A simple criterion for non-relative hyperbolicity and one-endedness of groups

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Abstract

We give a combinatorial criterion that implies both the non-strong relative hyperbolicity and the one-endedness of a finitely generated group. We use this to show that many important classes of groups do not admit a strong relatively hyperbolic group structure and have one end. Applications include surface mapping class groups, the Torelli group, (special) automorphism and outer automorphism groups of most free groups, and the three-dimensional Heisenberg group. Our final application is to Thompson’s group $F$.

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1 Introduction

In recent years, the notion of relative hyperbolicity has become a powerful method for establishing analytic and geometric properties of groups, see for example Dadarlat and Guentner [10], Dahmani [11], Osin [26], [27], Ozawa [28], Yaman [31]. Relatively hyperbolic groups, first introduced by Gromov [18] and then elaborated on by various authors (see Farb [15], Szczepański [30], Bowditch [5]), provide a natural generalization of hyperbolic groups and geometrically finite Kleinian groups.

When a finitely generated group $G$ is strongly hyperbolic relative to a finite collection $L_1, L_2, \ldots, L_p$ of proper subgroups, it is often possible to deduce that $G$ has a given property provided the subgroups $L_j$ have the same property. Examples of such properties include finite asymptotic dimension, see Osin [26], exactness, see Ozawa [28], and uniform embeddability in Hilbert space, see Dadarlat and Guentner [10]. In light of this, identifying a strong relatively hyperbolic group structure for a given group $G$, or indeed deciding whether or not one can exist, becomes an important objective.

One of the main results of this note, Theorem 2 in Section 3 asserts that such a structure cannot exist whenever the group $G$ satisfies a simple combinatorial property, namely that its *commutativity graph* with respect to some generating set $S$ is connected. We describe this graph in Section 3.

The other main result of this note, Theorem 9 in Section 4 asserts that whenever $G$ has connected commutativity graph with respect to some set of generators, it has one end.

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Using our main results, we deduce that many well-known groups do not admit a strong relatively hyperbolic group structure and have one end. These examples include all but finitely many surface mapping class groups, the Torelli group of a closed surface of genus at least 3, the (special) automorphism and outer automorphism groups of almost all free groups and the three-dimensional Heisenberg group. We remark that the one-endedness of the surface mapping class groups and (special) automorphism and outer automorphism groups of free groups considered herein was previously established by Culler and Vogtmann [9] using Bass-Serre theory. In Section 4.5, we prove that Thompson’s group $F$ has one end and is not strongly relatively hyperbolic using a minor variation of our main argument.

During the preparation of this work we were informed that Behrstock, Drutu, and Mosher [3] have found an alternative argument for the non-strong relative hyperbolicity of some of the groups treated here, using asymptotic cones and the description of relative hyperbolicity due to Drutu and Sapir [13].

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2 Relatively hyperbolic groups

We assume that all groups appearing in this note are infinite, unless otherwise explicitly stated.

There are two related but inequivalent definitions of relative hyperbolicity that are commonly used, one due to Farb [15], and the other developed by Gromov [18], Szczepański [30], and Bowditch [5]. As we only go into as much detail as is required for us to state our results, we refer the interested reader to the cited papers for a more extensive treatment.

We first give the definition given by Farb [15] and refer to this as weak relative hyperbolicity. For a group $G$, a finite generating set $S$ and a finite family of proper finitely generated subgroups $\{L_1, L_2, \ldots, L_m\}$, we form an augmentation Cayley graph $\text{Cay}^*(G, S)$ of the Cayley graph $\text{Cay}(G, S)$ as follows: Give $\text{Cay}(G, S)$ the path-metric obtained by declaring each edge to have length one. Then, for each $1 \leq j \leq m$, adjoin to $\text{Cay}(G, S)$ a new vertex $v_{gL_j}$ for each coset $gL_j$ of $L_j$ and declare the distance between each new vertex $v_{gL_j}$ and each vertex in the associated coset $gL_j$ to be one. We say that $G$ is weakly hyperbolic relative to $L_1, L_2, \ldots, L_m$ if the resulting metric on $\text{Cay}^*(G, S)$ is hyperbolic in the sense of Gromov. Farb [15] shows this definition does not depend on the choice of generating set.

In the same paper, Farb introduces the notion of bounded coset penetration (BCP), which is a weak local finiteness property satisfied by many important examples of weakly relatively hyperbolic groups. Roughly speaking, BCP imposes certain fellow-travelling conditions on pairs of quasi-geodesics on $\text{Cay}(G, S)$ with the same endpoints that enter cosets of the subgroups $L_j$, $1 \leq j \leq m$.

