Advanced extraction of the deuteron charge radius from electron-deuteron scattering data

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To extract the charge radius of the proton, \( r_p \), from the electron scattering data, the PRad collaboration at Jefferson Lab has developed a rigorous framework for finding the best functional forms - the fitters - for a robust extraction of \( r_p \) from a wide variety of sample functions for the range and uncertainties of the PRad data. In this work we utilize and further develop this framework. Herein we discuss methods for searching for the best fitter candidates as well as a procedure for testing the robustness of extraction of the deuteron charge radius, \( r_d \), from parameterizations based on elastic electron-deuteron scattering data. The ansatz proposed in this paper for the robust extraction of \( r_d \), for the proposed low-\( Q^2 \) DRad experiment at Jefferson Lab, can be further improved once there are more data.

1. INTRODUCTION

Nucleons (protons and neutrons) are the building blocks of atomic nuclei, the structure of which provides an excellent laboratory to advance our understanding about how quantum chromodynamics (QCD) – the theory of strong interaction – works in the non-perturbative region quantitatively where currently our knowledge is rather poor. The proton root-mean-square (rms) charge radius – defined as

\[ r_p = r_{p,rms} \equiv \sqrt{\langle r^2 \rangle} = \left( -6 \frac{dG_E^p(Q^2)}{dQ^2} \bigg|_{Q^2=0} \right)^{1/2}, \tag{1} \]

with \( G_E^p \) being the proton electric form factor and \( Q^2 \) the 4-momentum transfer squared measured in lepton scattering experiments - also has a major impact on bound state quantum electrodynamics (QED) calculations of atomic energy levels. As such the proton charge radius defined in the same way as in lepton scattering experiments [1] can be determined from hydrogen spectroscopic measurements However, there are distinct discrepancies in the measurement results, observed among three types of experiments. The discrepancies mostly arose after 2010, when high-precision muonic hydrogen (\( \mu H \)) spectroscopy experiments reported two values of \( r_p \) to be 0.8409 ± 0.0004 fm [2,3]. On the other hand, the world-average value from CODATA-2014 – \( r_p = 0.8751 ± 0.0061 \) fm [4] – determined from atomic hydrogen (\( eH \)) spectroscopy experiments, and the results from electron-proton (\( e-p \)) scattering experiments until 2010, mostly agreed with each other. The challenge stemming from such a difference between the \( r_p \) values, measured from different types of the experiments is known as proton charge radius puzzle [5-7].

Nonetheless, two other recent measurements from the \( eH \) spectroscopy [3,9] reported \( r_p \) values, which within experimental uncertainties are consistent with the \( \mu H \) spectroscopy results.

Such an agreement is also observed from \( r_p \) measured by the PRad collaboration at Jefferson Lab [10] - \( r_p = 0.831 ± 0.007_{\text{stat}} ± 0.012_{\text{syst}} \) fm - that used a magnetic-spectrometer-free, calorimeter-based method in an unpolarized elastic \( e-p \) scattering experiment at very low \( Q^2 \), down to \( 2.1 \times 10^{-4} \) GeV\(^2\)/c\(^2\) [11,12].

The situation becomes similarly interesting and challenging if we move on to discuss measurements of the rms charge radius of the deuteron, \( r_d \), in electron-deuteron (\( e-d \)) experiments as well as in \( eD \) and \( \mu D \) spectroscopy. In particular, the CREMA collaboration has reported a deuteron charge radius – \( r_d = 2.12562 ± 0.00078 \) fm – from a muonic spectroscopy-based measurement of three \( 2P \to 2S \) transitions in \( \mu D \) atoms [13], which is 2.7 times more accurate but 7.5-\( \sigma \) smaller than the CODATA-2010 world-average value [14]. The radius from [13] is also 3.5-\( \sigma \) smaller than the \( r_d \) value, 2.1415±0.0045 fm, extracted from an electronic spectroscopy-based measurement [15] of \( 1S \to 2S \) transitions in \( eD \) atoms, after these transi-
tions have already been measured in [10].

Thereby, one also observes discrepancies from \( r_d \) measurements (like in the case of \( r_p \)) that have given rise to another challenge, dubbed as "deuteron charge radius puzzle." The PRad collaboration has proposed a low-\( Q^2 \) unpolarized elastic \( e - d \) scattering experiment named as DRad – basically anchored upon the PRad’s experimental setup – for a model-independent extraction of \( r_d \) with a sub-percent (\( \leq 0.25\% \)) precision, in order to address this newly developed puzzle [17].

Thus, given the importance of measuring not only \( r_p \) but also \( r_d \), our goal is to show how one can robustly extract \( r_d \) and control its uncertainties in a fitting procedure, using four parameterizations of the deuteron charge form factor, \( G_d^C \) [18–22]. In this work we apply and extend the ansatz used in Ref. [24], in which a comprehensive and systematic method is presented for choosing mathematical functions that can robustly extract \( r_p \) from a broad set of input functions describing the proton electric form factor, \( G_p^E \).

The rest of the paper is presented as follows. Sec. 2 has a brief discussion on the deuteron form factors and the radius extraction. In Sec. 3 we describe the general fitting procedure on how to extract \( r_d \) from generated \( G_d^C \) pseudo-data in the DRad kinematics and define some quantities to compare the properties of different fitters. In Sec. 4 we introduce the pseudo-data generation from the \( G_d^C \) parameterizations and discuss the method for searching for a fitter that will be able to extract \( r_d \) by using the available elastic \( e - d \) scattering data. In Sec. 5 we show a conservative way to estimate the bias for \( r_d \) extraction. We conclude on our work and discuss its prospects at the end. Also, in the Appendices we discuss the results of testing a few theoretical models and provide another robust fitter candidate which is analogous to the one considered in Sec. 4.

