RINGEL’S GENERALIZED EARTH-MOON PROBLEM

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1. Introduction

A graph $G$ has surface thickness $n$ ($S$-thickness $n$) with respect to the surface $S$, if $G$ can be decomposed into $n$ and no fewer subgraphs by making $n$ copies of the vertex set of $G$ and then assigning each edge of $G$ to one of the $n$ copies so that $n$ graphs result; each resulting subgraph of $G$ must be $S$ embeddable. The chromatic number of a graph $G$, denoted $\chi(G)$, is simply the fewest number of colors needed to color the vertex set of $G$ such that no two adjacent vertices receive the same color. Define $\chi_n(S)$ to be the chromatic number of the surface $S$, that is, the minimum number of colors needed to properly color all $S$-thickness $n$ graphs. Let $E(S)$ denote the Euler characteristic of the surface $S$. Let $S_k$ denote the orientable surface of genus $k$ and $N_k$ the nonorientable surface obtained by adding $k$ crosscaps to the sphere; $S_0$ denotes the two-dimensional sphere. It is known that $E(S_k) = 2 - 2k$ and $E(N_k) = 2 - k$.

In 1959 [1, p.233], Ringel asked: What is the chromatic number of the sphere for graphs of thickness two? The bounds are known to be: $9 \leq \chi_2(S_0) \leq 12$. See [2] for a delightful exposition of the spherical case. In [1], Ringel and Jackson ask: What is $\chi_n(S)$ for any surface $S$? There [p.241] the following upper bound is given for any surface $S$, except the sphere:

$$\chi_n(S) \leq \left\lfloor \frac{(1 + 6n + \sqrt{(1 + 6n)^2 - 24nE(S)})}{2} \right\rfloor (1)$$

The main results in this note are three elementary arguments that establish $\chi_n(S)$ for two new surfaces: $N_3$, also known as Dyck’s surface, and the Moebius band which will be denoted by $M$. The arguments are based on previous results appearing in [1]. The results are not especially surprising, but they do provide good lower bounds for $\chi_n(N_{2k+1})$ once the value of $\chi_n(S_k)$ is known for $k > 0$; in the case $k = 1$, the exact value of $\chi_n(N_3)$ is obtained.

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2. **Main Results**

**Lemma 1.** \( \left\lfloor \frac{1+6n + \sqrt{(1+6n)^2 - 24nE(N_3)}}{2} \right\rfloor \leq 6n+1 \) for all \( n \geq 2 \).

**Proof.** Suppose for contradiction that the claim is false. Then since \( E(N_3) = -1 \), there exists an integer \( n \geq 2 \) such that:

\[
\left\lfloor \frac{1+6n + \sqrt{(1+6n)^2 + 24n}}{2} \right\rfloor > 6n + 1 \rightarrow
\]

\[
1 + 6n + \sqrt{(1+6n)^2 + 24n} \geq 12n + 4 \rightarrow
\]

\[
\sqrt{(1+6n)^2 + 24n} \geq 6n + 3 \rightarrow
\]

\[
(1+6n)^2 + 24n \geq (6n+3)^2 \rightarrow
\]

\[1 \geq 9. \] This is an obvious contradiction, hence, the original assumption was false and the lemma is true. \( \Box \)

**Theorem 1.** \( \chi_n(N_3) = 6n + 1 \) for all \( n \geq 2 \).

**Proof.** In [1, p.238], it is shown that \( \chi_n(S_1) = 6n + 1 \) for all \( n \geq 2 \). Let \( G \) be any graph embedded on \( S_1 \). Choose any face of \( S_1 \) created by the embedding of \( G \) and add a crosscap, what can be obtained is an embedding of \( G \) on the surface \( N_3 \) since \( N_3 \) is homeomorphic to the surface \( S_1 + \text{crosscap} \) and homeomorphism preserves graph isomorphism as well as graph embedding. Combining this with the previous lemma gives the desired result. \( \Box \)

For a complete classification of surfaces see Conway’s ZIP proof in [3].

**Theorem 2.** \( \chi_n(M) = 6n \) for all \( n \geq 2 \).

**Proof.** \( M \) is homeomorphic to \( N_1 \) minus an open disk, to see this look at the figure below; a square minus the region \( X \) (an open disk) with opposite ends identified. Notice that coupled ends have opposite orientation and so a half-twist is needed to join each pair correctly. When opposite ends are joined a Moebius band results. Now, fill in the two missing regions denoted by \( X \). If sides of the resulting square receive the same identifications induced by the labeling of the figure below, \( N_1 \), the projective plane, is obtained. Now, given any graph \( G \) embedded on \( N_1 \), select any face induced by the embedding and delete it, what results is an embedding of \( G \) on a Moebius band. Inductively it follows that \( \chi_n(M) \geq 6n \) for all \( n \geq 2 \) since in [1, p.240] it is proven that \( \chi_n(N_1) = 6n \) for all \( n \geq 2 \). Now, inequality (1) gives an upper bound of \( \chi_n(M) \leq 6n + 1 \) for all \( n \geq 2 \) since \( E(M) = 0 \). To show that this upper bound cannot be obtained, the case \( \chi_2(M) \) is shown, from which the general result easily follows. Suppose for contradiction that \( \chi_2(M) = 13 \). Then there exists a graph \( G \) of \( M \)-thickness two such that \( \chi(G) = 13 \). Since the square minus an open disk appearing in the picture below is homeomorphic to \( M \), \( G \) biembeds on this version of a Moebius band. Now fill in the deleted disk labeled \( X \) in the figure below to obtain a biembedding of the graph \( G \) on \( N_1 \). This implies that \( \chi_2(N_1) = 13 \) which is a contradiction since \( \chi_2(N_1) = 12 \). \( \Box \)
Hence, Ringel’s generalized earth-moon problem has been solved for two new surfaces $M$ and $N_3$, also known as the Moebius band and Dyck’s surface, respectively. The following list summarizes known results for Ringel’s generalized earth-moon problem:

For all $n \geq 2$,
- $\chi_n(S_2) = 6n + 2$ (see [5])
- $\chi_n(S_1) = 6n + 1$ (see [1], [5])
- $\chi_n(N_1) = 6n$ (see [1], [5])
- $\chi_n(M) = 6n$
- $\chi_n(N_3) = 6n + 1$.

Until very recently the $N_2$-thickness of $K_{13}$ was not known, however, it appears that Thom Sulanke has determined it is three using an exhaustive computer search [4]. The $N_2$-thickness of $K_{13}$ has been a long standing obstruction to extending Beineke’s results appearing in [5] for the Klein Bottle thickness of $K_n$. A similar obstruction for determining the $S_3$-thickness of $K_n$ is the $S_3$-thickness of $K_{16}$ [5, p.996]; Sulanke has determined that $\chi_2(S_3) = 16$, although this result has not been published. Through personal communication Thom has also told me that he has been able to determine $\chi_n(N_4)$ and $\chi_n(N_5)$. He also pointed out to me that Lemma 1, appearing here, generalizes for all integers $n > 0$ and $E(S) < 0$. Using the exact same arguments given in the proof of Lemma 1, one can show that the left side of inequality (1) is $\leq 6n - E(S)$ for $n > 0$ and $E(S) < 0$. Naturally, the next two surfaces to consider are $S_3$ and $N_2$. This motivates the following problem:

**Problem 1.** Determine $\chi_n(S_3)$ and $\chi_n(N_2)$.

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3. References

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