ATTRAVERS FOR A CLASS OF DELAYED REACTION-DIFFUSION EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. In this paper we study the asymptotic behavior of solutions for a class of nonautonomous reaction-diffusion equations with dynamic boundary conditions possessing finite delay. Under the polynomial conditions of reaction term, suitable conditions of delay terms and a minimal conditions of time-dependent force functions, we first prove the existence and uniqueness of solutions by using the Galerkin method. Then, we ensure the existence of pullback attractors for the associated process to the problem by proving some uniform estimates and asymptotic compactness properties (via an energy method). With an additional condition of time-dependent force functions, we prove that the boundedness of pullback attractors in smoother spaces.

1. Introduction. Partial differential equations with dynamic boundary conditions arise for example in hydrodynamics and the heat transfer theory. For instance, they allow to model heat flow inside the considered domain subject to nonlinear heating or cooling at the boundary, or heat transfer in a solid in contact with a moving fluid, in thermoelasticity, diffusion phenomena, heat transfer in two mediums, etc. [2, 8]. The long-time behavior of solutions for reaction-diffusion equations with dynamic boundary conditions has been studied extensively (see [1, 3, 7, 15, 16, 17, 18]).

Delayed differential equations arise in many realistic models of problems in science and engineering where there is a time lag or after-effect. In particular, the parabolic case represents some issues in mathematical biology and the time lags are often seen as maturation time for population dynamics. Let us introduce some relevant literatures in [9, 10, 13]. It is naturally to study time-dependent partial differential equations with dynamic boundary conditions concerning with delays.
especially the reaction-diffusion equations with dynamic boundary conditions involving delays.

The long-time behavior of solutions of the delayed reaction-diffusion equation with Dirichlet boundary condition were studied in [5, 11, 12]. To the best of our knowledge, there is only paper [14] considered the long-time behavior of solutions for the delayed equations with dynamic boundary conditions. More precisely, the authors studied the existence of pullback attractors for a $p$-Laplacian nonautonomous problem with dynamic boundary conditions and infinite delay. The $p$-Laplace operator becomes the Laplace operator when $p = 2$, but in the problem considered in [14], the reaction terms disappeared. So, up to now, there is no result concerning with the delayed reaction-diffusion equations with dynamic boundary conditions.

In this paper, we study the long-time behavior by analyzing the existence of pullback attractors for a class of reaction-diffusion equations with dynamic boundary conditions and finite delay. We also mention here that, in [14], the authors considered the case of $p$-Laplace operator with infinite delay. Thus, there are some differences in the proofs of the existence of solutions and pullback attractor, especially in the proof of the asymptotic compactness since the phase space is needed to be considered is different to the phase space in this paper.

Let $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}^+$, be an open bounded domain with smooth boundary $\Gamma = \partial \Omega$. We consider the following reaction-diffusion equation equations with dynamic boundary conditions and finite delay:

$$\begin{cases}
\partial_t u - \Delta u + f(u) = g(t, u^t) + \rho(x, t) & \text{in } \Omega \times (\tau, \infty), \\
\partial_t u + \partial_n u + \kappa u + f_T(u) = g_T(t, u^t) + \rho_T(x, t) & \text{on } \Gamma \times (\tau, \infty), \\
u(x, \theta) = \phi(x, \theta - \tau), & \theta \in [\tau - h, \tau], x \in \Omega, \\
u(x, \theta) = \phi_T(x, \theta - \tau), & \theta \in [\tau - h, \tau], x \in \Gamma,
\end{cases}$$

(1)

where $\tau \in \mathbb{R}$, $n$ denotes the outward normal vector at $\Gamma$, $\kappa > 0$, $\phi \in C([-h, 0]; L^2(\Omega))$, $\phi_T \in C([-h, 0]; L^2(\Gamma))$ are the initial datum, $h > 0$ being the length of the delay effect, and where for each $t \geq \tau$, we denote by $u^t(\theta) = u(t + \theta), \theta \in [-h, 0]$.

We consider (1) with the following conditions:

(h1) $f, f_T : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions satisfying

$$c_1|s|^p - c_0 \leq f(s)s \leq c_2|s|^p + c_0, \forall s \in \mathbb{R}, p \geq 2, \quad (2)$$

$$f(s) - f(r)(s - r) \geq -\ell|s - r|^2, \forall s, r \in \mathbb{R}, \quad (3)$$

and

$$c_1|s|^q - c_0 \leq f_T(s)s \leq c_2|s|^q + c_0, \forall s \in \mathbb{R}, q \geq 2, \quad (4)$$

$$f_T(s) - f_T(r)(s - r) \geq -\ell|s - r|^2, \forall s, r \in \mathbb{R}, \quad (5)$$

where $c_0, c_1, c_2$ and $\ell$ are positive constants.

(h2) The delay functions $(g, g_T) : \mathbb{R} \times C([-h, 0]; L^2(\Omega) \times L^2(\Gamma)) \rightarrow L^2(\Omega) \times L^2(\Gamma)$ satisfy

i) $\forall (\xi, \bar{\xi}) \in C([-h, 0]; L^2(\Omega) \times L^2(\Gamma))$,

$$\mathbb{R} \ni t \mapsto (g(t, \xi), g_T(t, \bar{\xi})) \in L^2(\Omega) \times L^2(\Gamma)$$

is measurable,

ii) $\forall t \in \mathbb{R}, g(t, 0) = 0, g_T(t, 0) = 0$,

iii) $\exists L_0 > 0$ such that $\forall \tau \in \mathbb{R}, \forall (\xi, \bar{\xi}), (\eta, \bar{\eta}) \in C([-h, 0]; L^2(\Omega) \times L^2(\Gamma))$,

$$\|g(t, \xi) - g(t, \eta)\|^2_{L^2(\Omega)} + \|g_T(t, \xi) - g_T(t, \bar{\eta})\|^2_{L^2(\Gamma)}$$
Here, we follow the conditions of delay functions which were studied in [5] for delayed reaction-diffusion equations with homogeneous Dirichlet boundary conditions.

From (2) and (4) then there exist two positive constants $c_3, c_4$ such that

$$|f(s)| \leq c_3(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R},$$

$$|f_T(s)| \leq c_4(1 + |s|^{q-1}), \quad \forall s \in \mathbb{R}. \quad (7)$$

We denote the primitive functions of $f$ and $f_T$ by $F(s) = \int_0^s f(r)dr$ and $F_T(s) = \int_0^s f_T(r)dr$. From conditions (2) and (4), there exist positive constants $\tilde{c}_0, \tilde{c}_1$ and $\tilde{c}_2$ such that

$$\tilde{c}_1|s|^p - \tilde{c}_0 \leq F(s) \leq \tilde{c}_2|s|^p + \tilde{c}_0, \quad \forall s \in \mathbb{R}, \quad p \geq 2,$$

$$\tilde{c}_1|s|^q - \tilde{c}_0 \leq F_T(s) \leq \tilde{c}_2|s|^q + \tilde{c}_0, \quad \forall s \in \mathbb{R}, \quad q \geq 2. \quad (9)$$

The rests of paper is organized as follows. In the next section we present some preliminaries. In Section 3, with above condition and a minimal condition of time-dependent force functions, we prove the existence and uniqueness of solution by using the Galerkin method. In the last section, after recall some abstract results of pullback attractors theory [6], we analyze conditions in order to obtain two different families of minimal pullback attractors, namely, those of fixed bounded sets but also for a class of time-dependent families (universe) given by a tempered condition, for the natural (nonautonomous) dynamical system associated to the problem through the previous result. Finally, we also prove the minimal pullback attractor is bounded in some smoother spaces under an additional condition of time-dependent force functions.

2. Preliminaries. Notations. We denote the inner product of $L^2(\Omega)$ by $(\cdot, \cdot)_{\Omega}$ with the norm $| \cdot |_{\Omega}$ and $L^2(\Gamma)$ by $(\cdot, \cdot)_{\Gamma}$ with the norm $| \cdot |_{\Gamma}$. With $1 \leq r \neq 2$, we denote $| \cdot |_{r, \Omega}$ (respectively, $| \cdot |_{r, \Gamma}$), the norm of $L^r(\Omega)$ (respectively, $L^r(\Gamma)$).

