Pseudo Periodic Higgs Inflation

I. G. MáríaN, 1 N. Defenu, 2 A. Trombettoni, 3, 4 and I. Nándoril, 5, 6
1University of Debrecen, P.O. Box 105, H-4010 Debrecen, Hungary
2Institut für Theoretische Physik, Universität Heidelberg, D-69120 Heidelberg, Germany
3CNR-IOM DEMOCRITOS Simulation Center, Via Bonomea 265, I-34136 Trieste, Italy
4SISSA and INFN, Sezione di Trieste, Via Bonomea 265, I-34136 Trieste, Italy
5MTA-DE Particle Physics Research Group, P.O.Box 51, H-4001 Debrecen, Hungary
6MTA Atomki, P.O. Box 51, H-4001 Debrecen, Hungary

There is a strong interest in finding a link between Higgs scalar fields and inflationary physics. A good Higgs-inflation potential should have a form as simple as possible, provide agreement with observations and be used, once its parameters are determined from experimental data, to make predictions. In literature the presence of one or possibly more minima of several class of potentials has been discussed: here we focus on a potential having infinitely many non-degenerate minima, with a tunable energy difference between them. Such potential has the advantage of having a unique ground-state, but at the same time with other minima that can excited. The potential having such property, that we term pseudo periodic potential, reads \( V(\phi) = \frac{1}{2} m^2 \phi^2 + u \left[ 1 - \cos(\beta \phi) \right] \) is a combination of quadratic monomial and periodic terms, known as massive sine-Gordon model. We show that it provides an excellent agreement with observations on the fluctuations of cosmic microwave background radiation, contains two adjustable parameters (the ratio \( u/m^2 \) and the frequency \( \beta \)) and can be considered as a convenient reparametrization of \( \phi^{\pm n} \) models. We discuss the applicability of the model for inflationary cosmology and for Higgs-inflation. Finally, motivated by the need of extract properties independent for the specific form of the potential, we perform a renormalization group (RG) running in the post-inflation period and we investigate the consequences of the RG running on convexity for the considered pseudo periodic potential. The implications of the obtained results for a more systematic use of RG are finally discussed.

PACS numbers: 98.80.-k,14.80.Cp,11.10.Hi

Introduction. — Early universe and Higgs physics are examples where scalar fields find a natural role to play in standard models (SM) of cosmology and particle physics. Since scalar fields can mimic the equation of state required for exponential expansion of the early universe, various types of scalar potentials have been proposed in inflationary cosmology. The simplest of these scenarios is provided by the slow-roll single field models with minimal kinetic terms [1]. A good candidate for inflationary potential should have a small number of free parameters which serves as the first condition for a reliable model, and the primary example is the well studied quadratic, large field inflationary (LFI) potential \( V = \frac{1}{2} m^2 \phi^2 \).

The progresses in model building are tested and motivated by the continuous comparison with observations and experiments such as the Planck mission [2], which measures thermal fluctuations of cosmic microwave background radiation (CMBR). Using reduced Planck units (\( c \equiv h \equiv 1 \) and \( m_p^2 = 1/(8\pi G) \equiv 1 \)), the conditions \( \epsilon \ll 1 \) and \( \eta \ll 1 \), with \( \epsilon \equiv V''/(2V) \) and \( \eta \equiv V''/V \) have to be fulfilled by a reliable potential for a prolonged exponential inflation with slow roll down. In addition the e-fold number \( N \equiv -\int_{\phi_0}^{\phi} d\phi V''/V \) should be in the range \( N = 50 - 60 \). These data are encoded in expressions for the scalar tilt \( n_s - 1 \approx 2\eta - 6\epsilon \) and for the tensor-to-scalar ratio \( r \approx 16\epsilon \), which can be directly compared to CMBR data. Of course, a reliable model should provide agreement with observations. For example, the quadratic LFI model gives \( n_s - 1 \approx -2/N \) and \( r \approx 8/N \) which is almost excluded by recent results of the Planck mission, see Fig. 1.

