Remarks on a limiting case of Hardy type inequalities

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Abstract

The classical Hardy inequality holds in Sobolev spaces $W^{1,p}_0$ when $1 \leq p < N$. In the limiting case where $p = N$, it is known that by adding a logarithmic function to the Hardy potential, some inequality which is called the critical Hardy inequality holds in $W^{1,N}_0$. In this note, in order to give an explanation of appearance of the logarithmic function at the potential, we derive the logarithmic function from the classical Hardy inequality with the best constant via some limiting procedure as $p \nearrow N$. And we show that our limiting procedure is also available for the classical Rellich inequality in second order Sobolev spaces $W^{2,p}_0$ with $p \in (1, \frac{N}{2})$ and the Poincaré inequality.

Keywords: Hardy inequality, limiting case, Sobolev embedding, Extrapolation, pointwise estimate of radial functions

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1. Introduction

Let $B_1 \subset \mathbb{R}^N$ be the unit ball, $1 < p < N$ and $N \geq 2$. The classical Hardy inequality

$$\left( \frac{N - p}{p} \right)^p \int_{B_1} \frac{|u|^p}{|x|^p} dx \leq \int_{B_1} |
abla u|^p dx,$$  \hspace{1cm} (1)

holds for all $u \in W^{1,p}_0(B_1)$, where $W^{1,p}_0(B_1)$ is a completion of $C^\infty_c(B_1)$ with respect to the norm $||\nabla(\cdot)||_{L^p(B_1)}$. Note that the inequality (1) expresses the embedding

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\[ W^{1,p}(B_1) \hookrightarrow L^p(B_1; |x|^{-p}dx) \] which is equivalent to \( W^{1,p}_0(B_1) \hookrightarrow L^{p^*}(B_1) \) thanks to the rearrangement technique, where \( p^* = \frac{Np}{N-p} \) and \( L^{p^*} \) is the Lorentz space. Therefore by a property of Lorentz spaces, we see that for any \( q > p \)

\[ W^{1,p}_0 \hookrightarrow L^{p^*}(B_1) \hookrightarrow L^{p^*,q}(B_1) \hookrightarrow L^{p^*,\infty}. \]

The variational problems and partial differential equations associated with the best constants of the inequalities related to the embeddings as above are classical interesting topics, see [31, 25, 4, 8, 33], to name a few.

On the other hand, in the limiting case where \( p = N \) the classical Hardy inequality \( \Box \) does not have its meaning due to the best constant becomes zero. Instead of it, by adding a logarithmic function at the Hardy potential, the following inequality (1) does not have its meaning due to the best constant becomes zero.

\[ \left( \frac{N-1}{N} \right)^N \int_{B_1} \frac{|u|^N}{|x|^{N(\log \frac{|u|}{|x|})^N}} dx \leq \int_{B_1} |\nabla u|^N dx \quad (2) \]

holds for all \( u \in W^{1,N}_0(B_1) \) and \( a \geq 1 \), see \([24, 23]\). The inequality (2) expresses the embedding: \( W^{1,N}_0(B_R) \hookrightarrow L^N(B_1; |x|^{-N(\log \frac{|u|}{|x|})^N} dx) \). Since for large \( a \) the weight functions \( |x|^{-N(\log \frac{|u|}{|x|})^N} \) are radially decreasing with respect to \( |x| \), the embedding with \( a > 1 \) is equivalent to \( W^{1,N}_0 \hookrightarrow L^{\infty,N}(\log L)^{-1} \) thanks to the rearrangement technique. Here \( L^{p,q}(\log L)^{r} \) is the Lorentz-Zygmund space which is given by

\[ L^{p,q}(\log L)^{r} = \left\{ u : B_1 \rightarrow \mathbb{R} \text{ measurable} \left| \|u\|_{L^{p,q}(\log L)^{r}} < \infty \right. \right\} \]

\[ \|u\|_{L^{p,q}(\log L)^{r}} = \begin{cases} \left( \int_{B_1} |(1 + \log \frac{|u|}{|x|})^q u^* (s)^q ds \right)^{\frac{1}{q}} & \text{if} \ 1 \leq q < \infty, \\ \sup_{0 < s < |B_1| |x| \frac{p}{q}} (1 + \log \frac{|u|}{|x|})^{\frac{p}{q}} u^*(s) & \text{if} \ q = \infty, \end{cases} \]

where \( u^* \) be the Schwartz symmetrization of \( u \). Note that \( L^{p,q}(\log L)^{0} \) becomes the Lorentz space \( L^{p,q} \) and \( L^{\infty,\infty}(\log L)^{0} \) becomes the Zygmund space \( Z^{-r} \) which can be equivalent to Orlicz space \( L_{\exp^L}^{\frac{1}{r}} = \exp L^{-\frac{1}{r}} \) in some sense (see \([6, 7, 13]\)). By a property of Lorentz-Zygmund spaces (see e.g. \([7]\) Theorem 9.5.), we see that for any \( q > N \)

\[ W^{1,N}_0 \hookrightarrow L^{\infty,N}(\log L)^{-1} \hookrightarrow L^{\infty,q}(\log L)^{-1+\frac{1}{q}-\frac{1}{p}} \hookrightarrow L^{\infty,\infty}(\log L)^{-1+\frac{1}{q}} = \exp L^{\frac{r}{r'}}. \]

And variational problems associated with the best constants of the inequalities related to embeddings as above are also studied, see \([3, 10, 1, 18, 13, 21, 29]\).

