Group Activity Selection with Few Agent Types

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Abstract

The Group Activity Selection Problem (GASP) models situations where a group of agents needs to be distributed to a set of activities while taking into account preferences of the agents w.r.t. individual activities and activity sizes. The problem, along with its two previously proposed variants sGASP and gGASP, has been studied in the parameterized complexity setting with various parameterizations, such as number of agents, number of activities and solution size. However, the complexity of the problem parameterized by the number of types of agents, a parameter motivated and proposed already in the paper that introduced GASP, has so far remained open.

In this paper we establish the complexity map for GASP, sGASP and gGASP when the number of types of agents is the parameter. Our positive results, consisting of one fixed-parameter algorithm and one XP algorithm, rely on a combination of novel Subset Sum machinery (which may be of general interest) and identifying certain compression steps which allow us to focus on solutions which are “acyclic”. These algorithms are complemented by matching lower bounds, which among others answer an open question of Gupta, Roy, Saurabh and Zehavi (2017). In this direction, the techniques used to establish W[1]-hardness of sGASP are of particular interest: as an intermediate step, we use Sidon sequences to show the W[1]-hardness of a highly restricted variant of multi-dimensional Subset Sum, which may find applications in other settings as well.

1 Introduction

In this paper we consider the GROUP ACTIVITY SELECTION PROBLEM (GASP) together with its two most prominent variants, the SIMPLE GROUP ACTIVITY SELECTION PROBLEM (sGASP), and the GROUP ACTIVITY SELECTION PROBLEM WITH GRAPH STRUCTURE (gGASP) [6][13]. Since their introduction, these problems have become the focal point of extensive research [3][4][5][9][15][16]. In GASP one is given a set of agents, a set of activities, and a set of preferences for each agent in the form of a complete transitive relation (also called the preference list) over the set of pairs consisting of an activity a and a number s, expressing the willingness of the agent to participate in the activity a if it has s participants. The aim is to find a “good” assignment from agents to activities subject to certain rationality and stability conditions. Specifically, an assignment is individually rational if agents that are assigned to an activity prefer this outcome over not being assigned to any activity, and an assignment is (Nash) stable if every agent prefers its current assignment over moving to any other activity. In this way GASP captures a wide range of real-life situations such as event organization and work delegation.

sGASP is a simplified variant of GASP where the preferences of agents are expressed in terms of approval sets containing (activity,size) pairs instead of preference lists. In essence sGASP is GASP where each preference list has only two equivalence classes: the class of the approved (activity,size) pairs (which contains all pairs that are preferred over not being assigned to any activity), and the class of disapproved
(activity, size) pairs (all possible remaining pairs). On the other hand, gGASP is a generalization of GASP where one is additionally given an undirected graph (network) on the set of all agents that can be employed to model for instance acquaintanceship or physical distance between agents. Crucially, in gGASP one only considers assignments for which the subnetwork induced by all agents assigned to some activity is connected. Note that if the network forms a complete graph, then gGASP is equivalent to the underlying GASP instance.

**Related Work.** SGASP, GASP, and gGASP, are known to be NP-complete even in very restricted settings [6, 18, 15, 16]. It is therefore natural to study these problems through the lens of parameterized complexity [8, 2]. Apart from parameterizing by the solution size (i.e., the number of agents assigned to any activity in a solution) [19], the perhaps most prominent parameterizations thus far have been the number of activities, the number of agents, and in the case of gGASP structural parameterizations tied to the structure of the network such as treewidth [6, 18, 15, 9]. Consequently, the parameterized complexity of all three variants of GASP w.r.t. the number of activities and/or the number of agents is now almost completely understood.

Computing a stable assignment for a given instance of GASP is known to be W[1]-hard and contained in XP parameterized by either the number of activities [6, 18] or the number of agents [18] and known to be fixed-parameter tractable parameterized by both parameters [18]. Even though it has never been explicitly stated, the same results also hold for gGASP when parameterizing by the number of agents as well as when using both parameters. This is because both the XP algorithm for the number of agents as well as the fixed-parameter algorithm for both parameters essentially brute-force over every possible assignment and are hence also able to find a solution for gGASP. Moreover, the fact that gGASP generalizes GASP implies that the W[1]-hardness result for the number of agents also carries over to gGASP. On the other hand, if we consider the number of activities as a parameter then gGASP turns out to be harder than GASP: Gupta et al. ([15]) showed that gGASP is NP-complete even when restricted to instances with a single activity. The hardness of gGASP has inspired a series of tractability results [15, 18] obtained by employing additional restrictions on the structure of the network. One prominent result in this direction has been recently obtained by Gupta et al. ([15]), showing that gGASP is fixed-parameter tractable parameterized by the number of activities if the network has constant treewidth. For SGASP, it was recently shown that the problem is also W[1]-hard when parameterized by the number of activities [9], and hence the only small gap left was the complexity of this problem parameterized by the number of agents. For completeness, we resolve this question in our concluding remarks by giving a fixed-parameter algorithm.

Already with the introduction of GASP [6] the authors argued that instead of putting restrictions on the total number of agents, which can be very large in general, it might be much more useful to consider the number of distinct types of agents. It is easy to imagine a setting with large groups of agents that share the same preferences (for instance due to inherent limitations of how preferences are collected). In contrast to the related parameter number of activity types, where it is known that SGASP remains NP-complete even for a constant number of activity types [6], the complexity of the problems parameterized by the number of agent types (with or without restricting the number of activities) has remained wide open thus far.

**Our Results.** In this paper we obtain a complete classification of the complexity of GASP and its variants SGASP and gGASP when parameterized by the number of agent types (t) alone, and also when parameterized by t plus the number of activities (a). In particular, for each of the considered problems and parameterizations, we determine whether the problem is in FPT, or W[1]-hard and in XP, or paraNP-hard. One distinguishing feature of our lower- and upper-bound results is that they make heavy use of novel Subset-Sum machinery. Below, we provide a high-level summary of the individual results presented in the paper.

**Result 1.** SGASP is fixed-parameter tractable when parameterized by t + a.
This is the only fixed-parameter tractability result presented in the paper, and is essentially tight: it was recently shown that SGASP is $\mathsf{W}[1]$-hard when parameterized by $a$ alone [9], and the $\mathsf{W}[1]$-hardness of the problem when parameterized by $t$ is obtained in this paper. Our first step towards obtaining the desired fixed-parameter algorithm for SGASP is to show that every YES-instance has a solution which is acyclic—in particular, a solution with no cycles formed by interactions between activities and agent types (captured in terms of the incidence graph $G$ of an assignment). This is proved via the identification of certain compression steps which can be applied on a solution in order to remove cycles.

Once we show that it suffices to focus on acyclic solutions, we branch over all acyclic incidence graphs (i.e., all acyclic edge sets of $G$); for each such edge set, we can reduce the problem of determining whether there exists an assignment realizing this edge set to a variant of SUBSET SUM embedded in a tree structure. The last missing piece is then to show that this problem, which we call TREE SUBSET SUM, is polynomial-time tractable; this is done via dynamic programming, whereas each step boils down to solving a simplified variant of SUBSET SUM.

**Result 2.** SGASP is $\mathsf{W}[1]$-hard when parameterized by $t$.

Our second result complements Result 1. As a crucial intermediate step towards Result 2, we obtain the $\mathsf{W}[1]$-hardness of a variant of SUBSET SUM with three distinct “ingredients”:

1. **Partitioning**: items are partitioned into sets, and precisely one item must be selected from each set,

2. **Multidimensionality**: each item is a $d$-dimensional vector ($d$ being the parameter) where the aim is to hit the target value for each component, and

3. **Simplicity**: each vector contains precisely one non-zero component.

We call this problem SIMPLE MULTIDIMENSIONAL PARTITIONED SUBSET SUM (SMPSS). Note that SMPSS is closely related to MULTIDIMENSIONAL SUBSET SUM (MSS), which (as one would expect) merely enhances SUBSET SUM via Ingredient 2. MSS has recently been used to establish $\mathsf{W}[1]$-hardness for parameterizations of EDGE DISJOINT PATHS [13] and BOUNDED DEGREE VERTEX DELETION [12]. However, Ingredient 1 and especially Ingredient 3 are critical requirements for our reduction to work, and establishing the $\mathsf{W}[1]$-hardness of SMPSS was the main challenge on the way towards the desired lower-bound result for SGASP. Since MSS has already been successfully used to obtain lower-bound results and SMPSS is a much more powerful tool in this regard, we believe that SMPSS will find applications in establishing lower bounds for other problems in the future.

**Result 3.** GASP is in $\mathsf{XP}$ when parameterized by $t$.

This is the only $\mathsf{XP}$ result required for our complexity map, as it implies $\mathsf{XP}$ algorithms for SGASP parameterized by $t$ and for GASP parameterized by $t + a$. We note that the techniques used to obtain Result 3 are disjoint from those used for Result 1; in particular, our first step is to reduce GASP parameterized by $t$ to solving “$\mathsf{XP}$-many” instances of SGASP parameterized by $t$. This is achieved by showing that once we know a “least preferred alternative” for every agent type that is active in an assignment, then the GASP instance becomes significantly easier—and, in particular, can be reduced to a (slightly modified version of) SGASP. It is interesting to note that the result provides a significant conceptual improvement over the known brute force algorithm for GASP parameterized by the number of agents which enumerates all possible assignments of agents to activities [17, Theorem 3] (see also [16]): instead of guessing an assignment for all agents, one merely needs to guess a least preferred alternative for every agent type.

The second part of our journey towards Result 3 focuses on obtaining an $\mathsf{XP}$ algorithm for SGASP parameterized by $t$. This algorithm has two components. Initially, we show that in this setting one can
reduce SGASP to the problem of finding an assignment which is individually rational (i.e., without the stability condition) and satisfies some additional minor properties. To find such an assignment, we once again make use of SUBSET SUM: in particular, we develop an XP algorithm for the MPSS problem (i.e., SUBSET SUM enhanced by ingredients 1 and 2) and apply a final reduction from finding an individually rational assignment to MPSS.

**Result 4.** GASP is \(W[1]\)-hard when parameterized by \(t + a\).

**Result 5.** gGASP is \(W[1]\)-hard when parameterized by \(t + a\) and the vertex cover number \([11]\) of the network.

The final two results are hardness reductions which represent the last pieces of the presented complexity map. Both are obtained via reductions from PARTITIONED CLIQUE (also called MULTICOLORED CLIQUE in the literature [2]), and both reductions essentially use \(k + \binom{k}{2}\) activities whose sizes encode the vertices and edges forming a \(k\)-clique. The main challenge lies in the design of (a bounded number of) agent types whose preference lists ensure that the chosen vertices are indeed endpoints of the chosen edges. The reduction for gGASP then becomes even more involved, as it can only employ a limited number of connections between the agents in order to ensure that vertex cover of the network is bounded.

We note that Result 5 answers an open question raised by Gupta, Roy, Saurabh and Zehavi [15], who showed that gGASP is fixed-parameter tractable parameterized by the number of activities if the network has constant treewidth and wondered whether their result can be improved to a more efficient fixed-parameter algorithm parameterized by the number of activities and treewidth. In this sense, our hardness result represents a substantial shift of the boundaries of (in)tractability: in addition to the setting of Gupta et al., it also rules out the use of agent types as a parameter and replaces treewidth by the more restrictive vertex cover number.

An overview of our results in the context of related work is provided in Table 1.

**Organization of the Paper.** After introducing the required preliminaries in Section 2, we present all of our Subset Sum machinery in the dedicated Section 3. Each subsequent Section \(i \leq 8\) then focuses on obtaining Result \(i - 3\).

## 2 Preliminaries

For an integer \(i\), we let \([i] = \{1, 2, \ldots, i\}\) and \([i]_0 = [i] \cup \{0\}\). We denote by \(\mathbb{N}\) the set of natural numbers, by \(\mathbb{N}_0\) the set \(\mathbb{N} \cup \{0\}\). For a set \(S\) and an integer \(k\), we denote by \(S^k\) and \(2^S\) the set of all \(k\) dimensional vectors over \(S\) and the set of all subsets of \(S\), respectively. For a vector \(\bar{p}\) of integers, we use \(|\bar{p}|\) to denote the sum of its elements.

We refer to the handbook by Diestel (7) for standard graph terminology. The vertex cover number of a graph \(G\) is the size of a minimum vertex cover of \(G\).

### 2.1 Parameterized Complexity

In parameterized algorithmics [8, 2, 20] the run-time of an algorithm is studied with respect to a parameter \(k \in \mathbb{N}_0\) and input size \(n\). The basic idea is to find a parameter that describes the structure of the instance such that the combinatorial explosion can be confined to this parameter. In this respect, the most favourable complexity class is \(\text{FPT} (\text{fixed-parameter tractable})\) which contains all problems that can be decided by an algorithm running in time \(f(k) \cdot n^{O(1)}\), where \(f\) is a computable function. Algorithms with this running time
are called fixed-parameter algorithms. A less favourable outcome is an XP algorithm, which is an algorithm running in time $O(n^{f(k)})$; problems admitting such algorithms belong to the class XP.

To obtain our lower bounds, we will need the notion of a parameterized reduction. Formally, a parameterized problem is a subset of $\Sigma^* \times \mathbb{N}$, where $\Sigma$ is the input alphabet. Let $L_1 \subseteq \Sigma_1^* \times \mathbb{N}$ and $L_2 \subseteq \Sigma_2^* \times \mathbb{N}$ be parameterized problems. A parameterized reduction (or FPT-reduction) from $L_1$ to $L_2$ is a mapping $P : \Sigma_1^* \times \mathbb{N} \to \Sigma_2^* \times \mathbb{N}$ such that (i) $(x, k) \in L_1$ iff $P(x, k) \in L_2$, (ii) the mapping can be computed by an FPT-algorithm w.r.t. parameter $k$, and (iii) there is a computable function $g$ such that $k' \leq g(k)$, where $(x', k') = P(x, k)$.

Finally, we introduce the complexity class used to describe our lower bounds. The class $W[1]$ captures parameterized intractability and contains all problems that are FPT-reducible to INDEPENDENT SET (parameterized by solution size).

### 2.2 Group Activity Selection.

The task in the **GROUP ACTIVITY SELECTION PROBLEM** (GASP) is to compute a stable assignment $\pi$ from a given set $N$ of agents to a set $A$ of activities, where each agent participates in at most one activity in $A$. The assignment $\pi$ is (Nash) stable if and only if it is individually rational and no agent has an NS-deviation to any other activity (both of these stability rules are defined in the next paragraph). We use a dummy activity $a_0$ to capture all those agents that do not participate in any activity in $A$ and denote by $A^*$ the set $A \cup \{a_0\}$. Thus, an assignment $\pi$ is a mapping from $N$ to $A^*$, and for an activity $a \in A$ we use $\pi^{-1}(a)$ to denote the set of agents assigned to $a$ by $\pi$; we set $|\pi^{-1}(a_0)| = 1$ if there is at least one agent assigned to $a_0$ and 0 otherwise.

| Parameterization | Lower Bound | Upper Bound |
|------------------|-------------|-------------|
| sGASP            |             | FPT$^0$     |
| GASP             | $n$         | W[1]$_a$    | XP$_a$ |
| gGASP            |             | W[1]       | XP$_a$ |
| sGASP            |             | $W[1]_b$   | XP     |
| GASP             | $a$         | W[1]       | XP$_a$ |
| gGASP            |             |             | paraNP$_c$ |
| sGASP            |             | $n + a$    | FPT$_d$ |
| GASP             |             |             |         |
| gGASP            |             |             |         |
| sGASP            |             | $t$        | W[1]$_2$ | XP    |
| GASP             |             | W[1]       | XP$_3$  |
| gGASP            |             |             | paraNP$_c$ |
| sGASP            |             | $t + a$    | W[1]$_1$ | XP    |
| GASP             |             |             | paraNP$_c$ |
| gGASP            |             | $t + a + \nu c$ (or $t + a + tw$) | W[1]$_5$ | XP$_c$ |

Table 1: Lower and upper bounds for sGASP, GASP, and gGASP parameterized by the number of agents ($n$), agent types ($t$), and the number of activities ($a$). In the case of $g$GASP, also the parameters vertex cover number ($\nu$) and treewidth ($tw$) of the network are considered. Entries in bold are shown in this paper, and the numbers 1 to 5 in the upper index are used to identify results 1 to 5. The result marked with $^0$ is provided in the concluding remarks. References: $a$ is $[18]$, $b$ is $[9]$, $c$ is $[15]$, $d$ is folklore.
The set $X$ of alternatives is defined as $X = (A \times |N|) \cup \{(a_0, 1)\}$. Each agent is associated with its own preferences defined on the set $X$. In the case of the standard GASP problem, an instance $I$ is of the form $(N, A, (\succeq_n)_{n \in N})$ where each agent $n$ is associated with a complete transitive preference relation (list) $\succeq_n$ over the set $X$. An assignment $\pi$ is individually rational if for every agent $n \in N$ it holds that if $\pi(n) = a$ and $a \neq a_0$, then $(a, \pi^{-1}(a)) \succeq_n (a_0, 1)$ (i.e., $n$ weakly prefers staying in $a$ over moving to $a_0$). An agent $n$ where $\pi(n) = a$ is defined to have an NS-deviation to a different activity $a'$ in $A$ if $(a', |\pi^{-1}(a')| + 1) \succ_n (a, |\pi^{-1}(a)|)$ (i.e., $n$ prefers moving to an activity $a'$ over staying in $a$). The task in GASP is to compute a stable assignment.

$gGASP$ is defined analogously to GASP, however where one is additionally given a set $L$ of links $L \subseteq \{(n, n') \mid n, n' \in N \wedge n \neq n'\}$ between the agents on the input; specifically, $L$ can be viewed as a set of undirected edges and $(N, L)$ as a simple undirected graph. In $gGASP$, the task is to find an assignment $\pi$ which is not only stable but also connected; formally, for every $a \in A$ the set of agents $\pi^{-1}(a)$ induces a connected subgraph of $(N, L)$. Moreover, an agent $n \in N$ only has an NS-deviation to some activity $a \neq \pi(n)$ if (in addition to the conditions for NS-deviations defined above) $n$ has an edge to at least one agent in $\pi^{-1}(a)$.

In $sGASP$, an instance $I$ is of the form $(N, A, (P_n)_{n \in N})$, where each agent has an approval set $P_n \subseteq X \setminus \{(a_0, 1)\}$ of preferences (instead of an ordered preference list). We denote by $P_n(a)$ the set $\{i \mid (a, i) \in P_n\}$ for an activity $a \in A$. An assignment $\pi : N \to A^*$ is said to be individually rational if every agent $n \in N$ satisfied the following: if $\pi(n) = a$ and $a \neq a_0$, then $|\pi^{-1}(a)| \in P_n(a)$. Further, an agent $n \in N$ where $\pi(n) = a_0$, is said to have an NS-deviation to an activity $a$ in $A$ if $(|\pi^{-1}(a)| + 1) \in P_n(a)$.

We now introduce the notions and definitions required for our main parameter of interest, the “number of agent types”. We say that two agents $n$ and $n'$ in $N$ have the same agent type if they have the same preferences. To be specific, $P_n = P_n'$ for $sGASP$ and $\succeq_n = \succeq_n'$ for GASP and $gGASP$. In the case of $sGASP$ and GASP $n$ and $n'$ are indistinguishable, while in $gGASP$ $n$ and $n'$ can still have different links to other agents. For a subset $N' \subseteq N$, we denote by $T(N')$ the set of agent types occurring in $N'$. Note that this notation requires that the instance is clear from the context. If this is not the case then we denote by $T(I)$ the set $T(N)$ if $N$ is the set of agents for the instance $I$ of $sGASP$, GASP, or $gGASP$.