We will denote $\| \cdot \|_\Omega$ and $\| \cdot \|_{r, \Omega}$ the norm of $H^1(\Omega)$ and its dual space $H^{-1}(\Omega)$ respectively. The duality between $H^1(\Omega)$ and $H^{-1}(\Omega)$ will be written as $(\cdot, \cdot)_{\Omega}$. We also denote $\| \cdot \|_\Gamma$ and $\| \cdot \|_{r, \Gamma}$ the norms of $H^r(\Gamma) = \gamma_0(H^1(\Omega))$ and its dual space $H^{-r}(\Gamma)$ respectively. The duality between $H^r(\Gamma)$ and $H^{-r}(\Gamma)$ will be written as $(\cdot, \cdot)_{\Gamma}$. Here, $\gamma_0 \in L(H^1(\Omega), H^\frac{1}{2}(\Gamma))$ denotes the trace operator $u \mapsto \gamma_0(u) = u|_\Gamma$ with the norm $\| \gamma_0 \|$. Moreover we have

$$\| \gamma_0(u) \|_\Gamma \leq \| \gamma_0 \|_\Omega \| u \|_\Omega, \quad \forall u \in H^1(\Omega). \quad (11)$$

Moreover, we denote by $H = L^2(\Omega) \times L^2(\Gamma)$ and $V = H^1(\Omega) \times H^\frac{1}{2}(\Gamma)$. The dual space of $V$ is denoted by $V^*$. We need the following Poincaré-Trace inequality (see [4]). For any $\kappa \geq 0$ then there exists an optimal constant $C^*_\kappa > 0$ such that

$$|\nabla \varphi|_{\Omega}^2 + \kappa|\gamma_0(\varphi)|_{\Gamma}^2 \geq C^*_\kappa|\varphi|_{\Omega}^2, \quad \forall \varphi \in H^1(\Omega), \quad (12)$$

where $C^*_\kappa$ is continuous and non-decreasing with respect to $\kappa$, and $C^*_0 = 0$. Moreover, the following constant

$$C^*_\kappa = \begin{cases} \kappa(2d/\text{diam}(\Omega) - \kappa) & \text{if } \kappa \in \left[0, \frac{d}{\text{diam}(\Omega)} \right], \\ \frac{d^2}{(\text{diam}(\Omega))^2} & \text{if } \kappa \in \left[\frac{d}{\text{diam}(\Omega)}, \infty \right), \end{cases}$$

also fulfills (12).
We consider the operator $A$ has a bounded inverse $A^{-1}$, then from (13) the operator $A$ is coercive and then by the Lax-Milgram theorem, $A$ has a bounded inverse $A^{-1} : V^* \to V$ with its restriction to $H$ is a compact operator since the compactness of the embedding $V \subset H$. Moreover, from (14) then $A : D(A) \to H$ is positive. Hence, since $A$ is self-adjoint, positive operator with compact resolvent, there exists an orthonormal basis $\{(w_j, \gamma_0(w_j))\}_{j=1}^\infty \subset D(A)$ in $H$, consisting of eigenfunctions of $A$ with corresponding eigenvalues $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}^+$, forms an increasing sequence converging to infinity. Moreover $\{(w_j, \gamma_0(w_j))\}_{j=1}^\infty$ is an orthogonal basis in $V$.

3. Existence and uniqueness of weak solutions.

**Definition 3.1.** Let $T > \tau$, $(\phi, \phi_T) \in C([-h, 0]; H)$ and $(\rho, \rho_T) \in L^2_{\text{loc}}(\mathbb{R}, H)$. A weak solution to (1) in $[\tau, T]$ is a vector function $(u, u_T)$ such that

$$(u, u_T) \in C([\tau - h, T]; H) \cap L^2(\tau, T; V) \cap (L^p(\tau, T; L^p(\Omega))) \times L^2(\tau, T; L^2(\Gamma))),$$

$$(\partial_t u, \partial_t u_T) \in L^2(\tau, T; V^*)$$

and

$$\begin{align*}
\frac{d}{dt} ((u, v)_{\Omega} + (u_T, \gamma_0(v))_\Gamma) + (\nabla u, \nabla v)_{\Omega} + \kappa(u_T, \gamma_0(v))_\Gamma \\
+ (f(u), u)_{\Omega} + (f_T(u_T), \gamma_0(v))_\Gamma \\
= (g(t, u^j), v)_{\Omega} + (g_T(t, u_T^j), \gamma_0(v))_\Gamma + \langle \rho(t), v \rangle_\Omega + \langle \rho_T(t), \gamma_0(v) \rangle_\Gamma \quad \text{in } D'(\tau, T),
\end{align*}$$

for all $v \in H^1(\Omega) \cap L^p(\Omega)$ such that $\gamma_0(v) \in L^2(\Gamma)$.

We prove the existence and uniqueness of weak solutions to (1) by using the Galerkin approximation method.

**Theorem 3.2.** Let $(\rho, \rho_T) \in L^2_{\text{loc}}(\mathbb{R}, V^*)$, $f, f_T, g, g_T$ satisfy (h1)-(h2) and $(\phi, \phi_T) \in C([-h, 0]; H)$ given. Then there exists a unique weak solution $(u, u_T)$ to (1) and the solution depends continuously on the initial data.

**Proof.** Step 1. Existence. We first prove the existence of weak solution to (1) by using the Galerkin method.

**Step 1.1. The Galerkin approximation.** Let $P_m$ be the orthonormal projector from $H$ to $\text{span}\{ (w_1, \gamma_0(w_1)), \ldots, (w_m, \gamma_0(w_m)) \}$, where $\{(w_j, \gamma_0(w_j))\}_{j=1}^\infty$ is the
basis of all the eigenfunctions of the operator $A$ which is orthonormal in $H$ and orthogonal in $V$. We consider the approximation solution
\[
u_{m}(t) = \sum_{j=1}^{m} \chi_{mj}(t)w_{j},\quad \nu_{m,T}(t) = \gamma_{0}(\nu_{m}(t)) = \sum_{j=1}^{m} \chi_{mj}(t)\gamma_{0}(w_{j}),
\]
where $\chi_{mj}(t)$ are required to satisfy the following differential equation system with finite delay:
\[
\begin{aligned}
\frac{d}{dt} ((u_{m}, w_{j})_{\Omega} + (\gamma_{0}(u_{m}), \gamma_{0}(w_{j}))_{\Gamma}) + (\nabla u_{m}, \nabla w_{j})_{\Omega} + \kappa (\gamma_{0}(u_{m}), \gamma_{0}(w_{j}))_{\Gamma} \\
+ (f(u_{m}), w_{j})_{\Omega} + (f_{\Gamma}(\gamma_{0}(u_{m})), \gamma_{0}(w_{j}))_{\Gamma} \\
= \langle g(t, u_{m}^{t} + \rho(t), w_{j})_{\Omega} + (g_{\Gamma}(t, \gamma_{0}(u_{m}^{t} + \rho_{\Gamma}(t), \gamma_{0}(w_{j}))_{\Gamma}, j = 1, \ldots, m, \\
(u_{m}(\theta), \gamma_{0}(u_{m})(\theta)) = P_{m}(\phi(\theta - \tau), \phi_{\Gamma}(\theta - \tau)), \theta \in [\tau - h, \tau].
\end{aligned}
\]

(15)

It is well-known that the above finite-dimensional delayed system is well-posed [10], at least locally. Hence, we conclude that the approximation solution $u_{m}(t)$ to (15) exists unique locally on $[\tau, T_{m}]$ with $\tau \leq T_{m} \leq T$. Next, we will obtain a priori estimates and ensure that the solutions $u_{m}$ exists in the whole interval $[\tau - h, T]$.

**Step 1.2. A priori estimates.** Multiplying the first equation in (15) by $\chi_{mj}(t)$ and summing up from $j = 1$ to $m$, we obtain
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} (|u_{m}|^{2}_{\Omega} + |\gamma_{0}(u_{m})|^{2}_{\Gamma}) + |\nabla u_{m}|^{2}_{\Omega} + \kappa |\gamma_{0}(u_{m})|^{2}_{\Gamma} \\
+ (f(u_{m}), u_{m})_{\Omega} + (f_{\Gamma}(\gamma_{0}(u_{m})), \gamma_{0}(u_{m}))_{\Gamma} \\
= \langle g(t, u_{m}^{t} + \rho(t), u_{m})_{\Omega} + (g_{\Gamma}(t, \gamma_{0}(u_{m}^{t} + \rho_{\Gamma}(t), \gamma_{0}(u_{m}))_{\Gamma}, \rho(t), u_{m})_{\Omega} + (\rho_{\Gamma}(t), \gamma_{0}(u_{m}))_{\Gamma}.
\end{aligned}
\]

Using the conditions (h1)-(h2), the Cauchy inequality and using (13), we have
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} (|u_{m}|^{2}_{\Omega} + |\gamma_{0}(u_{m})|^{2}_{\Gamma}) + |\nabla u_{m}|^{2}_{\Omega} + \kappa |\gamma_{0}(u_{m})|^{2}_{\Gamma} \\
+ c_{0}(|\Omega| + |\Gamma|) + L_{g}||u_{m}^{t}||_{C([\tau - h, 0]; L^{2}(\Omega))} |u_{m}|_{\Omega} + L_{g}||\gamma_{0}(u_{m})||_{C([\tau - h, 0]; L^{2}(\Gamma))} |\gamma_{0}(u_{m})|_{\Gamma} \\
+ \frac{1}{2\mu_{1}} (||\rho(t)||^{2}_{\Omega} + ||\rho_{\Gamma}(t)||^{2}_{\Gamma}) + \frac{1}{2\mu_{1}} (||u_{m}|^{2}_{\Omega} + |\gamma_{0}(u_{m})|^{2}_{\Gamma}) \\
\leq c_{0}(|\Omega| + |\Gamma|) + L_{g}||u_{m}^{t}, \gamma_{0}(u_{m}^{t})||_{C([\tau - h, 0]; H)}^{2} + \frac{1}{2\mu_{1}} (||\rho(t)||^{2}_{\Omega} + ||\rho_{\Gamma}(t)||^{2}_{\Gamma}) \\
+ \frac{1}{2} (|\nabla u_{m}|^{2}_{\Omega} + \kappa |\gamma_{0}(u_{m})|^{2}_{\Gamma})
\end{aligned}
\]

