OPTIMAL CONTROL OF A RATE-INDEPENDENT SYSTEM
CONstrained TO PARAMETRIZED BALANCED VISCOSITY SOLUTIONS

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Abstract. We analyze an optimal control problem governed by a rate-independent system in an abstract infinite-dimensional setting. The rate-independent system is characterized by a nonconvex stored energy functional, which depends on time via a time-dependent external loading, and by a convex dissipation potential, which is assumed to be bounded and positively homogeneous of degree one.

The optimal control problem uses the external load as control variable and is constrained to normalized parametrized balanced viscosity solutions (BV solutions) of the rate-independent system. Solutions of this type appear as vanishing viscosity limits of viscously regularized versions of the original rate-independent system. Since BV solutions in general are not unique, as a main ingredient for the existence of optimal solutions we prove the compactness of solution sets for BV solutions.

1. Introduction

In this paper, we focus on the optimal control of a rate-independent system. This system is given in terms of a state variable \( z : [0, T] \to \mathcal{Z} \), a time-dependent external load \( \ell \), a stored energy functional \( \mathcal{E} \) depending on \( \ell \) and \( z \), and a dissipation potential \( \mathcal{R} : \mathcal{Z} \to [0, \infty) \), which captures the dissipation due to internal friction. To be more precise, we assume that the state space \( \mathcal{Z} \) is a separable Hilbert space which fulfills the embedding \( \mathcal{Z} \subset \mathcal{V} \subset \mathcal{X} \) for another separable Hilbert space \( \mathcal{V} \) and a Banach space \( \mathcal{X} \) and choose \( \ell \in H^1(0, T; \mathcal{V}^*) \). We are working with a semilinear model,
i.e., we assume that there are a linear operator $A \in \text{Lin}(\mathbb{Z}, \mathbb{Z}^*)$ and a nonlinearity $\mathcal{F} : \mathbb{Z} \to [0, \infty)$ such that $\mathcal{E} : [0, T] \times \mathbb{Z} \to \mathbb{R}$ is given by

$$
\mathcal{E}(t, z) := \frac{1}{2} \langle Az, z \rangle_{\mathbb{Z}^*, \mathbb{Z}} + \mathcal{F}(z) - \langle \ell(t), z \rangle_{\mathbb{V}^*, \mathbb{V}} = \mathcal{J}(z) - \langle \ell(t), z \rangle_{\mathbb{V}^*, \mathbb{V}},
$$

where

$$
\mathcal{J} : \mathbb{Z} \to \mathbb{R}, \quad \mathcal{J}(z) := \frac{1}{2} \langle Az, z \rangle_{\mathbb{Z}^*, \mathbb{Z}} + \mathcal{F}(z)
$$

depends solely on the state $z$. $\mathcal{F}$ is supposed to be nonconvex and of lower order with respect to $A$. The precise assumptions on $A$, $\mathcal{F}$ and $\ell$ can be found in section 2. Rate-independence means that the system is invariant w.r.t. time rescaling in the sense that, given a time rescaling, the solutions of the rescaled system are exactly the rescaled solutions of the original system. In order to obtain rate-independence, the dissipation potential $\mathcal{R}$ is assumed not only to be continuous and convex, but also positively homogeneous of degree one. In this paper, we are dealing with a bounded dissipation potential, meaning that we also assume that there are constants $c, C > 0$ such that

$$
\text{for all } z \in \mathcal{X} : \quad c\|z\|_{\mathcal{X}} \leq \mathcal{R}(z) \leq C\|z\|_{\mathcal{X}}. \tag{1.1}
$$

With these ingredients, the evolution of the state variable $z$ can be described by means of the doubly nonlinear equation

$$
0 \in \partial \mathcal{R}(\dot{z}(t)) + D_z \mathcal{E}(t, z(t)) \quad \text{for a.a. } t \in [0, T], \tag{1.2}
$$

where $D_z \mathcal{E}$ is the Gâteaux derivative of $\mathcal{E}$ w.r.t. $z$ and $\partial \mathcal{R} : \mathbb{Z} \rightrightarrows \mathbb{Z}^*$ denotes the convex subdifferential of $\mathcal{R}$. The aim of the paper is to show existence of a globally optimal solution of an optimal control problem of the type

$$
\min_{\tilde{z} \in \tilde{M}_{ad}} \left\{ \|\tilde{z} - z_{\text{des}}\| + \alpha\|\ell\|_{H^1(0, T; \mathbb{V}^*)} : \tilde{z} \in \tilde{M}_{ad} \right\}, \tag{1.3}
$$

where the external load $\ell$ is the control variable, $\alpha > 0$ is a fixed Tikhonov parameter, and $z_{\text{des}}$ is a given desired state. We restrain the problem to an admissible set $\tilde{M}_{ad}$ consisting of all solutions of (1.2) in the sense of so-called parametrized BV solutions.

It is well known that rate-independent systems with nonconvex energy $\mathcal{E}$ in general do not admit solutions that are continuous in time. Several solution concepts are available in the literature that allow for discontinuous solutions. We mention here the meanwhile classical global energetic solutions (GES) first proposed in [MT04] and later refined for instance in [MRS16]. Due to a global stability criterion, GES tend to jump as early as possible even though a local stability criterion might predict a different behavior. In contrast to that, BV solutions tend to jump as late as possible. We refer to [MR15] for more details and an overview on further solution concepts. Independently of the chosen solution concept, solutions of rate-independent systems with nonconvex energies in general are not unique. This is a major challenge when it comes to optimal control of such systems.

The literature concerning the optimal control of rate-independent systems with nonconvex energies formulated on infinite dimensional spaces is rather scant. We mention here [Rin08], where the existence of optimal solutions to a variant of the problem (1.3) constrained to global energetic solutions is shown. In [MR09] and [Rin09], the authors proved a reverse approximation property for global energetic solutions via time incremental solutions and used this property to show that global minimizers of optimal control problems governed by GES can be approximated by solutions of special time discrete optimal control problems. To the best of our knowledge, no existence results are available in the literature for optimal control problems constrained to BV solutions.
The basic solution concept in this paper are normalized, p-parametrized balanced viscosity solutions, see Definition 3.1. Thereby, the solutions are represented with respect to an artificial arc length parameter so as to reformulate the viscous system with respect to an artificial arc length parameter for choosing the reparametrization. For our purpose, the reparametrization based on the so-called arc length parameter as in [EM06] or in [MRS16, Definition 4.2]. Let us go into more details concerning the type of solutions and the results of this paper.

The existence of BV solutions can be shown via a vanishing viscosity approach. Namely, the equation (1.2) is approximated by a sequence of equations

\[ 0 \in \partial \mathcal{R}(\hat{z}_\varepsilon(t)) + \varepsilon \nabla \hat{z}_\varepsilon(t) + D_1 \mathcal{E}(t, z_\varepsilon(t)) \] for a.a. \( t \in [0, T] \),

where \( \mathcal{V} \in \text{Lin}(\mathcal{V}, \mathcal{V}^*) \) is an elliptic and symmetric operator. These types of viscous systems have been analyzed in the past (see, e.g., [MRS13]) and are known to have absolutely continuous solutions \( z_\varepsilon \in W^{1,1}(0, T; \mathcal{V}) \). In order to identify the limit as the viscosity \( \varepsilon \) tends to zero, one option is to reformulate the viscous system with respect to an artificial arc length parameter so that the trajectory \( t \mapsto (t, z_\varepsilon(t)) \) is rewritten as \( s \mapsto (\hat{t}_\varepsilon(s), \hat{z}_\varepsilon(s)) \). There are several possibilities for choosing the reparametrization. For our purpose, the reparametrization based on the so-called vanishing viscosity contact potential \( p(\cdot, \cdot) \) is the most appropriate one, [MRS10]. Here, one sets

\[ s_\varepsilon(t) := t + \int_0^t p(\hat{z}_\varepsilon(\tau), -D_1 \mathcal{E}(\tau, z_\varepsilon(\tau)))d\tau \text{ with } p(v, \xi) := \mathcal{R}(v) + \|v\|_\mathcal{V} \text{ dist}_\mathcal{V}(\xi, \partial \mathcal{R}(0)) \] (1.5)

and chooses \( \hat{t}_\varepsilon \) as the inverse function of \( s_\varepsilon \). Thereby, \( \| \cdot \|_\mathcal{V} \) denotes the equivalent norm induced by \( \mathcal{V} \) on \( \mathcal{V} \) and \( \text{dist}_\mathcal{V}(\eta, \partial \mathcal{R}(0)) \) is the distance of an element \( \eta \in \mathcal{V}^* \) to the subdifferential \( \partial \mathcal{R}(0) \) measured by the corresponding norm on the dual space \( \mathcal{V}^* \). Defining \( \hat{z}_\varepsilon = z_\varepsilon \circ \hat{t}_\varepsilon : [0, S_\varepsilon] \to \mathcal{Z} \), it is then possible to pass to the limit for vanishing viscosity (i.e., for \( \varepsilon \to 0 \)) and obtain limits \( S \in [0, \infty) \) of \( S_\varepsilon \), \( \hat{z} \in \text{AC}(0, S; \mathcal{X}) \) of \( \hat{z}_\varepsilon \) and \( \hat{t} \in W^{1,\infty}(0, S; \mathbb{R}) \) of \( \hat{t}_\varepsilon \). Simultaneously passing to the limit in the reparametrized energy dissipation balance associated with (1.4), one also obtains the energy dissipation balance fulfilled by \( \hat{t}, \hat{z} \), which reads

\[ \mathcal{E}(\hat{t}(s), \hat{z}(s)) + \int_0^s \mathcal{R}(\hat{z}(r)) dr + \int_{(0, s) \cap G} \|\hat{z}(r)\|_\mathcal{V} \text{ dist}_\mathcal{V}(-D_1 \mathcal{E}(\hat{t}(r), \hat{z}(s)), \partial \mathcal{R}(0)) dr \]

\[ = \mathcal{E}(0, z_0) - \int_0^s \langle \hat{t}'(r), \hat{z}(r) \rangle dr. \] (1.6)

Here, it is possible to show that \( \hat{z} \in \text{AC}_{\text{loc}}(G; \mathcal{V}) \) is differentiable almost everywhere on the set

\[ G = \{ s \in [0, S] \mid \text{dist}_\mathcal{V}(-D_1 \mathcal{E}(\hat{t}(s), \hat{z}(s)), \partial \mathcal{R}(0)) > 0 \}, \]

so that the second integrand is defined almost everywhere. Normalized, p-parametrized BV solutions then are defined as triples \( (S, \hat{t}, \hat{z}) \) with certain regularities that satisfy the energy dissipation identity (1.0) and that are normalized in the sense of (3.4) in Section 3. One advantage of using the parametrization (1.5) is that limits of solutions \( (\hat{t}_\varepsilon, \hat{z}_\varepsilon)_\varepsilon \) are automatically normalized, a property that we will also exploit in the analysis for the optimal control problem. Since the focus of this paper is on the optimal control problem, we do not include a detailed proof of existence of parametrized BV solutions.

As already mentioned, BV solutions typically are not unique. Hence, for the purpose of optimal control one needs to show the sequential closedness of the graph of the set-valued solution operator and a compactness property. This is the contents of Theorem 3.12. For the proof of Theorem 3.12 the main challenge will be to derive a priori estimates for the driving forces \( D_1 \mathcal{E}(\hat{t}, \hat{z}) \) on the set \( G \). In order to obtain these, we first show that for each parametrized BV solution, there exists a Lagrange parameter \( \lambda : (0, S) \to [0, \infty) \) with \( \lambda(s) = 0 \) on \( (0, S) \setminus G \) such that the inclusion

\[ 0 \in \partial \mathcal{R}(\hat{z}'(s)) + \lambda(s) \nabla \hat{z}'(s) + D_1 \mathcal{E}(\hat{z}(s)) - \ell(\hat{t}(s)) \] (1.8)
is fulfilled almost everywhere on $G$. For each connected component of $G$, we subsequently choose a reparametrization in such a way that the transformed functions are solutions of the system

$$0 \in \partial \mathcal{R}(\dot{z}(t)) + \nabla z(t) + D\mathcal{J}(z(t)) - \ell_* \quad \text{for } t > 0$$  \hspace{1cm} (1.9)

for a constant load $\ell_* \in \mathcal{V}^*$. We then prove existence and a priori estimates for solutions of (1.9) by formally differentiating the inclusion (1.9) w.r.t. $t$ and testing by $\dot{z}$. This is made rigorous relying on a regularization approach. Namely, for $\delta > 0$, we analyze the system

$$0 \in \partial \mathcal{R}_\delta(\dot{z}_\delta(t)) + D\mathcal{J}(z_\delta(t)) - \ell_* \quad \text{for } t > 0$$

with the augmented dissipation potential

$$\mathcal{R}_\delta(z) = \mathcal{R}(z) + \frac{1}{2} \langle Vz, z \rangle_{\mathcal{V}} + \frac{\delta}{2} \langle Az, z \rangle_{\mathcal{Z}}.$$

This can be done relying mainly on ODE-arguments. We finally transfer the a priori estimates thus obtained for (1.9) to the original system (1.8) by means of a change of variable. These essential estimates for parametrized BV solutions then allow us to show compactness of solution sets of the type

$$M_\varrho := \left\{ (S, \hat{t}, \hat{z}) \mid (S, \hat{t}, \hat{z}) \text{ is a parametrized BV solution for } (z_0, \ell) \text{ with } \|z_0\|_\mathcal{Z} + \|\ell\|_{H^1((0,T);\mathcal{V}^*)} \leq \varrho \right\},$$

see Theorem 3.12.

In the final section of the paper, we turn to an optimal control problem governed by (1.2), which is constrained to the admissible set

$$M_{ad} := \left\{ (S, \hat{t}, \hat{z}, \ell) \mid (S, \hat{t}, \hat{z}) \text{ is a parametrized BV solution for } (z_0, \ell) \right\}.$$

We are then in the position to prove an existence result for an optimal control problem of the type (1.10), which now reads

$$\min_{(S, \hat{t}, \hat{z}, \ell) \in M_{ad}} J(S, \hat{t}, \hat{z}, \ell) := j(\hat{z}(S)) + \alpha \|\ell\|_{H^1((0,T);\mathcal{V}^*)}$$

s.t. \hspace{1cm} (1.10)

Here, $\alpha > 0$ is again a fixed Tikhonov parameter and $j : \mathcal{V} \to \mathbb{R}$ is bounded from below and continuous, e.g. $j(z) := \|z - z_{\text{des}}\|_{\mathcal{V}}$ for a desired end state $z_{\text{des}} \in \mathcal{V}$.

**Plan of the paper:** In Section 2 we list basic assumptions and estimates for the energy functional $E$ and the dissipation potential $\mathcal{R}$. In Section 3 we then give a definition and cite an existence result for parametrized BV solutions. We further provide basic properties of solutions, like for example the differential inclusion (1.8). We next derive uniform estimates for the driving forces $D\mathcal{E}$ by analyzing the system (1.9) and transferring the results to (1.8) by means of a rescaling argument. The section closes with the compactness result for the sets $M_\varrho$. The paper is concluded in Section 4 with the existence result for the optimal control problem (1.10). In the Appendix, we collect convergence results for the load term, lower semicontinuity properties of some functionals, results for Banach space valued absolutely continuous functions, a combined Helly and Ascoli-Arzelà theorem, and a chain rule.

2. Basic assumptions and estimates

The analysis will be carried out for the semilinear system introduced in [MZ14], compare also [MR15] Example 3.8.4 and [Kne18].

