UNIVERSAL CONSISTENCY OF THE $k$-NN RULE IN METRIC SPACES AND NAGATA DIMENSION*, **

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Abstract. The $k$ nearest neighbour learning rule (under the uniform distance tie breaking) is universally consistent in every metric space $X$ that is sigma-finite dimensional in the sense of Nagata. This was pointed out by Cérou and Guyader (2006) as a consequence of the main result by those authors, combined with a theorem in real analysis sketched by D. Preiss (1971) (and elaborated in detail by Assouad and Quentin de Gromard (2006)). We show that it is possible to give a direct proof along the same lines as the original theorem of Charles J. Stone (1977) about the universal consistency of the $k$-NN classifier in the finite dimensional Euclidean space. The generalization is non-trivial because of the distance ties being more prevalent in the non-euclidean setting, and on the way we investigate the relevant geometric properties of the metrics and the limitations of the Stone argument, by constructing various examples.

Résumé. La règle d’apprentissage des $k$ plus proches voisins (sous le bris uniforme d’égalité des distances) est universellment consistente dans chaque espace métrique séparable de dimension sigma-finie au sens de Nagata. Comme indiqué par Cérou et Guyader (2006), le résultat fait suite à une combinaison du théorème principal de ces auteurs avec un théorème d’analyse réelle esquisse par D. Preiss (1971) (et élaboré en détail par Assouad et Quentin de Gromard (2006)). Nous montrons qu’il est possible de donner une preuve directe dans le même esprit que le théorème original de Charles J. Stone (1977) sur la consistence universelle du classificateur $k$-NN dans l’espace euclidien de dimension finie. La généralisation est non-triviale, car l’égalité des distances est plus commune dans le cas non-euclidien, et pendant l’élaboration de notre preuve, nous étudions des propriétés géométriques pertinentes des métriques et tests des limites de l’argument de Stone, en construisant quelques exemples.

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Introduction

The \( k \)-nearest neighbour classifier, in spite of being arguably the oldest supervised learning algorithm in existence, still retains his importance, both practical and theoretical. In particular, it was the first classification learning rule whose (weak) universal consistency (in the finite-dimensional Euclidean space) was established, by Charles J. Stone in [16].

Stone’s result is easily extended to all finite-dimensional normed spaces, see, e.g., [6]. However, the \( k \)-NN classifier is no longer universally consistent already in the infinite-dimensional Hilbert space \( \ell^2 \). A series of examples of this kind, obtained in the setting of real analysis, belongs to Preiss, and the first of them [13] is so simple that it can be described in a few lines. We will reproduce it in the article, since the example remains virtually unknown in the statistical machine learning community.

There is sufficient empirical evidence to support the view that the performance of the \( k \)-NN classifier greatly depends on the chosen metric on the domain (see e.g. [8]). There is an algorithm, Large Margin Nearest Neighbour Classifier (LMNN), based on the idea of optimizing the \( k \)-NN performance over all Euclidean metrics on a finite dimensional vector space [17]. At the same time, it appears that a theoretical foundation for such an optimization over a set of distances is still lacking. The first question to address in this connection, is of course to characterize those metrics (generating the original Borel structure of the domain) for which the \( k \)-NN classifier is (weakly) universally consistent.

While the problem in this generality remains still open, a great advance in this direction was made by Cérou and Guyader in [2]. They have shown that the \( k \)-NN classifier is consistent under the assumption that the regression function \( \eta(x) \) satisfies the weak Lebesgue–Besicovitch differentiation property:

\[
\frac{1}{\mu(B_r(x))} \int_{B_r(x)} \eta(x) \, d\mu(x) \rightarrow \eta(x),
\]

where the convergence is in measure, that is, for each \( \epsilon > 0 \),

\[
\mu \left\{ x \in \Omega : \left| \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \eta(x) \, d\mu(x) - \eta(x) \right| > \epsilon \right\} \rightarrow 0 \text{ when } r \downarrow 0.
\]

The proof extended the ideas of [4], where it was previously observed that Stone’s universal consistency can be deduced from the classical Lebesgue–Besicovitch differentiation theorem: every \( L^1(\mu) \)-function \( f \) on \( \mathbb{R}^d \) satisfies Eq. (1), even in the strong sense (convergence almost everywhere). See also [7].

Those separable metric spaces in which the weak Lebesgue–Besicovitch differentiation property holds for every Borel probability measure (equivalently, for every sigma-finite locally finite Borel measure) have not yet been characterized. But the separable metric spaces in which the strong Lebesgue–Besicovitch differentiation property holds for every such measure as above have been described by Preiss [14]: they are exactly those spaces that are \textit{sigma-finite dimensional} in the sense of Nagata [10][12]. (For finite dimensional spaces in the sense of Nagata, the sketch of a proof by Preiss, in the sufficiency direction, was elaborated by Assouad and Quentin de Gromard in [1].) In particular, it follows that every sigma-finite dimensional separable metric space satisfies the weak Lebesgue–Besicovitch differentiation property for every probability measure.

Combining the result of Preiss with that of Cérou–Guyader, one concludes that the \( k \)-NN classifier is universally consistent in every sigma-finite dimensional separable metric space, as was noted in [2].

The authors of [2] mention in their paper that “[Stone’s theorem] is based on a geometrical result, known as Stone’s Lemma. This powerful and elegant argument can unfortunately not be generalized to infinite dimension.” The aim of this article is to show that at least Stone’s original proof, including Stone’s geometric lemma as its main tool, can be extended from the Euclidean case to the sigma-finite dimensional metric spaces. In fact, as we will show, the geometry behind the Stone lemma, even if it appears to be essentially based on the Euclidean structure of the space, is captured by the notion of Nagata dimension, which is a purely metric concept. In this
way, the geometric Stone lemma and indeed the original Stone’s proof of the universal consistency of the $k$-NN classifier, become applicable to a wide range of metric spaces.

In the absence of distance ties (that is, in case where every sphere is a $\mu$-negligeable set with regard to the underlying measure $\mu$), the extension is quite straightforward, indeed almost literal. However, this is not so in the presence of distance ties: an example shows that the conclusion of Stone’s geometric lemma may not hold. Another example shows that even in the compact metric spaces of Nagata dimension zero, the distance ties may be everpresent. We also show that an attempt to reduce the case to the situation without distance ties by learning in the product of $\Omega$ with the unit interval (an additional random variable used for tie-breaking) cannot work, because the product of finite dimensional spaces in the sense of Nagata can, paradoxically, have an infinite dimension. The geometric Stone’s lemma has to be modified, to parallel the Hardy–Littlewood inequality in the geometric measure theory.

We do not touch upon the subject of strong universal consistency in general metric spaces. The main open question left is whether every metric space in which the $k$-NN classifier is universally consistent is necessarily sigma-finite dimensional. A positive answer, modulo the work of [2] and [14], would also answer in the affirmative an open question in real analysis going back to Preiss: suppose a metric space $X$ satisfies the weak Lebesgue–Besicovitch differentiation property for every sigma-finite locally finite Borel measure, will it satisfy the strong Lebesgue–Besicovitch differentiation property for every such measure?

1. Setting for statistical learning

Here we will remind the standard probabilistic model for statistical learning theory. The domain, $\Omega$, means a standard Borel space, that is, a set equipped with a sigma-algebra which coincides with the sigma-algebra of Borel sets generated by a suitable separable complete metric. (Recall that the Borel structure generated by a metric $\rho$ on a set $\Omega$ is the smallest sigma-algebra containing all open subsets of the metric space $(\Omega, \rho)$.) The distribution laws for datapoints, both unlabelled and labelled, are Borel probability measures defined on the corresponding Borel sigma-algebra.

Since we will be dealing with the $k$-NN classifier, the domain, $\Omega$, will actually be a metric space, which we also assume to be separable.

Labelled data pairs $(x, y)$, where $x \in \Omega$ and $y \in \{0, 1\}$, will follow an unknown probability distribution $\tilde{\mu}$, that is, a Borel probability measure on $\Omega \times \{0, 1\}$. We denote the corresponding random element $(X, Y) \sim \tilde{\mu}$. Define two Borel measures on $\Omega$, $\mu_i$, $i = 0, 1$, by $\mu_i(A) = \tilde{\mu}(A \times \{i\})$. In this way, $\mu_0$ is governing the distribution of the elements labelled 0, and similarly for $\mu_1$. The sum $\mu = \mu_0 + \mu_1$ (the direct image of $\tilde{\mu}$ under the projection from $\Omega \times \{0, 1\}$ onto $\Omega$) is a Borel probability measure on $\Omega$, the distribution law of unlabelled data points. Clearly, $\mu_i$ is absolutely continuous with regard to $\mu$, that is, if $\mu(A) = 0$, then $\mu_i(A) = 0$ for $i = 0, 1$. The corresponding Radon-Nikodým derivative in the case $i = 1$ is just the conditional probability for a point $x$ to be labeled 1:

$$\eta(x) = \frac{d\mu_1}{d\tilde{\mu}}(x) = P[Y = 1|X = x].$$

In statistical terminology, $\eta$ is the regression function.

