Minimal p-divisible groups

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Introduction. A $p$-divisible group $X$ can be seen as a tower of building blocks, each of which is isomorphic to the same finite group scheme $X[p]$. Clearly, if $X_1$ and $X_2$ are isomorphic then $X_1[p] \cong X_2[p]$; however, conversely $X_1[p] \cong X_2[p]$ does in general not imply that $X_1$ and $X_2$ are isomorphic. Can we give, over an algebraically closed field in characteristic $p$, a condition on the $p$-kernels which ensures this converse? Here are two known examples of such a condition: consider the case that $X$ is ordinary, or the case that $X$ is superspecial ($X$ is the $p$-divisible group of a product of supersingular elliptic curves); in these cases the $p$-kernel uniquely determines $X$.

These are special cases of a surprisingly complete and simple answer:

if $G$ is “minimal”, then $X_1[p] \cong G \cong X_2[p]$ implies $X_1 \cong X_2$,

see (1.2); for a definition of “minimal” see (1.1). This is “necessary and sufficient” in the sense that for any $G$ that is not minimal there exist infinitely many mutually non-isomorphic $p$-divisible groups with $p$-kernel isomorphic to $G$; see (4.1).

Remark (motivation). You might wonder why this is interesting.

EO In [7] we have defined a natural stratification of the moduli space of polarized abelian varieties in positive characteristic: moduli points are in the same stratum if and only if the corresponding $p$-kernels are geometrically isomorphic. Such strata are called EO-strata.

Fol In [8] we define in the same moduli spaces a foliation: moduli points are in the same leaf if and only if the corresponding $p$-divisible groups are geometrically isomorphic; in this way we obtain a foliation of every open Newton polygon stratum.

Fol $\subset$ EO The observation $X \cong Y \Rightarrow X[p] \cong Y[p]$ shows that any leaf in the second sense is contained in precisely one stratum in the first sense; the main result of this paper, “$X$ is minimal if and only if $X[p]$ is minimal”, shows that a stratum (in the first sense) and a leaf (in the second sense) are equal if we are in the minimal, principally polarized situation.

In this paper we consider $p$-divisible groups and finite group schemes over an algebraically closed field $k$ of characteristic $p$.

An apology. In (2.5) and in (3.5) we fix notations, used for the proof of (2.2), respectively (3.1); according to the need, the notations in these two different cases are different. We hope this difference in notations in Section 2 versus Section 3 will not cause confusion.
Group schemes considered are supposed to be commutative. We use covariant Dieudonné module theory. We write $W = W_\infty(k)$ for the ring of infinite Witt vectors with coordinates in $k$. Finite products in the category of $W$-modules are denoted “×” or by “∏”, while finite products in the category of Dieudonné modules are denoted by “⊕”; for finite products of $p$-divisible groups we use “×” or “∏”. We write $F$ and $V$, as usual, for “Frobenius” and “Verschiebung” on commutative group schemes; we write $\mathcal{F} = \mathcal{D}(V)$ and $\mathcal{V} = \mathcal{D}(F)$, see [7], 15.3, for the corresponding operations on Dieudonné modules.

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1 Notations and the main result.

(1.1) Some definitions and notations.
$H_{m,n}$. We define the $p$-divisible group $H_{m,n}$ over the prime field $F_p$, in case $m$ and $n$ are coprime non-negative integers, see [2], 5.2. This $p$-divisible group $H_{m,n}$ is of dimension $m$, its Serre-dual $X^t$ is of dimension $n$, it is isosimple, and its endomorphism ring $\text{End}(H_{m,n} \otimes \overline{\mathbb{F}_p})$ is the maximal order in the endomorphism algebra $\text{End}^0(H_{m,n} \otimes \overline{\mathbb{F}_p})$ (and these properties characterize this $p$-divisible group over $\overline{\mathbb{F}_p}$). We will use the notation $H_{m,n}$ over any base $S$ in characteristic $p$, i.e. we write $H_{m,n}$ instead of $H_{m,n} \times_{\text{Spec}(\mathbb{F}_p)} S$, if no confusion can occur.

The ring $\text{End}(H_{m,n} \otimes \mathbb{F}_p) = R'$ is commutative; write $L$ for the field of fractions of $R'$. Consider integers $x, y$ such that for the coprime positive integers $m$ and $n$ we have $x \cdot m + y \cdot n = 1$. In $L$ we define the element $\pi = \mathcal{F}^y \cdot \mathcal{V}^x \in L$. Write $h = m + n$. Note that $\pi^h = p$ in $L$. Here $R' \subset L$ is the maximal order, hence $R'$ integrally closed in $L$, and we conclude that $\pi \in R'$. This element $\pi$ will be called the uniformizer in this endomorphism ring. In fact, $W_\infty(\mathbb{F}_p) = \mathbb{Z}_p$, and $R' \cong \mathbb{Z}_p[\pi]$. In $L$ we have:

$$m + n =: h, \quad \pi^h = p, \quad \mathcal{F} = \pi^n, \quad \mathcal{V} = \pi^m.$$ 

For a further description of $\pi$, of $R = \text{End}(H_{m,n} \otimes k)$ and of $D = \text{End}^0(H_{m,n} \otimes k)$ see [2], 5.4; note that $\text{End}^0(H_{m,n} \otimes k)$ is non-commutative if $m > 0$ and $n > 0$. Note that $R$ is a “discrete valuation ring” (terminology sometimes also used for non-commutative rings).

Newton polygons. Let $\beta$ be a Newton polygon. By definition, in the notation used here, this is a lower convex polygon in $\mathbb{R}^2$ starting at $(0,0)$, ending at $(h,c)$ and having break points with integral coordinates; it is given by $h$ slopes in non-decreasing order; every slope $\lambda$ is a rational number, $0 < \lambda \leq 1$.

To each ordered pair of nonnegative integers $(m,n)$ we assign a set of $m + n = h$ slopes equal to $n/(m+n)$; this Newton polygon ends at $(h,c = n)$.

