GENERATORS AND SPLITTING FIELDS OF CERTAIN ELLIPTIC K3 SURFACES

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ABSTRACT. Let \( k \subset \mathbb{C} \) be a number field and \( \mathcal{E} \) be an elliptic curve that is isomorphic to the generic fiber of an elliptic surface defined over the rational function field \( k(t) \) of the projective line \( \mathbb{P}^1_k \). The set \( \mathcal{E}(K) \) of \( K \)-rational points of \( \mathcal{E} \) is known to be a finitely generated abelian group, \( \mathcal{G} \), called the Mordell-Weil lattices. By the splitting field \( \mathcal{E}(C) \) of \( \mathcal{E}(K) \) we mean the smallest subfield \( \mathcal{K} \) of \( \mathbb{C} \) which is a finite extension of \( k \) and \( \mathcal{E}(\mathbb{C}) = \mathcal{E}(\mathcal{K}) \).

In this paper, we consider the elliptic K3 surfaces defined over \( k = \mathbb{Q} \) with the generic fiber given by the Weierstrass equation \( \mathcal{E}_n : y^2 = x^3 + t^n + 1/t^n \). We will determine the splitting field \( \mathcal{K}_n \) and find an explicit set of independent generators for \( \mathcal{E}_n(\mathcal{K}_n(t)) \) for \( 1 \leq n \leq 6 \).

1. INTRODUCTION AND THE MAIN RESULTS

Let \( \mathcal{E} \) be an elliptic curve defined over a number field \( k \subset \mathbb{C} \) that is isomorphic to the generic fiber of an elliptic surface over the function field \( k(t) \) of the projective line \( \mathbb{P}^1_k \). Given any subfield \( K \subset \mathbb{C}(t) \), the set \( \mathcal{E}(K) \) of \( K \)-rational points of \( \mathcal{E} \) is known to be a finitely generated abelian group and has a lattice structure, called the Mordell-Weil lattices. By the splitting field \( \mathcal{E}(k(t)) \) of \( \mathcal{E}(K) \) we mean the smallest subfield \( \mathcal{K} \) of \( \mathbb{C} \), which is a finite extension of \( k \) and one has \( \mathcal{E}(\mathbb{C}) = \mathcal{E}(\mathcal{K}) \). It is a well-known fact that \( \mathcal{K}|k \) is a Galois extension with the finite Galois group \( G = \text{Gal}(\mathcal{K}|k) \) and the \( G \)-invariant elements of \( \mathcal{E}(\mathcal{K}) \) are the \( \mathcal{E}(k(t)) \)-rational points \([1]\).

In \([2]\), T. Shioda introduced the theory of Mordell–Weil lattice associated to an elliptic curve \( \mathcal{E} \) defined over \( \mathbb{C}(t) \), which is known to be isomorphic with \( \mathcal{E}(\mathbb{C}(t)) \). The splitting field can also be defined in relation with Mordell–Weil lattices as follows. There is a natural action of \( \text{Gal}(\mathbb{C}|k) \) on \( \mathcal{E}(\mathbb{C}(t)) \) that preserves the height pairing \( \langle , \rangle \) and gives a Galois representation

\[
\rho : \text{Gal}(\mathbb{C}|k) \rightarrow \text{Aut}(\mathcal{E}(\mathbb{C}(t)), \langle , \rangle).
\]

Since the height pairing \( \langle , \rangle \) on \( \mathcal{E}(\mathbb{C}(t)) \) is positive definite up to torsion subgroup, the image \( \text{Im}(\rho) \) of \( \rho \) is a subgroup of the finite group \( \text{Aut}(\mathcal{E}(\mathbb{C}(t)), \langle , \rangle) \). Hence, in the terminology of Galois theory, the splitting field \( \mathcal{K} \) is exactly the extension of \( k \) which corresponds to \( \ker(\rho) \), and we have \( \text{Gal}(\mathcal{K}|k) = \text{Im}(\rho) \). One can see more on the general theory of Mordell–Weil lattices in \([1–3]\) and the associated Galois representations in \([1, 4–6]\).

In this paper, we consider the elliptic K3 surfaces over \( \mathbb{Q}(t) \) with a generic fiber given by the equation

\[
(1.1) \quad \mathcal{E}_n : y^2 = x^3 + t^n + 1/t^n, \text{ for } 1 \leq n \leq 6.
\]

The structure of these lattices are studied by T. Shioda in \([7, 8]\) and by A. Kumar and M. Kuwata in \([9]\) with a more general setting. It is remarkable that \( \mathcal{E}_n \) is a special member, considering \( \alpha = \beta = 0 \), of the generic fiber of a more general family K3 surface defined by

\[
I^{(n)}_{\alpha, \beta} : y^2 = x^3 - 3\alpha x + \left( t^n + \frac{1}{t^n} - 2\beta \right).
\]

In particular, the invariants of the Mordell–Weil lattices of \( \mathcal{E}_n \) are determined by T. Shioda in \([7, \text{Theorem } 2.4]\), and described a generic form of their generators in \([7, \text{Theorem } 2.6]\). For the convenience of reader, we gathered those results as Theorem 2.1 in Section 2.

The main aims of this paper are to determine the splitting field \( \mathcal{K}_n \subset \mathbb{C} \) of \( \mathcal{E}_n \) and to provide an explicit set of independent generators of \( \mathcal{E}_n(\mathcal{K}_n(t)) \) for each \( 1 \leq n \leq 6 \). In order to do this, we denote by \( r_n \) the rank
of $E_n(C(t))$ and fix the following roots of unity throughout the paper:

$$i = \sqrt{-1}, \quad \zeta_3 = \frac{i\sqrt{3} - 1}{2}, \quad \zeta_5 = \frac{\sqrt{5} - 1 + i\sqrt{2}\sqrt{5} + \sqrt{5}}{4},$$

$$\zeta_6 = \frac{1 + i\sqrt{3}}{2}, \quad \zeta_8 = \frac{\sqrt{2} (1 + i)}{2}, \quad \zeta_{12} = \frac{i + \sqrt{3}}{2}.$$

In below, we describe the main results of this paper.

**Theorem 1.1.** For the cases $m = 1, 2$, we have:

(i) The Mordell–Weil lattice $E_1(C(t))$ is of rank $r_1 = 0$;

(ii) The Mordell–Weil lattice $E_2(C(t))$ is isomorphic to $E_2(K_2(t))$ with $r_2 = 4$, where $K_2 \cong \mathbb{Q}(\zeta_3, 2^{1/3})$ and a set of independent generators of $E(K_2(t))$ includes the following points:

$$P_1 = \left(2^{1/3}, t + 1/t\right), \quad P_2 = \left(\zeta_3^{2^{1/3}}, t + 1/t\right),$$

$$P_3 = \left(-2^{1/3}, t - 1/t\right), \quad P_4 = \left(-\zeta_3^{2^{1/3}}, t - 1/t\right).$$

**Theorem 1.2.** The Mordell–Weil lattice $E_3(C(t))$ is isomorphic to $E_3(K_3(t))$ with $r_3 = 8$, where

$$K_3 = \mathbb{Q}\left(\zeta_{12}, 3^\frac{1}{2}, c_t^\frac{1}{2}\right), \quad \text{with } \epsilon_1 = 2 + \sqrt{3}.$$

Moreover, a set of eight independent generators of $E(K_3(t))$ includes the following points:

$$P_j = (x_j(t), y_j(t)) = \left(\frac{a_j t^2 + b_j t + a_j}{t}, \frac{c_j t^2 + d_j t + c_j}{t}\right),$$

and $P_{j+4} = (x_j(\zeta_3 t), y_j(\zeta_3 t))$ for $j = 1, \ldots, 4$, where $a_j, b_j, c_j, d_j$ are given in Subsection 3.2.

For $n = 4, 6$, we consider the automorphism of $E_n(C(t))$ given by

$$\phi_n : (x(t), y(t)) \rightarrow (-x(\zeta_2 n t), i y(\zeta_2 n t)).$$

**Theorem 1.3.** The Mordell–Weil lattice $E_4(C(t))$ is isomorphic to $E_4(K_4(t))$ with $r_4 = 12$, where

$$K_4 = \mathbb{Q}\left(\zeta_{12}, 2^\frac{1}{2}, c_j^\frac{1}{2}, c_j^\frac{1}{2}\right), \quad \text{with } \epsilon_2 = 11\sqrt{2} + 9\sqrt{3}, \quad \text{and } \epsilon_3 = \sqrt{2} + 5i.$$

Moreover, a set of twelve independent generators of $E_4(K_4(t))$ includes the following points:

$$P_j = \left(\frac{a_j t^2 + b_j t + a_j}{t}, \frac{t^4 + c_j t^3 + d_j t^2 + c_j t + 2 t^2 + 1}{t^2}\right),$$

and $P_{j+6} = \phi_4(P_j)$ for $j = 1, \ldots, 6$, where $a_j, b_j, c_j, d_j$ are given in Subsection 4.1.

**Theorem 1.4.** The Mordell–Weil lattice $E_5(C(t))$ is isomorphic to $E_5(K_5(t))$ with $r_5 = 16$, where

$$K_5 = \mathbb{Q}\left(\zeta_5, \zeta_{12}, 5^\frac{1}{4}, (c_4 \epsilon_3)^\frac{1}{2}\right), \quad \text{with } \epsilon_4 = 1 - \zeta_{12}, \quad \epsilon_5 = \left(\frac{\sqrt{5} - \sqrt{3}}{2}\right)(\zeta_{12} + \zeta_{12}^9).$$

and a set of sixteen independent generators of $E_5(K_5(t))$ includes $P_j = (x_j(t), y_j(t))$ with

$$x_j(t) = \frac{t^4 + a_j t^3 + (b_j + 2) t^2 + a_j t + 1}{u_j^2 t^2},$$

$$y_j(t) = \frac{t^6 + c_j t^5 + (d_j + 3) t^4 + (2 c_j + e_j) t^3 + (d_j + 3) t^2 + c_j t + 1}{u_j^3 t^3},$$

and $P_{j+8} = (x_j(\zeta_5 t), y_j(\zeta_5 t))$ for $j = 1, \ldots, 8$, where $a_j, b_j, c_j, d_j, e_j$ and $u_j$’s are given in Subsection 5.1.
Theorem 1.5. The Mordell–Weil lattice $\mathcal{E}_6(\mathbb{C}(t)) \cong \mathcal{E}_6(\mathcal{K}_6(t))$ is isomorphic to $\mathcal{E}_6(\mathcal{K}_6(t))$ with $r_6 = 16$, where

$$\mathcal{K}_6 = \mathbb{Q}(\zeta_8, \zeta_{12}, 2^{1/6}, 3^{1/12}, 5^{1/24}, \nu_7^2),$$

and $\nu_7$ is given by (6.4). Moreover, a set of sixteen independent generators includes $P_j = (x_j(t), y_j(t))$ with

$$x_j(t) = \frac{A_{j,4} t^4 + A_{j,3} t^3 + A_{j,2} t^2 + A_{j,1} t + A_{j,0}}{t^2},$$

$$y_j(t) = \frac{B_{j,6} t^6 + B_{j,5} t^5 + B_{j,4} t^4 + B_{j,3} t^3 + B_{j,2} t^2 + B_{j,1} t + B_{j,0}}{t^3},$$

with

$$A_{j,4} = A_{j,0} = a_j, \quad A_{j,3} = A_{j,1} = b_j - 2\sqrt{2}a_j,$$
$$A_{j,2} = g_j + 4a_j - \sqrt{2}b_j,$$
$$B_{j,6} = B_{j,6} = c_j, \quad B_{j,5} = B_{j,1} = c_j + d_j - 3\sqrt{2},$$
$$B_{j,4} = B_{j,2} = d_j + 9c_j + e_j - 2\sqrt{2}, \quad B_{j,3} = h_j - 8\sqrt{2}c_j - \sqrt{2}e_j + 4d_j,$$

and the points $P_{j+8} = \phi_6(P_j)$ for $j = 1, \ldots, 8$, where $a_j, b_j, c_j, d_j, e_j, g_j$ and $h_j$ are given in Subsection 6.1.

