Quantum entanglement in carbon nanotubes

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With the surge of research in quantum information, the issue of producing entangled states has gained prominence. Here, we show that judiciously bringing together two systems of strongly interacting electrons with vastly differing ground states - the gapped BCS superconductor and the Luttinger liquid, - can result in quantum entanglement. We propose three sets of measurements involving single-walled metallic carbon nanotubes (SWNT) which have been shown to exhibit Luttinger liquid physics, to test our claim and as nanoscience experiments of interest in and of themselves.

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Entangled pairs are quantum entities consisting of two components sharing a common wave function; a measurement on one component predetermined the state of the other. Such pairs are a basic resource for quantum information processing, and recent years have begun to see many promising approaches to their production in small numbers (e.g. \textsuperscript{3}). For large scale implementation of quantum information technology, a realization of quantum entanglement in solid state systems and an appropriate means of transporting the components of the entangled pair over long distances are essential. While the constituent Cooper pairs of a gapped BCS superconductor have been studied as a possible natural source of such pairs \textsuperscript{3}, the question of separating and transporting them requires further investigation. Here we show how electron-electron interactions in one dimension enables sequential injection of entangled pairs from a superconductor into two nanotubes. The SWNTs, in turn, would allow for transport of entangled states over appreciable distances.

SWNTs, essentially long conducting cylinders of nanoscale diameters and lengths 'L' of several microns, are indeed well-suited as carriers for coherent spin states. They are extremely pure systems with large Fermi velocities of $v_F \sim 10^6 m/s$, and are known to exhibit ballistic transport over very long distances. In particular, at low energies compared to the subband spacing $\epsilon_0 \sim 1eV$, transport is characterized by four ballistic modes propagating with linear dispersion. At low temperatures, $T \lesssim T_\phi = \hbar v_F/k_B L$, electrons can thus travel the entire length of the tube without losing coherence due to thermal effects. Moreover, nanotubes are expected to be nearly ideal spin conductors \textsuperscript{3}, as indeed supported by recent experiments \textsuperscript{3}. In addition, they can be grown in small numbers (e.g. \textsuperscript{2}). For large scale implementation of quantum information technology, a realization of quantum entanglement in solid state systems and an appropriate means of transporting the components of the entangled pair over long distances are essential. While the constituent Cooper pairs of a gapped BCS superconductor have been studied as a possible natural source of such pairs \textsuperscript{3}, the question of separating and transporting them requires further investigation. Here we show how electron-electron interactions in one dimension enables sequential injection of entangled pairs from a superconductor into two nanotubes. The SWNTs, in turn, would allow for transport of entangled states over appreciable distances.

The basic set-up we consider consists of two nanotubes A and B, end-contacted well within a coherence length of each other to a gapped singlet-paired superconductor as in Fig.1. Each wire can be within a coherence length of each other to a gapped singlet-paired superconductor as in Fig.1. Each wire can be described in bosonized language by a four channel Luttinger liquid Hamiltonian

$$H_i = \sum_a \int_0^\infty dx v_a \left[ g^{-1}_a (\partial_x \theta^\dag_a)^2 + g_a (\partial_x \phi^\dag_a)^2 \right],$$

where $i = A, B$ denotes the wires, and $a = \rho^{\pm}, \sigma^{\pm}$ correspond to the four free sectors of the theory where, by linear transformations we have made a change of basis from the spin-channel indices $(1 \uparrow, 1 \downarrow, 2 \uparrow, 2 \downarrow)$ to the original chiral right-/left-moving fermionic fields $\psi^\dag_R/Ln_a$ is expressed through the Bosonization procedure

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{setup.png}
\caption{Set-up of two nanotubes A and B end-contacted to a superconductor. Voltage drops $V_A$ and $V_B$ may be preferentially applied across tubes A and B respectively, and currents through each of them may be measured.}
\end{figure}
via the transformation \( \psi_{R/L}^{\dagger}(x) \sim e^{i\phi_{\mu}(x)\pm \theta_{\mu}(x)} \). The parameter \( g_a \) captures the strength of interactions; \( g_a = 1 \) for the non-interacting channels \( a = (\rho, -\sigma, \sigma) \), while for the charge sector \( g_{p+} \) has the value \( g_{p+} = g \approx 0.25 \). The velocities of the free modes \( v_a \) are given by \( v_a = v_F/g_a \).