2. FORM FACTORS AND CHARGE RADIUS FROM UNPOLARIZED ELASTIC ELECTRON-DEUTERON CROSS SECTION

The understanding of the electromagnetic properties of the deuteron is of fundamental importance in nuclear physics, given that the deuteron is the only bound two-nucleon system. It is expected that at low-\( Q^2 \) region, where the relativistic effects and non-nucleonic degrees of freedom are expected to be negligible, the deuteron form factors are dominated by part of its wave function for which the two constituent nucleons are far apart. Theoretical calculations of \( r_d \) are considered to be reliable since they are independent of the nucleon-nucleon potential (for a broad class of potentials), and depend mostly on the binding energy and neutron-proton scattering length [20]. This makes \( r_d \) a perfect observable for a theory-experiment comparison.

So far three experiments have been conducted for determination of \( r_d \) from unpolarized elastic \( e - d \) scattering at low-\( Q^2 \) [27–29], the cross section of which in the one-photon exchange approximation is given by

\[
\frac{d\sigma}{d\Omega}(E,\theta) = \sigma_{NS} \left( A_d(Q^2) + B_d(Q^2) \tan^2\left(\frac{\theta}{2}\right) \right), \tag{2}
\]

where \( \sigma_{NS} \) is the differential cross section for the elastic scattering from a point-like and spinless particle at a scattering angle \( \theta \) and an incident energy \( E \). The 4-momentum transfer squared carried by the exchanged virtual photon is defined in terms of the four-momenta of the incident (\( k \)) and scattered (\( k' \)) electrons: \( Q^2 = -(k - k')^2 \). In this case the deuteron structure functions in Eq. (2) are related to its charge, \( G_d^C \), magnetic dipole, \( G_d^M \), and electric quadrupole, \( G_d^Q \), form factors via [31–33]

\[
A_d(Q^2) = (G_d^C(Q^2))^2 + \frac{2}{3} \tau (G_d^M(Q^2))^2 + \frac{8}{9} \tau^2 (G_d^Q(Q^2))^2,
\]

\[
B_d(Q^2) = \frac{4}{3} \tau (1 + \tau) (G_d^M(Q^2))^2,
\]

with \( \tau = Q^2/4M_d^2 \), where \( M_d \) is the deuteron mass. Also, there are the following additional relations:

\[
G_d^C(0) = 1, \quad \frac{G_d^Q(0)}{\mu_Q^d} = 1, \quad \frac{G_d^M(0)}{\mu_M^d} = 1,
\]

with the given deuteron electric quadrupole moment, \( \mu_Q^d \), and magnetic dipole moment, \( \mu_M^d \).

At very low but experimentally accessible \( Q^2 \) such as \( \sim 10^{-4} \) (GeV/c)^2, the contributions from \( G_d^Q \) and \( G_d^M \) to the scattering process are negligible. By choosing different \( G_d^M \) and \( G_d^Q \) form factors [18–22] from four data-driven models discussed in Appendix A (and throughout the paper) for extracting \( G_d^C \) from the cross section, the effects of the choice of the form factor models on the deuteron radius are found to be 0.03\% and 0.009\%, respectively. Thereby, in order to extract the deuteron rms charge radius from \( e - d \) scattering data, one should fit \( G_d^C \) to the experimental data as a function of \( Q^2 \), and calculate the slope of this function at \( Q^2 = 0 \), according to

\[
r_d \equiv r_d,\text{rms} = \sqrt{\langle r^2 \rangle} = \left( -6 \left| \frac{dG_d^C(Q^2)}{dQ^2} \right|_{Q^2=0} \right)^{1/2}, \tag{4}
\]

in analogy to how \( r_p \) is obtained.

1 Throughout the text we use dimensionless \( \mu_M^d \equiv (\mu_M^d/\mu_N) = 0.8574 \) and \( \mu_Q^d \equiv (\mu_Q^d/\mu_N) = 0.2859 \) [34].
3. THE FITTING PROCEDURE AND ROBUSTNESS

3.1. The general procedure

Ref. [24] gives a general framework with input form-factor functions and various fitting functions, for finding functional forms (fitters) that allow a robust extraction of an input proton radius. Analogously, we can find robust fitters for extraction of \( r_d \) by

(i) generating sets of \( G_C \) pseudo-data values with user-defined certain fluctuations at given \( Q^2 \) bins, using some \( G_C \) charge form factor models as input;

(ii) systematically fitting each set of generated pseudo-data (see below) - in a repetitive manner - with all combinations of input functions and fitting functions;

(iii) calculating the \( r_d \) values according to Eq. (4), and studying the distributions of the results that come from a developed fitting routine.

When the program library generates bin-by-bin type fluctuations added to the pseudo-data, it occurs according to the user-defined random Gaussian distribution at each bin. Stated otherwise, in order to mimic the bin-by-bin fluctuations (\( Q^2 \)-independent) of the data, the pseudo-data should be smeared by shifting the \( G_C \) central value at each \( Q^2 \) bin with a random number following the Gaussian distribution, \( \mathcal{N}(\mu, \sigma^2) \), given by

\[
\mathcal{N}(\mu, \sigma_g^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-((G^d_C - \mu)^2/(2\sigma^2))}, \tag{5}
\]

In this work we take \( \mu = 0 \) and \( \sigma_g = \delta G^d_C \), where \( \delta G^d_C \) comes from the estimated statistical and/or systematic uncertainties in the \( e - d \) (DRad) experiment. The produced tables of \( G_C \) vs. \( Q^2 \) with fluctuations are fitted with a number of fitters for extracting \( r_d \) (see Fig. 1).

3.2. The robustness and goodness of the fitters

The meaning of the robustness of a fitter in the charge radius extraction is that it can precisely extract the radius in the presence of higher order effects in the \( e - d \) (or \( e - p \)) data. The higher order effects basically change the curvature of the charge form factor, and influence its extrapolation to the very low \( Q^2 \) range, which makes it harder to obtain the slope in the limit of \( Q^2 = 0 \) with only few parameters.

