Redundancy Scheduling with Locally Stable Compatibility Graphs

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Abstract

Redundancy scheduling is a popular concept to improve performance in parallel-server systems. In the baseline scenario any job can be handled equally well by any server, and is replicated to a fixed number of servers selected uniformly at random. Quite often however, there may be heterogeneity in job characteristics or server capabilities, and jobs can only be replicated to specific servers because of affinity relations or compatibility constraints. In order to capture such situations, we consider a scenario where jobs of various types are replicated to different subsets of servers as prescribed by a general compatibility graph. We exploit a product-form stationary distribution and weak local stability conditions to establish a state space collapse in heavy traffic. In this limiting regime, the parallel-server system with graph-based redundancy scheduling operates as a multi-class single-server system, achieving full resource pooling and exhibiting strong insensitivity to the underlying compatibility constraints.

Keywords: redundancy scheduling, affinity relations, state space collapse, heavy traffic, parallel-server systems

1 Introduction

Redundancy scheduling has emerged as a crucial mechanism to improve performance in parallel-server systems \cite{2,16–20,27,28,31,40}. The key feature of redundancy scheduling is that replicas are created for each arriving job, which are then assigned to different servers. As soon as the first of these replicas either starts service or completes service, the remaining ones are abandoned (referred to as ‘cancel-on-start’ and ‘cancel-on-completion’, respectively).

Dispatching replicas of the same job to several servers increases the chance for one of the replicas to find a short queue and thus start service fast. Indeed, the ‘cancel-on-start’ version in effect amounts to a Join-the-Shortest-Workload (JSW) policy with partial selection of servers \cite{45}. The ‘cancel-on-completion’ version additionally increases the chance for one of the replicas to have a short execution time (assuming independence among run times on different servers). On the flip side, the possible concurrent execution of replicas comes with a risk of wastage of capacity (depending on run time distributions), and has sparked a strong interest in the impact of redundancy on stability conditions \cite{3,4,24,33}.

In the present paper we consider a scenario with independent and exponentially distributed run times where stability is not an issue, and pursue a different performance question which has hardly received any attention so far. In the bulk of the literature on redundancy scheduling, it is assumed that

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the replicas are assigned to different servers that are selected uniformly at random. This implicitly assumes that any job can be handled equally well by any server, which is for example not the case in multi-skill environments with compatibility constraints. Even when jobs or servers are not intrinsically different, some servers may be better equipped to process particular jobs because of affinity relations or data locality issues. In these scenarios replicas can only be assigned to specific subsets of servers.

The product-form distributions for redundancy systems, as derived in the seminal paper [20], allow for completely arbitrary compatibility constraints between jobs and servers. However, the product-form distributions do not lend themselves easily for computational purposes, and do not yield any explicit insights in the impact of the replication options on the performance in terms of queue lengths and delays. Gaining such understanding is not only valuable in evaluating the performance in a given system deployment, but also highly relevant for examining various design trade-offs. For example, installing servers with broader capabilities to handle jobs will expand replication options and tend to improve the performance but may also generate additional expenses or implementation overhead, and it is not evident how to achieve a good balance between these conflicting goals.

Main contributions. In the present paper we examine the impact of the replication options on the performance of redundancy systems in terms of queue lengths and delays. We demonstrate that the performance impact tends to be fairly limited, provided the replication options offer sufficient flexibility for a mild and natural resource pooling condition to hold. Under the latter condition, the replication options provide maximum stability, meaning that no subset of servers is overloaded and the total load can be handled as long as it is less than the total service capacity. More specifically, we establish that as long as the latter condition holds and traffic is Markovian, the system occupancy exhibits state space collapse in heavy traffic. In the limit the system occupancy behaves just like in a multi-type M/M/1 queue where the various types correspond to jobs that are replicated to the same subset of servers. Informally speaking, the numbers of jobs of the various types remain in strict proportion to the arrival rates in the limit while the total number of jobs randomly varies over time. The latter number, properly scaled, tends to an exponential random variable, and thus the number of jobs of each type has an exponential limiting distribution as well. By virtue of the distributional form of Little’s law, this means that job delay, after scaling, also has an exponential limiting distribution.

In order to prove the above results, we start from the available product-form distributions and adopt a specific enumeration of all the possible job configurations to write the joint probability generating function in a convenient form that facilitates a heavy-traffic analysis.

It is worth observing that the resource pooling condition can be fulfilled even with severely restricted replication options, as long as they are well balanced in the sense that local overload is avoided. The heavy-traffic results thus suggest that very extensive replication options do not yield particularly significant performance gains, unless there is uncertainty in the distribution of the overall load across the various job types.

In a light-traffic scenario, the performance impact of the replication options is minor in absolute terms as well, as the probability of a non-empty system goes to zero in any circumstance. However, in relative terms, the performance is sensitive in the sense that the rescaled probability of a particular number of jobs residing in the system is critically dependent on the replication options. In particular, we illustrate that when any job can be replicated to (at most) two servers, the best-case scenario is where an arbitrary job is replicated to any pair of servers with equal probability, as in the celebrated power-of-two policy. This optimality property reflects the performance gains from highly diverse replication options, although the above-mentioned heavy-traffic results imply that these are not spectacular.

Related literature. As mentioned above, we exploit the crucial product-form distributions for redundancy systems obtained by Gardner et al. in [20]. Product-form results for redundancy systems with a cancel-on-start rather than cancel-on-completion mechanism are presented in [45]. Related product-form distributions for similar systems with compatibility constraints between jobs and servers and so-called Assign-to-Longest-Idle-Server-First (ALIS) policies were obtained in [1]. Notably, all these results turn out to be related to product-form distributions for so-called Order-
Independent queues \[30\], the concept of Balanced Fairness \[9, 10\] and token-based central queues \[7\].

While the product-form distributions are specified in closed form, these expressions unfortunately do not yield immediate insight in the impact of the compatibility constraints on the performance in terms of queue lengths and delays. In fact, it involves considerable effort to arrange these expressions in a suitable form that facilitates the derivation of heavy-traffic limits.

Over the last few decades, asymptotic analysis and particularly heavy-traffic limits have emerged as key methodologies in the area of stochastic networks. Indeed, heavy-traffic limits have been established for a wide variety of multi-class parallel-server systems, with a vast range of both job assignment (routing, load balancing) policies and server allocation (scheduling, sequencing) strategies \[6, 8, 11, 22, 23\]. In particular, the phenomenon of state space collapse has also emerged as a paramount characteristic in heavy-traffic conditions. However, redundancy systems, with concurrent execution of jobs, seem to fall outside the realm of the canonical settings that are adopted in the heavy-traffic literature, such as input-queued/output-queued models and generalized switches \[33, 42, 43\], stochastic processing networks \[5, 14, 22, 23, 32, 37\], and switched networks \[38, 39, 41\]. To the best of our knowledge, no heavy-traffic limits or state space collapse properties for redundancy systems have been established so far. As alluded to above, the heavy-traffic analysis in the present paper is fundamentally different from a methodological standpoint in the sense that it is based on explicit, be it somewhat unwieldy, expressions for the stationary distribution. In contrast, typically one resorts to a heavy-traffic approach when an exact analysis is intractable, and then considers so-called process-level limits over finite time intervals as opposed to the stationary distribution. The latter difference translates into distinct convergence notions, and raises the interchange-of-limits between the stationary regime (time tending to infinity) and the heavy-traffic regime (load approaching unity) as a highly intricate issue.

Similarly, we use the product-form distributions as a basis for the light-traffic analysis, while usually a light-traffic approach is only considered when explicit formulas are lacking, and then based on the powerful framework developed by Reiman & Simon \[36\]. For example, Izagirre & Makowski \[26\] adopt that framework to examine the performance impact of server heterogeneity in the context of power-of-two load balancing algorithms.

The performance impact of compatibility constraints does not only pertain to redundancy systems but also to parallel-server systems with different job assignment policies such as Join-the-Shortest-Queue (JSQ). The JSQ policy is similar in spirit to the JSW policy which in fact corresponds to redundancy scheduling with a cancel-on-start mechanism as mentioned above. The performance impact of compatibility constraints has been examined for JSQ type policies in a many-server regime, particularly in scenarios where these constraints can be represented in terms of some graph structure, see for instance Budhiraja et al. \[13\], Gast \[21\], Mukherjee et al. \[34\] and Turner \[44\]. The results of \[21, 44\] for certain fixed-degree graphs, in particular ring topologies, demonstrate that the performance sensitively depends on the underlying graph topology, and that sampling from a fixed set of neighbors typically does not match the performance of re-sampling the same number of alternate servers across the entire system. The results in \[13, 34\] establish conditions in terms of the density and topology of the sampling graph in order for the performance to be asymptotically similar in a many-server regime as in a standard scenario without any compatibility constraints at all. While this also embodies a certain asymptotic insensitivity property, the actual behavior is totally different since queues largely vanish in a many-server regime, and in particular no state space collapse occurs.

