NEW INTEGRABLE TWO-CENTRE PROBLEM ON SPHERE IN DIRAC MAGNETIC FIELD

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Abstract. We present a new integrable version of the two-centre problem on two-dimensional sphere in the presence of the Dirac magnetic monopole. The new system can be written on the dual space of Lie algebra $e(3)$ and is integrable both in classical and quantum case.

1. Introduction

The celebrated Euler two-centre problem [8] was one of the first non-trivial mechanical systems integrated completely since the solution of famous Kepler problem by Newton. In its two-dimensional version the Hamiltonian has the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{\mu}{r_1} - \frac{\mu}{r_2},$$

where

$$r_1 = \sqrt{(q_1 + c)^2 + q_2^2}, \quad r_2 = \sqrt{(q_1 - c)^2 + q_2^2}$$

are the distances from the two centres fixed at the points $(\pm c, 0)$. In the confocal coordinates $u_1 = r_1 + r_2, u_2 = r_1 - r_2$ the variables in the corresponding Hamilton-Jacobi equation can be separated, leading to the explicit solution of the system in quadratures (see Arnold [3]).

Its natural generalisation to the spaces of constant curvature was originally found in 1895 by Killing [11] and rediscovered by Kozlov and Harin [12] as part of a general family of systems, separable in spherical elliptic coordinates. More of the history of this problem can be found in Borisov and Mamaev [4], who also discussed various integrable generalisations of this system.

The main observation is that for the system on unit sphere $S^2$ the analogue of the potential $\mu/r$ is $\mu \cot \theta$, where $\theta$ is the angle between $q \in S^2$ and the fixed centre (which also the distance in the spherical geometry). As a result the corresponding potential

$$U = -\mu \cot \theta_1 - \mu \cot \theta_2,$$

where $\theta_1$ and $\theta_2$ are the spherical distances from the fixed centres, has actually four singularities, which can be interpreted as two antipodal pairs of centres with opposite charges $\pm \mu$ (see Fig. 1). So we should probably call Killing’s version as a 4-centre problem and consider the interaction as electric Coulomb rather than gravitational Newtonian one.
To write down the explicit formula for the Hamiltonian and the additional integral it is convenient to use the canonical Lie-Poisson bracket on the dual space $e(3)^*$ of the Lie algebra of the group of motion of the Euclidean space. The corresponding variable $M_i, q_i, i = 1, 2, 3$ have the Poisson brackets

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, q_j\} = \epsilon_{ijk} q_k, \quad \{q_i, q_j\} = 0. \quad (1)$$

We have two Casimir functions

$$C_1 = |q|^2, \quad C_2 = (M, q).$$

The symplectic leaf with $C_1 = |q|^2 = 1, C_2 = (M, q) = 0$ is symplectically isomorphic to the cotangent bundle of the unit sphere $T^*S^2$.

In the coordinates $M, q$ the Hamiltonian of the spherical analogue of the Euler two-centre problem is

$$H = \frac{1}{2}|M|^2 - \mu \frac{\beta q_3 - \alpha q_1}{\sqrt{q_2^2 + (\alpha q_3 + \beta q_1)^2}} - \mu \frac{\beta q_3 + \alpha q_1}{\sqrt{q_2^2 + (\alpha q_3 - \beta q_1)^2}},$$

where $\mu, \alpha, \beta$ are parameters such that $\alpha^2 + \beta^2 = 1$.

We have 4 fixed centres at $(\pm \alpha, 0, \pm \beta)$, two of which at $(\pm \alpha, 0, \beta)$ for $\mu > 0$ are attractive, while their antipodes $(\pm \alpha, 0, -\beta)$ are repulsive (see Fig. 1).

The explicit form of the additional integral at the special level $(M, q) = 0$ was found by Mamaev \[14\] (see also \[4\] \[1\])

$$F = \alpha^2 M_1^2 - \beta^2 M_3^2 - 2\alpha \beta \left( \mu \frac{\beta q_1 - \alpha q_3}{\sqrt{q_2^2 + (\beta q_1 - \alpha q_3)^2}} + \mu \frac{\alpha q_1 + \beta q_3}{\sqrt{q_2^2 + (\alpha q_3 + \beta q_1)^2}} \right).$$

For the recent detailed analysis of the orbits in this system see the paper by Gonzalez Leon et al \[13\].

\[1\]There is a couple of sign typos in the explicit form of the integral in these papers, which were kindly corrected for us by Ivan Mamaev.
Note that on the symplectic leaves with \((M, q) \neq 0\) the system is considered to be not integrable, although we are not sure if this was rigorously proved.

