RELATIVISTIC ENTROPY AND RELATED BOLTZMANN KINETICS

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It is well known that the particular form of the two-particle correlation function, in the collisional integral of the classical Boltzmann equation, fix univocally the entropy of the system, which turn out to be the Boltzmann-Gibbs-Shannon entropy.

In the ordinary relativistic Boltzmann equation, some standard generalizations, with respect its classical version, imposed by the special relativity, are customarily performed. The only ingredient of the equation, which tacitly remains in its original classical form, is the two-particle correlation function, and this fact imposes that also the relativistic kinetics is governed by the Boltzmann-Gibbs-Shannon entropy. Indeed the ordinary relativistic Boltzmann equation admits as stationary and stable solution, the exponential Juttner distribution.

Here, we show that the special relativity laws and the maximum entropy principle, suggest a relativistic generalization also of the two-particle correlation function and then of the entropy. The so obtained, fully relativistic Boltzmann equation, obeys the H-theorem and predicts a stationary stable distribution, presenting power-law tails in the high energy region. The ensued relativistic kinetic theory preserves the main features of the classical kinetics, which recovers in the $c \to \infty$ limit.

I. INTRODUCTION

In experimental high energy physics, the power law tailed probability distribution functions, have been observed systematically (plasmas, cosmic rays, particle production processes etc).

A mechanism, frequently used to explain the occurrence of non-exponential distributions, is based on certain non-linear evolution equations, mainly considered in the Fokker-Planck picture, but recently also in the Boltzmann picture. Clearly, the correctness of the analytic expression of a given distribution, used to describe a statistical system, is strongly related to the validity of its generating mechanism.

The classical Boltzmann equation, due to the particular form of the two-particle correlation function, in the collisional integral, fix univocally the entropy of the system, which turn out to be Boltzmann-Gibbs-Shannon entropy. The latter entropy imposes the exponential form to the probability distribution function, emerging as the stationary and stable solution of the equation.

In the ordinary relativistic Boltzmann equation, some standard generalizations, with respect its classical version, imposed by the special relativity, are customarily performed. The only ingredient of the equation, which tacitly remains in its original classical form, is the two-particle correlation function, and this fact imposes that also the relativistic kinetics is governed by the classical Boltzmann-Gibbs-Shannon entropy. Indeed the ordinary relativistic Boltzmann equation admits as stationary stable distribution, the exponential Juttner distribution.

In order to obtain a relativistic Boltzmann equation admitting as stationary and stable solution a probability distribution, different from the exponential one, the only possibility we have, is to modify the expression of the classical two-particle correlation function. As a consequence a modification of the system entropy emerges, and this important fact suggests that such modification must be confined possibly within the ordinary physics (special relativity, maximum entropy principle) and without invoking any additional assumption or postulate.

Main goal of the present paper is to show that it is possible to obtain a fully relativistic Boltzmann equation admitting as stationary and stable solution a distribution function presenting power-law tails.

The new relativistic equation, can be obtained starting from the ordinary relativistic Boltzmann equation, by properly generalizing the two-particle correlation function in the collisional integral.

Such generalization, can be obtained in a self-consistent manner, by employing (i) the special relativity laws and (ii) the maximum entropy principle, and without invoking any extra principle.

The new relativistic entropy and relevant probability distribution function have very simple expressions, which result to be one-parameter (light speed in dimensionless form) generalizations of the corresponding classical concepts, just as happen for all the physical quantities (energy, momentum etc) in special relativity. Clearly, the relativistic entropy and the relativistic distribution function, in the classical limit, reduce to the Boltzmann-Gibbs-Shannon entropy and to the Maxwell-Boltzmann distribution respectively.

Within the present theoretical framework, the power-law tails in the distribution functions, in high energy physics, emerge as a purely relativistic effect (those tails in the classical limit $c \to \infty$ deform into exponential tails). In other words the power-law tails represent the...
signature of the relativistic nature of the system.

The paper is organized as follows:

In Sect. II we consider briefly the basic concepts of relativistic dynamics by using dimensionless variables.

In Sect. III we consider the \( \kappa \)-differential calculus by giving particular emphasis to its physical origin.

In Sect. IV we present the main properties of the function \( \kappa \)-exponential.

In Sect. V we present the main properties of the function \( \kappa \)-logarithm.

In Sect. VI we introduce the velocity representation of the relativistic nature of the system.

In Sect. VII we consider Boltzmann relativistic kinet-

In the present section, for simplicity of the exposition, we introduce the dimensionless velocity

\[
\frac{v}{u} = \frac{p}{m\kappa} = \sqrt{\frac{E}{m\kappa^2}} = \kappa c = v_* ,
\]

so that the composition law of any variable can be de-

Clearly we can introduce further function \( h = h(q) \)
which define new variables. For instance the Minkovski

\[
\rho(q) = \frac{1}{\kappa} \arcsinh(\kappa q) .
\] (2.5)

The above definition of the rapidity variable is justified by the fact that in the classical limit it results:

\[
\rho = q = u = \sqrt{2W}.
\]

Let us consider in the inertial one-dimension spatial frame \( \Sigma \), two identical, non-interacting, and free particles \( A \) and \( B \), of rest mass \( m \). We suppose that the particle \( A \) moves toward right while the particle \( B \) moves toward left. The dimensionless variables for the particle \( A \) are given by \( q_A, v_A, E_A, W_A \) and \( \rho_A \). The absolute values of the momentum, velocity and rapidity of the particle \( B \) are given by \( q_B, v_B, \rho_B \), while its total energy and kinetic energy are given by \( E_B, W_B \) respectively.

In the rest frame \( \Sigma' = \Sigma_B \) of the particle \( B \), which moves toward left, with velocity \( v_B \) with respect \( \Sigma \), the above considered dimensionless variables, for the particle \( B \) are given by \( q'_B = 0, v'_B = 0, E'_B = 1/\kappa^2, W'_B = 0 \) and \( \rho'_B = 0 \).

According to the Lorentz transformations, the dy-

\[
q'_A = q_A + q_B ,
\]

(2.6)

follows directly from the Lorentz transformation of the four-vector energy-momentum.

Regarding the composition law of an arbitrary variable \( h(q) \), after posing \( h_A = h(q_A), h_B = h(q_B) \) and \( h'_A = h(q'_A) \), one obtains

\[
h'_A = h_A \oplus h_B ,
\]

(2.8)

with

\[
h_A \oplus h_B = h( q_A \oplus q_B ) ,
\]

(2.9)

so that the composition law of any variable can be de-

In particular starting from the general composition law defined in Eq. (2.2) and after taking into account Eqs. (2.2)-(2.5) we can obtain the composition laws for the velocity, total energy, kinetic energy and rapidity as follows

\[
u'_A = u_A \oplus u_B ,
\]

(2.10)

\[
E'_A = E_A \oplus E_B ,
\]

(2.11)

\[
W'_A = W_A \oplus W_B ,
\]

(2.12)

\[
\rho'_A = \rho_A \oplus \rho_B ,
\]

(2.13)
being
\[ u_A \oplus u_B = \frac{u_A + u_B}{1 + \kappa^2 u_A u_B}, \quad (2.14) \]
\[ \mathcal{E}_A \oplus \mathcal{E}_B = \kappa^2 \mathcal{E}_A \mathcal{E}_B + \frac{1}{\kappa^2 \sqrt{(\kappa^4 \mathcal{E}_A^2 - 1)(\kappa^4 \mathcal{E}_B^2 - 1)}}, \quad (2.15) \]
\[ \mathcal{W}_A \oplus \mathcal{W}_B = \mathcal{W}_A + \mathcal{W}_B + \kappa^2 \mathcal{W}_A \mathcal{W}_B + \sqrt{\mathcal{W}_A \mathcal{W}_B (2 + \kappa^2 \mathcal{W}_A) (2 + \kappa^2 \mathcal{W}_B)}, \quad (2.16) \]
\[ \rho_A \oplus \rho_B = \rho_A + \rho_B. \quad (2.17) \]

It is remarkable that the momentum composition law \( q_A \oplus q_B \), is a generalized sum having the properties: i) it is associative, ii) it is commutative, iii) the 0 is its neutral element, iv) the opposite element of \( q \) is \(-q\). Also the velocity composition law \( u_A \oplus u_B \), results to be a generalized sum, while the rapidity composition law \( \rho_A \oplus \rho_B \) is the ordinary sum.