For every agent type $t \in T(I)$, we denote by $N_t$ the subset of $N$ containing all agents of type $t$; observe that $\{N_t \mid t \in T(I)\}$ forms a partition of $N$. For an agent type $t \in T(I)$ we denote by $P_t$ ($sGASP$) or $\succeq_t$ (GASP) the preference list assigned to all agents of type $t$ and we use $P_t(a)$ (for an activity $a \in A$) to denote $P_t$ restricted to activity $a$, i.e., $P_t(a)$ is equal to $P_n(a)$ for any agent $n$ of type $t$. For an assignment $\pi : N \to A^*$, $t \in T(I)$, and $a \in A$ we denote by $\pi_{t,a}$ the set $\{n \mid n \in N_t \wedge \pi(n) = a\}$ and by $\pi_t$ the set $\bigcup_{a \in A} \pi_{t,a}$. Further, $\pi(t)$ is the set of all activities that have at least one agent of type $t$ participating in it. We say that $\pi$ is a perfect assignment for some agent type $t \in T(I)$ if $\pi(n) \neq a_0$ for every $n \in N_t$. We denote by $PE(I, \pi)$ the subset of $T(I)$ consisting of all agent types that are perfectly assigned by $\pi$, and say that $\pi$ is a perfect assignment if $PE(I, \pi) = T(I)$.

One notion that will appear throughout the paper is that of compatibility: for a subset $Q \subseteq T(I)$, we say that $\pi$ is compatible with $Q$ if $PE(I, \pi) = Q$. We conclude this section with a technical lemma which provides a preprocessing procedure that will be used as a basic tool for obtaining our algorithmic results. In particular, Lemma 1 allows us to reduce the problem of computing a stable assignment for a $sGASP$ instance compatible with $Q$ to the problem of finding an individually rational assignment.

**Lemma 1.** Let $I = (N, A, (P_n)_{n \in N})$ be an instance of $sGASP$ and $Q \subseteq T(I)$. Then in time $O(|N|^2|A|)$ one can compute an instance $\gamma(I, Q) = (N, A, (P'_n)_{n \in N})$ and $A \neq \emptyset(I, Q) \subseteq A$ with the following property: for every assignment $\pi : N \to A^*$ that is compatible with $Q$, it holds that $\pi$ is stable for $I$ if and only if $\pi$ is individually rational for $\gamma(I, Q)$ and $\pi^{-1}(a) \neq \emptyset$ for every $a \in A \neq \emptyset(I, Q)$.

**Proof.** Let $\pi : N \to A^*$ be an assignment that is compatible with $Q$. Then there is an agent of type $t \in T(N)$ assigned to $a_0$ if and only if $t \notin Q$. Hence $\pi$ is stable for $I$ if and only if $\pi$ is individually rational and
additionally it holds that for every agent type \( t \in T(N) \setminus Q \) and every activity \( a \in A, |\pi^{-1}(a)| + 1 \notin P_t(a) \). This naturally leads us to a certain set of “forbidden” sizes for activities, and we will obtain the desired instance \( \gamma(I, Q) \) by simply removing all tuples from all preference lists that would allow activities to reach a forbidden size. Formally, we obtain the desired instance \( \gamma(I, Q) \) from \( I \) removing all tuples \( (a, n) \) from every preference list \( P_t \), where \( t \in T(N) \) such that there is an agent type \( t' \in T(N) \setminus Q \) with \( n + 1 \in P_{t'}(a) \). This construction prevents the occurrence of all forbidden sizes of activities except for forbidding activities of size 0; that is where we use the set \( A_{\neq \emptyset}(I, Q) \). Formally, the set \( A_{\neq \emptyset}(I, Q) \) consists of all activities \( a \) such that there is an agent type \( t \in T(N) \setminus Q \) with \( 1 \in P_t(a) \). It is now straightforward to verify that \( \gamma(I, Q) \) and \( A_{\neq \emptyset}(I, Q) \) satisfy the claim of the lemma.

Finally the running time of \( O(|N|^2 |A|) \) for the algorithm can be achieved as follows. In a preprocessing step we first compute for every activity \( a \in A \) the set of all forbidden numbers, i.e., the set of all numbers \( n \) such that there is an agent type \( t \in T(N) \setminus Q \) with \( (a, n + 1) \in P_t(a) \). For every activity \( a \), we store the resulting set of numbers in such a way that determining whether a number \( n \) is contained in the set for activity \( a \) can be achieved in constant time; this can for instance be achieved by storing the set for each activity \( a \) as a Boolean array with \( |N| \) entries, whose \( i \)-th entry is \( \text{TRUE} \) if and only if \( i \) is contained in the set of numbers for \( a \). This preprocessing step takes time at most \( O(|N|^2 |A|) \) and after it is completed we can use the computed sets to test for every agent type \( t \in T(N) \), every activity \( a \in A \), and every \( n \in P_t(a) \), whether there is an agent type \( t' \in T(N) \setminus Q \) such that \((a, n + 1) \in P_{t'}(a)\) in constant time. If so we remove \( n \) from \( P_t(a) \), otherwise we continue. This shows that \( \gamma(I, Q) \) can be computed in \( O(|N|^2 |A|) \) time. The computation of \( A_{\neq \emptyset} \) only requires to check for every activity \( a \in A \) whether 1 is contained in the set of forbidden numbers for \( a \); if so \( a \) is contained in \( A_{\neq \emptyset} \) and otherwise it is not. After preprocessing, this can be achieved in time \( O(|A|) \).

\[ \square \]

3 Subset Sum Machinery

In this section we introduce the subset sum machinery required for our algorithms and lower bound results. In particular, we introduce three variants of \( \text{SUBSET SUM} \), obtain algorithms for two of them, and provide a \( \text{W}[1] \)-hardness result for the third. We note that it may be helpful to read the following three subsections in the context of the individual sections where they are used: in particular, Subsection 3.1 is used to obtain Result 1 (Section 4). Subsection 3.2 is used as a preprocedure for Result 3 (Section 5) and Subsection 3.3 (which is by far the most difficult of the three) is a crucial step in the reduction used for Result 2 (Section 5).

3.1 Tree Subset Sum

Here we introduce a useful generalization of \( \text{SUBSET SUM} \), for which we obtain polynomial-time tractability under the assumption that the input is encoded in unary. Intuitively, our problem asks us to assign values to edges while meeting a simple criterion on the values of edges incident to each vertex.

**Tree Subset Sum (TSS)**

| Input: | A vertex-labeled undirected tree \( T \) with labeling function \( \lambda: V(T) \to 2^\mathbb{N} \). |
| Question: | Is there an assignment \( \alpha: E(T) \to \mathbb{N} \) such that for every \( v \in V(T) \) it holds that \( \sum_{e \in E(T) \land e \in \alpha} \alpha(e) \in \lambda(v) \). |

Let us briefly comment on the relationship of TSS with \( \text{SUBSET SUM} \). Recall that given a set \( S \) of natural numbers and a natural number \( t \), the \( \text{SUBSET SUM} \) problem asks whether there is a subset \( S' \) of \( S \) such that \( \sum_{s \in S'} s = t \). One can easily construct a simple instance \( (G, \lambda) \) of TSS that is equivalent to a given instance \( (S, t) \) of \( \text{SUBSET SUM} \) as follows. \( G \) consists of a star having one leaf \( l_s \) for every \( s \in S \) with \( \lambda(l_s) = \{0, s\} \) and \( \lambda(c) = \{t\} \) for the center vertex \( c \) of the star. Given this simple reduction from \( \text{SUBSET SUM} \)
SUM to TSS it becomes clear that TSS is much more general than SUBSET SUM. In particular, instead of a star TSS allows for the use of an arbitrary tree structure and moreover one can use arbitrary subsets of natural numbers to specify the constrains on the vertices. The above reduction in combination with the fact that SUBSET SUM is weakly NP-hard implies that TSS is also weakly NP-hard.

In the remainder of this section we will show that TSS (like SUBSET SUM) can be solved in polynomial-time if the input is given in unary.

Let \( I = (T, \lambda) \) be an instance of TSS. We denote by \( \max(I) \) the value of the maximum number occurring in any vertex label. The main idea behind our algorithm for TSS is to apply leaf-to-root dynamic programming. In order to execute our dynamic programming procedure, we will need to solve a special case of TSS which we call PARTITIONED SUBSET SUM; this is the problem that will later arise when computing the dynamic programming tables for TSS.

**PARTITIONED SUBSET SUM**

| Input: | A target set \( R \) of natural numbers and \( \ell \) source sets \( S_1, \ldots, S_\ell \) of natural numbers. |
| Question: | Compute the set \( S \) of all natural numbers \( s \) such that there are \( s_1, \ldots, s_\ell \), where \( s_i \in S_i \) for every \( i \) with \( 1 \leq i \leq \ell \), satisfying \((\sum_{1 \leq i \leq \ell} s_i) + s \in R\). |

For an instance \( I = (R, S_1, \ldots, S_\ell) \) of PARTITIONED SUBSET SUM, we denote by \( \max(I) \) the value of the maximum number occurring in \( R \).

**Lemma 2.** An instance \( I = (R, S_1, \ldots, S_\ell) \) of PARTITIONED SUBSET SUM can be solved in time \( O(\ell \cdot \max(I)^2) \).

**Proof.** Here we also use a dynamic programming approach similar to the approach used for the well-known SUBSET SUM problem [14]. Let \( I = (R, S_1, \ldots, S_\ell) \) be an instance of PARTITIONED SUBSET SUM.

We first apply a minor modification to the instance which will allow us to provide a cleaner presentation of the algorithm. Namely, let \( P_0, P_1, \ldots, P_\ell \) be sets of integers defined as follows: \( P_0 = R \), and for every \( i \in [\ell] \), we set \( P_i = \{ -s \mid s \in S_i \} \). Then the solution \( S \) for \( I \) is exactly the set of all natural numbers \( n \) for which there are \( p_0, \ldots, p_\ell \) with \( p_i \in P_i \) for every \( i \) with \( 0 \leq i \leq \ell \) such that \( \sum_{0 \leq i \leq \ell} p_i = n \).

In order to compute the solution \( S \) for \( I \) (employing the above characterization for \( S \)), we compute a table \( D \) having one binary entry \( D[i, n] \) for every \( i \) and \( n \) with \( 0 \leq i \leq \ell \) and \( 0 \leq n \leq \max(I) \) such that \( D[i, n] = 1 \) if and only if there are \( p_0, \ldots, p_i \) with \( \sum_{0 \leq j \leq i} p_j = n \). Note that the solution \( S \) for \( I \) can be obtained from the table \( D \) as the set of all numbers \( n \) such that \( D[\ell, n] = 1 \). It hence remains to show how \( D \) can be computed.

We compute \( D[i, n] \) via dynamic programming using the following recurrence relation. We start by setting \( D[0, n] = 1 \) for every \( n \) with \( 0 \leq n \leq \max(I) \) if and only if \( n \in P_0 \). Moreover, for every \( i \) with \( 1 \leq i \leq \ell \) and every \( n \) with \( 0 \leq n \leq \max(I) \), we set \( D[i, n] = 1 \) if and only if there is an \( n' \) with \( n \leq n' \leq \max(I) \) and a \( p \in P_i \) such that \( n' + p = n \) and \( D[i-1, n'] = 1 \).

The running time of the algorithm is \( O(\ell \cdot \max(I)^2) \) since we require \( O(\ell \cdot \max(I)) \) to initialize the table \( D \) and each of the \( \ell \) recursive steps requires time \( O(\max(I)^2) \).

With Lemma 2 in hand, we can proceed to a polynomial-time algorithm for TSS.

**Lemma 3.** An instance \( I = (T, \lambda) \) of TSS can be solved in time \( O(|V(T)|^2 \cdot \max(I)^2) \).

**Proof.** As mentioned earlier, the main idea behind our algorithm for TSS is to use a dynamic programming algorithm working from the leaves to an arbitrarily chosen root \( r \) of the tree \( T \). Informally, the algorithm computes a set of numbers for each non-root vertex \( v \) of \( T \) representing the set of all assignments of the edge from \( v \) to its parent that can be extended to a valid assignment of all edges in the subtree of \( T \) rooted
at \( v \). Once this set has been computed for all children of the root we can construct a simple Partitioned Subset Sum instance (given below) to decide whether \( I \) has a solution.

More formally, for a vertex \( v \) of \( T \) we denote by \( T_v \) the subtree of \( T \) rooted at \( v \) and by \( T_v^* \) the subtree of \( T \) consisting of \( T_v \) plus the edge between \( v \) and its parent in \( T \); for the root \( r \) of \( T \) it holds that \( T_r = T_r^* \). For every non-root vertex \( v \) with parent \( p \) we will compute a set \( R(v) \) of numbers. Informally, \( R(v) \) contains all numbers \( n \) such that the assignment setting \( \{p, v\} \) to \( n \) can be extended to an assignment for all the edges in \( T_v^* \) satisfying all constrains given by the vertices in \( T_v \). More formally, \( n \in R(v) \) if and only if there is an assignment \( \alpha : E(T_v^*) \to \mathbb{N} \) with \( \alpha(\{p, v\}) = n \) such that for every \( v \in V(T_v) \) it holds that \( \sum_{e \in E(T_v^*)} \alpha(e) = \sum_{e \in E(T_v)} \alpha(e) \in \lambda(v) \).

As stated above we will compute the sets \( R(v) \) via a bottom-up dynamic programming algorithm starting at the leaves of \( T \) and computing \( R(v) \) for every inner node \( v \) of \( T \) using solely the computed sets \( R(c) \) of each child \( c \) of \( v \) in \( T \). Note that having computed \( R(c) \) for every child \( c \) of the root \( r \) of \( T \) we can decide whether \( I \) has a solution as follows. Let \( c_1, \ldots, c_l \) be the children of \( r \) in \( T \); then \( I \) has a solution if and only if the solution set for the instance \((\lambda(r), R(c_1), \ldots, R(c_l))\) of Partitioned Subset Sum contains \( 0 \).

It remains to show how to compute \( R(v) \) for the leaves and inner nodes of \( T \). If \( v \) is a leaf then \( R(v) \) is simply equal to \( \lambda(v) \). Moreover, if \( v \) is an inner node with children \( c_1, \ldots, c_l \), then \( R(v) \) is equal to the solution set for the instance \((\lambda(v), R(c_1), \ldots, R(c_l))\) of Partitioned Subset Sum. This completes the description of the algorithm.

The running time of the algorithm is at most \( O(|V(T)|^2 \cdot \max(I)^2) \) since the time required at a leaf \( q \) of \( T \) is at most \( O(\max(\lambda(q))) \) and the time required at any none-leaf node \( t \) of \( T \) with children \( t_1, \ldots, t_l \) is at most the time required to solve the instance \((\lambda(t), R(t_1), \ldots, R(t_l))\) of Partitioned Subset Sum, which is at most \( O(|V(T)| \cdot \max(\lambda(v))^2) \) due to Lemma 2.

### 3.2 Multidimensional Partitioned Subset Sum

Our second generalization of Subset Sum is a multi-dimensional variant of the problem that allows to separate the input set of numbers into several groups, and restricts the solution to take at most 1 vector from each group. For technical reasons, we will only search for solutions of size at most \( r \).

**Multidimensional Partitioned Subset Sum (MPSS)**

**Input:** \( k \in \mathbb{N}, r \in \mathbb{N}_0 \), and a family \( \mathcal{P} = \{P_1, \ldots, P_l\} \) of sets of vectors over \( \mathbb{N}_0^k \).

**Question:** Compute the set of all vectors \( \tilde{t} \in \{0, \ldots, r\}^k \) such that there are \( \tilde{p}_1, \ldots, \tilde{p}_l \) with \( \tilde{p}_i \in P_i \) for every \( i \) with \( 1 \leq i \leq l \) such that \( \sum_{1 \leq i \leq l} \tilde{p}_i = \tilde{t} \).

It is easy to see that Subset Sum is a special case of MPSS: given an instance of Subset Sum, we can create an equivalent instance of MPSS by setting \( r \) to a sufficiently large number and simply making each group \( P_i \) contain two vectors: the all-zero vector and the vector that is equal to the \( i \)-th number of the Subset Sum instance in all entries.

**Lemma 4.** An instance \( I = (k, r, \mathcal{P}) \) of MPSS can be solved in time \( O(|\mathcal{P}| \cdot r^k) \).

**Proof.** We use a dynamic programming procedure similar to the approach used for the well-known Subset Sum problem [14]. Let \( I = (k, r, \mathcal{P}) \) with \( \mathcal{P} = \{P_1, \ldots, P_l\} \) be an instance of MPSS.

We solve \( I \) by computing a table \( D \) having one binary entry \( D[i, \tilde{t}] \) for every \( i \) and \( \tilde{t} \) with \( 0 \leq i \leq l \) and \( \tilde{t} \in \{0, \ldots, r\}^k \) such that \( D[i, \tilde{t}] = 1 \) if and only if there are \( \tilde{p}_1, \ldots, \tilde{p}_l \) with \( \tilde{p}_j \in P_j \) for every \( j \) with \( 1 \leq j \leq i \) such that \( \sum_{1 \leq j \leq i} \tilde{p}_j = \tilde{t} \). Note that the solution for \( I \) can be obtained from the table \( D \) as the set of all vectors \( \tilde{t} \in \{0, \ldots, r\}^k \) such that \( D[\tilde{t}] = 1 \). It hence remains to show how to compute the table \( D \).

We compute \( D[i, \tilde{t}] \) via dynamic programming using the following recurrence relation. We start by setting \( D[1, \tilde{t}] = 1 \) for every \( \tilde{t} \in \{0, \ldots, r\}^k \) if and only if \( \tilde{t} \in P_1 \). Moreover, for every \( i \) with \( 1 \leq i \leq l \) and every \( \tilde{t} \in \{0, \ldots, r\}^k \), we set \( D[i, \tilde{t}] = 1 \) if and only if there is a \( \tilde{p}_j \in P_j \) with \( \tilde{p}_j \leq \tilde{t} \) such that \( D[i - 1, \tilde{t} - \tilde{p}_j] = 1 \).
The running time of the algorithm is \( O(|P| \cdot r^k) \) since we require \( O(|P| \cdot r^k) \) to initialize the table \( D \) and each of the \(|P| \) recursive steps requires time \( O(r^k) \).

3.3 Simple Multidimensional Partitioned Subset Sum

In this section, we are interested in a much more restrictive version of MPSS, where all vectors (apart from the target vector) are only allowed to have at most one non-zero component. Surprisingly, we show that the \( W[1] \)-hardness of the previously studied MULTIDIMENSIONAL SUBSET SUM problem [13][12] carries over to this more restrictive variant using an intricate and involved reduction.

To formalize, we say that a set \( P \) of vectors in \( \mathbb{N}_0^d \) is simple if each vector in \( P \) has exactly one non-zero component and the values of the non-zero components for any two distinct vectors in \( P \) are distinct.

**Lemma 5.** SMPSS is strongly \( W[1] \)-hard.

*Proof.* We will employ a parameterized reduction from the PARTITIONED CLIQUE problem, which is well-known to be \( W[1] \)-complete [21].

**The PARTITIONED CLIQUE.**

- **Input:** An integer \( k \), a \( k \)-partite graph \( G = (V, E) \) with partition \( \{V_1, \ldots, V_k\} \) of \( V \) into sets of equal size.

- **Parameter:** \( k \).

- **Question:** Does \( G \) have a \( k \)-clique, i.e., a set \( C \subseteq V \) of \( k \) vertices such that \( \forall u, v \in C \), with \( u \neq v \) there is an edge \( \{u, v\} \in E \)?

We denote by \( E_{i,j} \) the set of edges of \( G \) that have one endpoint in \( V_i \) and one endpoint in \( V_j \) and we assume w.l.o.g. that \( |V_i| = \{v_i^1, \ldots, v_i^n\} = n \) and \( |E_{i,j}| = m \) for every \( i \) and \( j \) with \( 1 \leq i < j \leq k \) (see the standard parameterized complexity textbook for a justification of these assumptions [2]).

Given an instance \((G, k)\) of PARTITIONED CLIQUE with partition \( V_1, \ldots, V_k \), we construct an equivalent instance \( I = (d, \ell, \mathcal{P}) \) of SMPSS in polynomial time, where \( d = k(k-1) + \binom{k}{2} \) and \(|\mathcal{P}| = \binom{k}{2} + nk(2k-3)\). We will also make use of the following notation. For \( i \) and \( j \) with \( 1 \leq i \leq k \) and \( 1 \leq j < k \), we denote by \( \text{indJ}(i, j) \) the \( j \)-th smallest number in \( [k] \setminus \{i\} \) and we denote by \( \text{indMin}(i) \) and \( \text{indMax}(i) \) the numbers \( \text{indJ}(i, 1) \) and \( \text{indJ}(i, k-1) \), respectively.