There is a strong interest to find a link between these scalar fields of the Higgs and inflationary physics [3, 4]. As a consequence of the Brout-Englert-Higgs mechanism [5, 6], three degrees of freedom of the Higgs scalar field (out of the four) mix with weak gauge bosons. The remaining degree of freedom becomes the Higgs boson discovered at CERN’s Large Hadron Collider [7, 8]. The complete Lagrangian for the Higgs sector of the SM with
the single real scalar field \( h \) reads
\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} h \partial^{\mu} h - \frac{1}{2} M_{h}^{2} h^{2} - \frac{M_{h}^{2}}{2 v^{2}} h^{4} - \frac{M_{h}^{2}}{8 v^{2}} h^{4} - \left( M_{W}^{2} W_{\mu}^{+} W_{\mu}^{-} + \frac{1}{2} M_{Z}^{2} Z_{\mu} Z_{\mu} \right) \left( 1 + \frac{2 h}{v} + \frac{h^{2}}{v^{2}} \right).
\]
with \( v = 246 \text{ GeV} \) known from low-energy experiments and \( M_{h} = \sqrt{-2 \mu^{2}} = \sqrt{2 \mu v^{2}} \). The measured value for the Higgs mass \( M_{h} = 125.6 \text{ GeV} \) implies \( \lambda = 0.13 \). Incidentally we note that the latter value is close to the predicted value based on an assumption of the absence of new physics between the Fermi and Planck scales and the asymptotic safety of gravity \[9\].

Inflationary potentials. — Extrapolating the SM of particle physics up to very high energies lead to interpret the Higgs boson as the inflaton. Therefore, the most “economic” choice would be to use the same scalar field for Higgs and inflationary physics. In case of a large non-minimal coupling to gravity \[3\], this approach results in the following action in the Jordan frame
\[
S = \int d^{4}x \sqrt{-g} \frac{m_{p}^{2}}{2} \left( F(h) \bar{h} - \tilde{g}^{\mu\nu} \partial_{\mu} \tilde{h} \partial_{\nu} \tilde{h} - 2 U(\tilde{h}) \right)
\]
with \( F(h) = 1 + \xi \tilde{h}^{2}, \quad U(\tilde{h}) = m_{p}^{2} \sqrt{2 \cos \beta \phi_{0}} \left( \tilde{h}^{2} + \frac{2 e^{2 \beta} \beta^{2} \phi_{0}^{2}}{m_{p}^{2}} \right) \) (2)
where \( \tilde{g}^{\mu\nu} \) is the metric in the Jordan frame, \( \xi \) is a new parameter, \( \tilde{h} \) is the dimensionless Higgs scalar and \( U(\tilde{h}) \) is the quartic SM Higgs potential. \[1\]

In order to perform the slow-roll study, the action is usually rewritten in the Einstein frame and takes the form
\[
S = \int d^{4}x \sqrt{-g} \left[ m_{p}^{2} \bar{h}^{2} - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right]
\]
with \( V(\phi) = m_{p}^{2} U / F^{2} \), \( \frac{d\phi}{dt} = m_{p} \sqrt{1 + (1 + 6 \phi^{2})^{1/2}} \) (3)
(the metric tensor being denoted by \( g^{\mu\nu} \)). For \( \xi = 0 \) one finds \( \phi = m_{p} \tilde{h} \), and the potentials in (2) and (3) have the same shape. For \( \xi \neq 0 \), the Higgs-\( \xi \) inflaton potential reads
\[
V(\phi) = \frac{m_{p}^{4} \lambda}{4 \xi^{2}} \left( 1 - e^{-\sqrt{2/3 \beta} \phi / m_{p}} \right)^{2}, \quad (4)
\]
which is considered as a zero parameter model since the overall factor of the potential is entirely determined by the amplitude of the CMBR anisotropies. However, this inflationary scenario is problematic since perturbative unitarity is violated which can be healed by introducing new degrees of freedom but than minimality is lost.

Another proposal to “economically” build up the scalar sector is the Higgs inflation from false vacuum where \( \xi = 0 \), but the SM Higgs potential is extended and assumed to develop a second (or more) minimum \[3\]. The difficulty is to achieve an exit from the inflationary phase: one may introduce new fields, but than the attractive minimality of the model would be lost.