In our computations, we mostly used the packages PolynomialTools and PolynomialIdeals in the Mathematical software Maple [10], as well as Number Theoretic software Pari/Gp included in Sagemath [11].

The organization of this paper is as follows. Prior to proving the main results, we provide the preliminary facts on the Mordell–Weil lattice of $\mathcal{E}_n$ in the next section. Then, we prove Theorems 1.1 and 1.2 in Section 3. In the last three sections, we demonstrate the proof of Theorems 1.3, 1.4 and 1.5 respectively.

2. Preliminaries on the Mordell–Weil Lattice of $\mathcal{E}_n$

In this section, we recall some of the known results on the elliptic K3 surface $\mathcal{E}_n$ defined over $\mathbb{Q}(t)$ from [12]. For a given lattice $(L, (,))$ and an integer $m \geq 2$, we let $L[m]$ be a lattice with the height pairing $m \cdot (,)$.

We denote by $M_n$ the Mordell–Weil lattice $\mathcal{E}_n(\mathbb{C}(t))$, which is a torsion free lattice, see [7, Lemma 5.2]. It is clear that $\mathcal{E}_n$ is obtained from $\mathcal{E}_1$ by the base change $t \to t^n$. Hence, we let $N_n = M_1[n]$ for each $2 \leq n \leq 6$.

In order to study on the lattice $M_n$, we will consider the Mordell–Weil lattice $M'_n = \mathcal{E}'_n(\mathbb{C}(s))$ of the rational elliptic surface $\mathcal{E}'_n : y^2 = x^3 + f_n(s)$ and $f_n(s)$ is a polynomial defined as follows,

$$f_n(s) = \begin{cases} 
  s^2 - 2 & n = 2, \\
  s^3 - 3s & n = 3, \\
  s^4 - 4s^2 + 2 & n = 4, \\
  s^5 - 5s^3 + 5s & n = 5, \\
  s^6 - 6s^4 + 9s^2 - 2 & n = 6.
\end{cases}$$

We denote by $\mathcal{K}'_n$ the splitting field of the elliptic surface $\mathcal{E}'_n$ over $\mathbb{Q}(s)$ for $1 \leq n \leq 6$ which will be determined in the next sections. The singular fibers of $\mathcal{E}'_n$ are of type $II$ over the roots of $f_n$, and are of type $IV^*, IV^*, IV, II, II$ at $s = \infty$, and hence applying the Shioda–Tate’s formula one can see that rank of $M'_n$ is $2, 4, 6, 8, 8$ for $n = 2, \ldots, 6$ respectively. Indeed, we have $M'_n \cong \{0\}, A_2^1, D_6^1, E_6^1, E_8, E_8, E_8, E_8$, for $n = 1, 2, 3, 4, 5, 6$, respectively. Here, $A_2^1$ indicates the dual lattice of root lattice $A_2$, etc. The minimal norms of these lattices are $0, 2/3, 1, 4/3, 2, 2$, respectively. We refer the reader to [7] for proof of the above results as well as the following theorem.

Theorem 2.1. With the above notations, the invariants of $M_n = \mathcal{E}_n(\mathbb{C}(t))$ are given in the following table:

| $n$  | 1   | 2   | 3   | 4   | 5   | 6   |
|------|-----|-----|-----|-----|-----|-----|
| $r_n$| 0   | 4   | 8   | 12  | 16  | 16  |
| $\text{det}(M_n)$ | $2^2/3^3$ | $3^3/4^2$ | $4^2/3^2$ | $5^2$ | $6^2$ |
| $\mu_n$ | -   | 4/3 | 2   | 8/3 | 4   | 4   |
where \( \mu_n \) denotes the length of minimal sections. Moreover, the lattice \( M_n \) is generated by the points \( P = (x(t), y(t)) \) with the coordinates

\[
x(t) = \frac{A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4}{t^2}, \quad (A_i \in \mathbb{C}) \\
y(t) = \frac{B_0 + B_1 t + B_2 t^2 + B_3 t^3 + B_4 t^4 + B_5 t^5 + B_6 t^6}{t^3}, \quad (B_j \in \mathbb{C}).
\]

More precisely, for \( n = 2 \), a set of independent generators of \( M_2 \) is given by \((\alpha, t + 1/t)\) and \((\alpha', t - 1/t)\), where \( \alpha \) and \( \alpha' \) run over the roots of cubic polynomials \( u^3 - 2 \) and \( u^3 + 2 \), respectively. For \( n > 2 \), the lattice \( M_n \) is generated by the following set of points:

1. \((x'(t + \frac{1}{t}), y'(t + \frac{1}{t})) \) and \((x'(\zeta_n t + \frac{1}{\zeta_n t}), y'(\zeta_n t + \frac{1}{\zeta_n t}))\) for \( n = 3, 5 \),
2. \((x'(t + \frac{1}{t}), y'(t + \frac{1}{t})) \) and \((-x'(\zeta_{2n} t + \frac{1}{\zeta_{2n} t}), y'(\zeta_{2n} t + \frac{1}{\zeta_{2n} t}))\) for \( n = 4, 6 \),

where \((x'(s), y'(s))\) belongs to a generating set of \( M_n' \) with the coordinates:

\[
x'(s) = a_0 + a_1 s + a_2 s^2, \quad y'(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3, \quad \text{with } a_i, b_j \in \mathbb{C}.
\]

Here is an sketch of the proof of the above result. Letting \( T = t^n, w = T + 1/T, \) and \( L_n = M_n'[2] \) for \( 1 \leq n \leq 6 \), considering the elliptic \( E_0 : y^2 = x^3 + w \) over \( \mathbb{C}(w) \) and ignoring the lattice structures, we have \( E_0 \cong E_1 \) and so \( M_n = E_0(\mathbb{C}(t)) \), \( L_n = E_0(\mathbb{C}(s)) \), and \( N_n = E_0(\mathbb{C}(T)) \). We note that \( \mathbb{C}(t) \) is a Galois extension of \( \mathbb{C}(w) \) with Galois group \( G = \langle \tau_0, \tau_n \rangle \) with \( \tau_0 : t \to 1/t \) and \( \tau_n : t \to \zeta_n t \), where \( \zeta_n \) is an \( n \)-th root of the unity. In the terminology of Galois Theory, the fields \( \mathbb{C}(s) \) and \( \mathbb{C}(T) \) correspond to the subgroups \( \langle \tau_0 \rangle \) and \( \langle \tau_n \rangle \), and the invariant sublattices of \( M_n \) are \( L_n \) and \( N_n \), respectively.

By [7, Lemmata. 7.2, 7.3], we have \( L_n \cap N_n = \{0\} \) and \( L_n \oplus N_n \) is an orthogonal direct sum of lattices. Moreover, if we let \( \bar{L}_n = \tau_n(L_n) \subseteq M_n \), then \( \bar{L}_n = E_0(\mathbb{C}(s')) \) with \( s' = \tau_n(s) = \zeta_n t + \frac{1}{\zeta_n t} \) such that \( L_n \cap \bar{L}_n = \{0\} \) for odd \( n \) and \( L_n \cap \bar{L}_n \cong M_n' \) otherwise. In [7, Lemma 7.4], it is proved that \( M_n = L_n + \bar{L}_n \) for \( n = 3, 5 \) and \( \det(M_n) \) is equal to \( 3^4/4^2 \) for \( n = 3 \), and \( 5^4 \) for \( n = 5 \). In the case of \( n = 4, 6 \), denoting the fourth root of the unity by \( \i \), redefining \( \bar{L}_n \) as the image of \( L_n \) by the following automorphism of \( M_n \),

\[
\phi_n : (x(t), y(t)) \to (-x(\zeta_2 t), \i y(\zeta_2 t)),
\]

and using [7, Lemma 7.5], we have \( L_n \cap \bar{L}_n = \{0\} \) and \( \det(L_n + \bar{L}_n) = 4^4/3^2 \) for \( n = 4 \) and \( 6^4 \) for \( n = 6 \). Therefore, one may conclude that \( N_n \oplus L_n \oplus \bar{L}_n \) is a sublattice of finite index in \( M_n \) for \( n = 4, 6 \) We refer the readers to Theorems 2.4 and 2.6 in [7] to see a detailed proof.

3. THE CASES \( \mathcal{E}_1, \mathcal{E}_2, \) AND \( \mathcal{E}_3 \)

In this section, we consider the Mordell–Weil lattices of the simple cases \( \mathcal{E}_1, \mathcal{E}_2, \) and \( \mathcal{E}_3 \).

3.1. **Proof of Theorem 1.1.** The cases \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are treated in [13, Theorems 1.1, 1.2 and 6.1] and also [12, Theorem 7.1]. In the case of \( \mathcal{E}_2 \), by Theorem 2.1, a set of independent generators can be found between points of the form \((a, bt + c + d/t)\). Substituting into the equation of \( \mathcal{E}_2 \) leads to \( c = 0, b, d \in \{\pm 1\} \); if \( b \) and \( d \) have same sign, then \( a^3 - 2 = 0 \) and otherwise \( a^3 + 2 = 0 \). Hence, there are six points totally and one can check that the Gram matrix of the points \( P_1, \ldots, P_4 \) given in Theorem 1.1 is

\[
R_2 = \frac{2}{3} \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix},
\]

which has the determinant \( 2^4/3^2 \) as desired.
3.2. Proof of Theorem 1.2. We consider the rational elliptic surface \( E'_3 \) : \( y^2 = x^3 - (s^3 - 3s) \) with discriminant \( 27s^2(s^2 - 3)^2 \). The singular fibers of \( E'_3 \) are of type II over \( s = 0, \pm \sqrt[3]{3} \), and of type I\(_0\) over \( s = \infty \). Applying the Shioda–Tate’s formula shows that the rank of \( E'_3(\mathbb{Q}(s)) \) is equal to 4 and \( E'_3(\mathbb{Q}(s)) \cong D'_4 \).

