A quantum isomonodromy equation and its application to $\mathcal{N} = 2$ $SU(N)$ gauge theories

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Abstract
We give an explicit differential equation which is expected to determine the instanton partition function in the presence of the full surface operator in $\mathcal{N} = 2$ $SU(N)$ gauge theory. The differential equation arises as a quantization of a certain Hamiltonian system of isomonodromy type discovered by Fuji, Suzuki and Tsuda.

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1. Introduction
In [1], Alday and Tachikawa formulated a combinatorial formula for the instanton partition function $Z_{\text{inst}}$ in the presence of the full surface operator in $\mathcal{N} = 2$ $SU(N)$ gauge theory, based on a result [2] about the affine Laumon space. Furthermore, they observed an interesting relation between $Z_{\text{inst}}$ for $SU(2)$ theory and the KZ equation with affine $SL(2)$ symmetry as an extension of the AGT relation [6]. A similar relation to Virasoro CFT including the irregular singularities was then examined in [3, 4]. For the $SU(N)$ case, the relation to affine $SL(N)$ conformal blocks was studied in [5].

In this paper, we study the differential equation satisfied by $Z_{\text{inst}}$ from a point of view slightly different from KZ equations or $W_N$ algebras. Our basic strategy is to use the isomonodromy equations. It is known that some aspects of the AGT relation have a natural interpretation [4, 7, 8] through the isomonodromy systems (or Painlevé equations for $SL(2)$ cases). That is, the 2d CFTs (= quantum isomonodromy systems) can be viewed as non-autonomous and quantum deformation of the Hitchin systems [9, 10] (= Seiberg–Witten theory) arising through the $\Omega$-deformation [11].

1 We will recapitulate it in section 2, equation (7).
At the classical level, the relation between the Seiberg–Witten theory and the isomonodromy equation is directly recognized by looking at the curves. For instance, the $SU(2)$ $N_f = 4$ Seiberg–Witten curve in Gaiotto form [12],

$$x^2 = \left( \frac{\mu_1}{z} + \frac{\mu_2}{z-1} + \frac{\mu_3}{z-t} \right)^2 - \frac{\kappa z + u}{z(z-1)(z-t)} - \left( \sum_{i=1}^{3} \mu_i \right),$$  

(1)

coincides with the Hamiltonian of the sixth Painlevé equation,

$$H_{VI} = q(q-1)(q-t) p^2 - 2[\mu_1(q-1)(q-1) + \mu_2q(q-t) + \mu_3q(q-1)]p - \kappa q,$$  

(2)

due to the fact that $f(z,v) = \kappa$.

We want to generalize this kind of correspondence to $SU(N)$ $N_f = 2N$ cases at the quantum level. The first problem is to look for the suitable isomonodromy system with higher rank symmetries. Fortunately, a nice candidate appeared in the recent work [15, 17]. The system, which we call the Fuji–Suzuki–Tsuda (FST) equation, can be described as an isomonodromic deformation of the $N \times N$ Fuchsian connection on $\mathbb{P}^1$ with regular singularity at $z = 0, 1, t, \infty$:

$$D = \partial_z - A, \quad A = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t},$$  

(4)

with the following spectral type (= eigenvalue multiplicity of the residue matrices): $A_0$ and $A_\infty = -A_0 - A_1 - A_t$ are of type $(1^N)$, while $A_1$ and $A_t$ are of type $(N-1, 1)^3$. We will recall the explicit form of the isomonodromy equation in section 3. Here, we look at the classical spectral curve in order to see its relation to the gauge theory. The curve for equation (4) is given by

$$\det(v - zA) \propto \det((z-1)(z-t)(v-A_0) - z(z-t)A_1 - z(z-1)A_t)$$

$$= (z-1)(z-t) f(z,v) = 0,$$  

(5)

where $v = z\theta$ is considered as a commuting variable. The factorization in the second line is due to the fact that $A_1, A_t$ are of rank 1. Hence $f(z,v)$ is of bi-degree $(2, N)$ in variables $(z, v)$ and has the form

$$f(z,v) = t \prod_{i=1}^{N}(v - m_i) + z \left( \sum_{i=0}^{N-1} a_i v^i - (1+t)v^N \right) + z^2 \prod_{i=1}^{N}(v - \tilde{m}_i),$$  

(6)

where $A_0 \sim \text{diag}(m_1, \ldots, m_N)$, $A_\infty \sim -\text{diag}(\tilde{m}_1, \ldots, \tilde{m}_N)$. This is the desired form as the Seiberg–Witten curve for $SU(N)$ with $N_f = 2N$.