Bowditch [5] gives two equivalent dynamical notions of relative hyperbolicity, of which we recall the second. We will refer to this notion as strong relative hyperbolicity. We say that a group $G$ is strongly hyperbolic relative to the family $L_1, L_2, \ldots, L_m$ of proper finitely generated subgroups if $G$
admits an action on a connected, hyperbolic graph \( G \) such that \( G \) is fine (that is, for each \( n \in \mathbb{N} \), each edge of \( G \) belongs to only finitely many circuits of length \( n \)), there are only finitely many \( G \)-orbits of edges, each edge stabiliser is finite, and the stabilisers of vertices of infinite valence are precisely the conjugates of the \( L_j \).

We note that strong relative hyperbolicity (with respect to some finite collection of proper finitely generated subgroups) is equivalent to weak relative hyperbolicity plus BCP (with respect to the same collection of subgroups), see Szczepański [30] and Dahmani [11]. The BCP property is crucial: as noted in Szczepański [30], the group \( \mathbb{Z} \oplus \mathbb{Z} \) is weakly, but not strongly, hyperbolic relative to the diagonal subgroup \( \{ (m, m) \mid m \in \mathbb{Z} \} \).

We mention here that a characterisation of strong relative hyperbolicity in terms of relative presentations and isoperimetric inequalities is given in Osin [27].

3 The commutativity graph

We begin by defining a graph which attempts to capture the notion that a group is well generated by large abelian subgroups.

**Definition 1 (Commutativity graph)** Let \( G \) be a group and let \( S \) be a (possibly infinite) generating set for \( G \), all of whose elements have infinite order. The commutativity graph \( K(G, S) \) for \( G \) with respect to \( S \) is the simplicial graph whose vertex set is \( S \) and in which distinct vertices \( s, s' \) are connected by an edge if and only if there are non-zero integers \( n_s, n_{s'} \) so that \( \langle s^{n_s}, (s')^{n_{s'}} \rangle \) is abelian.

As long as there is no danger of confusion, we will use the same notation for elements of \( S \) and vertices of \( K(G, S) \).

Notice that for any \( g \in G \), we have that \( K(G, S) \) is connected if and only if \( K(G, gSg^{-1}) \) is connected. Typically we shall only consider commutativity graphs in which adjacent vertices, rather than powers of adjacent vertices, commute.

Our main result about non-strong relative hyperbolicity may be stated as follows. Recall that the rank of a finitely generated abelian group \( A \) is the rank of some (and hence every) free abelian subgroup \( A_0 \) of finite index in \( A \).

**Theorem 2** Let \( G \) be a finitely generated group. Suppose there exists a (possibly infinite) generating set \( S \) of cardinality at least two such that every element of \( S \) has infinite order and \( K(G, S) \) is connected. Suppose further that there exist adjacent vertices \( s, s' \) of \( K(G, S) \) and non-zero integers \( n_s, n_{s'} \) so that \( \langle s^{n_s}, (s')^{n_{s'}} \rangle \) is rank 2 abelian. Then, \( G \) is not strongly hyperbolic relative to any finite collection of proper finitely generated subgroups.

The remainder of this section is dedicated to the proof of Theorem 2. The main tool we use is the following theorem on virtual malnormality for strongly relatively hyperbolic groups, which is contained in the work of Farb [15] and Bowditch [5], and is explicitly stated in Osin [27] (Theorems 1.4 and 1.5).
**Theorem 3** Let $G$ be a finitely generated group that is strongly hyperbolic relative to the proper finitely generated subgroups $L_1, \ldots, L_p$. Then,

1. For any $g_1, g_2 \in G$, the intersection $g_1 L_j g_1^{-1} \cap g_2 L_k g_2^{-1}$ is finite for $1 \leq j \neq k \leq p$.

2. For $1 \leq j \leq m$, the intersection $L_j \cap g L_j g^{-1}$ is finite for any $g \not\in L_j$.

Note that this immediately implies that if $g \in G$ has infinite order and if $g^k$ lies in a conjugate $hL_j h^{-1}$ of some $L_j$, then $g$ lies in that same conjugate $hL_j h^{-1}$ of $L_j$, since the intersection $hL_j h^{-1} \cap ghL_j h^{-1} g^{-1}$ then contains $(g^k)$, which is infinite.

We also need the following lemma, which follows directly from Theorems 4.16 and 4.19 of Osin [24]. We note that while this lemma is implicit in the literature, it has never (to the best of our knowledge) been stated in this form, and so we include it as it may be of independent interest.

