The Murnaghan-Nakayama rule and some virtual $S_n$ characters

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Abstract. We construct certain virtual characters for the symmetric groups, then compute a formula which calculates the values of these virtual characters.

1 Introduction

Partitions are denoted here by $\lambda, \mu, \nu$, etc., and a partition is identified with its Young diagram. As usual, $\lambda'$ denotes the conjugate partition of $\lambda$. We write $\lambda \vdash n$ if $\lambda$ is a partition of $n$. When the characteristic of the base field is zero, the partitions $\lambda \vdash n$ are in a one-to-one correspondence with the irreducible $S_n$-characters, denoted $\chi^\lambda$, [1], [2], [4]. An integer-combination of irreducible characters is called a virtual character.

We construct certain virtual characters $\psi_{\nu,n}$ of the symmetric group $S_n$. Here $\nu$ is a partition of $k$ where $k$ is much smaller than $n$. These virtual characters are alternating sums of certain irreducible $S_n$-characters. The main result here is, that the values $\psi_{\nu,n}(\mu)$ of these virtual characters on the partitions $\mu \vdash n$ are given by one character formula. This is Theorem 3.1 below. This formula shows that the character tables of the symmetric groups satisfy many relations and identities.

2 The virtual $S_n$-characters $\psi_{\nu,n}$

Let $\nu = (\nu_1, \nu_2, \ldots) \vdash k$, and $n \geq 2k + 2$. First form $\nu^{(1)} = (\nu_1 + n - k, \nu_2, \nu_3, \ldots)$. This is the diagram $\nu$ with $n - k$ additional boxes attached to its first row. Now pull these added $n - k$ boxes down and left around the diagram $\nu$ as follows. Think of these $n - k$ boxes as
Thus Definition 2.3.

2.1 The general construction

Let \( n \geq k \), \( \nu \vdash k \), and \( \eta \vdash n \). We shall assume that \( n \geq 2k + 2 \). Assume \( \nu \leq \eta \) and that \( S \) is a part of the rim of \( \eta \) such that \( \eta \setminus S = \nu \), then we write \( \eta = \nu \ast S \). Let \( h(S) \) denote the height of \( S \).

For example, verify that \( \nu^{(2)} = (n - k + \nu_2 - 1, \nu_1 + 1, \nu_3, \nu_4, \ldots) \). Note that since \( n \geq 2k + 2 \) and \( \nu \vdash k \), it follows that \( n - k + \nu_2 - 1 \geq \nu_1 + 1 \), and therefore \( \nu^{(2)} \) is indeed a partition.

Definition 2.1. (The virtual character \( \psi_{\nu,n} \)) Given \( \nu \vdash k \) and \( n \geq k \) (we usually require that \( n \geq 2k + 2 \)), with the partitions of \( n \) obtained by going around \( \nu \), these partitions are ordered as first, second, third, etc. We define \( \psi_{\nu,n} \) to be the alternating sum of the corresponding irreducible \( S_n \) characters.

Example 2.2. 1.

Going around the empty diagram \( \nu = \emptyset \) we get the following sequence of diagrams:

\[
(n) \rightarrow (n - 1, 1) \rightarrow (n - 2, 1^2) \rightarrow \cdots \rightarrow (2, 1^{n-2}) \rightarrow (1^n).
\]

Thus

\[
\psi_{\emptyset,n} = \sum_{j=0}^{n-1} (-1)^j \chi^{(n-j,1^j)}.
\]

2. Going around \( \nu = (1) \). Here \( k = 1 \), so \( n \geq 4 \). Get the diagrams

\[
(n) \rightarrow (n - 2, 2) \rightarrow (n - 3, 2, 1) \rightarrow (n - 4, 2, 1^2) \rightarrow \cdots \rightarrow (3, 2, 1^{n-5}) \rightarrow (2, 2, 1^{n-4}) \rightarrow (1^n).
\]

Therefore

\[
\psi_{(1),n} = \chi^{(n)} + \sum_{j=0}^{n-4} (-1)^{j+1} \chi^{(n-2-j,2,1^j)} + (-1)^n \chi^{(1^n)}.
\]