In order to find a set of independent generators, we consider the points \( Q = (as + b, cs + d) \) and substitute its coordinates into the equation of \( E'_3 \) : \( y^2 = x^3 - (s^3 - 3s) \) to obtain the following equalities:

\[
a^3 = -1, \quad c^2 - 3a^2b = 0, \quad -3ab^2 + 2cd + 3 = 0, \quad d^2 - b^3 = 0.
\]

Form the second and third equities, we get \( b = c^2/3a^2 \), and \( d = -(c^4 + 1)/6c \). Then, the equality \( b^3 = d^2 \) gives the polynomial \( c^8 - 54c^4 - 243 \), whose roots are as:

\[
\pm 3^{1/8} \epsilon_1^{1/4}, \quad \pm 3^{1/8} (\epsilon_1')^{1/4}, \quad \pm 3^{1/8} (\epsilon_1')^{1/4} \epsilon_2, \quad \pm 3^{1/8}(\epsilon_1')^{1/4} \epsilon_2',
\]

where

\[
\epsilon_1 = 2 + \sqrt{3}, \quad \epsilon_1' = 2 - \sqrt{3}, \quad \epsilon_2 = \frac{\sqrt{2(1+i)}}{2}, \quad \epsilon_2' = \frac{\sqrt{2(1-i)}}{2}.
\]

The above eight roots together with the three roots of \( a^3 + 1 \), say \( a = -1, (1\pm i\sqrt{3})/2 \), determine 24 points on \( E'_3(\mathbb{Q}(s)) \).

The points with \( a = -1 \) generate a sublattice isomorphic to the unit matrix of degree four. Letting \( u = 1/\sqrt[3]{p_1}(P) = d/b \), we have \( b = u^2, \quad d = u^3 \) and hence \( u^5 - 6u^4 - 3 = 0 \). It is a factor of the fundamental polynomial of \( E'_3 \) over \( \mathbb{Q}(s) \), say

\[
\Phi'_3(u) = u^{24} - 270u^{12} - 27 = (u^8 - 6u^4 - 3)(u^{16} + 6u^{12} + 39u^8 - 18u^4 + 9),
\]

The first factor has roots \( u = i^\ell 3^{1/8} \epsilon_1^{1/4}, \) or \( i^\ell 3^{1/8} (\epsilon_1')^{1/4} \) for \( \ell = 0, 1, 2, 3 \) with \( \epsilon_1 = 2 + \sqrt{3} \) and \( \epsilon_1' = 2 - \sqrt{3} \); The second factor has roots \( u = i^\ell \alpha^{1/4} \) for \( \ell = 0, 1, 2, 3 \) with \( \alpha \) in the set \( \{ \zeta_3 \sqrt[3]{3}, -\zeta_3 \sqrt[3]{3}, \zeta_3' \sqrt[3]{3}, -\zeta_3' \sqrt[3]{3}, \zeta_6 \sqrt[3]{3}, \zeta_6' \sqrt[3]{3} \} \).

By the straight computations and similar argument as given in [14, Section 6], one can check that the following four roots

\[
u_1 = 3^{1/8} \epsilon_1^{1/4}, \quad u_2 = -3^{1/8} \epsilon_1^{1/4}, \quad u_3 = i 3^{1/8} (\epsilon_1')^{1/4}, \quad u_4 = \zeta_3 3^{1/8} \epsilon_1^{1/4}
\]

of \( \Phi'_3(u) \) leads to the generators of \( E'_3(\mathbb{Q}(s)) \), say \( Q_j = (a_j s + b_j, c_j s + d_j) \) for \( j = 1, \ldots, 4 \), where

\[
a_1 = -1, \quad b_1 = 3^{1/4} \epsilon_1^{1/2}, \quad c_1 = 3^{1/4} \epsilon_1^{1/2}, \quad d_1 = -3^{1/4} \epsilon_1^{1/2},
\]

\[
a_2 = -1, \quad b_2 = -3^{1/4} \epsilon_1^{1/2}, \quad c_2 = i 3^{1/4} \epsilon_1^{1/2}, \quad d_2 = i 3^{1/4} \epsilon_1^{1/2},
\]

\[
a_3 = -1, \quad b_3 = i 3^{1/4} (\epsilon_1')^{1/2}, \quad c_3 = \frac{\sqrt{2(1+i)}}{2} 3^{1/4} (\epsilon_1')^{1/2}, \quad d_3 = -\frac{\sqrt{2(1-i)}}{2} 3^{1/4} (\epsilon_1')^{1/2},
\]

\[
a_4 = \zeta_6, \quad b_4 = \zeta_3 3^{1/4} \epsilon_1^{1/2}, \quad c_4 = 3^{1/4} \epsilon_1^{1/2}, \quad d_4 = -3^{1/4} \epsilon_1^{1/2}.
\]

The Gram matrix of these points has determinant \( 1/4 \) and is given by

\[
R'_3 = \frac{1}{2} \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 2 & 0 & 1 \\
0 & 0 & 2 & 1 \\
1 & 1 & 1 & 2
\end{pmatrix}.
\]

Thus the splitting field \( K'_3 \) of \( E'_3 \) over \( \mathbb{Q}(t) \) is determined by the roots of \( a^3 + 1 \) and \( c^8 - 54c^4 - 243 \), hence

\[
K'_3 \cong \mathbb{Q}(\zeta_6, 3^{1/4}, \epsilon_1^{1/4}, \epsilon_2^{1/4}).
\]

Using Theorem 2.1, and substituting \( s = t + 1/t \) and \( s = \zeta_3 t + \frac{1}{\zeta_3 t} \) into the coordinates of \( Q_j \)'s for \( j = 1, \ldots, 4 \), we obtain

\[
P_j = (x(t), y(t)) = \left( \frac{a_j t^2 + b_j t + a_j}{t}, \frac{c_j t^2 + d_j t + c_j}{t} \right),
\]

\[
P_{j+4} = (x(\zeta_3 t), y(\zeta_3 t)) = \left( \frac{a_j \zeta_3^2 t^2 + b_j \zeta_3 t + a_j}{\zeta_3 t}, \frac{c_j \zeta_3^2 t^2 + d_j \zeta_3 t + c_j}{\zeta_3 t} \right).
\]
By the properties of the height pairing and knowing that $K_3(t)$ is a quadratic extension of $K_3(s)$, where $K_3 = K'_3(ζ_3) = K'_3$, we have

$$\langle P_i, P_{i+j} \rangle = -\frac{1}{2} \langle P_i, P_j \rangle \quad (1 \leq i, j \leq 4).$$

Using this fact and the matrix $R'_3$, one can see that the Gram matrix of the eight points $P_1, \ldots, P_8$ is

$$R_3 = \frac{1}{2} \begin{pmatrix}
2 & 0 & 0 & 1 & -1 & 0 & 0 & -\frac{1}{2} \\
0 & 2 & 0 & 1 & 0 & -1 & 0 & -\frac{1}{2} \\
0 & 0 & 2 & 1 & 0 & 0 & -1 & -\frac{1}{2} \\
1 & 1 & 1 & 2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1 \\
-1 & 0 & 0 & -\frac{1}{2} & 2 & 0 & 0 & 1 \\
0 & -1 & 0 & -\frac{1}{2} & 0 & 2 & 0 & 1 \\
0 & 0 & -1 & -\frac{1}{2} & 0 & 0 & 2 & 1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & 1 & 1 & 2
\end{pmatrix},$$

and its determinant is $3^4/4^2$ as given by Theorem 2.1.

4. The Case of $E_4$

In this section, we prove Theorem 1.3 using the following result.

**Theorem 4.1.** The splitting field $K'_4$ of the rational elliptic surface $E'_4 : y^2 = x^3 - (s^4 - 4s^2 + 2)$ is

$$K'_4 = \mathbb{Q} \left( \zeta_{12}, 2^{\frac{1}{2}}, \frac{\zeta}{2}, \frac{\zeta}{3} \right)$$

with $c_2 = 11\sqrt{2} + 9\sqrt{3}$, $c_3 = \sqrt{2} + 5i$ and the Mordell–Weil lattice $E'_4(K'_4(s))$ is generated by the points

$$Q_j = (a_j s + b_j, s^2 + c_j s + d_j),$$

for $j = 1, \ldots, 6$, where $a_j, b_j, c_j, d_j$ are given in Subsection 4.1.

4.1. **Proof of Theorem 4.1.** Since the determinant of $E'_4$ is $-27(s^4 - 4s^2 + 2)^2$, the singular fibers of $E'_4$ are of type $II$ over the roots of $s^4 - 4s^2 + 2$, and $IV$ over $s = \infty$. Then, the Shioda–Tate’s formula shows that the Mordell–Weil rank of $E'_4(\mathbb{C}(s))$ is equal to 6 and $E'_4(\mathbb{C}(s)) \cong E_6$. Based on [14, Theorem 10.5], a set of six independent generators of $E'_4(\mathbb{C}(s))$ can be found between the set of 27 rational points $Q = (as + b, s^2 + cs + d)$. Substituting these into the equation of $E'_4$ leads to the following equalities:

$$2c - a^3 = 0,$$
$$3a^2b - c^2 - 2d - 4 = 0,$$
$$3ab^2 - 2cd = 0,$$
$$b^3 - d^2 + 2 = 0.$$

From the first two qualities, we get

$$c = \frac{a^3}{2}, \quad d = -\frac{1}{8}(a^6 - 12a^2b + 16),$$

and two polynomials in $b$ with coefficients in the ring $\mathbb{Q}[a]$ as

$$p_1 := b^2 - 12a^5b + 16a^3 + 24a^a, \quad p_2 := -64b^3 + 144a^4b^2 - 24a^2(a^6 + 16b + a^{12} + 32a^6 + 128).$$

Taking their resultant respect to $b$ of $p_1$ and $p_2$ gives a polynomial of degree 27 in terms of $a$ as follows:

$$\Phi(a) = a^3 \left( a^{12} - 352a^6 - 128 \right) \left( a^{12} - 32a^6 + 3456 \right).$$

