}{\sqrt{\mu}}, \partial^a f \right) \right| + \left| \left( \frac{\partial^a \tilde{G}_y}{\sqrt{\mu}}, \partial^a f \right) \right| \]
\[ \leq C \eta \varepsilon^{a-1} \| \partial^a f \|_\sigma^2 + C \eta \varepsilon^{\frac{7}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{4}{12} \varepsilon^{\frac{3}{2} - \frac{2}{a}} D_2(\tau), \]
according to (1.9), (5.25), the Sobolev imbedding inequality, Lemma 5.3, (3.3) and (3.4). Hence, by taking \( \eta > 0 \) small enough, we deduce from (3.56) and the above related estimates that
\[ \sum_{|\alpha|=1} \frac{d}{d\tau} \| \partial^\alpha f \|^2 + c \varepsilon^{a-1} \sum_{|\alpha|=1} \| \partial^\alpha f \|_\sigma^2 \]
\[ \leq C \varepsilon^{1-a} \sum_{|\alpha|=1} \left\{ \| \partial^\alpha (\tilde{\rho}_y, \tilde{u}_y, \tilde{\theta}_y) \|^2 + \| \partial^\alpha f_y \|_\sigma^2 \right\} \]
\[ + C \varepsilon^{\frac{7}{2} + \frac{1}{12} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \left( \eta_0 + k \frac{4}{12} \varepsilon^{\frac{3}{2} - \frac{2}{a}} \right) D_2(\tau). \] (3.57)
Integrating (3.57) with respect to $\tau$ and using (3.2) with (3.3), then by a suitable linear combination of the resulting equation and (3.55), we get

\[
\sum_{|\alpha|=1} \left( \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 + \| \partial^\alpha f \|^2 \right) + \varepsilon^{1-a} \sum_{|\alpha|=2} \int_0^\tau \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 \, ds + \varepsilon^{a-1} \sum_{|\alpha|=1} \int_0^\tau \| \partial^\alpha f \|^2 \, ds \leq C \varepsilon^2 (1-a) \left( \| \tilde{\rho} \|_{yy} \right)^2 + Ck \frac{1}{\varepsilon^{1/2}} + C \varepsilon^a \sum_{|\alpha|=2} \int_0^\tau \| \partial^\alpha f \|^2 \, ds + C \left( \eta_0 + \frac{1}{\varepsilon^{3/2}} + \frac{1}{\varepsilon^{3/2-a}} \right) \int_0^\tau D_2(s) \, ds. \tag{3.58}
\]

Finally, we denote our attention to the higher order derivative estimates of $F$. In terms of (2.7), (2.8) and (2.11), one has

\[
\left( \frac{F}{\sqrt{\mu}} \right)_\tau + v_1 \left( \frac{F}{\sqrt{\mu}} \right)_y - \varepsilon^{a-1} \mathcal{L} f = \varepsilon^{a-1} \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right) + \varepsilon^{a-1} \Gamma \left( M - \mu, f \right) + \varepsilon^{a-1} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) + \frac{1}{\sqrt{\mu}} P_1 v_1 M \left\{ \frac{|v - u|^2}{2 R^2 \theta^2} + \frac{(v - u) \cdot \tilde{u}}{R \theta} \right\}. \tag{3.59}
\]

Taking the derivative $\partial^\alpha$ of (3.59) with $|\alpha| = 2$ and then taking the inner product of the resulting equation with $\frac{\partial^\alpha F}{\sqrt{\mu}}$ over $\mathbb{R}_y \times \mathbb{R}_v^3$, we obtain

\[
\frac{1}{2} \frac{d}{d\tau} \left\| \frac{\partial^\alpha F}{\sqrt{\mu}} \right\|^2 - \varepsilon^{a-1} \left( \mathcal{L} \frac{\partial^\alpha F}{\sqrt{\mu}}, \frac{\partial^\alpha F}{\sqrt{\mu}} \right) = \varepsilon^{a-1} \left( \partial^\alpha \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) + \varepsilon^{a-1} \left( \partial^\alpha \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) + \varepsilon^{a-1} \left( \partial^\alpha \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) + \left( \frac{1}{\sqrt{\mu}} \partial^\alpha P_1 v_1 M \left\{ \frac{|v - u|^2}{2 R^2 \theta^2} + \frac{(v - u) \cdot \tilde{u}}{R \theta} \right\} \right). \tag{3.60}
\]

In order to dominate to the second term on the left hand side of (3.60). Recall $F = M + G + \sqrt{\mu} f$, we first have from (5.5) that

\[
-\varepsilon^{a-1} \left( \mathcal{L} \frac{\partial^\alpha f}{\sqrt{\mu}}, \frac{\partial^\alpha f}{\sqrt{\mu}} \right) \geq \sigma_1 \varepsilon^{a-1} \left\| \frac{\partial^\alpha f}{\sqrt{\mu}} \right\|^2.
\]
Due to $\mathcal{L} f = \Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu})$, we get from (5.7) that

$$
\varepsilon^{a-1} \left| \left( \mathcal{L} \partial^a f, \frac{\partial^a G}{\sqrt{\mu}} \right) \right| \leq C \varepsilon^{a-1} \left\| \partial^a f \right\|_\sigma \left\| \frac{\partial^a G}{\sqrt{\mu}} \right\|_\sigma \\
\leq \eta \varepsilon^{a-1} \left\| \partial^a f \right\|_\sigma^2 + C \eta \left\{ \left\| \partial^a [\tilde{u}_{1y}, \tilde{\theta}_y] \right\|^2 + \cdots + \left\| [\tilde{u}_{1y}, \tilde{\theta}_y] \right\|_{L^\infty}^2 \left\| \partial^a [u, \theta] \right\|^2 \right\} \\
\leq \eta \varepsilon^{a-1} \left\| \partial^a f \right\|_\sigma^2 + C \eta \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{\sqrt{\mu}} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} \mathcal{D}_2(\tau),
$$

according to (5.25), the Sobolev imbedding inequality, Lemma 5.3, (2.16) and (3.3). For $|\alpha| = 2$, we can see that

$$
\partial^a M = M \left( \frac{\partial^a \rho}{\rho} - \frac{3 \partial^a \theta}{2 \theta} + \frac{(v - u)^2 \partial^a \theta}{2 R^2} + \sum_{i=1}^3 \frac{\partial^a u_i(v_i - u_i)}{R^2} \right) + \cdots \\
= \left( \mu + (M - \mu) \right) \left( \frac{\partial^a \rho}{\rho} - \frac{3 \partial^a \theta}{2 \theta} + \frac{(v - u)^2 \partial^a \theta}{2 R^2} + \sum_{i=1}^3 \frac{\partial^a u_i(v_i - u_i)}{R^2} \right) + \cdots \\
= J_1^a + J_2^a + J_3^a.
$$

(3.61)

Here the terms $J_1^a$ and $J_2^a$ are the high order derivatives of $(\rho, u, \theta)$ with $\mu$ and $M - \mu$ and $J_3^a$ is the low order derivatives with $M$. Since $J_2^a \frac{1}{\sqrt{\mu}} \in \ker \mathcal{L}$, it follows that $(\mathcal{L} f, J_2^a \frac{1}{\sqrt{\mu}}) = 0$. For the terms $J_1^a \frac{1}{\sqrt{\mu}}$ and $J_3^a \frac{1}{\sqrt{\mu}}$, we use (5.7), Lemma 5.3 and (3.3) to get

$$
\varepsilon^{a-1} \left| \left( \mathcal{L} \partial^a f, \frac{J_3^a}{\sqrt{\mu}} \right) \right| \leq C \varepsilon^{a-1} \left\| \partial^a f \right\|_\sigma \left\| \frac{J_3^a}{\sqrt{\mu}} \right\|_\sigma \\
\leq C \left( \eta_0 + k \frac{1}{\sqrt{\mu}} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} \right) \varepsilon^{a-1} \left( \left\| \partial^a f \right\|_\sigma^2 + \left\| \partial^a (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \right\|^2 \right) \\
+ C \varepsilon^{a-1} \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}},
$$

and

$$
\varepsilon^{a-1} \left| \left( \mathcal{L} \partial^a f, \frac{J_1^a}{\sqrt{\mu}} \right) \right| \leq C \varepsilon^{a-1} \left\| \partial^a f \right\|_\sigma \left\| \frac{J_1^a}{\sqrt{\mu}} \right\|_\sigma \\
\leq \eta \varepsilon^{a-1} \left\| \partial^a f \right\|_\sigma^2 + C \eta \varepsilon^{a-1} \varepsilon^{\frac{7}{2} + \frac{1}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} \\
+ C \eta \varepsilon^{a-1} k \frac{1}{\sqrt{\mu}} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} \mathcal{D}_2(\tau),
$$

where we have used $\left\| (v)^m \mu^{-\frac{1}{2}} M \right\|^2 \leq C$ for any $m \geq 0$ by (3.5). Owing to these, we thereby obtain

$$
\varepsilon^{a-1} \left| \left( \mathcal{L} \partial^a f, \frac{\partial^a M}{\sqrt{\mu}} \right) \right| \leq C \left( \eta + \eta_0 + k \frac{1}{\sqrt{\mu}} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} \right) \varepsilon^{a-1} \left( \left\| \partial^a f \right\|_\sigma^2 + \left\| \partial^a (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \right\|^2 \right) \\
+ C \eta \varepsilon^{a-1} k \frac{1}{\sqrt{\mu}} \varepsilon^{\frac{3}{2} - \frac{2}{3} a} \mathcal{D}_2(\tau).$$
As a consequence, the second term on the left-hand side of (3.60) is controlled by

\[
\epsilon^{a-1} \left( L \partial^\alpha f, \frac{\partial^\alpha F}{\sqrt{\mu}} \right) \leq -\sigma_1 \epsilon^{a-1} \| \partial^\alpha f \|_\sigma^2 + C \left( \eta + \eta_0 + k \frac{1}{12} \epsilon^{3-2a} \right)
\]

\[
\epsilon^{a-1} \left( \| \partial^\alpha f \|_\sigma^2 + \| \partial^\alpha (\nabla, \tilde{u}, \tilde{\theta}) \|^2 \right)
\]

\[
+ C_\eta \epsilon^{a-1} \epsilon^{7+6} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C_\eta \epsilon^{a-1} k \frac{1}{12} \epsilon^{3-2a} D_2(\tau).
\]