Thus,
\[
\begin{aligned}
\frac{d}{dt} (|u_{m}|^{2}_{\Omega} + |\gamma_{0}(u_{m})|^{2}_{\Gamma}) + |\nabla u_{m}|^{2}_{\Omega} + \kappa |\gamma_{0}(u_{m})|^{2}_{\Gamma} &+ 2c_{1}(|u_{m}|^{2}_{\Omega} + |\gamma_{0}(u_{m})|^{2}_{\Gamma}) \\
&\leq 2c_{0}(|\Omega| + |\Gamma|) + 2L_{g}||u_{m}^{t}, \gamma_{0}(u_{m}^{t})||_{C([\tau - h, 0]; H)}^{2} \\
&+ \frac{1}{\mu_{1}} (||\rho(t)||^{2}_{\Omega} + ||\rho_{\Gamma}(t)||^{2}_{\Gamma}).
\end{aligned}
\]

(16)

Using (13) then we get from (16) that for any $t \geq \tau$,
\[
|u_{m}(t)|^{2}_{\Omega} + |\gamma_{0}(u_{m}(t))|^{2}_{\Gamma} \leq 2c_{0}(|\Omega| + |\Gamma|) + 2L_{g}||u_{m}^{t}, \gamma_{0}(u_{m}^{t})||_{C([\tau - h, 0]; H)}^{2} \\
+ \frac{1}{\mu_{1}} (||\rho(t)||^{2}_{\Omega} + ||\rho_{\Gamma}(t)||^{2}_{\Gamma}) ds
\]
+ 2c_1 \int_\tau^t |u_m(s)|^p_{\Omega} ds + 2c_1 \int_\tau^t |\gamma_0(u_m(s))|_{q, t}^{\gamma} ds \\
i |u_m(\tau)|_{\Omega}^2 + |\gamma_0(u_m(\tau))|_{\Omega}^2 + 2c_0(|\Omega| + |\Gamma|)(t - \tau) \\
+ 2L_g \int_\tau^t \|(u_m^*, \gamma_0(u_m^*))\|_{C([\tau, t]; H)}^2 ds + \frac{1}{\mu_1} \int_\tau^t (\|\rho(s)\|_{L^\infty(\Omega)}^2 + \|\rho_\Gamma(s)\|_{L^\infty(\Gamma)}^2) ds. \quad (17)

In particular,

\[ |u_m(t)|_{\Omega}^2 + |\gamma_0(u_m(t))|_{\Omega}^2 \leq |\phi(0)|_{\Omega}^2 + |\phi_\Gamma(0)|_{\Gamma}^2 + 2c_0(|\Omega| + |\Gamma|)(t - \tau) \\
+ 2L_g \int_\tau^t \|(u_m^*, \gamma_0(u_m^*))\|_{C([\tau, t]; H)}^2 ds \\
+ \frac{1}{\mu_1} \int_\tau^t (\|\rho(s)\|_{L^\infty(\Omega)}^2 + \|\rho_\Gamma(s)\|_{L^\infty(\Gamma)}^2) ds.

And then

\[ \|(u_m^*, \gamma_0(u_m^*))\|_{C([\tau, t]; H)}^2 \leq \|(\phi, \phi_\Gamma)\|_{C([\tau, t]; H)}^2 + 2c_0(|\Omega| + |\Gamma|)(t - \tau) \\
+ 2L_g \int_\tau^t \|(u_m^*, \gamma_0(u_m^*))\|_{C([\tau, t]; H)}^2 ds \\
+ \frac{1}{\mu_1} \int_\tau^t (\|\rho(s)\|_{L^\infty(\Omega)}^2 + \|\rho_\Gamma(s)\|_{L^\infty(\Gamma)}^2) ds.

Using the Gronwall inequality, we deduce that

\[ \|(u_m^*, \gamma_0(u_m^*))\|_{C([\tau, t]; H)}^2 \leq e^{2L_s(t - \tau)} \left( \|(\phi, \phi_\Gamma)\|_{C([\tau, t]; H)}^2 + 2c_0(|\Omega| + |\Gamma|)(t - \tau) \\
+ \frac{1}{\mu_1} \int_\tau^t (\|\rho(s)\|_{L^\infty(\Omega)}^2 + \|\rho_\Gamma(s)\|_{L^\infty(\Gamma)}^2) ds \right), \forall t \geq \tau.

From this and (17) we conclude that

\{ (u_m, \gamma_0(u_m)) \} is uniformly bounded in \( L^\infty(\tau - h, T; H) \cap L^2(\tau, T; V) \), \quad (18)
\{ (u_m, \gamma_0(u_m)) \} is uniformly bounded in \( L^p(\tau, T; L^p(\Omega)) \times L^q(\tau, T; L^q(\Gamma)) \). \quad (19)

Using (7)-(8) and (19) then

\{ (f(u_m), f_\Gamma(u_m)) \} is uniformly bounded in \( L^{p'}(\tau, T; L^{p'}(\Omega)) \times L^{q'}(\tau, T; L^{q'}(\Gamma)) \), \quad (20)

where \( p' = \frac{p}{p - 1} \), \( q' = \frac{q}{q - 1} \). From these bounds we get from the first and the second equations in (1) that

\{ \partial_t u_m \} is uniformly bounded in \( L^2(\tau, T; H^{-1}(\Omega)) \cap L^{p'}(\tau, T; L^{p'}(\Omega)) \), \quad (21)
\{ \partial_t \gamma_0(u_m) \} is uniformly bounded in \( L^2(\tau, T; H^{-\frac{1}{2}}(\Gamma)) \cap L^{q'}(\tau, T; L^{q'}(\Gamma)) \). \quad (22)

**Step 1.3. Passing to the limits.**

From bounds (18), (19), (20), (21) and (22), there exists a subsequence of \{ u_m \} (still denoted by \{ u_m \}), a function \( u \in C([\tau - h, T]; L^2(\Omega)) \) with \( \gamma_0(u) \in C([\tau - h, T]; L^2(\Omega)) \), and a function \( \gamma_0(u) \in C([\tau - h, T]; H^{-\frac{1}{2}}(\Gamma)) \).
From the bounds (18) and (21)-(22), it follows from the Aubin-Lions compact lemma that
\[ (u_m, \gamma_0(u_m)) \rightarrow (u, \gamma_0(u)) \] in \( L^2(\tau, T; H) \).
Thus, up to a subsequence, we have
\[ (u_m, \gamma_0(u_m)) \rightarrow (u, \gamma_0(u)) \] a.e. in \( \bar{\Omega} \times (\tau, T) \).
So from the continuity of \( f, f_\Gamma \) we conclude that
\[ (f(u_m), f_\Gamma(\gamma_0(u_m))) \rightarrow (f(u), f_\Gamma(\gamma_0(u))) \] a.e. in \( \bar{\Omega} \times (\tau, T) \).
And then we have that \( \chi = f(u), \zeta = f_\Gamma(\gamma_0(u)) \).
We now have to passing to the limit of the nonlinear function related to delay term. To do this, we show that
\[ (u_m^t, \gamma_0(u_m^t)) \rightarrow (u^t, \gamma_0(u^t)) \] in \( C([\tau-h, 0]; H) \), \( \forall t \in (\tau, T) \).  \hfill (23)
We have
\[
\|(u_m^t - u^t, \gamma_0(u_m^t - u^t))\|_{C([\tau-h, 0]; H)} \\
\leq \max \left\{ \sup_{-h \leq t \leq T} |u_m(t + \theta) - u(t + \theta)|_\Omega + |\gamma_0(u_m(t + \theta) - u(t + \theta)|_\Gamma, \\
\sup_{\theta \geq \tau - t} |u_m(t + \theta) - u(t + \theta)|_\Omega + |\gamma_0(u_m(t + \theta) - u(t + \theta)|_\Gamma \right\} \\
\leq \max \left\{ \|(P_m(\phi) - \phi, P_m(\phi_\Gamma) - \phi_\Gamma)\|_{C([-h, 0]; H)}, \|(u_m - u, \gamma_0(u_m - u))\|_{C([\tau, T]; H)} \right\}.
\]
We first see that
\[ \|(P_m(\phi) - \phi, P_m(\phi_\Gamma) - \phi_\Gamma)\|_{C([-h, 0]; H)} \rightarrow 0 \] as \( m \rightarrow \infty \).
Thus, to obtain (23), we need show that
\[ \|(u_m - u, \gamma_0(u_m - u))\|_{C([\tau, T]; H)} \rightarrow 0 \] as \( m \rightarrow \infty \).  \hfill (24)
Indeed, let \( (u_n, \gamma_0(u_n)) \) and \( (u_m, \gamma_0(u_m)) \) satisfy (15) then \( w = u_n - u_m \) satisfies
\[
\frac{1}{2} \frac{d}{dt} \left( |u_n - u_m|_\Omega^2 + |\gamma_0(u_n - u_m)|_\Omega^2 \right) + |\nabla(u_n - u_m)|_\Omega + \kappa |\gamma_0(u_n - u_m)|_\Gamma^2 \\
+ (f(u_n) - f(u_m), u_n - u_m)_\Omega + (f_\Gamma(\gamma_0(u_n)) - f_\Gamma(\gamma_0(u_m)), \gamma_0(u_n - u_m))_\Gamma \\
= (g(t, u_n^t) - g(t, u_m^t), u_n^t - u_m^t)_\Omega \\
+ (g_\Gamma(t, \gamma_0(u_n^t)) - g_\Gamma(t, \gamma_0(u_m^t)), \gamma_0(u_n - u_m))_\Gamma.
\]  \hfill (25)
Using conditions (3), (5) and (6) we deduce from (25) that
\[
\frac{1}{2} \frac{d}{dt} \left( |u_n - u_m|_\Omega^2 + |\gamma_0(u_n - u_m)|_\Omega^2 \right) + |\nabla(u_n - u_m)|_\Omega^2 + \kappa |\gamma_0(u_n - u_m)|_\Gamma^2 \\
\leq \ell \left( |u_n - u_m|_\Omega^2 + |\gamma_0(u_n - u_m)|_\Omega^2 \right) + L_g \|(u_n^t - u_m^t, \gamma_0(u_n^t - u_m^t))\|_{C([-h, 0]; H)}^2.
\]
This shows that

