DV and WDVV

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We prove that the quasiclassical tau-function of the multi-support solutions to matrix models, proposed recently by Dijkgraaf and Vafa to be related to the Cachazo-Intriligator-Vafa superpotentials of the $\mathcal{N}=1$ supersymmetric Yang-Mills theories, satisfies the Witten-Dijkgraaf-Verlinde-Verlinde equations.

1 Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [1] in the most general form can be written [2] as system of algebraic relations

$$F_I F_J^{-1} F_K = F_K F_J^{-1} F_I, \quad \forall I, J, K$$

(1)

for the third derivatives

$$\|F\|_{JK} = \frac{\partial^3 F}{\partial T_I \partial T_J \partial T_K} \equiv F_{IJK}$$

(2)

of some function $F(T)$. Have been appeared first in the context of topological string theories [1], they were rediscovered later on in much larger class of physical theories where the exact answer for a multidimensional theory could be expressed through a single holomorphic function of several complex variables [2, 3, 4, 5, 6, 7, 8].

Recently, a new example of similar relations between the superpotentials of $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions and free energies of matrix models in the planar limit was proposed [9, 10]. It has been realized that superpotentials in some $\mathcal{N}=1$ four-dimensional Yang-Mills theories can be expressed through a single holomorphic function [9] that can be further identified with free energy of the multi-support solutions to matrix models in the planar limit [10]. A natural question which immediately arises in this context is whether these functions – the quasiclassical tau-functions, determined by multi-support solutions to matrix models, satisfy the WDVV equations? In the case of positive answer this is rather important, since multi-support solutions to the matrix models can play

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a role of “bridge” between topological string theories and Seiberg-Witten theories [11] which give rise to two different classes of solutions to the WDVV equations (see, e.g., [12] and [13] [14]).

This question was already addressed in [15], where it was shown that the multicut solution to one-matrix model satisfies the WDVV equations. However, this was verified only perturbatively and, what is even more important, for a particular non-canonical! (and rather strange) choice of variables.

In this paper, we demonstrate that the quasiclassical tau-function of the multi-support solution satisfies the WDVV equations as a function of canonical variables identified with the periods and residues of the generating meromorphic one-form $dS$ [16]. An exact proof of this statement consists of two steps. The first, most difficult step is to find the residue formula for the third derivatives (2) of the matrix model free energy. Then, using an associative algebra, we immediately prove that free energy of multi-support solution satisfies WDVV equations, upon the number of independent variables is fixed to be equal to the number of critical points in the residue formula.

In sect. 2 we define the free energy of the multi-support matrix model in terms of the quasiclassical tau-function [16] along the line of [17] [18] [19]. In sect. 3 we derive the residue formula for the third derivatives of the quasiclassical tau-function for the variables associated both with the periods $\{S\} = \{S_i\}$ and residues $\{t\} = \{t_i\}$ of the generating differential $dS$. In sect. 4 we prove that the free energy of the multisupport solution $F(T)$ solves the WDVV equations [11] as a function of the full set of variables $\{T\} = \{S, t\}$ whose total number should be fixed to be equal to the number of critical points in the residue formula for the third derivatives (2). In sect. 5 we verify this statement explicitly for the first nontrivial case where the total number of variables is equal to four. 1 Finally we present several concluding remarks and discuss possible generalizations.

We restrict ourselves by the $\mathcal{N} = 1$ supersymmetric theories without flavours originally considered in [9]. The results can be easily generalized. Note that the literature on the subject is already quite extensive [20] [21], and different interesting developments of the issues discussed in this paper can be immediately obtained.

2 Tau-function of multi-support matrix model

We first exactly define what we call below the multi-support free energy of the matrix model in the planar limit. We are mostly doing with one-matrix integrals 2 of the form

$$Z = \int d\Phi \ e^{\frac{i}{\hbar} \ln W(\Phi)} \quad (3)$$

where the potential $W(\Phi)$ is supposed to be a polynomial of a degree $(n + 1)$. The free energy in the planar limit of (3) can be defined as the first term in the expansion

$$F(t, t_0) = \lim (\log Z(t)) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t, t_0) \quad (4)$$

implying $N \to \infty, \hbar \to 0$ with $N\hbar = t_0$ being fixed. In what follows we are only interested in the first term of this expansion, $F_0(t, t_0)$. In fact, we deal with another quantity, $F(t, t_0, S)$, where $S_i = \hbar N_i, \sum S_i = t_0$ are extra variables – the filling numbers of (metastable) vacua. In order to get from this quantity $F_0(t, t_0)$, one needs to minimize the free energy with respect to the filling numbers, $\frac{\partial F}{\partial S_i} = 0$. However, one can still preserve $S_i$ as free parameters introducing the "chemical potentials".

The origin of the new variables $S_i$ becomes rather transparent after one says that instead of direct computation of (3) this problem is replaced by the saddle point approximation – finding the extremum of the functional $F_0[\rho(\lambda)] \propto \int W \rho - \int \rho(\lambda_1) \log |\lambda_1 - \lambda_2| \rho(\lambda_2) + \sum \Pi_0(f \rho - t_0)$ where $\Pi_0$ is just a

1 Let us point out that the WDVV equations [11] are nontrivial only for the functions of at least three independent variables. However, as we see below, the structure of residue formula for the matrix model free energy requires the minimal number of independent variables to be at least four! From this point of view, the origin of the “experimental observation” of [13] valid for a function of three variables still remains unclear to us.

2 The generalization to the two-matrix case [19] is rather straightforward and will be discussed in the last section.
Figure 1: Cuts in the $\lambda$- or “eigenvalue” plane for the planar limit of 1-matrix model. The eigenvalues are supposed to be located “on” the cuts. The distribution of eigenvalues is governed by the period integrals $S = \oint A \rho(\lambda) d\lambda$ along the corresponding cycles and the dependence of partition function on “distributions” $S_i$ is given by the quasiclassical tau-function $\frac{\partial F}{\partial S_i} = \oint B \rho(\lambda) d\lambda$.

