An Exact Solution of Induced Large-$N$
Lattice Gauge Theory at Strong Coupling

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Abstract

I show that the strong coupling solution of the Kazakov–Migdal model with a
general interaction potential $V(\Phi)$ in $D$ dimensions coincides at large $N$ with that
of the hermitean one-matrix model with the potential $\hat{V}(\Phi)$:

$$(2D - 1)\hat{V}' = (D - 1)V' + D\sqrt{(V')^2 + 4(1 - 2D)\Phi^2},$$

whose solution is known. The proof is given for an even potential $V(\Phi) = V(-\Phi)$
by solving loop equations.

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1 Introduction

Solvable matrix models are usually associated with $D \leq 1$ dimensional theories. Kazakov and Migdal have recently proposed that the model (KMM) defined by the partition function

$$Z_{KMM} = \int \prod_{x,\mu} dU_{\mu}(x) \prod_{x} d\Phi_{x} e^{\sum_{x} N_{c} \text{tr} \left( -V(\Phi_{x}) + \sum_{\mu=1}^{D} \Phi_{x} U_{\mu}(x) \Phi_{x+\mu} U_{\mu}^{\dagger}(x) \right) },$$

where the scalar field $\Phi_{x}$ is in the adjoint representation of the gauge group $SU(N_{c})$ and the link variable $U_{\mu}(x)$ belongs to the gauge group, is solvable in the limit of large number of colors, $N_{c}$, for $D > 1$. Since stringy phenomena are associated with the strong coupling phase (see for a review), one is interested in the solution of KMM at strong coupling.

The original idea to solve KMM is based on the fact that the scalar field can be diagonalized by a (local) gauge transformation so that only $O(N_{c})$ degrees of freedom are left and the saddle-point method is applicable as $N_{c} \to \infty$. Migdal proposed to solve the saddle point equation by the Riemann–Hilbert method and derived a master field equation to determine the $N_{c} = \infty$ solution in the strong coupling phase. An explicit solution of this equation for the quadratic potential was found by Gross. A surprising property of the master field equation (not yet completely understood) is that it admits self-consistent scaling solutions with non-trivial critical indices.

There are two other approaches to solving KMM at strong coupling. The first one is based on loop equations which were derived and solved for KMM with the quadratic potential by the author recovering the solution of Ref. . The second one, proposed by Boulatov, relies on the relation of KMM to a matrix model on the Bethe lattice, whose “naive” continuum limit is equivalent for $D > 1$ to a one-matrix model with the upside-down potential.

In the present paper I extend the approach based on loop equations to the case of KMM with an arbitrary potential. On the one hand, this provides a way to solve the model at strong coupling which is an alternative to the Riemann–Hilbert method. On the other hand, this method is along the line of modern studies of matrix models of 2D quantum gravity by means of loop equations (for a review, see Ref. ).

The main results of this paper are as follows. The loop equations are drastically simplified in the strong coupling phase at $N_{c} = \infty$ due to the fact that the averages of closed Wilson loops vanish except for the loops with vanishing minimal area. The resulting equations are satisfied for an even potential $V(\Phi) = V(-\Phi)$ by the ansatz which reduces KMM in $D$ dimensions to a hermitean one-matrix model with the potential $\tilde{V}(\Phi)$:

$$\tilde{V}'(\Phi) = \frac{D-1}{2D-1} V'(\Phi) + \frac{D}{2D-1} \sqrt{(V'(\Phi)')^{2} + 4(1-2D)\Phi^{2}}.$$

The solution of this model with an arbitrary potential is well-known.
In Section 2 I derive the exact loop equations for KMM with an arbitrary potential and any $N_c$. In Section 3 I show how these equations are simplified in the strong coupling phase at $N_c = \infty$. In Section 4 I define the ansatz and obtain the exact solution of the $N_c = \infty$ loop equations in the case of the even potential. Section 5 is devoted to the explanation of the form of the solution from the viewpoint of the large mass expansion. In Section 6 I discuss some properties of the solution and show how it can be compared to the one obtained by the Riemann–Hilbert method. Appendix A contains the details of derivation of the loop equations. In Appendix B I analyze the large mass expansion of the one-link correlator of the gauge fields at $N_c = \infty$.