We assign to every vertex \( v \) of \( G \) a unique number \( S(v) \) from a Sidon sequence \( S \) of length \(|V(G)| [10]\). A Sidon sequence is a sequence of natural numbers such that the sum of each pair of numbers is unique; it can be shown that it is possible to construct such sequences whose maximum value is bounded by a polynomial in its length [1][10].

To simplify the description of \( I \), we will introduce names and notions to identify both components of vectors and sets in \( \mathcal{P} \). Every vector in \( I \) has the following components:

- For every \( i \) and \( j \) with \( 1 \leq i, j \leq k \) and \( i \neq j \), the vertex component \( c^i_j \). We set \( \ell[c^i_j(j)] \) to:
  - \( n^6 + n^4 \) if \( j = \text{indMin}(i) \),
  - \( (n-1)n^8 + n^6 + n^4 + \sum_{t=1}^{n}(\ell + \ell n^2) \) if \( j > \text{indMin}(i) \) and \( j < \text{indMax}(i) \), and
\( (n-1)n^8 + n^6 + \sum_{\ell=1}^n (\ell), \) otherwise.

- For every \( i \) and \( j \) with \( 1 \leq i < j \leq k \), the edge component \( c_E(i, j) \) with \( \tilde{\ell}[c_E(i, j)] = \sum_{v \in V_i \cup V_j} S(v) \).

Note that the total number of components \( d \) is equal to \( k(k-1) + \binom{k}{2} \) and that for every \( i \) with \( 1 \leq i \leq k \), there are \( k-1 \) vertex components, i.e., the components \( c_{EV}^i(\text{ind}J(i, 1)), \ldots, c_{EV}^i(\text{ind}J(i, k-1)) \), which intuitively have the following tasks. The first component, i.e., the component \( c_{EV}^i(\text{ind}J(i, 1)) \) identifies a vertex \( v \in V_i \) that should be a part of a \( k \)-clique in \( G \). Moreover, every component \( c_{EV}^i(\text{ind}J(i, j)) \) (including the first component), is also responsible for: (1) Signalling the choice of the chosen vertex \( v \in V_i \) to the next component, i.e., the component \( c_{EV}^i(\text{ind}J(i, j+1)) \) and (2) Signalling the choice of the vertex \( v \in V_i \) to the component \( c_E(i, j) \) that will then verify that there is an edge between the vertex chosen for \( V_i \) and the vertex chosen for \( V_j \). This interplay between the components will be achieved through the sets of vectors in \( \mathcal{P} \) that will be defined and explained next.

| \( i \) | \( P_{EV}^1(2, \ell) \) | \( P_{EV}^2(2, \ell) \) | \( P_{EV}^1(3, \ell) \) | \( P_{EV}^1(4, \ell) \) | \( \tilde{\ell} \) |
|---|---|---|---|---|---|
| \( c_{EV}^1(2) \) | \( n^6 + \ell \) | \( n^4 - \ell \) | \( n^6 + \ell \) | \( n^6 + \ell \) | \( n^6 + n^2 \) |
| \( c_{EV}^1(3) \) | \( n^8 + \ell + \ell n^2 \) | \( n^6 + \ell \) | \( n^4 + \ell n^2 \) | \( n^6 + \ell \) | \( Z + n^4 + \sum_{\ell=1}^n (\ell n^2) \) |
| \( c_{EV}^1(4) \) | \( S(v^1) \) | \( S(v^1) \) | \( S(v^1) \) | \( S(v^1) \) | \( \sum_{v \in V_i \cup V_j} S(v) \) |
| \( c_E(1, 2) \) | | | | | \( \sum_{v \in V_i \cup V_j} S(v) \) |
| \( c_E(1, 3) \) | | | | | \( \sum_{v \in V_i \cup V_j} S(v) \) |
| \( c_E(1, 4) \) | | | | | \( \sum_{v \in V_i \cup V_j} S(v) \) |

Table 2: An illustration of the vectors contained in the sets \( P_{EV}^1(2, \ell), \ldots, P_{EV}^1(4, \ell), P_{EV}^1(2, \ell), \ldots, P_{EV}^1(3, \ell) \). For example the column for the set \( P_{EV}^1(2, \ell) \) shows that the set contains two vectors, one whose only non-zero component \( c_{EV}^1(2) \) has the value \( n^6 + \ell \) and a second one whose only non-zero component \( c_E(1, 2) \) and has the value \( S(v^1) \). The last column provides the value for the target vector for the component given by the row. Finally, the value \( Z \) is equal to \( (n-1)n^8 + n^6 + \sum_{\ell=1}^n (\ell) \).

| \( i \) | \( P_{EV}^1(j, \ell) \) | \( P_{EV}^1(i, \ell) \) | \( P_E(i, j) \) | \( \tilde{\ell} \) |
|---|---|---|---|---|
| \( c_{EV}^1(j) \) | \( n^6 + \ell \) | \( n^6 + \ell \) | \( n^6 + \ell \) | \( \sum_{v \in V_i \cup V_j} S(v) \) |
| \( c_{EV}^1(i) \) | \( S(v^1) \) | \( S(v^1) \) | \( \{S(v) + S(u) \mid \{v, u\} \in E_{1,j}\} \) | \( \sum_{v \in V_i \cup V_j} S(v) \) |
| \( c_E(i, j) \) | \( S(v^1) \) | \( S(v^1) \) | \( \{S(v) + S(u) \mid \{v, u\} \in E_{i,j}\} \) | \( \sum_{v \in V_i \cup V_j} S(v) \) |

Table 3: An illustration of the vectors contained in the sets \( P_{EV}^1(j, \ell), P_{EV}^1(i, \ell), P_E(i, j) \) and their interplay with the components \( c_{EV}^1(j), c_{EV}^1(i), \) and \( c_{EV}(i, j) \). For the conventions used in the table please refer to Table 2. Additionally, note that the column for \( P_E(i, j) \) indicates that the set contains one vector for every edge \( \{v, u\} \in E_{i,j} \), whose only non-zero component \( c_E(i, j) \) has the value \( S(v) + S(u) \).

\( \mathcal{P} \) consists of the following sets, which are illustrated in Table 2 and 3:

- For every \( i, j', \) and \( \ell \) with \( 1 \leq i \leq k, 1 \leq j' \leq k-2, \) and \( 1 \leq \ell \leq n \), the vertex set \( P_{EV}^1(j, \ell) \), where \( j = \text{ind}J(i, j') \), containing two vectors \( \tilde{v}_{i,j',\ell}^+ \) and \( \tilde{v}_{i,j',\ell}^- \) defined as follows:
  - if \( j' = 1 \), then \( \tilde{v}_{i,j',\ell}^+ \) \( [c_{EV}^i(j)] = n^4 - \ell \) and \( \tilde{v}_{i,j',\ell}^- \) \( [c_{EV}^i(\text{ind}J(i, j' + 1))] = n^4 + \ell + \ell n^2 \) or
  - if \( 1 < j' < k-2 \), then \( \tilde{v}_{i,j',\ell}^+ \) \( [c_{EV}^i(j)] = n^4 + \ell n^2 \) and \( \tilde{v}_{i,j',\ell}^- \) \( [c_{EV}^i(\text{ind}J(i, j' + 1))] = n^4 + \ell + \ell n^2 \) or
  - if \( j' = k-2 \), then \( \tilde{v}_{i,j',\ell}^+ \) \( [c_{EV}^i(j)] = n^4 + \ell n^2 \) and \( \tilde{v}_{i,j',\ell}^- \) \( [c_{EV}^i(\text{ind}J(i, j' + 1))] = n^4 + \ell \).
We denote by $P^i_V(j)$, $P^i_{V+}(j)$, and $P^i_{V-}(j)$ the sets $\bigcup_{\ell=1}^{n}(P^i_V(j,\ell))$, $P^i_V(j) \cap \{ \bar{a}^+_{i,j,\ell} \mid 1 \leq \ell \leq n \}$, and $P^i_V(j) \setminus P^i_{V+}(j)$, respectively.

- For every $i, j, \ell$ with $1 \leq i, j \leq k, i \neq j$, and $1 \leq \ell \leq n$, the vertex incidence set $P^i_{EV}(j, \ell)$, which contains the two vectors $\bar{a}^+_{i,j,\ell}$ and $\bar{a}^-_{i,j,\ell}$ such that $\bar{a}^+_{i,j,\ell}[c^i_V(j)] = n^\ell + \ell$ and $\bar{a}^-_{i,j,\ell}[c_E(i,j)] = S(v_i)$.

We denote by $P^i_{EV}(j)$, $P^i_{EV+}(j)$, and $P^i_{EV-}(j)$ the sets $\bigcup_{\ell=1}^{n}(P^i_{EV}(j,\ell))$, $P^i_{EV}(j) \cap \{ \bar{a}^+_{i,j,\ell} \mid 1 \leq \ell \leq n \}$, and $P^i_{EV}(j) \setminus P^i_{EV+}(j)$, respectively.

- For every $i, j$ with $1 \leq i < j \leq k$, the edge set $P_E(i,j)$, which for every $e \in \{v, u\} \in E_{i,j}$ contains the vector $\bar{e}$ such that $\bar{e}^i_{[c_E(i,j)]} = S(v) + S(u)$; note that $P_E(i,j)$ is indeed a simple set, because $S$ is a Sidon sequence.

Note that altogether there are $nk(k-2) + \binom{k}{2} + nk(k-1) = \binom{k}{2} + nk(2k-3)$ sets in $\mathcal{P}$.

Informally, the two vectors $\bar{v}^+_{i,j,\ell}$ and $\bar{v}^-_{i,j,\ell}$ in $P^i_V(j,\ell)$ represent the choice of whether or not the vertex $v_j$ should be included in a $k$-clique for $G$, i.e., if a solution for $I$ chooses $v^+_{i,j,\ell}$ then $v^+_{i,j,\ell}$ should be part of a $k$-clique and otherwise not. The component $c^i_V(j)$, more specifically the value for $\bar{c}^i_{V}(j)$, now ensures that a solution can choose at most one such vector in $P^i_{EV+}(j)$. Moreover, the fact that all but one of the vectors $\bar{v}^-_{i,j,1}, \ldots, \bar{v}^-_{i,j,n}$ need to be chosen by a solution for $I$ signals the choice of the vertex for $V_i$ to the next component, i.e., either the component $c^i_{V}(j+1)$ if $j+1 \neq i$ or the component $c^i_{V}(j+2)$ if $j+1 = i$. Note that we only need $k-2$ sets $P^i_{EV}(j)$ for every $i$, because we need to copy the vertex choice for $V_i$ to only $k-1$ components. A similar idea underlies the two vectors $\bar{a}^+_{i,j,\ell}$ and $\bar{a}^-_{i,j,\ell}$ in $P^i_{EV}(j,\ell)$, i.e., again the component $c^i_{V}(j)$ now also ensures that the choice made for the sets in $P^i_{EV}(j)$ is the same as the choice made for the sets in $P^i_{EV+}(j)$. Moreover, the choice made for the sets in $P^i_{EV}(j)$ is now propagated to the component $c_E(i,j)$ (instead of the next vertex component). Finally, the vectors in the set $P_E(i,j)$ represent the choice of the edge used in a $k$-clique between $V_i$ and $V_j$ and the component $c_E(i,j)$ ensures that only an edge, whose endpoints are the two vertices signalled by the sets $P^i_{EV}(j)$ and $P^i_{EV+}(j)$ can be chosen. We are now ready to give a formal proof for the equivalence between the two instances.

This completes the construction of $I$. It is straightforward to verify that all sets in $\mathcal{P}$ are simple, $I$ can be constructed in polynomial-time, and all the component values of all vectors are bounded by a polynomial in $n$. It hence only remains to show that $(G, k)$ has a solution if and only if so does $I$.

Towards showing the forward direction, let $c_1, \ldots, c_k$ with $c_i \in V_i$ be the vertices of a $k$-clique of $G$ and for every $i$ and $j$ with $1 \leq i < j \leq k$, let $e_{i,j}$ be the edge between $c_i$ and $c_j$ in $G$. We obtain a solution $S \subseteq \mathcal{P}$ for $I$ with $\sum_{S \subseteq \mathcal{S}} s = \bar{t}$ and $|S \cap \mathcal{P}| = 1$ for every $P \in \mathcal{P}$ by choosing the following vectors:

- For every $i$ and $j$ with $1 \leq i, j \leq k$ and $i \neq j$, we choose the vector $\bar{a}^+_{i,j,\ell}$ from the set $P^i_{EV}(j,\ell)$ if $v^+_{i,\ell} = c_i$ and otherwise we choose the vector $\bar{a}^-_{i,j,\ell}$.

- For every $i$ and $j$ with $1 \leq i, j \leq k$ and $i \notin \{ i, \text{indMax}(i) \}$, we choose the vector $\bar{v}^+_{i,j,\ell}$ from the set $P^i_{V+}(j,\ell)$ if $v^+_{i,\ell} = c_i$ and otherwise we choose the vector $\bar{v}^-_{i,j,\ell}$.

- For every $i$ and $j$ with $1 \leq i < j \leq k$, we choose the vector $\bar{e}_{i,j}$ from the set $P_E(i,j)$.

It is straightforward to verify that the above choices constitute a solution for $I$.

Towards showing the reverse direction, let $S \subseteq \bigcup_{P \in \mathcal{P}} P$ be a solution for $I$, i.e., $\sum_{S \subseteq \mathcal{S}} s = \bar{t}$ and $|\mathcal{S} \cap \mathcal{P}| = 1$ for every $P \in \mathcal{P}$. We show the reverse direction using the following series of claims.

(C1) For every $i$ and $j$ with $1 \leq i, j \leq k$ and $i \neq j$, it holds that $|S \cap P^i_{EV+}(j)| = 1$ and $|S \cap P^i_{EV-}(j)| = n - 1$. In the following let $\bar{v}^i_{EV}(j)$ be the unique vector in $S \cap P^i_{EV+}(j)$.
(C2) For every $i$ and $j'$ with $1 \leq i \leq k$ and $1 \leq j' \leq k - 2$, it holds that $|S \cap P_{V^+(j')}| = 1$ and $|S \cap P_{V^-(j')}| = n - 1$, where $j = \text{ind}J(i, j')$. In the following let $\overline{v}_i(j)$ be the unique vector in $S \cap P_{V^+(j)}$.

(C3) For every $i, j$ and $j'$ with $1 \leq i, j, j' \leq k$, $j \neq i, j', j' \neq j$, it holds that $\overline{v}_{EV}(j)[\overline{c}_V(j)] = \overline{v}_{EV}(j')[\overline{c}_V(j')]$. In other words for $i$ as above, there exists a unique value $\ell_i$ such that $\overline{v}_{EV}(j)[\overline{c}_V(j)] = n^6 + \ell_i$ for every $j$ as above.

(C4) For every $i$ and $j$ with $1 \leq i < j \leq k$, $G$ contains an edge between $v^i_{\ell_i}$ and $v^j_{\ell_j}$.

(C5) The vertices $v^1_{\ell_1}, \ldots, v^k_{\ell_k}$ induce a clique in $G$.

Towards showing (C1) consider the component $c^i_V(j)$. Then $\overline{v}_i(j)$ contains the term $n^6$ and moreover the only vectors of $I$ having a non-zero component at $c^i_V(j)$ (apart from the vectors in $P_{V^+(j)}$) are the vectors in $P_{V^-(j)}$ (only if $j > \text{ind}Min(i)$), where $j' = j - 1$ if $j = i$ and $j' = j - 2$ otherwise, and the vectors in $P_{V^+(j')}$ (only if $j < \text{ind}Max(i)$). The former all have values larger than $n^8 \geq n^6$ and the sum of all values of the latter is at most $\sum_{t=1}^{n} t^4 + \ell n^2 \leq 2n^5 < n^6$. Hence the only vectors that can contribute the term $n^6$ are the vectors in $P_{V^+(j)}$ and since $n^6$ appears exactly once in $\overline{v}_i(j)$, (C1) follows.

Towards showing (C2) consider the component $c^i_V(j)$. Then $\overline{v}_{EV}(j)$ contains the term $n^4$ and moreover the only vectors of $I$ having a non-zero component at $c^i_V(j)$ (apart from the vectors in $P_{V^+(j)}$) are the vectors in $P_{V^-(j)}$ (only if $j > \text{ind}Min(i)$), where $j' = j - 1$ if $j = i$ and $j' = j - 2$ otherwise, and the vectors in $P_{V^+(j')}$ (only if $j < \text{ind}Max(i)$). The former all have values larger than $n^8$ and $n^6$, respectively. Hence the only vectors that can contribute the term $n^4$ are the vectors in $P_{V^+(j)}$ and since $n^4$ appears exactly once in $\overline{v}_i(j)$, (C2) follows.

Towards showing (C3), we show that $\overline{v}_{EV}(j)[\overline{c}_V(j)] = \overline{v}_{EV}(j')[\overline{c}_V(j')]$, where $j = \text{ind}J(i, r)$ and $j' = \text{ind}J(i, r + 1)$ for every $r$ with $1 \leq r < k$. Since we can assume that w.l.o.g. $k > 3$, we only need to distinguish the following three cases:

(A) $r = 1$ and $r + 1 < k - 1$,

(B) $r > 1$ and $r + 1 < k - 1$,

(C) $r > 1$ and $r + 1 = k - 1$.

For the case (A), consider the component $c^i_V(j)$. Note that due to (C1) and (C2), the vectors $\overline{v}_{EV}(j)$ and $\overline{v}_V(j)$ are the only vectors in $S$, for which the component $c^i_V(j)$ is non-zero. Hence, $\overline{v}_{EV}(j)[\overline{c}_V(j)] = \overline{v}_V(j)[\overline{c}_V(j)] = n^6 + n^4$, which is only possible if $\overline{v}_{EV}(j) = \overline{a}_{\ell, \ell_1}$ and $\overline{v}_V(j) = \overline{a}_{\ell_1, \ell_1}$ for some $\ell_1$ with $1 \leq \ell_1 \leq n$. Now consider the component $c^i_V(j')$. Because of (C2), we obtain that $\sum_{s \in \mathbb{P}_{V^-(j)}} s[\overline{c}_V(j)] = (\sum_{t=1}^{n} t^4 + \ell n^2) - (n^8 + \ell_1 + \ell_1 n^2)$. Moreover, because of (C1) and (C2), we obtain that $\sum_{t=1}^{n} t^8 + \ell n^2 - (n^8 + \ell_1 + \ell_1 n^2) + \overline{v}_{EV}(j') + \overline{v}_V(j') = \overline{v}_{EV}(j')$, which is only possible if $\overline{v}_{EV}(j') = \overline{a}_{\ell_1, \ell_1, \ell_1}$ and $\overline{v}_V(j') = \overline{a}_{\ell_1, \ell_1, \ell_1}$. Hence $\overline{v}_{EV}(j)[\overline{c}_V(j)] = \overline{v}_{EV}(j')[\overline{c}_V(j')]$, as required.

The proof for the cases (B) and (C) is analogous.

Towards showing (C4) consider the component $c_E(i, j)$. Note that the set $P_{E}(j)$, $P_{E}(i)$, and $P_{E}(i, j)$ are the only sets in $\mathbb{P}$ containing vectors that are non-zero at $c_E(i, j)$. Moreover, because of (C1) it holds that $\sum_{s \in \mathbb{P}_{E}(i, j)} s[\overline{c}_E(i, j)] = (\sum_{t=1}^{n} t^4) - S(v_{\ell_1})$ and similarly $\sum_{s \in \mathbb{P}_{E}(i, j)} s[\overline{c}_E(i, j)] = (\sum_{t=1}^{n} t^4) - S(v_{\ell_1})$. Since $\ell[\overline{c}_E(i, j)] = \sum_{v \in V \cup V'} S(v)$, we obtain that the unique vector $\overline{e} \in S \cap P_E(i, j)$ must satisfy $\overline{e}[\overline{c}_E(i, j)] = S(v_{\ell_1}) + S(v_{\ell_1})$, which due the properties of Sidon sequences is only possible if $e$ is an edge between $v^i_{\ell_1}$ and $v^j_{\ell_1}$ in $G$. Finally, (C5) follows immediately from (C3) and (C4).
4 Result 1: Fixed-Parameter Tractability of SGASP

In this section we will establish that SGASP is FPT when parameterized by the number of agent types and the number of activities by proving Theorem 6.