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In this note, in order to give an explanation of appearance of the logarithmic function in the limiting case $p = N$ of the classical Hardy inequality, we shall derive the logarithmic function in (2) from the classical Hardy inequality with the best constant by some limiting procedure as $p \nearrow N$ based on extrapolation. Giving an explanation of it is corresponding to giving a reason why we consider Lorentz-Zygmund space $L^{p,q}(\log L)^r$ in the embedding of the critical Sobolev space $W^{1,N}_0$. In this viewpoint, considering a limiting procedure for the classical Hardy inequality is significant. Main Theorem is as follows.

**Theorem 1.** The following non-sharp critical Hardy inequality (3) can be derived by a limiting procedure for the classical Hardy inequality (1) as $p \nearrow N$.

$$C \int_{B_1} \frac{|u|^N}{(\log \frac{|x|}{|y|})^p} \, dx \leq \int_{B_1} \left( \nabla u \cdot \frac{x}{|x|} \right)^N \, dx \quad (u \in C^1_c(B_1)).$$

Here $\beta > 2N, a > 1$, and the constant $C = C(\beta, a, N) > 0$ is independent of $u$.

Note that the inequality (3) does not have the optimal exponent $\beta$ and the optimal constant $C$, and itself is already well-known. However, our limiting procedure for the classical Hardy inequality is new and giving some explanation of appearance of the logarithmic function at the Hardy potential in the limiting case $p = N$. Our limiting procedure can be regarded as an analogue of Trudinger’s argument in [32] for the Sobolev inequality as $p \nearrow N$, see also [9] Theorem 1.7.

For several limiting procedures, we refer [5] (The Sobolev inequality as $N \nearrow \infty$), [34], [35] XII 4.41. (L$^p$ boundedness of the Hilbert transformation as $p \searrow 1$ or $p \nearrow \infty$), see also [28] Corollary 3.2.4 (The Sobolev inequality is derived from the Nash inequality).

A few comments on Theorem 1 are in order. Very recently, Ioku [19] showed the following improved Hardy inequality (5) on $B_1$ which is equivalently connected to the classical Hardy inequality on $\mathbb{R}^N$ via the following transformation (4) at the radial setting.

$$w(|y|) = u(|x|), \text{ where } |y|^{-\frac{N-p}{p}} = |x|^{-\frac{N-p}{p}} - 1$$

$$\int_{\mathbb{R}^N} |\nabla w|^p \, dy - \left( \frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} |w|^p \, dy$$

$$= \int_{B_1} |\nabla u|^p \, dx - \left( \frac{N-p}{p} \right)^p \int_{B_1} \frac{|u|^p}{|x|^p \left( 1 - |x|^{-\frac{N-p}{p}} \right)^p} \, dx \geq 0. \quad (5)$$

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And also, Ioku showed the improved Hardy inequality (5) yields the critical Hardy inequality (2) with the best constant by taking the limit $p \to N$. However, in the higher order or fractional order case, these beautiful and simple structures and transformations might be nothing. In this note, we also treat the second order case. We observe that the transformation (4) connects two singular solutions of $p$–Laplace equations: $-\text{div}(|\nabla u|^{p-2}\nabla u)=0$ on $B_1$ and $\mathbb{R}^N$. On the other hand, by considering a transformation which connects two singular solutions of $p$–Laplace equation and $N$–Laplace equation, the authors [30] showed an equivalence between the classical Hardy inequality (1) and the critical Hardy inequality (2), for the Sobolev type inequalities, see [36, 29]. Furthermore, for other transformations and another interesting equivalence, see [18, 20, 14].

This paper is organized as follows: In section 2, necessary preliminary facts are presented. In section 3, we give the limiting procedure as $p \to N$ for the Hardy inequality in Theorem 1. We also apply our limiting procedure for the Rellich inequality. In section 4, we consider a limit as $|\Omega| \to 0$ for the Poincaré inequality via our limiting procedure.

2. Preliminaries

In this section, we prepare several lemmas to show Theorems. The following pointwise estimate is well-known. We omit the proof.