In the following section, we formulate our main conjecture that a differential equation (12) determines the instanton partition function $Z_{\text{inst}}$. The isomonodromic origin of our equation (12) is discussed in section 3.

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2 This equation (called the $P_N$-chain in [15]) was first considered by Tsuda in 2008, as a similarity reduction of his ‘UC-hierarchy’ (certain generalization of KP hierarchy). Independently, in the context of the Drinfeld–Sokolov hierarchy, it was obtained by Fuji–Suzuki [14] in the case of $N = 3$ and generalized in [17]. Although our description in section 3 will follow the notation of [17], the isomonodromic picture considered here is naturally understood from the UC-hierarchy. Tsuda’s construction contains also the Garnier-type extension with spectral type $(N-1,1), (N-1,1), (1^N)$, $(1^N)$ (see [16]).

3 Then, without loss of generality, one can assume that the eigenvalues of $A_1$ are $(0, \ldots, 0, c)$ and similar for $A_t$.

4 Another indication comes from the special solutions. In both theories, generalized hypergeometric series $\mathcal{F}_{N-1}$ appears [18–22].
2. Main conjecture

The instanton partition function $Z_{\text{inst}} = Z_{\text{inst}}(y; a, m, \tilde{m})$ in the presence of the full surface operator in $SU(N)$ $N_f = 2N$ superconformal gauge theory is a function of $N$-variables $y = (y_1, \ldots, y_N)$ depending on $3N - 1$ parameters $m = (m_1, \ldots, m_N)$, $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_N)$ and $a = (a_1, \ldots, a_N)$, $\sum_{i=1}^N a_i = 0$.

Let us recall the combinatorial formula for $Z_{\text{inst}}$ following [1, 5]:

$$Z_{\text{inst}} = \sum_{\lambda} Z(\lambda) \prod_{i=1}^N \chi_{\lambda}^{i}(\lambda_{\bar{i}}).$$

(7)

Here the sum is taken over all $N$-tuples $\lambda = (\lambda_1, \ldots, \lambda_N)$ of partitions $\lambda_i = (\lambda_1^i \geq \lambda_2^i \geq \cdots \geq \lambda_{t_i}^i > 0)$. The indices of $\lambda_i$ will be extended to $\mathbb{Z}$ by $\lambda_i = \lambda_i^{+N}$ and $\lambda_{t_i}^j = 0$ ($j \leq 0$ or $j > t_i$). We put $|\lambda| = \sum_{i=1}^N \lambda_i^j = \sum_{i=1}^N \sum_{j \geq 1} \lambda_i^j$. The exponents $k_i(\lambda)$ are given by

$$k_i(\lambda) = \sum_{j \geq 1} \lambda_i^j - j - 1.$$  

(8)

The coefficients $Z(\lambda)$ are defined as

$$Z(\lambda) = \frac{n_f(a, \lambda, m) n_f(a, \lambda, \tilde{m})}{n_w(a, \lambda)},$$

(9)

where

$$n_f(a, \lambda, m) = n_{\text{bif}}(m, a, (\phi)^N, \lambda, 0), \quad n_f(a, \lambda, \tilde{m}) = n_{\text{bif}}(a, \tilde{m}, \lambda, (\phi)^N, 0),$$

$$n_w(a, \lambda) = n_{\text{bif}}(a, a, \lambda, \lambda, 0), \quad n_{\text{bif}}(a, b, \lambda, \mu, x) = \prod_{i=1}^{[|\lambda|]} (w_i - x),$$

