Lower Bound on the Sum-rate of Decremental Beam Selection Algorithm for Beamspace MIMO Systems

Naveed Iqbal, Student Member, IEEE, Waqas Ahmad, Christian Schneider and Reiner S. Thomä, Fellow, IEEE

Abstract—In general, the zero-forcing (ZF) precoding suffers from a severe receive signal-to-noise ratio (SNR) degradation in the high interference regime. However, recent evidences from realistic measurements demonstrated that millimeter wave (mmWave) systems are mainly noise-limited as high gain antennas behave as spatial filters to the interference signal. This makes ZF precoding equally attractive as that of other linear precoding counterparts. Considering ZF precoding, this paper aims to derive a lower bound on the sum-rate achieved by a decremental beam selection (BS) algorithm in a beamspace MIMO (B-MIMO) system operating at mmWave frequencies. This bound relates Frobenius norms of precoding matrices of full and reduced dimensional (i.e. after BS) B-MIMO systems through a deterministic square-hyperbolic function. Note that, both ZF precoding and decremental BS are not new concepts. However, the derived sum-rate bound provides a new insight to the topic. Given a particular full dimensional B-MIMO channel, the presented bound can be used to understand limits of BS algorithms.

Index Terms—Beam space MIMO, beam selection, mmWave communications, large scale MIMO systems, multiuser precoding

I. INTRODUCTION

Millimeter wave (mmWave) systems have recently been realized as promising candidates for next generations of the wireless technologies. MmWave systems are high beamforming gain MIMO systems, as a large number of antenna elements can be packed in a small aperture area. However, this poses a serious restriction on the implementation of fully digital and optimal beamforming which requires one radio frequency (RF) chain per antenna and results in a high power consumption and hardware cost. Alternatively, sub-optimal approaches proposed in the literature consider either fully digital precoding with low resolution digital-to-analog converters (DACs) [1] or hybrid digital/analog precoding [2] with a reduced number of RF chains and high resolution DACs. Both approaches have their associated advantages and disadvantages as discussed in [1].

The B-MIMO [3] is a recent hybrid beamforming concept which reduces the RF complexity dramatically without any significant performance loss. In contrast to a complex phase shifter/combiner network, the analog beamforming in a B-MIMO system is done by a simple discrete lense antenna array (DLA). The DLA acts as a discrete Fourier transform (DFT) filter and it transforms the spatial multi-user MIMO (MUMIMO) channel into a sparse beamspace channel. Each beam in a B-MIMO system is connected to a single RF chain. For the RF complexity reduction, B-MIMO system employ a digital precoder to the reduced dimensional beamspace channel followed by a BS module [3] see Fig. 4. Therefore, in order to maximize the sum-rate, BS studies have recently gained considerable attention [4]–[7]. In general, the BS problem is similar to the antenna/user selection problem which have been extensively studied in literature. In [4], BS with more than one beam allocated per user have been studied for different linear precoding schemes. Results show that ZF precoding is spectrally inefficient in high interference scenarios. However, realistic measurement based studies in [8] and references therein reveal that due to high gain antenna systems, mmWave networks are not interference limited. Amadori et al. [5], investigated concepts from antenna selection literature and proposed incremental and decremental BS algorithms based on the ZF precoding. Different from [4], algorithms proposed in [5]–[7] select one beam per user and provide close approximation to results obtained from a full dimensional B-MIMO system.

Despite a considerable work on BS algorithm development, theoretical studies on performance bounds merely exist. An upper bound introduced in [4] is based on a non-realistic assumption of perfectly orthogonal channels. As the B-MIMO channels at mmWave frequencies are expected to be sparse; therefore, the upper bounds derived in the antenna selection literature e.g. [9], [10] which are based on the Rayleigh fading assumption cannot be extended to beamspace channels. Note that the bounds based on the Rayleigh/Rice fading assumption rely on the fact that inphase and quadrature components in these channels are mutually independent random processes offering two degrees of freedom. However, this is not a case in the sparse multipath channels [11]. This leads us to the primary motivation of this work. The foundation of this work is laid by the paper of Hoog and Mattheij [12], which provides an extensive mathematical treatment on the maximum volume subset selection from real matrices. We intend to derive a lower bound on the sum-rate achieved by a ZF-based decremental BS algorithm without any particular assumption on the fading envelope of the channel.