The aim of this paper is to introduce a new integrable system on \(e(3)^*\) with the Hamiltonian
\[
H = \frac{1}{2}|M|^2 - \mu \frac{|q|}{\sqrt{R(q)}}, \\
R(q) = Aq_2^2 + Bq_1^2 + (A + B)q_3^2 - 2\sqrt{AB}|q|q_3
\]
with parameters \(\mu, A, B\) satisfy \(A > B > 0\).

This system can be interpreted as the motion on the unit sphere with very special potential having two singularities of Coulomb/Newtonian type in the external field of Dirac magnetic monopole (see more detail in the next section).

We show that the new system is integrable in both classical and quantum case for all values of parameters and for all magnetic charges. We discovered this system as a special case in our classification of the integrable generalisations of Dirac magnetic monopole [21].

2. New system

Consider now the general symplectic leaves in \(e(3)^*\), which are the coadjoint orbits of the Euclidean motion group \(E(3)\) determined by
\[
(p, p) = R^2, \quad (l, p) = \nu R.
\]
Novikov and Schmelzer [13] introduced the variables
\[
L_i = M_i - \frac{\nu}{R} q_i, \quad i = 1, 2, 3
\]
to identify the coadjoint orbits with \(T^* S^2\) : \((q, q) = R^2, (L, q) = 0\).

Assuming here for the convenience that the radius of sphere is 1, we have in the new variables the Poisson brackets are
\[
\{L_i, L_j\} = \epsilon_{ijk} (L_k - \nu q_k), \quad \{L_i, q_j\} = \epsilon_{ijk} q_k, \quad \{q_i, q_j\} = 0
\]
and the corresponding symplectic form becomes
\[
\omega = dP \wedge dQ + \nu dS,
\]
where \(dP \wedge dQ\) is the standard symplectic form on \(T^* S^2\) and \(dS\) is the area form on \(S^2\) (see [13]). As it was pointed out in [13] the second term corresponds to the magnetic field of the Dirac monopole: \(H = \nu dS\).

Consider the following functions on \(e(3)^*\):
\[
H = \frac{1}{2}|M|^2 - \mu \frac{|q|}{\sqrt{R(q)}},
\]
\[
F = AM_1^2 + BM_2^2 + \frac{2\sqrt{AB}}{|q|} (M, q) M_3 - 2\mu \sqrt{AB} \frac{q_3}{\sqrt{R(q)}},
\]
where
\[
R(q) = Aq_2^2 + Bq_1^2 + (A + B)q_3^2 - 2\sqrt{AB}|q|q_3
\]
and $\mu, A, B$ are parameters satisfying $A > B > 0$.

The graph of the potential $U = -\mu \frac{|q|}{\sqrt{R(q)}}$ after stereographic projection from the North pole is shown on Fig. 2. We can interpret it as the Coulomb-like potential with two fixed charged centres (repulsing when $\mu < 0$, and attracting when $\mu > 0$).

Note that our system with positive $\mu$ admits also Newtonian gravitational interpretation in contrast to Killing’s version.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{repulsive_potential}
\caption{Graph of the repulsive potential $U$ with $\mu < 0$ after the stereographic projection}
\end{figure}

**Theorem 1.** The Poisson bracket of $H$ and $F$ is identically zero:

$$\{F, H\} \equiv 0.$$  

On the symplectic leaf with $|q|^2 = 1$, $(M, q) = \nu$ we have a new integrable system on unit sphere with the potential having two Coulomb-like singularities with charge $\mu/\sqrt{A - B}$ fixed at the points $(\pm \alpha, 0, \beta)$ with

$$\alpha = \sqrt{\frac{A - B}{A}}, \quad \beta = \sqrt{\frac{B}{A}}$$

in the external field of Dirac magnetic monopole with charge $\nu$.

The proof is by direct calculation. The main problem was to derive these global formulae from the local calculations in [21].

