Accelerating Universe via Spatial Averaging

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We present a model of an inhomogeneous universe that leads to accelerated expansion after taking spatial averaging. The model universe is the Tolman-Bondi solution of the Einstein equation and contains both a region with positive spatial curvature and a region with negative spatial curvature. We find that after the region with positive spatial curvature begins to re-collapse, the deceleration parameter of the spatially averaged universe becomes negative and the averaged universe starts accelerated expansion. We also discuss the generality of the condition for accelerated expansion of the spatially averaged universe.

Keywords: inhomogeneous universe; dark energy; back reaction

Introduction.— The existence of the dark energy component is considered to be necessary to explain present acceleration of the Universe indicated by recent observational data[1, 2]. Although a cosmological constant or a negative pressure fluid are candidates for the dark energy, we do not know the true character of them and they are left as black boxes.

More conservative approach to explain the acceleration of the Universe without introduction of exotic fields is to utilize a backreaction effect due to inhomogeneities of the Universe[4, 5, 6, 7, 8, 9]. In this approach, we have to fit a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) universe to an inhomogeneous universe[10]. This is done by taking spatial average of fluctuations of metric and matter fields. The effect of the inhomogeneities modifies the evolution of spatially averaged “background” FRW universe.

Many researchers have tried to apply this idea to the dark energy problem, no one has yet succeeded to obtain an workable example that explains the accelerated expansion of the Universe. This is due to the non-linearity of the Einstein equation. Usually, the backreaction effect in the Universe is evaluated by cosmological perturbation theory. However, after density fluctuation in the Universe grows to be non-linear and begins to re-collapse, perturbative expansion breaks down and we cannot obtain reliable results beyond this time based on perturbative calculation[13]. Thus, we expect that non-perturbative feature of inhomogeneity is crucial to obtain a workable model that explains the accelerated expansion of the Universe by the backreaction effect.

In this letter, we consider a model of inhomogeneous universe that is an exact solution of the Einstein equation with dust field. By taking spatial average of this model explicitly, we obtained an accelerated expansion of the spatially averaged universe provided that the following two conditions are satisfied:

1. The universe contains both a region with positive spatial curvature and a region with negative spatial curvature.
2. The region with positive spatial curvature is in contracting phase.

The degree of the acceleration of the averaged universe depends on the scale of the initial inhomogeneity. We use units in which \( c = h = 8\pi G = 1 \) throughout the letter.

Averaging of Tolman-Bondi model.— We consider a spherically symmetric solution of the Einstein equation with dust field. This is the Lemaitre-Tolman-Bondi solution[12] and the metric is given by

\[
 ds^2 = -dt^2 + \frac{(R_r)^2}{1 + 2E(r)}dr^2 + R^2d\Omega^2, \tag{1}
\]

where \( E(r) \) and \( M(r) \) are arbitrary functions of \( r \). The solution of Eq. (2) can be written parametrically by using a variable \( \eta = \int dt/R \),

\[
 R(\eta, r) = \frac{M(r)}{-2E(r)} \left[ 1 - \cos \left( \sqrt{-2E(r)} \eta \right) \right],
\]

\[
 t(\eta, r) = \frac{M(r)}{-2E(r)} \left[ \eta - \frac{1}{\sqrt{-2E(r)}} \sin \left( \sqrt{-2E(r)} \eta \right) \right]. \tag{3}
\]

By introducing the following variables

\[
 a(t, r) = \frac{R(t, r)}{r}, \quad k(r) = -\frac{2E(r)}{r^2}, \quad \rho_0(r) = \frac{6M(r)}{r^3}, \tag{4}
\]

the metric and the evolution equation for the scale factor \( a(t, r) \) become

\[
 ds^2 = -dt^2 + a^2 \left[ \left(1 + \frac{a_r}{a}\right)^2 \frac{dr^2}{1 - k(r)r^2} + r^2d\Omega^2 \right], \tag{5}
\]

\[
 \left( \frac{\dot{a}}{a} \right)^2 = -\frac{k(r)}{a^2} + \frac{\rho_0(r)}{3a^3}. \tag{6}
\]

Eq. (6) is same as the Friedmann equation with dust and we can regard the Tolman-Bondi solution as a model of inhomogeneous universe of which local behavior is equivalent to a FRW universe with a curvature constant \( k(r) \).

