Interacting 2D Field-Theoretic Model for Hodge Theory

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Abstract: We take up the Stückerberg-modified version of the two (1+1)-dimensional (2D) Proca theory, in interaction with the Dirac fields, to study its various continuous and discrete symmetry transformations and show that this specific interacting 2D field-theoretic model provides a tractable example for the Hodge theory because its symmetries (and corresponding conserved charges) provide the physical realizations of the de Rham cohomological operators of differential geometry at the algebraic level. The physical state of this theory is chosen to be the harmonic state (of the Hodge decomposed state) in the quantum Hilbert space which is annihilated by the conserved and nilpotent (anti-)BRST as well as (anti-)co-BRST charges. A physical consequence of this study is an observation that the 2D anomaly, at the quantum level, does not lead to any problem as far as the consistency and unitarity of our present 2D theory is concerned. In other words, our present 2D field-theoretic model is amenable to particle interpretation despite the presence of the local chiral symmetry (which is associated with the nilpotent (anti-)co-BRST symmetry transformations) besides the presence of the nilpotent (anti-)BRST symmetries (which are connected with the local gauge symmetry). The physicality condition with the (anti-)co-BRST charges implies that the 2D anomaly term is trivial in our present theory. Hence, our 2D theory is consistent, unitary and amenable to particle interpretation.

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1 Introduction

Mathematics is the language of all the key branches of modern-day scientific pursuits. Its creative usefulness and glorious presence are undeniable truths in the domain of theoretical physics. The applications and realizations of the mathematical ideas and concepts in the realm of theoretical physics have played very important and decisive roles in the development of the latter. In this context, it is pertinent to point out that it has been possible to show, within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism [1-4], that the Abelian $p$-form ($p = 1, 2, 3,...$) gauge theories in $D = 2p$ dimensions of spacetime provide the field-theoretic examples of Hodge theory (see, e.g. [5-7] and references therein) where the discrete and continuous symmetries (and the conserved charges) have been able to provide the physical realizations of the de Rham cohomological operators of differential geometry [8-12] at the algebraic level. Furthermore, it has been found that some of the $\mathcal{N} = 2$ supersymmetric (SUSY) quantum mechanical models are also examples of Hodge theory (see, e.g. [13-17]) at the algebraic level (without the central extension(s)). Whereas the above Abelian $p$-form ($p = 1, 2, 3,...$) gauge theories are massless theories due to the presence of gauge symmetries, the $\mathcal{N} = 2$ SUSY quantum mechanical models are massive and they respect two nilpotent SUSY transformations which are not absolutely anticommuting in nature. However, the (anti-)BRST symmetries (corresponding to the gauge symmetry transformations of the Abelian $p$-form gauge theory) are nilpotent of order two but absolutely anticommuting in nature. These properties are very sacrosanct.

Against the backdrop of the above, we have shown that the St"uckelberg-modified 2D Proca as well as the 4D Abelian 2-form gauge theories (without any interaction with matter fields) provide the massive models of Hodge theories where the mass and the quantum (anti-)BRST as well as (anti-)co-BRST symmetries co-exist together [18-20]. The central purpose of our present investigation is to establish that the 2D modified Proca theory, in interaction with the Dirac fields, provides a field-theoretic example of Hodge theory where the continuous symmetry transformations (and corresponding conserved Noether charges) provide the physical realizations of the cohomological operators of differential geometry and the existence of a couple of discrete symmetry transformations is at the heart of the physical realizations of the Hodge duality $\ast$ operation of differential geometry. Thus, our present 2D theory is a novel example of a massive and interacting model of Hodge theory within the framework of BRST formalism. The existence of the nilpotent (anti-)BRST and (anti-)co-BRST symmetries provides the physical realizations of the nilpotent (co-) exterior derivatives of differential geometry. The (anti-)BRST symmetry transformations correspond to the local gauge symmetry in the theory. On the other hand, it is the local chiral symmetry transformation (in the massless limit of the Dirac fields) that leads to the existence of a proper set of (anti-)co-BRST transformations in the theory.

The study of the field-theoretic models of Hodge theory, within the ambit of BRST formalism, is physically important on the following grounds. First, it has been possible to show that the (non-)Abelian 1-form gauge theories, in two $(1+1)$-dimensions of spacetime, provide a new [21] type of topological field theory (TFT) that captures a few key aspects of the Witten-type TFT [22] and some salient features of the Schwarz-type TFT [23]. Second, we have been able to establish that the 4D Abelian 2-form and 6D Abelian 3-form gauge theories are tractable field-theoretic examples of Hodge theory and they provide the models
of quasi-TFTs where the topological invariants and their perfect recursion relations exist \[5, 24, 25\]. Third, it has been established that the massive models of Hodge theory (without any interaction with matter fields) allow the existence of fields with negative kinetic terms \[18-20\] which have been called as the “phantom” fields in the domain of cosmology. Such kinds of exotic fields have played important roles in the context of cyclic, bouncing and self-accelerated models of Universe (see, e.g. \[26-30\]). In our earlier works \[18-20\], we have laid the emphasis on the relevance of such fields (with negative kinetic terms) in the context of providing a set of possible candidates for the dark matter and dark energy within the framework of quantum field theory. Finally, in our present endeavor, we establish that the massive and interacting 2D modified version of the Proca theory with the Dirac fields shows that the vector and axial-vector currents (and their corresponding charges) are conserved together because the 2D anomaly term is trivial when we choose the physical state as the harmonic state (of the Hodge decomposed state in the total quantum Hilbert space) which is annihilated by the conserved and nilpotent BRST and co-BRST charges together.

The following key factors have been at the heart of our present investigation. First, in the physical four \((3 + 1)\)-dimensions of QED with the Dirac fields, it has been well-established that the vector and axial-vector currents can not be conserved together due to the presence of the anomaly term \((\sim \vec{E} \cdot \vec{B})\) which is a pseudo-scalar constructed with the polar-vector electric field \((\vec{E})\) and the axial-vector magnetic field \((\vec{B})\) (see, e.g. \[31\] for details). However, in the case of 2D, there is presence of only the electric field \((E)\) which becomes a pseudo-scalar and the anomaly term. We have been motivated to check the status of this anomaly term within the framework of BRST formalism. Our present endeavor is an attempt in that direction. Second, we have been working on the idea of dual-BRST symmetry for quite sometime and we have shown \[32, 33\] that the 2D QED with the Dirac fields (interacting with a massless photon) leads to the definition of a conserved co-BRST charge which contains the pseudo-scalar electric field \((E)\). We have been curious to establish a connection between the co-BRST charge and the anomaly term \((E)\) in 2D. We have accomplished this goal in our present endeavor. Finally, this is, for the first time, we have proven a massive as well as an interacting gauge theory to be an example of the Hodge theory. We have already established the existence of the field-theoretic models of Hodge theory with mass and massless gauge fields in our earlier works \[18-20, 5-7\]. Our present 2D theory is also a perfect model of a duality invariant theory \[34\] because of the existence of the discrete symmetry transformations (see, Eq. (10) below).

Our present interacting theory respects six continuous and a couple of discrete symmetry transformations cf. Eq. (10) below. Out of these, at the most economical level, three symmetries are very fundamental. These are the BRST, co-BRST and discrete symmetries. The conserved and nilpotent BRST charge annihilates the physical (i.e. harmonic) state which implies that the operator forms of the first-class constraints must annihilate the physical state. The decisive presence of the discrete symmetry transformations [cf. Eq. (10) below] ensures that our theory is a perfect model (cf. Secs. 5, 6 below) of the duality invariant theory \[34\]. Hence, the dual versions of the above first-class constraints must also annihilate the physical state. Thus, the annihilation of the physical state by the first-class constraints and their dual is very sacrosanct and it must be respected at the tree as well as at the loop-level diagrams. These dual versions of constraints [cf. Eq. (47) below] emerge
when we demand that the physical state must be annihilated by the co-BRST charge which is nilpotent but conserved only at the tree-level diagrams because it is derived from the co-BRST current [cf. Eq. (13) below] which is also conserved at the tree-level diagrams. However, it is the sanctity of the requirements of a perfect duality-invariant theory that (i) the 2D anomaly term, and (ii) its invariance under the time-evolution, must also annihilate the physical state of our theory [cf. Eq. (48) below]. This implies, primarily, that the vector and axial-vector currents are found to be conserved together for our 2D interacting theory with the Dirac fields. Hence, the 2D anomaly term becomes trivial.

The theoretical contents of our present endeavor are organized as follows. First, to set the notations and convention, we recapitulate the bare essentials of the (anti-)BRST symmetries of the D-dimensional Lagrangian density for the modified version of Proca (i.e. massive Abelian 1-form $A^{(1)} = d x^\mu A_\mu$) gauge theory where there is a coupling between the gauge field $A_\mu$ (with $\mu = 0, 1, 2 \ldots D - 1$) and the conserved current constructed with the fermionic Dirac fields in Sec. 2. Our Sec. 3 is devoted to the discussion on the nilpotent (anti-)co-BRST symmetry transformations for the same [i.e. (anti-)BRST invariant] Lagrangian density in two (1+1)-dimensions of spacetime. We derive the bosonic and ghost-scale symmetry transformations in our Sec. 4. Our Sec. 5 deals with the derivation of the algebras obeyed by the above (anti-)BRST, (anti-)co-BRST, a unique bosonic and the ghost-scale continuous symmetries and corresponding conserved charges. In our Sec. 6, we elaborate on the connection between the algebraic structures of the above symmetries (and corresponding conserved charges) and the de Rham cohomological operators of differential geometry. In Sec. 7, we dwell a bit on the irrelevance of the 2D anomaly term as far as our discussion on the importance of the (anti-)dual-BRST [i.e. (anti-)co-BRST] transformations (nd corresponding charges) is concerned. Finally, in Sec. 8, we summarize our results and lay emphasis on the physical importance of our present work.

In our Appendices A, B and C, we perform some explicit computations which corroborate some of the claims that have been made in the main body of our text.

Convention and Notations: We adopt the background 2D Minkowskian flat spacetime manifold that is endowed with the metric tensor $\eta_{\mu\nu} = \text{diag} (+1, -1)$ so that the dot product $(P \cdot Q)$ between the two non-null vectors $P_\mu$ and $Q_\mu$ is defined as: $P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i$ where the Greek indices $\mu, \nu, \lambda \ldots = 0, 1$ correspond to the spacetime directions and $i, j, k, \ldots = 1$ stand for the space direction. The antisymmetric $(\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu})$ Levi-Civita tensor $\varepsilon_{\mu\nu}$ is chosen such that $\varepsilon_{01} = +\varepsilon^{10} = 1$ and $\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2!$, $\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -1! \delta_5$, etc. We denote the (anti-)BRST symmetry transformations by $s_{(a)b}$ and (anti-)co-BRST symmetry transformations by $s_{(a)d}$ in the whole body of our text. We take the convention of the left-derivative w.r.t. the fermionic fields ($\bar{\psi}, \psi, \bar{C}, C$) which anticommute among themselves but commute with all the bosonic fields of our theory. We choose the $2 \times 2$ Dirac $\gamma$-matrices in terms of the traceless and hermitian $2 \times 2$ Pauli-matrices as: $\gamma_0 = \sigma_1, \gamma_1 = i \sigma_2$ which satisfy $\{\gamma_\mu, \gamma_\nu\} = 2 \eta_{\mu\nu}$ and lead to the definition of $\gamma_5 = \gamma_0 \gamma_1 \equiv -\gamma_3$ which is the diagonal matrix [like the $\gamma_5$ matrix in the Weyl representation (see, e.g. [35] for details) of Dirac’s matrices] and hermitian (i.e. $\gamma_5^\dagger = \gamma_5$). We also have: $(\gamma_5)^2 = I, \gamma_5 \gamma_\mu = \varepsilon_{\mu\nu} \gamma^\nu, \gamma_5 = -(1/2!) \varepsilon_{\mu\nu} \gamma^\mu \gamma^\nu$ and $\{\gamma_5, \gamma_\mu\} = 0$ where $\gamma_0 = \gamma^0$ and $\gamma^1 = -\gamma_1$ due to the choice of the signatures of our 2D flat metric tensor.
Standard Definitions: We recapitulate the following definitions of the differential geometry [8-12] which will be useful in our discussions (as far as the connection between the physical aspects and mathematical ideas of our present endeavor are concerned).

