ON THE EQUICONTINUITY REGION OF DISCRETE SUBGROUPS OF $PU(1,n)$

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Abstract. Let $G$ be a discrete subgroup of $PU(1,n)$. Then $G$ acts on $\mathbb{P}^n_\mathbb{C}$ preserving the unit ball $\mathbb{H}^n_\mathbb{C}$, where it acts by isometries with respect to the Bergman metric. In this work we determine the equicontinuity region $Eq(G)$ of $G$ in $\mathbb{P}^n_\mathbb{C}$: It is the complement of the union of all complex projective hyperplanes in $\mathbb{P}^n_\mathbb{C}$ which are tangent to $\partial \mathbb{H}^n_\mathbb{C}$ at points in the Chen-Greenberg limit set $\Lambda_{CG}(G)$, a closed $G$-invariant subset of $\partial \mathbb{H}^n_\mathbb{C}$, which is minimal for non-elementary groups. We also prove that the action on $Eq(G)$ is discontinuous.

Introduction

Let $PU(1,n) \subset PSL(n+1,\mathbb{C})$ be the group of automorphisms of $\mathbb{P}^n_\mathbb{C}$ that preserve the ball

$$\{[z_0 : z_1 : \ldots : z_n] \in \mathbb{P}^n_\mathbb{C} : |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 < |z_0|^2 \}.$$  

We equip this ball with the Bergman metric, so we get a model for the complex hyperbolic space $\mathbb{H}^n_\mathbb{C}$, with $PU(1,n)$ as its group of holomorphic isometries. If $G \subset PU(1,n)$ is a discrete subgroup, then its limit set $\Lambda_{CG}(G)$ was defined by Chen-Greenberg in [3] as the set of accumulation points of the $G$-orbits in $\mathbb{H}^n_\mathbb{C}$. As for conformal Kleinian groups, this limit set is contained in the “sphere at infinity” $\partial \mathbb{H}^n_\mathbb{C}$, and it is a closed invariant set, which either has cardinality $\leq 2$ or else it has infinitely many points, all orbits in it are dense and it is the unique minimal closed $G$-invariant set in $\mathbb{H}^n_\mathbb{C}$. The action of $G$ on $\mathbb{H}^n_\mathbb{C}$, being by isometries, is discontinuous and equicontinuous.

We notice that $G$ is by definition also a subgroup of $PSL(n+1,\mathbb{C})$ so it acts on all of $\mathbb{P}^n_\mathbb{C}$, and it is natural to look for information about its action on all of $\mathbb{P}^n_\mathbb{C}$, where the action is no longer isometric. This is analogous to considering a classical fuchsian group $\Gamma \subset PSL(2,\mathbb{R})$ and thinking of it as acting in $\mathbb{P}^1_\mathbb{C}$ preserving a ball, which serves as model for the real hyperbolic plane.

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In this article we are interested in studying the region of equicontinuity in $\mathbb{P}^n_\mathbb{C}$ of discrete subgroups of $PU(1, n)$. This is interesting, among other reasons, because equicontinuity is a gate for the analytic study of the dynamics of discrete subgroups of $PSL(n+1, \mathbb{C})$.

We prove:

**Theorem 1.** Let $G \subset PU(1, n)$ be a discrete subgroup and let $Eq(G)$ be its equicontinuity region in $\mathbb{P}^n_\mathbb{C}$. Then $\mathbb{P}^n_\mathbb{C} \setminus Eq(G)$ is the union of all complex projective hyperplanes tangent to $\partial \mathbb{H}^n_\mathbb{C}$ at points in $\Lambda_{CG}(G)$, and $G$ acts discontinuously on $Eq(G)$. Furthermore, the set of accumulation points of the $G$-orbit of every compact set $K \subset Eq(G)$ is contained in $\Lambda_{CG}(G)$.

The proof of this theorem relies on quasi-projective transformations, introduced by Furstenberg (see [1, 4]), which provide a completion of the non-compact Lie group $PSL(n+1, \mathbb{C})$.

