Impact of Bias on School Admissions and Targeted Interventions

Yuri Faenza
Columbia University, New York, NY
yf2414@columbia.edu

Swati Gupta
Georgia Institute of Technology, Atlanta, GA
swatig@gatech.edu

Xuan Zhang
Columbia University, New York, NY
xz2569@columbia.edu

There is an inherent problem in the way students are evaluated for admissions - be it standardized testing, interviews or essays. These evaluation criteria cannot be adjusted to account for the impact of implicit bias, socio-economic status or even opportunities available to the students. Motivated by this, we present, to the best of our knowledge, the first mathematical analysis of the impact of implicit bias and deficiencies in testing on the rank of schools that students get matched to. In particular, we first analyze a double continuous model of schools and students, where all the students have a unanimous ranking for all the schools, and schools observe the potential of students and then accept the best students from the available applicant pool dependent on their ranking. To account for bias in evaluations, we consider a bias model pioneered by Kleinberg and Raghavan (Kleinberg and Raghavan 2018) where the evaluators (i.e., schools) can only observe a discounted potential $\beta Z$ ($0 \leq \beta \leq 1$) for a subset of the candidates (i.e., students), instead of their actual potential $Z$.

We show that when the observed evaluations of students incorporate a group-specific bias, under a natural matching mechanism, the ranking of the matched schools of both unbiased group ($G_1$) and biased group ($G_2$) of students are affected, with some of the latter being heavily penalized even for relatively small values of bias (i.e., $\beta$ close to 1). Further, we find that schools have little incentive to change their evaluation mechanism, if their goal is the maximize the total potential of accepted students. Armed with this basic model and inferences on the impact on students as well as schools, we show that the students who are most in need of additional training/resources (or even community-building exercises) to achieve their true potential are the middle-performing students (average students), as opposed to those with high scores, thus questioning existing scholarship/aide mechanisms focusing on top performers. We further show, using computational experiments, that the qualitative take-aways from our model remain the same even if some of the assumptions are relaxed and we move, for instance, from a continuous to a discrete model and allow the bias factor to vary for each student.

Key words: bias, school matchings, disparate impact, interventions, vouchers, community, demographic, mistreatment, fairness
1. Introduction

In 1995, Steele and Aronson conducted multiple experiments to show that when students perceive frustration while doing a test (i.e., when they feel that the level of questions is beyond their competence), and if there exists a negative group-stereotype that they are aware of, then their performance is significantly undermined (Steele and Aronson 1995). This led them to hypothesize that stereotype-threat can cause the test scores to be low for African American students. Such studies are not unique to race-based groups, but have also been conducted to show the impact of stereotype-threat in groups with low socio-economic status (Lovaglia et al. 1998) and for women in Mathematics (Schmader 2002). Besides performances on tests, even when candidates from different groups look identical on paper, recent studies have found that there is a significant disparity in the perception of these candidates by the evaluators, which is manifested in differences in the number of offers and in salaries (Moss-Racusin et al. 2012).

Bias and the disparity in opportunities between different candidates are also believed to play a major role in the access to education at different levels (Quinn Capers et al. 2017). For instance, minorities often tend to cluster in middle and high schools of lower quality (Boschma and Brownstein 2016) and are underrepresented in higher education programs (Ashkenas et al. 2017). Our goal in this work is to consider a model of school admission that takes bias into consideration, and investigate mathematically justified policies to mitigate the effect of bias.

The most common way to model admission to schools is through a two-sided market, with the two sides being schools and students, respectively, and each agent having an ordered preference on the agents from the other side of the market that are considered acceptable. This model has been used to match doctors to hospitals by the National Residency Matching Program since 1960s, but has gained widespread notoriety when Abdulkadirouglu et al. (Abdulkadiroğlu et al. 2005) reformed the admission process to high schools in the New York City School System in 2003. New York City has a very extensive public school system, in which over a million students are currently enrolled. In particular, every year roughly 80,000 students wish to join one of the 700 high school programs. Since 2003, admission is centralized and (essentially) governed by the classical Gale–Shapley / Deferred Acceptance algorithm (Gale and Shapley 1962). The simplicity of the algorithm, as well as the drastic improvement over the quality of the matches it provides when compared to the pre-2003 method, led to academic and public acclaim, and spur applications in many other systems (see, e.g., (Biró 2008)). However, this mechanism does not naturally address problems like school segregation and class diversity, which have worsened and become more and more of a concern in recent years (Kucsera and Orfield 2014, Shapiro 2019b, Shapiro and Lai 2019). The scientific community has reacted by incorporating in the mathematical model quotas and proportionality constraints (Biró et al. 2010, Nguyen and Vohra 2019, Tomoeda 2018), but there
is evidence that adding such constraints may sometimes even hurt minorities they were meant to help (Hafalir et al. 2013).

In this work, we take a different route, introducing a novel mathematical model aimed at quantifying the effect of bias in a simple two-sided matching markets, and propose interventions with limited resources to mitigate this effect. To the best of our knowledge, this is the first approach in the literature that analyzes the impact of bias in school matchings. While one can see a natural application to the school assignment problem, our framework more generally models a two-sided market where a candidate may not be perceived at his/her true potential, and is therefore assigned to a lower level than the one (s)he deserves.

**The model.** We give here a high-level view of our model, and defer to Section 2 for details. We assume that both sides of the markets (to which we refer as schools and students henceforth) form continuous sets. Every student is assigned a real number, to which we refer as the (true) potential of the student. The schools are however only aware of a different number, to which we refer as the perceived potential. We assume the population of students is divided into two sets, $G_1$ and $G_2$. For students from $G_1$, the potential and the perceived potential coincide, while for students from $G_2$, the perceived potential equals to the potential multiplied by a constant $\beta \in (0, 1]$. Students from $G_2$ form a $p$ fraction of the total population, with $p \in [0, 1]$. We mostly think of $p$ being quite small – say, less than .25 – but many of our statements hold more generally.

As we alluded to in the introduction, there could be multiple causes why this bias is perceived: the scholastic career of the student, their attitude, their prior access to high-quality education, the tools schools use to evaluate them, etc.

We assume that schools, when evaluating students, only observe their perceived potential and not the true potential. The schools then rank students only based on their perceived potentials. Hence, all schools share the same ranking of students. One possible interpretation for the perceived potential is the student’s standardized test score. For instance, New York City’s top 8 public schools select candidates solely based on their scores in the Specialized High School Admissions Test (SHSAT) (New York City Department of Education NYC DOE). These scores are known to be impacted by socio-economic status of students (Lovaglia et al. 1998), or test preparation received in middle school (Corcoran and Baker-Smith 2018, Shapiro 2019a), with around 50% (resp. 80%) of students admitted to the top public high schools in New York City coming from only 5% (resp. 15%) of the middle schools (Corcoran and Baker-Smith 2018). Similarly to these top 8 high schools, universities in China and India rank students in an unaminous order according to their test scores in the college-entrance exam (Wikipedia contributors 2019).

Following the recent work on bias by Kleinberg and Raghavan (Kleinberg and Raghavan 2018), we assume that the potential of students is distributed following a Pareto (power law) distribution
with support $[1, \infty)$. There has been much empirical work (Clauset et al. 2009) supporting the statement that the achievement of individuals in many professions can be well approximated using such distributions with a small, positive power-law exponent, denoted in the following by $\alpha$.

We also assume that there is a unique, strict ranking of schools, that every student agrees on. This could be interpreted as one of the many available ranking of schools, or as a shared perception of which schools are the best. Although this assumption abstracts out many considerations that may be important for students (e.g. proximity of a school (Burgess et al. 2015) or constraints on the size of preference lists (Calsamiglia et al. 2010)), it is known that rankings have a huge impact on the colleges that students select. For instance, the authors of (Luca and Smith 2013) observe that U.S. News College have become so influential because it “makes the information so simple ... U.S. News did the work of aggregating the information into an easy-to-use ranking ... students tend to ignore the underlying details even though these details carry more information than the overall rank.” As another example, it is currently largely acknowledged that, among top public scientific high schools in New York City, Stuyvesant is the most selective and demanded, followed by Bronx Science and Brooklyn Tech (Corcoran and Baker-Smith 2018).

Because of the structure of our model, most classical concepts of efficiency and fairness$^1$ are satisfied by the matching where students are admitted to schools in order of potential, while we assume that in the biased setting, students are matched according to their perceived potential (see Section 2 for formal definitions). This paper is devoted to understanding how different those two matchings are (in terms of a variety of measures) and which interventions can be put in place to reduce this difference through the reduction of bias. We interpret these interventions as costly activities that can reveal the true potential for certain students, e.g., interventions can include conducting interviews, offering additional training, or investing resources to develop communities of students.

**Modeling assumptions.** Traditionally, both sides of a matching market are assumed to be discrete (Gale and Shapley 1962, Roth and Sotomayor 1992). This is a faithful description of reality, where goods and buyers mostly form discrete sets. There has been however, in recent years, an interest for models where one or both sides of the markets are continuous (Arnosti 2019, Azevedo and Leshno 2016). This is justified by the fact that, in many applications, markets are large, hence predictions in continuous markets often translate with a good degree of accuracy to discrete ones. Moreover, continuous markets are often analytically more tractable than discrete ones, see

---

$^1$ One could actually formally define e.g. stability in our double-continuous model following similar ideas to (Azevedo and Leshno 2016), and then take the proportion of blocking pairs as a measure of unfairness. We postpone this analysis to a later work.
again (Arnosti 2019, Azevedo and Leshno 2016). Our case is no exception: the continuous model allows us to deduce precise mathematical formulas, while we show through experiments that those formulas are a good approximation even in the discrete case. We also provide additional experiments evaluating the sensitivity of our results to other assumptions, like that of a unique bias value for all students in $G_2$. We remark that the goal of this study is not to provide a mechanism to admits students to schools, for which the assumption of all rankings of schools as well as of students being the same would be too simplistic. On the contrary, as we want to understand the impact of bias at a macroscopic level, we believe our approximation to be meaningful and useful since as in our model any reasonable mechanism would output the same assignment.

Results. We next discuss our key results on quantifying the impact of bias on school admissions, and propose targeted interventions that can reduce the negative impact of bias on $G_2$ students in the system. We specifically consider the following questions and give quantitative as well as qualitative answers to each of them. Recall that $\beta \in [0,1]$ is the discount factor given by bias, $p$ is the fraction of $G_2$ students, and $\alpha$ is the power-law exponent.

**Q1. How costly is bias for students? Can a student be placed in a school much worse than the one (s)he deserves, because of bias?**

We formally investigate this question in Section 3.1. We observe that all $G_2$ students are assigned to schools that are worse than what they actually deserve, with some $G_2$ students being assigned to significantly lower ranked school. Quantitatively, as students’ true potentials decrease, such discrepancy increases initially and then drops to 0. Thus, presence of bias impacts the average students the most - where they are pushed to significantly lower ranked schools. One can think of this intuitively in the following way. Starting from the top school, $G_1$ students gradually take more seats than they deserve, and thus gradually push $G_2$ students to worse schools than what they deserve. This process stops once all $G_1$ students are assigned to schools, and the only students that remain to be assigned are $G_2$ students. As a result, in lower ranked schools, all students are $G_2$ students. Therefore, the difference in ranks of the schools $G_2$ students are matched to decreases towards the end.

Let us call, for any student, the difference between the rank of the matched school in the biased setting and the rank of the matched school in the unbiased setting (when schools observe the true potentials) by the term *displacement*. We characterize the displacement of students in $G_1$ and $G_2$ and show that:

\[\text{displacement} = \text{rank of matched school in biased setting} - \text{rank of matched school in unbiased setting}\]

\[\text{displacement} \rightarrow 0\]

In the classical discrete model, when schools and students have unique ranking, there is only one stable assignment, and it is Pareto-optimal for students. A similar statement holds for the appropriate translations of those concepts to our model.
Proposition 1. (informal) The maximum displacement of \((1 - p)(1 - \beta^a)\) is experienced by a \(G_2\) student with potential \(1/\beta\). In contrast, the displacement of any \(G_1\) student is negative, i.e., they are always assigned to better schools in the biased setting. The most negative displacement \(-p(1 - \beta^a)\) is experienced by the \(G_1\) student with potential 1.

In other words, the maximum displacement (for \(G_2\) students) is worse when the proportion of \(G_2\) students, \(p\), is small and when the bias that \(G_2\) students experience is large (i.e., \(\beta\) is small).

Q2. How costly is bias for schools? Would the average potentials of assigned students (i.e., the utility of schools) be much lower when \(G_2\) students are perceived with lower potentials? In addition, how does the presence of bias affect diversity at schools?

We give a formal answer to the first two questions in Section 3.2. Qualitatively, we show that in contrast to the negative impact of bias on the school matches of \(G_2\) students, the impact on the utilities of schools is negligible for all schools other than the lowest ranked schools. This is because for each school, although the average potential of assigned \(G_1\) students is lower than it should be, its assigned \(G_2\) students have much higher true potential. And thus, the toll on the utility due to unqualified \(G_1\) students is partially canceled out by the overqualified \(G_2\) students and the net effect is minimal. On the other hand, some lower ranked schools that only admit \(G_2\) students fare better in the biased setting (since they admit over-qualified \(G_2\) candidates).

Proposition 2. (informal) Let \(s \in [0, 1]\) denote a school that is ranked at the \(s \times 100\%\) position among all schools\(^3\) and assume \(p \leq 0.5\). Consider \(s^* = 1 - p + p\beta^a\). Then schools ranked higher than \(s^*\) have a small loss in utility in the biased setting. The schools ranked after \(s^*\) only admit \(G_2\) students and see some increase in utility, with the highest utility increase right after \(s^*\).

Proposition 2 suggests that, since the average quality of admitted students does not seem to be affected much by bias, schools may have less of an incentive to remove it, especially when \(G_2\) forms the minority group.

We next consider in Section 3.3 the effect of bias on the diversity of students in differently ranked schools. When there is no bias, the proportion of \(G_2\) students at every school is uniformly \(p\), since the distributions of potentials are the same for both \(G_1\) and \(G_2\) students. However, when there is bias, the proportion of \(G_2\) students at top rank schools is reduced significantly.

Proposition 3. (informal) Consider again \(s^* = 1 - p + p\beta^a\) as in Proposition 2. In the biased setting, the proportion of \(G_2\) students is reduced to \(\frac{p\beta^a}{1 - p + p\beta^a}\) for schools ranked higher than \(s^*\), and for all schools ranked lower than \(s^*\) the proportion of \(G_2\) students is 1.

\(^3\)Note that for two schools \(s\) and \(s'\), if \(s < s'\), \(s\) is ranked higher than \(s'\). Thus, a school \(s\) is a top school, or higher ranked school, if \(s\) is small and vice versa.
Q3. What can schools do to remove bias (i.e., help $G_2$ students reveal their true potentials)? How much can a school’s utility change from such interventions?

This question is investigated in Section 4.1. We consider the intervention where schools conduct extensive analysis of students who are slightly below their admission criteria (i.e., students who are originally assigned to slightly lower ranked schools), with the goal of reviewing their true potential. We call this process interviewing. Given that conducting interviews takes resources, we assume that every school has certain interview capacities. We analyze the change in schools’ utility under such interviewing scheme. The key idea in computing the updated utilities is to express the cut-offs schools have for both $G_1$ and $G_2$ students in terms of true potentials after conducting interviews. Then, we can write down a school’s utility by combining the cutoffs it has for $G_1$ and $G_2$ students.

**Proposition 4 (informal)** The change in the utility of schools due to targeted interviews can be characterized in three cases: (a) If all schools interview, the gain that each school has is negligible, (b) From a single school $s$’s perspective, if all other schools interview students but school $s$, then its utility will not change much, (c) If $s$ is the only school that decides to interview students, then its gain in utility is significant, especially if it does not belong neither to the best, nor to the worst schools.

Therefore, when all schools conduct interviews, schools do not gain much. However, there is some gain for schools that are average in the ranking, and it is the only school interviewing. This may justify why, for instance, in New York City, the central (in terms of quality) bulk of schools do not only rely on students’ grade at schools, but also conduct on-site interviews (New York City Department of Education NYC DOE).

In general, the decentralized process of interviewing does not look very promising since the gain in utility for each individual school is minimal. Moreover, given that there is an overlap in the students that similarly ranked schools interview, students have to go through many interviews and reveal their true potential to the schools separately. We next explore a more societal level, centralized form of intervention, which can be thought of, for instance, as providing extra training/resources, developing problem solving groups, or team building exercises.

