MINIMAL DEFORMATIONS OF THE COMMUTATIVE ALGEBRA AND THE LINEAR GROUP GL(n)

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Abstract

We consider the relations of generalized commutativity in the algebra of formal series $M_q(x^i)$, which conserve a tensor $I_q$-grading and depend on parameters $q(i,k)$. We choose the $I_q$-preserving version of differential calculus on $M_q$. A new construction of the symmetrized tensor product for $M_q$-type algebras and the corresponding definition of minimally deformed linear group $QGL(n)$ and Lie algebra $qgl(n)$ are proposed. We study the connection of $QGL(n)$ and $qgl(n)$ with the special matrix algebra $\text{Mat}(n,Q)$ containing matrices with noncommutative elements. A definition of the deformed determinant in the algebra $\text{Mat}(n,Q)$ is given. The exponential parametrization in the algebra $\text{Mat}(n,Q)$ is considered on the basis of Campbell-Hausdorf formula.
1 Introduction. The principle of minimal deformation

Mathematical formalism of the quantum inverse scattering problem can be constructed on the basis of a concept of quantum group, which is developed intensively as a new branch of modern mathematics [1-4]. Quantum groups are considered as parametric deformations of the classical groups. Representations of quantum groups are closely connected with deformations of commutative algebras. Let \( x^1, \ldots, x^n \) are generators of the formal-series algebra over a field of complex numbers \( \mathbb{C} \). A deformation of the commutative algebra can be defined with the help of the following bilinear relations

\[
x^i x^j = \hat{R}_{kl}^{ij} x^k x^l
\]

where \( \hat{R} \) is \( n^2 \times n^2 \) matrix which satisfies the constant Yang-Baxter equation:

\[
\hat{R}_{rt}^{ij} \hat{R}_{pt}^{kn} \hat{R}_{mk}^{pl} = \hat{R}_{mr}^{ip} \hat{R}_{pt}^{kn} \hat{R}_{kl}^{rt}
\]

These relations can be written in terms of the \( R \)-matrix

\[
R_{kl}^{ij} = (P \hat{R})_{kl}^{ij} = \hat{R}_{kl}^{ij}
\]

We shall consider a limited class of deformations corresponding to the diagonal \( R \)-matrices

\[
R_{kl}^{ij} = q(i, j) \delta_k^i \delta_l^j = \hat{R}_{kl}^{ij}
\]

where \( q(i, j) \in \mathbb{C} \) are the parameters of deformations.

Let us define the \( q \)-deformed algebra \( M_q(n) \) (quantum space or \( q \)-commutative algebra) as the formal-series algebra with generators \( x^i \) satisfying the following relations

\[
x^i x^j = q(i, j) x^j x^i = [ij] x^j x^i
\]

\[
q(i, i) = 1, \quad q(i, j) q(j, i) = 1
\]

Here the notation \( [ij] = q(i, j) \) is introduced. The algebra \( M_q(n) \) has \( (n^2 - n)/2 \) independent parameters.

Note, that we shall use a summation convention for coinciding upper and low indices only. We do not treat symbols in parentheses as indeces, so summation is not used in Eqs \([1.5\) \( 1.6\).

The standard \( \mathbb{Z} \)-grading in the algebra of formal series corresponds to a decomposition in degrees \( p \) of the monomials

\[
x^{i_1} x^{i_2} \ldots x^{i_p} \overset{\text{def}}{=} x^{I(p)}
\]

Consider a set \( I \) of the totally symmetric multiindices

\[
I(p) = (i_1, i_2, \ldots, i_p)
\]

It is easy to introduce a commutative multiplication of multiindices in \( I \)

\[
I(p) \ast I(r) = I(p + r) = I(r) \ast I(p)
\]

A unit element \( I(0) \) (zero multiindex) of the commutative semigroup \( I \) corresponds to absence of indices.
Consider a generalized tensor $I_q$-grading as decomposition of the algebra $M_q(n)$ in a direct sum of vector spaces $M_q(I(p))$ corresponding to monomials \((1.7)\) with an arbitrary order of indeces

\[
M_q(n) = \bigoplus_{p=0}^{\infty} \bigoplus_{I(p)} M_q(I(p)) = \bigoplus_{p=0}^{\infty} \bigoplus_{I(p)} \{k x^{I(p)}\}
\]

\[
M_q(I(p)) M_q(I(r)) \subset M_q(I(p + r)) \tag{1.10}
\]

$I_q$-grading in the $q$-commutative algebra $M_q(n)$ is consistent with a simple generalization of the relation \((1.3)\) to the arbitrary monomials

\[
x^{I(p)} x^{K(r)} = q(I, K) x^{K(r)} x^{I(p)} \tag{1.11}
\]

\[
q(I, K) \overset{\text{def}}{=} [I(p)|K(r)] = \prod_{\alpha=1}^{p} \prod_{\beta=1}^{r} [i_{\alpha} k_{\beta}] \tag{1.12}
\]

\[
q(I, K) q(K, I) = 1, \quad q(I, I) = 1
\]

\[
q(I \ast J, K) = q(I, K) q(J, K) \tag{1.13}
\]

