Kinematics and dynamics of quantum walks in terms of systems of imprimitivity

Radhakrishnan Balu

Army Research Laboratory Adelphi, MD, 21005-5069, United States of America
Computer Science and Electrical Engineering, University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, United States of America

E-mail: radhakrishnan.balu.civ@mail.mil and radbalu1@umbc.edu

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Abstract

We build systems of imprimitivity (SI) in the context of quantum walks and provide geometric constructions for their configuration space. We consider three systems, an evolution of unitaries from the group $SO_3$ on a low dimensional de Sitter space where the walk happens on the dual of $SO_3$, a standard quantum walk whose SI live on the orbits of stabilizer subgroups (little groups) of semidirect products describing the symmetries of $1+1$ spacetime, and automorphisms (walks are specific automorphisms) on distance-transitive graphs as an application of the constructions.

Keywords: group representations, quantum walk, systems of imprimitivity, quantum graphs, fiber bundles

1. Introduction

Systems of imprimitivity (SI) for a group can be used to describe a class of systems within unitary equivalence in a concise manner. The simplest example is to describe the Weyl commutation relation of a class of systems undergoing Shr"odinger evolution. The SI in this case is defined with respect to the additive group of the real line to describe the symmetry of the time variable of the dynamics. After setting up the notions of SI formally we will discuss this example in detail. Some early examples for finite groups appeared in the works of von Neumann [1], which are ergodic systems. Wigner used these notions informally in his study on the theory of representations of the inhomogeneous Lorentz (Poincaré) group [2], which are transitive systems that are dynamical systems with trajectories dense in the configuration space. Mackey [3] provided the systematic development of SI for systems with infinite degrees of freedom. There are many fruitful applications of SI, including the establishment of equivalence between matrix and wave mechanics, an elegant description of the relativistic wave equation, and an unified approach to the theory of representations of the commutation
rules. Our interest in SI is primarily due to their application to infinite unitary representations of the Poincaré group, which are induced by subgroups or little groups in physics literature, in the context of quantum walks. The concept of localization, where the position operator is properly defined in a manifold, and covariance in a relativistic sense of systems can be completely characterized by systems of imprimitivity. These are representations of a group induced by representations of subgroups, more specifically stabilizer subgroups at a point in the orbit of the subgroup. Now we have a recipe called a Mackey machine to synthesize SI, a class of unitarily equivalent systems, using representations induced by subgroups, which in our primary example will be the little groups of the inhomogeneous Lorentz group. Systems of imprimitivity are a more fundamental characterization of dynamical systems, when the configuration space of a quantum systems is described by a group, from which infinitesimal forms in terms of differential equations (Shrödinger, Heisenberg, Dirac, etc), and the canonical commutation relations can be derived. For example, using SI arguments it can be shown [27] that massless elementary particles with spin less than or equal to 1 cannot have well defined position operators, that is photons are not localizable. As discrete quantum walks [7] can lead to Dirac evolution in the continuum, with proper choice of coin parameters and specific initial conditions [19] and [11], in this work we construct systems of imprimitivity for such quantum walks and describe the governing differential equations. Our treatment of this subject is based on the work of Varadarajan [5] where the Dirac equation is derived in the most rigorous fashion. Varadarajan constructed this program first by establishing quantum logics and geometry on orthomodular lattices and further developed the configuration space of a Dirac particle, endowed with a topology, measure space, and a $G$-space of a group $G$ (Poincaré) providing a comprehensive view of the subject that exploits symmetry and is formulated in geometric terms.

Let us first define the notion of SI and an important theorem by Mackey that characterizes such systems in terms of induced representations.

**Definition 1 ([5]).** A $G$-space of a Borel group $G$ is a Borel space $X$ with a Borel automorphism $\forall g \in G, t_g : x \rightarrow g \cdot x, x \in X$ such that

1. $t_e$ is an identity
2. $t_{g_1 \cdot g_2} = t_{g_1} \cdot t_{g_2}$.

The group $G$ acts on $X$ transitively if $\forall x, y \in X, \exists g \in G \ni x = g \cdot y$.

**Definition 2 ([5]).** A system of imprimitivity for a group $G$ acting on a Hilbert space $\mathcal{H}$ is a pair $(U, P)$ where $P : E \rightarrow P_E$ is a projection valued measure defined on the Borel space $X$ with projections defined on the Hilbert space and $U$ is a representation of $G$ satisfying

$$U_g P_E U_g^{-1} = P_{g \cdot E}.$$  

As an example let us express the Weyl commutation relation in terms of SI to firm up this notion.

**Example 3 ([9]).** Let $\mathbb{R}$ be the additive group of real numbers and define unitary operators that satisfy the Weyl commutative relation as follows:
\[ U_t f(x) = e^{-ixt}f(x), x \in L^2(\mathbb{R}) \text{ position operator.} \]
\[ V_t f(x) = f(x-t), t \in \mathbb{R} \text{ momentum operator.} \]
\[ V_t U_s f(x) = e^{i(t-s)ft} f(x), s, t \in \mathbb{R}. \]

A quantum observable such as the position operator has three equivalent descriptions as a self-adjoint operator, a stone generator of a one parameter family of unitaries, and as a spectral measure. The unitary operator \( U_t \) can be expressed in terms of a canonical spectral measure \( \mu \) in \( L^2(\mathbb{R}) \) as
\[
U_t = \int \text{e}^{-ixt} \mu(dx). \tag{4}
\]

Now, the Weyl commutation can be expressed in terms of SI as follows:
\[
V_t \mu(E)V_t^{-1} = \mu(E + t), \text{ for all Borel sets } E \subset \mathbb{R}. \tag{5}
\]

In the above example the entire real line forms the configuration space and the Fourier transform of the operators satisfies the canonical commutation relation. In quantum systems where the positions are confined to the positive half of the real line \( \mathbb{R}^+ \) the conjugate variables to consider will be \( x \) and \( xp \) satisfying the commutation relation \([x, xp] = ix \) [28]. In general, when the configuration space of a quantum system is not a complete manifold such as \( \mathbb{R}^n \) then the quantization is not straightforward. Let us see how the notion of SI helps with some generalization in this regard with an example that involves Lie groups and a pair of conjugate operators with one of them having an infinite spectral resolution and the other momentum-like features.