(10)

and finally, the weights $w_i = w_i(a, b, \lambda, \mu)$ are determined by the formula [2]

$$\chi(a, b, \lambda, \mu) = \sum_{t=1}^{[|\lambda|]} \sum_{k=1}^{N} e^{a_k - b_j - x_{\lambda}^k} \left[ \epsilon \left( \frac{e^{a_k - b_j - x_{\lambda}^k}}{x_{\lambda}^k} \right) \sum_{s=1}^{\mu^j} \epsilon^{s} \right]$$

$$- \sum_{t=1}^{[|\lambda|]} \sum_{k=1}^{N} \sum_{j=1}^{t_1} \sum_{l \geq t_1} e^{a_k - b_j - x_{\lambda}^k} \left( \frac{e^{a_k - b_j - x_{\lambda}^k}}{x_{\lambda}^k} \right) \left( \epsilon^{s} \right) \left( \frac{e^{s} - 1}{x_{\lambda}^k} \right)$$

$$+ \sum_{t=1}^{[|\lambda|]} \sum_{k=1}^{N} \sum_{j=1}^{t_1} \sum_{l \geq t_1} e^{a_k - b_j - x_{\lambda}^k} \left( \frac{e^{a_k - b_j - x_{\lambda}^k}}{x_{\lambda}^k} \right) \left( \epsilon^{s} \right) \left( \frac{e^{s} - 1}{x_{\lambda}^k} \right)$$

$$+ \sum_{t=1}^{[|\lambda|]} \sum_{k=1}^{N} \sum_{j=1}^{t_1} \sum_{l \geq t_1} e^{a_k - b_j - x_{\lambda}^k} \left( \frac{e^{a_k - b_j - x_{\lambda}^k}}{x_{\lambda}^k} \right) \sum_{s=1}^{\lambda_{t_1}^j - 1} \epsilon^{s} (x_{\lambda}^k)^{s},$$

(11)

where $[x]$ is the largest integer such that $[x] \leq x$. Formula (11) is applicable to periodic parameters $a_{i+N} = a_i, b_{i+N} = b_i$. However, another convention is possible, where all terms $\epsilon^s \left( \frac{e^{a_k - b_j - x_{\lambda}^k}}{x_{\lambda}^k} \right)$ in the exponentials in equation (11) are taken away in exchange for assuming quasi-periodicity $a_{i+N} = a_i + \epsilon_2, b_{i+N} = b_i + \epsilon_2$. In what follows, we will adopt the second option.

$^5$ $a, b \in \mathbb{C}^N$, and $\lambda, \mu$ are $N$-tuples of partitions. The constraints $\sum_{i=1}^N a_i = 0, \sum_{i=1}^N b_i = 0$ will be considered after the substitutions (10).
As compared with the above complicated formulas for $Z_{\text{inst}}$, the differential equation we propose is rather simple and defined as

$$DZ(y) = \left( \Delta_N + \sum_{i=1}^{N} u_i \partial_i \right) + \left( \sum_{i=1}^{N} y_j \right) \left( \Delta_N + \sum_{i=1}^{N} v_i \partial_i + \sum_{i=1}^{N} r_i s_i \right) + \cdots$$

$$= \left\{ \left( \Delta_N + \sum_{i=1}^{N} u_i \partial_i \right) + \left( \sum_{i=1}^{N} y_j \right) \left( \Delta_N + \sum_{i=1}^{N} v_i \partial_i + \sum_{i=1}^{N} r_i s_i \right) \right\} Z = 0,$$  \quad (12)

where $y_{i+N} = y_i$, $\partial_i = \frac{\partial}{\partial y_i}$, $\Delta_N = \frac{1}{2} \sum_{i=1}^{N} (\partial_i - \partial_{i+1})^2$. The parameters $u_i, v_i, r_i, s_i$ ($1 \leq i \leq N$) will be set as

$$u_i = \frac{-\alpha_i + \alpha_{i+1}}{\epsilon_1}, \quad v_i = \frac{\alpha_i + \alpha_{i+1} - \alpha_{i+2} + \tilde{m}_i - \tilde{m}_{i+1}}{\epsilon_1},$$

$$r_i = \frac{\alpha_i - \tilde{m}_i}{\epsilon_1}, \quad s_i = \frac{-a_i - \tilde{m}_i}{\epsilon_1},$$  \quad (13)

where $x_{i+N} = x_i + \epsilon_2$ for $x = a, m, \tilde{m}$ (while $u_i, v_i, r_i, s_i$ are periodic).

The main claim in this paper is as follows:

**Conjecture 1.** The instanton partition function $Z_{\text{inst}}$ is characterized as the unique formal power series solution of the form $Z = 1 + O(y)$ for the differential equation (12) with parameters (13).