Regarding the total energy composition law \( \mathcal{E}_A \oplus \mathcal{E}_B \) we note that i) it is associative, ii) it is commutative, iii) it has as neutral element the quantity 1/\( \kappa^2 \), iv) it does not exist the opposite element of \( \mathcal{E} \). Due to the latter property, the total energy composition law is not a generalized sum. Also the kinetic energy composition law \( \mathcal{W}_A \oplus \mathcal{W}_B \) is not a generalized sum because it does not exist the opposite element of \( \mathcal{W} \).

### III. THE \( \kappa \)-DIFFERENTIAL CALCULUS

#### A. Lorentz invariant integration

Let us consider the following integral in the three dimension momentum space, within the classical physics framework
\[ I_{cl} = \int \frac{d^4p}{p^3} F, \quad (3.1) \]
being \( F \) an arbitrary function and \( p_* = mc\kappa \). In dimensionless variables, the latter integral becomes
\[ I_0 = \int d^3q F. \quad (3.2) \]
Whenever \( F \) depends only on \( q = |q| \), the above integral can be reduced to the following one dimension integral
\[ I_0 = \int_0^\infty dq \ 4\pi q^2 F. \quad (3.3) \]
In the framework of a relativistic theory it is well known that the three-dimension integral \((3.1)\) must be replaced by the four-dimension Lorentz invariant integral
\[ I_{rel} = \int \frac{d^4p}{p^3 mc} \ 2\theta(p_0) (p^\mu p_\mu - m^2 c^2) F. \quad (3.4) \]
After introducing the dimensionless variables according to \( q^\mu = p^\mu/p_* = (q_0, \mathbf{q}) = (\kappa \mathbf{E}, \mathbf{q}) \), the latter integral becomes
\[ I_\kappa = \int d^4q \ 2\theta(q_0) \delta (\kappa^2 q^\mu q_\mu - 1) F, \quad (3.5) \]
or alternatively
\[ I_\kappa = \int d^3q \int d(\kappa^2 \mathcal{E}) \ 2\theta(\kappa^2 \mathcal{E}) \delta (\kappa^4 \mathcal{E}^2 - \kappa^2 q^2 - 1) F \quad (3.6) \]
with \( q = |\mathbf{q}| \).

In order to reduce the integral \((3.6)\) in a three-dimension integral, we introduce the new integration variable \( Z = \kappa^4 \mathcal{E}^2 \). After observing that \( d(\kappa^2 \mathcal{E}) = dZ/2\sqrt{Z} \) and \( \theta(\kappa^2 \mathcal{E}) = \theta(Z) \) one obtains
\[ I_\kappa = \int d^3q \int_0^\infty \frac{dZ}{2\sqrt{Z}} 2\delta (Z - \kappa^2 q^2 - 1) F \quad (3.7) \]
and finally
\[ I_{rel} = \int \frac{d^3p}{p^3} \frac{mc}{p_0} F \quad (3.8) \]
The latter three-dimension integral in dimensional variables becomes
\[ I_{rel} = \int \frac{d^3p}{p^3} \frac{mc}{p_0} F \quad (3.9) \]

We remark that in Eq. \((3.1)\) the integration element \( d^4p \) is a scalar because the Jacobian of the Lorentz transformation is equal to unity. Then \( I_{rel} \) transforms as \( F \). For this reason in Eq. \((3.5)\) the integration element \( d\kappa^2 \mathcal{E}/\sqrt{1 + \kappa^2 q^2} \) is a scalar.

Whenever \( F \) depends only on \( q = |\mathbf{q}| \), the integral \((3.6)\) can be reduced to the following one dimension integral
\[ I_\kappa = \int_0^\infty dq \ rac{4\pi q^2 F}{\sqrt{1 + \kappa^2 q^2}}, \quad (3.10) \]
which after introducing the \( \kappa \)-differential
\[ dq = \frac{dq}{\sqrt{1 + \kappa^2 q^2}}, \quad (3.11) \]
assumes the form
\[ I_\kappa = \int_0^\infty d_\kappa q \ 4\pi q^2 F \quad (3.12) \]
In the classical limit \( \kappa \to 0 \), the latter integral reproduces the corresponding classical one given by Eq. \((3.3)\).

We focus now our attention to the classical and relativistic expression of the one dimension integrals given by \((3.3)\) and \((3.12)\) respectively. One immediately observes that the relativistic integral is obtained directly from the classical one, by making the substitution \( dq \to d_\kappa q \).
The equivalence of the Lorentz invariant integration in four-dimension \( \int d^4p \) with the \( \kappa \)-integration in one-dimension \( \int_0^\infty d_\kappa q \), according to
\[
\int d^4p \ \theta (p_0) \ \delta (p^\mu p_\mu - m^2c^2) \ F(p) \propto \int_0^\infty d_\kappa q \ q^2 F(q) \ ,
\]
permits to explain better the relativistic origin of the \( \kappa \)-integration.

**B. Kinetic Energy and Work**

In classical mechanics the kinetic energy \( W \) is defined as the work produced by the external force \( f \)
\[
W_{cl}(p) = \int_0^p f \ dx \ .
\] (3.14)

After taking into account the Newton law and using dimensionless variables the above definition assumes the form
\[
W_0(q) = \int_0^q q \ dq \ .
\] (3.15)

and yields \( W_0(q) = q^2/2 \).

In special relativity, by using the same definition, the kinetic energy is given by
\[
W_{rel}(p) = \int_0^p f \ dx = \int_0^p \frac{dp}{dt} \ dx = \int_0^p v \ dp = \int_0^p \frac{p/m_0}{\sqrt{1 + p^2/m_0^2 c^2}} \ dp \ .
\] (3.16)

In dimensionless variables, the kinetic energy becomes
\[
W_\kappa(q) = \int_0^q q \ \frac{dq}{\sqrt{1 + \kappa^2 q^2}} \ ,
\] (3.17)

and after introducing the \( \kappa \)-differential can be written in the form
\[
W_\kappa(q) = \int_0^q q \ d_\kappa q \ ,
\] (3.18)

obtaining in this way the well known relativistic expression \( W_\kappa(q) = (\sqrt{1 + \kappa^2 q^2} - 1)/\kappa^2 \).

It is remarkable that the replacement of the ordinary integration by the \( \kappa \)-integration, in the classical definition (3.15), is sufficient to recover the relativistic expression of the kinetic energy.

Eq. (3.19) linking \( W \) and \( q \) through the \( \kappa \)-integral, can be written also in the following differential form
\[
\frac{d}{d_\kappa q} W_\kappa(q) = q \ ,
\] (3.19)

involving the \( \kappa \)-derivative
\[
\frac{d}{d_\kappa q} = \sqrt{1 + \kappa^2 q^2} \ \frac{d}{dq} \ .
\] (3.20)

Eq. (3.19) after integration with the condition \( W_\kappa(0) = 0 \), yields the relativistic expression of the kinetic energy. In the \( \kappa \rightarrow 0 \) limit the differential equation (3.19), reduces to the classical one
\[
\frac{d}{dq} W_0(q) = q \ .
\] (3.21)

**C. Momentum generalized sum and \( \kappa \)-differential calculus**

Let us consider in the inertial one-dimension spatial frame \( \Sigma \), two identical, non interacting, free particles \( A \) and \( B \) of rest mass \( m \) which move toward right. We suppose that the dimensionless momenta for the two particle are given by \( q_A = q + dq, \ q_B = q \) respectively.

In the rest frame \( \Sigma' = \Sigma_B \), of the particle \( B \) the particle momenta are given by \( q'_A = 0 \) and \( q'_B = (q + dq) \oplus (-q) \) respectively.

The infinitesimal difference of the particle momenta, in the frame \( \Sigma \) is \( dq \), while in the frame \( \Sigma' \) is given by
\[
d_\kappa q = (q + dq) \oplus (-q) = (q + dq) \otimes q \ ,
\] (3.22)

and results to be
\[
d_\kappa q = \frac{dq}{\sqrt{1 + \kappa^2 q^2}} \ .
\] (3.23)

In order to better understand the origin of the expression of the \( \kappa \)-differential, we recall that the variable \( q \) is a dimensionless momentum. Then the quantity \( \gamma(q) = \sqrt{1 + \kappa^2 q^2} \) is the Lorentz factor in the momentum representation, so that the \( \kappa \)-differential can be written as
\[
\frac{dq}{\gamma(q)} \ .
\] (3.24)

In other words if \( dq \) is the infinitesimal difference of two particle momenta in the inertial frame \( \Sigma \), this difference if observed in rest frame \( \Sigma' \) of one of the two particles, becomes \( d_\kappa q \) and results to be contracted by the Lorentz factor.