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2 we present a detailed model description and discuss some broader context and preliminaries. The main results are stated in Section 3. Sections 4 and 5 are devoted to the proofs of the results in the heavy-traffic and light-traffic regimes, respectively, with some proof details deferred to the appendices. Finally, some broader discussions and concluding remarks are provided in Section 6.
2 Model description and preliminaries

We consider a redundancy system with $N$ parallel servers with speeds $\mu_1, \ldots, \mu_N$. Jobs arrive as a Poisson process of rate $N\lambda$, so $\lambda$ is the arrival rate normalized by the number of servers. When a job arrives, with probability $p_S$, replicas are assigned to each of the servers in the (non-empty) set $S \subseteq \{1, \ldots, N\}$, hence jobs whose replicas are assigned to these servers arrive at rate $\lambda_S := N\lambda p_S$. For compactness, such a job will be referred to as a type-$S$ job, and we denote by $S = \{S \in 2^{\{1, \ldots, N\}} : p_S > 0\}$ the collection of job types. The sizes of the replicas are independent and exponentially distributed with unit mean. As soon as the first of the replicas of a particular job finishes, the remaining replicas are instantaneously abandoned. Each of the servers follows a First-Come First-Served (FCFS) discipline.

The system occupancy at time $t$ may be represented in terms of a vector $(c_1, \ldots, c_{M(t)})$, with $M(t)$ denoting the total number of jobs in the system at time $t$ and $c_m \in S$ indicating the type of the $m$th oldest job at that time. It is easily verified that the system occupancy evolves as a Markov process by virtue of the exponential traffic assumptions.

As stated in the introduction, a crucial issue in this context is how the performance of the system depends on the probabilities $p_S$. The first-order performance criterion is the stability condition. It was shown in [20] that

\[
N \lambda \sum_{S \in T} p_S < \sum_{n \in \bigcup_{S \in T} S} \mu_n
\]

for all (non-empty) $T \subseteq S$, or equivalently,

\[
N \lambda \sum_{S : S \subseteq U} p_S < \sum_{n \in U} \mu_n
\]

for all (non-empty) $U \subseteq \{1, \ldots, N\}$, is a necessary and sufficient condition for the system to be stable. In particular, taking $T = S$ or $U = \{1, \ldots, N\}$, we see that $\lambda < \mu$ is a necessary condition with $\mu := \frac{1}{N} \sum_{n=1}^{N} \mu_n$ denoting the average speed across all servers. Note that the left-hand side of (1) represents the total arrival rate of job types $S \in T$, while the right-hand side measures the aggregate service rate of the servers that can help in handling jobs of these types. Likewise, the right-hand side of (2) represents the aggregate service rate of the servers in the set $U$, while the left-hand side captures the total arrival rate of job types that can be handled by these servers only.

Moreover, it was shown in [20] that, under the above stability conditions, the stationary distribution of the system occupancy is

\[
\pi(c_1, \ldots, c_M) = C \prod_{i=1}^{M} \frac{N \lambda p_{c_i}}{\mu(c_1, \ldots, c_i)},
\]

with $C$ a normalization constant and

\[
\mu(c_1, \ldots, c_i) = \sum_{n \in \bigcup_{m=1}^{M} \{c_m\}} \mu_n.
\]

In Section 3 we will focus on the scenario where any job can be replicated to (at most) two servers, i.e., $p_S = 0$, unless $|S| = 2$. The replication options can in this case be visualized in a replication graph with $N$ vertices. The probabilities $p_S$ may then also be interpreted as edge selection probabilities $p_{(i,j)}$ in this graph; a type-$\{i,j\}$ job will be replicated to the servers corresponding to the vertices $i$ and $j$ incident to this edge. The special case of uniform edge probabilities, i.e., $p_{(i,j)} = p(N) = 1/E$ for all $i \neq j$, with $E = \binom{N}{2}$ the number of different server pairs, corresponds to the commonly considered power-of-two policy. Henceforth, we refer to this scenario as the graph model. When jobs are replicated to (at most) $d > 2$ servers, i.e., $p_S = 0$, unless $|S| = d$, one can think of $p_S$ as the edge selection probability of hyper-edge $S$ in a hyper-graph with $N$ vertices where each edge is incident to
Figure 1 visualizes the above-mentioned graph model and two different representations of a particular state for a system with $N = 4$ identical servers arranged in a ring structure. The probability of selecting server pairs incident to the same edge to replicate an arriving job, are given by $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})$ as can be seen in Figure 1a. This implies that the arrival rate $\lambda_{i,i+1}$ is equal to $4\lambda p_{(i,i+1)}$, which is $\frac{1}{2}\lambda$ if $i$ is odd and $\frac{3}{2}\lambda$ if $i$ is even. Consider, for instance, the state $c = ([1, 2], [1, 4], [2, 3], [1, 4])$. The oldest job in the system is replicated to servers 1 and 2, the second oldest job is replicated to servers 1 and 4 and so on. On the one hand, Figure 1b shows how the replicas, belonging to the jobs in state $c$, are stored in separate queues in front of each server. Moreover, the arrival streams to the different server pairs are indicated. On the other hand, Figure 1c visualizes the same state $c$ from a modelling perspective where all the jobs are stored in a fictitious central queue in order of arrival. This representation is completely equivalent with the more natural representation with a separate queue in front of each server; the next job replica a server can process after a service completion or abandonment can be extracted by scanning the central queue in order to find the next compatible job in line. When $\mu_n = \mu$ for all $n = 1, 2, 3, 4$ the stationary probability of state $c$ is given by $\pi(c) = C(\frac{3}{2})^4 \cdot \frac{p_{(1,2)} p_{(3,4)} p_{(2,3)} p_{(1,4)}}{p_{(1,1)} p_{(4,4)} p_{(2,2)} p_{(3,3)}}$ according to the general expression in (3).

**Remark 2.1.** Note that the job-server compatibility constraints can also be visualized as a bipartite graph with disjoint sets $S$ and $\{1, \ldots, N\}$ as the vertices $[1, 10, 20]$. This visualization can be recognized on the top of Figure 11 for the above-mentioned example. Henceforth, we will focus on a scenario, mainly in Section 6 where the replication options can be derived from a graph structure consisting of $N$ vertices for each of the servers. Therefore we will always refer to above-defined compatibility constraints and no longer to the bipartite compatibility graph to avoid overlap in terminology.

## 3 Main results

We now provide an overview of the main results, relegating the proofs to later sections. We first introduce some useful notation. Let $(Q_S)_{S \in S}$ be a random vector with the stationary distribution of the numbers of jobs of the various types. Let $(R_i)_{i=1,\ldots,N}$, with $R_i = \sum_{S \in S} Q_S$, be a random vector with the stationary distribution of the numbers of replicas assigned to the various servers. The random variables with the stationary distribution of the sojourn time and waiting time of an arbitrary type-$S$ job are denoted by $V_S$ and $W_S$, respectively.
If the probabilities $p_S$ satisfy the inequalities

$$\sum_{S \in T} p_S < \frac{1}{N\mu} \sum_{n \in \bigcup_{S \in T} S} \mu_n$$

(4)

for all (non-empty) $T \subset S$ with $\bigcup_{S \in T} S \neq \{1, \ldots, N\}$ and $\mu := \frac{1}{N} \sum_{n=1}^{N} \mu_n$ the average service rate, or equivalently,

$$\sum_{S : S \subseteq U} p_S < \frac{1}{N\mu} \sum_{n \in U} \mu_n$$

(5)

for all (non-empty) $U \subset \{1, \ldots, N\}$, then the stability conditions in (1) and (2) are satisfied for any $\lambda < \mu$. (Note that we restrict to subsets $T \subset S$ with $\bigcup_{S \in T} S \neq \{1, \ldots, N\}$ and strict subsets $U \subset \{1, \ldots, N\}$, as (4) and (5) hold with equality in case $\bigcup_{S \in T} S = \{1, \ldots, N\}$ and $U = \{1, \ldots, N\}$, respectively, by definition of the average service rate $\mu$. More specifically, the strict inequalities in (4) ensure that as $\lambda$ approaches $\mu$, for all (non-empty) $T \subset S$, the total arrival rate of the job types $S \in T$ remains bounded away from the aggregate service rate of the servers that can help in handling jobs of these types. Similarly, the strict inequalities in (5) guarantee that as $\lambda$ approaches $\mu$, for all (non-empty) $U \subset \{1, \ldots, N\}$, the aggregate service rate of the servers on the set $U$ remains bounded away from the total arrival rate of job types that can be handled by these servers only.) Hence, the local stability conditions in (4) and (5) ensure that as $\lambda$ approaches $\mu$, only the inequalities in (1) and (2) for $T = S$ and $U = \{1, \ldots, N\}$, respectively, are tight in the limit. In other words, only the boundary of the capacity region corresponding to the global capacity constraint is approached in the limit, meaning that a suitable notion of complete resource pooling applies in the heavy-traffic regime.

