A DYCK PATH FORMULA FOR CERTAIN ELEMENTS OF UPPER CLUSTER ALGEBRAS

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Abstract. We develop an elementary formula for certain non-trivial elements of upper cluster algebras in terms of Dyck paths. These elements have positive coefficients. Using this formula, we show that each non-acyclic skew-symmetric cluster algebras of rank 3 is properly contained in its upper cluster algebra.

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [7]. A (skew-symmetrizable) cluster algebra $A$ is a subalgebra of a rational function field with a distinguished set of generators, called cluster variables, that are generated by an iterative procedure called mutation. By construction cluster variables are rational functions, but it is shown in loc. cit. that they are Laurent polynomials with integer coefficients. Moreover, these coefficients are known to be non-negative for skew-symmetric cluster algebras [11].

Each cluster algebra $A$ also determines an upper cluster algebra $U$, where $A \subseteq U$ [4]. It is believed, especially in the context of algebraic geometry, that $U$ is better behaved than $A$ (for instance, see [5, 9]). Matherne and Muller [12] gave a general algorithm to compute generators of $U$. Plamondon [14, 15] obtained a (not-necessarily positive) formula for certain elements of skew-symmetric upper cluster algebras using quiver representations. However a directly computable and manifestly positive formula for (non-trivial) elements in $U$ is not available yet.

In this paper we develop an elementary formula for upper cluster algebra elements in terms of Dyck paths. These elements have positive coefficients. For an equioriented quiver of type $A$, these elements form a canonical basis [3]. For an acyclic cluster algebra, we conjecture that these elements form a basis. Using this formula, we show that $A \neq U$ for all non-acyclic skew-symmetric cluster algebras of rank 3.

The paper is organized as follows. In Section 2 we review definitions of cluster algebras and upper cluster algebras. Section 3 is devoted to the construction of a family of elements in $U$, and Section 4 to the proof of $A \neq U$ for non-acyclic rank 3 skew-symmetric cluster algebras.

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2. Cluster Algebras and Upper Cluster Algebras

A square integer matrix $B = (b_{ij})$ is said to be sign-skew-symmetric if either $b_{ij} = b_{ji} = 0$, or else $b_{ij}$ and $b_{ij}$ are of opposite sign; in particular, $b_{ii} = 0$ for all $i$. Furthermore, $B$ is said to be skew-symmetric if $b_{ij} = -b_{ji}$ for all $i, j$. If there exists a diagonal matrix $D$ with positive integer entries $d_i$ such that $DB$ is skew-symmetric then $B$ is said to be skew-symmetrizable. That is $d_i b_{ij} = -d_j b_{ji}$ for all $i, j$.

Let $(\mathbb{P}, +, \cdot)$ be a semifield, that is an abelian multiplicative group with a binary operation of auxiliary addition $\oplus$ which is commutative associative, and distributive with respect to the multiplication in $\mathbb{P}$. The group $\mathbb{P}$ is torsion free, so we will use its group ring $\mathbb{Z}\mathbb{P}$ as the ground ring for our cluster algebra. As an ambient field for our cluster algebra, we take a field $F$ which is isomorphic to the field of rational fractions in $m$ variables with coefficients in $\mathbb{Q}\mathbb{P}$.

One choice for $\mathbb{P}$ that we will use in this paper is the tropical semifield. Let $\text{Trop}(u_1, \ldots, u_m)$ be a multiplicative abelian group freely generated by the $u_j$. We define $\oplus$ in $\text{Trop}(u_1, \ldots, u_m)$ by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}.$$

When we choose $\mathbb{P} = \text{Trop}(u_1, \ldots, u_m)$ we say that the cluster algebra is of geometric type. It is also important to note that in this case $\mathbb{Z}\mathbb{P}$ is the ring of Laurent polynomials in the variables $u_i$.

A seed $\Sigma = (x, y, B)$ is a triple where $x = \{x_1, \ldots, x_n\}$ is an $n$-tuple of elements of $\mathcal{F}$ that form a free generating set over $\mathbb{Q}\mathbb{P}$, $y = \{y_1, \ldots, y_n\}$ is an $n$-tuple of elements in $\mathbb{P}$ and $B$ is an $n \times n$ integer sign-skew-symmetric matrix. We call $x$ a cluster, $y$ a coefficient tuple, $B$ the exchange matrix, and the elements of a cluster cluster variables. The rank of the seed is the number of columns of the exchange matrix, or equivalently the number of the cluster variables in a cluster.

For any integer $a$, let $[a]_+ := \max(0, a)$. Given a seed $(x, y, B)$ and a specified index $1 \leq k \leq n$, we define mutation of $(x, y, B)$ at $k$, denoted $\mu_k(x, y, B)$, to be a new seed $(x', y', B')$ where

$$x_i' = \begin{cases} (y_k \prod_{b_{jk} > 0} x_j^{b_{jk}} + \prod_{b_{jk} < 0} x_j^{-b_{jk}})((1 \oplus y_k)x_k)^{-1} & \text{if } i = k; \\ x_i & \text{otherwise.} \end{cases}$$

$$y_j' = \begin{cases} y_j^{-1} & \text{if } j = k; \\ y_j y_k^{[b_{kj}]} (1 \oplus y_k)^{-b_{kj}} & \text{otherwise.} \end{cases}$$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{b_{ik} b_{kj} + b_{ik} b_{kj}}{2} & \text{otherwise.} \end{cases}$$
If $B'$ is also sign-skew-symmetric, we say that the mutation is well-defined. Note that well-defined mutation is an involution, that is mutating $(x', y', B')$ at $k$ will return our original seed $(x, y, B)$.

Two seeds $\Sigma_1$ and $\Sigma_2$ are said to be mutation-equivalent or in the same mutation class if $\Sigma_2$ can be obtained by a sequence of well-defined mutations from $\Sigma_1$. This is obviously an equivalence relation. A seed $\Sigma$ is said to be totally mutable if every sequence of mutations from $\Sigma$ consists of well-defined ones.