3. Going around (2). Here \( k = 2 \) so \( n \geq 6 \). Get the partitions

\[
(n) \rightarrow (n - 3, 3) \rightarrow (n - 4, 3, 1) \rightarrow (n - 5, 3, 1^2) \rightarrow (n - 6, 3, 1^3) \rightarrow \cdots \rightarrow (4, 3, 1^{n-7}) \rightarrow \rightarrow (3, 3, 1^{n-6}) \rightarrow (2^2, 1^{n-4}) \rightarrow (2, 1^{n-2}).
\]

Thus

\[
\psi_{(2),n} = \chi^{(n)} + \sum_{j=0}^{n-6} (-1)^{j+1} \chi^{(n-3-j,3,1^j)} + (-1)^{n-1} \chi^{(2,2,1^{n-4})} + (-1)^n \chi^{(2,1^{n-2})}.
\]

2.1 The general construction

Definition 2.3. 1. Let \( n \geq k \), \( \nu \vdash k \), and \( \eta \vdash n \). We shall assume that \( n \geq 2k + 2 \). Assume \( \nu \leq \eta \) and that \( S \) is a part of the rim of \( \eta \) such that \( \eta \setminus S = \nu \), then we write \( \eta = \nu \ast S \). Let \( h(S) \) denote the height of \( S \).
2. Given such $\eta = \nu \ast S$, we say that "$S$ covers $\nu$" if $\nu_1' \leq h(S)$. Otherwise $S$ covers only an upper part of $\nu$.

3. Given $\nu \vdash k$, we start by constructing $\nu \ast S_1$. Here $S_1$ is the one row of length $n - k$, added to the first row of $\nu$. Clearly, $h(S_1) = 1$. Continue and construct the sequence of partitions $\nu \ast S_1, \nu \ast S_2, \ldots, \nu \ast S_{n-k}$. Lemma 2.5 shows that $h(S_j) = j$ for $1 \leq j \leq n-k$, and the process stops at $j = n-k$, namely after $n-k$ steps.

4. Define the virtual $S_n$ character $\psi_{\nu,n}$ as follows:

$$\psi_{\nu,n} := \sum_{j=1}^{n-k} (-1)^{j+1} \chi^\nu S_j.$$ (2)

5. We say that $\nu \ast S$ "has a tail" if $\nu \ast S = \mu = (\mu_1, \mu_2, \ldots)$ where $\mu_1 \geq \nu_1 + 1$.

The following unique-decomposition lemma is crucial here.

**Lemma 2.4.** Let $n \geq 2k + 2$. For $i = 1, 2$ let $\nu^{(i)} \vdash k$, and let $S^{(i)}$ be a rim of $\nu^{(i)} \ast S^{(i)}$ of length $n - k$.

If $\nu^{(1)} \ast S^{(1)} = \nu^{(2)} \ast S^{(2)}$ then $\nu^{(1)} = \nu^{(2)}$ and $S^{(1)} = S^{(2)}$.

Note that Example 2.6 below shows that the condition $n \geq 2k + 2$ is necessary.

**Proof.** Denote $\eta = \nu^{(1)} \ast S^{(1)} = \nu^{(2)} \ast S^{(2)}$. There are three cases to consider.

Case 1: $S^{(1)}$ covers an upper part of $\nu^{(1)}$ (and has a North-East tail).

case 2: The conjugate of case 1.

case 3: $S^{(1)}$ completely covers $\nu^{(1)}$ (and might have a North-East and/or South-East tails).

Case 1: Here the rim of $\nu^{(1)} \ast S^{(1)}$ contains $S^{(1)}$ and possibly an additional part which is a part of $\nu^{(1)}$. Assume $\eta = \nu^{(1)} \ast S^{(1)} = \nu^{(2)} \ast S^{(2)}$ with $\nu^{(1)} \neq \nu^{(2)}$. Then $\nu^{(2)}$ contains a cell of $S^{(1)}$. This cell splits $S^{(1)}$ into two parts: the North-East part of $S^{(1)}$ and the South-East part (which contains the South-East part of $S^{(1)}$). Let $\tilde{S}$ denote the South-East part of $S^{(1)}$ together with the lower part of the rim of $\nu^{(2)}$. We need to show that neither the North-East part of $S^{(1)}$, nor $\tilde{S}$ can be $S^{(2)}$.