**Example 4 ([5]).** Let \( G \) be a Lie group (Poincaré, \( SU_n \), etc) that describes the symmetry of the quantum system of interest and acts on differentiably on \( C^\infty \) manifold \( M \). Then, the SI relation can be stated as:
\[
U_g^{-1} P_g U_g = P_{g(E)}, g \in G, \text{ for all Borel sets } E \subset M. \tag{6}
\]

Let \( g \) be the Lie algebra associated with the group and for \( X \in g \) and \( t \in \mathbb{R} \) there is a one parameter group of diffeomorphisms of \( M \). Let us denote by \( \tau_X \) the vector field, smoothly varying tangent spaces, of this one-parameter group. By definition of a vector field we have
\[
(\tau_X f)(x) = \frac{d}{dt} (f(\exp(tX) \cdot x))_{t=0}, f \in C^\infty(M). \tag{7}
\]

Let the self-adjoint but not necessarily bounded operator \( A_f \) have a the spectral resolution \( dP \) that satisfies the SI relation.
\[
A_f = \int_M f dP, \quad U_g A_f U_g^{-1} = A_{(f \circ g)}. \tag{8}
\]

We can now choose the conjugate operator to be \( A_f \) by considering the one-parameter group \( t \to U_{(\exp tX)} \) and its Stone generator \( B_X \) that satisfies the relation \( \exp(itB_X) = U_{(\exp tX)} \). Let us now derive the commutator relation between \( A_f \) and \( B_X \). The dynamics of the system is expressed as a differential equation as
Applying equations (7) and (8) we get the commutation as \([B_X, A_\tau] = iA_\tau\). In this quantization the observable \(A_\tau\) may be thought of as a position operator with an infinite spectra and the vector field \(\tau\) induced observable is like a momentum operator.

The notion of SI can be generalized in several useful ways. When the action of \(G\) on \(X\) is transitive it is called transitive system of imprimitivity. In a more generalized setting when \(P\) is instead a projection operator valued measure (POVM), a resolution of identity, it is called a system of covariance [22]. One of the important results by Mackey is stated here, which is the basis for the recipe Mackey machine mentioned earlier.

**Theorem 5 (Mackey’s imprimitivity theorem [22]).** Let \(U, P\) be a transitive system of imprimitivity, based on the homogeneous (transitive) space \(X\) of the locally compact group \(G\). Then, there exist a closed subgroup \(H\) of \(G\), the Hilbert space \(H\), and a continuous unitary representation \(V\) of \(H\) on \(H\), such that the given system is unitarily equivalent to the canonical system of imprimitivity \((\tilde{U}, \tilde{P})\), arising from representation \(\tilde{U}\) of \(G\) induced from \(V\).

**Theorem 6 ([22]).** If \((U, P)\) is a transitive system of covariance, then \(U\) is a subrepresentation of an induced representation.

The next step is to consider semidirect product of groups that naturally describe the dynamics system of quantum walks and use the representation of the subgroup to induce a representation in such a way that it is an SI.

**Definition 7.** Let \(A\) and \(H\) be two groups and for each \(h \in H\) let \(t_h : a \rightarrow h[a]\) be an automorphism (defined below) of the group \(A\). Further, we assume that \(h \rightarrow t_h\) is a homomorphism of \(H\) into the group of automorphisms of \(A\) so that

\[
h[a] = hah^{-1}, \forall a \in A. \tag{11}\]

\[
h = e_H, \text{ the identity element of } H. \tag{12}\]

\[
t_{h_1 h_2} = t_{h_1} t_{h_2}. \tag{13}\]

Now, \(G = H \rtimes A\) is a group with the multiplication rule of \((h_1, a_1)(h_2, a_2) = (h_1 h_2, a_1 t_{h_1}[a_2])\). The identity element is \((e_H, e_A)\) and the inverse is given by \((h, a)^{-1} = (h^{-1}, h^{-1}[a^{-1}])\). When \(H\) is the homogeneous Lorentz group and \(A\) is \(\mathbb{R}^4\) we get the Poincaré group via this construction.

### 2. Coin space ensembles

The group \(SO_3\) is a compact Lie group for which we can apply the above construction to build systems of imprimitivity to describe a relativistic quantum walk on de Sitter space. In this walk evolution the coins are drawn from the compact group \(SO_3\) and the statistics of the
coins (affected by the walker DOF due to spin–orbit coupling) are considered. Alternately, an evolution \( \{ Un, g \in SO_3 \} \) can be thought of as a walk on a non-commutative space whose three dimensional axes would correspond to the generators of the Lie algebra so_3 [20].

In [24] the author uses an elegant technique that takes the difference between Lie algebra operators of two representations of the group SO_3 that are manifestly covariant under the actions of the group. This leads to a simple derivation of a Dirac equation on curved spacetime, on two spaces with each symmetries preserving the metrics \((-1, -1, -1, +1, +1)\) and \((-1, -1, -1, +1, +1)\), described by the equation \( x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = L \) in space \( R^5 \) that has three space-like and two time-like coordinates. We adopt that method for a similar construction on \( R^3 \) with a space coordinate and two time-like coordinates.