In this way a Newton polygon corresponds with a set of ordered pairs; such a set we denote symbolically by $\sum_i (m_i, n_i)$; conversely such a set determines a Newton polygon. Usually we consider only coprime pairs $(m_i, n_i)$; we write $H(\beta) := \times_i H_{m_i,n_i}$ in case $\beta = \sum_i (m_i, n_i)$. A $p$-divisible group $X$ over a field of positive characteristic defines a Newton polygon where $h$ is the height of $X$ and $c$ is the dimension of its Serre-dual $X^t$. By the Dieudonné-Manin classification, see [5], Th. 2.1 on page 32, we know: \textit{two $p$-divisible groups over an algebraically closed field of positive characteristic are isogenous if and only if their Newton polygons are equal.}
Definition. A $p$-divisible group $X$ is called minimal if there exists a Newton polygon $\beta$ and an isomorphism $X_k \cong H(\beta)_k$, where $k$ is an algebraically closed field.

Note that in every isogeny class of $p$-divisible groups over an algebraically closed field there is precisely one minimal $p$-divisible group.

Truncated $p$-divisible groups. A finite group scheme $G$ (finite and flat over some base, but in this paper we will soon work over a field) is called a $\text{BT}_1$, see [1], page 152, if $G[F] := \ker F_G = \text{Im} V_G := V(G)$ and $G[V] = F(G)$ (in particular this implies that $G$ is annihilated by $p$). Such group schemes over a perfect field appear as the $p$-kernel of a $p$-divisible group, see [1], Prop. 1.7 on page 155. The abbreviation “$\text{BT}_1$” stand for “1-truncated Barsotti-Tate group”; the terms “$p$-divisible group” and “Barsotti-Tate group” indicate the same concept.

The Dieudonné module of a $\text{BT}_1$ over a perfect field $K$ is called a $\text{DM}_1$; for $G = X[p]$ we have $\mathbb{D}(G) = \mathbb{D}(X)/p\mathbb{D}(X)$. In other terms: such a Dieudonné module $M_1 = \mathbb{D}(X[p])$ is a finite dimensional vector space over $K$, on which $F$ and $V$ operate (with the usual relations), with the property that $M_1[V] = F(M_1)$ and $M_1[F] = V(M_1)$.

Definition. Let $G$ be a $\text{BT}_1$ group scheme; we say that $G$ is minimal if there exists a Newton polygon $\beta$ such that $G_k \cong H(\beta)[p]_k$. A $\text{DM}_1$ is called minimal if it is the Dieudonné module of a minimal $\text{BT}_1$.

(1.2) Theorem. Let $X$ be a $p$-divisible group over an algebraically closed field $k$ of characteristic $p$. Let $\beta$ be a Newton polygon. Then

$$X[p] \cong H(\beta)[p] \implies X \cong H(\beta).$$

In particular: if $X_1$ and $X_2$ are $p$-divisible groups over $k$, with $X_1[p] \cong G \cong X_2[p]$, where $G$ is minimal, then $X_1 \cong X_2$.

Remark. We have no a priori condition on the Newton polygon of $X$, nor do we a priori assume that $X_1$ and $X_2$ have the same Newton polygon.

Remark. In general an isomorphism $\varphi_1 : X[p] \rightarrow H(\beta)[p]$ does not lift to an isomorphism $\varphi : X \rightarrow H(\beta)$.

(1.3) Here is another way of explaining the result of this paper. Consider the map

$$[p] : \{X \mid \text{a } p\text{-divisible group}\}/ \cong_k \rightarrow \{G \mid \text{a } \text{BT}_1\}/ \cong_k, \quad X \mapsto X[p].$$

This map is surjective, e.g. see [1], 1.7; also see [7], 9.10.

- By results of this paper we know: For every Newton polygon $\beta$ there is an isomorphism class $X := H(\beta)$ such that the fiber of the map $[p]$ containing $X$ consists of one element.

- For every $X$ not isomorphic to some $H(\beta)$ the fiber of $[p]$ containing $X$ is infinite; see (4.1)

Convention. The slope $\lambda = 0$, given by the pair $(1,0)$, defines the $p$-divisible group $G_{1,0} = \mathbb{G}_m[p^{\infty}]$, and its $p$-kernel is $\mu_p$. The slope $\lambda = 1$, given by the pair $(0,1)$, defines the $p$-divisible group $G_{0,1} = \mathbb{Q}_p/\mathbb{Z}_p$ and its $p$-kernel is $\mathbb{Z}/p\mathbb{Z}$. These $p$-divisible groups and their $p$-kernels split off naturally over a perfect field, see [6], 2.14. The theorem is obvious for these minimal $\text{BT}_1$ group schemes over an algebraically closed field. Hence it suffices to prove the theorem in case all group schemes considered are of local-local type, i.e. all slopes considered are strictly between 0 and 1; from now on we make this assumption.
We give already one explanation about notation and method of proof. Let \( m, n \in \mathbb{Z}_{>0} \) be coprime. Start with \( H_{m,n} \) over \( \mathbb{F}_p \). Let \( Q' = \mathbb{D}(H_{m,n} \otimes \mathbb{F}_p) \). In the terminology of [2], 5.6 and Section 6, a semi-module of \( H_{m,n} \) equals \([0, \infty) = \mathbb{Z}_{\geq 0} \). Choose a non-zero element in \( Q'/\pi Q' \), this is a one-dimensional vector space over \( \mathbb{F}_p \), and lift this element to \( A_0 \in Q' \).

Write \( A_i = \pi^i A_0 \) for every \( i \in \mathbb{Z}_{\geq 0} \). Note that

\[
\pi A_i = A_{i+1}, \quad FA_i = A_{i+n}, \quad VA_i = A_{i+m}.
\]

Fix an algebraically closed field \( k \); we write \( Q = \mathbb{D}(H_{m,n} \otimes k) \). Clearly \( A_i \in Q' \subset Q \), and the same relations as given above hold. Note that \( \{ A_i \mid i \in \mathbb{Z}_{\geq 0} \} \) generate \( Q \) as a \( W \)-module. The fact that a semi-module of the minimal \( p \)-divisible group \( H_{m,n} \) does not contain “gaps” is the essential (but sometimes hidden) argument in the proofs below.