By (4.1) and using the roots of $\Phi(a)$, we obtain the coefficients of 27 points of the form $Q = (as + b, s^2 + cs + d)$ in $E'_4(\mathbb{C}(t))$. The factors of degree 12 of $\Phi(a)$ can be decomposed as follows:

$$\Phi_1(a) = a^{12} - 352a^6 - 128 = \prod_{\ell=0}^{5} \left( a - \zeta_{12}^{2\ell} \cdot 2^{\frac{1}{2}} \cdot \zeta_{2}^{\frac{1}{2}} \right) \prod_{\ell=0}^{5} \left( a - \zeta_{12}^{2\ell+1} \cdot 2^{\frac{1}{2}} \cdot \zeta_{2}^{\frac{1}{2}} \right),$$

$$\Phi_2(a) = a^{12} - 32a^6 + 3456 = \prod_{\ell=0}^{5} \left( a - \zeta_{12}^{2\ell} \cdot 2^{\frac{1}{2}} \cdot \zeta_{3}^{\frac{1}{2}} \right) \prod_{\ell=0}^{5} \left( a - \zeta_{12}^{2\ell+1} \cdot 2^{\frac{1}{2}} \cdot \zeta_{3}^{\frac{1}{2}} \right).$$
where

\[(4.3) \quad c_2 = 11\sqrt{2} + 9\sqrt{3}, \quad c_2 = -11\sqrt{2} + 9\sqrt{3}, \quad c_3 = \sqrt{2} + 5i, \quad c_3 = \sqrt{2} - 5i.\]

Thus, the splitting field of $\mathcal{E}_4'$ is equal to $\mathcal{K}_4' = \mathbb{Q}(\sqrt[3]{2}, 2^{\frac{1}{2}}, \epsilon_2^\frac{1}{2}, \epsilon_3^\frac{1}{2})$, for which the polynomial $\Phi(a)$ can be written as a product of linear factors. Thus, $\mathcal{E}_4'(\mathbb{C}(s)) = \mathcal{E}_4'(\mathcal{K}_4'(s))$ and by straight height computations we obtained its six independent generators $Q_j = (a_j s + b_j, s^2 + c_j s + d_j)$ with the coefficients as follows:

\[
\begin{align*}
  a_1 &= 0, & b_1 &= 2^{\frac{1}{3}}, & c_1 &= 0, & d_1 &= -2, \\
  a_2 &= 2^{\frac{1}{12}} \epsilon_2^\frac{1}{2}, & b_2 &= 2\sqrt[3]{2 + 3}, & c_2 &= 2^{\frac{1}{5}} \epsilon_2^\frac{1}{2}, & d_2 &= 3\sqrt{2}(\sqrt{2} + \sqrt{3}), \\
  a_3 &= 2^{\frac{1}{12}} \epsilon_3^\frac{1}{2}, & b_3 &= \frac{2^{\frac{1}{5}} \epsilon_3^\frac{1}{2} (\epsilon_3 + 1)}{9}, & c_3 &= 2^{\frac{1}{5}} \epsilon_3^\frac{1}{2}, & d_3 &= 2 + i\sqrt{2}, \\
  a_4 &= 0, & b_4 &= \epsilon_3 2^{\frac{1}{3}}, & c_4 &= 0, & d_4 &= -2, \\
  a_5 &= \zeta_{12} \cdot 2^{\frac{1}{12}} \cdot (\epsilon_2^\frac{1}{2}), & b_5 &= \zeta_{12} \cdot 2^{\frac{1}{2}} \cdot (\epsilon_2^\frac{1}{2}) (\sqrt{2} + \sqrt{3}), & c_5 &= 2^{\frac{3}{2}} (\epsilon_2^\frac{1}{2}), & d_5 &= 3\sqrt{2}(\sqrt{2} - \sqrt{3}), \\
  a_6 &= \zeta_{12} 2^{\frac{1}{12}} \epsilon_3^\frac{1}{2}, & b_6 &= \frac{-2^{\frac{1}{5}} \epsilon_3^\frac{1}{2} (\epsilon_3 - 1)}{9}, & c_6 &= -2^{\frac{1}{5}} \epsilon_3^\frac{1}{2}, & d_6 &= 2 + i\sqrt{2}.
\end{align*}
\]

The Gram matrix of the points $Q_1, \ldots, Q_6$ is given by

\[
R_4' = \frac{1}{3} \begin{pmatrix}
4 & -2 & 1 & -2 & 1 & -2 \\
-2 & 4 & 1 & 1 & 1 & 1 \\
1 & 1 & 4 & -2 & 1 & 1 \\
-2 & 1 & -2 & 4 & -2 & 1 \\
1 & 1 & 1 & -2 & 4 & -2 \\
-2 & 1 & 1 & 1 & -2 & 4
\end{pmatrix},
\]

which is of determinant 1/3 as desired. Therefore, they are independent generators of $\mathcal{E}_4'(\mathcal{K}_4'(s))$.

4.2. Proof of Theorem 1.3. Considering Theorem 2.1 and substituting $s = t + 1/t$ into the coordinates of points $Q_j = (a_j s + b_j, s^2 + c_j s + d_j) \in \mathcal{E}_4'(\mathcal{K}_4'(s))$, we obtain

\[
P_j = \left( a_j t^2 + b_j t + a_j, \frac{t^4 + c_j t^3 + d_j t^2 + c_j t + 2t^2 + 1}{t^2} \right),
\]

and their images $P_{j+6} = \phi_4(P_j)$ under the automorphism $\phi_4$ of $\mathcal{E}_4$ with coordinates

\[
x(P_{j+6}) = -\frac{(1 + i) a_j t^2 + 2b_j t + (2 - 2i) a_j}{2t},
\]

\[
y(P_{j+6}) = \frac{it^4 + (1 + i) c_j t^3 + (4 + 2d_j) t^2 + (2 - 2i) c_j t - 4i}{2t^2},
\]

for $j = 1, \ldots, 6$, which all together generate $\mathcal{E}_4(\mathbb{C}(t)) = \mathcal{E}_4(\mathcal{K}_4(t))$, where $\mathcal{K}_4 = \mathcal{K}_4'(\zeta_3)$. The Gram matrix of the points $P_1, \ldots, P_{12} \in \mathcal{E}_4(\mathcal{K}_4(t))$ is given by
5. The case of $\mathcal{E}_5$

In this section, we prove Theorem 1.4 using the following result on the splitting field and a set of independent generators of $\mathcal{E}_5'$: $y^2 = x^3 + s^5 - 5s^3 + 5s$ over $\mathbb{C}(s)$. The reader can compare this result with [15, Corollary 6].

**Theorem 5.1.** The splitting field of $\mathcal{E}_5'$ is equal to $\mathcal{K}_5^* = \mathbb{Q}\left(\zeta_{12}, 5^{1/3}, (\epsilon_4\epsilon_5^{1/2})\right)$, with $\epsilon_4 = 1 - \zeta_{12}$, and $\epsilon_5 = (\sqrt{3} - \sqrt{5}) (\zeta_{12} + \zeta_{12}^{10})/2$. The lattice $\mathcal{E}_5'(\mathcal{K}_5^*(s))$ is generated by the points

$$Q_j = (x_j(s), y_j(s)) = \left(\frac{s^2 + a_j s + b_j}{u_j^2}, \frac{s^3 + c_j s^2 + d_j s + e_j}{u_j^3}\right),$$

for $j = 1, \ldots, 8$, where $a_j, b_j, c_j, d_j, e_j$ are given at the end of Subsection 5.1, and the constants $u_j$'s are as follow:

- $u_1 = i5^{1/3} (\epsilon_4\epsilon_5^{1/2})^{1/2}$,
- $u_2 = i5^{1/3} (\epsilon_4^{-1}\epsilon_5^{1/2})^{1/2}$,
- $u_3 = i5^{1/3} (\epsilon_4\epsilon_5^{-1})^{1/2}$,
- $u_4 = i5^{1/3} (\epsilon_4\epsilon_5^{-1})^{-1/2}$,
- $u_5 = i5^{1/3} (\epsilon_4\epsilon_5^{-1})^{1/2} \epsilon_6$,
- $u_6 = i5^{1/3} (\epsilon_4^{-1}\epsilon_5^{1/2})^{-1/2} \epsilon_6$,
- $u_7 = i5^{1/3} (\epsilon_4\epsilon_5^{1/2})^{1/2} \epsilon_6^{-1}$,
- $u_8 = i5^{1/3} (\epsilon_4\epsilon_5\epsilon_6)^{-1} \epsilon_6 = \frac{1 + \sqrt{5}}{2^{1/2}} \zeta_{12}$.

5.1. **Proof of Theorem 5.1.** Since $\mathcal{E}_5'(\mathbb{C}(s))$ is isomorphic to $E_8$, so there are 240 points $Q \in \mathcal{E}_5'(\mathbb{C}(s))$, corresponding to the 240 minimal roots of $E_8$, of the form:

$$Q = \left(\frac{s^2 + as + b}{u^2}, \frac{s^3 + cs^2 + ds + e}{u^3}\right),$$

for suitable constants $a, b, c, d, e, u \in \mathbb{C}$. Substituting the coordinates of $Q$ into the equation of $\mathcal{E}_5'$ and letting $U = u^6$, we get the following six relations:

$$\begin{align*}
2c - 3a - U &= 0, \\
2d - 3a^2 + c^2 - 3b &= 0, \\
2e - a^3 - 6ab + 2cd + 5U &= 0, \\
3b^2 + 3a^2b - 2ce - d^2 &= 0, \\
3ab^2 + 5U - 2de &= 0, \\
(b^3 - c^2) &= 0. 
\end{align*}$$

(5.2)
By the first three relations, we obtain $c, d, e$ in terms of $a, b$, and $U$ as:

\[(5.3) \quad c = \frac{3a + U}{2}, \quad d = \frac{3a^2 - c^2 + 3b}{2}, \quad e = \frac{a^3 + 6ab - 2cd - 5U}{2}.
\]

Substituting these into the other three ones, we obtain three relations among $a, b, U$ of the respective degrees $2, 2, 3$ in $b$. Taking the polynomial remainder of the quotient of one quadratic polynomial in $b$ by the other leads to a linear relation in $b$. Thus, eliminating $b$, we get two relations $\psi_1(a, U) = 0$ and $\psi_2(a, U) = 0$ among $a$ and $U$ of degrees 8 and 12 in $a$ (as well as in $U$). By taking the resultant of these relations with respect to $a$, up to a constant, we achieve finally a polynomial of degree 40 in $U$ that is equal to

\[
\Phi(U) = U^{40} - 75600 U^{38} + 1211326200 U^{36} - 6272129430000 U^{34} - 98750071954102500 U^{32} + 1436649623797470000 U^{30} - 143601692046338062995000 U^{28} + 17522835032723727101850000 U^{26} + 106262025664409273130343750 U^{24} - 82253763657541101030063750000 U^{22} + 2021939032512238541864405250000 U^{20} + 10710959483616788860204048750000 U^{18} + 55227097819116401769937600937500 U^{16} + 136561340826684494930298906250000 U^{14} - 3108446951885930154911783203125000 U^{12} + 402307798349967583894468750000 U^{10} + 1092957929961799504134526172890625 U^8 + 287123077484861926647307500000000 U^6 + 35826210962402514220000000000000000 U^4 - 319035872136360000000000000000000 U^2 + 8707129344000000000000.
\]

