Mapping cone of $k$-entanglement breaking maps

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Abstract
In Christandl et al. (Ann Henri Poincaré 20(7):2295–2322, 2019), the authors introduced $k$-entanglement breaking linear maps to understand the entanglement breaking property of completely positive maps on taking composition. In this article, we do a systematic study of $k$-entanglement breaking maps. We prove many equivalent conditions for a $k$-positive linear map to be $k$-entanglement breaking, thereby study the mapping cone structure of $k$-entanglement breaking maps. We discuss examples of $k$-entanglement breaking maps and some of their significance. As an application of our study, we characterize the completely positive maps that reduce Schmidt number on taking composition with another completely positive map. Finally, we extend a spectral majorization result for separable states.

Keywords Mapping cones · Dual cone · Tensor products · Positive maps · Completely positive maps · Entanglement breaking · Schmidt number

Mathematics Subject Classification 47L07 · 81P40 · 81P42

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1 Introduction

Completely positive maps that are PPT or entanglement breaking are of great importance in Quantum Information theory. In general, the problem of deciding whether a given linear map is entanglement breaking or not is computationally a hard one [11, 12]. Entanglement breaking maps are PPT-maps, but the converse is generally not true. It is a well-known conjecture [7] that the square of a PPT channel is entanglement breaking, and is known to have the affirmative answer in some cases ([5, 8]). To understand the entanglement property of maps on taking composition, in [8], the notion of $k$-entanglement breaking maps is introduced, where $k \in \mathbb{N}$. In this article, we study $k$-entanglement breaking maps in more detail. We generalize several known results about entanglement breaking maps into the setup of $k$-entanglement breaking maps. We hope that our study will help understanding entanglement breaking maps better and hence be of substantial interest.

We organize this article into seven sections. In Sect. 2, we recall some notations, basic definitions and known-results useful for later sections. In Sect. 3, we establish various characterizations of $k$-entanglement breaking maps. Making use of these, we prove (Theorem 3.10) that the set $\mathcal{EB}_k$ and $\mathcal{EBCP}_k$ of all $k$-entanglement breaking linear maps and $k$-entanglement breaking completely positive maps, respectively, form mapping cones. Further, we show that (Theorem 3.19) both the cones serve as examples of non-symmetric, untypical mapping cones.

Section 4 discuss examples of $k$-entanglement breaking maps and some of their significance in determining separability (Theorem 4.13) and entanglement (Corollary 4.14). Though there is no close relation between PPT-maps and $k$-entanglement breaking linear maps (Proposition 3.13 and Example 4.12), Theorem 4.10 provides a sufficient condition for a trace preserving positive map (in particular, PPT-map) to becomes a $k$-entanglement breaking map.

As an application, in Sect. 5, we exhibit the Schmidt number reducing property of 2-entanglement breaking completely positive maps. Motivated from work done in [29], we ask the following question, which is closely related to PPT-square conjecture: Suppose $\Phi$ is a non-entanglement breaking completely positive map on $\mathbb{M}_d$. Under what sufficient conditions on $\Phi$ does there exist $N \in \mathbb{N}$ such that $\Phi^N$ is entanglement breaking? In such cases, $1 = SN(\Phi^N) < SN(\Phi)$. This inequality motivates us to search for those completely positive maps, which reduce the Schmidt number after composing a finite number of times. In Theorem 5.2, we show that precisely those completely positive maps that are 2-entanglement breaking reduce the Schmidt number on composing with another completely positive map. Further, if $\Phi$ is a $k$-entanglement breaking completely positive map, Corollary 5.6 provides an upper bound for $N$.  

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In Sect. 6, we discuss a spectral majorization result for $k$-entanglement breaking maps. Section 7 discuss one open problem. In the “Appendix”, we prove the separability of a particular class of positive matrices, which we use in Sect. 4. As the proof involves a few technical lemmas, we write it as a separate section.

2 Notation and preliminaries

Throughout this article, we fix $d, d_1, d_2, d_3 \in \mathbb{N}$. Unless mentioned otherwise, $\{e_i\}_{i=1}^d \subseteq \mathbb{C}^d$ always denotes the standard orthonormal basis. We let $\mathcal{M}_{d_1 \times d_2}$ denote the space of all $d_1 \times d_2$ complex matrices and $I = I_d \in \mathcal{M}_d = \mathcal{M}_{d \times d}$ be the diagonal matrix with diagonals equals 1. By writing $A = [a_{ij}] \in \mathcal{M}_{d_1 \times d_2}$, we mean $A$ is a $d_1 \times d_2$ complex matrix with $(i, j)^{th}$ entry equals $a_{ij}$ for all $1 \leq i \leq d_1, 1 \leq j \leq d_2$.

Further, we let $T = T_d : \mathcal{M}_d \rightarrow \mathcal{M}_d$ denote the transpose map, $\text{tr} : \mathcal{M}_d \rightarrow \mathbb{C}$ the trace map, and $\text{id} = \text{id}_d : \mathcal{M}_d \rightarrow \mathcal{M}_d$ the identity map. The cone of all positive (semidefinite) matrices in $\mathcal{M}_d$ is denoted by $\mathcal{M}_d^+$. If $A \in \mathcal{M}_d^+$, then we write $A \geq 0$. Given $x \in \mathbb{C}^{d_1}, y \in \mathbb{C}^{d_2}$, define the mapping $\langle x \rangle : \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_1}$ by $\langle x \rangle (z) := x(y, z)$ for all $z \in \mathbb{C}^{d_2}$. Note that $|x(x) \geq 0$.

We let $\Omega_d = \sum_{i=1}^d e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d$. Given a unit vector $\xi \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ there always exist orthonormal sets $\{u_i\}_{i=1}^d \subseteq \mathbb{C}^{d_1}$ and $\{v_i\}_{i=1}^d \subseteq \mathbb{C}^{d_2}$ and scalars $\lambda_i \in [0, 1]$ with $\sum_i \lambda_i^2 = 1$ such that $\xi = \sum_{i=1}^d \lambda_i (u_i \otimes v_i)$, called a Schmidt decomposition [32] of the vector $\xi$, where $d = \min(d_1, d_2)$. The number of non-zero coefficients in any two Schmidt decomposition of $\xi$ is the same, and this unique number is called the Schmidt rank of $\xi$ and is denoted by $SR(\xi)$.

Definition 2.1 Let $X \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+$. If there exist $A_i \in \mathcal{M}_{d_1}^+$ and $B_i \in \mathcal{M}_{d_2}^+$, $1 \leq i \leq n$, such that $X = \sum_{i=1}^n A_i \otimes B_i$, then $X$ is said to be separable, otherwise called entangled.

Though sufficient and necessary conditions for separability are known [16], determining whether a given positive matrix is separable or not is very hard in general. In this article, we often use Schmidt number techniques to handle this situation. The Schmidt number [40] of a positive matrix $X \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+$ is defined as

$$SN(X) := \min \left\{ \max \left\{ |\phi(\xi_i) : X = \sum_{i=1}^n |\xi_i \rangle \langle \xi_i | \right\} : \xi_i \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}, n \geq 1 \right\}.$$

Clearly $SN(|\phi(\xi) \rangle \langle \xi |) = SR(\xi)$ for all $\xi \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, and $SN(\sum_{i=1}^n X_i) \leq \max_i (SN(X_i))$, where $X_i \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+$, $1 \leq i \leq n$. Note that $X$ is separable if and only if $SN(X) = 0$. From the definition of Schmidt number, $SN(X) \leq \min(d_1, d_2)$.

A linear map $\Phi : \mathcal{M}_{d_1} \rightarrow \mathcal{M}_{d_2}$ is said to be positive if $\Phi(\mathcal{M}_{d_1}^+) \subseteq \mathcal{M}_{d_2}^+$. Given $k \geq 1$, $\Phi$ is said to be $k$-positive if $\text{id}_k \otimes \Phi : \mathcal{M}_k \otimes \mathcal{M}_{d_1} \rightarrow \mathcal{M}_k \otimes \mathcal{M}_{d_2}$ is a positive map. If both $\Phi$ and $T \circ \Phi$ (equivalently $\Phi \circ T$) are $k$-positive, then $\Phi$ is called a $k$-PPT map. If $\Phi$ is $k$-positive for every $k \geq 1$, then $\Phi$ is called a completely positive (CP-)map. Every CP-map $\Phi : \mathcal{M}_{d_1} \rightarrow \mathcal{M}_{d_2}$ has a Choi-Kraus decomposition [6, 24], i.e., there exist $V_1, V_2, \ldots, V_n \in \mathcal{M}_{d_1 \times d_2}$, called Kraus operators, such that $\Phi = \sum_{i=1}^n \text{Ad}_{V_i}$, where...
Ad_{V}(X) := V^{*}XV \text{ for all } X \in \mathbb{M}_{d_{1}} \text{ and } V \in \mathbb{M}_{d_{1} \times d_{2}}. \text{ A linear map } \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \text{ is said to be completely co-positive (co-CP) if } T \circ \Phi \text{ (equivalently } \Phi \circ T) \text{ is a CP-map. A linear map that is both CP and co-CP is called a PPT-map.}

Given \( A = [a_{ij}] \in \mathbb{M}_{d_{1}} \) and \( B = [b_{ij}] \in \mathbb{M}_{d_{2}} \), we let \( A \otimes B = [a_{ij} B] \) and thereby identify \( \mathbb{M}_{d_{1}} \otimes \mathbb{M}_{d_{2}} = \mathbb{M}_{d_{1}}(\mathbb{M}_{d_{2}}) \). To every linear map \( \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \) associate the matrix

\[
C_{\Phi} := (\text{id}_{d_{1}} \otimes \Phi) \left( \sum_{i,j=1}^{d_{1}} E_{ij} \otimes E_{ij} \right) = [\Phi(E_{ij})] \in \mathbb{M}_{d_{1}} \otimes \mathbb{M}_{d_{2}} = \mathbb{M}_{d_{1}}(\mathbb{M}_{d_{2}}),
\]

where \( E_{ij} = |e_{i}\rangle \langle e_{j}| \in \mathbb{M}_{d_{1}}, \ 1 \leq i, j \leq d_{1} \), are the standard matrix units. The map \( \Phi \mapsto C_{\Phi} \) is an isomorphism (called Choi-Jamiolkowski isomorphism [6, 21]) from the space \( \{ \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \} \) onto the space \( \mathbb{M}_{d_{1}} \otimes \mathbb{M}_{d_{2}} = \mathbb{M}_{d_{1}}(\mathbb{M}_{d_{2}}) \). Further, \( \Phi \) is a CP-map if and only if \( \Phi \) is \( d_{1} \)-positive if and only if \( C_{\Phi} \in (\mathbb{M}_{d_{1}} \otimes \mathbb{M}_{d_{2}})^{+} \); and \( \Phi \) is PPT if and only if \( C_{\Phi} \) is a positive matrix with positive partial transpose (i.e., \( (\text{id}_{d_{1}} \otimes T)(C_{\Phi}) \geq 0 \)). The Schmidt number of a CP-map \( \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \) is denoted and defined \([9, 19]\) as \( SN(\Phi) := SN(C_{\Phi}) \). It is known \([9]\) that

\[
SN(\Phi) = \min \left\{ \max \left\{ \text{rank}(V_{i}) : \Phi = \sum_{i=1}^{n} AdV_{i} \right\} : V_{i} \in \mathbb{M}_{d_{1} \times d_{2}}, n \geq 1 \right\}.
\]

Given a linear map \( \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \), we let \( \Phi^{*} \) denotes the dual of \( \Phi \) w.r.t the Hilbert Schmidt inner product \( \langle X, Y \rangle := \text{tr}(X^{*}Y) \). From the Choi-Kraus decomposition, it follows that

\[
\begin{align*}
& \bullet \ SN(\Phi) = SN(\Phi^{*}), \\
& \bullet \ SN(\Phi \circ \Psi) \leq \min \left\{ SN(\Phi), SN(\Psi) \right\}, \\
& \bullet \ SN(\Phi + \Psi) \leq \max \left\{ SN(\Phi), SN(\Psi) \right\},
\end{align*}
\]

where \( \Phi, \Psi \) are CP-maps.

**Definition 2.2** A linear map \( \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \) is said to be entanglement breaking (EB) if \( (\text{id}_{k} \otimes \Phi)(X) \) is separable for all \( X \in (\mathbb{M}_{k} \otimes \mathbb{M}_{d_{1}})^{+} \) and \( k \geq 1 \).

Note that an EB-map is necessarily a CP-map; in fact, they are PPT-maps. Various characterizations of EB-maps are known in the literature.

**Theorem 2.3** ([14, 18]) Let \( \Phi : \mathbb{M}_{d_{1}} \rightarrow \mathbb{M}_{d_{2}} \) be a CP-map. Then the following conditions are equivalent:

(i) \( \Phi \) is EB.
(ii) \( \Gamma \circ \Phi \) is CP for all positive maps \( \Gamma : \mathbb{M}_{d_{2}} \rightarrow \mathbb{M}_{n}, n \geq 1 \).
(iii) \( \Gamma \circ \Phi \) is CP for all positive maps \( \Gamma : \mathbb{M}_{d_{2}} \rightarrow \mathbb{M}_{d_{1}} \).
(iv) \( \Phi \circ \Gamma \) is CP for all positive maps \( \Gamma : \mathbb{M}_{n} \rightarrow \mathbb{M}_{d_{1}}, n \geq 1 \).
(v) \( \Phi \circ \Gamma \) is CP for all positive maps \( \Gamma : \mathbb{M}_{d_{2}} \rightarrow \mathbb{M}_{d_{1}} \).
(vi) \( C_{\Phi} \in (\mathbb{M}_{d_{1}} \otimes \mathbb{M}_{d_{2}})^{+} \) is separable.
(vii) \( \Phi \) admits a set of rank one Kraus operators.
(viii) (Holevo form:) There exist \( F_i \in M_{d_1}^+ \) and \( R_i \in M_{d_2}^+ \) such that \( \Phi(X) = \sum_{i=1}^m \text{tr}(X F_i) R_i \) for all \( X \in M_{d_1}^+ \).

One may also consider CP-maps that have Kraus operators of rank less than or equal to some \( k > 1 \). A CP-map \( \Phi : M_{d_1} \rightarrow M_{d_2} \) is said to be \( k \)-partially entanglement breaking (\( k \)-PEB) if it has Kraus operators of rank less than or equal to \( k \) (equivalently \( SN(\Phi) \leq k \)). Refer [9] for details. EB-maps and \( k \)-PEB maps are also known as superpositive maps and \( k \)-superpositive maps, respectively, in the literature (See [1, 34]).

**Definition 2.4** ([33, 35]) Let \( \mathcal{P}(d_1, d_2) \) denotes the cone of all positive linear maps from \( M_{d_1} \) into \( M_{d_2} \). A mapping cone is a closed convex cone \( C \subseteq \mathcal{P}(d_1, d_2) \) such that \( \Phi \circ \Theta \circ \Psi \in C \) for all \( \Theta \in \mathcal{C} \), and CP-maps \( \Psi : M_{d_1} \rightarrow M_{d_1} \) and \( \Phi : M_{d_2} \rightarrow M_{d_2} \). The dual cone of a mapping cone \( C \) is defined by

\[
C^\circ := \{ \Gamma \in \mathcal{P}(d_1, d_2) : \text{tr}(C \Gamma C_\Theta) \geq 0 \text{ for all } \Theta \in \mathcal{C} \}.
\]

A mapping cone is said to be invariant if \( \Gamma_2 \circ \Theta \circ \Gamma_1 \in C \) for all \( \Theta \in \mathcal{C} \) and \( \Gamma_i \in \mathcal{P}(d_1, d_1), i = 1, 2 \).