The North-East part is properly contained in $S^{(1)}$ hence has length strictly less than $|S^{(1)}| = n - k$, thus this part cannot be $S^{(2)}$.

Since we are in case 1, the part $\tilde{S}$ is a part of $\nu^{(2)}$. But $\nu^{(2)} \vdash k$, hence $\tilde{S}$ is of length $|\tilde{S}| \leq k < n - k$, hence this part cannot be $S^{(2)}$ either.

So case 1 is impossible. By conjugation, case 2 is also impossible.

Case 3. The argument here is similar: A cell of $\nu^{(2)}$ on $S^{(1)}$ splits the rim of $\nu^{(1)} \ast S^{(1)}$ (namely $S^{(1)}$) into two parts, each of length strictly less than $n - k$, hence neither can be $S^{(2)}$. Thus case 3 is also impossible, and the proof follows. \[\square\]
Lemma 2.5. Let $\nu \vdash k$ and let $n \geq 2k + 2$. As in Definition 2.3, construct the sequence of partitions $\nu * S_1, \nu * S_2, \ldots, \nu * S_{n-k}$. Then $h(S_j) = j$ for $1 \leq j \leq n - k$.

Here also, Example 2.6 shows that the condition $n \geq 2k + 2$ is necessary.

Proof. Analogue to the height $h(S)$, we also define the width $w(S)$ (which is the height of the conjugate rim $S'$). By projecting $S$ on the axes (see also (1.7) in [2]), it follows that

$$|S| = h(S) + w(S) - 1. \quad (3)$$

Note that $n - k \geq \nu_1 + \nu'_1 + 1$. Indeed, $k \geq \nu_1 + \nu'_1 - 1$, so $n - k \geq k + 2 \geq \nu_1 + \nu'_1 + 1$.

As in Definition 2.3, construct the sequence $\nu * S_1, \nu * S_2, \ldots$. For the first $\nu'_1 + 1$ steps, by construction and induction, $h(S_j) = j, 1 \leq j \leq \nu'_1 + 1$ (since $|S| = n - k$, while the number of boxes in $S$ except those in the first row of $\nu * S$ is $\leq \nu_1 + \nu'_1 \leq k$). After $\nu'_1 + 1$ steps, $\nu$ is ”covered”, with $\nu_1 + \nu'_1 + 1$ out of the $n - k$ cells are covering $\nu$. Since $n - k \geq \nu_1 + \nu'_1 + 1$, at that stage there still is a ”tail” of added boxes in the first (i.e. top) row of $\nu * S_j, j = \nu'_1 + 1$.

For the remaining steps we consider the conjugate construction. Then, applying (3) together with the above argument, the proof follows.

Example 2.6. Counter examples when $n \not\geq 2k + 2$.

1. Let $k = 3, n = 6, \eta = (3, 3), \nu^{(1)} = (3), \nu^{(2)} = (2, 1)$. Let $S^{(i)}, i = 1, 2, s$ satisfy

$$(3, 3) = \nu^{(1)} * S^{(1)} = \nu^{(2)} * S^{(2)}.$$

Then clearly $S^{(1)} \neq S^{(2)}$, as well as $\nu^{(1)} \neq \nu^{(2)}$.

2. Again let $k = 3, n = 6$. Let $\nu = (3)$ and construct the sequence of partitions $\nu * S_1, \nu * S_2, \nu * S_3 \ldots$ Then $h(S_2) = 1 \neq 2$.

We proceed with the general case.

Corollary 2.7. Let $\nu \vdash k$, and let $n \geq 2k + 2$, then

$$\psi_{\nu,n} = \sum_{S, \nu * S \vdash n} (-1)^{h(S) + 1} \chi^{\nu * S} = \sum_{j=1}^{n-k} (-1)^{h(S_j) + 1} \chi^{\nu * S_j}.$$

Proof. This follows from Equation (2) (which defines $\psi_{\nu,n}$) and from Lemma 2.5.
3  A formula for $\psi_{\nu,n}(\mu)$

Our aim is to prove the following formula.