In order to determine the robustness of a fitter based upon the general procedure already discussed, one can compare the size of the bias (bias \( \equiv \delta r_d = r_d[\text{mean}] - r_d[\text{input}] \)) with the variance \( \sigma \) (the rms value of the radius distribution). The bias comes from the “inconsistency” between a chosen fitter and would-be data in the limit of \( Q^2 = 0 \). The variance reflects the influence of the \( G^d_C \), bin-by-bin uncertainties on the radius. If \( \delta r_d < \sigma_{\text{stat}} \) (statistical variance) for most of the input form factor models, the given fitter will be considered as robust. In the case of an experiment, the goal of which is to minimize the overall uncertainty, we should also consider the

\[\text{FIG. 1.} \quad (\text{Color online}) \quad \text{The upper plot shows an example of one fit using the Abbott1 model (see Sec. 4.1) as input and Rational (1,1) (see Sec. 3.3) as the fitting function. The lower plot shows an example of } r_d[\text{fit}] \text{ distribution obtained by following the above-mentioned pseudo-data and fitting procedure. A Gaussian function, similar to that in Eq. (5), is used to fit the distribution.} \]

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2 A C++ coded program library has been created for generating, adding fluctuations to, and fitting the pseudo-data [24, 25] (see Sec. 4). The bin-by-bin and overall type fluctuations are assumed to imitate the binning and random uncertainty of a given set of real data. For fitting purposes the library uses the Minuit package of CERN ROOT [36, 37].

3 If the higher order effects are neglected, it should be a straight line. In this case one can fit it with a linear function, and always obtain the correct answer regardless of what the input radius is.
bias and variance together, using the Root Mean Square Error (RMSE)\footnote{The RMSE discussed throughout this paper is somewhat different from that discussed in \cite{23}, where the authors have considered $\sigma_{\text{stat}}$ in the formula of RMSE.}

$$\text{RMSE} = \sqrt{\delta r_d^2 + \sigma_{\text{total}}^2},$$

where $\sigma_{\text{total}}$ includes both bin-by-bin statistical and systematic uncertainties. The RMSE is a standard way of quantifying goodness of fits that allows to determine appropriate fitting function(s). The smaller the RMSE is, the better the corresponding fitter is. Eventually, we need to find a fitter(s) that can extract the precise deuteron radius, for all possible higher order effects existing in the unpolarized elastic $e - d$ data, for a given $Q^2$ range of the DRad kinematics.

### 3.3. Initial studies and motivation

Ref. \cite{23} takes into account different kinds of higher order effects by using nine different $G_E^p$ form factor parameterizations to generate pseudo-data in the PRad $Q^2$ range. The studies show that the two-parameter rational function, Rational (1,1), is robust and the best fitter for extraction of $r_p$ for the range and uncertainties of the PRad data, represented by

$$f_{\text{Rational (1,1)}}(Q^2) \equiv \text{Rational (1,1)} = p_0 G_E^p (Q^2) = p_0 \frac{1 + p_1^{(a)} Q^2}{1 + p_1^{(b)} Q^2},$$

where $p_0$ is a floating normalization parameter, $p_1^{(a)}$ and $p_1^{(b)}$ are two free fitting parameters. The radius is determined by $r_p = \sqrt{6 \left( p_1^{(b)} - p_1^{(a)} \right)}$. The other two robust fitters are the two-parameter continued fraction and the second order polynomial expansion of the so-called $Z$-transformation \cite{24,38}, which can extract the input proton radius regardless of the input electric form factor functions.

Eq. (7) is actually a special case from the class of the multi-parameter rational function of $Q^2$ given by

$$f_{\text{Rational (N,M)}}(Q^2) \equiv \text{Rational (N,M)} = p_0 G_E^p (Q^2) = p_0 \frac{1 + \sum_{i=1}^N p_i^{(a)} Q^2}{1 + \sum_{j=1}^M p_j^{(b)} Q^2},$$

where the orders $N$ and $M$ are defined by the user.

All the fitters studied in Ref. \cite{23} have been tested here by fitting pseudo-data generated using the four $G_E^p$ parameterizations from \cite{18,20,22} (see Sec. 4.1). For this test we took the DRad kinematics range of $2 \times 10^{-4}$ (GeV/c)$^2 < Q^2 < 0.05$ (GeV/c)$^2$, using bin-by-bin statistical uncertainties from 0.02% to 0.07% and systematic uncertainties from 0.06% to 0.16%. The bias and $\sigma_{\text{stat}}$ values of the five best fitters are shown in Fig. 2. Although the four-parameter Polynomial Z gives the smallest bias, it also gives the largest variance and RMSE amongst them. The RMSE value of Rational (1,1) is the smallest one, though it gives larger bias compared to the others.

However, given the limited number of $G_E^p$ parameterizations, the robustness of the fitters cannot be convincingly determined from these results. In this case, we can not mimic different kinds of higher order effects as comprehensively as it can be done for the proton $G_E^p$ models. We have also studied some theory-based models (discussed in Appendix B), and found that those models have large discrepancies with the experimental data.

The three-parameter continued fraction (CF (3)) and Polynomial Z (4) are from the classes of the CF expansion and multi-parameter polynomial expansion of $Z$, respectively. For their explicit expressions we refer the reader to Ref. \cite{23}. The CF (3) has the same functional form as Rational (1,2).
which show that the testing method of robustness applied to PRad is no longer suitable for the deuteron radius extraction. Based on our studies, the bias is a non-negligible source of the \( r_d \) systematic uncertainty estimated for the DRad experiment. This observation was our motivation for looking into other potentially better fitters for DRad, which might give similar variance but smaller bias as compared to those of Rational (1,1). At the same time, by having limited \( G_C^d \) parameterizations at our disposal, we also need to develop a more comprehensive method to estimate the bias when using various fitters.

4. SEARCHING FOR A ROBUST FITTER CANDIDATE

4.1. Pseudo-data generation

Here we give some specific details on the pseudo-data generation and fitting procedure described in the previous section:

A) Generators: Four \( G_C^d \) parametrizations based on available experimental data (names as Abbott1, Abbott2, Parker and SOG) are used to generate \( G_C^d \) values at given \( Q^2 \) bins. The details of these parameterizations are discussed in Appendix A.

B) Generating pseudo-data with fluctuations and fitting: Our binning choice for DRad is based on the binning of PRad. There are thirty bins from 0.8° to 6.0° at 1.1 GeV beam energy, and thirty seven bins from 0.7° to 6.0° at 2.2 GeV [39].

