Plane Jacobian conjecture for simple polynomials

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Abstract

A non-zero constant Jacobian polynomial map $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ has a polynomial inverse if the component $P$ is a simple polynomial, i.e., if, when $P$ extended to a morphism $p : X \to \mathbb{P}^1$ of a compactification $X$ of $\mathbb{C}^2$, the restriction of $p$ to each irreducible component $C$ of the compactification divisor $D = X - \mathbb{C}^2$ is either degree 0 or 1.

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1. Let $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map, $P, Q \in \mathbb{C}[x, y]$, and denote $JF := P_yQ_x - P_xQ_y$ the Jacobian of $F$. The mysterious Jacobian conjecture (JC) (See [4] and [2]), posed first by Keller in 1939 and still open, asserts that $F$ has a polynomial inverse if the Jacobian $JF$ is a non-zero constant. In 1979 by an algebraic approach Razas [17] proved this conjecture for the most simple geometrical case when $P$ is a rational polynomial, i.e., the generic fiber of $P$ is a punctured sphere, and all fibers $P = c, c \in \mathbb{C}$, are irreducible. In attempt to understand the geometrical nature of (JC), this case was also reprovved by Heitmann [5] and Lê and Weber [10] in some other approaches. In fact, as observed by Neumann and Norbudy in [12], every rational polynomial with all irreducible fibers is equivalent to the coordinate

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polynomial. Most recent, Lê in [7] and [8] present the following observation, which was announced in the Hanoi conference, 2006, and the Kyoto conference, 2007.

**Theorem 1.** (Theorem 3.2 and Corollary 3.8 in [8]) A non-zero constant Jacobian polynomial map \( F = (P, Q) \) has a polynomial inverse if \( P \) is a simple rational polynomial.

Here, following [11], a polynomial map \( P : \mathbb{C}^2 \to \mathbb{C} \) is simple if, when extended \( P \) to a morphism \( p : X \to \mathbb{P}^1 \) of a compactification \( X \) of \( \mathbb{C}^2 \), the restriction of \( p \) to each irreducible component \( \ell \) of the compactification divisor \( D = X - \mathbb{C}^2 \) is either of degree 0 or 1. In fact, as in the proof in [8] of Theorem 1, if a component of non-zero constant Jacobian map \( F = (P, Q) \) is a simple rational polynomial, then this component determines a locally trivial fibration.

In this short paper we would like to present another explanation for Theorem 1 from viewpoint of the geometry of the non-proper value set of the map \( F \). In fact, we shall prove

**Theorem 2.** A non-zero constant Jacobian polynomial map \( F = (P, Q) \) has a polynomial inverse if \( P \) is a simple polynomial.

In any meaning, the addition condition on the simple polynomial component in a non-zero constant Jacobian polynomial map may be viewed as a kind of “good” local conditions at infinity, but it seems to be not a global one. A completed classification of all simple rational polynomials was presented in [11].

2. Given a polynomial map \( F = (P, Q) \) of \( \mathbb{C}^2 \). Following [6], the non-proper value set \( A_F \) of \( F \) is the set of all values \( a \in \mathbb{C}^2 \) such that there exists a sequence \( \mathbb{C}^2 \ni b_i \to \infty \) with \( F(b_i) \to a \). This set \( A_F \) is either empty or an algebraic curve in \( \mathbb{C}^2 \) for which every irreducible component is the image of a non-constant polynomial map from \( \mathbb{C} \) into \( \mathbb{C}^2 \). Our argument in the proof of Theorem 2 here is based on the following facts, that was presented in [13] and can be reduced from [3] (see also [14] and [15] for other refine versions).

**Theorem 3.** Support \( F = (P, Q) \) is a polynomial map with non-zero constant Jacobian. If \( A_F \neq \emptyset \), then every irreducible component of \( A_F \) can be parameterized by polynomial maps \( \xi \mapsto (\varphi(\xi), \psi(\xi)) \) with

\[
\deg \varphi / \deg \psi = \deg P / \deg Q.
\]
This theorem together with the Abhyankar-Moh Theorem \[1\] on the embeddings of the line to the plane allows us to obtain:

**Theorem 4.** A polynomial map \(F\) of \(\mathbb{C}^2\) must have singularities if its non-proper value set \(A_F\) has an irreducible component isomorphic to the line.

A simple proof of Theorem 4 recently presented in \[15\] gives a description in terms of Newton-Puiseux data how the singularity occurs in this situation.