We ought to mention that this work was inspired by Navarrete’s theorem in [8], where the author studies discrete subgroups of $PU(1, 2)$ and compares the aforementioned Chen-Greenberg limit set with a different notion of limit set, due to R. Kulkarni [7], which has the property of granting that the action on its complement $\Omega_{Kul}(G)$ is discontinuous. The theorem in [8] says that the Kulkarni region of discontinuity $\Omega_{Kul}(G)$ is the complement of the union of all projective lines in $\mathbb{P}^2_\mathbb{C}$ which are tangent to $\partial \mathbb{H}^2_\mathbb{C}$ at points in the Chen-Greenberg limit set. That proof also shows implicitly that for $n = 2$ the regions $Eq(G)$ and $\Omega_{Kul}(G)$ coincide: a fact which is not known in higher dimensions. Actually, for $n = 2$ an additional dimensional argument, together with Theorem 1 give a simpler proof of the theorem in [8]; the details are given in [2].

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1. **Preliminaries on projective and complex hyperbolic geometry**

   We recall that the complex projective space $\mathbb{P}^n_\mathbb{C}$ is defined as:
   $$\mathbb{P}^n_\mathbb{C} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$
   where the non-zero complex numbers are acting coordinate-wise. This is a compact connected complex $n$-dimensional manifold.

   If $\left[ \right] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_\mathbb{C}$ is the quotient map, then a non-empty set $H \subset \mathbb{P}^n_\mathbb{C}$ is said to be a projective subspace of dimension $k$ (i.e., $dim_\mathbb{C}(H) = k$) if there is a $\mathbb{C}$-linear subspace $\tilde{H}$ of dimension $k+1$ such that $[\tilde{H}]_n = H$. Hyperplanes are the projective subspaces of dimension $n-1$. Given distinct points $p, q \in \mathbb{P}^n_\mathbb{C}$, there is a unique complex projective subspace of dimension 1 passing through $p$ and $q$. Such a subspace will be called a complex (projective) line.
and denoted by $\overrightarrow{p, q}$; this is the image under $[\cdot]_n$ of a two-dimensional linear subspace of $\mathbb{C}^{n+1}$.

It is clear that every linear automorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{P}_n$, and it is well-known that every automorphism of $\mathbb{P}_n$ arises in this way. Thus one has that the group of projective automorphisms is:

$$PSL(n + 1, \mathbb{C}) := GL(n + 1, \mathbb{C})/(\mathbb{C}^*)^{n+1} \cong SL(n + 1, \mathbb{C})/\mathbb{Z}_{n+1},$$

where $(\mathbb{C}^*)^{n+1}$ is being regarded as the subgroup of diagonal matrices with a single non-zero eigenvalue, and we consider the action of $\mathbb{Z}_{n+1}$ (regarded as the roots of unity) on $SL(n + 1, \mathbb{C})$ given by the usual scalar multiplication. Then $PSL(n + 1, \mathbb{C})$ is a Lie group whose elements are called projective transformations.

We denote also by $[\cdot]_n : SL(n + 1, \mathbb{C}) \to PSL(n + 1, \mathbb{C})$ the quotient map, which indeed is restriction of a map defined on the general linear group $GL(n + 1, \mathbb{C})$. Given $\gamma \in PSL(n + 1, \mathbb{C})$ we say that $\tilde{\gamma} \in GL(n + 1, \mathbb{C})$ is a lift of $\gamma$ if there is an scalar $r \in \mathbb{C}^*$ such that $r \tilde{\gamma} \in SL(n, \mathbb{C})$ and $[r \tilde{\gamma}]_n = \gamma$.

Notice that $PSL(n + 1, \mathbb{C})$ acts transitively, effectively and by biholomorphisms on $\mathbb{P}_n$, taking projective subspaces into projective subspaces.

