Resolutions of $p$-Modular TQFT’s and Representations of Symmetric Groups

Thomas Kerler
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1. INTRODUCTION AND SURVEY OF RESULTS

In this article we bring together several areas of representation theory in a series of interrelated results. The first is the rather established theory of $p$-modular representations of the symmetric groups, followed by the representation theory of groups of Lie type over the finite field $\mathbb{F}_p$, and, finally, the area of topological quantum field theories (TQFT’s) over $\mathbb{F}_p$. The latter have been our original motivation since they appear as constant order reduction in the Reshetikhin Turaev Theories. In fact, as we shall outline in more detail at the end of in this section, the identities we will find in this article will, for example, imply relations between the Lescop invariant and the Reshetikhin Turaev invariant for 3-manifolds for $p = 5$.

Our results in each area concern resolutions and expansions of $p$-modular representations and invariants into their respective characteristic zero counterparts, and, thus, naturally build on each other. Let us state the results in order, beginning with the case of the symmetric groups.

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The Symmetric Groups: The representation theory of the symmetric group $S_n$ in $n$ letters over $\mathbb{Q}$ (or $\mathbb{Z}$) is well known. The theory is semisimple and the simple representations are isomorphic to the Specht modules $S^\tau$, where $\tau$ is a Young diagram with $n$ boxes. They have a natural basis given by Young tableaux, and the $S_n$-action preserves the free $\mathbb{Z}$-modules (or lattices) $S^\tau = \mathbb{Z}[S_n]/\mathbb{Z}[S^n]$ generated by these basis vectors. Passing to $S^\tau_p = S^\tau_{Z/p}\mathbb{Z}$ we thus obtain representations of the same rank over the field $\mathbb{F}_p$ for any given prime $p \geq 3$. The $S^\tau_p$, however, are not longer irreducible, but they have a unique simple quotient $D^\tau_p$ obtained from canonical inner forms on the Specht modules, see \cite{4}. The representations $S^\tau_p$ and $D^\tau_p$ define characters $\chi^\tau_p$ and $\varphi^\tau_p$ on $S_n$ with values in $\mathbb{Z}$ and $\mathbb{F}_p$, respectively. We also denote by $\chi^\tau_p$ the $p$-reduction of $\chi^\tau$, which may, alternatively, be thought of as the character associated to $S^\tau_p$.

The relationship between the “ordinary” representations $S^\tau_{Z/p}\mathbb{Z}$ and “ordinary” characters $\chi^\tau_{Z/p}\mathbb{Z}$ and their $p$-modular counterparts $D^\tau_p$ and $\varphi^\tau_p$ is, even after decades of research, still intensely investigated with many open questions remaining. While several algorithms exist for expressing the $\chi^\tau_p$ in terms of the $\varphi^\tau_p$, fewer results exist for the converse relations, and even fewer results relate the modules themselves. The exact modular structure of the $S^\tau_p$ in terms of a modular ordering of the $D^\tau_p$-components has only very recently been uncovered by Kleshchev and Sheth in \cite{12} for the special case of Young diagrams $\tau$ with two rows.

The first result of this article may be thought of as the inverse relation of the result in \cite{12}, and it implies a similarly inverse relation for the characters.

**Theorem 1** Let $p \geq 3$ be a prime and $\tau = [a,b]$ be the two row Young diagram with row lengths $a$ and $b$ with $0 \leq a - b \leq p - 2$. Consider the sequence of Young diagrams $\tau_j$ given by $\tau_{2i} = [a + ip, b - ip]$ and $\tau_{2i-1} = [b + ip - 1, a - ip + 1]$, whenever defined. (That is, $\tau_0 = \tau = [a,b]$, $\tau_1 = [b + p - 1, a - p + 1]$, $\tau_2 = [a + p, b - p]$, $\tau_3 = [b + 2p - 1, a - 2p + 1]$, etc.)

1. There is a resolution of the modular, simple representation $D^\tau_p$ of $S_n$ in terms ordinary representations given by an exact sequence of $S_n$-equivariant maps as follows.

\[ \ldots \longrightarrow S^\tau_{2i} \longrightarrow S^\tau_{2i-1} \longrightarrow S^{\tau_0}_p \longrightarrow D^\tau_p \longrightarrow 0. \quad (1) \]

2. We have the following expansion of $p$-modular characters in terms of ordinary characters.

\[ \varphi^\tau_p = \sum_{i \geq 0} (-1)^i \chi^\tau_i, \quad (2) \]

It is an intriguing fact that the maps in the sequence \cite[4] are powers of generators of $\mathfrak{sl}_2$ acting dually on $(\mathbb{Z}^2)^{\otimes n}$, which, as an $S_n$-module, contains the permutation modules $M^\tau$. The precise action is constructed in Corollary \cite{8} of Section 4, where we prove that it yields a well defined complex.

The proof of exactness of this complex uses the results of \cite{12} and occupies most of Section 5. A generalization of Theorem \cite{4} to $n$-row diagrams using a dual $\mathfrak{sl}_n$ is likely to yield more insights the structure of $p$-modular Specht modules for general Young diagrams. The character identity \cite[4] is stated as an immediate consequence in Corollary \cite{12} of Section 6.

**The Symplectic Groups:** The symmetric groups typically appear as or within Weyl groups of groups of Lie type. In this article we are particularly interested in the symplectic groups $\text{Sp}(2g, \mathbb{M})$ where $\mathbb{M}$ may be $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{F}_p$. There are obvious generalizations of our results to most other groups of Lie type, which we leave to the reader.
Let $H = H_1(\Sigma_g, \mathbb{Z})$ be the fundamental representation of $\text{Sp}(2g, \mathbb{Z})$ with symplectic basis $\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle$, and denote by $V(\varpi_k) \subset \wedge^k H$ the subrepresentation generated by the highest weight vector $a_1 \wedge \ldots \wedge a_k$. Over $\mathbb{R}$ this is the irreducible representation of fundamental heighest weight $\varpi_j = \epsilon_1 + \ldots + \epsilon_j$. Denote by $V_p(\varpi_j)$ the respective $p$-modular reduction, and by $\overline{V}_p(\varpi_j)$ the unique irreducible subquotient over $\mathbb{F}_p$ generated by the unique highest weight vector. Let $g - p + 2 \leq l \leq g$ and $\tilde{l} = 2(g - p + 1) - l$. From Theorem 1 now we derive resolutions of $\text{Sp}$-representations that are given by exact sequences as follows.

$$
\ldots \rightarrow V_p(\varpi_{l-4p}) \rightarrow V_p(\varpi_{l-2p}) \rightarrow V_p(\varpi_l) \rightarrow V_p(\varpi_l) \rightarrow \overline{V}_p(\varpi_l) \rightarrow 0. \quad (3)
$$

Evidently, (3) implies similar expansions of $\text{Sp}$-characters, and such expansions also exist for other groups of Lie type. The generalization more interesting to us, which in fact implies (3), are resolutions of topological quantum field theories (TQFT's).

In Lemmas 2 and 4 of Section 2 we establish the relation between the $\text{Sp}(2g, \mathbb{Z})$ weight spaces of $V(\varpi_k)$ and the Specht modules and the respective explicit actions of the $\mathfrak{sp}_{2g}$-generators on the Young diagrams is derived in Section 3.

**Homological TQFT's:** Recall that a TQFT is a functor $\mathcal{V} : \text{Cob}_3 \rightarrow \mathcal{M}$–mod from a category of 2+1-dimensional cobordisms into a category of free modules over a commutative ring $\mathcal{M}$. Specifically, $\mathcal{V}$ assigns to a surface $\Sigma$ a free $\mathcal{M}$-module $\mathcal{V}(\Sigma)$ and to a cobordism between two surfaces an $\mathcal{M}$-linear map between the respective $\mathcal{M}$-modules.

In [2] Frohman and Nicas construct a TQFT $\mathcal{V}^{FN}$ over $\mathcal{M} = \mathbb{Z}$, where the free $\mathbb{Z}$-module for a surface is given by the (co)homology of its Jacobian, specifically, $\mathcal{V}^{FN}(\Sigma) = H^*(\text{Hom}(\pi_1(\Sigma), U(1)), \mathbb{Z}) = \wedge^* H_1(\Sigma, \mathbb{Z})$. The basics of this construction are reviewed in the beginning of Section 2. Further, in Section 3, we will recall the decomposition of this TQFT into its irreducible components $\mathcal{V}^{(j)}_\mathbb{Z}$, with $j = 1, 2, \ldots$, of $\mathcal{V}^{FN}$. They are again TQFT's over $\mathbb{Z}$ and are obtained in [3] from a dual Lefschetz $\mathfrak{sl}_2$-action.

Composing $\mathcal{V}^{(j)}_\mathbb{Z}$ with the canonical (rank-preserving) functor $\mathcal{M}$–mod $\rightarrow \mathcal{F}_p$–mod for primes $p \geq 3$ we obtain a family of TQFT’s $\mathcal{V}_p^{(j)}$ over $\mathcal{M} = \mathcal{F}_p$. As before, the $p$-modular TQFT's are generally highly reducible. However, they have a unique irreducible TQFT subquotient $\overline{\mathcal{V}}^{(j)}_p$.

**Theorem 2** For any prime $p \geq 3$ and integer $k$ with $0 < k < p$ we have an exact sequence of natural transformations of TQFT's

$$
\ldots \rightarrow \mathcal{V}_p^{((i+1)p+k_i+1)} \rightarrow \mathcal{V}_p^{(ip+k_i)} \rightarrow \ldots \rightarrow \mathcal{V}_p^{(2p-k)} \rightarrow \mathcal{V}_p^{(k)} \rightarrow \overline{\mathcal{V}}^{(k)}_p \rightarrow 0, \quad (4)
$$

where we set $k_i = k$ for even $i$ and $k_i = p - k$ for odd $i$.

The sequence is constructed from its symmetric group summands in Corollary 3 of Section 4. Exactness follows in Section 5 from the respective results for Specht modules.

In order to see why (3) is indeed a special case of Theorem 2 observe that the mapping class group $\Gamma_g$ is identical with the group of invertible cobordisms in $\text{Cob}_3$ from a surface $\Sigma_g$ to itself. Hence, any TQFT $\mathcal{V}$ entails for every $g$ a representation of $\Gamma_g$ on $\mathcal{V}(\Sigma)$, which, in the case of $\mathcal{V}^{FN}$, factors through the symplectic quotient $\Gamma_g \rightarrow \text{Sp}(2g, \mathbb{Z})$. The sequence in (3) is now
that the invariant $V$ the following relation between these invariants over $\mathbb{Z}$ polynomial. Combining this observation with Theorem 2 we derive in the end of Section 6.

Note, that all TQFT’s constructed

Johnson-Morita-Extensions and TQFT-Overview:

the Johnson homomorphism $\Delta_\varphi(M, \varphi)$ as follows. Pick any two-sided surface $\Sigma \subset M$ which is dual to $\varphi$, and define $C_{\Sigma} = M - \Sigma$ seen as a cobordism from $\Sigma$ to itself. The value $V(M, \varphi) = \text{trace}(\mathcal{V}(C_{\Sigma}))$ is now independent of the choice of $\Sigma$.

As an important example, Frohman and Nicas extracted in §3 the Alexander polynomial $\Delta_\varphi(M) \in \mathbb{Z}[x, x^{-1}]$ be the Alexander Polynomial of a closed, compact, oriented 3-manifold $M$ with respect to an epimorphism $\varphi : H_1(M) \to \mathbb{Z}$.

Let $\Delta_{\varphi,p}(M)$ be the reduction of $\Delta_\varphi(M)$ obtained by substituting $x = \pm \zeta_p$, with $\zeta_p$ a $p$-th root of unity, and taking the $\mathbb{Z}$ coefficients modulo $p$. Then

$$\overline{\Delta}_{\varphi,p}(M) = \sum_{k=1}^{p-1} (\pm 1)^{k-1} [k] \zeta_p \overline{\mathcal{V}}_p^{(k)}(M, \varphi) \in \mathbb{F}_p[\zeta_p].$$

**Theorem 3** Let $\Delta_\varphi(M) \in \mathbb{Z}[x, x^{-1}]$ be the Alexander Polynomial of a closed, compact, oriented 3-manifold $M$ with respect to an epimorphism $\varphi : H_1(M) \to \mathbb{Z}$.

Let $\Delta_{\varphi,p}(M)$ be the reduction of $\Delta_\varphi(M)$ obtained by substituting $x = \pm \zeta_p$, with $\zeta_p$ a $p$-th root of unity, and taking the $\mathbb{Z}$ coefficients modulo $p$. Then

**Johnson-Morita-Extensions and TQFT-Overview:** Note, that all TQFT’s constructed up to this point have representations of $\Gamma_g$ that factor through $\text{Sp}(2g, \mathbb{Z})$, that is, they contain the Torelli group $\mathcal{I}_g$ in their kernels. In Section 7 of this article we will also consider extensions of these TQFT’s with slightly smaller kernels, at least at the level of representations of $\Gamma_g$.

In §6 Morita constructs a homomorphism $\tilde{k} : \Gamma_g \to \text{Sp}(2g, \mathbb{Z}) \ltimes \frac{1}{2}U$ which has a smaller kernel $K_g \subset \mathcal{I}_g$, and whose image $Q_g \cong \Gamma_g/K_g$ has finite index in $\text{Sp}(2g, \mathbb{Z}) \ltimes \frac{1}{2}U$. It extends the Johnson homomorphism $\mathcal{J}_g \to U$ previously constructed in §3, where $U$ denote the free abelian group $U \cong \mathbb{Z}^3/H/\omega \wedge H \cong V(\omega_3)$. As before we denote $H = H_1(\Sigma_g)$, considered as an $\text{Sp}$-module, and we let $\omega \in \mathbb{A}^2 H$ be the standard $\text{Sp}$-invariant symplectic form.

Non-trivial representations of $Q_g$ are readily obtained in Proposition §3 of Section 7 as the extension of a pair of representations $V(\omega_i)$ and $V(\omega_{i+3})$ using the (up to scale unique) $\text{Sp}$-equivariant map $U \to \text{Hom}(V(\omega_i), V(\omega_{i+3}))$. We prove that these extension also exist for the irreducible $p$-modular representations $\overline{\mathcal{V}}_p(\omega_i)$. In TQFT language this translates to the following result.

**Theorem 4** For $0 < k < p - 3$ there are indecomposable representations $\overline{\mathcal{U}}_p^{(k)}(\Sigma_g)$ of the mapping class group $\Gamma_g$ that factor through $Q_g = \Gamma_g/K_g$ but represent the Torelli group $\mathcal{I}_g$ non-trivially. There is a short, non-split exact sequence of $\Gamma_g$-equivariant maps as follows:

$$0 \to \overline{\mathcal{V}}_p^{(k+3)}(\Sigma_g) \to \overline{\mathcal{U}}_p^{(k)}(\Sigma_g) \to \overline{\mathcal{V}}_p^{(k)}(\Sigma_g) \to 0.$$

We know from Theorem §3 below that $\overline{\mathcal{U}}_p^{(1)}$ does on fact extend to a TQFT. However, the question whether or how the $\overline{\mathcal{U}}_p^{(k)}$ and $\mathcal{U}_p^{(k)}$ descend from TQFT’s for general $p$ and $k$ is still open, and will be discussed in future work.
We summarize the different TQFT’s of this paper, the constructions connecting them, and the included modules of the symmetric groups \( S_n \) in the following table:

| Construction                     | TQFT                        | Modules                             |
|----------------------------------|-----------------------------|-------------------------------------|
| \( \nu^F_{FN} \)                  | Jacobian TQFT \( \nu^F_{FN} \) | Permutation Modules                 |
| \( SL(2, \mathbb{R}) \)-Lefschetz Decomposition | Fully reducible over \( \mathbb{Z} \) |                                     |
| \( \nu^{(k)}_{\mathbb{Z}} \)    | Lefschetz Component, irreducible over \( \mathbb{Z} \) | Specht Modules \( S^\mathbb{Z}_k \) |
| \( \nu^{(k)}_{p} \)             | Reducible over \( \mathbb{F}_p \) | \( p \)-Specht Modules \( S^p_k \) |
| Null space quotient              |                             | Simple \( S_n \)-modules \( D^p_k \) |
| \( \overline{\nu}^{(k)}_p \)    | Irreducible over \( \mathbb{F}_p \) | Not \( \mathbb{Z} \)-liftable, but resolutions in \( \nu^{(j)}_p \)'s |
| Johnson-Morita Extension         |                             |                                     |
| \( \overline{U}^{(k)}_p \)      | Indecomposable with two factors \( \overline{V}^{(k)}_p \) and \( \overline{V}^{(k+3)}_p \) |                                     |

**Relations to the Reshetikhin Turaev Theory:** The original motivation of this article comes from the study of the TQFT’s \( \nu^F_{\text{RT}} \) constructed by Reshetikhin and Turaev in [21] from the quantum group \( \mathcal{U}_{\zeta_p}(\mathfrak{so}_3) \) for a \( p \)-th root of unity \( \zeta_p \). Thus, in order to put the TQFT’s of this article into their broader context and illustrate their relevance let us briefly sketch the pertinent relations and results from other work in quantum topology.

