COHOMOLOGY OF UNIPOTENT GROUP SCHEMES

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Abstract. We investigate the rational cohomology algebra of various unipotent group schemes defined over an algebraically closed field $k$ of characteristic $p > 0$. For the $r$-th Frobenius kernel $U(r)$ of many unipotent algebraic groups $U$, we construct an algebra $S^*(U(r))$ given by explicit generators and relations together with a map $\eta_{U,r} : S^*(U(r)) \to H^*(U(r), k)$ of graded $k$-algebras which serves as a good approximate model. Our work is motivated by the challenge of understanding the stabilization with respect to $r$ of the cohomology $H^*(U(r), k)$ and the natural map $H^*(U, k) \to \lim_{\leftarrow r} H^*(U(r), k)$.

0. Introduction

The rational cohomology $H^*(G, k)$ of a (connected, reduced) simple algebraic group $G$ over a field $k$ of characteristic $p > 0$ is trivial (cf. [6]). On the other hand, the rational cohomology $H^*(\mathbb{G}_a, k)$ of the additive group $\mathbb{G}_a$ can be identified with the exterior algebra on a countable vector space in degree 1 tensor a symmetric algebra on the (Bockstein applied to) the same vector space placed in degree 2 if $p \neq 2$ (see [8]). This suggests that unipotent algebraic groups have interesting cohomology. Indeed, our original motivation in pursuing calculations of the cohomology of unipotent algebraic groups was to relate this cohomology to our recent theory of support varieties for such groups (see [7], [8]). The computations provided in this paper show that rational cohomology $H^*(G, k)$ is inadequate for a theory of support varieties not only for $G$ simple but also for unipotent algebraic groups.

The cohomology of groups has played an important role in various aspects of topology, number theory, and algebraic geometry. Our initial interest was generated by the foundational work of D. Quillen [17], [18] and the connections with algebraic K-theory as also developed by Quillen [19]. Subsequently, thanks to the work of many mathematicians beginning with J. Alperin - L. Evens [2], J. Carlson [4], and E. Cline - B. Parshall - L. Scott (e.g., [5]), cohomology of groups has evolved into a useful tool ("support varieties") for the study of representations of finite group schemes. We would be amiss not to mention work of the author with B. Parshall (e.g., [9]), A. Suslin (e.g., [11]), and J. Pevtsova (e.g., [10]). Despite the efforts of many, there is a dearth of explicit computations of the cohomology of infinitesimal group schemes of height $> 1$.

In the papers [21], [22], Andrei Suslin, Christopher Bendel, and the author provide general qualitative information about the rational cohomology $H^*(G(r), k)$ of the $r$-th Frobenius kernel $G(r) = \ker \{ F^r : G \to G^{(r)} \}$ of an affine algebraic

Date: August 22, 2018.
2010 Mathematics Subject Classification. 20G05, 20C20, 20G10.
Key words and phrases. rational cohomology, unipotent algebraic groups.
* partially supported by the NSA and the Simons Foundation.
group $G$. However, this information gives little information about the restriction map $H^*(G_{(r+1)}, k) \rightarrow H^*(G_r, k)$ induced by $G_r \subset G_{(r+1)}$ or the relationship to $H^*(G, k)$ or specific calculations for a given cohomological degree.

In this paper, we build an “explicit model” $\eta_{U_J, r} : \overline{S}((U_J)_{(r)}) \rightarrow H^*((U_J)_{(r)}, k)$ for the cohomology of the unipotent radical $U_J$ of a parabolic subgroup $P_J$ of a simple algebraic group $G$ over $k$ of classical type (and more generally for any term $\Gamma_i(U_J)$ of the descending central series of $U_J$). Our first approach to this model is given in Section 3.13 relying upon the construction of the universal classes in $H^*(GL_N(r), k)$ by A. Suslin and the author in 11. One can view this constructions as establishing the existence of specific cohomology classes in $H^*((U_r, k))$.

