RESEARCH ARTICLE

Spectrally accurate energy-preserving methods for the numerical solution of the “good” Boussinesq equation

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In this paper we study the geometric numerical solution of the so called “good” Boussinesq equation. This goal is achieved by using a convenient space semi-discretization, able to preserve the corresponding Hamiltonian structure, then using energy-conserving Runge–Kutta methods in the Hamiltonian boundary value method class for the time integration. Numerical tests are reported, confirming the effectiveness of the proposed method.

KEYWORDS
blended iteration, energy-conserving methods, “good” Boussinesq equation, Hamiltonian boundary value methods, Hamiltonian PDEs, spectral methods

1 | INTRODUCTION

We here consider the efficient numerical solution of the “good” Boussinesq equation,

\[ w_{tt}(x, t) = -w_{xxxx}(x, t) + w_{xx}(x, t) + (w^2(x, t))_{xx}, \quad (x, t) \in [a, b] \times [0, \infty), \tag{1} \]

commonly used to describe small amplitude long waves propagation on the surface of shallow water. It is for this reason that the equation is often considered in several physical contexts, such as ocean and coastal engineering (as stressed, e.g., in [1, 2]). Moreover, the equation provides a balance between dispersion and nonlinearity that may lead to either the existence of solitons, or blowup solutions [3–9]. It is customary to simplify Equation (1), by defining the shifted variable

\[ u(x, t) = w(x, t) + \frac{1}{2}. \tag{2} \]
which transforms it into:

\[ u_{tt} = -u_{xxxx} + (u^2)_{xx}, \quad (x, t) \in [a, b] \times [0, \infty). \quad (3) \]

Equation (3) is further cast as the first order system (see, e.g., [10–12]),

\[ \begin{align*}
  u_t & = v_x, \\
  v_t & = -u_{xxx} + (u^2)_x, \\
  (x, t) & \in [a, b] \times [0, \infty),
\end{align*} \quad (4) \]

which is completed with the initial conditions

\[ \begin{align*}
  u(x, 0) & = u_0(x), \\
  v(x, 0) & = v_0(x), \\
  x & \in [a, b],
\end{align*} \quad (5) \]

and periodic boundary conditions. Hereafter, we shall assume that \( u_0(x) \) and \( v_0(x) \) are such that the solution of problem (4) and (5) is regular enough, as a periodic function on \([a, b]\), for all \( t \geq 0 \).

The numerical solution of (1), (3) or (4) has been developed along different directions, ranging from the pseudo-spectral or splitting approach [13–19, 46], up to finite-difference and finite-element schemes [20–24, 47], as well as structure-preserving methods [10, 25, 26] and energy-preserving methods [27, 28]. In particular, [11, 12] consider an energy-conserving strategy based on the Hamiltonian boundary value methods (HBVMs) for the “good” Boussinesq and the improved Boussinesq equation, respectively, while a second-order symplectic method preserving the energy and the momentum is considered in [29].

Hereafter, we shall focus on the geometric numerical solution of the first order form (4), where, by geometric it is meant that we will provide a numerical solution able to retain important geometric properties of the continuous one. In particular, we shall see that the system (4) has an Hamiltonian structure, which can be preserved by a suitable space semi-discretization. The time integration will be then performed by using energy-conserving methods in the HBVMs class [30–37], and this will allow us to retain many geometric properties of the solution, as later specified: as matter of fact, this paper follows a systematic study of the application of HBVMs for efficiently solving Hamiltonian partial differential equations (PDEs) [30, 31, 38–40].

It must be stressed that such methods become highly efficient only provided that one is able to fully exploit the specific form of the problem at hand. For this reason, all the specific implementation issues of the proposed method are worked-out, with a main focus on the iterative solution of the resulting nonlinear system of equations to be solved at each time-step. As a result, a spectrally accurate numerical method, both in space and time, able to retain relevant geometric properties, is derived.

With these premises, the structure of the paper is as follows: in Section 2 we study the geometric properties of (4); in Section 3 we study a convenient space semi-discretization; in Section 4 we sketch the main facts concerning HBVMs, along with their efficient implementation; in Section 5 we report some numerical tests aimed at assessing the geometric properties of the resulting method; at last, a few concluding remarks are given in Section 6.

2 | HAMILTONIAN FORMULATION

We here study the geometric properties, in particular the invariants, of the solution of problem (4) and (5), when equipped with periodic boundary conditions. For this purpose, let us define, for a generic function \( f(x, t) \), the linear functional

\[ \mathcal{L}[f](t) := \int_{a}^{b} f(x, t) dx, \quad \forall t \geq 0. \quad (6) \]

The following result then holds true.
Theorem 1 Assume that the solution of (4) and (5) is $C^3[a,b]$ as a periodic function. Then, with reference to (6), for all $t \geq 0$ one has:

$$\mathcal{L}[u](t) \equiv \int_a^b u_0(x) dx, \quad \mathcal{L}[v](t) \equiv \int_a^b v_0(x) dx.$$ \hfill (7)

Proof. Taking the integral in space over $[a,b]$ of both sides of the two equations in (4) one obtains, because of the periodicity in space of $u, v,$ and their space derivatives,

$$\int_a^b u_t(x,t) dx \equiv \dot{\mathcal{L}}[u](t) = 0, \quad \int_a^b v_t(x,t) dx \equiv \dot{\mathcal{L}}[v](t) = 0,$$

from which (7) follows, since

$$\mathcal{L}[u](0) = \int_a^b u_0(x) dx, \quad \mathcal{L}[v](0) = \int_a^b v_0(x) dx.$$

System (4) can be formally written in a more compact way as:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \left| J_2 \right| \otimes \partial_x \delta \mathcal{H}[u,v],$$ \hfill (8)

with

$$J_2 = \begin{pmatrix} -1 & 1 \end{pmatrix} \quad \text{and} \quad \delta \mathcal{H}[u,v] = \begin{pmatrix} \delta_u \mathcal{H}[u,v] \\ \delta_v \mathcal{H}[u,v] \end{pmatrix}$$ \hfill (9)

the vector of the functional derivatives of the Hamiltonian functional

$$\mathcal{H}[u,v] = \frac{1}{2} \int_a^b \left( v^2 + \frac{2}{3} u^3 + u_x^2 \right) dx =: \int_a^b L(v,u,u_x) dx.$$ \hfill (10)

In fact, one has:

$$\delta_u \mathcal{H}[u,v] = (\delta_x - \partial_x \delta_x) L(v,u,u_x) = u^2 - u_{xx},$$
$$\delta_v \mathcal{H}[u,v] = \partial_y L(v,u,u_x) = v.$$

Therefore, the “good” Boussinesq equation is an instance of a second order Hamiltonian PDE.

Theorem 2 Assume that the solution of (4) is $C^3[a,b]$ as a periodic function. Then, the Hamiltonian functional (10) is constant along the solution of (4).

Proof. In fact, by the hypotheses on $u$ and $v$, one has:

$$\mathcal{H}[u,v] = \frac{1}{2} \int_a^b \left[ 2vv_t + 2u^2 u_t + 2u_x u_{xt} \right] dx = \int_a^b \left[ v(-u_{xxx} + 2uu_x) + u^2 v_x + u_x v_{xx} \right] dx$$
$$= \int_a^b \left( 2uvu_x + u^2 v_x + u_x v_{xx} - vu_{xxx} \right) dx = \left[ u^2 v + v_x u_x - vu_{xx} \right]_{x=a}^{x=b} = 0,$$

because of the periodicity in space of the functions $u$ and $v$, as well as their derivatives w.r.t. $x$. \hfill □
We also consider the quadratic functional

$$\mathcal{M}[u, v] = \int_a^b uv \, dx,$$

(11)
corresponding to the momentum (or the impulse), for which the following result holds true.