(i) The de Rham Cohomological Operators: The set of three operators \((d, \delta, \Delta)\) constitute the de Rham cohomological operators (defined the D-dimensional compact manifold without a boundary) where \(d = d x^\mu \partial_\mu\) (with \(\mu = 0, 1, 2...D - 1\)) is the exterior derivative, \(\delta = \mp \ast d \ast\) is the co-exterior (or dual-exterior) derivative and \(\Delta = (d+\delta)^2\) is the Laplacian operator. Here \(\ast\) is the Hodge duality operation on the above-mentioned compact manifold without a boundary. For the even dimensions of spacetime \(\delta = -\ast d \ast\). These operators obey the algebra: \(d^2 = \delta^2 = 0\), \(\Delta = (d + \delta)^2 = d \delta + \delta d = \{d, \delta\}\), \([\Delta, d] = 0\), \([\Delta, \delta] = 0\) which is popularly known as the Hodge algebra (that is not a Lie algebra).

(ii) The Hodge Decomposition Theorem: On a compact manifold without a boundary, any arbitrary form \(f_n\) (of non-zero degree \(n\)) can be uniquely written as the sum of the harmonic form \((h_n)\), an exact form \((dg_{n-1})\) and a co-exact form \((\delta k_{n+1})\) as: \(f_n = h_n + dg_{n-1} + \delta k_{n+1}\) where the harmonic form \(h_n\) satisfies: \(dh_n = \delta h_n = 0\). In other words, we have \(\Delta h_n = 0\). Thus, the harmonic form \((h_n)\) is closed and co-closed together.

2 Preliminaries: (Anti-)BRST Symmetry Invariant Lagrangian Density in Any Arbitrary Dimension

We begin with the following D-dimensional (anti-)BRST invariant interacting Lagrangian density in the ’t Hooft gauge (see, e.g. [36] for details)

\[
\mathcal{L}_B = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m A_\mu \partial^\mu \phi + \bar{\psi} (i \gamma^\mu \partial_\mu - m') \psi
- e \bar{\psi} \gamma^\mu A_\mu \psi + B (\partial \cdot A + m \phi) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C. \tag{1}
\]

where the kinetic term for the massive photon has its origin in the exterior derivative \((d = d x^\mu \partial_\mu\) with \(d^2 = 0\)) of the differential geometry [8-12] because we note that \(d A^{(1)} = \frac{1}{2} (d x^\mu \wedge d x^\nu) F_{\mu\nu}\). Here \(A^{(1)} = d x^\mu A_\mu\) (with \(\mu = 0, 1, 2...D - 1\)) defines the massive vector field \((A_\mu)\) with rest mass \(m\) and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the field strength tensor. The pure scalar field \(\phi\) has been invoked because of Stückelberg’s formalism and it has positive kinetic term. The fermionic \((\psi^2 = \bar{\psi}^2 = 0, \bar{\psi} \bar{\psi} + \bar{\psi} \psi = 0)\) Dirac fields (with the rest mass \(m'\)) have interaction with the massive gauge field \(A_\mu\) through minimal interaction \((- e \bar{\psi} \gamma^\mu \psi A_\mu\) coupling. The gauge-fixing term for the gauge field owes its origin to the exterior derivative \((\delta = -\ast d \ast)\) because \(\delta A^{(1)} = (\partial \cdot A)\) and we have added the \(m \phi\) term to it on the dimensional ground. In our theory, \(B\) is the Nakanishi-Lautrup auxiliary field and the fermionic \((C^2 = \bar{C}^2 = 0, C \bar{C} + \bar{C} C = 0)\) (anti-)ghost fields \((\bar{C})\) \(C\) have been invoked to preserve the unitarity in the theory at any order of perturbative computations. The above Lagrangian density \((\mathcal{L}_B)\) respects the following off-shell nilpotent \([s^2_{(a)b} = 0]\) and absolutely
anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) (anti-)BRST transformations \([s_{(a)b}]\), namely:

\[
s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} \phi = m \bar{C}, \quad s_{ab} \psi = -i e \bar{C} \psi, \quad s_{ab} \bar{\psi} = -i e \psi \bar{C},
\]
\[
s_{ab} \bar{C} = 0, \quad s_{ab} C = -i B, \quad s_{ab} B = 0,
\]
\[
s_b A_\mu = \partial_\mu C, \quad s_b \phi = m C, \quad s_b \psi = -i e C \psi, \quad s_b \bar{\psi} = -i e \psi C,
\]
\[
s_b C = 0, \quad s_b \bar{C} = i B, \quad s_b B = 0,
\]

because the Lagrangian density \((\mathcal{L}_B)\) transforms to

\[
s_{ab} \mathcal{L}_B = \partial_\mu [B \partial^\mu \bar{C}], \quad s_b \mathcal{L}_B = \partial_\mu [B \partial^\mu C],
\]

thereby rendering the action integral \(S = \int d^D x \mathcal{L}_B\) invariant [i.e. \(s_{(a)b} S = 0\)] for the physical fields that vanish-off as \(x \to \pm \infty\) due to Gauss’s divergence theorem. As a consequence of the celebrated Noether’s theorem, we have the conserved currents \([J_{(a)b}^\mu]\) corresponding to the off-shell nilpotent, absolutely anticommuting, infinitesimal and continuous (anti-)BRST symmetry transformations \(s_{(a)b}\) as:

\[
J_{(a)b}^\mu = B \partial^\mu \bar{C} + m \bar{C} \partial^\mu \phi - F^{\mu\nu} \partial_\nu \bar{C} - m^2 A^\mu \bar{C} - e \bar{\psi} \gamma^\mu \bar{C} \psi,
\]
\[
J_{(b)}^\mu = B \partial^\mu C + m C \partial^\mu \phi - F^{\mu\nu} \partial_\nu C - m^2 A^\mu C - e \psi \gamma^\mu C \bar{\psi}.
\]  

The conservation law \([\partial_\mu J_{(a)b}^\mu = 0]\) can be proven readily by using the following Euler-Lagrange (EL) equations of motion (EOMs) from the Lagrangian density \((\mathcal{L}_B)\), namely:

\[
\begin{align*}
\partial_\mu F^{\mu \nu} + m^2 A^\nu - \partial^\nu B - m \partial^\nu \phi - e \bar{\psi} \gamma^\nu \psi &= 0, \\
(\Box + m^2) C &= 0
\end{align*}
\]

\(\Box = \partial^\mu \partial_\mu\) is the space Laplacian.

The conserved (anti-)BRST charges \((Q_{ab} = \int d^{D-1} x J_{(a)b}^0, \quad Q_b = \int d^{D-1} x J_{(b)}^0)\) are

\[
Q_{ab} = \int d^{D-1} x \left[ B \dot{\bar{C}} + m \dot{\phi} \bar{C} - F^{0i} \partial_i \bar{C} - m^2 A^0 \bar{C} - e \bar{\psi} \gamma^0 \bar{C} \psi \right]
\]
\[
\equiv \int d^{D-1} x (B \dot{\bar{C}} - \dot{B} \bar{C}),
\]
\[
Q_b = \int d^{D-1} x \left[ B \dot{C} + m \dot{\phi} C - F^{0i} \partial_i C - m^2 A^0 C - e \psi \gamma^0 C \bar{\psi} \right]
\]
\[
\equiv \int d^{D-1} x (B \dot{C} - \dot{B} C),
\]

where the final expressions for the concise form of the conserved charges have been obtained due to the following EL-EOM that has been chosen from (5), namely,

\[
\partial_i F^{0i} = m^2 A^0 - \dot{B} - m \dot{\phi} - e \bar{\psi} \gamma^0 \psi.
\]

In other words, first of all, we have expressed \((- F^{0i} \partial_i C \) and \((- F^{0i} \partial_i C \) as \([[(\partial_i F^{0i}) C - \partial_i (F^{0i} \bar{C})]\) and \([(\partial_i F^{0i}) C - \partial_i (F^{0i} C)]\), respectively, and applied the Gauss divergence theorem to drop the total space derivative terms. It is straightforward to note that the
conserved (anti-)BRST charges are off-shell nilpotent \([Q_{(a)b}^2 = 0]\) of order two and absolutely anticommuting \((Q_b Q_{ab} + Q_{ab} Q_b = 0)\) in nature due to the following observations in terms of the conserved charges \((6)\) and the nilpotent (anti-)BRST transformations \((2)\), namely;

\[
\begin{align*}
  s_b Q_b &= -i \{Q_b, Q_b\} = 0 & \Rightarrow & Q_b^2 = 0, \\
  s_{ab} Q_{ab} &= -i \{Q_{ab}, Q_{ab}\} = 0 & \Rightarrow & Q_{ab}^2 = 0, \\
  s_b Q_{ab} &= -i \{Q_{ab}, Q_b\} = 0 & \Rightarrow & \{Q_{ab}, Q_b\} = 0, \\
  s_{ab} Q_b &= -i \{Q_b, Q_{ab}\} = 0 & \Rightarrow & \{Q_b, Q_{ab}\} = 0, \\
\end{align*}
\]

where the l.h.s. of the above relationships have been computed by directly applying the (anti-)BRST symmetry transformations [cf. Eq. \((2)\)] on the expressions for the nilpotent (anti-)BRST charges [cf. Eq. \((6)\)] where the basic principle behind the continuous symmetry transformations and their generators (as the conserved Noether charges) has been exploited to establish the off-shell nilpotency \((\{Q_{(a)b}^2 = 0\) and absolute anticommutativity \((\{Q_{b}, Q_{ab}\} = 0)\) of the conserved \(\dot{Q}_{(a)b} = 0\) (anti-)BRST charges \([Q_{(a)b}]\).

We wrap-up this section with the following crucial and decisive remarks. First, we note that the kinetic term of the gauge field \((-\frac{1}{4} F_{\mu \nu} F^{\mu \nu})\) remains invariant (i.e. \(s_{(a)b} F_{\mu \nu} = 0\)) under the (anti-)BRST symmetry transformations \([s_{(a)b}]\). Second, as pointed out earlier, the kinetic term has its origin in the exterior derivative of differential geometry. Third, we observe that the ghost number is increased by one by the BRST symmetry transformation when it applies on a generic field (e.g. \(s_b A_\mu = \partial_\mu C, s_b \bar{C} = i B, \text{etc.}\) in contrast to the anti-BRST symmetry transformations that decreases (e.g. \(s_{ab} A_\mu = \partial_\mu \bar{C}, s_{ab} C = -i B, \text{etc.}\) the ghost number by one. Fourth, the absolute anticommutativity (i.e. \(Q_b Q_{ab} + Q_{ab} Q_b = 0, s_b s_{ab} + s_{ab} s_b = 0\)) of the (anti-)BRST charges and the continuous fermionic \([s_{(a)b}^2 = 0]\) symmetry transformations ensure that the (anti-)BRST symmetries (and their corresponding conserved charges) can not be together identified with the exterior derivative of differential geometry. Fifth, the absolute anticommutativity property ensures that the nilpotent (anti-)BRST symmetry transformations are different from the \(N = 2\) SUSY symmetry transformations which are nilpotent but not absolutely anticommuting in nature. Sixth, it is evident that, for the identification of the co-exterior derivative, we have to invoke and discuss another kind of nilpotent symmetries for our theory.

