On the iterative solution of the gap equation in the Nambu-Jona-Lasinio model

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Abstract. In this work we revise the standard iterative procedure to find the solution of the gap equation in the Nambu–Jona-Lasinio model within the most popular regularization schemes available in literature in the super-strong coupling regime. We observe that whereas for the hard cut-off regularization schemes, the procedure smoothly converges to the physically relevant solution, Pauli-Villars and Proper-Time regularization schemes become chaotic in the sense of discrete dynamical systems. We call for the need of an appropriate interpretation of the non-convergence of this procedure to the solution of the gap equation.

1. Introduction
The Nambu-Jona-Lasinio (NJL) model [1, 2] was one of the earliest attempts to describe strong interactions among protons and neutrons. Spontaneous breaking of chiral symmetry is presented in the model through an analogy with the phenomenon of superconductivity in condensed matter physics. The model has a single coupling constant for two kinds of 4-Fermi interactions and is non-renormalizable, which demands for a regulator. Nevertheless, up to date, the model continues being used as an effective description of quarks dynamics either as originally proposed, where fermion fields are now regarded as quark fields, or with slightly modifications [3, 4, 5] to account for confinement. NJL model has been widely used to sketch the quantum chromodynamics (QCD) phase diagram (see, for instance, Ref. [6] for a review) and its magnetized extension [7] from a complementary point of view of lattice [8] and other quantum field theoretical approaches [9, 10].

Several regularization schemes have been used in literature within the context of NJL model. Perhaps the simplest regulator one can think of is to establish a hard cut-off in the four-momentum integrals, or over the spatial momentum integrals, which thus allows a straightforward incorporation of temperature and density effects within the Matsubara formalism [11]. Pauli-Villars regularization has also been used in NJL studies due to the advantages of the scheme. Nevertheless, external magnetic field effects are best included through the Schwinger Proper-Time representation of the fermion propagator. This observation has been exploited to address the problem of magnetic catalysis [12] and inverse magnetic catalysis observed in lattice simulations [13, 14] and confirmed by several approaches [7, 8, 15]. Under this environment, the effective coupling of the model can reach very large values, and therefore,
exploring the validity of these regularization schemes in this super-strong coupling regime is a 
natural question to address.

In this contribution, we revise the commonly used iterative strategy to solve the gap equation 
for the NJL model in vacuum under each of the above mentioned regulation schemes. For this 
purpose, we have organized the remaining of the manuscript as follows: In Sect. 2 we present 
the model and the gap equation in each regularization scheme. Section 3 is devoted to illustrate 
the iterative procedure to self consistently find the dynamically generated mass as a function of 
the parameters of the model. A summary and outlook is presented in Sect. 4.

2. NJL model and the gap equation

The Lagrangian for the NJL model proposed in Ref. [1, 2] has the form

\[ \mathcal{L} = \bar{\psi} (i \partial - m_q) \psi + G \left\{ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \vec{\tau} \psi)^2 \right\}, \]

where in the modern applications, \( \psi \) is the quark field, \( m_q \) is the current quark mass, \( \vec{\tau} \) are the 
Pauli matrices acting on isospin space and \( G \) is the coupling constant. Within the Hartree-Fock 
approximation, the gap equation is written as

\[ m = m_q - 2G \langle \bar{\psi} \psi \rangle, \]

where \( m \) is the dynamically generated mass and \( -\langle \bar{\psi} \psi \rangle \) is the chiral condensate, defined as

\[ -\langle \bar{\psi} \psi \rangle = \int \frac{d^4k}{(2\pi)^4} \text{Tr}[iS(k)], \]

with the dressed quark propagator

\[ S(k) = \frac{(k + m)}{k^2 - m^2}. \]

It should be noted that the dynamical mass predicted by the NJL model is momentum 
independent; in vacuum, the dynamical mass depends only on the coupling constant strength 
once a regulator has been appropriately chosen. As mentioned before, the NJL model is non-
renormalizable and thus the integrals in the gap equation Eq. (2), require a regularization. For 
the purposes of this work, we apply 3D-cut–off (3D), 4D-cut–off (4D), Pauli-Villars (PV) and 
Proper-Time (PT) regularizations. Within each prescription, the gap equation acquires the form

\[ m = \frac{GN_f N_c}{\pi^2} m \left[ \Lambda_{3D}^2 \sqrt{\Lambda_{3D}^2 + m^2} - m^2 \ln \left( \frac{\Lambda_{3D} + \sqrt{\Lambda_{3D}^2 + m^2}}{m} \right) \right], \]

\[ m = \frac{GN_f N_c}{2\pi^2} m \left[ \Lambda_{4D}^2 - m^2 \ln \left( \frac{\Lambda_{4D}^2 + m^2}{m^2} \right) \right], \]

\[ m = \frac{GN_f N_c}{2\pi^2} m \left[ \Lambda_{PV}^2 - m^2 + m^2 \ln \left( \frac{m^2}{\Lambda_{PV}^2} \right) \right], \]

\[ m = \frac{GN_f N_c}{2\pi^2} m \int_{\Lambda_{PT}}^{\infty} ds e^{-m^2 s/s^2}. \]

In the above expressions, \( N_f \) and \( N_c \) are the number of families we are considering and the 
number of colors, respectively. Finally, for the purposes of the analysis in this contribution, we 
transform the gap equations Eqs. (5), (6), (7) and (8) into their dimensionless form
\[ M = G'M \left\{ \sqrt{1 + M^2} - M^2 \ln \left[ \frac{1 + \sqrt{1 + M^2}}{M^2} \right] \right\}, \quad (3D \text{ cut-off}) \quad (9) \]

\[ M = G'M \left\{ 1 - M^2 \ln \left[ 1 + M^{-2} \right] \right\}, \quad (4D \text{ cut-off}) \quad (10) \]

\[ M = G'M \left\{ 1 - M^2 + M^2 \ln \left[ M^2 \right] \right\}, \quad \text{(Pauli-Villars)} \quad (11) \]

\[ M = G'M \int_{1}^{\infty} d\tau e^{-M^2\tau}, \quad \text{(Proper-Time)} \quad (12) \]

where we performed the substitution \( M = m/\Lambda \) for 3D, 4D and PV regularizations and \( M^2 = m^2\Lambda \) for PT regularization, whereas \( G' \) absorbs numerical factors to render all equations in similar footing.

