Sympathetic cooling and growth of a Bose-Einstein condensate

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We study two sets of rate equations for sympathetic cooling of harmonically trapped Bose gases. Calculations for mixtures of Na-Rb and Li-Cs show that both sets yield similar results for the cooling times. The equilibration rates are in fair agreement with each other and differ considerably from classical rates. The onset of Bose–Einstein condensation is rather sudden and non–exponential in time, and the growth of the condensate differs for the two different mixtures we studied.

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I. INTRODUCTION

The cooling of atoms in traps is an important tool in the study of ultracold Bose or Fermi gases. Usually, the last step in the cooling process is evaporative cooling. This technique, however, can only be used for sufficiently strongly interacting gases since thermodynamic equilibrium of the cooled gas must be (nearly) attained during the entire cooling process. There is a number of gases in which the interaction is too weak for evaporative cooling to work. Moreover, because of the exclusion principle the method fails at low temperature for identical Bose particles. This technique, however, can only be used for sufficiently strongly interacting gases since thermodynamic equilibrium of the cooled gas must be (nearly) attained during the entire cooling process. There is a number of gases in which the interaction is too weak for evaporative cooling to work. Moreover, because of the exclusion principle the method fails at low temperature for identical Bose particles: Another gas is cooled evaporatively and acts as the cooling agent for the gas to be cooled.

The method of sympathetic cooling was proposed almost twenty years ago and has since found widespread application (see Refs. [1, 2]). Exciting recent applications include the sympathetic cooling of atoms of different species: 41K and 87Rb [3], 7Li and 144Cs [4], of different isotopes of the same species: the bosonic species 85Rb cooled by 87Rb [5] and 6Li Fermions in a bath of 7Li Bosons [6, 7], and the production of dual Bose–Einstein condensates (BEC) with sympathetic cooling [8, 9]. Only a few theoretical descriptions of sympathetic cooling exist. The simplest descriptions are purely classical while more sophisticated employ the quantum Boltzmann equation or master equations. Delannoy et al. [10] and Mosk et al. [11] derived a classical formula for the time dependence of the temperature and a classical formula for the cooling rate. Geist et al. [12] have formulated sympathetic cooling of Fermi gases in terms of a quantum Boltzmann equation. The tremendous simplification of the collision matrix elements achieved in this way precludes, however, a quantitative description of the cooling process. Lewenstein et al. [13] derived a master equation for sympathetic cooling. Unfortunately, that master equation is too complex to be practically useful. Recently, the present authors simplified this master equation by means of decoherence and ergodicity arguments [14] and obtained two different sets of rate equations that both describe sympathetic cooling of Bose or Fermi gases. These rate equations are simple enough to permit practical calculations while they contain detailed information about the quantum scattering processes between the cooled gas and the bath.

The purpose of this article is twofold. First, we want to compare the solutions of our rate equations with the much simpler expression for the classical cooling rate obtained by Delannoy et al. [10] and Mosk et al. [11]. Second, we use the rate equations to study the growth of a Bose–Einstein condensate for a system of 23Na cooled by 87Rb, and for 7Li in a bath of 133Cs.

This article is organized as follows. In Section II we present the rate equations that we use. The results, i.e. the comparison between quantum and classical cooling rates and the study of the growth of the condensate, are presented in Sect. III. Finally we give our summary.

II. RATE EQUATIONS FOR SYMPATHETIC COOLING

In this Section we present two different sets of rate equations that are derived in ref. [16]. Both sets of rate equations have been derived from a master equation [15] for sympathetic cooling by means of decoherence and ergodicity arguments. It is assumed that the cooling agent has a considerably larger mass than the gas to be cooled. The latter consists of $N_A$ (very weakly interacting) bosons in a spherically harmonic trap of frequency $\nu$. Single-particle trap orbitals have energies $E_n = \hbar \nu (m_x + m_y + m_z)$ where $m = (m_x, m_y, m_z)$ denotes the set of Cartesian quantum numbers, and the single-particle ground state energy is set to zero.