This first approach is not very satisfying for several reasons: the construction is neither explicit nor intrinsic, the model constructed is only for the reduced algebra structure, and the model does not appear to be suitable for computations. In Section 3, we introduce a more computational approach in the special case of the Heisenberg group $U_3$. The construction of $\eta_{U_3, r} : \overline{S}((U_3)_{(r)}) \rightarrow H^*((U_3)_{(r)}, k)$ is given and many good properties are established; for example, Theorem 2.17 states that the associated graded map of $\eta_{U_3, r}$ is an isomorphism modulo squares.

Unfortunately, even in this “simplest non-trivial example” of $U = U_3$, our computations do not fully determine $H^*((U_3)_{(r)}, k)$ modulo nilpotents. Much of the challenge of our computations as well as achieving better computations in the future involves the detection of non-nilpotent classes in $H^*(U_r, k)$ and $H^*(U, k)$. For example, we show in Theorem 1.0 that $H^*(U, k) \rightarrow \lim_{\leftarrow r} H^*(U_r, k)$ is an isomorphism for a wide class of unipotent groups $U$, but we do not know even for the special case of $U = U_3$ whether or not there exist “mock nilpotent classes” $\zeta \in H^*(U, k)$ which are not nilpotent but whose restriction $\zeta_r \in H^*(U_r, k)$ is nilpotent for each $r > 0$. An analogous challenge prevents us from proving the surjectivity of $\eta_{U_3, r} : \overline{S}((U_r, k)) \rightarrow H^*(U_r, k)$.

Section 3 is dedicated to extending the techniques and results employed for $U_3$ in the previous section. Applications of the Lyndon-Hochschild-Serre spectral sequence enable an inductive approach, one for which we must consider the quotient groups $U_J/\Gamma_i(U_J)$, where $\Gamma_i(U_J)$ is the $i$-th stage of the descending central series for $U_J \subset G$, $J \subset \Pi$ is a subset of the simple roots determined by a choice of Borel subgroup $B \subset G$. In order to specify our model $\eta_{U_J, r} : S((U_J)_{(r)}) \rightarrow H^*((U_J)_{(r)}, k)$, we also utilize the Andersen-Jantzen spectral sequence of [1]. Most of the properties proved for $\eta_{U_3, r} : \overline{S}((U_3)_{(r)}) \rightarrow H^*((U_3)_{(r)}, k)$ are extended to $\eta_{U_J, r} : \overline{S}((U_J)_{(r)}) \rightarrow H^*((U_J)_{(r)}, k)$ in Theorems 3.9 and 3.10.

Finally, in Section 4 we investigate the restriction maps in cohomology induced by the natural embeddings $U_r \subset U_{(r+1)} \subset U_{(r+2)} \subset \cdots$. As we show, most of the cohomology exhibited in previous sections for $H^*((U_J)_{(r)}, k)$ does not “survive” in the (inverse) limit $\lim_{\leftarrow s \geq r} H^*((U_J)_{(s)}, k)$. We conclude with the suggestive Theorem 4.1 which states that the natural map $H^*(V, k) \rightarrow \lim_{\leftarrow r} H^*(V_r, k)$, an isomorphism for unipotent groups $V$ with a suitable torus action.

In what follows, $k$ denotes an algebraically closed field of characteristic $p > 2$. For a commutative algebra $A$, we denote by $A_{red}$ the quotient of $A$ by its nilradical (the ideal of all nilpotent elements of $A$). We denote by $H^*(G, k)$ the (Hochschild) cohomology of an affine group scheme $G$ over $k$ and by $H^*(G, k) \subset H^*(G, k)$ the commutative subalgebra of cohomology classes of even degree. We use $V^\#$
to denote the $k$-linear dual of $k$-vector space $V$. Until we consider the effect of increasing $r$ in the final section, we fix an arbitrary positive integer $r$.

We thank Robert Guralnick for helpful discussions.

