Local solvability and stability of the generalized inverse Robin-Regge problem with complex coefficients

Xiao-Chuan Xu, Natalia Pavlovna Bondarenko

Abstract. We prove local solvability and stability of the inverse Robin-Regge problem in the general case, taking eigenvalue multiplicities into account. We develop the new approach based on the reduction of this inverse problem to the recovery of the Sturm-Liouville potential from the Cauchy data.

Keywords: Robin-Regge problem; Cauchy data; Inverse spectral problem; Local solvability; Stability; Multiple eigenvalues

2010 Mathematics Subject Classification: 34A55; 34B08; 34L40; 35R30

1. Introduction

Consider the following generalized Robin-Regge problem $L(q, h, \alpha, \beta)$:

\[
- y''(x) + q(x)y(x) = \lambda^2 y(x), \quad 0 < x < a, \quad (1.1)
\]

\[
y'(0) - hy(0) = 0, \quad (1.2)
\]

\[
y'(a) + (i\lambda\alpha + \beta)y(a) = 0, \quad (1.3)
\]

where $\lambda$ is spectral parameter, the complex-valued potential $q$ belongs to $L^2(0, a)$, $h, \beta \in \mathbb{C}$ and $\alpha > 0$.

The problem $L(q, h, \alpha, \beta)$ arises in various models of mathematical physics, such as the problem of small transversal vibrations of a smooth inhomogeneous string subject to viscous damping [16, 17], the resonance scattering problem [23], and the problem of determining the sharp of human vocal tract [1].

This paper is concerned with the inverse spectral problem that consists in recovery of the potential $q(x)$ and the coefficients of the boundary conditions $(1.2)-(1.3)$ from the eigenvalues of $L(q, h, \alpha, \beta)$. In the theory of inverse spectral problems, the most complete results were obtained for operators induced by the Sturm-Liouville equation $(1.1)$ with boundary conditions independent of the spectral parameter (see the monographs [9, 13, 14, 19] and references therein). In particular, Borg [5] has proved that the real-valued potential $q$ is uniquely specified by the two spectra $\{\lambda_{n, \nu}\}, \nu = 0, 1$, of the problems $L_{\nu}(q, h)$ given by $(1.1)-(1.2)$ and the boundary condition $y^{(\nu)}(a) = 0, \nu = 0, 1$. Moreover, Borg [5] obtained local solvability and stability of this inverse problem. Recently, the results of Borg were generalized by Buterin and Kuznetsova [7] to the case of the complex-valued potential $q$. The latter case is more difficult for investigation, since

1School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, Jiangsu, People’s Republic of China, Email: xcxu@nuist.edu.cn
2Department of Applied Mathematics and Physics, Samara National Research University, Moskovskoye Shosse 34, Samara 443086, Russia,
3Department of Mechanics and Mathematics, Saratov State University, Astrakhanskaya 83, Saratov 410012, Russia, Email: bondarenkomp@info.sgu.ru
the spectra \( \{\lambda_{n,\mu}\} \) can contain multiple eigenvalues, which can split under a small perturbation.

However, the presence of the spectral parameter \( \lambda \) in the boundary condition causes a significant qualitative difference of problem (1.1)-(1.3) from the classical Sturm-Liouville problems. Namely, in order to recover the potential \( q \) of the problem \( L(q, h, \alpha, \beta) \), one needs only one spectrum instead of two spectra. This can be easily shown by the reduction of the inverse Robin-Regge problem to the Borg inverse problem by two spectra (see, e.g., [24]). Nevertheless, the method of reduction to the Borg problem is inconvenient for studying various issues of the inverse problem theory, in particular, of local solvability and stability of the inverse problem. Therefore, the Regge-type problems require development of new methods for their investigation.

Some aspects of the inverse Regge-type problems were studied in the earlier papers [10, 11, 20]. Important advances in the theory of the Regge-type problems have been achieved by Yurko [25], who considered various types of inverse problems with linear and also with polynomial dependence on the spectral parameter in the boundary conditions. For the problem \( L(q, h, \alpha, \beta) \) with real coefficients, Möller and Pivovarchik [16] proved the uniqueness and existence of the inverse problem solution. In [18], the Dirichlet-Regge inverse problem was studied with the boundary condition (1.2) replaced by \( y(0) = 0 \). Later on, Xu [24] considered the problem \( L(q, h, \alpha, \beta) \) with complex coefficients, where the uniqueness theorems are proved with reconstruction algorithms being provided. In addition, Xu [24] studied local solvability and stability of the inverse problem under some restrictions on eigenvalue perturbations.

In this paper, we suggest a new approach to the inverse Regge-type problems. We reduce the inverse Robin-Regge problem to the recovery of the Sturm-Liouville potential by the so-called Cauchy data, by using the special exponential Riesz basis. The ideas of this approach appeared in the papers by Bondarenko [3, 4]. Our method is convenient for dealing with multiple eigenvalues. As it was pointed out in [12], our approach, in fact, provides the first constructive algorithm for interpolation of the Weyl function by its values in a countable set of points. We also mention that the reduction to the inverse problem by the Cauchy data has been recently applied to the inverse transmission eigenvalue problem by Buterin et al [6].

The main result of this paper is the following theorem on the local solvability and stability of the inverse Robin-Regge problem. Denote

\[
\mathbb{Z}_0 = \mathbb{Z}, \quad \mathbb{Z}_1 = \mathbb{Z} \setminus \{0\}, \quad \mathbb{Z}_j = \mathbb{Z} \setminus \{1\}, \quad j = 0, 1.
\]

It was known [16, 24] that the eigenvalues, which can be denoted by \( \{\lambda_n\}_{n \in \mathbb{Z}_j} \), of the problem \( L(q, h, \alpha, \beta) \) with \( (-1)^{j+1}(\alpha - 1) < 0 \) have the following asymptotics

\[
\lambda_n = \frac{(|n| - j + 1)\pi}{a} \text{sgn} n + \frac{i}{2a} \ln \left| \frac{\alpha + 1}{1 - \alpha} \right| + \frac{P}{n} + \frac{\gamma_{j,n}}{n}, \quad j = 0, 1, \quad (1.4)
\]
where \( \{ \gamma_{j,n} \} \in l_2 \), and

\[
P = \frac{1}{\pi} \left( \omega - \frac{\beta}{\alpha^2 - 1} \right), \quad \omega = h + \frac{1}{2} \int_0^a q(s)ds.
\] (1.5)

In our notations, we agree that \( j = 0 \) corresponds to the case \( \alpha > 1 \) and \( j = 1 \), to the case \( \alpha < 1 \). Consider the inverse problem that consists in recovery of \( q, h \) and \( \alpha \neq 1 \) from the known \( \beta \) and the set \( \{ \lambda_n \}_{n \in \mathbb{Z}^{-j}} \) of all the eigenvalues except one. Note that the numeration of the eigenvalues is not uniquely fixed by the asymptotics (1.4), so every eigenvalue can be excluded.