Q4. What can the government or other public agencies do to remove bias, i.e., what are some school-independent interventions? How much can students benefit from such interventions? Who should such interventions target given limited resources?

We generically call this societal interventions vouchers. We assume that only $G_2$ students will use vouchers, and this will have the effect of removing their bias. This is justified by the fact that $G_1$ students should already have had access to, say, extra training (this is why they were $G_1$
students in the beginning). We investigate two measures of unfairness or mistreatment (i.e., positive displacement) of \( G_2 \) students motivated by axiomatically justified notions of (in)equality (Kumar and Kleinberg 2000, Bertsimas et al. 2011), and for each one, we give explicit formula on who are the \( G_2 \) students that should be de-biased in order to minimize this measure of mistreatment. The two measures we consider are 1) the maximum mistreatment (in absolute value or in lexicographical order) and 2) the average mistreatment of all mistreated \( G_2 \) students. In both cases, informally speaking, the students we should target are those whose potentials are around the potential of the most misplaced student. For the first measure, we show the following proposition.

**Theorem 1.** (informal) The maximum mistreatment without any intervention is \((1 - p)(1 - \beta^a)\). Let \( \hat{c} \) denote the proportion of \( G_2 \) students to whom we can provide vouchers. Assuming \( p < 1 - \beta^a \), the maximum reduction in maximum mistreatment increases piecewise linearly with respect to \( \hat{c} \) and depending on the magnitude of \( \hat{c} \), the speed of reduction varies (see Figure 6).

For the second measure, we show the following proposition.

**Theorem 2.** (informal) The average mistreatment of all mistreated students without any intervention is \( \frac{1}{2}(1 - p)(1 - \beta^a) \). The maximum reduction in average mistreatment under this intervention increases about linearly with respect to the amount of resources given, as shown in Figure 6, assuming \( p < 1 - \beta^a \) and \( p < 0.5 \).

The above results complement the growing stream of papers that characterize trade-offs between multiple metrics of fairness, specifically for statistical measures of fairness such as calibration across groups, balance for positive class, and balance for negative class (Kleinberg et al. 2017). In contrast to statistical measures, where there exist impossibility results of satisfying multiple fairness metrics simultaneously (even approximately), we show that the average as well as maximum mistreatment is minimized in our problem by targeting a similar set of students.

We next comment on the applicability of our model under relaxed assumptions:

**Q5.** Can the assumptions made in our model for mathematical tractability can be relaxed, while keeping the results qualitatively similar? In particular, what if we move from a continuous to a discrete model, assume \( \beta \) non-constant, or assume that not all students that are offered a voucher will use it?

We investigate this question empirically. In particular, we compare the average mistreatment when, for a fixed \( \hat{c} \), the set of students given by Theorem 2 vs. the optimal continuous interval of students (obtained numerically) is debiased. Our numerical experiments, reported in Section 5, show that results in those two case are very similar, even if all assumptions are relaxed at the same time. This gives, we believe, evidence that our result extend well-beyond the simple model studied...
in this paper, and can be impactful in more realistic settings, where our assumptions hold only approximately.

In practice, the resources available to meaningful interventions in an existing system are limited, and there is resistance to change. Thus, our focus is on understanding the impact of minimally invasive and targeted resources (as opposed to changing the matching mechanism itself). Summing up, our key takeaways are: the disparity in admissions is experienced much more by groups that face biases/stereotypes, compared to the marginal advantage of the non-stereotyped group; decentralized interventions at the individual school level are not as impactful in terms of decreasing mistreatment, compared to centralized debiasing interventions focusing on average students in the overall student pool.

2. A continuous matching market
We introduce a simple matching market, where students rank schools following a unique strict order, and schools rank students following a unique order. Both schools and students are continuous sets. We let the student population be a set \( \Theta \), and we associate to \( \Theta \) a probability distribution on the potentials of students. For each student \( \theta \in \Theta \), we use \( Z(\theta) \) to denote his or her potential. Throughout the paper, we assume the distribution to be the Pareto distribution with parameters \((1, \alpha)\). Hence, we write \( Z(\Theta) \sim \text{Pareto}(1, \alpha) \), and each student has potential at least 1. We let \( \mu: \Theta \to [0, 1] \) be the ranking function based on students’ potentials, with rankings normalized to be between 0 and 1, and a student \( \theta \) with a smaller value of \( \mu(\theta) \) is a student who has higher potential. Hence for \( \theta, \theta' \in \Theta \), whenever \( Z(\theta') > Z(\theta) \), we have \( \mu(\theta') < \mu(\theta) \). Students with the same potential are ranked at the same level.

We assume that \( p \) fraction of the students belong to Group 2 (\( G_2 \)). \( G_2 \) students are the ones whose perceived potential is biased by a constant multiplicative factor \( \beta \in (0, 1] \). Students who are not in Group 2 are called Group 1 (\( G_1 \)) students. Their perceived potential is exactly their true potential and they account for \( 1 - p \) proportion of the population.

We let \( \hat{Z}(\theta) \) denote the perceived potential of a student \( \theta \in \Theta \). That is, if \( \theta \in G_1 \), then \( \hat{Z}(\theta) = Z(\theta) \); otherwise, \( \hat{Z}(\theta) = \beta Z(\theta) \). For \( G_1 \) students, the cdf (cumulative distribution function) of their perceived (or equivalently true) potentials is denoted by \( F_1 \). Similarly, let \( F_2 \) denote the cdf of the perceived potentials of \( G_2 \) students. Since \( Z(\Theta) \sim \text{Pareto}(1, \alpha) \), we have

\[
F_1(t) = 1 - \left(\frac{1}{t}\right)^\alpha; \quad F_2(t) = 1 - \left(\frac{\beta}{t}\right)^\alpha.
\]

Note that both function \( F_1(\cdot) \) and \( F_2(\cdot) \) take in the value of perceived potentials. Moreover, the domain of \( F_1 \) is \([1, \infty)\), whereas the domain of \( F_2 \) is \([\beta, \infty)\).
Let $\hat{\mu} : \Theta \to [0,1]$ be the function that ranks students based on their perceived potentials, with rankings normalized to be between 0 and 1. It is also convenient to define the ccdf (tail distribution) of students’ perceived potentials. That is, let $\bar{F}_1 := 1 - F_1$ and $\bar{F}_2 := 1 - F_2$ be the ccdf of their corresponding distributions. Then, for a student $\theta \in \Theta$, $\hat{\mu}(\theta)$ counts the fraction of students whose perceived potential is higher than that of $\theta$. That is:

$$\hat{\mu}(\theta) = \begin{cases} (1-p)\bar{F}_1(Z(\theta)) + p\bar{F}_2(Z(\theta)) & \text{if } \theta \in G_1, \\ (1-p)\bar{F}_1(\beta Z(\theta) \vee 1) + p\bar{F}_2(\beta Z(\theta)) & \text{if } \theta \in G_2. \end{cases}$$

(1)

where $\vee$ is the maximum operator. We say that student $\theta$ is assigned to school $\hat{\mu}(\theta)$. Note that, when $\beta = 1$ (i.e., no bias), formula (1) computes $\mu(\theta)$: $\mu(\theta) = \bar{F}_1(Z(\theta))$ for $\theta \in \Theta$.

Intuitively, $\mu(\theta)$ represents the school that student $\theta$ should be assigned to, whereas $\hat{\mu}(\theta)$ is the school where student $\theta$ is actually assigned to due to bias. For $\gamma \in \{\mu, \hat{\mu}\}$ and $s \in [0,1]$, we denote by $\gamma^{-1}(s)$ the set of students assigned to school $s$ under matching $\gamma$. In addition, we denote by $u_\gamma(s)$ the utility of school $s$ under matching $\gamma$, which is defined to be the average true potential of its assigned students. That is, $u_\gamma(s) := \int_{\theta \in \gamma^{-1}(s)} Z(\theta) dF_1(Z(\theta))$. To be mathematically precise, we define a matching in this market to be a surjective measurable function $\gamma$ from $\Theta$ to $[0,1]$ (i.e., students to schools), such that the mass of students mapped to a set of school $S \subseteq [0,1]$ coincides with the standard Lebesgue measure of $S$. One can easily check that $\mu$ and $\hat{\mu}$ defined above are matchings.

---

In formula, any surjective function $\gamma$ from $\Theta$ to $[0,1]$ is a matching if $\nu(\gamma^{-1}(S)) := (1-p)\int_{\theta \in \gamma^{-1}(S) \cap G_1} dF_1(Z(\theta)) + p\int_{\theta \in \gamma^{-1}(S) \cap G_2} dF_1(Z(\theta))$ is equal to the standard Lebesgue measure of $S$ for all $S \subseteq [0,1]$. 
3. Analysis of the market

Our first goal is to understand how much bias affects agents in the market. In particular, we would like to answer the following question: what is the loss of efficiency for agents of the market when all students $\theta \in \Theta$ are assigned to school $\hat{\mu}(\theta)$ instead of $\mu(\theta)$? We propose to measure this loss of efficiency in three ways: first, for students; second, for schools; and third, as a diversity measure.

3.1. Mistreatment of students

Formally, we define $\hat{\mu}(\theta) - \mu(\theta)$ to be the displacement of a student $\theta \in \Theta$. Note that if $\theta \in G_1$, the displacement is non-positive, and if $\theta \in G_2$, it is non-negative. The displacement can be easily calculated using the formulas for $\mu$ and $\hat{\mu}$ given in Section 2:

**Proposition 1.** For any student $\theta \in G_2$, the displacement $\hat{\mu}(\theta) - \mu(\theta)$ is given by:

$$\hat{\mu}(\theta) - \mu(\theta) = \begin{cases} (1-p)(\frac{1}{Z(\theta)})^\alpha(\frac{1}{\beta^\alpha} - 1) & \text{if } Z(\theta) \geq \frac{1}{\beta}, \\ (1-p)(1-(\frac{1}{Z(\theta)})^\alpha) & \text{if } Z(\theta) \leq \frac{1}{\beta}. \end{cases}$$

Moreover, for any student $\theta \in G_1$, we have $\hat{\mu}(\theta) - \mu(\theta) = (-p + p\beta^\alpha)(\frac{1}{Z(\theta)})^\alpha$. Thus, the maximum displacement of $(1-p)(1-\beta^\alpha)$ is experienced by a $G_2$ student with potential $\frac{1}{\beta}$. In contrast, the most significant displacement of any $G_1$ student is $-p(1-\beta^\alpha)$ experienced by a student with potential 1.

In the following, we mostly focus on $G_2$ students since they experience a positive displacement (i.e., the school they are matched to in the biased setting has lower rank compared to school they are matched to in the unbiased setting) and we will henceforth refer to the displacement experienced by $G_2$ students as mistreatment.

The $\hat{\mu} - \mu$ function is plotted for one set of parameters in Figure 1a. We give another perspective on those function in Figure 1b. X-axis are schools they should attend in the unbiased setting, whereas y-axis are schools they actually attend. The lower left corner is the best school and the upper right corner is the school that is ranked the lowest. The green dashed line is a line of slope one, representing the place a student should be placed on the figure if there are no bias in the system. The magenta line represents $G_1$ students and thus they are below the green line; the blue line represent $G_2$ students and thus they are above the green line. Note that the unequal impact of bias on the two groups is immediately evident from Figure 1b, which then strongly advocates the need for bias-reducing interventions, as we discuss in Section 4.

3.2. The schools’ perspective

We discuss next the impact of bias on the total potential of students accepted by any school. Let $s \in [0,1]$ denote school that is ranked in the $s \times 100\%$ position among the continuous range of schools.
Figure 2 Comparing the utilities of schools in the biased and unbiased setting. G₂ students form a minority group \((p < 0.05)\) (left). G₂ students form a majority group \((p > 0.05)\). In this case, the impact of bias on school utilities may no longer be negligible (right).

**Proposition 2.** For school \(s\), its utility under the unbiased model is
\[
u(\hat{s}) = \mu(s) = s - \frac{\alpha}{\beta} - 1
\]
and in the biased model is
\[
u(\hat{s}) = \frac{1 - p + p\beta^\alpha}{1 - p + p\beta^\alpha + 1} \left( \frac{s}{1 - p + p\beta^\alpha} \right)^{-\frac{\alpha}{\beta}}
\]
if \(s \leq 1 - p + p\beta^\alpha\), and
\[
u(\hat{s}) = \frac{1 - p + p\beta^\alpha}{1 - p + p\beta^\alpha + 1} \left( \frac{s}{1 - p + p\beta^\alpha} \right)^{-\frac{\alpha}{\beta}}
\]
if \(s > 1 - p + p\beta^\alpha\).

**Proof.** In order for a student \(\theta\) to be assigned to a school that is at least as good as \(s\), his or her perceived potential \(\tilde{Z}(\theta)\) needs to be high enough to satisfy
\[
(1 - p)\hat{F}_1(1 + \tilde{Z}(\theta)) + p\hat{F}_2(\tilde{Z}(\theta)) \leq s.
\]
That is, we need
\[
\tilde{Z}(\theta) \geq d(s) := \begin{cases} 
\frac{s}{1 - p + p\beta^\alpha} & \text{if } s \leq 1 - p + p\beta^\alpha \\
\frac{1 - p + p\beta^\alpha + 1}{\beta} \left( \frac{s}{1 - p + p\beta^\alpha} \right)^{-\frac{\alpha}{\beta}} & \text{if } s > 1 - p + p\beta^\alpha 
\end{cases}
\]
We call \(d(s)\) the cutoff for school \(s\). With the cutoffs, we can compute the utilities of schools.

First, we consider the case where there is bias against \(G_2\) students. Note that using Bayes rule, given that a student has perceived potential \(\tilde{Z}(\theta) \geq 1\), the probability that \(\theta\) is a \(G_1\) student is \(\frac{1 - p - p\beta^\alpha}{1 - p + p\beta^\alpha + 1}\). Also note that for a school \(s\) such that \(s \geq \tilde{\mu}(\frac{1}{\beta}) = 1 - p + p\beta^\alpha\), it is only assigned with \(G_2\) students. Thus, when \(s \leq 1 - p + p\beta^\alpha\), we have
\[
u(\hat{s}) = \frac{1 - p + p\beta^\alpha + 1}{1 - p + p\beta^\alpha + 1} d(s) = \frac{1 - p + p\beta^\alpha}{1 - p + p\beta^\alpha + 1} \left( \frac{s}{1 - p + p\beta^\alpha} \right)^{-\frac{\alpha}{\beta}}.
\]
And when \(s > 1 - p + p\beta^\alpha\), we have
\[
u(\hat{s}) = d(s)/\beta = \left( \frac{s}{1 - p} \right)^{-\frac{\alpha}{\beta}}.
\]
One the other hand, when there is no bias against \(G_2\) students, we simply have \(\nu(\hat{s}) = s^{-\frac{1}{\beta}}\).

In Figure 2, we plot functions \(\nu(\hat{s})\) and \(\nu(\hat{s})\) as a visualization of the impact of bias in evaluations on schools. When \(p \ll 1\), unlike the negative impact biases place on students, from schools’
perspective, the negative impact is negligible. Hence, from a operational perspective, it may be hard to convince schools to autonomously put in place mechanisms to alleviate the effect of bias. We discuss this more in Section 4.1.

3.3. Diversity in schools

We now investigate how bias affects diversity in schools. Let \( pr(s) \) (resp. \( \hat{pr}(s) \)) be the proportion of \( G_2 \) students assigned to school \( s \) when there is no bias (resp. there is bias) against \( G_2 \) students.

**Proposition 3.** Under the unbiased setting, we have \( pr(s) = p \). Under the biased setting, we have that \( \hat{pr}(s) = \frac{p\beta^\alpha}{1-p+p\beta^\alpha} \) if \( s \leq 1 - p + p\beta^\alpha \), and \( \hat{pr}(s) = 1 \) otherwise.

**Proof.** Note that \( pr(s) = p \) for all \( s \in [0,1] \) follows from the fact that the distribution of potentials is the same for \( G_1 \) and \( G_2 \) students. The formula for \( \hat{pr}(s) \) follows from the discussion in Proof of Proposition 1. \( \square \)

A visual comparison of \( pr(s) \) and \( \hat{pr}(s) \) can be found in Figure 3 for different values of \( \beta \) and \( p \).