**Theorem 3**  In the same hypotheses of Theorem 2, the quadratic functional (11) is constant along the solution of (4).

**Proof.** In fact, using arguments similar to those used in the previous theorem, one has:

$$\dot{\mathcal{M}}[u, v] = \int_a^b (u_t v + uv_t) \, dx = \int_a^b [v v_x + u (-u_{xxx} + 2 u u_x)] \, dx $$

$$= \int_a^b \left( \frac{1}{2} v^2 + \frac{2}{3} u^3 + \frac{1}{2} u_x^2 - uu_{xx} \right) \, dx = \left[ \frac{1}{2} v^2 + \frac{2}{3} u^3 + \frac{1}{2} u_x^2 - uu_{xx} \right]_{x=a}^{x=b} = 0,$$

by virtue of the periodicity in space of the functions $u$ and $v$, as well as their derivatives w.r.t. $x$. □

In particular, the conservation of the functionals $\mathcal{L}[u], \mathcal{L}[v], \mathcal{H}[u, v],$ and $\mathcal{M}[u, v]$ in (7), (10), and (11) represents the relevant geometric properties of the solution we are interested in, which we shall reproduce in the discrete approximation.

### 3  SPACE DISCRETIZATION

Since the solution is periodic in space, we now discretize the space variable along the following orthonormal basis for periodic $L^2[a, b]$ functions:

$$c_j(x) = \sqrt{\frac{2 - \delta_{j0}}{b - a}} \cos \left( 2\pi j \frac{x - a}{b - a} \right), \quad j \geq 0, \quad s_j(x) = \sqrt{\frac{2}{b - a}} \sin \left( 2\pi j \frac{x - a}{b - a} \right), \quad j \geq 1. \quad (12)$$

In fact, one verifies that, for all allowed indexes $i, j$:

$$\int_a^b c_i(x)c_j(x) \, dx = \delta_{ij} = \int_a^b s_i(x)s_j(x) \, dx, \quad \int_a^b s_i(x)c_j(x) \, dx = 0,$$

(13)

where, as is usual, $\delta_{ij}$ denotes the Kronecker delta. Consequently, for suitable time dependent coefficients $\alpha_j(t), \beta_j(t), \xi_j(t), \eta_j(t)$, the following expansions are derived:

$$u(x, t) = \alpha_0(t)c_0(x) + \sum_{j \geq 1} (\alpha_j(t)c_j(x) + \beta_j(t)s_j(x)), $$

$$v(x, t) = \xi_0(t)c_0(x) + \sum_{j \geq 1} (\xi_j(t)c_j(x) + \eta_j(t)s_j(x)).$$

(14)

The following result holds true.

**Theorem 4**  In order to conserve $\mathcal{L}[u]$ and $\mathcal{L}[v]$, see (6) and (7), in the expansions (14) one must have:

$$\alpha_0(t)c_0(x) \equiv \frac{1}{b - a} \int_a^b u_0(x) \, dx =: \hat{u}_0, \quad \xi_0(t)c_0(x) \equiv \frac{1}{b - a} \int_a^b v_0(x) \, dx =: \hat{v}_0.$$

(15)
Proof. The statements follow from the fact that \( \int_a^b c_j(x)dx = \int_a^b s_j(x)dx = 0 \), for all \( j = 1, 2, \ldots \) and, consequently, from (14) one obtains:

\[
\int_a^b u(x, t)dx = (b-a)\hat{u}_0 = \int_a^b u_0(x)dx, \quad \int_a^b v(x, t)dx = (b-a)\hat{v}_0 = \int_a^b v_0(x)dx.
\]

Taking into account (15), the previous expansions (14) become:

\[
\begin{align*}
    u(x, t) &= \hat{u}_0 + \sum_{j \geq 1} (\alpha_j(t)c_j(x) + \beta_j(t)s_j(x)) \equiv \hat{u}_0 + \omega(x)^T q(t), \\
    v(x, t) &= \hat{v}_0 + \sum_{j \geq 1} (\xi_j(t)c_j(x) + \eta_j(t)s_j(x)) \equiv \hat{v}_0 + \omega(x)^T p(t),
\end{align*}
\]

having set the infinite vectors

\[
\omega(x) = \begin{pmatrix} s_1(x) \\ c_1(x) \\ s_2(x) \\ c_2(x) \\ \vdots \end{pmatrix}, \quad q(t) = \begin{pmatrix} \beta_1(x) \\ \alpha_1(x) \\ \beta_2(x) \\ \alpha_2(x) \\ \vdots \end{pmatrix}, \quad p(t) = \begin{pmatrix} \eta_1(x) \\ \xi_1(x) \\ \eta_2(x) \\ \xi_2(x) \\ \vdots \end{pmatrix}.
\]

Moreover, by setting \( I_2 \) the identity matrix of dimension 2, \( J_2 \) the skew-symmetric and orthogonal matrix defined in (9), and the infinite matrix

\[
D = \frac{2\pi}{b-a} \begin{pmatrix} 1 & 2 & 3 & \cdots \end{pmatrix},
\]

the required partial derivatives of \( u(x, t) \) and \( v(x, t) \) can be easily computed as follows:

\[
\begin{align*}
    v_x(x, t) &= \omega(x)^T p(t), \quad v_{xx}(x, t) = [(D \otimes J_2)\omega(x)]^T p(t), \\
    u_x(x, t) &= \omega(x)^T q(t), \quad u_{xx}(x, t) = [(D \otimes J_2)\omega(x)]^T q(t), \\
    u_{xxx}(x, t) &= [(D \otimes J_2)^2 \omega(x)]^T q(t), \quad u_{xxxx}(x, t) = [(D \otimes J_2)^3 \omega(x)]^T q(t),
\end{align*}
\]

due to the fact that

\[
\omega'(x) = (D \otimes J_2)\omega(x).
\]

Consequently, by also considering that

\[
J_2^2 = -J_2, \quad J_2^3 = -I_2, \quad \int_a^b \omega(x)dx = 0, \quad \int_a^b \omega(x)\omega(x)^T dx = I,
\]

with \( 0 \) the zero vector and \( I \) the identity operator, the following result can be proved.

**Theorem 5** System (4) can be cast in Hamiltonian form as

\[
\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \left( D \otimes J_2^T \right) p + \int_a^b \omega(x)(\tilde{u}_0 + \omega(x)^T q) dx \equiv (|J_2| \otimes D \otimes J_2^T)\nabla H(q, p),
\]

with Hamiltonian

\[
H(q, p) = \frac{1}{2} \left( p^T q + q^T (D^2 \otimes I_2) q + \frac{2}{3} \int_a^b (\tilde{u}_0 + \omega(x)^T q)^3 dx \right).
\]

This latter, in turn, is equivalent, up to a constant, to the functional (10), via the transformations (15)–(19).
Finally, by using similar arguments, the following result can be proved.