**Theorem 3.1.** Let $\nu = (\nu_1, \nu_2, \ldots) \vdash k$, $n \geq 2k + 2$. Let $\mu = (\mu_1, \mu_2, \ldots) \vdash n$, and denote $\bar{\mu} = (\mu_2, \mu_3, \ldots)$, so $\mu_1 = n - k$ if and only if $\bar{\mu} \vdash k$. Then

$$
\psi_{\nu,n}(\mu) = \begin{cases} 
\chi^\nu(\bar{\mu}) \cdot (n - k) & \text{if } \mu_1 = n - k \\
0 & \text{if } \mu_1 \neq n - k
\end{cases}
$$

(4)

The proof is given below.

Let $n \geq 2k + 2$ and let $\nu \vdash k$, so $(n - k, \nu)$ is a partition of $n$. Let $\lambda \vdash n$ and assume $\chi^\lambda_{(n-k,\nu)} \neq 0$, then, by the Murnaghan-Nakayama (M-N) rule, $\lambda$ can be written, probably in several ways, as $\lambda = \rho \ast S$ where $\rho \vdash k$, $S$ is part of the rim of $\lambda$ and $\vert S \vert = n - k$. By Lemma 2.4 this decomposition, with $\vert \rho \vert = k$, is unique, hence we can write $\lambda \leftrightarrow (\rho, S)$. Again by the M-N rule, with $\lambda = \rho \ast S$,

$$
\chi^\lambda_{(n-k,\nu)} = (-1)^{h(S)+1} \chi^{\rho}_\nu = (-1)^{h(S)+1} \chi^{\rho}\chi^\nu.
$$

(5)

Lemma 2.4 allows us to prove the following formula.

**Proposition 3.2.** Let $\nu \vdash k$ and let $n \geq 2k + 2$. Then

$$
\sum_{\lambda \vdash n} \chi^\lambda_{(n-k,\nu)} \chi^\lambda = \sum_{\rho \vdash k} \chi^\nu \cdot \psi_{\rho,n}.
$$

Proof. We just saw that

$$
\{\lambda \vdash n\} = \{\rho \ast S \vdash n \mid \rho \vdash k, \vert S \vert = n - k\} \cup \{\lambda \vdash n \mid \chi^\lambda_{(n-k,\nu)} = 0\},
$$

and by Lemma 2.4 we have the bijection

$$
\{\rho \ast S \vdash n \mid \rho \vdash k, \vert S \vert = n - k\} \leftrightarrow \{(\rho, S) \mid \rho \vdash k, \vert S \vert = n - k \text{ and } \rho \ast S \vdash n\}.
$$

We denote

$$
A_{k,n} = \{(\rho, S) \mid \rho \vdash k, \vert S \vert = n - k \text{ and } \rho \ast S \vdash n\}.
$$

By (5)

$$
\sum_{\lambda \vdash n} \chi^\lambda_{(n-k,\nu)} \chi^\lambda = \sum_{(\rho, S) \in A_{k,n}} (-1)^{h(S)+1} \chi^{\rho}\chi^\nu = \sum_{\rho \vdash k} \sum_{S \vdash n} (-1)^{h(S)+1} \chi^{\rho}_{\nu} \cdot \chi^{\rho \ast S} = \sum_{\rho \vdash k} \chi^{\rho}_{\nu} \sum_{S \vdash n} (-1)^{h(S)+1} \chi^{\rho \ast S} = \sum_{\rho \vdash k} \chi^{\rho}_{\nu} \cdot \psi_{\rho,n}.
$$

The last equality applied corollary 2.7. 

Remark 3.3.
Let \( d = |\{Par(k)\}| \). Note that in matrix form, Proposition 3.2 can be written as follows:

\[
[\chi_{\rho}^\rho] [\psi_{\rho,n}] = \left[ \sum_{\lambda \vdash n-\nu} \chi_{\nu}^{\lambda} \cdot \chi_{\mu}^{\lambda} \right].
\]

Here \([\chi_{\rho}^\rho]\) is the \(d \times d\) character table of \(S_k\), and \([\psi_{\rho,n}]\) is a column of height \(d\). Of course, the locations of the entries of both \([\chi_{\rho}^\rho]\) and \([\psi_{\rho,n}]\) depend on how we order \(Par(k)\). Applying (6) on \(\mu \vdash n\) we get

\[
[\chi_{\rho}^\rho] [\psi_{\rho,n}(\mu)] = \left[ \sum_{\lambda \vdash n-\nu} \chi_{\nu}^{\lambda} \cdot \chi_{\mu}^{\lambda} \right]
\]

(7)

Recall the classical column-orthogonality-relations for the \(S_n\) characters.