4. Mitigating the effect of bias

In this section, we discuss several alternatives for moderating the effect of bias. We call these actions *bias mitigations*, and investigate two of them. We remark that, in both those interventions, no student \( \theta \in G_1 \) will be matched to a school worse than \( \mu(\theta) \). This is because our interventions are focused at debiasing the potential of (certain) \( G_2 \) students, hence for any \( G_1 \) student \( \theta \), no student with potential lower than \( Z(\theta) \) can have a perceived potential higher than \( \hat{Z}(\theta) = Z(\theta) \).

4.1. Interviewing

One way schools can act to mitigate the negative effects of bias is to spend resources that could reveal the true potential of students. We call such a process *interviewing*. We assume that, when a school \( s \) decides to conduct interviews, it is able to observe the true potentials of all \( G_2 \) students.
that were original matched to schools from \( s \) to \( s + iv \) for some constant \( iv \) that represents the amount of students a school is able to interview. In this section, we investigate the incentives of conducting interviews from schools’ perspective: does interviewing allow a school to drastically improve the quality of the students it is matched to?

The first situation we consider is when all schools conduct interviews and all schools have the same interview capacity (i.e., \( iv = iv \) for all schools \( s \)). Let \( d_i^1(s) \) and \( d_i^2(s) \) be the cutoffs in terms of true potentials each school \( s \) has for \( G_1 \) and \( G_2 \) students respectively, and let \( \bar{\mu} \) denote the resulting matching.

**Proposition 4.** The utilities of schools when all schools have interview capacity \( iv \) is:

\[
u_{\bar{\mu}}(s) = \begin{cases} 
\left( \frac{1-p}{d_1^1(s)^{\alpha+1}} d_1^1(s) + \frac{p}{d_2^2(s)^{\alpha+1}} d_2^2(s) \right) / \left( \frac{1-p}{d_1^1(s)^{\alpha+1}} + \frac{p}{d_2^2(s)^{\alpha+1}} \right) & \text{if } d_1^1(s) > 1 \\
\text{otherwise}, \end{cases}
\]

where

\[
\begin{cases} 
d_1^1(s) = d_2^2(s) = \bar{F}_1^{-1}(s) & \text{if } \bar{F}_1(d(\min(1,s+iv))/\beta) \geq s \\
d_2^2(s) = d(\min(1,s+iv))/\beta & \text{if } \bar{F}_1(d(\min(1,s+iv))/\beta) \leq s \\
d_1^1(s) = \bar{F}_1^{-1}\left( \frac{s-pF_1(d_2^2(s))}{1-p} \right) & \text{if } \bar{F}_1(d(\min(1,s+iv))/\beta) \leq s
\end{cases}
\]

**Proof.** Fix a school \( s \), under the continuous model, the amount of students school \( s \) takes has Lebesgue measure 0 and thus, it is more useful to consider the cutoffs or admission criteria in terms of all the schools that are at least as good as \( s \). This is what we mean by cutoffs in the following. If \( \bar{F}_1(d(\min(1,s+iv))/\beta) \geq s \), that is, if school \( s \) interviews more \( G_2 \) students than it is willing to admit, then the cutoffs for \( G_1 \) and \( G_2 \) students are the same. On the other hand, if \( \bar{F}_1(d(\min(1,s+iv))/\beta) \leq s \), then school \( s \) did not interview enough \( G_2 \) students and as a result, it will admits all the \( G_2 \) interviewees who are not yet admitted by schools better than \( s \), and fill in the rest with \( G_1 \) students. This gives rise to the formula for cutoffs \( d_1^1(s) \) and \( d_2^2(s) \) in the proposition.

The utilities of schools when conducting interviews, i.e., \( u_{\bar{\mu}}(s) \), with \( iv = 0.1 \) are plotted in Figure 4. Two additional lines are drawn in the figure. The dashed line on top represents the utility
of each school $s$ if $s$ is the only school that interviews students. In this case, school $s$’s utility is $d(s)/\beta$. On the contrary, the dotted on the bottom captures the utility of school $s$ when $s$ is the only school that does not interview students. In this case, its utility is $d_i(s)$. □

The utilities of schools when conducting interviews, i.e., $u_{\tilde{\mu}}(s)$, with $iv = 0.1$ are plotted in Figure 4. Two additional lines are drawn in the figure. The dashed line on top represents the utility of each school $s$ if $s$ is the only school that interviews students. In this case, school $s$’s utility is $d(s)/\beta$. On the contrary, the dotted on the bottom captures the utility of school $s$ when $s$ is the only school that does not interview students. And in this case, its utility is $d_i(s)$.

### 4.2. Vouchers

Suppose a central agency can select a subsets of students to debias. This can be achieved, for instance, by giving to (a limited amount of) students free vouchers to attend preparatory classes to high-school entry exams or by spending resources to build a community of students that explores learning as a group. Given a certain mass of available vouchers, we want to investigate which is the subset of students to whom these vouchers should be offered.

Recall that $\mu(\theta)$ and $\tilde{\mu}(\theta)$ denote the school a student $\theta$ is assigned to when the bias is equal to 1 (i.e., no bias), and $\beta$, respectively. Now let $\tilde{\mu}: \Theta \to [0,1]$ be the ranking of students after the bias mitigation. We consider two measures. The first is the most mistreated student: $mm(\tilde{\mu}) := \sup_{\theta \in \Theta}(\tilde{\mu} - \mu)(\theta)$. The second is the positive area under the curve (PAUC): $\sigma(\tilde{\mu}) := \int_{\theta \in \Theta}\max(\tilde{\mu} - \mu, 0)dF_1(Z(\theta))$. Note that this latter measure does not take into account students that are assigned to a school better than the one they would deserve. This is because we are interested in quantifying the improvement for mistreated $G_2$ students, rather than the penalization for $G_1$ students. In 1979 Tanner Lectures on Human Values, Amartya Sen argued that defining the basis of any desired notion of equality is important (Sen 1979). We take the view of achieving equality by minimizing the mistreatment of $G_2$ students, while satisfying a baseline treatment for $G_1$ students (since the capacities of the schools are fixed, if a $G_2$ student is admitted to a low rank school after debiasing, they must displace a $G_1$ student.) Moreover, the maximum displacement in all the students corresponds to the axiomatically established and well-studied min-max notion of fairness (Kumar and Kleinberg 2000), and the positive area under curve corresponds to total mistreatment of group $G_2$, i.e., it is a group notion of fairness (Conitzer et al. 2019, Dwork and Ilvento 2018, Marsh and Schilling 1994).

As we show next, the choice is quite similar, whether we want to minimize the most mistreated student or the PAUC. Before stating the result formally, we need to introduce new notation. For $\tilde{c} \in [0,1]$, let $\mathcal{T}(\tilde{c})$ to be all sets $T \subseteq [0,1]$ such that $\int_{t \in T}dF_1(t) \leq \tilde{c}$. Here, $\tilde{c}$ models the amount of resources or vouchers, and each $T \in \mathcal{T}(\tilde{c})$ represents the potentials of $G_2$ students to whom vouchers
are provided so that their true potential will be revealed. That is, for \( \theta \in G_2 \) such that \( Z(\theta) \in T \), we now have \( \hat{\theta} = \theta \).

Let \( \mu_T \) be the ranking of the students after \( G_2 \) students whose real potential lies in \( T \) have been debiased. Also, let \( \mathcal{T}_{mm}(\hat{c}) \) be the collection of sets \( T \) such that \( \sup(\mu_T - \mu) \) is minimized. Next result gives an explicit characterization of those sets, assuming

\[
p < 1 - \beta^\alpha.
\]

(2)

This is a reasonable assumption because of the following. First, \([2, 3.5]\) is a reasonable range to assume for the parameter \( \alpha \) in terms of academic achievements due to (Clauset et al. 2009). Then, if we, for example, assume \( \alpha = 3 \), even with mild biases towards \( G_2 \) students, say with \( \beta = .9 \), we only need \( p < 1 - .9^3 = .271 \). And with stronger bias, say \( \beta = .8 \), the requirement on \( p \), which is \( p < 1 - .8^3 = .488 \), is more relaxed.
THEOREM 1. Assume \( p < 1 - \beta^\alpha \). Then there exists a set \( T = [Z_1^*, Z_2^*] \in \mathcal{T}_{\text{mm}}(\hat{c}) \) such that all other sets from \( \mathcal{T}_{\text{mm}}(\hat{c}) \) differ from \( T \) on a set of measure zero. If \( \hat{c} \geq \frac{(1-p)(1-\beta^\alpha)}{1-p+1-\beta^\alpha} \), then:

\[
Z_1^* = \left( \frac{(1-p) + (\frac{1}{\beta^\alpha} - 1)\hat{c}}{\frac{1}{\beta^\alpha} - p} \right)^{-\frac{1}{\beta^\alpha}} \quad \text{and} \quad Z_2^* = \left( \frac{(1-p)(1-\hat{c})}{\frac{1}{\beta^\alpha} - p} \right)^{-\frac{1}{\beta^\alpha}},
\]

and the maximum mistreatment is reduced to \( \sup(\mu|Z_1^*, Z_2^*| - \mu) = (1-p)(1-\beta^\alpha)\frac{1-\hat{c}}{1-p+1-\beta^\alpha} \) from \( \sup(\hat{\mu} - \mu) = (1-p)(1-\beta^\alpha) \). Conversely, if \( \hat{c} \leq \frac{(1-p)(1-\beta^\alpha)}{1-p+1-\beta^\alpha} \), then:

\[
Z_1^* = \left( \frac{(1-p-\hat{c})\beta^\alpha}{1-p} + \hat{c} \right)^{-\frac{1}{\beta^\alpha}} \quad \text{and} \quad Z_2^* = \left( \frac{(1-p-\hat{c})\beta^\alpha}{1-p} \right)^{-\frac{1}{\beta^\alpha}},
\]

and the maximum mistreatment is reduced to \( \sup(\mu|Z_1^*, Z_2^*| - \mu) = (1-p-\hat{c})(1-\beta^\alpha) + \hat{p}\hat{c} \).

We include a proof of Theorem 1 in the appendix, Section EC.1. The main idea is to first assume that the set \( T \) of true potentials of students who receive vouchers forms a connected set. Then, we can express the resultant maximum mistreatment (after de-biasing) as a function of the endpoints of \( T \) and work out the minimizing interval. We then drop the assumption that \( T \) is connected and show that the optimal set of students to de-bias remains the same. The analysis we give is actually more general, and presents results under which vouchers improve the mistreatment of students lexicographically. Interestingly, it also shows that, if vouchers are not distributed carefully, one may actually worsen the most mistreated students.

A pictorial representation of Theorem 1 is given in Figure 5, each for one choice of \( \hat{c} \). Moreover, Figure 6a shows how much \( \sup(\mu|Z_1^*, Z_2^*| - \mu) \) decreases as \( \hat{c} \), the amount of resources, increases.

Besides minimizing the maximum of students’ mistreatment, we also investigate the objective of minimizing the aggregate amount of mistreatment received by all \( G_2 \) students. In this case, we restrict our attention to debiasing \( G_2 \) students whose potential lies in a connected set - which is a justifiable implementation in practice (otherwise a student might feel they are treated unfairly as someone with better potential than them as well as someone with worse potential than them received vouchers). This assumption also makes our analysis more tractable. In particular, let \( \mathcal{T}^c(\hat{c}) \subseteq \mathcal{T}(\hat{c}) \) be all the connected sets \( T \in \mathcal{T}(\hat{c}) \). That is, \( \mathcal{T}^c(\hat{c}) := \{ [t_1, t_2] : \hat{F}_1(t_1) - \hat{F}_1(t_2) \leq \hat{c} \} \). In addition, let \( \mathcal{T}^c_{\text{auc}}(\hat{c}) \) be the collection of sets \( T \in \mathcal{T}^c(\hat{c}) \) such that \( \sigma(\mu_T - \mu) \) is minimized. The next result gives an explicit description of the set \( \mathcal{T}^c_{\text{auc}}(\hat{c}) \) when assuming, again, (2) and additionally \( p < 0.5 \).

THEOREM 2. Assume \( p < 1 - \beta^\alpha \) and \( p < 0.5 \). Then \( \mathcal{T}^c_{\text{auc}}(\hat{c}) \) is made of a unique set \( T = [Z_1^*, Z_2^*] \). If \( \hat{c} \geq \frac{(1-p)(1-\beta^\alpha)}{2p - \beta^\alpha + p\beta^\alpha + p\beta^\alpha} \), then:

\[
Z_2^* = \left( \frac{(1-p)(1-\hat{c})}{p\beta^\alpha + \frac{1}{\beta^\alpha} - 2p} \right)^{-\frac{1}{\beta^\alpha}} \quad \text{and} \quad Z_1^* = \left( \frac{(1-p)(1-\hat{c})}{p\beta^\alpha + \frac{1}{\beta^\alpha} - 2p + \hat{c}} \right)^{-\frac{1}{\beta^\alpha}}.
\]
In this case, the PAUC drops to $\sigma(\mu[Z_1^*, Z_2^*] - \mu) = \frac{1}{2}(1-p)(1-\beta^\alpha)\left(\frac{1-p}{p\beta^\alpha + 1-p}\right)$ from $\sigma(\hat{\mu} - \mu) = \frac{1}{2}(1-p)\left(1-\beta^\alpha\right)$. Conversely, if $\hat{c} \leq \frac{(1-p)(1-\beta^\alpha)}{2-p-\beta^\alpha-p\beta^\alpha+p\beta^\alpha}$, then:

$$Z_2^* = \left(\frac{(p\beta^\alpha-1)\hat{c} + (1-p)}{(1-p)\frac{1}{\beta^\alpha}}\right)^{-\frac{1}{\beta^\alpha}} \quad \text{and} \quad Z_1^* = \left(\frac{(p\beta^\alpha-1)\hat{c} + (1-p)}{(1-p)\frac{1}{\beta^\alpha} + \hat{c}}\right)^{-\frac{1}{\beta^\alpha}}.$$

In this case, the PAUC is reduced to $\frac{1}{2}(1-p)(1-\hat{c})^2 - \frac{1}{2}\beta^\alpha \left[\frac{(p\beta^\alpha-1)\hat{c} + (1-p)^2}{1-p} + p\hat{c}^2\right]$.

A pictorial representation of Theorem 2 is given in Figure 7. Figure 6b shows how much $\sigma(\mu[Z_1^*, Z_2^*] - \mu)$ decreases as $\hat{c}$, the amount of resources, increases. The proof of Theorem 2 is given in Section EC.2. Table EC.2 in the appendix (Section EC.3) shows that, for reasonable choices of the parameters, Theorem 1 and Theorem 2 give very close intervals.

We remark that the current distribution of vouchers violates individual fairness (Dwork et al. 2012). It does not treat similar individuals similarly, if an individual is at the boundary of the de-biasing interval. However one can potentially get around this by randomization or distributing “partial” vouchers. This would not change the interval to be targeted, however potentially smooth-out the boundaries of the interval. We leave this as an extension for future work.

5. Empirical validation of the robustness of our conclusions

In this section, we provide an empirical investigation of the robustness of results from Theorem 2. In particular, we consider models where one or more of the following modifications are implemented: the continuous sets representing schools and students are replaced by discrete sets, with schools having finite capacities; we allow $G_2$ students to have non-constant bias factor; we assume that only a percentage of students receiving a voucher will use it (and hence be debiased). For each of those models, given a fixed set of resources $\hat{c}$, we compare the PAUC when we debias students...
following Theorem 2 vs. the best continuous debiasing interval, obtained numerically (by using a sliding window). Results are reported in Figure 8, Figure 9a, Figure 9b (when only one assumption is relaxed) and Figure 10 (when all three assumptions are relaxed). They show that, depending on the deviation from our model, the predictions from Theorem 2 stay between close to and very close to the optimum, and in all cases provide a very significant improvement over the status quo. In the following, we discuss more in detail each experiment.

**Figure 8**  Assuming a constant number of 100 students per school, we compare the PAUC when debiasing students in the range computed in the formula given in Theorem 2 versus the PAUC under the best range of $G_2$ students to debias empirically. The PAUC without intervention is presented as a baseline.

**From continuous to discrete.** We first investigate how accurately the results on our model translate to a model where both sides of the market are discrete. Assuming a constant number of 100 students per school, we carried out a set of experiments where the number of schools ranges from 10 to 200. Due to the randomness resulting from discretization, for each experiment, we conducted 100 simulations and report here their mean and standard error of the mean in Figure 8. In fact, the standard errors are very small and thus they do not show in the figure.

From Figure 8, we can see that when the number of schools is more than 50, the formula presented in Theorem 2 for the continuous-continuous model works for for discrete models as well. And regardless of instance size, the theoretical formula is able to reduce PAUC significantly compared to the situation without any intervention (close to 50%).

