A FIXED-POINT THEOREM FOR LOCAL OPERATORS WITH APPLICATIONS TO STOCHASTIC EQUATIONS

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ABSTRACT. We study weak and strong solutions of nonlinear non-compact operator equations in abstract spaces of adapted random points. The main result of the paper is similar to Schauder’s fixed-point theorem for compact operators. The illustrative examples explain how this analysis can be applied to stochastic differential equations.

1. Introduction

Nonlinear operators studied in this paper possess the property of locality. A mapping between two spaces consisting of continuous functions has this property if the value of the output function at a given point is completely determined by the values of the input function in an arbitrarily small neighbourhood of this point. Typical examples are superposition operators, differentiations and their combinations. However, this definition is no longer valid for spaces of measurable functions, where elements are equivalence classes and not individual functions. In the latter case, the formal definition of locality was suggested by I. V. Shragin in the paper [15]. It says that if two equivalence classes coincide on a set \( A \), then their images must coincide on the same set. If we replace the equivalent classes with their representatives, then we have to add the expression “almost everywhere” to this definition.

It can be shown that Shragin’s definition covers the above mentioned examples. On the other hand, it also covers stochastic integrals if they can be defined as limits of finite sums and, by this, stochastic operators, which do not directly contain global characteristics of stochastic processes, like expectation, covariance, distributions etc.

This paper develops a fixed-point theory for general local operators defined on spaces of adapted random points in abstract separable Banach spaces. The main result of the paper states, roughly speaking, that if a local and continuous operator, defined on a special set of adapted random points, maps this set into its tight subset, then it has at least one weak fixed point, a random point on an expanded probability space. This fixed-point theorem was first announced in the author’s paper [10], although without a proof. The main objective of this report is to provide a detailed proof of this result.

Note that this theory is not about a simple special case of local operators given by superpositions \( x(\cdot) \rightarrow F(\cdot, x(\cdot)) \) that are generated by random, a.s. continuous maps \( F(\omega, \cdot) \). Combining the theory of compact operators with the technique of measurable selections it is not difficult to develop a fixed-point theory for such operators, but they do not cover most interesting stochastic differential equations. The local operators considered in this report do cover stochastic integrals and equations, and this is demonstrated in a number of examples.

The paper is organized as follows. In Section 2 a simplified version of the main fixed-point theorem, which can be directly used in applications, is formulated. Here we replace abstract Banach spaces by two examples of functional spaces. The general case is considered in Section 5 while Sections 3 and 4 contain necessary definitions and auxiliary results, the proofs of which are moved to Appendix B. An overview of the terminology and the notation can be found in Appendix A. Appendix D contains illustrative examples, some of which are based on the propositions collected and proven in Appendix C.
2. Local operators and a simplified version of the fixed-point theorem

Let
\[ S = (\Omega, \mathcal{F}, P) \]
be a complete probability space. The expectation, the integral w.r.t. the measure \( P \), is denoted by \( E \), and the abbreviation a.s. means almost surely (i.e. almost everywhere) with respect to \( P \).

We will use the following notation: \( I_A \) is the indicator of a set \( A \); \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space with some norm \( |.| \); \( X, Y \) are separable Banach spaces with the norms \( ||.||_X \) and \( ||.||_Y \), respectively; we also put \( B_r = \{ x \in X : ||x||_X \leq r \} \).

For any separable metric space \( M \) the set \( \mathcal{P}(M, S) \) consists of all equivalence classes \([x]\) of \( \mathcal{F}\)-measurable functions \( x : \Omega \to M \) (also referred to as random points in \( M \)). Convergence in probability defines a metrizable topology on \( \mathcal{P}(M, S) \). In this paper we will use the metric
\[ d_M(x, y) = E \min(\rho(x, y); 1), \]
where \( \rho \) is the distance on \( M \). If \( M \) is complete, then \( \mathcal{P}(M, S) \) is complete as well. Any convergent in probability sequence contains an a.s. convergent subsequence. In particular, this implies that the topology on \( \mathcal{P}(M, S) \) does not depend on the choice of any equivalent distance on \( M \). If \( M = X \) is a separable Banach space \( X \), then the set \( \mathcal{P}(X, S) \) is a linear metric, but not locally convex, space.

Below we usually disregard the difference between the equivalence classes \([x]\) and their particular representatives \( x \) writing (somewhat unprecisely) \( x \in \mathcal{P}(M, S) \) instead of \([x]\) \( \in \mathcal{P}(M, S) \). We will also sometimes write \( \mathcal{P}(M) \) instead of \( \mathcal{P}(M, S) \) if the probability space \( S \) is fixed and if it does not cause misunderstandings.

Notice that if \( V \) is a closed resp. convex subset of \( X \), then \( \mathcal{P}(V, S) \) is a closed resp. convex subset of \( \mathcal{P}(X, S) \). Bounded subsets \( A \) of the space \( \mathcal{P}(X, S) \) can be described as follows: for any \( \varepsilon > 0 \) there is a ball \( B_r \) in \( X \) such that \( P\{ x \notin B_r \} < \varepsilon \) for any \( x \in A \).

It is assumed in the definition below that two equivalence classes \([x],[y]\) \( \in \mathcal{A} \) coincide on a set \( A \subset \Omega \), i.e. \( [x]_{\mathcal{A}} = [y]_{\mathcal{A}} \) if \( x(\omega) = y(\omega) \) for almost all \( \omega \in A \). Clearly, this definition is independent of the choice of the representatives \( x \) and \( y \).

**Definition 2.1.** Let \( A \subset \mathcal{P}(X, S) \). An operator \( h : A \to \mathcal{P}(Y, S) \) is called local if
\[ [x]_{\mathcal{A}} = [y]_{\mathcal{A}} \quad \text{implies} \quad h[x]_{\mathcal{A}} = h[y]_{\mathcal{A}} \]
for any \([x],[y]\) \( \in \mathcal{A} \) and \( A \subset \Omega \).

**Remark 2.1.** If the property of locality is only valid for all \( A \in \mathcal{F} \), then it also valid for all \( A \subset \Omega \). Indeed, for any representatives \( x \) and \( y \) of the classes \([x],[y]\) respectively, the set \( B = \{ \omega \in \Omega : x(\omega) = y(\omega) \} \) belongs to \( \mathcal{F} \) and satisfies \( P(A - B) = 0 \). Hence, the equivalence classes \( h[x] \) and \( h[y] \) coincide on \( B \) and thus on \( A \).

Note that any local operator \( h \) can be naturally defined on the set of all representatives of the equivalence classes belonging to \( A \) if we put \( hx \) to be an arbitrary representative of the class \( h[x] \).

The property of locality becomes then
\[ x(\omega) = y(\omega) \quad \text{for} \quad \omega \in A \quad \text{a.s. implies} \quad hx(\omega) = hy(\omega) \quad \text{for} \quad \omega \in A \quad \text{a.s.} \quad (\forall A \subset \Omega). \]

**Remark 2.2.** In this report, we only study local operators that are continuous in the topology of convergence in probability. Throughout the paper we will use the abbreviation LC for such operators.

**Remark 2.3.** The superposition operator
\[ h_f : \mathcal{P}(X, S) \to \mathcal{P}(Y, S), \quad \text{defined by} \quad (h_fx)(\omega) = f(\omega, x(\omega)), \]
where \( f : \Omega \times X \to Y \) is a random function, is local. Clearly, this operator is well-defined, as \( x(\omega) = \tilde{x}(\omega) \) a.s. implies \((h_f\tilde{x})(\omega) = (h_f\tilde{x})(\omega) \) a.s. If, in addition, the function \( f \) is Carathéodory, i.e. it is measurable in \( \omega \in \Omega \) for all \( x \in X \) and continuous in \( x \in X \) for almost all \( \omega \in \Omega \), then \( h_f \) is continuous in probability.

The Itô integral is another example of an LC operator. More examples can be found in Appendix D.

In this paper \( T \) usually stands for an arbitrary linearly ordered set containing its maximal element, see Appendix A. A typical example is \( T = [a,b] \), and this is assumed in the remaining part of this section. In addition, we suppose that \( X = C(T) \) or \( X = L^r(T) \) \((1 \leq r < \infty)\).
Let
\[(\mathcal{F}_t)_{t \in T}\]be a filtration on the probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), i.e. a nondecreasing family of \(\sigma\)-subalgebras of \(\mathcal{F}\), all \(\sigma\)-subalgebras being complete w.r.t. \(P\), i.e. containing all subsets of zero measure. The probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) with a filtration \([2]\) on it is usually called a stochastic basis.

An \(\mathcal{F} \otimes \text{Bor}(\Omega)\)-measurable stochastic process \(\xi(t) = \xi(\omega, t), t \in T\), is called \(\mathcal{F}_t\)-adapted \([\mathcal{3}]\) if \(\xi(\cdot, t)\) is \(\mathcal{F}_t\)-measurable for all \(t \in T\). Given a stochastic basis \(\mathcal{B}\), we denote by \(\mathcal{P}(\mathcal{X}, \mathcal{B})\) the set of all (equivalence classes of) \(\mathcal{F}_t\)-adapted stochastic processes whose trajectories a.s. belong to the space \(X = L^r(\Omega)\) or \(C(T)\). Any equivalence class consists in this case of all indistinguishable stochastic processes \([\mathcal{3}]\). The inclusion \(\mathcal{P}(\mathcal{X}, \mathcal{B}) \subset \mathcal{P}(\mathcal{X}, \mathcal{S})\) induces a natural topology on the aforementioned space.

Recall that a set \(\mathcal{K} \subset \mathcal{P}(\mathcal{X}, \mathcal{S})\) is called tight if for any \(\epsilon > 0\) there exists a compact set \(Q \subset X\) such that \(P\{\omega : x(\omega) \not\in Q\} < \epsilon\) whenever \(x \in \mathcal{K}\). We say that an operator \(h : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{X}, \mathcal{S})\) (\(\mathcal{A} \subset \mathcal{P}(\mathcal{X}, \mathcal{S})\)) is tight-range if 1) it maps \(\mathcal{A}\) into a tight subset of \(\mathcal{P}(\mathcal{X}, \mathcal{S})\) and 2) it is uniformly continuous on any tight subset of \(\mathcal{A}\). If \(h\) only maps bounded subsets of \(\mathcal{A}\) into tight subsets, then the operator \(h\) is called tight.

This definition generalises the notions of compact and compact-range operators: if \(\Omega\) shrinks into a singleton, then the space \(\mathcal{P}(\mathcal{X})\) coincides with \(X\) and tight subsets become precompact in \(X\). In this case, uniform continuity on compact subsets (and thus on their subsets) follows from continuity.

**Definition 2.2.** A stochastic basis \(\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)\) is an expansion of the stochastic basis \(\mathcal{B} = (\Omega, \mathcal{F}, \mathcal{F}_t, P)\) if there exists a \((\mathcal{F}^*, \mathcal{F})\)-measurable surjection \(c : \Omega^* \rightarrow \Omega\) such that

1. \(P^* c^{-1} = P\);
2. \(c^{-1}(\mathcal{F}_t) \subset \mathcal{F}_t^*\) (\(\forall t\)).

Note that \(\mathcal{P}(\mathcal{X}, \mathcal{B})\) can be naturally identified with a linear topological subspace of the space \(\mathcal{P}(\mathcal{X}, \mathcal{B}^*)\).

Expansions preserving the martingale property are of key interest in the theory of weak solutions of stochastic differential equations \([\mathcal{3}]\). In particular, for the standard Wiener process \(W(t)\) on \(\mathcal{B}\) the process \(W^*(t) = W(t) \circ c\) remains Wiener on any such expansion. In this paper, we only use a special version of the expansions preserving the martingale property, which we call Young expansions. In this case, the disintegration of the probability measure \(P^*\) is a Young measure, i.e. the weak limit of random Dirac measures generated by adapted random points, see Definition \([\mathcal{1}]\).

Given an LC operator \(h : \mathcal{P}(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{P}(\mathcal{X}, \mathcal{S})\) and an expansion \(\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)\) of the given stochastic basis \(\mathcal{B} = (\Omega, \mathcal{F}, \mathcal{F}_t, P)\), we say that an LC operator \(h^*\) defined on \(\mathcal{P}(\mathcal{X}, \mathcal{B}^*)\) is an extension of the operator \(h\) if the restriction of \(h^*\) to \(\mathcal{P}(\mathcal{X}, \mathcal{B})\) coincides with \(h\). Only extensions preserving locality and continuity are studied in this paper.

If an LC operator \(h\) admits an LC extension, then this extension is unique, see Theorem \([\mathcal{1}]\).

Existence of LC extensions is a more delicate issue, see Subsection \([\mathcal{2}]\).

Let \(h : \mathcal{P}(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{P}(\mathcal{X})\), where \(X = C(T)\) or \(X = L^r(T)\) (\(1 \leq r < \infty\)), be an LC operator. If there exists a Young expansion \(\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)\) of the stochastic basis \(\mathcal{B} = (\Omega, \mathcal{F}, \mathcal{F}_t, P)\) and \(x^* \in \mathcal{P}(\mathcal{X}, \mathcal{B}^*)\) such that \(h^* x^* = x^*\) \(P^*\)-a.s., then \(x^*\) will be called a weak fixed point of \(h\).

The following fixed-point theorem is a particular case of the main results presented in Section \([\mathcal{5}]\).

**Theorem 2.1.** Let \(X = C(T)\) or \(X = L^r(T)\) (\(1 \leq r < \infty\)) and \(h : \mathcal{P}(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{P}(\mathcal{X}, \mathcal{B})\) be a local and tight-range operator. Then \(h\) has at least one weak fixed point \(x^* \in \mathcal{P}(\mathcal{X}, \mathcal{B}^*)\) for some Young expansion \(\mathcal{B}^*\) of the stochastic basis \(\mathcal{B}\).

If the operator \(h\) has at most one weak fixed point for any Young expansion \(\mathcal{B}^*\) of \(\mathcal{B}\), then each weak fixed point will be equivalent to a unique strong, i.e. belonging to the space \(\mathcal{P}(\mathcal{X}, \mathcal{B})\), solution of the equation \(hx = x\).

**Proof.** See Corollary \([\mathcal{3}]\).
3.1. Adapted random points in abstract Banach spaces. Let $T$ be a linearly ordered set containing a maximal element $b$. Line intervals $[a, b]$ serve as examples of such sets. We are also given a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_t$ on it, i.e., a nondecreasing family of complete $\sigma$-algebras $\mathcal{F}_t \subset \mathcal{F}$ ($t \in T$) indexed by elements of the set $T$. The quadruple

$$B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P)$$

(3)
is addressed as a stochastic basis on the probability space $(\Omega, \mathcal{F}, P)$ over the set $T$.

Definition 3.3. By a projective system of linear topological spaces over $T$ we understand a triple $\mathcal{X} = (X_t, p^u_t, T)$, where $X_t$ are linear topological spaces ($t \in T$) and $p^u_t : X_t \to X_u$ ($t, u \in T$, $t \geq u$) are linear continuous surjective maps satisfying the property

$$p^u \circ p^t = p^u$$

for all $t, u, v \in T$, $t \geq u \geq v$.

Remark 3.1. The following complements Definition 3.3:

1. Projective systems are also known as inverse systems in the literature; the maps $p^u_t$ are usually called bonding maps.
2. The definition implies that $p^u_t$ are the identity maps on the respective spaces $X_t$ for all $t \in T$.
3. In most propositions below we consider projective systems of separable Banach spaces, but in connection with expansions of probability spaces projective systems of separable Fréchet spaces may be necessary, see Example D.19.
4. Below we systematically use the simplifications $X \equiv X_b$ and $p^t \equiv p^t_b$.
5. As for any $x \in \mathcal{P}(\mathcal{X}, \mathcal{B})$, the map $p^u \circ x = x$ must be $\mathcal{F}_t$-measurable, we can always assume, if necessary, that $\mathcal{F} = \mathcal{F}_b$.

An important example of a projective system is described in

Definition 3.2. If $T = T_m = \{0, \ldots, m\}$, $X_t \equiv E_i$ ($t = i \in T_m$, $\dim E_i = i$) are linear subspaces of the $m$-dimensional Euclidean space $E = E_m$, $E_j \subset E_i$ ($j \leq i$) and $p^u_t \equiv p^u_i$ are the orthogonal projections of $E_i$ onto $E_j$, then the projective system $E = (E_i, p^i_j, T_m)$ will be addressed as a Euclidean projective system.

The functional spaces like $L'(T)$ and $C(T)$ give rise to natural projective systems over the line intervals $T$, see Subsection D.3 in Appendix D.

Definition 3.3. Let $B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P)$ be a stochastic basis and $\mathcal{X} = (X_t, p^u_t, T)$ be a projective system of separable Banach spaces. A random point $x \in \mathcal{P}(X, S)$ ($X = X_b$) is called adapted with respect to $B$ and $\mathcal{X}$ if $p^t(x) \equiv p^t_b(x) : \Omega \to X_t$ is $\mathcal{F}_t$-measurable for all $t \in T$.

Definition 3.4. Let $B$ be a stochastic basis and $\mathcal{X}$ be a projective system of separable Banach spaces. The linear topological subspace of $\mathcal{P}(X, S)$ consisting of all (equivalence classes of) adapted random points with respect to $B$ and $\mathcal{X}$ will be denoted by $\mathcal{P}(\mathcal{X}, \mathcal{B})$. If $\mathcal{X}$ or/and $B$ are fixed, then the notation $\mathcal{P}(\mathcal{X}, \mathcal{B})$ may be simplified to $\mathcal{P}(\mathcal{X}, \mathcal{B})$ or $\mathcal{P}(\mathcal{X})$.

This notation is easy to see to be consistent with the one used in Section 2 for the case of the spaces $C[a, b]$ and $L^r[a, b]$.

Example 3.1. If $\mathcal{F}_t = \mathcal{F}$ for all $t \in T$, then $\mathcal{P}(\mathcal{X}, \mathcal{B}) = \mathcal{P}(X, S)$, so that all statements proven for the spaces of adapted random points are automatically true for the spaces of all random points. However, the converse statements are in many cases not true. For instance, the representation theorem D.4 for LC operators is only valid for $\mathcal{P}(X, S)$, but not necessarily for $\mathcal{P}(\mathcal{X}, \mathcal{B})$.

3.2. Uniform continuity and tightness of local operators. Let $X$ and $Y$ be separable Banach spaces. The canonical uniformity [14, p. 12-19] on the associated linear topological spaces of random points is understood in agreement with the topologies and the linear operations on these spaces. In particular, we have the following definition of uniform continuity of an operator $h : A \to \mathcal{P}(Y, S)$, $A \subset \mathcal{P}(X, S)$.

For any $\varepsilon > 0$, $\sigma > 0$ there exist $\delta > 0$, $\rho > 0$ such that

$$P(\|x_1 - x_2\|_X \geq \rho) < \delta (x_1, x_2 \in A) \Rightarrow P(\|h(x_1) - h(x_2)\|_Y \geq \sigma) < \varepsilon.$$

The translation invariant metric $d_X(x_1, x_2) = E \min\{\|x_1 - x_2\|_X; 1\}$ gives rise to the same canonical uniformity on $\mathcal{P}(X, S) \times \mathcal{P}(X, S)$, as it is generated by translations of the same set of neighborhoods of the origin in $\mathcal{P}(X, S)$. This applies, of course, to the space $\mathcal{P}(Y, S)$ as well. Therefore
the property of uniform continuity can be rewritten as

For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$dx(x_1, x_2) < \delta \implies dy(h(x_1), h(x_2)) < \varepsilon.$$ 

By technical reasons it may be convenient to combine these two definitions:

For any $\varepsilon > 0$ there exist $\rho > 0, \delta > 0$ such that

$$P\{|x_1 - x_2|, x_1, x_2 \in A\} < \delta \implies dy(h(x_1), h(x_2)) < \varepsilon.$$ 

In the next definition we generalize the classical notion of a Volterra operator as the one "only depending on the past": $(\phi u)(s) = (\phi v)(s)$ ($a \leq s \leq t$) if $u(s) = v(s)$ ($a \leq s \leq t$) for any $a \leq t \leq b$. The operator here acts on functions defined on the line interval $T = [a, b]$. In the case of an arbitrary linearly ordered $T$ this definition can be extended in the following manner:

**Definition 3.5.** Let $L = (L_t, l^u, T)$ be a projective system of separable Fréchet (in particular, Banach) spaces. We call an operator $\phi : L \to L$ a generalized Volterra operator (map) with respect to $L$ if it generates a family of continuous operators $\phi^t : L_t \to L_t$ ($t \in T$) satisfying the properties $\phi = \phi^b$ and $l^u \circ \phi^t = \phi^u \circ l^u$ for all $t, u \in T, t \geq u$.

**Remark 3.2.** The superposition operators generated by Volterra maps transform adapted random points to adapted random points. Indeed, if for $x \in P(L, S)$ the random point $l^b(x)$ in $L$ is $F_{\mathcal{T}}$-measurable, then $l^u(\phi x) = \phi^t(l^b x)$ will be $F_{\mathcal{T}}$-measurable as well due to continuity of $\phi^t$. This observation is important for our analysis, where the superpositions generated by finite dimensional Volterra maps are used to approximate LC operators defined on the adapted random points: it is essential that the domain and the range of the operators are invariant under approximations.

This remark explains

**Definition 3.6.** The projective system $L = (L_t, l^u, T)$ of separable Fréchet (in particular, Banach) spaces satisfies Property (II) if there exists a sequence $\pi_n : L \to L$ ($L = L_n$) of linear, continuous and finite dimensional generalized Volterra maps, which strongly converges to the identity map in $L$ as $n \to \infty$.

The property described in the definition is satisfied for most linear functional spaces used in applications, for instance, for $L^r[a, b]$ and $C[a, b]$, as it is shown in Example [10.2].

The next definition is a reminder.

**Definition 3.7.** A set $K \subset P(X, F)$ is called tight if for any $\varepsilon > 0$ there exists a compact set $Q \subset X$ such that $P\{\omega : x(\omega) \notin Q\} < \varepsilon$ whenever $x \in K$.

**Remark 3.3.** An equivalent, and sometimes more convenient, description of tightness says that $K$ is tight if and only if for any $\sigma > 0, \varepsilon > 0$ there exists a compact set $Q \subset X$ such that $P\{\omega : x(\omega) \notin Q, \sigma\} < \varepsilon$ whenever $x \in K$, where $Q, \sigma$ as the $\sigma$-neighborhood of the set $Q$.

The theorem below is an important technical result.

**Theorem 3.1.** Suppose that $X = (X_t, p^u, T)$ is a projective system of separable Banach spaces satisfying Property (II), $Y$ is another separable Banach space and $h : P_a(X, B) \to P(Y, S)$ is a local operator. Then the following statements are equivalent:

1. $h : P_a(X, B) \to P(Y, S)$ is uniformly continuous on each tight subset $K \subset P_a(X, B)$;
2. for any compact subset $Q$ of $X$ and any $\varepsilon > 0$, there is $\rho > 0$ such that $\|u - v\|_X \leq \rho$ a.s. implies $dy(hu, hv) < \varepsilon$ for all $u, v \in P_a(X, B) \cap P(Q, S)$; this is e.g. fulfilled if $h$ is uniformly continuous on any subset $P_a(X, B) \cap P(Q, S)$, where $Q \subset X$ is an arbitrary compact;
3. there exists a function $O(\gamma) > 0$ ($\gamma > 0$), $\lim_{\gamma \to 0} O(\gamma) = 0$ such that for any compact $Q \subset X$ and any $\varepsilon > 0$ there is $\delta > 0$ satisfying the property:

$$dy(x, y) < \delta \implies dy(hx, hy) < \varepsilon + O(\gamma)$$ 

for all $x, y \in P_a(X, B)$, $P\{x \notin Q\} < \gamma$, $P\{y \notin Q\} < \gamma$.

**Proof.** See Appendix B, Subsection [10.2].

In the case of continuous superposition operators, Property (II) in Theorem [11] can be omitted, see Example [11.3] in Appendix D.

The next definition introduced in [10] generalises the notion of a compact operator.
Definition 3.8. Let $X$ and $Y$ be separable Banach space and $h : A \rightarrow \mathcal{P}(Y,S)$, where $A \subset \mathcal{P}(X,S)$.

1) The operator $h$ is called tight if 1) it maps bounded subsets of $A$ into tight subsets of $\mathcal{P}(Y,S)$ and 2) it is uniformly continuous on any tight subset of $A$.

This definition yields the class of (continuous) compact and compact-range operators if $\Omega$ is single-pointed. On the other hand, local operators are almost never compact. For instance, it can be proven that $h : \mathcal{P}(X,S) \rightarrow \mathcal{P}(Y,S)$ is local and compact if and only if either $P$ assumes finitely many values, or $Y$ contains finitely many points.

For nontrivial examples of tight operators see Subsection D.4 in Appendix D.

4. Extensions of local operators

4.1. Expansions of stochastic bases. Expansions/changes of the underlying probability space are e.g. used if this space is not rich enough to host solutions of stochastic equations. Not all expansions preserve basic properties of stochastic integrals, and hence a fortiori we cannot hope that general LC operators can be extended to an arbitrary expansion of the original probability space. In this paper we use what we call Young expansions, which is sufficient for our purposes.

Let $\Omega^* = \Omega \times Z$, $Z$ be a Polish (e.g. separable Banach) space and $\mu$ be a measure on $F \otimes \text{Bor}(Z)$, whose marginal coincides with $P$: $\mu(A \times Z) = P A$ for any $A \in F$.

The disintegration of the measure $\mu$ [3] p. 19] is a random measure $\mu_\omega$ on $\text{Bor}(Z)$ for almost all $\omega \in \Omega$ such that

$$
\int_{\Omega \times Z} g(\omega, z) d\mu(\omega, z) = \int_{\Omega} \int_{Z} g(\omega, z) d\mu_\omega(z) dP(\omega)
$$

holds for every bounded measurable function $g : \Omega \times Z \rightarrow R$. The corresponding differential form reads as $d\mu(\omega, z) = d\mu_\omega(z) dP(\omega)$, which can be conveniently abbreviated to $d\mu = d\mu_\omega dP$.

