On finite groups in which commutators are covered by Engel subgroups

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Abstract. Let $m, n$ be positive integers and $w$ a multilinear commutator word. Assume that $G$ is a finite group having subgroups $G_1, \ldots, G_m$ whose union contains all $w$-values in $G$. Assume further that all elements of the subgroups $G_1, \ldots, G_m$ are $n$-Engel in $G$. It is shown that the verbal subgroup $w(G)$ is $s$-Engel for some $\{m, n, w\}$-bounded number $s$.

1 Introduction

Given a group-word $w = w(x_1, \ldots, x_k)$, we think of it primarily as a function of $k$ variables defined on any group $G$. We denote by $w(G)$ the verbal subgroup of $G$ corresponding to the word $w$, that is, the subgroup generated by the $w$-values in $G$. When the set of all $w$-values in $G$ is contained in a union of subgroups, we wish to know whether the properties of the covering subgroups have impact on the structure of the verbal subgroup $w(G)$. The reader can consult the articles [1, 2, 4–6, 15] for results on countable coverings of $w$-values in profinite groups.

The purpose of this paper is to prove the following result.

Theorem 1.1. Let $m, n$ be positive integers and $w$ a multilinear commutator word. Assume that $G$ is a finite group having subgroups $G_1, \ldots, G_m$ whose union contains all $w$-values in $G$. Furthermore, assume that all elements of the subgroups $G_1, \ldots, G_m$ are $n$-Engel in $G$. Then $w(G)$ is $s$-Engel for some $\{m, n, w\}$-bounded number $s$.

Here and throughout the article, we use the expression “$\{a, b, \ldots\}$-bounded” to abbreviate “bounded from above in terms of $a, b, \ldots$ only”.

Recall that multilinear commutators are words which are obtained by nesting commutators, but using always different variables. More formally, the word $w(x) = x$ in one variable is a multilinear commutator; if $u$ and $v$ are multilinear

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commutators involving different variables, then the word \( w = [u, v] \) is a multilinear commutator, and all multilinear commutators are obtained in this way. The number of variables involved in a multilinear commutator \( w \) is called the weight of \( w \).

Also, recall that a group \( G \) is called an Engel group if, for every \( x, y \in G \), the equation \( [y, x, x, \ldots, x] = 1 \) holds, where \( x \) is repeated in the commutator sufficiently many times depending on \( x \) and \( y \). The long commutators \( [y, x, \ldots, x] \), where \( x \) occurs \( i \) times, are denoted by \( [y, i \, x] \). An element \( x \in G \) is (left) \( n \)-Engel if \( [y, n \, x] = 1 \) for all \( y \in G \). A group \( G \) is \( n \)-Engel if \( [y, n \, x] = 1 \) for all \( x, y \in G \). Currently, finite \( n \)-Engel groups are understood fairly well. A theorem of Zorn says that finite Engel groups are nilpotent (see [12, Theorem 12.3.4]). More specific properties of finite \( n \)-Engel groups can be found for example in a theorem of Burns and Medvedev quoted as Theorem 3.5 in Section 3 of this paper. The interested reader is referred to the survey [17] and references therein for further results on finite and residually finite Engel groups.

In the next section, we describe the Lie-theoretic machinery that will be used in the proof of Theorem 1.1. The proof of the theorem is given in Section 3.

2 Associating a Lie ring to a group

There are several well-known ways to associate a Lie ring to a group \( G \) (see [8, 9, 13]). For the reader’s convenience, we will briefly describe the construction that we are using in the present paper.

A series of subgroups

\[
G = G_1 \geq G_2 \geq \cdots \tag{*}
\]

is called an \( N \)-series if it satisfies \( [G_i, G_j] \leq G_{i+j} \) for all \( i \) and \( j \). Obviously, any \( N \)-series is central, i.e., \( G_i / G_{i+1} \leq Z(G / G_{i+1}) \) for any \( i \). Given an \( N \)-series \((*)\), let \( L^*(G) \) be the direct sum of the abelian groups \( L_i^* = G_i / G_{i+1} \), written additively. Commutation in \( G \) induces a binary operation \([ \cdot, \cdot ]\) in \( L^*(G) \). For homogeneous elements \( xG_{i+1} \in L_i^*, yG_{j+1} \in L_j^* \), the operation is defined by

\[
[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}^*
\]

and extended to arbitrary elements of \( L^*(G) \) by linearity. It is easy to check that the operation is well-defined and that \( L^*(G) \) with the operations \(+\) and \([ \cdot, \cdot ]\) is a Lie ring.