II. SYSTEM AND CHANNEL MODEL

We consider downlink communication from an access point (AP) equipped with a DLA having a maximum of $n_B$ signal
space dimensions. Theoretically, a DLA can be modeled with a critically spaced uniform linear array (ULA) i.e. $n_B = \frac{2\lambda}{L}$, where $L$ and $\lambda$ denote the physical length of the ULA and wavelength of the carrier frequency [3], respectively. At a particular time instant, the AP can communicate with $n_U \leq n_B$ user terminals; each equipped with an omni-directional antenna. The antenna domain multi-user MIMO (MU-MIMO) system with a ULA at the AP is defined as

$$\hat{y} = \hat{H}\hat{F}x + \hat{w}, \hat{H} = [\hat{h}_1 \hat{h}_2 \cdots \hat{h}_{n_U}]^T \in \mathbb{C}^{n_B \times n_B}$$

(1)

where, $\hat{H}$ is a MU-MIMO channel matrix, $\hat{F} \in \mathbb{C}^{n_B \times n_B}$ is a transmit precoding matrix and $x \in \mathbb{C}^{n_U}$ is a transmit signal vector with $E[xx^H] = I_{n_U}$. Let $P$ be the total transmit power, then $x$ satisfies the average power constraint $E\left[\|\hat{F}x\|^2\right] \leq P$. Finally, $\hat{w} \sim \mathcal{CN}(0, \sigma^2 I_{n_B})$. Assuming a propagation channel with a Line-Of-Sight (LOS) path followed by $L$ reflected non-LOS multipath components (MPCs), a generic narrow-band channel model for the $k^{th}$ user is given as

$$\hat{h}_k = \beta_k(0) a_0(\theta_k(0)) + \sum_{l=1}^{L} \beta_k(l) a_l(\theta_k(l)).$$

(2)

In (2), $\beta_k(\cdot)$ corresponds to MPC gains. For the ULA, the array steering vector $a$$ \in \mathbb{C}^{n_B \times 1}$ is defined as $a(\phi_k) = \frac{1}{\sqrt{m}} \left[\exp\left(-j2\pi \phi_k\right)ight]_{i \in I(n_B)}$ where, $\phi_k = 0.5 \sin \left(\frac{\phi_k}{\lambda} \right)$ is the spatial frequency induced by the physical angle $\phi_k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $I(n_B) = \{i - (n_B - 1)/2, i = 0, 1, \ldots, n_B - 1\}$ is a symmetric set of indices centered around zero. Since, DLA acts as a DFT filter, the spatial MU-MIMO channel $\hat{H}$ can be transformed into the beamspace channel $H$ by multiplying it with a unitary DFT matrix $U = \frac{1}{\sqrt{m}} a(\frac{\pi}{\lambda}) \in \mathbb{C}^{n_B \times n_B}$, where, the columns $u_i \in U$ are the analog orthogonal precoding vectors corresponding to $n_B$ spatial modes pointed towards fixed predefined directions. An equivalent beamspace representation of (1) is given by

$$y = HFx + w, \quad H = \hat{H}U, \quad F = U^H$$

(3)

where, $H \in \mathbb{C}^{n_B \times n_B}$ is a full dimensional (i.e. without any BS) B-MIMO channel. In this paper, we assume that $H$ is known at the AP.