1. The map $\Omega_{U,r} : k[V_r(U)] \to H^\bullet(U_{(r)}, k)$

In this section, we extend the formulation of the map of $k$-algebras

$$(1.0.1) \quad \phi_{GL_N,r} : S^\bullet(\otimes_{\ell=0}^{p-1-r}(gl_N(r)[2p^\ell-\ell-1])) \to H^\bullet(GL_N(r), k), \quad 0 \leq \ell < r$$
given by A. Suslin and the author in [11] to the Frobenius kernels $U_{(r)}$ of various unipotent subgroups $U \subset GL_N$. In contrast to our subsequent constructions, this extension involves little computation. Throughout this discussion, $r$ will denote an arbitrary positive integer.

Recall that for any linear algebraic group $G$ over $k$, the $r$-th iterate $F^r : G \to G^{(r)}$ of the Frobenius map $F : G \to G^{(1)}$ admits a scheme theoretic kernel $G_{(r)}$ which is an infinitesimal group scheme of height $r$. The coordinate algebra $k[G_{(r)}]$ of $G$ equals the finite dimensional commutative Hopf algebra $k[G/I^{p^r}]$ where $I$ is the maximal ideal at the identity of $G$ and where $I^{p^r}$ is the ideal generated by $\{f^{p^r}, f \in I\}$. A (rational) $G_{(r)}$-module is a comodule for $k[G_{(r)}]$ or, equivalently, a module for $kG_{(r)} \equiv (k[G_{(r)}])^\#$ (the $k$-linear dual of $k[G_{(r)}]$ with its inherited Hopf algebra structure). For $G$ defined over $F_p$, we may view the Frobenius map as an endomorphism of $G$ and $G_{(r)} \subset G$ as the kernel of $F^r : G \to G$.

The universal, $GL_N$-invariant classes $e_{r-\ell} \in H^{2p^\ell-\ell-1}(GL_N, gl_N^{(r-\ell)})$ of [11] and their $\ell$-th Frobenius twist $e_{r-\ell}^{(\ell)} \in H^{2p^\ell-\ell-1}(GL_N, gl_N^{(r)})$ (i.e., pull-back along the $\ell$-th iterate $F^\ell$ of the Frobenius morphism $F : GL_N \to GL_N$) are elements in the rational cohomology of the (reductive) algebraic group $GL_N$. The restriction of $e_{r-\ell}^{(\ell)}$ to $GL_N(r)$,

$$(e_{r-\ell}^{(\ell)})_{(r)} \in H^{2p^\ell-\ell-1}(GL_N(r), gl_N^{(r)}) \simeq H^{2p^\ell-\ell-1}(GL_N(r), k) \otimes gl_N^{(r)},$$
can be identified with a $GL_N$-equivariant map

$$gl_N^{(r)}[2p^\ell-\ell-1] \to H^{2p^\ell-\ell-1}(GL_N(r), k)$$

(vanishing on the dual trace class $T_{r}(r) \in gl_N^{(r)}$), thereby determining the $GL_N$-equivariant map of commutative $k$-algebras (1.0.1). For $\ell < r$, the Frobenius map $F^\ell$ restricts to $F^\ell : GL_N(r) \to GL_N(r)$ and factors as

$$GL_N(r) \to GL_N(r)/GL_N(\ell) \simeq GL_N(r-\ell) \subset GL_N(r).$$

The Frobenius twist $(e_{r-\ell}^{(\ell)})_{(r)}$ can thus be realized as the pull-back along $GL_N(r) \to GL_N(r-\ell)$ of $e_{r-\ell} \in H^{2p^\ell-\ell-1}(GL_N(r-\ell), gl_N^{(r-\ell)})$.

