A Tight Bound of Hard Thresholding

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Abstract

This paper is concerned with the hard thresholding technique which sets all but the $k$ largest absolute elements to zero. We establish a tight bound that quantitatively characterizes the deviation of the thresholded solution from a given signal. Our theoretical result is universal in the sense that it holds for all choices of parameters, and the underlying analysis only depends on fundamental arguments in mathematical optimization. We discuss the implications for the literature:

**Compressed Sensing.** On account of the crucial estimate, we bridge the connection between restricted isometry property (RIP) and the sparsity parameter of $k$ for a vast volume of hard thresholding based algorithms, which renders an improvement on the RIP condition especially when the true sparsity is unknown. This suggests that in essence, many more kinds of sensing matrices or fewer measurements are admissible for the data acquisition procedure.

**Machine Learning.** In terms of large-scale machine learning, a significant yet challenging problem is producing sparse solutions in online setting. In stark contrast to prior works that attempted the $\ell_1$ relaxation for promoting sparsity, we present a novel algorithm which performs hard thresholding in each iteration to ensure such parsimonious solutions. Equipped with the developed bound for hard thresholding, we prove *global linear convergence* for a number of prevalent statistical models under mild assumptions, even though the problem turns out to be non-convex.

**Keywords:** Sparsity, Hard Thresholding, Compressed Sensing, RIP, Stochastic Optimization

1 Introduction

Over the last two decades, pursuing a sparse representation has emerged as a fundamental technique throughout bioinformatics [OF97], statistics [Tib96, EHJT04], engineering and mathematical science [CDS98, Don06], to name just a few. In order to obtain a sparse solution, numerous practical algorithms have been presented, among which two prominent examples are greedy pursuit and convex relaxation [TW10]. For instance, as one of the earliest algorithms, orthogonal matching pursuit (OMP) repeatedly picks a coordinate as the potential support of a solution and its performance was analyzed in [Tro04, TG07]. Compressive sampling matching pursuit (CoSaMP) [NT09] and subspace pursuit (SP) [DM09] improved OMP by selecting multiple coordinates followed by a pruning step in each iteration, and the recovery condition was framed under the restricted isometry property (RIP) [CT05]. The iterative hard thresholding (IHT) algorithm [BD08, BD09] begins with an initial guess and iteratively refines the iterates by gradient descent followed by hard thresholding.

Since the sparsity constraint counts the non-zero components which renders the problem non-convex, the $\ell_1$ norm was suggested as a convex relaxation dating back to Lasso [Tib96] where it was utilized for variable selection. Interestingly, under the RIP condition, a series of works showed that exact sparse recovery by $\ell_1$ program is possible as soon as the observation noise vanishes [CT05, Can08, CWX10, Fou12].

1This paper was submitted to COLT 2016. We thank the reviewers’ encouraging and insightful comments.
In this paper, we are interested in the hard thresholding (HT) operation underlying a large body of the developed algorithms in compressed sensing (e.g., IHT, CoSaMP, SP) as well as in machine learning (e.g., [YZ13, BRB13]). Formally, for a general vector $b \in \mathbb{R}^d$, the hard thresholded signal $\mathcal{H}_k(b)$ is formed by setting all but the largest (in magnitude) $k$ elements of $b$ to zero. Ties are broken lexicographically. Hence, hard thresholding always guarantees a $k$-sparse signal, i.e., the number of non-zero components does not exceed $k$. Moreover, the resultant signal $\mathcal{H}_k(b)$ is a best $k$-sparse approximation to $b$ in terms of any $\ell_p$ norm ($p \geq 1$). That is, for all $k$-sparse signals $a \in \mathbb{R}^d$, we are guaranteed that
\[
\|\mathcal{H}_k(b) - b\|_2 \leq \|a - b\|_2. \tag{1.1}
\]
In view of the above inequality, a broadly used bound in the literature for the deviation of the thresholded signal is as follows:
\[
\|\mathcal{H}_k(b) - a\|_2 \leq 2\|b - a\|_2. \tag{1.2}
\]
To gain the intuition for its utility, consider two concrete examples:

**Example 1** (Compressed Sensing). In this context, we want to (exactly) recover a true sparse signal $a$ from its linear measurements. Recall that the IHT algorithm iteratively performs gradient descent followed by hard thresholding so as to gradually approximate the true signal. Hence, the vector $b$ can be interpreted as the proxy after gradient update and $\mathcal{H}_k(b)$ is the iterate that is expected to converge to $a$. Then (1.2) justifies that after hard thresholding, the distance of the iterate to $a$ is bounded by a multiple of 2 to the one before. Due to the specific problem structure (i.e., linear model with least-squares loss), the term on right-hand side can be expanded and the factor of 2 will be downscaled by the restricted isometry constant (RIC) which is the crux for establishing the convergence of IHT.

**Example 2** (Machine Learning). Now let us consider a more general setting for the observation model and loss function. For instance, consider the logistic regression problem where we are to optimize the negative log-likelihood function for which $a$ is the empirical minimizer of the given program. For such machine learning problems where RIC is not applicable, efforts are devoted to characterizing the problem nature by introducing RIP-like condition in order to scale the factor of 2 to a much smaller value [YLZ13, BRB13]. Otherwise, we can imagine that in the worst case, $\mathcal{H}_k(b)$ is two times far away from the optima than $b$ is, and such error will be propagated with exponential rate. Thereby we have no hope to obtain the optima. This issue turns out to be more serious in online setting (i.e., stochastic gradient method), where the iterate $b$ itself contains variance.

These two illustrative examples have spelled out the importance of offering a tight bound on the deviation resulted by hard thresholding for compressed sensing and machine learning, especially when there is memory budget and henceforth stochastic gradient evaluation is inevitable. To be more detailed, for compressed sensing, a tighter bound relaxes the RIC requirement which permits more kinds of sensing matrices or fewer measurements to be used during the data acquisition paradigm. We remark that improving the RIC is virtually challenging and is one of the most popular lines in the literature. For instance, [Fou12] showed that the restricted isometry constant $\delta_{3k} < 0.25$ is sufficient for IHT, and [Fou11] obtained $\delta_{3k} < 0.29$ by a lengthy proof which is the best one to date. More works can be found in, e.g., [Can08, Fou10, CWX10]. Beyond compressed sensing, [BRB13, YLZ13] considered analogous conditions to RIC for general machine learning problems, where HT is the core technique to produce sparse solutions. Essentially, the less deviation the HT operation brings, the more statistical models the algorithms can cope with. Moreover, besides these mentioned batch methods where no variance occurs, the challenging case of stochastic gradient evaluation usually carries out variance for updating the iterates. Thereby, to mitigate this issue, we have to seek a better bound to control the precision of the thresholded solution, which will facilitate the development of hard thresholding in online setting.


1.1 Summary of Contributions

In this work, we make three contributions:

1. We examine the tightness of (1.2) and provably show that the equality therein will never be attained. We then improve this bound and quantitatively characterize that the deviation is inversely proportional to the value of $k$. Our bound is universal, which holds for all choices of $k$-sparse signal $a$ and general signal $b$. Our bound is tight, in the sense that the equality we built can be attained for specific signals, hence cannot be improved if no additional information is available.

2. Owing to the developed bound, we show that the RIP (or RIP-like) condition required by a wide range of hard thresholding based algorithms can be relaxed. In the context of compressed sensing, this means in essence, many more kinds of sensing matrices or less measurements can be utilized for data acquisition. For machine learning, it suggests the currently devised algorithms are capable of handling various statistical models in addition to those that have been investigated.

3. Finally, we present a novel algorithm which applies hard thresholding in online setting and is guaranteed to converge to the global optima with a geometric rate. Our analysis demonstrates that only when the deviation is controlled below the multiple of 1.15 can such algorithm succeed. This immediately implies the conventional bound (1.2) is not applicable in the challenging case.

1.2 Notation

Before delivering the algorithm and main theoretical results, let us instate several pieces of notation that are involved throughout the paper. We use bold lowercase letters, e.g., $v$ to denote a vector and its $i$th element is denoted by $v_i$. The $\ell_2$ norm of a vector $v$ is denoted by $\|v\|_2$. We write bold capital letters such as $M$ for matrices and its $(i, j)$-th entry is denoted by $m_{ij}$.

For an integer $d$, suppose $\Omega$ is a subset of \{1, 2, …, $d$\}. Then for a vector $v \in \mathbb{R}^d$, we define the orthogonal projection to the support set of $\Omega$ as follows:

$$P_{\Omega}(v) = \begin{cases} v_i, & i \in \Omega, \\ 0, & \text{otherwise}. \end{cases} \quad (1.3)$$

In particular, let $\Gamma$ be the support set indexing the $k$ largest absolute components of $v$. In this way, the hard thresholding operator is given by

$$H_k(v) = P_{\Gamma}(v). \quad (1.4)$$

We will also manipulate the orthogonal projection of a vector $v$ onto an $\ell_2$ ball with radius $\omega$. That is,

$$\Pi_\omega(v) = \frac{v}{\max\{1, \|v\|_2 / \omega\}}. \quad (1.5)$$

Finally, the support set of a vector $v$ is denoted by $\text{supp}(v)$ whose cardinality is denoted by $|\text{supp}(v)|$ or $\|v\|_0$. We write the capital upright letter $C$ and its subscript variants (e.g., $C_0$, $C_1$) for some absolute constants that may change from appearance to appearance.

1.3 Roadmap

We present the key tight bound for hard thresholding in Section 2, along with a justification that the conventional bound (1.2) is not tight. We then discuss the implications to compressed sensing and machine learning.
learning in Section 3, which shows that the RIP or RIP-like condition can be improved for a number of popular algorithms. Thanks to our new estimation, Section 4 develops a novel stochastic algorithm which performs hard thresholding for each iterate and establishes the global linear convergence. Some preliminary empirical results are demonstrated in Section 5. We conclude the paper in Section 6 and the proofs are deferred to the appendix.

2 The Key Bound

We argue that the conventional bound (1.2) is not tight, in the sense that the equality therein can hardly be attained. To see this, recall how the bound was derived (see, e.g., [NT09]):

$$\|H_k(b) - a\|_2 = \|\langle H_k(b) - b, a \rangle \|_2 \leq \xi \|H_k(b) - b\|_2 + \|b - a\|_2 \leq 2\|b - a\|_2,$$  \hspace{1cm} (2.1)

where the last inequality is owing to the fact that $H_k(b)$ is a best $k$-sparse approximation to $b$. The major issue occurs in $\xi$ since it does not take the sparsity structure of the signals into account. Note that in order to fulfill the equality in $\xi$, we must have $H_k(b) - b = t(b - a)$ for some $t \geq 0$, that is,

$$H_k(b) = (t + 1)b - ta.$$  \hspace{1cm} (2.2)

One may verify that the above equality holds if and only if

$$a = b = H_k(b).$$  \hspace{1cm} (2.3)

To see this, let $\Omega$ be the support set of $H_k(b)$ and $\overline{\Omega}$ be the complement. Let $b_1 = \mathcal{P}_\Omega(b) = H_k(b)$ and $b_2 = \mathcal{P}_{\overline{\Omega}}(b)$. Likewise, we define $a_1$ and $a_2$ as the components of $a$ supported on $\Omega$ and $\overline{\Omega}$ respectively. Hence, (2.2) indicates $a_1 = b_1$ and $a_2 = (1 + t^{-1})b_2$ where we assume $t > 0$ since $t = 0$ immediately implies the equality of (1.2) does not hold. If $\|b_1\|_0 < k$, then we have $a_2 = b_2 = 0$ since $b_1$ contains the $k$ largest absolute elements of $b$. Otherwise, the fact that $\|a\|_0 \leq k$ and $a_1 = b_1$ implies $a_2 = 0$, and hence $b_2$. Therefore, we obtain (2.3).

When (2.3) happens, however, in reality have $\|H_k(b) - a\|_2 = \|b - a\|_2 = 0$, hence the factor of 2 in (1.2) can essentially be replaced with an arbitrary constant! In this sense, we conclude the bound (1.2) is not tight. Our estimate for hard thresholding is as follows:

**Theorem 1** (Tight Bound for Hard Thresholding). Let $b$ be an arbitrary $d$-dimensional vector and $a \in \mathbb{R}^d$ be any $k$-sparse signal. Denote $k^* = \|a\|_0 \leq k$. Then, we have the following universal bound:

$$\|H_k(b) - a\|_2^2 \leq \nu \|b - a\|_2^2, \quad \nu = 1 + \frac{\rho + \sqrt{(4 + \rho)\rho}}{2}, \quad \rho = \frac{\min\{k^*, d - k\}}{k - k^* + \min\{k^*, d - k\}}.$$  \hspace{1cm} (2.4)

In particular, our bound is tight in the sense that there exist specific vectors of $b$ and $a$ such that the equality holds, hence the bound cannot be improved without further information on signals of $b$ and $a$.

**Remark 2** (Maximum $\nu$). In contrast to previous constant bound (1.2), our result asserts that the deviation resulted by hard thresholding is inversely proportional to $k$ (when $k^* \leq d - k$) in a universal manner. When $k$ tends to $d$, $\rho$ is given by $(d - k)/(d - k^*)$ which is still decreasing with respect to $k$. Thus, the maximum value of $\rho$ equals one. Even in this case, we find that $\sqrt{\nu} = 1.62 < 2$.

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2Note that, even if $a$ is not exactly $k$-sparse, we can always bound the error by $\|H_k(b) - a\|_2 \leq \|H_k(b) - H_k(a)\|_2 + \|H_k(a) - a\|_2$. Thus, without loss of generality, we can assume that $a$ is $k$-sparse.
Remark 3. If we allow \( k = d \), we obtain \( \| \mathcal{H}_k (b) - a \|_2 \leq \| b - a \|_2 \), which fulfills the non-expansiveness of convex proximal operator widely employed in convex stochastic optimization algorithms (see, for example, [DBL14, XZ14]). Yet, we will show in Section 4 a perhaps amazing result, which says \( k \) can be significantly smaller than \( d \) while still preserves the optimality if we replace the convex proximal operator (which encourages sparsity) with hard thresholding (which guarantees sparsity).

Remark 4. More importantly, although for some algorithms such as IHT, CoSaMP, the constant bound \((1.2)\) is sufficient to establish the convergence due to specific condition, we show in Section 4 that such bound cannot ensure global optimality for stochastic algorithms with generic loss function.

Remark 5. To intuitively understand the tightness of the bound, consider for example, the case of \( k = 4k^* \) and \( d = 50k^* \). The conventional bound is

\[
\| \mathcal{H}_k (b) - a \|^2_2 \leq 4 \| b - a \|^2_2,
\]

while our estimate reduces the factor of 4 above to 1.64.