**Lemma 1.** For any radial functions $u \in C^1(B_R) \cap C(\overline{B_R})$ satisfying $u|_{\partial B_R} = 0$, for any $r \in (0, R)$ the following estimate holds.

$$|u(r)| \leq \begin{cases} \omega_{N-1}^{-1} \|
abla u\|_{L^p(B_R)} r^{-(N-1)} & \text{if } p = 1, \\ \left( \frac{p-1}{|N-p|} \right)^{\frac{1}{p}} \omega_{N-1}^{-\frac{1}{p}} \|
abla u\|_{L^p(B_R)} \left| r^{\frac{N-p}{p}} - R^{\frac{N-p}{p}} \right|^{\frac{p-1}{p}} & \text{if } 1 < p \neq N, \\ \omega_{N-1}^{-\frac{1}{p}} \|
abla u\|_{L^N(B_1)} \left( \log \frac{R}{r} \right)^{\frac{N-p}{p}} & \text{if } p = N. \end{cases}$$

When the potential function is not radially decreasing, we can not apply rearrangement technique. Instead of rearrangement, we prepare the following lemma which can reduces a problem to the radial setting.
Lemma 2. Let $1 < q < \infty$, $V = V(x)$ be a radial function on $B_R$. If there exists $C > 0$ such that for any radial functions $u \in C^1_c(B_R)$ the inequality

$$C \int_{B_R} |u|^q V(x) \, dx \leq \int_{B_R} |\nabla u|^q \, dx < \infty$$

holds, then for any functions $w \in C^1_c(B_R)$ the inequality

$$C \int_{B_R} |w|^q V(x) \, dx \leq \int_{B_R} \left| \nabla w \cdot \frac{x}{|x|} \right|^q \, dx < \infty$$

holds.

Proof. For any $w \in C^1_c(B_R)$, define a radial function $W$ as follows.

$$W(r) = \left( \omega_{N-1}^{-1} \int_{B_{1}} |w(r\omega)|^q \, dS_{\omega} \right)^{\frac{1}{q}} \quad (0 \leq r \leq R).$$

Then we have

$$|W'(r)| = \omega_{N-1}^{-\frac{1}{q}} \left( \int_{B_{1}} |w(r\omega)|^q \, dS_{\omega} \right)^{-\frac{1}{q}} \left( \int_{\partial B_{1}} \frac{\partial w}{\partial r} \, dS_{\omega} \right) \leq \omega_{N-1}^{-\frac{1}{q}} \left( \int_{\partial B_{1}} \left| \frac{\partial w}{\partial r}(r\omega) \right|^q \, dS_{\omega} \right)^{\frac{1}{q}}.$$

Therefore we have

$$\int_{B_R} |\nabla W|^q \, dx \leq \int_{B_R} \left| \nabla w \cdot \frac{x}{|x|} \right|^q \, dx,$$  \hspace{1cm} (8)

$$\int_{B_R} |W|^q V(x) \, dx = \int_{B_R} |w|^q V(x) \, dx.$$  \hspace{1cm} (9)

From (6) for $W$, (8), and (9), we obtain (7) for any $w$. □

We show the pointwise estimates for radial functions and their derivative in $W^{2,p}_{0}(B_R)$ when $N \geq 3$ and $p \geq 1$. When $p = 1$ or $N = 2$, the pointwise estimates are already shown by [11, 12].
Lemma 3. Let $N \geq 3$ and $u \in C^2(B_R) \cup C(\overline{B_R})$ be a radial function satisfying $u|_{\partial B_R} = 0$. Then the following pointwise estimates hold for any $r \in (0, R)$.

\[
|u(r)| \leq \begin{cases}
\frac{p}{|N-2p|} \omega_{N-1}^{\frac{1}{p}} N^{\frac{1}{p}-1} \|\Delta u\|_{L^p(B_r)} \left| r^{\frac{N-2p}{p}} - R^{\frac{N-2p}{p}} \right| & \text{if } 1 \leq p \neq \frac{N}{2}, \\
\omega_{N-1}^{\frac{1}{p}} N^{\frac{1}{p}-1} \|\Delta u\|_{L^{N/2}(B_R)} \log \frac{R}{r} & \text{if } p = \frac{N}{2}.
\end{cases}
\]  

(10)

\[
|u'(r)| \leq \frac{\|\Delta u\|_{L^p(B_r)}^{\frac{p}{p}}}{\omega_{N-1}^{\frac{1}{p}} N^{\frac{1}{p}-1}} r^{-\frac{N+p}{p}} \quad \text{for any } p \geq 1.
\]  

(11)

Proof. Consider the following transformation (ref. [12]):

\[ w(t) = Au(r), \quad \text{where } r = R(t+1)^{-\frac{1}{N-2}} \text{ and } A^p = \omega_{N-1} R^{N-2p} (N-2)^{p-1}. \]  

(12)

Then we have

\[
w''(t) = \frac{AR^2}{(N-2)^2} (t+1)^{-2N-1} \left( u''(r) + \frac{N-1}{r} u'(r) \right) = \frac{AR^2}{(N-2)^2} (t+1)^{-2N-1} \Delta u
\]  

(13)

which yields that

\[
\int_{B_R} |\Delta u|^p \, dx = \int_0^{\infty} |w''(t)|^p (t+1)^{2(N-1)(p-1)} \, dt.
\]