(i) Thirty \( G_C^d \) pseudo-data points at 1.1 GeV and thirty seven \( G_C^d \) points at 2.2 GeV are generated from the two deuteron models in Eq. (14) and/or Eq. (15);

(ii) To add statistical fluctuations, the total sixty seven pseudo-data points generated in (i) are smeared by sixty seven different random numbers according to Eq. (6);

(iii) In this step a set of pseudo-data is fitted by a specific fitter \( f_d(Q^2) \). The data points at 1.1 GeV and 2.2 GeV are combined and fitted by that fitter with two different floating normalization parameters corresponding to these two beam energies. The other fitting parameters in the fitter are required to be the same for the two energies;

(iv) Then the fitted radius is calculated from the fitted function in (iii), with

\[
 r_d[\text{fit}] = \left( -\frac{6}{4} \frac{df_d(Q^2)}{dQ^2} \right)_{Q^2=0}^{1/2}; \tag{9} \]

(v) The above step is repeated (10000 times) until the \( r_d[\text{fit}] \) distribution is stable for obtaining 10000 sets of \( G_C^d \) pseudo-data (diluted by Eq. (5)), and 10000 \( r_d[\text{fit}] \) values are calculated as well. We note that \( r_d[\text{mean fit}] \) is the mean value, and the variance \( \sigma \) is the rms value of the 10000 \( r_d[\text{fit}] \) distribution.

4.2. A Data-driven method

As described in Sec 3.3, our studies have shown that the bias is an important source of systematic uncertainty in the extraction of \( r_d \). Hence, to better control and/or minimize the bias in the \( r_d \) extraction, such as that obtained by the Rational (1,1) fitter, we propose a data-driven approach to search for a new robust fitter candidate. The existing experimental data sets such as the one shown in the table 1 of Ref. [18] provides \( G_C^d \) and \( \delta G_C^d \) at fixed \( Q^2 \) values that are typically higher than the values of the \( Q^2 \) range of the proposed DRad experiment. We can use a fitter with four free parameters and one floating normalization parameter to fit this set of data. In order to control the variance when extracting \( r_d \), initially, two of the four parameters of the fitter are fixed and used to fit the pseudo-data at low-\( Q^2 \) range of DRad.

In principle, if some fitter functions have fitting uncertainties in their fixed parameters, those parameters should be smeared using a uniform distribution bounded by the size of the uncertainty of the parameter. We repeat this step in the fitting procedure until the \( r_d[\text{fit}] \) distribution is stable (again doing 10000 repetitions as discussed in Sec. 4.1).

The Rational (1,3) is a function with four free parameters that has been used in [40] to fit \( G_C^d \). Compared to the Rational (1,1), it has good asymptotic behaviors satisfying not only \( G_C^d = 1 \) at \( Q^2 \to 0 \) but also \( G_C^d = 0 \) at \( Q^2 \to \infty \). This fitter function is given by

\[
 f_{\text{Rational (1,3)}}(Q^2) = \text{Rational (1, 3)} = p_{01} G_C^d(Q^2) = p_{01} \frac{1 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4 + b_3 Q^6}, \tag{10} \]

where \( a_1, b_1, b_2, b_3 \) are free parameters, \( p_{01} \) is a floating normalization parameter.

Using this fitter for fitting the data at high-\( Q^2 \) region, we determine \( b_2 = 0.0416 \pm 0.0152 \) and \( b_3 = 0.00474 \pm 0.000892 \). Then fixing these values for fitting the pseudo-data in the (low-\( Q^2 \)) DRad range will render a fitter, which we refer to as fixed Rational (1,3) or

\[\text{6}\] The same mean value and the rms of the distribution can be obtained with any other 10000 sets of \( r_d[\text{fit}] \), with the precision of \( 10^{-4} \) fm.
fRational (1,3):
\[
f_{\text{fixed Rational (1,3)}}(Q^2) \equiv f_{\text{Rational (1,3)}} = 1 + a_1 Q^2 + b_1 Q^2 + b_2, \text{fixed} Q^2 + b_3, \text{fixed} Q^6, \quad (11)
\]
where the uncertainties in the fixed parameters are also taken into account when we calculate the bias.

To compare the differences between Rational (1,1), fRational (1,3) and other fitters shown in Fig. 2, all the functions are plotted in the Abbott1/Abbott2 model range (from \(Q^2 = 3 \times 10^{-2} \text{ (GeV/c)}^2 \) to 1.5 (GeV/c)^2). The parameters in these fitters are determined by fitting pseudo-data generated from the Abbott1 model in the DRad \(Q^2 \) range. The results from the Abbott2, Parker and SOG models are very similar, therefore we do not show them here. As shown in Fig. 3, all the fitters describe the data quite well in the low-\(Q^2 \) range (\(Q^2 < 0.15 \text{ (GeV/c)}^2 \)), while Polynomial Z (4) and CF (3) diverge. At high-\(Q^2 \) range, the fRational (1,3) describes the data much better than the other fitters, which means that the fRational (1,3) has a better asymptotic behavior at high-\(Q^2 \). Based on this observation, the fRational (1,3) may also have a potential to describe the data in the low-\(Q^2 \) range better than the Rational (1,1). Other than the fRational (1,3) functional form, we have also studied another fitter, which has similar properties and is capable of extracting \(r_d \) robustly. The details on our studies for this fitter are presented in Appendix C.

5. A CONSERVATIVE WAY TO ESTIMATE THE BIAS IN DEUTERON CHARGE RADIUS EXTRACTION

5.1. Smearing procedure

After the candidate fitter is found, the robustness for the deuteron radius extraction needs to be tested. Being limited by the number of \(G_C^d \) parameterizations, in order to mimic different extrapolations at low-\(Q^2 \), the parameters in the two Abbott as well as in the Parker and SOG models are smeared. Once they are smeared, the functional forms describing the models are different, and are used to perform a variety of extrapolations at low-\(Q^2 \). Overall, this test is a \(\chi^2 \) test, which consists of the following steps.