The first set of rate equations describes how the mean occupation number $n_{\vec{m}_0}$ of orbital $\vec{m}_0$ changes due to interactions with the cooling agent:

$$\frac{d}{dt} n_{\vec{m}_0}(t) = 2 \sum_{\vec{m} \neq \vec{m}_0} \Gamma_{\vec{m}_0, \vec{m}} n_{\vec{m}}(t) (n_{\vec{m}_0}(t) + 1) \nu$$

$$- 2 \sum_{\vec{m} \neq \vec{m}_0} \Gamma_{\vec{m}, \vec{m}_0} n_{\vec{m}}(t + 1) n_{\vec{m}_0}(t). \quad (1)$$
The form of these equations is intuitively obvious. Note that the rate equations (1) are of mean-field type and neglect correlations between occupancies of different single-particle orbitals. Particle number conservation implies that this approximation may fail when the occupancy of the ground state \( \tilde{m} = 0 \) approaches the total number of particles in the gas to be cooled. Particular solutions to these mean-field type rate equations are discussed in Refs. 16.

The rate coefficients \( \Gamma^{\tilde{m},\tilde{n}}_{\tilde{m},\tilde{n}} \) are input to the rate equations (1). For the computation of the rate coefficients \( \Gamma \) we assume that the bath particles are Boltzmann-distributed. A practical recipe for their computation is given in Ref. 14. We express the \( \Gamma \)’s in units of

\[
\omega = (32\pi^4)^{-1}\Lambda_B^3n_B(a/l_0)^2[(M + m)^2/(Mm)]\nu. \tag{2}
\]

Here \( n_B \) is the density of bath particles, \( \Lambda_B = (2\pi\hbar^2/\beta_B)^{1/2} \) their thermal de-Broglie wave length, \( \beta_B \) the inverse temperature, \( l_0 \) the oscillator length, and \( M \) and \( m \) are the mass of the cooling agent and the gas to be cooled, respectively. The interaction between bath and system is described in terms of the \( s \)-wave scattering length \( a \).

The second set of rate equations is based on a microcanonical approach and describes how the probability \( p_M \) of having the gas to be cooled at total energy \( M\hbar\nu \) changes with time:

\[
\frac{dp_M}{dt} = 2\sum_{\tilde{m}\neq\tilde{n}}\Gamma^{\tilde{m},\tilde{n}}_{\tilde{m},\tilde{n}} \times \left( p_{M+\alpha}\langle n_{\tilde{m}}[n_{\tilde{m}} + 1]\rangle_M + p_M\langle n_{\tilde{n}}[n_{\tilde{n}} + 1]\rangle_M - p_M\langle n_{\tilde{m}}[n_{\tilde{m}} + 1]\rangle_M \right). \tag{3}
\]

Here, \( \alpha = \epsilon_{\tilde{m}} - \epsilon_{\tilde{n}} \) is the energy transfer, \( n_{\tilde{m}} \) is the occupancy of single-particle orbital \( \tilde{m} \), and \( \langle n_{\tilde{m}}[n_{\tilde{m}} + 1]\rangle_M \) denotes the mean value of \( n_{\tilde{m}}(n_{\tilde{m}} + 1) \) taken over the many-body states with fixed energy \( M\hbar\nu \). They are thus microcanonical averages of occupancies. It is straightforward to check that \( \sum_M dp_M(t)/dt = 0 \). In Eq. (3), the \( p_M \)’s are the unknowns while the rate coefficients \( \Gamma \) and the quantities \( \langle n_{\tilde{m}}(n_{\tilde{m}} + 1)\rangle_M \) are input quantities defined in the framework of our model. The rate coefficients are identical to those used in the mean-field type rate equations (1). To compute the quantities \( \langle n_{\tilde{m}}[n_{\tilde{m}} + 1]\rangle_M \) we neglect correlations by putting \( \langle n_{\tilde{m}}[n_{\tilde{m}} + 1]\rangle_M \approx \langle n_{\tilde{m}}\rangle_M\langle n_{\tilde{m}} + 1\rangle_M \) and replace the microcanonical average by the grand canonical average

\[
\langle n_{\tilde{m}}\rangle_M = n_{\tilde{m}}(z,T) = \frac{z(E)\exp[-\epsilon_{\tilde{m}}/\kappa T(E)]}{1 - z(E)\exp[-\epsilon_{\tilde{m}}/\kappa T(E)]}|_{E=M\hbar\nu}. \tag{4}
\]

Expressions for the energy-dependent fugacity \( z(E) \) and temperature \( T(E) \) can be obtained numerically from the grand canonical partition function of non-interacting bosons in a harmonic trap. For details, we refer the reader to Ref. 14. Note that the the microcanonical rate equations are expected to be valid also in the regime of BEC.