Let us check that the potential $U(q) = -\mu/\sqrt{R(q)}$ has the described singularities. For this re-write $R(q)$ as

$$R(q) = (A - B)q_3^2 + (\sqrt{A}q_3 - \sqrt{B}|q|)^2.$$
Indeed, we have

\[(A - B)q_2^2 + (\sqrt{A}q_3 - \sqrt{B}|q|)^2 = (A - B)q_2^2 + Aq_3^2 - 2\sqrt{AB}|q|q_3 + B(q_1^2 + q_2^2 + q_3^2)\]

\[= Aq_2^2 + Bq_1^2 + (A + B)q_3^2 - 2\sqrt{AB}|q|q_3.\]

Since \(A > B > 0\) the equality

\[(A - B)q_2^2 + (\sqrt{A}q_3 - \sqrt{B}|q|)^2 = 0\]

implies that both \(q_2 = 0\) and \(\sqrt{A}q_3 - \sqrt{B}|q| = 0\), which gives two points

\[q_1 = \pm \sqrt{\frac{A - B}{A}}, \quad q_2 = 0, \quad q_3 = \sqrt{\frac{B}{A}}.\]

One can check that near the singularities we have

\[R \approx \sqrt{A - B}\rho,\]

where \(\rho \ll 1\) is the spherical distance from the singularity, which means that

\[U \approx -\frac{\mu}{R\sqrt{A - B}}\]

is Coulomb-like as claimed.

Note that at the antipodal points \((\pm \alpha, 0, -\beta)\) the potential is smooth since

\[R(\pm \alpha, 0, -\beta) = -2\sqrt{B} \neq 0.\]

3. NEW SYSTEM IN SPHERICAL ELLIPTIC COORDINATES

Consider the unit sphere given in the Cartesian coordinates \(q_1, q_2, q_3\) by the equation

\[q_1^2 + q_2^2 + q_3^2 = 1,\]

and introduce the spherical elliptical coordinates \([16, 17]\) as the roots \(u_1, u_2\) of the quadratic equation

\[\Phi(q) = \frac{q_1^2}{A - u} + \frac{q_2^2}{B - u} + \frac{q_3^2}{C - u} = 0,\] (9)

where \(C = 0\) and \(A > B > 0\) are the same as before. Expressing the function

\[\Phi(q) = \frac{(u - u_1)(u - u_2)}{-q(A - q)(B - q))}\] (10)

in terms of partial fractions we have the formula for the Cartesian coordinates in terms of the elliptic coordinates

\[q_1^2 = \frac{(A - u_1)(A - u_2)}{A(A - B)}, \quad q_2^2 = \frac{(B - u_1)(B - u_2)}{B(B - A)}, \quad q_3^2 = \frac{u_1u_2}{AB}.\] (11)

The metric on the sphere in the elliptic coordinates takes the form

\[ds^2 = \frac{u_1 - u_2}{f(u_1)} du_1^2 + \frac{u_2 - u_1}{f(u_2)} du_2^2.\] (12)

where

\[f(u) = -4u(u - A)(u - B).\] (13)
The graph of \( f(u) \) and the positions of the variables
\[
0 \leq u_1 \leq B \leq u_2 \leq A
\]
are shown on Fig. 2.

\[\text{Figure 3. Choice of roots } u_1, u_2\]

From (9) we have
\[
(u - u_1)(u - u_2) = u^2 - (Bq_1^2 + Aq_2^2 + (A + B)q_3^2)u + ABq_3^2,
\]
implying
\[
u_1 + u_2 = Bq_1^2 + Aq_2^2 + (A + B)q_3^2, \quad u_1u_2 = ABq_3^2. \quad (14)
\]
In particular,
\[
\left(\sqrt{u_1} - \sqrt{u_2}\right)^2 = u_1 + u_2 - 2\sqrt{u_1u_2} = Bq_1^2 + Aq_2^2 + (A + B)q_3^2 - 2\sqrt{AB}q_3
\]
\[
= B(q_1^2 + q_2^2 + q_3^2) + (A - B)q_2^2 + Aq_3^2 - 2\sqrt{AB}q_3 = (A - B)q_2^2 + (\sqrt{A}q_3 - \sqrt{B})^2,
\]
so
\[
R(q) = (A - B)q_2^2 + (\sqrt{A}q_3 - \sqrt{B})^2 = (\sqrt{u_1} - \sqrt{u_2})^2.
\]
Thus the (electric) potential of our new system is
\[
U(q) = -\frac{\mu}{\sqrt{R(q)}} = -\frac{\mu}{\sqrt{u_2} - \sqrt{u_1}}.
\]
Note that we have chosen here \( q_3 = \sqrt{u_1u_2}/\sqrt{AB} \) to be positive, which leads to the singularities corresponding to \( u_1 = u_2 = B \) lying in the upper half-space.