The narrow topology on the set $\text{Pr}_\omega(Z)$ of all random measures on $\text{Bor}(Z)$ with the marginal $P$ is generated by the maps

$$
\mu \mapsto \mu(f) = E \int_{Z} f d\mu \equiv E \int_{Z} f(\omega, z) d\mu_\omega(z),
$$

where $f : \Omega \times Z \rightarrow R$ is an arbitrary bounded Carathéodory function [3] p. 25].

A neighborhood $U_{f_1,\ldots,f_m,\delta}(\mu)$ of $\mu \in \text{Pr}_\omega(Z)$ in the narrow topology consists of all $\nu \in \text{Pr}_\omega(Z)$ such that

$$
|E \int_{Z} f_i d\mu - E \int_{Z} f_i d\nu| < \delta \quad (i = 1,\ldots,m).
$$

Here $f_i$ are bounded Carathéodory functions and $\delta > 0$.

Definition 22 of an expansion of a stochastic basis is too general for our purposes. Therefore we introduce the notion of a Young expansion starting with probability spaces.

Definition 4.1. A Young expansion $S^* = (\Omega^*, F^*, P^*)$ of the probability space $S = (\Omega, F, P)$ satisfies the properties:

1) $\Omega^* = \Omega \times Z$, $Z$ being a Polish (e.g. separable Fréchet or Banach) space;
2) $P^*$ is a probability measure on $F \otimes \text{Bor}(Z)$ with the marginal $P$;
3) the disintegration $F^*_\omega$ of $P^*$ is the limit (in the narrow topology) of a sequence of random Dirac measures $\{\delta_{\alpha_{\omega}}(\omega)\}$, where $\alpha_{\omega} \in \mathcal{P}(Z,S)$;
4) $F^*$ is the $P^*$-completion of the $\sigma$-algebra $F \otimes \text{Bor}(Z)$. 
Remark 4.1. Property (3) in Definition 4.1 can be rewritten in terms of the measure $P\alpha_n$ and the measures $P\alpha_n^\perp$, defined by
\[ dP\alpha_n^\perp(\omega, z) = \delta_{\alpha_n(\omega)}(z) dP(\omega), \]
for any bounded Carathéodory function $g$ \[ \text{in the following way:} \]
\[ E^* g = \int_{\Omega^*} g(\omega, z) dP\alpha_n^\perp(\omega, z) = \lim_{n \to \infty} \int_{\Omega^*} g(\omega, z) dP\alpha_n^\perp(\omega, z) \]
for any bounded Carathéodory function $g : \Omega \times Z \to R$. Strictly speaking, we should have written $P(\text{Gr} \alpha_n^\perp)$ and not $P\alpha_n^\perp$, but we will keep the latter notation for the sake of simplicity.

In the next definition we replace Polish spaces used in Definition 4.1 by separable Fréchet spaces, because we want to construct expansions utilizing projective systems from Definition 3.1.

Definition 4.2. Suppose that $Z = (Z_t, q^u, T)$ is a projective system of separable Fréchet spaces. A Young expansion $B^* = (\Omega^*, F^*, (F^*_t)_{t \in T}, P^*)$ of the stochastic basis $B = (\Omega, F, (F_t)_{t \in T}, P)$, generated by $Z$, satisfies the following properties:

1. $\Omega^* = \Omega \times Z$, where $Z = Z_h$;
2. $P^*_t$ is the limit (in the narrow topology) of a sequence of random Dirac measures $\{\delta_{\alpha_n(\omega)}\}$, where $\alpha_n \in \mathcal{P}(Z, \mathcal{B})$;
3. $F^*$ is the $P^*$-completion of the $\sigma$-algebra $F \otimes \text{Bor}(Z)$;
4. $F^*_t$ is the $P^*$-completion of the $\sigma$-algebra $F_t \otimes (q^u)^{-1}(\text{Bor}(Z_t))$ for any $t \in T$.

In particular, the probability space $(\Omega^*, F^*, P^*)$ is a Young expansion of the probability space $(\Omega, F, P)$.

Some examples of Young expansions can be found in Subsection 4.5.

Remark 4.2. The mapping $c : \Omega^* \to \Omega$ from Definition 2.2 is defined as $c(\omega, z) = \omega$ in Definitions 4.1 and 4.2. Clearly, $c$ is $(F^*, F)$-measurable resp. $(F^*_t, F_t)$-measurable for any $t \in T$.

Given $S \in F \otimes \text{Bor}(Z)$ we put $S(\omega) = \{z \in Z : (\omega, z) \in S, \text{cl} S = \{(\omega, z) : z \in \text{cl} S(\omega)\}$, where $\text{cl}(B)$ is the closure of a set $B \subseteq Z$, and $\partial S = \text{cl} S \cap \text{cl}(\Omega^* - S)$ is the random boundary of the random set $S$. It can be shown (see e.g. [4, p. 10]) that $\text{cl} S \in F \otimes \text{Bor}(Z)$ if $Z$ is a Polish space. Therefore $\partial S \in F \otimes \text{Bor}(Z)$ as well.

The next two definitions play a key role in the proof of the fixed-point theorem in Section 5.

Definition 4.3. Let $P^*$ be a Young probability measure defined on the $\sigma$-algebra $F \otimes \text{Bor}(Z)$, where $Z$ is a Polish space. A set $S \in F \otimes \text{Bor}(Z)$ is called a continuity set of the measure $P^*$ (or simply, a $P^*$-continuity set) if $P^*(\partial B) = 0$.

In the next definition we assume that $B^* = (\Omega^*, F^*, (F^*_t)_{t \in T}, P^*)$ is a Young expansion of the stochastic basis $B = (\Omega, F, (F_t)_{t \in T}, P)$. The corresponding expansion of the underlying probability space $S = (\Omega, F, P)$ is denoted by $S^*$.

Definition 4.4. Let $X = (X_t, p^u, T)$ be a projective system of separable Banach spaces and $X = X_h$.

1. A random point $x \in \mathcal{P}(X, S^*)$ is $P^*$-a.s. continuous if there exists a subset $A \in \Omega^*$ of zero measure $P^*$ such that the maps $x(\omega, \cdot) : Z \to X$ are continuous on $Z(\omega) - A(\omega)$.
2. An adapted random point $x \in \mathcal{P}(X, B^*)$ is simple if $x = \bigcup_{i=1}^s \alpha_i I_{A_i}$ for some $\alpha_i \in \mathcal{P}(X, B)$ and disjunct $P^*$-continuity subsets $A_i \in F^*$ ($i = 1, \ldots, s$), $\bigcup_{i=1}^s A_i = \Omega^*$.
3. The set of all simple points will be denoted by $S_a(X, B^*)$.

Clearly, the random points from $S_a(X, B^*)$ are $P^*$-continuous.

Remark 4.3. Note that conditions on the random function $g$ in Remark 4.2 can be relaxed [6]: the equality (6) holds, in fact, for any $P^*$-a.s. continuous $g : \Omega \times Z \to R$. In particular,
\[ P^* A = \lim_{n \to \infty} (P\alpha_n^\perp)(A) = P\{\alpha_n \in A\} \]
for any $P^*$-continuity subset $A \in F^*$.

The approximation result below is used in the proof of the main results.
Theorem 4.1. Assume that $X = (X_t, p^{ut}, T)$ and $Z = (Z_t, q^{ut}, T)$ are two projective systems of separable Banach and Fréchet spaces, respectively, both satisfying Property (II), and $B^* = (\Omega^*, F^*, (F^*_t)_{t \in T}, P^*)$ is a Young expansion of the stochastic basis $B = (\Omega, F, (F_t)_{t \in T}, P)$ generated by $Z$. Then for any $x, y \in \mathcal{P}(X, B^*)$, satisfying $x|_A = y|_A$ $P^*$-a.s. for some $A \in F^*$, there exist $x_n, y_n \in \mathcal{S}(\mathcal{A}, \mathcal{B}^*)$ and $P^*$-continuity subsets $A_n \in F^*$, for which

1. $x_n|_{A_n} = y_n|_{A_n}$ $P^*$-a.s.,
2. $x_n \to x, y_n \to y$ in probability $P^*$ and
3. $\lim_{n \to \infty} P^*(A_n \Delta A) = 0$.

Proof. See Appendix B, Subsection B.3. \hfill \Box

4.2. Construction of LC extensions. In this subsection we assume that $X$ is a projective system of separable Banach spaces, $Y$ is another separable Banach space, $B$ is a stochastic basis and $B^*$ is its Young expansion generated by a projective system of separable Fréchet spaces $Z = (Z_t, q^{ut}, T)$, see Definition 4.2. Recall that in this case $c: \Omega^* \equiv \Omega \times Z \rightarrow \Omega$ is the projection on the first factor. To simplify the notation, we put $\mathcal{P}(Y, S) \equiv \mathcal{P}(Y), \mathcal{P}^*(Y) \equiv \mathcal{P}^*(Y, S^*), \mathcal{P}(X) \equiv \mathcal{P}(X, B), \mathcal{P}^*(X) \equiv \mathcal{P}(X, B^*)$ and $\mathcal{P}(Z) \equiv \mathcal{P}(Z, B)$.

Note that the linear homeomorphism $x \mapsto x \circ c$ naturally identifies the linear topological subspaces $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ with the respective linear topological subspaces of $\mathcal{P}^*(X)$ and $\mathcal{P}^*(Y)$. This justifies

Definition 4.5. Let $h: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ be an LC operator. We say that an LC operator $h^*: \mathcal{P}^*(X) \rightarrow \mathcal{P}^*(Y)$ is an LC extension of the operator $h$ if the restriction of $h^*$ to $\mathcal{P}(X)$ coincides with $h$.

In the case of Young expansions generated by Dirac measures, the extension of local operators can be constructed explicitly, as it is shown in the following remark.

Remark 4.4. Let $B^* = (\Omega^*, F^*, (F^*_t)_{t \in T}, P^*)$ be the Young expansion of the stochastic basis $B = (\Omega, F, (F_t)_{t \in T}, P)$ where the Young measure $P^*$ is generated by a random Dirac measure $P = P_{\alpha^{-1}}$ for some $\alpha \in \mathcal{P}(Z)$, i.e. $P^*(A) = P\{\omega \in \Omega: \alpha(\omega) \in A(\omega)\}$. In this case, the measure preserving map $\alpha: \Omega \rightarrow (\omega, \alpha(\omega))$ gives rise to the linear topological isomorphism $\alpha_Y : y \mapsto y \circ \alpha$ between the spaces $\mathcal{P}(Y)$ and $\mathcal{P}^*(Y)$. Evidently, the inverse map is then given by $\alpha_Y^{-1} : y \mapsto y \circ c$. Moreover, according to Example 4.18 the map $\alpha_X : x \mapsto x \circ \alpha$ is a linear topological isomorphism between the spaces $\mathcal{P}(X)$ and $\mathcal{P}^*(X)$. Let us, therefore, put

$$h^* x = (h(x \circ \alpha) \circ c) = \alpha_Y^{-1} h \alpha_X.$$ 

Then $h^*: \mathcal{P}^*(X) \rightarrow \mathcal{P}^*(Y)$ is continuous, and we claim that $h^*$ is a local extension of $h$. Indeed, $h^*(x \circ \alpha) = h(x \circ c \circ \alpha) \circ c = h x \circ c$, because $\alpha(x)(\omega) = \omega$ for any $\omega \in \Omega$, so that $h^*$ is an extension of $h$. To prove that $h^*$ is local, take $x, y \in \mathcal{P}^*(X)$ and $A \in F \otimes \text{Bor}(Z)$ such that $x|_A = y|_A$ (it is sufficient to prove locality for such $A$). Then $B = \alpha^{-1}(A) \in F$ and

$$x \circ \alpha|_B = y \circ \alpha|_B \Rightarrow h(x \circ \alpha)|_B = h(y \circ \alpha)|_B \quad P - a.s.$$ 

$$\Rightarrow h(x \circ \alpha) \circ c|_{c^{-1}B} = h(y \circ \alpha) \circ c|_{c^{-1}B} \quad P^* - a.s. \Rightarrow h^* x|_{c^{-1}B} = h^* y|_{c^{-1}B} \quad P^* - a.s.$$ 

But

$$P^*(A \Delta c^{-1}B) = P\{\alpha \in A \Delta c^{-1}B\} = P\{\{\alpha \in A\} \Delta \{\alpha \in c^{-1}B\}\} = P(B \Delta B) = 0,$$

so that $h^* x|_A = h^* y|_A$ $P^* - a.s.$

In particular, this example shows that if $h$ is uniformly continuous on tight subsets of the space $\mathcal{P}(X)$ and the Young expansion $B^*$ of $B$ is generated by a random Dirac measure, i.e. $P^* = P_{\alpha^{-1}}$ ($\alpha \in \mathcal{P}(Z)$), then the extension $h^*$ of $h$ is uniformly continuous on tight subsets of the space $\mathcal{P}^*(X)$ as well. This follows from the fact that the linear topological isomorphisms $\alpha_X : \mathcal{P}(X) \rightarrow \mathcal{P}^*(X)$ and $\alpha_Y : \mathcal{P}(Y) \rightarrow \mathcal{P}^*(Y)$ preserve tight sets, because the map $\omega \mapsto (\omega, \alpha(\omega))$ is measure preserving.

The case of general Young expansions is considered in Theorem 4.3.

The uniqueness property of LC extensions can be easily proven in a rather general setting.

Theorem 4.2. Let $B^* = (\Omega^*, F^*, (F^*_t)_{t \in T}, P^*)$ be a Young expansion of the stochastic basis $B = (\Omega, F, (F_t)_{t \in T}, P)$ and $X = (X_t, p^{ut}, T)$ be a projective system of separable Banach spaces satisfying Property (II). If an LC operator $h: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ admits a continuous extension $h^*: \mathcal{P}^*(X) \rightarrow \mathcal{P}^*(Y)$, then this extension is unique.
Proof. If \( x \in S_0(\mathcal{X}, \mathcal{B}^*) \), then there exist \( \alpha_i \in \mathcal{P}_0(\mathcal{X}) \) and disjunct subsets \( A_i \in \mathcal{F}^* \ (i = 1, \ldots, s) \), \( \bigcup_{i=1}^{s} A_i = \Omega^* \), such that \( x = \sum_{i=1}^{s} \alpha_i I_{A_i} \). The property of locality implies that any two extensions \( h^*_1 \) and \( h^*_2 \) of the operator \( h \) must satisfy
\[
  h^*_1 x = h^*_2 x = \sum_{i=1}^{s} h(\alpha_i) I_{A_i} \quad P^* \text{-a.s.}
\]
By Theorem 4.1, the set \( S_0(\mathcal{X}, \mathcal{B}^*) \) is dense in \( \mathcal{P}_0^*(\mathcal{X}) \). Therefore \( h^*_1 x = h^*_2 x \) for all \( x \in \mathcal{P}_0^*(\mathcal{X}) \), as both are continuous in the topology of this space.

The next result generalizes the one considered in Remark 4.1.

**Theorem 4.3.** Let \( \mathcal{X} = (X_t, \mu^u, T) \) be a projective system of separable Banach spaces and \( \mathcal{B}^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t \in T}, P^*) \) be a Young expansion of the stochastic basis \( \mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P) \) generated by a projective system of separable Fréchet spaces \( \mathcal{Z} = (Z_t, \mu^u, T) \). Assume that \( \mathcal{X} \) and \( \mathcal{Z} \) satisfy Property (II). Then any local operator \( h : \mathcal{P}_0(\mathcal{X}) \to \mathcal{P}(\mathcal{Y}) \), which is uniformly continuous on tight subsets, admits an LC extension \( h^* : \mathcal{P}_0^*(\mathcal{X}) \to \mathcal{P}^*(\mathcal{Y}) \), which is also uniformly continuous on tight subsets.

**Proof.** See Appendix B, Subsection B.4. □

5. **Main Results**

In this section we justify the general infinite dimensional fixed-point theorem formulated in [10] without a proof. The first step in this direction will be a finite dimensional fixed-point theorem for LC operators.

It is still assumed that \( \mathcal{B} \) is a stochastic basis on a complete probability space \( \mathcal{S} \).

**Theorem 5.1.** Let \( \mathcal{X} = (X_t, \mu^u, T) \) be a projective system of finite dimensional spaces. If \( U \) is a closed, convex, bounded and nonempty subset of \( \mathcal{X} \), \( \mathcal{P}_0(U) \equiv \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(U, \mathcal{S}) \) and \( h : \mathcal{P}_0(U) \to \mathcal{P}_0(U) \) is an LC operator, then \( h \) has at least one fixed point.

**Proof.** See Appendix B, Subsection B.5. □

In the rest of the section we assume \( \mathcal{X} = (X_t, \mu^u, T) \) to be a projective system of arbitrary separable Banach spaces. The following definition of a weak fixed point generalizes the one briefly described in Section 2.

**Definition 5.1.** Let \( h : \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \to \mathcal{P}(\mathcal{X}, \mathcal{S}) \) be an LC operator. If there exists an expansion \( \mathcal{B}^* \) of the stochastic basis \( \mathcal{B} \), an LC extension \( h^* : \mathcal{P}_0(\mathcal{X}, \mathcal{B}^*) \to \mathcal{P}(\mathcal{X}, \mathcal{S}^*) \) of the operator \( h \) and a random point \( x^* \in \mathcal{P}_0(\mathcal{X}, \mathcal{B}^*) \) such that \( h^* x^* = x^* \ P^*\text{-a.s.} \), then \( x^* \) is called a weak fixed point of the operator \( h \).

Note that \( h^* \) in Definition 5.1 does exist if \( \mathcal{B}^* \) is a Young expansion and \( h \) is local and uniformly continuous on every tight subset of its domain, see Theorem 4.3.

**Remark 5.1.** The notion of a weak solution is well-known in stochastic analysis. It is also a well-established practice to call solutions strong if they are defined on the original probability space \( \mathcal{S} \). Following this terminology we call any fixed point of the operator \( h \) belonging to the space \( \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \) a strong fixed point.

Now we are able to formulate the main result of the paper.

**Theorem 5.2.** Let the projective system of separable Banach spaces \( \mathcal{X} = (X_t, \mu^u, T) \) satisfy Property (II) and \( h : \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \to \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \) be a local operator which is uniformly continuous on tight subsets of its domain.

1. If for some convex, closed and nonempty set \( V \subset X \) the operator \( h \) maps \( \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(V, \mathcal{S}) \) into its tight subset, then \( h \) has at least one weak fixed point \( x^* \in \mathcal{P}_0(\mathcal{X}, \mathcal{B}^*) \cap \mathcal{P}(V, \mathcal{S}^*) \) for some Young expansion \( \mathcal{B}^* \) of \( \mathcal{B} \) defined on the probability space \( \mathcal{S}^* \).
2. If for any Young expansion \( \mathcal{B}^* \) of \( \mathcal{B} \) the associated (unique) LC extension \( h^* \) of the operator \( h \) has at most one fixed point in \( \mathcal{P}_0(\mathcal{X}, \mathcal{B}^*) \), then each weak fixed point of the operator \( h \) will be equivalent to a unique strong, i.e. belonging to the space \( \mathcal{P}_0(\mathcal{X}, \mathcal{B}) \), solution of the equation \( hx = x \).
Proof. We will use the simplified notation for the spaces of random points. Given $U \subset X$ we put

$$\mathcal{P}a(U) \equiv \mathcal{P}a(X, \mathcal{B}) \cap \mathcal{P}(U, \mathcal{S}) \quad \text{and} \quad \mathcal{P}a^*(U) \equiv \mathcal{P}a(X, \mathcal{B}^c) \cap \mathcal{P}(U, \mathcal{S}^c).$$

In particular,

$$\mathcal{P}a(X) \equiv \mathcal{P}a(X, \mathcal{B}) \quad \text{and} \quad \mathcal{P}a^*(X) \equiv \mathcal{P}a(X, \mathcal{B}^c).$$

The Young expansion in the first part of the proof will be generated by the projective family $Z = (Z_t, q^{ut}, T)$ coinciding with $\mathcal{X}$:

$$Z = \mathcal{X} = (X_t, p^{ut}, T). \quad (7)$$

In accordance with the notational agreement from Remark 3.1 we write $Z = Z_b$, and $X = X_b$, so that $X = Z$. The Young measure $P^*$ on $\Omega^* = \Omega \times Z = \Omega \times X$ will be constructed in the course of the proof.

Existence of a weak fixed point.

Let $h(\mathcal{P}a(V)) \subset K$ for some tight subset $K \subset \mathcal{P}a(V)$. For any $n \in N$ there exists a compact subset $Q^n \subset V$ such that $P\{x \notin Q^n\} < 1/n$ for all $x \in K$. As $V$ is convex we may assume that $Q^n$ is convex, too.

Consider the sequence $\pi_n : X \to X$ of finite dimensional linear Volterra maps converging strongly to the identity map, which exists due to Property (II). From Remark 5.2 we know that the continuous superposition operators $h_{\pi_n}$ generated by $\pi_n$ map the space $\mathcal{P}a(X)$ into itself. Moreover, the strong convergence of the sequence $\{\pi_n\}$ is uniform on compacts, and it is therefore easy to check that $h_{\pi_n}(x) = \pi_n(x) \to x$ uniformly on $K$. That is why we may assume, without loss of generality, that $P\{\|\pi_n(x) - x\| \geq 1/n\} < 1/n$ for all $x \in K$.

For any $n \in N$ let us define the finite dimensional projective system by

$$\mathcal{X}^n = (X^n_t, p^{ut}_n, T), \quad \text{where} \quad X^n = \pi_n(X), \quad X^n_t = p^{tb}(X^n), \quad p^{ut}_n = p^{ut}|_{X^n_t}. \quad (8)$$

If $t \geq u \geq v$ and $x = p^{tb}y \in X^n_t$ (for some $y \in X^n$), then

$$(p^{tn}_n \circ p^{ut}_n)(x) = (p^{tn}_n \circ p^{ut}|_{X^n_t})(x) = (p^{tn}_n \circ p^{ut})(x) = p^{tv}x = (p^{tv}|_{X^n_t})(x) = p^{tn}_n x,$$

as $p^{ut} = (p^{ut} \circ p^{tb})(y) = p^{tb}y \in X^n$. Therefore, $\mathcal{X}^n$ is a projective system. Evidently, $\mathcal{P}a(X^n) \subset \mathcal{P}a(X)$ and $h_{\pi_n}(\mathcal{P}a(X^n)) \subset \mathcal{P}a(X^n)$.

By Lemma 4.2 there are continuous projections $\phi_n : X^n \to X^n \cap Q^n$ such that the associated superposition operators satisfy $h_{\phi_n}(\mathcal{P}a(X^n)) \subset \mathcal{P}a(X^n \cap Q^n)$. By construction,

$$P\{\|\phi_n \circ \pi_n(x) - x\| \geq 1/n\} < 2/n \quad \text{for all} \quad x \in K. \quad (9)$$

For the LC operators $h_n \equiv h_{\phi_n} \circ h_{\pi_n} \circ h$ we have $h_n(\mathcal{P}a(X^n \cap Q^n)) \subset \mathcal{P}a(X^n \cap Q^n)$. Hence by the finite dimensional fixed-point theorem 5.4

$$\exists \alpha_n \in \mathcal{P}a(X^n \cap Q^n) \quad \text{such that} \quad h_n \alpha_n = \alpha_n \quad P \text{-a.s.} \quad (10)$$

As $\alpha_n \in \mathcal{P}a(Q^n) \subset \mathcal{P}a(V)$, we have $h \alpha_n \in K$ for all $n \in N$. From (11) it then follows that

$$P\{\|h \alpha_n - \alpha_n\| \geq 1/n\} = P\{\|h \alpha_n - h_n \alpha_n\| \geq 1/n\}$$

$$= P\{\|h \alpha_n - (h_{\phi_n} \circ h_{\pi_n} \circ h) \alpha_n\| \geq 1/n\} < 2/n \quad \text{for all} \quad n \in N.$$

Hence

$$\|h \alpha_n - \alpha_n\|_X \to 0 \quad \text{in probability} \quad P \text{ as } n \to \infty. \quad (11)$$

Moreover, the set $\{h \alpha_n\}_{n \in N}$ is tight, so that the set $\{\alpha_n : n \in N\}$ satisfies the assumptions described in Remark 5.3, according to which the latter set is tight, too. Thus, the sequence of the random Dirac measures $\{\delta_{\alpha_n}(\omega)\}$ is precompact in the narrow topology of the space $\mathcal{P}F_1(Z)$, and we may assume, without loss of generality, that $\{\delta_{\alpha_n}(\omega)\}$ converges to a random probability measure $\mu_\omega$ on this $\sigma$-algebra. Using $Z = X$ let us define $P^*$ by

$$dP^*(\omega, x) = d\mu_\omega(x) dP(\omega). \quad (12)$$

In particular, we obtain

$$E^* g = \lim_{n \to \infty} E(g \circ \alpha_n) = \lim_{n \to \infty} \int_{\Omega^*} g \, dP \alpha_n^{-1}. \quad (13)$$

which holds for any bounded, $P^*$-a.s. continuous function $g$, see Remark 4.3. Here $E$ resp. $E^*$ is the expectation with respect to the probability measure $P$ resp. $P^*$ and the measure $P \alpha_n^{-1}$ is defined by $P \alpha_n^{-1}(A \times B) = \{\omega \in A : \alpha(\omega) \in B\}$, see Remark 4.1.
By Theorem 4.3, the operator \( h \) admits a unique LC extension \( h^* : \mathcal{P}a^*(X) \to \mathcal{P}a^*(X) \). We claim that the random point

\[
x^* : \Omega^* \to X \text{ defined as } x^*(\omega, x) = x, \text{ where } x \in X = Z,
\]

is a fixed point of the operator \( h^* \), i.e. \( h^*x^* = x^* \) \( P^*\text{-a.s.} \).