**Lemma 4** Let $G$ be a finitely generated group that is strongly hyperbolic relative to the proper finitely generated subgroups $L_1, L_2, \ldots, L_p$. If $A$ is an abelian subgroup of $G$ of rank at least two, then $A$ is contained in a conjugate of one of the $L_j$.

We are now ready to prove Theorem 2.

**Proof** [of Theorem 2] Suppose for contradiction that $G$ is strongly hyperbolic relative to the finite collection $L_1, L_2, \ldots, L_p$ of proper finitely generated subgroups. We first show that no conjugate of any $L_j$ can contain a non-zero power of an element of $S$. So, suppose there is some $g \in G$, some $s_0 \in S$, and some $k \neq 0$ so that $s_0^k \in g L_j g^{-1}$ for some $1 \leq j \leq p$. (Note that, by the comment immediately following Theorem 3 this implies that $s_0 \in g L_j g^{-1}$ as well.)

Let $s_1$ be any vertex of $K(G, S)$ adjacent to $s_0$. As there are non-zero integers $n_0, n_1$ so that $\langle s_0^{n_0}, s_1^{n_1} \rangle$ is abelian, we see that $\langle s_0^{n_0} s_1^{n_1} \rangle \subseteq g L_j g^{-1} \cap s_1^{n_1} g L_j g^{-1} s_1^{-n_1}$. However, since the subgroup $\langle s_0^{n_0} \rangle$ of $G$ is infinite, Theorem 3 implies that $s_1^{n_1} \in g L_j g^{-1}$. By the comment immediately following Theorem 3 we see that $s_1 \in g L_j g^{-1}$.

Now, let $s$ be any element of $S$. By the connectivity of $K(G, S)$, there is a sequence of elements $s_0, s_1, \ldots, s_n = s$ of $S$ such that $s_{k-1}$ and $s_k$ are adjacent in $K(G, S)$ for each $1 \leq k \leq n$. The argument we have given above implies that if $s_{k-1} \in g L_j g^{-1}$, then $s_k \in g L_j g^{-1}$. In particular, we have that $s \in g L_j g^{-1}$. Since $G$ is generated by $S$, it follows that $G$ and $g L_j g^{-1}$ are equal, contradicting the assumption that the subgroup $L_j$ is proper. We conclude that if $G$ is strongly hyperbolic relative to $L_1, L_2, \ldots, L_p$ then no conjugate of any $L_j$ can contain a non-zero power of an element of $S$.

Now, by assumption there exist adjacent vertices $t$ and $t'$ of $K(G, S)$ for which there exist non-zero integers $n_t, n_{t'}$ so that $A = \langle t^{n_t}, (t')^{n_{t'}} \rangle$ is rank 2 abelian. By Lemma 4 we see that $A$ must lie in some conjugate of some $L_j$. In particular, this conjugate of $L_j$ contains a non-zero power of an element of $S$, and we have a final contradiction. \[QED\]
4 Ends of groups

In this section, we use the same argument as that used in the proof of Theorem 2 to show that groups with connected commutativity graph have one end. We refer to Lyndon and Schupp [23] for standard facts about amalgamated free products and HNN extensions.

We do not give here a complete and formal definition of ends of groups; for this, the interested reader is referred to Epstein [14], Stallings [29], and Dicks and Dunwoody [12]. Let $G$ be a finitely generated group (which we have assumed to be infinite). Say that $G$ has one end if for some (and hence for every) finite generating set $S$, the Cayley graph Cay($G, S$) has the property that the complement \( \text{Cay}(G, S) \setminus B_n(e) \) is connected, where $B_n(e)$ is the ball of radius $n$ about the identity $e$. Say that $G$ has two ends if $G$ is virtually $\mathbb{Z}$.

Say that a finitely generated group $G$ has infinitely many ends if and only if $G$ admits either a non-trivial amalgamated free product decomposition $G = A \ast_C B$ or a non-trivial HNN decomposition $G = A \ast_C$, where (in either case) $C$ is finite. Here, by a trivial amalgamated free product decomposition, we mean a decomposition of the form $G = A \ast_C B$ where $C$ is finite and $[A : C] = [B : C] = 2$, and by a trivial HNN extension we mean an HNN extension of the form $G = C \ast_C$ where $C$ is finite. Note that, in these two cases, $C$ is a finite normal subgroup of $G$ and $G/C$ is infinite cyclic, and it follows $G$ has two ends.

For an infinite group, these are the only possibilities for the number of ends and these possibilities are mutually exclusive. It is known that a finitely generated group $G$ and a finite index subgroup $H$ of $G$ have the same number of ends, since the number of ends is a quasi-isometry invariant.