**Definition 2.1.** Given a totally mutable seed $(x, y, B)$, the cluster algebra $A(x, y, B)$ is the subring of $\mathcal{F}$ generated by

$$\bigcup_{(x', y', B') \text{ mutation-equivalent to } (x, y, B)} x',$$

where the union runs over all seeds $(x', y', B')$ that are mutation-equivalent to $(x, y, B)$. The seed $(x, y, B)$ is called the initial seed of $A(x, y, B)$.

Any seed in the same mutation class will generate the same cluster algebra up to isomorphism.

For any $n \times n$ sign-skew-symmetric matrix $B$, we associate a directed graph $Q_B$ with vertices $1, \ldots, n$ and arrows from vertex $i$ to vertex $j$ for $b_{ij} > 0$. We call $B$ acyclic if there are no oriented cycles in $Q_B$. We say the cluster algebra $A(x, y, B)$ is non-acyclic if and only if no matrix in its mutation class is acyclic.

**Definition 2.2.** Given a cluster algebra $A$, the upper cluster algebra $U$ is defined as

$$U = \bigcap_{x=(x_1, \ldots, x_n)} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

where $x$ runs over all clusters of $A$.

In certain cases it is sufficient to consider only the clusters of the initial seed and the seeds that are a single mutation away from it, rather than all the seeds in the entire mutation class.

**Definition 2.3.** A seed $(x, y, B)$ is coprime if the columns of $B$ are pairwise linearly independent. A cluster algebra is totally coprime if every seed is coprime.

For a cluster $x$, $U_x$ is defined as the intersection in $\mathbb{Z}[x_1, \ldots, x_n]$ of the $n + 1$ Laurent rings corresponding to $x$ and its one-step mutations.

$$U_x := \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cap \left( \bigcap_i \mathbb{Z}[x_1^{\pm 1}, \ldots, x_i'^{\pm 1}, \ldots, x_n^{\pm 1}] \right).$$

**Theorem 2.4.** [4, 13] We have $A \subseteq U \subseteq U_x$. Moreover,

(i) If $A$ is locally acyclic (in particularly, if $A$ is acyclic), then $A = U$.

(ii) If $A$ is totally coprime, then $U = U_x$ for any seed $(x, y, B)$. In particular, this holds when $A$ is of geometric type and the matrix $B$ (defined in §3) has full rank.
3. Construction of Some Elements in the Upper Cluster Algebra

In this section, we consider a cluster algebra $\mathcal{A}$ of geometric type. Let $n$ be the rank of $\mathcal{A}$, and $m$ be an integer such that $m \geq n$. Assume the coefficient semifield is $\mathbb{P} = \text{Trop}(x_{n+1}, \ldots, x_m)$, $\hat{B} = (b_{ij})$ ($1 \leq i \leq m$, $1 \leq j \leq n$) is the $m \times n$ matrix whose principal part $B = (b_{ij})$ ($1 \leq i, j \leq n$) is sign-skew-symmetric, and $b_{ij}$ with $i > n$ are used to encode the coefficient $y_i$ as $y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}}$. Equivalently, the exchange relation is given by

$$x'_k = x_k^{-1}(\prod_{i=1}^{m} x_i^{[b_{ik}]} + \prod_{i=1}^{m} x_i^{-[b_{ik}]}).$$

Since $y$ is determined by $\hat{B}$, we also denote $\mathcal{A}(x, \hat{B}) = \mathcal{A}(x, y, B)$. We define $b_{ij} = -b_{ji}$ for $1 \leq i \leq n$, $n + 1 \leq j \leq m$, and define $b_{ij} = 0$ if $i, j > n$. Define

$$E = \{(i, j) \mid 1 \leq i, j \leq m, \; b_{ij} > 0\}.$$

Let $(a_1, a_2)$ be a pair of nonnegative integers. Let $c = \min(a_1, a_2)$. The maximal Dyck path of type $a_1 \times a_2$, denoted by $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$, is a lattice path from $(0, 0)$ to $(a_1, a_2)$ that is as close as possible to the diagonal joining $(0, 0)$ and $(a_1, a_2)$, but never goes above it. A corner is a subpath consisting of a horizontal edge followed by a vertical edge.

**Definition 3.1.** Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the set of horizontal (resp. vertical) edges of a maximal Dyck path $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. We label $\mathcal{D}$ with the corner-first index in the following sense:

(a) edges in $\mathcal{D}_1$ are indexed as $u_1, \ldots, u_{a_1}$, such that $u_i$ is the horizontal edge of the $i$-th corner for $i \in [1, c]$ and $u_{c+i}$ is the $i$-th of the remaining horizontal ones for $i \in [1, a_1 - c]$,

(b) edges in $\mathcal{D}_2$ are indexed as $v_1, \ldots, v_{a_2}$ such that $v_i$ is the vertical edge of the $i$-th corner for $i \in [1, c]$ and $v_{c+i}$ is the $i$-th of the remaining vertical ones for $i \in [1, a_2 - c]$.

(Here we count corners from bottom left to top right, count vertical edges from bottom to top, and count horizontal edges from left to right.)

**Definition 3.2.** Let $S_1 \subseteq \mathcal{D}_1$ and $S_2 \subseteq \mathcal{D}_2$. We say that $S_1$ and $S_2$ are locally compatible with respect to $\mathcal{D}$ if and only if no horizontal edge in $S_1$ is the immediate predecessor of any vertical edge in $S_2$ on $\mathcal{D}$.

**Remark 3.3.** Notice that we have: $|\mathcal{P}(\mathcal{D}_1) \times \mathcal{P}(\mathcal{D}_2)|$ possible pairs for $(S_1, S_2)$, where $\mathcal{P}(\mathcal{D}_1)$ denotes the power set of $\mathcal{D}_1$ and $\mathcal{P}(\mathcal{D}_2)$ denotes the power set of $\mathcal{D}_2$. Further, when either $S_1$ or $S_2 = \emptyset$, any arbitrary choice of the other will yield local compatibility by our above definition.