As it is desirable to inject entangled pairs individually, we focus on the case of high resistance contacts where successive Cooper pairs hop sequentially from or into the superconductor. This limit corresponds to almost perfect backscattering at the superconductor-nanotube interface, whence \( \psi_{\mu_{Da}}^{\dagger}(0) = \psi_{\mu_{Ra}}^{\dagger}(0) \) where \( i = A, B \) denotes the wire and \( n \) refers to the channel indices 1 and 2. In the bosonized language, these boundary conditions become \( \theta_{\mu}(0) = 0 \) for \( i = A, B \) and all \( a = \rho_{\pm}, \sigma_{\pm} \).

In this set-up we analyze perturbatively the effects of a small amount of Cooper pair tunneling between the superconductor and the two wires. The corresponding Hamiltonian \( H_t \) for such processes is given by \( H_t = \sum_{i=A,B} H_{ii}^t + H_{AB}^t \) where \( H_{ii}^t \) describes the tunneling of whole pairs into wire \( 'i' \), and \( H_{AB}^t \) describes processes in which one electron of the pair tunnels into the tube A and the other one into the tube B. Thus

\[
H_{ii}^t(x) = v_F T_{ii}[\psi_{i\sigma}^{\dagger}(0)\psi_{i\sigma}(0)] - \psi_{i\sigma+b}^{\dagger}(0) \psi_{i\sigma}(0) + (h.c.),
\]

\[
H_{AB}^t(x) = v_F T_{AB}[\psi_{A\sigma+b}^{\dagger}(0)\psi_{B\sigma}(0)] - \psi_{A\sigma+b}^{\dagger}(0) \psi_{B\sigma}(0) + (h.c.),
\]

with \( \psi_{i\sigma+b}^{\dagger} \) are creation operators for electrons in wire \( 'i' \) with corresponding spin \( \sigma \) and flavor \( b = 1R, 1L, 2R \) and 2L. The coefficients \( T_{ii} \) and \( T_{AB} \) are the bare tunneling amplitudes.

As the common wave-function for each Cooper pair in the bulk is peaked at its center of mass, in a non-interacting model the transmission probability for a pair to enter e.g. wire \( A \), \( t_{AA} \propto |T_{AA}|^2 \), is much higher than the transmission probability for the pair-splitting process, \( t_{AB} \propto |T_{AB}|^2 \), i.e. \( t_{AA} \gg t_{AB} \). Naively, then, one might expect each Cooper pair to tunnel entirely into one tube or the other. However, due to interactions, tunneling of charge into the ends of the nanotubes involves more than the mere overlap of single-particle electronic wavefunctions between the tube and the superconductor. Addition of one extra electron into a tube involves the coherent rearrangement of all electrons in its bulk. As a consequence of this Luttinger liquid physics, one can show that the single electron tunneling density of states at a low energy \( E \) compared to \( \epsilon_0 \) goes as \( \rho_{\sigma}(E) \sim \epsilon_0^{-1}(E/\epsilon_0)^{1/2} \), while the density of states available to tunnel in a Cooper pair is \( \rho_{2e}(E) \sim \epsilon_0^{-1}(E/\epsilon_0)^{1/2} \). If the two nanotubes are raised to a voltage, \( V \ll k_BT/e \) above the superconductor, Fermi’s Golden Rule reveals that the rate \( \Gamma_{AA} \), at which entire Cooper pairs tunnel from the superconductor into the end of one tube, is proportional to \( eV\rho_{2e} \), and at any given temperature \( T \), has the dependence \( \Gamma_{AA} \sim (eV/h)(k_BT/\epsilon_0)^{1/2} \). However, the rate \( \Gamma_{AB} \), at which split entangled pairs are injected into both tubes is proportional to the convolution of their one particle tunneling densities of states and has the dependence \( \Gamma_{AB} \sim (eV/h)(k_BT/\epsilon_0)^{1/2} \). Remarkably, at low temperature, \( \Gamma_{AB} \gg \Gamma_{AA} \), and thus almost all charge transfer occurs as split entangled pairs. We now turn to three sets of measurements that capture these principles in a concrete manner.