To write down the kinetic term we need to introduce the potential of the Dirac magnetic monopole on sphere, which is known to be impossible to choose non-singular for non-zero magnetic charge \( \nu \) by topological reasons (see e.g. Wu-Yang [22]). If we make two punctures at North and South poles, then we can use, for example,
\[
A = -\nu q_3 \frac{q_1dq_2 - q_2dq_1}{q_1^2 + q_2^2}. \quad (15)
\]
Indeed, one can check that on the sphere \(|q|^2 = 1\)
\[
dA = \nu(q_1dq_2 \wedge dq_3 + q_2dq_3 \wedge dq_1 + q_3dq_1 \wedge dq_2) = \nu dS
\]
(where \( dS \) is the area form on the unit sphere), which is the magnetic form of the Dirac monopole with charge \( \nu \).

Let

\[
A = A_1(u)du_1 + A_2(u)du_2
\]

be any such 1-form (e.g. given by (15)) written in the elliptic coordinates, so that

\[
dA = \nu dS = B(u)du_1 \wedge du_2,
\]

where

\[
B(u) = \nu \frac{u_2 - u_1}{\sqrt{-f(u_1)f(u_2)}},
\]

is the density of the Dirac magnetic field in the elliptic coordinates.

Define the magnetic momenta by

\[
\tilde{p}_i = p_i - A_i(u), \quad i = 1, 2.
\]

The corresponding Poisson brackets are

\[
\{\tilde{p}_1, \tilde{p}_2\} = B(u), \quad \{\tilde{p}_1, u_1\} = \{\tilde{p}_2, u_2\} = 1,
\]

with all other to be zero.

The Hamiltonian and the integral of the new system can be written now as

\[
H = \frac{f(u_1)}{u_1 - u_2} \tilde{p}_1^2 + \frac{f(u_2)}{u_2 - u_1} \tilde{p}_2^2 + \frac{\mu}{\sqrt{u_2 - u_1}}
\]

\[
F = u_2 f(u_1) \tilde{p}_1^2 + u_1 f(u_2) \tilde{p}_2^2 + \phi_1(u) \tilde{p}_1 + \phi_2(u) \tilde{p}_2 + V(u),
\]

where

\[
\phi_1 = -\nu \frac{\sqrt{-f(u_1)f(u_2)}}{\sqrt{u_1 u_2 + u_2}}, \quad \phi_2 = -\nu \frac{\sqrt{-f(u_1)f(u_2)}}{\sqrt{u_1 u_2 + u_1}}
\]

and

\[
V = \frac{\mu \sqrt{u_1 u_2}}{\sqrt{u_2 - u_1}} - \nu^2 (\sqrt{u_1} - \sqrt{u_2})^2.
\]

Note that the electric potential can be written in Stäckel form as

\[
U = -\frac{\mu}{\sqrt{u_2 - u_1}} = \frac{\mu(\sqrt{u_2} + \sqrt{u_1})}{u_2 - u_1},
\]

so, when the magnetic charge \( \nu = 0 \), we can take \( A = 0 \), \( \tilde{p}_i = p_i \) and the variables in the corresponding Hamilton-Jacobi equation

\[
\frac{f(u_1)}{u_1 - u_2} \left( \frac{\partial S}{\partial u_1} \right)^2 + \frac{f(u_2)}{u_2 - u_1} \left( \frac{\partial S}{\partial u_2} \right)^2 - \mu \frac{\sqrt{u_2} + \sqrt{u_1}}{u_2 - u_1} = h
\]

can be separated (similarly to the classical two-centre problem, see Kozlov-Harin \[12\]).

What makes our case special is that the integrability holds for general \( \nu \), although the separation of variables does not work, at least immediately (see Concluding remarks).
Another nice property of the new system is that it has a natural integrable quantum version.

4. Quantum version

Let us first recall the geometric quantisation of the Dirac magnetic monopole following Kemp and one of the authors [10].

Let

$$X_1 = q_3 \partial_2 - q_2 \partial_3, \quad X_2 = q_1 \partial_3 - q_3 \partial_1, \quad X_3 = q_2 \partial_1 - q_1 \partial_2$$

be the vector fields generating rotations of $S^2$ given by $q_1^2 + q_2^2 + q_3^2 = 1$ and $\nabla_{X_j}$ be the corresponding covariant derivatives with respect to the Dirac $U(1)$-connection $iA$ with $A$ given, for example, by (15).