Proof. (Sketch) Our bound follows from fully exploring the sparsity pattern of the signals and fundamental arguments in mathematical optimization. Let \( \Omega \) be the support set of \( w \) \( \overset{\text{def}}{=} \mathcal{H}_k (b) \) and let \( \bar{\Omega} \) be its complement. We immediately have \( P_{\Omega} (b) = w \). Let \( \Omega' \) be the support set of \( a \). Define

\[
b_1 = P_{\Omega \setminus \Omega'} (b), \quad b_2 = P_{\Omega' \setminus \Omega} (b), \quad b_3 = P_{\bar{\Omega} \setminus \Omega'} (b), \quad b_4 = P_{\bar{\Omega} \setminus \Omega} (b).
\]

Likewise, we define \( a_i \) and \( w_i \) for \( 1 \leq i \leq 4 \). Due to construction, we have \( w_1 = b_1, w_2 = b_2, w_3 = w_4 = a_1 = a_3 = 0 \). Our goal is to estimate the maximum value of \( \| w - a \|^2_2 / \| b - a \|^2_2 \). It is easy to show that when attaining the maximum, \( \| b_3 \|^2_2 \) must be zero. Denote

\[
t = \frac{\| w - a \|^2_2}{\| b - a \|^2_2} = \frac{\| b_1 \|^2_2 + \| b_2 - a_2 \|^2_2 + \| a_4 \|^2_2}{\| b_1 \|^2_2 + \| b_2 - a_2 \|^2_2 + \| b_4 - a_4 \|^2_2}.
\]

Note that the the variables here only involve \( a \) and \( b \). Arranging the equation we obtain

\[
(t - 1) \| b_2 - a_2 \|^2_2 + t \| b_4 - a_4 \|^2_2 - \| a_4 \|^2_2 + (t - 1) \| b_1 \|^2_2 = 0.
\]

It is evident that for specific choices of \( b \) and \( a \), we have \( t = 1 \). Since we are interested in the maximum of \( t \), we assume \( t > 1 \) below. Fix \( b \), we can view the left-hand side of the above equation as a function of \( a \). One may verify that the function has positive definite Hessian matrix and thus it attains the minimum at stationary point given by

\[
a_2^* = b_2, \quad a_4^* = \frac{t}{t - 1} b_4.
\]

On the other hand, \( 2.8 \) implies the minimum values should not be greater than zero. Plugging the stationary point back gives

\[
\| b_1 \|^2_2 t^2 - (2 \| b_1 \|^2_2 + \| b_4 \|^2_2) t + \| b_1 \|^2_2 \leq 0.
\]

Solving the above inequality with respect to \( t \), we obtain

\[
t \leq 1 + \left( 2 \| b_1 \|^2_2 \right)^{-1} \left( \| b_4 \|^2_2 + \sqrt{4 \| b_1 \|^2_2 + \| b_4 \|^2_2 \| b_4 \|^2_2} \right).
\]
To derive an upper bound that is uniform over the choice of \( b \), we recall that \( b_1 \) contains the largest absolute elements of \( b \) while \( b_4 \) has smaller values. In particular, we have

\[
\| b_4 \|_2^2 / \| b_1 \|_0 \leq \| b_1 \|_2^2 / \| b_1 \|_0. \tag{2.12}
\]

Note that \( \| b_1 \|_0 = k - \| b_2 \|_0 = k - (k^* - \| b_4 \|_0) \). Hence, combining with the fact that \( 0 \leq \| b_4 \|_0 \leq \min\{k^*, d-k\} \) and optimizing over \( \| b_4 \|_0 \) gives

\[
\| b_4 \|_2^2 \leq \frac{\min\{k^*, d-k\}}{k-k^* + \min\{k^*, d-k\}} \| b_1 \|_2^2. \tag{2.13}
\]

Finally, we arrive at a uniform upper bound for \( t \):

\[
t \leq 1 + \frac{\rho + \sqrt{(4 + \rho)\rho}}{2}, \quad \rho = \frac{\min\{k^*, d-k\}}{k-k^* + \min\{k^*, d-k\}}. \tag{2.14}
\]

See Appendix B for the full proof.

\( \square \)

Remark 6 (Tightness). We construct proper vectors of \( b \) and \( a \) to establish the tightness of our bound by a backward induction. Note that \( t \) equals \( \nu \) if and only if \( \| b_4 \|_2^2 = \rho \| b_1 \|_2^2 \). Hence, we pick

\[
\| b_4 \|_2^2 = \rho \| b_1 \|_2^2, \quad a_2 = b_2, \quad a_4 = \frac{\nu}{\nu - 1} b_4, \tag{2.15}
\]

where \( a_2 \) and \( a_4 \) are actually chosen as the stationary point as in (2.9). We note that the quantity of \( \nu \) only depends on \( p, k \) and \( k^* \), not on the components of \( b \) or \( a \). Plugging the above back to (2.7) justifies \( t = \nu \).

It remains to show that our choices in (2.15) do not violate the definition of \( b_i \)’s, i.e., we need to ensure that elements in \( b_1 \) or \( b_2 \) should be no less than those in \( b_3 \) or \( b_4 \). Note that there is no such constraint for the vector \( a \). Let us consider the case \( k^* < d-k \) and \( \| b_4 \|_0 = k^* \), so that \( \| b_1 \|_0 = k \) and \( \rho = k^*/k \). Thus, the first equality of (2.15) holds as soon as all the entries of \( b \) have same magnitude. \( \| b_4 \|_0 = k^* \) also implies \( \Omega' \) is a subset of \( \Omega \) due to the definition of \( b_4 \) and the sparsity of \( a \), hence we have \( a_2 = 0 = b_2 \). Finally, picking \( a_4 \) as we did completes the reasoning since it fulfills the sparsity constraint on \( a \).

### 3 Implications to Compressed Sensing and Beyond

Let us take a further step to investigate what Theorem 1 implies for the literature of compressed sensing and machine learning. Since nearly all of the HT based algorithms (e.g., IHT, CoSaMP, SP, GraSP [BRB13] and GraHTP [YLZ13]) utilize the deviation bound (2.5) to derive the convergence condition, they can be improved by our new bound. We exemplify the power of our theorem on two popular algorithms in compressed sensing: IHT [BD08, BD09] and CoSaMP [NT09], and one interesting work GraSP [BRB13] which extends CoSaMP to general machine learning setting (e.g., sparse logistic regression).

We proceed with a brief review of the problem setting in compressed sensing. Compressed sensing algorithms aim to recover a sparse signal \( x^* \in \mathbb{R}^d \) from a set of (perhaps noisy) measurements

\[
y = Ax^* + \varepsilon, \tag{3.1}
\]

where \( \varepsilon \in \mathbb{R}^d \) is some observation noise and \( A \) is a given \( n \times d \) sensing matrix with \( n \ll d \), hence the name compressive sampling. In general, the problem is under-determined due to the number of unknowns is much larger than that of the equations on hand. Yet, the prior knowledge that \( x^* \) is sparse radically changes the premise. That is, if the geometry of the sparse signal is preserved under the action of the sampling matrix \( A \) for a restricted set of directions, then it is possible to invert the sampling process. Such novel idea was
quantified as the $k$th restricted isometry property (RIP) of $A$ by [CT05], which requires that there exists a (non-negative) constant $\delta$, such that for all $k$-sparse signals $x$, the following inequalities hold:

$$(1 - \delta) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta) \|x\|_2^2.$$  

(3.2)

The $k$th restricted isometry constant (RIC) $\delta_k$ is then defined as the smallest one that satisfies the above inequalities. Note that $\delta_{2k} < 1$ is the minimum requirement for identifying all $k$-sparse signals from the measurements. To date, there are three quintessential examples known to exhibit profound restricted isometry behavior as long as the number of measurements is proportional to the product of the sparsity and a polynomial logarithm of the dimension: Gaussian matrices (optimal RIP, i.e., very small $\delta_k$), partial Fourier matrices (fast computation) and Bernoulli ensembles.

Equipped with the (standard) RIP condition, numerous efficient algorithms were developed and exact sparse recovery was established. A partial list includes $\ell_1$ norm based linear programming, IHT, CoSaMP, SP, stagewise OMP [DTDS12], regularized OMP [NV10], along with many interesting works devoted to improving the RIP condition, e.g., [WS12, MS12, CZ13, Dan13, Mo15]. We revisit two popular algorithms of IHT and CoSaMP which employ the hard thresholding operator for the sake of sparsity and illustrate how our new bound (2.4) easily improves the RIP condition.

**Iterative Hard Thresholding.** The IHT algorithm recovers the underlying signal by iteratively performing a full gradient descent on the least-squares loss of $\|y - Ax\|_2^2$ followed by a hard thresholding step. That is, at the $t$-th iteration, given the previous iterate $x^{t-1}$, IHT updates the new solution as follows:

$$x^t = H_k \left( x^{t-1} + A^\top (y - Ax^{t-1}) \right).$$  

(3.3)

To establish global convergence with the rate of $0.5$, [BD09] assumed the RIP condition

$$\delta_{2k+k^*} \leq 0.18,$$  

(3.4)

where $k^*$ is the cardinality of the support of $x^*$ and applied the bound (2.5). As we have shown, this bound is actually not tight and hence, their results, especially the requirement of restricted isometry constant can be improved by Theorem 1.

**Theorem 7.** Let $x^*$ be the sparse signal of interest and denote $k^* = \|x^*\|_0$. Assume the measurements are collected by (3.1) for a given sensing matrix $A$. Pick $k \geq k^*$. Then, under the RIP condition of $\delta_{2k+k^*} \leq 1/\sqrt{8n}$, the sequence of the iterates produced by the IHT algorithm (3.3) converges to the true signal $x^*$ with a geometric rate of $0.5$.

Let us first study the vanilla case $k = k^*$. [BD09] required $\delta_{3k^*} \leq 0.18$ whereas our analysis shows $\delta_{3k^*} \leq 0.22$ suffices. Note that as we mentioned in Section 1, even a little relaxation on RIC is challenging and may require several pages of induction. In contrast, our improvement comes from a direct application of Theorem 1 which only modifies several lines of the proof. See Appendix C for details. To see why relaxing RIC is of central interest, note that standard result (see, e.g., [BDDW08]) asserts that to guarantee the RIP condition

$$\delta_r \leq \delta$$  

(3.5)

with overwhelming probability, it suffices to choose

$$n \geq C_0\delta^{-2} r \log(d/r).$$  

(3.6)

In view of (3.6), we find that the number of measurements is dramatically reduced with a factor of $0.67$. 

What is of more interest is a characterization on the RIP condition and the quantity of projection sparsity.

For $\text{BD09}$, when gradually tuning $k$ larger than $k^\ast$, it always requires $\delta_{2k+k^\ast} \leq 0.18$. Note that due to the monotonicity of RIC, i.e., $\delta_r \leq \delta_{r'}$ if $r \leq r'$, the condition turns out to be more and more stringent. Compared to their result, since $\nu$ is inversely proportional to $k$, Theorem 7 is powerful especially when $k$ becomes larger. For example, suppose $k = 20k^*$. In this case, Theorem 7 justifies IHT admits the linear convergence as soon as $\delta_{41k^*} \leq 0.32$. Such property is appealing in practice, in that among various real-world applications, the true sparsity is indeed unknown and we would like to estimate a conservative upper bound on it.

### Compressive Sampling Matching Pursuit

The CoSaMP algorithm is one of the most efficient algorithms for sparse recovery. Compared to IHT which performs hard thresholding after gradient update, CoSaMP prunes the gradient at the beginning of each iteration, followed by solving a least-squares program restricted to a small support set. In particular, in the last step, CoSaMP applies hard thresholding to form a $k$-sparse iterate for future update. The analysis on CoSaMP consists of bounding the estimation error in each step. Owing to Theorem 1, we advance the theoretical result of CoSaMP by improving the error bound for its last step, and hence the RIP condition.

**Theorem 8.** Let $x^\ast$ be the sparse signal of interest and denote $k^\ast = \|x^\ast\|_0$. Assume the measurements are collected by (3.1) for a given sensing matrix $A$. Pick $k \geq k^\ast$. Then, under the RIP condition that

$$\delta_{3k+k^\ast} \leq \left(\frac{\sqrt{32\nu + 49} - 9}{4\sqrt{\nu - 1}}\right)^{1/2},$$

(3.7)

the sequence of the iterates produced by CoSaMP converges to $x^\ast$ with a geometric rate of 0.5.

Roughly speaking, the bound is still inversely proportional to $\sqrt{\nu}$. Hence, it is monotonically increasing with respect to $k$, indicating our theorem is more effective for a large quantity of $k$. In fact, for the CoSaMP algorithm, our bound above is superior to the best known result even when $k = k^\ast$. To see this, we have the RIP condition $\delta_{4k^\ast} \leq 0.31$. In comparison, $\text{NT09}$ derived a bound $\delta_{4k} \leq 0.1$ and $\text{FR13}$, Theorem 6.27 improved to $\delta_{4k} < 0.29$ for a geometric rate of 0.5. Thus, our bound here is the best known bound. Moreover, the RIC can further be relaxed to 0.33 if we pick $k = 20k^\ast$.

### Gradient Support Pursuit

The GraSP algorithm generalizes CoSaMP in the sense that GraSP supports a wide class of loss functions rather than the least-squares. To characterize the problem structure, a property called stable restricted Hessian (SRH) was introduced in $\text{BRB13}$ which is quantitatively equal to the condition number of the Hessian matrix when restricted to sparse directions. In special cases, SRH reduces to RIP. The hard thresholding operation also emerges in their last step, and hence the SRH requirement can be relaxed like we have done for CoSaMP.

**Theorem 9.** Suppose the objective function that GraSP minimizes is a twice continuously differentiable function that has $\mu_{3k+k^\ast}$-SRH and the smallest eigenvalue of the Hessian matrix restricted on $(3k + k^\ast)$-sparse vectors is not smaller than $\varepsilon$. Then GraSP converges with a geometric rate of 0.5 as soon as

$$\mu_{3k+k^\ast} \leq 1 + \frac{(\sqrt{\nu} + 1)^{1/2} - 1}{\sqrt{\nu}}.$$  

(3.8)

### 4 Hard Thresholding in Online Setting

Now we move on to the machine learning setting where our focus is pursuing an optimal sparse solution that minimizes a given objective function based on a set of training samples $Z_1^n := \{Z_i\}_{i=1}^n$. Formally, we
are to optimize the following program:

$$\min_{x \in \mathbb{R}^d} F(x; Z_i^n), \quad \text{s.t. } \|x\|_0 \leq k. \quad (4.1)$$

We presume that the objective function is decomposable, i.e.,

$$F(x; Z_i^n) = \frac{1}{n} \sum_{i=1}^{n} f(x; Z_i). \quad (4.2)$$

That is, the empirical risk incurred on the training set of a solution $x$ can be interpreted as a sum of loss on each individual sample. This is quite a mild condition and most of the popular machine learning models fulfills it. Typical examples include (but not limited to):

- **Sparse Linear Regression**: For all $1 \leq i \leq n$, we have $Z_i = (a_i, y_i) \in \mathbb{R}^d \times \mathbb{R}$ and the generative model is $y = Ax^* + \epsilon$ for a $k$-sparse signal $x^*$ of interest and some observation noise $\epsilon$. The loss function $F(x; Z_i^n) = \frac{1}{2n} \|Ax - y\|_2^2$ is the least-squares and can be explained by $f(x; Z_i) = \frac{1}{2} \|a_i \cdot x - y_i\|_2^2$.