For later use we observe that the conditions in (4) may also be written in the less intuitive but equivalent form

$$\frac{1}{N\mu} \sum_{n \in \left( \bigcup_{S \in T'} S \right)^c} \mu_n < \sum_{S \in T'} p_S$$

for all (non-empty) $T' = T^c \subset S$. These inequalities reflect that for all (non-empty) $T'$ the total arrival rate of job types $S \in T$ as $\lambda \uparrow \mu$ should become strictly greater than the aggregate service rate of the servers that are able to handle jobs of these types only. In particular, taking $T' = \{S_0\}$, we obtain

$$\sum_{n \in N_{S_0}} \mu_n < N\mu p_{S_0},$$

(6)

with $N_{S_0} = \bigcup_{S \in S \setminus S_0} S$. Likewise, the conditions in (5) may be rewritten as

$$\frac{1}{N\mu} \sum_{n \in U'} \mu_n < \sum_{S : S \cap U' \neq \emptyset} p_S$$

for all (non-empty) $U' = U^c \subset \{1, \ldots, N\}$. These inequalities indicate that for all (non-empty) $U' \subset \{1, \ldots, N\}$, the aggregate service rate of the servers in the set $U$ should become strictly lower than the total arrival rate of the job types that can be handled by at least one of these servers as $\lambda \uparrow \mu$. In particular, taking $U' = \{n_0\}$, we see that any server $n_0$ (with a strictly positive speed $\mu_{n_0} > 0$) must be able to handle at least one job type $S$ (with a strictly positive arrival probability $p_S > 0$).

From the above observations, we also conclude that all choices of probabilities $p_S$ that satisfy the inequalities in (4) and (5) provide maximum stability. There are still many such choices, so this leaves the question how detailed performance metrics such as the (mean) queue lengths and delays depend on the probabilities $p_S$ if we impose these restrictions. The next theorem establishes that even such performance metrics are not strongly affected by the exact values of the probabilities $p_S$ as long as the (strict) inequalities in (4) and (5) hold.
Theorem 3.1. If the (strict) inequalities in (4) and (5) hold, then
\[(1 - \frac{\lambda}{\mu}) (Q_S)_{S \in S} \rightarrow \text{Exp}(1)(p_S)_{S \in S},\]
as \lambda \uparrow \mu, with \text{Exp}(1) representing a unit-mean exponentially distributed random variable.

The above theorem shows that the joint distribution of the numbers of jobs of the various types exhibits state space collapse in a heavy-traffic regime, and coincides in the limit with the joint distribution of a multi-type M/M/1 queue with arrival rate $N\lambda$, service rate $N\mu$, and type probabilities $p_S$, provided the local stability conditions in (4) and (5) are satisfied. In particular, the total number of jobs, properly scaled, tends to an exponential random variable in the limit, and the number of jobs of each type has an exponential limiting distribution as well.

Theorem 3.1 may be interpreted as follows. Even though the parallel-server system is not work-conserving, it is highly unlikely that any server is idling when the total number of jobs is large because of the natural diversity in job types and the fact that any server is able to handle at least one job type as noted above. In other words, the system will operate at the full aggregate service rate $N\mu$ with high probability when the total number of jobs is sufficiently large. This explains why the total number of jobs behaves asymptotically the same as in an M/M/1 queue with arrival rate $N\lambda$ and service rate $N\mu$, and in particular follows the well-known scaled exponential distribution in the limit. What is far less evident though, is the state space collapse, with the proportions of jobs of the various types coinciding with the corresponding arrival probabilities like in a multi-type M/M/1 queue with a FCFS discipline. In order to provide an informal explanation, suppose that for a particular type $S_0$ the proportion of jobs is significantly lower than the corresponding arrival probability $p_{S_0}$ while the total number of jobs is large. This means that a large number of type-$S_0$ jobs have been completed that arrived after jobs of other types that are still in the system. This in turn implies that type-$S_0$ jobs will only receive a non-vanishing fraction of the capacity of the servers in $N_{S_0}$, with $N_{S_0} = \cup_{S \in S \setminus \{S_0\}} S$, that cannot handle jobs of any other types. Thus the aggregate service rate of the servers in $N_{S_0}$ is no less than the arrival rate of the type-$S_0$ jobs, i.e., the inequality in (6) is violated, which yields a contradiction. Hence, we conclude that for none of the job types the proportion of jobs can be significantly lower than the corresponding arrival probability, meaning that the state space collapse occurs.

Corollary 3.1. If the (strict) inequalities in (4) and (5) hold, then
\[(1 - \frac{\lambda}{\mu}) (R_n)_{n=1,\ldots,N} \rightarrow \text{Exp}(1)(q_n)_{n=1,\ldots,N},\]
as \lambda \uparrow \mu, with $q_n = \sum_{S: n \in S} p_S$, $n = 1, \ldots, N$, and \text{Exp}(1) representing a unit-mean exponentially distributed random variable.

By virtue of the distributional form of Little’s law [29], the sojourn time and waiting time, properly scaled, also have an exponential distribution in the limit which does not depend on the probabilities $p_S$ in any way, as long as the (strict) inequalities in (4) and (5) hold, as stated in the next corollary.

Corollary 3.2. If the (strict) inequalities in (4) and (5) hold, then
\[(1 - \frac{\lambda}{\mu}) V_S \rightarrow \frac{1}{N\mu} \text{Exp}(1), \quad (1 - \frac{\lambda}{\mu}) W_S \rightarrow \frac{1}{N\mu} \text{Exp}(1),\]
as \lambda \uparrow \mu, for all $S \in S$, with \text{Exp}(1) representing a unit-mean exponentially distributed random variable.

The proof of Theorem 3.1 relies on a specific enumeration of all possible job configurations, which yields a particularly convenient form of the joint probability generating function of the numbers of jobs of the various types as provided in the next proposition.
Proposition 3.1.
\[
E \left[ \prod_{S \in \mathcal{S}} z^Q_S \right] = \frac{f(z)}{f(1)},
\]
(7)

where \( z \) and \( 1 \) are \(|S|\)-dimensional vectors with entries \(|z| \leq 1\) and \( f(z) \) is given by
\[
1 + \sum_{m=1}^{|S|} \sum_{S \in \mathcal{S}_m} \prod_{j=1}^m \mu(S_1, \ldots, S_j) \left( 1 - \frac{N \lambda p_S z_S}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^j p_{S_i} z_{S_i} \right)^{-1}.
\]

The \( m \)-dimensional vector \( S \) consists of \( m \) different job types, and the set consisting of all these vectors is denoted by \( \mathcal{S}_m \).

We will also consider a light-traffic regime for the scenario where all server speeds are identical and any job can be replicated to (at most) two servers, i.e., \( p_S = 0 \), unless \(|S| = 2\). As described in Section 2, the replication options can then be visualized as a replication graph with \( N \) vertices. The probabilities \( p_S \) may also be interpreted as edge selection probabilities \( p_{(i,j)} \) in this graph and the job is replicated to the two servers corresponding to the two vertices incident to the selected edge. Let \( Q(P) \) be a random variable with the stationary distribution of the total number of jobs in the system with edge selection probabilities \( P = (p_{(i,j)}) \) and let \( Q^* \) be that random variable for the special case of uniform edge probabilities corresponding to the power-of-two policy. The next proposition states that uniform edge selection probabilities yield the best-case scenario in light-traffic regime.

Proposition 3.2. For any choice of edge selection probabilities \( p_{(i,j)} \),
\[
\lim_{\lambda \to 0} \frac{\mathbb{P}(Q^* \geq q)}{\mathbb{P}(Q(P) \geq q)} \leq 1
\]
for \( q = 1, 2 \).

Although the absolute performance differences for various replication graphs will be marginal as the system will be empty with probability one in the light-traffic regime, this result contrasts with the insensitivity result obtained in Theorem 3.1 in the heavy-traffic regime. As argued before, all servers will constantly be busy in this regime, while only very few servers will be simultaneously active when the load on the system is low. When one job is present in the system, the total service rate offered by the servers will always be equal to \( 2 \mu \), irrespective of the edge selection probabilities. However, when two jobs are present the total service rate can be equal to \( 2 \mu \), \( 3 \mu \) or \( 4 \mu \) if the selected edges are, identical, have one common endpoint or have no common endpoints, respectively. This last option is preferred as it fully exploits the use of replicas in redundancy scheduling, and it turns out that for uniform edge selection probabilities this option will occur more often than for any other set of edge selection probabilities. This is one of the key ingredients for proving Proposition 3.2.

4 Heavy-traffic regime

The proof of the heavy-traffic result stated in Theorem 3.1 relies on a well-suited expression for the joint probability generating function of the numbers of jobs of the various types. This expression, which may be of independent interest, is provided in Proposition 3.1 and involves a specific enumeration of all possible job configurations, as explained in the proof below.

