LOCALLY $n$-CONNECTED COMPACTA AND $UV^n$-MAPS

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Abstract. We provide a machinery for transferring some properties of metrizable ANR-spaces to metrizable $LC^n$-spaces. As a result, we show that for complete metrizable spaces the properties $ALC^n$, $LC^n$ and $WLC^n$ coincide to each other. We also provide the following spectral characterizations of $ALC^n$ and cell-like compacta: A compactum $X$ is $ALC^n$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p_\beta^\alpha, \alpha < \beta < \tau\}$ consisting of compact metrizable $LC^n$-spaces $X_\alpha$ such that all bonding projections $p_\beta^\alpha$, as well all limit projections $p_\alpha$, are $UV^n$-maps. A compactum $X$ is a cell-like (resp., $UV^n$) space if and only if $X$ is the limit space of a $\sigma$-complete inverse system consisting of cell-like (resp., $UV^n$) metric compacta.

1. Introduction

Following [7], we say that a space $M$ is weakly locally $n$-connected (briefly, $WLC^n$) in a space $Y$ if $M \subset Y$ is closed and for every point $x \in M$ and its open neighborhood $U$ in $M$ there exists a neighborhood $V$ of $x$ in $M$ such that any map from $S^k$, $k \leq n$, into $V$ is null-homotopic in $\widetilde{U}$, where $\widetilde{U}$ is any open in $Y$ set with $\widetilde{U} \cap M = U$ (in such a case we say that $\widetilde{U}$ is an open extension of $U$ in $Y$). Dranishnikov [5] also suggested the following notion: a space $M$ is approximately locally $n$-connected (briefly, $ALC^n$) in a space $Y$ if $M \subset Y$ is closed and for every point $x \in M$ and its open neighborhood $U$ in $M$ there exists a neighborhood $V$ of $x$ in $M$ such that for any open in $Y$ extension $\widetilde{U}$ of $U$ there exists and open in $Y$ extension $\widetilde{V}$ of $V$ with any map from $S^k$, $k \leq n$, into $\widetilde{V}$ being null-homotopic in $\widetilde{U}$. One can show that if $M$ is metrizable (resp., compact), then $M$ is $WLC^n$ in a given space $Y$, where $Y$ is a metrizable (resp., compact) ANR, if and only if $M$
is $WLC^n$ in any metrizable (resp., compact) $ANR$ containing $M$ as a closed set. The same is true for the property $ALC^n$. So, the definitions of $WLC^n$ and $ALC^n$ don’t depend on the $ANR$-space containing $M$, and we say that $M$ is $WLC^n$ (resp., $ALC^n$).

Dranishnikov [7] proved that both properties $WLC^n$ and $LC^n$ are identical in the class of metrizable compacta. It also follows from Gutev [10] and Dugundji-Michael [9] that this remains true for complete metrizable spaces. One of the main result in this paper, Theorem 2.7, shows that all properties $WLC^n$, $LC^n$ and $ALC^n$ coincide for completely metrizable spaces. The proof of Theorem 2.7 is based on the technique, developed in Section 2, for transferring properties of metrizable $ANR$’s to $LC^n$-subspaces (in this way well known properties of metrizable $LC^n$-spaces can be obtained from the corresponding properties of metrizable $ANR$’s, see for example Proposition 2.2). Section 2 contains also a characterization of metrizable $LC^n$-spaces whose analogue for $ANR$’s was established by Nhu [12].

It is well known that the class of metrizable $LC^n$-spaces are exactly absolute neighborhood extensors for $(n + 1)$-dimensional paracompact spaces, and this is not valid for non-metrizable spaces. Outside the class of metrizable spaces we have the following characterization of $ALC^n$ compacta (Theorem 3.1): A compactum $X$ is $ALC^n$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable $LC^n$-spaces $X_\alpha$ such that all bonding projections $p_\alpha^\beta$, as a well all limit projections $p_\alpha$, are $UV^n$-maps. A similar spectral characterization is obtained for cell-like or $UV^n$ compacta, see Theorem 3.3. Both Theorem 3.1 and Theorem 3.3 provide different classes of compacta $C$ and corresponding classes of maps $M$ adequate with $C$ in the following sense (see Shchepin [18]): A compactum $X$ belongs to $C$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable $X_\alpha \in C$ with all bonding projections $p_\alpha^\beta$ being from $M$. For example, according to Theorem 3.1, the class of $ALC^n$ compacta is adequate with the class of $UV^n$-maps.

Recall that a closed subset $A \subset X$ is $UV^n$ in $X$ (resp., cell-like in $X$) if every neighborhood $U$ of $A$ in $X$ contains a neighborhood $V$ of $A$ such that, for each $0 \leq k \leq n$, any map $f: S^k \to V$ is null-homotopic in $U$ (resp., $A$ is contractible in every neighborhood of $A$ in $X$). A space $X$ is said to be $UV^n$ (resp., cell-like) provided it is $UV^n$ (resp., cell-like) in some $ANR$-space containing $X$ as a closed set. It is well known that for metrizable or compact $X$ this definition does not depend on the $ANR$-spaces containing $X$ as a closed set, see for example [17]. A map
f: X → Y between compact spaces is called UV^n (resp., cell-like) if all fibres of f are UV^n (resp., cell-like).

Theorem 3.1 yields that any compact LC^n-space is ALC^n. It is interesting to find an example of an ALC^n-compactum which is not LC^n (obviously, such a compactum should be non-metrizable).

2. METRIZABLE ALC^n-SPACES

We are going to establish some properties of metric LC^n-spaces using the corresponding properties of metric ANR-spaces. Recall that a map f: X → Y is n-invertible if for every Tychonoff space Z with dim Z ≤ n and any map g: Z → Y there is a map h: Z → X such that g = f ∘ h.

The next theorem follows from a stronger result due to Pasynkov [15, Theorem 6], and its proof is based on Dranishnikov results [6, Theorem 1] and [7, Theorem 1.2] (such a result concerning the extension dimension with respect to quasi-finite complexes was established in [14, Proposition 2.7]). We provide here a proof based on factorization theorems.

Theorem 2.1. Every Tychonoff space M is the image of a Tychonoff space X with dim X ≤ n under a perfect n-invertible map. In case M is metrizable, X can be supposed to be also a metrizable space with w(X) = w(M).

Proof. Let M be a Tychonoff space of weight τ. Consider all couples (Zα, fα), where Zα is a Tychonoff space of weight w(Zα) ≤ w(βM), dim Zα ≤ n and fα is a map from Zα into M (here βM is the Čech-Stone compactification of M). Denote by Z the disjoint sum of all spaces Zα. Obviously, there is a natural map f: Z → M such that f|Zα = fα for all α. Let ˜f: βZ → βM be the extension of f. Then, by the Mardešić’s factorization theorem [11], there exists a compactum ˜X of weight w(˜X) ≤ w(βM) with dim ˜X ≤ dim βZ = n and maps h: βZ → ˜X, ˜g: ˜X → βM such that ˜g ∘ h = ˜f. Let X = ˜g^{-1}(M) and g = ˜g|X. According to Corollary 6 and Main Theorem from [15], dim X ≤ n. To show that g is n-invertible, suppose f₀: Z₀ → M is a map with dim Z₀ ≤ n. Applying again the Mardešić’s factorization theorem for the map ˜f₀: βZ₀ → βM, we obtain a compactum K and maps h₁: βZ₀ → K and ˜f₂: K → βM such that dim K ≤ n, w(K) ≤ w(βM) and ˜f₂ ∘ h₁ = ˜f₀. Then, as above, Z’ = ˜f₂^{-1}(M) is a space of dimension ≤ n and weight ≤ w(βM). So, there exists α* and a homeomorphism j: Z’ → Z such that j(Z’) = Z₀* and ˜f₀ ∘ j = ˜f₂|Z’. Consequently, h ∘ j ∘ h₁ is a map from Z₀ to X with f₀|Z₀ = g ∘ h ∘ j ∘ h₁ = f₀.
If $M$ is metrizable, the proof is simpler. Indeed, in this case $P = \tilde{f}^{-1}(M)$ is a space of dimension $\leq n$ and the restriction $\tilde{f}|P$ is a perfect map. So, by Pasynkov’s factorization theorem [16], there exists a metrizable space $X$ and maps $h: P \to X$, $g: X \to M$ such that $g \circ h = \tilde{f}$, $w(X) \leq w(M)$ and $\dim X \leq n$. Then $g$ is a perfect map because so is $f|P$, and according to the above arguments, $g$ is $n$-invertible.

