ALGEBRAS OF MINORS

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To Jürgen Herzog on the occasion of his 60th birthday

INTRODUCTION

Let $I$ be an ideal of a Noetherian ring $R$. The Rees algebra of $I$ is the graded $R$-algebra $\bigoplus_{k=0}^{\infty} I^k$. The study of the properties of Rees algebras (and of the other blow-up algebras) has attracted the attention of many researchers in the last three decades. For a detailed account on the theories that have been developed and on the results that have been proved the reader should consult the monograph of Vasconcelos [V]. In this paper we treat a special and interesting case: the Rees algebras of determinantal ideals and their special fibers.

Let $X = (x_{ij})$ be a generic matrix of size $m \times n$ over a field $K$ and let $S$ be the polynomial ring $K[x_{ij}]$. Let $I_t$ be the ideal of $S$ generated by the minors of size $t$ of $X$. Finally, let $R_t$ be the Rees algebra of $I_t$, and let $A_t$ be the subalgebra of $S$ generated by the $t$-minors of $X$. In the case of maximal minors, i.e. $t = \min(m, n)$, the algebra $A_t$ is nothing but the homogeneous coordinate ring of the Grassmann variety which is known to be a Gorenstein factorial domain; for instance, see Bruns and Vetter [BV]. Furthermore, for maximal minors, Eisenbud and Huneke proved in [EH] that $R_t$ is normal and Cohen-Macaulay. Their approach is based on the notion of algebra with straightening law. Later Bruns proved that $R_t$ and $A_t$ are Cohen-Macaulay and normal for any $t$ if $\text{char } K = 0$ using invariant theory methods; see [B]. In [BC1] we have shown, by using Sagbi basis deformation and the Knuth-Rowinson-Schensted correspondence, that $R_t$ and $A_t$ are Cohen-Macaulay and normal domains for any $t$ and for any non-exceptional characteristic. Recall that (in our terminology) $K$ has non-exceptional characteristic if either $\text{char } K = 0$ or $\text{char } K > \min(t, m - t, n - t)$.

For exceptional characteristic $R_t$ and $A_t$ can be (and perhaps are always) very far from being Cohen-Macaulay; see the example in [B].

For maximal minors the divisor class group and the canonical class of $R_t$ have been determined. They are the “expected” ones, that is, those of the Rees algebra of a prime ideal with primary powers in a regular local ring: $\text{Cl}(R_t)$ is free of rank 1 with generator $\text{cl}(I_t R_t)$ and the canonical class is $(2 - \text{height } I_t) \text{cl}(I_t R_t)$; see Herzog and Vasconcelos [HV] or Bruns, Simis and Trung [BST]. On the other hand, for non-maximal minors, $\text{Cl}(R_t)$ is free of rank $t$ [B].

Our goal is to determine the canonical class of $R_t$, the divisor class group of $A_t$ and the canonical class of $A_t$ in all the remaining cases, that is, $t < \min(m, n)$. The case $t = 1$ is anyway easy, and also the study of $A_t$ in the case $m = n = t + 1$ is trivial since that ring is a polynomial ring.
Our main tools are the $\gamma$-functions that allow us to describe all relevant ideals in $R_t$ and $A_t$ and the Sagbi deformation by which we will lift the canonical module from the initial algebras of $R_t$ and $A_t$ to the algebras $R_t$ and $A_t$ themselves. In the following we consider $A_t$ and $R_t$ as subalgebras of the polynomial ring $S[T]$, $A_t$ being generated by the elements of $M_tT$, where $M_t$ is the set of $t$-minors, and $R_t$ being generated by $M_tT$ and the entries of $X$.

It turns out that the canonical class of $R_t$ is given by

$$\text{cl}(\omega(R_t)) = \sum_{i=1}^{t} \left( 2 - (m - i + 1)(n - i + 1) + t - i \right) \text{cl}(P_i)$$

$$= \text{cl}(I_tR_t) + \sum_{i=1}^{t} (1 - \text{height } I_i) \text{cl}(P_i)$$

Here $P_i$ is the prime ideal of $R_t$ determined by $\gamma_i$; it has the property $P_i \cap S = I_i$, and it is well-known that $\text{height } I_i = (m - i + 1)(n - i + 1)$. Moreover, $\text{cl}(I_tR_t) = \sum_{i=1}^{t} (t - i + 1) \text{cl}(P_i)$. The classes $\text{cl}(P_i)$ generate $\text{Cl}(R_t)$ freely.

The divisor class group of $A_t$ is also free; its rank is 1 with generator the class $\text{cl}(q)$ of the ideal $q = (f)S[T] \cap A_t$ where $f$ is a minor of size $t + 1$ of $X$. The canonical class of $A_t$ is given by

$$(mn - mt - nt) \text{cl}(q).$$

It follows that, apart from the trivial or known cases listed above, $A_t$ is Gorenstein if and only if $mn = t(m + n)$.

There are many open questions concerning the algebras $R_t$ and $A_t$. Just to mention few of them:

(a) In general we do not know (the degrees of) the defining equations of $R_t$ and $A_t$. It is known that $R_t$ and $A_t$ are defined by quadrics in the maximal minors case or if $t = 1$ or if $m = n = t + 1$ and that in general higher degree relations are needed. Since $A_t$ has a rational singularity, it has a negative $a$-invariant. This implies that the defining equations of $A_t$ have degree $\leq \dim A_t$ and $\dim A_t = mn$ for non-maximal minors. But this bound is not at all sharp. There are indications that quadrics and cubics are enough for $A_t$ and $R_t$ if $t = 2$.

(b) The Hilbert function of $A_t$ can be computed by the hook length formula (for example, see Stanley [St, 7.21.6]). We have in fact computed these Hilbert functions for a substantial number of cases, but we do not know an explicit formula, not even, say, for the multiplicity of $A_t$.

(c) For maximal minors the algebra $A_t$ has an isolated singularity since the Grassmann variety is a homogeneous variety and, hence, smooth; for instance see [BV]. But in general we do not know (the dimension of) the singular locus of $A_t$.

In principle, the results of this paper can be extended to the other generic determinantal ideals (minors of a symmetric matrix, pfaffians of an alternating matrix, etc.). In the last section we do this in detail for a generic Hankel matrix.

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1. Notation and generalities

We keep the notation of the introduction. We will always assume that the characteristic of the field $K$ is not exceptional, which means that it is either 0 or larger than $\min(t, m - t, n - t)$. We will study the algebras $R_t$ and $A_t$. As mentioned above, both algebras can be embedded into the polynomial ring $S[T]$ where $T$ is an auxiliary indeterminate. Denote by $M_t$ the set of the $t$-minors of $X$. Then $A_t$ is (isomorphic to) the $K$-subalgebra of $S[T]$ generated by the elements of the set $M_tT$, and $R_t$ is the subalgebra generated by the elements of $X$ and of $M_tT$. Clearly, $A_t \subset R_t$ and both algebras are $\mathbb{N}$-graded in a natural way. Moreover $R_t/(X) \simeq A_t$ where $(X)$ is the ideal of $R_t$ generated by the $x_{ij}$. Denote by $V_t$ the $K$-subalgebra of $S[T]$ generated by monomials that have degree $t$ in the variables $x_{ij}$ and degree 1 in $T$. Note that $A_t \subset V_t$ and that $V_t$ is a normal semigroup ring isomorphic to the $t$-th Veronese subring of the polynomial ring $S$. The rings $R_t$ and $A_t$ are known to be normal Cohen-Macaulay domains. The goal of the paper is to determine their divisor class group and canonical class and to discuss the Gorenstein property of these rings. As we have mentioned in the introduction these invariants are already known in some special cases. Therefore we will concentrate our attention to the remaining cases, that is, we make the following

**Assumptions.** (a) When studying the Rees algebra $R_t$, we will assume that $t < \min(m, n)$,

(b) when studying $A_t$, we will assume that $1 < t < \min(m, n)$ and that $m \neq n$ if $t = \min(m, n) - 1$.