**Theorem 3.4.** (The column-orthogonality-relations)

\[
\sum_{\lambda \vdash n} \chi_{\eta}^{\lambda} \chi_{\mu}^{\lambda} = \begin{cases} |Z_{S_n}(\eta)| & \text{if } \mu = \eta \\ 0 & \text{if } \mu \neq \eta, \end{cases}
\]

(8)

where \(\lambda, \eta, \mu \vdash n\), and \(Z_{S_n}(\eta)\) is the centralizer of \(\eta\) in \(S_n\) (i.e, the centralizer of \(\pi \in S_n\) with cycle structure \(\eta\)). Let \(K = K^{(n)} = [\chi_{\eta}^{\theta}]\) denote the character table of \(S_n\), then (8) can be written as

\[
KK^T = \text{diag}(|Z_{S_n}(\eta)|, |\eta \vdash n|).
\]

(9)

Then numbers \(|Z_{S_n}(\eta)|\) are calculated through the following well known formula.

**Theorem 3.5.** Let \(\eta \vdash n\), \(\eta = (1^{m_1}, 2^{m_2}, \ldots)\), and let \(Z_{S_n}(\eta)\) denote the centralizer in \(S_n\) of \(\sigma \in S_n\) where \(\eta\) is the cycle structure of \(\sigma\). Then

\[
|Z_{S_n}(\sigma)| = |Z_{S_n}(\eta)| = \prod_i (i^{m_i} \cdot m_i!).
\]

**Remark 3.6.** Let \(\nu \vdash k\), \(n \geq 2k + 2\). Then \((n-k, \nu)\) is a partition of \(n\), and

\[
|Z_{S_n}(n-k, \nu)| = |Z_{S_n}(\nu)| \cdot (n-k).
\]

This follows from Theorem 3.5 since \(n-k\) is strictly larger than any component of \(\nu\). Together with (8) this implies

**Corollary 3.7.** Let \(\nu \vdash k\), \(n \geq 2k + 2\), \(\mu = (\mu_1, \mu_2, \ldots) \vdash n\) and let \(\bar{\mu} = (\mu_2, \mu_3, \ldots)\). Then

\[
\sum_{\lambda \vdash n} \chi_{(n-k,\nu)}^{\lambda} \chi_{\mu}^{\lambda} = \begin{cases} |Z_{S_k}(\nu)| \cdot (n-k) & \text{if } \bar{\mu} = \nu \\ 0 & \text{if } \mu \neq \eta. \end{cases}
\]

(10)
3.1 The proof of Theorem 3.1

Apply (7) and Corollary 3.7. With \( \rho \) and \( \nu \) denoting partitions of \( k \) and \( \mu \) partitions of \( n \) we have

\[
[x_\nu^\rho] [\psi_{\rho,n}(\mu)] = \sum_{\lambda \vdash n-k, \nu} \chi_{(n-k,\nu)}^\lambda \cdot \chi_\mu^\lambda = \begin{cases} |Z_{S_k}(\nu)| \cdot (n-k) & \text{if } \bar{\mu} = \nu \\ 0 & \text{if } \bar{\mu} \neq \nu. \end{cases}
\]

and by (9)

\[
KM = KK^T = D = \text{diag}(|Z_{S_k}(\bar{\mu}^{(1)})|,...,|Z_{S_k}(\bar{\mu}^{(d)})|).
\]

Thus we have \( KM = KK^T \cdot (n-k) \).

