ON EIGENVECTOR STATISTICS
IN THE SPHERICAL AND TRUNCATED UNITARY ENSEMBLES

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Abstract

We study the overlaps between right and left eigenvectors for random matrices of the spherical and truncated unitary ensembles. Conditionally on all eigenvalues, diagonal overlaps are shown to be distributed as a product of independent random variables. This enables us to prove that the scaled diagonal overlaps, conditionally on one eigenvalue, converge in distribution to a heavy-tail limit, namely, the inverse of a $\gamma_2$ distribution. These results are analogous to what is known for the complex Ginibre ensemble. We also provide formulae for the conditional expectation of diagonal and off-diagonal overlaps, with respect to all eigenvalues.

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1 Introduction

1.1 Spherical and Truncated Unitary Ensembles

This work considers two ensembles of random matrices defined as follows.

(i) The spherical ensemble consists of products $G_1 G_2^{-1}$, where $G_1, G_2$ are i.i.d. complex Ginibre matrices. We denote the $N \times N$ complex Ginibre ensemble by CGE($N$) and the corresponding spherical ensemble by Sph($N$). The name spherical comes from a geometric description of the eigenvalues, stated as Proposition 2.5 and illustrated on Figure 1.

![Figure 1 – Scaled eigenvalues of CGE(1000) and Sph(1000); the third picture is the preimage of the latter by the stereographic projection (2.9).](image)

(ii) The truncated unitary ensemble consists of truncations of unitary matrices distributed according to the Haar measure (CUE). It therefore depends on two parameters determining the size of the original CUE matrix and the size of the truncation. We denote by TUE($N, M$) the ensemble of truncations of size $N$ of matrices distributed according to CUE($N + M$). Our results are only valid when $N \leq M$, that is, when the truncated matrix is at most half as large as the original matrix. Both parameters are assumed to go to infinity.

![Figure 2 – Eigenvalues of TUE($N, M$) for $N = 500$ and $M = 500, 1000, 1500$.](image)

The reason for treating these two ensembles in the same paper is the strong analogy between them, underlined and exemplified by [9], that extends to the overlap distribution. All results are presented in details for the spherical case in Section 2, while the corresponding results in the truncated unitary case are found in Section 3 – with less detail whenever the two computations are exactly the same.

2
1.2 Results

The matrix of overlaps associated to the bi-orthogonal family of left and right eigenvectors of a non-Hermitian random matrix has been introduced and studied by Chalker & Mehlig in [4,5], then more recently in a series of paper involving a variety of methods [1–3,6,8,10,13,15]. It is defined as follows: for a given matrix $G \in \mathcal{M}_N(\mathbb{C})$ with simple spectrum $\{\lambda_1, \ldots, \lambda_N\}$ (note that the random spectra we consider are almost surely simple), if $R_j = |R_j\rangle$ is the right eigenvector associated to $\lambda_j$ and $L_j = \langle L_j|$ the left eigenvector associated to the same eigenvalue, chosen such that for every $j$,

$$GR_j = \lambda_j R_j, \quad L_j G = \lambda_j L_j, \quad \langle L_i | R_j \rangle = L_i R_j = \delta_{ij},$$

we define the matrix of overlaps $\mathcal{O}$ by

$$\mathcal{O}_{i,j} = \langle L_i | L_j \rangle \langle R_j | R_i \rangle = (L_i L_j^*)(R_j^* R_i).$$

Chalker & Mehlig computed the conditional expectation of the overlaps in the complex Ginibre ensemble and conjectured several of their properties. For a more detailed presentation, see the introduction of [3] and the appendix of [8].

The results we obtain in the spherical and truncated unitary cases are analogous to some of the results obtained in [3] for the complex Ginibre ensemble $\text{CGE}(N)$. We recall these results, and point out which statement of the present paper corresponds to each one.

(i) A decomposition of the distribution of diagonal overlaps. The first notable fact is that, conditionally on the spectrum $\Lambda \in \mathbb{C}^N$, diagonal overlaps can be decomposed as a product of independent variables. In the complex Ginibre ensemble, Theorem 2.2 from [3] states that, conditionally on the event $\{\Lambda = (\lambda_1, \ldots, \lambda_N)\}$, the distribution of diagonal overlaps is given by

$$\mathcal{O}_{\text{CGE}(N)}^{1,1} \overset{d}{=} \prod_{i=2}^{N} \left(1 + \frac{|Z_i|^2}{N|\lambda_i - \lambda_1|^2}\right),$$

where $(Z_i)_{i=2}^{N}$ are i.i.d. standard complex Gaussian. Instead of Gaussian variables, the analogous statements in the spherical and truncated unitary ensembles involve i.i.d. variables whose distribution is specific to each case. Namely, in $\text{Sph}(N)$, we have

$$\mathcal{O}_{\text{Sph}(N)}^{1,1} \overset{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} X_{N}^{(k)}\right),$$

where the $X_{N}^{(k)}$ are i.i.d. variables whose distribution is defined in (2.5); and in $\text{TUE}(N,M)$,

$$\mathcal{O}_{\text{TUE}(N,M)}^{1,1} \overset{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1 - |\lambda_1|^2)(1 - |\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} Y_{M}^{(k)}\right),$$

where the $Y_{M}^{(k)}$ are i.i.d. variables whose distribution is defined in (3.5). These decompositions are stated as Theorem 2.6 and 3.5 respectively.
(ii) **A limit theorem for diagonal overlaps.** In the complex Ginibre ensemble, Theorem 1.1 from [3] states that conditionally on the event \( \{\lambda_1 = z\} \) with \( z \in \mathbb{D} \), the scaled diagonal overlap \( \Theta_{1,1} \) converges to the inverse of a \( \gamma_2 \) distribution:

\[
\frac{1}{N} \Theta_{1,1}^{\text{CGE}(N)} \xrightarrow{d N \to \infty} (1 - |z|^2) \frac{1}{\gamma_2}.
\]

(1.6)

This heavy-tail limit\(^1\) appears to be universal.

In particular, the exact same convergence holds at the origin for the spherical and truncated unitary ensembles, which is stated as Proposition 2.8 and 3.7 respectively. Unlike the complex Ginibre case, where \( \Theta_{1,1}^{-1} \) follows a beta distribution when \( \{\lambda_1 = 0\} \), the distribution of the overlap for fixed \( N \) does not take an especially simple form here; nevertheless, the asymptotical result can be worked out in an analogous way.

Figure 3 – Histograms of scaled diagonal overlaps for CGE\((N)\), Sph\((N)\) and TUE\((N, N)\) respectively, with \( N = 1000 \) and over 30 experiments (for each experiment, the overlaps of all eigenvalues in a given domain, chosen arbitrarily inside the bulk, have been considered).

The specific structure of the spherical ensemble allows one to extend this result to the whole complex plane, yielding the following Theorem.

**Theorem 1.1.** *Conditionally on the event \( \{\lambda_1 = z\} \) with \( z \in \mathbb{C} \),

\[
\mathbb{E}(\Theta_{1,1}^{\text{Sph}(N)}) = N \quad \text{and} \quad \frac{1}{N} \Theta_{1,1}^{\text{Sph}(N)} \xrightarrow{d N \to \infty} \frac{1}{\gamma_2}.
\]

(1.7)

It is to be expected that a similar statement holds for TUE\((N, M)\) in the bulk of its limit density of eigenvalues, with a scaling parameter coherent with the expressions of [2].

(iii) **Conditional expectations of overlaps.** In the complex Ginibre ensemble, it follows from (1.3) that the expectation of diagonal overlaps of CGE\((N)\) takes the following form:

\[
\mathbb{E}_{\text{CGE}(N)}^{\Theta_{1,1}} = \prod_{k=2}^{N} \left( 1 + \frac{1}{N |\lambda_i - \lambda_k|^2} \right),
\]

(1.8)

\(^1\)or heavy-tail distributions of the same family, such as \( \gamma_1^{-1} \) and \( \gamma_4^{-1} \) (see [10] and [8]).
which had been obtained earlier by Chalker & Mehlig [4,5] by a direct computation. Analogous identities derive from equations (1.4) and (1.5); they are stated in Theorem 2.6 and 3.5 respectively. Moreover, expressions of the same kind can be obtained for off-diagonal overlaps, although no decomposition in independent variables is known in that case. In the Ginibre ensemble, this yields

$$E_{\Lambda}^{CGE(N)}(\theta_{1,2}) = -\frac{1}{|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left( 1 + \frac{1}{N(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right).$$  \hspace{1cm} (1.9)

The analogous results for Sph(N) and TUE(N,M) are stated as Theorem 2.11 and 3.8 respectively.

(iv) **Conditional expectation of mixed moments.** The conditional expectation of $\text{Tr} G^*G$ with respect to $\Lambda$ also exhibits a remarkable decomposition in all three ensembles. One reason for considering this particular quantity, which is the simplest 'mixed moment', is that it is obtained from the eigenvalues and the overlaps by the identity:

$$\text{Tr} G^*G = \sum_{i,j=1}^{N} \lambda_i \bar{\lambda}_j \theta_{i,j}. \hspace{1cm} (1.10)$$

More general mixed moments are linked to the generalized overlaps considered in [6] by similar relations.

In the complex Ginibre case, it suffices to write

$$\text{Tr} G^*G = \text{Tr}(TT^*) = \sum_{i \leq j} |T_{i,j}|^2 = \sum_{i=1}^{N} |\lambda_i|^2 + \sum_{i<j} |T_{i,j}|^2. \hspace{1cm} (1.11)$$

The conditional expectation follows immediately, using the fact that the upper-diagonal entries of the Schur transform are Gaussian and independent of the eigenvalues.

The spherical and truncated unitary ensembles yield slightly more intricate expressions, stated as Proposition 2.12 and 3.9 respectively.