As a specific case, we assume the following spatial distribution of spatial curvature,

\[
 k(r) = \frac{1}{L^2} \left[ 2\theta(r - r_0) - 1 \right], \quad 0 \leq r \leq L, \quad 0 \leq r_0 \leq L \tag{7}
\]
and assume that $\rho_0(r) = \rho_0 = \text{constant}$. In the region $1: 0 \leq r < r_0$, the solution is that of a spatially open FRW universe and in the region $2: r_0 < r \leq L$, the solution is that of a spatially closed FRW universe. The spatial volume of the comoving region $D: 0 \leq r \leq L$ is

$$V_D(t) = \int_0^L dr \frac{r^2}{\sqrt{1-k(r)r^2}} \left(1 + \frac{a(r)}{a^3}\right)$$

$$= L^3 \left(c_1 a_1^3 + c_2 a_2^3\right),$$

where

$$c_1 = \int_0^{r_0/L} \frac{x^2}{1+x^2} dx, \quad c_2 = \int_{r_0/L}^1 \frac{x^2}{1-x^2} dx$$

and the scale factor $a_1, a_2$ obey the following Friedmann equations:

$$\left(\frac{\dot{a}_1}{a_1}\right)^2 = + \frac{1}{L^2 a_1^2} + \frac{\rho_0}{3a_1^2}, \quad \left(\frac{\dot{a}_2}{a_2}\right)^2 = - \frac{1}{L^2 a_2^2} + \frac{\rho_0}{3a_2^2}.$$ (10)

We define a spatially averaged scale factor of the region $D$ by its physical volume

$$a_D \equiv \left(\frac{V_D(t)}{V_D(t_*)}\right)^{1/3},$$

where we have normalized $a_D = 1$ at a fixed time $t = t_*$. We can evaluate the deceleration parameter of the spatially averaged scale factor

$$q_D = -\frac{\ddot{a}_D a_D}{\dot{a}_D^2}.$$ (12)

Fig. 1 shows evolution of $q_D$ for different values of $r_0/L$. About $t \sim 0$, the averaged universe behaves as the spatially flat FRW model with dust. The averaged universe enters the accelerating phase at a certain time after the closed FRW region begins to re-collapse at $t = \pi r_0 L^3/6$ [14]. As the ratio $r_0/L$ becomes smaller, the averaged universe enters the accelerating phase earlier. For too small ratio $r_0/L < 0.4$, the averaged universe re-collapse and we do not have the accelerated expansion.

The condition for accelerated expansion.— To obtain the condition for accelerated expansion of the spatially averaged universe, we consider a model of inhomogeneous universe of which local scale factor $a(t, x)$ obeys the following Einstein equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k(x)}{a^2} + \frac{\rho}{3},$$

$$\ddot{a} - \frac{\rho}{6} \dot{a} + 3\left(\frac{\dot{a}}{a}\right)\rho = 0.$$ (14)

This model includes the Tolman-Bondi solution presented previously. We assume that the volume of the spatial region $D$ is given by

$$V_D = \int_D d^3 x \sqrt{h} a^3(t, x)$$

where $\sqrt{h}$ is a time independent function of spatial coordinate $x$ and its explicit form is not necessary for the following analysis. The spatially averaged scale factor is defined by Eq. (11) and the spatial average of a quantity $f(t, x)$ is defined by

$$\langle f \rangle = \frac{1}{V_D} \int_D d^3 x \sqrt{h} a^3 f.$$ (16)

The important property of the operation of the spatial averaging is that it does not commute with the time derivative [3]:

$$\langle f \rangle' = \langle \dot{f} \rangle + 3 \left[ \langle f H \rangle - \langle H \rangle \langle f \rangle \right].$$ (17)

The average of the local Hubble parameter $H = \dot{a}/a$ is

$$H_D \equiv \langle H \rangle = \frac{\dot{a}_D}{a_D}.$$ (18)

The Einstein equation for the averaged scale factor becomes

$$\left(\frac{\dot{a}_D}{a_D}\right)^2 = \frac{\rho_D}{3} - \mathcal{R} - Q,$$ (19)

$$\frac{\ddot{a}_D}{a_D} = -\frac{\rho_D}{6} + 2Q, \quad \rho_D + 3H_D \rho_D = 0,$$ (20)

where we have defined

$$\mathcal{R} = \langle \frac{k(x)}{a^2} \rangle, \quad Q = \langle H^2 \rangle - H_D^2 = \frac{\rho_D}{3} - \mathcal{R} - H_D^2.$$ (21)