**Theorem 6.** SGASP can be solved in time $O(2^{T(N)(1+|A|)}((|N| + |A|)|N|)^2)$.

Let $I = (N, A, (P_n)_{n \in N})$ be a SGASP instance and let $\pi : N \to A^*$ be an assignment of agents to activities. We denote by $G_I(\pi)$ the incidence graph between $T(N)$ and $A$, which is defined as follows. $G_I(\pi)$ has vertices $T(N) \cup A$ and contains an edge between an agent type $t \in T(N)$ and an activity $a \in A$ if $\pi_{t,a} \neq \emptyset$. We say that $\pi$ is acyclic if $G_I(\pi)$ is acyclic.

Our first aim towards the proof of Theorem 6 is to show that if $I$ has a stable assignment, then it also has an acyclic stable assignment (Lemma 9). We will then show in Lemma 11 that finding a stable assignment whose incidence graph is equal to some given acyclic pattern graph can be achieved in polynomial-time via a reduction to the TSS problem defined and solved in Subsection 3.1. Since the number of (acyclic) pattern graphs is bounded in our parameters, we can subsequently solve SGASP by enumerating all acyclic pattern graphs and checking for each of them whether there is an acyclic solution matching the selected pattern.

A crucial notion towards showing that it is sufficient to consider only acyclic solutions is the notion of (strict) compression. We say that an assignment $\tau$ is a compression of $\pi$ if it satisfies the following conditions:

(C1) for every $t \in T(N)$ it holds that $|\pi_t| = |\tau_t|$,  
(C2) for every $a \in A$ it holds that $|\pi^{-1}(a)| = |\tau^{-1}(a)|$, and  
(C3) for every $a \in A$ it holds that the set of agent types $\tau$ assigns to $a$ is a subset of the agent types $\pi$ assigns to $a$.

Moreover, if $\tau$ additionally satisfies:

(C4) $|E(G_I(\tau))| < |E(G_I(\pi))|$,  
we say that $\tau$ is a strict compression of $\pi$.

Intuitively, an assignment $\tau$ is a compression of $\pi$ if it maintains all the properties required to preserve stability and compatibility with a given subset $Q \subseteq T(N)$. We note that condition (C3) can be formalized as $T(\pi^{-1}(a)) \subseteq T(\tau^{-1}(a))$. Observe that if $\tau$ is a strict compression then there is at least one activity $a \in A$ such that $T(\tau^{-1}(a)) \subseteq T(\pi^{-1}(a))$. The following lemma shows that every assignment that is not acyclic admits a strict compression.

**Lemma 7.** If $\pi$ is not acyclic, then there exists an assignment $\tau$ that strictly compresses $\pi$.

**Proof.** Let $C = (t_1, a_1, \ldots, t_l, a_l, t_1)$ be a cycle of $G_I(\pi)$. Consider the following modification (M) of our instance: reassign one (arbitrary) agent in $\pi_{t_1,a_1}$ to $a_1$, and for every $i$ with $1 < i \leq l$ reassign one (arbitrary) agent in $\pi_{t_i,a_i}$ to $a_{i-1}$. An illustration of (M) is also provided in Figure 11.

First, we show that the assignment $\kappa$ obtained from $\pi$ after applying modification (M) satisfies (C1)–(C3). Towards showing (C1), observe that $|\pi_t| = |\kappa_t|$ for any $t \in T(N) \setminus \{t_1, \ldots, t_l\}$. Moreover, for every
Figure 1: Illustration of the modification (M) in the proof of Lemma 7 for a cycle of length six. A label of +1 on an edge \{t, a\} of $G_I(\pi)$ means that $|\kappa_{t,a}|$ is by one larger than $|\pi_{t,a}|$. Similarly, a label of −1 on an edge \{t, a\} of $G_I(\pi)$ means that $|\kappa_{t,a}|$ is by one smaller than $|\pi_{t,a}|$.

$i$ with $1 < i \leq l$, we have:

$$|\kappa_{t_i}| = \sum_{a \in A} |\kappa_{t_i,a}|$$

$$= \left( \sum_{a \in (A \setminus \{a_i, a_{i-1}\})} |\kappa_{t_i,a_i}| + |\kappa_{t_i,a_i}| + |\kappa_{t_i,a_{i-1}}| \right)$$

$$= \left( \sum_{a \in (A \setminus \{a_i, a_{i-1}\})} |\pi_{t_i,a_i}| + (|\pi_{t_i,a_i}| - 1) + (|\pi_{t_i,a_{i-1}}| + 1) \right)$$

$$= \sum_{a \in A} |\pi_{t_i,a}|$$

$$= |\pi_{t_i}|$$

and for $i = 1$ we have:

$$|\kappa_{t_1}| = \sum_{a \in A} |\kappa_{t_1,a}|$$

$$= \left( \sum_{a \in (A \setminus \{a_1, a_l\})} |\kappa_{t_1,a_1}| + |\kappa_{t_1,a_1}| + |\kappa_{t_1,a_l}| \right)$$

$$= \left( \sum_{a \in (A \setminus \{a_1, a_l\})} |\pi_{t_1,a_1}| + (|\pi_{t_1,a_1}| - 1) + (|\pi_{t_1,a_l}| + 1) \right)$$

$$= \sum_{a \in A} |\pi_{t_1,a}|$$

$$= |\pi_{t_1}|$$

Towards showing (C2) note that $|\pi^{-1}(a)| = |\kappa^{-1}(a)|$ for every $a \in A \setminus \{a_1, \ldots, a_l\}$ and moreover for every
and for $i = l$ we have:

$$|\kappa^{-1}(a_l)| = \sum_{t \in T(N)} |\kappa_{t,a_l}|$$

$$= ( \sum_{t \in T(N) \setminus \{t_i,t_{i+1}\}} |\pi_{t,a_l}| + (|\pi_{t_i,a_l}| - 1) + (|\pi_{t_{i+1},a_l}| + 1)$$

$$= \sum_{t \in T(N)} |\pi_{t,a_l}|$$

$$= |\pi^{-1}(a_l)|$$

Because $|\kappa_{t,a}|$ and $|\pi_{t,a}|$ can only differ if $|\pi_{t,a}| \neq 0$, i.e., $|\kappa_{t,a}|$ will never be non-equal to zero if $|\pi_{t,a}| = 0$, we obtain that $|\kappa|$ satisfies also condition (C3).

Having settled that (M) does not violate (C1)–(C3), we observe that if $G_I(\kappa)$ still contains the cycle $C$, we can apply modification (M) again to $\kappa$ and the obtained assignment still satisfies conditions (C1)–(C3). Hence we can repeatedly apply modification (M) as long as the cycle $C$ is not destroyed in the resulting assignment. Namely let $m = \min\{\pi_{t_1,a_1}, \ldots, \pi_{t_l,a_l}\}$, let $i$ be an index with $m = \pi_{t_i,a_i}$, and let $\tau$ be the assignment obtained after $m$ applications of modification (M) to $\pi$. Note that $m$ applications of modification (M) are possible since the cycle $C$ remains preserved up to the $m - 1$-th modification of (M). Furthermore, since $|\tau_{t_i,a_i}| = 0$, we obtain that $\tau$ satisfies (C4).

The following lemma shows that any assignment can be compressed into an acyclic assignment.

**Lemma 8.** Let $\pi : N \rightarrow A^*$ be an assignment for $I$. Then there exists an acyclic assignment $\pi'$ that compresses $\pi$.

**Proof.** The lemma follows via an exhaustive application of Lemma[7] Namely, we start by checking whether $\pi$ is acyclic. If yes, then $\pi$ itself is the acyclic assignment that compresses $\pi$. If not, then we apply Lemma[7] to $\pi$ and obtain the assignment $\pi'$ that strictly compresses $\pi$. If $\pi'$ is acyclic, we are done; otherwise, we repeat the above procedure with $\pi'$ instead of $\pi$. Because at every step $|E(G_I(\pi'))| < |E(G_I(\pi'))|$ and $G_I(\pi)$ has at most $|T(N)| \cdot |A|$ edges, this process concludes after at most $|T(N)| \cdot |A|$ steps and results in an acyclic assignment that compresses $\pi$.

The following lemma provides the first cornerstone for our algorithm by showing that it is sufficient to consider only acyclic solutions. Intuitively, it is a consequence of Lemma[8] along with the observation that compression preserves stability and individual rationality.
Lemma 9. If \( I \) has a stable assignment, then \( I \) has an acyclic stable assignment.

Proof. Let \( \pi \) be a stable assignment for \( I \). Because of Lemma 8, there is an acyclic assignment \( \pi' \) that compresses \( \pi \). We claim that \( \pi' \) is also a stable assignment. Conditions (C2) and (C3) together with the fact that \( \pi \) is individually rational imply that also \( \pi' \) is individually rational. Moreover, it follows from Condition (C1) that \( PE(I, \pi) = PE(I, \pi') \), which together with Condition (C2) and the stability of \( \pi \) implies the stability of \( \pi' \).

Our next step is the introduction of terminology related to the pattern graphs mentioned at the beginning of this section. Let \( G \) be a bipartite graph with bi-partition \( \{ T(N), A \} \). We say that \( G \) models an assignment \( \pi : N \rightarrow A^* \) if \( G_\pi(\pi) = G \); in this sense every such bipartite graph can be seen as a pattern (or model) for assignments. For a subset \( Q \subseteq T(N) \) we say that \( G \) is compatible with \( Q \) if every vertex in \( Q \) and every vertex in \( A_{\neq \emptyset}(I, Q) \) (recall the definition of \( A_{\neq \emptyset}(I, Q) \) given in Lemma 1) has at least one neighbour in \( G \); note that if \( G \) is compatible with \( Q \) then any assignment \( \pi \) modelled by \( G \) satisfies \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset}(I, Q) \). Intuitively, the graph \( G \) captures information about which types of agents are mapped to which activities (without specifying numbers), while \( Q \) captures information about which agent types are perfectly (i.e., “completely”) assigned.

Let \( Q \subseteq T(N) \) and let \( G \) be a bipartite graph with bi-partition \( \{ T(N), A \} \) that is compatible with \( Q \). The following simple lemma shows that, modulo compatibility requirements, finding a stable assignment for \( I \) can be reduced to finding an individually rational assignment for \( \gamma(I, Q) \) (recall the definition of \( \gamma(I, Q) \) given in Lemma 1).

Lemma 10. Let \( Q \subseteq T(N) \) and let \( G \) be a bipartite graph with bi-partition \( \{ T(N), A \} \) that is compatible with \( Q \). Then for every assignment \( \pi : N \rightarrow A^* \) modelled by \( G \) and compatible with \( Q \), it holds that \( \pi \) is stable for \( I \) if and only if \( \pi \) is individually rational for \( \gamma(I, Q) \).

Proof. Since \( \pi \) is compatible with \( Q \), it follows from Lemma 1 that \( \pi \) is stable for \( I \) if and only if \( \pi \) is individually rational for \( \gamma(I, Q) \) and \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset}(I, Q) \). Consider an arbitrary \( a \in A_{\neq \emptyset}(I, Q) \). Because \( G \) is compatible with \( Q \) it holds that every \( a \) has at least one neighbour in \( G \) and since \( \pi \) is modelled by \( G \), we obtain that \( \pi^{-1}(a) \neq \emptyset \), as required.

The next, final lemma forms (together with Lemma 9) the core component of our proof of Theorem 6.

Lemma 11. Let \( Q \subseteq T(N) \) and let \( G \) be an acyclic bipartite graph with bi-partition \( \{ T(N), A \} \) that is compatible with \( Q \). Then one can decide in time \( O((|N| + |A|)^2|N|^2) \) whether \( I \) has a stable assignment which is modelled by \( G \) and compatible with \( Q \).

Proof. By Lemma 10, \( I \) has a stable assignment that is modelled by \( G \) and compatible with \( Q \) if and only if \( \gamma(I, Q) \) has an individually rational assignment that is modelled by \( G \) and compatible with \( Q \). To determine whether \( \gamma(I, Q) \) has an individually rational assignment that is compatible with \( Q \) and modelled by \( G \), we will employ a reduction to the TSS problem, which can be solved in polynomial-time by Lemma 3.

The reduction proceeds as follows. We will construct an instance \( I' = (T, \lambda) \) of TSS that is a YES-instance if and only if \( \gamma(I, Q) = (N, A, (P_n^v)_{n \in N}) \) has an individually rational assignment that is compatible with \( Q \) and modelled by \( G \). We set \( T \) to be the graph \( G \) and let \( \lambda \) be defined for every \( v \in V(G) \) as follows:

- if \( v \in Q \), then \( \lambda(v) = \{|N_v|\} \),
- if \( v \in T(N) \setminus Q \), then \( \lambda(v) = \{0, \ldots, |N_v| - 1\} \),
- otherwise, i.e., if \( v \in A \), then \( \lambda(v) = \bigcap_{t \in V(G) \setminus \{v, t\}} E(G) P_t^v(v) \).
Note that the reduction can be achieved in time $O((|E(G)|+|N|))$, assuming that the sets $P_l(a)$ for every $t \in T(N)$ and $a \in A$ are given in terms of a data structure that allows to test containment in constant time; such a data structure could for instance be a Boolean array with $|N|$ entries, whose $i$-th entry is TRUE if and only if $i$ is contained in $P_l(a)$.

Because the time to construct $\gamma(I, Q)$ and $A \neq \emptyset(I, Q)$ is at most $O((|N| \cdot |A|)|N|)$ (Lemma 2) and the time to solve $I'$ is at most $O(|V(T)|^2 \cdot |N|^2) = O(|V(G)|^2 \cdot |N|^2) = O((|N|+|A|)^2|N|^2)$ (Lemma 3), we obtain $O((|N|+|A|)^2|N|^2)$ as the total running time of the algorithm.

It remains to show that $I'$ is a $\text{Yes}$-instance if and only if $I$ has an individually rational assignment that is compatible with $Q$ and modelled by $G$. Towards showing the forward direction let $\alpha : E(T) \to \mathbb{N}$ be a solution for $I'$. We claim that the assignment $\pi : N \to A^*$ that for every $e \in \{t,a\}$ of $G$ assigns exactly $\alpha({t,a})$ agents of type $t$ to activity $a$ and assigns all remaining agents (if any) to $a_0$ is an individually rational assignment for $\gamma(I, Q)$ that is compatible with $Q$ and modelled by $G$.

First observe that for every $t \in T(N)$ and every $a \in A$ it holds that $\pi_{t,a} = \alpha({t,a})$ if $\{t,a\} \in E(T)$ and $\pi_{t,a} = 0$ otherwise. Hence $G(I(\pi)) = G = T$ which implies that $\pi$ is modelled by $G$. We show next that $\pi$ is also compatible with $Q$, i.e., $\pi$ satisfies:

(P1) for every $t \in Q$, it holds that $\sum_{a \in A} |\pi_{t,a}| = |N_t|$.

(P2) for every $t \in T(N) \setminus Q$, it holds that $\sum_{a \in A} |\pi_{t,a}| < |N_t|$.

Towards showing (P1) first note that because $G$ is compatible with $Q$, it holds that every $t \in Q$ is adjacent to at least one edge in $G$ and thus also in $T$. Moreover, because $\alpha$ is a solution for $I'$, we obtain that $\sum_{e=\{t,a\} \in E(T)} \alpha(e) = \in \lambda(t) = \{\{N_t\}\}$. Since $\sum_{e=\{t,a\} \in E(T)} \alpha(e) = \sum_{a \in A} |\pi_{t,a}|$, we obtain (P1).

Towards showing (P2) let $t \in T(N) \setminus Q$. If $t$ is isolated in $T$ then $\sum_{a \in A} |\pi_{t,a}| = 0$ but $N_t \neq \emptyset$ and hence $\sum_{a \in A} |\pi_{t,a}| < |N_t|$, as required. If on the other hand $t$ is not isolated in $T$ then because $\alpha$ is a solution for $I'$, we obtain that $\sum_{e=\{t,a\} \in E(T)} \alpha(e) = \in \lambda(t) = \{0, \ldots, |N_t| - 1\}$. Since $\sum_{e=\{t,a\} \in E(T)} \alpha(e) = \sum_{a \in A} |\pi_{t,a}|$, we obtain (P2).

Finally it remains to show that $\pi$ is individually rational for $\gamma(I, Q)$, i.e., for every $a \in A$ and $t \in T(N)$, it holds that if $\pi_{t,a} \neq 0$ then $|\pi^{-1}(a)| \in P_l'(a)$. Let $a \in A$. If $a$ has no neighbour in $T$ then $\pi_{t,a} = 0$ for every $t \in T(N)$ and the claim holds. Hence let $t_1, \ldots, t_l$ be the neighbours of $a$ in $T$. Then because $\alpha$ is a solution for $I'$, we obtain:

$$|\pi^{-1}(a)| = \sum_{1 \leq i \leq l} |\pi_{t_i,a}|$$

$$\quad = \sum_{1 \leq i \leq l} \alpha({t_i,a})$$

$$\quad \in \lambda(a) = \bigcap_{1 \leq i \leq l} P_{t_i}'(a)$$

Hence for every $1 \leq i \leq l$ it holds that $|\pi^{-1}(a)| \in P_{t_i}'(a)$, as required.

Towards showing the reverse direction let $\pi$ be an individually rational assignment for $\gamma(I, Q)$ that is compatible with $Q$ and modelled by $G$. We claim that the assignment $\alpha : E(G) \to \mathbb{N}$ with $\alpha({t,a}) = \pi_{t,a}$ for every $\{t,a\} \in E(T)$ is a solution for $I'$. Observe that because $\pi$ is modelled by $G$, we have that $\pi_{t,a} \neq 0$ if and only if $\{t,a\} \in E(G)$. Hence $|\pi_t| = \sum_{a \in A} |\pi_{t,a}| = \sum_{\{t,a\} \in E(T)} \alpha(t,a)$ for every $t \in T(i)$ and $|\pi^{-1}(a)| = \sum_{t \in T(N)} |\pi_{t,a}| = \sum_{\{t,a\} \in E(T)} \alpha({t,a})$. Since $\pi$ is compatible with $Q$ we obtain that $\pi_t = |N_t|$ for every $t \in Q$ and hence $\alpha(t) = |N_t| \in \lambda(t)$. Moreover for every $t \in T(N) \setminus Q$ it holds that $\pi_t < |N_t|$ and hence $\alpha(t) \in \lambda(t)$. Because $\pi$ is an individually rational assignment, it holds that $|\pi^{-1}(a)| \in \bigcap_{t \in T(\pi^{-1}(a))} P_{t}'(a)$ and thus $\sum_{\{t,a\} \in E(T)} \alpha({t,a}) \in \lambda(a)$, as required. □
We are now ready to establish Theorem 6.

Proof of Theorem 6. Let $I = (N, A, (P_n)_{n \in N})$ be the given instance of SGASP. It follows from Lemma 2 that it suffices to decide whether $I$ has an acyclic stable assignment. Observe that every acyclic stable assignment $\pi$ is compatible with $PE(I, \pi)$ and the acyclic bipartite graph $G_I(\pi)$. Hence there is an acyclic stable assignment $\pi$ if and only if there is a set $Q \subseteq T(N)$ and an acyclic bipartite graph $G$ with bi-partition \{\{T(N), A\} compatible with $Q$ such that there is a stable assignment for $I$ modelled by $G$ and compatible with $Q$.

Consequently, we can determine the existence of a stable assignment for $I$ by first branching over every $Q \subseteq T(N)$, then over every acyclic bipartite graph $G$ with bi-partition \{\{T(N), A\} compatible with $Q$, and checking whether $I$ has a stable assignment that is compatible with $Q$ and modelled by $G$. Since there are $2^{|T(N)|}$ many subsets $Q$ of $T(N)$ and at most $2^{|T(N)|+|A|}$ (acyclic) bipartite graphs $G$, and we can determine whether $I$ has a stable assignment compatible with $Q$ and modelled by $G$ in time $O((|N| + |A|)^2 N^2)$ (see Lemma 11), it follows that the total running time of the algorithm is at most $O(2^{|T(N)|(|N| + |A|)^2 N^2})$. 

\[\Box\]

5 Result 2: Lower Bound for SGASP

In this subsection we complement Theorem 6 by showing that if we drop the number of activities in the parameterization, then SGASP becomes W[1]-hard. We achieve this via a parameterized reduction from SMPSS that we have shown to be strongly W[1]-hard in Theorem 5.