Proof Proposition 3.1. Using the stationary distribution given in (3), the joint probability generating function of the numbers of jobs of the various types may be written as
\[
E \left[ \prod_{S \in \mathcal{S}} z^Q_S \right] = \sum_{M=0}^{\infty} \sum_{(c_1, \ldots, c_M) \in \mathcal{S}^M} \pi(c_1, \ldots, c_M) \prod_{S \in \mathcal{S}} z^Q_S,
\]
(8)
with \( z \) an \(|S|\)-dimensional vector with entries \(|z_S| \leq 1\) and \(q_S^z\) the total number of type-\(S\) jobs in state \(c = (c_1, \ldots, c_M)\). The summation on the right-hand side in (8) over all possible states can be conducted in three steps:

(i) Fix the number of different job types that occur in the state \(c, m = 1, \ldots, |S|\).

(ii) Fix \(m\) distinct job types and the order in which they occur, \(S = [S_1, \ldots, S_m]\). This implies that the oldest job in the system is of type \(S_1\), the possible following jobs are of the same type, the first time a different type is observed it will be of type \(S_2\), etc. The set containing all these vectors of length \(m\) is denoted by \(S_m\).

(iii) Sum over all states with this particular order. For instance, for the vector \([S_1, S_2, S_3] \in S_3\) with only three job types one has to sum over all states

\[
c = (S_{k_1}, S_{k_2}, \times, \times, S_{k_3}, \circ, \circ, \circ),
\]

where \(\times\) denotes jobs of types \(S_1\) and/or types \(S_2\) and \(\circ\) denotes jobs of types \(S_1\) and \(S_2\) and/or \(S_3\). The values \(k_1, k_2\) and \(k_3\) can be any natural number.

The third step might warrant some illustration. For example, the contribution of the ordered vector \([S_1, S_2, S_3]\) to (8) can be computed by first determining how many jobs there are present in total, \(M \geq 3\). Then, the values of \(k_1, k_2\) and \(k_3 = M - k_1 - k_2 - 3\) are set. Finally, the \(k_2\) and \(k_3\) intermediate jobs are labeled as type-\(S_1\) or type-\(S_2\) jobs and type-\(S_1\), type-\(S_2\) or type-\(S_3\) jobs, respectively. Relying on (3), the total contribution to (8) is then given by

\[
C \sum_{M=3}^{\infty} \frac{\Lambda}{\mu(S_1)} p_{S_1} z_{S_1} \left[ \sum_{k_1=0}^{M-3} \left( \frac{\Lambda \lambda p_{S_1} z_{S_1}}{\mu(S_1)} \right)^{k_1} \right] \times \left[ \sum_{k_2=0}^{M-3-k_1} \left( \frac{\Lambda}{\mu(S_1)} \right)^{k_2} \left( p_{S_1} z_{S_1} \right)^{k_2} \left( p_{S_2} z_{S_2} \right)^{k_2-l} \times \left[ \sum_{k_3=0}^{k_3-l-1} \left( p_{S_1} z_{S_1} \right)^{l_1} \left( p_{S_2} z_{S_2} \right)^{l_2} \left( p_{S_3} z_{S_3} \right)^{k_3-l_1-l_2} \right] \right].
\]

Applying the multinomial of Newton leads to

\[
C \prod_{j=1}^{3} \frac{\Lambda}{\mu(S_1, \ldots, S_j)} p_{S_j} z_{S_j} \left[ \sum_{k_1=0}^{\infty} \left( \frac{\Lambda}{\mu(S_1)} \right)^{k_1} \left( p_{S_1} z_{S_1} \right)^{k_1} \right] \times \left[ \sum_{k_2=0}^{\infty} \left( \frac{\Lambda}{\mu(S_1, S_2)} \right)^{k_2} \left( p_{S_1} z_{S_1} + p_{S_2} z_{S_2} \right)^{k_2} \right] \times \left[ \sum_{k_3=0}^{\infty} \left( \frac{\Lambda}{\mu(S_1, S_2, S_3)} \right)^{k_3} \left( p_{S_1} z_{S_1} + p_{S_2} z_{S_2} + p_{S_3} z_{S_3} \right)^{k_3} \right].
\]

Interchanging the order of summation results in

\[
C \prod_{j=1}^{3} \frac{\Lambda}{\mu(S_1, \ldots, S_j)} p_{S_j} z_{S_j} \left[ \sum_{k_1=0}^{\infty} \left( \frac{\Lambda}{\mu(S_1, S_2)} \right)^{k_1} \left( \frac{\Lambda}{\mu(S_1, S_2, S_3)} \right)^{k_2} \left( p_{S_1} z_{S_1} + p_{S_2} z_{S_2} + p_{S_3} z_{S_3} \right)^{k_3} \right] \times \left[ \sum_{k_3=0}^{\infty} \left( \frac{\Lambda}{\mu(S_1, S_2, S_3)} \right)^{k_3} \left( p_{S_1} z_{S_1} + p_{S_2} z_{S_2} + p_{S_3} z_{S_3} \right)^{k_3} \right].
\]

Due to the stability conditions in (11) and (22), the expression for the infinite geometric sum may be applied to obtain:

\[
C \prod_{j=1}^{3} \frac{\Lambda}{\mu(S_1, \ldots, S_j)} \prod_{j=1}^{3} \left( 1 - \frac{\Lambda}{\mu(S_1, \ldots, S_j)} (p_{S_1} z_{S_1} + \ldots + p_{S_j} z_{S_j}) \right)^{-1}.
\]
Generalizing the above reasoning and applying the above-mentioned three steps will give an expression for (8), namely
\[ C \left[ 1 + \sum_{m=1}^{\left| S \right|} \sum_{S \in S_m} \prod_{j=1}^{m} \frac{N \lambda p_s z_s}{\mu(S_1, \ldots, S_j)} \prod_{j=1}^{m} \left( 1 - \frac{N \lambda}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^{j} p_s z_s \right) \right]^{-1}. \]

The three steps only consider states with at least one job and the contribution of the empty state can be seen in the additional '1'. Now, substituting \( z_s = 1 \) in (8) for all \( S \in S \) should give 1 as a result, and this yields an expression for the normalization constant \( C \). This concludes the derivation of the joint probability generating function (7).

The expression provided in Proposition 3.1 allows us to prove Theorem 3.1.

**Proof of Theorem 3.1.** We will prove the following heavy-traffic behavior of the moment generating function of the numbers of jobs of the various types \( (Q_S)_{S \in S} \).

\[ E \left[ \exp \left( - \left( 1 - \frac{\lambda}{\mu} \right) \sum_{S \in S} t_S Q_S \right) \right] \rightarrow \left( 1 + \sum_{S \in S} p_S t_S \right)^{-1}, \tag{9} \]

as \( \lambda \uparrow \mu \) and \( t_S \geq 0 \) for all \( S \in S \). Moreover, it can easily be seen that the moment generating function of the random vector \( \exp(1)(p_S)_{S \in S} \) is given by
\[ E \left[ \prod_{S \in S} e^{-t_S p_S \exp(1)} \right] = E \left[ e^{-\left( \sum_{S \in S} p_S t_S \right) \exp(1)} \right] = \left( 1 + \sum_{S \in S} p_S t_S \right)^{-1}. \tag{10} \]

The non-negative random vector \( \left( 1 - \frac{\lambda}{\mu} \right) (Q_S)_{S \in S} \) converges weakly to the random vector \( \exp(1)(p_S)_{S \in S} \) when its moment generating function converges point wise to the moment generating function given in (10). Hence, it is sufficient to show that (9) holds, in order to conclude the result stated in Theorem 3.1.

To obtain the moment generating function of \( (1 - \frac{\lambda}{\mu})(Q_S)_{S \in S} \) define \( z_S := \exp\left( -(1 - \frac{\lambda}{\mu}) t_S \right) \) and use the expression for the probability generating function in (7). As we allow \( t_S \geq 0 \), it follows that \( \left| z_S \right| \leq 1 \). For any \( m = 1, \ldots, |S| \), for any \( S = [S_1, \ldots, S_m] \in S_m \) and for any \( j = 1, \ldots, m \) it can be observed that
\[ \lim_{\lambda \uparrow \mu} \frac{N \lambda}{\mu(S_1, \ldots, S_j)} p_S z_s = \frac{N \mu}{\mu(S_1, \ldots, S_j)} p_S \in (0, \infty), \]
this yields
\[ \lim_{\lambda \uparrow \mu} \prod_{j=1}^{m} \frac{N \lambda}{\mu(S_1, \ldots, S_j)} p_S z_s = \prod_{j=1}^{m} \frac{N \mu}{\mu(S_1, \ldots, S_j)} p_S \in (0, \infty). \]