Another possible drawback of the Higgs-inflaton potential to its applicability is the vacuum stability \[10\]. The measured Higgs mass is close to the lower limit, 126 GeV, ensuring absolute vacuum stability within the SM \[10\]. Stability studies of various polynomial types of Higgs potentials have been performed by using functional renormalization group (RG) technique \[11\] and reported no stability problems.

\( \phi^{2} \) or/and not-\( \phi^{2} \). — An important qualitative issue is provided by the task of understanding the structure of the inflaton potential. It is clear that a potential \( \phi^{2} \) has a single minimum, and that adding higher-order powers \( \phi^{2n} \) one has more minima. In \[12\] the \( \phi^{2} \) or not-\( \phi^{2} \) issue was tested on the simplest inflationary potential: constraints were obtained and the relevance of non-Gaussianity discussed. From the opposite point of view, a very much “not-\( \phi^{2n} \)” potential is the one having infinitely many minima, at the same energy. In this logic one can explore a periodic potential of the form
\[
V_{SG}(\phi) = u \left[ 1 - \cos(\beta \phi) \right], \quad (5)
\]
which for finite Fourier amplitude is of course always bounded from above and below \[13\]. The proposal of \[5\] has also the advantage that one can also think to construct a very economic scalar sector by incorporating the periodic scalar axion potential too \[13\].

The periodic potential \[5\] is usually denoted as natural inflation or Pseudo-Numbo-Goldstone boson (PNGB) model, while in field theory and condensed matter is denoted as the sine-Gordon model \[14\]. It also been proposed and studied as a viable inflationary scenario \[12 \[13 \[15 \[16\]. It was shown that the the potential \[5\] is able to produce better agreement to Planck results than the simplest quadratic LFI potential \[12\] and that in \( d = 4 \) it has a single phase \[13\].

We discuss now the the applicability of the potential \[5\] for inflationary cosmology. In \( d = 4 \) dimensions the scalar field carries a dimension: \( \phi = k^{(d-2)/2} \phi \), where \( \phi \) is dimensionless and \( k \) is an arbitrary chosen momentum scale convenient to take at the planck mass \( k = m_{p} \). Thus, the corresponding dimensionless parameters are \( \beta = m_{p}^{-1} \beta \) and \( u = m_{p}^{2} u \). In this work we use the reduced Planck units where \( m_{p} = 1 \) still the tilde superscript has been kept for a better understanding. The parameters \( \epsilon, \eta \) and \( N \) reads as
\[
\epsilon = \frac{\beta^{2}}{2} \cot^{2} \left( \frac{\beta \phi}{2} \right), \quad \eta = \frac{\beta^{2}}{2} \frac{\cos(\beta \phi)}{\sin^{2} \left( \frac{\beta \phi}{2} \right)}, \quad N = - \frac{2}{\beta^{2}} \log \cos \left( \frac{\beta \phi}{2} \right) \mid_{\phi_i} \phi_{f}, \quad (6)
\]
from which the scalar tilt and tensor-to-scalar ratio
\[
n_{s} - 1 \approx \frac{\beta^{2}}{2} \left[ 1 - 2 \sin^{-2} \left( \frac{\beta \phi}{2} \right) \right], \quad r \approx 8 \beta^{2} \cot \left( \frac{\beta \phi}{2} \right)
\]
which imply the relation $n_s - 1 + \frac{r}{2} = -\beta^2$ which is very similar to that of obtained for the quadratic monomial potential but having a dependence on the frequency $\beta$, thus, it appears as an additional parameter which can be tuned to achieve a good agreement with the Planck data, see Fig. 2. For decreasing frequencies (from bottom to the top) it reduces to the quadratic monomial model which is almost excluded by experimental data. For increasing frequencies (from top to the bottom) there is a region where it finds a good agreement with the Planck data ($\beta \sim 0.15$) but too large frequencies are not supported.

**The massive sine-Gordon model.** — Despite the agreement with Planck results of the potential (5) is better than the simplest quadratic LFI potential, we verified that the agreement is not as good as for the linear potential alone. The combination of polynomial and periodic terms could be even more reliable. For example in [17] a linear term is added to the periodic one and proposed as a viable inflationary potential. However in this case the potential is unbounded from below. Thus, a better choice would be to extend periodic potential in such a way that:

1. the potential is always bounded from below;
2. the model has $Z_2$ symmetry;
3. the model has infinitely-many non-degenerate minima, separated in energy by a tunable amount.