First of all, let us check that \( x^* \in \mathcal{P}a^*(V) \) \( P^*\text{-a.s.} \). Indeed, for any \( t \in T \) and any \( B \in \text{Bor}(X_t) \), the set

\[
\{ (\omega, x) \in \Omega \times X : (p^t x^*)(\omega, x) \in B \} = \{ (\omega, x) \in \Omega \times X : p^t(x) \in B \} = \Omega \times (p^t)^{-1}(B)
\]

belongs to the \( \sigma \)-algebra \( \mathcal{F}_t^* \), see Definition 4.2. Thus, \( x^* \in \mathcal{P}a^*(X) \). To see that \( x^* \) takes values in \( V \) \( P^*\text{-a.s.} \),\( x^* \in \mathcal{P}a^*(V) \) \( P^*\text{-a.s.} \). Indeed, for any \( n \in N \), we observe that by construction \( \alpha_n \in V_{\text{a.s.}} \), so that \( P_{\alpha_n}^{-1}(\Omega \times V) = 1 \) for all \( n \in N \).

By Theorem 4.3, the operator \( h \) admits a unique LC extension \( h^* : \mathcal{P}a^*(X) \to \mathcal{P}a^*(X) \). We shall prove that \( H^*x^* = 0 \) \( P^*\text{-a.s.} \).

The operator \( H = h - id \), where \( id \) is the identity map on \( \mathcal{P}a(X) \). From (15) we have

\[
\|H\alpha_n\|_X \to 0 \quad \text{in probability } P \quad \text{as } n \to \infty.
\]

Evidently, \( H^* \equiv h^* - id^* : \mathcal{P}a^*(X) \to \mathcal{P}a^*(X) \) is the LC extension of the LC operator \( H \), where \( id^* \) is the identity map on \( \mathcal{P}a^*(X) \). We shall prove that \( H^*x^* = 0 \) \( P^*\text{-a.s.} \).

By Theorem 4.3, there exists a function \( O(\gamma) > 0 \) \( \gamma > 0 \), \( \lim_{\gamma \to +0} O(\gamma) = 0 \), such that for any compact \( Q \subset X \) and any \( \varepsilon > 0 \) there is \( \delta > 0 \) satisfying the property:

\[
d_X(x, y) < \delta \implies d_X(Hx, Hy) < \varepsilon + O(\gamma)
\]

for all \( x, y \in \mathcal{P}a(X) \), \( P\{x \notin Q\} < \gamma \), \( P\{y \notin Q\} < \gamma \).

By Theorem 4.1, there exists a sequence \( \{x_\nu\} \subset S_a(X, \mathcal{B}^*) \), for which \( d_X(x_\nu, x^*) \to 0 \) as \( \nu \to \infty \), so that

\[
d_X(x_\nu, x^*) < \delta \quad \text{for all } \nu \geq \nu_1.
\]

Pick an arbitrary \( \varepsilon > 0 \), \( \gamma > 0 \) and find \( \delta \) satisfying (16). Due to tightness of the sets \( \{\alpha_n : n \in N\} \subset \mathcal{P}a(X) \) and \( \{x_\nu : \nu \in N\} \subset \mathcal{P}a^*(X) \) there is a convex compact \( K_\gamma \subset X \) such that

\[
P\{\alpha_n \notin K_\gamma\} < \gamma \quad \text{for any } n \in N
\]

and

\[
P\{x_\nu \notin K_\gamma\} < \gamma \quad \text{for any } \nu \in N.
\]

By continuity of the operator \( H^* \), the sequence \( \{H^*x_\nu\} \) converges to \( H^*x^* \) in the space \( \mathcal{P}a^*(X) \).

Hence there exists \( m_0 \in N \) for which

\[
d_X(H^*x^*, H^*x_\nu) < \varepsilon \quad \text{for all } \nu \geq \nu_2.
\]

Let \( m = \max\{\nu_1; \nu_2\} \) and put \( y = x_m \). The set \( \{y \notin K_\gamma\} \) is a \( P^*\text{-continuity set} \), as \( y \) is \( P^*\text{-a.s.} \) continuous. Therefore, by (19) and Remark 4.3

\[
P\{y \circ \alpha_n \notin K_\gamma\} = P\alpha_n^{-1}\{x_m \notin K_\gamma\} < \gamma \quad \text{for all } n \geq n_1.
\]

Let \( E^n \) be the expectation with respect to the measure \( P\alpha_n^{-1} \). As the function \( \min\{\|y - x^*\|_X; 1\} \) is \( P^*\text{-a.s.} \) continuous, we get

\[
d_X(y \circ \alpha_n, \alpha_n) = E \min\{\|y \circ \alpha_n - \alpha_n\|_X; 1\} = E \min\{\|y \circ \alpha_n - x^* \circ \alpha_n\|_X; 1\} = E^n \min\{\|y - x^*\|_X; 1\} < \delta \quad \text{for all } n \geq n_2
\]

by (17), where \( x_m = y \). Combining (18), (21) and (22) and applying the estimate (19) yield

\[
d_X(H(y \circ \alpha_n), \alpha_n) < \varepsilon + O(\gamma) \quad \text{for all } n \geq \max\{n_1; n_2\}.
\]

Minding (15) we find \( n_3 \in N \) such that \( d_X(H\alpha_n, 0) < \varepsilon \) for all \( n \geq n_3 \). Therefore

\[
d_X(H(y \circ \alpha_n), 0) < 2\varepsilon + O(\gamma) \quad \text{for all } n \geq \max\{n_1; n_2; n_3\}.
\]
By construction, \( y = x_m \in S a(X, B^*) \). Therefore, \( y \) can be represented as \( y = \sum_{i=1}^{s} c_i I_{A_i} \) for some disjunct \( P^* \)-continuity sets \( A_i \in F \otimes \text{Bor}(X), \bigcup_{i=1}^{s} A_i = \Omega^*, c_i \in \mathcal{P}a(X) \) (\( i = 1, ..., s \)), see Definition 4.4. Then, by the representation (B.14), we obtain \( H^*y = \sum_{i=1}^{s} h(c_i)I_{A_i} \) \( P^* \)-a.s. On the other hand,

\[
H(y \circ \alpha_n) = h \left( \sum_{i=1}^{s} c_i I_{\{\alpha_n \in A_i\}} \right) = \sum_{i=1}^{s} Hc_i I_{\{\alpha_n \in A_i\}} \quad \text{P} - \text{a.s.}
\]

by the property of locality of the operator \( H \). As the random point \( H^*y \) is \( P^* \)-a.s. continuous, we obtain from (23) that

\[
d^*_X(H^*y, 0) = E^* \min \{ \left\| \sum_{i=1}^{s} h(c_i)I_{A_i} \right\|_Y ; 1 \}
\]

\[
= \lim_{n \to \infty} E^* \min \{ \left\| \sum_{i=1}^{s} h(c_i)I_{A_i} \right\|_Y ; 1 \} = \lim_{n \to \infty} E \min \{ \left\| H(y \circ \alpha_n) \right\|_Y ; 1 \} \leq 2\varepsilon + O(\gamma)
\]

and minding (17), where \( x_n = y \), we arrive at the estimate \( d^*_X(H^*x^*, 0) < 3\varepsilon + O(\gamma) \). As \( \varepsilon > 0 \) and \( \gamma > 0 \) were arbitrary and \( \lim_{\gamma \to 0} O(\gamma) = 0 \), we see that \( d^*_X(H^*x^*, 0) = 0 \), so that \( H^*x^* = 0 \) and \( h^*x^* = x^* \) \( P^* \)-a.s. This completes the proof of the first part of the theorem.

**Existence of a strong fixed point.**

Let \( x^* \) be the only weak fixed point of the operator \( h \) defined on a Young expansion \( B^* = (\Omega^*, F^*, F_t^*, P^*) \), where \( \Omega^* = \Omega \times Z \) for some separable Fréchet space \( Z \).

Consider two copies of the Young expansion \( B^* \):

\[
B_t = (\Omega^t, F_t^*, F_t^*, P^*) = (\Omega^*, F^*, F_t^*, P^*) + B^* \quad (i = 1, 2),
\]

so that, in particular, \( Z = Z_1 = Z_2 \), as well as their product

\[
B^{**} = (\Omega^{**}, F^{**}, F_t^{**}, P^{**}),
\]

defined by

1. \( \Omega^{**} = \Omega \times Z \times Z; \)
2. \( dP^{**}(\omega, z_1, z_2) = dP^*_\omega(z_1, z_2)dP(\omega), \text{ where } P^*_\omega(z_1, z_2) = P^*_\omega(z_1) \otimes P^*_\omega(z_2), z_1 \in Z \quad (i = 1, 2); \)
3. \( F^{**} \) is the \( P^{**} \)-completion of the \( \sigma \)-algebra \( F \otimes \text{Bor}(Z) \otimes \text{Bor}(Z) \)
4. \( F_t^{**} \) is the \( P^{**} \)-completion of the \( \sigma \)-algebra \( F_t \otimes (q^t)^{-1}(\text{Bor}(Z_t)) \otimes (q^t)^{-1}(\text{Bor}(Z_t)) \) for any \( t \in T \), where \( Z_t = q^t(Z), q^t = q^b \).

Denote

\[
\mathcal{P}a^t(X) \equiv \mathcal{P}a(X, B_t^*) = \mathcal{P}a^*(X) \quad \text{and} \quad \mathcal{P}a^{**}(X) \equiv \mathcal{P}(X, B^{**})
\]

and let \( h^t \) and \( h^{**} \) be the LC extensions of the operator \( h \) to the spaces \( \mathcal{P}a^t(X) \) and \( \mathcal{P}a^{**}(X) \), respectively. Due to the uniqueness of LC extensions (Theorem 4.2), \( h^{**} \) is an LC extension of each of the LC operators \( h^t : \mathcal{P}a^t(X) \to \mathcal{P}a^t(X) \) to the space \( \mathcal{P}a^{**}(X) \).

By construction, \( h^t x_i = x_i \) \( P^t \)-a.s. (\( i = 1, 2 \)), where \( x_i \) is a copy of \( x^* \) if \( Z \) is replaced by \( Z^t \).

Put \( x_i^t(\omega, z_1, z_2) = x_i(\omega, z_1) \) and observe that \( x_i^t \in \mathcal{P}a^t(X) \). Therefore,

\[
h^{**}x_i^t = h^t x_i = x_i = x_i^{**} \quad \text{P} - \text{a.s.} \quad (i = 1, 2).
\]

By uniqueness, \( x_1^{**} = x_2^{**} \) \( P^{**} \)-a.s., so that

\[
(P^*_\omega \otimes P^*_\omega)\{ (z_1, z_2) : x^*(\omega, z_1) = x^*(\omega, z_2) \} = 1 \quad P - \text{a.s.}
\]

Hence, there must exist \( \alpha : \Omega \to X \) such that \( P^*_\omega = \delta_\alpha(\omega) \) \( P \)-a.s. To verify that \( \alpha \in \mathcal{P}a(X) \) we note that \( P^*_\omega \) is a Young measure, so that it is the limit (in the narrow topology) of the sequence of the Dirac measures \( \{ \delta_{\alpha_n(\omega)} \} \) for some \( \alpha_n \in \mathcal{P}a(X) \). Then \( \alpha = \lim_{n \to \infty} \alpha_n \) in the topology of the space \( \mathcal{P}a(X) \), which proves that \( \alpha \in \mathcal{P}a(X) \). Finally,

\[
\|H\alpha\|_X = \lim_{n \to \infty} \|H\alpha_n\|_X = \lim_{n \to \infty} \|h\alpha_n - \alpha\|_X = 0
\]

in probability \( P \) due to continuity of \( H = h - id \). Thus, \( h\alpha = \alpha \) \( P \)-a.s. On the other hand, \( \alpha = x^* \) \( P^* \)-a.s. by construction. Therefore, \( x^* \) is \( P^* \)-equivalent to \( \alpha \), which means the weak solution \( x^* \) is, in fact, strong. The theorem is proven.

For \( V = X \) we obtain the following generalization of Theorem 2.1.
Corollary 5.1. Let the projective system of separable Banach spaces $X = (X_t, p^{ut}, T)$ satisfy Property (II) and $h: P_a(X) \to P_a(X)$ be a local and tight-range operator. Then $h$ has at least one weak fixed point $x^* \in P_a(X)$ for some Young expansion $B^*$ of the stochastic basis $B$.

If for any Young expansion $B^*$ of $B$, the operator $h$ has at most one weak fixed point in $P_a(X)$, then each weak fixed point of the operator $h$ will be equivalent to a unique strong, i.e. belonging to the space $P_a(X, B)$, solution of the equation $hx = x$.

For some applications of this theorem see Subsection 13.7 in Appendix D.

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The projective system \( \mathcal{L} = (L_t, t^{ul}, T) \) of separable Fréchet spaces satisfies Property (II) if there exists a sequence \( \pi_n : L \to L (L = L_b) \) of linear, continuous and finite dimensional generalized Volterra maps, which strongly converges to the identity map in \( L \) as \( n \to \infty \), see Definition 3.9.

A random point \( x \in \mathcal{P}(X, \mathcal{S}) \) is called adapted with respect to the stochastic basis \( \mathcal{B} \) and the projective system \( \mathbf{X} \) if \( p^i(x) : \Omega \to X_t \) is \( \mathcal{F}_t \)-measurable for all \( t \in T \), see Definition 3.6.

The projective system \( \mathcal{P}(X, \mathcal{S}) \) is the linear topological subspace of the space \( \mathcal{P}(X, \mathcal{S}) \) consisting of all (equivalence classes of) adapted points with respect to \( \mathcal{B} \) and \( \mathbf{X} \); if \( \mathbf{X} \) and/or \( \mathcal{B} \) are fixed, then the notation \( \mathcal{P}(\mathbf{X}, \mathcal{B}) \) can be simplified to \( \mathcal{P}(\mathbf{X}) \) or \( \mathcal{P}(\mathcal{X}) \).

A local operator is characterized by the property \( x|_A = y|_A \) a.s. \( \Rightarrow \) \( hx|_A = hy|_A \) a.s. for all \( A \subset \Omega \), see Definition 4.1.

The superposition operator \( h_f \) is defined by \( (h_f x)(\omega) = f(\omega, x(\omega)) \), where \( f : \Omega \times X \to Y \) is a given random function; any superposition operator is local.

An operator is called tight (resp. tight-range) if 1) it maps bounded subsets of its domain (resp. the entire domain) into tight subsets of its range and 2) it is uniformly continuous on tight subsets of its domain, see Definition 3.8.

Given \( \alpha \in \mathcal{P}(\mathcal{F}, \mathcal{Z}) \), the measure \( P\alpha^{-1} \) on the \( \sigma \)-algebra \( \mathcal{F} \otimes \text{Bor}(\mathcal{Z}) \) is defined by

\[
P\alpha^{-1}(A \times B) = \{ \omega \in A : \alpha(\omega) \in B \}
\]

\( P_{\Omega}(\mathcal{Z}) \) is the set of all random measures on \( \text{Bor}(\mathcal{Z}) \) with the marginal \( P ; \mathcal{Z} \) is a Polish space, see Subsection 4.1.

The narrow topology on the set \( P_{\Omega}(\mathcal{Z}) \) of all random measures \( \mu_\omega \) on \( \text{Bor}(\mathcal{Z}) \) with the marginal \( P \) is generated by the maps

\[
\mu \mapsto \mu(f) = E \int_Z f(\omega, z) d\mu_\omega(z),
\]

where \( f : \Omega \times \mathcal{Z} \to R \) is an arbitrary bounded Carathéodory function, see Subsection 4.1.

Given a projective system of separable Fréchet spaces \( \mathcal{Z} = (Z_t, t^{ul}, T) \), a Young expansion \( \mathcal{B}^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t \in T}, \mathcal{B}^* \) of the stochastic basis \( \mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, \mathcal{P}) \), generated by \( \mathcal{Z} \), satisfies the following properties: 1) \( \Omega^* = \Omega \times \mathcal{Z} \), where \( Z = Z_0 \); 2) \( \mathcal{F}_t^* \) is the limit point (in the narrow topology) of a sequence of random Dirac measures \( \delta_{\alpha_t(\omega)} \), where \( \alpha_t \in \mathcal{P}(\mathcal{Z}, \mathcal{B}) \); 3) \( \mathcal{F}^* \) is the \( \mathcal{F}^* \)-completion of the \( \sigma \)-algebra \( \mathcal{F} \otimes (\text{Bor}(\mathcal{Z})) \) and 4) \( \mathcal{F}_t^* \) is the \( \mathcal{F}^* \)-completion of the \( \sigma \)-algebra \( \mathcal{F}_t \otimes (q^t)^{-1}(\text{Bor}(\mathcal{Z})) \) for any \( t \in T \), see Definition 3.2.

**Appendix B. Proof of the auxiliary results**

We start this section with some technical results.

**B.1. Lemmata.**

**Lemma B.1.** Let \( \mathcal{B} \) be a stochastic basis on the probability space \( \mathcal{S} \) and \( \mathbf{X} = (X_t, p^{ul}, T) \) be a projective system of finite dimensional Banach spaces. Then, given a linear bijection \( G : X \to E_m, E_m \) being the \( m \)-dimensional Euclidean space, there exists a finite stochastic basis \( \mathcal{B}_m = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T(m)}, \mathcal{P}) \) on \( \mathcal{S} \), for which the superposition operator \( (h_G x)(\omega) = G(x(\omega)) \) defines a linear isomorphism between the linear topological spaces \( \mathcal{P}(\mathbf{X}, \mathcal{B}) \) and \( \mathcal{P}(\mathcal{X}, \mathcal{B}_m) \), where \( \mathcal{X} \) is a Euclidean projective system from Definition 3.7.

**Proof.** First of all, we notice that \( X \) can be identified with \( E_m \) if we replace the bonding maps \( p^i \equiv p^b \) (\( b \) is the maximal element in \( T \)) with \( p^i \circ G^{-1} \) and leave the remaining bonding maps unchanged. In this case, \( G \) becomes the identity map, and we have to prove that \( \mathcal{P}(\mathbf{X}, \mathcal{B}) = \mathcal{P}(\mathcal{X}, \mathcal{B}_m) \) for some finite stochastic basis \( \mathcal{B}_m \). Let us construct it. To this end, consider the nonincreasing family of subspaces \( \text{Ker} p^b \) of the space \( X \). For each \( t \in T \) we define \( E_t \) to be the orthogonal complement of \( \text{Ker} p^b \) in the space \( X \). Let \( T' \subset \{0, ..., m\} \) be the set of indices \( i \) such that there exists \( E_i \) for which \( i = \dim E_i \). We define \( E_i = E_t \) for these \( t \) and \( p^{ii} \) to be the induced
linear maps from $E_i$ onto $E_j$, ($i \geq j$) defined as $\kappa^{-1}_u \circ p^{u\iota} \circ \kappa_\iota$ ($t \in T_i$, $u \in T_j$), where $\kappa_t \equiv p^t|_{E_k}$ ($s \in T_k$) is the linear isomorphism between the spaces $E_k$ and $X_s$ of the same dimension.

Changing the basis in $E_m$ we may always assume that $E_i = \{(x_1, \ldots, x_i, 0, \ldots, 0)\}$ and $p^{ij}$ ($i, j \in T'$, $i \geq j$) is the orthogonal projection, which removes the coordinates $(x_{j+1}, \ldots, x_i)$.

This defines a Euclidean projective system $\mathcal{E}' = (E_i, p^{ij}, T')$.

Putting

$$\mathcal{F}_i = \bigcap_{\dim X_i = i} \mathcal{F}_i$$

results in the finite stochastic basis $\mathcal{B}' = (\Omega, \mathcal{F}, (\mathcal{F}_i)_{i \in T'}, P)$.

By construction, the map $p^i|_{E_i}$ is a linear isomorphism onto $X_i$ if $\dim X_i = i$. Therefore $p^i x$ is $\mathcal{F}_i$-measurable for all $t$ satisfying $\dim X_t = i$ if and only if $p^i x$ is $\mathcal{F}_i$-measurable. Thus, we have proven that $x \in P \mathcal{A}(\mathcal{X}, \mathcal{B})$ if and only if $p^i x$ is $\mathcal{F}_i$-measurable for any $i \in T'$. Hence $P \mathcal{A}(\mathcal{X}, \mathcal{B}) = P \mathcal{A}(\mathcal{E}', \mathcal{B}')$.

To extend the projective system $\mathcal{E}'$ from the subset $T' \subset \{0, 1, \ldots, m\}$ to the entire set $T_m = \{0, 1, \ldots, m\}$ let us take any $k \in T_m - T'$, put $E_k = \{(x_1, \ldots, x_k, 0, \ldots, 0)\}$ and define $p^{ik} : E_k \to E_i$ ($k, l \in T_m$, $k \geq l$) to be the orthogonal projection removing the coordinates $(x_{k+1}, \ldots, x_k)$. This yields the projective system $\mathcal{E}_m = (E_i, p^{ij}, T_m)$. The corresponding filtration $(\mathcal{F}_i)_{i \in T_m}$ coincides with the previous one if $i \in T'$, while for $i \in T_m - T'$ we put $\mathcal{F}_i = \mathcal{F}_k$, where $k$ is the least number from $T'$ which exceeds $i$.

Evidently, for any $x : \Omega \to E$, the random point $p^i(x) \equiv p^{im}(x)$ is $\mathcal{F}_i$-measurable for any $i \in T(m)$ if and only if $p^i x$ is $\mathcal{F}_i$-measurable for any $i \in T$, so that $P \mathcal{A}(\mathcal{X}, \mathcal{B}) = P \mathcal{A}(\mathcal{E}', \mathcal{B}') = P \mathcal{A}(\mathcal{E}_m, \mathcal{B}_m)$.

**Lemma B.2.** Let $\mathcal{X} = (X_i, p^{ii}, T)$ be a projective system of finite dimensional Banach spaces and $\mathcal{B}$ be a stochastic basis on a probability space $S$. Then for any nonempty, convex and compact subset $U$ of $X = X_b$ there exists a continuous projection $\phi : X \to U$, for which $h_\phi(P \mathcal{A}(\mathcal{X}, \mathcal{B})) = P \mathcal{A}(\mathcal{X}, \mathcal{B}) \cap P(U, S)$.

**Proof.** Step 1. We first replace $\mathcal{X}$ and $\mathcal{B}$ with the Euclidean projective system $\mathcal{E} = (E_i, p^{ij}, T_m)$ and a finite stochastic basis $\mathcal{B}_m$. The set $U \subset X$ will be at this step replaced by a nonempty, convex and compact subset $W \subset E \equiv E_m$.

Redefining the coordinates we may assume that $E_i = \{(x_1, \ldots, x_i, 0, \ldots, 0)\}$ and $p^{ij}$ are the orthogonal projection, which removes the coordinates $(x_{j+1}, \ldots, x_i)$. Let $p^i \equiv p^{im}$, $W_i = p^i(V)$ and construct a Volterra projection $\psi^i : E_i \to W_i$ by induction. For $i = 1$, the set $W_1$ is a closed, bounded interval $[a, b]$, so that we simply put $\psi^1 = \pi_{[a, b]}$, where

$$\pi_{[a, b]}(x_i) = x_i \text{ if } x_i \in [a, b], \quad \pi_{[a, b]}(x_i) = a \text{ if } x_i < a \text{ and } \pi_{[a, b]}(x_i) = b \text{ if } x_i > b.$$

Assuming that $\psi^{k-1} : E_{k-1} \to W_{k-1}$ is constructed, we observe that for each $x^{k-1} \in W_{k-1}$ the set $\{x^{k-1}, x_k\} \cap W_k$ is again a closed, bounded interval $[a(x^{k-1}), b(x^{k-1})]$, where the functions $a(\cdot) \leq b(\cdot)$ are continuous on $W_{k-1}$, as $W_k$ is convex and compact. Put

$$\psi^k((x^{k-1}, x_k)) = \left(\psi^{k-1}(x^{k-1}), \pi_{[a(x^{k-1}), b(x^{k-1})]}(x_k)\right).$$

Then $\psi^k : E_k \to W_k$ is continuous and by construction satisfies $p^{k-1,k} \circ \psi^k = \psi^{k-1} \circ p^{k-1,k}$. Therefore, $\psi^k$ is Volterra, and this completes the induction argument. Note that the superposition operator $h_\phi$ maps adapted points into adapted points, see Remark 5.2. Thus, we have proven the lemma for the case of $\mathcal{E}$ and $\mathcal{B}_m$.