We now state standard facts about amalgamated free products and HNN extensions that correspond to Theorem 3 and Lemma 4. We begin with the virtual malnormality results, corresponding to Theorem 3. These results easily follow from the existence of normal forms for amalgamated free products and HNN extensions, see Chapter IV of Lyndon and Schupp [23].

Lemma 5 Let $G$ be a finitely generated group that admits a non-trivial amalgamated free product decomposition $G = A \ast_C B$, where $C$ is finite.

1. If $g, h \in G$, then $gAg^{-1} \cap hBh^{-1}$ is finite;
2. If $g \in G \setminus A$, then $A \cap gAg^{-1}$ is trivial (and if $h \in G \setminus B$, then $B \cap hBh^{-1}$ is trivial).

As a consequence, if $g \in G$ has infinite order and if $g^n \in A$ for some $n \geq 2$, then $g \in A$ (and similarly for $h \in B$).

Lemma 6 Let $G$ be a finitely generated group that admits a non-trivial HNN decomposition $G = A \ast_C$, where $C$ is finite. If $g \in G \setminus A$, then $A \cap gAg^{-1}$ is trivial.

As a consequence, if $g \in G$ has infinite order and if $g^n \in A$ for some $n \geq 2$, then $g \in A$.

We now state the lemmas corresponding to Lemma 4. The first follows from H. Neumann’s generalization of the Kurosh theorem for amalgamated free products, see Lyndon and Schupp [23], Chapter I, Proposition 11.22. The second follows from Britton’s Lemma given in Lyndon and Schupp [23], Chapter IV, Section 2.
Lemma 7 Let \( G \) be a finitely generated group that admits a non-trivial amalgamated free product decomposition \( G = A *_C B \), where \( C \) is finite. If \( K \) is an abelian subgroup of \( G \) of rank at least two, then \( K \) is contained in a conjugate of either \( A \) or \( B \).

Lemma 8 Let \( G \) be a finitely generated group that admits a non-trivial HNN decomposition \( G = A*_{C} \), where \( C \) is finite. If \( K \) is an abelian subgroup of \( G \) of rank at least two, then \( K \) is contained in a conjugate of \( A \).

We are now able to state and prove the analogue of Theorem 2 for the one-endedness of groups with connected commutativity graph. This proof follows very much the same line of argument as the proof of Theorem 2.

Theorem 9 Let \( G \) be a finitely generated group which is not virtually \( \mathbb{Z} \). Suppose there exists a (possibly infinite) generating set \( S \) of cardinality at least two such that every element of \( S \) has infinite order and \( K(G, S) \) is connected. Suppose further that there exist adjacent vertices \( s, s' \) of \( K(G, S) \) and non-zero integers \( n, n' \) so that \( \langle s^n, (s')^{n'} \rangle \) is rank 2 abelian. Then, \( G \) has one end.

Proof [of Theorem 9] Since \( G \) is assumed not to be virtually \( \mathbb{Z} \), either \( G \) has one end or \( G \) has infinitely many ends. Suppose for contradiction that \( G \) has infinitely many ends, so that either \( G \) admits a non-trivial amalgamated free product decomposition \( G = A*_{C} B \) with \( C \) finite, or \( G \) admits a non-trivial HNN extension \( G = A*_{C} \) with \( C \) finite. We give full details for the amalgamated free product case; the details in the HNN extension case are analogous.

We first show that no conjugate of either \( A \) (or of \( B \), the details are the same) can contain a non-zero power of an element of \( S \). So, suppose there is some \( g \in G \), some \( s_0 \in S \), and some \( k \neq 0 \) so that \( s_0^k \in gAg^{-1} \). (Note that, by the comment immediately following Lemma 5, this implies that \( s_0 \in gAg^{-1} \) as well.)

Let \( s_1 \) be any vertex of \( K(G, S) \) adjacent to \( s_0 \). As there are non-zero integers \( n_0, n_1 \) so that \( \langle s_0^{n_0}, s_1^{n_1} \rangle \) is abelian, we see that \( \langle s_0^{n_0} \rangle \subseteq gAg^{-1} \cap s_1^{n_1} gAg^{-1} s_1^{-n_1} = g(A \cap g^{-1}s_1^{n_1} gAg^{-1} s_1^{-n_1} g)g^{-1} \). However, since \( s_0 \) has infinite order, the second part of Lemma 5 implies \( g^{-1}s_1^{n_1} g \in A \). By the comment immediately following Lemma 5 we see that \( s_1 \in gAg^{-1} \) as well.