3 (Anti-)Dual-BRST Symmetries: Modified Interacting 2D Proca Theory with Dirac Fields

As we have modified the mass term \(\left(\frac{m^2}{2} A_\mu A^\mu\right)\) by exploiting the theoretical tricks of the Stäckelberg formalism [36], in exactly similar fashion, we can modify the 2D kinetic term \((-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} = \frac{1}{2} E^2)\) of the gauge field by invoking a pseudo-scalar field \(\tilde{\phi}\) along with a linearizing Nakanishi-Lautrup type auxiliary field \(B\) as (see, e.g. [19] for details):

\[
\begin{align*}
  \mathcal{L}_B &= B (E - m \tilde{\phi}) - \frac{1}{2} B^2 + m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\
  &- m A_\mu \partial^\mu \phi + \bar{\psi} (i \gamma^\mu \partial_\mu - m') \psi - e \bar{\psi} \gamma^\mu A_\mu \psi + B (\partial \cdot A + m \phi) + \frac{1}{2} B^2 \\
  &- i \partial_\mu C \partial^\mu C + i m^2 C C.
\end{align*}
\]
It should be noted that the kinetic term \(-\frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}\) for the pseudo-scalar field \(\tilde{\phi}\) carries a negative sign with it. This is due to the fact that we have the following discrete symmetry in our theory (modulo total derivatives) provided we have the positive and negative signs for the kinetic terms for the pure scalar (\(\phi\)) and pseudo-scalar (\(\tilde{\phi}\)) fields, respectively, namely:

\[
A_\mu \rightarrow \mp i \varepsilon_{\mu\nu} A^\nu, \quad \phi \rightarrow \mp i \tilde{\phi}, \quad \tilde{\phi} \rightarrow \mp i \phi, \quad C \rightarrow \pm i \tilde{C}, \quad \tilde{C} \rightarrow \pm i C,
\]

\[
B \rightarrow \mp i B, \quad B \rightarrow \mp i \tilde{B}, \quad \psi \rightarrow \bar{\psi}, \quad \bar{\psi} \rightarrow \psi, \quad e \rightarrow \pm i e \gamma_5,
\]

\[(\partial \cdot A) \rightarrow \pm i E, \quad E \rightarrow \pm i (\partial \cdot A), \quad -m A_\mu \partial^\mu \phi \rightarrow +m E \tilde{\phi}.
\]  

(10)

We would like to mention that the mass term \((\frac{m^2}{2} A_\mu A^\mu)\) for the gauge field \(A_\mu\) remains invariant under \((A_\mu \rightarrow \mp i \varepsilon_{\mu\nu} A^\nu)\). Hence, the masses associated with \(\phi\) and \(\tilde{\phi}\) fields are the same [otherwise the discrete symmetry (10) will be violated]. We shall see later that the above two discrete symmetry transformations (10) are very important for our whole discussion because they provide the physical realizations of the Hodge duality * operation of differential geometry [8-12] as the analogue of relationship: \(\delta = \pm * d \pm * \) in terms of the off-shell nilpotent (anti-)BRST and (anti-)co-BRST transformations (cf. Sec. 5).

It is very interesting to point out that the 2D Lagrangian density (9) respects the following (anti-)co-BRST symmetry transformations \([s_{(a)d}],\) namely;

\[
s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C, \quad s_{ad} \tilde{\phi} = -m C, \quad s_{ad} \phi = i e C \gamma_5 \psi, \quad s_{ad} \bar{\psi} = -e \bar{\psi} \gamma_5 C,
\]

\[
s_{ad} \tilde{C} = i B, \quad s_{ad} C = 0, \quad s_{ad} B = s_{ad} \tilde{B} = s_{ad} \phi = 0, \quad s_{ad} E = \Box C,
\]

\[
s_{d} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \tilde{C}, \quad s_{d} \tilde{\phi} = -m \tilde{C}, \quad s_{d} \phi = i e \tilde{C} \gamma_5 \psi, \quad s_{d} \bar{\psi} = -e \bar{\psi} \gamma_5 \tilde{C},
\]

\[
s_{d} \tilde{C} = 0, \quad s_{d} C = -i B, \quad s_{d} B = s_{d} \tilde{B} = s_{d} \phi = 0, \quad s_{d} E = \Box \tilde{C},
\]  

(11)

because [for the massless \((m' = 0)\) fermions], we have the following

\[
s_{ad} \mathcal{L}_B = \partial_\mu \left[ B \partial^\mu C + m \varepsilon^{\mu\nu} (m C A_\nu + \phi \partial_\nu C) + m \tilde{\phi} \partial^\mu \tilde{C} \right],
\]

\[
s_{d} \mathcal{L}_B = \partial_\mu \left[ B \partial^\mu C + m \varepsilon^{\mu\nu} (m C A_\nu + \phi \partial_\nu C) + m \tilde{\phi} \partial^\mu \tilde{C} \right],
\]  

(12)

which render the action integral \(S = \int d^2 x \mathcal{L}_R\) invariant for the physical fields that vanish-off as \(x \rightarrow \pm \infty\) due to Gauss’s divergence theorem. According to Noether’s theorem, the above continuous symmetry transformations lead to the derivation of the Noether conserved currents \([J^{\mu}_{(r)}, r = d, ad]\) at the classical (i.e. tree Feynman diagram) level as:

\[
J^{\mu}_{(ad)} = B \partial^\mu C + m C \partial^\mu \tilde{\phi} - \varepsilon^{\mu\nu} \left[ B \partial_\nu C + m^2 C A_\nu + m \phi \partial_\nu C \right] + e \bar{\psi} \gamma^\mu \gamma_5 C \psi,
\]

\[
J^{\mu}_{(d)} = B \partial^\mu \tilde{C} + m \tilde{C} \partial^\mu \tilde{\phi} - \varepsilon^{\mu\nu} \left[ B \partial_\nu \tilde{C} + m^2 \tilde{C} A_\nu + m \phi \partial_\nu \tilde{C} \right] + e \bar{\psi} \gamma^\mu \gamma_5 \tilde{C} \psi.
\]  

(13)

The conservation law for the above currents requires that \(\partial_\mu J^{\mu}_{(5)} = \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) = 0\) which is true only at the tree-level Feynman diagram (when the massless limit of the Dirac fields are taken into account). These tree-level conserved currents lead to the definition of the
conserved \((\dot{Q}_{(r)} = 0\) with \(r = d, ad\)\) charges \(Q_{(r)} = \int d x J_{\mu}^{(r)}\) (with \(r = d, ad\)) as:

\[
Q_{ad} = \int d x \left[ B \dot{C} + m \dot{\phi} C + B \partial_1 C + m^2 A_1 C + m \phi \partial_1 C + e \bar{\psi} \gamma^0 \gamma_5 C \psi \right],
\]

\[
Q_d = \int d x \left[ B \dot{C} + m \dot{\phi} C + B \partial_1 C + m^2 A_1 C + m \phi \partial_1 C + e \bar{\psi} \gamma^0 \gamma_5 C \psi \right].
\]

It is worthwhile to mention here that the conservation law \((\partial_\mu J_{\mu}^{(r)} = 0, r = d, ad)\) for the Noether conserved currents (13) and conserved \((\dot{Q}_{(r)} = 0, r = d, ad)\) charges (14) can be readily proven, at the classical (i.e. tree Feynman diagram) level, by using the following EL-EOMs that are derived from the Lagrangian density (9):

\[
\varepsilon^{\mu \nu} [\partial_\nu B + m \partial_\nu \dot{\phi}] = m^2 A^\mu - \partial^\mu B - m \partial_\nu \phi - e \bar{\psi} \gamma^\mu \psi,
\]

\[
\square \dot{\phi} + m E - m B = 0, \quad \square \phi - m (\partial \cdot A) - m B = 0,
\]

\[
i (\partial_\mu \bar{\psi}) \gamma^\mu + m' \bar{\psi} + e \bar{\psi} \gamma^\mu A_\mu = 0, \quad i \gamma^\mu \partial_\mu \psi - m' \psi - e \gamma^\mu A_\mu \psi = 0,
\]

\[
(\square + m^2) C = 0, \quad (\square + m^2) \bar{C} = 0, \quad B = - (\partial \cdot A + m \phi)
\]

\[
B = (E - m \dot{\phi}) \equiv - (\varepsilon^{\mu \nu} \partial_\mu A_\nu + m \dot{\phi}).
\]

A close look and keen observation of these equations imply that we have: \((\square + m^2) \phi = 0, (\square + m^2) \dot{\phi} = 0, (\square + m^2) B = 0, (\square + m^2) \bar{B} = 0\). The last entry [i.e. \((\square + m^2) \bar{B} = 0\)] is true only at the classical level (i.e. tree-level Feynman diagrams) in the massless limit \((m' = 0)\) of the Dirac fields. This is due to the fact that we use: \(\partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) = 0\). We shall offer more comments on it in Sec. 7 where the 2D anomaly term will be discussed. It is also worthwhile to mention that in the derivation of \((\square + m^2) B = 0\), we have used the conservation of vector current (i.e. \(\partial_\mu J_{\mu}^{(m)} = e \partial_\mu \bar{\psi} \gamma^\mu \psi = 0\) that is true at the classical as well as quantum level. Furthermore, we can derive the simpler forms of the conserved charges \(Q_r (r = d, ad)\) using the Gauss’s divergence theorem and the following EL-EOM that has been taken from the top entry of Eq. (15), namely:

\[
\dot{B} = \partial_1 B + m (\partial_1 \phi) + e \bar{\psi} \gamma_1 \psi - m^2 A_1 - m \dot{\phi},
\]

which leads to the following concise expressions

\[
Q_{ad}^{(1)} = \int d x \left( B \dot{C} - \dot{\bar{B}} \bar{C} \right),
\]

\[
Q_d^{(1)} = \int d x \left( B \dot{C} - \dot{\bar{B}} \bar{C} \right).
\]

It is now crystal clear, from the (anti-)co-BRST symmetry transformations (11) and the above concise expressions for the conserved (anti-)co-BRST charges (17), that we have the following explicit relationships, namely:

\[
s_d Q_d^{(1)} = - i \{ Q_d^{(1)}, Q_d^{(1)} \} = 0 \quad \Rightarrow \quad [Q_d^{(1)}]^2 = 0,
\]

\[
s_{ad} Q_{ad}^{(1)} = - i \{ Q_{ad}^{(1)}, Q_{ad}^{(1)} \} = 0 \quad \Rightarrow \quad [Q_{ad}^{(1)}]^2 = 0,
\]

\[
s_d Q_{ad}^{(1)} = - i \{ Q_{ad}^{(1)}, Q_{ad}^{(1)} \} = 0 \quad \Rightarrow \quad Q_{ad}^{(1)} Q_d^{(1)} + Q_d^{(1)} Q_{ad}^{(1)} = 0,
\]

\[
s_d Q_{ad}^{(1)} = - i \{ Q_{ad}^{(1)}, Q_{ad}^{(1)} \} = 0 \quad \Rightarrow \quad Q_d^{(1)} Q_{ad}^{(1)} + Q_{ad}^{(1)} Q_d^{(1)} = 0.
\]
The above equations capture the off-shell nilpotency (i.e. \([Q^{(r)}_d]^2 = 0, \ r = d, ad\) and absolute anticommutativity (i.e. \(Q^{(d)}_d Q^{(d)}_d + Q^{(d)}_ad Q^{(d)}_d = 0\)) of the (anti-)co-BRST charges where we have exploited the theoretical strength of the continuous symmetry transformations and their generators as the conserved Noether charges. It is straightforward to compute the l.h.s. of Eq. (18) by directly applying the (anti-)co-BRST symmetry transformations (11) on the concise expressions for the conserved \((\dot{Q}^{(a)}_d) = 0\) (anti-)co-BRST charges that have been quoted in Eq. (17) to check the sanctity of Eq. (18).

We wrap-up this section with the following key and decisive remarks. First of all, we note that the (anti-)co-BRST symmetry transformations (11) are off-shell nilpotent (i.e. \(s^2_d = 0\)) of order two and they are absolutely anticommuting (i.e. \(s_d s_{ad} + s_{ad} s_d = 0\)) in nature. Second, we observe that the total gauge-fixing term [i.e. \((\partial A + m \phi)\)] remains invariant under the (anti-)co-BRST symmetry transformations [i.e. \(s(a)_d (\partial A) = 0, s(a)_d \phi = 0\)]. Third, as already mentioned earlier, the gauge-fixing term \((\partial A)\) of the gauge field \((A_\mu)\) owes its origin to the co-exterior derivative because \(\delta A^{(1)} = -\ast d \ast (d x^\mu A_\mu) = (\partial A)\). In the total gauge-fixing term \((\partial A + m \phi)\), it is evident that the term \(m \phi\) has been added on the dimensional ground (in the natural units where \(\hbar = c = 1\)). Fourth, to be precise, we have four forms of Lagrangian densities for our 2D theory as discussed thoroughly in our earlier work [19]. However, we have chosen only one for the sake of brevity in Eq. (9). We have provided a synopsis of the varieties of Lagrangian densities in our Appendix A. Fifth, the absolute anticommutativity \((s_d s_{ad} + s_{ad} s_d = 0)\) of the (anti-)co-BRST symmetry transformations (11) (and their intimate connection with the invariance of the gauge-fixing term owing its origin to the co-exterior derivative \(\delta = -\ast d \ast\)) demonstrates that only one of them can be identified with \(\delta\). Sixth, we note that the co-BRST symmetry transformations decrease the ghost number by one (i.e. \(s_d A_\mu = -\epsilon_{\mu \nu} \partial^\nu \bar{C}, s_d C = -i B, etc.\)). On the contrary, the anti-co-BRST symmetry transformations increase (i.e. \(s_{ad} A_\mu = -\epsilon_{\mu \nu} \partial^\nu \bar{C}, s_{ad} \bar{C} = i B, etc.\)) the ghost number by one for the fields on which they act. Finally, we note that the conservation of the currents in (13) and charges [cf. Eqs. (14), (17)] is valid only at the classical (i.e. tree Feynman diagram) level. We shall discuss more on this issue in our Sec. 7 to show that the conservation law persists at the loop-level, too.