**Theorem 6** The quadratic invariant (11) is equivalent, up to a constant, to

\[ M(q, p) = q^\top p. \]  

(24)
Proof. From (11), (16), and (21), one has:

\[
\mathcal{M}[u, v] = \int_a^b uvdx = \int_a^b (\hat{u}_0 + q^T \omega(x))(\hat{v}_0 + \omega(x)^Tp)dx
\]

\[
= \int_a^b \hat{u}_0\hat{v}_0dx + (\hat{u}_0p + \hat{v}_0q)^T \int_a^b \omega(x)dx + q^T \int_a^b \omega(x)\omega(x)^T dxp
\]

\[
= (b - a)\hat{u}_0\hat{v}_0 + q^T p = (b - a)\hat{u}_0\hat{v}_0 + M(q, p).
\]

As is clear, in order to obtain a computational method, the infinite series in (16) have to be truncated at a convenient index \(N\). In so doing, the infinite vectors in (17) and matrix \(D\) in (18) become of dimension \(2N\) and \(N\), respectively, that is,

\[
\omega(x) = \begin{pmatrix} s_1(x) \\ c_1(x) \\ \vdots \\ s_N(x) \\ c_N(x) \end{pmatrix}, \quad q(t) = \begin{pmatrix} \beta_1(t) \\ \alpha_1(t) \\ \vdots \\ \beta_N(t) \\ \alpha_N(t) \end{pmatrix}, \quad p(t) = \begin{pmatrix} \eta_1(t) \\ \xi_1(t) \\ \vdots \\ \eta_N(t) \\ \xi_N(t) \end{pmatrix}, \quad D = \frac{2\pi}{b - a} \begin{pmatrix} 1 & \cdots & N \end{pmatrix},
\]

which will be assumed hereafter. Consequently, (19) continues formally to hold, as well as the result of Theorem 4, even though now the truncated approximations to \(u\) and \(v\) do not satisfy Equations (4) anymore. However, in the spirit of Galerkin methods, by imposing the residual be orthogonal to the functional space spanned by the entries of \(\omega(x)\), also the results of Theorems 5 and 6 continue formally to hold, with the only difference that now the truncated versions of \(H(q, p)\) and \(M(q, p)\) do not coincide, up to a constant, with the functionals (10) and (11), respectively. Nevertheless, it is known that, upon regularity assumptions on \(u\) and \(v\), the truncated series \((\hat{u}_0 + \omega(x)^T q)\) and \((\hat{v}_0 + \omega(x)^T p)\) converge more than exponentially to them, as well as the truncated version of \(H(q, p)\) and \(M(q, p)\) to the corresponding functionals, as \(N \to \infty\). This phenomenon is usually referred to as spectral accuracy (see, e.g., [41]).

For completeness, we mention that, in order to obtain a fully semi-discrete problem, the integrals appearing in (22) and (23) need to be evaluated. In the present case, since the Hamiltonian is a polynomial of degree 3, this can be done exactly (see, e.g., [39, Theorem 7]) by using a composite trapezoidal rule based at the evenly spaced points:

\[
x_i = a + i \frac{b - a}{m}, \quad i = 0, 1, \ldots, m,
\]

with \(m = 2N + 1\) (for (22)) and \(m = 3N + 1\) (for (23)), respectively.

4 | HAMILTONIAN BOUNDARY VALUE METHODS

In this section, we recall the main facts about HBVMs, which constitute a class of energy-conserving Runge–Kutta methods for Hamiltonian problems. Moreover, we study their efficient implementation for solving problem (22)–(25). HBVMs have been investigated in a series of papers (e.g., [32–34], see also the monograph [30]) for solving Hamiltonian problems, and have been developed in a number of directions (see, e.g., the recent review paper [31] and references therein). More recently, they have been successfully used to solve Hamiltonian PDEs [30, 31, 38–40], and this paper belongs to this last field of investigation.
In more detail, for all $k \geq s$, the HBVM($k$, $s$) method is the $k$-stage Runge–Kutta method with Butcher tableau given by

$$
\begin{array}{c|cccc}
  c & I_sP_s^T\Omega \\
  \hline
  b^T
\end{array}
$$

(27)

where, by setting \{\$P_j\$\} the Legendre polynomials shifted and scaled in order to be orthonormal on the interval $[0, 1]$,

$$
c = (c_1, \ldots, c_k)^T, \quad b = (b_1, \ldots, b_k)^T, \quad \Omega = \begin{pmatrix}
  b_1 \\
  \vdots \\
  b_k
\end{pmatrix},
$$

$$
I_s = \begin{pmatrix}
  \int_0^{c_1} P_0(x)dx & \cdots & \int_0^{c_1} P_{s-1}(x)dx \\
  \vdots & \ddots & \vdots \\
  \int_0^{c_k} P_0(x)dx & \cdots & \int_0^{c_k} P_{s-1}(x)dx
\end{pmatrix}, \quad P_s = \begin{pmatrix}
  P_0(c_1) & \cdots & P_{s-1}(c_1) \\
  \vdots & \ddots & \vdots \\
  P_0(c_k) & \cdots & P_{s-1}(c_k)
\end{pmatrix},
$$

(28)

with $(c_i, b_i)$ the Legendre abscissae and weights of the Gauss interpolatory quadrature formula of order $2k$ (i.e., $P_k(c_i) = 0, \ i = 1, \ldots, k$).

When applied for solving the ODE-IVPs

$$
\dot{y} = f(y), \quad y(0) = y_0 \in \mathbb{R}^m,
$$

(29)

with a stepsize $h$, the Runge–Kutta method (27) and (28) implicitly defines a polynomial approximation $\sigma \in \Pi_s$ such that

$$
\sigma(0) = y_0, \quad \sigma(h) = y_1 \approx y(h),
$$

(30)

providing, in the case of Hamiltonian problems, relevant conservation properties, as is specified by the following theorem [30, 31, 34].

**Theorem 7** For all $k \geq s$, a HBVM($k$, $s$) method used with stepsize $h$:

- is symmetric and $y_1 - y(h) = O(h^{2s+1})$;
- when $k = s$, it coincides with the symplectic $s$-stage Gauss method.

Moreover, when solving an Hamiltonian system, that is, in (29) $f(y) = J^T\nabla H(y)$ with $f^T = -J$:

- it is energy-conserving when the Hamiltonian $H$ is a polynomial and $\deg H \leq 2k/s$;
- conversely, $H(y_1) - H(y_0) = O(h^{2k+1})$.

**Remark 1** From the last two points in Theorem 7, one has that, by choosing $k$ large enough, either an exact energy-conservation can be gained, in the polynomial case, or a “practical” energy-conservation can be obtained in the general case. In fact, in the latter case, it is enough that the energy error falls within the round-off error level.

As a consequence, one has the following conservation result.

**Corollary 1** For all $k \geq \frac{3}{2}s$, the HBVM($k$, $s$) method is energy-conserving and of order $2s$, when used for solving the Hamiltonian problem (22) and (23).

**Proof.** In fact, in such a case, $\deg H = 3 \leq 2k/s$, for all $k \geq \frac{3}{2}s$. ■
We now sketch the efficient implementation of HBVMs, in view of their application for solving the semi-discrete problem (22)–(25). To begin with, one of the main features of a HBVM($k$, $s$) method is that the discrete problem generated by the application of the method has (block) dimension $s$, independently of the number of stages $k$. This feature, in turn, allows the use of possibly much larger values of $k$, w.r.t. $s$. The key point for this [42] stems from the fact that the polynomial approximation (30) has degree $s$, and the discrete problem can be cast in terms of its unknown coefficients. In more details, the equation for the $k$ stages $Y_1, \ldots, Y_k$ of the Runge–Kutta method (27) solving (29) can be written as

$$Y = e \otimes y_0 + hI_s P_s^T \Omega \otimes I_m f(Y),$$  \hspace{1cm} (31)

having set

$$Y = \left( \begin{array}{c} Y_1 \\ \vdots \\ Y_k \end{array} \right), \quad f(Y) = \left( \begin{array}{c} f(Y_1) \\ \vdots \\ f(Y_k) \end{array} \right), \quad e = \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) \in \mathbb{R}^k,$$

with the new approximation given by

$$y_1 = y_0 + h \sum_{i=1}^{k} b_if(Y_i).$$  \hspace{1cm} (32)

By defining the block vector of dimension $s$:

$$\gamma = \left( \begin{array}{c} y_0 \\ \vdots \\ y_{s-1} \end{array} \right) := P_s^T \Omega \otimes I_m f(Y),$$  \hspace{1cm} (33)

one then obtains that (31) can be re-written as:

$$Y = e \otimes y_0 + hI_s \otimes I_m \gamma,$$  \hspace{1cm} (34)

which, substituted at the right-hand side in (33), provides us with the following equation,

$$\gamma = P_s^T \Omega \otimes I_m f(e \otimes y_0 + hI_s \otimes I_m \gamma),$$  \hspace{1cm} (35)

having block dimension $s$. It can be easily seen that:

- the polynomial $\sigma$ mentioned in (30) is given by

$$\sigma(ch) = y_0 + h \sum_{j=0}^{s-1} \int_0^c P_j(x) dx y_j, \quad c \in [0, 1];$$  \hspace{1cm} (36)

- the stages of the $k$-stage HBVM($k$, $s$) method are defined by

$$Y_i = \sigma(ch), \quad i = 1, \ldots, k,$$

- the new approximation (32) is given by, setting $c = 1$ in (36):

$$y_1 = y_0 + h \gamma_0.$$  \hspace{1cm} (37)

Consequently, in order to implement the step of an HBVM($k$, $s$) method, one needs to solve the discrete problem (35), that is, the equation

$$F(\gamma) := \gamma - P_s^T \Omega \otimes I_m f(e \otimes y_0 + hI_s \otimes I_m \gamma) = 0.$$  \hspace{1cm} (38)
This equation, which could in principle be solved by using a straightforward fixed-point iteration,

\[ \gamma^{\ell+1} = P_\ell^T \Omega I_m \left( e \otimes y_0 + h I_s \otimes I_m \gamma^{\ell} \right), \quad \ell = 0, 1, \ldots, \]

actually requires, in the case of problem (22)–(25), the use of a Newton-type iteration, in order to avoid the use of very small stepsizes (indeed, of the order of \( \| D \|^{-3} \propto N^{-3} \), which becomes very small when large values of \( N \) are considered). For this purpose, let us consider the simplified Newton iteration for solving (38) which, by considering that (see, e.g., [30, 42])

\[ \Sigma = I_m - h \rho \; f'(y_0), \quad \text{with} \quad \rho_s = \min_{\lambda \in \sigma(X_s)} | \lambda |, \]

(42)

with \( f' \) the Jacobian of the function \( f \) in (29). This iteration, in turn, requires the factorization of the matrix

\[ [I - h X_s \otimes f'(y_0)] \]

(41)

having dimension \( s \) times larger than that of \( f'(y_0) \). It can be proved that this iteration can be conveniently replaced by a corresponding blended iteration [30, 42, 43] which, having set \( \Sigma = I_m - h \rho_s f'(y_0) \),

\[ \eta^\ell = -F(\gamma^\ell), \quad \eta_1^\ell = (\rho_s X_s^{-1} \otimes I_m) \eta^\ell, \]

(43)

\[ \gamma^{\ell+1} = \gamma^\ell - I_s \otimes \Sigma^{-1} [\eta_1^\ell + I_s \otimes \Sigma^{-1} (\eta^\ell - \eta_1^\ell)], \quad \ell = 0, 1, \ldots, \]

Consequently, only the factorization of matrix \( \Sigma \) in (42) is needed, independently of \( s \). This, in turn, allows the use of relatively large values of \( s \). Moreover, in the case of the problem (22)–(25), one has the further simplification that the Jacobian of the right-hand side can be conveniently approximated by the linear part alone, that is,

\[ \left( \begin{array}{cc} D^3 \otimes J^T_2 \\ \tau(D \otimes J^T_2) \\ -\tau(D \otimes J^T_2) \end{array} \right), \]

so that matrix \( \Sigma \) in (42) becomes, by setting hereafter \( I \in \mathbb{R}^{N \times N} \) the identity matrix and \( h \) the used time step,

\[ \Sigma = \left( \begin{array}{cc} I \otimes I_2 & -\tau (D \otimes J^T_2) \\ -\tau (D \otimes J^T_2) & I \otimes I_2 \end{array} \right), \quad \text{where} \quad \tau = \rho_s h. \]

(43)

Consequently, its inverse has to be computed only once for all.

It must be stressed that the particular structure of the Jacobian matrix and, therefore, of matrix \( \Sigma \), is of paramount importance to derive an efficient numerical method, based on the use of HBVMs. Clearly, this structure is strictly related to the equation at hand, and this fact has been investigated for many Hamiltonian PDEs [38–40]. In particular, for the “good” Boussinesq equation, the following result holds true.

**Theorem 8** With reference to matrix \( \Sigma \) defined in (43), one has:

\[ \Sigma^{-1} = \left( \begin{array}{cc} D_1 \otimes I_2 & \tau(D_1 D \otimes J^T_2) \\ \tau(D_1 D \otimes J^T_2) & D_1 \otimes I_2 \end{array} \right), \quad \text{with} \quad D_1 = (I + \tau^2 D^4)^{-1}. \]
FIGURE 1   Plot of $\frac{1}{2} - u(x, t)$ for problem (4), and (48) and (49)

Proof. In fact, one has:

$$\begin{pmatrix}
I \otimes I_2 \\
I \otimes J_2^T
\end{pmatrix}
\begin{pmatrix}
I \otimes I_2 \\
I \otimes J_2
\end{pmatrix}
= (I \otimes I_2) \begin{pmatrix}
I \\
-\tau D^3 \\
\end{pmatrix}

and

$$\begin{pmatrix}
I \\
\tau D^3 \\
I
\end{pmatrix}
^{-1}
= (I_2 \otimes D_1) \begin{pmatrix}
I \\
-\tau D^3 \\
I
\end{pmatrix},$$

with $D_1$ defined as in (44). Consequently,

$$\Sigma^{-1} = (I_2 \otimes D_1) \begin{pmatrix}
I \\
\tau D^3 \\
I
\end{pmatrix}
\begin{pmatrix}
I \otimes I_2 \\
\tau(D \otimes J_2^T) \\
I \otimes I_2
\end{pmatrix}
= (D_1 \otimes I_2) \begin{pmatrix}
D_1 \otimes I_2 \\
\tau(D_1 D \otimes J_2^T) \\
D_1 \otimes I_2
\end{pmatrix}.$$  

As a consequence of the previous theorem, one has that matrix $\Sigma^{-1}$ in (44) can be computed with a cost which is linear in the dimension of the problem, since it has blocks with a diagonal structure. Moreover, it can be conveniently stored by using 3 vectors of dimension $N$ (containing the diagonal entries of $D_1$, $\tau D_1 D$, and $\tau D_1 D^3$).

4.1 | Spectral HBVMs

The previous blended implementation of HBVMs is particularly interesting, since it allows the use of relatively large values of $s$. This, in turn, allows to use HBVMs as spectral methods in time [44, 45], so that one obtains, for the used finite precision arithmetic, the maximum possible accuracy compatible with the considered time-step. We here sketch the use of HBVMs as spectral methods (which we shall refer to as spectral HBVMs or, in short, SHBVMs): further details can be found in the previous references [44, 45].
To begin with, let us consider the problem (29) which, by considering the time interval $[0, h]$ and expanding the right-hand side along the Legendre basis, can be rewritten as:

$$\dot{y}(c h) = \sum_{j \geq 0} P_j(c) \gamma_j(y), \quad c \in [0, 1], \quad \gamma_j(y) = \int_0^1 P_j(c) f(y(c h)) dc, \quad j = 0, 1, \ldots.$$  

Then, integrating term by term and imposing that $y(0) = y_0$, one obtains that the solution is formally given by:

$$y(c h) = y_0 + h \sum_{j \geq 0} \int_0^c P_j(x) dx \gamma_j(y), \quad c \in [0, 1].$$

