A NOTE ON THE INVARIANTLY GENERATING GRAPH OF A FINITE GROUP

DANIELE GARZONI

ABSTRACT. We prove that the invariably generating graph of a finite group can have an arbitrarily large number of connected components of order at least 2.

1. Introduction

Given a finite group $G$ and a set $X = \{C_1, \ldots, C_t\}$ of conjugacy classes of $G$, we say that $X$ invariably generates $G$ if $\langle x_1, \ldots, x_t \rangle = G$ for every $x_1 \in C_1, \ldots, x_t \in C_t$. We write in this case $\langle X \rangle_I = G$. This concept was introduced by Dixon [Dix92] with motivations from computational Galois theory, and has been widely studied in recent years.

In [Gar20], the following definition was given. For a finite group $G$, the invariably generating graph $\Lambda(G)$ of $G$ is the undirected graph whose vertices are the conjugacy classes of $G$ different from $\{1\}$, and two vertices $C$ and $D$ are adjacent if $\langle C, D \rangle_I = G$. If $G$ is not invariably 2-generated, $\Lambda(G)$ is the empty graph. Even when $G$ is invariably 2-generated, the graph $\Lambda(G)$ can have isolated vertices (e.g., when $G$ is not cyclic, the classes contained in the Frattini subgroup); define $\Lambda^+(G)$ as the graph obtained by removing the isolated vertices of $\Lambda(G)$. The purpose of this note is to prove the following result.

**Theorem 1.1.** For every positive integer $n$, there exists a finite group $G$ such that $\Lambda^+(G)$ has more than $n$ connected components.

Theorem 1.1 should be seen in comparison to the analogous graph for the case of usual generation; see Subsection 1.1.

In the proof of Theorem 1.1, $G$ is a suitable direct power of a nonabelian finite simple group $S$. We use $S = \text{PSL}_2(q)$, although there are other possible choices. A crucial ingredient is that $\Lambda^+(S)$ is bipartite, which follows from the fact that $S$ admits a 2-covering (Lemma 2.4). See Section 3 for definitions and further comments in this direction, related to clique number and chromatic number of $\Lambda^+(S)$.

We will also give a suitable lower bound to the number of connected components of $\Lambda^+(G)$ in our examples (see Theorem 2.8), which is not strictly necessary for the mere proof of Theorem 1.1. We will get the bound as a consequence of the following result.

**Theorem 1.2.** Let $S = \text{PSL}_2(q)$, and let $C_1, C_2$ be conjugacy classes of $S$ chosen uniformly at random. Then
\[
P(\langle C_1, C_2 \rangle_I = S) = 1/2 + O(1/q).
\]
The proof of Theorem 1.2 is straightforward, since subgroups and conjugacy classes of \(PSL_2(q)\) are known very explicitly. It is interesting that the asymptotic behaviour of \(P((C_1, C_2)_I = S)\) is equal to the asymptotic behaviour of \(P((x_1^S, x_2^S)_I = S)\), where \(x_1, x_2 \in S\) are random elements. For this, see [GM20 Subsection 6.1].

1.1. Comparison to usual generation. For a finite group \(G\), the generating graph \(\Gamma(G)\) of \(G\) is the undirected graph whose vertices are the nonidentity elements of \(G\), and two vertices \(x\) and \(y\) are adjacent if \(\langle x, y \rangle = G\). This graph has been intensively studied in the last two decades; see Burness [Bur19] and Lucchini–Maróti [LM09] for many results in this context.

Again, the graph \(\Gamma(G)\) can have isolated vertices, and we consider the graph \(\Gamma^+(G)\) obtained by removing the isolated vertices of \(\Gamma(G)\). It is known that \(\Gamma^+(G)\) is connected in several cases (see Burness–Guralnick–Harper [BGH20], Crestani–Lucchini [CL13a, CL13b]). By contrast, no example of \(G\) is known for which \(\Gamma^+(G)\) is disconnected, which determines a sharp difference with respect to Theorem 1.1.

We note that this difference does not occur for nilpotent groups. Indeed, in a finite nilpotent group every maximal subgroup is normal, therefore the concepts of generation and invariable generation coincide.

The organization of the paper is simple. In Section 2 we prove Theorems 1.1 and 1.2, and in Section 3 we make further comments and propose some problems.

Acknowledgements. The author thanks Andrea Lucchini for a discussion about the literature on the generating graph of a finite group, and Daniela Bubboloni for a discussion about the literature on normal coverings of finite simple groups.

