Tracking Dark Energy from Axion-Gauge Field Couplings

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We propose a toy model of Dark Energy in which the degrees of freedom currently dominating the energy density of the universe are described by a pseudo-scalar “axion field” linearly coupled to the Pontryagin density, $\text{tr}(F \wedge F)$, i.e., the exterior derivative of the Chern-Simons form, of a gauge field. We assume that the axion has self-interactions corresponding to an exponential potential. We argue that a non-vanishing magnetic helicity of the gauge field leads to slow-rolling of the axion at field values far below the Planck scale. Our proposal suggests a “Tracking Dark Energy Scenario” in which the contribution of the axion energy density to the total energy density is constant (and small), during the early radiation phase, until a secular growth term proportional to the Pontryagin density of the gauge field becomes dominant. The initially small contribution of the axion field to the total energy density is related to the observed small baryon-to-entropy ratio.

I. INTRODUCTION

As is well known, the Strong CP Problem related to the vacuum structure of QCD, as described by the vacuum angle $\theta$, can be solved by promoting $\theta$ to a pseudo-scalar field, the axion $\text{1}$. This field gives rise to a new species of light particles. It can be interpreted as the phase, related to a $U(1)$-symmetry, of a complex scalar field $\text{2}$. A non-vanishing vacuum expectation value of the scalar field leads to the spontaneous breaking of this symmetry. The field quanta of the axion are the Goldstone bosons accompanying the breaking of the $U(1)$-symmetry. They may acquire a mass through instanton effects and can be made “invisible” by choosing the symmetry breaking scale to be sufficiently high $\text{3}$. The axion may then be a candidate for dark matter; (see e.g. $\text{4}$). Some time ago, it has been suggested $\text{5}$ that, besides the QCD axion, there could exist an effective axion field conjugate to the anomalous axial vector current in QED. The time derivative of this axion field would then play the role of a space-time dependent chemical potential for the axial charge density in QED and, through the chiral anomaly, would give rise to an instability triggering the growth of low-frequency magnetic fields with non-trivial helicity; see also $\text{6}$. Possible applications of this observation to early universe cosmology, and in particular to the issue of the generation of primordial magnetic fields, have been discussed in $\text{7}$; (see also $\text{8}$, $\text{9}$).

In this paper, we explore the possibility that an axion field, $\phi$, linearly coupled to the Pontryagin- or “instanton” density, $\text{10} \text{11}$, of a non-abelian gauge field, $A_\mu$, could contribute to the dark energy of the Universe. We assume that the axion field has non-trivial self-interactions described by a potential term, $V(\phi)$, in the action functional. As has previously been observed in the context of inflationary models in $\text{12}$ $\text{13}$ (see also $\text{14}$), the coupling of the axion to the instanton density of $A_\mu$ can lead to slow-rolling of $\phi$ also for values of $\phi$ much smaller than the Planck mass. We show that slow-rolling of $\phi$ leads to a “tracking solution” with the property that the energy density of $\phi$ tracks that of the radiation-dominated background of the early Universe up to a time $t_c$ when a secular growth term in the magnetic helicity of the gauge field starts to dominate. From this time on, the contribution of $\phi$ to the energy density of the Universe starts to grow until it might actually dominate it at some late time. Choosing parameter values motivated by the observed small baryon-to-entropy ratio, we arrive at a scenario in which the currently observed dark energy in the Universe may come from the axion field $\phi$. Thus, our mechanism might represent a realization of the tracking dark energy scenario previously discussed in $\text{15}$; (see also $\text{16}$).

In the following section we describe some key features of our scenario. One such feature is a secular growth in cosmological time of the electric component of the gauge field tensor. This is discussed in more detail in Section III, where we derive the gauge field equations of motion in the presence of a term coupling the Pontryagin density, i.e., the exterior derivative of the Chern-Simons 3-form, to the axion field. We then attempt to find solutions of these equations that yield a homogeneous and isotropic Pontryagin density. It turns out that such solutions only exist for non-abelian gauge fields. In Section IV we consider an exponential potential for the self-interactions of $\phi$ and try to find out under what conditions tracking dark energy arises. In Section V we discuss tentative particle physics connections of our scenario. Some conclusions are presented in Section VI. An interesting variant of our scenario involving a complex scalar field whose phase plays the role of the new axion field introduced in the present paper will be discussed in forthcoming work.
A word on our notation: Our space-time metric has signature \((-, +, +, +)\). We work in units in which the speed of light, Planck’s constant and Boltzmann’s constant are all set to 1. The cosmological scale factor is denoted by \(a(t)\), where \(t\) is physical time. The Hubble expansion rate is \(H(t) = \frac{\dot{a}}{a}(t)\), and \(t_{eq}\) denotes the time of equal matter and radiation.