### Table 1: Numerical results with time-step $h = 80/n$ for the solitary wave problem (4), and (48) and (49)

| Gauss 1 | n   | Time | $e_u$   | Rate | $e_H$     | Rate | $e_M$     | Rate | –       |
|---------|-----|------|---------|------|-----------|------|-----------|------|---------|
| 8,000   | 34.3| 3.87e-06 | –      | 2.80e-12 | –   | 2.78e-15 | –    | –       |
| 9,600   | 40.8| 2.69e-06 | 2.0    | 1.37e-12 | 3.9 | 2.55e-15 | –    | –       |
| 11,200  | 47.8| 1.97e-06 | 2.0    | 7.62e-13 | 3.8 | 2.78e-15 | –    | –       |
| 12,800  | 54.4| 1.51e-06 | 2.0    | 4.62e-13 | 3.8 | 2.22e-15 | –    | –       |
| 14,400  | 61.6| 1.19e-06 | 2.0    | 3.04e-13 | 3.6 | 3.77e-15 | –    | –       |
| 16,000  | 76.6| 9.67e-07 | 2.0    | 2.10e-13 | 3.5 | 4.22e-15 | –    | –       |

| Gauss 2 | n   | Time | $e_u$   | Rate | $e_H$     | Rate | $e_M$     | Rate | –       |
|---------|-----|------|---------|------|-----------|------|-----------|------|---------|
| 1,600   | 19.0| 1.01e-09 | –      | 3.73e-14 | –   | 1.67e-15 | –    | –       |
| 2,400   | 28.3| 1.99e-10 | 4.0    | 3.55e-14 | ** | 1.67e-15 | –    | –       |
| 3,200   | 33.3| 6.30e-11 | 4.0    | 3.73e-14 | ** | 1.78e-15 | –    | –       |
| 4,000   | 41.8| 2.58e-11 | 4.0    | 4.44e-14 | ** | 1.89e-15 | –    | –       |

| HBVM(2,1) | n   | Time | $e_u$   | Rate | $e_H$     | Rate | $e_M$     | Rate | –       |
|-----------|-----|------|---------|------|-----------|------|-----------|------|---------|
| 8,000     | 54.3| 3.97e-06 | –      | 1.24e-14 | –   | 2.97e-12 | –    | –       |
| 9,600     | 66.0| 2.76e-06 | 2.0    | 1.42e-14 | 3.9 | 1.45e-12 | –    | –       |
| 11,200    | 76.8| 2.03e-06 | 2.0    | 1.24e-14 | 3.9 | 8.01e-13 | –    | –       |
| 12,800    | 87.7| 1.55e-06 | 2.0    | 1.24e-14 | 3.7 | 4.89e-13 | –    | –       |
| 14,400    | 98.6| 1.23e-06 | 2.0    | 1.24e-14 | 3.7 | 3.17e-13 | –    | –       |
| 16,000    | 109.1| 9.93e-07 | 2.0    | 1.42e-14 | 3.4 | 2.23e-13 | –    | –       |

| HBVM(3,2) | n   | Time | $e_u$   | Rate | $e_H$     | Rate | $e_M$     | Rate | –       |
|-----------|-----|------|---------|------|-----------|------|-----------|------|---------|
| 1,600     | 24.0| 9.96e-10 | –      | 1.07e-14 | –   | 3.97e-14 | –    | –       |
| 2,400     | 32.1| 1.97e-10 | 4.0    | 1.07e-14 | 3.9 | 3.96e-14 | –    | –       |
| 3,200     | 37.8| 6.22e-11 | 4.0    | 1.42e-14 | ** | 4.25e-14 | –    | –       |
| 4,000     | 47.3| 2.55e-11 | 4.0    | 1.24e-14 | ** | 4.02e-14 | –    | –       |

| SHBVM ($k = 15, s = 10$) | n   | Time | $e_u$   | Rate | $e_H$     | Rate | $e_M$     | Rate | –       |
|-------------------------|-----|------|---------|------|-----------|------|-----------|------|---------|
| 80                      | 9.1 | 4.70e-14 | –      | 8.88e-15 | –   | 3.29e-14 | –    | –       |

** when it cannot be evaluated.
Furthermore, for a suitably regular function \( f \), one has that
\[
\gamma_j(y) \to 0, \quad \text{as} \quad j \to \infty. \quad (45)
\]
Consequently, \( y(ch) \) can be approximated within machine accuracy by the polynomial (36), provided that the quadrature \((c_i, b_i)\) is accurate enough (i.e., \( k \) is large enough), and \( s \) is the first index such that
\[
\|\gamma_{s-1}\| \leq tol \cdot \max_{j=0,\ldots,s-1} \|\gamma_j\|, \quad (46)
\]
with \( tol \sim u \), \( u \) being the machine epsilon of the considered finite precision arithmetic.

In the practice, however, the numerical evaluation of the coefficients \( \gamma_j \approx \gamma_j(y) \) makes them “stagnate” (in norm) around a small value, rather than tending to 0, according to (45): in such a case, the tolerance \( tol \) in (46) is more conveniently chosen in order to avoid using the coefficients with a stagnating norm, since this means that they are not reliably computed. This criterion will always be used in the sequel, when implementing SHBVMs.

5 | NUMERICAL TESTS

In this section, we report a few numerical tests concerning the numerical solution of problem (22)–(25) with initial conditions given by (see (5)):
\[
q(0) = \int_a^b \omega(x)u_0(x)\,dx, \quad p(0) = \int_a^b \omega(x)v_0(x)\,dx. \quad (47)
\]
In particular, we compare the following methods:

- the symplectic \( s \)-stage Gauss methods (which we shall denote Gauss \( s \)), having order \( 2s \), for \( s = 1, 2 \). Such methods are expected to conserve the quadratic invariant (24) but only approximately the Hamiltonian (23);
- the energy-conserving HBVM(2,1) and HBVM(3,2) methods, having respectively order 2 and 4. Such method conserve the Hamiltonian (23) but only approximately the quadratic invariant (24);
- the spectral HBVM (SHBVM) method, using a value of \( s \) and \( k = \lceil 1.5s \rceil \) large enough so that the maximum possible accuracy is gained. As sketched in Section 4.1, the value of \( s \) is obtained
TABLE 2 Numerical results with time-step $h = 50/n$ for the spread of solitary waves problem (4)–(51)

| Gauss 1 | Time | $e_u$     | Rate | $e_H$     | Rate | $e_M$ | –   |
|---------|------|-----------|------|-----------|------|-------|-----|
| 5,000   | 21.3 | 1.31e-07  | –    | 1.66e-10  | 0.0  | 4.23e-17 | –   |
| 6,000   | 25.6 | 9.13e-08  | 2.0  | 1.15e-10  | 2.0  | 4.08e-17 | –   |
| 7,000   | 29.9 | 6.71e-08  | 2.0  | 8.48e-11  | 2.0  | 3.96e-17 | –   |
| 8,000   | 34.2 | 5.13e-08  | 2.0  | 6.49e-11  | 2.0  | 4.04e-17 | –   |
| 9,000   | 38.7 | 4.06e-08  | 2.0  | 5.13e-11  | 2.0  | 3.92e-17 | –   |
| 10,000  | 42.7 | 3.29e-08  | 2.0  | 4.16e-11  | 2.0  | 3.94e-17 | –   |

| Gauss 2 | Time | $e_u$     | Rate | $e_H$     | Rate | $e_M$ | –   |
|---------|------|-----------|------|-----------|------|-------|-----|
| 1,000   | 10.3 | 1.81e-11  | –    | 7.11e-14  | –    | 3.97e-17 | –   |
| 1,500   | 15.4 | 3.57e-12  | 4.0  | 6.04e-14  | **  | 4.00e-17 | –   |
| 2,000   | 17.7 | 1.14e-12  | 4.0  | 6.57e-14  | **  | 4.02e-17 | –   |
| 2,500   | 22.0 | 4.80e-13  | 3.9  | 6.93e-14  | **  | 4.10e-17 | –   |

