QUASIMODULARITY OF THE $k$TH RESIDUAL CRANKS

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Abstract. We establish quasimodularity for a family of residual crank generating functions defined on overpartitions. We also show that the second moments of these $k$th residual cranks admit a combinatorial interpretation as weighted overpartition counts.

The partition crank function was developed by Andrews and Garvan [Gar88, AG88] with the goal of giving a combinatorial proof of Ramanujan’s congruence [Ram21]

\[ p(11n + 6) \equiv 0 \mod 11, \]

where $p(n)$ denotes the number of partitions of $n$. Recall, a partition $\lambda$ of $n$ is a non-increasing sequence of positive integers which sum to $n$. We denote this relation by $\lambda \vdash n$.

Given $\lambda$, let $\omega(\lambda)$ denote the number of occurrences of 1 as a part of $\lambda$. The crank of $\lambda$, which we denote by $cr(\lambda)$, is then defined according to the value of $\omega(\lambda)$. If $\omega(\lambda) = 0$, then $cr(\lambda)$ is the largest part of $\lambda$. Otherwise, $cr(\lambda)$ is equal to the number of parts of $\lambda$ which exceed $\omega(\lambda)$, minus $\omega(\lambda)$. Let $M(m, n)$ denote the number of partitions $\lambda \vdash n$ with $cr(\lambda) = m$. Andrews and Garvan proved that

\[ M(m, 11n + 6) = \frac{1}{11} p(11n + 6), \]

which is sufficient to demonstrate (1). The two-variable generating series for the cranks of partitions is given by [AG88]

\[ C(z; q) := \sum_{n \geq 0} \sum_{m \geq 1} M(m, n) z^m q^n = \frac{(q; q)_\infty}{(zq, q/z; q)_\infty}, \]

where

\[ (a_1, \ldots, a_k; q)_n = \prod_{i=0}^{n-1} (1 - a_1 q^i) \cdots (1 - a_k q^i), \]

\[ (a_1, \ldots, a_k; q)_\infty = \lim_{n \to \infty} (a_1, \ldots, a_k; q)_n \]

is the $q$-Pochhammer symbol. From (2) we observe that

\[ M(m, n) = M(-m, n). \]

The crank function has gone on to play an interesting role in the theory of partitions, some examples being the moments of the crank function, and weighted counts of partitions [ACK13, CKL09]. Let $C_\ell(q)$ denote the generating series for the $\ell$th moments of the crank function,

\[ C_\ell(q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} m^\ell M(m, n) q^n. \]
Further, take $\delta_q$ to be the usual differential operator $q\frac{d}{dq}$, and $P(q) := \frac{1}{(q;q)_\infty}$ the generating function for partitions. Atkin and Garvan established quasimodular properties of the function $C_\ell(q)$ and its images under $\delta_q$.

**Theorem 1** (Theorem 4.2 of [AC03]). For $m \geq 0$ and $j \geq 1$, the function $\delta_q^m (C_{2j})$ is an element of the space

$$P \cdot \mathcal{W}_{m+j} (\Gamma),$$

where $\mathcal{W}_k (\Gamma)$ is the space of all quasimodular forms of weight at most $2k$ on $\Gamma$ with no constant term.

Loosely speaking, a quasimodular form is the product of a modular form which transforms under a group of Möbius transformations and the non-modular Eisenstein series $E_2(z)$. We postpone precise definitions until Section II.

Of particular note is the notion of a residual crank, wherein the ordinary crank function is applied to a subset of the parts of $\lambda$. Jennings-Shaffer [JS15] studied the functions

$$C1(z; q) = \frac{(q^2; q^4)_\infty (q; q)_\infty}{(zq, q/z; q)_\infty},$$

$$C2(z; q) = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(zq^2, q^2/z; q^2)_\infty},$$

$$C4(z; q) = \frac{(q^4; q^4)_\infty}{(q^2; q^2)_\infty (zq^4, q^4/z; q^4)_\infty},$$

as the generating series for residual crank-like functions for partitions without repeated odd parts. If we let $C1_1(q), C2_1(q)$, and $C4_1(q)$ denote the generating functions of the $\ell$th moments of these residual cranks, respectively, then these generating functions exhibit quasimodular properties.

