Gravitational field of a global defect

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Global topological defects described by real scalar field in (3,1) dimensions coupled to gravity are analyzed. We consider a class of scalar potentials with explicit dependence with distance, evading Derrick’s theorem and leading to defects with spherical symmetry. The analysis shows that the defects have finite energy on flat space, contrary to the observed for the global monopole. With the aim to study the gravitational field produced by such defects, after an Ansatz for the static metric with spherical symmetry, we obtain the coupled system of Einstein and field equations. On the Newtonian approximation, we numerically find that the defects have a repulsive gravitational field. This field is like one generated by a negative mass distributed on a spherical shell. In the weak gravity regime a relation between the Newtonian potential and one of the metric coefficients is obtained. The numerical analysis in this regime leads to a spacetime with a deficit solid angle on the core of the defect.

I. INTRODUCTION

Topological defects are solutions of the classical field equations, which are similar to particles. In quantum theory, they correspond to extended particles, composed of the elementary particles in each particular model. Topological defects are found very frequently in condensed matter physics. For instance, the vortex, the simplest soliton in gauge theory with scalars occur in type-2 superconductors. In the cosmological context we have cosmic strings, and some models use them as seeds for the formation of structures in the early universe. In this way topological defects are of great interest in high energy physics.

Topological defects can be global or local. The global defects arise in models with a continuum but global symmetry, and have been investigated for instance in condensed matter physics. For instance, the vortex, the simplest soliton in gauge theory with scalars occur in type-2 superconductors. In the cosmological context we have cosmic strings, and some models use them as seeds for the formation of structures in the early universe. In this way topological defects are of great interest in high energy physics.

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As is well known, however, there is a theorem which puts a limit on the dimensionality of systems constructed only with scalar fields, in order to ensure the presence of topological defects. Along the years, several distinct routes have been constructed to evade this theorem, such as: i) to consider constraints on the scalar fields, as in the nonlinear O(3) model; ii) to add gauge fields, as with the ’tHooft-Polyakov monopole; iii) to include higher-order derivatives on the Lagrangian; iv) to introduce time-dependent and non-dissipative solutions (see ref. [11] pp.141-150 and ref. [12]); v) to suppose non-local solutions, such as global monopoles - this means models with a divergent energy that can be limited by a cutoff in a cosmological context due to the presence of another defect.

Recently, another possibility was considered on searching for defects described by only one scalar field in (3,1) dimensions. For this were introduced models described by a dimensionless Lagrangian density with an explicit dependence with distance:

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \tilde{\phi} \partial^{\mu} \tilde{\phi} - \frac{1}{2} f(\tilde{x}^2) \left( \frac{\partial W}{\partial \phi} \right)^2,$$

where $f(\tilde{x}^2)$ is in principle an arbitrary function of the dimensionless coordinate $\tilde{x}$, $\phi$ a real dimensionless scalar field and $W = W(\tilde{\phi})$ a smooth function of $\phi$.

The equation of motion for static field is, for a specific choice of $f(\tilde{x}^2)$,

$$\partial_i \partial^i \tilde{\phi} = \nabla^2 \tilde{\phi} = \frac{1}{\tilde{r}^4} \frac{\partial W}{\partial \phi} \frac{\partial^2 W}{\partial \phi^2},$$

with $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{1/2}$ a radial dimensionless quantity.

Following Ref. [13] we look for solutions depending only on the radial coordinate $\tilde{r}$, we change the variable $\tilde{r} \rightarrow 1/x$, obtaining $\nabla^2 \tilde{\phi} = 1/x^4 (d^2 \tilde{\phi} / dx^2)$. Substituting this in Eq. (2), we have

$$\frac{d^2 \tilde{\phi}}{dx^2} = \frac{\partial W}{\partial \phi} \frac{\partial^2 W}{\partial \phi^2},$$

a 1-dim equation on the $x$ variable. Note that the solutions of the first-order differential equation

$$\frac{d \tilde{\phi}}{dx} = \frac{\partial W}{\partial \phi}$$

are also solutions of the equation of motion.