**Theorem 1.1.** Let \( \{ \lambda_n \}_{n \in \mathbb{Z}^{-j}} \ (j = 0, 1) \) be the eigenvalues of the problem \( L(q, h, \alpha, \beta) \) with complex-valued \( q \in L^2(0, a) \), \( h, \beta \in \mathbb{C} \) and \( (-1)^{j+1}(\alpha - 1) < 0 \). Then there exists \( \varepsilon > 0 \) (depending on the problem \( L(q, h, \alpha, \beta) \)) such that for any sequence \( \{ \tilde{\lambda}_n \}_{n \in \mathbb{Z}^{-j}} \) satisfying

\[
\Lambda := \sqrt{\sum_{n \in \mathbb{Z}^{-j}} (n^2 + 1)|\lambda_n - \tilde{\lambda}_n|^2} \leq \varepsilon,
\] (1.6)

there exist unique \( \tilde{q} \in L^2(0, a) \) and \( \tilde{h} \in \mathbb{C} \) such that \( \{ \tilde{\lambda}_n \}_{n \in \mathbb{Z}^{-j}} \) are the eigenvalues of the problem \( L(\tilde{q}, \tilde{h}, \alpha, \beta) \). Moreover,

\[
\| \tilde{q} - q \|_{L^2} \leq C\Lambda, \quad |\tilde{h} - h| \leq C\Lambda,
\] (1.7)

where \( C > 0 \) depends only on the problem \( L(q, h, \alpha, \beta) \).

An important difference of this theorem comparing with the results of [24] is that in [24] the following stability estimates are obtained:

\[
\| \tilde{q} - q \|_{L^2} < C\Lambda^{1/p}, \quad |\tilde{h} - h| \leq C\Lambda^{1/p}
\]

with the additional constant \( p \geq 1 \) depending on \( q(x) \) and \( h \). Moreover, the proofs in [24] contain a mistake related with eigenvalue multiplicities. In fact, the results of [24] are valid only in the special case when the multiplicities of \( \{ \tilde{\lambda}_n \} \) coincide with the multiplicities of \( \{ \lambda_n \} \). But under a small perturbation, multiple eigenvalues of the problem \( L(q, h, \alpha, \beta) \) can split into smaller groups. In the present paper, we take this effect into account and prove Theorem 1.1 in the general case, without any restrictions on the eigenvalue multiplicities. Moreover, our new method allows us to obtain the improved estimate (1.7) without \( p \).

The paper is organized as follows. In Section 2, we provide the definition of the Cauchy data and prove the local solvability and stability of the inverse problem by the Cauchy data (Theorem 2.1). This theorem plays an auxiliary role in this paper, but also can be considered as a separate result. In Section 3, the proof of the main Theorem 1.1 is provided.
2. Inverse Problem by the Cauchy Data

In this section, we prove an auxiliary theorem on the local solvability and stability of the inverse problem by the Cauchy data.

Let \( \varphi(x, \lambda) \) be the solution of (1.1) with the initial values \( \varphi(0, \lambda) = 1, \varphi'(0, \lambda) = h \). It is well known that

\[
\varphi(x, \lambda) = \cos(\lambda x) + \int_{0}^{x} K(x, t) \cos(\lambda t) dt,
\]

where \( K(x, t) \) is a two variable continuous function with first partial derivatives, satisfying \( K_t(a, \cdot), K_x(a, \cdot) \in L^2(0, a) \), and \( K(a, a) = \omega \). Using (2.1), we have

\[
\varphi(a, \lambda) = \cos(\lambda a) + \frac{\omega \sin(\lambda a)}{\lambda} - \int_{0}^{a} K_t(a, t) \frac{\sin(\lambda t)}{\lambda} dt,
\]

\[
\varphi'(a, \lambda) = -\lambda \sin(\lambda a) + \frac{\omega \cos(\lambda a)}{\lambda} + \int_{0}^{a} K_x(a, t) \cos(\lambda t) dt.
\]

The set \( \{K_t(a, t), K_x(a, t), \omega\} \) is called the Cauchy data for \( q \) and \( h \). We shall consider the following inverse problem.

**Inverse Problem 2.1.** Given the Cauchy data \( \{K_t(a, t), K_x(a, t), \omega\} \), find the potential \( q(x) \) and \( h \).

We remark here, when \( q \) and \( h \) are real, Rundell and Sacks [21] gave the numerical reconstruction algorithm for Problem 2.1 and applied the technique to the inverse resonance problem [22]. We shall consider the local solvability and stability for Problem 2.1 with complex \( q \) and \( h \).

**Theorem 2.1.** Let \( q(x) \) be a fixed complex-valued function from \( L^2(0, a) \), and let \( h \in \mathbb{C} \) be a fixed number. Denote by \( \{K_1, K_2, \omega\} \) the corresponding Cauchy data. Then there exists \( \varepsilon > 0 \) (depending only on \( q \) and \( h \)) such that, for any functions \( \{\tilde{K}_1, \tilde{K}_2\} \) satisfying

\[
\Xi := \max\{\|\tilde{K}_1 - K_1\|_{L^2(0,a)}, \|\tilde{K}_2 - K_2\|_{L^2(0,a)}\} \leq \varepsilon,
\]

there exists a unique function \( \tilde{q} \in L^2(0, a) \) such that \( \{\tilde{K}_1, \tilde{K}_2, \omega\} \) are the Cauchy data for \( \tilde{q} \) and \( \tilde{h} = \omega - \frac{1}{2} \int_{0}^{a} \tilde{q}(x) dx \). Moreover,

\[
\|\tilde{q} - q\|_{L^2(0,a)} \leq C\Xi, \quad |\tilde{h} - h| \leq C\Xi,
\]

where \( C \) depends only on \( q \) and \( h \).

We remark that the analog of Theorem 2.1 for the case of the Dirichlet boundary condition \( y(0) = 0 \) was proved in [3].