Lagrange multiplier to fix the total normalization of the eigenvalue density. This latter condition means the saddle point equation is non-trivial only on the support of $\rho$. For one matrix model, this support can be presented as a set $\{D_i\}$ of cuts in complex eigenvalue plane, see fig. 1. Then, one should add to this functional the term $\sum \Pi_i \left( \oint_{D_i} \rho - S_i \right)$, which via Lagrange multipliers, controls the filling numbers at each cut, i.e. to consider

$$\mathcal{F}(t, t_0, S) \propto \int W \rho - \int \int \rho(\lambda_1) \log |\lambda_1 - \lambda_2| \rho(\lambda_2) + \Pi_0 \left( \int \rho - t_0 \right) + \sum ' \Pi_i \left( \oint_{D_i} \rho - S_i \right)$$

Then, the extra variables appear due to the extra information hidden in (5) compare to (3) – the structure of nontrivial eigenvalue supports. It is well-known that at “critical” densities $\frac{\partial F}{\partial \rho} = 0$, (5) is a (logarithm) of quasiclassical tau-function [16] (see, e.g., [18, 19]).

In principle, in order to compare $\mathcal{F}$ with the matrix model quantity $F_0$, one needs to put further restrictions to get rid of the metastable vacua. This would lead to shrinking part of the cuts into the double points (see discussions of these issues, say, in [18]). Here we would forget this issue and consider smooth curves (6) with only two marked points at infinities on two $\lambda$-sheets of the curve (6).

Note also that in (5) one can make two different natural choices for the set of new independent variables: the first choice corresponds to independent filling of all cuts, then $t_0 = \sum S_i$, while the second choice corresponds to choosing as independent $t_0$ and all $S_i$ except of corresponding to one of the cuts (that is why the corresponding sum in (5) is denoted as $\sum '$). These choices are related by the linear change of variables which does not influence the WDVV equations (see [2, 30]). The first choice is more “symmetric” while the second one corresponds to the canonical choice of variables in the sense of [16] or to the homology basis on smooth curve (6) with added marked points at two infinities. We use both of them below depending on convenience.

The complex curve of one-matrix model “comes from” the loop equations (see, for example, [22]) and can be written in the form

$$y^2 = W'(\lambda)^2 + f(\lambda) \equiv R(\lambda)$$

with the matrix model potential [3] parameterized as

$$W(\lambda) = \sum_{l \geq 0} t_{l+1} \lambda^{l+1}$$

or

$$W'(\lambda) = \sum_{l = 0}^{n} (l + 1) t_{l+1} \lambda^l$$

$$3$$
being the polynomial of \(n\)-th degree in our conventions. The coefficients of the function

\[
f(\lambda) = \sum_{k=0}^{n-1} f_k \lambda^k
\]

are related to the extra data (the filling numbers) of the multicut solution. The eigenvalue density \(\rho(\lambda)\) is the imaginary part of \(y(\lambda)\) and vanishes outside the cuts. Therefore, the eigenvalue distribution \(S\) can be fixed by the periods

\[
S_i = \oint_{A_i} dS
\]

taken around the eigenvalue supports to be identified (except for one of the supports) with the canonical \(A\)-cycles. Then

\[
\frac{\partial dS}{\partial S_i} = d\omega_i
\]

\[
\oint_{A_i} d\omega_j = \delta_{ij}
\]

when the derivatives are taken at fixed coefficients \(\{t_i\}\) of the potential \(\mathcal{S}\). One can show that the Lagrangian multipliers in \(\mathcal{S}\) are given by integrals of the same generating differential \(\mathcal{S}^{\text{gen}}\) over the dual contours (see fig. 1)

\[
\Pi_i = \oint_{B_i} dS
\]

To the set of parameters \(\mathcal{S}^{\text{gen}}\) one should also add \(^3\) the total number of eigenvalues \(N\hbar = t_0\)

\[
\text{res}_\infty (dS) = \frac{f_{n-1}}{2(n+1)t_{n+1}} \equiv t_0
\]

and the parameters of the potential \(\mathcal{S}, \mathcal{S}\), which can be equivalently written as

\[
t_k = \frac{1}{k} \text{res}_\infty (\lambda^{-k} dS)
\]

\[
k = 1, \ldots, n
\]

while the leading term coefficient \(t_{n+1}\) is supposed to be fixed (we will discuss this issue in detail below). Then

\[
d\Omega_0 = \frac{\partial dS}{\partial t_0} = (n+1)t_{n+1} \frac{\lambda^{n-1} d\lambda}{y} + \frac{1}{2} \sum_{k=0}^{n-2} \frac{\partial f_k \lambda^k d\lambda}{y}
\]

and the dependence of \(\{f_k\}\) with \(k = 0, 1, \ldots, n-2\) on \(t_0\) is fixed by the condition

\[
\oint_{A_i} \left( (n+1)t_{n+1} \frac{\lambda^{n-1} d\lambda}{y} + \sum_{k=0}^{n-2} \frac{\partial f_k \lambda^k d\lambda}{y} \right) = 0
\]

which for \(i = 1, \ldots, n-1\) gives exactly \(n-1\) relations on the derivatives of \(f_0, f_1, \ldots, f_{n-2}\) w.r.t. \(t_0\).