II. KEY FEATURES OF OUR SCENARIO

In this section we introduce our dark energy model, postponing a discussion of its origins in particle physics to Section V.

A key element of our model is a pseudo-scalar axion field, \(\phi\), that couples linearly to the Pontryagin density, \(\text{tr}(F \wedge F)\), of a (massive) non-abelian gauge field, \(A_\mu\).

The dynamics of the axion field \(\phi\), the gauge field \(A_\mu\) and the space-time metric \(g_{\mu\nu}\) is determined by the following action functional:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \mathcal{L}_m \right],
\]

where the matter Lagrangian is given by

\[
\mathcal{L}_m = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}_{\alpha} - \frac{\lambda}{f} \phi F_{\mu\nu} \tilde{F}^{\mu\nu}_{\alpha} + \text{mass terms}.
\]

The second but last term, henceforth called “magnetic helicity term,” can be understood as arising from coupling the gradient of \(\phi\) to an anomalous axial vector current and then invoking the chiral anomaly [13]. We will discuss possible particle physics origins of the field \(\phi\), of an anomalous axial vector current, and of a heavy gauge field (with field strength denoted by \(F\)) in Section V. In Eq. (2), repeated indices are to be summed over, the index \(\alpha\) is a gauge group index, \(\mu\) and \(\nu\) are space-time indices, \(\lambda\) is a dimensionless coupling constant, and \(f\) is a reference field value that also appears in the axion potential \(V(\phi)\). In this paper we consider an exponential potential:

\[
V(\phi) = \mu^4 e^{\phi/f},
\]

where \(\mu\) sets the energy scale of the potential. This choice of \(V(\phi)\) leads to an explicit breaking of parity and time-reversal invariance. (To avoid this, one might consider replacing \(\exp(\phi/f)\) by \(\cosh(\phi/f)^{-1}\) in Eq. (3).) A more natural choice of self-interactions not breaking these symmetries explicitly will be considered in forthcoming work.

A basic feature of our model is related to the expectation that \(\phi\) is very slowly rolling at sub-Planckian field values, due to its coupling to the gauge field. Assuming spatial homogeneity, the field equation of motion for \(\phi\) is given by

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = \frac{\lambda}{8f} \vec{E}_a \cdot \vec{B}_a,
\]

where the prime denotes a derivative of \(V\) with respect to \(\phi\). Following a hypothesis introduced in the context of inflationary models in [9] and in [10], we assume that the term proportional to the Pontryagin is responsible for slow-rolling of \(\phi\), in the sense that the terms in (4) proportional to first and second time derivatives of \(\phi\) are negligible as compared to the two remaining terms. If this assumption can be justified the equation of motion for \(\phi\) reduces to

\[
V'(\phi) \simeq \frac{\lambda}{8f} \vec{E}_a \cdot \vec{B}_a,
\]

an equation that determines the time-dependence of \(\phi\), once one knows the time-dependence of \(\vec{E}_a \cdot \vec{B}_a\). We note that slow rolling can arise at sub-Planckian field values, [9], in contrast to the usual slow-roll in large-field inflationary scenarios, which requires super-Planckian values.

In the context of inflation, a scenario based on the two basic features of our model described so far is sometimes called chromo-natural inflation [10].

A third key feature of our scenario concerns the secular growth of a spatially homogeneous configuration of the electric field \(E_a\), in excess of its usual dynamics. This growth is induced by the coupling of the gauge field to the axion field \(\phi\), as in (2). Under the assumption that the coupling constant \(\lambda\) is sufficiently large, we will show that secular growth of \(E_a\), when combined with Eq. (5), yields an axion field configuration that gives rise to tracking dark energy.

The main point is that a non-vanishing magnetic helicity, which originates from the coupling of the gauge field to the axion as expressed by the “magnetic helicity term,” acts as an extra “friction term” that ensures that \(\phi\) will slowly roll down its potential – even at sub-Planckian field values. Thus, the resulting equation of state for the energy density of the axion is dominated by the potential energy term, which yields a contribution to dark energy. Equations (3) and (5) then tell us that if the Pontryagin density \(\vec{E}_a \cdot \vec{B}_a\) exhibits secular growth, the contribution of \(\phi\) to the total energy density of the Universe can become important at late times.