Together with the Borel probability measure $\mu$ on $\Omega$, the regression function allows for an alternative, and often more convenient, description of the joint law $\tilde{\mu}$. Namely, given $A \subseteq \Omega$,

$$\mu_1(A) = \int_A \eta(x) \, d\mu,$$

and

$$\mu_0(A) = \int_A (1 - \eta(x)) \, d\mu,$$

which allows to reconstruct the measure $\tilde{\mu}$ on $\Omega \times \{0, 1\}$. 
Let $B(\Omega, \{0,1\})$ be the collection of all Borel measurable binary functions on the domain, that is, essentially, the family of all Borel subsets of $\Omega$. Given such an $f: \Omega \rightarrow \{0,1\}$ (a classifier), the misclassification error is defined by

$$\text{err}_\mu(f) = \tilde{\mu}\{ (x,y): f(x) \neq y \} = P[f(X) \neq Y].$$

The Bayes error is the infimal misclassification error taken over all possible classifiers:

$$\ell^* = \ell^*(\tilde{\mu}) = \inf_f \text{err}_\mu(f).$$

It is a simple exercise to verify that the Bayes error is achieved on some classifier (and thus is the minimum), which is called a Bayes classifier. For instance, every classifier satisfying

$$T_{bayes}(x) = \begin{cases} 1, & \text{if } \eta(x) > \frac{1}{2}, \\ 0, & \text{if } \eta(x) < \frac{1}{2}, \end{cases}$$

is a Bayes classifier.

The Bayes error is zero if and only if the learning problem is deterministic, that is, the regression function $\eta$ is equal almost everywhere to the indicator function, $\chi_C$, of a concept $C \subseteq \Omega$, a Borel subset of the domain.

A learning rule is a family $\mathcal{L} = (\mathcal{L}_n)_{n=1}^{\infty}$, where

$$\mathcal{L}_n: \Omega^n \times \{0,1\}^n \rightarrow \mathcal{B}(\Omega, \{0,1\}), \quad n = 1, 2, \ldots$$

and the functions $\mathcal{L}_n$ satisfy the following measurability assumption: the associated maps

$$\Omega^n \times \{0,1\}^n \times \Omega \ni (\sigma, x) \mapsto \mathcal{L}_n(\sigma)(x) \in \{0,1\}$$

are Borel (or even just universally measurable). Here $\sigma = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a labelled learning sample.

The data is modelled by a sequence of independent identically distributed random elements $(X_n, Y_n)$ of $\Omega \times \{0,1\}$, following the law $\tilde{\mu}$. Denote $\varsigma$ an infinite sample path. In this context, $\mathcal{L}_n$ only gets to see the first $n$ labelled coordinates of $\varsigma$. A learning rule $\mathcal{L}$ is weakly consistent, or simply consistent, if $\text{err}_\mu(\mathcal{L}_n(\varsigma)) \rightarrow \ell^*$ in probability as $n \rightarrow \infty$. If the convergence occurs almost surely (that is, along almost all sample paths $\varsigma \sim \tilde{\mu}^\infty$), then $\mathcal{L}$ is said to be strongly consistent. Finally, $\mathcal{L}$ is universally (weakly / strongly) consistent if it is weakly / strongly consistent under every Borel probability measure $\tilde{\mu}$ on the standard Borel space $\Omega \times \{0,1\}$.

The learning rule we study is the $k$-NN classifier, defined by selecting the label $\mathcal{L}_n(\sigma)(x) \in \{0,1\}$ by the majority vote among the values of $y$ corresponding to the $k = k_n$ nearest neighbours of $x$ in the learning sample $\sigma$. 

![Figure 1. Labeled domain and the projection $\pi: \Omega \times \{0,1\} \rightarrow \Omega$.](image_url)
If \( k \) is even, then a voting tie may occur. This is of lesser importance, and can be broken in any way. For instance, by always assigning the value 1 in case of a voting tie, or by choosing the value randomly. The consistency results usually do not depend on it. Intuitively, if voting ties keep occurring asymptotically at a point \( x \) along a sample path, it means that \( \eta(x) = 1/2 \) and so any value of the classifier assigned to \( x \) would do.

It may also happen that the smallest closed ball containing \( k \) nearest neighbours of a point \( x \) contains more than \( k \) elements of a sample (distance ties). This situation is more difficult to manage and requires a consistent tie-breaking strategy, whose choice may affect the consistency results.

Given \( k \) and \( n \geq k \), we define \( r_{k,\text{NN}}^\sigma(x) \) as the smallest radius of a closed ball around \( x \) containing at least \( k \) nearest neighbours of \( x \) in the sample \( \varsigma_n \):

\[
r_{k,\text{NN}}^\sigma(x) = \min \{ r \geq 0 : \sharp \{ i = 1, 2, \ldots, n : x_i \in \bar{B}_r(x) \} \geq k \}.
\]  

As the corresponding open ball around \( x \) contains at most \( k - 1 \) elements of the sample, the ties may only occur on the sphere.

If \( \sigma \in \Omega^n \) and \( \sigma' \in \Omega^k \), \( k \leq n \), the symbol

\[ \sigma' \sqsubset \sigma \]

means that there is an injection \( f : [k] \rightarrow [n] \) such that

\[ \forall i = 1, 2, \ldots, k, \; \sigma'_i = \sigma_{f(i)}. \]

A \textit{k nearest neighbour map} is a function

\[ k\text{-NN}^\sigma : \Omega^n \times \Omega \rightarrow \Omega^k \]

with the properties

1. \( k\text{-NN}^\sigma(x) \sqsubset \sigma \), and
2. all points \( x_i \) in \( \sigma \) that are at a distance strictly less than \( r_{k,\text{NN}}^\sigma(x) \) to \( x \) are in \( k\text{-NN}^\sigma(x) \).

The mapping \( k\text{-NN}^\sigma \) can be deterministic or stochastic, in which case it will depend on an additional random variable, independent of the sample path.

An example of the former kind is based on the natural order on the sample, \( x_1 < x_2 < \ldots < x_n \). In this case, from among the points belonging to the sphere of radius \( r_{k,\text{NN}}^\sigma(x) \) around \( x \) we choose points with the smaller index: \( k\text{-NN}^\sigma(x) \) contains all the points of \( \sigma \) in the open ball, \( B_{r_{k,\text{NN}}^\sigma(x)}(x) \), plus a necessary number (at least one) of points of \( \sigma \cap S_{r_{k,\text{NN}}^\sigma(x)}(x) \) having smallest indices.

An example of the second kind is to use a similar procedure, after applying a random permutation of the indices first. A random learning input will consist of a pair \( (W_n, P_n) \), where \( W_n \) is a random \( n \)-sample and \( P_n \) is a random element of the group of permutations of rank \( n \). An equivalent (and more common) way would be to use a sequence of i.i.d. random elements \( Z_n \) of the unit interval or the real line, distributed according to the uniform (resp. gaussian) law, and in case of a tie, give a preference to a realization \( x_i \) over \( x_j \) whose value \( z_i \) is smaller than \( z_j \).

Now, a formal definition of the \( k\text{-NN} \) learning rule can be given as follows:

\[
L_n^{k,\text{NN}}(\sigma, \epsilon)(x) = \theta \left[ \frac{1}{k} \sum_{x_i \in k\text{-NN}^\sigma(x)} \epsilon_i - \frac{1}{2} \right]
\]

\[
= \theta \left[ \mathbb{E}_{\mu_{k,\text{NN}^\sigma(x)}} \epsilon - \frac{1}{2} \right].
\]
Here, $\theta$ is the Heaviside function, the sign of the argument:

$$\theta(t) = \begin{cases} 
1, & \text{if } t \geq 0, \\
0, & \text{if } t < 0.
\end{cases}$$

The empirical measure $\mu_{kNN}(x)$ is a uniform measure supported on the set of $k$ nearest neighbours of $x$ within the sample $\sigma$, and the label $\epsilon$ is seen as a function $\epsilon: \{x_1, x_2, \ldots, x_n\} \to \{0, 1\}$.

The expression appearing under the argument,

$$\eta_{n,k} = \frac{1}{k} \sum_{x_i \in kNN(x)} \epsilon_i,$$

is the empirical regression function. In the presence of a law of labelled points, it is a random variable, and so we have the following immediate, yet important, observation.

**Proposition 1.1.** Let $(\mu, \eta)$ be a learning problem in a separable metric space $(\Omega, d)$. If the values of the empirical regression function, $\eta_{n,k}$, converge to $\eta$ in probability (resp. almost surely) in the region

$$\Omega_\eta = \left\{ x \in \Omega : \eta(x) \neq \frac{1}{2} \right\},$$

then the $k$-NN classifier is consistent (resp. strongly consistent) under $(\mu, \eta)$.

