LETTER TO THE EDITOR

On the nonlinearity of subsidiary systems

Helmut Friedrich

Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Golm, Germany

Received 28 April 2005, in final form 13 June 2005
Published 4 July 2005
Online at stacks.iop.org/CQG/22/L77

Abstract
In hyperbolic or other reductions of the Einstein equations the evolution of gauge conditions or constraint quantities is controlled by subsidiary systems. We point out a class of nonlinearities in these systems which may have the potential of generating catastrophic growth of gauge respectively constraint violations in numerical calculations.

PACS numbers: 04.20.Ex, 04.25.Dm

1. Introduction
Most numerical calculations of solutions to Einstein’s field equations are being plagued by an undesirably fast growth of constraint violations. In fact, many workers in the field report on seemingly unmotivated catastrophic blow-ups of numerical calculations at a stage of the numerical time evolution where coordinate or curvature singularities were not to be expected. Consequently, quite a number of different types of reduced equations have been derived from Einstein’s equations with the hope of finding versions with stable propagation properties and there have been suggested modifications of the equations which were hoped to force back the solution to the constraint manifold (cf [1] for an example and [9] for further discussion and references). More recently, there have been performed stability analyses of the subsidiary systems which control the evolution of the gauge conditions or the constraint quantities (cf [4, 11] and the references given therein). These led to requirements on the coefficients of the equations which in the case of [4] resulted in geometric conditions on the foliation underlying the evolution by the main system. Nevertheless, the field still appears to be wide open, most of the remedies suggested are experimental, and the cause of the problems is not understood. In [4, 11] the subsidiary systems have been considered as linear systems on given spacetimes. In contrast, we wish to draw in this letter attention to the nonlinearities of the subsidiary systems, which appear to have a potential of generating catastrophic constraint violations. Understanding to what extent these nonlinearities may affect the numerical construction of spacetimes will require further studies.
2. The hyperbolic reduction

We shall need a few facts about the hyperbolic reduction procedure by which the geometric initial value problem for Einstein’s field equations is reduced to a Cauchy problem for a hyperbolic system. There exist now many versions of this process; their general underlying structure is, however, more or less identical. Because it leads to concise expressions, we will use the metric coefficients as basic unknowns and consider representations of the field equations in which the evolution is governed by systems of wave equations. To emphasize the independence of our discussion of any particular coordinate system we shall employ the notion of a gauge source function introduced in [5].

Let \((M, g)\) denote a smooth four-dimensional Lorentz space with smooth space-like Cauchy hypersurface \(S\) and \(U \ni x^\lambda \mapsto F^\mu(x^\lambda) \in V\) a smooth map of an open subset \(U\) into another subset \(V\) of \(\mathbb{R}^4\). With functions \(x^\mu\) and their (four-dimensional) differentials \(dx^\nu\) suitably prescribed on some open subset \(W\) of \(S\) one can solve near \(W\) the Cauchy problem for the semi-linear system of wave equations

\[
\nabla_\nu \nabla_\nu x^\mu = -F^\mu(x^\lambda).
\]

If the \(dx^\mu\) have been chosen linearly independent on \(W\) the solution will provide a smooth coordinate system \(x^\lambda\) on some neighbourhood of \(W\) in \(M\). In terms of these coordinates the equations above take the form

\[
-\Gamma^\mu(x^\lambda) = -F^\mu(x^\lambda),
\]

where the \(\Gamma^\mu\) denote the contracted Christoffel symbols \(\Gamma^\mu = g^{\nu\rho} \Gamma^\mu_{\nu\rho}\) of \(g_{\mu\nu}\).

This shows (ignoring subtleties arising in situations of low differentiability) that the contracted Christoffel symbols can locally be made to agree with any prescribed set of functions \(F^\mu\) and that these functions, to which we refer as gauge source functions, determine, together with the initial data for the coordinates, the coordinates uniquely. Any arbitrary coordinate system can be characterized this way, harmonic coordinates, defined by the conditions \(F^\mu = \Gamma^\mu = 0\), represent just one particular choice.