The simplest experimental signature of the splitting of Cooper pairs may be obtained from the transconductance measured for two nanotubes as shown in Fig.1. In response to a voltage difference between the nanotubes and the superconductor, we compute the resulting currents flowing into the two wires. We start from the Hamiltonian of Eq. (3) with the appropriate boundary conditions and, along the lines of Ref. [3], we use a nonequilibrium Keldysh technique [9] perturbative in the amount of Cooper pair tunneling described by Eq. (3). To bring out the physics of the Cooper pair splitting, we first consider the specific case of applying a voltage drop \( V_A \) across tube \( A \) and none across tube \( B \) under the condition \( k_BT < eV_A < \epsilon_0, \Delta \), where \( \Delta \) is the superconducting gap. In tube \( A \), the applied voltage would produce a current with two components – one due to entire pairs tunneling in \( A \), \( I_{AA} \sim t_{AA}^2(2e^2/h)(eV_A/\epsilon_0)^2V_A \), and another due to the splitting of pairs into the two tubes, \( I_{AB} \sim t_{AB}^2(2e^2/h)(eV_A/\epsilon_0)^2V_A \), where \( \alpha = (1/g - 1)/2 \approx 1.5 \). Strikingly, the current \( I_{AB} \) runs equally in both tubes, in spite of the absence of a voltage drop across tube \( B \). The non-linear behavior of current is a reflection of the power laws in the density of states, and despite the fact that \( t_{AB} < t_{AA} \), the contribution from the pair split process clearly dominates at low voltages.

More generally, at finite temperature \( T \), when voltage drops \( V_A \) and \( V_B \) are present across both tubes, one can define an associated conductance matrix \( G_{ij} = \partial I_j/\partial V_I \), where \( i, j \) stand for the tubes A and B. It has the form

\[
G_{AA} = t_{AA}^2 \frac{(2e)^2}{h} \frac{k_BT}{\epsilon_0} \left[ \frac{g_{eV_A}}{k_BT} \right] F_{2\alpha} \left[ \frac{g_{eV_A}}{k_BT} \right] + t_{AB}^2 \frac{(2e)^2}{h} \frac{k_BT}{\epsilon_0} \left[ \frac{g_{V_A+V_B}}{k_BT} \right] F_{2\alpha} \left[ \frac{g_{V_A+V_B}}{k_BT} \right],
\]

\[
G_{AB} = t_{AB}^2 \frac{(2e)^2}{h} \frac{k_BT}{\epsilon_0} \left[ \frac{g_{eV_A}}{k_BT} \right] F_{2\alpha} \left[ \frac{g_{eV_A}}{k_BT} \right],
\]

with \( G_{BB} \) obtained from \( G_{AA} \) by interchanging \( A \) and \( B \). Here, the scaling function \( F_{\alpha}(x) = \partial_x \left[ \sinh(x/2) \Gamma(1 + \alpha/2 + ix/2\pi) \right]^2 \) (where \( \Gamma(z) \) is the gamma function) and has the limits \( F_{\alpha}(x) \rightarrow x \rightarrow \infty \). The dominance of split Cooper pair injection in charge transfer is directly seen in Eqs. (3),
which implies that for \( k_B T < \epsilon_0 \), the two differential conductances \( G_{AA} \approx G_{AB} \).

The transconductance experiment directly measures the fact that charge is simultaneously injected into both nanotubes. It is not, however, sensitive to the spin state of these electrons. We next consider a Josephson current measurement which verifies that spin \( 1/2 \) is added to each wire. Here, as in Fig. 2, we consider two nanotubes A and B, meeting at points X and Y separated by distance ‘L’ along the tubes.