We summarize all results relative to (iii) and (iv) in the table below, Section 1.3. It follows from (1.10) that the third column is related to the first two by elementary linear relations – a fact which is not directly seen from the quenched expressions.
### 1.3 Synoptic table of conditional expectations

|                | \( \mathbb{E}_\Lambda (\mathcal{O}_{1,1}) \) | \( \mathbb{E}_\Lambda (\mathcal{O}_{1,2}) \) | \( \mathbb{E}_\Lambda \left( \frac{1}{N} \operatorname{Tr} GG^* \right) \) |
|----------------|--------------------------------|--------------------------------|--------------------------------|
| **Complex Ginibre**<br>CGE\((N)\) | \( \prod_{k=2}^{N} \left( 1 + \frac{1}{N|\lambda_1 - \lambda_k|^2} \right) \) | \( -\frac{1}{N|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left( 1 + \frac{1}{N(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right) \) | \( \frac{1}{N} \sum_{k=1}^{N} |\lambda_k|^2 + \frac{N-1}{2N} \) |
| Chalker & Mehlig [4,5] | \text{Chalker & Mehlig [4,5]} | \text{Chalker & Mehlig [4,5]} | \text{From eq. (1.11)} |
| **Spherical Ensemble**<br>Sph\((N)\) | \( \prod_{k=2}^{N} \left( 1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)}{N|\lambda_1 - \lambda_k|^2} \right) \) | \( -\frac{1}{N|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left( 1 + \frac{(1 + \lambda_1 \lambda_2)(1 + |\lambda_k|^2)}{N(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right) \) | \( \prod_{k=1}^{N} \left( 1 + \frac{1 + |\lambda_k|^2}{N} \right) - 2 \) |
| Theorem 2.6 | Theorem 2.11 | Proposition 2.12 |
| **Truncated Unitary**<br>TUE\((N,M)\) | \( \prod_{k=2}^{N} \left( 1 + \frac{(1 - |\lambda_1|^2)(1 - |\lambda_k|^2)}{M|\lambda_1 - \lambda_k|^2} \right) \) | \( -\frac{1}{M|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left( 1 + \frac{(1 - \lambda_1 \lambda_2)(1 - |\lambda_k|^2)}{M(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right) \) | \( \prod_{i=1}^{N} \left( 1 + \frac{1 - |\lambda_i|^2}{M} \right) - \left( 1 + \frac{N}{M} \right) \) |
| Theorem 3.5 | Theorem 3.8 | Proposition 3.9 |

Table 1 – Quenched expectations.

The above table contains the expression of quenched expectations (that is, conditional expectations with respect to \( \Lambda = \{\lambda_1, \ldots, \lambda_N\} \)) of overlaps in the three relevant integrable ensembles. The first column is the consequence of an identity in distribution involving independent variables; not so for off-diagonal overlaps (second column). The third column is related to the first two by the linear relations implied by (1.10).
1.4 Method, notations and conventions

1.4.1 Overlaps and Schur form. We recall here only what is needed in order to follow the method we apply to the spherical and truncated unitary cases.

We first note that the conditions (1.1) can be achieved by choosing \( R_i \) as the columns of \( P \) and \( L_i \) as the rows of \( P^{-1} \) for a given diagonalization \( G = P\Delta P^{-1} \); the overlaps are independent of this choice. Moreover, overlaps are unchanged by an unitary change of basis, and therefore one can study directly the overlaps of the Schur form

\[
T = U^* GU = \begin{pmatrix}
\lambda_1 & T_{1,2} & \ldots & T_{1,N} \\
0 & \lambda_2 & \ldots & T_{2,N} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \lambda_N
\end{pmatrix}
\]

(1.12)

By exchangeability of the eigenvalues, we can also limit ourselves to studying the variables \( O_{1,1} \) and \( O_{1,2} \), whose definitions only involve the first two left and right eigenvectors of \( T \), chosen such that

\[
R_1 = (1, 0, \ldots, 0)^t, \quad R_2 = (a, 1, 0, \ldots, 0)^t, \\
L_1 = (b_1, \ldots, b_N), \quad L_2 = (d_1, \ldots, d_N).
\]

Biorthogonality (1.1) gives \( b_1 = 1, \ d_1 = 0, \ d_2 = 1 \) and \( a = -b_2 \). Thanks to the upper-triangular form of \( T \), the coefficients \( b_i, d_i \) are obtained according to a straightforward recurrence. Indeed, if we consider the sequences of sub-vectors:

\[
B_k = (1, b_2, \ldots, b_k) \quad \text{so that} \quad L_1 = B_N, \\
D_k = (0, 1, d_3, \ldots, d_k) \quad \text{so that} \quad L_2 = D_N, \\
u_k = (T_{1,k}, \ldots, T_{k-1,k})^t \quad \text{(subset of the} \ k \text{th column of} \ T).
\]

The recurrence formula is

\[
\begin{cases}
    b_{n+1} = \frac{1}{\lambda_1 - \lambda_{n+1}} B_n u_{n+1}, \quad n \geq 1, \\
    d_{n+1} = \frac{1}{\lambda_2 - \lambda_{n+1}} D_n u_{n+1}, \quad n \geq 2.
\end{cases}
\]

(1.13)

The first overlaps, according to (1.2), are then given by the expressions

\[
\begin{align*}
    O_{1,1} &= \sum_{i=1}^{N} |b_i|^2, \\
    O_{1,2} &= -b_2 \sum_{i=1}^{N} b_i d_i.
\end{align*}
\]

(1.14)

The reason why the recurrence (1.13) leads to a decomposition in distribution (resp. a decomposition of the conditional expectation with respect to all eigenvalues) of the overlaps in different ensembles is that the distribution of the Schur form is known and allows to perform such a computation explicitly. For instance, in the complex Ginibre case, the upper-triangular entries \((T_{i,j})_{i<j}\) are i.i.d. complex Gaussian variables with variance \(1/N\), so that \(u_{k+1}\) is a \(k\)-dimensional Gaussian vector with independent coordinates, and independent of \(u_2, \ldots, u_k\). The Schur forms of Sph(\(N\)) and TUE(\(N,M\)) have explicit densities expressed in the form of a determinant; a structure which allows an analogous analysis.
1.4.2 Notations and conventions. Throughout the paper, \( N \) is the size of the system (i.e. the number of eigenvalues); the spectrum is \( \Lambda = \Lambda_N = (\lambda_1, \ldots, \lambda_N) \). For any \( n \leq N \), we denote by \( T_n \) the \( n \times n \) top-left submatrix of the Schur form \( T \), and by \( u_n \) the first \( n - 1 \) coordinates of the last column vector of \( T_n \), so that

\[
T_n = \begin{pmatrix} T_{n-1} & u_n \\ 0 & \lambda_n \end{pmatrix}, \quad T = T_N.
\]

\( \mathbb{E}_A \) denotes the conditional expectation with respect to \( A \) (if \( A \) is a random variable or a sigma algebra), or the expectation for the conditional probability (if \( A \) is an event); the context should prevent any ambiguity to arise. In particular, \( \mathbb{E}_\Lambda \) is the conditional expectation with respect to the spectrum \( \Lambda \). When conditioning on \( \Lambda \), we will also use the following filtration, adapted to the nested structure of the Schur transform:

\[
\mathcal{F}_n = \sigma(u_k, 2 \leq k \leq n) = \sigma(T_{i,j}, 1 \leq i < j \leq n).
\]

(This convention differs from the one chosen in [3]. In particular, \( b_2 = \frac{T_{12}}{\lambda_2 - \lambda_1} \in \mathcal{F}_2 \), and \( \mathcal{F}_1 \) is trivial.) With any suitable function \( V \), the generalized Gamma and Meijer functions are defined as

\[
\Gamma_V(\alpha) := \int_{\mathbb{R}_+} t^{\alpha-1} e^{-V(t)} dt, \quad G_V(k) = \Gamma_V(1) \cdots \Gamma_V(k).
\]

We also define the partial sums

\[
e^{(m)}_V(X) = \sum_{k=0}^{m} \frac{X^k}{\Gamma_V(k+1)},
\]

and the generalized Gamma distributions \( \gamma_V(\alpha) \), with density

\[
\frac{1}{\Gamma_V(\alpha)} t^{\alpha-1} e^{-V(t)} 1_{\mathbb{R}_+}
\]

with respect to the Lebesgue measure. We will use the fact, established for instance in [7,11,12], that a point process in \( \mathbb{C} \) with joint density given by

\[
\frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 e^{-\sum_{i=1}^{N} V(|\lambda_i|^2)}
\]

where \( Z_N = G_V(N) \), is such that the following identity in distribution holds:

**Proposition 1.2** (Kostlan’s property). \( \{|\lambda_1|^2, \ldots, |\lambda_N|^2\} \overset{d}{=} \{\gamma_V(1), \ldots, \gamma_V(N)\} \) where the latter variables are independent, and \( \gamma_V(k) \) is distributed according to (1.15) with \( \alpha = k \).

What we need here is a specific form of Kostlan’s property, obtained by applying Proposition 1.2 to the conditioned measure.

**Proposition 1.3.** Conditionally on the event \( \{\lambda_1 = 0\} \), \( \{|\lambda_2|^2, \ldots, |\lambda_N|^2\} \overset{d}{=} \{\gamma_V(2), \ldots, \gamma_V(N)\} \) where the latter variables are independent, and \( \gamma_V(k) \) is distributed according to (1.15) with \( \alpha = k \).

In the complex Ginibre ensemble, the scaled variables \( N \gamma_V \) follow usual gamma distributions.

Other notations or conventions relatives specifically to the spherical or truncated unitary case are mentioned in the corresponding section.
This section contains the proof of all claims related to the spherical ensemble $\text{Sph}(N)$. These proofs rely on a few estimates that are found in Subsection 2.3.

