Abstract

The equations for various spin particles with oscillator-like interactions are discussed in this talk. Contents: 1. Comment on "The Klein-Gordon Oscillator"; 2. The Dirac oscillator in quaternion form; 3. The Dirac-Dowker oscillator; 4. The Weinberg oscillator; 5. Note on the two-body Dirac oscillator.
NOTES ON OSCILLATOR-LIKE INTERACTIONS OF VARIOUS SPIN RELATIVISTIC PARTICLES

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1 Comment on ”The Klein-Gordon Oscillator”

[By S. Bruce and P. Minning, Nuovo Cim. 106A (1993) 711]

Concept of relativistic harmonic oscillator with intrinsic spin structure has been proposed long ago [1]. However, in connection with the publications of Moshinsky et al. [2] the interest in this simple model with a $j = 1/2$ Hamiltonian that is linear in both momenta and coordinates has grown recently [3]. Analogous type of interaction has been considered for the case of $j = 0$ and $j = 1$ Duffin-Kemmer field [4] and for the case of $j = 0$ Klein-Gordon field [5].

In the paper [5] the operators $\vec{Q}$, coordinate, and $\vec{P}$, momentum, have been represented in $n \otimes n$ matrix form

$$\vec{Q} = \hat{\eta} \vec{q}, \quad \vec{P} = \hat{\eta} \vec{p},$$

(1.1)

with $\hat{\eta}^2 = 1$. The interaction in the Klein-Gordon equation has been introduced in the following way:

$$\vec{P} \rightarrow \vec{P} - im \hat{\gamma} \hat{\Omega} \cdot \vec{Q},$$

(1.2)

where for the sake of completeness $\hat{\Omega}$ is chosen by $3 \otimes 3$ matrix with coefficients $\hat{\Omega}_{ij} = \omega_i \delta_{ij}$ (the case of anisotropic oscillator). The $\hat{\gamma}$ matrix obeys the following anticommutation relations:

$$\{\hat{\gamma}, \hat{\eta}\} = 0, \quad \hat{\gamma}^2 = 1.$$  

(1.3)

The Klein-Gordon equation for $\Psi(\vec{q},t)$, the wave function which could be expanded in two-component form, is then

$$-\frac{\partial^2}{\partial t^2} \Psi(\vec{q},t) = \left(\vec{p}^2 + m^2 \vec{q} \cdot \vec{\Omega}^2 \cdot \vec{q} + m \gamma \text{tr} \Omega + m^2\right) \Psi(\vec{q},t),$$

(1.4)

what gives the energy spectrum [5]

$$E_{(a),N_i}^2 - m^2 = 2m (\omega_1 N_1 + \omega_2 N_2 + \omega_3 N_3), \quad N_1, N_2, N_3 = 0, 1, 2 \ldots$$

$$E_{(b),N_i}^2 - m^2 = 2m (\omega_1 (N_1 + 1) + \omega_2 (N_2 + 1) + \omega_3 (N_3 + 1)).$$

(1.5)

It becomes, up to an additive constant, the spectrum of the anisotropic oscillator in the non-relativistic limit.

However, the physical sense of implementing the matrices $\hat{\eta}$ and $\hat{\gamma}$ in [5] is obscure. In this Section we try to attach some physical foundations to this procedure.

It is well-known some ways to recast the Klein-Gordon equation in the Hamiltonian form [6]-[10]. First of all, the Klein-Gordon equation could be re-written to the system of two coupled equations [8,p.98]

$$i \frac{\partial}{\partial t} \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \kappa \Xi \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}, \quad i \frac{\partial}{\partial t} \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = -\kappa \Psi,$$

(1.6)

where $\kappa = mc/\hbar$ (in the following we use the system where $c = \hbar = 1$). By means of redefining the components they are easy to present in the matrix Hamiltonian form (cf. with [11])

$$i \frac{\partial}{\partial t} \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 \end{pmatrix} + m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} + \begin{pmatrix} \phi \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix},$$

(1.7)

provided that

$$\begin{cases} \phi = i \partial_t \Psi + m \Psi \\ \chi_i = -i \nabla_i \Psi = \vec{p}_i \Psi. \end{cases}$$