- **Sparse Logistic Regression**: For all $1 \leq i \leq n$, we have $Z_i = (a_i, y_i) \in \mathbb{R}^d \times \{+1, -1\}$ and the negative log-likelihood is penalized, i.e., $F(x; Z_i^n) = n^{-1} \sum_{i=1}^{n} \log (1 + \exp (-y_i a_i \cdot x))$ for which $f(x; Z_i) = \log (1 + \exp (-y_i a_i \cdot x))$.

Throughout the paper, we use $\hat{x}$ to denote the global optima of (4.1), i.e.,

$$\hat{x} = \arg \min_{\|x\|_0 \leq k} F(x; Z_i^n). \quad (4.3)$$

Without loss of generality, we assume that $\hat{x}$ and $x^*$ lie in an $\ell_2$ ball with a constant radius $\omega$, i.e.,

$$\|\hat{x}\|_2 \leq \omega, \quad \|x^*\|_2 \leq \omega. \quad (4.4)$$

Note that we did not assume the magnitude of $\omega$. To ease notation, we will often write $F(x; Z_i^n)$ as $F(x)$ and $f(x; Z_i)$ as $f_i(x)$ for $i = 1, 2, \cdots, n$.

The major contribution of this section is a novel algorithm termed HT-SVRG to optimize (4.1), tackling one of the most important problems in large-scale machine learning: producing sparse solutions in *online setting*. We emphasize that the formulation (4.1) is in stark contrast to the $\ell_1$ regularized convex programs considered by previous stochastic solvers such as Prox-SVRG [XZ14] and SAGA [DBL14] in machine learning. We target here a stochastic algorithm for the non-convex problem that is less exploited in the literature. From the theoretical aspect, (4.1) is much harder to analyze but it always produces $k$-sparse solutions in practice, whereas performance guarantee for convex programs is easier to be delivered but one cannot characterize the sparsity of the obtained solution. In online setting, the $\ell_1$ formulation becomes much less effective in terms of sparsity even with a proximal operation, naturally owing to the randomness. See [LLZ09, Xia10, DS09] for more discussion on the issue. We also remark that existing works such as [YLZ13, BRB13] only investigated the sparsity constrained problem (4.1) in batch scenario, which is not practical for large-scale learning problems on account of memory cost. The perhaps most related work to our new algorithm is [NWW14]. Nevertheless, the optimization error of their algorithm does not vanish and it is not clear how far the iterate is from the optima.

Our main result shows global linear convergence to the empirical risk minimizer $\hat{x}$ with vanishing optimization error. We illustrate such appealing behavior by two prevalent statistical models, hence bridging the gap of the lack in theoretical analysis and the practical elegance of online sparsity constrained solvers. We
find that the global convergence is attributed to both the tight bound and the variance reduction technique, and examining the necessity of them is an interesting future work. We would also like to remind that readers should distinguish the optimal solution \( \hat{x} \) and the true parameter \( x^* \). For instance, assume that the response \( y \) of sparse linear regression model is generated by the true parameter \( x^* \), but minimizing (4.1) does not mean to recover \( x^* \). In fact, the convergence to \( x^* \) of the iterates is only guaranteed to an accuracy reflected by the statistical precision of the problem, i.e., \( \| x^* - \hat{x} \|_2 \), which is the best one can hope for any statistical model [ANW12]. There are numerous insightful works devoted to the minimax rate of such statistical error [Zha10, RWY11], whereas the focus of the paper is establishing convergence to \( \hat{x} \).

**Algorithm 1** Hard Thresholded SVRG (HT-SVRG)

**Require:** Training samples \( \{ Z_i \}_{i=1}^n \), maximum stage count \( S \), update frequency \( m \), projection sparsity \( k \), radius \( \omega \), learning rate \( \eta \), initial solution \( \bar{x}^0 \).

**Ensure:** Optimal solution \( \bar{x}^S \).

1: for \( s = 1 \) to \( S \) do
2:    Set \( \bar{x} = \bar{x}^{s-1} \), \( \mu = n^{-1} \sum_{i=1}^n \nabla f_i(\bar{x}) \), \( x^0 = \bar{x} \).
3: for \( t = 1 \) to \( m \) do
4:    Uniformly pick \( i_t \in \{ 1, 2, \cdots, n \} \) and update the solution
5:        \( b^t = x^{t-1} - \eta \left( \nabla f_{i_t}(x^{t-1}) - \nabla f_{i_t}(\bar{x}) + \mu \right) \), \( (4.5) \)
6:        \( r^t = \mathcal{H}_k( b^t ) \), \( (4.6) \)
7:        \( x^t = \Pi_{\omega}(r^t) \). \( (4.7) \)
8: end for

9: Uniformly choose \( j^s \in \{ 0, 1, \cdots, m - 1 \} \) and set \( \bar{x}^s = x^{j^s} \).

**4.1 Algorithm Overview**

Our algorithm applies the framework of SVRG [JZ13], whose main idea is to leveraging past gradients for the current gradient update for the sake of variance reduction. At each stage \( s \), HT-SVRG repeatedly draws a random sample and performs a variance reduced gradient update. To guarantee each iterate is \( k \)-sparse, it then invokes the hard thresholding operation. Our algorithm also requires an estimation \( \omega \) which upper bounds the \( \ell_2 \) norm of the global optima, which facilitates the theoretical analysis as well as ensures good behavior for early stages. Note that the orthogonal projection for \( r^t \) will not change the support set, and hence \( x^t \) is still \( k \)-sparse.

**Challenges.** The most difficult part of establishing global convergence comes from the hard thresholding operation \( \mathcal{H}_k( r^t ) \). To see this, note that it is \( b^t \) which reduces the objective value in expectation. If \( b^t \) is not \( k \)-sparse (which is the common case), \( x^t \) is not equal to \( b^t \) so it does not decrease the objective function. In addition, compared with the convex proximal operator [DBL14] which guarantees the non-expansiveness of the distance to the optima, the hard thresholding step can enlarge the distance up to a multiple of 4 (if using the bound (2.5)). What makes it **more serious** is that such inaccurate estimation \( x^t \) will be used for the next gradient update, and hence the error might be progressively propagated at an exponential rate.

**Key Idea.** The key idea in our analysis is combining the tight bound of Theorem 1 and the curvature of the objective function on sparse directions. To be more detailed, by assuming standard conditions, we can bound the curvature from below and above, meaning that the iterates will not proceed too fast (large variance) or too slowly (little improvement). Then, by picking a properly chosen projection sparsity \( k \), we are able to...
manipulate the deviation of the iterates from the optima. Note that $\nu$ is always greater than one, hence the curvature bound is necessary to downscale the distance of the hard thresholded iterate to the optima. This is analogous to assuming RIP condition and using the RIC to establish convergence [BD09, NT09]. Yet, our conditions are shown to be more general for machine learning problems. Also note that the conventional bound does not suffice in such online setting as we will show in the sections to follow.

4.2 Assumptions

Our analysis depends on two assumptions on the curvature of the objective function which have been standard in the literature (see, e.g., [RWY11, ANW12, JTK14]):

(A1) $F(x)$ satisfies the restricted strong convexity (RSC) condition with parameter $\alpha_{k+k^\prime}$.

(A2) For all $1 \leq i \leq n$, $f_i(x)$ satisfies the restricted smoothness (RSS) condition with parameter $L_{3k+k^\prime}$.

Here, $\hat{k} = \|\bar{x}\|_0$ and the RSC and RSS conditions are defined as follows:

**Definition 10** (Restricted Strong Convexity). A differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to satisfy the property of restricted strong convexity with positive parameter $\alpha_r$, if for all vectors $x, x' \in \mathbb{R}^d$ with $|\text{supp}(x) \cup \text{supp}(x')| \leq r$, it holds that

$$f(x') - f(x) - \langle \nabla f(x), x' - x \rangle \geq \frac{\alpha_r}{2} \|x' - x\|_2^2.$$

**Definition 11** (Restricted Smoothness). A differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to satisfy the property of restricted smoothness with parameter $L_r$, if for all vectors $x, x' \in \mathbb{R}^d$ with $|\text{supp}(x) \cup \text{supp}(x')| \leq r$, it holds that

$$\|\nabla f(x') - \nabla f(x)\|_2 \leq L_r \|x' - x\|_2.$$

Note that if we assume $f(x)$ is twice continuously differentiable (which holds for various loss functions such as least-squares, logistic loss), the RSC and RSS conditions amount to bounding the curvature of $f(x)$ from below and above when restricted to directions with sparse support.

4.3 Performance Guarantee

For simplicity, let us denote

$$L = L_{3k+k^\prime}, \quad \alpha = \alpha_{k+k^\prime}, \quad c = L/\alpha,$$

$$T = \max_{\Omega} \{ \|P_{\Omega}(\nabla F(\bar{x}))\|_2 : \text{supp}(\bar{x}) \subseteq \Omega, |\Omega| \leq 3k + \hat{k} \},$$

$$\kappa = 4\nu\eta^2T(2L\omega + T)m + 2T\omega/\alpha,$$

where we recall that $\omega$ is a universal constant that upper bounds the $\ell_2$ norm of the global optima. Note that the values of $T$ and $\kappa$ do not depend on the iterates produced by HT-SVRG. Also note that the quantity $c$ is virtually the condition number of the problem when restricted on sparse directions. For a stage $s$, we denote $\mathcal{I}^s = \{i_1, i_2, \cdots, i_m\}$, i.e., the samples randomly chosen for updating the solution.

**Theorem 12** (Linear Convergence of HT-SVRG). Assume (A1) and (A2). Pick the step size $\eta < 1/(4L)$. If $\nu < 4L/(4L - \alpha)$, then for sufficiently large $m$, the sequence of iterates $\{\bar{x}^s\}_{s \geq 1}$ converges to the global optima $\bar{x}$ of (4.1) in the sense that

$$\mathbb{E} \left[ F(\bar{x}^s) - F(\bar{x}) \right] \leq \beta^s \left[ F(\bar{x}^0) - F(\bar{x}) \right] + \tau,$$

where $\beta^s$ is the $s$-th iterate of the sequence and $\tau$ is a constant depending on the problem.
where $\beta < 1$ is the convergence coefficient and the expectation is taken over $\{I^1, j^1, I^2, j^2, \ldots, I^s, j^s\}$. In particular, for $1/(1 - \eta \alpha) < \nu < 4L/(4L - \alpha)$, we have

$$
\beta_1 = \frac{1}{2 \nu \eta \alpha - 2 \nu \eta^2 \alpha L - \nu + 1} \frac{1}{\alpha \kappa} + \frac{2 \nu \eta^2 \alpha L}{2 \nu \eta \alpha - 2 \nu \eta^2 \alpha L - \nu + 1}, \\
\tau_1 = \frac{2 (2 \nu \eta \alpha - 2 \nu \eta^2 \alpha L - \nu + 1) (1 - \beta_1) m}{2(2 \nu \eta \alpha - 2 \nu \eta^2 \alpha L - \nu + 1)}.
$$

(4.14) (4.15)

For $\nu < 1/(1 - \eta \alpha)$, we have

$$
\beta_2 = \frac{1}{\nu \eta \alpha (1 - 2 \eta L) m + 2 \eta L} \frac{1}{1 - 2 \eta L}, \quad \tau_2 = \frac{\kappa}{2 \nu \eta \alpha (1 - 2 \eta L)(1 - \beta_2) m}.
$$

(4.16)

**Remark 13.** From the theorem, we know that in order to guarantee the convergence, the quantity of $\nu$ should be less than $4/3$ due to $L \geq \alpha$. Hence, the conventional bound (2.5) is not applicable. In contrast, Theorem 1 asserts such requirement can be fulfilled by tuning $k$ slightly larger than $\hat{k}$.

**Remark 14 (Projection Sparsity).** For the sake of success of our algorithm, we require that $\nu$ is smaller than $4c/(4c - 1)$, which implies $\rho < 1/(16c^2 - 4c)$. Recall that $\rho$ is given in Theorem 1. In general, we are interested in the regime where $\hat{k} \leq k \ll d$. Hence, we have $\rho = k/k$ and the minimum requirement for the projection sparsity is $k > (16c^2 - 4c) \hat{k}$, which agrees with [JTK14] where a batch projected gradient descent algorithm was considered.

**Remark 15 (Update Frequency).** To examine the update frequency $m$, let us consider the second case for example. In order to ensure $\beta_2 < 1$, we must have $m > 1/\{(\nu \eta \alpha (1 - 4 \eta L)) > (4L - \alpha)/(4 \eta L (1 - 4 \eta L) \alpha)\}$. Picking $\eta = \mathcal{O}(1/L)$, we find that the update frequency $m$ scales as $m \geq \mathcal{O}(c)$, which is of the same order as in the convex case [JZ13].

**Remark 16 (RSC and RSS).** Compared to the convex algorithms such as SAG [RSB12], SVRG [JZ13] and SAGA [DBL14] that assume strong convexity and smoothness everywhere, we only require the restricted version. This is not only milder but also more practical in the high dimensional regime where the Hessian matrix could be degenerate [ANW12].

**Remark 17 (Tradeoff between $\beta$ and $k$).** It is interesting to analyze the tradeoff between the projection sparsity $k$ and the convergence coefficient $\beta$. Note that due to $\eta < 1/(4L)$, $\beta_1$ is monotonically increasing with respect to $\nu$. By Theorem 1, we know that $\nu$ is decreasing with respect to $k$. Thus, we conclude that a larger quantity of $\hat{k}$ results in a smaller value of $\beta_1$, and hence a faster rate. Interestingly, for the second case, we obtain a contrary result: the smaller the $k$ tends, the faster the algorithm converges. Combining these two cases, we have the following conclusion:

**Proposition 18 (Optimal $\beta$).** Fix $\eta$ and $m$. Then the optimal choice of $\nu$ in Theorem 12 is $\nu = 1/(1 - \eta \alpha)$ in the sense that the convergence coefficient $\beta$ attains the minimum quantity. In this case, we have $k = \hat{k}(1 - \eta \alpha)/(\eta^2 \alpha^2)$ which is proportional to $(L/\alpha)^2$ if we pick $\eta = \mathcal{O}(1/L)$.