2.1 SCHUR FORM AND EIGENVALUES

We first present a few general results in order to illustrate the method; the tools and definitions that follow are specific to the spherical case. We recall that the Schur transform $T$ of a matrix from $\text{Sph}(N)$ is distributed with density proportional to

$$
\prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \frac{1}{\det(I_N + TT^*)^{2N}}
$$

with respect to the Lebesgue measure on all complex matrix elements, diagonal ($d\Lambda = d\lambda_1 \cdots d\lambda_N$) and upper-triangular ($du_2 \cdots du_n$). We introduce the Hermitian, definite-positive matrices

$$
H_n := I_n + T_n T_n^*, \quad S_{n-1} := (1 + |\lambda_n|^2)^{1/2} H_{n-1}^{1/2}.
$$

The following lemma is the essential tool used in [9].

**Lemma 2.1.** The determinant of $H_n = I_n + T_n T_n^*$ can be recursively decomposed as

$$
det(H_n) = (1 + |\lambda_n|^2) \det(H_{n-1}) \left( 1 + \frac{1}{1 + |\lambda_n|^2} u_n^* H_{n-1}^{-1} u_n \right).
$$

**Proof.** We first write

$$
det(H_n) = \left| \begin{array}{ccc}
I_{n-1} + T_{n-1} T_{n-1}^* + u_n u_n^* & \sum_n u_n \\
\lambda_n u_n^* & 1 + |\lambda_n|^2
\end{array} \right|.
$$

Elementary operations on columns brings this matrix to an upper-triangular form, so that

$$
det(H_n) = (1 + |\lambda_n|^2) \det(I_{n-1} + T_{n-1} T_{n-1}^* + \frac{1}{1 + |\lambda_n|^2} u_n u_n^*)
$$

$$
= (1 + |\lambda_n|^2) \det(H_{n-1}) \det(I_{n-1} + \frac{1}{1 + |\lambda_n|^2} u_n u_n^* H_{n-1}^{-1})
$$

The claim follows by Sylvester’s identity, $\det(I + AB) = \det(I + BA)$.

For any $p > n$, we denote by $V_p^{(n)}$ a random vector with density

$$
\frac{1}{C_{n,p}} \frac{1}{(1 + v^* v)^p}
$$

with respect to the Lebesgue measure on $\mathbb{C}^n$; the value of $C_{n,p}$ is given by (2.21). For any $m \geq 0$, we denote by $X_m$ a real random variable with density

$$
\frac{m + 1}{(1 + x)^{m+2}} 1_{\mathbb{R}^+}
$$
with respect to the Lebesgue measure. In particular \( E X_m = \frac{1}{m} \), and if \( v_i \) is a coordinate of \( V_p^{(n)} \), it follows from Lemma 2.16 that

\[
|v_i|^2 \overset{d}{=} X_{p-n-1}.
\]

Note that the i.i.d. variables that appear in Theorem 2.6 follow the distribution of \( X_m \) with \( m = N \).

**Lemma 2.2.** Identity holds between the following expressions, for \( p \geq n \) and \( f, g \) integrable functions of the matrix elements:

\[
\int \frac{f(\Lambda_n, u_2, \ldots, u_{n-1}) g(u_n)}{\det(H_n)^p} dT_n = C_{n-1,p} \int \frac{f(\Lambda_n, u_2, \ldots, u_{n-1}) E (g(S_{n-1} V_p^{(n-1)}))}{(1 + |\lambda_n|^2)^{p-n+1} \det(H_{n-1})^{p-1}} dT_{n-1} d\lambda_n,
\]

where \( H_n, S_{n-1}, V_p^{(n)} \) are defined in (2.2) and (2.4).

**Proof.** Lemma 2.1 and the change of variable \( u_n = S_{n-1} v_n \) bring the left hand side to the form

\[
\int \frac{f(\Lambda_n, u_2, \ldots, u_{n-1}) g(S_{n-1} v_n)}{(1 + |\lambda_n|^2)^{p-n+1} \det(H_{n-1})^{p-1}(1 + v_n^* v_n)^p} dT_{n-1} d v_n d\lambda_n.
\]

Recall that \( u_n \), and therefore \( v_n \), are column vectors of size \( n - 1 \). The claim follows by definition of the random vector \( V_p^{(n-1)} \).

A first relevant fact that can be deduced from the above Lemma is the distribution of every top-left submatrix of the Schur form \( T \).

**Proposition 2.3.** Conditionally on \( \Lambda \) and for \( 2 \leq n \leq N \), the submatrix \( T_n \) of the Schur transform is distributed with density proportional to

\[
\frac{1}{\det(I_n + T_n T_n^*)^{N+n}}.
\]

with respect to the Lebesgue measure on upper-triangular matrix elements \((d u_2 \cdots d u_n)\).

**Proof.** The claim is known for \( n = N \). We deduce it for all \( n \) by a backward recurrence; indeed, as long as \( n - 1 \geq 2 \), the claim for \( n - 1 \) follows from the claim for \( n \) by Lemma 2.2 with \( g = 1 \) and generic \( f \).

We can also derive the joint eigenvalue density of the spherical ensemble from the density of its Schur form, as was done in [9].

**Theorem 2.4.** The joint density of eigenvalues for the spherical ensemble is proportional to

\[
\prod_{i<j} |\lambda_i - \lambda_j|^2 
\frac{1}{\prod_{i=1}^{N}(1 + |\lambda_i|^2)^{N+1}}
\]

with respect to the Lebesgue measure on \( \mathbb{C}^N \).
Proof. Let \( h \) be a bounded and continuous function of the spectrum \( \Lambda_n \). We use Lemma 2.1 with \( p = 2N, g = 1 \) and
\[
f_0(\Lambda_n, u_2, \ldots, u_n) := \prod_{i<j} |\lambda_i - \lambda_j|^2 h(\Lambda_n),
\]
which yields
\[
E(h(\Lambda_n)) = C_{N-1,2N} \int \frac{f_n(\Lambda_n, u_2, \ldots, u_{n-1})}{(1 + |\lambda_n|^{2N+1})} \det(H_{n-1})^{2N-1} dT_{n-1} d\lambda_n
\]
we then use Lemma 2.1 again with
\[
f_{n-1}(\Lambda_{n-1}, u_2, \ldots, u_{n-1}) := \int \frac{f_n(\Lambda_n, u_2, \ldots, u_{n-1})}{(1 + |\lambda_n|^{2N+1})} d\lambda_n,
\]
and so on; this recurrence leads to the expression
\[
E(h(\Lambda_N)) = C \int \prod_{i<j} |\lambda_i - \lambda_j|^2 \frac{h(\Lambda_N)}{\prod_{i=1}^N (1 + |\lambda_i|^{2N+1})} d\Lambda_N,
\]
which is equivalent to the claim. \( \square \)

Theorem 2.4 can be rephrased by saying that the eigenvalues of \( \text{Sph}(N) \) are distributed according to (1.16) with potential \( V(t) = (N + 1) \ln(1 + t) \). A straightforward computation shows that
\[
\gamma_V(\alpha) = \frac{\beta_{N+1-\alpha, \alpha}}{\beta_{N+1, \alpha}} - 1. \tag{2.8}
\]

**Origin of the name spherical.** The stereographic projection from \( S^2 \) to \( \mathbb{C} \) is defined by
\[
\rho(w) = \tan \left( \frac{\phi}{2} \right) e^{i\theta}, \quad \text{where} \quad w = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix} \in S^2, \tag{2.9}
\]
and its inverse map from \( \mathbb{C} \) to \( S^2 \) is given by :
\[
p(\lambda) = \frac{1}{1 + |\lambda|^2} \begin{pmatrix} 2 \text{Re}\lambda \\ 2 \text{Im}\lambda \\ |\lambda|^2 - 1 \end{pmatrix} \in S^2, \quad \text{where} \quad \lambda \in \mathbb{C}. \tag{2.10}
\]
The reason for the name spherical is that the following identity in distribution holds.

**Proposition 2.5.** Let \( \{w_1, \ldots, w_N\} = \{p(\lambda_1), \ldots, p(\lambda_N)\} \) be the images of the eigenvalues of \( G_1G_2^{-1} \) by the map (2.10). This point process on \( S^2 \) has joint density proportional to
\[
\prod_{i<j} ||w_i - w_j||_{\mathbb{R}^3}. \tag{2.11}
\]
In other terms, the eigenvalues of \( \text{Sph}(N) \) can be described as the stereographic projection of a one-component plasma on \( S^2 \), with respect to a uniform potential\(^2\).

\(^2\)Note that the appropriate convention for the stereographic projection here is such that the unit circle is mapped to the equator of \( S^2 \). In particular, the average proportion of eigenvalues of \( \text{Sph}(N) \) falling in the unit disk is \( \frac{1}{2} \).
Proof. This is obtained by a change of variable applied to the density (2.7), noting that

\[ \|p(\lambda) - p(\mu)\|_{\mathbb{R}^3}^2 = \frac{4|\lambda - \mu|^2}{(1 + |\lambda|^2)(1 + |\mu|^2)} \]  

and that the Jacobian of \( p \) at \( \lambda \) is \( \frac{4}{(1 + |\lambda|^2)^2} \).

2.2 DISTRIBUTION AND CONDITIONAL EXPECTATION OF OVERLAPS

We now give the proof of the claims concerning diagonal and off-diagonal overlaps in the spherical ensemble. Some results hold conditionally on the whole spectrum \( \Lambda_N \), whereas others only imply a condition on one eigenvalue.