The \( \kappa \)-derivative of the function \( f(q) \), is defined through
\[
\frac{df}{d_\kappa q} = \lim_{z \rightarrow q} \frac{f(z) - f(q)}{z \oplus q} = \frac{f(q + dq) - f(q)}{(q + dq) \oplus q} \ .
\] (3.26)

We observe that \( df(q)/d_\kappa q \), which reduces to \( df(q)/dq \) as the deformation parameter \( \kappa \rightarrow 0 \), can be written in the form
\[
\frac{df(q)}{d_\kappa q} = \sqrt{1 + \kappa^2 q^2} \ \frac{df(q)}{dq} \ .
\] (3.27)
From the latter equation follows that the generalized derivative obeys the Leibniz’s rules of the ordinary derivative. After introducing the Lorentz factor \( \gamma(q) \), the derivative operator can be written also in the form:

\[
\frac{d}{d_{\kappa} q} = \gamma(q) \frac{d}{dq}.
\]  

(3.28)

The \( \kappa \)-integral is defined through

\[
\int d_{\kappa} q \ f(q) = \int \frac{d q}{\gamma(q)} \ f(q),
\]  

(3.29)

and obeys the same rules of the ordinary integral, which recovers when \( \kappa \to 0 \).

IV. THE \( \kappa \)-EXPONENTIAL FUNCTION

A. Definition

In the present section, the independent variable is a dimensionless momentum and is indicated by \( x \), \( y \) or \( z \). We recall that the ordinary exponential \( f(x) = \exp(x) \) emerges as solution both of the functional equation \( f(x + y) = f(x)f(y) \) and of the differential equation \( (d/dx)f(x) = f(x) \).

The question to determine the solutions of the generalized equations

\[
f(x \odot y) = f(x)f(y),
\]  

(4.1)

\[
\frac{d f(x)}{d_{\kappa} x} = f(x),
\]  

(4.2)

reducing in the \( \kappa \to 0 \) limit, to the ordinary exponential, naturally arises. The latter two equations admit the same solution, which represents an one-parameter generalization of the ordinary exponential.

**Solution of Eq. (4.2):** We write this equation explicitly

\[
f(x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2}) = f(x)f(y),
\]  

(4.3)

and after performing the change of variables \( f(x) = \exp(g(\kappa x)) \), \( z_1 = \kappa x \), \( z_2 = \kappa y \) it transforms into the following equation

\[
g(z_1 \sqrt{1 + z_2^2} + z_2 \sqrt{1 + z_1^2}) = g(z_1) + g(z_2),
\]  

(4.4)

which admits the solution \( g(x) = A \text{arsinh}x \). Then, it results that \( f(x) = \exp(A \text{arsinh} \kappa x) \). The arbitrary constant \( A \) can be fixed through the condition \( \lim_{\kappa \to 0} f(x) = \exp(x) \), obtaining \( A = 1/\kappa \). Therefore \( f(x) \) assumes the form \( f(x) = \exp_{\kappa}(x) \) being

\[
\exp_{\kappa}(x) = \exp\left(\frac{1}{\kappa} \text{arsinh} \kappa x\right).
\]  

(4.5)

**Solution of Eq. (4.2):** After performing the change of variable \( \rho = \kappa^{-1} \text{arsinh} \kappa x \), it obtains \( d_{\kappa} x = d\rho \) and Eq. (4.9) assumes the form

\[
\frac{df}{d\rho} = f .
\]  

(4.6)

From the solution of the latter equation with the condition \( f(0) = 1 \), follows immediately that \( f(x) = \exp_{\kappa}(x) \) with

\[
\exp_{\kappa}(x) = \exp\left(\frac{1}{\kappa} \text{arsinh} \kappa x\right).
\]  

(4.7)

By taking into account that \( \text{arsinh} x = \ln(\sqrt{1 + x^2} + x) \) we can write \( \exp_{\kappa}(x) \) in the form

\[
\exp_{\kappa}(x) = \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{1/\kappa},
\]  

(4.8)

which will be used in the following. We remark that \( \exp_{\kappa}(x) \) given by Eq. (4.8), is solution both of the Eqs. (4.1) and (4.2) and therefore represents a generalization of the ordinary exponential.

In particular according to Eq. (4.2) the \( \kappa \)-exponential is defined as eigenfunction of the \( \kappa \)-derivative i.e.

\[
\frac{d}{d_{\kappa} x} \exp_{\kappa}(x) = \exp_{\kappa}(x).
\]  

(4.9)

B. Basic Properties

From the definition (4.8) of \( \exp_{\kappa}(x) \), follows that

\[
\exp_{0}(x) \equiv \lim_{\kappa \to 0} \exp_{\kappa}(x) = \exp(x),
\]  

(4.10)

\[
\exp_{-\kappa}(x) = \exp_{\kappa}(x).
\]  

(4.11)

Like the ordinary exponential, \( \exp_{\kappa}(x) \) has the properties

\[
\exp_{\kappa}(x) \in C^{\infty}(\mathbb{R}),
\]  

(4.12)

\[
\frac{d}{dx} \exp_{\kappa}(x) > 0,
\]  

(4.13)

\[
\exp_{\kappa}(-\infty) = 0^{+},
\]  

(4.14)

\[
\exp_{\kappa}(0) = 1,
\]  

(4.15)

\[
\exp_{\kappa}(+\infty) = +\infty,
\]  

(4.16)

\[
\exp_{\kappa}(x) \exp_{\kappa}(-x) = 1.
\]  

(4.17)

The property (4.17) emerges as particular case of the more general one

\[
\exp_{\kappa}(x) \exp_{\kappa}(y) = \exp_{\kappa}(x \odot y).
\]  

(4.18)

Furthermore \( \exp_{\kappa}(x) \) has the property

\[
\left[ \exp_{\kappa}(x) \right]^{r} = \exp_{\kappa/r}(rx)
\]  

(4.19)

with \( r \in \mathbb{R} \), which in the limit \( \kappa \to 0 \) reproduces one well known property of the ordinary exponential.
We remark the following convexity property
\[
\frac{d^2}{dx^2} \exp_\kappa(x) > 0 \ ; \ x \in \mathbb{R} \ ,
\] (4.20)
holding when \( \kappa^2 < 1 \).

Undoubtedly one of the more interesting properties of \( \exp_\kappa(x) \), is its power law asymptotic behavior
\[
\exp_\kappa(x) \sim_{x \to \pm \infty} \left| 2\kappa x^{\pm 1/|\kappa|} \right| .
\] (4.21)

It is remarkable that the first three terms in the Taylor expansion of \( \exp_\kappa(x) \) are the same as those of the ordinary exponential
\[
\exp_\kappa(x) = 1 + x + \frac{x^2}{2} + (1 - \kappa^2) \frac{x^3}{3!} + \ldots .
\] (4.22)

This latter result is a particular case of a more general property of the relativistic dynamics. Indeed, the Taylor expansion up to the second order, of any relativistic formula, coincides with the corresponding classical formula.

V. THE \( \kappa \)-LOGARITHM FUNCTION

A. Definition

The function \( \ln_\kappa(x) \) is defined as the inverse function of \( \exp_\kappa(x) \), namely
\[
\ln_\kappa(\exp_\kappa x) = \exp_\kappa(\ln_\kappa x) = x ,
\] (5.1)
and is given by
\[
\ln_\kappa(x) = \frac{1}{\kappa} \sinh (\kappa \ln x) ,
\] (5.2)
or more properly
\[
\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} .
\] (5.3)

B. Basic properties

It results that
\[
\ln_0(0) \equiv \lim_{\kappa \to 0} \ln_\kappa(x) = \ln(x) ,
\] (5.4)
\[
\ln_{-\kappa}(x) = \ln_\kappa(x) .
\] (5.5)

The function \( \ln_\kappa(x) \), just as the ordinary logarithm, has the properties
\[
\ln_\kappa(x) \in C^\infty(\mathbb{R}^+),
\] (5.6)
\[
\frac{d}{dx} \ln_\kappa(x) > 0 ,
\] (5.7)
\[
\ln_\kappa(0^+) = -\infty ,
\] (5.8)
\[
\ln_\kappa(1) = 0 ,
\] (5.9)
\[
\ln_\kappa(+\infty) = +\infty ,
\] (5.10)
\[
\ln_\kappa(1/x) = -\ln_\kappa(x) .
\] (5.11)

Furthermore \( \ln_\kappa(x) \) has the two properties
\[
\ln_\kappa(x^r) = r \ln_\kappa(x) , \quad (5.12)
\]
\[
\ln_\kappa(xy) = \ln_\kappa(x) \oplus \ln_\kappa(y) , \quad (5.13)
\]
with \( r \in \mathbb{R} \). Note that the property (5.11) follows as particular case of the property (5.12).