If \( C \) is a mapping cone, then the dual cone \( C^\circ \) is also a mapping cone. Further, if \( C \) is invariant, then so is \( C^\circ \). The following are some well-known examples of mapping cones of \( \mathcal{P}(d_1, d_2) \):

\[
\begin{align*}
\mathcal{P}_k(d_1, d_2) &:= \{ k \text{-positive maps from } M_{d_1} \text{ into } M_{d_2} \}, \\
\mathcal{CP}(d_1, d_2) &:= \{ \text{CP-maps from } M_{d_1} \text{ into } M_{d_2} \}, \\
\mathcal{PEB}_k(d_1, d_2) &:= \{ k \text{-PEB maps from } M_{d_1} \text{ into } M_{d_2} \}, \\
\mathcal{EB}(d_1, d_2) &:= \{ \text{EB-maps from } M_{d_1} \text{ into } M_{d_2} \},
\end{align*}
\]

where \( k > 1 \). (If \( d_1 = d_2 = d \), then we write \( \mathcal{P}_k(d), \mathcal{CP}(d), \mathcal{PEB}_k(d), \mathcal{EB}(d) \), respectively.) It is known that \( (C^\circ)^\circ = C \) for every mapping cone \( C \), and

\[
\begin{align*}
\mathcal{P}(d_1, d_2)^\circ = \mathcal{EB}(d_1, d_2) & \quad \text{(hence } \mathcal{EB}(d_1, d_2)^\circ = \mathcal{P}(d_1, d_2) \text{)} \quad (2.1) \\
\mathcal{P}_k(d_1, d_2)^\circ = \mathcal{PEB}_k(d_1, d_2) & \quad \text{(hence } \mathcal{PEB}_k(d_1, d_2)^\circ = \mathcal{P}_k(d_1, d_2) \text{)} \quad (2.2) \\
\mathcal{CP}(d_1, d_2)^\circ = \mathcal{CP}(d_1, d_2) & \quad \text{(2.3)}
\end{align*}
\]

The following much more generalized result is known. See [33, 34, 36, 37] for details.

**Theorem 2.5** Let \( C \subseteq \mathcal{P}(d_1, d_2) \) be a mapping cone and \( \Gamma \in \mathcal{P}(d_1, d_2) \). Then the following conditions are equivalent:

(i) \( \Gamma \in C^\circ \).

(ii) \( \Gamma \circ \Theta^* \in \mathcal{CP}(d_2) \) for all \( \Theta \in C \).

(iii) \( \Theta^* \circ \Gamma \in \mathcal{CP}(d_1) \) for all \( \Theta \in C \).

Further, if \( d_1 = d_2 = d \) and \( C \) is \( * \)-invariant, then the above conditions are equivalent to:

(iv) \( \Gamma \circ \Theta \in \mathcal{CP}(d) \) for all \( \Theta \in C \).

(v) \( \Theta \circ \Gamma \in \mathcal{CP}(d) \) for all \( \Theta \in C \).
3 Mapping cone of \( k \)-entanglement breaking maps

**Definition 3.1** ([8]) Let \( k \in \mathbb{N} \). A linear map \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) is said to be \( k \)-entanglement breaking (\( k \)-EB) if \( \Phi \) is a \( k \)-positive map and \( (\text{id}_k \otimes \Phi)(X) \) is separable for every \( X \in (\mathbb{M}_k \otimes \mathbb{M}_{d_1})^+ \).

Note that a linear map \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) is EB if and only if \( \Phi \) is \( 1 \)-EB for every \( k \geq 1 \). Further, from Theorem 2.3(vi), this is equivalent to saying that \( \Phi \) is \( d_1 \)-EB.

**Remark 3.2** Let \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) be a \( k \)-EB map.

(i) If \( m < k \), then \( \Phi \) is \( m \)-EB also. For, let \( X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+ = (\mathbb{M}_m (\mathbb{M}_{d_1}))^+ \).

Then there exist \( A_i \in \mathbb{M}_k^+ \) and \( B_i \in \mathbb{M}_{d_2}^+ \) such that

\[
\begin{bmatrix}
(id_m \otimes \Phi)(X) & 0 \\
0 & 0_{k-m}
\end{bmatrix} = (id_k \otimes \Phi) \begin{bmatrix}
X & 0 \\
0 & 0_{k-m}
\end{bmatrix} = \sum A_i \otimes B_i.
\]

Writing \( A_i = \begin{bmatrix}
A_{11}^{(i)} & A_{12}^{(i)} \\
A_{21}^{(i)} & A_{22}^{(i)}
\end{bmatrix} \in \mathbb{M}_k^+ \), where \( A_{11}^{(i)} \in \mathbb{M}_m^+ \) and \( A_{22}^{(i)} \in \mathbb{M}_{k-m}^+ \), from above we get

\[(id_m \otimes \Phi)(X) = \sum A_{11}^{(i)} \otimes B_i \in (\mathbb{M}_m \otimes \mathbb{M}_{d_2})^+,
\]

so that \( (id_m \otimes \Phi)(X) \) is separable.

(ii) Let \( m \geq 1 \). Then, from the definition, it follows that both \( \Gamma_2 \circ \Phi \) and \( \Phi \circ \Gamma_1 \) are \( k \)-EB maps for all \( \Gamma_2 \in \mathcal{P}(d_2, m) \) and \( \Gamma_1 \in \mathcal{P}(k, d_1) \).

In this section, our main aim is to prove an analog of Theorem 2.3 in the context of \( k \)-EB maps. First, we prove some technical lemmas. Though it did not explicitly state, the following lemma was observed in [8].

**Lemma 3.3** Let \( X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+ \). Then for \( j = 1, 2, \ldots \), \( S \mathcal{N}(X) \) there exist family of isometries \( V_{ji} : \mathbb{C}^j \to \mathbb{C}^m \) and vectors \( \psi_{ji} \in \mathbb{C}^j \otimes \mathcal{C}_{d_1} \) such that

\[X = \sum_{i,j} (V_{ji} \otimes I_{d_1}) \langle \psi_{ji} \rangle |\psi_{ji} \rangle (V_{ji} \otimes I_{d_1})^* \text{.}\]

**Proof** Let \( r := S \mathcal{N}(X) \). Then there exist \( z_i \in \mathbb{C}^m \otimes \mathcal{C}_{d_1} \) with \( S \mathcal{R}(z_i) \leq r \) such that

\[X = \sum_i |z_i \rangle \langle z_i| = \sum_{j=1}^r \sum_{SR(z_i)=j} |z_i \rangle \langle z_i| \text{.} \tag{3.1}\]

Note that whenever \( S \mathcal{R}(z_i) = j \), from [8, Lemma 1.2], there exist isometry \( V_{ji} : \mathbb{C}^j \to \mathbb{C}^m \) and a vector \( \psi_{ji} \in \mathbb{C}^j \otimes \mathcal{C}_{d_1} \) such that \( z_i = (V_{ji} \otimes I_{d_1}) \psi_{ji} \). Thus

\[X = \sum_{j=1}^r \sum_i (V_{ji} \otimes I_{d_1}) |\psi_{ji} \rangle |\psi_{ji} \rangle (V_{ji} \otimes I_{d_1})^* \text{.}\]
In (3.1), if there is no \( z_i \) with \( SR(z_i) = j \), then the corresponding term is taken to be zero, and take \( \psi_{ji} = 0 \).  

The following result is known in literature (c.f \([31, 33, 34, 40]\)) but we could not find out in the following form (except \((i) \Leftrightarrow (vi)\)), hence providing proof.

**Lemma 3.4** Let \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) be a positive map and \( k \geq 1 \). Then the following conditions are equivalent:

(i) \( \Phi \) is \( k \)-positive.

(ii) \((\text{id}_m \otimes \Phi)(X) \in (\mathbb{M}_m \otimes \mathbb{M}_{d_2})^+ \) for all \( X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+ \) with \( SN(X) \leq k \) and \( m \geq 1 \).

(iii) \((\text{id}_{d_2} \otimes \Phi)(X) \in (\mathbb{M}_{d_2} \otimes \mathbb{M}_{d_2})^+ \) for all \( X \in (\mathbb{M}_{d_2} \otimes \mathbb{M}_{d_2})^+ \) with \( SN(X) \leq k \).

(iv) \( \Phi \circ \Psi \) is CP for every \( k \)-PEB map \( \Psi : \mathbb{M}_m \to \mathbb{M}_{d_1} \) and \( m \geq 1 \).

(v) \( \Phi \circ \Psi \) is CP for every \( k \)-PEB map \( \Psi : \mathbb{M}_{d_2} \to \mathbb{M}_{d_2} \).

**Proof** The implications \((ii) \Rightarrow (i), (ii) \Rightarrow (iii), (iv) \Rightarrow (v)\) are trivial, and \((v) \Rightarrow (i)\) follows from \([33, \text{Theorem } 3]\). To prove \((i) \Rightarrow (ii)\), let \( X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+ \). If \( m \leq k \) there is nothing to prove. So assume \( m > k \), and let \( r := SN(X) \leq k \). Then, from Lemma 3.3, there exist isometries \( V_{ji} : \mathbb{C}^j \to \mathbb{C}^m \) and a vectors \( \psi_{ji} \in \mathbb{C}^j \otimes \mathbb{C}^d_1 \), where \( j \leq r \) such that

\[
(\text{id}_m \otimes \Phi)(X) = (\text{id}_m \otimes \Phi) \left( \sum_{i,j} (V_{ji} \otimes I_{d_1}) |\psi_{ji}\rangle \langle \psi_{ji}| (V_{ji} \otimes I_{d_1})^* \right).
\]

\[
= \sum_{i,j} (V_{ji} \otimes I_{d_2}) ((\text{id}_j \otimes \Phi) (|\psi_{ji}\rangle \langle \psi_{ji}|)) (V_{ji} \otimes I_{d_2})^*. \tag{3.2}
\]

Since \( \Phi \) is \( k \)-positive, from the above equation, we conclude that \((\text{id}_m \otimes \Phi)X \) is positive. Finally, \((ii) \Leftrightarrow (iv)\) and \((iii) \Leftrightarrow (v)\) follows from the Choi-Jamiolkowski isomorphism as \( C_{\Phi \circ \Psi} = (\text{id}_m \otimes \Phi)(C_{\Psi}) \) and \( SN(C_{\Psi}) = SN(\Psi) \) whenever \( \Psi \in CP(m, d_1) \).  

Now we are ready to prove the following analog of Theorem 2.3.

**Theorem 3.5** Let \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) be a \( k \)-positive map, where \( k > 1 \). Then the following conditions are equivalent:

(i) \( \Phi \) is \( k \)-entanglement breaking.

(ii) \( \Phi \circ \Psi \) is entanglement breaking for all CP-maps \( \Psi : \mathbb{M}_k \to \mathbb{M}_{d_1} \).

(iii) \( \Phi \circ \Psi \) is entanglement breaking for all CP-maps \( \Psi : \mathbb{M}_m \to \mathbb{M}_{d_1} \) and \( 1 \leq m \leq k \).

(iv) \( \Phi \circ \Psi \) is entanglement breaking for all \( k \)-PEB maps \( \Psi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_1} \).

(v) \( \Phi \circ \Psi \) is entanglement breaking for all \( k \)-PEB maps \( \Psi : \mathbb{M}_m \to \mathbb{M}_{d_1} \) and \( m \geq 1 \).

(vi) \( \Phi \circ \text{Ad}_P \) is entanglement breaking for all projections \( P \in \mathbb{M}_{d_1} \) with \( \text{rank}(P) = \min[k, d_1] \).

(vii) \( SN((\text{id}_m \otimes \Phi)(X)) = 1 \) for all \( X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+ \) with \( SN(X) \leq k \) and \( m \geq 1 \).

(viii) \( SN((\text{id}_{d_1} \otimes \Phi)(X)) = 1 \) for all \( X \in (\mathbb{M}_{d_1} \otimes \mathbb{M}_{d_1})^+ \) with \( SN(X) \leq k \).
(ix) $\Gamma \circ \Phi$ is $k$-positive for all positive maps $\Gamma : M_{d_2} \to M_m$ and $m \geq 1$.

(x) $\Gamma \circ \Phi$ is $k$-positive for all positive maps $\Gamma : M_{d_2} \to M_{d_1}$.

(xi) $\text{tr}(C_{\Phi} C_{\Gamma \circ \Psi}) \geq 0$ for all $k$-PEB maps $\Psi : M_{d_1} \to M_{d_1}$ and positive maps $\Gamma : M_{d_1} \to M_{d_2}$.

**Proof** (i)$\Rightarrow$(iii). Let $\Psi \in CP(m, d_1)$. Since $\Phi$ is $m$-EB $C_{\Phi \circ \Psi} = (\text{id}_m \otimes \Phi)(C_{\Psi})$ is separable, so that $\Phi \circ \Psi$ is an EB-map.

(iii)$\Rightarrow$(ii). Clear.

(ii)$\Rightarrow$(i) Let $X \in (M_k \otimes M_{d_1})^+$ and $\Psi \in CP(k, d_1)$ be such that $C_{\Psi} = X$. From assumption, $(\text{id}_k \otimes \Phi)(X) = C_{\Phi \circ \Psi}$ is separable, so that $\Phi$ is a $k$-EB map.

(i)$\Rightarrow$(vii) Let $X \in (M_m \otimes M_{d_1})^+$ with $r := SN(X) \leq k$. Then, by Lemma 3.4, $(\text{id}_m \otimes \Phi)(X)$ is positive. Further, by Lemma 3.3, there exist isometries $V_{ji} : C^j \to C^m$ and vectors $\psi_{ji} \in C^j \otimes C^{d_1}$, $1 \leq j \leq r$ such that

$$SN((\text{id}_m \otimes \Phi)(X)) = SN\left(\sum_{i,j} (V_{ji} \otimes I_{d_2})(\text{id}_j \otimes \Phi)(|\psi_{ji}\rangle\langle\psi_{ji}|)(V_{ji} \otimes I_{d_2})^*\right)$$

$$\leq \max_{i,j} \{SN((V_{ji} \otimes I_{d_2})(\text{id}_j \otimes \Phi)(|\psi_{ji}\rangle\langle\psi_{ji}|)(V_{ji} \otimes I_{d_2})^*)\}$$

$$\leq \max_{i,j} \{SN((\text{id}_j \otimes \Phi)(|\psi_{ji}\rangle\langle\psi_{ji}|))\}$$

$$= 1.$$

The last equality follows as $\Phi$ is $j$-entanglement breaking for all $1 \leq j \leq k$.

(vii)$\Leftrightarrow$(viii) Clear.

(viii)$\Rightarrow$(iv) Follows from Choi-Jamiolkowski isomorphism.

(iv)$\Rightarrow$(i) We prove this in two cases.

Case (1): Suppose $k \geq d_1$. Then $\Psi = \text{id}_{d_1}$ is a $k$-PEB map, hence by assumption $\Phi = \Phi \circ \Psi$ is an EB-map. In particular, $\Phi$ is a $k$-EB map.

Case (2): Suppose $k < d_1$. Let $X \in (M_k \otimes M_{d_1})^+$. Without loss of generality assume that $X$ is of rank one; hence there exists $V \in M_{k \times d_1}$ such that $X = C_{\text{Adv}_V}$. Let $\iota : C^k \to C^{d_1}$ be the inclusion map. Note that $SN(\text{Adv}_{\iota V}) = \text{rank}(\iota \circ V) \leq k$. Thus, $\text{Adv}_{\iota V} \in PEB_k(d_1)$, hence by assumption $C_{\Phi \circ \text{Adv}_{\iota V}}$ is separable. Therefore,

$$(\text{id}_k \otimes \Phi)(X) = C_{\Phi \circ \text{Adv}_{\iota V}} = (\iota \otimes I_{d_2})^* C_{\Phi \circ \text{Adv}_{\iota V}} (\iota \otimes I_{d_2})$$

is separable. Since $X \in (M_k \otimes M_{d_1})^+$ is arbitrary we conclude that $\Phi$ is $k$-EB.

(vii)$\Leftrightarrow$(v) Follows from Choi-Jamiolkowski isomorphism and Lemma 3.4.