## 2 Loop equations for arbitrary potential

The loop equations of KMM relate the closed adjoint Wilson loops

$$W_A(C) = \left\langle \frac{1}{N_c^2} \left( |\text{tr} U(C)|^2 - 1 \right) \right\rangle$$

(2.1)

to the open ones with the matter field attached at the ends:

$$G_\lambda(C_{xy}) = \left\langle \frac{1}{N_c} \text{tr} \left( \Phi_x U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle.$$  

(2.2)

The loop equations result from the invariance of the measure in Eq. (1.1) under an arbitrary shift of $\Phi$ and read

$$\left\langle \frac{1}{N_c} \text{tr} \left( V'(\Phi) U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle - \sum_{\mu=-D}^D G_\lambda(C_{(x+\mu)x} C_{xy}) = \delta_{xy} \left\langle \frac{1}{N_c} \text{tr} \left( U(C_{xy}) \frac{1}{\lambda - \Phi_y} \right) \frac{1}{N_c} \text{tr} \left( \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle$$

(2.3)

where the path $C_{(x+\mu)x} C_{xy}$ on the l.h.s. is obtained by attaching the link $(x, \mu)$ to the path $C_{xy}$ at the end point $x$ as is depicted in Fig. 1. The details of derivation are presented in Appendix A. I have omitted here and below additional contact terms which arise at finite $N_c$ due to the fact that $\Phi$ belongs to the adjoint representation, so that Eq. (2.3) is written for the hermitean matrices. This difference should disappear, however, as $N_c \to \infty$ which can be easily proven for the even potential when $\left\langle \frac{1}{N_c} \text{tr} V'(\Phi) \right\rangle$ vanishes.

The path $C_{xy}$ on the r.h.s. of Eq. (2.3) is always closed due to the presence of the delta-function. The explicit equation for the vanishing contour $C_{xx} = 0$ at large $N_c$, when the factorization holds, reads

$$\int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\lambda - \omega} E_{\omega} - 2DG_\lambda(1) = E^2_\lambda$$

(2.4)
Fig. 1: The graphic representation for $G\lambda(C_{xy})$ (a) and $G\lambda(C_{(x+\mu)x}C_{xy})$ (b) entering Eq. (2.3). The empty circles represent $\Phi_x$ or $\Phi_{x+\mu}$ while the filled ones represent $\frac{1}{\lambda-\Phi_y}$. The oriented solid lines represent the path-ordered products $U(C_{xy})$ and $U(C_{(x+\mu)x}C_{xy})$. The color indices are contracted according to the arrows.

where

$$E\lambda \equiv \left\langle \frac{1}{N_c} \text{tr} \left( \frac{1}{\lambda-\Phi_x} \right) \right\rangle = \frac{1}{\lambda} (G\lambda(0) + 1)$$  \hspace{1cm} (2.5)

with $G\lambda$ defined by Eq. (2.2). I have denoted the one-link average by

$$G\lambda(1) = G\lambda(C_{x+\mu})$$ \hspace{1cm} (2.6)

since the r.h.s. does not depend on $x$ and $\mu$ due to the invariance under translations by a multiple of the lattice spacing and/or rotations by a multiple of $\pi/2$ on the lattice. The contour $C_1$ encircles singularities of $E_\omega$ so that the integration over $\omega$ on the l.h.s. of Eq. (2.4) plays the role of a projector picking up negative powers of $\lambda$.

