ON GROUPS ACTING ON CONTRACTIBLE SPACES WITH STABILIZERS OF PRIME POWER ORDER

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Abstract. Let \( \mathcal{F} \) denote the class of finite groups, and let \( \mathcal{P} \) denote the subclass consisting of groups of prime power order. We study group actions on topological spaces in which either (1) all stabilizers lie in \( \mathcal{P} \) or (2) all stabilizers lie in \( \mathcal{F} \). We compare the classifying spaces for actions with stabilizers in \( \mathcal{F} \) and \( \mathcal{P} \), the Kropholler hierarchies built on \( \mathcal{F} \) and \( \mathcal{P} \), and group cohomology relative to \( \mathcal{F} \) and to \( \mathcal{P} \). In terms of standard notations, we show that \( \mathcal{F} \subset H_1 \mathcal{P} \subset H_1 \mathcal{F} \), with all inclusions proper; that \( H_2 \mathcal{F} = H_2 \mathcal{P} \); that \( H^*(G; -) = \mathcal{P} H^*(G; -) \); and that \( E_\mathcal{P} G \) is finite-dimensional if and only if \( E_\mathcal{F} G \) is finite-dimensional and every finite subgroup of \( G \) is in \( \mathcal{P} \).

1. Introduction

Let \( \mathcal{F} \) denote a class of groups, by which we mean a collection of groups which is closed under isomorphism and taking subgroups. A \( G \)-CW-complex \( X \) is said to be a model for \( E_\mathcal{F} G \), the classifying space for actions of \( G \) with stabilizers in \( \mathcal{F} \), if for each \( H \leq G \), one has that the fixed point set \( X^H \) is contractible for \( H \in \mathcal{F} \) and is empty for \( H \notin \mathcal{F} \). The most common classes considered are the class of trivial groups and the class \( \mathcal{F} \) consisting of all finite groups. In these cases \( E_\mathcal{F} G \) is often denoted \( EG \) and \( EG \) respectively. Note that \( EG \) is the total space of the universal principal \( G \)-bundle, or equivalently the universal covering space of an Eilenberg-Mac Lane space for \( G \). The space \( EG \) is called the classifying space for proper actions of \( G \). Recently there has been much interest in finiteness conditions for the spaces \( E_\mathcal{F} G \), especially for \( EG \). Milnor and Segal’s constructions of \( EG \) both generalize easily to construct models for any \( E_\mathcal{F} G \), and one can show that any two models for \( E_\mathcal{F} G \) are naturally equivariantly homotopy equivalent.

For some purposes the structure of the fixed point sets for subgroups in \( \mathcal{F} \) is irrelevant. For example, a group is in Kropholler’s class \( h_1 \mathcal{F} \) if there is any finite-dimensional contractible \( G \)-CW-complex \( X \) with

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all stabilizers in \( \mathcal{F} \). The class \( \mathbf{h}_1 \mathcal{F} \) is the first stage of a hierarchy whose union is Kropholler’s class \( \mathbf{h} \mathcal{F} \) of hierarchically decomposable groups \([10]\). (These definitions were first considered for the class \( \mathfrak{F} \) of all finite groups, but work for any class \( \mathcal{F} \).)

A priori, the class \( \mathbf{h}_1 \mathcal{F} \) may contain groups \( G \) that do not admit a finite-dimensional model for \( E_\mathcal{F} G \), and we shall give such examples in the case when \( \mathcal{F} = \mathfrak{P} \), the class of groups of prime power order. By contrast, in the case when \( \mathcal{F} = \mathfrak{F} \), no group \( G \) is known to lie in \( \mathbf{h}_1 \mathfrak{F} \) without also admitting a finite-dimensional model for \( E G \). A construction due to Serre shows that every group \( G \) in \( \mathbf{h}_1 \mathfrak{F} \) that is virtually torsion-free has a finite-dimensional \( E G \) \([4]\), and the authors have given examples of \( G \) for which the minimal dimension of a contractible \( G \)-CW-complex is lower than the minimal dimension of a model for \( E G \) \([14]\). These examples \( G \) also have the property that they admit a contractible \( G \)-CW-complex with finitely many orbits of cells, but that they do not admit any model for \( E G \) with finitely many orbits of cells.