Our approach to the study of the algebras under investigation makes use of Sagbi bases deformations and of the straightening law for generic minors. For generalities on the former we refer the reader to Conca, Herzog and Valla [CHV]. As for the latter, the reader can consult Bruns and Vetter [BV], Doubilet, Rota and Stein [DRS], or De Concini, Eisenbud and Procesi [DEP].

Here we just recall its main features. The minor of $X$ with row indices $a_1, \ldots, a_t$ and column indices $b_1, \ldots, b_t$ is denoted by $[a_1, \ldots, a_t|b_1, \ldots, b_t]$.

We will assume that the row and the column indices are given in ascending order $a_i < a_{i+1}$ and $b_i < b_{i+1}$. The set of the minors of $X$ is partially ordered in a natural way:

$$[a_1, \ldots, a_t|b_1, \ldots, b_t] \leq [c_1, \ldots, c_s|d_1, \ldots, d_s]$$

if and only if $t \geq s$ and $a_i \leq c_i$, $b_i \leq d_i$ for all $i = 1, \ldots, s$.

Let $\delta_1, \delta_2, \ldots, \delta_p$ be minors of size $t_1, t_2, \ldots, t_p$ respectively. Consider the product $\Delta = \delta_1 \cdots \delta_p$. The shape of $\Delta$ is by definition the sequence of numbers $t_1, t_2, \ldots, t_p$ and $\Delta$ is said to be a standard monomial if $\delta_1 \leq \cdots \leq \delta_p$. The straightening law asserts the following:

**Theorem 1.1.** The standard monomials form a $K$-vector space basis of $S$. 
We recall now the definition of the functions $\gamma_t$. Given a sequence of numbers $s = s_1, \ldots, s_p$ and a number $t$ we define

$$\gamma_t(s) = \sum_{i=1}^p \max(s_i + 1 - t, 0).$$

Then we extend the definition to products of minors by setting:

$$\gamma_t(\Delta) = \gamma_t(s)$$

where $s$ is the shape of $\Delta$. Finally, for ordinary monomials $M$ of $S$ we define:

$$\rho_t(M) = \sup\{\gamma_t(\Delta) : \Delta \text{ is a product of minors of } X \text{ with } \text{in}(\Delta) = M\}$$

where $\text{in}(f)$ denotes the initial term of a polynomial $f$ with respect to a diagonal term order, i.e. a term order such that $\text{in}(\delta) = x_{a_1b_1} \cdots x_{a_tb_t}$ for every minor $\delta = [a_1, \ldots, a_t | b_1, \ldots, b_t]$ of $X$. In our previous papers [BC1, BC2], the function $\rho_t$ was also denoted by $\gamma_t$; this could create some ambiguities in the present paper, so that we have changed the notation.

The functions $\gamma_t$ were introduced in [DEP] to describe the symbolic powers of $I_t$ and the primary decomposition of the powers of $I_t$. We have shown in [BC1] that the functions $\rho_t$ describe instead the initial ideals of the symbolic and ordinary powers of $I_t$. Let us recall the precise statements:

**Theorem 1.2.**

(a) $I_t^{(k)}$ is $K$-vector space generated by the standard monomials $\Delta$ with $\gamma_t(\Delta) \geq k$, and it contains all products of minors $\Delta$ with $\gamma_t(\Delta) \geq k$.

(b) The initial ideal $\text{in}(I_t^{(k)})$ is the $K$-vector space generated by the ordinary monomials $M$ with $\rho_t(M) \geq k$.

**Theorem 1.3.**

(a) $I_t^k = \bigcap_{l=1}^t I_t^{k(t+1-i)}$ is a primary decomposition of $I_t^k$. In particular, $I_t^k$ is generated by the standard monomials $\Delta$ with $\gamma_t(\Delta) \geq k(t + 1 - i)$ for all $i = 1, \ldots, t$, and it contains all products of minors $\Delta$ satisfying these conditions. Moreover this primary decomposition is irredundant if $t < \min(m, n)$ and $k \geq (v - 1)/(v - t)$ where $v = \min(m, n)$.

(b) The initial ideal $\text{in}(I_t^k)$ is the $K$-vector space generated by the ordinary monomials $M$ with $\rho_t(M) \geq k(t + 1 - i)$ for all $i = 1, \ldots, t$.

We may define the value of function $\gamma_t$ for any polynomial $f$ of $S$ as follows. Let $f = \sum_{i=1}^p \lambda_i \Delta_i$, the unique representation of $f$ as a linear combination of standard monomials $\Delta_i$ with coefficients $\lambda_i \neq 0$. Then we set

$$\gamma_t(f) = \inf\{\gamma_t(\Delta_i) : i = 1, \ldots, p\}$$

and $\gamma_t(0) = +\infty$. This definition is consistent with the one above in the sense that if $\Delta$ is a product of minors of shape $s$ which is non-standard, then $\gamma_t(\Delta) = \gamma_t(s)$.

The function $\gamma_t$ is indeed a discrete valuation on $S$ (with values in $\mathbb{N}$), that is, the following conditions are satisfied for every $f$ and $g$ in $S$:

(a) $\gamma_t(f + g) \geq \min(\gamma_t(f), \gamma_t(g))$, and equality holds if $\gamma_t(f) \neq \gamma_t(g)$,
(b) $\gamma_t(fg) = \gamma_t(f) + \gamma_t(g)$.

Note that (a) follows immediately from the definition, and (b) from the fact that the associated graded ring of a symbolic filtration is a domain if the base ring is regular. We may further extend $\gamma_t$ to the field of fractions $Q(S)$ of $S$ by setting

$$\gamma_t(f/g) = \gamma_t(f) - \gamma_t(g)$$

so that $S$ is a subring of the valuation ring associated with each $\gamma_t$.

The next step is to extend the valuation $\gamma_t$ to the polynomial ring $S[T]$ and to its field of fractions. We want to do this in a way such that the subalgebras $R_t$ and $A_t$ of $S[T]$ will then be described in terms of these functions. So, from now on, let us fix a number $t$ with $1 \leq t \leq \min(m, n)$. For every polynomial $F = \sum_{j=0}^{p} f_j T^j \neq 0$ of $S[T]$ we set

$$\gamma_i(F) = \inf \{\gamma_i(f_j) - j(t + 1 - i) : j = 0, \ldots, p\};$$

in particular

$$\gamma_i(T) = -(t + 1 - i).$$

Then we have:

**Proposition 1.4.** The function $\gamma_i$ defines a discrete valuation on the field of fractions $Q(S[T])$ of $S[T]$.

**Proof.** This is a general fact, see Bourbaki [Bo, Ch. VI, §10, no. 1, lemme 1].

Note that $F = \sum f_j T^j \in S[T]$ has $\gamma_i(F) \geq 0$ if and only if $f_j \in I^{i(j(t+1-i))}_t$ for every $j$. It follows from 1.3 that the Rees algebra $R_t$ of $I_t$ has the following description:

**Lemma 1.5.**

$$R_t = \{F \in S[T] : \gamma_i(F) \geq 0 \text{ for } i = 1, \ldots, t\}.$$ 

Similarly $A_t = \{F \in V_t : \gamma_i(F) \geq 0 \text{ for } i = 2, \ldots, t\}$. But this description is redundant:

**Lemma 1.6.**

$$A_t = \{F \in V_t : \gamma_2(F) \geq 0\}.$$ 

**Proof.** We have to show that if $\Delta$ is a standard monomial with $\deg \Delta = kt$ and $\gamma_2(\Delta) \geq k(t-1)$, then $\gamma_i(\Delta) \geq k(t+1-i)$ for all $j = 3, \ldots, t$. For $i = 1, \ldots, m$ let $a_i$ denote the number of the factors of $\Delta$ which are minors of size $i$. By assumption we have that

(a) $\sum_{i=1}^{m} ia_i = kt$ and

(b) $\sum_{i=2}^{m} (i-1)a_i \geq k(t-1)$.