2. Proof of Theorems 1.1 and 1.2

2.1. Direct powers of finite simple groups. Throughout this subsection, \(S\) denotes a nonabelian finite simple group. We review some properties of invariable generation of direct powers of \(S\), which reflect some interesting properties of the corresponding invariably generating graphs. The key tool is an elementary criterion due to Kantor and Lubotzky [KL90], which we recall.

Denote by \(\Psi_2(S)\) the set of all pairs \((C_1, C_2)\), where \(C_i\) is a conjugacy class of \(S\), and \(C_1\) and \(C_2\) invariably generate \(S\). Guralnick–Malle [GM12] and Kantor–Lubotzky–Shalev [KLS11] independently proved that \(\Psi_2(S) \neq \emptyset\).

Let now \(t\) be a positive integer, and let \(C\) and \(D\) be conjugacy classes of \(S^t\), with \(C = C_1 \times \cdots \times C_t\) and \(D = D_1 \times \cdots \times D_t\) (and \(C_i\) and \(D_i\) are conjugacy classes of \(S\)). Consider the matrix

\[
A_{C,D} = \begin{pmatrix}
C_1 & C_2 & \cdots & C_t \\
D_1 & D_2 & \cdots & D_t
\end{pmatrix}.
\]

Lemma 2.1. We have that \(\langle C, D \rangle_I = S^t\) if and only if the following conditions are both satisfied:

(i) Each column of \(A_{C,D}\) belongs to \(\Psi_2(S)\), and

(ii) No two columns of \(A_{C,D}\) lie in the same orbit for the diagonal action of \(\text{Aut}(S)\) on \(\Psi_2(S)\).

Proof. See [KL90 Proposition 6], and also [DL15 Lemma 20].
Let \( \beta = \beta(S) \) be the largest integer for which \( S^\beta \) is invariably 2-generated. Lemma 2.1 implies that \( \beta(S) \) is equal to the number of orbits for the diagonal action of \( \text{Aut}(S) \) on \( \Psi_2(S) \). We note the following fact.

**Lemma 2.2.** We have

\[
\frac{|\Psi_2(S)|}{|\text{Out}(S)|} \leq \beta(S) \leq |\Psi_2(S)|.
\]

**Proof.** The second equality is clear, and the first follows from the fact that \( \text{Inn}(S) \cong S \) acts trivially in the relevant action, hence each orbit has size at most \( |\text{Out}(S)| \). \( \square \)

We expect \( |\text{Out}(S)| \) to be much smaller than \( |\Psi_2(S)| \) for every sufficiently large nonabelian finite simple group \( S \). Therefore, \( |\Psi_2(S)| \) should be, in some sense, a good approximation for \( \beta(S) \).

We make a one-paragraph digression in order to compare to the case of classical generation. Let \( \delta = \delta(S) \) be the largest integer for which \( S^\delta \) is 2-generated. Unlike for \( \beta(S) \), there is an exact formula for \( \delta(S) \), namely, \( \delta(S) = \phi_2(S)/|\text{Aut}(S)| \), where \( \phi_2(S) \) denotes the number of ordered pairs \((x, y) \in S^2\) such that \( (x, y) = S \). (This goes back to Hall [Hal30] in 1930s and has been widely used.) The difference is that the diagonal action of \( \text{Aut}(S) \) on the set of generating pairs of elements is semiregular (i.e., only the identity fixes a generating pair), while this needs not be the case for the action of \( \text{Aut}(S) \) on the set of invariably generating pairs of conjugacy classes.

Lemma 2.1 describes quite precisely the graph \( \Lambda^+(S^\beta) \). Indeed, any arc in the graph is obtained as follows (and only in this way). Construct a \( 2 \times \beta \) matrix, in which the columns form a set of representatives for the \( \text{Aut}(S) \)-orbits on \( \Psi_2(S) \). Then the first row is adjacent to the second row in \( \Lambda^+(S^\delta) \) (here we are identifying a conjugacy class \( C_1 \times \cdots \times C_\beta \) of \( S^\beta \) with a row vector \((C_1, \ldots, C_\beta)\)). Since \( \text{Aut}(S^\beta) \cong \text{Aut}(S) \wr \text{Sym}(\beta) \) acts by automorphisms on \( \Lambda^+(S^\beta) \), we also see that \( \Lambda^+(S^\beta) \) is arc-transitive.