Note that \( K \) is the character table of \( S_k \), hence is invertible. Left cancelation of \( K \) implies that \( M = K^T \cdot (n-k) \), so \( M = (\text{the character table of } S_k) \cdot (n-k) \). This completes the proof of Theorem 3.1.
3.2 Applications

Example 3.8. 1. In Example 2.2.1 we saw that \( \psi_{\emptyset, n} = \sum_{j=0}^{n-1} (-1)^j \chi^{(n-j,1^j)} \). Theorem 3.1 with \( k = 0 \) then implies that
\[
\psi_{\emptyset, n}(\mu) = \sum_{j=0}^{n-1} (-1)^j \chi^{(n-j,1^j)}(\mu) = \begin{cases} n & \text{if } \mu = (n) \\ 0 & \text{if } \mu \neq (n) \end{cases} \tag{13}
\]

2. Similarly, by Example 2.2.2, \( \psi_{(1), n} = \chi^{(n)} + \sum_{j=0}^{n-4} (-1)^{j+1} \chi^{(n-2-j,2,1^j)} + (-1)^n \chi^{(1^n)} \), and we get
\[
\psi_{(1), n}(\mu) = \chi^{(n)}(\mu) + \sum_{j=0}^{n-4} (-1)^{j+1} \chi^{(n-2-j,2,1^j)}(\mu) + (-1)^n \chi^{(1^n)}(\mu) = \begin{cases} n-1 & \text{if } \mu = (n-1,1) \\ 0 & \text{if } \mu \neq (n-1,1) \end{cases} \tag{14}
\]

Clearly, there are infinitely many identities that can be deduced this way.

3. Consider the first column \( \{f^\lambda \mid \lambda \vdash n\} \) in the character table of \( S_n \). Let \( n \geq 2k + 2 \), fix some \( \nu \vdash k \), then construct the sequence of partitions of \( n \): \( \nu \ast S_1, \nu \ast S_2, \ldots \). Finally, form the corresponding alternating sum, then always
\[
\sum_{j=1}^{n-k} (-1)^j f^{\nu \ast S_j} = 0. \tag{15}
\]

This follows from Theorem 3.1 since this corresponds to \( \mu = (1^n) \), so \( \mu_1 = 1 \neq n-k \). When \( k = 0 \) this is the well known identity
\[
\sum_{j=1}^{n} (-1)^j \binom{n}{j} = 0, \tag{16}
\]
see (13). Thus, (15) can be seen as a generalization of (16).

Remark 3.9. In [3] a formula is proved for the values
\[
\sum_{i \geq 0} \chi^{(n-i,1^i)}(\mu), \quad \mu \vdash n.
\]

Of course, here we deduced a formula for the values for the corresponding alternating sum
\[
\sum_{i \geq 0} (-1)^i \chi^{(n-i,1^i)}(\mu), \quad \mu \vdash n.
\]
Adding, we get a formula for
\[ \sum_{i \text{ even}} \chi^{(n-i,1^1)}(\mu), \quad \mu \vdash n, \]
hence also for
\[ \sum_{i \text{ odd}} \chi^{(n-i,1^1)}(\mu), \quad \mu \vdash n. \]
We leave the details for the reader.

4 Final remarks

Remark 4.1. Let \( n \geq 4 \) and let \( A_n \) be the set of partitions obtained by walking around \( \nu = (1) \):
\[ A_n = \{(n), (1^n)\} \cup \{(r, 2, 1^{n-2-r}) \mid 2 \leq r \leq n-2\}. \]
Then
\[ \sum_{\lambda \in A_n} f^{\lambda} = (n - 4) \cdot 2^{n-2} + 4. \quad (17) \]

Proof. Let \( \lambda = (r, 2, 1^{n-2-r}) \), then
\[ f^{\lambda} = \frac{n!}{(r-2)!(n-2-r)!r(n-r)(n-1)} = \frac{n(n-2)(n-3)}{r(n-r)} \frac{(n-4)}{(r-2)}. \]

Thus, omitting \( \lambda \in \{(n), (1^n)\} \), we need to show that
\[ \sum_{r=2}^{n-4} \frac{n(n-2)(n-3)}{r(n-r)} \frac{(n-4)}{(r-2)} = (n - 4) \cdot 2^{n-2} + 2. \quad (18) \]

This can be proved by the Zeilberger Algorithm [5], and is implemented in Maple (function SumTools[Hypergeometric][Zeilberger]).

Here is a direct proof.