Let \( s \) be any element of \( S \). By the connectivity of \( K(G, S) \), there is a sequence of elements \( s_0, s_1, \ldots, s_n = s \) of \( S \) such that \( s_{k-1} \) and \( s_k \) are adjacent in \( K(G, S) \) for each \( 1 \leq k \leq n \). The argument we have given above implies that if \( s_{k-1} \in gAg^{-1} \), then \( s_k \in gAg^{-1} \). In particular, we have that \( s \in gAg^{-1} \). Since \( G \) is generated by \( S \), it follows that \( G \) and \( gAg^{-1} \) are equal, contradicting the fact that the subgroup \( A \) is proper. We conclude that if \( G \) admits a non-trivial amalgamated free product decomposition \( G = A*_{C} B \) with \( C \) finite, then no conjugate of either \( A \) or \( B \) can contain a non-zero power of an element of \( S \).

Now, by assumption there exist adjacent vertices \( t \) and \( t' \) of \( K(G, S) \) for which there exist non-zero integers \( n, n' \) so that \( D = \langle t^n, (t')^{n'} \rangle \) is rank 2 abelian. By Lemma 7 we see that \( D \) must lie in some conjugate of \( A \) (or of \( B \)). In particular, this conjugate of \( A \) contains a non-zero power of an element of \( S \), and we have a final contradiction. QED
5 Applications

In this section, we apply Theorems 2 and 9 to a selection of finitely generated groups, and deduce that each is not strongly hyperbolic relative to any finite collection of proper finitely generated subgroups (we will just say that such a group is not strongly relatively hyperbolic) and has one end.

5.1 Mapping class groups

Let \( \Sigma \) be a connected, orientable surface without boundary, of finite topological type and negative Euler characteristic. As such, \( \Sigma \) is the complement in a closed, orientable surface of a (possibly empty) finite set of points. The mapping class group \( \text{MCG}(\Sigma) \) associated to \( \Sigma \) is the group of all homotopy classes of orientation preserving self-homeomorphisms of \( \Sigma \). For a thorough account of these groups, we refer the reader to Ivanov [21]. It is known that every mapping class group \( \text{MCG}(\Sigma) \) is finitely presentable and can be generated by Dehn twists. Masur and Minsky [24] prove that \( \text{MCG}(\Sigma) \) is weakly hyperbolic relative to a finite collection of curve stabilisers.

Now let \( S \) be the collection of primitive Dehn twists about all elements of \( \pi_1(\Sigma) \) that are represented by simple closed curves on \( \Sigma \). (Here, an element of a group is primitive if it is not a proper power of another element of the group.) The associated commutativity graph is precisely the 1-skeleton of the curve complex, introduced in Harvey [20]; this follows from the observation that two Dehn twists commute if and only if their associated curves are disjoint. Moreover, the Dehn twists associated to any pair of adjacent vertices in the curve complex generate a rank 2 free abelian group. Such a graph is connected provided \( \Sigma \) is not a once-punctured torus or a four-times punctured sphere. Hence, we have the following:

**Proposition 10** Let \( \Sigma \) be a connected, orientable surface without boundary, of finite topological type and negative Euler characteristic. If \( \Sigma \) is not a once-punctured torus or a four-times punctured sphere, then the mapping class group \( \text{MCG}(\Sigma) \) of \( \Sigma \) is not strongly relatively hyperbolic.

This answers Question 6.24 of Behrstock [2] in the negative. Note that, when \( \Sigma \) is a once-punctured torus or a four-times punctured sphere, its mapping class group is isomorphic to \( \text{PSL}(2, \mathbb{Z}) \) which is a hyperbolic group. We remark that the result of Proposition 10 was previously obtained by Bowditch [4], using arguments based on convergence groups.

Since the mapping class group of a punctured sphere can be viewed as a braid group, the braid group \( B_n \) on \( n \) strings is not strongly relatively hyperbolic whenever \( n \geq 5 \). This also follows by considering the usual presentation for \( B_n \) and its corresponding commutativity graph.

We also have a corresponding result about ends.

**Proposition 11** Let \( \Sigma \) be a connected, orientable surface without boundary, of finite topological type and negative Euler characteristic. If \( \Sigma \) is not a once-punctured torus or a four-times punctured sphere, then the mapping class group \( \text{MCG}(\Sigma) \) of \( \Sigma \) has one end.
We note that Proposition 11 is implicit in the work of Harer [19], as it is proven there that \( \text{MCG}(\Sigma) \) is a virtual duality group with virtual cohomological dimension greater than one, and such groups are known to have one end by a standard yoga. It also is contained in Culler and Vogtmann [9].