Due to this result, for the following experiments, simulations are run on instances with 10,000 students and 100 schools.

**Non-constant bias.** We next investigate the assumption of constant discount factor $\beta$ for all $G_2$ students. In this set of experiments, we allow $\beta$ to vary by different amounts. Results are in Figure 9a.
(a) For each variation level $\varepsilon$, a $G_2$ student’s discount factor $\beta$ is drawn uniformly randomly from the interval $[0.8 - \varepsilon, 0.8 + \varepsilon]$.

(b) For each level of debiasing probability $q$, a $G_2$ student who is offered a voucher is able to reveal his true potential with probability $q$.

Figure 9 All simulations are carried out on instances with 10,000 students and 100 schools, where each school has 100 seats. We compare the PAUC when debiasing students in the range computed in the formula given in Theorem 2 versus the PAUC under the best range of $G_2$ students to debias empirically. The PAUC without intervention is presented as a baseline.

As shown in the figure, when variation in $\beta$ is small, the PAUC obtained using the formula in Theorem 2 is close the empirically optimal value. However, as the amount of variation in $\beta$ increases, the discrepancy between the theoretical optimal value the empirical one increases. Regardless, debiasing $G_2$ students in the theoretically optimal range can reduce the PAUC significantly from the situation without any intervention.

Students may not use vouchers. We then investigate the assumption that once a $G_2$ student is offered the voucher, he/she is able to reveal his true potential with 100% probability. In this last set of experiments, we vary this probability from 50% to 100%. Results are presented in Figure 9b.

As the probability decreases, the optimal PAUC, theoretically as well as empirically, increases. This makes sense because with smaller debiasing probability, many vouchers are essentially “wasted”. However, the discrepancy between the theoretical optimal value the empirical one is relatively small even when the debiasing probability is only 50%. Again, standard errors are very small and thus do not appear in the figure.

Simultaneous relaxation of all the three modeling assumptions. As a final step, we combine the three above-mentioned situations together. The results are summarized in Figure 10. Regardless of the magnitude of the debiasing probability and/or the number of schools, the comparison between the theoretically optimal PAUC and the empirically optimal PAUC under intervention by vouchers remains the same. This can be seen by comparing the solid lines and dashed lines of the same color in reference to Figure 9a.
Figure 10 Assuming each school has 100 seats, we carried out four sets of experiments, where in each experiment, the variation $\varepsilon$ in the discount factor $\beta$ varies from 0 to 0.20. These experiments differ in debiasing probability, either 60% or 90%, and/or the number of schools, either 50 or 150. We compare the PAUC when debiasing students in the range computed in the formula given in Theorem 2 versus the PAUC under the best range of $G_2$ students to debias empirically. The PAUC without intervention is presented as a baseline.

For a fixed level of $\varepsilon$, the distances between the solid lines and dashed lines are similar for all different colors (i.e., different combination of low or high debiasing probability and small or large number of schools). This implies that the differences between the theoretically optimal PAUC and the empirically optimal PAUC can be explained mostly by the variation in the discount factor that models the bias. In other words, varying debiasing probability and/or varying the number of schools do not, in general, result in a notably different discrepancy between the theoretically optimal solution and the empirically optimal solution.

6. Conclusion

Our key qualitative result is that, assuming that the potential of each student is drawn from a power-law distribution, bias in evaluations results in significant mistreatment of (biased) $G_2$ students compared to the marginal benefits of (unbiased) $G_1$ students. This motivates the urgent need for interventions. Mathematically, we show that, while schools do not seem to have an incentive to remove the bias, the maximum displacement of $G_2$ students as well as total displacement of $G_2$ students is minimized by targeting limited resources towards the top students in low ranked schools.

We note that the interventions we consider do not require the knowledge of group membership of students, which helps us deviate from the conundrum of how groups should be defined in practice. However, perhaps one can achieve a greater efficiency in targeting resources if group membership was known. We also showed empirically that many of the assumptions made in our model for the sake of mathematical tractability can be relaxed without modifying the qualitative message, and with only minor modifications to the quantitative statements.

Future work include further relaxations of the assumptions, such assuming that students have slightly different rankings of schools, investigating different distributions of students’ potential, as
well as debiasing mechanisms that satisfy individual fairness (see the discussion at the end of Section 4.2). Moreover, although our exposition in this paper is focused on school-student matchings, it would be interesting to apply/extend these results to two-sided markets in other domains as well. For instance, in ride-sharing systems like Uber and Lyft, there is evidence of bias in the ratings of drivers (e.g., socio-economic status is often correlated to the conditions of the car or the accents that people speak in) (Rosenblat et al. 2017). These ratings guide the quality and frequency of matches to riders among the drivers that are within a bounded radius from a new ride request. Our approach could then shed light on how the platforms (like Uber/Lyft) themselves might target drivers and de-bias their ratings, by potentially using a neutral evaluator.

References
Abdulkadiroğlu, A., Pathak, P. A., and Roth, A. E. (2005). The new york city high school match. American Economic Review, 95(2):364–367.
Arnosti, N. (2019). A continuum model of stable matchings with finite capacities. Talk at Simons Institute for the Theory of Computing.
Ashkenas, J., Park, H., and Pearce, A. (2017). Even with affirmative action, blacks and hispanics are more underrepresented at top colleges than 35 years ago. New York Times, pages 1–18.
Azevedo, E. M. and Leshno, J. D. (2016). A supply and demand framework for two-sided matching markets. Journal of Political Economy, 124(5):1235–1268.
Bertsimas, D., Farias, V. F., and Trichakis, N. (2011). The price of fairness. Operations research, 59(1):17–31.
Biró, P. (2008). Student admissions in hungary as gale and shapley envisaged. University of Glasgow Technical Report TR-2008-291.
Biró, P., Fleiner, T., Irving, R. W., and Manlove, D. F. (2010). The college admissions problem with lower and common quotas. Theoretical Computer Science, 411(34-36):3136–3153.
Boschma, J. and Brownstein, R. (2016). The concentration of poverty in american schools. The Atlantic, 29.
Burgess, S., Greaves, E., Vignoles, A., and Wilson, D. (2015). What parents want: School preferences and school choice. The Economic Journal, 125(587):1262–1289.
Calsamiglia, C., Haeringer, G., and Klijn, F. (2010). Constrained school choice: An experimental study. American Economic Review, 100(4):1860–74.
Clauset, A., Shalizi, C. R., and Newman, M. E. (2009). Power-law distributions in empirical data. SIAM review, 51(4):661–703.
Conitzer, V., Freeman, R., Shah, N., and Vaughan, J. W. (2019). Group fairness for the allocation of indivisible goods. In Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI).

5 https://www.businessinsider.com/leaked-charts-show-how-ubers-driver-rating-system-works-2015-2.
Corcoran, S. P. and Baker-Smith, E. C. (2018). Pathways to an elite education: Application, admission, and matriculation to New York City’s specialized high schools. *Education Finance and Policy*, 13(2):256–279.

Dwork, C., Hardt, M., Pitassi, T., Reingold, O., and Zemel, R. (2012). Fairness through awareness. In *Proceedings of the 3rd Innovations in Theoretical Computer Science conference*, pages 214–226. ACM.

Dwork, C. and Ilvento, C. (2018). Group fairness under composition.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

Hafalir, I. E., Yenmez, M. B., and Yildirim, M. A. (2013). Effective affirmative action in school choice. *Theoretical Economics*, 8(2):325–363.

Kleinberg, J., Mullainathan, S., and Raghavan, M. (2017). Inherent trade-offs in the fair determination of risk scores. In *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

Kleinberg, J. and Raghavan, M. (2018). Selection problems in the presence of implicit bias. *arXiv preprint arXiv:1801.03533*.

Kucsera, J. and Orfield, G. (2014). New York state’s extreme school segregation: Inequality, inaction and a damaged future.

Kumar, A. and Kleinberg, J. (2000). Fairness measures for resource allocation. In *Proceedings 41st Annual Symposium on Foundations of Computer Science*, pages 75–85. IEEE.

Lovaglia, M. J., Lucas, J. W., Houser, J. A., Thye, S. R., and Markovsky, B. (1998). Status processes and mental ability test scores. *American Journal of Sociology*, 104(1):195–228.

Luca, M. and Smith, J. (2013). Salience in quality disclosure: Evidence from the US news college rankings. *Journal of Economics & Management Strategy*, 22(1):58–77.

Marsh, M. T. and Schilling, D. A. (1994). Equity measurement in facility location analysis: A review and framework. *European Journal of Operational Research*, 74(1):1–17.

Moss-Racusin, C., D., Brescoll, V., Graham, M., and Handelsman, J. (2012). Science faculty’s subtle gender biases favor male students. *Proceedings of the National Academy of Sciences*, 109:16474–16479.

New York City Department of Education (NYC DOE) (2019). 2019 NYC High School Directory. [https://bigappleacademy.com/wp-content/uploads/2018/06/HSD_2019_ENGLISH_Web.pdf](https://bigappleacademy.com/wp-content/uploads/2018/06/HSD_2019_ENGLISH_Web.pdf).

Nguyen, T. and Vohra, R. (2019). Stable matching with proportionality constraints. *Operations Research*.

Quinn Capers, I., Clinchot, D., McDougle, L., and Greenwald, A. G. (2017). Implicit racial bias in medical school admissions. *Academic Medicine*, 92(3):365–369.

Rosenblat, A., Levy, K. E., Barocas, S., and Hwang, T. (2017). Discriminating tastes: Uber’s customer ratings as vehicles for workplace discrimination. *Policy & Internet*, 9(3):256–279.
Roth, A. E. and Sotomayor, M. (1992). Two-sided matching. *Handbook of game theory with economic applications*, 1:485–541.

Schmader, T. (2002). Gender identification moderates stereotype threat effects on women’s math performance. *Journal of Experimental Social Psychology*, 38(2):194–201.

Sen, A. (1979). Equality of what? *The Tanner lecture on human values*, 1.

Shapiro, E. (February 06, 2019a). Racist? fair? biased? asian-american alumni debate elite high school admissions. *The New York Times Magazine.*

Shapiro, E. (March 26, 2019b). Segregation has been the story of new york city’s schools for 50 years. *The New York Times Magazine.*

Shapiro, E. and Lai, K. K. R. (June 03, 2019). How new york’s elite public schools lost their black and hispanic students. *The New York Times Magazine.*

Steele, C. M. and Aronson, J. (1995). Stereotype threat and the intellectual test performance of african americans. *Journal of Personality and Social Psychology*, 69:797–811.

Tomoeda, K. (2018). Finding a stable matching under type-specific minimum quotas. *Journal of Economic Theory*, 176:81–117.

Wikipedia contributors (2019). List of admission tests to colleges and universities — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=List_of_admission_tests_to_colleges_and_universities&oldid=914002454. [Online; accessed 9-September-2019].
Omitted Proofs

EC.1. Proof of Theorem 1 and related facts

The analysis we give is actually more general, and has the goal of investigating conditions under which giving vouchers can improve over the status quo. More formally, for bounded functions $f,g : G_2 \to \mathbb{R}$, we write $f \succ g$ if we can partition $G_2$ in two sets $S,S'$ (with possibly $S' = \emptyset$) so that $f(\theta) = g(\theta)$ for $\theta \in S'$ and $\sup_{\theta \in S} f(\theta) > \sup_{\theta \in S} g(\theta)$. Note that $\succ$ is transitive and antisymmetric, and can be interpreted as a continuous equivalent of the classical lexicographic ordering for discrete vectors. In particular, if we let $f = \gamma - \mu$ and $g = \gamma' - \mu$ for matchings $\gamma, \gamma'$, then $\sup_{\theta \in G_2}(\gamma - \mu)(\theta) > \sup_{\theta \in G_2}(\gamma' - \mu)(\theta)$ implies $f \succ g$ (taking $S = G_2$). Now suppose we debias student in $T = [Z_1, Z_2]$ for some $T \in T(\bar{c})$, and let $f := (\tilde{\mu} - \mu)(\theta), g := (\mu_T - \mu)(\theta)$. Table EC.1 provides conditions under which $f \succ g$. In particular it shows that for certain combinations of the data and the choice of $Z_1$ and $Z_2$, giving vouchers may actually lead to worse (according to $\succ$) matchings. It is easy to see that under assumption (2) all the conditions given in Table EC.1 are verified.

| CASE | subcase | condition for $\tilde{\mu} - \mu \prec \tilde{\mu} - \mu$ |
|------|---------|--------------------------------------------------|
| I. $\beta Z_2 \geq Z_1$ | 1. $1 \leq \beta Z_1$ | $p < 1 - (\frac{Z_1}{Z_2})^\alpha$ |
| | 2. $\beta Z_1 \leq 1 \leq \beta Z_2$ | $p < 1 - (\frac{1}{\beta Z_2})^\alpha$ |
| II. $\beta Z_2 \leq Z_1$ | 1. $1 \leq \beta Z_1$ | $p < 1 - \beta^\alpha$ |
| | 2. $\beta Z_1 \leq 1 \leq \beta Z_2$ | $p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha) < (1-p)(\frac{1}{\beta^\alpha} - 1)(\beta^\alpha - (\frac{1}{Z_2})^\alpha)$ |
| | 3. $\beta Z_2 \leq 1$ | Not possible: $g \succ f$ in this case. |

Table EC.1 Sufficient conditions for $(\tilde{\mu} - \mu) \succ (\mu_T - \mu)$ by cases, where $T = [Z_1, Z_2]$. Each strict inequality, when replaced with its non-strict counterpart, gives instead a necessary condition.

In this first part of the proof, we proceed as follows. First, we assume that $T \in T^c(\bar{c})$. That is, we assume $T = [Z_1, Z_2]$ with extreme points $Z_1 < Z_2$. For simplicity, we let $\tilde{\mu}$ denote $\mu_T$. We then compare $f := \tilde{\mu} - \mu$ and $g := \tilde{\mu} - \mu$ using the relation $\succ$ defined in Section 4.2.

Note that, if we let $S$ be the set of students in $G_2$ with potential in $[Z_1, Z_2/\beta]$ and $S' := G_2 \setminus S$, we have $f(\theta) = g(\theta)$ for $\theta \in S'$. That is, only students from $G_2$ whose true potential lies in range $[Z_1, Z_2/\beta]$ are affected by the voucher correction. Hence, $\sup_{\theta \in S} f > \sup_{\theta \in S} g$ if and only if $f \succ g$.

We divide the analysis in the following two major cases: the first case is when $\beta Z_2 \geq Z_1$ (i.e., when $[\beta Z_1, \beta Z_2]$ and $[Z_1, Z_2]$ overlap) and the second case is when $\beta Z_2 \leq Z_1$. For both major cases, we will consider two subcases: $\beta Z_1 \geq 1$, $\beta Z_1 \leq 1 \leq \beta Z_2$. And for the second major case, we also need to consider the subcase where $\beta Z_2 \leq 1$. The results for all cases are summarized in the Table EC.1.
CASE I: \( \beta Z_2 \geq Z_1 \)

The dashed lines represent perceived potential equaling to 1, and the two dashed lines represent the two subcases we will consider.