3 Loop equations at large $N_c$

The loop equations for non-vanishing contours $C_{xy} \neq 0$ are drastically simplified at $N_c = \infty$ in the strong coupling region where the closed adjoint Wilson loops (2.1) vanish except the contractable loops (i.e. those with vanishing minimal area $A_{\text{min}}(C)$ which are equivalent to $C_{xx} = 0$ due to the unitarity of $U$’s):

$$W_A(C) = \delta_{0A_{\text{min}}(C)} + \mathcal{O}\left(\frac{1}{N_c^2}\right).$$  \hspace{1cm} (3.1)
While the averages of a new kind arise on the r.h.s. of Eq. (2.3) for \( C_{xx} \neq 0 \), they obey at \( N_c = \infty \) the following analogue of Eq. (3.1)

\[
\left\langle \frac{1}{N_c} \text{tr} \left( U(C_{xx}) \frac{1}{\lambda - \Phi_x} \right) \frac{1}{N_c} \text{tr} \left( U^\dagger(C_{xx}) \frac{1}{\lambda - \Phi_x} \right) \right\rangle = \delta_{0,A_{min}(C)} E_\lambda^2 + O\left( \frac{1}{N_c^2} \right)
\]

(3.2)
i.e. vanish for \( C_{xx} \neq 0 \).

Hence, the strong coupling loop equation for \( C_{xy} \neq 0 \) at \( N_c = \infty \) reads

\[
\left\langle \frac{1}{N_c} \text{tr} \left( V'(\Phi_x) U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle = \sum_{\mu=0}^D \sum_{\mu=0}^D G_\lambda(C(x+\mu)xC_{xy}) = 0
\]

(3.3)
independently of whether \( C_{xy} \) is closed or open.

Therefore, the r.h.s. of the loop equation in nonvanishing at \( N_c = \infty \) only for \( C_{xy} = 0 \) (modulo backtracking) when the proper equation is given by Eq. (2.4). This property of the strong coupling loop equations at \( N_c = \infty \) allows to find a simple solution.

### 4 The strong coupling solution

Let us solve the set (2.4), (3.3) of the \( N_c = \infty \) loop equations at strong coupling by the following ansatz in the case of the even potential

\[
\left\langle \frac{1}{N_c} \text{tr} \left( F(\Phi_x) U(x) \Phi_{x+\mu} U^\dagger(x) \right) \right\rangle = \left\langle \frac{1}{N_c} \text{tr} \left( F(\Phi_x) \Phi_x \Lambda(\Phi_x) \right) \right\rangle
\]

(4.1)

where \( F(\Phi) \) is arbitrary. The function \( \Lambda(\omega) \) is analytic at \( \omega = 0 \):

\[
\Lambda(\omega) = \sum_{k=0}^\infty \Lambda_k \omega^k.
\]

(4.2)

For \( G_\lambda(1) \) which is defined by Eq. (2.4), Eq. (4.1) can be written as

\[
G_\lambda(1) = \int_{C_1} \frac{d\omega}{2\pi i} \frac{\omega \Lambda(\omega)}{\lambda - \omega} E_\omega
\]

(4.3)

where the contour \( C_1 \) encircles singularities of \( E_\omega \), i.e. the same as in Eq. (2.4).

The formula (4.3) extends to the general potential the one

\[
G_\lambda(1) = \Lambda_0 \lambda E_\lambda \quad \text{(quadratic potential)}
\]

(4.4)

for the quadratic potential which is associated with \( [3] \)

\[
\Lambda(\omega) = \Lambda_0 \quad \text{(quadratic potential)},
\]

(4.5)
i.e. \( \Lambda_k = 0 \) for \( k \geq 1 \).
The substitution of the ansatz (4.3) into Eq. (2.4) yields
\[
\int_{C_1} \frac{d\omega}{2\pi i} \frac{\tilde{V}'(\omega)}{\lambda - \omega} E_\omega = E_\lambda^2
\] (4.6)
where
\[
\tilde{V}'(\omega) = V'(\omega) - 2D\omega \Lambda(\omega)
\] (4.7)
which coincides with the loop equation for the hermitean one-matrix model with the potential \( \tilde{V} \).