**Theorem 2** (Theorem 1.1 of [JS15]). For $k \geq 1$, the functions in

$$\{ \delta_q^m (C_{12j}) : m \geq 0, 1 \leq j \leq k, j + m \leq k \},$$

$$\{ \delta_q^m (C_{22j}) : m \geq 0, 1 \leq j \leq k, j + m \leq k \},$$

$$\{ \delta_q^m (C_{42j}) : m \geq 0, 1 \leq j \leq k, j + m \leq k \},$$

are in the space $(-q; q^2)_\infty P (q^2) \mathcal{W}_k (\Gamma_0(4)).$

The crank function has been extended to the more general setting of overpartitions. An overpartition $\lambda$ is a non-increasing sequence of positive integers in which the first occurrence of any integer part may be overlined. We retain the notation $\lambda \vdash n$ to indicate that the sum of the integer parts of $\lambda$ is $n$. As an example, the overpartitions $\lambda \vdash 3$ are

$$(3), (3), (2, 1), (2, \overline{1}), (2, \overline{2}), (2, \overline{2}), (2, 1, 1), (\overline{1}, 1), (\overline{1}, 1),$$

which includes the three ordinary partitions of 3. These objects were introduced by Corteel and Lovejoy [CL04] in order to provide a combinatorial interpretation of $q$-hypergeometric series.

Two residual crank functions have been defined in the overpartition setting by Bringmann, Lovejoy, and Osburn [BLO09]. To calculate the first residual crank of the overpartition $\lambda$, take $\lambda'$ to be the partition whose parts are the non-overlined parts of $\lambda$. We then define $\overline{\sigma}_1 (\lambda) := cr(\lambda')$.

Similarly, to calculate the second residual crank of the overpartition $\lambda$, take $\lambda'$ to be the partition whose parts are the non-overlined even parts of $\lambda$, divided by two. We then define $\overline{\sigma}_2 (\lambda) := cr(\lambda')$. As an example, $\overline{\sigma}_1 (4, 2, 1) = cr(2, 1) = 0$. 


and $\overline{cr}_2(4, \overline{3}, 1) = cr(2) = 2$. From (2), we see that the two variable generating series for the first and second residual cranks of overpartitions are given by
\[
\overline{C}[1](z; q) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \overline{M}[1](m, n) z^m q^n = \frac{(-q; q^2)_\infty}{(zq/q^2; q^2)_\infty} \tag{4}
\]
and
\[
\overline{C}[2](z; q) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \overline{M}[2](m, n) z^m q^n = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(zq^2/q^2; q^2)_\infty} \tag{5}
\]
respectively.

Let $\overline{C}[1]_l$ and $\overline{C}[2]_l$ denote the generating series for the $l$th moment of the first and second residual crank functions, respectively. Also, let $\overline{P}(q) := (-q; q^\infty) P(q)$ denote the generating function for overpartitions. Bringmann, Lovejoy, and Osburn established quasimodular properties for these moment generating functions.

**Theorem 3** (Theorem 1.1 of [BLO09]). For $k \geq 1$, the functions in
\[
\left\{ \delta^m_q \left( \overline{C}[1]_{2j} \right) : m \geq 0, 1 \leq j \leq k, j + m \leq k \right\},
\left\{ \delta^m_q \left( \overline{C}[2]_{2j} \right) : m \geq 0, 1 \leq j \leq k, j + m \leq k \right\},
\]
are in the space $\overline{P} \cdot \overline{W}_k(\Gamma_0(2))$.

