Average transmission probability of a random stack

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Abstract

The transmission through a stack of identical slabs that are separated by gaps with random widths is usually treated by calculating the average of the logarithm of the transmission probability. We show how to calculate the average of the transmission probability itself with the aid of a recurrence relation and derive analytical upper and lower bounds. The upper bound, when used as an approximation for the transmission probability, is unreasonably good and we conjecture that it is asymptotically exact.

1. Introduction

We revisit a classical problem: the transmission through a linear array of many identical slabs (glass plates, plastic transparencies, or the like) with random separation, as depicted in figure 1. The transmission probability that Stokes derived in 1862 \cite{stokes1862} on the basis of ray-optical arguments (thereby improving on an earlier attempt by Fresnel in 1821; see \cite{fresnel1821} and \cite{berry1997} for the history of the subject) is not correct because there are crucial interference effects that require a proper wave-optical treatment. Just that was given by Berry and Klein in 1997 \cite{berry1997} who found that the average of the logarithm of the transmission probability through $N$ slabs is equal to $N$ times the logarithm of the single-slab transmission probability,

$$\langle \log \tau_N \rangle = N \log \tau_1.$$ \hfill (1)

Here, $\tau_1$ is the probability of transmission through a single slab and $\tau_N$ denotes the transmission probability for $N$ slabs. Its implicit dependence on the random phases that originate in the random spacing of the slabs is averaged over, indicated by the $\langle \cdots \rangle$ notation. As emphasized in \cite{berry1997}, the disorder is crucial; without it, most wavelength components would be transmitted, and
the stack should then appear rather transparent, but this is not the case as a simple experiment with a stack of transparencies demonstrates [3, 4].

It is indeed common to average logarithms because they are known to be ‘self-averaging’ [5], and the exact result (1) is truly remarkable. But one should realize what it tells us about the average transmission probability \( \langle \tau_N \rangle \) itself. As a consequence of the inequality

\[
\langle \log \tau_N \rangle \leq \log \langle \tau_N \rangle,
\]

the Berry–Klein relation (1) amounts to a lower bound on the average transmission probability,

\[
\langle \tau_N \rangle \geq \tau_1^N. \tag{3}
\]

As we shall see below, this bound is not particularly tight because there is a very large range of individual \( \tau_N \) values. We also note that the ray-optics result [3]

\[
\langle \tau_N \rangle_{\text{ray}} = \frac{\tau_1}{\tau_1 + N(1 - \tau_1)} \tag{4}
\]

is consistent with (3).

It is the objective of the present contribution to report good wave-optics estimates for \( \langle \tau_N \rangle \) and closely related quantities. In particular, we will improve on the lower bound of (3) and supplement it with an upper bound. We observe that the upper bound, when used as an approximation for \( \langle \tau_N \rangle \), is unreasonably good and seems to give us the exact asymptotic values of quantities such as \( \langle \tau_N+1 \rangle / \langle \tau_N \rangle \) or \( \langle \tau_N \rangle^{1/N} \). At present, this coincidence of the upper bound with exact asymptotic values is a poorly understood mystery.

Our approach will be valuable for all who lecture about transport through random media, for which the situation of figure 1 is a standard example in the classroom, treated invariably by a variant of the method described in [3]—our treatment goes much beyond that. The presentation is such that an undergraduate with some basic knowledge about one-dimensional scattering can follow the argument and master the technical details.

2. Single slab: the transfer matrix

For a wave of wavelength \( 2\pi / k \), the wavefunctions to the left and to the right of the \( n \)th slab are

\[
\begin{align*}
\psi_n^{(\text{left})}(x) &= u_{n-1} e^{ik(x-x_n)} + v_{n-1} e^{-ik(x-x_n)}, \\
\psi_n^{(\text{right})}(x) &= u_n e^{ik(x-x_n-\ell)} + v_n e^{-ik(x-x_n-\ell)},
\end{align*}
\]