Let $\mathcal{X}$ be a Banach space and $\mathcal{Z}, \mathcal{V}$ be separable Hilbert spaces that are densely and compactly resp. continuously embedded in the following way:

$$\mathcal{Z} \subset \mathcal{V} \subset \mathcal{X}.$$  \hspace{1cm} (2.1)
Let further $A \in \text{Lin}(Z, Z^*)$ and $V \in \text{Lin}(V, V^*)$ be linear symmetric, bounded $Z$- and $V$-elliptic operators, i.e. there exist constants $\alpha, \gamma > 0$ such that
\begin{equation}
\forall z \in Z, \forall v \in V: \quad \langle Az, z \rangle \geq \alpha \|z\|_Z^2, \quad \langle Vv, v \rangle \geq \gamma \|v\|_V^2, \quad (2.2)
\end{equation}
and $\langle Az_1, z_2 \rangle = \langle Az_2, z_1 \rangle$ for all $z_1, z_2 \in Z$ (and similar for $V$). Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairings in $Z$ and $V$, respectively. We define $\|v\|_V := (\langle Vv, v \rangle)^{\frac{1}{2}}$, which is a norm that is equivalent to the Hilbert space norm $\|\cdot\|_V$. Let further
\begin{equation}
\mathcal{F} \in C^2(Z; \mathbb{R}) \text{ with } \mathcal{F} \geq 0.
\end{equation}
The functional $\mathcal{F}$ shall play the role of a possibly nonconvex lower order term (cf. [MR15, Section 3.8]). Hence, we assume that
\begin{equation}
D_z \mathcal{F} \in C^1(Z; V^*), \quad \|D_z^2 \mathcal{F}(z)v\|_{V^*} \leq C(1 + \|z\|_Z^q) \|v\|_Z \quad (2.4)
\end{equation}
for some $q \geq 1$. Let $\ell \in H^1((0, T); V^*)$. Energy functionals of the following type are considered
\begin{equation}
\mathcal{J} : Z \to \mathbb{R}, \quad \mathcal{J}(z) := \frac{1}{2} \langle Az, z \rangle + \mathcal{F}(z). \quad (2.5)
\end{equation}
\begin{equation}
\mathcal{E} : [0, T] \times Z \to \mathbb{R}, \quad \mathcal{E}(t, z) = \mathcal{J}(z) - \langle \ell(t), z \rangle. \quad (2.6)
\end{equation}
Clearly, $\mathcal{J} \in C^1(Z; \mathbb{R})$. If not otherwise stated, in the whole paper we assume that the initial datum $z_0 \in Z$ and the load $\ell$ are compatible in the following sense
\begin{equation}
z_0 \in Z, \quad \ell \in L^1((0, T); V^*) \text{ and } D\mathcal{E}(0, z_0) = D\mathcal{J}(z_0) - \ell(0) \in V^*. \quad (2.7)
\end{equation}
The dissipation functional $\mathcal{R} : X \to [0, \infty)$ is assumed to be convex, continuous, positively homogeneous of degree one and
\begin{equation}
\exists c, C > 0 \forall x \in X: \quad c \|x\|_X \leq \mathcal{R}(x) \leq C \|x\|_X. \quad (2.8)
\end{equation}
\cite[Lemma 1.1]{Kne18}
\begin{Lemma}
Assume \( (2.1), (2.3), (2.8) \) and \( (2.4) \). For every $\rho > 0$ and $\kappa > 0$ there exists $C_{\rho, \kappa} > 0$ such that for all $z_1, z_2 \in Z$ with $\|z_i\|_Z \leq \rho$ we have
\begin{equation}
\|D\mathcal{J}(z_1) - D\mathcal{J}(z_2), z_1 - z_2\| \leq \kappa \|z_1 - z_2\|_Z^2 + C_{\rho, \kappa} \min\{\mathcal{R}(z_1 - z_2), \mathcal{R}(z_2 - z_1)\} \|z_1 - z_2\|_V. \quad (2.9)
\end{equation}
\end{Lemma}
As a consequence, $\mathcal{E}$ is $\lambda$-convex on sublevels. To be more precise, we have the following estimate: For every $\rho > 0$ there exists $\lambda = \lambda(\rho) > 0$ such that for all $t \in [0, T]$ and all $z_1, z_2 \in Z$ with $\|z_i\|_Z \leq \rho$ we have
\begin{equation}
\langle D_z \mathcal{E}(t, z_1) - D_z \mathcal{E}(t, z_2), z_1 - z_2 \rangle \geq \frac{\lambda}{2} \|z_1 - z_2\|_Z^2 \quad (2.10)
\end{equation}
and
\begin{equation}
\mathcal{J}(z_2) - \mathcal{J}(z_1) \geq \langle D_z \mathcal{J}(z_1), z_2 - z_1 \rangle \geq \frac{\lambda}{2} \|z_1 - z_2\|_Z^2 - \lambda \|z_2 - z_1\|_V. \quad (2.11)
\end{equation}
Finally, we assume that
\begin{equation}
\mathcal{F} : Z \to \mathbb{R} \text{ and } D_z \mathcal{F} : Z \to Z^* \text{ are weak-weak continuous.} \quad (2.12)
\end{equation}
3. Parametrized BV-solutions and properties of the solution set

3.1. Definition of parametrized balanced viscosity solutions. We use the following notation: for $\xi \in V^*$,
\[
\text{dist}_V(\xi, \partial R(0)) := \inf \{ \|\xi - \sigma\|_{V^{-1}} : \sigma \in \partial R(0) \},
\]
where $\|\xi\|^2_{V^{-1}} := \langle \xi, V^{-1}\xi \rangle$. Moreover, for $\ell \in V^*$, $z \in \mathbb{Z}$ we define
\[
m(\ell, z) := \text{dist}_V(-D\hat{\ell}(z) + \ell, \partial R(0)).
\]

**Definition 3.1.** Let $z_0 \in \mathbb{Z}$ and $\ell \in H^1((0, T); V^*)$. A triple $(\ell, \hat{\ell}, \hat{z})$ with $S > 0$, $\hat{\ell} \in W^{1,\infty}((0, S); \mathbb{R})$, $\hat{z} \in AC^\infty([0, S]; \mathbb{X}) \cap L^\infty((0, S); \mathbb{Z})$ is a normalized, $p$-parametrized balanced viscosity solution of the rate-independent system $(\mathcal{I}, \mathcal{R})$ with data $z_0, \ell$, if there exists a (relatively) open set $G \subset [0, S]$ such that $\hat{z} \in W^{1,1}_{\text{loc}}(G; \mathbb{V})$, $D\hat{\ell}(\cdot, \hat{z}(\cdot)) \in L^\infty_{\text{loc}}(G; V^*)$ and such that $m(\hat{\ell}(s), \hat{z}(s)) > 0$ on $G$ and $m(\hat{\ell}(s), \hat{z}(s)) = 0$ on $[0, S]\setminus G$. Let $\tilde{\ell} := \ell \circ \hat{\ell}$ and $\tilde{\mathcal{E}}(s, v) := \mathcal{E}(v) - (\hat{\ell}(s), v)$. In addition to the above, the following relations shall be satisfied:

- **Complementarity and normalization condition:**
  \[
  \tilde{\ell}(s) \geq 0, \quad \tilde{\ell}(S) = T, \quad \hat{z}(0) = z_0, \quad \tilde{\mathcal{E}}(s, \hat{z}(s)) = \tilde{\mathcal{E}}(0, z_0) = 0.
  \]

- **1**
  \[
  1 = \begin{cases} 
  \tilde{\ell}'(s) + R[\hat{z}'](s) & \text{if } s \notin G, \\
  \tilde{\ell}'(s) + R[\hat{z}'](s) + \|\hat{z}'(s)\|_{V^*} \text{dist}_V(-D\hat{\ell}(s, \hat{z}(s)), \partial R(0)) & \text{if } s \in G.
  \end{cases}
  \]

- **Energy-dissipation balance:**
  \[
  \tilde{\mathcal{E}}(s, \hat{z}(s)) + \int_0^s R[\hat{z}'](r) \, dr + \int_{(0, s) \cap G} \|\hat{z}'(r)\|_{V^*} \text{dist}_V(-D\hat{\ell}(r, \hat{z}(r)), \partial R(0)) \, dr = \tilde{\mathcal{E}}(0, z_0) - \int_0^s (\tilde{\ell}'(r), \hat{z}(r)) \, dr.
  \]

With $\mathcal{L}(z_0, \ell)$ we denote the set of normalized, $p$-parametrized balanced viscosity solutions associated with the pair $(z_0, \ell)$.

**Remark 3.2.** This definition is an adapted version of Definition 4.2 from [MRS16].

**Theorem 3.3.** Under the conditions formulated in Section 2, for every compatible $z_0 \in \mathbb{Z}$ and $\ell \in H^1((0, T); V^*)$ (see 2.7), there exists at least one normalized, $p$-parametrized balanced viscosity solution of the rate-independent system $(\mathcal{I}, \mathcal{R})$. In other words, $\mathcal{L}(z_0, \ell)$ is not the empty set.

Remarks on the proof: Assuming more regularity on $\ell$, namely $\ell \in C^1((0, T); V^*)$, this theorem is a special case of [Mie11], [MRS16] Theorem 4.3, see also [Kne18], where the situation of the present article is discussed. For the case $\ell \in BV((0, T); V^*)$ we refer to [KZ18]. A typical strategy to prove the existence of solutions is to follow a vanishing viscosity approach. This means that in a first step the existence of solutions of the viscously regularized systems
\[
0 \in \partial R(\hat{z}_\varepsilon(t)) + \varepsilon V\hat{z}_\varepsilon(t) + D\hat{\ell}(t, \hat{z}_\varepsilon(t)), \quad \hat{z}_\varepsilon(0) = z_0, \quad t \in (0, T), \quad \varepsilon > 0
\]
is shown and a priori estimates are derived that are uniform with respect to the viscosity parameter $\varepsilon$. In a second step, the viscous solutions are reparametrized and the passage to the limit $\varepsilon \to 0$ is carried out in the reparametrized setting. In order to obtain the above introduced parametrized solutions, one uses the change of variables
\[
s_\varepsilon(t) := t + \int_0^t p(\hat{z}_\varepsilon(\tau), -D\hat{\ell}(\tau, \hat{z}_\varepsilon(\tau))) \, d\tau \quad \text{with} \quad p(v, \xi) := R(v) + \|v\|_{V^*} \text{dist}_V(\xi, \partial R(0)).
\]
and defines \( t_\varepsilon := s_\varepsilon^{-1} \) as the inverse function and \( \tilde{z}_\varepsilon := z_\varepsilon \circ t_\varepsilon \). Reformulating the energy-dissipation balance associated with (3.7) in the new variables and passing to the limit in this expression ultimately leads to a normalized, \( p \)-parametrized balanced viscosity solution of the rate-independent system \((\mathcal{I}, \mathcal{R})\) in the sense of Definition 3.1. The quantity \( p(\cdot, \cdot) \) is the so-called vanishing viscosity contact potential, [MRS16]. Since the aim of this paper is to discuss an optimal control problem taking parametrized solutions as constraints, we do not go into further details concerning the existence of solutions.

### 3.2. Alternative representation of BV solutions and basic uniform estimates.

#### Proposition 3.4. Assume (2.7).

Every normalized, \( p \)-parametrized balanced viscosity solution \((S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell)\) of the rate-independent system satisfies

1. The mapping \( s \mapsto \hat{E}(s, \hat{z}(s)) \) belongs to \( AC^2([0, S]; \mathbb{R}) \).
2. \( \hat{t} \) is constant on the closure of each connected component of \( G \) and there exists a measurable function \( \lambda : (0, S) \to [0, \infty) \) with \( \lambda(s) = 0 \) on \((0, S)\setminus G\) such that on each connected component \((a, b) \subseteq G\) the differential inclusion

   \[
   0 \in \partial \hat{E}(\hat{z}'(s)) + \lambda(s) \nu \hat{z}'(s) + D\hat{E}(s, \hat{z}(s))
   \]

   is satisfied, for almost all \( s \in (a, b) \).

For almost all \( s \in G \) we have \( \lambda(s) = \text{dist}(\cdot, \hat{E}(0, S); \nu) \).

3. Basic energy estimates: There exists a constant \( c > 0 \) (depending on the ellipticity constant \( \alpha \) in (2.2) and embedding constants, only) such that for all \((z_0, \ell)\) with (2.7) and all \((S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell)\) it holds

\[
\| \hat{z} \|_{L^{\infty}(0, S; \mathbb{R})} \leq c(1 + |\mathcal{E}(0, S_0)| + \| \ell \|_{W^{1, 1}(0, T; \nu^*)}) \tag{3.8}
\]

\[
S = \int_0^S \hat{E}[\hat{z}](s) \, ds + \int_{(0, S) \cap G} \| \hat{z}'(s) \|_{\nu^*} \, \text{dist}(\cdot, \hat{E}(s, \hat{z}(s)), \partial \hat{E}(0)) \, ds
\]

\[
\leq c(1 + |\mathcal{E}(0, S_0)| + \| \ell \|_{W^{1, 1}(0, T; \nu^*)})^2. \tag{3.9}
\]

**Proof.** Claim (1): Taking into account the normalization condition (3.4), from the energy dissipation balance (3.5) we obtain for every \( s < \sigma \in [0, S] \):

\[
|\hat{E}(\sigma, \hat{z}(\sigma)) - \hat{E}(s, \hat{z}(s))| \leq \int_s^\sigma \left( |\dot{\hat{t}}(r), \hat{z}(r)| + 1 \right) \, dr.
\]

Thanks to Proposition 1.1 we have \( \hat{z} \in C([0, S]; \mathbb{V}) \) and \( \dot{\hat{t}} \in H^1((0, S); \mathbb{V}^*) \) thanks to Lemma A.1. Hence, the integrand belongs to \( L^2((0, S); \mathbb{R}) \) from which claim (1) ensues.

Claim (2) is a standard property of nondegenerate parametrized solutions, cf. [Mic11] and we give the proof here for completeness. Since \( m(\dot{\hat{t}}(s), \hat{z}(s)) > 0 \) on \( G \), from the complementarity condition (3.6) we deduce that \( \hat{t} \) is constant on each connected component of \( G \). In order to verify (3.9), let \([a, b] \subseteq G\). Since by assumption \( \hat{z} \in W^{1, 1}((a, b); \mathbb{V}) \) we have \( \mathcal{R}[\hat{z}](s) = \mathcal{R}(\hat{z}'(s)) \) for almost all \( s \in (a, b) \), cf. [AGS05] Remark 1.1.3. Thus, localizing the energy dissipation identity (3.5) (where we apply the chain rule formulated in Proposition 1.1) yields

\[
\mathcal{R}(\hat{z}'(s)) + \langle D\hat{E}(s, \hat{z}(s)), \hat{z}'(s) \rangle_{\nu^*, \nu} + \| \hat{z}'(s) \|_{\nu^*} \, \text{dist}(-D\hat{E}(s, \hat{z}(s)), \partial \hat{E}(0)) = 0 \tag{3.10}
\]

which is valid for almost all \( s \in (a, b) \). Since \( \hat{t} \) is constant on \((a, b)\), from (3.4) it follows that \( \hat{z}'(s) \neq 0 \) almost everywhere on \((a, b)\). Hence, with

\[
\lambda(s) = \begin{cases}
\text{dist}(\cdot, -D\hat{E}(s, \hat{z}(s)), \partial \hat{E}(0))/ \| \hat{z}'(s) \|_{\nu^*}, & \text{if } \hat{z}'(s) \neq 0, \\
0, & \text{otherwise}
\end{cases}
\]
we have \( \|\hat{z}'(s)\|_V \text{dist}(\hat{D}(s, \hat{z}(s)), \partial R(0)) = \langle \lambda(s) V \hat{z}'(s), \hat{z}'(s) \rangle \) and \( (3.7) \) follows from \( (3.10) \) and the one-homogeneity of \( R \). This finishes the proof of claim (2) in Lemma 3.4.