We conclude this section by reminding an important technical tool.

**Theorem 1.2** (Cover-Hart lemma [3]). Let $\Omega$ be a separable metric space, and let $\mu$ be a Borel probability measure on $\Omega$. Almost surely, the function $r_{kNN}^n$ (Eq. 2) converges to zero uniformly over any precompact subset $K \subseteq \text{supp } \mu$.

**Proof.** Let $A$ be a countable dense subset of $\text{supp } \mu$. A standard argument shows that, almost surely, for all $a \in A$ and each rational $\epsilon > 0$, the open ball $B_\epsilon(a)$ contains an infinite number of elements of a sample path. Consequently, the functions $r_{kNN}^n: \Omega \to \mathbb{R}$ almost surely converge to zero pointwise on $A$ as $n \to \infty$. Since these functions are easily seen to be 1-Lipschitz and in particular form a uniformly equicontinuous family, we conclude. \qed

2. Example of Preiss

Here we will discuss an example of [13], showing in particular that the $k$-NN learning rule is not universally consistent in the infinite-dimensional separable Hilbert space $\ell^2$.

We adopt the combinatorial notation $[n] = \{1, 2, \ldots, n\}$. Let $(N_k)$ be a sequence of positive natural numbers $\geq 2$, to be selected later. Denote by

$$Q = \prod_{k=1}^\infty [N_k]$$

the Cartesian product of finite discrete spaces equipped with the product topology. It is a Cantor space (the unique, up to a homeomorphism, totally disconnected compact metrizable space without isolated points).

Let $\pi_k$ denote the canonical coordinate projections of $Q$ on the $k$-dimensional cubes $Q_k = \prod_{i=1}^k [N_i]$. Denote $Q^* = \bigcup_{k=1}^\infty Q_k$ a disjoint union of the cubes $Q_k$, and let $\mathcal{H} = \ell^2(Q^*)$ be a Hilbert space spanned by an orthonormal basis $(e_n)$ indexed by elements $\bar{n}$ of this union.

For every $\bar{n} = (n_1, \ldots, n_i, \ldots) \in Q$ define

$$f(\bar{n}) = \sum_{i=1}^\infty 2^{-i} e_{(n_1, \ldots, n_i)} \in \mathcal{H}.$$
The map $f$ is continuous and injective, thus a homeomorphism onto its image. Denote $\nu$ the Haar measure on $Q$ (the product of the uniform measures on all $[N_k]$). Let $\mu_1 = f_\ast(\nu)$ be the direct image of $\nu$, a compactly-supported Borel probability measure on $H$. If $r > 0$ satisfies $2^{-k} \leq r^2 < 2^{-k+1}$, then for each $\bar{n} = (n_1, n_2, \ldots) \in Q$,

$$\mu_1(B_r(f(\bar{n}))) = \nu(\pi_{k+1}^{-1}(\bar{n})) = (N_1N_2\ldots N_{k+1})^{-1}.$$ 

Now, for every $k$ and each $\bar{n} = (n_1, \ldots, n_k) \in Q_k \subseteq Q_k^*$ define in a similar way

$$f(\bar{n}) = \sum_{i=1}^{k} 2^{-i}e(n_1, \ldots, n_i) \in H.$$ 

Note that the closure of $f(Q^*)$ contains $f(Q)$ (as a proper subset). Now define a purely atomic measure $\mu_0$ supported on the image of $Q_k$ under $f$, having the following special form:

$$\mu_0 = \sum_{k=1}^{\infty} \sum_{\bar{n} \in Q_k} a_k \delta_{\bar{n}}.$$ 

The weights $a_k > 0$ are chosen so that the measure is finite:

$$\sum_{k=1}^{\infty} a_k < \infty. \quad (3)$$ 

Since for $r$ satisfying $2^{-k} \leq r^2 < 2^{-k+1}$ and $\bar{n} \in Q$ the ball $B_r(f(\bar{n}))$ contains in particular $f(n_1n_2, \ldots, n_k)$, we have

$$\mu_0(B_r(f(\bar{n}))) \geq a_k.$$ 

Assuming in addition that

$$a_kN_1N_2\ldots N_kN_{k+1} \to \infty \text{ as } k \to \infty, \quad (4)$$

we conclude:

$$\frac{\mu_1(B_r(f(\bar{n})))}{\mu_0(B_r(f(\bar{n})))} \leq \frac{(N_1N_2\ldots N_{k+1})^{-1}}{a_k} \to 0 \text{ when } r \downarrow 0.$$ 

Clearly, the conditions $[3]$ and $[4]$ can be simultaneously satisfied by a recursive choice of $(N_k)$ and $(a_k)$.

Now renormalize the measures $\mu_0$ and $\mu_1$ so that $\mu = \mu_0 + \mu_1$ is a probability measure, and interpret $\mu_i$ as the distribution of points labelled $i = 0, 1$. Thus, the regression function is deterministic, $\eta = \chi_C$, where we are learning the concept $C = f(Q) = \text{supp} \mu_1$, $\mu_1(C) > 0$.

For a random element $X \in H$, $X \sim \mu$, the distance $r_k(X)$ to the $k$-th nearest neighbour within an i.i.d. $n$-sample goes to zero almost surely when $k/n \to 0$, according to a lemma of Cover and Hart, and the convergence is uniform on the precompact support of $\mu$. It follows that the probability of one of the $k$ nearest neighbours to a random point $X \in H$ to be labelled one, conditionally on $r_{k,NN}^\infty = r$, converges to zero. The $k$-NN learning rule will almost surely predict an identically zero classifier, and so is not consistent.

3. Classical theorem of Charles J. Stone

3.1. The case of continuous regression function

Proposition 1.1 and the Cover–Hart lemma 1.2 together imply that the $k$-NN classifier is universally consistent in a separable metric space whenever the regression function $\eta$ is continuous. Indeed, it is enough to make the following observation.
Lemma 3.1. Let $(\Omega, \mu)$ be a separable metric space equipped with a Borel probability measure, and let $\eta$ be a continuous regression function. Then

$$\eta_{n,k} \to \eta$$

in probability, when $n, k \to \infty$, $k/n \to 0$.

Proof. It follows from the Cover–Hart lemma that the set $k$-NN$^{\eta_n}(x)$ of $k$ nearest neighbours of $x$ almost surely converges to $x$, for almost all $x \in \text{supp}\mu$, and since $\eta$ is continuous, the set of values $\eta(k$-NN$(x))$ converges to $\eta(x)$ in an obvious sense: for every $\varepsilon > 0$, there exists $N$ such that

$$\forall n \geq N, \ \eta(k$-NN$(x)) \subseteq (\eta(x) - \varepsilon, \eta(x) + \varepsilon), \quad (5)$$

where $k = k_n$ depends on $n$. Let $\varepsilon > 0$ and $N$ be fixed, and denote $P_{\varepsilon,N}$ the set of pairs $(\varsigma, x)$ consisting of a sample path $\varsigma$ and a point $x \in \Omega$ satisfying Eq. (5). Select $N$ with the property $\mu(P_{\varepsilon,N}) > 1 - \varepsilon$. Let $\epsilon = (\epsilon_i)_{i=1}^{\infty}$ denote the sequence of labels for $\varsigma$, which is a random variable with the joint law $\otimes_{i=1}^{\infty} \{\eta(x_i), 1 - \eta(x_i)\}$. By the above, whenever $(\varsigma, x) \in P_{\varepsilon,N}$ and $n \geq N$, if $x_i$ is one of the $k$ nearest neighbours of $x$ in $s_n$, we have $\mathbb{E} \epsilon_i = \eta(x_i) \in (\eta(x) - \varepsilon, \eta(x) + \varepsilon)$. According to a version of the Law of Large Numbers with Chernoff’s bounds, the probability of the event

$$\frac{\sum_{x_i \in k$-NN$(x)} \epsilon_i}{k} \notin (\eta(x) - 2\varepsilon, \eta(x) + 2\varepsilon) \quad (6)$$

is exponentially small, bounded above by $2 \exp(-2\varepsilon^2 k)$. Thus, when $n \geq N$, $P[|\eta_{n,k} - \eta| \geq \varepsilon] < \varepsilon + 2 \exp(-2\varepsilon^2 k_n)$, and we conclude. \qed

Remark 3.2. In the most general case (with the uniform tie-breaking) we can only infer the almost sure convergence if $k$ grows fast enough as a function in $n$, for otherwise the series $\sum_{n=1}^{\infty} 2 \exp(-2\varepsilon^2 k_n)$ may be divergent.

