Representation-independent manipulations with Dirac matrices and spinors

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Abstract

Dirac matrices, also known as gamma matrices, are defined only up to a similarity transformation. Usually, some explicit representation of these matrices is assumed in order to deal with them. In this article, we show how it is possible to proceed without any explicit form of these matrices. Various important identities involving Dirac matrices and spinors have been derived without assuming any representation at any stage.

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1 Introduction

In order to obtain a relativistically covariant equation for the quantum mechanical wave function, Dirac introduced a Hamiltonian that is linear in the momentum operator. In modern notation, it can be written as

\[ H = \gamma^0 \left( \gamma \cdot p_{op} + m \right), \quad (1.1) \]

where \( m \) is the mass of the particle and \( p_{op} \) the momentum operator. We will throughout use natural units with \( c = \hbar = 1 \) so that \( \gamma^0 \) and \( \gamma \) are dimensionless. Because of their anticommutation properties that we mention in §2 they have to be matrices. The four matrices are written together as

\[ \gamma^\mu \equiv \{ \gamma^0, \gamma^i \}, \quad (1.2) \]
where we have put a Lorentz index in the left hand side. We will also define the corresponding matrices with lower indices in the usual way:

\[ \gamma_\mu = g_{\mu\nu} \gamma^\nu, \] (1.3)

where \( g_{\mu\nu} \) is the metric tensor, for which our convention has been stated in Eq. (A.1). Of course, the Lorentz indices on the gamma matrices do not imply that the matrices transform as vectors. They are, in fact, constant matrices which are frame-independent. The Lorentz index in \( \gamma^\mu \) only indicates that the four quantities obtained by sandwiching these matrices between fermionic fields transform as components of a vector.

Some properties of the Dirac matrices follow directly from their definition in Eq. (1.1), as shown in §2. However, these properties do not specify the elements of the matrices uniquely. They only define the matrices up to a similarity transformation. Since spinors are plane-wave solutions of the equation

\[ i \frac{\partial \psi}{\partial t} = H \psi, \] (1.4)

and \( H \) contains the Dirac matrices which are not uniquely defined, the solutions also share this non-uniqueness.

In physics, whenever there is an arbitrariness in the definition of some quantity, it is considered best to deal with combinations of those quantities which do not suffer from the arbitrariness. For example, components of a vector depend on the choice of the axes of co-ordinates. Physically meaningful relations can either involve things like scalar products of vectors which do not depend on the choice of axes, or are in the form of equality of two quantities (say, two vectors) both of which transform the same way under a rotation of the axes, so that their equality is not affected. Needless to say, it is best if we can follow the same principles while dealing with Dirac matrices and spinors. However, in most texts dealing with them, this approach is not taken. Most frequently, one chooses an explicit representation of the Dirac matrices and spinors, and works with it.

Apart from the fact that an explicit representation is aesthetically less satisfying, it must also be said that dealing with them can also lead to pitfalls. One might use some relation which holds in some specific representation but not in general, and obtain a wrong conclusion.

In this article, we show how, without using any explicit representation of the Dirac matrices or spinors, one can obtain useful relations involving them. The article is organized as follows. In §2 we define the basic properties of Dirac matrices...
and spinors and mention the extent of arbitrariness in the definitions. In §3 we recall some well-known associated matrices which are useful in dealing with Dirac matrices. In §4 we derive some identities involving the Dirac matrices and associated matrices in a completely representation-independent way. In §5 we show how spinor solutions can be defined in a representation-independent fashion and identify their combinations on which normalization conditions can be imposed. We derive some important relations involving spinors in §6 and involving spinor bilinears in §7. Concluding remarks appear in §9.

2 Basic properties of Dirac matrices and spinors

Some properties of the Dirac matrices are immediately derived from Eq. (1.1). First, the relativistic Hamiltonian of a free particle is given by

\[ H^2 = p^2 + m^2, \tag{2.1} \]

and Eq. (1.1), when squared, must yield this relation. Assuming \( \gamma_0 \) and \( \gamma \) commute with the momentum operator, this gives a set of relations which can be summarized in the form

\[ \gamma_\mu, \gamma_\nu \] + = 2g_\mu\nu I, \tag{2.2} \]

where \( g_\mu\nu \) is the metric defined in Eq. (A.1), and \( I \) is the unit matrix which will not be always explicitly written in the subsequent formulas. This relation requires that the Dirac matrices are at least \( 4 \times 4 \) matrices, and we take them to be \( 4 \times 4 \).

Hermiticity of the Hamiltonian of Eq. (1.1) gives some further conditions on the Dirac matrices, namely that \( \gamma_0 \) must be hermitian, and so should be the combinations \( \gamma_0 \gamma_i \). Both these relations can be summarized by writing

\[ \gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \tag{2.3} \]

in view of the anticommutation relations given in Eq. (2.2).

Eqs. (2.2) and (2.3) are the basic properties which define the Dirac matrices. With these defining relations, the arbitrariness can be easily seen through the following theorems.

**Theorem 1** For any choice of the matrices \( \gamma_\mu \) satisfying Eqs. (2.2) and (2.3), if we take another set defined by

\[ \tilde{\gamma}_\mu = U \gamma_\mu U^\dagger \tag{2.4} \]
for some unitary matrix $U$, then these new matrices satisfy the same anticommutation and hermiticity properties as the matrices $\gamma_{\mu}$.

The proof of this theorem is straightforward and trivial. The converse is also true:

**Theorem 2** If two sets of matrices $\gamma_{\mu}$ and $\tilde{\gamma}_{\mu}$ both satisfy Eqs. (2.2) and (2.3), they are related through Eq. (2.4) for some unitary matrix $U$.

The proof is non-trivial [2, 3] and we will not give it here. The two theorems show that the Dirac matrices are defined only up to a similarity transformation with a unitary matrix.

To obtain the defining equation for the spinors, we multiply both sides of Eq. (1.4) by $\gamma_0$ and put $p_{\text{op}} = -i\nabla$ into the Hamiltonian of Eq. (1.1). This gives the Dirac equation:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0.$$  
(2.5)

There are two types of plane-wave solutions:

$$\psi \sim \begin{cases} u((p))e^{-ip\cdot x}, \\ v((p))e^{+ip\cdot x}. \end{cases}$$  
(2.6)

Here and later, we indicate functional dependence in double parentheses so that it does not get confused with multiplicative factors in parentheses. The objects $u((p))$ and $v((p))$ are 4-component column vectors, and will be called “spinors”. The 4-vector $p^\mu$ is given by

$$p^\mu \equiv \{E_p, p\},$$  
(2.7)

where $E_p$ is the positive energy eigenvalue:

$$E_p = +\sqrt{p^2 + m^2}.$$  
(2.8)

Putting Eq. (2.6) into Eq. (2.5), we obtain the equations that define the $u$ and $v$-spinors:

$$(\gamma_\mu p^\mu - m)u((p)) = 0,$$  
(2.9a)

$$(\gamma_\mu p^\mu + m)v((p)) = 0.$$  
(2.9b)

Obviously, if we change $\gamma^\mu$ to $\tilde{\gamma}^\mu$ through the prescription given in Eq. (2.4) and also change the spinors to

$$\tilde{u}((p)) = Uu((p)), \quad \tilde{v}((p)) = Uv((p)),$$  
(2.10)

Eq. (2.9) is satisfied by the new matrices and the new spinors. Eq. (2.10) shows that the spinors themselves are representation-dependent.
3 Some associated matrices

In order to proceed, we recall the definitions of some matrices associated with the Dirac matrices. These definitions can be obtained in any textbook dealing with Dirac particles or fields, but are compiled here for the sake of completeness.

The sigma-matrices are defined as

\[ \sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] . \]  
(3.1)

The matrices \( \frac{1}{2} \sigma_{\mu\nu} \) constitute a representation of the Lorentz group. The subgroup of rotation group has the generators \( \frac{1}{2} \sigma_{ij} \), with both spatial indices. We define the spin matrices:

\[ \Sigma^i = \frac{1}{2} \varepsilon^{ijk} \sigma_{jk} , \]  
(3.2)

so that \( \frac{1}{2} \Sigma^i \) represent the spin components. From Eq. (2.3), it is easy to check that the matrices \( \sigma_{0i} \) are anti-hermitian, whereas the matrices \( \sigma_{ij} \), and therefore the matrices \( \Sigma^i \), are hermitian.