4 Unique Bosonic and Ghost-Scale Continuous Symmetry Transformations: Conserved Charges

It is evident that, so far, we have discussed four fermionic [i.e. (anti-)BRST and (anti-)co-BRST] symmetry transformations for our theory. We have also noted that \(s_b s_{ab} + s_{ab} s_b = 0\) and \(s_d s_{ad} + s_{ad} s_d = 0\) due to the absolute anticommutativity properties of the (anti-)BRST and (anti-)co-BRST symmetry transformations. With the above observations as inputs, we note that we have the following bosonic symmetry transformations \((s_w, s_{\bar{w}})\), namely:

\[
\begin{align*}
  s_w &= \lbrace s_b, s_d \rbrace, \\
  s_{\bar{w}} &= \lbrace s_{ad}, s_{ab} \rbrace,
\end{align*}
\]
that are constructed with the existing four fermionic symmetry transformations \( s_{(a)b} \) and \( s_{(a)d} \) of our theory. These explicit symmetry transformations are

\[
\begin{align*}
    s_w A_\mu &= -i \varepsilon_{\mu
u} \partial^\nu B - i \partial_\mu B, \\
    s_w \phi &= -i m B, \\
    s_w \tilde{\phi} &= -i m B, \\
    s_w \psi &= e B \gamma_5 \psi - e B \psi, \\
    s_w \tilde{\psi} &= e B \tilde{\psi} \gamma_5 + e B \tilde{\psi}, \\
    s_w C &= s_w \tilde{C} = s_w B = s_w \overline{B} = 0, \\
    s_w^2 &\neq 0,
\end{align*}
\]  

(20)

where we have considered all the basic and auxiliary fields of our chosen Lagrangian density in Eq. (9). A close look at (20) and (21) demonstrates that the bosonic symmetry in Eq. (9). A close look at (20) and (21) demonstrates that the bosonic symmetry transformations \( s_w \) and \( s_{\bar{w}} \) are not independent of each-other as they satisfy the following

\[
(s_w + s_{\bar{w}}) \Sigma = 0, \quad \Sigma = A_\mu, \phi, \tilde{\phi}, C, \bar{C}, B, \overline{B}, \psi, \tilde{\psi},
\]  

(22)

where \( \Sigma \) is the generic field of the Lagrangian density (9). In other words, we have a unique bosonic symmetry in the theory. The sanctity of the above statement can be verified by the following transformation properties of the Lagrangian density (9):

\[
\begin{align*}
    s_w \mathcal{L}_B &= \partial_\mu \left[ i m \varepsilon^{\mu\nu} \right. \\
    &\left. \left( \phi (\partial_\nu B) + m B A_\nu \right) + i m \tilde{\phi} \partial_\mu B + i B \partial^\mu B - i B \partial^\mu B \right], \\
    s_{\bar{w}} \mathcal{L}_B &= -\partial_\mu \left[ i m \varepsilon^{\mu\nu} \right. \\
    &\left. \left( \phi (\partial_\nu B) + m B A_\nu \right) + i m \tilde{\phi} \partial_\mu B + i B \partial^\mu B \\
    &- i B \partial^\mu B \right].
\end{align*}
\]  

(23)

This observation, once again, corroborates the fact that there is an existence of a unique boson-like \( s_w \) bosonic symmetry \( (s_w + s_{\bar{w}}) \mathcal{L}_B = 0 \) in our theory.

According to the Noether theorem, the above continuous bosonic symmetry leads to the derivation of the following conserved \( (\partial_\mu J^\mu_{(w)} = 0) \) Noether current \( (J^\mu_{(w)}) \):

\[
\begin{align*}
    J^\mu_{(w)} &= i \varepsilon^{\mu\nu} \left[ B \partial_\nu B - B \partial_\nu B + m \tilde{\phi} \partial_\nu B - m \phi \partial_\nu B - m^2 B A_\nu \right] \\
    &+ i m B \partial^\mu \tilde{\phi} - i m B \partial^\mu \phi + i m^2 B A^\mu + i e B \tilde{\psi} \gamma^\mu \gamma_5 \psi - i e B \tilde{\psi} \gamma^\mu \psi.
\end{align*}
\]  

(24)

The conservation law \( (\partial_\mu J^\mu_{(w)} = 0) \) can be proven by using the beauty and strength of the EL-EOMs that have been derived in Eq. (15) from the Lagrangian density (9). The bosonic conserved charge \( Q_w = \int d x J^0_{(w)} \) (with \( \gamma_0 \gamma_5 = \gamma_1 \)) is as follows:

\[
\begin{align*}
    Q_w &= \int d x \left[ i B \partial_1 B - i B \partial_1 B + i m B \tilde{\phi} - i m B \phi + i m \phi \partial_1 B - i m \tilde{\phi} \partial_1 B \\
    &+ i m^2 B A_1 + i m^2 B A_0 + i e B \tilde{\psi} \gamma_1 \psi - i e B \tilde{\psi} \gamma_0 \psi \right].
\end{align*}
\]  

(25)

It is straightforward to check that the above conserved charge \( (Q_w) \) is the generator of the continuous and infinitesimal bosonic symmetry transformations in Eq. (20).
We have a ghost-scale symmetry in our theory where only the fermionic \((C^2 = \bar{C}^2 = 0, \ C \bar{C} + \bar{C} C = 0)\) (anti-)ghost fields \((\bar{C})\ C\) transform continuously by a scale factor:

\[
C \rightarrow C = e^{\Omega} C, \quad \bar{C} \rightarrow \bar{C} = e^{-\Omega} \bar{C}, \quad \Sigma \rightarrow e^0 \Sigma,
\]

(26)

In the above, the field \(\Sigma = A_\mu, \phi, \tilde{\phi}, B, B, \psi, \tilde{\psi}\) stands for the generic field of \(L_B\) \([\text{without the fermionic (anti-)ghost fields}]\) and \(\Omega\) in the exponentials of Eq. (26) is the global \(\text{(i.e. spacetime independent)}\) scale factor. The signs in the exponents of \(C\) and \(\bar{C}\) denote the ghost numbers \((\pm 1)\), respectively. The infinitesimal versions of (26) are

\[
s_g C = +C, \quad s_g \bar{C} = -\bar{C}, \quad s_g \Sigma = 0,
\]

(27)

where, for the sake of brevity, we have taken \(\Omega = 1\). The invariance of \(L_B\) under the continuous and infinitesimal \textit{global} ghost-scale symmetry transformations (27) leads to the existence of the following Noether’s conserved current:

\[
J^\mu_{(g)} = i \left[ \bar{C} (\partial^\mu C) - (\partial^\mu \bar{C}) C \right].
\]

(28)

The conservation law \((\partial_\mu J^\mu_{(g)} = 0)\) can be proven by using the EL-EOMs that have been quoted in Eq. (15). The conserved charge \(Q_g = \int d x \ J^0_{(g)}\) is as follows:

\[
Q_g = i \int d x \ (\bar{C} \dot{C} - \dot{\bar{C}} \bar{C}).
\]

(29)

The subscripts \(g\) in Eqs. (27), (28) and (29) denote the infinitesimal transformations, conserved current and conserved charge corresponding to the presence of an infinitesimal and continuous \textit{global} ghost-scale symmetry transformation in our theory.

We end this section with the following key comments. First of all, we note that the (anti-)ghost fields \((\bar{C})\ C\) do \textit{not} transform under the bosonic symmetry transformations [cf. Eq. (20)]. Second, out of the four fermionic symmetries \([\text{i.e.} \ s_{(a)b} \text{ and } s_{(a)d}]\) of our theory, it can be checked that the anticommutators: \(\{s_b, s_{ab}\}, \{s_d, s_{ad}\}, \{s_b, s_{ad}\}, \{s_d, s_{ab}\}\) turn out to be \textit{absolutely} zero. Third, only two \textit{non-trivial} anticommutators \((\text{i.e.} \ s_w = \{s_b, s_d\}, \ s_{\bar{w}} = \{s_{ad}, s_{ab}\}\) exist. However, only one of them turns out to be \textit{independent} which defines a \textit{unique} bosonic symmetry transformation \((s_w)\) for our theory [cf. Eq. (20)] because of our observation: \(s_w + s_{\bar{w}} = 0\) [cf. Eqs. (20), (21), (23)]. Fourth, under the ghost-scale symmetry transformations, only the (anti-)ghost fields transform by a global scale factor. However, all the rest of the fields do \textit{not} transform at all. Finally, we observe the following interesting relationships, namely:

\[
\begin{align*}
\ s_w Q_r & = -i \left[ Q_r, Q_w \right] = 0, & r = b, ab, d, ad, g, w, \\
\ s_g Q_s & = -i \left[ Q_s, Q_g \right] = +Q_s, & s = b, ad, \\
\ s_g Q_t & = -i \left[ Q_t, Q_g \right] = -Q_t, & t = d, ab,
\end{align*}
\]

(30)

which demonstrate that the conserved charge \(Q_w\) is the Casimir operator for the whole algebra constructed with the conserved charges \(Q_r\) \((r = b, ab, d, ad, g, w)\). Furthermore, we note that the ghost number for the conserved charges \(Q_b\) and \(Q_{ad}\) is +1 and the \textit{same} for the conserved charges \(Q_d\) and \(Q_{ab}\) is \(-1\). We shall corroborate these statements very clearly in our forthcoming sections 5 and 6, respectively, where these observations will be shown to be \textit{important} in the proof of our present theory to be a model for Hodge theory.
5 Algebraic Structures: Symmetries and Charges

In this section, we assimilate all the continuous symmetry transformations for our 2D theory and point out their algebraic structure where they act like operators. In other words, when we say that \( \{s_b, s_{ab}\} = 0 \), it implies that we have: \( \{s_b, s_{ab}\} \Sigma = 0 \) where \( \Sigma = A_\mu, C, \bar{C}, B, \bar{B}, \psi, \bar{\psi}, \phi, \bar{\phi} \) is the generic field. It is straightforward to note that we obtain the following extended BRST-algebra that is obeyed by the infinitesimal and continuous symmetry transformation operators \( s_r \) \((r = b, ab, d, ad, g, w)\) of our theory, namely:

\[
\begin{align*}
\{s_b, s_{ab}\} &= 0, & \{s_d, s_{ad}\} &= 0, & s^{(a)b} &= 0, & s^{(a)d} &= 0, \\
\{s_b, s_{ad}\} &= 0, & \{s_d, s_{ab}\} &= 0, & \{s_d, s_b\} &= -\{s_{ab}, s_{ad}\} = s_w, \\
[s_w, s_r] &= 0, & r &= b, ab, d, ad, g, w, \\
[s_g, s_b] &= +s_b, & [s_g, s_{ad}] &= +s_{ad}, & [s_g, s_{ab}] &= -s_{ab}, & [s_g, s_d] &= -s_d,
\end{align*}
\]

which shows that the bosonic symmetry transformation \( s_w \) commute with all the other continuous and infinitesimal symmetry transformations \( s_r \) \((r = b, ab, d, ad, g)\) of our 2D theory. On the other hand, the commutator of \( s_g \) with \( s_r \) \((r = b, ad)\) produces the transformations \( s_r \) with positive sign (which is just the opposite of the commutators of \( s_g \) with \( s_r \) \((r = d, ab)\) where there is a negative sign on the r.h.s.).