### III. BEAM SELECTION

Considering ZF precoding $F = H^*$, where $(\cdot)^*$ denotes the pseudo-inverse of the matrix $H$, then the achievable sum-rate of a full-dimensional B-MIMO system with $n_B$ RF chains is described as [5, 6]

$$R_{\text{Full}} = n_U \log_2 \left(1 + \frac{P}{\sigma^2 \|H^*\|^2_F}\right),$$

(4)

where, $\|H^*\|^2_F = \text{Tr}\left(HH^H\right)^{-1}$ is Frobenius norm of the pseudo-inverse of $H$. In contrast to the full RF complexity B-MIMO system (i.e. with $n_B$ RF chains), we assume that AP is equipped with $K$ RF chains allowing a selection of $n_U \leq K \leq n_B$ beams. Let us decompose the full complexity B-MIMO channel matrix $H$ into a reduced dimensional beamspace channel $H_s \in \mathbb{C}^{n_B \times K}$ of the selected beams and a set of discarded beams $H_d \in \mathbb{C}^{n_B \times (n_B - K)}$ as

$$H = \begin{bmatrix} H_s & H_d \end{bmatrix}$$

(5)

where, $\Pi$ is a permutation matrix. Let $F_s = H^*_s$ be the ZF precoding matrix of the reduced-dimensional B-MIMO system, then sum-rate $R_s \leq R_{\text{Full}}$ with $K$ selected beams is defined as

$$R_s = n_U \log_2 \left(1 + \frac{P}{\sigma^2 \|H^*_s\|^2_F}\right).$$

(6)

From (5), it is clear that for sum-rate maximization, a BS algorithm is supposed to minimize $\|H^*_s\|^2_F$. An optimal solution to this problem is computationally prohibitive as it requires exhaustive pseudo-inverse evaluations over $\binom{n_B}{K}$ combinations.

#### A. Decremental Beam Selection Algorithm

A greedy algorithm presented here intends to minimize $\|H^*_s\|^2_F$ by successive elimination of beams from the full dimensional B-MIMO channel matrix $H$. Using rank – 1 update of the matrix inverse, a beam index $j_1$ is dropped such that

$$j_1 = \arg \left\{ \min_{j_1 = 1 \ldots n_B} \text{Tr}\left(HH^H - h_jh_j^H\right)^{-1}\right\}.$$  

(7)

Let $H_1 \in \mathbb{C}^{n_B \times (n_B - 1)}$ be a B-MIMO channel matrix obtained after dropping a beam $j_1$, then in the second iteration, a beam $j_2$ is dropped such that

$$j_2 = \arg \left\{ \min_{j_2 = 1 \ldots n_B} \text{Tr}\left(H_1H_1^H - h_jh_j^H\right)^{-1}\right\}.$$  

(8)

and so on till $K$ beams are left. A low complexity implementation of the above decremental algorithm can be done using Sherman-Morrison formula for rank – 1 update which requires $O(n_B n_U (n_B - K))$ operations [13]. Under the condition $n_B \gg n_U$, this operation count is much lower than the decremental algorithm proposed in [5] which requires $O(n_B^3)$ operations. The algorithm in [7] and [8] results in the following bound on the Frobenious norm of $H_s^*$.

#### Theorem 3.1: For $H \in \mathbb{C}^{n_B \times n_B}$ with $n_U \leq n_B$, there exist a permutation matrix $\Pi$ such that (5) holds for $K \geq n_U$ with

$$\|H_s^*\|^2_F \leq \frac{(n_B - n_U + 1)}{(K - n_U + 1)} \|H^*\|^2_F.$$  

(9)

**Proof:** Let $h_j$ be a beamspace (column) channel vector to be eliminated from $H$ at the first iteration, i.e., $K = n_B - 1$. Then from (7), the $\text{Tr}\left(HH^H - h_jh_j^H\right)^{-1}$ would be lowest as compared to any other column $h_m \neq h_j$ in $H$. Therefore,

$$\text{Tr}\left(HH^H - h_jh_j^H\right)^{-1} \leq \text{Tr}\left(HH^H - h_mh_m^H\right)^{-1} \leq \sum_{m=1}^{n_B} \text{Tr}\left(HH^H - h_mh_m^H\right)^{-1}.$$  

