Energy distribution in Kerr-Newman space-time in Bergmann-Thomson formulation

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Abstract

We obtain the energy distribution in the Kerr-Newman metric with the help of Bergmann-Thomson energy-momentum complex. We find that the energy-momentum definitions prescribed by Einstein, Landau-Lifshitz, Papapetrou, Weinberg, and Bergmann-Thomson give the same and acceptable result and also support the Cooperstock Hypothesis for energy localization in general relativity. The repulsive effect due to the electric charge and rotation parameters of the metric is also reflected from the energy distribution expression.

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I. INTRODUCTION

Energy-momentum is an important conserved quantity whose definition has been a focus of many investigations. Unfortunately, there is still no generally agreed definition of energy and momentum in general relativity (GR). This dilemma in GR is highlighted in an important paper by Penrose[1] in the following way:

“It is perhaps ironic that energy conservation, paradigmatic physical concept arising initially from Galileo’s (1638) studies of the motion of bodies under gravity, and which now has found expression in the (covariant) equation

\[ \nabla_a T^{ab} = 0 \]

a cornerstone of Einstein’s (1915) general relativity—should nevertheless have found no universally applicable formulation, within Einstein’s theory, incorporating the energy of gravity itself.”

Indeed, conservation laws of energy-momentum together with the equivalence principle, played a significant role in guiding Einstein’s search for his generally covariant field equations. The numerous attempts aimed at finding an expression for describing energy-momentum distribution due to matter, nongravitational and gravitational fields gave rise to a large number of energy-momentum complexes whose physical meaning have been questioned (see [2, 3, 4, 5, 6, 7]).

The absence of a generally accepted definition of energy distribution in curved space-times has led to doubts regarding the idea of energy localization. Over the past two decades considerable effort has been put in trying to define an alternative concept of energy, the so-called quasi-local energy. The idea in this case is to determine the effective energy of a source by measurements on a two-surface. These masses are obtained over a two-surface as opposed to an integral spanning over a three-surface of a local density as is the case for pseudocomplexes. To date a large number of definitions of quasi-local mass have been proposed (see in Brown and York[8] and Hayward[9]). Bergqvist[10] considered quasi-local mass definitions of Komar, Hawking, Penrose, Ludvigsen-Vickers, Bergqvist-Ludvigsen, Kulkarni et al., and Dougan-Mason and concluded that no two of these definitions give agreed results for the Kerr as well as Reissner-Nordström space-times. On the contrary, the pioneering work of Virbhadra and his collaborators (notably, Nathan Rosen of Einstein-Rosen gravitational waves fame), and others have demonstrated with several examples that
for a given spacetime, the energy-momentum complexes of Einstein, Landau and Lifshitz, Papapetrou, and Weinberg (ELLPW), show a high degree of consistency in giving the same and acceptable energy and momentum distributions (see for instance 11, 12, 13, 14, 15, 16, 17, 18). Aguirregabiria et al. [19] showed that the ELLPW energy-momentum complexes “coincide” for any metric of Kerr-Schild class.

The stationary axially symmetric and asymptotically flat electrovac Kerr-Newman (KN) solution, characterized by constant parameters mass $M$, charge $Q$, and angular momentum $a$, is the most general black hole solution to the Einstein-Maxwell equations. Virbhadra [20] and then following him Cooperstock and Richardson [21] showed (up to the third order and seventh order, respectively, of rotation parameter) that the energy-momentum complexes of Einstein and Landau-Lifshitz give the same and reasonable energy distribution in KN space-time. Further Aguirregabiria et al. [19] performed exact computations for the energy distribution in KN space-time in Kerr-Schild Cartesian coordinates. They showed that the energy distribution in the prescriptions of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg (ELLPW) gave the same result. In this paper we investigate the energy-momentum distribution of the Kerr-Newman space-time using the Bergmann-Thomson prescription and compare the results obtained by Aguirregabiria et al.

## II. ENERGY AND MOMENTUM DISTRIBUTIONS IN BERGMANN-THOMSON FORMULATION

In this Section we first write the energy-momentum complex formulated by Bergmann and Thomson and then use it to compute energy and momentum distributions in Kerr-Newman spacetime.

The energy-momentum complex of Bergmann-Thomson is

$$ B^{jk} = \frac{1}{16\pi} B^{jkl} d, \quad (1) $$

where

$$ B^{jkl} = g^{ji} V_{i}^{kl} \quad (2) $$

with

$$ V_{i}^{kl} = -V_{i}^{lk} = \frac{g_{im}}{\sqrt{-g}} \left[ -g \left( g^{kn} g^{lm} - g^{ln} g^{km} \right) \right]_{,m}. \quad (3) $$
The Latin indices take values 0 to 3. The Bergmann-Thomson energy-momentum complex satisfies the local conservation laws

$$\frac{\partial B^{jk}}{\partial x^k} = 0,$$  \(\text{(4)}\)

in any coordinate system; however, \(B^{jk}\) itself does not transform as a tensor under a general coordinate transformation. The energy and energy current (momentum) density components are respectively represented by \(B^{00}\) and \(B^{\alpha 0}\).