The three generators, two boosts and a rotation of the Lie algebra so_3 are \( J_{ab}, a, b \in \{ 1, 2, 3 \} \) with the only possible Casimir operator \( J^2 \) (corresponding to spin angular momentum) for this algebra. The invariant operators (Casimir) are the center of the Lie algebra so_3. Let us induce an unitary representation of it by using a finite dimensional representation \( \Sigma \) of SO_3, which could be any one of the countably infinitely many unitary irreducible representations afforded by Plancherel decomposition [8], restricted to its subgroup SO_2. Now, we have a representation of \( SO_3 = U^j \times \Sigma \) that is a product of the one induced by the trivial representation of SO_2 (which is just the regular representation) and \( \Sigma \).

The infinitesimal (differential), \( M_{ab} \) in representation \( U^j \) and \( \sigma_{ab} \) that of \( \Sigma \), form of the generator and Casimir operators for the Lie algebra of the group \( U^j \times \Sigma \) can be written as the angular momentum operators that are more fundamental than linear momentum operators in de Sitter space [10]

\[
M_{ab} = i\{ x_a \left( \frac{\partial}{\partial x_b} \right) - x_b \left( \frac{\partial}{\partial x_a} \right) \}. \tag{14}
\]

\[
J_{ab} = M_{ab} \times I + I \times \sigma_{ab} \tag{15}
\]

\[
J^2_{ab} = M_{ab}M_{ab} + 2M_{ab}\sigma_{ab} + \sigma_{ab}\sigma_{ab}. \tag{16}
\]

The invariant operator \( J^2 \) can be used to derive the Klein–Gordon equation; to establish the Dirac equation using first-order operators we need the following procedure that uses induced representations for building SI to guarantee covariance. The expression \( M_{ab} \) commutes with \( J_{ab} \) (they are on different Hilbert spaces) and so the remainder \( 2M_{ab}\sigma_{ab} + \sigma_{ab}\sigma_{ab} \) of \( J^2_{ab} \) also commutes with every operator of the algebra giving rise to the following Dirac equation with first order operators as a result of Schur’s lemma:

\[
\begin{pmatrix}
M_{ab}\sigma_{ab} + \frac{1}{2}\sigma_{ab}\sigma_{ab}
\end{pmatrix} \Psi = \lambda \Psi. \tag{17}
\]

In the neighborhood of \((0,0,R^2)\) the above equation may be written as

\[
\{ p_2 + \sigma_{j}p_1 - m\sigma_{j}\sigma_{z} \} \Psi = 0. \tag{18}
\]

Later, we will provide a geometric interpretation for this construction in terms of fiber bundles.
3. Quantum walks

Next, let us apply the constructions to quantum walk evolutions. Quantum walks are unitary evolutions that involve a coin Hilbert space and a walker Hilbert space where the dynamics happens \([4, 13, 15]\). Let \(\mathcal{C}^2\) (complex space) and \(\mathbb{Z}\) (set of integers) correspond to Hilbert spaces of the coin and walker respectively. The dynamics of the quantum walk is described by the unitary operator \(U\) composed of a rotation on the Bloch sphere and a translation on the integer line

\[
L^\pm(x) = x \pm 1, x \in \mathbb{Z}
\]

\[
\mathcal{C}^2 = \Pi_0 \oplus \Pi_1, \Pi_i = |i\rangle \langle i|
\]

\[
S(x) = \Pi_0 \otimes L^+ + \Pi_1 \otimes L^-
\]

\[
U(x) = S(x)T.
\]

That is, the walker takes a step to the right on \(\mathbb{Z}\) if the outcome of projecting the Hadamard coin

\[
T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]

on \(\mathcal{C}^2\) is 0 and moves a step left otherwise. A variation of the above unitary evolution split-step quantum walk \([18]\) can be defined as follows:

\[
S(x)^+ = \Pi_0 \otimes L^+ + \Pi_1 \otimes I
\]

\[
S(x)^- = \Pi_0 \otimes I + \Pi_1 \otimes -
\]

\[
W(x) = S^-(x)T(\theta_2)S^+(x)T(\theta_1).
\]

The split-step quantum walk has an effective Hamiltonian \(H_{\text{eff}}\) with several symmetries:

\[
\Gamma_\theta = e^{i\pi A_\theta \mathcal{Z}}.
\]

\[
\Gamma_\theta^{-1}H_{\text{eff}}\Gamma_\theta = -H_{\text{eff}}. \quad \text{chiral symmetry.}
\]

\[
P_h = K \quad \text{complex conjugation.}
\]

\[
P_hH_{\text{eff}}P_h = -H_{\text{eff}}. \quad \text{Particle-hole symmetry.}
\]

\[
T = \Gamma_\theta P_h.
\]

\[
TH_{\text{eff}}T^{-1} = H_{\text{eff}}. \quad \text{time-reversal symmetry.}
\]

In this work we describe the kinematics of quantum walks in terms induced by representations of groups that act on the configuration space of the walker and derive systems of imprimitivity.