Finally, another key feature of our scenario is that the initial value of the energy density of \(\phi\) is proportional to a small number in cosmology, such as the baryon to entropy ratio \(n_b/s\), \((n_b\) and \(s\) being the baryon and photon number densities, respectively). As far as relating a late-time cosmological observable to the small baryon to entropy ratio (via a term in the Lagrangian coupling the axion to an anomalous current) is concerned there are similarities of our work to the one in [15], where the tensor-to-scalar ratio, (i.e., the ratio of the strength of gravitational waves to that of scalar cosmological fluctuations), is related to \(n_b/s\).
III. GAUGE FIELD DYNAMICS IN THE PRESENCE OF THE “ANOMALY TERM”

The equation of motion for the field strength tensor of the gauge field in the presence of a Chern-Simons term (but neglecting mass terms) is given by

$$D^{ab}_\alpha E^{\alpha \beta}_{a} - \frac{4\lambda}{f} \epsilon^{\mu \nu \alpha} \partial^{\alpha} (\phi F_{\mu \nu}) = 0 \ . \ (6)$$

In this equation $D$ denotes the covariant derivative, which is defined by

$$D^{ab}_\alpha \equiv \delta^{ab} \nabla_\alpha - g^{abc} A^c_\alpha \ , \ (7)$$

where $\nabla_\alpha$ is the space-time covariant derivative, $g$ is the gauge coupling constant, and the $f^{abc}$ are the structure constants of the gauge group.

We write the equation of motion for the gauge field in terms of the “electric” and “magnetic” components of the field tensor,

$$E^a_\mu = F^a_{\mu \nu} u^\nu \ , \ (8)$$

$$B^a_\mu = -\frac{1}{2} \epsilon^{\mu \nu \rho} F_{a \nu \rho} u^\nu \ , \ (9)$$

where $u^\mu = (1, 0, 0, 0)$ is the four-velocity of a comoving observer in an FRW spacetime. In manifestly covariant form, the equations of motion are

$$u^\alpha D^{ab}_\alpha E^{b\alpha} + 2HE^{a} - u_\mu \epsilon^{\mu \alpha \beta} D^{ab}_\alpha B^b_{\beta} = -8\lambda \left( u_\mu \epsilon^{\mu \alpha \beta} \partial_\alpha \phi E^a_\beta + \epsilon^{a \beta \gamma} \partial_\alpha \phi B^{\beta \gamma} \right) \ ,$$

$$D^{ab}_\alpha E^{b\alpha} = \frac{8\lambda}{f} \partial_\alpha \phi B^{a\alpha} \ . \ (10)$$

In standard three-vector form, the equations are

$$D^{ab}_0 E^b + 2HE^a - \frac{1}{a} D^{ab} \times B^b = 0 \ (10)$$

$$= -8\lambda \left( \frac{1}{a} \nabla_0 \phi + E_0 + \phi B^0 \right) \ ,$$

$$D^{ab} \cdot E^b = \frac{8\lambda}{f} \nabla_0 \phi \cdot B \ (11)$$

where $D^{ab}$ is the spatial part of $D^{ab}_\alpha$.

Using the general definition of the electric and magnetic components of the field tensor,

$$E_{ai} = \partial_0 A_i^a - D^a_i (A) A_0^b \ (12)$$

and

$$B_{ai} = \epsilon_{ijk} \left( \partial_0 A_k^a - \frac{g}{2} \epsilon_{abc} A_j^b A_k^c \right) \ (13)$$

the field equations for the gauge field coupled to the axion field take the form

$$\frac{\partial}{\partial t} E^i_a - \frac{g}{2} \epsilon_{abc} A_{0k}^a E^c_k + 2HE^i_a - \epsilon^{ijk} \nabla_j B_{ak} \ (14)$$

and

$$\nabla_i E^i_a = -\frac{\lambda}{f} \nabla_i \phi E_a^i \ , \ (15)$$

with

$$\epsilon^{ijk} \nabla_j E_{ak} = -\frac{\partial}{\partial t} B^i_a \ (16)$$

In the following we will consider a spatially homogeneous gauge field configuration. Of course, in general non-vanishing electric and magnetic background fields break rotational invariance; but spatial homogeneity can be preserved in gauge-invariant combinations of $\vec{E}$ and $\vec{B}$. As a result of a non-vanishing instanton condensate, $CP$ is broken, which may have interesting consequences that we will return to elsewhere. The instanton condensate $\vec{E}_a \cdot \vec{B}_a$ can be non-zero, homogeneous and isotropic, i.e., translation- and rotation invariant.