(1.8)
Using matrices

\[
\vec{\alpha} = \begin{pmatrix}
0_{1 \otimes 1} & i & j & k \\
i & 0_{3 \otimes 3}
\end{pmatrix}, \quad \beta = \begin{pmatrix}
0_{1 \otimes 1} & 0 \\
0 & -\mathbb{1}_{3 \otimes 3}
\end{pmatrix}
\]  

(1.9)

(i, j, k are the orth-vectors of the Euclidean basis) and substituting analogously [2a], i.e. \( \vec{p} \rightarrow \vec{p} - i m \omega \beta \vec{r} \), we come to the equation for upper component

\[
(E^2 - m^2)\phi = [p^2 + m^2 \omega^2 r^2 - 3 m \omega] \phi
\]

what coincides with Eq. (10a) of ref. [5] in the case \( \omega_1 = \omega_2 = \omega_3 \).

The similar formulation also originated from the Duffin-Kemmer approach [12]. In this case the wave function \( \Phi = \text{column}(\phi_1, \phi_2, \chi_i) \) is five-dimensional and its components are

\[
\begin{cases}
\phi_1 = (i \partial_t \Psi + m \Psi) / \sqrt{2} \\
\phi_2 = (i \partial_t \Psi - m \Psi) / \sqrt{2} \\
\chi_i = -i \nabla_i \Psi = \vec{p}_i \Psi.
\end{cases}
\]  

(1.11)

It satisfies the equation

\[
\frac{i}{\partial t} \Phi = \left( \vec{B} \vec{p} + m \beta_0 \right) \Phi, \quad B_\mu = [\beta_0, \beta_\mu]
\]

(1.12)

(our choice of 5 \( \otimes \) 5 dimension \( \beta \)-matrices corresponds to ref. [4b]). As shown there, the substitution \( \vec{p} \rightarrow \vec{p} - i m \omega \eta_0 \vec{r} \), where

\[
\eta = \begin{pmatrix}
\mathbb{1}_{2 \otimes 2} & 0 \\
0 & -\mathbb{1}_{3 \otimes 3}
\end{pmatrix}
\]

(1.13)

leads to the equation (1.10) for both \( \phi_1 \) and \( \phi_2 \). Let us remark, in both the approach based on Eq. (1.7) and the Duffin-Kemmer approach, Eq. (1.12), we have the equation

\[
(E^2 - m^2)\chi_i = (p_i - i m \omega x_i) (p_j + i m \omega x_j) \chi_j
\]

(1.14)

for down component, which seems to not be reduced to oscillator-like equation.

Then, Sakata-Taketani approach [9, 10] is characterized by the equation which we write in the form:

\[
\begin{align*}
\frac{i}{\partial t} \Phi &= \left\{ \vec{p} (\tau_3 + i \tau_2) \vec{p} \right\} / 2m + m \tau_3 \\
\Phi,
\end{align*}
\]

(1.15)

with \( \tau_i \) being the Pauli matrices. \( \Phi = \text{column}(\phi, \chi) \) is the two-component wave function with components which could be written as following:

\[
\begin{cases}
\phi = (\Psi + \frac{i}{m} \partial_t \Psi) / \sqrt{2} \\
\chi = (\Psi - \frac{i}{m} \partial_t \Psi) / \sqrt{2}.
\end{cases}
\]  

(1.16)

From the previous experience we learned that in order to get the oscillator-like equation we need to do substitution with matrix which anticommutes with matrix structure of the momentum part of the equation. In our case the matrix which has this property is \( \tau_1 \) matrix. Therefore, we do the substitution

\[
\vec{p} \rightarrow \vec{p} - i m \omega \tau_1 \vec{r}
\]

(1.17)

and come to

\[
E^2 \xi = [p^2 + m^2 \omega^2 r^2 - 3 m \omega + m^2] \xi,
\]

(1.18)
Meantime, in the biquaternion formulation of Morita \[14\] the norm
\[
\left| \psi \right|^2 = \psi^\dagger \psi = \left( \psi_1 \right)^2 + \left( \psi_2 \right)^2 + \left( \psi_3 \right)^2 + \left( \psi_4 \right)^2
\]
leads to the tachionic solutions of the Dirac equation. For a gauge theory, involving quaternion-valued fields, see \[19\]-\[21\].

\[\begin{align*}
\alpha &= \tau_x \\
\beta &= \tau_z
\end{align*}\]

In the case of the choice of \(\gamma\)-matrices as in \[14\], i. e.
\[
\gamma_0 = \tau_1, \quad \gamma_1 = i\tau_3,
\]
in order to obtain the Dirac oscillator it is necessary to do substitution \(1.17\).

