Goldman flows on the Jacobian

Lisa C. Jeffrey and David B. Klein
Mathematics Department, University of Toronto
Toronto, Ontario, Canada M5S 2E4
jeffrey@math.toronto.edu, dklein@math.toronto.edu

Abstract

We show that the Goldman flows preserve the holomorphic structure on the moduli space of homomorphisms of the fundamental group of a Riemann surface into $U(1)$, in other words the Jacobian.

1 Introduction

This note concerns the moduli space $\mathcal{M}(G)$ of conjugacy classes of homomorphisms of the fundamental group of a compact orientable 2-manifold $\Sigma$ into a Lie group $G$.

This object has recently attracted a great deal of interest in symplectic and algebraic geometry and mathematical physics. In mathematical physics it appears as the space of gauge equivalence classes of flat connections on a 2-manifold. In algebraic geometry it appears as the moduli space of holomorphic bundles on a Riemann surface.

The smooth locus of the space $\mathcal{M}(G)$ has a symplectic structure; see, for instance, [1] or [7]. If the 2-manifold is equipped with a complex structure, the space $\mathcal{M}(G)$ inherits a complex structure compatible with the symplectic structure. When $G = U(1)$, the moduli space of a Riemann surface coincides with its Jacobian, which is a complex torus whose complex dimension is the genus of the surface.

In [2], W. Goldman studied the Hamiltonian flows of certain natural functions on the moduli space. These functions are constructed from functions that send a flat connection to its holonomy around a specific simple closed curve $C$ in the 2-manifold.

In this paper we show that when the gauge group is $U(1)$ the Goldman flows preserve the complex structure on $\mathcal{M}(U(1))$. When the gauge group is $SU(2)$, the Goldman flows are ill defined on a set of real codimension 3, (see [6]), which is inconsistent with these flows preserving the complex structure.

2 The $U(1)$ Goldman flow

Let $G = U(1)$. The Goldman flow on $\mathcal{M}(G)$ is a periodic $\mathbb{R}$-action $\{\Xi_s\}_{s \in \mathbb{R}}$ associated to a simple closed curve $C$. Since the Lie group is abelian, the moduli space is $\text{Hom}(\pi, G)$. 

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Choose a symplectic basis of $H_1(\Sigma, \mathbb{Z})$, in other words a collection of cycles $\{\lambda_1, \ldots, \lambda_{2g}\}$ in which all the intersections are empty except for $\lambda_{2j-1}$ and $\lambda_{2j}$, which intersect once transversely with positive intersection index. If the curve $C$ is nonseparating then we let $\lambda_2 = C$ in this symplectic basis; if the curve $C$ is separating then we assume that the cycles $\lambda_1, \ldots, \lambda_{2g}$ do not intersect $C$. Choose a basepoint on $C$ for the fundamental group of $\Sigma$, and lift the cycles in the symplectic basis to loops $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{2g} \in \pi_1(\Sigma)$ that only intersect $C$ at their endpoints. See Figure 1.

For $G = U(1)$, identify $\Phi \in \text{Hom}(\pi_1(\Sigma), G)$ with $(\Phi_1, \ldots, \Phi_{2g}) \in G^{2g}$ by letting $\Phi_j = \Phi(\tilde{\lambda}_j)$. If the simple closed curve $C$ is nonseparating then the associated Goldman flow on $\text{Hom}(\pi, G) = G^{2g}$ is

$$\Xi_s(\Phi) = (e^{2\pi is} \Phi_1, \Phi_2, \ldots, \Phi_{2g}),$$

for $s \in \mathbb{R}/\mathbb{Z}$. If $C$ is separating then the Goldman flow is trivial, $\Xi_s(\Phi) = \Phi$.