**Theorem 8.** The split-step quantum walk described by the equation (25) is a transitive system of imprimitivity.
Proof. It is well known that the inhomogeneous Lorentz transformations are performed on a continuous spacetime configurations. However, as Boozer has shown [26] there are discrete spacetime lattices that are Lorentz invariant. As the Hamiltonian of the walk is translational invariant simultaneously with respect to time and the \( X \) axis the configuration space of the \( 1 + 1 \) spacetime quantum walker can be seen to satisfy the criteria for periodic lattices in Minkowski space [26]. Boozer has listed viable spacetime lattices that are Lorentz invariant specified by parameters such as the ratio between the lattice constants of space and time dimensions, and one feasible value is \( \sqrt{3} \). The rest of the parameters are concerned with choosing their values for the frames of references for Lorentz invariance and so the discrete time quantum walk has a spacetime configuration that is a Boozer lattice and we will establish the relativistic features of the walk in terms of systems of imprimitivity. Let us observe that the \( 1 + 1 \) spacetime configuration of the walker is \( \mathbb{Z}^2 \) whose Pontryagin dual (Fourier space) is a two-torus that has the symmetry described by the semidirect product of two groups, the locally compact discrete Lorentz group \( O(1, 1) \) and the abelian two-torus \( \mathbb{T}^2 \). The same spacetime possesses symmetry described by the semidirect product of two groups, the locally compact discrete Lorentz group \( SO_3 \), whose universal cover is \( SU_2 \), and the abelian two-torus \( \mathbb{T}^2 \) as the generic Lorentz transformation can be written as a sum of a transformations on coordinate and internal degrees of freedom [14]. The internal DOFs may be the spinors in the case of massive particles and for massless photons they are rotation around momentum and gauge transformations. The discrete Lorentz group \( O(1, 1) \) [21] with boost, with a generic element \( \Lambda = \begin{bmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{bmatrix} \) is the only transformation apart from translations in this space being an automorphism group for the two-torus and so the semidirect product \( O(1, 1) \rtimes \mathbb{T}^2 \) is well defined.

Let \( \mathcal{H} \) be a separable Hilbert space and \( P \) a projection valued measure based on \( \mathbb{T}^2 \) acting in \( \mathcal{H} \).

\[
\langle U_a f, f_1 \rangle = \int_{\mathbb{T}^2} x(a) d\nu_{f_1}(x), x \in \mathbb{T}^2, x(a) \text{ is the value of character at } a
\]

\[
\nu_{f_1}(E) = \langle Pf_1, f_1 \rangle
\]

\[
U_a = \int_{\mathbb{T}^2} x(a) dP(x).
\]

The group representation \( S \) of equation (22) acts on \( \mathbb{Z}^2 \) gives rise to a system of imprimitivity. Suppose \( V \) is a unitary representation of \( O(1, 1) \). Then, by equation (11) we have,

\[
hah^{-1} = h[a],
\]

\[
V_h U_a V_h^{-1} = U_{h[a]} h \in O(1, 1), a \in \mathbb{T}^2 \text{ replacing the above by its unitary representation.}
\]

Using the spectral resolution of \( U_a \) we can write the left-hand side of the above equation as a representation of \( a \in \mathbb{T} \) and so the representation of \( E \) is \( V_h P_E V_h^{-1} \). Similarly, the mapping \( a \to h[a] \) gives the representation \( E \to P_{h[E]} \) and we have the following system of imprimitivity relationship

\[
V_h P_E V_h^{-1} = P_{h[E]}.
\]

\[\blacksquare\]
4. Little groups (stabilizer subgroups)

Some of the different systems of imprimitivity that live on the orbits of the stabilizer subgroups are described below. It is good to keep in mind the picture that SI are an irreducible unitary representation of Poincaré group $\mathcal{P}^+$ induced from the representation of a subgroup such as $SO_3$ as $(U_m(g)\psi(k)) = e^{iA} \psi(R^{-1}p k)$ where $g$ belongs to the Poincaré group and $R$ is a member of the rotation group and the expression is in momentum space.

The stabilizer subgroups of the Poincaré group $\mathcal{P}^+$ can be described as follows [14]. Time-like quantum walker: there is a reference frame in which the two-component momentum is proportional to $(0, m)$ and the stabilizer subgroup is $SO_3$, which describes the spin. The walker, a massive particle, is at rest in this frame. Let us denote the eigenvector of the Casimir operator $P^\mu P^\mu$ as $|0\lambda\rangle$. Then, we can describe the invariant space under the Poincaré group $\mathcal{P}^+$ as $\Lambda_p |0\lambda\rangle = |p\lambda\rangle$ by applying the Lorentz boost [32]. Any Lorentz operator operating on this space can be shown to be by a rotation (spin in our case). In other words, quantum walk is covariant if a proper basis is chosen to describe the spacetime grid such as the Boozer lattices [26] on which the walker evolves.

Space-like walker: the Lorentz frame in which the walker is at rest has momentum proportional to $(Q, 0)$ and the little group is again $SO_3$ and this time the rotations will change the helicity. In this imaginary mass case the little group is rotations around the axis and the analysis above carries through.

Light-like walker: there is no frame in which the walker is at rest but the frame where the momentum is proportional to $(\omega, \omega)$ has the stabilizer subgroup with elements of the form $J_1$, which is a rotation around the first component of momentum and the boost $\Lambda_p$ in the spatial direction [14]. These two operators commute and the induced representation can be constructed as above.