(i)$\Rightarrow$(ix) Let $\Gamma \in P(d_2, m)$, where $m \geq 1$ and $X \in (M_k \otimes M_{d_1})^+$. By assumption $(\text{id}_k \otimes \Phi)(X)$ is separable, hence there exist $A_i \in M_k^+$ and $B_i \in M_{d_1}^+$ such that $(\text{id}_k \otimes \Phi)(X) = \sum_i A_i \otimes B_i$. Then, $(\text{id}_k \otimes \Gamma \circ \Phi)(X) = \sum_i A_i \otimes (\Gamma(B_i)) \geq 0$. We conclude that $\Gamma \circ \Phi$ is $k$-positive.

(ix)$\Rightarrow$(x) Clear.

(x)$\Rightarrow$(vi) Let $P \in M_{d_1}$ with $\text{rank}(P) = \text{min}(k, d_1)$. Then $\text{Adv}_P \in PEB_k(d_1)$. If $\Gamma \in P(d_2, d_1)$, then by assumption $\Gamma \circ \Phi \in P_k(d_1)$ and hence from (2.2) and Theorem 2.5 we have $\Gamma \circ \Phi \circ \text{Adv}_P \in CP(d_1)$. Since $\Gamma$ is arbitrary, from Theorem 2.3, we conclude that $\Phi \circ \text{Adv}_P$ is EB.
(vi) ⇒ (iv) If \( k \geq d_1 \), then by assumption \( \Phi = \Phi \circ \text{Ad}_{d_1} \) is EB so that \( \Phi \circ \Psi \) is also EB for every \( \Psi \in \mathcal{PB}(m, d_1) \) and \( m \geq 1 \). Now suppose \( k < d_1 \). We show that \( \Phi \circ \text{Ad}_V \) is EB for all \( V \in \mathbb{M}_{d_1} \) with rank\( (V) \leq k \). By singular value decomposition there exists unitaries \( U, U' \in \mathbb{M}_{d_1} \) and a diagonal matrix \( D \in \mathbb{M}_r \) such that \( V = U^* \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U' \), where \( r = \text{rank}(V) \). Then \( V = XP \), where \( X = U^* \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U' \) and \( P = U^* \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} U' \), which is a projection of rank \( k \). By assumption \( \Phi \circ \text{Ad}_P \) is EB, and hence \( \Phi \circ \text{Ad}_V = \Phi \circ \text{Ad}_P \circ \text{Ad}_X \) is also EB. Since \( V \in \mathbb{M}_{d_1} \) is arbitrary we have \( \Phi \circ \Psi \) is EB for all \( \Psi \in \mathcal{PB}(d_1) \).

\[ \square \]

**Remark 3.6** From (iv) it follows that if \( \Phi \) is \( k \)-EB, then \( \Phi \circ \text{Ad}_P \) is entanglement breaking for all projections \( P \in \mathbb{M}_{d_1} \) with rank less than or equal to \( k \). The converse is also true because of (vi). This equivalence was observed in [3, Lemma 6.1].

From Theorem 2.3, we have \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) is an EB-map if and only if the Choi matrix \( C_\Phi = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij}) \) is separable, where \( \{E_{ij}\}_{i,j=1}^{d_1} \) is the standard matrix units in \( \mathbb{M}_{d_1} \). Next we prove an analogue of this result in the context of \( k \)-EB maps. Suppose \( \{F_{ij}\}_{i,j=1}^{d_1} \) is a complete set of matrix units in \( \mathbb{M}_{d_1} \) (i.e., there exists an orthonormal basis \( \{f_i\}_{i=1}^{d_1} \) for \( \mathbb{C}^d \) such that \( F_{ij} = |f_i \rangle \langle f_j| \) and \( U \in \mathbb{M}_{d_1} \) is a unitary such that \( UF_{ij}U^* = E_{ij} \), for all \( 1 \leq i, j \leq d_1 \). Then, by [38, Lemma 4.1.2], the Choi matrix \( C_\Phi^F \) w.r.t \( \{F_{ij}\}_{i,j} \) is given by

\[
C_\Phi^F := \sum_{i,j=1}^{d_1} F_{ij} \otimes \Phi(F_{ij}) = \text{Ad}_{U \otimes 1_2} C_{\Phi \circ \text{Ad}_U}.
\]

**Definition 3.7** Suppose \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) is a linear map and \( \{F_{ij}\}_{i,j=1}^{d_1} \) is a complete set of matrix units in \( \mathbb{M}_{d_1} \). Given \( 1 \leq k \leq d_1 \), \( i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d_1\} \) the matrix

\[
C_{\Phi(i_1, \ldots, i_k)}^F := \sum_{p,q=1}^k E_{ij}^{(k)} \otimes \Phi(F_{ip}^{(k)}) = \begin{bmatrix} \Phi(F_{i_1i_1}) & \Phi(F_{i_1i_2}) & \cdots & \Phi(F_{i_1i_k}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(F_{i_ki_1}) & \Phi(F_{i_ki_2}) & \cdots & \Phi(F_{i_ki_k}) \end{bmatrix}
\]

in \( \mathbb{M}_k \otimes \mathbb{M}_{d_2} = \mathbb{M}_k(\mathbb{M}_{d_2}) \) is called a principal \((k \times k)\)-block submatrix of \( C_\Phi^F \), where \( E_{ij}^{(k)} \in \mathbb{M}_k \) is the standard matrix units. The principal \((k \times k)\)-block submatrix \( C_{\Phi(i_1, \ldots, i_k)}^E \) of \( C_\Phi \) w.r.t the standard matrix units \( \{E_{ij}\}_{i,j} \subseteq \mathbb{M}_{d_1} \) is denoted simply by \( C_{\Phi(i_1, \ldots, i_k)} \).

Let \( \{e_i^{(k)}\}_{i=1}^k \) be the standard orthonormal basis for \( \mathbb{C}^k \) and \( \{f_i\}_{i=1}^{d_1} \) be an orthonormal basis for \( \mathbb{C}^{d_1} \) such that \( F_{ij} = |f_i \rangle \langle f_j| \). Then

\[
C_{\Phi(i_1, \ldots, i_k)}^F = (id_k \otimes \Phi) \left( \sum_{p,q=1}^k E_{pq}^{(k)} \otimes F_{ip}^{(k)} \right) = (id_k \otimes \Phi) \left( \sum_{p=1}^k e_p^{(k)} \otimes f_p \right) \left( \sum_{p=1}^k e_p^{(k)} \otimes f_p \right).
\]
Note that \( \sum_{p=1}^{k} e_p^{(k)} \otimes f_i \) is positive (resp. separable) if \( \Phi \) is \( k \)-positive (resp. \( k \)-EB).

**Theorem 3.8** Let \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) be a \( k \)-positive map, where \( 1 \leq k \leq d_1 \). Then the following conditions are equivalent:

(i) \( \Phi \) is \( k \)-EB.

(ii) Given any complete set of matrix units \( \{ F_{ij} \}_{i,j=1}^{d_1} \subseteq \mathbb{M}_{d_1} \), every principal \( (k \times k) \)-block submatrix \( C_{\Phi(i_1, \ldots, i_k)}^{F} \) of the Choi matrix \( C_{\Phi}^{F} \) is separable.

(iii) Given any complete set of matrix units \( \{ F_{ij} \}_{i,j=1}^{d_1} \subseteq \mathbb{M}_{d_1} \), the principal \( (k \times k) \)-block submatrix \( C_{\Phi(1, 2, \ldots, k)}^{F} \) of the Choi matrix \( C_{\Phi}^{F} \) is separable.

**Proof** Clearly (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii). To prove (iii) \( \Rightarrow \) (i). So let \( P \in \mathbb{M}_{d_1} \) be a projection of rank \( k \) and \( \tilde{\Phi} = \Phi \circ \text{Ad} \). Suppose \( \{ f_i \}_{i=1}^{d_1} \subseteq \mathbb{C}^{d_1} \) is an orthonormal basis such that \( P = \sum_{i=1}^{k} | f_i \rangle \langle f_i | \). Let \( F_{ij} = | f_i \rangle \langle f_j | \) and \( U \in \mathbb{M}_{d_1} \) be a unitary such that \( UF_{ij}U^* = E_{ij} \in \mathbb{M}_{d_1} \) for all \( 1 \leq i, j \leq d_1 \). Since \( \{ F_{ij} \} \) is a complete set of matrix units, from (3.3), we have

\[
C_{\tilde{\Phi} \circ \text{Ad} U} = \text{Ad} U^* \otimes I_{d_2} C_{\Phi}^{F} = \text{Ad} U^* \otimes I_{d_2} \left( \sum_{i,j=1}^{d_1} F_{ij} \otimes \tilde{\Phi} (F_{ij}) \right)
\]

\[
= \text{Ad} U^* \otimes I_{d_2} \left( \sum_{i,j=1}^{k} F_{ij} \otimes \Phi (F_{ij}) \right).
\]

Now let \( W \in \mathbb{M}_{d_1 \times k} \) be such that \( W (e_i^{(k)}) = f_i \) for all \( 1 \leq i \leq k \). Then

\[
C_{\Phi \circ \text{Ad} U_{W}} = \text{Ad} U_{W}^* \otimes I_{d_2} (W \otimes I_{d_2}) \left( \sum_{i,j=1}^{k} E_{ij}^{(k)} \otimes \Phi (F_{ij}) \right) (W \otimes I_{d_2})^*
\]

\[
= (U W \otimes I_{d_2}) C_{\Phi(1, 2, \ldots, k)}^{F} (U W \otimes I_{d_2})^*.
\]

Since \( C_{\Phi(1, 2, \ldots, k)}^{F} \) is separable we conclude that \( C_{\Phi \circ \text{Ad} U_{W}}^{F} \) is separable. Hence \( \Phi \circ \text{Ad} P = \tilde{\Phi} \circ \text{Ad} U_{W} \circ \text{Ad} U_{W}^* \) is an EB-map. Further, since \( P \) is arbitrary, by Theorem 3.5, \( \Phi \) is \( k \)-EB. \( \square \)

**Remark 3.9** If \( \Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2} \) is a \( k \)-EB map, where \( k \geq 1 \), then \( T \circ \Phi \circ T \) is also a \( k \)-EB map. For, let \( \Psi \in \mathcal{CP}(k, d_1) \). Since \( T \circ \Psi \circ T \in \mathcal{CP}(k, d_1) \), by Theorem 3.5, \( \Phi \circ (T \circ \Psi \circ T) \) is EB. Hence \( T \circ \Phi \circ T \circ \Psi = T \circ (\Phi \circ T \circ \Psi \circ T) \circ T \) is also an EB-map. Since \( \Psi \) is arbitrary, by Theorem 3.5, \( T \circ \Phi \circ T \) is \( k \)-EB.

**Theorem 3.10** Let \( \mathcal{EB}_{k}(d_1, d_2) \) and \( \mathcal{EBCP}_{k}(d_1, d_2) \) be the set of all \( k \)-entanglement breaking linear maps and \( k \)-entanglement breaking CP-maps from \( \mathbb{M}_{d_1} \) into \( \mathbb{M}_{d_2} \), respectively. Then both \( \mathcal{EB}_{k}(d_1, d_2) \) and \( \mathcal{EBCP}_{k}(d_1, d_2) \) are mapping cones.
Proof Suppose \( \{\Theta_n\}_{n=1}^{\infty} \) is a sequence in \( \mathcal{EB}_k(d_1, d_2) \) such that \( \Theta_n \to \Theta \in \mathcal{P}(d_1, d_2) \). Given any \( \Psi \in \mathcal{CP}(k, d_1) \), by Theorem 3.5, \( \Theta_n \circ \Psi \) is EB for every \( n \geq 1 \), and hence so is \( \Theta \circ \Psi = \lim_n \Theta_n \circ \Psi \). Thus \( \Theta \in \mathcal{EB}_k(d_1, d_2) \), and concludes that \( \mathcal{EB}_k(d_1, d_2) \) is closed. Further, from Remark 3.2, it follows that \( \mathcal{EB}_k(d_1, d_2) \) is a mapping cone. Now, since \( \mathcal{CP}(d_1, d_2) \) is a mapping cone the intersection \( \mathcal{EB}_k(d_1, d_2) \cap \mathcal{CP}(d_1, d_2) = \mathcal{EBCP}_k(d_1, d_2) \) is also a mapping cone. \( \square \)

The smallest closed convex cone containing a set \( S \) will be denoted by \( \text{conv}(S) \).

Theorem 3.11 Let \( k \geq 1 \). Then\(^1\)

\[
\mathcal{EB}_k(d_1, d_2)^o = \text{conv}[\Gamma \circ \Psi : \Gamma \in \mathcal{P}(d_1, d_2), \Psi \in \mathcal{PEB}_k(d_1)]
= \text{conv}[\Gamma \circ \Psi : \Gamma \in \mathcal{P}(k, d_2), \Psi \in \mathcal{CP}(d_1, k)].
\] (3.4)

Proof Let \( S = \{ \Gamma \circ \Psi : \Gamma \in \mathcal{P}(d_1, d_2), \Psi \in \mathcal{PEB}_k(d_1) \} \). By Theorem 3.5(xi), we have \( S \subseteq \mathcal{EB}_k(d_1, d_2)^o \), and hence \( \text{conv}(S) \subseteq \mathcal{EB}_k(d_1, d_2)^o \). Now to prove the reverse inclusion, it is enough to prove that \( \text{conv}(S)^o \subseteq \mathcal{EB}_k(d_1, d_2)^o \). So let \( \Phi \in (\text{conv}(S))^o \). Then

\[
\text{tr}(C_{\Phi \circ \Psi} C_\Gamma) = \text{tr}(C_\Gamma C_{\Phi \circ \Psi}) \geq 0, \quad \forall \Gamma \in \mathcal{P}(d_1, d_2), \Psi \in \mathcal{PEB}_k(d_1).
\]

Hence, from (2.1), \( \Phi \circ \Psi \in \mathcal{EB}(d_1, d_2) \). Since \( \Psi \) is arbitrary, by Theorem 3.5(iv), \( \Phi \in \mathcal{EB}_k(d_1, d_2) \). This completes the proof of (3.4).

To prove (3.5), let

\[
C_k(d_1, d_2) := \text{conv}[\Gamma \circ \Psi : \Gamma \in \mathcal{P}(k, d_2), \Psi \in \mathcal{CP}(d_1, k)].
\]

To show that \( C_k(d_1, d_2) \subseteq \mathcal{EB}_k(d_1, d_2)^o \), note that \( \text{tr}(C_{\Gamma \circ \Psi} C_\Phi) = \text{tr}(C_\Gamma (\text{id}_k \otimes \Phi)(C_{\Psi^*})) \geq 0 \) for all \( \Phi \in \mathcal{EB}_k(d_1, d_2) \), \( \Gamma \in \mathcal{P}(k, d_2) \) and \( \Psi \in \mathcal{CP}(d_1, k) \). To show the reverse inequality it is enough to show that \( C_k(d_1, d_2)^o \subseteq \mathcal{EB}_k(d_1, d_2) \). To see this, consider \( \Phi \in C_k(d_1, d_2)^o \). By definition, we know that \( \text{tr}(C_{\Phi \circ \Psi} C_\Gamma) = \text{tr}(C_{\Phi} C_{\Gamma \circ \Psi^*}) \geq 0 \) for all \( \Gamma \in \mathcal{P}(k, d_2) \) and \( \Psi \in \mathcal{CP}(k, d_1) \). We conclude that \( \Phi \circ \Psi \in \mathcal{EB}(k, d_2) \) for every \( \Psi \in \mathcal{CP}(k, d_1) \), which is equivalent to \( \Phi \in \mathcal{EB}_k(d_1, d_2) \). \( \square \)

Remark 3.12 Using [39, Corollary 4], we can compute the dual of \( \mathcal{EBCP}_k(d_1, d_2) \) as

\[
\mathcal{EBCP}_k(d_1, d_2)^o = (\mathcal{EB}_k(d_1, d_2) \cap \mathcal{CP}(d_1, d_2))^o = \mathcal{EB}_k(d_1, d_2)^o \cap \mathcal{CP}(d_1, d_2)^o.
\]

Here, \( C_1 \vee C_2 \) denotes the smallest closed convex cone containing the union of two mapping cones \( C_1 \) and \( C_2 \). Note that \( C_1 \vee C_2 = C_1 + C_2 \), where \( C_1 + C_2 := \{ \Theta_1 + \Theta_2 : \Theta_i \in C_i, i = 1, 2 \} \).