By specifying admissible values to $\eta$, $\nu$ and $m$, and assuming the vanishing gradient of $F(x)$ evaluated at $\hat{x}$, we obtain the following corollary for Theorem 12:

**Corollary 19.** Assume (A1), (A2) and $\nabla F(\hat{x}) = 0$. Pick $k = 25c^2 \hat{k}$. Then, the sequence of the $k$-sparse solutions $\{\bar{x}^s\}_{s \geq 1}$ produced by Algorithm 1 with $m \geq 12.5(5c-1)$ and $\eta = 1/(5L)$ converges to the global optima of (4.1) with a geometric rate of $0.8$, i.e., $\mathbb{E}[F(\bar{x}^s) - F(\hat{x})] \leq 0.8^s [F(\bar{x}^0) - F(\hat{x})]$.  

12
Remark 20 (Vanishing Gradient). The last assumption in the corollary says the gradient of the objective function vanishes at the optima. This implies that if one is solving a convex problem without the sparsity constraint but the optimal solution happens to be sparse, it is safe to perform hard thresholding without loss of optimality. We exemplify this with another convex algorithm SAGA [DBL14] in Appendix E.

For the compressed sensing community where \( y = Ax^* \), the corollary guarantees that HT-SVRG exactly recovers the underlying true signal \( x^* \) when \( F(x) \) is chosen as the least-squares loss. This is simply due to the fact that we have \( \hat{x} = x^* \) and \( \nabla F(x^*) = A^\top(Ax^* - y) = 0 \).

4.4 Optimization Error and Condition Number

It remains to characterize the optimization error \( \tau \) in Theorem 12 which measures the distance of the iterates to the global optima. Another crucial component is the condition number, which should be reasonably small so that we have a small projection sparsity \( k \), and hence an efficient computation. Intuitively, the magnitude of \( \tau \) indicates how close the solution of (4.1) is to that without the sparsity constraint. We will show that a broad class of popular statistical models admits a small, or even vanishing estimation error. Under these settings, we also give a constant bound on the condition number which controls the projection sparsity.

Before formal analysis, we point out that the estimation error \( \tau \) in Theorem 12 is dominated by the quantity \( T \). To see this, we consider the choice \( \eta = \eta' / L \) for some constant \( \eta' < 1/4 \) and \( \nu = 1/(1 - \eta \alpha) \) as an example. Note that such choice only simplifies the constants and does not hinder the merit of our analysis. One may derive similar results for other \( \nu \) as long as the step size is inversely proportional to \( L \), which is actually a typical practice in the literature [Nes04]. By Remark 15, we know that the update frequency \( m \) has to be large enough, i.e.,

\[
m > \frac{c - \eta'}{\eta' (1 - 4 \eta')},
\]

Let us pick \( m \) such that

\[
1 - 4 \eta' - \frac{c - \eta'}{\eta' m} = \frac{\omega}{\alpha \min\{1, L\}}.
\]

In this way and pick \( \omega \) larger than 0.5, we have

\[
\tau = \left[ (1 - 4 \eta') \min\{1, L\} - \frac{1}{\alpha} \right] T + 4 \eta' \min\{1, L\} \cdot T + \frac{2 \eta' \min\{1, L\}}{L \omega} T^2 < T^2 + T.
\]

Thus, it is evident that the optimization error is determined by the quantity of \( T \), i.e., the maximum \( \ell_2 \) norm of the gradient evaluated at the optima \( \hat{x} \) when restricted on a sparse direction.

However, what we assumed on the objective function is only the curvature which is too general to analyze \( T \). Thereby, for the purpose of a concrete result, we focus on a sub-Gaussian design, which is a popular statistical model in the literature [RWY11, ANW12]. There are several ways to (equivalently) define a sub-Gaussian random variable. Below we provide a two of them:

Definition 21 (Sub-Gaussian Random Variable). A centered random variable \( \theta \in \mathbb{R} \) is called \( \sigma \)-sub-Gaussian if for all \( z \in \mathbb{R} \),

\[
\mathbb{E}_\theta[\exp(\theta z)] \leq \exp\left(\frac{\sigma^2 z^2}{2}\right),
\]

or equivalently

\[
\mathbb{P}(|\theta| > z) \leq 2 \exp\left(-\frac{z^2}{2 \sigma^2}\right).
\]
Classical examples of sub-Gaussian random variables are Gaussian, Bernoulli and all bounded ones. Note that the parameter $\sigma$ is usually interpreted as the variance [Ver10]. In the sections to follow, we assume that:

(A3) The entries of the design matrix $A$ are i.i.d. $\sigma$-sub-Gaussian.

### 4.4.1 Sparse Linear Regression

Sparse linear regression is one of the fundamental statistical models for high dimensional inference [ANW12]. Assume that there is a true parameter $x^*$ that generates the observations:

$$y = Ax^* + \varepsilon,$$

where $A$ is the design matrix and $\varepsilon$ is some observation noise. Then, one may optimize an $\ell_2$ regularized least-squares function in order to estimate the parameter:

$$\min_x F(x) \overset{\text{def}}{=} \frac{1}{2n} \sum_{i=1}^{n} \|y_i - a_i \cdot x\|_2^2 + \frac{\gamma}{2} \|x\|_2^2, \quad \text{s.t. } \|x\|_0 \leq k. \quad (4.23)$$

**Lemma 22.** Assume (A3) and the generative model is given by (4.22). Further assume that the elements of the random noise $\varepsilon$ are i.i.d. $\sigma_0$-sub-Gaussian that is independent of $A$. Then as soon as $n \geq C_0 \log d$ for some absolute constant $C_0$, with probability at least $1 - 2d^{-1}$,

$$T \leq \sigma^2 \|\hat{x} - x^*\|_2 \sqrt{\frac{C_0(3k + \hat{k}) \log d}{n}} + \sigma_0 \sqrt{\frac{C_0(3k + \hat{k}) \log d}{n}} + \gamma \|\hat{x}\|_2. \quad (4.24)$$

**Proposition 23** (Condition Number). Assume (A3). Then there exists some absolute constant $C_1$, such that with probability at least $1 - 2\exp(-n/(8C_1))$, $\alpha = 0.5\sigma^2 + \gamma$ and $L = 1.5\sigma^2 + \gamma$ are admissible RSC and RSS parameters for (4.23) respectively, provided that $n \geq 8C_1(3k + \hat{k}) \log(ed/(3k + \hat{k}))$. In particular, the condition number $c$ of (4.23) is bounded by the constant 3 with the same probability.

**Proposition 24** (Optimization Error). Assume (A1), (A2) and (A3). Then with probability at least $1 - 2d^{-1}$, the sequence of the iterates produced by HT-SVRG converges linearly to the global optima $\hat{x}$ with vanishing optimization error $\tau = O\left(\max\{\sigma^2, \sigma_0\} \sqrt{(3k + \hat{k})n^{-1} \log d}\right)$, provided that $n \geq C_0 \log d$ for some absolute constant $C_0$ and the regularization parameter $\gamma$ is not greater than $O\left(\sqrt{(3k + \hat{k})n^{-1} \log d}\right)$.

**Remark 25.** The assumptions (A1) and (A2) can be removed in the price of a lower success probability of $1 - 2d^{-1} - 2\exp(-n/(8C_1))$ and a sample complexity stated in Proposition 23.

### 4.4.2 Sparse Logistic Regression

Logistic regression is one of the most popular models in machine learning for classification. It assumes that the label ($+1$ or $-1$) is determined by a conditional probability as follows:

$$\mathbb{P}(y_i \mid a_i; x^*) = \frac{\exp(2y_i a_i^\top x^*)}{1 + \exp(2y_i a_i^\top x^*)}, \quad \forall 1 \leq i \leq n. \quad (4.25)$$

It then learns the parameter by minimizing $\ell_2$ regularized negative log-likelihood:

$$\min_x F(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \exp(-2y_i a_i^\top x)\right) + \frac{\gamma}{2} \|x\|_2^2, \quad \text{s.t. } \|x\|_0 \leq k. \quad (4.26)$$
Proposition 26 (Condition Number). Assume (A3). Then there exist absolute constant $C_0$ and $C_1$, such that the condition number of $(4.26)$ is upper bounded by $8.16$ with probability at least $\exp \left(1 - \frac{C_0}{\sigma^2 \omega^2}\right) - 2 \exp \left(-\frac{n}{8C_1}\right)$, provided that $n > 8C_1(3k + \hat{k}) \log \left(\frac{ed}{3k + \hat{k}}\right)$.

Proposition 27 (Optimization Error). Assume (A1), (A2) and (A3) for the problem $(4.26)$. Then with probability at least $1 - 4d^{-1}$, the sequence of the iterates produced by HT-SVRG converges linearly to the global optima with vanishing optimization error $\tau = \mathcal{O}\left(\sigma \sqrt{(3k + \hat{k})n^{-1} \log d}\right)$, provided that $\gamma$ is not larger than $\mathcal{O}\left(\sqrt{(3k + \hat{k})n^{-1} \log d}\right)$.

4.5 Proof Sketch for Theorem 12

Proof. Our analysis starts with bounding the Euclidean distance of $x^t$ and $\hat{x}$ (in expectation) in the high dimensional regime. For a fixed stage $s$, let us denote

$$v^t = \nabla f_i(x^{t-1}) - \nabla f_i(\hat{x}) + \bar{\mu}.$$  \hspace{1cm} (4.27)

Let $\Omega$ be a support set such that $\text{supp}(x^t) \subseteq \Omega$, then it follows that

$$r^t = H_k(b^t) = H_k(P_{\Omega}(b^t)).$$  \hspace{1cm} (4.28)

Thus, the Euclidean distance of $x^t$ and $\hat{x}$ can be bounded as follows:

$$\|x^t - \hat{x}\|^2 \leq \|r^t - \hat{x}\|^2 = \|H_k(P_{\Omega}(b^t)) - \hat{x}\|^2 \leq \|P_{\Omega}(b^t) - \hat{x}\|^2,$$  \hspace{1cm} (4.29)

where the factor $\nu$ can be estimated by Theorem 1. Specifying $\Omega = \text{supp}(x^{t-1}) \cup \text{supp}(x^t) \cup \text{supp}(\hat{x}) \cup \text{supp}(\hat{x})$ gives

$$\|P_{\Omega}(b^t) - \hat{x}\|^2 = \|x^{t-1} - \hat{x} - \eta P_{\Omega}(v^t)\|^2$$  \hspace{1cm} (4.30)

$$= \|x^{t-1} - \hat{x}\|^2 + \eta^2 \|P_{\Omega}(v^t)\|^2 - 2\eta \langle x^{t-1} - \hat{x}, v^t \rangle,$$  \hspace{1cm} (4.31)

where the last equality applies Lemma 28. The first term will be preserved for mathematical induction. The third term is easy to manipulate thanks to the unbiasedness of $v^t$. For the second term, we will use Lemma 30 to bound it. Put them together for (4.29), we have

$$\mathbb{E}_{i,|x^{t-1}} \|x^t - \hat{x}\|^2 \leq \nu(1 - \eta \alpha) \|x^{t-1} - \hat{x}\|^2 - 2\nu \eta(1 - 2\eta L) [F(x^{t-1}) - F(\hat{x})]$$

$$+ 4\nu\eta^2 L [F(\hat{x}) - F(\hat{x})] + 4\nu\eta^2 T(2L\omega + T),$$  \hspace{1cm} (4.32)

where we apply Lemma 30, Assumption (A1) and the fact $\|x^t\|^2 \leq \omega$.

Summing over the inequalities over $t = 1, 2, \cdots, m$, we have

$$\mathbb{E}_{x^s,\hat{x}} \|x^m - \hat{x}\|^2 \leq \left[\nu(1 - \eta \alpha) - 1\right] m \mathbb{E}_{x^s,\hat{x}} \||\bar{x}^s - \hat{x}\|^2 + (2/\alpha + 4\nu^2 L m) [F(\bar{x}) - F(\hat{x})]$$

$$- 2\nu \eta(1 - 2\eta L) m \mathbb{E}_{x^s,\hat{x}} [F(\bar{x}^s) - F(\hat{x})] + \kappa,$$  \hspace{1cm} (4.33)

where we recall that $j^s$ is the randomly chosen index used to determine $\bar{x}^s$ (see Algorithm 1).

Now we discuss two cases according to the sign of $\nu(1 - \eta \alpha) - 1$. If it is non-positive, we immediately obtain

$$\nu \eta(1 - 2\eta L) m \mathbb{E}_{x^s,\hat{x}} [F(\bar{x}^s) - F(\hat{x})] \leq \left(1/\alpha + 2\nu^2 L m\right) [F(\bar{x}) - F(\hat{x})] + \kappa/2.$$  \hspace{1cm} (4.34)
For the sake of convergence, we require
\[ 1 - 2\eta L > 0, \quad \frac{2\nu \eta^2 L m}{\nu \eta (1 - 2\eta L) m} < 1. \tag{4.35} \]

For the case of \( \nu (1 - \eta \alpha) - 1 > 0 \), we obtain
\[
(2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1) m \mathbb{E}_{x, j \sim \hat{x}} [F(\tilde{x}^s) - F(\hat{x})] \leq (1 + 2\nu \eta^2 \alpha L m) [F(\tilde{x}) - F(\hat{x})] + \alpha \kappa/2. \tag{4.36}
\]

Likewise, we require
\[
2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1 > 0, \quad \frac{2\nu \eta^2 \alpha L m}{2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1} < 1. \tag{4.37}
\]

Combining these two cases completes the proof. See the full proof in Appendix D.

5 Experiments on Sparse Recovery

To understand the practical behavior of our algorithm as well as justify the theoretical analysis, we perform numerical experiments for sparse recovery.

Data Generation. We follow the setting in [TG07], i.e., the data dimension \( d \) is fixed as 256 and we generate an \( n \times d \) Gaussian random sensing matrix \( A \) whose entries are independently and identically distributed with zero mean and variance \( 1/n \). Then 1000 \( K \)-sparse signals are independently generated, where the support of each signal is uniformly distributed over \( \{1, 2, \ldots, d\} \). That is, we run our algorithm and baselines for 1000 trials. The measurements \( y \) for each signal \( x \) is obtained by \( y = Ax \).