In view of (a), condition (b) can be rewritten as

(c) $\sum_{i=1}^{m} a_i \leq k$. 

We have to show that
\[(d) \sum_{i=j}^{m} a_i(i+1-j) \geq k(t+1-j).\]

Note that
\[
\sum_{i=j}^{m} a_i(i+1-j) = \sum_{i=1}^{m} a_i(i+1-j) - \sum_{i=1}^{j-1} a_i(i+1-j)
= kt + \sum_{i=1}^{m} a_i(1-j) + \sum_{i=1}^{j-1} a_i(j-i-1).
\]
Therefore (d) is equivalent to
\[
(k - \sum_{i=1}^{m} a_i)(j-1) + \sum_{i=1}^{j-1} a_i(j-i-1) \geq 0
\]
which is true since the left hand side is the sum of non-negative terms. \(\square\)

Similarly the initial algebras of \(\mathcal{R}_t\) and \(\mathcal{A}_t\) have a description in terms of the functions \(\rho_i\). To this end we extend the definition of the function \(\rho_i\) to monomials of \(S[T]\) by setting
\[
\rho_i(MT^k) = \rho_i(M) - k(t+1-i)
\]
where \(M\) is a monomial of \(S\). Furthermore, for a polynomial \(F = \sum_{j=1}^{p} \lambda_j N_j\) of \(S[T]\) where the \(N_i\) are monomials and the \(\lambda_j\) are non-zero elements of \(K\), we set
\[
\rho_i(F) = \inf\{\rho_i(N_j) : j = 1, \ldots, p\}.
\]
Now Theorem 1.3 implies:

**Lemma 1.7.**

1. The initial algebra \(\text{in}(\mathcal{R}_t)\) of \(\mathcal{R}_t\) is given by
\[
\text{in}(\mathcal{R}_t) = \{F \in S[T] : \rho_i(F) \geq 0 \text{ for } i = 1, \ldots, t\}
\]
The set of the monomials \(N\) of \(S[T]\) such that \(\rho_i(N) \geq 0 \text{ for } i = 1, \ldots, t\) form a \(K\)-vector space basis of \(\text{in}(\mathcal{R}_t)\).
2. The initial algebra \(\text{in}(\mathcal{A}_t)\) of \(\mathcal{A}_t\) is given by:
\[
\text{in}(\mathcal{A}_t) = \{F \in V_t : \rho_2(F) \geq 0\}
\]
The set of the monomials \(N\) of \(S[T]\) such that \(\rho_1(N) = 0 \text{ and } \rho_2(N) \geq 0\) form a \(K\)-vector space basis of \(\text{in}(\mathcal{A}_t)\).

The major difference between the functions \(\gamma_i\) and \(\rho_i\) is that the latter is not a valuation. Nevertheless, it will turn out that \(\rho_i\) is an “intersection” of valuations, see Section 3.
2. The divisor class groups of $\mathcal{R}_t$ and $A_t$

The divisor class group of $\mathcal{R}_t$ has already been determined in [B, Cor. 2.4]. In this section we will present a (slightly) different approach to it, which will also be used to determine the divisor class group of $A_t$. We will need the following general facts:

**Lemma 2.1.** Let $A$ be a normal domain. Suppose $v_1, \ldots, v_t$ are discrete valuations on $A$. Let $B = \{x \in A : v_i(x) \geq 0 \text{ for } i = 1, \ldots, t\}$ and $P_i = \{x \in B : v_i(x) \geq 1\}$. Then:

1. Assume that there exist elements $y_1, \ldots, y_t \in A$ with $v_i(y_i) < 0$ and $v_j(y_i) \geq 0$ for $j \neq i$. Then $P_i$ is a height one ideal of the Krull domain $B$ whose symbolic power $P_i^{(j)}$ is $\{x \in B : v_i(x) \geq j\}$ for every $j \geq 1$.

2. Assume that height($P_iA$) $> 1$ for $i = 1, \ldots, t$. Then $A = \bigcap B_Q$, where the intersection is taken over all height 1 prime ideals $Q$ of $B$ with $Q \not\in \{P_1, \ldots, P_t\}$.

**Proof.** (1) Let $V_i$ be the valuation ring of $v_i$. Then evidently

$$B = V_1 \cap \cdots \cap V_t \cap \bigcap_{\text{height } Q = 1} A_Q,$$

and $B$ is obviously a Krull domain. All essential valuation overrings of $B$ occur in the above intersection. Since, by assumption, $y_i$ is in $A$ and in all the $V_j$ with $j \neq i$, but not in $B$, we have that $V_i$ is not redundant in the above description of $B$. Then $V_i$ is an essential valuation overring of $B$. The rest is clear by standard results about essential valuations of Krull domains.

(2) Let $f$ be an arbitrary element of $A$. Let $Q$ be a height 1 prime ideal of $B$ with $Q \not\in \{P_1, \ldots, P_t\}$. We choose an element $g \in P_1 \cap \cdots \cap P_t \setminus Q$. Since $v_i(g) \geq 1$ for all $i$, there is a positive integer $r$ such that $v_i(fg^r) = v_i(f) + rv_i(g) \geq 0$ for all $i$. It follows that $fg^r \in B$. Hence $f \in B_Q$. So we get $A \subseteq B_Q$. To prove $A \supseteq \bigcap B_Q$, we first note that $A = \bigcap A_\Omega$, where $\Omega$ is taken over all height 1 prime ideal $\Omega$ of $A$. So we only need to show that if $Q = \Omega \cap B$, then $Q$ is a height 1 prime ideal of $B$ and $Q \not\in \{P_1, \ldots, P_t\}$. That $Q \not\in \{P_1, \ldots, P_t\}$ follows from the assumption height($P_iA$) $> 1$. To prove height($Q$) $= 1$, it is sufficient to show that $B_Q = A_\Omega$. Let $f \in A \setminus \Omega$. Then $f \not\in QB_Q$ because otherwise there is an element $g \in B \setminus Q$ such that $fg \in Q \subset \Omega$, contradicting the facts that $f \not\in \Omega$, $g \not\in \Omega$. Thus, $f$ is an invertible element in $B_Q$. Hence we can conclude that $A_\Omega \subseteq B_Q$ so that $A_\Omega = B_Q$. \hfill $\Box$

We are grateful to the referee for the above lemma, which has simplified our original treatment. For $i = 1, \ldots, t$ we set $P_i = \{F \in \mathcal{R}_t : \gamma_i(F) \geq 1\}$.

**Lemma 2.2.** The ideal $P_j$ is a height 1 prime ideal of $\mathcal{R}_t$ for $i = 1, \ldots, t$ and $P_i^{(j)} = \{F \in \mathcal{R}_t : \gamma_i(F) \geq j\}$ for every $j > 0$.