We remark the following concavity properties
\[
\frac{d^2}{dx^2} \ln_\kappa(x) < 0 ,
\] (5.14)
\[
\frac{d^2}{dx^2} x \ln_\kappa(x) < 0 .
\] (5.15)

A very interesting property of this function is its power law asymptotic behavior
\[
\ln_\kappa(x) \sim \begin{cases} \frac{1}{2 |\kappa|} x^{-|\kappa|} & , \quad x \to 0^+ \\ \frac{1}{2 |\kappa|} x^{|\kappa|} & , \quad x \to \pm \infty \\ \end{cases}
\] (5.16)
(5.17)

After recalling the integral representation of the ordinary logarithm
\[
\ln(x) = \frac{1}{2} \int_{1/x}^{x} \frac{1}{t} \, dt ,
\] (5.18)
one can verify that the latter relationship can be generalized easily in order to obtain \( \ln_\kappa(x) \), by replacing the integrand function \( y_0(t) = t^{-1} \) by the new function \( y_\kappa(t) = t^{-1-\kappa} \), namely
\[
\ln_\kappa(x) = \frac{1}{2} \int_{1/x}^{x} \frac{1}{t^{1+\kappa}} \, dt .
\] (5.19)

The first terms of the Taylor expansion related to \( \kappa \)-logarithm, are
\[
\ln_\kappa(1 + x) = x - \frac{x^2}{2} + \left( 1 + \frac{\kappa^2}{2} \right) \frac{x^3}{3} - \ldots .
\] (5.20)

C. The \( \ln_\kappa(x) \) as solution of functional equations

**First functional equation:** The logarithm \( f(x) = \ln(x) \) is the only existing function, except for a multiplicative constant, which results to be solution of the function equation \( f(x_1 x_2) = f(x_1) + f(x_2) \). Let us consider now the generalization of this equation, obtained by substituting the ordinary sum by the momentum generalized sum
\[
f(x_1 x_2) = f(x_1) \oplus f(x_2) .
\] (5.21)

We proceed by solving this equation, which assumes the explicit form
\[
f(x_1 x_2) = f(x_1) \sqrt{1 + \kappa^2 f(x_2)^2} + f(x_2) \sqrt{1 + \kappa^2 f(x_1)^2} .
\] (5.22)
After performing the substitution \( f(x) = \kappa^{-1} \sinh \kappa g(x) \) we obtain that the auxiliary function \( g(x) \) obeys the equation \( g(x_1 + x_2) = g(x_1) + g(x_2) \), and then is given by \( g(x) = A \ln x \). The unknown function becomes \( f(x) = \kappa^{-1} \sinh(\kappa \ln x) \) we have set \( A = 1 \) in order to recover, in the limit \( \kappa \to 0 \), the classical solution \( f(x) = \ln(x) \). Then we can conclude that the solution of Eq. \( (5.21) \) is given by

\[
f(x) = \ln_\kappa(x) \ . \quad (5.23)
\]

**Second functional equation:** The following first order differential-functional equation emerges in statistical mechanics within the context of the maximum entropy principle

\[
\frac{d}{dx} [x f(x)] = \lambda f(x/\alpha) \ , \quad (5.24)
\]

\[
f(1) = 0 \ , \quad (5.25)
\]

\[
f'(1) = 1 \ , \quad (5.26)
\]

\[
f(1/x) = -f(x) \ , \quad (5.27)
\]

\( \alpha \) and \( \lambda \) being two arbitrary constants. The latter problem admits two solutions. The first is given by \( f(x) = \ln(x) \) and \( \alpha = 1/e, \lambda = 1 \). The second solution is obtained after tedious but straightforward calculations [18], and is given by

\[
f(x) = \ln_\kappa(x) \ , \quad (5.28)
\]

and

\[
\alpha = \left( \frac{1 - \kappa}{1 + \kappa} \right)^{1/2\kappa} \ , \quad (5.29)
\]

\[
\lambda = \sqrt{1 - \kappa^2} \ . \quad (5.30)
\]

**D. The Entropy**

A physically meaningful link between the functions \( \ln_\kappa(x) \) and \( \exp_\kappa(x) \) is given by the following variational principle.

Let be \( h(q) \) an arbitrary real function and \( f(q) \) a real positive function of the variable \( q \in A \). The solution of the variational equation

\[
\frac{\delta}{\delta f(q)} \left[ - \int_A dq \ f(q) \ln_\kappa f(q) + \int_A dq \ f(q) \ h(q) \right] = 0 \ ,
\]

is unique and is given by

\[
f(q) = \alpha \ \exp_\kappa (h(q)/\lambda) \ , \quad (5.32)
\]

\( \alpha \) and \( \lambda \) being the constants defined by Eqs. \( (5.29) \) and \( (5.30) \). The solution of the variational equation \( (5.31) \) is trivial and employs Eq. \( (5.24) \).

This important result permits us to interpret the functional

\[
S_\kappa = - \int_A dq \ f(q) \ln_\kappa f(q) \ , \quad (5.33)
\]

which can be written also in the form

\[
S_\kappa = \int_A dq \ \frac{f(q)^{1-\kappa} - f(q)^{1+\kappa}}{2\kappa} \ , \quad (5.34)
\]

as the entropy associated to the function \( \exp_\kappa (x) \). It is remarkable that in the \( \kappa \to 0 \) limit, as \( \ln_\kappa(x) \) and \( \exp_\kappa(x) \) approach \( \ln(x) \) and \( \exp(x) \) respectively, the new entropy reduces to the old Boltzmann-Shannon entropy.

It is shown that the entropy \( S_\kappa \) has the standard properties of Boltzmann-Shannon entropy: is thermodynamically stable, is Lesche stable, obeys the Khinchin axioms of continuity, maximality, expandability and generalized additivity.

**VI. OTHER REPRESENTATIONS OF THE \( \kappa \)-EXPONENTIAL**

**A. Rapidity variable**

We define the rapidity as \( r = c \arctanh(\nu/c) \), so that in the classical limit it reduces to the particle velocity. The rapidity in dimensionless form \( \rho = r/c\kappa \), is given by

\[
\rho(u) = \frac{1}{\kappa} \arctanh(\kappa u) \ . \quad (6.1)
\]

Clearly the above expression of the rapidity is written in terms of the velocity, but it can be written also in terms of other variables. For instance, after taking into account the standard formulas of relativistic dynamics e.g. \( u = q/\sqrt{1 + \kappa^2 q^2} \) etc, the rapidity can be expressed in terms of the momentum, of the total energy, and of the kinetic energy, according to

\[
\rho(q) = \frac{1}{\kappa} \arcsinh(\kappa q) \quad (6.2)
\]

\[
\rho(\mathcal{E}) = \frac{1}{\kappa} \arccosh(\kappa^2 \mathcal{E}) \quad (6.3)
\]

\[
\rho(W) = \frac{1}{\kappa} \arccosh(1 + \kappa^2 W) \ , \quad (6.4)
\]

and obviously it results in

\[
\rho = \rho(q) = \rho(u) = \rho(\mathcal{E}) = \rho(W) \ . \quad (6.5)
\]

**B. The function \( \kappa \)-exponential in the velocity representation**

We recall that the composition laws of the momenta and of the velocities are two generalized sums. For this reason we will consider more in detail the rapidity variable in the momentum representation and in the velocity representation. From Eq. \( (6.5) \) we have

\[
\exp(\rho) = \exp(\rho(q)) = \exp(\rho(u)) \ , \quad (6.6)
\]

or equivalently

\[
\exp(\rho) = \exp_\kappa(q) = \exp^\kappa(u) \ . \quad (6.7)
\]
The functions $\exp_\kappa(q)$ and $\exp^\kappa(u)$ are defined through
\[ \exp_\kappa(q) = \exp \left( \frac{1}{\kappa} \arcsinh \kappa q \right), \quad (6.8) \]
\[ \exp^\kappa(u) = \exp \left( \frac{1}{\kappa} \arctanh \kappa u \right), \quad (6.9) \]
or equivalently
\[ \exp_\kappa(q) = \left( \sqrt{1 + \kappa^2 q^2 + \kappa q} \right)^{1/\kappa}, \quad (6.10) \]
\[ \exp^\kappa(u) = \left( \frac{1 + \kappa u}{1 - \kappa u} \right)^{1/2\kappa}. \quad (6.11) \]

The explicit relationships linking $\exp_\kappa(q)$, and $\exp^\kappa(u)$ are given by
\[ \exp^\kappa(u) = \exp_\kappa(u \gamma(u)), \quad (6.12) \]
\[ \exp_\kappa(q) = \exp^\kappa(q/\gamma(q)), \quad (6.13) \]
\[ \gamma(u) = \frac{1}{\sqrt{1 - \kappa^2 u^2}}, \quad (6.14) \]
\[ \gamma(q) = \sqrt{1 + \kappa^2 q^2}, \quad (6.15) \]
respectively.