The set \( \{ A_0, \ldots, A_{m+n-1} \} \) is a \( W \)-basis for \( Q \). If \( m \geq n \) we see that \( \{ A_0, \ldots, A_{n-1} \} \) is a set of generators for \( Q \) as a Dieudonné module; the structure of this Dieudonné module can be described as follows; for this set of generators we consider another numbering \( \{ C_1, \ldots, C_n \} = \{ A_0, \ldots, A_{n-1} \} \) and we define positive integers \( \gamma_i \) by: \( C_1 = A_0 \) and \( F^0 C_1 = VC_2, \ldots, F^{n-1} C_n = VC_1 \) (note that we assume \( m \geq n \)), which gives a “cyclic” set of generators for \( Q/pQ \) in the sense of [3]. These notations will be repeated and explained more in detail in (2.5) and (3.5).

## 2 A slope filtration

\[(2.1)\] We consider a Newton polygon \( \beta \) given by \( r_1(m_1, n_1), \ldots, r_t(m_t, n_t) \); here \( r_1, \ldots, r_t \in \mathbb{Z}_{\geq 0} \), and every \( (m_j, n_j) \) is an ordered pair of coprime positive integers; we write \( h_j = m_j + n_j \) and we suppose the ordering is chosen in such a way that \( \lambda_1 := n_1/h_1 < \cdots < \lambda_t := n_t/h_t \).

Write

\[
H := H(\beta) = \prod_{1 \leq j \leq t} (H_{m_j, n_j})^{r_j}; \quad G := H(\beta)[p].
\]

The following proposition uses this notation; suppose that \( t > 0 \).

\[(2.2)\] **Proposition.** Suppose \( X \) is a \( p \)-divisible group over an algebraically closed field \( k \). Suppose that \( X[p] \cong H(\beta)[p] \). Suppose that \( \lambda_1 = n_1/h_1 \leq 1/2 \). Then there exists a \( p \)-divisible subgroup \( X_1 \subset X \) and isomorphisms

\[
X_1 \cong (H_{m_1, n_1})^{r_1} \quad \text{and} \quad (X/X_1)[p] \cong \prod_{j>1} (H_{m_j, n_j}[p])^{r_j}.
\]

\[(2.3)\] **Remark.** The condition that \( X[p] \) is minimal is essential; e.g. it is easy to give an example of a \( p \)-divisible group \( X \) which is isosimple, such that \( X[p] \) is decomposable; see [9].

\[(2.4)\] **Corollary.** For \( X \) with \( X[p] \cong H(\beta)[p] \), with \( \beta \) as in (2.1), there exists a filtration by \( p \)-divisible subgroups

\[
X_0 := 0 \subset X_1 \subset \cdots \subset X_t = X \quad \text{such that} \quad X_j/X_{j-1} \cong (H_{m_j, n_j})^{r_j}, \quad \text{for} \quad 1 \leq j \leq t.
\]

**Proof of the corollary.** Assume by induction that the result has been proved for all \( p \)-divisible groups where \( Y[p] = H(\beta')[p] \) is minimal such that \( \beta' \) has at most \( t-1 \) different slopes; induction starting at \( t-1 = 0 \), i.e. \( Y = 0 \). If on the one hand the smallest slope of
$X$ is at most 1/2, the proposition gives $0 \subset X_1 \subset X$, and using the induction hypothesis on $Y = X/X_1$ we derive the desired filtration. If on the other hand all slopes of $X$ are bigger than 1/2, we apply the proposition to the Serre-dual of $X$, using the fact that the Serre-dual of $H_{m,n}$ is $H_{n,m}$; dualizing back we obtain $0 \subset X_{t-1} \subset X$, and using the induction hypothesis on $Y = X_{t-1}$ we derive the desired filtration. Hence we see that the proposition gives the induction step; this proves the corollary. \footnote{(2.2)$\Rightarrow$ (2.4)}

\textbf{(2.5)} We use notation as in (2.1) and (2.2), and we fix further notation which will be used in the proof of (2.2).

Let $M = \mathbb{D}(X)$. We write $Q_j = \mathbb{D}(H_{m_j,n_j})$. Hence

$$M/pM \cong \bigoplus_{1 \leq j \leq t} (Q_j/pQ_j)^{r_j}.$$  

Using this isomorphism we construct a map

$$v : M \to \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$ 

We use notation as in (1.1) and in (1.4). Let $\pi_j$ be the uniformizer of $\text{End}(Q_j)$. We choose $A_{i,s}^{(j)} \in Q_j$ with $i \in \mathbb{Z}_{\geq 0}$ and $1 \leq s \leq r_j$ (which generate $Q_j$) such that $\pi_j \cdot A_{i,s}^{(j)} = A_{i+1,s}^{(j)}$, $F \cdot A_{i,s}^{(j)} = A_{i+n_j,s}^{(j)}$ and $V \cdot A_{i,s}^{(j)} = A_{i+m_j,s}^{(j)}$. We have $Q_j/pQ_j = \times_{0 \leq i < h_j} k \cdot (A_{i,s}^{(j)} \pmod{pQ_j})$. We write

$$A_{i,s}^{(j)} = (A_{i,s}^{(j)} | 1 \leq s \leq r_j) \in (Q_j)^{r_j}$$

for the vector with coordinate $A_{i,s}^{(j)}$ in the summand on the $s$-th place.

For $B \in M$ we uniquely write

$$B \pmod{pM} = a = \sum_{j, 0 \leq i < h_j, \ 1 \leq s \leq r_j} b_{i,s}^{(j)} \cdot (A_{i,s}^{(j)} \pmod{pQ_j}), \ b_{i,s}^{(j)} \in k;$$

if moreover $B \notin pM$ we define

$$v(B) = \min_{j, i, s, b_{i,s}^{(j)} \neq 0} \frac{i}{h_j}.$$ 

If $B' \in p^3M$ and $B' \notin p^{3+1}M$ we define $v(B') = \beta \cdot v(p^{-\beta} \cdot B')$. We write $v(0) = \infty$. This ends the construction of $v : M \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$.

For any $\rho \in \mathbb{Q}$ we define

$$M_\rho = \{B \mid v(B) \geq \rho\};$$

note that $pM_\rho \subset M_{\rho+1}$. Let $T$ be the least common multiple of $h_1, \ldots, h_t$. Note that, in fact, $v : M - \{0\} \to \frac{T}{p}\mathbb{Z}_{\geq 0}$. Note that, by construction, $v(B) \geq d \in \mathbb{Z}$ if and only if $p^d$ divides $B$ in $M$. Hence $\bigcap_{\rho \to \infty} M_\rho = \{0\}$.