## 2 The Dirac oscillator in quaternion form

The quaternion (and conjugated to it) with real coefficient is defined as
\[
q = q_0 + iq_1 + jq_2 + kq_3 \\
\bar{q} = q_0 - iq_1 - jq_2 - kq_3.
\]

The basis vectors satisfy the equations
\[i^2 = j^2 = k^2 = -1\] and \(ij = -ji = k\) with cyclic permutations.

Consider the two-component quaternionic spinor (or \(SL(2,H)\) spinor) one could write the free Dirac equation as following, ref. \[14c,d\]
\[
\tilde{\Gamma} \cdot \partial \Psi - m\tau_3 \Psi k = 0.
\]

Anticommutation relations for \(\tilde{\Gamma}\) are given in \[14d,p.222\]. In Pauli representation \((i \to -\sqrt{-1}\tau_1, j \rightarrow -\sqrt{-1}\tau_2 \text{ and } k \rightarrow -\sqrt{-1}\tau_3)\) it goes through to usual Dirac equation and its complex conjugate. As mentioned in \[14\] it is convenient to diagonalize the matrices entering in Eq. \(2.4\) using matrix
\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{-1} \\ 1 & \sqrt{-1} \end{pmatrix}.
\]

In such a way we come to biquaternionic formulation \((q_i \in \mathbb{C}, \text{ i. e. the coefficients are the complex numbers})\):
\[
\partial \psi_L + im\bar{\psi}_R^\dagger = 0 \\
\partial \psi_R^\dagger + im\bar{\psi}_L = 0,
\]
where \(\psi_L \equiv \psi_{p^+}, \psi_R \equiv \psi_{p^-}\). This decomposition of \(\Psi\) into left ideals is carried out by means of the projection operators \(p_\pm = (b_0 \pm b_3)/2\). New basis is \(b_0 \equiv 1, b_1 \equiv \sqrt{-1}i, b_2 \equiv \sqrt{-1}j, b_3 \equiv \sqrt{-1}k\) and

\[\begin{align*}
\bar{q} = -\tilde{\eta}q^2 \tilde{q}^2 \equiv (q_0)^2 - (q_1)^2 - (q_2)^2 - (q_3)^2 = 0.
\end{align*}\]

See footnote on p. 1651 of ref. \[14a\] where the necessity of this suggestion is discussed.
\( \bar{b}_\alpha = b_\alpha, b_\alpha = -b_\alpha. \) Introducing interaction in the form \( \partial_t \rightarrow \partial_t + \tau_3 V_i(\vec{x}) \), V is the compensating field for this type of \( Sp(1, Q) \) transformations, and taking into account that the vectors of biquaternionic basis anticommute \( b_\alpha \bar{b}_\beta + b_\beta \bar{b}_\alpha = -2\eta_{\alpha \beta}, \eta_{\alpha \beta} = diag(-1, 1, 1) \), we come to the equations for the left and right spinor-quoternians in the following form:

\[
\begin{align*}
(E^2 - m^2)\psi_L &= \left[ (p^2 + k^2 \vec{x}^2) - 3k - 2\epsilon_{ijk} b_k x_i p_j \right] \psi_L \quad (2.8) \\
(E^2 - m^2)\psi^\dagger_R &= \left[ (p^2 + k^2 \vec{x}^2) + 3k + 2\epsilon_{ijk} b_k x_i p_j \right] \psi^\dagger_R \quad (2.9)
\end{align*}
\]

if we choose \( V_i(\vec{x}) = kx_i \). Eqs. (2.8) and (2.9) are the Dirac oscillator equations in the Pauli rep, \( b_k \rightarrow \tau_k \). Analogous equations for \( \psi_R \) and \( \psi^\dagger_L \) could be obtained if we start from (2.4) and (2.5) with the opposite signs at the mass terms. In the above, we assume, following for Morita [14], that the imaginary unit \( \sqrt{-1} \) commutes with Hamilton's basis vectors \( i, j, k \), what is not obviously. Consideration of \( \sqrt{-1} \) on an equal footing with the Hamilton's units would also be interesting.