### 2.2. The case \( S = \text{PSL}_2(q) \)

In this subsection we choose \( S = \text{PSL}_2(q) \), with \( q \geq 4 \). For reader’s convenience, we recall some well known facts. See [Suz82 Chapter 3.6] for the description of the maximal subgroups of \( S \).

**Lemma 2.3.** Let \( S = \text{PSL}_2(q) \), where \( q \geq 4 \) is a power of the prime \( p \). Set \( d = (2, q - 1) \).

1. \( S \) contains a unique conjugacy class of involutions and, for \( p \) odd, two conjugacy classes of elements of order \( p \).
2. Assume \( 3 \leq \ell \mid (q + 1)/d \). There are \( \phi(\ell)/2 \) conjugacy classes of elements of order \( \ell \) in \( S \), where \( \phi \) is Euler’s totient function.
3. The number of conjugacy classes of \( S \) is \( (q - 1)/d + 3 - \delta_{p,2} \) (where \( \delta_{p,2} = 1 \) if \( p = 2 \), and \( \delta_{p,2} = 0 \) otherwise).

**Proof.** We sketch a proof. (1) Assume first \( q \) is odd, and let us deal with involutions. Let \( \varepsilon = 1 \) if \( q \equiv 1 \mod 4 \), and \( \varepsilon = -1 \) otherwise. By explicit matrix computation, we find that the number of involutions of \( S \) is \( q(q + \varepsilon)/2 \). This coincides with the number of conjugates of a dihedral subgroup of order \( q - \varepsilon \), so we deduce that all involutions of \( S \) are conjugate. Next we deal with elements of order \( p \) (also in the case \( q \) even). The image in \( S \) of the subgroup of \( \text{SL}_2(q) \) consisting of the upper unitriangular matrices is a Sylow \( p \)-subgroup \( P \) of \( S \). This is contained in
a subgroup $B$, consisting of the image of the upper triangular matrices of $\text{SL}_2(q)$. Let $1 \neq x \in P$. We verify that $x^g \cap P = x^g$, and we compute that conjugating $x$ by elements of $B$ can only multiply the upper-right entry by every nonzero square of $F_q$. This proves (1).

(2) Assume $3 \leq \ell \mid (q \pm 1)/d$. All cyclic subgroups of order $\ell$ are conjugate in $S$. Assume $x \in S$ has order $\ell$. We have that $N_S(\langle x \rangle)$ is dihedral of order $2(q \pm 1)/d$, from which $x^{S} \cap \langle x \rangle = \{x, x^{-1}\}$. This proves (2).

(3) An element of $S$ has either order $p$, or order dividing $(q \pm 1)/d$. Then (3) follows from (1), (2), and the formula

$$\sum_{\ell \mid (q \pm 1)/d} \frac{\phi(\ell)}{2} = \frac{q \pm 1}{2d}.$$ 

The statement is proved. \hfill \Box

The following lemma represents the main observation regarding $\Lambda^+(S)$.

**Lemma 2.4.** The graph $\Lambda^+(S)$ is bipartite.

**Proof.** It is well known that $S$ admits a 2-covering, that is, a pair of proper subgroups $(H, K)$ such that

$$S = \bigcup_{g \in S} H^g \cup \bigcup_{g \in S} K^g.$$ 

We take $H$ a dihedral subgroup of order $2(q+1)/d$, and $K$ a Borel subgroup of order $q(q-1)/d$, with $d = (2, q-1)$. Write for convenience $\widetilde{H} = \cup_{g \in S} H^g$ and $\widetilde{K} = \cup_{g \in S} K^g$. A conjugacy class contained in $\widetilde{H} \cap \widetilde{K}$ is isolated in $\Lambda(S)$, and a class contained in $\widetilde{H} \setminus \widetilde{K}$ can be adjacent in $\Lambda^+(S)$ only to a class contained in $\widetilde{K} \setminus \widetilde{H}$. This gives a partition of $\Lambda^+(S)$ in two parts. \hfill \Box