The next proposition shows that Theorem 2.1 allows some properties of metrizable ANR-spaces to be transferred to metrizable $\mathcal{L}C^n$-spaces.

**Proposition 2.2.** Let $M$ be a metrizable $\mathcal{L}C^n$-space and $\alpha$ an open cover of $M$. Then there exists an open cover $\beta$ of $M$ refining $\alpha$ such that for any two $\beta$-near maps $f, g: Z \to M$ defined on a metrizable space $Z$ of dimension $\leq n$ any $\beta$-homotopy $H: A \times [0, 1] \to M$ between $f|A$ and $g|A$, where $A$ is closed in $Z$, can be extended to an $\alpha$-homotopy $H: Z \times [0, 1] \to M$ connecting $f$ and $g$.

**Proof.** We embed $M$ as a closed subset of a metrizable ANR-space $P$ and let $p: Y_P \to P$ be a perfect $(n + 1)$-invertible surjection such that $Y_P$ is a metrizable space of dimension $\leq n + 1$ (see Theorem 2.1). Since $M$ is $\mathcal{L}C^n$, there is an open set $G$ in $Y_P$ containing $p^{-1}(M)$ and a map $q: G \to M$ extending the restriction $p|p^{-1}(M)$. Then there exists an open set $W \subset P$ containing $M$ with $p^{-1}(W) \subset G$ (recall that $p$ is a perfect map). Obviously $W$ is also an ANR-space containing $M$ as a closed set. So, without losing generality, we may assume that $W = P$, $G = Y_P$ and $q$ is a map from $Y_P$ onto $M$. Now, for every open $U \subset M$, let $\tilde{U} = P \setminus (p(q^{-1}(M \setminus U)))$. The set $\tilde{U}$ is non-empty and open in $P$, $\tilde{U} \cap M = U$ and $p^{-1}(\tilde{U}) \subset q^{-1}(U)$.

If $\alpha$ is an open cover of $M$ consisting of proper subsets of $M$, the set $L = \bigcup\{\tilde{U} : U \in \alpha\}$ is open in $P$ and contains $M$. So, $L$ is also an ANR and $\bar{\alpha} = \{\tilde{U} : U \in \alpha\}$ is an open cover of $L$. According to the properties of metrizable ANR’s (see for example [13 chapter IV, Theorem 1.2]), there exists an open cover $\bar{\beta}$ of $L$ with the following property: for any two $\beta$-near maps $h_1, h_2: Z \to M$ defined on a metrizable space $Z$ any $\beta$-homotopy $H: A \times [0, 1] \to M$ between $h_1|A$ and $h_2|A$, where $A$ is closed in $Z$, can be extended to an $\alpha$-homotopy $F: Z \times [0, 1] \to M$ connecting $h_1$ and $h_2$. Then $\beta = \{V \cap M : V \in \bar{\beta}\}$ is an open cover of $M$ refining $\alpha$ and has the desired property. Indeed, suppose $f, g: Z \to M$ are two $\beta$-near maps and $H: A \times [0, 1] \to M$ is a $\beta$-homotopy between $f|A$ and $g|A$, where $Z$ is a metrizable space of dimension $\leq n$ and $A$ is closed in $Z$. According to the choice of $\bar{\beta}$, $H$ can be extended to an $\bar{\alpha}$-homotopy
Let \( U \subseteq P \) be a perfect \((n+1)\)-invertible surjection such that \( U \) is a metrizable space of dimension \( \leq n+1 \), and \( q: p^{-1}(W) \to M \) extends the restriction \( p|p^{-1}(W) \to M \) of \( p \). Since \( \dim Z \leq n \), and \( h: Z \to U \) is a map, there exists a map \( h_1: Z \to M \) such that \( h \) and \( h_1 \) are homotopic in \( P \).

**Proof.** Let \( p: Y_P \to P \) be a perfect \((n+1)\)-invertible surjection such that \( Y_P \) is a metrizable space of dimension \( \leq n+1 \), and \( q: p^{-1}(W) \to M \) extends the restriction \( p|p^{-1}(W) \to M \) of \( p \). Finally, \( \tilde{H} = q \circ F_1 \) is an \( \alpha \)-homotopy between \( f \) and \( g \). \( \square \)

We also need the following property of metrizable \( LC^n \)-spaces.

**Proposition 2.3.** Suppose both \( M \) and \( P \) are metrizable \( LC^n \)-spaces with \( M \subset P \) being closed. Then there exists an open set \( U \subseteq P \) containing \( M \) with the following property: If \( Z \) is a metrizable space with \( \dim Z \leq n \) and \( h: Z \to U \) is a map, there exists a map \( h_1: Z \to M \) such that \( h \) and \( h_1 \) are homotopic in \( P \).

**Proof.** Let \( p: Y_P \to P \) be a perfect \((n+1)\)-invertible surjection such that \( Y_P \) is a metrizable space of dimension \( \leq n+1 \), and \( q: p^{-1}(W) \to M \) extends the restriction \( p|p^{-1}(W) \to M \) of \( p \). Finally, \( \tilde{H} = q \circ F_1 \) is an \( \alpha \)-homotopy between \( f \) and \( g \). \( \square \)

**Corollary 2.4.** If \( M \) and \( P \) are metrizable \( LC^n \)-spaces such that \( M \subset P \) is closed, then \( M \) is \( ALC^n \) in \( P \).

**Proof.** Since \( M \) is \( LC^n \), for every \( x \in M \) and its open neighborhood \( U \) in \( M \) there exists a neighborhood \( V \) of \( x \) in \( M \) such that any map of a \( k \)-sphere, \( k \leq n \), into \( V \) is contractible in \( U \). Let \( \bar{U} \subseteq P \) and \( G \subseteq \bar{U} \) be open in \( P \) extensions of \( U \) and \( V \), respectively. Then both \( V \) and \( G \) are \( LC^n \) (as open subsets of \( LC^n \)-spaces), and \( V \) is closed in \( G \). So, there is an open extension \( \tilde{V} \subseteq G \) satisfying the conclusion from Proposition 2.4. Consequently, any map \( g: S^k \to \tilde{V} \), \( 0 \leq k \leq n \), is homotopic in \( G \) to a map \( g_1: S^k \to V \). Since \( g_1 \) is homotopic in \( U \) to a constant map, we obtain that \( g \) is homotopic in \( \bar{U} \) to a constant map. \( \square \)

**Proposition 2.5.** Let \( P \) be metrizable and \( M \subset P \) be a closed and \( LC^n \)-set. Then every closed set \( A \subseteq M \) is \( UV^n \) in \( M \) provided \( A \) is \( UV^n \) in \( P \).