Assume now that \((M, g)\) is to be obtained by solving a Cauchy problem for Einstein’s vacuum field equations. We shall derive the properties which will help us formulate this problem as a Cauchy problem for hyperbolic equations. The contracted Christoffel symbols are of particular interest to us because the Ricci tensor of \(g\) can be written in the form

\[
R_{\mu\nu} = -\frac{1}{2} g^{\rho\lambda} g_{\mu\nu,\lambda\rho} + \nabla_\mu \Gamma_\nu + \Gamma_{\lambda\mu\nu} g^{\delta\nu} \Gamma_{\rho\delta} + 2 \Gamma_{\lambda\mu\nu} g^{\delta\nu} g_{\delta\rho} \Gamma_{\lambda\rho} + 2 \Gamma_{\lambda\mu\nu} g^{\delta\nu} g_{\delta\rho} \Gamma_{\lambda\rho}.
\]

(2.1)

Here the contracted Christoffel symbols (and the functions \(F^\mu\) considered in the following) are being formally treated as if they defined a vector field (which, of course, they do not). Thus \(\Gamma_\nu = g_{\mu\nu} \Gamma^\mu\) and \(\nabla_\mu \Gamma_\nu = \partial_\mu \Gamma_\nu = \Gamma_{\mu\nu}\), etc.

The discussion above suggests replacing in (2.1) the functions \(\Gamma_\nu\) by freely chosen gauge source functions \(F_\nu\) so that the resulting expression will depend in general on the coordinates \(x^\lambda\) not any longer only through the \(g_{\mu\nu}\). With this replacement the vacuum field equations take the form

\[
0 = R_{\mu\nu}^F = -\frac{1}{2} g^{\rho\lambda} g_{\mu\nu,\lambda\rho} + \nabla_\mu (F_\nu) + \Gamma_{\lambda\mu\nu} g^{\delta\nu} \Gamma_{\rho\delta} + 2 \Gamma_{\lambda\mu\nu} g^{\delta\nu} g_{\delta\rho} \Gamma_{\lambda\rho}.
\]

(2.2)

of a system of quasi-linear wave equations for the \(g_{\mu\nu}\). We refer to (2.2) as the main evolution system or the reduced equations. For this system the Cauchy problem for \(g_{\mu\nu}\) with data satisfying the constraints on a space-like hypersurface \(S\) is well posed.

Suppose that \(g_{\mu\nu}\) is a solution of this problem near \(S\). Since equation (2.2) is in fact of the form

\[
R_{\mu\nu} = \nabla_\mu (F_\nu),
\]

(2.3)
where $Q_\mu = \Gamma_\mu - F_\mu$ with the $\Gamma_\mu$ calculated from $g_{\mu\nu}$, it is not clear \textit{a priori} whether the solution $g$ will indeed satisfy the gauge condition $\Gamma_\mu = F_\mu$ and thus the vacuum field equation $R_{\mu\nu} = 0$. Equation (2.3) implies $2\nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = \nabla_\mu \nabla^\mu Q_\nu + R^\mu_\nu Q_\mu$. It thus follows from the twice contracted Bianchi identity, which holds for any metric, that

$$\nabla_\mu \nabla^\mu Q_\nu + R^\mu_\nu Q_\mu = 0. \tag{2.4}$$

We refer to this system of wave equations for the quantities $Q_\nu$ as the \textit{subsidiary system} of our reduction procedure.

The detailed analysis shows that if the Cauchy problem for the main evolution system is arranged so that the initial data satisfy the constraints and the gauge condition $Q_\nu = 0$ on $S$, it follows for the corresponding solution of the main evolution system that also the coordinate time derivative $\partial_t Q_\nu$ transverse to $S$ vanishes on $S$. The uniqueness property of the subsidiary system therefore implies that the solution to the main evolution system does indeed satisfy $Q_\nu = 0$ on the domain of dependence of $S$ with respect to $g$. This reduces the local Cauchy problem for Einstein’s field equations to the problem of solving equation (2.2), which thus takes the central role in the analytic discussion. Of the subsidiary system only the homogeneity and the uniqueness property have been used here.

The above argument was given for the particular case of harmonic coordinates by Y Chouquet–Bruhat, who also showed the well posedness of the Cauchy problem for equations (2.2) in that case [3]. The more general case of arbitrary gauge source functions is not much different. Equations of the type (2.2) can be cast into the form of quasi-linear symmetric hyperbolic systems [2] and there exists a well-developed existence theory for such systems (cf [8]). The various steps of setting up the initial problem for (2.2) and showing the well posedness of this problem, i.e. proving the existence and uniqueness of solutions and their stable dependence on the initial data, as well as the well posedness of the geometric initial value problem for Einstein’s field equations have been discussed in detail in [6].