In what follows $\mathbb{C}^{1,n}$ is a copy of $\mathbb{C}^{n+1}$ equipped with a Hermitian form of signature $(1, n)$ that we assume is given by:

$$\langle u, v \rangle = -u_0\overline{v_0} + \sum_{j=1}^{n} u_j\overline{v_j},$$

where $u = (u_0, u_1, \ldots, u_n)$ and $v = (v_0, v_1, \ldots, v_n)$. A vector $v$ is called negative, null or positive depending (in the obvious way) on the value of $\langle v, v \rangle$; we denote the set of negative, null or positive vectors by $N_-, N_0$ and $N_+$ respectively. Thus one has:

$$N_- = \{ u \in \mathbb{C}^{1,n} \mid u_0\overline{v_0} > \sum_{j=1}^{n} u_j\overline{v_j} \}.$$

The image $\mathbb{B}$ of $N_-$ in $\mathbb{P}_n$ under the map $[\cdot]_n$ is diffeomorphic to a ball of real dimension $2n$, with boundary a sphere $S^{2n-1}$, which is the image of $N_0$.

If we let $U(1, n) \subset GL(n+1, \mathbb{C})$ be the subgroup consisting of the elements that preserve the above Hermitian form, then its projectivization $[U(1, n)]_n$ is a subgroup of $PSL(n + 1, \mathbb{C})$ that we denote by $PU(1, n)$.

It is easy to see that $PU(1, n)$ acts transitively on $\mathbb{B}$ with isotropy $U(n)$. Let $0$ denote the center of the ball $\mathbb{B}$, consider the space $T_0 \mathbb{B} \cong \mathbb{C}^n$ tangent to $\mathbb{B}$ at $0$, and put on it the usual Hermitian metric on $\mathbb{C}^n$. Now we use the action of $PU(1, n)$ to spread the metric, using that the action is transitive and the isotropy is $U(n)$, which preserves the usual metric on $\mathbb{C}^n$. We thus get a Hermitian metric on $\mathbb{B}$, which is clearly homogeneous. This metric is, up to scaling, the Bergman metric, and gives the model we use for complex
hyperbolic $n$-space, that we denote by $\mathbb{H}^n_C$. It is clear from this construction that $PU(1,n)$ is the group of holomorphic isometries of $\mathbb{H}^n_C$. Its elements are classified as follows [3]:

**Definition 1.1.** The non-trivial elements of $PU(n,1)$ fall into three general conjugacy types, depending on the number and location of their fixed points:

(i) Elliptic elements have a fixed point in $\mathbb{H}^n_C$;
(ii) parabolic elements have a single fixed point on the boundary of $\mathbb{H}^n_C$;
(iii) loxodromic elements have exactly two fixed points on the boundary of $\mathbb{H}^n_C$;

This exhausts all possibilities, see [3] for details.

**Definition 1.2.** Set $\mathbb{H}^n_C = \mathbb{H}^n_C \cup \partial \mathbb{H}^n_C$, and let $G$ be a discrete subgroup of $PU(1,n)$. The region of discontinuity of $G$ in $\mathbb{H}^n_C$ is the set $\Omega = \Omega(G)$ of all points in $\mathbb{H}^n_C$ which have a neighborhood that intersects only finitely many copies of its $G$-orbit.

Since $\mathbb{H}^n_C$ with the Bergman metric is a connected, metric space where each closed ball is compact and $PU(1,n)$ acts by isometries, the Arzelà-Ascoli theorem yields [9]:

**Theorem 1.3.** Let $G$ be a subgroup of $PU(1,n)$. The following three conditions are equivalent:

(i) The subgroup $G \subset PU(1,n)$ is discrete.
(ii) The region of discontinuity of $G$ in $\mathbb{H}^n_C$ is all of $\mathbb{H}^n_C$.
(iii) The region of discontinuity of $G$ in $\mathbb{H}^n_C$ is non-empty.

The following characterization of finite groups will be useful later, see [3] [8].

**Proposition 1.4.** Let $G \subset PU(1,n)$ be a discrete group, then $G$ is finite if and only if every element $\gamma \in G$ has finite order $o(\gamma)$.