Step 2. Applying Lemma B.1, we can reduce the general case to the one considered in step 1. Assume that the linear map $G : X \to E_m$ induces the linear topological isomorphism $h_G : P \mathcal{A}(\mathcal{X}, \mathcal{B}) \to P \mathcal{A}(\mathcal{E}, \mathcal{B}_m)$, put $W = G(U) \subset E$ and define $\phi = G^{-1} \circ \psi \circ G$, where $\psi : E \to W$ is a Volterra projection. By construction, $\phi$ is a continuous projection from $X$ unto $U$. Note that $h_\phi = h_{G^{-1} \circ \psi \circ G} = h_{G^{-1} \circ \psi \circ G}$. As the mapping $h_\phi : P \mathcal{A}(\mathcal{E}, \mathcal{B}_m) \to P \mathcal{A}(\mathcal{E}, \mathcal{B}_m) \cap P(W, S)$ is a continuous projection, then so is the mapping $h_\phi : P \mathcal{A}(\mathcal{X}, \mathcal{B}) \to P \mathcal{A}(\mathcal{X}, \mathcal{B}) \cap P(U, S)$. □

**Lemma B.3.** Let $P^* \circ \mathcal{B}$ be a Young probability measure defined on the $\sigma$-algebra $\mathcal{F} \otimes \text{Bor}(Z)$, where $Z$ is a Polish space. Suppose that $A = \bigcup_{i=1}^n A_i$, where $A_i \in \mathcal{F} \otimes \text{Bor}(Z)$ are disjunct subsets and
$\varepsilon > 0$. Then there exist disjunct $P^*$-continuity subsets $B_i \in \mathcal{F} \otimes \text{Bor}(Z)$ such that $B = \bigcup_{i=1}^s B_i$ and 
\[ \sum_{i=1}^s P^*(A_i \triangle B_i) < \varepsilon. \]

\textbf{Proof.} Step 1. We first prove this result for $s = 1$, i.e. for a given $A \in \mathcal{F} \otimes \text{Bor}(Z)$ and $\varepsilon > 0$ we shall find a $P^*$-continuity subset $B \in \mathcal{F} \otimes \text{Bor}(Z)$ such that $P^*(A \triangle B) < \varepsilon$. Indeed, there exist $\Omega_j \in \mathcal{F}$ and closed subsets $C_j \subset Z$ ($j = 1, \ldots, J$) such that the set $A_{\varepsilon} = \bigcup_j (\Omega_j \times C_j)$ satisfies $P^*(\partial(\Omega_j \times C_j)) = 0$ for all $j = 1, \ldots, J$ and $n \in N$. On the other hand, $\bigcap_j (\Omega_j \times C_j(\delta_{n,j})) = \Omega_j \times C_j$, so that there is a number $k \in N$ such that $P^*((\Omega_j \times C_j(\delta_{n,j})) - (\Omega_j \times C_j)) < \varepsilon/2J$ for all $j = 1, \ldots, J$. The set $B = \bigcup_j (\Omega_j \times C_j(\delta_{n,j}))$ is a continuity set of the measure $P^*$ and

\[ P^*(A \triangle B) \leq P^*(A \triangle A_{\varepsilon}) + \sum_{j=1}^J P^*((\Omega_j \times C_j(\delta_{n,j}))) = \varepsilon. \]

Step 2. Consider the case of $s = 2$. Let $A_1, A_2 \in \mathcal{F} \otimes \text{Bor}(Z)$, $A_1 \cap A_2 = \emptyset$, $A = A_1 \cup A_2$. It follows from step 1 that for any $\varepsilon > 0$ there is a $P^*$-continuity subset $B \in \mathcal{F} \otimes \text{Bor}(Z)$ such that $P^*(A \triangle B) < \varepsilon$. We shall find two $P^*$-continuity subsets $B_1, B_2$ such that

\[ B_1 \cap B_2 = \emptyset, \quad B_1 \cup B_2 = B, \quad P^*(A_1 \triangle B_1) + P^*(A_2 \triangle B_2) < \varepsilon. \]

Put $C_1 = A_1 \cap B$ and $C_2 = B - C_1 = B - A_1$. Clearly, $C_1 \cap C_2 = C_1 \cap A_2 = C_2 \cap A_1 = \emptyset$ and $C_1 \cup C_2 = B$. Therefore,

\[ (A_1 \triangle C_1) \cap (A_2 \triangle C_2) = \emptyset \quad \text{and} \quad (A_1 \triangle C_1) \cup (A_2 \triangle C_2) = A \triangle B, \]

so that

\[ P^*(A_1 \triangle C_1) + P^*(A_2 \triangle C_2) = P^*(A \triangle B) < \varepsilon. \]

Let $\sigma = \varepsilon - P^*(A \triangle B) > 0$. Applying the result from step 1 for $A_1$ and $A_2$ yields two $P^*$-continuity subsets $B'_1, B'_2 \in \mathcal{F} \otimes \text{Bor}(Z)$ such that $P^*(A_1 \triangle B'_1) < \sigma/2$ and $P^*(A_2 \triangle B'_2) < \sigma/2$. Define $B_1 = B'_1 \cap B$ and $B_2 = B - B_1$. Then

\[ P^*(B_1 \triangle C_1) = P^*((B_1 \cap B) \triangle C_1) \leq P^*(B'_1 \triangle C_1) < \sigma/3, \]

as $C_1 \subset B$. Moreover,

\[ P^*(B_2 \triangle C_2) = P^*((B - B_1) \triangle (B - C_1)) = P^*(B_1 \triangle C_1) < \sigma/3, \]

as $B_1, C_1 \subset B$. Summarizing we obtain

\[
\begin{align*}
P^*(A_1 \triangle B_1) + P^*(A_2 \triangle B_2) &
\leq P^*(A_1 \triangle C_1) + P^*(A_2 \triangle C_2) \\
&+ P^*(B_1 \triangle C_1) + P^*(B_2 \triangle C_2) = P^*(A \triangle B) + \frac{2\sigma}{3} < \varepsilon, \\
\end{align*}
\]

which concludes the proof if $s = 2$.

Step 3. The general case is treated by induction. Suppose that the statement is proven for $s - 1$, define $A^1 = \bigcup_{i=1}^{s-1} A_i$, $A^2 = A_s$ and construct, as in step 2, two disjunct $P^*$-continuity subsets $B_1^1, B_2^2$, for which $\varepsilon^1 + \varepsilon^2 < \varepsilon$, where $\varepsilon^1 = P^*(A^1 \triangle B^1)$ and $\varepsilon^2 = P^*(A^1 \triangle B^1)$. Applying the induction hypothesis, we get disjunct $P^*$-continuity subsets $B_1, \ldots, B_{s-1}$ such that

\[ \sum_{i=1}^{s-1} P^*(A_i \triangle B_i) < \varepsilon_1 \quad \text{and} \quad \bigcup_{i=1}^{s-1} B_i = B^1. \]

Adding $B_s \equiv B_2^2$ to $B_1, \ldots, B_{s-1}$ yields $s$ a set of disjunct $P^*$-continuity subsets satisfying

\[ \sum_{i=1}^s P^*(A_i \triangle B_i) = \sum_{i=1}^{s-1} P^*(A_i \triangle B_i) + P^*(A^2 \triangle B^2) < \varepsilon_1 + \varepsilon_2 < \varepsilon. \]
Lemma B.4. Suppose that \( U(\omega) (\omega \in \Omega) \) is a random closed, convex, bounded and nonempty subset of \( \mathbb{R}^n \) such that
\[
\text{Gr} U \equiv \{(\omega, U(\omega)) : \omega \in \Omega\} \in \mathcal{F} \otimes \text{Bor}(\mathbb{R}^m).
\]
Let
\[
A = \mathcal{P}(U) = \mathcal{P}(\mathbb{R}^n) \cap \{x : x(\omega) \in U(\omega) \text{ a.s}\}
\]
and \( h : A \to A \) be an LC operator. Then \( h \) has at least one fixed point in \( A \).

Proof. The proof is based on the generalization of the Nemytskii conjecture. The latter states that the Carathéodory conditions on \( F \) are not only sufficient, but also necessary for the superposition operator \( h_F \) to be continuous in measure. This conjecture, in a slightly adjusted form, was proven in \([9]\), together with its generalization for arbitrary LC operators. More precisely, the main result in \([9]\) says that for an LC operator \( h : A \to A \) there exists a Carathéodory function \( f : \text{Gr} U \to \mathbb{R}^m \) such that \( hx = h_f x \) \( \text{P.a.s.} \) for any \( x \in A \). Evidently, \( f(\omega, \cdot) \) leaves the set \( U(\omega) \) \( \text{a.s.} \) invariant. By Brouwer’s fixed-point theorem, the set \( \text{Fix}(\omega) \) consisting of all fixed points \( x_\omega \in U(\omega) \) of the map \( f(\omega, \cdot) : U(\omega) \to U(\omega) \) \( \text{a.s.} \) nonempty. On the other hand, the function \( F(\omega, x) = f(\omega, x) - x \) is Carathéodory and hence \( \mathcal{F} \otimes \text{Bor}(\mathbb{R}^m) \)-measurable. Therefore, \( \{(\omega, \text{Fix}(\omega)) : \omega \in \Omega\} = G^{-1}(0) \in \mathcal{F} \otimes \text{Bor}(\mathbb{R}^n) \) and by the measurable selection theorem (see e.g. \([4]\) p. 10) there exists a \( \mathcal{F} \)-measurable function \( x : \Omega \to \mathbb{R}^n \) such that \( x(\omega) \in U(\omega) \) \( \text{a.s.} \). Thus, \( x \in A \) and, by construction, \( hx = h_f x = x \text{ a.s.} \). \( \square \)

Let us remark that the representation theorem from \([9]\) is not valid for all subsets \( A \subset \mathcal{P}(\mathbb{R}^n, \mathcal{S}) \). On the other hand, the fixed-point result from Lemma B.4 is not valid either for arbitrary closed, convex, bounded and nonempty subsets of \( \mathcal{P}(\mathbb{R}^n, \mathcal{S}) \), see \([12]\).
B.3. Proof of Theorem 4.1

We split up the proof into 5 steps. In steps 1-4 we prove

The simplified version of Theorem 4.1. For any \( x \in \mathcal{P}a(\mathcal{X}, \mathcal{B}^*) \) such that \( x|_A = 0 \) \( P^* \)-a.s. on the set \( A \in \mathcal{F}^* \) and any \( \varepsilon > 0 \), there exist \( y \in \mathcal{S}a(\mathcal{X}, \mathcal{B}^*) \) and a \( P^* \)-continuity subset \( B \in \mathcal{F}^* \), for which

1. \( y|_B = 0 \) \( P^* \)-a.s.,
2. \( d^*_\mathcal{X}(x, y) < \varepsilon \), and
3. \( P^*(A \triangle B) < \varepsilon \).

In the course of the proof the random point \( x \) and the set \( A \) will be successively simplified by constructing special approximations with an arbitrary precision. This will be done in steps 1 - 3.

In step 4 the proof of the simplified version of Theorem 4.1 will be completed. Here we will use Lemma B.3 and simplifications from steps 1-3. The proof of Theorem 4.1 will be finished in step 5.

Here and in the sequel \( d^*_\mathcal{X}(u, v) \) is the distance on the metric space \( \mathcal{P}(\mathcal{X}, \mathcal{S}^*) \), where \( \mathcal{S}^* \) is the probability space hosting the stochastic basis \( \mathcal{B}^* \).

Step 1. The random point \( x \) may be assumed to take values in a finite dimensional subspace.

Pick an arbitrary \( \varepsilon > 0 \) and denote by \( \pi_\nu : X \to X \) linear finite dimensional Volterra maps strongly converging to the identity map in \( X \) as \( \nu \to \infty \). We use the index \( \nu \) instead of \( n \) is this proof, as \( n \) is already included in the formulation of Theorem 4.1.

Let \( \mathcal{X}^\nu \) be the finite dimensional projective systems defined in (3) by means of \( \pi_\nu \). Note that \( \pi_\nu x \in \mathcal{P}(\mathcal{X}^\nu, \mathcal{B}^*) \subset \mathcal{P}(\mathcal{X}, \mathcal{B}^*) \) due to Remark 5.2. Evidently, \( \pi_\nu x|_A = 0 \) \( P^* \)-a.s. and the strong convergence of the sequence \( \{\pi_\nu\} \) to the identity map in \( X \) implies convergence of \( \{\pi_\nu x\} \) to \( x \) in probability \( P^* \) as \( \nu \to \infty \). Therefore, \( x \) can be approximated, with an arbitrary precision, by \( \pi_\nu x \) for sufficiently large \( \nu \), the set \( A \) being unchanged. All this means that the projective system \( \mathcal{X} \) can be replaced by its finite dimensional approximation \( \mathcal{X}^\nu \). Moreover, utilizing the construction from the proof of Lemma B.3 we can replace \( \mathcal{X}^\nu \) and \( \mathcal{B}^* \) with the projective system \( \mathcal{E} = (E_i, \pi^i, T_m) \) and the finite stochastic basis \( \mathcal{B}^*_{m} = (\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_i)_{i \in T_m}, P) \), respectively, constructed as follows:

1) \( E_i = \{ (x_1, ..., x_i, 0, ..., 0) \} \) and \( \pi^i (i \geq j) \) is the orthogonal projection, which removes the coordinates \( (x_{j+1}, ..., x_i) \);
2) \( T_i = \{ t : \dim X_t = k \} \), where \( k = \min \{ j \geq i : T_j \neq \emptyset \} \);
3) \( \mathcal{F}^*_i = \bigcap_{t \in T_i} \mathcal{F}_t \) (\( i \in T_m \)).

Evidently, \( \mathcal{F}^*_i \) is the \( P^* \)-completion of the \( \sigma \)-algebra \( \mathcal{F}_i \subset \bigcap_{t \in T_i} (q^t)^{-1}(\text{Bor}(Z_t)) \), where \( \mathcal{F}_i = \bigcap_{t \in T_i} \mathcal{F}_t \). Due to Lemma B.3 there exists a local linear isomorphism between the topological spaces \( \mathcal{P}(\mathcal{X}^\nu, \mathcal{B}^*) \) and \( \mathcal{P}(\mathcal{E}, \mathcal{B}^*_{m}) \), so that the latter can replace the former in the next steps of the proof.

Step 2. The random point \( x \) may be assumed to take finitely many values.

We proceed with assuming that \( x \in \mathcal{P}(\mathcal{E}, \mathcal{B}^*_{m}) \), which is easy to see to be equivalent to the representation \( x = \sum_{i=1}^{m} \alpha_i e_i \), a.s., where \( (e_1, ..., e_m) \) is the standard basis in \( E = E_m \) and \( \alpha_i : \Omega \to R \) (\( i = 1, ..., m \)) is \( \mathcal{F}^*_i \)-measurable (\( i = 1, ..., m \)). From the property \( x|_A = 0 \) \( P^* \)-a.s., we conclude that \( \alpha_i|_A = 0 \) \( P^* \)-a.s. for all \( i = 1, ..., m \). Putting \( \Omega^*_i = \{ \omega^* \in \Omega^* : \alpha_i(\omega^*) = 0 \} \in \mathcal{F}^*_i \), we obtain

\[
P^*(A \triangle (\bigcap_{i=1}^m \Omega^*_i)) = 0.
\]

Using standard approximation technique for the \( \mathcal{F}^*_i \)-measurable, real valued functions \( \alpha_i \) we can find, for arbitrary \( \varepsilon \) and each \( i \), sets \( \Omega_{ij} \in \mathcal{F}_i, B_{ij} \in \bigcap_{t \in T_i} (q^t)^{-1}(\text{Bor}(Z_t)) \), real constants \( a_{ij} (1 \leq i \leq m, 1 \leq j \leq s) \) and a natural number \( 1 \leq r \leq s \) satisfying the following properties:

\[
P^*(\Omega^* \triangle (\bigcup_{j=1}^s (\Omega_{ij} \times B_{ij}))) < \varepsilon, \quad a_{ij} = 0 \quad (1 \leq i \leq m, 1 \leq j \leq r)
\]

\[
d^*_R(\alpha, \alpha') < \frac{\varepsilon}{m}, \quad P^*(\Omega^*_i \triangle \Omega^*_i \triangle \Omega^*_i) < \varepsilon, \quad (B.4)
\]

\[
\Omega^*_i = \bigcup_{j=1}^r (\Omega_{ij} \times B_{ij}) \quad \text{and} \quad \alpha'_i = \sum_{j=1}^r a_{ij} I_{\Omega_{ij} \times B_{ij}}, \quad \alpha'_i|_{\Omega^*_i} = 0.
\]

Here \( d^*_R \) is the following metric on the space of \( \mathcal{F}^*_i \)-measurable random points:

\[
d^*_R(\alpha, \alpha') = E^*(\min\{|\alpha - \alpha'|; 1\}). \quad (B.5)
\]
In what follows we assume, by technical reasons, that the norm in the finite dimensional space $E$ is defined as $\|x\|_E = \sum_{i=1}^{m} |\alpha_i|$. In this case,

$$d^*_{E}(x, x') = E^*(\min\{||x - x'||_E; 1\}) \leq \sum_{i=1}^{m} E^*(\min\{|\alpha_i - \alpha'_i|; 1\}) < \varepsilon,$$

where $x' = (\alpha'_1, \ldots, \alpha'_m)$. As $\varepsilon > 0$ is arbitrary, we can redefine $x = (\alpha_1, \ldots, \alpha_m)$ and $A$ to be

$$x = x' = (\alpha'_1, \ldots, \alpha'_m) \quad \text{and} \quad A = \bigcap_{i=1}^{m} \Omega_i^{*,-2}, \quad (B.6)$$

where $\alpha'_i$, $\beta'_i$ and $\Omega_i^{*,-2}$ are defined in (B.4). By construction, $x$, so redefined, assumes finitely many values and $x|_A = 0$ on the new subset $A$. This simplification is used in Step 3.

**Step 3. The random points $x$ and $y$ can be assumed to be measurable with respect to the $\sigma$-algebra of random cylinder sets.**

Examples show that the $\sigma$-algebras $\bigcap_{i \in T} (q_i^{-1})(\text{Bor}(Z_i))$ may not necessarily be the Borel $\sigma$-algebras on some Polish space, so that Lemma B.3 cannot be directly used in connection with these $\sigma$-algebras. However, Property (II) for the projective system $Z$ helps to avoid this problem by replacing the $\sigma$-algebras $F_i^*$ by their finite dimensional approximations based on the finite dimensional projective systems $Z^\nu = (Z^\nu, q^\nu \mid_{Z^\nu}, T)$, where $Z^\nu = q^\nu(Z)$, so that the corresponding intersections of cylinder $\sigma$-algebras will be Borel on some Polish space.

Let $\tau^\nu: Z \to Z$ be finite dimensional Volterra maps, which strongly converge to the identity map in $Z$ as $\nu \to \infty$, and $\tau^\nu_i: Z_i \to Z_i$ be the maps generated by $q^\nu$, see Definition B.3. For any $i = 1, \ldots, m$, $t \in T_i$ consider

$$\phi_{t,\nu} = q^t \circ \tau^\nu = \tau^\nu \circ q^t : Z \to Z^\nu, \quad \text{and the associated measure} \quad P_{t,\nu}^* = P \phi_{t,\nu}^{-1} \text{ defined on the } \sigma\text{-algebra } F_i \otimes (q^t)^{-1}(\text{Bor}(Z^\nu)) \text{ by}$$

$$P_{t,\nu}^*(\Delta) = P^*[(\text{id} \times \phi_{t,\nu})^{-1}(\Delta)],$$

where $\text{id} : \Omega \to \Omega$ is the identity map.

Denote by $F_{t,\nu}^*$ the completion of the $\sigma$-algebra $F_i \otimes \text{Bor}(Z^\nu)$ w.r.t. the measure $P_{t,\nu}^*$ and put, for any $B_{ij}$ from Step 2, $B_{ij}^\nu = \phi_{t,\nu}(B_{ij})$. As $B_{ij} \in \text{Bor}(Z)$, its image $B_{ij}^\nu$ under the continuous map $\phi_{t,\nu}$ can be obtained by an $\alpha$-operation from the closed subsets of the space $Z^\nu_t$, see e.g. [3] Th. 2.4.2. Then, using the same $\alpha$-operation we obtain the set $\Omega_{ij} \times B_{ij}^\nu$ from the subsets belonging to the family

$$\Sigma \equiv \{\Omega_{ij} \times \text{closed subsets of } Z^\nu_i\},$$

which is closed under countable intersections and finite unions. Therefore, by [3] Th. 2.2.9 the set $\Omega_{ij} \times B_{ij}^\nu$ is $F_{t,\nu}^*$-measurable for all $1 \leq i \leq m, 1 \leq j \leq r$.

Let us pick some $t_i \in T_i$ ($i = 1, \ldots,m$). The strong convergence of the sequence $\tau^\nu$ to the identity map in $Z$ implies that

$$\bigcap_{\nu=1}^{\infty} (\Omega_{ij} \times \phi_{t_i,\nu}^{-1}(B_{ij}^\nu \nu)) = \Omega_{ij} \times B_{ij} \quad (1 \leq i \leq m, 1 \leq j \leq s).$$

Therefore, for any $\varepsilon > 0$, there exists $\nu \in N$ such that

$$P^* \{(\Omega_{ij} \times \phi_{t_i,\nu}^{-1}(B_{ij}^\nu \nu)) \Delta (\Omega_{ij} \times B_{ij})\} < \frac{\varepsilon}{2\delta^2m} \quad (B.7)$$

for all $1 \leq i \leq m, 1 \leq j \leq s$. By the Volterra property,

$$\phi_{t_i,\nu}^{-1}(B_{ij}^\nu \nu) = (q^u \phi_{t_i,\nu}^{-1}(B_{ij}^\nu \nu))$$

if $u < t_i$, $u \in T_i$. Therefore, the estimates in (B.7) hold true for any $u \leq t_i$, $u \in T_i$.

As $\dim q^t(Z_u) < \infty$, there is $u_i \in T_i$, $u_i \leq t_i$, for which $\dim q^u(Z_u) \leq \dim q^t(Z_u)$ for all $t \in T_i$. Then $q^u \phi_{t_i,\nu}$ is the identity map for all $u \leq u_i$, $u \in T_i$, so that $F_i \otimes \text{Bor}(Z^\nu_u) = F_i \otimes \text{Bor}(Z^\nu_{u_i})$ for all $u \in T_i, u \leq u_i$.

Denote

$$C_{ij} = B_{ij}^\nu \phi_{t_i,\nu}^{-1}(B_{ij}^\nu \nu) \in F_{t,\nu}^*,$$

$$B_{ij}^\nu \phi_{t_i,\nu}^{-1}(C_{ij}) \in F_{t,\nu}^*,$$

$$\tilde{B}_{ij} = \Omega_{ij} \times B_{ij}, \quad \tilde{B}_{ij}^\nu = \Omega_{ij} \times B_{ij}^\nu, \quad C_{ij}^\nu = \Omega_{ij} \times C_{ij}^\nu.$$
for all $1 \leq i \leq m$, $1 \leq j \leq s$. By definition, $B'_{ij}$ are $C'_{ij}$ are random cylinder sets, for which

$$
\bigcup_{j=1}^{s} C'_{ij} = \bigcup_{j=1}^{s} B'_{ij}
$$

for all $i = 1, \ldots, m$.

We claim that

$$
P^\ast\left( (\Omega_{ij} \times B_{ij}) \Delta (\Omega_{ij} \times C'_{ij}) \right) = P^\ast\left( \tilde{B}_{ij} \Delta \tilde{C}_{ij} \right) < \frac{\varepsilon}{sm} \quad (1 \leq i \leq m, 1 \leq j \leq s). \quad (B.8)
$$

Indeed, minding $\tilde{C}_{ij} \subset \tilde{B}_{ij}$, $\tilde{B}_{ik} \cap \tilde{B}_{ij} = \emptyset$ we obtain

$$
P^\ast\left( \tilde{B}_{ij} \Delta \tilde{C}_{ij} \right) = P^\ast\left( \tilde{B}_{ij} - (\tilde{B}_{ij} - \bigcup_{k<j} B'_{ik}) \right) = P^\ast\left( \bigcup_{k<j} P^\ast(B'_{ik} \cap \tilde{B}_{ij}) \right) \leq \sum_{k<j} P^\ast(B'_{ik} \cap \tilde{B}_{ij})
$$

by $(B.7)$. Therefore,

$$
P^\ast\left( \tilde{B}_{ij} \Delta \tilde{C}_{ij} \right) \leq P^\ast\left( \tilde{B}_{ij} \Delta \tilde{B}_{ij} \right) + P^\ast\left( \tilde{B}_{ij} \Delta \tilde{C}_{ij} \right) < \frac{\varepsilon}{2s^2m} + \frac{\varepsilon}{2sm} \leq \frac{\varepsilon}{sm},
$$

which justifies $(B.8)$.

Put now

$$
\alpha'' = \sum_{j=1}^{s} a_{ij} I_{\Omega_{ij} \times C'_{ij}}; \quad \Omega_{ij}^{*3} = \bigcup_{j=1}^{r} (\Omega_{ij} \times C'_{ij}).
$$

By construction,

$$
\alpha''_{ij} = 0 \quad \text{and} \quad P^\ast(\Omega_{ij}^{*3} \Delta \Omega_{ij}^{*2}) \leq \sum_{j=1}^{r} P^\ast\left( \tilde{B}_{ij} \Delta \tilde{C}_{ij} \right) < \frac{r \varepsilon}{sm} \leq \frac{\varepsilon}{m},
$$

where $\Omega_{ij}^{*2} = \bigcup_{j=1}^{r} (\Omega_{ij} \times B_{ij})$, as it was defined in $(B.4)$.

Observe that

$$
d^p_{\mathbb{R}}(\alpha'_{ij}, \alpha''_{ij}) \leq \sum_{j=1}^{s} P^\ast\left( \tilde{B}_{ij} \Delta \tilde{C}_{ij} \right) < \frac{s \varepsilon}{sm} < \frac{\varepsilon}{m}.
$$

Hence, as in step 2, we obtain that $d^p_{\mathbb{R}}(x, x'') < \varepsilon$, where $(\alpha''_{ij})_{i,j}$.

In addition, we have

$$
P^\ast\left( A \Delta \bigcap_{i=1}^{m} \Omega_{ij}^{*3} \right) = P^\ast\left( \left( \bigcap_{i=1}^{m} \Omega_{ij}^{*3} \right) \Delta \left( \bigcap_{i=1}^{m} \Omega_{ij}^{*3} \right) \right) \leq \sum_{i=1}^{m} P^\ast(\Omega_{ij}^{*3} \Delta \Omega_{ij}^{*2}) < \varepsilon.
$$

As $\varepsilon > 0$ is arbitrary, we can again redefine $x$ and $A$ to be

$$
x = x'' = (\alpha''_{ij}), \quad \text{and} \quad A = \bigcap_{i=1}^{m} \Omega_{ij}^{*3}, \quad \text{respectively}. \quad (B.9)
$$

The great advantage of $(B.9)$ compared with $(B.6)$ is that the sets $C'_{ij} = \Omega_{ij} \times \phi_{u,v}^{-1}(C_{ij})$ are random cylinder sets for all $u \in T$, $u \leq u_i$, so that the set $A$ and the random point $x$ can be, without loss of generality, assumed to belong to the $P^\ast$-completion of the cylinder $\sigma$-algebra $\mathcal{F}_i \otimes (\phi_{u,v})^{-1}(\text{Bor}(Z_{w}^u))$ and be measurable with respect to this $\sigma$-algebra, respectively.

This enables us to apply Lemma $(B.3)$, which is done in the final step of the proof.

