On the Time–Modulation of the $\beta^+–$Decay Rate of H–like $^{140}$Pr$^{58+}$ Ion

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According to recent experimental data at GSI, the rates of the number of daughter ions, produced by the nuclear K–shell electron capture (EC) decays of the H–like ions $^{140}$Pr$^{58+}$ and $^{142}$Pm$^{60+}$, are modulated in time with periods $T_{EC} \approx 7$ sec and amplitudes $a_{EC} \approx 0.20$. Study of a possible time–dependence of the nuclear positron $\beta^+$ decay rate of the H–like $^{140}$Pr$^{58+}$ ion. We show that the time–dependence of the $\beta^+$ decay rate of the H–like $^{140}$Pr$^{58+}$ ion as well as any H–like heavy ions cannot be observed. This result can be used as a prediction for future analysis of the time–dependence of the $\beta^+$ decay rates of the H–like heavy ions $^{140}$Pr$^{58+}$ and $^{142}$Pm$^{60+}$ at GSI for the test of the measuring method.

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INTRODUCTION

The experimental investigation of the $EC$–decays of the H–like ions $^{140}$Pr$^{58+}$ and $^{142}$Pm$^{60+}$, $^{140}$Pr$^{58+}$ $\rightarrow$ $^{140}$Ce$^{58+}$ $+$ $\nu$ and $^{142}$Pm$^{60+}$ $\rightarrow$ $^{142}$Nd$^{60+}$+$\nu$, carried out at the Experimental Storage Ring (ESR) at GSI in Darmstadt [1], showed modulation in time with periods $T_{EC} \approx 7$ s if the rates of the number $N_d^{EC}(t)$ of daughters ions $^{140}$Ce$^{58+}$ and $^{142}$Nd$^{60+}$, respectively (see Fig.1).

By the number of daughter ions is defined by

$$dN_d^{EC}(t)/dt = \lambda_{EC}(t) N_m(t), \quad \lambda_{EC}(t) = \lambda_{EC} \left\{ 1 + a_{EC} \cos \left( 2\pi \frac{t}{T_{EC}} + \phi \right) \right\},$$

where $\phi$ and $a_{EC}$ are the phase and the amplitude of the time–dependent term, which are not well–measured quantities [2, 3].

In turn, periods of the time–modulation $T_{EC}$ are measured well and equal to $T_{EC} = 7.06(8)$ s and $T_{EC} = 7.10(22)$ s for $^{140}$Ce$^{58+}$ and $^{142}$Nd$^{60+}$, respectively [1]. Such a periodic dependence has been explained in [4, 5, 6] in terms of the massive neutrino mixing with the period $T_{EC}$ equal to

$$T_{EC} = \frac{4\pi \gamma M_m}{(\Delta m^2_{21})_{GSI}},$$

where $M_m$ is the mass of the mother ion [6], $\gamma = 1.45$ is the Lorentz factor [1] and $(\Delta m^2_{21})_{GSI} = 2.22(4) \times 10^{-4} eV^2$, calculated in [4] for the experimental value $T_{EC} = 7\ s$ [1]. A relation of the value $(\Delta m^2_{21})_{GSI} = 2.22(4) \times 10^{-4} eV^2$ to the KamLAND experimental data $(\Delta m^2_{21})_{KamLAND} = 0.20 \pm 0.05 \times 10^{-4} eV^2$ [5] has been investigated and obtained in [6].

The $EC$–decay rate of the H–like $^{140}$Pr$^{58+}$ ion, averaged over time $\langle \lambda_{EC}(t) \rangle = \lambda_{EC}$, as well as the $\beta^+$–decay rate of the $^{140}$Pr$^{58+}$ $\rightarrow$ $^{140}$Ce$^{58+}$ + $e^+ + \nu$ decay have been also measured at GSI [8]: $\lambda_{EC} = 0.00219(6)\ s^{-1}$ and $\lambda_{EC} = 0.00161(10)\ s^{-1}$ with the ratio $R_{EC/\beta^+} = 1.36(9)$.