Work of Al-Saedi, Swisher, and the first author [ASMS0] extends the first and second residual cranks by defining the $k$th residual crank functions for all $k > 1$. To calculate the $k$th residual crank of the overpartition $\lambda$, take $\lambda'$ to be the partition whose parts are the non-overlined parts of $\lambda$ which vanish modulo $k$, divided by two. We then define $\overline{cr}_k(\lambda) := cr(\lambda')$.

This definition generalizes the two-variable generating functions (4) and (5) with the family
\[
\overline{C}[k](z; q) := (q^k; q^k)_\infty \overline{P}(q) C(z; q^k) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \overline{M}[k](m, n) z^m q^n,
\]
whose moment generating functions are calculated via
\[
\overline{C}[k]_{2l}(q) := \delta^l_z \left( \overline{C}[k](z; q) \right) \big|_{z=1} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} m' \overline{M}[k](m, n) q^n.
\]

Following the works of [AG03, JS15, BLO09], we will prove the following generalization of Theorem 3.

**Theorem 4.** For $l \geq 1$ and $k \geq 1$, the functions in
\[
\left\{ \delta^m_q \left( \overline{C}[k]_{2j} \right) : m \geq 0, 1 \leq j \leq l, j + m \leq l \right\}
\]
are in the space $\overline{P} \cdot \overline{W}_l(\Gamma_0(\text{lcm}\,(2, k)))$, where $\text{lcm}\,(2, k)$ is the least common multiple of 2 and $k$.

Of course, quasimodularity is not the only focus in the study of crank functions. Let $\overline{M}[1]_l(n)$ and $\overline{M}[2]_l(n)$ denote the $l$th crank moments of the first and second residual crank functions,
\[
\overline{M}[1]_l(n) = \sum_{m=-\infty}^{\infty} m^l \overline{M}[1](m, n),
\]
\[
\overline{M}[2]_l(n) = \sum_{m=-\infty}^{\infty} m^l \overline{M}[2](m, n).
\]
Larson, Rust, and Swisher proved in the following inequality for these moments in the case \( \ell = 1 \).

**Theorem 5 ([LRS14]).** For all \( n \geq 2 \) we have \( M[1]_1(n) > M[2]_1(n) \).

This result generalizes to all the \( k \)th residual cranks of overpartitions.

**Theorem 6 ([ASMS0]).** For all \( k, \ell \geq 1 \),
\[
M[k + 1]_\ell(n) \leq M[k]_\ell(n) + \ell(n),
\]
with inequality if and only if \( n < 2k \), in which case \( M[k]_\ell(n) = 0 \).

Our combinatorial interpretation of \( M[k]_\ell(n) \) allows us to improve Theorem 6 in the case \( \ell = 2 \).

**Corollary 1.** For all \( k, m \geq 1 \),
\[
m \cdot M[md]_2(n) \leq M[k]_2(n),
\]
with inequality if and only if \( n < 2k \), in which case \( M[d]_\ell(n) = 0 \).

The rest of this paper is organised as follows. In Section 1, we review properties of quasimodular forms on the congruence group \( \Gamma_0(N) \). Section 2 is devoted to the proof of Theorem 4. In Section 3, we give a combinatorial interpretation of the moments \( M[k]_\ell(n) \) and explore some corollaries. Finally, we close in Section 4 with remarks for future study.

1. Quick Review of Modular and Quasimodular Forms

We begin with a review of quasimodular forms, themselves an extension of the more ordinary modular forms. More complete details on the theory of modular forms may be found in Apostol [Apo90] for an introduction, and Miyake or Ono [Miy06, Ono04] for more advanced material.