The investigation of interactions of quaternionic Dirac field deserves further elaboration.

3 The Dirac-Dowker oscillator

In this Section we start from the equation for any spin given by Dirac [22], see also [23], in the form written down by Corson, ref. [8,p.154], (here we use Corson’s notation)

\[
\begin{align*}
\partial^A B v_B (k + \frac{1}{2}, l - \frac{1}{2}) - m \left( \frac{2k + 1}{2k + 1} \right)^{1/2} v^A (l) \psi (k, l) &= 0 \\
\partial_B A v^A (l) \psi (k, l) + m \left( \frac{2l + 1}{2l + 1} \right)^{1/2} v_B (k + \frac{1}{2}, l - \frac{1}{2}) &= 0
\end{align*}
\]

(3.1)

where \( v_A \) and \( v^A \) are the rectangular spinor-matrices of 2\( k \) rows and 2\( k+1 \) columns (see, e. g., section 17b of ref. [8] for details). The wave function \( \psi (k, l) \) belongs to the \( (k, l) \) representation of the homogeneous Lorentz group. The choice \( l = 1/2 \) and \( k = j - 1/2, j \) is the spin of a particle, permits one to reduce a number of subsidiary conditions. Moreover, the equations (3.1) are shown by Dowker [24] to recast to the matrix form which is similar to the well-known Dirac equation for \( j = 1/2 \) particle

\[
\begin{align*}
\alpha^\mu \partial_\mu \Phi &= m\Upsilon, \\
\bar{\alpha}^\mu \partial_\mu \Upsilon &= -m\Phi
\end{align*}
\]

(3.2)

The 4\( j \)- component function \( \Phi \) could be identified with the wave function in \((j, 0) \oplus (j-1, 0)\) representation. Then, \( \Upsilon \), which also has 4\( j \) components, is written down

\[
\Upsilon = (-1)^{2j/(2j)} \left( \frac{v^A (j - \frac{1}{2}) \otimes v^A (\frac{1}{2})}{v_A (j) \otimes v^A (\frac{1}{2})} \right) \psi (j, 1, \frac{1}{2}, \frac{1}{2}).
\]

(3.3)

and it belongs to \((j-1/2, 1/2)\) representation. The matrices \( \alpha^\mu \) and \( \bar{\alpha}^\mu = \alpha_\mu \) obeys the anticommutation relations of Pauli matrices

\[
\bar{\alpha}^\mu (\alpha^\nu) = g^{\mu \nu}.
\]

(3.4)

This set of matrices has been investigated in details in ref. [24b,c] and \( \alpha^\mu \) was proved there to satisfy all the algebraic relations of the Pauli matrices except for completeness.

Defining \( p_\mu = -i\partial_\mu \) and the analogs of \( \gamma \)- matrices as following:

\[
\gamma^\mu = \begin{pmatrix} 0 & -i\bar{\alpha}^\mu \\ i\alpha^\mu & 0 \end{pmatrix}
\]

(3.5)
the set of equations (3.2) is written down to the form of the Dirac equation

\[(p_{\mu} \gamma^{\mu} - m) \begin{pmatrix} \Upsilon \\ \Phi \end{pmatrix} = 0. \tag{3.6}\]

However, let us not forget that \(\Phi\) and \(\Upsilon\) are 2-spinors only in the case of \(j = 1/2\).