**Theorem 2.6.** Conditionally on \( \{\Lambda = (\lambda_1, \ldots, \lambda_N)\} \), the diagonal overlaps of \( \text{Sph}(N) \) are distributed as

\[ \Theta_{1,1} \overset{d}{=} \prod_{k=2}^{N} \left( 1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} X_N^{(k)} \right) \tag{2.13} \]

where the \( X_N^{(k)} \) are i.i.d. distributed according to (2.5) with \( m = N \). In particular, the quenched expectation is given by

\[ \mathbb{E}_{\Lambda} (\Theta_{1,1}) = \prod_{k=2}^{N} \left( 1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)}{N|\lambda_1 - \lambda_k|^2} \right) \tag{2.14} \]

Proof. For \( 1 \leq d \leq N \), we define the partial sums

\[ \Theta_{1,1}^{(d)} := \sum_{i=1}^{d} |b_i|^2. \]

It follows from the general facts presented in Section 1.2 that \( \Theta_{1,1}^{(1)} = 1 \), and for any \( d \geq 1 \),

\[ \Theta_{1,1}^{(d+1)} = \Theta_{1,1}^{(d)} + |b_{d+1}|^2 = \Theta_{1,1}^{(d)} \left( 1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|B_d u_{d+1}|^2}{\|B_d\|^2} \right) \]

In order to characterize the distribution of this factor, we use our preliminary results in the following order:

- Proposition 2.3 gives the distribution of \( T_{d+1} \), so that \( p = N + d + 1 \) in the following steps.
- Lemma 2.2 with \( n = d + 1 \), generic \( f \) and \( g(u_{d+1}) := h(|B_d u_{d+1}|^2) \) with generic \( h \) gives that

\[ |B_d u_{d+1}|^2 \overset{d}{=} |B_d S_d V_{N+d+1}^{(d)}|^2 \]

and is independent of \( F_d \).
- Lemma 2.16 with \( a = b = B_d^* \) and \( S = S_d \) yields

\[ |B_d u_{d+1}|^2 \overset{d}{=} \|S_d B_d^* X_N = (1 + |\lambda_{d+1}|^2)(B_d (I_d + T_d T_d^*) B_d^*) X_N \tag{2.15} \]

where \( X_N \) is distributed according to (2.5) with parameter \( m = N \), and independent of \( F_d \).
We notice that, as $T$ is triangular and $T_d, B_d$ are obtained from $T$ and $L_1$,

$$B_dT_d = \lambda_1 B_d$$

which implies that $B_d(I_d + T_dT_d^*)B_d^* = (1 + |\lambda_1|^2)\|B_d\|^2$. It follows that

$$\theta_{1,1}^{(d+1)} = \theta_{1,1}^{(d)} \left(1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_{d+1}|^2)}{|\lambda_1 - \lambda_{d+1}|^2} X_N\right),$$

where $X_N$ is independent of $\mathcal{F}_d$; we denote this variable by $X_{N}^{(d+1)}$ in order to avoid confusion between the different variables $X_N$. This implies the claim, as $\theta_{1,1} = \theta_{1,1}^{(N)}$. □

Diagonal overlap are (deterministically) larger than one, and typically of order $N$. The following proposition states that in the spherical ensemble the expectation of the diagonal overlap for an eigenvalue conditioned to be at the origin is exactly $N$, as is also the case in the complex Ginibre and truncated unitary ensembles.

**Proposition 2.7.** Conditionally on $\{\lambda_1 = 0\}$, the expectation of the diagonal overlap $\theta_{1,1}$ in the spherical ensemble $\text{Sph}(N)$ is

$$\mathbb{E}_{\{\lambda_1 = 0\}} \theta_{1,1} = N.$$  

**Proof.** We know from Proposition 1.2 that the squared radii are distributed like independent variables with distributions $\gamma_{V,k}$ with $V(x) = (N + 1) \log(1 + x)$ and $2 \leq k \leq N$. We have

$$\mathbb{E}_{\{\lambda_1 = 0\}} \theta_{1,1} = \prod_{k=2}^{N} \mathbb{E} \left(1 + \frac{1}{N} + \frac{1}{N \gamma_{V,k}}\right),$$

and according to Lemma 2.13,

$$\mathbb{E} \left(\frac{1}{\gamma_{V,k}}\right) = \frac{\beta(N + 2 - k, k - 1)}{\beta(N + 1 - k, k)} = \frac{N + 1 - k}{k - 1},$$

so that the expectation is given by the telescopic product

$$\mathbb{E}_{\{\lambda_1 = 0\}} \theta_{1,1} = \prod_{k=2}^{N} \frac{k}{k - 1} = N$$

as was claimed. □

**Proposition 2.8.** Conditionally on $\{\lambda_1 = 0\}$, the following convergence in distribution takes place:

$$\frac{1}{N} \theta_{1,1} \xrightarrow{d} \frac{1}{\gamma_2} \frac{1}{N \to \infty}$$

The proof relies on the following elementary Lemma.
Lemma 2.9. Let \( \left( u_{k,n}^{(m)} \right)_{1 \leq k \leq n, m \in \mathbb{N}} \) be a countable family of double-indexed real positive sequences such that
\[
\forall m, k \geq 1, \quad u_{k,n}^{(m)} \xrightarrow{n \to \infty} 0.
\]
Then there exists a sequence \( (k_n)_{n \geq 1} \) such that \( 1 \leq k_n \leq n, k_n \xrightarrow{n \to \infty} \infty \), and for any \( m \in \mathbb{N} \),
\[
\sum_{i=1}^{k_n} u_{i,n}^{(m)} \xrightarrow{n \to \infty} 0.
\]

Proof of Lemma 2.9. We first prove the statement for one double-indexed sequence \( (u_{k,n}^{(m)})_{1 \leq k \leq n} \). We define, for \( 1 \leq k \leq n \), the partial sums \( S_{k,n} = \sum_{i=1}^{k} u_{i,n} \), and the following sequence, iteratively:
\[
n_1 := 1, \quad n_{j+1} := \min \left\{ l \mid \forall n \geq l, S_{j+1,n} \leq \frac{1}{2} S_{j,n} \right\}.
\]
By assumption on \( u_{k,n}^{(m)} \), the sequence \( (n_j)_{j \geq 1} \) is well defined, increasing, and goes to infinity. Moreover, by construction we see that \( S_{j,n_j} \) converges to zero. It is straightforward to check that the sequence
\[
k_n := \max \{ j \in [1, n] \mid n_j \leq n \}
\]
is such that \( 1 \leq k_n \leq n, k_n \xrightarrow{n \to \infty} \infty \), and
\[
\forall n \in [n_j, n_{j+1} - 1], \quad S_{k_n,n} = S_{j,n} \leq \frac{1}{2} S_{j-1,n_{j-1}},
\]
so that \( S_{k_n,n} \) converges to 0; thus, the Lemma is established for one sequence. We extend this to a countable family of double-indexed sequences \( u_{k,n}^{(m)} \) by defining \( v_{k,n} := \sum_{m=1}^{k} u_{k,n}^{(m)} \), which converges to 0 for every fixed \( k \); by the above argument, there exists a sequence \( k_n \) such that
\[
\sum_{j=1}^{k_n} v_{j,n} \xrightarrow{n \to \infty} 0, \quad \text{so that} \quad \forall m \in \mathbb{N}, \sum_{j=1}^{k_n} u_{j,n}^{(m)} \xrightarrow{n \to \infty} 0.
\]
Indeed, every term being positive, as \( k_n \to \infty \), the latter sum can be bounded by the first one as soon as \( k_n \geq m \). This concludes the proof of Lemma 2.9.

The following argument uses the multiplicative version of Lemma 2.9; namely, if a countable family of double-indexed sequences \( p_{k,n}^{(m)} \) is such that \( p_{k,n}^{(m)} \to 1 \) for every fixed \( k \) and \( m \), then there exists a sequence \( (k_n)_{n \geq 1} \), going to infinity, such that for every \( m \)
\[
\prod_{j=1}^{k_n} p_{j,n}^{(m)} \to 1.
\]
Note that this existential statement does not give any estimate on the growth rate of \( (k_n) \).

Proof of Proposition 2.8. We first recall how convergence to \( \gamma^{-1}_2 \) arises for the complex Ginibre ensemble; part of the argument then relies on comparison with this case, treated in [3]. The reason
why this situation is more tractable is that the distribution of the diagonal overlap yields an exact expression: using a few classical identities of the beta and gamma distributions, we see that

\[ N \sigma_{1,1}^{-1} \overset{d}{=} N \prod_{k=2}^{N} \left( 1 + \frac{\gamma_1^{(k)}}{\gamma_k} \right)^{-1} \overset{d}{=} N \prod_{k=2}^{N} \beta_{k,1} = N \beta_{2, N-1} \overset{d}{\rightarrow} \gamma_2. \]

Now, for any sequence of integers \((k_n)_{n \geq 1}\) such that

\[ 1 \leq k_n \leq n, \quad k_n \overset{n \rightarrow \infty}{\longrightarrow} \infty, \tag{2.17} \]

the same product can be decomposed as

\[ k_N \prod_{k=2}^{k_N} \left( 1 + \frac{\gamma_1^{(k)}}{\gamma_k} \right)^{-1} \times N \prod_{k=k_N+1}^{N} \left( 1 + \frac{\gamma_1^{(k)}}{\gamma_k} \right)^{-1} \overset{d}{=} k_N \beta_{2, k_N - 1} \times \frac{N}{k_N} \beta_{k_N + 1, N-1} \]

It is straightforward to check that

\[ k_N \beta_{2, k_N - 1} \overset{d}{\rightarrow} \gamma_2, \quad \frac{N}{k_N} \beta_{k_N + 1, N} \overset{d}{\rightarrow} 1. \]

In other words, the limit distribution \(\gamma_2\) essentially depends on the first \(k_N\) factors, provided \(k_N\) goes to infinity. Similarly in the spherical case, using Theorem 2.6 and Proposition 1.3, we write:

\[ N \sigma_{1,1}^{-1} \overset{d}{=} k_N \prod_{k=2}^{k_N} \left( 1 + \frac{\gamma_1^{(V)}(k)}{\gamma_{V}(k)} \right)^{-1} \times N \prod_{k=k_N+1}^{N} \left( 1 + \frac{\gamma_1^{(V)}(k)}{\gamma_{V}(k)} \right)^{-1} \overset{d}{=} F(2, k_N) \times F(k_N + 1, N) \]

We will prove that the first factor \(F(2, k_N)\) converges to \(\gamma_2\) for a suitable sequence \(k_N\) that allows comparison with the complex Ginibre case, whereas the second factor \(F(k_N + 1, N)\) converges to 1. By the identity (2.8), the independent variables involved are distributed as follows:

\[ F_{N,k} := 1 + \frac{\gamma_{V}(k)}{\gamma_{V}(k)} X_N^{(k)} \overset{d}{=} 1 + \frac{X_N}{\beta_{k,N+1-k}} \]

where \(X_N\) is defined by (2.5).