\[
\{u(t) - u_m(t)\}_{t=0}^\infty + |\gamma(\tilde{u}(t) - u_m(t))|_1 \leq (t + L_g) \int_\tau^t \|u^n_s - u^n_m, \gamma(\tilde{u}_n(t) - u_m(t))\|_{C([-\theta, 0]; H)} ds.
\]

Integrating this inequality over \([\tau, t]\), we get

\[
|u_n(t) - u_m(t)|_1^2 + |\gamma(\tilde{u}(t) - u_m(t))|_1^2 \leq |u_n(\tau) - u_m(\tau)|_1^2 + |\gamma(\tilde{u}(\tau) - u_m(\tau))|_1^2 + 2(\ell + L_g) \int_\tau^t \|u^n_s - u^n_m, \gamma(\tilde{u}_n(t) - u_m(t))\|_{C([-\theta, 0]; H)} ds.
\]

Thus,

\[
\|u(t) - u_m(t)\|_{C([-\theta, 0]; H)}^2 \leq \|P_n(\phi, \phi_T) - P_m(\phi, \phi_T)\|_{C([-\theta, 0]; H)}^2 + 2(\ell + L_g) \int_\tau^t \|u^n_s - u^n_m, \gamma(\tilde{u}_n(t) - u_m(t))\|_{C([-\theta, 0]; H)} ds.
\]

By the Gronwall inequality we obtain that

\[
\|u - u_m\|_{C([-\theta, 0]; H)}^2 \leq e^{2(\ell + L_g)(t - \tau)} \|P_n(\phi, \phi_T) - P_m(\phi, \phi_T)\|_{C([-\theta, 0]; H)}^2.
\]

This shows that \(\{u_n - u_m, \gamma(\tilde{u}_n - u_m)\}\) is a Cauchy sequence in \(C([\tau, t]; H)\). So, we get (24). And consequence we get (23) and

\[
(g(t, u_m^t), g(t, \gamma(\tilde{u}_m^t)) \to (g(t, u^t), g(t, \gamma(\tilde{u}^t))) \in \mathcal{L}^2(\tau, T; H).
\]

Therefore, we can pass to the limits to get that \((u, \gamma(u))\) is the solution of (1).

**Step 2. The uniqueness and dependence continuously on initial datum.**

Suppose that \((u, u_T)\) and \((v, v_T)\) are two weak solutions to (1) with initial datum \((\phi, \phi_T)\) and \((\psi, \psi_T)\) respectively. Then \((z, z_T) = (u - v, u_T - v_T)\) satisfies:

\[
\begin{aligned}
\partial_t z - \Delta z + f(u) - f(v) &= g(t, u^t) - g(t, v^t) &\text{in } \Omega \times (\tau, \infty), \\
\partial_t z_T + \partial_n z_T + \kappa z_T + f_T(u_T) - f_T(v_T) &= g_T(t, u_T^t) - g_T(t, v_T^t) &\text{on } \Gamma \times (\tau, \infty), \\
z(x, \theta) &= \phi(x, \theta - \tau) - \psi(x, \theta - \tau), &\theta \in [\tau - h, \tau], x \in \Omega, \\
z_T(x, \theta) &= \phi_T(x, \theta - \tau) - \psi_T(x, \theta - \tau), &\theta \in [\tau - h, \tau], x \in \Gamma.
\end{aligned}
\]

(26)

Multiplying the first equation in (26) by \(z\), integrating over \(\Omega\) and using (3), (5), (6), we have

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} (|z|^2_{\Omega} + |\gamma(z)|^2_{\Gamma}) + |\nabla z|_{\Omega}^2 + \kappa |\gamma(z)|_{\Gamma}^2 &= -(f(u) - f(v), z)_{\Omega} - (f_T(\gamma(u)) - f_T(\gamma(v)), \gamma(z))_{\Gamma} \\
&\quad + (g(t, u) - g(t, v)), z)_{\Omega} + (g_T(t, \gamma(u)) - g_T(t, \gamma(v))), z)_{\Gamma} \\
&\leq \ell(|z|^2_{\Omega} + |\gamma(z)|^2_{\Gamma}) + L_g |z|^2_{C([-\theta, 0]; L^2(\Omega))} + |\gamma(z)|_{\Omega} + L_g |\gamma(z)|_{C([-\theta, 0]; L^2(\Gamma))} |\gamma(z)|_{\Gamma}.
\end{aligned}
\]

Thus,

\[
|z(t)|_{\Omega}^2 + |\gamma(z(t))|_{\Gamma}^2 \leq (t + L_g) \int_\tau^t |(z^s, \gamma(z^s))|_{C([-\theta, 0]; H)} ds.
\]

Hence

\[
|z|^2_{C([-\theta, 0]; H)} \leq |(\phi - \psi, \gamma(\phi - \psi))|_{C([-\theta, 0]; H)}^2.
\]
By using the Gronwall inequality, we deduce the uniqueness and continuous dependence on initial datum of solutions to (1).

$\Box$

4. Existence of pullback attractors.

4.1. Some concepts of pullback attractors. For the convenience of the reader, we recall in this section some concepts and results on the theory of pullback $\mathcal{D}$-attractors (see [6]), which will be used in the paper.

Let $X$ be a metric space with metric $d_X$. A process in $X$ is a mapping $U(t, \tau) : X \to X$ such that $U(t, \tau) = Id$ and $U(t, s)U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau, \tau \in \mathbb{R}$. A process $\{U(t, \tau)\}_{t \geq \tau}$ is said to be continuous if for any $\tau \leq t$, the map $U(t, \tau) : X \to X$ is continuous. The process $\{U(t, \tau)\}_{t \geq \tau}$ is said to be closed if for any $\tau \leq t$, any sequence $\{x_m\} \subset X$ with $x_m \to x \in X$ and $U(t, \tau)x_m \to y \in X$, then $U(t, \tau)x = y$. We note that if a process is continuous then it is closed.

Denote by $\mathcal{P}(X)$ the set of all nonempty subsets of $X$ and consider a family of nonempty sets $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

**Definition 4.1.** We say that a process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$, any any sequence $\{\tau_m\}$ with $\tau_m \leq t$ for all $m$ satisfying $\tau_m \to -\infty$, any sequence $x_m \in D(\tau_m)$ for all $m$, the sequence $\{U(t, \tau_m)x_m\}$ is relatively compact in $X$.

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Denote

$$\Lambda_X(\hat{\mathcal{D}}, t) = \bigcap_{s \leq \tau \leq s} U(t, \tau)D(\tau) \subset X, \quad t \in \mathbb{R}.$$  

Given two subset of $X$, $\mathcal{O}_1, \mathcal{O}_2$, we denote by $\text{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in $X$ between them.

**Definition 4.2.** A process $\{U(t, \tau)\}_{t \geq \tau}$ on $X$ is called pullback $\mathcal{D}$-asymptotically compact if it is pullback $\hat{\mathcal{D}}$-asymptotically compact for any $\hat{\mathcal{D}} \in \mathcal{D}$.

It is said that $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$ in $X$ if for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \leq t$ such that $U(t, \tau_0)D(\tau_0) \subset D_0(t)$ for all $\tau \leq \tau_0(\hat{\mathcal{D}}, t)$.

**Theorem 4.3.** Let $\{U(t, \tau)\}_{t \geq \tau}$ be a closed process $U(t, \tau) : X \to X$. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{D}}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ and $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}_0$-asymptotically compact. Then $\{U(t, \tau)\}_{t \geq \tau}$ has the minimal pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}} = \{A_\mathcal{D}(t) : t \in \mathbb{R}\}$ defined by $A_\mathcal{D}(t) = \bigcup_{\hat{\mathcal{D}} \in \mathcal{D}} \Lambda_X(\hat{\mathcal{D}}, t)$ which satisfies

1. for any $t \in \mathbb{R}$, the set $A_\mathcal{D}(t)$ is a nonempty compact subset of $X$, and $A_\mathcal{D}(t) \subset \Lambda_X$;
2. $A$ is invariant; i.e., $U(t, \tau)A(\tau) = A(t)$, for all $t \geq \tau$;
3. $A$ is pullback $\mathcal{D}$-attracting; i.e.,

$$\lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), A(t)) = 0,$$

for all $\hat{\mathcal{D}} \in \mathcal{D}$ and all $t \in \mathbb{R}$;
If $\hat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_D(t) = \Lambda_X(\hat{D}_0(t)) \subset \overline{D(t)}^X$ for all $t \in \mathbb{R}$.