For a non-Abelian gauge group, such as $SU(2)$, we make the ansatz of a spatially homogeneous background gauge field. Following [16] (see also [17]) we take

$$A_0 = 0 \ , \ A_i(t) = a(\psi(t)) \delta^a_i J_a \ (17)$$

where $J_a$ are the generators of the non-Abelian $SU(2)$ gauge group and $\delta^i_a$ is a Kronecker delta symbol combining an upper internal (Lie-algebra) index, $a$, with a lower spatial index, $i$. The field strength tensor elements are then

$$F^a_{0i} = a(\psi) \delta^a_i J_a \ ,$$

$$F^a_{ij} = -g(a) \epsilon^a_{ij} \ , \ (18)$$

which in particular implies that the electric field can be written as

$$E_i^a \sim E(t) \delta^a_i \ . \ (19)$$

Before the gauge field acquires a mass, the equation of motion for $\psi$ is (see [10])

$$\ddot{\psi} + 3H \dot{\psi} + (\dot{H} + 2H^2) \psi + 2\psi^3 = \tilde{g} \lambda \phi \delta^{a3/2} \psi \ . \ (20)$$

where the term on the right hand side of the equation comes from the magnetic helicity (instanton) term in the action which leads to a coupling of the axion to the background gauge field. The coefficient of the term linear in $\psi$ vanishes in the radiation epoch.

The equation (20) displayed above holds at all times for an unbroken gauge theory. However, we are more interested in a gauge theory that is spontaneously broken at a fairly large mass scale $m$. The gauge field of the theory acquires its mass after the symmetry breaking phase transition, a transition occurring when the temperature, $T(t)$, of the Universe is of the order of $m$. After the phase transition, at temperatures below the transition temperature, the energy density of the gauge field scales like
that of matter, i.e., $\rho_{\text{gauge}} \sim a(t)^{-3}$, for times greater than $t_m$, where $t_m$ is determined by

$$T(t_m) \approx m,$$  \hspace{1cm} (21)

where $T$ denotes the temperature of radiation. This corresponds to the scaling

$$E(t), B(t) \sim a(t)^{-3/2}.$$  \hspace{1cm} (22)

Once the gauge field acquires a mass, there will be an extra mass term in the equation of motion for $\psi$, yielding

$$\ddot{\psi} + 3H \dot{\psi} + (\dot{H} + 2H^2) \psi + m^2 \psi + 2\dot{\phi}^2 \psi^3 = \frac{g}{f} \phi a^{3/2} \psi^2.$$  \hspace{1cm} (23)

Since we are assuming that the mass will be much larger than the value of $H$ at the time of equal matter and radiation, the third term on the left hand side of (23) is negligible even in the matter era. The nonlinear term proportional to $\psi^3$ becomes increasingly unimportant compared to the $\sim m^2 \psi$ term as time goes on since the amplitude of $\psi$ decreases. The approximate form of (23) then becomes

$$\ddot{\psi} + 3H \dot{\psi} + m^2 \psi = \frac{g}{f} \phi a^{3/2} \psi^2.$$  \hspace{1cm} (24)

In terms of electric and magnetic fields, this equation corresponds to (suppressing the gauge group index $a$)

$$\frac{\partial}{\partial t} E_i + \frac{3}{2} H E_i = - \frac{\lambda}{f} [\dot{\phi} B_i]$$

$$\dot{B}_i + \frac{3}{2} H B_i = 0$$  \hspace{1cm} (25)

The key point is that the coupling of the gauge field to the axion field $\phi$ has the consequence that the electric field decays less rapidly than it would in the absence of $\phi$. This effect entails that the magnetic helicity and the energy density, $\frac{1}{2} (\dot{E}^2 + \dot{B}^2)$, of the gauge field, which, initially, scale like that of radiation, then like the one of matter, turn out to grow relative to the energy density of matter, at late times.

In the absence of the magnetic helicity term proportional to $\dot{\phi}$, or if the $\phi$-field is time-independent, the equations (24) or (25) imply the behavior $E_i \sim a^{-3/2}$ and $B_i \sim a^{-3/2}$. Hence they imply that the energy density of the gauge field (i.e., of the $\dot{E}$- and $\dot{B}$- fields) scales like that of matter (For times $t < t_m$, the energy density of the $\dot{E}$- and $\dot{B}$- fields decays like that of radiation. This corresponds to a scaling of these fields proportional to $a^{-2}$. Let us denote the $\psi$ field in the absence of the “magnetic helicity term” by $\psi_0(t)$.

We may then determine the effects caused by the “magnetic helicity term” using the Green function method. We write

$$\psi(t) = \psi_0(t) + \psi_1(t),$$  \hspace{1cm} (26)

where $\psi_0(t)$ is the solution without the magnetic helicity term which satisfies the given initial conditions at an initial time $t_i$, and $\psi_1(t)$ is the first order Born approximation calculated from

$$\ddot{\psi}_1 + 3H \dot{\psi}_1 + m^2 \psi_1 = \frac{g}{f} \phi a^{3/2} \psi_0^2 \equiv S(t),$$  \hspace{1cm} (27)

where $S(t)$ stands for the “source” term in the Born approximation. The solution can be written as

$$\psi_1(t) = \int_{t_i}^{t} dt' G(t, t') S(t'),$$  \hspace{1cm} (28)

where $G(t, t')$ is the Green function which is given by

$$G(t, t') = w(t')^{-1} (u_1(t)u_2(t') - u_2(t)u_1(t')),$$  \hspace{1cm} (29)

where $u_1$ and $u_2$ are the two basis solutions of the homogeneous equation of motion, and

$$w(t) = \dot{u}_1(t)u_2(t) - \dot{u}_2(t)u_1(t)$$  \hspace{1cm} (30)

is the Wronskian.