We have checked this conjecture for $N \leq 5$ up to total degree 5 in $y$-variables (in some cases by specializing the parameters to numerical values).

Under a degeneration limit $y_i \to \epsilon^2 y_i$, $m_i (\tilde{m}_i) \to \epsilon^{-1} \Lambda$ ($\epsilon \to 0$), the differential equation (12) reduces to the Toda equation

$$\left( \Delta_N + \sum_{i=1}^{N} u_i \partial_i + \frac{\Lambda^2}{\epsilon_1} \sum_{i=1}^{N} y_i \right) Z = 0,$$  \quad (14)

whose relation to $\mathcal{N} = 2$ SU($N$) pure gauge theory has already been established by Braverman–Etingof [24] (see also [25–27]).

3. Origin of the differential equation

In this section, we explain an isomonodromic origin of our differential equation (12). As already mentioned in the introduction, equation (12) is a quantization of the Fuji–Suzuki–Tsuda (FST) equation [14, 15, 17]. The FST equation can be written as a Hamiltonian system for $2(N-1)$ variables $q = (q_1, \ldots, q_{N-1})$ and $p = (p_1, \ldots, p_{N-1})$

$$t(t-1) \frac{dq_i}{dr} = \frac{\partial H}{\partial p_i}, \quad t(t-1) \frac{dp_i}{dr} = -\frac{\partial H}{\partial q_i},$$  \quad (15)

with parameters $\eta$ and $\alpha = (\alpha_0, \ldots, \alpha_{2N-1})$, $\sum_{j=0}^{2N-1} \alpha_j = 1$. The Hamiltonian $H = H(q, p, t; \eta, \alpha)$ is given by

$$H = \sum_{i=1}^{N-1} H_{V1}(q_i, p_i; a_i, b_i, c_i, d_i) + \sum_{1 \leq j < N-1} (q_i - 1)(q_j - 1)(q_i p_j + \alpha_{2-j})p_j + p_j(q_j p_j + \alpha_{2-j}),$$  \quad (16)

This is a kind of coupled system of the Painlevé VI. Another equation of such type first found by Sasano [23] will also be important for some superconformal gauge theories.
where $H_{VI}$ is the Hamiltonian of sixth Painlevé equation
\[ H_{VI}(q, p; a, b, c, d) = q(q - 1)(q - t)p^2 - {aq(q - 1) + bq(q - t)} + c(q - 1)(q - t)p + dq, \]
and $a_i = \sum_{j=1}^{N-1} a_{ij}, \quad b_i = \sum_{j=0}^{i-1} a_{2j}, \quad c_i = \sum_{j=0}^{N-1} a_{2j+i} - a_{2i-1} - \eta, \quad d_i = \eta a_{2i-1}.$

A quantization of the system is given by the Schrödinger equation\(^7\)
\[ t(t - 1) \frac{\partial}{\partial t} - H(q_i, \frac{\partial}{\partial q_i}) \Psi(q_1, \ldots, q_{N-1}, t) = 0. \] (18)

To connect this equation with $Z_{\text{inst}}$, we make a gauge transformation $\hat{\Psi}(q, t) = y_1 q_1 \cdots q_N \Psi(y)$, together with the variables' change
\[ y_1 = q_1, \quad y_2 = \frac{q_2}{q_1}, \quad \ldots, \quad y_{N-1} = \frac{q_{N-1}}{q_{N-2}}, \quad y_N = \frac{t}{q_{N-1}}, \]
\[ q_i \frac{\partial}{\partial q_i} = \partial_i - \partial_{i+1} \ (i = 1, \ldots, N-1), \quad t \frac{\partial}{\partial t} = \partial_N \]
\[ (\partial_i = y_i \frac{\partial}{\partial y_i}). \] Then equation (18) takes the form (in the following, we will concentrate on the $N = 3$ case)
\[ D \Psi(y) = \{ \Delta_3 + u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3 + M_0 + y_1 (\partial_{31} + x_1)(\partial_{12} + r'_1) + y_2 (\partial_{12} + x_2)(\partial_{23} + r'_2) + y_3 (\partial_{23} + x_3)(\partial_{31} + r'_3) + y_2 y_3 (\partial_{12} + x_2)(\partial_{31} + x_1)(\partial_{12} + r'_1) + y_1 y_2 (\partial_{31} + x_3)(\partial_{23} + r'_2) + y_1 y_2 y_3 (\Delta_3 + v_1 \partial_1 + v_2 \partial_2 + v_3 \partial_3 + M_1) \} \Psi(y) = 0, \] (20)
where $\partial_{ij} = \partial_i - \partial_j$, and $r'_i, s_i, u_i, v_i, M_0$, and $M_1$ are some constants depending only on the parameters $\eta, \alpha$ and $k_1, \ldots, k_N$ (their precise expressions are not necessary). We can and will choose $k_N$ so that $M_0 = 0$. Then, equation (20) has a unique formal series solution of the form
\[ \Psi(y) = \sum_{i,j,k=0}^{\infty} c_{ijk} y_1^i y_2^j y_3^k \quad (c_{000} = 1). \] (21)