There is a very beautiful relationship between the fermionic symmetry transformations and the discrete symmetry transformations (10) of our 2D theory. We very clearly observe the validity of the following relationships:

\[
s_d = \pm \ast s_b \ast, \quad s_{ab} = \pm \ast s_{ad} \ast,
\]

where \( \ast \), in the above, stands for the discrete symmetry transformations (10) of our theory. A close look at (32) demonstrates that we have captured the mathematical relationship: \( \delta = \pm \ast d \ast \) of differential geometry in the terminology of the symmetry transformations where the continuous and discrete symmetry transformations are intertwined together and play a very important and decisive role. Following the basic concepts behind the duality invariant theories (see, e.g. [34] for details), we observe that the \((\pm)\) signs in the relationship: \( s_d = \pm \ast s_b \ast \) are governed by the double operations of \( \ast \) operator [i.e. the discrete symmetry transformations] on a specific field. In other words, we note that for the generic field \( \Psi \):

\[
\ast (\ast \Psi) \equiv -\Psi, \quad \Psi = C, \bar{C}, A_\mu, B, \bar{B}, \phi, \bar{\phi}, \\
\ast (\ast \Psi) \equiv +\Psi, \quad \Psi = \psi, \bar{\psi}.
\]

The above observations show that, only for the Dirac fields \((\psi, \bar{\psi})\), we have the relationships: \( s_d \psi = + \ast s_b \ast \psi, s_d \bar{\psi} = + \ast s_b \ast \bar{\psi} \). On the contrary, for the rest of the fields of our theory, we have: \( s_d = - \ast s_b \ast \) where there is a negative sign on the r.h.s. We have provided more discussions on it in our Appendix B.

According to the basic principle behind the Noether theorem, we know that the continuous symmetry transformations and the Noether conserved charges (i.e. symmetry generators) are deeply connected with one-another. Hence, it turns out that the algebra (31), obeyed by the symmetry transformation operators, is also respected and replicated by the
conserved charges $Q_r$ ($r = b, ab, d, ad, g, w$). In other words, the algebra satisfied by the conserved charges (which are responsible for the existence of the continuous and infinitesimal symmetry transformations in our theory) respect the following:

$$
Q_{(a)b}^2 = 0, \quad Q_{(a)d}^2 = 0, \quad Q_w = \{Q_b, Q_d\} = - \{Q_{ad}, Q_{ab}\},
$$

$$
[Q_w, Q_r] = 0, \quad r = b, ab, d, ad, g, w,
$$

$$
\{Q_w, Q_{ab}\} = \{Q_d, Q_{ad}\} = \{Q_b, Q_{ab}\} = Q_{ab}, Q_d = 0,
$$

$$
i [Q_g, Q_b] = + Q_b, \quad i [Q_g, Q_{ad}] = + Q_{ad},
$$

$$
i [Q_g, Q_d] = - Q_d, \quad i [Q_g, Q_{ab}] = - Q_{ab}.
$$

It is crystal clear that the bosonic conserved charge $Q_w$, constructed from the off-shell nilpotent (i.e. fermionic) conserved charges, is the Casimir operator for the whole algebra.

A close and clear look at (34) also shows, in a subtle manner, that the ghost numbers for the set of charges $(Q_b, Q_d, Q_w)$ are $(+1, -1, 0)$, respectively. On the other hand, the set of charges $(Q_{ad}, Q_{ab}, Q_w)$ carries the ghost numbers equal to $(+1, -1, 0)$, respectively, too. Hence, we observe that there are two sets of conserved charges, in the algebra (34), which carry the ghost numbers $(+1, -1, 0)$ in our theory. This observation would play a very important role in the next section where we shall establish the connection of the algebra (obeyed by the physical conserved charges and continuous as well as discrete symmetries of our theory) with that of the algebra respected by the de Rham cohomological operators of the differential geometry [8-12] which are purely mathematical in nature.

## 6 Physical Symmetries, Conserved Charges and Cohomological Operators: Algebraic Connection

In this section, we establish the precise connection between the discrete and continuous physical symmetries (and conserved charges) of our field-theoretic model and the de Rham cohomological operators of differential geometry [8-12] at the level of algebra. In this context, first of all, we recall that the following explicit algebra [8-12]

$$
d^2 = 0, \quad \delta^2 = 0, \quad \Delta = d \delta + \delta d = \{d, \delta\},
$$

$$
[\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad \Delta^2 \neq 0,
$$

is satisfied by the de Rham cohomological operators $(d, \delta, \Delta)$ where the (co-)exterior derivatives $(\delta)$ obey: $\delta = \pm \ast d \ast$. Here the symbol $\ast$ stands for the Hodge duality operator on the compact manifold without a boundary. Furthermore, we note that when $d$ operates on an arbitrary form $(f_n)$ of degree $n$, it raises the degree by one (i.e. $df_n \sim f_{n+1}$). On the other hand, we have: $\delta f_n \sim f_{n-1}$ which demonstrates that the operation of $\delta$ on a form decreases the degree of the form by one. It is worthwhile to mention that here (i.e. $\delta f_n \sim f_{n-1}$) the degree of the form is non-zero (i.e. $n = 1, 2, 3\ldots$). The degree of a given form $(f_n)$ remains unchanged when it is operated upon by the Laplacian operator $\Delta$.

Against the backdrop of the above paragraph and equation (35), it is very important for us to provide the physical realizations of (35) and capture the key properties of the operations of the cohomological operators on a given differential form $(f_n)$ of a non-zero
degree \( n \) in the language of the physical symmetries and conserved charges (in the quantum Hilbert space). In this context, a clear and close look at the algebra (34) demonstrates that there exists a deep connection between the cohomological operators and conserved charges of our theory at the algebraic level. In fact, we find that the following two-to-one mapping exists between the conserved charges and the cohomological operators:

\[
(Q_b, Q_{ad}) \rightarrow d, \quad (Q_d, Q_{ab}) \rightarrow \delta, \\
Q_w = \{Q_b, Q_d\} \equiv -\{Q_{ad}, Q_{ab}\} \rightarrow \Delta. \tag{36}
\]

As pointed out earlier, the absolute anticommutativity (i.e. \( \{s_b, s_{ab}\} = 0, \{s_d, s_{ad}\} = 0 \)) properties of the (anti-)BRST and (anti-)co-BRST symmetry transformations (and their corresponding conserved charges) imply that only one of these symmetries (and corresponding conserved charge) can be identified with \( d \) and \( \delta \), respectively. It turns out that, when \( Q_b \) is identified with the exterior derivative \( d \), the corresponding dual-BRST charge \( Q_d \) is identified with \( \delta \). As a consequence, we find that the Laplacian operator \( \Delta \) is identified with \( Q_w = \{Q_b, Q_d\} \) due to the fact that \( \Delta = \{d, \delta\} = (d + \delta)^2 \). In exactly similar fashion, we find that when \( Q_{ad} \) provides the physical realization of \( d \), then, the co-exterior derivative is identified with the anti-BRST charge \( Q_{ab} \). Hence, the Laplacian operator turns out to be \( Q_w = -\{Q_{ad}, Q_{ab}\} \). The negative sign appears here due to the fact that \( Q_w \) is also defined as: \( Q_w = \{Q_d, Q_b\} \) where there is a plus sign on the r.h.s. (in view of \( s_w + s_{\bar{w}} = 0 \)).

As far as the change in the degree of the form, due to the operations of the cohomological operators is concerned, we point out that this property can be realized in the quantum Hilbert space of states where we define (in terms of the conserved ghost charge \( Q_g \)) any arbitrary quantum state with a non-zero ghost number \( n \) as:

\[
i Q_g | \psi >_n = n | \psi >_n. \tag{37}\]

The algebra (34) now immediately leads to the following interesting and important observations as far as the ghost numbers of some specific states [constructed with \( (Q_w, Q_b, Q_d) \) and \( (Q_w, Q_{ad}, Q_{ab}) \)] in the quantum Hilbert space are concerned, namely:

\[
i Q_g Q_b | \psi >_n = (n + 1) Q_b | \psi >_n, \\
i Q_g Q_d | \psi >_n = (n - 1) Q_d | \psi >_n, \\
i Q_g Q_w | \psi >_n = n Q_w | \psi >_n. \tag{38}\]

The above equations imply that the quantum states \( Q_b | \psi >_n, Q_d | \psi >_n \) and \( Q_w | \psi >_n \) have the ghost numbers \( (n + 1), (n - 1), n \), respectively. In other words, the ghost number increases by one due to the operation of \( Q_b \) but decreases by one due to the operation of \( Q_d \). On the contrary, the ghost number of a state remains intact due to the operation of \( Q_w \) on it. In exactly similar fashion, given equation (37) as an input, we have the following mathematical relationships, namely:

\[
i Q_g Q_{ad} | \psi >_n = (n + 1) Q_{ad} | \psi >_n, \\
i Q_g Q_{ab} | \psi >_n = (n - 1) Q_{ab} | \psi >_n, \\
i Q_g Q_w | \psi >_n = n Q_w | \psi >_n. \tag{39}\]
which demonstrates that the ghost numbers are \((n + 1), (n - 1)\) and \(n\) for the states \(Q_{ad} | \psi >_n, Q_{ab} | \psi >_n\) and \(Q_w | \psi >_n\), respectively. Hence, the operations of the set of charges \((Q_{ad}, Q_{ab}, Q_w)\) on a quantum state of the ghost number \(n\), are exactly like the operations of the set of de Rham cohomological operators \((d, \delta, \Delta)\) on a form of degree \(n\). We can capture mathematically, the observations made in Eqs. (38) and (39), as follows:

\[
Q_b | \psi >_n \sim | \psi >_{n+1}, \quad Q_{ad} | \psi >_n \sim | \psi >_{n+1},
Q_d | \psi >_n \sim | \psi >_{n-1}, \quad Q_{ab} | \psi >_n \sim | \psi >_{n-1},
Q_w | \psi >_n \sim | \psi >_n, \quad Q_w | \psi >_n \sim | \psi >_n.
\]

Thus, we observe that the changes in the degree of a given form, due to the operations of \((d, \delta, \Delta)\) are exactly like the changes in the ghost numbers of a suitably chosen quantum state due to the operations of \((Q_b, Q_d, Q_w)\) and/or \((Q_{ad}, Q_{ab}, Q_w)\). In equations (37) to (40), we have denoted the ghost numbers of the states by the subscripts on these states.

At this crucial juncture, we are in the position to capture the Hodge decomposition theorem of differential geometry [8-12] in the language of the conserved charges of our 2D interacting theory in the quantum Hilbert space of states, namely:

\[
| \psi >_n = | \omega >_n + Q_b | \chi >_{n-1} + Q_d | \lambda >_{n+1}
\equiv | \omega >_n + Q_{ad} | \sigma >_{n-1} + Q_{ab} | \kappa >_{n+1},
\]

(41)

where \(| \psi >_n\) is any arbitrary quantum state in the Hilbert space with the ghost number \(n\). For the sake of generality, we have chosen the non-null states \(| \chi >_{n-1}, | \sigma >_{n-1}, | \lambda >_{n+1}\) and \(| \kappa >_{n+1}\) in the Hodge decomposed state \(| \psi >_n\) in terms of the two sets of charges \((Q_w, Q_b, Q_d)\) and \((Q_w, Q_{ad}, Q_{ab})\), respectively. We note that the states \(| \chi >_{n-1}\) and \(| \sigma >_{n-1}\) are the BRST-exact and anti-co-BRST-exact, respectively. On the other hand, we have the states \(| \lambda >_{n+1}\) and \(| \kappa >_{n+1}\) which are co-BRST-exact and anti-BRST-exact, respectively. In the Hodge-decomposed state (41), we have the quantum state \(| \omega >_n\) as the harmonic state which is the most symmetric state in the whole theory because it is annihilated by all the fermionic conserved charges. In other words, we have the following:

\[
Q_{(a)b} | \omega >_{n} = 0, \quad Q_{(a)d} | \omega >_{n} = 0.
\]

(42)

We choose the harmonic state as the physical state of the theory which is annihilated by the conserved and nilpotent BRST charge as well as the co-BRST charge. We have, purposely, not written the anti-BRST as well as the anti-co-BRST charges because these do not lead to any new restrictions on the physical state of the theory. To be economical and precise, we demand that the physical state (i.e. \(| phys >\)) is the one which is annihilated by, at least, the conserved BRST and co-BRST charges. Thus, the physical space of our 2D theory is a subspace of the total quantum Hilbert space which satisfies the physicality criteria:

\[
Q_b | phys >= 0, \quad Q_d | phys >= 0.
\]

