A NEW APPROACH TO THE THEORY
OF ORDINARY DIFFERENTIAL EQUATIONS
WITH SMALL PARAMETER \(^1\)

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In this paper we study both the periodic problem and the Cauchy problem associated to the system of ordinary differential equations described by

\[ \dot{x} = \varepsilon \phi(t, x) + \psi(t, x), \]

where \(\phi, \psi : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k\) are continuously differentiable \(T\)-periodic, with respect to time \(t\), functions and \(\varepsilon\) is a small positive parameter. Systems of the form (1) represent a classic topic of the theory of differential equations depending on a small parameter and such systems have been investigated by a number of different methods. Here we devote the attention only to the methods and results which are directly related to the aim of the present work. In the papers [5], [8], [9] this problem is tackle by means of the theory of the rotation number of vector fields or by theorems based on Poincaré index. In this case for the \(T\)-periodic problem one usually assumes that the system (1) has isolated \(T\)-periodic solution \(x_T\) of non-zero topological index at \(\varepsilon = 0\), while for the Cauchy problem one assumes the uniqueness of the solution \(x_0\) defined on \([0, d]\). This second condition ensures that the topological index of \(x_0\) is also non-zero (see [10]). After this, by using results concerning the continuous dependence of solutions on the operator equations on \(\varepsilon\) (see, for instance, [9]), it is shown the existence and convergence of the solutions to (1) when \(\varepsilon \geq 0\) sufficiently small to \(x_T\) and \(x_0\) respectively. Observe, that for the Cauchy problem the closeness of the solutions is proved on the interval where it is assumed the existence and uniqueness of the solution \(x_0\). An other approach to deal with the periodic and Cauchy problem for system (1) is the averaging method proposed by N. N. Bogolubov - N. M. Krilov (see [2]). To use this method system (1) is first reduced to the standard form

\[ \dot{x} = \varepsilon f(t, x), \]

where function \(f\) is \(T\)-periodic with respect to the first variable. Then, topological methods and vector field theory can be applied to investigate system (2), (see, for instance, [1], [6], [12], [14]). For this, the following auxiliary system is considered

\[ \dot{x} = \varepsilon f_0(x), \]

where

\[ f_0(\xi) = \frac{1}{T} \int_0^T f(s, \xi)ds, \quad \xi \in \mathbb{R}^k. \]

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The isolated equilibrium states of system (3) which have non-zero topological index with respect to the vector field $-f_0$ give rise to $T$-periodic solutions of system (2). While the solutions of the Cauchy problem for system (2) are close to the corresponding solutions of the Cauchy problem for system (3) on the interval of length $(1/\varepsilon)d$. The main method for reducing system (1) to standard form (2) consists in the following change of variable

$$z(t) = \Omega(0, t, x(t)),$$

where $\Omega(\cdot, t_0, \xi)$ denotes the solution $x$ of (1) with the initial condition $x(t_0) = \xi$ and $\varepsilon = 0$. Therefore, it is necessary to assume that the change of variable (4) is $T$-periodic with respect to $t$ for every $T$-periodic function $x$ in order to use the classical averaging principle (see [13], [3]).

In the present paper we do not require such $T$-periodicity condition for (4). In fact, we assume only that the boundary of some open set $U \subset \mathbb{R}^k$ when $\varepsilon = 0$ represents the initial values of $T$-periodic solutions of system (1). This assumption implies that the $T$-periodic solutions of system (1) at $\varepsilon = 0$ are not isolated and so it is impossible to use directly results concerning the continuous dependence of the solutions of the operator equations on $\varepsilon$. This situation takes place, for instance, if system (1) for $\varepsilon = 0$ is autonomous and it has an isolated cycle $x_0$. In this case, as set $U$ can be taken the interior of $x_0$. Such systems are very important in the applications, but the investigation of the existence of $T$-periodic solutions is difficult since the topological index of the set of $T$-periodic solutions which arise from cycle $x_0$ is equal to zero (see [4]).

In order to investigate the behaviour of solutions of system (1) when $\varepsilon \geq 0$ is small we introduce the following linear system

$$\dot{y} = \phi(t, \Omega(t, 0, \xi)) + \psi(t, \Omega(t, 0, \xi))y,$$

where $\xi \in \mathbb{R}^k$. We will establish our results in terms of system (5). The classical results of the averaging theory both of the $T$-periodic problem and the Cauchy problem will follow from our results. We will also show that the result from [7] in the case when $\psi(t, x) = Ax$, where the matrix $A$ has two simple eigenvalue $\pm i\frac{T}{2\pi}$, is a consequence of our results.