For \(|\alpha| = 2,\) recalling \(F = M + \nabla + \sqrt{\mu} f,\) we get from (5.25), (3.61), the Sobolev imbedding inequality, Lemma 5.3, (3.3) and (3.4) that

\[
\left\| \frac{\partial^\alpha F}{\sqrt{\mu}} \right\|_\sigma^2 \leq \left\| \frac{\partial^\alpha \sqrt{\mu} f}{\sqrt{\mu}} \right\|_\sigma^2 + \left\| \frac{\partial^\alpha G}{\sqrt{\mu}} \right\|_\sigma^2 + \left\| \frac{\partial^\alpha M}{\sqrt{\mu}} \right\|_\sigma^2
\]

\[
\leq C \left( \| \partial^\alpha f \|_\sigma^2 + \| \partial^\alpha (\nabla, \tilde{u}, \tilde{\theta}) \|^2 \right) + C_\epsilon \epsilon^{7+6} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C_\epsilon k \frac{1}{12} \epsilon^{3-2a} D_2(\tau).
\]

(3.62)

For the first term on the right-hand side of (3.60), we directly apply (5.7) to get

\[
\epsilon^{a-1} \left| \left( \frac{\partial^\alpha \Gamma (M - \mu, f, \frac{\partial^\alpha F}{\sqrt{\mu}}) \right) \right|
\]

\[
\leq C \epsilon^{a-1} \sum_{|\alpha_1| \leq |\alpha|} \int_{\mathbb{R}} \left| \partial^{\alpha_1} \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right| \left| \partial^{\alpha - \alpha_1} f \right| \left| \frac{\partial^\alpha F}{\sqrt{\mu}} \right| \text{ dy.} \quad \text{(3.63)}
\]

For \(|\alpha| = 0\) in (3.63), we use (5.12) and (3.62) as well as the smallness of \(\eta_0\) to obtain

\[
\epsilon^{a-1} \int_{\mathbb{R}} \left| \partial^{\alpha_1} \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right| \left| \partial^{\alpha - \alpha_1} f \right| \left| \frac{\partial^\alpha F}{\sqrt{\mu}} \right| \text{ dy}
\]

\[
\leq C \left( \eta_0 + k \frac{1}{12} \epsilon^{3-2a} \right) \epsilon^{a-1} \| \partial^\alpha f \|_\sigma \left| \frac{\partial^\alpha F}{\sqrt{\mu}} \right|_\sigma
\]

\[
\leq C \left( \eta_0 + k \frac{1}{12} \epsilon^{3-2a} \right) \epsilon^{a-1} (\| \partial^\alpha f \|_\sigma^2 + \| \partial^\alpha (\nabla, \tilde{u}, \tilde{\theta}) \|^2)
\]

\[
+ C \epsilon^{a-1} \epsilon^{7+6} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C \epsilon^{a-1} k \frac{1}{12} \epsilon^{3-2a} D_2(\tau).
\]

(3.64)

For \(1 \leq |\alpha_1| \leq |\alpha|\) in (3.63), it holds that

\[
\epsilon^{a-1} \int_{\mathbb{R}} \left| \partial^{\alpha_1} \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right| \left| \partial^{\alpha - \alpha_1} f \right| \left| \frac{\partial^\alpha F}{\sqrt{\mu}} \right| \text{ dy}
\]

\[
\leq C \epsilon^{a-1} \left\{ \sum_{|\alpha'| = 1} \| \partial^{\alpha_1} (\rho, u, \theta) \|_\sigma \left\| \frac{\partial^\alpha F}{\sqrt{\mu}} \right\|_\sigma \right\}
\]

\[
\leq \eta \epsilon^{a-1} \left( \| \partial^\alpha f \|_\sigma^2 + \| \partial^\alpha (\nabla, \tilde{u}, \tilde{\theta}) \|_\sigma^2 \right) + C \eta \epsilon^{a-1} \epsilon^{7+6} (\delta + \epsilon^a \tau)^{-\frac{4}{3}}
\]

\[
+ C \eta \epsilon^{a-1} k \frac{1}{12} \epsilon^{3-2a} D_2(\tau).
\]

(3.65)
The term $\varepsilon^{-1}(\partial^\alpha \Gamma(f, \frac{M-\mu}{\sqrt{\mu}}, \frac{\partial^\alpha F}{\sqrt{\mu}}))$ can be treated in the similar way as in (3.63). We thus get from (3.64) and (3.65) that

$$\varepsilon^{-1} \left| \left( \partial^\alpha \Gamma(f, \frac{M-\mu}{\sqrt{\mu}}, \frac{\partial^\alpha F}{\sqrt{\mu}}) \right) \right| \leq C \left( \eta + \eta_0 + k \frac{1}{\varepsilon^3} \right)^{3-\frac{2}{3}} \varepsilon^{-1} \left( \|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|^2 \right)$$  
$$+ C_\eta \varepsilon^{a-1} \varepsilon^{\frac{7}{3} + \frac{4}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_\eta \varepsilon^{a-1} k \frac{1}{\varepsilon^3} \varepsilon^{-\frac{2}{3}} D_2(\tau).$$

Due to $G = \bar{G} + \sqrt{\mu} f$, one has

$$\Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) = \Gamma \left( \frac{\bar{G}}{\sqrt{\mu}}, \frac{\bar{G}}{\sqrt{\mu}} \right) + \Gamma \left( \frac{\bar{G}}{\sqrt{\mu}}, f \right) + \Gamma \left( f, \frac{\bar{G}}{\sqrt{\mu}} \right) + \Gamma(f, f).$$

For the second term on the right-hand side of (3.60), we apply the Cauchy–Schwarz inequality, (3.62) and perform a method similar to that of (5.26) and (5.27) to obtain

$$\varepsilon^{-1} \left| \left( \partial^\alpha \Gamma \left( \frac{\bar{G}}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) \right| \leq \eta \varepsilon^{-1} \left( \|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|^2 \right)$$  
$$+ C_\eta \varepsilon^{a-1} \varepsilon^{\frac{7}{3} + \frac{4}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_\eta \varepsilon^{a-1} k \frac{1}{\varepsilon^3} \varepsilon^{-\frac{2}{3}} D_2(\tau).$$  

Following a method similar to that used in (5.28), we have, by using (3.62), that

$$\varepsilon^{-1} \left| \left( \partial^\alpha \Gamma(f, f), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) \right| + \varepsilon^{-1} \left| \left( \partial^\alpha \Gamma \left( f, \frac{\bar{G}}{\sqrt{\mu}} \right), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) \right| \leq C_\eta \varepsilon^{a-1} \left( \|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|^2 \right)$$  
$$+ C_\eta \varepsilon^{a-1} \varepsilon^{\frac{7}{3} + \frac{4}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_\eta \varepsilon^{a-1} k \frac{1}{\varepsilon^3} \varepsilon^{-\frac{2}{3}} D_2(\tau).$$

With (5.7) and the Sobolev imbedding inequality in hand, we get from (3.3), (3.4) and (3.62), as well as the Cauchy–Schwarz inequality, that

$$\varepsilon^{-1} \left| \left( \partial^\alpha \Gamma(f, f) \right) \right| \leq C_\eta \varepsilon^{a-1} \left( \|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|^2 \right)$$  
$$+ C_\eta \varepsilon^{a-1} \varepsilon^{\frac{7}{3} + \frac{4}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_\eta \varepsilon^{a-1} k \frac{1}{\varepsilon^3} \varepsilon^{-\frac{2}{3}} D_2(\tau).$$  