Throughout this paper, \( \mathfrak{F} \) will denote the class of finite groups, and \( \mathfrak{P} \) will denote the class of finite groups of prime power order. We compare the classifying space for \( G \)-actions with stabilizers in \( \mathfrak{P} \) with the more well-known \( E \mathfrak{F} G \), and we compare the Kropholler hierarchies built on \( \mathfrak{F} \) and \( \mathfrak{P} \). We show that a finite group \( G \) that is not of prime power order cannot admit a finite-dimensional \( E_\mathfrak{P} G \), but that every finite group is in \( \mathbf{h}_1 \mathfrak{P} \). We also construct a group that is in \( \mathbf{h}_1 \mathfrak{F} \) but not in \( \mathbf{h}_1 \mathfrak{P} \), and we show that \( \mathbf{h} \mathfrak{P} = \mathbf{h} \mathfrak{P} \).

In the final section we shall contrast this with cohomology relative to all finite subgroups. The relative cohomological dimension can be viewed as a generalisation of the virtual cohomological dimension, since for virtually torsion free groups these are equal, see \([17]\). By a result of Bouc \([2, 12]\) it follows that groups belonging to \( \mathbf{h}_1 \mathfrak{F} \) have finite relative cohomological dimension, but the converse it not known. In contrast to our results concerning classifying spaces, we show that cohomology relative to subgroups in \( \mathfrak{F} \) is naturally isomorphic to cohomology relative to subgroups in \( \mathfrak{P} \).

2. Classifying spaces for actions with stabilizers in \( \mathfrak{P} \)

**Theorem 2.1.** Let \( G \) be a finite group. Then \( G \) has a finite dimensional model for \( E_\mathfrak{P} G \) if and only if \( G \) has prime power order.

**Proof.** If \( G \) has prime power order, then a single point may be taken as a model for \( E_\mathfrak{P} G \). Now let \( G \) be an arbitrary finite group, let \( p \) be a prime dividing the order of \( G \), and assume that there is a \( p \)-subgroup
$P < G$, such that $N_G(P)$ is not a $p$-group. Then the Weyl-group $WP = N_G(P)/P$ contains a subgroup $H$ of order prime to $p$. Assume $G$ has a finite dimensional model for $E_{\mathfrak{P}}G$, $X$ say. Then the augmented cellular chain complex of the $P$-fixed point set, $X^P$, is a finite length resolution of $\mathbb{Z}$ by free $H$-modules. This gives a contradiction, since $\mathbb{Z}$ has infinite projective dimension as an $H$-module for any non-trivial finite group $H$.

Therefore we may suppose that for each subgroup $P \leq G$ which lies in $\mathfrak{P}$, the normalizer $N_G(P)$ is also in $\mathfrak{P}$. It follows from the Frobenius normal $p$-complement theorem [7, 5.26] that $G$ has a normal $p$-complement for each prime $p$. Hence $G$ is nilpotent and equal to the direct product of its Sylow subgroups. If $P$ is a non-trivial Sylow subgroup of $G$, it follows that $G = N_G(P)$ is in $\mathfrak{P}$. □

Remark 2.2. The above proof was suggested to the authors by Yoav Segev. It is considerably shorter than our original proof, which did not quote the Frobenius normal $p$-complement theorem.

Corollary 2.3. For a group $G$, the following are equivalent.

(i) $G$ admits a finite-dimensional $E_{\mathfrak{P}}G$;
(ii) Every finite subgroup of $G$ is in $\mathfrak{P}$ and $G$ admits a finite-dimensional $E_G$.