Subcase 1. \( 1 \leq \beta Z_1 \). Under the status quo \( \hat{\mu} \), the student in \( S \) that is most unfairly treated is the one whose true potential is \( Z_1 \). This can be easily observed from Figure 1a. Thus,

\[
\sup\{(\hat{\mu} - \mu)(S)\} = (\hat{\mu} - \mu)(Z_1) = (1 - p)(\frac{1}{Z_1})^\alpha(\frac{1}{\beta^\alpha} - 1). \tag{EC.1}
\]

Now, for the corrected ranking \( \tilde{\mu} \), \( G_2 \) students whose true potentials are within \( [Z_1, Z_2] \) (i.e., those whose true potentials are revealed through vouchers) are not harmed under \( \tilde{\mu} \). That is, for all \( G_2 \) student \( \theta \) such that \( \theta \in [Z_1, Z_2] \), we have \( (\tilde{\mu} - \mu)(\theta) \leq 0 \), with equality achieved for \( G_2 \) students whose true potentials are in \( [Z_1, \beta Z_2] \). Thus, the \( G_2 \) student that is the mostly unfairly treated must be in \( [Z_2, Z_2/\beta] \). In particular, for every \( G_2 \) student \( \theta \in S \), we have

\[
\tilde{\mu}(\theta) = p\tilde{F}_2(\beta Z(\theta)) + (1 - p)\tilde{F}_1(\beta Z(\theta)) + p(\tilde{F}_2(\beta^2 Z(\theta)) - \tilde{F}_2(\beta Z_2)),
\]

where the three components correspond to \( G_2 \) students with better perceived potentials before voucher correction, \( G_1 \) students with better perceived potentials, and additional \( G_2 \) students with better perceived potentials after voucher correction. Thus,

\[
(\tilde{\mu} - \mu)(\theta) = p(\frac{1}{Z(\theta)})^\alpha + (1 - p)(\frac{1}{\beta Z(\theta)})^\alpha + p\left(\frac{1}{\beta Z(\theta)}\right)^\alpha - \left(\frac{1}{Z_2}\right)^\alpha - \left(\frac{1}{Z(\theta)}\right)^\alpha
\]

\[
= \left(\frac{1}{Z(\theta)}\right)^\alpha \left( p + (1 - p)\frac{1}{\beta^\alpha} + p\frac{1}{\beta^\alpha} - 1 \right) - p\left(\frac{1}{Z_2}\right)^\alpha
\]

\[
= \left(\frac{1}{Z(\theta)}\right)^\alpha \left( p + \frac{1}{\beta^\alpha} - 1 \right) - p\left(\frac{1}{Z_2}\right)^\alpha.
\]

Thus,

\[
\sup\{(\tilde{\mu} - \mu)(S)\} = (\tilde{\mu} - \mu)(Z_2) = \left(\frac{1}{\beta^\alpha} - 1\right)\left(\frac{1}{Z_2}\right)^\alpha \tag{EC.2}
\]

and \( f \succ g \) if

\[
(1 - p)(\frac{1}{Z_1})^\alpha(\frac{1}{\beta^\alpha} - 1) > \left(\frac{1}{\beta^\alpha} - 1\right)\left(\frac{1}{Z_2}\right)^\alpha \iff p < 1 - (\frac{Z_2}{Z_1})^\alpha
\]

See Figure EC.1 for a comparison of \( \tilde{\mu} - \mu \) and \( \hat{\mu} - \mu \) under this case, with two different values of \( p \).
Subcase 2. $\beta Z_1 \leq 1 \leq \beta Z_2$. Before the correction, the $G_2$ student that is mistreated the most is the one whose perceived potential is 1. That is,

$$\sup\{ (\hat{\mu} - \mu)(S) \} = (\hat{\mu} - \mu)(1) = (1-p)(1-\beta^\alpha).$$

(EC.3)

Note that $Z_1 \geq 1$ since $Z_1$ denotes the true potential. Thus, it is not hard to see that the previous analysis remains valid. In particular, equation (EC.2) remains true. Therefore, $f \succ g$ if

$$(1-p)(1-\beta^\alpha) > \left( \frac{1}{\beta^\alpha} - 1 \right) \left( \frac{1}{Z_2^\alpha} \right) \iff p < 1 - \left( \frac{1}{\beta Z_2^\alpha} \right).$$

See Figure EC.2 for a demonstration of scenarios under this case.
EC.1.2. CASE II: $\beta Z_2 \leq Z_1$

As in the previous case, the dashed lines represent perceived potential equal to 1, and dashed lines represent the different subcases we will consider. Note that besides the two cases in the previous case, we also have a third additional case to consider.

![Diagram of subcases](https://example.com/diagram)

**Subcase 1.** $1 \leq \beta Z_1$. Equation (EC.1) holds true for $\tilde{\mu} - \mu$. And as for the voucher correction, with the same argument, it is clear that $\sup\{ (\tilde{\mu} - \mu)(S) \}$ is achieved for some $\theta$ with $Z(\theta) \in [Z_2, Z_2/\beta]$. However, to have a complete analysis, we need to consider the two ranges, $[Z_2, Z_1/\beta]$ and $[Z_1/\beta, Z_2/\beta]$, separately as the expressions for $\tilde{\mu}$ differ.

**First Range** $[Z_1/\beta, Z_2/\beta]$. Here, $\tilde{\mu}$ has the same expression as in case 1.1 and thus,

$$\sup\{ (\tilde{\mu} - \mu)([Z_1/\beta, Z_2/\beta]) \} = (\tilde{\mu} - \mu)(Z_1/\beta).$$

**Second Range** $[Z_2, Z_1/\beta]$. In this range, we have

$$\tilde{\mu}(\theta) = p\tilde{F}_2(\beta Z(\theta)) + (1 - p)\tilde{F}_1(\beta Z(\theta)) + p(\tilde{F}_2(Z_1) - \tilde{F}_2(Z_2))$$

and

$$(\tilde{\mu} - \mu)(\theta) = p\left( \frac{1}{Z(\theta)} \right)^\alpha + (1 - p)\left( \frac{1}{\beta Z(\theta)} \right)^\alpha + p\left( \frac{1}{Z_1} \right)^\alpha - \left( \frac{1}{Z_2} \right)^\alpha - \left( \frac{1}{Z(\theta)} \right)^\alpha$$

$$= \left( \frac{1}{Z(\theta)} \right)^\alpha \left( p + (1 - p) \frac{1}{\beta^\alpha} - 1 \right) + p(\left( \frac{1}{Z_1} \right)^\alpha - \left( \frac{1}{Z_2} \right)^\alpha)$$

$$(=1-p)(\frac{1}{\beta^\alpha}-1)>0$$

Thus,

$$\sup\{ (\tilde{\mu} - \mu)([Z_2, Z_1/\beta]) \} = (\tilde{\mu} - \mu)(Z_2).$$

All together, since $\tilde{\mu} - \mu$ is continuous on $[Z_2, Z_2/\beta]$, we have

$$\sup\{ (\tilde{\mu} - \mu)(S) \} = \sup\{ (\tilde{\mu} - \mu)([Z_1, Z_2/\beta]) \} = (\tilde{\mu} - \mu)(Z_2)$$

$$= p\left( \frac{1}{Z_1} \right)^\alpha + \left( \frac{1}{Z_2} \right)^\alpha \left( (1 - p) \frac{1}{\beta^\alpha} - 1 \right)$$

(EC.4)

and $f > g$ if

$$(1 - p)\left( \frac{1}{Z_1} \right)^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) > p\left( \frac{1}{Z_1} \right)^\alpha + \left( \frac{1}{Z_2} \right)^\alpha \left( (1 - p) \frac{1}{\beta^\alpha} - 1 \right) \Leftrightarrow p < 1 - \beta^\alpha.$$
(a) when correction is better in the affected region

(b) when correction is worse in the affected region

Figure EC.3 Mistreatment of $G_2$ student before and after voucher correction for CASE II-1.

See Figure EC.3 for a demonstration of this case.

**Subcase 2.** $\beta Z_1 \leq 1 \leq \beta Z_2$. As in case II.2, since $\beta \in S$, Equation (EC.3) expresses $\sup\{\hat{(\mu - \mu)}(S)\}$.

The analysis for $\mu - \mu$ is the same as in the previous section and equation (EC.4) holds true. Thus, $f \succ g$ if

$$(1 - p)(\frac{1}{Z_1})^\alpha (\frac{1}{\beta^\alpha} - 1) > p(\frac{1}{Z_1})^\alpha + (\frac{1}{Z_2})^\alpha ((1 - p)\frac{1}{\beta^\alpha} - 1)$$

$\Leftrightarrow$

$$p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha) < (1 - p)(\frac{1}{\beta^\alpha} - \frac{1}{Z_2})^\alpha).$$

See Figure EC.4 below which demonstrates the scenarios under this case.

(a) when correction is better in the affected region

(b) when correction is worse in the affected region

Figure EC.4 Mistreatment of $G_2$ student before and after voucher correction for CASE II-2.
Subcase 3. $\beta Z_2 \leq 1$. Again, since $\frac{1}{\beta} \in S$, we can refer to Equation (EC.3) for the expression of $\sup\{(\hat{\mu} - \mu)(S)\}$. For voucher correction, as in subcase 2.1, it is clear that $\sup\{(\hat{\mu} - \mu)(S)\}$ is obtained for some $\theta$ with $Z(\theta) \in [Z_2, Z_2/\beta]$. To have a complete analysis, we will consider three ranges, $[Z_2, 1/\beta]$, $[1/\beta, Z_1/\beta]$ and $[Z_1/\beta, Z_2/\beta]$, separately as the expressions for $\tilde{\mu}$ differ, the same reason as before.

First Range $[Z_1/\beta, Z_2/\beta]$. As in Case 2.1, in this range, $\tilde{\mu}$ has the same expression as in case I.1. Thus, $\tilde{\mu} - \mu$ is decreasing in this range and

$$\sup\{(\tilde{\mu} - \mu)([Z_1/\beta, Z_2/\beta])\} = (\tilde{\mu} - \mu)(Z_1/\beta).$$

Second Range $[1/\beta, Z_1/\beta]$. The analysis here is the same as that of the second range in Case II.1. Therefore, $\tilde{\mu} - \mu$ is decreasing in this range as well and we have

$$\sup\{(\tilde{\mu} - \mu)([1/\beta, Z_1/\beta])\} = (\tilde{\mu} - \mu)(1/\beta).$$

Third Range $[Z_2, 1/\beta]$. Lastly, for this range, we have

$$\tilde{\mu}(\theta) = p\tilde{F}_2(\beta Z(\theta)) + (1 - p) + p(\tilde{F}_2(Z_1) - \tilde{F}_2(Z_2))$$

and

$$(\tilde{\mu} - \mu)(\theta) = p\left(\frac{1}{Z(\theta)}\right)^\alpha + (1 - p) + p\left(\frac{1}{Z_1}\right)^\alpha -\left(\frac{1}{Z_2}\right)^\alpha$$

$$= \left(\frac{1}{Z(\theta)}\right)^\alpha (p - 1) + (1 - p) + p\left(\frac{1}{Z_1}\right)^\alpha -\left(\frac{1}{Z_2}\right)^\alpha$$

Thus, $\tilde{\mu} - \mu$ is increasing in this range and

$$\sup\{(\tilde{\mu} - \mu)([Z_2, 1/\beta])\} = (\tilde{\mu} - \mu)(1/\beta).$$

All together, by continuity of $\tilde{\mu} - \mu$ on $[Z_2, Z_2/\beta]$, we have

$$\sup\{(\tilde{\mu} - \mu)(S)\} = (\tilde{\mu} - \mu)(1/\beta) = (1 - p)(1 - \beta^\alpha) + p\left(\frac{1}{Z_1}\right)^\alpha -\left(\frac{1}{Z_2}\right)^\alpha$$

Hence, $f \succ g$ only if

$$(1 - p)(1 - \beta^\alpha) \geq (1 - p)(1 - \beta^\alpha) + p\left(\frac{1}{Z_1}\right)^\alpha -\left(\frac{1}{Z_2}\right)^\alpha.$$}

However, this is impossible since $Z_1 < Z_2$. Hence, $g \succ f$ in this case, as shown in Figure EC.5 with two different values of $p$. 


EC.1.3. Computing the optimal range for vouchers

We now prove Theorem 1 under the additional restriction that sets in $T(\bar{c})$ are connected, i.e., of the form $[Z_1, Z_2]$ (we shall show later that this condition can be relaxed).

Observation EC.1. If there is a range $[Z_1, Z_2]$, of either case I.2 or case II.2, such that $\tilde{\mu} - \mu \prec \hat{\mu} - \mu$ with $S = G_2$, then the optimal range must be of case I.2 or case II.2. This is because for any range $[Z_1', Z_2']$ that is not of case I.2 or case II.2, we have $\sup\{\mu[ Z_1', Z_2'] - \mu\} \geq \sup\{\hat{\mu} - \mu\} > \sup\{\mu[ Z_1, Z_2] - \mu\}$.

As it turns out, indeed, the optimal range will be either case I.2 or case II.2, and exactly which one the optimal solution is depends on the amount of resources, i.e., the magnitude of $\hat{c}$.

We now show the first half of Theorem 1, i.e., we assume $\bar{c} \geq \frac{(1-p)(1-\rho^\alpha)}{1-p+1-\beta\rho}$. We proceed as follows:

(1). We first show that $[Z_1^*, Z_2^*]$ is of case I.2. That is, we show $\beta Z_2^* \geq Z_1^*$ and $Z_1^* \leq \frac{1}{\beta} \leq Z_2^*$.

From the analysis in Section EC.1.1 and Section EC.1.2, one can see that for such $[Z_1, Z_2]$ of case I.2 or case II.2, $\mu(Z_1, Z_2) - \mu$ increases on $[1, Z_1]$, decreases on $[Z_2, \infty]$, and it is non-positive on $[Z_1, Z_2]$. This means $\sup\{\mu[ Z_1, Z_2] - \mu\}$ is achieved either at $Z_1$ or $Z_2$.

(2). Next, we show that $[Z_1^*, Z_2^*]$ is an exact range, that is, $\frac{1}{Z_1^*} - \frac{1}{Z_2^*} = \bar{c}$. Moreover, let $\theta_1^*$ and $\theta_2^*$ be the $G_2$ students whose potentials are $Z_1^*$ and $Z_2^*$ respectively. Then, $(\mu(Z_1, Z_2) - \mu)(\theta_1^*) = (\mu(Z_1, Z_2) - \mu)(\theta_2^*)$ and therefore, they are both equal to $\sup\{\mu[ Z_1, Z_2] - \mu\}$.

Together with assumption (2), we have $\sup_{\theta \in \Theta}\{\mu[ Z_1^*, Z_2^*] - \mu\} \leq \sup_{\theta \in \Theta}\{\hat{\mu} - \mu\}$. Thus, due to Observation EC.1, it is sufficient to compare $[Z_1^*, Z_2^*]$ only with regions $[Z_1, Z_2]$ of case I.2 and case II.2 (i.e., when $\beta Z_1 \leq 1 \leq \beta Z_2$). Since $[Z_1^*, Z_2^*]$ is exact, we must either have $Z_1 > Z_1^*$ or $Z_2 < Z_2^*$.
(3). Lastly, we show that for any other feasible range \([Z_1, Z_2]\) of case I.2 or case II.2, we must have \(\sup_{\theta \in \Theta} \mu[Z_1, Z_2] - \mu > \sup_{\theta \in \Theta} \mu[Z_1, Z_2] - \mu\). Let \(\theta_1\) and \(\theta_2\) be the \(G_2\) students whose potentials are \(Z_1\) and \(Z_2\). It suffices to show

i). if \(Z_1 > Z_1^*\), then \((\mu[Z_1, Z_2] - \mu)(\theta_1) > (\mu[Z_1, Z_2] - \mu)(\theta_1^*)\)

ii). if \(Z_2 < Z_2^*\), then \((\mu[Z_1, Z_2] - \mu)(\theta_2) > (\mu[Z_1, Z_2] - \mu)(\theta_2^*)\)

STEP (1). We will start by showing \(\beta Z_2^* \geq Z_1^*\) through a sequence of equivalence relations:

\[
\beta Z_2^* \geq Z_1^* \iff \left( \frac{1}{\beta Z_2^*} \right)^{\alpha} \leq \left( \frac{1}{Z_1^*} \right)^{\alpha} \\
\iff \frac{1}{\beta^\alpha} (1 - p)(1 - \tilde{c}) \leq (1 - p) + \left( \frac{1}{\beta^\alpha} - 1 \right) \tilde{c} \\
\iff (1 - p)(1 - \tilde{c}) \leq (1 - p) \beta^\alpha + (1 - \beta^\alpha) \tilde{c} \\
\iff \tilde{c}(1 - \beta^\alpha + 1 - p) \geq (1 - p)(1 - \beta^\alpha).
\]

Here, the last inequality is implied by the condition we have on \(\tilde{c}\). Next, we will show \(Z_1^* \leq \frac{1}{\beta}\). Note that

\[
Z_1^* \leq \frac{1}{\beta} \iff \left( \frac{1}{\beta} \right)^{\alpha} \leq 1 \iff \beta^\alpha \cdot \frac{\frac{1}{\beta^\alpha} - p}{(1 - p) + (\frac{1}{\beta^\alpha} - 1) \tilde{c}} \leq 1 \\
\iff 1 - p \beta^\alpha \leq (1 - p) + (\frac{1}{\beta^\alpha} - 1) \tilde{c} \\
\iff (\frac{1}{\beta^\alpha} - 1) \tilde{c} \geq p(1 - \beta^\alpha) \iff \tilde{c} \geq p \beta^\alpha.
\]

To show \(\tilde{c} \geq p \beta^\alpha\), it suffices to show that \(\frac{(1 - p)(1 - \beta^\alpha)}{1 - p + 1 - \beta^\alpha} \geq p \beta^\alpha\) because of the condition we assumed on \(\tilde{c}\). Let \(a := 1 - p\) and \(b := 1 - \beta^\alpha\). Note that \(a, b \leq 1\) and \(a + b = 1 - p + 1 - \beta^\alpha \geq 1 - p + p = 1\). Hence, essentially we want to show that if \(a, b \leq 1\) and \(a + b \geq 1\), we must have

\[
\frac{ab}{a + b} \geq (1 - a)(1 - b).
\]

Indeed, this is true because

\[
\frac{ab}{a + b} \geq (1 - a)(1 - b) \iff ab \geq (a + b)(1 - (a + b) + ab) \\
\iff ab \geq (a + b)(1 - (a + b)) + (a + b)ab \\
\iff ab(1 - (a + b)) \geq (a + b)(1 - (a + b)).
\]

When \(a + b = 1\), this is clearly true as both sides equal to 0. When \(a + b > 1\), this is equivalent to \(ab \leq a + b\), which is also clearly true since \(ab \leq 1\).