(5)
where $x_n$ is the position of the left edge and $\ell$ is the thickness of the slab; see figure 2. The incoming amplitudes are related to the outgoing amplitudes by the unitary scattering matrix $S$:

$$\begin{pmatrix} u_n \\ v_{n-1} \end{pmatrix} = S \begin{pmatrix} u_{n-1} \\ v_n \end{pmatrix} = \begin{pmatrix} a & b' \\ b & a' \end{pmatrix} \begin{pmatrix} u_{n-1} \\ v_n \end{pmatrix}, \tag{6}$$

where the entries of $S$ are restricted by

$$|a|^2 = |a'|^2 = \tau_1, \quad |b|^2 = |b'|^2 = 1 - \tau_1, \quad a^* b' + b^* a' = 0, \tag{7}$$

which account for the single-slab transmission probability $\tau_1$ and the unitary nature of $S$. The particular values of the complex phases of $a, b, a'$ and $b'$ are of secondary interest, but we note that we have $a = a' = e^{ik\ell}$ and $b = b' = 0$ for a completely transparent, non-scattering slab, for which $\psi_{\mu}^{\text{left}}(x) = \psi_{\mu}^{\text{right}}(x)$.

The transfer matrix $T$ is used to express the amplitudes on the right in terms of the amplitudes on the left,

$$\begin{pmatrix} u_n \\ v_{n-1} \end{pmatrix} = T \begin{pmatrix} u_{n-1} \\ v_n \end{pmatrix}. \tag{8}$$

The one-to-one relation between $S$ and $T$ implies that the transfer matrix is of the form

$$T = e^{i\alpha} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} e^{i\beta'} & 0 \\ 0 & e^{-i\beta'} \end{pmatrix}, \tag{9}$$

where

$$\cosh \theta = \frac{1}{\sqrt{\tau_1}}, \quad \tau_1 = \frac{2}{\cosh(2\theta) + 1}, \tag{10}$$

and $\alpha, \beta, \beta'$ are phase factors that have fixed values which, however, are largely irrelevant for what follows.

The transfer matrix for the gap of length $L_n$ between the $n$th slab and the $(n+1)$th slab is the diagonal phase matrix,

$$D(kL_n) = \begin{pmatrix} e^{ikL_n} & 0 \\ 0 & e^{-ikL_n} \end{pmatrix}. \tag{11}$$
Phase matrices of the same structure sandwich the central $\theta$-dependent matrix in (9), so that we have
\[ T = e^{i\alpha D(\beta)} t(\theta) D(\beta') \quad \text{with} \quad t(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \] (12)
as a more useful way of writing $T$.

The product of two transfer matrices is another transfer matrix, whereby the relevant observation is the composition law
\[ t(\theta_1) D(\phi) t(\theta_2) = D(\gamma) t(\theta_{1\&2}) D(\gamma') \] (13)
with $\theta_{1\&2}$ determined by
\[ \cosh(2\theta_{1\&2}) = \cosh(2\theta_1) \cosh(2\theta_2) + \cos(2\phi) \sinh(2\theta_1) \sinh(2\theta_2) \] (14)
and the phases $\gamma$ and $\gamma'$ by
\[ e^{2i\gamma} \sinh(2\theta_{1\&2}) = \sinh(2\theta_1) \cosh(2\theta_2) \]
\[ + \cos(2\phi) \cosh(2\theta_1) \sinh(2\theta_2) \]
\[ + i \sin(2\phi) \sinh(2\theta_1), \]
\[ e^{2i\gamma'} \sinh(2\theta_{1\&2}) = \cosh(2\theta_1) \sinh(2\theta_2) \]
\[ + \cos(2\phi) \sinh(2\theta_1) \cosh(2\theta_2) \]
\[ + i \sin(2\phi) \sinh(2\theta_1). \]

Whereas (15) is of no consequence for the following considerations, the composition rule (14) is of central importance.

3. Many slabs: a recurrence relation

We now turn to the situation of figure 1, where we have $N$ identical slabs separated by gaps $L_1, L_2, \ldots, L_{N-1}$ that are not controlled on the scale set by the wavelength $2\pi/k$. Therefore, we regard the phase factors $e^{i k L_n}$ as random with a uniform distribution on the unit circle in the complex plane.