For \( N \in \mathbb{N} \), we define \( \Gamma_0(N) \) to be the group of transformations given by
\[
\Gamma_0(N) := \left\{ \tau \mapsto \frac{a \tau + b}{c \tau + d} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \pmod{N} \right\}. \tag{6}
\]

Here, each element \( \gamma \in \Gamma_0(N) \) is a Möbius transformation acting on the complex upper half plane \( \mathbb{H} = \{ \tau : \Im \tau > 0 \} \), and may be represented as an integral matrix:
\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

In this notation, \( \Gamma := \Gamma_0(1) \) is the modular group. For \( N > 1 \), we call \( \Gamma_0(N) \) the principal congruence subgroup of level \( N \). It will be useful to recall that \( \Gamma \) is generated by the transformations
\[
\tau \mapsto \tau + 1 \quad \tau \mapsto -\frac{1}{\tau}.
\]

Next, let \( k \) be an integer. A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is a modular form of weight \( k \) on \( \Gamma_0(N) \) if the following conditions are met. First,
\[
f(\gamma(\tau)) = (c\tau + d)^k f(\tau)
\]
for all \( \tau \in \mathbb{H} \) and \( \gamma \in \Gamma_0(N) \). Second, \( f \) has a representation as a power series in the variable \( q := e^{2\pi i \tau} \), e.g.,
\[
f(\tau) = \sum_{n=0}^{\infty} a_n q^n.
\]

\( ^2\)This representation is taken modulo \( -I \).
Note that this imposes growth conditions on \( f \) as \( \tau \to i\infty \).

It is a fact that if \( f \) is a modular form of weight \( k \), then \( k \) is even. If \( k = 0 \), then \( f \) is constant by Liouville’s theorem, and if \( k = 2 \), then \( f \equiv 0 \). Nonconstant modular forms \( f \) must have weight \( k \geq 4 \), see Apostol [Apo90, Theorem 6.2]. We define \( \mathcal{M}_k (\Gamma_0(N)) \) to be the vector space of modular forms of weight \( 2k \) on \( \Gamma_0(N) \), where \( k \) is a non-negative integer.

Following Atkin and Garvin [AG03], the Eisenstein series \( E_{2k}(\tau) \) is defined for \( k \in \mathbb{N} \) by

\[
E_{2k}(\tau) = 1 + \frac{(2\pi)^{2k}}{(-1)^k (2k - 1)! \zeta(2k)} \Phi_{2k-1}(q),
\]

where

\[
\Phi_l(q) := \sum_{n=1}^{\infty} \frac{n^l q^n}{1 - q^n} = \sum_{n=1}^{\infty} \sum_{d|n} d^l q^n.
\]

For all \( k \geq 2 \), we have \( E_{2k} \in \mathcal{M}_k(\Gamma) \). In fact, these functions generate all modular forms, as

\[
\{ E_a^b E_0^b : a, b \in \mathbb{N}_0, 2a + 3b = k \}
\]

forms a basis for \( \mathcal{M}_k(\Gamma) \), as demonstrated by Apostol [Apo90, Section 6.5].

Note that \( E_2 = 1 - 24\Phi_1 \) is not a modular form, since

\[
E_2 \left( -\frac{1}{\tau} \right) = \tau^2 E_2(\tau) + \frac{6}{\pi^2} \tau \neq (\tau)^2 E_2(\tau).
\] (7)

Quasimodular forms were first defined in work of Kaneko and Zagier [KZ95]. Here we extend the definition of weight by declaring the weight of \( E_2 \) to be 2. We can then define the space of quasimodular forms of weight \( 2k \) on \( \Gamma_0(N) \) by

\[
\overline{M}_k (\Gamma_0(N)) := \left\{ \sum_{j=0}^{k} f_j E_j^2 : f_j \in \mathcal{M}_{k-j} (\Gamma_0(N)) , f_k \neq 0 \right\}.
\]

We see immediately that \( \mathcal{M}_k (\Gamma_0(N)) \) is a subspace of \( \overline{M}_k (\Gamma_0(N)) \). Because \( \delta_q (E_2) = \frac{1}{\pi} (E_3^2 - E_4) \), we have \( \delta_q (E_2) \in \overline{M}_2(\Gamma) \).