Baselines. We mainly compare with two closely related algorithms in the literature: IHT [BD09] and Projected Gradient Descent (PGD) [JTK14]. Both of them compute the full gradient of the least-squares loss followed by hard thresholding. Yet, PGD is more general, in the sense that it allows the projection sparsity \( k \) to be larger than the true sparsity \( K \) (\( k = K \) for IHT) and a flexible step size \( \eta \) (\( \eta = 1 \) for IHT). Hence, PGD can be viewed as a batch counterpart of our method HT-SVRG.

Evaluation Metric. We say a signal \( x \) is successfully recovered by \( \hat{x} \) if
\[
\frac{\| \hat{x} - x \|_2}{\| x \|_2} < 10^{-3}. \tag{5.1}
\]

In this way, we can compute the percentage of success over the 1000 trials for each algorithm.

Parameter Setting. If not specified, we set the update frequency \( m = 3n \), \( k = 9K \), \( S = 10000 \) for HT-SVRG. We use the heuristic step size \( \eta = 2/\text{svds}(AA^\top) \) for HT-SVRG and PGD, where \( \text{svds}(AA^\top) \) returns the largest singular value of the matrix \( AA^\top \). Note that the RSS parameter \( L \) is smaller than it. Since for each stage, HT-SVRG computes the full gradient for \( (2m/n + 1) \) times, we run the IHT and PGD for \( (2m/n + 1)S \) iterations for fair comparison, i.e., they have the same number of full gradient evaluation.

Our first simulation aims at offering a big picture on the recovery performance. Hence, we vary the number of measurements \( n \) from 1 to 256, roughly with a step size 8. We also study the true sparsity parameter \( K \), which ranges from 1 to 26, roughly with step size 2. The results are illustrated in Figure 1, where a brighter block means a higher percentage of success and the brightest ones indicate exact sparse recovery. It is apparent that PGD and HT-SVRG require fewer measurements for an accurate recovery than IHT, possibly due to the flexibility in projection sparsity and step size. We also observe that as an online
algorithm, HT-SVRG performs comparably to PGD. This suggests that HT-SVRG is an appealing solution to large-scale sparse learning problems since its memory cost is independent of sample size. In Figure 2, we exemplify some of the results of Figure 1 as a collection of curves, so that one can examine the quantity of the percentage of success.

Figure 1: Percentage of exact signal recovery under various sparsity and number of measurements. Values range from 0 to 100, where the brighter color means higher percentage of success.

![Figure 1](image1.png)

Figure 2: Percentage of Success against the number of measurements and sparsity.

Based on the results in Figure 1, we have an estimation on the minimum requirement of the number of measurements which ensures accurate (or exact) recovery. Now we are to investigate how many measurements are needed to guarantee a success percentage of 95% and 99%. To this end, for each sparsity parameter $K$, we find the number of measurements $n_0$ from Figure 1 where 90 percent of success is achieved. Then we carefully enlarge $n_0$ with step size 1 and run the algorithms. The empirical results are recorded in Figure 3, where the circle markers represent the empirical results with different colors indicating different algorithm, e.g., red circle for empirical observation of HT-SVRG. Then we fit these empirical results by linear regression, which is plotted as solid lines, e.g., the green line is a fitted model for IHT. We find that $n$ is nearly linear with $K$. Especially, the curve of HT-SVRG is almost on top of that of PGD, which again verifies HT-SVRG is an attractive alternative to the batch method.

6 Conclusion

In this paper, we have provided a tight bound on the deviation resulted by hard thresholding operation, which underlies a vast volume of developed algorithms for sparsity constrained problems. Our derived bound is universal over all choices of parameters and we have proved that it cannot be improved without further message carried out. We discuss the implications of our result to the community of compressed sensing and
Figure 3: **Minimum measurements to achieve 95% and 99% percentage of success.** Red equation indicates the linear regression of HT-SVRG.

machine learning, and have demonstrated that the theoretical results of a number of popular algorithms in the literature can be advanced. We also devised a novel algorithm which tackles the problem of sparse learning in online setting. We have elaborated that equipped with the tight bound, our algorithm is guaranteed to produce global optimal solution for prevalent statistical models.
A Technical Lemmas

We present some useful lemmas that will be invoked by subsequent analysis.

**Lemma 28.** Let \( \Omega \subseteq \{1, 2, \cdots, d\} \) be a support set. For all \( x, x' \in \mathbb{R}^d \), it follows that

\[
\langle x, P_\Omega (x') \rangle = \langle P_\Omega (x), x' \rangle = \langle P_\Omega (x), P_\Omega (x') \rangle.
\]  

(A.1)

In particular, if \( \text{supp} (x) \subseteq \Omega \), we have

\[
\langle x, P_\Omega (x') \rangle = \langle x, x' \rangle.
\]  

(A.2)

**Proof.** The results can be established by simple algebra. \( \square \)

**Lemma 29** ([NNW14], Corollary 8). Assume that \( f(x) \) is a convex function and satisfies the RSS condition with parameter \( L_r \). Then for any support set \( \Omega \) with \( |\Omega| \leq r \), and all \( x, x' \in \mathbb{R}^d \) such that \( \text{supp} (x) \cup \text{supp} (x') \subseteq \Omega \), we have

\[
\|P_\Omega (\nabla f(x') - \nabla f(x))\|_2^2 \leq 2L_r \|f(x') - f(x) - \langle \nabla f(x), x' - x \rangle\|_2. \tag{A.3}
\]

**Lemma 30.** Let \( \Omega \) be a support set such that \( \text{supp} (x^{t-1}) \cup \text{supp} (x) \cup \text{supp} (\bar{x}) \subseteq \Omega \). Put \( r = |\Omega| \). Assume that for all \( 1 \leq i \leq n \), the function \( f_i(x) \) has the RSS property with parameter \( L_r \). For a fixed stage \( 1 \leq s \leq S \) and \( 1 \leq t \leq m \), denote \( v^t = \nabla f_i(x^{t-1}) - \nabla f_i(\bar{x}) + \bar{\mu} \). Then, conditioned on \( x^{t-1} \), we have the following expectation on the energy of \( v^t \) restricted on \( \Omega \):

\[
\mathbb{E}_{x^{t-1}} \|P_{\Omega} (v^t)\|_2^2 \leq 4L_r \|F(x^{t-1}) - F(\bar{x})\| + 4L_r \|F(\bar{x}) - F(\bar{x})\| - 4L_r \langle \nabla F(\bar{x}), x^{t-1} + \bar{x} \rangle + 4 \|P_{\Omega} (\nabla F(\bar{x}))\|_2^2. \tag{A.4}
\]

**Proof.** We have

\[
\|P_{\Omega} (v^t)\|_2^2 = \|P_{\Omega} (\nabla f_i(x^{t-1}) - \nabla f_i(\bar{x}) + \bar{\mu})\|_2^2
\]

\[
\leq 2 \|P_{\Omega} (\nabla f_i(x^{t-1}) - \nabla f_i(\bar{x}))\|_2^2 + 2 \|P_{\Omega} (\nabla f_i(\bar{x}) - \nabla f_i(\bar{x}) - \bar{\mu})\|_2^2 \tag{A.5}
\]

\[
= 2 \|P_{\Omega} (\nabla f_i(x^{t-1}) - \nabla f_i(\bar{x}))\|_2^2 + 2 \|P_{\Omega} (\nabla f_i(\bar{x}) - \nabla f_i(\bar{x}))\|_2^2
\]

\[
+ 2 \|P_{\Omega} (\bar{\mu})\|_2^2 - 4 \langle P_{\Omega} (\nabla f_i(\bar{x}) - \nabla f_i(\bar{x}), \bar{\mu}) \rangle \tag{A.6}
\]

\[
\overset{\xi_1}{\leq} 2 \|P_{\Omega} (\nabla f_i(x^{t-1}) - \nabla f_i(\bar{x}))\|_2^2 + 2 \|P_{\Omega} (\nabla f_i(\bar{x}) - \nabla f_i(\bar{x}))\|_2^2
\]

\[
+ 2 \|P_{\Omega} (\bar{\mu})\|_2^2 - 4 \langle \nabla f_i(\bar{x}) - \nabla f_i(\bar{x}), \bar{\mu} \rangle \tag{A.7}
\]

\[
\overset{\xi_2}{\leq} 4L_r \left[ f_i(x^{t-1}) - f_i(\bar{x}) - \langle \nabla f_i(\bar{x}), x^{t-1} - \bar{x} \rangle \right]
\]

\[
+ 4L_r \left[ f_i(\bar{x}) - f_i(\bar{x}) - \langle \nabla f_i(\bar{x}), \bar{x} - \bar{x} \rangle \right]
\]

\[
+ 2 \|P_{\Omega} (\bar{\mu})\|_2^2 - 4 \langle \nabla f_i(\bar{x}) - \nabla f_i(\bar{x}), \bar{\mu} \rangle, \tag{A.8}
\]

where \( \xi_1 \) is due to Lemma 28, \( \xi_2 \) applies Lemma 29 and the fact that \( |\Omega| = r \).
Taking the expectation over \( i_t \) conditioned on \( x^{t-1} \), we obtain the following bound by algebra:

\[
\mathbb{E}_{i_t|x^{t-1}} \left\| \mathcal{P}_i \left( v^{i} \right) \right\|_2^2 \leq 4L_r \left[ F(x^{t-1}) - F(\bar{x}) \right] + 4L_r \left[ F(\bar{x}) - F(x) \right] \\
- 4L_r \left\langle \nabla F(\bar{x}), x^{t-1} + \bar{x} - 2\bar{x} \right\rangle + 2 \left\langle 2\mathcal{P}_i \left( \nabla F(\bar{x}) \right) - \mathcal{P}_i \left( \bar{\mu} \right), \mathcal{P}_i \left( \bar{\mu} \right) \right\rangle
\]

(A.10)

\[
= 4L_r \left[ F(x^{t-1}) - F(\bar{x}) \right] + 4L_r \left[ F(\bar{x}) - F(x) \right] \\
- 4L_r \left\langle \nabla F(\bar{x}), x^{t-1} + \bar{x} - 2\bar{x} \right\rangle + \left\| 2\mathcal{P}_i \left( \nabla F(\bar{x}) \right) - \mathcal{P}_i \left( \bar{\mu} \right) \right\|_F^2 \\
- \left\| \mathcal{P}_i \left( \nabla F(\bar{x}) \right) \right\|_F - \left\| \mathcal{P}_i \left( \bar{\mu} \right) \right\|_F^2
\]

(A.11)

\[
\leq 4L_r \left[ F(x^{t-1}) - F(\bar{x}) \right] + 4L_r \left[ F(\bar{x}) - F(x) \right] \\
- 4L_r \left\langle \nabla F(\bar{x}), x^{t-1} + \bar{x} - 2\bar{x} \right\rangle + 4 \left\| \mathcal{P}_i \left( \nabla F(\bar{x}) \right) \right\|_2^2
\]

(A.12)

\[
= 4L_r \left[ F(x^{t-1}) - F(\bar{x}) \right] + 4L_r \left[ F(\bar{x}) - F(x) \right] \\
- 4L_r \left\langle \nabla F(\bar{x}), x^{t-1} + \bar{x} \right\rangle + 4 \left\| \mathcal{P}_i \left( \nabla F(\bar{x}) \right) \right\|_2^2,
\]

(A.13)

where the last equality applies the fact that when restricted on the support set of \( \bar{x} \), \( \nabla F(\bar{x}) \) equals zero due to RSC.

\[\square\]

**Corollary 31.** Assume same conditions as in Lemma 30. If \( \nabla F(\bar{x}) = 0 \), then we have

\[
\mathbb{E}_{i_t|x^{t-1}} \left\| \mathcal{P}_i \left( v^{i} \right) \right\|_2^2 \leq 4L_r \left[ F(x^{t-1}) + F(\bar{x}) - 2F(\bar{x}) \right].
\]

(A.14)

**Lemma 32 ([YLZ13], Prop. 2).** Assume that the entries of \( A \) are i.i.d. \( \sigma \)-sub-Gaussian. Then with probability at least \( 1 - 4d^{-1} \),

\[
T \leq 4\sqrt{\frac{(3k + k) \log d}{n}} + \gamma \left\| \bar{x} \right\|_2.
\]

(A.15)

**Lemma 33 ([FR13], Theorem 9.2).** Let \( M \) be an \( n_1 \times n_2 \) sub-Gaussian matrix with parameter \( \sigma \). Then there exists a constant \( C_1 \), such that the restricted isometry constant of \( (\sigma \sqrt{n_1})^{-1} M \) satisfies \( \delta_r \leq \delta \) with probability at least \( 1 - 2 \exp(-\delta^2 n_1/(2C_1)) \) provided that \( n_1 \geq 2C_1 \delta^{-2} r \log(\text{env}_{n_2}/r) \).

**Lemma 34.** Assume \( \theta_1 \) and \( \theta_2 \) are two centered independent sub-Gaussian random variables with parameter \( \sigma_1 \) and \( \sigma_2 \) respectively. Then, the following holds:

1. \( \theta_1^2 \) is a sub-exponential random variable with parameter \( \sigma_1^2 \);
2. For any real value \( x \) and \( y \), \( x \theta_1 + y \theta_2 \) is a centered sub-Gaussian random variable with parameter \( \sqrt{x^2 \sigma_1^2 + y^2 \sigma_2^2} \);
3. \( \theta_1 \theta_2 \) is a sub-exponential random variable with parameter \( \sigma_1 \sigma_2 \).

**Proof.** We will mainly use the results stated in [Ver10] to establish the lemma. In particular, we apply the equivalent definitions for sub-Gaussian and sub-exponential interchangeably.

We know from (4.21) that for all scalars \( z \in \mathbb{R} \),

\[
\mathbb{P}(\theta_1^2 > z^2) \leq 2 \exp(-\frac{z^2}{2\sigma_1^2}),
\]

(A.16)

which is equivalent to

\[
\mathbb{P}(|\theta_1^2| > z) \leq 2 \exp(-\frac{z}{2\sigma_1^2}) \leq \exp(1 - \frac{z}{2\sigma_1^2}), \quad \forall z \geq 0.
\]

(A.17)
Hence, $\theta_1^2$ is a sub-exponential random variable with parameter $\sigma_1^2$ [Ver10, Section 5.2.4].

By [Ver10, Lemma 5.5], we known that there exists absolute constant $C$, such that
\[
\mathbb{E}[\exp(\beta x \theta_1)] \leq \exp(Cz^2 \sigma_1^2), \quad \mathbb{E}[\exp(\beta y \theta_2)] \leq \exp(Cz^2 y^2 \sigma_2^2), \quad \forall \, z \in \mathbb{R}.
\] (A.18)

This gives
\[
\mathbb{E}[\exp(z(x \theta_1 + y \theta_2))] \leq \exp(Cz^2 (x^2 \sigma_1^2 + y^2 \sigma_2^2)), \quad \forall \, z \in \mathbb{R}.
\] (A.19)

Hence the second claim.