2. Quasi-projective maps and equicontinuity

Let us construct a “completion” of the Lie group $PSL(n,\mathbb{C})$ which is known as the space of quasi-projective maps. These were introduced by Furstenberg and they are studied in [1].

Let $\tilde{M} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a non-zero linear transformation which is not necessarily invertible. Let $Ker(\tilde{M})$ be its kernel and let $Ker(M)$ denote its projectivization. That is, $Ker(M) := [Ker(\tilde{M}) \setminus \{0\}]_n$ where $[\cdot]_n$ is the projection $\mathbb{C}^{n+1} \to \mathbb{P}^n_C$. Then the quasi-projective transformation induced by $\tilde{M}$ is the map $M : \mathbb{P}^n_C \setminus Ker(M) \to \mathbb{P}^n_C$ given by:

$$M([v]) = [M(v)]_n;$$
Notice that \(|\gamma_{m}|\) is independent of the choice of lift in \(SL(n, \mathbb{C})\), and that \(|\gamma_{m}|^{-1}\gamma_{m}\) is again a lift of \(\gamma_{m}\). Since every bounded sequence in \(\mathbb{C}\) has a convergent subsequence we deduce that there is a subsequence of \((|\gamma_{m}|^{-1}\gamma_{m})\), still denoted by \((|\gamma_{m}|^{-1}\tilde{\gamma}_{m})\), and a non-zero \((n \times n)\)-matrix \(\tilde{\gamma} = (\tilde{\gamma}_{ij}) \in \text{Gr}(n, \mathbb{C})\), such that:

\[
|\gamma_{m}|^{-1}\gamma_{ij} \xrightarrow{m \to \infty} \tilde{\gamma}_{ij}.
\]

This implies that \(|\gamma_{m}|^{-1}\tilde{\gamma}_{m} \xrightarrow{m \to \infty} \tilde{\gamma}\) uniformly on compact sets of \(\mathbb{C}^{n}\), regarded as linear transformations.

Now let \(K \subset \mathbb{P}_{\mathbb{C}}^{n} - \text{Ker}(\gamma)\) be a compact set, so \(\hat{K} = \{|k|^{-1}k : [k]_{n} \in K\}\) is a compact set which satisfies \([\hat{K}]_{n} = K\). Thus

\[
(2.1) \quad |\gamma_{m}|^{-1}\tilde{\gamma}_{m} \xrightarrow{m \to \infty} \tilde{\gamma}\text{ uniformly on }\hat{K}.
\]

From this equation and the facts \(|\gamma_{m}|^{-1}\gamma_{m}|_{n} = \gamma_{m}\) and \([\hat{K}]_{n} = K\), we conclude that:

\[
\gamma_{m} \xrightarrow{m \to \infty} \gamma\text{ uniformly on }K,
\]

where \(\gamma = [\hat{\gamma}]_{n}\).
In what follows we will say that the sequence \((\gamma_m) \subset PSL(n, \mathbb{C})\) to \(\gamma \in QP(n, \mathbb{C})\) in the sense of quasi-projective transformations if \(\gamma_m \xrightarrow{m \to \infty} \gamma\) uniformly on compact sets of \(\mathbb{P}^n_{\mathbb{C}} \setminus Ker(\gamma)\).

We now recall:

**Definition 2.2.** The *equicontinuity region* for a family \(G\) of endomorphisms of \(\mathbb{P}^n_{\mathbb{C}}\), denoted \(Eq(G)\), is defined to be the set of points \(z \in \mathbb{P}^n_{\mathbb{C}}\) for which there is an open neighborhood \(U\) of \(z\) such that \(G|_U\) is a normal family. (Where normal family means that every sequence of distinct elements has a subsequence which converges uniformly on compact sets.)