We denote the set of all PPT-maps from \( M_{d_i} \) into \( M_{d_j} \) by \( \mathcal{PPT}(d_1, d_2) \). From Holevo form, it follows that every EB-map is a PPT-map. But the converse is not true in general. In [15], Horodecki proved that there exists \( \Psi \in \mathcal{PPT}(2, 4) \) which is not

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\( ^1 \) The identity (3.5) was proved by Alexander Müller-Hermes and communicated to us privately after we upload the preprint in arXiv. This proof leads us to prove the identity (3.4).
EB (equivalently, not 2-EB). Now, given \( d > 4 \) let \( W = [I_4 \ 0_{4 \times (d-4)}] \in \mathbb{M}_{4 \times d} \). Then

the map \( \Psi' := Ad_W \circ \Psi : \mathbb{M}_2 \rightarrow \mathbb{M}_d \) is PPT but not EB since \( Ad_W \circ \Psi' = \Psi \) is not

EB. Thus, for every \( d \geq 4 \) there is a PPT-map \( \Psi' : \mathbb{M}_2 \rightarrow \mathbb{M}_d \) that is not EB.

**Proposition 3.13** Let \( d \geq 4 \). Then there exists a PPT map \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_d \) which is not 2-EB. Moreover, we can choose \( \Phi \) such that \( SN(\Phi) = SN(\Phi \circ T) = 2 \).

**Proof** Choose \( \Psi \in \mathcal{PT}(2, d) \) that is not EB. Let \( V = \begin{bmatrix} I_2 \\ 0_{(d-2) \times 2} \end{bmatrix} \in \mathbb{M}_{d \times 2} \). Then

the map \( \Phi := \Psi \circ Ad_V \) defined on \( \mathbb{M}_d \) is PPT, and both \( SN(\Phi) \) and \( SN(\Phi \circ T) = SN(T \circ \Phi) \) are less than or equal to \( SN(Ad_V) = 2 \). Since \( \Phi = Ad_V \circ \Psi \) is not an

EB-map, by Theorem 3.5, \( \Phi \) is not 2-EB. Observe that \( \Phi \) and \( \Phi \circ T \) has Schmidt number strictly greater than one as they are not EB. This completes the proof.

From the above discussion, we saw that for all \( d \geq 4 \) there exists a \( \Psi \in \mathcal{PT}(d, 2) \) which is not EB. The following proposition, which is a generalization of [16, Theorem 3], says that such a map will always be 3-EB.

**Proposition 3.14** Let \( d \geq 1 \).

(i) If \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_2 \) is a PPT-map, then \( \Phi \) is 3-EB.

(ii) If \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_3 \) is a PPT-map, then \( \Phi \) is 2-EB.

**Proof** (i) Let \( \Psi \in CP(3, d) \). Then \( \Phi \circ \Psi \in \mathcal{PT}(3, 2) \), and hence by [16, Theorem 3], it is EB. Since \( \Psi \) is arbitrary, from Theorem 3.5, it follows that \( \Phi \) is 3-EB.

(ii) Follows from the fact that \( \Phi \circ \Psi \in \mathcal{PT}(2, 3) \), and hence EB for every \( \Psi \in CP(2, d) \).

A mapping cone \( C \subseteq \mathcal{P}(d) := \mathcal{P}(d, d) \) is said to be symmetric [37] if \( \Theta^* \circ T \circ \Theta \circ T \in C \) for all \( \Theta \in C \). If \( C \) is a symmetric mapping cone, then \( C^c \) is also symmetric. The mapping cones \( \mathcal{P}(d), \mathcal{P}_k(d), CP(d), \mathcal{PEB}_k(d), \mathcal{EB}(d) \), where \( k > 1 \), are known to be symmetric. The following theorem says that, in general, the mapping cone \( \mathcal{EB}_k(d) \) (and hence \( \mathcal{EBCP}_k(d) \)) is not symmetric for \( d \geq 4 \).

**Theorem 3.15** Let \( d \geq 4 \).

(i) There exists a 3-EB map \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_2 \) such that \( \Phi^* \) is not 2-EB (and hence not 3-EB).

(ii) There exists a 3-EB map \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_d \) such that \( \Phi^* \) is not 2-EB (and hence not 3-EB).

**Proof** (i) Let \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_2 \) be a linear map such that \( \Phi^* \) is PPT-map but not EB. Since \( \Phi \) is PPT, by Proposition 3.14, \( \Phi \) is 3-EB. But \( \Phi^* : \mathbb{M}_2 \rightarrow \mathbb{M}_d \) is not 2-EB (equivalently not EB).

(ii) Let \( \Phi' \in \mathcal{EB}_3(d, 2) \) be such that \( \Phi'^* \) is not 2-EB. Take \( \Phi = \iota \circ \Phi' \), where \( \iota : \mathbb{M}_2 \rightarrow \mathbb{M}_d \) is the inclusion map. Observe that \( \Phi \in \mathcal{EB}_3(d) \). Now, if \( \Phi^* \) is 2-EB, then \( \Phi^* \circ \iota = \Phi'^* \circ \iota \circ \iota = \Phi^* \) is also 2-EB, which is not possible. Hence \( \Phi^* \) is not 2-EB.
Remark 3.16 From Remark 3.2 it follows that $\mathcal{EB}_k(d_1, d_2)$ is a left-invariant mapping cone in the sense that it is invariant under composition by positive maps from the left side. However, $\mathcal{EB}_k(d_1, d_2)$ is not right-invariant in general. If $\mathcal{EB}_k(d_1, d_2)$ is right-invariant, then from Theorem 3.5(ix) it follows that the adjoint of a $k$-EB map is always $k$-EB which is not true in general, as seen in the above theorem.

Remark 3.17 We know that every $d_1$-EB map $\Phi : M_{d_1} \to M_{d_2}$ is EB. However, from the above Theorem, we observe that a $d_2$-EB map $\Phi : M_{d_1} \to M_{d_2}$ is not necessarily an entanglement breaking mapping.

Remark 3.18 In Theorem 3.5(iv), it is not enough to consider $\Psi \in \mathcal{PEB}_k(d_2, d_1)$ as in Lemma 3.4(v). For, consider a PPT-map $\Phi : M_{d_2} \to M_{d_2}$ which is not EB (as in Proposition 3.13). Then for every $\Psi \in \mathcal{PEB}_k(2, d)$ we have $\Phi \circ \Psi \in \mathcal{PPT}(2) = \mathcal{EB}(2)$. But $\Phi$ is not $d$-EB as $\Phi$ is not EB.

Let $C, C_1, C_2$ be mapping cones of $\mathcal{P}(d_1, d_2)$. If $C$ is any of $\mathcal{P}, \mathcal{P}_k, \mathcal{CP}, \mathcal{PEB}_k, \mathcal{EB}$, then

$$C \circ T := \{ \Phi \circ T : \Phi \in C \}$$

is a mapping cone. A mapping cone arises from $\mathcal{P}, \mathcal{P}_k, \mathcal{CP}, \mathcal{PEB}_k, \mathcal{EB}$, via the operations

$$C \mapsto C \circ T, (C_1, C_2) \mapsto C_1 \cap C_2, \text{ or } (C_1, C_2) \mapsto C_1 \lor C_2$$

is called typical, otherwise called untypical [33]. The author posed a question in [33] that: Are there any untypical mapping cones? In [22], the authors discussed examples and properties of untypical mapping cones on $M_d$, emphasizing the case $d = 2$. From [8, Theorem 3.1] we have $\mathcal{EB}_2(3) = \mathcal{P}_2(3) \cap \mathcal{P}_2 \circ T(3)$ so that $\mathcal{EB}_2(3)$ is typical. Observe that typical mapping cones are symmetric. By Theorem 3.15, $\mathcal{EB}_k(d)$ and $\mathcal{EBCP}_k(d)$, $k = 2, 3$ are not symmetric for $d \geq 4$, hence untypical. In fact much more is true, as the following theorem shows.

Theorem 3.19 Let $d \geq 4$ and $1 < k < d$. Then the mapping cones $\mathcal{EB}_k(d)$ and $\mathcal{EBCP}_k(d)$ are untypical.

Proof By Proposition 3.13, there exists a map $\Phi \in \mathcal{PEB}_2(2) \cap (\mathcal{PEB}_2(d) \circ T)$ which is not 2-EB, hence $\Phi \notin \mathcal{EB}_k(d)$ and $\Phi \notin \mathcal{EBCP}_k(d)$. Now, from [22, Lemma 12], it follows that $\mathcal{EB}_k(d)$ and $\mathcal{EBCP}_k(d)$ are not typical.

4 Examples

Given $A = [a_{ij}] \in M_d$ let $S_A : M_d \to M_d$ be the Schur product map defined by $S_A(X) = [a_{ij}x_{ij}]$ for all $X = [x_{ij}] \in M_d$. It is well known (c.f. [26]) that $S_A$ is CP if and only if $S_A$ is positive if and only if $A$ is positive. Further, $S_A$ is PPT if and only if $S_A$ is EB if and only if $A$ is positive diagonal. See [23, 29] for details.
Proposition 4.1 Let $A \in \mathbb{M}_d^+$. If $S_A$ is $2$-EB, then it is EB.

Proof Given $1 \leq i < j \leq d$, let $V_{ij} \in \mathbb{M}_{2 \times d}$ be the matrix with $(1, i)$ and $(2, j)$ entry equals one and zero elsewhere. By Theorem 3.5(ii), $S_A \circ Ad_{V_{ij}}$ is EB, and hence

$$(\text{id} \otimes \mathcal{T})C_{S_A \circ Ad_{V_{ij}}} = \begin{bmatrix} a_{ii} E_{ii} & a_{ij} E_{jj} \\ a_{ji} E_{ij} & a_{jj} E_{jj} \end{bmatrix} \in (\mathbb{M}_2 \otimes \mathbb{M}_d)^+,$$

where $E_{kl} \in \mathbb{M}_d$ are the matrix units. But this is possible only if $a_{ij} = 0 = a_{ji}$. Thus $A$ is a diagonal matrix and hence $S_A$ is EB. □

Proposition 4.2 Let $V \in \mathbb{M}_{d_1 \times d_2}$. If $Ad_V$ is $2$-EB, then it is EB.

Proof If possible assume that $k = \text{rank}(V) > 1$. Let $W \in \mathbb{M}_{2 \times d_1}$ be such that $\text{rank}(WV) > 1$. Then $Ad_{WV} : \mathbb{M}_2 \to \mathbb{M}_{d_2}$ is not EB. But, Theorem 3.5(ii) implies that $Ad_V \circ Ad_W = Ad_{WV}$ is EB, which is a contradiction. Hence $\text{rank}(V) = 1$ concluding that $Ad_V$ is EB. □

In the following, we consider linear maps satisfying some equivariance property, which is significant in studying the $k$-positivity of linear maps defined on matrix spaces [10] and entanglement detection [2]. A linear map $\Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2}$ is said to be equivariant if, for every unitary $U \in \mathbb{M}_{d_1}$ there exists a (not necessarily unitary) matrix $V(U) \in \mathbb{M}_{d_2}$ such that $\Phi \circ Ad_U = Ad_{V(U)} \circ \Phi$.

Proposition 4.3 ([10]) Let $\Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2}$ be an equivariant map and $1 \leq k \leq d_1$. Then the following conditions are equivalent:

(i) $\Phi$ is $k$-positive.
(ii) The principle $(k \times k)$-block submatrix $C_{\Phi(1,\ldots,k)}$ is positive.
(iii) $\Phi \circ Ad_P$ is CP, where $P = [I_k 0_{k \times d_1-k}] \in \mathbb{M}_{k \times d_1}$.

The characterization (iii) is not there in [10]. But through the same lines of proof of the following theorem, which is an analogue of the above result for $k$-EB maps, we can get a simple alternative proof of Proposition 4.3. We leave the details to the reader.

Theorem 4.4 Let $\Phi : \mathbb{M}_{d_1} \to \mathbb{M}_{d_2}$ be an equivariant $k$-positive map, where $1 \leq k \leq d_1$. Then the following conditions are equivalent:

(i) $\Phi$ is $k$-EB.
(ii) The principal $(k \times k)$-block submatrix $C_{\Phi(1,\ldots,k)} \in (\mathbb{M}_k \otimes \mathbb{M}_{d_2})^+$ is separable.
(iii) $\Phi \circ Ad_P$ is EB, where $P = [I_k 0_{k \times d_1-k}] \in \mathbb{M}_{k \times d_1}$.

Proof (i) $\Rightarrow$ (ii) Follows from Theorem 3.8.
(ii) $\Rightarrow$ (iii) Follows from the identity

$$C_{\Phi \circ Ad_P} = \sum_{i,j=1}^{k} E_{ij}^{(k)} \otimes \Phi(P^* E_{ij}^{(k)} P) = \sum_{i,j=1}^{k} E_{ij}^{(k)} \otimes \Phi(E_{ij}^{(d_1)}) = C_{\Phi(1,\ldots,k)}.$$
where $E_{ij}^{(d_1)} \in M_{d_1}$ and $E_{ij}^{(k)} \in M_k$ are the standard matrix units.

(iii) $\Rightarrow$ (i) Let $Q \in M_{k \times d_1}$. By singular value decomposition, there exist unitary matrices $W \in M_k$, $U \in M_{d_1}$ and a rectangular matrix $\Sigma = [D_{0_k \times (d_1-k)}] \in M_{k \times d_1}$, where $D \in M_k$ is a diagonal matrix, such that $Q = W \Sigma U$. Set $\tilde{W} = WD \in M_k$. Then $\tilde{W}PU = W \Sigma U = Q$. Let $V(U) \in M_{d_2}$ be such that $\Phi \circ Ad_U = Ad_{V(U)} \circ \Phi$. Since $\Phi \circ Ad_P$ is EB we have

$$
\Phi \circ Ad_Q = \Phi \circ Ad_U \circ Ad_P \circ Ad_{\tilde{W}} = Ad_{V(U)} \circ \Phi \circ Ad_P \circ Ad_{\tilde{W}}.
$$

is also EB. Since $Q \in M_{k \times d_1}$ is arbitrary, by Theorem 3.5, $\Phi$ is $k$-EB. \hfill $\Box$

**Remark 4.5** In the above theorem, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds for every $\Phi \in \mathcal{P}_k(d_1, d_2)$.

Next, we consider a particular example of an equivariant map, namely the Holevo-Werner map [41]. Given $\lambda \in \mathbb{R}$ the Holevo-Werner map $W_\lambda : M_d \rightarrow M_d$ is given by

$$
W_\lambda(X) = \text{tr}(X)I - \lambda X^T.
$$

It is well-known [8, 41] that

- $W_\lambda$ is CP $\iff$ $\lambda \in [-1, 1]$
- $W_\lambda$ is PPT $\iff$ $W_\lambda$ is EB $\iff$ $\lambda \in [-1, 1/d]$
- $W_\lambda$ is 2-EB $\iff$ $\lambda \in [-1, 1/2]$.

In the following, we discuss when do they become $k$-EB for $k > 2$. First, we prove the following technical lemma.

**Lemma 4.6** Let $\Phi_i : M_d \rightarrow M_{d_i}$, $i = 1, 2$ and $\Phi : M_d \rightarrow M_{d_1 + d_2}$ be linear maps such that

$$
\Phi(X) = \begin{bmatrix}
\Phi_1(X) & 0 \\
0 & \Phi_2(X)
\end{bmatrix}
$$

for all $X \in M_d$. Then $\Phi$ is $k$-EB if and only if $\Phi_i$’s are $k$-EB.