III. RESULTS

We study the cooling process for different temperatures of the bath and for different combinations of bath and system. The system always consists of \( N_A = 400 \) Bosons and the harmonic trap is cut off beyond the single-particle orbitals with energy \( K\hbar\nu, K = 21 \). In the microcanonical approach, the homogeneous system of linear rate equations (1) can be put into matrix form. For stability reasons, the real parts of the eigenvalues of this matrix must be zero or negative. The equilibrium solution is determined by the eigenvalue zero. The equilibration rate is given by the modulus of the real part of the eigenvalue with largest negative real part. The leading eigenvalues are computed using the sparse matrix solver ARPACK 15. Within our numerical accuracy we found one zero eigenvalue and no eigenvalues with positive real parts. The results for the equilibration rate \( \gamma_{eq} \) are listed in Table I. The temperatures are chosen such that \( \kappa T = 7\hbar\nu \) is close to the critical temperature while \( \kappa T = 3\hbar\nu \) yields a BEC at the end of the cooling process (\( \kappa \) denotes Boltzmann’s constant).

We turn to the rate equations (1). We took the Bosons of the system \( A \) to be initially equally distributed over the degenerate single-particle orbitals with energy \( K\hbar\nu \), all other orbitals being empty. This initial configuration has the highest energy that can be realized in the finite trap we are considering. While details of the initial conditions are reflected in the short-time evolution they cannot affect the long-time behavior and the equilibration rate. The rate equations (1) need to be modified in the regime of BEC; we therefore restrict the computation to the case \( \kappa T = 7\hbar\nu \). We solve the rate equations numerically and extract the equilibration rate \( \gamma_{eq} \) from the solution at large times. The results are listed in Table I. Note that the equilibration rates \( \gamma_{eq} \) and \( \gamma_{eq} \) depend only very weakly on the composition of system and bath for \( \kappa T = 7\hbar\nu \). Comparing the rate \( \gamma_{eq} \) obtained in the microcanonical approach with the rate \( \gamma_{eq} \) from factorization, we note that the former is about a factor \( \approx 1.7 \) larger than the latter. The time evolution however shows that the loss of energy at short times is practically identical in both approaches. Most of the energy is removed from system \( A \) during the early part of the cooling process. Therefore, both approaches yield comparable predictions for the cooling time.

It is difficult to compare our full time dependence with that of the classical theory 12, 13 since we assumed a finite trap and, thus, different initial conditions. It is however meaningful to compare equilibration rates since these do not depend on initial conditions. In order to compare the equilibration rates, we have adjusted the formulas of Ref. 12, 13 to the case of a homogeneous bath. In Table II the classical rate \( \gamma_{eq} \) (which decreases quadratically with decreasing temperature) is seen to be about a factor 2 larger than the quantum–mechanical rate \( \gamma_{eq} \) below the condensation temperature, and a factor 1.7 to 3 smaller in the regime of BEC. This casts some
Bath - System | $\kappa T / \hbar \nu$ | $\gamma_{eq}/10^4 \omega$ | $\bar{\gamma}_{eq}/10^4 \omega$ | $\tilde{\gamma}_{eq}/10^4 \omega$
--- | --- | --- | --- | ---
$^{87}$Rb - $^{23}$Na | 7.0 | 2.7 | 1.6 | 5.84
$^{87}$Rb - $^{23}$Na | 3.0 | 3.1 | – | 1.07
$^{133}$Cs - $^7$Li | 7.0 | 2.5 | 1.7 | 4.97
$^{133}$Cs - $^7$Li | 3.0 | 1.66 | – | 0.913

**TABLE I:**
Comparison of quantum–mechanical equilibration rates $\gamma_{eq}$ (obtained from the solution of Eq. (3)) and $\tilde{\gamma}_{eq}$ (obtained from the solution of Eq. (1)) with the classical rate $\bar{\gamma}_{eq}$ for several systems and temperatures. The critical temperature for onset of BEC is just below $\kappa T / \hbar \nu = 7$. The classical equilibration rates differ considerably from the quantum mechanical rates.

doubt on the possibility to describe sympathetic cooling in terms of the much simpler classical theory.