**Proposition 2.3.** Let \((\gamma_m) \subset PSL(n, \mathbb{C})\) be a sequence which converges to \(\gamma \in QP(n, \mathbb{C})\), with \(Ker(\gamma)\) being a hyperplane. Let \(p \in Ker(\gamma) \setminus Im(\gamma)\), let \(U\) be a neighborhood of \(p\) and \(\ell\) a line such that \(Ker(\gamma) \cap \ell = \{p\}\). Then there is a subsequence of \((\gamma_m)\), still denoted \((\gamma_m)\), and a line \(\ell_p\), such that for every open neighborhood \(W\) of \(p\) with compact closure in \(U\), the set of cluster points of \(\{\gamma_m(W \cap \ell)\}\) is \(\ell_p\).

**Proof.** Let \(Gr_2(\mathbb{C}^{n+1})\) be the grassmannian of complex 2-planes in \(\mathbb{C}^{n+1}\). Since \(Gr_2(\mathbb{C}^{n+1})\) is compact, there is as subsequence of \((\gamma_m)\), still denoted \((\gamma_m)\), and a line \(\ell_p\) such that \(\gamma_m(\ell) \xrightarrow{m \to \infty} \ell_p\). On the other hand, since the sequence \((\gamma_m)\) converges to \(\gamma\) and \(\ell \cap Ker(\gamma) = \{p\}\) we conclude that \(\ell_p \cap Im(\gamma) = Im(\gamma)\) is a point, say \(q\). Let \(x \in \ell_p \setminus \{q\}\), then there is a sequence \((y_m) \subset \ell\) such that \((y_m)\) is convergent and \(\gamma_m(y_m) \xrightarrow{m \to \infty} x\). This implies that the limit point of \((y_m)\) lies in \(Ker(\gamma)\), thence such a point is \(p\). In short \(y_m \xrightarrow{m \to \infty} p\) and \(\gamma_m(y_m) \xrightarrow{m \to \infty} x\). \(\square\)

As an immediate consequence one has:

**Corollary 2.4.** Let \((\gamma_m) \subset PSL(n, \mathbb{C})\) be a sequence which converges to \(\gamma \in QP(n, \mathbb{C})\). If \(Ker(\gamma)\) is a hyperplane, then the equicontinuity set is:

\[Eq(\{\gamma_m : m \in \mathbb{N}\}) = \mathbb{P}^n_{\mathbb{C}} \setminus Ker(\gamma)\]

3. The limit set according to Chen and Greenberg

The main result of this section is Lemma 3.2 which is useful for proving properties about subgroups of \(PU(1, n)\). This lemma is used in the following Section 4 for proving Theorem 1. We use 3.2 also in this section, to give direct proofs of several important results from 3 used in the sequel.

**Definition 3.1.** Let \(G\) be a discrete subgroup of \(\text{Iso}(\mathbb{H}^n)\). The *limit set of \(G\) in the sense of Chen-Greenberg*, denoted \(\Lambda_{CG}(G)\) or simply \(\Lambda_{CG}\), is the set of accumulation points in \(\mathbb{H}^n_{\mathbb{C}}\) of orbits of points in \(\mathbb{H}^n_{\mathbb{C}}\).

**Lemma 3.2.** Let \(G \subset PU(1, n)\) be a discrete group, \((\gamma_m)_{m \in \mathbb{N}} \subset G\) a sequence of distinct elements and \(\gamma \in QP(n, \mathbb{C})\) such that \((\gamma_m)\) converges to \(\gamma\) in the sense of quasi-projective transformations, then:
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(i) The image $\text{Im}(\gamma)$ is a point in $\partial \mathbb{H}_C^n$.
(ii) The kernel $\text{Ker}(\gamma)$ is a hyperplane tangent to $\partial \mathbb{H}_C^n$.
(iii) One has $\text{Ker}(\gamma) \cap \partial \mathbb{H}_C^n \in \Lambda_{CG}(G)$.

Proof. Let us prove by contradiction (i). Since $\gamma$ is holomorphic the set $\gamma(\mathbb{H}_C^n \setminus \text{Ker}(\gamma))$ is an open set in the projective subspace $\text{Im}(\gamma)$. On the other hand, by 1.3 the set $\gamma(\mathbb{H}_C^n \setminus \text{Ker}(\gamma))$ is contained in $\text{Im}(\gamma) \cap \partial \mathbb{H}_C^n$, which has empty interior (whenever $\text{Im}(\gamma)$ is not a point), which is a contradiction. In the rest of the proof the unique element in $\text{Im}(\gamma)$ will be denoted by $q$.