**Proof** If $\Phi$ is $k$-EB, then from Remark 3.2(ii) we get $\Phi_i = Ad_{V_i^*} \circ \Phi$, $i = 1, 2$ are $k$-EB maps, where $V_1 = [I_{d_1} \ 0_{d_1 \times d_2}]$ and $V_2 = [0_{d_2 \times d_1} \ I_{d_2}]$. Conversely, assume that both $\Phi_1$ and $\Phi_2$ are $k$-EB, and let $\Psi \in \mathcal{CP}(k, d)$. Then using the Holevo form of the EB-map $\Phi_i \circ \Psi$, $i = 1, 2$ we conclude that $\Phi \circ \Psi$ is EB. Since $\Psi$ is arbitrary $\Phi$ is $k$-EB. \hfill $\Box$

**Theorem 4.7** Given $1 \leq k \leq d$ the following conditions are equivalent:

(i) $W_\lambda$ is a $k$-entanglement breaking CP-map.

(ii) $W_\lambda$ is a $k$-PPT map.

(iii) $\lambda \in [-1, 1/k]$. 

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Theorem 4.10  Let \( \mathcal{W}_\lambda \) is CP if and only if \( \lambda \in [-1, 1] \). Also since \( \mathcal{W}_\lambda \) is equivariant (with \( V(U) = U \)), by Theorem 4.4, \( \mathcal{W}_\lambda \) is \( k \)-EB if and only if \( \mathcal{W}_\lambda \circ \text{AdP} \) is EB, where \( P = [I_k \ 0_{k \times (d-k)}] \in \mathbb{M}_{k \times d} \). But

\[
\mathcal{W}_\lambda \circ \text{AdP}(X) = \begin{pmatrix} \text{tr}(X)I_k - \lambda X^T & 0 \\ 0 & \text{tr}(X)I_{d-k} \end{pmatrix}
\]

(4.1)

for all \( X \in \mathbb{M}_k \). Since the map \( \mathbb{M}_k \ni X \mapsto \text{tr}(X)I_{d-k} \in \mathbb{M}_{d-k} \) is always EB, by Lemma 4.6, \( \mathcal{W}_\lambda \circ \text{AdP} \) is EB if and only if the Holevo-Werner map \( \mathbb{M}_k \ni X \mapsto \text{tr}(X)I_k - \lambda X^T \in \mathbb{M}_k \) is EB, and which is true if and only if \( \lambda \in [-1, \frac{1}{k}] \).

\( (ii) \iff (iii) \) Since \( \mathcal{W}_\lambda \) is equivariant, by Proposition 4.3, \( \mathcal{W}_\lambda \) is \( k \)-PPT if and only if \( \mathcal{W}_\lambda \circ \text{AdP} \) is PPT, where \( P = [I_k \ 0_{k \times (d-k)}] \in \mathbb{M}_{k \times d} \). But, Eq. (4.1) implies that \( \mathcal{W}_\lambda \circ \text{AdP} \) is PPT if and only if the Holevo-Werner map \( \mathbb{M}_k \ni X \mapsto \text{tr}(X)I_k - \lambda X^T \in \mathbb{M}_k \) is PPT if and only if \( \lambda \in [-1, \frac{1}{k}] \).

Equivalence of \( (i) \) and \( (iii) \) in the above theorem has been shown in [3] also.

Example 4.8 ([3]) The Holevo-Werner map \( \mathcal{W}_1 : \mathbb{M}_d \rightarrow \mathbb{M}_d \) is a \( k \)-entanglement breaking CP-map for \( 1 < k < d \). The composition \( \mathcal{W}_1 \circ \Gamma = \Gamma \circ \mathcal{W}_1 \) is \( k \)-entanglement breaking but not \( (k+1) \)-positive, hence it is not \( (k+1) \)-entanglement breaking.

Theorem 4.9  Let \( 1 < k \leq d \) and \( \Gamma : \mathbb{M}_d \rightarrow \mathbb{M}_d \) be a positive map. If \( \lambda \in [-\frac{1}{k\|\Gamma\|}, \frac{1}{k\|\Gamma\|}] \), then \( \mathcal{W}_\lambda, \Gamma : \mathbb{M}_d \rightarrow \mathbb{M}_d \) given by

\[
\mathcal{W}_\lambda, \Gamma(X) := \text{tr}(X)I_d + \lambda \Gamma(X)
\]

(4.2)

is a \( k \)-EB map.

Proof  Let \( \lambda \in [-\frac{1}{k\|\Gamma\|}, \frac{1}{k\|\Gamma\|}] \). Since \( \alpha := -\lambda \|\Gamma\| \in [-1, \frac{1}{k}] \) Remark 3.2(ii) and Theorem 4.7 implies that \( \Gamma \circ T \circ \mathcal{W}_\alpha \) is a \( k \)-EB map. Let \( \Phi \in \mathcal{E}B(d) \) be the map given by \( \Phi(X) := \text{tr}(X)R \), where \( R = \|\Gamma\| I_d - \Gamma(I_d) \in \mathbb{M}_{d}^+ \). Now it follows that \( \mathcal{W}_\lambda, \Gamma = \|\Gamma\| \left( \Phi + \Gamma \circ T \circ \mathcal{W}_\alpha \right) \) is a \( k \)-EB map. \( \square \)

The following is a generalization of [8, Theorem 3.3].

Theorem 4.10  Let \( 1 < k \leq d \) and \( \Gamma : \mathbb{M}_d \rightarrow \mathbb{M}_d \) be a trace preserving positive map such that

\[
\|\text{tr}(X)I_d - \Gamma(X)\| \leq \frac{1}{k} \|X\|
\]

for all \( X \in \mathbb{M}_d \). Then \( \Gamma \) is a \( k \)-entanglement breaking map.

Proof  Let \( \tilde{\Gamma} := \mathcal{W}_{-1, \Gamma} : \mathbb{M}_d \rightarrow \mathbb{M}_d \), which is a positive map. Note that \( \|\tilde{\Gamma}\| \leq \frac{1}{k} \), hence by Theorem 4.9, \( \mathcal{W}_{-1, \tilde{\Gamma}} \) is \( k \)-EB. But \( \mathcal{W}_{-1, \tilde{\Gamma}} = \Gamma \). \( \square \)
In the rest of this section, we consider a particular case of the maps defined by (4.2).
Given \( \lambda \in \mathbb{R} \) define \( \Phi_{\lambda,d} : \mathbb{M}_d \to \mathbb{M}_d \) by

\[
\Phi_{\lambda,d}(X) := \text{tr}(X)I + \lambda(X + X^T) \tag{4.3}
\]

for all \( X \in \mathbb{M}_d \). Clearly \( \Phi_{\lambda,d} = \Phi_{\lambda,d} \circ T \), hence \( \Phi_{\lambda,d} \) is CP if and only if \( \Phi_{\lambda,d} \) is PPT. Note that \( \Phi_{\lambda,d} \) is unital if and only if \( d + 2\lambda = 1 \) and only if \( \Phi_{\lambda,d} \) is trace preserving. It is not hard to see that \( \Phi_{\lambda,d} \) is positive if and only if \( \lambda \in [-\frac{1}{2}, \infty) \).

**Theorem 4.11** Let \( \lambda \in \mathbb{R}, 1 < k \leq d \). Then the following statements holds.

(i) \( \Phi_{\lambda,d} \) is EB if and only if \( \Phi_{\lambda,d} \) is CP if and only if \( \lambda \in [-\frac{1}{d+1}, 1] \).

(ii) \( \Phi_{\lambda,d} \) is k-EB implies \( \Phi_{\lambda,k} \) is EB (and hence \( \lambda \in [-\frac{1}{d+1}, 1] \)). Conversely if \( \lambda \in [-\frac{1}{k^2}, 1] \), then \( \Phi_{\lambda,d} \) is k-EB.

**Proof** (i) Clearly, \( \Phi_{\lambda,d} \) is EB implies it is CP. Now assume that \( \Phi_{\lambda,d} \) is a CP-map. Note that

\[
C_{\Phi_{\lambda,d}} = (I_d \otimes I_d) + \lambda(|\Omega_d\rangle \langle \Omega_d| + (\text{id} \otimes T)|\Omega_d\rangle \langle \Omega_d|).
\]

Then \( 0 \leq \langle \Omega_d | C_{\Phi_{\lambda,d}} | \Omega_d \rangle = d(1 + (d + 1)\lambda) \), hence \( 1 + (d + 1)\lambda \geq 0 \), i.e., \( \lambda \geq -\frac{1}{d+1} \). Also, since the principal submatrix \[
\begin{bmatrix}
I + 2\lambda E_{11} & \lambda(E_{12} + E_{21}) \\
\lambda(E_{21} + E_{12}) & I + 2\lambda E_{22}
\end{bmatrix}
\]

of \( C_{\Phi_{\lambda,d}} \) is positive we have \( \lambda^2 \leq 1 \), hence \( \lambda \leq 1 \). Conversely, assume that \( \lambda \in [-\frac{1}{d+1}, 1] \). Then, by Corollary A.7, the Choi matrix \( C_{\Phi_{\lambda,d}} \) is separable. Therefore, \( \Phi_{\lambda,d} \) is EB.

(ii) Assume that \( \Phi_{\lambda,d} \) is k-EB. Let \( P = [I_k \ 0_{k \times (d-k)}] \in \mathbb{M}_{k \times d} \). Then, by Theorem 3.5, \( \Phi_{\lambda,d} \circ AdP \) is EB, and hence \( AdP \circ \Phi_{\lambda,d} \circ AdP = \Phi_{\lambda,k} \) is also EB. So, from (i), it follows that \( \lambda \in [-\frac{1}{k+1}, 1] \). Conversely, assume that \( \lambda \in [-\frac{1}{k^2}, 1] \). If \( \lambda \in [0, 1] \), then from (i), we have \( \Phi_{\lambda,d} \) is EB and hence k-EB. Now if \( \lambda \in [-\frac{1}{k}, 0] \), then consider the positive map \( \Gamma(X) = X + X^T \) on \( \mathbb{M}_d \). As \( \|\Gamma\| = 2 \), by Theorem 4.9, \( \Phi_{\lambda,d} = \mathcal{W}_{\lambda,\Gamma} \) is k-EB.

In the above theorem, we believe that \( \Phi_{\lambda,d} \) is k-EB if and only if \( \Phi_{\lambda,k} \) is EB if and only if \( \lambda \in [-\frac{1}{k+1}, 1] \). But, we could not prove our claim.

**Example 4.12** Given \( k < \frac{d+1}{2} \), choose \( \lambda \in [-\frac{1}{2k}, -\frac{1}{d+1}] \). Then the map \( \Phi_{\lambda,d} : \mathbb{M}_d \to \mathbb{M}_d \) is k-EB but not CP.

Let \( \text{tr}_1, \text{tr}_2 \) be the partial trace maps on \( \mathbb{M}_{d_1} \otimes \mathbb{M}_{d_2} \), i.e., \( \text{tr}_1(A \otimes B) := \text{tr}(A)B \) and \( \text{tr}_2(A \otimes B) := \text{tr}(B)A \) for all \( A \in \mathbb{M}_{d_1} \) and \( B \in \mathbb{M}_{d_2} \). It is well-known that if \( X \in (\mathbb{M}_{d_1} \otimes \mathbb{M}_{d_2})^+ \) is separable then \( \text{tr}_1(X), (\text{id}_{d_1} \otimes T)X, (T \otimes \text{id}_{d_2})X \geq 0 \). If \( \text{rank}(X) = \text{max}\{d_1, d_2\} \), then \( X \in (\mathbb{M}_{d_1} \otimes \mathbb{M}_{d_2})^+ \) is separable if and only if \( (T \otimes \text{id})X \geq 0 \). If \( X \in (\mathbb{M}_{d_1} \otimes \mathbb{M}_{d_2})^+ \) and \( d_1 d_2 \leq 6 \), then \( X \) is separable if and only if \( (T \otimes \text{id})X \geq 0 \). See [16, 17, 27] for details. We shall next obtain a necessary condition for separability. Note that, from the above known results, it holds trivially for \( \lambda \geq 0 \).
Theorem 4.13 Let $X \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+$ be separable. Then
\[
\lambda X + \lambda(T \otimes \text{id}_d)X + (I_d \otimes tr_1(X)) \geq 0 \\
\lambda X + \lambda(\text{id}_d \otimes T)X + (tr_2(X) \otimes I_d) \geq 0
\]
for all $\lambda \in [-\frac{1}{2}, \infty)$.

Proof Suppose $X = \sum A_i \otimes B_i$ for some $A_i, B_i \in \mathbb{M}_d^+$. Since $\Gamma(X) = tr(X)I + \lambda(X + X^\dagger)$ is a positive map for all $\lambda \in [-\frac{1}{2}, \infty)$, we get
\[0 \leq (\Gamma \otimes \text{id}_d)X = (I_d \otimes tr_1(X)) + \lambda X + \lambda(T \otimes \text{id}_d)X.
\]
Similarly by considering $(\text{id}_d \otimes \Gamma)(X)$ we can get the second inequality. \qed

Corollary 4.14 Let $\Phi : \mathbb{M}_d \rightarrow \mathbb{M}_d$ be an entanglement breaking CP-map. Then
\[
(Id \otimes \Phi(Id)) + \lambda(C_{\Phi} + C_{\Phi \circ T}) \geq 0 \\
(tr_2(C_{\Phi} \otimes I_d) + \lambda(C_{\Phi} + C_{T \circ \Phi}) \geq 0
\]
for all $\lambda \in [-\frac{1}{2}, \infty)$.

5 Schmidt number reducing CP-maps

Suppose $\Phi \in CP(d_1, d_2)$. Then from the definition of Schmidt number, it follows that
\[SN(\Phi \circ \Psi) \leq SN(\Psi)
\]
for all $\Psi \in CP(m, d_1)$, or equivalently,
\[SN(\text{id}_m \otimes \Phi)(X)) \leq SN(X)
\]
for all $X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+$, where $m \geq 1$. We want to know as to when strict inequality occurs in the above inequalities? Note that if $\Psi$ is an EB-map or $SN(X) = 1$, then the above inequalities become equality.

Lemma 5.1 ([8, Lemma 2.1]) Given a $k$-positive map $\Phi : \mathbb{M}_{d_1} \rightarrow \mathbb{M}_{d_2}$ the following conditions are equivalent:

(i) $\Phi$ is $n$-entanglement breaking for $n \leq \min\{k, d_2\}$.
(ii) $SN((\text{id}_m \otimes \Phi)(X)) \leq \max\{m - n + 1, 1\}$, for all $X \in (\mathbb{M}_m \otimes \mathbb{M}_{d_1})^+$ and for all $m \leq \min\{k, d_2\}$.

Theorem 5.2 Let $d_1, d_2 > 1$ and $\Phi : \mathbb{M}_{d_1} \rightarrow \mathbb{M}_{d_2}$ be a CP-map. Then the following conditions are equivalent:

(i) $\Phi$ is 2-entanglement breaking.
(ii) $SN(\Phi \circ \Psi) < SN(\Psi)$ for all non-entanglement breaking $\CP$-maps $\Psi : M_m \rightarrow M_{d_1}$ and $m \geq 2$.

(iii) $SN(\Phi \circ \Psi) < SN(\Psi)$ for all non-entanglement breaking $\CP$-maps $\Psi : M_m \rightarrow M_{d_1}$ and $2 \leq m \leq d_2$.

(iv) $SN(\Phi \circ \Psi) < SN(\Psi)$ for all non-entanglement breaking $\CP$-maps $\Psi : M_2 \rightarrow M_{d_1}$.

Proof (i) $\Rightarrow$ (ii) We prove by induction on $m$. If $m = 2$, then from Theorem 3.5(ii), it follows that $SN(\Phi \circ \Psi) < SN(\Psi)$ for all non-entanglement breaking maps $\Psi \in \CP(m, d_1)$. Now assume that $SN(\Phi \circ \Psi) < SN(\Psi)$ for all non-entanglement breaking $\Psi \in \CP(l, d_1)$ and $2 \leq l \leq m - 1$. Let $\Psi \in \CP(m, d_1)$ be non-entanglement breaking and $r = SN(\Psi)$. To show that $SN(\Phi \circ \Psi) < SN(\Psi)$.

