Quantum Information and Wave Function Collapse

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Abstract

Information-theoretical restrictions on the information transferred in quantum measurements are regarded for the measurement of object \( S \) performed via its interaction with information system \( O \). This information restrictions, induced by Heisenberg commutation relations, are derived in the formalism of inference maps in Hilbert space. \( O \) restricted states \( \xi^O \) are calculated from \( S, O \) quantum dynamics and the structure of \( O \) observables set (algebra). It’s shown that this principal constraints on the information transfer result in the appearance of stochasticity in the measurement outcomes. Consequently, \( \xi^O \) describes the random ‘pointer’ outcomes \( q_j \), which correspond to the collapse of \( S \) pure state. \( O \) decoherence by its environment is studied for some \( S, O \) systems, it’s shown that it doesn’t change this results principally.

1 Introduction

Despite its significant achievements, Quantum mechanics (QM) still contains some open questions concerned with its internal consistency[1, 2, 3]. The most famous and oldest of them is the State Collapse or Quantum Measurement problem[4, 5]. In this paper this problem will be considered mainly within the framework of Information Theory [6, 7, 8]. Really, the measurement of any system \( S \) includes the transfer of information from \( S \) to the information system \( O \) (Observer) which processes and memorizes it. Correspondingly, in Information Theory, any measuring system (MS) can be described as the information channel, which transfers the information about \( S \) state to \( O \) [6]. Plainly, if some restrictions on the information transfer via such channel exist, they can influence, in principle, the outcomes of measurement events percepted by \( O \). Recently, it was shown that such constraints, induced mainly by Heisenberg commutation relations, result in the significant information losses for typical information channels [9]. Our calculations for some simple MS models evidence that for MS such restrictions induce the unavoidable stochasticity in the measurements of \( S \) pure states, due to the loss of information about the rate of their purity [8]. Here we develop our formalism and present some new results and their discussion.
For MS dynamics we shall exploit here the standard approach of measurement theory in which the evolution of all physical objects, including \( O \), is described by the quantum state (density matrix) \( \rho(t) \) which obeys Schrödinger-Liouville equation in arbitrary reference frame (RF). The information-theoretical formalism of systems’ self-description or ‘measurement from inside’ is applied to the description of information acquisition by \( O \) in quantum measurements\(^7\). In Schrödinger QM framework, it is realized by means of the formalism of inference maps in Hilbert space \(^5\). Overall, we shall demonstrate for simple MS model that such approach allows to construct the consistent measurement theory, which results in the stochasticity of measurement outcomes without exploit of additional Reduction Postulate in Schrödinger QM scheme. It’s instructive to note beforehand that even if \( O \) is the human brain, in our theory the observer’s consciousness doesn’t play any role and will not be referred to at all; some aspects of this problem will be discussed in conclusion. Yet for the illustrative purposes some terms characteristic for conscious perception of signals will be used in the discussion of our model.

## 2 Model of Quantum Measurements

Here our measurement model will be described and some aspects of QM measurements theory will be reviewed. We shall also formulate the main ideas of our approach in semi-qualitative form, the detailed mathematical formalism will be described in the next chapter. In our model MS consists of the studied system \( S \), detector \( D \) and the information system \( O \), which acquires and processes the incoming information. The effect of MS decoherence by the environment\(^10\) will be considered in the final part of our paper, it will be shown that in our theory its influence is inessential for the measurement process and so it will be analyzed separately. \( S \) is taken to be the particle with the spin \( \frac{1}{2} \) and the measurement of its projection \( S_z \) will be studied. Its \( u, d \) eigenstates denoted \( |s_1\rangle, |s_2\rangle \), so that the measured \( S \) pure state is:

\[
\psi_s = a_1 |s_1\rangle + a_2 |s_2\rangle
\]