Theorem 12. SGASP is W[1]-hard parameterized by the number of agent types.

Proof. Let $(d, \bar{t}, \mathcal{P})$ with $\mathcal{P} = (P_1, \ldots, P_m)$ be an instance of SMPSS. Because SMPSS is strongly W[1]-hard, we can assume that all numbers of the instance $(d, \bar{t}, \mathcal{P})$, i.e., the values of all components of the vectors in $\{\bar{t}\} \cup \bigcup_{P \in \mathcal{P}} P$, are encoded in unary. We will also assume that all non-zero components of the vectors $\{\bar{t}\} \cup \bigcup_{P \in \mathcal{P}} P$ are at least 3 (this can for instance be achieved by multiplying every vector in $\{\bar{t}\} \cup \bigcup_{P \in \mathcal{P}} P$ with the number 3). We will now construct the instance $I = (N, A, (P_n)_{n \in N})$ of SGASP in polynomial time with $|T(I)| = d + 3$ such that $(d, \bar{t}, \mathcal{P})$ has a solution if and only if so does $I$. $I$ has one agent type $t_i$ for every $1 \leq i \leq d$ that comes with $\bar{t}[i]$ agents as well as three additional agent types $t_p, t^1_1 \neq \emptyset$, and $t^2_2 \neq \emptyset$ having one agent each. The last three agent types will be employed to ensure that every agent type $t_i$ must be perfectly assigned and every activity must be non-empty. Moreover, $I$ has one activity $a_\ell$ for every $\ell$ with $1 \leq \ell \leq m$ as well as one activity $a_P$, which will be used in conjunction with the agent type $t_P$ to ensure that all agent types $t_i$ must be perfectly assigned. The approval set for the agent type $t_i$ w.r.t. an activity $a_\ell$ is given by $P_{t_i}(a_\ell) = \{\bar{v} | \bar{v} \in P_\ell \land \bar{v}[i] \neq 0\}$. Note that because all agents in $\mathcal{P}$ are simple, it holds that $P_{t_i}(a_\ell) \cap P_{t_i}(a_{\ell'}) = \emptyset$ for every $i, \ell, j \leq i, j \neq i, j$, and $1 \leq \ell \leq m$ and hence every such activity $a_\ell$ is populated by agents of at most one type in any individual rational assignment for $I$. Finally, we set $P_{t_i}(a_P) = \{2\}$ for every $1 \leq i \leq d$, $P_{t_P}(a_P) = \{1, 3\}$, $P^1_{t^1_1}(a) = \{1\}$ and $P^2_{t^2_2}(a) = \{2\}$ for every activity $a \in A \setminus \{a_P\}$. Note that indeed $|T(I)| = d + 3$ and $I$ can be constructed in polynomial-time (recall that we assumed that all numbers of $(d, \bar{t}, \mathcal{P})$ are encoded in unary). It remains to show that $(d, \bar{t}, \mathcal{P})$ has a solution if and only if so does $I$.

Towards showing the forward direction, let $p_1, \ldots, p_m$ with $p_\ell \in P_\ell$ and $\sum_{\ell=1}^m p_\ell = \bar{t}$ be a solution for $(d, \bar{t}, \mathcal{P})$. Let $\pi : N \to A^*$ be the assignment defined as follows:

- For every agent type $t_i$ and every vector $\bar{p}_\ell$ such that $\bar{p}_\ell[i] \neq \emptyset, \pi$ assigns exactly $\bar{p}_\ell[i]$ agents to activity $a_j$.
- $\pi(n_P) = \{a_P\}$, where $n_P$ is the unique agent of type $t_P$. 

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\[ \pi(n^1_{\neq \emptyset}) = \pi(n^2_{\neq \emptyset}) = a_0, \text{ where } n^1_{\neq \emptyset} \text{ and } n^2_{\neq \emptyset} \text{ are the unique agents having type } t^1_{\neq \emptyset} \text{ and } t^2_{\neq \emptyset}, \text{ respectively.} \]

We claim that \( \pi \) is a stable assignment for \( I \), which we will show using the following sequence of observations:

(O1) Due to the construction of \( \pi \), we obtain that \( |\pi_{t_i,a_t}| = \bar{p}_\ell[i] \) for every \( i \) and \( j \) with \( 1 \leq i \leq d \) and \( 1 \leq \ell \leq m \).

(O2) Because of (O1) and the fact that \( \sum_{\ell=1}^m \bar{p}_\ell = \bar{t} \), we obtain that \( |\pi_{t_i}| = \bar{t}[i] \) for every \( i \) with \( 1 \leq i \leq d \). Since furthermore \( I \) has exactly \( \bar{t}[i] \) agents of type \( t_i \), we obtain that the agents of type \( t_i \) are perfectly assigned by \( \pi \).

(O3) Because every set in \( \mathcal{P} \) is simple, it also holds that \( \pi^{-1}(a_\ell) \) consists of exactly \( \bar{p}_\ell[i] \) agents of type \( i \), where \( i \) is the unique non-zero component of \( \bar{p}_\ell \).

(O4) Since \( \pi^{-1}(n_P) = \{n_P\} \) the assignment \( \pi \) is stable for the agent \( n_P \).

(O5) Since \( \pi(n^1_{\neq \emptyset}) = \pi(n^2_{\neq \emptyset}) = a_0 \) and \( |\pi^{-1}(a)| \geq 2 \) for every activity \( a \in A \setminus \{a_P\} \) (because of (O3)), it holds that the assignment \( \pi \) is stable for the agents \( n^1_{\neq \emptyset} \) and \( n^2_{\neq \emptyset} \).

(O6) Consider an agent \( n \) of type \( t_i \). Because of (O2), we have that \( \pi(n) \neq a_0 \). Hence \( \pi(n) = a_\ell \) for some \( \ell \) with \( 1 \leq \ell \leq m \) and since \( |\pi^{-1}(a_\ell)| = \bar{p}_\ell[i] \) (because of (O3)), we have that \( |\pi^{-1}(a_\ell)| \in P_n(a_\ell) \), which implies that \( \pi \) is a stable assignment for \( n \).

Consequently, \( \pi \) is a stable assignment for \( I \).

Towards showing the reverse direction, let \( \pi : N \to A^* \) be a stable assignment for \( I \). We start by showing that \( \pi \) satisfies the following two properties:

(P1) for every \( i \) with \( 1 \leq i \leq d \), all agents of type \( t_i \) are assigned to some activity in \( A \setminus \{a_P\} \).

(P2) for every activity \( a \in A \setminus \{a_P\} \), it holds that \( \pi^{-1}(a) \neq \emptyset \) and moreover \( T(\pi^{-1}(a)) \subseteq \{t_1, \ldots, t_d\} \) and \( |T(\pi^{-1}(a))| = 1 \).

We will show (P1) and (P2) using the following series of claims that hold for any stable assignment \( \pi : N \to A \) for \( I \):

(C1) \( \pi(n^2_{\neq \emptyset}) = a_0 \) for the unique agent \( n^2_{\neq \emptyset} \) of type \( t^2_{\neq \emptyset} \).

(C2) \( \pi(n^1_{\neq \emptyset}) = a_0 \) for the unique agent \( n^1_{\neq \emptyset} \) of type \( t^1_{\neq \emptyset} \).

(C3) \( \pi(n_P) = a_P \) for the unique agent \( n_P \) of type \( t_P \).

(C4) \( \pi^{-1}(a_P) = \{n_P\} \).

Towards showing (C1), assume for the contrary that \( \pi(n^2_{\neq \emptyset}) = a \) for some \( a \in A \). Then \( a \in A \setminus \{a_P\} \) and \( |\pi^{-1}(a)| = 2 \). However, this is not possible since \( n^2_{\neq \emptyset} \) is the only agent in \( I \) that approves size 2 for any activity in \( A \setminus \{a_P\} \).

Towards showing (C2), assume for the contrary that \( \pi(n^1_{\neq \emptyset}) = a \) for some \( a \in A \). Then \( a \in A \setminus \{a_P\} \) and \( |\pi^{-1}(a)| = 1 \). Moreover, because of (C1), we have that \( \pi(n^2_{\neq \emptyset}) = a_0 \) and hence \( n^2_{\neq \emptyset} \) would prefer \( a \) over his current assignment, contradicting the stability of \( \pi \).

Towards showing (C3), assume for the contrary that \( \pi(n_P) \neq a_P \). Then \( \pi(n_P) = a_0 \). Moreover, due to the approval set of \( n_P \), it must hold that \( \pi^{-1}(a_P) \neq \emptyset \). Since \( P_\ell(a_P) \in \{\emptyset, \{2\}\} \) for every agent type
in \( T(I) \setminus \{t_P\} \), it follows that \( |\pi^{-1}(a_P)| = 2 \). However, this contradicts the stability of \( \pi \), since \( n_P \) would prefer \( a_P \) over his current assignment. (C4) is a direct consequence of (C3) since \( P_{n_P}(a_P) = \{1, 3\} \) and moreover \( n_P \) is the only agent with \( 3 \in P_{n_P}(a_P) \).

We are now ready to show (P1) and (P2). Towards showing (P1) assume for a contradiction that there is an agent \( n \) whose type is in \( \{t_1, \ldots, t_d\} \) such that \( \pi(n) \in \{a_0, a_P\} \). Because of (C4), we obtain that \( \pi(n) = a_0 \). Moreover, since \( 2 \in P_n(a_P) \) and \( |\pi^{-1}(a_P)| = 1 \) (because of (C4)), it follows that \( n \) would prefer \( a_P \) over his current assignment, which contradicts the stability of \( \pi \).

Towards showing (P2) assume for a contradiction that there is an activity \( a \in A \setminus \{a_P\} \) with \( \pi^{-1}(a) = \emptyset \). Consider the agent \( n_1 \neq \emptyset \), i.e., the only agent of type \( t_1 \neq \emptyset \), then because of (C2), we have \( \pi(n_1) = a_0 \). Moreover, since \( 1 \in P_{n_1}(a) \) for every \( a \in A \setminus \{a_P\} \), the agent \( n_1 \neq \emptyset \) would prefer \( a \) over his current assignment, which contradicts the stability of \( \pi \). Hence \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A \setminus \{a_P\} \). Furthermore, since the agent types in \( \{t_1, \ldots, t_d, t_1^1 \neq \emptyset, t_2^1 \neq \emptyset\} \) are the only types that approve of an activity in \( A \setminus \{a_P\} \) and it follows from (C1) and (C2) that neither \( n_1^1 \neq \emptyset \) nor \( n_2^1 \neq \emptyset \) are assigned to an activity in \( A \setminus \{a_P\} \), we obtain that \( T(\pi^{-1}(a)) \subseteq \{t_1, \ldots, t_d\} \). It remains to show that \( |T(\pi^{-1}(a))| = 1 \) for every \( 1 \leq \ell \leq m \). Assume for a contradiction that there are two distinct agent types \( t \) and \( t' \) in \( \{t_1, \ldots, t_d\} \) with \( t, t' \in T(\pi^{-1}(a)) \). Since \( \pi \) is individual rational, we obtain that \( |\pi^{-1}(a_t)| \in P_t(a_t) \) and \( |\pi^{-1}(a_{t'})| \in P_t(a_{t'}). \) Hence the set \( P_t \) in \( P \) contains two vectors that share the same value at their non-zero component, which contradicts our assumption that \( P_t \) is a simple set. This concludes the proof for (P1) and (P2) and we are now ready to complete the proof of the reverse direction.

Consider an activity \( a_q \) for some \( 1 \leq \ell \leq m \). Because of (P2), we obtain that all agents in \( \pi^{-1}(a_q) \) have the same type say \( t_i \). Because \( \pi \) is stable it holds that \( |\pi^{-1}(a_q)| \in P_t(a_q) \) and hence there is a vector, say \( \bar{p}_t \), in \( P_t \) with \( \bar{p}_t[i] = |\pi^{-1}(a_q)| \). We claim that the vectors \( \bar{p}_1, \ldots, \bar{p}_m \) chosen in this way form a solution for \( (d, t, P) \). Consider a component \( i \) with \( 1 \leq i \leq d \), then because of (P1), we obtain that \( \sum_{j=1}^m \bar{p}_j[i] \) is equal to the number of agents of type \( t_i \), which in turn is equal to \( \ell[i] \) by the construction of \( I \). Hence \( \sum_{j=1}^m \bar{p}_j = \ell \) and \( \bar{p}_1, \ldots, \bar{p}_m \) is a solution for \( (d, t, P) \).

6 Result 3: XP Algorithms for SGASP and GASP

In this section, we present our XP algorithm for GASP parameterized by the number of agent types. In order to obtain this result, we observe that the stability of an assignment for GASP can be decided by only considering the stability of agents that are assigned to a “minimal alternative” w.r.t. their type. We then show that once one guesses (i.e., branches over) a minimal alternative for every agent type, the problem of finding a stable assignment for GASP that is compatible with this guess can be reduced to the problem of finding a perfect and individual rational assignment for a certain instance of sGASP, where one additionally requires that certain activities are assigned to at least one agent. Our first task will hence be to obtain an XP algorithm which can find such a perfect and individually rational assignment for SGASP.

6.1 An XP Algorithm for SGASP

The aim of this section is twofold. First of all, we obtain Lemma [3] which allows us to find certain individually rational assignments in SGASP instances and forms a core part of our XP algorithm for GASP.

Lemma 13. Let \( I = (N, A, (P_n)_{n \in N}) \) be an instance of SGASP, \( Q \subseteq T(N) \), and \( A_{\neq \emptyset} \subseteq A \). Then one can decide in time \( O(|A| \cdot (|N|)^{|T(N)|}) \) whether \( I \) has an individual rational assignment \( \pi \) that is compatible with \( Q \) such that \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \).

Proof. Let \( k = |T(N)| \) and \( r = \max_{i \in T(N)} |N_i| \). We construct an instance \( I' = (k, P, t) \) of MPSS such that \( I' \) is a YES-instance if and only if \( I = (N, A, (P_n)_{n \in N}) \) has an individually rational assignment.
\( \pi : N \to A^* \) that is compatible with \( Q \) and \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \). Let \( T(N) = \{ t_1, \ldots, t_k \} \). The set \( P \) contains one set \( P_a \) for every activity \( a \in A \), defined as follows. For every number \( p \in \bigcup_{t \in T(N)} P_t(a) \) the set \( P_a \) contains the set of all vectors \( \bar{p} \in [\ast]_0 \) such that \( |\bar{p}| = p \) and \( \bar{p}[i] = 0 \) for every \( i \) with \( 1 \leq i \leq k \) such that \( p \notin P_t(a) \). Moreover, if \( a \notin A_{\neq \emptyset} \), then the set \( P_a \) additionally contains the all-zero vector 0. This completes the construction of \( I' \).

We claim that \( I \) has an individually rational assignment \( \pi : N \to A^* \) that is compatible with \( Q \) and \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \) if and only if the solution \( T \) for \( I' \) contains a vector \( \bar{t} \) with \( \bar{t}[i] = |N_{t_i}| \) for every \( t_i \in Q \) and \( \bar{t}[i] < |N_{t_i}| \) otherwise (i.e., for every \( t_i \in T(N) \setminus Q \)). Note that establishing this claim completes the proof of the lemma since \( I' \) can be constructed in time \( O(r^k) \) and solved in time \( O(|A| \cdot r^k) \) by Lemma 4.

Towards showing the forward direction, let \( \pi : N \to A^* \) be an individually rational assignment for \( I \) that is compatible with \( Q \) and \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \). For every \( a \in A \) let \( \bar{p}_a \) be the vector with \( \bar{p}_a[i] = |\pi_{t_i,a}| \) for every \( i \) with \( 1 \leq i \leq k \). Note that if \( \bar{p}_a \neq 0 \) then \( \bar{p}_a \in P_a \) for every \( a \in A \) because \( \pi \) is individually rational. On the other hand, if \( \bar{p}_a = 0 \) then \( a \notin A_{\neq \emptyset} \) and we also obtain \( \bar{p}_a \in P_a \). Hence the vector \( \bar{t} = \sum_{a \in A} \bar{p}_a \) is in the solution for \( I' \) and moreover \( \bar{t}[i] = \sum_{a \in A} \bar{p}_a[i] = \sum_{a \in A} |\pi_{t_i,a}| = |\pi_{t_i}| \) for every \( i \) with \( 1 \leq i \leq k \). Finally, because \( \pi \) is compatible with \( Q \), we obtain that \( \bar{t}[i] = |\pi_{t_i}| = |N_{t_i}| \) for every \( i \) with \( t_i \in Q \) and also \( \bar{t}[i] = |\pi_{t_i}| < |N_{t_i}| \) for every \( i \) with \( t_i \in T(N) \setminus Q \), as required.

Towards showing the reverse direction, assume that the solution \( T \) for \( I' \) contains a vector \( \bar{t} \) with \( \bar{t}[i] = |N_{t_i}| \) for every \( t_i \in Q \) and \( \bar{t}[i] < |N_{t_i}| \) otherwise (i.e., for every \( t_i \in T(N) \setminus Q \)) and for every \( a \in A \) let \( \bar{p}_a \) be the vector in \( P_a \) such that \( \sum_{a \in A} \bar{p}_a = \bar{t} \). We claim that the assignment \( \pi : N \to A^* \) that for every \( 1 \leq i \leq k \) and every \( a \in A \) assigns exactly \( \bar{p}_a[i] \) agents of type \( t_i \) to activity \( a \) and all remaining agents to activity \( a_{\emptyset} \) is an individually rational assignment for \( I \) that is compatible with \( Q \) and \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \). First observe that for every \( i \) with \( 1 \leq i \leq k \), it holds that \( |\pi_{t_i}| = \sum_{a \in A} |\pi_{t_i,a}| = \sum_{a \in A} \bar{p}_a[i] = \bar{t}[i] \). Note also that because \( \bar{t}[i] \leq |N_{t_i}| \), i.e., there is a sufficient number of agents for every agent type, we know that it is possible to assign the agents according to \( \pi \). Since in addition it holds that \( \bar{t}[i] = |N_{t_i}| \) if \( t_i \in Q \) and \( \bar{t}[i] < |N_{t_i}| \) if \( t_i \notin Q \), it follows that \( \pi \) is compatible with \( Q \). Moreover, because \( \emptyset \notin P_a \) for every \( a \in A_{\neq \emptyset} \), it also holds that \( |\pi^{-1}(a)| = \sum_{1 \leq i \leq k} |\pi_{t_i,a}| = \sum_{1 \leq i \leq k} \bar{p}_a[i] \neq 0 \) for every such \( a \). It remains to show that \( \pi \) is individually rational for \( I \). By the definition of \( \pi \) it holds that whenever \( \pi \) assigns an agent of type \( t_i \) to some activity \( a \in A \), then \( \bar{p}_a[i] \neq 0 \). Moreover, by the construction of \( I' \) it holds that if \( \bar{p}_a[i] \neq 0 \) then \( |\bar{p}_a| \in P_t(a) \). Hence because \( |\pi^{-1}(a)| = |\bar{p}_a| \), we obtain that \( |\pi^{-1}(a)| \in P_t(a) \).

As a secondary result, we can already obtain an XP algorithm for sGASP parameterized by the number of agent types. This may also be of interest, as the obtained running time is strictly better than that of the algorithm obtained for the more general GASP. The last thing we need for this result is the following corollary, obtained as a direct consequence of Lemma 4 and Lemma 13.

**Corollary 14.** Let \( I = (N, A, (P_n)_{n \in N}) \) be an instance of sGASP and \( Q \subseteq T(N) \). Then one can decide in time \( O(|N|^2 \cdot |A| + |A| \cdot (|N|)^{|T(N)|}) \) whether \( I \) has a stable assignment compatible with \( Q \).

We can now prove the following.

**Theorem 15.** An instance \( I = (N, A, (P_n)_{n \in N}) \) of sGASP can be solved in time \( |A| \cdot |N|^{|O(|T(N)|)} \).