At each junction, a superconducting lead makes a point contact with both tubes. To probe the spin state of the electrons, tube A (tube B) is subjected to a magnetic field \( \vec{B} = B_{A} \hat{x} \) (\( \vec{B} = B_{B} \hat{x} \)). It may be convenient to choose the field axis \( \hat{x} \) parallel to the junction so as to minimize orbital effects. For this particular set-up the Hamiltonian for each wire is similar to the one described in Eq. (1) with the limits of integration extending over the entire range in the position space, as opposed to the imposition of hard boundary conditions of the previous set-up. We compute the free energy of the system perturbatively in the tunneling of Cooper pairs at points X and Y described by a tunneling term similar (but not identical) to Eq. (2). We find that, at low temperature (when the thermal coherence length \( L_\phi = \hbar v_F / k_B T > L \)), the nanotubes act as a Josephson weak link with associated critical current

\[
I_c = \frac{e v_F}{L} \left\{ \sum_{i=A,B} \left[ i_i^{(1)} \left( \frac{d}{L} \right)^2 + i_i^{(2)} \left( \frac{d}{L} \right)^4 \cos \left( \frac{\mu_B B_i L}{\hbar v_F} \right) \right] \right. \\
\left. + \left( \frac{d}{L} \right)^2 \left[ i_i^{(1)} \cos \left( \frac{\mu_B B_i^{(1/2)} L}{2 \hbar v_F} \right) + i_i^{(2)} \cos \left( \frac{\mu_B B_i^{(1/2)} L}{2 \hbar v_F} \right) \right] \right\},
\]

where \( \delta B = B_A - B_B \), \( B_T = B_A + B_B \) and \( \beta = (g+1/g)/4 - 1/2 \). Also \( d \) is the diameter of the tube and is of the order of a few nanometers, \( g \) is the Landé factor, and \( \mu_B \) is the Bohr magneton. The dimensionless constants \( i_i^{(1/2)} \) and \( i_i^{(1/2)} \) are proportional to the (small) bare transmission probabilities of Cooper pairs from the superconductors to the nanotubes in the unsplit and split processes, respectively. The index ‘1’ refers to injection of pairs of electrons with the same chirality (two right movers or two left movers) into the wires, while the index ‘2’ refers to injection of pairs consisting of a left-moving and a right-moving electron. Notice the highly anomalous length dependence compared to the case of non-interacting wires, where the Josephson current is inversely proportional to the separation length \( L \). Because they have the largest power-law prefactor, the pair-splitting \( (i_i^{(1/2)}) \) terms are enhanced relative to the next largest contribution by a factor of \( (L/d)^{0.6} > 63 \) for typical nanotube parameters. Strikingly, as a function of either one of the applied magnetic fields \( B_A \) and \( B_B \), contributions from Cooper pair split processes oscillate with half the frequency of those generated by unsplit pairs \( (i_i^{(2)}) \). We estimate the period of these magnetic field oscillations to be in the Tesla range. In an actual experiment, in which some flux between the nanotubes is inevitable, the above critical current will give the envelope for much faster Aharonov-Bohm oscillations (on the scale of a few Gauss) in the critical current, but the two types of oscillations can be easily distinguished by their very different periodicities.

While the proposed measurements establish that charge enters from the superconductor in the form of separated electrons with spin \( 1/2 \) each, they do not establish that these electrons are actually in the (maximally) entangled singlet state \( | \uparrow>_A | \downarrow>_B - | \downarrow>_A | \uparrow>_B \rangle /\sqrt{2} \). This entanglement is encoded in correlations between the injected spins. As a simple but revealing example, consider the joint probability \( P_{iT} \) that both electrons in a given pair have spin “up” along a selected axis \( \hat{z} \). In the singlet case, illustrating the EPR ‘paradox’, once spin up is measured in tube A, spin down is automatically selected in tube B, and the probability is zero. Contrast this with the measurement of an unentangled up-down pair \( | \uparrow>_A | \downarrow>_B \rangle \) along an arbitrary direction \( \hat{m} \). If one attempts to preserve spin-rotational invariance on average by choosing the axis \( \hat{m} \) with uniform probability on the unit sphere, the probability \( P_{iT} > 0 \) is non-vanishing, due to pairs that are not oriented along \( \hat{z} \).