It is shown by Gilmer in [3] that, for a restricted set of cobordisms, \( \nu^F_{\text{RT}} \) can be written as a TQFT over the ring of cyclotomic integers \( \mathbb{Z}[\zeta_p] \). For the Reshetikhin Turaev invariants of closed 3-manifolds this integrality property was previously proved in [17, 13].

Applying the ring reduction \( \mathbb{Z}[\zeta_p] \rightarrow \mathbb{F}_p \), with \( \zeta_p \rightarrow 1 \), we obtain a TQFT \( \nu^F_p \) over the finite field \( \mathbb{F}_p \) from \( \nu^F_{\text{RT}} \) for a given \( p \). It is now natural to ask whether or not there exists a relationship between the \( \nu^F_p \) and the \( \overline{V}^{(j)}_p \), since they are both TQFT’s over \( \mathbb{F}_p \) and are conjectured to share a list of other features.

Identifications between the TQFT’s \( \nu^F_p \) obtained from quantum groups on the one hand and the ones obtained from the homological theory \( \nu^F_{FN} \) (and their Johnson Morita extensions) on the other hand will entail many insights into the relation between classical and quantum invariants as well as establish natural bases for the quantum theory in the language of the tensor calculus of the symplectic groups. Moreover, we expect this to lead to a deeper understanding of the geometric interpretations of the higher order terms in the cyclotomic integer expansions of the Reshetikhin Turaev theories, which include the Ohtsuki and, particularly, the Casson-Walker-Lescop invariants.

Since the \( \nu^F_{\text{RT}} \) are “unitarizable” theories and the dimensions of the vector spaces do not match any combination of ordinary Sp-representations we must expect the simple quotients \( \overline{V}^{(j)}_p \) to enter the relations rather than the reducible TQFT’s \( \nu^{(j)}_p \). This is precisely the point where the main results of this article are crucially needed. The resolutions of \( p \)-modular TQFT’s provide the missing link between the classical invariants defined over \( \mathbb{Z} \) and the quantum invariants defined over \( \mathbb{F}_p \).
Thus far we are able to understand these relations and their applications in a rather detailed manner for \(p = 5\). We state next the main result from \([10]\).

**Theorem 5** (\([10]\)) *We have the following isomorphism of \(\Gamma_g\)-representations over \(\mathbb{F}_5\):*

\[
\mathcal{V}_5^I(\Sigma_g) \cong \overline{U}_5^{(1)}(\Sigma_g) .
\]

*In particular, for pairs \((M, \varphi)\) as in Theorem 3 we have*

\[
\mathcal{V}_5^I(M, \varphi) = \overline{\mathcal{V}}_5^{(1)}(M, \varphi) + \overline{\mathcal{V}}_5^{(4)}(M, \varphi) = -2 \cdot \lambda_{\text{Lescop}}(M) \mod 5 \quad (7)
\]

The first equality in (7) follows directly from (6). The second equality is now a consequence of the identification with the Alexander polynomial at a root of unity from Theorem 3. It is used in \([9]\) to identify \(\mathcal{V}_5^I(M, \varphi)\) with the \(\mathbb{F}_5\)-reduction of the Lescop invariant \(\lambda_{\text{Lescop}}(M)\). This identity is special to \(p = 5\) and does not hold for \(p > 5\).

Observe also that (7) implies that the invariant \(\mathcal{V}_5^I(M, \varphi)\) is really independent of \(\varphi\). This is not surprising since it is also equal to the lowest oder non-vanishing coefficient of the cyclotomic integer expansion of \(\mathcal{V}_{\zeta_5}^{RT}(M)\). Nevertheless, this raises an interesting question, namely, which linear or polynomial combinations of the \(\mathcal{V}_p^{(j)}(M, \varphi)\) are independent of the choice of \(\varphi\) and, thus, yield true invariants of closed 3-manifolds.

**Dimensions, Combinatorics, and Asymptotics:** An important special case for any character formula is the implied relation for the dimensions of modules, that is, the characters evaluated at the unit element. The dimensions of the irreducible modules over \(\mathbb{F}_5\) in this article are Fibonacci numbers, while the dimensions of the corresponding modules over \(\mathbb{Z}\) are Catalan numbers. As a result we obtain in (54) and (55) of Section 6 identities that express the Fibonacci numbers as 5-periodic, alternating sums in these Catalan numbers. Despite the simplicity of these relations they appear to be unknown in the available literature.

We also determine in Section 6 the asymptotics of the dimensions of the TQFT modules for \(g \to \infty\), which is summarized in the following lemma.

**Lemma 1** *For a fixed prime \(p \geq 3\) and fixed \(j\), the dimensions of the vector spaces grow for large \(g\) as follows:*

\[
\dim(\mathcal{V}_p^{(j)}(\Sigma_g)) \sim \text{const} \cdot \|\mathbf{F}_p\|^g \quad \text{and} \quad \dim(\overline{\mathcal{V}}_p^{(j)}(\Sigma_g)) \sim \text{const} \cdot \|\mathbf{f}_p\|^g \quad (8)
\]

where \(\|\mathbf{F}_p\| = \frac{p}{4 \sin^2(\frac{\pi}{p})}\) and \(\|\mathbf{f}_p\| = 4 \cos^2(\frac{\pi}{2p})\).

Moreover, there are polynomials \(R_j(f) \in \mathbb{Z}[f]\) of degree \(\deg(R_j) = \frac{p-3}{2}\) with

\[
\|\mathbf{F}_p\| = R_p(\|\mathbf{f}_p\|) .
\]

The polynomial dependence between the asymptotic behaviors from (8) suggests a similar “polynomial” dependence between the TQFT’s. If the products of the dimensions are replaced by tensor products of the respective TQFT’s, this suggests that \(\sim R_p(\|\mathbf{f}_p\|)^g\) describes the asymptotics of parts of an \(\mathbb{F}_p^{\frac{p-3}{2}}\)-fold tensor product of the \(\overline{\mathcal{V}}_p^{(j)}\). We are thus led to the following conjecture on the constant order structure of the Reshetikhin Turaev theory, which can be verified for the genus=1 mapping class group (SL(2,\(\mathbb{Z}\)) representations.

**Conjecture 6**

The irreducible components of \(\mathcal{V}_p^{(j)}\) can be found as irreducible components of \(\bigotimes \overline{\mathcal{V}}_p^{(j)}\).
Acknowledgements: I am much indebted to Alexander Kleshchev for pointing out his results in [12], and to Gordon James and Alain Reuter for checking the arguments in Section 5. I thank Ronald Solomon, Stephen Rallis, Adalbert Kerber, and Charles Frohman for comments and interest. Also, I have benefited much from discussions with Pat Gilmer on related integral TQFT’s and their applications. Finally, I would like to express my thanks to Tomotada Ohtsuki and Hitoshi Murakami for hospitality at the R.I.M.S., Kyoto, where final parts of this paper were completed, and for the opportunity to present and discuss the results.

2. Frohman-Nicas TQFT’s over \( \mathbb{Z} \)

In [2] Frohman and Nicas construct a TQFT that allows the interpretation of the Alexander Polynomial as a weighted Lefschetz trace. The vector spaces are the cohomology rings of the Jacobians of the surfaces. In [8] we extract a natural Lefschetz \( SL(2, \mathbb{R}) \)-action on these spaces, with respect to which this TQFT is equivariant. Let us describe in this sections the basic ingredients of the Frohman Nicas theory, reviewing also the conventions of [8].

We denote by \( \text{Cob}^{22}_{fr} \) the category of evenly framed cobordisms between connected standard surfaces \( \Sigma_g \). These are essentially the cobordisms obtained by an even number of surgeries, see Lemma 10, [8]. The Frohman Nicas TQFT is then given as a functor

\[
\mathcal{V}_F N : \text{Cob}^{22}_{fr} \longrightarrow SL(2, \mathbb{R}) - \text{mod}_\mathbb{R},
\]

for which

\[
\mathcal{V}_F N(\Sigma_g) = \bigwedge^* H_1(\Sigma_g)
\]

for the surface \( \Sigma_g \) of genus \( g \). For each surface we pick a symplectic basis \( \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \) for \( H_1(\Sigma_g) \) and introduce the inner form \( \langle \cdot, \cdot \rangle \), for which this basis orthonormal. We denote \( i_* : H_1(\Sigma_g) \rightarrow H_1(\Sigma_{g+1}) \) the inclusion map compatible with the choice of bases and the construction of \( \Sigma_{g+1} \) from \( \Sigma_g \) by handle addition.

The bases naturally provide lattice bases for \( \bigwedge^* H_1(\Sigma_g, \mathbb{Z}) \), given by monomials in the \( a_i \) and \( b_i \). Moreover, the inner form extends to to the exterior product to make the monomial basis an orthonormal one. We also denote by \( \omega_g = \sum_{i=1}^g a_i \wedge b_i \in \bigwedge^2 H_1(\Sigma_g, \mathbb{Z}) \) the symplectic form. The \( \mathfrak{sl}(2, \mathbb{R}) \)-action on \( \bigwedge^* H_1(\Sigma_g) \) is given in terms of the standard generators \( E, F, \) and \( H \) as

\[
H\alpha = (j - g)\alpha \quad \forall \alpha \in \bigwedge^* H_1(\Sigma_g), \quad E\alpha = \alpha \wedge \omega_g, \quad F = E^*.
\]

From the explicit actions of the generators in [8] we see that the lattices \( \bigwedge^* H_1(\Sigma_g, \mathbb{Z}) \) are preserved by the maps \( \mathcal{V}_F N(M) \) for cobordisms \( M \) in \( \text{Cob}^{22}_{fr} \). Moreover, also the standard generators of \( \mathfrak{sl}(2, \mathbb{R}) \) preserve the lattices. Hence also the universal enveloping algebra over \( \mathbb{Z} \) generated by the operators in [11], which we shall (abusively) denote by \( \mathfrak{sl}(2, \mathbb{Z}) \). We denote the respective functor into \( \mathbb{Z} \)-modules by:

\[
\mathcal{V}_Z : \text{Cob}^{22}_{fr} \longrightarrow \mathfrak{sl}(2, \mathbb{Z}) - \text{mod}_\mathbb{Z},
\]

More specifically, the extension of the mapping class group in \( \text{Cob}^{22}_{fr} \) factors under \( \mathcal{V}_Z \) through a split \( \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} \) extension of \( \text{Sp}(2g, \mathbb{Z}) \), acting in the natural way on \( \bigwedge^* H_1(\Sigma_g, \mathbb{Z}) \). We also denote by \( \mathfrak{sp}(2g, \mathbb{Z}) \) the algebra over \( \mathbb{Z} \) generated by the standard \( \text{Sp}(2g, \mathbb{R}) \) Lie algebra generators. It is not hard to show that in this representation it coincides with the group algebra.
\(Z[\text{Sp}(2g, Z)]\). Besides the mapping class groups the other generators of \(\text{Cob}^{2fr}_{3}\) are the handle attachment cobordisms \(H^+_g : \Sigma_g \to \Sigma_{g+1}\) and \(H^-_g : \Sigma_{g+1} \to \Sigma_g\) with actions given by

\[
\mathcal{V}_Z(H^+_g) : \alpha \mapsto i_*(\alpha) \wedge a_{g+1} \quad \text{and} \quad \mathcal{V}_Z(H^-_g) = \mathcal{V}_Z(H^+_g)^* ,
\]

where we use the notation \(i_*\) also for the inclusion \(H_1(\Sigma_g) \hookrightarrow H_1(\Sigma_{g+1})\) extended to the exterior powers.

The structure of \(\mathcal{V}_Z\) and the dual \(\mathfrak{sl}(2, Z)\)-action can be better understood if we decompose the lattices \(\mathcal{V}_Z(\Sigma_g)\) according to \(\mathfrak{sp}(2g, Z)\)-weights. We write an \(\mathfrak{sp}(2g)\)-weight in the standard basis \(\lambda = \sum_{i=1}^g \lambda_i \epsilon_i\) as given in \(12\). Specifically, we have \(ha_i = \langle \epsilon_i, h \rangle a_i\) and \(hb_i = -\langle \epsilon_i, h \rangle b_i\) for \(h \in \mathfrak{h}\), the diagonal matrices in \(\mathfrak{sp}(2g, Z)\).

We denote by \(\nabla_g = \{\lambda \in \mathfrak{h}^* : \lambda_i \in \{-1, 0, 1\}\}\) the set of possible weights of vectors in \(\Lambda^* H_1(\Sigma_g)\). This yields a decomposition into weight spaces denoted as follows.

\[
\mathcal{V}_Z(\Sigma_g) = \bigoplus_{\lambda \in \nabla_g} \mathcal{W}_Z(\lambda, g) \tag{14}
\]

For a given weight \(\lambda \in \nabla_g\) let \(N(\lambda) = \{i : \lambda_i = 0\} \subset \{1, \ldots, g\}\) and \(n(\lambda) = |N(\lambda)|\). A special vector \(w(\lambda) \in \mathcal{W}_Z(\lambda, g)\) is given by

\[
w(\lambda) = w_1(\lambda_1) \wedge \ldots \wedge w_g(\lambda_g) \in \Lambda^{g-n(\lambda)} H_1(\Sigma_g) \quad \text{where} \quad \begin{cases} w_i(1) = a_i \\
w_i(-1) = b_i \\
w_i(0) = 1 \end{cases}.
\]

Let \(e_\pm\) be generators of a 2-dimensional lattice \(\langle e_-, e_+ \rangle_Z\), and \(L^n_Z = \langle e_-, e_+ \rangle_Z^{\otimes n}\) the lattice of rank \(2^n\). Given \(N(\lambda) = \{j_1, \ldots, j_{n(\lambda)}\}\) with \(j_1 < \ldots < j_{n(\lambda)}\) we thus define maps

\[
\Upsilon_\lambda : L^n_Z(\Sigma_g) \xrightarrow{\cong} \mathcal{W}_Z(\lambda, g) \tag{15}
\]

\[
: e_{\epsilon_1} \otimes \ldots \otimes e_{\epsilon_n} \mapsto w(\lambda) \wedge o_{j_1}(\epsilon_1) \wedge \ldots \wedge o_{j_n}(\epsilon_n)
\]

\[
\text{where } n = n(\lambda) \quad \text{and} \quad \begin{cases} o_j(+) = a_j \wedge b_j \\
o_j(-) = 1 \end{cases}.
\]

The lattices \(L^n_Z\) also have a natural inner product for which the monomials \(e_{\epsilon_1} \otimes \ldots \otimes e_{\epsilon_n}\) form an orthonormal basis, and carry a natural \(\mathfrak{sl}(2, Z)\)-action given by \(Ee_- = e_+, Fe_+ = e_-, He_\pm = \pm e_\pm\), and \(Ee_+ = Fe_- = 0\), for which also \(E^* = F\) and \(H^* = H\).