One can formulate mnemonic rule for the calculation of the function $q(I, K)$: **The multiplier $q(i, k)$ appears when the index $i$ moves by the index $k$ from the left to right.** This rule is analogous to the rule of signs in $\mathbb{Z}_2$-graded algebras \([5,6]\), which is a key principle in the supersymmetric generalization of the “commutative” algebra and analysis. In further considerations this rule can be generalized taking into account the introducing of covariant (low) indeces and other extensions of $I_q$-grading.

In Ref\([7]\) and other works there were considered $(G, f)$-graded algebras which correspond to Abelian groups $G$ and commutation functions $f(g, h)$ satisfying the Eq\((1.13)\)-type restrictions. We shall not use the accepted in mathematical works name “coloured” for these groups and algebras, because this word is widely used in the quantum chromodynamics.

According to the results of Ref\([7]\) $(G, f)$-Lie algebras for the finite groups $G$ can be reduced to ordinary Lie algebras or Lie superalgebras if the restrictions $f(g, g) = \pm 1$ is used.

Note, that deformations of formal Lie groups and Lie algebras corresponding to a general $R$-matrix and condition $R^2 = 1$ was investigated in Ref\([8]\).

We shall treat the algebra $M_q(n)$ as **minimal deformation** of the commutative formal-series algebra $C(x^i)$ and shall use the principle of minimal deformation for constructing the theories, which consistent with $I_q$-grading. One can see from Refs\([7-11]\) that there exists some uncertainty in the constructing of a differential calculus and the action of a quantum linear group on the algebra $M_q(n)$ . It seems to us very natural to build these theories on a basis of the minimal deformation (MD) principle.

One can made a linear similarity transformation with the complex matrix $T^i_k$ \([14]\) in the algebra $M_q(n)$ , which does not conserve $I_q$ -grading. It is evident that this transformation generate undiagonal solution of Eq\((1.2)\), which is similar to the solution \((1.4)\). Note that the diagonal solution \((1.4)\) is invariant under the transformation $T^i_k = t(i) \delta^i_k$ . An arbitrary transformation $T^i_k$ conserves the following relations for the matrix \((1.4)\):

\[
\hat{R}_{kl}^{ij} \hat{R}_{mn}^{kl} = \delta^i_m \delta^j_n, \quad \hat{R}_{kj}^{ki} = \delta^i_k = \hat{R}_{jk}^{ik} \\
\hat{R}_{lm}^{ij} \hat{R}_{ik}^{lp} = \delta^i_k \delta^p_m = \hat{R}_{ml}^{ij} \hat{R}_{ki}^{pl} \tag{1.14}
\]
A generalization of $I_q$-grading and a differential calculus on the algebra $M_q(n)$ are considered in section 2. Section 3 contains a definition of the minimal deformation $QGL(n)$ for the linear group and its connection with the deformed Lie algebra $qgl(n)$. An alternative definition of the quantum group $QGL(n)$ in terms of the special matrix algebra $\text{Mat}(n, Q)$ is suggested in section 5.

It should be stressed, that our main purpose is the constructive discussion of the simple formalism of deformations, which may be convenient for physical applications, so we do not give a detailed review of references on quantum groups and construct proofs of basic statements on the level of strictness accepted in theoretical physics. The standard definition of quantum groups is based on the original formulation of the inverse scattering problem [1]. We do not know how to use minimal deformations of the linear group in physics, but a search of corresponding applications seems to us very interesting.

In the conclusion we discuss a solution of the quantum Yang-Baxter equation, which depends on the functions $q(i, k, u, h)$, where $u$ is a spectral parameter and $h$ is a quasi-classical parameter.

Note, that minimal deformations of the linear supergroup $GL(p, q)$ can be constructed by the analogy with $QGL(n)$.

## 2 Differential calculus on minimally deformed formal-series algebras

Consider the algebra $M_q^*(n)$ with generations $y_i$ satisfying the relation

$$y_i y_j = [ij] y_j y_i \quad (2.1)$$

The $I_q$-grading of monomials $y_{k_1} \ldots y_{k_p} = y_{K(p)}$ is determined by covariant (low) multi-index $K(p)$. A multiplication in the algebra $M_q^*(n)$ is consistent with the principle of minimal deformation

$$y_{K(p)} y_{K(r)} = [K(p)]K(r) \ y_{K(r)} y_{K(p)} \quad (2.2)$$

where the commutation function can be determined by Eq(1.12).