By the estimates from (3.66) to (3.67), we can conclude that

$$\varepsilon^{-1} \left( \partial^\alpha \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), \frac{\partial^\alpha F}{\sqrt{\mu}} \right) \leq C_\eta \varepsilon^{a-1} \left( \|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|^2 \right)$$  
$$+ C_\eta \varepsilon^{a-1} \varepsilon^{\frac{7}{3} + \frac{4}{3} a} (\delta + \varepsilon^a \tau)^{-\frac{4}{3}} + C_\eta \varepsilon^{a-1} k \frac{1}{\varepsilon^3} \varepsilon^{-\frac{2}{3}} D_2(\tau).$$
For the last term on the right-hand side of (3.60), one has from (2.14) and (3.62), that
\[
\left| \frac{1}{\sqrt{\mu}} \partial^\alpha P_1 v_1 M \left\{ \frac{|v-u|^2 \theta_y}{2 R \theta^2} + \frac{(v-u) \cdot \bar{u}_y}{R \theta} \right\} \right| = C \left\| \langle v \rangle \frac{1}{\sqrt{\mu}} \partial^\alpha P_1 v_1 M \left\{ \frac{|v-u|^2 \theta_y}{2 R \theta^2} + \frac{(v-u) \cdot \bar{u}_y}{R \theta} \right\} \right\| \left\| \langle v \rangle - \frac{1}{2} \partial^\alpha F \right\|_{\sqrt{\mu}} \\
\leq C \eta \epsilon^{a-1} \left( \| \partial^\alpha f \|_{\sigma}^2 + \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|_{2}^2 \right) + C \eta \epsilon^{a-1} \epsilon^{2+\frac{1}{15}a} (\delta + \epsilon^a \tau)^{-\frac{3}{5}} \\
+ C \eta \epsilon^{a-1} k^{\frac{1}{12}} \epsilon^{\frac{3}{5}-\frac{2}{5}a} \mathcal{D}_2 (\tau).
\]

Hence, it holds by the estimates above and for any \( \eta > 0 \) and \( \eta_0 > 0 \) small enough that
\[
\frac{1}{2} \frac{d}{d\tau} \sum_{|\alpha|=2} \| \partial^\alpha F \|_{\sqrt{\mu}}^2 + C \epsilon^{a-1} \sum_{|\alpha|=2} \| \partial^\alpha f \|_{\sigma}^2 \\
\leq C (\eta + \eta_0 + k^{\frac{1}{12}} \epsilon^{\frac{3}{5}-\frac{2}{5}a}) \epsilon^{a-1} \sum_{|\alpha|=2} \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|_{2}^2 \\
+ C \eta \epsilon^{a-1} \epsilon^{2+\frac{1}{15}a} (\delta + \epsilon^a \tau)^{-\frac{3}{5}} + C \eta \epsilon^{a-1} k^{\frac{1}{12}} \epsilon^{\frac{3}{5}-\frac{2}{5}a} \mathcal{D}_2 (\tau). \tag{3.68}
\]

Integrating (3.68) with respect to \( \tau \) and then multiplying the resulting equation by \( \epsilon^{2(1-a)} \) with \( a \in [\frac{3}{5}, 1] \), we can obtain
\[
\epsilon^{2(1-a)} \sum_{|\alpha|=2} \left( \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|_{2}^2 + \| \partial^\alpha f \|_{\sigma}^2 \right) + \epsilon^{1-a} \sum_{|\alpha|=2} \int_0^\tau \| \partial^\alpha f \|_{\sigma}^2 \, ds \\
\leq C (\eta + \eta_0) \epsilon^{1-a} \sum_{|\alpha|=2} \int_0^\tau \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|_{2}^2 \, ds + C \eta k^{\frac{1}{12}} \epsilon^{\frac{6}{5}-\frac{4}{5}a} \\
+ C \eta k^{\frac{1}{12}} \epsilon^{\frac{3}{5}-\frac{2}{5}a} \int_0^\tau \mathcal{D}_2 (s) \, ds. \tag{3.69}
\]

Here we used \( F = M + \tilde{G} + \sqrt{\mu} f \), (3.2), (3.61), (3.3), (3.4) and the Sobolev imbedding inequality to get
\[
\epsilon^{2(1-a)} \sum_{|\alpha|=2} \left( \| \partial^\alpha F (0) \|_{\sqrt{\mu}}^2 \right) \leq C k^{\frac{1}{5}} \epsilon^{\frac{6}{5}-\frac{4}{5}a},
\]
and
\[
\epsilon^{2(1-a)} \sum_{|\alpha|=2} \left( \| \partial^\alpha F (\tau) \|_{\sqrt{\mu}}^2 \right) \geq C \epsilon^{2(1-a)} \sum_{|\alpha|=2} \left( \| \partial^\alpha (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|_{2}^2 + \| \partial^\alpha f \|_{\sigma}^2 \right) - C k^{\frac{1}{5}} \epsilon^{\frac{6}{5}-\frac{4}{5}a}.
\]

By a suitable linear combination of (3.58) and (3.69), we can obtain (3.32) by choosing \( \eta \) and \( \eta_0 \) small enough. This then completes the proof of Lemma 3.2. \( \square \)
3.3. Weighted Energy Estimates

In this subsection, we will derive the weighted mixed derivative estimates of the function $f$ in order to close a priori estimates.

Lemma 3.3. Let (3.3), (3.4) and (3.5) hold, then one has

$$\sum_{|\alpha|+|\beta|\leq 2, |\beta|\geq 1} \left\| \partial_\alpha^\beta f (\tau) \right\|_{L^2, |\beta|}^2 + \varepsilon a^{-1} \int_0^\tau \left\| \partial_\alpha^\beta f (s) \right\|_{L^2, \sigma, |\beta|}^2 \, ds \right\|_{L^2, |\beta|}^2 + C \varepsilon^{1-a}$$

$$\leq C \varepsilon a^{-1} \int_0^\tau \left\| \partial_\alpha^\beta f (s) \right\|_{L^2, |\beta|}^2 \, ds + C \varepsilon^{1-a}$$

$$\sum_{1\leq |\alpha|\leq 2} \int_0^\tau \left\{ \| \partial_\alpha^\beta f (s) \|_{L^2}^2 + \| \partial_\alpha^\beta (\tilde{\rho}, \tilde{u}, \tilde{\theta}) (s) \|_{L^2}^2 \right\} \, ds$$

$$+ C k^2 \varepsilon \frac{2}{3} \varepsilon + C \left( \eta_0 + k \frac{1}{2} \varepsilon \frac{2}{3} \right) \int_0^\tau D_2 (s) \, ds$$

for any $\tau \in [0, \tau_1]$.

Proof. Applying $\partial_\beta^\alpha f$ to (2.10) with $|\alpha| + |\beta| \leq 2$ and $|\beta| \geq 1$, for $e_1 = (1, 0, 0)$, one has

$$\partial_\beta^\alpha f_\tau + v_1 \partial_\beta^\alpha f_y + C_{\beta}^\beta \partial_\beta^{\beta-e_1} f_y - \varepsilon a^{-1} \partial_\beta^\alpha \mathcal{L} f$$

$$= e a^{-1} \partial_\beta^\alpha \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right) + e a^{-1} \partial_\beta^\alpha \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right)$$

$$+ \varepsilon a^{-1} \partial_\beta^\alpha \Gamma \left( G \frac{G}{\sqrt{\mu}}, f \right) + \partial_\beta^\alpha \left\{ \frac{P_0 (v_1 \sqrt{\mu} f)}{\sqrt{\mu}} \right\}$$

$$- \partial_\beta^\alpha \left\{ \frac{1}{\sqrt{\mu}} P_1 v_1 M \left( \frac{|v - u|^2 \tilde{\gamma}_y}{2 R^2} + \frac{(v - u) \cdot \tilde{n}_y}{R^2} \right) \right\}$$

$$- \partial_\beta^\alpha \left\{ \frac{P_1 (v_1 \tilde{G}_y)}{\sqrt{\mu}} \right\} - \partial_\beta^\alpha \left\{ \tilde{G}_\tau \right\}.$$  \hspace{1cm} (3.71)

We take the inner product of (3.71) with $u^{2|\beta|} \partial_\beta^\alpha f$ over $\mathbb{R}_y \times \mathbb{R}_v^3$ and estimate each term. First of all, by the integration by parts, we obtain

$$\left( \partial_\beta^\alpha f_\tau + v_1 \partial_\beta^\alpha f_y, u^{2|\beta|} \partial_\beta^\alpha f \right) = \frac{1}{2} \frac{d}{d\tau} \left\| \partial_\beta^\alpha f \right\|_{L^2, |\beta|}^2.$$  \hspace{1cm} (3.72)

From the Hölder inequality and Cauchy inequality, it follows that

$$\left| \left( \partial_\beta^{\beta-e_1} f_y, u^{2|\beta|} \partial_\beta^\alpha f \right) \right| \leq C \left\| u^{2^{|\beta|-1}} \partial_\beta^{\beta-e_1} f_y \right\| \left\| u^{2|\beta|} \partial_\beta^\alpha f \right\|$$

$$= C \left\| u^{2^{|\beta|-1}} \partial_\beta^{\beta-e_1} f_y \right\| \left\| u^{2^{|\beta|}} \partial_\beta^\alpha f \right\|$$

$$\leq \eta \varepsilon a^{-1} \left\| \partial_\beta^\alpha f \right\|_{L^2, |\beta|}^2 + C \eta \varepsilon^{1-a} \left\| \partial_\beta^{\beta-e_1} f_y \right\|_{L^2, |\beta-e_1|}^2.$$  \hspace{1cm} (3.73)
Here we have used the fact that $|\beta - \epsilon| = |\beta| - 1$ and $\|w^{\frac{1}{2}} u^{|\beta|} \partial_y^a f\| \leq C \|\partial_{\frac{\mu}{\sqrt{\mu}}}^2 f\|_{\sigma,|\beta|}$ for $w = \langle v \rangle^{y+2}$ by (2.14). Due to (5.6), we can see that

\[
- \epsilon^{-1} \left| \left( \partial_{\frac{\mu}{\sqrt{\mu}}} \left( \frac{M - \mu}{\sqrt{\mu}}, f \right), w^{2|\beta|} \partial \partial_y^a f \right) \right| \geq \epsilon^{-1} \left| \left( \partial_{\frac{\mu}{\sqrt{\mu}}} \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), w^{2|\beta|} \partial \partial_y^a f \right) \right|
\]