**Convergence of \(F(2, k_N)\) to \(\gamma_2\), for a suitable sequence \((k_N)\).** For fixed \(k\), each term \(F_{N,k}\) converges to its analog in the complex Ginibre case. Indeed,

\[ N X_N \overset{d}{\rightarrow} \gamma_1, \quad \text{and} \quad N \beta_{k, N-k+1} \overset{d}{\rightarrow} \gamma_k, \]

so that

\[ F_{N,k} \overset{d}{=} 1 + \frac{N X_N}{N \beta_{k, N-k+1}} \overset{d}{\rightarrow} 1 + \frac{\gamma_1}{\gamma_k}. \]

The function \(x \mapsto x^{-m}\) being smooth and bounded on \((1, \infty)\) for any integer \(m\), we have that

\[ \mathbb{E} F_{N,k}^{-m} \overset{N \rightarrow \infty}{\rightarrow} \mathbb{E} \left( 1 + \frac{\gamma_1}{\gamma_k} \right)^{-m} \]
and so, by the multiplicative version of Lemma 2.9 applied to the appropriate fraction of moments, there exists a sequence $k_n$ verifying (2.17), such that for every $m$,

$$
\prod_{k=2}^{k_N} \mathbb{E} \left( \frac{EF^{-m}_{N,k}}{(1 + \frac{\gamma_1}{\gamma_k})^{-m}} \right) \rightarrow 1
$$

which implies, by comparison with the product arising in the complex Ginibre case,

$$
\mathbb{E} \left( F(2, k_N)^m \right) = k_N^m \mathbb{E} \prod_{k=2}^{k_N} \left( \frac{EF^{-m}_{N,k}}{(1 + \frac{\gamma_1}{\gamma_k})^{-m}} \right) \rightarrow (m + 1)! = \mathbb{E} \gamma_2^m,
$$

so that we have

$$
F(2, k_N) \xrightarrow{d}{N \to \infty} \gamma_2.
$$

**Convergence of $F(k_N + 1, N)$ to the constant 1.** Let $k_n$ be the sequence of integers used in the first part of the argument; in particular, it satisfies (2.17). We check that this is enough to ensure the convergence of $F(k_N + 1, N)$ to 1. A straightforward computation, similar to the one performed in Proposition 2.7, yields

$$
\mathbb{E} F_{N,k} = \frac{k}{k-1}, \quad \mathbb{E} F_{N,k}^2 = \frac{k}{k-2}
$$

so that, thanks to telescopic products, we obtain the following expressions

$$
\mathbb{E} \left( \prod_{k=k_N+1}^{N} F_{N,k} \right) = \frac{N}{k_N}, \quad \mathbb{E} \left( \prod_{k=k_N+1}^{N} F_{N,k}^2 \right) = \frac{N(N-1)}{k_N(k_N-1)}.
$$

As $k_N$ verifies condition (2.17),

$$
\mathbb{E} \left( F(k_N + 1, N)^{-1} \right) = 1, \quad \text{Var} \left( F(k_N + 1, N)^{-1} \right) = \frac{N - k_N}{N(k_N - 1)} \to 0,
$$

which proves that $F(k_N + 1, N)^{-1} \xrightarrow{L^2} 1$, and in particular $F(k_N + 1, N) \xrightarrow{d}{N \to \infty} 1$, concluding the second half of the proof. The claim of the Theorem follows by Slutsky’s theorem.

The following proposition relies on the spherical structure of Sph($N$) and has no analog in Section 3.

**Proposition 2.10.** The distribution of $\Theta_{1,1}$ conditionally on the event $\{\lambda_1 = z\}$, for $z \in \mathbb{C}$, does not depend on $z$.

**Proof.** Recall that the Jacobian of $p$ at $\lambda \in \mathbb{C}$ is $\frac{4}{(1 + |\lambda|^2)^2}$ and that, for any $\lambda, \mu \in \mathbb{C}$, identity (2.12) holds. For any continuous and bounded function $F$ of $N - 1$ variables, evaluated in

$$
l_k := \frac{4|\lambda_1 - \lambda_k|^2}{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)} \quad k = 2, \ldots, N
$$

we have for any $z \in \mathbb{C}$, by a straightforward change of variables,

$$
\mathbb{E}_{\{\lambda_1 = z\}} \left( F(l_2, \ldots, l_N) \right) = \mathbb{E}_{\{w_1 = p(z)\}} \left( F \left( ||w_1 - w_2||^2, \ldots, ||w_1 - w_N||^2 \right) \right),
$$

(2.18)
where \((w_1,\ldots,w_N)\) is a point process on the sphere with density proportional to (2.11). As the
expectation on the right hand side does not depend on \(z\) (by invariance under orthogonal trans-
formations), neither does the one on the left hand side. The claim follows by noting that for any
continuous and bounded function \(G\), by the tower property of conditional expectation,
\[
E_{\{\lambda_1=z\}} G(\theta_{1,1}) = E_{\{\lambda_1=z\}} F(l_2,\ldots,l_N)
\]
where \(F(l_2,\ldots,l_N) := \mathbb{E}_A G(\theta_{1,1})\) is indeed a function of the variables \(l_2,\ldots,l_N\).

Clearly, Propositions 2.7, 2.8 and 2.10 provide together a full proof of Theorem 1.1.

**Theorem 2.11.** The quenched expectation of off-diagonal overlaps in the spherical ensemble is given
by the formula
\[
E_{\Lambda} (\theta_{1,2}) = -\frac{1}{N|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left(1 + \frac{(1 + \lambda_1 \lambda_2)(1 + |\lambda_k|^2)}{N(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right)
\]  
(2.19)

**Proof.** Similarly to the diagonal case, we define the partial sums
\[
\theta^{(d)}_{1,2} := -\overline{b_2} \sum_{i=2}^{d} b_i d_i.
\]
It follows from the facts presented in Section 1.2 that
\[
\theta^{(2)}_{1,2} = -|b_2|^2 = \frac{-|u_2|^2}{|\lambda_1 - \lambda_2|^2}.
\]
One can check, following the proof of Theorem 2.6, that \(|u_2|^2 \overset{d}{=} X_N\), so that
\[
\mathbb{E}|u_2|^2 = \frac{1}{N} \quad \text{and} \quad \mathbb{E}_A \theta^{(2)}_{1,2} = \frac{-1}{N|\lambda_1 - \lambda_2|^2},
\]
which initiates the recurrence. We now compute the conditional expectation of \(b_{n+1}\overline{d}_{n+1}\) by integrating out the vector \(u_{n+1}\). We use Proposition 2.3 and (2.25) from Lemma 2.15 with \(a = B^*_n\),
\(b = D^*_n\) and \(S = S_n\) such that \(S^2_n = (1 + |\lambda_n|^2)(I_{n-1} + T_{n-1}T_n)\). It follows that
\[
\mathbb{E}_{\Lambda, \mathcal{F}_{n-1}} b_{n+1}\overline{d}_{n+1} = \frac{1}{N(\lambda_1 - \lambda_{n+1})(\lambda_2 - \lambda_{n+1})} B_n S^2 D^*_n = \frac{(1 + |\lambda_{n+1}|^2)}{N(\lambda_1 - \lambda_{n+1})(\lambda_2 - \lambda_{n+1})}(B_n D^*_n + B_n TT^* D^*_n)
\]
We notice that, as \(T\) is triangular and \(B_n, D_n\) are subvectors of \(L_1\) and \(L_2\),
\[
B_n T_n = \lambda_1 B_n, \quad D_n T_n = \lambda_2 D_n,
\]
which gives
\[
-\overline{b_2} \mathbb{E}_{\Lambda, \mathcal{F}_n} b_{n+1}\overline{d}_{n+1} = \frac{(1 + |\lambda_{n+1}|^2)(1 + \lambda_1 \lambda_2)}{N(\lambda_1 - \lambda_{n+1})(\lambda_2 - \lambda_{n+1})} \theta^{(n)}_{1,2}.
\]
The factorization follows.
Proposition 2.12. The conditional expectation of $\frac{1}{N} \operatorname{Tr} GG^*$ with \(G\) distributed according to $\text{Sph}(N)$ is given by the formula:

$$
\mathbb{E}_\Lambda \left( \frac{1}{N} \operatorname{Tr} GG^* \right) = \prod_{i=1}^{N} \left( 1 + \frac{1 + |\lambda_i|^2}{N} \right) - 2.
$$

Proof. It is clear that $\operatorname{Tr} GG^* = \operatorname{Tr} T_n^* T_n$, and that for any $n \leq N$,

$$
\operatorname{Tr} T_n^* T_n = |\lambda_n|^2 + \|u_n\|^2 + \operatorname{Tr} T_{n-1}^* T_{n-1},
$$

so that defining

$$
v_{N,n} = v_{N,n} (\lambda_1, \ldots, \lambda_n) := \mathbb{E}_{N,A} \operatorname{Tr} T_n^* T_n,
$$

yields a recursion with $v_{N,1} = |\lambda_1|^2$ and, using Proposition 2.3 and (2.26) from Lemma 2.15,

$$
v_{N,n+1} = v_{N,n} \left( 1 + \frac{1 + |\lambda_{n+1}|^2}{N} \right) + |\lambda_{n+1}|^2 + \frac{n}{N} \left( 1 + \frac{|\lambda_{n+1}|^2}{N} \right).
$$