1. We consider first the problem of the existence of $T$-periodic solutions. We denote by $\eta(\cdot, s, \xi)$ the solution $y$ of system (5) satisfying $y(s) = 0$. In this section we will use the classical notion of the rotation number of a continuous map $F : \bar{U} \to \bar{U}$ defined in the closure of a bounded set $U \subset \mathbb{R}^k$ (see [9]) and we denote it by $\gamma(F, U)$.

**Theorem 1.**

Let the set $U \subset \mathbb{R}^k$ be open and bounded. Assume, that the following conditions hold

(A0) $\Omega(T, 0, \xi) = \xi$, $\xi \in \partial U$,
(A1) $\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0$, $s \in [0, T]$, $\xi \in \partial U$,
(A2) $\gamma(\eta(T, 0, \cdot), U) \neq 0$. 

2
Then there exists \( \varepsilon_0 \geq 0 \) such, that system (1) has at least one \( T \)-periodic solution belonging to the set \( X = \{ x : \Omega(0, t, x(t)) \in U, \ t \in [0, T] \} \) for all \( \varepsilon \in (0, \varepsilon_0) \).

We would like to point out that there are many cases for which it is enough to verify condition (A2) for some particular function \( \phi \), which does not necessarily coincide with the one given in (1).

**Theorem 2.** Assume condition (A0). Then for every functions \( \phi_1 \) and \( \phi_2 \), such that the function \( \phi_\lambda = \lambda \phi_1 + (1-\lambda)\phi_2 \) satisfies condition (A1) for all \( \lambda \in [0, 1] \), the rotations \( \gamma(\eta_1(T, 0, \cdot), U) \) and \( \gamma(\eta_2(T, 0, \cdot), U) \) coincide.

Assume that the dimension of the phase space \( k \) is equal to 2 and the system (1) has a \( T \)-periodic cycle, that is it has \( T \)-periodic solution \( x_0 \), such that function \( x_0(t+\theta) \) on \( t \) is a solution of (1) for \( \varepsilon = 0 \) and \( \theta \in [0, T] \). Assume that the cycle \( x_0 \) is simple, that is curve \( x_0 \) has no self-joints. Than by Djordan theorem the curve \( x_0 \) restricts an one-connected domain \( U \) of the space \( \mathbb{R}^2 \). By using the fact that the rotation number of the field of tangents and the rotation number of the field of normals are equal to 1 on the cycle we derive in the sequel an existence result for \( T \)-periodic solutions. Since domain \( U \) is one-connected than it homeomorphic to the unique circle \( B_1 \). Denote by \( g \) some homeomorphism of \( U \) on \( B_1 \). For an arbitrary \( \delta \in [0, 1] \) define \( \delta \)-contraction \( W_\delta(U) \) of the domain \( U \) by the formula \( W_\delta(U) = g^{-1}((1-\delta)g(U)) \). Now if

(A3) \( T \)-periodic system \( \dot{y} = \psi^{(2)}_\eta(t, x_0(t+\theta))y \) has 1 as a simple Floquet exponent for all \( \theta \in [0, T] \)

than the points of boundary of the set \( W_{-\delta}(U) \) are the points of \( T \)-irreversibility (see [8]) of the system (1) for \( \varepsilon = 0 \). Thus by condition (A3) the system (1) has \( T \)-periodic solution acting to \( W_{-\delta}(U) \) for small \( \varepsilon > 0 \) and \( \delta > 0 \) (see [8], theorem 6.1). The theorem 6.1 from [8] is not applicable when the condition (A3) does not hold generally speaking. In this case a condition of existence of \( T \)-periodic solutions for system (1) can be derived by the ground of theorems 1 and 2. In fact, observing than in this case \( \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)), \dot{x}_0(\theta) \perp \neq 0 \) for such and only such \( s, \theta \in [0, T] \) that

\[
(A3)_1 \int_0^T e^{-s \int_0^r \psi^{(2)}_\eta(r, x_0(r))dr} < \phi(t - \theta, x_0(t)), \dot{x}_0(t) \perp > dt \neq 0,
\]

where \( < \cdot, \cdot > \) is a scalar product in \( \mathbb{R}^2 \) and \( \dot{x}_0(\theta) \perp \) is a rotation of \( \mathbb{R}^2 \) and \( \dot{x}_0(\theta) \perp \) against clockwise we can derive the following establishment.

**Theorem 3.** Let \( x_0 \) is a simple cycle of the system (1) and \( U \) is it’s interior. Assume the condition \((A3)_1\) holds for all \( s, \theta \in [0, T] \). Then for sufficiently small \( \varepsilon > 0 \) the system (1) has \( T \)-periodic solution acting in \( U \).