Claim (3): The verification of \( (3.8) - (3.9) \) takes the energy dissipation estimate as a starting point. Indeed, for all \( b \in [0, S] \) from the energy dissipation balance we obtain

\[
\hat{E}(b, \hat{z}(b)) \leq \mathcal{E}(0, z_0) + c_2 \left\| \hat{t} \right\|_{L^1((0,T);V)} \left\| \hat{z} \right\|_{L^\infty((0,S);Z)},
\]

where the constant \( c_2 \) is related with the embedding \( Z \subset V \). On the other hand, due to the structure of \( \mathcal{E} \), we have \( \hat{E}(b, \hat{z}(b)) \geq \frac{\sigma}{2} \| \hat{z}(b) \|_Z^2 - c_\sigma c_\omega \| \ell \|_{L^\infty((0,T);V^*)}^2 \). Combining these estimates yields \( (3.8) \). Estimate \( (3.9) \) now is immediate. \( \square \)

3.3. A uniform estimate for the driving forces \( D\mathcal{E} \) and a viscous model on \( \mathbb{R}_+ \). By assumption, parametrized solutions \( (S, \hat{t}, \hat{z}) \) satisfy \( D\mathcal{E}(\hat{t}, \hat{z}(\cdot)) \in L^\infty_{\text{loc}}(G;V^*) \). Moreover, for \( s \in [0, S] \setminus G \) we have \( m(\hat{t}(s), \hat{z}(s)) = 0 \) which implies that \(-D\mathcal{E}(s, \hat{z}(s)) \in \partial \mathcal{R}(0) \). Since \( \partial \mathcal{R}(0) \) is a bounded subset of \( V^* \) we obtain \( D\mathcal{E}(\hat{t}, \hat{z}(\cdot)) \in L^\infty((0,S);G;V^*) \). The goal of this section is to show that \( D\mathcal{E}(\hat{t}, \hat{z}(\cdot)) \in L^\infty((0,S);V^*) \) and to derive estimates that are uniform on sets of the type

\[
M_\varrho := \{ (S, \hat{t}, \hat{z}) : (S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell) \text{ for } (z_0, \ell) \text{ with } (2.7) \text{ and } \|z_0\|_Z + \|\ell\|_{H^{1}((0,T), V')} \leq \varrho \}.
\]

For the proof of such uniform estimates we consider the inclusion \( (3.7) \) on connected components of \( G \) and reparametrize it in such a way that the transformed function \( \tilde{z} \) satisfies

\[
0 \in \partial \mathcal{R}(\tilde{z}'(r)) + V\tilde{z}'(r) + D\mathcal{J}(\tilde{z}(r)) - \ell_*, \ r > 0
\]

with a constant load \( \ell_* \). The essential estimates will be derived for this system and subsequently transferred to the original one.

3.3.1. An autonomous viscously regularized rate-independent system on \( \mathbb{R}_+ \). The aim of this section is to derive regularity properties and estimates for solutions of the system

\[
0 \in \partial \mathcal{R}(\hat{z}(t)) + V\hat{z}(t) + D\mathcal{J}(\hat{z}(t)) - \ell_*, \ t > 0
\]

being defined on \( \mathbb{R}_+ = (0, \infty) \) with a constant load \( \ell_* \in V^* \).

Theorem 3.5. (1) Uniqueness of solutions: For every \( \ell_* \in V^* \) and \( z_0 \in Z \) there exists at most one function \( z \in L^\infty(\mathbb{R}_+;Z) \) with \( \hat{z} \in L^1((0,a);V) \) for every \( a > 0 \) that satisfies \( z(0) = z_0 \) and the inclusion \( (3.11) \) for almost all \( t > 0 \).

(2) Existence of solutions and regularity: For every \( \ell_* \in V^* \) and \( z_0 \in Z \) with \( D\mathcal{J}(z_0) \in V^* \) there exists a unique function \( z \in L^\infty(\mathbb{R}_+;Z) \) with \( \hat{z} \in L^2(\mathbb{R}_+;V) \) that satisfies \( z(0) = z_0 \) and the inclusion \( (3.11) \) for almost all \( t > 0 \). Moreover, this solution belongs to \( W^{1,\infty}(\mathbb{R}_+;V) \) with \( \text{Var}_z(z;[0, \infty)) < \infty \) and \( D\mathcal{J}(z(\cdot)) \in L^\infty(\mathbb{R}_+;V^*) \).

(3) Uniform estimates: There exist functions \( m_1, m_2 : Z \times V^* \to [0, \infty) \) that map bounded sets on bounded sets such that for all \( \ell_* \in V \) and all \( z_0 \in Z \) with \( D\mathcal{J}(z_0) \in V^* \) it holds: let \( z \) be the solution of \( (3.11) \) corresponding to \( (z_0, \ell_*) \). Then

\[
\|z\|_{L^\infty(\mathbb{R}_+;Z)} \leq m_1(z_0, \ell_*),
\]

\[
\|\hat{z}\|_{L^\infty(\mathbb{R}_+;V)} + \text{Var}_z(z;[0, \infty)) \leq m_2(z_0, \ell_*), \text{dist}_V(-D\mathcal{J}(z_0) + \ell_*, \partial \mathcal{R}(0)) + m_1(z_0, \ell_*)),
\]

\[
\|D\mathcal{J}(z(\cdot))\|_{L^\infty(\mathbb{R}_+;V^*)} \leq \text{diam}_V(\partial \mathcal{R}(0)) + \|\ell_*\|_{V^*} + c_V \|\hat{z}\|_{L^\infty(\mathbb{R}_+;V)}. \]

Remark 3.6. Let \( z_0 \in Z, \ell_* \in V^* \) and assume that \( -D\mathcal{J}(z_0) + \ell_* \in \partial \mathcal{R}(0) \). Then the constant function \( z(t) = z_0, t > 0 \), is the unique solution of \( (3.11) \). If \( -D\mathcal{J}(z_0) + \ell_* \notin \partial \mathcal{R}(0) \), then along the whole solution curve we have \( -D\mathcal{J}(z(t)) + \ell_* \notin \partial \mathcal{R}(0) \).

For deriving the uniform estimates \( (3.10) - (3.14) \) one formally takes the derivative of the inclusion \( (3.11) \) with respect to \( t \) and chooses \( \hat{z} \) as a test function. This can be made rigorous on a time-discrete level or alternatively by an argument relying on a regularized version of \( (3.11) \). In the presentation here, we choose the latter approach.
For $\delta > 0$ and $v \in \mathbb{Z}$ let $\mathcal{R}(v) := \mathcal{R}(v) + \frac{\delta}{2} (Av, v)_{Z^*, Z}$ with $\mathcal{R}(v) = \mathcal{R}(v) + \frac{1}{2} (Vv, v)$ and with the operator $A$ from (2.2) (any operator of that type would do).

**Proposition 3.7.** For every $\delta > 0$ and every $z_0 \in \mathbb{Z}$, $\ell_* \in V^*$ there exists a unique function $z_\delta \in W^{2,\infty}(\mathbb{R}_+; \mathbb{Z})$ satisfying $z_\delta(0) = z_0$ and

$$0 \in \partial \mathcal{R}(z_\delta(t)) + D\mathcal{J}(z_\delta(t)) - \ell_*.$$  \hspace{1cm} (3.15)

Moreover, $z_\delta$ fulfills the energy-dissipation balance

$$\mathcal{E}(t, z_\delta(t)) + \int_0^t \mathcal{R}(\dot{z}_\delta(\tau)) + \frac{\delta}{2} (A\dot{z}_\delta(\tau), \dot{z}_\delta(\tau)) + \mathcal{R}_*^*(\mathcal{D}(\tau, z_\delta(\tau)))d\tau = \mathcal{E}(0, z_0).$$  \hspace{1cm} (3.16)

with $\mathcal{E}(t, z) := \mathcal{J}(z) - \langle \ell_*, z \rangle$. Finally, there exists a function $m : \mathbb{Z} \times V^* \to [0, \infty)$ that maps bounded sets to bounded sets such that for all $z_0 \in \mathbb{Z}$, $\ell_* \in V^*$ and $\delta > 0$ the corresponding solution $z_\delta$ satisfies

$$\|z_\delta\|_{L^\infty(\mathbb{R}_+; \mathbb{Z})} \leq m(z_0, \ell_*),$$  \hspace{1cm} (3.17)

$$\int_0^\infty \mathcal{R}(\dot{z}_\delta(\tau)) + \mathcal{R}_*^*(\mathcal{D}(\tau, z_\delta(\tau)))d\tau \leq m(z_0, \ell_*).$$  \hspace{1cm} (3.18)

**Proof.** Existence of solutions: The arguments follow closely those presented in Section 4 of the preprint version [KRZ11] of [KRZ13]. The operator $\mathcal{G}_\delta := (\partial \mathcal{R})^{-1} : Z^* \to Z$ in fact is single valued and Lipschitz continuous. Thus, the differential inclusion (3.15) is equivalent to the abstract ordinary differential equation

$$\dot{z}_\delta(t) = \mathcal{G}_\delta(- D\mathcal{J}(z_\delta(t)) + \ell_*)$$

living in the space $\mathbb{Z}$. Since $D\mathcal{J} : \mathbb{Z} \to Z^*$ is locally Lipschitz continuous, by the Picard–Lindelöf Theorem, for every initial value $z_0 \in \mathbb{Z}$ there exists a unique local solution $z_\delta \in W^{2,\infty}([0, T_0]; \mathbb{Z})$. By standard convex analysis and chain rule arguments, see for instance [MR15] Sec. 1.3.4, it follows that this solution satisfies the energy dissipation balance (3.16) on $[0, T_0]$. In order to show that there is a global in time solution, assume that there exists $T_* > 0$ such that the solution cannot be extended beyond $T_*$. By the energy-dissipation estimate it follows that $\|z_\delta\|_{L^\infty([0, T_*]; \mathbb{Z})} < \infty$ as well as $\dot{z}_\delta \in H^1((0, T_*); \mathbb{Z})$, implying in particular that $\dot{z}_\delta \in C([0, T_*]; \mathbb{Z})$. Applying again the Picard-Lindelöf theorem with the new initial value $\dot{z}_\delta(T_*)$ we obtain a contradiction to the definition of $T_*$. The proof of (3.17)–(3.18) is an immediate consequence of the energy-dissipation balance, compare also the proof of Proposition 3.3. \hfill \square

For solutions $z_\delta$ and $t > 0$ we define

$$\nu_\delta(t) := \left( \|\dot{z}_\delta(t)\|_V^2 + \delta \|\dot{z}_\delta(t)\|_A^2 \right)^\frac{1}{2},$$

where $\|v\|_A := \sqrt{\langle Av, v \rangle}$ is a norm on $\mathbb{Z}$ that is equivalent to the standard norm on $\mathbb{Z}$. Since $\dot{z}_\delta \in W^{1,\infty}((0, T); \mathbb{Z})$, the function $\nu_\delta$ is well defined for all $t \in [0, \infty)$.

**Proposition 3.8.** Under the assumptions of Proposition 3.7 for $\delta > 0$, $\ell_* \in V^*$, $z_0 \in \mathbb{Z}$ with $D\mathcal{J}(z_0) \in V^*$ let $z_\delta \in W^{2,\infty}(\mathbb{R}_+; \mathbb{Z})$ with $z_\delta(0) = z_0$ be the unique solution of (3.15). Then

$$\nu_\delta(0) \leq \text{dist}_V(- D\mathcal{J}(z_0) + \ell_*, \partial \mathcal{R}(0)).$$  \hspace{1cm} (3.19)

Moreover, there exists a function $m : \mathbb{Z} \times V^* \to [0, \infty)$ mapping bounded sets on bounded sets such that for all $\delta > 0$, $\ell_* \in V^*$, $z_0 \in \mathbb{Z}$ with $D\mathcal{J}(z_0) \in V^*$ the corresponding solution satisfies

$$\|\dot{z}_\delta\|_{L^\infty(\mathbb{R}_+; V)} + \sqrt{\delta} \|\dot{z}_\delta\|_{L^2(\mathbb{R}_+; \mathbb{Z})} + \|\dot{z}_\delta\|_{L^1(\mathbb{R}_+; \mathbb{Z})} \leq m(z_0, \ell_*)(1 + \text{dist}_V(- D\mathcal{J}(z_0) + \ell_*, \partial \mathcal{R}(0))).$$  \hspace{1cm} (3.20)

**Proof.** Thanks to the regularity $z_\delta \in W^{2,\infty}(\mathbb{R}_+; \mathbb{Z})$ and the continuity of the quantities appearing in (3.15), relation (3.19) in particular is valid for $t = 0$. Let $\mu \in \partial \mathcal{R}(0)$ such that
\(\text{dist}_\mathcal{V}(-\text{D}\mathcal{E}(0, z_0), \partial \mathcal{R}(0)) = \|\text{D}\mathcal{E}(0, z_0) + \mu\|_{\mathcal{V}^{-1}}\). Choosing \(\dot{z}_\delta(0)\) as a test in (3.15) for \(t = 0\) and exploiting the one-homogeneity of \(\mathcal{R}\) we find
\[
\mathcal{R}(\dot{z}_\delta(0)) + \nu_\delta(0)^2 = -\langle \text{D}\mathcal{E}(0, z_0), \dot{z}_\delta(0) \rangle \\
= -\langle \text{D}\mathcal{E}(0, z_0) + \mu, \dot{z}_\delta(0) \rangle + \langle \mu, \dot{z}_\delta(0) \rangle \\
\leq \|\text{D}\mathcal{E}(0, z_0) + \mu\|_{\mathcal{V}^{-1}} \|\dot{z}_\delta(0)\|_{\mathcal{V}} + \mathcal{R}(\dot{z}_\delta(0)),
\]
where in the last term we have exploited the one-homogeneity of \(\mathcal{R}\). Applying Young’s inequality on the right hand side and absorbing corresponding terms yields (3.19).

Following the difference quotient arguments in [KRZ11] Section 4, the preprint version of [KRZ13], it follows that for almost all \(t > 0\) we have
\[
\langle \mathcal{V}\dot{z}_\delta(t), \dot{z}_\delta(t) \rangle + \delta \langle A\dot{z}_\delta(t), \dot{z}_\delta(t) \rangle + \left(\frac{d}{dt} \mathcal{D}\mathcal{J}(\dot{z}_\delta(t)), \dot{z}_\delta(t) \right) = 0.
\tag{3.21}
\]
Observe that the first two terms on the left hand side coincide with \(\frac{1}{2} \frac{d}{dt} \nu_\delta(t)^2\). With the same interpolation argument as in the proof of Lemma 2.1, see also [Kne18], Lemme 1.1, we obtain
\[
\|D^2\mathcal{J}(z_\delta(t))\|_{\mathcal{Z}}^2 \leq c(1 + \|z_\delta(t)\|_{\mathcal{Z}}^2) \|\dot{z}_\delta(t)\|_{\mathcal{V}} \|z_\delta(t)\|_{\mathcal{Z}} \\
\leq \frac{2}{\nu_\delta(t)} \|\dot{z}_\delta(t)\|_{\mathcal{Z}}^2 + c_\alpha c_1 \|z_\delta(t)\|_{\mathcal{Z}^{\alpha \infty}(\mathbb{R}_+, \mathcal{Z})} \mathcal{R}(\dot{z}_\delta(t)) \|\dot{z}_\delta(t)\|_{\mathcal{V}}.
\]
Taking into account the uniform bound (3.17) and going back to (3.21) we have shown that there exists a function \(\tilde{m} : \mathcal{Z} \times \mathcal{V}^* \to [0, \infty)\) mapping bounded sets on bounded sets such that for all \(\delta > 0\), \(z_0 \in \mathcal{Z}\), \(\ell_\star \in \mathcal{V}^*\) we have
\[
\frac{1}{2} \frac{d}{dt} \nu_\delta(t)^2 + \nu_\delta(t) \|\dot{z}_\delta(t)\|_{\mathcal{Z}} \leq \tilde{m}(z_0, \ell_\star) \mathcal{R}(\dot{z}_\delta(t)) \|\dot{z}_\delta(t)\|_{\mathcal{V}} \tag{3.22}
\]
Clearly, \(\|\dot{z}_\delta(t)\|_{\mathcal{V}} \leq \nu_\delta(t)\). Moreover, since \(\mathcal{Z}\) is continuously embedded in \(\mathcal{V}\), for \(\delta \leq 1\) we have \(\|\dot{z}_\delta(t)\|_{\mathcal{Z}} \geq c(\|z_\delta(t)\|_{\mathcal{Z}}^2 + \|\dot{z}_\delta(t)\|_{\mathcal{V}}^2) \geq c \nu_\delta(t)\), and \(c\) is independent of \(\delta > 0\) and \(\dot{z}_\delta\). Hence, (3.22) can be rewritten as follows
\[
\frac{1}{2} \frac{d}{dt} \nu_\delta(t)^2 + c \|\dot{z}_\delta(t)\|_{\mathcal{Z}} \nu_\delta(t) \leq \tilde{m}(z_0, \ell_\star) \mathcal{R}(\dot{z}_\delta(t)) \nu_\delta(t). 
\tag{3.23}
\]
Next we follow the arguments from [Mic11] Section 4.4. For \(t > 0\) with \(\nu_\delta(t) \neq 0\) we find
\[
\dot{\nu}_\delta(t) + c \|\dot{z}_\delta(t)\|_{\mathcal{Z}} \nu_\delta(t) \leq \tilde{m}(z_0, \ell_\star) \mathcal{R}(\dot{z}_\delta(t)),
\]
while for \(t\) with \(\nu_\delta(t) = 0\) the previous inequality is trivially satisfied. Integration with respect to \(t\) yields
\[
\forall t > 0 \quad \nu_\delta(t) + \int_0^t \|\dot{z}_\delta(\tau)\|_{\mathcal{Z}} d\tau \leq c \left(\nu_\delta(0) + \tilde{m}(z_0, \ell_\star) \int_0^\infty \mathcal{R}(\dot{z}_\delta(\tau)) d\tau\right) .
\]
Combining this with estimate (3.19) and (3.18) we finally have shown that there exists a further function \(m : \mathcal{Z} \times \mathcal{V}^* \to [0, \infty)\) mapping bounded sets on bounded sets such that (3.24) is valid. □