Since $w(0) = w'(\infty) = 0$, we have

\[
w(t) = -\int_0^t \int_s^{\infty} w''(u) \, du \, ds
\]

\[
\leq \int_0^t \left( \int_s^{\infty} |w''(u)|^p (u+1)^{\frac{2(N-1)(p-1)}{N-2}} \, du \right)^{\frac{1}{p}} \left( \int_s^{\infty} (u+1)^{-\frac{2N-1}{N-2}} \, du \right)^{\frac{p-1}{p}} \, ds
\]

\[
= \left( \frac{N-2}{N} \right)^{\frac{p-1}{p}} \|\Delta u\|_{L^p(B_R)} \int_0^t (s+1)^{\frac{2N-1}{N-2p}} \, ds
\]

\[
= \left( \frac{N-2}{N} \right)^{\frac{p-1}{p}} \left( \frac{(N-2)p}{N-2p} \|\Delta u\|_{L^p(B_R)} ((t+1)^{\frac{N-2p}{N-2p}} - 1) \right) \quad \text{if } p \neq \frac{N}{2},
\]

\[
= \left( \frac{N-2}{N} \right)^{1-\frac{N}{2}} \|\Delta u\|_{L^{N/2}(B_R)} \log(t+1) \quad \text{if } p = \frac{N}{2}.
\]
Therefore we obtain (10). On the other hand, since
\[ w'(t) = -\int_t^\infty w''(u) du \leq \left( \frac{N-2}{N} \right) \frac{p-1}{p} \|\Delta u\|_{L^p(B_k)} (t+1)^{\frac{N(p-1)}{N+p}} \]
and \( w'(t) = -Au'(r) \frac{R}{N-2}(t+1)^{\frac{N-1}{N}} \), we also obtain (11). \( \Box \)

For much higher order case, see Proposition 1 in §3.

3. A limiting procedure for the Hardy type inequalities

3.1. Proof of Theorem 1

The Hardy inequality

First, we prepare for making the optimal constant \( \left( \frac{N-p}{p} \right)^p \) for \( p \not\to N \) compete with \( \int_{B_1} |u|^p dx \) as \( p \not\to N \), in general.

Let \( p_k = N - \frac{1}{k} \) for \( k \in \mathbb{N}, f \in C^1(-\infty, \infty) \) be a monotone-decreasing function which satisfies \( \lim_{t \to \pm \infty} f(t) = 0 \), and \( \{\phi_k\}_{k \in \mathbb{Z}} \subset C^\infty_c(\mathbb{R}^N \setminus \{0\}) \) be radial functions which satisfy

\begin{align*}
(i) \quad & \sum_{k=-\infty}^{+\infty} \phi_k(x)^N = 1, \quad 0 \leq \phi_k(x) \leq 1 \quad (\forall x \in \mathbb{R}^N \setminus \{0\}), \\
(ii) \quad & \text{supp} \phi_k \subset B_{f(k)} \setminus B_{f(k+2)}.
\end{align*}

For any radial function \( u \in C^1_c(B_1) \), set \( u_k = u \phi_k \) and \( A_k = \text{supp} u_k \subset B_1 \cap \left( B_{f(k)} \setminus B_{f(k+2)} \right) \). In order to obtain a limit for the classical Hardy inequality (1) as \( p \not\to N \), the left-hand side of (1) for \( u_k \) and \( p_k \) must not be vanishing as \( k \to \infty \).

We shall determine such \( f \). Note that if \( x \in A_k \), then \( f(k+2) \leq |x| \leq f(k) \) and \( k \leq f^{-1}(|x|) \leq k+2 \). By Lemma 1 we have

\begin{align*}
& \left( \frac{N-p_k}{p_k} \right)^p \int_{A_k} |u_k|^p_{|x|^{p_k}} \, dx = p_k^{-p_k} \int_{A_k} \left( \frac{|u_k(x)|}{|x|^{p_k}} \right)^{N-\frac{1}{2}} \, dx \\
& \geq C \int_{A_k} \frac{|u_k(x)|^N}{|x|^N (f^{-1}(|x|))^N} \left( \frac{|x|}{|u_k(x)|} \right)^{\frac{k}{2}} \, dx \\
& \geq C \|\nabla u_k\|_{L^N(A_k)}^{-\frac{k}{2}} \int_{A_k} \frac{|u_k(x)|^N}{|x|^N (f^{-1}(|x|))^N} \left( f(k+2) \left( \log \frac{f(k)}{f(k+2)} \right)^{N-1} \right)^{\frac{1}{2}} \, dx. \quad (14)
\end{align*}
Therefore, if for any \( k \in \mathbb{N} \) the function \( f \) satisfies
\[
\left( f(k + 2) \left( \log \frac{f(k)}{f(k + 2)} \right)^{-\frac{N-1}{p_k}} \right)^{\frac{1}{p_k}} \geq C > 0, \tag{15}
\]
then the information on the left-hand side of the classical Hardy inequality \((1)\) is not vanishing in this limiting procedure. From \((15)\) and l’Hôpital’s rule, we have an ordinary differential inequality for \( f \) as follows:
\[
\frac{d}{dt} f(t) \geq -C f(t)
\]
whose solution satisfies \( f(t) \geq e^{-Ct} \). Thus \( f^{-1}(t) \geq \frac{1}{C} \log \frac{1}{t} \). We believe that the above calculation and consideration give some explanation of appearance of the logarithmic function at the Hardy potential in the limiting case \( p = N \).