Denote the special tensor product of the algebras $M_q$ and $M_q^*$ by the symbol $M_q \otimes_q M_q^* = M_q(x^i, y_k)$. Let us treat $M_q(x, y)$ as the formal-series algebra over $\mathbb{C}$ with $2n$ generators $x^i, y_k$, so the symbol $\otimes_q$ can be omitted in many cases. A multiplication of generators $x^i, y_k$ can be determined by Eqs(1.3, 2.1) and the relation

$$y_k x^i = [ik] \ x^i y_k \quad (2.3)$$

Special tensor products of several algebras $M_q \otimes_q \ldots \otimes_q M_q \otimes_q M_q^* \otimes_q \ldots \otimes_q M_q^* = M(x^i_\alpha, y_{k\beta})$ are defined as the algebra of formal series with several sets of generators $x^i_1 \ldots x^i_p, y_{k1} \ldots y_{kr}$ and the commutation relations independent of the additional indices $\alpha, \beta$

$$x^i_\alpha x^j_\beta = [ij] \ x^j_\beta x^i_\alpha, \ y_{i\alpha} y_{j\beta} = [ij] \ y_{j\beta} y_{i\alpha}$$

$$x^i_\alpha y_{k\beta} = [ki] \ y_{k\beta} x^i_\alpha \quad (2.4)$$

One can analogously define the special tensor product of $I_q$-graded moduli over the algebra $M_q$. 

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3
A standard definition of tensor product $\otimes$ is not consistent with MD-principle

$$(x^i \otimes x^k)(x^l \otimes x^j) = (x^i x^l \otimes x^k x^j)$$

$$(x^i \otimes y_k)(x^l \otimes y_j) = (x^i x^l \otimes y_k y_j)$$

(2.5)

It should be stressed, that the covariant definition of a tensor product in $\mathbb{Z}_2$-graded algebras is consistent with the sign rule [6].

The choice of the relations (2.1, 2.3) follows from the requirement that product of corresponding generators with equal indices $x^1 y_1, x^2 y_2 \ldots$ must commute with any generators $x^i, y_k$:

$$[x^i, x^1 y_1] = 0 = [x^i, x^2 y_2] = \ldots$$

$$[y_k, x^1 y_1] = 0 = [y_k, x^2 y_3] = \ldots$$

(2.6)

A generalized $\tilde{I}_q$-grading for the homogeneous element $z^I_K = x^{I(p)} y_{K(r)}$ of $M_q(x, y)$ can be defined by a table containing upper and low multiindices

$$I \left( \frac{p}{r} \right) = \left[ \frac{I(p)}{K(r)} \right] = \left[ \frac{i_1 \ldots i_p}{k_1 \ldots k_r} \right]$$

(2.7)

A commutation relation for the elements $z^I_K$ has the following form

$$z^I_K z^J_L = \left[ \frac{I|J}{K|L} \right] z^I_K z^J_L$$

$$\left[ \frac{I|J}{K|L} \right] = [I|J] [J|K] [K|L] [L|I]$$

(2.8, 2.9)

where the notation of repeating products of $[ij]$ multipliers Eq(1.12) is used.

A semigroup $\tilde{I}_q$ is the set of pairs $I \left( \frac{p}{r} \right)$ of the symmetrized multiindices (2.7) and the multiplication in $\tilde{I}_q$ can be defined with the help of Eq (1.9):

$$I \left( \frac{p}{r} \right) * I \left( \frac{s}{t} \right) = I \left( \frac{p + s}{r + t} \right)$$

(2.10)

One can introduce the equivalence relation in the algebra $M_q(x, y)$. Let us speak that the elements $z^I_K$ and $z^K_I$ are equivalent if the following relations are fulfilled for any pair of multiindices $J, L$:

$$\left[ \frac{I|J}{K|L} \right] = \left[ \frac{P|J}{N|L} \right]$$

(2.11)

It is easy to show that in a general case the equivalent elements differ by addition of one or several pairs of coinciding upper and low indeces, for example

$$z^i_j \sim z^i_{j1} \sim z^i_{j13} \sim \ldots$$

(2.12)

In particular, all diagonal elements with coinciding upper and low indeces belong to the centre $C(x, y)$ of the algebra $M_q(x, y)$. Note, that the equivalence of elements in $M_q(x, y)$ does not mean the equivalence of their cotransformation laws in the quantum group. A connection of different tensor representation can be realized with the help of contraction operation.
One can see from Refs \[9-11\] that an introduction of partial derivatives in the algebra $M_q(n)$ is uncertain procedure, because there is no accepted generalization of the Leibnitz rule for differentiation on the algebra of functions. Using of the MD-principle removes this uncertainty

\[
\partial_i x^k = \delta^k_i + [ki] x^k \partial_i \\
\partial_i \partial_k = [ik] \partial_k \partial_i 
\]

(2.13)