The class of models

$$V_p(\tilde{\phi}) = \frac{1}{2} \tilde{\phi}^2 (\tilde{\phi}^2 - \frac{1}{2} \tilde{\phi}^2)^2 = \frac{1}{2} \left( \frac{\partial W}{\partial \phi} \right)^2$$

is described by the parameter $p$ which drives the way the field self-interacts. For $p = 1$ we have the usual $\phi^4$...
theory. For \( p = 3, 5, \ldots \) the models describe potentials supporting minima at \( \tilde{\phi} = 0 \) and \( \pm 1 \), and the solutions of Eq. (4) connect the minima \( \phi = \pm 1 \) crossing \( \phi = 0 \), on the form of a 2-kink on the \( x \) variable. It was shown in [12] that the \( p = 4, 6, \ldots \) cases, where the potential has minima at \( \phi = 0 \) and 1, lead to stable solutions for Eq. (4), valid for \( \phi \in [0, \infty) \), in the form

\[
\tilde{\phi}(\tilde{r}) = \tanh^{p}\left[ \frac{1}{p} \phi \right].
\]

These solutions with even \( p, p \geq 4 \), have spherical symmetry, with a central core growing with \( p \). Here we are interested on analyzing the gravitational properties of one of these solutions with spherical symmetry. We fix from here on \( p = 4 \), whose solution on the flat space given by Eq. (4), leads to a smooth interpolation between the minima \( \tilde{\phi} = 1 \) and \( \tilde{\phi} = 0 \) for \( \tilde{r} \) going from 0 to \( \infty \), respectively.

For this case we calculate the energy of the configuration at a distance \( \tilde{R} \) from the defect. The result, obtained for flat space and depicted in Fig. 1, shows that the energy of the defect grows until a plateau is achieved. In this way there is no the characteristic divergence obtained for the global monopole. This shows that this defect, contrary to the global monopole, does not need a distance cutoff in order to achieve finite energy.

As this defect was obtained in flat space, a next step is to analyze its gravitational field. With this aim in Sec. II we obtain the Einstein and field equations. We show that the system of equations need four boundary conditions, two of them related to the behaviour of the scalar potential at the center of the core and at the infinity and two other ones coming from imposing a Minkowski spacetime far from the defect. The possible ways to solve completely this system of equations are discussed. In Sec. III we consider the Newtonian approximation which shows the repulsive character of the defect. We also present an analysis of the weak field approximation, leading as a result that the metric inside the core of the defect presents a deficit solid angle. Our conclusions are presented in Sec. IV.

II. EINSTEIN AND FIELD EQUATIONS

In curved spacetime the action for the defect structure is

\[
S = \int d^{4}x \sqrt{\gamma} \left[ -\frac{1}{4} R + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{r^{4}} V(\phi) \right],
\]

with the usual relation between the spherical coordinates \( r, \theta, \phi \) and the Cartesian coordinates \( x^{i}, i = 1, 2, 3 \) and where \( g = -\det(g_{\mu \nu}) \) and \( \mu, \nu = 0, 1, 2, 3 \). The potential is

\[
V(\phi) = \frac{1}{\lambda \eta^{p/2}} \frac{1}{2} \phi^{3/2}(\eta^{1/2} - \phi^{1/2})^{2},
\]

which is the potential from Eq. (4) for \( p = 4 \), after restoring the dimensional quantity \( \eta \), the vacuum expectation value (v.e.v) of the scalar field \( \phi \), and the coupling constant \( \lambda \).

As an Ansatz, we try a general static metric with spherical symmetry:

\[
ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

Defining \( \phi = \phi/\eta \) and \( \tilde{r} = r/\delta \), with \( \delta = \lambda^{-1/2} \eta^{-1} \), the equation of motion for the scalar field can be written as

\[
\frac{d^2 \tilde{\phi}}{d\tilde{r}^2} + \left( \frac{1}{2B} \frac{dB}{d\tilde{r}} - \frac{1}{2A} \frac{dA}{d\tilde{r}} + \frac{2}{\tilde{r}} \right) \frac{d\tilde{\phi}}{d\tilde{r}} - \frac{A}{4\delta^4}(1 - \tilde{\phi}^{1/2})(3\tilde{\phi}^{1/2} - 5\tilde{\phi}) = 0
\]

The energy-momentum tensor of the defect is related to the variation of the Lagrangian density of the matter field with respect to the metric $\tilde{g}_{\mu \nu}$:

\[
T_{\mu \nu} = \frac{2}{\sqrt{\tilde{g}}} \frac{\partial \mathcal{L}_{\text{mat}}}{\partial \tilde{g}^{\mu \nu}}.
\]