**Proof.** Let us prove Theorem 2.1 by showing several auxiliary propositions. Let \( z = \lambda^2 \). Define the functions

\[
\eta_1(z) := \cos(\lambda a) + \frac{\omega \sin(\lambda a)}{\lambda} - \int_{0}^{a} K_1(t) \frac{\sin(\lambda t)}{\lambda} dt,
\]
\[ \eta_2(z) := -\lambda \sin(\lambda a) + \omega \cos(\lambda a) + \int_0^a K_2(t) \cos(\lambda t) \, dt. \] (2.7)

By the standard method related to the Rouché’s theorem, one can easily obtain the asymptotics of the zeros of the function \( \eta_2(z) \).

**Proposition 2.1.** Let \( K_2(t) \) be an arbitrary complex-valued function in \( L^2(0, a) \). Then the zeros \( \{ z_n \}_{n \geq 0} \) with \( |z_{n+1}| \geq |z_n| \) of the function \( \eta(z) \) have the asymptotics

\[ \rho_n := \sqrt{z_n} = \frac{n\pi}{a} + O \left( \frac{1}{n} \right). \] (2.8)

In view of the asymptotic formula (2.8), we can find the smallest integer \( n_1 \geq 1 \) such that the zeros \( \{ z_n \}_{n \geq n_1} \) are simple and \( |z_{n_1}| > |z_{n_1-1}| \). Consider the disk \( \Gamma_0 = \{ z : |z| \leq (|z_{n_1}| + |z_{n_1-1}|)/2 \} \). Obviously, the zeros \( \{ z_n \}_{n=0}^{n_1-1} \subset \text{int} \, \Gamma_0 \), and the zeros \( \{ z_n \}_{n \geq n_1} \) lie strictly outside \( \Gamma_0 \).

Denote by \( k_n \) the multiplicity of the value \( z_n \) in the sequence \( \{ z_n \}_{n \geq 0} \), and assume that multiple \( z_n \)'s are neighboring: \( z_0 = z_{n+1} = \cdots = z_{n+k_n-1} \). Define \( I_0 := \{ n \geq 1, z_n \neq z_{n-1} \} \cup \{ 0 \} \). Introduce the Weyl function \( M(z) \) and the sequence \( \{ M_n \}_{n \geq 0} \) as follows:

\[ M(z) := \frac{\eta_1(z)}{\eta_2(z)}, \quad M_n := \text{Res}(z - z_n)^n M(z), \quad n \in I, \quad v = 0, 1, \ldots, k_n - 1. \]

In the following discussion, we agree that, if a certain symbol \( \gamma \) denotes an object constructed by \( \{ K_1, K_2, \omega \} \), then the symbol \( \hat{\gamma} \) with tilde denotes the analogous object constructed by \( \{ \hat{K}_1, \hat{K}_2, \omega \} \).

**Lemma 2.1.** Let \( K_1, K_2 \) be fixed complex-valued functions in \( L^2(0, a) \), and let \( \omega \in \mathbb{C} \). Then, there exists \( \varepsilon > 0 \) (depending on \( K_1, K_2, \omega \)) such that, for any \( \hat{K}_1, \hat{K}_2 \in L^2(0, \pi) \) satisfying (2.4), the zeros \( \{ \hat{z}_n \}_{n=0}^{n_1-1} \) of \( \hat{\eta}_2(z) \) lie strictly inside \( \Gamma_0 \) and

\[ \max_{z \in \partial \Gamma_0} |M(z) - \hat{M}(z)| \leq C \Xi. \] (2.9)

For \( n \geq n_1 \), we have \( \bar{k}_n = 1 \) and

\[ \left( \sum_{n=n_1}^{\infty} (n\xi_n)^2 \right)^{1/2} \leq C \Xi \] (2.10)

where \( \xi_n := |\rho_n - \bar{\rho}_n| + |M_n - \hat{M}_n| \). Here the positive constant \( C \) in (2.9) and (2.10) depends only on \( K_1, K_2, \) and \( \omega \).

**Proof.** In the proof, we denote by \( C_i \) (\( i = 1, \ldots, 20 \)) positive constants, which depend only on \( K_1, K_2, \) and \( \omega \). From the conditions of the lemma, we see that

\[ |\eta_2(z)| \geq C_1, \quad |\eta_2(z) - \hat{\eta}_2(z)| \leq C_2 \Xi, \quad z \in \partial \Gamma_0. \] (2.11)

It follows that

\[ |\eta_2(z) - \hat{\eta}_2(z)|/|\eta_2(z)| < 1, \quad z \in \partial \Gamma_0, \] (2.12)
for sufficiently small $\varepsilon > 0$. Thus, we have from the Rouché’s theorem that the function $\tilde{\eta}_2(z)$ has the same number of zeros as $\eta_2(z)$ inside $\Gamma_0$. According to our notations, these zeros of $\tilde{\eta}_2(z)$ are $\{\tilde{z}_n\}_{n=0}^{\infty}_{n=0}$. Again, using (2.12), we have

$$|\tilde{\eta}_2(z)| \geq |\eta_2(z)| - |\eta_2(z) - \tilde{\eta}_2(z)| \geq C_3, \quad z \in \partial \Gamma_0$$

(2.13)

for sufficiently small $\varepsilon > 0$. Using the definition of $M(z)$ together with (2.11) and (2.13), and noting that $\eta_i(z)$ ($i = 1, 2$) are bounded on $\Gamma_0$, we obtain

$$|M(z) - \tilde{M}(z)| \leq \frac{|\eta_1(z) - \tilde{\eta}_1(z)| |\eta_2(z)| + |\eta_2(z) - \tilde{\eta}_2(z)||\eta_1(z)|}{|\eta_2(z)|} \leq C \Xi, \quad z \in \partial \Gamma_0,$$

which implies (2.9).