With (5.9) and (5.18) in hand, one can show that

\[
\epsilon^{-1} \left| \left( \partial_{\frac{\mu}{\sqrt{\mu}}} \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), w^{2|\beta|} \partial \partial_y^a f \right) \right| \leq C \eta \epsilon^{-1} \|\partial_{\frac{\mu}{\sqrt{\mu}}}^2 f\|_{\sigma,|\beta|}^2 + C_{\eta} \left( \eta_0 + k^{\frac{1}{2}} \epsilon^{\frac{3}{2} - \frac{2}{\gamma}} \right) D_2(\tau),
\]

and

\[
\epsilon^{-1} \left| \left( \partial_{\frac{\mu}{\sqrt{\mu}}} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), w^{2|\beta|} \partial \partial_y^a f \right) \right| \leq C \eta \epsilon^{-1} \|\partial_{\frac{\mu}{\sqrt{\mu}}}^2 f\|_{\sigma,|\beta|}^2 + C_{\eta} \epsilon^{\frac{7}{2} + \frac{1}{\gamma} + a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C_{\eta} k^{\frac{1}{2}} \epsilon^{\frac{3}{2} - \frac{2}{\gamma}} D_2(\tau).
\]

By using (1.9), (3.5), (2.14), the Sobolev imbedding inequality, Lemma 5.3, (3.3) and (3.4), we can obtain

\[
\left| \left( \partial_{\frac{\mu}{\sqrt{\mu}}} \left( \frac{P_0(v_1 \sqrt{\mu} f_y)}{\sqrt{\mu}} \right), w^{2|\beta|} \partial \partial_y^a f \right) \right| \leq \left\| \sum_{j=0}^4 \left( \langle v \rangle^{\frac{1}{2}} w^{\frac{1}{2} |\beta|} \partial \partial_y^a f \left( \langle v_1 \sqrt{\mu} f_y, \frac{X_j}{M} \right), \langle v \rangle^{\frac{1}{2} |\beta|} \partial \partial_y^a f \right) \right\| \leq \eta \epsilon^{-1} \|\partial_{\frac{\mu}{\sqrt{\mu}}}^2 f\|_{\sigma,|\beta|} + C_{\eta} \epsilon^{1-a} \|\partial_y f_y\|_{\sigma}^2
\]

\[
+ C_{\eta} \epsilon^{\frac{7}{2} + \frac{1}{\gamma} + a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C_{\eta} k^{\frac{1}{2}} \epsilon^{\frac{3}{2} - \frac{2}{\gamma}} D_2(\tau),
\]

and

\[
\left| \left( \partial_{\frac{\mu}{\sqrt{\mu}}} \left( \frac{1}{\sqrt{\mu}} v_1 M \left\{ \frac{v - u}{2 R^2} \partial^2 \gamma + \frac{(v - u) \cdot \tilde{u}_y}{R \theta^2} \right\} \right), w^{2|\beta|} \partial \partial_y^a f \right) \right| \leq C \left\| \langle v \rangle^{\frac{1}{2}} w^{\frac{1}{2} |\beta|} \partial \partial_y^a f \left( \frac{1}{\sqrt{\mu}} v_1 M \left\{ \frac{v - u}{2 R^2} \partial^2 \gamma + \frac{(v - u) \cdot \tilde{u}_y}{R \theta^2} \right\} \right) \right\| \leq \eta \epsilon^{-1} \|\partial_{\frac{\mu}{\sqrt{\mu}}}^2 f\|_{\sigma,|\beta|} + C_{\eta} \epsilon^{1-a} \|\partial_y \left[ \tilde{u}_y, \tilde{\gamma}_y \right]\|_{\sigma}^2
\]

\[
+ C_{\eta} \epsilon^{\frac{7}{2} + \frac{1}{\gamma} + a} (\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C_{\eta} k^{\frac{1}{2}} \epsilon^{\frac{3}{2} - \frac{2}{\gamma}} D_2(\tau),
\]

where we have used the fact that $\langle v \rangle^m \mu^{-\frac{1}{2}} \partial \partial_y M |_{2} \leq C$ for any $m \geq 0$ and $\beta \geq 0$ by (3.5).
Notice that the last two terms of (3.71) can be dominated by

\[
\left| \left( \partial_\beta^a \left( \frac{P_1(v_1 \tilde{G}_y)}{\sqrt{\mu}} \right), w^{2|\beta|} \partial_\beta^a f \right) \right|
\leq C \left( \| v \|^{1/2} |w|^{|\beta|} \partial_\beta^a \left( \frac{P_1(v_1 \tilde{G}_y)}{\sqrt{\mu}} \right) \right) + \| v \|^{1/2} |w|^{|\beta|} \partial_\beta^a f \| \| v \|^{-1/2} \| w \|^{2|\beta|} \partial_\beta^a f \|
\leq \eta e^{a-1} \| \partial_\beta^a f \|_{\sigma,|\beta|}^2 + C \eta e^{\frac{7}{15} + \frac{4}{15} a}(\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C \eta k \frac{1}{|\alpha|} \epsilon^\frac{3}{5} \tilde{a} D_2(\tau),
\]

given by (1.9), (5.25), (2.14), Lemma 5.3 and the elementary inequalities.

Hence, for \(|\alpha| + |\beta| \leq 2 \) and \(|\beta| \geq 1\), we have by the above related estimates and \(\eta > 0\) small enough that

\[
\frac{1}{2} \frac{d}{d\tau} \| \partial_\beta^a f \|_{2,|\beta|}^2 + C \epsilon^{a-1} \| \partial_\beta^a f \|_{\sigma,|\beta|}^2
\leq C \epsilon^{a-1} \| \partial_\beta^a f \|_{\sigma}^2 + C \epsilon^{1-a} \left\{ \| \partial_\beta^a f_y \|_{\sigma}^2 + \| \partial_\beta^a [\tilde{u}_y, \tilde{\theta}_y] \|^2 \right\} + C \epsilon^{1-a} \| \partial_\beta^a f_y \|_{\sigma,|\beta|-1}^2 + C \epsilon^{a-1} \sum_{|\beta| < 1} \| \partial_\beta^a f \|_{\sigma,|\beta|}^2
\leq C \epsilon^{a-1} \sum_{|\alpha| \leq 1} \| \partial_\beta^a f \|_{\sigma}^2 + C \epsilon^{1-a} \sum_{1 \leq |\alpha| \leq 2} \left\{ \| \partial_\beta^a f \|_{\sigma}^2 + \| \partial_\beta^a (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 \right\} + C \epsilon^{\frac{7}{5} + \frac{4}{15} a}(\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C \left( \eta_0 + k \frac{1}{|\alpha|} \epsilon^\frac{3}{5} \tilde{a} \right) D_2(\tau). \tag{3.72}
\]

Notice that the coefficients on the third line of (3.72) is large and \(|\beta| < |\beta|\). We will use the induction in \(|\beta|\) to control these terms. By the suitable linear combinations, we can obtain

\[
\frac{1}{2} \frac{d}{d\tau} \| \partial_\beta^a f \|_{2,|\beta|}^2 + C \epsilon^{a-1} \| \partial_\beta^a f \|_{\sigma,|\beta|}^2
\leq C \epsilon^{a-1} \sum_{|\alpha| \leq 1} \| \partial_\beta^a f \|_{\sigma}^2 + C \epsilon^{1-a} \sum_{1 \leq |\alpha| \leq 2} \left\{ \| \partial_\beta^a f \|_{\sigma}^2 + \| \partial_\beta^a (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \|^2 \right\} + C \epsilon^{\frac{7}{5} + \frac{4}{15} a}(\delta + \epsilon^a \tau)^{-\frac{4}{3}} + C \left( \eta_0 + k \frac{1}{|\alpha|} \epsilon^\frac{3}{5} \tilde{a} \right) D_2(\tau). \tag{3.73}
\]

Here we have required that

\[
\epsilon^{a-1} \geq \epsilon^{1-a}, \quad \text{that is} \quad a \leq 1. \tag{3.74}
\]

Integrating (3.73) with respect to \(\tau\) and using (3.2) as well as (3.3) gives (3.70). We consequently finish the proof of Lemma 3.3. \(\square\)

4. Stability and Convergence Rate

Based on the energy estimates derived in Lemmas 3.1, 3.2 and 3.3, in this section we are now in a position to complete the
Proof of Theorem 1.1. By a suitable linear combination of (3.6), (3.32) and (3.70), we can obtain

\[
E_2(\tau) + \int_0^\tau \| \sqrt{u_{1y}}(\tilde{\rho}, \tilde{u}_1, \tilde{\theta}) \|^2 \, ds + \int_0^\tau D_2(s) \, ds \\
\leq C k^{\frac{1}{2}} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a + C \left( \eta_0 + k \frac{1}{12} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a + k \frac{1}{12} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a \right) \int_0^\tau D_2(s) \, ds. \tag{4.1}
\]