Using the result $\partial g/\partial g^{\mu \nu} = -g_{\mu \nu}$ (v. [11], p.364), we have for a Lagrangian with an usual \((1/2)\partial_{\mu} \phi \partial^{\mu} \phi^{a}\) kinetic term:

\[
T_{\mu \nu} = \partial_{\mu} \phi^{a} \partial_{\nu} \phi^{a} - g_{\mu \nu} \mathcal{L}_{\text{mat}}^\text{flat}.
\]

where $\mathcal{L}_{\text{mat}}^\text{flat}$ is the usual Lagrangian of matter field in flat space. This gives for our model the following non-null components for the energy-momentum tensor:

\[
T_{i}^{i} = T_{\theta}^{\theta} = T_{\phi}^{\phi} = \frac{\phi^{2}}{2A} + \frac{1}{r^{4}} V(\phi)
\]

\[
T_{r}^{r} = -\frac{1}{2A} \phi^{2} + \frac{1}{r^{4}} V(\phi),
\]
where the prime denotes derivation with respect to $r$. The Einstein equations are

$$ R_{\mu}^{\nu} - \frac{1}{2} R g_{\mu}^{\nu} \equiv G_{\mu}^{\nu} = 8\pi G T_{\mu}^{\nu}, $$

where $R_{\mu}^{\nu}$ and $G_{\mu}^{\nu}$ are, respectively, the Riemann and Einstein tensors.

The components of the Einstein tensor are easily obtained using the computer algebra systems GRTensorII (for Maple):

$$ G_{t}^{t} = \frac{A'}{A^2 r} + \frac{1}{r^2} - \frac{1}{A r^2}, $$
$$ G_{r}^{r} = \frac{B'}{AB r} - \frac{1}{r^2} + \frac{1}{A r^2}, $$
$$ G_{\theta}^{\theta} = G_{\phi}^{\phi} = \frac{B'}{2AB r} - \frac{A'}{2A^2 r} - \frac{B'}{4A^2 B} + \frac{B''}{2AB} - \frac{B'^2}{4AB^2}. $$

With the components of Einstein and energy-momentum tensors, and changing to the $\tilde{r}$ variable we get explicitly:

$$ dA \over dr = A^2 \tilde{r} \left[ \frac{1}{Ar^2} - \frac{1}{\tilde{r}^2} \right] + \Delta \left( \frac{1}{2A} \left( \frac{d\tilde{\phi}}{d\tilde{r}} \right)^2 + \frac{1}{2A^4} \tilde{\phi}^{3/2} (1 - \tilde{\phi}^{1/2})^2 \right), $$
$$ dB \over dr = -dA B \over d\tilde{r} A + \Delta B \tilde{r} \left( \frac{d\tilde{\phi}}{d\tilde{r}} \right)^2 $$

and

$$ \frac{B'}{2AB r} - \frac{A'}{2A^2 r} - \frac{B'}{4A^2 B} + \frac{B''}{2AB} - \frac{B'^2}{4AB^2} = \Delta \left( \frac{1}{2A} \left( \frac{d\tilde{\phi}}{d\tilde{r}} \right)^2 + \frac{1}{2A^4} \tilde{\phi}^{3/2} (1 - \tilde{\phi}^{1/2})^2 \right). $$

with $\Delta \equiv \eta^2 / M_4^2$. We see that the Einstein equations (Eqs. [20] - [25]) together with the equation of motion (Eq. [10]) for the $\tilde{\phi}$ field form a highly nonlinear set of equations.

The solution $\tilde{\phi}(\tilde{r})$ for flat space (Eq. [6]), with $p = 4$ impose the boundary conditions (b.c.) $\tilde{\phi}(0) = 1$ and $\tilde{\phi}(\infty) = 0$. Also, as the matter content (related with the $\phi$ field) decreases with distance, we expect that far from the defect the spacetime tend to be four-dimensional Minkowski, $M_4$. This leads to the additional conditions $A(\infty) = 1, B(\infty) = 1$.

From the b.c. we see that the set of equations for $A, B$, $\tilde{\phi}$ is over-determined, and one of the Einstein equations is used as a consistency check. In a numerical search for a complete set of solutions the b.c. favor the use of Eqs. [20] - [21] together with Eq. [10]. The hybrid nature of the b.c. signals that the use of relaxation (or shooting) method for Eq. [10] together with a Runge-Kutta method for Eqs. [20] - [21] can be used in a search for a solution. This study is in progress. In the present work, however, as we are interested in a weak field limit, we will use Eqs. [20] and [22] together with the solution for the $\tilde{\phi}$ field obtained in flat space (Eq. [10] for $p = 4$).