**Step 4. Final approximation of $x$ and $A$.**

According to Step 3, we may assume that

$$
x = (\alpha_{1}, \ldots, \alpha_{m}),
$$

where

$$
\alpha_{i} = \sum_{j=1}^{s} a_{ij} I_{\Omega_{ij} \times (\phi_{u,v})^{-1}(C_{ij})}; \quad \alpha_{i}|_{\Omega_{ij}^{*3}} = 0; \quad \Omega_{ij}^{*3} \in \mathcal{F}_i \otimes (\phi_{u,v})^{-1}(\text{Bor}(Z_{w}^u)) \quad (B.10)
$$

for some $\nu \in N$ and all $1 \leq i \leq m$, so that, in particular, $x|_{A} = 0$, where $A = \bigcap_{i=1}^{m} \Omega_{ij}^{*3}$. Moreover, $u_i \in T$, can be chosen in such a way that $\mathcal{F}_i \otimes (\phi_{u,v})^{-1}(\text{Bor}(Z_{w}^u)) = \mathcal{F}_i \otimes (\phi_{u,v})^{-1}(\text{Bor}(Z_{w}^u))$ for all $u \leq u_i$, $u \in T_i$.
Consider the probability spaces \((\Omega \times Z_n^\nu, \mathcal{F}, \mathbb{B})\), the \(\mathcal{F}_i \otimes \mathbb{B}(Z_n^\nu)\)-measurable random variable
\[
\hat{\alpha}_i = \sum_{j=1}^s a_{ij} I_{\Omega_{ij} \times C_{ij}}, \quad \hat{x} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)
\]
and the \(\mathcal{F}_i \otimes \mathbb{B}(Z_n^\nu)\)-measurable set \(\hat{\Omega}_i = \bigcup_{j=1}^r (\Omega_{ij} \times C_{ij})\). By construction, \(\alpha_i = \hat{\alpha}_i \circ (\phi_{u_i} \circ \nu)^{-1}\)
and \((\text{id} \times \phi_{u_i} \circ \nu)^{-1}(\hat{\Omega}_i) = \Omega_i^\nu\), so that \(\hat{\alpha}_i|_{\hat{\Omega}_i} = 0\).

We pick an arbitrary \(\varepsilon > 0\) and apply Lemma 1.3 to the disjunct sets \(A_{ij} \times C_{ij}\) and \((\Omega \times Z_n^\nu) - \bigcup_{j=1}^s A_{ij} \times C_{ij}\). By this, we arrive at disjunct \(P^*_{u_i, \nu}\)-continuity sets \(\hat{\Omega}^\nu_{ij} \in \mathcal{F}_i \otimes \mathbb{B}(Z_n^\nu)\), for which
\[
\sum_{j=1}^s P^*_{u_i, \nu}((A_{ij} \times C_{ij}) \triangle \hat{\Omega}^\nu_{ij}) < \frac{\varepsilon}{m} \quad \text{for all } i = 1, \ldots, m.
\]
Therefore,
\[
P^*_{u_i, \nu}(A_i^\nu \triangle \hat{\Omega}_i) \leq \sum_{j=1}^s P^*_{u_i, \nu}((A_{ij} \times C_{ij}) \triangle \hat{\Omega}^\nu_{ij}) < \frac{\varepsilon}{m}, \quad \text{where } A_i^\nu = \bigcup_{j=1}^r \Omega_i^\nu.
\]
Define \(\alpha_i^\nu = a_{ij}\) on the disjoint sets \(\hat{\Omega}^\nu_{ij}\), \(i = 1, \ldots, m\). By construction, \(\alpha_i^\nu\) are \(\mathcal{F}_i \otimes \mathbb{B}(Z_n^\nu)\)-measurable random variables and \(\alpha_i^\nu|_{A_i^\nu} = 0\), \(i = 1, \ldots, m\). In addition,
\[
P^*_{u_i, \nu}(\alpha_i^\nu \neq \hat{\alpha}_i) \leq P^*_{u_i, \nu}(A_i^\nu \triangle \hat{\Omega}_i) < \frac{\varepsilon}{m},
\]
so that
\[
d_{E^\nu}(\hat{\alpha}_i, \alpha_i^\nu) = E^\nu[\min|\alpha_i - \alpha_i^\nu|; 1] \leq P^*_{u_i, \nu}(\alpha_i^\nu \neq \hat{\alpha}_i) < \frac{\varepsilon}{m}
\]
and
\[
d_{E}(\hat{x}, \hat{x}^\nu) = E[\min\|\alpha_i - \alpha_i^\nu\|; 1] < \varepsilon,
\]
where \(E^\nu\) is the expectation associated with the probability \(P^*_{u_i, \nu}\) and \(\hat{x}^\nu = (\alpha_1^\nu, \ldots, \alpha_m^\nu)\).

In Step 3 we proved that \(\mathbb{B}(Z_n^\nu) = \mathbb{B}(Z_n^\nu)\) for all \(u \in T_i\), \(u \leq u_i\). Hence the random variables \(\alpha_i^\nu \circ \nu^\nu_{u_i}\) are measurable with respect to \(\bigcap_{u \leq u_i, u \in T_i} \mathcal{F}_i \otimes \mathbb{B}(Z_u) = \mathcal{F}^\nu\), which means that
\[
y = \sum_{i=1}^m (\alpha_i^\nu \circ \phi_{u_i, \nu}) e_i \in \mathcal{S}(\mathcal{X}, \mathcal{B}^\nu).
\]
From the definitions of the measure \(P^*_{u_i, \nu}\), the random point \(y\) and the estimates (B.11)-(B.12) we obtain
\[
d_{E}^\nu(x, y) = d_{E}(\hat{x}, \hat{y}) < \varepsilon, \quad y|_{B} = 0,
\]
where
\[
B = \bigcap_{i=1}^m (\text{id} \times \Phi_{u_i, \nu})^{-1}(A_i^\nu) \in \mathcal{F}^\nu
\]
is a \(P^*\)-continuity set.

Finally,
\[
P^*(A \triangle B) = P^* \left( \bigcap_{i=1}^m \Omega_i^\nu \triangle B \right) = P^* \left( \bigcap_{i=1}^m (\text{id} \times \Phi_{u_i, \nu})^{-1}(\Omega_i^\nu) \right) \triangle \left( \bigcap_{i=1}^m (\text{id} \times \Phi_{u_i, \nu})^{-1}(A_i^\nu) \right)
\]
\[
= P^*_{u_i, \nu} \left( \bigcup_{i=1}^m \hat{\Omega}_i \triangle \bigcup_{i=1}^m A_i^\nu \right) \leq \sum_{i=1}^m P^*_{u_i, \nu}(\hat{\Omega}_i \triangle A_i^\nu) < \frac{m \varepsilon}{m} = \varepsilon.
\]
Now we return to the projective system \(\mathcal{X}^\nu\), which in step 1 was replaced by \(\mathcal{S}(\mathcal{X}, \mathcal{B}^\nu)\). We see that \(y \in \mathcal{S}(\mathcal{X}^\nu, \mathcal{B}^\nu) \subset \mathcal{S}(\mathcal{X}, \mathcal{B}^\nu)\). From (B.13) we obtain \(d_{\mathcal{X}}(x, y) < \varepsilon\) and \(y|_{B} = 0\), where \(B\) is a \(P^*\)-continuity set satisfying \(P^*(A \triangle B) < \varepsilon\). The proof of the simplified version of Theorem 4.1 is complete.

**Step 5. Proof of the full version of Theorem 4.1.**

Let \(x, y \in \mathcal{P}(\mathcal{X}, \mathcal{B}^\nu)\) and \(x|_A = y|_A\) \(P^*-\text{a.s.}\) for some \(A \in \mathcal{F}^\nu\). From the already proven simplified version we can deduce density of \(\mathcal{S}(\mathcal{X}, \mathcal{B}^\nu)\) in the space \(\mathcal{P}(\mathcal{X}, \mathcal{B}^\nu)\) simply by picking \(A = \emptyset\). Pick any sequence \(x_n \in \mathcal{S}(\mathcal{X}, \mathcal{B}^\nu)\), \(x_n \to x\) in probability \(P^*\) \((n \to \infty)\) and find, using Lemma 1.3 a sequence of \(P^*\)-continuity sets \(A_n \in \mathcal{F}^\nu\) such that \(P^*(A_n \triangle A) \to 0\) \((n \to \infty)\). Applying the simplified version of Theorem 4.1 to \(z = x - y\), \(z|_A = 0\) \(P^*-\text{a.s.}\) we find a sequence
By Definition 4.4, any \( x \) precompact set. We will use the simplified notation from Subsection 4.2 in the proof.

**Remark B.1.** From steps 1-4 it follows that if the values of \( x \) a.s. belong to some compact \( Q_0 \subset X \), then the values of its approximation \( y \) may be chosen to belong to the closed convex hull \( Q \) of the precompact set \( \bigcup_{\nu=1}^{\infty} \pi_{\nu}(Q_0) \). This remark will be used in the proof of Theorem 4.3.

**B.4. Proof of Theorem 4.3.**

We will use the simplified notation from Subsection 4.2 in the proof.

We start with constructing an extension of \( h \) to the subspace \( S_{0}(X, B^\ast) \) of the space \( \mathcal{P}a^\ast(X) \). By Definition 4.4 any \( x \) from this space can be written as \( x = \sum_{i=1}^{s} c_i I_{A_i} \) for some \( c_i \in \mathcal{P}a(X) \) and disjunct subsets \( A_i \in F^\ast \) \((i = 1, ..., s)\), \( \bigcup_{i=1}^{s} A_i = \Omega^\ast \). Define

\[
h'^{0} x = \sum_{i=1}^{s} h(c_i) I_{A_i}, \tag{B.14}\]

and consider another element \( y \in S_{0}(X, B^\ast) \) coinciding with \( x \) on some subset \( C \subset \Omega^\ast \). Then \( y \) can be represented as \( y = \sum_{k=1}^{\sigma} d_k I_{B_k} \) for some \( d_k \in \mathcal{P}a(X) \) and disjunct subsets \( B_k \in F^\ast \) \((k = 1, ..., \sigma)\), \( \bigcup_{k=1}^{\sigma} B_k = \Omega^\ast \). By assumption, \( x = c_i = d_k \) \( P \)-a.s. on each subset \( A_i \cap B_k \cap C \), so that \( h(c_i) = h(d_k) \) \( P \)-a.s. on \( A_i \cap B_k \cap C \) by locality of \( h \). Then

\[
h'^{0} x|_{C} = \sum_{i=1}^{s} h(c_i) I_{A_i \cap C} = \sum_{k=1}^{\sigma} \sum_{i=1}^{s} h(c_i) I_{A_i \cap B_k \cap C} \]
\[
= \sum_{i=k=1}^{\sigma} h(d_k) I_{A_i \cap B_k \cap C} = \sum_{k=1}^{\sigma} h(d_k) I_{B_k \cap C} \quad P^\ast \text{ a.s.} \tag{B.15}\]

If \( C = \Omega^\ast \), i.e. \( x = y \) \( P^\ast \)-a.s., then equality (B.15) means that definition (B.14) is up to a set \( P^\ast \)-zero measure independent of the alternative representation of \( x \). If \( C \) is an arbitrary subset of \( \Omega^\ast \), then (B.15) proves locality of \( h'^{0} \) on its domain.

Next we prove uniform continuity of \( h'^{0} \) on tight subsets of the set \( S_{0}(X, B^\ast) \). For this purpose, we fix a sequence \( \alpha_{\nu} \in \mathcal{P}a(\mathcal{Z}, B) \) \((\nu \in N)\) such that the disintegration \( P_{\nu}^\circ \) of the measure \( P^\ast \) is the limit of the random Dirac measures \( \{\delta_{\alpha_{\nu}}\} \) in the narrow topology and define the auxiliary probability spaces and the stochastic bases by

\[
S_{\nu} = (\Omega^\ast, F^\ast, P_{\alpha_{\nu}}^{-1}) \quad \text{and} \quad B_{\nu} = (\Omega^\ast, F^\ast, (F_{i})_{i \in T}, P_{\alpha_{\nu}}^{-1}). \tag{B.16}\]

As it was shown in Remark 4.4, for every \( \nu \in N \) there exists an LC operator \( h' : \mathcal{P}a(X, B_{\nu}) \rightarrow \mathcal{P}(Y, S_{\nu}) \), which extends the operator \( h \). By Theorem 4.2, this extension is unique, so that by the construction from Remark 4.4

\[
h'^{0} x = h(x \circ \alpha_{\nu}) = h \sum_{i=1}^{s} h(c_i) I_{A_i} P_{\alpha_{\nu}}^{-1} \quad \text{a.s.} \tag{B.17}\]

for any \( x = \sum_{i=1}^{s} c_i I_{A_i} \), where \( c_i \in \mathcal{P}a(X) \) and \( A_i \in F^\ast \) \((i = 1, ..., s)\) are disjunct sets with the property \( \bigcup_{i=1}^{s} A_i = \Omega^\ast \). This means, in particular, that \( h'^{0} x \) defined in (B.14) is \( P_{\alpha_{\nu}}^{-1} \)-equivalent to \( h'^{0} x \).

Pick an arbitrary tight subset \( \mathcal{K} \subset S_{0}(X, B^\ast) \) and arbitrary \( \varepsilon > 0 \). Then there is a compact \( Q_0 \subset X \) such that

\[
P^\ast \{ x \notin Q_0 \} < \varepsilon \quad \text{for all} \quad x \in \mathcal{K}. \]

The closed convex hull \( Q \) of the set \( \bigcup_{\nu=1}^{\infty} \pi_{\nu}(Q_0) \) is compact as well. As \( h \) is uniformly continuous on the tight set \( \mathcal{P}a(Q) \equiv \mathcal{P}a(X) \cap \mathcal{P}(Q, B) \), we can find \( \rho > 0 \) and \( \delta > 0 \), \( \delta < \varepsilon \) such that

\[
\|u - v\|_{1} \leq \rho \quad \text{implies} \quad \forall u, v \in \mathcal{P}a(Q), \quad dv(h u, h v) < \varepsilon.
\]
where $d_{Y}$ is the metric on $P(Y)$. Choose arbitrary $x, y \in K$ satisfying $P^*\{\|x - y\| \geq \frac{\rho}{3}\} < \frac{\delta}{3}$ and find a sufficiently large $n_0 \in N$ such that

$$P^*\{\|\pi_n x - x\| \geq \frac{\rho}{3}\} < \frac{\delta}{3} \quad \text{and} \quad P^*\{\|\pi_n y - y\| \geq \frac{\rho}{3}\} < \frac{\delta}{3} \quad (\forall n \geq n_0).$$

Evidently, $P^*\{\|\pi_n x - \pi_n y\| \geq \rho\} < \delta \quad (n \geq n_0).

Now we use the construction from the proof of Theorem 3.1, see Subsection 3.2, and consider the direct product $E''$ of two copies of the finite dimensional projective subsystem $X''$ defined in [S], the compact convex subset $W'' = \{\pi_n(x) \times \pi_n(y) : x, y \in Q, \|x - y\| \leq \rho\}$ and the continuous projection $\phi_n : \pi_n(x) \times \pi_n(y) \to W''$ such that the corresponding superposition operator $h_{\phi_n}$ maps $Pa(E'', B^*)$ to $Pa(E'', B^*) \cap P(W'', B^*)$. Such a projection exists due to Lemma [B.2]

Put $(u, v) = h_{\phi_n}(\pi_n x, \pi_n y)$. By construction, $\|u - v\| \leq \rho$ and

$$P^*\{u \neq \pi_n x \& v \neq \pi_n y\} = P^*\{\pi_n x, \pi_n y \notin W''\} \leq P^*\{\pi_n x \notin Q\} + P^*\{\pi_n y \notin Q\} + P^*\{\|\pi_n x - \pi_n y\| \geq \rho\} \leq 2P^*\{x \notin Q_0\} + \delta \quad (B.18)$$

$$< 2\varepsilon + \delta < 3\varepsilon \quad (n \geq n_0).$$

As $x$ and $y$ and hence $\pi_n x, \pi_n y, u$ and $v$ belong to $Sa(X, B^*)$, the sets $\{\pi_n x \neq u\}$ and $\{\pi_n y \neq v\}$ are $P^*$-continuity subsets of $\Omega^*$, so that by Remark [B.3]

$$P\{\pi_n(x \circ \alpha_{\nu}) \neq u \circ \alpha_{\nu}\} = P\frac{\alpha_{\nu}}{\nu}^{-1}\{\pi_n x \neq u\} < \delta \quad \text{and} \quad P\{\pi_n(y \circ \alpha_{\nu}) \neq v \circ \alpha_{\nu}\} = P\frac{\alpha_{\nu}}{\nu}^{-1}\{\pi_n y \neq v\} < \delta \quad (\forall \nu \geq n_0)$$

for sufficiently large $n_0$.

For any $\nu$ the random points $u \circ \alpha_{\nu}$ and $v \circ \alpha_{\nu}$ belong to $Pa(Q)$ and satisfy $\|u \circ \alpha_{\nu} - v \circ \alpha_{\nu}\| \leq \rho$, so that

$$E_{\nu} \min\{\|h^* u - h^* v\|; 1\} = E \min\{\|h(u \circ \alpha_{\nu}) - h(v \circ \alpha_{\nu})\|; 1\} < \varepsilon \quad (\nu \geq n_0),$$

where $E_{\nu}$ is the expectation with respect to the measure $P\frac{\alpha_{\nu}}{\nu}^{-1}$. Making use of representations [B.14] and [B.17] and Remark [B.3], we can let $\nu \to \infty$ in the last estimate giving

$$d^*(h^* u, h^* v) = E^* \min\{\|h^0 u - h^0 v\|; 1\} \leq \varepsilon,$$

where $E^*$ is the expectation with respect to the measure $P^*$. By locality of $h^*$ and estimate (B.18),

$$d^*(h^0(\pi_n x), h^0(\pi_n y)) \leq d^*(h^0 u, h^0 v) + P^*\{\pi_n x \neq u \& \pi_n y \neq v\} < \varepsilon + 3\varepsilon = 4\varepsilon \quad (\forall n \geq n_0).$$

On the other hand, $h^0(\pi_n x) = \sum_{i=1}^{\infty} \lambda(\pi_n(c_i))I_{A_i}$ $P^*$-a.s. for some $c_i \in Pa(X)$, see (B.14). As $\pi_n(c_i) \to c_i$ in the topology of the space $Pa(X, B^*)$, we obtain $h^0(\pi_n x) \to h^0 c$ ($n \to \infty$) in the topology of the space $Pa^*(X)$. Similarly, $h^0(\pi_n y) \to h^0 y$ ($n \to \infty$) in this topology, so that

$$d^*(h^0 u, h^0 v) \leq 4\varepsilon,$$

which yields uniform continuity of $h^0$ on the tight set $K$.

In the final part of the proof, we use Theorem 4.1 according to which the set $Sa(X, B^*)$ is dense in the space $Pa^*(X)$. The operator $h^0 : Sa(X, B^*) \to P^*(Y)$ is uniformly continuous on tight and, thus, on precompact subsets of $Sa(X, B^*)$. Therefore, the operator $h^0$ admits a unique continuous extension $h^* : Pa^*(X) \to P^*(Y)$.

To show locality of $h^*$ we pick $x, y \in Pa^*(X)$, $x|A = y|A$ for some $A \in F^*$ and find two sequences $x_n, y_n \in Sa(X, B^*)$, again using Theorem 4.1, such that $x_n \to x, y_n \to y$ in probability and $x_n|A_n = y_n|A_n$, where $P^*(A_n \uparrow A) \to 0$ as $n \to \infty$. Then $h^* x_n \to h^* x, h^* y_n \to h^* y$ and $I_{A_n \uparrow A} \to I_A$ in probability, so that $(h^* x_n) I_{A_n \uparrow A} \to (h^* x) I_A$ and $(h^* y_n) I_{A_n \uparrow A} \to (h^* y) I_A$ in probability as $n \to \infty$. Therefore, $h x|A = h y|A$ $P^*$-a.s., and the operator $h^*$ is local.

Due to Theorem 4.1 uniform continuity of $h^*$ on arbitrary tight subsets of the space $Pa^*(X)$ is equivalent to uniform continuity on any set $K_0 = Pa^*(X) \cap P(Q_0, S^*)$, where $Q_0 \subset X$ is compact.

The closed convex hull $Q$ of the set $\bigcup_{n=1}^{\infty} \pi_n(Q_0)$ is again compact in $X$, so that $h^0$ is uniformly continuous on the set $K = Sa(X, B^*) \cap P(Q, S^*)$ and hence on its closure in the topology of the space $P^*(X)$. On the other hand, this closure contains the set $K_0$, because by Theorem 4.1 and Remark [B.1] for any $x \in K_0$ there exist

$$x_n \in Sa(X, B^*) \cap P(\pi_n(Q), F^*) \subset K$$

that converges to $x$. Hence $h^*$ is uniformly continuous on $K_0$. The theorem is proven.
B.5. Proof of Theorem 5.1

Using the LC operator $h_\phi$ from Lemma B.2 we can replace $h$ by $h \circ h_\phi$, thus obtaining an LC operator mapping the space $\mathcal{P}(\mathcal{X}, \mathcal{B})$ to the subset $\mathcal{P}(U)$. Moreover, taking advantage of the construction used in the proof of Lemma B.3 we can replace the operator $h_\phi$ by the LC operator $h_{h_\phi}^{-1} \circ h \circ h_{h_\phi}$, where the linear topological isomorphism $h_{h_\phi} : \mathcal{P}(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{P}(\mathcal{E}, \mathcal{B}_m)$ generated by the linear isomorphism $G : X \cong X_h \rightarrow E_m \equiv E$, $\mathcal{E} = \{E_1, p^1(T(m))\}$ is a projective system of Euclidean spaces $E_1 = \{(x_1, ..., x_i, 0, ..., 0)\}$, $p^j(i \geq j)$ is the orthogonal projection, which removes the coordinates $(x_{j+1}, ..., x_i)$, and $\mathcal{B}_m = \{\Omega, F, (\mathcal{F}_i)_{i \in \mathbb{N}}, \mathcal{P}\}$ is a finite stochastic basis. Therefore, we can assume that $\Omega \subset E$ and $h : \mathcal{P}(E) \rightarrow \mathcal{P}(U)$, where $\mathcal{P}(E) \equiv \mathcal{P}(\mathcal{E}, \mathcal{B}_m)$. 

The idea of the proof is to study the spaces $\mathcal{P}(E_i, \mathcal{F}_i) \ (i = 0, ..., m)$ by induction applying Lemma B.4 at each step, so that we obtain the statement of the theorem at $i = m$. The problem here is that the operator $h$ is not supposed to be Volterra, so that a priori there exist no "truncated versions" $h_{i}$ of it defined on $\mathcal{P}(E_i, \mathcal{F}_i)$. However, we will show that using locality of $h$ gives us opportunity to partially define the operators $h_{i}$ and thus find a sequence of "partial" fixed points in the subspaces $\mathcal{P}(E_i, \mathcal{F}_i)$.

Step 1: Coincidence set of two $\sigma$-algebras. Let $\mathcal{G} \subset \mathcal{G}' \subset \mathcal{F}$ be two complete $\sigma$-algebras with respect to the measure $\mathcal{P}$. Consider the family $\Omega$ of subsets $A \in \mathcal{G}$ for which $\mathcal{G} \cap A = \mathcal{G}' \cap A$ and let $\gamma = sup \mathcal{P} A$. Pick any sequences $\gamma_n \rightarrow \gamma$ for which there is $A_n \in \mathcal{O}$ with $PA_n = \gamma_n$ and define $\Omega = \bigcup_{n=1}^{\infty} A_n$. Clearly, $\mathcal{P} \mathcal{G} = \mathcal{P} \mathcal{G}' \cap \mathcal{G}$. If $B \in \mathcal{O}$ is arbitrary, then $\mathcal{G} \cap B \in \mathcal{O}$, so that $\mathcal{P} \mathcal{G} \cap B \leq \mathcal{P} \mathcal{G}' \cap B$ and hence $\mathcal{P} \mathcal{G} \cap B \cap B = 0$. We have proven that $\mathcal{G}$ is the largest (up to a zero measure) set belonging to $\mathcal{O}$. We will call this set the coincidence set of the $\sigma$-algebras $\mathcal{G} \subset \mathcal{G}'$. By the definition, if $C \in \mathcal{G} \setminus \mathcal{G}'$ and $\mathcal{P} C > 0$, then there is a subset $C \in \mathcal{G}'$ such that $\mathcal{P} C > 0$ and $\mathcal{G} \cap C = \emptyset$. For the pair $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ from the above finite filtration we fix $\Omega_i \in \mathcal{F}_i \ (i = 0, ..., m)$ to be one of the realisations of the coincidence sets of the $\sigma$-algebras $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$. We also assume by definition that $\Omega_m = \emptyset$, which is formally possible to achieve if we define $\mathcal{F}_{m+1} = \mathcal{F}_m \otimes \mathcal{B} [0, 1]$ and equip $\mathcal{B} [0, 1]$ with the Lebesgue measure.