Here $E_2(\tau)$ and $D_2(\tau)$ are defined by (2.15) and (2.16), respectively. At the moment, one has to require that the second term on the right hand side of (4.1) should be absorbed by the left hand side. Thus, this leads us to impose

\[
\frac{3}{5} - \frac{2}{5} a \geq 0 \quad \text{and} \quad \frac{3}{5} a - \frac{2}{5} \geq 0, \quad \text{that is} \quad \frac{2}{3} \leq a \leq \frac{3}{2}.
\tag{4.2}
\]

Due to (3.74) and (4.2), we need to require that $2 \frac{3}{5} \leq a \leq 1$ for the choice of the parameter $a$ in the scaling transformation (2.1) which we start with. Hence, by using the smallness of $k > 0$, $\eta_0 > 0$ and $\varepsilon > 0$, we have from (4.1) that

\[
E_2(\tau) + \int_0^\tau \| \sqrt{u_{1y}}(\tilde{\rho}, \tilde{u}_1, \tilde{\theta}) \|^2 \, ds + \frac{1}{2} \int_0^\tau D_2(s) \, ds \leq C k^{\frac{1}{2}} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a < \frac{1}{2} k^{\frac{1}{2}} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a. \tag{4.3}
\]

Then (4.3) implies that for $a \in [\frac{2}{3}, 1]$ and $\tau_1 \in (0, +\infty)$, one has

\[
\sup_{0 \leq \tau \leq \tau_1} E_2(\tau) < \frac{1}{2} k^{\frac{1}{2}} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a, \tag{4.4}
\]

which is strictly stronger than (3.4). Thus the a priori assumption (3.4) can be closed. Therefore, by the uniform a priori estimates and the local existence of the solution, the standard continuity argument gives the existence and uniqueness of global solutions to the Landau equation (1.1) with initial data (3.1). Moreover, the desired estimate (1.29) holds true.

We are going to justify the convergence rate as in (1.30). By (4.4), (2.15) and the Sobolev imbedding inequality, we get

\[
\sup_{0 \leq \tau \leq +\infty} \left\{ \| (\tilde{\rho}, \tilde{u}, \tilde{\theta}) (\tau) \|_{L^\infty_y} + \| f(\tau) \|_{L^\infty_y L^1_v} \right\} \leq C k^{\frac{1}{2}} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a. \tag{4.5}
\]

On the other hand, we have by using (5.24), Lemma 5.3 and $\delta = \frac{1}{k} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a$ with $2 \frac{3}{5} \leq a \leq 1$ that

\[
\sup_{0 \leq \tau \leq +\infty} \left\| \frac{G(\tau)}{\sqrt{\mu}} \right\|_{L^\infty_y L^2_v} \leq C \varepsilon^{1-a} \sup_{0 \leq \tau \leq +\infty} \| \tilde{u}_{1y}, \tilde{\theta}_y \|_{L^\infty_y} \leq C k^{\frac{1}{2}} \varepsilon^{\frac{3}{5}} - \frac{2}{5} a \tag{4.6}
\]
It follows from (4.5) and (4.6) that
\[
\sup_{0 \leq \tau \leq +\infty} \left\| \frac{F - M_{\bar{\rho}, \bar{u}, \bar{\theta}}}{\sqrt{\mu}} \right\|_{L_x^\infty L_v^2} \\
\leq C \sup_{0 \leq \tau \leq +\infty} \left\{ \left\| \frac{M - M_{\bar{\rho}, \bar{u}, \bar{\theta}}}{\sqrt{\mu}} \right\|_{L_x^\infty L_v^2} + \left\| f \right\|_{L_x^\infty L_v^2} + \left\| \frac{G}{\sqrt{\mu}} \right\|_{L_x^\infty L_v^2} \right\} \\
\leq Ck^1 \varepsilon^{3 - \frac{2}{5}a},
\]
where we have used the facts that \( F = M + \overline{G} + \sqrt{\mu} f \) and (3.5). By Lemma 5.2 and \( \delta = \frac{1}{k} \varepsilon^{\frac{3}{5} - \frac{2}{5}a} \) with \( k \) independent of \( \varepsilon \) satisfying \( \varepsilon \ll k \), we have for \( t > 0 \) that
\[
\left\| (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) - (\rho_R, u_R, \theta_R) \left( \frac{x}{t} \right) \right\|_{L_x^\infty} \leq C \frac{1}{k} t^{-1} \varepsilon^{\frac{3}{5} - \frac{2}{5}a} [\ln(1 + t) + |\ln \varepsilon|].
\]

With (4.7) and (4.8) in hand, for any given constant \( l > 0 \) and all \( t \in [l, +\infty) \), there exists a constant \( C_{l, k} > 0 \), independent of \( \varepsilon \), such that
\[
\left\| \frac{F(t, x, v) - M_{[\rho^R, u^R, \theta^R]}(x/t)(v)}{\sqrt{\mu}} \right\|_{L_x^\infty L_v^2} \\
\leq \left\| \frac{F - M_{\bar{\rho}, \bar{u}, \bar{\theta}}}{\sqrt{\mu}} \right\|_{L_x^\infty L_v^2} + \left\| M_{[\bar{\rho}, \bar{u}, \bar{\theta}]} - M_{[\rho^R, u^R, \theta^R]} \right\|_{L_x^\infty L_v^2} \\
\leq C_{l, k} \varepsilon^{\frac{3}{5} - \frac{2}{5}a} [\ln \varepsilon].
\]

This gives (1.30) and then completes the proof of Theorem 1.1. \( \square \)

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5. Appendix

In this section, we will give some basic estimates, which have been used in the previous energy estimates. We first list some properties for the rarefaction wave defined by (1.22) and (1.23). Then, we give some properties of the Burnett
functions and the fast velocity decay of $\bar{G}$ to overcome the difficulties due to the term involving $L_M^{-1}$ and the slow time decay of the term $||[\tilde{u}_y, \tilde{\theta}_y]||^2$. Lastly, we recall some basic properties of the collision operators and prove some linear and nonlinear estimates in the previous energy analysis.

We now give the properties of the solution $\omega_\delta(t, x)$ to Burgers equation (1.22) and the smooth approximate 3-rarefaction wave $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x)$ constructed by (1.23). The properties of $\omega_\delta(t, x)$ can be proved by the characteristic method, (cf. [24,32,45]).

**Lemma 5.1.** The Burgers equation (1.22) has a unique smooth global solution $\omega_\delta(t, x)$ such that

1. $\omega_- < \omega_\delta(t, x) < \omega_+$, $\partial_x \omega_\delta(t, x) > 0$, $\forall \ x \in \mathbb{R}$, $t \geq 0$.
2. The following estimates hold for any $t > 0$, $\delta > 0$ and $p \in [1, +\infty]$:
\[
\|\partial_x \omega_\delta(t, x)\|_{L^p(\mathbb{R}_x)} \leq C(\delta + t)^{-1 + \frac{1}{p}}, \\
\|\partial_x^j \omega_\delta(t, x)\|_{L^p(\mathbb{R}_x)} \leq C \delta^{-j + 1 + \frac{1}{p}}(\delta + t)^{-1}, \ j \geq 2.
\]
3. There exists a constant $\delta_0 \in (0, 1)$ such that, for $\delta \in (0, \delta_0)$ and $t > 0$,
\[
\|\omega_\delta(t, x) - \omega_R^R(\frac{x}{t})\|_{L^\infty(\mathbb{R}_x)} \leq C \delta t^{-1}[\ln(1 + t) + |\ln \delta|].
\]

The properties of $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x)$ can be found in [45, Section 2] as well as [24, Lemma 2.3].

**Lemma 5.2.** Let $\delta := |\rho_+ - \rho_-| + |u_+ - u_-| + |\theta_+ - \theta_-|$ is the wave strength of the approximate 3-rarefaction wave $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x)$ defined in (1.23), then $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x)$ is smooth in $t$ and $x$, and the following things holds true:

1. $\tilde{u}_2 = \tilde{u}_3 = 0$, $\tilde{u}_{1x} > 0$, $\rho_- < \tilde{\rho} < \rho_+$, and $\tilde{\theta}_x = \sqrt{\frac{2}{3}}\tilde{\theta}_x^1 \tilde{u}_{1x}$, $\forall \ x \in \mathbb{R}$, $t \geq 0$.
2. The following estimates hold for any $t > 0$, $\delta > 0$ and $p \in [1, +\infty]$:
\[
\|\partial_x (\tilde{\rho}, \tilde{u}_1, \tilde{\theta}')(t, x)\|_{L^p(\mathbb{R}_x)} \leq C(\delta + t)^{-1 + \frac{1}{p}}, \\
\|\partial_x^j (\tilde{\rho}, \tilde{u}_1, \tilde{\theta}')(t, x)\|_{L^p(\mathbb{R}_x)} \leq C \delta^{-j + 1 + \frac{1}{p}}(\delta + t)^{-1}, \ j \geq 2.
\]
3. There exists a constant $\delta_0 \in (0, 1)$ such that, for $\delta \in (0, \delta_0)$ and $t > 0$,
\[
\|(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t, x) - (\rho_R^R, u_R^R, \theta_R)(\frac{x}{t})\|_{L^\infty(\mathbb{R}_x)} \leq C \delta t^{-1}[\ln(1 + t) + |\ln \delta|].
\]

Since the scaling transformation $y = \varepsilon^{-a}x$ and $\tau = \varepsilon^{-a} t$ is considered through the proof, the following lemma is equivalent to Lemma 5.2 (ii), which will be used frequently in the previous energy estimates.
Lemma 5.3. Let \((\bar{\rho}, \bar{u}, \bar{\theta})(t, x)\) be the smooth approximate 3-rarefaction wave defined in (1.23), then it holds that
\[
\|\partial_y (\bar{\rho}, \bar{u}, \bar{\theta})(\varepsilon^a \tau, \varepsilon^a y)\|_{L^p(R_y)} \leq C \varepsilon^{a(1-\frac{1}{p})}(\delta + \varepsilon^a \tau)^{-1+\frac{1}{p}},
\]
\[
\|\partial_y^j (\bar{\rho}, \bar{u}, \bar{\theta})(\varepsilon^a \tau, \varepsilon^a y)\|_{L^p(R_y)} \leq C \varepsilon^{a(j-\frac{1}{p})}\delta^{-j+1+\frac{1}{p}}(\delta + \varepsilon^a \tau)^{-1}, \quad j \geq 2,
\]
for any \(\tau > 0, \delta > 0\) and \(p \in [1, +\infty]\).