□

Remark 2.4. We conclude the section with a remark on the type of $E_{\mathfrak{P}}G$. It can proved analogously to Lück’s proof for $E_G$ [15] that a group $G$ admits a finite type model for $E_{\mathfrak{P}}G$ if and only if $G$ has finitely many conjugacy classes of groups of prime power order and the Weyl-groups $N_G(P)/P$ for all subgroups $P$ of prime power order are finitely presented and of type $FP_\infty$. Hence any group admitting a finite type $E_G$ also admits a finite type $E_{\mathfrak{P}}G$. Recall that a finite extension of a group admitting a finite model for $EG$ always has finitely many conjugacy classes of subgroups of prime power order [4, IX.13.2]. Hence the groups exhibited in [14, Example 7.4] are groups admitting a finite type $E_{\mathfrak{P}}G$ which do not admit a finite type $E_G$.

This behaviour is in stark contrast to that of $E_{\mathcal{VC}}G$, the classifying space with virtually cyclic isotropy. Any group admitting a finite dimensional model for $E_{\mathcal{VC}}G$ admits a finite dimensional model for $E_G$, see [16] and the converse also holds for a large class of groups including all polycyclic-by-finite and all hyperbolic groups [8, 16]. Furthermore, any group admitting a finite type model for $E_{\mathcal{VC}}G$ also admits a finite type model for $E_G$ [9], but it is conjectured [8] that any group admitting a finite model for $E_{\mathcal{VC}}G$ has to be virtually cyclic. This has been
shown for a class of groups containing all hyperbolic groups [8] and for elementary amenable groups [9].

3. The hierarchies $\mathcal{H}$ and $\mathcal{P}$

**Proposition 3.1.** Let $X$ be a finite dimensional contractible $G$-CW-complex such that all stabilizers are finite. If there is a bound on the orders of the stabilizers then there exists a finite dimensional contractible $G$-CW-complex $Y$ and an equivariant map $f : Y \to X$ such that $Y^H = \emptyset$ if $H$ is not a $p$-group.

**Proof.** Using the equivariant form of the simplicial approximation theorem, we may assume that $X$ is a simplicial $G$-CW-complex. To simplify notation the phrase ‘$G$-space’ shall mean ‘simplicial $G$-CW-complex’ and ‘$G$-map’ will mean ‘$G$-equivariant simplicial map’ throughout the rest of the proof. The space $Y$ will be a $G$-space in this sense and the map $f : Y \to X$ will be a $G$-map in this sense. The $G$-space $Y$ is constructed in two stages. Firstly, for each finite $K \leq G$ we build a finite-dimensional contractible $K$-space $Y_K$ with the property that all simplex stabilizers in $Y_K$ lie in $\mathcal{P}$.

Suppose for now that each such $K$-space $Y_K$ has been constructed. Using the $G$-equivariant form of the construction used in [11, Section 8] the space $Y$ is constructed as follows. Let $I$ be an indexing set for the $G$-orbits of vertices in $X$. For each $i \in I$, let $v_i$ be a representative of the corresponding orbit, and let $K_i$ be the stabilizer of $v_i$. Let $X^0$ denote the 0-skeleton of $X$. Define a $G$-space $Y^0$ by

$$Y^0 = \coprod_{i \in I} G \times_{K_i} Y_{K_i},$$

and define a $G$-map $f : Y^0 \to X^0$ by $f(g, y) = g.v_i$ for all $i \in I$, for all $g \in G$ and for all $y \in Y_{K_i}$. For each vertex $w$ of $X$, let $Y(w) = f^{-1}(w) \subset Y^0$. Each $Y(w)$ is a contractible subspace of $Y^0$, and the stabilizer of $w$ acts on $Y(w)$.