Letting $V = U^2$, the polynomial $\Phi(V)$ decomposes into four irreducible factors in $\mathbb{Z}[V]$, namely,

\[
\Phi_1(V) = V^4 - 56700 V^3 - 1204210 V^2 - 283500 V + 25,
\]

\[
\Phi_2(V) = V^4 + 6660 V^3 - 936810 V^2 - 91320300 V + 25,
\]

\[
\Phi_3(V) = V^4 - 1260 V^3 + 1178590 V^2 - 4592700 V + 13286205,
\]

\[
\Phi_4(V) = V^{12} - 24300 V^7 + 280192930 V^6 - 18498253500 V^5 + 569262158025 V^4 + 591944112000 V^3 + 28673969152000 V^2 + 796262400000 V + 10485760000.
\]

Then, one can decompose all polynomials $\Phi_1, \Phi_2, \Phi_3$ and $\Phi_4$ over $k_0 = \mathbb{Q}(i, \sqrt{3}, \sqrt{5}) = \mathbb{Q}(\zeta_{12}, \sqrt{5})$ into the product of linear factors as given below,

\[
\Phi_1(V) = (V - v_1)(V - v_2)(V - v_3)(V - v_4),
\]

\[
\Phi_2(V) = (V - v_5)(V - v_6)(V - v_7)(V - v_8),
\]

\[
\Phi_3(V) = (V - v_9)(V - v_{10})(V - v_{11})(V - v_{12}),
\]

\[
\Phi_4(V) = (V - v_{13})(V - v_{14})(V - v_{15})(V - v_{16})(V - v_{17})(V - v_{18})(V - v_{19}),
\]

where $\bar{z}$ gives the conjugate of any complex number $z$, and the maps $\sigma$ and $\tau$ change respectively the signs of $\sqrt{3}, \sqrt{5}$, and $v_1, v_2, v_3, v_4$ are as follows:

\[
v_1 = \left(3660 \sqrt{5} - 8190\right) \sqrt{5} - 3644 \sqrt{5} + 14175,
\]

\[
v_2 = \left(-420 \sqrt{5} - 990\right) \sqrt{5} - 784 \sqrt{5} - 1665,
\]

\[
v_3 = 315 - 440 i + (140 - 198 i) \sqrt{5},
\]

\[
v_4 = \frac{3510 - 3300 i - (1560 - 1485 i) \sqrt{5}}{2} \sqrt{5} + \frac{6075}{2} - 28601 - (1350 - 1287 i) \sqrt{5}.
\]

Hence, the 240 roots of the fundamental polynomial $\Phi(u^{12})$ of $\mathcal{E}'_5$ are of the form $u = \zeta_{12}^\ell u^{11/12}$ for $\ell = 0, 1, \ldots, 11$, where $v$ varies on the set of 20 roots of $\Phi(V)$. For each root $u$ of $\Psi(u) := \Phi(u^{12})$, there is at least one common zero $a$ of the relations $\psi_1(a, U) = 0$ and $\psi_2(a, U) = 0$. When $u$ and $a$ are determined in this way, one can find at least one $b$ which satisfies the three relations among $a, b, U$ mentioned above. Therefore, by determining $c, d, e$ using (5.3), each root $u$ of the fundamental polynomial corresponds to at least one rational point $Q \in \mathcal{E}_5' (\mathcal{K}_5'(s))$ such that $s_{p, \infty}(Q) = u$, where $s_{p, \infty}$ is the specializing map of $\mathcal{E}_5' (\mathcal{K}_5'(s))$ to the
additive group of $\mathcal{K}_5'$. Indeed, it maps $Q \in \mathcal{E}_5'(\mathcal{K}_5'(s))$ to the intersection point of the section $(Q)$ and the fiber over $\infty$ which lies in the smooth part of the additive singular fiber $\pi^{-1}(\infty)$.

Since $\mathcal{E}_5'$ has no reducible fiber and all the 240 sections $Q_j$'s corresponding to the roots of $\Psi(u)$ are points in $\mathcal{E}_5'(\mathcal{K}_5'(s))$ with polynomial coordinates, we have $(Q_j, Q_j) = 2$ and $(Q_j, Q_{j'}) = 1 - (Q_j \cdot Q_{j'})$, where $(Q_j, Q_{j'})$ denotes the intersection number for any $1 \leq j_1 \neq j_2 \leq 2$. Assuming $x_j = x(Q_j)$ and $y_j = y(Q_j)$, the number $(Q_j, Q_{j'})$ can be computed by the following formula:

$$(5.4) \quad (Q_j, Q_{j'}) = \deg(\gcd(x_j - x_{j'}, y_j - y_{j'})) + \min\{2 - \deg(x_j - x_{j'}), 3 - \deg(y_j - y_{j'})\}.$$  

Using this and a direct searching between 240 sections, we obtain a subset of eight points with unimodular height paring matrix. In order to describe, let $\epsilon_4, \epsilon_5, \epsilon_6$ be as in the statement of Theorem 5.1. Then, one can check that

$$v_1 = \sqrt[5]{5} \epsilon_4^6 \epsilon_5, \quad v_2 = \sqrt[5]{5} \epsilon_4^6 \epsilon_5^{-6} \epsilon_6^6, \quad v_3 = \sqrt[5]{5} \epsilon_4^6 \epsilon_5 \epsilon_6^6,$$  

$$v_4 = \sqrt[5]{5} \epsilon_4^6 \epsilon_5^6 \epsilon_6^{-6}, \quad v_5 = \sqrt[5]{5} \epsilon_4 \epsilon_5 \epsilon_6^6,$$  

$$v_6 = \sqrt[5]{5} \epsilon_4^6 \epsilon_5 \epsilon_6^6, \quad v_7 = \sqrt[5]{5} \epsilon_4 \epsilon_5 \epsilon_6^6.$$  

Hence, the roots $v_1, v_2, v_3, v_4, v_5, v_6$ and $v_7$ correspond to the eight points as follows:

$$(5.5) \quad Q_j = \left(\frac{s^2 + a_js + b_j}{v_j^2}, \frac{s^3 + c_js^2 + d_js + e_j}{v_j^2}\right) \in \mathcal{E}_5'(\mathcal{K}_5'(s)),$$

for $j = 1, \ldots, 8$, where $a_j, b_j, c_j, d_j, e_j$ are listed at the end of this subsection and $u_j$'s are given as follows:

$$u_1 = (v_1)^{\frac{1}{5}}, \quad u_2 = (v_2)^{\frac{1}{5}}, \quad u_3 = (v_3)^{\frac{1}{5}}, \quad u_4 = (v_4)^{\frac{1}{5}},$$  

$$u_5 = (v_5)^{\frac{1}{5}}, \quad u_6 = (v_6)^{\frac{1}{5}}, \quad u_7 = (v_7)^{\frac{1}{5}}, \quad u_8 = (v_8)^{\frac{1}{5}}.$$  

Applying the specialization map $sp_{\infty} : \mathcal{E}_5'(\mathcal{K}_5') \rightarrow (\mathcal{K}_5')^+$ to these points and dividing the images by $u_1$, we obtain the following subset of the field $\mathcal{K}_5'$,

$$\left\{1, \frac{u_j}{u_1} : j = 2, \ldots, 8\right\} = \left\{1, \epsilon_4^{-1}\epsilon_6, \epsilon_6, \epsilon_4^{-1}, \epsilon_5, \epsilon_4^{-1}\epsilon_5\epsilon_6, \epsilon_5^{-1}\epsilon_6, \epsilon_4^{-1}\epsilon_5^{-1}\right\},$$

which is easy to see that they are linearly independent over $\mathbb{Q}$. Thus, the points $Q_1, \ldots, Q_8$ form a linearly independent subset generating a sublattice of rank 8 in $\mathcal{E}_5'(\mathcal{K}_5'(s))$. It is easy to check that the Gram matrix of these eight points is equal to the following unimodular matrix:

$$R_5 = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$  

Thus, the points $Q_j$'s for $j = 1, \ldots, 8$ generate the whole group $\mathcal{E}_5'(\mathcal{K}_5'(s))$ as desired. Hence, the specializing map is an isomorphism and the splitting field $\mathcal{K}_5'$ is obtained by adjoining one of the $u_j$'s, for example $u_3$, to the field $k_0$. In other words, we have $\mathcal{K}_5' = k_0(u_3) = \mathbb{Q}(\xi_{12}, 5^{\frac{1}{5}}, (\epsilon_4\epsilon_5)^{\frac{1}{5}})$. Therefore, the proof of Theorem 5.1 is finished.
Here, we list the coefficients \(a_j, b_j, c_j, d_j, e_j\) of the points \(Q_1, \ldots, Q_8\). To ease the notations, we set
\[
\begin{align*}
    a_1 &= (8\sqrt{5} - 18)\sqrt{3} + 14\sqrt{5} - 31, & a_2 &= (843 - 377\sqrt{5})\sqrt{3} - 653\sqrt{5} + 460, \\
    b_1 &= (4\sqrt{5} - 10)\sqrt{3} - 8\sqrt{5} + 15, & e_2 &= \frac{(1845 - 825\sqrt{5})\sqrt{3} - 1429\sqrt{5} + 3195}{2}, \\
    c_1 &= (12\sqrt{5} - 27)\sqrt{3} + 21\sqrt{5} - 46, & c_2 &= \frac{(2529 - 1131\sqrt{5})\sqrt{3} - 1959\sqrt{5} + 4381}{2}, \\
    d_1 &= (60 - 27\sqrt{5})\sqrt{3} + 46\sqrt{5} - 105, & d_2 &= \frac{(3\sqrt{5} + 15)\sqrt{3} + 7\sqrt{5} + 15}{2}.
\end{align*}
\]

Then the coefficients are as follows:
\[
\begin{align*}
    a_1 &= i5^4 c_{(1,-1,0)}^3 \cdot a(1), & b_1 &= b(1), & c_1 &= i5^4 c_{(1,-1,0)}^3 \cdot c(1), & d_1 &= d(1), & e_1 &= i5^4 c_{(1,-1,0)}^3, \\
    a_2 &= i5^4 c_{(1,-1,0)}^3 \cdot a(2), & b_2 &= b(2), & c_2 &= i5^4 c_{(1,-1,0)}^3 \cdot c(2), & d_2 &= d(2), & e_2 &= i5^4 c_{(1,-1,0)}^3, \\
    a_3 &= i5^4 c_{(1,1,0)}^3 \cdot a(3), & b_3 &= b(3), & c_3 &= i5^4 c_{(1,1,0)}^3 \cdot c(3), & d_3 &= d(3), & e_3 &= i5^4 c_{(1,1,0)}^3, \\
    a_4 &= i5^4 c_{(1,1,0)}^3 \cdot a(4), & b_4 &= b(4), & c_4 &= i5^4 c_{(1,1,0)}^3 \cdot c(4), & d_4 &= d(4), & e_4 &= i5^4 c_{(1,1,0)}^3, \\
    a_5 &= i5^4 c_{(1,-1,2)}^3 \cdot a(5), & b_5 &= -\sqrt{5}, & c_5 &= \frac{i5^4}{2} c_{(1,-1,2)}^3 \cdot c(5), & d_5 &= d(5), & e_5 &= \frac{i5^4}{2} c_{(1,-1,2)}^3 \cdot c(5), \\
    a_6 &= i5^4 c_{(1,-1,2)}^3 \cdot a(6), & b_6 &= -\sqrt{5}, & c_6 &= \frac{i5^4}{2} c_{(1,-1,2)}^3 \cdot c(6), & d_6 &= d(6), & e_6 &= \frac{i5^4}{2} c_{(1,-1,2)}^3 \cdot c(6), \\
    a_7 &= i5^4 c_{(1,-1,2)}^3 \cdot a(7), & b_7 &= \sqrt{5}, & c_7 &= \frac{i5^4}{2} c_{(1,-1,2)}^3 \cdot c(7), & d_7 &= d(7), & e_7 &= \frac{i5^4}{2} c_{(1,-1,2)}^3 \cdot c(7), \\
    a_8 &= i5^4 c_{(1,1,2)}^3 \cdot a(8), & b_8 &= \sqrt{5}, & c_8 &= \frac{i5^4}{2} c_{(1,1,2)}^3 \cdot c(8), & d_8 &= d(8), & e_8 &= \frac{i5^4}{2} c_{(1,1,2)}^3 \cdot c(8),
\end{align*}
\]
where \(e_{i_1,i_2,i_3}^m = e_{i_1}^{m_{i_1}} e_{i_2}^{m_{i_2}} e_{i_3}^{m_{i_3}}\) for integers \(i_1, i_2, i_3\) and \(m\).