The next important matrix is defined from the observation that the matrices \( -\gamma_{\mu}^\top \) satisfy the same anticommutation and hermiticity properties as \( \gamma_{\mu} \). By Theorem 2, there must then exist a unitary matrix \( C \),

\[ C^\dagger = C^{-1} , \]  
(3.3)

such that

\[ C^{-1} \gamma_{\mu} C = -\gamma_{\mu}^\top . \]  
(3.4)

Note that the definitions in Eqs. (3.1) and (3.4) imply the relation

\[ C^{-1} \sigma_{\mu\nu} C = -\sigma_{\mu\nu}^\top . \]  
(3.5)

Another important matrix is \( \gamma_5 \), defined as

\[ \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 , \]  
(3.6)

or equivalently as

\[ \gamma_5 = \frac{i}{4!} \varepsilon_{\mu\nu\lambda\rho} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho , \]  
(3.7)

where \( \varepsilon_{\mu\nu\lambda\rho} \) stands for the completely antisymmetric rank-4 tensor, with \( \varepsilon_{0123} = 1 . \)  
(3.8)
From Eq. (2.2), it is easily seen that
\[
(\gamma_5)^2 = \mathbb{1}.
\] (3.9)

It is also easy to see that \(\gamma_5\) anticommutes with all \(\gamma_\mu\)'s and commutes with all \(\sigma_{\mu\nu}\)'s:
\[
[\gamma_\mu, \gamma_5] = 0,
\] (3.10)
\[
[\sigma_{\mu\nu}, \gamma_5] = 0.
\] (3.11)

Also, Eqs. (3.6) and (3.4) imply the relation
\[
C^{-1}\gamma_5 C = \gamma_5^\dagger.
\] (3.12)

There is another property of \(\gamma_5\) which is of interest. Consider the trace of this matrix. Using the anticommutation properties of the Dirac matrices, we can write
\[
\text{Tr} \left( \gamma_5 \right) = i \text{Tr}(\gamma_0 \gamma_1 \gamma_2 \gamma_3) = -i \text{Tr}(\gamma_0 \gamma_1 \gamma_3 \gamma_2) = +i \text{Tr}(\gamma_0 \gamma_3 \gamma_1 \gamma_2) = -i \text{Tr}(\gamma_3 \gamma_0 \gamma_1 \gamma_2).
\] (3.13)

Now, since \(\text{Tr}(M_1 M_2) = \text{Tr}(M_2 M_1)\) for any two matrices \(M_1\) and \(M_2\), we can take \(M_1 = \gamma^3\) and \(M_2 = \gamma^0 \gamma^1 \gamma^2\), use this cyclic property, and get
\[
\text{Tr} \left( \gamma_5 \right) = -i \text{Tr}(\gamma_0 \gamma^1 \gamma_2 \gamma^3) = -\text{Tr} \left( \gamma_5 \right),
\] (3.14)
which means that
\[
\text{Tr} \left( \gamma_5 \right) = 0.
\] (3.15)

Using the argument that leads to the existence of the matrix \(C\), we can define some other associated matrices \([3]\). For example, given any representation of the matrices \(\gamma_\mu\), the matrices \(\gamma_\mu^\dagger\) also satisfy the same anticommutation and hermiticity properties as the matrices \(\gamma_\mu\), and therefore there must be a unitary matrix \(A\) that satisfies the relation
\[
A^{-1} \gamma_\mu A = \gamma_\mu^\dagger.
\] (3.16a)

Similarly, there exist unitary matrices \(A'\), \(B\), \(B'\) and \(C'\) which satisfy the relations
\[
A'^{-1} \gamma_\mu A' = -\gamma_\mu^\dagger,
\] (3.16b)
\[
C'^{-1} \gamma_\mu C' = \gamma_\mu^\dagger,
\] (3.16c)
\[
B^{-1} \gamma_\mu B = \gamma_\mu^*,
\] (3.16d)
\[
B'^{-1} \gamma_\mu B' = -\gamma_\mu^*.
\] (3.16e)
Fortunately, there is no need to discuss these matrices separately. Eq. (2.3), used in Eq. (3.16a), gives the relation

\[ [\gamma_\mu, A\gamma_0] = 0. \]  

(3.17)

Thus \( A\gamma_0 \) must be a multiple of the unit matrix, i.e., \( A = \alpha\gamma_0 \) for some number \( \alpha \). Since \( A \) will have to be unitary, \( \alpha \) can only be a phase. Thus we conclude that

\[ A = \gamma_0 \]  

(3.18)

apart from a possible overall phase. Similarly, we can see that \( A'\gamma_0 \) must anticommute with any \( \gamma_\mu \), and so we must have

\[ A' = \gamma_5\gamma_0 \]  

(3.19)

up to a phase factor. By a similar argument, we find

\[ C' = \gamma_5C \]  

(3.20)

up to an overall phase. As for \( B \) and \( B' \), we note that

\[ \gamma_\mu^* = (\gamma_\mu^\dagger)^\top = (\gamma_0\gamma_\mu\gamma_0\gamma_5) = -C^{-1}\gamma_0\gamma_\mu\gamma_0C = -(\gamma_0C)^{-1}\gamma_\mu(\gamma_0C). \]  

(3.21)

Thus, up to overall phase factors, one obtains

\[ B' = \gamma_0C, \]  

(3.22a)

\[ B = \gamma_5\gamma_0C. \]  

(3.22b)

The following properties follow trivially, using the explicit form for the matrix \( B' \):

\[ \sigma_{\mu\nu}^* = (\gamma_0C)^{-1}\sigma_{\mu\nu}(\gamma_0C), \]  

(3.23a)

\[ \gamma_5^* = -(\gamma_0C)^{-1}\gamma_5(\gamma_0C), \]  

(3.23b)

\[ (\gamma_\mu\gamma_5)^* = (\gamma_0C)^{-1}\gamma_\mu\gamma_5(\gamma_0C). \]  

(3.23c)

### 4 Identities involving Dirac matrices

#### 4.1 Trace identities

Previously, we have shown that the matrix \( \gamma_5 \) is traceless. We can also try to find the trace of any of the Dirac matrices \( \gamma_\mu \). Using Eq. (3.9), we can write

\[ \text{Tr} (\gamma_\mu) = \text{Tr} (\gamma_\mu\gamma_5\gamma_5). \]  

(4.1)
Then, using the cyclic property of traces and Eq. (3.10), we obtain
\[
\text{Tr} (\gamma_{\mu}) = \text{Tr} (\gamma_{5}\gamma_{\mu}\gamma_{5}) = -\text{Tr} (\gamma_{\mu}\gamma_{5}\gamma_{5}) .
\] (4.2)

Comparing the two equations, we obtain
\[
\text{Tr} (\gamma_{\mu}) = 0 .
\] (4.3)

The same technique can be employed to prove that the trace of the product of any odd number of Dirac matrices is zero. For the product of even number of Dirac matrices, we can use the result
\[
\text{Tr} (\gamma_{\mu}\gamma_{\nu}) = 4g_{\mu\nu} ,
\] (4.4)
\[
\text{Tr} (\gamma_{\mu}\gamma_{\nu}\gamma_{\lambda}\gamma_{\rho}) = 4\left( g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda} \right) ,
\] (4.5)
and so on. We do not give the details of the proofs because they are usually proved in a representation-independent manner in textbooks.

### 4.2 Contraction identities

First, there are the contraction formulas involving only the Dirac matrices, e.g.,
\[
\gamma^{\mu}\gamma_{\mu} = 4 ,
\] (4.6a)
\[
\gamma^{\mu}\gamma_{\nu}\gamma_{\mu} = -2\gamma_{\nu} ,
\] (4.6b)
\[
\gamma^{\mu}\gamma_{\nu}\gamma_{\lambda}\gamma_{\mu} = 4g_{\nu\lambda} ,
\] (4.6c)
\[
\gamma^{\mu}\gamma_{\nu}\gamma_{\lambda}\gamma_{\rho}\gamma_{\mu} = -2\gamma_{\rho}\gamma_{\lambda}\gamma_{\nu} ,
\] (4.6d)
and so on for longer strings of Dirac matrices, which can be proved easily by using the anticommutation relation of Eq. (2.2). There are also similar formulas involving contractions of the sigma matrices, like
\[
\sigma^{\mu\nu}\sigma_{\mu\nu} = 12 ,
\] (4.7a)
\[
\sigma^{\mu\nu}\sigma^{\lambda\rho}\sigma_{\mu\nu} = -4\sigma^{\lambda\rho} ,
\] (4.7b)
and some other involving both gamma matrices and sigma matrices:
\[
\sigma^{\mu\nu}\gamma^{\lambda}\sigma_{\mu\nu} = 0 ,
\] (4.8a)
\[
\gamma^{\lambda}\sigma^{\mu\nu}\gamma_{\lambda} = 0 .
\] (4.8b)

All of these can be easily proved by using the definition of the sigma matrices and the contraction formulas for the gamma matrices.
4.3 Identities from linear independence

There are many other identities involving the Dirac matrices which are derived from the fact that the 16 matrices

\[ \mathbb{1}, \gamma_\mu, \sigma_{\mu\nu} (\text{for } \mu < \nu), \gamma_\mu \gamma_5, \gamma_5 \]

constitute a complete set of $4 \times 4$ matrices. In other words, any $4 \times 4$ matrix $M$ can be expressed as a linear superposition of these 16 matrices:

\[ M = a \mathbb{1} + b^\mu \gamma_\mu + c^{\mu\nu} \sigma_{\mu\nu} + d^\mu \gamma_\mu \gamma_5 + e \gamma_5. \tag{4.10} \]

In particular, any product of any number of these basis matrices can also be written in the form proposed in Eq. (4.10), with suitable choices of the co-efficients $a, b^\mu, c^{\mu\nu}, d^\mu$ and $e$. One example of this kind of relation is the identity

\[ \gamma_\mu \gamma_\nu = g_{\mu\nu} \mathbb{1} - i \sigma_{\mu\nu}, \tag{4.11} \]

which follows trivially from Eqs. (2.2) and (3.1). Eqs. (4.7) and (4.8) are also examples of this general theme. To see more examples of this kind, let us consider the combination $\varepsilon^{\mu\nu\lambda\rho} \sigma_{\mu\nu} \gamma_5$. We can use the definition of $\gamma_5$ from Eq. (3.7), and use the product of two Levi-Civita symbols given in Eq. (A.2). This gives

\[ \varepsilon^{\mu\nu\lambda\rho} \sigma_{\mu\nu} \gamma_5 = -\frac{i}{4!} \sigma_{\mu\nu} \left( \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho + (-1)^P \text{(permutations)} \right), \tag{4.12} \]

where the factor $(-1)^P$ is +1 if the permutation is even, and −1 if the permutation is odd. There are 24 possible permutations. Each of them can be simplified by using one or other of the contraction formulas given above, and the result is

\[ \varepsilon^{\mu\nu\lambda\rho} \sigma_{\mu\nu} \gamma_5 = 2i \sigma^\lambda \rho, \tag{4.13} \]

or equivalently

\[ \sigma^\lambda \rho \gamma_5 = -\frac{i}{2} \varepsilon^{\mu\nu\lambda\rho} \sigma_{\mu\nu}. \tag{4.14} \]

A very useful identity can be derived by starting with the combination $\varepsilon_{\mu\nu\lambda\rho} \gamma^\rho \gamma_5$, and using Eqs. (3.7) and (A.2), as was done for deducing Eq. (4.13). The final result can be expressed in the form

\[ \gamma_\mu \gamma_\nu \gamma_\lambda = g_{\mu\nu} \gamma_\lambda + g_{\nu\lambda} \gamma_\mu - g_{\lambda\mu} \gamma_\nu - i \varepsilon_{\mu\nu\lambda\rho} \gamma^\rho \gamma_5. \tag{4.15} \]
With this identity, any string of three or more gamma matrices can be reduced to strings of smaller number of gamma matrices.