(43)

In the next section, we shall discuss the physical consequences of (43), in detail, to demonstrate that our present study is physically interesting and useful.
7 Harmonic State as Physical State: Consequences

As pointed out in the previous section, it is the harmonic state in the Hodge decomposed state (of the total quantum Hilbert space) which is the most symmetric state of our entire theory. This state has to be annihilated by the conserved and nilpotent BRST and co-BRST charges (which is the most economical and precise requirement). In this context, we have the following (from the condition: \( Q_b \mid \text{phys} = 0 \)), namely;

\[
\Pi^0 \equiv B \mid \text{phys} = 0 \quad \Rightarrow \quad -(\partial \cdot A + m \phi) \mid \text{phys} = 0, \\
\dot{\Pi}^0 \equiv \dot{B} \mid \text{phys} = 0 \quad \Rightarrow \quad -\partial_b (\partial \cdot A + m \phi) \mid \text{phys} = 0, \\
\equiv \quad (\nabla \cdot \vec{E} - m \Pi_\phi - e \psi^\dagger \psi) \mid \text{phys} = 0, \tag{44}
\]

where we have used the concise expressions for charge from Eq. (6) and assumed that the (anti-)ghost fields are not physical fields. The above restrictions on the physical state are consistent with Dirac’s quantization condition where the physical state of a quantum gauge theory must be annihilated by the operator form of the first-class constraints (of the corresponding classical gauge theory). It is clear from the Lagrangian density (1) [and/or (9)] that we have: \( \Pi^0 = -(\partial \cdot A + m \phi) \equiv B \) and \( \dot{\Pi}^0 = \dot{B} \equiv \nabla \cdot \vec{E} - m \Pi_\phi - e \psi^\dagger \psi \)

where \( \Pi_\phi = \dot{\chi} - m A_0 \) is the canonical conjugate momentum w.r.t. the pure scalar field \( \phi \) (which is nothing but the St"uckelberg-field in the context of our 2D Proca theory). We would like to point out that \( \Pi^0 \) is the conjugate momentum w.r.t. \( A_0 \) field and \( \dot{\Pi}^0 = \dot{B} \equiv \nabla \cdot \vec{E} - m \Pi_\phi - e \psi^\dagger \psi \) emerges out from the EL-EOMs (5) and/or (15). In the 2D case, we have \( \nabla \cdot \vec{E} = \partial_1 E \) and, therefore, the Gauss-law of divergence is: \( \dot{B} = \partial_1 E - m \Pi_\phi - e \psi^\dagger \psi \) which is derived from the top entry of Eq. (15) with the inputs: \( B = -(\partial \cdot A + m \phi) \) and \( \mathcal{B} = E - m \tilde{\phi} \). In more precise language, within the ambit of BRST formalism, we have the restrictions on the physical state [cf. Eq. (44)] as the quantum generalizations of the first-class constraints (\( \Pi^0 \approx 0 \) and \( \nabla \cdot \vec{E} - m \Pi_\phi - e \psi^\dagger \psi \approx 0 \)) on the classical gauge theory which become operators at the quantum level and they annihilate the physical state.

Let us focus on the proof of the conservation law (\( \dot{Q}_b = 0 \)) for the concise form of the nilpotent BRST charge (\( Q_b \)) that is quoted in Eq. (6). In this context, we note that the conserved current \( J^\mu_{(0)} \) [cf. Eq. (4)] corresponding to the infinitesimal and continuous BRST symmetry transformations (2) defines the BRST charge (\( Q_b \)) in a straightforward manner. A close look at the expression of \( J^\mu_{(0)} \) in Eq. (4) shows that, for the proof of the conservation law \( \partial_\mu J^\mu_{(0)} = 0 \), we invoke the validity of the conservation law \( \partial_\mu J^\mu_{(m)} = \partial_\mu [\bar{\psi} \gamma^\mu \psi] = 0 \) where \( J^\mu_{(m)} \) is the polar-vector current constructed with the Dirac fields (which are the interacting matter fields in our theory). There is no problem for the proof of \( \partial_\mu J^\mu_{(m)} = 0 \) as the EL-EOM of Eq. (15) lead to it. It is straightforward to note that we have the following when we take a direct time derivative on (6), namely;

\[
\dot{Q}_b = \int d^{D-1} x \left[ B \hat{C} - \hat{B} C \right]. \tag{45}
\]

Using the EL-EOMs from (5) and/or (15), we note that \( (\Box + m^2) C = 0 \) and \( (\Box + m^2) B = 0 \). In the derivation of the latter, we have used \( (\Box + m^2) \phi = 0 \) and the conservation of the
matter current \[ i.e. \partial_{\mu} J_{(m)}^{\mu} \equiv e \partial_{\mu} (\bar{\psi} \gamma^{\mu} \psi) = 0 \] which is valid due to the local gauge and/or BRST symmetry invariance in our theory. As a consequence of the above EL-EOMs, we have: \( \dot{\mathcal{C}} = \nabla^{2} C - m^{2} C \) and \( \dot{\mathcal{B}} = \nabla^{2} B - m^{2} B \). Substitutions of these into (45) lead to the following explicit form of the conservation law, namely;

\[
\dot{Q}_b = \int d^{D-1} x \left[ \tilde{\nabla} \cdot (B \nabla C - C \nabla B) \right] \rightarrow 0. \tag{46}
\]

Thus, we find that the conservation of BRST charge is true in any arbitrary dimension of spacetime because we see that \( \dot{Q}_b \rightarrow 0 \) due to Gauss’s divergence theorem. For 2D case, we have \( \dot{Q}_b = \int d x \left[ \partial_{1} (B \partial_{1} C - C \partial_{1} B) \right] \rightarrow 0 \). We conclude that the physicality criterion with the nilpotent and conserved BRST charge: \( Q_{b} \mid \text{phys} \geq 0 \) leads to the annihilation of the physical state by the operator forms of the primary constraint \( (\Pi^{0} \approx 0) \) and the secondary constraint: \( (\nabla \cdot \tilde{E} - m \Pi_{\phi} - e \psi^\dagger \psi \approx 0) \) on our 2D St"{u}ckelberg’s modified Proca theory. It is clear that both these constraints are first-class.

Due to the presence of a set of discrete symmetry transformations (10) and our observations in Secs. 5 and 6, it is obvious that our present 2D theory is (i) a perfect model of a duality-invariant theory [34], and (ii) an example of Hodge theory. As far as the duality property is concerned, it will be noted that the discrete symmetry transformation on the gauge field (i.e. \( A_{\mu} \rightarrow \mp i \varepsilon_{\mu\nu} A^{\nu} \)) owes its origin to the self-duality (see, e.g. [5, 7] for details) condition [i.e. \( *A^{(1)} = * (dx^{\mu} A_{\mu}) \)] on the 1-form \( (A^{(1)} = dx^{\mu} A_{\mu}) \) gauge field \( A_{\mu} \) where the \( * \) is the Hodge duality operation. Rest of the transformations in (10) are consistent with this self-duality requirement. For a perfect duality invariant gauge theory, it is very sacrosanct requirement that the dual to the first-class constraints [cf. Eq. (44)] must also annihilate the physical state of the theory. In this context, it is interesting to point out that \( (E - m \tilde{\phi}) \) is the dual of the gauge-fixing term \( (\partial \cdot A + m \phi) \) due to the discrete symmetry transformations (10). Similarly, we note that \( \partial_{0} (E - m \tilde{\phi}) \) is also dual to \( [\partial_{0} (\partial \cdot A + m \phi)] \).

The dual restrictions emerge out from the co-BRST charge \( Q_{d}^{(1)} \) as discussed below. We lay emphasis on the fact that operator forms of the first-class constraints and their dual must annihilate the physical state at the level of tree and loop diagrams. In other words, these constraints/restrictions on the physical state are universal for a perfect duality invariant gauge theory and they must be respected at the classical as well as quantum level.

Against the backdrop of the above, the physicality criterion: \( Q_{d}^{(1)} \mid \text{phys} \geq 0 \) imposes exactly the conditions on the physical state due to the duality considerations, namely;

\[
\begin{align*}
B \mid \text{phys} \geq 0 & \Rightarrow (E - m \tilde{\phi}) \mid \text{phys} \geq 0, \\
\bar{B} \mid \text{phys} \geq 0 & \Rightarrow \partial_{0} (E - m \tilde{\phi}) \mid \text{phys} \geq 0 \\
& \equiv (\partial_{1} \Pi^{0} + m \Pi_{\phi} + e \psi^\dagger \gamma_{5} \psi + m \partial_{1} \phi - m^{2} A_{1}) \mid \text{phys} \geq 0, \tag{47}
\end{align*}
\]

where we have used the concise expression for the conserved co-BRST charge \( Q_{d}^{(1)} \) [cf. Eq. (17)] and EL-EOMs from Eq. (15) in the derivation of \( \partial_{0} (E - m \tilde{\phi}) \). We have also used the expressions: \( \Pi^{0} = B, \Pi_{\phi} = - \dot{\phi}, B = E - m \tilde{\phi} \). Let us now concentrate on the physical

\footnote{The conservation of the matter current \( (J_{(m)}^{\mu} = e \bar{\psi} \gamma^{\mu} \psi) \), constructed with the Dirac fields \( (\bar{\psi}, \psi) \), is sacrosanct because this leads to the conservation of the electric charge which is universal and true at the classical as well as quantum level (in any physically allowed process).}
consequence that emerges out from the top entry of (47). A close and careful look at the 2D (anti-)BRST and (anti-)co-BRST invariant Lagrangian density (9) demonstrates that the pseudo-scalar field $\tilde{\phi}$ has appeared in the theory with a negative kinetic term. Such fields have been christened as the “phantom” fields in the realm of cosmology and they have been found to be useful in the context of cyclic, bouncing and self-accelerated models of Universe (see e.g. [26-30] for details). In a nut-shell, such exotic fields (with negative kinetic terms) are unphysical in the sense that they have not yet been detected by the experiments. Thus, just like the (anti-)ghost fields, such kinds of fields (e.g. the pseudo-scalar field $\tilde{\phi}$) do not impose any restriction on the physical state of a quantum gauge theory. This leads us to draw the conclusion that physically, the restrictions in (47), imply the following (from the physicality requirement $Q^{(1)}_{\dot{a}} \mid \text{phys} >= 0$), namely;

$$
\mathcal{B} \mid \text{phys} >= 0 \quad \Rightarrow \quad (E - m \tilde{\phi}) \mid \text{phys} >= 0 \quad \Rightarrow \quad E \mid \text{phys} >= 0,

\dot{\mathcal{B}} \mid \text{phys} >= 0 \quad \Rightarrow \quad \partial_0 (E - m \tilde{\phi}) \mid \text{phys} >= 0 \quad \Rightarrow \quad \dot{E} \mid \text{phys} >= 0. \quad (48)
$$

At this crucial juncture, we recall that the electric field $E = -\frac{1}{2} \varepsilon^{\mu\nu} F_{\mu\nu} \equiv -\varepsilon^{\mu\nu} \partial_{\mu} A_{\nu}$ is also the anomaly term in 2D because $\partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) = -\frac{1}{2} \varepsilon^{\mu\nu} F_{\mu\nu} = \alpha E$ where $\alpha$ is a constant factor (that is not very important for our present discussion). Thus, the anomaly term and the requirement of its time-evolution invariance must annihilate the physical state.