The dependence of \( \Lambda(\omega) \) on the potential \( V \) can be determined from Eq. (3.3). The simplest way to do this is to consider Eq. (3.3) in the case when \( C_{xy} \) is just one link \( (x, \mu_0) \) and to take the \( 1/\lambda^2 \) term of the \( 1/\lambda \) expansion. The resulting equation reads explicitly
\[
\left\langle \frac{1}{N_c} \text{tr} \left( V'(\Phi_x) U_{\mu_0}(x) \Phi_{x+\mu_0} U_{\mu_0}^\dagger(x) \right) \right\rangle - \left\langle \frac{1}{N_c} \text{tr} \Phi_{x+\mu_0}^2 \right\rangle \\
- \sum_{\mu=0}^{D} \left\langle \frac{1}{N_c} \text{tr} \left( \Phi_{x-\mu} U_{\mu}(x-\mu) U_{\mu_0} (x) \Phi_{x+\mu_0} U_{\mu_0}^\dagger(x-\mu) \right) \right\rangle = 0
\] (4.8)
which reduces after the substitution of the ansatz (4.3) to
\[
\int_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\omega} \Lambda(\omega) - 1 - (2D - 1)\Lambda^2(\omega) \omega^2 E_\omega = 0.
\] (4.9)
This equation is satisfied provided that \( \Lambda(\omega) \) obeys the quadratic equation
\[
\frac{V'(\omega)}{\omega} - \frac{1}{\Lambda(\omega)} - (2D - 1)\Lambda(\omega) = 0
\] (4.10)
for the generic potential
\[
V(\omega) = \sum_{n=1}^{\infty} t_{2n} \omega^{2n}
\] (4.11)
and \( t_2 \equiv m_0^2 \).

The solution to Eq. (4.10) reads
\[
\Lambda(\omega) = \frac{2}{V'(\omega) + \sqrt{\left( \frac{V'(\omega)}{\omega} \right)^2 + 4(1 - 2D)}}
\] (4.12)
which recovers the one \[8\] for the quadratic potential when
\[
V'(\omega) = 2m_0^2 \omega \quad \text{(quadratic potential)}
\] (4.13)
and Eq. (4.5) holds. The fact that the ansatz (4.3) with \( \Lambda(\omega) \) given by Eq. (4.12) satisfies Eq. (3.3) means that it is indeed a solution providing Eq. (4.6) with
\[
\frac{\tilde{V}'(\omega)}{\omega} = \frac{D - 1}{2D - 1} \frac{V'(\omega)}{\omega} + \frac{D}{2D - 1} \sqrt{\left( \frac{V'(\omega)}{\omega} \right)^2 + 4(1 - 2D)}
\] (4.14)
\[^2\text{For a review, see Ref. \[8\].}\]
is satisfied.

The one-cut solution to this equation for an arbitrary potential is well-known \[9\]

\[ E_{\lambda} = \int_{C_1} \frac{d\omega}{4\pi i} \frac{\tilde{V}'(\omega)}{\lambda - \omega} \sqrt{\frac{(\lambda - x)(\lambda - y)}{(\omega - x)(\omega - y)}} \] (4.15)

where \(x\) and \(y\) are expressed via \(\tilde{V}\) by

\[ \int_{C_1} \frac{d\omega}{2\pi i} \frac{\tilde{V}'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} = 0, \quad \int_{C_1} \frac{d\omega}{2\pi i} \frac{\omega \tilde{V}'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} = 2. \] (4.16)

One gets \(x = -y\) for the even potential.

The formulas (4.3), (4.14), (4.15) and (4.16) completes the solution of the strong coupling loop equations of KMM at \(N_c = \infty\). The saddle point value of \(\Phi_x\) is totally determined to be \(x\)-independent and is described, modulo a gauge transformation, by the spectral density

\[ \rho(\lambda) = \frac{1}{2\pi^2} \int_{y}^{x} dt \frac{\tilde{V}'(t) - \tilde{V}'(\lambda)}{t - \lambda} \sqrt{\frac{(x - \lambda)(\lambda - y)}{(x - t)(t - y)}} \text{ for } y < \lambda < x \] (4.17)

with support \(y < \lambda < x\). Properties of this solution are discussed in the next section.