For the comparison, the incoming \( u, d \) ’test’ mixture with the same \( \bar{S}_z \) should be regarded also. This is \( S \) ensemble described by the gemenge \( W^s = \{|s_i\}, P_i\} \), where \( P_i = |a_i|^2 \) are the probabilities of \( |s_i\rangle \) in this ensemble \(^1\), its density matrix denoted \( \rho_m^s \). Analogously to \( S \), in our model \( D \) state in \( O \) RF is described by Dirac vector \( |D\rangle \) in two-dimensional Hilbert space \( \mathcal{H}_D \). Its basis is constituted by \( |D_{1,2}\rangle \) eigenstates of \( Q \) ‘pointer’ observable with eigenvalues \( q_{1,2} \). The initial \( D \) state is: \( |D_0\rangle = \frac{|D_{1}\rangle + |D_{2}\rangle}{\sqrt{2}} \). \( S, D \) interaction \( \hat{H}_{S,D} \) starts at \( t_0 \) and finishes effectively at some \( t_1 \); for Zurek Hamiltonian \( H_{S,D} \) with suitable parameters\(^{10}\) it would result in \( S, D \) entangled final state:

\[
\Psi_{S,D} = \sum a_i |s_i\rangle |D_i\rangle
\]

in \( O \) RF, it follows that \( \bar{Q} = |a_1|^2 - |a_2|^2 \). The measurement of \( S \) eigenstate \( |s_{1,2}\rangle \) results in factorized state \( \Psi^{1,2} = |s_{1,2}\rangle |D_{1,2}\rangle \) in which \( Q \) has the eigenvalue \( q_{1,2} \). If \( a_{1,2} \neq 0 \), then \( D \) is also described by the quantum state \( R_D \), but due to \( S, D \)
entanglement, it can’t be completely factorized from $S$ state, so it’s instructive to use in the calculations $\Psi_{S,D}$ in place of $R_D$. In our model $D$, $O$ interaction starts at $t > t_1$ and finishes at some $t_2$, during this time interval the information about $D$ state is transferred to $O$. Here we suppose that $O$ can acquire all essential $D$ information copiously, this assumption will be proved in sect. 3.

Let’s remind how the measurement process is described in Information Theory [11]. The signal induced by the measured state and registrated by $O$ in event $n$ is characterized by the array of real parameters called information pattern (IP) $J(n) = \{e_1, ..., e_i\}$. The set of all possible $J$ constitutes the independent ‘information space’, which define $O$ recognition of measured states [7]. In quantum case, some IP parameters, in principle, can be uncertain, but this feature will be shown to be unimportant for our problem. Consider, as the example, the measurement of $S$ eigenstate $|s_{1,2}\rangle$. $O$ supposedly percepts $|D_{1,2}\rangle$ state in event $n$ as IP: $J^D_{1,2} = q_{1,2}$. $Q$ eigenvalues $q_{1,2}$ are $D$ real properties[1], they correspond to the orthogonal projectors $P^D_{1,2}$. Hence for $O$ the difference between this $D$ states is the objective or Boolean Difference (BD) [3]. It is equivalent to the distinction between the logical operands $Yes/No$, or that’s the same between the values $1/0$ of some parameter $L_g$.

For example, if the parameter $L_g = 1/0$ for $|D_{1,2}\rangle$, then it corresponds to projector $P^D_1$.

Now let’s regard the measurement in case when $a_{1,2} \neq 0$, i.e. $\psi_s$ is $|s_i\rangle$ superposition. In this case, $\Psi_{S,D}$ state is different from $\Psi^{1,2}$ but it doesn’t mean automatically that $O$ can discriminate them as the different signals, this question demands the careful analysis. Remind that the standard or ‘Pedestrian’ interpretation[2, 3] (PI) of QM claims that without the inclusion of Reduction Postulate into QM formalism, $O$ would percept $S$, $D$ entangled state $\Psi_{S,D}$ as the superposition of $|D_{1,2}\rangle$ states and so can discriminate them from any of $|D_{1,2}\rangle$. It supposed sometimes that it corresponds the simultaneous observation of $J^D_{1,2}$, more realistically, one can expect that at least $\Psi_{S,D}$ is percepted by $O$ as some IP $J^s$, which should differ from $J^D_{1,2}$. This assumption is the essence of famous ‘Schroedinger Cat’ Paradox [3].