Now for $\sigma = (w_0, \ldots, w_n)$ an $n$-simplex of $X$, define a space $Y(\sigma)$ as the join

$$Y(\sigma) = Y(w_0) * Y(w_1) * \cdots * Y(w_n).$$

Each vertex of $Y(\sigma)$ is already a vertex of one of the $Y(w_i)$, and so the map $f : Y^0 \to X^0$ defines a unique simplicial map $f : Y(\sigma) \to \sigma$. By construction, whenever $\tau$ is a face of $\sigma$, the space $Y(\tau)$ is identified with a subspace of $Y(\sigma)$. This allows us to define $Y$ and $f : Y \to X$ as the colimit over the simplices $\sigma$ of $X$ of the subspaces $Y(\sigma)$, and to define $f : Y \to X$, which is a $G$-map of $G$-spaces. Since each $Y(\sigma)$ is
contractible, it follows that $f$ is a homotopy equivalence, and hence $Y$ is also contractible (see [11, Corollary 8.6]).

It remains to build the $K$-space $Y_K$ for each finite group $K \leq G$. In the case when $K \in \Psi$ we may take a single point to be $Y_K$, and so we may suppose that $K \notin \Psi$. Fix such a subgroup $K$, and suppose that we are able to construct a finite-dimensional contractible $K$-space $Z_K$ in which each stabilizer is a proper subgroup of $K$. We may assume by induction that for each $L < K$ we have already constructed the $L$-space $Y_L$. The $K$-space $Y_K$ can now be constructed from $Z_K$ and the spaces $Y_L$ using a process similar to the construction of $Y$ from $X$ and the spaces $Y_K$. It remains to construct the $K$-space $Z_K$.

An explicit construction of an $K$-space $Z_K$ with the required properties is given in [13]. We therefore provide only a sketch of the argument. We may assume that $K$ is not in $\Psi$. Let $S$ be the unit sphere in the reduced regular complex representation of $K$, so that $S$ is a topological space with $K$-action such that the stabilizer of every point of $S$ is a proper subgroup of $K$. Since $K$ is not in $\Psi$, there are $K$-orbits in $S$ of coprime lengths. Using this property, it can be shown that the sphere $S$ admits an $K$-equivariant self-map $g : S \to S$ of degree zero. The $K$-space $Z_K$ is defined to be the infinite mapping telescope (suitably triangulated) of the map $g$. □

**Corollary 3.2.** If $G$ is in $h_1\mathfrak{F}$ and there is a bound on the orders of the finite subgroups of $G$, then $G$ is in $h_1\Psi$. □

**Remark 3.3.** In Proposition 3.1, the bound on the orders of the stabilizers of $X$ is used only to give a bound on the dimensions of the spaces $Y_K$. In Theorem 3.8 we shall show that $h_1\mathfrak{F} \neq h_1\Psi$.

**Remark 3.4.** The construction in Proposition 3.1 does not preserve cocompactness, because for most finite groups $K$, the space $Y_K$ used in the construction cannot be chosen to be finite. A result similar to Proposition 3.1 but preserving cocompactness can be obtained by replacing $\Psi$ by a larger class $\mathfrak{O}$ of groups. Here $\mathfrak{O}$ is defined to be the class of $\Psi$-by-cyclic-by-$\Psi$-groups. A theorem of Oliver [20] implies that any finite group $K$ that is not in $\mathfrak{O}$ admits a finite contractible $K$-CW-complex $Z'_K$ in which all stabilizers are proper subgroups of $K$. Applying the same argument as in the proof of Proposition 3.1, one can show that given any contractible $G$-CW-complex $X$ with all stabilizers in $\mathfrak{F}$, there is a contractible $G$-CW-complex $Y'$ with all stabilizers in $\mathfrak{O}$ and a proper equivariant map $f' : Y' \to X$. (By proper, we mean that the inverse image of any compact subset of $X$ is compact.)
For $X$ a $G$-CW-complex with stabilizers in $\mathfrak{P}$, and $p$ a prime, let $X_{\text{sing}(p)}$ denote the subcomplex consisting of points whose stabilizer has order divisible by $p$. For $G$ a group and $p$ a prime, let $S_p(G)$ denote the poset of non-trivial finite $p$-subgroups of $G$.

**Proposition 3.5.** Suppose that $X$ is a finite-dimensional contractible $G$-CW-complex with all stabilizers in $\mathfrak{P}$. For each prime $p$, the mod-$p$ homology of $X_{\text{sing}(p)}$ is isomorphic to the mod-$p$ homology of the (realization of the) poset $S_p(G)$.