An important identity can be derived by multiplying Eq. (4.11) by $\gamma_5$, and using Eq. (4.14). This gives

$$\gamma_\mu \gamma_\nu \gamma_5 = g_{\mu\nu} \gamma_5 - \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \sigma^{\lambda\rho}.$$  \hspace{1cm} (4.16)

In particular, if the index $\mu$ is taken to be in the time direction and the index $\nu$ to be a spatial index, we obtain

$$\gamma_0 \gamma_i \gamma_5 = -\frac{1}{2} \epsilon_{0ijk} \sigma^{jk}.$$  \hspace{1cm} (4.17)

Taking the convention for the completely antisymmetric 3-dimensional tensor in such a way that $\epsilon_{0ijk} = \epsilon_{ijk}$, we can rewrite this equation by comparing the right hand side with the definition of the spin matrices in Eq. (3.2):

$$\Sigma_i = -\gamma_0 \gamma_i \gamma_5.$$  \hspace{1cm} (4.18)

With this form, it is easy to show that

$$\left[\Sigma_i, \Sigma_j \right]_+ = 2 \delta_{ij},$$  \hspace{1cm} (4.19)

by using anticommutation properties of the gamma matrices.

### 4.4 Antisymmetry of $C$

Taking the transpose of Eq. (3.4) that defines the matrix $C$, we obtain

$$\gamma_\mu = -C^T \gamma_\mu (C^{-1})^T = C^T C^{-1} \gamma_\mu C C^{-1} = C^T C^{-1} \gamma_\mu (C^T C^{-1})^{-1},$$  \hspace{1cm} (4.20)

which can be rewritten in the form

$$\left[\gamma_\mu, C^T C^{-1} \right]_+ = 0.$$  \hspace{1cm} (4.21)

Because the matrix $C^T C^{-1}$ commutes with each Dirac matrix, it must be a multiple of the unit matrix. So we write

$$C^T = \lambda C$$  \hspace{1cm} (4.22)

for some number $\lambda$. Taking transpose of both sides of this equation, we obtain

$$C = \lambda C^T,$$  \hspace{1cm} (4.23)
and therefore $\lambda^2 = 1$.

We now go back to Eq. (3.4) and rewrite it in the form

$$\gamma_\mu C = -C\gamma_\mu^\dagger = -\lambda(\gamma_\mu C)^\dagger.$$  \hspace{1cm} (4.24)

This means that the matrices $\gamma_\mu C$ are all antisymmetric if $\lambda = +1$, and symmetric if $\lambda = -1$. Using the definition of the sigma-matrices from Eq. (3.1), it is easy to show that the matrices $\sigma_{\mu\nu} C$ have the same properties as well:

$$\sigma_{\mu\nu} C = -\lambda(\sigma_{\mu\nu} C)^\dagger.$$  \hspace{1cm} (4.25)

In addition, note that the four matrices $\gamma_\mu C$ and the six matrices $\sigma_{\mu\nu} C$ are linearly independent, because the ten matrices $\gamma_\mu$ and $\sigma_{\mu\nu}$ are. There cannot be ten linearly independent antisymmetric $4 \times 4$ matrices. Thus, the matrices $\gamma_\mu C$ and $\sigma_{\mu\nu} C$ must be all symmetric, implying

$$\lambda = -1,$$  \hspace{1cm} (4.26)

i.e.,

$$C^\dagger = -C.$$  \hspace{1cm} (4.27)

The matrix $C$ is therefore antisymmetric [4].

Once the choice of $\lambda$ has been determined, it is easy to see, using Eq. (3.12), that the matrix $\gamma_5 C$ and the four matrices $\gamma_\mu \gamma_5 C$ are antisymmetric. Thus, any symmetric $4 \times 4$ matrix can be written as a linear superposition of the ten matrices $\gamma_\mu C$ and $\sigma_{\mu\nu} C$, whereas any antisymmetric $4 \times 4$ matrix can be written as a linear superposition of the six matrices $C$, $\gamma_5 C$ and $\gamma_\mu \gamma_5 C$. In this sense, this collection of 16 matrices is a better basis, compared to that given in Eq. (4.9), for writing an arbitrary $4 \times 4$ matrix.

5 Spinors

5.1 Eigenvectors of $\gamma_0$

Consider the matrix $\gamma_0$. It is a $4 \times 4$ matrix, so it has four eigenvalues and eigenvectors. It is hermitian, so the eigenvalues are real. In fact, from Eq. (2.2) we know that its square is the unit matrix, so that its eigenvalues can only be $\pm 1$. Since $\gamma_0$
is traceless, as we have proved in Eq. (4.3), there must be two eigenvectors with
eigenvalue +1 and two with −1:

\[ \gamma_0 \xi_s = \xi_s, \quad \gamma_0 \chi_s = -\chi_s. \quad (5.1) \]

The subscripts on \( \xi \) and \( \chi \) distinguishes two different eigenvectors of each kind. Of

\[ \xi^\dagger_s \chi_{s'} = 0, \quad (5.2) \]

since they belong to different eigenvalues. But since the two \( \xi \)'s are degenerate and

\[ \chi^\dagger_s \chi_{s'} = 0 \]

so are the two \( \chi \)'s, there is some arbitrariness in defining them even for a given form

\[ [\sigma_{12}, \gamma_0] = 0. \quad (5.3) \]

Thus, we can choose the eigenstates of \( \gamma_0 \) such that they are simultaneously eigen-

\[ (\sigma_{12})^2 = 1, \quad (5.4) \]

so that the eigenvalues of \( \sigma_{12} \) are \( \pm 1 \) as well. Therefore, let us choose the eigenvectors

\[ \sigma_{12} \xi_s = s \xi_s, \quad \sigma_{12} \chi_s = s \chi_s, \quad (5.5) \]

with \( s = \pm \). Once we fix the spinors in this manner, the four eigenvectors are mutually orthogonal, i.e., in addition to Eq. (5.2), the following relations also hold:

\[ \xi^\dagger_s \xi_{s'} = \delta_{ss'}, \quad \chi^\dagger_s \chi_{s'} = \delta_{ss'}. \quad (5.6) \]

One might wonder, why are we spending so much time in discussing the eigenvectors of \( \gamma_0 \)? To see the reason, let us consider Eq. (2.9) for vanishing 3-momentum. In this case \( E_p = m \), so that the equations reduce to

\[ \begin{align*}
(\gamma_0 - 1) u((0)) &= 0, \\
(\gamma_0 + 1) v((0)) &= 0.
\end{align*} \quad (5.7a, b) \]

This shows that, at zero momentum, the \( u \)-spinors and the \( v \)-spinors are simply eigenstates of \( \gamma_0 \) with eigenvalues +1 and −1. Thus we can define the zero-
momentum spinors as

\[ u_s((0)) \propto \xi_s, \quad v_s((0)) \propto \chi_{-s}, \quad (5.8) \]

apart from possible normalizing factors which will be specified later.
5.2 Spinors and their normalization

We now want to find the spinors for any value of $p$. We know that these will have to satisfy Eqs. (2.9a) and (2.9b), and, for $p = 0$, should have the forms given in Eq. (5.8). With these observations, we can try the following solutions:

$$u_s(p) = N_p (\gamma_\mu p^\mu + m) \xi_s, \quad (5.9a)$$
$$v_s(p) = N_p (-\gamma_\mu p^\mu + m) \chi_{-s}, \quad (5.9b)$$

where $N_p$ is a normalizing factor. One might wonder why we have put $\chi_{-s}$ and not $\chi_s$ in the definition of $v_s$. It is nothing more than a convention. It turns out that when we do quantum field theory, this convention leads to an easy interpretation of the subscript $s$. This issue will not be discussed here.