### 3. The nonlinearity of the subsidiary equation

The main evolution system is also central in numerical discussions. The evolution properties of the subsidiary system now become important too, however, because the quantities $Q_\mu$ and $\partial_t Q_\nu$ come with a numerical error on the initial hypersurface $S$ or develop errors in the numerical evolution of the metric by the reduced equations.

There is no way to relate a numerical solution to the solution of the continuum problem one wants to approximate. To get some idea of how errors are seeping through the various systems, we consider an analogy accessible to analytic methods. We assume the main evolution system to be satisfied by fields $g_{\mu\nu}$ of class $C^3$ with an error term $E_{\mu\nu}$ of class $C^1$ so that

$$R_{\mu\nu} = \nabla_\mu (\nabla_\nu Q_v) + E_{\mu\nu}. \tag{3.1}$$

Nothing will be assumed here about the origin and structure of this error and the errors in the initial data $Q_{\mu}$ and $\partial_t Q_\nu$ on $S$.

Using the Bianchi identity with (3.1) gives the analogue

$$\nabla_\mu \nabla^\mu Q_v + R^\mu_\nu Q_\mu = -2 \nabla^\mu \left( E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E_\rho^\rho \right), \tag{3.2}$$

of the subsidiary equation (2.4). This equation will in general not be homogeneous any longer. There may be a way to avoid this problem. Assume a splitting of the form

$$E_{\mu\nu} = \nabla_\mu (e_\nu) + f_{\mu\nu},$$

with some $C^2$ vector field $e_\nu$ and some symmetric $C^1$ tensor field $f_{\mu\nu}$. With (3.1) this gives

$$R_{\mu\nu} = \nabla_\mu (Q'_\nu) + f_{\mu\nu}$$

with $Q'_\nu = Q_\nu + e_\nu$. \tag{3.3}
If \( f_{\mu\nu} \) vanishes this allows us to take advantage of our formalism by absorbing the error into the gauge source function
\[
F_{\mu} \rightarrow F'_{\mu} = F_{\mu} - e_{\mu},
\]
which may be quite harmless. Moreover, we would get a homogeneous wave equation for \( Q'_\mu \). If \( E_{\mu\nu} \) is known but not of the form \( \nabla(\mu, e_{\nu}) \), a homogeneous system can be obtained by choosing the splitting suitably. If we require \( e_{\mu} \) to solve the system of wave equations
\[
\nabla_{\mu}\nabla^{\alpha} e_{\nu} + R_{\mu\nu} e_{\alpha} = 2\nabla^{\alpha} \left( E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E_{\rho\sigma} \right),
\]
it follows from the splitting above that
\[
\nabla^{\alpha} \left( f_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f_{\rho\sigma} \right) = 0,
\]
and the Bianchi identity implies for \( Q'_\mu \) the homogeneous equation
\[
\nabla_{\mu}\nabla^{\alpha} Q'_\nu + R_{\mu\nu} Q'_\alpha = 0.
\]
(3.4)

One could now try to analyse how errors in \( Q_\mu \) and \( \partial_t Q_\nu \) are propagated, in dependence of \( E_{\mu\nu} \), by (3.2) or how errors in \( Q'_\mu \) and \( \partial_t Q'_\nu \), are propagated by (3.4). Assuming in this analysis \( E_{\mu\nu}, g_{\mu\nu} \) and \( R_{\mu\nu}[g] \) as suitably bounded but given with no further information, standard energy estimates admit but cannot exclude exponential growth of \( Q_\mu \) resp. \( Q'_\mu \) (cf [8]). One might be able to work out conditions on \( g_{\mu\nu}, R_{\mu\nu} \) which would allow one to keep the errors under control. By itself this may be of limited use, however, because it is not clear whether these conditions will be respected by the main system.

Inserting instead (3.1) into (3.2) and, observing the splitting, into (3.4), gives
\[
\nabla_{\mu}\nabla^{\alpha} Q_\nu + Q^{\alpha}(\nabla_{\mu} Q_\nu + E_{\nu\lambda}) = -2\nabla^{\alpha} \left( E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E_{\rho\sigma} \right),
\]
and
\[
\nabla_{\mu}\nabla^{\alpha} Q'_\nu + Q^{\alpha}(\nabla_{\mu} Q'_\nu + f_{\nu\lambda}) = 0,
\]
respectively. The relative size of the errors \( E_{\nu\lambda} \) and \( f_{\nu\lambda} \) is not known and it is not clear which of these systems is more useful. In any case, most important for our discussion is the nonlinearity exhibited by this version of the subsidiary system.