The family $\mathcal{A}_D$ is minimal in the sense that if $\hat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\hat{D} \in \mathcal{D}$, $\lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), C(t)) = 0$, then $\mathcal{A}_D(t) \subset C(t)$.

Remark 1. If $\mathcal{A}_D \in \mathcal{D}$ then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (2)-(3).

A sufficient condition for $\mathcal{A}_D \in \mathcal{D}$ is to have that $\hat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and $\mathcal{D}$ is inclusion closed (i.e., if $\hat{D} \in \mathcal{D}$, and $\hat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all $t$, then $\hat{D}' \in \mathcal{D}$).

Let $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of $X$, i.e., the class of all families $\hat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with $D$ a fixed nonempty bounded subset of $X$.

Corollary 1. Under the assumptions of Theorem 4.3, if $\mathcal{D}$ contains $\mathcal{D}_F(X)$ then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_D$ exist and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_D(t)$ for all $t \in \mathbb{R}$.

Moreover, if for some $T \in \mathbb{R}$, the set $\bigcup_{t \leq T} D_0(t)$ is bounded subset of $X$, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_D(t)$, for all $t \leq T$.

4.2. Existence of pullback attractors. From Theorem 3.2 we can define a process $U(t, \tau) : C([-h, 0]; H) \to C([-h, 0]; H)$ for $t \geq \tau$, given by $U(t, \tau)(\phi, \phi_\tau) := (u^t, u^\tau_0)$, where $(u, u_\tau)$ is the unique weak solution to (1).

For any $\sigma > 0$, we consider $\mathcal{D}_\sigma(C([-h, 0]; H))$ the class of all families of nonempty subsets $\hat{D} = \{D(t) = D : t \in \mathbb{R}\}$ such that

$$\lim_{\tau \to -\infty} \sup_{\varphi \in \mathcal{D}(\tau)} \|\varphi\|_{C([-h, 0]; H)}^2 = 0.$$ We also denote by $\mathcal{D}_F(C([-h, 0]; H))$ the class of families $\hat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with $D$ a fixed nonempty bounded set of $C([-h, 0]; H)$.

Remark 2. Observe that for any $\sigma > 0$, $\mathcal{D}_F(C([-h, 0]; H)) \subset \mathcal{D}_\sigma(C([-h, 0]; H))$ and that $\mathcal{D}_\sigma(C([-h, 0]; H))$ is inclusion closed.

We denote $\overline{B}_X(R)$ is the closed ball in $X$ centered at 0 with radius $R$.

Lemma 4.4 (Pullback absorbing set). Under assumptions of Theorem 3.2 and we assume that $\underline{\gamma} := \mu_2 - 2L_0 e^{\mu_2 h} > 0$ and

$$\int_{-\infty}^0 e^{\underline{\gamma}s} \left(\|\rho(s)\|_{\Sigma, \Omega}^2 + \|\rho_{\tau}(s)\|_{\Sigma, \Gamma}^2\right) ds < \infty.$$ Then the family $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ with $D_0(t) = \overline{B}_{C([-h, 0]; H)}(R_H(t))$, is pullback $\mathcal{D}_\sigma(C([-h, 0]; H))$-absorbing set in $C([-h, 0]; H)$ for the process $\{U(t, \tau)\}_{t \geq \tau}$, where

$$R_H(t) = \left(1 + \frac{2c_0(|\Omega| + |\Gamma|)}{\underline{\gamma}} e^{\mu_2 h} + \frac{e^{\mu_2 h}}{\mu_2} \int_{-\infty}^t e^{-\underline{\gamma}(t-s)}(\|\rho(s)\|_{\Sigma, \Omega}^2 + \|\rho(s)\|_{\Sigma, \Gamma}^2) ds\right)^{\frac{1}{2}}.$$ (27)

Proof: We first have the following estimate as in (16) for the solution $(u, u_\tau)$

$$\frac{d}{dt} \left(\|u\|_{\Sigma, \Omega}^2 + |\gamma_0(u)|^2\right) + |\nabla u\|_{\Sigma, \Omega}^2 + \kappa |\gamma_0(u)|^2 + 2c_1(|u|_{\Sigma, \Omega}^2 + |\gamma_0(u)|_{\Sigma, \Omega}^2)
\leq 2c_0(|\Omega| + |\Gamma|) + 2L_9\|(u, \gamma_0(u))\|_{C([-h, 0]; H)}^2 + \frac{1}{\mu_1}(\|\rho(t)\|_{\Sigma, \Omega}^2 + \|\rho_{\tau}(t)\|_{\Sigma, \Gamma}^2).$$ (28)
Using (14), (28) becomes
\[
\frac{d}{dt} \left( |u(t)|^2 + |\gamma_0(u(t))|^2 \right) + \mu_2 \left( |u(t)|^2 + |\gamma_0(u(t))|^2 \right) \\
\leq 2c_0(|\Omega| + |\Gamma|) + 2L_g \| (u^t, \gamma_0(u^t)) \|_{C([-h,0];H)}^2 + \frac{1}{\mu_1} \left( \| \rho(t) \|^2_{\ast,\Omega} + \| \rho(t) \|^2_{\ast,\Gamma} \right).
\]
And therefore,
\[
\frac{d}{dt} \left[ e^{\mu_2 t} \left( |u(t)|^2 + |\gamma_0(u(t))|^2 \right) \right] \leq 2c_0(|\Omega| + |\Gamma|) e^{\mu_2 t} + 2L_g e^{\mu_2 t} \| (u^t, \gamma_0(u^t)) \|_{C([-h,0];H)}^2 + \frac{e^{\mu_2 t}}{\mu_1} \left( \| \rho(t) \|^2_{\ast,\Omega} + \| \rho(t) \|^2_{\ast,\Gamma} \right).
\]
Hence,
\[
e^{\mu_2 t} \left( |u(t)|^2 + |\gamma_0(u(t))|^2 \right) \leq e^{\mu_2 \tau} \left( |u(\tau)|^2 + |\gamma_0(u(\tau))|^2 \right) + \frac{1}{\mu_1} \int_{\tau}^{t} e^{\mu_2 s} \left( \| \rho(s) \|^2_{\ast,\Omega} + \| \rho_\Gamma(s) \|^2_{\ast,\Gamma} \right) ds \\
+ \frac{2c_0}{\mu_2} \left( |\Omega| + |\Gamma| \right) (e^{\mu_2 t} - e^{\mu_2 \tau}) + 2L_g \int_{\tau}^{t} e^{\mu_2 s} \| (u^s, \gamma_0(u^s)) \|_{C([-h,0];H)}^2 ds.
\]
Replacing \( t \) by \( t + \theta \) with \( \theta \in [-h,0] \) then we get
\[
e^{\mu_2 t} \| (u^t, \gamma_0(u^t)) \|_{C([-h,0];H)}^2 \\
\leq e^{\mu_2 (\tau + h)} \| (\phi, \phi_\Gamma) \|_{C([-h,0];H)}^2 + \frac{e^{\mu_2 h}}{\mu_1} \int_{\tau}^{t} e^{\mu_2 s} \left( \| \rho(s) \|^2_{\ast,\Omega} + \| \rho_\Gamma(s) \|^2_{\ast,\Gamma} \right) ds \\
+ \frac{2c_0}{\mu_2} \left( |\Omega| + |\Gamma| \right) (e^{\mu_2 (t + h)} - e^{\mu_2 (\tau + h)}) \\
+ 2L_g e^{\mu_2 h} \int_{\tau}^{t} e^{\mu_2 s} \| (u^s, \gamma_0(u^s)) \|_{C([-h,0];H)}^2 ds. \tag{29}
\]
Using the Gronwall inequality and after some simple computations, we obtain from (29) that
\[
\| (u^t, \gamma_0(u^t)) \|_{C([-h,0];H)}^2 \leq e^{\mu_2 h - (\mu_2 - 2L_g e^{\mu_2 h})(\tau - t)} \left( (\phi, \phi_\Gamma) \right)_{C([-h,0];H)}^2 + \frac{2c_0(|\Omega| + |\Gamma|) e^{\mu_2 h}}{\mu_2 - 2L_g e^{\mu_2 h}} \left( 1 - e^{-(\mu_2 - 2L_g e^{\mu_2 h})(\tau - t)} \right) + \frac{e^{\mu_2 h}}{\mu_1} \int_{\tau}^{t} e^{-(\mu_2 - 2L_g e^{\mu_2 h})(s - \tau)} \| \rho(s) \|^2_{\ast,\Omega} + \| \rho_\Gamma(s) \|^2_{\ast,\Gamma} ds. 
\]
From this inequality, for any \( t \in \mathbb{R} \), any \( \hat{D} \in \mathcal{D}_\theta(C([-h,0];H)) \), there exists a time \( \tau_0 = \tau_0(\hat{D}, t) \leq t \) such that
\[
\| U(t, \tau)(\phi, \phi_\Gamma) \|_{C([-h,0];H)} = \| (u^t, \gamma_0(u^t)) \|_{C([-h,0];H)} \leq R_H(t), \tag{30}
\]
for all \( (\phi, \phi_\Gamma) \in D(\tau) \), \( \forall \tau \leq \tau_0 \), where \( R_H(t) \) is defined in (27). This completes the proof. \( \square \)

**Lemma 4.5.** Under assumptions of Lemma 4.4, then the process \( \{ U(t, \tau) \}_{t \geq \tau} \) is pullback \( \mathcal{D}_\theta(C([-h,0];H)) \)-asymptotically compact.