Hereinafter we set \( f(t) = e^{-t} \).

**Proof of Theorem 1.** From Lemma 2, it is enough to show the inequality \((3)\) for any radial functions \( u \in C_c^1(B_1) \). Applying the classical Hardy inequality \((1)\) for \( u_k \) and \( p_k \) for \( k \geq 1 \), we have
\[
\left( \frac{N - p_k}{p_k} \right)^{p_k} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{p_k}} dx \leq \int_{A_k} |\nabla u_k|^{p_k} dx \leq |A_k|^{-\frac{p_k}{N}} \|\nabla u_k\|_{N^{-\frac{1}{p}}}. \tag{14}
\]
By \((14)\) and \((15)\), for \( k \geq 1 \)
\[
C \int_{A_k} \frac{|u_k|^N}{|x|^N \left( \log \frac{1}{|x|} \right)^N} \, dx \leq \int_{A_k} |\nabla u_k|^N \, dx.
\]
Since \( k \leq \log \frac{1}{|x|} \) for \( x \in A_k \),
\[
C \int_{A_k} \frac{|u_k|^N}{|x|^N \left( \log \frac{1}{|x|} \right)^\beta} \, dx \leq b_k \int_{A_k} |\nabla u_k|^N \, dx \tag{16}
\]
for \( k \geq 1, a > 1, \) and \( \beta > 2N \), where \( b_k \) is given by
\[
b_k = \begin{cases} 
  k^{N-\beta} & \text{if } k \geq 1, \\
  1 & \text{if } k = 0, -1, \\
  0 & \text{if } k \leq -2.
\end{cases}
\]
Here, note that the inequalities (16) with \( k = 0, -1 \) come from the Poincaré inequality and the boundedness of the function \(|x|^{-N} (\log \frac{|x|}{a})^{-\beta}\) on \( A_0 \cup A_{-1} \subset B_1 \setminus B_{e^{-2}}\). Summing both sides on (16), we have

\[
C \sum_{k \in \mathbb{Z}} \int_{B_1} \frac{|u\phi_k|^N}{|x|^N (\log \frac{|x|}{a})^\beta} dx \leq \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\nabla (u\phi_k)|^N dx
\]

which yields that

\[
C \int_{B_1} \frac{|u|^N}{|x|^N (\log \frac{|x|}{a})^\beta} dx \leq 2^{N-1} \sum_{k \in \mathbb{Z}} b_k \int_{A_k} \phi_k^N |\nabla u|^N + |u|^N |\nabla \phi_k|^N dx
\]

\[
\leq 2^{N-1} \int_{B_1} |\nabla u|^N dx + C \sum_{k=1}^{+\infty} b_ke^{kN} \int_{A_k} |u|^N dx. \quad (17)
\]

By Lemma 1 we have

\[
b_k e^{kN} \int_{A_k} |u|^N dx \leq C b_k e^{kN} \|\nabla u\|^N_{A_k} \int_{A_k} \left( \log \frac{1}{|x|} \right)^{N-1} dx
\]

\[
\leq C b_k e^{kN} \|\nabla u\|^N_{A_k} \int_{k+2}^{\infty} s^{-N-1} e^{-sN} ds \leq C b_k k^{-1} \|\nabla u\|^N_{A_k}.
\]

From (17) we have

\[
C \int_{B_1} \frac{|u|^N}{|x|^N (\log \frac{|x|}{a})^\beta} dx \leq C \int_{B_1} |\nabla u|^N dx + C \left( \sum_{k=1}^{+\infty} k^{-1-(\beta-2N)} \right) \int_{B_1} |\nabla u|^N dx
\]

\[
\leq C \int_{B_1} |\nabla u|^N dx.
\]

\[
\square
\]

### 3.2. The Rellich inequality

Let \( 1 < p < \frac{N}{2} \). The classical Rellich inequality:

\[
\left( \frac{N(p - 1)(N - 2p)}{p} \right)^{\frac{1}{p}} \int_{B_1} \frac{|u|^p}{|x|^{2p}} dx \leq \int_{B_1} |\Delta u|^p dx \quad (18)
\]

holds for all \( u \in W_{0}^{2,p}(B_1) \), where \( W_{0}^{2,p}(B_1) \) is a completion of \( C_c^\infty(B_1) \) with respect to the norm \( \|\Delta \cdot\|_{L^p(B_1)} \) (see \([27], [15], [26]\)). In this section, we apply our limiting procedure in §3.1 to the Rellich inequality (18) as \( p \nearrow \frac{N}{2} \).
Theorem 2. The following non-sharp critical Rellich inequality (19) can be derived by a limiting procedure for the classical Rellich inequality (18) as \( p \to \frac{N}{2} \).

\[
C \int_{B_1} \frac{|u|^\frac{N}{2}}{|x|^N \left( \log \frac{N}{|x|} \right)^\beta} \, dx \leq \int_{B_1} |\Delta u|^\frac{N}{2} \, dx \quad (u \in C^2_{c,rad}(B_1)).
\tag{19}
\]

Here \( \beta > N + 2, a > 1 \), and the constant \( C = C(\beta, a, N) > 0 \) is independent of \( u \).