**Proof.** Let \( I = (N, A, (P_n)_{n \in N}) \) be the given instance of sGASP. The algorithm loops through every \( Q \subseteq T(N) \) and in each branch checks whether \( I \) has a stable assignment that is compatible with \( Q \). Note that due to Corollary 4, this can be achieved in time \( O(|N|^2 \cdot |A| + |A| \cdot (|N|)^{|T(N)|}) \) for every \( Q \subseteq T(N) \). Since there are \( 2^{|T(N)|} \) subsets of \( T(N) \), we obtain \( O(2^{|T(N)|} \cdot (|N|^2 \cdot |A| + |A| \cdot (|N|)^{|T(N)|})) = |A| \cdot |N|^{|O(|T(N)|)} \) as the total running time of the algorithm. □
6.2 An XP algorithm for GASP

Our aim here is to use Lemma\textsuperscript{13} to obtain an XP algorithm for GASP. To simplify the presentation of our algorithm, we start by introducing the notion of an NS\textsuperscript{*}-deviations that combines and unifies individual rationality and NS-deviations. Namely, let $I = (N, A, (\succeq_n)_{n \in N})$ be a GASP instance, $\pi : N \to A^{*}$ be an assignment, and $n \in N$. Then we say that $n$ has an NS\textsuperscript{*}-deviation to an activity $a' \in A^{*} \setminus \{\pi(n)\}$ if
\[ (a', |\pi^{-1}(a')| + 1) \succ_{T(n)} (a, |\pi^{-1}(a)|). \]
Note that in order to deal with the case that $a' = a_{\emptyset}$, we define $(a_{\emptyset}, i + 1)$ to be equal to $(a_{\emptyset}, 1)$ for every $i$.

Observation 16. An assignment $\pi$ for $I$ is stable if and only if no agent $n \in N$ has an NS\textsuperscript{*}-deviation to any activity in $A^{*} \setminus \{\pi(n)\}$.

Let $I$ and $\pi$ be as above and let $t \in T(I)$. We denote by $\pi_i^t$ the set of activities $\pi_i$ if $t$ is perfectly assigned by $\pi$ and $\pi_i \cup \{a_{\emptyset}\}$, otherwise. We say an activity $a \in \pi_i^t$ is minimal with respect to $t$ if $(a', |\pi^{-1}(a')|) \succeq_t (a, |\pi^{-1}(a)|)$ for each $a' \in \pi_i^t$ and we address the alternative $(a, |\pi^{-1}(a)|)$ as a minimal alternative with respect to $t$. The following lemma provides our first key insight, by showing that only agents assigned to a minimal activity need to be checked for having an NS-deviation.

Lemma 17. Let $\pi$ be an assignment and let $n$ be an agent of type $t$ that has an NS\textsuperscript{*}-deviation. Then there is an agent $n'$ of type $t$ that also has an NS\textsuperscript{*}-deviation and is assigned to a minimal activity w.r.t. $t$.

Proof. Let $a = \pi(n)$, then because $n$ has an NS\textsuperscript{*}-deviation, there is an activity $a' \in A^{*} \setminus \{a\}$ such that $(a', |\pi^{-1}(a')| + 1) \succ_t (a, |\pi^{-1}(a)|)$. Let $a_m$ be a minimal activity w.r.t. $t$. Then because $(a, |\pi^{-1}(a)|) \succeq_t (a_m, |\pi^{-1}(a_m)|)$ any agent assigned to $a_m$ does also have an NS\textsuperscript{*}-deviation to $a_m$. $\square$

The following lemma is a direct consequence of Observation\textsuperscript{16} and Lemma\textsuperscript{17} and allows us to characterize the stability condition of an assignment in terms of minimal activities for each agent type.

Lemma 18. An assignment $\pi$ for $I$ is stable if and only if for each $t \in T(N)$ and each $a \in A^{*} \setminus \{a_m\}$, it holds that $(a_m, |\pi^{-1}(a_m)|) \succeq_t (a, |\pi^{-1}(a)| + 1)$, where $a_m$ is a minimal activity w.r.t. $t$.

Proof. It follows from Lemma\textsuperscript{17} and Observation\textsuperscript{16} that $\pi$ is not stable if and only if there is an agent $n$ participating in a minimal activity $a_m$ w.r.t. $T(n)$ that has an NS\textsuperscript{*}-deviation. This implies that $(a, |\pi^{-1}(a)| + 1) \succ_{T(n)} (a_m, |\pi^{-1}(a_m)|)$ for some $a \in A^{*} \setminus \{a_m\}$. $\square$

The following theorem now employs the above lemma to construct an instance $I'$ of SGASP together with a subset $A_{\not=\emptyset}$ of activities such that for every function $f_{\min} : T(I) \to X$ (or in other words for every guess of minimal alternatives in an assignment), it holds that $I$ has a stable assignment such that $f_{\min}(t)$ is a minimal alternative w.r.t. $t$ for every $t \in T(I)$ if and only if $I'$ has a perfect and individual rational assignment $\pi$ such that $\pi^{-1}(a) \not= \emptyset$ for every $a \in A_{\not=\emptyset}$. For brevity, we will say that an assignment $\pi$ is compatible with $f_{\min}$ if and only if $f_{\min}(t)$ is a minimal alternative w.r.t. $t$ for every $t \in T(I)$.

Theorem 19. Let $I = (N, A, (\succeq_n)_{n \in N})$ be an instance of GASP and let $f_{\min} : T(N) \to X$, which informally represents a guess of a minimal alternative for every agent type. Then one can in time $O(|N|^2 |A|)$ construct an instance $I' = (N, A \cup \{a_{\emptyset}\}, (P_n)_{n \in N})$ of SGASP together with a subset $A_{\not=\emptyset}$ of activities such that $|T(I')| \leq 2|T(I)|$ and $I$ has a stable assignment compatible with $f_{\min}(t)$ if and only if $I'$ has a perfect individual rational assignment $\pi$ with $\pi^{-1}(a) \not= \emptyset$ for every $a \in A_{\not=\emptyset}$.

Proof. As a first step we remove all alternatives from $X$ that violate the stability condition of an assignment $\pi$ for $I$ having $f_{\min}(t)$ as a minimal alternative w.r.t. $t$ for every $t \in T(I)$ given in Lemma\textsuperscript{18}. Namely, let $X'$ be the set of alternatives obtained from $X \setminus \{(a_{\emptyset}, 1)\}$ after removing all alternatives $(a, i) \in X$ for
which there is an agent type \( t \in T(I) \) such that \((a, i + 1) \succ_t f_{\min}(t)\). We will also assume w.l.o.g. that \( f_{\min}(t) \succeq_t (a_\emptyset, 1) \) and if \((a_\emptyset, 1) \succeq_t f_{\min}(t) \succeq_t (a_\emptyset, 1)\), then \( f_{\min}(t) = (a_\emptyset, 1)\).

Note that the activity \( a_\emptyset \) of \( I' \) will take the place of the activity \( a_\emptyset \) of \( I \). We define the preferences \((P_n)_{n \in N} \) for \( I' \) as follows. Let \( t \in T(I) \), let \((a_m, i_m) = f_{\min}(t)\) and let \( n_t \in N_t \) be an arbitrary agent of type \( t \). Then we set \( P_{n_t} = \{ (a_m, i_m) \} \) if \( a_m \neq a_\emptyset \) and \( P_{n_t} = \{ (a_j, j) \mid 1 \leq j \leq |N| \} \) otherwise. Informally, this will ensure that \( \pi(n_t) = a_m \) and \((a_m, \pi^{-1}(a_m)) = (a_m, i_m)\) if \( a_m \neq a_\emptyset \) and \( \pi(n_t) = a_\emptyset \) otherwise for any perfect assignment \( \pi \) for \( I' \).

Moreover, for every other agent \( n \) of type \( t \), i.e., \( n \in N_t \) and \( n \neq n_t \), we set \( P_n = \{ (a, i) \in X' \mid (a, i) \succeq_t f_{\min}(t) \} \) if \( a_m \neq a_\emptyset \) and \( P_n = \{ (a, i) \in X' \mid (a, i) \succeq_t f_{\min}(t) \} \cup \{ (a_j, j) \mid 1 \leq j \leq |N| \} \} \) otherwise. Informally, this ensures that only alternatives that are at least as preferred as \( f_{\min}(t) \) and that if \( a_m = a_\emptyset \), then the agents can be assigned to \( a_\emptyset \) for any size. Finally, the set \( A \neq \emptyset \) contains all activities \( a \in A \) for which there is an agent type \( t \) such that \((a, 1) \succ_t f_{\min}(t)\), i.e., all activities that must be non-empty for any stable assignment \( \pi \) for \( I \).

Clearly the construction of \( I' \) can be achieved in time \( \mathcal{O}(|N|^2|A|) \) and moreover \( |T(I')| \leq 2|T(I)| \). It remains to show that for every choice of \( f_{\min} \), \( I \) has a stable assignment \( \pi \) compatible with \( f_{\min}(t) \) if and only if \( I' \) has a perfect individual rational assignment \( \pi \) with \( \pi^{-1}(a) \neq \emptyset \) for every \( a \in A \neq \emptyset \).

Towards showing the forward direction, let \( \pi \) be a stable assignment for \( I \) compatible with \( f_{\min}(t) \). Note that, w.l.o.g., we can assume that \( \pi(n_t) = a_m \), where \((a_m, i_m) = f_{\min}(t)\) for every \( t \in T(I) \) since otherwise we can switch the assignment for \( n_t \) with some agent that is assigned to \( a_m \), which exists because \( \pi \) is compatible with \( f_{\min} \). We claim that \( \pi_1 : N \to (A \cup \{a_\emptyset\})^* \) such that for every \( n \in N \), \( \pi_1(n) = \pi(n) \) if \( \pi(n) \neq a_\emptyset \) and \( \pi_1(n) = a_\emptyset \) otherwise, is a perfect individual rational assignment for \( I' \) such that \( \pi_1^{-1}(a) \neq \emptyset \) for every \( a \in A \neq \emptyset \). The assignment \( \pi_1 \) is obviously perfect. Towards showing that \( \pi_1 \) is a perfect individual rational assignment for a contradiction that there is an agent \( n \in N \) say of type \( t \) with \( \pi_1(n) = a \neq a_\emptyset \) such that \( |\pi_1^{-1}(a)| \neq P_{n_t}(a) \). If \( n = n_t \) then by our assumption on \( \pi \), we have that \( \pi(n) = a_m \) and \( |\pi^{-1}(a_m)| = i_m \), where \((a_m, i_m) = f_{\min}(t)\). Hence, if \( a_m \neq a_\emptyset \) then \( \pi_1(n) = a_m \) and \( |\pi_1^{-1}(a_m)| = i_m \), which because \( P_{n_t} = \{ (a_m, i_m) \} \), implies that \( |\pi_1^{-1}(a_m)| = i_m = P_{n_t}(a_m) \). If, on the other hand, \( a_m = a_\emptyset \), then \( \pi_1(n) = a_\emptyset \), which because \( P_{n_t} = \{ (a_\emptyset, j) \mid 1 \leq j \leq |N| \} \} \) implies that \( |\pi_1^{-1}(a_\emptyset)| = |\pi_1^{-1}(a_\emptyset)| \in P_{n_t}(a_\emptyset) \).

Finally consider the case that \( n \neq n_t \). Note that if \( \pi_1(n) = a_\emptyset \), then \( \pi(n) = a_\emptyset \), which implies that \( (a_\emptyset, 1) = f_{\min}(t) \) and hence because \( P_n = \{ (a_\emptyset, j) \mid 1 \leq j \leq |N| \} \}, we obtain \( (a_\emptyset, |\pi_1^{-1}(a_\emptyset)|) \in P_n \). Moreover, if \( \pi_1(n) = a \neq a_\emptyset \), then \( |\pi_1^{-1}(a)| \neq P_{n_t}(a) \) implies that either \( (a, |\pi_1^{-1}(a)|) \notin X' \) or \( f_{\min}(t) \succeq_t (a, |\pi_1^{-1}(a)|) \) in the former case \( \pi \) is not stable and in the latter case \( \pi \) is not compatible with \( f_{\min} \). Finally, it remains to show that \( |\pi_1^{-1}(a)| \neq \emptyset \) for every \( a \in A \neq \emptyset \). Assume for a contradiction that there is an activity \( a \in A \neq \emptyset \) such that \( \pi_1^{-1}(a) = \emptyset \). Note that then also \( \pi^{-1}(a) = \emptyset \) and because of the definition of \( A \neq \emptyset \), it follows that there is an agent type \( t \) such that \((a, 1) \succ_t f_{\min}(t) = (a_m, i_m) \). But this contradicts the stability of \( \pi \) since now the at least one agent assigned to the minimal activity \( a_m \) has an NS\(^*\)-deviation to \( a \) in \( I \).

Towards showing the reverse direction, let \( \pi_1 \) be a perfect individual rational assignment for \( I' \) such that \( \pi_1^{-1}(a) \neq \emptyset \) for every \( a \in A \neq \emptyset \). We claim that the assignment \( \pi \) with \( \pi(n) = \pi_1(n) \) if \( \pi_1(n) \neq a_\emptyset \) and \( \pi(n) = a_\emptyset \) otherwise is a stable assignment for \( I \) that is compatible with \( f_{\min} \); note that because \( \pi_1 \) is perfect it cannot be the case that \( \pi(n) = a_\emptyset \). We start by showing that \( \pi \) is compatible with \( f_{\min} \), for every \( t \in T(I) \). We distinguish two cases depending on whether or not \((a_\emptyset, 1) = f_{\min}(t) \). If \((a_\emptyset, 1) = f_{\min}(t) \), then \( P_{n_t} = \{ (a_\emptyset, j) \mid 1 \leq j \leq |N| \} \} \) and hence because \( \pi_1 \) is perfect, we obtain that \( \pi_1(n_t) = a_\emptyset \). It follows that \( \pi(n_t) = a_\emptyset \), which implies that \( f_{\min}(t) = (a_\emptyset, 1) \) is activated in \( \pi \). Finally, because \( P_n = \{ (a, i) \in X' \mid (a, i) \succeq_t f_{\min}(t) \} \cup \{ (a_\emptyset, j) \mid 1 \leq j \leq |N| \} \} \} \} \) for every \( n \in N \) with \( n \neq n_t \), it holds that if \( \pi_1(n) = a \neq a_\emptyset \), then \((a, |\pi_1^{-1}(a)|) \in (a, |\pi_1^{-1}(a)|) \) is activated in \( \pi \). As required. If, on the other hand, \((a_m, i_m) = f_{\min}(t) \) and \( a_m \neq a_\emptyset \), then because \( P_{n_t} = \{ (a_m, i_m) \} \), we have that \( \pi(n_t) = a_m \) and \( |\pi_1^{-1}(a_m)| = i_m \) and hence \( f_{\min}(t) \) is activated in \( \pi \). Moreover, for every \( n \in N \) with \( n \neq n_t \), we obtain that \( \pi_1(n) \neq a_\emptyset \) and \( \pi_1(n) \neq a_\emptyset \), because \( \pi_1 \) is perfect and \( P_n = \{ (a, i) \in X' \mid (a, i) \succeq_t f_{\min}(t) \} \). Hence
\( (\pi(n), |\pi^{-1}(\pi(n))|) = (\pi_1(n), |\pi_1^{-1}(\pi_1(n))|) \) and since \( (\pi_1(n), |\pi_1^{-1}(\pi_1(n))|) \in \{(a, i) \in X' \mid (a, i) \geq t f_{\text{min}}(t)\} \), we obtain that \( (\pi(n), |\pi^{-1}(\pi(n))|) \geq t f_{\text{min}}(t) \), as required.

Towards showing that \( \pi \) is also stable assume for a contradiction that this is not the case, then because of Lemma [18] there is an agent type \( t \) and an activity \( a \in A^* \setminus \{a_m\} \) such that \( (a, |\pi^{-1}(a)| + 1) \geq t f_{\text{min}} = (a_m, i_m) \). Note that if \( |\pi^{-1}(a)| = 0 \), then \( a \in A_{\neq \emptyset} \) contradicting our assumption that \( \pi_1^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \). Moreover, if \( a = a_\emptyset \), then this contradicts our assumption that \( f_{\text{min}}(t) \geq_t (a_\emptyset, 1) \).

Hence \( a \neq a_\emptyset \) and \( |\pi^{-1}(a)| \neq 0 \), which implies that there is an agent \( n \) with \( \pi(n) = a \) and furthermore \( \pi(n) = \pi_1(n) \). But then, because \( \pi_1 \) is individual rational, we obtain that \( (|\pi_1^{-1}(a)|) \in \mathcal{P}_n(a) \), which implies that \( (a, |\pi_1^{-1}(a)|) \in X' \) contradicting our choice of \( X' \).

We can now proceed to the main result of this section.

**Theorem 20.** An instance \( I = (N, A, (\geq_n)_{n \in N}) \) of GASP can be solved in time \( (|A| \cdot |N|)^{O(|T(I)|)} \).

**Proof.** Given an instance \( I = (N, A, (\geq_n)_{n \in N}) \) of GASP, the algorithm enumerates all of the at most \( (|A| \cdot |N|)^{|T(I)|} \) possible functions \( f_{\text{min}} \) and for each such function \( f_{\text{min}} \) the algorithm uses Theorem [19] to construct the instance \( I' = (N, A \cup \{a_\emptyset\}, (P_n)_{n \in N}) \) of sGASP with \( |T(I')| \leq |T(I)| \) together with the set \( A_{\neq \emptyset} \) of activities in time \( O(|N|^2 |A|) \). It then uses Lemma [13] to decide whether \( I' \) has a perfect individual rational assignment \( \pi_1 \) such that \( \pi_1^{-1}(a) \neq \emptyset \) for every \( a \in A_{\neq \emptyset} \) in time \( O((|A| + 1)(|N|)^{|T(I')|}) = O((|A| + 1)(|N|)^{|T(I)|}) \). If this is true for at least one of the functions \( f_{\text{min}} \), the algorithm returns that \( I \) has a solution, otherwise the algorithm correctly returns that \( I \) has no solution. The total running time of the algorithm is hence \( O(|A|^{|T(I)|} \cdot |N|^2 |T(I)|) = (|A| \cdot |N|)^{O(|T(I)|)} \).

7 Result 4: Lower Bound for GASP

This section presents our hardness result for GASP. In particular, we show that GASP is unlikely to be fixed-parameter tractable parameterized by both the number of activities and the number of agent types.

**Theorem 21.** GASP is \( W[1] \)-hard parameterized by the number of activities and the number of agent types.

**Proof.** We will employ a parameterized reduction from the PARTITIONED CLIQUE problem, which is well-known to be \( W[1] \)-complete [21].

**PARTITIONED CLIQUE**

**Input:** An integer \( k \), a \( k \)-partite graph \( G = (V, E) \) with partition \( \{V_1, \ldots, V_k\} \) of \( V \) into sets of equal size.

**Parameter:** \( k \)

**Question:** Does \( G \) have a \( k \)-clique, i.e., a set \( C \subseteq V \) of \( k \) vertices such that \( \forall u, v \in C \), with \( u \neq v \) there is an edge \( \{u, v\} \in E \)?

We denote by \( E_{i,j} \) the set of edges of \( G \) that have one endpoint in \( V_i \) and one endpoint in \( V_j \) and we assume w.l.o.g. that \( |V_i| = n \) and \( |E_{i,j}| = m \) for every \( i \) and \( j \) with \( 1 \leq i < j \leq k \) (see the standard parameterized complexity textbook for a justification of these assumptions [2]).

Given an instance \( (G, k) \) of PARTITIONED CLIQUE with partition \( V_1, \ldots, V_k \), we construct an equivalent instance \( I = (N, A, (\geq_n)_{n \in N}) \) of GASP in polynomial time with \((k \choose 2) + k \) activities and \( 2k + 1 \) agent types.

The instance \( I \) has the following activities:

- For every \( i \) with \( 1 \leq i \leq k \) the activity \( a_i \), whose size in a stable assignment for \( I \) will be used to identify the vertex in \( V_i \) chosen to be part of a \( k \)-clique in \( G \).
For every \( i \) and \( j \) with \( 1 \leq i < j \leq k \) the activity \( a_{i,j} \), whose size in a stable assignment for \( I \) will be used to identify the edge in \( E_{i,j} \) chosen to be part of a \( k \)-clique in \( G \).