In the radiation epoch, i.e. for $t < t_{eq}$ we have

$$u_1(t) = \left( \frac{t_i}{t} \right)^{3/4} \cos(mt)$$

$$u_2(t) = \left( \frac{t_i}{t} \right)^{3/4} \sin(mt).$$  \hspace{1cm} (31)

Hence the leading term in the Wronskian gives

$$w(t') \approx m \left( \frac{t_i}{t} \right)^{3/2}.$$  \hspace{1cm} (32)

As we will see later

$$\frac{1}{f} |\dot{\phi}| = \hat{n} \frac{1}{t},$$  \hspace{1cm} (33)

where $\hat{n}$ is a number of order one. Computing (28), using (29), (31), and (33), yields a term $\psi_0(t)$, multiplied by a certain integral denoted $G(t)$,

$$\psi_1(t) = \psi_0(t) G(t).$$  \hspace{1cm} (34)

Here $G(t)$ is a “secular growth factor” given by

$$G(t) \approx \hat{g} \hat{n} \lambda m^{-1} \psi_0(t_i) \log \left( \frac{t_i}{t} \right).$$  \hspace{1cm} (35)

The secular growth of $\psi$ leads to a corresponding secular growth of the magnetic helicity $\text{tr}(\dot{E} \cdot \dot{B})$. The analysis for $t > t_{eq}$ is analogous, except that the mode functions $u_1$ and $u_2$ now scale as $t^{-1}$.

Based on the above analysis we are able to determine the scaling of the magnetic helicity term. Taking into account the fact that $B(t)$ scales as $a(t)^{-2}$, for $t < t_m$, and as $a(t)^{-3/2}$, for $t > t_m$, we find that

$$\dot{E}_a \cdot \dot{B}_a \sim a(t)^{-4}$$  \hspace{1cm} (36)

for $t < t_m$, as

$$\dot{E}_a \cdot \dot{B}_a \sim a(t)^{-3}$$  \hspace{1cm} (37)

for $t_m < t < t_c$, and as

$$\dot{E}_a \cdot \dot{B}_a \sim a(t)^{-3} G(t),$$  \hspace{1cm} (38)

for $t > t_c$. 
IV. LATE TIME ACCELERATION FOR AN EXPONENTIAL POTENTIAL

Next, we analyze how the dynamics of the axion $\phi$ is affected by gauge field configurations with non-vanishing magnetic helicity. We recall that, for a spatially homogeneous configuration of axions, the field equation of $\phi$ is given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{\lambda}{8f} \vec{E}_a \cdot \vec{B}_a,$$  \hfill (39)

where the prime indicates a derivative of $V$ with respect to $\phi$. As in the context of inflationary models in \cite{23}, we assume that the term on the right side of (39) generates slow-rolling of $\phi$, namely that the terms $\phi$ and $3H\dot{\phi}$ in (39) are negligible as compared to the two remaining terms. We will check the self-consistency of this assumption below. The evolution of $\phi$ is then determined by

$$V'(\phi) = \frac{\lambda}{8f} \vec{E}_a \cdot \vec{B}_a$$  \hfill (40)

For an exponential potential,

$$V(\phi) = \mu^4 e^{\phi/f},$$  \hfill (41)

Eq. (40) yields

$$V(\phi) = \frac{\lambda}{8} \vec{E}_a \cdot \vec{B}_a$$  \hfill (42)

The proportionality of $V(\phi)$ to $\text{tr} (\vec{E} \cdot \vec{B})$ is a special feature of the exponential potential. For polynomial and periodic potentials, $V(\phi)$ ends up being proportional to a power of $\text{tr} (\vec{E} \cdot \vec{B})$ greater than 1, and hence would decay faster in time, in an expanding universe. This makes it more difficult to interpret $\phi$ as a dark-energy candidate. We will study such types of potentials in the context of a different (possibly more natural) model in a follow-up paper.