We have the conservation ($\dot{Q}_b = 0$) of the BRST charge due to the conservation ($\partial_\mu J^\mu_{(b)} = 0$) of the BRST invariant ($s_b J^\mu_{(b)} = 0$) Noether current ($J^\mu_{(b)}$) which, in a precise language, implies the following in terms of the physical state (see, e.g. [36] for details)

$$
< \text{phys} \mid \partial_\mu J^\mu_{(b)} \mid \text{phys} >= 0 \quad \Rightarrow \quad < \text{phys} \mid \dot{Q}_{(b)} \mid \text{phys} >= 0. \quad (49)
$$

Against this as a backdrop, we now concentrate on the proof of the conservation ($\dot{Q}^{(1)}_{\dot{a}} = 0$) of the concise form of the co-BRST charge in Eq. (17). It is straightforward to note that we have the following explicit expression for ($\dot{Q}^{(1)}_{\dot{a}} \equiv dQ^{(1)}_{\dot{a}} / dt$), namely;

$$
\dot{Q}^{(1)}_{\dot{a}} = \int d x \left[ \mathcal{B} \ddot{C} - \ddot{\mathcal{B}} C \right], \quad (50)
$$

where we have applied directly the time derivative on (17). We have the EL-EOMs: ($\Box + m^2) \dot{C} = 0$ and ($\Box + m^2) \mathcal{B} = 0$. The latter is true at the classical level (where there are no loop-diagrams and the massless limit of the Dirac fermions is taken into account). To be precise, we lay emphasis on the fact that we find that ($\Box + m^2) \mathcal{B} = e \partial_\mu \left[ \bar{\psi} \gamma^\mu \gamma_5 \psi \right]$ where the r.h.s. is precisely equal to zero in the classical limit as stated above. When we consider the triangle Feynman diagram, we obtain the ABJ anomaly term on the r.h.s. where $\partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) = \alpha E$ for the 2D theory (with $\alpha$ as a constant factor). Thus, we find that Eq. (50) reduces to the following explicit form:

$$
\dot{Q}^{(1)}_{\dot{a}} = \int d x \left[ \mathcal{B} (\partial_1^2 \dot{C} - m^2 \dot{C}) - (\partial_1^2 \mathcal{B} - m^2 \mathcal{B} + \alpha E) \dot{C} \right]

\equiv \int d x \left[ \partial_1 \left\{ \mathcal{B} \partial_1 \dot{C} - (\partial_1 \mathcal{B}) \dot{C} \right\} - \alpha \int d x E \dot{C}, \quad (51)
$$
where we have used \( \ddot{\bar{C}} = \partial_1 \bar{C} - m^2 \bar{C}, \) \( \ddot{\bar{B}} = \partial_2^2 \bar{B} - m^2 \bar{B} + \alpha E. \) It is straightforward to note that the first term of (51) goes to zero as \( x \to \pm \infty \) due to Gauss’s divergence theorem. However, the second term (due to 2D anomaly) remains intact. In view of (49), we note that: \(< \text{phys} \mid \partial_\mu J_{(d)}^\mu \mid \text{phys} > \Rightarrow < \text{phys} \mid \dot{Q}_{d}^{(1)} \mid \text{phys} >. \) However, as pointed out earlier, our theory is a perfect model of duality invariant theory [34] because \( A_\mu \to \mp i \varepsilon_{\mu\nu} A^\nu \) [cf. Eq. (10)] arises due to the self-duality restriction (see, e.g. [5, 7] for details). Thus, the restrictions (48) are valid at the tree-level as well as loop-level Feynman diagrams. At this crucial stage, we find the following explicit expressions (see, e.g. [36] for details)

\[
< \text{phys} \mid \partial_\mu J_{(d)}^\mu \mid \text{phys} > = \alpha < \text{phys} \mid E \mid \text{phys} >= 0 \Rightarrow \\
< \text{phys} \mid \dot{Q}_{d}^{(1)} \mid \text{phys} > = \alpha \int < \text{phys} \mid E \bar{C} \mid \text{phys} > dx = 0. 
\]  

(52)

The r.h.s. of the above expressions are zero due to the fact that, in view of Eq. (48), we have: \( E \mid \text{phys} >= 0. \) In the proof of the conservation of co-BRST charge, we note that the anti-ghost field \( \bar{C} \) does not impose any restriction on the physical state. Hence, in 2D, the anomaly term is trivial in the sense that the vector and axial-vector currents (and corresponding Noether charges) are conserved together as far as the physical state of our 2D interacting theory is concerned. This is the basic reason behind the consistency and unitarity of the 2D anomalous Abelian 1-form gauge theories (see, e.g. [37-39] for details).

8 Summary and Outlook

In our present investigation, we have demonstrated that an interacting 2D modified version of Proca theory (in interaction with the Dirac fields) represents a tractable field-theoretic model for the Hodge theory where the discrete and continuous symmetries of this theory provide the physical realizations of the de Rham cohomological operators of differential geometry at the algebraic level. Whereas the infinitesimal and continuous symmetries (and corresponding conserved charges) of our present 2D theory provide the physical realizations of the cohomological operators, the discrete symmetry transformations correspond to the Hodge duality * operation of differential geometry. The ghost number considerations of the quantum states in the total Hilbert space, in terms of the conserved charges, have been able to provide the physical analogue of the operations of cohomological operators on the differential form of a given degree and the changes (in the degree) that ensue.

For our 2D interacting theory, we have been able to demonstrate that (i) the nilpotent (anti-)BRST symmetry transformations (and conserved charges) exist [cf. Eqs. (2), (6)] corresponding to the local gauge symmetry in the theory which is generated by the first-class constraints, and (ii) the nilpotent (anti-)co-BRST symmetries (and corresponding conserved charges) are present [cf. Eqs. (11), (17)] in the theory due to the existence of the

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1It is the very elegant interplay between the co-BRST symmetry transformations (11) and the discrete symmetry transformations (10) that ensures the conservation of the co-BRST charge at the tree as well as the loop-level diagrams (when we choose the physical state as the harmonic state).

2Under the discrete symmetry transformations (10), the kinetic term \((\bar{\psi} i \gamma^\mu \partial_\mu \psi)\) and the mass term \((-m \bar{\psi} \psi)\) of the Dirac fields remain trivially invariant. The interaction term \((-e \bar{\psi} \gamma^\mu A_\mu \psi)\) is also found to respect the discrete symmetry transformations (10) [cf. Appendix C for details].
local chiral symmetry (in the massless limit of the Dirac fields). The latter symmetry exists due to the presence of a duality symmetry in the theory [cf. Eq. (10)]. As is well-known, there is presence of the ABJ anomaly whenever the local gauge and chiral symmetries co-exist together in a theory with Dirac fields (in interaction with a gauge field). However, at the classical level (in the massless limit of the Dirac fields), this anomaly term is zero (at the tree level Feynman diagrams). For the QED in 2D, the anomaly term is nothing but the electric field (i.e. \( \frac{\alpha}{2} \varepsilon^{\mu\nu} F_{\mu\nu} = \alpha E \) with \( \alpha \) as a constant factor) which appears in the conservation law of the chiral current (i.e. \( \partial_{\mu} J_{(5)}^\mu = \partial_{\mu} (\bar{\psi} \gamma^{\mu} \gamma_5 \psi) = \alpha E \)).

As per Dirac’s quantization condition, a physical state of the total quantum Hilbert space must be annihilated by the operator form of the constraints of a given classical theory. In our present 2D theory, this key feature is incorporated in the requirement that the physical state must be annihilated by the conserved and nilpotent BRST charge [cf. Eq. (44)]. However, if a theory is perfectly duality invariant, the dual version of these constraints must also annihilate the physical state at the quantum level [cf. Eq. (47)]. Our present 2D theory is not only a perfect field-theoretic example of Hodge theory, it is also a beautiful model of the duality invariant theory [34] due to the presence of the discrete symmetry transformations (10). This is the reason that we have the restrictions (44) and (47) on the physical state of our theory which are very sacrosanct and they must be respected at the classical as well as the quantum level. In other words, the restrictions (44) and (47) are true at the tree as well as loop-level Feynman diagrams for our 2D interacting theory. This is precisely the reason that the 2D anomaly term becomes trivial and, as a consequence, the vector \( (J_{(b)}^\mu) \) and axial-vector \( (J_{(d)}^\mu) \) currents are conserved together leading to the conservation of the BRST and co-BRST charges at the tree as well as at the loop-level diagrams.

We end this section with a few crucial remarks. First of all, the simultaneous conservation of the BRST and co-BRST charges (and corresponding Noether currents) are true only in the case of 2D theory of QED with Dirac fields. Second, in our earlier works [32, 33] on the dual-BRST symmetries (without mass term for the photon), we have shown that the physical state is annihilated by the electric field \( E \) and its time-evolution \( \dot{E} \) invariance (i.e. \( E \mid \text{phys} >= 0, \dot{E} \mid \text{phys} >= 0 \)). However, in such theories, the infrared divergence problem exists. In contrast, in our present theory of massive photon, such divergences do not exist (see, e.g. [36]). Third, if QED with massive photon (along with a pseudo-scalar field) is taken into account, our conjecture is the observation that the anomaly term will be: \( (E - m \phi) \). Fourth, our present analysis provides the basic reason behind the consistency and unitarity of the 2D anomalous Abelian gauge theory with Dirac fields (see, e.g. [37-39]). Finally, the existence of the negative kinetic term for the fields is not a problem in the modern-day cosmology where the cyclic, bouncing and self-accelerated models of Universe require the existence of such kind of “phantom” fields and particles (see, e.g. [26-30]). This pseudo-scalar field and an axial-vector field (with negative kinetic terms) have also been shown to provide a possible set of candidates for dark matter and dark energy [18-20]. Within the framework of (SUSY) quantum field theories, the Stuckelberg-boson has been able to shed light on the infrared problem in QED and it has been shown to play an important role in providing a possible candidate for the ultralight dark matter. These issues have been recently discussed in a set of very interesting papers [40-42]. It will be nice future endeavor to apply BRST approach to these (SUSY) field theoretic models.
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Appendix A: On Different Forms of Lagrangian Density

The central theme of our present Appendix is to highlight that we have discussed a few different forms of the (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities (without the Dirac fields) in our earlier work [19] besides the Lagrangian density (9) that has been taken for discussion in our present endeavor. For instance, we have taken into account the following Lagrangian densities (without Dirac fields), namely:

\[
L_{(B_1)} = \frac{1}{2} (E - m \tilde{\phi}) + m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi
\]

\[-m A_\mu \partial^\mu \phi - \frac{1}{2} (\partial \cdot A + m \phi)^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,
\]

\[
L_{(B_2)} = \frac{1}{2} (E + m \tilde{\phi}) - m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi
\]

\[+m A_\mu \partial^\mu \phi - \frac{1}{2} (\partial \cdot A - m \phi)^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,
\]

which are intimately connected with each-other by the discrete symmetry transformations:

\[A_\mu \rightarrow A_\mu, \ C \rightarrow C, \ \bar{C} \rightarrow \bar{C}, \ \phi \rightarrow -\phi, \ \tilde{\phi} \rightarrow -\tilde{\phi}.
\]

Thus, they do not describe two different physical systems. It is straightforward to check that the following on-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations \([s_{(a)b}]\)

\[
s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} C = i (\partial \cdot A + m \phi), \quad s_{ab} \phi = m \bar{C},
\]

\[
s_{ab} (\bar{C}, E, \tilde{\phi}) = 0, \quad s_{ab} (\partial \cdot A + m \phi) = (\Box + m^2) \bar{C},
\]

\[s_b A_\mu = \partial_\mu C, \quad s_b \bar{C} = -i (\partial \cdot A + m \phi), \quad s_b \phi = m C,
\]

\[s_b (C, E, \tilde{\phi}) = 0, \quad s_b (\partial \cdot A + m \phi) = (\Box + m^2) C,
\]

(A.3)

transform the Lagrangian density \(L_{B_1}\) to a total spacetime derivative thereby rendering the action integral \(S = \int d^4 x L_{B_1}\) invariant. In exactly similar fashion, we can write the on-shell nilpotent and absolutely anticommuting (anti-)BRST transformations for \(L_{B_2}\) from (A.3) by the replacements: \(A_\mu \rightarrow A_\mu, \ C \rightarrow C, \ \bar{C} \rightarrow \bar{C}, \ \phi \rightarrow -\phi, \ \tilde{\phi} \rightarrow -\tilde{\phi}.
\]

The Lagrangian density (A.1) also respects an on-shell nilpotent and absolutely anticommuting set of (anti-)co-BRST symmetry transformations \(s_{(a)d}\), namely:

\[
s_{ad} A_\mu = -\varepsilon_{\mu
u} \partial^\nu C, \quad s_{ad} \bar{C} = i (E - m \tilde{\phi}), \quad s_{ad} \tilde{\phi} = -m C,
\]