Since the (strict) inequalities (4) and (5) hold \( \lim_{\lambda \uparrow \mu} \left( 1 - \frac{\lambda}{\mu} \right) \sum_{i=1}^{j} p_s z_s \right)^{-1} \) results in infinity when \( j = m = |S| \) and otherwise this limit results in
\[ \left( 1 - \frac{N \mu}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^{j} p_s z_s \right)^{-1} \in (0, \infty). \]
Therefore the dominating terms in both the numerator and denominator of (7) in the heavy-traffic regime are those with $m = |S|$, the states in which all different job types occur. This leads to

$$\lim_{\lambda \uparrow \mu} \mathbb{E} \left[ \exp \left( - \left( 1 - \frac{\lambda}{\mu} \right) \sum_{S \in \mathcal{S}} t_S Q_S \right) \right] = \lim_{\lambda \uparrow \mu} \frac{\sum_{S \in \mathcal{S}_{|S|}} |S| \prod_{j=1}^{|S|} \frac{N \lambda p_S z_S}{\mu(S_1, \ldots, S_j)} \prod_{j=1}^{|S|} \left( 1 - \frac{N \Lambda}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^j p_S z_S \right)^{-1}}{\sum_{S \in \mathcal{S}_{|S|}} |S| \prod_{j=1}^{|S|} \frac{N \lambda p_S z_S}{\mu(S_1, \ldots, S_j)} \prod_{j=1}^{|S|} \left( 1 - \frac{N \Lambda}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^j p_S \right)^{-1}}.$$  

Since $\mu(S_1, \ldots, S_{|S|}) = N \mu$ and $\sum_{S \in \mathcal{S}} ^p S = 1$, the above fraction can be rewritten as

$$\lim_{\lambda \uparrow \mu} \frac{\sum_{S \in \mathcal{S}_{|S|}} |S| \prod_{j=1}^{|S|} \frac{N \lambda p_S z_S}{\mu(S_1, \ldots, S_j)} \prod_{j=1}^{|S|-1} \left( 1 - \frac{N \Lambda}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^j p_S z_S \right)}{\sum_{S \in \mathcal{S}_{|S|}} |S| \prod_{j=1}^{|S|-1} \frac{N \lambda p_S z_S}{\mu(S_1, \ldots, S_j)} \prod_{j=1}^{|S|-1} \left( 1 - \frac{N \Lambda}{\mu(S_1, \ldots, S_j)} \sum_{i=1}^j p_S \right)^{-1}} \cdot \lim_{\lambda \uparrow \mu} \frac{1 - \frac{\lambda}{\mu} \sum_{S \in \mathcal{S}} p_S z_S}{1 - \frac{\lambda}{\mu} \sum_{S \in \mathcal{S}} p_S}.$$  

Due to the above observations, this is equal to

$$\lim_{\lambda \uparrow \mu} \frac{1 - \frac{\lambda}{\mu} \sum_{S \in \mathcal{S}} p_S z_S}{1 - \frac{\lambda}{\mu} \sum_{S \in \mathcal{S}} p_S}.$$  

After applying l'Hôpital’s rule, the limit indeed evaluates to

$$\left( 1 + \sum_{S \in \mathcal{S}} p_S t_S \right)^{-1}.$$  

This concludes the proof of Theorem 3.1.  

The result stated in Corollary 3.1 follows from straightforward manipulations of the definitions of the moment generating functions of $(R_i)_{i=1, \ldots, N}$ and $(Q_S)_{S \in \mathcal{S}}$. The proof of Corollary 3.2 is deferred to Appendix A.

5 Light-traffic regime

We now turn attention to the light-traffic regime. In the previous section we showed that the underlying job-server compatibility constraints did not influence the system performance in the heavy-traffic regime, provided that the (strict) inequalities (14) and (15) hold. In the light-traffic regime, a similar statement holds in the absolute sense, even without any restrictions on the job-server compatibility constraints, since the system will be empty with probability one in the limit regardless. However, the scaled occupancy probabilities do sensitively depend on the probabilities $p_S$, as we will demonstrate in this section for the graph model with identical server speeds.

Let $Q(P)$ be a random variable with the stationary distribution of the total number of jobs in the system with edge selection probabilities $P = (p_{ij})_{i,j}$. Let $Q^*$ be that random variable for the special case of uniform edge selection probabilities. We refer to this replication graph as the uniform complete graph with $N$ vertices. With $C$ the normalization constant of the stationary distribution in (3) equal to $\mathbb{P}(Q_\Lambda(P) = 0)$, it can easily be seen that

$$\mathbb{P}(Q_\Lambda(P) = q) = \mathbb{P}(Q_\Lambda(P) = 0) \cdot \alpha_q(P) \cdot \left( \frac{N \Lambda}{\mu} \right)^q,$$  

(11)
for \( q \geq 1 \). The value of \( \alpha_q(P) \) is given by

\[
\sum_{(c_1, \ldots, c_q) \in \mathcal{S}} \prod_{i=1}^{q} \frac{P_{c_i}}{|\bigcup_{j=1}^{q} c_j|}.
\]  

(12)

Note that \( \alpha_1(P) \) is always equal to 1/2 since \( \mu(c) = 2\mu \) for all \( c \in \mathcal{S} \) in this scenario. However, the value \( \alpha_2(P) \) depends on the edge selection probabilities, since \( \mu(c_1, c_2) \) is equal to \( 2\mu \), \( 3\mu \) or \( 4\mu \) whenever the edges \( c_1 \) and \( c_2 \) are the same edges, edges with one common endpoint or edges without common endpoints, respectively. Therefore, \( \alpha_2(P) \), with \( q \geq 2 \), will determine the relative performance of various replication graphs. It makes sense to believe that more uniformity, more flexibility, will improve the overall performance once the number of servers a job can be replicated to is fixed. This is confirmed by Proposition 3.2 and its proof relies on the above observations.

**Proof of Proposition 3.2.** From the Taylor expansion it can be seen that

\[
\mathbb{P}\{Q_\lambda(P) = 0\} = 1 + o(1)
\]

as \( \lambda \downarrow 0 \). Then, from the observation in (11) it follows that

\[
\mathbb{P}\{Q_\lambda(P) \geq q\} = \alpha_q(P) \left( \frac{N\lambda}{\mu} \right)^q + o(\lambda^q)
\]

(13)

as \( \lambda \downarrow 0 \). Now, \( \alpha_1(P) = 1/2 \) for any set of edge selection probabilities \( P \). As mentioned before, when a second job arrives to the system, the total service rate becomes equal to \( 2\mu \), \( 3\mu \) or \( 4\mu \), where the last option is preferred as it fully exploits the use of replicas. Let \( \{i_1, j_1\} \) be the edge of the job that is already present in the system. When a second job arrives selecting edge \( \{i_2, j_2\} \), the total rate at which the system processes the two jobs is given by \( \mu(\{i_1, j_1, i_2, j_2\}) = \mu(\{i_1, j_1, i_2, j_2\})^{-1} \). The expected time till one of these jobs finishes its service and leaves the system is given by \( \mu(\{i_1, j_1\}, \{i_2, j_2\})^{-1} \). Considering all possible edge pairs one can select, this expected time will be minimized when all edges are selected uniformly at random. More formally, it is shown in (12) that the value \( \alpha_2(P) \) is minimized for the uniform complete graph with \( N \) vertices, hence \( \alpha_2(P) \geq \alpha_2^* \) for any set of edge selection probabilities \( P \). Therefore,

\[
\lim_{\lambda \downarrow 0} \frac{\mathbb{P}\{Q_\lambda^* \geq q\}}{\mathbb{P}\{Q_\lambda(P) \geq q\}} = \frac{\alpha_q^* + o(1)}{\alpha_q(P) + o(1)} \leq 1
\]

for \( q = 1, 2 \). This concludes the proof.

Note that, the second-order occupancy probabilities can also be derived using the framework of Reiman & Simon [36] or using the joint probability generating function in (7). These methods are often used when the stationary distribution as in [8] is not available.

Furthermore, it is believed that the result in Proposition 3.2 can be generalized even further to all \( q \geq 1 \). Equation (13) holds for all \( q \geq 1 \), hence

\[
\frac{\mathbb{P}\{Q_\lambda^* \geq q\}}{\mathbb{P}\{Q_\lambda(P) \geq q\}} = \frac{\alpha_q^* + o(1)}{\alpha_q(P) + o(1)}
\]

Now, if we knew that \( \alpha_q(P) \geq \alpha_q^* \) for any set of edge selection probabilities \( P \) and any \( q \geq 2 \), then the above fraction would tend to a value smaller than or equal to one in the light-traffic regime. In order to illustrate that \( \alpha_q(P) \geq \alpha_q^* \) could hold, we will focus on a particular replication graph that is much sparser than the complete graph, so its performance is expected to be inferior. Next to the complete
graph, the ring is another graph structure that scales in a straightforward way when increasing the number of servers $N$. Let $\epsilon \in (0, 1)$ and $N$ be even and define the edge selection probabilities as

$$P_{(i, i+1)} = \begin{cases} \frac{\epsilon^2}{2N} & \text{if } i \text{ is even} \\ \frac{(1-\epsilon)^2}{2N} & \text{if } i \text{ is odd} \end{cases}$$

where the edge indices are calculated modulo $N$. When $\epsilon = 1/2$, all probabilities are equal to $N^{-1}$, on average the arrival rate to each of the edges is given by $\lambda$ and therefore it is referred to as the homogeneous ring. In all other cases we refer to the replication graph as the heterogeneous ring. Let $Q_{N}^{\text{hom}}$ and $Q_{N}^{\text{het}}$ denote the number of jobs in the systems consisting of $N$ servers with the homogeneous and heterogeneous ring graphs, respectively. Figure 2 presents the values of $\alpha_q^*/\alpha_q(P)$ for various settings of the above-mentioned ring graphs. Observe that for any fixed number of servers $N$ the intuitive performance ordering is preserved. The homogeneous ring performs better than the heterogeneous ring, but worse than the uniform complete graph (all values of $\alpha_q^*/\alpha_q(P)$ are less than one). This becomes more pronounced when $\epsilon$ deviates from $1/2$. This implies that, even if the graph structure is sparse, uniform edge selection probabilities will optimize performance. Moreover, for a fixed value of $q$ it can be observed that $\alpha_q^*/\alpha_q(P)$ decreases for the homogeneous ring for increasing $N$. This indicates that a sparser replication graph is disadvantageous for the system performance.