Lastly, we will show \(Z_2^* \geq \frac{1}{\beta}\). That is, we want to show \((\beta Z_2)^{\alpha} \geq 1\). Plugging in the value of \(Z_2^*\), we want to show \((1 - p)(1 - \tilde{c}) \leq 1 - \beta^\alpha p \iff \tilde{c}(p - 1) \leq p(1 - \beta^\alpha).\) This is true simply because \(p - 1 < 0\) and \(1 - \beta^\alpha > 0\).
STEP (2). The claims follow simply from plugging in the values. First,
\[
\left(\frac{1}{Z_1}\right)^\alpha - \left(\frac{1}{Z_2}\right)^\alpha = \frac{(1-p) + (\frac{1}{\beta^\alpha} - 1)\hat{c} - (1-p)(1-\hat{c})}{\frac{1}{\beta^\alpha} - p} = \frac{(1-p) - (1-p)\hat{c}}{\frac{1}{\beta^\alpha} - p} = \hat{c}.
\]
Then, since \(Z_1^* < \frac{1}{\beta}\), we have \((\mu_{[Z_1^*,Z_2^*]} - \mu)(\theta_1^*) = (\hat{\mu} - \mu)(\theta_1^*) = (1-p)(1-((\frac{1}{\beta^\alpha})^\alpha)\). For the other side, using the formula in Equation (EC.2), we have \((\mu_{[Z_1^*,Z_2^*]} - \mu)(\theta_2^*) = (\frac{1}{\beta^\alpha} - 1)(\frac{1}{\beta^\alpha} - p)^\alpha\). Therefore, we simply want to show
\[
(1-p)(1-\left(\frac{1}{Z_1}\right)^\alpha) = (\frac{1}{\beta^\alpha} - 1)(\frac{1}{Z_2}\alpha)
\]
and plugging in the values, we have
\[
LHS = (1-p)(\frac{1}{\beta^\alpha} - 1) - (\frac{1}{\beta^\alpha} - 1)(\hat{c}) = (1-p)(\frac{1}{\beta^\alpha} - 1)(1-\hat{c})
\]
\[
RHS = (\frac{1}{\beta^\alpha} - 1)(1-p)(1-\hat{c}) = LHS.
\]

STEP (3.i). Note that \((\mu_{[Z_1,z_2]} - \mu)(\theta_1) = (\hat{\mu} - \mu)(\theta_1)\), \((\mu_{[Z_1^*,Z_2^*]} - \mu)(\theta_1^*) = (\hat{\mu} - \mu)(\theta_1^*)\). Since \(Z_1 < \frac{1}{\beta}\), from Figure 1a, it is clear that if \(Z_1 > Z_1^*\), we must have \((\hat{\mu} - \mu)(\theta_1) > (\hat{\mu} - \mu)(\theta_1^*)\).

STEP (3.ii). Note that \((\mu_{[Z_1,z_2]} - \mu)(\theta_2) = (\frac{1}{\beta^\alpha} - 1)(\frac{1}{\beta^\alpha} - p)^\alpha\) and \((\mu_{[Z_1^*,Z_2^*]} - \mu)(\theta_2^*) = (\frac{1}{\beta^\alpha} - 1)(\frac{1}{\beta^\alpha} - p)^\alpha\). Since \((\frac{1}{\beta^\alpha} > (\frac{1}{\beta^\alpha})^\alpha\) by assumption, the claim follows clearly.

This concludes the case when \(\hat{c} \geq \frac{1-p}{1-p+1-\beta^\alpha}(1-\beta^\alpha)\). For the second half of the theorem, we will follow similar steps and reasoning. Step (3) is essentially the same as before and is thus omitted.

The first two steps are outlined below.

1. We will first show that \([Z_1^*, Z_2^*]\) is of case II.2. That is to show \(\beta Z_2^* \leq Z_1^*\) and \(Z_1^* \leq \frac{1}{\beta} \leq Z_2^*\).

2. We will check that \([Z_1^*, Z_2^*]\) is an exact range. That is, let \(\theta_1^*\) and \(\theta_2^*\) be the \(G_2\) students whose potentials are \(Z_1^*\) and \(Z_2^*\) respectively, we have \((\mu_{[Z_1,z_2]} - \mu)(\theta_1) = (\mu_{[Z_1,z_2]} - \mu)(\theta_2)\), which implies both are sup\{\(\mu_{[Z_1^*,Z_2^*]} - \mu\)\}.

3. Next, we will show \(\mu_{[Z_1^*,Z_2^*]} - \mu \prec \hat{\mu} - \mu\), which, unlike in the previous case, is not immediately implied by (2).

Again, due to Observation EC.1, it is sufficient to compare \([Z_1^*, Z_2^*]\) only with regions \([Z_1, Z_2]\) of case I.2 and case II.2 (i.e., when \(\beta Z_1 \leq 1 \leq \beta Z_2\)).

4. As before, we will show two cases, which is enough because \([Z_1^*, Z_2^*]\) is exact and one of the two cases is bound to happen. Again, let \(\theta_1\) and \(\theta_2\) be the \(G_2\) students whose potentials are \(Z_1\) and \(Z_2\) respectively. We want to show
   i). if \(Z_1 > Z_1^*\), then \((\mu_{[Z_1,z_2]} - \mu)(\theta_1) > (\mu_{[Z_1^*,Z_2^*]} - \mu)(\theta_1^*)\)

ii). if \(Z_1 < Z_1^*\), then \((\mu_{[Z_1,z_2]} - \mu)(\theta_2) > (\mu_{[Z_1^*,Z_2^*]} - \mu)(\theta_2^*)\)
ii). otherwise, we must have \( Z_2 < Z_2^* \), and then \((\mu_{[z_1, z_2]} - \mu)(\theta_2) > (\mu_{[z_1^*, z_2^*]} - \mu)(\theta_2^*)\)

**STEP (1).** We will start by showing \( \beta Z_2^* \leq Z_1^* \). Note that

\[
\beta Z_2^* \leq Z_1^* \Leftrightarrow \left( \frac{1}{\beta Z_2} \right)^\alpha \geq \left( \frac{1}{Z_1} \right)^\alpha \Leftrightarrow \frac{1}{\beta \alpha} \left[ (1 - p - \hat{c}) \beta^\alpha + (1 - p) \hat{c} \right] \geq (1 - p - \hat{c}) \beta^\alpha \\
\Leftrightarrow \hat{c} [1 - \frac{1 - p}{\beta^\alpha} - \beta^\alpha] \leq (1 - p)(1 - \beta^\alpha)
\]

The last inequality is clearly true because under the assumption \( p \leq 1 - \beta^\alpha \) we have \( \frac{1 - p}{\beta^\alpha} \geq 1 \) and thus \( 1 - \frac{1 - p}{\beta^\alpha} - \beta^\alpha \leq -\beta^\alpha < 0 \). Next, we will show \( Z_1^* \leq \frac{1}{\beta} \Leftrightarrow \left( \frac{1}{Z_1^*} \right)^\alpha \geq \beta^\alpha \). Indeed,

\[
\left( \frac{1}{Z_1^*} \right)^\alpha = \frac{(1 - p - \hat{c}) \beta^\alpha}{1 - p} + \hat{c} = \beta^\alpha + (1 - \frac{\beta^\alpha}{1 - p}) \hat{c} \geq \beta^\alpha.
\]

Lastly, we want to show \( Z_2^* \geq \frac{1}{\beta} \Leftrightarrow \left( \frac{1}{Z_2^*} \right)^\alpha \leq \beta^\alpha \). Plugging in the formula, we have

\[
\left( \frac{1}{Z_2^*} \right)^\alpha = \frac{(1 - p - \hat{c})}{1 - p} \beta^\alpha = \beta^\alpha \leq \beta^\alpha.
\]

**STEP (2).** From the formula of \( Z_1^* \) and \( Z_2^* \), it is clear that \( \left( \frac{1}{Z_1^*} \right)^\alpha - \left( \frac{1}{Z_2^*} \right)^\alpha = \hat{c} \) and thus, \([Z_1^*, Z_2^*] \) is an exact range. For the second part, note that as in the previous case, since \( Z_1^* < \frac{1}{\beta} \), we have \((\mu_{[z_1^*, z_2^*]} - \mu)(\theta_1^*) = (\hat{\mu} - \mu)(\theta_1^*) = (1 - p)(1 - \left( \frac{1}{Z_1^*} \right)^\alpha) \). And for the other side, plugging in the formula in Equation (EC.4), we have \((\mu_{[z_1^*, z_2^*]} - \mu)(\theta_1^*) = p\left( \frac{1}{Z_1^*} \right)^\alpha + \left( \frac{1}{Z_2^*} \right)^\alpha (1 - p) \frac{1}{\beta^\alpha} - 1 \). Hence, it suffices to show that

\[
(1 - p) \left( 1 - \left( \frac{1}{Z_1^*} \right)^\alpha \right) = p\left( \frac{1}{Z_1^*} \right)^\alpha + \left( \frac{1}{Z_2^*} \right)^\alpha \left( 1 - p \right) \frac{1}{\beta^\alpha} - 1 \\
\Downarrow \\
1 - p - \left( \frac{1}{Z_1^*} \right)^\alpha = \left( \frac{1}{Z_2^*} \right)^\alpha (1 - p) \frac{1}{\beta^\alpha} - 1.
\]

Plugging in the values, indeed, we have

\[
LHS = 1 - p - \frac{(1 - p - \hat{c}) \beta^\alpha}{1 - p} - \hat{c} \\
RHS = (1 - p - \hat{c}) - \frac{(1 - p - \hat{c}) \beta^\alpha}{1 - p} = LHS.
\]

**STEP (3).** To show \( \mu_{[z_1^*, z_2^*]} - \mu \leq R \hat{\mu} - \mu \), we want to use the condition given in Case II.2. That is, we want to show \( p\hat{c} \leq (1 - p)(\frac{1}{\beta^\alpha} - 1)(\beta^\alpha - (\frac{1}{Z_2^*})^\alpha) \). Simpilying plugging in the formula for \( Z_2^* \), we have

\[
RHS = (1 - p)\left( \frac{1}{\beta^\alpha} - 1 \right) \beta^\alpha (1 - \frac{1 - p - \hat{c}}{1 - p}) = (1 - p)(1 - \beta^\alpha) \frac{\hat{c}}{1 - p} = (1 - \beta^\alpha) \hat{c} \geq p\hat{c}.
\]
STEP (4.i). This is exactly the same as in the previous case.

STEP (4.ii). In this case, we assume $Z_i < \bar{Z}_i$ for $i = 1, 2$. Note that

$$
\begin{align*}
(\mu_{[Z_1, Z_2]} - \mu)(\theta_2) &= p\left(\frac{1}{Z_1}\right)^\alpha + \left(\frac{1}{Z_2}\right)^\alpha \left(1 - p\right)\frac{1}{\beta^\alpha} - 1 \\
(\mu_{[\bar{Z}_1, \bar{Z}_2]} - \mu)(\theta_2^*) &= p\left(\frac{1}{Z_1}\right)^\alpha + \left(\frac{1}{Z_2}\right)^\alpha \left(1 - p\right)\frac{1}{\beta^\alpha} - 1
\end{align*}
$$

and the claim follows directly under assumption (2) as it implies $(1 - p)\frac{1}{\beta^\alpha} - 1 \geq 0$.

**EC.1.3.1. Optimality of $[Z_1^*, Z_2^*]$** Now let $T^* \in \mathcal{T}(\hat{c})$ be the optimal solution without the restriction that sets in $\mathcal{T}(\hat{c})$ are connected. We will show that $T^*$ differs from $[Z_1^*, Z_2^*]$ in a set of measure zero. First, in order to have $\sup(\mu_{T^*} - \mu) \leq \sup(\mu_{[Z_1^*, Z_2^*]} - \mu) =: s$, in $T^*$, we must debias all students $\theta$ whose mistreatment $(\hat{\mu} - \mu)(\theta)$ is greater than $s$. That is, we must have $T^*_1 := [Z_1^*, Z_{(1)}] \subseteq T^*$, where $Z_{(1)} := Z(\theta^{(1)}) \geq 1/\beta$ and $(\hat{\mu} - \mu)(\theta^{(1)}) = s$. Geometrically, this cuts off the peak of $\hat{\mu} - \mu$ in Figure EC.6. However, now, there is a $G_2$ student $\theta^{(2)}$ such that $Z_{(2)} := Z(\theta^{(2)}) > Z_{(1)}$ and $(\hat{\mu} - \mu)(\theta^{(2)}) = s$ (see Figure EC.6). We have moreover that $(\mu_{T^*_1} - \mu)(\theta) \geq s$ for all $\theta \in G_2$ such that $Z(\theta) \in [Z_{(1)}, Z_{(2)}]$. Thus, we must also have $[Z_{(1)}, Z_{(2)}] \in T^*$. Let $T^*_2 := [Z_1^*, Z_{(2)}]$. We can repeat the argument and observe that there is a $G_2$ student $\theta^{(3)}$ such that $Z_{(3)} := Z(\theta^{(3)}) > Z_{(2)}$ and $(\mu_{T^*_2} - \mu)(\theta) \geq s$ for $\theta \in G_2$ such that $Z(\theta) \in [Z_{(2)}, Z_{(3)}]$ and conclude that $T^*_3 := [Z_1^*, Z_{(3)}]$ must be contained in $T^*$. Continuously applying the same argument, we have $\lim_{n \to \infty} Z(\theta^{(n)}) = Z_2^*$ and thus the claim follows.

**Figure EC.6** Plots of $\mu_T - \mu$ for sets $T$ constructed in Section EC.1.3.1.
EC.2. Proof of Theorem 2
Assume \( T = [Z_1, Z_2] \) is the range of true potentials of \( G_2 \) students we want to debias. For simplicity, as in previous sections, let \( \tilde{\mu} \) denote \( \mu_T \). In order to obtain the minimizer of \( \sigma(\tilde{\mu} - \mu) \), first, we want to compute \( \sigma(\tilde{\mu} - \mu) \) for each of the cases in Section EC.1. To start with, under the status quo, we have

\[
\sigma(\tilde{\mu} - \mu) = (1 - p)(\frac{1}{\beta^\alpha} - 1) \int_1^\infty (\frac{1}{t})^\alpha \frac{\alpha}{t^{\alpha+1}} dt + (1 - p) \int_1^\frac{1}{\tilde{\mu}} (1 - (\frac{1}{t})^\alpha) \frac{\alpha}{t^{\alpha+1}} dt
\]

\[
= (1 - p)(\frac{1}{\beta^\alpha} - 1) \frac{1}{2} (\beta^{2\alpha} - 0) + (1 - p)(1 - \beta^\alpha - \frac{1}{2}(1 - \beta^{2\alpha}))
\]

\[
= \frac{1}{2}(1 - p)(1 - \beta^\alpha)
\]

For \( 1 \leq t_1 \leq t_2 \in \mathbb{R} \cup \{+\infty\} \), let \( \sigma_{t_1}^{t_2}(f) := \int_{t_1}^{t_2} \max(f(t), 0) dF_1(t) \) for any function \( f : [1, \infty] \to [0, 1] \). When \( t_1 = 1 \) and \( t_2 = \infty \), we simply write \( \sigma(f) \), which is consistent with previous notations. Note that with \( \sigma(\tilde{\mu} - \mu) \) as a reference, it actually suffices to compute only \( \sigma_{Z_1}^{Z_2}(\tilde{\mu} - \mu) \), because minimizing \( \sigma(\tilde{\mu} - \mu) \) is equivalent to maximizing \( \sigma_{Z_1}^{Z_2}(\tilde{\mu} - \mu) - \sigma_{Z_1}^{Z_2}(\bar{\mu} - \mu) \) given that \( (\bar{\mu} - \mu)(\theta) = (\bar{\mu} - \mu)(\theta) \) for all \( \theta \in G_2 \) such that \( Z(\theta) \notin [Z_1, Z_2/\beta] \).