Note that in the quantum case the charge $\nu$ of the Dirac magnetic monopole must be quantised, as it was pointed out already by Dirac [7]. Geometrically this corresponds to the integrality of the Chern class of $U(1)$-bundle over sphere:

$$\frac{1}{2\pi} \int_{S^2} \nu dS = 2\nu \in \mathbb{Z}.$$

Then one can check [10] that

$$\widehat{\nabla}_j := i \nabla_{X_j}$$

and the operators $\hat{q}_j$ of multiplication by $q_j$ satisfy the commutation relations

$$[\widehat{\nabla}_k, \widehat{\nabla}_l] = i \epsilon_{klm} (\widehat{\nabla}_m - \nu \hat{q}_m)$$

and thus can be considered as quantization of Novikov-Schmelzer variables. The quantum versions of the original variables

$$\hat{M}_j = \widehat{\nabla}_j + \nu q_j,$$

satisfy the standard angular momentum relations

$$[\hat{M}_k, \hat{M}_m] = i \epsilon_{kmm} \hat{M}_n, \quad [\hat{M}_k, \hat{q}_m] = i \epsilon_{kmm} \hat{q}_n,$$

and coincide with Fierz’s modification of the angular momentum in the presence of the Dirac magnetic monopole [9].

The quantum Hamiltonian of the Dirac monopole can be written in terms of magnetic angular momentum $\hat{M}$ as

$$\hat{H} = \frac{1}{2} (\hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2).$$

Since the operator $\hat{H}$ is a Casimir operator for $SO(3)$ it acts on every irreducible representation of $SO(3)$ as a scalar, which allows to use the representation theory for an explicit computation of the spectrum of Dirac magnetic monopole on the Hilbert space of functions $F(S^2)$ (see the details in [10]).

The quantum Hamiltonian of our system can now be defined by the same formula

$$\hat{H} = \frac{1}{2} (\hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2) - \mu \frac{|q|}{\sqrt{R(q)}},$$

(21)
where as before $R(q) = (A - B)q^2 + (\sqrt{A}q_3 - \sqrt{B}|q|)^2$. Here slightly abusing notations, we mean by $f(q)$ the operator of multiplication by $f(q)$ on the space of functions $\mathcal{F}(S^2)$. Define similarly the quantum version of $F$ by

$$\hat{F} = A\hat{M}_1^2 + B\hat{M}_2^2 + \frac{2\sqrt{AB}}{|q|}(\hat{M}, q)\hat{M}_3 - 2\mu \sqrt{AB} \frac{q_3}{\sqrt{R(q)}}.$$  \hspace{1cm} (22)

Note that there is no ordering problem here since $\hat{M}_3$ commutes with the operator $(\hat{M}, q)|q|^{-1}$.

**Theorem 2.** Operators $\hat{H}$ and $\hat{F}$ given by (21), (22) commute:

$$[\hat{H}, \hat{F}] = 0,$$

so $F$ is the second quantum integral of the system, ensuring its integrability.

The proof is again by straightforward check. When the magnetic charge $\nu = 0$ the Hamiltonian and integral become

$$\hat{H} = \frac{1}{2}\Delta - \mu \frac{|q|}{\sqrt{R(q)}}, \quad \hat{F} = A\hat{M}_1^2 + B\hat{M}_2^2 - 2\mu \sqrt{AB} \frac{q_3}{\sqrt{R(q)}},$$

where $\Delta$ is the Laplace-Beltrami operator on the unit sphere.

5. **Hyperbolic version**

Replacing the Euclidean group of motion $E(3)$ by the group $E(2,1)$ of motion of the pseudo-Euclidean space $\mathbb{R}^{2,1}$ we come to the following natural hyperbolic version of our system.