This suggests the introduction of $w_{N,n} = v_{N,n} + N + n$, for which we see that

$$
w_{N,1} = N \left( 1 + \frac{1 + |\lambda_1|^2}{N} \right) \quad \text{and} \quad w_{N,n+1} = w_{N,n} \left( 1 + \frac{1 + |\lambda_{n+1}|^2}{N} \right),
$$

so that for every $n \leq N$,

$$
\frac{1}{N} v_{N,n} = \prod_{i=1}^{N} \left( 1 + \frac{1 + |\lambda_i|^2}{N} \right) \left( 1 + \frac{n}{N} \right)
$$

which is equivalent to the statement, when $n = N$. \qed

2.3 Constants and integrals

Lemma 2.13. The normalization constant for generalized gamma variables $\gamma_{V,k}$ with potential $V(x) = M \log(1 + x)$ and $1 \leq k \leq M - 1$ is

$$
\int_{\mathbb{R}^+} \frac{x^{k-1}}{(1 + x)^M} dx = \beta(M - k, k),
$$

and $\gamma_{V,k} \overset{d}{=} \frac{1}{\beta_{M-k,k}} - 1$. Moreover, the associated function $e^{(M-2)}_V$ is given by

$$
e^{(M-2)}_V = (M - 1)(1 + X)^{M-2}.
$$

Proof. Let us compute, for any suitable function $f$,

$$
\int_{\mathbb{R}^+} \frac{x^{k-1}}{(1 + x)^M} f(x) dx = \int_{0}^{1} x^{M-k-1} (1 - x)^{k-1} f \left( \frac{1}{x} - 1 \right) dx = \beta(M - k, k) \mathbb{E} f \left( \frac{1}{\beta_{M-k,k}} - 1 \right),
$$

which implies the first claim. As

$$
\frac{1}{\Gamma_V(k)} = \frac{1}{\beta(M - k, k)} = \frac{\Gamma(M)}{\Gamma(M - k)\Gamma(k)} = (M - k) \binom{M-1}{k-1},
$$
Lemma 2.15. For any \( p > n \),
\[
C_{n,p} := \int_{z \in \mathbb{C}^n} \frac{1}{(1 + \sum_{i=1}^{n} |z_i|^2)^p} \, dm(z_1) \ldots dm(z_n) = \pi^n \frac{(p-n-1)!}{(p-1)!}, \tag{2.21}
\]
and for \( p > n + 1 \),
\[
C_{n,p}^{(1)} := \int_{z \in \mathbb{C}^n} \frac{|z|^2}{(1 + \sum_{i=1}^{n} |z_i|^2)^p} \, dm(z_1) \ldots dm(z_n) = \frac{1}{p-(n+1)} C_{n,p}. \tag{2.22}
\]
Proof. We first compute \( C_{n,p} \) by induction on \( n \). For \( n = 1, p > 1 \),
\[
C_{1,p} = \int_{z \in \mathbb{C}} (1 + |z|^2)^{p-1} \, dm(z) = \pi \int_{r=1}^{\infty} \frac{1}{r^p} \, dr = \frac{\pi}{p-1},
\]
and one can note that for any \( \alpha > 0 \),
\[
\int_{z \in \mathbb{C}} \frac{1}{(1 + |z|^2)^{p}} \, dm(z) = \frac{\pi \alpha}{p-1}.
\]
For general \( n \), using the above equalities with \( \alpha_n = 1 + \sum_{i=1}^{n-1} |z_i|^2 \),
\[
C_{n,p} = \int_{z \in \mathbb{C}^n} \frac{1}{(1 + \sum_{i=1}^{n-1} |z_i|^2)^p} \times \frac{1}{(1 + \alpha_n |z_n|^2)^p} \, dm(z_1) \ldots dm(z_n)
\]
\[
= \frac{\pi}{p-1} \int_{z \in \mathbb{C}^{n-1}} \frac{1}{(1 + \sum_{i=1}^{n-1} |z_i|^2)^{p-1}} \, dm(z_1) \ldots dm(z_{n-1}) = \frac{\pi}{p-1} C_{n-1,p-1}.
\]
Equation (2.21) follows. A similar induction can be performed on \( C_{n,p}^{(1)} \). The only difference is that the last step involves the following identity: for \( p > 2 \),
\[
C_{1,p}^{(1)} = \int_{z \in \mathbb{C}} \frac{|z|^2}{(1 + |z|^2)^{p}} \, dm(z) = \pi \int_{r=1}^{\infty} \frac{r-1}{r^p} \, dr = \pi \left( \frac{1}{1-p} - \frac{1}{2-p} \right) = \frac{\pi}{(p-1)(p-2)},
\]
which, in general, yields the extra factor \( \frac{1}{p-(n+1)} \) in (2.22).

Note that when we begin the recursion from [9] with \( n = N - 1, p = 2N \), the extra factor is \( \frac{1}{N} \) at every step.

Lemma 2.16. For any \( p > n \), \( a, b \in \mathbb{C}^n \) and any Hermitian positive-definite matrix \( S \),
\[
\int_{\mathbb{C}^n} \frac{1}{(1 + u^* S^{-2} u)^p} \, du = C_{n,p} |\det S|^2, \tag{2.23}
\]
\[
\int_{\mathbb{C}^n} \frac{a^* u}{(1 + u^* S^{-2} u)^p} \, du = 0, \tag{2.24}
\]
\[
\int_{\mathbb{C}^n} \frac{(a^* u)(a^* b)}{(1 + u^* S^{-2} u)^p} \, du = C_{n,p} |\det S|^2 \frac{a^* S^2 b}{p - (n+1)}, \tag{2.25}
\]
\[
\int_{\mathbb{C}^n} \frac{\|u\|^2}{(1 + u^* S^{-2} u)^p} \, du = C_{n,p} |\det S|^2 \frac{\text{Tr} S^2}{p - (n+1)}, \tag{2.26}
\]

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where the constant $C_{n,p}$ is explicitly computed in Lemma 2.14.

Proof. Integral (2.23) was computed in [9]. (2.24) is zero by symmetry. For (2.25), the change of variables $u = Sv$ yields

$$|\det S|^2 \int \frac{(a^* Sv)(v^* Sb)}{(1 + v^* v)^p} dv.$$

We notice that

$$(a^* Sv)(v^* Sb) = v^* (Sba^* S)v = v^* Av$$

where $A = Sba^* S$ is a matrix of rank 1. If we express $v = \sum v_i e_i$ in a unitary basis such that the vectors $(e_2, \ldots, e_n)$ form a basis of $\ker(A)$ and denote $Ae_1 = \lambda_1(A)e_1 + \sum_{i \geq 2} \alpha_i e_i$,

$$v^* Av = \lambda_1(A)v_1^2 + \sum_{i \geq 2} \alpha_i v_1 v_i$$

Therefore, after a unitary change of basis the integral becomes, using Lemma 2.14 and the fact that cross-terms $v_1 v_i$ vanish by symmetry,

$$\int dv \frac{\lambda_1(A)v_1^2}{(1 + v_1^2 + \cdots + v_n^2)^p} = \frac{\lambda_1(A)}{p - (n + 1)} C_{n,p}.$$

The value of $\lambda_1(A)$ can be obtained by writing

$$\lambda_1(A) = \text{Tr} Sba^* S = a^* S^2 b,$$

from which the claim (2.25) follows. The same technique applied to (2.26) yields

$$|\det S|^2 \int dS \frac{\|Sv\|^2}{(1 + v^* v)^p}.$$

and a unitary change of variable to a basis that diagonalizes $S$, together with Lemma 2.14, gives

$$\int dv \frac{\lambda_1(S^2)v_1^2 + \cdots + \lambda_n(S^2)v_n^2}{(1 + v_1^2 + \cdots + v_n^2)^p} = (\lambda_1(S^2) + \cdots + \lambda_n(S^2)) \frac{1}{p - (n + 1)} C_{n,p},$$

concluding the proof of the last claim.

Lemma 2.16. For any $p > n$, $a \in \mathbb{C}^n$ and any Hermitian positive-definite matrix $S$, if $u \in \mathbb{C}^n$ is distributed with density

$$\frac{1}{C_{n,p} |\det S|^2} \frac{1}{(1 + u^* S^{-2} u)^p}$$

with respect to the Lebesgue measure on $\mathbb{C}^n$, then the following identity in distribution holds:

$$|a^* u|^2 \overset{d}{=} \|Sa\|^2 X_{p-n-1}.$$
This section contains the proof of all claims concerning the truncated unitary ensembles TUE\((N, M)\) when \(N \leq M\). Almost every step in this study is analogous to what was done in the spherical case; we therefore refer constantly to the corresponding parts of Section 2.

### 3.1 Schur form and eigenvalues

As in Section 2, we first present a few general results in order to illustrate the method, as well as a few tools and definitions that are specific to the truncated unitary case. We first recall that the Schur transform \(T\) is distributed with density proportional to

\[
\prod_{i<j} |\lambda_i - \lambda_j|^2 \det(I_N - TT^*)^{M-N} \mathbb{1}_{TT^* < 1}
\]

with respect to the Lebesgue measure on all complex matrix elements, diagonal \((d\Lambda = d\lambda_1 \cdots d\lambda_N)\) and upper-triangular \((du_2 \cdots du_n)\).

Provided \(TT^* < 1\) (which implies the same condition on every submatrix \(T_n\)), we introduce the Hermitian, definite-positive matrices

\[
H_n := I_n - T_n T_n^*, \quad S_{n-1} := (1 - |\lambda_n|^2)^{1/2} H_{n-1}^{1/2}.
\]

Note that the only differences with the matrices \(H_n, S_{n-1}\) used in the spherical case are the minus sign and the condition on the eigenvalues of \(TT^*\).