It is easy to see that our choices for the spinors satisfy Eq. (2.9) since

$$(\gamma_\mu p^\mu - m)(\gamma_\nu p^\nu + m) = p^2 - m^2 = 0 . \quad (5.10)$$

It is also easy to see that in the zero-momentum limit, these solutions reduce to the eigenvalues of $\gamma_0$, apart from a normalizing factor. For example, putting $p = 0$ and $E_p = m$ into Eq. (5.9a), we obtain

$$u_s(0) = N_0 m (\gamma_0 + 1) \xi_s = 2m N_0 \xi_s . \quad (5.11)$$

In order to determine a convenient normalization of the spinors, let us rewrite Eq. (5.9a) more explicitly:

$$u_s(p) = N_p (\gamma_0 E_p - \gamma_i p_i + m) \xi_s = N_p (E_p + m - \gamma_i p_i) \xi_s , \quad (5.12)$$

using Eq. (5.1) in the last step. Similarly, we obtain

$$v_s(p) = N_p (E_p + m + \gamma_i p_i) \chi_{-s} . \quad (5.13)$$

Recalling that $\gamma_i$'s are anti-hermitian matrices, we then obtain

$$u_s^\dagger(p) = N_p^* \xi_s^\dagger (E_p + m + \gamma_i p_i) , \quad (5.14a)$$
$$v_s^\dagger(p) = N_p^* \chi_{-s}^\dagger (E_p + m - \gamma_i p_i) . \quad (5.14b)$$

Thus,

$$u_s^\dagger(p) u_s(p) = |N_p|^2 \xi_s^\dagger \left( (E_p + m)^2 - \gamma_i \gamma_j p_i p_j \right) \xi_{s'} . \quad (5.15)$$
Since $p_ip_j = p_jp_i$, we can write

$$\gamma_i \gamma_j p_i p_j = \frac{1}{2} \left[ \gamma_i, \gamma_j \right] + p_i p_j = -\delta_{ij} p_i p_j = -\mathbf{p}^2.$$  

(5.16)

Using Eq. (2.8) then, we obtain

$$u^\dagger_s(p) u_{s'}(p) = 2E_p (E_p + m) \left| N_p \right|^2 \xi_s^\dagger \xi_{s'}.$$  

(5.17)

Choosing

$$N_p = \frac{1}{\sqrt{E_p + m}}$$  

(5.18)

and using Eq. (5.6), we obtain the normalization conditions in the form

$$u^\dagger_s(p) u_{s'}(p) = 2E_p \delta_{ss'}.$$  

(5.19)

Through a similar procedure, one can obtain a similar condition on the $v$-spinors:

$$v^\dagger_s(p) v_{s'}(p) = 2E_p \delta_{ss'}.$$  

(5.20)

We now need a relation that expresses the orthogonality between a $u$-spinor and a $v$-spinor. In obtaining Eqs. (5.19) and (5.20), the linear terms in $\gamma_i p_i$, appearing in Eqs. (5.12) and (5.14a) or in the similar set of equations involving the $v$-spinors, cancel. The same will not work in combinations of the form $u^\dagger_s(p) v_{s'}(p)$ because the $\gamma_i p_i$ terms have the same sign in both factors. However we notice that if we reverse the 3-momentum in one of the factors, these problematic terms cancel. We can then follow the same steps, more or less, and use Eq. (5.2) to obtain

$$u^\dagger_s(-p) v_{s'}(p) = v^\dagger_s(-p) u_{s'}(p) = 0.$$  

(5.21)

Eq. (5.21) can be expressed in an alternative form by using bars rather than daggers, where $\bar{w} = w^\dagger \gamma_0$ for any spinor. Multiplying Eq. (2.9a) from the left by $\bar{v}_{s'}(p)$ we obtain

$$\bar{v}_{s'}(p) (\gamma_\mu p^\mu - m) u_s(p) = 0.$$  

(5.22)

Multiplying the hermitian conjugate of the equation for $v_{s'}(p)$ by $u_s(p)$ from the right, we get

$$\bar{v}_{s'}(p) (\gamma_\mu p^\mu + m) u_s(p) = 0.$$  

(5.23)
Subtracting one of these equations from another, we find that

$$\bar{v}_{s'}(p)u_s(p) = 0$$

(5.24)

provided \(m \neq 0\). Similarly, one can also obtain the equation

$$\bar{u}_{s'}(p)v_s(p) = 0.$$  

(5.25)

We will also show, in §7.1, that Eqs. (5.19) and (5.20) are equivalent to the relations

$$\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}$$  

(5.26a)

$$\bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}.$$  

(5.26b)

Unless \(m = 0\), these can be taken as the normalization conditions on the spinors.

### 5.3 Spin sums

The spinors also satisfy some completeness relations, which can be proved without invoking their explicit forms [6]. Consider the sum

$$A_u(p) \equiv \sum_s u_s(p)\bar{u}_s(p).$$

(5.27)

Note that, using Eq. (5.26a), we get

$$A_u(p)u_{s'}(p) = \sum_s u_s(p)\left[\bar{u}_s(p)u_{s'}(p)\right] = 2mu_{s'}(p).$$

(5.28)

And, using Eq. (5.26), we get

$$A_u(p)v_{s'}(p) = 0.$$  

(5.29)

Recalling Eqs. (2.9a) and (2.9b), it is obvious that on the spinors \(u_s(p)\) and \(v_s(p)\), the operation of \(A_u(p)\) produces the same result as the operation of \(\gamma_\mu p^\mu + m\). Since any 4-component column vector can be written as a linear superposition of the basis spinors \(u_s(p)\) and \(v_s(p)\), it means that the action of \(A_u(p)\) and of \(\gamma_\mu p^\mu + m\) produces identical results on any 4-component column vector. The two matrices must therefore be the same:

$$\sum_s u_s(p)\bar{u}_s(p) = \gamma_\mu p^\mu + m.$$  

(5.30)

Similar reasoning gives

$$\sum_s v_s(p)\bar{v}_s(p) = \gamma_\mu p^\mu - m.$$  

(5.31)
6 Relations involving spinors

We now show some non-trivial properties of the spinors. In all textbooks, they are deduced in the Dirac-Pauli representation of the $\gamma$-matrices. Using Eq. (2.4), one can show that if they hold in one representation, they must hold in other representations as well. Here we derive them without using any representation at any stage of the proofs.

6.1 What $\gamma_0$ does on spinors

We first consider the effect of $\gamma_0$ acting on the spinors. From Eq. (5.12), we find

\[ \gamma_0 u_s(p) = N_p \gamma_0 (E_p + m - \gamma_i p_i) \xi_s = N_p (E_p + m + \gamma_i p_i) \gamma_0 \xi_s, \]

using the anticommutation relations and Eq. (5.1). This shows that

\[ \gamma_0 u_s(p) = u_s(-p). \] (6.2)

Following the same procedure, we can obtain the result

\[ \gamma_0 v_s(p) = -v_s(-p). \] (6.3)

Eqs. (6.2) and (6.3) are very important relations for deducing behavior of fermions under the parity transformation. These relations can be used to deduce Eqs. (5.24) and (5.25) from Eq. (5.21), or vice versa.

6.2 Conjugation relations

Let us now deduce another set of relations, which plays an important role in deriving charge conjugation properties of fermions. To build up to these relations, let us first consider the object

\[ \hat{\xi}_s = \gamma_0 C \xi_s^*, \] (6.4)

where the matrix $C$ was defined in Eq. (3.4). To find out about the nature of $\hat{\xi}_s$, we first consider the action of $\gamma_0$ on it:

\[ \gamma_0 \hat{\xi}_s = \gamma_0 \gamma_0 C \xi_s^* = -\gamma_0 C \gamma_0^\top \xi_s^*, \] (6.5)
using Eq. (3.4) again. However, the complex conjugate of Eq. (5.1) implies that
\[ \gamma_0^\dagger \xi_s^* = \xi_s^*, \]  
(6.6)
since
\[ \gamma_0^* = \gamma_0^\dagger \]  
(6.7)
because of the hermiticity of the matrix \( \gamma_0 \). Putting this in, we obtain
\[ \gamma_0 \hat{\xi}_s = -\gamma_0 C \xi_s^* = -\hat{\xi}_s, \]  
(6.8)
showing that \( \hat{\xi}_s \) is an eigenvector of \( \gamma_0 \) with eigenvalue \(-1\). Therefore, it must be a combination of the \( \chi_s \)'s.

To determine which combination of the \( \chi_s \)'s occur in \( \hat{\xi}_s \), we use Eq. (3.5) and recall that \( \sigma_{12} \) commutes with \( \gamma_0 \) to obtain
\[ \sigma_{12} \hat{\xi}_s = \gamma_0 \sigma_{12} C \xi_s^* = -\gamma_0 C \sigma_{12} \xi_s^*. \]  
(6.9)
It can be easily seen from Eqs. (2.3) and (3.1) that \( \sigma_{12} \) is hermitian. So, from Eq. (5.5), we obtain
\[ \sigma_{12}^\dagger \xi_s^* = \left( \sigma_{12} \xi_s \right)^* = s \xi_s^*, \]  
(6.10)
which gives
\[ \sigma_{12} \hat{\xi}_s = -s \gamma_0 C \xi_s^* = -s \hat{\xi}_s. \]  
(6.11)
This shows that \( \hat{\xi}_s \) is also an eigenstate of \( \sigma_{12} \), with eigenvalue \(-s\). Recalling the result we found earlier about its eigenvalue of \( \gamma_0 \), we conclude that \( \hat{\xi}_s \) must be proportional to \( \chi_{-s} \). Since both \( \gamma_0 \) and \( C \) are unitary matrices and \( \xi_s \) is normalized to have unit norm, the norm of \( \hat{\xi}_s \) is also unity, so the proportionality constant can be a pure phase, of the form \( e^{i\theta} \). But notice that the definition of the matrix \( C \) in Eq. (3.4) has a phase arbitrariness as well. In other words, given a set of matrices \( \gamma_\mu \), the matrix \( C \) can be obtained only up to an overall phase from Eq. (3.4). We can choose the overall phase of \( C \) such that the relation
\[ \gamma_0 C \xi_s^* = \chi_{-s} \]  
(6.12)
is obeyed. One can then see that
\[ \gamma_0 C \chi_s^* = \gamma_0 C \left( \gamma_0 C \xi_{-s}^* \right)^* = \gamma_0 C \gamma_0^\dagger \left( C^\dagger \right)^\dagger \xi_{-s}, \]  
(6.13)
using Eq. (6.17). At this stage, using Eqs. (3.4) and (4.27), we can write
\[ \gamma_0 C \chi_s^* = \gamma_0 \gamma_0 C C^\dagger \xi_{-s} . \] (6.14)

Since \( C \) is unitary and \( \gamma_0 \) squares to the unit matrix, we obtain
\[ \gamma_0 C \chi_s^* = \xi_{-s} , \] (6.15)
similar to Eq. (6.12).