3. Solving with iterations

To introduce the iterations to the gap equations, we define.

\[ y = m, \]
\[ y = G'f(m), \quad (13) \]

where \( f(m) \) is the RHS of any of the gap equations in Eqs. (5), (6), (7) and (8). Plotting these two equations for a fixed value \( G' = 1 \), for example, we get Fig. 1. Next, we perform an iteration of these two equations, first using an starting value \( m_0 \) then plugging into \( f(m) \) to get \( m_1 = f(m_0) \) and so on. Graphically, this iteration procedure can be depicted by a cobweb plot, in the language of discrete dynamical systems [16], as shown in Fig. 2.

![Figure 1](image)

Figure 1. Plot of the system of Eqs. (13), were we have used the RHS of Eq. (8) in place of \( G'f(m) \). The solution of the gap equation is the intersection of the two curves, \( M_s \).

For each value of the coupling constant \( G' \), the curves in Fig. 1 change as well as their intersection and thus, the iteration procedure converges to a different point, and the generated mass for that particular value of \( G' \) changes as well. Below we discuss results corresponding to each of the gap Eqs. (5), (6), (7) and (8).

In Fig. 3 we show the last 200 steps of an iterative procedure with a total of 500 steps for each value of \( G' \). We observe that for \( G' < 1 \), the procedure converges to \( M_s = 0 \), which is consistent...
Figure 2. Graphic representation of the iteration process (coweb plot). In this case, we can see that the sequence of iterations tends to converge to the intersection of the curves.

Figure 3. Iterations plot for Eq. (9). In the vertical axes, the point at where the iterations converges. At the horizontal axes the various values of the coupling constant $G'$. 

with the known feature of NJL model: Unless the coupling exceeds a critical value, the gap equation would have only the trivial solution $m = 0$. The critical value is therefore $G_c' = 1$. It is said that at this critical value, the gap equation bifurcates away from the trivial solution, because above this critical value, the iterative procedure always converges to $M_s = m > 0$, that grows smoothly as $G'$ increases. Such a behavior is expected and, in fact, corroborated in Fig. 4, where a similar procedure was performed on the gap Eq. (10). We emphasize that the behavior is independent of the initial choice $m_0$ and settles the best with a larger number of iteration steps. The critical value remains the same, though the growth of $m$ with $G'$ is slightly different.

From Figs. 5 and 6, corresponding to the PV (11) and PT (12) regularizations, we observe that iterations fail to converge to a fixed value $M_s$ after a second critical coupling is reached at.
Figure 4. Iterations plot for Eq. (10). In the vertical axes, the point at where the iterations converge. At the horizontal axes the various values of the coupling constant $G'$.  

Figure 5. Iterations plot for Eq. (11). In the vertical axes, the point at where the iterations converge. In this case for $G' \approx 2.8$ and above, the iterations stop converging and start to bounce between two values of $M_n$. For $G' > 3.6$ the iterations go into a chaotic regime. At the horizontal axes the various values of the coupling constant $G'$.  

This happens for $G' \approx 2.8$ for PV and $G' \approx 3.9$ for PT. In both cases, a bifurcation develops, meaning that the iterations do not converge and they keep bouncing between two values, as shown in Fig. 7. Furthermore, as the coupling gets stronger, each branch bifurcates once again and so on until the iterations become chaotic. The iteration diagram resembles the *logistic map* of discrete dynamical systems [17].

4. Summary and Outlook

In this contribution we have confirmed that applying an iteration scheme to solve the gap equation in the NJL model is still an effective method to explore the dependence of the generated
Figure 6. Iterations plot for Eq. (12). In the vertical axes, the point at which the iterations converge. In this case for $G' \approx 3.9$ and above the iterations stop converging and start to oscillate between two values of $M_n$. Then for $G' > 5.75$ the iterations go into a chaotic regime. At the horizontal axes the various values of the coupling constant $G'$.

Figure 7. Example of iterations that do not converge and start bouncing between four values of $M_n$ at fixed $G' = 3.51$ in PV regularization. In this case, the bouncing among the four values of $M_n$ correspond to the second bifurcation in Fig. 5.

mass on the coupling constant and other relevant parameters, at least in a broad range of values of $G'$ which contains the typical values used to describe hadron physics phenomenology. For 3D and 4D cut-off regularization schemes, the iterative succession converges quickly for every value of the coupling constant, even at the super-strong coupling regime. On the other hand, applying an iterative procedure to solve Eqs. (7) or (8) can result in ill solutions for super-strong couplings. The procedure fails to converge after a second critical value for the coupling constant is reached at: iterative steps bounce between two values, four values for larger coupling and so on, till the behavior becomes completely chaotic, as shown in Figs. 5 and 6. This observation implies
that special care should be taken when choosing the coupling in any of these two regularization schemes, which are preferred, for instance, to insert the influence of an external magnetic that could render the coupling super-strong.

Furthermore, unless a neat physical interpretation of the partial iterative results can be given, as the one suggested in Ref. [18] as a resummation of bubble insertions into the self-energy, the chaotic behavior might hint a failure of the procedure. This and other considerations are under scrutiny and shall be presented elsewhere [19].

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