**Definition 3.4.** Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, $a_{n+1} = \cdots = a_m = 0$.

(i) Denote $\mathcal{D}^{[a]}_+ \times [a]_+$ by $\mathcal{D}^{(i,j)}$ for $(i, j) \in E$. We label $\mathcal{D}^{(i,j)}$ with the corner-first index (Definition ), whose horizontal edges are denoted $u_{1}^{(i,j)}, \ldots, u_{[a_1]}^{(i,j)}$, and vertical edges are denoted by $v_{1}^{(i,j)}, \ldots, v_{[a_2]}^{(i,j)}$. We say that the collection of $S_{1}^{(i,j)}$ and $S_{2}^{(i,j)}$, where $S_{1}^{(i,j)} \subseteq \mathcal{D}_{1}^{(i,j)}$ and $S_{2}^{(i,j)} \subseteq \mathcal{D}_{2}^{(i,j)}$, for all $(i, j) \in E$ is globally compatible if and only if

1. $S_{1}^{(i,j)}$ and $S_{2}^{(i,j)}$ are locally compatible with respect to $\mathcal{D}^{(i,j)}$ for all $(i, j) \in E$;
(2) If \((j, i)\) and \((i, k)\) are in \(E\), then \(v^{(j, i)}_r \in S_2^{(j, i)} \iff u^{(i, k)}_r \notin S_1^{(i, k)}\) for all \(r \in [1, [a_i]_+]\);
(3) If \((i, j)\) and \((i, k)\) are in \(E\), then \(u^{(i, j)}_r \in S_1^{(i, j)} \iff u^{(i, k)}_r \in S_1^{(i, k)}\) for all \(r \in [1, [a_i]_+]\);
(4) If \((j, i)\) and \((k, i)\) are in \(E\), then \(v^{(j, i)}_r \in S_2^{(j, i)} \iff v^{(k, i)}_r \in S_2^{(k, i)}\) for all \(r \in [1, [a_i]_+]\).

(ii) Define \(\tilde{x}[a]\) to be the Laurent polynomial

\[
\tilde{x}[a] := \left( \prod_{i=1}^{n} x_i^{-a_i} \right) \sum \left( \prod_{(i, j) \in E} b_{i,j} \left| S_2^{(i, j)} \right| x_i^{a_i} x_j^{-b_{i,j} \left| S_1^{(i, j)} \right|} \right),
\]

where the sum runs over all globally compatible collections (abbreviated GCCs).

Next, we give an equivalent definition of \(\tilde{x}[a]\) based on the following simple observation: for a fixed vertex \(i\) in \(Q\), any one of \(S_2^{(j, i)}\) with \((j, i) \in E\) and \(S_1^{(i, k)}\) with \((i, k) \in E\) determines the rest, and they are determined by a sequence \(s_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,[a_i]_+}) \in \{0, 1\}^{[a_i]_+}\) in the following way \((1 \leq r \leq [a_i]_+)\):

1. If \((j, i) \in E\), then \(u^{(j, i)}_r \in S_2^{(j, i)}\) if and only if \(s_{i,r} = 0\).
2. If \((i, k) \in E\), then \(u^{(i, k)}_r \in S_1^{(i, k)}\) if and only if \(s_{i,r} = 1\).

We define the complement of \(s_i\) to be

\[
\tilde{s}_i = (\tilde{s}_{i,1}, \ldots, \tilde{s}_{i,[a_i]_+}) = (1 - s_{i,1}, \ldots, 1 - s_{i,[a_i]_+}).
\]

By convention, if \(s_i = () \in \{0, 1\}^0\), then \(\tilde{s}_i = () = s_i\). For sequences \(t = (t_1, \ldots, t_a)\) and \(t' = (t'_1, \ldots, t'_b)\) we denote

\[
|t| = \sum_{r=1}^{a} t_r, \quad t \cdot t' = \sum_{r=1}^{\min(a,b)} t_r t'_r.
\]

It is easy to check that Definition 3.4 is equivalent to the following.

**Definition 3.5.** Let \(a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n\), \(a_{n+1} = \cdots = a_m = 0\).

1. Let \(s = (s_1, \ldots, s_m)\) wherever \(s_i \in \{0, 1\}^{[a_i]_+}\) for \(i = 1, \ldots, m\). Let \(S_{\text{all}}\) be the set of all such \(s\). Let \(S_{\text{gcc}} = \{s \in S_{\text{all}} \mid s_i \cdot \tilde{s}_j = 0\) for every \((i, j) \in E\}\).

2. Define \(\tilde{x}[a]\) to be the Laurent polynomial

\[
\tilde{x}[a] := \left( \prod_{i=1}^{n} x_i^{-a_i} \right) \sum_{s \in S_{\text{gcc}}} \left( \prod_{(i, j) \in E} b_{i,j} \left| \tilde{s}_i \cdot \tilde{s}_j \right| x_i^{a_i} x_j^{-b_{i,j} \left| s_i \right|} \right).
\]

**Definition 3.6.** Let \(a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n\). For \(n + 1 \leq i \leq m\), let \(\ell_i\) be the lowest degree of \(x_i\) in all the monomials in \(\left( \prod_{i=1}^{n} x_i^{a_i} \right) \tilde{x}[a]\). Define \(\tilde{x}_{\text{red}}[a]\) to be the Laurent polynomial

\[
\tilde{x}_{\text{red}}[a] := \tilde{x}[a] \prod_{i=n+1}^{m} x_i^{-\ell_i}.
\]
Example 3.7. (a) Using Definition 3.5 to compute $\tilde{x}[a]$ for $n = 2$, $m = 3$, $a = (1, 1)$, and

$$\tilde{B} = \begin{bmatrix} 0 & a \\ -a' & 0 \\ c & -b \end{bmatrix}, \quad a, a', b, c > 0.$$ 

Then $E = \{(1, 2), (2, 3), (3, 1)\}$, $S_{gcc} = \{((0), (0), (0)), (0), (1), ((1), (1), (1))\}$. Note that $s = ((1), (0), (0))$ is not in $S_{gcc}$ because $(1, 2) \in E$ but $s_1 \cdot \tilde{s}_2 = 1(1 - 0) = 1 \neq 0$. Thus

$$\tilde{x}[a] = x_1^{-1}x_2^{-1} \left((x_1^a x_2^b)(x_2 x_3^c)(x_3^d x_1^e) + (x_1^f x_2^g)(x_2 x_3^h)(x_3^i x_1^j) + (x_1^k x_2^l)(x_2 x_3^m)(x_3 x_1^n)\right) = x_1^{-1}x_2^{-1} \left(x_1^a x_3^c + x_2^b x_3^c + x_2^a x_2^b \right).$$ 