**Proof.** Fix a prime $p$, and let $S$ denote the realization of the poset $S_p(G)$. For $P$ a non-trivial $p$-subgroup of $G$, let $X^P$ denote the points fixed by $P$, and let $S_{\geq P}$ denote the realization of the subposet of $S_p(G)$ consisting of all $p$-subgroups that contain $P$. By the P. A. Smith theorem [3], each $X^P$ is mod-$p$ acyclic. Each $S_{\geq P}$ is contractible since it is equal to a cone with apex $P$. Let $P$ and $Q$ be $p$-subgroups of $G$, and let $R = (P, Q)$, the subgroup of $G$ generated by $P$ and $Q$. If $R$ is a $p$-group then $X^P \cap X^Q = X^R$, and otherwise $X^P \cap X^Q$ is empty. Similarly, $S_{\geq P} \cap S_{\geq Q} = S_{\geq R}$ if $R$ is a $p$-group and $S_{\geq P} \cap S_{\geq Q}$ is empty if $R$ is not a $p$-group.

Since each $X^P$ is mod-$p$ acyclic, the mod-$p$ homology $H_*(X_{\text{sing}(p)})$ is isomorphic to the mod-$p$ homology of the nerve of the covering $X_{\text{sing}(p)} = \bigcup P X^P$. Similarly, the mod-$p$ homology $H_*(S)$ is isomorphic to the mod-$p$ homology of the nerve of the covering $S = \bigcup P S_{\geq P}$. By the remarks in the first paragraph, these two nerves are isomorphic. $\square$

**Proposition 3.6.** Let $k$ be a finite field, and let $G$ be the group of $k$-points of a reductive algebraic group over $k$, whose commutator subgroup has $k$-rank $n$. (For example, $G = SL_{n+1}(k)$, or $GL_{n+1}(k)$.) Any finite-dimensional contractible $G$-CW-complex with stabilizers in $\mathfrak{P}$ has dimension at least $n$.

**Proof.** The hypotheses on $G$ imply that $G$ acts on a spherical building $\Delta$ of dimension $n - 1$ [1, 5, Appendix on algebraic groups]. Any such building is homotopy equivalent to a wedge of $(n - 1)$-spheres. Quillen has shown that $\Delta$ is homotopy equivalent to the realization of $S_p(G)$, where $p$ is the characteristic of the field $k$ [21, Proposition 2.1 and Theorem 3.1]. It follows that $S_p(G)$ is homotopy equivalent to a wedge of $(n - 1)$-spheres, and in particular the mod-$p$ homology group $H_{n-1}(S_p(G))$ is non-zero.

Now suppose that $X$ is a finite-dimensional contractible $G$-CW-complex with stabilizers in $\mathfrak{P}$. Using Proposition 3.5, one sees that
the mod-$p$ homology group $H_{n-1}(X_{\text{sing}(p)})$ is non-zero. It follows that $X$ must have dimension at least $n$. □

Remark 3.7. In [22] it is shown that if $G$ is a finite simple group of Lie type, of Lie rank $n$, then any contractible $G$-CW-complex of dimension strictly less than $n$ contains a point fixed by $G$. (Theorem 1 of [22] contains the additional hypothesis that the $G$-CW-complex should be finite, but this is not used in the proof.) A similar argument to that used in [22, Theorem 2] was used in [13] to show that when $G = SL_{n+1}(\mathbb{F}_p)$, every contractible $G$-CW-complex without a global fixed point has dimension at least $n$. Note that Proposition 3.6 applies in greater generality than these results. For example, the Conner-Floyd construction [6] shows that whenever the multiplicative group of $k$ does not have prime-power order, there is, for any $n \geq 1$, a 4-dimensional contractible $GL_n(k)$-CW-complex without a global fixed point.