| HBVM(2,1) | Time | $e_u$     | Rate | $e_H$     | Rate | $e_M$ | Rate |
|-----------|------|-----------|------|-----------|------|-------|------|
| 5,000     | 34.0 | 1.31e-07  | –    | 3.20e-14  | 3.88e-17 | –    |      |
| 6,000     | 40.9 | 9.09e-08  | 2.0  | 3.73e-14  | 4.05e-17 | **  |      |
| 7,000     | 47.7 | 6.68e-08  | 2.0  | 4.62e-14  | 3.98e-17 | **  |      |
| 8,000     | 54.3 | 5.11e-08  | 2.0  | 3.38e-14  | 4.13e-17 | **  |      |
| 9,000     | 61.3 | 4.04e-08  | 2.0  | 3.91e-14  | 4.10e-17 | **  |      |
| 10,000    | 68.2 | 3.27e-08  | 2.0  | 3.55e-14  | 4.00e-17 | **  |      |

| HBVM(3,2) | Time | $e_u$     | Rate | $e_H$     | Rate | $e_M$ | Rate |
|-----------|------|-----------|------|-----------|------|-------|------|
| 1,000     | 11.1 | 1.78e-11  | –    | 3.38e-14  | 3.97e-17 | –    |      |
| 1,500     | 16.6 | 3.53e-12  | 4.0  | 3.91e-14  | 4.11e-17 | **  |      |
| 2,000     | 19.1 | 1.13e-12  | 4.0  | 3.20e-14  | 3.89e-17 | **  |      |
| 2,500     | 23.9 | 4.80e-13  | 3.8  | 3.55e-14  | 3.85e-17 | **  |      |

| SHBVM ($k = 15, s = 10$) | Time | $e_u$     | Rate | $e_H$     | Rate | $e_M$ | Rate |
|--------------------------|------|-----------|------|-----------|------|-------|------|
| 50                       | 3.5  | 5.58e-14  | 1.60e-14 | 4.07e-17 | –    |      |      |

** when it cannot be evaluated.

by appropriately choosing the tolerance $tol$ in (46). Such methods are expected to conserve both the Hamiltonian (23) and the momentum (24), as well as to provide a solution error within the round-off error level.

We observe that the conservation of the linear invariants (6) and (7) is implicitly gained, by virtue of Theorem 4, through the definition of the semi-discrete problem obtained by truncating the series in (15) and (16). As matter of fact, their conservation is satisfied by all the above methods and, therefore, it will not be checked further. Consequently, for each considered problem, we shall compare the methods in terms of:

- the maximum solution error $e_u$;
• the maximum Hamiltonian error $e_H$;
• the maximum momentum error $e_M$;
• the execution time (in seconds);
• moreover, when appropriate, we also estimate the numerical rate of convergence.

All numerical tests have been performed on a 3.1 GHz quad-core Intel i7 computer with 16GB of memory, running Matlab 2017b. Moreover, the same Matlab code implements all the above methods, so that the comparisons are fair. In all cases, the blended iteration previously described has been used.

5.1 Solitary wave

Let us at first consider the solitary wave solution [10] of (4) given, by taking into account (2), by

$$u(x,t) = \frac{1}{2} - A \cdot \text{sech}^2 \left( \sqrt{\frac{A}{6}} (x + ct - \xi_0) \right), \quad c = \pm \sqrt{1 - \frac{2}{3}A}. \quad (48)$$

Consequently, the initial conditions (5) at $t=0$ are given by:

$$u_0(x) = \frac{1}{2} - A \cdot \text{sech}^2 \left( \sqrt{\frac{A}{6}} (x - \xi_0) \right), \quad v_0(x) = c \left( u_0(x) - \frac{1}{2} \right). \quad (49)$$

We consider the values $\xi_0 = 0$, $A = 3/8$, and the positive value of $c$. We integrate in time until $T = 80$, so that if we consider the space interval $[-120, 80]$ both $u$ and $v$ can be assumed to be approximately periodic.\footnote{The expansions (16) have been truncated at $N = 300$, providing spectral accuracy in space. As matter of fact, the spatial semi-discretization error, measured on the initial conditions (see (47)), which is defined as

$$e_0 := \max \{ \| u_0(x) - \hat{u}_0 - \omega(x)^T q(0) \|, \| v_0(x) - \hat{v}_0 - \omega(x)^T p(0) \| \},$$

is $5.06 \times 10^{-14}$. The solution (48) of the problem is depicted in Figure 1, whereas in Table 1 we list the obtained numerical results, as explained above, by using a time-step $h = 80/n$. For the SHBVM method, we used a tolerance $\text{tol} \sim 10^{-11}$ in (46), providing $s = 10$ (and, therefore, $k = 15$). From the results reported in Table 1, one infers that the latter method (SHBVM) is the most effective one among
those considered, able to numerically conserve all the invariants, while providing a negligible solution error, with a very small execution time.

5.2 Spread of two solitary waves

In general, the superposition of solitary waves as (48) is no more a solution of (4). Nevertheless, it provides an approximate solution configuration for that equation. As an example, the following initial conditions:

\[
u_0(x) = \frac{1}{2} - A \cdot \text{sech}^2 \left( \sqrt{\frac{A}{6}} x \right), \quad \nu_0(x) \equiv 0,
\]

**TABLE 3** Numerical results with time-step \( h = 120/n \) for the collision of solitary waves problem (4)–(52)

| Gauss 1                  | \( n \) | Time | \( e_u \)    | Rate | \( e_H \)   | Rate | \( e_M \)   | –     |
|-------------------------|--------|------|--------------|------|-------------|------|-------------|-------|
|                         | 1,200  | 7.6  | 7.62e-04     | –    | 3.15e-05    | –    | 1.60e-14    | –     |
|                         | 2,400  | 13.0 | 1.90e-04     | 2.0  | 7.87e-06    | 2.0  | 1.59e-14    | –     |
|                         | 3,600  | 27.0 | 8.46e-05     | 2.0  | 3.50e-06    | 2.0  | 1.60e-14    | –     |
|                         | 4,800  | 35.8 | 4.76e-05     | 2.0  | 1.97e-06    | 2.0  | 1.60e-14    | –     |
|                         | 6,000  | 44.5 | 3.05e-05     | 2.0  | 1.26e-06    | 2.0  | 1.59e-14    | –     |

| Gauss 2                  | \( n \) | Time | \( e_u \)    | Rate | \( e_H \)   | Rate | \( e_M \)   | –     |
|-------------------------|--------|------|--------------|------|-------------|------|-------------|-------|
|                         | 1,200  | 16.9 | 3.16e-08     | –    | 7.98e-10    | –    | 1.62e-14    | –     |
|                         | 2,400  | 30.3 | 1.97e-09     | 4.0  | 4.99e-11    | 4.0  | 1.65e-14    | –     |
|                         | 3,600  | 40.7 | 3.90e-10     | 4.0  | 9.86e-12    | 4.0  | 1.63e-14    | –     |
|                         | 4,800  | 49.4 | 1.23e-10     | 4.0  | 3.11e-12    | 4.0  | 1.64e-14    | –     |
|                         | 6,000  | 61.8 | 5.05e-11     | 4.0  | 1.27e-12    | 4.0  | 1.64e-14    | –     |

| HBVM(2,1)                | \( n \) | Time | \( e_u \)    | Rate | \( e_H \)   | Rate | \( e_M \)   | Rate |
|-------------------------|--------|------|--------------|------|-------------|------|-------------|------|
|                         | 1,200  | 16.2 | 7.66e-04     | –    | 1.42e-14    | 1.64e-14 | –           |
|                         | 2,400  | 28.9 | 1.91e-04     | 2.0  | 1.60e-14    | 1.64e-14 | –           |
|                         | 3,600  | 38.8 | 8.51e-05     | 2.0  | 1.95e-14    | 1.65e-14 | **          |
|                         | 4,800  | 49.2 | 4.78e-05     | 2.0  | 1.78e-14    | 1.64e-14 | **          |
|                         | 6,000  | 64.1 | 3.06e-05     | 2.0  | 1.78e-14    | 1.65e-14 | **          |