In this paper, we use the above construction in the cases where \((*)\) is either the lower central series of \( G \) or the \( p \)-dimension central series, also known under the name of Zassenhaus–Jennings–Lazard series (see [8, p. 250] for details). In the former case, we denote the associated Lie ring by \( L(G) \). In the latter case, \( L^*(G) \)
On finite groups in which commutators are Engel

1051

can be viewed as a Lie algebra over the field with \( p \) elements. We write \( L_p(G) \) for the subalgebra generated by the first homogeneous component \( G_1/G_2 \). Usually, nilpotency of \( L^*(G) \) has a strong effect on the structure of \( G \). In particular, \( L(G) \) is nilpotent of class \( c \) if and only if the group \( G \) is nilpotent of class \( c \). Nilpotency of \( L_p(G) \) also leads to strong conclusions about \( G \). The proof of the following theorem can be found in [10].

**Theorem 2.1.** Let \( P \) be a \( d \)-generated finite \( p \)-group, and suppose that \( L_p(G) \) is nilpotent of class \( c \). Then \( P \) has a powerful characteristic subgroup of \( \{ p, c, d \} \)-bounded index.

Recall that powerful \( p \)-groups were introduced by Lubotzky and Mann in [11]. They have many nice properties, some of which are listed in the next section.

Thus criteria of nilpotency of Lie algebras provide effective tools for applications in group theory.

Let \( X \) be a subset of a Lie algebra \( L \). By a commutator in elements of \( X \) we mean any element of \( L \) that can be obtained as a Lie product of elements of \( X \) with some system of brackets. If \( x, y \) are elements of \( L \), we define inductively

\[
[x, \_0 y] = x \quad \text{and} \quad [x, _i y] = [[x, _{i-1} y], y] \quad \text{for all positive integers} \ i.
\]

As usual, we say that an element \( a \in L \) is ad-nilpotent if there exists a positive integer \( n \) such that \( [x, na] = 0 \) for all \( x \in L \). If \( n \) is the least integer with the above property, then we say that \( a \) is ad-nilpotent of index \( n \).

The next theorem is a deep result of Zelmanov with many applications to group theory. It was announced by Zelmanov in [19, 20]. A detailed proof was published in [21].

**Theorem 2.2.** Let \( L \) be a Lie algebra over a field, and suppose that \( L \) satisfies a polynomial identity. If \( L \) can be generated by a finite set \( X \) such that every commutator in elements of \( X \) is ad-nilpotent, then \( L \) is nilpotent.

Theorem 2.2 admits the following quantitative version (see, for instance, [10]).

**Theorem 2.3.** Let \( L \) be a Lie algebra over a field \( K \). Assume that \( L \) is generated by \( m \) elements such that each commutator in the generators is ad-nilpotent of index at most \( n \). Suppose that \( L \) satisfies a polynomial identity \( f \equiv 0 \). Then \( L \) is nilpotent of \( \{ f, K, m, n \} \)-bounded class.

As usual, \( \gamma_i(L) \) denotes the \( i \)th term of the lower central series of \( L \). The following Lie-ring variation on the theme of Theorem 2.2 is a particular case of [16, Proposition 2.6].
Theorem 2.4. Let $L$ be a Lie ring satisfying a polynomial identity $f \equiv 0$. Assume that $L$ is generated by $m$ elements such that every commutator in the generators is ad-nilpotent of index at most $n$. Then there exist positive integers $e$ and $c$ depending only on $f$, $m$ and $n$ such that $e c \gamma_e(L) = 0$.

3 Proof of the main theorem

It will be convenient first to prove Theorem 1.1 in the particular case where $w = \delta_k$ is a derived word. Recall that the derived words $\delta_k$, on $2^k$ variables, are defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^k-1}), \delta_{k-1}(x_{2^k-1+1}, \ldots, x_{2^k})] \quad \text{for } k \geq 1.$$ 

The verbal subgroup corresponding to the word $\delta_k$ in a group $G$ is the familiar $k$th term of the derived series of $G$ denoted by $G^{(k)}$.

Lemma 3.1. Let $m, n, k$ be positive integers, and let $G$ be a finite group with subgroups $G_1, \ldots, G_m$ whose union contains all $\delta_k$-values in $G$. If all elements of the subgroups $G_1, \ldots, G_m$ are $n$-Engel in $G$, then $G^{(k)}$ is $s$-Engel for some $\{k, m, n\}$-bounded number $s$.

A subset $X$ of a group $G$ is called commutator-closed if $[x, y] \in X$ whenever $x, y \in X$. The fact that, in any group, the set of all $\delta_k$-values is commutator-closed will be used without explicit reference.