Combining (41) and (42), we find the following expression for the axion field $\phi$:

$$\frac{\dot{\phi}}{f} = \log\left(\frac{\lambda}{8} \mu^4 \vec{E}_a \cdot \vec{B}_a\right),$$  \hfill (43)

which leads to

$$\frac{1}{f} |\dot{\phi}| = \frac{n}{t},$$  \hfill (44)

where the number $n$ is $n = 2$, for $t < t_m$ and for $t > t_{eq}$, and equals $n = 3/2$, for $t_m < t < t_{eq}$. This expression is important for the evaluation of the magnitude of the secular growth term (45). In fact, inserting (44) into (45) and taking into account that $B$ and $E_0$ scale the same way as a function of time we find that

$$E_a(t) = E_{a0}(t) \left[1 + \frac{\bar{n} \lambda B_n(t_i) t}{E_a(t_i) \log\left(t/t_i\right)}\right].$$  \hfill (45)

Thus, the secular growth term grows logarithmically in time.

We now search for conditions implying that $\phi$ is a viable candidate for dark energy. First of all, $\phi$ has to be slowly rolling as a function of time in order for the equation of state of $\phi$ to be that of dark energy. Second, we have to show that the energy density of $\phi$ has the potential to dominate over the background energy density shortly before the present time. To complete our analysis, we need to make sure that the energy density of the gauge field which the axion field $\phi$ couples to remains subdominant.

Under the assumption that the slow-rolling conditions are satisfied, Eq. (42) immediately leads to an expression for the contribution, $\Omega_\phi$, of the $\phi$-field to the total energy density of the universe. From the above discussion it follows that, for $t < t_m$, the energy density of $\phi$ scales like that of radiation and hence leads to a constant contribution to $\Omega_\phi$. For $t_m < t < t_{eq}$, the potential energy of $\phi$ decreases less fast than the background radiation density, leading to a contribution to $\Omega_\phi$ that grows linearly in $a(t)$. We should emphasize, however, that the contribution of $\phi$ to $\Omega$ scales in the same way as the contribution of dark matter to $\Omega$. Once $t > t_{eq}$, but before $t = t_c$, both the background density and the energy density of $\phi$ scale as $a(t)^{-3}$, and hence the contribution of $\phi$ to $\Omega$ is constant. Finally, once $t > t_c$, $\Omega_\phi$ increases in time. Specifically, for late times $t > t_{eq}$, we obtain that

$$\Omega_\phi(t) \simeq \frac{V(\phi(t))}{\rho_0(t)} \bigg[1 + \frac{n \lambda B(t_i) t}{E(t_i) \log\left(t/t_i\right)}\bigg],$$  \hfill (46)

where $\rho_0(t)$ is the background energy density at time $t$, and $\rho_r(t_i)$ is the energy density of radiation at the initial time $t_i$, (which is approximately equal to the total energy density at that time, since we have assumed that $t_i$ is chosen to belong to the radiation period). Equivalently, we can express $\Omega_\phi$ in terms of the background matter density, $\rho_m$, at the initial time

$$\Omega_\phi(t) \simeq \frac{V(\phi(t))}{\rho_0(t)} \bigg[1 + \frac{n \lambda B(t_i) t}{E(t_i) \log\left(t/t_i\right)}\bigg],$$  \hfill (47)

Figure 1 presents a sketch of the time evolution of $\Omega_\phi$.

Since the energy density of the new gauge field is larger than $\text{tr}(\vec{E} \cdot \vec{B})$, (by the Schwarz inequality), a necessary condition for $\phi$ to be a dark energy candidate is that

$$V(\phi) \gg \text{tr}(\vec{E} \cdot \vec{B}),$$  \hfill (48)

which, by (42), can only hold provided

$$\lambda \gg 1.$$  \hfill (49)
For $\phi$ to be a good *tracking quintessence* candidate, we need the secular growth term in (35) to become dominant at a time $t_c$, with

$$t_{eq} < t_c < t_0, \quad (50)$$

where $t_0$ is the present time. This leads to the requirement that $B(t_i)/E(t_i)$ needs to be slightly smaller than $\lambda^{-1}$, which is a tuning condition we need to impose.

Above we have assumed that the slow-roll conditions

$$\ddot{\phi} \ll V'(\phi) \quad \text{and} \quad 3H\dot{\phi} \ll V'(\phi), \quad (51)$$

are satisfied, and that the equation of state of $\phi$ leads
to acceleration. It is easy to check that the slow-roll conditions are satisfied, provided
\[ f \ll m_{pl} . \]  
(52)

It is not hard to check that the equation of state for \( \phi \) is dominated by the potential energy if condition (52) is satisfied. Thus, the field \( \phi \) is indeed a candidate for tracking dark energy.