**Lemma 2** The \(\Upsilon_\lambda\) are \(\mathfrak{sl}(2, Z)\)-equivariant isomorphisms of lattices with inner forms.

**Proof:** They are obviously isomorphisms of lattices as they map orthonormal bases to each other. It is also easy to see that the \(H\)-weight for a monomial is \(-n(\lambda) + 2 \sum_{i=1}^{n(\lambda)} \epsilon_i\) on both sides of \(15\). Now, \(E\) is multiplication with \(\omega_g = \sum_j a_j \wedge b_j\). Clearly, \(w(\lambda) \wedge a_j \wedge b_j = 0\) if \(j \notin N(\lambda)\) so we multiply only with \(\omega_\lambda = \sum_{i=1}^{n(\lambda)} a_{j_i} \wedge b_{j_i} = \sum_{i=1}^{n(\lambda)} o_{j_i}(+)\). The \(E\)-equivariance then follows from \(o_j(+) \wedge o_j(+) = 0\) and \(o_j(-) \wedge o_j(+) = o_j (+)\). \(F\)-equivariance follows from \(F^* = E\). \(\blacksquare\)

Although not of immediate necessity for the main result let us record here also how the actions of morphisms \(\mathcal{V}_Z(M)\) on the lattices \(L^n_Z\) look like. We give them in terms of generators of \(\mathfrak{sp}(2g, Z)\) and the handle attaching maps. We introduce \(\mathfrak{sl}(2, Z)\)-equivariant morphisms

\[
coev_k = I_{k-1} \otimes coev \otimes I_{n-k} : L^n_Z \to L^{n+2}_Z \tag{16}
\]


We have natural subgroups $\hat{\lambda}$ representations of $\hat{S}$ symmetric group in $g$. The obey relation

$$ev_k = -(coev_k)^* : L_{\mathbb{Z}}^{n+2} \rightarrow L_{\mathbb{Z}}^n$$

The obey relation

$$ev_{k+1} \circ coev_k = \mathbb{I} \quad \text{and} \quad ev_k \circ coev_k = -2 \cdot \mathbb{I}.$$ 

For a map $\phi : W_{\mathbb{Z}}(\lambda, g) \rightarrow W_{\mathbb{Z}}(\lambda', g')$ we denote $\phi^\mathcal{Y} = \Phi_{\lambda'}^{-1} \cdot \phi \cdot \Phi_{\lambda} : L_{\mathbb{Z}}^{n(\lambda)} \rightarrow L_{\mathbb{Z}}^{n(\lambda')}$. Moreover, as in $\mathcal{Y}$, we denote the standard standard generators $e_{\alpha_i}$ and $f_{\alpha_j}$ of $\mathfrak{sp}(2g, \mathbb{Z})$ for simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < g$ and $\alpha_g = 2\varepsilon_g$. We have $e_{\alpha_i}a_{i+1} = a_i$, $e_{\alpha_i}b_i = -b_{i+1}$, and $e_{\alpha_i}v = 0$ for all other basis vectors $v$ if $i < g$. Also $e_{\alpha_g}b_g = e_g$ and $e_{\alpha_g}v = 0$ for all others. This further determines the action of the other generators with $f_{\alpha_i} = e_{\alpha_i}^*$.

Denote now by $e_{\alpha, \lambda}$ the restriction $e_{\alpha} : W_{\mathbb{Z}}(\lambda, g) \rightarrow W_{\mathbb{Z}}(\lambda + \alpha, g)$, where we put $e_{\alpha, \lambda} = 0$ of $\lambda + \alpha \not\in \nabla_g$. Let us also denote the restriction of the handle attaching map $H_{\lambda} = V_{\mathbb{Z}}(H_g^+)$ : $W_{\mathbb{Z}}(\lambda, g) \rightarrow W_{\mathbb{Z}}(\lambda + g+1, g+1)$. The following is the result of a straightforward calculation.

**Lemma 3** For $\lambda \in \nabla_g$ and $n = n(\lambda)$ we have, when $i < g$,

$$e_{\alpha_i, \lambda}^\mathcal{Y} = \begin{cases} \mathbb{I}_n & \text{for } (\lambda_i, \lambda_{i+1}) = (0, 1) \\ -\mathbb{I}_n & \text{for } (\lambda_i, \lambda_{i+1}) = (-1, 0) \\ coev_k & \text{for } (\lambda_i, \lambda_{i+1}) = (-1, 1) \quad \text{with } j_{k-1} < i < j_k \\ -ev_k & \text{for } (\lambda_i, \lambda_{i+1}) = (0, 0) \quad \text{with } i = j_k, i + 1 = j_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$e_{\alpha_g, \lambda} = \begin{cases} \mathbb{I}_n & \text{for } \lambda_g = -1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H_{\lambda}^\mathcal{Y} = \mathbb{I}_n.$$

Another prominent, and for our purposes more important, action on the lattices $V_{\mathbb{Z}}(\Sigma_g)$ is that of subgroups of the Weyl group $W_g \cong (\mathbb{F}_2)^g \times S_g$ of $\mathfrak{sp}(2g, \mathbb{Z})$, where $S_g$ denotes the symmetric group in $g$ letters. The $j$-th $\mathbb{F}_2$-generator of $W_g$ acts on weights by changing the sign of $\lambda_j$. It is realized as a subgroup $\hat{W}_g \subset \mathfrak{sp}(2g, \mathbb{Z})$ by an extension $1 \rightarrow \mathbb{F}_2 \rightarrow \hat{W}_g \rightarrow W_g \rightarrow 1$, with $\hat{W}_g \cong (\mathbb{F}_4)^g \times S_g$. The $\mathbb{F}_4$-generators are given by the “$S$-matrices” $S_j \in \mathfrak{sp}(2g, \mathbb{Z})$, see $\mathfrak{S}$, defined by $S_ja_j = -b_j$, $S_jb_j = a_j$ and $S_ja_i = a_i$ and $S_jb_i = b_i$ for $i \neq j$. We specify two relevant representations of $\hat{W}_g$:

1. $L_{\mathbb{Z}}^k \cong (\mathbb{Z}^2)^{\otimes k}$: This action factors through $\hat{W}_k \rightarrow S_k$ the symmetric group, which acts canonically on the lattice by permutation of factors.

2. $M_{\mathbb{Z}}^k \cong (\mathbb{Z})^{\otimes k}$: Here the $j$-th $\mathbb{F}_4$-factor is represented by the matrix $S_j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ acting on the $j$-th factor of the tensor product. The action of $S_k$ on $M_{\mathbb{Z}}^k$ is the canonical representation multiplied by the alternating representation, i.e., $\sigma(v_1 \otimes \ldots \otimes v_k) = sign(\sigma)v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(k)}$.

We have natural subgroups $\hat{W}_g \cong \hat{W}_g \times \hat{W}_g \subset \hat{W}_g$, for which the right coset

$$C_g^\mathfrak{S} = \hat{W}_g / \hat{W}_g \times \hat{W}_g = S_g / S_g \times S_n,$$
is identified with the set of subsets $A \subset \{1, \ldots, g\}$ of size $|A| = n$. We denote
\begin{equation}
W_{Z}(n,g) = \bigoplus_{A \in C_{n}^{g}} W_{Z}(A,g) = \bigoplus_{\lambda \in \nabla_{g,n}(\lambda) = n} W_{Z}(\lambda,g) ,
\end{equation}
where
\begin{equation}
W_{Z}(A,g) = \bigoplus_{\lambda \in \nabla_{g,N(\lambda) = A}} W_{Z}(\lambda,g) .
\end{equation}

Clearly, the summands of $W_{Z} = \bigoplus_{n} W_{Z}(n,g)$ are invariant under the $\hat{W}_{g}$-action for each $n$. These subrepresentations are identified next as induced representations.

**Lemma 4** For every $n$ with $0 \leq n \leq g$ there is a natural isomorphism of $\hat{W}_{g}$-modules
\[ \Upsilon : \text{Ind}_{\hat{W}_{g-n} \times \hat{W}_{n}} \left( M_{Z}^{g-n} \otimes L_{Z}^{n} \right) \xrightarrow{\cong} W_{Z}(n,g) . \]
This map is an $\mathfrak{sl}(2,\mathbb{Z})$-equivariant isometry.

**Proof:** For $N_{n} = \{ g-n+1, \ldots, g \}$ the $\hat{W}_{g-n} \times \hat{W}_{n}$-module is readily identified via the isomorphism
\[ M_{Z}^{g-n} \otimes L_{Z}^{n} \xrightarrow{\cong} W_{Z}(N_{n},g) : e_{\lambda_{1}} \otimes \cdots \otimes e_{\lambda_{n-g}} \otimes l \mapsto \Upsilon_{\{\lambda_{1}, \ldots, \lambda_{n-g}, 0, \ldots, 0\} \{l\}} . \]
with the submodule $W_{Z}(N_{n},g) \subset \wedge H_{1}(\Sigma_{g})$, where the action is defined by restricting the action of $\hat{W}_{g}$. We next define a natural section
\begin{equation}
\pi : C_{n}^{g} \rightarrow S_{g} \subset \hat{W}_{g} : A \mapsto \pi_{A} \quad (19)
\end{equation}
as follows. Let $A \subset \{1, \ldots, g\}$ with $n = |A|$. There is a unique permutation $\pi_{A} \in S_{g}$ such that $A = \pi_{A}(\{g-n+1, \ldots, g\})$, $\pi_{A}(1) < \pi_{A}(2) < \ldots < \pi_{A}(g-n)$, and $\pi_{A}(g-n+1) < \pi_{A}(g-n+2) < \ldots < \pi_{A}(g)$. Clearly, we have
\[ \pi_{A} : W_{Z}(N_{n},g) \xrightarrow{\cong} W_{Z}(A,g) . \]
The induced representation by definition the space of all maps $f : \hat{W}_{g} \rightarrow M_{Z}^{g-n} \otimes L_{Z}^{n}$ such that $f(\sigma \eta) = \eta^{-1}(f(\sigma))$ for $\eta \in \hat{W}_{g-n} \times \hat{W}_{n}$, equipped with the left regular $\hat{W}_{g}$-action $(\sigma f)(\sigma') = f(\sigma^{-1}\sigma')$. Now, every $\sigma \in \hat{W}_{g}$ has a unique decomposition $\sigma = \pi_{\sigma(N_{n})}\eta_{\sigma}$, with $\eta_{\sigma} \in \hat{W}_{g-n} \times \hat{W}_{n}$. Thus we may identify the induced representation with the space of maps $\overline{f} : C_{n}^{g} \rightarrow M_{Z}^{g-n} \otimes L_{Z}^{n}$, setting $\overline{f}(A) = f(\pi_{A})$ and hence $f(\sigma) = \eta_{\sigma}^{-1} \overline{f}(\sigma(N_{n}))$. The isomorphism is now given by
\[ \text{Ind} \cong \text{Map}(C_{n}^{g}, M_{Z}^{g-n} \otimes L_{Z}^{n}) \rightarrow W_{Z}(n,g) : \overline{f} \mapsto \Upsilon(\overline{f}) = \bigoplus_{\pi_{A}(\overline{f}(A))} \pi_{A} \Upsilon(\overline{f}(A)) . \quad (20) \]
An inverse is obtained by mapping $v \in W_{Z}(A,g)$ to $\pi^{-1}_{A}(v) \otimes \delta_{\{A\}}$, where $\delta_{\{A\}}(B) = 1$ for $A = B$ and 0 elsewise. In order to show that it is equivariant let $\overline{f}$ be an arbitrary map $C_{n}^{g} \rightarrow W_{Z}(N_{n},g)$, $\sigma \in \hat{W}_{g}$, and $\overline{f} = \sigma(\overline{f})$. Let $A \in C_{n}^{g}$ and define $\eta_{\sigma A} \in \hat{W}_{g-n} \times \hat{W}_{n}$ by $\sigma^{-1}_{A} \pi_{A} = \pi_{\sigma^{-1}(A)}\eta_{\sigma A}^{-1}$. Hence we have $\overline{f}(A) = f'(\pi_{A}) = (\sigma f)(\pi_{A}) = f(\sigma^{-1}\pi_{A}) = f(\pi_{\sigma^{-1}(A)}\eta_{\sigma A}^{-1}) = \eta_{\sigma A}f(\pi_{\sigma^{-1}(A)})$. Thus
\[ \Upsilon(\overline{f}) = \bigoplus_{A \in C_{n}^{g}} \pi_{A} \overline{f}(A) = \bigoplus_{A \in C_{n}^{g}} \pi_{A} \eta_{\sigma A} \overline{f}(\sigma^{-1}A) = \bigoplus_{A \in C_{n}^{g}} \sigma \pi_{\sigma^{-1}(A)} \overline{f}(\sigma^{-1}A) = \bigoplus_{B \in C_{n}^{g}} \pi_{B} \overline{f}(B) = \sigma \bigoplus_{B \in C_{n}^{g}} \pi_{B} \overline{f}(B) = \sigma(\bigoplus_{B \in C_{n}^{g}} \pi_{B} \overline{f}(B)) = \sigma(\Upsilon(\overline{f})) , \]
which is what we needed to show. Isometry of $\Upsilon$ is with respect to the natural inner product on $\text{Map}(C^g_n, M^g_{-n} \otimes L^n_Z)$ given by

$$\langle \overline{f}, \overline{h} \rangle = \sum_{A \in C^g_n} \langle f(A), g(A) \rangle$$  \hspace{1cm} (21)

given the inner form on $M^g_{-n} \otimes L^n_Z$. Also, as $L^n_Z$ is an $\mathfrak{sl}(2, \mathbb{Z})$-module also $\text{Map}(C^g_n, M^g_{-n} \otimes L^n_Z)$ is. Both, equivariance and isometry, follows immediately from the form of the isomorphism in (20) and the fact that the $\pi_A$ are isometric equivariant maps.