Now, let us prove (2.10). We shall first prove the inequality for the part of $|\rho_n - \tilde{\rho}_n|$. For $n \geq n_1$, consider the disks $\gamma_{n,\delta} := \{\lambda : |\lambda - \rho_n| \leq \delta\}$, where $\delta > 0$ is fixed and so small that $\delta \leq \frac{|\rho_n - \rho_{n+1}|}{2}$ for all $n \geq n_1$. Then the function $\eta_2(\lambda^2)$ has exactly one zero $\rho_n \in \text{int} \gamma_{n,\delta}$ in the $\lambda$-plane for every $n \geq n_1$. It follows from (2.7) that

$$|\eta_2(\lambda^2)| \leq nC_4, \quad \lambda \in \gamma_{n,\delta}, \quad |\tilde{\eta}_2(\rho_n^2)| \geq nC_5, \quad n \geq n_1$$

(2.14)

where $\tilde{\eta}_2(\lambda^2) := \frac{d\eta_2(\lambda^2)}{d\lambda}$. For $\lambda \in \text{int} \gamma_{n,\delta}$, we have the Taylor formula

$$\eta_2(\lambda^2) = \eta_2(\rho_n^2) + \tilde{\eta}_2(\rho_n^2)(\lambda - \rho_n) + \frac{(\lambda - \rho_n)^2}{2i} \int_{\partial \gamma_{n,\delta}} \frac{\eta_2(\rho^2) d\rho}{(\rho - \rho_n)^2 (\rho - \lambda)}.$$

Using (2.14) and (2.15), we obtain

$$|\eta_2(\lambda^2)| \geq nC_5|\lambda - \rho_n| - \frac{nC_4}{\delta^2(\delta - \delta_1)}|\lambda - \rho_n|^2 \geq nC_6|\lambda - \rho_n|, \quad \lambda \in \gamma_{n,\delta_1},$$

(2.16)

where $\delta_1 \in (0, \delta)$ is sufficiently small and fixed.

For sufficiently small $\varepsilon > 0$, we have

$$|\eta_2(\lambda^2) - \tilde{\eta}_2(\lambda^2)| \leq C_7 \Xi, \quad \lambda \in \partial \gamma_{n,\delta_1}, \quad n \geq n_1.$$  

(2.17)

Using (2.17), and noting $|\eta_2(\lambda^2)| \geq C_8$ for $\lambda \in \partial \gamma_{n,\delta_1}$ for $n \geq n_1$, we obtain that for sufficiently small $\varepsilon > 0$ there holds

$$|\eta_2(\lambda^2) - \tilde{\eta}_2(\lambda^2)| < |\eta_2(\lambda^2)|, \quad \lambda \in \partial \gamma_{n,\delta_1}, \quad n \geq n_1.$$  

It follows from the Rouché’s theorem that the function $\tilde{\eta}_2(\lambda^2)$ has exactly one zero $\tilde{\rho}_n \in \text{int} \gamma_{n,\delta_1}$ for each $n \geq n_1$. Using (2.16) and (2.17), we get

$$|\tilde{\rho}_n - \rho_n| \leq \frac{1}{nC_6} |\eta_2(\rho_n^2) - \tilde{\eta}_2(\rho_n^2)| \leq \frac{C_9}{n} \left| \int_0^a \tilde{K}_2(t) \cos(\tilde{\rho}_nt) dt \right|.$$  

(2.18)
where \( \hat{K}_2 := \tilde{K}_2 - K_2 \). Using (2.24) and the asymptotic formula (2.8) of \( \tilde{\rho}_n \), we have
\[
\left| \int_0^a \hat{K}_2(t) \cos (\tilde{\rho}_n t) \, dt \right| \leq \left| \int_0^a \hat{K}_2(t) \cos \left( \frac{n \pi t}{a} \right) \, dt \right| + \left| \int_0^a \hat{K}_2(t) \left( \cos (\tilde{\rho}_n t) - \cos \left( \frac{n \pi t}{a} \right) \right) \, dt \right|
\leq \left| \tilde{K}_{2,n} \right| + \frac{C_{10} \Xi}{n}, \quad n \geq n_1, \quad \tilde{K}_{2,n} := \int_0^a \hat{K}_2(t) \cos \left( \frac{n \pi t}{a} \right) \, dt.
\]  
(2.19)

It follows from (2.18) and (2.19) that
\[
n|\tilde{\rho}_n - \rho_n| \leq C_0 \left| \tilde{K}_{2,n} \right| + \frac{C_{11} \Xi}{n}.
\]

Using the Bessel inequality for the Fourier coefficients \( \{K_{2,n}\}_{n \geq n_1} \) together with (2.4), we have
\[
\sqrt{\sum_{n=n_1}^{\infty} n^2 |\tilde{\rho}_n - \rho_n|^2} \leq C_{12} \Xi.
\]  
(2.20)

Let us prove the inequality (2.10) for the part of \( |M_n - \tilde{M}_n| \). Note that \( \{z_n\}_{n \geq n_1} \) are simple zeros of \( \eta_2(z) \). Thus we have
\[
M_n := \text{Res } M(z) = \frac{\eta_1(z_n)}{\eta_2'(z_n)}, \quad n \geq n_1.
\]  
(2.21)

For the sufficiently small \( \varepsilon > 0 \), the analogous relation is valid for \( \tilde{M}_n \) for \( n \geq n_1 \). Thus we have
\[
\tilde{M}_n - M_n = \frac{(\tilde{\eta}_1(\tilde{z}_n) - \eta_1(z_n)) \eta_2'(z_n) + \eta_1(z_n) (\eta_2'(z_n) - \tilde{\eta}_2'(\tilde{z}_n))}{\eta_2'(z_n)\tilde{\eta}_2'(\tilde{z}_n)}, \quad n \geq n_1
\]  
(2.22)

From (2.6) and (2.7), we know that
\[
|\eta_1(z_n)| \leq C_{13}, \quad |\eta_2'(z_n)| \geq C_{14}, \quad |\eta_1(z_n) - \tilde{\eta}_1(\tilde{z}_n)| \leq C_{15} \left( \frac{|\tilde{K}_{1,n}|}{n} + \frac{\Xi}{n^2} \right)
\]  
\[
|\tilde{\eta}_2'(\tilde{z}_n)| \geq C_{16}, \quad |\eta_2'(z_n) - \tilde{\eta}_2'(\tilde{z}_n)| \leq C_{17} \left( \frac{|\tilde{K}_{2,n}|}{n} + \frac{\Xi}{n^2} \right), \quad n \geq n_1,
\]  
(2.23)

where
\[
\tilde{K}_{1,n} = \int_0^a [\tilde{K}_1(t) - K_1(t)] \sin \frac{n \pi t}{a} \, dt, \quad \tilde{K}_{2,n} = \int_0^a t[K_2(t) - \tilde{K}_2(t)] \sin \frac{n \pi t}{a} \, dt.
\]

Using (2.22), (2.23) and the second inequality in (2.14), we have
\[
|\tilde{M}_n - M_n| \leq C_{18} (|\tilde{K}_{1,n}| + |\tilde{K}_{2,n}|) + \frac{C_{19} \Xi}{n^2}.
\]  
(2.24)

Similarly to (2.20), we get
\[
\sqrt{\sum_{n=n_1}^{\infty} n^2 |\tilde{M}_n - M_n|^2} < C_{20} \Xi.
\]  
(2.25)
Together with (2.20) and (2.25), we arrive at (2.10). The proof of Lemma 2.1 is complete. □

In [2, 8], the following inverse problem is considered.