**Proof of Theorem 2.10 Uniqueness of solutions:** For \(i \in \{1, 2\}\) let \(z_i \in L^\infty(\mathbb{R}_+; \mathcal{Z})\) with \(z_i \in L^1((0, a); \mathcal{V})\) for all \(a > 0\) and such that (3.11) is satisfied. Testing (3.11) with the difference 
\(z_1(t) - z_2(t)\), by the monotonicity of the operator \(\partial \mathcal{R}\) we obtain
\[
0 \geq \langle \mathcal{V}(z_1(t) - z_2(t)), z_1(t) - z_2(t) \rangle + \langle \mathcal{D}\mathcal{J}(z_1(t)) - \mathcal{D}\mathcal{J}(z_2(t)), z_1(t) - z_2(t) \rangle.
\]
Thanks to the \(\lambda\)-convexity estimate (2.10) this implies
\[
0 \geq \|z_1(t) - z_2(t)\|_{\mathcal{V}}^2 + \frac{\lambda}{2} \|z_1(t) - z_2(t)\|_{\mathcal{V}} - \lambda \|z_1(t) - z_2(t)\|_{\mathcal{V}}^2
\]
for some \(\lambda > 0\). Integration with respect to \(t\) and applying the Gronwall estimate we conclude.

**Existence and regularity of solutions:** Let \(z_0 \in \mathcal{Z}\) with \(\mathcal{D}\mathcal{J}(z_0) \in \mathcal{V}^*\) and \(\ell_\star \in \mathcal{V}^*\). For \(\delta > 0\) let \(\dot{z}_\delta\) denote the corresponding solution of (3.15). Thanks to the uniform estimates provided in Proposition 3.7 and Proposition 3.8, there exists a vanishing sequence \((\delta_n)\) (i.e. \(\delta_n \to 0\) for
such that the following convergences are available:

\[ z_{\delta_n} \rightharpoonup z \text{ weakly* in } L^\infty(\mathbb{R}^+; \mathbb{Z}), \]

\[ \hat{z}_{\delta_n} \rightharpoonup \hat{z} \text{ weakly* in } L^\infty(\mathbb{R}^+; \mathbb{V}) \cap L^2(\mathbb{R}^+; \mathbb{V}), \]

\[ \liminf_{\delta_n} \| \hat{z}_{\delta_n} \|_{L^1(\mathbb{R}^+; \mathbb{Z})} = \liminf_{\delta_n} \text{Var}_\mathbb{Z}(z_{\delta_n}, [0, \infty)) \geq \text{Var}_\mathbb{Z}(z, [0, \infty)). \]

for all \( t \geq 0 \):

\[ z_{\delta_n}(t) \rightharpoonup z(t) \text{ weakly in } \mathbb{Z}. \]

The last two assertions are a consequence of the Banach space valued version of Helly’s selection principle, [BP86]. Moreover, by lower semicontinuity, the uniform bounds derived in Proposition 3.8 carry over to the limit function \( z \), and Proposition 3.9 follows. In the following we omit the index \( n \).

Let us next prove that for almost all \( t > 0 \) the function \( z \) satisfies the inclusion (3.11). For that purpose we start from the energy dissipation balance (3.16). Thanks to (3.27) and since \( DJ(z_{\delta_n}(\cdot)) \) is weakly continuous, pointwise weak convergence in \( \mathbb{Z}^* \) of the terms \( DJ(z_{\delta_n}(\cdot)) \) ensues. Hence, by [KRZ11, Lemma A.1], see also Lemma B.3 in the appendix, we obtain for all \( t > 0 \)

\[ \liminf_{\delta} \mathcal{R}_{\lambda}^\tau(-DJ(z_{\delta}(t)) + \ell_*) \geq \mathcal{R}_{\lambda}^\tau(-DJ(z(t)) + \ell_*). \]

Hence, by lower semicontinuity and Fatou’s Lemma it follows that the limit function \( z \) satisfies the energy-dissipation estimate

\[ \forall t \geq 0 \quad \mathcal{E}(t, z(t)) + \int_0^t \mathcal{R}_\mathcal{V}(\dot{z}(\tau)) + \mathcal{R}_\mathcal{V}(-DE(\tau, z(\tau))d\tau \leq \mathcal{E}(0, z_0). \]

Standard arguments relying on the chain rule (Proposition 3.1) show that we in fact have an equality in (3.28). Applying again the chain rule and localizing this energy-dissipation identity shows that \( z \) satisfies the inclusion (3.11). Since \( \partial \mathcal{R}(0) \) is a bounded subset of \( \mathcal{V}_* \), together with the estimate (3.13) we finally conclude that \( DJ(z(\cdot)) \) belongs to \( L^\infty(\mathbb{R}^+; \mathcal{V}_*) \) and satisfies (3.14). This finishes the proof.

### 3.3.2. A uniform estimate for the driving force \( D\hat{E} \)

We now turn back to the properties of the parametrized BV solutions introduced in Definition 3.1.

**Theorem 3.9.** There exists a function \( m : \mathbb{Z} \times H^1((0, T); \mathcal{V}_*) \to [0, \infty) \) mapping bounded sets to bounded sets such that for all \( z_0 \in \mathbb{Z}, \ell \in H^1(0, T; \mathcal{V}_*) \) satisfying (2.7) and all \( (S, \bar{t}, \bar{z}) \in \mathcal{L}(z_0, \ell) \) we have \( D\hat{E}(\cdot, \bar{z}(\cdot)) \in L^\infty(0, S; \mathcal{V}_*) \) and

\[ \| D\hat{E}(\cdot, \bar{z}(\cdot)) \|_{L^\infty(0, S; \mathcal{V}_*)} + \| \lambda V\bar{z}' \|_{L^\infty(G; \mathcal{V}_*)} \leq m(z_0, \ell) \]

and \( DJ(\bar{z}(\cdot)) \in C_{\text{weak}}([0, S]; \mathcal{V}_*) \).

**Remark 3.10.** As a byproduct, in the proof of Theorem 3.9 we show that the function \( \lambda \) from (2.7) is positive almost everywhere on \( G \), that the function \( s \mapsto 1/\lambda(s) \) belongs to \( L^1_{\text{loc}}(G) \) but that it is not integrable on any connected component of \( G \), see also Remark 3.11 for further consequences of this observation.

**Proof.** As already stated at the beginning of this section, we have \( D\hat{E}(\cdot, \bar{z}(\cdot)) \in L^\infty((0, S)\setminus G; \mathcal{V}_*) \) with \( \| D\hat{E}(\cdot, \bar{z}(\cdot)) \|_{L^\infty((0, S);G; \mathcal{V}_*)} \leq \text{diam} V \cdot (\partial \mathcal{R}(0)) \) and it remains to study the behavior on the set \( G \). For that purpose we start from the differential inclusion (3.7). We recall that by the definition of p-parametrized solutions the set \( G \) is a relatively open subset of \([0, S]\). Let \((a, b) \subset G\) be a maximal connected component of \( G \). By Proposition 3.4 \( \bar{t} \) is constant on \((a, b)\). Hence, \( \bar{t} \) is constant on \((a, b)\) as well and we denote its value with \( \ell_\star \). For each compact set \( K \subset (a, b) \) we have \( \bar{z} \in W^{1,1}(K; \mathcal{V}) \) and \( D\hat{E}(\cdot, \bar{z}(\cdot)) \in L^\infty(K; \mathcal{V}_*) \) which implies that \( D\hat{E}(\cdot, \bar{z}(\cdot)) \in C_{\text{weak}}(K; \mathcal{V}_*) \). Thus, by lower
semicontinuity, there exists \( c_K > 0 \) such that for \( m(\cdot, \cdot) \) from (3.1) it holds \( m(\hat{\ell}(\cdot), \hat{z}(\cdot)) \geq c_K > 0 \) for all \( s \in K \). The normalization condition (3.4) now implies that \( \|\hat{z}'(s)\|_\gamma \leq c_K^{-1} \) almost everywhere on \( K \) and hence \( \lambda(s) \geq c_K^{-1} > 0 \) almost everywhere on \( K \), where we used the representation of \( \lambda \) from Proposition [3.3]. This observation was already made in [MRS16].

The next aim is to perform a change of variables \( s \mapsto r \) and \( (a, b) \to (0, \Lambda) \) such that (3.7) rewritten in the new variable is of the form (3.11).

Assume first that there is \( s_0 \in (a, b) \) such that \( 1/\lambda \notin L^1((a, s_0)) \). The above considerations imply that for every \( \varepsilon > 0 \) there is a constant \( c_{\varepsilon} > 0 \) such that \( \lambda^{-1} \in (a+\varepsilon, s_{0}) \) \( \leq c_{\varepsilon} \). Hence, since \( \lambda^{-1} \) is not integrable on \((a, s_{0})\), \( \lambda^{-1} \) is unbounded in a neighborhood of \( a \). To be more precise, for every \( n \in \mathbb{N} \) the set \( \Sigma_n := \{ s \in (a, a + 1/n) : 1/\lambda(s) \geq n \} \) has positive Lebesgue measure. From the normalization property and the structure of \( \lambda \) we therefore deduce that

\[
\text{for all } n \in \mathbb{N} \text{ and almost all } s \in \Sigma_n: \quad \text{dist}_\gamma(-D\hat{\lambda}(s, \hat{z}(s)), \partial \mathcal{R}(0)) \leq \frac{1}{\sqrt{n}}. \tag{3.29}
\]

Let now \( s_n \in \Sigma_n \) with \( \lambda(s_n)^{-1} \geq n \) and such that \( \text{dist}_\gamma(-D\hat{\lambda}(s_n, \hat{z}(s_n)), \partial \mathcal{R}(0)) \leq \frac{1}{\sqrt{n}} \). Clearly, \( \lim_n s_n = a \) and without loss of generality we may assume that the sequence \( (s_n)_n \) is decreasing. We next study the system (3.7) on the intervals \((s_n, b)\). For \( s \in (s_n, b) \) let \( \Lambda_n(s) := \int_{s_n}^{s} \frac{1}{\lambda(s)} \, ds \). The above considerations show that \( \Lambda_n \) is well defined and \((s_n, b)\). Moreover, \( \Lambda_n \) is strictly increasing and the inverse function \( \Lambda_n^{-1} : [0, \Lambda_n(b)) \to [s_n, b) \) exists. We remark that \( \Lambda_n(b) = \infty \) is not excluded. For \( r \in [0, \Lambda_n(b)) \) let \( \tilde{z}_n(r) := \Lambda_n^{-1}(r) \). Observe that \( \tilde{z}_n \in W^{1,1}(0, \Lambda_n(b-\delta); \mathcal{V}^*) \) for every \( \delta > 0 \). The function \( \tilde{z}_n \) solves the Cauchy problem

\[
0 \in \partial \mathcal{R}(\tilde{z}_n'(r)) + \mathcal{V}\tilde{z}_n'(r) + D\mathcal{J}(\tilde{z}_n(r)) - \ell_s, \quad r \in (0, \Lambda(b)), \tag{3.30}
\]

\[
\tilde{z}_n(0) = \hat{z}(s_n). \tag{3.31}
\]

Hence, Theorem [5.3] is applicable and implies in particular that \( D\mathcal{J}(\tilde{z}_n(\cdot)) \in L^\infty(0, \Lambda_n(b); \mathcal{V}^*) \) with

\[
\|D\mathcal{J}(\tilde{z}_n(\cdot))\|_{L^\infty(0, \Lambda_n(b); \mathcal{V}^*)} \leq m_2(\hat{z}(s_n), \ell_s)(\text{dist}_\gamma(-D\hat{\lambda}(s_n, \hat{z}(s_n)), \partial \mathcal{R}(0)) + m_1(\hat{z}(s_n), \ell_s)),
\]

where \( m_1, m_2 : \mathbb{Z} \times \mathcal{V}^* \to [0, \infty) \) are functions that map bounded sets on bounded sets and that do not depend on \( n \). This immediately translates into \( D\mathcal{J}(\tilde{z}(\cdot)) \in L^\infty(s_n, b; \mathcal{V}^*) \) along with the estimate

\[
\|D\mathcal{J}(\tilde{z}(\cdot))\|_{L^\infty(s_n, b; \mathcal{V}^*)} \leq m_2(\hat{z}(s_n), \ell_s)(\text{dist}_\gamma(-D\hat{\lambda}(s_n, \hat{z}(s_n)), \partial \mathcal{R}(0)) + m_1(\hat{z}(s_n), \ell_s)) \tag{3.32}
\]

\[
\leq \tilde{m}_2(z_0, \ell)(\frac{1}{\sqrt{n}} + \tilde{m}_1(z_0, \ell)), \tag{3.33}
\]

where \( \tilde{m}_1, \tilde{m}_2 : \mathbb{Z} \times H^1((0, T); \mathcal{V}^*) \to [0, \infty) \) are functions that map bounded sets on bounded sets and depend on \( \mathcal{J}, \mathcal{R} \) and embedding constants, only. The previous estimate is of the structure \( \alpha_n \leq \beta_n \) with an increasing sequence \( (\alpha_n)_n \) and a decreasing sequence \( (\beta_n)_n \). Hence, for \( s_n \downarrow 0 \) we obtain \( D\mathcal{J}(\tilde{z}(\cdot)) \in L^\infty(a, b; \mathcal{V}^*) \) along with a bound that ultimately depends on \( \|z_0\|_{L} \) and \( \|\ell\|_{H^1(0,T;\mathcal{V}^*)} \), only.

Assume next that \( \lambda^{-1} \in L^1(a, s_{0}) \) for every \( s_{0} \leq b \). In this case, we use the transformation \( \Lambda(s) := \int_{s_0}^{s} \frac{1}{\lambda(s)} \, ds \) and the transformed function \( \tilde{z} \) satisfies (3.11) on \((0, \Lambda(b))\) with the initial condition \( \tilde{z}(0) = \hat{z}(a) \). Since \( G \) is open, \( a \) does not belong to \( G \) and hence, \( -D\hat{\lambda}(a, \hat{z}(a)) \in \partial \mathcal{R}(0) \).