(10)
Now we evaluate the right-hand side (R.H.S.) of (10), which after re-arranging becomes

\[ HH^H - h_m h_m^H \]

\[ = HH^H - h_m^H (H H^H)^{1/2} (H H^H)^{1/2} \]

\[ = (H H^H)^{1/2} - h_m^H (H H^H)^{1/2} (H H^H)^{1/2} \]

\[ = (H H^H)^{1/2} Q (H H^H)^{1/2} \]  (11)

where,

\[ Q = I - (H H^H)^{1/2} h_m^H (H H^H)^{1/2} \]

\[ = I - q_m q_m^H, \quad \text{and} \quad q_m = (H H^H)^{1/2} h_m. \]  (12)

Now the inverse of rank-1 updated matrix in (11) becomes

\[ (H H^H - h_m h_m^H)^{-1} = (H H^H)^{-1/2} Q^{-1} (H H^H)^{-1/2} \]  (13)

Using the Sherman-Morrison formula [13] for rank-1 update, \( Q^{-1} \) can be written as

\[ Q^{-1} = (I - q_m q_m^H)^{-1} = I + q_m q_m^H. \]  (14)

Plugging (14) in (13) and substituting the scalar \( \tilde{q} = 1 - ||q_m||^2 \) for simplification purpose, one can write (13) as

\[ \tilde{q}(H H^H)^{-1} + (H H^H)^{-1/2} (q_m q_m^H) (H H^H)^{-1/2} \]

\[ = (H H^H)^{-1} (H H^H)^{-1} \]  (15)

Substituting \( \tilde{q} = 1 - ||q_m||^2 \) back in (15), and applying the trace operator followed by summation as on the R.H.S. of (10), we get

\[ \sum_{m=1}^{n_B} \left( 1 - ||q_m||^2 \right) Tr \left( (H H^H - h_m h_m^H)^{-1} \right) \]

\[ = \sum_{m=1}^{n_B} \left( 1 - ||q_m||^2 \right) Tr \left( H H^H \right)^{-1} \]

\[ + \sum_{m=1}^{n_B} \left[ Tr(H H^H)^{-1} h_m h_m^H (H H^H)^{-1} \right] \]  (16)

Note that,

\[ \sum_{m=1}^{n_B} ||q_m||^2 = Tr \left( \sum_{m=1}^{n_B} q_m q_m^H \right) \]

\[ = Tr \left( H H^H \right)^{-1} \sum_{m=1}^{n_B} h_m h_m^H (H H^H)^{-1} \]

\[ = Tr \left( H H^H \right)^{-1} \sum_{m=1}^{n_B} h_m h_m^H (H H^H)^{-1} \]

\[ = n_B \]  (17)

and

\[ \sum_{m=1}^{n_B} ||q_m||^2 \]

\[ \leq \sum_{m=1}^{n_B} \left( 1 - ||q_m||^2 \right) \]

\[ = n_B \times n_U. \]

\[ \sum_{m=1}^{n_B} \left( 1 - ||q_m||^2 \right) Tr \left( (H H^H - h_m h_m^H)^{-1} \right) \]

\[ = \sum_{m=1}^{n_B} \left( 1 - ||q_m||^2 \right) Tr \left( H H^H \right)^{-1} \]

\[ + \sum_{m=1}^{n_B} \left[ Tr(H H^H)^{-1} h_m h_m^H (H H^H)^{-1} \right] \]  (18)

Plugging (19) in (10) shows that (9) holds for \( K = n_B - 1 \). Proceeding in the same way and deleting columns at each iteration, the result follows by induction. □

Note that, due to summation on R.H.S. of (10), the bound in (9) is likely to be loose when \( n_B \gg K \). Since, the mmWave systems are supposed to operate with large dimensional antenna arrays, and we know from (13) that a selection of \( K = n_U \) beams (i.e., one beam per user) results in a considerable sum-rate loss as compared to the full RF complexity system, particularly when \( n_B \gg n_U \). Therefore, in order to obtain a comparable performance as that of full RF complexity system, study of increased RF complexity system (i.e., \( K > n_U \) more than one beam per user) is of fundamental importance.