We now define \( \mathcal{W}_k (\Gamma_0(N)) \) to be the space of quasimodular forms of weight at most \( 2k \) on \( \Gamma_0(N) \), that is

\[
\mathcal{W}_k (\Gamma_0(N)) := \left\{ \sum_{j=0}^{k} f_j E_j^2 : f_j \in \sum_{l=0}^{k-j} \mathcal{M}_l (\Gamma_0(N)) \right\}.
\]

Then \( \overline{M}_k (\Gamma_0(N)) \) is a vector subspace of \( \mathcal{W}_k (\Gamma_0(N)) \), and we see that the set

\[
\{ E_a^b E_0^b E_0^c : a, b, c \in \mathbb{N}_0, a + 2b + 3c \leq k \}
\]

forms a basis for \( \mathcal{W}_k (\Gamma) \). Finally, define the space \( \overline{\mathcal{W}}_k (\Gamma_0(N)) \) via

\[
\overline{\mathcal{W}}_k (\Gamma_0(N)) = \{ f = \sum_{n=0}^{\infty} a_n q^n \in \overline{\mathcal{W}}_k (\Gamma_0(N)) \mid a_0 = 0 \}.
\]

We can see that \( \Phi_{2k-1}(q) \in \overline{\mathcal{W}}_k (\Gamma) \).

2. Proof of Theorem

We next give some useful lemmas towards the proof of Theorem. These lemmas are well-known to specialists, but we provide their proofs for the sake of completeness.

Lemma 1. Let \( n \in \mathbb{N} \) and \( f(\tau) \in \mathcal{M}_k (\Gamma_0(N)) \). Then \( f(n\tau) \in \mathcal{M}_k (\Gamma_0(nN)) \).
We break into three cases.

2. It suffices to prove that \( g(\tau) := \frac{a\tau + b}{c\tau + d} \), where

\( ad - bc = 1 \) and \( c \equiv 0 \pmod{nN} \) for integers \( a, b, c, d \). There exists an integer \( c' \) such that \( c = c' nN \). Then we have \( n\gamma(\tau) = \gamma_1(n\tau) \), where \( \gamma_1(\tau) := (a\tau + nb) / (c' n\tau + d) \). Because \( \gamma_1 \in \Gamma_0(N) \), we obtain

\[
 f(\gamma_1(n\tau)) = (c' n\tau + d)^{2k} f(n\tau),
\]

which is our claim.

A special case of Lemma 1 is that \( f(\tau) \in \mathcal{M}_k(\Gamma) \) implies \( f(n\tau) \in \mathcal{M}_k(\Gamma_0(n)) \).

We also deduce that \( \Phi_{2k-1}(q^n) \in \overline{\mathcal{M}}_k(\Gamma_0(n)) \).

**Lemma 2.** Let \( f \in \mathcal{M}_k(\Gamma_0(N)) \). Then \( 12\delta_q(f) - 2kE_2f \in \mathcal{M}_{k+1}(\Gamma_0(N)) \).

**Proof.** For brevity, we let

\[
g(\tau) := 12\delta_q(f(\tau)) - 2kE_2f(\tau) = \frac{6}{\pi^1} f'(\tau) - 2kE_2f(\tau).\]

It suffices to prove that \( g \) is modular of weight \( k+1 \) for both generators of \( \Gamma \). We already have \( g(\tau + 1) = g(\tau) \).

Let \( \tau = -1/\tau' \). Using the fact that \( f(-1/\tau') = \tau^{2k} f(\tau') \) and \( \Phi \) gives us

\[
f'(\tau) = \tau^{2k} \frac{d}{d\tau} \left( \tau^{2k} f(\tau') \right) = (2kf(\tau') + \tau f'(\tau')) \tau^{2k+1},
\]

\[
E_2(\tau)f(\tau) = \left( \tau' E_2(\tau') f(\tau') + \frac{6}{\pi^1} \right) \tau^{2k+1}.
\]

Therefore, \( g(-1/\tau) = \tau^{2k+2} g(\tau) \).