The bipole differential \(\mathcal{S}^{\text{gen}}\) can be also presented as

\[
d\Omega_0 = d \log \left( \frac{E(P, \infty)}{E(P, \infty_-)} \right)
\]

\(^3\) By the \(\infty\)-point in what follows we call for short the point \(\infty_+\) or \(\lambda = \infty\) on the “upper” sheet of hyperelliptic Riemann surface \(\mathcal{S}\) corresponding to the positive sign of the square root, i.e. to \(y = +\sqrt{W'(\lambda)^2 + f(\lambda)}\).
where $E(P, P')$ is the Prime form. Differential (18) obviously obey the properties

$$\text{res}_\infty d\Omega_0 = -\text{res}_\infty d\Omega_0 = 1$$

$$\oint_{A_i} d\Omega_0 = 0, \quad i = 1, \ldots, n - 1 \quad (19)$$

For the derivatives w.r.t. parameters of the potential (15), one gets

$$d\Omega_k = \frac{\partial dS}{\partial t_k} = \frac{W'(\lambda)k\lambda^{k-1}d\lambda}{y} + \frac{1}{2} \sum_{j=0}^{n-2} \frac{\partial f_j(\lambda)}{\partial t_k} \frac{\lambda^k d\lambda}{y} \quad (20)$$

obeying

$$\oint_{A_i} d\Omega_k = \oint_{A_i} \frac{W'(\lambda)k\lambda^{k-1}d\lambda}{y} + \frac{1}{2} \sum_{j=0}^{n-2} \frac{\partial f_j(\lambda)}{\partial t_k} \oint_{A_i} \frac{\lambda^k d\lambda}{y} = 0 \quad (21)$$

and this is again a system of linear equations on $\frac{\partial f_j(\lambda)}{\partial t_k}$. To complete the setup one should also add to (13) the following formulas $^4$:

$$\Pi_0 = \int_{-\infty}^{\infty} dS \quad (22)$$

(we again remind that, instead of $t_0$, the parameter $S_n = t_0 - \sum_{i=1}^{n-1} S_i$ can be used equivalently) and

$$v_k = \text{res}_\infty \left( \lambda^k dS \right), \quad k > 0 \quad (23)$$

On genus $g = n - 1$ smooth Riemann surface (6), there are $2g = 2n - 2$ independent noncontractable contours which can be split into the so-called $A \equiv \{A_i\}$ and $B \equiv \{B_i\}$, $i = 1, \ldots, g$, cycles with the intersection form $A_i \circ B_j = \delta_{ij}$. The canonical holomorphic differentials (12) are normalized to the $A$-cycles, and their integrals along the $B$-cycles give the period matrix,

$$\oint_{B_i} d\omega_j = T_{ij} \quad (24)$$

To check integrability of (13) and (22) one needs to verify the symmetricity of the second derivatives. For the part related with the derivatives only w.r.t. the variables (10), this is just a symmetricity of the period matrix of (6) and follows from the Riemann bilinear relations for the canonical holomorphic differentials (12)

$$0 = \int_{\Sigma} d\omega_i \wedge d\omega_j = \sum_k \left( \oint_{A_k} d\omega_i \oint_{B_k} d\omega_j - \oint_{A_k} d\omega_j \oint_{B_k} d\omega_i \right) = T_{ij} - T_{ji} \quad (25)$$

Analogously

$$0 = \int_{\Sigma} d\omega_i \wedge d\Omega_0 = \sum_k \left( \oint_{A_k} d\omega_i \oint_{B_k} d\Omega_0 - \oint_{A_k} d\Omega_0 \oint_{B_k} d\omega_i \right) +$$

$$+ \text{res}_\infty (d\omega_i) \int_{-\infty}^{\infty} d\Omega_0 - \text{res}_\infty (d\Omega_0) \int_{-\infty}^{\infty} d\omega_i = \oint_{B_i} d\Omega_0 - \int_{-\infty}^{\infty} d\omega_i \quad (26)$$

Formula (25) means that

$$\frac{\partial \Pi_i}{\partial S_j} = T_{ji} = T_{ij} = \frac{\partial \Pi_j}{\partial S_i} \quad (27)$$

$^4$ Naively understood the integral in (22) is divergent and should be supplemented by some proper regularization. In what follows we ignore this subtlety since it does not influence the residue formulas for the third derivatives, those one really needs for the WDVV equations (1). The simplest way to avoid these complications is to think of the pair of marked points $\infty$ and $\infty_-$ as of degenerate handle; then the residue (14) comes from degeneration of the extra $A$-period, while the integral (22) from degeneration of the extra $B$-period.
Figure 2: Cut Riemann surface with boundary $\partial \Sigma$. The integral over the boundary can be divided into several pieces (see formula (32)). In the process of computation we use the fact that the boundary values of Abelian integrals $v_j^\pm$ on two copies of the cut differ by period integral of the corresponding differential $d\omega_j$ over the dual cycle.

while from (26) one gets

$$\frac{\partial \Pi_j}{\partial t_0} = \frac{\partial \Pi_0}{\partial S_j}$$

(28)

This allows one to introduce the function $F(T) = F(S, t_0, t)$ such that

$$\frac{\partial F}{\partial S_j} = \Pi_j, \quad \frac{\partial F}{\partial t_0} = \Pi_0, \quad \frac{\partial F}{\partial t_k} = v_k$$

(29)

The integrability of the last relation can be checked similar to (27), (28) with the help of Riemann bilinear relations involving the Abelian integrals $\Omega_k = \int^P d\Omega_k$, for example (cf. e.g. with [23], where similar relations were used for the quasiclassical tau-function of the Seiberg-Witten theory):

$$\text{res}_\infty (\Omega_k d\omega_i) = \oint_{\partial \Sigma} \Omega_k d\omega_i = \sum_l \left( \int_{A_l} \Omega_k^+ d\omega_i - \int_{A_l} \Omega_k^- d\omega_i \right) - \sum_l \left( \int_{B_l} \Omega_k^+ d\omega_i - \int_{B_l} \Omega_k^- d\omega_i \right) =$$

$$= \sum_l \left( \oint_{B_l} d\Omega_k \oint_{A_l} d\omega_i - \oint_{A_l} d\Omega_k \oint_{B_l} d\omega_i \right) = \oint_{B_l} d\Omega_k$$

(31)

where $\partial \Sigma$ is the cut Riemann surface (6) (see fig. 2), and in the last equality we used (21).