According to Remark 3.6 the unique solution of the transformed system is given by the constant function \( \tilde{z}(r) = \hat{z}(a) \) for all \( r \in (0, \Lambda(b)) \). But this implies in particular that \( \hat{z} \) is constant on \((a, b)\), a contradiction to the normalization condition. As a consequence, \( \lambda^{-1} \) is not bounded close to \( a \). Similarly, again taking into account Remark 3.6 it follows that the values \( \Lambda_n(b) \) from above and the value \( \Lambda(b) \) in the situation discussed here, are not finite, since otherwise \( -D\mathcal{J}(\tilde{z}_n(\Lambda_n(b))) + \ell_s \in \partial \mathcal{R}(0) \). Summarizing this shows that the function \( s \mapsto 1/\lambda(s) \) is not integrable on \((a, b)\) and unbounded towards \( a \) and \( b \).
If \( 0 \notin G \), then the proof of Theorem 3.9 is finished. Otherwise let \([0, b)\) be a maximal connected component of \(G\). But now we can argue exactly in the same way as before with 0 instead of \(s_n\) in (3.32).

**Remark 3.11.** Reinterpreting the arguments of the previous proof we have shown the following: Let \((S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell)\) and let \((a, b) \subset G\) be a maximal connected component of \(G\). Let \(s_* := (a + b)/2\) and define \(\Lambda_*(s) := \int_a^s \frac{1}{W(r)} \, dr\). The arguments from the previous proof show that \(\Lambda_*\) is well defined, strictly monotone and that \(\Lambda_*((a, b)) = \mathbb{R}\) with \(\lim_{s \to a} \Lambda_*(s) = -\infty\) and \(\lim_{s \to b} \Lambda_*(s) = \infty\). Moreover, \(\tilde{z} := \hat{z} \circ \Lambda_*^{-1}\) (inverse function) satisfies (3.11) on \(\mathbb{R}\) with \(\lim_{r \to -\infty} \tilde{z}(r) = \tilde{z}(a)\) and \(\lim_{r \to \infty} \tilde{z}(r) = \tilde{z}(b)\) (strong convergence in \(V\) since \(\hat{z} \in C([0, S); V])\)). The limit points \(\hat{z}(a), \hat{z}(b)\) are stable in the sense that \(-\Delta \hat{z}(s) + \ell_s \in \partial \mathcal{R}(0)\) for \(z_s \in \{\hat{z}(a), \hat{z}(b)\}\). Hence, \(\tilde{z}\) can be interpreted as a heteroclinic orbit for (3.11), connecting \(\hat{z}(a)\) and \(\hat{z}(b)\).

### 3.4. Compactness of solution sets.

The aim of this section is to derive compactness properties of the sets

\[
M_\rho := \{ (S, \hat{t}, \hat{z}) ; (S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell) \text{ for } (z_0, \ell) \text{ with (3.1)} \text{ and } \|z_0\|_Z + \|\ell\|_{H^1([0,T); V^*)} \leq \rho \}. \tag{3.34}
\]

for arbitrary \(\rho \geq 0\). These properties will be based on the uniform estimates derived in the previous two sections.

**Theorem 3.12.** Let \(\rho > 0\) and \(z_0 \in Z\). Then the set \(M_\rho\) is compact in the following sense: For every sequence \((S_n, \hat{t}_n, \hat{z}_n)_{n \in \mathbb{N}} \subseteq M_\rho\) with \((S_n, \hat{t}_n, \hat{z}_n) \in \mathcal{L}(z_0, \ell_n)\) and such that \((z_0, \ell_n)\) satisfy (2.7), there exists a subsequence (denoted by the same symbols for simplicity) and limit elements \(\ell \in H^1(0, T; V^*)\) and \((S, \hat{t}, \hat{z}) \in \mathcal{L}(z_0, \ell)\) such that \((z_0, \ell)\) comply with (2.7) and

\[
S_n \to S \text{ in } \mathbb{R}, \quad \hat{t}_n \rightharpoonup \hat{t} \text{ in } W^{1,\infty}(0, S), \quad \hat{t}(S) = T, \quad \ell_n \to \ell \text{ in } H^1(0, T; V^*), \tag{3.35}
\]

\[
\hat{z}_n \rightharpoonup \hat{z} \text{ in } L^\infty(0, S; Z) \text{ and } \hat{z}_n \to \hat{z} \text{ uniformly in } C([0, S], V), \tag{3.36}
\]

\[
\hat{z}_n(S_n) \to \hat{z}(S) \text{ strongly in } V, \tag{3.37}
\]

\[
\partial \mathcal{J}(\hat{z}_n) \to \partial \mathcal{J}(\hat{z}) \text{ in } L^\infty(0, S; V^*), \tag{3.38}
\]

and for every \(s \in [0, S]\), it holds that

\[
\hat{t}_n(s) \to \hat{t}(s), \quad \hat{z}_n(s) \to \hat{z}(s) \text{ in } Z, \quad \partial \mathcal{J}(\hat{z}_n(s)) \to \partial \mathcal{J}(\hat{z}(s)) \text{ in } V^*, \tag{3.39}
\]

\[
\hat{z}_n(s) \to \hat{z}(s) \text{ strongly in } Z. \tag{3.40}
\]

Furthermore, the map \(s \mapsto \partial \mathcal{J}(\hat{z}(s))\) is continuous w.r.t. the weak topology on \(V^*\).

**Proof.** Let \((S_n, \hat{t}_n, \hat{z}_n)_{n \in \mathbb{N}} \subseteq M_\rho\) be a sequence as in the proposition and for \(n \in \mathbb{N}\) let \(G_n \subset [0, S]\) be the corresponding open sets according to Definition 3.11. Thanks to Proposition 3.4 the estimates (3.8) and (3.9) hold uniformly for \(n \in \mathbb{N}\) as we infer the first of (3.35). If \(S > S_n\), we extend all functions \(\hat{z}_n\) and \(\hat{t}_n\) constantly to \([0, S]\) by their value at \(S_n\) and thus obtain the first of (3.36). Due to (3.2) and the normalization condition (3.3), the second of (3.35) ensues, and since \(W^{1,\infty}(0, S)\) is compactly embedded into \(C([0, S], V))\), also the third of (3.35) as well as the first of (3.39). Combining the a priori estimate (3.8) and the normalization condition (3.3), we conclude uniform convergence of \(\hat{z}_n\) to \(\hat{z}\) in \(V\) and pointwise weak convergence in \(Z\) along a subsequence by means of Proposition 3.11. We also obtain (3.37) by means of the following estimate:

\[
\|\hat{z}_n(S_n) - \hat{z}(S)\|_V \leq \|\hat{z}_n(S_n) - \hat{z}_n(S)\|_V + \|\hat{z}_n(S) - \hat{z}(S)\|_V \to 0,
\]

where for the convergence of the first term we exploit the equicontinuity of the sequence \((\hat{z}_n)_n\) (cf. the proof of Proposition 3.11) and the second summand tends to zero due to the uniform convergence (3.36).
In order to show \(3.38\), we first note that thanks to the a priori estimate in Theorem 3.9 there are \(\xi \in V^*\) such that \(\mathcal{D}(\hat{z}_n(s)) \rightharpoonup \xi\) in \(L^\infty(0, S; V^*)\) as well as pointwise limits such that \(\mathcal{D}(\hat{z}_n(s)) \to \mu(s)\) in \(V^*\) for all \(s \in [0, S]\) along a subsequence. Now, since we also have \(\hat{z}_n(s) \to \hat{z}(s)\) in \(\mathbb{Z}\), and \(\mathcal{D}\) is supposed to be weakly continuous (cf. \(2.12\)), we also know that \(\mathcal{D}(\hat{z}_n(s)) \to \mathcal{D}(\hat{z}(s))\) in \(\mathbb{Z}^*\), whence \(3.38\) and the third of \(3.39\) ensue along a subsequence. A standard argument by contradiction shows convergence along the entire sequence. By the same arguments, we obtain the weak continuity of \(s \mapsto \mathcal{D}(\hat{z}(s))\).

It remains to show that \((S, \hat{t}, \hat{z}) \in \mathcal{L}(0, \ell)\). As a first step, we show that the complementarity identity \(3.33\) is valid. To this end, we note that \(\hat{t}_n \to \hat{t}\) in \(L^1(0, S)\). Furthermore, we have \(\ell_n(\hat{t}_n(s)) \to \ell(\hat{t}(s))\) in \(V^*\) for all \(s \in [0, S]\) according to Lemma \(A.1\). Together with the weak convergence of \(\mathcal{D}(\hat{z}_n(s))\) according to \(3.39\) and the weak lower semicontinuity of \(\text{dist}_V(\cdot, \partial \mathcal{R}(0))\), this implies

\[
m(\ell(\hat{t}(s)), \hat{z}(s)) \leq \liminf_{n \to \infty} m(\ell_n(\hat{t}_n(s)), \hat{z}_n(s)) \quad \text{for all} \quad s \in [0, S]
\]

with \(m(\cdot, \cdot)\) from \(3.41\). This allows us to conclude by means of Lemma \(3.2\) that we have

\[
0 \leq \int_0^S \ell'(s)m(\ell(\hat{t}(s)), \hat{z}(s)) \, ds \leq \liminf_{n \to \infty} \int_0^S \ell_n'(s)m(\ell_n(\hat{t}_n(s)), \hat{z}_n(s)) \, ds = 0,
\]

and since the integrand is nonnegative, \(3.41\) ensues.

Next, we want to show that \(3.39\) is valid with \(\leq\) instead of \(=\). For every \(n \in \mathbb{N}\) and \(s \in [0, S]\), it holds with \(\hat{\mathcal{E}}_n(s, v) := \mathcal{I}(v) - \langle \ell_n(s), v \rangle\) and \(\hat{\ell}_n := \ell_n \circ \hat{t}_n\) that

\[
\hat{\mathcal{E}}_n(s, \hat{z}_n(s)) + \int_0^s \mathcal{R}[\hat{z}'_n](r) \, dr + \int_{[0, s] \cap G_n} \|\hat{z}'_n(r)\| \mathcal{M}(\hat{\ell}_n(r), \hat{z}_n(r)) \, dr
\]

\[
= \hat{\mathcal{E}}_n(0, z_0) - \int_0^s \langle \hat{\ell}_n'(r), \hat{z}_n(r) \rangle \, dr.
\]

Now, the second of \(3.39\) together with the lower semicontinuity of \(v \mapsto \mathcal{I}(v)\) w.r.t. the weak topology on \(\mathbb{Z}\), as well as Lemma \(A.1\) imply for all \(s \in [0, S]\) that

\[
\liminf_{n \to \infty} \hat{\mathcal{E}}_n(s, \hat{z}_n(s)) \geq \hat{\mathcal{E}}(s, \hat{z}(s)) \quad \text{and} \quad \lim_{n \to \infty} \hat{\mathcal{E}}_n(0, z_0) = \hat{\mathcal{E}}(0, z_0).
\]

For the first dissipation integral, it follows by means of Helly’s selection principle, \(\text{[MM05]}\) Theorem 3.2, for all \(s \in [0, S]\) that

\[
\liminf_{n \to \infty} \int_0^s \mathcal{R}[\hat{z}'_n](r) \, dr \geq \int_0^s \mathcal{R}[\hat{z}'](r) \, dr.
\]

According to Lemma \(A.1\) the load term fulfills the convergence

\[
\int_0^s \langle \hat{\ell}'(r), \hat{z}(r) \rangle \, dr = \lim_{n \to \infty} \int_0^s \langle \hat{\ell}_n'(r), \hat{z}_n(r) \rangle \, dr,
\]

and it remains to study the second dissipation term. Let \(G := \{s \in [0, S] : m(\ell(s), \hat{z}(s)) > 0\}\). First, we show that \(G\) is an open set. To this end, let \((s_k)_{k \in \mathbb{N}} \subset [0, S] \setminus G\) be a sequence converging to an element \(s \in [0, S]\). By the weak continuity of \(s \mapsto \mathcal{D}(\hat{z}(s))\), we obtain \(0 = \liminf_{n \to \infty} m(\ell(s_n), \hat{z}(s_n)) \geq m(\ell(s), \hat{z}(s)) = 0\), whence \(s \in [0, S] \setminus G\) and \(G\) is indeed open. Next, we are going to show the improved regularity of \(\hat{z}\) on \(G\). Let \(K \subset G\) be compact. By the same arguments as above, we conclude that \(c := \liminf_{s \in K} m(\ell(s), \hat{z}(s)) > 0\). Thus, for every \(s \in K\), there exists \(N_0 \in \mathbb{N}\) such that for all \(n \geq N_0\) we have \(m(\ell_n(s), \hat{z}_n(s)) \geq \frac{c}{2}\), and a proof by contradiction shows that \(N_0\) can be chosen independently of \(s \in K\). Therefore, the normalization condition \(3.40\) shows that \(\sup_{n \geq N_0} \|\hat{z}'_n\|_{L^\infty(K; V)} \leq \frac{2}{c}\), whence it follows in combination with \(3.40\)
that \( \tilde{z}_n \rightarrow \tilde{z} \in W^{1,\infty}(K;V) \). Now, by means of Proposition 3.1 and having in mind 3.41, we may conclude that

\[
\liminf_{n \to \infty} \int_K \| \tilde{z}_n'(r) \| v_m(\hat{\ell}_n(r), \tilde{z}_n(r)) \, dr \geq \int_K \| \tilde{z}'(r) \| v_m(\hat{\ell}(r), \tilde{z}(r)) \, dr,
\]

and thus, 3.5 is valid with \( \leq \) instead of = and also 3.4 with \( \geq \) instead of =.

In order to show the opposite estimates, we follow the ideas from [KZ18].

We first show that \( s \mapsto \mathcal{J}(\tilde{z}(s)) \) is continuous on \([0, S]\) and hence uniformly continuous. From \( \tilde{z} \in C([0, S]; V) \) and \( I_s \) we obtain \( \tilde{z} \in C_{weak}(\omega_0, S; Z) \). Hence, thanks to 2.12, \( \mathcal{J}(\tilde{z}(\cdot)) \) is continuous on \([0, S]\) and \( \mathcal{D} \tilde{z}(\cdot) \) belongs to \( C_{weak}(\omega_0, S; \mathcal{V}^*) \). Since the same is true for \( \mathcal{D} \tilde{z}(\cdot) \), we conclude that \( A \tilde{z}(\cdot) \) is continuous with respect to the weak topology in \( \mathcal{V}^* \), as well. But this ensures the continuity of the term \( s \mapsto \langle A \tilde{z}(s), \tilde{z}(s) \rangle_{\mathcal{V}^*, \mathcal{V}} \) and ultimately the continuity of \( \mathcal{J}(\tilde{z}(\cdot)) \).