Case (1): Suppose $m - 1 \leq d_2$. If $r = m$, then by Lemma 5.1, we have $SN(\Phi \circ \Psi) = SN((id_m \otimes \Phi)(C_\Psi)) \leq m - 1 < SN(\Psi)$. So assume that $1 < r < m$. Then, by Lemma 3.3, for all $1 \leq j \leq r$ there exist vectors $\psi_{ji} \in C^j \otimes C^{d_1}$ such that

$$SN(\Phi \circ \Psi) = SN((id_m \otimes \Phi)(C_\Psi)) \leq \max_{ij} SN((id_j \otimes \Phi)(\psi_{ji})) = \max_{ij} \{SN(\Phi \circ \Psi_{ji})\},$$

where $\Psi_{ji} \in \CP(j, d_1)$ is such that $C_{\Psi_{ji}} = \psi_{ji} \langle \psi_{ji} |$. If $\Psi_{ji}$ is EB, then $SN(\Phi \circ \Psi_{ji}) = 1 < SN(\Psi)$. If $\Psi_{ji}$ is not an EB-map, from induction hypothesis, $SN(\Phi \circ \Psi_{ji}) < SN(\Psi_{ji}) \leq j \leq SN(\Psi)$. So, from the above equation, we get $SN(\Phi \circ \Psi) < SN(\Psi)$.

Case (2): Suppose $m - 1 > d_2$. If $r > d_2$, then $SN(\Phi \circ \Psi) \leq SN(\Phi) \leq d_2 < SN(\Psi)$. So assume that $r \leq d_2$. By Lemma 3.3, for all $1 \leq j \leq r$ there exist $\psi_{ji} \in \CP(j, d_1)$ and isometries $V_{ji} : C^j \rightarrow C^m$ such that $C_{\Psi} = \sum_{i,j}(V_{ji} \otimes I_{d_1})C_{\Psi_{ji}}(V_{ji}^* \otimes I_{d_1})^*$. Observe that not all $\Psi_{ji}$ are EB as $\Psi$ is not an EB-map. Since $j \leq d_2 < m - 1$, by induction hypothesis

$$SN(\Phi \circ \Psi) \leq \max_{i,j} SN(\Phi \circ \Psi_{ji}) < \max_{i,j} SN(\Psi_{ji}) \leq j \leq SN(\Psi).$$

Thus we have $SN(\Phi \circ \Psi) < SN(\Psi)$ for all non-entanglement breaking map $\Psi \in \CP(m, d_1)$. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Obvious.

(iii) $\Rightarrow$ (i) To prove that $\Phi$ is 2-EB it is enough to show that $\Phi \circ \Psi$ is EB for every $\Psi \in \CP(2, d_1)$. If $\Psi$ is EB, then we are done. Otherwise, by assumption $SN(\Phi \circ \Psi) < SN(\Psi) \leq 2$. Thus, $\Phi \circ \Psi$ has Schmidt number one, and hence it is EB. \hfill \square

One would also ask, given a non-entanglement breaking $\Psi \in \CP(d_2, d_1)$, does there exist a map $\Phi \in \CP(d_1, d_2)$ such that $SN(\Phi \circ \Psi) = SN(\Psi)$? If yes, then such a map $\Phi$ cannot be 2-EB.

Remark 5.3 Suppose $\Psi \in \CP(d_2, d_1)$ is a non-entanglement breaking map and $d_1 \leq d_2$. Let $\Phi = Ad_V$, where $V \in M_{d_1 \times d_2}$ is any co-isometry (i.e., $VV^* = id_{d_1}$). Then

$$SN(\Psi) = SN(Ad_{V^*} \circ Ad_V \circ \Psi) \leq SN(\Phi \circ \Psi) \leq SN(\Psi).$$
Thus, there exists \( \Phi \in \mathcal{CP}(d_1, d_2) \) such that \( SN(\Phi \circ \Psi) = SN(\Psi) \). But if \( d_1 > d_2 \) this is not the case. For example, consider a 3-entanglement breaking CP-map \( \Psi_0 : \mathbb{M}_d \to \mathbb{M}_2 \) as given by Theorem 3.15, where \( d \geq 4 \). Since \( \Psi_0^* \) is not 3-EB there exists \( \Psi_1 \in \mathcal{CP}(3, 2) \) such that \( \Psi = \Psi_0^* \circ \Psi_1 \in \mathcal{CP}(3, d) \) is not EB. Now, if possible assume that there exists \( \Phi \in \mathcal{CP}(d, 3) \) such that \( SN(\Phi \circ \Psi) = SN(\Psi) \). Since \( \Psi_0 \) is 3-EB we have \( \Psi_0 \circ \Phi^* \) and hence \( \Psi_1 \circ \Psi_0 \circ \Phi^* \) are EB-maps. Therefore, \( 1 = SN(\Phi \circ \Psi) = SN(\Psi) \), which is a contradiction. Thus there is no \( \Phi \in \mathcal{CP}(d, 3) \) such that \( SN(\Phi \circ \Psi) = SN(\Psi) \).

**Theorem 5.4** Let \( 2 \leq m, n \leq d \) and \( \Phi, \Psi : \mathbb{M}_d \to \mathbb{M}_d \) be n-entanglement breaking CP-map and m-entanglement breaking CP-map, respectively. Then \( \Phi \circ \Psi \) is \( (n+m-1) \)-entanglement breaking CP-map.

**Proof** Case (1): Suppose \( d \leq n + m - 1 \). Let \( X \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ \). Clearly \( Y := (\text{id}_d \otimes \Psi)(X) \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ \) and hence, by Lemma 5.1, \( SN(Y) \leq \max\{d-m+1, 1\} = n \). Further, since \( \Phi \) is n-EB, by Theorem 3.5(viii), \( SN((\text{id}_d \otimes \Phi \circ \Psi)(X)) = SN((\text{id}_d \otimes \Phi)(Y)) = 1 \). Since \( X \) is arbitrary we conclude that \( \Phi \circ \Psi \) is EB, and in particular, \( (n+m-1) \)-EB.

Case (2): Suppose \( n + m - 1 < d \). Let \( \Gamma \in \mathcal{CP}(n + m - 1, d) \). We shall show that \( \Phi \circ \Psi \circ \Gamma \) is EB so that, by Theorem 3.5(ii), \( \Phi \circ \Psi \) is \( (n + m - 1) \)-EB. Since \( \Psi \) is \( m \)-EB, by Lemma 5.1, \( SN(\Psi \circ \Gamma) = SN((\text{id}_{n+m-1} \otimes \Psi)C) \leq n \). Thus \( \Psi \circ \Gamma \in \mathcal{PEB}_n(n + m - 1, d) \). Since \( \Phi \) is n-EB, by Theorem 3.5(v), \( \Phi \circ \Psi \circ \Gamma \) is EB.

The following improved version of [8, Theorem 2.1] is an immediate consequence of the above theorem.

**Corollary 5.5** Let \( 2 \leq n_i \leq d \) and \( k \in \mathbb{N} \). If \( \Phi_i : \mathbb{M}_d \to \mathbb{M}_d \) are \( n_i \)-entanglement breaking CP-maps for \( 1 \leq i \leq k \), then the composition \( \Phi_1 \circ \Phi_2 \cdots \circ \Phi_k \) is \( (n-k+1) \)-entanglement breaking CP-map, where \( n = \sum_i n_i \).

**Corollary 5.6** Let \( 2 \leq k \leq d \) and \( \Phi : \mathbb{M}_d \to \mathbb{M}_d \) be a k-entanglement breaking CP-map. Then \( \Phi^m \) is entanglement breaking, where \( m = \min\{SN(\Phi), \lceil \frac{d-1}{k-1} \rceil \} \).

**Proof** Let \( k_1 = SN(\Phi) \). Since \( \Phi \) is 2-EB, by Theorem 5.2, \( SN(\Phi^{k_1}) = 1 \). Meanwhile, if \( k_2 \) is such that \( kk_2 - k_2 + 1 = d \), then Corollary 5.5 implies that \( SN(\Phi^{k_2}) = 1 \). Now we conclude that \( SN(\Phi^m) = 1 \), where \( m = \min\{SN(\Phi), \lceil \frac{d-1}{k-1} \rceil \} \).

### 6 Majorization

Given \( x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d \), we denote by \( x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \ldots, x_d^\downarrow)^T \in \mathbb{R}^d \) the vector with the same components, but sorted in descending order. Thus, \( x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_d^\downarrow \). Given \( x, y \in \mathbb{R}^d \), we say that \( x \) is weakly majorized by \( y \) (and write \( x \prec_w y \)), if

\[
\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow, \quad \forall \ 1 \leq k \leq d.
\]
If \( x \prec_w y \) and \( \sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i \), then we say that \( x \) is majorized by \( y \), and write \( x \prec y \). If the dimensions of \( x \) and \( y \) are different, we define \( x \prec y \) and \( x \prec_w y \) similarly by appending extra zeros to the smaller vector to equalize their dimensions.

If \( A, B \) are two positive matrices, then we say \( A \) is (weakly) majorized by \( B \) if \( \sigma(A) \) is (weakly) majorized by \( \sigma(B) \), where \( \sigma(X) \in \mathbb{R}^d \) denotes the vector of all eigenvalues of \( X \in \mathbb{M}_d^+ \) arranged in the decreasing order.

A matrix \( D = [d_{ij}] \in \mathbb{M}_d \) is said to be doubly sub-stochastic if \( d_{ij} \geq 0 \) for all \( 1 \leq i, j \leq d \) and

\[
\sum_{i=1}^{d} d_{ij} \leq 1, \quad \forall 1 \leq j \leq d \quad \text{and} \quad \sum_{j=1}^{d} d_{ij} \leq 1, \quad \forall 1 \leq i \leq d.
\]

In the above, if \( \sum_{i=1}^{d} d_{ij} = \sum_{j=1}^{d} d_{ij} = 1 \), then \( D \) is called doubly stochastic. It is well-known that \( x \prec y \) (resp. \( x \prec_w y \)) if and only if \( x = Dy \) for some doubly stochastic (resp. sub-stochastic) matrix \( D \in \mathbb{M}_d \).

It is known [13, 25] that \( X \) is majorized by both \( \text{tr}_1(X) \) and \( \text{tr}_2(X) \) whenever \( X \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ \) is separable. Equivalently, if \( \Phi : \mathbb{M}_d \rightarrow \mathbb{M}_d \) is an EB-map, then \( C_\Phi \) is majorized by both \( \text{tr}_1(C_\Phi) \) and \( \text{tr}_2(C_\Phi) \). We prove an analogue of this result for \( k \)-EB maps, and the proof is almost same as that in [13].

Given \( X \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ = \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)^+ \) decompose \( \mathbb{C}^d = \ker(\text{tr}_2(X))^\perp \oplus \ker(\text{tr}_2(X)) \). Then with respect to the decomposition

\[
\mathbb{C}^d \otimes \mathbb{C}^d = \left( \ker(\text{tr}_2(X))^\perp \otimes \mathbb{C}^d \right) \oplus \left( \ker(\text{tr}_2(X)) \otimes \mathbb{C}^d \right)
\]

we can write \( X = \begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix} \), where \( x_1 \) acts on \( \ker(\text{tr}_2(X))^\perp \otimes \mathbb{C}^d \). See [13, Lemma 2] for details.

The following is a generalization of [13, Theorem 1].

**Theorem 6.1** Let \( X \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ \) and \( k \geq 1 \).

(i) If \( (id_d \otimes \mathcal{W}_{-\frac{1}{k}, T})(X) \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ \), then \( X \) is weakly majorized by \( k \text{tr}_2(X) \).

(ii) If \( (\mathcal{W}_{-\frac{1}{k}, T} \otimes id_d)(X) \in (\mathbb{M}_d \otimes \mathbb{M}_d)^+ \), then \( X \) is weakly majorized by \( k \text{tr}_1(X) \).

(Here \( \mathcal{W}_{-\frac{1}{k}, T} \) is given by (4.2).)

**Proof** Without loss of generality assume that \( \text{tr}_2(X) \) is invertible and diagonal, say \( \text{tr}_2(X) = \sum_{i=1}^{d} \alpha_i E_{ii} \), where \( \alpha_i \in (0, \infty) \). Since \( (\text{tr}_2(X) \otimes I_d) - \frac{X}{k} = (id_d \otimes \mathcal{W}_{-\frac{1}{k}, T})(X) \geq 0 \), by [13, Lemma 1], there exists a \( C \in \mathbb{M}_d \otimes \mathbb{M}_d \) with \( \|C\| \leq 1 \) such that

\[
X^\frac{1}{2} = \sqrt{k}(\text{tr}_2(X)^\frac{1}{2} \otimes I_d)C. \tag{6.1}
\]

Suppose \( \sigma(X) = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1d}, \ldots, \lambda_{d1}, \ldots, \lambda_{dd})^T \in \mathbb{R}^{d^2} \). Let \( U \in \mathbb{M}_d \otimes \mathbb{M}_d \) be a unitary such that

\[
U^* X^\frac{1}{2} U = \text{diag} \left( \sqrt{\lambda_{11}}, \sqrt{\lambda_{12}}, \ldots, \sqrt{\lambda_{1d}}, \ldots, \sqrt{\lambda_{d1}}, \ldots, \sqrt{\lambda_{dd}} \right). \tag{6.2}
\]
Let \( R = CU \). Then from (6.1), (6.2) we get

\[
X = k (\text{tr}_2(X)^{\frac{1}{2}} \otimes I_d)CC^* (\text{tr}_2(X)^{\frac{1}{2}} \otimes I_d)
\]

\[
diag(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{dd}) = k R^* (\text{tr}_2(X) \otimes I_d)R.
\]

From (6.3), for all \( 1 \leq i, j \leq d \), we have

\[
\langle e_i \otimes e_j, Xe_i \otimes e_j \rangle = k (\text{tr}_2(X)^{\frac{1}{2}} \otimes I_d) CC^* (\text{tr}_2(X)^{\frac{1}{2}} \otimes I_d) (e_i \otimes e_j)
\]

\[
= k \langle \sqrt{\alpha_i} e_i \otimes e_j, CC^* (\sqrt{\alpha_i} e_i \otimes e_j) \rangle
\]

\[
= k \alpha_i \sum_{p, q} |\langle e_i \otimes e_j, C(e_p \otimes e_q) \rangle|^2
\]

(6.5)

Similarly, from (6.4), we get

\[
\lambda_{ij} = k d \sum_{p, q} \alpha_p |\langle e_p \otimes e_q, R(e_i \otimes e_j) \rangle|^2
\]

(6.6)