The coefficients $c_{ijk}$ are written in terms of the very-well-poised, balanced hypergeometric series
\[ F(a_0; a_1, \ldots, a_s) = \sum_{k=0}^{\infty} \frac{(a_0 + 2k)}{a_0} \prod_{i=0}^{s} \frac{(a_i)_k}{(a_0 + 1 - a_i)_k}, \] (22)
as
\[ c_{ijk} = \frac{(r'_i - j)\Gamma(r'_i + s_i)\Gamma(-s_i)}{i!(j + 1 - j)!(-u_i + 1, u_i - s_i, -u_2 - j)} F(u_1 - j; -i, -j, u_1 - r'_1 + 1, u_1 - s_2, -u_2 - j), \] (23)
where $(x)_i = \Gamma(x + i)/\Gamma(x)$ is the Pochhammer symbol (see the appendix for the proof). Similarly, $c_{0ij}$ and $c_{i0}$ are given by cyclic shifts of parameters $x_i \rightarrow x_{i+1 \mod 3}$ ($x = r, s, u$).

On the other hand, the corresponding coefficients of the instanton partition function
\[ Z_{\text{inst}}(y) = \sum_{i,j,k=0}^{\infty} c_{ijk}^L y_1^i y_2^j y_3^k \] (24)
\[ \text{Here we will not consider the problem of operator ordering seriously since the ambiguities can be absorbed by shifts of parameters.} \]
are obtained (at least for the first several terms) as
\[
c_{ij0}^L = \sum_{k=0}^{\min(i,j)} Z([i, k], [j - k], \phi) = (-1)^{i-j} \prod_{l=0}^{i-1} \frac{1}{\epsilon_l} \frac{(\epsilon_l - a_1 \times m_1)}{\epsilon_l} \frac{(\epsilon_l - a_2 \times m_2)}{\epsilon_l} \frac{(\epsilon_l - a_i \times m_i)}{\epsilon_l} \frac{(\epsilon_l - a_j \times m_j)}{\epsilon_l} \frac{(\epsilon_l - a_k \times m_k)}{\epsilon_l} \\
\times F \left( \frac{-\epsilon_l}{\epsilon_l} : -i, -j, -\epsilon_l j + a_2 - a_3, \epsilon_l - a_1 + m_3, -a_1 + \tilde{m}_2 \right),
\]
(25)
together with similar formulas for \(c_{ijk}^L, c_{ij0}^L\) obtained by the shifts \(x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1 + \epsilon_2\) \((x = a, m, \tilde{m})\). Comparing the coefficients \(c_{ij0}^L\) and \(c_{ij0}^L\), etc., we find that they almost coincide if we put
\[
\begin{align*}
  r_1 &= \frac{a_2 - m_3}{\epsilon_1}, & r_2 &= \frac{a_3 - (m_1 + \epsilon_2)}{\epsilon_1}, & r_3 &= \frac{a_1 - m_2}{\epsilon_1}, & s_1 &= \frac{a_1 - \tilde{m}_1}{\epsilon_1}, \\
  s_2 &= \frac{a_2 - \tilde{m}_2}{\epsilon_1}, & s_3 &= \frac{a_3 - \tilde{m}_3}{\epsilon_1}, & u_1 &= \frac{a_2 - a_1}{\epsilon_1}, & u_2 &= \frac{a_3 - a_2}{\epsilon_1}, \\
  u_3 &= \frac{(a_1 + \epsilon_2) - a_3}{\epsilon_1}.
\end{align*}
\]
(26)
In fact, under this parameter identification, the ratio of the coefficients \(c_{ij0}/c_{ij0}^L\), etc. are simply given by
\[
c_{ij}^{ij} = (r_1')_{i-j} (r_2')_{i-j} (r_3')_{i-j}.
\]
(27)
Moreover, this relation (27) is satisfied also for \((ijk) = (111), (211), (121), (112)\) and further (as far as we checked), by putting