For \( s \in [0, S] \), let \( \mu(s) \in \partial R(0) \) such that \( \| - D_s \hat{\ell}(s, \tilde{z}(s)) - \mu(s) \|_{\mathcal{V}^*} = m(\hat{\ell}(s), \tilde{z}(s)) \). An application of 2.11 yields for every \( s \in [0, S] \) and \( 0 < h < S - s \)

\[
\mathcal{J}(\tilde{z}(s + h)) - \mathcal{J}(\tilde{z}(s)) \geq (D_s \hat{\ell}(s, \tilde{z}(s)), \Delta_h \tilde{z}(s)) + \langle \hat{\ell}(s), \Delta_h \tilde{z}(s) \rangle - \lambda R(\Delta_h \tilde{z}(s)) ||\Delta_h \tilde{z}(s)||_{\mathcal{V}}
\]

\[
= (D_s \hat{\ell}(s, \tilde{z}(s)) + \mu(s), \Delta_h \tilde{z}(s)) + \langle \hat{\ell}(s), \Delta_h \tilde{z}(s) \rangle - \langle \mu(s), \Delta_h \tilde{z}(s) \rangle
\]

\[
- \lambda R(\Delta_h \tilde{z}(s)) ||\Delta_h \tilde{z}(s)||_{\mathcal{V}},
\]

where we abbreviate \( \Delta_h \tilde{z}(s) := \tilde{z}(s + h) - \tilde{z}(s) \). Now, thanks to the choice of \( \mu(s) \), we can estimate the first term on the right hand side by

\[
- (D_s \hat{\ell}(s, \tilde{z}(s)) + \mu(s), \Delta_h \tilde{z}(s)) \leq \| D_s \hat{\ell}(s, \tilde{z}(s)) + \mu(s) \|_{\mathcal{V}^*} ||\Delta_h \tilde{z}(s)||_{\mathcal{V}} = m(\hat{\ell}(s), \tilde{z}(s)) ||\Delta_h \tilde{z}(s)||_{\mathcal{V}},
\]

and the third term by \( \langle \mu(s), \Delta_h \tilde{z}(s) \rangle \leq R(\Delta_h \tilde{z}(s)) \). Therefore, rearrangement of the terms leads to the estimate

\[
\mathcal{J}(\tilde{z}(s + h)) - \mathcal{J}(\tilde{z}(s)) + m(\hat{\ell}(s), \tilde{z}(s)) ||\Delta_h \tilde{z}(s)||_{\mathcal{V}} + (1 + \lambda ||\Delta_h \tilde{z}(s)||_{\mathcal{V}}) R(\Delta_h \tilde{z}(s)) \geq \langle \hat{\ell}(s), \Delta_h \tilde{z}(s) \rangle,
\]

which we divide by \( h > 0 \) and integrate w.r.t. \( s \) to obtain for every \( 0 \leq \sigma_1 < \sigma_2 \leq S - h \)

\[
\int_{\sigma_1}^{\sigma_2} \frac{1}{h} (\mathcal{J}(\tilde{z}(s + h)) - \mathcal{J}(\tilde{z}(s))) \, ds
\]

\[
+ \int_{\sigma_1}^{\sigma_2} m(\hat{\ell}(s), \tilde{z}(s)) ||\Delta_h \tilde{z}(s)||_{\mathcal{V}} \, ds + \int_{\sigma_1}^{\sigma_2} (1 + \lambda ||\Delta_h \tilde{z}(s)||_{\mathcal{V}}) R(\Delta_h \tilde{z}(s)) \, ds
\]

\[
\geq \int_{\sigma_1}^{\sigma_2} \langle \hat{\ell}(s), \frac{1}{h} \Delta_h \tilde{z}(s) \rangle \, ds.
\]

(3.44)

Now, since \( s \mapsto \mathcal{J}(\tilde{z}(s)) \) is uniformly continuous (as shown above), the first integral converges to \( \mathcal{J}(\tilde{z}(\sigma_2)) - \mathcal{J}(\tilde{z}(\sigma_1)) \) with \( h \to 0 \). For the second integral, we have to distinguish the cases \( s \in [0, S) \backslash G \) where we have \( m(\hat{\ell}(s), \tilde{z}(s)) = 0 \), and \( s \in G \), where we can argue as follows: Since \( \tilde{z} \in W^{1,\infty}_{\text{loc}}(G; \mathcal{V}) \), we find by the Dominated Convergence Theorem for every \( K \in G \)

\[
\lim_{h \to 0} \int_{(\sigma_1, \sigma_2) \cap K} m(\hat{\ell}(s), \tilde{z}(s)) ||\Delta_h \tilde{z}(s)||_{\mathcal{V}} \, ds = \int_{(\sigma_1, \sigma_2) \cap K} m(\hat{\ell}(s), \tilde{z}(s)) ||\tilde{z}'(s)||_{\mathcal{V}} \, ds.
\]

Furthermore, since we have \( \tilde{z} \in C([0, S]; \mathcal{V}) \), it follows that \( \Delta_h \tilde{z}(s) \to 0 \) strongly in \( \mathcal{V} \) and uniformly in \( s \), and since \( \tilde{z} \in AC^\infty([0, S]; \mathcal{X}) \), we infer by means of the results in Appendix C that \( R(\frac{1}{h} \Delta_h \tilde{z}(s)) \to R[\tilde{z}'](s) \) for almost every \( s \in [0, S] \). Keeping in mind that \( \mathcal{R}[\tilde{z}'](s) \leq 1 \) due to the normalization inequality, the Dominated Convergence Theorem implies that

\[
\lim_{h \to 0} \int_{\sigma_1}^{\sigma_2} (1 + \lambda ||\Delta_h \tilde{z}(s)||_{\mathcal{V}}) R(\frac{1}{h} \Delta_h \tilde{z}(s)) \, ds = \int_{\sigma_1}^{\sigma_2} \mathcal{R}[\tilde{z}'](s) \, ds.
\]
Finally, for the term on the right hand side of (3.44), we find
\[
\int_{\sigma_1}^{\sigma_2} \langle \ell(s), \frac{1}{h} \Delta_h \hat{z}(s) \rangle \, ds = \frac{1}{h} \int_{\sigma_1 + h}^{\sigma_2 + h} \langle \ell(s) - \ell(s - h), \hat{z}(s) \rangle \, ds + \langle \ell(s), \hat{z}(s) \rangle - \langle \ell(s - h), \hat{z}(s - h) \rangle \, ds \\
= - \int_{\sigma_1 + h}^{\sigma_2 + h} \langle \ell(s) - \ell(s - h), \hat{z}(s) \rangle \, ds + \frac{1}{h} \int_{\sigma_1}^{\sigma_2} \langle \ell(s), \hat{z}(s) \rangle \, ds - \frac{1}{h} \int_{\sigma_1}^{\sigma_1 + h} \langle \ell(s), \hat{z}(s) \rangle \, ds \\
= - \int_{\sigma_1}^{\sigma_2} \langle \hat{z}(s + h) - \hat{z}(s), \hat{z}(s + h) \rangle \, ds + \frac{1}{h} \int_{\sigma_1}^{\sigma_2 + h} \langle \hat{\ell}(s), \hat{z}(s) \rangle \, ds - \frac{1}{h} \int_{\sigma_1}^{\sigma_1 + h} \langle \hat{\ell}(s), \hat{z}(s) \rangle \, ds.
\]
(3.45)

In order to show convergence, we apply (2) in Lemma A.2 to \(v = \hat{\ell} \) and (3) in the same Lemma to \(v = \hat{z} \) and obtain for the first term on the right hand side of (3.45) the convergence
\[
\lim_{h \to 0} \int_{\sigma_1}^{\sigma_2} \langle \frac{\ell(x + h) - \ell(x)}{h}, \hat{z}(x + h) \rangle \, dx = \lim_{h \to 0} \int_{\sigma_1}^{\sigma_2} \langle L_h \ell(x), S_h \hat{z}(x) \rangle \, dx = \int_{\sigma_1}^{\sigma_2} \langle \hat{\ell}(x), \hat{z}(x) \rangle \, dx,
\]
while the second and third term converge to \(\langle \hat{\ell}(\sigma_2), \hat{z}(\sigma_2) \rangle \) and \(\langle \hat{\ell}(\sigma_1), \hat{z}(\sigma_1) \rangle \), respectively, for almost all \(\sigma_1, \sigma_2\). In fact, since both \(\hat{\ell} \in H^1(0, S; V^*) \) and \(\hat{z} \in C([0, S], V) \) are continuous, the product \(s \mapsto \langle \hat{\ell}(s), \hat{z}(s) \rangle \) is uniformly continuous on \([0, S] \), and we have convergence for all \(\sigma_1, \sigma_2 \in [0, S] \). Altogether, we can now pass to the limit in (3.44) and obtain the opposite estimate in (3.5), which is therefore valid as an identity.

We can now proceed to show that the estimates (3.42) and (3.43) can be improved to equalities by standard arguments. Observe first that by arguments similar to the proof of the continuity of \(s \mapsto \langle \hat{\ell}(s), \hat{z}(s) \rangle \), exploiting (3.38) and Lemma A.1 it follows that \(\hat{E}_n(s, \hat{z}_n(s)) \to \hat{E}(s, \hat{z}(s)) \) for all \(s \in [0, S] \). Hence, from the energy dissipation balance written in the following way,
\[
\lim_{n \to \infty} \left( \int_{0}^{\sigma} \mathcal{R}[\hat{z}_n'](s) \, ds + \int_{(0, \sigma) \cap G_n} m(\ell_n(s), \hat{z}_n(s)) \| \hat{z}_n'(s) \|_V \, ds + \hat{E}_n(s, \hat{z}_n(s)) \right) = \hat{E}(0, \hat{z}(0)) + \int_{0}^{\sigma} \langle \hat{\ell}(s), \hat{z}(s) \rangle \, ds \\
= \int_{0}^{\sigma} \mathcal{R}[\hat{z}'](s) \, ds + \int_{(0, \sigma) \cap G} m(\ell(s), \hat{z}(s)) \| \hat{z}'(s) \|_V \, ds + \hat{E}(\sigma, \hat{z}(\sigma)),
\]
we may conclude that in fact
\[
\lim_{n \to \infty} \int_{0}^{\sigma} \mathcal{R}[\hat{z}_n'](s) \, ds = \int_{0}^{\sigma} \mathcal{R}[\hat{z}'](s) \, ds
\]
and
\[
\lim_{n \to \infty} \int_{(0, \sigma) \cap G_n} m(\ell_n(s), \hat{z}_n(s)) \| \hat{z}_n'(s) \|_V \, ds = \int_{(0, \sigma) \cap G} m(\ell(s), \hat{z}(s)) \| \hat{z}'(s) \|_V \, ds
\]
are valid for all \(\sigma \in [0, S] \). Now, writing \(\int_{0}^{\sigma} \mathcal{R}[\hat{z}'](s) \, ds + \int_{(0, \sigma) \cap G} m(\ell(s), \hat{z}(s)) \| \hat{z}'(s) \|_V \, ds = \int_{0}^{\sigma} (1 - \hat{\ell}_n(s)) \, ds \), the above convergences yield the normalization condition (3.3).

Finally, exploiting the \(\lambda\)-convexity property (2.11) and keeping in mind that \(\mathcal{D}(\hat{z}(s)) \in V^* \) for all \(s \), for every \(s \in [0, S] \) we deduce that \(\hat{z}_n(s) \to \hat{z}(s) \) strongly in \(\mathcal{Z} \), which is (3.40). \(\square\)

4. The optimal control problem

We now turn to the optimal control problem governed by (1.2). Our control variable is \(\ell \in H^1(0, T; V^*) \) and the admissible set \(M_{ad} \) consists of all normalized, p-parametrized BV solutions of the system (1.2) with data \(z_0 \) and \(\ell \). To be more precise, we define
\[
M_{ad} := \{(S, \hat{\ell}, \hat{z}, \ell) \in \mathbb{R}_+ \times W^{1, \infty}(0, S) \times AC([0, S]; X) \times H^1(0, T; V^*) \mid (z_0, \ell) \text{ comply with (2.7), and } (S, \hat{\ell}, \hat{z}) \in \mathcal{L}(z_0, \ell) \}.
\]
Then, the optimal control problem under consideration reads as follows:

$$
\begin{align*}
\min \quad & J(S, \hat{z}, \ell) := j(\hat{z}(S)) + \alpha \|\ell\|_{H^1(0,T;V^*)} \\
\text{s.t.} \quad & (S, \hat{t}, \hat{z}, \ell) \in M_{ad}.
\end{align*}
$$

(4.1)

Herein, $\alpha > 0$ is a fixed Tikhonov parameter and $j : V \to \mathbb{R}$ is bounded from below and continuous, e.g. $j(z) := \|z - z_{des}\|_V$ for a desired end state $z_{des} \in V$.

We now have the following existence result:

**Theorem 4.1.** Let $\alpha > 0$ be a fixed Tikhonov parameter, $z_0 \in Z$ be chosen such that there exists $\ell \in H^1(0,T;V^*)$ such that $(z_0, \ell)$ complies with (2.7) and let $j : V \to \mathbb{R}$ be bounded from below and continuous. Then, the optimal control problem (4.1) has a globally optimal solution.

**Proof.** Since $j$ is prerequisite to be continuous and bounded from below, we find that $I := \inf \{J(S, \hat{z}, \ell) \mid (S, \hat{t}, \hat{z}, \ell) \in M_{ad}\} > -\infty$. We choose an infimizing sequence $((S_n, \hat{t}_n, \hat{z}_n, \ell_n))_{n \in \mathbb{N}} \subset M_{ad}$, i.e.

$$
I = \lim_{n \to \infty} J(S_n, \hat{z}_n, \ell_n).
$$

Due to the boundedness assumption on $j$, we find that

$$
R := \sup_{n \in \mathbb{N}} \|\ell_n\|_{H^1(0,T;V^*)} < \infty,
$$

whence $((S_n, \hat{t}_n, \hat{z}_n))_{n \in \mathbb{N}} \subset M_{\|\ell_0\|_H + R}$ with $M_\rho$ as in (3.33). According to Theorem 3.12 this set is compact. Thus, there exists a subsequence (not relabeled for simplicity) and limit elements $\ell_* \in H^1(0,T;V^*)$ and $(S_*, t_*, z_*) \in L(z_0, \ell_*)$ such that $(z_0, \ell_0)$ comply with (2.7) and we have in particular the convergences (cf. (3.33))

$$
\ell_n \to \ell_* \text{ in } H^1(0,T;V^*) \text{ and } \hat{z}_n(S_n) \to z_*(S_*) \text{ in } V.
$$

Therefore, $(S_*, t_*, z_*, \ell_*) \in M_{ad}$, and since $j$ is assumed to be continuous, we infer that

$$
I \leq J(S_*, z_*, \ell_*) \leq \liminf_{n \to \infty} \left( j(\hat{z}_n(S_n)) + \alpha \|\ell_n\|_{H^1(0,T;V^*)} \right) = \lim_{n \to \infty} J(S_n, \hat{z}_n, \ell_n) = I,
$$

whence $(S_*, t_*, z_*, \ell_*)$ is indeed a minimizer of $J$ on the admissible set $M_{ad}$. \hfill $\square$

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**Appendix A. Convergence of the load term**

The following convergence results are used Section 3.4 to show convergence of the load term in the energy-dissipation balance (3.5).

**Lemma A.1.** (1) Let $\ell \in H^1(0,T;V^*)$ and $\hat{t} \in W^{1,\infty}(0,S)$ with $\hat{t}(s) \in [0,T]$ for all $s \in [0,S]$, $\hat{t}(0) = 0$, and $\hat{t}(S) = T$. Then, it holds that $\ell \circ \hat{t} \in H^1(0,S;V^*)$ with

$$
\ell \circ \hat{t} = \ell \cdot (\hat{t}(s)) \quad \text{f.a.a. } s \in [0,S],
$$

(A.1)

$$
\|\ell \circ \hat{t}\|_{L^2(0,S;V^*)} \leq C \|\ell\|_{H^1(0,T;V^*)},
$$

(A.2)

$$
\|\ell \circ \hat{t}\|_{L^2(0,S;V^*)} \leq \|\ell\|_{L^2(0,T;V^*)} \frac{1}{\|\hat{t}\|_{L^\infty(0,S)}}
$$

(A.3)

for a constant $C > 0$ depending on the space $V^*$ only.
(2) Let \((\ell_n)_{n \in \mathbb{N}} \subset H^1(0; T; V^*)\) and \((\hat{\ell}_n)_{n \in \mathbb{N}} \subset W^{1, \infty}(0, S)\) be sequences fulfilling
\[
\hat{\ell}_n(0) = 0, \quad \hat{\ell}_n(S) = T, \quad f.a.s. \ s \in [0, S] : 0 \leq \hat{\ell}_n'(s) \leq 1, \tag{A.4}
\]
as well as
\[
\ell_n \to \ell \text{ in } H^1(0, T; V^*) \quad \text{and} \quad \hat{\ell}_n \rightharpoonup \hat{\ell} \text{ in } W^{1, \infty}(0, S). \tag{A.5}
\]
Set \(\hat{\ell}_n := \ell_n \circ \hat{\ell} \) for \(n \in \mathbb{N}\) and \(\hat{\ell} := \ell \circ \hat{\ell} \). Then it holds that
\[
\hat{\ell}_n \to \hat{\ell} \text{ in } H^1(0, S; V^*),
\]
for all \(s \in [0, S] \): \(\hat{\ell}_n(s) \rightharpoonup \hat{\ell}(s) \) weakly in \(V^*\).