Step 2: Construction of auxiliary (truncated) local operators. Let us first introduce truncated spaces of adapted random points. For any $i = 0, ..., m$ we put $\Omega_i = \Omega - \Omega_i \in \mathcal{F}_i$ and let $\gamma_i$ consist of all $x : \Omega_i \rightarrow E_i$ for which there exists $\bar{x} \in \mathcal{P}(E)$ such that $x = \gamma_i(x)$, where $x \equiv p^i(\hat{x})_{|\Omega_i}$ and $x \in \mathcal{P}(E)$. Below we show that the operator $h$ induces LC operators $h_i : \mathcal{P}_i \rightarrow \mathcal{P}_i$ by the formula

$$h_i x = (p^i \circ h)(\hat{x})_{|\Omega_i}, \quad \text{where} \quad x = p^i(\hat{x})_{|\Omega_i} \quad \text{and} \quad x \in \mathcal{P}(E). \quad \text{(B.19)}$$

Let us check the following property:

$$p^i(\hat{x})_{|\mathcal{A}} = p^i(\hat{y})_{|\mathcal{A}}, \quad (\hat{x}, \hat{y}) \in \mathcal{P}(E), \quad A \subset \mathcal{F}_i, \quad A \subset \Omega_i \quad \text{implies} \quad p^i \circ h(\hat{x})_{|\mathcal{A}} = p^i \circ h(\hat{y})_{|\mathcal{A}} \text{a.s.} \quad \text{(B.20)}$$

Assume, on the contrary, that the last equality is not fulfilled on a subset of $A$ on a positive measure. By the definition of the set $\Omega_i$ as the complement of the coincidence set $\Omega_i$, there exists a set $B \subset A$, $B \subset \mathcal{F}_{i+1} - \mathcal{F}_i$, of a positive measure such that

$$p^i(\hat{x})(\omega) = p^i(\hat{y})(\omega) \quad \text{and} \quad (p^i \circ h)(\hat{x})(\omega) \neq (p^i \circ y)(\hat{y})(\omega) \quad (\omega \in B). \quad \text{(B.21)}$$

Put $\hat{z} = \hat{x}$ on $B \in \mathcal{F}_i$ and $\hat{z} = \hat{y}$ on $\Omega - B$. We claim that $\hat{z} \in \mathcal{P}(E_m)$. Indeed, if $k > i$, then $p^k(\hat{z}) = p^k(\hat{x})$ on $B \in \mathcal{F}_i$ and $p^k(\hat{z}) = p^k(\hat{y})$ on $\Omega - B$. Hence $p^k(\hat{z}) = \mathcal{F}_i$-measurable for $k > i$. If $j \leq i$, then $p^j(\hat{z}) = p^j(p^i(\hat{y}))$ on $\Omega - B$ and $p^j(\hat{z}) = p^j(p^i(\hat{x})) = p^j(p^i(\hat{y}))$ on $\Omega$. Therefore, $(p^i \circ h)(\hat{z}) \neq (p^i \circ h)(\hat{x})$ a.s. on $\mathcal{F}_i - B$ by (B.21). Hence

$$D = \{(w, u) : w \in B, \quad y = (p^i \circ h)(\hat{x})(\omega)\} = \{(w, u) : w \in A, \quad y = (p^i \circ h)(\hat{x})(\omega)\} \cap \{(w, u) : w \in A, \quad y = (p^i \circ h)(\hat{z})(\omega)\} \subset \mathcal{F}_i \otimes \mathcal{B}(E_i) \text{a.s.}$$

By the well-known measurable projection theorem [10], $B = \{\omega \in \Omega : \exists u \in E_i \mid (\omega, u) \in B\} \subset \mathcal{F}_i$. But this contradicts the assumption $B \subset \mathcal{F}_{i+1} - \mathcal{F}_i$. We have proven (B.20).

This property with $A = \Omega_i$ guarantees that the following operator is well-defined on the set $\mathcal{P}_i$:

$$x \in \mathcal{P}_i \Rightarrow h_i x = p^i \circ h(\hat{x})_{|\Omega_i}, \quad \text{where} \quad x = p^i(\hat{x})_{|\Omega_i} \quad \text{and} \quad \hat{x} \in \mathcal{P}(E). \quad \text{(B.22)}$$

The same property with an arbitrary $A \subset \Omega_i$ yields locality of the operator $h_i$ on $\mathcal{P}_i$. Continuity of $h_i$ follows from the fact that $\mathcal{P}_i$ is topologically embedded in $\mathcal{P}(E, \mathcal{S})$ via the map $x = (\eta_1, ..., \eta_i) \mapsto (\eta_1, ..., \eta_i, 0, ..., 0)_{|\Omega_i}$. Note that by construction, $\Omega_m = \emptyset$, so that $\Omega_m = \Omega$ and $h_m = h$. 

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From (B.19) and (B.20) we also have the following property:
\[ h_j(x_j)(\omega) = p^{ij}h_i(x_i)(\omega) \text{ a.s. on } \Omega_i \cap \Omega_j \text{ if } x_j = p^{ij}x_i \text{ (} i \geq j \text{).} \] (B.22)

**Step 3. Construction of partial fixed points.**

The theorem to be proven by induction:

For any \( i (0 \leq i \leq m) \) there exist \( \tilde{x}_j \in \mathcal{P}(E) (0 \leq j \leq i) \) such that \( x_{ij} \equiv p^j(\tilde{x}_i)|_{\Omega_j} \) is a fixed point of the operator \( h_j \) for all \( 0 \leq j \leq i \).

The statement is trivial for \( i = 0 \) and it is equivalent to Theorem 5.1 if \( i = m \). Assume that it is true for some \( 0 \leq i < m \) and check that it is also true for \( i + 1 \).

Define the sets
\[ W_j(\omega) = \begin{cases} p^{i+1}(V) & \text{if } \omega \in \Omega_j = \Omega - \Omega_j \\ \left(p^{(i+1)}|^{(i)}(x_{ij})(\omega)\right) & \text{if } \omega \in \Omega_j \end{cases} \] (B.23)
and put \( U_{i+1}(\omega) = \bigcap_{j=0}^{i} W_j(\omega) \). By construction, \( \{\omega, W_j(\omega) : \omega \in \Omega\} \in \mathcal{F}_{i+1} \otimes \text{Bor}(E_{i+1}) \).

For the set \( \mathcal{P}(U_{i+1}) \) consisting of all \( \mathcal{F}_{i+1} \)-measurable random points \( \omega \rightarrow S_{i+1}(\omega) \) we check that \( \mathcal{P}(S_{i+1}) = \mathcal{P}(E_{i+1}, B_{i+1}) \), where \( S_{i+1} \equiv (\Omega, (\mathcal{F}_j)_{0 \leq j \leq i+1}, \mathcal{F}, P) \) is the truncated stochastic basis.

Pick any \( z \in \mathcal{P}(S_{i+1}) \) and \( j \leq k \leq i \). Then for any \( \omega \in \Omega_k \), we have
\[ p^{k,i+1}(z(\omega)) = x_{ik}(\omega) = p^k(\tilde{x}_i(\omega)), \]
and therefore
\[ p^{i,i+1}(z(\omega)) = (p^k \circ p^{k+1})(z(\omega)) = (p^k \circ p^k)(\tilde{x}_i(\omega)) = p^i(\tilde{x}_i(\omega)) \]
(\( \omega \in \mathcal{O_j} \equiv \bigcup_{j \leq k \leq i} \Omega_k \)). The set \( \mathcal{O}_j \) belongs to \( \mathcal{F}_j \), because the \( \sigma \)-algebras \( \mathcal{F}_j \) (\( j \leq k \leq i+1 \)) coincide, by construction, on its complement \( \Omega - \mathcal{O}_j = \bigcap_{j \leq k \leq i} \Omega_k \). Therefore, \( p^{i,i+1}(z)|_{\mathcal{O}_j} = p^i(\tilde{x}_i)|_{\mathcal{O}_j} \) is \( \mathcal{F}_j \)-measurable for any \( j \leq i \). On the other hand, \( p^{i,i+1}(z)|_{\Omega - \mathcal{O}_j} \) is \( \mathcal{F}_j \)-measurable as well: as it was already mentioned, \( \mathcal{F}_{i+1} = \mathcal{F}_i = \ldots = \mathcal{F}_j \) on this subset. We have proven that \( z \in \mathcal{P}(E_{i+1}, B_{i+1}) \).

From (B.22) it follows that the LC operator
\[ (Hz)(\omega) = \begin{cases} z(\omega) & \text{if } \omega \in \Omega_{i+1} \\ (h_{i+1}z)(\omega) & \text{if } \omega \in \Omega_{i+1} \end{cases} \]
leaves the subset \( \mathcal{P}(S_{i+1}) \) invariant.

Finally, we observe that the operator \( H \) satisfies the assumption of Lemma B.3, so that there exists \( \tilde{x}_{i+1} \in \mathcal{P}(S_{i+1}) = \mathcal{P}(E_{i+1}, B_{i+1}) \subset \mathcal{P}(E) \), for which \( H(x_{i+1}) = x_{i+1} \text{ a.s.} \). By construction, \( x_{i+1}|_{\Omega_{i+1}} \) is a fixed point of the operator \( h_{i+1} \). On the other hand, due to (B.23) we have that
\[ p^i(\tilde{x}_i)|_{\Omega_j} = (p^{i,i+1} \circ p^{i+1})(\tilde{x}_i)|_{\Omega_j} = x_{ij}|_{\Omega_j}, \]
which by assumption is a fixed point of the operator \( h_j \). The induction argument, and hence the proof of Theorem 5.1 is complete.

**Appendix C. Some additional properties of local operators**

Unlike the results of Appendix B, the propositions collected in Appendix C are only used in the examples of Appendix D and not in the proof of the fixed-point theorems for LC operators.

Below it is assumed that \( \mathcal{S} \) is a complete probability space (P), \( \mathcal{B} \) is a stochastic basis (3) on it, \( \mathcal{X} = (X_t, p^m, T) \) is a projective system of separable Banach spaces satisfying Property (P), the associated finite dimensional linear Volterra operators being \( \pi_n \).

**Proposition C.1.** \( \text{A local operator } h : \mathcal{P}(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{P}(\mathcal{Y}, \mathcal{S}) \text{ is tight if it is tight on any subset } \mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(\mathcal{B}_r, \mathcal{S}) \text{ where } \mathcal{B}_r = \{ x \in \mathcal{X} : \|x\|_\mathcal{X} \leq r \}. \)

**Proof.** Due to Proposition B.1 we only have to prove that the set \( h(\mathcal{M}) \) is tight for any bounded \( \mathcal{M} \subset \mathcal{P}(\mathcal{X}) \). Taking arbitrary \( \varepsilon, \sigma > 0 \) we can find \( r > 0 \) such that the inequality \( P\{x \notin \mathcal{B}_r \} < \varepsilon \) holds for all \( x \in \mathcal{M} \). There exists \( n \) (depending on \( x \)) such that
\[ P\|h(\pi_n x) - h(x)\|_\mathcal{Y} \geq \sigma < \varepsilon. \]

By Lemma B.2, there exists \( x_r \in \mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(\pi_n(B_r), \mathcal{S}) \) with the property \( (\pi_n x)(\omega) = x_r(\omega) \) if \( (\pi_n x)(\omega) \in \pi_n(B_r) \). Therefore, \( P\{x_r \neq \pi_n x\} < \varepsilon. \) This property and locality of \( h \) yield \( P\{h x_r \neq h(\pi_n x)\} < \varepsilon \). The strong convergence of \( \{\pi_n\} \) to the identity map in \( \mathcal{X} \) implies boundedness of the
set $D_r = \bigcup_{n \in N} \pi_n(B_r)$. By assumption, $h$ maps the set $\mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(D_r, \mathcal{S})$ into a tight subset of $\mathcal{P}(\mathcal{X}, \mathcal{B})$. Therefore, there exists a compact $G \subset X$ for which
\[
P\{h \notin G\} < \varepsilon \quad \text{for all } y \in \mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(D_r, \mathcal{S}).
\]
In particular, this is satisfied for $y = x_r$ and, denoting the $\sigma$-neighborhood of $G$ by $G_{\varepsilon}$, we get
\[
P\{hx \notin G_{\varepsilon}\} \leq P\{hx_r \notin G\} + P\{\|\pi_{n,x} - hx_r\|_{Y} \geq \sigma\} + P\{hx_r \neq \pi_{n,x}\} < 3\varepsilon.
\]
This property and Remark 3.3 yield tightness of the set $h\mathcal{M}$. \qed

**Proposition C.2.** Suppose that the sequence of local and tight operators $h_n : \mathcal{P}(\mathcal{X}, \mathcal{B}) \to \mathcal{P}(Y, S)\ (n \in N)$ converges to an operator $h : \mathcal{P}(\mathcal{X}, \mathcal{B}) \to \mathcal{P}(Y, S)$ uniformly on any subset $\mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(B_r, S)$ as $n \to \infty$. Then $h$ is local and tight as well.

**Proof.** If $x, y \in \mathcal{P}(\mathcal{X}, \mathcal{B})$ and $xI_A = yI_A$ for some $A \in \mathcal{F}$, then $(h_n)_xI_A = (h_n)_yI_A$ for all $n \in N$, as all $h_n$ are local. Therefore, $(hx)_I = (hy)_I$, because $(h_n)_xI_A$ and $(h_n)_yI_A$ converge in probability to $(hx)_I$ and $(hy)_I$, respectively. Hence $h$ is local.

The operators $h_n$ are tight and hence uniformly continuous on any subset $C_r \equiv \mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(B_r, S)$ (which is tight). Then so is the operator $h$, as the sequence $(h_n)_x$ converges uniformly to $h$ uniformly on $C_r$. Applying Proposition C.1 yields uniform continuity of $h$ on an arbitrary tight subset of its domain.

It remains to prove that $h$ maps bounded subsets of $\mathcal{P}(\mathcal{X}, \mathcal{B})$ into tight subsets of $\mathcal{P}(Y, S)$. According to Proposition C.1, it is sufficient to check that $h(C_r)$ is tight for all $r > 0$. Using again uniform converegence of $(\{h_n\})$ on $C_r$, we find, for any $\varepsilon > 0$ and $\sigma > 0$, a number $m \in N$ such that $P(\|hx - h_nx\|_{Y} \geq \sigma) < \varepsilon$ whenever $x \in C_r$. As $h_n$ is tight, there exists a compact subset $K \subset Y$ such that $P\{h_nx \notin K\} < \varepsilon$ for all $x \in C_r$. Therefore, $P\{hx \notin K_{\varepsilon}\} < 2\varepsilon$, where $K_{\varepsilon}$ is the $\sigma$-neighborhood of $K$. As $\varepsilon > 0$ and $\sigma > 0$ were arbitrary, the set $h(C_r)$ is tight by Remark 3.3. \qed

**Proposition C.3.** If a local operator $h : \mathcal{P}(\mathcal{X}, \mathcal{B}) \to \mathcal{P}(Y, S)$ is uniformly continuous on any tight subset of its domain, then $h$ maps tight sets into tight ones.

**Proof.** Step 1. Assume first that $X$ is finite dimensional and prove that $h(K_0)$ is tight for any $K_0 = \mathcal{P}(\mathcal{X}, \mathcal{B}) \cap \mathcal{P}(Q, S)$, where $Q \subset X$ is compact. Observe that by Lemma B.1, any linear bijection $G : X \to E$ induces the linear isomorphism $h_G$ between the spaces $\mathcal{P}(\mathcal{X}, \mathcal{B})$ and $\mathcal{P}(\mathcal{E}, \mathcal{B}_m)$, where $E = (E_i, p^i, T_m)$, $E_i = \{(x_1, ..., x_i, 0, ..., 0)\}$, $E = E_m$, $B_m$ is a finite stochastic basis with a filtration $(\mathcal{F}_i)_{i \in T_m}$ and $p^i$ are the orthogonal projections, which remove the coordinates $(x_{i+1}, ..., x_m)$. Observe that $x = (x_1, ..., x_m) \in \mathcal{P}(\mathcal{E}, \mathcal{B}_m)$. If and only if $x_i$ is a $\mathcal{F}_i$-measurable random variable.

From now on we replace $X$ with $E$, so that $Q \subset E$. We write $\mathcal{P}(H)$ for $\mathcal{P}(\mathcal{E}, \mathcal{B}') \cap \mathcal{P}(H, S)$ if $H \subset E$.

Choose a sufficiently large $m$-dimensional cube $\Pi_m = \{(x_1, ..., x_m) : E = -r \leq x_i \leq r, i = 1, ..., m\}$ containing $Q$ and put $\Pi_i = p^i(\Pi_m)$. Each $\Pi_i$ is an $i$-dimensional cube. For arbitrary $\sigma > 0$ and $\varepsilon > 0$ find $\rho > 0$ such that
\[
\|x - x'\|_E < \rho \quad \text{a.s.} \quad \Rightarrow \quad P(\|hx - hx'\|_{Y} \geq \sigma) < \varepsilon
\]
for all $x, x' \in \mathcal{P}(\Pi_m)$. This follows from uniform continuity of $h$ on $\mathcal{P}(\Pi_m)$. We want to construct a finite subset $F \subset \Pi_m$ satisfying the following condition: for any $x \in \mathcal{P}(\Pi_m)$ there exists $x' \in \mathcal{P}(F)$ for which $\|x - x'\|_E < \rho$. For this purpose, we divide the interval $[-r, r]$ into disjoint intervals $[-r, -r + \xi], [-r + \xi, -r + 2\xi], ..., (r - \xi, r]$, where $\xi < \frac{\rho}{\sqrt{m}}$. This induces the partition of the cubes $\Pi_i$ into disjoint cubic cells $\Pi(J_i) (J_i$ is an associated $i$-dimensional multi-index) of equal size and the diameter less than $\rho$. Let $c(\Pi(J_i))$ be the center of the cubic cell $\Pi(J_i)$, let the finite set $F_i$ consist of all these centers and put $F = F_m$. By construction, the projection $p^i$ maps each cell $\Pi(J_i)$ onto some cell $\Pi(J_i)$, and in this case $p^i(c(\Pi(J_i))) = c(\Pi(J_i))$.

Given $x \in \mathcal{P}(\Pi_m)$ define
\[
A(J_i) \equiv \{\omega \in \Omega : p^i(x)(\omega) \in \Pi(J_i)\} \in \mathcal{F}_i \quad \text{and} \quad x'_i(\omega) = c(\Pi(J_i)) \quad \text{if} \quad \omega \in A(J_i)
\]
and put $x' = x'_m$. Evidently, $\|x - x'\|_E < \rho$ a.s. and $p^m x' = x'_i$ for any $i \in T_m$, which implies that $x' \in \mathcal{P}(F)$.

Let $F = \{f_1, ..., f_s\}$ and $h(f_k) = y_k \ (1 \leq k \leq s)$. The set $h(F) = \{y_k : 1 \leq k \leq s\} \subset \mathcal{P}(Y, S)$ contains finitely many random points, so that there exists a compact $C \subset Y$ and a set $B \subset \Omega$ such that $PB \geq 1 - \varepsilon$ and $y_k(\omega) \in C$ for all $1 \leq k \leq s$ and almost all $\omega \in B$. On the other hand, arbitrary $u \in \mathcal{P}(F)$ can be represented as $u = \sum_{k=1}^{s} f_k I_B(k)$ for some measurable subsets $B(k)$. 
By locality of $h$, we then obtain $hu = \sum_{k=1}^{n} h(f(k))I_{B(k)} = \sum_{k=1}^{n} ykI_{B(k)}$. Hence $(hu)(\omega) \in C$ for almost all $\omega \in B$, so that $P(u \notin C) < \varepsilon$ for all $u \in Pa(F)$. Now, for an arbitrary $x \in Pa(\Pi_m)$ we put $u = x'$ and minding (C.1) yields $P(hz \notin C_\sigma) < 2\varepsilon$, where $C_\sigma$ is the $\sigma$-neighborhood of $C$. By Remark 3.3, it means that $h(\mathcal{K}_0) \subset h(Pa(\Pi_m))$ is tight.

Step 2. Consider now the case of a general $X$. Let $\mathcal{K}_0$ be an arbitrary tight subset of $Pa(X, \sigma$ and $\varepsilon$ two positive numbers and $Q_0$ be a compact subset of $X$ satisfying the property $P(x \notin Q_0) < \varepsilon$. By uniform continuity of $h$ on tight subsets, we can find $\rho > 0$ and $\delta > 0$ such that

$$P(\|x - y\| \geq \rho) < \delta \implies P(\|h(h - hy)\|_{\gamma} \geq \sigma) < \varepsilon \quad \forall x \in \mathcal{K}.$$

As $\pi_n \to \pi \in$ in probability, we can find an $n \in N$ with the property $P(\|x - \pi_n x\|_{X} \geq \rho) < \delta$. According to Step 1, the set $h(Pa(\pi_n(Q_0)))$ is tight in $(Y, \sigma)$, so that there exist a compact $C \subset Y$ such that $P(hz \notin C) < \varepsilon$ for all $z \in Pa(\pi_n(Q_0))$. For any $x \in \mathcal{K}_0$ we put $y = \pi_n x \in \mathcal{K}$. By Lemma 3.3 there exists $z \in Pa(\pi_n(Q_0))$ such that $y(\omega) = z(\omega)$ as long as $y(\omega) \in \pi_n(Q_0)$, so that $P(y \notin C) < \varepsilon$. Thus, for the $\sigma$-neighborhood $C_\sigma$ of $C$ we obtain

$$P(x \notin C_\sigma) \leq P(hz \notin C) + P(\|h(x - hz)\|_{Y} \geq \sigma) \leq \varepsilon + P(\|h(x - hy)\|_{Y} \geq \sigma) + P(hy \neq hz) < 2\varepsilon + P(y \neq z) < 3\varepsilon,$$

because $\{hy \neq hz\} \subset \{y \neq z\}$ due to locality of $h$. By Remark 3.3, the set $h(\mathcal{K})$ is tight. \hfill \Box

**Proposition C.4.** Let $X, Y$ and $U$ be projective system of separable Banach spaces. Let the operators $h_1 : Pa(X, B) \to Pa(U, B)$ and $h_2 : Pa(U, B) \to Pa(Y, S)$ be local and uniformly continuous on tight subsets of the corresponding domains. Then the operator $h = h_1 \circ h_2$ will be local and tight if either

1) $h_2$ is bounded (i.e. it maps bounded sets into bounded ones) and $h_1$ is tight, or
2) $h_2$ is tight and the projective system $U$ satisfies Property (II).

**Proof.** Evidently, the superposition of local operators is local. Now, the first statement follows directly from the definitions, while the second statement follows from Proposition C.3. \hfill \Box

**Remark C.1.** Properties of the operators in Propositions C.2, C.4 mimic to some extent the corresponding properties of deterministic operators: 1) the limit of a sequence of compact operators is compact if the convergence is uniform on bounded subsets; 2) continuous operators map compact sets into compact sets; 3) the superposition $h_1 \circ h_2$ of two continuous operators is compact if either $h_2$ is bounded and $h_1$ is compact or $h_2$ is compact.

The property of locality is essential for the results in this section: none of them is, in general, true if at least one of the involved operators is not local.

### Appendix D. Examples

#### D.1. Examples of projective systems.

**Example D.1.** Euclidean projective systems $E = (E_t, p^t, T_m)$, see Definition 3.2.

**Example D.2.** Let $t \in T \equiv [a, b], X_t = C[a, t], p^u : C[a, t] \to C[a, u]$ be the restriction maps. We prove that the projective system $X = (C[a, t], p^u, T)$ satisfies Property (II).

Due to the linear rescaling of the variable $t$, it suffices to consider $T = [0, 1]$. For $n \in N$ we put $\delta_n = \frac{1}{n}$ and define

$$\hat{x} = \frac{1}{n+1} \sum_{k=0}^{n} \left[ (x(k\delta_n) - x((k-1)\delta_n)) (nx - k) + x((k-1)\delta_n) \right] I_{[k\delta_n, (k-1)\delta_n]}$$

for any $x \in C[0, 1]$ ($x(-\delta_n) = 0$). As

$$p^u(x(k\delta_n + 0)) = p^u(x(k\delta_n) - x((k-1)\delta_n)) (n \cdot k\delta_n - k) + x((k-1)\delta_n) = x((k-1)\delta_n)$$

and

$$p^u(x(k\delta_n - 0)) = p^u(x(k\delta_n) - x((k-1)\delta_n)) (n \cdot (k - 1)\delta_n - k) + x((k-2)\delta_n) = x((k-2)\delta_n),$$

The piecewise linear function $\pi_n x$ is continuous for all $t \in T$. On the other hand, if $x(s) = y(s)$ ($0 \leq s \leq t$) and $t \in [k\delta_n, (k-1)\delta_n]$, then $(\pi_n x)(s) = (\pi_n y)(s)$ ($0 \leq s \leq t$) due to (D.1). Evidently, this implies the Volterra property of $\pi_n$ in the sense of Definition 3.2. Finally, uniform continuity of $x \in C[0, 1]$ implies $\|\pi_n x - x\|_{C[0, 1]} \to 0$ ($n \to \infty$), so that the sequence $\{\pi_n\}$ strongly converges to the identity operator in the space $C[0, 1]$. Property (II) is verified.
Example D.3. Let $1 \leq r < \infty$, $t \in T = [a,b]$, $X_t = L^r[a,t]$, $p^u: \mathcal{L}^r[a,u] \to \mathcal{L}^r[a,t]$ be the restriction maps. The projective system $\mathcal{X} = (\mathcal{L}^r[a,t], p^u, T)$ satisfies Property (P) as well. To check this, we observe that the sequence of operators

$$
(\tau_n x)(t) = \int_a^t \Delta_n(t-s)x(s)ds \quad (n \in \mathbb{N}),
$$

where $\Delta_n(u) \geq 0$ for all $u \in R$ is a continuous function satisfying the properties $\Delta_n(u) = 0$ outside $[a,a+\frac{1}{n}]$ and $\int_a^R \Delta(u)du = 1$ strongly converges to the identity operator in the space $\mathcal{L}^r[a,b]$ (due to the standard argument). On the other hand, $\tau_n(\mathcal{L}^r[a,b]) \subset \mathcal{C}[a,b]$ and since the topology on $\mathcal{C}[a,b]$ is stronger, than the topology on $\mathcal{L}^r[a,b]$, the sequence of finite dimensional Volterra maps $\tau_n \circ \pi_n : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ were defined in the previous example strongly converges to the identity operator in the space $\mathcal{L}^r[a,b]$. By construction, this sequence satisfies all requirements needed for Property (P).

Remark D.1. From Examples D.2, D.3 and Corollary 5.1 we deduce Theorem 2.4, the "light version" of the fixed-point theorem for LC operators, as Young expansions preserve the martingale property, which is proven below in Lemma D.4.

D.2. Examples of adapted random points.