3 Residue formula

3.1 Holomorphic differentials

Let us now derive the formulas for the third derivatives of $F$, following the way proposed by I.Krichever [16, 25]. We first note that the derivatives of the elements of period matrix (in this section, for simplicity, we set the coefficients of potential (8) to be fixed) can be expressed through the integral over the “boundary” $\partial \Sigma$ of cut Riemann surface $\Sigma$ (see fig. 2)

$$\frac{\partial T_{ij}}{\partial S_k} = \partial_k T_{ij} = \int_{B_j} \partial_k d\omega_i = \int_{\partial \Sigma} \omega_j \partial_k d\omega_i$$

(31)
where $\omega_j = \int \omega_j d\omega_j$ are the Abelian integrals, whose values on two copies of cycles on the cut Riemann surface (see also fig. 2) are denoted below as $\omega_j^\pm$. Indeed, the computation of the r.h.s. of (31) gives

$$\int_{\partial \Sigma} \omega_j \partial_k d\omega_i = \sum_l \left( \int_{B_l} \omega_j^+ \partial_k d\omega_i - \int_{B_l} \omega_j^- \partial_k d\omega_i \right) - \sum_l \left( \int_{A_l} \omega_j^+ \partial_k d\omega_i - \int_{A_l} \omega_j^- \partial_k d\omega_i \right) =$$

$$= \sum_l \int_{B_l} \left( \int_{A_l} d\omega_j \right) \partial_k d\omega_i - \sum_l \int_{A_l} \left( \int_{B_l} d\omega_j \right) \partial_k d\omega_i =$$

$$= \sum_l \left( \int_{A_l} d\omega_j \right) \int_{B_l} \partial_k d\omega_i - \sum_l \left( \int_{B_l} d\omega_j \right) \int_{A_l} \partial_k d\omega_i = \partial_k T_{ij}$$

One can now rewrite (31) as

$$\partial_k T_{ij} = \int_{\partial \Sigma} \omega_j \partial_k d\omega_i = - \int_{\partial \Sigma} \partial_k \omega_j d\omega_i = \sum \text{res}_{\lambda=0} \left( \partial_k \omega_j d\omega_i \right)$$

where the sum is taken over all residues of the integrand, i.e. over all residues of $\partial_k \omega_j$ since $d\omega_j$ are holomorphic. In order to investigate these singularities and understand the last equality in (33), we discuss first how to take derivatives $\partial_k$ w.r.t. moduli, or introduce the corresponding connection.

To this end, we introduce a covariantly constant function – the hyperelliptic co-ordinate $\lambda$, i.e. such a connection that $\partial_k \lambda = 0$. Roughly speaking, the role of covariantly constant function can be played by one of two co-ordinates – in the simplest possible description of complex curve by a single equation on two complex variables. Then, using this equation, one may express the other co-ordinate as a function of $\lambda$ and moduli. Any Abelian integral $\omega_j$ can be then expressed in terms of $\lambda$, and in the vicinity of critical points $\{\lambda_\alpha\}$ where $d\lambda = 0$ (for a general (non-singular) curve this is always true) we get an expansion

$$\omega_j(\lambda) = \omega_{j\alpha} + c_{ja} \sqrt{\lambda - \lambda_\alpha} + \ldots$$

whose derivative

$$\partial_k \omega_j \equiv \partial_k \omega_j |_{\lambda=\text{const}} = - \frac{c_{ja}}{2\sqrt{\lambda - \lambda_\alpha}} \partial_k \lambda_\alpha + \text{regular}$$

(35) gives first order poles at $\lambda = \lambda_\alpha$ up to regular terms which do not contribute to (33). The exact coefficient in (35) can be computed for $\lambda$ related with the generating differential $dS = y d\lambda$. Then, using

$$y(\lambda) \rightarrow \lambda_\alpha : \Gamma_\alpha \sqrt{\lambda - \lambda_\alpha} + \ldots$$

where $\Gamma_\alpha = \sqrt{\prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta)}$ or

$$\frac{\partial}{\partial t_k} y(\lambda) = - \frac{\Gamma_\alpha}{2\sqrt{\lambda - \lambda_\alpha}} \frac{\partial \lambda_\alpha}{\partial t_k} + \text{regular}$$

(37) together with

$$dy = \frac{\Gamma_\alpha}{2\sqrt{\lambda - \lambda_\alpha}} d\lambda + \text{regular}$$

(38)

and

$$d\omega_j = \frac{c_{ja}}{2\sqrt{\lambda - \lambda_\alpha}} d\lambda + \ldots$$

(39)

and, following from (11) and (12)

$$d\omega_k = \partial_k dS = - \frac{\Gamma_\alpha \partial_k \lambda_\alpha}{2\sqrt{\lambda - \lambda_\alpha}} d\lambda + \text{regular}$$

(40)

one finally gets for (33)

$$\sum \text{res} (\partial_k \omega_j d\omega_i) = \sum \text{res} \left( \frac{c_{ja} \partial_k \lambda_\alpha}{2\sqrt{\lambda - \lambda_\alpha}} d\omega_i \right) = \sum \text{res} \left( \frac{d\omega_j}{d\lambda} d\omega_i \partial_k \lambda_\alpha \right) = \sum \text{res} \left( \frac{d\omega_i d\omega_j d\omega_k}{d\lambda dy} \right)$$

(41)
In hyperelliptic situation, the derivation presented above is equivalent to using the Fay formula \[24\]

\[
\frac{\partial T_{ij}}{\partial \lambda_\alpha} = \tilde{\omega}_i(\lambda_\alpha)\tilde{\omega}_j(\lambda_\alpha)
\] (42)

where \(\tilde{\omega}_i(\lambda_\alpha) = \left.\frac{d\omega_i(\lambda)}{d\sqrt{\lambda-\lambda_\alpha}}\right|_{\lambda=\lambda_\alpha}\) is the “value” of canonical differential at a critical point.