To see the implication of these relations between the eigenvectors of \( \gamma_0 \), we take the complex conjugate of Eq. (5.12). Remembering that the matrices \( \gamma_i \) are anti-hermititan so that \( \gamma_i^* = -\gamma_i^\dagger \), we obtain
\[ u^*_s((p)) = N_p(E_p + m + \gamma_i^\dagger p_i) \xi_s^* = N_p(E_p + m - C^{-1}\gamma_i C p_i) \xi_s^* , \] (6.16)
using the definition of the matrix \( C \) from Eq. (3.4). Multiplying from the left by \( \gamma_0 C \), we obtain
\[ \gamma_0 C u^*_s((p)) = N_p[(E_p + m)\gamma_0 C \xi_s^* - \gamma_0 \gamma_i C p_i \xi_s^*] . \] (6.17)

Since \( \gamma_0 \) anticommutes with \( \gamma_i \), this can be written as
\[ \gamma_0 C u^*_s((p)) = N_p[(E_p + m) + \gamma_i p_i] \gamma_0 C \xi_s^* = N_p[(E_p + m) + \gamma_i p_i] \chi_{-s} . \] (6.18)

Using Eq. (5.13), we now obtain
\[ \gamma_0 C u^*_s((p)) = u_s((p)) . \] (6.19)

This is an important relation. Following similar steps, we can also prove the relation
\[ \gamma_0 C v^*_s((p)) = u_s((p)) . \] (6.20)

Because \( C \) appears in the conjugation properties of the spinors, we will sometimes refer to it as the conjugation matrix.

### 6.3 What \( \gamma_5 \) does on spinors

Multiplying both sides of Eq. (2.9a) by \( \gamma_5 \) from the left and using the anticommutation of \( \gamma_5 \) with all Dirac matrices, we obtain the equation
\[ (\gamma_5 p^\mu + m) \gamma_5 u((p)) = 0 , \] (6.21)
which clearly shows that $\gamma_5 u$ is a $v$-spinor. Similarly, $\gamma_5 v$ must be a $u$-spinor. However, this simple argument does not say whether $\gamma_5 u_+ = v_+$, or $v_-$, or a linear combination of the two.

To settle the issue, we note that

$$\gamma_0 \gamma_5 \xi_s = -\gamma_5 \gamma_0 \xi_s = -\gamma_5 \xi_s, \quad (6.22)$$

since $\gamma_5$ anticommutes with $\gamma_0$. This equation shows that $\gamma_5 \xi_s$ is an eigenvector of $\gamma_0$ with eigenvalue $-1$, i.e., it must be some combination of the $\chi$-eigenvectors defined in Eq. (5.1). Moreover, since $\gamma_5$ commutes with $\sigma_{12}$, we observe that

$$\sigma_{12} \gamma_5 \xi_s = \gamma_5 \sigma_{12} \xi_s = s \gamma_5 \xi_s. \quad (6.23)$$

This means that $\gamma_5 \xi_s$ is an eigenstate of $\sigma_{12}$ with eigenvalue $s$. Combining this information about the eigenvalues of $\gamma_0$ and $\sigma_{12}$, we conclude that $\gamma_5 \xi_s$ must be equal to $\chi_s$ apart from a possible constant phase factor. Let us therefore write

$$\gamma_5 \xi_s = \eta_s \chi_s, \quad (6.24)$$

with $|\eta_s| = 1$. Because of Eq. (5.9), this would also imply

$$\gamma_5 \chi_s = \eta_s^* \xi_s. \quad (6.25)$$

However, the two phase factors $\eta_s$ (for $s = \pm$) cannot be chosen in a completely arbitrarily way, since we have utilized the freedom in imposing Eqs. (6.12) and (6.15). For example, we see that,

$$\chi_{-s} = \gamma_0 C \xi_s^* = \gamma_0 C \left(\eta_s \gamma_5 \chi_s\right)^* = \eta_s^* \gamma_0 C \gamma_5 \chi_s^*, \quad (6.26)$$

using the hermiticity of the matrix $\gamma_5$. Now, using Eq. (3.12), we can further simplify this expression and write

$$\chi_{-s} = \eta_s^* \gamma_0 \gamma_5 C \chi_s^* = -\eta_s^* \gamma_5 \gamma_0 C \chi_s^* = -\eta_s^* \gamma_5 \xi_{-s} = -\eta_s^* \eta_{-s} \chi_{-s}, \quad (6.27)$$

using Eqs. (6.15) and (6.24) on the way. This means that, the choice of phases implied in writing Eqs. (6.12) and (6.15) forces us to impose the relation

$$\eta_s^* \eta_{-s} = -1. \quad (6.28)$$

One possible way of assuring this relation is to take

$$\eta_s = (-1)^{(s-1)/2}, \quad (6.29)$$
although we will not use the specific choice in what follows.

The action of $\gamma_5$ on the spinors can now be calculated easily. For example, one finds

$$\gamma_5 u_s(\mathbf{p}) = N_p \gamma_5 (\gamma_\mu p^\mu + m) \xi_s = N_p (-\gamma_\mu p^\mu + m) \gamma_5 \xi_s$$

$$= N_p (-\gamma_\mu p^\mu + m) \eta_s \chi_s = \eta_s u_{-s}(\mathbf{p}).$$

(6.30)

Through similar manipulations or through the use of Eq. (3.9), we can get

$$\gamma_5 v_s(\mathbf{p}) = -\eta_s^* u_{-s}(\mathbf{p}).$$

(6.31)

### 6.4 Alternative forms

The results obtained above can be combined to obtain some other relations. For example, multiply both sides of Eq. (6.19) from the left by $C^{-1}$. Using Eq. (3.4), the result can be written as

$$-\gamma^T_0 u^*_s(\mathbf{p}) = C^{-1} v_s(\mathbf{p}).$$

(6.32)

But $\gamma^T_0 = \gamma^*_0$, so the left hand side is the complex conjugate of $\gamma_0 u_s(\mathbf{p})$. Using Eq. (6.2), we can then write

$$C^{-1} v_s(\mathbf{p}) = -u^*_s(\mathbf{-p}).$$

(6.33)

Similar manipulations give the complimentary result,

$$C^{-1} u_s(\mathbf{p}) = v^*_s(\mathbf{-p}).$$

(6.34)

We can also combine this result with the identities of Eqs. (6.30) and (6.31) to obtain

$$C^{-1} \gamma_5 v_s(\mathbf{p}) = -\eta_s^* v^*_s(\mathbf{-p}),$$

$$C^{-1} \gamma_5 u_s(\mathbf{p}) = -\eta_s u^*_s(\mathbf{-p}).$$

(6.35)

The matrix $C^{-1} \gamma_5$ plays a crucial role in the time-reversal properties of a fermion field [6].

### 7 Spinor bilinears

Whenever fermion fields have to be used in Lorentz invariant combinations, we must encounter pairs of them in order that the overall combination conserves angular momentum. For this reason, fermion field bilinears deserve some attention. In momentum space, one encounters bilinears involving spinors, which is what we discuss in this section.
7.1 Identities involving bilinears

A vector $p_\lambda$ can be rewritten as

$$p_\lambda = g_{\lambda\rho}p^\rho = (\gamma_\lambda\gamma_\rho + i\sigma_{\lambda\rho})p^\rho = \gamma_\lambda\not{p} + i\sigma_{\lambda\rho}p^\rho.$$  \hspace{1cm} (7.1)

Alternatively, we can write

$$p'_\lambda = g_{\lambda\rho}p'^\rho = (\gamma_\rho\gamma_\lambda + i\sigma_{\rho\lambda})p'^\rho = \not{p}'\gamma_\lambda - i\sigma_{\lambda\rho}p'^\rho.$$  \hspace{1cm} (7.2)

Adding these two equations, sandwiching the result between two spinors, and using Eq. (2.9a) and its hermitian conjugate, we obtain the relation

$$\bar{u}(\not{p}')\gamma_\lambda u(\not{p}) = \frac{1}{2m}\bar{u}(\not{p}')\left[Q_\lambda - i\sigma_{\lambda\rho}q^\rho\right]u(\not{p}),$$  \hspace{1cm} (7.3)

where

$$Q = p + p', \quad q = p - p'.$$  \hspace{1cm} (7.4)

This result is called the Gordon identity.

Variants of this identity can be easily derived following the same general technique. For example, suppose the two spinors on the two sides belong to different particles, with masses $m$ and $m'$. In this case, it is easy to see that

$$\bar{u}(\not{p}')\left[Q_\lambda - i\sigma_{\lambda\rho}q^\rho\right]u(\not{p}) = (m' + m)\bar{u}(\not{p}')\gamma_\lambda u(\not{p}).$$  \hspace{1cm} (7.5)

Similarly, one can obtain the identity

$$\bar{u}(\not{p}')\left[Q_\lambda - i\sigma_{\lambda\rho}q^\rho\right]\gamma_5u(\not{p}) = (m' - m)\bar{u}(\not{p}')\gamma_\lambda\gamma_5u(\not{p}).$$  \hspace{1cm} (7.6)

It should be noted that the normalization relations of Eqs. (5.19) and (5.20) can be written in an alternative form by using the Gordon identity. For this, we put $p = p'$ in Eq. (7.3) and take only the time component of the equation. This gives

$$2m u^s_\lambda(\not{p})u_s(\not{p}) = 2E_p\bar{u}_{s'}(\not{p})u_s(\not{p}),$$  \hspace{1cm} (7.7)

where we have put the indices $s, s'$ on the spinors in order to distinguish the different solutions. This shows that Eqs. (5.19) and (5.26a) are equivalent. The proof of the equivalence of Eqs. (5.20) and (5.26b) is similar.
7.2 Non-relativistic reduction

In field-theoretical manipulations, sometimes we encounter expressions which can be interpreted easily by making a non-relativistic reduction. For example, in Quantum Electrodynamics (QED), the matrix element of the electromagnetic current operator turns out to be superposition of two bilinears of the form $\bar{u}(p')\gamma_\lambda u(p)$ and $\bar{u}(p')\sigma_{\lambda\rho}q^\rho u(p)$, and an intuitive feeling for these bilinears can be obtained by going to the non-relativistic limit. With this in mind, here we give the non-relativistic reduction of all possible fermion bilinears.