Remark 1. In the limiting case \( p = \frac{N}{2} \), the inequality (19) with the optimal exponent \( \beta \) and its best constant is already known, see [16], [2].

Proof. We shall show (19) for any \( u \in C^2_{c,rad}(B_1) \). The strategy of the proof is the same as in §3.1.

Let \( p_k = \frac{N}{2} - \frac{1}{2k} \) for \( k \geq 2 \) and only condition (i) of \( \phi_k \) in §3.1 is changed to \( \sum_{k=-\infty}^{\infty} \phi_k(x) = 1 \) for any \( x \in \mathbb{R}^N \setminus \{0\} \). Applying the classical Rellich inequality (18) for \( u_k = u \phi_k \) for \( u \in C^2_{c,rad}(B_1) \) and \( p_k \) for \( k \geq 2 \), we have

\[
\left( (N - 2p_k)(p_k - 1)N \right)^{\frac{1}{p_k}} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{2p_k}} \, dx \leq \int_{A_k} |\Delta u_k|^{p_k} \, dx.
\tag{20}
\]

On the left-hand side of (20), by (10) in Lemma 3 we have

\[
\left( (N - 2p_k)(p_k - 1)N \right)^{\frac{1}{p_k}} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{2p_k}} \, dx \geq C \int_{A_k} \left( \frac{|u_k(x)|}{|x|^2k} \right)^{\frac{N}{2} - \frac{1}{p_k}} \, dx
\]
\[
\geq C \int_{A_k} \frac{|u_k(x)|^{\frac{N}{2}}}{|x|^N} \left( \frac{|x|^2k}{|u_k(x)|} \right)^{\frac{N}{2}} \, dx
\]
\[
= C |\Delta u_k| \frac{1}{L^{\frac{N}{2}}(A_k)} \int_{A_k} \frac{|u_k(x)|^{\frac{N}{2}}}{|x|^N} \left( f(k + 2) \left( \log \frac{f(k)}{f(k + 2)} \right)^{-1} \right)^{\frac{1}{p_k}} \, dx.
\]

If we choose \( f(t) = e^{-t} \), then the left-hand side of (20) is not vanishing as \( k \to \infty \). Thus we set \( f(t) = e^{-t} \) hereinafter. In the similar way to it in §3.1 for \( a > 1, k \in \mathbb{Z} \), and \( \beta > N + 2 \) we have

\[
C \int_{A_k} \frac{|u_k|^{\frac{N}{2}}}{|x|^N \left( \log \frac{N}{|x|} \right)^\beta} \, dx \leq b_k \int_{A_k} |\Delta u_k|^{\frac{N}{2}} \, dx,
\tag{21}
\]
where $b_k$ is given by

$$b_k = \begin{cases} 
  k^{\frac{\gamma}{2} - \beta} & \text{if } k \geq 2, \\
  1 & \text{if } k = 1, 0, -1, \\
  0 & \text{if } k \leq -2.
\end{cases}$$

Note that we used the second order Poincaré inequality: $C\|u\|_q \leq \|\Delta u\|_q$ to show (21) in the case where $k \leq 1$, see e.g. [17]. Then we have

$$C \sum_{k \in \mathbb{Z}} \int_{B_1} \frac{|u\phi_k|^\frac{N}{2}}{|x|^N \left(\log \left(\frac{|x|}{a} \right)\right)^\beta} \, dx \leq \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\Delta \phi_k|^\frac{N}{2} \, dx$$

which yields that

$$C \int_{B_1} \frac{|u|^\frac{N}{2}}{|x|^N \left(\log \left(\frac{|x|}{a} \right)\right)^\beta} \, dx \leq \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\Delta \phi_k|^\frac{N}{2} \, dx$$

$$= C \sum_{k = 2}^{\infty} (I_1 + I_2 + I_3). \quad (22)$$

Since $|\Delta \phi_k(x)| \leq C e^{2(k+1)}$ for $x \in A_k$, by (10) in Lemma 3 we have

$$I_1 \leq C k^{\frac{N}{2} - \beta} e^{N(k+1)} \int_{A_k} |u|^\frac{N}{2} \, dx$$

$$\leq C k^{\frac{N}{2} - \beta} e^{N(k+1)} \|\Delta u\|_N^{\frac{N}{2}} \int_{A_k} \left(\log \left(\frac{1}{|x|} \right)\right)^\frac{N}{2} \, dx$$

$$\leq C k^{\frac{N}{2} - \beta} e^{N(k+1)} \|\Delta u\|_N^{\frac{N}{2}} \int_{k}^{k+2} \int_k^{t} e^{-Nt} \, dt$$

$$\leq C k^{N+1-\beta} \|\Delta u\|_N^{\frac{N}{2}}.$$ 

In the similar way, we obtain the following estimates of $I_2$ and $I_3$.