The integrability condition [3] of Eq. (19) and Eq. (20) is

$$a_D^2 a_D^2 \mathcal{R}' + (a_D^2 Q)' = 0$$ (22)

and this equation does not yield independent evolution equation. Thus, we can integrate Eqs. (19) and (20) by assuming that the scale factor dependence of the averaged spatial curvature $\mathcal{R} = \mathcal{R}(a_D)$. After all, the Einstein equation for the averaged scale factor becomes

$$\left(\frac{\dot{a}_D}{a_D}\right)^2 = \frac{\rho_D}{3} - \frac{4}{a_D^2} \int_1^{a_D} dx x^5 \mathcal{R}(x),$$ (23)

$$\frac{\ddot{a}_D}{a_D} = -\frac{\rho_D}{6} - 2\mathcal{R}(a_D) + \frac{8}{a_D^2} \int_1^{a_D} dx x^5 \mathcal{R}(x).$$ (24)
The averaged dust density is \( \rho_D = \rho_* / a_D^3 \) where \( \rho_* = \rho_D(a_D = 1) \). We can solve these equations if we obtain the scale factor dependence of the averaged spatial curvature \( R \).

For the Tolman-Bondi model introduced previously, the averaged dust density is given by

\[
\rho_D = \left( \frac{\rho_0 L^3}{V_*} \right) \frac{c_1 + c_2 a_2}{a_D^3}, \quad V_* = V_D(a_D = 1)
\]  

and the averaged spatial curvature is

\[
R = \frac{L}{V_*} \left( -\frac{c_1 a_1 + c_2 a_2}{a_D^2} \right).
\]

We evaluate \( R \) about the time at which the closed FRW region (region 2) begins to re-collapse. The scale factor of the region 2 stays nearly constant \( a_2 \approx \rho_0 L^2 / 3 \ll a_1 \) around the maximal expansion and the averaged spatial curvature is approximated to be \([15]\)

\[
R \approx \frac{K}{a_D^2} + \frac{C}{a_D^4},
\]

where

\[
K = -\left( \frac{c_1}{V_*} \right)^{2/3}, \quad C = \frac{\rho_0 L^3 c_2}{3V_*}.
\]

By substituting \([27]\) to Eqs. \([23]\) and \([24]\), we obtain

\[
\left( \frac{\dot{a}_D}{a_D} \right)^2 = \left( \frac{\rho_0}{3} - \frac{4C}{3} \right) \frac{1}{a_D^2} - K \frac{1}{a_D^2} + \left( K + \frac{4C}{3} \right) \frac{1}{a_D^2} - \frac{\rho_{\text{eff}}}{3},
\]

\[
\dot{a}_D = \left( \frac{\rho_0}{6} + \frac{2C}{3} \right) \frac{1}{a_D^2} - 2 \left( K + \frac{4C}{3} \right) \frac{1}{a_D^2} - \frac{1}{6}(\rho_{\text{eff}} + 3p_{\text{eff}}),
\]

and for \( a_D \gg 1 \),

\[
\rho_{\text{eff}} \approx \rho_* - \frac{4C}{a_D^3} - \frac{3K}{a_D^2}, \quad p_{\text{eff}} \approx \frac{K}{a_D^2}.
\]

Asymptotically, the equation of state of the averaged universe is

\[
w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} \approx -\frac{1}{3} \left( \frac{4C - \rho_*}{-3K} \right) \frac{1}{a_D}.
\]

The condition for the accelerated expansion of the universe is \( \rho_{\text{eff}} > 0 \) and \( \rho_{\text{eff}} + 3p_{\text{eff}} < 0 \) and this yields the following relation between constants contained in our model:

\[
K < 0, \quad 0 < 4C - \rho_* < 3(-K).
\]

For the Tolman-Bondi model, the first condition \( K < 0 \) is automatically satisfied. This condition means that the average of spatial curvature in the inhomogeneous universe must be negative asymptotically in time. That is, the comoving region \( D \) must contains a spatially open region. The condition \( 0 < 4C - \rho_* \) corresponds to violation of the strong energy condition of the averaged universe, and it yields

\[
c_1 < c_2 / 3.
\]