As mentioned in, e. g., ref. [8,p.33,124], in the case of spin \(j = 1/2\) the set of \(\gamma\)-matrices in representation (3.5)

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & -i \mathbb{1}_{2\otimes 2} \\ i \mathbb{1}_{2\otimes 2} & 0 \end{pmatrix}, \\
\gamma^1 &= \begin{pmatrix} 0 & i \tau^1 \\ i \tau^1 & 0 \end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix} 0 & i \tau^2 \\ i \tau^2 & 0 \end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix} 0 & i \tau^3 \\ i \tau^3 & 0 \end{pmatrix}
\end{align*}
\tag{3.7}\]

is defined up to the unitary transformation and Eq. (3.6) could be recast to the Hamiltonian form given by Dirac (with \(\alpha_k\) and \(\beta\) matrices) by means of the unitary matrix. It is easy to carry out the same procedure \((\alpha^k = S^0 \gamma^k S^{-1} \text{ and } \beta = S^0 \gamma S^{-1})\) for \(\gamma\) matrices (Eq. (3.5)) and functions of arbitrary spin \((\Psi = S^{-1} \Phi)\). For our aims it is convenient to choose the unitary matrix as following:

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_{4j\otimes 4j} & i \mathbb{1}_{4j\otimes 4j} \\ i \mathbb{1}_{4j\otimes 4j} & -\mathbb{1}_{4j\otimes 4j} \end{pmatrix}.
\tag{3.8}\]

After standard substitution \(\vec{p} \rightarrow \vec{p} - im \omega \gamma^0 \vec{r}\) we obtain

\[
\begin{align*}
E\phi &= -i \left[ \alpha_0 (\vec{p} \vec{p}) + im \omega (\vec{e} \vec{e}) \right] \nu + m \alpha_0 \phi, \\
E\nu &= i \left[ \alpha_0 (\vec{p} \vec{p}) - im \omega (\vec{e} \vec{e}) \right] \phi - m \alpha_0 \nu.
\end{align*}
\tag{3.9}\]

Since it follows from the anticommutation relations \((4.4)\) that \(\alpha_i \alpha_0 = \alpha_0 \alpha_i\) we have the equations which coincide with Eq. (8) of ref. [2a] or Eqs. (3.6) and (3.12) of ref. [2h] except for \(\tau \mu \rightarrow \alpha \mu\), i.e. their explicit forms,

\[
\begin{align*}
(E^2 - m^2)\phi &= \left[ \vec{p}^2 + m \omega r^2 - 3 \alpha_0 m \omega - m \omega \alpha_0 \alpha_i \alpha_j r^i \nabla_j \right] \phi \\
(E^2 + m^2)\nu &= \left[ \vec{p}^2 + m \omega r^2 + 3 \alpha_0 m \omega + m \omega \alpha_0 \alpha_i \alpha_j r^i \nabla_j \right] \nu.
\end{align*}
\tag{3.10, 3.11}\]

Thus, we convinced ourselves that we got the same oscillator-like interaction and the similar spectrum as for the case of \(j = 1/2\) particles in [2a].

### 4 The Weinberg oscillator

The principal equation of 2(2j + 1)-component approach [25] in the case of spin \(j = 1\) is

\[
(\gamma_{\omega} p_{\mu} p_{\nu} + M^2) \Psi^{(j=1)}(x) = 0, \tag{4.1}\]

with \(\gamma_{\alpha \beta}\) being defined by the formulae

\[
\begin{align*}
\gamma_{ij} &= \begin{pmatrix} 0 & \delta_{ij} - S_i S_j - S_j S_i \\ \delta_{ij} - S_i S_j - S_j S_i & 0 \end{pmatrix}, \\
\gamma_{4i} &= \begin{pmatrix} 0 & i S_i \\ -i S_i & 0 \end{pmatrix}, \\
\gamma_{44} &= \begin{pmatrix} 0 & \mathbb{1}_{3\otimes 3} \\ \mathbb{1}_{3\otimes 3} & 0 \end{pmatrix}.
\end{align*}
\tag{4.2, 4.3}\]

5
(\(S_i\) are the spin matrices for a vector particle).

The \(j = 1\) Hamiltonian has been given in refs. [26, 27]:

\[
\mathcal{H} = \frac{2E^2}{2E^2 - M^2}(\bar{\alpha}\bar{p}) + \beta \left[ E - \frac{2E}{2E^2 - M^2}(\bar{\alpha}\bar{p})^2 \right],
\]

where

\[
\bar{\alpha} = \begin{pmatrix} \hat{S} & 0 \\ 0 & -\hat{S} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \mathbb{1}_{3\otimes3} \\ \mathbb{1}_{3\otimes3} & 0 \end{pmatrix}.
\]

Though this way of description is a little bit antique, attention has again been paid to it recently, e. g. [28, 29]. This formalism presents oneself the example of the Bargmann-Wightman-Wigner type quantum field theory [28b]. The remarkable feature is the fact that boson and its antiboson have the opposite parities.