If we further assume $b \geq c$, then

$$\tilde{x}_{\text{red}}[a] = \tilde{x}[a] x_3^{-c} = x_1^{-1}x_2^{-1} \left(x_1^a x_3^c + x_2^b x_3^c + x_2^a x_2^b \right).$$

(b) For a non-acyclic seed, $\tilde{x}[a]$ is less interesting for certain choices of $a$: take $n = m = 3$, $a = (1, 1, 1)$ and

$$\tilde{B} = \begin{bmatrix} 0 & a & -c' \\ -a' & 0 & b \\ c & -b' & 0 \end{bmatrix}, \quad a, a', b, b', c, c' > 0.$$ 

Then $E = \{(1, 2), (2, 3), (3, 1)\}$, $S_{gcc} = \{((0), (0), (0)), (0), (1), ((1), (1), (1))\}$. Thus

$$\tilde{x}[a] = \frac{x_1^a x_2^b x_3^c + x_2^a x_3^b x_1^c}{x_1 x_2 x_3} = x_1^{-a}x_2^{-b}x_3^{-c} + x_2^{-a}x_3^{-b}x_1^{-c}.$$ 

Remark 3.8. For the cluster algebra with an acyclic seed $(x, y, B)$, it is known [4, Theorem 1.16] that the set of monomials in $x_1, \ldots, x_n, x_1', \ldots, x_n'$ which contains no product of the form $x_j x_j'$ is a basis. This is called a standard monomial basis. Let $x[a]$ be defined in the same way as $\tilde{x}[a]$ in Definition 3.4 except that we drop the local compatibility condition (i.e. drop (1) but keep (2)–(4)). Then $x[a]$ are precisely the standard monomial basis elements.

Remark 3.9. The local compatibility condition in Definition 3.4(1) is weaker than compatibility given in [10]. In particular, not all cluster variables are of the form $\tilde{x}[a]$ even for rank 2 cluster algebras.

Theorem 3.10. For each $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, $\tilde{x}[a]$ are in the upper cluster algebra; moreover if the initial seed is acyclic, then $(\prod_{i=1}^n x_i^{a_i}) \tilde{x}[a]$ is not divisible by $x_k$ for any $1 \leq k \leq n$. As a consequence, the same holds for $\tilde{x}_{\text{red}}[a]$.

Proof. We first show that if the initial seed is acyclic and $1 \leq k \leq n$, then $(\prod_{i=1}^n x_i^{a_i}) \tilde{x}[a]$ is not divisible by $x_k$, or equivalently, there exists $s = (s_1, \ldots, s_m) \in S_{gcc}$ such that

$$\prod_{(i,j) \in E} x_i^{b_j|s_j|} x_j^{-b_j|s_i|}$$

is not divisible by $x_k$. So we need to find $s$ such that $|\tilde{s}_j| = 0$ if $(k, j) \in E$, and $|s_i| = 0$ if $(i, k) \in E$. Such an $s$ can be constructed as follows: since the initial seed is acyclic, $E$
determines a partial order generated by "\(i \prec j\) if \((i, j) \in E\)". Define
\[
s_j = \begin{cases} 
(1, \ldots, 1), & \text{if } k \prec j, \\
(0, \ldots, 0), & \text{otherwise.}
\end{cases}
\]
To check that \(s\) is in \(S_{gcc}\), note that if \(s_i \cdot s_j \neq 0\) for some \((i, j) \in E\), then there exists \(r \leq [a_j]_+\) that \(s_i, r = 1, s_j, r = 0\). By our choice of \(s\), we have \(k \prec i, k \not\prec j\). This contradicts with \(i \prec j\).

Next, we show that \(\bar{x}[a]\) is in the upper cluster algebra.

Denote \(a_+ = ([a_1]_+, [a_2]_+, \ldots, [a_n]_+)\). Then
\[
\bar{x}[a] = \left( \prod_{i=1}^{n} x_i^{-[a_i]_+} \right) \bar{x}[a_+].
\]
Since \(\prod_{i=1}^{n} x_i^{-[a_i]_+}\) is in the upper cluster algebra, it suffices to show that \(\bar{x}[a_+]\) is in the upper cluster algebra. Thus without loss of generality we assume that \(a \in \mathbb{Z}_{\geq 0}^n\).

For a sequence \(t = (t_1, \ldots, t_a) \in \{0, 1\}^a\), we can regard it as an infinite sequence \((t_1, \ldots, t_a, 0, 0, \ldots)\). Then \(\bar{t} = (f_r(a) - t_r)_{r=1}^{\infty}\) where
\[
f_r(a) = \begin{cases} 
1, & \text{if } r \leq a, \\
0, & \text{otherwise.}
\end{cases}
\]
The sum and dot product in (1) extend naturally.

Then \(\bar{x}[a]\) is the coefficient of \(z^0\) in the following polynomial in \(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}][z]\):
\[
\left( \prod_{i=1}^{n} x_i^{-a_i} \right) \sum_{s \in S_{alt}} \left( \prod_{(i,j) \in E} \frac{b_{ij} s_j}{x_i^{b_{ij} s_j + b_{ji} s_i - z^{s_i, s_j}}} \right)
\]
\[
= \left( \prod_{i=1}^{n} x_i^{-a_i} \right) \sum_{s \in S_{alt}} \prod_{k \geq 1} \left( \prod_{(i,j) \in E} \frac{b_{ij} s_j, k}{x_i^{b_{ij} s_j, k + b_{ji} s_i, k - z^{s_i, s_j, k}}} \right)
\]
\[
= \left( \prod_{i=1}^{n} x_i^{-a_i} \right) \sum_{s \in S_{alt}} \prod_{k \geq 1} \left( \prod_{(i,j) \in E} \frac{b_{ij} (f_k(a_j) - s_j, k)}{x_i^{b_{ij} (f_k(a_j) - s_j, k) + b_{ji} s_i, k - z^{s_i, s_j, k}}} \right)
\]
\[
= \left( \prod_{i=1}^{n} x_i^{-a_i} \right) \prod_{k \geq 1} \left( \sum_{s^k \in S_{alt}^k} \prod_{(i,j) \in E} \frac{b_{ij} (f_k(a_j) - s_j, k)}{x_i^{b_{ij} (f_k(a_j) - s_j, k) + b_{ji} s_i, k - z^{s_i, s_j, k}}} \right)
\]
where \(S_{alt}^k\) is the set of all possible \(s^k = (s_{1,k}, \ldots, s_{m,k})\) with \((s_1, \ldots, s_m)\) running through \(S_{alt}\), or equivalently,
\[
S_{alt}^k = \{ s^k = (s_{1,k}, \ldots, s_{m,k}) \mid 0 \leq s_{i,k} \leq f_k(a_i) \text{ for } i = 1, \ldots, m \}.
\]
Meanwhile, denote \( f_k(a) = (f_k(a_1), \ldots, f_k(a_n)) \in \{0, 1\}^a \). Then \( \bar{x}[f_k(a)] \) is the coefficient of \( z^0 \) of
\[
\left(\prod_{i=1}^{n} x_i^{-f_k(a_i)}\right) \sum_{s \in S_{\text{all}}} \left( \prod_{(i,j) \in E} x_i^{b_{ij}(f_k(a_j)-s_{j,k})} x_j^{-b_{ij}s_{i,k}} z^{s_{i,k}(f_k(a_j)-s_{j,k})} \right).
\]