The $I_q$-deformed external algebra $\Lambda(M_q(n)) = \Lambda_q(n)$ can be defined as an algebra with generators $x^i, \xi^i = dx^i$, which satisfy the following relations

\[
\xi^i \xi^k = - [ik] \xi^k \xi^i \\
x^i \xi^k = [ik] \xi^k x^i 
\]

(2.14)

The algebra $\Lambda_q(n)$ can be treated as a modulus over $M_q(n)$ which has an additional $\mathbb{Z}_n$-grading correspondingly to degrees of $\xi^i$. An operator of external derivation $d = dx^i \partial_i$ \[4,9\] is defined by relations

\[
d^2 = 0, \; \; d(fg) = dfg + (-1)^{(s(f)} f dg 
\]

(2.15)

These relations are consistent with the formula

\[
\partial_k \xi^i = [ik] \xi^i \partial_k 
\]

(2.16)

Basic operators $i_k = \partial/\partial \xi^k$ of an inner derivation in the algebra $\Lambda_q(n)$ satisfy the following relations

\[
i_k \xi^l = \delta^l_k - [lk] \xi^l i_k \\
i_k x^l = [lk] x^l i_k \\
i_k i_l = - [kl] i_l i_k 
\]

(2.17)

A Lie derivative is the operator of zero degree in $\Lambda_q(n)$:

\[
L_k = L(\partial_k) = i_k d + d i_k 
\]

(2.18)

The basic elements in a tangent vector space $D_1(M_q)$ are

\[
x^{i_1} x^{i_2} \ldots x^{i_p} \partial_k = x^{I(p)} \partial_k = D_k^{I(p)} 
\]

(2.19)

Let us define a minimal deformation of commutator in $D_1(M_q)$, which we shall call the $q$-commutator

\[
[D_k^{I(p)}, D_l^{J(r)}]_q = D_k^{I_j} D_l^{J_r} - \left[ \frac{I}{k} \frac{J}{l} \right] D_l^{I_j} D_k^{J_r} 
\]

(2.20)

It is not hard to verify that this $q$-commutator can be decomposed in terms of the sum of basic differential operators. Thus, the space $D_1(M_q)$ with the operation (2.20) can be treated as minimal deformation of the infinite-dimensionlal Lie algebra $Diff(R_n)$. We shall use the name Lie $q$-algebras for the algebras of this type.
3 Minimal deformation of the group GL(n)

Consider a $q$-commutative algebra $A$ over $\mathbb{C}$ with generators $a_k^i$, which satisfy the relation

$$\left[a_k^i, a_k^j\right]_q = a_k^i a_k^j - [ij] [jk] [kl] [li] a_k^j a_k^i = 0 \quad (3.1)$$

The algebra $A(a_k^i)$ has a natural $\tilde{I}_q$-grading

$$\prod_{\alpha=1}^{p} a_{k^\alpha}^i = a_{K(p)}^{I(p)} \rightarrow \left[\frac{I(p)}{K(p)}\right] \quad (3.2)$$

Define a special tensor product $A \otimes_q M_q$ as the $q$-commutative algebra with generators $a_k^i$ and $x^i$

$$a_k^i \otimes_q x^l - [il] [lk] x^l \otimes_q a_k^i = 0 \quad (3.3)$$

Further we shall omit the symbol $\otimes_q$ in multiplication of $a_k^i$ and $x^l$.

Consider the $I_q$-grading-conserving homomorphism $\psi$ of $q$-commutative algebras which we shall call a linear cotransformation:

$$\psi : M_q(n) \rightarrow A \otimes_q M_q(n)$$

$$\psi(x^i) = x^i = a_k^i \otimes_q x^k = a_k^i x^k \quad (3.4)$$

The widespread definition of linear cotransformation on $M_q(n)$ is based on using of the symmetrized tensor product

$$\phi(x^i) = c_k^i \otimes x^k = x^k \otimes c_k^i$$

$$c_k^i c_l^j - [ij] [lk] c_l^j c_k^i = 0 \quad (3.5)$$

This definition is not consistent with $MD$-principle. Note, that a condition on the components $c_k^i$ can be written in the standard form $R c_1 c_2 = c_2 c_1 R$, but a restriction on the components $a_k^i$ in our approach cannot be transformed to the standard form.

Denote a minimal deformation of the linear group by the symbol $QGL(n)$. Define a comultiplication $\Delta$ in the algebra $A(a_k^i)$

$$\Delta(a_k^i) = a_k^i \otimes_q a_k^i = [ik] [kj] [ji] a_j^k \otimes_q a_k^i \quad (3.6)$$

One can define also the following homomorphisms

$$s : A \rightarrow A, \quad s(a_k^i) = \delta_i^i$$

$$\varepsilon : A \rightarrow \mathbb{C}, \quad \varepsilon(a_k^i) = \delta_k^i \quad (3.7)$$

It should be remarked that the homomorphisms $\Delta, s$ and $\varepsilon$ conserve the $I_q$-grading. A standard system of axioms in terms of commutative diagrams is valid for these homomorphisms. We shall call the $q$-commutative bialgebras of this type by the name Hopf $q$-algebras.