5 Relation to the large mass expansion

The peculiar form (4.3) of the strong coupling solution at \(N_c = \infty\) can be understood in the framework of the large mass expansion. To this aim let us consider the one-link correlator of the gauge fields

\[ \left\langle \frac{1}{N_c} \text{tr} \left(t^a U \Phi_{x+\mu} U^\dagger\right) \right\rangle_U = \int dU \frac{e^{N_c \text{tr} \left(\Phi_x U \Phi_{x+\mu} U^\dagger\right)}}{N_c} \frac{1}{N_c} \text{tr} \left(t^a U \Phi_{x+\mu} U^\dagger\right) \] (5.1)

where the averaging is only w.r.t. \(U\) while \(\Phi_x\) and \(\Phi_{x+\mu}\) play the role of external fields. \(t^a\) \((a = 1, \ldots, N_c^2 - 1)\) stand for generators of \(SU(N_c)\) which are normalized by

\[ \frac{1}{N_c} \text{tr} t^a t^b = \delta^{ab}. \] (5.2)

As was proposed in Refs. \[\[1, 10\] the following formula holds at \(N_c = \infty\):

\[ \left\langle \frac{1}{N_c} \text{tr} \left(t^a U \Phi_{x+\mu} U^\dagger\right) \right\rangle_U = \sum_{m=0}^{\infty} \Lambda_m \frac{1}{N_c} \text{tr} \left(t^a \Phi_x^{m+1}\right). \] (5.3)

It is shown in Appendix B how this formula can be obtained in the large mass expansion.
Eq. (5.3) allows to explain the solution of the previous section as follows. Let us multiply both sides of Eq. (5.3) by $\text{tr}(t^a F(\Phi_x))$ which gives, using the completeness condition (A.4),

$$
\frac{1}{N_c} \text{tr} \left( F(\Phi_x) U \Phi_{x+\mu} U^\dagger \right) = \\
\frac{1}{N_c} \text{tr} \left( F(\Phi_x) \Phi_x \Lambda(\Phi_x) \right) + \frac{1}{N_c} \text{tr} \left( F(\Phi_x) \right) \frac{1}{N_c} \text{tr} \left( \Phi_{x+\mu} - \Phi_x \Lambda(\Phi_x) \right)
$$

(5.4)

where $\Lambda(\Phi)$ is defined by Eq. (4.2). The second term on the r.h.s. which is due to the difference between the adjoint representation and the hermitean matrices vanishes for the even potential at $N_c = \infty$. Hence, Eq. (5.4) for $\Phi_x$ and $\Phi_{x+\mu}$ given by the saddle point matrix $\Phi_S$ recovers Eq. (4.1). On the other hand, the solution of the previous section allows to calculate $\Lambda_m$ in Eq. (5.3) as the coefficients on the expansion of (4.12) in $\omega$.

### 6 Discussion

The above strong coupling solution is realized at given $D$ only in some region of the couplings $t_{2k}$’s entering the potential (4.11). At $D = 0$ it coincides with the well-known solution of the hermitean one-matrix model. At any $D$ but $t_2 \neq 0$, $t_{2k} = 0$ for $k \geq 1$, the solution coincides with the one for the quadratic potential [4]. For this reason I expect that it is realized in some region around this point. The condition is that Eq. (4.16) should yield real $x$ and $y$ and the spectral density (4.17), which describes the distribution of eigenvalues of the saddle point matrix $\Phi_S$, should be positive. This is a restriction on the one-cut solution which is satisfied in some region of values of the couplings $t_{2k}$’s. It is well known that for the simple quartic potential

$$
\tilde{V}(\omega) = \tilde{t}_2 \omega^2 + \tilde{t}_4 \omega^4
$$

(6.1)

the one-cut spectral density is positive for $|\tilde{t}_4| \leq \tilde{t}_2/12$. This is, however, a nontrivial restriction on the potential $V$ since

$$
V'(\omega) = D\sqrt{(\tilde{V}'(\omega))^2 + 4\omega^2} - (D - 1)\tilde{V}'(\omega).
$$

(6.2)

One more restriction on the one-cut solution is given by the requirement that the expression under the square root in Eq. (4.14) must be positive for any $\omega$ which belongs to the support of the spectral density. If this expression becomes negative for some values of $t_{2k}$’s this simply means that the one-cut solution is not realized and one should look for a more sophisticated support (multi-cut solutions). This is quite standard for the large-$N$ phase transitions which occur at the values of couplings where the behavior of the spectral density changes.