A basic property of $e_{r-\ell}$ is that its restriction to $GL_N(1)$, $(e_{r-\ell})_{(1)} \in H^{2p^\ell-\ell-1}(GL_N(1), k) \otimes gl_N^{(r)}$, is non-zero for any $r \geq 1$. The injectivity of twisting thereby implies that

$$(1.0.2) \quad (e_{r-\ell}^{(\ell)})_{(\ell+1)} \neq 0 \in H^{2p^\ell-\ell-1}(GL_N(\ell+1), k) \otimes gl_N^{(r)},$$

whereas $(e_{r-\ell}^{(\ell)})_{(\ell)} = 0 \in H^{2p^\ell-\ell-1}(GL_N(\ell), k) \otimes gl_N^{(r)}$ because the composition of $GL_N(\ell) \subset GL_N \xrightarrow{F^\ell} GL_N$ is trivial. A second basic property of $e_{r-\ell} \in H^{2p^\ell-\ell-1}(GL_N, gl_N^{(r-\ell)})$ is that its restriction via the “standard inclusion” $GL_{N-1} \subset GL_N$ equals $e_{r-\ell} \in H^{2p^\ell-\ell-1}(GL_{N-1}, gl_N^{(r-\ell)})$. 
Following [21], we use the following notation: we identify $S^*(\bigoplus_{t=0}^{r-1}(\mathfrak{gl}_N^r[2p^r-t-1]))$ with the affine space $A^{rN^2} = \prod_{t=0}^{r-1} M_{n,n}$, identifying $X^{i,j}(\ell) \in \mathfrak{gl}_N^r[2p^r-t-1]$ with the $(i,j)$ coordinate function of the $\ell$-th factor.

For any affine group scheme $G$ over $k$, we use the notation $V_r(G)$ for the affine scheme of $1$-parameter subgroups of $G$ of height $r$ (i.e., homomorphisms $G_{a(r)} \to G$ of group schemes over $k$) with coordinate algebra $k[V_r(G)]$ as in [21].

We state two theorems of Suslin-Friedlander-Bendel.

**Theorem 1.1.** ([21] 5.1) The map $\phi_{GL_{N,r}}$ of (1.0.1) factors as

$$\phi_{GL_{N,r}} = \overline{\phi}_{GL_{N,r}} \circ q : S^*(\bigoplus_{t=0}^{r-1}(\mathfrak{gl}_N^r[2p^r-t-1])) \to k[V_r(GL_N)] \to H^\bullet(GL_{N(r)}, k).$$

Here, $q$ is the quotient by the ideal generated by the relations

$$R_{i,j,\ell,\ell'} = \sum X^{i,t}(\ell) \cdot X^{i,j}(\ell') - X^{i,t}(\ell') \cdot X^{i,j}(\ell)$$

$$S_{i,j,\ell} = \sum_{t_1, \ldots, t_p=1} X^{i,t_1}(\ell) \cdot X^{i,t_2}(\ell) \cdots X^{t_p,j}(\ell)$$

for all $i,j,\ell,\ell'$ as in [21] 5.1. Thus, $V_r(GL_N)$ is identified with the $k$-scheme of $r$-tuples of $p$-nilpotent matrices (because of conditions $\{S_{i,j,\ell}\}$) which are pairwise commuting (because of conditions $\{R_{i,j,\ell,\ell'}\}$).

Theorem 1.1 can be viewed as a complement (in the special case $G = GL_{N(r)}$) of the following theorem.

**Theorem 1.2.** ([22] 5.2, [21] 5.2] Fix some integer $r \geq 1$. Then for any infinitesimal group scheme $H$ of height $\leq r$, there is a natural homomorphism of commutative $k$-algebras

$$\psi : H^\bullet(H, k) \to k[V_r(H)]$$

whose kernel is nilpotent and whose image contains all $p^r$-th powers of elements of $k[V_r(H)]$. If $H = G_{a(r)}$, the $r$-th Frobenius kernel of some linear algebraic group $G$, then we denote $\psi$ by $\psi_{G,r} : H^\bullet(G_{a(r)}, k) \to k[V_r(G)]$.

The map $\psi_{G,r}$ is $G$-equivariant.

In the special case of $G = GL_{N(r)}$, the composition

$$\psi_{GL_{N,r}} \circ \overline{\phi}_{GL_{N,r}} : k[V_r(GL_N)] \to k[V_r(GL_N)]$$

is the $r$-th iterate of the Frobenius map. In particular, $\psi_{GL_{N,r}}(\phi_{GL_{N,r}}(X^{i,j}(\ell))) = (X^{i,j}(\ell))^{p^r}$.