In addition to giving an explicit formula for \( \sigma_{Z_1}^{Z_2}(\tilde{\mu} - \mu) - \sigma_{Z_1}^{Z_2}(\bar{\mu} - \mu) \) in each case, we also analyze how this value changes (increase or decrease) with respect to \( Z_1 \) and \( Z_2 \).

EC.2.1. CASE I: \( \beta Z_2 \geq Z_1 \)

**Subcase 1.** \( 1 \leq Z_1 \). To compute the PAUC, we need to express \( \bar{\mu} - \mu \) for all \( \theta \) with \( Z(\theta) \in [1, \infty) \) such that \( (\bar{\mu} - \mu)(\theta) > 0 \). In fact, all such pieces are computed in the previous section. All pieces together, we have

\[
(\bar{\mu} - \mu)(\theta) = \begin{cases} 
(1 - p)(\frac{1}{Z(\theta)})^\alpha \left( \frac{1}{\tilde{\mu}} - 1 \right) & \text{if } 1 \leq Z(\theta) \leq \frac{1}{\beta} \\
(1 - p)(\frac{1}{Z(\theta)})^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) & \text{if } \frac{1}{\beta} \leq Z(\theta) \leq Z_1 \\
(1 - p)(\frac{1}{Z(\theta)})^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) & \text{if } Z_1 \leq Z(\theta) \leq Z_2/\beta \\
(1 - p)(\frac{1}{Z(\theta)})^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) & \text{if } Z(\theta) \geq Z_2/\beta \\
0 & \text{if } Z(\theta) = \infty
\end{cases}
\]

The only necessary piece to integrate here is

\[
\sigma_{Z_2}^{Z_2}(\tilde{\mu} - \mu) = (p + \frac{1}{\beta^\alpha} - 1) \int_{Z_2}^{Z_2} (\frac{1}{t})^\alpha \frac{\alpha}{t^{\alpha+1}} dt - p(\frac{1}{Z_2})^\alpha \int_{Z_2}^{Z_2} \frac{\alpha}{t^{\alpha+1}} dt
\]

\[
= (p + 1 - \frac{1}{\beta^\alpha} - 1) \frac{1 - \beta^{2\alpha}}{2} (\frac{1}{Z_2})^{2\alpha} - p(\frac{1}{Z_2})^{2\alpha} (1 - \beta^\alpha)
\]

Moreover, in this case, we have

\[
\sigma_{Z_1}^{Z_1}(\tilde{\mu} - \mu) = (1 - p)(\frac{1}{\beta^\alpha} - 1) \int_{Z_1}^{Z_1} (\frac{1}{t})^\alpha \frac{\alpha}{t^{\alpha+1}} dt
\]

\[
= (1 - p)(\frac{1}{\beta^\alpha} - 1) \frac{1}{2} \left( (\frac{1}{Z_1})^{2\alpha} - (\frac{1}{Z_2})^{2\alpha} \right)
\]
Thus,

\[ \sigma_{Z_1}^{\alpha}(\hat{\mu} - \mu) - \sigma_{Z_1}^{\alpha}(\tilde{\mu} - \mu) = \frac{(1-p)(\frac{1}{\beta^\alpha} - 1)}{2} - \frac{1}{Z_1^2} + \frac{p - p\beta^\alpha - \frac{1}{\beta^\alpha} + 1}{Z_2^2} \]

Now, to analyze how this quantity changes with \( Z_1 \) and \( Z_2 \), we first simplify some of the terms, which will also be used in later sections. Let \( x = (\frac{1}{Z_1})^\alpha \in [0, 1] \) and let \( (\frac{1}{Z_1})^\alpha = c + x \in [0, 1] \) for some \( c \leq \tilde{c} \). Also, let \( g(x, c) := \sigma_{Z_1}^{\alpha}(\hat{\mu} - \mu) - \sigma_{Z_1}^{\alpha}(\tilde{\mu} - \mu) \).

Next, rearranging the terms, we have

\[ g(x, c) = \frac{(1-p)(\frac{1}{\beta^\alpha} - 1)}{2} (c + x)^2 + \frac{p - p\beta^\alpha - \frac{1}{\beta^\alpha} + 1}{2} x^2 \]

We want to show that \( g(x, c) \) increases as \( x \) increases (or equivalently, as \( Z_2 \) decreases) and as \( c \) increases (meaning that the constraint \( (\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha \leq \tilde{c} \) is effectively \( (\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha = \tilde{c} \)). First,

\[ \frac{\partial g(x, c)}{\partial c} = (1-p)(\frac{1}{\beta^\alpha} - 1)(c + x) \geq 0. \]

Next, rearranging the terms, we have

\[ g(x, c) = \frac{p}{2}(2 - \frac{1}{\beta^\alpha} - \beta^\alpha)x^2 + (1-p)(\frac{1}{\beta^\alpha} - 1)cx + \text{const} \]

Let

\[ h(c) := \frac{(1-p)(\frac{1}{\beta^\alpha} - 1)c}{p(2 - \frac{1}{\beta^\alpha} - \beta^\alpha)} \]

be the peak of the parabola \( g(x, c) \). For each given \( c \), we want to show that \( h(c) \) is always greater than the maximum possible \( x \) in this case. If so, we have that \( g(x, c) \) is an increasing function on \( x \). Since \( \beta Z_2 \geq Z_1 \), we have \( c = (\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha \geq (\frac{1}{Z_2})^\alpha (\frac{1}{\beta^\alpha} - 1) = x(\frac{1}{\beta^\alpha} - 1) \). Thus, it is enough to show that

\[ \frac{(1-p)(\frac{1}{\beta^\alpha} - 1)^2}{p(\frac{1}{\beta^\alpha} + \beta^\alpha - 2)} \geq 1 \quad \iff \quad (1-p)(\frac{1}{\beta^\alpha} - 1)^2 \geq p(1 - \beta^\alpha)(\frac{1}{\beta^\alpha} - 1) \]

\[ \iff \quad (1-p)(\frac{1}{\beta^\alpha} - 1) \geq p(1 - \beta^\alpha) \]

And the last inequality is true because when assuming \( p < .5 \), we have \( 1 - p > p \) and moreover, \( \frac{1}{\beta^\alpha} + \beta^\alpha - 2 \geq 0 \) implies \( \frac{1}{\beta^\alpha} - 1 \geq 1 - \beta^\alpha \).

**Subcase 2.** \( \beta Z_1 \leq 1 \leq \beta Z_2 \). The formula for \( \tilde{\mu} - \mu \) is similar to the previous case

\[
(\tilde{\mu} - \mu)(\theta) = \begin{cases} 
(\tilde{\mu} - \mu)(\theta) = (1-p)(1-(\frac{1}{Z(\theta)})^\alpha) > 0 & \text{if } 1 \leq Z(\theta) \leq Z_1 \\
0 & \text{if } Z_1 \leq Z(\theta) \leq \beta Z_2 \\
< 0 & \text{if } \beta Z_2 \leq Z(\theta) \leq Z_2 \\
(\frac{1}{Z(\theta)})^\alpha (p + \frac{1}{\beta^\alpha} - 1) - p(\frac{1}{Z_2})^\alpha > 0 & \text{if } Z_2 \leq Z(\theta) \leq Z_2/\beta \\
(\tilde{\mu} - \mu)(\theta) = (1-p)(\frac{1}{Z(\theta)})^\alpha (\frac{1}{\beta^\alpha} - 1) > 0 & \text{if } Z(\theta) \geq \beta Z_2 
\end{cases}
\]
However, here, for the status quo baseline, we have
\[
\sigma_{Z_1}^{2/\beta} (\hat{\mu} - \mu) = (1 - p)(\frac{1}{\beta^2} - 1) \int_{1/\beta}^{2/\beta} \frac{1}{t^{\alpha+1}} \alpha dt + (1 - p) \int_{Z_1}^{1/\beta} (1 - \frac{1}{t^{\alpha+1}}) \frac{1}{t^{\alpha+1}} dt
\]
\[
= \frac{1}{2} (1 - p)(\beta^2 - \beta^{2\alpha}) + (1 - p) \left[ \frac{1}{2} \left( \frac{1}{Z_1} \right)^{\alpha} - \beta^\alpha - \frac{1}{2} \left( \frac{1}{Z_1} \right)^{2\alpha} \right]
\]
and thus,
\[
\sigma_{Z_1}^{2/\beta} (\hat{\mu} - \mu) - \sigma_{Z_2}^{2/\beta} (\hat{\mu} - \mu) = \frac{1}{2} (1 - p)(\beta^\alpha + (1 - p)(\frac{1}{Z_1} \right)^{\alpha} - \frac{1}{2} \left( \frac{1}{Z_1} \right)^{2\alpha})
\]
\[
+ p - p\beta^\alpha - \frac{1}{\beta^2} + 1 \left( \frac{1}{Z_2} \right)^{2\alpha}.
\]
Now, for the analysis, similarly, write
\[
g(x, c) = \text{const} + (1 - p)((c + x) - \frac{1}{2}(c + x)^2) + \frac{p - p\beta^\alpha - \frac{1}{\beta^2} + 1}{2} x^2.
\]
Then,
\[
\frac{\partial g(x, c)}{\partial c} = (1 - p)(1 - (c + x)) \geq 0
\]
and rearranging the terms, we have
\[
g(x, c) = \frac{2p - p\beta^\alpha - \frac{1}{\beta^2}}{2} x^2 + (1 - p)(1 - c)x + \text{const}.
\]
We will show that \(2p - p\beta^\alpha - \frac{1}{\beta^2} \leq 0\) by showing \(\beta^\alpha p(2 - \beta^\alpha) \leq 1\). This is true under assumption (2), which is \(p \leq 1 - \beta^\alpha\), because
\[
\beta^\alpha p(2 - \beta^\alpha) \leq \beta^\alpha(1 - \beta^\alpha)(2 - \beta^\alpha) \leq 1,
\]
where the last inequality is due to the fact that \(\beta^\alpha \leq 1\). Hence, let
\[
h_1(c) = \frac{(1 - p)(1 - c)}{p\beta^\alpha + \frac{1}{\beta^2} - 2p}
\]
then \(g(x, c)\) is an increasing function w.r.t. \(x\) on \([0, h_1(c)]\) and is a decreasing function on \([h_1(c), 1]\).

EC.2.2. CASE II: \(\beta Z_2 \leq Z_1\)

Subcase 1. \(1 \leq \beta Z_1\). Again, we want to express \(\hat{\mu} - \mu\) for all \(\theta\) such that \(Z(\theta) \in [1, \infty)\) such that \((\hat{\mu} - \mu)(\theta) > 0\). All pieces together, we have
\[
(\hat{\mu} - \mu)(\theta) = \begin{cases} 
(\hat{\mu} - \mu)(\theta) = (1 - p)(1 - \left( \frac{1}{Z(\theta)} \right)^\alpha) & \text{if } 1 \leq Z(\theta) \leq \frac{1}{\beta} \\
(\hat{\mu} - \mu)(\theta) = (1 - p)(\left( \frac{1}{Z(\theta)} \right)^\alpha(\frac{1}{\beta^2} - 1)) & \text{if } \frac{1}{\beta} \leq Z(\theta) \leq Z_1 \\
< 0 & \text{if } Z_1 \leq Z(\theta) \leq Z_2 \\
(\left( \frac{1}{Z(\theta)} \right)^\alpha(\frac{1}{\beta^2} - 1) + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha)) & \text{if } Z_2 \leq Z(\theta) \leq Z_1 / \beta \\
(\left( \frac{1}{Z(\theta)} \right)^\alpha(\frac{1}{\beta^2} - 1) - p((\frac{1}{Z_2})^\alpha) & \text{if } Z_1 / \beta \leq Z(\theta) \leq Z_2 / \beta \\
(\hat{\mu} - \mu)(\theta) = (1 - p)(\left( \frac{1}{Z(\theta)} \right)^\alpha(\frac{1}{\beta^2} - 1) > 0 & \text{if } Z(\theta) \leq \beta Z_2 \\
\end{cases}
\]
As subcase 1. of Section EC.2.1, for $\hat{\mu} - \mu$, we have

$$\sigma_{Z_1}^{Z_2/\beta} (\hat{\mu} - \mu) = (1 - p) \left( \frac{1}{\beta^\alpha} - 1 \right) \frac{1}{2} \left( (\frac{1}{Z_1})^{2\alpha} - (\frac{\beta}{Z_2})^{2\alpha} \right)$$

and for $\bar{\mu} - \mu$, we need to compute the integral of the following two pieces.

$$\sigma_{Z_2}^{Z_1/\beta} (\bar{\mu} - \mu) = (1 - p) \left( \frac{1}{\beta^\alpha} - 1 \right) \int_{Z_2}^{Z_1/\beta} \left( \frac{1}{t} \right)^\alpha \frac{1}{\Gamma(t)} \alpha dt + p \left( \frac{1}{Z_1} \right)^\alpha - (\frac{1}{Z_2})^\alpha \right) \int_{Z_2}^{Z_1/\beta} \frac{1}{\Gamma(t)} \alpha dt$$

$$= \frac{(1 - p) \left( \frac{1}{\beta^\alpha} - 1 \right)}{2} \left( (\frac{1}{Z_2})^{2\alpha} - \left( \frac{\beta}{Z_2} \right)^{2\alpha} + p \left( \frac{1}{Z_1} \right)^{2\alpha} - \left( \frac{1}{Z_2} \right)^{2\alpha} \right);$$

$$\sigma_{Z_1}^{Z_2/\beta} (\hat{\mu} - \mu) = (p + \frac{1}{\beta^\alpha} - 1) \int_{Z_1/\beta}^{Z_2/\beta} \left( \frac{1}{t} \right)^\alpha \frac{1}{\Gamma(t)} \alpha dt - p \left( \frac{1}{Z_2} \right)^\alpha \left( \frac{1}{Z_1} \right)^\alpha - (\frac{1}{Z_2})^\alpha \right) \int_{Z_1/\beta}^{Z_2/\beta} \frac{1}{\Gamma(t)} \alpha dt$$

$$= \frac{(p + \frac{1}{\beta^\alpha} - 1)}{2} \beta \left( (\frac{1}{Z_1})^{2\alpha} - (\frac{1}{Z_2})^{2\alpha} \right) - \beta \left( \frac{1}{Z_1} \right)^\alpha \left( \frac{1}{Z_2} \right)^\alpha.$$

Hence,

$$\sigma_{Z_1}^{Z_2/\beta} (\hat{\mu} - \mu) - \sigma_{Z_1}^{Z_2/\beta} (\bar{\mu} - \mu) = \left( \frac{1}{Z_1} \right)^{2\alpha} \left( \frac{(1 - p)(\frac{1}{\beta^\alpha} - 1) + p \beta^\alpha}{2} \right)$$

$$- \left( \frac{1}{Z_2} \right)^{2\alpha} \left( \frac{(1 - p)(\frac{1}{\beta^\alpha} - 1) + p \beta^\alpha}{2} \right)$$

$$+ p \left( \frac{1}{Z_1} \right)^\alpha - \left( \frac{1}{Z_2} \right)^\alpha.$$

Now, for the analysis, let $A = \frac{(1-p)(\frac{1}{\beta^\alpha} - 1) + p \beta^\alpha}{2} \geq 0$. Then,

$$g(x, c) = A(c + x)^2 - Ax^2 + px^2 - p(c + x)(x).$$

First, we have

$$\frac{\partial g(x, c)}{\partial c} = 2A(c + x) - px = 2Ac + (2A - p)x.$$

Under the assumption that $1 - p > \beta^\alpha$, we have $2A - p \geq \beta^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) + p \beta^\alpha + \beta^\alpha - 1 = p \beta^\alpha \geq 0$. Therefore, $\frac{\partial g(x, c)}{\partial c} \geq 0$. Next, rearranging the terms, we have

$$g(x, c) = c(2A - p)x + \text{const},$$

and thus, $g(x, c)$ is an increasing function on $x$.