Theorem 3.8. There are the following strict containments and equalities between classes of groups:

(i) $\mathfrak{F} \subseteq H_1 \mathfrak{P}$;
(ii) $H_1 \mathfrak{P} \subseteq H_1 \mathfrak{F}$;
(iii) $H_\mathfrak{F} = H_\mathfrak{P}$.

Proof. Corollary 3.2 shows that $\mathfrak{F} \subseteq H_1 \mathfrak{P}$. The free product of two cyclic groups of prime order is in $H_1 \mathfrak{P}$ and is not finite. The claim that $H_\mathfrak{F} = H_\mathfrak{P}$ follows from the inequalities $\mathfrak{P} \subseteq \mathfrak{F} \subseteq H_1 \mathfrak{P}$, and the claim $H_1 \mathfrak{P} \subseteq H_1 \mathfrak{F}$ follows from $\mathfrak{P} \subseteq \mathfrak{F}$.

It remains to exhibit a group $G$ that is in $H_1 \mathfrak{F}$ but not in $H_1 \mathfrak{P}$. Let $G = SL_\infty(\mathbb{F}_p)$, the direct limit of the groups $G_n = SL_n(\mathbb{F}_p)$, where $G_n$ is included in $G_{n+1}$ as the ‘top corner’. As a countable locally-finite group, $G$ acts with finite stabilizers on a tree. (Explicitly, the vertex set $V$ and edge set $E$ are both equal as $G$-sets to the disjoint union of the sets of cosets $G/G_1 \cup G/G_2 \cup \cdots$, with the edge $gG_i$ joining the vertex $gG_i$ to the vertex $gG_{i+1}$.) It follows that $G \in H_1 \mathfrak{F}$. By Proposition 3.6, $G$ cannot be in $H_1 \mathfrak{P}$. □

Remark 3.9. Let $G$ be a group in $H_\mathfrak{F}$ that is also of type $FP_\infty$. By a result of Kropholler [10], there is a bound on the orders of finite subgroups of $G$, and Kropholler-Mislin show that $G$ is in $H_1 \mathfrak{F}$ [11]. Corollary 3.2 shows that $G$ is in $H_1 \mathfrak{P}$.

4. Cohomology relative to a class of groups

Let $\Delta$ denote a $G$-set, and let $\mathbb{Z}\Delta$ denote the corresponding $G$-module. For $\delta \in \Delta$, we write $G_\delta$ for the stabilizer of $\delta$. A short
exact sequence $A \inject B \onto C$ of $G$-modules is said to be $\Delta$-split if and only if it splits as a sequence of $G_\delta$-modules for each $\delta \in \Delta$. Equivalently, the sequence is $\Delta$-split if and only if the following sequence of $\mathbb{Z}G$-modules splits: $A \otimes \mathbb{Z} \Delta \inject B \otimes \mathbb{Z} \Delta \onto C \otimes \mathbb{Z} \Delta$ [18].

We say a $G$ module is $\Delta$-projective if it is a direct summand of a $G$-module of the form $N \otimes \mathbb{Z} \Delta$, where $N$ is an arbitrary $G$-module. $\Delta$-projectives satisfy analogue properties to ordinary projectives. Furthermore, for each $\delta$, and each $G_\delta$-module $M$, the induced module $\text{Ind}_{G_\delta}^G M$ is $\Delta$-projective. Given two $\Delta$-sets $\Delta_1$ and $\Delta_2$ and a $G$-map $\Delta_1 \rightarrow \Delta_2$ then $\Delta_1$-projectives are $\Delta_2$-projective and $\Delta_2$-split sequences are $\Delta_1$-split. For more detail the reader is referred to [18].