**Inverse Problem 2.2.** Given the data \( \{ z_n, M_n \}_{n=0}^{\infty} \), find \( q \) and \( h \).

In [2], Bondarenko proved the local solvability and stability for the above Inverse Problem 2.2.

**Proposition 2.2.** Let \( q \in L^2(0, a) \) and \( h \in \mathbb{C} \) be fixed. Then, there exists \( \varepsilon > 0 \) (depending on \( q \) and \( h \)) such that, for any complex numbers \( \{ \tilde{z}_n, \tilde{M}_n \}_{n=0}^{\infty} \) satisfying the estimate

\[
\Omega := \max \left\{ \max_{\lambda \in \partial \Omega} |M(\lambda) - \tilde{M}(\lambda)|, \left( \sum_{n=1}^{\infty} (n\xi_n)^2 \right)^{1/2} \right\} \leq \varepsilon
\]

there exist the unique complex-valued function \( \tilde{q} \in L^2(0, a) \) and \( \tilde{h} \in \mathbb{C} \) being the solution of Inverse Problem 2.2 for \( \{ \tilde{z}_n, \tilde{M}_n \}_{n=0}^{\infty} \). Moreover,

\[
\| \tilde{q} - q \|_{L^2(0, a)} \leq C\Omega, \quad |\tilde{h} - h| \leq C\Omega,
\]

where the constant \( C \) depends only on \( q \) and \( h \).

Using Lemma 2.1 and Proposition 2.2, we finish the proof of Theorem 2.1. □

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 on the local solvability and stability theorem for the inverse Robin-Regge problem.

Note that the eigenvalues of the problem \( L(q, h, \alpha, \beta) \) coincide with the zeros of the characteristic function

\[
\Delta(\lambda) = \varphi'(a, \lambda) + (i\lambda \alpha + \beta)\varphi(a, \lambda).
\]

(3.1)

Denote by \( m_k \) the multiplicity of the value \( \lambda_k \) in the sequence \( \{ \lambda_n \}_{n \in \mathbb{Z}_j^-} \). In view of the asymptotics (1.4), there are at most finitely many multiple eigenvalues. Therefore, \( m_n = 1 \) for all \( |n| \geq n_0 \) for some \( n_0 > 0 \). Define the set

\[
S_j := \{ n \in \mathbb{Z}_j^- : \lambda_n \neq \lambda_k, \forall k \in \mathbb{Z}_j^- : k < n \}, \quad j = 0, 1.
\]

Clearly, the sequence \( \{ \lambda_n \}_{n \in S_j} \) consists of elements of \( \{ \lambda_n \}_{n \in \mathbb{Z}_j^-} \) being taken only once.

Without loss of generality, impose the following assumption.

**Assumption (N):** The multiple values \( \lambda_n \) in the sequence \( \{ \lambda_n \}_{n \in \mathbb{Z}_j^-} \) are neighboring: \( \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m_n-1} \) for all \( n \in S_j \).

Introduce the functions

\[
u_{n+\nu}(t) := (it)^{\nu}e^{i\lambda_n t}, \quad n \in S_j, \quad \nu = 0, 1, \ldots, m_n - 1.
\]

(3.2)
Lemma 3.1 (See [24]). Suppose that the sequence \( \{\lambda_n\}_{n \in \mathbb{Z}_j^-} \) \( (j = 0, 1) \) satisfies the asymptotics (1.4) and assumption (\( N \)). Then the system \( \{u_n(t)\}_{n \in \mathbb{Z}_j^-} \) is a Riesz basis in \( L^2(-a, a) \).

Define the inner product in \( L^2(-a, a) \) as
\[
(g_1, g_2) := \int_{-a}^{a} g_1(t)g_2(t)dt, \quad \forall g_1, g_2 \in L^2(-a, a).
\]

Substituting (2.1), (2.2) and (2.3) into (3.1), we have
\[
\Delta(\lambda) = f(\lambda) + \frac{1}{2} \int_{-a}^{a} [M(t) + \alpha N(t)]e^{i\lambda t}dt = \frac{1}{2} (M(t) + \alpha N(t), e^{i\lambda t}),
\]
where
\[
f(\lambda) := -\lambda[\sin(\lambda a) - i\alpha \cos(\lambda a)] + (\omega + \beta) \cos(\lambda a) + i\alpha \omega \sin(\lambda a),
\]
\[
M(t) = \begin{cases} 
K_x(a, t) + \beta K(a, t), & t \in (0, a), \\
K_x(a, -t) + \beta K(a, -t), & t \in (-a, 0), 
\end{cases}
\]
\[
N(t) = \begin{cases} 
-K_t(a, t), & t \in (0, a), \\
K_t(a, -t), & t \in (-a, 0).
\end{cases}
\]

It is obvious that \( M(t) \) is even and \( N(t) \) is odd. Denote
\[
w_{n+\nu} := -f^{(\nu)}(\lambda_n), \quad n \in \mathcal{S}_j, \quad \nu = 0, 1, \ldots, m_n - 1.
\]

Then we have
\[
\frac{1}{2} (M + \alpha N, u_n) = w_n, \quad n \in \mathbb{Z}_j^-.
\]

To deal with the multiple eigenvalues, we need the following lemma from [15].

Lemma 3.2. Assume that \( f(z) \) is an entire function, and \( z_1, \ldots, z_m \) (not necessarily distinct) are in the disk \( \{z : |z - z_0| \leq r < 1/2\} \). Let \( p(z) \) be the unique polynomial of degree at most \( m - 1 \) which interpolates \( f(z) \) and its derivatives in the usual way at the points \( z_j \), \( j = 1, \ldots, m \): namely, if \( z_j \) appears \( m_j \) times, then \( p^{(m_j)}(z_j) = f^{(n)}(z_j) \) for \( n = 0, \ldots, m_j - 1 \). Then for each \( j = 0, \ldots, m - 1 \),
\[
|f^{(j)}(z_0) - p^{(j)}(z_0)| \leq Cr^{m-j} \sup_{|z-z_0|=1} |f(z)|,
\]
where the constant \( C \) depends only on \( m \).