The corresponding Lie algebra of $so(2,1)$ consists of $3 \times 3$ matrices $X$ satisfying $X : XJ + JXT = 0$, where

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have a natural basis $M_1, M_2, M_3$, defined by

$$X = \begin{pmatrix} 0 & a_3 & a_2 \\ -a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix} = a_1 M_1 + a_2 M_2 + a_3 M_3,$$

with the commutation relations between themselves and with natural generators of translations $q_1, q_2, q_3$:

$$[M_1, M_2] = M_3, \quad [M_2, M_3] = -M_1, \quad [M_3, M_1] = -M_2, \quad (23)$$

$$[M_1, q_2] = q_3 = -[M_2, q_1], \quad [M_1, q_3] = q_2 = -[M_3, q_1], \quad [M_2, q_3] = -q_1 = -[M_3, q_2].$$

The Casimir functions are

$$C_1 = (q, Jq) = -q_1^2 - q_2^2 + q_3^2 := ||q||^2,$$

defining the pseudo-Euclidean structure on $\mathbb{R}^{2,1}$, and

$$C_2 = \langle M, q \rangle := (M, Jq) = -M_1 q_1 - M_2 q_2 + M_3 q_3.$$
The relation \( C_1 = \|q\|^2 = 1 \) now defines the two-sheeted hyperboloid, one sheet of which presenting a model of the hyperbolic plane (see Fig. 4).

\[
H = \frac{1}{2} (M_1^2 + M_2^2 - M_3^2) + \frac{\mu \|q\|}{\sqrt{R(q)}}, \tag{24}
\]

where

\[
R(q) = -A q_2^2 - B q_1^2 + (A + B) q_3^2 - 2 \sqrt{AB} \|q\| q_3, \tag{25}
\]

which can be rewritten as

\[
R(q) = (B - A) q_2^2 + (\sqrt{A} q_3 - \sqrt{B} \|q\|)^2.
\]

Assuming now that \( B > A > 0 \), we see that the potential has two singularities when \( q_2 = 0, \sqrt{A} q_3 - \sqrt{B} \|q\| = 0 \), or, if we assume that \( \|q\|^2 = 1 \), at two points \((\pm \frac{\sqrt{B-A}}{\sqrt{A}}, 0, \frac{B}{\sqrt{A}})\). The corresponding additional integral has the form

\[
F = AM_1^2 + BM_2^2 - 2 \frac{\sqrt{AB}}{\|q\|} (M, q) M_3 + 2 \frac{\mu \sqrt{AB} q_3}{\sqrt{R(q)}}. \tag{26}
\]

**Theorem 3.** The functions \( H \) and \( F \) given by (24)-(26) commute with respect to the Lie-Poisson bracket (23) on the dual of Lie algebra so(2,1)∗.

At the symplectic leaf \( \|q\|^2 = 1 \), \( (M, q) = \nu \) this gives a new integrable two-centre problem on the hyperbolic plane with constant magnetic field of charge \( \nu \). The same is true for the natural quantum versions \( \hat{H} \) and \( \hat{F} \).

The formulae in the corresponding hyperbolic elliptic coordinates \( u_1, u_2 \), defined as the roots

\[
\frac{q_1^2}{A - u} + \frac{q_2^2}{B - u} + \frac{q_3^2}{u} = 0, \quad \|q\|^2 = -q_1^2 - q_2^2 + q_3^2 = 1,
\]

are similar to the spherical case.

Note that on the symplectic leaves with \( \|q\|^2 = -1 \), determining a one-sheeted hyperboloid, we have a “de Sitter” version of the problem using the same formulae with \( B > 0 > A \).
6. Concluding remarks

The geometry of the new system is still to be properly studied, but the most important problem is to study the corresponding dynamics in the classical case and spectral properties in the quantum case.

For the detailed analysis of the classical two-centre problem on sphere we refer to the work of Albouy and Stuchi [1, 2], Borisov and Mamaev [4, 5] and Gonzalez Leon et al [13].

In our case this looks much more difficult because of the presence of the magnetic field, which usually creates a lot of problem for the separation of variables.

A famous example of this sort is the classical Clebsch system, describing special integrable case of rigid body motion in an ideal fluid [6]. It can be written as Euler equation on $e(3)^*$ with

$$H = \frac{1}{2} |M|^2 + \frac{1}{2} (Aq_1^2 + Bq_2^2 + Cq_3^2),$$

(see [18]) and can be interpreted as the harmonic oscillator on sphere with additional Dirac magnetic field [20]. One can show that Clebsch and our new system are the only two potential extensions of the Dirac magnetic monopole having an additional integral, which is quadratic in momenta [21].

Recently there was a substantial progress in separation of variables for the Clebsch system due to Magri and Skrypnyk [15, 19]. It would be interesting to see if these new ideas can be applied in our case as well.

7. Acknowledgements

We are very grateful to Alexey Bolsinov for many useful and stimulating discussions.

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