In general, the upper and down components of 6- component wave function do not uncouple neither under the interaction \(\vec{p} \rightarrow \vec{p} - \text{im}\omega\beta\vec{r}\) nor under \(\gamma_5,_{\mu\nu}u_{\mu}r_{\nu}\). However, if we introduce the Dirac oscillator interaction so that the conditions of the longitudity of \(\Psi = \text{column} (\phi_i, \chi_i)\) respective to \(\vec{r}\)

\[
\vec{r} \times \vec{\phi} = 0, \quad \vec{r} \times \vec{\chi} = 0
\]

are fulfilled, we come to more simple equations (\(\xi = \phi - \chi, \eta = \phi + \chi\))

\[
(2E^2 - M^2)\xi = E(\vec{S}\vec{p})\eta + \left[ (\vec{S}\vec{p}) - k(\vec{S}\vec{r}) \right] (\vec{S}\vec{p})\xi,
\]

\[
E(\vec{S}\vec{p})\xi = \left[ (\vec{S}\vec{p}) + k(\vec{S}\vec{r}) \right] (\vec{S}\vec{p})\eta
\]

which could be uncoupled to the following form (\(k = \text{im}\omega\))

\[
(\vec{S}\vec{p})(2E^2 - M^2)(\vec{S}\vec{p})\xi = (\vec{S}\vec{p}) \left[ \vec{p}^2 + m\omega^2r^2 + 3m\omega + 4m\omega \vec{S}(\vec{r} \times \vec{p}) \right] (\vec{S}\vec{p})\xi
\]

\[
(\vec{S}\vec{p})(2E^2 - M^2)(\vec{S}\vec{p})\eta = (\vec{S}\vec{p}) \left[ \vec{p}^2 + m\omega^2r^2 - 3m\omega - 4m\omega \vec{S}(\vec{r} \times \vec{p}) \right] (\vec{S}\vec{p})\eta - \text{im}\omega(2E^2 - M^2)(\vec{S}\vec{r})(\vec{S}\vec{p})\eta
\]

These equations can be considered as the extension of the equations with Dirac oscillator interaction to the \(j = 1\) case, for the components \((\vec{S}\vec{p})\xi\) and \((\vec{S}\vec{p})\eta\). However, remark that one has the additional spin-orbit term acting as earlier at \(\eta\).

5 Note on the two-body Dirac oscillator

The two-body Dirac Hamiltonian with oscillator-like interaction is given by (see, e. g., refs. [2d,f])

\[
i \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \psi = \mathcal{H}\psi =
\]

\[
= \left[ \frac{1}{\sqrt{2}}(\vec{\alpha}_1 + \vec{\alpha}_2) \cdot \vec{P} + \frac{1}{\sqrt{2}}(\vec{\alpha}_1 - \vec{\alpha}_2) \cdot \vec{p} - \frac{i}{\sqrt{2}}(\vec{\alpha}_1 - \vec{\alpha}_2) \cdot \vec{r}\vec{B} + m(\beta_1 + \beta_2) \right] \psi,
\]

where the matrices are given by the direct products

\[
\vec{\alpha}_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_{2\otimes2} & 0 \\ 0 & \mathbb{1}_{2\otimes2} \end{pmatrix}, \quad \vec{\alpha}_2 = \begin{pmatrix} \mathbb{1}_{2\otimes2} & 0 \\ 0 & \mathbb{1}_{2\otimes2} \end{pmatrix} \otimes \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix},
\]
where in symmetric and antisymmetric parts using the above-mentioned complete system of matrices symmetric properties under oscillator-like potentials, see also [30, p. 31]. The wave function is decomposed to the Proca equations (see, e.g., [30, p. 30-31]). The 16-component wave function of two-body Dirac oscillator equations in the form (5.8)-(5.14) could be uncoupled on the set containing only functions $\phi$ and (5.12). The two-body Dirac oscillator equations in the form (5.8)-(5.14) could be uncoupled on the complete set of matrices: $(\gamma^\mu C)$, $(\sigma^{\mu\nu} C)$ and $C$, $(\gamma^5 C)$, and $(\gamma^5\gamma^\mu C)$. We consider the system multiplied by $C$, the matrix of charge conjugation, in order to trace for symmetric properties under oscillator-like potentials, see also [30, p. 31]. The wave function is decomposed in symmetric and antisymmetric parts using the above-mentioned complete system of matrices