$$P\{Q_4^* = q\} = \frac{1}{9} \left( 1 - \frac{\lambda}{\mu} \right) \left( 3 - \frac{\lambda}{\mu} \right) \left( 3 - 2\frac{\lambda}{\mu} \right) \left\{ -4 \left( \frac{2\lambda}{3\mu} \right)^q + \frac{1}{2} \left( \frac{\lambda}{3\mu} \right)^q + \frac{9}{2} \left( \frac{\lambda}{\mu} \right)^q \right\}$$

and

$$P\{Q_4^{\text{hom}} = q\} = \frac{(1 - \frac{\lambda}{\mu}) (2 - \frac{\lambda}{\mu}) (3 - 2\frac{\lambda}{\mu})}{(6 - \frac{\lambda}{\mu})} \left\{ -6 \left( \frac{2\lambda}{3\mu} \right)^q + 2 \left( \frac{\lambda}{2\mu} \right)^q + 5 \left( \frac{\lambda}{\mu} \right)^q \right\}.$$
the occupancy probabilities of the uniform complete graph and the heterogeneous ring becomes. For completeness, the occupancy probabilities for the heterogeneous ring for $q \geq 0$ are given by

$$P\{Q_{4}^{het} = q\} = \frac{1 - \frac{\lambda}{\mu}}{3 - 2\frac{\lambda}{\mu} + (1 - \epsilon)\frac{\lambda}{\mu}} \left(1 - \frac{1}{2} \frac{\lambda}{\mu}\right) \left(1 - \frac{3 - 2\lambda}{2\lambda}\right) \left(\frac{6\epsilon(1 - \epsilon)}{2 - 9\epsilon + 9\epsilon^2} \frac{2\lambda}{3\mu}\right)^q + \left(\frac{1 - \epsilon^2}{\epsilon^2}\right)^q \left(\frac{\epsilon}{\epsilon - 3\epsilon}\right)^q \left(\frac{\epsilon}{\epsilon - 3\epsilon}\right)^q \left(\frac{1}{\epsilon - 3\epsilon}\right)^q.$$ 

The derivations of the above stationary distributions, along with the stochastic dominance proof, are deferred to Appendix B.

Figure 3: The occupancy probabilities compared for replication graphs with $N = 4$ servers: uniform complete graph, homogeneous ring, heterogeneous ring with $\epsilon = 0.7$ and $\epsilon = 0.9$, $\lambda/\mu = 0.8$.

For larger values of $N$ and more asymmetric replication graphs it becomes very quickly notationally inconvenient to derive explicit expressions for the stationary probabilities. It makes sense to compare the total number of jobs in various systems with different replication graphs, but obtaining the stationary distribution from the detailed stationary distribution requires a summation over all possible job-type sequences of a particular length to obtain an expression for $\alpha_q(P)$ as stated in (12), possibly with distinct edge selection probabilities. Moreover, the order in which jobs arrive plays a crucial role. This can be seen in the denominator of (3), e.g. $\mu(c_1)\mu(c_1, c_2)\mu(c_1, c_2, c_3)$ might give another value than $\mu(c_1)\mu(c_1, c_3)\mu(c_1, c_3, c_2)$. So, the number of different job sequences to consider grows exponentially with the number of servers and the asymmetry one allows in the replication graph. This did not cause any problems when deriving the probability generating function in (7), since the generating function sums over all possible job sequences of all lengths. Obtaining the probability to observe exactly $q$ jobs in the system seems to be a huge combinatorial challenge, without an insightful solution so far. Establishing that the number of jobs in the system is stochastically minimized for the uniform complete graph thus remains as a challenge for further research.

6 Discussion and conclusion

We now discuss some of the fundamental implications of the main results as stated in Section 3, and also mention a few suggestions for further research. In Theorem 3.1 we have established a state space collapse property of a parallel-server system with redundancy scheduling based on general

For larger values of $N$ and more asymmetric replication graphs it becomes very quickly notationally inconvenient to derive explicit expressions for the stationary probabilities. It makes sense to compare the total number of jobs in various systems with different replication graphs, but obtaining the stationary distribution from the detailed stationary distribution requires a summation over all possible job-type sequences of a particular length to obtain an expression for $\alpha_q(P)$ as stated in (12), possibly with distinct edge selection probabilities. Moreover, the order in which jobs arrive plays a crucial role. This can be seen in the denominator of (3), e.g. $\mu(c_1)\mu(c_1, c_2)\mu(c_1, c_2, c_3)$ might give another value than $\mu(c_1)\mu(c_1, c_3)\mu(c_1, c_3, c_2)$. So, the number of different job sequences to consider grows exponentially with the number of servers and the asymmetry one allows in the replication graph. This did not cause any problems when deriving the probability generating function in (7), since the generating function sums over all possible job sequences of all lengths. Obtaining the probability to observe exactly $q$ jobs in the system seems to be a huge combinatorial challenge, without an insightful solution so far. Establishing that the number of jobs in the system is stochastically minimized for the uniform complete graph thus remains as a challenge for further research.

6 Discussion and conclusion

We now discuss some of the fundamental implications of the main results as stated in Section 3, and also mention a few suggestions for further research. In Theorem 3.1 we have established a state space collapse property of a parallel-server system with redundancy scheduling based on general
replication options. As mentioned earlier, the replication options can represent data locality issues, affinity relations, or broader compatibility constraints between jobs and servers which are increasingly common as the heterogeneity in job characteristics and server capabilities continues to increase. As it turns out, in heavy traffic the system operates as a multi-type single-server system, achieving full resource pooling and exhibiting insensitivity to the underlying compatibility constraints, under fairly mild local stability conditions.

This result has some crucial implications, and in particular suggests that neither the density nor the structure of the replication graph strongly affect the performance as long as the local stability conditions are satisfied. The latter conditions can in fact be satisfied in extremely sparse graphs, provided the replication options are judiciously chosen. For example, suppose that
\[ \mu_n = \mu \quad \text{for all} \quad n = 1, \ldots, N, \]
and
\[ p_n = \frac{1 - \epsilon}{N} \quad \text{for all} \quad n = 1, \ldots, N \]

and
\[ p_{(i,j)} = \frac{\epsilon}{N-1} \quad \text{for all} \quad \{i, j\} \in E, \]

with \( E \) the edge set of some connected tree with vertex set \( V = \{1, \ldots, N\} \). Then the average replication degree is \( 1 + \epsilon \) while for all (non-empty) subsets \( U \subset \{1, \ldots, N\} \) we have for any \( \epsilon > 0 \),
\[
\sum_{S: S \subseteq U} p_S = \sum_{n \in U} p_n + \sum_{i,j \in U: (i,j) \in E} p_{(i,j)} \leq |U| \frac{1 - \epsilon}{N} + (|U| - 1) \frac{\epsilon}{N-1} < |U| \frac{1 - \epsilon}{N} + |U| \frac{\epsilon}{N} = \frac{1}{N} \mu \sum_{n \in U} \mu_n,
\]

implying that the local stability conditions in (5) are satisfied. As a further example, suppose that \( \mu_n = \mu \) for all \( n = 1, \ldots, N \), and we set the probabilities as
\[ p_n = \frac{N - 1 - \epsilon}{N(N - 1)} \quad \text{for all} \quad n = 1, \ldots, N \]
and
\[ p_{\{1, \ldots, N\}} = \frac{\epsilon}{N-1}. \]