The condition \( 4C - \rho_* < 3(-K) \) means the averaged universe does re-collapse before it enters the accelerating phase and yields

\[
(c_2 / 3 - c_1)c_1^{-2/3} < \frac{3}{\rho_0 L^2} \left( \frac{V_*}{L^3} \right)^{1/3}.
\]

Thus, the conditions \([34]\) and \([35]\) constrain the size of spatially open region. For \( \rho_0 L^2 = 1, V_* = L^3 \), we have

\[
0.4 < r_0 / L < 0.9,
\]

and this range of the parameter is consistent with the behavior of the Tolman-Bondi model. If the spatially open region is too large, the averaged universe can not enter accelerating phase before the spatially closed region re-collapse. On the other hand, if the spatially open region is too small, the averaged universe will re-collapse before it enters the accelerating phase.

Conclusion.— In this letter, we have presented an example of inhomogeneous universe that shows the accelerated expansion after taking spatial averaging. The necessary condition to realize accelerated expansion is that the inhomogeneous universe contains both a spatially open region (with negative spatial curvature) and a spatially closed region (with positive spatial curvature). Although if the whole universe starts its evolution with positive expansion, the region with positive spatial curvature will re-collapse after all. The scale factor dependence of the averaged curvature \( R \) in Eq. \([27]\), that is crucial to derive the accelerated expansion, is the result of co-existence of the expanding spatial region and the contracting spatial region. This is the non-perturbative feature of inhomogeneity and it is not possible to derive the same result based on the ordinary perturbative calculation. Even though we have checked this form of the averaged spatial curvature only for a spherically symmetric case, we expect that this form of the averaged spatial curvature will hold for more general case without symmetry and the acceleration of the averaged universe is a general feature of inhomogeneous universe with non-linear fluctuations.

As an application of our model to the Universe, it is possible to fix the scale of inhomogeneity \( r_0, L \) by using the present value of the cosmological parameters. We will report on this subject in a forthcoming paper.

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[1] N. A. Bahcall, J. P. Ostriker, S. Perlmutter, and P. J. Steinhardt, Science 284, 1481 (1999).
[2] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003).
[3] T. Buchert, Gen. Rel. Grav. 32, 105 (2000).
[4] V. F. Mukhanov, L. R. Abramo, and R. H. Brandenberger, Phys. Rev. Lett. 78, 1624 (1997).
[5] Y. Nambu, Phys. Rev. D 65, 104013 (2002).
[6] S. Räsänen, astro-ph/0311257 (2003).
[7] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, hep-th/0503117 (2005).
[8] E. W. Kolb, S. Matarrese, and A. Riotto, astro-ph/0506534 (2005).
[9] T. Buchert, gr-ac/0507028 (2005).
[10] G. F. R. Ellis and W. Stoeger, Class. Quant. Grav. 4, 1687 (1987).
[11] Y. Nambu and Y. Yamaguchi, Phys. Rev. D 60, 104011 (1999).
[12] A. Krasiński, Inhomogeneous cosmological models (Cambridge University Press, 1997), ISBN 0-521-48180-5.
[13] The renormalization group method improve the secular divergence of the perturbative expansion that appears in course of evolution of density fluctuations and circumvents the situation.[11]

[14] The reader might wonder why the existence of the contracting region could help accelerate the (averaged ) universe. This is however the case, as seen from the following identity

\[ X^2 \ddot{X} = 2b_1 b_2 \frac{(b_1 b_2 - b_1 \dot{b}_2)^2}{b_1^2 + b_2^2} + b_1^2 \ddot{b}_1 + b_2^2 \ddot{b}_2, \]

where \( X = (b_1^3 + b_2^3)^{1/3} \), and \( b_1 \) and \( b_2 \) are arbitrary functions of time. In this identity if we regard \( b_i = L(c_i/V)^{1/3}\alpha_i \) (\( i = 1, 2 \)), \( X \) corresponds to the averaged scale factor \( a_D \), and one can easily see that even if each acceleration \( \ddot{b}_i \) is negative, the acceleration \( \ddot{X} \) for the averaged one will become positive if \( b_1 \) and \( b_2 \) have the opposite signs (meaning one is expanding and the other is contracting) and the first positive term in the identity becomes large. This is exactly what is happening in the model.

[15] The present approximation serves a good estimate of the coefficients \( K \) and \( C \) for smaller \( r_0/L \). When this ratio is closer to unity, we can determine the coefficients exactly by means of a comparison of Taylor expansion.