If \( Q'_\mu \) and \( \partial_t Q'_\nu \) vanish on \( S \), it still follows by general analytic arguments that equation (3.6) implies in the continuum model the vanishing of \( Q'_\mu \) in the domain of dependence. The following observation shows, however, that (3.6) can imply a growth of the solution \( Q'_\mu \) worse than exponential if \( Q'_\mu \) and \( \partial_t Q'_\nu \) do not both vanish on \( S \).

We assume for simplicity that \( g = dt^2 - \delta_{ab} dx^a dx^b \) (so that (3.6) decouples from (3.3)), that \( f_{\nu\lambda} = 0 \), and that the initial data satisfy for \( a = 1, 2, 3, 4 \),
\[
Q'_a = 0, \quad \partial_t Q'_a = 0, \quad \partial_\nu \partial_\alpha Q'_\nu = 0, \quad \partial_\alpha \partial_\beta Q'_\nu = 0 \quad \text{on } \{ t = 0 \},
\]
so that the error resides only in the constant functions \( a = Q'_a, b = \partial_t Q'_0 \) on \( \{ t = 0 \} \).

Equation (3.6) then implies \( Q'_a = 0 \) and reduces in fact to \( \partial_t Q'_0 = \frac{1}{2}(c - Q'_0) \) with \( c = 2b + a^2 \). The integration gives
\[
Q'_0 = a = \text{const.} \quad \text{if } \quad b = 0,
\]
\[
Q'_0 = \sqrt{c} \left\{ \frac{a + \sqrt{c} \tan \left( \frac{\sqrt{c}}{2} \right)}{\sqrt{c} + a \tan \left( \frac{\sqrt{c}}{2} \right)} \right\} \quad \text{if } \quad 0 \neq 2b > -a^2,
\]
\[
Q'_0 = \frac{2a}{at + 2} \quad \text{if } \quad a^2 = -2b,
\]
\[
Q'_0 = \sqrt{|c|} \tan \left\{ -\frac{\sqrt{|c|}t}{2} + \arctan \left( \frac{a}{\sqrt{|c|}} \right) \right\} \quad \text{if } \quad 2b < -a^2.
\]
The solutions thus remain bounded for $t \geq 0$ if $b \geq 0$ or if $a \geq 0$ and $-a^2 < 2b < 0$, while they develop poles at some $t_* > 0$ if $b < 0$ and $a \leq 0$ or if $2b < -a^2$ and $a > 0$. If the corresponding initial data would be modified outside the intersection of the hypersurface \( \{ t = 0 \} \) with the backward light cone of the point \((t_*, x^a)\), the solution would still become singular at \((t_*, x^a)\). We also note that in general any Killing vector field \( K \) of a vacuum solution \( g_{\mu \nu} \) satisfies equation (2.4) with \( Q^\mu = K^\mu \) in the coordinates \( x^\mu \). In the present case this gives solutions which grow linearly.

For us the following is important:

For a given smooth metric \( g_{\mu \nu} \) (extending sufficiently far into the future) consider the second of equations (4.1) as an independent system. The discussion above suggests then that arbitrarily small local errors in \( Q_\alpha \) and \( \partial_\tau Q_\alpha \) at some time may induce the solutions to become unbounded after some finite coordinate time \( t_* > 0 \).

The time \( t_* \) will be quite large for very small errors in \( Q_\alpha \) and \( \partial_\tau Q_\alpha \), but the observation above indicates that after some relatively stable numerical evolution of the metric by the first of equations (4.1) the nonlinear effects of the second equation may take over and cause a collapse of the calculation.

4. Concluding remarks

The only purpose of the discussion above was to indicate the behaviour of solutions to the subsidiary system. It would be interesting to know whether the singular data set, i.e. the subset of data which determine solutions of equation (2.4) which become singular in the future of the initial hypersurface, is open or of lower dimension in the set of all data if the metric \( g \) is given or whether it can be characterized in other useful ways (the fact that the nonlinear subsidiary system is manifestly not time symmetric may become important here). Such characterizations might help avoid entering the singular sector in numerical calculations.