3.2. Geometric Stone lemma for $\mathbb{R}^d$

In the case of a general Borel regression function $\eta$, which can easily be discontinuous almost everywhere, the classical Luzin theorem of real analysis says that for any $\varepsilon > 0$ there is a compact set $K$ of measure $\mu(K) > 1 - \varepsilon$ upon which $\eta$ is continuous. Now we have control over the behaviour of those $k$-nearest neighbours of a point $x$ that belong to $K$: the mean value of the regression function $\eta$ at those $k$-nearest neighbours will converge to $\eta(x)$. However, we have no control over the behaviour of the values of $\eta$ at the $k$-nearest neighbours of $x$ that belong to the open set $U = \Omega \setminus K$. The problem is therefore to limit the influence of the remaining $\approx \varepsilon n$ sample points belonging to $U$. Intuitively, as the example of Preiss shows, in infinite dimensions the influence of the few points outside of $K$ can become overwhelming, no matter how close the measure of $K$ is to one.

In the Euclidean case, this goal is achieved with the help of the geometric Stone lemma, which uses the finite-dimensional euclidean structure of the space in a beautiful way.

Lemma 3.3 (Geometric Stone’s lemma for $\mathbb{R}^d$). For every natural $d$, there is an absolute constant $C = C(d)$ with the following property. Let

$$\sigma = (x_1, x_2, \ldots, x_n), \ x_i \in \mathbb{R}^d, \ i = 1, 2, \ldots, n,$$

be a finite sample in $\ell^2(d)$ (possibly with repetitions), and let $x \in \ell^2(d)$ be any. Given $k \in \mathbb{N}_+$, the number of $i$ such that $x \neq x_i$ and $x$ is among the $k$ nearest neighbours of $x_i$ inside the sample

$$x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n$$

is limited by $Ck$. \(7\)
Figure 2. To the proof of geometric Stone’s lemma (case $k = 2$).

Proof. Cover $\mathbb{R}^d$ with $C = C(d)$ cones of central angle $< \pi/3$ with vertices at $x$. Inside each cone mark the maximal possible number $\leq k$ of the nearest neighbours of $x$. (The strategy for possible distance tie-breaking is unimportant.) In this way, up to $Ck$ points are marked. Let now $i$ be any, such that $x_i \neq x$ as a point. If $x_i$ has not been marked, this means the cone containing $x_i$ has $k$ points that have been marked. Consider any of the marked points inside the same cone, say $y$. A simple argument of planimetry, inside an affine plane passing through $x$, $x_i$, and $y$, shows that

$$\|x_i - x\| > \|x_i - y\|,$$

and so the $k$ nearest neighbours of $x_i$ inside the sample in Eq. (7) will all be among the marked points. \hfill $\square$

Remark 3.4. Note that in the statement of the geometric Stone lemma neither the order of the sample $x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n$, nor the tie-breaking strategy are of any importance.

Remark 3.5. If the cones have central angle $\pi/3 = 60^\circ$, then the inequality in the conclusion of the lemma is no longer strict. This is less convenient in case of distance ties.

3.3. Proof of the Stone theorem

Theorem 3.6 (Charles J. Stone, 1977). Let $k, n \to \infty$, $k/n \to 0$. Then the $k$-NN classification rule in the finite-dimensional Euclidean space $\mathbb{R}^d$ is universally consistent.

Let $K$ be a compact subset of the domain $\Omega$, and let $U = \Omega \setminus K$. Let us estimate the expected number of the random elements $X_i$ of an $n$-sample that (1) belong to $U$, and (2) are among the $k$ nearest neighbours of a random element $X$ belonging to $K$. To this end, let $X, X_1, X_2, \ldots, X_n$ be i.i.d. random elements of the domain. We apply the symmetrization with a transposition of the coordinates $\tau_i: X \leftrightarrow X_i$ to obtain:
\[ E \frac{1}{k} \sum_{i = 1, 2, \ldots, n} \{ i : X_i \in k\text{-NN}(X), \ X \in K, \ X_i \notin K \} \]
\[ = E \frac{1}{k} \sum \{ \chi_U(X_i) : X_i \in k\text{-NN}(X), \ X \in K \} \]
\[ \leq E \frac{1}{k} \sum \{ \chi_U(X_i) : X_i \in k\text{-NN}(X), \ X_i \neq X \} \]
\[ = E \frac{1}{k} \sum \{ \chi_U(X) : X \in k\text{-NN}(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)(X_i), \ X_i \neq X \} \]
\[ = E \frac{1}{k} \chi_U(X) \frac{1}{C} \mu(U) \]
\[ = C \mu(U). \] (8)

Let \( \varepsilon > 0 \) be any. Use Luzin’s theorem to choose a compact set \( K \) with \( \mu(K) > 1 - \varepsilon \) and \( \eta|_K \) continuous. Denote \( U = \Omega \setminus K \). The function \( \eta|_K \) extends to a continuous \([0, 1]\)-valued function \( \psi \) on all of \( \Omega \). (All this can be done in an arbitrary separable metric space.)

We have:

\[ E |\eta - \eta_n| \leq E |\eta - \psi| + E |\psi - \psi_{n,k}| + E |\psi_{n,k} - \eta_n|, \]

where (I) \( \leq \mu(U) < \varepsilon \), (II) \( \to 0 \) in probability by virtue of Lemma 3.1 and

\[ (III) \]
\[ \leq \frac{1}{k} \left\{ i = 1, 2, \ldots, n : X_i \in k\text{-NN}(X), \ X \in K, \ X_i \notin K \right\} + \mu(U) \]
\[ = (C + 1) \mu(U) \]
\[ < (C + 1) \varepsilon. \]

Since \( \varepsilon > 0 \) is as small as desired, we conclude that \( \eta_{n,k}(X) \to \eta(X) \) in probability, and so the \( k\)-NN classifier in \( \ell^2(d) \) is universally (weakly) consistent in \( \ell^2(d) \).

4. Nagata dimension of a metric space

Recall that a family \( \gamma \) of subsets of a set \( \Omega \) has multiplicity \( \leq \delta \) if the intersection of more than \( \delta \) different elements of \( \gamma \) is always empty. In other words,

\[ \forall x \in \Omega, \sum_{V \in \gamma} \chi_V(x) \leq \delta. \]

**Definition 4.1.** Let \( \delta \in \mathbb{N} \), \( s \in (0, +\infty] \). We say that a metric space \((\Omega, d)\) has **Nagata dimension** \( \leq \delta \text{ on the scale } s > 0 \), if every finite family \( \gamma \) of closed balls of radii \( < s \) admits a subfamily \( \gamma' \) of multiplicity \( \leq \delta + 1 \) which covers the centres of all the balls in \( \gamma \). A space \( \Omega \) has Nagata dimension \( \delta \) if it has Nagata dimension \( \delta \) on a suitable scale \( s \in (0, +\infty] \). Notation: \( \dim_{Nag}^*(\Omega) = \delta \), or simply \( \dim_{Nag}(\Omega) = \delta \).

Sometimes the following reformulation is more convenient.
Proposition 4.2. A metric space \((\Omega, d)\) has Nagata dimension \(\leq \delta\) on the scale \(s > 0\) if and only if it satisfies the following property. Given a sequence \(x_1, \ldots, x_{\delta+2} \in \mathcal{B}_r(x), r < s\), there are \(i, j, i \neq j\), such that \(d(x_i, x_j) \leq \max\{d(x, x_i), d(x, x_j)\}\).

Proof. Necessity \((\Rightarrow)\): from the family of closed balls \(B_{d(x, x_j)}(x_i), i = 1, 2, \ldots, \delta + 2\), all having radii \(< s\), extract a family of \(\delta + 1\) balls covering the centres. One of those balls, say with centre at \(x_1\), must contain some \(x_j\) with \(i \neq j\), which means \(d(x_i, x_j) \leq d(x_i, x) \leq \max\{d(x, x_i), d(x, x_j)\}\).

Sufficiency \((\Leftarrow)\): let \(\gamma\) be a finite family of closed balls of radii \(< s\). Suppose it has multiplicity \(> \delta + 1\). Then there exist a point \(x \in \Omega\) and \(\delta + 2\) balls in \(\gamma\) with centres that we denote \(x_1, \ldots, x_{\delta+2}\), all containing \(x\). Denote \(r = \max\{d(x, x_i)\}\). Then \(r < s\), and by the hypothesis, there are \(i, j, i \neq j\), with \(d(x_i, x_j) \leq \max\{d(x, x_i), d(x, x_j)\}\). Without loss in generality, assume \(d(x_i, x_j) \leq d(x_i, x)\), that is, \(x_j\) belongs to the ball with centre in \(x_i\). Now the ball centred at \(x_j\) can be removed from the family \(\gamma\), with the remaining family still covering all the centres and having the cardinality \(|\gamma| - 1\). After finitely many steps, we arrive at a subfamily of multiplicity \(\leq \delta + 1\) covering all the centres. \(\square\)

Example 4.3. The property of a metric space \(\Omega\) having Nagata dimension zero on the scale \(+\infty\) is equivalent to \(\Omega\) being a non-archimedean metric space, that is, a metric space satisfying the strong triangle inequality, \(d(x, z) \leq \max\{d(x, y), d(y, z)\}\).