Again by [Ver10, Lemma 5.5], we known that there exists absolute constant $C$, such that
\[
\mathbb{E}[|\theta_1|^p] \leq (C \sigma_1)^{p/p^2}, \quad \mathbb{E}[|\theta_2|^p] \leq (C \sigma_2)^{p/p^2}.
\] (A.20)

Since $\theta_1$ and $\theta_2$ are independent, we immediately have
\[
\mathbb{E}[|\theta_1 \theta_2|^p] \leq (C^2 \sigma_1 \sigma_2)^{p/p^2},
\] (A.21)

implying
\[
(\mathbb{E}[|\theta_1 \theta_2|^p])^{1/p} \leq (C^2 \sigma_1 \sigma_2)^{p/p}.
\] (A.22)

Hence, $\theta_1 \theta_2$ is a sub-exponential random variable with parameter $\sigma_1 \sigma_2$. \qed

\section*{B Proofs of Section 2}

\subsection*{B.1 Proof of Theorem 1}

Proof. For the trivial case that $b$ is a zero vector, we have
\[
\|x - a\|_2^2 = \|b - a\|_2^2.
\] (B.1)

In the following, we assume that $b$ is not a zero vector. Denote $w = H_k (b)$. Let $\Omega$ be the support set of $w$ and let $\Omega'$ be its complement. We immediately have $P_{\Omega'} (b) = w$.

Let $\Omega'$ be the support set of $a$. For the sake of simplicity, let us split the vector $b$ as follows:
\[
b_1 = P_{\Omega \setminus \Omega'} (b), \quad b_2 = P_{\Omega \cap \Omega'} (b), \quad b_3 = P_{\Omega \setminus \Omega'} (b), \quad b_4 = P_{\Omega \cap \Omega'} (b).
\] (B.2)

Likewise, we denote
\[
w_1 = P_{\Omega \setminus \Omega'} (w), \quad w_2 = P_{\Omega \cap \Omega'} (w), \quad w_3 = P_{\Omega \setminus \Omega'} (w) = 0, \quad w_4 = P_{\Omega \cap \Omega'} (w) = 0,
\] (B.4)
\[
a_1 = P_{\Omega \setminus \Omega'} (a) = 0, \quad a_2 = P_{\Omega \cap \Omega'} (a), \quad a_3 = P_{\Omega \setminus \Omega'} (a) = 0, \quad a_4 = P_{\Omega \cap \Omega'} (a).
\] (B.5)

Due to the hard thresholding, we have
\[
w_1 = b_1, \quad w_2 = b_2.
\] (B.6)
In this way, by simple algebra we have

\[ \|w - a\|^2_2 = \|b_1\|^2_2 + \|b_2 - a_2\|^2_2 + \|a_4\|^2_2, \quad (B.7) \]
\[ \|b - a\|^2_2 = \|b_1\|^2_2 + \|b_2 - a_2\|^2_2 + \|b_3\|^2_2 + \|b_4 - a_4\|^2_2. \quad (B.8) \]

Our goal is to estimate the maximum of \( \|w - a\|^2_2 / \|b - a\|^2_2 \). It is easy to show that when attaining the maximum value, \( \|b_3\|^2_2 \) must be zero since otherwise one may decrease this term to make the objective larger. Hence, maximizing \( \|w - a\|^2_2 / \|b - a\|^2_2 \) amounts to estimating the upper bound of the following over all choices of \( a \) and \( b \):

\[ t \overset{\text{def}}{=} \frac{\|b_1\|^2_2 + \|b_2 - a_2\|^2_2 + \|a_4\|^2_2}{\|b_1\|^2_2 + \|b_2 - a_2\|^2_2 + \|b_4 - a_4\|^2_2}. \quad (B.9) \]

Firstly, we consider the case of \( \|b_1\|^2_2 = 0 \), which means \( \Omega = \Omega' \) implying \( t = 1 \). In the following, we consider \( \|b_1\|^2_2 \neq 0 \). In particular, we consider \( t > 1 \) since we are interested in the maximum value of \( t \).

Arranging \((B.9)\) we obtain

\[ (t - 1) \|b_2 - a_2\|^2_2 + t \|b_4 - a_4\|^2_2 - \|a_4\|^2_2 + (t - 1) \|b_1\|^2_2 = 0. \quad (B.10) \]

Let us fix \( b \) and define the function

\[ G(a_2, a_4) = (t - 1) \|b_2 - a_2\|^2_2 + t \|b_4 - a_4\|^2_2 - \|a_4\|^2_2 + (t - 1) \|b_1\|^2_2. \quad (B.11) \]

Thus, \((B.10)\) indicates that \( G(a_2, a_4) \) can attain the objective value of zero. Note that \( G(a_2, a_4) \) is a quadratic function and its gradient and Hessian matrix can be computed as follows:

\[ \frac{\partial}{\partial a_2} G(a_2, a_4) = 2(t - 1)(a_2 - b_2), \quad (B.12) \]
\[ \frac{\partial}{\partial a_4} G(a_2, a_4) = 2t(a_4 - b_4) - 2a_4, \quad (B.13) \]
\[ \nabla^2 G(a_2, a_4) = 2(t - 1)I, \quad (B.14) \]

where \( I \) is the identity matrix. Since the Hessian matrix is positive definite, \( G(a_2, a_4) \) attains the global minimum at the stationary point, which is given by

\[ a_2^* = b_2, \quad a_4^* = \frac{t}{t - 1} b_4, \quad (B.15) \]

resulting in the minimum objective value

\[ G(a_2^*, a_4^*) = \frac{t}{1 - t} \|b_4\|^2_2 + (t - 1) \|b_1\|^2_2. \quad (B.16) \]

In order to guarantee the feasible set of \((B.10)\) is non-empty, we require that

\[ G(a_2^*, a_4^*) \leq 0, \quad (B.17) \]

implying

\[ \|b_1\|^2_2 t^2 - (2 \|b_1\|^2_2 + \|b_4\|^2_2) t + \|b_1\|^2_2 \leq 0. \quad (B.18) \]
Solving the above inequality with respect to \( t \), we obtain

\[
t \leq 1 + \frac{\|b_4\|_2^2 + \sqrt{\left(4 \|b_1\|_2^2 + \|b_4\|_2^2\right) \|b_1\|_2^2}}{2 \|b_1\|_2^2}.
\]  
(B.19)

To derive an upper bound that is uniform over the choice of \( b \), we recall that \( b_1 \) contains the largest absolute elements of \( b \) while \( b_4 \) has smaller values. In particular, the averaged value of \( b_4 \) is no greater than that of \( b_1 \) in magnitude, i.e.,

\[
\frac{\|b_4\|_2^2}{\|b_1\|_2^2} \leq \frac{\|b_1\|_2^2}{\|b_1\|_2^0}.
\]  
(B.20)

Note that \( \|b_1\|_0 = k - \|b_2\|_0 = k - (k^* - \|b_4\|_0) \). Hence, combining with the fact that \( 0 \leq \|b_4\|_0 \leq \min\{k^*, d - k\} \) and optimizing over \( \|b_4\|_0 \) gives

\[
\|b_4\|_2^2 \leq \frac{\min\{k^*, d - k\}}{k - k^* + \min\{k^*, d - k\}} \|b_1\|_2^2.
\]  
(B.21)

Plugging back to (B.19), we finally obtain a uniform upper bound for \( t \):

\[
t \leq 1 + \frac{\rho + \sqrt{(4 + \rho)\rho}}{2}, \quad \rho = \frac{\min\{k^*, d - k\}}{k - k^* + \min\{k^*, d - k\}}.
\]  
(B.22)
\[\square\]

## C Proofs of Section 3

### C.1 Proof of Theorem 7

**Proof.** We follow the proof procedure of [BD09] and only remark the difference of our proof and theirs, i.e., where Theorem 1 applies. In case of possible confusion due to notation, we follow the symbols in Blumensath and Davies. One may refer to that article for a complete proof.

The first difference occurs in [BD09, Eq. (22)], where they reached

\[
\text{(Old) } \left\| x^s - x^{[n+1]} \right\|_2 \leq 2 \left\| x^s_{B^{n+1}} - a_{B^{n+1}}^{[n+1]} \right\|_2,  \tag{C.1}
\]

while Theorem 1 gives

\[
\text{(New) } \left\| x^s - x^{[n+1]} \right\|_2 \leq \sqrt{\nu} \left\| x^s_{B^{n+1}} - a_{B^{n+1}}^{[n+1]} \right\|_2.  \tag{C.2}
\]

Combining this new inequality and Eq. (23) therein, we obtain

\[
\left\| x^s - x^{[n+1]} \right\|_2 \leq \sqrt{\nu} \left\| (I - \Phi_{B^{n+1}}^\top \Phi_{B^{n+1}}) r_{B^{n+1}}^{[n]} \right\|_2 + \sqrt{\nu} \left\| (\Phi_{B^{n+1}}^\top \Phi_{B^{n+1} \setminus B^{n+1}}) r_{B^{n+1} \setminus B^{n+1}}^{[n]} \right\|_2.  \tag{C.3}
\]

By noting the fact that \( |B^n \cup B^{n+1}| \leq 2s + s^* \) where \( s^* \) denotes the sparsity of the global optima and following their reasoning of Eq. (24) and (25), we have a new bound for Eq. (26):

\[
\text{(New) } \left\| r^{[n+1]} \right\|_2 \leq \sqrt{2\nu \delta_{2s+s^*}} \left\| r^{[n]} \right\|_2 + \sqrt{(1 + \delta_{s+s^*})\nu} \|e\|_2.  \tag{C.4}
\]

Now our result follows by setting the coefficient of \( \|r^{[n]}\|_2 \) to 0.5. Note that specifying \( \nu = 4 \) gives the result of [BD09].  \[\square\]
C.2 Proof of Theorem 8

Proof. We follow the proof technique of [FR13, pp. 164, Theorem 6.27] which gives the best known RIP condition for the CoSaMP algorithm to date. Since most of the reasoning is similar, we only point out the difference of our proof and theirs, i.e., where Theorem 1 applies. In case of confusion by notation, we follow the symbols used in [FR13] for the time being. The reader may refer to that book for a complete proof.

The first difference is in Eq. (6.49). Note that to derive this inequality, Foucart and Rauhut invoked the conventional bound (2.5), which gives

\[ \| x_s - x^{n+1} \|_2 \leq \| (x_S - u^{n+1})_{U^{n+1}} \|_2^2 + 4 \| (x_S - u^{n+1})_{U^{n+1}} \|_2^2, \]  

(C.5)

while utilizing Theorem 1 gives

\[ \| x_s - x^{n+1} \|_2 \leq \| (x_S - u^{n+1})_{U^{n+1}} \|_2^2 + \nu \| (x_S - u^{n+1})_{U^{n+1}} \|_2^2. \]  

(C.6)

Combining this new inequality with Eq. (6.50) and Eq. (6.51) therein, we obtain

\[ \| x_s - x^{n+1} \|_2 \leq \sqrt{2} \delta_{3s+s^*} \sqrt{1 + (\nu - 1) \delta_{3s+s^*}^2} \| x^n - x_s \|_2 \]  

\[ + \sqrt{2} \delta_{3s+s^*} \sqrt{1 + (\nu - 1) \delta_{3s+s^*}^2} \| (A^* e')_{(S \cup S^c) \Delta \bar{T}^{n+1}} \|_2 \]  

\[ + \frac{2}{1 - \delta_{3s+s^*}} \| (A^* e')_{U^{n+1}} \|_2, \]  

(C.9)

where \( s^* \) denotes the sparsity of the optima. Our new bound follows by setting the coefficient of \( \| x^n - x_S \|_2 \) to 0.5 and solving the resultant equation. Note that setting \( \nu = 4 \) gives the old bound of Foucart and Rauhut.

C.3 Proof of Theorem 9

Proof. The first difference is in [BRB13, pp. 831, Eq. (19)]:

\[ \| b_s - \hat{x} \|_2 \leq 2 \| \hat{x}| \bar{T} - b \|_2 + \| \hat{x}| \bar{T}^c \|_2, \]  

(Old)

\[ \| b_s - \hat{x} \|_2 \leq \sqrt{\nu} \| \hat{x}| \bar{T} - b \|_2 + \| \hat{x}| \bar{T}^c \|_2. \]  

(New)

(C.10)

(C.11)

In this way, Lemma 3 in [BRB13] can be improved to:

\[ \| b_s - \hat{x} \|_2 \leq \frac{\gamma_{3s+s^*}(\hat{x}, \hat{x})}{\beta_{3s+s^*}(\hat{x}, \hat{x})} \left( 1 + \frac{\sqrt{\nu} \gamma_{3s+s^*}(b, \hat{x})}{\beta_{3s+s^*}(b, \hat{x})} \right) \| \hat{x} - \hat{x} \|_2 \]

\[ + \left( 1 + \frac{\sqrt{\nu} \gamma_{3s+s^*}(b, \hat{x})}{\beta_{3s+s^*}(b, \hat{x})} \right) \frac{\| \nabla f(\hat{x})| \bar{T} \|_2 \| \nabla f(\hat{x})| \bar{T}^c \|_2 + \sqrt{\nu} \| \nabla f(\hat{x})| \bar{T} \|_2}{\beta_{3s+s^*}(\hat{x}, \hat{x})} + \frac{\sqrt{\nu} \| \nabla f(\hat{x})| \bar{T} \|_2}{\beta_{3s+s^*}(b, \hat{x})}. \]  

(C.12)

Then, following their proof of Theorem 1 on page 832, we have

\[ \| \hat{x}^{(i)} - \hat{x} \|_2 \leq \left( \mu_{3s+s^*} - 1 \right) \left( 1 + \frac{\sqrt{\nu}}{\bar{\varepsilon}} \mu_{3s+s^*} - \frac{\sqrt{\nu}}{\bar{\varepsilon}} \right) \frac{\| \hat{x}^{(i-1)} - \hat{x} \|_2}{\varepsilon} \]  

+ \left( \frac{2}{\varepsilon} + \frac{\sqrt{\nu}}{\varepsilon} \mu_{3s+s^*} \right) \| \nabla f(\hat{x})| \bar{T} \|_2. \]  

(C.13)

Setting the coefficient of \( \| \hat{x}^{(i-1)} - \hat{x} \|_2 \) to 0.5 completes the proof.
D Proofs of Section 4