A) Smearing of the parameters and calculation of \(\chi^2 \): First, we smear all the parameters for \(\pm 10\% \), following a uniform distribution in Eq. (14) and/or Eq. (15). Then we generate the corresponding \(G_C^d \) with respect to its value at the same \(Q^2 \) bin in the (\(G_S^d \), \(G_C^d \), \(\delta G_C^d \)) data set from the table 1 of Ref. [18]. Afterwards, we calculate \(\chi^2 \) by
\[
\chi^2 = \sum \frac{(G_C^d - G_C^d, \text{fit})^2}{\delta G_C^d}. \quad (12)
\]

B) Checking of the acceptable region: The definition of an acceptable \(\chi^2 \) region is that the probability of the calculated \(\chi^2 \) (after the parameters are smeared) with a specific degree of freedom is “acceptable” when it is larger than 99.7% in the \(\chi^2 \) probability distribution. This requirement restricts the value of \(\chi^2 \), which means that the smeared model should not be far away from the real experimental data. With the specific degree of freedom \(\nu \), the \(\chi^2 \) probability distribution is defined as
\[
f(\chi^2) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-\chi^2/2} (\chi^2)^{(\nu/2)-1}. \quad (13)
\]
Integrating the function in Eq. (13), from 0 to \(\chi^2_0 \), will result in the probability for \(\chi^2_0 \). The number of degrees of freedom.
freedom (NDF) and the critical $\chi^2_3$ value for each of the four smeared data-based models are shown in Table I.

| Model      | NDF | $\chi^2_3$ |
|------------|-----|------------|
| Abbott1    | 16  | 35.9       |
| Abbott2    | 7   | 21.6       |
| Parker     | 16  | 35.9       |
| SOG        | 11  | 28.2       |

TABLE I. The number of degrees of freedom and the critical $\chi^2_3$ value for each of the four smeared data-based models.

If the calculated $\chi^2$ is smaller than the above numbers for each smeared model, then we keep the given smeared model and go to the next step. For each smeared model there is a new $r_d[\text{input}]$, which is calculated by Eq. (1) with the slope of a smeared model at $Q^2 = 0$. If $\chi^2$ is unacceptable, the parameters of the model are re-smeared and the whole procedure is repeated.

C) Generating pseudo-data: If the smeared models pass the step B), in this case these models can be utilized to generate sets of pseudo-data in the DRad $Q^2$ range using the binning discussed in Sec. 4.

D) Fitting and calculating the bias: After the pseudo-data is generated, we use the selected fitter to fit and obtain the quantity $r_d[\text{fit}]$.

E) Repeating and obtaining the relative bias: In this step the above procedure for each model is repeated 10000 times for obtaining 10000 values of relative bias, which is defined as $\delta r_d/r_d[\text{input}]$.

F) Finalization: To analyze the results, we combine the relative bias values obtained from the four smeared Abbott1, Abbott2, Parker and SOG models independently. From the relative bias distribution, we select the rms value to calculate $\delta r_d$ in Eq. (6).

5.2. Proof of the robustness test using the proton form factor models

The parameter smearing approach for deuteron form factor models helps us better calculate the bias, by imitating a variety of extrapolations to low-$Q^2$, when the number of models is limited. In order to verify that this approach is valid and applicable, several proton electric form factor $G_E^p$ models can be tested in turn. Namely, we consider such parameterization models, including Kelly [10], Arrington1 [11], Arrington2 [12], Arrington-Sick [13], Ye [14], Alarcon and Bernauer-2014. The Alarcon model is our refit based upon [15-17], and the Bernauer-2014 model is our refit of data from [18]. By smearing the parameters in the proton $G_E^p$ models, we determine whether or not the smearing method can mimic the low-$Q^2$ extrapolation behavior of those models.

Following the same steps shown in the previous section, the bias values obtained from fitting the Rational (1,1) with pseudo-data generated by the $G_E^p$ models, before and after smearing, have been found and are displayed in Table II. The non-smeared bias in the table is the relative bias obtained by fitting pseudo-data generated from the smeared models following the procedure in the previous section. We take the non-smeared largest relative bias (0.476% from the Ye model) as an upper bound of the low-$Q^2$ extrapolation behavior. As shown in the table, the smeared bias is larger than the non-smeared bias for each model. Besides, only the smeared bias from the Alarcon and Bernauer-2014 models exceed 0.476%, which means that only these two models can mimic the upper bound when using the smearing method.

| Model       | non-smeared bias | smeared bias  |
|-------------|------------------|---------------|
| Kelly       | 0.002 %          | 0.007 %       |
| Arrington1  | 0.005 %          | 0.028 %       |
| Arrington2  | 0.009 %          | 0.019 %       |
| Arrington-Sick | 0.001 %       | 0.012 %       |
| Alarcon     | 0.166 %          | 1.091 %       |
| Ye          | 0.476 %          | 0.621 %       |
| Bernauer-2014 | 0.271 %      | 0.515 %       |

TABLE II. The relative bias obtained from fitting the Rational (1,1) with pseudo-data generated by non-smeared and smeared seven proton $G_E^p$ models.

In Fig. 4, we show a band of each model by smearing all the parameters (again in each model) for $\pm 10\%$, and restricting the values of $\chi^2$ with respect to their degrees of freedom based on available data. One can also see that all the models and the superimposed PRad data are covered by most of the bands except for the band from the Arrington-Sick model, which means that the smearing method generates the pseudo-data in a reasonable range.

By looking at both Table II and Fig. 4, we conclude that although the smearing method used with limited models can not precisely mimic other models, it can exhibit more conservatively how a fitter can control the bias.

CONCLUSIONS AND OUTLOOK

In this section we summarize and conclude on our findings exhibited in the paper (including both appendices). Besides, we briefly discuss the prospects that this work may have in the future.