**Remark 1.3.** The assertion of Theorem 1.2 of $G$-equivariance of $\psi_{G,r}$ arises from the naturality of $\psi$, in particular the commutativity of the first displayed square of the proof of Theorem 1.14 of [21].

As shown in [21], $V_r(G)$ has a natural grading given by the monoid action of (right) composition $V_r(G_{a(r)})$ on $V_r(G)$; namely, one restricts this action of $A^r \simeq V_r(G_{a(r)})$ to the linear polynomials $A^1 \subset A^r$. With this grading, $X^{i,j}(\ell) \in k[GL_{N(r)}]$ has grading $p^r-t-1$ mapping via $\phi_{GL_{N,r}}$ to a cohomology class of degree $2p^r-t-1$, then further mapping via $\psi_{GL_{N,r}}$ to $F^\bullet(X^{i,j}(\ell))$ which has degree $p^r \cdot p^r-t-1$. In general, the map $\psi_{G,r} : H^\bullet(G_{a(r)}, k) \to k[V_r(G)]$ multiplies degree by $p^r$. Finally, we remark that $k[V_r(H)]$ is not necessarily reduced.
We thank R Guralnick for explaining the following result of S. Garibaldi given in [12].

**Proposition 1.4.** [12 Prop 8.1] If $G$ is a simple algebraic group for which $p > 2$ is a very good prime (i.e., for type $A_{n-1}$, $p$ does not divide $n$; for type $G_2, F_4, E_6, E_7$, $p > 3$; for type $E_8$, $p > 5$), then there exist a closed embedding $i : G \to GL_N$ such that the induced map $i : g \to gl_N$ admits a unique $G$-equivariant splitting $\tau : g|_N \to g$.

We use Proposition 1.4 to extend the map $\phi_{GL_N, r}$ of (1.0.1).

**Proposition 1.5.** Let $G$ be a simple algebraic group and assume that $p$ is very good for $G$; let $B \subset G$ be a Borel subgroup with maximal torus $T$. Choose some embedding $i : G \subset GL_N$ as in Proposition 1.4 and let $\tau : gl_N \to g = Lie(G)$ be a splitting of $i : g \to gl_N$. Consider some $T$-stable unipotent subgroup $U \subset B$ with associated map Lie algebras $j : u \to g$. There exists a unique $G$-equivariant map $\phi_{G, r} : S^*(\oplus_{t=0}^r (g^{\#(r)}[2p^{r-t-1}])) \to H^*(G(r), k)$ and a unique $T$-equivariant map $\phi_{U, r} : S^*(\oplus_{t=0}^r (u^{\#(r)}[2p^{r-t-1}])) \to H^*(U(r), k)$ fitting in the following commutative diagram (whose vertical maps are restriction maps):

\[
\begin{array}{ccc}
S^*(\oplus_{t=0}^r (g^{\#(r)}[2p^{r-t-1}])) & \xrightarrow{\phi_{GL_N, r}} & H^*(GL_N, k) \\
\downarrow i^* & & \downarrow i^* \\
S^*(\oplus_{t=0}^r (g^{\#(r)}[2p^{r-t-1}])) & \xrightarrow{\phi_{G, r}} & H^*(G(r), k) \\
\downarrow j^* & & \downarrow j^* \\
S^*(\oplus_{t=0}^r (u^{\#(r)}[2p^{r-t-1}])) & \xrightarrow{\phi_{U, r}} & H^*(U(r), k) 
\end{array}
\]

**Proof.** We define $\phi_{G, r}$ to be $i^* \circ \phi_{GL_N, r} \circ \tau^*$ on generating spaces $g^{\#(r)}[2p^{r-t-1}]$ and extend multiplicatively. The commutativity of the upper square of (1.5.1) follows from the fact that $\tau$ is a splitting of $g \to gl_N$; uniqueness follows from the uniqueness of such a splitting.