Interestingly, the bound in (9) demonstrates that \( ||H^+||^2_F \) and \( ||H^+||^2_F \) are related by a square-hyperbolic function \( y = \sqrt{a} \), where \( a = n_B - n_U + 1 \) and \( x = K - n_U + 1, \forall K \geq n_U \) with center located at the point \( (n_U - 1, 0) \). Fig. 1 shows that for \( K > n_U \), the lower bound in (9) hyperbolically becomes tighter as \( K \) approaches \( n_U \). Notice that, the vertex of the square-hyperbola in Fig. 1 lies at the point \( (n_U - 1 + \sqrt{a}, \sqrt{a}) \). This shows that, a small increment of \( \sqrt{a} \) in \( K = n_U - 1 \) beams reduces maximum euclidean distance between \( ||H^+||^2_F \) and \( ||H^+||^2_F \) from \( a \) to \( \sqrt{a} \) which demonstrates a very fast convergence with an increase in \( K \).

B. Improving sharpness of (9)

Assuming that \( n_U \) and \( K \) are constants, then sharpness of (9) can be further improved by reducing \( n_B \). This can be done by using the following update,

\[ Q = I - (H H^H)^{1/2} h_m^H (H H^H)^{1/2} \]

\[ = I - q_m q_m^H, \quad \text{and} \quad q_m = (H H^H)^{1/2} h_m. \]  (12)

Theorem (16) simplifies to

\[ \sum_{m=1}^{n_B} Tr \left( (H H^H - h_m h_m^H)^{-1} \right) \]

\[ = Tr \left( (H H^H)^{-1} \sum_{m=1}^{n_B} h_m h_m^H (H H^H)^{-1} \right) \]

\[ = Tr \left( (H H^H)^{-1} H H^H (H H^H)^{-1} \right) \]

\[ = ||H^+||^2_F. \]  (19)

Therefore (16) simplifies to

\[ \sum_{m=1}^{n_B} Tr \left( (H H^H - h_m h_m^H)^{-1} \right) \]

\[ = \left( n_B - n_U + 1 \right) ||H^+||^2_F. \]  (19)

Fig. 1. Sharpness analysis of the bound in (9) as function of \( K \).
be done by pre-selecting a set of candidate beams $n_c \leq n_B$ prior to the start of decremental BS. For this, we employ subspace sampling technique introduced in [14] to determine influence score of each beam to be selected as $\pi_i = \frac{|\|v_i\|^2|}{n_i}$, where, $v_i$ is the $i^{th}$ right singular vector of $H$. Finally, the submatrix $H_c \in \mathbb{C}^{n_c \times n_c}$ of $H$ with $n_c$ candidate beams is selected such that the probability of each selected beam satisfies the criterion $p_i = \min(1, n_B \pi_i)$. Notice that, the subspace sampling technique above selects $n_c$ beams while assuring that $\|H_c^e\|_F^2 = (1-\epsilon)\|H^e\|_F^2$, where $\epsilon$ is a small number. Therefore, (9) becomes

$$\|H_c^e\|_F^2 \leq \frac{(n_c - n_c + 1)}{(K - n_U + 1)} \|H_c^e\|_F^2,$$

where, $\|H_c^e\|_F^2 \approx \|H^e\|_F^2$ and $n_c \ll n_B$ particularly when $H$ is ill-conditioned. These assumptions on the channel model are same as the ones used in [6]. Considering different channel conditions, results in [5] already provided an extensive comparison of decremental BS with other greedy channel subset selection strategies known from antenna selection literature. Therefore, results in Fig. 2 are restricted to analysis of derived sum-rate bounds only. For the comparison purpose, we have also shown sum-rate results obtained from (9). Fig. 2 shows that a reduced RF complexity system with $K = 64$ RF chains provides similar sum-rate guarantees as that of full RF complexity system and the sum-rate bound achieved from (20) for $K = 64$ is also quite tight. However, for $K = 32$, the sum-rate obtained from both (9) and (20) are quite loose, which is due to the summation on R.H.S. in (10).