5. Quantum graphs

Quantum walks may be set up on any connected graph other than the usual integer line as investigated by Liu and Balu [15] where the evolutions are fashioned after Markov chains and Simms et al [16] who designed propagators for the walk using the adjacency matrix of finite graphs. In Simms’ work the quantum walk propagator, a unitary matrix, is used to resolve the cospectral problem, same set of eigenvalues of similar but different finite graphs, instead of the traditional adjacency and Laplacian matrices, as the propagator encodes information to control probability amplitudes during the dynamical evolution. The quantum walk propagator has a richer structure, encodes superposition information dynamically, as opposed to the usual Laplacians of random walks, and such techniques are applicable to graphs with finite nodes. In our algebraic approach we set up product rules to combine graphs, which is very convenient in treating growing graphs, and then construct the propagators using the adjacency matrix of the resulting graph. For examples of walks other than on the integer line, a particular formulation of a Grover walk [17] is a dynamics on a three-regular graph (each vertex has three nodes) using a walker with three internal degrees of freedom such as a qutrit. Extra degrees of freedom in the coin space can lead to additional modes to the usual bimodal (horn-like) distribution. These additional modes may be utilized to create localization, concentration of probability amplitudes, and at higher dimensions graphs can produce a rich structure of topologically protected states. Depending upon the number of branches at a node of the graph the internal degrees of freedom of the walker can be designed to fashion the walks to branch out. As a result, instead of interference between two pathways we will get more complex superposition
of multiple pathways. Non-trivial position space may also introduce redundancies in terms of equivalent pathways and we need to quotient them out. Investigating the dynamics of such evolutions is better carried out at the right level of abstraction by treating the walk propagations as automorphisms on the graphs and casting them within the systems of the imprimitivity framework. The SI formulation will enable identifying the equivalent configurations (gauges) using the associated cohomology groups of cocycles in addition to fashioning relativistic walks on graphs. Unlike the regular walk, which may be thought of as an evolution in the Schrödinger picture the quantum walks on graphs are easy to describe in the Heisenberg picture. The positions of the walker correspond to points on the integer line or nodes of a connected graph but the actual Hilbert space is the square of the summable functions, for example the probability amplitudes, on them. These functions may be complex and that is how the conjugation enters into the picture. In this work we consider growing graphs that have a finite number of generators, whose adjacency matrix can be used to set up the propagator for the dynamics, but an infinite number of nodes. The adjacency matrix $A$ describes the connectivity of the graph and the product $A \ast A$ corresponds to connectivity at two-distance between the nodes, which depends upon the stratification of the graph, and so on. The configuration of the walk at the second step is $A \ast A$ and at the step $n$ is $A^n$ and so the algebraic approach provides a concise description including the asymptotic limit of $n \rightarrow \infty$ and the corresponding central limit theorems. For detailed discussions on different central limit theorems that arise in the quantum context based on various notions of stochastic independence we recommend the work by Hora and Obata [31]. We can create an algebra of these matrices and using conjugation it becomes a $*$-algebra, the self-adjoint members of which make up the observables of the system, which forms the basis for a quantum probability space [6] with an addition of a quantum state. There are rigorous results that ensure the existence of quantum states in such algebraic quantum probability spaces [5] via GNS construction that associates a Hilbert space to the algebra. Since our description is in the Heisenberg picture we can choose to work with the ground state, the starting position of the walker can be used to define a state as shown in the following sections. Our starting point is an arbitrary graph, with some regularity constraints, for the quantum walk and we want to describe the most general dynamics leading us to the algebraic and quantum probability based approach, which provides the required tools. The $*$-algebra we can endow on a graph is of Bose–Mener type with rich structures that may be used to construct entangled Markov chains [12]. In the following we will formalize the notions before establishing the first step towards the cohomology treatment.

The central ideas of classical probability consist of random variables and measures that have quantum analogues in self-adjoint operators and trace mappings. Let us define the non-commutative quantum probability space formally and make some comments comparing it with the classical probability space.

**Definition 9.** A finite dimensional quantum probability (QP) space is a tuple $(\mathcal{H}, \mathcal{A}, \mathbb{P})$ where $\mathcal{H}$ is a finite dimensional Hilbert space, $\mathcal{A}$ is a $*$-algebra of operators, and $\mathbb{P}$ is a trace class operator, specifically a density matrix, denoting the quantum state.

This is analogous to a classical probability (CP) space which is a tuple $(\Omega, \mathbb{F}, \rho)$ where $\Omega$ is the set of outcomes of a random experiment, $\mathbb{F}$ is the $\sigma$-algebra of events, and $\rho$ is a probability measure. Random variables in a CP are stochastically equivalent to observables in a Hilbert space $\mathcal{H}$. These are self-adjoint operators with a spectral resolution $X = \sum \lambda_i E_i^X$ where the $\lambda_i$ are the eigenvalues of $X$ and each $E_i^X$ is interpreted as the event $X$ takes the value $\lambda_i$. States are positive operators with unit trace and denoted by $\mathbb{P}$. In this framework, the expectation of an observable $X$ in the state $\mathbb{P}$ is defined using the trace as $\text{tr} \mathbb{P}(X)$. The observables when measured are equivalent to random variables on a probability space, and a collection of such
classical spaces constitutes a quantum probability (QP) space. If all the observables of interest commute with each other then the classical spaces can be composed to a product probability space, and the equivalence $\text{CP} = \text{QP}$ holds. The main feature of a QP is the admission of possibly non-commuting projections and observables of the underlying Hilbert space within the same setting.

There is a canonical way to create quantum probability spaces from their classical counterparts. The process involves creating a Hilbert space from the square integrable functions with respect to the classical probability measure. The $\ast$-algebra of interest is usually defined in terms of creation, conservation, and annihilation operators. Classical probability measures become quantum states in a natural way through Gleason’s theorem \cite{25}. In this case, unitary operators forming a quantum stochastic process can be defined similarly to stochastic processes in a classical probability space.

Given a graph $G = (V,E)$ with finite number of vertices $V$ and set of edges $E$ let us consider the Hilbert space $\mathcal{H} = \ell^2(V)$, the square summable functions on $V$ with the inner product $\langle f,g \rangle = \sum_{x \in G} f(x)\overline{g(x)}, f,g \in \ell^2(G)$. An edge $(x,y) \in E, x,y \in V$ is denoted by $x \sim y$ and the distance function $\partial(x,y), x,y \in V$ is defined as the shortest path connecting the two vertices.