The relation of the isomorphisms in Lemma 4 to the ones in Lemma 2 is given by the restrictions of $\Upsilon^{-1}$ to the weight spaces

$$\pi^{-1}_{N(\lambda)} = \Upsilon_\lambda \circ \Upsilon^{-1}_\lambda : \mathcal{W}(\lambda, g) \longrightarrow \mathcal{W}(\lambda^\pi, g) \subset \mathcal{W}(N_n, g)$$  \hspace{1cm} (22)

Here $\lambda^\pi$ denotes the the “sign-content” of a weight $\lambda \in \nabla_g$ defined by

$$\lambda^\pi := \pi^{-1}_{N(\lambda)} \lambda = \sum_{j=1}^{g-n(\lambda)} \lambda_{\pi_N(\lambda)(j)} \epsilon_j = \pm \epsilon_1 + \ldots + \pm \epsilon_{g-n(\lambda)} .$$

3. Lefschetz Decompositions and Specht Modules

As in [8] we consider the decomposition of the Frohman Nicas TQFT according to $SL(2, \mathbb{R})$-representations:

$$\mathcal{V}^{FN}_Z = \bigoplus_{j \geq 1} V_j \otimes \mathcal{V}^{(j)} ,$$

where $V_j$ is the $j$-dimensional irreducible representation of $SL(2, \mathbb{R})$. Note, that we the convention we use here for the superscript in $\mathcal{V}^{(j)}$ is shifted by one from the one used in [8]. For weights we follow the notations of [4]. The sublattices of the irreducible TQFT components can be defined as the $SL(2, \mathbb{R})$ lowest weight spaces

$$\mathcal{V}^{(j)}_Z(\Sigma) = \{ v \in \mathcal{V}_Z(\Sigma) : Fv = 0 \text{ and } Hv = -(j-1)v \} = \Lambda^{g-j+1}H_1(\Sigma, \mathbb{Z}) \cap \ker(F) .$$

The representation of $\text{Sp}(2g, \mathbb{Z})$ on $\mathcal{V}^{(j)}_Z(\Sigma)$ is irreducible of fundamental heighest weight $w_{g-j+1} = \epsilon_1 + \ldots + \epsilon_{g-j+1}$ with heighest weight vector

$$w(w_{g-j+1}) = a_1 \wedge \ldots \wedge a_{g-j+1} = \Upsilon \omega_{g-j+1}(e_1^{\otimes j+1}) .$$

The possible weights in this representation are given by

$$\nabla_g^{(j)} = \{ \lambda \in \nabla_g : n(\lambda) \geq j - 1 \text{ and } n(\lambda) \equiv j - 1 \text{ mod } 2 \} .$$

We obtain an analogous weight space decomposition

$$\mathcal{V}^{(j)}_Z(\Sigma_g) = \bigoplus_{\lambda \in \nabla_g^{(j)}} \mathcal{W}^{(j)}_Z(\lambda, g) \longrightarrow \bigoplus_{\lambda \in \nabla_g^{(j)}} L^n_Z(\lambda) \cap \ker(F) \cap \ker(H + j - 1) .$$  \hspace{1cm} (23)
Lemma 5 The spaces $W_{ij}^j(\lambda, g)$ are as $S_{n(\lambda)}$-modules isomorphic to the standard irreducible Specht modules $S^{[a, b]}$ for the two-row Young-diagram

$$[a, b] = \left[\frac{n(\lambda) + j - 1}{2}, \frac{n(\lambda) - j + 1}{2}\right].$$

Proof: Although this appears to be standard we shall provide a proof to fix conventions. We largely follow here the definitions and notations of [1]. First we note that $\ker(H + j - 1)$ is naturally isomorphic to the permutation module $M^{[a, b]}$, where $n(\lambda) = a + b$ and $j = a - b + 1$. The isomorphism $M^{[a, b]} \cong \ker(H + j - 1) \cap L_n^{(\lambda)}$ maps a tabloid $\{t\} = \frac{t_1, t_2 \ldots t_a}{j_1 \ldots j_b}$ to the basis vector $e_{\{t\}} = \epsilon_{\epsilon_1} \otimes \epsilon_{\epsilon_2} \otimes \ldots \otimes \epsilon_{\epsilon_N}$ with $\epsilon_k = +$ if $k \in \{j_1, \ldots, j_b\}$ and $\epsilon_k = -$ if $k \in \{i_1, \ldots, i_a\}$. It is obvious that the $e_{\{t\}}$ indeed span ker$(H + j - 1)$. Consider a tableau $t = \frac{j_1}{i_1} \ldots \frac{j_b}{i_a}$ of shape $[a, b]$. Let $C_t$ be the column stabilizer group and

$$\kappa_t = \sum_{\pi \in C_t} \text{sign}(\pi)\pi = \prod_{k=1}^b (1 - (i_k, j_k))$$

the signed column sum. We set $e_t = \kappa_t e_{\{t\}}$. The Specht module $S^{[a, b]}$ is the space generated by these $e_t$ and we want to show that it coincides with ker$(F)$. The easy part is to show $S^{[a, b]} \subset$ ker$(F)$. Using that $F$ commutes with $\kappa_t$, we compute $F e_t = F \kappa_t e_{\{t\}} = \kappa_t F e_{\{t\}} = \sum_k \kappa_t e_{\{t_k\}}$, where $\{t_k\}$ is the tabloid of shape $[a + 1, b - 1]$, in which we have removed $j_k$ from the bottom row and added to the top row. As a result $(1 - (i_k, j_k))\{t_k\} = 0$, hence $\kappa_t \{t_k\} = 0$ so that $F e_t = 0$.

In order to prove ker$(F) \subset S^{[a, b]}$ we proceed by induction. ker$(F)$ on $M^{[a, b]}$ is given by ker$(F^2)$ on $M^{[a-1, b]}$. We have a map

$$\ker(F^2) \cap M^{[a-1, b]} \longrightarrow \ker(F) \cap M^{[a, b]} : x \mapsto e_- \otimes x - e_+ \otimes F x.$$

It is easy to see that this map is an isomorphism. We use first that $Fx \in \ker(F) \cap M^{[a-1, b]}$ and hence by induction $Fx = \sum_{t \in T} b_t e_t$. Here $T$ is the set of tableaux of the form $t = \frac{i_1}{j_1} \ldots \frac{i_b}{j_b} \ldots \frac{i_a}{j_a}$ with numbers from $\{2, 3, \ldots, a + b\}$. Now as $b - 1 < a$ we have that $i_a$ is not permuted by $C_t$. Thus if we denote $E_j = \mathbb{I}^{[a-1]} \otimes \mathbb{I}^N = \mathbb{I}^{[N]}$, we have that $E_{i_a}$ commutes with $\kappa_t$. Let $h_t = \kappa_t E_{i_a} e_{\{t\}}$, now $F h_t = F E_{i_a} \kappa_t e_{\{t\}} = [F, E_{i_a}] \kappa_t e_{\{t\}} + E_{i_a} F \kappa_t e_{\{t\}} = -H_{i_a} \kappa_t e_{\{t\}}$ since $e_t \in$ ker$(F)$ as shown above. Now, $H_{i_a}$ commutes with $\kappa_t$ and $H_{i_a} e_{\{t\}} = -e_{\{t\}}$ by construction. Hence $F h_t = e_t$. In other words $S^{[a-1, b]} \subset \text{im}(F)$. Consider now $y = x - \sum_{t \in T} b_t h_t$.

We thus have $y \in \ker(F) \cap M^{[a-1, b]}$ so that by induction $y = \sum_{s \in S} c_s e_s$, where $S$ denotes the tableau of the form $s = \frac{k_1}{t_1} \ldots \frac{k_b}{t_b} \ldots \frac{k_a}{t_a}$ with all numbers in $\{2, 3, \ldots, a + b\}$. Inserting everything we find

$$z = e_- \otimes x - e_+ \otimes F x = \sum_{s \in S} c_s e_- \otimes e_s + \sum_{t \in T} b_t (e_- \otimes h_t - e_+ \otimes e_t).$$

Now, it is not hard to see that $e_s = (e_- \otimes e_s)$ with $s = \frac{k_1}{t_1} \ldots \frac{k_b}{t_b} \ldots \frac{k_a}{t_a}$ given that $a - 1 \geq b$ so that 1 is not permuted by $C_s$. Moreover, $(e_- \otimes h_t - e_+ \otimes e_t) = (1 - (1, i_a))(e_- \otimes h_t)(1 - (1, i_a))\kappa_t E_{i_a} e_{\{t\}} = \kappa_t e_{\{t\}} = e_{\{t\}}$, where $\hat{t} = \frac{1}{i_{a+1}} \ldots \frac{i_a}{t_b} \ldots \frac{i_{a-1}}{j_{b-1}}$.

Thus $z = \sum_s c_s e_s + \sum_{t} b_t e_{\{t\}}$ so that $z \in S^{[a, b]}$. \qed
Combining Lemma 3 with Lemma 5 and using notation $W_n^{(j)}(A, g)$ and $W_n^{(j)}(n, g)$ with $n \geq j - 1$ and $n \equiv j - 1 \mod 2$ analogous to (17) so that $V_n^{(j)}(\Sigma_g) = \bigoplus_n W_n^{(j)}(n, g)$ we find the following structure.

**Corollary 7** For every $1 \leq j - 1 \leq n \leq g$ with $n \equiv j - 1 \mod 2$ there is an isomorphism of $\mathfrak{m}_g$-modules

$$\mathcal{T}^{(j)} : \text{Ind}_{\mathfrak{m}_g}^{\mathfrak{m}_g - n \times \mathfrak{m}_g}(M_{\mathfrak{m}_g}^{g-n} \otimes S^{[\mathfrak{m}_g^{-1} - 1, \mathfrak{m}_g^{-1} + 1]}) \xrightarrow{\cong} W_n^{(j)}(n, g).$$

Let us also describe the $\mathfrak{sp}(2g, \mathbb{Z})$-generators on the vectors $e_t$ spanning the Specht modules $W_n^{(j)}(\lambda, g)$. To this end it is convenient to use tableaux in which entries are taken from the set $N(\lambda)$ rather than $\{1, \ldots, n(\lambda)\}$, related to the standard ones by application of $\pi_N(\lambda)$. We denote the set of these tableaux by $T^{(j)}(\lambda)$.

**Lemma 6** The $\mathfrak{sp}(2g, \mathbb{Z})$-generators act on the tableau vectors of Specht modules in the $V_n^{(j)}$-TQFT as follows. For the $e_{\alpha_i, \lambda}$ with $1 \leq i \leq g - 1$ we have:

1. If $(\lambda_i, \lambda_{i+1}) = (0, 1)$ then $e_{\alpha_i, \lambda} e_t = e_s$ where $s \in T^{(j)}(\lambda + \alpha_i)$ is obtained from $t \in T^{(j)}(\lambda)$ by replacing the label $i \in N(\lambda)$ in $t$ by the label $i + 1 \in N(\lambda + \alpha_i)$.

2. If $(\lambda_i, \lambda_{i+1}) = (-1, 0)$ then $e_{\alpha_i, \lambda} e_t = -e_s$ where $s \in T^{(j)}(\lambda + \alpha_i)$ is obtained from $t \in T^{(j)}(\lambda)$ by replacing the label $i + 1 \in N(\lambda)$ in $t$ by the label $i \in N(\lambda + \alpha_i)$.

3. If $(\lambda_i, \lambda_{i+1}) = (-1, 1)$ we have $\{i, i+1\} = N(\lambda + \alpha_i) - N(\lambda)$ and $e_{\alpha_i, \lambda} e_t = e_s$ where $s \in T^{(j)}(\lambda + \alpha_i)$ by adding a column $\begin{array}{c} k \\ i \\ k \\ i+1 \\ k \\ i \\ k \\ i+1 \\ k \\ i \\ k \end{array}$ to $t \in T^{(j)}(\lambda)$.

4. If $(\lambda_i, \lambda_{i+1}) = (0, 0)$ so that $\{i, i+1\} = N(\lambda) - N(\lambda + \alpha_i)$ then $e_{\alpha_i, \lambda} e_t$, with $t \in T^{(j)}(\lambda)$, is

(a) $0$ if the labels $i$ and $i + 1$ occur in columns of height 1.

(b) $2e_s$ if $t$ is given by adding the column $\begin{array}{c} k \\ i \\ k \end{array}$ to $s$.

(c) $e_s$ if $i$ and $i + 1$ occur in different columns of height 2.

Here $s$ is obtained from $t$ by replacing the double column $\begin{array}{c} k \\ i \\ k \\ i+1 \end{array}$ by $\begin{array}{c} k \\ i \\ k \\ i+1 \end{array}$.

(d) $e_s$ if $i$ is in column of height 2 and $i + 1$ in columns of height 1, where $s$ is obtained from $t$ by deleting the column $\begin{array}{c} k \\ k \end{array}$ and replacing $\begin{array}{c} i+1 \\ k \end{array}$ by $\begin{array}{c} i+1 \\ k \end{array}$.

(e) $e_s$ if $i + 1$ is in column of height 2 and $i$ in columns of height 1, where $s$ is obtained from $t$ by deleting the column $\begin{array}{c} k \\ i+1 \\ k \end{array}$ and replacing $\begin{array}{c} i \\ k \end{array}$ by $\begin{array}{c} i \\ k \end{array}$.

All other cases of positions of $i$ and $i + 1$ follow from the symmetry properties of the vectors $e_t$ under permutations of columns or within columns. Since $N(\lambda + \alpha_g) = N(\lambda)$ if $\lambda + \alpha_g, \lambda \in \nabla_g$ we have that $e_{\alpha_g, \lambda}$ acts as identity also on the vectors $e_t$. Similarly, $H_\lambda$ acts as identity.
4. $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$-Reductions and the Sequences $\mathcal{C}_{p,k}$

For the remainder of this article let $p$ be an odd prime number. Since the TQFT’s $\mathcal{V}_g^{(j)}$ are defined over free $\mathbb{Z}$-modules (lattices) we naturally obtain TQFT’s $\mathcal{V}_p^{(j)}$ over the number field $\mathbb{F}_p$ by setting

$$\mathcal{V}_p^{(j)}(\Sigma_g) = \mathcal{V}_g^{(j)}(\Sigma_g) / p\mathcal{V}_g^{(j)}(\Sigma_g)$$

Now, each $\mathcal{V}_g^{(j)}(\Sigma_g)$ inherits a non-degenerate inner product as a sublattice of $\mathcal{N}H_1(\Sigma_g, \mathbb{Z})$. This, however, will in general degenerate if we consider the the $p$-reduction $\langle , \rangle_p : (\mathcal{V}_g^{(j)}(\Sigma_g))^\otimes 2 \rightarrow \mathbb{F}_p$. We denote the corresponding null space as follows.

$$\mathcal{V}_p^{(j)}(\Sigma_g) \rightarrow \{ v \in \mathcal{V}_p^{(j)}(\Sigma_g) : \langle v, w \rangle_p = 0 \quad \forall \ w \in \mathcal{V}_p^{(j)}(\Sigma_g) \}$$

The elements are represented by vectors $v \in \mathcal{V}_p^{(j)}(\Sigma_g)$ for which $\langle v, w \rangle_p \in p\mathbb{Z}$ for all $w \in \mathcal{V}_p^{(j)}(\Sigma_g)$ although $v \notin p\mathcal{V}_p^{(j)}(\Sigma_g)$.

**Lemma 7** There are well defined TQFT’s $\mathcal{V}_p^{(j)}$ and $\mathcal{V}_p^{(j)}$ which assign to a surface $\Sigma_g$ the $\mathbb{F}_p$-vectors spaces

$$\mathcal{V}_p^{(j)}(\Sigma_g) \quad \text{and} \quad \mathcal{V}_p^{(j)}(\Sigma_g) = \mathcal{V}_p^{(j)}(\Sigma_g) / \mathcal{V}_p^{(j)}(\Sigma_g)$$

**Proof:** We note that since for a cobordism $M$ the map $\mathcal{V}_g(M)$ commutes with $E$ we have that $\mathcal{V}_g(M)^*$ commutes with $F = E^*$ and hence also maps the $\mathcal{V}_g^{(j)}(\Sigma_g)$ to themselves. Thus if $v_i \in \mathcal{V}_g^{(j)}(\Sigma_i)$ for $i = 1, 2, M$ is a cobordism from $\Sigma_{g_1}$ to $\Sigma_{g_2}$, and $v_1$ represents a vector in $\mathcal{V}_p^{(j)}(\Sigma_1)$, then $\langle v_2, \mathcal{V}_g^{(j)}(M)v_1 \rangle = \langle \mathcal{V}_g^{(j)}(M)^*v_2, v_1 \rangle \in p\mathbb{Z}$ as $\mathcal{V}_g^{(j)}(M)^*v_2 \in \mathcal{V}_g^{(j)}(\Sigma_1)$. \hfill \blacksquare

We extend the previous notations to the weight spaces $\hat{\mathcal{W}}_p^{(j)}(\lambda, g), \bar{\mathcal{W}}_p^{(j)}(\lambda, g), \hat{\mathcal{W}}_p^{(j)}(n, g)$, etc.. Since the weight spaces are all orthogonal to each other these subspaces can be defined also as the null spaces from the respective restriction of the inner forms. Now, also the Specht modules $S^\tau$ for a diagram $\tau = [a, b]$ inherit an inner form from the permutation module $M^\tau$, which is via the isometry $\Upsilon$ compatible with the one on the weight spaces. As in the standard literature, e.g., [3], we set

$$D^\tau_p = S^\tau_p / \hat{S}^\tau_p \quad \text{where} \quad \hat{S}^\tau_p = S^\tau_p \cap S^\tau_p$$

and $S^\tau_p$ is the $p$-reduction of $S^\tau$. They are related to irreducible TQFT’s as follows.