V. CONCLUSION

A greedy decremental beam selection algorithm with ZF precoding is analytically studied for the B-MIMO systems and a lower sum-rate bound of the algorithm is derived. It has been shown that the maximum euclidean distance between the Frobenious norms of the full and reduced dimensional (i.e. after beam selection) ZF precoding matrices is related by a square-hyperbolic function $\approx \frac{\pi^2}{K - n_U + 1}$. This result shows that the derived bound hyperbolically becomes tighter as $K \geq n_u$.

Fig. 2. Sum-rate analysis of decremental BS and derived lower bounds.

REFERENCES

[1] K. Roth, H. Pirzadeh, A. L. Swindlehurst, and J. A. Nossek, “A comparison of hybrid beamforming and digital beamforming with low-resolution ADCs for multiple users and imperfect CSI,” IEEE Journal of Selected Topics in Signal Processing, vol. 12, no. 3, pp. 484–498, June 2018.

[2] O. E. Ayach, S. Rajagopal, S. Abu-Surra, Z. Pi, and R. W. Heath, “Spatially sparse precoding in millimeter wave MIMO systems,” IEEE Transactions on Wireless Communications, vol. 13, no. 3, pp. 1499–1513, March 2014.

[3] J. Brady, N. Behdad, and A. M. Sayeed, “Beamspace MIMO for millimeter-wave communications: System architecture, modeling, analysis, and measurements,” IEEE Transactions on Antennas and Propagation, vol. 61, no. 7, pp. 3814–3827, July 2013.

[4] A. Sayeed and J. Brady, “Beamspace MIMO for high-dimensional multuser communication at millimeter-wave frequencies,” in 2013 IEEE Global Communications Conference (GLOBECOM), Dec 2013, pp. 3679–3684.

[5] P. V. Amadori and C. Masouros, “Low RF-Complexity Millimeter-Wave Beamspace-MIMO Systems by Beam Selection,” IEEE Transactions on Communications, vol. 63, no. 6, pp. 2212–2223, June 2015.

[6] X. Gao, L. Dai, Z. Chen, Z. Wang, and Z. Zhang, “Near-optimal beam selection for beamspace mmwave massive MIMO systems,” IEEE Communications Letters, vol. 20, no. 5, pp. 1054–1057, May 2016.

[7] R. Pal, K. V. Srinivas, and A. K. Chaitanya, “A beam selection algorithm for millimeter-wave multi-user mimo systems,” IEEE Communications Letters, vol. 22, no. 4, pp. 852–855, April 2018.

[8] N. Iqbal, J. Luo, Y. Xin, R. Mueller, S. Haefner, and R. S. Thomae, “Measurements Based Interference Analysis at Millimeter Wave Frequencies in an Indoor Scenario,” in IEEE Globecom Workshops (GC Wkshps), Dec 2017, pp. 1–5.

[9] A. F. Molisch, M. Z. Win, Y.-S. Choi, and J. H. Winters, “Capacity of MIMO systems with antenna selection,” IEEE Transactions on Wireless Communications, vol. 4, no. 4, pp. 1759–1772, July 2005.

[10] S. Y. Park and D. J. Love, “Capacity limits of multiple antenna multicasting using antenna subset selection,” IEEE Transactions on Signal Processing, vol. 56, no. 6, pp. 2524–2534, June 2008.

[11] M. Pätzold and G. Rafiq, “Sparse multipath channels: Modelling, analysis, and simulation,” in 2013 IEEE 24th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC), Sept 2013, pp. 30–35.

[12] F. de Hoog and R. Mattheij, “Subset selection for matrices,” Linear Algebra and its Applications, vol. 422, no. 2, pp. 349 – 359, 2007.

[13] W. W. Hager, “Updating the inverse of a matrix,” SIAM Rev., vol. 31, no. 2, pp. 221–239, Jun. 1989.

[14] M. W. Mahoney and P. Drineas, “CUR matrix decompositions for improved data analysis,” Proceedings of the National Academy of Sciences, vol. 106, no. 3, pp. 697–702, 2009.