The Goldman flows can be described using gauge theory as follows, (cf. [3]). If $A$ is a flat connection on the trivial $G$-bundle over $\Sigma$, and $A_j$ is the holonomy of $A$ along the loop $\tilde{\lambda}_j$, then the map that sends $A$ to $(A_1, \ldots, A_{2g}) \in G^{2g} = \text{Hom}(\pi_1(\Sigma), G)$ identifies the space of gauge equivalence classes of flat connections with the moduli space; see [5]. Let $\hat{\Sigma}$ be the complement of $C$ in $\Sigma$, and let $U_- \cup U_+$ be the intersection of $\hat{\Sigma}$ with a tubular
neighbourhood of \( C \) in \( \Sigma \). The open sets \( U_- \) and \( U_+ \) can be thought of as neighbourhoods of the two “boundary components” of \( \hat{\Sigma} \), as shown in Figure 2. For a flat connection \( A \) on \( \Sigma \), define
\[
\Xi_s(A) = A^{g_s},
\]
where \( g_s \) is a gauge transformation on \( \hat{\Sigma} \) with \( g_s = 1 \) on \( U_+ \) and \( g_s = e^{2\pi i s} \) on \( U_- \). Here, since \( g_s \) is constant on both \( U_+ \) and \( U_- \) and since the gauge group is abelian, the flat connection \( A^{g_s} \) on \( \hat{\Sigma} \) extends (by \( A \)) to a flat connection \( \Xi_s(A) \) on \( \Sigma \). If the curve \( C \) separates \( \Sigma \) into two components then \( \Xi_s(A) = A \) because \( g_s \) may be chosen to be a locally constant gauge transformation on \( \hat{\Sigma} \), which acts trivially since \( G \) is abelian. If the curve \( C \) is nonseparating, however, then the gauge transformations \( g_s \) on \( \hat{\Sigma} \) act nontrivially and do not come from gauge transformations on \( \Sigma \) for \( s \not\in \mathbb{Z} \), so in general the \( \Xi_s(A) \) are distinct elements of the moduli space \( \mathcal{M}(G) \), (although they are the same when viewed as elements of the moduli space of \( \hat{\Sigma} \)).

3 The Jacobian

When \( G = U(1) \), the moduli space \( \mathcal{M}(U(1)) \) is the Jacobian \( \text{Jac}(\Sigma) \cong U(1)^{2g} \). The Jacobian inherits a complex structure from the Riemann surface \( \Sigma \), and identifies with \( \mathbb{C}^g/\Lambda \) as a complex manifold for a lattice \( \Lambda \) described below. See, for example, [4].

The Jacobian is defined as
\[
\text{Jac}(\Sigma) = \frac{H^0(\Sigma, K)^*}{H_1(\Sigma, \mathbb{Z})},
\]
where \( H_1(\Sigma, \mathbb{Z}) \) maps to \( H^0(\Sigma, K)^* \) by integration: a class \( \lambda \in H_1(\Sigma, \mathbb{Z}) \) sends \( \omega \in H^0(\Sigma, K) \) to \( \int_{\lambda} \omega \in \mathbb{C} \). Explicitly, choose a basis \( \{\omega_1, \ldots, \omega_g\} \) of \( H^0(\Sigma, K) \), and use the dual basis to identify \( H^0(\Sigma, K)^* \) with \( \mathbb{C}^g \). Let \( F \) be the resulting map from \( H_1(\Sigma, \mathbb{Z}) \) to \( \mathbb{C}^g \),
\[
F(\lambda) = \left( \int_{\lambda} \omega_1 \atop \vdots \atop \int_{\lambda} \omega_g \right),
\]
and equate $H_1(\Sigma, \mathbb{Z}) \subset H^0(\Sigma, K)^*$ with the lattice

$$\Lambda = \{ F(\lambda) : \lambda \in H_1(\Sigma, \mathbb{Z}) \} \subset \mathbb{C}^g.$$ 

A choice of basis $\{\lambda_1, \ldots, \lambda_{2g}\}$ of $H_1(\Sigma, \mathbb{Z})$ identifies $\mathbb{C}^g/\Lambda$ with $U(1)^{2g}$ as follows. Viewed as a real vector space, $\mathbb{C}^g$ is spanned by $\{F(\lambda_1), \ldots, F(\lambda_{2g})\}$,

$$\mathbb{C}^g = \left\{ \sum_{j=1}^{2g} v_j F(\lambda_j) : v_j \in \mathbb{R} \right\}.$$ 

Identify $\mathbb{C}^g/\Lambda$ with $U(1)^{2g}$ by the group isomorphism that maps

$$\left[ \sum_{j=1}^{2g} v_j F(\lambda_j) \right] \in \mathbb{C}^g/\Lambda$$

to

$$(\exp 2\pi i v_1, \ldots, \exp 2\pi i v_{2g}).$$

We define $z_j = \exp 2\pi i v_j$.

4 Goldman flows on the Jacobian

The Goldman flow on the Jacobian is defined as follows. If $C$ is a nonseparating simple closed curve then choose $C$ as the generator $\lambda_2$ in a symplectic basis $\{\lambda_1, \ldots, \lambda_{2g}\}$ of $H_1(\Sigma, \mathbb{Z})$. So the Goldman flow associated to $C$ is

$$(z_1, z_2, \ldots, z_{2g}) \mapsto (e^{2\pi i s} z_1, z_2, \ldots, z_{2g})$$

for $s \in \mathbb{R}/\mathbb{Z}$. This corresponds to translation by $s F(\lambda_1)$ in $\mathbb{C}^g$, which is clearly a holomorphic self map of $\mathbb{C}^g/\Lambda$.

If $C$ is a separating simple closed curve then the Goldman flow on the Jacobian is trivial (because the gauge group is abelian).

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