**Lemma 8** Let $p \geq 3$ be a prime. The TQFT’s $\bar{\mathcal{W}}_p^{(j)}$ are irreducible over $\mathbb{F}_p$ and the weight spaces are identified as $\hat{\mathcal{W}}_g$-modules by equivariant isomorphisms:

$$\Upsilon_p^{(j)} : \text{Ind}_{\mathfrak{m}_g}^{\mathfrak{m}_g} (M_{p^{-n}} \otimes T_p^{[\lambda^{a-1} \lambda^{a-1}]}) \xrightarrow{\cong} \bar{\mathcal{W}}_p^{(j)}(n, g). \quad (25)$$

**Proof:** The isomorphism in (25) follows from the definitions and properties of $\Upsilon^{(j)}$. We first show that the spaces $\bar{\mathcal{W}}_p^{(j)}(n, g)$ are irreducible with respect to the semidirect product $\mathfrak{X}_g$ of the Cartan algebra $\mathbb{Z}[\mathfrak{h}_g] \subset \mathfrak{sl}(2g, \mathbb{Z})$ and the algebra of the Weyl group $\mathbb{Z}[\mathfrak{m}_g] \subset \mathbb{Z}[\text{Sp}(2g, \mathbb{Z})]$. For any vector $v = \sum \lambda v_\lambda \in \bar{\mathcal{W}}_p^{(j)}(n, g)$ the action of $\mathfrak{h}_g$ shows that each weight component $v_\lambda \in \bar{\mathcal{W}}_p^{(j)}(\lambda, g)$ has to lie in $\mathfrak{X}_g v$. Each $\bar{\mathcal{W}}_p^{(j)}(n, g)$ is a module of a symmetric group
where $S_n \subset \hat{W}_g \subset \mathfrak{X}_g$ which is equivalent to $D_p^{|n-j-1| + \frac{n-j+1}{2}}$. It now follows from Theorem 4.9 in [3] that these representations are irreducible. In fact, as $p \geq 3$ any two-row diagram is $p$-regular so that these representations are never zero, see Theorem 11.1 in [3]. In particular, if $v_\lambda \neq 0$ then $\mathbb{Z}[S_n]v_\lambda$ is the entire module. Hence, for $v \neq 0$ we must have $\hat{W}_p^{(j)}(\lambda, g) \subset \mathfrak{X}_g v$ for at least one $\lambda \in \nabla_g$ with $n(\lambda) = n$. Since $\hat{W}_g$ acts transitively on all of such weights and provides isomorphisms between the weight spaces we thus have $\hat{W}_p^{(j)}(n, g) = \mathfrak{X}_g v$, which implies irreducibility since $v$ was arbitrary. A submodule of $\hat{V}_p^{(j)}(\Sigma_g)$ must therefore be a direct sum of the $\hat{W}_p^{(j)}(n, g)$. Each of these contains a special vector $w_n^g = \Upsilon_\omega a_{g-j+1}(e_{t(n,j)})$ with $t(n,j) = \begin{array}{cccc} 1 & 3 & \cdots & n-1 \downarrow \downarrow \downarrow \\ 2 & 4 & \cdots & m \end{array}$ where $m = n - j + 1$. Using $\kappa^2_{t(n,j)} = 2^m \kappa_{t(n,j)}$ for the antisymmetrizer [24], we find $\langle w_n^g, w_n^g \rangle_p = \langle e_{t(n,j)}, e_{t(n,j)} \rangle_p = 2^m \langle e_{t(n,j)} \rangle_p$ for $p \geq 3$ so that all of these vectors are non-zero in $\hat{W}_p^{(j)}(n, g)$. It can be computed from the rules 3. and 4. (b) in Lemma 3 that $e_{a_{g-n-1}} S_{g-n-1} w_n^g = w_n^g$ and $e_{a_{g+n+1}} w_n^g = 2 w_{n+2}^g$, where $S_t \in \hat{W}_g$ maps $a_j$ to $-b_j$ so that with $2 \neq 0 \mod p$ we have non-trivial maps between all of the irreducible $\mathfrak{X}_g$-components. Consequently, the $\hat{V}_p^{(j)}(\Sigma_g)$ are irreducible as $\mathfrak{sp}(2g, \mathbb{Z})$-representations and any sub-TQFT must assign either this space or 0 to a surface $\Sigma_g$. As in [3] we easily check that the handle attachment maps are non-trivial between these spaces so that $\hat{V}_p^{(j)}$ does in fact contain no proper sub-TQFT.

We next construct a sequence of maps between the $p$-reductions of the Specht modules, using the $\mathfrak{sl}(2, \mathbb{Z})$-actions. As before we fix $n \in \mathbb{N}$ and denote the Specht module

$$S^{(c)}_Z \equiv L^n_Z \cap \ker(F) \cap \ker(H + c - 1) \cong S^\tau,$$

where $c = a - b + 1$ with tableau $\tau = [a, b] = \left[ \frac{n+c-1}{2}, \frac{n-c+1}{2} \right]$.

**Lemma 9** Let $p \geq 3$, $c \not\equiv 0 \mod p$, and $c_0 \in \{1, \ldots, p-1\}$ such that $c \equiv c_0 \mod p$. Then

$$E^{c_0}(S^{(c)}_Z) \subset S^{(c-2c_0)}_Z + p L^n_Z.$$

Moreover,

$$E^{c_0}(S^{(c)}_Z) \not\subset p L^n_Z \quad \text{and} \quad E^p(L^n_Z) \subset p L^n_Z.$$

**Proof:** Now, with notation as in the proof of Lemma 25, $e_{t(n,c-1)} \in S^{(c)}_Z$ is a cyclic vector so it suffices to show that $E^{c_0}(e_{t(n,c-1)}) \in (S^{(c-2c_0)}_Z + p L^n_Z) - p L^n_Z$. Furthermore, $e_{t(n,c-1)} = e_{t(m,0)} \otimes e_{c-1}$ and $E e_{t(m,0)} = 0$ so that we really need to show that $E^{c_0}(e_{c-1}) \in (\ker(F) + p L^{c-1}_Z) - p L^{c-1}_Z$, where $c = c_0 + kp$. We do this by induction in $c_0$.

For $c_0 = 1$ we have $c-1 = kp$ and, using $\mathfrak{sl}_2$-relations, $F e_{c-1} = -H e_{c-1} + E F e_{c-1} = k p e_{c-1}$. Hence, if we set $w = k e_{c-1} \otimes e_+$, we find $F w = e_{c-1}$ so that $E e_{c-1} = k p e_{c-1}$ in $\ker(F) + p L^{c-1}_Z$. Next assume the assertion is true for $c_0$ so that $E^{c_0}(e_{c-1}) = y + p z$ with $F y = 0$ and $-H y = (c - 2 c_0 - 1) y$. We have by binomial formula and $E e_+ = e_+$ and $E^2 e_+ = 0$ that $E^{c_0+1}(e_{c-1}) = E^{c_0+1}(e_{c-1} \otimes e_+) = (E^{c_0+1} e_{c-1}) \otimes e_+ + (c_0 + 1) E e_{c-1} \otimes e_+ = (E y) \otimes e_+ + (c_0 + 1) y \otimes e_+ + p z$. Thus we need to show $t = (E y) \otimes e_+ + (c_0 + 1) y \otimes e_+ \in \ker(F) \cap p L^n_Z$. We compute $F t = (E F y) \otimes e_+ + (c_0 + 1) y \otimes e_+ = (c y) \otimes e_+ + (c_0 + 1) y \otimes e_+ = (c y) \otimes e_+ + (c_0 + 1) y \otimes e_+ = k p y \otimes e_+$. Also $F (y \otimes e_+) = y \otimes e_+$ so that $t - k p y \otimes e_+ \in \ker(F)$ and hence $t \in \ker(F) \cap p L^n_Z$. 15
Finally, it is not hard to see that \( (e_{c_{0}}^\otimes \otimes e_{c-c_{0}-1}^\otimes, E_{c_{0}}^\otimes e_{c-1}^\otimes) = c_{0}! \not\equiv 0 \mod p \) if \( c_{0} < p \) so that \( E_{c_{0}}^\otimes e_{c-1}^\otimes \not\in pL_{Z}^\otimes c_{0} \) and hence \( E_{c_{0}}e_{t(n,c-1)}^\otimes \not\in pL_{Z}^n \). Also, the binomial formula yields \( E^{p} = \sum_{j} \binom{p}{j}E^{p-j} \otimes E^{j} \) so that we can conclude \( E^{p} = 0 \) by a similar induction argument. 

In particular Lemma 8 implies that we have well defined, non-zero maps \( E_{c_{0}} : S_{p}^{(c)} \rightarrow S_{p}^{(c-2c_{0})} \) on the respective \( p \)-reductions \( S_{p}^{(c)} = S_{Z}^{(c)} / pS_{Z}^{(c)} = S_{Z}^{(c)} / pL_{Z}^{n} \) using that \( pS_{Z}^{(c)} = S_{Z}^{(c)} \cap pL_{Z}^{n} \).

**Corollary 8** For \( p \) and \( n \) as above and \( k = 1, \ldots, p-1 \) with \( k \equiv n+1 \mod 2 \) there is a sequence \( C_{p,k} \) of Specht modules over \( \mathbb{F}_{p} \) as follows:

\[
0 \rightarrow S_{p}^{(n+1-2l)} \rightarrow \ldots \rightarrow S_{p}^{(2p+k)} \xrightarrow{E^{k}} S_{p}^{(2p-k)} \xrightarrow{E^{p-k}} S_{p}^{(k)} \rightarrow D_{p}^{(k)} \rightarrow 0. \tag{26}
\]

All maps (except the first and last one) are non-zero, and any two consecutive maps compose to zero.

More precisely, we have that the \( i \)-th component of this sequence is \( C_{p,k}^{(i)} = S_{p}^{i(p+k_{i})} \), where \( k_{i} = k \) if \( i \) is even and \( k_{i} = p-k \) if \( i \) is odd. The maps are \( E^{k_{i}} : C_{p,k}^{(i)} \rightarrow C_{p,k}^{(i-1)} \) so that two consecutive maps compose as \( E^{k_{i}}E^{k_{i+1}} = E^{p} = 0 \).

We have that \( C_{p,k}^{(0)} = S_{p}^{(k)} \rightarrow C_{p,k}^{(-1)} = D_{p}^{(k)} \) is the (non-zero) quotient map. Now, it is clear that \( \text{im}(E) \subset \ker(F)^{\perp} \subset S_{p}^{(k)} \) given that \( E^{*} = F \). Hence also the composite \( C_{p,k}^{(1)} \rightarrow C_{p,k}^{(0)} \rightarrow C_{p,k}^{(-1)} \) is zero. In order to characterize the last index write \( \frac{n+1+k}{2} = ph + q \) with \( h \in \mathbb{Z} \) and \( q = 0, \ldots p-1 \). We have

\[
l = \begin{cases} 
q - k & \text{if } q \geq k \\
k & \text{if } q < k
\end{cases}.
\tag{27}
\]

The maps in Corollary 8 thus extend to a sequence of the \( p \)-reductions of the induced representations from Corollary 4 as well as the weight spaces \( \mathcal{W}_{p}^{(c)}(n,g) \). The respective maps on the vector spaces \( \mathcal{V}_{p}^{(c)}(\Sigma_{g}) \) are, by equivariance of the \( Y^{(j)} \), given by the restriction and \( p \)-reductions of the maps \( E_{c_{0}} \) on \( \tilde{H}_{i}^{*}(\Sigma_{g}) \). In particular, these maps commute with the TQFT-images of the cobordisms by equivariance of \( \mathcal{V}_{Z} \).

**Corollary 9** For \( p, n \) and \( k \) as above there is a sequence of natural transformations of TQFT’s

\[
\ldots \rightarrow \mathcal{V}_{p}^{(i+1)p+k_{i+1}} \rightarrow \mathcal{V}_{p}^{(ip+k_{i})} \rightarrow \ldots \rightarrow \mathcal{V}_{p}^{(2p-k)} \rightarrow \mathcal{V}_{p}^{(k)} \rightarrow \overline{\mathcal{V}}_{p}^{(k)} \rightarrow 0
\tag{28}
\]

with \( k_{i} = k, p-k \) as above and any two consecutive transformations compose to zero.

5. **Exactness of \( C_{p,k} \)**

Exactness of the sequence \( C_{p,k} \) defined in Corollary 8 follows from the modular structure of the involved Specht modules. The irreducible factors of \( S_{p}^{(i)} \) for two row diagrams \( \tau \) are determined in Theorem 24.15 of [3]. For our proof we will, however, have to make use also of the precise submodule structure, which turns out to be rather rigid. We use the recent result of Kleshchev and Sheth in [12] that describes this structure precisely. In order to state it we need to introduce some more conventions and definitions.
For a Young diagram \( \tau = [a, b] \) we set \( c = a - b + 1 \), and consider the \( p \)-adic expansion
\[
c = c_0 + c_1 p + c_2 p^2 + \ldots + c_r p^r \quad \text{with} \quad c_j \in \{0, 1, \ldots, p-1\}.
\] (29)

As in \([12]\) we denote by \( \hat{A}_r \) the family of sets of integers of the form
\[
I = [i_1, i_2) \cup [i_3, i_4) \cup \ldots \cup [i_{2u-1}, i_{2u})
\] (30)

such that
\[
i_1 < i_2 < \ldots < i_{2u} \quad \text{and} \quad c_{i_{2j-1}} \not\equiv 0 \quad \text{and} \quad c_{i_{2j}} \not\equiv p - 1.
\] (31)

For such a set \( I \in A_r \) we define as in \([12]\) the number
\[
\delta^r_I = \sum_{i \in I} (p-1-c_i)p^i + \sum_{j=1}^u p^{i_{2j}-1} = \sum_{j=1}^u \delta^r_{i_{2j-1}, i_{2j}},
\] (32)

where \( \delta^r_{i,u} = \sum_{i \in u} (p-1-c_i)p^i + p^u \). Also as in \([12]\) we introduce the smaller set \( A_r \subset \hat{A}_r \) given by
\[
A_r = \{ I \in \hat{A}_r : \delta^r_I \leq b \},
\] (33)

as well as for any \( I \in A_r \) the function \( \nu_I \) on Young diagrams \( \tau = [a, b] \) defined as
\[
\nu_I(\tau) = [a + \delta^r_I, b - \delta^r_I].
\] (34)

As before we denote by \( D_{\mu}^a \) the irreducible quotient of the Specht module \( S^a_{\mu} \) over \( \mathbb{F}_p \) for any two row diagram \( \mu \). Also denote by \( F(M) \) the set of irreducible factors that occur in a composition series of a representation \( M \). Further, let \( M^a_{\mu} \) be the smallest sub module of \( S^a_{\mu} \) such that \( D_{\mu}^a \in F(M^a_{\mu}) \). The description of the submodule structure in \([12]\) uses the partial order on \( F(S^a_{\mu}) \) defined as \( D_{\mu_1}^{a_1} \leq \tau D_{\mu_2}^{a_2} \) if and only if \( M^a_{\mu_1} \subseteq M^a_{\mu_2} \) i.e., if and only if \( D_{\mu_1}^{a_1} \subseteq D_{\mu_2}^{a_2} \).

**Theorem 10 (Corollary 3.4 of \([12]\))**

1. All multiplicities of \( D_{\mu}^a \) in \( S^a_{\mu} \) are zero or one.

2. \( F(S^a_{\mu}) = \{ D_{\mu}^{a_I}(\tau) : I \in A_r \} \)

3. \( D_{\mu}^{a_I}(\tau) \geq \tau D_{\mu}^{a_J}(\tau) \) if and only if \( J \subseteq I \).