Fix \( \{\lambda_n\}_{n \in \mathbb{Z}_j^-} \) to be the subspectrum of the problem \( L(q, h, \alpha, \beta) \). To prove Theorem 1.1, we shall use the data \( \beta \) and \( \{\tilde{\lambda}_n\}_{n \in \mathbb{Z}_j^-} \) to construct \( \tilde{q} \) and \( \tilde{h} \). We agree that, if a certain symbol \( \delta \) denotes an object related to the problem \( L(q, h, \alpha, \beta) \), then \( \tilde{\delta} \) will denote an analogous object related to the sequence \( \{\tilde{\lambda}_n\}_{n \in \mathbb{Z}_j^-} \). The notation \( C \) may stand for different positive constants depending only on the problem \( L(q, h, \alpha, \beta) \) and on the subspectrum \( \{\lambda_n\}_{n \in \mathbb{Z}_j^-} \).
By virtue of (1.6), the sequence \( \{\tilde{\lambda}_n\}_{n \in \mathbb{Z}_j^-} \) also has the asymptotics (1.4). Consequently, \( \tilde{\alpha} = \alpha \) and \( \tilde{P} = P \). Put \( \tilde{\beta} = \beta \) and \( \tilde{\omega} = \omega \). Note that multiplicities of \( \lambda_n \) and \( \tilde{\lambda}_n \) may be distinct. However, for sufficiently small \( \varepsilon > 0 \), the inclusion \( S_j \subseteq \tilde{S}_j \) holds. In particular, \( \tilde{m}_n = 1 \) for \( |n| \geq n_0 \).

Denote
\[
\tilde{u}_{n+\nu}(t) := (it)^\nu e^{i\tilde{\lambda}_n t}, \quad \tilde{w}_{n+\nu} := -f^{(\nu)}(\tilde{\lambda}_n), \tag{3.10}
\]
for \( n \in \tilde{S}_j \) and \( \nu = 0, 1, \ldots, \tilde{m}_n - 1 \). Consider the system of equations
\[
\frac{1}{2} \left( \tilde{M} + \alpha \tilde{N}, \tilde{u}_n \right) = \tilde{w}_n, \quad n \in \mathbb{Z}_j^- \tag{3.11}
\]
where the unknown functions \( \tilde{M}(t) \) and \( \tilde{N}(t) \) are respectively even and odd.

Fix \( k \in (-n_0, n_0) \cap S_j \), and assume that the eigenvalue \( \lambda_k \) with multiplicity \( m_k \) corresponds to the numbers \( \{\tilde{\lambda}_n\}_{n \in M_k} \), where \( M_k := \{k, k + 1, \ldots, k + m_k - 1\} \). Define \( \tilde{S}_k^j = \tilde{S}_j \cap M_k \). It is obvious that the relation (3.11) for \( n \in M_k \) can be rewritten as
\[
\frac{1}{2} \left( \tilde{M}(t) + \alpha \tilde{N}(t), (it)^\nu e^{i\tilde{\lambda}_n t} \right) = -f^{(\nu)}(\tilde{\lambda}_n), \quad n \in \tilde{S}_k^j, \quad \nu = 0, 1, \ldots, \tilde{m}_n - 1. \tag{3.12}
\]
For each fixed \( t \in [-a, a] \), let \( E_k(t, \lambda), F_k(\lambda) \) be the unique polynomials of degree at most \( m_k - 1 \), respectively, interpolating \( e^{i\lambda t} \) and \( -f(\lambda) \) and their derivatives in the usual way at the points \( \{\tilde{\lambda}_n\}_{n \in M_k} \). Namely,
\[
E_k^{(\nu)}(t, \tilde{\lambda}_n) = (it)^\nu e^{i\tilde{\lambda}_n t}, \quad F_k^{(\nu)}(\tilde{\lambda}_n) = -f^{(\nu)}(\tilde{\lambda}_n), \quad \nu = 0, 1, \ldots, \tilde{m}_n - 1.
\]
It follows from (3.12) that
\[
\frac{1}{2} \left( \tilde{M}(\cdot) + \alpha \tilde{N}(\cdot), E_k^{(\nu)}(\cdot, \tilde{\lambda}_n) \right) = F_k^{(\nu)}(\tilde{\lambda}_n), \quad n \in \tilde{S}_k^j, \quad \nu = 0, 1, \ldots, \tilde{m}_n - 1. \tag{3.13}
\]
Since \( E_k(t, \lambda), F_k(\lambda) \) are the polynomials of degree at most \( m_k - 1 \), we have
\[
\frac{1}{2} \left( \tilde{M}(\cdot) + \alpha \tilde{N}(\cdot), E_k(\cdot, \lambda) \right) = F_k(\lambda), \quad \lambda \in \mathbb{C}. \tag{3.14}
\]
In particular, we have
\[
\frac{1}{2} \left( \tilde{M}(\cdot) + \alpha \tilde{N}(\cdot), E_k^{(\nu)}(\cdot, \lambda_k) \right) = F_k^{(\nu)}(\lambda_k), \quad \nu = 0, 1, \ldots, m_k - 1. \tag{3.15}
\]
Define the sequence \( \{\tilde{u}_n\}_{n \in \mathbb{Z}_j^-} \) as follows
\[
\tilde{u}_{n+\nu}(t) = E_n^{(\nu)}(t, \lambda_n), \quad \tilde{w}_{n+\nu} = F_n^{(\nu)}(\lambda_n), \quad |n| < n_0, n \in S_j, \quad \nu = 0, 1, \ldots, m_n - 1,
\]
\[
\tilde{u}_n(t) = e^{i\tilde{\lambda}_n t}, \quad \tilde{w}_n = \tilde{w}_n, \quad |n| \geq n_0. \tag{3.16}
\]
Then the system (3.11) is equivalent to
\[
\frac{1}{2} \left( \tilde{M} + \alpha \tilde{N}, \tilde{u}_n \right) = \tilde{w}_n, \quad n \in \mathbb{Z}_j^- \tag{3.17}
\]
Let us prove the following two lemmas successively.
Lemma 3.3. There exists $\varepsilon > 0$ such that, for any sequence $\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}}$ satisfying (1.6), the following estimates hold

$$\sqrt{\sum_{n \in \mathbb{Z}} n^2\|u_n - \tilde{u}_n\|_{L^2(-a,a)}^2} \leq CA,$$  \hspace{1cm} (3.18)

$$\sqrt{\sum_{n \in \mathbb{Z}} |w_n - \tilde{w}_n|^2} \leq CA.$$  \hspace{1cm} (3.19)