**Proof.** Let \( p: Y_P \to P \) be a perfect \((n+1)\)-invertible surjection such that \( Y_P \) is a metrizable space of dimension \( \leq n+1 \), and \( q: p^{-1}(W) \to M \) extends the restriction \( p|p^{-1}(W) \to M \) of \( p \).
containing $M$. Suppose $U \subset M$ is an open set containing $A$ and let $	ilde{U} = W \backslash p(q^{-1}(M \backslash U))$. Obviously, $\tilde{U} \subset W$ is an open extension of $U$. Since $A \in UV^n(P)$, there is an open set $\tilde{V} \subset \tilde{U}$ containing $A$ such that any map from $S^k$ with $0 \leq k \leq n$ to $\tilde{V}$ can be extended to a map from $B^{k+1}$ to $\tilde{U}$. Let $V = M \cap \tilde{V}$ and $g : S^k \to V$ be a map. Then extend $g$ to a map $g_1 : B^{k+1} \to \tilde{U}$ and take a lifting $g_2 : B^{k+1} \to p^{-1}(\tilde{U})$ of $g_1$. Finally, $\tilde{g} = q \circ g_2$ is a map from $B^{k+1}$ to $U$ extending $g$. □

Next lemma is a non-compact analogue of Lemma 2.1 from [7].

**Lemma 2.6.** Suppose both $M$ and $P$ are metric $LC^{n-1}$-spaces such that $M$ is a closed subset of $P$. Then for every $\epsilon > 0$ there exists a neighborhood $U_\epsilon(M)$ in $P$ such that for any map $\varphi : (Q,Q_0) \to (U_\epsilon(M),M)$, where $(Q,Q_0)$ is a polyhedral pair, there exists a map $\psi : (Q^{(n)},Q_0) \to M$ such that $\psi|Q_0 = \varphi|Q_0$ and both $\varphi$ and $\psi$ are $\epsilon$-close.

**Proof.** The proof is similar to that one of Proposition 2.5, the only difference is that the map $p : Y_P \to P$ is $n$-invertible and $\dim Y_P \leq n$. We take an open cover $\omega$ of $M$ with each $V \in \omega$ having a diameter $< \epsilon$. For every $V \in \omega$ consider the set $\tilde{V} = P \backslash p(q^{-1}(M \backslash V))$ and let $U_\epsilon(M) = \bigcup \{\tilde{V} : V \in \omega\}$. If $\varphi : (Q,Q_0) \to (U_\epsilon(M),M)$, we first lift $\varphi|Q^{(n)}$ to a map $\varphi_1 : Q^{(n)} \to Y_P$ and define $\psi : (Q^{(n)},Q_0) \to M$ to be the map with $\psi|Q_0 = \varphi|Q_0$ and $\psi|Q^{(n)} = q \circ \varphi_1$. Obviously, $\psi$ satisfies the required conditions. □

Now, we are in a position to prove the main theorem in this section.

**Theorem 2.7.** For a complete metric space $(M,d)$ the following are equivalent:

(i) $M$ is $LC^n$;
(ii) $M$ is $ALC^n$;
(iii) $M$ is $WLC^n$.

**Proof.** Implication (i) $\Rightarrow$ (ii) follows from Corollary 2.4, and implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). We are going to prove this implication by induction. Since $M$ is $LC^{-1}$ (there is no such thing as a $(-1)$-sphere), we can suppose that $M$ is $LC^{n-1}$ and $WLC^n$. We embed $M$ as a closed subset of a complete metric ANR-space $P$ and consider the following relation between the open subsets of $M$: $V \alpha U$ if $V \subset U$ and every map from $S^k$, $k \leq n$, into $V$ is null-homotopic in $\tilde{U}$, where $\tilde{U}$ is any open extension of $U$ in $P$. It follows from [9, Theorem 1] the existence
a complete metric $\rho$ on $M$ generating its topology such that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ with $B_{\delta(\epsilon)}(x)\cap B_{\epsilon}(x)$ for all $x \in M$, where $B_{\epsilon}(x) = \{y \in M : \rho(x, y) < \epsilon\}$. The metric $\rho$ can be extended to a complete metric $\bar{\rho}$ on $P$, see [1]. We fix a sequence $\{G_k\}_{k \geq 1}$ of open subsets of $P$ containing $M$ with $\bigcap_{k=1}^{\infty} G_k = M$, and define by induction a decreasing sequence $\{W_k\}_{k \geq 1}$ of open subsets of $P$ such that $W_k \subset U_{2^{-k}}(M) \cap G_k$, where $U_{2^{-k}}(M)$ is a neighborhood of $M$ in $P$ corresponding to $2^{-k}$ (see Lemma 2.6). Let $\{g_k\}_{k \geq 1}$ be a sequence of continuous functions $g_k : P \to [0, 1]$ such that $g_k(M) = 1$ and $g_k(P \setminus W_k) = 0$.

Then the equality $g(x, y) = \bar{\rho}(x, y) + \sum_{k=1}^{\infty} \frac{|g_k(x) - g_k(y)|}{2^k}$ provides a metric on $P$. It easily seen that $g$ is a complete metric generating the topology of $P$. Moreover, $g(x, y) = \bar{\rho}(x, y) = \rho(x, y)$ for all $x, y \in M$.

For any $x \in M$ and $y \in P \setminus W_k$ we have

$$g(x, y) = \bar{\rho}(x, y) + \sum_{j=1}^{j=k-1} \frac{|1 - g_j(y)|}{2^j} + \sum_{j=k}^{\infty} \frac{1}{2^j} \geq \frac{1}{2^{k-1}}.$$

Hence, $g(M, P \setminus W_k) \geq 2^{-k+1}$ for all $k$. For every $\epsilon > 0$ let $k_\epsilon = \min\{k : 2^{-k} < \epsilon\}$, $\eta(\epsilon) = 2^{-k_\epsilon+1}$ and $B_{\epsilon}^o(M) = \{y \in P : g(y, M) < \epsilon\}$.

**Claim 1.** Let $(Q, Q_0)$ be a polyhedral pair and $\epsilon > 0$. Then for any map $\varphi : Q \to B_{\eta(\epsilon)}^o(M)$ with $\varphi(Q_0) \subset M$ there exists a map $\psi : Q^{(n)} \cup Q_0 \to M$ such that $\psi|Q_0 = \varphi|Q_0$ and $\psi$ is $\epsilon$-close to $\varphi$.

Since $g(M, P \setminus W_k) \geq 2^{-k+1} = \eta(\epsilon)$, $B_{\eta(\epsilon)}^o(M) \subset W_k$. Hence, $B_{\eta(\epsilon)}^o(M) \subset U_{2^{-k}}$, and according to Lemma 2.6, there exists a map $\psi : Q^{(n)} \cup Q_0 \to M$ such that $\psi|Q_0 = \varphi|Q_0$ and $\psi$ is $(2^{-k_\epsilon})$-close to $\varphi$. Then the inequality $2^{-k_\epsilon} < \epsilon$ completes the proof of Claim 1.