### 3.2 Meromorphic differentials

Almost in the same way the residue formula can be derived for the meromorphic differentials \[20\]. One gets

\[
\frac{\partial F}{\partial t_k} = \text{res}_\infty \left(\lambda^k dS\right), \quad k > 0
\] (43)

therefore

\[
\frac{\partial^2 F}{\partial t_k \partial t_n} = \text{res}_\infty \left(\lambda^k d\Omega_n\right) = \text{res}_\infty ((\Omega_k) d\Omega_n)
\] (44)

where \((\Omega_k)\) is the singular part of the integral of 1-form \(d\Omega_k\). Further

\[
\frac{\partial}{\partial t_m} \text{res}_\infty \left(\lambda^k d\Omega_n\right) = \text{res}_\infty \left(\lambda^k \frac{\partial d\Omega_n}{\partial t_m}\right) = -\text{res}_\infty \left(\frac{d\Omega_n}{\partial t_m}\right)
\] (45)

The last expression can be rewritten as

\[
-\text{res}_\infty \left(\frac{d\Omega_n}{\partial t_m}\right) = \oint_{\Omega} \frac{\partial d\Omega_n}{\partial t_m} + \sum_{\lambda} \text{res}_\alpha \left(\frac{d\Omega_n}{\partial t_m}\right) = \sum_{\lambda} \text{res}_\alpha \left(\frac{d\Omega_n}{\partial t_m}\right)
\] (46)

since \(\oint_{\Omega} \frac{\partial d\Omega_n}{\partial t_m} = 0\) due to \(\oint_{A_i} \frac{d\Omega_n}{\partial t_m} = 0\), (cf. with \[32\]):

\[
\oint_{\Omega} \frac{\partial d\Omega_i}{\partial t_k} = \sum_i \left(\oint_{B_j} \Omega_j^+ \frac{\partial}{\partial t_k} d\Omega_i - \oint_{B_j} \Omega_j^- \frac{\partial}{\partial t_k} d\Omega_i\right) - \sum_i \left(\oint_{A_i} \Omega_i^+ \frac{\partial}{\partial t_k} d\Omega_i - \oint_{A_i} \Omega_i^- \frac{\partial}{\partial t_k} d\Omega_i\right) =
\]

\[
= \sum_i \oint_{B_j} \left(\oint_{A_i} d\Omega_i\right) \frac{\partial}{\partial t_k} d\Omega_i - \sum_i \oint_{A_i} \left(\oint_{B_j} d\Omega_i\right) \frac{\partial}{\partial t_k} d\Omega_i =
\]

\[
= \sum_i \left(\oint_{A_i} d\Omega_i\right) \oint_{B_j} \frac{\partial}{\partial t_k} d\Omega_i - \sum_i \left(\oint_{B_j} d\Omega_i\right) \oint_{A_i} \frac{\partial}{\partial t_k} d\Omega_i \oint_{A_i} d\Omega_i = 0
\] (47)

Now, as in the holomorphic case one takes

\[
\Omega_n(\lambda) \rightarrow \Omega_n + \gamma_{n\alpha} \sqrt{\lambda-\lambda_\alpha} + \ldots
\] (48)

and, therefore

\[
\frac{\partial}{\partial t_k} \Omega_j \equiv \left.\frac{\partial}{\partial t_k} \Omega_j\right|_{\lambda=\text{const}} = -\frac{\gamma_{j\alpha}}{2\sqrt{\lambda-\lambda_\alpha}} \frac{\partial \lambda_\alpha}{\partial t_k} + \text{regular}
\] (49)

Then, using \[36\], \[37\] and \[38\] together with

\[
d\Omega_j = \frac{\gamma_{j\alpha}}{2\sqrt{\lambda-\lambda_\alpha}} d\lambda + \ldots
\] (50)

and the relation, following from \[20\], \[37\]

\[
d\Omega_k = \frac{\partial}{\partial t_k} dS = -\frac{\Gamma_\alpha d\lambda}{2\sqrt{\lambda-\lambda_\alpha}} \frac{\partial \lambda_\alpha}{\partial t_k} + \text{regular}
\] (51)
one gets for (52)

$$\frac{\partial^3 F}{\partial\Omega_k \partial\Omega_n \partial\Omega_m} - \sum_{\alpha} \text{res}_{\alpha} \left( d\Omega_k \frac{\partial\Omega_n}{\partial\Omega_m} \right) = - \sum_{\alpha} \text{res}_{\alpha} \left( d\Omega_k \frac{\gamma_{\alpha}}{2\sqrt{\lambda - \lambda_{\alpha}}} \frac{\partial\lambda_{\alpha}}{\partial\Omega_m} \right) =$$

$$= - \sum_{\alpha} \text{res}_{\alpha} \left( d\Omega_k \frac{d\Omega_n}{d\lambda} \frac{\partial\lambda_{\alpha}}{\partial\Omega_m} \right) = \sum_{\alpha} \text{res}_{\alpha} \left( \frac{d\Omega_k d\Omega_n d\Omega_m}{d\lambda dy} \right)$$