5.2. **Proof of Theorem 1.4.** First, we note that the splitting field of the elliptic \(K3\) surface \(\mathcal{E}_5\) over \(\mathbb{Q}(t)\) is equal to \(K_5 = K_5'(\zeta_5)\) where \(K_5'\) is the splitting field of \(\mathcal{E}_5'\) over \(\mathbb{Q}(s)\). Letting \(s = t + 1/\ell\), the rational elliptic surface \(\mathcal{E}_5'\) over \(K_5(s)\) is isomorphic to \(\mathcal{E}_5\) over \(K_5(t)\) as a quadratic extension of \(K_5(s)\). Hence, the independent generators
\[
Q_j = \left( \frac{s^2 + a_j s + b_j}{u_j^2}, \frac{s^3 + c_j s^2 + d_j s + e_j}{u_j^3} \right)
\]
of \(\mathcal{E}'(K_5(s))\) give the points \(P_j = (x_j(t), y_j(t)) \in \mathcal{E}_5(K_5(t))\) where
\[
\begin{align*}
    x_j(t) &= \frac{t^4 + a_j t^3 + (b_j + 2) t^2 + a_j t + 1}{u_j^2 t^2}, \\
    y_j(t) &= \frac{t^6 + c_j t^5 + (d_j + 3) t^4 + (2 c_j + e_j) t^3 + (d_j + 3) t^2 + c_j t + 1}{u_j^3 t^3},
\end{align*}
\]
the constants \(a_j, b_j, c_j, d_j, e_j\) and \(u_j\)'s for \(j = 1, \ldots, 8\), are listed in the previous subsection. Furthermore, by letting \(s = \zeta_5 t + \frac{1}{\zeta_5 t}\) and the same argument as above, we obtain the points \(P_{j+8} = (x_j(\zeta_5 t), y_j(\zeta_5 t))\) for \(j = 1, \ldots, 8\). We note that the points \(P_j' = (t^2 x(P_j), t^3 y(P_j))\) belong to the Mordell–Weil lattice of \(\mathcal{E} : y^2 = x^3 + t(t^{10} + 1)\), which is birational to \(\mathcal{E}_5\) over \(K_5(t)\). Since \(P_j'\)'s have no intersection with the zero section of \(\mathcal{E}\), we have \(P_j' \cdot P_j' = 4, \langle P_j', P_j' \rangle = 2 - \langle P_j', P_j' \rangle\), and for any \(1 \leq j_1 \neq j_2 \leq 16\), the intersection number \(\langle P_j', P_j' \rangle\) can be computed by
\[
\langle P_j', P_j' \rangle = \gcd(x_j - x_{j_2}, y_j - y_{j_2}) + \min\{4 - \gcd(x_j - x_{j_2}), 6 - \gcd(y_j - y_{j_2})\}.
\]
Thus, we obtain the Gram matrix of the height pairing for points $P_j$’s and hence $P_j$’s as follows:

$$
R_5 = \begin{pmatrix}
4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\
0 & 4 & 2 & 0 & 0 & 0 & -2 & 2 & 0 & -2 & -1 & 1 & 0 & 2 & 2 & -1 \\
0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 2 & 0 \\
2 & 0 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 4 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & -1 & 1 \\
0 & -2 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 2 & 2 & 0 & 0 & -1 & -2 & 0 \\
0 & 2 & 0 & 0 & 2 & 0 & 0 & 4 & 1 & -1 & 0 & 2 & 0 & 1 & 0 & -2 \\
-2 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & -2 & -1 & 1 & 0 & 2 & 2 & -1 & 0 & 4 & 2 & 0 & 0 & 0 & -2 & 2 \\
0 & -1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 4 & 2 & 2 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 2 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 2 & 0 & 4 & 2 & 0 \\
0 & 2 & 2 & 0 & 0 & -1 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & 4 & 0 \\
1 & -1 & 0 & 2 & 0 & 1 & 0 & -2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 4
\end{pmatrix}.
$$

One can check that its determinant is equal to $5^4$ as desired, which shows that the points $P_j$’s for $j = 1, \ldots, 16$ form a set of independent generators of $\mathcal{E}_5$ over $K_5(t)$.

6. The case of $\mathcal{E}_6$

In this section we prove Theorem 1.5 on the elliptic $K$-3 surface $\mathcal{E}_6 : y^2 = x^3 + (t^6 + 1/t^6)$ over $\mathbb{C}(t)$. To do this, first we determine the splitting field $K_6$ and find a set of independent generators for the Mordell–Weil lattice of the rational elliptic surface $\mathcal{E}_6' : y^2 = x^3 + f_6(s)$, where

$$f_6(s) = s^6 - 6s^4 + 9s^2 - 2 = (s^2 - 2)(s^4 - 4s^2 + 1).$$

To ease the computations, we change the coordinate $\tilde{s} = s - \sqrt{2}$ to obtain the rational elliptic surface

$$\mathcal{E}_6' : y^2 = x^3 + g_6(\tilde{s}), \quad g_6(\tilde{s}) = \tilde{s}^2 - 2\sqrt{2}(\tilde{s}^2 - \sqrt{2}\tilde{s} - 1)(\tilde{s}^2 - 3\sqrt{2}\tilde{s} + 3),$$

which is birational to $\mathcal{E}_6'$ over $\mathbb{Q}(\sqrt{2})$. Since $\mathcal{E}_6'(\mathbb{C}(\tilde{s})) \cong \mathcal{E}_6'(\mathbb{C}(s)) \cong E_8$, there exist exactly 240 points in $\mathcal{E}_6'(\mathbb{C}(\tilde{s}))$ of the form

$$\tilde{Q} = (a\tilde{s}^2 + b\tilde{s} + g, \quad cs^2 + d\tilde{s}^2 + e\tilde{s} + h),$$

corresponding to the points $Q = (x, y) \in \mathcal{E}_6'(\mathbb{C}(s))$ with

$$x(s) = as^2 + (b - 2\sqrt{2}a)s + g + (2a - \sqrt{2}b),$$

$$y(s) = cs^2 + (d - 3\sqrt{2}c)s^2 + (6c - 2\sqrt{2}d + e)s + h - \sqrt{2}(2c - \sqrt{2}d + e).$$

It is clear that the splitting field $K_6'$ of $\mathcal{E}_6'$ is a quadratic extension by $\sqrt{2}$ of the splitting field of $\mathcal{E}_6'$, which is denoted by $K_6'$ and contains $\mathbb{Q}(\sqrt{2})$ as a subfield. We have the following result on $K_6'$ and the set of independent generators of $\mathcal{E}_6'(K_6'(\tilde{s}))$.

**Theorem 6.1.** The splitting field $K_6'$ of the rational $\mathcal{E}_6'$ is $K_6' = \mathbb{Q}(i, \beta_0, \beta_1, u_7^{\pm})$, where $u_8$ is given by (6.4) and $\beta_0 := 2^{\pm}, \quad \beta_1 := (3 + 2\sqrt{3})^{\pm}$. Moreover, the lattice $\mathcal{E}_6'(K_6'(\tilde{s}))$ is generated by

$$\tilde{Q}_j = (a_j\tilde{s}^2 + b_j\tilde{s} + g_j, \quad c_j\tilde{s}^2 + d_j\tilde{s}^2 + e_j\tilde{s} + h_j), \quad (j = 1, \ldots, 8)$$

where $a_j, b_j, c_j, d_j$ are given in Subsection 6.1.
In the next subsection, we prove the above theorem and in the last one we provide the complete proof of Theorem 1.5.

6.1. Proof of Theorem 6.1. We first determine the fundamental polynomial of \( \tilde{E}_0' \) over \( \mathbb{Q}(\sqrt{2}) \). Substituting the coordinates of \( \tilde{Q} \in \tilde{E}_0' (K_0' (s)) \), given by (6.2), into the equation (6.1) of \( \tilde{E}_0' \) and letting \( g = u^2, h = u^4 \), we get the following six relations:

\[
\begin{align*}
3a^2 b - 2cd - 6\sqrt{2} &= 0, \\
6abu^2 - 2cu^3 + b^3 - 2de - 16\sqrt{2} &= 0, \\
3b^4u^2 - 2du^3 - e^2 - 3 &= 0.
\end{align*}
\]

Using the Maple[10] or a similar argument given in the previous section, one can compute the fundamental polynomial \( \Phi(v) \) of the above equations, which is a polynomial of degree 240 in terms of \( u \) up to a constant.