We remark that the temporal derivatives of \((\bar{\rho}, \bar{u}, \bar{\theta})(t, x)\) in Lemma 5.2 (ii) and 5.3 obviously hold in terms of Euler system (1.24) and the elementary inequalities.

Recall the Burnett functions, cf. [3, 4, 8, 20, 46], defined as
\[
\hat{A}_j(v) = \frac{|v|^2 - 5}{2} v_j \quad \text{and} \quad \hat{B}_{ij}(v) = v_i v_j - \frac{1}{3} \delta_{ij} |v|^2 \quad \text{for} \quad i, j = 1, 2, 3.
\]

Noting that \(\hat{A}_j\left(\frac{v-u}{\sqrt{R\theta}}\right)M\) and \(\hat{B}_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)M\) are orthogonal to the null space \(\mathcal{N}\) of \(L_M\), we can define functions \(A_j\left(\frac{v-u}{\sqrt{R\theta}}\right)\) and \(B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\) such that \(P_0 A_j = 0, \ P_0 B_{ij} = 0\) and
\[
A_j\left(\frac{v-u}{\sqrt{R\theta}}\right) = L_M^{-1} \left[ \hat{A}_j\left(\frac{v-u}{\sqrt{R\theta}}\right) M \right]
\]
\[
B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right) = L_M^{-1} \left[ \hat{B}_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right) M \right].
\]

The following some elementary but important properties of the Burnett functions is borrowed from [12, Lemma 6.2]. Readers also refer to [3, 4, 8, 20, 46]:

Lemma 5.4. The Burnett functions have the following properties:
- \(\langle \hat{A}_i, A_i \rangle\) is positive and independent of \(i\);
- \(\langle \hat{A}_i, A_j \rangle = 0\) for any \(i \neq j\); \(\langle \hat{A}_i, B_{jk} \rangle = 0\) for any \(i, j, k\);
- \(\langle \hat{B}_{ij}, B_{kl} \rangle = \langle \hat{B}_{kl}, B_{ij} \rangle = \langle \hat{B}_{ji}, B_{kj} \rangle\), which is independent of \(i, j\) for fixed \(k, l\);
- \(\langle \hat{B}_{ij}, B_{ij} \rangle\) is positive and independent of \(i, j\) when \(i \neq j\);
- \(\langle \hat{B}_{ii}, B_{jj} \rangle\) is positive and independent of \(i, j\) when \(i = j\);
- \(\langle \hat{B}_{ij}, B_{ij} \rangle\) is positive and independent of \(i\);
- \(\langle \hat{B}_{ij}, B_{kl} \rangle = 0\) unless either \((i, j) = (k, l)\) or \((l, k)\), or \(i=j\) and \(k=l\);
- \(\langle \hat{B}_{ii}, B_{ij} \rangle = 2\langle \hat{B}_{ij}, B_{ii} \rangle\) holds for any \(i \neq j\).

In terms of the properties of Burnett functions, the viscosity coefficient \(\mu(\theta)\) and heat conductivity coefficient \(\kappa(\theta)\) can be represented by
\[
\mu(\theta) = -R\theta \int_{\mathbb{R}^3} \hat{B}_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right) B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right) \, dv > 0, \quad i \neq j,
\]
\[
\kappa(\theta) = -R^2\theta \int_{\mathbb{R}^3} \hat{A}_j\left(\frac{v-u}{\sqrt{R\theta}}\right) A_j\left(\frac{v-u}{\sqrt{R\theta}}\right) \, dv > 0.
\]
Lemma 5.5. Suppose that $U(v)$ is any polynomial of $\frac{v-\hat{\mu}}{\sqrt{R\theta}}$ such that $U(v)\hat{M} \in (\ker L_\hat{M})^\perp$ for any Maxwellian $\hat{M} = M_{[\hat{\rho}, \hat{\mu}, \hat{\theta}]}(v)$ where $L_\hat{M}$ is as (1.14). For any $\varepsilon \in (0, 1)$ and any multi-index $\beta$, there exists constant $C_\beta > 0$ such that

$$|\partial_\beta L^{-1}_\hat{M}(U(v)\hat{M})| \leq C_\beta \hat{M}^{1-\varepsilon}.$$  

In particular, if the assumptions of (3.5) hold, there exists constant $C_\beta > 0$ such that

$$\left|\partial_\beta A_j \left(\frac{v-\mu}{\sqrt{R\theta}}\right)\right| + \left|\partial_\beta B_{ij} \left(\frac{v-\mu}{\sqrt{R\theta}}\right)\right| \leq C_\beta M^{1-\varepsilon}.$$  

In what follows we shall turn to summarize some refined estimates for the collision operators $L$ and $\Gamma_1$ defined as (2.7). We first recall the properties of the linearized operators $L$. Note that the null space $N_1$ of $L$ is spanned by the functions $\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$ in $[10, 21]$. Moreover, for any $g \in N_1^\perp$, there exists $\sigma_1 > 0$ such that

$$-\langle Lg, g \rangle \geq \sigma_1 |g|^2.$$  

We also recall the weighted coercivity estimates on the linearized operators $L$.

Lemma 5.6. Let $|\beta| > 0$ and $w$ defined in (2.12). Then for any $\eta > 0$, there exists $C_\eta > 0$ such that

$$-\langle \partial_\beta Lg, w^{2|\beta|}\partial_\beta g \rangle \geq |\partial_\beta g|_\sigma^2|\beta| - \eta \sum_{|\beta_1| \leq |\beta|} |\partial_{\beta_1} g|_{\sigma,|\beta_1|} - C_\eta |g|^2.$$  

Proof. The proof of (5.6) can be found in [21] and we omit the proof here for brevity.

In addition, the weighted estimates on the nonlinear collision operators $\Gamma$ can be stated as follows.

Lemma 5.7. Let $w$ defined in (2.12) and $\ell \geq 0$, for arbitrarily large constant $b > 0$, one has

$$|\langle \partial_\beta \Gamma(g_1, g_2), \partial_\beta g_3 \rangle| \leq C \sum_{|\alpha_1| \leq |\alpha|} |\langle v \rangle^{-b}\partial^{\alpha_1} g_1|_2 |\partial^{\alpha-\alpha_1} g_2|_\sigma |\partial^{\alpha} g_3|_\sigma,$$  

and

$$|\langle \partial_\beta \Gamma(g_1, g_2), w^{2\ell}\partial_\beta g_3 \rangle| \leq C \sum_{|\alpha_1| \leq |\alpha|} \sum_{|\alpha'| \leq |\beta|} \sum_{|\beta_1| \leq |\beta|} |\langle v \rangle^{-b}\partial^{\alpha_1} g_1|_2 |\partial^{\alpha-\alpha_1} g_2|_\sigma, |\partial^{\alpha'} g_3|_\sigma, |\partial^{\alpha} g_3|_\sigma, |\partial_\beta g_3|_\sigma,.$$  

Proof. The proof of (5.7) and (5.8) can be found in [41, Proposition 1].

Finally, we prove some linear and nonlinear estimates, which are used in Section 3. The first estimates involving the linear terms $\Gamma(M-\mu, f)$ and $\Gamma(f, M-\mu).$
\textbf{Lemma 5.8.} Let $|\alpha| + |\beta| \leq 2$ with $|\beta| \geq 1$ and $w$ defined in (2.12). Suppose that (3.4), (3.3) and (3.5) hold. If we choose $\eta_0$ in (3.5) and $k$ in (3.3) small enough, for any $\eta > 0$, we get

$$\varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right), w^2 |\beta| \partial^a h \right) \right| + \varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), w^2 |\beta| \partial^a h \right) \right| \leq C \eta \varepsilon^{a-1} \| \partial^a h \|_{\sigma, |\beta|}^2 + C \eta \left( \eta_0 + k \frac{1}{\tau} \varepsilon^{3-\frac{2}{3}a} \right) D_2(\tau).$$

(5.9)

Moreover, for $|\alpha| \leq 1$, one has

$$\varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right), \partial^a h \right) \right| + \varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( f, \frac{M - \mu}{\sqrt{\mu}} \right), \partial^a h \right) \right| \leq C \eta \varepsilon^{a-1} \| \partial^a h \|_{\sigma}^2 + C \eta \left( \eta_0 + k \frac{1}{\tau} \varepsilon^{3-\frac{2}{3}a} \right) D_2(\tau).$$

(5.10)