We recall that $U$ is a product of root subgroups $U_\alpha \subset B$. This implies that $j : u \subset g$ is the $T$-stable subspace spanned by $u_\alpha = Lie(U_\alpha)$. Since each $u_\alpha$ is the $T$-weight space of $g$ of weight $\alpha$, $j$ admits a unique $T$-invariant splitting $\tau_U : g \to u$. We define $\phi_{U, r}$ to be $j^* \circ \phi_{G, r} \circ \tau_U^*$ on generating spaces $u^{\#(r)}[2p^{r-t-1}]$ and extend multiplicatively.

We next proceed to verify that the maps $\phi_{U, r}$ of Proposition 1.5 satisfy properties similar to those for $\phi_{GL_N, r}$ given in Theorems 1.3 and 1.2 provided that $U \subset G \subset GL_N$ is an embedding of exponential type. Our starting point is the fact proved in [21] that every infinitesimal 1-parameter subgroup $\psi : G_{\alpha(r)} \to GL_N$ is uniquely of the form

\[
\prod_{s=0}^{r-1} \exp_{B_s} \circ F^* : G_{\alpha(r)} \to GL_N, \quad t \mapsto \prod_{s=0}^{r-1} \exp(t \cdot B_s)
\]

for some $r$-tuple $B = (B_0, \ldots, B_{r-1})$ of $p$-nilpotent, pair-wise commuting elements of $gl_N$.

We recall that a closed embedding $U \to GL_N$ of a linear algebraic group $U$ is said to be of exponential type if the map of schemes $exp : G_{\alpha} \times N_p(gl_N) \to V_r(GL_N)$
given by the usual truncated exponential map restricts to $\mathcal{E} : G_a \times N_{\mathbb{p}}(u) \to V_r(U)$. Here, $N_{\mathbb{p}}(u) \subset u$ denotes the subvariety whose $k$ points are elements $X \in g$ such that $X^{[p]} = 0$. For $U$ equipped with such an embedding, every infinitesimal 1-parameter subgroup $G_{u(r)} \to U$ is uniquely of the form $\prod_{s=0}^{r-1} \mathcal{E} B_s \circ F_s$ for some $B_s \in C_r(N_{\mathbb{p}}(u))$, the variety of $r$-tuples $(B_{0}, \ldots, B_{r-1})$ of $p$-nilpotent, pairwise commuting elements of $u$ as shown in [20 2.5].

**Proposition 1.6.** Let $i : U \to GL_N$ be a closed embedding of exponential type for some linear algebraic group $U$. Then the following square is a cartesian square of closed immersions

$$
\begin{array}{ccc}
V_r(U) & \longrightarrow & V_r(GL_N) \\
\downarrow & & \downarrow \\
\mathfrak{u}^r & \longrightarrow & \mathfrak{g}^r.
\end{array}
$$

(1.6.1)

In other words, we have a cocartesian square of quotient maps of $k$-algebras

$$
\begin{array}{ccc}
S^*(\oplus_{\ell=0}^{r-1}(\mathfrak{g}_{N}^{\#(r)}[2p^{r-\ell-1}])) & \longrightarrow & \mathbb{GL}_N \\
\downarrow & & \downarrow \\
S^*(\oplus_{\ell=0}^{r-1}(u^{\#(r)}[2p^{r-\ell-1}])) & \longrightarrow & \mathbb{V}_r(U).
\end{array}
$$

(1.6.2)

**Proof.** The condition that $i : U \to GL_N$ is of exponential type enables us to identify the embedding $V_r(U) \to V_r(GL_N)$ of schemes representing height $r$ infinitesimal 1-parameter subgroups with the embedding of schemes of $r$-tuples of $p$-nilpotent, pairwise commuting elements of respective Lie algebras. Using this, we verify that (1.6.1) arises as a cartesian square of representable functors. Namely, we verify that the defining relations $\{R_{i,j,\ell,k}, S_{i,j,\ell}\}$ in $S^*(\oplus_{\ell=0}^{r-1}(\mathfrak{g}_{N}^{\#(r)}[2p^{r-\ell-1}]))$ (for $V_r(GL_N) \subset (\mathfrak{g}_{N})^r$) have image in $S^*(\oplus_{\ell=0}^{r-1}(u^{\#(r)}[2p^{r-\ell-1}]))$ (i.e., restrictions to $(\mathfrak{u})^r \subset (\mathfrak{g}_{N})^r$) which generate defining relations for $V_r(u) \subset (\mathfrak{g}_{N})^r$. This follows from the observation for $X, Y \in \mathfrak{u}$ that the condition that $X, Y$ commute in $\mathfrak{u}$ is the same as the condition that their images commute in $\mathfrak{g}_{N}$, and the condition that $X^{[p]} = 0$ is the condition that the image of $X$ in $\mathfrak{g}_{N}$ has $p$-th power $0$.