D.1 Proof of Theorem 12

Proof. Fix a stage $s$, let us denote

$$v^t = \nabla f_{i_t}(x^{t-1}) - \nabla f_{i_t} (\tilde{x}) + \tilde{\mu},$$

so that

$$b^t = x^{t-1} - \eta v^t.$$  \hfill (D.1)

Specifying $\Omega = \text{supp} (x^{t-1}) \cup \text{supp} (x^t) \cup \text{supp} (\tilde{x}) \cup \text{supp} (\tilde{\alpha})$, it follows that

$$r^t = \mathcal{H}_k (b^t) = \mathcal{H}_k (P_\Omega (b^t)).$$ \hfill (D.3)

Thus, the Euclidean distance of $x^t$ and $\tilde{x}$ can be bounded as follows:

$$\|x^t - \tilde{x}\|_2^2 \leq \|r^t - \tilde{x}\|_2^2 = \|\mathcal{H}_k (P_\Omega (b^t)) - \tilde{x}\|_2^2 \leq \nu \|P_\Omega (b^t) - \tilde{x}\|_2^2,$$ \hfill (D.4)

where the first inequality holds because $x^t = \Pi_{\omega}(r^t)$ and $\|\tilde{x}\|_2 \leq \omega$. We also have

$$\|P_\Omega (b^t) - \tilde{x}\|_2^2 = \|x^{t-1} - \tilde{x} - \eta P_\Omega (v^t)\|_2^2$$

$$= \|x^{t-1} - \tilde{x}\|_2^2 + \eta^2 \|P_\Omega (v^t)\|_2^2 - 2\eta \langle x^{t-1} - \tilde{x}, v^t \rangle,$$ \hfill (D.6)

where the second equality uses the fact that $\langle x^{t-1} - \tilde{x}, P_\Omega (v^t) \rangle = \langle x^{t-1} - \tilde{x}, v^t \rangle$ (Lemma 28). The first term will be preserved for mathematical induction. The third term is easy to manipulate thanks to the unbiasedness of $v^t$. For the second term, we use Lemma 30 to bound it. Put them together, conditioned on $x^{t-1}$ and taking the expectation over $i_t$ for (D.4), we have

$$\mathbb{E}_{i_t|x^{t-1}} \|x^t - \tilde{x}\|_2^2$$

$$\leq \nu \|x^{t-1} - \tilde{x}\|_2^2 + 4\nu \eta^2 L [F(x^{t-1}) - F(\tilde{x}) + F(\tilde{x}) - F(\tilde{x})] - 2\nu \eta \langle x^{t-1} - \tilde{x}, \nabla F(x^{t-1}) \rangle$$

$$- 4\nu \eta^2 L \langle \nabla F(\tilde{x}), x^{t-1} + \tilde{x} \rangle + 4\nu \eta^2 \|P_\Omega (\nabla F(\tilde{x}))\|_2^2$$ \hfill (D.7)

$$\leq \nu (1 - \eta_\alpha) \|x^{t-1} - \tilde{x}\|_2^2 - 2\nu \eta (1 - 2\eta L) [F(x^{t-1}) - F(\tilde{x})] + 4\nu \eta^2 L [F(\tilde{x}) - F(\tilde{x})]$$

$$+ 4\nu \eta^2 L \|P_\Omega (\nabla F(\tilde{x}))\|_2 \cdot \|x^{t-1} + \tilde{x}\|_2 + 4\nu \eta^2 \|P_\Omega (\nabla F(\tilde{x}))\|_2^2$$ \hfill (D.8)

$$\leq \nu (1 - \eta_\alpha) \|x^{t-1} - \tilde{x}\|_2^2 - 2\nu \eta (1 - 2\eta L) [F(x^{t-1}) - F(\tilde{x})]$$

$$+ 4\nu \eta^2 L [F(\tilde{x}) - F(\tilde{x})] + 4\nu \eta^2 T (2L\omega + T).$$ \hfill (D.9)

where $\xi_1$ applies Lemma 30 and $\xi_2$ applies Assumption (A1).

Now summing over the inequalities over $t = 1, 2, \cdots, m$, conditioned on $\tilde{x}$ and taking the expectation
on \( \mathcal{I} = \{i_1, i_2, \ldots, i_m\} \), we have

\[
\mathbb{E}_{I^* | \tilde{x}} \| w^m - \tilde{x}\|^2 \leq [\nu(1 - \eta \alpha) - 1] \mathbb{E}_{I^* | \tilde{x}} \sum_{t=1}^{m} \|x^{t-1} - \tilde{x}\|^2 + \|w^0 - \tilde{x}\|^2 + 4\nu \eta^2 T(2L \omega + T)m
- 2\nu \eta(1 - 2\eta L)\mathbb{E}_{I^* | \tilde{x}} \sum_{t=1}^{m} [F(x^{t-1}) - F(\tilde{x})] + 4\nu \eta^2 Lm \left[F(\tilde{x}) - F(\bar{x})\right] \tag{D.10}
= [\nu(1 - \eta \alpha) - 1] m \mathbb{E}_{I^* | j^*| \tilde{x}} \|\tilde{x}^s - \tilde{x}\|^2 + \|\tilde{x} - \tilde{x}\|^2 + 4\nu \eta^2 T(2L \omega + T)m
- 2\nu \eta(1 - 2\eta L)m \mathbb{E}_{I^* | j^*| \tilde{x}} [F(\tilde{x}^s) - F(\tilde{x})] + 4\nu \eta^2 Lm \left[F(\tilde{x}) - F(\bar{x})\right] \tag{D.11}
\leq [\nu(1 - \eta \alpha) - 1] m \mathbb{E}_{I^* | j^*| \tilde{x}} \|\tilde{x}^s - \tilde{x}\|^2 + \left(\frac{2}{\alpha} + 4\nu \eta^2 Lm\right) \left[F(\tilde{x}) - F(\bar{x})\right]
- 2\nu \eta(1 - 2\eta L)m \mathbb{E}_{I^* | j^*| \tilde{x}} [F(\tilde{x}^s) - F(\tilde{x})] + \kappa, \tag{D.12}
\]

where we recall that \( j^s \) is the randomly chosen index used to determine \( \tilde{x}^s \) (see Algorithm 1). The last inequality holds due to the RSC condition and \( \|x^t\|_2 \leq \omega \).

Based on (D.10), we discuss two cases to examine the convergence of the algorithm.

\begin{itemize}
    \item [(I)] \( \nu(1 - \eta \alpha) \leq 1 \). This immediately results in
        \[
        \mathbb{E}_{I^* | \tilde{x}} \| w^m - \tilde{x}\|^2 \leq \left(\frac{2}{\alpha} + 4\nu \eta^2 Lm\right) \left[F(\tilde{x}) - F(\bar{x})\right] - 2\nu \eta(1 - 2\eta L)m \mathbb{E}_{I^* | j^*| \tilde{x}} [F(\tilde{x}^s) - F(\tilde{x})] + \kappa, \tag{D.13}
        \]
        which implies
        \[
        \nu \eta(1 - 2\eta L)m \mathbb{E}_{I^* | j^*| \tilde{x}} [F(\tilde{x}^s) - F(\tilde{x})] \leq \left(\frac{1}{\alpha} + 2\nu \eta^2 Lm\right) \left[F(\tilde{x}) - F(\bar{x})\right] + \frac{\kappa}{2}. \tag{D.14}
        \]
        Assume that
        \[
        1 - 2\eta L > 0, \tag{D.15}
        \]
        we obtain
        \[
        \mathbb{E}_{I^* | j^*| \tilde{x}} [F(\tilde{x}^s) - F(\tilde{x})] \leq \left(\frac{1}{\nu \eta \alpha (1 - 2\eta L)m} + \frac{2\eta L}{1 - 2\eta L}\right) \left[F(\tilde{x}) - F(\bar{x})\right] + \frac{\kappa}{2\nu \eta \alpha (1 - 2\eta L)m}. \tag{D.16}
        \]
    \item [(II)] \( \nu(1 - \eta \alpha) > 1 \). The convergence coefficient here is
        \[
        \beta = \frac{1}{\nu \eta \alpha (1 - 2\eta L)m} + \frac{2\eta L}{1 - 2\eta L}. \tag{D.19}
        \]
\end{itemize}
Thus, we have
\[
\mathbb{E} [F(\bar{x}^s) - F(\bar{x})] \leq \beta^s \left[ F(\bar{x}^0) - F(\bar{x}) \right] + \frac{\kappa(1 - \beta^s)}{2\nu \eta \alpha (1 - 2\eta L)(1 - \beta)m}, \tag{D.20}
\]
where the expectation is taken over \( \{I^1, j^1, I^2, j^2, \ldots, I^s, j^s\} \).

\( \textbf{(II)} \) \( \nu(1 - \eta \alpha) > 1 \). In this case, \( (D.10) \) implies
\[
\mathbb{E}_{I^s,j^s} \left\| w^m - \hat{\bar{x}} \right\|_2^2 \leq \left( \frac{2}{\alpha} + 4\nu \eta^2 Lm \right) [F(\bar{x}) - F(\hat{\bar{x}})] + \kappa \\
+ \left( \frac{2}{\alpha} \left[ \nu(1 - \eta \alpha) - 1 \right] m - 2\nu \eta (1 - 2\eta L)m \right) \mathbb{E}_{I^s,j^s} [F(\bar{x}^s) - F(\hat{\bar{x}})]. \tag{D.21}
\]
Arranging the terms gives
\[
(2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1) m \mathbb{E}_{I^s,j^s} [F(\bar{x}^s) - F(\hat{\bar{x}})] \leq (1 + 2\nu \eta^2 \alpha Lm) [F(\bar{x}) - F(\hat{\bar{x}})] + \frac{\alpha \kappa}{2}. \tag{D.22}
\]
To ensure the convergence, we require
\[
2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1 > 0, \tag{D.23}
\]
\[
2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1 > 2\nu \eta^2 \alpha L. \tag{D.24}
\]
That is,
\[
4\nu L \eta^2 - 2\nu \alpha \eta + \nu - 1 < 0. \tag{D.25}
\]
We need to guarantee the feasible set of the above inequality is non-empty for the positive variable \( \eta \). Thus, we require
\[
4\nu^2 \alpha^2 - 4 \times 4\nu \alpha L(\nu - 1) > 0, \tag{D.26}
\]
which is equivalent to
\[
\nu < \frac{4L}{4L - \alpha}. \tag{D.27}
\]
Combining it with \( \nu(1 - \eta \alpha) > 1 \) gives
\[
\frac{1}{1 - \eta \alpha} < \nu < \frac{4L}{4L - \alpha}. \tag{D.28}
\]
To ensure the above feasible set is non-empty, we impose
\[
\frac{1}{1 - \eta \alpha} < \frac{4L}{4L - \alpha}, \tag{D.29}
\]
so that
\[
\eta < \frac{1}{4L}, \quad \frac{1}{1 - \eta \alpha} < \nu < \frac{4L}{4L - \alpha}. \tag{D.30}
\]
The convergence coefficient for this case is
\[
\beta = \frac{1}{(2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1) m} + \frac{2\nu \eta^2 \alpha L}{2(2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1)(1 - \beta^*)}.
\] (D.31)

Thus,
\[
\mathbb{E}[F(\bar{x}^*) - F(\bar{x})] \leq \beta^* \left[ F(\bar{x}^0) - F(\bar{x}) \right] + \frac{\alpha \kappa (1 - \beta^*)}{2(2\nu \eta \alpha - 2\nu \eta^2 \alpha L - \nu + 1)(1 - \beta^*)m}.
\] (D.32)

Combining (D.18) and (D.30), the minimum requirement for \(\eta\) and \(\nu\) is:
\[
\eta < \frac{1}{4L}, \quad \nu < \frac{4L}{4L - \alpha}.
\] (D.33)

\[\square\]

**D.2 Proof of Lemma 22**

**Proof.** Let \(l(x) = \frac{1}{2n} \sum_{i=1}^{n} \|y_i - a_i \cdot x\|^2\) and let \(\Omega\) be the support set which maximizes \(T\). By algebra, we know that
\[
T = \|P_\Omega (\nabla F(\bar{x}))\|_2 \leq \|P_\Omega (\nabla l(\bar{x}))\|_2 + \gamma \|P_\Omega (\bar{x})\|_2 \leq \sqrt{3k + \bar{k}} \|\nabla l(\bar{x})\|_\infty + \gamma \|\bar{x}\|_2.
\] (D.34)

Also it follows that
\[
\frac{\partial l(x)}{\partial x_j}|_{x = \bar{x}} = \frac{1}{n} \sum_{i=1}^{n} (a_i \cdot \bar{x} - y_i)a_{ij} = \frac{1}{n} \sum_{i=1}^{n} [a_i \cdot (\bar{x} - x^*) - \varepsilon_i] a_{ij},
\] (D.35)

where the last equality applies the fact that \(y\) is generated by \(y = Ax^* + \varepsilon\).