**Proof.** We apply 2.1(1) with $A = S[T]$ and $v_i = \gamma_i$. For $i = 1, \ldots, t$ set

$$y_i = g^a f^{t-i}T^{m-i}$$
where \( g \) is an \((i-1)\)-minor, \( f \) is an \( m\)-minor and \( a \) is a large enough integer. The reader may check that these elements have the right \( \gamma \)-values, i.e. \( \gamma_j(y_i) \geq 0 \) for \( j = 1, \ldots, t \) and \( j \neq i \) and \( \gamma_i(y_i) < 0 \).

We can now describe the polynomial ring as a subintersection of the Rees algebra:

**Proposition 2.3.** Set \( R = \mathcal{R}_t \). Then
\[
S[T] = \bigcap R_Q
\]
where the intersection is extended over all the height 1 prime ideals \( Q \) of \( \mathcal{R}_t \) different from \( P_1, \ldots, P_t \).

**Proof.** We apply 2.1(2) with \( A = S[T] \) and \( v_i = \gamma_i \). To this end it suffices to note that \( I_i S[T] \subset P_i S[T] \) for all \( i \). \( \square \)

The ideal \( I_t \mathcal{R}_t \) is a height 1 ideal of \( \mathcal{R}_t \). This follows, for instance, from the fact that \( \mathcal{R}_t \) has dimension \( \dim S + 1 \) and the associated graded ring \( \mathcal{R}_t/I_t \mathcal{R}_t \) has dimension \( \dim S \). Furthermore \( I_t \mathcal{R}_t \) is divisorial since it is isomorphic to \( \bigoplus_{k>0} I_t^k T^k \), which is a height 1 prime ideal.

As a consequence of 2.2 and 1.3 (or [B, Thm. 2.3]) we obtain

**Proposition 2.4.** One has
\[
I_t \mathcal{R}_t = \bigcap_{i=1}^t P_i^{(t-i+1)},
\]
and this is an irredundant primary decomposition of \( I_t \mathcal{R}_t \). In particular, \( P_i \) is a height 1 prime ideal of \( \mathcal{R}_t \) for \( i = 1, \ldots, t \).

**Proof.** By virtue of 2.2 the equality \( I_t \mathcal{R}_t = \bigcap_{i=1}^t P_i^{(t-i+1)} \) is simply a re-interpretation of 1.3. Since we assume that \( t < \min(m, n) \), the primary decomposition of \( I_t^k \) given in 1.3 is irredundant for \( k \gg 0 \). It follows that the primary decomposition of \( I_t \mathcal{R}_t \) is also irredundant. \( \square \)

**Remark 2.5.** Since one can prove directly that \( P_1 \) is a height 1 prime ideal with primary powers, one can also use the standard localization argument in order to show 2.2 and 2.4 (see [B]). However, this argument is not available for the Hankel matrices discussed in Section 6.

**Theorem 2.6.** The divisor class group \( \text{Cl}(\mathcal{R}_t) \) of \( \mathcal{R}_t \) is free of rank \( t \),
\[
\text{Cl}(\mathcal{R}_t) \cong \mathbb{Z}^t
\]
with basis \( \text{cl}(P_1), \ldots, \text{cl}(P_t) \).

**Proof.** The conclusion follows from 2.4 and from a general result of Simis and Trung [ST, Thm. 1.1]. Let us mention that one can derive the result also directly from Fossum [F, Thm. 7.1] and 2.3. \( \square \)

We have similar constructions and results for \( \mathcal{A}_t \). First define
\[
p = \{ F \in \mathcal{A}_t : \gamma_2(F) \geq 1 \}.
\]
It is clear that \( p \) is a prime ideal of \( \mathcal{A}_t \).
Lemma 2.7. The ideal \( p \) is prime of height 1. Moreover, one has \( p^{(j)} = \{ F \in A_t : \gamma_2(F) \geq j \} \).

Proof. We apply 2.1(1) where \( A = V_t \) and \( v_1 = \gamma_2 \). It suffices to take \( f = x_{11}^t T \in V_t \) and note that \( \gamma_2(f) = -t + 1 < 0 \).

Proposition 2.8. Set \( A = A_t \). Then

\[
V_t = \bigcap A_p
\]

where the intersection is extended over all the height 1 prime ideals \( P \) of \( A_t \) with \( P \neq p \).

Proof. We apply 2.1(2) where \( A = V_t \) and \( v_1 = \gamma_2 \). It suffices to prove that \( pA \) has height > 1. To this end take \( f_1 \) and \( f_2 \) distinct \( (t + 1) \)-minors (they exist by our assumptions). Set \( g_j = f_j^t T^{t+1} \). Then \( \gamma_2(g_j) = 1 \) and hence \( g_j \in pV_t \). Since the \( f_i \) are prime elements, \( (f_1^t, f_2^t)S \) has height 2. Since \( V_t \) is a direct summand of the polynomial ring it follows that \( (g_1, g_2)V_t \) has height 2.

Let \( f \) be a \( (t + 1) \)-minor of \( X \). Set \( g = f^t T^{t+1} \). By construction, \( g \in V_t \) and \( \gamma_2(g) = 1 \) so that \( g \in A_t \). Set

\[
q = (f)S[T] \cap A_t.
\]

In other words, \( q = \{ fa \in A_t : a \in S[T] \} \). Since \( f \) is a prime element in \( S[T] \), the ideal \( q \) is prime. Furthermore we have \( pq^t \subset (g) \subset p \cap q \). The second inclusion is trivial. As for the first, note that any generator of \( pq^t \) can be written in the from \( gb \), and then just evaluate \( \gamma_2 \) to show that \( b \) is in \( A_t \).

Note also that:

Corollary 2.9. The ideal \( q \) is a height 1 prime ideal of \( A_t \). Furthermore \( q^{(j)} = (f^j)S[T] \cap A_t \).

Proof. It suffices to check that \( q \) does not contain \( p \). Let \( f_1 \) be a \( (t + 1) \)-minor of \( X \) different from \( f \) (it exist by our assumptions). Then \( f_1^t T^{t+1} \) is in \( p \), but not in \( q \). The second statement is easy.

It follows that:

Theorem 2.10. The divisor class group of \( A_t \) is free of rank 1,

\[
\text{Cl}(A_t) \cong \mathbb{Z}
\]

with basis \( \text{cl}(q) \). Furthermore we have \( \text{cl}(p) = -t \text{cl}(q) \).

Proof. By [F, Thm. 7.1] and 2.8 there is an exact sequence:

\[
0 \to \mathbb{Z} \text{cl}(p) \to \text{Cl}(A_t) \to \text{Cl}(V_t) \to 0.
\]

It is well-known that \( \text{Cl}(V_t) \) is isomorphic to \( \mathbb{Z}/t\mathbb{Z} \) and that it is spanned by the class of the prime ideal \( P \) generated by the elements of the form \( x_{i_j} \mu T \) where \( \mu \) is a monomial in the \( x_{i_j} \) of degree \( t - 1 \). Note that \( \text{cl}(p) \) is a torsion free element in \( \text{Cl}(A_t) \). This is because \( p^{(j)} = \{ F \in A_t : \gamma_2(F) \geq j \} \) cannot be principal; in fact, it contains all the elements of the form \( g^j \) where \( g = f^t T^{t+1} \) and \( f \) is a \( (t + 1) \)-minor.
Now fix a \((t + 1)\)-minor \(f\) and set \(P_1 = (f)S[T] \cap V_i\), that is \(P_1\) is the ideal generated by all the elements of the form \(f \mu T^2\) where \(\mu\) is a monomial of degree \(t - 1\). Evidently \(P_1\) is isomorphic to \(P\) and hence \(\text{cl}(P_1)\) generates \(\text{Cl}(V_i)\). But \(q\) is \(P_1 \cap A_t\) and hence \(\text{cl}(q)\) is the preimage of \(\text{cl}(P_1)\) with respect to the map \(\text{Cl}(A_t) \to \text{Cl}(V_i)\). It follows that \(\text{cl}(p)\) and \(\text{cl}(q)\) generate \(\text{Cl}(A_t)\). By evaluating the function \(\gamma_2\) one shows that

\[
q^{(t)} \cap p = (g)A_t.
\]