For every \( i \) and \( j \) with \( 1 \leq i < j \leq k \) let \( \alpha_i \) be a bijection from \( V_i \) to the set \( \{3,5,\ldots, (2n-1)+1, 2n+1 \} \) and similarly let \( \alpha_{i,j} \) be a bijection from \( E_{i,j} \) to the set \( \{1,3,\ldots, 2m-1 \} \). The main ideas behind the reduction are as follows. First the reduction ensures that for every stable assignment \( \pi : N \rightarrow A^* \) for \( I \) the size \( s = |\pi^{-1}(a_i)| \) of activity \( a_i \) uniquely identifies a vertex in \( V_i \), i.e., the vertex \( \alpha^{-1}_i(s) \), and the size \( s = |\pi^{-1}(a_{i,j})| \) of activity \( a_{i,j} \) uniquely identifies an edge in \( E_{i,j} \), i.e., the edge \( \alpha^{-1}_{i,j}(s) \). Employing a set of “special agents” and their associated preference lists, the reduction will then ensure that the vertices identified by the sizes of the activities \( a_1, \ldots, a_k \) are endpoints of all the edges identified by the sizes of the activities \( a_{1,2}, a_{1,3}, \ldots, a_{k-1,k} \), which implies that these vertices form a \( k \)-clique in \( G \).

For a pair \( i, j \) of numbers, we denote by \( o(i,j) \), the (ordered) pair \( i, j \) if \( i \leq j \) and the (ordered) pair \( j, i \) otherwise. We will now introduce the equivalence classes required for the definition of our preference lists. Namely, we define the following equivalence classes:

- For \( x \in \{1,2\} \) we define the set \( C^x_A = \{(a_i,x) \mid 1 \leq i \leq k \} \),

- For every \( i \) with \( 1 \leq i \leq k \), we define the following sets:
  - \( C_V(i) = \{(a_i, \alpha_i(v)) \mid v \in V_i \} \),
  - \( C^+_V(i) = \{(a_i, \alpha_i(v) + 1) \mid v \in V_i \} \),

- For every \( i \) and \( j \) with \( 1 \leq i < j \leq k \), we define the set \( C_E(i,j) = \{(a_{i,j}, \alpha_{i,j}(e)) \mid e \in E_{i,j} \} \),

- For every \( i \) and \( v \in V_i \), we define the set \( C_I(i,v) = \{(a_i, \alpha_i(v)) \} \cup \{(a_{o(i,j)}, \alpha_{o(i,j)}(v) + 1) \mid 1 \leq j \leq k \land j \neq i \land e \in E_{i,j} \land v \in e \} \), i.e., \( C_I(i,v) \) contains the tuple \((a_i, \alpha_i(v))\) and all tuples \((a_{o(i,j)}, \alpha_{o(i,j)}(v) + 1)\) such that \( j \neq i \) and the edge \( e \in E_{o(i,j)} \) is incident to \( v \).

Now ready to define the required preference lists. When defining a preference list we will only list the equivalence classes that are more or equally preferred to the alternative \((a_0,1)\) and assume that all remaining alternatives, i.e., all alternatives that are not listed, are less preferred than \((a_0,1)\).

- The validity preference list, denoted by \( P_{VAL} \), defined as \( C_V \cup C_E \cup C^2_A \cup C^1_A > (a_0,1) \). Informally, \( P_{VAL} \) is crucial in ensuring that \( |\pi^{-1}(a_i)| \in \{ \alpha_i(v) \mid v \in V_i \} \) and \( |\pi^{-1}(a_{i,j})| \in \{ \alpha_{i,j}(e) \mid e \in E_{i,j} \} \) for every stable assignment \( \pi \) for \( I \) and every \( i \) and \( j \) with \( 1 \leq i < j \leq k \).

- For every \( i \) with \( 1 \leq i \leq k \), let \( v_1, \ldots, v_n \) be the unique ordering of the vertices in \( V_i \) in ascending order w.r.t. \( \alpha_i \). We define the following two preference lists for every \( i \) with \( 1 \leq i \leq k \):
  - The forward-vertex preference list, denoted by \( P^{+}_V(i) \), defined as \( C^{+}_V(i) > C_I(i,v_1) > C_I(i,v_2) > \cdots > C_I(i,v_{n-1}) > C_I(i,v_1) > (a_0,1) \). Informally, \( P^{+}_V(i) \) is crucial to ensure that for every \( j \) with \( 1 \leq j \leq k \) and \( j \neq i \) the edge \( e \) with \( \alpha_{o(i,j)}(e) = |\pi^{-1}(a_{i,j})| \) is not adjacent with any vertex \( v' \in V_i \) such that \( \alpha_i(v') > \alpha_i(v) \) for the vertex \( v \) with \( \alpha_i(v) = |\pi^{-1}(a_i)| \). This intuition will be made precise in Claim [1].
  - The backward-vertex preference list, denoted by \( P^{-}_V(i) \), is defined as \( C^{+}_V(i) > C_I(i,v_1) > C_I(i,v_2) > \cdots > C_I(i,v_n) > (a_0,1) \). Informally, \( P^{-}_V(i) \) is crucial to ensure that for every \( j \) with \( 1 \leq j \leq k \) and \( j \neq i \) the edge \( e \) with \( \alpha_{o(i,j)}(e) = |\pi^{-1}(a_{i,j})| \) is not adjacent with any vertex \( v' \in V_i \) such that \( \alpha_i(v') < \alpha_i(v) \) for the vertex \( v \) with \( \alpha_i(v) = |\pi^{-1}(a_i)| \). This intuition will be made precise in Claim [1].
Informally, $P_V^{-2}(i)$ and $P_V^{+2}(i)$ together ensure that for every $j$ with $1 \leq j \leq k$ and $j \neq i$, the edge $e$ with $\alpha_{o(i,j)}(e) = |\pi^{-1}(a_{o(i,j)})|$ is adjacent with the vertex $v$ with $\alpha_i(v) = |\pi^{-1}(a_i)|$. This intuition will be made precise in Claim 1.

We are now ready to define the set $N$ of agents:

- for every $i$ with $1 \leq i \leq k$:
  - one agent $n_i^\rightarrow$ with preference list $P_V^{-2}(i)$ and
  - one agent $n_i^\leftarrow$ with preference list $P_V^{+2}(i)$.
- a set $N_V$ of $(\binom{k}{2})(2m - 1) + k(2n + 1) + 1$ agents with preference list $P_{VAL}$.

This completes the construction of the instance $I$. Clearly the given reduction can be achieved in polynomial-time. Moreover, since $I$ has exactly $(\binom{k}{2}) + k$ activities and exactly $2k + 1$ distinct types of preference lists, both parameters are bounded by a function of $k$, as required. It remains to show that $G$ has a $k$-clique if and only if $I$ has a stable assignment.

Towards showing the forward direction let $C = \{v_1, \ldots, v_k\}$ be a $k$-clique of $G$ such that $v_i \in V_i$ for every $i$ with $1 \leq i \leq k$ and for every $i$ and $j$ with $1 \leq i < j \leq k$ let $e_{i,j}$ be the edge between $v_i$ and $v_j$ in $G$. We claim that the assignment $\pi : N \rightarrow A^*$ defined in the following is a stable assignment for $I$. We set:

- $\pi(n_i^\rightarrow) = \pi(n_i^\leftarrow) = a_i$ and
- for every $i$ and $j$ with $1 \leq i < j \leq k$, $\pi$ assigns exactly $\alpha_{i,j}(e_{i,j})$ agents from $N_V$ to activity $a_{i,j}$,
- for every $i$ with $1 \leq i \leq k$, $\pi$ assigns exactly $\alpha_i(v_i) - 2$ agents from $N_V$ to activity $a_i$,
- all remaining agents (which are only in $N_V$) are assigned to $a_0$.

Note that $|\pi^{-1}(a_{i,j})| = \alpha_{i,j}(e_{i,j})$ and $|\pi^{-1}(a_i)| = \alpha_i(v_i) - 2 + 2 = \alpha_i(v_i)$ for every $i$ and $j$ with $1 \leq i < j \leq k$.

Towards showing that the assignment $\pi$ is stable, we consider any agent $n$ and distinguish the following cases:

- if $n$ is one of $n_i^\rightarrow$ or $n_i^\leftarrow$ for some $i$ with $1 \leq i \leq k$, then $\pi$ is stable w.r.t. to $n$ because for every $j$ with $1 \leq j \leq k$ and $j \neq i$, the edge $e_{o(i,j)}$ is incident with $v_i$.
- if $u \in N_V$, we consider the following cases:
  - $(\pi(u) = a_i)$ In this case the assignment is stable w.r.t. $n$ because the tuple $(a_i, |\pi^{-1}(a_i)|) = (a_i, \alpha_i(v_i))$ is in the largest equivalence class of $P_V$.
  - $(\pi(u) = a_{i,j})$ In this case the assignment is stable w.r.t. $n$ because the tuple $(a_{i,j}, |\pi^{-1}(a_{i,j})|) = (a_{i,j}, \alpha_{i,j}(e_{i,j}))$ is in the largest equivalence class of $P_V$.
  - $(\pi(u) = a_0)$ In this case the assignment is stable w.r.t. $n$ because the tuples $(a_i, \alpha_i(v_i) + 1)$ and $(a_{i,j}, |\pi^{-1}(a_{i,j})| + 1) = (a_{i,j}, \alpha_{i,j}(e_{i,j}) + 1)$ are less preferred than the tuple $(a_0, 1)$ in the preference list $P_V$ for $n$ (for every $i$ and $j$ with $1 \leq i < j \leq k$).

Towards showing the backward direction, we start by formalizing the intuition given above about the preference lists $P_V^{-2}(i)$ and $P_V^{+2}(i)$.

**Claim 1.** Let $i$ be an integer with $1 \leq i \leq k$ and let $\pi$ be a stable assignment for $I$ satisfying:

(A1) $|\pi^{-1}(a_i)| \in \{ \alpha_i(v) \mid v \in V_i \}$ and
(A2) for every \( j \) with \( 1 \leq j \leq k \) and \( j \neq i \) it holds that \( |\pi^{-1}(a_{o(i,j)})| \in \{ \alpha_{o(i,j)}(e) \mid e \in E_{o(i,j)} \} \).

Then the following holds for \( \pi \):

(C1) \( \pi(n_i^+) = \pi(n_i^-) = a_i \).

(C2) For every \( j \) with \( 1 \leq j \leq k \) and \( j \neq i \), the unique edge \( e_{o(i,j)} \in E_{o(i,j)} \) with \( \alpha_{o(i,j)}(e) = |\pi^{-1}(a_{o(i,j)})| \) is incident with the unique vertex \( v \in V_i \) such that \( \alpha(v) = |\pi^{-1}(a_{o(i,j)})| \).

Towards showing (C1) assume for a contradiction that this is not the case, i.e., one of the agents \( n_i^+ \) or \( n_i^- \), in the following denoted by \( n \), is not assigned to \( a_i \). Because \( (a_i, \alpha(v) + 1) \in C_V^+(i) \) and \( C_V^+(i) \) only contains alternatives for activity \( a_i \), the agent \( n \) would prefer to change from his current activity to activity \( a_i \) contradicting our assumption that \( \pi \) is stable.

Towards showing (C2) assume for a contradiction that this is not the case and let \( j \) be an index witnessing this, i.e., \( v \notin e_{o(i,j)} \). Let \( v' \) be the vertex in \( V_i \) that is incident with \( e_{o(i,j)} \). We distinguish two analogous cases (note that \( v' \neq v \)): (1) \( \alpha(v') > \alpha(v) \) and (2) \( \alpha(v') < \alpha(v) \). In the former case \( \pi \) would not be stable because the agent \( n_i^+ \) would prefer to join activity \( a_{i,j} \) over his current activity \( a_i \); this is because \( \alpha(v') > \alpha(v) \) and hence the equivalence class \( C_I(i, v') \), which contains the tuple \( (a_{o(i,j)}, \alpha_{o(i,j)}(e) + 1) \), is more preferred in \( P_{o(i,j)}^+(i) \) than the equivalence class \( C_I(i, v) \), which contains the tuple \( (a_i, \alpha_i(v)) \). The proof for the latter case is analogous, using the agent \( n_i^- \) instead of the agent \( n_i^+ \).

Note that once we show that the assumptions (A1) and (A2) hold for every \( i \) with \( 1 \leq i \leq k \), property (C2) ensures that the vertices \( \alpha^{-1}(|\pi^{-1}(a_1)|), \ldots, \alpha^{-1}(|\pi^{-1}(a_k)|) \) form a \( k \)-clique in \( G \). This is achieved with the following claim.

**Claim 2.** For every stable assignment \( \pi : N \rightarrow A^+ \) for \( I \), it holds that:

(A0) \( \pi(u) = a_0 \) for at least one agent \( u \in N_V \).

(A1) \( \pi^{-1}(a_i) \in \{ \alpha_i(v) \mid v \in V_i \} \) for every \( i \) with \( 1 \leq i \leq k \).

(A2) \( \pi^{-1}(a_{i,j}) \in \{ \alpha_{i,j}(e) \mid e \in E_{i,j} \} \) for every \( i \) and \( j \) with \( 1 \leq i < j \leq k \).

Towards showing (A0), assume for a contradiction that this is not the case, i.e., all \( \binom{k}{2}(2m-1)+k(2n+1)+1 \) agents in \( N_V \) are assigned to one of the \( \binom{k}{2}+k \) activities in \( A \). Then there either exists an activity \( a_{i,j} \) such that more than \( 2m-1 \) agents in \( N_V \) are assigned to \( a_{i,j} \) by \( \pi \) or there exists an activity \( a_i \) such that more than \( 2n+1 \) agents in \( N_V \) are assigned to \( a_i \) by \( \pi \). In the former case let \( u \in N_V \) be an agent with \( \pi(u) = a_{i,j} \). Then the assignment is not stable because \( n \) would prefer being assigned to \( a_0 \) over its current assignment to \( a_{i,j} \). The latter case is analogous. This completes the proof for (A0).

Because of (A0) there is at least one agent \( u \in N_V \) such that \( \pi(u) = a_0 \). Hence because of the preference list \( P_V \) for \( n \), we obtain that \( |\pi^{-1}(a_i)| \notin \{0,1,2\} \cup \{\alpha_i(v) - 1 \mid v \in V_i \} \), since otherwise \( n \) would prefer activity \( a_i \) over \( a_0 \). It follows that either \( |\pi^{-1}(a_i)| \in \{ \alpha_i(v) \mid v \in V_i \} \) or \( |\pi^{-1}(a_i)| > \max\{\alpha_i(v) \mid v \in V_i \} = 2n+1 \). In the former case (A1) holds, so assume that the latter case applies. Since we can assume w.l.o.g. that \( 2n+1 > 2k \) (and there are only \( 2k \) agents in \( N \setminus N_V \)), we obtain that there is at least one agent \( u \in N_V \) such that \( \pi(u) = a_i \). But then because of the preference list \( P_V \) of \( n \), \( n \) would prefer activity \( a_0 \) over \( a_i \), contradicting our assumption that \( \pi \) is a stable assignment. The completes the proof of (A1). The proof of (A2) is analogous to the proof of (A1).

8 Result 5: Lower Bound for gGASP

From our previous result in conjunction with the equivalence between GASP and gGASP on complete networks, we can immediately conclude that gGASP is also \( \mathcal{W}[1] \)-hard parameterized by the number of agent
types and the number of activities. However, here we strengthen this result by showing that the problem remains \( W[1] \)-hard even when one additionally parameterizes by the vertex cover number of the network.

As noted in the introduction, this also implies the \( W[1] \)-hardness of the problem when parameterized by the treewidth of the network, a question raised in previous work \cite{15}; in fact, the presented lower-bound result not only shows the (conditional) fixed-parameter intractability of the problem with a more restrictive graph parameter, but also when additionally parameterizing by the number of agent types.

**Theorem 22.** \( gGASP \) is \( W[1] \)-hard parameterized by the number of activities, the number of agent types, and the vertex cover number of the network.

**Proof.** The proof is via a parameterized reduction from PARTITIONED CLIQUE, i.e., given an instance \((G, k)\) of PARTITIONED CLIQUE with partition \( V_1, \ldots, V_k \), we construct an equivalent instance \( I = (N, A, (\geq_n)_{n \in N}, L) \) of \( gGASP \) in polynomial time with \( \binom{k}{2} + k \) activities, \( \binom{k}{2} + 3k \) agent types, and whose network \((N, L)\) has vertex cover number at most \( \binom{k}{2} + 2k \).

The main ideas behind the reduction are quite similar to the reduction used in the proof of Theorem 21. The main differences is that to ensure that the network \((N, L)\) has a small vertex cover number, it is necessary to split the set \( N \) of agents used in the previous reduction, into sets \( N_i \) and \( N_{i,j} \) for every \( i \) and \( j \) with \( 1 \leq i < j \leq k \). For every \( i \) and \( j \) with \( 1 \leq i < j \leq k \) let \( \alpha_i \) be a bijection from \( V_i \) to the set \( \{3, 5, \ldots, 2(n - 1) + 1, 2n + 1\} \) and similarly let \( \alpha_{i,j} \) be a bijection from \( E_{i,j} \) to the set \( \{3, 5, \ldots, 2m + 1\} \).

Let \( I' \) be the instance of GASP as defined in the proof of Theorem 21. Since the instance \( I \) is defined quite similar to the instance \( I' \), we will refer to \( I' \) for the construction of \( I \). In particular, \( I \) has the same activities as \( I' \), i.e., \( I \) has one activity \( a_i \) for every \( i \) with \( 1 \leq i \leq k \) and one activity \( a_{i,j} \) for every \( i \) and \( j \) with \( 1 \leq i < j \leq k \).

For the definition of the preference lists, we will mainly use the equivalence classes defined in the proof of Theorem 21 i.e., the classes \( C_V(i), C_V^{+1} \), and \( C_E(i, j), C_I(i, v) \). Apart from those we will also need a slightly modified version of the equivalence class \( C_V^{+1}(i) \), which we denote by \( C_V^{+1,2}(i) \) and set to \( \{(a_i, 2)\} \cup C_V^{+1}(i) \).

For each activity \( a \), we define the following two preference lists for every \( i \) with \( 1 \leq i \leq k \):

- The vertex-validity preference list, denoted by \( P_V^{\text{val}}(i) \), defined as \( C_V(i) > (a_i, 2) > (a_i, 1) > (a_0, 1) \).
- The forward-validity preference list, denoted by \( P_V^+(i) \), defined as \( C_V^{+1,2}(i) > C_I(i, v_n) > \cdots > C_I(i, v_1) > (a_0, 1) \).
- The backward-validity preference list, denoted by \( P_V^-(i) \), defined as \( C_V^{+1,2}(i) > C_I(i, v_2) > \cdots > C_I(i, v_n) > (a_0, 1) \).

For every \( i \) and \( j \) with \( 1 \leq i < j \leq k \) the edge-validity preference list, denoted by \( P_{\text{val}}^E(i,j) \), and defined as \( C_E(i,j) > (a_{i,j}, 2) > (a_{i,j}, 1) > (a_0, 1) \).

We are now ready to define the set \( N \) of agents:

- for every \( i \) with \( 1 \leq i \leq k \):
  - one agent \( n_i^+ \) with preference list \( P_v^+(i) \),
  - one agent \( n_i^- \) with preference list \( P_v^-(i) \),
  - one agent \( n_{i,j}^+ \) with preference list \( P_v^+(i,j) \),
  - one agent \( n_{i,j}^- \) with preference list \( P_v^-(i,j) \),
  - one agent \( n_{i,j,0}^+ \) with preference list \( P_v^+(i,j,0) \),
  - one agent \( n_{i,j,0}^- \) with preference list \( P_v^-(i,j,0) \),
  - one agent \( n_{i,j,1}^+ \) with preference list \( P_v^+(i,j,1) \),
  - one agent \( n_{i,j,1}^- \) with preference list \( P_v^-(i,j,1) \),
  - one agent \( n_{i,j,2}^+ \) with preference list \( P_v^+(i,j,2) \),
  - one agent \( n_{i,j,2}^- \) with preference list \( P_v^-(i,j,2) \).
Figure 2: An illustration of the network \((N, L)\) obtained in the reduction of Theorem 22 for the case that \(k = 3\).

- one agent \(n_i^+\) with preference list \(P^{V^+}(i)\), and
- a set \(N_i\) of \(2n + 3\) agents with preference list \(P^{V\text{VAL}}(i)\).

- for every \(i\) and \(j\) with \(1 \leq i < j \leq k\), a set \(N_{i,j}\) of \(2m + 3\) agents with preference list \(P^{E\text{VAL}}(i, j)\). In the following let \(n_{i,j}\) be one of the agents in \(N_{i,j}\).