Fig. 5 shows the rms values of the bias for the given five fitters, derived from fitting pseudo-data generated by the four smeared Abbott1, Abbott2, Parker and SOG models (see Sec. 5.1), along with the bin-by-bin total uncertainties (see Sec. 3.1). According to the definition of the robustness discussed in Sec. 3.2, the five fitters are all robust (bias[rms] < $\sigma_{\text{stat}}$). Although the Rational (1,1) and
fRational (1,3) have larger bias values compared to those of the other three fitters; they can control the RMSE better because their variance are smaller than those of the others.
By comparing the bias and variance ($\sigma_{\text{total}}$) in that figure, our understanding is that the RMSE (overall uncertainty) in the DRad experiment will be dominated by the bin-by-bin uncertainties rather than by the bias obtained in the fitting procedure. Based on our results, we propose to use the $f_{\text{Rational}}(1,3)$ as the primary fitter in the deuteron charge radius extraction for this planned experiment, noting that it also has a better asymptotic behavior compared to that of Rational (1,1). Nonetheless, the $f_{\text{Rational}}(1,3)$ is determined based on the data-driven method. Since it only has constraints from deuteron charge form factor data at high-$Q^2$, its extrapolation may not be very accurate, when it is used for fitting generated pseudo-data in a lower $Q^2$ range. Once we have more data at low-$Q^2$, one can better determine the fixed parameters in this fitter, in which case we will be able to extract the $r_d$ value more precisely. This might be done, for example, with possible upcoming new data from the A1 Collaboration at Mainz Microtron (MAMI). On the other hand, if we consider the results shown in Fig. 3, Fig. 5 and Fig. 9 together, in this case we find that (i) the $R_{\text{Rational}}(1,3)$ and (ii) the $R_{\text{modRational}}(1,1)$ are currently our best fitters for the robust extraction of $r_d$. In addition, we note that the above-mentioned conclusions are anchored upon our studies for the DRad experiment. One should first account for the trade-off between the bias and variance, then select the best fitter stemming from the latest estimation of experimental uncertainties. If it turns out that the bin-by-bin uncertainties during the DRad experiment are much smaller (at least ten times) than what we have already evaluated, in this case we may search for another potentially robust fitter, which can minimize the bias and simultaneously will also have good asymptotics.

The radius extraction methods discussed so far depend on specific functional forms. In Ref. [19], different extraction of the charge radius of the proton is discussed. The so-called cubic spline method is used to interpolate form factor data, by which a smooth function is obtained afterwards. Then the radius could be extracted with an extrapolation using that smooth function. This method may also be applicable by us for the robust extraction of the deuteron charge radius in the near future, as an independent way for cross checking our results coming from the ansatz provided in this paper.

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**APPENDIX**

Appendix A. Details on the Abbott1, Abbott2, Parker and SOG models

In this appendix we concisely discuss the parameterizations describing the Abbott1, Abbott2, Parker and SOG models.

Parameterization I (Abbott1 model) [15]: In the first parameterization, each charge form factor is represented by

$$G_{C,0}^d(Q^2) = G_{C,0} \times \left[ 1 + \frac{Q}{Q_{C,0}^0} \right]^{-\frac{1}{2}},$$

where $G_{C,0}$ is a normalizing factor fixed by the deuteron charge, $Q_{C,0}^0$ and $a_{C_i}$ are all together six free parameters that can be found on the website from [15].

Parameterization II (Abbott2 model) [15, 20]: The second parameterization is given by

$$G_{C}^d(Q^2) = \frac{G^2(Q^2)}{(2\tau + 1)} \left[ \left( 1 - \frac{2}{3} \tau \right) g_{00}^+ + \frac{8}{3}\sqrt{2\tau} g_{10}^+ + \right.$$  

$$+ \frac{2}{3}(2\tau - 1) g_{0}^- \right],$$

$$G_{C,0}^d(Q^2) = \frac{G_{C,0}^d}{\left( 2\tau + 1 \right)} \left[ \left( 1 - \frac{2}{3} \tau \right) g_{00}^+ + \frac{8}{3}\sqrt{2\tau} g_{10}^+ + \right.$$  

$$+ \frac{2}{3}(2\tau - 1) g_{0}^- \right],$$

FIG. 5. (Color online) This figure shows the rms values of the bias for the shown fitters, derived from fitting pseudo-data generated by the four smeared Abbott1, Abbott2, Parker and SOG models (Sec. 5.1). The error bars reflect the effects of the bin-by-bin total uncertainties of $G_{C}^d$ (Sec. 3.1).
where
\[ g_{00}^+ = \sum_{i=1}^{n} \frac{a_i}{\alpha_i^2 + Q^2}, \quad g_{+0}^+ = Q \sum_{i=1}^{n} \frac{b_i}{\beta_i^2 + Q^2}, \]
\[ g_{++}^+ = Q^2 \sum_{i=1}^{n} \frac{c_i}{\gamma_i^2 + Q^2}. \]  
(16)

\[ G(Q^2) \] in Eq. (15) is a dipole form factor given by
\[ G(Q^2) = \left( 1 + \frac{Q^2}{\delta^2} \right)^{-2}, \]  
(17)

where \( \delta \) is a parameter of the order of the nucleon mass.

The twenty four parameters \( a_i, b_i, c_i, \alpha_i^2, \beta_i^2, \gamma_i^2 \) can also be found on the website of [18]. They are constrained by the following twelve relations:

\[ \sum_{i=1}^{n} \frac{a_i}{\alpha_i^2} = 1, \quad \sum_{i=1}^{n} b_i = 0, \quad \sum_{i=1}^{n} \frac{b_i}{\beta_i^2} = \frac{2 - \mu_M^d}{2\sqrt{2}M_d}, \]
\[ \sum_{i=1}^{n} c_i = 0, \quad \sum_{i=1}^{n} \frac{c_i}{\gamma_i^2} = 0, \quad \sum_{i=1}^{n} \frac{c_i}{\gamma_i^2} = \frac{1 - \mu_M^d - \mu_Q^d}{4M_d^2}, \]
\[ \alpha_i^2 = 2M_d \mu^{(\alpha)}, \quad \beta_i^2 = \alpha_i^2 + \frac{\alpha_i^2 - \alpha_i^2}{n-1}(i-1), \]  
for \( i = 1, \ldots, n, \) \( \)  
(18)

where the parameter \( \mu^{(\alpha)} \) has the dimension of energy and is of the order of \( A \sqrt{\rho} \sim 0.2 \) MeV. In total, there are twelve free parameters in this model.