\textbf{Proof.} We only consider the first term on the left-hand side of (5.9) and the other term can be handled in the same way. First of all, we have from (5.8) that

$$\varepsilon^{a-1} \left| \left( \partial^a \Gamma \left( \frac{M - \mu}{\sqrt{\mu}}, f \right), w^2 |\beta| \partial^a h \right) \right| \leq C \varepsilon^{a-1} \sum_{|\alpha| \leq |\alpha|} \sum_{|\beta| \leq |\beta|} \int_{\mathbb{R}^3} |\langle v \rangle^{-b} \partial^{a_1}_\mu \left( \frac{M - \mu}{\sqrt{\mu}} \right) |_2 |\partial^{a_1}_\mu f| |\alpha| \partial^{a_1}_\mu h|_{\sigma, |\beta|} \, dv. \quad (5.11)$$

due to the fact that $w^2 |\beta| \leq w^2 |\beta - \beta'|$ for $|\beta - \beta'| \leq |\beta|$. Recalling that (2.14), then for any $\bar{\beta} \geq 0$ and $m > 0$, there exists a small constant $\varepsilon_1 > 0$ such that

$$\left| \langle v \rangle^m \partial^a \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right|_{\sigma}^2 + \left| \langle v \rangle^m \partial^a \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right|_{2}^2 \leq C_m \sum_{|\bar{\beta}| \leq |\beta|, |\beta'| \leq |\bar{\beta}| + 1} \int_{\mathbb{R}^3} \mu^{-\varepsilon_1} |\partial^{a'} \left( \frac{M - \mu}{\sqrt{\mu}} \right) |^2 \, dv.$$

For $\eta_0 > 0$ in (3.5), there exists a suitably large constant $R > 0$ such that

$$\int_{|v| \geq R} \mu^{-\varepsilon_1} |\partial^{a'} \left( \frac{M - \mu}{\sqrt{\mu}} \right) |^2 \, dv \leq C \left( \eta_0 + k \frac{1}{\tau} \varepsilon^{3-\frac{2}{3}a} \right)^2.$$

We also have from (3.5) and a direct calculation that

$$\int_{|v| \leq R} \mu^{-\varepsilon_1} |\partial^{a'} \left( \frac{M - \mu}{\sqrt{\mu}} \right) |^2 \, dv \leq C \left( |\rho - 1| + |\mu - 0| + |\theta - \frac{3}{2}| \right)^2 \leq C \left( \eta_0 + k \frac{1}{\tau} \varepsilon^{3-\frac{2}{3}a} \right)^2.$$

Therefore, for any $\bar{\beta} \geq 0$ and $m > 0$, we deduce from the above three estimates that

$$\left| \langle v \rangle^m \partial^a \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right|_{\sigma}^2 + \left| \langle v \rangle^m \partial^a \left( \frac{M - \mu}{\sqrt{\mu}} \right) \right|_{2}^2 \leq C \left( \eta_0 + k \frac{1}{\tau} \varepsilon^{3-\frac{2}{3}a} \right)^2. \quad (5.12)$$
Since $|\alpha| + |\beta| \leq 2$ with $|\beta| \geq 1$, then $|\alpha| \leq 1$. It follows that $|\alpha_1| \leq |\alpha| \leq 1$ in (5.11). If $|\alpha_1| = 0$, we use (5.12), (2.16), the Cauchy–Schwarz inequality and the smallness of $\eta_1$ to get

$$
epsilon^{a-1} J_1 \leq C \left( \eta_0 + k \frac{1}{\varepsilon^{\frac{3}{5} - \frac{2}{5} a}} \right) \varepsilon^{a-1} \| \partial^{\alpha}_{\beta - \beta_1} f \|_{\sigma, |\beta - \beta_1|} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|} \\
\leq \eta \varepsilon^{a-1} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|}^2 + C_\eta \left( \eta_0 + k \frac{1}{\varepsilon^{\frac{3}{5} - \frac{2}{5} a}} \right) D_2 (\tau) .
$$

(15.13)

If $|\alpha_1| = |\alpha| = 1$, by using (3.5), the Sobolev imbedding inequality, the Cauchy–Schwarz inequality, Lemma 5.3 and (2.16), we deduce that

$$
epsilon^{a-1} J_1 \leq C \varepsilon^{a-1} \int |G^{\alpha} (\rho, u, \theta)| \| \partial^{\alpha}_{\beta - \beta_1} f \|_{\sigma, |\beta - \beta_1|} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|} \text{d} y \\
\leq \eta \varepsilon^{a-1} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|} \| \partial^{\alpha}_{\beta} (\rho, u, \theta) \| \| \partial^{\alpha}_{\beta - \beta_1} f \|_{\sigma, |\beta - \beta_1|} \\
\leq \eta \varepsilon^{a-1} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|} + C_\eta \left( \eta_0 + k \frac{1}{\varepsilon^{\frac{3}{5} - \frac{2}{5} a}} \right) D_2 (\tau) .
$$

(15.14)

Here we have used (3.4) and (3.3) such that $\mathcal{E}_2 (\tau) \leq \frac{1}{\varepsilon^{\frac{3}{5} - \frac{2}{5} a}} \leq k \frac{1}{\varepsilon^{3} - \frac{5}{2} a}$ and

$$
\varepsilon (\delta + e^a \tau)^{-1} \leq \varepsilon (\delta - 1) = k e \varepsilon^{\frac{5}{3} - \frac{3}{2} a} \leq k \frac{1}{\varepsilon^{3} - \frac{5}{2} a} .
$$

(15.15)

It follows from (5.13), (15.14) and (5.11) that

$$
\varepsilon^{a-1} \left| \left( \partial^{\alpha}_{\beta} \Gamma \left( \frac{M - \mu}{\sqrt{\mu}} , f \right) , w^{2|\beta|} \partial^{\alpha}_{\beta} h \right) \right| \\
\leq C \eta \varepsilon^{a-1} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|}^2 + C_\eta \left( \eta_0 + k \frac{1}{\varepsilon^{3} - \frac{5}{2} a} \right) D_2 (\tau) .
$$

(15.16)

On the other hand, similar arguments as (15.16) imply

$$
\varepsilon^{a-1} \left| \left( \partial^{\alpha}_{\beta} \Gamma \left( f , \frac{M - \mu}{\sqrt{\mu}} \right) , w^{2|\beta|} \partial^{\alpha}_{\beta} h \right) \right| \\
\leq C \eta \varepsilon^{a-1} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|}^2 + C_\eta \left( \eta_0 + k \frac{1}{\varepsilon^{3} - \frac{5}{2} a} \right) D_2 (\tau) .
$$

(15.17)

Estimate (5.9) thus follows from (15.16) and (15.17). By (5.7) and the similar arguments as (15.16) and (15.17), we can prove that (5.10) holds and we omit the details for brevity. This ends the proof of Lemma 5.8. \(\square\)

The second estimates are concerned with the nonlinear term $\Gamma (\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}})$.

**Lemma 5.9.** Let $|\alpha| + |\beta| \leq 2$ with $|\beta| \geq 1$ and $w$ defined in (2.12). Suppose that (3.4), (3.3) and (3.5) hold. Then for any $\eta > 0$, one has

$$
\varepsilon^{a-1} \left| \left( \partial^{\alpha}_{\beta} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) , w^{2|\beta|} \partial^{\alpha}_{\beta} h \right) \right| \\
\leq C \eta \varepsilon^{a-1} \| \partial^{\alpha}_{\beta} h \|_{\sigma, |\beta|}^2 + C_\eta \left( \eta_0 + k \frac{1}{\varepsilon^{\frac{2}{3} + \frac{1}{5} a}} \right) (\delta + e^a \tau)^{-\frac{4}{3}} + C_\eta k \frac{1}{\varepsilon^{\frac{3}{5} - \frac{2}{5} a}} D_2 (\tau) .
$$

(5.18)
Moreover, for $|\alpha| \leq 1$, it holds that
\[
\varepsilon^{a-1} \left| \partial^{a} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), \partial^{a} h \right| \leq C \eta \varepsilon^{a-1} \| \partial^{a} h \|_{\sigma}^{2} + C \eta \varepsilon^{\frac{2}{3} + \frac{1}{3} a} (\delta + \varepsilon^{a} \tau)^{-\frac{2}{3}} + C \eta \kappa \varepsilon^{\frac{2}{3} - \frac{2}{3} a} D_{2} (\tau). 
\]  
(5.19)

**Proof.** Recalling $G = \overline{G} + \sqrt{\mu} f$ by (2.2), a simple computation shows that
\[
\Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right) = \Gamma \left( \frac{\overline{G}}{\sqrt{\mu}} - \frac{\overline{G}}{\sqrt{\mu}}, f \right) + \Gamma \left( f, \frac{G}{\sqrt{\mu}} \right) + \Gamma \left( f, f \right). 
\]  
(5.20)

From (5.8), one can see that
\[
\varepsilon^{a-1} \left| \partial^{a} \Gamma \left( \frac{\overline{G}}{\sqrt{\mu}}, \frac{\overline{G}}{\sqrt{\mu}} \right), u^{2} |\beta| \partial^{a} h \right| \leq C \varepsilon^{a-1} \sum_{|\alpha| \leq |\alpha|} \sum_{|\beta| \leq |\beta|} \int_{\mathbb{R}} \left| \partial^{a} \left( \frac{\overline{G}}{\sqrt{\mu}} \right) \right|_{2,|\beta|} \left| \partial^{a-\alpha} \left( \frac{\overline{G}}{\sqrt{\mu}} \right) \right|_{|\alpha|,|\beta|} \partial^{a} h_{|\alpha|,|\beta|} \, dy. 
\]  
(5.21)