So we conclude that
\[
\bar{x}[a] = \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \prod_{k \geq 1} \left( \bar{x}[f_k(a)] \prod_{i=1}^{n} x_i^{f(k,a_i)} \right)
= \left(\prod_{i=1}^{n} x_i^{-a_i+\sum_{k \geq 1} f_k(a_i)}\right) \prod_{k \geq 1} \bar{x}[f_k(a)] = \prod_{k \geq 1} \bar{x}[f_k(a)].
\]

So \( \bar{x}[a] \) is in the upper cluster algebra by Lemma 3.11(ii).

**Lemma 3.11.** For \( a = (a_1, a_2, \ldots, a_n) \in \{0, 1\}^n \), \( a_{n+1} = \cdots = a_m = 0 \), denote by \( S \) the set of all \( n \)-tuples \( s = (s_1, \ldots, s_m) \in \{0, 1\}^m \) such that \( 0 \leq s_i \leq a_i \) for \( i = 1, \ldots, m \), and \( (s_i, a_j - s_j) \neq (1, 1) \) for every \( (i, j) \in E \). Then

(i) \( \bar{x}[a] \) can be written as
\[
\left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{s \in S} \prod_{i=1}^{m} x_i^{\sum_{j=1}^{n}(a_j-s_j)[b_{ij}]_+ + s_j[-b_{ij}]_+}.
\]

(ii) \( \bar{x}[a] \) is in the upper cluster algebra.

**Proof.** (i) By Definition 3.5
\[
\bar{x}[a] = \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{s \in S} \left( \prod_{(i,j) \in E} x_i^{(a_j-s_j)b_{ij}} x_j^{-s_i(-b_{ij})} \right)
= \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{s \in S} \left( \prod_{i,j=1}^{m} x_i^{(a_j-s_j)[b_{ij}]_+} x_j^{-s_i[-b_{ij}]_+} \right)
= \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{s \in S} \left( \prod_{i,j=1}^{m} x_i^{(a_j-s_j)[b_{ij}]_+} \prod_{i,j=1}^{m} x_j^{s_i[-b_{ij}]_+} \right)
= \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{s \in S} \left( \prod_{i,j=1}^{m} x_i^{(a_j-s_j)[b_{ij}]_+} \prod_{i,j=1}^{m} x_j^{s_i[-b_{ij}]_+} \right) = \text{[3.3]}.
\]

(ii) We introduce \( n \) extra variables \( x_{m+1}, \ldots, x_{m+n} \). Let \( \mathbb{P}' = \text{Trop}(x_{n+1}, \ldots, x_{m+n}) \). Then the group ring \( \mathbb{Z} \mathbb{P}' \) is the ring of Laurent polynomials in the variables \( x_{n+1}, \ldots, x_{m+n} \).
Let
\[
\bar{B}' = \begin{bmatrix} \bar{B} \\ I_n \end{bmatrix}
\]
be the \((m + n) \times n\) matrix that encodes a new cluster algebra \(\mathcal{A}'\). Assume \(a_{n+1} = \cdots = a_{m+n} = 0\) and define \(E', S', \bar{x}'[\mathbf{a}]\) for the coefficient semifield \(\mathbb{P}'\) similarly as the definition of \(E, S, \bar{x}[\mathbf{a}]\) for the coefficient semifield \(\mathbb{P}\). So
\[
\bar{x}'[\mathbf{a}] = \left( \prod_{i=1}^{n} x_i^{-a_i} \right) \sum_{\mathbf{s} \in S'} P_{\mathbf{s}}, \quad P_{\mathbf{s}} := \prod_{i=1}^{m+n} x_i^{\sum_{j=1}^{n} (a_{j-s_j})[b_{ij}] + s_j[-b_{ij}]}. 
\]
We will show that \(\bar{x}'[\mathbf{a}]\) is in the upper bound
\[
\mathcal{U}'_x = \mathbb{Z}\mathbb{P}[x^{\pm 1}] \cap \mathbb{Z}\mathbb{P}[x_1^{\pm 1}] \cap \cdots \mathbb{Z}\mathbb{P}[x_n^{\pm 1}]
\]
(\(x = \{x_1, \ldots, x_n\}\) and for \(1 \leq k \leq n\), the adjacent cluster \(x_k\) is defined by \(x_k = x - \{x_k\} \cup \{x'_{k}\}\)), therefore \(\bar{x}'[\mathbf{a}]\) is in the upper cluster algebra \(\mathcal{U}'\), thanks to the fact that \(\mathcal{U}'_x = \mathcal{U}'\) when \(\bar{B}'\) is of full rank \([4, \text{Corollary 1.7 and Proposition 1.8}]\). Then we substitute \(x_{m+1} = \cdots = x_{m+n} = 1\) and conclude that \(\bar{x}[\mathbf{a}]\) is in the upper cluster algebra \(\mathcal{U}\).