**Lemma 3.** Let \( k \geq 1 \). If \( \delta_q(f) \in \overline{\mathcal{M}}_k(\Gamma_0(N)) \), then \( \delta_q(f) \in \overline{\mathcal{M}}_{k+1}(\Gamma_0(N)) \).

**Proof.** Let \( f \in \overline{\mathcal{M}}_k(\Gamma_0(N)) \). Then there exist modular forms \( f_j \in \mathcal{M}_{k-j}(\Gamma_0(N)) \) such that \( f = \sum_{j=0}^k f_j E_j^\gamma \) and \( f_k \neq 0 \). By Lemma 2 we have \( \delta_q(f_j) = \delta_q(f_j) = \frac{6}{\pi^1} f_j E_j^\gamma \) for some modular forms \( f_j \in \mathcal{M}_{k+1-j}(\Gamma_0(N)) \).

We need to show that \( \delta_q(f) = \sum_{j=0}^k f_j E_j^\gamma \) for some modular forms \( f_j \in \mathcal{M}_{k+1-j}(\Gamma_0(N)) \).

We break into three cases.

Suppose \( k = 1 \). We have \( \tilde{f}_0 = f_{0,0} - \frac{1}{12} f_1 E_4 \), \( \tilde{f}_1 = f_{0,1} + \frac{1}{6} f_0 \) and \( \tilde{f}_2 = \frac{1}{2} f_1 \).

Suppose \( k = 2 \). We have \( \tilde{f}_0 = f_{0,0} - \frac{1}{12} f_1 E_4 \), \( \tilde{f}_1 = f_{0,1} + \frac{1}{12} f_0 - \frac{1}{6} f_2 E_4 \), \( \tilde{f}_2 = \frac{1}{2} f_1 \).

Otherwise, \( k \geq 3 \). We have \( \tilde{f}_0 = f_{0,0} - \frac{1}{12} f_1 E_4 \), \( \tilde{f}_1 = f_{0,1} + \frac{k}{2} f_0 - \frac{1}{6} f_2 E_4 \),

\[
\tilde{f}_j = \sum_{l=2}^{k-1} f_{0,l} + \frac{2k + l - 1}{12} f_{l-1} - \frac{l + 1}{12} f_{l+1} E_4
\]

for \( 2 \leq j \leq k - 1 \), \( \tilde{f}_k = f_{0,k} + \frac{k-1}{2} f_{k-1} \) and \( \tilde{f}_{k+1} = \frac{k}{2} f_k \).

**Corollary 2.** If \( f \in \mathcal{W}_k(\Gamma_0(N)) \), then \( \delta_q(f) \in \overline{\mathcal{W}}_{k+1}(\Gamma_0(N)) \).

Note that the moment generating function \( \overline{\mathcal{C}}_{[k]_j}(q) \) vanished for odd \( \ell \), since \( \overline{\mathcal{M}}_k(m,n) = \overline{\mathcal{M}}_k(-m,n) \).

Following [AG03], we can obtain the representation

\[
\overline{\mathcal{C}}_{[k]}(q) = 2\overline{\mathcal{P}}(q) \sum_{a_1 + 2a_2 + \cdots + ja_j = j} \alpha_{a_1, \ldots, a_j} \Phi_{a_1}^{\alpha_1}(q^k) \Phi_{a_2}^{\alpha_2}(q^k) \cdots \Phi_{a_j}^{\alpha_j}(q^k),
\]