The basic assumption $X[p] \cong H(\beta)[p]$ of (1.2) is:

$$M/pM = \bigoplus_{1 \leq j \leq t, \ 1 \leq s \leq r_j} \prod_{0 \leq i < h_j} k \cdot (A_{i,s}^{(j)} \pmod{pQ_j})$$

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We write eventually if it is not already a Dieudonné module.

Define $B_{i,s}^{(j)} = p^j \cdot B_{i,s}$. By construction we have: $v(B_{i,s}^{(j)}) = i/h_j$ for all $i \geq 0$, all $j$ and all $s$. Note that $M_p$ is generated over $W = W_\infty(k)$ by all elements $B_{i,s}^{(j)}$ with $v(B_{i,s}^{(j)}) \geq \rho$. As a short-hand we will write $B_i^{(j)}$ for the vector $(B_{i,s}^{(j)} | 1 \leq s \leq r_j) \in M^{r_j}$. 

We write $P \subset M$ for the sub-$W$-module generated by all $B_{i,s}^{(j)}$ with $j \geq 2$ and $i < h_j$; we write $N \subset M$ for the sub-$W$-module generated by all $B_{i,s}^{(1)}$ with $i < h_1$. Note that $M = N \times P$, a direct sum of $W$-modules. Note that $M_p = (N \cap M_p) \times (P \cap M_p)$.

In the proof the $W$-submodule $P \subset M$ will be fixed; its $W$-complement $N \subset M$ will change eventually if it is not already a Dieudonné submodule.

We write $m_1 = m$, $n_1 = n$, $h = h_1 = m + n$, and $r = r_1$. Note that we assumed $0 < \lambda_1 \leq 1/2$, hence $m \geq n > 0$. For $i \geq 0$ we define integers $\delta_i$ by:

$$i-h \leq \delta_i < i \cdot m + (i+1) \cdot n = ih + n$$

and non-negative integers $\gamma_i$ such that

$$\delta_0 = 0, \quad \delta_1 = \gamma_1 + 1, \ldots, \delta_i = \gamma_1 + 1 + \gamma_2 + 1 + \ldots + \gamma_i + 1, \ldots;$$

note that $\delta_n = h = m + n$; hence $\gamma_1 + \ldots + \gamma_n = m$. For $1 \leq i \leq n$ we write

$$f(i) = \delta_{i-1} \cdot n - (i-1) \cdot h;$$

this means that $0 \leq f(i) < n$ is the remainder of dividing $\delta_{i-1} \cdot n$ by $h$; note that $f(1) = 0$. As $\gcd(n,h) = 1$ we see that

$$f : \{1, \ldots, n\} \rightarrow \{0, \ldots, n-1\}$$

is a bijective map. The inverse map $f'$ is given by:

$$f' : \{0, \ldots, n-1\} \rightarrow \{1, \ldots, n\}, \quad f'(x) \equiv 1 - \frac{x}{h} \pmod{n}, \quad 1 \leq f'(x) \leq n.$$ 

In $(Q_1)^r$ we have the vectors $A_i^{(1)}$. We choose $C_i^{(1)} := A_i^{(1)}$ and we choose $\{C_1^{(i)}, \ldots, C_n^{(i)}\} = \{A_0^{(i)}, \ldots, A_{n-1}^{(i)}\}$ by

$$C_i^{(i)} := A_{f(i)}^{(i)}, \quad C_{f(i)}^{(i)} = A_{x}^{(1)};$$

this means that:

$$\mathcal{F}^{\gamma} C_i^{(i)} = \mathcal{V} C_{i+1}^{(i)}, \quad 1 \leq i < n, \quad \mathcal{F}^{\infty} C_n^{(i)} = \mathcal{V} C_1^{(i)}, \quad \mathcal{F}^h C_i^{(i)} = p^j C_{i+1}^{(i)}, \quad 1 \leq i < n;$$

note that $\mathcal{F}^{h} C_i^{(i)} = p^\ell C_i^{(i)}$. With these choices we see that

$$\{\mathcal{F}^{j} C_i^{(i)} | 1 \leq i \leq n, 0 \leq j \leq \gamma_i\} = \{A_{f(i)}^{(i)} | 0 \leq \ell < h\}.$$ 

For later reference we state:
(2.6) Suppose $Q$ is a nonzero Dieudonné module with an element $C \in Q$, such that there exist coprime integers $n$ and $n + m = h$ as above such that $\mathcal{F}^h C = p^n C$ and such that $Q$ as a $W$-module is generated by $\{p^{-[n/h]}\mathcal{F}^j C \mid 0 \leq j < h\}$, then $Q \cong \mathbb{D}(H_{m,n})$. This is proved by explicitly writing out the required isomorphism. Note that $\mathcal{F}^n$ is injective on $Q$, hence $\mathcal{F}^h C = p^n C$ implies $\mathcal{F}^m C = \mathcal{V}^m C$.

(2.7) Accordingly we choose $C_{i,s} := B_{f(i),s}^{(1)} \in M$ with $1 \leq i \leq n$. Note that

$$\{\mathcal{F}^j C_{i,s} \mid 1 \leq i \leq n, \; 0 \leq j \leq \gamma_i \; 1 \leq s \leq r\} \text{ is a } W\text{-basis for } N,$$

$$\mathcal{F}^n C_{i,s} - \mathcal{V} C_{i+1,s} \in pM, \; 1 \leq i < n, \; \mathcal{F}^n C_{n,s} - \mathcal{V} C_{1,s} \in pM.$$

We write $C_i = (C_{i,s} \mid 1 \leq s \leq r)$. As a reminder, we sum up some of the notation constructed:

$$N \subset M = \bigoplus_j (Q_j)^{r_j}$$

$$M/pM = \bigoplus_j (Q_j/pQ_j)^{r_j},$$

$$B_{i,s}^{(j)} \in M, \quad A_{i,s}^{(j)} \in Q_j \subset (Q_j)^{r_j},$$

$$C_{i,s} \in N, \quad C_{i,s}^{(r)} \in Q_1 \subset (Q_1)^{r_1}.$$

(2.8) Lemma. Use the notation fixed up to now.