In a similar way, one proves the residue formula for the mixed derivatives, so that we finally conclude

$$\frac{\partial^3 F}{\partial T_I \partial T_J \partial T_K} = \sum_{\alpha} \text{res}_{\alpha} \left( \frac{dH_I dH_J dH_K}{d\lambda dy} \right) = \sum_{\alpha} \text{res}_{\alpha} \left( \frac{\phi_I \phi_J \phi_K}{d\lambda dy} \right) =$$

$$= \sum_{\alpha} \Gamma^2_{\alpha} \phi_I(\lambda) \phi_J(\lambda) \phi_K(\lambda) = \sum_{\lambda} \frac{\tilde{H}_I(\lambda) \tilde{H}_J(\lambda) \tilde{H}_K(\lambda)}{\prod_{\beta \neq \alpha} (\lambda_{\alpha} - \lambda_{\beta})^2}$$

for the whole set \( \{T_I\} = \{t_k, t_0, S_i\} \) and \( \{dH_I\} = \{d\Omega_k, d\Omega_0, d\omega_i\} \). In formula (53) we have introduced the meromorphic functions

$$\phi_I(\lambda) = \frac{dH_I}{d\lambda} = \frac{\tilde{H}_I(\lambda)}{R(\lambda)}$$

for any (meromorphic or holomorphic) differential \( dH_I = \tilde{H}_I(\lambda) \frac{d\lambda}{dy} \) and \( R(\lambda) = W'(\lambda)^2 + f(\lambda) \). The derivation of the residue formula for the set of parameters including \( t_0 \) corresponding to the third-kind Abelian differential (16) can be performed in a similar way.

### 4 Proof of WDVV

Having the residue formula (53), the proof of the WDVV equations (1) is reduced to solving the system of linear equations (7, 26, 14), which requires only fulfilling the two conditions:

- The “matching” condition

$$\#(I) = \#(\alpha)$$

and

- nondegeneracy of the matrix built from (53):

$$\det_{\alpha} \|\phi_I(\lambda_{\alpha})\| \neq 0$$

Under these conditions, the structure constants \( C_{IJ}^K \) of the associative algebra

$$(C_I)_L^K (C_J)_M^L = (C_J)_L^K (C_I)_M^L$$

$$(C_I)_J^K \equiv C_{IJ}^K$$

responsible for the WDVV equations can be found from the system of linear equations

$$\phi_I(\lambda_{\alpha}) \phi_J(\lambda_{\alpha}) = \sum_K C_{IJ}^K \phi_K(\lambda_{\alpha}), \quad \forall \lambda_{\alpha}$$

with the solution

$$C_{IJ}^K = \sum_{\alpha} \phi_I(\lambda_{\alpha}) \phi_J(\lambda_{\alpha}) (\phi_K(\lambda_{\alpha}))^{-1}$$

To make it as general, as in [2, 14, 16], one may consider an associative isomorphic algebra (again \( \forall \lambda_{\alpha} \))

$$\phi_I(\lambda_{\alpha}) \phi_J(\lambda_{\alpha}) = \sum_K C_{IJ}^K(\xi) \phi_K(\lambda_{\alpha}) \cdot \xi(\lambda_{\alpha})$$
which instead of (59) leads to

\[ C^K_{IJ}(\xi) = \sum_{\alpha} \frac{\phi_I(\lambda_\alpha)\phi_J(\lambda_\alpha)}{\xi(\lambda_\alpha)} (\phi_K(\lambda_\alpha))^{-1} \tag{61} \]

The rest of the proof is consistency of this algebra with relation

\[ F_{IJK} = \sum_{L} C^L_{IJ}(\xi)\eta_{KL}(\xi) \tag{62} \]

with

\[ \eta_{KL}(\xi) = \sum_{A} \xi_{A} F_{KLA} \tag{63} \]

expressing structure constants in terms of the third derivatives and, thus, leading to (1). It is easy to see that (62) is satisfied if \( F_{KLA} \) are given by residue formula (53).

Indeed, requiring only matching \#(\alpha) = #(I), one gets

\[ \sum_{K} C^K_{IJ}(\xi)\eta_{KL}(\xi) = \sum_{K,\alpha,\beta} \frac{\phi_I(\lambda_\alpha)\phi_J(\lambda_\alpha)}{\xi(\lambda_\alpha)} \cdot (\phi_K(\lambda_\alpha))^{-1} \cdot \phi_K(\lambda_\beta)\phi_L(\lambda_\beta)\xi(\lambda_\beta)\Gamma_\beta \tag{64} \]

and finally

\[ \sum_{K} C^K_{IJ}(\xi)\eta_{KL}(\xi) = \sum_{\alpha} \frac{\phi_I(\lambda_\alpha)\phi_J(\lambda_\alpha)}{\xi(\lambda_\alpha)} \phi_L(\lambda_\alpha)\xi(\lambda_\alpha)\Gamma_\alpha = \sum_{\alpha} \Gamma_\alpha \phi_I(\lambda_\alpha)\phi_J(\lambda_\alpha)\phi_L(\lambda_\alpha) = F_{IJK} \tag{65} \]

Hence, for the proof of (1) one has to adjust the number of parameters \( \{T_I\} \) according to (55). The number of critical points \#(\alpha) = 2n since \( d\lambda = 0 \) for \( y^2 = R(\lambda) = 0 \). Thus, one have to take a codimension one subspace in the space of all parameters \( \{T_I\} \), a natural choice will be to fix the eldest coefficient of \( S \). Then the total number of parameters \#(I), including the periods \( S \), residue \( t_0 \) and the rest of the coefficients of the potential will be \( g + 1 + n = (n - 1) + 1 + n = 2n \), i.e. exactly equal to \#(\alpha). In sect. 5 we present an explicit check of the WDVV equations for this choice, using the expansion of free energy computed in [9] [31].

Note that equations (60) are basically equivalent to the algebra of forms (or differentials) considered in [3] [4]. In this particular case one may take the basis of 1-differentials \( d\omega_i, d\Omega_k \) with multiplication given by usual (not wedge!) multiplication modulo \( dS = yd\lambda \). Then, one can either directly check that the algebra with this multiplication is associative (similar to how it was done in [4] [6]), or, using hyperelliptic parameterization, remove the factor \( \frac{d\lambda}{y} \) in order to reduce the algebra to the ring of polynomials with multiplication modulo the polynomial ideal \( y^2 = W'^2(\lambda) + f(\lambda) \), which is obviously associative.