Finally, we study the magnitude of the contribution of \( \phi \) to the dark energy budget. Evaluating (46) at the present time \( t_0 \) and assuming \( t_0 > t_c \) we obtain (dropping Lie-algebra indices on \( \vec{E} \) and \( \vec{B} \))
\[ \Omega_{\phi}(t_0) \simeq \frac{\lambda (\vec{E} \cdot \vec{B})(t_i)}{8 \rho_c(t_i) \left( \frac{a(t_m)}{a(t_m)} \right)} \lambda \frac{B(t_i)}{E(t_i)} \log \left( \frac{t_0}{t_i} \right) . \]  
(53)

If we do not want to introduce a new mass hierarchy into our model, it is natural to assume that \( t_m \sim t_i \). In this case, an initial value of \( \text{tr} (\vec{E} \cdot \vec{B}) \) comparable to the initial matter density is required in order for the order of magnitude of (53) today to be close to unity. This is ensured if, initially, at time \( t_i \), the ratio between the instanton density and the energy density of radiation is proportional to the ratio between baryon- and entropy density, i.e.,
\[ \frac{\text{tr} (\vec{E} \cdot \vec{B})(t_i)}{\rho_c(t_i)} \sim \frac{n_B}{s}(t_i), \]  
(54)

where \( n_B \) is the baryon number density and \( s \) the entropy density. Hence, the smallness of the initial contribution of \( \phi \) to dark energy is guaranteed by the observed small baryon to entropy ratio. This factor is believed to be of the order \( 10^{-10} \).

V. PARTICLE PHYSICS CONNECTIONS

A) An axion coupling to an anomalous matter current:

The standard axion field, \( a(x, t) \), along with the Peccei-Quinn symmetry has been introduced to solve the strong CP problem of QCD; see [2]. The mechanism leading to the spontaneous breaking of the Peccei-Quinn symmetry involves a complex scalar field with a standard symmetry breaking potential whose angular variable is the axion field \( a \) [11]. The coefficient of the Pontryagin density, \( \text{tr}(F \wedge F) \), in the QCD Lagrangian then becomes a dynamical variable. At the perturbative level, the axion has a flat potential. Non-perturbative instanton effects create however a non-trivial potential, \( V(a) \), for the axion. This potential is periodic in \( a \), which is an “angular variable.” The periodicity of the potential is not unproblematic, since it could give rise to an axion domain-wall problem.

The axion of QCD is a candidate for dark matter [4], but cannot be a candidate for dark energy, since it interacts too strongly with electromagnetism. Any viable candidate scalar field for dark energy needs to couple very weakly to standard model matter [18].

The idea underlying our proposal is that the field \( \phi \) responsible for dark energy could be a new axion field conjugate to an anomalous matter current [14]. A possible example would be an anomalous lepton current, in which case the gauge field would be a weak \( SU(2) \) field (see, e.g., [19]). The gradient of \( \phi \) can then be linearly coupled to the anomalous axial vector current \( J_5^\mu \), introducing a term proportional to
\[ \partial_\mu \phi \cdot J_5^\mu \]  
(55)
in the Lagrangian of the theory. Apparently, the time derivative of \( \phi \) then plays the role of a space-time dependent axial chemical potential for an axial charge density \( \tilde{\rho} \), (e.g., the 0-component of the left-handed lepton current). This may furnish an ingredient in a mechanism responsible for the observed matter-antimatter asymmetry. Thanks to the anomaly equation [14], the term (55) is equivalent to a term proportional to
\[ \phi(F \wedge F + \frac{1}{2} \sum_j m_j \tilde{\psi}_j \gamma_5 \psi_j) , \]  
(56)
with \( j \) labeling fermion species, where species \( j \) has mass \( m_j \), and is described by a spinor field \( \psi_j \).

Instanton effects are usually expected to lead to a potential for \( \phi \) that is periodic in \( \phi \), and this possibility is studied in forthcoming work. One may imagine, however, that axion shift-symmetry breaking effects might generate an exponential potential. The value of the parameter \( f \) is related to the symmetry breaking scale, and the energy-scale parameter \( \mu \) is set by the strength of the instanton effects.

B) Universal axion of string theory:

Axions arise naturally in superstring theory [20]. Specifically, string compactifications generate Peccei-Quinn type symmetries often broken at the string scale [21]. For example [22], there is an axion field \( a \) that is in the same chiral superfield \( S \) as the four dimensional dilaton \( \varphi \)
\[ S = e^{-\varphi} + ia . \]  
(57)
In addition, there is an axion field \( \tilde{a} \) in the superfield \( \tilde{S} \) of the volume scalar \( \rho \):
\[ \tilde{S} = e^\rho + i\tilde{a} . \]  
(58)

The Peccei-Quinn symmetries of string theory are always broken by stringy instanton effects, leading to a coupling of the axion to some \( \text{tr}(F \wedge F) \)-term. This can be shown explicitly by reducing the ten-dimensional supergravity action to four space-time dimensions via compactification on some internal Calabi-Yau manifold; (see e.g. [22]). Such a compactification also generates potentials for the superfields to which the axions belong.
These potentials are typically exponential in the radial direction, but a remnant of the exponential potential may also affect the potential in the axion direction; especially if stringy effects lead to a breaking of the shift symmetry in the axion direction, as happens in axion-monodromy models \[23, 24\]. For some explicit constructions of exponential potentials see \[25\].