Now, from the definition of \( \text{tr}_2(X) \) and (6.5), we have

\[
\alpha_p = \langle e_p, \text{tr}_2(X) e_p \rangle = \sum_{l=1}^d \langle e_p \otimes e_l, X(e_p \otimes e_l) \rangle
\]

\[
= k \alpha_p \sum_{l, m, n=1}^d |\langle e_p \otimes e_l, C(e_m \otimes e_n) \rangle|^2,
\]

\[
i.e., \quad \frac{1}{k} = \sum_{l, m, n=1}^d |\langle e_p \otimes e_l, C(e_m \otimes e_n) \rangle|^2
\]

for all \( 1 \leq p \leq d \). Let \( S = [S_{ij}] \in M_d \), where \( S_{ij} = \sum_{q=1}^d |\langle e_j \otimes e_q, R(e_1 \otimes e_i) \rangle|^2 \geq 0 \) for all \( 1 \leq i, j \leq d \). Note that

\[
\sum_{i=1}^d S_{ij} = \sum_{i, q=1}^d |\langle e_j \otimes e_q, R(e_1 \otimes e_i) \rangle|^2 \leq \sum_{q, m, n=1}^d |\langle e_j \otimes e_q, R(e_m \otimes e_n) \rangle|^2 = \frac{1}{k} < 1
\]

for all \( 1 \leq j \leq d \). Also since \( \|R\| \leq 1 \)

\[
\sum_{j=1}^d S_{ij} = \sum_{j, q=1}^d |\langle e_j \otimes e_q, R(e_1 \otimes e_j) \rangle|^2 = \{e_1 \otimes e_j, R^* R(e_1 \otimes e_j)\} \leq 1
\]
for all $1 \leq i \leq d$. Thus $S$ is a doubly sub-stochastic matrix. Now from (6.6),

$$
\begin{bmatrix}
\lambda_{11} \\
\lambda_{12} \\
\vdots \\
\lambda_{1d}
\end{bmatrix} = S \begin{bmatrix}
k\alpha_1 \\
k\alpha_2 \\
\vdots \\
k\alpha_d
\end{bmatrix},
$$

and hence $(\lambda_{11}, \ldots, \lambda_{1d})^T \prec_w (k\alpha_1, k\alpha_2, \ldots, k\alpha_d)^T$ i.e.,

$$
\sum_{j=1}^n \lambda_{1j} \leq \sum_{i=1}^n k\alpha_i \quad \forall \ 1 \leq n \leq d.
$$

Note that

$$
\sum_{i,j=1}^d \lambda_{ij} = \text{tr}(X) = \text{tr}(\text{tr}_2(X)) = \sum_{i=1}^d \alpha_i \leq \sum_{i=1}^d k\alpha_i.
$$

Take $\alpha_i = 0$ for all $d < i \leq d^2$ and $(\beta_1, \beta_2, \ldots, \beta_{d^2})^T = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1d}, \ldots, \lambda_{dd})^T$. Then for all for all $d < n \leq d^2$ we have

$$
\sum_{i=1}^n \beta_i \leq \sum_{i,j=1}^d \lambda_{ij} = \sum_{i=1}^d \alpha_i = \sum_{i=1}^n \alpha_i.
$$

Now from (6.7), (6.8) and (6.9) we conclude that $\sigma(X) \prec_w \sigma(k \text{tr}_2(X))$. Similarly we can prove that $\sigma(X) \prec_w \sigma(k \text{tr}_1(X))$. \hfill $\Box$

**Theorem 6.2** Let $\Phi : M_d \to M_d$ be a $k$-EB map, where $1 \leq k \leq d$. Then $C_\Phi$ is weakly majorized by both $(d-k+1)\text{tr}_1(C_\Phi)$ and $(d-k+1)\text{tr}_2(C_\Phi)$.

**Proof** Let $r = d - k + 1$. By Theorem 4.9, $\mathcal{W}_{-\frac{1}{r}, T}$ on $M_d$ is $(d-k+1)$-entanglement breaking CP-map, and hence, by Lemma 5.1, $\mathcal{W}_{-\frac{1}{r}, T}$ is $k$-PEB map. From Theorem 3.5(xi) it follows that $k$-PEB maps are in the dual of $k$-EB maps. Hence, by Theorem 2.5, $\mathcal{W}_{-\frac{1}{r}, T} \circ \Phi = (\Phi^* \circ \mathcal{W}_{-\frac{1}{r}, T})^*$ is CP. Thus $\left( I_d \otimes \mathcal{W}_{-\frac{1}{r}, T} \right)(C_\Phi) \geq 0$ and hence, from the above theorem, $\frac{1}{d-k+1} C_\Phi$ is weakly majorized by $\text{tr}_2(C_\Phi)$. Similarly $C_\Phi$ is is weakly majorized by $(d-k+1)\text{tr}_2(C_\Phi)$ also. \hfill $\Box$

**7 Discussion**

Recall that a positive map is said to be decomposable if it can be written as a sum of a CP-map and a co-CP-map. Characterization of CP-maps and PPT-maps in terms of decomposable maps is known in the literature. See [28, Theorem], [39, Proposition 8]. Note that, by Theorem 2.5, a non-zero linear map $\Phi \in \mathcal{P}(d_1, d_2)$ is EB if and only if
\(\Gamma \circ \Phi\) is CP for all \(\Gamma \in \mathcal{P}(d_2, d_1)\), and by [30, Theorem 3.1], this is in turn equivalent to \(\text{id}_k \otimes \Gamma \circ \Phi\) is decomposable for every positive maps \(\Gamma \in \mathcal{P}(d_2, d_1)\) and for some \(k \geq 2\).

**Remark 7.1** Given any non-zero \(\Gamma \in \mathcal{P}(d_1, d_2)\) and \(k \geq 1\) the map \(\text{id}_k \otimes \Gamma\) is not 2-EB. For, suppose \(A \in (\mathbb{M}_2 \otimes \mathbb{M}_k)^+\) is entangled and \(B \in \mathbb{M}_d^+\). Then \((\text{id}_2 \otimes \text{id}_k \otimes \Gamma)(A \otimes B) = A \otimes \Gamma(B)\) is not separable in \(\mathbb{M}_2 \otimes (\mathbb{M}_k \otimes \mathbb{M}_d)\). For, if \(A \otimes \Gamma(B)\) is separable, then

\[
\text{tr}(\Gamma(B))A = (\text{id}_2 \otimes \text{tr}^{(d_1)})(A \otimes \Gamma(B)) \in \mathbb{M}_2^+ \otimes \mathbb{M}_k^+,
\]

which is not possible as \(A\) is entangled; here \(\text{tr}^{(d_1)} : \mathbb{M}_k \otimes \mathbb{M}_{d_1} \rightarrow \mathbb{M}_k\) is the partial trace map. Note that \(\text{id}_k \otimes \Gamma\) is decomposable if and only if \(\Gamma\) is CP [30].

**Problem 1** Is every 2-EB map \(\Phi : \mathbb{M}_{d_1} \rightarrow \mathbb{M}_{d_2}\) decomposable?

It is known [42] that if \(d_1d_2 \leq 6\), then every 2-positive map \(\Gamma : \mathbb{M}_{d_1} \rightarrow \mathbb{M}_{d_2}\) is decomposable. But when \(4 \leq \max(d_1, d_2) \leq 9\) or \(d_1, d_2 \geq 10\) [20, 30], then there exists 2-positive map which is not decomposable.

**Theorem 7.2** Suppose the Problem-1 has an affirmative answer. If \(\Phi : \mathbb{M}_d \rightarrow \mathbb{M}_d\) is a 2-entanglement breaking PPT-map, then \(\Phi^2\) is entanglement breaking.

**Proof** Suppose \(\Phi^2\) is not an EB-map. Then there exists a \(\Gamma \in \mathcal{P}(d)\) such that \(\Gamma \circ \Phi^2\) is not CP. Note that \(\Gamma \circ \Phi\) is 2-EB and hence decomposable. So there exist \(\Phi_1, \Phi_2 \in \mathcal{C}\mathcal{P}(d)\) such that \(\Gamma \circ \Phi = \Phi_1 + \Phi_2 \circ T\). Hence

\[
\Gamma \circ \Phi^2 = (\Phi_1 + \Phi_2 \circ T) \circ \Phi = \Phi_1 \circ \Phi + \Phi_2 \circ (T \circ \Phi)
\]

is CP, which is a contradiction. \(\square\)

**Example 7.3** If \(\lambda \in [-1, \frac{1}{2}]\), then the Holevo-Werner map \(\mathcal{W}_\lambda\) (and hence \(T \circ \mathcal{W}_\lambda\)) is 2-EB. Note that \(\mathcal{W}_\lambda\) (resp. \(T \circ \mathcal{W}_\lambda\)) is a CP-map (resp. co-CP), and hence decomposable.

**Example 7.4** Let \(\lambda \in [-\frac{1}{4}, 1]\). Then the map \(\Phi_{\lambda,d} : \mathbb{M}_d \rightarrow \mathbb{M}_d\) given by (4.3) is 2-EB. If \(\lambda \in [-\frac{1}{4}, \frac{1}{2}]\), then \(\Phi_{\lambda,d} = \frac{1}{2}(\mathcal{W}_{-2\lambda} + \mathcal{W}_{-2\lambda} \circ T)\), hence we conclude that \(\Phi_{\lambda,d}\) is decomposable. Now if \(\lambda \in [\frac{1}{2}, 1]\), then \(\Phi_{\lambda,d}\) is EB, and in particular decomposable.

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Appendix A

Let $a, b, c \in \mathbb{C}$ and define $\Phi : \mathbb{M}_d \rightarrow \mathbb{M}_d$ by $\Phi(X) = a \text{tr}(X) I + bX + cX^T$. We want to know for what values of $a, b, c \in \mathbb{C}$ the map $\Phi$ is EB; equivalently when the corresponding Choi matrix

$$a(I_d \otimes I_d) + b |\Omega_d\rangle \langle \Omega_d| + c \Delta_d \in \mathbb{M}_d \otimes \mathbb{M}_d$$

is separable? Here $\Delta_d = (\text{id}_d \otimes T) |\Omega_d\rangle \langle \Omega_d| = \sum_{i,j=1}^{d} E_{ij} \otimes E_{ji}$. We answer this through similar lines as Stormer [38] proved the entanglement property of the map $\text{tr}(\cdot) \Gamma(I) + \Gamma(\cdot)$, where $\Gamma \in \mathcal{P}(d)$. Through out we let $G$ denotes the compact group

$$G = \{ \text{Ad}_U \otimes U : U \in \mathbb{M}_d \text{ real orthogonal} \}$$

where $d > 1$. Let $\mu$ denotes the normalized Haar measure on $G$ and let

$$\text{Fix}(G) = \{ A \in \mathbb{M}_d \otimes \mathbb{M}_d : \text{Ad}_U \otimes U(A) = A \text{ for all real orthogonal matrices } U \in \mathbb{M}_d \}.$$ 

Clearly $I_d \otimes I_d \in \text{Fix}(G)$. Observe that, for every $x = (x_1, \ldots, x_d)^T, y = (y_1, \ldots, y_d)^T \in \mathbb{C}^d$

$$\Delta_d(x \otimes y) = \sum_{ij} x_j e_i \otimes y_i e_j = \left( \sum_i y_i e_i \right) \otimes \left( \sum_j x_j e_j \right) = y \otimes x.$$

Consequently, $\text{Ad}_U \otimes U(\Delta_d)(x \otimes y) = y \otimes x = \Delta_d(x \otimes y)$ for every real orthogonal matrices $U \in \mathbb{M}_d$. In other words, $\Delta_d \in \text{Fix}(G)$. Note that

$$\text{Ad}_U \otimes U(|\Omega_d\rangle \langle \Omega_d|) = \text{Ad}_U \otimes U((\text{id}_d \otimes T) \Delta_d) = (\text{id}_d \otimes T)(\text{Ad}_U \otimes U(\Delta_d)) = (\text{id}_d \otimes T)(\Delta_d) = |\Omega_d\rangle \langle \Omega_d|,$$

for every real orthogonal matrices $U \in \mathbb{M}_d$. Thus $|\Omega_d\rangle \langle \Omega_d| \in \text{Fix}(G)$.

Lemma A.1 Let $d = 2$. Then $\text{Fix}(G) = \{ \alpha(I_2 \otimes I_2) + \beta |\Omega_2\rangle \langle \Omega_2| + \gamma \Delta_2 : \alpha, \beta, \gamma \in \mathbb{C} \}$.

Proof Let $A = [a_{ij}] \in \text{Fix}(G) \subseteq \mathbb{M}_2(\mathbb{M}_2)$. Consider $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{M}_2$. Then $\text{Ad}_U \otimes U(A) = A$ implies that

$$A = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix}.$$
Similarly considering \( U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) we conclude that

\[
A = \begin{bmatrix}
da_{11} & 0 & 0 & a_{14} \\
0 & a_{22} & a_{23} & 0 \\
0 & a_{23} & a_{22} & 0 \\
a_{14} & 0 & 0 & a_{11}
\end{bmatrix} = \begin{bmatrix}
a_0 & a_{22} \\
a_{22} & a_0
\end{bmatrix} + a_{14} \begin{bmatrix} 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} = (a_0 - a_{22})(E_{11} \otimes E_{11} + E_{22} \otimes E_{22}) + a_{22}(I_2 \otimes I_2) + a_{14}(|\Omega_2\rangle \langle \Omega_2|) + a_{23}\Delta_2,
\]

where \( a_0 = a_{11} - (a_{23} + a_{14}) \). Since \( A, I_2 \otimes I_2, \Delta_2, |\Omega_2\rangle \langle \Omega_2| \) are in \( \text{Fix}(G) \), from above we get \( B = (a_0 - a_{22})(E_{11} \otimes E_{11} + E_{22} \otimes E_{22}) \in \text{Fix}(G) \). Take \( U = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \)

Then \( \text{Ad}_{U \otimes U}(B) = B \) implies that \( a_0 - a_{22} = 0 \). Therefore, \( A = a_{22}(I_2 \otimes I_2) + a_{14}(|\Omega_2\rangle \langle \Omega_2|) + a_{23}\Delta_2 \). \( \square \)

**Lemma A.2** Let \( d > 2 \). Given \( m \neq n \) let

\[
P_{[m,n]} = \sum_{i,j \in \{m,n\}} |e_i \otimes e_j\rangle \langle e_i \otimes e_j|
\]

be the projection of \( \mathbb{C}^d \otimes \mathbb{C}^d \) onto \( \text{span}\{e_i \otimes e_j : i, j \in \{m,n\}\} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \). Then for every \( A \in \text{Fix}(G) \) there exist unique \( \alpha_{mn}, \beta_{mn}, \gamma_{mn} \in \mathbb{C} \) such that

\[
P_{[m,n]}AP_{[m,n]}^* = \alpha_{mn}(I_2 \otimes I_2) + \beta_{mn} |\Omega_2\rangle \langle \Omega_2| + \gamma_{mn}\Delta_2.
\]

**Proof** Let \( U \in \mathbb{M}_2 \) be a real orthogonal matrix. Then \( U \) induce a linear map on \( \mathcal{W} = \text{span}\{e_m, e_n\} \). Decompose \( \mathbb{C}^d = \mathcal{W} \oplus \mathcal{W}^\perp \) and write \( A = \begin{bmatrix} A_{mn,11} & A_{mn,12} \\
A_{mn,21} & A_{mn,22} \end{bmatrix} \in \mathbb{M}_d(\mathbb{M}_d) \),

where \( A_{mn,11} \in \mathbb{M}_2 \otimes \mathbb{M}_2 \). Consider the real orthogonal matrix \( \tilde{U} = \begin{bmatrix} U & 0 \\
0 & I_{\mathbb{W}^\perp} \end{bmatrix} \in \mathbb{M}_d \). Then \( \text{Ad}_{\tilde{U} \otimes \tilde{U}}(A) = A \) implies that \( \text{Ad}_{U}(A_{mn,11}) = A_{mn,11} \). Since \( U \in \mathbb{M}_2 \) is arbitrary, from Lemma A.1, there exist \( \alpha_{mn}, \beta_{mn}, \gamma_{mn} \in \mathbb{C} \) such that

\[
\alpha_{mn}(I_2 \otimes I_2) + \beta_{mn} |\Omega_2\rangle \langle \Omega_2| + \gamma_{mn}\Delta_2 = A_{mn,11} = P_{[m,n]}AP_{[m,n]}^*.
\]

Uniqueness of \( \alpha_{mn}, \beta_{mn}, \gamma_{mn} \) follows from the linear independence of \( I_2 \otimes I_2, |\Omega_2\rangle \langle \Omega_2|, \Delta_2 \). \( \square \)

**Proposition A.3** Let \( d > 2 \). Then \( \text{Fix}(G) = \{\alpha(I_d \otimes I_d) + \beta |\Omega_d\rangle \langle \Omega_d| + \gamma\Delta_d : \alpha, \beta, \gamma \in \mathbb{C}\} \).