As with Proposition 10 the cases of the once-punctured torus and the four-times punctured sphere are anomalous; as noted above, in both of these cases, the mapping class group is isomorphic to \( \text{PSL}(2, \mathbb{Z}) \), which is a free product and as such has infinitely many ends.

5.2 The Torelli group

The Torelli group \( \mathcal{I}(\Sigma) \) of a connected, orientable surface \( \Sigma \) is the kernel of the natural action of the mapping class group \( \text{MCG}(\Sigma) \) on the first homology group \( H_1(\Sigma, \mathbb{Z}) \). It is of continued interest, given its connections with homology 3-spheres and the number of basic open questions it carries. If \( \Sigma \) is compact and has genus at least 3, \( \mathcal{I}(\Sigma) \) is generated by all Dehn twists around separating simple closed curves and all double twists around pairs of disjoint simple closed nonseparating curves (called bounding pairs) that together separate (see [22]).

Farb and Ivanov [16] introduce a graph they call the Torelli geometry. The vertices of this graph comprise all separating curves and bounding pairs in \( \Sigma \), with two distinct vertices declared adjacent if their corresponding curves or bounding pairs are disjoint. Whenever \( \Sigma \) has genus at least three this graph is connected (this holds even when \( \Sigma \) has non-empty boundary, see [23]). For this reason, let us take \( S \) to be the collection of primitive Dehn twists about separating curves and double twists around bounding pairs. The corresponding commutativity graph \( K(\mathcal{I}(\Sigma), S) \) is precisely the Torelli geometry. Also, as is the case with mapping class groups, adjacent vertices generate a rank 2 free abelian subgroup of \( \mathcal{I}(\Sigma) \). Thus, we have:

**Proposition 12** If \( \Sigma \) is a closed, orientable surface of genus at least three, then the Torelli group \( \mathcal{I}(\Sigma) \) of \( \Sigma \) is not strongly relatively hyperbolic.

We conjecture this extends to all surfaces \( \Sigma \) of genus \( g \) and \( n \) punctures with \( 2g + n - 4 \geq 1 \) (to exclude small surfaces). For this, one would need to establish the connectivity of the Torelli geometry.

We have also the following result on ends of the Torelli group. Since \( \mathcal{I}(\Sigma) \) has infinite index in \( \text{MCG}(\Sigma) \), the fact that \( \mathcal{I}(\Sigma) \) has one end is independent of Section 5.1.

**Proposition 13** If \( \Sigma \) is a closed, orientable surface of genus at least three, then the Torelli group \( \mathcal{I}(\Sigma) \) of \( \Sigma \) has one end.

5.3 The special automorphism group of a free group

In this subsection, we use the notation and basic results from Gersten [17] (without further reference). Let \( \mathbf{F}_n \) be the free group on \( n \) generators and consider the automorphism group \( \text{Aut}(\mathbf{F}_n) \) of \( \mathbf{F}_n \). Abelianisation gives a surjective homomorphism

\[
\text{Aut}(\mathbf{F}_n) \to \text{Aut}(\mathbb{Z}^n) = \text{GL}(n, \mathbb{Z}).
\]
Composing with the sign of determinant map

\[ \text{GL}(n, \mathbb{Z}) \to \{ \pm 1 \}, \]

we obtain a surjective homomorphism

\[ \varphi : \text{Aut}(F_n) \to \{ \pm 1 \}, \]

which we call the determinant map.

The special automorphism group of \( F_n \) is \( \text{Aut}^+(F_n) = \ker(\varphi) \), and has the following finite presentation in terms of Nielsen maps: Let \( X \) be a free basis for \( F_n \) and let \( E = X \cup X^{-1} \). Given \( a, b \in E \) with \( a \neq b, b^{-1} \), define the Nielsen map \( E_{ab} \) for \( a, b \) by \( E_{ab} : F_n \to F_n \), where \( E_{ab}(a) = ab \) and \( E_{ab}(c) = c \) for \( c \neq a, a^{-1} \).

Gersten [17] shows that \( \text{Aut}^+(F_n) \) is generated by the finite set

\[ S = \{ E_{ab} \mid a, b \in E \text{ with } a \neq b, b^{-1} \} \]

and that the following relation holds:

\[ [E_{ab}, E_{cd}] = 1 \text{ if } a \neq c, d, d^{-1} \text{ and } b \neq c, c^{-1}. \]

(We suppress the full set of relations).

Note that if \( E_{ab}, E_{cd} \) are distinct and commute, they generate a rank 2 abelian subgroup of \( \text{Aut}^+(F_n) \), since both of them have infinite order and neither is a power of the other. We then have:

**Proposition 14** With \( S \) as above, \( K(\text{Aut}^+(F_n), S) \) is connected for \( n \geq 5 \). In particular, \( \text{Aut}^+(F_n) \) is not strongly relatively hyperbolic for \( n \geq 5 \).