The collection of functions below form an orthonormal basis of $\ell^2(G)$.

$$\delta_o(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Let $C_0(V)$ be the dense subspace of $\ell^2(G)$ spanned by $\{\delta_v\}$ and the adjacency algebra $L(C_0(V))$ be the $\ast$-algebra generated by linear operators on $C_0(V)$. The operator $T$ is said to be locally finite if the following conditions are satisfied:

$$|\{x \in V : T_{xy} = 0\}| < \infty, \forall y \in V, |\{y \in V : T_{xy} = 0\}| < \infty, \forall x \in V$$

(32) when $G$ is finite the matrix elements of $T \in L(C_0(V))$ are determined as $T_{xy} = \langle \delta_x, T\delta_y \rangle$ and its adjoint as $T_{xy}^\dagger = \langle T\delta_x, \delta_y \rangle$.

**Definition 10.** Given a graph $\mathcal{G} = (G,E)$ and an adajacency algebra $\mathcal{A}(\mathcal{G})$ a vacuum state $\delta_o$ at a fixed origin of the graph $o \in V$ is defined as $\langle \delta_o, a\delta_o \rangle, a \in \mathcal{A}$.

It is easy to verify the fact $(A^m)_{xy} = \langle \delta_x, A^m\delta_y \rangle$, which is the number of $m$-step walks connecting $x$ and $y$ vertices.

**Definition 11.** The adjacency matrix of a graph $(G,E)$ is defined as

$$A_{xy} = \begin{cases} 1 & \text{if } (x,y) \in E. \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $A$ is symmetric, taking values in $\{0,1\}$, with vanishing diagonals.

**Definition 12.** A stratification of a graph $\mathcal{G} = (V,E)$ with a fixed origin $o \in V$ is defined as a disjoint union of strata

$$V = \bigcup_{n=0}^{\infty} V_n, V_n = \{x : \partial(o,x) = n\}.$$  \hspace{1cm} (33)

It is easy to verify that

$$\forall x,y \in V, x \sim y \text{ and } x \in V_n \Rightarrow y \in V_{n-1} \cup V_n \cup V_{n+1}.$$  \hspace{1cm} (34)
We define the following matrices wrt the stratification equation (33) and the quantum decomposition of $A$

$$\forall y \in V_n, (A_\epsilon) = A_{xy}, x \in V_{n+\epsilon}, \epsilon \in \{\alpha, +, -\}$$

(35)

$= 0$ o.t.

(36)

$$A = A^\alpha + A^+ + A^-$$

(37)

$$(A^\dagger)^* = A^-$$

(38)

$$(A^\dagger)^* = A^+$$

(39)

$$\langle A^\dagger f, g \rangle = \langle f, A^- g \rangle, \forall f, g \in C_\theta V$$

(40)

$$(A^\dagger)^* = A^\alpha.$$  

(41)

Given a stratification of a graph with origin $o \in V$ the degree $k(x)$ of a vertex $x \in V$ can be decomposed as follows

$$\omega_o(x) = |\{y \in V; x \sim y, \partial(o, y) = \partial(o, x) + \epsilon\}|$$

(42)

$$k(x) = \omega_o(x) + \omega_+(x) + \omega_-(x).$$

(43)

The $*$-algebra generated by $A^\alpha$ is non-commutative. The vectors $\Phi_n = \sum_{x \in V_n} \delta_x$ span a space denoted by $\Gamma(G)$.

**Definition 13.** A graph $\mathcal{G} = (V, E)$ is called distance-regular if for any choice of $x, y \in V$ with $\partial(x, y) = k$ the number of vertices $z \in V$ such that $\partial(x, z) = i$ and $\partial(y, z) = j$ is independent of the choice of $x$ and $y$. Then, the intersection numbers $i, j$ and $k$ are defined as

$$p^k_{ij} = |\{z \in V: \partial(x, z) = i, \partial(y, z) = j, \partial(x, y) = k\}|.$$  

(44)

**Definition 14.** A $k$th distance adjacency matrix of a graph $\mathcal{G} = (V, E)$ is defined as

$$(A^k)_{xy} = 1 \text{ if } \partial(x, y) = k.$$  

(45)

$= 0$ o.t.

(46)

$$\sum_k A^k = J, \text{ where } (J)_{xy} = 1.$$  

(47)

**Lemma 15.** Let $\mathcal{G} = (V, E)$ be a distance-regular graph with intersection numbers $p^k_{ij}$. Then the following holds:

$$A_i A_j = \sum_{|i-j|} p^k_{ij} A_k.$$  

(48)

Association schemes are naturally related to graphs and the resulting quantum probability space. Specifically, the Bose–Mesner algebra can be identified with distance regular graphs [31] that leads to our next result.
Theorem 16. Let \((\chi = V, A_i, 0 \leq i \leq d, R)\) be an association scheme of Bose–Mesner type and the corresponding adjacency matrix is defined on a distance-regular graph \(\mathcal{G} = (V, E)\) whose vertices \(V\) with cardinality \(|V| = d\) such that

\[(A_i)_{xy} = 1 \text{ if } (x, y) \in R_i, = 0 \text{ o.t..}

Let \(o \in V\) be the origin of the graph and \(\{\Phi_i, i = 1, \ldots, n\}\) as defined above. Then, the unitary regular representation

\[U_g \Phi_n U_g^\dagger(f(y)) = U_g \Phi_n (f(gy)), f \in L^2(V), g, y \in G\]

(49)

\[= U_g \left( \sum_{x \in V_n} \delta_x \right) \left( \sum_{x \in V_n} \delta_x \right)^\dagger (f(gy))\]

(50)

\[= U_g \left( \sum_{xg^{-1} \in V_n} \delta_x \right) \left( \sum_{xg^{-1} \in V_n} \delta_x \right)^\dagger (f(y))\]

(51)

\[= \left( \sum_{xg^{-1} \in V_n} \delta_x \right) \left( \sum_{xg^{-1} \in V_n} \delta_x \right)^\dagger (f(y))\]

(52)

\[= P_{g^{-1}} \Phi_n.\]

(53)

6. Geometric interpretation and the Dirac equation

The states of freely evolving relativistic quantum particles are described by unitary irreducible representations of a Poincaré group that has a geometric interpretation in terms of fiber bundles. To express quantum walks in a similar fashion let us recall the related definitions.