**Proof** Let \( A \in \text{Fix}(G) \). Assume that \( A = \sum_{m,n,p,q} a_{(m,p),(n,q)}E_{mp} \otimes E_{nq} \). Thus,

\[
|e_m \otimes e_n, A(e_p \otimes e_q)\rangle = a_{(m,p),(n,q)}|e_m \otimes e_n, (\text{id}_d \otimes T)A(e_p \otimes e_q)\rangle = |e_m \otimes e_q, A(e_p \otimes e_n)\rangle,
\]

for all \( 1 \leq m, n, p, q \leq d \). Now let \( m, n, p, q \in \{1, 2, \ldots, d\} \).

Case (1): \( m \neq n \) or \( p \neq q \).
Subcase (i): \( \{m, n\} \neq \{p, q\} \). Suppose \( m, n, p, q \) are distinct. Given \( r \in \{m, n, p, q\} \) define the real orthogonal matrix

\[
U_r = \left( \sum_{i=1, \ldots, d} |e_i\rangle \langle e_i| \right) - |e_r\rangle \langle e_r| \in \mathbb{M}_d.
\] (A.1)

Note that \( U(e_r) = -e_r \) and \( U(e_i) = e_i \) for all \( i \neq r \). Since \( \text{Ad}_{U_r} \otimes U_r(A) = A \), we get

\[
\langle e_m \otimes e_n, A(e_p \otimes e_q) \rangle = \langle U_r e_m \otimes U_r e_n, A(U_r e_p \otimes U_r e_q) \rangle = -\langle e_m \otimes e_n, A(e_p \otimes e_q) \rangle.
\]

Hence \( a_{(m, p), (n, q)} = 0 \). Now suppose \( m = p \) or \( m = q \) or \( n = p \) or \( n = q \). Without loss of generality assume that \( m = p \). Since \( \{m, n\} \neq \{p, q\} \) we have \( n \neq q \). Fix \( r \in \{n, q\} \) and define the real orthogonal matrix \( U_r \in \mathbb{M}_d \) by (A.1). Then also

\[
\langle e_m \otimes e_n, A(e_p \otimes e_q) \rangle = \langle U_r e_m \otimes U_r e_n, A(U_r e_p \otimes U_r e_q) \rangle = -\langle e_m \otimes e_n, A(e_p \otimes e_q) \rangle
\]

so that \( a_{(m, p), (n, q)} = 0 \).

Subcase (ii): \( \{m, n\} = \{p, q\} \). From Lemma A.2, there exists scalars \( \alpha_{mn}, \beta_{mn}, \gamma_{mn} \in \mathbb{C} \) such that

\[
P_{[m, n]}AP_{[m, n]}^* = \alpha_{mn}(I_2 \otimes I_2) + \beta_{mn} |\Omega_2\rangle \langle \Omega_2| + \gamma_{mn} \Delta_2.
\]

Therefore,

\[
a_{(m, p), (n, q)} = \langle e_m \otimes e_n, A(e_p \otimes e_q) \rangle
\]

\[
= \langle P_{[m, n]}^* e_m \otimes e_n, A P_{[m, n]}^* (e_p \otimes e_q) \rangle
\]

\[
= \langle e_m \otimes e_n, P_{[m, n]} A P_{[m, n]}^* (e_p \otimes e_q) \rangle
\]

\[
= \langle e_m \otimes e_n, (\alpha_{mn}(I_2 \otimes I_2) + \beta_{mn} |\Omega_2\rangle \langle \Omega_2| + \gamma_{mn} \Delta_2)(e_p \otimes e_q) \rangle
\]

\[
= \begin{cases} 
\alpha_{mn} & \text{if } p = m \neq n = q \\
\gamma_{mn} & \text{if } q = m \neq n = p.
\end{cases}
\]

Case (2): \( m = n \) and \( p = q \). If \( m = n = p = q \), then choose any \( r \neq m \). Then, by Lemma A.2,

\[
a_{(m, p), (n, q)} = \langle e_m \otimes e_m, A(e_m \otimes e_m) \rangle
\]

\[
= \langle e_m \otimes e_m, P_{rm} A P_{rm}^* (e_m \otimes e_m) \rangle
\]

\[
= \alpha_{rm} \langle e_m \otimes e_m, e_m \otimes e_m \rangle + \beta_{rm} \langle e_m \otimes e_m, |\Omega_2\rangle \langle \Omega_2| (e_m \otimes e_m) \rangle
\]

\[
+ \gamma_{rm} \langle e_m \otimes e_m, \Delta_2 (e_m \otimes e_m) \rangle
\]

\[
= \alpha_{rm} + \beta_{rm} + \gamma_{rm}.
\]
Thus \( a_{(m,m),(m,m)} = \alpha_{rm} + \beta_{rm} + \gamma_{rm} \) for any \( r \neq m \). Now suppose \( m = n \neq p = q \). Note that \( n \neq p \) and \( m \neq q \) but \( \{m, q\} = \{n, p\} \). Hence

\[
a_{(m,p),(n,q)} = \begin{cases} 
\alpha_{rm} + \beta_{rm} + \gamma_{rm} & \text{if } m = n = p = q \text{ (where } r \neq m) \\
\alpha_{mn} & \text{if } p = m \neq n = q \\
\beta_{mq} & \text{if } m = n \neq p = q \\
\gamma_{mn} & \text{if } q = m \neq n = p \\
0 & \text{otherwise}.
\end{cases}
\]

(A.2)

Now given any \( m \neq n \) and \( p \neq q \) consider a real orthogonal matrix \( U \in \overline{\mathbb{M}}_d \) such that \( U(e_m) = e_p, U(e_n) = e_q \) and \( U^2 = I_d \). Clearly, then \( U(e_p) = e_m \) and \( U(e_q) = e_n \). Further,

\[
Ad_{U \otimes U}(A) = A \implies P_{(m,n)}Ad_{U \otimes U}(A)P_{(m,n)}^* = P_{(m,n)}AP_{(m,n)}^*
\]

\[
\implies \alpha_{pq} = \alpha_{mn}, \beta_{pq} = \beta_{mn}, \gamma_{pq} = \gamma_{mn}.
\]

Hence, there exists \( \alpha, \beta, \gamma \in \mathbb{C} \) such that (A.2) becomes

\[
a_{(m,p),(n,q)} = \begin{cases} 
\alpha + \beta + \gamma & \text{if } m = n = p = q \text{ (where } r \neq m) \\
\alpha & \text{if } p = m \neq n = q \\
\beta & \text{if } m = n \neq p = q \\
\gamma & \text{if } q = m \neq n = p \\
0 & \text{otherwise}.
\end{cases}
\]

(A.3)

Thus \( A = \alpha(I_d \otimes I_d) + \beta(|\Omega_d\rangle \langle \Omega_d|) + \gamma \Delta_d \). \( \square \)

**Definition A.4** Define \( P : \mathbb{M}_d \otimes \mathbb{M}_d \to \mathbb{M}_d \otimes \mathbb{M}_d \) by

\[
P(A) = \int_G Ad_{U \otimes U}(A)d\mu(U).
\]
Observation A.5 We make the following observations:

(i) \( P \) is a unital positive projection with
\[
\text{range}(P) = \text{Fix}(G) = \{ \alpha (I_d \otimes I_d) + \beta |\Omega_d\rangle \langle \Omega_d| + \gamma \Delta_d : \alpha, \beta, \gamma \in \mathbb{C} \}.
\] (A.4)

Further, \( \text{tr}(P(A)) = \text{tr}(A) \) for all \( A \in \mathbb{M}_d \otimes \mathbb{M}_d \).

(ii) Let \( \mathcal{C} \subseteq \mathcal{P}(d) \) be a mapping cone and let \( \Phi \in \mathcal{C} \). Let \( \Psi : \mathbb{M}_d \to \mathbb{M}_d \) be the linear map such that \( C_\Psi = P(C_\Phi) \). Then for any \( \Gamma \in \mathcal{C}^\circ \) we have
\[
\text{tr}(C_\Psi C_\Gamma) = \text{tr}(P(C_\Phi) C_\Gamma) = \int_G \text{tr} \left( C_{\Delta d U \Phi \Phi^* d U^*} C_\Gamma \right) d\mu(U) \geq 0,
\]
where the last inequality follows because \( \text{Ad}_{U} \circ \Phi \circ \text{Ad}_{U^*} \in \mathcal{C} \). Since \( \mathcal{P} \) is a positive projection we conclude that
\[
P(\{ C_\Phi : \Phi \in \mathcal{C} \}) = \{ C_\Psi : \Psi \in \mathcal{C} \} \cap \text{range}(P).
\]

In particular, considering \( \mathcal{C} = \mathcal{E}B(d) \), we have
\[
P \left( (\mathbb{M}_d \otimes \mathbb{M}_d)^{+}_{\text{sep}} \right) = (\mathbb{M}_d \otimes \mathbb{M}_d)^{+}_{\text{sep}} \cap \text{range}(P), \quad (A.5)
\]
where \( (\mathbb{M}_d \otimes \mathbb{M}_d)^{+}_{\text{sep}} \) denotes the set of separable positive matrices.

(iii) Let \( X, Y \in \mathbb{M}_d^+ \). Since \( \text{Ad}_{U \otimes U}(\Delta_d) = \Delta_d \) for every real orthogonal \( U \in \mathbb{M}_d \) we have
\[
\text{tr} \left( P(X \otimes Y) \Delta_d \right) = \int_G \text{tr} \left( \text{Ad}_{U \otimes U}(X \otimes Y) \Delta_d \right) d\mu(U)
\]
\[
= \int_G \text{tr} \left( (X \otimes Y) \text{Ad}_{U^* \otimes U^*}(\Delta_d) \right) d\mu(U)
\]
\[
= \int_G \text{tr} \left( (X \otimes Y) \Delta_d \right) d\mu(U)
\]
\[
= \text{tr} \left( (X \otimes Y) \Delta_d \right).
\]

Similarly, \( \text{tr} \left( P(X \otimes Y) |\Omega_d\rangle \langle \Omega_d| \right) = \text{tr} \left( (X \otimes Y) |\Omega_d\rangle \langle \Omega_d| \right) \). Thus
\[
\text{tr} \left( P(Z) \Delta_d \right) = \text{tr}(Z \Delta_d) \quad (A.6)
\]
\[
\text{tr} \left( P(Z) |\Omega_d\rangle \langle \Omega_d| \right) = \text{tr}(Z |\Omega_d\rangle \langle \Omega_d|) \quad (A.7)
\]
for all \( Z \in (\mathbb{M}_d \otimes \mathbb{M}_d)^{+}_{\text{sep}} \).

(iv) From (A.4) and (A.5), we have \( a(I_d \otimes I_d) + b |\Omega_d\rangle \langle \Omega_d| + c \Delta_d \) is separable if and only if
\[
a(I_d \otimes I_d) + b |\Omega_d\rangle \langle \Omega_d| + c \Delta_d = P(Z)
\]
for some $Z \in (\mathbb{M}_d \otimes \mathbb{M}_d)^{\text{sep}}$. Now since $\Delta_d^2 = I_d \otimes I_d$, $\Delta_d |\Omega_d\rangle \langle \Omega_d| = |\Omega_d\rangle \langle \Omega_d| \Delta_d$ and $|\Omega_d\rangle \langle \Omega_d| = d |\Omega_d\rangle \langle \Omega_d|$, from (A.6) and (A.7), we get

$$ad + bd + cd^2 = \text{tr}(Z \Delta_d)$$  \hspace{1cm} (A.8)

$$ad + bd^2 + cd = \text{tr}(Z |\Omega_d\rangle \langle \Omega_d|).$$  \hspace{1cm} (A.9)

Further, $\text{tr}(P(Z)) = \text{tr}(Z)$ implies that

$$ad^2 + bd + cd = \text{tr}(Z).$$  \hspace{1cm} (A.10)

Solving (A.8), (A.9) and (A.10) for $a$, $b$, $c$ we get

$$a = \frac{1}{d^3 + d^2 - 2d}((d + 1) \text{tr}(Z) - \text{tr}(Z \Delta_d) - \text{tr}(Z |\Omega_d\rangle \langle \Omega_d|))$$  \hspace{1cm} (A.11)

$$b = \frac{1}{d^3 + d^2 - 2d}((d + 1) \text{tr}(Z |\Omega_d\rangle \langle \Omega_d|) - \text{tr}(Z \Delta_d) - \text{tr}(Z))$$  \hspace{1cm} (A.12)

$$c = \frac{1}{d^3 + d^2 - 2d}((d + 1) \text{tr}(Z \Delta_d) - \text{tr}(Z |\Omega_d\rangle \langle \Omega_d|) - \text{tr}(Z)).$$  \hspace{1cm} (A.13)

**Theorem A.6** Let $x, y \in \mathbb{C}^d$. Then

$$P\left(|x\rangle \langle x| \otimes |y\rangle \langle y|\right) = a(I_d \otimes I_d) + b(\Delta_d + |\Omega_d\rangle \langle \Omega_d|)$$

is separable, where $a, b \in \mathbb{R}$ are given by

$$a = \frac{1}{d^3 + d^2 - 2d}((d + 1) \|x\|^2 \|y\|^2 - \|\langle x, y\rangle\|^2 - \|\langle x, \overline{y}\rangle\|^2)$$

$$b = \frac{1}{d^3 + d^2 - 2d}((d + 1) \|\langle x, \overline{y}\rangle\|^2 - \|\langle x, y\rangle\|^2 - \|x\|^2 \|y\|^2)$$

$$c = \frac{1}{d^3 + d^2 - 2d}((d + 1) \|\langle x, y\rangle\|^2 - \|\langle x, \overline{y}\rangle\|^2 - \|x\|^2 \|y\|^2).$$

(Note that $a \geq 0$.)

**Proof** Let $x, y \in \mathbb{C}^d$ and set $X = |x\rangle \langle x|$ and $Y = |y\rangle \langle y|$. Then, from (A.5), $P(X \otimes Y)$ is separable and there exist $a, b, c \in \mathbb{C}$ such that

$$P(X \otimes Y) = a(I_d \otimes I_d) + b \Delta_d + c |\Omega_d\rangle \langle \Omega_d|.$$

From (A.11), (A.13) and (A.12) we get $a, b, c$ as required. \hfill \Box

**Corollary A.7** If $\lambda \in \left[\frac{1}{d+1}, 1\right]$, then $(I_d \otimes I_d) + \lambda(|\Omega_d\rangle \langle \Omega_d| + \Delta_d) \in \mathbb{M}_d \otimes \mathbb{M}_d$ is separable.
\textbf{Proof} Let \( x_1 = r_1 e_1, \ y_1 = r_1 e_2, \ x_2 = y_2 = r_2 e_1 \), where \( r_1 = \left( \frac{d^3 + d^2 - 2d}{d + 1} \right)^{\frac{1}{4}} \) and \( r_2 = \left( \frac{d^3 + d^2 - 2d}{d - 1} \right)^{\frac{1}{4}} \). From the above theorem,\[ P(|x_1\rangle\langle x_1| \otimes |y_1\rangle\langle y_1|) = (I_d \otimes I_d) - \frac{1}{d + 1}(|\Omega_d\rangle\langle \Omega_d| + \Delta_d) \]
and\[ P(|x_2\rangle\langle x_2| \otimes |y_2\rangle\langle y_2|) = (I_d \otimes I_d) + (|\Omega_d\rangle\langle \Omega_d| + \Delta_d) \]
are separable. Now let \( \lambda \in \left[ -\frac{1}{d+1}, 1 \right] \). Then \( \lambda = t(\frac{-1}{d+1}) + (1-t) \) for some \( t \in [0, 1] \), and hence\[ tP(|x_1\rangle\langle x_1| \otimes |y_1\rangle\langle y_1|) + (1-t)P(|x_2\rangle\langle x_2| \otimes |y_2\rangle\langle y_2|) = (I_d \otimes I_d) + \lambda(|\Omega\rangle\langle \Omega| + \Delta_d) \]
is separable. \( \square \)

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