Now let us prove (ii). Assume that $\text{Ker}(\gamma) \cap \mathbb{H}_C^n$ is not empty. Thus we can choose $p \in \text{Ker}(\gamma) \setminus \text{Im}(\gamma) \cap \mathbb{H}_C^n$. Applying Proposition 2.3 to $(\gamma_m)$, $\gamma$, $p$, $\mathbb{H}_C^n$ it followed that there is line $\ell_p$ contained in $\mathbb{H}_C^n$, which is a contradiction, since $\mathbb{H}_C^n$ does not contains complex lines. Therefore $\text{Ker}(\gamma) \cap \mathbb{H}_C^n = \emptyset$.

Assume now that $\text{Ker}(\gamma) \cap \mathbb{H}_C^n = \emptyset$. From this and (i) of the present lemma we conclude that $\gamma_m$ converges uniformly to the constant function $q$ on $\mathbb{H}_C^n$. Let $x \in \mathbb{H}_C^n$ and $U$ be a neighborhood of $p$ such that $U \cap \mathbb{H}_C^n \subset (\mathbb{H}_C^n \setminus \{x\})$. The uniform convergence implies that there is a natural number $n_0$ such that $\gamma_m(\mathbb{H}_C^n) \subset U \cap \mathbb{H}_C^n \subset \mathbb{H}_C^n \setminus \{x\}$ for each $m > n_0$. This is a contradiction since each $\gamma_m$ is a homeomorphism.

Let us prove (iii). By 2.1 we can assume that there is $\tau \in \text{QP}(n)$ such that $(\gamma_m^{-1})$ converges to $\tau$ in the sense of quasi-projective transformations. Thus by (i) of the present lemma we have that $\text{Im}(\tau)$ is a point $p$ in $\Lambda_{Kul}(G)$. We claim that $\{p\} = \text{Im}(\tau) = \text{Ker}(\gamma) \cap \partial \mathbb{H}_C^n$. Assume this does not happen; let $x \in \mathbb{H}_C^n$, then $\gamma_m^{-1}(x) \xrightarrow[m \to \infty]{} p$, thus $\{\gamma_m^{-1}(x) : m \in \mathbb{N}\} \cup \{p\}$ is a compact set which lies in $\mathbb{P}_C^n \setminus \text{Ker}(\gamma)$, thus $x = \gamma_m(\gamma_m^{-1}(x)) \xrightarrow[m \to \infty]{} q$. Which is a contradiction.

□

From the proof of the previous result one gets:

Corollary 3.3. Let $G \subset PU(1, n)$ be a discrete group, $(\gamma_m)_{m \in \mathbb{N}} \subset G$ a sequence of distinct elements and $\gamma \in \text{QP}(n\mathbb{C})$ such that $(\gamma_m)$ converges to $\gamma$ in the sense of quasi-projective transformations. Then there are a subsequence of $(\gamma_m)$, still denoted $(\gamma_m)$, and an element $\tau \in \text{QP}(n, \mathbb{C})$ such that:

(i) The sequence $(\gamma_m^{-1})$ converges to $\tau$ in the sense of quasi-projective transformations;
(ii) The image $\text{Im}(\tau)$ is a point in $\partial \mathbb{H}_C^n$ and $\text{Ker}(\tau)$ is a hyperplane tangent to $\partial \mathbb{H}_C^n$.
(iii) One has $\text{Im}(\tau) = \text{Ker}(\gamma) \cap \partial \mathbb{H}_C^n$;
(iv) Also $\text{Im}(\gamma) = \text{Ker}(\tau) \cap \partial \mathbb{H}_C^n$.