(A.4)
It is straightforward to note that the Lagrangian density (A.1) transforms to a total space-time derivative under the on-shell nilpotent (anti-)dual BRST symmetry transformations $s_{(a)d}$. As a consequence, the action integral $S = \int dx \mathcal{L}_B$, remains invariant under $s_{(a)d}$. It goes without saying that the (anti-)co-BRST symmetries for the Lagrangian density $\mathcal{L}_B$ can be obtained from (A.4) by the replacements: $A_\mu \rightarrow A_\mu$, $C \rightarrow C$, $\bar{C} \rightarrow \bar{C}$, $\phi \rightarrow -\phi$, $\bar{\phi} \rightarrow -\bar{\phi}$. As far as the off-shell nilpotent (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities (without the Dirac fields) are concerned, we have the following [19]:

$$
\mathcal{L}_B = \mathcal{B}(E - m \bar{\phi}) - \frac{1}{2} B^2 + m E \bar{\phi} - \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi
$$

(A.5)

$$
\mathcal{L}_B = \mathcal{B}(E + m \bar{\phi}) - \frac{1}{2} B^2 - m E \bar{\phi} - \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi
$$

(A.6)

We point out that we have invoked new type of Nakanishi-Lautrup type auxiliary fields $(\mathcal{B}, \bar{\mathcal{B}})$ in the Lagrangian density (A.6) for the linearization of the kinetic and gauge-fixing terms. In our present endeavor, we have concentrated only on the Lagrangian density (A.5) with the Dirac fields $(\bar{\psi}, \psi)$ in interaction with the massive gauge field $(A_\mu)$ for the sake of brevity. However, one can also focus on the Lagrangian density (A.6) for which the (anti-)BRST as well as the (anti-)co-BRST symmetry transformations have been written in our earlier work [19] on the modified 2D Proca theory without Dirac fields. It can be trivially checked that (A.6) can also be generalized with Dirac fields $(\bar{\psi}, \psi)$ in interaction with massive gauge field $(A_\mu)$ as we have done in Eq. (9).

**Appendix B: On the Physical Analogue of $\delta = \pm \ast d \ast$**

As discussed in Sec. 5, it is a well-known fact that the (co-)exterior derivatives $(\delta) d$ of the differential geometry are connected to each-other by the precise relationship: $\delta = \pm \ast d \ast$ where $(\pm)$ signs are dictated by the dimensionality of the compact manifold without a boundary and the degree of the form that is invoked in the inner product [8-12]. The central purpose of our present Appendix is to provide the physical realization of this relationship in the terminology of the discrete and continuous symmetry transformations of our present 2D theory. In this context, we take a few explicit examples to demonstrate that all the key
ingredients of the algebraic relationship $\delta = \pm \ast d\ast$ can be physically realized. We note that, for the Dirac fields $(\bar{\psi}, \psi)$ and the gauge field $A_\mu$, we have the following [34]

$$\ast (\ast \psi) = +\psi, \quad \ast (\ast \bar{\psi}) = +\bar{\psi}, \quad \ast (\ast A_\mu) = -A_\mu,$$  \hspace{1cm} (B.1)

where $\ast$ stands for the discrete symmetry transformations (10). First of all, we can verify that the following relationships are correct, namely;

$$s_d \psi = + \ast s_b \ast \psi, \quad s_d \bar{\psi} = + \ast s_b \ast \bar{\psi}, \quad s_d A_\mu = - \ast s_b \ast A_\mu, \hspace{1cm} (B.2)$$

where the signs on the r.h.s. of (B.2) have been dictated by the signs in (B.1) as per the principles behind a duality invariant theory (see, e.g. [34] for details). In Eq. (B.2), the symbol $\ast$ stands, once again, for the discrete symmetry transformations (10). The nilpotent BRST symmetry transformations $s_b$ are given in Eq. (2) and the co-BRST symmetry transformations $s_d$ are quoted in Eq. (11). It is crystal clear that (B.2) provides the physical realization of $\delta = \pm \ast d\ast$ where there is a mapping: $\delta \leftrightarrow s_d$ and $d \leftrightarrow s_b$. Furthermore, the Hodge duality $\ast$ operation of differential geometry is realized in terms of the discrete symmetry transformations (10). We note that, exactly like Eq. (B.2), we have the validity of the following relationships between $s_{ab}$ and $s_{ad}$, namely;

$$s_{ab} \psi = + \ast s_{ad} \ast \psi, \quad s_{ab} \bar{\psi} = + \ast s_{ad} \ast \bar{\psi}, \quad s_{ab} A_\mu = - \ast s_{ad} \ast A_\mu, \hspace{1cm} (B.3)$$

which demonstrate that we have the mappings between the mathematical quantities ($\delta$) $d$ and the physical symmetry transformations ($s_{ab}$) $s_{ad}$ as: $\delta \leftrightarrow s_{ab}$, $d \leftrightarrow s_{ad}$. In addition, the notation $\ast$ in (B.3) denotes, once again, the discrete symmetry transformations (10).

We end this Appendix with the remarks that, it is the peculiarity of the specific two $(1 + 1)$-dimensional (2D) theory that we have also the validity of the inverse relationships corresponding to (B.2) and (B.3) as:

$$s_b \psi = + \ast s_d \ast \psi, \quad s_b \bar{\psi} = + \ast s_d \ast \bar{\psi}, \quad s_b A_\mu = - \ast s_d \ast A_\mu, \quad s_{ad} \psi = + \ast s_{ab} \ast \psi, \quad s_{ad} \bar{\psi} = + \ast s_{ab} \ast \bar{\psi}, \quad s_{ad} A_\mu = - \ast s_{ab} \ast A_\mu. \hspace{1cm} (B.4)$$

Thus, it appears that we have also the sanctity of the mappings: $d \leftrightarrow (s_d, s_{ad})$ and $\delta \leftrightarrow (s_b, s_{ab})$. However, this is not true as is evident from our discussions on the ghost number considerations (cf. Sec. 6) in the total quantum Hilbert space in terms of conserved charges corresponding to the continuous symmetry transformations generated by $s_b, s_{ab}, s_d, s_{ad}, s_w$ and $s_g$. It turns out that the ghost number increases by one when we apply the charges ($Q_b, Q_{ad}$) on a state with the ghost number $n$. On the contrary, the ghost number decreases by one due to the applications of ($Q_d, Q_{ab}$) on a state with the ghost number $n$ in the total quantum Hilbert space of states. Hence, the following identifications, namely;

$$d \leftrightarrow (s_b, s_{ad}), \quad \delta \leftrightarrow (s_d, s_{ab}), \hspace{1cm} (B.5)$$

are true as the conserved charges corresponding to the continuous symmetry transformations ($s_b, s_{ad}$) and ($s_d, s_{ab}$) capture the real and sacrosanct properties of $d$ and $\delta$. Furthermore, we note that we have not taken other fields: $\phi, \bar{\phi}, C, \bar{C}, B, \bar{B}$ of our theory in our present discussion. However, as pointed out earlier, these fields will obey exactly the same kind of relationships as has been respected by the $A_\mu$ field [cf. Eq. (33)].

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Appendix C: On the Discrete Symmetries in Component Form

We demonstrate here the invariance of the Dirac fields in interaction with a massive gauge field (i.e. \(-e \bar{\psi} \gamma^\mu A_\mu \psi\)) under the discrete symmetry transformations (10) where we take into account the component forms of the Dirac fields as well as the gauge field \((A_\mu)\). With the choices: \(\gamma_0 = \sigma_1, \gamma^0 = \gamma_0\), we note that we have the following for our 2D theory:

\[
\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*) \sigma_1 \equiv (\psi_2^*, \psi_1^*). \tag{C.1}
\]

It is elementary to verify that the kinetic term for the fermionic Dirac Lagrangian density (i.e. \(\bar{\psi} i \gamma^\mu \partial_\mu \psi\)) remains invariant under (10) because: \(\psi \to \bar{\psi}, \bar{\psi} \to \psi\). Written in the component form, the interaction term \((-e \bar{\psi} \gamma^\mu A_\mu \psi\)) in 2D is as follows:

\[
-e \bar{\psi} \gamma^\mu A_\mu \psi = -e \bar{\psi} \gamma^0 A_0 \psi - e \bar{\psi} \gamma^1 A_1 \psi
\equiv -e \psi_2^* A_0 \psi_1 + e \psi_2^* A_1 \psi_2 - e \psi_1^* A_1 \psi_1. \tag{C.2}
\]

In the above, we have used (C.1) and taken into account \(\gamma^0 = \gamma_0 = \sigma_1, \gamma^1 = -\gamma_1 = -i \sigma_2\). We have also considered the column vectors: \(\psi = (\psi_1, \psi_2)^T\) and \(A_\mu = (A_1, A_2)^T\). It is straightforward to check that the latter satisfies: \(A_\mu \to \mp i \varepsilon_{\mu\nu} A^\nu\) correctly in the matrix form, too. In other words, we obtain: \(A_0 \to \pm i A_1, A_1 \to \pm i A_0\) from the matrix forms of \(\varepsilon_{\mu\nu}\) and \(A_\mu\) where \(\varepsilon_{\mu\nu}\) is a 2 \(\times\) 2 matrix and \(A_\mu\) is a 2 \(\times\) 1 matrix.

To check the invariance of the interaction term \((-e \bar{\psi} \gamma^\mu A_\mu \psi\)) under (10), first of all, we note that we have the following transformations under (10), namely:

\[
-e \bar{\psi} \gamma^\mu A_\mu \psi \to -(\pm i e \gamma_5) \bar{\psi} \gamma^\mu (\mp i \varepsilon_{\mu\nu} A^\nu) \psi
= e \bar{\psi} \gamma_5 \gamma^\mu \varepsilon_{\mu\nu} A^\nu \psi. \tag{C.3}
\]

In the above, we have used: \(\gamma_5 \bar{\psi} = -\bar{\psi} \gamma_5\) (as \(\gamma_5 \gamma_0 = -\gamma_0 \gamma_5\)). Using the straightforward relationships: \(\gamma^\mu \varepsilon_{\mu\nu} = -\gamma_5 \gamma^\nu, \gamma_5^2 = I\), we observe that the explicit form of (C.3) is nothing but the l.h.s. of (C.2) which proves the point. However, we take into account the explicit expression (C.3) in the component form (with \(\gamma^0 = \gamma_0, \gamma^1 = -\gamma_1, A^0 = A_0, A^1 = -A_1\)) as

\[
\gamma^\mu \varepsilon_{\mu\nu} A^\nu = \gamma^0 \varepsilon_{01} A^1 + \gamma^1 \varepsilon_{10} A^0 = -\gamma_0 A_1 + \gamma_1 A_0, \tag{C.4}
\]

to obtain the following form of the interaction term [cf. Eq. (C.3)]

\[
-e \bar{\psi} \gamma_5 \gamma_0 A_1 \psi + e \bar{\psi} \gamma_5 \gamma_1 A_0 \psi. \tag{C.5}
\]

Substitutions of the component forms: \(\bar{\psi} = (\psi_2^*, \psi_1^*)\), \(\psi = (\psi_1, \psi_2)^T\), \(\gamma_5 = -\sigma_3, \gamma_0 = \sigma_1, \gamma_1 = i \sigma_2\) leads to the derivation of the following from (C.5):

\[
-e \bar{\psi} \gamma_5 \gamma_0 A_1 \psi = e \psi_2^* A_1 \psi_2 - e \psi_1^* A_1 \psi_1, \\
e \bar{\psi} \gamma_5 \gamma_1 A_0 \psi = -e \psi_2^* A_0 \psi_2 - e \psi_1^* A_0 \psi_1. \tag{C.6}
\]

The sum of the above two terms [cf. Eqs. (C.5), (C.6)] is nothing but the Eq. (C.2). Thus, even in the component form, the interaction term remains invariant under (10). There is a simpler way to verify the same thing. If we substitute \(\gamma_5 = \gamma_0 \gamma_1\) into (C.5) and use \(\gamma^0 = \gamma_0, \gamma^1 = -\gamma_1, \gamma_0^2 = I, \gamma_1^2 = -I\) and \(\gamma_0 \gamma_1 = -\gamma_1 \gamma_0\), we observe that Eq. (C.5) reduces to: \((-e \bar{\psi} \gamma^0 A_0 \psi - e \bar{\psi} \gamma^1 A_1 \psi\) which was the starting point in Eq. (C.2).
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