Indeed, \(\dim_{\text{Nag}}^{\infty}(\Omega) = 0\) means exactly that for any sequence of \(\delta + 2 = 2\) points, \(x_1, x_2\), contained in a closed ball \(B_r(x)\), we have \(d(x_1, x_2) \leq \max\{d(x, x_1), d(x, x_2)\}\).

Example 4.4. It follows from Proposition 4.2 that \(\dim_{\text{Nag}}(\mathbb{R}) = 1\). Let \(x_1, x_2, x_3\) be three points contained in a closed ball, that is, an interval \([x-r, x+r]\). Without loss in generality, assume \(x_1 < x_2 < x_3\). If \(x_2 \leq x\), then \(|x_1 - x_2| \leq |x_1 - x|\), and if \(x_2 \geq x\), then \(|x_3 - x_2| \leq |x_3 - x|\).

The following example suggests that the Nagata dimension is relevant for the study of the \(k\)-NN classifier, as it captures in an abstract context the geometry behind the Stone lemma.

Example 4.5. The Nagata dimension of the euclidean space \(\ell^2(d)\) is finite, and it is bounded by \(C(d) - 1\), where \(C(d)\) is the value of the constant in the geometric Stone lemma.

Indeed, let \(x_1, \ldots, x_{C(d)+1}\) be points belonging to a ball with centre \(x\). Using the argument in the proof of the geometric Stone lemma with \(k = 1\), mark \(\leq C(d)\) points \(x_i\) belonging to the \(\leq C(d)\) cones with apex at \(x\). At least one point, say \(x_j\), has not been marked; it belongs to some cone, which therefore already contains a marked point, say \(x_i\), different from \(x_j\), and \(\|x_i - x_j\| \leq \|x_j - x\|\).

Example 4.6. A similar argument shows that every finite-dimensional normed space has finite Nagata dimension.

Remark 4.7. In \(\mathbb{R}^2 = \mathbb{C}\) the family of closed balls of radius one centred at the vectors \(\exp(2\pi ki/5), k = 1, 2, \ldots, 5\), has multiplicity 5 and admits no proper subfamily contains all the centres. Therefore, the Nagata dimension of \(\ell^2(2)\) is at least 5. Since the plane can be covered with 6 cones of the angle \(\pi/3\), Example 4.5 implies that \(\dim_{\text{Nag}}(\ell^2(2)) = 5\).

Remark 4.8. The problem of calculating the Nagata dimension of the Euclidean space \(\ell^2(d)\) is mentioned as “possibly open” by Nagata [11], p. 9 (where the value \(\dim_{\text{Nag}}+2\) is called the “crowding number”). Nagata also mentions that \(\dim_{\text{Nag}}(\mathbb{R}^7) = 1\) and \(\dim_{\text{Nag}}(\ell^2(3)) = 5\) (without a proof).

Remark 4.9. Notice that the property of the Euclidean space established in the proof of the geometric Stone lemma is strictly stronger than the finiteness of the Nagata dimension. There exists a finite \(\delta\) (in general, higher than the Nagata dimension) such that, given a sequence \(x_1, \ldots, x_{\delta+2} \in \mathcal{B}_r(x), r < s\), there are \(i, j, i \neq j\), such that \(d(x_i, x_j) < \max\{d(x, x_i), d(x, x_j)\}\). The inequality here is strict. This is exactly the property that removes the problem of distance ties in the Euclidean space. However, adopting this as a definition in the general case would be too restrictive, removing from consideration a large class of metric spaces in which the \(k\)-NN classifier is still universally consistent, such as all non-archimedean metric spaces.
Example 4.10. The convergent sequence $(1/n)e_n$, $n \geq 0$, in the Hilbert space $\ell^2$, together with the limit 0, equipped with the induced metric, has infinite Nagata dimension on every scale $s > 0$. This is witnessed by the family of closed balls $B_{1/n}(a/n)e_n$, having zero as the common point, and having the property that every centre belongs to exactly one ball of the family. Realizing $\mathbb{R}$ as a continuous curve without self-intersections passing through all elements of the sequence as well as the limit leads to a metric on $\mathbb{R}$ having infinite Nagata dimension on each scale.

Remark 4.11. The Nagata–Ostrand theorem [10][12] states that the Lebesgue covering dimension of a metrizable topological space is the smallest Nagata dimension of a compatible metric on the space (and in fact this is true on every scale $s > 0$, [9]). This is the historical origin of the concept of the metric dimension.

Metric spaces of finite Nagata dimension admit an almost literal version of the Stone geometric lemma in case where the sample has no distance ties, that is, the values of the distances $d(x_i, x_j)$, $i \neq j$, are all pairwise distinct.

Lemma 4.12 (Geometric Stone lemma, finite Nagata dimension, no ties). Let $\Omega$ a metric space of Nagata dimension $\delta < \infty$ on a scale $s > 0$. Let  
$$\sigma = (x_1, x_2, \ldots, x_n), \ x_i \in \mathbb{R}^d, \ i = 1, 2, \ldots, n,$$
be a finite sample in $\Omega$, and let $x \in X$ be any. Suppose there are no distance ties inside the sample  
$$x, x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n,$$
and $k$ is such that, inside the above sample, $r_{k, \text{NN}}(x_i) < s$ for all $i$. The number of $i$ having the property that $x \neq x_i$ and $x$ is among the $k$ nearest neighbours of $x_i$ inside the sample above is limited by $(k+1)(\delta + 1)$.

Proof. Suppose that $x_i, i = 1, 2, \ldots, m$ has $x$ among the $k$ nearest neighbours. The family $\gamma$ of closed balls $B_{r_{k, \text{NN}}(x_i)}(x_i), i \leq m$, admits a subfamily $\gamma'$ of multiplicity $\leq \delta + 1$ covering all the points $x_i, i \leq m$. Since there are no distance ties, every ball belonging to $\gamma$ contains $\leq k + 1$ points. It follows that $\sharp \gamma' \geq m/(k + 1)$. All the balls in $\gamma'$ contain $x$, and we conclude: $\sharp \gamma' \leq \delta + 1$. The result follows.

Now the same argument as in the original proof of Stone shows that the $k$-NN classifier is consistent under each distribution $\mu$ on $\Omega \times \{0, 1\}$ with the property that the distance ties are improbable. Since we are going to give a proof of a more general result, we will not repeat the argument here, only mention that due to the Cover–Hart lemma, if $n$ is sufficiently large, then with arbitrarily high probability, the $k$ nearest neighbours of a random point inside a random sample will all lie at a distance $< s$.

5. Distance ties

In this section we will construct a series of examples to illustrate the difficulties arising in the presence of distance ties in general metric spaces that are absent in the Euclidean case. The fundamental difference between the two situations is the inequality in the equivalent definition of the Nagata dimension (proposition 4.2) that is, unlike in the Euclidean space, no longer strict.

As we have already noted (remark 3.4), the conclusion of geometric Stone’s lemma [5,3] remains valid even if we allow the adversary to break the distance ties and pick the $k$ nearest neighbours. Our first example shows that it is no longer the case in a metric space of finite Nagata dimension.

Example 5.1. Consider a finite set $\sigma = \{x_1, x_2, \ldots, x_n\}$ with $n \geq k$ points, and assume that in the metric space $\sigma \cup \{x\}$ all $n + 1$ points are pairwise at a distance one from each other. The Nagata dimension of the metric space $\sigma \cup \{x\}$ is equal to $\delta = 0$. Indeed, if a family $\gamma$ of closed balls contains any ball of radius $\geq 1$, it already covers $\sigma$ on its own. Otherwise, we choose one ball of radius $< 1$ (that is, a singleton) for each centre. The multiplicity of the selected subfamily is 1 in each case.
Now let us discuss the distance ties. For any element $x_i$ of $\sigma$, the remaining $n$ points of $\sigma \cup \{x\}$ are tied between themselves as the possible $k$ nearest neighbours. The adversary may decide to always select $x$ among them, thus invalidating the conclusion of the geometric Stone lemma.

However, the problem is easily resolved if we break distance ties using a uniform distribution on the nearest neighbour candidates. In this case, the expected number of indices $i$ such that $x$ is chosen as one of the $k$ nearest neighbours of $x_i$ within the sample $\{x_1, x_2, \ldots, x, \ldots, x_k\}$ is obviously $k$.

**Remark 5.2.** It is worth observing that in the euclidean case $\Omega = \mathbb{R}^d$ the size of a sample inheriting a 0-1 distance will be limited from above by the dimension, $d$.

The next example shows that the geometric Stone lemma in finite dimensional metric spaces cannot be saved even with the uniform tie-breaking.

**Example 5.3.** Given $N \in \mathbb{N}$, there exists a finite metric space $\sigma = \{x_1, \ldots, x_n\}$ of Nagata dimension 0, with the property that for $x = x_1$ the expected number of points $x_i \neq x_1$ having $x$ as the nearest neighbour under the uniform tie-breaking is $\geq N$.