A general bilinear is of the form

$$\bar{u}_{s'}(p') Fu_s(p)$$ (7.8)

for some matrix $F$. Any such matrix can be written as a superposition of the 16 basis matrices shown in Eq. (4.9). So it is enough to obtain non-relativistic reduction with the bilinears involving these basis matrices only.

We will keep terms up to linear order in the 3-momenta. The spinor, to this order, can be written as

$$u_s(p) \approx \sqrt{2m} \left(1 - \frac{\gamma_ip_i}{2m}\right) \xi_s,$$ (7.9)

using Eqs. (5.12) and (5.18), where the ‘approximate equal to’ sign ($\approx$) will be used throughout this section to imply that all terms of the order 3-momentum squared have been omitted. Then

$$\bar{u}_{s'}(p') \approx \sqrt{2m}\xi_{s'}^\dagger \left(1 - \frac{\gamma_i p_i'}{2m}\right),$$ (7.10)

using the hermiticity and anticommutation properties of the gamma matrices, and the fact that

$$\xi_{s'}^\dagger \gamma_0 = \xi_{s'}^\dagger$$ (7.11)

which follows from Eq. (5.1). For the general bilinear, then, we obtain

$$\bar{u}_{s'}(p') Fu_s(p) \approx \xi_{s'}^\dagger \left(2mF - \frac{1}{2}Q_j [F, \gamma_j]_+ - \frac{1}{2}q_j [F, \gamma_j]\right)\xi_s,$$ (7.12)

using the sum and difference of momenta introduced in Eq. (7.4). The three terms on the right side of this equation will be referred to as the momentum-independent term, the anticommutator term and the commutator term respectively. We now evaluate these terms for the five types of basis matrices shown in Eq. (4.9).
7.2.1 Scalar bilinear

This corresponds to the case $F = \mathbb{I}$, so Eq. (7.12) for this case reads

$$\bar{u}_{s'}(\langle p' \rangle) Fu_s(\langle p \rangle) \approx \xi_{s'}^\dagger \left(2m \mathbb{I} - Q_j \gamma_j\right) \xi_s.$$  \hfill (7.13)

In the second term on the right side, one can use the definition of $\xi_s$ from Eqs. (5.1) and (7.11) as well as the anticommutation of $\gamma_0$ with all $\gamma_i$'s to write

$$\xi_{s'}^\dagger \gamma_i \xi_s = \xi_{s'}^\dagger \gamma_0 \gamma_i \xi_s = -\xi_{s'}^\dagger \gamma_i \gamma_0 \xi_s = -\xi_{s'}^\dagger \gamma_i \xi_s,$$  \hfill (7.14)

so that

$$\xi_{s'}^\dagger \gamma_i \xi_s = 0.$$  \hfill (7.15)

The momentum-independent term can be easily written down using Eq. (5.6), and one obtains

$$\bar{u}_{s'}(\langle p' \rangle) u_s(\langle p \rangle) \approx 2m \delta_{ss'}.$$  \hfill (7.16)

Recall that with $p = p'$, this is the equality of Eq. (5.26a). This equation shows that even if $p \neq p'$, the corrections appear only in the second order of the momenta.

7.2.2 Vector bilinears

These corresponds to $F = \gamma_\lambda$ for some index $\lambda$. Consider first the case when $\lambda$ is a spatial index. Note that the momentum-independent term for this case vanishes due to the identity of Eq. (7.15). Thus we obtain

$$\bar{u}_{s'}(\langle p' \rangle) \gamma_i u_s(\langle p \rangle) \approx \xi_{s'}^\dagger \left[\delta_{ij} Q_j + i \sigma_{ij} q_j\right] \xi_s,$$  \hfill (7.17)

using the anticommutator and commutator of the gamma matrices from Eqs. (2.2) and (3.1). In fact, this result follows directly from the Gordon identity, Eq. (7.3), if we keep only terms up to first order in momenta. Using the definition of the spin matrices, Eq. (3.2), we can also write the equation in the form

$$\bar{u}_{s'}(\langle p' \rangle) \gamma_i u_s(\langle p \rangle) \approx Q_i \delta_{ss'} + i \varepsilon_{ijk} q_j \xi_{s'}^\dagger \Sigma_k \xi_s.$$  \hfill (7.18)

We now turn to the temporal part of the matrix element, i.e., the case with $F = \gamma_0$ in Eq. (7.12). Since $\gamma_0$ anticommutes with all $\gamma_j$, we obtain

$$\bar{u}_{s'}(\langle p' \rangle) \gamma_0 u_s(\langle p \rangle) \approx \xi_{s'}^\dagger \left(2m \gamma_0 - \frac{1}{2} (\gamma_0 \gamma_j - \gamma_j \gamma_0) q_j\right) \xi_s.$$  \hfill (7.19)

Using Eqs. (5.1) and (7.11), we find that the term linear in momenta vanishes, so that, to the order specified, we obtain

$$\bar{u}_{s'}(\langle p' \rangle) \gamma_0 u_s(\langle p \rangle) \approx 2m \delta_{ss'}.$$  \hfill (7.20)
7.2.3 Tensor bilinears

For tensor bilinears, \( F = \sigma_{\lambda\rho} \). Individual terms that appear in the commutator and anticommutator involving \( F \) appearing in Eq. (7.12) are products of three gamma matrices. All such terms can be reduced by using Eq. (4.15), and one obtains

\[
\begin{align*}
\left[ \sigma_{\lambda\rho}, \gamma_\mu \right] &= 2i(g_{\mu\rho}\gamma_\lambda - g_{\mu\lambda}\gamma_\rho), \\
\left[ \sigma_{\lambda\rho}, \gamma_\mu \right]_+ &= 2\varepsilon_{\lambda\mu\nu}\gamma^\nu\gamma_5.
\end{align*}
\] (7.21a)

In particular, if we consider the sigma matrices with one temporal index, we need the relations

\[
\begin{align*}
\left[ \sigma_{0i}, \gamma_j \right] &= 2ig_{ji}\gamma_0 = -2i\delta_{ij}\gamma_0, \\
\left[ \sigma_{0i}, \gamma_j \right]_+ &= 2\varepsilon_{0ijk}\gamma^k\gamma_5 = -2\varepsilon_{ijk}\gamma_k\gamma_5 = 2\varepsilon_{ijk}\gamma_0\Sigma_k,
\end{align*}
\] (7.22a)

using Eq. (4.18) on the way. Thus, from Eq. (7.12), we obtain

\[
\bar{u}_{s'}(p')\sigma_{0i}u_s(p) \approx \xi_{s'}^\dagger\left(2m\sigma_{0i} - \varepsilon_{ijk}Q_j\gamma_0\Sigma_k + iq_i\gamma_0\right)\xi_s.
\] (7.23)

Using Eq. (5.1) throughout, we see that the momentum-independent term on the right side vanishes, and we are left with

\[
\bar{u}_{s'}(p')\sigma_{0i}u_s(p) \approx iq_i\delta_{ss'} - Q_j\varepsilon_{ijk}\xi_{s'}^\dagger\Sigma_k\xi_s.
\] (7.24)

A different kind of non-relativistic limit is obtained if both indices on the sigma-matrix are spatial. The matrix \( \sigma_{ij} \) is essentially a spin matrix, as mentioned in Eq. (3.2). The momentum-independent term in \( \bar{u}_{s'}(p')\sigma_{ij}u_s(p) \) is then the matrix element of the spin operator. In particular, if the spinors on the two sides have \( s = s' \), then the bilinear is the expectation value of spin in that state. In order to evaluate the terms linear in the momenta, we need the following relations which follow from Eq. (7.21):

\[
\begin{align*}
\left[ \sigma_{ij}, \gamma_k \right] &= 2i(g_{jk}\gamma_i - g_{ik}\gamma_j), \\
\left[ \sigma_{ij}, \gamma_k \right]_+ &= 2\varepsilon_{ijk}\gamma^\nu\gamma_5 = -2\varepsilon_{ijk}\gamma_0\gamma_5.
\end{align*}
\] (7.25a)

Obviously, the commutator term does not give a non-zero contribution to the bilinear of Eq. (7.12) because of Eq. (7.15). Since \( \gamma_5 \) anticommutes with \( \gamma_0 \), we can use the steps shown in Eq. (7.14), with \( \gamma_i \) replaced by \( \gamma_5 \), to show that

\[
\xi_{s'}^\dagger\gamma_5\xi_s = 0.
\] (7.26)
So even the anticommutator term does not contribute. Only the momentum-independent term survives to this order, and the result is
\[
\bar{u}_{s'}(p')\sigma_{ij}u_s(p) \approx 2m\varepsilon_{ijk}\xi^\dagger_{s'}\Sigma_k\xi_s.
\] (7.27)

### 7.2.4 Pseudoscalar bilinear

This corresponds to the case \(F = \gamma_5\). The momentum-independent term vanishes because of Eq. (7.26), and the anticommutator is also zero, so that we are left with
\[
\bar{u}_{s'}(p')\gamma_5\gamma_\xi u_s(p) \approx q_j\xi_{s'}^\dagger\gamma_j\gamma_5\xi_s,
\] (7.28)

Using Eq. (4.18), this expression can be written in the form
\[
\bar{u}_{s'}(p')\gamma_5u_s(p) \approx -Q_j\xi_{s'}^\dagger\Sigma_j\xi_s,
\] (7.29)

recalling the definition of \(\xi\) in Eq. (5.1).