C) Axion monodromy:
Indeed, it has recently been realized that stringy effects break the shift symmetry of the axion. The axion ceases to be an angular variable and, instead, has an infinite range of values. Monodromy induces an axion potential rising without bound, as \(\phi\) increases to \(\infty\); see, e.g., \[24\]. At large field values, the axion potential may be linear. To make contact with our scenario we need to assume that the potential is exponential at small field values.

We are not the first to connect an axion with a potential induced by stringy monodromy effects with dark energy. In \[26\] it was in fact suggested that a stringy axion may play the role of a quintessence field. The construction in \[26\] makes use of standard slow-roll inflation and thus requires super-Planckian field values. It must still be shown that such field values are consistent from the point of view of string theory, since for other axion models they are not \[27\]. In our construction, the axion field values are sub-Planckian, because slow-rolling is induced by the coupling of the axion to the Chern-Simons term of a gauge field.

VI. CONCLUSIONS AND DISCUSSION

We have studied a model of tracking dark energy in which dark energy arises from an axion field \(\phi\) linearly coupled to the Pontryagin density of a gauge field, i.e., to a term \(\text{tr}(F \wedge F)\). Thanks to this coupling, the axion is rolling slowly even for sub-Planckian field values. It thus has the right equation of state to account for dark energy. We have considered the example of an exponential potential for the axion. The coupling between the axion and the gauge fields leads to a secular growth term in the electric field. At early times, the energy density in \(\phi\) tracks that of the background matter; but when the secular growth term becomes important the contribution of \(\phi\) to the density parameter \(\Omega\) starts to increase. We have studied the evolution of \(\Omega_\phi\) (the fraction of the total energy density required for a spatially flat universe due to the axion field \(\phi\)) as a function of time and found that it is constant for early times \(t < t_m\), where \(t_m\) is the time when the mass of the gauge field becomes important. It grows linearly in the scale factor between time \(t_m\) and the time \(t_{eq}\) of equal matter and radiation. After time \(t_{eq}\), the value of \(\Omega_\phi\) ceases to grow until the time when the secular growth term becomes dominant, after which it will start to grow again.

In order for \(\phi\) to be a successful candidate for dark energy, the time when \(\Omega_\phi\) approaches \(\Omega = 1\) has to be close to the present time \(t_0\). This is only the case if, at the initial time, the \(\phi\)-field energy is a small contribution to the total energy density. Our proposal is that this small initial value of \(\Omega_\phi\) is linked to the small value of the lepton to entropy ratio. This would imply that the secular growth term becomes important only at rather late times. Thus, our model represents an implementation of the “tracking quintessence” scenario of \[12\]. We have shown that, in order to obtain the currently observed value of dark energy in our model, it suffices to require a fairly mild tuning of dimensionless coupling constants.

In this paper we have neglected the coupling of the axion field \(\phi\) in \[56\] to pseudo-scalar mass terms, \(m_j \bar{\psi}_j \gamma_5 \psi_j\), of fermionic matter fields. Taking such couplings into account would lead to extra terms on the right side of the axion equation of motion \[39\]. For \(H > \max_j m_j\), these terms will decay as radiation, and, for \(H < \min_j m_j\), they decay as matter. If \(m \equiv \max_j m_j < m\), there is a time interval \(t_m < t < t_{eq}\) when the contribution due to the mass terms on the right side of \[55\] decays rapidly, relative to the one of the \(F \wedge F\) term. Hence, as long as \(m > \bar{m}\), the extra terms in \[55\] will not change our conclusions.

It has been pointed out that if the field responsible for dark energy is a pseudo-scalar field then it could couple to visible matter, and this leads to rather stringent constraints. A discussion of the coupling of an axion to visible matter has been given in \[18\], where it has been assumed that the axion couples to the \(\bar{E} \cdot \vec{B}\)-term of electromagnetism. This would lead to a rotation of the direction of polarization of light emitted by distant radio sources. The constraints resulting from this effect are quite restrictive and would potentially rule out our model if our axion were to couple to the electromagnetic field. However, we have assumed that our axion does not interact with the photon and thus evades the bounds presented in \[18\] and in related work. In a future paper, we will investigate collider signals due to a possible coupling of the axion field \(\phi\) to W- and Z- bosons.

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