$$I_2 + I_3 \leq C k^{\frac{N}{2} - \beta} \|\Delta u\|_N^{\frac{N}{2}}.$$

Here we used (11) in Lemma 3 to show the estimate of $I_3$. From (22) and the estimates of $I_i$ ($i = 1, 2, 3$) we have

$$C \int_{B_1} \frac{|u|^\frac{N}{2}}{|x|^N \left(\log \left(\frac{|x|}{a} \right)\right)^\beta} \, dx \leq C \sum_{k = 2}^{\infty} k^{N+1-\beta} \int_{B_1} |\Delta u|^\frac{N}{2} \, dx \leq C \int_{B_1} |\Delta u|^\frac{N}{2} \, dx.$$
Let $1 < p < \frac{N}{m}$ and $m \geq 2$. The higher order Rellich inequality

$$C_{m,p}^p \int_{B_R} \frac{|u|^p}{|x|^{mp}} \, dx \leq |u|_{m,p}^p$$

holds for all $u \in W^{m,p}_0(B_R)$ (see [27], [15], [26]). Here we set

$$|u|_{m,p}^p = \begin{cases} \int_{B_R} |\Delta^\ell u|^p \, dx & \text{if } m = 2\ell, \\ \int_{B_R} |\nabla (\Delta^\ell u)|^p \, dx & \text{if } m = 2\ell + 1, \end{cases}$$

$$C_{m,p} = \begin{cases} p^{-2\ell} \prod_{j=1}^\ell (N - 2pj)(Np - 1) + 2p(j-1) & \text{if } m = 2\ell, \\ \frac{(N-p)}{p^{(2\ell+1)}} \prod_{j=1}^\ell (N - 2j + 1)p \{Np - 1 + (2j - 1)p\} & \text{if } m = 2\ell + 1, \end{cases}$$

for $m, \ell \in \mathbb{N}, \ell \geq 1$.

In the higher order case where $m \geq 3$, it is difficult to show the pointwise estimate corresponding to Lemma 3 by the same method in Lemma 3. Due to the lack of good pointwise estimate for radial functions, our limiting procedure as $p \nearrow \frac{N}{m}$ cannot work well in the higher order case. However, we can show at least the following pointwise estimates for radial functions in $W^{m,p}_0(B_R)$ for $m \geq 2$ via iteration method. The following pointwise estimates are not optimal. We expect that the pointwise estimates in Proposition 1 will be applicable somewhere.

**Proposition 1.** Let $N, m \geq 3, p \in [1, \frac{N}{2})$ if $m$ is even and $p \in [1, N)$ if $m$ is odd, $u \in C_0^\infty(B_R)$ be a radial function, and $C$ be a constant which is independent of $u$. Then the following pointwise estimates hold for any $r \in (0, R)$.

$$|u(r)| \leq C|u|_{m,p}^{2-N}.$$  \hspace{1cm} (23)

**Proof.** We shall show (23) for $p \in [1, N)$ and odd number $m$ inductively. First we show the case where $m = 3$. By the transformation (12) for radial function $u$ and the pointwise estimate for radial function $v := \Delta u \in W^{1,p}_0$, we obtain

$$|v(r)| \leq C\|\nabla v\|_p r^{\frac{N-p}{2}} \leq C\|\nabla \Delta u\|_p (t+1)^{\frac{N-p}{2}}.$$  \hspace{1cm} (12)

By (13) we have

$$|w''(t)| \leq C\|\nabla \Delta u\|_p (t+1)^a,$$
where \( a = \frac{N-(2N-1)p}{(N-2)p} < -1 \). Therefore we have
\[
|w(t)| \leq \int_0^t \int_0^t |w''(u)| \, du \, ds \\
\leq C\|\nabla \Delta u\|_p \int_0^t \int_0^t (u+1)^a \, du \, ds \\
\leq C\|\nabla \Delta u\|_p \max \{ (t+1)^{a+2}, t+1 \} \leq C\|\nabla \Delta u\|_p (t+1).
\]
Thus we obtain (23) for \( m = 3 \). Next we assume that (23) holds for \( m = 2\ell + 1 \).
And we shall show that (23) also holds for \( m = 2(\ell + 1) + 1 \). For a radial function \( u \in C_c^{2\ell+3} \), set \( v := \Delta u \in C_c^{2\ell+1} \). Applying (23) for \( v \), we have
\[
|v(r)| \leq C\|\nabla \Delta u\|_{L^p(B_R)r^{2-N}}.
\]
By (12) and (13), we have
\[
|w''(t)| \leq C\|\nabla \Delta u\|_{L^p(B_R)} (t+1)^b,
\]
where \( b = \frac{-2N}{N-2} < -1 \). Therefore we have
\[
|w(t)| \leq \int_0^t \int_0^t |w''(u)| \, du \, ds \\
\leq C\|\nabla \Delta u\|_p \int_0^t \int_0^t (u+1)^b \, du \, ds \\
\leq C\|\nabla \Delta u\|_p \max \{ (t+1)^{b+2}, t+1 \} \leq C\|\nabla \Delta u\|_p (t+1).
\]
Therefore we observe that (23) holds for \( m = 2(\ell + 1) + 1 \).