The calculation of the $EC$ and $\beta^+$ decay rates of the H–like $^{140}$Pr$^{58+}$ ion as well as the He–like $^{140}$Pr$^{57+}$ ion has been carried out in [8]

$$\lambda_{EC} = \frac{1}{2F + 1} \left( \frac{3}{2} |M_{GT}|^2 \right) \langle |\psi_{1s}^{(Z)}|^2 \rangle \frac{Q_{\beta^+}^2}{\pi},$$

$$\lambda_{\beta^+} = \frac{2}{2F + 1} \left( \frac{|M_{GT}|^2}{4\pi^3} \right) f(Q_{\beta^+}, Z - 1),$$

where $F = 1/2$ is the total angular momentum of the H–like $^{140}$Pr$^{58+}$, $Q_{H} = (3348 \pm 6)\ keV$ and $Q_{\beta^+} = (3396 \pm 6)\ keV$ are the $Q$–values of the $^{140}$Pr$^{58+}$ $\rightarrow$ $^{140}$Ce$^{58+}$ + $e^+ + \nu$ and $^{140}$Pr$^{58+}$ $\rightarrow$ $^{140}$Ce$^{57+}$ + $e^+ + \nu$ decays, respectively; $M_{GT}$ is the nuclear matrix element of the Gamow–Teller transition

$$M_{GT} = -2g_AG_FV_{ud} \int d^3x \overline{\Psi_d(r)} \Psi_m(r),$$

where $\Psi_d(r)$ and $\Psi_m(r)$ are the wave functions of the daughter and mother nuclei, respectively, and $\Psi_d(r) \approx \rho(r)$ is the nuclear matter density $\rho(r)$ having the Woods–Saxon shape with $R = 1.1 A^{1/3}\ fm$ and diffuseness parameter $a = 0.50\ fm$ [10]. Then, $\psi_{1s}^{(Z)}$ is the Dirac wave function of
the bound electron in the ground state, \( Z = 59 \) is the electric charge of the mother nucleus \(^{140}\text{Pr}^{59+}\).

The average value of the Dirac wave function of the bound electron \( \langle \psi_{1s}(Z) \rangle \) is defined by \( \bar{a} \)

\[
\langle \psi_{1s}(Z) \rangle = \int d^3x \frac{\psi_{1s}(Z)}{\sqrt{\rho(r)}} = \frac{1.66}{\bar{a}_B} 
\]

where \( \bar{a}_B = 1/m_e Z \alpha = 897 \text{ fm} \) for the electron mass \( m_e = 0.511 \text{ MeV} \) and the fine-structure constant \( \alpha = 1/137.036 \). In the \( \beta^+ \)-decay rate \( f(Q_{\beta^+}, Z - 1) = (2.21 \pm 0.03) \text{ MeV}^5 \) is the Fermi integral \( \bar{a} \). The theoretical prediction for the ratio \( R_{EC/\beta^+}^{th} \) is \[ R_{EC/\beta^+}^{th} = \frac{3\pi^2 Q_H^2}{f(Q_{\beta^+}, Z - 1)} = 1.40(4), \]

which agrees well with the experimental data \( R_{EC/\beta^+}^{exp} = 1.36(9) \).

According to \( \bar{a} \), a time-dependence of the rate of the number of daughter ions in the \( \beta^+ \)-decay of the H-like \(^{140}\text{Pr}^{58+}\) ion has not been studied experimentally until now.

In this paper we apply a theoretical approach, developed in \[4, 5, 6\] for the analysis of the time-modulation in the \( EC \)-decay of the H-like \(^{140}\text{Pr}^{58+}\), to the study of the time-dependence of the \( \beta^+ \)-decay rate of the H-like \(^{140}\text{Pr}^{58+}\) ion. Its experimental investigation should be a stringent test of the applied single-ion Schottky mass-measurement method.

### Amplitudes of the \( \beta^+ \)-Decay of the H-Like \(^{140}\text{Pr}^{58+}\) Ion

Following \[4\], for the calculation of the time-modulation of the \( \beta^+ \)-decay rate of the H-like \(^{140}\text{Pr}^{58+}\) ion we use time-dependent perturbation theory. The Hamilton operator \( H_W(t) \) of the weak interactions is given by

\[
H_W(t) = \frac{G_F}{\sqrt{2}} V_{ud} \sum_j U_{ej} 
\]

\[
\times \int d^3 \rho [\bar{\psi}_n(x) \gamma^\mu (1-gA\gamma^5) \psi_p(x)] 
\]

\[
\times [\bar{\psi}_\nu_j(x) \gamma_\mu (1-\gamma^5) \psi_{\nu^-}(x)] 
\]

with standard notations \[4\]. In our analysis neutrinos \( \nu_j \) \((j = 1, 2, 3) \) are Dirac particles with masses \( m_j \) \((j = 1, 2, 3) \) \[4\].