We will construct $\sigma$ recurrently. Let $\sigma_1 = \{x_1\}$. Add $x_2$ at a distance 1 from $x_1$, and set $\sigma_2 = \{x_1, x_2\}$. If $\sigma_n$ has been already defined, add $x_{n+1}$ at a distance $2^n$ from all the existing points $x_i$, $i \leq n$, and set $\sigma_{n+1} = \sigma_n \cup \{x_{n+1}\}$. It is clear that the distance so defined is a metric.

We will verify by induction in $n$ that $\dim_{\text{Nag}}(\sigma_n) = 0$. For $n = 1$ this is trivially true. Assume the statement holds for $\sigma_n$, and let $\gamma$ be a family of closed balls in $\sigma_{n+1}$. If one of those balls contains all the points, there is nothing to prove. Assume not, that is, all the balls elements of $\gamma$ have radii smaller than $2^{n+1}$. Choose a subfamily of multiplicity 1 consisting of balls centred in elements of $\sigma_n$ and covering them all, and add one ball centred in $x_{n+1}$ (which is a singleton).

Finally, let us show that if $n$ is sufficiently large, then the expected number of indices $i$ such that $x = x_1$ is the nearest neighbour of $x_i$ under a uniform tie-breaking is as large as desired. With this purpose, for each $i \geq 2$ we will calculate the expectation of the event $x_1 \in NN(x_i)$. For $x_2$, the unique nearest neighbour within $\sigma_n$ is $x_1$, therefore $\mathbb{E}[x_1 \in NN(x_2)] = 1$. For $x_3$, there are two points in $\sigma_n$ at a distance 2 from $x_3$, which can be chosen each with probability 1/2, namely $x_1$ and $x_2$, therefore $\mathbb{E}[x_1 \in NN(x_3)] = 1/2$. For arbitrary $i$, in a similar way, $\mathbb{E}[x_1 \in NN(x_i)] = 1/i$. We conclude:

$$\mathbb{E}\left[\sharp\{i = 1, \ldots, n: x_1 \in NN(x_i)\}\right] = \sum_{i=1}^{n} \frac{1}{i},$$

and the sum of the harmonic series converges to $+\infty$ as $n \to \infty$.

Can it be that the distance ties are in some sense extremely rare? Even this expectation is unfounded.

**Example 5.4.** Given a value $\delta > 0$ (risk) and a sequence $n_k' \uparrow +\infty$, there exist a compact metric space of Nagata dimension zero (a Cantor space with a suitable compatible metric) equipped with a non-atomic probability measure, and a sequence $n_k \geq n_k'$, $k/n_k \to 0$, with the following property. With confidence $> 1 - \delta$, for every $k$, a random element $X$ has $\geq n_k$ distance ties among its $k$ nearest neighbours within a random $n_{k+1}$-sample $\sigma$.

The space $\Omega$, just like in the Preiss example (Sect. 2), is the direct product $\prod_{k=1}^{\infty} [N_k]$ of finite discrete spaces, whose cardinalities $N_k \geq 2$ will be chosen recursively, and $[N_k] = \{1, 2, \ldots, N_k\}$. The metric is given by the rule

$$d(\sigma, \tau) = \begin{cases} 0, & \text{if } \sigma = \tau, \\ 2^{-\min\{i: \sigma_i \neq \tau_i\}}, & \text{otherwise}. \end{cases}$$

This metric induces the product topology and is non-archimedian, so the Nagata dimension of $\Omega$ is zero (example 1.3). The measure $\mu$ is the product of uniform measures $\mu_{N_k}$ on the spaces $[N_k]$. This measure is non-atomic, and in particular, $\mu$-almost all distance ties occur at a strictly positive distance from a random element $X$. 
Choose a sequence \( (\delta_i) \) with \( \delta_i > 0 \) and \( 2 \sum \delta_i = \delta \). Choose \( N_1 \) so large that, with probability \( > 1 - \delta_1 \), \( n_1 = n'_1 \) independent random elements following a uniform distribution on the space \([N_1]\) are pairwise distinct. Now let \( n_2 \geq n'_2 \) be so large that with probability \( > 1 - \delta_1 \), if \( n_2 \) independent random elements follow a uniform distribution on \([N_1]\), then each element of \([N_1]\) appears among them at least \( n_1 \) times.

Suppose that \( n_1, N_1, n_2, N_2, \ldots, n_k \) have been chosen. Let \( N_k \) be so large that, with probability \( > 1 - \delta_k \), \( n_k \) i.i.d. random elements uniformly distributed in \([N_k]\) are pairwise distinct. Choose \( n_{k+1} \geq n'_{k+1} \) so large that, with probability \( > 1 - \delta_k \), if \( n_{k+1} \) i.i.d. random elements are uniformly distributed within \( \prod_{i=1}^k [N_i] \), then each element of \( \prod_{i=1}^k [N_i] \) will appear among them at least \( n_k \) times.

Let \( k \) be any. Choose \( n_{k+1} + 1 \) i.i.d. random elements \( X, X_1, \ldots, X_{n_{k+1}} \) of \( \Omega \), following the distribution \( \mu \). With probability \( > 1 - 2\delta_k \), the following occurs: there are \( n_k \) elements in the sample \( X_1, X_2, \ldots, X_n \), which have the same \( i \)-th coordinates as \( X, i = 1, 2, 3, \ldots, k \), yet the \((k+1)\)-coordinates of \( X, X_1, \ldots, X_n \) are all pairwise distinct. In this way, the distances between \( X \) and all those \( n_k \) elements are equal to \( 2^{-k-1} \). We have \( n_k \) distance ties between \( k \) nearest neighbours of \( X \) (which are all at the same distance as the nearest neighbour of \( X \)), and \( n_k \geq n'_{k+1} \), as desired.

Now, it would be tempting to try and reduce the general case to the case of zero probability of ties, as follows. Recall that the \( \ell^1 \)-type direct sum of two metric spaces, \((X,d_X)\) and \((Y,d_Y)\), is the direct product \( X \times Y \) equipped with the coordinatewise sum of the two metrics:

\[
d(x,y) = d_X(x_1,y_1) + d_Y(x_2,y_2).
\]

Notation: \( X \oplus_1 Y \).

Let \( \Omega \) be a domain, that is, a metric space equipped with a probability measure \( \mu \) and a regression function, \( \eta \). Form the \( \ell^1 \)-type direct sum \( \Omega \oplus_1 [0,\varepsilon] \), and equip it with the product measure \( \mu \otimes \lambda \) (where \( \lambda \) is the normalized Lebesgue measure on the interval) and the regression function \( \eta \circ \pi_1 \), where \( \pi_1 \) is the projection on the first coordinate. It is easy to see that the probability of distance ties in the space \( \Omega \oplus_1 [0,\varepsilon] \) is zero, and every uniform distance tie breaking within a given finite sample will occur for a suitably small \( \varepsilon > 0 \). In this way, one could derive the consistency of the classifier by conditioning. However, we will now give an example of two metric spaces of Nagata dimension 0 and 1 respectively, whose \( \ell^1 \)-type sum has infinite Nagata dimension. This is quite an unexpected phenomenon, again very different from what happens in the Euclidean case.

**Example 5.5.** Fix \( \alpha > 0 \). Let \( \Omega = \{x_n: n \in \mathbb{N}\} \), equipped with the following distance:

\[
d(x_i,x_j) = \begin{cases} 0, & \text{if } i = j, \\ \sum_{k=1}^j \alpha^k, & \text{if } i < j. \end{cases}
\]

For \( i < j < k \),

\[
d(x_i,x_k) = \sum_{m=1}^k \alpha^m = d(x_j,x_k) > d(x_i,x_j) = \sum_{m=1}^j \alpha^m,
\]

from where it follows that \( d \) is an ultrametric. Thus, \( \Omega \) is a metric space of Nagata dimension 0.

The interval \( \mathbb{I} = [0,1] \) has Nagata dimension 1. Now let us consider the \( \ell^1 \)-type sum \( \Omega \oplus_1 \mathbb{I} \). Let \( 0 < \beta < \alpha < 1 \), and \( \beta < 1/2 \). Consider the infinite sequence

\[
z_i = (x_i,\beta^i) \in \Omega \oplus_1 \mathbb{I}
\]

and the point

\[
z = (x_0,0).
\]
Whenever $i < j$, 

$$
\begin{align*}
    d(z_i, z_j) &= d(x_i, x_j) + \beta^i - \beta^j \\
    &\geq d(x_i, x_0) + \alpha^j + \beta^i - \beta^j \\
    &> d(x_i, x_0) + \beta^i \\
    &= d(z_i, z),
\end{align*}
$$

and also 

$$
\begin{align*}
    d(z_i, z_j) &= d(x_i, x_j) + \beta^i - \beta^j \\
    &> d(x_j, x_0) + \beta^j \\
    &= d(z_j, z).
\end{align*}
$$

Together, the properties imply: for all $i \neq j$, 

$$
    d(z_i, z_j) > \max\{d(z_i, z_0), d(z_j, z_0)\}.
$$

Thus, the Nagata dimension of the $\ell^1$-type sum $\Omega \oplus I$ is infinite.