Parameterization III (Parker model) [21]: The third parameterization is essentially based on the re-made fits from the first two parameterizations, however, with constraints to prevent singularities in the functional forms of the \( G_{C}^{d}, G_{Q}^{d} \) and \( G_{M}^{d} \) form factors:

\[ G_{C}^{d}(Q^2) = G_{C,0} \times \left[ 1 - \left( \frac{Q}{Q_{C,0}} \right)^2 \right] \times \left[ \prod_{i=1}^{5} (1 + |a_i| Q^2) \right]^{-1}, \]  
(19)

where the values of \( G_{C,0} \) and \( Q_{C,0}^0 \) are the same as the ones shown in Eq. (14). \( a_i \) are all together five free parameters determined from fitting the data from the website of [18].

Parameterization IV (Sum-of-Gaussians (SOG) model) [18, 22, 23]: The fourth parameterization utilizes the SOG method, by which \( G_{C}^{d}(Q^2) \) reads as

\[ G_{C}^{d}(Q^2) = G_{C,0} \times e^{-\frac{1}{2}Q^2\gamma^2} \times \sum_{i=1}^{N} \frac{A_i}{1 + 2R_i^2 / \gamma^2}, \]
\[ \left[ \cos(QR_i) + \frac{2R_i^2 \sin(QR_i)}{\gamma^2} \right]. \]  
(20)

In the configuration space this parameterization corresponds to a density \( \rho(s) \) given in terms of a sum of Gaussians located at arbitrary radii \( R_i \), with amplitudes \( A_i \) fitted to the data and with a fixed width \( \gamma \), where \( \gamma \sqrt{3/2} = 0.8 \) fm.

In our fitting we take \( N = 12 \). There are eleven free fitting parameters: ten Gaussian amplitudes \( A_1, A_2, \ldots, A_{10} \) that correspond to ten \( R_i < 4 \) fm, and one overall amplitude \( A_{11} \) corresponding to the range of \( R_{11} \) from 4 fm to 10 fm. For obtaining the normalization, there is one more amplitude \( A_{12} \) with \( R_{12} = 0.4 \) fm. All the amplitudes satisfy the condition \( \sum_{i=1}^{12} A_i = 1 \). To determine the parameters \( A_i \), a set of \( R_i \) is randomly generated in the range mentioned above, then the function in Eq. (20) is fitted to the \( G_{C}^{d} \), data set from the table 1 of Ref. [18]. The sets of \( R_i \) are generated repeatedly until the \( \chi^2 \) value is minimized and converged. With eleven fixed \( R_i \) and eleven free parameters \( A_i \), a fit to the data set is obtained with \( \chi^2/NDF = 1.63 \) [23].

Appendix B. A fitter test based upon using theory-based models

Except for the four data-based deuteron charge form factor models under consideration, we also test some theory-based models by following the same method developed in Ref. [24]. Besides, the data generation and fitting procedure is already described in Sec. 3.1 with statistical fluctuations included. Here we use the following \( G_{C}^{d} \) models as additional generators:

i) The IA (relativistic impulse approximation), IAMEC (relativistic IA plus meson exchange current), RSC (relativistic IA with Reid Soft Core), RSCMEC (relativistic IA plus meson exchange current with Reid Soft Core) are the parameterizations to the theoretical calculations discussed in [25].

ii) The quadratic and cubic models are the second and third order polynomial fits to theoretical points calculated by using the model in [26], where the parameters in this model are given in the table 12 of [26].

iii) The Gaussian, Monopole and Dipole are naive models that imitate possible extrapolations at \( Q^2 \to 0 \). Their functional forms can be found in [24].

Fig. 6 shows the statistical variance, bias and their quadratic sum (RMSE) from fitting various fitters with pseudo-data (including statistical fluctuations) generated by eleven different deuteron models. One can see that the bias is much larger than the statistical variance.

7 The density \( \rho \) is a function of the distance \( s \), which is the distance of the nucleons to the deuteron center-of-mass.
We can use the method to test the robustness in the charge radius extraction of the proton from [24] is not suitable for the deuteron’s case in the DRad kinematics. When we investigate the properties of the fitters for DRad, we would need to have more data to decide whether we should take any of these existing theory-based models into consideration.

Appendix C. Searching for other robust fitter candidates

The modified Rational (1,1) function: Except for the fRational (1,3) fitter function discussed in the paper, for the deuteron charge radius extraction we have also studied a modified and generalized version of Rational (1,1), which we designate as modified Rational (1,1) (or simply as modRational (1,1)):

\[
f_{\text{modRational}}(1,1)(Q^2) \equiv \text{modRational}(1,1) = p_{02} \left( \frac{1 + p_1^{(a)} Q^2}{1 + p_1^{(b)} Q^2} \right)^A.
\]

where \(p_{02}\) is a floating normalization parameter, \(p_1^{(a)}\) and \(p_1^{(b)}\) are two free fitting parameters. To control the variance, we need to limit the number of the free parameters. The deuteron rms charge radius is calculated by \(r_d = \sqrt{\frac{6}{3} \left( B \times p_1^{(b)} - A \times p_1^{(a)} \right)} \). It is obvious that Eq. (21) reduces to Eq. (7) at \(A = 1\) and \(B = 1\). Both powers \(A\) and \(B\) can be fixed and given by different methods that we discuss below.

To search for the best combination of \(A\) and \(B\) in Eq. (21), for the purpose of extracting \(r_d\) robustly within the scope of the data-based models, we use a scanning approach. In this approach, \(A\) and \(B\) (each) are varied from 0 to 10 with the step equal to 0.1, and the fitter is fitted with the pseudo-data generated by the two models in the DRad kinematics range of \(2 \times 10^{-4}\) (GeV/c)c < \(Q^2 < 0.05\) (GeV/c)c. Using the scanning approach, we obtain \(A = 3.0 - 4.2\) and \(B = 0.8\) to be the best \(A\) range and \(B\) value to minimize the bias.