**Lemma 3.1.** The determinant of \(H_n = I_n - T_n T_n^*\) can be recursively decomposed as

\[
\det(H_n) = (1 - |\lambda_n|^2) \det(H_{n-1}) \left(1 - \frac{1}{1 - |\lambda_n|^2} u_n^* H_{n-1}^{-1} u_n\right).
\]

The proof is analogous to the proof of Lemma 2.1.

For any \(p \geq 0\), we denote by \(W_p^{(n)}\) a random vector with density

\[
\frac{1}{C_{n,p}} (1 - v^* v)^p \mathbb{1}_{v^* v < 1}
\]

with respect to the Lebesgue measure on \(\mathbb{C}^n\); the value of \(C_{n,p}\) is given by (3.14). For any \(m \geq 2\), we denote by \(Y_m\) a real random variable with density

\[
(m - 1)(1 - y)^{m-2} \mathbb{1}_{(0,1)}
\]

with respect to the Lebesgue measure, i.e. it follows a \(\beta_{1,m-1}\) distribution; in particular \(\mathbb{E}Y_m = \frac{1}{m}\). If \(w_i\) is a coordinate of \(W_p^{(n)}\), it follows from Lemma 3.12 that

\[
|w_i|^2 \overset{d}{=} Y_{p+n+1}.
\]

Note that the i.i.d. variables that appear in Theorem 3.5 follow the above distribution with \(m = M\).
Lemma 3.2. Identity holds between the following expressions, for \( p \geq n \) and \( f, g \) integrable functions of the matrix elements:

\[
\int f(T_{n-1}, \lambda_n)g(u_n)\det(H_n)^p \mathbb{1}_{T_n T_n^* < 1} dT_n = C_{n-1, p} \int f(T_{n-1}, \lambda_n)E(g(S_n-1W_{1, n}^{(n-1)}))(1-|\lambda_n|^2)^{p+n-1} \det(H_{n-1})^{p+1} \mathbb{1}_{T_{n-1} T_{n-1}^* < 1} dT_{n-1} d\lambda_n,
\]

where \( H_n, S_n-1, W_{1, n}^{(n)} \) are defined in (2.2) and (2.4).

We deduce from the above Lemma the distribution of every top-left submatrix of the Schur form, analogously to Proposition 2.3.

Proposition 3.3. Conditionally on \( \Lambda \) and for \( 2 \leq n \leq N \), the submatrix \( T_n \) of the Schur transform is distributed with density proportional to

\[
\det(I_n - T_n T_n^*)^{M-n} \mathbb{1}_{T_n T_n^* < 1}. \tag{3.6}
\]

with respect to the Lebesgue measure on upper-triangular matrix elements \((du_2 \cdots du_n)\).

We also derive the joint eigenvalue density of the truncated unitary ensemble from the density of its Schur form, as was done in [9]. The result itself was first proven in [14].

Theorem 3.4 (Życzkowski & Sommers). The joint density of eigenvalues for the truncated unitary ensemble when \( M \geq N \) is proportional to

\[
\frac{1}{Z_{M,N}} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \prod_{i=1}^{N} (1-|\lambda_i|^2)^{M-1} \mathbb{1}_{\mathbb{D}}(\lambda_i) \tag{3.7}
\]

with respect to the Lebesgue measure on \( \mathbb{C}^N \).

The proof is analogous to the one of Theorem 2.4.

Theorem 3.4 can be rephrased by saying that the eigenvalues of TUE\((N, M)\) are distributed according to (1.16) with potential \( V(t) = -(M - 1) \ln(1-t) \mathbb{1}_{(0,1)} \). A straightforward computation shows that

\[
\gamma_V(\alpha) \overset{d}{=} \beta_{\alpha,M}. \tag{3.8}
\]

Thus, Kostlan’s theorem in that case asserts that the set of squared radii is distributed as a set of independent \( \beta \) variables. Namely,

\[
\{|\lambda_1|^2, \ldots, |\lambda_k|^2\} \overset{d}{=} \{\beta_{1,M}, \ldots, \beta_{k,M}\}.
\]

3.2 DISTRIBUTION AND CONDITIONAL EXPECTATION OF OVERLAPS

Theorem 3.5. Conditionally on \( \{\Lambda = (\lambda_1, \ldots, \lambda_N)\} \), diagonal overlaps in the truncated unitary ensemble TUE\((N, M)\) are distributed as

\[
\Theta_{1,1} \overset{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1-|\lambda_1|^2)(1-|\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} V_{M}^{(k)}\right) \tag{3.9}
\]
where the $Y_M^{(k)}$ are i.i.d. distributed according to (3.5) with $m = M$. In particular, the quenched expectation is given by the formula

$$\mathbb{E}_\Lambda (\mathcal{O}_{1,1}) = \prod_{k=2}^{N} \left( 1 + \frac{(1 - |\lambda_1|^2)(1 - |\lambda_k|^2)}{M|\lambda_1 - \lambda_k|^2} \right)$$

(3.10)

**Proof.** It is similar to the one of Theorem 2.6; we sketch it again to see where the differences lie. We first write

$$\mathcal{O}_{1,1}^{(d+1)} = \mathcal{O}_{1,1}^{(d)} + |b_{d+1}|^2 = \mathcal{O}_{1,1}^{(d)} \left( 1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|B_d b_{d+1}|^2}{\|B_d\|^2} \right)$$

In order to characterize the distribution of this factor, we use Proposition 3.3, then Lemma 3.2 and Lemma 3.12 with $a = b = \mathcal{B}_d$ and $S = S_{d+1}$ such that $S_{d+1}^2 = (1 - |\lambda_{d+1}|^2)(I_d - T_d T_d^*)$. This yields

$$|B_d b_{d+1}|^2 \overset{d}{=} (1 - |\lambda_{d+1}|^2)\|B_d - T_d T_d^*\|_{\mathcal{B}_d}^2 Y_N$$

(3.11)

where $Y_N$ is distributed according to (3.5) with $m = M$, and independent of $\mathcal{F}_d$; we denote this variable by $Y_N^{(d+1)}$ to avoid confusion. The last steps of the proof follow accordingly.

**Proposition 3.6.** Conditionally on $\{\lambda_1 = 0\}$, the expectation of the diagonal overlap $\mathcal{O}_{1,1}$ in the truncated unitary ensemble $\text{TUE}(N, M)$ is

$$\mathbb{E}_{\{\lambda_1 = 0\}} \mathcal{O}_{1,1} = N.$$

Note that the same statement, which is an exact identity for any $N$, holds in the complex Ginibre ensemble and spherical ensemble respectively.

**Proof.** We know from Proposition 1.3 that the squared radii, conditionally on the event $\{\lambda_1 = 0\}$, are distributed like independent variables with distributions $\gamma_{V,k}$ with $V(x) = -(M-1) \log(1-x) \mathbb{1}_{(0,1)}$ and $2 \leq k \leq N$. We already noticed that $\gamma_{V,k} \overset{d}{=} \beta_{k,M}$. A straightforward computation follows:

$$\mathbb{E}_{\{\lambda_1 = 0\}} \mathcal{O}_{1,1} = \mathbb{E} \prod_{k=2}^{N} \left( 1 + \frac{1 - |\lambda_k|^2}{M|\lambda_k|^2} \right) = \prod_{k=2}^{N} \mathbb{E} \left( 1 - \frac{1}{M} + \frac{1}{M \beta_{k,M}} \right).$$

For any $k \geq 2$,

$$\mathbb{E} \left( \frac{1}{\beta_{k,M}} \right) = \frac{\beta(k - 1, M)}{\beta(k, M)} = \frac{M + k - 1}{k - 1},$$

so that the expectation is given by the telescopic product

$$\mathbb{E}_{\{\lambda_1 = 0\}} \mathcal{O}_{1,1} = \prod_{k=2}^{N} \frac{k}{k - 1} = N$$

as was claimed.

**Proposition 3.7.** Conditionally on $\{\lambda_1 = 0\}$, the following convergence in distribution takes place:

$$\frac{1}{N} \mathcal{O}_{1,1} \overset{d}{\longrightarrow} \frac{1}{\gamma_2} \text{ as } N \to \infty.$$
Note that \( N \to \infty \) implies \( M \to \infty \), as we study the truncated unitary ensemble in the regime where \( N \leq M \). The rate at which \( N, M \) go to infinity does not have any impact on the following proof (although it is expected to play a role when conditioning on a generic \( z \) in the bulk).

**Proof.** The technique is similar to the proof of Proposition 2.8. We decompose the distribution obtained by Theorem 3.5 in two factors

\[
N \sigma_{1,1}^{-1} \overset{\text{d}}{=} k_N \prod_{k=2}^{k_N} \left( 1 + \frac{1 - \gamma_V(k)}{\gamma_V(k)} Y^{(k)}_M \right)^{-1} \times N \prod_{k=k_N+1}^{N} \left( 1 + \frac{1 - \gamma_V(k)}{\gamma_V(k)} Y^{(k)}_M \right)^{-1} \overset{\text{d}}{=} G(2, k_N) \times G(k_N + 1, N).
\]

As \( \gamma_V(k) \overset{\text{d}}{=} \beta_{k,M} \), we have

\[
G_{M,k} := 1 + \frac{1 - \gamma_V(k)}{\gamma_V(k)} Y^{(k)}_M \overset{d}{=} 1 + \left( \frac{1}{\beta_{k,M}} - 1 \right) Y_M,
\]

where \( Y_M \) is defined by (3.5). The proof then proceeds in two separate parts.

**Convergence of** \( G(2, k_N) \) **to** \( \gamma_2 \) **for a suitable sequence** \( k_N \). It is straightforward to check that for every \( k \), the term \( G_{M,k} \) converges to the factor playing an analogous role in the complex Ginibre case. Indeed,

\[
MY_M \overset{d}{\to} \gamma_1 \quad \text{and} \quad M\beta_{k,M} \overset{d}{\to} \gamma_k,
\]

so that

\[
G_{M,k} \overset{d}{=} 1 + \left( \frac{1}{M\beta_{k,M}} - 1 \right) MY_M \overset{\text{d}}{\to} 1 + \frac{\gamma_1}{\gamma_k},
\]

The argument then proceeds exactly as in Proposition 2.8: by Lemma 2.9, there exists a sequence \( k_N \) that verifies (2.17) and such that we can derive the convergence

\[
G(2, k_N) \overset{\text{d}}{\to} \gamma_2
\]

by comparison with the complex Ginibre case.