Definition 17. A fiber bundle is a triple \((E, \pi, M)\) where \(E\) is a space, \(\pi : E \rightarrow M\) is a projection from \(E\) to a manifold \(M\). A principle fiber bundle is isomorphic to \((E, \rho, E/G)\) where \(E\) is a right \(G\)-space of a Lie group, \(G\) acts on \(E\) freely, and the quotient group \(E/G\) is the orbit.
space of the $G$-action on $E$. A bundle can be thought of as a product space with a twist that can encode information such as topological invariants or Casimir operators.

**Example 18.** Let $G$ be the compact Lie group $SO_3$ and $H$ is the closed subgroup $SO_2$, then the group acts freely on the $H$-orbits space, which is four-sphere, and $(G, \pi, G/H)$ is a principle $H$-bundle. The bundle can be viewed as a twisted product of $H$ and $G/H$.

**Definition 19.** A cross-section of a bundle $(E, \pi, M)$ is a map $s : M \to E$ such that the image of each point $x \in M$ lies in the fiber $\pi^{-1}(x)$ over $x$ as $\pi \circ s = id_M$. When these functions are square integrable with respect to an appropriate measure they form a Hilbert space to describe the states of quantum systems. In general, fiber bundles may not have cross sections but for vector bundles they can be constructed. An important theorem states that principle bundles have smooth cross sections only when they are trivial, that is the twist is a regular product \[28\].

**Definition 20.** Let $X$ and $Y$ be $G$-spaces, then a $G$-product $\times_G$ is the equivalence class $(x, y) \equiv (x', y')$ if $\exists g \in G, \exists x' = gx, y' = gy$ and is denoted by $X \times_G Y, [x, y]$. They are the $G$-orbits contained in the product space.

**Definition 21.** Let $\eta = (P, \pi, M)$ be a principle $G$-bundle and $F$ be a left $G$-space. Define the $F$-orbits as $PF = P \times_F G$ where the left action is $g(p, v) = (pg, g^{-1}v)$ and a projection $\pi_F : PF \to M$ by the map $\pi_F([p, v]) := \pi(p)$. Then $\eta[F] = (PF, \pi_F, M)$ is the fiber bundle associated with the principle bundle $\eta$ via the action of the group $G$ on $F$. It is easy to prove that $\forall x \in M$ the space $\pi_F^{-1}(x)$, the fibers, is homeomorphic to $F$. The specific left action on the $F$-orbits are required to make sure that the associated bundle projection $\pi_F$ is well defined. This can be seen with if $[p_1, v_1] = [p_2, v_2]$ then $\exists g \in G$ such that $(p_2, v_2) = (p_1g, g^{-1}v_1)$. This implies that

$$\pi_F(p_2, v_2) = \pi(p_2) = \pi(p_1g) = \pi(p_1) = \pi_F(p_1, v_1). \quad (54)$$

This way a family of new fiber bundles can be constructed out of a principle $G$-bundle using $G$-spaces that act as fibers.

**Definition 22.** A vector bundle is a special case of an associated bundle in which the fiber is a vector space. Furthermore, $\forall x \in M$ there must exist some neighborhood $U \subset M$ of $x$ and a local trivialization $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that, $\forall Y \in U, h : [y] \times \mathbb{R}^n \to \pi^{-1}(y)$ is a linear map.

The discrete time quantum walk described above leads to entanglement between the internal spin and walker DOFs and when the initial state of the walker is at a highly localized (Compton wavelength) single site of the 1D lattice with positive-energy it leads to relativistic propagation \[23\]. The $1 + 1$ spacetime discrete Lorentz group $\hat{O}(1, 1)$-orbits of the momentum space $\mathbb{T}^2 = S^1 \times S^1$, where the systems of imprimitivity established above will live, described by the symmetry $\hat{O}(1, 1) \times \mathbb{T}^2$. That is, the rest frames of the walker are related by Lorentz boosts where the position operator is defined up to Compton wavelength. The orbits have an invariant measure $\alpha^+_m$ whose existence is guaranteed as the groups and the stabilizer groups concerned are unimodular, and in fact it is the Lorentz invariant measure $\frac{dp}{p_0}$ for the case of forward mass hyperboloid. The orbits are defined as:

$$X^+_m = \{ p : p_0^2 - p_1^2 = m^2, p_0 > 0 \}, \text{ forward mass hyperboloid : positive-energy} \quad (55)$$
$\hat{X}_m^+ = \{ p : p_0^2 - p_1^2 = m^2, p_0 < 0 \}$, backward mass hyperboloid

$\hat{X}_{00} = \{0\}$, origin.