| HBVM(3,2)                | \( n \) | Time | \( e_u \)    | Rate | \( e_H \)   | Rate | \( e_M \)   | Rate |
|-------------------------|--------|------|--------------|------|-------------|------|-------------|------|
|                         | 1,200  | 19.4 | 3.13e-08     | –    | 1.60e-14    | 1.62e-14 | –           |
|                         | 2,400  | 34.3 | 1.96e-09     | 4.0  | 1.42e-14    | 1.64e-14 | **          |
|                         | 3,600  | 45.2 | 3.87e-10     | 4.0  | 1.95e-14    | 1.64e-14 | **          |
|                         | 4,800  | 60.9 | 1.22e-10     | 4.0  | 1.78e-14    | 1.63e-14 | **          |
|                         | 6,000  | 67.8 | 5.02e-11     | 4.0  | 2.31e-14    | 1.63e-14 | **          |

| SHBVM \((k = 18, s = 12)\) | \( n \) | Time | \( e_u \)    | –     | \( e_H \)   | –     | \( e_M \)   | –     |
|---------------------------|--------|------|--------------|------|-------------|------|-------------|------|
|                           | 60     | 11.4 | 7.86e-14     | 1.07e-14 | 1.59e-14    | –     |

** when it cannot be evaluated.
Hamiltonian error for the 1-stage Gauss method (dashed line), 2-stage Gauss method (dotted line), using a
time-step $h = 0.1$. The Hamiltonian errors close to round-off are those of HBVM(2,1) and HBVM(3,2), using a time-step $h = 0.1$, and SHBVM, using time-step $h = 2$

provide a single wave that, after a transient phase, approximately generates two solitary waves moving in opposite directions. We choose the parameters $A = 3/32$, the space interval $[-150, 150]$, and integrate until $T = 50$. The expansions (16) have been truncated at $N = 300$, providing a spectral accuracy in space, with a spatial semi-discretization error (50) of $5.00 \times 10^{-14}$. The corresponding solution is depicted in Figure 2. In Table 2 we list the obtained numerical results, when using a time-step $h = 50/n$. For the SHBVM method, we used a tolerance $tol \sim 10^{-10}$ in (46), again providing $s = 10$ (and $k = 15$).

As in the previous example, this latter method turns out to be the most effective one, among those considered here, able to numerically conserve all the invariants and providing a negligible solution error, with a very small execution time.

### 5.3 Collision of two solitary waves

The last test problem we consider is provided by the following initial conditions,

$$
u_0(x) = \frac{1}{2} - A \text{sech}^2 \left( \sqrt{\frac{A}{6}}(x - \xi_2) \right) - A \text{sech}^2 \left( \sqrt{\frac{A}{6}}(x - \xi_1) \right),$$

$$v_0(x) = c \left[ A \text{sech}^2 \left( \sqrt{\frac{A}{6}}(x - \xi_2) \right) - A \text{sech}^2 \left( \sqrt{\frac{A}{6}}(x - \xi_1) \right) \right],$$

which, when choosing the parameters $A = 0.369$, $c = \sqrt{1 - \frac{2}{3}A}$, $\xi_2 = -\xi_1 = 50$, provide two waves, colliding at about $t \approx 60$. We choose the space interval as $[-150, 150]$ and integrate until $T = 120$. The corresponding solution is depicted in Figure 3. The expansions (16) have been truncated at $N = 300$, providing spectral accuracy in space, with a spatial semi-discretization error (50) of $5.01 \times 10^{-14}$. In
Table 3 we list the obtained numerical results, when using a time-step $h = 120/n$. For the SHBVM method, we used a tolerance $tol \sim 10^{-11}$ in (46), providing $s = 12$ (and, then, $k = 18$). As in the previous cases, this latter method turns out to be the most effective one, conserving all the invariant and with a negligible solution error, and a small execution time.

It is worth mentioning that, as is shown in Figure 4, for this problem the symplectic methods exhibit a growth in the Hamiltonian error, when the two waves collide, unless the time-step is very small. Conversely, the energy conserving HBVMs and the SHBVM method always provide a uniformly small Hamiltonian error.

6 | CONCLUSIONS

In this paper we have studied the efficient numerical solution of the “good” Boussinesq equation with periodic boundary conditions. The equation has, at first, been cast into Hamiltonian form, then using a spectrally accurate Fourier space semi-discretization. Time integration has then been carried out by considering the energy conserving HBVM\(\left(\left\lceil \frac{3}{5} s\right\rceil, s\right)\) methods. In particular, when $s$ is suitably large, such methods can be regarded as spectral methods in time (SHBVMs). A very efficient implementation of such methods, relying on their so-called blended implementation, has been then considered, providing a very efficient numerical method for solving the “good” Boussinesq equation, with spectral accuracy both in space and time. A few numerical tests duly confirm this conclusion, showing that SHBVMs provide, for the problem at hand, a geometric integrator able to preserve all the invariants of the problem, as well as to provide a negligible solution error. These results further confirm the effectiveness of SHBVMs for solving Hamiltonian PDEs [44].

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NOTES

* Hereafter, for sake of brevity we shall omit the arguments of the functions, unless they are needed.
† As is usual, $\sigma(X_s)$ denotes the spectrum of matrix $X_s$.
‡ This feature will be very important, as we are going to see in the sequel.
§ As it will made clear in the sequel, the leading Fourier coefficients $\gamma_0(y), \ldots, \gamma_{s-1}(y)$ can be adequately approximated by the discrete counterparts (35), provided that $s$ is large enough and $k$ is chosen according to Corollary 1.
¶ Actually, “exactly” periodic, when using the double precision IEEE.
|| The reference solution has been computed by using the SHBVM on a doubled time mesh.

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REFERENCES

[1] M. B. Abd-el-Malek et al., New solutions for solving Boussinesq equation via potential symmetries method, Appl. Math. Comput. vol. 251 (2015) pp. 225–232.

[2] A. Mohebbi, Z. Asgari, Efficient numerical algorithms for the solution of “good” Boussinesq equation in water wave propagation, Comput. Phys. Commun. vol. 182 (2011) pp. 2464–2470.

[3] V. S. Manoranjan, A. R. Mitchell, J. L. Morris, Numerical solutions of the good Boussinesq equation, SIAM J. Sci. Stat. Comput. vol. 5 (1984) pp. 946–957.

[4] V. S. Manoranjan, T. Ortega, J. M. Sanz-Serna, Soliton and antisoliton interactions in the “good” Boussinesq equation, J. Math. Phys. vol. 29 (1988) pp. 1964–1968.

[5] L. T. K. Nguyen, Soliton solution of good Boussinesq equation, Vietnam J. Math. vol. 44 (2016) pp. 375–385.

[6] X. Runzhang et al., Global existence and blowup of solutions for the multidimensional sixth-order “good” Boussinesq equation, Z. Angew. Math. Phys. vol. 66 (2015) pp. 955–976.

[7] R. L. Sachs, On the blow-up of certain solutions of the “good” Boussinesq equation, Appl. Anal. vol. 36 (1990) pp. 145–152.

[8] X. Zha et al., Soliton interactions of the “good” Boussinesq equation on a nonzero background, Commun. Theor. Phys. vol. 64 (2015) pp. 367–371.

[9] W. Zhang, Y. Zhao, A. Chen, The elastic-fusion-coupled interaction for the Boussinesq equation and new soliton solutions of the KP equation, Appl. Math. Comput. vol. 295 (2015) pp. 251–257.

[10] M. Chen, L. Kong, Y. Hong, Efficient structure-preserving schemes for good Boussinesq equation, Math. Methods Appl. Sci. vol. 41 (2018) pp. 1743–1752.