We can conclude that the three functions $\exp(\rho)$, $\exp_\kappa(q)$, and $\exp^\kappa(u)$ are the same function in three different representations.

Hereafter we discuss briefly the function $\exp^\kappa(u)$ which has been introduced firstly in the appendix of ref. [19]. Starting from the velocity generalized sum we can obtain easily the $\kappa$-differential in the velocity representation as follows
\[ d^\kappa u = \frac{du}{1 - \kappa^2 u^2}. \quad (6.16) \]

The $\kappa$-derivative and the $\kappa$-integral of the function $f(u)$ with respect the dimensionless velocity, assume respectively the forms
\[ \frac{df(x)}{d^\kappa u} = (1 - \kappa^2 u^2) \frac{df(u)}{du}, \quad (6.17) \]
\[ \int d^\kappa u \ f(u) = \int \frac{du}{1 - \kappa^2 u^2} \ f(u). \quad (6.18) \]

The function $f(u) = \exp^\kappa(u)$ can be obtained as solution of the two following equations
\[ f(u_1 \oplus u_2) = f(u_1)f(u_2), \quad (6.19) \]
\[ \frac{d f(u)}{d^\kappa u} = f(u). \quad (6.20) \]

It is easy to verify that $\exp^\kappa(u)$ is a monotonic, continuous function and is defined in the interval $-1/|\kappa| < u < 1/|\kappa|$. The inverse function of $\exp^\kappa(u)$ namely $\ln^\kappa(x)$ is given by
\[ \ln^\kappa(x) = \frac{1}{\kappa} \tanh(\kappa \ln x), \quad (6.21) \]
\[ \ln^\kappa(x) = \frac{1}{\kappa} \frac{x^\kappa - x^{-\kappa}}{x^\kappa + x^{-\kappa}}. \quad (6.22) \]

The relationships linking $\ln^\kappa(x)$ and $\ln_\kappa(x)$ are the following
\[ \ln^\kappa(x) = \frac{\ln_\kappa(x)}{\sqrt{1 + \kappa^2 [\ln_\kappa(x)]^2}}, \quad (6.23) \]
\[ \ln_\kappa(x) = \frac{\ln^\kappa(x)}{\sqrt{1 - \kappa^2 [\ln^\kappa(x)]^2}}. \quad (6.24) \]

The properties of the functions $\exp^\kappa(x)$ and $\ln^\kappa(x)$ follows easily from the ones of the functions $\exp_\kappa(x)$ and $\ln_\kappa(x)$. For instance, we obtain
\[ \exp^\kappa(x) \exp^\kappa(-x) = 1, \quad (6.25) \]
\[ \ln^\kappa(1/x) = - \ln^\kappa(x), \quad (6.26) \]
and so on.

In ref. [18] a general procedure to deform the ordinary mathematics starting from an arbitrary generalized sum has been proposed. This procedure have been adopted in ref. [19] to construct the deformed mathematics related to the momentum generalized sum and then, based on the function $\exp_\kappa(q)$.

Clearly starting from the velocity generalized sum we can construct the related deformed mathematics based on the function $\exp^\kappa(u)$.

For instance we can introduce the deformed hyperbolic functions $\sinh^\kappa(u)$ and $\cosh^\kappa(u)$ according to
\[ \sinh^\kappa(u) = \frac{\exp^\kappa(u) - \exp^\kappa(-u)}{2}, \quad (6.27) \]
\[ \cosh^\kappa(u) = \frac{\exp^\kappa(u) + \exp^\kappa(-u)}{2}. \quad (6.28) \]

These two functions defines a $\kappa$-deformed hyperbolic trigonometry which is isomorphic with respect the ordinary hyperbolic trigonometry.

The deformed cyclic functions $\sin^\kappa(u)$ and $\cos^\kappa(u)$ given by
\[ \sin^\kappa(u) = \frac{\exp^\kappa(\text{i}u) - \exp^\kappa(-\text{i}u)}{2\text{i}}, \quad (6.29) \]
\[ \cos^\kappa(u) = \frac{\exp^\kappa(\text{i}u) + \exp^\kappa(-\text{i}u)}{2}, \quad (6.30) \]
defines the $\kappa$-deformed cyclic trigonometry which is isomorphic with respect the ordinary cyclic trigonometry.

The three mathematical structures based on the ordinary sum, on the momentum generalized sum, and on
the velocity generalized sum, result to be mutually isomorphic.

Regarding the possibility to consider further representations of the exponential function in special relativity we recall that the composition laws of the total energy \( \mathcal{E} \), and of the kinetic energy \( \mathcal{W} \), are not generalized sums and for these reason we can’t use the variables \( \mathcal{E} \) and \( \mathcal{W} \) to introduce new representation of the exponential function. Let us consider for instance the function

\[
 f(\mathcal{W}) = \exp \left( \rho(\mathcal{W}) \right) = \exp_\kappa \left( q(\mathcal{W}) \right) .
\] (6.31)

Clearly this function can’t be considered as a representation of the exponential function. Indeed it results in

\[
 f(\mathcal{W}) f(-\mathcal{W}) \neq 1 .
\] (6.32)

In general, a deformation of the ordinary exponential i.e. \( \exp(h(x)) \) is not an its representation. In order to have a new representation of the exponential (the mathematics underlying the new exponential, must be isomorphic with respect the ordinary mathematics, underlying the ordinary exponential) the function \( h(x) \) must have specific properties \([18]\).

It is important to note that in the classical limit, the composition law for the relativistic kinetic energies, given by Eq. \((2.16)\), reduces to the following expression of Galilean relativity

\[
 W_1 \oplus W_2 = W_1 + W_2 + 2\sqrt{W_1 W_2} .
\] (6.33)

Then, a non trivial composition law, which is not a generalized sum, appears also in classical physics when we change the particle observation inertial frame. The latter composition law never has been used in classical physics to introduce new representations of the exponential function. We stress that in classical physics, the ordinary exponential can be viewed as a function emerging in the momentum or equivalently in the velocity representation. Indeed, according to the Galilean relativity the composition laws for the classical momenta and velocities are the ordinary sum and this ordinary sum generates the ordinary exponential and the ordinary mathematics.

In conclusion in special relativity emerge only two non trivial representations of the exponential function namely the deformed exponentials \( \exp_\kappa(q) \) and \( \exp^\kappa(u) \) defined by Eqs. \((6.10)\) and \((6.11)\) respectively. There are several reasons to select the function \( \exp_\kappa(q) \) as the more proper representation of the exponential function in order to construct the relativistic statistical mechanics.

A first reason is related to the fact that the momentum is the more proper variable to formulate the one-particle relativistic dynamics. Indeed the dynamical Lorentz transformations involve the momentum and no the velocity. Also the relativistic Newton equation in terms of the momentum, assumes a very simple form, similar to the classical Newton equation. On the contrary the relativistic Newton equation, if expressed in terms of velocity and acceleration, results to be a non-linear equation very different with respect the classical Newton equation.

A second reason is related to the fact that also in the formulation of the many body relativistic theory the momentum results to be the more proper variable. For instance in relativistic kinematics but also in relativistic field theories, the Lorentz invariant integration involves the momentum and no the velocity.

The third and more important reason is related to the maximum entropy principle, the cornerstone of statistical mechanics. Indeed, the couple of functions \( \exp_\kappa(q) \) and \( \ln_\kappa(q) \) are linked through the variational principle described in the subsection D of the section V. On the contrary the functions \( \exp^\kappa(q) \) and \( \ln^\kappa(q) \) can’t be connected by the same variational principle.

\[ \text{VII. RELATIVISTIC KINETICS} \]

\[ \text{A. The } \kappa\text{-Product and } \kappa\text{-Sum of functions} \]

Let us consider the set of the non negative real functions \( \mathcal{D} = \{ f, h, w, ... \} \).

**Proposition 1:** The composition law \( \otimes \) defined through

\[
 \ln_\kappa(f \otimes h) = \ln_\kappa f + \ln_\kappa h ,
\] (7.1)

which reduces to the ordinary product as \( \kappa \to 0 \), namely \( f \otimes_h = f \cdot h \), is a generalized product and the algebraic structure \( (\mathcal{D} - \{0\}, \otimes) \) forms an abelian group.