Theorem 3.4. Let $G$ be as above. Then the limit set $\Lambda_{CG}(G)$ is independent of the choice of orbit.
Proof. Let \( x, y \in \mathbb{H}_n^\mathbb{C} \), and let \( z \) be a cluster point of \( G y \). Then there exists a sequence \( (g_m) \subset \hat{G} \) such that \( g_m(y) \) converges to \( z \). It follows from lemma \( \ref{lem:cluster-point} \) that \( z \) is also a cluster point of \( (g_m(x)) \), which ends the proof. \( \square \)

It is clear from the definitions that the limit set \( \Lambda_{CG}(G) \) is a closed, \( G \)-invariant set, and it is empty if and only if \( G \) is finite (since every sequence in a compact set contains convergent subsequences). Moreover one has:

**Theorem 3.5.** Let \( G \) be a discrete group such that \( \Lambda_{CG}(G) \) has more than two points, then it has infinitely many points.

**Proof.** Assume that \( \Lambda_{CG}(G) \) is finite with at least 3 points. Then

\[
\hat{G} = \bigcap_{x \in \Lambda_{CG}(G)} \text{Isot}(x, G)
\]

is a normal subgroup of \( G \) with finite index. Moreover, by \( \ref{lem:finite-index} \) each element in \( PU(1, n) \) has at most 2 fixed points in \( \partial \mathbb{H}_n^\mathbb{C} \). Hence \( \hat{G} \) is trivial and therefore \( G \) is finite, which is a contradiction. \( \square \)

**Definition 3.6.** The group \( G \) is elementary if \( \Lambda_{CG}(G) \) has at most two points.

**Theorem 3.7.** If \( G \subset PU(1, n) \) is a non-elementary discrete group, then \( \Lambda_{CG}(G) \) is the unique closed minimal set.

**Proof.** Let \( S \) be a closed invariant set and \( z \in S \). By \( \ref{lem:cluster-point} \) we know there is an accumulation point \( \tilde{z} \in S \) of \( Gz \) such that \( \tilde{z} \in \Lambda_{CG}(G) \). Let \( y \in \Lambda_{CG}(G) \), then there is a sequence \( (g_m) \subset G \) and a point \( p \in \mathbb{H}_n^\mathbb{C} \) such that \( g_m(p) \) converges to \( y \). By Lemma \( \ref{lem:cluster-point} \) there are points \( p, q \in \partial \mathbb{H}_n^\mathbb{C} \) such that we can assume that \( g_m \) converges uniformly to \( y \) in compact sets of \( \mathbb{H}_n^\mathbb{C} \setminus \{p\} \). Now, it is well know (see Theorem 3.1 in [6]) that there is a transformation \( g \in G \) such that \( g(p) \neq p \). Hence we can assume that \( \tilde{z} \neq q \), so we conclude that \( g_m(\tilde{z}) \) converges to \( y \). \( \square \)

**Remark 3.8.** Notice that the previous result implies that if \( G \) non-elementary, then \( \Lambda_{CG}(G) \) is a nowhere dense perfect set. In other words \( \Lambda_{CG}(G) \) has empty interior and every orbit in \( \Lambda_{CG}(G) \) is dense in \( \Lambda_{CG}(G) \).

4. ON THE EQUICONTINUITY REGION

The sphere \( \partial \mathbb{H}_n^\mathbb{C} \) has real codimension 1 in \( \mathbb{P}_n^\mathbb{C} \) and its tangent bundle contains a maximal subbundle which is a complex subbundle of \( T\mathbb{P}_n^\mathbb{C}|_{\partial \mathbb{H}_n^\mathbb{C}} \). In fact this subbundle defines the canonical contact structure on the sphere. We let \( \mathcal{C}(\partial \mathbb{H}_n^\mathbb{C}) \) be the union of all complex projective hyperplanes tangent to \( \partial \mathbb{H}_n^\mathbb{C} \). Given a discrete subgroup \( G \subset PU(1, n) \) set:

\[
\mathcal{C}(G) := \mathcal{C}(\partial \mathbb{H}_n^\mathbb{C})|_{\Lambda_{CG}(G)} = \bigcup_{p \in \Lambda_{CG}(G)} \mathcal{H}_p,
\]
where $\mathcal{H}_p$ denotes the hyperplane tangent to $\partial \mathbb{H}^n_\mathbb{C}$ at a point $p \in \Lambda_{CG}(G)$. It is clear that $\mathcal{C}(G)$ is a closed $G$-invariant subset of $\mathbb{P}^n_\mathbb{C}$.