The amplitude of the \( \beta^+ \)-decay of the H-like \(^{140}\text{Pr}^{58+}\) ion with undetected neutrino we define as a coherent sum of the amplitudes of the \( \beta^+ \)-decays \(^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{57+} + e^+ + \nu_j \) \[4, 6\]

\[
M_{F_{M_F} \rightarrow F'}(t) = -i \sum_j \int_{-\infty}^t dt \times \langle \nu_j(\vec{k}_j)e^+(p_+)d(\vec{q})|H_W(\tau)|m(0)d\rangle d\tau, \]

where \( m \) and \( d \) are the mother ion \(^{140}\text{Pr}^{58+}\) and the daughter ion \(^{140}\text{Ce}^{57+}\), respectively. The mother ion is taken in the state \(^{140}\text{Pr}^{58+}=1/2^+\) with \( F = 1/2 \) and \( M_F = \pm 1/2 \) and in the rest frame \[4\]. In turn, the daughter \(^{140}\text{Ce}^{57+}\) ion is a H-like ion in the state with \( F' = 1/2 \) and \( M_{F'} = \pm 1/2 \).

The wave function of the neutrino \( \nu_j \) we define in the form of a wave packet \[4, 6\]

\[
\psi_{\nu_j}(\vec{r}, t) = (2\pi \delta^2)^{3/2} \int \frac{d^3 k}{(2\pi)^3} e^{-\frac{i}{2} \delta^2 (\vec{k} - \vec{k}_j)^2} \times e^{i\vec{k} \cdot \vec{r} - i E_j(\vec{k})t} u_{\nu_j}(\vec{k}, \sigma_j). \]

where a spatial smearing of the neutrino \( \nu_j \) is determined by the parameter \( \delta \), \( \vec{k}_j \) is the neutrino momentum and \( E_j(\vec{k}) = \sqrt{\vec{k}^2 + m_j^2} \) is the energy of a plane wave with the momentum \( \vec{k} \), \( u_{\nu_j}(\vec{k}, \sigma_j) \) is the Dirac bispinor of the neutrino \( \nu_j \) \[4\]. In the limit \( \delta \rightarrow \infty \), due to the relation

\[
(2\pi \delta^2)^{3/2} e^{-\frac{1}{2} \delta^2 (\vec{k} - \vec{k}_j)^2} \rightarrow (2\pi)^3 \delta(3)(\vec{k} - \vec{k}_j),\]
the wave function \( \Phi \) reduces to the form of a plane wave \([1]\). 

Following \([1] \) and \([9]\) we obtain the amplitudes of the \( \beta^+ \)-decay

\[
\mathcal{M}_{\frac{1}{2}+, \frac{1}{2}, \frac{1}{2}}(t) = -\sqrt{2M_mE_d}\mathcal{M}_{\text{GT}} \frac{2\pi\delta^2}{2\sqrt{2}} \times e^{it} \sum U_{e\gamma} e^{-\frac{i}{2}(\bar{k}_d + \bar{p}_+ + k_f^2)} \left( \frac{1}{\Delta E_j(\bar{k}_j)} \right) \frac{1}{i\varepsilon} \right) 
\]

\[
\mathcal{J}_{\frac{1}{2}+, \frac{1}{2}, \frac{1}{2}}(t) = -\sqrt{2M_mE_d}\mathcal{M}_{\text{GT}} \frac{2\pi\delta^2}{2\sqrt{2}} \times e^{it} \sum U_{e\gamma} e^{-\frac{i}{2}(\bar{k}_d + \bar{p}_+ + k_f^2)} \left( \frac{1}{\Delta E_j(\bar{k}_j)} \right) \frac{1}{i\varepsilon} \right) 
\]

For the calculation of the amplitudes Eq.(11) we have carried out the integration over \( \bar{k} \) with the \( \delta \)-function \((2\pi\delta^2)^3(\vec{k} + \vec{k}_d + \vec{p}_+)\) and denoted \( \Delta E_j(\bar{k}_j) = E_d(\bar{k}_d) + E_+(\bar{p}_+) + E_j(\bar{k}_j) - M_m \), where \( E_d(\bar{k}_d) \), \( E_+(\bar{p}_+) \) and \( E_j(\bar{k}_j) \) are energies of the daughter ion, positron and neutrino \( \nu_j \), \( \vec{k}_d \), \( \vec{p}_+ \) and \( \vec{k}_j \) are their momenta and \( M_m \) is the mother ion mass. For the calculation of the integral over time we have used the \( \varepsilon \)-regularization \([3]\). Finally the parameter \( \varepsilon \) should be taken in the limit \( \varepsilon \rightarrow 0 \).