\[
\begin{align*}
  M_1 &= (r_1')_1 s_1 + (r_2')_1 s_2 + (r_3')_1 s_3, & v_1 &= r_1' - r_3' - s_1 + s_2 - u_1, \\
  v_2 &= r_2' - r_1' - s_2 + s_3 - u_2, & v_3 &= r_3' - r_1' - s_3 + s_1 - u_3 - \epsilon_2.
\end{align*}
\]
(28)
In order to remove the factor in equation (27), we make a ‘gauge transformation’ (in momentum space) defined by
\[
D' \rightarrow D = V^{-1} D', \quad V = (r_1')_{\delta_1} (r_2')_{\delta_2} (r_3')_{\delta_3}.
\]
(29)
Under this transformation, the Euler derivatives remain invariant \(\partial_i \rightarrow V^{-1} \partial_i V = \partial_i\). On the other hand, by using the relation \(\partial_i y_j = y_j (\partial_i + \delta_{ij})\), the multiplication operators \(y_i\) are transformed as
\[
\begin{align*}
  y_1 &\rightarrow V^{-1} y_1 V = \frac{y_1 (r_1')_{\delta_1} (r_2')_{\delta_2} (r_3')_{\delta_3}}{(r_1')_{\delta_1} (r_2')_{\delta_2} (r_3')_{\delta_3}} = y_1 \frac{\partial_1 + r_1' - 1}{\partial_1 + r_1'}, \\
  y_2 &\rightarrow y_2 \frac{\partial_2 + r_2' - 1}{\partial_2 + r_2'}, \quad y_3 \rightarrow y_3 \frac{\partial_3 + r_3' - 1}{\partial_3 + r_3'},
\end{align*}
\]
(30)
and hence
\[
\begin{align*}
  y_1 y_2 &\rightarrow y_1 \frac{\partial_1 + r_1' - 1}{\partial_1 + r_1'} y_2 \frac{\partial_2 + r_2' - 1}{\partial_2 + r_2'} = y_1 y_2 \frac{\partial_1 + r_1' - 1}{\partial_1 + r_1'} \frac{\partial_2 + r_2' - 1}{\partial_2 + r_2'}.
\end{align*}
\]
(31)
Then the differential operator \(D'\) in equation (20) is transformed into
\[
D = V^{-1} D' V = \Delta_3 + u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3 \\
+ y_1 (\partial_3 + s_1)(\partial_3 + r_1) + y_2 (\partial_1 + s_2)(\partial_1 + r_2) + y_3 (\partial_2 + s_3)(\partial_2 + r_3) \\
+ y_2 y_3 (\partial_2 + s_2)(\partial_2 + r_2) + y_3 y_1 (\partial_3 + s_3)(\partial_3 + r_3) + y_1 y_3 (\partial_3 + s_1)(\partial_3 + r_1) \\
+ y_1 y_2 y_3 (\partial_3 + s_1 \partial_1 + \partial_2 \partial_2 + \partial_3 \partial_3 + M_1),
\]
(32)
where \((r_1, r_2, r_3) = (r_1' - 1, r_1' - 1, r_1' - 1)\). Thus, we arrived at equation (12) for the \(N = 3\) case.
4. Summary and discussions

In this paper, we formulated an explicit differential equation (12) which is expected to determine the instanton partition function $Z_{\text{inst}}$ in the presence of the full surface operator in $\mathcal{N} = 2$, $SU(N)$ gauge theory with $N_f = 2N$. The differential equation is derived as a quantization of the FST equation of isomonodromy type.