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the parts of $\Lambda^+(S)$ given in the proof of Lemma 2.4. We note that, for every conjugacy class $C$ of $S$ and for every $\sigma \in \text{Aut}(S)$, $\{C_1, C_\sigma\}$ does not invariably generate $S$. (A way to see this is that the sets $\widetilde{H}$ and $\widetilde{K}$ from the proof of Lemma 2.4 are preserved by every automorphism of $S$.) In particular, for every $(C_1, C_2) \in \Psi_2(S)$, $(C_1, C_2)$ and $(C_2, C_1)$ belong to different $\text{Aut}(S)$-orbits. We also note that the parts $\mathcal{P}_1$ and $\mathcal{P}_2$ are invariant under the action of $\text{Aut}(S)$. We deduce the following

**Lemma 2.5.** $\beta = \beta(S)$ is even, and for each vertex $C = C_1 \times \cdots \times C_\beta$ of $\Lambda^+(S^2)$, there exists a subset $\Omega = \Omega(C)$ of $\{1, \ldots, \beta\}$ of size $\beta/2$ such that for every $i \in \Omega$, $C_i \in \mathcal{P}_1$, and for every $i \not\in \Omega$, $C_i \in \mathcal{P}_2$.

We can finally prove the key result.

**Theorem 2.6.** The graph $\Lambda^+(S^2)$ has at least $\frac{1}{2} \cdot (\beta/\beta) = \frac{\beta}{2}$ connected components.

**Proof.** For a vertex $C = C_1 \times \cdots \times C_\beta$ of $\Lambda^+(S)$, let $\Omega(C)$ be the set from Lemma 2.5. Then, $C$ can be adjacent only to vertices $D$ such that $\Omega(D) = \{1, \ldots, \beta\} \setminus \Omega(C)$. In particular, the number of connected components of $\Lambda^+(S^2)$ is at least half the number of $\beta/2$-subsets of $\{1, \ldots, \beta\}$, which proves the statement. \hfill \Box

It is not difficult to establish that $\beta(S)$ tends to infinity as $|S| \to \infty$ (that is, $q \to \infty$), thereby proving Theorem 1.1. In the next subsection we will obtain a better estimate for $\beta(S)$.
2.3. **Bounds.** We want to estimate $\beta(S)$, where $S = \text{PSL}_2(q)$. We will find the asymptotic behaviour of $|\Psi_2(S)|$, and then apply Lemma 2.2. In the following, $f = O(g)$ means that $|f| \leq Cg$ for some constant $C$ (so $f$ might also be negative).

**Theorem 2.7.** Let $S = \text{PSL}_2(q)$ and $d = (2, q-1)$. We have

$$|\Psi_2(S)| = \frac{q^2}{2d^2} + O(q).$$

(For $q$ odd the first term of the expression is not an integer, but still the statement makes sense.)

**Proof.** In this proof, when we say that a conjugacy class $C$ intersects a subgroup $H$, we mean $C \cap H \neq \emptyset$. We refer to [Suz82, Chapter 3.6] for the description of the maximal subgroups of $S$. We need to count the pairs of conjugacy classes $(C_1, C_2)$ which invariably generate $S$. We ignore the pairs where either $C_1$ or $C_2$ is made of elements of order $p$, or of order at most 2. By Lemma 2.3 the number of these pairs is $O(q)$.

By this choice, up to swapping the indices, $C_1$ intersects a cyclic subgroup of order $(q-1)/d$, and $C_2$ intersects a cyclic subgroup of order $(q+1)/d$. Given $C_1$ and $C_2$ with this property, we have that $C_1$ and $C_2$ invariably generate $S$ unless one of the following occurs:

(i) $C_1$ and $C_2$ intersect a certain maximal subgroup of order at most 60, and there are at most five possibilities for such subgroup.

(ii) $C_1$ and $C_2$ intersect a maximal subgroup conjugate to $\text{PSL}_2(q^{1/r})$ where $r$ is an odd prime (and $q$ is an $r$-th power).

(In (ii), we are not considering subgroups $\text{PGL}_2(q^{1/2})$. Indeed, any class of elements of $\text{PGL}_2(q^{1/2})$ of order prime to $q$ intersects a cyclic subgroup of $S$ order $(q-1)/d$; and this cannot occur for $C_2$. Clearly there are $O(1)$ possibilities for $(C_1, C_2)$ satisfying (i). The number of conjugacy classes of $\text{PSL}_2(q^{1/r})$ is $O(q^{1/r})$; therefore, for fixed $r$, the number of possibilities for the pair $(C_1, C_2)$ satisfying (ii) is $O(q^{2/r})$.

Summing through odd prime $r$, we get $O(q^{2/3})$ (note that $r \leq \log \log q$).