It is interesting to compare our solution with that obtained by the Riemann–Hilbert method. It is easy to identify the function $T_\lambda(z)$ which determines the solution of Ref. [3].
with the one-link correlator
\[ \mathcal{T}_\lambda(z) - 1 = \left\langle \frac{1}{N_c} \text{tr} \left( \frac{1}{z - \Phi_S} U \frac{1}{\lambda - \Phi_S} U^\dagger \right) \right\rangle_U \] (6.3)
which is defined by the same average as in Eq. (5.1). The proof is based on the following
extension of the Migdal procedure. Let us define \[ G_\lambda(\Phi_x) \equiv \frac{1}{I(\Phi_x, \Phi_{x+\mu})} \frac{1}{\lambda - \frac{1}{N_c} \partial_\Phi} I(\Phi_x, \Phi_{x+\mu}) \] (6.4)
where \( I \) stands for the Itzykson–Zuber integral
\[ I(\Phi_x, \Phi_{x+\mu}) = \int dU e^{N_c \text{tr} \left( \Phi_x U \Phi_{x+\mu} U^\dagger \right)}. \] (6.5)
By a direct differentiation of Eq. (6.5) one gets
\[ \frac{1}{N_c} \text{tr} \left( \frac{1}{z - \Phi_x} G_\lambda(\Phi_x) \right) = \left\langle \frac{1}{N_c} \text{tr} \left( \frac{1}{z - \Phi_x} U \frac{1}{\lambda - \Phi_{x+\mu}} U^\dagger \right) \right\rangle_U. \] (6.6)
Rewriting the l.h.s. via the spectral density, substituting for \( \Phi_x \) and \( \Phi_{x+\mu} \) the saddle point value \( \Phi_S \) and remembering the definition of \( T_\lambda(z) \) \[ T_\lambda(z) = 1 + \int d\mu \frac{\rho(\mu) G_\lambda(\mu)}{\mu - \lambda}, \] (6.7)
one proves Eq. (6.3).

From Eq. (6.6) it is easy to see alternatively that \( N_c = \infty \)
\[ \mathcal{T}_\lambda(z) - 1 = \left\langle \frac{1}{N_c} \text{tr} \left( \frac{1}{z - \Phi_x} U_{\mu}(x) \frac{1}{\lambda - \Phi_{x+\mu}} U^\dagger_{\mu}(x) \right) \right\rangle \] (6.8)
where the average is w.r.t. the same measure as in Eq. (1.1). Therefore, we get asymptotically
\[ \mathcal{T}_\lambda(z) = 1 + \frac{E_\lambda}{z} + \frac{G_\lambda(1)}{z^2} + \ldots \quad \text{as } z \to \infty \] (6.9)
so that our \( G_\lambda(1) \) is to be compared with the \( O(z^{-2}) \) term in the expansion of \( \mathcal{T}_\lambda(z) \).
It would be very interesting to calculate exactly the correlator on the r.h.s. of Eq. (6.8),
which should be manifestly symmetric w.r.t. \( \lambda \) and \( z \), by solving the loop equations in
order to compare with \( T_\lambda(z) \) of Ref. 3.

It is worth mentioning that in finding the solution I did not use the fact that \( \Phi_x \) can
be diagonalized. For this reason a solution analogous to that of this paper exists for
the adjoint fermion model \[ which reduces to the complex one-matrix model \[. This result, as well as an analysis of the \( D \leq 1 \) case, will be published elsewhere. The most
interesting question is whether the solution of this paper admits a continuum limit for
\( D > 1 \).