Then the average replication degree is \( 1 + \epsilon \) while for all (non-empty) subsets \( U \subset \{1, \ldots, N\} \) we have for any \( \epsilon > 0 \),
\[
\sum_{S: S \subseteq U} p_S = \sum_{n \in U} p_n = |U| \frac{N - 1 - \epsilon}{N(N - 1)} < \frac{|U|}{N} = \frac{1}{N} \mu \sum_{n \in U} \mu_n,
\]

implying that the local stability conditions in (5) are satisfied. Note that for \( \epsilon = 0 \) both above-described example scenarios reduce to systems of \( N \) independent M/M/1 queues with arrival rate \( \lambda \) and service rate \( \mu \). In that case the number of jobs at each server has a geometric stationary distribution with parameter \( \lambda/\mu \), and when scaled with \( 1 - \lambda/\mu \), tends to a unit-exponential distribution as \( \lambda \) approaches \( \mu \). Thus the scaled total number of jobs converges to the sum of \( N \) independent unit-exponential random variables. In contrast, for any \( \epsilon > 0 \), Theorem 3.1 implies that in both example scenarios the total number of jobs tends to a single unit-exponential random variable, and is hence smaller by a factor \( N \). The observation that the performance at high load is roughly similar for all values \( \epsilon > 0 \), and significantly better than for \( \epsilon = 0 \), is reminiscent of the finding that assigning replicas to \( d = 2 \) servers selected uniformly at random provides a significant improvement over assigning jobs to just a single server selected uniformly at random in a regime where the number of servers \( N \) grows large while \( \lambda \) remains fixed \[18\]. The fact that the heavy-traffic regime and a many-server scenario point to similar behavior also suggests that it would be interesting to explore a possible interchange of these two limits.
While the above heavy-traffic results reflect strong insensitivity to the underlying compatibility relations, we also demonstrated that in a light-traffic regime the suitably scaled distribution of the total number of jobs does depend on the network structure and in particular is optimal for strictly uniform random replication, given the average degree of replication. We conjecture that this notion of stochastic optimality may in fact hold for all values of $\lambda$, and proving this remains as a challenging topic for further research.

It is also worth emphasizing that the network design in the above-described example scenarios is highly fragile in the sense that for values of $\epsilon$ close to zero, the local stability conditions would immediately be violated if the service rate values $\mu_n$ were not all strictly equal, even if the overall service rate still equals $N\mu$. In other words, even if higher replication degrees may not necessarily yield significant performance gains, they may ensure that the local stability conditions continue to be satisfied when arrival rates of the various job types and speeds of the servers vary or deviate from estimated values, and hence help improve the overall robustness.

It is worth observing that heavy-traffic analysis is typically motivated by the lack of a tractable exact solution, and that state space collapse properties are then usually formulated in the form of so-called process-level limits over finite time intervals. In contrast, an explicit product-form stationary solution is available for the systems considered in the present paper, and in fact provides the starting point for the heavy-traffic analysis, but its complicated structure and state representation yields no immediate insight which renders the derivation of the limiting distribution quite challenging.

On a related account, we note that similar but different product-form distributions are available for the so-called ‘cancel-on-start’ version of redundancy scheduling [45] where redundant replicas are already discarded as soon as the first replica starts execution, as opposed to the ‘cancel-on-completion’ version considered in the present paper. It would be interesting to investigate whether these product-form solutions can be leveraged to derive similar heavy-traffic limits and insensitivity results in cancel-on-start scenarios.

Redundancy scheduling policies with cancel-on-start operation are in fact equivalent to "Join-the-Smallest-Workload" (JSW) type assignment policies, where an arriving job is assigned to the server with the smallest workload among the servers in the subset $S$ with probability $p_S$. Hence they are also strongly related to "Join-the-Shortest-Queue" (JSQ) assignment policies, where an arriving job is assigned to the server with the shortest queue among the servers in the subset $S$ with probability $p_S$.

The latter kind of policies have been extensively investigated in the case where $p_S = 1/M(d)$ for all $M(d) = \binom{N}{d}$ subsets of $d$ servers, and are commonly referred to as ‘power-of-$d$’ or JSQ($d$) policies in that case. Even though no product-form solutions are available for these policies, it is natural to expect that similar state space collapse properties and insensitivity results hold in a heavy-traffic regime. State space collapse properties have in fact been proved for power-of-$d$ policies in the form of process-level limits [3], and also reveal some form of asymptotic insensitivity to the diversity parameter $d$ as long as stability is maintained. However, to the best of our knowledge no heavy-traffic insensitivity results have been established for JSQ policies with general compatibility constraints, which in fact seem to have received hardly any attention at all. The only available results are for many-server scenarios rather than heavy-traffic regimes, and there are some scattered results suggesting that strictly uniform sampling is optimal for a given degree of sampling [21, 44]. More recent results point to a more restrictive notion of insensitivity where the optimality of the full JSQ policy with sampling all servers may be asymptotically matched in sampling graphs that are relatively sparse as long as the structure is sufficiently random [34].

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A Heavy-traffic regime

The distributional form of Little’s law [29] allows us to derive the result in Corollary 3.2 for the sojourn and waiting time in a heavy-traffic regime. Some details are described below.

Proof of Corollary 3.2 Fix a job type $S \in \mathcal{S}$. Then the probability generating function of $Q_S$ is equal to

$$G_{Q_S}(z) = \mathbb{E}[z^{Q_S}] = \mathbb{E}\left[z^{Q_S} \prod_{S \in \mathcal{S} \setminus S} 1^{Q_{\tilde{S}}}ight]$$

(15)

with $|z| \leq 1$. The moment generating function of the sojourn time is defined as $L_{V_S}(s) = \mathbb{E}[e^{-s V_S}]$. Due to Theorem 1 in [29] it is known that

$$G_{Q_S}(z) = L_{V_S}(\lambda_S(1 - z)),$$

with $\lambda_S = N\lambda p_S$. In order to deduce the limiting behavior of $\left(1 - \frac{\lambda}{\mu}\right) V_S$, observe the following intermediate steps:

$$L_{(1 - \frac{\lambda}{\mu}) V_S}(s) = L_{V_S}\left(1 - \frac{\lambda}{\mu}\right) s = G_{Q_S}\left(1 - \left(1 - \frac{\lambda}{\mu}\right) \frac{s}{\lambda_S}\right).$$

Using (15) and similar arguments as in the proof of Theorem 3.1 while taking the limit $\lambda \uparrow \mu$ results in

$$\lim_{\lambda \uparrow \mu} L_{(1 - \frac{\lambda}{\mu}) V_S}(s) = \frac{N\mu}{N\mu + s},$$

which may be recognized as the moment generating function of an exponentially distributed random variable with parameter $N\mu$.

In order to show that the waiting time of a job of type $S \in \mathcal{S}$ is also exponentially distributed with parameter $N\mu$, it is sufficient to note that the sojourn time of a type-$S$ job is upper bounded by

$$\lim_{\lambda \uparrow \mu} L_{(1 - \frac{\lambda}{\mu}) V_S}(s) = \frac{N\mu}{N\mu + s},$$
the sum of its waiting time and the service time offered by the first server where it starts its service. Note that this service time is independent of the waiting time and is stochastically smaller than an exponentially distributed random variable $X$ with parameter $\tilde{\mu} = \min\{\mu_i : i = 1, \ldots, N\}$. Hence, the sojourn time $V_S$ is lower bounded by the waiting time $W_S$, and upper bounded by $W_S + X$. This results in

$$L_{(1-\frac{\lambda}{\mu}) W_S(s)} \geq L_{(1-\frac{\lambda}{\mu}) W_S(s)} \geq L_{(1-\frac{\lambda}{\mu}) W_S(s)} L_{(1-\frac{\lambda}{\mu}) X(s)}.$$ 

Since

$$\lim_{\lambda \uparrow \mu} L_{(1-\frac{\lambda}{\mu}) X(s)} = \lim_{\lambda \uparrow \mu} \frac{\tilde{\mu}}{\tilde{\mu} + s (1 - \frac{\lambda}{\mu})} = 1,$$

it follows that

$$\lim_{\lambda \uparrow \mu} L_{(1-\frac{\lambda}{\mu}) W_S(s)} = \frac{N\mu}{N\mu + s}.$$ 

Since both $V_S$ and $W_S$ are non-negative random variables, convergence of the moment generating functions implies weak convergence to exponentially distributed random variables with parameter $N\mu$.

This concludes the proof of Corollary B.2.

### \textbf{B Light-traffic regime}

The stationary distributions of the total number of jobs in a system with $N = 4$ servers and the uniform complete graph and the heterogeneous ring are derived in Lemmas B.1 and B.2 respectively. Next, we show in Proposition B.1 that the total number of jobs in a system with the uniform complete graph is stochastically smaller than the total number of jobs in a system with the heterogeneous ring replication graph.

#### \textbf{Lemma B.1.}

The stationary distribution of the total number of jobs in a system with $N = 4$ servers and the uniform complete replication graph is given by

$$P\{Q^*_i = q\} = \frac{1}{9} \left(1 - \frac{\lambda}{\mu}\right) \left(3 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left\{-4 \left(\frac{2\lambda}{3\mu}\right)^q + \frac{1}{2} \left(\frac{\lambda}{3\mu}\right)^q + \frac{9}{2} \left(\frac{\lambda}{\mu}\right)^q \right\},$$

for $q \geq 0$ and $\lambda < \mu$.