### 7.2.5 Axial vector bilinears

Finally, we discuss the cases when \(F\) is of the form \(\gamma_i\gamma_5\). Two different cases arise, as in the case with vector or tensor bilinears. For \(F = \gamma_0\gamma_5\), we can use Eqs. (7.11) and (7.26) to write
\[
\xi_{s'}^\dagger\gamma_0\gamma_5\xi_s = \xi_{s'}^\dagger\gamma_5\xi_s = 0,
\] (7.30)

which means that the momentum-independent term vanishes. The commutator term is also zero, so that we are left with
\[
\bar{u}_{s'}(p')\gamma_0\gamma_5u_s(p) \approx -Q_j\xi_{s'}^\dagger\gamma_0\gamma_5\gamma_j\xi_s,
\] (7.31)

Using Eq. (4.18) now, this can be written as
\[
\bar{u}_{s'}(p')\gamma_0\gamma_5u_s(p) \approx -Q_i\xi_{s'}^\dagger\Sigma_i\xi_s,
\] (7.32)

On the other hand, for \(F = \gamma_i\gamma_5\), we find that the commutator appearing in Eq. (7.12) is
\[
[\gamma_i\gamma_5, \gamma_j] = -[\gamma_i, \gamma_j]_+\gamma_5 = 2\delta_{ij}\gamma_5,
\] (7.33)

whose matrix element vanishes because of Eq. (7.26). The anticommutator is
\[
[\gamma_i\gamma_5, \gamma_j]_+ = -2i\sigma_{ij}\gamma_5.
\] (7.34)
However, using Eqs. (3.2) and (4.18), we find
\[ \sigma_{ij} \gamma_5 = \varepsilon_{ijk} \Sigma_k \gamma_5 = \varepsilon_{ijk} \gamma_0 \gamma_k, \] (7.35)
whose matrix element also vanishes owing to Eqs. (7.11) and (7.15). Thus, only the
momentum-independent term survives to the proposed order, and the result can be
written as
\[ \bar{u}_{s'}(\mathbf{p'}) \gamma_i \gamma_5 u_s(\mathbf{p}) \approx -\xi_{s'}^\dagger \Sigma_i \xi_s, \] (7.36)
using Eqs. (4.18) and (7.11). Once again, these are matrix elements of the spin
operator, which reduce to the expectation value of spin if the two spinors on both
sides are the same.

### 7.2.6 A note on the momentum expansion of bilinears

We see that, in the momentum expansion, bilinears which have a momentum-
independent term do not have a term that is linear in the 3-momenta, and vice
versa. This feature can be explained by using the parity properties of the bilinears.
Here we outline a proof without invoking parity explicitly.

For this, consider two different representations of the Dirac matrices, one denoted
by a tilde sign and one without, which are related in the following way:
\[ \tilde{\gamma}_0 = \gamma_0, \quad \tilde{\gamma}_i = -\gamma_i. \] (7.37)
Obviously, if the \( \gamma_\mu \)'s satisfy the anticommutation relation, so do the \( \tilde{\gamma}_\mu \)'s. The
eigenvectors \( \xi_s \) will be identical in the two representations, since \( \gamma_0 \) is the same.
From Eq. (5.12), we see that the spinors in the tilded representation are given by
\[ \tilde{u}_s(-\mathbf{p}) = u_s(\mathbf{p}). \] (7.38)

The bilinears are representation-independent. Thus,
\[ \tilde{u}_{s'}(\mathbf{p'}) \tilde{F} \tilde{u}_s(\mathbf{p}) = \tilde{u}_{s'}(\mathbf{p'}) \tilde{F} u_s(\mathbf{p}), \] (7.39)
where \( \tilde{F} \) contains exactly the same string of Dirac matrices or associated matrices
of the tilded representation that are contained in \( F \), e.g., if \( F = \sigma_{0i} \) then \( \tilde{F} = \tilde{\sigma}_{0i} \). Using Eq. (7.38) now, we can write
\[ \tilde{u}_{s'}(-\mathbf{p'}) \tilde{F} u_s(-\mathbf{p}) = \tilde{u}_{s'}(\mathbf{p'}) \tilde{F} u_s(\mathbf{p}), \] (7.40)
If $F$ contains an even number of $\gamma_i$'s, then $F$ and $\tilde{F}$ are equal, and we see that the bilinear would contain only even order terms in the 3-momenta. This is the case if $F$ is $\mathbb{1}$, $\gamma_0$, $\sigma_{ij}$ or $\gamma_i\gamma_5$. Note that the definition of Eq. (3.6) implies that $\gamma_5$ contains an odd number of $\gamma_i$'s. On the other hand, if $F$ is any of the combinations $\gamma_i$, $\gamma_0\gamma_5$, $\sigma_{0i}$ and $\gamma_5$, the bilinears are odd in the 3-momenta. We have seen these features explicitly in the reductions of the bilinears above.

8 Spinor quadrilinears: Fierz identities

A quadrilinear is a product of two bilinears of spinors. This kind of objects appear in the low-energy limit of any theory where fermions interact through exchanges of bosons, e.g., in the Fermi theory of weak interactions. The important point is that there is some arbitrariness in the order of the spinors in writing quadrilinears, expressed through identities which are called Fierz identities [5]. This is the subject of discussion of this section.

We will denote spinors by $w_1$, $w_2$ etc. in this section. Here, the letter $w$ can stand for either $u$ or $v$, i.e., each of the spinors that appear in this section can be either a $u$-spinor or a $v$-spinor. The subscript 1,2 etc. stand for a certain 3-momentum and a certain mass. For example, $w_1$ can mean a $u$-spinor with momentum $p_1$ for a particle of mass $m_1$, and so on.

In order to pave the road for the Fierz identities, we first consider a product $w_2\bar{w}_1$. It is a $4 \times 4$ matrix, and therefore can be written in the form given in Eq. (4.10):

$$w_2\bar{w}_1 = a + b^\mu\gamma_\mu + c^{\mu\nu}\sigma_{\mu\nu} + d^\mu\gamma_\mu\gamma_5 + e\gamma_5.$$  
(8.1)

In order to evaluate the co-efficients $a$ through $e$, we first take the trace of this expression. Using the facts that the trace of any odd number of $\gamma$-matrices is zero, and the trace of $\sigma_{\mu\nu}$ is zero because it is a commutator, and Eq. (3.15), we obtain

$$\text{Tr} \left( w_2\bar{w}_1 \right) = a \text{Tr} \mathbb{1} = 4a.$$  
(8.2)

Since the trace operation is cyclic, we can write this equation as

$$a = \frac{1}{4} \text{Tr} \left( \bar{w}_1w_2 \right) = \frac{1}{4} \bar{w}_1w_2,$$  
(8.3)

using in the last step the fact that $\bar{w}_1w_2$, being just a $1 \times 1$ matrix, is the trace of itself. Exactly similarly, multiplying both sides of Eq. (8.1) by $\gamma_\lambda$ from the left.
before taking the trace, we would obtain

\[ b^\mu = \frac{1}{4} \left( \bar{w}_1 \gamma^\mu w_2 \right). \]  

We can continue this process to evaluate all co-efficients of Eq. (8.1), and the result is

\[ w_2 \bar{w}_1 = \frac{1}{4} \left[ \left( \bar{w}_1 w_2 \right) \mathbb{I} + \left( \bar{w}_1 \gamma^\mu w_2 \right) \gamma_\mu + \frac{1}{2} \left( \bar{w}_1 \sigma^{\mu\nu} w_2 \right) \sigma_{\mu\nu} 
- \left( \bar{w}_1 \gamma^\mu \gamma_5 w_2 \right) \gamma_\mu \gamma_5 + \left( \bar{w}_1 \gamma_5 w_2 \right) \gamma_5 \right]. \]  

This is the basic Fierz identity. The identities involving quadrilinears follow from it. For example, one can multiply both sides by \( \bar{w}_3 \) from the left and by \( w_4 \) from the right, obtaining

\[ \left( \bar{w}_3 w_2 \right) \left( \bar{w}_1 w_4 \right) = \frac{1}{4} \left[ \left( \bar{w}_1 w_2 \right) \left( \bar{w}_3 w_4 \right) + \left( \bar{w}_1 \gamma^\mu w_2 \right) \left( \bar{w}_3 \gamma_\mu w_4 \right) 
+ \frac{1}{2} \left( \bar{w}_1 \sigma^{\mu\nu} w_2 \right) \left( \bar{w}_3 \sigma_{\mu\nu} w_4 \right) - \left( \bar{w}_1 \gamma^\mu \gamma_5 w_2 \right) \left( \bar{w}_3 \gamma_\mu \gamma_5 w_4 \right) 
+ \left( \bar{w}_1 \gamma_5 w_2 \right) \left( \bar{w}_3 \gamma_5 w_4 \right) \right]. \]  