Energy and momentum components \(P^i\) are given by

$$P^i = \int \int \int B^{0i} dx^1 dx^2 dx^3.$$  \(\text{(5)}\)

Using Gauss’s theorem one obtains energy \(P^0\) and momentum components \(P^\alpha\) (\(\alpha\) takes values 1 to 3) as follows:

$$P^j = \frac{1}{16\pi} \int \int B^{j\alpha} n_\alpha ds.$$  \(\text{(6)}\)

The Kerr-Newman space-time in Kerr-Schild Cartesian coordinates \(\{t, x, y, z\}\) is expressed by the line-element \(\text{(22)}\):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2MR^3 - q^2R^2}{R^4 + a^2z^2} \times \left[ dt + \frac{z}{R} dz + \frac{R}{R^2 + a^2} (xdx + ydy) + \frac{a}{R^2 + a^2} (ydx - ydx) \right]^2,$$  \(\text{(7)}\)

where \(R\) is defined by

$$\frac{x^2 + y^2}{R^2 + a^2} + \frac{z^2}{R^2} = 1.$$  \(\text{(8)}\)

Note that \(R\) becomes the usual spherical radial coordinate \(r = \sqrt{x^2 + y^2 + z^2}\) for rotation parameter \(a = 0\).

Now we use equations (2), (3), (6), (7) and (8) to obtain energy and momentum distributions in Kerr-Schild Cartesian coordinates. In these coordinates, all the components of the metric tensor for the Kerr-Newman metric are nonvanishing and also have lengthy expressions. It is extremely difficult to perform these calculations manually. Therefore, we used Mathematica 4.0 \(\text{(23)}\) for computations. The energy and momentum components inside a 2-surface with constant \(R\) are respectively given by
\[ E(R) = M - \frac{q^2}{4R} \left[ 1 + \frac{(a^2 + R^2)}{aR} \arctan \left( \frac{a}{R} \right) \right] \] (9)

and

\[ P_x(R) = P_y(R) = P_z(R) = 0. \] (10)

It is striking that Bergmann-Thomson complex gives the same result as found by Aguirregabiria et al. ([19]) who used definitions proposed by Einstein, Landau and Lifshitz, Papapetrou, and Weinberg. The results found by us further support the importance of the energy-momentum complexes, as these yield consistent and meaningful results.

Now for our convenience we define

\[ E := \frac{E}{M}, \]
\[ S := \frac{a}{M}, \]
\[ Q := \frac{q}{M}, \]
\[ R := \frac{R}{M}. \] (11)

and plot \( \mathcal{E} \) against \( \mathcal{R} \) and \( \mathcal{S} \) for different values of \( Q \) (see Fig. 1). For the same value of \( \mathcal{R} \) and \( \mathcal{S} \), \( \mathcal{E} \) is higher for lower values of \( Q \). Linear momentum components \( \{P_x, P_y, P_z\} \) remain zero for all values of \( Q, S, \) and \( R \). This is also shown in the same figure. Note that the Kerr-Newman represents gravitational fields of black holes for \( M^2 \geq q^2 + a^2 \) and of naked singularities for \( M^2 < q^2 + a^2 \).

### III. CONCLUSION

The main weakness of energy-momentum complexes is that these restrict one to make calculations in ‘Cartesian coordinates’. The alternative concept of quasi-local mass appears more attractive because these are not restricted to the use of any special coordinate system. There is a large number of quasi-local masses. It has been shown [10] that for the Kerr as well as Reissner-Nordström space-times many quasi-local mass formulations do not give agreed results. On the other hand Aguirregabiria, Chamorro, and Virbhadra [19] showed that the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg give the same result for any metric of Kerr-Schild class if the computations are carried out in Kerr-Schild cartesian coordinates. The well-known spacetimes of the Kerr-Schild class are
FIG. 1: $\mathcal{E}$ is plotted on Z-axis against $R$ on X-axis and $S$ on Y-axis for $Q = 0.9$ (top surface) and for $Q = 2.0$ (middle surface). Momentum components which remain zero for all values of $Q, S$ and $R$ are also shown by a flat surface.

for example the Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman, Vaidya, Dybney et al., Kinnersley, Bonnor-Vaidya and Vaidya-Patel. Virbhadra[20] also showed that for a general nonstatic spherically symmetric spacetime of the Kerr-Schild class, the ELLPW give the same energy distribution as the Penrose quasi-local mass definition if calculations are performed in Kerr-Schild cartesian coordinates.

In the previous section we showed that the Bergmann-Thomson energy-momentum complex gives the same energy and momentum distribution in the Kerr-Newman spacetime as the ELLPW prescriptions. These results and the important paper of Chang et al[24], which dispels doubts expressed about the physical meaning of these energy-momentum complexes, give confidence to the use of energy-momentum complexes to compute energy-momentum distribution in a given spacetime.

The energy expression $E(R)$ in Eq. (9) can be treated as the effective gravitational mass that acts on a neutral (electric charge zero) test particle situated at a coordinate distance $R$ from the centre of the Kerr-Newman black hole. The repulsive effects of the electric charge
and rotation parameters are obvious. For sufficiently small $R$, $E(R)$ can be negative for large values of $q$ and $a$. For Reissner-Nordström metric, $E(r) < 0$ for $r < q^2/2M$. As expected, the total energy ($\lim_{R \to \infty} E(r)$) of the Kerr-Newman metric comes to be the well-known ADM mass $M$. It is independent of the other two (electric charge and rotation) parameters of the charged rotating black hole.

A most important implication of our result (expressed in Eq. (9)) is that it supports Cooperstock Hypothesis [25] for energy localization, which essentially states that energy in a curved space-time is localized in the region of nonvanishing energy-momentum tensor $T^{ik}$ of matter and non-gravitational fields. For the Kerr black hole space-time $T^{ik} = 0$ and therefore energy is confined to exterior of the black hole; however, for the Kerr-Newman black hole metric $T^{ik} \neq 0$ and the energy is distributed to the interior and exterior of the black hole.

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