Zeilberger. Note that
\[ \frac{1}{r(n-r)} = \frac{1}{n} \left( \frac{1}{r} + \frac{1}{n-r} \right), \]
so the sum in (18) equals
\[ (n - 2)(n - 3) \sum_{r=2}^{n-2} \frac{1}{r} \frac{(n-4)}{(r-2)} + (n - 2)(n - 3) \sum_{r=2}^{n-2} \frac{1}{n-r} \frac{(n-4)}{(r-2)}. \]
By symmetry these two summands are equal, so this equals
\[
2(n - 2)(n - 3) \sum_{r=2}^{n-2} \frac{1}{r} \binom{n - 4}{r - 2} = 
\]
\[
2(n - 2)(n - 3) \sum_{r=2}^{n-2} \left( \int_0^1 x^{r-1} dx \right) \binom{n - 4}{r - 2} = 
\]
\[
2(n - 2)(n - 3) \sum_{r=2}^{n-2} \left( \int_0^1 x^{r-1} \left( \frac{n - 4}{r - 2} \right) \right) = 
\]
\[
2(n - 2)(n - 3) \sum_{r=0}^{n-4} \left( \int_0^1 x^{r+1} \binom{n - 4}{r} \right) = 
\]
\[
2(n - 2)(n - 3) \int_0^1 x(1 + x)^{n-4} dx = 
\]
\[
2(n - 2)(n - 3) \left( \int_0^1 (1 + x)^{n-3} dx - \int_0^1 (1 + x)^{n-4} dx \right) = 
\]
\[
(n - 4)2^{n-2} + 2. 
\]

Here is a second, combinatorial proof [Zeilberger].

Let \( f(n) \) be the number of standard Young tableaux of shape \((r, 2, 1^{n-2-r})\) for some \(2 \leq r \leq n - 2\), and look at the location of \( n \).

Case 1: It is at the rightmost cell of the top row. Deleting it gives something counted by \( f(n - 1) \).

Case 2. The conjugate case, also yielding \( f(n - 1) \).

Case 3: \( n \) is in the \((2, 2)\) cell. Deleting it gives a tableau of strict hook shape (namely, not \((n - 1)\) nor \((1^{n-1})\)). Clearly, the number of these tableaux is \(2^{n-1} - 2\).

So we have the recurrence \( f(n) = 2f(n - 1) + 2^{n-1} - 2\). Since \( f(4) = 2 \), it follows by induction that \( f(n) = (n - 4)2^{n-2} + 2 \).

### 4.1 The polynomials \( p_{\nu, n}(t) \)

Recall Equation (15), replace minus 1 by \( t \) and get the polynomials

\[
p_{\nu, n}(t) = \sum_{j=1}^{n-k} f^{\nu} S_j \cdot t^j. 
\]

In the case \( \nu \) is empty we get \( p_{\emptyset, n}(t) = (t+1)^{n-1} \). Denote \( p_{(1), n}(t) = p_n(t) \) in the case \( \nu = (1) \).

By direct computations we got the following polynomials, for \( 1 \leq n \leq 9 \).
\[ p_4(t) = (t + 1)^2 \]
\[ p_5(t) = (t + 1)(t^2 + 4t + 1) \]
\[ p_6(t) = (t + 1)^2(t^2 + 7t + 1) \]
\[ p_7(t) = (t + 1)(t^4 + 13t^3 + 22t^2 + 13t + 1) \]
\[ p_8(t) = (t + 1)^2(t^4 + 18t^3 + 27t^2 + 18t + 1) \]
\[ p_9(t) = (t + 1)(t^6 + 26t^5 + 79t^4 + 110t^3 + 79t^2 + 26t + 1) \]
\[ p_{10}(t) = (t + 1)^2(t^6 + 33t^5 + 93t^4 + 131t^3 + 93t^2 + 33t + 1) \]

By (13) \( p_n(t) \) is divisible by \( t + 1 \) for all \( n \).

**Conjecture 4.2.** We conjecture that when \( n \) is even, the highest power of \( t + 1 \) which divides \( p_n(t) \) is 2, while when \( n \) is odd, the highest power on \( t + 1 \) which divides \( p_n(t) \) is 1.

One also conjectures positiveness and unimodality on the coefficients of these polynomials.

Additional conjecture is as follows. Denote \( p_{2k+1}(t) = (t + 1) \cdot q_{2k+1}(t) \), then \( q_{2k+1}(-1) = -2 \).

**References**

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