Example D.4. Let the projective system $\mathcal{X}$ be defined as in Examples D.2 or D.3, i.e. $X$ is either $\mathcal{C}[a,b]$ or $\mathcal{L}^r[a,b]$ ($1 \leq r < \infty$). Let $\mathcal{B}$ be a right-continuous stochastic basis, i.e. $\mathcal{F}_t = \bigwedge_{s \leq t} \mathcal{F}_s$. Then using the standard approximation procedure (see e.g. [8]) it is straightforward to see that $\mathcal{P}a(X,\mathcal{B})$ coincides with the space $\mathcal{F}_t$-adapted, $\mathcal{F} \otimes \text{Bor}(R)$-measurable and whose trajectories a.s. belong to the space $\mathcal{C}[a,b]$ and $\mathcal{L}^r[a,b]$, respectively.

D.3. Examples of local operators.

Example D.5. Any finite linear combination of local (resp. local and continuous) operators is again local (resp. local and continuous).

Example D.6. The superposition operator

$$
h_f : \mathcal{P}(X,\mathcal{S}) \to \mathcal{P}(Y,\mathcal{S}), \text{ defined by } (h_f x)(\omega) = f(\omega, x(\omega)),
$$

where $f: \Omega \times X \to Y$ is a given random function, is local, as $x(\omega) = y(\omega)$ a.s. on $A \subset \Omega$ implies $(h_f x)(\omega) = f(\omega, x(\omega)) = f(\omega, y(\omega)) = (h_f y)(\omega)$ a.s. on $A$.

If, in addition, $f: \Omega \times X \to Y$ is a Carathéodory map, i.e. $f(\cdot, x) \in \mathcal{P}(Y, \mathcal{S})$ for all $x \in X$ and $f(\omega, \cdot)$ is continuous for almost all $\omega \in \Omega$, then the superposition operator $h_f : \mathcal{P}(X,\mathcal{S}) \to \mathcal{P}(Y,\mathcal{S})$ is continuous in probability. The converse is true as well. If a local operator $h : \mathcal{P}(X,\mathcal{S}) \to \mathcal{P}(Y,\mathcal{S})$ is continuous in probability, then $h = h_f$ for some Carathéodory map $f : \Omega \times X \to Y$, see [9]. This result is also valid for random subsets of $X$, see Theorem B.4.

Example D.7. Let the projective system $\mathcal{X}$ be as in Examples D.2 or D.3. The Itô integral

$$
(J u)(s) = \int_a^t u(s) dW(s)
$$

is a LC operator acting from the space $\mathcal{P}a(X,\mathcal{B})$ to the space $\mathcal{P}a(Y,\mathcal{B})$ consisting of adapted stochastic processes with the trajectories belonging to $X = \mathcal{C}[a,b]$ or $\mathcal{L}^q[a,b]$ ($2 \leq r < \infty$) and $Y = \mathcal{C}[a,b]$ or $\mathcal{L}^q[a,b]$ ($1 \leq q < \infty$), respectively. In this example, the domain of the local operator is a proper subset of $\mathcal{P}(X,\mathcal{S})$, and the representation by a Carathéodory function is no longer true. Otherwise, the Itô integral would have been a Lebesgue-Stieltjes integral by the Riesz representation theorem.

Example D.8. This example generalises Example D.7.

The composition $(h_f x)(t) = \int_a^t F(s, x(s)) dW(s)$ of the Itô integral with a superposition operator is an LC operator acting from $\mathcal{P}a(X)$ to $\mathcal{P}a(Y)$, where $X = \mathcal{C}[a,b]$ and $Y = \mathcal{C}[a,b]$ or $Y = \mathcal{L}^q[a,b]$ ($1 \leq q < \infty$), provided that the following conditions are satisfied:

- $F(\cdot, x)$ is $\mathcal{F} \otimes \text{Bor}([a,b])$-measurable for all $x \in \mathbb{R}^n$
- $F(\cdot, t, x)$ is $\mathcal{F}_t$-adapted for any $t \in [a,b]$ and $x \in \mathbb{R}^n$. 

• $F(\omega, t, \cdot)$ is continuous for $P \otimes \mu$-almost all $(\omega, t) \in \Omega \times [a, b]$, where $\mu$ is the Lebesgue measure on $[a, b]$.
• $\int_0^\tau \sup_{|x| \leq r} |F(\omega, t, x)|^2 dt < \infty$ a.s.

Indeed, in this case the random map $f : \Omega \times X \to Y$, given by $(f(\omega, x(\cdot))(t) = F(\omega, t, x(t)))$, $t \in [a, b]$, is Carathéodory and due to the last condition maps $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, see e.g. [1]. Moreover, it maps $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ due to the second and third assumption, so that the claim follows from Examples D.6 and D.7.

If the last of the above conditions on $F$ is replaced by $\cdot$

• $|F(\omega, t, x)| \leq A(\omega, t) + C|x|^{p/q}$,

where $p \geq 2$, $1 \leq q \leq \infty$, $A$ is a measurable stochastic process with $L^q$-trajectories and $C \geq 0$ is a constant, then $h$ acts from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, where $X = L^q[a, b]$ and $Y = L^q[a, b]$ if $1 \leq q < \infty$ and $Y = C[a, b]$ if $q = \infty$. This follows from the continuity properties of the superposition operator in $L^p$-spaces.

Finally, the function $F$ can be replaced by a random continuous Volterra operator $V_\omega : X \to Y$ such that $V_\omega^t(x)$ is $\mathcal{F}_t$-measurable for any $x \in X$, where $V_\omega^t$ is the restriction of $V_\omega$ on the subspace $C[a, t]$, as in this case $h_\omega$ acts from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. Here $X$ and $Y$ are again one of the above functional spaces.

**Example D.9.** More general stochastic integrals are also $LC$ operators as long as they can be defined as limits in probability of finite dimensional approximations. However, in this case the domain and the range may be more complicated, see [11].

**Example D.10.** More nontrivial examples of local operators are given by the evolution operators $U^t_\omega$ constructed for finite or infinite dimensional stochastic differential equations with the existence and uniqueness property on some interval $[a, b]$, see e.g. [13 Prop. 5.1, 5.5].

Indeed, suppose that $x_0|_A = y_0|_A$ a.s. for some $A$. As $x_0$ and $y_0$ are $\mathcal{F}_a$-measurable, we may assume that $A \in \mathcal{F}_a$. Put $x(t) = U^t_\omega x_0$, $y(t) = U^t_\omega y_0$, $z(t) = x(t)|_A + y(t)|_{\Omega - A}$ ($t \geq a$) and observe that due to the locality of stochastic integrals (see Example D.7), $z(t)$ is a unique solution of the underlying equation, which satisfies $z(a) = x_0|_A + y_0|_{\Omega - A} = y_0$. By the uniqueness property, $z(t) = y(t)$ a.s. for all $a \leq t \leq b$. In particular, $x(t)|_A = y(t)|_A$ a.s. This yields locality of the evolution operator $U^t_\omega$ for $a \leq t \leq b$.

This example shows that the evolution operators are always $LC$ operators, and this property is a simple consequence of the well-posedness of the initial value problem for the underlying stochastic equation. In this respect, it is important to remark that evolution operators are not always generated by Carathéodory functions. For instance, the so-called "singular" delay differential equations do not produce Carathéodory evolution operators [2]. Another example is described in [13 Ex. 6.2].

**Example D.11.** Differentiation is also an example of a local operator which cannot be represented by a Carathéodory function.

**D.4. Examples of tight operators.**

**Example D.12.** Some general properties:

• Any finite linear combination of local and tight (resp. tight-range) operators is again local and tight (resp. tight-range).

• For operator superpositions check Proposition C.4.

• For uniform limits of sequences of tight operators check Proposition C.5.

**Example D.13.** For any separable Banach spaces $X$ and $Y$ and any Carathéodory map $f : \Omega \times X \to Y$, the superposition operator $h_f : \mathcal{P}(X, \mathcal{S}) \to \mathcal{P}(Y, \mathcal{S})$ is local and uniformly continuous on tight subsets of $\mathcal{P}(X)$.

We only have to prove the property of uniform continuity. Let $\mathcal{P}(X) \equiv \mathcal{P}(X, \mathcal{S})$ and $\mathcal{P}(Y) \equiv \mathcal{P}(Y, \mathcal{S})$ and $K$ be an arbitrary tight subset of $\mathcal{P}(X)$ and $\sigma > 0$, $\varepsilon > 0$. Choose a compact and convex set $K \subset X$ such that

$$P\{x \notin X \} < \varepsilon \quad \text{for any } x \in K$$

and put

$$\theta_\delta(\omega) \equiv \sup \{\|f(\omega, x) - f(\omega, y)\|_Y, \quad x, y \in K, \|x - y\|_X \leq \delta\}.$$  

Since $K$ is compact, $\delta \to 0$ implies that $\theta_\delta$ goes to zero a.s. and hence in probability. Thus, for any positive $\sigma, \varepsilon$ there is $\delta > 0$ such that $P\{\theta_\delta(\omega) \geq \sigma\} < \varepsilon$. Pick two arbitrary random points $x_1$
and $x_2$ from $K$ satisfying $P\{\|x_1 - x_2\|_X \geq \delta\} < \varepsilon$ and put $\hat{x}_i = \pi(x_i)$ $(i = 1, 2)$ where $\pi : X \to K$ is a continuous projection ($K$ is convex, closed and bounded). Let

$$\Omega = \{\omega \in \Omega : x_1(\omega) \in K \& x_2(\omega) \in K\}.$$  

Then $x_i(\omega) = \hat{x}_i(\omega)$ $(i = 1, 2)$, and therefore $(h_f x_i)(\omega) = (h_f \hat{x}_i)(\omega)$ $(i = 1, 2)$, if $\omega \in \hat{\Omega}$. On the other hand, $P\{x_i \in K\} < \varepsilon$ $(i = 1, 2)$. Therefore,

$$P\{\|h_f x_1 - h_f x_2\|_Y \geq \sigma\} \leq P\{\|\hat{x}_1 - \hat{x}_2\|_X \geq \delta\} + P\{\theta_\delta(\omega) \geq \sigma\} + P\{h_f x_1 \neq h_f \hat{x}_1\} + P\{h_f x_2 \neq h_f \hat{x}_2\} < 4\varepsilon,$$

which yields the uniform continuity of $h$ on $K$.

**Example D.14.** If a Carathéodory map $f : \Omega \times V \to Y$ is almost surely compact (resp. compact-range) operator from $V \subset X$ to $Y$, then the superposition operator $h_f : \mathcal{P}(V) \to \mathcal{P}(Y)$ is tight (resp. tight-range).

Consider an arbitrary tight subset $K \in \mathcal{P}(V)$ and arbitrary positive numbers $\varepsilon > 0$, $\sigma > 0$. Pick $r > 0$ for which $P\{x \not\in B_r \cap V\} < \varepsilon$ for all $x \in K$, where $B_r = \{z \in X : \|z\|_X \leq r\}$. Let also fix a countable set $\{z_i, i \in N\}$ which is dense in $B_r \cap V$. For each $\omega$ the set $H(\omega) = \{f(\omega, z_i), i \in N\}$ is precompact. Therefore, the measurable function

$$k_n(\omega) \equiv \sup_{v \in H(\omega)} \inf_{w \in H_n(\omega)} \|v - u\|_Y,$$

where $H_n(\omega) \equiv \{f(\omega, z_i), 1 \leq i \leq n\}$, tends to zero a.s. and hence in probability. Geometrically, it means that there exists a number $m \in N$ and a subset $\Omega_1^\omega \subset \mathcal{F}$, $P\{\Omega_1^\omega \geq 1 - \varepsilon\}$, such that the set $H(\omega)$ is contained in the $\sigma$-neighborhood of the finite set $H_m(\omega)$ if $\omega \in \Omega_1^\omega$. Let $K$ be a compact for which $x_i(\omega) \in K$ $(i = 1, ..., m)$ if $\omega \in \Omega_2^\omega$ and $P\{\Omega_2^\omega \geq 1 - \varepsilon\}$. Therefore, for each random point $z$ taking values in $\{z_i, i \in N\}$, $\omega \in \Omega_1^\omega \cap \Omega_2^\omega$,

$$(h_f z)(\omega) = f(\omega, z(\omega)) \in K_r \quad \text{if} \quad \omega \in \Omega_1^\omega \equiv \Omega_1^\omega \cap \Omega_2^\omega, \quad \text{(D.2)}$$

where $P\{\Omega_1^\omega \cap \Omega_2^\omega \geq 1 - 2\varepsilon\}$. The set of all such $z$ is dense in $\mathcal{P}(B_r \cap V)$, so that (D.2) holds true for all $z \in \mathcal{P}(B_r \cap V)$. Defining for any $x \in K$ the random point $z(\omega) = x(\omega)$ if $x(\omega) \in B_r$ and $z(\omega) = 0$ otherwise, we get $P\{x \not\in z\} < \varepsilon$, so that

$$P\{h_f x \not\in K_\sigma\} \leq P\{h_f z \not\in K_\sigma\} + P\{x \not\in z\} < 3\varepsilon.$$  

By Remark D.3, the set $h_f(K)$ is tight.

Deterministic integrals define compact operators in typical functional spaces. The next example shows that stochastic integrals define tight operators in typical spaces of stochastic processes. For the sake of simplicity we only consider Itô integrals. However, more general stochastic integrals give rise to tight operators as well, see e.g. [11].

**Example D.15.** Let the projective system $X$ be as in Example D.2 and let $K(t, s)$ be a continuous (deterministic) function on $[a, b] \times [a, b]$. Consider the Itô integral operator

$$(Ju)(s) = \int_a^t K(t, s)u(s)dW(s)$$

as a LC operator acting from the space $\mathcal{P}(X, \mathcal{B})$ to the space $\mathcal{P}(Y, \mathcal{B})$ consisting of adapted stochastic processes with the continuous or $p$-integrable trajectories. We claim that the operator $J$ is tight if one of the following conditions is fulfilled:

1. $X = C[a, b]$ or $X = L^r[a, b]$ $(2 \leq r < \infty)$ and $Y = L^q[a, b]$ $(1 \leq q < \infty)$;
2. $X = L^r[a, b]$ or $L^r[a, b]$ $(2 < r < \infty)$ and $Y = C[a, b]$.

To simplify the presentation we assume that $[a, b] = [0, 1]$ and $K(t, s) \equiv 1$. Let us first consider the case $X = L^2[a, b]$, where $r = 2$. Notice that the immersion $L^2[a, b]$ in $L^q[a, b]$ is a continuous map if $1 \leq q < 2$. It is sufficient, therefore, to consider the case $2 \leq q < \infty$. Put

$$g_t^q = \sum_{k=0}^{n-1} \frac{k}{n} f_{\left(\frac{k+1}{n}, \frac{k+1}{n}\right)}(t)$$

where $I_A$ is the indicator of the set $A$. Clearly, $g_t \leq t$. The standard estimates for stochastic integrals yield

$$E\left|\int_0^t I_{[g_t^q, t]}(s)u(s)dW(s)\right|^q \leq \text{const} \left(\int_0^t I_{[g_t^q, t]}(s)u(s)^2ds\right)^{\frac{q}{2}}.$$
Therefore,

\[
E \int_0^1 \left| \int_0^1 I_{[\varrho^*, t]}(s)u(s)dW(s) \right|^q dt = \int_0^1 dt E \left[ \int_0^1 I_{[\varrho^*, t]}(s)u(s)dW(s) \right]^q \leq \\
\leq \text{const} \int_0^1 dt E \left[ \int_0^1 I_{[\varrho^*, t]}(s)(s^2)ds \right]^q = \text{const} E \int_0^1 dt \left( \int_0^1 I_{[\varrho^*, t]}(s)(s^2)ds \right)^{\frac{q}{2}} \leq \\
\leq \text{const} E \left( \int_0^1 (u(s)^2)ds \right)^{\frac{q}{2}} \times \sup_{0 \leq s \leq 1} \int_0^1 I_{[\varrho^*, t]}(s)dt \leq \text{const} \frac{1}{n} E\|u\|_{L^2}^q
\]
due to the generalized Hölder inequality. Therefore,

\[
E \left[ \int_0^1 u(s)dW(s) - \int_0^{q_0(\cdot)} u(s)dW(s) \right]_{L^2}^q \leq \text{const} \frac{1}{n} E\|u\|_{L^2}^q,
\]
which means that a sequence of linear random finite dimensional (and therefore tight) operators converges to \( J \) uniformly on the sets \( \{ u \in \mathcal{P}a(X) : \|u\|_{L^2} \leq r \text{ a.s.} \} \) Applying Proposition C.2 completes the consideration of the case \( r = 2, 1 \leq q < \infty \).

Assume now that \( r > 2 \) and \( Y = C[a, b] \). Then (see e.g. [8])

\[
E \left( \sup_{0 \leq s \leq 1} \left| \int_0^t u(s)dW(s) \right|^2 \right) \leq 4E \int_0^1 u(s)^2ds \leq \text{const} E\|u\|_{L^2}^r
\]

and

\[
E \left| \int_0^t u(s)dW(s) - \int_0^u u(s)dW(s) \right|^2 = E\left[ \int_u^t u(s)dW(s) \right]^2 = E \int_u^t u(s)^2ds \leq \\
\leq E \left( \int_t u ds \right)^{\frac{q}{r}} \left( \int_u^t u(s)^{\frac{r}{q}}ds \right)^{\frac{q}{r}} \leq |t - u|^{\frac{q}{r}} \|u\|_{L^r}^q
\]

where \( r' = \frac{r}{r - 2} \). By Kolmogorov's criterion \( J \) maps subsets \( \{ u \in \mathcal{P}a(X) : \|u\|_{L^r} \} \) of the space into tight subsets of the space \( \mathcal{P}a(Y) \). By Theorem C.1 the operator \( J \) is tight.

All other cases follow from the two considered, as the space \( C[a, b] \) is continuously imbedded in any space \( L^r[a, b] \).

**Example D.16.** The composition of the Itô integral \( J \) with any of the superposition operators \( h_t \) and \( h_Y \) from Example D.8 is a tight local operator acting from \( \mathcal{P}a(X) \) to \( \mathcal{P}a(Y) \), where \( X = C[a, b] \) or \( X = L^r[a, b] \) \((2 < r < \infty)\) and \( Y = C[a, b] \) or \( Y = L^q[a, b] \) \((1 \leq q < \infty)\). This follows from the tightness properties of the operator \( J \), the properties of superposition operators from Example D.14 and Proposition C.4.

**Example D.17.** The evolution operators \( U(t) \) for stochastic differential equations with bounded delays are local and tight for sufficiently large \( t \), see [10] for the details.

D.5. Examples of Young expansions.

**Example D.18.** Suppose that \( B^* = (\Omega, F^*, F_t^*, P^*) \) is an expansion of the stochastic basis \( B = (\Omega, F, F_t, P) \) where the measure \( P^* = \text{Po}^{-1} \) is generated by a random Dirac measure \( \text{Po}^{-1} \) for some \( \alpha \in \mathcal{P}a(\Omega) \equiv \mathcal{P}a(\Omega, B) \), i.e. \( P^*(\omega) = P\{ \omega \in \Omega : \alpha(\omega) \in A(\omega) \} \). By Definition 4.2 this is a Young expansion of \( B \). We claim that the measure preserving map \( \omega \mapsto (\omega, \alpha(\omega)) \) generates a linear topological isomorphism between the spaces \( \mathcal{P}a(X) \) and \( \mathcal{P}a^*(X) \) defined by \( \alpha_X : x \mapsto x \circ \alpha \).

To see this, let us first check that \( x \in \mathcal{P}a^*(X) \equiv \mathcal{P}a(X, B^*) \) implies \( x \circ \alpha \in \mathcal{P}a(X) \). Below we will write \( \Delta(\omega) \) for the set \( \{ z \in U : (\omega, u) \in \Delta \} \) where \( \Delta \subset \Omega \times U \). Let \( t \in T \) and \( B \in \text{Bor}(Z_t) \), so that \( B^- \equiv (p^t x)^{-1}(B) \in F_t^* \). Then there exist \( B_1, B_2 \in F_t \) such that \( B_1 \subset B^- \subset B_2 \) and \( P^*(B_1) = P^*(B_2) = P^*(B^*) \), which by the definition of \( F_t^* \) means that the set \( \{ (\omega, q^t(B_1(\omega))) : \omega \in \Omega \} \in F_t \otimes \text{Bor}(Z_t) \) \((i = 1, 2)\). By the theorem of measurability of projections [10], the sets \( C_i = \{ \omega \in \Omega : q^t(\alpha(\omega)) \in q^t(B_i(\omega)) \} \) belong to \( F_t \). In addition,

\[
(p^t(x \circ \alpha))^{-1}(B) = \{ \omega \in \Omega : p^t(x(\alpha(\omega))) \in B \} = \{ \omega \in \Omega : \alpha(\omega) \in B^-(\omega) \} = \{ \omega \in \Omega : \alpha(\omega) \in B_2(\omega) \} - \Omega_0 = \{ \omega \in \Omega : \alpha(\omega) \in B_2(\omega) \} - \Omega_0,
\]
where \( \Omega_0 = \{ \omega \in \Omega : \alpha(\omega) \in B_2 - B^- \} \). The latter is of measure 0, because
\[
P(\alpha \in (B_2 - B^-)) \leq P(\alpha \in (B_2 - B_1)) = P'(B_2 - B_1) = 0
\]
and \( B_2 - B_1 \in \mathcal{F}^\times \). Therefore, \( C_2 - \Omega_0 \in \mathcal{F}_t \), so that \( x \circ \alpha \in \mathcal{P}a(X) \). Thus, the correspondence \( \alpha_x : x \mapsto x \circ \alpha \) is a linear isomorphism between \( \mathcal{P}a(X) \) and \( \mathcal{P}a^*(X) \) and since the map \( \omega \mapsto (\omega, \alpha(\omega)) \) is measure preserving, this correspondence is also a topological isomorphism.

**Example D.19.** A Young expansion \( B^{**} = (\Omega^{**}, \mathcal{F}^{**}, (\mathcal{F}^*_t)_{t \in T}, P^{**}) \) of any Young expansion \( B^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t \in T}, P^*) \) of a given stochastic basis \( B = (\Omega, \mathcal{F}, (\mathcal{F}^*_t)_{t \in T}, P) \) is again a Young expansion of this basis.

To see this, let us assume that \( \Omega^* = \Omega \times Z_1, \Omega^{**} = \Omega \times Z_2 \), where \( Z_1 \) and \( Z_2 \) are separable Frechét spaces, and \( P^{**}_\omega \) is the limit (in the narrow topology) of a sequence of random Dirac measures. Pick arbitrary \( \delta > 0 \) and arbitrary bounded random functions \( f_i : \Omega \times Z_1 \times Z_2 \to \mathbb{R} \) that are continuous in \( (z_1, z_2) \). This defines the neighborhood \( U_{f_1,\ldots,f_m,\delta} \) of the random measure \( P^{**}(\omega,z_1) \) in the space \( P_{\Omega^*}(Z_2) \), see Subsection [4.1]. Then there exists a random Dirac measure \( \delta_{(\omega,z_1)} \), where \( \beta \in \mathcal{P}a(Z_2, B^*) \), belonging to this neighborhood. By Theorem 4.1, the random points \( \beta : \Omega^* \to Z_2 \) can be assumed, without loss of generality, to be \( P^* \)-a.s. continuous in \( z_1 \in Z_1 \). This means that
\[
\left| \int_{\Omega^*} f_i dP^{**} - \int_{\Omega^*} f_i(\omega, z_1, \beta(\omega, z_1)) dP^* \right| < \delta, \quad i = 1, \ldots, m.
\]

On the other hand, \( P^{**}_\omega \) is the limit (in the narrow sense) of a sequence of random Dirac measures \( \{ \delta_{\alpha_n(\omega)} \} \), where \( \alpha_n \in \mathcal{P}a(Z_1, B) \). As the functions \( f_i(\omega, z_1, \beta(\omega, z_1)) \) are \( P^* \)-a.s. continuous in \( z_1 \), Remark 4.3 ensures that
\[
\lim_{n \to \infty} \int_{\Omega} f_i(\omega, \alpha_n(\omega), \beta(\omega, \alpha_n(\omega))) dP = \int_{\Omega^*} f_i(\omega, z_1, \beta(\omega, z_1)) dP^*.
\]

Therefore
\[
\left| \int_{\Omega} f_i(\omega, \alpha_n(\omega), \beta(\omega, \alpha_N(\omega))) dP - \int_{\Omega^*} f_i dP^{**} \right| < 2\delta
\]
for sufficiently large \( n \). The random point \( \gamma : \Omega \to Z_1 \times Z_2 \), defined as
\[
\gamma(\omega) = (\alpha_n(\omega), \beta(\omega, \alpha_n(\omega))),
\]
is easy to see to be \( B \)-adapted. By construction, it belongs to the neighborhood \( U_{f_1,\ldots,f_m,\delta} \) of the random measure \( P^{**}_\omega \) in the space \( P_{\Omega^*}(Z_2) \). Thus, \( B^{**} \) is a Young expansion of the stochastic basis \( B \).

Evidently, this construction can be iterated, so that finitely many consecutive Young expansions are all Young expansions of the original stochastic basis.

**Example D.20.** The previous example can be extended to the case of countably many iterations. More precisely, let
\[
B^*_{\nu} = (\Omega^\nu, \mathcal{F}^\nu, (\mathcal{F}^*_t)_{t \in T}, P^\nu), \quad \nu \in \mathbb{N} \cup \{0\}
\]
be a sequence of stochastic bases, where \( B^*_0 = B \) and \( B^*_\nu \) is a Young expansion of \( B^*_{\nu-1} \) for any \( \nu \in \mathbb{N} \). In particular,
\[
\Omega^\nu = \Omega \times \left( \bigoplus_{j=1}^{\nu} Z^j \right), \quad \nu \in \mathbb{N},
\]
where \( Z^j \equiv Z^j \) are separable Frechét spaces coming from the respective projective families \( Z^j = (Z^j, q^j, T) \) (\( b = \max T \)).