In a consistent discussion of the growth of \( Q_\mu \), one would have to consider the main system and the subsidiary system

\[
R_{\mu \nu} = \nabla_{(\mu} Q_{\nu)}, \quad \nabla_\mu \nabla^\mu Q_\nu + Q^\lambda \nabla_{(\lambda} Q_{\nu)} = 0, \quad (4.1)
\]

as a coupled system. The main equation can be studied independently. The subsidiary system, implicit in the main system, depends on the metric defined by the latter. If \( Q_\mu \) tends to grow, the main system will react to it, whether for the better or worse is not clear. Whether the nonlinearity of the subsidiary system can still imply a blow up of \( Q_\mu \) at a finite time if \( g_{\mu \nu} \) is changing needs to be analysed.

For this purpose it might be interesting to study under simplifying assumptions such as spherical symmetry whether the model system

\[
R_{\mu \nu} = \nabla_{(\mu} q_{\nu)}, \quad \nabla_\mu \nabla^\mu q_\nu + q^\lambda \nabla_{(\lambda} q_{\nu)} = 0,
\]

considered as Einstein equations coupled to a source field given by a vector field \( q_\mu \), will develop a blow up for suitable data. The hyperbolic reduction of these equations is obtained by a slight modification of the argument described above. The system can be simplified further by assuming \( q_\mu \) to be the differential \( q_\mu = \nabla_\mu f \) of some function \( f \). The second equation will then be implied if \( f \) satisfies

\[
\nabla_\mu \nabla^\mu f + \nabla_\mu f \nabla^\mu f = \text{const.},
\]

which is a wave map equation in the case where the constant on the right-hand side vanishes. In any case the results might lead to an identification of a mechanism responsible for the growth of constraint violations and to the subsequent development of methods to avoid them.
It has been suggested in [7] that adding terms built from $Q_\nu$ to the main system might reduce the growth of the constraint violations. In the further development of the work begun in [10] such terms have been used with good success. Adding such terms changes the nonlinearity of the subsidiary system and it might be interesting to compare the effects of different types of nonlinearities.

There are available now many different types of reductions. In spite of the different resulting main and subsidiary systems, nonlinearities are likely to occur in any subsidiary system. Again, it would be interesting to compare the effects of different types of nonlinearities.

In a numerical scheme for the second-order wave equations, the subsidiary system, which is of third order in the metric, can, of course, hardly be identified any longer as a kind of identity and the relations between the two systems are obscured. But if the nonlinearity of the subsidiary system can have for non-vanishing initial data $Q_\mu$ and $\partial_\tau Q_\mu$ drastic effects in the continuum model, they are likely to be reflected in numerical calculations.

References

[1] Brodbeck O, Fritelli S, Hübner P and Reula O 1999 Einstein’s equations with asymptotically stable constraint propagation J. Math. Phys. 40 909–23
[2] Courant R and Hilbert D 1962 Methods of Mathematical Physics (New York: Wiley)
[3] Fourès-Bruhat Y 1952 Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires Acta Math. 88 141–225
[4] Frauendiener J and Vogel T 2004 On the stability of constraint propagation Preprint gr-qc/0410100
[5] Friedrich H 1985 On the hyperbolicity of Einstein’s and other gauge field equations Commun. Math. Phys. 100 525–43
[6] Friedrich H and Rendall A 2000 The Cauchy problem for the Einstein equations Einstein’s Field Equations and Their Physical Implications (Lecture Notes in Physics) ed B Schmidt (Berlin: Springer)
[7] Gundlach C, Calabrese G and Hinder I 2005 Constraint damping in the Z4 formulation and harmonic gauge Preprint gr-qc/0504114
[8] Kato T 1975 The Cauchy problem for quasi-linear symmetric hyperbolic systems Arch. Ration. Mech. Anal. 58 181–205
[9] Lehner L and Reula O 2004 Status quo and open problems in the numerical construction of space-times The Einstein Equations and the Large Scale Behaviour of Gravitational Fields ed P T Chruściel and H Friedrich (Basel: Birkhäuser)
[10] Pretorius F 2005 Numerical relativity using a generalized harmonic decomposition Class. Quantum Grav. 22 425–51
[11] Yoneda G and Shinkai H 2003 Diagonalizability of constraint propagation matrices Class. Quantum Grav. 20 L31–6