Similarly, if one multiplies Eq. (8.5) by \( \bar{w}_3 \gamma^\lambda \) from the left and by \( \gamma^\lambda w_4 \) from the right, one obtains \( \left( \bar{w}_3 \gamma^\lambda w_2 \right) \left( \bar{w}_1 \gamma^\lambda w_4 \right) \) on the left side. On the right side, the bilinears of the form \( \bar{w}_3 \cdots w_4 \) that appear are the following:

\[ \bar{w}_3 \gamma^\lambda \gamma_\lambda w_4 = 4 \bar{w}_3 w_4, \]  
\[ \bar{w}_3 \gamma^\lambda \gamma_\mu \gamma_\lambda w_4 = -2 \bar{w}_3 \gamma_\mu w_4, \]  
\[ \bar{w}_3 \gamma^\lambda \sigma_{\mu\nu} \gamma_\lambda w_4 = 0, \]  
\[ \bar{w}_3 \gamma^\lambda \gamma_\mu \gamma_5 \gamma_\lambda w_4 = 2 \bar{w}_3 \gamma_\mu \gamma_5 w_4, \]  
\[ \bar{w}_3 \gamma^\lambda \gamma_5 \gamma_\lambda w_4 = -4 \bar{w}_3 \gamma_5 w_4, \]  

where various contraction formulas listed in Eq. (4.6) and Eq. (4.8b) have been used to simplify the left sides of these equations. Thus, the final form of this Fierz identity would be

\[ \left( \bar{w}_3 \gamma^\lambda w_2 \right) \left( \bar{w}_1 \gamma^\lambda w_4 \right) = \frac{1}{4} \left[ 4 \left( \bar{w}_1 w_2 \right) \left( \bar{w}_3 w_4 \right) - 2 \left( \bar{w}_1 \gamma^\mu w_2 \right) \left( \bar{w}_3 \gamma_\mu w_4 \right) 
- 2 \left( \bar{w}_1 \gamma^\mu \gamma_5 w_2 \right) \left( \bar{w}_3 \gamma_\mu \gamma_5 w_4 \right) - 4 \left( \bar{w}_1 \gamma_5 w_2 \right) \left( \bar{w}_3 \gamma_5 w_4 \right) \right]. \]  

Identities involving other kinds of bilinears on the left side can be easily constructed.
9 Concluding remarks

The aim of the article was to show that some important identities involving Dirac spinors can be proved without invoking any specific form for the spinors. As we mentioned earlier, the specific forms depend on the representation of the Dirac matrices. For the sake of elegance and safety, it is better to deal with the spinors in a representation-independent manner.

The analysis can be extended to quantum field theory involving Dirac fields. Properties of Dirac field under parity, charge conjugation and time reversal can be derived in completely representation-independent manner. This has been done at least in one textbook of quantum field theory [6], to which we refer the reader for details.

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Appendix A The metric tensor and the Levi-Civita symbol

Our convention for the metric tensor is:

\[ g_{\mu\nu} = \text{diag}(+1, -1, -1, -1). \]  

(A.1)

The Levi-Civita symbol is the completely antisymmetric rank-4 tensor, whose non-zero elements have been chosen by the convention given in Eq. (3.8). Product of two Levi-Civita symbols can be expressed in terms of the metric tensor:

\[ \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{\mu'\nu'\lambda'\rho'} = \left| \begin{array}{cccc} \delta^\mu_{\mu'} & \delta^\mu_{\nu'} & \delta^\mu_{\lambda'} & \delta^\mu_{\rho'} \\ \delta^\nu_{\mu'} & \delta^\nu_{\nu'} & \delta^\nu_{\lambda'} & \delta^\nu_{\rho'} \\ \delta^\lambda_{\mu'} & \delta^\lambda_{\nu'} & \delta^\lambda_{\lambda'} & \delta^\lambda_{\rho'} \\ \delta^\rho_{\mu'} & \delta^\rho_{\nu'} & \delta^\rho_{\lambda'} & \delta^\rho_{\rho'} \end{array} \right| \equiv -\delta^\mu_{[\mu'}\delta^\nu_{\nu'}\delta^\lambda_{\lambda'}\delta^\rho_{\rho']} , \]  

(A.2)

where the pair of two vertical lines on two sides of the matrix indicates the determinant of the matrix, and the square brackets appearing among the indices imply an antisymmetrization with respect to the enclosed indices. By taking successive
contractions of this relation, we can obtain the following relations:

\[
\begin{align*}
\varepsilon^\mu_\nu \lambda_\rho \varepsilon^\mu'_\nu' \lambda'_\rho' &= -\delta^\nu_\nu' \delta^\lambda_\lambda' \delta^\rho_\rho' \quad \text{,} \\
\varepsilon^\mu_\nu \lambda_\rho \varepsilon^\mu'_\nu' \lambda'_\rho' &= -2\delta^\lambda_\lambda' \delta^\rho_\rho' \quad \text{,} \\
\varepsilon^\mu_\nu \lambda_\rho \varepsilon^\mu_\nu \lambda_\rho' &= -6\delta^\rho_\rho' \quad \text{,} \\
\varepsilon^\mu_\nu \lambda_\rho \varepsilon^\mu_\nu \lambda_\rho &= -24 .
\end{align*}
\]

(A.3a)

(A.3b)

(A.3c)

(A.3d)

Appendix B  Note on a class of representations of Dirac matrices

In this appendix, we want to make a comment about a class of representations of the Dirac matrices where each matrix is either purely real or purely imaginary. Note that the hermiticity property of the 16 basis matrices mentioned in Eq. (4.9) are all determined by their definitions and through the hermiticity property of the $\gamma_\mu$'s given in Eq. (2.3). Thus, if any of these 16 basis matrices has either purely real or purely imaginary elements, it would be a symmetric or an antisymmetric matrix. However, the number of antisymmetric and symmetric $4 \times 4$ matrices must be 6 and 10 respectively. This property, invoked already in §4.4 to obtain the antisymmetry of the matrix $C$, can produce interesting constraints on possible representations of the Dirac matrices of this class.

Here is how it goes. In this class of representation, we can introduce the parameter $c_0$ by the definition

\[
\gamma_0^* = c_0 \gamma_0 .
\]

(B.1)

In other words, if $c_0 = +1$, the matrix $\gamma_0$ is real. Since $\gamma_0$ must be hermitian, it implies that it is symmetric in this case. On the other hand, if $c_0 = -1$, the matrix $\gamma_0$ is imaginary and therefore antisymmetric. Next, we suppose that, among the three matrices $\gamma_i$, there are $n$ matrices whose elements are all real. Of course $0 \leq n \leq 3$. Since the $\gamma_i$ matrices are antihermitian, this implies that $n$ of them should be antisymmetric. If $c_0 = +1$, this means that $n$ among the three $\sigma_{0i}$ matrices will be antisymmetric. On the other hand, if $c_0 = -1$, we will have $3 - n$ antisymmetric matrices among the $\sigma_{0i}$’s. Continuing the counting in this manner we obtain that, among the 16 basis matrices given in Eq. (4.9), the number of antisymmetric matrices is given by

\[
N_A = 8 - \binom{3}{n} + n + \left[ 1 + 2c_0(1 - n) \right] E ,
\]

(B.2)
where

\[ E = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \]  

(B.3)

More explicitly, the result of Eq. (B.2) can be written as follows:

| \( \gamma_0 \) is | Number of antisymmetric matrices for |
|-------------------|-------------------------------------|
| real              | \( n = 0 \) \quad \( n = 1 \) \quad \( n = 2 \) \quad \( n = 3 \) |
|                   | 10 \quad 6 \quad 6 \quad 10 |
| imaginary         | 6 \quad 6 \quad 10 \quad 10 |

(B.4)

Of course, 10 is an inadmissible solution. Thus, this table shows that for real \( \gamma_0 \), only one or two of the \( \gamma_i \)’s can be real. On the other hand, for imaginary \( \gamma_0 \), the number of real \( \gamma_i \)’s is either zero or one.

There is an interesting feature of the table in Eq. (B.4). It is possible to have all \( \gamma_\mu \)’s to be imaginary (i.e., \( n = 0 \) and \( c_0 = -1 \)), but not possible to have all of them to be real (i.e., \( n = 3 \) and \( c_0 = 1 \)). If all \( \gamma_\mu \)’s are taken to be imaginary, the differential operator that acts on the field \( \psi \) in Eq. (2.5) is real. It shows that it is possible to have real solutions of the Dirac equation in some representation. Such solutions for the field are called Majorana fields, and the representation in which all \( \gamma_\mu \)’s are imaginary is called the Majorana representation of the Dirac matrices.

However, since \( n = 3 \) produces inadmissible solutions, all \( \gamma_\mu \)’s cannot be taken to be real. Accordingly, the matrix multiplying the spinors in Eq. (2.9) cannot be real, and so the spinors can never be purely real in any representation.

References

[1] See, for example,

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b) C. Itzykson and J. B. Zuber, *Quantum Field Theory*, (McGraw-Hill, 1980);

c) F. Halzen and A. D. Martin, *Quarks and leptons*, (John Wiley & Sons, 1984);

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[2] See, e.g., S. S. Schweber, *Introduction to relativistic quantum field theory*, (Harper & Row, 1962).
[3] J. M. Jauch and F. Rohrlich, *The theory of photons and electrons*, (Springer-Verlag, 2nd edition, 1976).

[4] This proof appears in the following textbooks:

  a) A. I. Akhiezer and S. V. Peletminsky, *Fields and Fundamental Interactions*, (Naukova Dumka, Kiev, 1986), Section 1.5.5;

  b) A. Das, *Lectures on Quantum Field Theory*, (World Scientific, Singapore, 2008), Section 11.2.2.

I am grateful to E. Akhmedov for bringing this proof to my attention.

[5] M. Fierz, “Zur Fermischen Theorie des $\beta$-Zerfalls,” Z. Physik 104, 553-565 (1937). For more modern derivations including chiral identities as well, see, e.g., J. F. Nieves and P. B. Pal, “Generalized Fierz identities,” Am. J. Phys. 72 (2004) 1100–1108; C. C. Nishi, “Simple derivation of general Fierz-type identities”, Am. J. Phys. 73 (2005) 1160–1163.

[6] A. Lahiri and P. B. Pal, *A first book of Quantum Field Theory*, (Narosa Publishing House, New Delhi, 2nd edition 2004).