### III. LINEARIZED GRAVITY

Consider first the Newtonian approximation. The Newtonian gravitational potential $\Phi(r)$, is obtained as a solution of the Poisson equation

$$ \nabla^2 \Phi = 8\pi G (T_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} T) = 4\pi G (T_0^0 - T_i^i). $$

Substituting the stress-energy tensor on the Poisson equation, after setting $A = 1$ for the Newtonian approximation, we get

$$ \nabla^2 \Phi = - \frac{V(\phi)}{M_4^2 \tilde{r}^4} $$

or, in terms of the dimensionless variables $\tilde{\phi}$ and $\tilde{r}$,

$$ \frac{d^2 \Phi}{d\tilde{r}^2} + \frac{2}{\tilde{r}} \frac{d\Phi}{d\tilde{r}} = -\Delta \frac{\tilde{\phi}^{3/2}}{2\tilde{r}^4} (1 - \tilde{\phi}^{1/2})^2. $$

In this approximation (which for our problem means $\tilde{r} >> 1$), we can consider $\tilde{\phi}(\tilde{r})$ as the solution on flat space, $\tilde{\phi}(\tilde{r}) = \tanh^4 (1/(4\tilde{r}))$. A graphic of the r.h.s. of Eq. [25] as a function of $\tilde{r}$ indicates that the defect acts as a thick spherical shell with negative mass. In this way, we expect a potential with a constant value on the core and decreasing with the distance to the shell. As the potential is defined unless a constant, we numerically solved Eq. [25] using the second order Runge-Kutta method, with the initial conditions $\Phi(0) = 1$ e $\Phi'(0) = 0$. The corresponding gravitational force under a particle with unitary mass is depicted in Fig.2. This figure shows a localized aspect of the gravitational force, with the magnitude increasing with the parameter $\Delta$.

The next step is to investigate the influence of this defect over the spacetime geometry, on the linear approximation. With this purpose we consider $A$ and $B$ from Eq. [11] in the form

$$ A(r) = 1 + \epsilon \alpha(r), \quad B(r) = 1 + \epsilon \beta(r). $$

The $\theta \theta$ component of the Einstein equations gives, for this approximation,

$$ \frac{d^2 \beta}{d\tilde{r}^2} + \frac{2}{\tilde{r}} \beta = -\Delta \frac{\tilde{\phi}^{3/2}}{2\tilde{r}^4} (1 - \tilde{\phi}^{1/2})^2. $$

After comparison with the Poisson equation for the Newtonian gravitational potential, we are led to $\beta = 2\Phi + c$,
\[ F(r/\delta) = \frac{c}{r^2} \]

where \( c \) is a constant determined with the boundary condition \( B(\infty) = 1 \). The result is depicted in Fig. 3(a). Also, Eq. (21) in this approximation leads to

\[ \frac{d\alpha}{d\tilde{r}} = -\frac{d\beta}{d\tilde{r}} + \Delta \tilde{r} \left( \frac{d\phi}{d\tilde{r}} \right)^2. \]  

(28)

We numerically integrate this equation with \( \tilde{\phi} \) given by the solution on flat space, and with the result for \( \beta(\tilde{r}) \) obtained previously. In this way we obtain Fig. 3(b). Note that Figs. 3(a)-(b) show that near to the origin, the solutions for \( A \) and \( B \) tend to different values, indicating the appearance of a deficit solid angle.

IV. CONCLUSION

In this work we analyzed the gravitational field produced by a global defect with an explicit \( 1/r^2 \) dependence on the scalar potential. We found that with such a potential the defect produced has finite energy. This is an alternative to the global monopole \[4, 5\] for constructing a (3, 1)-dim defect, where no cutoff distance is needed. The analysis of the Newtonian potential shows no effect besides a repulsive gravitational field appearing near the core of the defect. The repulsive Newtonian force is proportional to the v.e.v. \( \eta \) of the scalar field. The analysis of the gravitational field on the linear regime shows a space with a deficit solid angle. The amount of this deficit is controlled by the v.e.v. of the scalar field. In fact, Figs. 3(a)-(b) show that inside the defect, as the parameter \( \Delta = \eta/M_\rho \) increases, the metric coefficients \( A \) and \( B \) are more departed from unity. Our weak field analysis shows the \( 1/r^4 \) dependence of the scalar potential fixes qualitatively the gravitational effects of the defect. The magnitude of these effects is governed mainly by the v.e.v. of the scalar potential. We expect that a further numerical analysis of the complete set of Einstein and field equations can reveal other aspects not presented in this work.

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[15] Note that $\mathcal{L}_{\text{mat}}$ differs from the usual Lagrangian of matter field in flat space $\mathcal{L}_{\text{mat}}^{\text{flat}}$ by the $\sqrt{g}$ term.