(1) For every $\rho \in \mathbb{Q}_{\geq 0}$ the map $p : M_\rho \to M_{\rho+1}$, multiplication by $p$, is surjective.

(2) For every $\rho \in \mathbb{Q}_{\geq 0}$ we have $\mathcal{F} M_\rho \subset M_{\rho+(n/h)}$.

(3) For every $i$ and $s$ we have $\mathcal{F} B_{i,s}^{(1)} \in M_{(i+n)/h}$; for every $i$ and $s$ and every $j > 1$ we have $\mathcal{F} B_{i,s}^{(j)} \in M_{(i/h)+h+(n/h)+(1/T)}$.

(4) For every $1 \leq i \leq n$ we have $\mathcal{F}^h C_i - p^i B_{f(i+1)}^{(1)} \in (M_{(u+1)/T})^{r_i'}$; moreover $\mathcal{F}^n C_1 - p^n C_1 \in (M_{(u+n+1)/T})^{r}$.

(5) If $u$ is an integer with $u > Tn$, and $\xi_N \in (N \cap M_{u/T})^{r}$, there exists

$$\eta_N \in N \cap (M_{(u/T) - n})^{r} \text{ such that } (\mathcal{F}^n - p^n)\eta_N \equiv \xi_N \pmod{(M_{(u+1)/T})^{r}}.$$

Proof. We know that $M_{\rho+1}$ is generated by the elements $B_{i,s}^{(j)}$ with $i/h_j \geq \rho + 1$; because $\rho \geq 0$ such elements satisfy $i \geq h_j$. Note that $p B_{i,s}^{(j)} = B_{i,s}^{(j)}$. This proves the first property. \hfill \Box (1)

At first we show $\mathcal{F} M \subset M_{n/h}$. Note that for all $1 \leq j \leq t$ and all $\beta \in \mathbb{Z}_{\geq 0}$

$$\beta h_j \leq i < \beta h_j + m_j \Rightarrow \mathcal{F} B_{i}^{(j)} = B_{i+n_j}^{(j)} \quad \text{(*)}$$

and

$$\beta h_j + m_j \leq i < (\beta + 1)h_j \Rightarrow B_{i}^{(j)} = \mathcal{V} B_{i-m_j}^{(j)} + p^{(\beta+1)}\xi, \; \xi \in M^{r_j} \quad \text{(**)}$$

from these properties, using $n/h \leq n_j/h_j$ we conclude: $\mathcal{F} M \subset M_{n/h}$.

Further we see: by (*) we have

$$v(\mathcal{F} B_{i,s}^{(j)}) = v(B_{i+n_j,s}^{(j)}) = (i + n_j)/h_j,$$

and
The factor space $\mathcal{B}_{i,s}$ exists if $j = 1$; $i + n_j / h_j > i + n / h$ if $j > 1$.

By (***) it suffices to consider only $m_j \leq i < h_j$, and hence $\mathcal{F}B^{(j)}_{i,s} = pB^{(j)}_{i-m_j,s} + p\mathcal{F}\xi$; so we have

$$v(\mathcal{F}B^{(j)}_{i,s}) \geq \min\left( v(pB^{(j)}_{i-m_j,s}), v(p\mathcal{F}\xi) \right);$$

for $j = 1$ we have $v(pB^{(1)}_{i-m_1,s}) = (i + n)/h \geq 1$ and $v(p\mathcal{F}\xi) \geq 1 + (n/h) > (i/h) + (n/h)$; for $j > 1$ we have $v(pB^{(j)}_{i-m_j,s}) > (i/h_j) + (n/h)$ and $(i/h_j) + (n/h) < 1 + (n/h) \leq v(p\mathcal{F}\xi)$; hence $v(\mathcal{F}B^{(j)}_{i,s}) > (i/h_j) + (n/h)$ if $j > 1$. This ends the proof of (3). Using (3) we see that (2) follows. □(2)+(3)

From $\mathcal{F}^n C_i = \mathcal{V}C_{i+1} + p\xi_i$ for $i < n$ and $\mathcal{F}^n C_n = \mathcal{V}C_1 + p\xi_n$, here $\xi_i \in M_\ell^r$ for $i \leq n$, we conclude:

$$\mathcal{F}\delta C_i = p^{i+1}C_{i+1} + \sum_{1 \leq \ell \leq i} p^\ell \mathcal{F}^{\ell-i}\delta \mathcal{F}\xi, \quad i < n,$$

and the analogous formula for $i = n$ (write $C_{n+1} = C_1$). Note that

$$ih \leq \delta_i n \quad \text{and} \quad \delta_n n < \ell m + (\ell + 1)n = \ell h + n;$$

this shows that

$$\ell h + (\delta_i - \delta) n + n > ih;$$

using (2) we conclude (4). □(4)

Note that $h = h_1$ divides $T$. If $\ell$ is an integer such that $(\ell - 1)/h < u/T < \ell/h$ then $u < u + 1 \leq \ell T/h$; in this case we see that $N \cap M_{u/T} = N \cap M_{(u+1)/T}$. In this case we choose $\eta_N = 0$.

Suppose that $\ell$ is an integer with $u/T = \ell/h$. Then $N \cap M_{u/T} = N_{\ell/h} \supset N_{(\ell+1)/h} = N \cap M_{(u+1)/T}$. We consider the image of $N \cap M_{(\ell/h)-n}$ under $\mathcal{F}^h - p^n$. We see, using previous results, that this image is in $N_{\ell/h} + M_{(u+1)/T}$ (here “+” stands for the span as $W$-modules). We obtain a factorization and an isomorphism

$$\mathcal{F}^h - p^n : N \cap M_{(\ell/h)-n} \longrightarrow (N_{\ell/h} + M_{(u+1)/T})/M_{(u+1)/T} \cong N_{\ell/h}/N_{(\ell+1)/h}.$$

We claim that this map is surjective. The factor space $N_{\ell/h}/N_{(\ell+1)/h}$ is a vector space over $k$ spanned by the residue classes of the elements $B^{(1)}_{\ell,s}$. For the residue class of $y_s B^{(1)}_{\ell,s}$ we solve the equation $x_s p^n = x_s = y_s$ in $k$; lifting these $x_s$ to $W$ (denoting the lifts by the same symbol), we see that $\eta_N := \sum s x_s B^{(1)}_{\ell-nh,s}$ has the required properties. This proves the claim, and it gives a proof of part (5) of the lemma. □(5),(2.8)

(2.9) **Lemma** (the induction step). Let $u \in \mathbb{Z}$ with $u \geq nT + 1$. Suppose $D_1 \subseteq M_\ell^r$ such that $D_1 \equiv C_1 \pmod{(M_1/T)^r}$, and such that $\xi := \mathcal{F}^h D_1 - p^n D_1 \in (M_{u/T})^r$. Then there exists $\delta \in (M_{(u/T)-n})^r$ such that for $E_1 := D_1 - \delta$ we have $\mathcal{F}^h E_1 - p^n E_1 \in (M_{(u+1)/T})^r$ and $E_1 \equiv C_1 \pmod{(M_{1/T})^r}$.