**Proof** Let \( E_{ab} \) and \( E_{cd} \) be any two vertices of \( K(\text{Aut}^+(F_n), S) \). We have that \( E_{ab} \) and \( E_{cd} \) are adjacent in \( K(\text{Aut}^+(F_n), S) \) unless \( a \in \{ c, d, d^{-1} \} \) or \( b \in \{ c, c^{-1} \} \). Let us consider \( a = c \) (the remaining cases are similar). We want to find a path from \( E_{ab} \) to \( E_{ad} \) in \( K(\text{Aut}^+(F_n), S) \). Since \( n \geq 5 \), there are \( e, f \in E \setminus \{ a^{\pm 1}, b^{\pm 1}, d^{\pm 1} \} \). It then follows, from the commutativity relation in \( \text{Aut}^+(F_n) \) mentioned above, that the sequence of generators \( E_{ab}, E_{cd}, E_{bf}, E_{ad} \) gives a path in \( K(\text{Aut}^+(F_n), S) \) from \( E_{ab} \) to \( E_{ad} \). \( \text{QED} \)

Let \( \text{Out}^+(F_n) = \text{Aut}^+(F_n)/\text{Inn}(F_n) \) be the special outer automorphism group of \( F_n \). It is immediate that the natural surjective homomorphism \( \text{Aut}^+(F_n) \) to \( \text{Out}^+(F_n) \) preserves the connectivity of our commutativity graph for \( \text{Aut}^+(F_n) \) and the hypotheses of Theorem 2. We therefore deduce the following:

**Corollary 15** \( \text{Out}^+(F_n) \) is not strongly relatively hyperbolic for \( n \geq 5 \).
Restricting the surjective homomorphism \( \text{Aut}(F_n) \to \text{GL}(n, \mathbb{Z}) \) to \( \text{Aut}^+(F_n) \), we obtain a homomorphism \( \text{Aut}^+(F_n) \to \text{SL}(n, \mathbb{Z}) \). The generating set \( S = \{ E_{ab} \mid a, b \in E \text{ with } a \neq b, b^{-1} \} \) projects onto a generating set \( \mathcal{S} \) for \( \text{SL}(n, \mathbb{Z}) \) whose elements have infinite order in \( \text{SL}(n, \mathbb{Z}) \), as immediately follows from the definition of the Nielsen maps. Also \( K(\text{SL}(n, \mathbb{Z}), \mathcal{S}) \) is connected, since \( K(\text{Aut}^+(F_n), S) \) is. Thus we have:

**Corollary 16** \( \text{SL}(n, \mathbb{Z}) \) is not strongly relatively hyperbolic for \( n \geq 5 \).

Similarly, we have the following result about the number of ends of these groups. We can extend the discussion to the automorphism and outer automorphism groups of \( F_n \) in this case, since the number of ends of a group is invariant under passing to finite index subgroups. We note that these results were earlier obtained by Culler and Vogtmann [9].

**Proposition 17** For \( n \geq 5 \), the groups \( \text{Aut}(F_n) \), \( \text{Aut}^+(F_n) \), \( \text{Out}(F_n) \), \( \text{Out}^+(F_n) \), and \( \text{SL}(n, \mathbb{Z}) \) are one-ended.

5.4 The Heisenberg group

Recall that the 3-dimensional Heisenberg group \( \mathcal{H} \) is given by the presentation

\[
\mathcal{H} = \langle a, b, c \mid [a, b] = c, [a, c] = 1 = [b, c] \rangle.
\]

Consider the generating set \( S = \{a, b, c\} \). It is evident from this presentation that the commutativity graph \( K(\mathcal{H}, S) \) is connected. Also, the group \( \langle a, c \rangle \) is rank 2 abelian.

**Proposition 18** The 3-dimensional Heisenberg group \( \mathcal{H} \) is not strongly relatively hyperbolic.

We also have the following, which was previously known, see for instance Apurara [1].