where \( \alpha_{a_1, \ldots, a_j} \) are integers.
Let $K := \text{lcm}(2, k)$. By \cite{[S]} and Lemma \cite{[1]} we have $\mathcal{C}_{[k]_2}(q) \in \mathcal{T} \cdot \mathcal{W}_j(\Gamma_0(K))$, and by Lemma \cite{[S]} also $\delta_q: \mathcal{W}_1(\Gamma_0(K)) \rightarrow \mathcal{W}_{l+1}(\Gamma_0(K))$. Because $\delta_q(\mathcal{T}) = 2\mathcal{T}(\Phi_1(q) - \Phi_1(q^2)) \in \mathcal{T} \cdot \mathcal{W}_1(\Gamma_0(K))$, we also have $\delta_q: \mathcal{T} \cdot \mathcal{W}_j(\Gamma_0(K)) \rightarrow \mathcal{T} \cdot \mathcal{W}_{j+1}(\Gamma_0(K))$. Therefore, $\delta_q^{\text{ov}}(\mathcal{C}_{[k]_2}(q)) \in \mathcal{T} \cdot \mathcal{W}_{j+m}(\Gamma_0(K)) \subset \mathcal{T} \cdot \mathcal{W}_l(\Gamma_0(K))$.

This proves Theorem \cite{[4]}

3. Weighted Overpartition Counts

We now establish a combinatoric interpretation for the coefficients of $\mathcal{C}_{[k]_2}(q)$.

Let $\text{nov}_k(n)$ denote the sum of all non-overlined parts which vanish modulo $k$, taken across all overpartitions $\lambda \vdash n$. For example, the overpartitions

\[
(3), (3), (2, 1), (2, 1), (1, 1, 1), (1, 1, 1).
\]

We see that $\text{nov}_2(3) = 6$. Similarly, let $\text{ov}_k(n)$ denote the sum of all overlined parts which vanish modulo $k$, taken across all $\lambda \vdash n$. Here, $\text{ov}_2(3) = 2$. Note that $\text{nov}_1(n) = \text{enov}(n)$ in the notation of \cite{[BLO09], p. 1768}.

**Theorem 7.** We have $\text{nov}_k(n) = \frac{k}{2} \cdot \text{M}[k]_2$.

**Proof.** We use Dyson’s result \cite{[Dys89]} that

$$ np(n) = \frac{1}{2} \cdot M_2(n). $$

In the case of residual cranks, we have

\[
\sum_{n=0}^{\infty} \text{nov}_k(n) q^n = \frac{(-q; q)_{\infty}(q^k; q^k)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} kn \cdot p(n) q^{kn} = \frac{(-q; q)_{\infty}(q^k; q^k)_{\infty}}{(q; q)_{\infty}} \frac{k}{2} \sum_{n=0}^{\infty} M_2(n) q^n = \frac{(-q; q)_{\infty}(q^k; q^k)_{\infty}}{(q; q)_{\infty}} \frac{k}{2} \delta_2 \left\{ \frac{(q^k; q^k)_{\infty}}{(zq^k; q/z^k; q^k)_{\infty}} \right\}_{z=1} = \frac{k}{2} \cdot \mathcal{C}_{[k]_2}(q).
\]

**Lemma 4.** We have

$$ \text{ov}_k(n) = \text{nov}_k(n) - \text{nov}_{2k}(n). $$

**Proof.** We claim there is a weight-preserving bijection $\phi_k$ between partitions into distinct parts which vanish modulo $k$, and partitions whose parts vanish modulo $k$ but not modulo $2k$. This map is simply a dilation of Euler’s map between partitions into distinct parts and partitions into odd parts \cite{[And76]}. Thus, summing the overlined parts which vanish modulo $k$, when taken over all $\lambda \vdash n$, is equivalent to summing the non-overlined parts which vanish modulo $k$ but not modulo $2k$, as desired.

**Corollary 3.** We have

$$ \text{ov}_k(n) = \frac{k}{2} \cdot \text{M}[k]_2(n) - k \cdot \text{M}[2k]_2(n). $$
Corollary 4. For all \( d, k \geq 1 \) and \( n \geq 0 \),
\[
dM[dk](n) \leq M[k](n),
\]
with equality if and only if \( n < k \).