In order to fulfill latter requirements it is possible to add a quadratic mass term to periodic potentials, leading to

$$V_{MSG}(\phi) = \frac{1}{2} m^2 \phi^2 + u \left[ 1 - \cos(\beta \phi) \right].$$

(7)

Potential (7) is called the massive sine-Gordon model (MSG). It contains two adjustable parameters (the ratio $u/m^2$ and the frequency $\beta$) and it can be considered as a reparametrised version of $\phi^{2n}$ models. We show in the following that it provides an excellent agreement with CMBR observations.

The parameters $\epsilon$, $\eta$ and $N$ using dimensionless quantities are given by

$$\epsilon = \frac{1}{2} \left( \frac{\tilde{u} \beta \sin(\beta \phi) + \phi}{\tilde{u} m^2 \left[ 1 - \cos(\beta \phi) \right] + \frac{1}{2} \phi^2} \right)^2$$

(8)

$$\eta = \frac{\tilde{u} \beta^2 \cos(\beta \phi) + 1}{\tilde{u} m^2 \left[ 1 - \cos(\beta \phi) \right] + \frac{1}{2} \phi^2}$$

(9)

$$N = - \int_{\phi_i}^{\phi_f} d\phi \frac{\tilde{u} m^2}{\tilde{u} \beta \sin(\beta \phi) + \phi}$$

(10)

which depend on the ratio $\tilde{u}/m^2$ and the frequency $\beta$. If the mass term is negligible compared to the periodic one, than one arrives back to the SG model. If the periodic term is negligible compared to the mass, than one gets the quadratic monomial inflationary model. This ratio plays a crucial role in the phase diagram of the MSG model as discussed later.

Let is first compare results obtained from the SG and MSG models for various values for the ratio $\tilde{u}/m^2$ but with fixed frequency. As it is shown in Fig. 3 the MSG model provides more reliable results. Moreover, the ratio $\tilde{u}/m^2$ and the frequency $\beta$ can be fixed by choosing the best fit to observations, see Fig. 4. In order to show how good the fit of the MSG model to Planck data is, the e-fold number is fixed for $N = 55$ in Fig. 5 and for $N = 50, 55, 60$ in Fig. 6 where we indicate regions of the parameter space of the MSG with different colours corresponding to different level of acceptance. The ratio and frequency taken from the dark blue region give the best fit to Planck data and the third parameter, the mass can be fixed by the power spectrum normalisation which gives dimensionful values

$$\tilde{u}/m^2 = \nu \beta^2/m^2 \approx 0.3^2/0.22^2 > 1, \quad m \approx 5.92 \times 10^{-6} m_p.$$
FIG. 4: Best fit of the MSG model to Planck data for $\tilde{u}/\tilde{m}^2 = 1/(0.22)^2$ and for various frequencies $\tilde{\beta}$ (orange full lines).

FIG. 5: Regions of the parameter space of the MSG model indicated by different colours corresponds to different level of acceptance for $N = 55$. Dark blue region give the best fit to Planck data.

From one side one could attribute the fact that there is an improved agreement of the MSG model with respect to the PNGB model, with Planck results to the simple fact that one has a parameter more. While this is certainly true, the exploration of different potentials [1] shows that adding fitting parameters often does not improve at all or significantly, the obtained results. Here is the inclusion of a mass term produces a better agreement with observations because it adds minima and at the same it makes the potential overall concave. Thus, one can conclude that convexity or non-convexity of the potential may be crucial, and then a good candidate for inflationary potential should have a concave region as it is indicated in Fig. 1. We discuss in the following the issue of concavity/convexity in the post-inflation period adopting a renormalization group (RG) point of view.