In the proof of the WDVV equations we used in this section, the associativity (57) is absolutely evident, being associativity of the usual multiplication, and the main point to check was to derive (62)-(63). When using instead the algebra of differentials, the main non-trivial point is to check its associativity, while the relations (62)-(63) appear as even more transparent than above corollary of the residue formula.

5 Explicit check of the WDVV equations

To convince ourselves that the general proof indeed works and to get some further insights, in this section we consider the explicit check of the WDVV equations (1) perturbatively. To do this, let us take the perturbative expansion of the prepotential (29) at small \( S_i \)’s (here we take the symmetric choice of variables with \( i = 1, \ldots, n = g + 1 \)), keeping the coefficients of the matrix model potential t’s arbitrary, and see if this perturbative expansion of \( \mathcal{F} \) satisfies the WDVV equations order by order. A general procedure of getting such perturbative expansion was proposed in [9].
We are going to check here only the simplest non-trivial case of cubic matrix model potential, which is reformulated in terms of elliptic curve \( \mathbb{C} \) with four branch points. According to\([\ref{9}]\) the corresponding solution to WDVV equations \( \mathbb{C} \) should depend exactly on four independent variables, and we are choosing them to consist of two filling numbers \( S_1 \) and \( S_2 \) and two coefficients of the potential \( \{t_1, t_2\} \). The perturbative expansion for this case was constructed in \( \mathbb{C} \) up to the fifth order in \( S_1 \)’s and was later discussed in many other places (see, e.g., \([\ref{32}]\) \([\ref{31}]\)). It reads

\[
F = -S_1 W(\mu_1) - S_2 W(\mu_2) + \frac{1}{2} S_1 \log \left( \frac{S_1}{\Delta} \right) + \frac{1}{2} S_2 \log \left( \frac{S_2}{\Delta} \right) + 2 S_1 S_2 \log(\Delta) +
\]

\[
\left( \frac{2}{3} S_1^3 - \frac{2}{3} S_3^3 + 5 S_1 S_2^2 - 5 S_1^2 S_2 \right) + \frac{1}{\Delta^3} \left( \frac{8}{3} S_1^4 + \frac{8}{3} S_2^4 - \frac{91}{3} S_1 S_2^3 - \frac{91}{3} S_1^3 S_2 - 59 S_1^2 S_2^2 \right) +
\]

\[
\frac{1}{\Delta^9} \left( \frac{56}{3} S_1^5 - \frac{56}{3} S_3^5 - \frac{871}{3} S_1^4 S_2 + \frac{871}{3} S_2^4 S_1 + \frac{2636}{3} S_1^3 S_2^2 - \frac{2636}{3} S_1^2 S_2^3 \right) + O(S^6)
\]

Here the matrix model potential \( \mathbb{C} \) is fixed to be \( W(\lambda) = \lambda^3/3 + t_2 \lambda^2/2 + t_1 \lambda \); and \( \mu_1, \mu_2 \) are the roots of the equation \( W'(\lambda) = 0 \), i.e.

\[
\mu_1 = -\frac{1}{2} t_2 + \frac{1}{2} \sqrt{t_2^2 - 4t_1}
\]

\[
\mu_2 = -\frac{1}{2} t_2 - \frac{1}{2} \sqrt{t_2^2 - 4t_1}
\]

\[
\Delta \equiv \mu_1 - \mu_2 = \sqrt{t_2^2 - 4t_1}
\]

The perturbative expansion of free energy \( \mathbb{C} \) is symmetric with respect to simultaneous transformation \( S_1 \leftrightarrow -S_2 \) and \( t_2 \leftrightarrow -t_2 \). However, since the whole prepotential but its linear part depends only on \( t_2^2 \), the transformation of \( t_2 \) is only essential for the linear term.

Note that in the context of supersymmetric field theories such formulas usually depend on an additional parameter \( \Lambda_{QCD} \) associated with the field theory scale \( \mathbb{C} \). However, this parameter here emerges only as regularization (see footnote \([\ref{4}]\)) and generally can be omitted within the matrix model approach. Moreover, since the terms depending on \( \Lambda_{QCD} \) are at most quadratic in \( S_1 \) and \( t_i \) (they are logarithmic or polynomial in \( \Lambda_{QCD} \), with the latter proportional to \( W(\Lambda_{QCD}) \)), they do not contribute to the WDVV equations and just renormalize the second derivatives of the prepotential (or period matrix of the curve similar to \( \mathbb{C} \) playing the role of the set of effective couplings in the field theory at low energies (cf. with \([\ref{11}]\)).

There is one subtlety with the perturbative prepotential \( \mathbb{C} \) – when calculating it the authors of \( \mathbb{C} \) were interested only in the terms depending on \( S_i \). There could be, in principle, some terms dependent only on \( t_i \)’s. This does not, however, happen in our case. Indeed, these terms would survive in the limit of all \( S_i = 0 \), i.e. when \( f(\lambda) = 0 \) in \( \mathbb{C} \) and \( y = W'(\lambda) \) is just a polynomial. Then from \( \mathbb{C} \) it follows that \( v_k = \frac{\partial F}{\partial t_k} = 0 \) in this limit, hence we miss at most linear in \( t_i \) terms.

In contrast to the weak coupling expansion of the Seiberg-Witten theory when logarithmic terms do satisfy the WDVV equations themselves exactly (see \([\ref{2}]\))\(^5\), some technical complication with perturbative check of whether the prepotential \( \mathbb{C} \) satisfies the WDVV equations emerges due to different orders of magnitude of different matrix elements of the matrix of third derivatives of the prepotential \( \mathbb{C} \). Say, matrix \( (F_{t_2})_{IJK} \) has all matrix elements except for \( (F_{t_2})_{t_j, t_k} \) of the order \( O(1) \), while the latter ones of the order of \( O(S_i) \). (We definitely put all \( S_i \)’s of the same order of magnitude.)