Since \(\bar{x}'[\mathbf{a}]\) is obviously in \(\mathbb{Z}\mathbb{P}[x^{\pm 1}]\) from its definition, we only need to show that \(\bar{x}'[\mathbf{a}]\) is in \(\mathbb{Z}\mathbb{P}[x_k^{\pm 1}]\) for \(1 \leq k \leq n\). Again from its definition we see that \(\bar{x}'[\mathbf{a}]\) is in \(\mathbb{Z}\mathbb{P}[x_k^{\pm 1}]\) when \(a_k = 0\). So we may assume \(a_k = 1\). Let \(N \subset S'\) contains those \(\mathbf{s}\) such that \(P_{\mathbf{s}}\) is not divisible by \(x_k\); equivalently,
\[
N = \{ \mathbf{s} \in S' | s_j = a_j \text{ if } b_{kj} > 0; s_j = 0 \text{ if } b_{kj} < 0 \}.
\]
Then it suffices to show that \(\sum_{\mathbf{s} \in N} P_{\mathbf{s}}\) is divisible by \(A\), where \(A\) is the binomial in the exchange relation
\[
x_k' = x_k^{-1} A, \quad A = \prod_{i=1}^{m+n} x_i^{[b_{ik}]} + \prod_{i=1}^{m+n} x_i^{-[b_{ik}]}.
\]
Write \(N\) into a partition \(N = N_0 \cup N_1\) where \(N_0 = \{ \mathbf{s} \in N | s_k = 0 \} \), \(N_1 = \{ \mathbf{s} \in N | s_k = 1 \}\). Define \(\varphi : N_0 \to N_1\) by
\[
\varphi(s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_n) = (s_1, \ldots, s_{k-1}, 1, s_{k+1}, \ldots, s_n).
\]
It is easy to check that \(\varphi\) is a well-defined bijection. Thus
\[
\sum_{\mathbf{s} \in N} P_{\mathbf{s}} = \sum_{\mathbf{s} \in N_0} P_{\mathbf{s}} + \sum_{\mathbf{s} \in N_1} P_{\mathbf{s}} = \sum_{\mathbf{s} \in N_0} (P_{\mathbf{s}} + P_{\varphi(\mathbf{s})})
\]
\[
= \prod_{i=1}^{m+n} \prod_{1 \leq j \leq n, j \neq k} x_i^{(a_j - s_j)[b_{ij}] + s_j[-b_{ij}]} (\prod_{i=1}^{m+n} x_i^{[b_{ik}]} + \prod_{i=1}^{m+n} x_i^{-[b_{ik}]})
\]
\[
= \prod_{i=1}^{m+n} \prod_{1 \leq j \leq n, j \neq k} x_i^{(a_j - s_j)[b_{ij}] + s_j[-b_{ij}]} A
\]
is divisible by \(A\).
\(\square\)
4. Non-acyclic Rank 3 Cluster Algebras

In this section, we consider non-acyclic skew-symmetric rank 3 cluster algebras of geometric type.

4.1. Definition and Properties of $\tau$. Let $m \geq 3$ be an integer, $\tilde{B} = (b_{ij})$ be an $m \times 3$ matrix whose principal part $B$ is skew-symmetric and non-acyclic (i.e., $b_{12}, b_{23}, b_{31}$ are of the same sign). Define the map

$$\tau(B) := ([b_{2,3}], [b_{3,1}], [b_{1,2}]) \in \mathbb{Z}_{\geq 0}^3.$$

Define three involutary functions $\mu_1, \mu_2, \mu_3 : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$$\mu_1 : (x, y, z) \mapsto (x, xz - y, z), \mu_2 : (x, y, z) \mapsto (x, y, xy - z), \mu_3(x, y, z) \mapsto (yz - x, y, z).$$

Let $\Gamma$ be the group generated by $\mu_1, \mu_2,$ and $\mu_3$. Consider the following two situations.

(M1) $(a, b, c) \leq \mu_i(a, b, c)$ for all $i = 1, 2, 3$.

(M2) $(a, b, c) \leq \mu_i(a, b, c)$ for precisely two indices $i$.

The following statement follows from [6, Theorem 1.2, Lemma 2.1] for a coefficient free cluster algebra, and it is easy to see that it holds more generally for a cluster algebra of geometric type.

**Theorem 4.1.** Let $B$ be as above, $(a, b, c) = \tau(B)$. Then the following are equivalent:

(i) $A(x, y, B)$ is non-acyclic.

(ii) $a, b, c \geq 2$ and $abc + 4 \geq a^2 + b^2 + c^2$.

(iii) $a, b, c \geq 2$ and there exists a unique triple in the $\Gamma$-orbit of $\tau(B)$ that satisfies (M1).

Moreover, under these conditions, triples in the $\Gamma$-orbit of $\tau(B)$ not satisfying (M1) must satisfy (M2).

Assume $A(x, y, B)$ is non-acyclic. We call the unique triple in Theorem 4.1(iii) the root of the $\Gamma$-orbit of $\tau(B)$. It follows from the above theorem that the $\Gamma$-orbit of $\tau(B)$ is identical to the set $\{\tau(B') : B'$ is in the mutation class of $B\}$.

**Definition 4.2.** Let $(x, y, z) \in \mathbb{R}^3$. We define a partial ordering “$\leq$” on $\mathbb{R}^3$ by $(x, y, z) \leq (x', y', z')$ if and only if $x \leq x'$, $y \leq y'$, and $z \leq z'$.

**Lemma 4.3.** Let $B$ be as above, $(a, b, c) = \tau(B)$.

(i) The root $(a, b, c)$ of the $\Gamma$-orbit of $\tau(B)$ is the minimum of the $\Gamma$-orbit, i.e., $(a, b, c) \leq (a', b', c')$ for any $(a', b', c')$ in the $\Gamma$-orbit.

(ii) If $\mu_k(a, b, c) = (a, b, c)$, then $(a, b, c)$ is the root of the $\Gamma$-orbit of $\tau(B)$.