To prove that $M$ is $LC^n$, we introduce the following notation: If $U \subset M$ is open and $\gamma > 0$, then $E_\gamma(U)$ denotes any open extension of $U$ in $P$ which is contained in the set $B_{\gamma}^o(M)$. Since $M$ is $WLC^n$, according to the choice of the function $\delta$, any map $\varphi : S^n \to M$ with $\text{diam}\varphi(S^n) < \delta(\epsilon)$ can be extended to a map $\tilde{\varphi} : \mathbb{B}^{n+1} \to E_\gamma(B_{\delta(\epsilon)}^o(\varphi(S^n)))$, where $\gamma > 0$ is arbitrary. Now, we proceed as in the proof of Lemma 2.5 from [7]. Fix $\epsilon > 0$ and a map $f : S^n \to M$ with $\text{diam} f(S^n) < \delta(\epsilon/2)$. We are going to show that $f$ can be extended to a map $\tilde{f} : \mathbb{B}^{n+1} \to M$ such that $\text{diam} f(\mathbb{B}^{n+1}) \leq 10\epsilon$. The map $\tilde{f}$ will be obtained as limit of a sequence $\{f_k : \mathbb{B}^{n+1} \to P\}$, and this sequence will be constructed by induction together with a sequence of triangulations $\{\tau_k\}$ of $\mathbb{B}^{n+1}$ such that $f_k(\tau_k^{(n)}) \subset M$ for all $k$. To start the induction, we choose $\tau_1$ to be
\( \mathbb{B}^{n+1} \), considered as one \((n+1)\)-dimensional simplex, and let \( f_1: \mathbb{B}^{n+1} \to E_{\eta(\delta(2^{-k}\epsilon))/3}(B_\epsilon^g(z)) \) be a map extending \( f \), where \( z \in f(S^n) \). Suppose a triangulation \( \tau_k \) of \( \mathbb{B}^{n+1} \) and a map \( f_k: \mathbb{B}^{n+1} \to B_{\eta(\delta(2^{-k}\epsilon))/3}(M) \) are already constructed with \( f_k(\tau_k^{(n)}) \subset M \). Denote by \( \tau_{k+1} \) any subdivision of \( \tau_k \) such that \( \text{diam} f_k(\sigma) < \delta(2^{-k}\epsilon)/3 \) for any simplex \( \sigma \in \tau_{k+1} \).

According to Claim 1, there exists a map \( g_k: \tau_{k+1}^{(n)} \to M \) such that \( g_k \) is \((\delta(2^{-k}\epsilon)/3)\)-close to the restriction \( f_k|_{\tau_k^{(n)}} \) and \( g_k|_{\tau_k^{(n)}} = f_k|_{\tau_k^{(n)}} \). Therefore, \( \text{diam} g_k(\partial \sigma) < \delta(2^{-k}\epsilon) \) for any \( \sigma \in \tau_{k+1} \). Hence, \( g_k|\partial \sigma \) can be extended to a map \( g_k^e: \sigma \to E_{\eta(\delta(2^{-k}\epsilon))/3}(B_{\epsilon/2k}(g_k(\partial \sigma))) \), \( \sigma \in \tau_{k+1} \).

Finally, we define \( f_{k+1}^e: \mathbb{B}^{n+1} \to B_{\eta(\delta(2^{-k}\epsilon))/3}(M) \), \( f_{k+1}|\sigma = g_k^e|\sigma \). Because \( \delta(\epsilon) \leq \epsilon \) and \( \eta(\epsilon) \leq \epsilon \), for every \( \sigma \in \tau_{k+1} \) we have \( \text{diam} f_{k+1}^e(\sigma) \leq \text{diam} g_k(\partial \sigma) + \epsilon/2k + \eta(\delta(2^{-k}\epsilon)/3) \leq \delta(2^{-k}\epsilon)/3 \leq \epsilon/2k \). Now, for every \( x \in \mathbb{B}^{n+1} \) take \( \sigma \in \tau_{k+1} \) containing \( x \) and \( y \in \partial \sigma \). Then \( \rho(f_k(x), f_{k+1}(x)) \leq \rho(f_k(x), f_{k+1}(y)) + \rho(f_k(y), f_k(x)) \). Consequently, \( \rho(f_k(x), f_{k+1}(x)) \leq 3\epsilon/2k + 2\delta(2^{-k}\epsilon)/3 \leq 4\epsilon/2k \) for all \( x \in \mathbb{B}^{n+1} \).

The last inequality implies that the sequence \( \{f_k(x)\} \) is convergent for all \( x \in M \). So, the limit map \( \tilde{f}: \mathbb{B}^{n+1} \to P \) is well defined and \( \rho(f_1(x), \tilde{f}(x)) \leq 4\epsilon \).

Then for every \( x, y \in \mathbb{B}^{n+1} \) we have \( \rho(\tilde{f}(x), \tilde{f}(y)) \leq \rho(\tilde{f}(x), f_1(x)) + \rho(f_1(x), f_1(y)) + \rho(f_1(y), \tilde{f}(y)) \leq 4\epsilon + \epsilon + \eta(\delta(\epsilon)/3) + 4\epsilon \leq 10\epsilon \).

Since \( f_k(\mathbb{B}^{n+1}) \subset B_{\eta(\delta(2^{-k}\epsilon))/3}(M) \) and \( \lim \eta(\delta(2^{-k}\epsilon)/3) = 0 \), \( \tilde{f}(\mathbb{B}^{n+1}) \subset M \). This completes the proof.

A sequence of open covers \( \mathcal{U} = (\mathcal{U}_k)_{k \in \mathbb{N}} \) of a metric space \((M, d)\) is called a zero-sequence if \( \lim_{k \to \infty} \text{mesh} \mathcal{U}_k = 0 \). For any such a sequence we define \( \text{Tel}(\mathcal{U}) = \bigcup_{k \in \mathbb{N}} N(\mathcal{U}_k \cup \mathcal{U}_{k+1}) \). Here \( N(\mathcal{U}_k \cup \mathcal{U}_{k+1}) \) is the nerve of \( \mathcal{U}_k \cup \mathcal{U}_{k+1} \) with \( \mathcal{U}_k \) and \( \mathcal{U}_{k+1} \) considered as disjoint sets. For any \( \sigma \in \text{Tel}(\mathcal{U}) \) let \( s(\sigma) = \max\{s : \sigma \in N(\mathcal{U}_k \cup \mathcal{U}_{k+1})\} \).

We complete this section by a characterization of metrizable \( LC^n \)-spaces similar to the characterization of metrizable \( ANR \)-spaces provided in [12] (see also [17, Theorem 6.8.1]).

**Proposition 2.8.** A metric space \((M, d)\) is \( LC^n \) if and only if it has a zero-sequence \( \mathcal{U} \) of open covers such that any map \( f_0: K(0) \to M \) with \( f(U) \in U, U \in \mathcal{K}(0) \), where \( K \) is a subcomplex of \( \text{Tel}(\mathcal{U}) \), extends to a map \( f: K^{(n+1)} \to M \) satisfying the following condition:

\((^*)\) For any sequence \( \{\sigma_k\} \) of simplexes of \( K^{(n+1)} \) with \( s(\sigma_k) \to \infty \) we have \( \lim_{k \to \infty} \text{diam}(f(\sigma_k)) = 0 \).