Finally, the links \(L\) between the agents are given by:

- for every \(i\) with \(1 \leq i \leq k\):
  - a link between \(n_i^+\) and \(n_i^-\),
  - for every \(u \in N_i\) a link between \(n_i^+\) and \(n\),
  - for every \(j\) with \(1 \leq j \leq k\) and \(j \neq k\) a link between \(n_i^+\) and \(n_{o(i,j)}\),
- for every \(i, j\) with \(1 \leq i < j \leq k\) and every \(u \in N_{i,j} \setminus \{n_{i,j}\}\) a link between \(n_{i,j}\) and \(n\).

An illustration of the network \((N, L)\) as defined above is given in Figure 2.

This completes the construction of the instance \(I\). Observe that the set \(\{n_i^+, n_i^- \mid 1 \leq i \leq k\} \cup \{n_{i,j} \mid 1 \leq i < j \leq k\}\) is a vertex cover of the network \((N, L)\) of size at most \(\binom{k}{2} + 2k\), and hence the network has vertex cover number at most \(\binom{k}{2} + 2k\). Clearly the given reduction can be achieved in polynomial-time. Moreover, since \(I\) has exactly \(\binom{k}{2} + k\) activities, exactly \(\binom{k}{2} + 3k\) distinct types of preference lists, and the vertex cover number of the network \((N, L)\) is at most \(\binom{k}{2} + 2k\), all parameters are bounded by a function of \(k\), as required. It remains to show that \(G\) has a \(k\)-clique if and only if \(I\) has a stable assignment. Towards showing the forward direction let \(C = \{v_1, \ldots, v_k\}\) be a \(k\)-clique of \(G\) such that \(v_i \in V_i\) for every \(i\) with \(1 \leq i \leq k\) and for every \(i\) and \(j\) with \(1 \leq i < j \leq k\) let \(e_{i,j}\) be the edge between \(v_i\) and \(v_j\) in \(G\). We claim that the assignment \(\pi : N \rightarrow A^*\) defined in the following is a stable assignment for \(I\). We set:

- \(\pi(n_i^+) = \pi(n_i^-) = a_i\),
- for every \(i\) with \(1 \leq i \leq k\), \(\pi\) assigns exactly \(\alpha_i(v_i) - 2\) agents from \(N_i\) to activity \(a_i\),
• for every $i$ and $j$ with $1 \leq i < j \leq k$, $\pi(n_{i,j}) = a_{i,j}$, and
• for every $i$ and $j$ with $1 \leq i < j \leq k$, $\pi$ assigns exactly $\alpha_{i,j}(e_{i,j}) - 1$ agents from $N_{i,j}$ to activity $a_{i,j}$.
• all remaining agents are assigned to $a_{\emptyset}$.

Note that for every $i$ and $j$ with $1 \leq i < j \leq k$ the agents assigned to activities $a_i$ and $a_{i,j}$ are connected and moreover $|\pi^{-1}(a_{i,j})| = \alpha_{i,j}(e_{i,j})$ and $|\pi^{-1}(a_i)| = \alpha_i(v_i)$. Let $n$ be an agent, we consider the following cases:

• if $n$ is one of $n_i^*$ or $n_i^-$ for some $i$ with $1 \leq i \leq k$, then the assignment $\pi$ is stable w.r.t. to $n$ because for every $j$ with $1 \leq j \leq k$ and $j \neq i$, the edge $e_{o(i,j)}$ is incident to $v_i$ in $G$ and hence the tuples $(a_i, \alpha_i(v_i))$ and $(a_{o(i,j)}, \alpha_{o(i,j)}(e_{o(i,j)}) + 1)$ are in the same equivalence class of $P_{V}^+$ and $P_{V}^-$.

• if $u \in N_i$ and $\pi(u) = a_i$, then the assignment $\pi$ is stable w.r.t. $n$ because the tuple $(a_i, \alpha_i(v_i))$ is in the most preferred equivalence class of $P_{VAL}^+(i)$.

• if $u \in N_i$ and $\pi(u) = a_{\emptyset}$, then the assignment $\pi$ is stable w.r.t. $n$ because $(a_{\emptyset}, 1)$ is preferred to $(a_i, \alpha_i(v_i) + 1)$ as well as to any tuple with any other activity in $P_{VAL}^+(i)$.

• if $u \in N_{i,j}$ and $\pi(u) = a_{i,j}$, then the assignment $\pi$ is stable w.r.t. $n$ because the the tuple $(a_{i,j}, \alpha_{i,j}(e_{i,j}))$ is in the most preferred equivalence class of $P_{VAL}^+(i,j)$.

• if $u \in N_{i,j}$ and $\pi(u) = a_{\emptyset}$, then the assignment $\pi$ is stable w.r.t. $n$ because $(a_{\emptyset}, 1)$ is preferred to $(a_{i,j}, \alpha_{i,j}(e_{i,j}) + 1)$ as well as to any tuple with any other activity in $P_{VAL}^+(i,j)$.

The reverse direction follows immediately from the following claim.

Claim 3. For every stable assignment $\pi : N \rightarrow A^+$ for $I$, it holds that:

(C1) for every $i$ with $1 \leq i \leq k$, there is an agent $u \in N_i$ such that $\pi(u) = (a_{\emptyset}, 1)$,

(C2) for every $i$ and $j$ with $1 \leq i < j \leq k$, there is an agent $u \in N_{i,j}$ such that $\pi(u) = (a_{\emptyset}, 1)$,

(C3) $|\pi^{-1}(a_i)| \in \{ \alpha_i(v) \mid v \in V_i \}$ for every $i$ with $1 \leq i \leq k$,

(C4) $\pi(n_i^*) = \pi(n_i^-) = a_i$ for every $i$ with $1 \leq i \leq k$,

(C5) $|\pi^{-1}(a_{i,j})| \in \{ \alpha_{i,j}(e) \mid e \in E_{i,j} \}$ for every $i$ and $j$ with $1 \leq i < j \leq k$,

(C6) $\pi(n_{i,j}) = a_{i,j}$ for every $i$ and $j$ with $1 \leq i < j \leq k$,

(C7) $\alpha_{i,j}^{-1}(|\pi^{-1}(a_{i,j})|) \in \alpha_{o(i,j)}^{-1}(|\pi^{-1}(a_{o(i,j)})|)$ for every $i$ and $j$ with $1 \leq i, j \leq k$ and $i \neq j$.

Towards showing (C1) suppose for a contradiction that this is not the case. Then because all agents in $N_i$ must either be assigned to $a_i$ or $a_{\emptyset}$ by $\pi$, we obtain that all $2n + 3$ agents in $N_i$ must be assigned to $a_i$. However such an assignment would not be stable because any tuple $(a_i, x)$ with $x \geq 2n + 3$ is less preferred than $(a_{\emptyset}, 1)$ in every preference list.

The proof of (C2) is very similar to the proof of (C1). Namely, suppose for a contradiction that this is not the case. Then all $2m + 3$ agents in $N_{i,j}$ must be assigned to $a_{i,j}$, however such an assignment would not be stable because any tuple $(a_{i,j}, x)$ with $x \geq 2m + 3$ is less preferred than $(a_{\emptyset}, 1)$ in every preference list.

Towards showing (C3) we first show that $|\pi^{-1}(a_i)| > 1$. Because of (C1) there is an agent $u \in N_i$ with $\pi(u) = \emptyset$. Hence $|\pi^{-1}(a_i)| \neq 0$ since otherwise the agent $n$ would prefer to switch to $a_i$. Now suppose for
Moreover, because all agents in $N_i$ are the only agents that prefer the tuple $(a_i, 1)$ over $(a_0, 1)$, it holds that $\pi^{-1}(a_i) \subseteq N_i$. But since the tuple $(a_i, 2)$ is a tuple that is in the highest equivalence class $C_E(i, 1, 2)(i)$ of the preference list $P_E(i)$ for $n_i^{-\rightarrow}$ and $n_i^{-\rightarrow}$ is linked with every vertex in $N_i$, the agent $n_i^{-\rightarrow}$ would prefer to switch to $a_i$, contradicting our assumption that $\pi$ is stable. Consequently $|\pi^{-1}(a_i)| > 1$ and we show next that $\pi^{-1}(a_i)$ contains $n_i^{-\rightarrow}$. Observe that the agents in $\{n_i^{-\rightarrow}, n_i^{-\rightarrow}\} \cap N_i$ are the only agents in $N$ that can be assigned to $a_i$; all other agents prefer the tuple $(a_0, 1)$ over any tuple involving the activity $a_i$. Since $|\pi^{-1}(a_i)| > 1$ the set $\pi^{-1}(a_i)$ can only be connected if it contains $n_i^{-\rightarrow}$. Hence we have $|\pi^{-1}(a_i)| > 1$ and $n_i^{-\rightarrow} \in \pi^{-1}(a_i)$. Because of (C1) there is an agent $u \in N_i$ with $\pi(u) = a_0$. Since $n$ is linked with $n_i^{-\rightarrow}$ and prefers $a_i$ over $a_0$, whenever $|\pi^{-1}(a_i)| \in \{0, 1\} \cup \{\alpha_i(v) - 1 \mid v \in V_i\} = \{0, 1, 2, 4, 6, \ldots, 2n\}$, we obtain that either $|\pi^{-1}(a_i)| \in \{\alpha_i(v) \mid v \in V_i\} = \{3, 5, \ldots, 2n + 1\}$ or $|\pi^{-1}(a_i)| > 2n + 1$. In the former case (C3) holds, so assume that the latter case applies. Since the agents in $\{n_i^{-\rightarrow}, n_i^{-\rightarrow}\} \cup N_i$ are the only agents in $N$ that can be assigned to $a_i$ and we can assume w.l.o.g. that $2n + 1 > 2$, we obtain that there is at least one agent $u \in N_i$ such that $\pi(u) = a_i$. But then because of the preference list $P_E(i)$ of $n, n$ would prefer activity $a_0$ over $a_i$, contradicting our assumption that $\pi$ is a stable assignment. The completes the proof of (C3).

Towards showing (C5) first observe that the agents in $\{n_i^{-\rightarrow}, n_i^{-\rightarrow} \mid 1 \leq i \leq k\} \cup N_{i,j}$ are the only agents in $N$ that can be assigned to $a_{i,j}$; all other agents prefer the tuple $(a_0, 1)$ over any tuple involving the activity $a_{i,j}$. Moreover because of (C4), we obtain that only the agents in $N_{i,j}$ can actually be assigned to $a_{i,j}$. Moreover because all agents in $N_{i,j}$ must either be assigned to $a_{i,j}$ or to $a_0$ and due to (C2) there is always an agent $n' \in N_{i,j}$ with $\pi(n') = a_0$, and since the set $N_{i,j}$ is connected by $L_i$, we obtain that there is always an agent $n \in N_{i,j}$ with $\pi(u) = a_0$ that is linked to an agent in $\pi^{-1}(a_{i,j})$. Hence it follows from the preference list $P_E(i, j)$ of the agents in $N_{i,j}$ that $|\pi^{-1}(a_{i,j})| \notin \{0, 1\} \cup \{\alpha_{i,j}(e) - 1 \mid e \in E_{i,j}\}$, since otherwise the agent $n$ would prefer to switch to $a_{i,j}$. Hence either $|\pi^{-1}(a_{i,j})| \in \{\alpha_{i,j}(v) \mid v \in E_{i,j}\} = \{3, 5, \ldots, 2m + 1\}$ or $|\pi^{-1}(a_{i,j})| > 2m + 1$. In the former case (C5) holds, so assume that the latter case applies. Then there is an agent $u \in N_{i,j}$ with $\pi(u) = a_{i,j}$, but since $|\pi^{-1}(a_{i,j})| > 2m + 1$ the agent would prefer to be assigned to $a_0$, contradicting our assumption that the assignment $\pi$ is stable. This completes the proof of (C5).

(C6) can be obtained as follows. Because of (C4), we have that $|\pi^{-1}(a_{i,j})| \geq 3$. Since (as observed already in the proof of (C5)) only the agents in $N_{i,j}$ can be assigned to activity $a_{i,j}$, we obtain that $\pi(n_{i,j}) = a_{i,j}$ since otherwise $\pi^{-1}(a_{i,j})$ would not be connected.

Towards showing (C7) assume for a contradiction that there are $i$ and $j$ with $1 \leq i, j \leq k$ and $i \neq j$ such that $v \notin e$, where $v = \alpha_{i,j}^{-1}(|\pi^{-1}(a_{i,j})|)$ and $e = \alpha_{o(i,j)}(\pi^{-1}(a_{o(i,j)}))$. Observe first that because of (C3) and (C4) $v$ and $e$ are properly defined and moreover $v \in V_i$ and $e \in E_{i,j}$. Let $v'$ be the endpoint of $e$ in $V_i$, which because $v \notin e$ is not equal to $v$. We distinguish two analogous cases: (1) $\alpha_i(v') < \alpha_i(v)$ and (2) $\alpha_i(v') > \alpha_i(v)$. In the former case consider the agent $n_i^{-\rightarrow}$. Because of (C4) and (C6), it holds that $\pi(n_i^{-\rightarrow}) = a_i$ and $\pi(n_{i,j}) = a_{o(i,j)}$, which implies that $n_i^{-\rightarrow}$ is linked with an agent, namely the agent $n_{i,j}$ assigned to $a_{o(i,j)}$ by $\pi$. Together with the facts that the equivalence class $C_E(i, v')$ contains the tuple $(a_{o(i,j)}, \alpha(e))$, the equivalence class $C_E(i, v)$ contains the tuple $(a_i, \alpha(v))$, and $C_E(i, v')$ is preferred over $C_E(i, v)$ in the preference list for $n_i^{-\rightarrow}$, we obtain that the agent $n_i^{-\rightarrow}$ prefers the activity $a_{o(i,j)}$ over its current activity $a_i$, contradicting the stability of $\pi$. The proof for the latter case is analogous using the agent $n_i^{-\rightarrow}$ instead of the agent $n_i^{-\rightarrow}$. This completes the proof for (C7).

9 Conclusion

We obtained a comprehensive picture of the parameterized complexity of Group Activity Selection problems parameterized by the number of agent types, both with and without the number of activities as an additional parameter. Our positive results suggest that using the number of agent types is a highly appealing parameter for GASP and its variants; indeed, this parameter will often be much smaller than the number of agents due to
the way preference lists are collected or estimated (as also argued in initial work on GASp [6]). For instance, in the large-scale event management setting of GASp (or SGASp), one would expect that preference lists for event participants are collected via simple questionnaires—and so the number of agent types would remain fairly small regardless of the size of the event.

We believe that the techniques used to obtain the presented results, and especially the Subset Sum tools developed to this end, are of broad interest to the algorithms community. For instance, Multidimensional Subset Sum (MSS) has been used as a starting point for \(W[1]\)-hardness reductions in at least two different settings over the past year \([13, 12]\), but the simple and partitioned variant of the problem (i.e., SMPSS) is much more restrictive and hence forms a strictly better starting point for any such reductions in the future. This is also reflected in our proof of the \(W[1]\)-hardness of SMPSS, which is significantly more involved than the analogous result for MSS. Likewise, we expect that the developed algorithms for Tree Subset Sum and Multidimensional Partitioned Subset Sum may find applications as subroutines for (parameterized and/or classical) algorithms in various settings.

Note that there is now an almost complete picture of the complexity of Group Activity Selection problems w.r.t. any combination of the parameters number of agents, number of activities, and number of agent types (see also Table 1). There is only one piece missing, namely, the parameterized complexity of SGASp parameterized by the number of agents, which we resolve for completeness with the following theorem.

**Theorem 23.** SGASp is fixed-parameter tractable parameterized by the number of agents.

**Proof.** Let \(I = (A, (P_n)_{n \in A})\) be a SGASp instance. The main idea behind the algorithm is to guess (i.e., branch over) the set \(M_0\) of agents that are assigned to \(a_0\) as well as a partition \(\mathcal{M}\) of the remaining agents, i.e., the agents in \(A \setminus M_0\), and then check whether there is a stable assignment \(\pi\) for \(I\) such that:

(P1) \(\pi^{-1}(a_0) = M_0\) and

(P2) \(\{\pi^{-1}(a) \mid a \in A\} \setminus \emptyset = \mathcal{M}\), i.e., \(\mathcal{M}\) corresponds to the grouping of agents into activities by \(\pi\).

Since there are at most \(n^n\) possibilities for \(M_0\) and \(\mathcal{M}\) and those can be enumerated in time \(O(n^n)\), it remains to show how to decide whether there is a stable assignment for \(I\) satisfying (P1) and (P2) for any given \(M_0\) and \(\mathcal{M}\). Towards showing this, we first consider the implications for a stable assignment resulting from assigning the agents in \(M_0\) to \(a_0\). Namely, let \(P'_n\) for every \(n \in A\) be the approval set obtained from \(P_n\) after removing all alternatives \((a, i)\) such that \(i \neq 0\) and there is an agent \(n_0 \in M_0\) with \((a, i+1) \in P'_n\). Moreover, let \(A_{\neq 0}\) be the set of all activities that cannot be left empty if the agents in \(M_0\) are assigned to \(a_0\), i.e., the set of all activities such that there is an agent \(n_0 \in M_0\) with \((a, 1) \in P'_n\). Now consider a set \(M \in \mathcal{M}\), and observe that the set \(A_M\) of activities that the agents in \(M\) can be assigned to in any stable assignment satisfying (P1) and (P2) is given by: \(A_M = \{a \mid |M| \in \bigcap_{n \in M} P'_n(a)\}\). Let \(B\) be the bipartite graph having \(\mathcal{M}\) on one side and \(A\) on the other side and having an edge between a vertex \(M \in \mathcal{M}\) and a vertex \(a \in A\) if \(a \in A_M\). We claim that \(I\) has a stable assignment satisfying (P1) and (P2) if and only if \(B\) has a matching that saturates \(\mathcal{M} \cup A_{\neq 0}\). Since deciding the existence of such a matching can be achieved in time \(O(\sqrt{|\mathcal{M}|}(|\mathcal{M}| + |A_{\neq 0}|)) = O(\sqrt{|A|}(|\mathcal{M}| + |A|))\) (see e.g. [13, Lemma 4]), establishing this claim is the last component required for the proof of the theorem.

Towards showing the forward direction, let \(\pi\) be a stable assignment for \(I\) satisfying (P1) and (P2). Then \(O = \{\{a, \phi^{-1}(a)\} \mid a \in A\}\) is a matching in \(B\) that saturates \(\mathcal{M} \cup A_{\neq 0}\). Note that \(O\) saturates \(\mathcal{M}\) due to (P2), moreover, \(O\) saturates \(A_{\neq 0}\) since otherwise there would be an activity \(a \in A_{\neq 0}\) with \(\pi^{-1}(a) = \emptyset\), which due to the definition of \(A_{\neq 0}\) and (P1) implies there is an agent \(n\) with \(\pi(n) = a_0\) such that \(1 \in P'_n(a)\), contradicting our assumption that \(\pi\) is stable.

Towards showing the reverse direction, let \(O\) be a matching in \(B\) that saturates \(\mathcal{M} \cup A\). Then the assignment \(\pi\) mapping all agents in \(M\) (for every \(M \in \mathcal{M}\)) to its partner in \(O\) and all other agents to \(a_0\) clearly already satisfies (P1) and (P2). It remains to show that it is also stable. Note that \(\pi\) is individually
rational because of the construction of $B$. Moreover, assume for a contradiction that there is an agent $n \in N \backslash \emptyset$ with $\pi(n) = a \in A$ such that $(a, |\pi^{-1}(a)| + 1) \in P_n$. If $|\pi^{-1}(a)| = 0$, then $a \in A \backslash \emptyset$ and hence $|\pi^{-1}(a)| > 0$ (because $O$ saturates $A \backslash \emptyset$), a contradiction. If on the other hand $|\pi^{-1}(a)| \neq 0$, then $\{M, a\} \in O$ (for some $M \in M$), but $(a, |\pi^{-1}(a)|) \notin P_n$ and hence $\{M, a\} \notin E(B)$, also a contradiction.

For future work, we believe that it would be interesting to see how the complexity map changes if one were to consider the number of activity types instead of the number of activities in our parameterizations.

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