**Subcase 2. $\beta Z_1 \leq 1 \leq \beta Z_2$.** Similar to the previous case, the formula for $\hat{\mu} - \mu$ is as follows:

$$\begin{align*}
(\hat{\mu} - \mu)(\theta) &= \begin{cases} 
(1 - p)(1 - \left( \frac{1}{Z(\theta)} \right)^\alpha) & \text{if } 1 \leq Z(\theta) \leq Z_1 \\
< 0 & \text{if } Z_1 \leq Z(\theta) \leq Z_2 \\
\left( \frac{1}{Z(\theta)} \right)^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) + p \left( \frac{1}{Z_1} \right)^\alpha - \left( \frac{1}{Z_2} \right)^\alpha & \text{if } Z_2 \leq Z(\theta) \leq Z_1/\beta \\
\left( \frac{1}{Z(\theta)} \right)^\alpha \left( p + \frac{1}{\beta^\alpha} - 1 \right) - p \left( \frac{1}{Z_2} \right)^\alpha & \text{if } Z_1/\beta \leq Z(\theta) \leq Z_2/\beta \\
(\hat{\mu} - \mu)(\theta) &= (1 - p)(\frac{1}{Z(\theta)} \right)^\alpha \left( \frac{1}{\beta^\alpha} - 1 \right) > 0 & \text{if } Z(\theta) \geq \beta Z_2 
\end{cases}
\end{align*}$$
Same as subcase 2 of Section EC.2.1, we have

\[
\sigma_{Z_1}^{2 \beta / \beta} (\bar{\mu} - \mu) = \frac{(1-p)(\beta^\alpha - \beta^\beta)}{2}(1 - \frac{1}{Z_2})^2 + (1-p)[(\frac{1}{Z_1})^\alpha - \beta^\alpha - \frac{1}{2}(\frac{1}{Z_1})^2 - \beta^\beta)].
\]

Note the components of interest for \( \bar{\mu} - \mu \) have the same expression as in the previous section. Thus, using the expression for the integrals from before, we have

\[
\sigma_{Z_1}^{2 \beta / \beta} (\bar{\mu} - \mu) - \sigma_{Z_1}^{2 \beta / \beta} (\bar{\mu} - \mu) = \frac{1}{2}(1-p)\beta^\alpha + \left(\frac{1}{Z_1}\right)^{2 \alpha} \left(-\frac{(1-p)+ \beta^\beta}{2}\right) + (1-p)(\frac{1}{Z_1})^\alpha
\]

\[
- \left(\frac{1}{Z_2}\right)^{2 \alpha} \left(\frac{(1-p)(\frac{1}{Z_1})^2 + \beta^\beta}{2}\right) + p(\frac{1}{Z_2})^{2 \alpha} - p(\frac{1}{Z_1})^\alpha \left(\frac{1}{Z_2}\right)^\alpha.
\]

For the analysis, again let \( A = \frac{(1-p)(\frac{1}{Z_1})^2 + \beta^\beta}{2} \geq 0 \) and \( B = \frac{(1-p) + \beta^\beta}{2}. \) Since \( p < .5, \) we have \( B \leq 0. \) Then, we can write

\[
g(x, c) = \text{const} + B(c + x)^2 + (1-p)(c + x) - Ax^2 + px^2 - p(c + x)(x).
\]

First, we have

\[
\frac{\partial g(x, c)}{\partial c} = 2B(c + x) + (1-p) - px = -(1-p + \beta^\alpha)(c + x) + [(1-p) - px].
\]

Since \( x = \left(\frac{1}{Z_2}\right)^\alpha \leq \beta^\alpha, \) we have \([(1-p) - px] \geq (1-p) - p\beta^\alpha \) and thus, together with the fact that \( c + x \leq 1, \) we have

\[
\frac{\partial g(x, c)}{\partial c} \geq ((1-p) - p\beta^\alpha)(1-(c + x)) \geq 0.
\]

Next, after rearranging the terms, we have

\[
g(x, c) = (B-A)x^2 + \left[2Bc + (1-p) - px\right] x + \text{const}
\]

\[
= \frac{-(1-p)\beta^\alpha}{2} x^2 + [(p\beta^\alpha - 1)c + (1-p)] x + \text{const}.
\]

We will show that in this case, we must have \([(p\beta^\alpha - 1)c + (1-p)] \geq 0. \) First, note that since \( \beta Z_2 \leq Z_1, \) and \( Z_2 \geq \frac{1}{\beta}, \) we have

\[
c = \left(\frac{1}{Z_1}\right)^\alpha - \left(\frac{1}{Z_2}\right)^\alpha \leq \left(\frac{1}{\beta^\alpha} - 1\right)(\frac{1}{Z_2})^\alpha \leq \left(\frac{1}{\beta^\alpha} - 1\right)\beta^\alpha = 1 - \beta^\alpha.
\]

Thus, to show \([(p\beta^\alpha - 1)c + (1-p)] \geq 0, \) it is enough to show that \([(p\beta^\alpha - 1)(1-\beta^\alpha) + (1-p)] \geq 0, \) which is equivalent to \( p(\beta^\alpha(1-\beta^\alpha) - 1) + \beta^\alpha \geq 0. \) Under the assumption that \( p < 1 - \beta^\alpha \) and with the fact that \( \beta^\alpha(1 - \beta^\alpha) - 1 \leq 0, \) we have

\[
p(\beta^\alpha(1 - \beta^\alpha) - 1) + \beta^\alpha \geq (1 - \beta^\alpha)(\beta^\alpha(1 - \beta^\alpha) - 1) + \beta^\alpha = (\beta^\alpha)^3 - 2(\beta^\alpha)^2 + 3\beta^\alpha - 1 \geq 0,
\]
where the last inequality is true because $f(t) := t^3 - 2t^2 + 3t - 1$ is nonnegative when $t \geq 0$. Now, let

$$h_{ll}(c) = \frac{(p \beta^\alpha - 1)c + (1 - p)}{(1 - p) \frac{1}{\beta^\alpha}},$$

then $g(x, c)$ is an increasing function on $[0, h_{ll}(c)]$ and is a decreasing function on $[h_{ll}(c), 1]$.

**Subcase 3.** $\beta Z_2 \leq 1$. Again, we want to express $\tilde{\mu} - \mu$ for all $\theta$ such that $Z(\theta) \in [1, \infty)$ such that $(\tilde{\mu} - \mu)(\theta) \geq 0$. Combining all pieces, we have

$$(\tilde{\mu} - \mu)(\theta) = \begin{cases} 
(\tilde{\mu} - \mu)(\theta) = (1 - p)(1 - (\frac{1}{Z(\theta)})^\alpha) & \text{if } 1 \leq Z(\theta) \leq Z_1 \\
\leq 0 & \text{if } Z_1 \leq Z(\theta) \leq Z_2 \\
(1 - p)(1 - (\frac{1}{Z(\theta)})^\alpha) + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha) & \text{if } Z_2 \leq Z(\theta) \leq 1/\beta \\
(\frac{1}{Z(\theta)})^\alpha(1 - p)(\frac{1}{\alpha} - 1) + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha) & \text{if } 1/\beta \leq Z(\theta) \leq Z_1/\beta \\
(\frac{1}{Z(\theta)})^\alpha(p + \frac{1}{\alpha} - 1) - p(\frac{1}{Z_2})^\alpha & \text{if } Z_1/\beta \leq Z(\theta) \leq Z_2/\beta \\
(\tilde{\mu} - \mu)(\theta) = (1 - p)((\frac{1}{Z(\theta)})^\alpha(\frac{1}{\alpha} - 1) > 0 & \text{if } Z(\theta) \geq \beta Z_2
\end{cases}$$

For $\tilde{\mu} - \mu$, as in subcase 2, we have

$$\sigma_{Z_1}^{Z_2/\beta}(\tilde{\mu} - \mu) = \frac{(1 - p)(\beta^\alpha - \beta^\alpha)}{2}((\frac{1}{Z_1})^\alpha(\frac{1}{Z_2})^\alpha) + (1 - p)[((\frac{1}{Z_1})^\alpha - \beta^\alpha - \frac{1}{2}((\frac{1}{Z_1})^\alpha(\frac{1}{Z_2})^\alpha - \beta^\alpha)].$$

For $\tilde{\mu} - \mu$, the component of $\theta \in [Z_1/\beta, Z_2/\beta]$ is the same as in subcase 1 of this section, which is

$$\sigma_{Z_1/\beta}^{Z_2/\beta}(\tilde{\mu} - \mu) = \frac{(p + \frac{1}{\beta^\alpha} - 1)(\beta^\alpha)}{2}((\frac{1}{Z_1})^\alpha(\frac{1}{Z_2})^\alpha) - p\beta^\alpha((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha)$$

and we only have to integrate the other two components:

$$\sigma_{Z_2/\beta}^{Z_2/\beta}(\tilde{\mu} - \mu) = (1 - p)\int_{Z_2}^{1/\beta} (1 - (\frac{1}{t})^\alpha) \frac{\alpha}{t^{\alpha+1}} dt + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha) \int_{Z_2}^{1/\beta} \frac{\alpha}{t^{\alpha+1}} dt
\hspace{1cm} = (1 - p)[((\frac{1}{Z_1})^\alpha - \beta^\alpha - \frac{1}{2}((\frac{1}{Z_2})^\alpha - \beta^\alpha)] + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha)((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha);$$

$$\sigma_{Z_1/\beta}^{Z_2/\beta}(\tilde{\mu} - \mu) = (1 - p)(\frac{1}{\beta^\alpha} - 1)\int_{1/\beta}^{Z_1/\beta} (\frac{1}{t})^\alpha \frac{\alpha}{t^{\alpha+1}} dt + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha) \int_{1/\beta}^{Z_1/\beta} \frac{\alpha}{t^{\alpha+1}} dt
\hspace{1cm} = (1 - p)(\frac{1}{\beta^\alpha} - 1)(\beta^\alpha - (\frac{1}{Z_1})^\alpha)\beta^\alpha - (\frac{1}{Z_2})^\alpha - \frac{1}{2}((\frac{1}{Z_2})^\alpha - \beta^\alpha)] + p((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha)((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha);$$

Putting them together, we have

$$\sigma_{Z_1/\beta}^{Z_2/\beta}(\tilde{\mu} - \mu) - \sigma_{Z_1/\beta}^{Z_2/\beta}(\tilde{\mu} - \mu) = \left(\frac{1}{Z_1}\right)^{2\alpha} \left(\frac{-(1 - p) + p\beta^\alpha}{2}\right) + (1 - p)\left(\frac{1}{Z_1}\right)^\alpha
\hspace{1cm} - \left(\frac{1}{Z_2}\right)^{2\alpha} \left(\frac{-(1 - p) + p\beta^\alpha}{2}\right) - (1 - p)\left(\frac{1}{Z_1}\right)^\alpha
\hspace{1cm} + p\left(\frac{1}{Z_2}\right)^{2\alpha} - p\left(\frac{1}{Z_1}\right)^\alpha(\frac{1}{Z_2})^\alpha.$$
Lastly, for the analysis, write
\[ g(x, c) = B(c + x)^2 + (1 - p)(c + x) - Bx^2 - (1 - p)x + px^2 - p(c + x)x \]

Rearranging the terms, we have
\[ g(x, c) = [2Bc + (1 - p) - (1 - p)pc]x = (-1 + p\beta^\alpha)cx \]

Thus, \( g(x, c) \) is a decreasing function in \( x \). In addition, taking the partial derivative w.r.t. \( c \), we have
\[ \frac{\partial g(x, c)}{\partial c} = 2B(c + x) + (1 - p) - px = (\frac{1}{Z_2} - (\frac{1}{Z_1})^\alpha) - (\frac{1}{Z_2})^\alpha - px + px^2 - p\beta^\alpha \]

It is actually not clear whether \( \frac{\partial g(x, c)}{\partial c} \) is positive or negative in this subcase. But for the purpose of finding the minimizer of \( \sigma(\tilde{\mu} - \mu) \), this is not important because for a fixed value of \( c \), \( g(x, c) \) achieves its maximum when \( x \) is of the value such that \( [Z_1, Z_2] \) is of subcase 2, of either case I or case II.

**EC.2.3. Computing the optimal range for vouchers**

In Figure EC.7, we demonstrate how the type of range \( [Z_1, Z_2] \) evolves as it moves to the right along the axis given that \( (\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha = \hat{c} \) is fixed.

![Figure EC.7](image)

**Figure EC.7** Change in type of \([Z_1, Z_2] \) as \( Z_1 \) increases given \((\frac{1}{Z_1})^\alpha - (\frac{1}{Z_2})^\alpha = \hat{c} \) and fixed parameters.

Although the evolution is different depending the value of \( \hat{c} \), for a fixed value of \( \hat{c} \), as \( Z_1 \) gets larger (or equivalently as \( Z_2 \) gets larger, or as \( x := (\frac{1}{Z_2})^\alpha \) gets smaller), the range \( [Z_1, Z_2] \) goes from type II to type I. In particular, for each value of \( \hat{c} \), such transition happens exactly when \( \beta Z_2 = Z_1 \).

That is, when
\[ \hat{c} = (\frac{1}{\beta^\alpha} - 1)(\frac{1}{Z_2})^\alpha \iff (\frac{1}{Z_2})^\alpha = \frac{\hat{c}\beta^\alpha}{1 - \beta^\alpha} \]

Now, for each fixed value of \( \hat{c} \), Figure EC.8 plots \( \sigma(\tilde{\mu} - \mu) - \sigma(\tilde{\mu} - \mu) \) against \( Z_1 \). It also shows that as \( Z_1 \) increases, how the interval \([Z_1, Z_2]\) changes by cases.

With simple algebra, one can easily check that when \( \hat{c} = (\frac{1}{Z_2})^\alpha \), we have that \( h_1(\hat{c}) = h_{\text{II}(\hat{c})} = \frac{\hat{c}\beta^\alpha}{1 - \beta^\alpha} \). Therefore,
Figure EC.8  For fixed values of $\hat{c}$, the function of $\sigma(\hat{\mu} - \mu) - \sigma(\bar{\mu} - \mu)$ on $Z_1$

- When $\hat{c} \geq \frac{(1-p)(1-\beta^a)}{2-p-\beta^a-p\beta^a+p^2\alpha}$, we have $h_I(\hat{c}) \leq \frac{\hat{c}\beta^a}{1-\beta^a}$ and $h_{II}(\hat{c}) \leq \frac{\hat{c}\beta^a}{1-\beta^a}$. Thus, the maximum of $\sigma(\hat{\mu} - \mu) - \sigma(\bar{\mu} - \mu)$ is achieved when $x = h_I(\hat{c})$.
- When $\hat{c} \leq \frac{(1-p)(1-\beta^a)}{2-p-\beta^a-p\beta^a+p^2\alpha}$, we have $h_I(\hat{c}) \geq \frac{\hat{c}\beta^a}{1-\beta^a}$ and $h_{II}(\hat{c}) \geq \frac{\hat{c}\beta^a}{1-\beta^a}$. Thus, the maximum of $\sigma(\hat{\mu} - \mu) - \sigma(\bar{\mu} - \mu)$ is achieved when $x = h_{II}(\hat{c})$.

EC.3. Comparison of the optimal ranges for voucher correction under two measures
| $\hat{c}$ | $T_{mm}(\hat{c}) = [Z_1, Z_2]$ | $T_{auc}(\hat{c}) = [Z'_1, Z'_2]$ | $Z_1 - Z'_1$ |
|---|---|---|---|
| 0.10 | [1.2252, 1.3111] | [1.2187, 1.3026] | 0.0065 |
| 0.20 | [1.2022, 1.3861] | [1.1903, 1.3653] | 0.0119 |
| 0.30 | [1.1802, 1.4803] | [1.1644, 1.4421] | 0.0158 |
| 0.40 | [1.1461, 1.5584] | [1.1346, 1.5203] | 0.0115 |
| 0.50 | [1.1156, 1.6560] | [1.1070, 1.6155] | 0.0086 |
| 0.60 | [1.0881, 1.7839] | [1.0819, 1.7403] | 0.0063 |
| 0.70 | [1.0632, 1.9635] | [1.0589, 1.9154] | 0.0043 |
| 0.80 | [1.0404, 2.2476] | [1.0377, 2.1926] | 0.0026 |

Table EC.2  The optimal ranges of $G_2$ students to debias under two measures of mistreatments, under parameters $\alpha = 3$, $\beta = .8$, and $p = .25$.  