However, no such exotic outcomes are registrated experimentally, in place of it, for the regarded kind of measurements $J^D_{1,2}$ are observed at random, from that it is usually concluded that Reduction Postulate should be added to QM formalism. Yet the situation isn’t so simple and doesn’t favor such prompt jump to the conclusions. Really, given PI implications are correct, $O$ should distinguish in a single event $\Psi_{S,D}$ from both $|D_{1,2}\rangle$ states. Hence the relation of corresponding $O$ IPs should be characterized by BD, i.e. $J^s \neq J^D_{1,2}$. So it should be at least one $D$ parameter (observable) $G^D$ which value $g_0$ for $\Psi_{S,D}$ is different from its values $g_{1,2}$ for $|D_{1,2}\rangle$. Roughly speaking, in this case it should be such $D$ parameter $G_D$ which is equal to 1/0 depending on the presence/absence of $D$, superpositions. Meanwhile, in QM all measurable parameters are related strictly to the observables represented by corresponding Hermitian Operators on $\mathcal{H}$ (or POV in general formalism). Consequently, to verify the proposed PI hypothesis for $\Psi_{S,D}$ and $|D_{1,2}\rangle$, one should check the set (algebra) of $D$ PV observables $\{G^D\}$ looking for the suitable candidates. Yet the simple analysis shows that there are no such quantum $D$ observbles. To demonstrate
it, suppose that such $G^O$ - Hermitian operator exists, then it follows:

$$G^D \Psi_{S,D} = a_1 |s_1\rangle G^D |D_1\rangle + a_2 |s_2\rangle G^D |D_2\rangle = g_0 \Psi_{S,D}$$  \hspace{1cm} (2)

As easy to see, such equality fulfilled only for $G^D = I$. Any $G^D$, which is sensitive to the presence of superpositions, corresponds to the nonlinear operator on $\mathcal{H}_D$, so the observation of such difference seems to be incompatible with standard QM formalism. Consequently, it seems impossible for $O$ to distinguish $|D_i\rangle$ from $\Psi_{S,D}$ of (1). From that it’s logical to conclude that $O$ would observe only one of $q_i$ outcomes in each event. For pure $S$ ensemble it’s reasonable to assume that QM expectation value $\bar{Q}$ will be obtained by $O$ from the measurement of $S$ ensemble with the number of events $N \to \infty$. To fulfill such relation, $O$ should observe the stochastic $q_{1,2}$ outcomes with probabilities $P_{1,2} = |a_{1,2}|^2$. This considerations put doubts on the necessity of independent Reduction Postulate in QM; the similar hypothesis was proposed by Wigner[12]. Note that the obtained results don’t mean that $R_D$ is the probabilistic mixture of $|D_{1,2}\rangle$, rather $R_D$ can be characterized as their ‘weak’ superposition, stipulated by the entanglement of $S, D$ states. The possible role of joint $S, D$ observables will be regarded below, but their account don’t change the situation principally.

### 3 Measurements and Systems’ Self-description

Now the information system $O$ will be consistently considered as the quantum object. In this case, MS is described by the quantum state $\rho_{MS}$ relative to some external RF $O'$. In our model $O$ pure state is a vector in two-dimensional Hilbert space $\mathcal{H}_O$. Analogously to $D$, we settle $O$ initial state $|O_0\rangle = \frac{|O_{1,2}\rangle}{\sqrt{2}}$ where $|O_{1,2}\rangle$ are eigenstates of $O$ ‘internal pointer’ observable $Q_O$ with eigenvalues $q^{O}_{1,2}$. For suitable $D, O$ Hamiltonian $H_{D,O}$ one obtains at $t > t_2$:

$$\Psi_{S,D,O} = \sum a_i |s_i\rangle |D_i\rangle |O_i\rangle$$

As easy to see, $D$ states only double $S$ states for this set-up, so for the simplicity $D$ can be dropped from further considerations. In such scheme $S$ directly interacts with $O$ by means of Hamiltonian $H_{S,O}$, which result in the final state:

$$\Psi_{MS} = \sum a_i |s_i\rangle |O_i\rangle$$  \hspace{1cm} (3)