Consider the $q$-commutative algebra of formal series $A(a_j^i, b_j^k)$. One can introduce a formal group law, which corresponds to the comultiplication $\Delta$

$$\Delta(a_j^i) = \mu_j^i(a, b) = a_k^i b_j^k \quad (3.8)$$
We shall study the automorphisms of the algebra $A(a^i_j)$ conserving $\tilde{I}_q$-grading. The simplest example of automorphism is connected with the following substitution of generators:

$$a^i_j = \delta^i_j + \alpha^i_j, \quad A(a) \rightarrow A(\alpha) \quad (3.9)$$

This substitution change the parametrization of the formal group law and the maps $s, \varepsilon$ in $QGL(n)$

$$\mu^i_j(\alpha, \beta) = \alpha^i_j + \beta^i_j + \alpha^i_k \beta^k_j$$

$$s^i_j(\alpha) = \sum_{p=1}^{\infty} (-1)^p (\alpha^p)^{i}_{j}, \quad \varepsilon(\alpha^i_j) = 0 \quad (3.10)$$

A validity of the $q$-group axioms can be verified in this parametrization

$$\mu(\mu(\alpha, \beta), \gamma) = \mu(\alpha, \mu(\beta, \gamma))$$

$$\mu(\alpha, s(\alpha)) = \mu(s(\alpha), \alpha) = 0$$

$$\mu(\alpha, 0) = \alpha, \quad \mu(0, \beta) = \beta \quad (3.11)$$

In accordance with the results of Ref[8] there exists a correspondence between deformations of formal Lie groups and deformed Lie algebras. It is convenient to define a linear Lie $q$-algebra in terms of the fundamental corepresentation of $QGL(n)$

$$\psi(x^i) = x^i + \alpha^i_k x^k = x^i + \delta(\alpha) x^i \quad (3.12)$$

$$\delta(\alpha) x^i = (\alpha^k_i L^l_k) x^i \quad (3.13)$$

where $L^l_k$ is the linear operator in the vector space spanned on generators $x^i$. The operator $\delta(\alpha)$ conserves the $I_q$-grading, but the operator $L^l_k$ change this grading. In the simplest representation $L^l_k$ has the form of the first-order differential operator

$$L^l_k = x^l \partial_k$$

$$[L^l_k, L^m_l] = \delta^l_k L^m_l - [ij] [jk] [ki] \delta^m_i L^j_k \quad (3.14)$$

The corresponding matrix representation has the following form

$$L^i_k x^l = (l^i_k)^l_j x^j = \delta^i_k \delta^l_j x^j \quad (3.16)$$

A validity of the commutation relations (3.15) for the matrices $l^i_k$ can be verified by a direct calculation. Denote by $qgl(n)$ the Lie $q$-algebra with generators $L^i_k$ and the commutation relation (3.15). Remark, that the matrices $l^i_k$ can be treated as a representation of the Lie algebra $gl(n)$ if the ordinary matrix commutator is considered instead of the $q$-commutator.

The contragradient representation of $qgl(n)$ can be realized on the algebra $M_q^*(y_i)$

$$\tilde{L}^i_k = -[kl] y_k \partial / \partial y_i$$

$$\tilde{L}^i_k y_l = (\tilde{l}^i_k)^l_j y_j = -[kl] \delta^i_l y_k \quad (3.17)$$

Define an action of the generators $L^i_k$ on tensor representations of $qgl(n)$ by analogy with the action of the differential operator $L^i_k + \tilde{L}^i_k$ on the monomials of the algebra $M_q(x) \otimes_q M_q^*(y)$
\[ L^i_k T^i_{j_1 \ldots j_p} \sim (L^i_k + \tilde{L}^i_k) x^{i_1} \ldots x^{i_p} y_{j_1} \ldots y_{j_p} \] (3.18)

For example, a transformation of the tensor \( T^{i_1i_2} \) can be determined with the help of a matrix

\[ (L^i_k)^{i_1i_2} = \delta^i_k \delta_{j_1}^i \delta_{j_2}^i + [i_1i_2] [i_1k] \delta_{j_1}^i \delta_{j_2}^i \delta_k^i \] (3.19)

It is evident that an arrangement of indices has an essential influence on transformational properties of the \( qgl(n) \)-tensors.