**Proof.** We write $\xi = \xi_N + \xi_P$ according to $M = N \times P$. We conclude that $\xi_N \in (N \cap M_{u/T})^r$ and $\xi_P \in (P \cap M_{u/T})^r$. Using (2.8), (5), we construct $\eta_N \in (N \cap M_{1/T})^r$ such that

(8)
\[(\mathcal{F}^h - p^r)\eta_N \equiv \xi_N \pmod{(M_{(u+1)/T})^r}.\] As \(M_{u/T} \subset M_n\) we can choose \(\eta_P := -p^{-n}\xi_P;\) we have \(\eta_P \in M'_{(u/T)-n} \subset (M_{1/T})^r.\) With \(\eta := \eta_N + \eta_P\) we see that
\[(\mathcal{F}^h - p^r)\eta \equiv \xi \pmod{(M_{(u+1)/T})^r} \text{ and } \eta \in (M_{1/T})^r.\]

Hence \((\mathcal{F}^h - p^r)(D_1 - \eta) \in (M_{(u+1)/T})^r\) and we see that \(E_1 := D_1 - \eta\) has the required properties. This proves the lemma.

\(\square(2.9)\)

(2.10) \textbf{Proof of (2.2)}. (1) There exists \(E_1 \in M^r\) such that \((\mathcal{F}^h - p^r)E_1 = 0\) and \(E_1 \equiv C_1 \pmod{(M_{1/T})^r}\).

\textbf{Proof}. For \(u \in \mathbb{Z}_{\geq nT+1}\) we write \(D_1(u) \in M^r\) for a vector such that
\[D_1(u) \equiv C_1 \pmod{(M_{1/T})^r} \text{ and } \mathcal{F}^hD_1(u) - p^nD_1(u) \in (M_{u/T})^r.\]

By (2.8), (4), the vector \(C_1 := D_1(nT + 1)\) satisfies this condition for \(u = nT + 1\). Here we start induction. By repeated application of (2.9) we conclude there exists a sequence
\[(\{D_1(u) \mid u \in \mathbb{Z}_{\geq nT+1}\}) \text{ such that } D_1(u) - D_1(u + 1) \in (M_{(u/T)-n})^r\]
satisfying the conditions above. As \(\cap_{\rho \to \infty} M_{\rho} = \{0\}\) this sequence converges. Writing \(E_1 := D_1(\infty)\) we achieve the conclusion.

\(\square(1)\)

(2) Choose \(E_1\) as in (1). For every \(j \geq 0\) we have
\[p^{-\left[\frac{jn}{h}\right]}\mathcal{F}^jE_1 \in M;\] define \(N' := \prod_{1 \leq j < h} W^{-p^{-\left[\frac{jn}{h}\right]}\mathcal{F}^jE_1} \subset M.\)

This is a Dieudonné submodule and it is a \(W\)-module direct summand of \(M\). Moreover there is an isomorphism\n\[\mathbb{D}((H_{m,n})^r) \cong N',\]
the map \(N'\prod P \to N' + P\) is an isomorphism of \(W\)-modules, and \(N' + P = M.\) This constructs \(X_1 \subset X,\) with
\[\mathbb{D}(X_1 \subset X) = (N' \subset M) \text{ such that } (X/X_1)[p] \equiv \prod_{j > 1} (M_{m_j,n_j})^r.\]

\textbf{Proof}. By (2.8), (2), we see that \(\mathcal{F}^jE_1 \in M_{[jn/k]},\) hence the first statement follows. As \(\mathcal{F}^hE_1 = p^nE_1\) it follows that \(N' \subset M\) is a Dieudonné submodule; using (2.6) this shows \(\mathbb{D}((H_{m,n})^r) \cong N'.\)

\textbf{Claim}. The images \(N' \to N' \otimes k = N'/pN' \subset M/pM\) and \(P \to P/pP \subset M/pM\) inside \(M/pM\) have zero intersection and \(N' \otimes k + P \otimes k = M/pM.\) Here we write \(- \otimes k = - \otimes_W(W/pW).\)

For \(y \in \mathbb{Z}_{\geq 0}\) we write \(g(y) := yn - h[\frac{yn}{n}];\) note that, in the notation in (2.5), we have
\[p^{-\left[\frac{jn}{h}\right]}\mathcal{F}^jC_1^r = A_g^{(1)};\]

Suppose
\[
\tau := \sum_{0 \leq j < h} \beta_{j,a}p^{-\left[\frac{jn}{h}\right]}\mathcal{F}^j(E_1 - a \mod pM) \in (N' \otimes k \cap P \otimes k) \subset M/pM, \text{ } \beta_j \in k.
\]
such that $\tau \neq 0$. Let $x, s$ be a pair of indices such that $\beta := \beta_{x,s} \neq 0$ and for every $y$ with $g(y) < g(x)$ we have $\beta_{y,s} = 0$. Project inside $M/pM$ on the factor $N_s$. Then

$$\tau_s \equiv \beta \cdot B_{g(x),s}^{(1)} \mod \left(\frac{M_{g(x)}}{k} + \frac{1}{k} + P\right),$$

which is a contradiction with the fact that $N \cap P = 0$ and with the fact that the residue class of $B_{g(x),s}^{(1)}$ generates $\left(\frac{M_{g(x)}}{k} + \frac{1}{k} + P\right)/N_{g(x),s}$. We see that $\tau \neq 0$ leads to a contradiction. This shows that $N' \otimes k \cap P \otimes k = 0$ and $N' \otimes k + P \otimes k = M/pM$. Hence the claim is proved.