\textbf{Proof.} \textbf{Proof of (1):} Let us first prove the chain rule \((A.1)\) in analogy to the finite dimensional case. Let \(s_0 \in [0, S]\) be such that \(\hat{\ell}\) is differentiable in \(s_0\) and \(\ell\) is differentiable in \(\hat{\ell}(s_0)\). According to [CH98] Thm. 1.4.35, this is the case almost everywhere in \([0, S]\). For \(t, t_0 \in [0, T]\), we define
\[
D(t, t_0) := \begin{cases} \ell(t) - \ell(t_0), & \text{if } t \neq t_0, \\ \ell'(t_0), & \text{if } t = t_0. \end{cases}
\]
Then, for those points \(t_0\) in which \(\ell\) is differentiable, the map \(D(\cdot, t_0) : [0, T] \to V^*\) is norm-continuous in \(t_0\). Therefore, it holds in \(V^*\) that
\[
\lim_{s \to s_0} \frac{\ell(\hat{\ell}(s)) - \ell(\hat{\ell}(s_0))}{s - s_0} = \lim_{s \to s_0} \left( \frac{D(\hat{\ell}(s), \hat{\ell}(s_0)) \cdot (\hat{\ell}(s) - \hat{\ell}(s_0))}{s - s_0} \right)
\]
\[
= \lim_{s \to s_0} \left( D(\hat{\ell}(s), \hat{\ell}(s_0)) \right) \cdot \lim_{s \to s_0} \left( \frac{\hat{\ell}(s) - \hat{\ell}(s_0)}{s - s_0} \right)
\]
\[
= D(\hat{\ell}(s_0), \hat{\ell}(s_0)) \hat{\ell}'(s_0)
\]
\[
= \ell'(\hat{\ell}(s_0)) \hat{\ell}'(s_0),
\]
where the second to last equation follows from the fact that \(\hat{\ell}\) is continuous, and we infer \((A.1)\).

Now, since \(H^1(0, T; V^*) \subset C([0, T], V^*)\) with a continuous embedding, it holds that
\[
\|\ell\|_{L^\infty(0, T; V^*)} = \|\ell\|_{C([0, T], V^*)} \leq C \|\ell\|_{H^1(0, T; V^*)} < \infty
\]
for the corresponding embedding constant \(C > 0\), whence
\[
\|\ell \circ \hat{\ell}\|_{L^\infty(0, S; V^*)} \leq \sqrt{S} \cdot \|\ell\|_{L^\infty(0, T; V^*)} \leq \sqrt{S} \cdot C \|\ell\|_{H^1(0, T; V^*)},
\]
which is \((A.2)\). Furthermore, due to the chain rule \((A.1)\), it holds that
\[
\int_0^S \| (\ell \circ \hat{\ell})'(s) \|_{V^*}^2 \; ds = \int_0^S \| \ell'(\hat{\ell}(s)) \|_{V^*}^2 \; ds
\]
\[
\leq \| \hat{\ell}' \|_{L^\infty(0, S)} \int_0^S \| \ell'(\hat{\ell}(s)) \|_{V^*} \| \hat{\ell}'(s) \| \; ds
\]
\[
= \| \hat{\ell}' \|_{L^\infty(0, S)} T \int_0^T \| \ell'(t) \|_{V^*}^2 \; dt,
\]
yielding \((A.3)\), and we obtain that \(\ell \circ \hat{\ell} \in H^1(0, S; V^*)\), which finishes the proof of (1).

\textbf{Proof of (2):} Let \((\ell_n)_{n \in \mathbb{N}} \subset H^1(0, T; V^*)\) and \((\hat{\ell}_n)_{n \in \mathbb{N}} \subset W^{1, \infty}(0, S)\) be sequences fulfilling the assumptions \((A.4)\) and \((A.5)\). We conclude by means of (1) that \((\ell_n)_{n \in \mathbb{N}} \subset H^1(0, S; V^*)\) and that \(\sup_{n \in \mathbb{N}} \| \ell_n \|_{H^1(0, S; V^*)} < \infty\), so that there is \(\hat{\ell} \in H^1(0, S; V^*)\) such that \(\hat{\ell}_n \rightharpoonup \hat{\ell} \) in \(H^1(0, S; V^*)\) along a subsequence. In order to identify the limit, we first employ the embedding \(H^1(0, T; V^*) \subset C^{0, \frac{1}{2}} \left([0, T]; V^*\right)\) (cf. [CH98] Cor. 1.4.38) to infer for every \(s \in [0, S]\) that
\[
\|\ell_n(\hat{\ell}_n(s)) - \ell_n(\hat{\ell}(s))\|_{V^*} \leq \|\ell_n\|_{C^{0, \frac{1}{2}} \left([0, T]; V^*\right)} \left|\hat{\ell}_n(s) - \hat{\ell}(s)\right|^\frac{1}{2} \leq C\|\hat{\ell}_n - \hat{\ell}\|_{\frac{1}{2}} \to 0.
\]
Moreover, the continuity of the above embedding also implies that for every \( t \in [0, T] \), the operator 
\[
\gamma_t : H^1(0, T; V^*) \to V^*, \quad \gamma_t(\xi) := \xi(t)
\]
is well-defined and continuous, whence we further have that 
\[
\ell_n(\ell(s)) - \ell(\ell(s)) \to 0 \quad \text{in } V^* \quad \text{for every } s \in [0, S].
\]
Altogether we have the pointwise weak convergence 
\[
\ell_n(s) - \ell(s) = \ell_n(\ell_n(s)) - \ell_n(\ell(s)) + \ell_n(\ell(s)) - \ell(\ell(s)) \to 0 \quad \text{in } V^*.
\]
Thus, it holds that \( \ell_n \rightharpoonup \ell = \ell \in H^1(0, S; V^*) \) along a subsequence. A standard proof by contradiction finally concludes the proof of Lemma A.1. 

\[\square\]

**Lemma A.2.** Let \( 0 < h < h_0 \). For \( v \in H^1(0, S; V^*) \), consider the constant continuation to the interval \([0, S + h]\) and define the operator \( L_h : H^1(0, S; V^*) \to L^2(0, S; V^*) \) by 
\[
L_h(v)(t) := \frac{1}{h} (v(t + h) - v(t)).
\]
Then the following are true

1. \( L_h \) is well defined, linear and continuous and for every \( v \in H^1(0, S; V^*) \), it holds 
   \[
   \|L_h v\|_{L^2(0, S; V^*)} \leq \|v\|_{L^2(0, S; V^*)}. 
   \]
2. For all \( v \in H^1(0, S; V^*) \), it holds that \( L_h v \to v' \) strongly in \( L^2(0, S; V^*) \).
3. For \( v \in L^2(0, S; V) \), consider again the constant continuation to the interval \([0, S + h]\) and define the operator \( S_h : L^2(0, S; V) \to L^2(0, S; V) \) by 
   \[
   S_h(v)(s) := v(s + h). 
   \]
   Then 
   \[
   \|S_h v \to v \text{ strongly in } L^2(0, S; V). 
   \]

**Proof.**

**Proof of (1):** First assume that \( v \in C^\infty([0, S]; V^*) \), then we have 
\[
\|L_h v\|_{L^2(0, S; V^*)}^2 = \int_0^S \int_0^1 \left\| v'(t + sh) \right\|_{V^*}^2 \, dt \, ds 
\]
\[
= \int_0^1 \int_0^S \left\| v'(t + sh) \right\|_{V^*}^2 \, dt \, ds = \int_0^1 \left\| v' \right\|_{L^2(0, S; V^*)}^2 \, ds 
\]
\[
\leq \left\| v' \right\|_{L^2(0, S; V^*)}^2.
\]
By density of \( C^\infty([0, S]; V^*) \) in \( H^1(0, S; V^*) \) (cf. [Emm04 Satz 8.1.9]), we obtain (1).

**Proof of (2):** Again assume that \( v \in C^\infty([0, S]; V^*) \). Then it holds that 
\[
\|L_h v - v'\|_{L^2(0, S; V^*)} \leq \int_0^S \left\| \frac{v(t + h) - v(t)}{h} - v'(t) \right\|_{V^*}^2 \, dt 
\]
\[
= \frac{E}{h} \to 0,
\]

since the integrand is dominated by \( C\|v'\|_{L^\infty(0, S; V^*)} \) for a constant \( C > 0 \). Now let \( v \in H^1(0, S; V^*) \) and \( \eta > 0 \) and choose \( \overline{v} \in C^\infty([0, S]; V^*) \) such that \( \|v - \overline{v}\|_{H^1(0, S; V^*)} \leq \frac{\eta}{3} \) and \( h_0 > 0 \) so small that for all \( 0 < h < h_0 \) it holds that \( \|L_h \overline{v} - \overline{v}'\|_{L^2(0, S; V^*)} < \frac{\eta}{3} \). Then we have for all \( 0 < h < h_0 \)
\[
\|L_h v - v'\|_{L^2(0, S; V^*)} \leq \|L_h v - L_h \overline{v}\|_{L^2(0, S; V^*)} + \|L_h \overline{v} - \overline{v}'\|_{L^2(0, S; V^*)} + \|\overline{v}' - v'\|_{L^2(0, S; V^*)} 
\]
\[
\leq 2\|\overline{v} - v'\|_{L^2(0, S; V^*)} + \|L_h \overline{v} - \overline{v}'\|_{L^2(0, S; V^*)} + \|\overline{v}' - v'\|_{L^2(0, S; V^*)} 
\]
\[
< \eta,
\]
which finishes the proof of (2).

**Proof of (3):** Let again \( v \in C^\infty([0, S]; V) \), then it holds that 
\[
\|S_h v - v\|_{L^2(0, S; V)}^2 = \int_0^S \|v(s + h) - v(s)\|_V^2 \, ds 
\]
\[
\leq C \int_0^{S-h} \|v(s + h) - v(s)\|_V^2 \, ds + \int_{S-h}^S \|v(S) - v(s)\|_V^2 \, ds 
\]
\[
\leq C \int_0^{S-h} \|v(s + h) - v(s)\|_V^2 \, ds + h\|v\|_{L^\infty(0, S; V)}^2 
\]
\[
\xrightarrow{h \to 0} 0,
\]

\[\square\]
since the first integrand converges to 0 and is bounded by $2\|v\|_{L^\infty(0, S; V)}$. Now, let $v \in L^2(0, S; V)$ and $\eta > 0$. Since $C^\infty([0, S], V)$ is dense in $L^2(0, S; V)$ according to \cite[Remark 2.2.4]{GP06}, we can choose $\bar{v} \in C^\infty([0, S]; V)$ such that $\|v - \bar{v}\|_{L^2(0, S; V)} < \frac{\eta}{4}$. We further find $h_1 > 0$ so small that for all $0 < h < h_1$, it holds $\|S_h \bar{v} - \bar{v}\|_{L^2(0, S; V)} < \frac{\eta}{4}$ as well as $h < \frac{\eta}{4\|\bar{v}(S) - \bar{v}(S_0)\|_{V}}$. This implies for all $0 < h < h_1$ that

$$
\|S_h v - v\|_{L^2(0, S; V)}^2 \\
\leq \|S_h v - S_h \bar{v}\|_{L^2(0, S; V)}^2 + \|S_h \bar{v} - \bar{v}\|_{L^2(0, S; V)}^2 + \|v - \bar{v}\|_{L^2(0, S; V)}^2 \\
\leq \|v - \bar{v}\|_{L^2(0, S; V)}^2 + h\|v(S) - \bar{v}(S)\|_{V}^2 + \|S_h \bar{v} - \bar{v}\|_{L^2(0, S; V)}^2 + \|v - \bar{v}\|_{L^2(0, S; V)}^2 \\
< \eta,
$$

and (3) is proven. \hfill \Box

\section*{Appendix B. Lower semicontinuity properties}

For the following Proposition we refer to \cite[Lemma 3.1]{MRS09}.

\begin{proposition}
Let $v_n, v \in L^\infty(0, S; V)$ with $v_n \rightharpoonup^* v$ in $L^\infty(0, S; V)$ and $\delta_n, \delta \in L^1(0, S; [0, \infty))$ with $\liminf_{n \to \infty} \delta_n(s) \geq \delta(s)$ for almost all $s$. Then

$$\liminf_{n \to \infty} \int_0^S \|v_n(s)\|_V \delta_n(s) \, ds \geq \int_0^S \|v(s)\|_V \delta(s) \, ds. \tag{B.1}$$

\end{proposition}

The next lemma that we cite from \cite[Lemma 4.3]{MRS12} is closely related to the previous proposition:

\begin{lemma}
Let $I \subset \mathbb{R}$ be a bounded interval and $f, g, f_n, g_n : I \to [0, \infty)$, $n \in \mathbb{N}$, measurable functions satisfying $\liminf_{n \to \infty} f_n(s) \geq f(s)$ for a.a. $s \in I$ and $g_n \rightharpoonup g$ weakly in $L^1(I)$. Then

$$\liminf_{n \to \infty} \int_I f_n(s) g_n(s) \, ds \geq \int_I f(s) g(s) \, ds. $$

\end{lemma}

A variant of the following lower semicontinuity property can be found in \cite[Lemma A.1]{KRZ11}.

\begin{lemma}
Let $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^*$ be sequences such that $\delta_n \searrow 0$ and $\eta_n \rightharpoonup \eta$ weakly in $\mathbb{Z}^*$. Then, it holds

$$\liminf_{n \to \infty} \mathcal{R}^*_n(\eta_n) \geq \mathcal{R}^*_0(\eta).$$

\end{lemma}

\begin{proof}
Recall the definition $\mathcal{R}_n^*(v) := \mathcal{R}_n(v) + \frac{\delta}{2}(Av, v)_{Z^*, Z} = \mathcal{R}(v) + \frac{1}{2}(Av, v) + \frac{\delta}{2}(Av, v)_{Z^*, Z} =: \mathcal{R}(v) + \mathcal{R}_2(v) + \mathcal{R}_{2, \delta}(v)$ for $v \in Z$. Since both $V \subset \operatorname{Lin}(V, \mathbb{V}^*)$ and $A \subset \operatorname{Lin}(Z, \mathbb{Z}^*)$ are supposed to be linear, continuous, symmetric and elliptic operators (cf. (2.2)), they define equivalent norms $\|\cdot\|_V$ and $\|\cdot\|_A$ on the spaces $V$ and $Z$, respectively. We denote the corresponding induced norms on the dual spaces $\mathbb{V}^*$ and $\mathbb{Z}^*$ by $\|\xi\|_{V^*} = \sqrt{\langle \xi, V^{-1}\xi \rangle}$ and $\|\cdot\|_{A^*} = \sqrt{\langle \xi, A^{-1}\xi \rangle}$, respectively. By standard arguments, we thus obtain for $\eta \in \mathbb{Z}^*$

$$\mathcal{R}^*_2(\eta) = \begin{cases} 0, & \text{if } \eta \in \partial \mathcal{R}(0), \\ \infty, & \text{if } \eta \in \mathbb{Z}^* \setminus \partial \mathcal{R}(0), \end{cases}$$

$$\mathcal{R}^*_3(\eta) = \begin{cases} \frac{1}{2}\|\eta\|_V^2, & \text{if } \eta \in \mathbb{V}^*, \\ \infty, & \text{if } \eta \in \mathbb{Z}^* \setminus \mathbb{V}^*. \end{cases}$$

Furthermore, an application of \cite[Theorem 3.3.4.1]{IT79} yields

$$\mathcal{R}_n^*(\eta) = \inf\{\mathcal{R}_n^*(\eta_1) + \mathcal{R}_n^*(\eta_2) + \mathcal{R}_{2, \delta}(\eta_3) \mid \eta_1 + \eta_2 + \eta_3 = \eta\}$$

$$= \inf\{\mathcal{R}_n^*(\eta_2) + \mathcal{R}_{2, \delta}(\eta_3) \mid \eta - \eta_2 - \eta_3 \in \partial \mathcal{R}(0)\}$$

$$= \min\{\mathcal{R}_n^*(\eta_2) + \mathcal{R}_{2, \delta}(\eta_3) \mid \eta - \eta_2 - \eta_3 \in \partial \mathcal{R}(0)\},$$

since $\partial \mathcal{R}(0)$ is a weakly closed subset of $\mathbb{V}^*$.