in external RF $O'$. Our aim is to find the relation between this state and the information acquired by $O$. Plainly, the measurement of arbitrary system $S'$ by an information system $O'$ can be considered as the mapping of $S'$ states set $N_S$ to the set $N_O$ of $O'$ internal states [6]. In Information Theory, this most general approach is described by the formalism of systems’ self-description called also ‘measurement from inside’ [7]. In its framwork, $O'$ considered as the subsystem of larger system $\Xi_T = S', O'$ with the states set $N_T$. The information acquired by $O'$ about $\Xi_T$ (including $O'$ itself) is described by $O'$ internal state $R_O$ called also $\Xi_T$ restricted state or restriction. For given $\Xi_T$ system $R_O$ is defined by the inference map $M_O$ of $\Xi_T$.
state to \( N_O \) set. In quantum case \( M_O \) derivation is the complicated problem for any realistic \( \Xi_T \) and now we shall turn to its detailed analysis.

The important property of inference map \( M_O \) is formulated by Breuer Theorem: if for two arbitrary \( \Xi_T \) states \( \Gamma, \Gamma' \) their restricted states \( R, R' \) coincide, then for \( O^I \) this \( \Xi_T \) states are indistinguishable, for any nontrivial \( S', O^I \) at least one such pair of states exist [5]. In classical case, the origin of this effect is obvious: \( O^I \) has less degrees of freedom than \( \Xi_T \) and hence can't discriminate all possible \( \Xi_T \) states [7]. For quantum systems \( M_O \) ansatz should be derived from first QM principles, however, Schroedinger QM formalism only doesn't permit to derive \( M_O(\Xi_T \to O^I) \) unambiguously and some additional inputs are needed for that. For that purpose Breuer assumed phenomenologically that for arbitrary \( \Xi_T \) its restricted state is equal to the partial trace of \( \Xi_T \) individual state over \( S' \), i.e. is \( \Xi_T \) partial state on \( O^I \). In our MS set-up for \( \Psi_{MS} \) of (3) it gives:

\[
R_O^B = Tr_s \rho_{MS} = \sum |a_i|^2 |O_i\rangle \langle O_i|
\]  

Plainly, this ansatz excludes beforehand any kind of stochastic \( R_O \) behavior. For MS mixed ensemble, induced by the corresponding \( W^s \) gemenge, the individual MS states differ from event to event:

\[
\varsigma_{MS}^{\text{O}}(n) = |O_i\rangle \langle O_i||s_l\rangle \langle s_l|
\]  

where the frequencies of random \( l(n) \) appearance in given event \( n \) are stipulated by the probabilistic distribution \( P_l = |a_l|^2 \). \( O \) restricted state for this mixed ensemble is also stochastic: in a given event

\[
R_O^{\text{mix}}(n) = \xi_1^O \text{ or } \xi_2^O.
\]

where \( \xi_i^O = |O_i\rangle \langle O_i| \) appears with the corresponding probability \( P_i \), so that the ensemble of \( O \) states described by the gemenge \( W_{\text{mix}}^O = \{\xi_i^O, P_i\} \). \( R_O^{\text{mix}}(n) \) differs from \( R_O^B \) in any event, hence for the restricted \( O \) individual states the main condition of cited theorem is violated. From that Breuer concluded that \( O \) can discriminate the individual pure/mixed MS states ‘from inside’, so the collapse of pure state can’t be observed by \( O \) [5].