**Time-dependence of the \( \beta^+ \)-Decay Rate of the H-Like \( ^{140}\text{Pr}^{58+} \) Ion**

According to \([3]\), the first step to the calculation of the time-dependent \( \beta^+ \)-decay rate \( \lambda_{\beta^+}(t) \) of the H-like \( ^{140}\text{Pr}^{58+} \) ion is the calculation of the rate of the neutrino spectrum \( \lambda_{\nu}(t) \). It is defined by \([3]\)

\[
\frac{dN_{\nu}(t)}{dt} = \frac{1}{2F + 1} \frac{1}{\pi^2 (\pi\delta^2)^{3/2}} \int \mathcal{M}_{F,M_F \rightarrow F',M_{F'}}(t)^2 \times F(\lambda - 1, E_+) \frac{d^3k_\nu}{(2\pi)^32E_d (\pi\delta^2)^{3/2}} \frac{d^3p_+}{(2\pi)^32E_+} \]

where \( F(\lambda - 1, E_+) \) is the Fermi function \([11]\) (see also \([9]\) describing the Coulomb repulsion between the positron and the nucleus \( ^{140}\text{Ce}^{58+} \). It is equal to \([11]\)

\[
F(\lambda - 1, E_+) = \left( 1 + \frac{1}{2} \gamma \right) \frac{4(2R_{\nu})^2\gamma}{\Gamma^2(3 + 2\gamma)} e^{-\frac{\pi(Z - 1)E_+}{p_+}} \right) \frac{\Gamma^2(1 + \gamma - i\alpha(Z - 1)E_+)}{\Gamma^2(1 + \gamma + i\alpha(Z - 1)E_+)} \right)^2, \]

where \( p_+ = \sqrt{E_d^2 - m_e^2} \), \( R = 5.712 \) fm, \( Z = 59 \) and \( \gamma = \sqrt{1 - ((Z - 1)\alpha)^2} - 1 \).

After the integration over the phase volume of the daughter ion, the directions of the positron momentum \( \vec{p}_+ \) and the limit \( \varepsilon \rightarrow 0 \) we arrive at the following expression for the rate of the neutrino spectrum

\[
\frac{dN_{\nu}(t)}{dt} = \frac{1}{2F + 1} \frac{\mathcal{M}_{\text{GT}}^2}{\pi^2 (\pi\delta^2)^{3/2}} \times \int_{m_N}^{Q_{\nu} - m_e} \frac{(2\pi)\delta(Q_{\nu} - m_e - E_+ - E_\nu) E_\nu}{(2\pi)^32E_d (\pi\delta^2)^{3/2}} \frac{d^3k_\nu}{(2\pi)^32E_d (\pi\delta^2)^{3/2}} \times \cos \left[ \left( \sqrt{E_\nu^2 + m_e^2} - \sqrt{E_d^2 + m_e^2} \right) t \right] \times F(\lambda - 1, E_+) \sqrt{E_d^2 - m_e^2} E_+ dE_+, \]

where \( \Delta \bar{k}_{ij} = (\bar{k}_i - \bar{k}_j)/2 \) and \( e^{-\delta^2(\Delta \bar{k}_{ij})^2} \) are kept as input parameters \([3]\).

For the calculation of the r.h.s of Eq.(14) we have set neutrino masses zero everywhere except in the energy difference \( E_i(\bar{k}_i) - E_j(\bar{k}_j) \) \([3]\). Then, due to the exponential function \( e^{-\delta^2(\bar{k}_i - \bar{k}_j)^2/2} \) the neutrino momenta are constrained by \( \bar{k}_i \approx \bar{k}_j = \bar{k} \) \([3]\). In such an approximation we get \( E_i(\bar{k}_i) \approx E_\nu = |\bar{k}| \) and \( E_i(\bar{k}_i) - E_j(\bar{k}_j) = \sqrt{E_\nu^2 + m_e^2} - \sqrt{E_d^2 + m_e^2} \).