**Proof:** Indeed this \( \kappa \)-product has the following properties

1) associative law: \( (f \otimes h) \otimes w = f \otimes (h \otimes w) \);
2) neutral element: \( f \otimes_\kappa 1 = 1 \otimes_\kappa f = f \);
3) inverse element: \( f \otimes_\kappa (1/f) = (1/f) \otimes_\kappa f = 1 \);
4) commutative law: \( f \otimes_\kappa h = h \otimes_\kappa f \).

Of course the \( \kappa \)-division \( \oslash \) can be defined as follows

\[ f \oslash_\kappa h = f \otimes_\kappa (1/h) . \]

The deformed \( \kappa \)-power \( f^{\otimes r} \) is defined through

\[
 \ln_\kappa \left( f^{\otimes r} \right) = r \ln_\kappa f ,
\] (7.2)

and generalizes the ordinary power \( f^r \). In particular, when \( r \) is integer one has \( f^{\otimes r} = f \otimes f \ldots f \) (\( r \) times).

The \( \kappa \)-product allows us to write the following property of the \( \kappa \)-exponential

\[
 \exp_\kappa(x) \otimes \exp_\kappa(y) = \exp_\kappa(x + y) .
\] (7.3)

**Proposition 2:** The algebraic structure \( (\mathcal{D}, \otimes) \) forms an abelian monoid.

**Proof:** Indeed the element 0 does not admit an inverse element.

Furthermore, just as in the case of the ordinary product, it results \( f \otimes_\kappa 0 = 0 \otimes_\kappa f = 0 \).

**Proposition 3:** The composition law \( \oslash \) defined through

\[
 \exp_\kappa \left( \ln_\kappa \left( f \oslash_\kappa h \right) \right) = \exp_\kappa \left( \ln_\kappa f \right) + \exp_\kappa \left( \ln_\kappa h \right) ,
\] (7.4)
which reduces to the ordinary sum as the deformation parameter approaches to zero, namely \( f \oplus h = f + h \), is a
generalized sum and the algebraic structure \((D, \otimes)\) forms
an abelian monoid.

Proof: Indeed this \( \kappa \)-sum has the following properties 
1) associative law: \( (f \oplus h) \oplus w = f \oplus (h \oplus w) \);
2) neutral element: \( f \oplus 0 = 0 \oplus f = f \);
3) commutative law: \( f \oplus h = h \oplus f \).

Proposition 4: The product \( \otimes \) and sum \( \oplus \) are distributive
operations
\[
w \otimes (f \oplus h) = (w \otimes f) \oplus (w \otimes h) \ . \quad (7.5)
\]

B. Evolution Equation

By using the standard notations of the relativistic theory
we denote with \( x = x^\nu = (ct, \mathbf{x}) \) the four-vector
position and with \( p = p^\nu = (p^0, \mathbf{p}) \) the four-vector
momentum, being \( p^0 = \sqrt{p^\nu p_\nu + mc^2} \) and employ the metric
\( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) [21].

Let us consider the following relativistic kinetic equation
\[
p^\nu \partial_\nu f - mF^\nu \frac{\partial f}{\partial p^\nu} = \int \frac{d^3p'}{p'^0} \frac{d^3p_1}{p_1^0} \frac{d^3p_2}{p_2^0} G \times [C(f', f_1) - C(f, f_1)] , \quad (7.6)
\]
where the distribution \( f = f(x, p) \) is a function of the four-vectors position and momentum, \( G \) is the transition
rate which depends only on the nature of the two body
particle interaction and \( C(f, f_1) \) is the two particle correlation
function with the same four-vector position \( x \), and
four-momenta \( p \) and \( p_1 \) respectively.

We note that the left hand side of Eq. (7.6) is 
the same as in the standard relativistic Boltzmann equation.
In the particular case where the two particle correlation
function is assumed to have the same expression like in
the classical Boltzmann equation i.e. \( C(f, f_1) = f f_1 \)
(Stosszahlansatz), the above equation reduces to the
ordinary relativistic Boltzmann equation [21], admitting
as stationary distribution an exponential distribution,
known as relativistic Maxwell-Boltzmann distribution or
as Juttner distribution.

Clearly in the case where \( C(f, f_1) \neq f f_1 \), Eq. (7.6)
describes a new relativistic kinetics, radically different from
the standard one. In the following we pose
\[
C(f, f_1) = (f/\alpha) \otimes (f_1/\alpha) \ . \quad (7.7)
\]
The origin and value of the constant \( \alpha \) will be discussed in
the following.

C. Stationary distribution

We consider now the steady states of Eq. (7.3) for
which the collision integral becomes equal to zero. Then
we have
\[
(f/\alpha) \otimes (f_1/\alpha) = (f'/\alpha) \otimes (f'_1/\alpha) \ , \quad (7.8)
\]
and after taking into account the definition (7.1) of the
\( \kappa \)-product, we obtain
\[
\ln_\kappa(f/\alpha) + \ln_\kappa(f_1/\alpha) = \ln_\kappa(f'/\alpha) + \ln_\kappa(f'_1/\alpha) \ . \quad (7.9)
\]
This last equation represents a conservation law and then
we can conclude that \( \ln_\kappa(f/\alpha) \) is a summational
invariant; in the most general case it is a linear combination
of the microscopic relativistic invariants, namely a constant
and the four-vector momentum. In the literature
it is shown that in presence of external electromagnetic
fields the more general microscopic relativistic invariant
has a form proportional to \( (p^\nu + q A^\nu/c) U_\nu + m^2 \) constant,
being \( U_\nu \) the hydrodynamic four-vector velocity
with \( U^\nu U_\nu = c^2 \). Then we can pose
\[
\ln_\kappa(f/\alpha) = -\beta \left[ (p^\nu + q A^\nu/c) U_\nu - mc^2 \right] + \frac{\beta \mu}{\lambda} \ . \quad (7.10)
\]
Consequently we obtain the following stationary distribution
\[
f = \alpha \exp_\kappa \left( -\beta \left[ (p^\nu + q A^\nu/c) U_\nu - mc^2 - \mu \right] \right) \ . \quad (7.11)
\]
At the moment \( \alpha, \lambda, \beta, \) and \( \mu \) remains arbitrary constants
which will be calculated and/or interpreted by using
the Maximum Entropy Principle imposing that the stationary distribution of the system (7.11) must
maximize the entropy of the system.

D. The maximum Entropy Principle

We define the four-vector entropy \( S^\nu = (S^0, \mathbf{S}) \) as follows
\[
S^\nu = -\int \frac{d^3p}{p^0} p^\nu f \ln_\kappa f \ . \quad (7.12)
\]
The identity \( d^3p/p^0 = d^4p \ 2 \theta(p^0) \delta(p^0 p_\mu - m^2 c^2) \) permits us to write \( S^\nu \) also in the form
\[
S^\nu = -\int d^4p \ 2 \theta(p^0) \delta(p^0 p_\mu - m^2 c^2) \ p^\nu f \ln_\kappa f \ . \quad (7.13)
\]
In the latter expression \( d^4p \) is a scalar because the Jacobian of the Lorentz transformation is equal to unit. Then
since \( p^\nu \) transforms as a four-vector, we can conclude that
\( S^\nu \) transforms as a four-vector.

The quantity \( \mathbf{S} \) is the entropy flow while \( S^0 = S \) is the
\( \kappa \)-entropy given by
\[
S = -\int d^3p f \ln_\kappa f \ . \quad (7.14)
\]
The maximization of the latter entropy under the constraints imposing the conservation of the norm of the


distribution $f$, and the a priori knowledge of the value of the more general microscopic invariant, conducts to the following variational equation

$$
\frac{\delta}{\delta f} \left\{ - \int d^3p \, f \ln \kappa f + \beta \mu \int d^3p \, f - \beta \int d^3p \, \left[ (p^\nu + q A^\nu/c) U_\nu - mc^2 \right] f \right\} = 0 ,
$$

(7.15)

$\beta$ and $\mu$ being the Lagrange multipliers.

The solution of the latter variational problem conduct to the stationary distribution (7.11) only thanks to the fact that the function $\ln \kappa f$ has the property

$$
\frac{d}{df} \left[ f \ln \kappa f \right] = \lambda \ln \kappa (f/\alpha) ,
$$

(7.16)

with

$$
\alpha = \left( \frac{1 - \kappa}{1 + \kappa} \right)^{1/2\kappa} ,
$$

(7.17)

$$
\lambda = \sqrt{1 - \kappa^2} .
$$

(7.18)

It is important at this point to remark that the function $\ln \kappa (x)$ does not possess the latter property. For this reason, we can’t use the generalized logarithm and exponential in the velocity representation, in order to construct a relativistic statistical theory.