To further estimate (5.21), we use (5.1) and (5.2) to rewrite (2.3) as
\[
\overline{G} = \varepsilon^{1-a} \sqrt{R \tilde{\theta}_{y} y} \varepsilon^{1-a} \tilde{u}_{1y} B_{11} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right), 
\]  
(5.22)

which implies that, for $\beta_{1} = (1, 0, 0),
\[
\partial^{a} G = \varepsilon^{1-a} \left\{ \sqrt{R \tilde{\theta}_{y} y} \partial_{v} A_{1} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right) + \tilde{u}_{1y} \partial_{v} B_{11} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right) \right\}. 
\]  

Similarly, we also have
\[
\tilde{y} \overline{G} = \varepsilon^{1-a} \left\{ \sqrt{R \tilde{\theta}_{y} y} \partial_{v} A_{1} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right) - \sqrt{R \tilde{\theta}_{y} y} \partial_{v} A_{1} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right) \right\} 
- \sqrt{R \tilde{\theta}_{y} y} \nabla_{v} A_{1} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right) - \sqrt{R \tilde{\theta}_{y} y} \nabla_{v} A_{1} \left( \frac{v - u}{\sqrt{R \tilde{\theta}}}, \frac{1}{\sqrt{R \tilde{\theta}}} \right) \right\}. 
\]  
(5.23)

And $\partial_{\tau} \overline{G}$ has the similar expression as (5.23). By using (5.4), (3.5), (2.14) and the similar expansion as the above, for any $|\tilde{\alpha}| \geq 1$ and $|\tilde{\beta}| \geq 0$, we get
\[
\left| \langle v \rangle^{m} \partial^{\tilde{\beta}} \left( \frac{\overline{G}}{\sqrt{\mu}} \right) \right|_{2,|\tilde{\beta}|} + \left| \langle v \rangle^{m} \partial^{\tilde{\beta}} \left( \frac{\overline{G}}{\sqrt{\mu}} \right) \right|_{|\alpha|,|\tilde{\beta}|} \leq C \varepsilon^{1-a} \| \tilde{u}_{1y}, \tilde{\theta}_{y} \|. 
\]  
(5.24)
and
\[
|\langle v \rangle^m \partial_{\beta}^{\alpha}(\frac{G}{\sqrt{\mu}})|_{2,|\beta|} + |\langle v \rangle^m \partial_{\beta}^{\alpha}(\frac{G}{\sqrt{\mu}})|_{\sigma,|\beta|} \leq C \varepsilon^{1-a}(|\partial^{\alpha} [\bar{u}_{1y}, \bar{\theta}_y]| + \cdots + |[\bar{u}_{1y}, \bar{\theta}_y]| |\partial^{\alpha} [u, \theta]|),
\]
(5.25)
due to the fact that $|\langle v \rangle^m w|^{\varepsilon} \mu^{1-\varepsilon} |_{2} \leq C$ for any $m \geq 0$ and $\varepsilon > 0$ small enough.

Since $|\alpha| + |\beta| \leq 2$ with $|\beta| \geq 1$, then $|\alpha| \leq 1$. It follows that $|\alpha_1| \leq |\alpha| \leq 1$ in (5.21). With the help of (5.24) and (5.25), we get from the Sobolev imbedding inequality, the Cauchy–Schwarz inequality, Lemma 5.3 and (2.16) that
\[
e^{a-1} \int_{\mathbb{R}} \left| \partial_{\beta}^{\alpha_1}(\frac{G}{\sqrt{\mu}}) \right|_{2,|\beta'|} \left| \partial_{\beta-\beta_1}^{\alpha-\alpha_1}(\frac{G}{\sqrt{\mu}}) \right|_{\sigma,|\beta-\beta_1|} |\partial_{\beta}^{\alpha} f|_{\sigma,|\beta|} \, dy
\]
\[
\leq C \varepsilon^{a-1} \int_{\mathbb{R}} \varepsilon^{1-a} (|\partial^{\alpha_1} [\bar{u}_{1y}, \bar{\theta}_y]| + |[\bar{u}_{1y}, \bar{\theta}_y]| |\partial^{\alpha} [u, \theta]|) \times \varepsilon^{1-a} (|\partial^{\alpha-\alpha_1} [\bar{u}_{1y}, \bar{\theta}_y]| + |[\bar{u}_{1y}, \bar{\theta}_y]| |\partial^{\alpha_1} [u, \theta]|) \|\partial_{\beta}^{\alpha} h\|_{\sigma,|\beta|} \, dy
\]
\[
\leq C \eta \varepsilon^{a-1} \|\partial_{\beta}^{\alpha} h\|_{\alpha,|\beta|}^2 + C \eta \varepsilon^{\frac{3}{2} + \frac{1}{4} a} (\delta + \varepsilon^{a} \tau)^{-\frac{3}{4}} + C \eta \varepsilon^{\frac{3}{2} + \frac{3}{2} a} D_{2}(\tau),
\]
(5.26)
according to (3.3) and (3.4). This, together with (5.21), gives that
\[
e^{a-1} \left| \partial_{\beta}^{\alpha} \Gamma \left( \frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), w^{2|\beta|} |\partial_{\beta}^{\alpha} h| \right|
\]
\[
\leq C \eta \varepsilon^{a-1} \|\partial_{\beta}^{\alpha} h\|_{\alpha,|\beta|}^2 + C \eta \varepsilon^{\frac{3}{2} + \frac{3}{2} a} D_{2}(\tau).
\]
(5.27)
For the second term of (5.20), by using (5.8), (5.25) and the Sobolev imbedding inequality, one can deduce from Lemma 5.3, (3.3) and (3.4) that
\[
e^{a-1} \left| \partial_{\beta}^{\alpha} \Gamma \left( \frac{G}{\sqrt{\mu}}, f \right), w^{2|\beta|} |\partial_{\beta}^{\alpha} h| \right|
\]
\[
\leq C \varepsilon^{a-1} \sum_{|\alpha_1| \leq |\alpha|} \sum_{|\beta_1| \leq |\beta|} \int_{\mathbb{R}} \varepsilon^{1-a} (|\partial^{\alpha_1} [\bar{u}_{1y}, \bar{\theta}_y]| + |[\bar{u}_{1y}, \bar{\theta}_y]| |\partial^{\alpha_1} [u, \theta]|) \|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f\|_{\sigma,|\beta-\beta_1|} \|\partial_{\beta}^{\alpha} h\|_{\sigma,|\beta|} \, dy
\]
\[
\leq \eta \varepsilon^{a-1} \|\partial_{\beta}^{\alpha} h\|_{\alpha,|\beta|}^2 + C \eta \varepsilon^{\frac{3}{2} + \frac{3}{2} a} D_{2}(\tau).
\]
(5.28)
Since the third term of (5.20) shares the same estimates as (5.28). Thus, we still deal with the last term of (5.20). In view of (5.8), the Sobolev imbedding inequality
and (3.4), one has
\[ e^{a-1} \left| \left( \frac{\partial^\alpha \Gamma[f, f]}{\sqrt{\mu}}, w^{2|\beta|} \frac{\partial^\beta h}{\sqrt{\mu}} \right) \right| \]
\[ \leq C e^{a-1} \sum_{|\alpha_1| \leq |\alpha|} \sum_{|\beta'| \leq |\beta|} \int_{\mathbb{R}} |\partial^{\alpha_1} f|_{L^1} |\partial^{\alpha_1 - \alpha} f|_{L^1} |\partial^{\alpha_1} h|_{L^1} dy \]
\[ \leq \eta e^{a-1} \| \partial^\beta h \|_{\sigma, \beta}^2 + C \eta E_2 (\tau) D_2 (\tau) \]
\[ \leq \eta e^{a-1} \| \partial^\beta h \|_{\sigma, \beta}^2 + C \eta \kappa^{\frac{1}{3}} \varepsilon^{-\frac{3}{5}} D_2 (\tau). \] (5.29)

Collecting the estimates of (5.27), (5.28) and (5.29), we can obtain
\[ e^{a-1} \left| \left( \frac{\partial^\alpha \Gamma}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}} \right), w^{2|\beta|} \frac{\partial^\beta h}{\sqrt{\mu}} \right| \]
\[ \leq C \eta e^{a-1} \| \partial^\beta h \|_{\sigma, \beta}^2 + C \eta \varepsilon^{\frac{3}{5}} + \frac{1}{5} (\delta + e^a \tau)^{-\frac{3}{4}} + C \eta \kappa^{\frac{1}{3}} \varepsilon^{-\frac{3}{5}} D_2 (\tau). \]

This completes the proof of (5.18). One can deduce (5.19) by employing (5.7) and argument similar to the above related estimates. This ends the proof of Lemma 5.9.

\[ \square \]

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Renjun Duan  
Department of Mathematics,  
The Chinese University of Hong Kong,  
Shatin  
Hong Kong  
People’s Republic of China.  
e-mail: rjduan@math.cuhk.edu.hk

and

Dongcheng Yang, Hongjun Yu  
School of Mathematical Sciences,  
South China Normal University,  
Guangzhou  
510631 People’s Republic of China.  
e-mail: dcmath@sina.com

and

Hongjun Yu  
e-mail: yuhj2002@sina.com