The above examples show that beyond the Euclidean setting, we have to put up with the possibility that some points in a sample will appear disproportionately often among $k$ nearest neighbours of other points. In data science, such points are known as “hubs” and the above (empirical) observation, as the “hubness phenomenon”, see e.g. [15] and further references therein. The geometric Stone lemma has to be generalized to allow for the possibility of a few of those “hubs”, whose number will be nevertheless limited. The lemma has to be reshaped in the spirit of the Hardy–Littlewood inequality in geometric measure theory.

To begin with, following Preiss [14], we will extend further our metric space dimension theory setting.

### 6. Sigma-finite dimensional metric spaces

**Definition 6.1.** Say that a metric subspace $X$ of a metric space $\Omega$ has *Nagata dimension* $\leq \delta \in \mathbb{N}$ on the scale $s > 0$ *inside of* $\Omega$ if every finite family of closed balls in $\Omega$ with centres in $X$ admits a subfamily of multiplicity $\leq \delta + 1$ in $\Omega$ which covers all the centres of the original balls. The subspace $X$ has a finite Nagata dimension in $\Omega$ if $X$ has finite dimension in $\Omega$ on some scale $s > 0$. Notation: $\dim_{Nag}^s(X, \Omega)$ or sometimes simply $\dim_{Nag}(X, \Omega)$.

Following Preiss, let us call a family of balls *disconnected* if the centre of each ball does not belong to any other ball. Here is a mere reformulation of the above definition.

**Proposition 6.2.** For a subspace $X$ of a metric space $\Omega$, one has 

$$
    \dim_{Nag}^s(X, \Omega) \leq \beta
$$

if and only if every disconnected family of closed balls in $\Omega$ of radii $< s$ with centres in $X$ has multiplicity $\leq \beta + 1$.

**Proof.** Necessity. Let $\gamma$ be a finite disconnected family of closed balls in $\Omega$ with centres in $X$. Since by assumption $\dim_{Nag}^s(X, \Omega) \leq \beta$, $\gamma$ admits a subfamily of multiplicity $\leq \beta + 1$ covering all the original centres. But only subfamily that contains centres is $\gamma$ itself.

Sufficiency. Let $\gamma$ be a finite family of closed balls in $\Omega$ with centres in $X$. Denote $C$ the set of centres of those balls. Among all the disconnected subfamilies of $\gamma$ (which exist, e.g. each family containing just one ball is such) there is one, $\gamma'$, with the maximal cardinality of the set $C \cap \cup \gamma'$. We claim that $C \subseteq \gamma'$, which will
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finish the argument. Indeed, if it is not the case, there is a ball, $B \in \gamma$, whose centre, $c \in C$, does not belong to $\cup \gamma'$. Denote $Y = C \cap B$. Remove from $\gamma'$ all the balls with centres in $C \setminus \{c\}$ and add $B$ instead. The new family, $\gamma''$, contains $C \cup \{c\}$, which contradicts the choice of $\gamma'$.

In the definition [6.1] as well as in the proposition [6.2], closed balls can be replaced with open ones. In fact, the statements remain valid if some balls in the families are allowed to be closed, other, open. We have the following.

**Proposition 6.3.** For a subspace $X$ of a metric space $\Omega$, the following are equivalent.

1. $\dim_{Nag}^X(X, \Omega) \leq \beta$,
2. every finite family of balls (some open, others closed) in $\Omega$ with centres in $X$ and radii $< s$ admits a subfamily of multiplicity $\leq \delta + 1$ in $\Omega$ which covers all the centres of the original balls,
3. every finite family of open balls in $\Omega$ having radii $< s$ with centres in $X$ admits a subfamily of multiplicity $\leq \delta + 1$ in $\Omega$ which covers all the centres of the original balls,
4. every disconnected family of open balls in $\Omega$ of radii $< s$ with centres in $X$ has multiplicity $\leq \beta + 1$,
5. every disconnected family of balls (some open, others closed) in $\Omega$ of radii $< s$ with centres in $X$ has multiplicity $\leq \beta + 1$.

**Proof.** [1] $\Rightarrow$ [2]: Let $\gamma$ be a finite family of balls in $\Omega$ with centres in $X$, of radii $< s$, where some of the balls may be open and others, closed. For every element $B \in \gamma$ and each $k \geq 2$, form a closed ball $B_k$ as follows: if $B$ is closed, then $B_k = B$, and if $B$ is open, then define $B_k$ as having the same centre and radius $r(1 - 1/m)$, where $r$ is the radius of $B$. Thus, we always have $B = \bigcap_{k=2}^\infty B_k$. Select recursively a chain of subfamilies

\[ \gamma \supseteq \gamma_1 \supseteq \gamma_2 \supseteq \cdots \supseteq \gamma_k \supseteq \cdots \]

with the properties that for each $k$, the family of closed balls $B_k, B \in \gamma_k$ has multiplicity $\leq \delta + 1$ in $\Omega$ and covers all the centres of the balls in $\gamma$. Since $\gamma$ is finite, starting with some $k$, the subfamily $\gamma_k$ stabilizes, and now it is easy to see that the subfamily $\gamma_k$ itself has the desired multiplicity, and of course covers all the original centres.

[2] $\Rightarrow$ [3]: Trivially true.

[3] $\Rightarrow$ [4]: Same argument as in the proof of necessity in proposition [6.2]

[4] $\Rightarrow$ [5]: Let $\gamma$ be a disconnected family of balls in $\Omega$, of radii $< s$ and centred in $X$. For each $B \in \gamma$ and $\varepsilon > 0$, denote $B_\varepsilon$ an open ball equal to $B$ if $B$ is open, and concentric with $B$ and of the radius $r + \varepsilon$, where $r$ is the radius of $B$, if $B$ is closed. For a sufficiently small $\varepsilon > 0$, the family $\{B_\varepsilon: B \in \gamma\}$ is disconnected, and its radii are all strictly less than $s$, therefore this family has multiplicity $\leq \beta + 1$ by assumption. The same follows for $\gamma$.

[5] $\Rightarrow$ [1]: Follows from proposition [6.2].

**Proposition 6.4.** Let $X$ be a subspace of a metric space $\Omega$, satisfying $\dim_{Nag}^X(X, \Omega) \leq \delta$. Then $\dim_{Nag}^X(\bar{X}, \Omega) \leq \delta$, where $\bar{X}$ is the closure of $X$ in $\Omega$.

**Proof.** Let $\gamma$ be a finite disconnected family of open balls in $\Omega$ of radii $< s$, centred in $\bar{X}$. Let $y \in \Omega$, and let $\gamma'$ consist of all balls in $\gamma$ containing $y$. Choose $\varepsilon > 0$ so small that the open $\varepsilon$-ball around $y$ is contained in every element of $\gamma'$. For every open ball $B \in \gamma'$, denote $y_B$ the centre and $r_B$ the radius. We can also assume that $\varepsilon < r_B$ for each $B \in \gamma'$. Denote $B'$ an open ball of radius $r_B - \varepsilon > 0$, centred at a point $x_B \in X$ satisfying $d(x_B, y_B) < \varepsilon$. Then $B' \subseteq B$, so the family $\{B': B \in \gamma'\}$ is disconnected, and has radii $< s$. Therefore, $y$ only belongs to $\leq \beta + 1$ balls $B', B \in \gamma'$, consequently the cardinality of $\gamma'$ is bounded by $\beta + 1$.

**Proposition 6.5.** If $X$ and $Y$ are two subspaces of a metric space $\Omega$, having finite Nagata dimension in $\Omega$ on the scales $s_1$ and $s_2$ respectively, then $X \cup Y$ has a finite Nagata dimension in $\Omega$, with $\dim_{Nag}(X \cup Y, \Omega) \leq \dim_{Nag}(X, \Omega) + \dim_{Nag}(Y, \Omega)$, on the scale $\min\{s_1, s_2\}$.

**Proof.** Given a finite family of balls $\gamma$ in $\Omega$ of radii $< \min\{s_1, s_2\}$ centred in $X \cup Y$, represent it as $\gamma = \gamma_X \cup \gamma_Y$, where the balls in $\gamma_X$ are centered in $X$, and the balls in $\gamma_Y$ are centred in $Y$. The rest is obvious.
Definition 6.6. A metric space $\Omega$ is said to be \textit{sigma-finite dimensional in the sense of Nagata} if $\Omega = \bigcup_{i=1}^{\infty} X_n$, where every subspace $X_n$ has finite Nagata dimension in $\Omega$ on some scale $s_n > 0$ (where the scales $s_n$ are possibly all different).