In the even case, the strategy of the proof is same as the odd case. In order to obtain (23) for \( m = 4 \), we use the pointwise estimate in Lemma 3 for radial function \( v := \Delta u \in C_c^3 \). We omit the proof. \(\Box\)

4. A limiting procedure for the Poincaré inequality

In this section, we apply our limiting procedure to the Poincaré inequality:
\[
\lambda(\Omega) \int_{\Omega} |u|^p \, dx \leq \int_{\Omega} |\nabla u|^p \, dx \quad (u \in C_c^1(\Omega), 1 \leq p < \infty).
\]
(24)
The Poincaré inequality (24) does not have a critical exponent with respect to \( p \) like the Hardy type inequalities. However the optimal constant \( \lambda(\Omega) \) goes to \( \infty \).
and \( \int_\Omega |u|^p \, dx \) goes to 0, as \( |\Omega| \searrow 0 \). This can be regarded as a kind of limiting situation. Recall that
\[
\lambda(\Omega) \geq \left( \frac{N}{p} |B_1| \right)^p |\Omega|^{-\frac{N}{p}} \tag{25}
\]
see e.g. [22]. By using this growth order of \( \lambda(\Omega) \) as \( |\Omega| \searrow 0 \) and our limiting procedure, we shall consider a **limit** for the Poincaré inequality as \( |\Omega| \searrow 0 \).

**Theorem 3.** Let \( 1 \leq p < \frac{N^2}{N-1} \). The following non-sharp classical Hardy inequality (26) can be derived by a limiting procedure for the Poincaré inequality (24) as \( |\Omega| \searrow 0 \).
\[
C \int_{B_1} \frac{|u|^p}{|x|^p} \, dx \leq \int_{B_1} |\nabla u|^p \, dx \quad (u \in C^1_c(B_1)). \tag{26}
\]
Here the constant \( C = C(\beta, p, N) > 0 \) is independent of \( u \) and \( \beta > 0 \) satisfies
\[
\begin{cases}
\beta < \frac{p}{N} & \text{if } 1 \leq p \leq N, \\
\beta < \frac{p}{N} + N - p & \text{if } N < p < \frac{N^2}{N-1}.
\end{cases}
\]

**Remark 2.** If \( p \geq \frac{N^2}{N-1} \), then we can not obtain any information which is better than the Poincaré inequality (24) by out limiting procedure as \( |\Omega| \searrow 0 \), since \( \beta = 0 \) in that case.

**Proof.** From Lemma [2] it is enough to show the inequality (26) for any radial functions \( u \in C^1_c(B_1) \). Let \( 1 \leq p < N \) and \( \{\phi_k\}_{k \in \mathbb{Z}} \subset C^\infty_c(\mathbb{R}^N \setminus \{0\}) \) be radial functions which satisfy
\[
(i) \quad \sum_{k=-\infty}^{+\infty} \phi_k(x)^p = 1, \quad 0 \leq \phi_k(x) \leq 1 \quad (\forall x \in \mathbb{R}^N \setminus \{0\}),
\]
\[
(ii) \quad \text{supp} \phi_k \subset B_{1/k} \setminus B_{1/(k+2)}.
\]
Set \( u_k = u \phi_k \) and \( A_k = \text{supp} \, u_k \subset B_1 \cap (B_{1/k} \setminus B_{1/(k+2)}) \). Applying the Poincaré inequality (24) for \( u_k \) and (25), we have
\[
Ck^p(k + 2)^\frac{N}{p} \int_{A_k} |u_k|^p \, dx \leq \int_{A_k} |\nabla u_k|^p \, dx.
\]
Since \( k \leq \frac{1}{|n|} \leq k + 2 \) for \( x \in A_k \),

\[
C \int_{A_k} \frac{|u_k|^p}{|x|^\beta} \, dx \leq b_k \int_{A_k} |\nabla u_k|^p \, dx
\]  \hfill (27)

for \( k \in \mathbb{Z} \), and \( \beta > 2N \), where \( b_k \) is given by

\[
b_k = \begin{cases} 
  k^{-p}(k+2)^{\beta-N} & \text{if } k \geq 1, \\
  1 & \text{if } k = 0, -1, \\
  0 & \text{if } k \leq -2.
\end{cases}
\]

Summing both sides on (27), we have

\[
C \sum_{k \in \mathbb{Z}} \int_{B_1} \frac{|u_k|^p}{|x|^\beta} \, dx \leq \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\nabla (u \phi_k)|^N \, dx.
\]

By applying Lemma 1 and calculating in the similar way to it in §3.1, we see that for \( \beta < \frac{p}{N} \) the desired inequality (26) can be obtained. In the case where \( N \leq p \), the desired inequality (26) can be obtained. In the case where \( N \leq p < \frac{N^2}{N-1} \), the proof is similar. Therefore we omit the proof in that case. \( \square \)

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