The distribution function (7.11) in the global rest frame where $U_\nu = (c, 0, 0, 0)$ and in absence of external forces i.e. $A^\nu = 0$ simplifies as

$$
f = \alpha \exp \left( -\beta \frac{W - \mu}{\lambda} \right) ,
$$

(7.19)

being $W$ the relativistic kinetic energy.

We observe that the latter distribution in the classical limit ($\kappa \to 0$, $W \to 0$) reduces to the classical Maxwell-Boltzmann distribution i.e. $f \approx (1/e) \exp (-\beta (W - \mu))$, while at relativistic energies ($W \to +\infty$) presents power law tails $f \propto W^{-1/\kappa}$, in accordance with the experimental evidence in several relativistic systems.

### E. The H-theorem

In the ordinary relativistic kinetics it is well known from the H-theorem that the production of entropy is never negative and in equilibrium conditions there is no entropy production. In the following we will demonstrate the H-theorem for the system governed by the kinetic equation (7.6) when the two-particle correlation function is given by (7.7).

By using the property (7.10) of $\ln \kappa (f)$ and the notation $g = f/\alpha$ one obtains

$$
\partial_\nu (f \ln \kappa f) = \left[ \frac{\partial}{\partial f} f \ln \kappa f \right] \partial_\nu f
$$

(7.20)

The entropy production $\partial_\nu S^\nu$ can be calculated starting from the definition of $S^\nu$ given by Eq. (7.12). By using the result (7.21) and the evolution equation (7.6) it obtains the following expression for the entropy production

$$
\partial_\nu S^\nu = -\lambda \alpha \int \frac{d^3p}{p^{\rho 0}} \ln \kappa (g) p^\rho \partial_\nu g
$$

(7.21)

$$
= -\lambda \int \frac{d^3p}{p^{\rho 0}} \ln \kappa (g) p^\rho \partial_\nu f
$$

$$
= -\lambda \int \frac{d^3p}{p^{\rho 0}} \frac{d^3p_1}{p_1^{\rho 1}} \frac{d^3p_1'}{p_1'^{\rho 1'}} \frac{d^3p}{p^{\rho}} G
$$

$$
\times \left( g' \otimes g_1' - g \otimes g_1 \right) \ln \kappa (g)
$$

$$
-\lambda \mu \int \frac{d^3p}{p^{\rho 0}} \ln \kappa (g) F^\nu \frac{\partial f}{\partial p^\nu} .
$$

(7.22)

Since the Lorentz force $F^\nu$ has the properties $p^\rho F_\nu = 0$ and $\partial F^\nu/\partial p^\nu = 0$ the last term in the above equation involving $F^\nu$ is equal to zero (21), therefore we have

$$
\partial_\nu S^\nu = -\lambda \int \frac{d^3p}{p^{\rho 0}} \frac{d^3p_1}{p_1^{\rho 1}} \frac{d^3p_1'}{p_1'^{\rho 1'}} \frac{d^3p}{p^{\rho}} G
$$

$$
\times \left( g' \otimes g_1' - g \otimes g_1 \right) \ln \kappa (g).
$$

(7.22)

Given the particular symmetry of the integral in the latter equation we can write the entropy production as follows

$$
\partial_\nu S^\nu = -\frac{1}{4} \lambda \int \frac{d^3p}{p^{\rho 0}} \frac{d^3p_1}{p_1^{\rho 1}} \frac{d^3p_1'}{p_1'^{\rho 1'}} \frac{d^3p}{p^{\rho}} G
$$

$$
\times \left( g' \otimes g_1' - g \otimes g_1 \right)
$$

$$
\times \left[ \ln \kappa (g) + \ln \kappa (g_1) \right] \ln \kappa (g_1') .
$$

(7.23)

Finally we set the latter equation in the form

$$
\partial_\nu S^\nu = \frac{1}{4} \lambda \int \frac{d^3p}{p^{\rho 0}} \frac{d^3p_1}{p_1^{\rho 1}} \frac{d^3p_1'}{p_1'^{\rho 1'}} \frac{d^3p}{p^{\rho}} G
$$

$$
\times \left( g' \otimes g_1' - g \otimes g_1 \right)
$$

$$
\times \left[ \ln \kappa (g' \otimes g_1') - \ln \kappa (g \otimes g_1) \right] .
$$

(7.24)

and after posing $h = g \otimes g_1$ and $h' = g' \otimes g_1'$ and taking into account that $\ln \kappa h$ is an increasing function, we note that $(h' - h) (\ln \kappa h' - \ln \kappa h) \geq 0$, $\forall h, h'$.

Consequently we can conclude that

$$
\partial_\nu S^\nu \geq 0 .
$$

(7.25)

This last relation is the local formulation of the relativistic H-theorem which represents the second law of the thermodynamics for the system governed by the evolution equation (7.6) and the two-particle correlation function (7.7).
F. On the definition of the entropy

Let us adopt for the two particle correlation function the definition

\[ C(f, f_1) = f \otimes f_1, \tag{7.26} \]

in place of the \( \{7.11\} \). It is straightforward to verify that one obtains the following expressions for the distribution function, the four-vector entropy and the scalar entropy

\[ f = \exp_\kappa \left( -\beta \left( \frac{p^\nu + q A^\nu / c}{U_\nu - mc^2 - \mu} \right) \right), \tag{7.27} \]

\[ S^\nu = -\int \frac{d^3 p}{p^0} \, p^\nu f \ln_\kappa (\alpha f), \tag{7.28} \]

\[ S = -\int d^3 p \, f \ln_\kappa (\alpha f), \tag{7.29} \]

in place of \( \{7.11\}, \{7.12\} \) and \( \{7.14\} \) respectively.

In particular, in the classical limits we have \( \lim_{\kappa \to 0} \alpha = 1/e \), and the entropy \( \{7.29\} \) reduces to the classical expression \( S_{\text{class}} = -\int d^3 p f \left[ \ln f - 1 \right] \), used some time in kinetic theory \( \{21\} \).

In order to explain better the differences between the two definitions \( \{7.14\} \) and \( \{7.29\} \) of the entropy we consider a system described by a discrete probability distribution \( f = \{ f_i, 1 \leq i \leq N \} \). We observe that the probability distribution \( f = \{ \delta_m, 1 \leq i \leq N \} \) being \( m \) a fixed integer with \( 1 \leq m \leq N \), describes a state of the system for which we have the maximum information. It is natural to set for this state \( S = 0 \). Clearly it is possible only if we adopt for the system entropy the definition \( \{7.14\} \). This interesting property of the entropy \( \{7.14\} \) makes it more appealing with respect the definition \( \{7.29\} \) which predicts a residual entropy \( S = -\ln_\kappa \alpha \) for the state corresponding to the maximum information. Anyway, the two above discussed choices for the entropy definition does not influence the physics of the system.

VIII. CONCLUSIONS

We have shown that the special relativity laws and the maximum entropy principle, suggest a relativistic generalization for the two-particle correlation function in the relativistic Boltzmann equation. This fact imply a relativistic generalization of the classical Boltzmann-Gibbs-Shannon entropy.

The so obtained, fully relativistic Boltzmann equation, obeys the H-theorem and predicts a stationary stable distribution, presenting power-law high-energy tails, according to the experimental evidence. The ensued relativistic kinetic theory preserves the main features of the classical kinetics which recovers in \( c \to \infty \) limit.

In the last few years the statistical theory based on the new entropy \( \{16, 17, 18, 19\} \), has been considered by various authors. Investigations related with the foundations of the theory include e.g. the H-theorem and the molecular chaos hypothesis \( \{22, 23\} \), the thermodynamic stability \( \{24, 25\} \), the Leščke stability \( \{26, 27, 28, 29\} \), the Legnèdre structure of the ensued thermodynamics \( \{30\} \) etc. On the other hand, specific applications of the theory, include e.g. the cosmic rays \( \{31\} \), relativistic \( \{31\} \) and classical \( \{32\} \) plasmas in presence of external electromagnetic fields, the relaxation in relativistic plasmas under wave-particle interactions \( \{33, 34\} \), astrophysical systems \( \{35, 36\} \), the kinetics of interacting atoms and photons \( \{37\} \), particle systems in external conservative force fields \( \{38\} \), the quark-gluon plasma formation \( \{39\} \) etc. Other applications regard dynamical systems at the edge of chaos \( \{40, 41\} \), fractal systems \( \{42\} \), the random matrix theory \( \{43\} \), the error theory \( \{44\} \), the game theory \( \{45\} \), the Information theory \( \{46\} \), etc. Also applications to economic systems have been considered e.g. to study the personal income distribution \( \{47, 48\} \), to model deterministic heterogeneity in tastes and product differentiation \( \{49, 50\} \) etc.