We define now a submodules \( S^a_{\mu} \) by choosing a special subset of of \( A_r \) and \( \hat{A}_r \). It is defined as
\[
\hat{A}^0_r = \{ I \in \hat{A}_r : 0 \not\in I \} \quad \text{and} \quad \hat{A}^+_r = \hat{A} - \hat{A}^0_r = \{ I \in \hat{A}_r : 0 \not\in I \}
\] (35)

as well as \( A^0_r = \hat{A}^0_r \cap A_r \) and \( A^+_r = \hat{A}^+_r \cap A_r \). Let us also introduce the number \( k_r = \min\{j \geq 1 : c_j \neq 0\} \) with \( k_r = \infty \) if \( c = c_0 \). Hence
\[
c = c_0 + c_{k_r} p^{k_r} + \ldots + c_r p^r \quad \text{and} \quad [0, k_r) \cap I = \emptyset \quad \text{for any} \ I \in \hat{A}^+_r.
\] (36)

The latter follows since \( 0 \not\in I \) we must have \( i_1 > 0 \). But with \( c_i \neq 0 \) we find \( i_1 \geq k_r \). For the following we assume \( c \neq 0 \mod p \), i.e., \( c_0 \neq 0 \). Given this we introduce for a diagram \( \tau = [a, b] \) with \( a - b = c - 1 \geq 2c_0 \) (so that \( k_r < \infty \)) the notation
\[
\tau' = [a - c_0, b + c_0] \quad \text{so that} \quad \tau = \nu_{[0,k_r]}(\tau')
\] (37)
For \( \tau' \) we thus have \( c' = c - 2c_0 \) and for the \( p \)-adic expansion of \( c' = \sum_j c'_jp^j \) we obtain
\[
\begin{align*}
c'_0 &= p - c_0 \\
c'_i &= p - 1 & \text{for } 1 \leq i < k_\tau \\
c'_k &= c_k - 1 \\
c'_i &= c_i & \text{for } i > k_\tau
\end{align*}
\]
From these equations it is clear that \( c'_0 \neq 0 \) and \( c'_k \neq p - 1 \) so that \([0,k_\tau)\) is an admissible interval for \( \hat{A}_{\tau'} \). In fact, it is the unique minimal interval of the special subset
\[
\hat{A}^0_{\tau'} = \{ I \in \hat{A}_{\tau'} : [0,k_\tau) \subseteq I \} \quad \text{and} \quad [0,k_\tau) \in A^0_{\tau'}.
\]

This is obvious since we must have \( i_1 = 0 \) for \( 0 \in I \) and then the next possible \( i_2 = k_\tau \). It is also easy to see that \( \delta_{[0,k_\tau)}^\tau = c_0 \), confirming the relation in (37) between \( \tau \) and \( \tau' \) via (34). We have the following simple but crucial observation.

**Lemma 10** With \( \tau \) and \( \tau' \) as above we have a well defined bijection with inverse
\[
\phi : \hat{A}^0_{\tau'} \longrightarrow \hat{A}^+_{\tau'} : I \mapsto I - [0,k_\tau) \quad \text{and} \quad \phi^{-1} : \hat{A}^+_{\tau'} \longrightarrow \hat{A}^0_{\tau'} : J \mapsto J \cup [0,k_\tau)
\]

**Proof:** We first show that \( \phi \) is well defined. From (35) we know that every \( I \in \hat{A}^0_{\tau'} \), contains the special interval and is thus of the form
\[
I = [0,i_2) \cup [i_3,i_4) \cup \ldots \cup [i_{2u-1},i_{2u}) \quad \text{with} \quad i_2 \geq k_\tau.
\]
Clearly, \( 0 \not\in \phi(I) \). We further distinguish the following two cases:

**Case** \( i_2 = k_\tau \) : In this situation \( \phi(I) = [i_3,i_4) \cup \ldots \). Now \( \phi(I) \in \hat{A}^+_{\tau'} \) since by (38) the coefficients of \( c \) and \( c' \) and hence the conditions on the \( i_j \) for \( j \geq 3 \) are the same.

**Case** \( i_2 > k_\tau \) : In this situation \( \phi(I) = [k_\tau,i_2) \cup R \) with \( R = [i_3,i_4) \cup \ldots \). We have \( c_{k_\tau} \neq 0 \) by assumption, and \( c_{i_2} = c'_{i_2} \neq p - 1 \) as \( i_2 > k_\tau \). Also all intervals in \( R \) are admissible. Hence \( \phi(I) \in \hat{A}^+_{\tau'} \) again.

In a similar fashion we show that \( \phi^{-1} \) is well defined using the fact that any \( L \in \hat{A}^+_{\tau'} \) is by (34) of the form \( L = [j_1,j_2) \cup [j_3,j_4) \cup \ldots \) with \( j_1 \geq k_\tau \). For \( j_1 > k_\tau \) we have \( \phi^{-1}(L) = [0,k_\tau) \cup [j_1,j_2) \cup [j_3,j_4) \cup \ldots \). This is \( A^0_{\tau'} \) as \( c'_0 \neq 0 \), \( c'_{k_\tau} \neq p - 1 \), and \( c'_j = c_j \) for \( j \geq j_1 \). In case that \( j_1 = k_\tau \) we have \( \phi^{-1}(L) = [0,j_2) \cup [j_3,j_4) \cup \ldots \), which is again in \( \hat{A}^0_{\tau'} \) since \( c'_j = c_j \) for \( j \geq j_2 > j_1 \). It is obvious that \( \phi \) and \( \phi^{-1} \) are inverses of each other given (39) and (37).

Next we consider how the differential from (32) changes under this bijection.

**Lemma 11** We have
\[
\delta_I^{\tau'} = \delta_{\phi(I)}^\tau + c_0
\]

**Proof:** This is a computation done according to the same cases as in the proof of Lemma 10.

**Case** \( i_2 = k_\tau \) : Then \( \delta_I^{\tau'} = \delta_{[0,k_\tau)}^\tau + \sum_{j \geq 2} \delta_{[i_{2j-1},i_{2j})}^{\tau'} \). Now, since \( c'_i = p - 1 \) for \( i = 1,\ldots,k_\tau - 1 \) we have \( \delta_{[0,k_\tau)}^{\tau'} = (p - 1 - c'_0) + p^0 = c_0 \). We also have \( \delta_{[i_{2j-1},i_{2j})}^{\tau'} = \delta_{[i_{2j-1},i_{2j})}^\tau \) since the \( c_j \) are all the same with \( i_j > k_\tau \). But \( \delta_{[0,k_\tau)}^{\tau'} = \sum_{j \geq 2} \delta_{[i_{2j-1},i_{2j})}^{\tau'} \) which implies the assertion.
We denote the modules $\delta^\tau_I = \sum_{i=0}^{i_2-1} (p-1-c_i')p^i + 1 + \delta^\tau_R$ via the identification.

Then we have $\delta^\tau_I = (p-1-c_0') + 1 + (p-1-c_k')p^{k_\tau} + \sum_{i>k_\tau} (p-1-c_i')p^i + \delta^\tau_R$.

Lemma 12: Let $\tau$ and $\tau'$ be as above. Suppose there is a non-zero map $\xi : S_p^\tau \to S_p^{\tau'}$ with $S_p^\tau \subseteq ker(\xi)$. Then we have $im(\xi) = \tilde{S}_p^{\tau'}$ and $ker(\xi) = \tilde{S}_p^{\tau}$. 

Corollary 11: The map $\phi$ restricts to a bijection $\phi : A^0_{\tau} \to A^+_\tau$ with the properties that it is monotonous with respect to inclusions and $\nu_I(\tau') = \nu_{\phi(I)}(\tau)$. 

Proof: From (37) we have $\delta^\tau_I \leq b'$ iff $\delta^\tau_{\phi(I)} + c_0 \leq b + c_0$ iff $\delta^\tau_{\phi(I)} \leq b$ iff $\phi(I) \in A^+_\tau$. The fact that $\phi(A) \subseteq \phi(B)$ iff $A \subseteq B$ is obvious. Relation (47) is immediate from Lemma 11, (34) and (37). 

Let us turn now to the analogous relations between the modules. The set $A^0_{\tau}$ contains a minimal element, namely $I_{\tau}^0 = [0, h_\tau)$, which is the smallest interval with $h_\tau > 0$ and $c_{h_\tau} \neq p-1$. We denote the modules $\tilde{S}_p^\tau = M_{\nu_{I^0}}(\tau)$ and $\tilde{S}_p^\tau = S_p^\tau / \tilde{S}_p^\tau$. 

By 3. of Theorem 10 we know that the modules in the composition series of $M_{\nu_{I^0}}(\tau)$ are of the form $D_{p}^{\nu_I(\tau)}$, where $I^0_p \subseteq J$, which is equivalent to $J \in A^0_{\tau}$. Hence

$$\mathcal{F}(\tilde{S}_p^\tau) = \{D_p^{\nu_I(\tau)} : I \in A^0_{\tau}\} \quad \text{and} \quad \mathcal{F}(\tilde{S}_p^\tau) = \{D_p^{\nu_I(\tau)} : I \in A^+_{\tau}\}. \quad (46)$$

With $h_\tau' = k_\tau$ and $I^0_{\tau'} = \phi(0)$ we thus have $\tilde{S}_p^{\tau'} = M_{\nu_{I^0}}(\tau')$. The bijection from Corollary 11 allows us now to show that the composition series of $\tilde{S}_p^{\tau'}$ and $\tilde{S}_p^\tau$ yield exactly the same set of irreducibles via the identification $\phi^* : \mathcal{F}(\tilde{S}_p^{\tau'}) \to \mathcal{F}(\tilde{S}_p^\tau) : D_p^{\nu_I(\tau')} \mapsto D_p^{\nu_{\phi(I)}(\tau')}$. \quad (47)
Proof: Consider the map \( \overline{\xi} : \overline{\mathcal{S}}_p^r \rightarrow \mathcal{S}_p^r \) defined on the quotient by \( \overline{\mathcal{S}}_p^r \) as well as the composite \( \overline{\xi} : \overline{\mathcal{S}}_p^r \rightarrow \overline{\mathcal{S}}_p^r \) with the projection. Now, since by 1. of Theorem \( 14 \) we have a disjoint union \( \mathcal{F}(\mathcal{S}_p^r) = \mathcal{F}(\mathcal{S}_p^r) \cup \mathcal{F}(\overline{\mathcal{S}}_p^r) \) so that by (17) \( \mathcal{F}(\overline{\mathcal{S}}_p^r) \cap \mathcal{F}(\overline{\mathcal{S}}_p^r) = \emptyset \). Hence \( \overline{\xi} \) is a map between modules with no common irreducibles in their composition series so that \( \overline{\xi} = 0 \).

Thus the image of \( \xi \) or \( \overline{\xi} \) must lie in \( \mathcal{S}_p^r \). Consider the sequence of maps

\[
\xi : \mathcal{M}_p^r = \mathcal{S}_p^r \rightarrow \overline{\mathcal{S}}_p^r \rightarrow \mathcal{S}_p^r = \mathcal{M}_p^r
\]

Since all multiplicities are one we have again a disjoint union \( \mathcal{F}(\mathcal{M}_p^r) = \mathcal{F}(\ker(\xi)) \cup \mathcal{F}(\operatorname{im}(\xi)) \) so that either \( \ker(\xi) \subseteq \mathcal{M}_p^r \) is a submodule that contains \( \mathcal{D}_p^r \) in its composition series so that by minimality \( \ker(\xi) = \mathcal{M}_p^r \). We had, however, assumed that \( \xi \neq 0 \). Hence \( \mathcal{D}_p^r \) must be contained in the composition series of \( \operatorname{im}(\xi) \subseteq \mathcal{M}_p^r \). Again it follows by minimality that \( \operatorname{im}(\xi) = \mathcal{M}_p^r \). This also means that \( \overline{\xi} \) is a surjective map. The fact that all multiplicities are one together with (17) implies that \( \dim(\overline{\mathcal{S}}_p^r) = \dim(\mathcal{S}_p^r) \). Hence \( \overline{\xi} \) must be an isomorphism.

We are finally in the position to establish following resolution of irreducible modules by Specht modules.

**Theorem 12** The sequences \( [\mathcal{S}_p^r] \) from Corollary 8 are exact.

**Proof:** We begin by deriving from Theorem \( 14 \) that the first term \( \mathcal{S}_p^{(n+1-2l)} \) in the sequence is irreducible. Here we have \( b = l \) and \( c = n + 1 - 2l = 2ph + 2q - k - 2l \) so that by (27)

\[
b = \begin{cases} q-k & \text{if } q \geq k \\ q & \text{if } q < k \end{cases} \quad \text{and} \quad c_0 = \begin{cases} k & \text{if } q \geq k \\ p-k & \text{if } q < k \end{cases}
\]

Now for a set \( I \in A_\tau \) with \( I \neq \emptyset \) we have \( \delta_I \geq p - c_0 \). We also need \( \delta_I \leq b \) and hence \( p - c_0 \leq b \). For \( q \geq k \) this condition reduces to \( p \leq q \), which is not possible, and for \( q < k \) we find \( q \geq k \), a contradiction as well. Hence \( A_\tau = \{ \emptyset \} \) so that \( \mathcal{S}_p^{(n+1-2l)} = \overline{\mathcal{S}}_p^{(n+1-2l)} = \mathcal{D}_p^{(n+1-2l)} \) and \( \mathcal{S}_p^{(n+1-2l)} = 0 \).

Next, we observe that all maps \( E^{k_i} : \mathcal{S}_p^{((i+1)p+k_{i+1})} \rightarrow \mathcal{S}_q^{(ip+k_i)} \) in the sequence are of the type of the ones in Lemma 12 with diagrams related by (17). More precisely, we have \( c_0 = k_{i+1} \) so that \( c' = c - 2c_0 = (i + 1)p + k_{i+1} - 2k_{i+1} = jp + (p - k_{i+1}) = ip + k_i \). Corollary 8 also implies that these maps are non-zero and equivariant. It now follows by induction, going from large to small \( i \), that

\[
\ker(E^{k_i}) = \overline{\mathcal{S}}_q^{((i+1)p+k_{i+1})} \quad \text{and} \quad \operatorname{im}(E^{k_i}) = \overline{\mathcal{S}}_q^{(ip+k_i)}.
\]

For the first two maps in the sequence this is clear by irreducibility of the first module and the fact that the next map is non-zero.

Once we have proved the relation for \( E^{k_{i+1}} \) we know from \( E^{k_i} E^{k_{i+1}} = 0 \) that \( \overline{\mathcal{S}}_q^{((i+1)p+k_{i+1})} \subseteq \ker(E^{k_i}) \). We can thus apply Lemma 12 to \( E^{k_i} \) and infer the statement for \( i \) from \( i+1 \). Hence exactness holds for the terms before \( \mathcal{S}_p^{(k_i)} \).

For this last Specht module we have \( c_0 = k \) and \( c_j = 0 \) for \( j > 0 \). Thus \( A_\tau^r = \{ \emptyset \} \) and \( A_\tau^l = \{ 0, j \} : p^l \leq \frac{n+k+1}{2} \) using that \( \delta_{(0,j)} = p^l - k \) and \( b = \frac{n-k+1}{2} \). Particularly, we have that \( \overline{\mathcal{S}}_p^{(k)} = \mathcal{D}_p^{(k)} \) is already irreducible. The kernel of the last map already contains \( \overline{\mathcal{S}}_p^{(k)} \), the
image of the previous map, in its kernel. If the kernel was bigger the map would thus have to be zero, which is not the case. Hence exactness also holds at $S_p^{(k)}$. Finally, the last map is by irreducibility onto so that exactness holds through out the sequence.