The proof of Lemma 3.1 will require the following two lemmas which were obtained in [1, Lemma 3.1] and [16, Lemma 4.1], respectively.

Lemma 3.2. Let $G$ be a nilpotent group generated by a commutator-closed subset $X$ which is contained in a union of finitely many subgroups $G_1, G_2, \ldots, G_m$. Then $G = G_1 G_2 \cdots G_m$.

Lemma 3.3. Let $G$ be a group generated by $m$ elements which are $n$-Engel. If $G$ is soluble with derived length $d$, then $G$ is nilpotent of $\{d, m, n\}$-bounded class.

The proof of Lemma 3.1 requires the concept of powerful $p$-groups. A finite $p$-group $P$ is said to be powerful if and only if $[P, P] \leq P^P$ for $p \neq 2$ (or $[P, P] \leq P^4$ for $p = 2$), where $P^i$ denotes the subgroup of $P$ generated by all $i$th powers. If $P$ is a powerful $p$-group, then the subgroups $\gamma_i(P)$, $P^{(i)}$ and $P^i$ are also powerful. Moreover, for given positive integers $n_1, \ldots, n_j$, it follows, by repeated applications of [11, Propositions 1.6 and 4.1.6], that

$$[P^{n_1}, \ldots, P^{n_j}] \leq \gamma_j(P)^{n_1 \cdots n_j}.$$
Furthermore, if a powerful \( p \)-group \( P \) is generated by \( d \) elements, then any subgroup of \( P \) can be generated by at most \( d \) elements and \( P \) is a product of \( d \) cyclic subgroups. For more details, we refer the reader to [9, Chapter 11].

**Proof of Lemma 3.1.** By the hypothesis, each \( \delta_k \)-value is \( n \)-Engel in \( G \). Hence Zorn’s theorem [12, Theorem 12.3.4] implies that \( G^{(k)} \) is nilpotent. Replacing \( G_i \) by \( G_i \cap G^{(k)} \) if necessary, we can assume that all subgroups \( G_i \) are contained in \( G^{(k)} \). Then, by Lemma 3.2, \( G^{(k)} = G_1 G_2 \cdots G_m \).

Choose arbitrary elements \( a, b \in G^{(k)} \). It is sufficient to show that the subgroup \( \langle a, b \rangle \) is nilpotent of \( \{m, n, k\}\)-bounded class. Write \( a = a_1 \cdots a_m, b = b_1 \cdots b_m \), where \( a_i \) and \( b_i \) belong to \( G_i \). Let \( H \) be the subgroup generated by the elements \( a_i, b_i \) for \( i = 1, \ldots, m \). Since the subgroup \( \langle a, b \rangle \) is contained in \( H \), it is enough to show that \( H \) is nilpotent of \( \{m, n, k\}\)-bounded class. Observe that the generators \( a_i, b_i \) of \( H \) are \( n \)-Engel elements. Thus, in view of Lemma 3.3, it is sufficient to prove that \( H \) has \( \{m, n, k\}\)-bounded derived length. Since \( H \) is nilpotent, we need to show that each Sylow \( p \)-subgroup \( P \) of \( H \) has \( \{m, n, k\}\)-bounded derived length.

Obviously, \( P \) can be generated by \( 2m \) elements, each of which is \( n \)-Engel. Set \( R = P^{(k)} \). We will now prove that \( R \) can be generated by \( \{m, n, k\}\)-boundedly many, say \( r \), elements. Note that, by the Burnside Basis Theorem [12, Theorem 5.3.2], it is sufficient to show that the Frattini quotient \( R/\Phi(R) \) has \( \{m, n, k\}\)-boundedly many generators. The quotient \( P/\Phi(R) \) has derived length at most \( k + 1 \). Thus Lemma 3.3 implies that \( P/\Phi(R) \) has \( \{m, n, k\}\)-bounded nilpotency class. It follows that \( R/\Phi(R) \) can be generated with \( \{m, n, k\}\)-boundedly many elements. This is also true for \( R \).