Alternatively, we find that the information-theoretical considerations permit to calculate MS restriction to \( O \) directly and unambiguously; as will be shown, the obtained results contradict to Breuer conclusion. To demonstrate it, consider the measurement of \( S \) eigenstate \( |s_{1,2}\rangle \), it results in MS individual \( \varsigma_{MS}^O \) state of (5), which restriction is \( \xi_{1,2}^O \) with eigenvalues \( q_{1,2}^O \). Hence it’s natural to conclude that \( O \) can identify this states as IP \( J_{1,2}^O = q_{1,2}^O \). The difference between \( \xi_i^O \) states is boolean (classical), because in QM formalism \( \xi_i^O \) eigenvalues \( q_i^O \) are \( O \) real properties [1]. Now let’s compare the detection by \( O \) of \( \xi_i^O \) and \( \Psi_{MS} \) of (3), i.e \( R_O \). Note that the formal difference of two \( O \) restricted states doesn’t mean, in general, that this difference will be detected by \( O \). Such difference is the necessary but not sufficient condition for that, there should be also the specific \( O \) observation \( G^O \), which indicate this difference. For \( R_O \) and \( \xi_i^O \) the check of this hypothesis is analogous to the
approach described by (2). Really, suppose that such $G^O$ exists, in QM framework, it should be Hermitian PV operator, from that $G^O$ should obey:

$$G^O \Psi_{MS} = a_1 |s_1\rangle G^O |O_1\rangle + a_2 |s_2\rangle G^O |O_2\rangle = g_0 \Psi_{MS}$$

(6)

Yet for $O$ observables such equality fulfilled only for $G^O = I$. In the regarded case, only the parameters corresponding to nonlinear operators can establish BD between $R_O$ and $\xi^O_i$ states for $O$, but their observability contradicts to standard QM axiomatic. Consequently, $O$ can’t distinguish $R_O$ and $\xi^O_i$ states and resulting MS restriction is equal to:

$$R_O = \xi^O_1 \text{or} \xi^O_2.$$

(7)

i.e. it coincides with $R^\text{mix}_O$ as the individual state. POV generalization of standard QM PV observables doesn’t change this conclusions.

It seems natural to expect that in the measurement of $S$ pure ensemble $O$ should obtain $Q^O_i$ expectation values, which agree with QM predictions for arbitrary $l$. To fulfill this condition, $O$ should observe the collapse of pure MS state to one of $q^O_i$, at random with probability $P_i = |a_i|^2$, i.e. the ensemble of $O$ states should be described by the gemenge $W^O = \{\xi^O_i, P_i\}$. It induces the corresponding $O$ IP ensemble $Z^O = \{J^O, P_i\}$ which describes the collapse of $S$ pure state. Note however, that the assumption that the probabilities of $q^O_i$ outcomes follow QM ansatz, admitted here, isn’t self-obvious [13]. In general, it should be proved in any new theory of measurements [1]. Leaving for the future the detailed proof for our theory, here we notice that for our MS, which consist of spin-$\frac{1}{2}$ objects only, such relations can be derived from QM invariance relative to the space reflections and rotations. For example, for $a_1 = a_2 e^{i\alpha}$ the equality $P_1 = P_2$ follows directly from QM reflection invariance.

The difference between the pure and mixed $S$ states with the same $S_z$ is indicated by 'interference term' (IT) observables. For our $S$ states they are $S_{x,y}$ linear forms. For example, if $\frac{a_1}{a_2}$ is real, the maximal distinction reveals $S_z$ with $S_z = \frac{1}{2} |a_1| |a_2|$ for pure states and $S_z = 0$ for the mixture. For MS entangled states such difference can be revealed only by joint $S, O$ observables. As the example, consider the symmetric IT for MS:

$$B = |O_1\rangle\langle O_2| |s_1\rangle\langle s_2| + j.c.$$  

(8)