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Appendix A Derivation of the loop equations

Let us consider an equation which results from the invariance of the measure over $\Phi$ in the open-loop average (2.2) under an infinitesimal shift

$$\Phi_x \rightarrow \Phi_x + \xi_x \quad (A.1)$$

of $\Phi_x$ at the given site $x$ with $\xi_x$ being an infinitesimal hermitean matrix. For KMM one should impose $\text{tr} \, \xi_x = 0$ in order for the shifted matrix to belong to the adjoint representation of $SU(N_c)$. Since this condition should be inessential as $N_c \rightarrow \infty$, I derive loop equations for the hermitean model, defined by the partition function (1.1) with the integration going over arbitrary hermitean matrices $\Phi_x$. $\xi_x$ in Eq. (A.1) is then arbitrary hermitean.

It is convenient to introduce $N_c^2$ generators

$$[t^A]_{ij} = \left( \delta_{ij}, [t^a]_{ij} \right) \quad (A.2)$$

with $t^a$ ($a = 1, \ldots, N_c^2 - 1$) being the standard generators of $SU(N_c)$. The generators (A.2) obey the following normalization

$$\frac{1}{N_c} \text{tr} \, t^A t^B = \delta^{AB} \quad (A.3)$$

and completeness condition

$$[t^A]_{ij} [t^A]_{kl} = N_c \delta_{il} \delta_{kj} \quad (A.4)$$

An arbitrary $N_c \times N_c$ hermitean matrix $\Phi$ can be represented as

$$\Phi = t^A \Phi^A \quad \text{where} \quad \Phi^A = \left( \frac{1}{N_c} \text{tr} \, \Phi, \frac{1}{N_c} \text{tr} \, t^a \Phi \right) \quad (A.5)$$

with $\Phi^0 = \frac{1}{N_c} \text{tr} \, \Phi$ vanishing if $\Phi$ is taken in the adjoint representation of $SU(N_c)$.

To derive the loop equation I apply a trick similar to that used in deriving loop equations of QCD [9]. Let us consider the loop average

$$\left\langle \frac{1}{N_c} \text{tr} \left( t^A U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle = 0 \quad (A.6)$$

where the averaging is taken with the same measure as in Eq. (1.1), which vanishes due to the gauge invariance. Performing the shift (A.1) of $\Phi_x$, using the invariance of the measure and calculating $\partial / \partial \Phi^B(x)$, one gets

$$\left\langle \frac{1}{N_c} \text{tr} \, \left( t^B V'(\Phi) \right) \frac{1}{N_c} \text{tr} \left( t^A U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle$$

$$- \sum_{\mu=1}^{D} \sum_{\mu \neq 0} \left\langle \frac{1}{N_c} \text{tr} \left( t^B U_{\mu}(x) \Phi_{x+\mu} U_{\mu}^\dagger(x) \right) \frac{1}{N_c} \text{tr} t^A U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right\rangle$$

$$= \delta_{xy} \left\langle \frac{1}{N_c^3} \text{tr} \left( t^A U(C_{xy}) \frac{1}{\lambda - \Phi_y} t^B \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle \quad (A.7)$$

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The l.h.s. of this equation results from the variation of the action while the r.h.s. represents
the commutator term resulting from the variation of the integrand.

The averaging over the gauge group picks up two nonvanishing invariant equations for
the hermitean matrices. The first one can be obtained contracting Eq. (A.7) by \(\delta^{AB}\)
\((A, B = 0, \ldots, N_c^2 - 1)\) while the second one is given by the \(A, B = 0\) component.

The first equation reads

\[
\left\langle \frac{1}{N_c} \text{tr} \left( V'(\Phi_x) U(C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle \\
- \sum_{\mu = -D}^{D} \left\langle \frac{1}{N_c} \text{tr} \left( \Phi_{x+\mu} U(C_{(x+\mu)x}C_{xy}) \frac{1}{\lambda - \Phi_y} U^\dagger(C_{(x+\mu)x}C_{xy}) \right) \right\rangle \\
= \delta_{xy} \left\langle \frac{1}{N_c} \text{tr} \left( U(C_{xy}) \frac{1}{\lambda - \Phi_y} \right) \frac{1}{N_c} \text{tr} \left( \frac{1}{\lambda - \Phi_y} U^\dagger(C_{xy}) \right) \right\rangle
\]

(A.8)

where the contour \(C_{(x+\mu)x}C_{xy}\) is obtained by attaching the link \((x, \mu)\) to the path \(C_{xy}\)
at the end point \(x\) as is depicted in Fig. 1. Using the definition (2.2), this equation can be
written finally in the form (2.3).