Using Lemma 2.3 we get the following formula for $|\Psi_2(S)|$ (the factor 2 at the beginning comes from the fact that we may also have $C_1$ intersecting a cyclic subgroup of order $(q+1)/d$, and $C_2$ intersecting a cyclic subgroup of order $(q-1)/d$).

$$|\Psi_2(S)| = 2 \cdot \sum_{\ell_1 | (q-1)/d, \ell_2 | (q+1)/d} \frac{\phi(\ell_1) \phi(\ell_2)}{2} + O(q)$$

$$= \frac{q^2 - 1}{2d^2} + O(q) = \frac{q^2}{2d^2} + O(q).$$

(In the first equality we used the formula $\sum_{d|n} \phi(d) = n$, and the fact that $\phi(\ell_1)\phi(\ell_2) = \phi(\ell_1 \ell_2)$ for coprime $\ell_1$ and $\ell_2$.)

We can rephrase Theorem 2.7 in probabilistic language, that is, we can prove Theorem 1.2.

**Proof of Theorem 1.2.** The statement follows from Lemma 2.3(3) and Theorem 2.7.

At this point we can estimate $\beta(S)$ and get a lower bound to the number of connected components of $\Lambda^+(S^3)$, thereby proving Theorem 1.1. For aesthetic
reasons, we denote \( 2^a = \exp_2 \{a\} \), and the symbol \( o(1) \) is understood with respect to the limit \( q \to \infty \).

**Theorem 2.8.** Let \( S = \text{PSL}_2(q) \). We have

\[
q^{2-o(1)} \leq \beta(S) \leq \frac{q^2}{2} + O(q).
\]

Let \( N(S) \) denote the number of connected components of \( \Lambda^+(S^t) \). Then

\[
N(S) \geq \exp_2 \left\{ q^{2-o(1)} \right\}.
\]

**Proof.** The order of \( \text{Out}(S) \) is at most \( 2 \log q \). By Lemma 2.2 and Theorem 2.7 we get

\[
q^{2-o(1)} \leq \beta \leq \frac{q^2}{2} + O(q),
\]

which proves the first part of the statement. By Theorem 2.6 Stirling’s approximation and (1), we get

\[
N(S) \geq \frac{1}{2} \left( \frac{\beta}{\beta/2} \right) = (1 + o(1)) \cdot \frac{2^\beta}{(2\pi\beta)^{1/2}} \geq \exp_2 \left\{ q^{2-o(1)} \right\},
\]

which proves the last part of the statement. \( \square \)

### 3. Further comments

Recall that \( \Gamma^+(G) \) is the graph obtained by removing the isolated vertices from the generating graph \( \Gamma(G) \) of \( G \). Crestani–Lucchini [CL13b] showed that, if \( G \) is a 2-generated direct power of a nonabelian finite simple group, then \( \Gamma^+(G) \) is connected.

In particular, Theorem 2.8 says that the result of [CL13b] does not hold for invariable generation. Nevertheless, a combinatorial proof along the lines of [CL13b, Theorem 3.1] might be feasible in order to show the following: If a finite simple group \( S \) is such that \( \Lambda^+(S) \) is connected and not bipartite, then \( \Lambda^+(S^t) \) is connected for every \( t \leq \beta(S) \). Unfortunately, the fact that \( \Lambda^+(S) \) is connected is known essentially only for alternating groups [Gar20], therefore the result at this stage would not be of wide use.

We also remark that we are currently unable to construct examples of soluble groups \( G \) for which \( \Lambda^+(G) \) is disconnected.

**Question 3.1.** Let \( G \) be a finite soluble group which is invariably 2-generated. Is the graph \( \Lambda^+(G) \) connected?

Crestani–Lucchini [CL13a] showed that this is true for the graph \( \Gamma^+(G) \) (and in particular Question 3.1 has a positive answer for nilpotent groups).