By taking \( v = u^2 \), we obtain a polynomial \( \Phi(v) \) in \( \mathbb{Z}[v] \) which can be decomposed into nine irreducible factors, namely,

\[
\Phi(v) = \prod_{i=1}^{9} \Phi_i(v).
\]

Letting \( \beta_0 = 2^\frac{1}{4}, \beta_1 = 3^\frac{1}{4} \epsilon_1^\frac{1}{2} \) and \( \beta_2 = 3^\frac{1}{4} (\epsilon_1')^\frac{1}{2} \), where \( \epsilon_1 = 2 + \sqrt{3}, \epsilon_1' = 2 + \sqrt{3} \), the first six factors of \( \Phi(v) \) can be decomposed as follows:

\[
\begin{align*}
\Phi_1(v) &= v^3 - 2 + (v - \beta_0^2)(v - \beta_0^2 \cdot \zeta_3)(v - \beta_0^2 \cdot \zeta_3^2), \\
\Phi_2(v) &= v^4 - 6v^2 - 3 = (v - \beta_2^4)(v + \beta_2^4)(v - i \cdot \beta_2^4)(v + i \cdot \beta_2^4), \\
\Phi_3(v) &= v^6 + 12v^4 + 12v^2 + 6 = (v - v_31)(v - v_32)(v - v_33), \\
\Phi_4(v) &= v^8 + 6v^6 + 39v^4 - 18v^2 + 9 \\
&= (v + \zeta_3 \beta_1^2)(v - \zeta_3 \beta_1^2)(v + \zeta_6 \beta_1^2)(v - \zeta_6 \beta_1^2), \\
\Phi_5(v) &= v^6 - 12v^5 + 132v^4 - 132v^3 + 72v^2 - 72v + 36, \\
&= (v - v_51)(v - v_52)(v - v_53)(v - v_54), \\
\Phi_6(v) &= v^8 - 48v^7 + 168v^6 - 912v^5 + 1272v^4 - 1152v^3 + 864v^2 - 576v + 144, \\
&= (v - v_61)(v - v_62)(v - v_63)(v - v_64)(v - v_65)(v - v_66)(v - v_67)(v - v_68).
\end{align*}
\]

\[
\begin{align*}
v_61 &= \sqrt{3} \beta_1^4 + 2i\sqrt{2} \beta_1^4 - (2\sqrt{3} + 1) \beta_1^2 - i\sqrt{2} (\sqrt{3} + 3) \beta_1, \\
v_62 &= \sqrt{3} \beta_1^4 + 2\sqrt{2} \beta_1^4 + (2\sqrt{3} + 1) \beta_1^2 + \sqrt{2} (\sqrt{3} + 3) \beta_1, \\
v_63 &= -\sqrt{3} \beta_1^4 + (1 + i) \sqrt{2} (3\sqrt{3} - 5) \beta_1^3 + i(5\sqrt{3} - 8) \beta_1^2 + (i - 1)\sqrt{2} (-3 + 2\sqrt{3}) \beta_1 + 12, \\
v_64 &= -\sqrt{3} \beta_1^4 + (1 - i) \sqrt{2} (3\sqrt{3} - 5) \beta_1^3 - i(5\sqrt{3} - 8) \beta_1^2 + (i + 1)\sqrt{2} (-3 + 2\sqrt{3}) \beta_1 + 12,
\end{align*}
\]

where \( \gamma \) denotes the automorphism that changes the sign of \( \sqrt{2} \). One can check that the other three factors of \( \Phi(v) \) can be completely decomposed over the field

\[
\mathbb{Q}(\zeta_8, \zeta_{12}, 2^{\frac{1}{4}}, 3^{\frac{1}{4}}, \epsilon_1^{\frac{1}{2}}).
\]

We only list those factors together with one root for each of them below. For example, the seventh factor is equal to

\[
\Phi_7(v) = v^{16} + 48v^{15} + 2136v^{14} + 6240v^{13} - 16824v^{12} + 32256v^{11} + 564480v^{10} + 815040v^9 + 477360v^8 \\
- 6912v^7 - 248832v^6 - 338688v^5 - 100224v^4 + 165888v^3 + 207360v^2 + 82944v + 20736,
\]

In the next subsection, we prove the above theorem and in the last one we provide the complete proof of Theorem 1.5.
and one of its roots is equal to

\begin{equation}
\varphi_7 = \frac{1}{2} \left( \varphi_{71} \beta_1^3 + \varphi_{72} \beta_1^2 + \varphi_{71} \beta_1 + \varphi_{70} \right),
\end{equation}

where

\begin{align*}
\varphi_{70} &= 3((1 + 2i)\sqrt{3} - (2 + 3i)), \quad \varphi_{71} = -\beta_0^3((5i - 1)\sqrt{3} + (3 - 9i)), \\
\varphi_{72} &= (5i - 8)\sqrt{3} + (15 - 8i), \quad \varphi_{73} = 2\beta_0^3((4 + i)\sqrt{3} - (7 + 2i)).
\end{align*}

The other two factors of \( \Phi(v) \) are the following polynomials:

\begin{align*}
\Phi_8(v) &= v^{24} + 240 v^{23} + 2760 v^{22} + 118416 v^{21} + 21672 v^{20} - 1081152 v^{19} + 5430816 v^{18} \\
&\quad + 65696832 v^{17} + 84093552 v^{16} + 87367680 v^{15} - 337063680 v^{14} + 193135104 v^{13} \\
&\quad - 11176704 v^{12} + 1220272128 v^{11} + 325721088 v^{10} + 1835882496 v^9 + 2322908928 v^8 \\
&\quad + 1997629296 v^7 + 43296768 v^6 - 653930496 v^5 - 559872000 v^4 - 179159040 v^3 \\
&\quad + 53747712 v^2 + 107495424 v + 2687356,
\end{align*}

with a root

\begin{equation}
\varphi_8 = \frac{\varphi_{83} \beta_1^3 + \varphi_{82} \beta_1^2 + \varphi_{81} \beta_1 + \varphi_{80}}{2},
\end{equation}

where

\begin{align*}
\varphi_{80} &= (4 + 5i)\beta_0^3 + (4 - 9i)\beta_0^2 - 10)\sqrt{3} + (5 + 12i)\beta_0^3 + (9 - 12i)\beta_0^2 - 20, \\
\varphi_{81} &= (i + 5)\beta_0^3 - 6i\beta_0^2 + 4(i - 1)\beta_0 \sqrt{3} + (5i - 3)\beta_0^3 - 6i\beta_0^2 + 4(i - 3)\beta_0, \\
\varphi_{82} &= -(3i + 1)\beta_0^4 + (4i - 1)\beta_0^2 + 8)\sqrt{3} - 3(1 + i)\beta_0^3 + (3i - 4)\beta_0^2 + 2, \\
\varphi_{83} &= -(1 + i)\beta_0^3 + 4i\beta_0^2 + 4\beta_0 \sqrt{3} - (3 + i)\beta_0^5 - 4i\beta_0,
\end{align*}

and

\begin{align*}
\Phi_9(v) &= v^{48} - 240 v^{47} + 54840 v^{46} - 425568 v^{45} - 20823912 v^{44} - 315344448 v^{43} \\
&\quad + 14232872448 v^{42} - 9903363264 v^{41} + 144956704464 v^{40} + 1108290767616 v^{39} \\
&\quad - 628479782784 v^{38} - 1642039485696 v^{37} + 120274931645568 v^{36} \\
&\quad - 49953561957888 v^{35} + 391697114707360 v^{34} - 4950416518437888 v^{33} \\
&\quad + 2601957234147840 v^{32} - 52685688536838144 v^{31} + 24908377659328512 v^{30} \\
&\quad + 35030358507028480 v^{29} + 258093710640691200 v^{28} - 38448999625605120 v^{27} \\
&\quad - 155769324643565568 v^{26} - 708851515142356992 v^{25} - 131816508850188288 v^{24} \\
&\quad + 893207293024604928 v^{23} - 689059891204521084 v^{22} - 1163607775758512384 v^{21} \\
&\quad - 1347789006222557184 v^{20} + 5596461768237907968 v^{19} + 70221202555797504 v^{18} \\
&\quad - 3288262670260843136 v^{17} + 98915288029986816 v^{16} - 3550573421298450432 v^{15} \\
&\quad + 49955436686723001568 v^{14} - 1911132695709926240 v^{13} + 1801602023865974784 v^{12} \\
&\quad - 1853784337623810048 v^{11} + 853357744877469696 v^{10} - 640026667502272512 v^{9} \\
&\quad + 561017200636329984 v^8 - 2928938712109869312 v^7 + 134811438777630720 v^6 \\
&\quad - 93164333583826944 v^5 + 37193513019899904 v^4 - 15407021574586368 v^3 \\
&\quad + 1011085790322304 v^2 - 2888816545234944 v + 72220413638736.
\end{align*}
has a root with a root \( v_9 = (v_{93} \beta_1 + v_{92} \beta_2 + v_{91} \beta_1 + v_{90})/2 \), where
\[
\begin{align*}
v_{90} &= (-8\beta_0^5 + (4 + 9i)\beta_0^3 + 5(1 - 2i) \sqrt{3} - 10\beta_0^4 + (9 + 12i)\beta_0^3 + (10 - 15i)\beta_0^2), \\
v_{91} &= (1 + 5i)\beta_0^5 + 3(1 - i)\beta_0^3 - 8\beta_0^2 \sqrt{3} + (5 + 3i)\beta_0^5 + 3(1 - 3i)i\beta_0^3 - 8\beta_0^2, \\
v_{92} &= (-2\beta_0^5 + (1 + 4i)\beta_0^3 + (4 - i) \sqrt{3} - 6\beta_0^4 + (4 + 3i)\beta_0^5 + (1 - 12i), \\
v_{93} &= ((1 + i)\beta_0^5 + 2\beta_0^3) \sqrt{3} + (1 + 3i)\beta_0^5 - 6i\beta_0^3 - 8i\beta_0. \\
\end{align*}
\]
By solving (6.1), for the following roots of the fundamental polynomial \( \Phi(u) \):
\[
\begin{align*}
u_1 &= \beta_0, & u_2 &= \zeta_6 \beta_0, & u_3 &= \beta_1, & u_4 &= \zeta_8 \beta_1, \\
u_5 &= (v_{32})^\frac{1}{2}, & u_6 &= \zeta_{12} \beta_1, & u_7 &= (v_{61})^\frac{1}{2}, & u_8 &= (v_7)^\frac{1}{2},
\end{align*}
(6.6)
one can get the generators of \( \mathcal{E}_6'(\mathbb{K}_6'(\tilde{s})) \) as follows:
\[
\tilde{Q}_j = (a_j s^2 + b_j \tilde{s} + g_j, c_j s^3 + d_j \tilde{s}^2 + e_j \tilde{s} + h_j), \quad (j = 1, \ldots, 8)
\]
where \( a_j, b_j, c_j, d_j, g_j, h_j \) are given in the sequel.