The overall transfer matrix
\[ T_{\text{tot}} = T D(kL_1) T D(kL_2) T \cdots T D(kL_{N-1}) T \]
\[ = e^{iN\alpha} D(\beta) \left( \prod_{n=1}^{N-1} t(\theta) D(\phi_n) \right) t(\theta) D(\beta') \]
\[ = e^{i\theta_{\text{tot}}} D(\beta_{\text{tot}}) t(\theta_{\text{tot}}) D(\beta'_{\text{tot}}) \] (16)
is characterized by $\theta_{\text{tot}}$, which is obtained by repeated application of the composition rule (14), whereby the phases $\phi_n = \beta + \beta' + kL_n$ have random values. Each experimental realization of the $N$-slab stack of figure 1 has different values for these random phases, and the transmission probability
\[ \tau_N = \frac{2}{\cosh(2\theta_{\text{tot}}) + 1} \] (17)
varies from one experiment to the next. We need to average over the $N-1$ random phases to find $\langle \tau_N \rangle$.

Let us consider a somewhat more general question: What is the average value $\langle f(\cosh(2\theta_{\text{tot}})) \rangle$ of a function of $\cosh(2\theta_{\text{tot}})$, and thus of a function of $\tau_N$? When the averaging is carried out successively, first averaging over $\phi_1$, then over $\phi_2$, and so forth, finally over $\phi_{N-1}$,
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we have an intermediate value $f_n(C')$ after averaging over the first $n-1$ random phases; see figure 3. Here $C'$ denotes the value of $\cosh(2\theta_n)$ for the stack of slabs $n$ to $N$ with its dependence on the remaining phases $\phi_n, \ldots, \phi_{N-1}$. We then have $f_1(C') = f(C')$ for the value prior to any averaging, and (14) tells us that we get $f_{n+1}(C')$ from $f_n(C')$ by means of

$$f_{n+1}(C') = \int \frac{d\phi}{2\pi} f_n(C'C' + SS' \cos \phi)$$

with $C = \cosh(2\theta) = \frac{2}{\tau_1} - 1$, $S = \sinh(2\theta) = \frac{2}{\tau_1} \sqrt{1 - \tau_1}$

and $S' = \sqrt{C'^2 - 1}$ with $C' \geq 1$, (18)

and the integration covers any convenient interval of $2\pi$. Eventually this takes us to

$$\langle \cosh(2\theta_{tot}) \rangle = \cosh(2\theta)$$

when the recursive averaging over the $N-1$ random phases is completed.

For illustration, we take $f_1(C') = C'$ as a first example. The recurrence relation (18) yields $f_n(C') = C^{n-1}C'$, so that

$$\langle \cosh(2\theta_{tot}) \rangle = \cosh(2\theta)^N$$

or

$$\langle 1/\tau_N \rangle = \frac{1}{2} + \frac{1}{2} (2/\tau_1 - 1)^N$$

when stated in terms of transmission probabilities. A second illustrating example is $f_1(C') = C'^2 - \frac{1}{2}$, for which

$$f_n(C') = \left[ \frac{1}{2} f_1(C) \right]^{n-1} f_1(C'),$$

$$f_N(C) = \left[ \frac{2}{3} f_1(C) \right]^N$$

(22)
and
\[
\langle \cosh(2\theta_{\text{tot}})^2 \rangle = \frac{1}{3} + \frac{2}{3} \left[ \frac{1}{2} (3 \cosh(2\theta)^2 - 1) \right]^N
\]
(23)
follows.

Taken together, (20) and (23) tell us that the normalized variance of \(\cosh(2\theta_{\text{tot}})\) grows exponentially with the number of slabs,
\[
\frac{\langle \cosh(2\theta_{\text{tot}})^2 \rangle}{\langle \cosh(2\theta_{\text{tot}}) \rangle^2} - 1 = \frac{1}{3} \cosh(2\theta)^{-2N} + \frac{2}{3} \left[ 1 + \frac{1}{2} \tanh(2\theta)^2 \right]^N - 1 \\
\approx \frac{2}{3} \left[ \frac{3}{2} - \frac{1}{2} \left( \tau_1 - \tau_{\text{ray}} \right)^2 \right]^N \quad \text{for} \quad N \gg 1.
\]
(24)

The values of \(\tau_N\) cover a correspondingly large range, and so we understand why the two sides of (2) differ much.

This brings us to the much more important \(\log \tau\) case of
\[
f(C') = \log \frac{2}{C' + 1}.
\]
(25)

Here,
\[
f_n(C') = (n - 1) f_1(C') + f_1(C'), \quad f_N(C) = N f_1(C)
\]
(26)
is a manifestation of the ‘self-averaging’ of the logarithm (not any logarithm though, but this particular one), and we get
\[
\langle \log \frac{2}{\cosh(2\theta_{\text{tot}}) + 1} \rangle = f_N(C) = N \log \tau_1.
\]
(27)

This is the Berry–Klein result (1), of course.