**Proof.** We consider a fixed times \( t_0 \in \mathbb{R}, \hat{D} = \{ D(t) : t \in \mathbb{R} \} \in \mathcal{D}_\theta(C([-h,0];H)) \), \( \{ \tau_m \} \subset \mathbb{R} \) with \( \tau_m < t_0 \) for all \( m \geq 1 \), \( \lim_{m \to \infty} \tau_m = -\infty \), and \( \{ (\phi^m, \phi^m_\Gamma) \} \subset D(\tau_m) \).
We now prove that \( \{(u_m^t, \gamma_0(u_m^t))\} \) is relatively compact in \( C([-h, 0]; H) \), where \((u_m, \gamma_0(u_m))\) is the weak solution to (1) with the initial condition \((\phi^m, \phi_0^m)\).

From Lemma 4.4 for \( T > h \) there exists \( m_0 = m_0(t_0, T) \) such that \( \tau_m < t_0 - T \) for all \( m \geq m_0 \) and
\[
\|(u_m^t, \gamma_0(u_m^t))\|_{C([-h, 0]; H)}^2 \leq R(t_0, T) \quad \forall t \in [t_0 - T, t_0], \ m \geq m_0,
\]
where
\[
R(t_0, T) = 1 + 2c_0(|\Omega| + |\Gamma|)e^{\mu_2 h} + e^{\mu_2 h + \sigma T} \int_{-\infty}^{t_0} e^{-\sigma(t_0-s)}(\|\rho(s)\|_{\ast, \Omega}^2 + \|\rho_T(s)\|_{\ast, T}^2)ds.
\]
In particular,
\[
|u_m(t)|_{\Omega}^2 + |\gamma_0(u_m(t))|_{\Gamma}^2 \leq R(t_0, T) \quad \forall t \in [t_0 - T, t_0], \ m \geq m_0. \tag{31}
\]
Now, to continue we set \( y_m(t) = u_m(t + t_0 - T) \) for \( t \in [0, T] \), we have that \( \{(y_m, \gamma_0(y_m))\}_{m \geq m_0} \) is uniformly bounded in \( L^\infty(0, T; H) \).

Furthermore, \((y_m, \gamma_0(y_m))\) is a weak solution on \([0, T]\) to an analogous problem to (1) with \( g, g_T, \rho, \rho_T \) are replaced respectively by
\[
\tilde{g}(t, \cdot) = g(t + t_0 - T, \cdot), \quad \tilde{g}_T(t, \cdot) = g_T(t + t_0 - T, \cdot),
\]
\[
\tilde{\rho}(t) = \rho(t + t_0 - T), \quad \tilde{\rho}_T(t) = \rho_T(t + t_0 - T), \quad t \in (0, T),
\]
and with \((y_0^0, \gamma_0(y_0^0)) = (u_0^{m_0-T}, \gamma_0(u_0^{m_0-T})), (y_m^T, \gamma_0(y_m^T)) = (u_m^{t_0}, \gamma_0(u_m^{t_0}))\).

We also have from (31) that
\[
\|(y_m, \gamma_0(y_m))\|_{C([-h, 0]; H)} \leq R(t_0, T), \ \forall m \geq m_0. \tag{32}
\]
Moreover, integrating (28) from \( t_0 - T \) to \( t_0 \), using (13) and (32), we get for all \( m \geq m_0, \)
\[
\mu_1 \|(y_m, \gamma_0(y_m))\|_{L_2^\infty(0, T; V)}^2 + 2c_1 \left( \|u_m\|_{L_p(0, T; L^p(\Omega))}^p + \|\gamma_0(u_m)\|_{L_q(0, T; L^q(\Gamma))}^q \right)
\]
\[
\leq K(t_0, T),
\]
where
\[
K(t_0, T) = R(t_0, T) + 2c_0(|\Omega| + |\Gamma|)T + 2L_gR(t_0, T)T
\]
\[
+ \frac{1}{\mu_1} \int_{t_0-T}^{t_0} (\|\rho(t)\|_{\ast, \Omega}^2 + \|\rho_T(t)\|_{\ast, T}^2)dt.
\]
Therefore,
\[
\{(y_m, \gamma_0(y_m))\}_{m \geq m_0} \text{ is uniformly bounded in } L^2(0, T; V),
\]
\[
\{(y_m, \gamma_0(y_m))\}_{m \geq m_0} \text{ is uniformly bounded in } L^p(0, T; L^p(\Omega)) \times L^q(0, T; L^q(\Gamma)),
\]
\[
\{f(y_m)\}_{m \geq m_0} \text{ is uniformly bounded in } L^p(0, T; L^p(\Omega)),
\]
\[
\{f_{\Gamma}(\gamma_0(y_m))\}_{m \geq m_0} \text{ is uniformly bounded in } L^q(0, T; L^q(\Gamma)),
\]
\[
\{\partial_t y_m\}_{m \geq m_0} \text{ is uniformly bounded in } L^2(0, T; H^{-1}(\Omega)) + L^p(0, T; L^p(\Omega)),
\]
\[
\{\partial_t \gamma_0(y_m)\}_{m \geq m_0} \text{ is uniformly bounded in } L^2(0, T; H^{-\frac{1}{2}}(\Gamma)) + L^q(0, T; L^q(\Gamma)).
\]
So, as same as the proof in the existence of solutions, we conclude that there exists \((y, \gamma_0(y))\) \( \in C([0, T]; H) \cap L^2(0, T; V) \cap (L^p(0, T; L^p(\Omega)) \times L^q(0, T; L^q(\Gamma))) \) with
\[
\partial_t y \in L^2(0, T; H^{-1}(\Omega)) + L^p(0, T; L^p(\Omega)),
\]
\[
\partial_t \gamma_0(y) \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma)) + L^q(0, T; L^q(\Gamma)),
\]
From the boundedness of \( \{(y_m, \gamma_0(y_m))\}_{m \geq m_0} \) in \( L^\infty(0, T; H) \) and the boundedness of \( \{ (\partial_t y_m, \partial_t \gamma_0(y_m))\}_{m \geq m_0} \) in
\[
\left( L^2(0, T; H^{-1}(\Omega)) + L^p(0, T; L^p(\Omega)) \right) \times \left( L^2(\tau, T; H^{-\frac{1}{2}}(\Gamma)) + L^q(0, T; L^q(\Gamma)) \right)
\]
and from the compactness of the injection of \( L^2(\Omega) \) in \( H^{-1}(\Omega) \) and \( L^2(\Gamma) \) in \( H^{-\frac{1}{2}}(\Gamma) \), then by applying the Arzelà-Ascoli theorem, we deduce that for any \( \{ t_m \} \subset [0, T] \) with \( t_m \to t_1 \), we have
\[
(y_{m_1}(t_m), \gamma_0(y_{m_1}(t_m))) \to (y(t_1), \gamma_0(y(t_1))) \text{ in } H. \tag{33}
\]
From conditions (h2) and (31), we have that
\[
\int_s^t \left( |\tilde{g}(r, y_m)|_{\Omega}^2 + |\tilde{g}_r(r, \gamma_0(y_m))|^2 \right) \, dr \leq C(t - s), \quad \forall 0 \leq s \leq t \leq T, \quad \forall m \geq m_0,
\]
with the constant \( C \) independent on \( m \). Thus, there exists a subsequence of \( \{(y_m, \gamma_0(y_m))\} \) and \( (\chi, \zeta) \in L^2(0, T; H) \) such that
\[
(\tilde{g}(r, y_{m_1}), \tilde{g}_r(r, \gamma_0(y_{m_1}))) \to (\chi, \zeta) \text{ in } L^2(0, T; H),
\]
and therefore,
\[
\int_s^t \left( |\chi(r)|^2 + |\zeta(r)|^2 \right) \, dr \leq C(t - s), \quad \forall 0 \leq s \leq t \leq T.
\]
From the above convergences we conclude that \( (y, \gamma_0(y)) \) is the weak solution to
\[
\begin{cases}
\partial_t u - \Delta u + f(u) = \chi + \tilde{\rho}(x, t) & \text{in } \Omega \times (0, T), \\
\partial_n u + \kappa u + f_r(u) = \zeta + \tilde{\rho}_r(x, t) & \text{on } \Gamma \times (0, T), \\
u(x, 0) = y(0), & x \in \Omega, \\
u(x, 0) = \gamma_0(y(0)), & x \in \Gamma.
\end{cases}
\]
We have the following energy estimate
\[
\frac{1}{2}(|z(t)|_{\Omega}^2 + |\gamma_0(z(t))|_{\Gamma}^2) - \frac{1}{2}(|z(s)|_{\Omega}^2 + |\gamma_0(z(s))|_{\Gamma}^2)
\]
\[
\leq \int_s^t \left( (\rho(r, z(r))_{\Omega} + (\rho_r(r, \gamma_0(z(r)))_{\Gamma}) \right) dr + C(t - s), \quad \forall s, t \in [0, T],
\]
where \( z = y_m \) or \( z = y \). Thus, the functionals \( J_m, J : [0, T] \to \mathbb{R} \) defined by
\[
J_m(t) = \frac{1}{2}(|y_m(t)|_{\Omega}^2 + |\gamma_0(y_m(t))|_{\Gamma}^2)
\]
\[
- \int_0^t \left( (\rho(r, y_m(r))_{\Omega} + (\rho_r(r, \gamma_0(y_m(r)))_{\Gamma}) \right) dr - Ct,
\]
\[ J(t) = \frac{1}{2}(g(y(t))_\Omega^2 + |\gamma_0(y(t))|^2) - \int_0^t ((\rho(r), y(r))_\Omega + (\rho_T(r), \gamma_0(u(r)))_T)dr - Ct \]

are non-increasing and continuous functions satisfying

\[ J_m(t) \to J(t) \text{ for a.e. } t \in [0, T]. \quad (34) \]