It is easy to define the transformations of tensors by the operators of \( \delta(\alpha) \)-type (3.13), which belong to a \( q \)-envelope of the algebra \( qgl(n) \). This object will be defined in section 4 by the analogy with a Grassman envelope of Lie superalgebras [5].

\[ \delta(\alpha) T^{i_1i_2} = \alpha_{i_1}^i T^{i_1i_2} + \alpha_{i_2}^i T^{i_1i_2} [i_1i_2] \] (3.20)

It should be stressed that the connection with the \( q \)-algebra \( qgl(n) \) arises also in the standard approach to a definition of the coaction of the quantum group \( GL_q(n) \) on \( M_q(n) \) (3.13) in terms of left-invariant differential operators.

Note, that the algebra \( qgl(2) \) is undeformed and isomorphic to the Lie algebra \( gl(2) \) in our approach. This fact was discovered in the investigation of a two-parametric family of the \( GL(2) \)-deformations by a standard method [10]. It is easy to show that our version of differential calculus on \( M_q(n) \) defined in section 2 is covariant under the action of \( qgl(n) \) and \( QGL(n) \).

A \( q \)-commutativity of the matrix elements \( a_k^i \) (3.11) is the sufficient condition for the preservation of \( L_q \)-grading under the linear cotransformation (3.4) on \( M_q(n) \). A necessity of the condition (3.11) follows from the requirement of \( QGL(n) \)-covariance of a tensor product \( \otimes_q \) or relations (2.3), (2.13) and (2.14).

It is useful to consider an additional restriction on the deformation parameters \( q(i,k) = [ik] \), which leads to disappearance of deformations \( qgl(n) \rightarrow gl(n) \) for any \( n \):

\[ [ij] [jk] = [ik] \] (3.21)

The imposing of this condition retains \( n - 1 \) independent parameters \( q(i,i+1) \), then the \( q \)-commutator (3.13) transforms into the ordinary commutator of \( gl(n) \). Even if we use the restriction (3.21) the action of \( gl(n) \)-generator \( L_k^i \) on \( M_q \) depends on the parameters \( [ik] \) and the commutative elements \( a_k^i \) of a formal \( GL(n) \)-transformation do not commute with \( x^i \).

## 4 Minimal deformation of the matrix algebra

In the theory of \( \mathbb{Z}_2 \)-graded superspaces and supergroups there exists a convenient for physical applications language of “points” which is equivalent to the formulation in terms of bundles and Hopf superalgebras [6, 12, 13]. This approach establishes the correspondence of the linear supergroup \( GL(p,q) \) and the matrix algebra \( \text{Mat}(p,q|\Lambda) \) with elements in some arbitrary commutative superalgebra \( \Lambda \) [5, 6].

We consider an analogous approach to a description of the \( q \)-deformed group \( QGL(n) \). Let \( A_k^i(X) \) form an arbitrary set of elements depending on some parameters \( X \) and \( Q(A) \) is the \( q \)-commutative algebra of formal series with generators \( A_k^i(X) \)

\[ [A_k^i(X), A_k^m(X')]_q = 0 \] (4.1)
Consider a set of $n \times n$ matrices $\text{Mat}(n, Q)$ with the elements in $Q(A)$. A structure of the matrix algebra $\text{Mat}(n, Q)$ can be defined with the help of matrix addition and multiplication

\[
(A + A')_k = A_k^i(X) + A_k^i(X')
\]

\[
(A A')_k = A_i^l(X) A_i^l(X')
\]

(4.2)

One can multiply $A_i^k(X)$ by the central elements of the algebra $Q(A)$. Note, that for the standard-type quantum matrices (3.5) $c_i^k$ and $c_i^k$ addition and matrix multiplication cannot be defined simultaneously.

There exists an uncertainty in the choice of $q$-matrix multiplication and one can define the another operation

\[
(A \ast A')_k = A_k^i(X') A_i^i(X) = [kl][li][ik] A_i^i(X) A_i^i(X')
\]

(4.3)

It is evident that these operations of matrix multiplication become identical by the imposing of the condition (3.21) when the algebra $\text{Mat}(n, Q)$ consists of matrices with commuting elements. In the general case we shall use the definition (4.2).

The group $GL(n, Q)$ can be defined as a set of invertible matrices from $\text{Mat}(n, Q)$. The elements $A_i^k(X) \in GL(n, Q)$ correspond to homomorphisms of the Hopf $q$-algebra $QGL(n)$ to the algebra $Q(A)$

\[
\phi_X(a_i^k) = A_i^k(X)
\]

(4.4)

A structure of the group $GL(n, Q)$ determines the homomorphisms $\Delta, s$ and $\varepsilon$ in the $q$-group $QGL(n)$.