Theorem 3 can be generalized to the case of systems (1) of even dimension \( k = 2p \), namely to the case of systems (1) consisting of \( p \) two-dimensional systems which have simple cycles \( x_1, \ldots, x_p \) of the same period \( T \). In this case one might take as \( U \) the cartesian product of the
interiors of this cycles in the corresponding spaces.

2. Let us now to study the Cauchy problem. In this section the initial condition for system (1) is fixed. Assume, that the limit
\[ \Phi(\xi) = -\lim_{n \to \infty} \frac{\eta(-nT, 0, \xi)}{nT} \]
exists uniformly with respect to \( \xi \in B(0, r) \) for every \( r > 0 \). The limit (6) exists, for instance, in the case when the system (5) is stable for \( t \to -\infty \) uniformly with respect to \( \xi \in B(0, r) \) for every \( r > 0 \).

**Theorem 4.** Assume that the system \( \dot{z} = \Phi(z) \) has an unique solution \( z_0 \) on the interval \([0, d]\), satisfing \( z(0) = \xi_0 \). Then for every \( \gamma > 0 \) there exists \( \varepsilon_0 > 0 \) such that (1) has a solution satisfying \( x_\varepsilon(0) = \xi_0 \) and such that
\[ \|x_\varepsilon(t) - \Omega(t, 0, z(\varepsilon t))\| \leq \gamma, \quad t \in \left[0, \frac{d}{\varepsilon}\right] \]
for every \( \varepsilon \in (0, \varepsilon_0) \).

Classical results of averaging theory given by N. N. Bogolubov - N. M. Krilov for \( T \)-periodic problem and for Cauchy problem (see [2]) follow from our results. To see this, it is sufficient to put \( \psi = 0 \) in (1) and use the formula
\[ \eta(\pm nT, s, \xi) - \eta(0, s, \xi) = \int_{s \pm nT}^{s} \phi(\tau, \xi) d\tau = \int_{s \pm nT}^{0} \phi(\tau, \xi) d\tau. \]
in order to compare the conditions of theorems 1 and 4 with the classical ones.

We now describe how from theorem 1 we can obtain a classical result from ([7], Theorem 3.1, p. 362) in the case when \( \psi(t, x) = Ax = (-x_2, x_1) \) and \( \phi(t, x) = (0, g(t, -x_1, x_2)) \), where the function \( g \) is \( 2\pi \)-periodic with respect to the first variable. Thus the dimension of the phase space is 2 and the matrix \( A \) has simple eigenvalues \( \pm i \). In this example we have choosen the notations in such a way that formulas from [7] and formulas from theorem 1 coincide. We have
\[ \eta(T, s, \xi(a, \theta)) - \eta(0, s, \xi(a, \theta)) = \begin{pmatrix} \cos \theta \sin \theta \\ -\sin \theta \cos \theta \end{pmatrix} H(a, \theta), \]
where \( \xi(a, \theta) = (-a \cos \theta, a \sin \theta) \) and
\[ H(a, \theta) = \int_{0}^{2\pi} \begin{pmatrix} (\sin \tau)f(\tau + \theta, a \cos \tau, -a \sin \tau) \\ (\cos \tau)f(\tau + \theta, a \cos \tau, -a \sin \tau) \end{pmatrix} d\tau. \]

It is assumed in [7] the existence of \( a_0, \theta_0 \in \mathbb{R} \) such that
\[ H(a_0, \theta_0) = 0 \quad \text{and} \quad \det|H'(a_0, \theta_0)| \neq 0. \]
But by this assumption there exists such the set \( V = (a_0 - \delta, a_0 + \delta) \times (\theta_0 - \delta, \theta_0 + \delta) \), that \( |\gamma(V)| = 1 \). Since vector fields \( \eta(T, s, \xi(\cdot, \cdot)) - \eta(0, s, \xi(\cdot, \cdot)) \) and \( H(\cdot, \cdot) \) are homotopic
on $\partial V$ for every $s \in [0, 2\pi]$ then these fields have the same rotation number on $V$. Without loss of generality we can consider $0 \leq \delta \leq a_0$ and $0 \leq \delta \leq \pi$. In this case the continuous function $\xi$ maps the open set $V$ on the open set $\xi(V)$. Hence $\gamma(\eta(T, 0, \cdot) - \eta(0, 0, \cdot), \xi(V)) = \gamma(\eta(T, s, \xi(\cdot, \cdot)) - \eta(0, s, \xi(\cdot, \cdot)), V)$ and by theorem 1 system (1) has $2\pi$-periodic solutions in the set $X = \{ x : (a, t + \theta) \in V, (a, \theta) = \xi^{-1}(x(t)), t \in [0, 2\pi] \}$ for sufficiently small $\varepsilon \geq 0$.

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