For simplicity, we write \(x := \bar{x} - x^*\). In this way, we have
\[
\frac{\partial l(x)}{\partial x_j}|_{x = \bar{x}} = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{l \neq j} a_{il}x_l - \varepsilon_i \right) a_{ij} + \frac{1}{n} \sum_{i=1}^{n} a_{ij}^2 x_j.
\] (D.36)

In the sequel, we bound the two terms above. We begin with the second summand. Note that as \(a_{ij}\) is centered \(\sigma\)-sub-Gaussian, \(a_{ij}^2\) is \(\sigma^2\)-sub-exponential (see Lemma 34). Also note that \(a_{ij}^2\)'s are mutually independent. Hence, Prop. 5.16 of [Ver10] applies, i.e.,
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} a_{ij}^2 \right| \geq z \right) \leq 2 \exp \left( -C_0 \min \left\{ \frac{n z^2}{\sigma^4}, \frac{n z}{\sigma^2} \right\} \right), \quad \forall \ z \geq 0,
\] (D.37)

for some absolute constant \(C_0\). The union bound then indicates
\[
P \left( \max_{1 \leq j \leq d} \left| \frac{1}{n} \sum_{i=1}^{n} a_{ij}^2 \right| \geq z \right) \leq 2d \exp \left( -C_0 \min \left\{ \frac{n z^2}{\sigma^4}, \frac{n z}{\sigma^2} \right\} \right), \quad \forall \ z \geq 0.
\] (D.38)

For the first summand in (D.36), we find that the random variable \(\sum_{l \neq j} a_{il}x_l - \varepsilon_i\) is \(\sqrt{\sum_{l \neq j} x_l^2 \sigma^2 + \sigma_0^2}\)-sub-Gaussian due to Lemma 34, which immediately follows that \(\left( \sum_{l \neq j} a_{il}x_l - \varepsilon_i \right) a_{ij}\) is a sub-exponential
random variable with parameter $\sigma \sqrt{\sum_{l \neq j} x_l^2 \sigma^2 + \sigma_0^2}$ due to the independence. Therefore, we have by [Ver10, Prop. 5.16]

$$
P\left( \left| \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{l \neq j} a_{il} x_l - \varepsilon_i \right) a_{ij} \right| \geq z \right) \leq 2 \exp \left( -C_1 \min \left\{ \frac{n z^2}{\sigma_j^2}, \frac{nz}{\bar{\sigma}_j} \right\} \right), \quad \forall \ z \geq 0, \quad (D.39)
$$

where $C_1$ is some absolute constant and

$$
\bar{\sigma}_j = \sigma \sqrt{\sum_{l \neq j} x_l^2 \sigma^2 + \sigma_0^2} \leq \sigma \|x\|_2 + \sigma_0 \sigma. \quad (D.40)
$$

Again, the union bound gives

$$
P\left( \max_{1 \leq j \leq d} \left| \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{l \neq j} a_{il} x_l - \varepsilon_i \right) a_{ij} \right| \geq z \right) \leq 2d \exp \left( -C_1 \min \left\{ \frac{n z^2}{\sigma^2}, \frac{nz}{\bar{\sigma}} \right\} \right), \quad \forall \ z \geq 0, \quad (D.41)
$$

where

$$
\bar{\sigma} = \max_{1 \leq j \leq d} \bar{\sigma}_j. \quad (D.42)
$$

Now let us pick

$$
z = \sigma^2 \sqrt{\frac{2 \log d}{C_0 n}} \quad (D.43)
$$

in (D.38). Assuming $n \geq 2C_0^{-1} \log d$ gives

$$
\max_{1 \leq j \leq d} \left| \frac{1}{n} \sum_{i=1}^{n} a_{ij}^2 x_j \right| \leq \sigma^2 \sqrt{\frac{2 \log d}{C_0 n}} \max |x_j| \leq \sigma^2 \|x\|_2 \sqrt{\frac{2 \log d}{C_0 n}} \quad (D.44)
$$

with probability at least $1 - d^{-1}$.

Let us further pick

$$
z = \bar{\sigma} \sqrt{\frac{2 \log d}{C_1 n}} \quad (D.45)
$$

in (D.41). Assuming $n \geq 2C_1^{-1} \log d$ gives

$$
\max_{1 \leq j \leq d} \left| \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{l \neq j} a_{il} x_l - \varepsilon_i \right) a_{ij} \right| \leq \max_{1 \leq j \leq d} \bar{\sigma}_j \sqrt{\frac{2 \log d}{C_1 n}} \leq \sigma^2 \|x\|_2 \sqrt{\frac{2 \log d}{C_1 n}} + \sigma_0 \sigma \sqrt{\frac{2 \log d}{C_1 n}} \quad (D.46)
$$

with probability at least $1 - d^{-1}$.

On the other hand, a simple algebra deduces

$$
\|\nabla l(\hat{x})\|_\infty \leq \max_{1 \leq j \leq d} \left| \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{l \neq j} a_{il} x_l - \varepsilon_i \right) a_{ij} \right| + \max_{1 \leq j \leq d} \left| \frac{1}{n} \sum_{i=1}^{n} a_{ij}^2 \right| \cdot |x_j|. \quad (D.47)
$$

29
Put together, and note that for any two events $A$ and $B$, $\mathbb{P}(AB) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$, we have with probability at least $1 - 2d^{-1}$,

$$\|\nabla l(\tilde{x})\|_\infty \leq \sigma^2 \|\tilde{x} - x^*\|_2 \sqrt{\frac{C_2 \log d}{n}} + \sigma_0 \sqrt{\frac{C_2 \log d}{n}}, \quad (D.48)$$

as soon as $n \geq C_2 \log d$.

Finally, we conclude that when $n \geq C_2 \log d$ for some absolute constant $C_2$, with probability at least $1 - 2d^{-1}$,

$$T \leq \sigma^2 \|\tilde{x} - x^*\|_2 \sqrt{\frac{C_2 (3k + \hat{k}) \log d}{n}} + \sigma_0 \sqrt{\frac{C_2 (3k + \hat{k}) \log d}{n}} + \gamma \|\tilde{x}\|_2. \quad (D.49)$$

D.3 Proof of Proposition 23

Proof. Note that for the sparse linear regression problem, we have the Hessian matrix

$$\nabla^2 F(x) = \frac{1}{n} A^\top A + \gamma I. \quad (D.50)$$

Due to Lemma 33 and the definition of RIC (3.2), the condition number is actually given by

$$c = \frac{\sigma^2 (1 + \delta_{3k + \hat{k}}) + \gamma}{\sigma^2 (1 - \delta_{3k + \hat{k}}) + \gamma} \leq 1 + \frac{\delta_{3k + \hat{k}}}{1 - \delta_{3k + \hat{k}}}. \quad (D.51)$$

Specifying $\delta_{3k + \hat{k}} = 0.5$ in Lemma 33 gives the result. \hfill \Box

D.4 Proof of Proposition 27

Proof. This is an immediate result by applying Lemma 32 and pick $\gamma = O\left(\sigma \sqrt{(3k + \hat{k}) \log d/n}\right)$.

D.5 Proof of Proposition 26

Proof. The Hessian matrix of (4.26) can be computed as follows:

$$\nabla^2 F(x) = \frac{1}{n} \sum_{i=1}^{n} (1 + \exp(2y_i a_i \cdot x))^{-1} a_i^\top a_i + \gamma I. \quad (D.52)$$

It remains to bound the coefficients. Recall that we perform an orthogonal projection for the iterates so that $\|x^t\|_2 \leq \omega$, where $\omega$ is an upper bound on the global optima of (4.26). Hence, (4.26) is equivalent to

$$\min_x F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \exp(-2y_i a_i \cdot x)\right) + \gamma \|x\|_2^2, \quad \text{s.t. } \|x\|_0 \leq k, \|x\|_2 \leq \omega. \quad (D.53)$$

Since $a_{ij}$’s are i.i.d. $\sigma$-sub-Gaussian, standard results (see e.g. [Ver10, Prop. 5.10]) tell us that

$$\mathbb{P}(\|2y_i a_i \cdot x\| \geq z) \leq \exp \left(1 - \frac{C_0 z^2}{4\sigma^2 \|x\|_2^2}\right) \leq \exp \left(1 - \frac{C_0 z^2}{4\sigma^2 \omega^2}\right), \quad (D.54)$$

30
Specifying \( z = 1 \) gives

\[
\nabla^2 F(x) \succeq \frac{4(1 + e)}{n} A^\top A + \gamma I \tag{D.55}
\]

with probability at least \( 1 - \exp \left( 1 - C_0/(4\sigma^2\omega^2) \right) \). The fact that \( (1 + \exp(2y_i a_i x))^{-1} < 1 \) provides an upper bound

\[
\nabla^2 F(x) \preceq \frac{4}{n} A^\top A + \gamma I. \tag{D.56}
\]

Invoking Lemma 33 to bound the eigenvalues of \( A^\top A \) when restricted on sparse support (as we did in Appendix D.3). Put them together concludes the proof.

## E HT-SAGA

We demonstrate that the hard thresholding step can be integrated into SAGA [DBL14] as shown in Algorithm 2. Note that the only difference of Algorithm 2 and the one proposed in [DBL14] is that we perform hard thresholding rather than proximal operator. Hence, our algorithm guarantees \( k \)-sparse solution. We have the following theorem which justifies the algorithm.

**Algorithm 2 SAGA with Sparsity Constraint (HT-SAGA)**

**Require:** The value of \( x^t \) and of each \( \nabla f_i(\phi^t_i) \) at the end of iteration \( t \), the step size \( \eta \).

**Ensure:** The new iterate.

1. Pick \( j \in \{1, 2, \cdots, n\} \) uniformly at random.
2. Take \( \phi^{t+1}_j = x^t \) and store \( \nabla f_j(\phi^{t+1}_j) \) in the table. All other entries in the table remain unchanged.
3. Update the new iterate \( x^{t+1} \) as follows:

\[
b^{t+1} = x^t - \eta \left[ \nabla f_j(\phi^{t+1}_j) - \nabla f_j(\phi^t_j) + \frac{1}{n} \sum_{i=1}^n \nabla f_i(\phi^t_i) \right], \tag{E.1}
\]

\[
x^{t+1} = H_k(b^{t+1}). \tag{E.2}
\]

**Theorem 35.** Assume same conditions as in [DBL14]. Further assume the optima of (4.1) without the sparsity constraint happens to be \( k \)-sparse. Then, the sequence of the solutions produced by Algorithm 2 converges to the optima with geometric rate for some properly chosen projection sparsity \( k \).

**Proof.** Define the Lyapunov function \( Z \) as follows:

\[
Z^t \overset{\text{def}}{=} Z(x^t, \{\phi^t_i\}) = \frac{1}{n} \sum_{i=1}^n f_i(\phi^t_i) - F(\tilde{x}) - \frac{1}{n} \sum_{i=1}^n \langle \nabla f_i(\tilde{x}), \phi^t_i - \tilde{x} \rangle + c \| x^t - \tilde{x} \|_2^2. \tag{E.3}
\]

We examine \( Z^{t+1} \). We have

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i} f_i(\phi^{t+1}_i) \right] = \frac{1}{n} F(x^t) + \left( 1 - \frac{1}{n} \right) \frac{1}{n} \sum_{i} f_i(\phi^t_i), \tag{E.4}
\]

\[
\mathbb{E} \left[ -\frac{1}{n} \sum_{i} \langle \nabla f_i(\tilde{x}), \phi^{t+1}_i - \tilde{x} \rangle \right] = -\frac{1}{n} \langle \nabla F(\tilde{x}), x^t - \tilde{x} \rangle - \left( 1 - \frac{1}{n} \right) \frac{1}{n} \sum_{i} \langle \nabla f_i(\tilde{x}), \phi^t_i - \tilde{x} \rangle. \tag{E.5}
\]

31
Also,
\[ c \| x^{t+1} - \hat{x} \|^2 \leq c v \| b^{t+1} - \hat{x} \|^2 = c v \| b^{t+1} - \hat{x} + \eta \nabla F(\hat{x}) \|^2. \quad (E.6) \]

For the first term, we have
\[
e v \mathbb{E} \| b^{t+1} - \hat{x} + \eta \nabla F(\hat{x}) \|^2 \\
\leq c v (1 - \eta \alpha) \| x^t - \hat{x} \|^2 + c v \left( (1 + \mu) \eta^2 - \frac{\eta}{L} \right) \mathbb{E} \| \nabla f_j(x^t) - \nabla f_j(\hat{x}) \|^2 \\
- \frac{2c v \eta (L - \alpha)}{L} \left[ F(x^t) - F(\hat{x}) - \langle \nabla F(\hat{x}), x^t - \hat{x} \rangle \right] - c v \eta^2 \mu \| \nabla F(x^t) - \nabla F(\hat{x}) \|^2 \\
+ 2c v (1 + \mu^{-1}) \eta^2 L \left[ \frac{1}{n} \sum_i f_i(\phi_i^t) - F(\hat{x}) - \frac{1}{n} \sum_i \langle \nabla f_i(\hat{x}), \phi_i^t - \hat{x} \rangle \right]. \quad (E.7) \]

Therefore,
\[
\mathbb{E}[Z^{t+1}] - Z^t \leq - \frac{1}{\kappa} Z^t + \left( \frac{1}{n} - \frac{2c v \eta (L - \alpha)}{L} - 2c v \eta^2 \alpha \mu \right) \left[ F(x^t) - F(\hat{x}) - \langle \nabla F(\hat{x}), x^t - \hat{x} \rangle \right] \\
+ \left( \frac{1}{\kappa} + 2c v (1 + \mu^{-1}) \eta^2 L - \frac{1}{n} \right) \left[ \frac{1}{n} \sum_i f_i(\phi_i^t) - F(\hat{x}) - \frac{1}{n} \sum_i \langle \nabla f_i(\hat{x}), \phi_i^t - \hat{x} \rangle \right] \\
+ \left( \frac{c}{\kappa} - c v \eta \alpha \right) \| x^t - \hat{x} \|^2 + \left( (1 + \mu) \eta - \frac{1}{L} \right) c v \eta \mathbb{E} \| \nabla f_j(x^t) - \nabla f_j(\hat{x}) \|^2. \quad (E.8) \]

In order to guarantee the convergence, we choose proper values for \( \eta, c, \kappa, \mu \) and \( \nu \) such that the terms in round brackets are non-positive. That is, we require
\[
\frac{c}{\kappa} - c v \eta \alpha \leq 0, \quad (E.9) \\
(1 + \mu) \eta - \frac{1}{L} \leq 0, \quad (E.10) \\
\frac{1}{n} - \frac{2c v \eta (L - \alpha)}{L} - 2c v \eta^2 \alpha \mu \leq 0, \quad (E.11) \\
\frac{1}{\kappa} + 2c v (1 + \mu^{-1}) \eta^2 L - \frac{1}{n} \leq 0. \quad (E.12) 
\]

Pick
\[
\eta = \frac{1}{2(\alpha n + L)}, \quad (E.13) \\
\mu = \frac{2\alpha n + L}{L}, \quad (E.14) \\
\kappa = \frac{1}{\nu \eta \alpha}, \quad (E.15) 
\]
we fulfill the first two inequalities. Pick
\[
c = \frac{1}{2 \eta (1 - \eta \alpha) n}. \quad (E.16) 
\]

Then by the last two equalities, we require
\[
1 - \eta \alpha \leq \nu \leq \frac{(1 - \eta \alpha) L}{\eta \alpha (1 - \eta \alpha) L n + 1}. \quad (E.17) 
\]
On the other hand, by Theorem 1, we have

\[ \nu > 1. \]  

(E.18)

Thus, we require

\[ 1 < \nu \leq \frac{(1 - \eta \alpha)L}{\eta \alpha(1 - \eta \alpha)Ln + 1}. \]  

(E.19)

By algebra, the above inequalities has non-empty feasible set provided that

\[ (6\alpha^2 - 8\alpha^2 L)n^2 + (14\alpha L - \alpha - 16\alpha L^2)n + 8L^2(1 - L) < 0. \]  

(E.20)

Due to \( \alpha \leq L \), we know

\[ n \geq \frac{14L + \sqrt{224L^3 + 1}}{2\alpha(8L - 6)} \]  

(E.21)

suffices where we assume \( L > 3/4 \). Picking

\[ \nu = \frac{(1 - \eta \alpha)L}{\eta \alpha(1 - \eta \alpha)Ln + 1} \]  

(E.22)

completes the proof. \( \square \)
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