Denote by $C(Q)$ a centre of the algebra $Q(A)$ and consider some maps $\text{Mat}(n, Q) \rightarrow C(Q)$. The simplest map of this kind is connected with a calculation of the trace of $q$-matrices

\[
\text{Tr} \; A = A_i^i
\]

\[
[A_k^i(X), \text{Tr} \; A] = 0
\]

(4.5)

The trace function on $q$-matrix products is invariant under the circular permutation

\[
\text{Tr} \; (A_1 A_2 \cdots A_p) = \text{Tr} \; (A_p A_1 A_2 \cdots A_{p-1})
\]

(4.6)

One should use in the proof the ordinary commutativity of the elements $(A_p)_i^k$ and $B_i^k = (A_1 \cdots A_{p-1})_i^k$ with corresponding values of indeces.

A standard form of the Hamilton-Cayley theorem [15] is valid for the matrix $A \in \text{Mat}(n, Q)$

\[
A^n = p_1 A^{n-1} + \cdots + p_{n-1} A + p_n
\]

(4.7)

where $p_r$ are coefficients of the characteristic polinomial which can be written in terms of the functions $\text{Tr} \; (A^k)$, for instance

\[
p_1 = \text{Tr} \; A
\]

(4.8)

A quantum determinant of the $q$-matrix $A$ ($q$-determinant) by definition is proportional to the coefficient $p_n$

\[
\text{qdet}_n A = (-1)^n p_n
\]

(4.9)
In the cases \( n = 2,3 \) quantum determinants have the following form:

\[
\begin{align*}
\text{qdet}_2 A &= p_2 = \frac{1}{2} (\text{Tr } A^2) - \frac{1}{2} \text{Tr } A^2 \\
\text{qdet}_3 A &= -p_3 = \frac{1}{3} \text{Tr } A^3 - \frac{1}{2} \text{Tr } A \text{Tr } A^2 + \frac{1}{6} (\text{Tr } A)^3
\end{align*}
\]  

(4.10) (4.11)

These formulas are completely equivalent to well-known expressions of \( p_n \) for the complex matrices. In the same time the explicit formula of \( \text{qdet}_3 \) in terms of the components \( A^i_k \) contains an essential distinction compared with a formula for the commutative \( \text{det}_3 \):

\[
\begin{align*}
\text{qdet}_3 A &= A^1_1 A^2_2 A^3_3 - A^2_1 A^3_2 A^1_3 + A^3_1 A^1_2 A^2_3 - \\
&- A^3_2 A^1_3 A^1_2 - A^1_3 A^2_3 A^1_2 + A^2_3 A^1_1 A^2_1 \\
&= A^1_1 A^2_2 A^3_3 - A^2_1 A^3_2 A^1_3 - A^3_1 A^2_3 A^1_2 + A^2_3 A^1_1 A^2_1
\end{align*}
\]  

(4.12)

In comparison with \( \text{det}_3 \) this formula has different order of elements in the last term but the rest of terms are completely identical.

Now we present a formula for the function \( \text{qdet}_n A \), which is useful for a proof of the multiplicity of this function in \( \text{Mat}(n,Q) \). Let us use for this purpose a linear cotransformation of the external form \( dx^i = \xi^i \)

\[
\xi^i = A^i_k \xi^k
\]  

(4.13)

Define a \( q \)-deformation of the antisymmetric symbol of \( n \)-th rank

\[
(\varepsilon_q)_{i_1 i_2 \cdots i_n} = \prod_{1 \leq \alpha < \beta} \sqrt{[i_\alpha i_\beta]} \varepsilon_{i_1 i_2 \cdots i_n}
\]  

(4.14)

where \( \varepsilon_{i_1 \cdots i_n} \) is the ordinary antisymmetric symbol. Properties of \( \varepsilon_q \)-symbol follow from the definition immediately, for example

\[
(\varepsilon_q)_{i_1 i_2 \cdots i_n} = -[i_1 i_2] (\varepsilon_q)_{i_2 i_1 \cdots i_n}
\]  

(4.15)

A contravariant symbol \( (\varepsilon_q)^{i_1 \cdots i_n} \) can be defined by analogy with (1.14). Consider the following identity:

\[
(\varepsilon_q)^{j_1 \cdots j_n} (\varepsilon_q)_{i_1 \cdots i_n} \overset{\text{def}}{=} n! \Pi^{j_1 \cdots j_n} = \\
\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_n}_{j_n} - [j_2 j_1] \delta^{i_2}_{j_1} \delta^{i_1}_{j_2} \cdots \delta^{i_n}_{j_n} + \cdots
\]  

(4.16)

\[
\text{Tr } \Pi = 1, \quad \Pi^2 = \Pi
\]  

(4.17)

where \( \Pi \) is a deformation of the antisymmetric proctional operator of the \( n \)-th rank.

\[
\xi^{i_1} \cdots \xi^{i_n} = \Pi^{i_1 \cdots i_n} \xi^{k_1} \cdots \xi^{k_n} = (\varepsilon_q)^{i_1 \cdots i_n} V_n
\]  

(4.18)