$$\psi = \psi_{\{\alpha\beta\}} + \psi_{[\alpha\beta]},$$

where

$$\psi_{\{\alpha\beta\}} = \gamma^\mu_{\alpha\eta} C_{\eta\beta} A_\mu + \sigma^{\mu\nu}_{\alpha\eta} C_{\eta\beta} F_{\mu\nu}$$

$$\psi_{[\alpha\beta]} = C_{\alpha\beta} \phi + \gamma^5_{\alpha\eta} C_{\eta\beta} \tilde{\phi} + \gamma^5_{\alpha\eta} \gamma^\mu_{\eta\beta} A_\mu.$$

In such a way we obtain the set of equations:

$$EA_0 = 0, \quad E\tilde{A}_0 = -2m\tilde{\phi}$$

$$E\phi = 2i\sqrt{2}(\tilde{p}_i - i\tilde{r}^i)F^{0i}$$

$$E\tilde{\phi} = -2m\tilde{A}_0 + \sqrt{2}\epsilon_{ijk}(\tilde{p}_j + i\tilde{r}^j)F^{jk}$$

$$E\tilde{A}^i = -i\sqrt{2}\epsilon_{ijk}(\tilde{p}_j \mp i\tilde{r}^j)A^k$$

$$EA^i = 4imF^{0i} + i\sqrt{2}\epsilon_{ijk}(\tilde{p}_j \pm i\tilde{r}^j)\tilde{A}^k$$

$$EF^{0i} = -2imA^i + i\sqrt{2}(\tilde{p}_i + i\tilde{r}^i)\phi$$

$$EF_{jk} = \frac{1}{\sqrt{2}}\epsilon_{ijk}(\tilde{p}_i - i\tilde{r}^i)\tilde{\phi}$$

Let us mention that for another type of Dirac oscillator-like interaction $\sim (\tilde{\sigma}_1 - \tilde{\sigma}_2)B\Gamma_5$ the only changes are the sign changes at the term $i\tilde{r}^i$ in the fourth and fifth equations of the above system, Eqs. (5.11) and (5.12). The two-body Dirac oscillator equations in the form (5.8)-(5.14) could be uncoupled on the set containing only functions $\phi$, $\tilde{\phi}$ and $\tilde{A}_0$ and the another one containing only $A_\mu$ and $F_{\mu\nu}$:

$$(E^2 - 8m^2)\phi = 4(\tilde{p}_i - i\tilde{r}^i)(\tilde{p}_i + i\tilde{r}^i)\phi - \frac{1}{E}\left\{16m\epsilon_{ijk}\tilde{r}^i\tilde{p}_j\right\}\tilde{A}^k$$

$$(E^2 - 4m^2)\tilde{\phi} = 2(\tilde{p}_i + i\tilde{r}^i)(\tilde{p}_i - i\tilde{r}^i)\tilde{\phi}$$

$$EA_0 = -2m\phi$$

$$(E^2 - 8m^2)\tilde{A}^i = 2(\tilde{p}_j \mp i\tilde{r}^j)(\tilde{p}_i \pm i\tilde{r}^i)\tilde{A}^j - 2(\tilde{p}_j \mp i\tilde{r}^j)(\tilde{p}_j \pm i\tilde{r}^j)\tilde{A}^i +$$

$$+ \frac{1}{E}\left\{16m\epsilon_{ijk}\tilde{r}^j\tilde{p}_k\right\}\phi$$

and

$$EA_0 = 0$$

$$(E^2 - 8m^2)F^{0i} = 4(\tilde{p}_i + i\tilde{r}^i)(\tilde{p}_j - i\tilde{r}^j)F^{0j} -$$

$$-4im(\tilde{p}_j \pm i\tilde{r}^j)(\tilde{p}_i \mp i\tilde{r}^i)A^j + 4im(\tilde{p}_j \pm i\tilde{r}^j)(\tilde{p}_j \mp i\tilde{r}^j)A^i$$