[1] G. Kaniadakis, Maximum Entropy Principle and Power-Law Tailed Distributions Eur. Phys. J. B 69, DOI: 10.1140/epjb/e2009-00161-0 (2009).
[2] A. Hasegawa, A.M. Kunioki, and M. Duong-van, Phys. Rev. Lett. 54, 2608 (1985).
[3] V.M. Vasyliunas, J. Geophys. Res. 73, 2839 (1968).
[4] P.L. Biemann, and G. Sigl, Physics and Astrophysics of Ultra-High-Energy Cosmic Rays, Lectures Notes in Physics 576, Spring-Verlag Berlin (2001).
[5] G. Wilk and, Z. Wlodarczyk, Phys. Rev. D 50, 2318 (1994).
[6] D.B. Walton, and J. Rafelski, Phys. Rev. Lett. 84, 31 (2000).
[7] G. Kaniadakis, P. Quarati, A set of stationary non-Maxwellian distributions, Physica A 192, 677 (1993).
[8] G. Kaniadakis, P. Quarati, Polynomial expansion of diffusion and drift coefficients for classical and quantum statistics, Physica A 237, 229 (1997).
[9] G. Kaniadakis, A. Lavagno, P. Quarati, Kinetic approach to fractional exclusion statistics, Nucl. Phys. B 466, 527 (1996).
[10] G. Kaniadakis, A. Lavagno, P. Quarati, Kinetic model for q-deformed bosons and fermions, Phys. Lett. A 227, 227 (1997).
[11] S. Abe, Generalized entropy optimized by a given arbitrary distribution, J. Phys. A: Math. Gen. 36, 8733-8738 (2003).
[12] T.D. Frank, Interpretation of Langrange multipliers of generalized maximum-entropy distributions, Phys. Lett. A 299, 153 (2002).
[13] P.-H. Chavanis, Gener. Thermod. and kinetic equations: Boltzmann, Landau, Kramers and Smoluchowski, Physica A 332, 89 (2004).
[14] T.D. Frank, Generalized multivariate Fokker-Planck
equations derived from kinetic transport theory and linear nonequilibrium thermodynamics, Phys. Lett. A 305, 150 (2002).

[15] V. Schwammle, E.M.F. Curado, F.D. Nobre, A general nonlinear Fokker-Planck equation and its associated entropy, Eur. Phys. J B 58, 159 (2007).

[16] G. Kaniadakis, Non-linear kinetics underlying generalized statistics, Physica A 296, 405 (2001).

[17] G. Kaniadakis, H-theorem and generalized entropies within the framework of nonlinear kinetics, Phys. Lett. A 288, 283 (2001).

[18] G. Kaniadakis, Statistical mechanics in the context of special relativity, Phys. Rev. E 66, 056125 (2002).

[19] G. Kaniadakis, Statistical mechanics in the context of special relativity II, Phys. Rev. E 72, 036108 (2005).

[20] T.S. Biro, G. Kaniadakis, Two generalizations of the Boltzmann equation, Eur. Phys. J. B 50, 3 (2006).

[21] S.R. de Groot, W.A. van Leeuwen, Ch.G. van Weert, Relativistic Kinetic Theory, North-Holland Publishing Company, Amsterdam (1980).

[22] R. silica, The relativistic statistical theory and Kaniadakis entropy: an approach through a molecular chaos hypothesis, Eur. Phys. J. B 54, 499 (2006).

[23] R. silica, The H-theorem in \( \kappa \)-statistics: influence on the molecular chaos hypothesis, Phys. Lett. A 352 17 (2006).

[24] T. Wada, Thermodynamic stabilities of the generalized Boltzmann entropies, Physica A 340, 126 (2004).

[25] T. Wada, Thermodynamic stability conditions for nonadditive composable entropies, Contin. Mechanics and Thermodynamics 16, 263 (2004).

[26] G. Kaniadakis, A.M. Scarfone, Lesch stability of \( \kappa \)-entropy, Physica A 340, 102 (2004).

[27] S. Abe, G. Kaniadakis and A.M. Scarfone, Stabilities of generalized entropy, J. Phys. A: Math. Gen. 37, 10513 (2004).

[28] J. Naudts, Deformed exponentials and logarithms in generalized thermostatistics, Physica A 316, 323 (2002).

[29] J. Naudts, Continuity of a class of entropies and relative entropies, Rev. Math. Phys. 16, 809 (2004).

[30] A.M. Scarfone, T. Wada, Canonical partition function for anomalous systems described by the kappa-entropy Progress of Theor. Phys. Suppl. 162 45 (2006).

[31] Guo Lina, Du Jiulin, and Liu Zhipeng. The property of \( \kappa \)-deformed statistics for a relativistic gas in an electromagnetic field: \( \kappa \) parameter and \( \kappa \)-distribution, Phys. Lett. A 367, 431-435 (2007).

[32] Guo Lina and Du Jiulin. The \( \kappa \) parameter and \( \kappa \)-distribution in \( \kappa \)-deformed statistics for the systems in an external field, Phys. Lett. A 362, 368-370 (2007).

[33] G. Lapenta, S. Markidis, A. Marocchino, and G. Kaniadakis, Relaxation of relativistic plasmas under the effect of wave-particle interactions, The Astrophysical Journal 666, 949-954 (2007).

[34] G. Lapenta, S. Markidis, G. Kaniadakis, Computer experiments on the relaxation of collisionless plasmas, Journal of Statistical Mechanics, P02024 (2009).

[35] J. C. Carvalho, R. Silva, J. D. do Nascimento jr., and J. R. De Medeiros, Power law statistics and stellar rotational velocities in the Pleiades, EPL 84, 59001 (2008).

[36] J. C. Carvalho, J.D. do Nascimento jr., R. Silva, and J. R. De Medeiros, Non-gaussian statistics and stellar rotational velocities of main sequence field stars, Astrophyiscal Journal Letters 696, L48 (2009).

[37] A. Rossani and A.M. Scarfone, Generalized kinetic equations for a system of interacting atoms and photons: Theory and Simulations, J. Phys. A 37, 4955 (2004)

[38] J.M. Silva, R. Silva, J.A.S. Lima, Conservative force fields in non-Gaussian statistics, Phys. Lett. A 372, 5754 (2008).

[39] A.M. Teweldeberhan, H.G. Miller, and R. Tegen, \( \kappa \)-deformed Statistics and the formation of a quark-gluon plasma, Int. J. Mod. Phys. E 12, 669 (2003).

[40] M. Coradu, M. Lissia, R. Tonelli, Statistical descriptions of nonlinear systems at the onset of chaos Physica A 365, 252 (2006).

[41] A. Celikoglu, U. Tirkaki, Sensitivity function and entropy increase rates for z-logistic map family at the edge of chaos, Physica A 372, 238 (2006).

[42] A.I. Olemskoi, V.O. Kharchenko, V.N. Borisyuk, Multifractal spectrum of phase space related to generalized thermostatistics, Physica A 387, 1895 (2008).

[43] A.Y. Abul-Magd, Nonextensive random-matrix theory based on Kaniadakis entropy, Phys. Lett. A 361, 450 (2007).

[44] T. Wada, H. Suyari, \( \kappa \)-generalization of Gauss’ law of error, Phys. Lett. A 348, 89 (2006).

[45] F. Topsoe, Entropy and equilibrium via games of complexity, Physica A 340 11 (2004).

[46] T. Wada, H. Suyari, A two-parameter generalization of Shannon-Khinchin Axioms and the uniqueness theorem, Phys. Lett. A 368, 199 (2007).

[47] F. Clementi, M. Gallegati, and G. Kaniadakis, \( \kappa \)-generalized statistics in personal income distribution, Eur. Phys. J. B 57, 187 (2007).

[48] F. Clementi, T. Di Matteo, M. Gallegati, G. Kaniadakis, The kappa-generalized distribution: A new descriptive model for the size distribution of incomes, Physica A 387, 3201 (2008).

[49] D. Rajaonarison, D. Bolduc, and H. Jayet, The K-deformed multinomial logit model, Econ. Lett. 86, 13-20 (2005).

[50] D. Rajaonarison, Deterministic heterogeneity in tastes and product differentiation in the K-logit model, Econ. Lett. 100, 396 (2008).