The second equation which is given by the \(A, B = 0\) component of Eq. (A.7) reads

\[
\left\langle \frac{1}{N_c} \text{tr} V''(\Phi_x) \frac{1}{N_c} \text{tr} \frac{1}{\lambda - \Phi_y} \right\rangle \\
- \sum_{\mu = -D}^{D} \left\langle \frac{1}{N_c} \text{tr} \Phi_{x+\mu} \frac{1}{N_c} \text{tr} \frac{1}{\lambda - \Phi_y} \right\rangle \\
= \left\langle \frac{1}{N_c^3} \text{tr} \frac{1}{(\lambda - \Phi_y)^2} \right\rangle.
\]

(A.9)

In the large-\(N_c\) limit when the factorization holds, Eq. (A.9) is automatically satisfied as
a consequence of the \(O(\lambda^{-1})\) term in Eq. (A.8).

### Appendix B  The one-link correlator at large \(N_c\)

Eq. (5.3) can be derived for \(\Phi\) given by the saddle point (master field) configuration
analyzing the large mass expansion which allows to calculate the one-link correlator (5.1)
in the strong coupling phase (i.e. before an expected large-\(N_c\) phase transition). To
calculate it, let us expand the numerator in powers of \(\Phi\) as is depicted in Fig. 2.

The idea is now not to calculate the complicated integral over \(U_\mu(x)\) in Eq. (5.1)
but rather substitute for \(\Phi_x\) and \(\Phi_{x+\mu}\) the saddle point value \(\Phi_S\) which is determined
by the future integration over \(\Phi\) according to Eq. (1.1). Notice, that the integrals over
\(\Phi_x\) and \(\Phi_{x+\mu}\) are independent to each order of the large mass expansion. Therefore, one
substitutes

\[
(\Phi^a_x)_S(\Phi^b_x)_S = K \frac{1}{N_c^2} \delta^{ab}
\]

(B.1)
Fig. 2: The graphic representation of the large mass expansion of the one-link correlator (5.1). The right filled circles represent $\Phi_{x+\mu}$ while the left ones represent $\Phi_x$. The empty circle represents $t^a$.

and

\[
(\Phi^a_x)_S(\Phi^b_{x+\mu})_S = 0 \tag{B.2}
\]

where

\[
K = \frac{1}{N_c} \text{tr} \Phi^2_S \tag{B.3}
\]

and (B.2) vanishes due to the gauge invariance.

After the use of Eqs. (B.1), (B.2) and the completeness condition (A.4) the color indices are contracted in such a way that all the matrices $U_{\mu}(x)$ disappear due to the unitarity. The simplest contractions are depicted in Fig. 3. Only the connected diagrams should be taken into account due to the presence of the denominator. The diagrams of the type of Fig. 3a are the only ones which emerge for the quadratic potential. They always result in $\Lambda_0$. The diagrams of the type Fig. 3b appear when the interaction is present. They result in $\Lambda_m$ with $m \geq 1$.

Finally, let us notice that this structure of $\Lambda_m$ is not spoiled by the fact that $\Phi_x$ enters $2D$ links emanating from the point $x$. This affect only combinatorics making $K$ to be $D$-dependent.
\[ t^a \Phi_x = K \frac{1}{N_c} \text{tr} t^a \Phi_x \]

\[ t^a \Phi_x = t_3 K^3 \frac{1}{N_c} \text{tr} t^a \Phi_x^2 \]

a) The diagram results in the contribution to \( \Lambda_0 \).
b) The diagram results in the contribution to \( \Lambda_1 \).

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