Proof. Using the Schwarz’s lemma (see, e.g., [9, p.51]), one can obtain

$$|e^{i\lambda_nt}| \leq C, \quad |e^{i\nu} - e^{i\tilde{\lambda}_nt}| \leq C|\lambda_n - \tilde{\lambda}_n|, \quad t \in [-a, a], \quad |n| \geq n_0.$$ \hspace{1cm} (3.20)

Substituting (3.20) into (3.2), we get

$$\|u_n - \tilde{u}_n\|_{L^2} \leq C|\lambda_n - \tilde{\lambda}_n|, \quad |n| \geq n_0,$$

which implies

$$\sum_{|n| \geq n_0} n^2\|u_n - \tilde{u}_n\|_{L^2(-a,a)}^2 < C^2\Lambda^2.$$ \hspace{1cm} (3.21)

Substituting (3.20) into (3.7), we have

$$|w_n - \tilde{w}_n| \leq Cn|\lambda_n - \tilde{\lambda}_n|, \quad |n| \geq n_0,$$

which implies

$$\sum_{|n| \geq n_0} |w_n - \tilde{w}_n|^2 < C^2\Lambda^2.$$ \hspace{1cm} (3.22)

Now let us consider $|n| < n_0$. By the definitions of $E_n(t, \lambda)$ and $F_n(\lambda)$, using Lemma 3.2 we have that for each fixed $k \in (-n_0, n_0) \cap \mathcal{S}_j$,

$$|E_k^{(\nu)}(t, \lambda_k) - (it)^\nu e^{i\lambda_k t}| \leq C \max_{n \in \mathcal{S}_k} |\tilde{\lambda}_n - \lambda_k|, \quad \nu = 0, 1, \ldots, m_k - 1,$$

$$|F_k^{(\nu)}(\lambda_k) + f^{(\nu)}(\lambda_k)| \leq C \max_{n \in \mathcal{S}_k} |\tilde{\lambda}_n - \lambda_k|, \quad \nu = 0, 1, \ldots, m_k - 1,$$

for sufficient small $\varepsilon > 0$. Thus

$$\sum_{n \in M_k} \|	ilde{u}_n - u_n\|_{L^2(-a,a)} \leq C \max_{n \in \mathcal{S}_k} |\tilde{\lambda}_n - \lambda_k|, \quad |k| < n_0, \quad k \in \mathcal{S}_j,$$ \hspace{1cm} (3.23)

$$\sum_{n \in M_k} |\tilde{w}_n - w_n| \leq C \max_{n \in \mathcal{S}_k} |\tilde{\lambda}_n - \lambda_k|, \quad |k| < n_0, \quad k \in \mathcal{S}_j.$$ \hspace{1cm} (3.24)

It follows that

$$\sum_{|n| < n_0} n^2\|u_n - \tilde{u}_n\|_{L^2(-a,a)}^2 < C^2\Lambda^2, \hspace{1cm} (3.25)$$

$$\sum_{|n| < n_0} n^2|w_n - \tilde{w}_n|^2 < C^2\Lambda^2.$$ \hspace{1cm} (3.26)
Together with (3.21), (3.25) and (3.22), (3.26), we arrive at (3.18) and (3.19), respectively.

\[\square\]

**Lemma 3.4.** There exists \(\varepsilon > 0\) such that, for any sequence \(\{\lambda_n\}_{n \in \mathbb{Z}_j}\) satisfying (1.6), there exists a unique pair of functions \(\tilde{M}(t)\) and \(\tilde{N}(t)\) in \(L^2(-a, a)\) satisfying the relation (3.11), where \(\tilde{M}(t)\) is odd and \(\tilde{N}(t)\) is even. Moreover,

\[
\|\tilde{M} - \tilde{M}\|_{L^2(-a,a)} + \|N - \tilde{N}\|_{L^2(-a,a)} \leq \Lambda. \tag{3.27}
\]

**Proof.** Using Proposition 4.1 from [24] together with Lemma 3.3, we conclude that there exists a unique function \(\tilde{U} \in L^2(-a, a)\) such that \((\tilde{U}, \tilde{u}_n) = \tilde{w}_n, n \in \mathbb{Z}_j\) and \(\|\tilde{U} - \frac{1}{2}(M + \alpha N)\| \leq \Lambda\). Denote

\[
\tilde{M}(t) := \tilde{U}(t) + \tilde{U}(-t), \quad \tilde{N}(t) := \frac{\tilde{U}(t) - \tilde{U}(-t)}{\alpha}.
\]

Then \(\tilde{U}(t) = \frac{1}{2} \left( \tilde{M}(t) + \alpha \tilde{N}(t) \right)\), and \(\tilde{M}(t)\) is even and \(\tilde{N}(t)\) is odd. Thus the system (3.17) is satisfied, which is equivalent to (3.11). By a direct calculation, we can obtain (3.27).

Define the functions \(\tilde{\Delta}(\lambda), \tilde{K}_x(a,t)\) and \(\tilde{K}_t(a,t)\) with the functions \(\tilde{M}(t)\) and \(\tilde{N}(t)\):

\[
\tilde{\Delta}(\lambda) = f(\lambda) + \frac{1}{2} \int_{-a}^{a} [\tilde{M}(t) + \alpha \tilde{N}(t)] e^{i\lambda t} dt, \tag{3.28}
\]

\[
\tilde{K}_x(a,t) := \tilde{M}(t) - \beta \int_{t}^{a} \tilde{N}(s) ds - \beta \omega, \quad \tilde{K}_t(a,t) := -\tilde{N}(t), \quad t \in [0,a]. \tag{3.29}
\]

Clearly, \(\tilde{K}_x(a,t)\) is even, and \(\tilde{K}_t(a,t)\) is odd. It follows from (3.11) and (3.28) that \(\{\lambda_n\}_{n \in \mathbb{Z}_j}\) (with multiplicities) are the zeros of \(\tilde{\Delta}(\lambda)\). By the Schwarz’s inequality, we calculate

\[
\int_{0}^{a} \left( \int_{0}^{a} (\tilde{N}(s) - N(s)) ds \right)^2 dt \leq a \left( \int_{0}^{a} \left| \tilde{N}(s) - N(s) \right| ds \right)^2 \leq a^2 \|\tilde{N} - N\|_{L^2(0,a)}^2.
\]