As a result we obtain that the sequence of TQFT’s in (28) of Corollary 9 is exact and this yields a resolutions of the TQFT’s $\tilde{\mathcal{V}}_p^{(k)}$ with $0 < k < p$, thus proving Theorem 12. The kernels and cokernels of these sequences also define TQFT’s, which we denote in analogy to the symmetric group representations by $\tilde{\mathcal{V}}_p^{(j)}$ and $\tilde{\mathcal{V}}_p^{(j)}$ so that we have short exact sequences

$$0 \to \tilde{\mathcal{V}}_p^{(j)} \to \mathcal{V}_p^{(j)} \to \tilde{\mathcal{V}}_p^{(j)} \to 0 .$$

(48)

6. Characters, Dimensions, and the Alexander Polynomial

An obvious application of Theorem 12 is that we can express the characters $\varphi_p^{\tau_k}$ and dimensions of the $D_p^{\tau_k} = D_p^{(k)}$ for diagrams $\tau_k = \left[ \frac{n+k-1}{2}, \frac{n-k+1}{2} \right]$ with $k \equiv n + 1 \mod 2$ and $0 < k < p$ in terms of the ordinary characters $\chi^\tau$ of Specht modules $S^\tau$.

Corollary 13 For $k \equiv n+1 \mod 2$ and $0 < k < p$ we have the following identity of $S_n$-characters

$$\varphi_p^{\tau_k} = \sum_{i \geq 0} (-1)^i \chi^{\tau_i},$$

where $j_i = ip + k_i$ and $k_i = k$ for $i$ even and $k_i = p - k$ for $i$ odd.

As an example we consider the special case $p = 5$, where $k = 1$ or 3 if $n$ is even and $k = 2$ or 4 if $n$ odd. In [22] Ryba constructs a family of irreducible so called Fibonacci representations $R_n$ and $R_n'$ of $S_n$ over $\mathbb{F}_5$ with Brauer characters $\varphi_n$ and $\varphi_n'$ respectively. It follows by straightforward computation from (49) and the formulae in Definition 2 of [22] that $\phi_n = \phi_5^{(r)}$ and $\phi_n' = \phi_5^{(r+1,r-1)}$ if $n = 2r$ is even, and $\phi_n' = \phi_5^{(r+2,r-1)}$ and $\phi_n = \phi_5^{(r+1,r)}$ if $n = 2r + 1$ is odd.

Corollary 14 The Fibonacci representations from [22] are

$$R_n \cong \begin{cases} D_5^{(r+1,r-1)} & \text{if } n = 2r \\ D_5^{(r+1,r)} & \text{if } n = 2r + 1 \end{cases}$$

and

$$R_n' \cong \begin{cases} D_5^{(r,r)} & \text{if } n = 2r \\ D_5^{(r+2,r-1)} & \text{if } n = 2r + 1 \end{cases}$$

We expect similar relations with the generalizations of these representations obtained by Kleshchev in [1].

The dimensions of the Specht modules are naturally given by the Catalan numbers $C(n,j) = \binom{n}{j} - \binom{n}{j-1}$. More precisely, $\dim(S^{(n-b,b)}) = C(n,b)$. Particularly, we find for the components of the sequence in [22] that $\dim(S^{(c,s)}) = C(n,b - sp)$ and $\dim(S^{(c,s+1)}) = -C(n,b + (s + 1)p)$, with $b = \frac{n+1-k}{2}$, where we also use that $C(n,j) = -C(n,n+1-j)$. The alternating sum of the Specht module dimensions comes out to be

$$d_k^n = \dim(D_p^{(n-b,b)}) = \sum_{s \in \mathbb{Z}} C(n,b + sp) \quad \text{provided } 0 < k = n - 2b + 1 < p .$$

(50)

Note that if we extend the above formula for $d_k^n$ to the next indices we find

$$d_0^n = 0 \text{ for odd } n , \quad \text{and} \quad d_p^n = 0 \text{ for even } n$$

(51)
In order to describe generating functions for these dimensions we introduce some notation. First we write \([n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathbb{Z}[x, x^{-1}]\) for the usual quantum integers. Moreover we consider now the ring of cyclotomic integers \(\mathbb{Z}[\zeta_p]\) obtained \(\mathbb{Z}[x, x^{-1}]\) by imposing the relation \(\sum_{j=0}^{p-1} x^j = 0\). It is free as a \(\mathbb{Z}\)-module of rank \(p - 1\). We also denote by \(A[\zeta_p] \subset \mathbb{Z}[\zeta_p]\) the subring invariant under conjugation \(\zeta_p \mapsto \zeta_p^{-1}\). This is a free \(\mathbb{Z}\)-module of rank \(\frac{p(p+1)}{2}\). It is not hard to see that the set of \([k]_{\zeta_p}\) restricted to either even \(k\) or to odd \(k\) yields a \(\mathbb{Z}\)-basis for \(A[\zeta_p]\).

**Lemma 13** We have the following identity in \(A[\zeta_p]\):

\[
[2]_{\zeta_p}^n = \sum_{0 < k < p \atop k \equiv n+1 \text{ mod } 2} d_k^n[1]_{\zeta_p} \quad (52)
\]

**Proof:** We have by Schur Weyl duality that \(L_n^n \cong V_2^{\otimes n} \cong \bigoplus_{j \equiv n-1 \text{ mod } 2} V_j \otimes S^{[\frac{n-1}{2}, \frac{n+1}{2}]}\). The operator \(x^H\) with \(H \in \mathfrak{s}(2, \mathbb{Z})\) is well defined and has trace \(\text{tr}_V(x^H) = [j]_x\). It thus follows that \([2]_x^n = \sum_{l \equiv 1, l \equiv n+1 \text{ mod } 2} [l]_x C(n, \frac{n-1}{2})\). Now any such \(l\) can be uniquely written in the form \(l = k + 2sp\) or \(l = -k + 2(s+1)p\) with \(s \geq 0\) and \(k \equiv n+1 \text{ mod } 2\) and \(0 < k < p\). Specializing to a root of unity \(x = \zeta_p\) we have then that \([l]_{\zeta_p} = [k]_{\zeta_p}\) and \([l]_{\zeta_p} = -[k]_{\zeta_p}\) respectively. Also \(C(n, \frac{n-1}{2}) = C(n, \frac{n-k+1}{2} - 2sp)\) and \(C(n, \frac{n-1}{2}) = C(n, \frac{n+k+1}{2} - (s+1)p)\) and \([k]_{\zeta_p} C(n, b + (s+1)p)\) for \(s \geq 0\) and \(b = \frac{n-k+1}{2}\), which with \(s \in \mathbb{Z}\) adds up to the expression in \((52)\). Using \([2]_x^n[k]_x = [k-1]_x + [k+1]_x\) and the bases of \(A[\zeta_p] \subset \mathbb{Z}[\zeta_p]\) we readily derive from \((52)\) the recursion \(d_{k+1}^n = d_k^n + d_{k+1}^n\). This translates for \(0 \leq a - b < p - 2\) to

\[
dim(D_p^{[a,b]}) = \begin{cases} 
\dim(D_p^{[a,a-1]}) & \text{if } a = b \\
\dim(D_p^{[a-1,b]}) + \dim(D_p^{[a,b-1]}) & \text{if } 0 < a - b < p - 2 \\
\dim(D_p^{[a-1,b-1]}) & \text{if } a - b = p - 2
\end{cases} \quad (53)
\]

It is easy to see from this form that the dimensions are indeed given by the number of paths through the set of diagrams with \(a - b \leq p - 2\) as described in \((14)\) for general diagrams.

In the case \(p = 5\) this recursion reduces to \(\dim(R_n) = \dim(R_{n-1}) + \dim(R_{n-1}')\) and \(\dim(R_n) = \dim(R_{n-1})\) so that the dimensions are given by Fibonacci numbers. More precisely, we have \(\dim(R_n) = f_n\) and \(\dim(R_n') = f_{n-1}\), where \(f_n\) are the Fibonacci numbers defined by \(f_0 = 0, f_1 = 1\) and \(f_{n+1} = f_n + f_{n-1}\). Note that together with \((51)\) we find interesting presentations of Fibonacci numbers in terms of alternating, 5-periodic sums of Catalan numbers:

\[
f_{2r} = C(2r, r - 1) - C(2r, r - 3) + C(2r, r - 6) - C(2r, r - 8) + \ldots = C(2r + 1, r - 1) - C(2r + 1, r - 2) + C(2r + 1, r - 6) - C(2r + 1, r - 7) + \ldots
\]

and

\[
f_{2r+1} = C(2r + 1, r) - C(2r + 1, r - 3) + C(2r + 1, r - 5) - C(2r + 1, r - 8) + \ldots = C(2r + 2, r + 1) - C(2r + 2, r - 3) + C(2r + 2, r - 4) - C(2r + 2, r - 8) + \ldots
\]

The reader is invited to check these identities independently via recursion relations such as \(C(n + 1, j) = C(n, j) + C(n, j - 1)\) or via the well known generating functions of Catalan and Fibonacci numbers.
Another useful tool in the determination of dimensions are fusion algebras or Verlinde algebras, see [1]. For the quantum group $U_q(\mathfrak{sl}_2)$ at a $p$-th root of unity the ring of irreducible representations yields the fusion algebra $\Phi_p$ generated by the irreducibles $\{1\}, \{2\}, \ldots, \{p-1\}$. Let us denote by $\text{mult}(k, R) \in \mathbb{N} \cup \{0\}$ the multiplicity of $\{j\}$ in an element $R \in \Phi_p$ so that $R = \sum_j \text{mult}(k, R)[k]$. We have relations $\{1\} \circ \{k\} = [k]$, $\{p-1\} \circ \{k\} = [p-k]$ and $\{2\} \circ \{k\} = [k+1] + [k-1]$ for $1 < k < p-1$. Comparing this to the recursion in (53) we find that

$$\text{dim}(D_p^{\frac{n+k-1}{2} - \frac{n+k+1}{2}}) = \text{mult}(k, [2]^n)$$

(56)

Now, as the $D^*_p$ are isomorphic to the weight spaces $\overline{\mathcal{V}}_p(k)(\lambda, g)$ with $n = n(\lambda)$ we find $\text{dim}(\overline{\mathcal{V}}_p(k)(\Sigma_g)) = \sum_{n=0}^{g} 2^{g-n}(\frac{g}{2})\text{dim}(D_p^{\frac{n+k-1}{2} - \frac{n+k+1}{2}})$ so that we find the following Verlinde type formula.

**Lemma 14**

$$\text{dim}(\overline{\mathcal{V}}_p(k)(\Sigma_g)) = \text{mult}(k, f^g_p) \quad \text{with} \quad f_p = 2[1] + [2] \in \Phi_p.$$  

(57)

Compare this to the TQFT’s $\mathcal{V}^\text{RT}_p$ and $\mathcal{V}^\text{RT}_p$ of Reshetikhin Turaev for the quantum groups $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{so}_3)$ respectively at a $p$-th root of unity. ($\mathcal{V}^\text{RT}_p$ is really a factor TQFT obtained from $\mathcal{V}^\text{RT}_p$ by restricting to odd dimensional representations). With $\mathcal{F}_p = \sum_j (2j + 1)^2$ and $\mathcal{F}^*_p = 2\mathcal{F}_p = \sum_k (k)^2$ we have

$$\text{dim}(\mathcal{V}^\text{RT}_p(\Sigma_g)) = \text{mult}(1, \mathcal{F}^g_p) = 2^g\text{mult}(1, \mathcal{F}^g_p) = 2^g\text{dim}(\mathcal{V}^\text{RT}_p(\Sigma_g))$$

(58)

In the case $p = 5$ these formulae allow us to efficiently compute and compare dimensions.

**Lemma 15** We have for the dimensions $D_g^{(k)} = \text{dim}(\overline{\mathcal{V}}_5^{(k-1)}(\Sigma_g))$.

1. For even $g$:

$$D_g^{(1)} = \frac{1}{2}(5\frac{2}{5}f_{g-1} + f_{g+1})$$

$$D_g^{(2)} = \frac{1}{2}(5\frac{2}{5}f_{g} + f_{2g})$$

$$D_g^{(3)} = \frac{1}{2}(5\frac{2}{5}f_{g} - f_{2g})$$

2. For odd $g$:

$$D_g^{(1)} = \frac{1}{2}(5\frac{2}{5}(f_{g-2} + f_{g}) + f_{2g+1})$$

$$D_g^{(2)} = \frac{1}{2}(5\frac{2}{5}(f_{g-1} + f_{g+1}) + f_{2g})$$

$$D_g^{(3)} = \frac{1}{2}(5\frac{2}{5}(f_{g-1} + f_{g+1}) - f_{2g})$$

and

$$\text{dim}(\mathcal{V}^\text{RT}_5(\Sigma_g)) = D_g^{(1)} + D_g^{(4)} \quad \forall g \geq 0.$$  

(59)

**Proof:** Let us use a more convenient notation $1 = \{1\}$, $\rho = \{3\}$, $\sigma = \{4\}$, and $\rho \circ \rho = \{2\}$ subject to relations $\rho \circ \rho = 1 + \rho$ and $\sigma^2 = 1$. These relations imply $\rho^n = f_{n-1} + f_n \rho$, $\mathcal{F}_5 = 2 + \sigma \circ \rho$ and $\mathcal{F}^g_5 = 2 + \rho$. We note now that $\mathcal{F}^g_5 = (2 + \rho)^2 = 4 + 4\rho + \rho^2 = 5(1 + \rho) = 5\rho^2$ so that $\mathcal{F}^g_5 = 5\frac{2}{5}(f_{g-1} + f_{g}\rho)$ if $g$ is even. From there we compute directly that for odd $g$ we have $\mathcal{F}^g_5 = 5\frac{2}{5}(f_{g-2} + f_{g}) + (f_{g-1} + f_{g+1})\rho$. Consider also $\eta = 1 - \rho$. We have $\eta^2 = 1 - 2\rho + \rho^2 = 2 - \rho = 1 + \eta$ and thus again $\eta^n = f_{n-1} + f_n \eta$. We find $(2-\rho)^g = \eta^{2g} = f_{2g-1} + f_{2g}\eta = f_{2g-1} - f_{2g}\rho$.

Now, we have $(1 + \sigma) \circ \mathcal{F}_5 = (1 + \sigma) \circ \mathcal{F}_5$ and thus $(1 + \sigma) \circ \mathcal{F}^g_5 = (1 + \sigma) \circ \mathcal{F}^g_5$. Similarly, we find $\mathcal{F}_5^g = (1 + \sigma) \circ \mathcal{F}_5^g$ and $\mathcal{F}_5^g = (2 - \rho)^g$. With the
previous results on $\mathbf{F}_q^2$ and $(2 - \rho)^q$ we thus find a formula for $\mathbf{f}_q^2$, which inserted into (57) yields the asserted formulæ.

The formula in (59) reflects the fact that $\overline{V}_3^{(1)} \oplus \overline{V}_3^{(4)}$ is the sum of the irreducible constituents of the $\mathbb{F}_p$-reductions of $\mathcal{V}_R^T$ as TQFT’s, see [14]. For larger primes $p$ it is, however, not possible that a $\mathbb{F}_p$-reduction $\mathcal{V}_R^T$ has only the $V_3^{(j)}$ as irreducible components. This is easily see by looking at the large $g$ asymptotics of the dimension expressions in (57) and (58). The operations on $\Phi_p$ given by multiplication by $\mathbf{f}_g$ or $\mathbf{f}_g$ are represented by matrices with non-negative integer coefficients. Perron-Frobenius theory implies that the matrix elements of $\mathbf{F}_g^2$ or $\mathbf{f}_g^2$, such as those in (57) and (58), grow like $\sim \|\mathbf{f}_p\|^q$ and $\sim \|\mathbf{F}_p\|^q$, respectively, where $\|\mathbf{F}_p\| = \frac{p}{4 \sin^2 \left(\frac{\pi}{p}\right)}$ and $\|\mathbf{f}_p\| = 4 \cos^2 \left(\frac{\pi}{p}\right)$ are the largest eigenvalues of the associate matrices. We thus obtain (59).