Now suppose that $\mathcal{F}$ is a class of groups closed under taking subgroups. We consider $G$-sets $\Delta$ satisfying the following condition, for all $H \leq G$:

\[ \Delta^H \neq \emptyset \iff H \in \mathcal{F}. \]

There are $G$-maps between any two $G$-sets satisfying condition (*), and so we may define an $\mathcal{F}$-projective module to be a $\Delta$-split module for any such $\Delta$. Similarly, an $\mathcal{F}$-split exact sequence of $G$-modules is defined to be a $\Delta$-split sequence. If $\Delta$ satisfies (*) and $M$ is any $G$-module, the module $M \otimes \mathbb{Z} \Delta$ is $\mathcal{F}$-projective and admits an $\mathcal{F}$-split surjection to $M$. This leads to a construction of homology relative to $\mathcal{F}$. An $\mathcal{F}$-projective resolution of a module $M$ is an $\mathcal{F}$-split exact sequence

\[ \cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0, \]

where all $P_i$ are $\mathcal{F}$-projective. Group cohomology relative to $\mathcal{F}$, denoted $\mathcal{F}H^*(G; N)$ can now be defined as the cohomology of the cochain complex $\text{Hom}_G(P_\bullet, N)$, where $P_\bullet$ is an $\mathcal{F}$-projective resolution of $\mathbb{Z}$.

We say that a module $M$ is of type $\mathcal{F}\text{FP}_n$ if $M$ admits an $\mathcal{F}$-projective resolution in which $P_i$ is finitely generated for $0 \leq i \leq n$. It has been shown that modules of type $\mathcal{F}\text{FP}_n$ are of type $\text{FP}_n$ [18]. We will say that a group $G$ is of type $\mathcal{F}\text{FP}_n$ if the trivial $G$-module $\mathbb{Z}$ is of type $\mathcal{F}\text{FP}_n$.

We now specialize to the cases when $\mathcal{F} = \mathfrak{F}$ and $\mathcal{F} = \mathfrak{P}$.

**Theorem 4.1.** The following properties hold.

(i) A short exact sequence of $G$-modules is $\mathfrak{F}$-split if and only if it is $\mathfrak{P}$-split.

(ii) A $G$ module is $\mathfrak{F}$-projective if and only if it is $\mathfrak{P}$-projective.

(iii) $\mathfrak{F}H^*(G, -) \cong \mathfrak{P}H^*(G, -)$

**Proof:** (i) It is obvious that any $\mathfrak{F}$-split sequence is $\mathfrak{P}$-split, and the converse follows from a standard averaging argument. Let $H$ be an
arbitrary finite subgroup of $G$. Then $|H| = \prod_{i=1}^{n} p_i^{a_i}$ where $p_i$ are distinct primes and $0 < a_i \in \mathbb{Z}$. For each $i$, let $n_i$ be the index $n_i = [H : P_i]$. Now consider a $\mathcal{P}$-split surjection $A \twoheadrightarrow B$. Let $\sigma_i$ be a $P_i$-splitting of $\pi$, and define a map $s_i$ by summing $\sigma_i$ over the cosets of $P_i$:

$$s_i(b) = \sum_{t \in H/P_i} t \sigma_i(t^{-1}b).$$

For each $P_i$ we obtain a map $s_i : B \rightarrow A$, such that $\pi \circ s_i = n_i \times id_B$. There exist $m_i \in \mathbb{Z}$ so that $\sum_i m_i n_i = 1$, and the map $s = \sum_i m_i s_i$ is the required $H$-splitting.

(ii) It is obvious that a $\mathcal{P}$-projective module is $\mathfrak{C}$-projective. Now let $P$ be $\mathfrak{C}$-projective. We may take a $\mathcal{P}$-split surjection $M \twoheadrightarrow P$ with $M$ a $\mathcal{P}$-projective. By (i) this surjection is $\mathfrak{C}$-split, and hence split. Thus $P$ is a direct summand of a $\mathcal{P}$-projective and so is $\mathcal{P}$-projective.

(iii) now follows directly from (i) and (ii). $\Box$

**Proposition 4.2.** A group $G$ is of type $\mathfrak{C}FP_0$ if and only if $G$ has only finitely many conjugacy classes of subgroups of prime power order.