It follows from (3.27) and (3.29) that

\[
\|\tilde{K}_x(a,\cdot) - K_x(a,\cdot)\|_{L^2(0,a)} + \|\tilde{K}_t(a,\cdot) - K_t(a,\cdot)\|_{L^2(0,a)} \leq \Lambda. \tag{3.30}
\]

Using Theorem 2.4 and (3.30), we get that there exist the unique pair of \(\tilde{q}\) and \(\tilde{h}\) such that \(\{\tilde{K}_x(a,t), \tilde{K}_t(a,t), \omega\}\) are the corresponding Cauchy data. Moreover,

\[
\|q - \tilde{q}\|_{L^2(0,a)} \leq \Lambda, \quad |\tilde{h} - h| \leq \Lambda.
\]

Define the functions

\[
\tilde{\varphi}(a, \lambda) = \cos(\lambda a) + \frac{\omega \sin(\lambda a)}{\lambda} - \int_{0}^{a} \tilde{K}_t(a,t) \frac{\sin(\lambda t)}{\lambda} dt, \tag{3.31}
\]

\[
\tilde{\varphi}'(a, \lambda) = -\lambda \sin(\lambda a) + \omega \cos(\lambda a) + \int_{0}^{a} \tilde{K}_x(a,t) \cos(\lambda t) dt. \tag{3.32}
\]
Using (3.28), (3.29), (3.31), (3.32), and (3.4), we obtain that the function $\tilde{\Delta}(\lambda)$ constructed in (3.28) has the expression

$$\tilde{\Delta}(\lambda) = \tilde{\varphi}'(a, \lambda) + (i\lambda \alpha + \beta) \tilde{\varphi}(a, \lambda).$$

(3.33)

The proof of Theorem 1.1 is complete.

Acknowledgments. The author Xu was supported by the National Natural Science Foundation of China (11901304). The author Bondarenko was supported by Grants 20-31-70005 and 19-01-00102 of the Russian Foundation for Basic Research.

REFERENCES

[1] T. Aktosun, A. Machuca, P. Sacks, Determining the shape of a human vocal tract from pressure measurements at the lips, Inverse Problems 33 (2017), 115002 (33pp).
[2] N.P. Bondarenko, Local solvability and stability of the inverse problem for the non-self-adjoint Sturm-Liouville operator, Boundary Value Problems 2020 (2020), 123 (13pp).
[3] N.P. Bondarenko, Inverse Sturm-Liouville problem with analytical functions in the boundary condition, Open Mathematics 18 (2020), no. 1, 512-528.
[4] N.P. Bondarenko, Solvability and stability of the inverse Sturm-Liouville problem with analytical functions in the boundary condition, Math. Meth. Appl. Sci. 43 (2020), no. 11, 7009-7021.
[5] G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Acta Math. 78 (1946), 1-96.
[6] S. A. Buterin, A. E. Choque-Rivero, M. A. Kuznetsova, On a regularization approach to the inverse transmission eigenvalue problem, Inverse Problems 36 (2020), 105002.
[7] S. Buterin, M. Kuznetsova, On Borg’s method for non-selfadjoint Sturm-Liouville operators, Anal. Math. Phys. 9 (2019), 2133-2150.
[8] S. Buterin, On inverse spectral problem for non-selfadjoint Sturm-Liouville operator on a finite interval, J. Math. Anal. Appl. 335 (2007), 739-749.
[9] G. Freiling, V. Yurko, Inverse Sturm-Liouville Problems and their Applications, NOVA Science Publishers, New York, 2001.
[10] M. G. Krein, A. A. Nudelman, Direct and inverse problems for frequencies of boundary dissipation of a nonuniform string (Russian), Dokl. Akad. Nauk SSSR 247 (1979), No. 5, 1046-1049.
[11] M. G. Krein, A. A. Nudelman, Some spectral properties of a nonhomogeneous string with a dissipative boundary condition (Russian), J. Operator Theory 22 (1989), 369-395.
[12] V. V. Kravchenko, S. M. Torba, A practical method for recovering Sturm-Liouville problems from the Weyl function, preprint (2021), arXiv:2101.08930 [math.CA].
[13] B. M. Levitan, Inverse Sturm-Liouville Problems, Nauka, Moscow, 1984 (Russian); English transl., VNU Sci. Press, Utrecht, 1987.
[14] V. Marchenko, Sturm-Liouville Operators and Applications, Publisher Birkhäuser, Boston, 1986.
[15] M. Marletta and R. Weikard, Weak stability for an inverse Sturm-Liouville problem with finite spectral data and complex potentials, Inverse Problems, 21 (2005) 1275-1290.
[16] M. Möller, V. Pivovarchik, Direct and inverse Robin-Regge problems, Electron. J. Differential Equations, 2017 (2017), No. 287, 1-18.
[17] M. Möller, V. Pivovarchik, Spectral Theory of Operator Pencils, Hermite-Biehler Functions, and Their Applications, OT 246, Birkhäuser, Cham, 2015.
[18] V. Pivovarchik, C. van der Mee, The inverse generalized Regge problem, Inverse Problems, 17 (2001), 1831-1845.
[19] J. Pöschel, E. Trubowitz, Inverse spectral theory, Academic Press, London, 1987.
[20] T. Regge, Construction of potentials from resonance parameters, Nuovo Cimento 9 (1958) 491-503.
[21] W. Rundell, P. Sacks, Reconstruction techniques for classical inverse Sturm-Liouville problems, Mathematics of Computation 58 (1992), 161-183.
[22] W. Rundell, P. Sacks, Numerical technique for the inverse resonance problem, J. Computational and Applied Mathematics 170 (2004), 337-347.
[23] B. Simon, Resonances in one dimension and Fredholm determinants, Journal of Functional Analysis 178 (2000), 396-420
[24] X.-C. Xu, Inverse spectral problems for the generalized Robin-Regge problem with complex coefficients, Journal of Geometry and Physics 159 (2021), 103936
[25] V.A. Yurko, On boundary value problems with a parameter in the boundary conditions, Soviet J. Contemporary Math. Anal. 19 (1984), 62-73.