**Proof.** Suppose \( M \) is \( LC^n \) and embed \((M, d)\) isometrically in a metric \( ANR \)-space \((P, \rho)\) as a closed subset. According to the proof of
Theorem 2.1, there are metrizable space $Y_p$ and two maps $p: Y_p \to P$ and $q: Y_p \to M$ such that $\dim Y_p \leq n + 1$, $p$ is $(n + 1)$-invertible and $q$ extends the map $p|p^{-1}(M)$. Using the proof of Nhu’s theorem \cite{12} Theorem 1.1 for ANR’s, we can find a zero-sequence $V = (V_k)_{k \in \mathbb{N}}$ of $P$ such that any map $h_0: K^{(0)} \to P$ with $h_0(V) \in V$ for each $V \in K^{(0)}$, where $K$ is a subcomplex of Tel($U$), extends to a map $h: |K| \to P$ such that $\lim_{k \to \infty} \text{diam}(h(\sigma_k)) = 0$ for any sequence $\{\sigma_k\}$ of simplex of $K$ with $s(\sigma_k) \to \infty$. Let us show that the sequence $(U_k)_{k \in \mathbb{N}}, U_k = \{V \cap M : V \in V_k\}$, is as required. Indeed, for each $U = V \cap M \in U_k$ define $W(U) = V \setminus (p^{-1}(M \setminus U))$ and consider the open families $W = \{W(U) : U \in U_k\}, k \in \mathbb{N}$. We may assume that each $U$ is a proper subset of $M$, so $W_k \neq \emptyset$ for all $k$. Note that $\mathcal{W}_k$ may not cover $P$, but any $\mathcal{W}_k$ covers $M$. Moreover, mesh$\mathcal{W}_k \leq$ mesh$\mathcal{W}_k$, so $\lim_{k \to \infty}$ mesh$\mathcal{W}_k = 0$. If $K$ is a subcomplex of Tel($U$), take any map $f_0: K^{(0)} \to M$ with $f_0(U) \in U$. Hence, $f_0$ extends to a map $g: |K| \to P$ such that $\lim_{k \to \infty} \text{diam}(g(\sigma_k)) = 0$ for any sequence $\{\sigma_k\}$ of simplex of $K$ with $s(\sigma_k) \to \infty$. Finally, let $f: |K^{(n+1)}| \to M$ be the map $g \circ \tilde{g}$, where $\tilde{g}: |K^{(n+1)}| \to Y_p$ is a lifting of $g$. To show that $f$ satisfies condition (1), fix an $\epsilon > 0$, a cover $\mathcal{W}_m$ with mesh$\mathcal{W}_m < \epsilon$ and a sequence $\{\sigma_k\} \subset K^{(n+1)}$ with $\lim_{k \to \infty} s(\sigma_k) = \infty$. Then $\lim_{k \to \infty}$ mesh$(g(\sigma_k)) = 0$, so there exists $k_0$ such that $g(\sigma_k) \subset \bigcup \mathcal{W}_m$ for all $k \geq k_0$ (recall that $g(U) \in M$ for all $U \in K^{(0)}$). Hence, for any two different points $x, y \in \sigma_k$ with $k \geq k_0$ we have $g(x) \in W(U_x)$ and $g(y) \in W(U_y)$ for some $W(U_x), W(U_y) \in \mathcal{W}_m$. So, according to the definition of $W(U), f(x) \in U_x$ and $f(y) \in U_y$. Therefore, $\rho(f(x), g(x)) \leq \text{diam}(W(U_x)) < \epsilon$ and, similarly, $\rho(f(y), g(y)) < \epsilon$. Consequently, $d(f(x), f(y)) < 2 \cdot \epsilon + \text{diam}(g(\sigma_k))$.

To prove the other implication, embed $M$ as a closed subset of a metrizable space $Z$ with $\dim Z \setminus M \leq n + 1$ and follow the proof of implication (iii) $\Rightarrow$ (i) from \cite{12} Theorem 1.1 to obtain that $M$ is a retract of a neighborhood $W_1$ of $M$ in $Z$ (the only difference is that in Fact 1.2 from \cite{12} we take the cover $\mathcal{V}$ of $W_1 \setminus M$ to be of order $\leq n + 1$, so the nerve $N(\mathcal{V})$ is a complex of dimension $\leq n + 1$). Then by \cite{13} chapter V, Theorem 3.1, $M$ is $LC^n$.

\section{UV$^n$-maps and $ALC^n$-spaces}

In this section we provide spectral characterizations of non-metrizable $ALC^n$-compacta and cell-like compacta. Recall that a map $f: X \to Y$ between compact spaces is said to be soft \cite{18} if for every compactum $Z$, its closed subset $A \subset Z$ and maps $h: A \to X$ and $g: Z \to Y$ with $f \circ h = g|A$ there exists a lifting $\overline{g}: Z \to X$ of $g$ extending $h$. 
Theorem 3.1. A compactum $X$ is $\text{ALC}^n$ if and only if $X$ is the limit space of a $\sigma$-complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable $\text{LC}^n$-spaces $X_\alpha$ such that all bonding projections $p_\alpha^\beta$, as well all limit projections $p_\alpha$, are $\text{UV}^n$-maps.

Proof. Suppose that $X$ is the limit space of an inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ such that each $X_\alpha$ is a metric $\text{LC}^n$-compactum and all $p_\alpha^\beta$ are $\text{UV}^n$-maps. We embed $X$ in a Tychonoff cube $\mathbb{I}^B$, where $\mathbb{I} = [0, 1]$ and the cardinality of $B$ is equal to $\tau$. According to Shchepin’s spectral theorem [18], we can assume that $B$ is the union of countable sets $B_\alpha$, $\alpha \in A$, such that $B_\alpha \subseteq B_\beta$ for $\alpha < \beta$, $B_\beta = \bigcup\{B_\gamma(k) : k = 1, 2, \ldots\}$ for any chain $\gamma(1) < \gamma(2) < \ldots$ with $\gamma = \sup\{\gamma(k) : k \geq 1\}$, and each $p_\alpha^\beta : X_\beta \to X_\alpha$ is the restriction of the projection $q_\alpha^\beta : \mathbb{I}^B_\beta \to \mathbb{I}^B_\alpha$. Since each $X_\alpha = q_\alpha(X)$ is a subset of $\mathbb{I}^B_\alpha$, where $q_\alpha$ denotes the projection $q_\alpha : \mathbb{I}^B \to \mathbb{I}^B_\alpha$. We also denote $q_\alpha|X$ by $p_\alpha$. Choose $x_0 \in X$ and its neighborhood $U \subset X$. There exists $\alpha_0 < \tau$ and an open set $U_0 \subset X_{\alpha_0}$ with $x_0 \in p_{\alpha_0}^{-1}(U_0) \subset p_{\alpha_0}^{-1}(U_0) \subset U$. Since $X_{\alpha_0}$ is $\text{LC}^n$, there exists a neighborhood $V_0$ of $x_{\alpha_0} = p_{\alpha_0}(x_0)$ such that for all $k \leq n$ the inclusion $j_0 : V_0 \hookrightarrow U_0$ generates trivial homomorphisms $j_0^*: \pi_k(V_0) \to \pi_k(U_0)$ between the homotopy groups. Let $V = p_{\alpha_0}^{-1}(V_0)$ and $\mathring{U}$ be an open set in $\mathbb{I}^B$ extending $U$. Choose a finite family $\omega = \{W_1, \ldots, W_m\}$ of open sets from the ordinary base of $\mathbb{I}^B$ such that $W = \bigcup_{i=1}^m W_i$ covers $p_{\alpha_0}^{-1}(V_0)$ and $W \subset \mathring{U}$. Then we can find $\alpha_1 > \alpha_0$ with $q_{\alpha_1}^{-1}(q_{\alpha_1}(W_i)) = W_i$, $i \leq m$. Denote $V_1 = (p_{\alpha_0}^\alpha)^{-1}(V_0)$ and $U_1 = (p_{\alpha_0}^\alpha)^{-1}(U_0)$. Obviously, $q_{\alpha_1}(W)$ is open in $\mathbb{I}^B_{\alpha_1}$ containing $U_1$. Take an open in $\mathbb{I}^B_{\alpha_1}$ extension $V_1$ of $V_1$ with $\mathring{V}_1 \subset q_{\alpha_1}(W)$. Since $\mathring{V}_1$ is an ANR and $V_1$ is $\text{LC}^n$ (as an open subset of $X_{\alpha_1}$), by Proposition 2.3, there exists an open in $\mathbb{I}^B_{\alpha_1}$ extension $G$ of $V_1$ which is contained in $\mathring{V}_1$ with the following property: for every map $h : Z \to G$, where $Z$ is at most $n$-dimensional metric space, there exists a map $h_1 : Z \to V_1$ such that $h$ and $h_1$ are homotopic in $\mathring{V}_1$. Finally, let $\mathring{V} = q_{\alpha_1}^{-1}(G)$. It is easily seen that $\mathring{V}$ is an open extension of $V$ and $\mathring{V} \subset W \subset \mathring{U}$. Consider a map $f : S^k \to \mathring{V}$, where $k \leq n$. Then there exists a map $g : S^k \to V_1$ such that $q_{\alpha_1} \circ f$ and $g$ are homotopic in $\mathring{V}_1$. We are going to show that $g$ is homotopic to a constant map in the set $U_1$. This will be done if the inclusion $j_1 : V_1 \hookrightarrow U_1$ generates a trivial homomorphism $j_1^*: \pi_k(V_1) \to \pi_k(U_1)$. 
To this end, we consider the following commutative diagram:

\[
\begin{array}{ccc}
\pi_k(V_1) & \xrightarrow{j_1^*} & \pi_k(U_1) \\
\downarrow (p_{a_1})^* & & \downarrow (p_{a_0})^* \\
\pi_k(V_0) & \xrightarrow{j_0^*} & \pi_k(U_0)
\end{array}
\]

Since \(X_{a_1}\) is \(LC^n\) and the map \(p_{a_1}^{\alpha_1}\) is \(UV^n\), each fiber of \(p_{a_0}^{\alpha_1}\) is an \(UV^n\)-set in \(X_{a_1}\), see Proposition 2.5. Then, according to [8, Theorem 5.3], both vertical homomorphisms from the above diagram are isomorphisms. This implies that \(j_1^*\) is trivial because so is \(j_0^*\). Hence, \(q_{a_1} \circ f\) is homotopic to a constant map in the set \(\tilde{V}_1 \cup U_1 \subset q_{a_1}(W)\) (recall that \(q_{a_1} \circ f\) is homotopic to \(g\) in \(\tilde{V}_1\) and \(g\) is homotopic to a constant map in \(U_1\)). Therefore, there is a map \(f_1: B^{n+1} \to q_{a_1}(W)\) extending \(q_{a_1} \circ f\). Since \(q_{a_1}\) is a soft map, and \(W = q_{a_1}^{-1}(q_{a_1}(W))\), \(f\) can be extended to a map \(\tilde{f}: B^{n+1} \to W\). Thus, the inclusion \(\tilde{V} \hookrightarrow \tilde{U}\) generates a trivial homomorphism between \(\pi_k(\tilde{V})\) and \(\pi_k(\tilde{U})\) for any \(k \leq n\). So, \(X\) is \(ALC^n\).

Now, suppose \(X\) is \(ALC^n\), and consider \(X\) as a subset of some \(\mathbb{B}^B\). Since the sets \(V\) and \(\tilde{V}\) in the \(ALC^n\) definition depend on the point \(x\), the set \(U\) and its open extension \(\tilde{U}\), respectively, we use the notations \(\lambda(x, U) = V\) and \(\lambda(x, U, \tilde{U}) = \tilde{V}\). First, we show that the sets \(V\) and \(\tilde{V}\) can be chosen to be functionally open in \(X\) and in \(\mathbb{B}^B\), respectively. Indeed, if \(U \subset X\) is a neighborhood of \(x \in X\), we take a functionally open in \(X\) neighborhood \(V^*\) of \(x\) with \(V^* \subset \lambda(x, U)\). Then for a given open in \(\mathbb{B}^B\) extension \(\tilde{U}\) of \(U\) and every \(y \in V^*\) choose a functionally open in \(\mathbb{B}^B\) neighborhood \(G(y)\) of \(y\) with \(G(y) \subset \lambda(x, U, \tilde{U}) \cap G\), where \(G\) is an open in \(\mathbb{B}^B\) extension of \(V^*\). Since \(V^*\), as a functionally open subset of \(X\), is Lindelöf, there exist countably many sets \(G(y_i)\) whose union covers \(V^*\). Obviously the set \(\tilde{G} = \bigcup_{i=1}^{\infty} W(y_i)\) is a functionally open in \(\mathbb{B}^B\) extension of \(V^*\) which is contained in \(\lambda(x, U, \tilde{U})\). So, every map from \(\mathbb{S}^k\) to \(\tilde{G}, k \leq n\), is homotopic in \(\tilde{U}\) to a constant map.

Therefore, for every open set \(U \subset X\) and every \(x \in U\) there exists a functionally open in \(X\) set \(V = \lambda(x, U)\) such that for any open in \(\mathbb{B}^B\) extension \(\tilde{U}\) of \(U\) we can find a functionally open in \(\mathbb{B}^B\) extension \(\lambda(x, U, \tilde{U})\) of \(V\) contained in \(\tilde{U}\) with all homomorphisms \(\pi_k(\lambda(x, U, \tilde{U})) \to \pi_k(\tilde{U}), k \leq n\), being trivial. If \(U\) is functionally open in \(X\), then it is Lindelöf and there are countably many \(x_i \in U\) such
that \( \{\lambda(x_i, U) : i = 1, 2, \ldots\} \) is a cover of \( U \). We fix such a countable cover \( \gamma(U) \) for any functionally open set \( U \subset X \).

Let \( A \subset B \) and \( W_0, W_1, \ldots, W_k \) be elements of the standard open base \( \mathcal{B}_A \) for \( \mathbb{I}^A \) such that
\[
\bigcup_{i=1}^{k} W_i.
\]
Here, \( X_A = q_A(X) \), where \( q_A : \mathbb{I}^B \rightarrow \mathbb{I}^A \) is the projection. We also denote by \([W_0, W_1, \ldots, W_k]_A \) the set \( q^{-1}_A((\bigcup_{i=1}^{k} W_i) \setminus (X_A \setminus W_0)) \). Observe that \([W_0, W_1, \ldots, W_k]_A \cap X = q^{-1}_A(W_0) \cap X \), so \([W_0, W_1, \ldots, W_k]_A \cap X \) is functionally open in \( X \). Moreover, if \( \gamma(q^{-1}_A(W_0) \cap X) = \{V(y_i) : i = 1, 2, \ldots\} \), where \( y_i \in q^{-1}_A(W_0) \cap X \) for all \( i \), we consider the sets \( V(y_i) = \lambda(y_i, q^{-1}_A(W_0) \cap X, [W_0, W_1, \ldots, W_k]_A) \). So, each \( V(y_i) \) is a functionally open extension of \( V(y_i) \) and all homomorphisms \( \pi_k(V(y_i)) \rightarrow \pi_k([W_0, W_1, \ldots, W_k]_A) \) are trivial. Denote the family \( \{V(y_i) : i = 1, 2, \ldots\} \) by \( \tilde{\gamma}([W_0, W_1, \ldots, W_k]_A) \). We say that a set \( A \subset B \) is admissible if the following holds:

1. \( q^{-1}_A(q_A(V)) = V \) for all \( V \in \tilde{\gamma}([W_0, W_1, \ldots, W_k]_A) \) and all finitely many elements \( W_0, W_1, \ldots, W_k \) of \( \mathcal{B}_A \) satisfying condition \( (w) \);
2. \( p^{-1}_A(p_A(V)) = V \) for all \( V \in \gamma(q^{-1}_A(W) \cap X) \) and \( W \in \mathcal{B}_A \), where \( p_A : X \rightarrow X_A \) is the restriction \( q_A(X) \).