$$E^2A^i = 2(\tilde{p}_j \pm i\tilde{r}^j)(\tilde{p}_i \mp i\tilde{r}^i)A^j - 2(\tilde{p}_j \pm i\tilde{r}^j)(\tilde{p}_j \mp i\tilde{r}^j)A^i + 4imEF^{0i}$$

$$(E^2 - 4m^2)F^{jk} = \epsilon_{ijk}\epsilon_{lmn}(\tilde{p}_i - i\tilde{r}^i)(\tilde{p}_l + i\tilde{r}^l)F^{mn}.$$
This fact proves the Dirac oscillator interaction, like the case of introduction of electrodynamic interaction in the Proca or the Bargmann-Wigner equations, does not mix $S = 1$ and $S = 0$ states.

Next, the interaction term of the following form:

$$ V_{int} = \frac{1}{r} \left( \frac{dV(r)}{dr} \right) (\vec{a}_1 - \vec{a}_2) B \Gamma_5 \vec{r} $$

has been deduced \[^{31}\] from the equation of Relativistic Quantum Constraint Dynamics (RQCR), e.g. refs. \[^{32, 33}\], or $N$-particle Barut equation \[^{34}\]. In \[^{31}\] it proved to lead to the Dirac oscillator-like interactions provided that the definite choice of the function $V(r)$. In connection with that let us remark the curious behavior of the another potential $V(r)$ which has been proposed in ref. \[^{35b,36}\]:

$$ V(r) = -g^2 \coth \left( \frac{\kappa r}{4\pi} \right) $$

(5.22)

It could be deduced from the one-boson exchange quasipotential $V(\vec{p}, \vec{k}; E) = -g^2(p - k)^{-2}$ by means of the transformation into the relativistic configurational representation (RCR) using the complete set of Shapiro plane-wave functions \[^{37}\]:

$$ \xi(\vec{p}, \vec{r}) = \left( p_0 - \vec{p} \vec{n}/m \right)^{-1-irm}, \quad p_0 = \sqrt{\vec{p}^2 + m^2}, \quad \vec{n} = \vec{r}/|\vec{r}|. $$

(5.23)

In the case of the quasipotential (5.22) the interaction term $V$, Eq. (5.21), has the different asymptotic behavior in three regions ($g^2/(4\pi) = 1$):

$$ V_{int} \simeq \frac{1}{r(r^2 - 1)} (\vec{a}_1 - \vec{a}_2) B \Gamma_5 \vec{r} \simeq$$

$$ \approx \begin{cases} 
(1/r^3)(\vec{a}_1 - \vec{a}_2) B \Gamma_5 \vec{r}, & \text{if } r >> \frac{1}{\kappa} \text{ and } r > 1 \\
- (1/r)(\vec{a}_1 - \vec{a}_2) B \Gamma_5 \vec{r}, & \text{if } \frac{1}{\kappa} << r < 1,
\end{cases} $$

(5.24)

behavior in the infrared region ($r >> \frac{1}{\kappa}$, large distances); and

$$ V_{int} \simeq -2\kappa(\vec{a}_1 - \vec{a}_2) B \Gamma_5 \vec{r}, \quad \text{if } r << \frac{1}{\kappa}, $$

(5.25)

in the ultraviolet region (small distances). We can convince ourselves that in one of the regions we have the Coulomb-like behavior, in the other region, the Dirac oscillator-like behavior. This fact could be some quantum field foundations for implementing the Dirac oscillator potential.

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\[^{b}\]In the quasipotential approach of Kadyshevsky \[^{35}\] it is convenient to pass through to the variables $\Delta_0 = (k_0 p_0 - \vec{k} \vec{p})/m$ and $\tilde{\Delta} = \vec{k}(-\vec{p}) = \vec{k} - \vec{p} \left( k_0 - \frac{\vec{p}}{p_0 + \vec{m}} \right)$, which have the physical sense of the momentum transfer in the Lobachevsky space, $p_0 - \vec{p}^2 = m^2$. Then, the transformation to the RCR is carried out by means of $\xi(\tilde{\Delta}, \vec{r})$. 

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