It is particularly interesting to study the growth of a BEC. As pointed out above, the rate equations (1) cannot be used in this regime. We chose $1 \kappa T = 3 \hbar \nu$ and followed the time evolution of the total energy $E(t) = \hbar \nu \sum_M M p_M(t)$ by solving the rate equations (3) numerically. As initial conditions we took $p_{KN_A} = 1$ and $p_M = 0$ for $M < KN_A$. This is the configuration with highest energy that might be realized in the open trap. Using the grand canonical partition function, we obtain the time-dependent temperature $T(E)$ for the system, and the time-dependent ground state occupation number $n_0$ from Eq. (4). Fig. 1 shows the result for two different systems. Note that the BEC grows faster in the Rb-Na system, which also has the larger equilibration rate [in units of $\omega$, see eq. (2)]. The reason for this behavior is not easy to understand: Table I shows that equilibration rates for both systems are quite similar at temperature $\kappa T / \hbar \nu = 7$ while they differ considerably below the critical temperature. Note that a naive (and classical) explanation would link the difference in rates to the smaller momentum transfer during Li-Cs collisions and thereby to the larger mass difference. Our quantum mechanical results show that the situation is more complex. The onset of BEC is rather sudden and non-exponential while the long time evolution is exponential. Qualitatively similar observations have been made in numerical simulations of evaporative cooling [19, 21, 22]. One does not obtain 100% BEC since the final temperature is only about half the condensation temperature and the system is not of macroscopic size.

IV. SUMMARY

In summary, we have used two sets of rate equations to describe sympathetic cooling of Bose gases by a much heavier gas. Practical calculations in the systems $^{23}$Na-$^{87}$Rb and $^7$Li-$^{133}$Cs show that the cooling times from both rate equations agree well with each other. Our quantum–mechanical equilibration rates are in fair agreement with each other but differ considerably from the classical rates. This is unfortunate since the classical approach is much simpler than the quantum–mechanical description presented in this work. We studied the growth of the Bose-Einstein condensate and found that the condensate in the system $^{23}$Na-$^{87}$Rb grows faster than in $^7$Li-$^{133}$Cs. At the onset of Bose–Einstein condensation we observe a rather sudden and non–exponential growth of the condensate.

FIG. 1: Relative ground-state occupation as function of time for Li in Cs bath (full line) and Na in Rb bath (dashed line) at temperature $\kappa T = 3 \hbar \nu$.

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[1] B. DeMarco et al., Phys. Rev. Lett. 82 (1999) 4208.
[2] D. J. Wineland, R. E. Drullinger, and F. L. Walls, Phys. Rev. Lett. 40 (1978) 1639.
[3] R. E. Drullinger, D. J. Wineland, and J. C. Berquist, Appl. Phys. 22 (1980) 365.
[4] D. J. Larson et al., Phys. Rev. Lett. 57 (1986) 70.
[5] Atomic Physics 9, edited by R. S. van Dyck, Jr., and E. N. Fortson, World Scientific, Singapore 1985.
[6] G. Modugno et al., Science 294, 1320 (2001).
[7] M. Mudrich et al., arXiv: physics/0111213.
[8] I. Bloch et al., Phys. Rev. A 64, 021402 (2001).
[9] A. G. Truscott et al., Science 291 (2001) 2570.
[10] F. Schreck et al., Phys. Rev. A 64 (2001) 011402(R).
[11] C. J. Myatt et al., Phys. Rev. Lett. 78 (1997) 586.
[12] G. Delannoy et al., Phys. Rev. A 63 (2001) 051602(R).
[13] A. Mosk et al., Appl. Phys. B 73, 791 (2001).
[14] W. Geist, L. You, and T. A. B. Kennedy, Phys. Rev. A 59 (1999) 1500.
[15] M. Lewenstein, J. I. Cirac, and P. Zoller, Phys. Rev. A 51 (1995) 4617.
[16] T. Papenbrock, A. N. Salgueiro, and H. A. Weidenmüller, Phys. Rev. A 65, 043601 (2002).
[17] S. J. Wang, M. C. Nemes, A. N. Salgueiro, H. A. Weidenmüller, cond-mat/0110293; cond-mat/0111413.
[18] R. B. Lehoucq, D. C. Sorensen, and C. Yang, ARPACK User’s Guide (SIAM 1998) ISBN 0-89871-407-9.
[19] O. J. Luiten, M. W. Reynolds, J. T. M. Walraven, Phys. Rev. A 53, 381 (1996).
[20] M. Holland, J. Williams, and J. Cooper, Phys. Rev. A 55, 3670 (1997).
[21] C. W. Gardiner et al., Phys. Rev. Lett. 79, 1793 (1997); C. W. Gardiner and P. Zoller, Phys. Rev. A 58, 536 (1998); C. W. Gardiner et al., Phys. Rev. Lett. 81, 5266 (1998); M. D. Lee and C. W. Gardiner, Phys. Rev. A 62, 033606 (2000); M. Köhl et al., Phys. Rev. Lett. 88, 080402 (2002).