As $(N' \cap P) \otimes k \subset N' \otimes k \cap P \otimes k = 0$ this shows $(N' \cap P) \otimes k = 0$. By Nakayama’s lemma this implies $N' \cap P = 0$. The proof of the remaining statements follows, in particular we see that $N'$ is a $W$-module direct summand of $M$. This finishes the proof of (2), and it ends the proof of the proposition. $\square$

3 Split extensions and proof of the theorem

In this section we prove a proposition on split extensions. We will see that Theorem (1.2) follows.

(3.1) Proposition. Let $(m, n)$ and $(d, e)$ be ordered pairs of pairwise coprime positive integers. Suppose that $n/(m + n) < e/(d + e)$. Let

$$0 \to Z := H_{m, n} \to T \to Y := H_{d, e} \to 0$$

be an exact sequence of $p$-divisible groups such that the induced sequence of the $p$-kernels splits:

$$0 \to Z[p] \to T[p] \to Y[p] \to 0.$$

Then the sequence of $p$-divisible groups splits: $T \cong Z \oplus Y$.

(3.2) Remark. It is easy to give examples of a non-split extension $T/Z \cong Y$ of $p$-divisible groups, with $Z$ non-minimal or $Y$ non-minimal, such that the extension $T[p]/Z[p] \cong Y[p]$ does split.

(3.3) Proof of (1.2). The theorem follows from (2.4) and (3.1). $\square$

(3.4) In order to show (3.1) it suffices to prove (3.1) under the extra condition that $\frac{1}{2} \leq e/(d + e)$.

In fact, if $n/(m + n) < e/(d + e) < \frac{1}{2}$, we consider the exact sequence

$$0 \to H^d_{d, e} = H_{d, d} \to T' \to H^d_{m, n} = H_{n, m} \to 0$$

with $\frac{1}{2} < d/(e + d) < m/(n + m)$. $\square$

From now on we assume that $\frac{1}{2} \leq e/(d + e)$. 10
(3.5) We fix notation which will be used in the proof of (3.1). We write the Dieudonné modules as: $\mathbb{D}(Z) = N$, $\mathbb{D}(T) = M$ and $\mathbb{D}(Y) = Q$; we obtain an exact sequence of Dieudonné modules $M/N = Q$, which is a split exact sequence of $W$-modules, where $W = W_\infty(k)$. We write $m + n = h$ and $d + e = g$. We know that $Q$ is generated by elements $A_i$, with $i \in \mathbb{Z}_{\geq 0}$ such that $\pi(A_i) = A_{i+1}$, where $\pi \in \text{End}(Q)$ is the uniformizer, and $\mathcal{V} \cdot A_i = A_{i+d}$, $\mathcal{F} \cdot A_i = A_{i+e}$; we know that $\{A_i \mid 0 \leq i < g = d + e\}$ is a $W$-basis for $Q$. Because $\frac{1}{2} \leq e/(d + e)$, hence $e \geq d$ we can choose generators for the Dieudonné module $Q$ in the following way. We choose integers $\delta_i$ by:

$$i \cdot g \leq \delta_i \cdot d < (i + 1) \cdot d + i \cdot e = ig + d$$

and integers $\gamma_i$ such that:

$$\delta_1 = \gamma_1 + 1, \ldots, \delta_i = \gamma_1 + 1 + \gamma_2 + 1 + \cdots + \gamma_i + 1;$$

note that $\delta_d = g = d + e$. We choose $C = C_0 = C_1$ and $\{C_1, \cdots, C_d\} = \{A_0, \cdots, A_{d-1}\}$ such that:

$$\mathcal{V}^\gamma C_i = \mathcal{F} C_{i+1}, \quad 1 \leq i < d, \quad \mathcal{V}^\delta C_d = \mathcal{F} C_1, \quad \text{hence} \quad \mathcal{V}^\delta C = p^i C_{i+1}, \quad 1 \leq i < d;$$

note that $\mathcal{V}^g C = p^d C$. With these choices we see that

$$\{ p^{-(\frac{ig}{d})} \mathcal{V}^j C \mid 0 \leq j < g \} = \{ \mathcal{V}^j C_i \mid 1 \leq i \leq d, \quad 0 \leq j \leq \gamma_i \} = \{ A_\ell \mid 0 \leq \ell < g \}.$$

Choose an element $B = B_1 \in M$ such that

$$M \rightarrow Q \quad \text{gives} \quad B_1 = B \mapsto (B \mod N) = C = C_1.$$ 

Let $\pi'$ be the uniformizer of $\text{End}(N)$. Consider the filtration $N = N^{(0)} \supset \cdots \supset N^{(i)} \supset N^{(i+1)} \supset \cdots$ defined by $(\pi')^j(N^{(0)}) = N^{(i)}$. Note that $\mathcal{F} N^{(i)} = N^{(i+n)}$, and $\mathcal{V} N^{(i)} = N^{(i+m)}$, and $p^i N = N^{(i+h)}$ for $i \geq 0$.

(3.6) Proof of (3.1).

(1) Construction of $\{B_1, \cdots, B_d\}$. For every choice of $B = B_1 \in M$ with $(B \mod N) = C$, and every $1 \leq i < d$ we claim that $\mathcal{V}^\delta B$ is divisible by $p^i$. Defining $B_{i+1} := p^{-i} \mathcal{V}^\delta B$, we see that $B_i \mod N = C_i$ for $1 \leq i \leq d$. Moreover we claim:

$$\mathcal{V}^g B - p^i B \in N^{(dh+1)}.$$ 

Choose $B''_i \in M$ with $B''_i \mod N = C_i$. Then $\mathcal{V}^\gamma B''_i - \mathcal{F} B''_{i+1} = : p^r \xi_i \in pN$; hence $\mathcal{V}^\gamma B''_i - p B''_{i+1} = p^r \xi_i \in p\mathcal{V}N$. For $1 < i \leq d$ we obtain that

$$\mathcal{V}^\delta B - p^i B = \sum_{1 \leq j < i} \mathcal{V}^i - \delta_j p^i \mathcal{V} \xi_j, \quad \xi_j \in N.$$ 

From $n/(m + n) < e/(d + e)$ we conclude $g/d > h/m$; using $\delta_i \cdot d \geq ig$ and $\delta_i d < (j + 1) d + je$ we see:

$$i > j \quad \text{implies} \quad \delta_i - \delta_j + 1 > (i - j) (g/d) > (i - j) (h/m);$$

hence

$$(\delta_i - \delta_j) m + j (m + n) + m > ih;$$

This shows

$$\mathcal{V}^{\delta_i - \delta_j} p^i \mathcal{V} \xi_j \in p^{i} N^{(i)}.$$ 

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As \( \delta_d = g \) we see that \( V^gB - p^dB \in p^dN^{(1)} = N^{(dh+1)} \).