Hence \(\text{cl}(p) = -t \text{cl}(q)\). Consequently \(\text{Cl}(A_t)\) is generated by \(\text{cl}(q)\), and \(\text{Cl}(A_t)\) is torsion free. \(\square\)

3. Canonical modules of \(\text{in}(R_t)\) and \(\text{in}(A_t)\)

The goal of this section is to describe the canonical modules of the semigroup rings \(\text{in}(R_t)\) and \(\text{in}(A_t)\). We know that \(\text{in}(R_t)\) and \(\text{in}(A_t)\) are normal (see [BC1]) and hence their canonical modules are the vector spaces spanned by all monomials represented by integral points in the relative interiors of the corresponding cones.

To simplify notation we identify monomials of \(S[T]\) with integral points of \(\mathbb{R}^{mn+1}\). To a subset \(G\) of the lattice \([1, \ldots, m] \times [1, \ldots, n]\) we associate the ideal \(P_G = (x_{ij} : (i, j) \not\in G)\) of \(S\) and a linear form \(\ell_G\) on \(\mathbb{R}^{mn}\) defined by \(\ell_G(x_{ij}) = 1\) if \((i, j) \not\in G\) and \(\ell_G(x_{ij}) = 0\) otherwise.

The initial ideal \(\text{in}(I_i)\) of \(I_i\) is the square free monomial ideal generated by the initial terms (i.e. main diagonal products) of the \(i\)-minors. Therefore \(\text{in}(I_i)\) is the Stanley-Reisner ideal of a simplicial complex \(\Delta_i\). Let \(F_i\) be the set of the facets of \(\Delta_i\). Then \(\text{in}(I_i) = \bigcap_{F \in F_i} P_F\). The elements of \(F_i\) are described by Herzog and Trung [HT] in terms of families of non-intersecting paths. It turns out that \(\Delta_i\) is a pure (even shellable) simplicial complex. We have shown in [BC2] that

\[
\text{in}(I_i^{(k)}) = \bigcap_{F \in F_i} P_F^k.
\]

and that

\[
\text{in}(I_i^k) = \bigcap_{i=1}^{t} \bigcap_{F \in F_i} P_F^{k(t+1-i)}.
\]

For every \(F \in F_i\) we extend the linear form \(\ell_F\) to the linear form \(L_F\) on \(\mathbb{R}^{mn+1}\) by setting \(L_F(T) = -(t + 1 - i)\). Then the equations (2) and (3) imply

**Lemma 3.1.** A monomial \(N\) belongs to \(\text{in}(R_t)\) if and only if it has non-negative exponents and \(L_F(N) \geq 0\) for every \(F \in F_i\) and \(i = 1, \ldots, t\).

This is the description of the semigroup in terms of linear homogeneous inequalities. It follows from the general theory (see Bruns and Herzog [BH, Ch. 6]) that the canonical module \(\omega(\text{in}(R_t))\) of \(\text{in}(R_t)\) is the semigroup ideal of \(R_t\) generated by the monomials \(N\) with all exponents \(\geq 1\) and \(L_F(N) \geq 1\) for every \(F \in F_i\) and \(i = 1, \ldots, t\).

Let \(X\) denote the product of all the variables \(x_{ij}\) with \((i, j) \in [1, \ldots, m] \times [1, \ldots, n]\). The canonical module \(\omega(\text{in}(R_t))\) has a description in terms of \(X\) and the functions \(\rho_i\):
Lemma 3.2. The canonical module $\omega(\text{in}(R_t))$ of $\text{in}(R_t)$ is the ideal
\[
\{ F \in S[T] : \mathcal{X}T \mid F \text{ in } S[T] \text{ and } \rho_i(F) \geq 1 \text{ for every } i = 1, \ldots, t \}
\]
of $\text{in}(R_t)$.

Proof. Let $N = MT^k$ be a monomial, where $M$ is a monomial in the variables $x_{ij}$. Then, for a given $i$, $L_F(N) \geq 1$ for every $F \in F_i$ if and only if $\ell_F(M) \geq k(t+1-i)+1$ for every $F \in F_i$. By (2) this is equivalent to $M \in \text{in}(I_i^{(k(t+1-i)+1)})$ which in turn is equivalent to $\rho_i(M) \geq k(t+1-i)+1$. Summing up, $L_F(N) \geq 1$ for every $F \in F_i$ and $i = 1, \ldots, t$ if and only if $\rho_i(N) \geq 1$ for every $i = 1, \ldots, t$. \hfill \Box

Similarly the canonical module $\omega(\text{in}(A_t))$ has a description in terms of the function $\rho_2$:

Lemma 3.3. The canonical module $\omega(\text{in}(A_t))$ of $\text{in}(A_t)$ is the ideal
\[
\{ F \in V_t : \mathcal{X}T \mid F \text{ in } S[T] \text{ and } \rho_2(F) \geq 1 \}
\]
of $\text{in}(A_t)$.

For later application we record the following lemma. Its part (2) asserts that $\mathcal{X}$ is a “linear” element for the functions $\rho_i$.

Lemma 3.4.

(1) $\rho_i(\mathcal{X}) = (m-i+1)(n-i+1)$.

(2) Let $M$ be any monomial in the $x_{ij}$’s. Then $\rho_i(\mathcal{X}M) = \rho_i(\mathcal{X}) + \rho_i(M)$ for every $i = 1, \ldots, \min(m,n)$.

Proof. Let $M$ be a monomial in the $x_{ij}$’s. We know that $\rho_i(M) \geq k$ if and only if $M \in \text{in}(I_i^{(k)})$. From equation (2) we may deduce that
\[
\rho_i(M) = \inf\{ \ell_F(M) : F \in F_i \}.
\]

Note that $\Delta_i$ is a pure simplicial complex of dimension equal to the dimension of the determinantal ring defined by $I_i$ minus 1. It follows that $\ell_F(\mathcal{X}) = (m-i+1)(n-i+1)$ for every facet $F$ of $\Delta_i$. In particular, $\rho_i(\mathcal{X}) = (m-i+1)(n-i+1)$.

Since $\ell_F(\mathcal{X}M) \geq \ell_F(N) + \ell_F(M)$ for all monomials $N, M$ and for every $F$, we have $\rho_i(\mathcal{X}M) \geq \rho_i(N) + \rho_i(M)$. Conversely, let $G$ be a facet of $\Delta_i$ such that $\rho_i(M) = \ell_G(M)$. Then $\ell_G(\mathcal{X}M) = \ell_G(\mathcal{X}) + \ell_G(M) = \rho_i(\mathcal{X}) + \rho_i(M)$. Hence $\rho_i(\mathcal{X}N) \leq \rho_i(N) + \rho_i(M)$, too. \hfill \Box

4. The deformation lemma

We have been able to identify the canonical modules of $\text{in}(R_t)$ and $\text{in}(A_t)$ and we would like to use this information to describe the canonical modules of $R_t$ and $A_t$. To this end we need a deformation lemma:

Lemma 4.1. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring equipped with a term order $\tau$ and with a grading induced by positive weights $\deg(x_i) = v_i$. Let $B$ be a finitely generated $K$-subalgebra of $R$ generated by homogeneous polynomials and $J$ be a homogeneous ideal of $B$. Denote by $\text{in}(B)$ and $\text{in}(J)$ the initial algebra and the initial ideal of $B$ and $J$ respectively. Then we have:
(1) If \( \text{in}(B) \) is finitely generated and \( \text{in}(B)/\text{in}(J) \) is Cohen-Macaulay, then \( B/J \) is Cohen-Macaulay.