3. To use Theorem 4 in the situation of simple polynomials, at first, we need to describe the the non-proper value curve \(A_F\) in terms of the regular extension of \(F\) in a convenience compatification \(X \supset \mathbb{C}^2\). Given a polynomial \(F = (P, Q)\), extend it to a map \(F : \mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) and resolve the points of indeterminacy to get a regular map \(f = (p, q) : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) that coincides with \(F = (P, Q)\) on \(\mathbb{C}^2 \subset X\). We call \(D = X - \mathbb{C}^2\) the divisor at infinity. The divisor \(D\) consists of a connected union of rational curves isomorphism to \(\mathbb{P}^1\) and the dual graph of the divisor \(D\) is a tree. An irreducible component \(\ell\) of \(D\) is a horizontal curve of \(P\) (or \(Q\)) if the restriction of \(p\) (res. \(q\)) to \(E\) is not a constant mapping. An irreducible component \(\ell\) of \(D\) is a dicritical curve of \(F\) if the restriction of \(f\) to \(\ell\) is not a constant mapping. A dicritical curve of \(F\) must be a horizontal curve of \(P\) or \(Q\). Although the compactification defined above is not unique, the horizontal curves of \(P\) or \(Q\) as well as the dicritical curves of \(F\) are essentially independent of choice. Further, by blowdown components of self-intersection \(-1\) corresponding to linear vertexes or endpoint in the dual graph of \(D\), but not of dicritical curves of \(F\), horizontal curves of \(P\) or \(Q\), we can work with minimal compactification \(X \supset \mathbb{C}^2\) on which \(F\) can be extended to a regular morphism \(f = (p, q) : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\).

Let us denote \(D_\infty := f^{-1}((\{\infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{\infty\}))\). The following description of the dual graph of the divisor \(D\) is well-known (see, for example, in \[18\], \[16\] and \[9\]).

**Proposition 1.** i) The dual graph of the divisor \(D\) is a tree;
   ii) The dual graph of the curve \(D_\infty\) is a tree;
   iii) The dual graph of each connected component of the closure of \((D - D_\infty)\) is a linear path of the form

\[
\bigcirc \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet
\]

in which the beginning vertex \(\bigcirc\) is a dicritical curve of \(F\) and the next possible vertexes \(\bullet\) are curves to which the restriction of \(f\) are finite constant mappings.
The following provides a description of the non-proper value set $A_F$ of $F$ in terms of regular extension of $F$ in a minimal compatification $X \supset \mathbb{C}^2$.

**Proposition 2.**  

i) 

$$A_F = \bigcup_{\text{dicritical curves } \ell \text{ of } F} f(\ell) \cap \mathbb{C}^2. \quad (2)$$

ii) Let $\ell$ be a dicritical curve of $F$. Then, $\ell$ and the curve $D_\infty$ have an unique common point. Let $\ell^* := \ell - D_\infty$. Then, the curve $\ell^*$ is isomorphic to $\mathbb{C}$ and

$$f(\ell^*) = f(\ell) \cap \mathbb{C}^2 \quad (3)$$

iii) 

$$A_F = \bigcup_{\text{dicritical curves } \ell \text{ of } F} f(\ell^*). \quad (4)$$

**Proof.** Conclusion (i) can be easy verified by the definition of $A_F$ and a simple topological argument. Conclusion (ii) follows from the fact in Proposition 1 that the dual graph of the divisor $D$ is a tree. Conclusion (iii) results from (i) and (ii). \hfill \square

4. Now, we consider the situation when the restriction of $p$ to a dicritical curve $\ell$ of $F$ is of degree 1.

**Lemma 1.** Let $\ell$ be a dicritical component of $F$. If the restriction of $p$ to $\ell$ is of degree 1, then the image $f(\ell^*)$ is isomorphic to the line $\mathbb{C}$.

**Proof.** Suppose $\ell$ is a dicritical component of $F$ and the degree of the restriction $p|_\ell$ equals 1. Then, $p|_\ell : \ell \rightarrow \mathbb{P}^1$ is injective, and hence, is bijective, since $\ell$ is isomorphic to $\mathbb{P}^1$. This ensures that the curve $f(\ell^*)$ intersects each line $\{(u,v) \in \mathbb{C}^2 : u = c\}$, $c \in \mathbb{C}$, at an unique point. Then, the polynomial $H(u,v)$ defining the curve $f(\ell^*) \subset \mathbb{C}^2$ can be chosen of the form $v + h(u)$, $h \in \mathbb{C}[u]$. So, the automorphism $A(u,v) := (u, v - h(u))$ maps isomorphically the curve $f(\ell^*)$ onto the line $v = 0$. \hfill \square

**Proof of Theorem** Suppose $F = (P,Q)$ with $JF \equiv c \neq 0$ and $P$ is a simple polynomial. Note that each dicritical curve of $F$ must be a horizontal curve of $P$ or $Q$. Since $JF \equiv c \neq 0$ and $P$ is simple, in view of Theorem
4 and Lemma 1, a horizontal curve of $P$ cannot be a dicritical curve of $F$. So, if $\ell$ is a dicritical curve of $F$, then $\ell$ must be a horizontal curve of $Q$ and the restriction $p|_{\ell}$ maps $\ell$ to a finite constant. Thus, for such $\ell$ the image $f(\ell^*)$ is a line $u = \text{const.}$. The last is impossible again by Theorem 4 as $JF \equiv c \neq 0$. Hence, $F$ has not any dicritical component. Then, $A_F = \emptyset$ by Proposition 2 and $F$ is a proper map by the definition of $A_F$. Therefore, by simple connectedness of $\mathbb{C}^2$ the local diffeomorphic map $F$ must be bijective. Thus, $F$ is an automorphism of $\mathbb{C}^2$.

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