#### \textbf{Proof.}

First of all, note that

$$P\{Q^*_i = q\} = C \sum_{(c_1, \ldots, c_q) \in S^q \mu(c_1, \ldots, c_q)} \prod_{i=1}^q \frac{N\lambda p_{c_i}}{\mu(c_1, \ldots, c_q)} = C \left(\frac{4\lambda}{6\mu}\right)^q \sum_{(c_1, \ldots, c_q) \in S^q \mu(c_1, \ldots, c_q)} \prod_{i=1}^q (|c_1 \cup \cdots \cup c_i|)^{-1},$$

because $P_{(i,j)} = 1/6$ for each edge in the setting under consideration and $|c_1 \cup \cdots \cup c_i|$ denotes the total number of busy servers processing jobs of types $c_1, \ldots, c_i$. Then, we can enumerate all server rate sequences $(|c_1|, |c_1 \cup c_2|, \ldots, |c_1 \cup \cdots \cup c_q|)$ and count how many states will result in these particular sequences. First, we consider the sequence (2, \ldots, 2), which can only be generated by states where all the jobs are of the same type, hence there are six such sequences. Second, the sequences (2, \ldots, 2, 3, \ldots, 3) with $q_1$ 3 entries are considered. For the first job in the state there are six options, all following $q - q_1 - 1$ jobs must be of the same type as the first job, say $\{i, j\}$. An arrival of the $(q - q_1 + 1)$st job increases the number of busy servers from two to three, there are four such job types that can do this, say edge $\{j, k\}$ is selected. Then, all remaining jobs can be of types $\{i, j\}, \{j, k\}$ or $\{i, k\}$, this results in $6 \cdot 4 \cdot 3^{q-1}$ different states that lead to this server rate sequence. Third, in a similar way one can see that there are $6^q$ states that will lead to the server rate sequence (2, \ldots, 2, 4, \ldots, 4) with $q_1$ 4 entries. The fourth server rate sequence is given by

$$\frac{(2, \ldots, 2, 3, \ldots, 3, 4, \ldots, 4)}{q_1, q_2, q_3}.$$
In total there are \(6 \cdot 4 \cdot 3^{q_1 - 1} \cdot 3 \cdot 6^{q_2 - 1} = 4 \cdot 3^{q_1} \cdot 6^{q_2}\) states that result in the above service rate vector. Summing over all possible values of \(q_1\) and \(q_2\), results in

\[
\mathbb{P}\{Q_4^* = q\} = \mathcal{C} \left( \frac{2 \lambda}{3 \mu} \right)^q \left\{ \frac{1}{2^q} + 8 \sum_{q_1=1}^{q-1} \frac{3^{q_1}}{2^{q-q_1} 3^{q_1}} + \sum_{q_1=1}^{q-2} \frac{6^{q_1}}{2^{q-q_1} 4^{q_1}} + 4 \sum_{q_1=1}^{q-2} \sum_{q_2=1}^{q-q_1-1} \frac{3^{q_1} \cdot 6^{q_2}}{2^{q-q_1-q_2} 3^{q_1} 4^{q_2}} \right\}
\]

\[
= \mathcal{C} \left\{ -4 \left( \frac{2 \lambda}{3 \mu} \right)^q + \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^q + \frac{9}{2} \left( \frac{\lambda}{\mu} \right)^q \right\}.
\]

Finally, the expression for the normalization constant \(\mathcal{C}\) can be obtained by summing the expression (16) for all values \(q \geq 0\), and this leads to

\[
\mathcal{C} = \frac{1}{9} \left( 1 - \frac{\lambda}{\mu} \right) \left( 3 - \frac{\lambda}{\mu} \right) \left( 3 - 2 \frac{\lambda}{\mu} \right).
\]

This concludes the proof.

Lemma B.2. The stationary distribution of the total number of jobs in a system with \(N = 4\) servers and the heterogeneous ring replication graph is given by

\[
\mathbb{P}\{Q_4^{\text{het}} = q\} = \mathcal{C} \left( \frac{4 \lambda}{\mu} \right)^q \left\{ \prod_{(c_1, \ldots, c_q) \in S_q} \frac{p_{c_i}}{\mathbb{E}[c_1 \cup \cdots \cup c_i]} \right\}
\]

\[
= \mathcal{C} \left( \frac{4 \lambda}{\mu} \right)^q \left\{ \frac{1}{2^q} \left( \frac{\epsilon}{2} \right)^q + \frac{1}{2^q} \left( 1 - \epsilon \right)^q \right\}
\]

\[
+ \varepsilon \sum_{q_1=1}^{q-1} \left( \frac{2^{q_1} \cdot \epsilon^{q-q_1}}{2^{q-q_1} 3^{q_1}} \right) \left( 1 - \frac{\epsilon}{2^q} \right)^{q-q_1} + \left( \frac{1 - \epsilon}{2^q} \right)^{q-q_1} \left( \frac{1 - \epsilon}{2^q} \right)^{q-q_1} \right\}
\]

\[
+ \varepsilon \sum_{q_1=1}^{q-2} \left( \frac{2^{q_1} \cdot \epsilon^{q-q_1+1}}{2^{q-q_1} 4^{q_1}} \right) \left( 1 - \frac{\epsilon}{2^q} \right)^{q-q_1+1} + \left( \frac{1 - \epsilon}{2^q} \right)^{q-q_1+1} \right\}
\]

\[
+ \varepsilon \sum_{q_1=1}^{q-2} \sum_{q_2=1}^{q-q_1-1} \left( \frac{2^{q_1} \cdot \epsilon^{q-q_1-q_2}}{2^{q-q_1-q_2} 3^{q_1} 4^{q_2}} \right) \left( 1 - \frac{\epsilon}{2^q} \right)^{q-q_1-q_2} + \left( \frac{1 - \epsilon}{2^q} \right)^{q-q_1-q_2} \right\}.
\]

Simplification of the previous expression and summing over all states to determine the normalization constant \(\mathcal{C}\) will eventually lead to the expression (17). This concludes the proof.

**Proposition B.1.** Let \(Q_4^*\) and \(Q_4^{\text{hom}}\) denote the total number of jobs in a system with \(N = 4\) servers and a uniform complete replication graph and homogeneous ring replication graph, respectively. Then, \(Q_4^*\) is stochastically smaller than \(Q_4^{\text{hom}}\), i.e.,

\[
Q_4^* \leq_{\text{st}} Q_4^{\text{hom}}.
\]

(18)
Proof. Using the occupancy probabilities derived in Lemmas 3.1 and 3.2 (with $\epsilon = 1/2$) it can be seen that for $q \geq 0$:

$$
P\{Q^*_4 \geq q\} = -\frac{4}{3} \left(1 - \frac{\lambda}{\mu}\right) \left(3 - \frac{\lambda}{\mu}\right) \left(\frac{2\lambda}{3\mu}\right)^q + \frac{1}{6} \left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left(\frac{\lambda}{3\mu}\right)^q + \frac{1}{2} \left(3 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^q$$

and

$$
P\{Q^\text{hom}_4 \geq q\} = -18 \left(\frac{1 - \frac{\lambda}{\mu}}{6 - \frac{\lambda}{\mu}}\right) \left(\frac{2\lambda}{3\mu}\right)^q + 4 \left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left(\frac{\lambda}{2\mu}\right)^q + 5 \left(3 - 2\frac{\lambda}{\mu}\right) \left(\frac{2}{3}\right)^q.$$

It is sufficient to establish that $P\{Q^\text{hom}_4 \geq q\} - P\{Q^*_4 \geq q\} > 0$ for any $q \geq 1$ in order to show that (18) holds. Dividing by common factors results in the following equivalent inequality:

$$
g_\lambda(q) := \frac{9}{2} \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{2 + \frac{\lambda}{\mu}}{6}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left(\frac{1}{2}\right)^q - \frac{3}{2} \left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left(\frac{1}{3}\right)^q - 6 \left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left(6 + \frac{\lambda}{\mu}\right) \left(\frac{2}{3}\right)^q > 0.$$

Now, it is clear that $g_\lambda(1) = \frac{\lambda}{\mu} \left(1 - \frac{\lambda}{\mu}\right) > 0$. Moreover, it can be seen that $g_\lambda(q + 1) - g_\lambda(q)$ is equal to

$$
\left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left\{ -18 \left(\frac{1}{2}\right)^q + 6 \left(\frac{1}{3}\right)^q + 2 \left(6 + \frac{\lambda}{\mu}\right) \left(\frac{2}{3}\right)^q \right\}.$$

If $q = 1$, then

$$g_\lambda(2) - g_\lambda(1) = \left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \frac{2}{3} > 0.$$

If $q \geq 2$, the difference $g_\lambda(q + 1) - g_\lambda(q)$ can be lower bounded as follows

$$
\left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left\{ -18 \left(\frac{1}{2}\right)^q + 5 \left(\frac{1}{3}\right)^q + 12 \left(\frac{2}{3}\right)^q \right\} \geq 6 \left(1 - \frac{\lambda}{\mu}\right) \left(3 - 2\frac{\lambda}{\mu}\right) \left\{ -3 \left(\frac{1}{2}\right)^q + 2 \left(\frac{2}{3}\right)^q \right\} > 0.$$

This concludes the proof.