(2) If \( \text{in}(B) \) is finitely generated and Cohen-Macaulay and \( \text{in}(J) \) is the canonical module of \( \text{in}(B) \) (up to shift) then \( B \) is Cohen-Macaulay and \( J \) is the canonical module of \( B \) (up to the same shift).

Proof. (1) Let \( f_1, \ldots, f_k \) be a Sagbi basis of \( B \) and \( g_1, \ldots, g_h \) a Gröbner basis of \( J \). We may assume that these polynomials are monic and homogeneous. Consider the presentation \( K[y_1, \ldots, y_k]/I \cong B \) of \( B \) obtained by mapping \( y_i \) to \( f_i \) and the presentation \( K[y_1, \ldots, y_k]/I_1 \cong \text{in}(B) \) of \( \text{in}(B) \) obtained by mapping \( y_i \) to \( \text{in}(f_i) \). Let \( h_1, \ldots, h_p \) a system of binomial generators for the toric ideal \( I_1 \), say \( h_i = y^{a_i} - y^{b_i} \). Then for each \( i \) we have expressions \( f^{a_i} - f^{b_i} = \sum \lambda_{ij} f^{c_{ij}} \) with \( \lambda_{ij} \in K \setminus \{0\} \) and \( \text{in}(f^{c_{ij}}) < \text{in}(f^{a_i}) \) for every \( i, j \). It is known that the polynomials

\[
y^{a_i} - y^{b_i} - \sum \lambda_{ij} y^{c_{ij}}
\]

(4)
generate \( I \). For each \( g_i \) we may take a presentation \( g_i = f^{d_i} + \sum \delta_{ij} f^{d_{ij}} \) with \( \delta_{ij} \in K \setminus \{0\} \) and \( \text{in}(f^{d_{ij}}) < \text{in}(f^{d_i}) \) for all \( i, j \). Then, by construction, the preimage of \( J \) in \( K[y]/I \) is generated by the elements

\[
y^{d_i} + \sum \delta_{ij} y^{d_{ij}},
\]

(5)
and the preimage of \( \text{in}(J) \) in \( K[y]/I_1 \) is generated by the \( y^{d_i} \). Hence \( B/J \) is the quotient of \( K[y] \) defined by the ideal \( H \) that is generated by the polynomials (4) and (5), and \( \text{in}(B)/\text{in}(J) \) is the quotient of \( K[y] \) defined by the ideal \( H_1 \) that is generated by the polynomials \( h_i \) and \( y^{d_i} \).

If we can find a positive weight \( w \) on \( K[y] \) such that \( \text{in}_w(H) = H_1 \), then there is 1-parameter flat family with special fiber \( \text{in}(B)/\text{in}(J) \) and general fiber \( B/J \). This implies that \( B/J \) is Cohen-Macaulay, provided \( \text{in}(B)/\text{in}(J) \) is. Let us define \( w \). First consider a positive weight \( \alpha \) on \( K[x] \) such that \( \text{in}_\alpha(f^{c_{ij}}) < \text{in}_\alpha(f^{a_i}) \) for every \( i, j \) and \( \text{in}_\alpha(f^{d_{ij}}) < \text{in}_\alpha(f^{d_i}) \) for every \( i, j \). Such an \( \alpha \) exists is a well-known property of monomial orders; for instance, see Sturmfels [S, Proof of Cor. 1.11]. Then we define \( w \) as the “preimage” of \( \alpha \) in the sense that we put \( w(y_i) = \alpha(\text{in}(f_i)) \).

It is clear, by construction, that the initial forms of the polynomials (4) and (5) with respect to \( w \) are exactly the \( h_i \) and the \( y^{d_i} \).

This proves that \( \text{in}_w(H) \) contains \( H_1 \). But \( H_1 \) and \( H \) have the same Hilbert function by construction, and \( H \) and \( \text{in}_w(H) \) have the same Hilbert function because they have the same initial ideal if we refine \( w \) to a term order; for instance see [S, Prop. 1.8]. Here we consider Hilbert functions with respect to the original graded structure induced by the weights \( v_i \). It follows that \( \text{in}_w(H) = H_1 \) and we are done.

(2) That \( B \) is Cohen-Macaulay follows from [CHV]. Since \( B \) is a Cohen-Macaulay positively graded \( K \)-algebra which is a domain, to prove that \( J \) is the canonical module of \( B \) it suffices to show that \( J \) is a maximal Cohen-Macaulay module whose Hilbert series satisfies the relation \( H_J(t) = (-1)^d t^d H_B(t) (t^{-1}) \) for some integer \( k \) where \( d = \dim B \) [BH, Thm. 4.4.5,Cor. 4.4.6].

The relation \( H_J(t) = (-1)^d t^d H_B(t) (t^{-1}) \) holds since by assumption the corresponding relation holds for the initial objects and Hilbert series do not change by taking
initial terms. So it is enough to show that \( J \) is a maximal Cohen-Macaulay module. But \( \text{in}(J) \) is a height 1 ideal since it is the canonical module [BH, Prop. 3.3.18], and hence also \( J \) has height 1. Therefore it suffices to show that \( B/J \) is a Cohen-Macaulay ring. But this follows from (1) since \( \text{in}(B)/\text{in}(J) \) is Cohen-Macaulay (it is even Gorenstein) [BH, Prop. 3.3.18]. \( \square \)

5. The canonical classes of \( \mathcal{R}_t \) and \( A_t \)

In this section we will describe the canonical modules of \( \mathcal{R}_t \) and of \( A_t \) and determine the canonical classes.

Assume for simplicity that \( m \leq n \). Let us consider a product of minors \( D \) such that \( \text{in}(D) = \mathcal{X} \) and \( \gamma_i(D) = \rho_i(\mathcal{X}) \). Since we have already computed \( \rho_i(\mathcal{X}) \) (see 3.4), we can determine the shape of \( D \), which turns out to be \( 1^2, 2^2, \ldots, (m - 1)^2, m^{(n-m+1)} \). In other words, \( D \) must be the product of 2 minors of size 1, 2 minors of size 2, \ldots , 2 minors of size \( m - 1 \) and \( n - m + 1 \) minors of size \( m \). It is then not difficult to show that \( D \) is uniquely determined, the 1-minors are \([m|1]\) and \([1|n]\), the 2-minors are \([m - 1, m|1, 2]\) and \([1, 2|n - 1, n]\) and so on.