A constructive definition of the \( q \)-determinant is connected with the transformation of the volume \( n \)-form \( V_n \) in the group \( GL(n,Q) \)

\[
\xi^{i_1} \cdots \xi^{i_n} = \text{qdet}_n A \xi^{i_1} \cdots \xi^{i_n}
\]  

(4.19)

Using the Eqs (4.13, 4.18, 4.19) and commutation rules of \( A^i_k \) with \( \xi^i \) we shall obtain the following relation

\[
(\varepsilon'_q)^{i_1 \cdots i_n} = (\varepsilon_q)^{i_1 \cdots i_n} \text{qdet}_n A =
\]  


\[
\prod_{\alpha=1}^{n-1} \prod_{\beta=\alpha+1}^{n} [k_\alpha i_\beta] A_{k_1}^{i_1} \cdots A_{k_n}^{i_n} (\varepsilon_q)^{k_1 \cdots k_n} \quad (4.20)
\]

A formula for \(q_{\text{det}}_n A\) can be obtained by contraction of this relation with \((\varepsilon_q)_{i_n \cdots i_1}\). The Eq (4.20) is the key relation for the proof of a multiplicativity of the function \(q_{\text{det}}_n\)

\[
q_{\text{det}}_n(AB) = q_{\text{det}}_n(A) q_{\text{det}}_n(B) \quad (4.21)
\]

In this proof one use the following formulas

\[
\prod_{\alpha=1}^{n} A_{j_\alpha}^{i_\alpha} B_{k_\alpha}^{j_\alpha} = \prod_{\alpha=1}^{n} A_{j_\alpha}^{i_\alpha} \prod_{\beta=1}^{n} B_{k_\beta}^{j_\beta} \prod_{\rho=1}^{n-1} \left[ \frac{j_\rho | i_{\rho+1} \cdots i_n}{k_\rho | j_{\rho+1} \cdots j_n} \right] \quad (4.22)
\]

An exponential parametrization in the group \(GL(n, Q)\) can be constructed on the basis of Campbell-Hausdorf formula

\[
e^u e^v = e^{H(u, v)}
\]

\[
H(u, v) = u + v + \frac{1}{2}[u, v] + \cdots \quad (4.23)
\]

This formula characterizes an exponential map of the Lie algebra \(gl(n, Q) = \text{Mat}(n, Q)\) to the group \(GL(n, Q)\). The Lie algebra \(gl(n, Q)\) is a \(q\)-envelope of the the Lie \(q\)-algebra \(qgl(n)\).

Other \(q\)-groups can be considered naturally as some subgroups of \(GL(n, Q)\). For instance, the group \(SL(n, Q)\) can be determined by imposing of the condition \(q_{\text{det}}_n A = 1\). Denote \(sl(n, Q)\) the corresponding Lie algebra of traceless \(q\)-matrices \(u\)

\[
q_{\text{det}}_n(e^u) = e^{\text{Tr} u} = 1 \quad (4.24)
\]

The quantum group of formal diffeomorphisms \(\text{Diff}(M_q(n))\) can be defined as a set of formal homomorphisms of the algebra \(M_q(n)\) preserving \(I_q\)-grading

\[
x^i = \sum_{p=0}^{\infty} \sum_{I(p)} a_i^{I(p)}(X) x^{I(p)} \quad (4.25)
\]

where \(X\) is an arbitrary variety of parameters. It is evident that this quantum group corresponds to the infinite-dimensional Lie \(q\)-algebra \(D_1(M_q)\).

5 Conclusion

Consider a partial example of the Zamolodchikov algebra \([16]\) which is a local analog of the algebra \(M_q(n)\):

\[
A^i(u) A^k(v) = q(i, k, u - v, h) A^k(v) A^i(u) = q(i, k, u - v, h) q(k, i, v - u, h) = 1 \quad (5.1)
\]

\[
q(i, i, 0, h) = 1, \quad q(i, k, u, 0) = 1 \quad (5.3)
\]
where $q(i, k, u, h)$ are arbitrary functions, $u, v$ are the values of a spectral parameter and $h$ is a quasi-classical parameter. An associativity of the algebra (5.1) is equivalent to validity of the parametric quantum Yang-Baxter equation (QYBE) for $R(u)$ with arbitrary functions $q(i, k, u, h)$.

It is shown in the quantum inverse scattering method (QISM) (see e.g. [14]) that the quantum integrable systems correspond to the wide class of QYBE- solutions $R(u)$. It seems to us very interesting to use the solution (5.1) in QISM.

Note, that $\mathbb{Z}_2$-graded and $(G, f)$-graded generalizations of the Yang-Baxter equation were discussed in Ref[17]. The physical applications of the field commutation relations with constant $R$-matrices and some examples of the diagonal $R$-matrices in QISM were considered in Ref[18].

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