Recall that for any functionally open set \( U \) in \( \mathbb{I}^B \) (resp., in \( X \)) there is a countable set \( s(U) \subset B \) such that \( q^{-1}_{s(U)}(q_{s(U)}(U)) = U \) (resp., \( p^{-1}_{s(U)}(p_{s(U)}(U)) = U \)).

**Claim 2.** For any set \( A \subset B \) there exists an admissible set \( C \subset B \) of cardinality \( |A|.\aleph_0 \) containing \( A \).

We construct by induction sets \( A = A_0 \subset A_1 \subset \ldots \subset A_k \subset A_{k+1} \subset \ldots \) of cardinality \( |A|.\aleph_0 \) such that:

1. \( s(V) \subset A_{k+1} \) for all \( V \in \gamma(q^{-1}_A(W) \cap X) \) and all \( W \in \mathcal{B}_{A_k} \);
2. \( s(V) \subset A_{k+1} \) for all \( V \in \tilde{\gamma}([W_0, W_1, \ldots, W_m]_A) \) and all finitely many \( W_0, W_1, \ldots, W_m \in \mathcal{B}_{A_k} \) satisfying condition \( (w) \).

The construction follows from the fact that the cardinality of each base \( \mathcal{B}_{A_k} \) is \( |A|.\aleph_0 \) and the families \( \gamma(q^{-1}_A(W) \cap X) \) and \( \tilde{\gamma}([W_0, W_1, \ldots, W_m]_A) \) are countable provided \( W \), \( W_0 \), \( W_1 \), \ldots, \( W_m \) are elements of \( \mathcal{B}_{A_k} \). It is easily seen that the set \( C = \bigcup_{k=1}^{\infty} A_k \) is as required.

**Claim 3.** \( X_A \) is an \( \text{ALC}^\infty \)-space for every admissible set \( A \subset B \).

Let \( y \in U \), where \( U \) is open in \( X_A \). Take \( x \in X \) and \( W_0 \in \mathcal{B}_A \) containing \( y \) such that \( W_0 \cap X_A \subset U \) and \( q_A(x) = y \). Then \( x \) belongs to some \( V_x \in \gamma(q^{-1}_A(W_0) \cap X) \). Since \( s(V_x) \subset A \) (recall that \( A \) is admissible), \( p^{-1}_A(p_A(V_x)) = V_x \). So, \( V = p_A(V_x) \) is a functionally open
in $X_A$ neighborhood of $y$, which is contained in $U$. Take any open
extension $\tilde{U} \subset \mathbb{A}^1$ of $U$, and finitely many $W_1, \ldots, W_k \in \mathcal{B}_A$ satisfying
$W_0 \cap X_A \subset \bigcup_{i=1}^{i=k} W_i \subset \tilde{U}$. Since $V_x \in \gamma(q_A^{-1}(W_0) \cap X)$, the set $\tilde{V}_x = \lambda(x, q_A^{-1}(W_0) \cap X, [W_0, W_1, \ldots, W_k]_A)$ is a functionally open extension of
$V_x$ with $\tilde{V}_x \in \gamma([W_0, W_1, \ldots, W_k]_A)$. Then $s(\tilde{V}_x) \subset A$ and $\tilde{V} = q_A(\tilde{V}_x)$ is a
functionally open in $\mathbb{A}^1$ extension of $V$. We are going to show that all homomorphisms $\pi_k(\tilde{V}) \to \pi_k(\tilde{U})$, $k \leq n$, are trivial. Indeed, every map $f: \mathbb{A} \to \tilde{V}$ can be lifted to a map $g: \mathbb{A} \to \tilde{V}$ because $q_A^{-1}(\tilde{V}) = \tilde{V}_x$ and $q_A$ is a soft map. Recall that $\tilde{V}_x$ belongs to $\gamma([W_0, W_1, \ldots, W_k]_A)$, so $g$ can be extended to a map $\tilde{g}: \mathbb{B}^{k+1} \to [W_0, W_1, \ldots, W_k]_A$. Finally, $q_A \circ \tilde{g} : \mathbb{B}^{k+1} \to q_A([W_0, W_1, \ldots, W_k]_A) \subset \tilde{U}$ is an extension of $f$. This completes the proof of Claim 3.

Claim 4. Let $A_2 \subset A_1$ be admissible subsets of $B$. Then each fiber
of $p_{A_1}^A : X_{A_1} \to X_{A_2}$ is $UV^n$.

Let $x \in X_{A_2}$ and $U \subset \mathbb{A}^1$ be an open set containing $F = (p_{A_2}^{A_1})^{-1}(x)$. Take $W_0 \in \mathcal{B}_{A_2}$ with $x \in W_0$ and $(p_{A_2}^{A_1})^{-1}(W_0) \cap X_{A_2} \subset U$. So,
$(q_{A_2}^{A_1})^{-1}(W_0) \cap X_{A_1} \subset U$. Next, choose $W_1, \ldots, W_k \in \mathcal{B}_{A_2}$ such that
$(q_{A_2}^{A_1})^{-1}(W_0) \cap X_{A_1} \subset \bigcup_{i=1}^{i=k} W_i \subset U$. Obviously, $W_0 = (q_{A_2}^{A_1})^{-1}(W_0) \in \mathcal{B}_{A_1}$ and $W_0 \cap X_{A_1} \subset \bigcup_{i=1}^{i=k} W_i$. Let $y \in X$ with $q_{A_2}(y) = x$. Then $y$
belongs to some $V_y \in \gamma(q_{A_1}^{A_1}(W_0) \cap X)$. Because $q_{A_1}^{A_1}(W_0) = q_{A_2}^{A_1}(W_0)$ and $A_2$ is admissible, we have $s(V_y) \subset A_2$. Hence, $p_{A_1}^{A_2}(p_{A_2}(V_y)) = V_y$ and
$p_{A_2}^{A_1}(x) = p_{A_1}^{A_1}(F) \subset V_y$. Then $\tilde{V}_y = \lambda(y, q_{A_1}^{A_1}(W_0) \cap X, [W_0, W_1, \ldots, W_k]_{A_1})$
is a functionally open extension of $V_y$ and $\tilde{V}_y \in \gamma([W_0, W_1, \ldots, W_k]_{A_1})$. Because $s(\tilde{V}_y) \subset A_1$, $q_{A_1}^{A_1}(q_{A_1}(\tilde{V}_y)) = \tilde{V}_y$ and $V = q_{A_1}(\tilde{V}_y)$ is an open
subset of $\mathbb{A}^1$ such that $F \subset V \subset \bigcup_{i=1}^{i=k} W_i \subset U$. Then, as in the proof
of Claim 3, we can show that the inclusion $V \hookrightarrow U$ generates trivial homomorphisms $\pi_k(V) \to \pi_k(U)$. Hence, $F$ is $UV^n$.