(2) The induction step. Suppose that for a choice \( B \in M \) with \((B \mod N) = C\), there exists an integer \( s \geq dh + 1 \) such that \( V^gB - p^dB \in N^{(s)} \); then there exists a choice \( B' \in M \) such that \( B' - B \in N^{(s-dh)} \) and

\[
V^gB' - p^dB' \in N^{(s+1)}.
\]

In fact, write \( p^dB - V^gB = p^d\xi \). Then \( \xi \in N^{(s-dh)} \). Choose \( B' := B - \xi \). Then:

\[
V^gB' - p^dB' = V^gB - p^dB - V^g\xi + p^d\xi = -V^g\xi \in N^{(gm-dh+s)};
\]

note that \( gm - dh > 0 \).

(3) For any integer \( r \geq d + 1 \), and \( w \geq rh \) there exists \( B = B_1 \) as in (3.5) such that \( V^gB - p^dB \in N^{(w)} = p^rN^{(w-rh)} \). This gives a homomorphism \( \varphi_{r-d} \)

\[
M/p^{r-d}M \leftarrow Q/p^{r-d}Q \quad \text{extending} \quad M/pM \leftarrow Q/pQ.
\]

The induction step (2) proves the first statement, induction starting at \( w = (d+1)h > dh + 1 \). Having chosen \( B_1 \), using (1) we construct \( B_{i+1} := p^{-1}V^gB_1 \) for \( 1 \leq i < d \). In that case on the one hand \( V^gB_d - F B_1 = p^{r-1}\xi_d \); on the other hand \( V^gB - p^dB \in N^{(w)} \subset p^rN \). Hence \( p^rV^g\xi_d \in p^rN \); hence \( p^r\xi_d \in p^rN \). This shows that the residue classes of \( B_1, \ldots, B_d \) in \( M/p^{−d}M \) generate a Dieudonné module isomorphic to \( Q/p^{r-d}Q \) which moreover by (3.5) extends the given isomorphism induced by the splitting.

By [8], 1.6 we see that for some large \( r \) the existence of \( M/p^{r-d}M \leftarrow Q/p^{r-d}Q \) as in (3) shows that its restriction \( M/pM \leftarrow Q/pQ \) lifts to a homomorphism \( \varphi \) of Dieudonné modules \( M \leftarrow Q \); in that case \( \varphi_1 \) is injective. Hence \( \varphi \) splits the extension \( M/N \cong Q \). Taking into account (3.4) this proves the proposition.

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Remark. Instead of the last step of the proof above, we could construct an infinite sequence \( \{B(u) \mid u \in \mathbb{Z}_{(d+1)}h\} \) such that \( V^gB(u) - p^dB \in N^{(u)} \) and \( B(u + 1) - B(u) \in N^{(u-dh)} \) for all \( u \geq (d+1)h \). This sequence converges and its limit \( B(\infty) \) can be used to define the required section.

4 Some comments

(4.1) Remark. For any \( G \), a BT\(_1\) over \( k \), which is not minimal there exist infinitely many mutually non-isomorphic \( p \)-divisible groups \( X \) over \( k \) such that \( X[p] \cong G \). Details will appear in a later publication, see [9].

(4.2) Remark. Suppose that \( G \) is a minimal BT\(_1\); we can recover the Newton polygon \( \beta \) with the property \( H(\beta)[p] \cong G \) from \( G \). This follows from the theorem, but there are also other ways to prove this fact.

(4.3) For BT\(_1\) group schemes we can define a Newton polygon; let \( G \) be a BT\(_1\) group scheme over \( k \), and let \( G = \times_i G_i \) be a decomposition into indecomposable ones, see [3]. Let \( G_i \) be of rank \( p^{h_i} \), and let \( n_i \) be the dimension of the tangent space of \( G_i^{\text{D}} \); here \( G_i^{\text{D}} \) stands for the Cartier dual of \( G_i \); define \( N_i(G_i) \) as the isoclinic polygon consisting of \( h_i \) slopes equal to \( n_i/h_i \); arranging the slopes in non-decreasing order, we have defined \( N_i(G) \). For a \( p \)-divisible
group $X$ we compare $\mathcal{N}(X)$ and $\mathcal{N}'(X[p])$. These polygons have the same endpoints. If $X$ is minimal, equivalently $X[p]$ is minimal, then $\mathcal{N}(X) = \mathcal{N}'(X[p])$. Besides this I do not see rules describing the relation between $\mathcal{N}(X)$ and $\mathcal{N}'(X[p])$. For Newton polygons $\beta$ and $\gamma$ with the same endpoints we write $\beta \prec \gamma$ if every point of $\beta$ is on or below $\gamma$. Note:

- There exists a $p$-divisible group $X$ such that $\mathcal{N}(X) \preceq \mathcal{N}'(X[p])$; indeed, choose $X$ isosimple, hence $\mathcal{N}(X)$ isoclinic, such that $X[p]$ is decomposable.
- There exists a $p$-divisible group $X$ such that $\mathcal{N}(X) \preceq \mathcal{N}'(X[p])$; indeed, choose $X$ such that $\mathcal{N}(X)$ is not isoclinic, hence $X$ not isosimple, all slopes strictly between 0 and 1 and $a(X) = 1$; then $X[p]$ is indecomposable, hence $\mathcal{N}'(X[p])$ is isoclinic.

Here we use $a(X) := \dim_k \text{Hom}(\alpha_p, X)$. It could be useful to have better insight in the relation between various properties of $X$ and $X[p]$.

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