Proof. Since \( \text{nov}_d(n) \leq \text{nov}_k(n) \), we have
\[
0 \leq \text{nov}_d(n) - \text{nov}_d(n) = \frac{k}{2} \left( M[k](n) - dM[dk](n) \right).
\]
The conditions for equality follow directly from Theorem 1 of [ASMS0]. ■

Finally, we show how the second moment of the \( k \)th residual crank relates to weighted counts of all overpartitions. For an overpartition \( \lambda \), let \( \omega_k(\lambda) \) denote the number of times \( k \) occurs non-overlined as a part in \( \lambda \).

Corollary 5.
\[
C[k]^2(q) = - \sum_{\lambda \in \mathcal{P}} \omega_k(\lambda) \text{cr}_k(\lambda) q^{\lambda'.}
\]

Proof. We use Chern’s result [Che20] that
\[
\sum_{\lambda \vdash n} \omega(\lambda) \text{cr}(\lambda) = -np(n).
\]
Then
\[
C[k]^2(q) = \frac{(-q; q)_{\infty} (q^k; q^k)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} M_2(n) q^k n
\]
\[
= \frac{(-q; q)_{\infty} (q^k; q^k)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} n \cdot p(n) q^{kn}
\]
\[
= \frac{(-q; q)_{\infty} (q^k; q^k)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{\lambda' \in \mathcal{P}} -\omega(\lambda') \text{cr}(\lambda') q^{kn},
\]
where the inner summation is taken over ordinary partitions \( \lambda' \). If we consider such \( \lambda' \) to be the residual partitions as used to define the \( k \)th residual crank, then we see that
\[
\frac{(-q; q)_{\infty} (q^k; q^k)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{\lambda' \in \mathcal{P}} -\omega(\lambda') \text{cr}(\lambda') q^{kn} = - \sum_{\lambda \in \mathcal{P}} \omega_k(\lambda) \text{cr}_k(\lambda) q^{\lambda'},
\]
as desired. ■

4. Future Study

Those familiar with the theory are likely to ask how the quasimodularity of the \( k \)th residual crank functions interacts with moment generating functions of any corresponding overpartition rank functions. It is common to study crank functions in relation to certain rank functions defined for partitions and overpartitions, which will be nearly quasimodular on the same group \( \Gamma_0(N) \) [AG03, JS15, BLO09]. For example, Bringmann, Lovejoy and Osburn established a partial differential equation between the first and second residual crank functions, and the Dyson rank and \( M_2 \)-rank functions [BLO10]. This work then led to establishing the quasimodularity of
functions such as
\[
(\ell^2 - 3\ell + 2)\mathcal{R}_\ell + 2 \sum_{i=1}^{\ell/2-1} \binom{\ell}{2i} (3^{2i} - 2^{2i} - 1)\delta_q \mathcal{R}_{\ell-2i} + \sum_{i=1}^{\ell/2-1} \binom{\ell}{2i+1} + 2 \binom{\ell}{2i+1} (1 - 2^{2i+1}) + \frac{1}{2} \binom{\ell}{2i+2} (3^{2i+2} - 2^{2i+2} - 1)\mathcal{R}_{\ell-2i},
\]
where \(\mathcal{R}_\ell = \mathcal{R}_\ell(q)\) is the generating series for the \(\ell\)th moment of the Dyson rank function for overpartitions [BLO09].

At this point in time, we are not aware of a suitable rank function to pair with the \(k\)th residual crank function for \(k > 2\). Although the first author’s \(M_k\)-ranks [Mor19] generalize the rank functions studied by Bringmann, Lovejoy, and Osburn, these fail to produce the expected \(spt\) relations. Where we would expect
\[
spt_k(n) = \frac{1}{2} (N[k]_2(n) - M[k]_2(n)),
\]
with \(spt_k(n)\) a positive-weighted count function of overpartitions, we have instead
\[
\frac{1}{2} (N[3]_2(4) - M[3]_2(4)) = -2.
\]

It may be fruitful to instead establish a theory of \(k\)th residual \(spt\) functions for overpartitions, then seek rank functions so that
\[
N[k]_2(n) = M[k]_2(n) - 2spt_k(n).
\]

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