[11] J. Yan, Z. Zhang, New energy-preserving schemes using Hamiltonian boundary value and Fourier pseudospectral methods for the numerical solution of the “good” Boussinesq equation, Comput. Phys. Commun. vol. 201 (2016) pp. 33–42.

[12] J. Yan et al., High-order energy-preserving schemes for the improved Boussinesq equation, Numer. Methods Partial Differ. Equ. vol. 34 (2018) pp. 1145–1165.

[13] J. Cai, Y. Wang, Local structure-preserving algorithms for the “good” Boussinesq equation, J. Comput. Phys. vol. 239 (2013) pp. 72–89.

[14] J. Chen, Multisymplectic geometry, local conservation laws and Fourier pseudospectral discretization for the “good” Boussinesq equation, Appl. Math. Comput. vol. 161 (2005) pp. 55–67.

[15] K. Cheng et al., A Fourier pseudospectral method for the “good” Boussinesq equation with second-order temporal accuracy, Numer. Methods Partial Differ. Equ. vol. 31 (2015) pp. 202–224.

[16] J. De Frutos, T. Ortega, J. M. Sanz-Serna, Pseudospectral method for the “good” Boussinesq equation, Math. Comput. vol. 57 (1991) pp. 109–122.

[17] M. Uddin, S. Haq, M. Ishaq, RBF-pseudospectral method for the numerical solution of good Boussinesq equation, Appl. Math. Sci. (Ruse) vol. 6 (2012) pp. 2403–2410.

[18] C. Zhang et al., On the operator splitting and integral equation preconditioned deferred correction methods for the “good” Boussinesq equation, J. Sci. Comput. vol. 75 (2018) pp. 687–712.

[19] C. Zhang et al., A second order operator splitting numerical scheme for the “good” Boussinesq equation, Appl. Numer. Math. vol. 119 (2017) pp. 179–193.

[20] A. G. Bratsos, Solitary-wave propagation and interactions for the ‘good’ Boussinesq equation, Int. J. Comput. Math. vol. 85 (2008) pp. 143–1440.

[21] H. El-Zoheiry, Numerical investigation for the solitary waves interaction of the “good” Boussinesq equation, Appl. Numer. Math. vol. 53 (2005) pp. 161–173.

[22] M. S. Ismail, F. Mosally, A fourth order finite difference method for the good Boussinesq equation, Abstr. Appl. Anal. Art. ID 323260 (2014) pp. 10.

[23] T. Ortega, J. M. Sanz-Serna, Nonlinear stability and convergence of finite-difference methods for the “good” Boussinesq equation, Numer. Math. vol. 58 (1990) pp. 215–229.

[24] A. K. Pani, H. Saranga, Finite element Galerkin method for the “good” Boussinesq equation, Nonlinear Anal. vol. 29 (1997) pp. 937–956.

[25] L. Huang, W. Zeng, M. Qin, A new multi-symplectic scheme for nonlinear “good” Boussinesq equation, J. Comput. Math. vol. 21 (2003) pp. 703–714.

[26] W. Zeng, L. Huang, M. Qin, The multi-symplectic algorithm for “good” Boussinesq equation, Appl. Math. Mech. (Engl. Ed.) vol. 23 (2002) pp. 835–841.

[27] C. Jiang et al., High order energy-preserving method of the “good” Boussinesq equation, Numer. Math. Theory Methods Appl. vol. 9 (2016) pp. 111–122.

[28] Q. Wang et al., Energy-preserving finite volume element method for the improved Boussinesq equation, J. Comput. Phys. vol. 270 (2014) pp. 58–69.

[29] A. Aydin, B. Karasözen, Symplectic and multisymplectic Lobatto methods for the “good” Boussinesq equation, J. Math. Phys. vol. 49 (2008) pp. 1–18.
[30] L. Brugnano, F. Iavernaro, *Line integral methods for conservative problems*, Chapman and Hall/CRC, Boca Raton, FL, 2016.

[31] L. Brugnano, F. Iavernaro, *Line integral solution of differential problems*, Axioms vol. 7(2) (2018). https://doi.org/10.3390/axioms7020036.

[32] L. Brugnano, F. Iavernaro, D. Trigiante, *Hamiltonian BVMs (HBVMs): A family of “drift-free” methods for integrating polynomial Hamiltonian systems*, AIP Conf. Proc. vol. 1168 (2009) pp. 715–718.

[33] L. Brugnano, F. Iavernaro, D. Trigiante, *Hamiltonian boundary value methods (energy preserving discrete line integral methods)*, J. Numer. Anal. Ind. Appl. Math. vol. 5 (2010) pp. 17–37.

[34] L. Brugnano, F. Iavernaro, D. Trigiante, *A simple framework for the derivation and analysis of effective one-step methods for ODEs*, Appl. Math. Comput. vol. 218 (2012) pp. 8475–8485.

[35] F. Iavernaro, B. Pace, *s-Stage trapezoidal methods for the conservation of Hamiltonian functions of polynomial type*, AIP Conf. Proc. vol. 936 (2007) pp. 603–606.

[36] F. Iavernaro, B. Pace, *Conservative block-boundary value methods for the solution of polynomial Hamiltonian systems*, AIP Conf. Proc. vol. 1048 (2008) pp. 888–891.

[37] F. Iavernaro, D. Trigiante, *High-order symmetric schemes for the energy conservation of polynomial Hamiltonian problems*, J. Numer. Anal. Ind. Appl. Math. vol. 4 (2009) pp. 87–101.

[38] L. Barletti et al., *Energy-conserving methods for the nonlinear Schrödinger equation*, Appl. Math. Comput. vol. 318 (2018) pp. 3–18.

[39] L. Brugnano, G. Frasca Caccia, F. Iavernaro, *Energy conservation issues in the numerical solution of the semilinear wave equation*, Appl. Math. Comput. vol. 270 (2015) pp. 842–870.

[40] L. Brugnano, C. Zhang, D. Li, *A class of energy-conserving Hamiltonian boundary value methods for nonlinear Schrödinger equation with wave operator*, Commun. Nonlinear Sci. Numer. Simul. vol. 60 (2018) pp. 33–49.

[41] L. N. Trefethen, *Spectral methods in Matlab*, SIAM, Philadelphia, PA, 2000.

[42] L. Brugnano, F. Iavernaro, D. Trigiante, *A note on the efficient implementation of Hamiltonian BVMs*, J. Comput. Appl. Math. vol. 236 (2011) pp. 375–383.

[43] L. Brugnano, G. Frasca Caccia, F. Iavernaro, *Efficient implementation of Gauss collocation and Hamiltonian boundary value methods*, Numer. Algorithms vol. 65 (2014) pp. 633–650.

[44] L. Brugnano et al., *Spectrally accurate space-time solution of Hamiltonian PDEs*, Numer. Algorithms (2018) pp. 1–20. https://doi.org/10.1007/s11075-018-0586-z.

[45] L. Brugnano, J. I. Montijano, L. Rández, *On the effectiveness of spectral methods for the numerical solution of multi-frequency highly-oscillatory Hamiltonian problems*, Numer. Algorithms (2018) pp. 1–32. https://doi.org/10.1007/s11075-018-0552-9.

[46] J. De Frutos, T. Ortega, J. M. Sanz-Serna, *A Hamiltonian explicit algorithm with spectral accuracy for the “good” Boussinesq system. Spectral and high order methods for partial differential equations (Como, 1989)*, Comput. Methods Appl. Mech. Eng. vol. 80 (1990) pp. 417–423.

[47] M. S. Ismail, H. A. Ashi, *A compact finite difference schemes for solving the coupled nonlinear Schrödinger–Boussinesq equations*, Appl. Math. vol. 7 (2016) pp. 605–615.

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