Using (34) we will prove that \( (y^m, \gamma_0(y^m)) \to (y(t), \gamma_0(y(t))) \) in \( C([\delta, T]; H) \) for any \( 0 < \delta < T \). Indeed, if it is not true then there exists \( 0 < \delta < T, \epsilon > 0 \), and a subsequence \( \{y_{m}, \gamma_0(y_{m})\} \) of \( \{(y_{m}, \gamma_0(y_{m}))\}_{m \geq m_0} \) and \( t_m \in [\delta, T] \) with\( t_m \to t_* \in [\delta, T] \) such that

\[ |y_{m}(t_m) - y(t_*)|_\Omega^2 + |\gamma_0(y_{m}(t_m))|_T^2 \geq \varepsilon_* \quad (35) \]

For fix \( \varepsilon > 0 \), from (34) and since \( J_m, J \) are continuous and non-increasing, there exists \( 0 < t_* < t_m \) such that

\[ J_m(t_m) = J(t_*). \]

Since \( t_m \to t_* \) then there exists \( m_\varepsilon > 0 \) such that \( t_* < t_m < t_m \) for \( m \geq m_\varepsilon \). So, we have

\[ J_m(t_m) - J(t_*) \leq J(t_m) - J(t_*) \leq |J_m(t_m) - J(t_*)| + |J(t_m) - J(t_*)| \]

\[ \leq |J(t_m) - J(t_*)| + \varepsilon, \quad \forall m \geq m_\varepsilon. \]

Thus, \( \limsup_{m \to \infty} J_m(t_m) \leq J(t_*) + \varepsilon. \) Since \( \varepsilon \) is arbitrary, we have

\[ \limsup_{m \to \infty} J_m(t_m) \leq J(t_*). \]

So,

\[ \limsup_{m \to \infty} (|y_{m}(t_m)|_\Omega^2 + |\gamma_0(y_{m}(t_m))|_T^2) \leq |y(t_*)|_\Omega^2 + |\gamma_0(y(t_*))|_T^2. \]

This and (33) we conclude that \( (y_{m}(t_m), \gamma_0(y_{m}(t_m))) \to (y(t_*) , \gamma_0(y(t_*))) \) in \( H \), and which is a contradiction with (35).

Hence we have proved that \( (y_m, \gamma_0(y_m)) \to (y, \gamma_0(y)) \) in \( C([\delta, T]; H) \) for any \( 0 < \delta < T \). And so, \( (u_{m}^0, \gamma_0(u_{m}^0)) \to (u^0, \gamma_0(u^0)) \) in \( C([-h, 0]; H) \) since \( T > h \) as we have chosen.

We now have the main theorem.

**Theorem 4.6 (Pullback attractor).** Under the assumptions of Lemma 4.4 then the process \( \{U(t, \tau)\}_{t \geq \tau} \) has the minimal pullback \( D_B(C([\delta, 0]; H)) \)-attractor

\[ \mathcal{A}_{D_B(C([\delta, 0]; H))} \]

and the minimal pullback \( D_F(C([-h, 0]; H)) \)-attractor \( \mathcal{A}_{D_F(C([-h, 0]; H))} \). The family \( \mathcal{A}_{D_F(C([-h, 0]; H))} \) belongs to \( D_B(C([-h, 0]; H)) \), and it holds that

\[ \mathcal{A}_{D_F(C([-h, 0]; H))}(t) \subset \mathcal{A}_{D_B(C([-h, 0]; H))}(t) \subset \overline{B_{C([-h, 0]; H)}}(0, R_H(t)), \forall t \in \mathbb{R}. \]

If, additionally, it holds that

\[ \sup_{r \leq 0} \int_{-\infty}^r e^{-\bar{\sigma}(r-s)} (\|\rho(s)\|_{s, \Omega}^2 + \|\rho_T(s)\|_{s, T}^2) ds < \infty, \]

then

\[ \mathcal{A}_{D_F(C([-h, 0]; H))}(t) = \mathcal{A}_{D_B(C([-h, 0]; H))}(t), \forall t \in \mathbb{R}. \]
Proof. Since \( \{U(t, \tau)\}_{\tau \geq \tau_0} \) is continuous and therefore closed, then from Lemmas 4.4-4.5, the existence of \( \mathcal{A}_{D_0}(C([-h,0];H)) \) is a consequence of Theorem 4.3.

By Remark 2 and Corollary 1, we immediately obtain the existence of
\[
\mathcal{A}_{D_0}(C([-h,0];H))(t).
\]
Then, we also deduce the first inclusion in (36).

Theorem 4.3 also implies the last inclusion in (36) and the fact that
\[
\mathcal{A}_{D_0}(C([-h,0];H)) \subseteq \mathcal{D}_{\sigma}(C([-h,0];H)),
\]
since \( D_0(t) \) is closed in \( C([-h,0];H) \) for \( t \in \mathbb{R} \), and \( D_0 \in \mathcal{D}_{\sigma}(C([-h,0];H)) \) and then the sufficient conditions in Remark 1 hold.

If (37) holds, then \( R_H(\cdot) \) given by (27) is uniformly bounded. Thus, (38) is obtained by applying Corollary 1.

4.3. Boundedness of pullback attractors in \( C([-h,0]; V) \cap (C([-h,0]; L^p(\Omega) \times L^q(\Gamma))) \).

**Theorem 4.7.** Under assumptions of Lemma 4.4 and assume that
\[
(\rho, \rho_T) \in L^2_{\text{loc}}(\mathbb{R}, H)
\]
such that
\[
\int_{-\infty}^0 e^{\hat{\sigma}s} (|\rho(s)|^2 + |\rho_T(s)|^2) ds < \infty.
\]
Then the family \( \mathcal{A}_{D_0}(C([-h,0];H)) \) is bounded in
\[
C([-h,0]; V) \cap (C([-h,0]; L^p(\Omega) \times L^q(\Gamma)))
\]
Proof. We first integrate (28) over \( [t-1, t] \) for \( t \geq \tau_0 + 1 \), where \( \tau_0 \) is given in proof of Lemma 4.4, we get
\[
|u(t)|^2_\Omega + |\gamma_0(u(t))|^2_\Gamma
+ \int_{t-1}^t \left( |\nabla u(s)|^2_\Omega + \kappa |\gamma_0(u(s))|^2_\Gamma + 2c_1(|u(s)|^p_{p,\Omega} + |\gamma_0(u(s))|_{q,\Gamma}) \right) ds
\leq |u(t-1)|^2_\Omega + |\gamma_0(u(t-1))|^2_\Gamma + 2c_0(|\Omega| + |\Gamma|)
+ 2L_g \int_{t-1}^t (u^*, \gamma_0(u^*))|\nabla u(s)|^2_{C([-h,0];H)} ds + \frac{1}{\mu_1} \int_{t-1}^t (\rho(s)|^2_{L^p,\Omega} + \rho_T(s)|^2_{L^q,\Gamma}) ds.
\]
Since
\[
|u(t-1)|^2_\Omega + |\gamma_0(u(t-1))|^2_\Gamma \leq (u^{t-1}, \gamma_0(u^{t-1}))_{C([-h,0];H)} \leq R_H(t-1) \leq R_H(t)
\]
for \( \tau \leq \tau_0 - 1 \). Hence,
\[
\int_{t-1}^t \left( |\nabla u(s)|^2_\Omega + \kappa |\gamma_0(u(s))|^2_\Gamma + 2c_1(|u(s)|^p_{p,\Omega} + |\gamma_0(u(s))|_{q,\Gamma}) \right) ds
\leq 2c_0(|\Omega| + |\Gamma|) + (2L_g + 1) R_H(t) + \frac{1}{\mu_1} \int_{t-1}^t (\rho(s)|^2_{L^p,\Omega} + \rho_T(s)|^2_{L^q,\Gamma}) ds. \quad (39)
\]
Now, multiplying the first equation in (1) by \( \partial_t u \), integrating over \( \Omega \) and using the second equation in (1), we get
\[
|\partial_t u|^2_\Omega + |\partial_t \gamma_0(u)|^2_\Gamma + \frac{d}{dt} (|\nabla u(s)|^2_\Omega + \kappa |\gamma_0(u(s))|^2_\Gamma + 2F(u) + 2F_T(\gamma_0(u)))
= (g(t, u^t), \partial_t u)_\Omega + (g_T(t, u^t), \partial_t (\gamma_0(u)))_\Gamma + (\rho(t), \partial_t u) + (\rho_T(t), \partial_t \gamma_0(u))_\Gamma. \quad (40)
\]
Integrating (43) with respect to 
for 
where 
Using condition (6) and the Cauchy inequality to deduce that 
\[ \frac{1}{2} \left( |\partial_t u(t)|^2 + |\partial_x \gamma_0(u(t))|^2 \right) + 2L_g \left( |u(t)|^2 \gamma_0(u(t)) \right) \|C_{[-\kappa,0];H}\| + |\rho(t)|^2 + |\rho_\Gamma(t)|^2. \] (41)