Finally, we turn to calculating \(\langle \tau_N \rangle\). The first few \(f_n(C')s\) are
\[
f_1(C') = \frac{2}{C' + 1},
\]
\[
f_2(C') = \frac{2}{C' + C},
\]
\[
f_3(C') = \frac{2}{\sqrt{(C' + 1)(2C^2 + C' - 1)}}.
\]
(28)
giving
\[
\langle \tau_2 \rangle = f_2(C) = \frac{1}{C} = \frac{\tau_1}{2 - \tau_1},
\]
\[
\langle \tau_3 \rangle = f_3(C) = \frac{2}{(C + 1)\sqrt{2C - 1}} = \frac{\tau_1}{\sqrt{4/\tau_1 - 3}}.
\]
(29)
and it is frustratingly difficult to go beyond \(n = 3\). But it is possible to evaluate the recurrence relation (18) numerically and so determine \(\langle \tau_N \rangle = f_N(C)\). In passing, we note that \(\langle \tau_2 \rangle_{\text{ray}} = \langle \tau_2 \rangle\) and \(\langle \tau_3 \rangle_{\text{ray}} > \langle \tau_3 \rangle\) for \(0 < \tau_1 < 1\); ray optics fails for \(N > 2\).

For \(\tau_1 = 0.85\), the outcome of such a computation is shown in the lin–log plot of figure 4 as the dotted curve ‘a’. The crosses near the curve were obtained by a Monte Carlo calculation in which 400 000 experiments were simulated with up to 200 slabs. The straight dashed line ‘b’ is the lower bound of (3). The solid lines are the upper and lower bounds discussed in the next section. Other values of \(\tau_1\) result in plots with the same general features.
Figure 4. Average transmission probability for a stack of \( N \) identical slabs. For \( \tau_1 = 0.85 \) and \( N \leq 200 \), the dotted curve ‘a’ shows the values of \( \log(\tau_N) \) computed by a numerical evaluation of the recurrence relation (18), commencing with the small-\( n \) functions of (28). The crosses that follow curve ‘a’ are values obtained by a Monte Carlo calculation that simulated 400,000 experimental realizations. The two solid lines are the upper and lower bounds of (32) and (36), respectively. The dashed line ‘b’ is the lower bound (3) derived by Berry and Klein [3].

4. Many slabs: upper and lower bounds

Since \( f_2(C') \leq 1/\sqrt{C'} \), we have

\[
f_3(C') \leq \frac{1}{\sqrt{C' C}} \int_{2\pi} d\phi \frac{1}{2\pi} \frac{1}{\sqrt{C' + S (S'/C') \cos \phi}}
\]

\[
\leq \frac{1}{\sqrt{C' C}} \int_{2\pi} d\phi \frac{1}{2\pi} \frac{1}{\sqrt{C' + S \cos \phi}}
\]

\[
= \frac{1}{\sqrt{C' C}} \Upsilon(\tau_1),
\]

where the second inequality recognizes that the integral in the first line is a monotonically increasing function of \( S'/C' \), so that the replacement \( S'/C' \to 1 \) increases its value. The integral defining \( \Upsilon(\tau_1) \) is of elliptic kind and its value is less than 1 if \( C > 1 \), that is: \( \Upsilon(\tau_1) < 1 \) if \( \tau_1 < 1 \). We conclude by induction that

\[
f_a(C') \leq \frac{1}{\sqrt{C' C}} \Upsilon(\tau_1)^{n-2}
\]

holds for \( n \geq 2 \). The upper bound

\[
\langle \tau_N \rangle \leq (\tau_2) \Upsilon(\tau_1)^{N-2}
\]

then follows. The ray-optics result (4) is inconsistent with this upper bound. Figure 5 shows \( \Upsilon(\tau_1) \) as a function of \( \tau_1 \).