Remark 6.7. Due to exercise 6.4 in the above definition we can assume the subspaces $X_n$ to be all closed, in particular Borel.

Remark 6.8. A good reference for a great variety of metric dimensions, including the Nagata dimension, and their applications to measure differentiation theorems, is the article \cite{1}.

Now we will develop a version of the geometric Stone lemma for general metric spaces of finite Nagata dimension.

7. From Stone to Hardy–Littlewood

Lemma 7.1. Let $\sigma = \{x_1, x_2, \ldots, x_n\}$ be a finite sample in a metric space $\Omega$, and let $X$ be a subspace of finite Nagata dimension $\delta$ in $\Omega$ on a scale $s > 0$. Let $\alpha \in (0, 1]$ be any. Let $\sigma' \subseteq \sigma$ be a sub-sample with $m$ points. Assign to every $x_i \in \sigma$ a ball, $B_i$ (which could be open or closed), centred at $x_i$, of radius $s > 0$. Then at most $\alpha^{-1}(\delta + 1)m$ points $x_i$ belonging to $X$ have the property that the proportion of $\geq \alpha$ points of $\sigma'$ is contained in the ball $B_i$:

$$\sharp\{i = 1, 2, \ldots, n: x_i \in X, \sharp(B_i \cap \sigma') \geq \alpha \sharp B_i\} \leq \alpha^{-1}(\delta + 1)m.$$ 

Proof. The family of all the balls $B_i$ having the properties $x_i \in X$ and

$$\sharp(B_i \cap \sigma') \geq \alpha \sharp B_i$$

admits a subfamily of multiplicity $\leq \delta + 1$ which covers all the centres. Each points of $\sigma'$ belongs, at most, to $\delta + 1$ balls from this subfamily. The sum of cardinalities of those balls, times $\alpha$, does not exceed the cardinality of $\sigma'$ times $\delta + 1$ (because each point of $\sigma'$ is counted at most $\delta + 1$ times), from which the conclusion follows. \hfill $\Box$

Remark 7.2. In applications of the lemma, $B_i = B_{r_{KNN}}(x)$ (sometimes the ball will need to be taken be open, sometimes closed, as dictated by the presence of distance ties).

Lemma 7.3. Let $\alpha, \alpha_1, \alpha_2 \geq 0$, $t_1, t_2 \in [0, 1]$, $t_2 \leq 1 - t_1$. Assume that $\alpha_1 \leq \alpha$ and

$$t_1 \alpha_1 + (1 - t_1) \alpha_2 \leq \alpha.$$

Then

$$\frac{t_1 \alpha_1 + t_2 \alpha_2}{t_1 + t_2} \leq \alpha.$$

Proof. If $\alpha_2 \leq \alpha$, the conclusion is immediate. Otherwise, $\alpha_2 > \alpha$, and it follows that

$$t_1 \alpha_1 + t_2 \alpha_2 \leq \alpha - (1 - t_1 - t_2) \alpha_2 \leq (t_2 + t_2) \alpha.$$

\hfill $\Box$

Lemma 7.4. Let $x, x_1, x_2, \ldots, x_n$ be a finite sample (possibly with repetitions), and let $\sigma' \subseteq \sigma$ be a subsample. Let $\alpha \geq 0$, and let $B$ be a closed ball around $x$ of radius $r_{KNN}(x)$ which contains $K$ elements of the sample,

$$\sharp\{i = 1, 2, \ldots, n: x_i \in B\} = K.$$
Suppose that the fraction of points of $\sigma'$ found in $B$ is no more than $\alpha$,
\[
\mathbb{P}\{i: x_i \in \sigma', x_i \in B\} \leq \alpha K,
\]
and that the same holds for the corresponding open ball, $B^o$,
\[
\mathbb{P}\{i: x_i \in \sigma', x_i \in B^o\} \leq \alpha \mathbb{P}\{i: x_i \in B^o\}.
\]
Under the uniform tie-breaking of the $k$ nearest neighbours, the expected fraction of the points of $\sigma'$ found among the $k$ nearest neighbours of $x$ is also equal to $\alpha$.

Proof. We apply lemma 7.3 with $\alpha_1$ and $\alpha_2$ being the fractions of the points of $\sigma'$ found in the closed ball $B$ and on the sphere $S = B \setminus B^o$ respectively, $t_1 = \mathbb{P}B^o/\mathbb{P}B$, and $t_2$ being the fraction of the points of the sphere $S = B \setminus B^o$ to be chosen uniformly and randomly as the $k$ nearest neighbours of $x$ that are still missing in the open ball $B^o$. Now it is enough to observe that the expected fraction of the points of $\sigma'$ among the $k$ nearest neighbours that belong to the sphere is also equal to $\alpha_2$, because they are being chosen randomly, following a uniform distribution.

Now we can give a promised alternative proof of the principal result along the same lines as Stone's original proof in the finite-dimensional Euclidean case.

**Theorem 7.5.** The $k$ nearest neighbour classifier under the uniform distance tie-breaking is universally consistent in every metric space having sigma-finite Nagata dimension, when $n, k \to \infty$ and $k/n \to 0$.

Proof. Represent $\Omega = \bigcup_{i=1}^{\infty} Y_n$, where $Y_n$ have finite Nagata dimension in $\Omega$. According to proposition 6.5, we can assume that $Y_n$ form an increasing chain, and proposition 6.4 allows to assume that $Y_n$ are Borel sets. Let $\mu$ and $\eta$ be any on $\Omega$. Given $\epsilon > 0$, there exists $l$ such that $\mu(Y_l) \geq 1 - \epsilon/2$, and there is a compact subset $K \subseteq Y_l$ such that $\eta|K$ is continuous and $\mu(K) \geq 1 - \epsilon$. The function $\eta|K$ extends to a uniformly continuous function $g$ over $\Omega$.

In the spirit of the proof of Stone’s theorem 3.6 it is enough to limit the term
\[
(B) = \mathbb{E}\left[ \sum_{i=1}^{l} \left\{ \mathbb{P}\{\eta(X_i) - g(X_i)\} : X_i \in k-NN(X), X_i \not\in K \right\} \right]
\]
where $\mu_i$ is the uniform measure on the set $\{0, 1, 2, \ldots, n\}$, and we denote $X_0 = X$. We will treat the term $(B)$ as the sum of two conditional expectations, $(B_1)$ and $(B_2)$, according to whether the $k$ nearest neighbours of $X_j$ inside the sample $\{X_0, X_1, \ldots, X_{j-1}, X, X_{j+1}, \ldots, X_n\}$ contain more or less than $\sqrt{\delta k}$ elements belonging to $U = \Omega \setminus K$.

Applying lemma 7.1 to the closed balls of radius $k$ nearest $X_j$ as well as the corresponding open balls, together with lemma 7.4 we get in the first case
\[
(B_1) = \mathbb{E}\sum_{i=1}^{l} \left( \mathbb{P}\{\eta(X_i) - g(X_i)\} : X_i \in k-NN(X_j), X_i \not\in K, \right) \geq k\sqrt{\delta}
\]
\[
\leq \mathbb{E}\frac{1}{k} 2e^{-1/2}(\delta + 1)^{1/2} \frac{1}{n} \mathbb{P}\{i: X_i \in k-NN(X_j), X_i \not\in K\} \geq k\sqrt{\delta}
\]
\[
\leq 2e^{-1/2}(\delta + 1)^{1/2} = 2\sqrt{\delta}(\delta + 1),
\]
where we have used the fact that the sum does not exceed $k$, as well as the Law of Large Numbers. In the second case,

$$(B_2) = \mathbb{E} \mathbb{E}_{j \sim \mu_j} \left[ \frac{1}{k} \sum \{ r(X_i) - g(X_i) : X_i \in k-\text{NN}(X_j), \: X_j \in K, \: X_i \notin K, \right.$$ 

$$i \in \{0, 1, \ldots, n\} \setminus \{j\} \mid \#\{i : X_i \in k-\text{NN}(X_j), \: X_i \notin K\} \leq k\sqrt{\epsilon} \right]$$

$$\leq \frac{1}{k} k\sqrt{\epsilon} = \sqrt{\epsilon}. \quad \square$$

**Open Question**

The following question remains open. Let $\Omega$ be a separable complete metric space in which the $k$-NN classifier is universally consistent. Does it follow that $\Omega$ is sigma-finite dimensional in the sense of Nagata?

A positive answer would imply, modulo the results of Cérou and Guyader [2] and of Preiss [14], that a separable metric space $\Omega$ satisfies the weak Lebesgue–Besicovitch differentiation property for every Borel sigma-finite locally finite measure if and only if $\Omega$ satisfies the strong Lebesgue–Besicovitch differentiation property for every Borel sigma-finite locally finite measure, which would answer an old question asked by Preiss in [14].

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