Therefore, a careful calculation in every matrix element is required to determine the order of the WDVV equations where the perturbative expansion contributes to. In practice, one suffices to rescale all \( S_i \)’s with a scale parameter \( \Lambda \). Then, all non-trivial matrix elements of same matrix WDVV equation (i.e. for concrete \( IJK \)) are of the same order in \( \Lambda \) (choosing other indices \( IJK \) in \( \mathbb{C} \) changes

\(^5\)Note that expansion \( \mathbb{C} \) that goes over positive powers of variables, \( S_i \)’s is rather similar not to the weak coupling expansion of the Seiberg-Witten theory but to its strong coupling expansion (see, for example, \([\ref{28}]\)). In the latter case, the WDVV equations trivially hold due to the duality argument \([\ref{30}]\).
this order). Say, for the choice \((I, J, K) = (t_2, S_1, t_1)\) the leading order of the equation \((1)\) is \(\Lambda^7\) and trivially vanishes. It is contributed only by the linear and logarithmic (in \(S_i\)’s) terms of \((66)\). The cubic terms of the prepotential contribute to \(\Lambda^8\)-order, the quartic terms to \(\Lambda^9\)-order etc. Using the program MAPLE, we have checked that the WDVV equations \((1)\) are satisfied with the prepotential \((66)\) up to the fifth order in \(S_i\).

Now, one can consider further non-trivial examples with matrix model potentials being polynomials of higher degree. One technical problem now is that \(W(\mu_i)\) is getting more and more involved function of \(t_i\)’s, since \(\mu_i\)’s become the roots of polynomial equations of increasing order. Therefore, we lose just a principal possibility of analytic analyzing \(W(\mu_i)\) if \(W(\lambda)\) is a polynomial of order 6 and higher. Nevertheless, one can still look at different limiting regimes with some of roots of \(W'(\lambda)\) approaching close to each other.

6 Conclusion

In this paper we have proven that the (generalized) free energy of the matrix model in planar limit satisfies the WDVV equations \((1)\). The proof is based on the residue formula for the quasiclassical tau-functions introduced in \([16]\) which was derived above for the whole basis of the first-kind or holomorphic \([12]\), second-kind \([20]\) and third-kind \([16]\) meromorphic Abelian differentials. The proof of the WDVV equations based on the residue formula requires the matching condition \((55)\) (or associativity of the algebra of differentials), and the first nontrivial example of our statement for the case of four variables we have checked explicitly for its perturbative expansion \((66)\).

The matching condition \((55)\) requires for the quasiclassical tau-function defined by \((6)\) and \((11)\) to satisfy the WDVV equations \((1)\) as a function of the set of parameters involving necessarily the coefficients of the matrix model potential and corresponding to the meromorphic differentials. In other words, we cannot restrict ourselves for the set of only holomorphic differentials, like it happens for the prepotentials of the Seiberg-Witten models \([2]\), associated as well in the sense of sect. \([2]\) to integrable systems \([27]\) (see also \([28]\) and references therein). This is similar to the case of softly broken \(N = 4\) Seiberg-Witten theory, where the WDVV equations are not satisfied by the prepotential, being a function of the Seiberg-Witten periods corresponding to the holomorphic differentials only \([4]\). In that case one should also necessarily add meromorphic differentials, but any physical sense of the corresponding parameters, in contrast to the situation considered in this paper, remains yet unclear.

Our statement can be generalized straightforwardly to the case of the two-matrix model and the corresponding non-hyperelliptic curve \([19]\). It was already suggested in \([19]\) that the WDVV equations should hold for the quasiclassical tau-function of the two-matrix model, whose construction is based on the non-hyperelliptic curve, and similar to \((11)\) meromorphic differential. One can use our line of reasoning of sect. \([3]\) and sect. \([4]\) as a general proof in the two-matrix case either, since our proof does not require any specific of the curve \([3]\) like existence of the hyperelliptic parameterization. The perturbative expansion of the multisupport solution of the two-matrix model is not known yet (it can be computed, say, using the diagrammatic expansion proposed in \([19]\)) and it would be interesting to calculate the free energy of the two matrix model explicitly and repeat for this case the explicit calculation of sect. \([5]\).

There is another, very important direction where one can easily extend the consideration of the present paper. In \([18]\) there was shown how one can introduce more Whitham times, preserving the same number of moduli \(S_i\). It is based on using potential \(W(\lambda)\) of higher degree and fixing some of the coefficients of \(f(\lambda)\) by the double point conditions. This procedure allows one to make the number of Whitham times and moduli independent in the WDVV equations. It would be interesting to check if the WDVV equations are satisfied in this case, and to construct the perturbative expansion for such a case.

Finally, let us point out that the fact that free energy of multi-support solution of matrix model does satisfy the WDVV equations \((1)\) can be thought of as a first step to write down the generalization of the (quasiclassical) Hirota equations which are known up to now only for the one-support solutions. A link between the WDVV equations and (generalized) Hirota relations \([7, 33]\) together with the
statement of our paper allows one to believe that the counterpart of the Hirota relations for nontrivial quasiclassical tau-functions can be written in an explicit form.

Acknowledgements

We are grateful to V.Kazakov, I.Krichever, A.Morozov and V.Rubtsov for valuable discussions. The work was partly supported by INTAS grant 00-561 (A.Mar, A.Mir. and D.V.), RFBR grants 01-01-00549 (L.Ch.), 02-02-16946 (A.Mar.), 01-01-00548 (A.Mir.) and 01-02-17488 (D.V.), by the Grants of Support of the Scientific Schools 00-15-96046 (L.Ch.), 00-15-96566 (A.Mar) and 96-15-96798 (A.Mir.), by Russian President’s grant 00-15-99296 (D.V.), by the Volkswagen-Stiftung (A.Mir. and D.V.) and by the program “Solitons” (L.Ch.). A.Mar. would like to thank the support of CNRS, Lab. de Physique Theorique of ENS, Paris, and IHES where an essential part of this work has been done.

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