We have:

**Theorem 5.1.**

1. The canonical module of \( \mathcal{R}_t \) is the ideal
   \[
   J = \{ F \in S[T] : DT \mid F \text{ in } S[T] \text{ and } \gamma_i(F) \geq 1 \text{ for } i = 1, \ldots, t \}.
   \]

2. The canonical module of \( A_t \) is the ideal
   \[
   J_1 = \{ F \in V_t : DT \mid F \text{ in } S[T] \text{ and } \gamma_2(F) \geq 1 \}.
   \]

**Proof.** By virtue of 4.1 it suffices to show that \( \text{in}(J) \) and \( \text{in}(J_1) \) are the canonical modules of \( \text{in}(\mathcal{R}_t) \) and \( \text{in}(A_t) \) respectively. A description of the canonical modules of \( \text{in}(\mathcal{R}_t) \) and \( \text{in}(A_t) \) has been given in Section 3. Therefore it is enough to check that \( \text{in}(J) \) is exactly the ideal described in 3.2 and \( \text{in}(J_1) \) is the ideal described in 3.3. Note that we may write
   \[
   J = DT\{ F \in S[T] : \gamma_i(F) \geq 1 - \gamma_i(DT) \text{ for } i = 1, \ldots, t \}
   \]
   and
   \[
   J_1 = DT\{ F \in S[T] : \gamma_2(F) \geq 1 - \gamma_2(DT) \} \cap V_t.
   \]
   Furthermore, by virtue of 3.4,
   \[
   \omega(\text{in}(\mathcal{R}_t)) = \mathcal{X}T\{ F \in S[T] : \rho_i(F) \geq 1 - \rho_i(\mathcal{X}T) \text{ for } i = 1, \ldots, t \}
   \]
   and
   \[
   \omega(\text{in}(A_t)) = \mathcal{X}T\{ F \in S[T] : \rho_2(F) \geq 1 - \rho_2(\mathcal{X}T) \} \cap V_t.
   \]
   Since, by the very definition of \( D \), we have \( \text{in}(D) = \mathcal{X} \) and \( \gamma_i(D) = \rho_i(\mathcal{X}) \), it suffices to show that
   \[
   \text{in}\{ F \in S[T] : \gamma_i(F) \geq j \} = \{ F \in S[T] : \rho_i(F) \geq j \}.
   \]
   But this has (essentially) been proved in [BC1]. \( \square \)

Now we determine the canonical class of \( \mathcal{R}_t \).
Theorem 5.2. The canonical class of $R_t$ is given by

$$\text{cl}(\omega(R_t)) = \sum_{i=1}^{t}(2 - (m - i + 1)(n - i + 1) + t - i) \text{cl}(P_i)$$

$$= \text{cl}(I_tR_t) + \sum_{i=1}^{t}(1 - \text{height } I_i) \text{cl}(P_i)$$

Proof. The second formula for $\omega(R_t)$ follows from the first, since height $I_i = (m - i + 1)(n - i + 1)$ and $\text{cl}(I_tR_t) = \sum_{i=1}^{t}(t - i + 1) \text{cl}(P_i)$ by 2.4.

We have seen that

$$\omega(R_t) = DT \{ F \in S[T] : \gamma_i(F) \geq 1 - \gamma_i(DT) \text{ for } i = 1, \ldots, t \}.$$ 

We can get rid of $DT$ and obtain a representation of $\omega(R_t)$ as a fractional ideal, namely,

$$\omega(R_t) = \{ F \in S[T] : \gamma_i(F) \geq 1 - \gamma_i(DT) \text{ for } i = 1, \ldots, t \}.$$ 

It follows that $\omega(R_t) = \cap I_i^{(1 - \gamma_i(DT))}$. As $\gamma_i(DT) = (m - i + 1)(n - i + 1) - (t + 1 - i)$, we are done. \qed

For $A_t$ the situation is slightly more difficult since $D \notin A_t$ in general. Therefore we need an auxiliary lemma:

Lemma 5.3. Let $f_j$ be a $j$-minor of $X$ and $q_j = (f_j)S[T] \cap A_t$. Then $q_j$ is a height 1 prime ideal of $A_t$ and $\text{cl}(q_j) = (j - t) \text{cl}(q)$.

Proof. Note that $q_{t+1} = q$ by definition. Let $\Delta$ be a product of minors. We say that $\Delta$ has tight shape if its degree is divisible by $t$ and it has exactly $\deg(\Delta)/t$ factors. In other words, $\Delta$ has tight shape if $\gamma_1(\Delta^k) = 0$ and $\gamma_2(\Delta^k) = 0$ where $k = \deg(\Delta)/t$. Let $\Delta$ be a product of minors with tight shape and $k = \deg(\Delta)/t$. Note that

$$(\Delta)S[T] \cap A_t = (\Delta^k)A_t.$$ 

Now fix $j \leq t$ and set $k = t - j + 1$; the product $\Delta = f_jf_{t+1}^{t-j}$ has tight shape and hence $q_j \cap q_{t+1}^{t-j} = (\Delta^k)A_t$. Note that, for obvious reasons, $q_j$ does not contain $q$. It follows that $q_j$ is a prime ideal of height 1 and that $\text{cl}(q_j) = (j - t) \text{cl}(q)$. Now take $j > t$ and set $k = j - t + 1$; the product $\Delta = f_jf_{t+1}^{t-j}$ has tight shape and hence, as above, we conclude that $q_j$ is prime of height 1 and $\text{cl}(q_j) = (t - j) \text{cl}(q_{t-1})$. Since we know already that $\text{cl}(q_{t-1}) = -\text{cl}(q)$, we are done. \qed

Now we can prove

Theorem 5.4. The canonical class of $A_t$ is given by

$$\text{cl}(\omega(A_t)) = (mn - tm - tn) \text{cl}(q).$$

Proof. Assume that $m \leq n$. Set $W = (DT)S[T] \cap A_t$. We have seen that

$$\omega(A_t) = W \cap p.$$ 

Consequently

$$\text{cl}(\omega(A_t)) = \text{cl}(W) + \text{cl}(p) = \text{cl}(W) - t \text{cl}(q).$$
Note that $W$ can be written as the intersection of $(D)S[T] \cap A_t$ and $(T)S[T] \cap A_t$. But the latter is the irrelevant maximal ideal of $A_t$, whence $W = (D)S[T] \cap A_t$. Further $(D)S[T] \cap A_t$ can be written as an intersection of ideals $q_j$. Taking into consideration the shape of $D$ and 5.3, we have

$$\text{cl}(W) = \left( \sum_{j=1}^{m-1} 2(j - t) + (n - m + 1)(m - t) \right) \text{cl}(q_j).$$

Summing up, we get the desired result. \hfill \square

As a corollary we have

**Theorem 5.5.** The ring $A_t$ is Gorenstein if and only if $mn = t(m + n)$.

Note that we assume $1 < t < \min(m, n)$ and $m \neq n$ if $t = \min(m, n) - 1$ in the theorem. We have observed in the introduction that $A_t$ is indeed Gorenstein (and even factorial) in the remaining cases.

**Remark 5.6.** It is also possible to derive Theorem 5.4 from Theorem 5.2 by a suitable generalization of [BV, (8.10)].

## 6. Algebras of minors of generic Hankel matrices

Let $S$ be the polynomial ring $K[x_1, \ldots, x_n]$ over some field $K$. Choose a Hankel matrix $X$ with distinct entries $x_1, \ldots, x_n$; this means that $X$ is an $a \times b$ matrix $(x_{ij})$ with $x_{ij} = x_{i+j-1}$ and $a + b - 1 = n$. Let $I_t$ be the ideal generated by the minors of size $t$ of $X$. It is known that $I_t$ does not depend on the size of the matrix $X$ (provided, of course, $X$ contains $t$-minors); it depends only on $t$ and $n$. For a given $n$ it follows that $t$ may vary from 1 to $m$, where $m = \lceil (n + 1)/2 \rceil$ is the integer part of $(n + 1)/2$. All the properties of the generic determinantal ideals that we have used in the previous sections to study their algebras of minors hold also for the determinantal ideals of Hankel matrices. This has been shown in Conca [C]. In particular:

(1) the symbolic powers $I_t^{(k)}$ and the primary decomposition of $I_t^k$ are described in terms of the $\gamma$-functions;

(2) the initial algebras are described in terms of the corresponding functions for monomials, the $\rho$-functions;

(3) the $\rho$-functions can be described in terms of the linear forms associated to the facets of the simplicial complex $\Delta_i$ of $\text{in}(I_i)$; moreover $\Delta_i$ is a pure simplicial complex;

(4) the determinantal ideal $I_t$ is prime and height $I_t = n - 2t + 2$; in particular the minors of $X$ are irreducible polynomials,

(5) the product of all the variables is a linear element for the $\rho$ functions, that is, if $M$ is a monomial and $X$ is the product of all the variables then $\rho_i(MX) = \rho_i(M) + \rho_i(X)$.