**Proof:** Suppose that $G$ has only finitely many conjugacy classes of subgroups in $\mathcal{P}$. Let $I$ be a set of representatives for the conjugacy classes of $\mathcal{P}$-subgroups and set

$$\Delta_0 = \bigsqcup_{P \in I} G/P.$$ 

This $G$-set satisfies condition $(\ast)$ for $\mathcal{P}$ and therefore the surjection $\mathbb{Z}\Delta_0 \twoheadrightarrow \mathbb{Z}$ is $\mathfrak{C}$-split and also $\mathbb{Z}\Delta_0$ is finitely generated.

To prove the converse we consider an arbitrary $\mathfrak{C}$-split surjection $P_0 \twoheadrightarrow \mathbb{Z}$ with $P_0$ a finitely generated $\mathfrak{C}$-projective. As in [18, 6.1] we can show that $P_0$ is a direct summand of a module $\bigoplus_{\delta \in \Delta_f} \text{Ind}^G_{G_\delta} P_\delta$, where $\Delta_f$ is a finite $G$-set, the $G_\delta$ are finite groups and $P_\delta$ are finitely generated $G_\delta$-modules. Therefore we might assume from now on that $P_0$ is of the above form. Since there is a $G$-map $\Delta_f \rightarrow \Delta$, where $\Delta$ satisfies condition $(\ast)$ the $\mathfrak{C}$-split surjection $P_0 \twoheadrightarrow \mathbb{Z}$ is also $\Delta_f$-split [18]. Consider now the following commutative diagram:

$$\begin{array}{ccc}
P_0 & \overset{\varepsilon}{\longrightarrow} & \mathbb{Z} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\mathbb{Z}\Delta_f & \overset{\varepsilon_f}{\longrightarrow} & \mathbb{Z}
\end{array}$$
That we can find such an $\alpha$ follows from the fact that $\varepsilon$ is $\Delta_f$-split, and $\beta$ exists since $P_0$ is $\Delta_f$-projective being a direct sum of induced modules, induced from $G_\delta$, $(\delta \in \Delta_f)$ to $G$.

As a next step we’ll show that $\varepsilon_f$ is $\mathfrak{F}$-split. Take an arbitrary finite subgroup $H$ of $G$ and show that $\varepsilon_f$ splits when restricted to $H$. Since $\varepsilon$ is split by $s$, say, when restricted to $H$ we can define the required splitting by $\beta \circ s$.

Now let $P$ be an arbitrary $p$-subgroup of $G$. Since the module $\mathbb{Z}[G/P]$ is $\mathfrak{F}$-projective, there exists a $G$-map $\varphi$, such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \\
\varphi \downarrow & & \downarrow \\
\mathbb{Z}[G/P] & \longrightarrow & \mathbb{Z}
\end{array}
\]

The image $\varphi(P)$ of the identity coset $P$ is a point of $\mathbb{Z}\Delta$ fixed by the action of $P$. If $H$ is any group and $\mathbb{Z}\Omega$ is any permutation module, then the $H$-fixed points are generated by the orbit sums $H.\omega$. Hence $P$ must stabilize some point of $\Delta_f$, since otherwise we would have that $p$ divides $\varepsilon_f\varphi(P) = \varepsilon\alpha(P) = 1$, a contradiction. It follows that $P$ is a subgroup of $G_\delta$ for some $\delta \in \Delta_f$. \qed

Note that being of type $\mathfrak{F}\text{FP}_0$ does not imply that there are finitely many conjugacy classes of finite subgroups. In fact, the authors have examples with infinitely many conjugacy classes of finite subgroups, see [14]. Nevertheless this gives rise to the following conjecture:

**Conjecture 4.3.** A group $G$ is of type $\mathfrak{F}\text{FP}_\infty$ if and only if $G$ is of type $\text{FP}_\infty$ and has finitely many conjugacy classes of $p$-subgroups.

It is shown in [18] that any $G$ of type $\mathfrak{F}\text{FP}_\infty$ is of type $\text{FP}_\infty$, which together with Proposition 4.2 proves one implication in the above conjecture.

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