Each of these orbits are discrete and invariant with respect to $\hat{O}(1, 1)$ and let us consider the stabilizer subgroup of the first orbit at $p = (m, 0)$. Now, assuming that the spin of the particle is $1/2$ let us define the corresponding fiber bundles (vector) for the positive mass hyperboloid that corresponds to the positive-energy states by building the total space as a product of the orbits and the group $SL(2, C)$ (more precisely the members of the Clifford algebra) and using the discrete form of Dirac equation in momentum space.
\[ \mathcal{B}^{+,+}_m = \{(p,v) \mid p \in \mathcal{X}^+_m, \quad v \in \mathfrak{J}^2, p_0 \sigma_z + p_1 \sigma_\times = mv \} \]  

(58)

\[ \hat{\pi} : (p,v) \rightarrow p \text{ projection from the total space } \mathcal{B}^{+,+}_m \text{ to the base } \mathcal{X}^+_m. \]  

(59)

In other words we have used a discretized Dirac operator to locally trivialize the fiber bundle and still it provides a representation of the group (figure 1). As a consequence of the trivialization we have spin–orbit coupling in the system. The states of the particles are defined on the Hilbert space \( \mathcal{H}^{+,+}_m \), square integrable functions on Borel sections of the bundle \( \mathcal{B}^{+,+}_m \) with respect to the invariant measure \( \alpha^+_m \), whose norm induced by the inner product is given below:

\[ \|\phi\|^2 = \int_{\mathcal{X}^+_m} p^0 \langle \phi p, \phi p \rangle \cdot d\alpha^+_m(p). \]  

(60)

The symmetries defined in equation (26) can be incorporated into the two-torus as a hierarchy of cells making the connection to \( k \)-theory and index theory of topological systems [33].

In the discrete configuration space a Dirac-like difference equation of motion can be defined only in the momentum representation. Now, let us consider the continuous version of the above evolution with the \( 1 \times 1 \) spacetime discrete Lorentz group \( O(1,1) \)-orbits of the momentum space \( \mathbb{R}^2 \), where the systems of imprimitivity established above will live in the continuum of \( 1 + 1 \) spacetime, described by the symmetry \( O(1,1) \times \mathbb{R}^2 \), with an invariant measure \( \alpha^+_m \) defined as:

\[ \mathcal{X}^+_m = \{p : p_0^2 - p_1^2 = m^2, p_0 > 0\}, \text{ forward mass hyperboloid} \]  

(61)

\[ \mathcal{X}^{-}_m = \{p : p_0^2 - p_1^2 = m^2, p_0 < 0\}, \text{ backward mass hyperboloid} \]  

(62)

\[ \mathcal{X}_{00} = \{0\}, \text{ origin}. \]  

(63)

\[ \mathcal{B}^{+,+}_m = \{(p,v) \mid p \in \mathcal{X}^+_m, \quad v \in \mathfrak{J}^2, p_0 \sigma_z + p_1 \sigma_\times = mv \} \]  

(64)

\[ \pi : (p,v) \rightarrow p, \text{ projection from the total space } \mathcal{B}^{+,+}_m \text{ to the base } \mathcal{X}^+_m. \]  

(65)

We have used the Dirac operator to locally trivialize the fiber bundle in this case. In the continuum an inverse transformation will provide the 1D Dirac equation in position space as well. The above construction is similar to that of the Poincaré group [5], which was based on the original work of Wigner [29] that now leads to a one dimensional Dirac equation after assuming the velocity of light, mass and the Planck constant as unity. The limit of the \( 1 + 1 \) lattice is the euclidean space \( \mathbb{R}^2 \) and so the torus based fiber bundle converges to the \( \mathbb{R}^2 \) based bundle (figure 2). Another way to look at the limit is the dense set of points of the hyperbola of the discrete walk become a continuous curve in the base space of the fiber bundle that is lifted to the bundle, making the difference equation a differential one.

The coin ensemble based on \( SO_3 \) constructed earlier can be described in the language of associated bundles as well. Starting with the principal \( G \)-bundle \( \eta = (SO_3, \pi, SO_3/\mathbb{Z}_2) \) and a vector space \( V \) on which \( SO_3 \) has a representation as the left \( G \)-space, fiber, let us construct the associated bundle \( \eta[V] = (SO_3 \times \mathbb{R}^2, \pi, SO_3/\mathbb{Z}_2) \). The left action of \( SO_3 \) is defined by \( g(p,v) = (ph^{-1}v), h \in SO_3 \). This associated bundle (figure 3) has a cross section, guaranteed by the theorem on page 149 of [28], \( S \) with respect to a mapping \( \phi : P(\eta) \rightarrow F \) satisfying \( \phi(pg) = g^{-1}\phi(p), \forall p \in P(\eta), g \in G \) defined as...
\[ S_{\phi}(x) = [p, \phi(p)], \text{ where, } p \text{ is any point on the fiber } \pi^{-1}(x). \quad (66) \]

The cross-section can be used to build a Hilbert space containing rays of the wave functions \( \Psi \) defined in equation (18).

7. Summary and conclusions

We derived the kinematics of split-step quantum walk using induced representations of groups and expressed them in terms of systems of imprimitivity. We developed a geometric picture of the time-discrete version of the evolution as a fiber bundle in the momentum space with a two-torus containing an hyperboloid as the base manifold. The continuous time counterpart is a bundle with the real plane as the base manifold and both the systems have the group \( \text{SL}(2, \mathbb{C}) \) (spinors) as the fibers. We also constructed systems of imprimitivity for walks based on \( SO_3 \) coins and derived Dirac equation on a de Sitter space. The automorphisms on distance-transitive graphs was shown as SI. The quantum walk framework being an important simulation tool it is imperative to cast the kinematics in terms of systems of imprimitivity to leverage the tools of induced representations for building more complex, for example gauges of cohomology, evolutions. We plan to build two-cocycles from SI for the quantum walks and using that to construct a semigroup whose dilation results in a Fock space where the non-interacting walker evolves freely.

ORCID iDs

Radhakrishnan Balu © https://orcid.org/0000-0003-1494-5681

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