We now propose a specific measurement to test the presence of entanglement through current correlations, i.e. noise. Specifically, in the transconductance set-up of Fig. 1, what is required is an experimental measurement of the currents \( I_{in} \) of electrons with a given spin orientation (along \( \hat{m} \)) in each nanotube, \( i = A,B \). Experimentally, this could be accomplished by a variety of spin-filtering techniques, e.g. by attaching two oppositely polarized half-metallic ferromagnets via ideal adiabatic contacts to each nanotube (many other schemes, e.g. Ref. [12] are possible.) Consider measuring spin-filtered currents along the \( \hat{z} \) axis in tube A and along an axis \( \hat{n} \) oriented at an angle \( \theta \) with respect to \( \hat{z} \) in tube B. When a finite voltage drop \( V_A \) is applied across tube A, the
most revealing measurements are those with no voltage drop across tube B, since then all the current in it is due to pair splitting processes.

In a manner similar to the one used to compute the conductance matrix for the system described in Fig.1 we make use of a perturbative Keldysh approach \cite{10,11} to derive forms for the spin-filtered currents and current-current correlations. For the case of singlet Cooper pair tunneling, the spin-filtered current correlations at zero temperature are found to be:

\[ \langle I_{A\pm\hat{z}} I_{B\pm\hat{n}} \rangle = e \sin^2 \theta/2 \langle I_{B\hat{n}} \rangle, \]

\[ \langle I_{A\pm\hat{z}} I_{B\mp\hat{n}} \rangle = e \cos^2 \theta/2 \langle I_{B\hat{n}} \rangle, \]

\[ \langle I_{B\hat{n}} \rangle = \langle I_{B\hat{n}} \rangle, \]

where \( \theta \) is the angle between the \( \hat{n} \) and \( \hat{z} \) axes. From Eq. (6) we see that when both measurements are made along the \( \hat{z} \)-axis (\( \theta = 0 \)), as expected, correlations between like spin currents in the two wires vanish. Also, measurements on B can be made in the x-y plane (\( \theta = \pi/2 \)). Then, as seen in Eq. (7), measuring spin-up (or spin-down) in tube A, provides a 50-percent chance of spin-up and a 50-percent chance of spin-down for tube B. By contrast, in the case considered above of tunneling of classically random unentangled spin-up spin-down pairs, one would expect zero temperature correlations of the form

\[ \langle I_{A\pm\hat{z}} I_{B\pm\hat{n}} \rangle = e/3 [1 + \sin^2 \theta/2] \langle I_{B\hat{n}} \rangle, \]

\[ \langle I_{A\pm\hat{z}} I_{B\mp\hat{n}} \rangle = e/3 [1 + \cos^2 \theta/2] \langle I_{B\hat{n}} \rangle. \]

Though the correlations show an angular dependence on \( \theta \), their form is very different from the entangled case of Eq. (6). Specifically, as anticipated for measurements along the \( \hat{z} \) axis in both tubes (\( \theta = 0 \)), we have \( < I_{A\pm\hat{z}} I_{B\pm\hat{z}} > = e/3 < I_{B\hat{n}} > \), which is non-zero, in stark contrast to the entangled case.

To conclude, in the vast search for physical realizations of entanglement, we have described one method of extracting singlet pairs from a superconducting source. If employing nanotubes for this purpose indeed proves tractable, the next stage in the realm of quantum information would involve new challenges such as probing information at the single electron level and building arrays of coupled logic gates. In the fields of nanoscience and Luttinger physics, attention, both theoretical and experimental, has fallen on bringing effectively one-dimensional systems into contact with superconductors \cite{13}. Here we have hoped to provide more food for thought in these fields by describing two nanotubes in contact with a gapped BCS superconductor.

While finalizing this work we became aware of a related independent proposal by P. Recher and D. Loss \cite{13}. The analysis of \cite{13} is based on a set-up similar to the one we describe here, yielding similar results with the ones we derive in the first part of our paper.

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