Substituting (41) into (40) to obtain that 
\[ \frac{d}{dt} \left( |\nabla u(s)|^2 + \kappa |\gamma_0(u(s))|^2 \right) + 2F(u) + 2F_\Gamma(\gamma_0(u)) \leq 2L_g \left( |u(t)|^2 \gamma_0(u(t)) \right) \|C_{[-\kappa,0];H}\| + |\rho(s)|^2 + |\rho_\Gamma(s)|^2. \] (42)

Integrating (42) over \([s, t]\) with \(s \in [t - 1, t]\), we have 
\[ |\nabla u(t)|^2 + \kappa |\gamma_0(u(t))|^2 \leq |\nabla u(s)|^2 + \kappa |\gamma_0(u(s))|^2 + 2F(u(s)) + 2F_\Gamma(\gamma_0(u(s))) \]
\[ + 2L_g \int_t^s \left( |u(t)|^2 \gamma_0(u(t)) \right) \|C_{[-\kappa,0];H}\| dr + 2 \int_s^t \left( |\rho(r)|^2 + |\rho_\Gamma(r)|^2 \right) dr. \] (43)

Integrating (43) with respect to \(s \in [t - 1, t]\), using (9), (10) and estimates (30), (39) when \(\tau \leq \tau_0 - 1\), we obtain 
\[ |\nabla u(t)|^2 + \kappa |\gamma_0(u(t))|^2 \leq |\nabla u(s)|^2 + \kappa |\gamma_0(u(s))|^2 + 2F(u(s)) + 2F_\Gamma(\gamma_0(u(s))) \]
\[ + 2L_g \int_t^s \left( |u(t)|^2 \gamma_0(u(t)) \right) \|C_{[-\kappa,0];H}\| dr ds + 2 \int_s^t \left( |\rho(r)|^2 + |\rho_\Gamma(r)|^2 \right) dr ds \]
\[ \leq c_3 \left( 2c_0(|\Omega| + |\Gamma|) \right) \left( 2L_g + 1 \right) R_{\Omega}(t) \]
\[ + \frac{1}{\mu_1} \int_{t-1}^t e^{\hat{\sigma}(s-t+1)} \left( |\rho(s)|^2_{C_{\Omega}} + |\rho_\Gamma(s)|^2_{C_{\Gamma}} \right) ds \]
\[ + 2c_0(|\Omega| + |\Gamma|) + 2L_g R_{\Omega}(t) + 2 L_g^2 R_{\Omega}^2(t) + 2 \int_{t-1}^t e^{\hat{\sigma}(s-t+1)} \left( |\rho(s)|^2_{C_{\Omega}} + |\rho_\Gamma(s)|^2_{C_{\Gamma}} \right) ds, \] (44)

where \(c_3 = \max\{1, \hat{\sigma}c_1, c_2\}\). By using (9) and (10) once again, we deduce from (44) for \(\tau \leq \tau_0 - 1\)
\[ |\nabla u(t)|^2 + \kappa |\gamma_0(u(t))|^2 \leq 2(2c_0 \left( 2L_g + 1 \right) R_{\Omega}(t) \]
\[ + c_3 \mu_1 \int_{t-1}^t e^{\hat{\sigma}(s-t+1)} \left( |\rho(s)|^2_{C_{\Omega}} + |\rho_\Gamma(s)|^2_{C_{\Gamma}} \right) ds \]
\[ + 2 \int_{t-1}^t e^{\hat{\sigma}(s-t+1)} \left( |\rho(s)|^2_{C_{\Omega}} + |\rho_\Gamma(s)|^2_{C_{\Gamma}} \right) ds. \] (45)
Using (13), then we obtain from (45) that
\[
\mu_2 \left( \|u(t)\|_{\Omega}^2 + \|\gamma_0(u(t))\|_{\Omega}^2 \right) + 2c_3 \left( \|u(t)\|_{p,\Omega}^p + \|\gamma_0(u(t))\|_{q,\Gamma}^q \right)
\leq 2(c_3c_0 + 2\tilde{c}_0)(|\Omega| + |\Gamma|) + (c_3(2L_g + 1) + 2L_g^2) R_H^2(t)
\]
\[+ c_3\mu_1^{-1} \int_{t-1}^{t} e^{\sigma(s-t+1)} \left( \|\rho(s)\|_{\Omega}^2 + \|\rho_T(s)\|_{\Gamma}^2 \right) ds
\]
\[+ 2 \int_{t-1}^{t} e^{\sigma(s-t+1)} \left( \|\rho(s)\|_{\Omega}^2 + \|\rho_T(s)\|_{\Gamma}^2 \right) ds.
\]
Replacing \( t \) by \( t + \theta, \theta \in [-h, 0] \) we have
\[
\mu_2 \left( \|u(t + \theta)\|_{\Omega}^2 + \|\gamma_0(u(t + \theta))\|_{\Omega}^2 \right) + 2c_3 \left( \|u(t + \theta)\|_{p,\Omega}^p + \|\gamma_0(u(t + \theta))\|_{q,\Gamma}^q \right)
\leq 2(c_3c_0 + 2\tilde{c}_0)(|\Omega| + |\Gamma|) + (c_3(2L_g + 1) + 2L_g^2) R_H^2(t + \theta)
\]
\[+ c_3\mu_1^{-1} \int_{t+\theta-1}^{t+\theta} e^{\sigma(s-t-\theta+1)} \left( \|\rho(s)\|_{\Omega}^2 + \|\rho_T(s)\|_{\Gamma}^2 \right) ds
\]
\[+ 2 \int_{t+\theta-1}^{t+\theta} e^{\sigma(s-t-\theta+1)} \left( \|\rho(s)\|_{\Omega}^2 + \|\rho_T(s)\|_{\Gamma}^2 \right) ds.
\]
Hence,
\[
\mu_2 \|u^t, \gamma_0(u^t)\|_{C([-h,0];\mathcal{V})}^2 + 2c_3 \left( \|u^t\|_{C([-h,0];L^p(\Omega))}^p + \|\gamma_0(u^t)\|_{C([-h,0];L^q(\Gamma))}^q \right)
\leq 2(c_3c_0 + 2\tilde{c}_0)(|\Omega| + |\Gamma|) + (c_3(2L_g + 1) + 2L_g^2) R_H^2(t)
\]
\[+ c_3\mu_1^{-1} e^{-\sigma(t-h-1)} \int_{t-1}^{t} e^{\sigma(s)} \left( \|\rho(s)\|_{\Omega}^2 + \|\rho_T(s)\|_{\Gamma}^2 \right) ds
\]
\[+ 2e^{-\sigma(t-h-1)} \int_{-\infty}^{t} e^{\sigma(s)} \left( \|\rho(s)\|_{\Omega}^2 + \|\rho_T(s)\|_{\Gamma}^2 \right) ds.
\]
This inequality implies the existence of a bounded pullback \( \mathcal{D}_\sigma(C([-h,0];H))-\)absorbing set in \( C([-h,0];\mathcal{V}) \) and in \( C([-h,0];L^p(\Omega) \times L^q(\Gamma)) \) for the process \( \{U(t,\tau)\}_{t \geq \tau} \). From this and note that \( A_{\mathcal{D}_\sigma(C([-h,0];H))} \in \mathcal{D}_\sigma(C([-h,0];H)) \) (see Theorem 4.6), then by the invariance of \( A_{\mathcal{D}_\sigma(C([-h,0];H))} \) we complete the proof of theorem. \( \square \)

**Remark 3.** We have just proved the pullback-attractor in \( C([-h,0];L^2(\Omega) \times L^2(\Gamma)) \) is also bounded in \( C([-h,0];\mathcal{V}) \) and in \( C([-h,0];L^p(\Omega) \times L^q(\Gamma)) \) via the existence of a pullback absorbing set in these spaces. However, it seems to be difficult to prove the asymptotic compactness of the process \( U(t,\tau) \) in \( C([-h,0];L^p(\Omega) \times L^q(\Gamma)) \) and \( C([-h,0];\mathcal{V}) \). The essential difficulty for the asymptotic compactness is that the process \( U(t,\tau) \) is now only continuous on the phase space \( C([-h,0];L^2(\Omega) \times L^2(\Gamma)) \).

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