We derive a lower bound by first observing that

\[
f_3(C') \geq f_2(C') \Delta(\tau_1),
\]

where \( \Delta(\tau_1) \) is a correction factor that depends on \( \tau_1 \). The lower bound

\[
\langle \tau_N \rangle \geq (\tau_2) \Upsilon(\tau_1)^{N-2}
\]

then follows.
Figure 5. Upper bound \( \Upsilon(\tau_1) \) and lower bound \( \Lambda(\tau_1) \) on \( (\langle \tau_N \rangle /\langle \tau_2 \rangle)^{1/(N-2)} \) as functions of \( \tau_1 \). The dashed straight line is the lower bound on \( \langle \tau_N \rangle \) of (3).

with

\[
\Lambda(\tau_1) = \min_{C'} \frac{f_2(C')}{f_2(C')} = \begin{cases} 
1 - \frac{\tau_1}{2} & \text{for } \tau_1 \geq 2 - \sqrt{2} \\
\sqrt{\tau_1 - \tau_1^2/4} & \text{for } \tau_1 \leq 2 - \sqrt{2}
\end{cases}
\]  

(34)

and then inferring by induction that

\[
f_n(C') \geq f_2(C')^n \Lambda(\tau_1)^{n-2}
\]  

(35)

holds for \( n \geq 2 \). The lower bound

\[
\langle \tau_N \rangle \geq \langle \tau_2 \rangle \Lambda(\tau_1)^{N-2}
\]  

(36)

then follows. The plot of \( \Lambda(\tau_1) \) as a function of \( \tau_1 \) in figure 5 shows that \( \Lambda(\tau_1) > \tau_1 \) for \( 0 < \tau_1 < 1 \) and, therefore, this lower bound is more stringent than (3), but it is not tight either. We are certain, however, that \( \langle \tau_N \rangle \) is bounded exponentially both from above and from below.

Figure 6 illustrates the two bounds

\[
\Lambda(\tau_1) \leq \frac{\langle \tau_N \rangle}{\langle \tau_2 \rangle}^{1/(N-2)} \leq \Upsilon(\tau_1)
\]  

(37)

for \( \tau_1 = 0.85 \). The values for curve ‘a’ are obtained by the numerical evaluation of the recurrence relation (18). Clearly all values are well within the two bounds, the horizontal dashed lines. This figure and analogous plots for other values of \( \tau_1 \) suggest that

\[
\left( \frac{\langle \tau_N \rangle}{\langle \tau_2 \rangle} \right)^{1/(N-2)} \rightarrow \Upsilon(\tau_1) \quad \text{as } N \to \infty.
\]  

(38)

The corresponding observation in figure 4 is that, for sufficiently large \( N \), line ‘a’ is parallel to the solid line for the upper bound. At present, (38) is no more than a conjecture that is supported by a body of numerical evidence.

Some of the evidence is curve ‘b’ in figure 6. Its values are obtained by an extrapolation that assumes that

\[
\left( \frac{\langle \tau_N \rangle}{\langle \tau_2 \rangle} \right)^{1/(N-2)} \approx A - B/N
\]  

(39)
Figure 6. Values of \( \langle (\tau_N)/(\tau_0) \rangle^{1/(N-2)} \) for \( \tau_1 = 0.85 \). The bounds of (37) are the two horizontal dashed lines. Curve ‘a’ shows the actual values. The extrapolation explained in the context of (38) and (39) gives curve ‘b’.

for large \( N \) with \( A \) and \( B \) slowly varying with \( N \). For two consecutive \( N \) values of curve ‘a’, we can get an estimate of \( A \) and \( B \), and curve ‘b’ represents the successive values of \( A \) thus extrapolated. The rapid and consistent approach of ‘b’ to the horizontal line of the upper limit feeds the expectation that the conjecture (38) could be true. We leave the matter at that.

5. Summary

We established the recurrence relation (18) that facilitates the calculation of the average value \( \langle f(\tau_N) \rangle \) of any function of \( \tau_N \), the transmission probability through the stack of \( N \) identical slabs with random gaps between them. We observed that the individual values of \( \tau_N \) are spread over a large range and, therefore, \( \langle \tau_N \rangle \) greatly exceeds \( e^{\langle \log \tau_N \rangle} = \tau_1^N \).

Further, we derived strict upper and lower bounds on \( \langle \tau_N \rangle \), both bounds being exponential functions of \( N \). The ray-optics prediction for \( \langle \tau_N \rangle \) is consistent with the lower bound but not with the upper bound. The upper bound, when used as an approximation for \( \langle \tau_N \rangle \), is of much better accuracy than its derivation suggests and, based on numerical evidence, we conjecture that it is asymptotically exact.

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