Next, we will show that \( R \) has \( \{m, n, k\}\)-bounded derived length. By the Burnside Basis Theorem, \( R \) is generated by \( r \) \( \delta_k \)-values which are \( n \)-Engel elements. Let \( L_1 = L(R) \) be the Lie ring associated to \( R \) using the lower central series. The proof of [18, Theorem 1] shows that, whenever a group \( K \) satisfies a commutator identity \( f \equiv 1 \), the Lie ring \( L(K) \) satisfies the linearized version of “the same” identity \( [y, n\delta_k(x_1, \ldots, x_{2^k})] \equiv 0 \). Since the identity \( [y, n\delta_k(x_1, \ldots, x_{2^k})] \equiv 1 \) is satisfied in \( R \), it follows that the Lie ring \( L_1 \) satisfies the linearized version of the identity \( [y, n\delta_k(x_1, \ldots, x_{2^k})] \equiv 0 \). Further, each commutator in the generators of \( L_1 \) corresponding to \( \delta_k \)-values in \( R \) is ad-nilpotent of index at most \( n \). By Theorem 2.4, there exist positive integers \( e \) and \( c \), depending only on \( k, m \) and \( n \), such that \( e\gamma_c(L_1) = 0 \). If \( p \) is not a divisor of \( e \), we have \( \gamma_c(L_1) = 0 \), and so the group \( R \) is nilpotent of class at most \( c - 1 \). In what follows, we assume that \( p \) is a divisor of \( e \). Note that, in this case, \( p \) is bounded in terms of \( k, m \) and \( n \).

Let \( L_2 = L_p(R) \) be the Lie algebra associated to \( R \) using the \( p \)-dimensional series. Applying Theorem 2.3, we deduce that \( L_2 \) is nilpotent with \( \{m, n, k\}\)-
bounded nilpotency class. Hence, by Theorem 2.1, \( R \) has a powerful subgroup \( N \) of \( \{m, n, k\} \)-bounded index. It is now sufficient to show that \( N \) has \( \{m, n, k\} \)-bounded derived length.

Since the index of \( N \) in \( R \) is \( \{m, n, k\} \)-bounded, it follows that \( N \) can be generated with \( \{m, n, k\} \)-boundedly many elements, say \( t \). Taking into account that \( N \) is powerful, we deduce that all subgroups of \( N \) can be generated by at most \( t \) elements, and the \( k \)th derived subgroup \( N^{(k)} \) is also powerful. We now look at the Lie ring \( L(N^{(k)}) \) associated to \( N^{(k)} \).

By Theorem 2.4, there exist positive integers \( e_1, c_1 \), depending only on \( k, m \) and \( n \), such that \( e_1 \gamma_{c_1}(L(N^{(k)})) = 0 \). Since \( P \) is a \( p \)-group, we can assume that \( e_1 \) is a \( p \)-power. Set
\[
R_1 = (N^{(k)})^{e_1^{2k}} = (N^{e_1})^{(k)}. \tag{**}
\]

Note that if \( p \neq 2 \), then
\[
[R_1, R_1] \leq [N^{(k)}, N^{(k)}]^{e_1^{2k}} e_1^{2k} \leq (N^{(k)})^{pe_1^{e_1^{2k}}} = R_1^{pe_1^{2k}}.
\]

If \( p = 2 \), then we have
\[
[R_1, R_1] \leq R_1^{4e_1^{2k}}.
\]

Since \( e_1 \gamma_{c_1}(L(R_1)) = 0 \), we deduce that \( \gamma_{c_1}(R_1)^{e_1} \leq \gamma_{c_1+1}(R_1) \). Taking into account that \( R_1 \) is powerful, if \( p \neq 2 \), we obtain that
\[
\gamma_{c_1}(R_1)^{e_1} \leq \gamma_{c_1+1}(R_1) = [R_1', c_1-1 R_1] \leq [R_1^{pe_1^{2k}} c_1-1 R_1] \leq \gamma_{c_1}(R_1)^{pe_1^{2k}}.
\]

If \( p = 2 \), we obtain that
\[
\gamma_{c_1}(R_1)^{e_1} \leq \gamma_{c_1}(R_1)^{4e_1^{2k}}.
\]

Hence \( \gamma_{c_1}(R_1)^{e_1} = 1 \). Since \( \gamma_{c_1}(R_1) \) is powerful and generated by at most \( t \) elements, we conclude that \( \gamma_{c_1}(R_1) \) is a product of at most \( t \) cyclic subgroups. Hence the order of \( \gamma_{c_1}(R_1) \) is at most \( e_1^{t} \). It follows that the derived length of \( R_1 \) is \( \{k, m, n\} \)-bounded. Recall that \( N^{(k)} \) is a powerful \( p \)-group and (**) it follows that the derived length of \( N^{(k)} \) is \( \{k, m, n\} \)-bounded. Hence the derived length of \( P \) is \( \{k, m, n\} \)-bounded, as required. The proof is now complete. \( \Box \)

The next lemma is well known (see, for example, [14, Lemma 4.1] for a proof).