#### 3.1. \( \Lambda^+(S) \) bipartite.

For the proof of Theorem 2.8 the only important fact about \( S = \text{PSL}_2(q) \) is that the graph \( \Lambda^+(S) \) is bipartite, which follows from the fact that \( S \) admits a 2-covering (see the proof of Lemma 2.4). Recall that, given a finite group \( G \), a 2-covering of \( G \) is a pair \( (H, K) \) of proper subgroups such that

\[
G = \bigcup_{g \in G} H^g \cup \bigcup_{g \in G} K^g.
\]
A NOTE ON THE INVARIANTLY GENERATING GRAPH OF A FINITE GROUP

The 2-coverings of the finite simple groups have been well studied; see Bubboloni \cite{Bub10}, Bubboloni–Lucido \cite{BL02}, Bubboloni–Lucido–Weigel \cite{BLW06, BLW11}, Pellegrini \cite{Pel13}. In particular, all finite simple groups admitting a 2-covering are known, except for some classical groups in small dimension.

We have the following clear implications:

\[ S \text{ admits a 2-covering} \implies \Lambda^+(S) \text{ is bipartite} \implies \Lambda^+(S) \text{ has no triangles} \tag{2} \]

(These implications are a particular case of the inequalities in (3) below.) The reverse of the first implication in (2) does not necessarily hold. For instance, \( A_9 \) does not admit a 2-covering (it was proved in \cite{Bub10} that \( A_n \) admits a 2-covering if and only if \( 4 \leq n \leq 8 \)). On the other hand, it is not difficult to show that \( \Lambda^+(A_9) \) is bipartite. This might be one of only finitely many exceptions.

**Problem 3.2.** Determine the finite simple groups \( S \) for which \( \Lambda^+(S) \) is bipartite (resp., contains no triangles). Up to finitely many cases, do the reverse implications in (2) hold?

These considerations can be viewed more generally as follows. For a noncyclic finite group \( G \), let \( \kappa(G) \) be the clique number of \( \Lambda^+(G) \), that is, the largest order of a complete subgraph of \( \Lambda^+(G) \). Let \( \tau(G) \) be the chromatic number of \( \Lambda^+(G) \), that is, the least number of colours needed to colour the vertices of \( \Lambda^+(G) \) in such a way that adjacent vertices get different colours. Let \( \gamma(G) \) be the normal covering number of \( G \), that is, the least number of proper subgroups of \( G \) such that each element of \( G \) lies in some conjugate of one of these subgroups. The following inequalities hold:

\[ \kappa(G) \leq \tau(G) \leq \gamma(G). \tag{3} \]

(These are “invariable” versions of inequalities studied for instance in \cite{LM09}.) The implications in (2) can be stated as follows for a general noncyclic finite group \( G \):

\[ \gamma(G) \leq 2 \implies \tau(G) \leq 2 \implies \kappa(G) \leq 2. \]

(We note that, for every finite group \( G \), \( \gamma(G) \geq 2 \) and, by \cite{GM12} and \cite{KLS11}, if \( S \) is nonabelian simple then \( \kappa(S) \geq 2 \).

Problem 3.2 asks whether, up to finitely many exceptions, \( \gamma(S) = 2 \iff \tau(S) = 2 \iff \kappa(S) = 2 \).

The invariants \( \kappa(G) \) and \( \gamma(G) \) have been studied; see for instance Britnell–Maróti \cite{BM13}, Bubboloni–Prager–Spiga \cite{BPS13} and Garonzi–Lucchini \cite{GL15}.

As a final remark, the fact that \( \Lambda(G) \) can have no triangles is somewhat strange, in comparison to classical generation. Indeed, for every 2-generated finite group \( G \) of order at least 3, the generating graph \( \Gamma(G) \) contains a triangle, and indeed “many” triangles. This follows from the fact that if \( \langle x, y \rangle = G \) then \( \langle x, xy \rangle = \langle xy, y \rangle = G \).

The fact that this property fails for invariable generation represents an annoying obstacle in order to extend results from the classical to the invariable setting. See the introduction of \cite{GL20} for comments in this direction.

**References**

\cite{BGH20} T. C. Burness, R. Guralnick, and S. Harper. The spread of a finite group. \textit{arXiv preprint arXiv:2006.01421}, 2020.

\cite{BL02} D. Bubboloni and M. S. Lucido. Coverings of linear groups. \textit{Communications in Algebra}, 30(5):2143–2159, 2002.

\cite{BLW06} D. Bubboloni, M. S. Lucido, and T. Weigel. Generic 2-coverings of finite groups of lie type. \textit{Rendiconti del Seminario Matematico della Università di Padova}, 115:209–252, 2006.
Daniele Garzoni, Dipartimento di Matematica “Tullio Levi-Civita”, Università degli Studi di Padova, Padova, Italy

E-mail address: daniele.garzoni@phd.unipd.it