The following lemma proves part of Theorem 1.

**Lemma 4.1.** The equicontinuity region of $G$ is:

$$\text{Eq}(G) = \mathbb{P}^n_\mathbb{C} \setminus \mathcal{C}(G).$$

**Proof.** Since $G$ is infinite and discrete, it contains at least a parabolic or a loxodromic element $\gamma$. Let $x_0$ be a fixed point of $\gamma$. By Corollary 3.3 we can ensure that the hyperplane $\mathcal{H}_{x_0}$ tangent to $\partial \mathbb{H}^n_\mathbb{C}$ at $x_0$ is contained in $\mathbb{P}^n_\mathbb{C} \setminus \text{Eq}(G)$. On the other hand by Theorem 3.7, the closure of the orbit of $x_0$ is $\Lambda_{CG}(G)$. Thus $\text{Eq}(G) \subset \mathbb{P}^n_\mathbb{C} \setminus \mathcal{C}(G)$. Let us now prove $\mathbb{P}^n_\mathbb{C} \setminus \mathcal{C}(G) \subset \text{Eq}(G)$. Let $p \in \mathbb{P}^n_\mathbb{C} \setminus \mathcal{C}(G)$ and $(\gamma_m)_{m \in \mathbb{N}}$ a sequence of distinct elements. By Lemma 3.2 there are points $p, q \in \Lambda_{CG}(G)$ and a subsequence of $(\gamma_m)_{m \in \mathbb{N}}$, still denoted $(\gamma_m)_{m \in \mathbb{N}}$, such that $\gamma_m \to q$ uniformly on compact sets of $\mathbb{P}^n_\mathbb{C} \setminus \mathcal{H}_p$, where $\mathcal{H}_p$ denotes the hyperplane tangent to $\partial \mathbb{H}^n_\mathbb{C}$ at $p$. This completes the proof. □

The following result completes the proof of Theorem 1.

**Corollary 4.2.** Let $G \subset \text{PU}(1, n)$ be a discrete subgroup. Then $G$ acts discontinuously on $\text{Eq}(G)$ and, moreover, for every compact set $K \subset \text{Eq}(G)$ the cluster points of the orbit $GK$ lie in $\Lambda_{CG}(G)$.

**Proof.** Assume on the contrary that $G$ does not act discontinuously on $\text{Eq}(\Gamma)$. Then there is a compact set $K$ and a sequence of distinct elements $(\gamma_m) \subset G$, such that $\gamma_m(K) \cap K \neq \emptyset$. By Proposition 2.1 there is a subsequence of $(\gamma_m)$, still denoted $(\gamma_m)$, and $\gamma \in \text{QP}(n, \mathbb{C})$, such that $(\gamma_m) \text{ converges to} \gamma$ in the sense of quasi-projective transformations. Moreover, by Lemma 3.2 $\text{Im}(\gamma)$ is a point $p$ in $\partial \mathbb{H}^n_\mathbb{C}$ and $\text{Ker}(\gamma)$ is a hyperplane tangent to $\partial \mathbb{H}^n_\mathbb{C}$. Therefore there is a neighborhood $U$ of $p$ disjoint from $K$ and a natural number $n_0$ such that $\gamma_m(K) \subset U$ for all $m > n_0$. This implies $\gamma_m(K) \cap K = \emptyset$, which is a contradiction. Therefore $\Gamma$ acts discontinuously on $\text{Eq}(G)$.

From the previous argument we deduce also that for every compact set $K \subset \text{Eq}(G)$ the cluster points of $GK$ lie in $\Lambda_{CG}(G)$. □

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