Now, let $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(\eta_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^*$ be sequences such that $\delta_n \searrow 0$ and $\eta_n \rightharpoonup \eta$ weakly in $\mathbb{Z}^*$. We denote the subsequence realizing the limit inferior with the same symbols for simplicity.
and choose \( \eta^o_2, \eta^o_3 \in \mathcal{Z}^* \) such that \( \eta^o_n - \eta^o_2 - \eta^o_3 \in \partial \mathcal{R}(0) \) and \( \mathcal{R}^*_n(\eta_n) = \mathcal{R}^*_2(\eta^o_2) + \mathcal{R}^*_{2,\delta_n}(\eta^o_3) \). Assume that
\[
\lim_{n \to \infty} \mathcal{R}^*_n(\eta_n) = \lim_{n \to \infty} \mathcal{R}^*_2(\eta^o_2) + \mathcal{R}^*_{2,\delta_n}(\eta^o_3) \geq 0.
\]
This implies that \( \sup_{n \in \mathbb{N}} \frac{1}{2\delta_n} \| \eta^o_3 \|_{\mathcal{V}^{-1}}^2 < \infty \), whence \( \eta^o_3 \to \eta_2 \) strongly in \( \mathcal{Z}^* \) as well as \( \sup_{n \in \mathbb{N}} \| \eta^o_n \|_{\mathcal{V}^{-1}} < \infty \). Thus, there exists a limit \( \eta_2 \in \mathcal{V}^* \) such that \( \eta^o_2 \to \eta_2 \) weakly in \( \mathcal{V}^* \) and \( \eta^o_2 \to \eta_2 \) strongly in \( \mathcal{Z}^* \). Again, the closedness of \( \partial \mathcal{R}(0) \) yields \( \eta_n - \eta^o_2 - \eta^o_3 \to \eta - \eta_2 \in \partial \mathcal{R}(0) \) strongly in \( \mathcal{Z}^* \). We can now use the weak lower semincontinuity of \( \mathcal{R}^*_2 \) on \( \mathcal{V}^* \) to infer that
\[
\lim_{n \to \infty} \mathcal{R}^*_n(\eta_n) = \lim_{n \to \infty} \left( \mathcal{R}^*_2(\eta^o_2) + \mathcal{R}^*_{2,\delta_n}(\eta^o_3) \right)
\]
\[
\geq \lim_{n \to \infty} \left( \mathcal{R}^*_2(\eta^o_2) + \mathcal{R}^*_2(\eta^o_3) \right)
\]
\[
\geq \mathcal{R}^*_2(\eta_2)
\]
\[
\geq \inf \{ \mathcal{R}^*_2(\tilde{\eta}) \mid \eta - \tilde{\eta} \in \partial \mathcal{R}(0) \}
\]
\[
= \mathcal{R}^*_2(\eta),
\]
where in the last step we have again used \( [\text{IT}79] \) Theorem 3.3.4.1 to determine \( \mathcal{R}^*_2 \). \( \square \)

**Appendix C. Absolutely continuous functions and BV-functions**

We follow \([\text{MRS}16] \) Section 2.2. Let \( X \) be a Banach space and let \( \mathcal{R} : X \to \mathbb{R} \) be convex, lower semicontinuous, positively homogeneous of degree one and with \( (2.8) \). For \( 1 \leq p \leq \infty \), we define the set of \( p \)-absolutely continuous functions (related to \( \mathcal{R} \)) as
\[
AC^p([a,b];X) := \{ z : [a,b] \to X ; \exists m \in L^p((a,b)), m \geq 0, \forall s_1 < s_2 \in [a,b] : \mathcal{R}(z(s_2) - z(s_1)) \leq \int_{s_1}^{s_2} m(r) \, dr \}.
\]

Observe that thanks to \( (2.8) \) this set coincides with the one defined with \( \| \cdot \|_X \) instead of \( \mathcal{R} \). Let \( z \in AC^p([a,b];X) \). It is shown in \([\text{RMS}08] \) Prop. 2.2, \([\text{AGS}05] \) Thm. 1.1.2] that for almost every \( s \in [a,b] \) the limits
\[
\mathcal{R}[z'](s) := \lim_{h \downarrow 0, h > 0} \mathcal{R}(z(s+h) - z(s))/h = \lim_{h \downarrow 0} \mathcal{R}((z(s) - z(s-h))/h)
\]
exist and are equal, that \( \mathcal{R}[z'] \in L^p((a,b)) \) and that \( \mathcal{R}[z'] \) is the smallest function for which the integral estimate in \( (C.1) \) is valid.

Let further \( \text{Var}_{\mathcal{R}}(z;[a,b]) \) denote the \( \mathcal{R} \)-variation of \( z : [a,b] \to X \), i.e.
\[
\text{Var}_{\mathcal{R}}(z;[a,b]) := \sup_{\text{partitions of } [a,b]} \sum_{i=1}^{m} \mathcal{R}(z(s_i) - z(s_{i-1})).
\]

**Lemma C.1.** For all \( p \in (1, \infty] \) and \( z \in AC^p([a,b];X) \) we have
\[
\text{Var}_{\mathcal{R}}(z;[a,b]) = \int_{a}^{b} \mathcal{R}[z'](s) \, ds.
\]

**Proof.** Since \( \mathcal{R}(z(s_i) - z(s_{i-1})) \leq \int_{s_{i-1}}^{s_i} \mathcal{R}[z'](s) \, ds \) the upper estimate \( \text{Var}_{\mathcal{R}}(z;[a,b]) \leq \int_{a}^{b} \mathcal{R}[z'](s) \, ds \) is immediate. In order to obtain the opposite estimate we follow the ideas in the proof of Corollary 3.3.4 in \([\text{AGS}05] \). Let \( z \in AC^p([a,b];X) \) and choose a sequence of equidistant partitions \( \pi_h \) of \([a,b]\) with fineness \( h > 0 \). Let \( z_h : [a,b] \to X \) be the linear interpolant of \( z \) with respect to \( \pi_h \). Since \( z \in AC^p([a,b];X) \), the sequence \( (z_h)_h \) converges pointwise in \( X \) to \( z \) as \( h \) tends to zero. Moreover, direct calculations show that for all \( h > 0 \) we have \( \| \mathcal{R}[z'_h] \|_{L^p((a,b))} \leq \| \mathcal{R}[z'] \|_{L^p((a,b))} \). Hence, there exists \( A \in L^p((a,b)) \) and a subsequence such that \( \mathcal{R}[z'_h] \to A \) weakly(*) in \( L^p((a,b)) \).
Now for almost every $s \in (a, b)$ we conclude $A(s) \geq \mathcal{R}[z'](s)$. This can be seen as follows: Let $s_1 < s_2 \in (a, b)$ be arbitrary. Then

$$
\mathcal{R}(z(s_2) - z(s_1)) \leq \liminf_{h \to 0} \mathcal{R}(z_h(s_2) - z_h(s_1)) \leq \limsup_{h \to 0} \int_{s_1}^{s_2} \mathcal{R}(z'_h(r)) \, dr = \int_{s_1}^{s_2} A(r) \, dr.
$$

Hence, for almost every $s \in (a, b)$ we have

$$
\mathcal{R}[z'](s) = \lim_{h \to 0} \mathcal{R}((z(s + h) - z(s))/h) \leq h^{-1} \int_s^{s+h} A(r) \, dr = A(s).
$$

On the other hand, for all $h > 0$ it holds $\text{Var}_{\mathcal{R}}(z; [a, b]) \geq \int_a^b \mathcal{R}(z'_h(r)) \, dr$. Taking the limit $h \to 0$ we ultimately arrive at $\text{Var}_{\mathcal{R}}(z; [a, b]) \geq \int_a^b A(r) \, dr \geq \int_a^b \mathcal{R}[z'](r) \, dr$. □

Appendix D. A Combination of Helly’s Theorem and the Ascoli-Arzelà Theorem

The arguments are closely related to those in [MRS16, AGS05].

**Proposition D.1.** Let $\mathbb{Z}$ be a reflexive Banach space, $\mathcal{V}$, $\mathcal{X}$ further Banach spaces such that (2.1) is satisfied and assume that $\mathcal{R} : \mathcal{X} \to [0, \infty)$ complies with (2.8).

(a) The set $AC^1([a, b]; \mathcal{X}) \cap L^\infty((a, b); \mathbb{Z})$ is contained in $C([a, b]; \mathcal{V})$ and there exists $C > 0$ such that for all $z \in AC^1([a, b]; \mathcal{X}) \cap L^\infty((a, b); \mathbb{Z})$ we have

$$
\|z\|_{C([a, b]; \mathcal{V})} \leq C(\|z\|_{L^\infty((a, b); \mathbb{Z})} + \|\mathcal{R}[z']\|_{L^1((a, b))}).
$$

(b) Let $(z_n)_n \subset AC^\infty([a, b]; \mathcal{X}) \cap L^\infty((a, b); \mathbb{Z})$ be uniformly bounded in the sense that $A := \sup_n \|z_n\|_{L^\infty((a, b); \mathbb{Z})} < \infty$ and $B := \sup_n \|\mathcal{R}[z']\|_{L^\infty((a, b))} < \infty$.

Then there exists $z \in AC^\infty([a, b]; \mathcal{X}) \cap L^\infty((a, b); \mathbb{Z})$ and a (not relabeled) subsequence $(z_n)_n$ such that

$$
z_n \to z \text{ uniformly in } C([a, b]; \mathcal{V}),
$$

$$
\forall t \in [a, b] \quad z_n(t) \to z(t) \text{ weakly in } \mathbb{Z}.
$$

(c) It is $L^\infty((a, b); \mathbb{Z}) \cap C([a, b]; \mathcal{V}) \subset C_{\text{weak}}([a, b]; \mathbb{Z})$.

**Proof.** In order to verify (a) let $z \in AC^1([a, b]; \mathcal{X}) \cap L^\infty((a, b); \mathbb{Z})$. By the Ehrling Lemma, [Wlo87], for every $\mu > 0$ there exists $C_\mu > 0$ such that for all $t, s \in [a, b]$ we have

$$
\|z(t) - z(s)\|_{\mathcal{V}} \leq \mu \|z(t) - z(s)\|_{\mathbb{Z}} + C_\mu \|z(t) - z(s)\|_{\mathcal{X}}
$$

$$
\leq 2\mu \|z\|_{L^\infty((a, b); \mathbb{Z})} + \tilde{C}_\mu \int_s^t \mathcal{R}[z'](r) \, dr.
$$

This implies that $z \in C([a, b]; \mathcal{V})$ together with the norm estimate and (a) is proved.

For (b) let $(z_n)_n \subset AC^\infty([a, b]; \mathcal{X}) \cap L^\infty((a, b); \mathbb{Z})$ as in part (b) of the Proposition. Again by Ehrling’ Lemma for every $\mu > 0$ there exists $C_\mu > 0$ such that for all $t > s \in [a, b]$ and $n \in \mathbb{N}$ we have

$$
\|z_n(t) - z_n(s)\|_{\mathcal{V}} \leq \mu \|z_n(t) - z_n(s)\|_{\mathbb{Z}} + C_\mu \|z_n(t) - z_n(s)\|_{\mathcal{X}}
$$

$$
\leq 2\mu A + \tilde{C}_\mu \int_s^t \mathcal{R}[z'_n](r) \, dr \leq 2\mu A + C_\mu B |t - s|.
$$

This implies the equicontinuity of the sequence $(z_n)_n$ in $C([a, b]; \mathcal{V})$. Indeed, for $\varepsilon > 0$ choose $\mu < \varepsilon/(4A)$ and $\delta < \varepsilon/(2BC_\mu)$. Then for all $n \in \mathbb{N}$, $s, t \in [a, b]$ with $|s - t| < \delta$ we have $\|z_n(s) - z_n(t)\|_{\mathcal{V}} \leq \varepsilon$. Together with $z_n(t) \in \mathcal{X}$ for all $t$ and $n$, by the classical version of the Arzelà-Ascoli Theorem, see e.g. [Die69], we obtain (D.1) for a subsequence. After possibly extracting a
Proposition E.1. \hspace{1em} Let $z \in H^1((0,T); \mathcal{V}) \cap L^\infty((0,T); \mathcal{Z})$ and $\mathcal{D}(z(\cdot)) \in L^\infty((0,T); \mathcal{V}^*)$. Then for almost all $t$, the mapping $t \mapsto \mathcal{I}(z(t))$ is differentiable and we have the identity
\[ \frac{d}{dt} \mathcal{I}(z(t)) = \langle Az(t), \dot{z}(t) \rangle_{\mathcal{V}^*, \mathcal{V}} + \langle \mathcal{D}(z(t)), \dot{z}(t) \rangle_{\mathcal{V}^*, \mathcal{V}}. \]

Integrated version of the chain rule: Let $z \in W^{1,1}((0,T); \mathcal{V}) \cap L^\infty((0,T); \mathcal{Z})$ with $\mathcal{D}(z(\cdot)) \in L^\infty((0,T); \mathcal{V}^*)$ and assume that $t \mapsto \mathcal{I}(z(t))$ is continuous on $[0,T]$. Then for all $t_1 < t_2 \in [0,T]$
\[ \mathcal{I}(z(t_2)) - \mathcal{I}(z(t_1)) = \int_{t_1}^{t_2} \langle \mathcal{D}(z(r)), \dot{z}(t) \rangle_{\mathcal{V}^*, \mathcal{V}} \, dr. \] \hspace{1em} (E.1)

Proof. For the proof of the integrated version of the chain rule let $t_1 < t_2 \in [0,T)$ and $h_0 > 0$ such that $t_2 + h_0 \leq T$. Then for all $0 < h \leq h_0$ the $\lambda$-convexity estimate \([4.11]\) implies
\[ h^{-1} \int_{t_1}^{t_2} \mathcal{I}(z(t+h)) - \mathcal{I}(z(t)) \, dt \geq \int_{t_1}^{t_2} \langle \mathcal{D}(z(t)), h^{-1}(z(t+h) - z(t)) \rangle_{\mathcal{V}^*, \mathcal{V}} \, dt - \frac{\lambda}{8} \int_{t_1}^{t_2} \|z(t+h) - z(t)\|_{\mathcal{V}}^2 \, dt, \]
where $\lambda > 0$ depends on $\|z\|_{L^\infty((0,T); \mathcal{Z})}$. Thanks to the continuity of $\mathcal{I}(z(\cdot))$, for the left hand side we obtain $\lim_{h \to 0} h^{-1} \int_{t_1}^{t_2} \mathcal{I}(z(t+h)) - \mathcal{I}(z(t)) \, dt = \mathcal{I}(z(t_2)) - \mathcal{I}(z(t_1))$. Since $z \in W^{1,1}((0,T); \mathcal{V})$, on each $(t_1, t_2) \subseteq (0, T)$ the difference quotients converge strongly in the following sense: $h^{-1}(z(\cdot + h) - z(\cdot)) \to \dot{z}(\cdot)$ strongly in $L^1((t_1, t_2); \mathcal{V})$, \([\text{CH98}]\) Cor. 1.4.39. Thus the first integral on the right hand side converges to $\int_{t_1}^{t_2} \langle \mathcal{D}(z(r)), \dot{z}(t) \rangle_{\mathcal{V}^*, \mathcal{V}} \, dr$, while the second integral on the right hand side converges to zero. A similar argument for $h < 0$ finally proves \([E.1]\). \hspace{1em} □

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