**Proposition 19** The 3-dimensional Heisenberg group \( \mathcal{H} \) has one end.

5.5 Thompson’s group \( F \)

Thompson’s group \( F \) is a torsion-free group of orientation-preserving, piecewise-linear homeomorphisms of the unit interval of the real line; see Brown and Geoghegan [6] or Cannon, Floyd and Parry [8] for a complete definition. Even though \( F \) is a finitely presented group, it is sometimes convenient to work with the infinite presentation

\[
F = \langle x_0, x_1, x_2, \ldots \mid x_{j+1} = x_ix_jx_i^{-1}, i < j \rangle.
\]

Let \( S = \{x_j \mid j \geq 0\} \). Clearly, the commutativity graph \( K(F,S) \) is far from being connected. So, we tinker: If we consider the generating set \( S' = S \cup \{x_0x_1^{-1}\} \), then the commutativity graph \( K(F,S') \) is still not connected, as \( x_0 \) and \( x_1 \) are isolated vertices. However, \( K(F,S') \setminus \{x_0, x_1\} \) is connected, since \( x_0x_1^{-1} \) commutes with \( x_i \) for all \( i \geq 2 \) (see, for instance, Burillo [7]). This is enough for us to modify our main argument and deduce:
Proposition 20  Thompson’s group $F$ is not strongly relatively hyperbolic.

Proof  Suppose, for contradiction, that $F$ were strongly hyperbolic relative to the proper finitely generated subgroups $L_1, \ldots, L_p$. Since all abelian subgroups of rank at least 2 are conjugate into some $L_m$ by Lemma 4 and since $\langle x_0x_1^{-1}, x_2 \rangle$ is abelian of rank 2, we find that $\langle x_0x_1^{-1}, x_2 \rangle \subset gL_mg^{-1}$ for some $m = 1, \ldots, p$ and some $g \in F$. For all $j \geq 2$, $x_0x_1^{-1}$ and $x_j$ commute and so $(x_0x_1^{-1}) \subset gL_mg^{-1} \cap x_jgL_mg^{-1}x_j^{-1}$. Since $\langle x_0x_1^{-1} \rangle$ is infinite, we have $x_j \in gL_mg^{-1}$, for all $j \geq 2$, by Theorem 6.

Now, $gL_mg^{-1}$ cannot contain both $x_0$ and $x_1$, since $S = \{x_j \mid j \geq 0\}$ generates $F$ and $gL_mg^{-1}$ is a proper subgroup of $F$ by assumption. So suppose $x_0 \notin gL_mg^{-1}$. (The case $x_1 \notin gL_mg^{-1}$ is similar.) From the presentation above, we see that $x_{j+1} = x_0x_jx_0^{-1}$ for all $j \geq 2$. Therefore $x_{j+1} \in x_0gL_mg^{-1}x_0^{-1}$, since we have shown that $x_j \in gL_mg^{-1}$, for all $j \geq 2$. Therefore $gL_mg^{-1} \cap x_0gL_mg^{-1}x_0^{-1}$ is infinite (as it contains $\langle x_j \rangle$ for any $j \geq 2$), contradicting Theorem 6. QED

Using the same style of argument, we also have the following.

Proposition 21  Thompson’s group $F$ has one end.

Proof  Suppose, for contradiction, that $F$ has more than one end. Since $F$ is not virtually $\mathbb{Z}$ and $F$ is torsion free, we see that $F$ admits a non-trivial free product splitting $F = A * B$. Since all abelian subgroups of rank at least 2 are conjugate into either $A$ or $B$ by Lemma 4 and since $\langle x_0x_1^{-1}, x_2 \rangle$ is abelian of rank 2, we find that $\langle x_0x_1^{-1}, x_2 \rangle \subset gAg^{-1}$ for some $g \in F$. (The details are similar if $\langle x_0x_1^{-1}, x_2 \rangle \subset gBg^{-1}$ for some $g \in F$.) For all $j \geq 2$, $x_0x_1^{-1}$ and $x_j$ commute and so $\langle x_0x_1^{-1} \rangle \subset gAg^{-1} \cap x_jgLAg^{-1}x_j^{-1}$. Since $\langle x_0x_1^{-1} \rangle$ is infinite, we have $x_j \in gAg^{-1}$, for all $j \geq 2$, by Lemma 5.

Now, $gAg^{-1}$ cannot contain both $x_0$ and $x_1$, since $S = \{x_j \mid j \geq 0\}$ generates $F$ and $gAg^{-1}$ is a proper subgroup of $F$ by assumption. So suppose $x_0 \notin gAg^{-1}$. (The case $x_1 \notin gAg^{-1}$ is similar.) From the presentation above, we see that $x_{j+1} = x_0x_jx_0^{-1}$ for all $j \geq 2$. Therefore $x_{j+1} \in x_0gLAg^{-1}x_0^{-1}$, since we have shown that $x_j \in gAg^{-1}$, for all $j \geq 2$. Therefore $gAg^{-1} \cap x_0gLAg^{-1}x_0^{-1}$ is infinite (as it contains $\langle x_j \rangle$ for any $j \geq 3$), contradicting Lemma 5. QED

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