Convexity and phase structure. — The MSG model was extensively studied in $d = 2$ dimensions by functional RG [19, 20]. Here, we extend these RG studies for $d = 4$ dimensions using an optimised regulator giving the best convergence and at the time the same functional form in any dimensions at level of the local potential approximation [19, 20]. The dimensionless equation has then the following form

$$
\left( d - \frac{d - 2}{2} \partial_\phi \partial_\phi + k \partial_k \right) \tilde{V}_k = \frac{2 \alpha_d}{d} \frac{1}{1 + \beta^2 \tilde{V}_k} \quad (12)
$$

where $\tilde{V}_k$ is the dimensionless scaling potential and $\alpha_d = \Omega_d/(2(2\pi)^d)$. Looking for the solution of Eq. (12) in the functional form of the MSG model (7), one should separate the periodic and non-periodic parts where the latter results in trivial scaling for the dimensionless frequency $\beta_k^2 = \beta^2 k^{d-2}$ and for the dimensionless mass $\tilde{m}_k^2 = m^2 k^{-2}$ while the corresponding dimensionful couplings $\beta^2$ and $m^2$ remain constant over the flow. Solving the periodic part of Eq. (12) one finds two phases controlled by the dimensionless quantity $\tilde{u}_k \beta_k^2 / \tilde{m}_k^2$ which tends to a constant in the IR limit. In the $(Z_2)$ symmetric phase the magnitude of this constant is arbitrary (and depends on the initial conditions) but always smaller than 1, i.e., $\lim_{k \to 0} |\tilde{u}_k \beta_k^2 / \tilde{m}_k^2| < 1$, see blue lines of Fig. 7. In

FIG. 6: Best acceptance regions for $N = 50, 55, 60$.

FIG. 7: RG flow of the MSG model showing two phases.

FIG. 2: Density of states for $d = 2$.

FIG. 3: Density of states for $d = 4$.
initial values) which serves as an upper bound, see red lines of Fig. 7. The black line separates the two phases.

Conclusions and outlook.— A good candidate for an inflationary potential should fulfill the following conditions, (i) to have as simple form as possible (with the smallest possible number of parameters), (ii) to provide the best agreement with observations, (iii) to be as "economic" as possible in term of the formulation of the theory. In this paper we propose the massive sine-Gordon model as an inflationary potential and we show that adding the mass term to the periodic potential produces a remarkably improved agreement with the Planck results. We attribute such improvement to the fact that it has infinitely many minima, but they are not-degenerate and with tunable controllable energy difference, providing a way to be as much as possible both $\phi^2$ and not-$\phi^2$. The crucial point emerging from a careful analysis of different potentials is that the inflationary potential should have a concave region as indicated in Fig. 1, and the mass term in the massive sine-Gordon potential add such an overall convexity in presence of the many minima.

To explore the issue of the convexity, we adopted a renormalization group (RG) point of view and we analyzed the phase diagram associated to the running determined by the slow-roll conditions $\left[ \frac{\dot{u}_k^2 u_k^2}{n_k^2} \right]$ having in mind the post-inflation period. The obtained values for the ratio $\left[ \frac{\dot{u}_k^2 u_k^2}{n_k^2} \right]$ are found in the phase of spontaneous symmetry breaking and very close to the critical line. Thus, the RG study of the MSG model shows that slow-roll conditions represent strong constraints on the RG running i.e., it stays in its broken phase. Moreover, the RG running produces us a convex (dimensionful) potential in agreement with the theoretical requirement of the convexity of the (dimensionful) effective potential.

We are of course aware that the performed RG analysis is applied to the specific model we are considering, the massive sine-Gordon potential, and that a plethora of models can be considered. However, the RG approach appears to be a method that can show the presence of robust properties independent from the specific choice of the inflationary potential and to obtain model-independent predictions to compare with available/future experimental data. Therefore it could be an excellent tool to show that, as we expect after the analysis of the present paper, any inflationary potential with concave regions should become a convex one if it is considered as an effective potential when quantum fluctuations are taken into account.

Acknowledgements.— This work was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Useful discussions with F. Bianchini, G. Gori, Z. Trocsanyi and G. Somogyi are gratefully acknowledged.

[1] J. Martin, C. Ringeval, V. Vennin, "Encyclopaedia Inflationaris", Phys. Dark Univ. 5-6 (2014) 75-235.