In [1, 5], it was claimed that the partition function $Z_{\text{inst}}$ is the conformal block of the affine Lie algebra $\text{SL}_N$ (with the insertion of the $K$-operators). It is known [28, 29] that the KZ equation satisfied by the conformal blocks can be interpreted as the quantization of a typical isomonodromy system, the Schlesinger equation. Hence, it is quite natural to expect a direct relation between the formulation of [1, 5] and the isomonodromy approach here. For instance, the specialization of the primary fields $V_{\chi}, \chi = \kappa/\Lambda_1, \kappa/\Lambda_{N-1}$ in [5] for the simple punctures agrees with the choice of the spectral type $(N - 1, 1)$. More precise relations between these two formulations, in particular the understanding of the mysterious $K$-operators, will be an important future problem.

Although we have considered the isomonodromy deformation of an operator of the form (4), it can also be formulated by a scalar differential operator

$$L = \partial_x^N + u_2 \partial_x^{N-2} + \cdots + u_N.$$ (33)

Then the relation to $W_N$-algebras is also naturally expected (see [30, 31] and references therein).

At present, our understanding of the relation between the 4d gauge theory and the isomonodromy equation is still extrinsic. In [32] it was noted that the linear action of the loop operators (monodromy of surface operators) on the chiral partition function is independent of the gauge coupling. This observation may be a key ingredient for more conceptual understanding of the isomonodromic nature of gauge theories and the AGT relation.

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Appendix A. Proof of equation (23)

From equation (20), the coefficients $c_{i,j,0}$ in (21) are determined by

$$c_{i-1,j,0}(i - 1 - j + r_1)(i - 1 - s_1) + c_{i-1,j-1,0}(j - 1 + r_2)(i - 1 - s_1)
- c_{i,j-1,0}(j - 1 + r_2)(1 + i - j + s_2) - c_{i,j,0}(i^2 - ij + j^2 + iu_1 + ju_2) = 0.$$ (A.1)

Plugging (23) into this equation and rewriting the parameters as $a_0 = u_1 - j, \ a_1 = -i, \ a_2 = -j, \ a_3 = -u_2 - j, \ a_4 = u_1 - r_1 + 1, \ a_5 = u_1 - s_2$, the relation we should prove reduces to

$$F - (a_0 - a_1) a_1 F^{a_1} + \frac{(1 + a_0) a_2 a_3 (1 + a_0 - a_1 - a_4) (1 + a_0 - a_1 - a_5)}{(1 + a_0 - a_1) (1 + a_0 - a_4) (1 + a_0 - a_5)} F^{a_0 a_2}$$
$$+ \frac{(1 + a_0) a_1 a_2 a_3}{(1 + a_0 - a_2) (1 + a_0 - a_3)} F^{a_0 a_1 a_2 a_3} = 0.$$ (A.2)
where $F = F(a_0, a_1, a_2, a_3)$ and $F^{iujv} = F_{|a_0 \to a_0 + 1, a_2 \to a_2 + 1}$. Expanding the series $F$ in each term, equation (A.2) can be written as

$$\sum_{k=0}^{\infty} \varphi_k \omega_k = 0,$$

(A.3)

where $\varphi_k = \prod_{i=0}^{k} \frac{(a_i)_k}{(a_{i-n})_k}$ and

$$\omega_k = (a_0 a_1 - a_1^2 - a_2 a_3)(a_0 + 2k) - (a_0 - a_1 + k)(a_1 + k)(a_0 + 2k) + (1 + a_0 - a_1 - a_2)(1 + a_0 - a_1 - a_2)(a_0 + k)(a_2 + k)(a_3 + k)(1 + a_0 + 2k)
\frac{(1 + a_0 - a_1 + k)(1 + a_0 - a_4 + k)(1 + a_0 - a_5 + k)}{(1 + a_0 - a_4 + k)(1 + a_0 - a_5 + k)} + \frac{(a_0 + k)(a_1 + k)(a_2 + k)(a_3 + k)(1 + a_0 + 2k)}{(1 + a_0 - a_4 + k)(1 + a_0 - a_5 + k)}.$$

(A.4)

Then equation (A.3) follows from an identity

$$\varphi_0 \omega_k = \varphi_{k+1} u_{k+1} - \varphi_k u_k, \quad u_k = k(k + a_0 - a_2)(k + a_0 - a_3),$$

(A.5)

since the infinite sum is terminating: $\varphi_k = 0$ for $k > \min(i, j)$.

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