The direct product \( Z^\infty = \prod_{j=1}^{\infty} Z^j \) is a separable Frechét space as well, and it gives rise to the stochastic basis
\[
B^\infty = (\Omega^\infty, \mathcal{F}^\infty, (\mathcal{F}^*_t)_{t \in T}, P^\infty),
\]
where \( \Omega^\infty = \Omega \times Z^\infty \) and \( P^\infty \) is defined to be the inverse (projective) limit of the sequence \( \{ P^\nu \} \), while \( \mathcal{F}^\infty \) and \( \mathcal{F}^*_t \) are constructed according to the recipes from Definition 4.2.
Let $\mathcal{P}^\nu : Z^\infty \to \prod_{j=1}^{\nu} Z^j$ be the natural projections ($\nu \in N$), take arbitrary $\delta > 0$ and random functions $f_i : \Omega \times Z^\infty \to R$ that are continuous in the second variable and bounded by 1. Since $Z^\infty$ is a separable metric space, the measure $P^\infty_\infty$ is a random Radon measure [3, Th. 3.1.10, p. 3056], so that there exists a compact $C \subset Z^\infty$ such that $P^\infty(\Omega \times C) \geq 1 - \delta$ and

$$\left| \int_{\Omega^\infty} f_k dP^\infty - \int_{\Omega \times C} f_k dP^\infty \right| < \delta, \ k = 1, \ldots, m.$$ 

Therefore, there exists a number $\nu$ and random functions $f_k^\nu : \Omega \times \prod_{j=1}^{\nu} Z^j \to R$ that are continuous in the second variable, bounded by 1 and satisfying

$$P\{\sup_C |f_k - (f_k^\nu \circ \mathcal{P}^\nu)| \geq \delta\} < \delta, \ k = 1, \ldots, m,$$

so that

$$\left| \int_{\Omega \times C} f_k dP^\infty - \int_{\Omega \times C} (f_k^\nu \circ \mathcal{P}^\nu) dP^\infty \right| < \delta, \ k = 1, \ldots, m.$$ 

Hence

$$\left| \int_{\Omega^\infty} f_k dP^\infty - \int_{\Omega^\infty} (f_k^\nu \circ \mathcal{P}^\nu) dP^\infty \right| < 4\delta, \ k = 1, \ldots, m,$$

as $|f_k| \leq 1$ and $P^\infty(\Omega^\infty - (\Omega \times C)) \leq \delta$. By the definition of the inverse product of probability measures [3],

$$\int_{\Omega^\infty} (f_k^\nu \circ \mathcal{P}^\nu) dP^\infty = \int_{\Omega^\nu} f_k^\nu dP^\nu, \ k = 1, \ldots, m.$$ 

Applying the proposition from Example D.19 yields a random Dirac measure $\delta_\gamma(\omega), \gamma \in \mathcal{P}(Z_\nu)$, which satisfies

$$\left| \int_{Z^\nu} f_k^\nu dP^\nu - E(f_k^\nu \circ \gamma) \right| < \delta.$$ 

Let $\tilde{\gamma}(\omega) = (\gamma(\omega), 0) \in \prod_{j=1}^{\nu} Z^j \times \prod_{j=\nu+1}^{\nu plus} Z^j$. Then $\tilde{\gamma} \in \mathcal{P}(Z^\infty, \mathcal{B})$ and

$$|E(f_k^\nu \circ \gamma) - E(f_k \circ \tilde{\gamma})| = |E(f_k^\nu \circ \mathcal{P}^\nu \circ \gamma) - E(f_k \circ \tilde{\gamma})| < 2\delta$$

due to D.3.

Summarizing we obtain

$$\left| \int_{\Omega^\infty} f_k dP^\infty - E(f_k \circ \tilde{\gamma}) \right| < 6\delta.$$ 

As $\delta > 0$ and random continuous functions $f_i : \Omega \times Z^\infty \to R$ bounded by 1 are arbitrary, we have proven that $P^\infty_\infty$ can be represented as the limit (in the narrow topology) of a sequence of random Dirac measures $\{\delta_{\tilde{\gamma}(\omega)}\}$, where $\tilde{\gamma}_n \in \mathcal{P}(Z^\infty, \mathcal{B})$. Thus, $\mathcal{B}^\infty_\infty$ is a Young expansion of the stochastic basis $\mathcal{B}_0 = \mathcal{B}$ in the sense of Definition 4.2.

This construction can be generalized to the case of a countable projective family $(\mathcal{B}_0^\ast, \mathcal{P}^{\nu \lambda}, \Lambda)$ of (partially ordered) Young expansions. The inverse limit of this family will be again a Young expansion of the stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P)$.

D.6. Examples of LC expansions.

**Example D.21.** Let $(\Omega^\ast, \mathcal{F}^\ast, P^\ast)$ be a probability space and $c : \Omega^\ast \to \Omega$ be a $(\mathcal{F}^\ast, \mathcal{F})$-measurable surjection such that $P^\ast c^{-1} = P$. Denote by $\mathcal{P}^\ast(X)$ the set of all (equivalence classes of) random points $x : \Omega^\ast \to X$.

If $f(\cdot, \cdot) : \Omega \times X \to Y$ is a Carathéodory function, then the associated (local and continuous) superposition operator $h_f : \mathcal{P}(X) \to \mathcal{P}(Y)$, $h_f(u)(\omega) = f(\omega, u(\omega))$ admits a unique LC extension $h_f^\ast : \mathcal{P}^\ast(X) \to \mathcal{P}^\ast(Y)$ given by $(h_f^\ast u)(\omega^\ast) = f(\omega, u(\omega^\ast))$. In other words, the extension $h_f^\ast$ of $h_f$ is the superposition operator generated by the same mapping $F$, but naturally extended to a bigger probability space. In this case, the expansion does not need to be a Young expansion.
In the next example we need

**Lemma D.1.** Let \( B^* = (\Omega^*, F^*, F^*_t, P^*) \) be a Young expansion of the stochastic basis \( B \) in the sense of Definition 4.3 and \( M(t), t \in [a, b] \), is a martingale on \( B \) (see e.g. [3]). Then the stochastic process \( M^*(t, \omega) = M^*(t, \omega, z) \equiv M(t, \omega) \) is a martingale on the stochastic basis \( B^* \).

**Proof.** First of all, we notice that \( M^*(t) \) is \( F_t \otimes (q^t)^{-1} \text{Bor} (Z_t) \)-measurable. Indeed, for any \( t \in [a, b] \) and \( B \in \text{Bor} (R) \), the set

\[
\{ M^*(t) \in B \} = \{ M(t) \in B \} \times Z = \{ M(t) \in B \} \times (q^t)^{-1}Z_t \in F_t \otimes (q^t)^{-1} \text{Bor} (Z_t).
\]

It remains, therefore, to prove the equality

\[
E^*(M^*(t)u) = E^*(M^*(s)u)
\]

for any \( s, a \leq s \leq t \) and any \( F_t \otimes (q^t)^{-1} \text{Bor} (Z_t) \)-measurable and bounded random variable \( u : \Omega^* \rightarrow R \). In fact, it is sufficient to check this equality for \( u = I_D, A \in D \), where \( D \) generates the \( \sigma \)-algebra \( F_t \otimes (q^t)^{-1} \text{Bor} (Z_t) \), in particular, for \( P^* \)-continuity sets of the form \( D = A \times C \), where \( A \in F_s \) and \( C = (q^t)^{-1}(C_0) \), \( C_0 \in \text{Bor} (Z_t) \). In this case, the function \( M^*u \) becomes \( P^* \)-a.s. continuous, which gives us the opportunity to assume, without loss of generality, that \( P^* \) is generated by a random Dirac measure \( \delta_\alpha(w), \alpha \in \mathcal{P}(Z) \), because \( P^* \) is a Young measure.

Under the above simplifications Eq. (D.3) becomes

\[
\int_\Omega M(t, \omega)I_A(\omega)I_C(\alpha(\omega))dP = \int_\Omega M(s, \omega)I_A(\omega)I_C(\alpha(\omega))dP.
\]

Notice that \( I_C \circ \alpha : \omega \rightarrow I_C(\alpha(\omega)) \) is \( F_s \)-measurable, because \( I_C \circ \alpha = I_{(q^t)^{-1} \cdot C_0} \circ \alpha = I_{C_0} \circ q^t \circ \alpha \) is the composition of the \( \text{Bor} (Z_t) \)-measurable function \( I_{C_0} \) and the \( F_s \)-measurable random point \( q^t \circ \alpha : \Omega \rightarrow Z_s \). Thus, (D.3) follows from the assumption that \( M(t) \) is a martingale on the stochastic basis \( B \). Therefore, the equality (D.3) is fulfilled as well, which means that \( M^*(t) \) is a martingale on \( B^* \).

**Example D.22.** Consider the LC operator \( J : \mathcal{P}(X, B) \rightarrow \mathcal{P}(X, B) \) given by

\[
(Jx)(t) = \int_a^t x(s)dW(s),
\]

where \( W(t) \) is the standard scalar Wiener process on \( [a, b] \) and \( X \) is either \( C[a, b] \) or \( L^r[a, b] \).

The operator \( J \) is linear and, therefore, uniformly continuous on its domain (adapted stochastic processes with square integrable trajectories). By Theorem 4.3, \( J \) admits a unique LC extension \( J^* \) for any Young expansion \( B^* \) of \( B \).

Let us check that \( W^*(t, \omega^*) = W^*(t, \omega, z) = W(t, \omega) \) remains the standard Wiener process on \( B^* \). Indeed, \( W^* \) is sample continuous and by Lemma D.1 it is a martingale with the zero mean (which coincides with \( W^*(a) = W(a) = 0 \)), and \( (W^*)^2 - t \) is a martingale as well by the same lemma. Thus, we have verified Lévy’s characterization of the standard scalar Wiener process.

By this, the well-defined LC operator \( \int_a^t x(s)dW^*(s) \) extends the operator \( J \). Applying the uniqueness property proven in Theorem 4.2 yields

\[
(J^*x)(t) = \int_a^t x(s)dW^*(s).
\]

A similar argument can be used for an arbitrary stochastic integral defined on an appropriate domain described e.g. in [11].

Let us also remark that the operator \( J \) cannot be extended to arbitrary expansions of \( B \), because \( W^*(t) \) has at least to be a semimartingale in order that stochastic integration is properly defined.

**Example D.23.** Combining Examples D.21 and D.22 we get the formula for the (unique) LC extension

\[
(h^*x)(t) = \int_a^t F(s, x(s))dW^*(s)
\]
of the nonlinear integral operator
\[(hx)(t) = \int_a^t F(s, x(s))dW(s),\]
which is valid for any Young expansion of the underlying stochastic basis and a function \(F : \Omega \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) satisfying the conditions from Example D.8. The function \(F\) can be again replaced by a Volterra operator described in the latter example.

D.7. Weak solutions of stochastic equations.

**Example D.24.** Consider the initial value problem for an ordinary stochastic differential equation

\[dx(t) = f^0(t, x(t))dt + \sum_{j=1}^m f^j(t, x(t))dW_j(t) \quad (t \in [a, b]) \quad \text{and} \quad x(a) = x_0, \tag{D.6}\]

where \(f^j\) satisfies the conditions that are similar to those listed in Example D.8:

1. \(f^j(\cdot, x)\) is \(\mathcal{F} \otimes \text{Bar}([a, b])\)-measurable for all \(x \in \mathbb{R}^n, j = 0, \ldots, m;\)
2. \(f^j(\cdot, t, x)\) is \(\mathcal{F}_t\)-adapted for any \(t \in [a, b]\) and \(x \in \mathbb{R}^n, j = 0, \ldots, m;\)
3. \(f^j(\omega, t, \cdot)\) \((j=0, \ldots, m)\) is continuous for \(\mathcal{P} \otimes \mu\)-almost all \((\omega, t) \in \Omega \times [a, b],\) where \(\mu\) is the Lebesgue measure on \([a, b],\)
4. \(|f^j(\omega, t, x)| \leq C_j(\omega, t) \mathcal{P} \otimes \mu\)-almost everywhere, where \(C_0(\omega, \cdot) \in L^{r_0}[a, b]\) a.s. and \(C_0(\omega, \cdot) \in L^{2r_j}[a, b]\) a.s. \((j = 1, \ldots, m)\) for some \(r_j > 1\) \((j = 0, \ldots, m)\).

and \(W_j\) are standard scalar Wiener processes (not necessarily independent) on the stochastic basis \([3].\)

The claims are that under the above assumptions on \(f^j\) the initial value problem (D.6) has at least one weak solution on the interval \([a, b]\) for any \(\mathcal{F}_a\)-measurable random point \(x_0\) and that this solution has continuous paths on \([a, b].\) If it is a priori known that the problem (D.6) has at most one weak solution for any Young expansion of the stochastic basis \([3],\) then \(x\) is, in fact, strong, i.e. it is defined on the stochastic basis \([3]\) for all \(t \in [a, b],\)

To prove these claims let us consider the operator

\[(hx)(t) = x_0 + \int_a^t f^0(s, x(s))ds + \sum_{j=1}^m \int_a^t f^j(s, x(s))dW_j(s) \tag{D.7}\]

in the space \(\mathcal{P}_a(X),\) where \(X = C[a, b],\) and check the assumptions of Corollary 5.1 (or Theorem 2.1, a particular case of this corollary).

Using the information from the examples of this section we obtain that

- the corresponding projective system, generated by the space \(X = C[a, b]\) satisfies Property (II), see Example D.3
- the integral superposition operator \((I_0x)(t) \equiv \int_a^t f^0(s, x(s))ds\) is an LC operator in the space \(\mathcal{P}(X)\) to \(\mathcal{P}(X),\) see Example D.4, as \(f^0(\cdot, \cdot, x)\) is adapted for each \(x \in \mathbb{R}^n,\) then \(I_0\) maps \(\mathcal{P}(X)\) into itself;
- the superposition operator \(I_0 : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) is tight-range, as the integral operator generating \(I_0\) is compact-range in the space \(C[a, b]\) due to assumption (4), see Example D.4;
- the integral operators \((I_{j}(x)(t) \equiv \int_a^t f^j(s, x(s))dW_j(s)\) are LC operators in the space \(\mathcal{P}(X)\) to \(\mathcal{P}(X),\) see Examples D.8;
- the operators \(I_{j} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) are tight-range, because \(I_{j}(\mathcal{P}(X)) = I_{j}(\mathcal{A}),\) where the set

\[\mathcal{A} = \{ x \in \mathcal{P}(X) : \|x(\omega)\|_{L^{2r_j}} \leq \|C_j(\omega)\| \text{ a.s.}\}\]

is bounded in the space \(\mathcal{P}(L^{2r_j})\) and \(r_j > 1,\) see Example D.15;
- the operator \(h : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) is a local and tight-range operator as a sum of such operators, see Example D.14.

Therefore, the operator \(h\) has at least one weak fixed point \(x^*\) in the space \(\mathcal{P}(X) = \mathcal{P}(C[a, b]).\) This fixed point will be a weak, path-continuous solution of the initial value problem (D.6) on the interval \([a, b].\) If, in addition, this initial value problem is known to have at most one local solution on the interval \([a, b]\) for any Young expansion of the stochastic basis \([3],\) then the operator \(h\) has
at most one weak fixed point in the space \( P[a,b] \). Applying Corollary 5.7, we get a unique strong solution of the problem \((D.7)\) on the interval \([a,b]\).

If the right-hand sides of the equation in \((D.6)\) are not bounded, then the solutions of it may not be defined on the entire interval \([a,b]\). In this case, we will need a notion of a local solution, i.e. a solution defined on some random subinterval. Such local solutions may be than extended either to the interval \([a,b]\), or they explode within a finite random interval. The proof is based on the iterated application of the fixed-point principle and, therefore, on an infinitely repeated Young expansions of the original stochastic basis, as it is constructed in Example D.20. Below we illustrate this procedure by using more general stochastic equations and the space \( L^1[a,b] \) instead of \( C[a,b] \), which allows to relax assumptions on the right-hand sides (because the tightness conditions are weaker in \( L^1[a,b] \), see Example D.15).

**Example D.25.** Consider the initial value problem

\[
dx(t) = (V^0x)(t)dt + \sum_{j=1}^{m} (V^j)(x(t))dW_j(t) \quad (t \in [a,b]) \quad \text{and} \quad x(a) = x_0,
\]

where \( W_j \) are the same as in the previous example, \( V^j \) are the superposition operators generated by random, continuous Volterra operators \( V^j : P[a,b] \to P[a,b] \) \( (j = 0, \ldots, m) \), which satisfy the measurability conditions with respect to the filtration \( \mathcal{F}_t \) from Example D.15 and \( X = C[a,b] \), \( Y = L^1[a,b] \) and \( Y^j = L^2[a,b] \) \( (j = 1, \ldots, m) \).

Then we have the following statements:

1. the initial value problem \((D.8)\) has at least one weak local, path-continuous solution for any \( \mathcal{F}_n \)-measurable random point \( x_0 \) in \( R^n \), i.e. a solution defined on some Young expansion of the stochastic basis \( \mathcal{B} \) and some random subinterval;
2. if the absolute values of all weak local solutions of \((D.8)\) are known to be bounded in probability, then these solutions are defined for all \( a \leq t \leq b \), i.e. \( \tau = b \) a.s.;
3. if for any Young expansion \( \mathcal{B}^* \) of the stochastic basis \( \mathcal{B} \) the initial value problem \((D.8)\) has at most one weak solution on \([a,b]\), then any such a is strong, i.e. defined on the original stochastic basis \( \mathcal{B} \) for all \( t \in [a,b] \).

The proof of statements (1)-(3) is based again on Corollary 5.4 (or Theorem 2.1, a particular case of this corollary). Let us start with the first statement. To define a tight-range LC operator in the space \( P[a,b] \) we first define the random, \( \mathcal{F}_n \)-measurable in \( \omega \) and continuous projections \( \kappa_n \) of the space \( R^n \) onto the ball \( B_1 \) of radius 1 centered at \( x_0(\omega) \) and define \( V_n^j \omega = V_n^j(\omega) \circ \kappa_n(\omega) \). By construction, the operator \( V_n^j \omega \) is random continuous Volterra operator acting from \( P[a,b] \) to \( P[a,b] \) \( (j = 0) \) and \( P(a,b) \) \( (j = 1, \ldots, m) \), respectively, and satisfying the same measurability conditions with respect to the filtration \( \mathcal{F}_t \) as the operators \( V^j \). Defining \( h^1 \) by

\[
(h^1x)(t) = x_0 + \int_a^t (V^0x)(s)ds + \sum_{j=1}^{m} \int_a^t (V^jx)(s)dW_j(s)
\]

and using the tightness property of the Itô integral from Example D.15, the compactness of the Lebesgue integral as an operator in \( L^1[a,b] \), together with Example D.14 we see that \( h^1 \) is a tight LC operator in the space \( P[a,b] \). Moreover, it is tight-range, as it maps the space \( P[a,b] \) onto the set \( h^1(A) \), where \( A = \{ x \in P[a,b] : \inf x(\omega) - x_0(\omega) \leq 1 \ a.s. \} \), which is bounded in the space \( P[a,b] \). By Theorem 2.1, there exists a Young expansion \( \mathcal{B}^1 = (\Omega^1, \mathcal{F}^1, \mathcal{F}^1_t, P^1) \) of the stochastic basis \( \mathcal{B} \), where \( \Omega^1 = \Omega \times Z \) and \( Z = L^1[a,b] \) and a weak fixed point \( x_1 \in P[a,b] \) of the operator \( h^1 \). Notice that \( x_1 - x_0 \leq 1 \) \( \mathcal{F}^1 \)-a.s. by construction and that this solution, in fact, has continuous trajectories. Hence the stopping time \( \tau_1(t) = \inf \{ t : |x_1(t) - x_0| > 1 \} \) is well-defined and \( \tau_1 > 0 \) a.s., so that the restriction of \( x_1 \) to the random interval \([a,\tau_1]\) solves the initial value problem \((D.5)\) on this interval. For the sake of simplicity, we may still denote this solution by \( x_1 \). This proves the first part of the theorem.

To prove the second statement, we iterate the above procedure by induction. If \( \nu \geq 2 \) and \( x_{\nu-1} \) is an already constructed weak solution defined on a Young expansion

\[
\mathcal{B}^{\nu-1} = (\Omega^{\nu-1}, \mathcal{F}^{\nu-1}, \mathcal{F}^{\nu-1}_t, P^{\nu-1})
\]
for all $t \in [a, \tau_{\nu-1}]$ and satisfying $|x_{\nu-1} - x_0| \leq \nu - 1$ $P^{\nu-1}$-a.s. Here $\tau_{\nu-1}$ is some stopping time on $B^{\nu-1}$ and $\Omega^{\nu-1}$ is the direct product of $\Omega$ and $\nu - 1$ copies of the space $Z = L^1[a, b]$. Put

$$\tilde{x}(t) = \begin{cases} x(t) & (t \geq \tau_{\nu-1}) \\ x_{\nu-1}(t) & (t < \tau_{\nu-1}) \end{cases}$$

and define the LC operator

$$(h^\nu x)(t) = x_0 + \int_a^t (V_a^0 x)(s) ds + \sum_{j=1}^m \int_a^t (V_j^a x)(s) dW^{\nu-1}_j(s),$$

where $W^{\nu-1}_j$ are the standard Wiener processes on $B^{\nu-1}$ and $V_j^a$ are random continuous Volterra operators given by

$$(V_j^a x)(t) = \begin{cases} (V_j^a(\tilde{x}_0 \circ \kappa_\nu^a))(t) & (t \geq \tau_{\nu-1}) \\ (V_j^a(x_{\nu-1})(\omega))(t) & (t < \tau_{\nu-1}) \end{cases}$$

the random continuous projections $\kappa_\nu^a$ of the space $R^n$ onto the ball $B_\nu$ of radius $\nu$ centered at $x_0(\omega)$. By construction, the operators $V_j^a$ satisfy the same measurability conditions with respect to the filtration $(F_a)$ as the operators $V_j$ do for the filtration $(F_t)$. Therefore, $V_0^\nu: \mathcal{P}(L^1[a, b], B^{\nu-1}) \rightarrow \mathcal{P}(L^1[a, b], B^{\nu-1})$ and $V_j^a: \mathcal{P}(L^1[a, b], B^{\nu-1}) \rightarrow \mathcal{P}(L^2[a, b], B^{\nu-1})$ ($j = 1, \ldots, m$).

The LC operator $h^\nu$ is tight-range exactly by the same reasons as the operator $h^1$, so that it has a weak fixed point $x_\nu$ defined on a Young expansion $B^\nu$ of the stochastic basis $B^{\nu-1}$. As before, $x_\nu$ has continuous paths, so that $\tau_{\nu} = \inf \{|x_\nu - x_0| > \nu\}$ is well-defined and satisfies $\tau_{\nu} > \tau_{\nu-1}$ $P^{\nu}$-a.s. Therefore, it gives rise to a local solution defined on $B^\nu$ for all $t \in [t_0, \tau_\nu]$. We denote this solution by $x_\nu$ as well. By construction it $P^{\nu}$-a.s. coincides with $x_{\nu-1}$ on the random interval $[a, \tau_{\nu-1}]$ and satisfies $|x_\nu - x_0| \leq \nu$ $P^{\nu}$-a.s. The induction argument is completed.

By letting $\nu \rightarrow \infty$ we obtain the stochastic basis

$$B^\ast = (\Omega^\ast, F^\ast, F^\ast_t, P^\ast)$$

to be the limit of the sequence of the stochastic bases $B^\nu$ in the sense of Example D.20. Clearly, all $\tau_{\nu}$ remain stopping times on $B^\ast$. Let us, therefore, put $x^\ast(t) = x_{\nu}(t)$ if $t \in (\tau_{\nu-1}, \tau_\nu)$, $\nu \in N$. Evidently, this stochastic process is sample continuous on the random interval $[t_0, \tau]$, where $\tau = \sup_{\nu \in N} \tau_{\nu}$ is a stopping time on $B^\ast$ and satisfies the initial value problem (D.8), where $W_j$ are replaced by the standard Wiener processes $W^\ast_j$ on $B^\ast$. Moreover, by construction $\{\tau = b\}$ if and only if $|x^\ast| < \infty P^\ast$-a.s. This means that if the sup-norm of all local weak solutions of the problem (D.8) on the interval $[a, b]$ is a priori known to be bounded in probability, then $x^\ast(t)$ is a.s. defined for all $a \leq t \leq b$.

Finally, we prove the third statement. To this end, let us assume that the problem (D.8) admits at most one weak solution on the interval $[a, b]$ for any Young expansion $B^\ast$ of the stochastic basis $B$. This means that the integral operator

$$(hx)(t) = x_0 + \int_a^t (V^0 x)(s) ds + \sum_{j=1}^m \int_a^t (V_j x)(s) dW_j(s)$$

which is local and uniformly continuous on tight subsets of the space $\mathcal{P}(L^1[a, b])$, has at most one weak fixed point in this space for any acceptable expansion $B^\ast$ of the stochastic basis $B$ as well, then by Theorem 2.3 any weak fixed point of this operator must be strong. This fixed point will be a strong solution of the initial value problem (D.8) defined for all $a \leq t \leq b$.

D.8. Counterexamples.

Example D.26. There exists a complete probability space $S$, a closed, convex, bounded and nonempty subset $\Xi$ of the space $\mathcal{P}(R^2, S)$ and an LC operator $h: \Xi \rightarrow \Xi$ such that the equation $hx = x$ has no solutions in $\Xi$. This explains why we need additional assumptions on the invariant subset $\Xi$ in the finite dimensional fixed-point theorem 5.7. For the proof of this result see 12.

Example D.27. There exists a tight-range LC operator $h: \mathcal{P}(C[a, b]) \rightarrow \mathcal{P}(C[a, b])$ with no strong fixed points in the space $\mathcal{P}(C[a, b])$. This justifies the notion of a weak solution, which always exists in this case (see Corollary 5.7). The existence of such $h$ follows from the results of the paper 2, where a stochastic ordinary differential equation with non-Lipschitz yet continuous coefficients and no strong solutions is constructed.