Applying the specialization map \( sp_0 : \mathcal{E}_6'(\mathbb{K}_6'(\tilde{s})) \rightarrow (\mathbb{K}_6')^+ \), defined by
\[
P \mapsto sp_0(P) = \frac{1}{u} \left. \left( \frac{x(P)}{y(P)} \right) \right|_{s=0},
\]
to the above points and multiplying the images by \( u_8 \), we obtain
\[
\left\{ 1, \frac{u_j}{u_1} : j = 2, \ldots, 8 \right\} \subset \mathbb{K}_6',
\]
which can be checked that they are linearly independent over \( \mathbb{Q} \). Thus the points \( \tilde{Q}_1, \ldots, \tilde{Q}_8 \) form a linearly independent subset generating a sublattice of rank 8 in \( \mathcal{E}_6'(\mathbb{K}_6'(\tilde{s})) \). Moreover, the Gram matrix of the points \( \tilde{Q}_1, \ldots, \tilde{Q}_8 \) is equal to the following unimodular matrix:
\[
R_6 = \begin{pmatrix}
2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & 1 & -1 & 1 \\
1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 2
\end{pmatrix}
\]
Finally, one can use (6.3) to get the points \( Q_j = (x_j, y_j) \in \mathcal{E}_6'(\mathbb{K}_6(s)) \) with
\[
\begin{align*}
x_j(s) &= a_j s^2 + (b_j - 2\sqrt{2}a_j)s + g_j + (2a_j - \sqrt{2}b_j), \\
y_j(s) &= c_j s^3 + (d_j - 3\sqrt{2}c_j)s^2 + (6c_j - 2\sqrt{2}d_j + e_j)s + h_j - \sqrt{2}(2c_j - \sqrt{2}d_j + e_j),
\end{align*}
(6.7)
where \( g_j = u_j^2, h_j = u_j^3 \) and the constants \( a_j, b_j, c_j, d_j, e_j \) are as follows:
\[
\begin{align*}
a_1 &= 0, & b_1 &= 0, & c_1 &= 1, & d_1 &= -3\beta_0^3, & e_1 &= 3, \\
a_2 &= 0, & b_2 &= 0, & c_2 &= -1, & d_2 &= 3\beta_0^3, & e_2 &= -3, \\
a_3 &= -1, & b_3 &= \beta_0^3, & c_3 &= 0, & d_3 &= -\sqrt{3}\beta_1, & e_3 &= \sqrt{3}\beta_0^3\beta_1, \\
a_4 &= -1, & b_4 &= \beta_0^3, & c_4 &= 0, & d_4 &= -4\beta_0^3\beta_1, & e_4 &= -4\beta_0^3\beta_1.
\end{align*}
\]
\[ a_5 = \frac{1}{6} (4(3 - 2\sqrt{3}) - 2(1 + i\sqrt{3})\beta_2^1 \beta_0^3 - (1 - i\sqrt{3})\beta_2^1 \beta_0^3)\beta_1^1, \]
\[ b_5 = \frac{2}{3} (-6 + 4\sqrt{3})\beta_0^3 + (1 - i\sqrt{3})\beta_2^1 \beta_0^3 + (1 + i\sqrt{3})\beta_2^1 \beta_0^3), \]
\[ c_5 = -(3i\sqrt{3} + 6) - \frac{1}{3} (3 - i\sqrt{3})\beta_2^1 \beta_0^3 + \frac{1}{3} (3 + i\sqrt{3})\beta_2^1 \beta_0^3), \]
\[ d_5 = -(2(3 + i\sqrt{3})\beta_2^1 + 9i(\sqrt{3} - 2)\beta_2^0 + (3 - i\sqrt{3})\beta_2^1 \beta_0^3)\beta_1^1, \]
\[ e_5 = (17i(2 - \sqrt{3}) - 2(3 - i\sqrt{3})\beta_2^1 \beta_0^3 + \frac{11}{6} (3 + i\sqrt{3})\beta_2^1 \beta_0^3)\beta_1^1, \]
\[ a_6 = \frac{i\sqrt{3} + 1}{2}, \]
\[ b_6 = -(1 + i\sqrt{3})\beta_0^3, \quad c_6 = 0, \quad d_6 = i\sqrt{3}\beta_1, \quad e_6 = -2i\sqrt{3}\beta_0^3 \beta_1, \]
\[ a_7 = 2 + \sqrt{3}(1 + \beta_1^2), \]
\[ b_7 = 2(2 + \sqrt{3})\beta_0^3 - (3 + \sqrt{3})\beta_1 - 2\sqrt{3}\beta_0^3 \beta_1^2 - 2\beta_1^3, \]
\[ c_7 = -3(2 + \sqrt{3}) - (3 + \sqrt{3})\beta_1^2, \]
\[ d_7 = 9(2 + \sqrt{3})\beta_0^3 + (7\sqrt{3} + 9)\beta_1 + 3\beta_0^3(\sqrt{3} + 3)\beta_1^2 + (3\sqrt{3} + 3)\beta_1^3, \]
\[ e_7 = -3(17 + 9\sqrt{3}) - 2(9 + 7\sqrt{3})\beta_0^3 \beta_1 - 3(8 + 3\sqrt{3})\beta_1^2 - 6\beta_0^3 (1 + \sqrt{3})\beta_1^3, \]
\[ a_8 = \frac{1}{6} (-3\sqrt{3} + (3\beta_1^2 - \beta_1^4)), \]
\[ b_8 = \frac{1}{6} ((\sqrt{3} + i)(\sqrt{3} - 1) (3\beta_1 - i(\sqrt{3} + 3)\beta_0^3)\beta_1^4 + \frac{1}{3} (3i - \sqrt{3})(3\beta_0^3 \beta_1^4 + \sqrt{3}\beta_1^4)), \]
\[ c_8 = -3(2 + \sqrt{3}) + (3 + \sqrt{3})\beta_1^2, \]
\[ d_8 = 9(2 + \sqrt{3})\beta_0^3 + i(9 + 7\sqrt{3})\beta_1 - 3(3 + \sqrt{3})\beta_0^3 \beta_1^2 - 3i(1 + \sqrt{3})\beta_1^3, \]
\[ e_8 = -3(17 + 9\sqrt{3}) - 2i(9 + 7\sqrt{3})\beta_0^3 \beta_1 + 3(8 + 3\sqrt{3})\beta_1^2 + 6i(1 + \sqrt{3})\beta_0^3 \beta_1^3. \]

6.2. **Proof of Theorem 1.5.** The splitting field of the elliptic K3 surface $E_6$ over $\mathbb{Q}(t)$ is equal to $K_6 = K_6'(\zeta_{12})$ where $K_6'$ is the splitting field of $E_6': y^2 = x^3 + f_6(s)$ over $\mathbb{Q}(s)$. Letting $s = t + 1/t$, the rational elliptic surface $E_6'$ over $K_6(s)$ is isomorphic to $E_6$ over $K_6(t)$ as a quadratic extension of $K_6(s)$. Hence, the eight independent generators $Q_j = (x_j(s), y_j(s)) \in E'(K_6(s))$ give the points $P_j = (x_j(t), y_j(t)) \in E_6(K_6(t))$ with

\[
\begin{aligned}
x_j(t) &= \frac{A_{j, 4} t^4 + A_{j, 3} t^3 + A_{j, 2} t^2 + A_{j, 1} t + A_{j, 0}}{t^2}, \\
y_j(t) &= \frac{B_{j, 6} t^6 + B_{j, 5} t^5 + B_{j, 4} t^4 + B_{j, 3} t^3 + B_{j, 2} t^2 + B_{j, 1} t + B_{j, 0}}{t^3},
\end{aligned}
\]

where

\[
\begin{aligned}
A_{j, 4} &= A_{j, 0} = a_j, \\
A_{j, 3} &= A_{j, 1} = b_j - 2\sqrt{2} a_j, \\
A_{j, 2} &= g_j + 4a_j - \sqrt{2} b_j, \\
B_{j, 5} &= B_{j, 1} = c_j + d_j - 3\sqrt{2}, \\
B_{j, 4} &= B_{j, 2} = d_j + 9c_j + e_j - 2\sqrt{2}, \\
B_{j, 3} &= h_j - 8\sqrt{2} c_j - \sqrt{2} e_j + 4d_j.
\end{aligned}
\]

The constants $a_j, b_j, c_j, d_j, e_j$ and $u_j$'s for $j = 1, \ldots, 8$, are listed in the previous subsection. Furthermore, letting $s = \zeta_{12} t + \frac{1}{\zeta_{12}} t$, same as above, we obtain the points $P_{j+s} = (x_{j+s}(t), y_{j+s}(t))$ with coordinates

\[
\begin{aligned}
x_{j+s}(t) &= \frac{A_{j+s, 4} t^4 + A_{j+s, 3} t^3 + A_{j+s, 2} t^2 + A_{j+s, 1} t + A_{j+s, 0}}{\zeta_{12}^4 t^2}, \\
y_{j+s}(t) &= \frac{B_{j+s, 6} t^6 + B_{j+s, 5} t^5 + B_{j+s, 4} t^4 + B_{j+s, 3} t^3 + B_{j+s, 2} t^2 + B_{j+s, 1} t + B_{j+s, 0}}{\zeta_{12}^3 t^3},
\end{aligned}
\]

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where
\[
A_{j+8,4} = \zeta_3 A_{j+8,0} = \zeta_3 a_j, \quad A_{j+8,3} = \zeta_6 A_{j+8,1} = \zeta_6 2\sqrt{2}a_j + b_j, \\
A_{j+8,2} = \zeta_6 (a_j + \sqrt{2}b_j + g_j), \quad B_{j+8,6} = -B_{j+8,0} = c_j, \\
B_{j+8,5} = B_{j+8,1} = \zeta_5^2(3\sqrt{2}c_j + d_j), \quad B_{j+8,4} = B_{j+8,2} = \zeta_5(9c_j + 2\sqrt{2}d_j + e_j), \\
B_{j+8,3} = i(8\sqrt{2}c_j + 4d_j + \sqrt{2}e_j + h_j),
\]
for \( j = 1, \ldots, 8 \). We note that the points \( P'_j = (\zeta_5^2 t^2 x(P_j), \zeta_5^3 t^3 y(P_j)) \) belong to the Mordell–Weil lattice of \( E : y^2 = x^3 + t^{12} + 1 \), which is birational to \( \mathcal{E}_5 \) over \( K_6(t) \). See [16] for more details. Having polynomial coordinates, the \( P'_j \)'s have no intersection with zero sections of \( E \), we get that \( (P'_j, P''_j) = 4 \), \( (P'_j, P'''_j) = 2 - (P'_j \cdot P'_j) \), and for any \( 1 \leq j_1 \neq j_2 \leq 16 \), the intersection number \( (P'_j \cdot P'_j) \) can be computed by
\[
(P'_j \cdot P'_j) = \deg (\gcd(x_{j_1} - x_{j_2}, y_{j_1} - y_{j_2})) + \min\{4 - \deg (x_{j_1} - x_{j_2}), 6 - \deg (y_{j_1} - y_{j_2})\}.
\]
Thus, we obtain the Gram matrix of the height pairing for \( P'_j \)'s and hence \( P_j \)'s as
\[
R_6 = \begin{pmatrix}
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 4 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 4 & -2 & 0 & -2 & 1 & 1 & 0 \\
0 & 1 & 0 & -2 & 4 & 0 & 1 & -2 & -1 & 0 \\
2 & 1 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & -1 \\
-1 & -2 & 0 & -2 & 1 & 0 & 4 & -2 & -1 & 1 \\
2 & 1 & 0 & 1 & -2 & 0 & -2 & 4 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 4 & -2 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}
\]
and its determinant is \( 2^4 3^4 \) as desired. Therefore, the points \( P_j \)'s for \( j = 1, \ldots, 16 \) form a set of independent generators of \( \mathcal{E}_5 \) over \( K_6(t) \).

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