Being measured by external RF $O'$ via its interaction with $S, O$, it gives $B = 0$ for any $|s_i\rangle$ incoming mixture, but $B \neq 0$ for entangled MS states of (4). For example, for incoming $S$ state $\psi^s_1$ with $a_{1.2} = \frac{1}{\sqrt{2}}$, the resulting MS state $\Psi^s_{MS}$ is $B$ eigenstate with eigenvalue $b_1 = 1$. Hence in that case, $S_z$ is mapped to $B$. However, $B$ and any other IT can’t be directly measured by $O$ 'from inside', at least simultaneously with $S_z$, because they don’t commute [5]. Note also that the pure/mixed MS states with the same $Q_O$ can be discriminated even by external $O'$ only statistically, since the corresponding distributions of $B$ values (or other ITs) overlap. For example, for $\Psi^s_{MS}$ the probability $P_B(b_{1,2}) = .5$ for $W^s$ mixture, so its $b$ distribution intersects largely with $\delta(b - b_1)$ distribution for $B$ eigenstate $\Psi^s_{MS}$. Consequently, even $O'$ can discriminate the pure/mixed MS states only statistically for MS ensemble with $N \to \infty$ but not in a single event.
It’s well known that the decoherence of pure states by its environment $E$ is the important effect in quantum measurements [1, 10], we find yet that $O$ decoherence by $E$ doesn’t play the principal role in our theory. However, its account stabilizes the described collapse mechanism additionally and defines unambiguously the preferred basis (PB) $\{\xi^O_i\}$ of $O$ final states used in our model. Really, for the typical Hamiltonian of $O,E$ interaction [10], it follows that $\Psi_{MS}$ of (3) decoheres into $MS,E$ final state:

$$\Psi_{MS,E} = \sum a|i\rangle|O_i\rangle \prod_{j} |E^j_i\rangle$$

where $E^j$ are $E$ elements (atoms), $N_E$ is $E^j$ total number. If an arbitrary $O$ pure state $\Psi_O$ is prepared, it will also decohere in a very short time into the analogous $|O_i\rangle$ combinations, entangled with $E$. Hence, of all pure $O$ states, only $\xi^O_i = |O_i\rangle$ are stable relative to $E$ decoherence. Consequently, it advocates the choice of such $O$ states as $O$ PB set, since in such environment $O$ simply can’t percept and memorize any other $O$ pure state during any sizable time interval.

4 Conclusion

Any consistent physical theory should not contain the logical contradictions, it should be true also for the predictions of measurement outcomes. Yet as was noticed by Wigner: 'The simultaneous observation of two opposite outcomes of quantum experiments is nonsense' [12]. It seems from our analysis that the structure of QM Observable Algebra by itself excludes such controversial observations even without the inclusion of Reduction Postulate into QM formalism. In addition, the formalism of systems’ self-description permits to resolve the old problem of Heisenberg cut in quantum measurements, by the inclusion of the information system into quantum formalism properly and on equal terms with other MS elements.

The most exciting and controversial question is whether this theory is applicable to the observations made by human observer $O$, in particular, whether in this case IP $J^O$ describes the true $O$ ’impressions’ about their outcomes? This is open problem, but at the microscopic level the human brain should obey QM laws as any other object, so we don’t see any serious reasons to make the exceptions. Note that in our theory the brain or any other processor $O$ plays only the passive role of signal receiver, the real effect of information loss, essential for collapse, occurs 'on the way', when the quantum signal passes through the information channel.

We conclude that standard Schrödinger QM formalism together with the theory of systems’ self-description permit to obtain the ‘subjective’ collapse of pure states without implementation of independent Reduction Postulate into QM axiomatic. In our approach the main source of stochasticity is the principal constraint on the transfer of specific information in $S \rightarrow O$ information channel. This information characterizes the purity of $S$ state, because of its loss, $O$ can’t discriminate the pure and mixed $S$ states. As the result of this information incompleteness, the stochasticity of measurement outcomes appear, which can be interpreted as the analog of fundamental 'white noise'.
The interesting feature of this theory is that the same MS state can be stochastic in $O_{\text{RF}}$, but evolve linearly in $O'_{\text{RF}}$. In particular, $\Psi_{\text{MS}}$ restriction to $O$ in $O_{\text{RF}}$ is stochastic state $R_O$ of (7), yet in $O'_{\text{RF}}$ $O$ partial state is $R_O^B$ of (4), i.e. is the ‘weak superposition’. The detailed explanation of this effect is given by the formalism unitarily nonequivalent representations of Algebraic QM [4]. Here we notice only that $O$ and $O'$ deal with different sets of MS observables, and so the transformation of MS states between them is nonunitary. Obtained results agree well with our calculations in $C^*$ Algebras formalism, in that approach the inference map $M_O$ corresponds to the restriction of MS observable algebra to $O$ (sub)algebra [8].

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