(iii) Assume that $\tau(B)$ is the root of the $\Gamma$-orbit of $\tau(B)$, $i_1, \ldots, i_l \in \{1, 2, 3\}$, $i_s \neq i_{s+1}$ for $1 \leq s \leq l - 1$. Let $B_0 = B$, $B_j = \mu_{i_j}(B_{j-1})$ for $j = 1, \ldots, l$. Then

$$\tau(B_0) \leq \tau(B_1) \leq \cdots \leq \tau(B_l).$$
Proof. It follows easily from Theorem 4.1. To see (iii): if it is false, then there are \(1 \leq j < j' \leq l\) such that
\[
\tau(B_j) < \tau(B_{j+1}) = \tau(B_{j+2}) = \cdots = \tau(B_{j'}) > \tau(B_{j'+1}).
\]
But this cannot hold because of (ii) and Theorem 4.1. \(\square\)

4.2. Grading on \(\mathcal{A}\). We now adapt the grading introduced in \cite{8} Definition 3.1] to the geometric type.

Definition 4.4. A graded seed is a triple \((x, y, B, G)\) such that

(i) \((x, y, B)\) is a seed of rank \(n\), and
(ii) \(G = [g_1, \ldots, g_n]^T \in \mathbb{Z}^n\) is an integer column vector such that \(BG = 0\).

Set \(\deg_G(x_i) = g_i\) and \(\deg(x_i^{-1}) = -g_i\) for \(i \leq n\), and set \(\deg(y_j) = 0\) for all \(j\). Extend the grading additively to Laurent monomials hence to the cluster algebra \(\mathcal{A}(x, y, B)\). In \cite{8} it is proved that under this grading every exchange relation is homogeneous and thus the grading is compatible with mutation.

Theorem 4.5. \cite{8} Corollary 3.4] The cluster algebra \(\mathcal{A}(x, y, B)\) under the above grading is a \(\mathbb{Z}\)-graded algebra.

In the rest of the paper we assume that \(B\) is a \(3 \times 3\) skew-symmetric non-acyclic matrix.

The following two propositions come from the work in \cite{1} to show that rank three non-acyclic cluster algebras have no maximal green sequences.

Theorem 4.6. \cite{1} Proposition 2.2] Suppose that \(B\) is a \(3 \times 3\) skew-symmetric non-acyclic matrix. Then the (column) vector \(G = \tau(B)^T\) satisfies \(BG = 0\).

Lemma 4.7. \cite{1} Lemma 2.3] For any graded seed \((x', y', B', G')\) in the mutation class of our initial graded seed, we have \(G' = \tau(B')\).

In other words, if \(\tau(B) = (a', b', c')\) then \(\deg(x_1') = b', \deg(x_2') = c', \deg(x_3') = a'\) for any seed \((x, y, B)\). Notice that if we look at the quiver associated to our exchange matrix we see that the grading we chose gives that the degree of the cluster variable associated to vertex \(i\) is the number of arrows on the edge not incident to \(i\). Furthermore, there is a canonical grading for a non-acyclic rank three cluster algebras since regardless of your choice of initial seed the grading imposed on \(\mathcal{A}\) is the same.

4.3. Construction of an element in \(\mathcal{U} \setminus \mathcal{A}\). Here we shall prove the following main theorem of the paper.

Theorem 4.8. A non-acyclic rank three skew-symmetric cluster algebra \(\mathcal{A}\) of geometric type is not equal to the upper cluster algebra \(\mathcal{U}\).
We give $\mathcal{A}$ the $\mathbb{Z}$-grading as in §4.2. By Theorem 4.1, we can assume that the initial seed
\[
\Sigma = (\{x_1, x_2, x_3\}, y, B, G), \quad \text{where } B = \begin{bmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{bmatrix}, \quad G = \begin{bmatrix} b \\ c \\ a \end{bmatrix}
\]
satisfies $a, b, c \geq 2$ and (M1). Then $\deg x_1 = b$, $\deg x_2 = c$, $\deg x_3 = a$, and we let $x_4, \ldots, x_m$ to be of degree 0. Furthermore, by permuting the indices if necessary, we assume $a \geq b \geq c$.

Define six degree-0 elements as follows:
\[
\alpha \pm_i := \prod_{j=4}^m x_j^{[\pm b, i]}, \quad i = 1, 2, 3.
\]
Looking at the seeds neighboring $\Sigma$ we obtain three new cluster variables. Namely,
\[
z_1 = \frac{\alpha^- x_2^a + \alpha^+ x_3^c}{x_1}, \quad z_2 = \frac{\alpha^+ x_1^a + \alpha^- x_3^b}{x_2}, \quad z_3 = \frac{\alpha^- x_1^a + \alpha^+ x_2^b}{x_3}
\]
which have degree $ac - b$, $ab - c$, and $bc - a$, respectively.

We also require the following theorem, whose proof in [12] is for coefficient free cluster algebra but clearly applies to a cluster algebra of geometric type.

**Theorem 4.9.** [12, Proposition 6.1.2] If $\mathcal{A}$ is a rank three skew-symmetric non-acyclic cluster algebra, then $\mathcal{A}$ is totally coprime.

Let $B'$ be the matrix obtained from $B$ by deleting the second column and interchanging rows 2 and 3, i.e. treat $x_2$ as a frozen variable. The first three rows of $B'$ are
\[
B'_{1,2,3} = \begin{bmatrix} 0 & -c \\ c & 0 \\ -a & b \end{bmatrix}.
\]
Computing $\tilde{x}[(1, 1)]/x_2^b$ in the cluster algebra determined by $B'$ similar to Example 3.7(a), then dividing by $x_2^b$, we obtain the following element in the rank 2 upper cluster algebra:
\[
\tilde{x}[(1, 1)]/x_2^b = \frac{\alpha^- \alpha_3^- x_1^a x_2^{a-b} + \alpha^- \alpha_3^+ x_2^a + \alpha^+ \alpha_3^- x_3^c}{x_1 x_3}.
\]
We are ready to prove the main theorem by showing that the element constructed above (but now treat $x_2$ as a non-frozen variable instead) is in $\mathcal{U} \setminus \mathcal{A}$.