**Convergence of** \( G(k_N + 1, N) \) **to** \( 1 \). It follows from the computation performed in the proof of Proposition 3.6 that

\[
\mathbb{E}G_{M,k} = \frac{k}{k-1},
\]

which is the same as the expectation of \( F_{N,k} \) (and does not depend on \( M \) nor \( N \)). We compute the second moment, using the values

\[
\mathbb{E}Y^2_M = \frac{2}{M(M+1)}, \quad \text{and} \quad \mathbb{E} \left( \frac{1}{\beta_{k,M}} - 1 \right)^2 = \frac{M(M+1)}{(k-1)(k-2)},
\]

and find, as for \( F_{N,k} \),

\[
\mathbb{E}G^2_{M,k} = \frac{k}{k-2}
\]

so that we obtain the exact same expressions as in the spherical case. The end of the argument (and of the whole proof) is strictly similar to what has been written in the proof of Proposition 2.8. \( \square \)
The analog of the spherical structure of Sph(N) for TUE(N, M) is the stereographic projection on the pseudosphere (see [9]). However, the symmetries of the pseudosphere do not allow to establish an exact equivalent to Proposition 2.10.

**Theorem 3.8.** The quenched expectation of off-diagonal overlaps in TUE(N, M) with $N \leq M$ is given by the formula

$$E_A(\mathcal{O}_{1,2}) = -\frac{1}{M|\lambda_1 - \lambda_2|^2} \prod_{k=3}^{N} \left(1 + \frac{(1 - \lambda_1 \overline{\lambda}_2)(1 - |\lambda_k|^2)}{M(\lambda_1 - \lambda_k)(\overline{\lambda}_2 - \lambda_k)}\right)$$  \hspace{1cm} (3.12)

**Proof.** As for the proof of theorem 2.11, we consider the partial sums $\mathcal{O}_{1,2}^{(d)}$ and proceed by induction. It follows from the proof of Theorem 2.6, that $|u_2|^2 = Y_M$, so that

$$E|u_2|^2 = \frac{1}{M} \quad \text{and} \quad E_A\mathcal{O}_{1,2}^{(2)} = \frac{-1}{M|\lambda_1 - \lambda_2|^2}.$$

We then compute the conditional expectation of $b_{n+1}d_{n+1}$ by integrating out the vector $u_{n+1}$, using Proposition 3.3 and (3.18) from Lemma 3.11 with $a = B_n^*$, $b = D_n^*$ and $S = S_n$. It follows that

$$E_{A,\mathcal{F}_{n-1}} b_{n+1}d_{n+1} = \frac{(1 - |\lambda_{n+1}|^2)}{M(\lambda_1 - \lambda_{n+1})(\lambda_2 - \lambda_{n+1})} \left(B_nD_n^* - B_nT_nT_n^*D_n^*\right)$$

As noted in the proof of Theorem 2.11, we have

$$B_nT_n = \lambda_1 B_n, \quad D_nT_n = \lambda_2 D_n,$$

and conclude that

$$E_{A,\mathcal{F}_{n}} b_{n+1}d_{n+1} = \frac{(1 - |\lambda_{n+1}|^2)(1 - \lambda_1 \overline{\lambda}_2)}{M(\lambda_1 - \lambda_{n+1})(\lambda_2 - \lambda_{n+1})} \mathcal{O}_{1,2}^{(n)}$$

and the factorization follows. \hfill \Box

**Proposition 3.9.** The quenched expectation of $\text{Tr} G^*G$ with $G$ distributed according to TUE(N, M) is given by the formula:

$$E_A\left(\frac{1}{N} \text{Tr} G^*G\right) = \prod_{i=1}^{N} \left(1 + \frac{1 - |\lambda_i|^2}{M}\right) - \left(1 + \frac{N}{M}\right).$$

**Proof.** As in the proof of Proposition 2.12, we define $v_{N,n} := E_{N,A} \text{Tr} T_nT_n^*$ and note that for any $n \leq N$,

$$\text{Tr} T_nT_n^* = |\lambda_n|^2 + \|u_n\|^2 + \text{Tr} T_{n-1}T_n^*.$$

Using (3.19) from Lemma 3.11 yields a induction with $v_{N,1} = |\lambda_1|^2$ and

$$v_{N,n+1} = v_{N,n} \left(1 + \frac{1 - |\lambda_{n+1}|^2}{M}\right) + |\lambda_{n+1}|^2 + \frac{n}{M} \left(1 - |\lambda_{n+1}|^2\right).$$  \hspace{1cm} (3.13)

This is an analogous recursion formula to the one obtained in Proposition 2.12 and it can be solved the same way, replacing $|\lambda_i|^2$ by $-|\lambda_i|^2$ and $N$ by $M$ in the denominators; this leads to the expression

$$\frac{1}{N}v_{N,n} = \prod_{i=1}^{n} \left(1 + \frac{1 - |\lambda_i|^2}{M}\right) - \left(1 + \frac{n}{M}\right)$$

which is equivalent to the statement, when $n = N$. \hfill \Box
3.3 Constants and integrals

Lemma 3.10. For any \( p \geq 0 \), with \( \mathcal{B}_n := \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \mid \sum |\lambda_i|^2 \leq 1 \} \),
\[
D_{n,p} := \int_{\mathcal{B}_n} \left( 1 - \sum_{i=1}^n |\lambda_i|^2 \right)^p \, dm(\lambda_1) \ldots dm(z_n) = \pi^n \frac{p!}{(p+n)!},
\] (3.14)
and
\[
D_{n,p}^{(1)} := \int_{\mathcal{B}_n} |\lambda_1|^2 \left( 1 - \sum_{i=1}^n |\lambda_i|^2 \right)^p \, dm(\lambda_1) \ldots dm(z_n) = \frac{1}{p+n+1} D_{n,p}.
\] (3.15)

Proof. We first compute \( D_{n,p} \) by induction on \( n \). For \( n = 1, p \geq 0 \),
\[
D_{1,p} = \int_{\mathbb{D}} (1 - |z|^2)^p \, dm(z) = \pi \int_0^1 r^p \, dr = \frac{\pi}{p+1}
\]
ote that for any \( \alpha > 0 \),
\[
\int_{|z|^2 < \alpha} (1 - \alpha^{-1}|z|^2)^p \, dm(z) = \frac{\pi \alpha}{p+1}.
\]
For general \( n \), using the above equalities with \( \alpha_n = 1 - \sum_{i=1}^{n-1} |\lambda_i|^2 \),
\[
D_{n,p} = \int_{\mathcal{B}_n} \left( 1 - \sum_{i=1}^{n-1} |\lambda_i|^2 \right)^p \times (1 - \alpha_n^{-1}|\lambda_n|^2)^p \, dm(\lambda_1) \ldots dm(\lambda_n)
\]
\[
= \frac{\pi}{p+1} \int_{\mathcal{B}_{n-1}} \left( 1 - \sum_{i=1}^{n-1} |\lambda_i|^2 \right)^{p+1} \, dm(\lambda_1) \ldots dm(\lambda_{n-1}) = \frac{\pi}{p+1} D_{n-1,p+1}.
\]
Equation (3.14) follows. A similar induction can be performed on \( D_{n,p}^{(1)} \). The only difference is that the last step involves the following identity: for any \( p \geq 0 \),
\[
D_{1,p}^{(1)} = \int_{\mathbb{D}} |z|^2 (1 - |z|^2)^p \, dm(z) = \pi \int_0^1 (r-1) r^p \, dr = \pi \left( \frac{1}{p+1} - \frac{1}{p+2} \right) = \frac{\pi}{(p+1)(p+2)},
\]
which in general yields the extra factor \( \frac{1}{p+n+1} \) in (3.15). \( \square \)

Note that when we begin the recursion from [9] with \( n = N - 1, p = M - N \), the extra factor is \( \frac{1}{N^2} \) at every step.

Lemma 3.11. For any \( p > n \), \( a, b \in \mathbb{C}^N \) and any Hermitian positive-definite matrix \( S \),
\[
\int_{S\mathcal{B}_n} (1 - u^* S^{-2} u)^p \, du = D_{n,p} |\det S|^2,
\] (3.16)
\[
\int_{S\mathcal{B}_n} (a^* u)(1 - u^* S^{-2} u)^p \, du = 0,
\] (3.17)
\[
\int_{S\mathcal{B}_n} (a^* u)(u^* b)(1 - u^* S^{-2} u)^p \, du = D_{n,p} |\det S|^2 \frac{a^* S^2 b}{n + p + 1},
\] (3.18)
\[
\int_{S\mathcal{B}_n} \|u\|^2 (1 - u^* S^{-2} u)^p \, du = D_{n,p} |\det S|^2 \frac{\text{Tr} S^2}{n + p + 1},
\] (3.19)
where the constant \( D_{n,p} \) is explicitly computed in Lemma 3.10.
Lemma 3.12. For any $p > n$, $a \in \mathbb{C}^n$ and any Hermitian positive-definite matrix $S$, if $u \in \mathbb{C}^n$ is distributed with density
\[
\frac{1}{C_{n,p} |\det S|^2} (1 - u^* S^{-2} u)^p
\]
with respect to the Lebesgue measure on $\mathbb{C}^n$, then the following identity in distribution holds:
\[
|a^* u|^2 \overset{d}{=} \|Sa\|^2 Y_{p+n+1}.
\]

The proofs of Lemmata 3.11 and 3.12 are exactly analogous to the proofs of their spherical counterpart, Lemmata 2.15 and 2.16.

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