Nonetheless, the outlined scanning method is model-dependent, which is limited by the number of the given

| Models       | \(\chi^2\) |
|--------------|------------|
| IA           | 183.09     |
| IAMEC        | 295.57     |
| RSC          | 125.74     |
| RSCMEC       | 196.71     |
| Abbott1      | 19.77      |
| Abbott2      | 38.59      |
| Parker       | 22.94      |
| SOG          | 17.88      |
| Gaussian     | 668.59     |
| Monopole     | 1.14 \times 10^4 |
| Dipole       | 5.83 \times 10^2 |
| Quadratic    | 2.65 \times 10^{12} |
| Cubic        | 1.15 \times 10^{15} |

TABLE III. The \(\chi^2\) value, obtained from Eq. (12), for each of the eleven models using the available data points from the table 1 of Ref. [18].
charge form factor models. In that case the fewer the reliable models are, the higher the model-dependency is. In order to avoid this issue, we have also tried a data-driven method described in Sec. 4.2. For the \text{modRational}(1,1), when \(A\) and \(B\) are also considered as free parameters, there are totally four free parameters. We use this fitter for fitting the form factor data at high-\(Q^2\) region listed in the table 1 of [18] – which gives \(A = 3.48668 \pm 0.01568\) and \(B = 0.75600 \pm 0.11313\) – then fix these values for fitting the pseudo-data in the low-\(Q^2\) range of the DRad kinematics. In this case we will have the fixed modified Rational (1,1) (or simply as the f\text{modRational}(1,1)):

\[
f_{\text{fixed modified Rational}(1,1)}(Q^2) = f_{\text{modRational}(1,1)} = \frac{p_{02} (1 + p_{1}^{(a')} Q^2)^{A,\text{fixed}}}{(1 + p_{1}^{(b')} Q^2)^B,\text{fixed}},
\]

where the uncertainties in the fixed parameters are taken also into account when we calculate the bias.

To compare the differences between the Rational (1,1), f\text{Rational}(1,3), \text{modRational}(1,1) and f\text{modRational}(1,1) in the Abbott1/Abbott2 model range (from \(3 \times 10^{-2}\) (GeV/c)^2 to 1.5 (GeV/c)^2), all the functions are plotted in this range. As an example, we pick up fixed values \(A = 3.4\) and \(B = 0.8\) in the \text{modRational}(1,1) in Eq. (21). The parameters in these different fitters are determined by fitting pseudo-data generated from the Abbott1 model in the DRad range. The results from the Abbott2, Parker and SOG models are quite similar, and are not shown here. As shown in Fig. 7 except for the Rational (1,1), the other fitters show good asymptotic behavior in the high-\(Q^2\) range.

**Similarity of the fitters \text{modRational}(1,1) and Rational (1,3):** The \text{modRational}(1,1) fitter kind of lacks a clear physical meaning. Meanwhile, one can show that the functional form of the \text{modRational}(1,1) is actually similar to the Rational (1,3). It could be demonstrated if we started with the fitted \text{modRational}(1,1) for generating a set of \(G_C^d\) pseudo-data, and then used the Rational (1,3) for fitting this set of generated pseudo-data. Thereby,

(i) the \text{modRational}(1,1) with \(A = 3.4\) and \(B = 0.8\) from Eq. (21) is used to fit pseudo-data generated by the Abbott1 model [18]. The fitted function comes out to be the following:

\[
\text{modRational}(1,1) = \frac{(1 - 0.0456785 Q^2)^{3.4}}{(1 + 0.718695 Q^2)^{0.8}},
\]

where the dimension of \(Q^2\) is in fm^-2;

(ii) to generate a set of \(G_C^d\) pseudo-data with reasonable bins and uncertainties, we choose both the DRad binning with its simulated statistical uncertainty and the Abbott binning from the table 1 of Ref. [18]. In total, there are eighty-two pseudo-data points that are generated by the function in Eq. (23), in the range of \(Q^2 = 0.006 - 21.344\) fm^-2;

(iii) the Rational (1,3) function as shown in Eq. (10) is used to fit those pseudo-data.

By fulfilling the above steps, we present the result in Fig. 8 where the black points are the pseudo-data points generated from Eq. (23), and the red curve is a fitted Rational (1,3). This figure shows that the mod-Rational (1,1) has a very similar behavior as the Rational (1,3) in the range of \(Q^2 < 21.5\) fm^-2 or equivalently of \(Q^2 < 0.84\) (GeV/c)^2.

At the end we wish to briefly discuss Fig. 9 which
shows the rms values of the bias for the given four fitters, derived from fitting pseudo-data generated by the four smeared Abbott1, Abbott2, Parker and SOG models (see Sec. 5.1), along with the bin-by-bin total uncertainties (see Sec. 3.1). For the modRational(1,1), the results of various tested combinations with $A = 3.0 - 4.2$ and $B = 0.8$ are stable inside the $A$ range. As shown in this figure, all the fitters are robust (bias [rms] $< \sigma_{\text{stat}}$) based on the definition in Sec. 3.2. The RMSE (overall uncertainty) values from the Rational (1,1), fRational (1,3) and modRational (1,1) are similar, which means that the modRational (1,1) can also be considered as a good fitter candidate for the deuteron charge radius extraction. However, taking also into account the $G_2^d$ behavior that we observe in Fig. 3 and Fig. 7, we consider both fRational (1,3) and modRational (1,1) as the best two fitters for the robust extraction of $r_d$ in the DRad experiment because the Rational (1,1) drops out in this combined picture.

FIG. 8. (Color online) The Rational (1,3) (red curve) fitted with the pseudo-data generated by Eq. (23) (black points).

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FIG. 9. (Color online) This figure shows the rms values of the bias for the shown fitters, derived from fitting pseudo-data generated by the four smeared Abbott1, Abbott2, Parker and SOG models (Sec. 5.1). The error bars reflect the effects of the bin-by-bin total uncertainties of $G_C^d$ (Sec. 3.1).

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