Note, that $\|\mathbf{F}_5\|$ = $\|\mathbf{f}_5\|$ but that $\|\mathbf{f}_p\| > \|\mathbf{f}_p\|$ if $p > 5$. Thus, a linear relation as Theorem 3 cannot generalize to $p > 5$. Instead, we can find polynomials $R_p(f) \in \mathbb{Z}[f]$ of degree $\text{deg}(R_p) = \frac{p - 3}{2}$ such that $\mathbf{F}_p = \mathbf{R}_p(\mathbf{f}_p)$ by using recursive relations in $\Phi_p$, which may be identified with a $\mathbb{F}_2$-extension of the real part of $\mathbb{Z}[\zeta_p]$. Using modified Tschebycheff polynomials, which we define by the recursion $P_{j+1}(x) + P_{j-1} = x \cdot P_j(x)$, with $R_0(x) = 1$ and $R_1(x) = x$, the $R_p$ can be written as follows.

\[ R_p(f) = \sum_{j=0}^{\frac{p-1}{2}} n_j \cdot P_j(f - 2) \ , \quad \text{with} \quad n_j = \begin{cases} \frac{p-1-j}{2} & \text{for } j \text{ even} \\ \frac{p-1-j}{2} & \text{for } j \text{ odd} \end{cases} \] (60)

For example, $R_5(f) = f$, $R_7(f) = 2 f^2 - 7 f + 7$, $R_9(f) = 2 f^3 - 9 f^2 + 9 f + 3$, $R_{11} = 3 f^4 - 22 f^3 + 55 f^2 - 55 f + 22$, and $R_{13}(f) = 3 f^5 - 26 f^4 + 78 f^3 - 91 f^2 + 26 f + 13$. These polynomials appear to play an important role in representation theoretic aspects of Conjecture 3.

The analogs of the character expansions given in Corollary 13 and relations as in Lemma 13 in the context of the corresponding TQFT’s attain a topological interpretation via the Alexander Polynomial $\Delta_\varphi(M) \in \mathbb{Z}[x, x^{-1}]$ for a compact, oriented 3-manifold $M$ with a selected epimorphism $\varphi : \tilde{H}_1(M, \mathbb{Z}) \rightarrow \mathbb{Z}$. Up to $S$-equivalence the cocycle $\varphi$ defines a two-sided embedded surface $\Sigma \subset M$ and cobordism $C_\Sigma : \Sigma \rightarrow \Sigma$ obtained by removing a neighborhood of $\Sigma$ from $M$. The Alexander Polynomial is then given (up to a sign, which is determined by the additional framing structure on $M$) by the following identity extracted in Section 11 of [13]:

\[ \Delta_\varphi(M) = \text{trace}(x^{-H} \mathcal{V}_\Sigma(C_\Sigma)) = \sum_{j \geq 1} [j]_{-x} \text{trace}(\overline{\mathcal{V}}_{Z}^{(j)}(C_\Sigma)) \] (61)

By inserting $x = \zeta_p$ or $x = -\zeta_p$, a $p$-th root of unity, and reducing the integer coefficients to $\mathbb{F}_p$, we consider the image of the Alexander Polynomial under the two so defined natural maps

\[ \mathbb{Z}[x, x^{-1}] \rightarrow \mathbb{F}_p[\zeta_p] : \quad x \mapsto \pm \zeta_p : \Delta_\varphi(M) \mapsto \overline{\Delta}_p^\pm(M) \] (62)

Now, if we insert $x = \zeta_p$ into (51) and use $[j]_p = (\pm 1)^{k-1} (-1)^j |k|_{\zeta_p}$, where $j_i = ip + k_i$ as before, we infer from the resolutions of TQFT’s in (28) the following identities

\[ \overline{\Delta}_p^{+}(M) = \sum_{k=1}^{|\Delta_p|} (-1)^{k-1} |k|_{\zeta_p} \left( \text{trace}(\overline{\mathcal{V}}_{p}^{(k)}(C_\Sigma)) + \text{trace}(\overline{\mathcal{V}}_{p}^{(p-k)}(C_\Sigma)) \right) \] (63)

\[ \overline{\Delta}_p^{-}(M) = \sum_{k=1}^{|\Delta_p|} |k|_{\zeta_p} \left( \text{trace}(\overline{\mathcal{V}}_{p}^{(k)}(C_\Sigma)) - \text{trace}(\overline{\mathcal{V}}_{p}^{(p-k)}(C_\Sigma)) \right) . \] (64)
This implies Theorem 3. At a 5-th root of unity $\bar{\Delta}^+_{\varphi,p}(M)$ depends only on traces of $C_\Sigma$ under $\mathcal{V}^{(1)}_5 \oplus \mathcal{V}^{(4)}_5$ and $\mathcal{V}^{(2)}_5 \oplus \mathcal{V}^{(3)}_5$. The former is in [10] identified with the integral semisimple reduction of the Reshetikhin Turaev theory $\mathcal{V}^{RT}_5$ and implies (5).

7. Johnson-Morita Extensions

As before we consider for $H = H_1(\Sigma)$ a standard basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ that is symplectic with respect to the standard skew form $(\cdot, \cdot)$ and orthonormal with respect to the inner form $(\cdot, \cdot)$. Let $J \in \text{Sp}(2g, \mathbb{Z})$ be the special element defined by $Ja_i = b_i$ and $Jb_i = -a_i$ so that $(x, y) = (x, Jy)$ and $Jg^{-1}J^{-1} = g^*$. Also denote by $\omega = \sum_j a_j \wedge b_j$ the standard invariant 2-form.

For any $x \in \Lambda^m H$ we can now define a degree-$m$ map $\nu(x) : \Lambda^m H \to \Lambda^{*+m} H$ by $\nu(x), y = x \wedge y$. From this we define another map as $\mu(x) = \nu(Jx)^* : \Lambda^m H \to \Lambda^{*-m} H$ of degree $-m$.

**Lemma 16** The maps $\nu$ and $\mu$ have the following properties:

1. Covariance: $g \nu(x)g^{-1} = \nu(gx)$ and $g \mu(x)g^{-1} = \mu(gx)$ for all $x \in \Lambda^\ast H$ and $g \in \text{Sp}(2g, \mathbb{Z})$.

2. Homomorphy: $\nu(x \wedge y) = \nu(x)\nu(y)$ and $\mu(x \wedge y) = \nu(y)\nu(x)$.

3. Generators: $\nu(\omega) = E$ and $\mu(\omega) = F$ so that $[E, \nu(x)] = [F, \mu(x)] = 0$ for all $x \in \Lambda^\ast H$.

4. Anticommutator: $\mu(x)\nu(y) + \nu(y)\mu(x) = (x, y)\mathbb{I}$ for all $x, y \in H$.

5. Commutator : $[E, \mu(x)] = \nu(x)$ and $[F, \nu(x)] = -\mu(x)$ for $x \in H$.

**Proof**: Covariance for $\nu$ is obvious. For $\mu$ consider $g \mu(x)g^{-1} = (g^{-1}\nu(Jx)g^*) = \nu(g^{-1}Jx)^* = \nu(Jgx)^* = \mu(gx)$. Also homomorphy is obvious and the fact that $\nu(\omega) = E$ follows by definition. This implies the zero-commutators since $\omega$ is even, hence central, and $F = E^*$. The anticommutator relation is readily translated to $\nu(x)\nu(y) + \nu(y)\nu(x)^* = (x, y)\mathbb{I}$. As a symmetric bilinear relation it suffices to prove this for a system of two orthonormal vectors $v$ and $w$. Then $\Lambda^\ast H = \Lambda^\ast L \otimes \Lambda^\ast L^\perp$, where $L$ is the space spanned by $v$ and $w$ and $L^\perp$ its orthogonal complement. Clearly, $\nu(v)$ and $\nu(w)$ act only on the $\Lambda^\ast L \cong (\mathbb{Z}^2)^{\otimes 2}$ with basis $\{1, v, w, v \wedge w\} = \{e_-, e_+ \otimes e_-, e_+ \wedge e_-, e_0 \otimes e_+, e_0 \wedge e_+\}$. In this form the operators take on the form $\nu(v) = E \otimes \mathbb{I}$ and $\nu(w) = -H \otimes E$, where the $\mathfrak{sl}_2$-generators $E$ and $H$ act as usual. The relation follows now from $HE + EH = 0$ and $E^*E + EE^* = FE + EF = \mathbb{I}$. The commutators follow by direct calculation. $E\mu(x) = \nu(\omega)\mu(x) = \sum_i \nu(a_i)\nu(b_i)\mu(x) = -\sum_i \nu(a_i)\mu(x)\nu(b_i) + \sum_i \nu(a_i)(x, b_i) = \sum_i \mu(x)\nu(a_i)\nu(b_i) - \sum_i (x, a_i)\nu(b_i) + \sum_i \nu(a_i)(x, b_i) = \mu(x)E + \nu\left(\sum_i a_i(x, b_i) - b_i(x, a_i)\right) = \mu(x)E + \nu\left(\sum_i a_i(x, a_i) + b_i(x, b_i)\right) = \mu(x)E + \nu(x)$. The second relation is just the conjugate of the first.

Equipped with these relations we can now construct maps between the $\text{Sp}(2g, \mathbb{Z})$-representation spaces from previous sections.

**Lemma 17** The map $\mu$ restricts and factors into an $\text{Sp}(2g, \mathbb{Z})$-covariant map

$$\mu : \frac{\Lambda^m H}{\omega \wedge \Lambda^{m-2} H} \to \text{Hom}(\mathcal{V}_Z^{(j)}(\Sigma), \mathcal{V}_Z^{(j+m)}(\Sigma)).$$

Moreover, for any $x \in \Lambda^m H$ we have that

$$\mu(x) : \text{im}(E^l+m) \to \text{im}(E^l).$$

(65)
Proof: Recall that $\mathcal{V}^{(j)}(\Sigma_g) = \ker(F) \cap \Lambda^{g-j+1} H$. Now $\mu(x)$ commutes with $F$ and thus maps $\ker(F)$ to itself and thus by counting degrees to $\mathcal{V}^{(j+m)}(\Sigma_g) = \ker(F) \cap \Lambda^{g-j-m+1} H$. Moreover, $\mu(\omega \wedge x) = \mu(x) \mu(\omega) = \mu(x) F = 0$ if restricted to $\ker(F)$. By the homomorphism property it suffices to show the second relation for $m = 1$. In this case $\mu(x) E^{l+1}z = E^{l+1} \mu(x) z + (l + 1) E^l \nu(x) z \in \im(E^l)$ by iteration of the commutator relation.

Consider now the following family of extensions of the symplectic group defined as a semidirect product.

$$JM_a(m, g) = \left( \frac{1}{a} \frac{\Lambda^m H}{\Lambda^{m-2} H} \right) \rtimes \text{Sp}(2g, \mathbb{Z})$$

(66)

**Proposition 15** For $a \not\equiv 0 \mod p$ we have well defined representations of $JM_a(m, g)$ on

$$\mathcal{U}_p^{(j)}(m, g) = \mathcal{V}_p^{(j)}(\Sigma_g) \oplus \mathcal{V}_p^{(j+m)}(\Sigma_g)$$

which decomposes as indicated if restricted to $\text{Sp}(2g, \mathbb{Z})$ and is given by $\mu$ if restricted to the abelian part. For $j \equiv k \mod p$ with $0 < k < p - m$ and using the notation from (43) we have that $\mathcal{U}_p^{(j)}(m, g) = \mathcal{V}_p^{(j)}(\Sigma_g) \oplus \mathcal{V}_p^{(j+m)}(\Sigma_g)$ is a proper submodule with subquotient

$$\overline{\mathcal{U}}_p^{(j)}(m, g) = \mathcal{V}_p^{(j)}(\Sigma_g) \oplus \mathcal{V}_p^{(j+m)}(\Sigma_g) \cong \mathcal{U}_p^{(j)}(m, g) / \mathcal{U}_p^{(j)}(m, g).$$

(67)

**Proof:** In general if $V$, $W$ and $M$ are $G$-modules, and $\mu : M \rightarrow \text{Hom}(V, W)$ a covariant map, we construct a module $V \oplus W$ of $M \rtimes G$ by letting $(m, g)$ act on the sum $V \oplus W$ by the block matrix

$$\begin{bmatrix} g & 0 \\ \mu(m) g & g \end{bmatrix}$$

so that $W \subset V \oplus W$ is a submodule with subquotient $V$. Thus the map $\mu$ from Lemma 17 defines such a module for $JM_1(m, g)$ over $\mathbb{Z}$. In the $\mathbb{F}_p$-reduction $a$ is invertible so that $\mu$ can be extended to $JM_a(m, g)$.

By exactness of the sequences in (25) we have that $\mathcal{V}_p^{(j)}(\Sigma_g) = \mathcal{V}_p^{(j)}(\Sigma_g) \cap \im(E^{p-k})$. Thus by (15) of Lemma 17 we have that $\mu(x)(\mathcal{V}_p^{(j)}(\Sigma_g)) \subset \mathcal{V}_p^{(j+m)}(\Sigma_g) \cap \im(E^{p-k-m}) = \mathcal{V}_p^{(j+m)}(\Sigma_g)$. This implies, by construction, that $\mathcal{V}_p^{(j)}(\Sigma_g) \oplus \mathcal{V}_p^{(j+m)}(\Sigma_g)$ is indeed a submodule.

Note that for $0 < k < p$ the factors in (67) are irreducible. We also write

$$\overline{\mathcal{U}}_p^{(j)}(m, g) = \mathcal{V}_p^{(j)}(\Sigma_g) \oplus \mathcal{V}_p^{(j+m)}(\Sigma_g).$$

(68)

The case that is topologically relevant is $m = 3$. For this case Morita constructed in (16) a homomorphism $\tilde{k} : \Gamma_g \rightarrow JM_2(3, g)$ on the mapping class group $\Gamma_g$. Its kernel is the group $\mathcal{K}_g$ generated by bounding cycles and its restriction to the Torelli group coincides with the Johnson homomorphism $\tau_2 : \iota_g \rightarrow \Lambda^3 H$ from (6).

**Theorem 16 (Johnson[3], Morita [16])** There is a finite index subgroup $Q_g \subset JM_2(3, g) = \frac{1}{2} \Lambda^3 H \rtimes \text{Sp}(2g, \mathbb{Z})$, and homomorphisms $\tau_2$ and $\tilde{k}$ such that the following diagram is commutative
Combining this result with Proposition 15 now yields Theorem 4, where we denoted the special module \( \overline{U}^{(j)}(\Sigma_g) = \overline{U}^{(j)}(3, g) \). Note, that we obtain from the sequences in (28) similar resolutions. Particularly, for \( g \geq 0, p \geq 5 \) a prime and \( 0 < k < p - 3 \) there is an exact sequence of maps of \( \mathbb{F}_p \)-modules as follows.

\[
\cdots \to U_p^{(i+1)p+k_{i+1}}(\Sigma_g) \to U_p^{(ip+k_i)}(\Sigma_g) \to \cdots \to U_p^{(2p+k)}(\Sigma_g) \to \\
\to U_p^{(2p-k-3)}(\Sigma_g) \to U_p^{(k)}(\Sigma_g) \to \overline{U}_p^{(k)}(\Sigma_g) \to 0,
\]

where now \( k_i = k \) for \( i \) is even, and \( k_i = p - i - 3 \) if \( i \) is odd.

The maps are alternatingly given by \( E^k \oplus E^{k+3} \) and \( E^{p-k} \oplus E^{p-k-3} \). Note, however, that the module extensions work alternatingly in opposite ways so that the maps cannot be \( Q_g \)-equivariant. It is true that by setting \( \tilde{R}(g) = R(\tilde{J}g^{-1}J^{-1})^* \), where \( J \in \Gamma_g \) is a representative of the \( J \in \text{Sp}(2g, \mathbb{Z}) \) above we can reverse the exact sequences that define an extension. Yet, even by flipping every second extension in (69) still does not yield equivariant maps.
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The Ohio State University, Department of Mathematics, 231 West 18th Avenue, Columbus, OH 43210, U.S.A.
E-mail: kerler@math.ohio-state.edu