In terms of the rate of the neutrino spectrum the time-dependent \( \beta^+ \)-decay rate is defined by \([3]\)

\[
\lambda_{\beta^+}(t) = \int \frac{d^3k}{(2\pi)^32E_d} \frac{1}{\pi^2 (\pi\delta^2)^{3/2}} \frac{dN_{\nu}(t)}{dt}. \]
FIG. 2: Time–dependence of the $\beta^+$–decay rate of the H–like $^{140}$Pr$^{58+}$ ion on the time–interval equal to $2T_{EC} = 14\text{s}$.

For $\theta_{13} = 0$ (see also [4]) we deal with two–neutrino mass–eigenstates and obtain

$$\frac{\lambda_{\beta^+}(t)}{\lambda_{\beta^+}} = 1 + R_{\beta^+}(t), \quad (16)$$

where $\lambda_{\beta^+}$ is given by Eq.(4) and $R_{\beta^+}(t)$ is equal to

$$\begin{align*}
R_{\beta^+}(t) & = \sin 2\theta_{12}^* \delta^2(\Delta E) \int_{m_e}^{Q_{\beta^+} - m_e} dE \frac{E}{E^*} \\
& \times \sqrt{E^2 - m_e^2} \left( Q_{\beta^+} - m_e - E^* \right)^2 F(Z-1, E^*) \\
& \times \cos \left[ \left( \sqrt{(Q_{\beta^+} - m_e - E^*)^2 + m_e^2} \right)^2 - \frac{1}{2} \right]. \quad (17)
\end{align*}$$

For the numerical calculations we use $\sin 2\theta_{12} = 0.20$ [4] and $Q_{\beta^+} = 3396(6)\text{keV}$ [4].

The time–dependent part of the $\beta^+$–decay rate of the H–like $^{140}$Pr$^{58+}$ ion on the time–interval equal to $2T_{EC} = 14\text{s}$, i.e. two periods of the time–modulation of the $EC$–decay rate of the H–like $^{140}$Pr$^{58+}$ ion, is shown in Fig. 2. For the calculation of $R_{\beta^+}(t)$ we have used neutrino masses $m_j(R)$, obtained in [8] and corrected by the interaction of massive neutrinos with the strong Coulomb field of the daughter nucleus $^{140}$Ce$^{58+}$ [3]. It is seen that the $\beta^+$–decay rate varies in time much faster than the $EC$–decay rate of the H–like $^{140}$Pr$^{58+}$ ion.

In the measurement of the time–dependence of the $\beta^+$–decay rate of the H–like $^{140}$Pr$^{58+}$ ions the time–spectrum of the decay is defined by the time–intervals $\Delta T = 5 \times \Delta T_{\text{bin}}$, caused by 5 bins with the length $\Delta T_{\text{bin}} = 64\text{ms}$ each. This leads to the experimental value of the $\beta^+$–decay rate, averaged over the time–interval $\Delta T = 5 \times \Delta T_{\text{bin}} = 0.32\text{s}$. Due to the rapid variations of $R_{\beta^+}(t)$ a modulation of the time–dependence of the $\beta^+$–decay rate $\lambda_{\beta^+}(t)$ of the H–like $^{140}$Pr$^{58+}$ ion is not observable in an experiment.

CONCLUSION

We have studied the time–dependence of the $\beta^+$–decay rate $\lambda_{\beta^+}(t)$ of the H–like $^{140}$Pr$^{58+}$ ion. For the calculation of $\lambda_{\beta^+}(t)$ we have followed the approach, proposed in [3]–[9], as applied to the explanation of the time–modulation of the $EC$–decay rate of the H–like $^{140}$Pr$^{58+}$ ion. We have found that the time–dependent term of the $\beta^+$–decay rate varies very rapidly in time, which makes such a time–dependence unobservable. This result can be used as a prediction for future analysis of the time–dependence of the $\beta^+$–decay rates of the H–like heavy ions $^{140}$Pr$^{58+}$ and $^{142}$Pm$^{58+}$ at GSI for the test of the measuring method.

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