Claim 5. The union of any increasing sequence of admissible subsets
of $B$ is also admissible.

This claim follows directly from the definition of admissible sets.

Now we can complete the proof of Theorem 3.1. According to Claim
2 and Claim 5, the set $B$ is covered by a family $\mathcal{S}$ of countable sets such that
$\mathcal{S}$ is stable with respect to countable unions. Then, by Claim 3,
each $X_A$, $A \in \mathcal{S}$, is a metric $ALC^n$-compactum. Hence, Proposition 2.7
yields that all spaces $X_A$, $A \in \mathcal{S}$, are $LC^n$. Moreover, the projections $p_{A_1}^A$ are $UV^n$-maps for any $A_1, A_2 \in \mathcal{S}$ with $A_2 \subset A_1$. Because
the set $B$ is admissible, it follows from Claim 4 that the limit projections
Proof. We embed $X$ in some $\mathbb{R}^B$ and let $A_0 \subset B$ be a countable set. According to the factorization theorem of Bogatyi-Smirnov [2, Theorem 3], there is a metric compactum $Y_1$ and maps $g_1: X \rightarrow Y_1$ such that $p_{A_0} = h_1 \circ g_1$ and all fibers of $g_1$ are $UV^n$-sets in $X$. Then, by [8, Theorem 5.4], $Y_1$ is $LC^n$. Since $g_1$ depends on countably many coordinates, there is a countable set $A_1 \subset B$ containing $A_0$ and a map $f_1: X_{A_1} \rightarrow Y_1$ such that $f_1 \circ p_{A_1} = g_1$. In this way we construct countable sets $A_k \subset A_{k+1} \subset B$ and $LC^n$ metric compacta $Y_k$ together with maps $g_k: X \rightarrow Y_k$, $f_k: X_{A_k} \rightarrow Y_k$ and $h_k: Y_k \rightarrow X_{A_{k-1}}$ such that $g_k = f_k \circ p_{A_k}$, $p_{A_{k-1}} = h_k \circ g_k$ and the fibers of each $g_k$ are $UV^n$-sets in $X$. Let $A$ be the union of all $A_k$. Then $X_A$ is the limit space of the inverse sequence $S = \{Y_k, s^{k+1}_k = f_k \circ h_k\}$. According to [8, Theorem 5.3], for all open sets $U \subset Y_k$ the group $\pi_m(U)$ is isomorphic to $\tau_m(g^{-1}_k(U))$, $m = 0, 1, ..., n$. This property of the maps $g_k$ implies that any $s^{k+1}_k: Y_{k+1} \rightarrow Y_k$ is an $UV^n$-map. Hence, by Theorem 3.1, $X_A$ is an $ALC^n$-compactum (as the limit of an inverse sequence of metric $LC^n$-compacta and bounding $UV^n$-maps). Finally, by Theorem 2.7, $X_A$ is also an $LC^n$-space. Moreover, for any $y \in X_A$ we have $p_A^{-1}(y) = \bigcap_{k \geq 1} g^{-1}_k(y_k)$, where $y_k = s_k(y)$ with $s_k: X_A \rightarrow Y_k$ being the projections of $S$. Because all $g^{-1}_k(y_k)$ are $UV^n$-sets in $X$, so is the set $p_A^{-1}(y)$. Therefore, every countable subset $A_0$ of $B$ is contained in an element of the family $\mathcal{A}$ consisting of all countable sets $A \subset B$ such that $X_A$ is $LC^n$ and the fibers of the map $p_A$ are $UV^n$-sets in $X$. It is easily seen that the union of an increasing sequence of elements of $\mathcal{A}$ is again from $\mathcal{A}$, and that $p_{A_0}^C: X_C \rightarrow X_A$ is an $UV^n$-map for all $A, C \in \mathcal{A}$ with $A \subset C$. So, the inverse system $\{X_A, p_{A_0}^C, A, C \in \mathcal{A}\}$ is $\sigma$-complete and consists of metric $LC^n$-compacta and $UV^n$-bounding maps. Then by Theorem 3.1, $X$ is $ALC^n$ (observe that the proof of the "if" part of Theorem 3.1 does not need the assumption that all limit projections are $UV^n$-maps).

Theorem 3.1 shows that the class of $ALC^n$-compacta is adequate to the class of $UV^n$-maps. Next theorem provides another classes of compacta adequate to all continuous maps.

**Theorem 3.3.** A compactum $X$ is a cell-like (resp., $UV^n$) space if and only if $X$ is the limit space of a $\sigma$-complete inverse system consisting of cell-like (resp., $UV^n$) metric compacta.
Proof. Suppose $X$ is a cell-like compactum. Because this means that $X$ has a shape of a point, we can apply Corollary 8.4.8 from [3] stating that if $\varphi$ is a shape isomorphism between the limit spaces of two $\sigma$-complete inverse systems $\{X_\alpha, p_\alpha^\beta, \alpha \in A\}$ and $\{Y_\alpha, q_\alpha^\beta, \alpha \in A\}$ of metric compacta, then the set of those $\alpha \in A$ for which there exist shape isomorphisms $\varphi_\alpha : X_\alpha \to Y_\alpha$ satisfying $\text{Sh}(q_\alpha) \circ \varphi = \varphi_\alpha \circ \text{Sh}(p_\alpha)$ is cofinal and closed in $A$. So, according to this corollary, $X$ is the limit space of a $\sigma$-complete inverse system consisting of metric cell-like compacta. In case $X$ is an $UV^n$-compactum, it has an $n$-shape of a point (this notion was introduced by Chigogidze in [4]), and the above arguments apply.

Suppose now that $X$ is the limit space of a $\sigma$-complete inverse system $\{X_\alpha, p_\alpha^\beta, \alpha \in A\}$ such that all $X_\alpha$ are metric cell-like compacta. As in the proof of Theorem 3.1, we can embed $X$ in a Tychonoff cube $\mathbb{I}^B$, where $B$ is the union of countable sets $B_\alpha$, $\alpha \in A$, such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$, $B_\gamma = \bigcup\{B_\gamma(k) : k = 1, 2, \ldots\}$ for any chain $\gamma(1) < \gamma(2) < \ldots$ with $\gamma = \sup\{\gamma(k) : k \geq 1\}$, and each $p_\alpha^\beta : X_\beta \to X_\alpha$ is the restriction of the projection $q_\alpha^\beta : \mathbb{I}^B_\beta \to \mathbb{I}^B_\alpha$. Then $X_\alpha = q_\alpha(X) \subset \mathbb{I}^B_\alpha$ with $q_\alpha$ being the projection from $\mathbb{I}^B$ onto $\mathbb{I}^B_\alpha$. If $U$ is a neighborhood of $X$ in $\mathbb{I}^B$, there is $\alpha$ and an open set $U_\alpha$ in $\mathbb{I}^B_\alpha$ such that $q_\alpha^{-1}(U_\alpha) \subset U$. Since $X_\alpha$ is a cell-like space, there exists a closed neighborhood $V_\alpha \subset \mathbb{I}^B_\alpha$ of $X_\alpha$ contractible in $U_\alpha$. Using that $q_\alpha$ is a soft map, we conclude that $q_\alpha^{-1}(V_\alpha)$ is contractible in $q_\alpha^{-1}(U_\alpha)$. Similarly, we can show that any limit space of a $\sigma$-complete inverse system of metric $UV^n$-compacta is also an $UV^n$-compactum. \qed

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