**Proof of Theorem 4.8** Define
\[
Y = \frac{\alpha^- \alpha_3^- x_1^a x_2^{a-b} + \alpha^- \alpha_3^+ x_2^a + \alpha^+ \alpha_3^- x_3^c}{x_1 x_3}.
\]
We first show that $Y \in \mathcal{U}$. Note that $\mathcal{A}$ is a rank 3 cluster algebra so it is totally coprime by Theorem 4.9. It then suffices to show that $Y \in \mathcal{U}_k$ by Theorem 2.4. Clearly $Y \in \mathbb{Z}[x_{1}^{\pm 1}, x_{2}^{\pm 1}, x_{3}^{\pm 1}]$ where $\mathbb{P} = \text{Trop}(x_{4}, \ldots, x_{m})$. Also,

$$Y = \frac{\alpha_{3}^{a} z_{3}^{c} + (\alpha_{1} x_{2}^{a} + \alpha_{3} x_{3}^{c})^{-1} x_{2}^{a-b}}{z_{3}^{c-1} x_{3}} \in \mathbb{Z}[z_{1}^{\pm 1}, z_{2}^{\pm 1}, z_{3}^{\pm 1}],$$

$$Y = \frac{\alpha_{1}^{a} x_{1}^{a} x_{3}^{c} x_{3}^{c} x_{3}^{c}}{z_{3}^{c-1}} \in \mathbb{Z}[x_{1}^{\pm 1}, z_{2}^{\pm 1}, z_{3}^{\pm 1}],$$

$$Y = \frac{\alpha_{1}^{a} x_{2}^{a-b} z_{3}^{c} + \alpha_{3}^{a} x_{1}^{a} x_{1}^{a} x_{1}^{a}}{z_{3}^{c-1}} \in \mathbb{Z}[x_{1}^{\pm 1}, z_{2}^{\pm 1}, z_{3}^{\pm 1}].$$

Therefore we conclude that $Y \in \mathcal{U}_k$.

Next, we show that $Y \notin \mathcal{A}$. With respect to our grading of $\mathcal{A}$, $Y$ is homogeneous of degree $ac - b - a$.

We claim that all cluster variables, except possibly $x_1, x_2, x_3, z_3$, have degree strictly larger than $ac - b - a$. Indeed, let $x$ be a cluster variable such that $x \neq x_1, x_2, x_3, z_3$. Then $x$ can be written as

$$x = \mu_{i_{1}} \cdots \mu_{i_{k}} x_{k}, \quad i_{1}, \ldots, i_{l}, k \in \{1, 2, 3\}, \text{ and } i_{s} \neq i_{s+1} \text{ for } 1 \leq s \leq l-1.$$

Let $B_{0} = B, B_{j} = \mu_{i_{j}}(B_{j-1})$ for $j = 1, \ldots, l$. Let $\tau(B_{j}) = (b_{j}, c_{j}, a_{j})$ for $j = 0, \ldots, l$. By Lemma 4.3 we can further require that

$$(b, c, a) = (b_{0}, c_{0}, a_{0}) \leq (b_{1}, c_{1}, a_{1}) \leq \cdots \leq (b_{l}, c_{l}, a_{l}).$$

We prove the claim in the following five cases.

**Case** $k = 1$. We let $r > 0$ be the smallest integer such that $i_{r} = k$ (which exists since $x \neq x_{1}, x_{2}, x_{3}$). It suffices to prove that the degree of $w = \mu_{i_{r}} \cdots \mu_{i_{k}} x_{k}$ is larger than $ac - b - a$. Indeed,

$$\deg w = b_{r} = a_{r-1} c_{r-1} - b_{r-1} = a_{r-1} c_{r-1} - b \geq ac - b > ac - b - a.$$

**Case** $k = 2$. The above proof still works:

$$\deg w = c_{r} = a_{r-1} b_{r-1} - c_{r-1} = a_{r-1} b_{r-1} - c \geq ab - c > ac - b - a.$$

**Case** $k = 3, i_{1} = 1$. We let $r > 0$ be the smallest integer such that $i_{r} = k$. Then $\mu_{i_{1}} x_{k} = (ac - b, c, a)$, and

$$\deg w = a_{r} = b_{r-1} c_{r-1} - a_{r-1} = b_{r-1} c_{r-1} - a \geq b_{1} c_{1} - a = (ac - b)c - a > ac - b - a.$$

**Case** $k = 3, i_{1} = 2$. Similar to the above case, $\mu_{i_{1}} x_{k} = (a, b, ab - c)$,

$$\deg w = a_{r} \geq b_{1} c_{1} - a = b(ab - c) - a \geq b^{2}(a - 1) - a > c(a - 1) - a \geq ac - b - a.$$

**Case** $k = 3, i_{1} = 3$. Then $\mu_{i_{1}} x_{k} = (b, c, bc - a), i_{2} \neq 3$. We let $r \geq 3$ be the smallest integer such that $i_{r} = k$ (which exists since $x \neq z_{3}$). It suffices to prove that the degree of $w = \mu_{i_{r}} \cdots \mu_{i_{k}} x_{k}$ is larger than $ac - b - a.$
If $i_2 = 1$, then $(b_2, c_2, a_2) = (c(bc - a) - b, c, bc - a)$,
\[
deg w = a_r = b_{r-1}c_{r-1} - (bc - a) \geq b_2c_2 - (bc - a) = (c(bc - a) - b)c - (bc - a) \\
= (c^2 - 2)(bc - a) - a \geq (c^2 - 2)a - a = a(c^2 - 3) \geq a(c - 1) > ac - b - a.
\]

If $i_2 = 2$, then $(b_2, c_2, a_2) = (b, (bc - a)b - c, bc - a)$, and $(bc - a)b - c \geq c(bc - a) - b$. Then
\[
deg w \geq b((bc - a)b - c) - (bc - a) \geq (c(bc - a) - b)c - (bc - a) > ac - b - a,
\]
where the last inequality follows from the computation in the previous case. This completes the proof of the claim that all cluster variables, except possibly $x_1, x_2, x_3, z_3$, have degree strictly larger than $ac - b - a$.

Therefore it is sufficient to check that $Y$ cannot be written as a linear combination of products of the cluster variables $x_1, x_2, x_3$, and $z_3$ since all other cluster variable have larger degree than $Y$. This is clear as the numerator of $Y$ is irreducible and none of these cluster variables have a factor of $x_1$ in the denominator. $\square$

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