Now fix a number $t$, $1 \leq t \leq m = \lceil (n + 1)/2 \rceil$. Denote by $R_t$ the Rees algebra of $I_t$ and by $A_t$ the algebra generated by the minors of size $t$. These algebras have been studied in [C, Section 4]. It turns out that $R_t$ and $A_t$ are Cohen-Macaulay algebras.
normal domains and that the dimension of $A_t$ is $n$, unless $t = m$. In the latter case, the minors are algebraically independent. The arguments of the previous sections apply also to the present situation. One deduces the following theorem:

**Theorem 6.1.**

1. If $t < m$, then the divisor class group $\text{Cl}(R_t)$ of $R_t$ is free of rank $t$ with basis $\text{cl}(P_1), \ldots, \text{cl}(P_t)$. Here $P_i$ is the prime ideal of $R_t$ associated with the valuation $\gamma_i$ and $P_i \cap S = I_i$. The canonical class is

$$\text{cl}(\omega(R_t)) = \sum_{i=1}^{t} (-n + t + i) \text{cl}(P_i)$$

$$= \text{cl}(I_tR_t) + \sum_{i=1}^{t} (1 - \text{height } I_i) \text{cl}(P_i)$$

2. If $t = m$ and $n$ is even, then $\text{Cl}(R_t)$ is free of rank 1 with basis $\text{cl}(P)$ where $P = I_mR_t$. In this case $R_t$ is a complete intersection and $\text{cl}(\omega(R_t)) = 0$.

3. If $t = m$ and $n$ is odd, then $I_m$ is principal.

To prove the theorem one argues like in the generic case. The only difference is that the product of minors $D$ such that $\text{in}(D) = \mathcal{X}$ and $\gamma_i(D) = \rho_i(\mathcal{X})$ now has shape $m, m - 1$ if $n$ is odd, and shape $m, m$ if $n$ is even.

For the algebra $A_t$ we have:

**Theorem 6.2.**

1. Assume that $1 < t < m$ and $n$ is even if $t = m - 1$. Then the divisor class group $\text{Cl}(A_t)$ of $A_t$ is free of rank 1 with basis $\text{cl}(q)$. Here $q$ is the prime ideal of $A_t$ defined as $q = (f)S[T] \cap A_t$ where $f$ is a minor of size $t + 1$. The canonical class is $\text{cl}(\omega(A_t)) = (n - 3t) \text{cl}(q)$.

2. If $t = m - 1$ and $n$ is odd, then $A_t$ is isomorphic to the coordinate ring $\text{Grass}(m - 1, m + 1)$ of the Grassmann variety of the subspaces of dimension $m - 1$ in a vector space of dimension $m + 1$. In particular $A_t$ is factorial.

3. If $t = m$ or $t = 1$, then $A_t$ is a polynomial ring.

The only assertion which still needs an argument is (2). To this end note that $I_t$ is the ideal of maximal minors of the Hankel matrix of size $t \times (n + 1 - t)$. This induces a surjective map from $\text{Grass}(t, n + 1 - t)$ to $A_t$. If we take $n$ odd and $t = m - 1$, then the dimension of $\text{Grass}(t, n + 1 - t)$ is $n$, which is also the dimension of $A_t$. Hence the map $\text{Grass}(t, n + 1 - t) \to A_t$ is an isomorphism.

**Theorem 6.3.** The ring $A_t$ is Gorenstein if and only if one of the following conditions holds:

1. $3t = n$,
2. $t = m - 1$ and $n$ is odd,
3. $t = m$ or $t = 1$.

Despite of the analogy, let us point out that there are some important differences between the Rees algebras $R_t$ and the algebras $A_t$ for generic matrices and those for
Hankel matrices. In the Hankel case the algebras $R_t$ and $A_t$ are defined by (Gröbner bases of) quadrics, one has $\text{in}(I_t^k) = \text{in}(I_t)^k$ for all $t$ and $k$, and all the results hold over an arbitrary field, no matter what the characteristic is.

**References**

[Bo] N. Bourbaki, *Algèbre commutative, Chap. I–IX*. Hermann, Masson, 1961–1983.

[B] W. Bruns, *Algebras defined by powers of determinantal ideals*, J. of Algebra 142 (1991), 150–163.

[BC1] W. Bruns and A. Conca, *KRS and powers of determinantal ideals*, Compositio Math. 111 (1998), 111–122.

[BC2] W. Bruns and A. Conca, *KRS and determinantal ideals*, preprint 2000.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.

[BST] W. Bruns, A. Simis, and N. V. Trung, *Blow-up of straightening-closed ideals in ordinal Hodge algebras*, Trans. Amer. Math. Soc. 326 (1991), 507–528.

[BV] W. Bruns and U. Vetter, *Determinantal rings*, Lect. Notes Math. 1327, Springer 1988.

[C] A. Conca, *Straightening law and powers of determinantal ideals of Hankel matrices*, Adv. Math. 138 (1998), 263–292.

[CHV] A. Conca, J. Herzog, and G. Valla, *Sagbi bases and application to blow-up algebras*, J. Reine Angew. Math. 474 (1996), 113–138.

[DEP] C. De Concini, D. Eisenbud, and C. Procesi, *Young diagrams and determinantal varieties*, Invent. math. 56 (1980), 129–165.

[DRS] P. Doubitel, G.C. Rota, and J. Stein, *On the foundations of combinatorial theory: IX, Combinatorial methods in invariants theory*, Studies in Applied Mathematics LIII (1974), 185–216.

[EH] D. Eisenbud and C. Huneke, *Cohen-Macaulay Rees algebras and their specialization*, J. Algebra 81 (1983), 202–224.

[F] R. Fossum, *The divisor class group of a Krull domain*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74. Springer-Verlag, New York-Heidelberg, 1973.

[HSV1] J. Herzog, A. Simis, and W. Vasconcelos, *On the canonical module of the Rees algebra and the associated graded ring of an ideal*, J. Algebra 105 (1987), 285–302.

[HSV2] J. Herzog, A. Simis, and W. Vasconcelos, *Arithmetic of normal Rees algebras*, J. Algebra 143 (1991), 269–294.

[HV] J. Herzog and W. Vasconcelos, *On the divisor class group of Rees-algebras*, J. Algebra 93 (1985), 182–188.

[HT] J. Herzog and N. V. Trung, *Gröbner bases and multiplicity of determinantal and pfaffian ideals*, Adv. in Math. 96 (1992), 1–37.

[ST] A. Simis and N. V. Trung, *The divisor class group of ordinary and symbolic blow-ups*, Math. Z. 198 (1988), 479–491.

[St] R. P. Stanley, *Enumerative combinatorics, Vol. 2*, Cambridge University Press 1999.

[S] B. Sturmfels, *Gröbner bases and convex polytopes*, AMS University Lecture Series, Vol. 8, Providence RI, 1995.

[V] W. V. Vasconcelos, *Arithmetic of blowup algebras*, London Mathematical Society Lecture Note Series, 195, Cambridge University Press, Cambridge, 1994.

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