**Lemma 3.4.** Let \( G \) be a group and \( w \) a multilinear commutator word of weight \( k \). Then every \( \delta_k \)-value in \( G \) is a \( w \)-value.
On finite groups in which commutators are Engel

The proof of Theorem 1.1 will require the following result, due to Burns and Medvedev [3].

**Theorem 3.5.** Let $n$ be a positive integer. There exist constants $c$ and $e$ depending only on $n$ such that if $G$ is a finite $n$-Engel group, then the exponent of $\gamma_c(G)$ divides $e$.

Another useful result which we will need is the next theorem [7, Theorem B].

**Theorem 3.6.** Let $w$ be a multilinear commutator word, and let $G$ be a soluble group. Then there exists a series of subgroups from 1 to $w(G)$ such that

- all subgroups of the series are normal in $G$,
- every section of the series is abelian and can be generated by $w$-values all of whose powers are also $w$-values.

Furthermore, the length of this series only depends on the word $w$ and on the derived length of $G$.

**Corollary 3.7.** Assume the hypotheses of Theorem 1.1, and suppose additionally that $G$ is soluble with derived length $k$. Then each element of $w(G)$ can be written as a product of $\{k, m\}$-boundedly many elements from the subgroups $G_1, \ldots, G_m$.

**Proof.** Let $1 = A_0 \leq A_1 \leq \cdots \leq A_u = w(G)$ be a series as in Theorem 3.6. Arguing by induction on $u$, it is sufficient to show that each element of $A_1$ can be written as a product of $\{k, m\}$-boundedly many elements from the subgroups $G_1, \ldots, G_m$. Since $A_1$ is abelian and generated by $w$-values, each of which lies in some $G_i$, we deduce that $A_1$ is the product of subgroups of the form $A_1 \cap G_i$. The result follows.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Recall that $w$ is a multilinear commutator word. Since each $w$-value in $G$ is $n$-Engel, Zorn’s theorem implies that the verbal subgroup $w(G)$ is nilpotent. Let $d$ be the weight of the word $w$. Combining Lemmas 3.4 and 3.1, we deduce that $G^{(d)}$ is $u$-Engel for some $\{m, n, d\}$-bounded number $u$. Theorem 3.5 shows that there exists an $\{m, n, d\}$-bounded number $c$ such that $\gamma_c(G^{(d)})$ has $\{m, n, d\}$-bounded exponent. It follows that there is an $\{m, n, d\}$-bounded number $k$ such that $M = G^{(k)}$ has $\{m, n, d\}$-bounded exponent.

Choose arbitrary elements $a, b \in w(G)$. We will show that the subgroup $\langle a, b \rangle$ is nilpotent of $\{m, n, w\}$-bounded class. Corollary 3.7 shows that any element in $w(G)/M$ can be written as a product of $\{m, n, w\}$-boundedly many, say $r$, elements.
$w$-values. Thus we can write $a = a_1 \cdots a_r m_1$ and $b = b_1 \cdots b_r m_2$, where $a_i, b_i$ are $w$-values for $i = 1, \ldots, r$ and $m_1, m_2$ belong to $M$. Let $H$ be the subgroup generated by all these elements, that is,

$$H = \langle a_1, \ldots, a_r, b_1, \ldots, b_r, m_1, m_2 \rangle.$$ 

Note that the subgroup $\langle a, b \rangle$ is contained in $H$, and therefore it is sufficient to show that $H$ is nilpotent of $\{m, n, w\}$-bounded class.

Set $N = M \cap H$, and let $\tilde{H}$ be the quotient group $H/\Phi(N)$. Note that the image of $N$ in $\tilde{H}$ is an abelian group, and so the images of $m_1, m_2$ in $\tilde{H}$ are 2-Engel. Note also that the derived length of $\tilde{H}$ is at most $k + 1$. Lemma 3.3 yields that the nilpotency class of $\tilde{H}$ is $\{m, n, w\}$-bounded. Thus we get that the image of $N$ in $\tilde{H}$ has $\{m, n, w\}$-boundedly many generators. Of course, this is true also for $N$. Recall that the exponent of $N$ is $\{m, n, w\}$-bounded, and so we obtain from the positive solution of the restricted Burnside problem [19, 20] that the order of $N$ is $\{m, n, w\}$-bounded. Since $H$ is nilpotent, there is an $\{m, n, w\}$-bounded number $t$ such that $N$ is contained in the $t$th term $Z_t(H)$ of the upper central series of $H$. Consequently, $H$ is nilpotent of $\{m, n, w\}$-bounded class, as required. The proof is now complete.

\[ \square \]

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On finite groups in which commutators are Engel

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