The cartesian square (1.6.1) is equivalent to the cocartesian square (1.6.2) of coordinate algebras thanks to the anti-equivalence of categories relating affine $k$-schemes and finitely generated commutative $k$-algebras. □

**Theorem 1.7.** As in Proposition 1.4 let $G$ be a simple algebraic groups such that $p$ is very good for $G$ and $U \subset B \subset G$ be a unipotent subgroup which is a product of root subgroups. Assume that there exists a closed embedding $i : G \to GL_N$ with $G$ satisfying the hypotheses of Proposition 1.4 such that the composition $U \subset G \subset GL_N$ is an embedding $i : U \to GL_N$ of exponential type defined over $\mathbb{F}_p$. Then

\[
(\phi_U \circ r)_{red} : S^*(\oplus_{\ell=0}^{r-1}(u^{\#(r)}[2p^{r-\ell-1}]))_{red} \to H^*(U_{(r)}, k)_{red}
\]

factors through $([k[V_r(U)]]_{red}$, thus equals to the composition

\[
(\phi_U \circ r)_{red} \circ (q_U)_{red} : S^*(\oplus_{\ell=0}^{r-1}(u^{\#(r)}[2p^{r-\ell-1}]))_{red} \to ([k[V_r(U)]]_{red} \to H^*(U_{(r)}, k)_{red}
\]

for a uniquely defined map $(\phi_U)_{red}$. 
Moreover, \((\psi_{U,r})_{red} \circ (\phi_{U,r})_{red} : k[V_r(U)]_{red} \rightarrow H^*(U^{(r)}, k)_{red} \rightarrow k[V_r(U)]_{red}\) is the \(r\)-th power of Frobenius.

Furthermore, if \(k[V_r(U)]\) is reduced, then \(\phi_{U,r} = \phi_{U,r} \circ q_U\) with

\[
\phi_{U,r} : k[V_r(U)] \rightarrow H^*(U^{(r)}, k)
\]
satisfying \(\psi_{U,r} \circ \phi_{U,r} = F_r^\tau\).

**Proof.** We consider the following diagram whose left square commutes by Proposition 1.6, whose square involving \(\phi_{GLN,r}\) and \(\phi_{U,r}\) commutes by Proposition 1.5 whose right square commutes by the naturality of \(\psi\) in Theorem 1.2 and whose square involving \(F_r^\tau\) commutes by the naturality of Frobenius:

\[
\begin{array}{ccc}
S^*(\oplus_{\ell=0}^{r-1}(gl_N^{\#(r)}[2p^r-\ell-1])) & \xrightarrow{q_{GLN}} & k[V_r(GLN)] \\
\downarrow \psi_U & & \downarrow \phi_{GLN,r} \\
S^*(\oplus_{\ell=0}^{r-1}(u^{\#(r)}[2p^r-\ell-1])) & \xrightarrow{q_U} & k[V_r(U)] \\
\downarrow \phi_{U,r} & & \downarrow \phi_{U,r} \\
& & k[V_r(U)]
\end{array}
\]

To prove the theorem, it suffices to complete (1.7.1) once one applies \((-)_{red}\) to the bottom row. To construct \((\phi_{U,r})_{red} : k[V_r(U)]_{red} \rightarrow H^*(U^{(r)}, k)_{red}\) factoring \(\phi_{U,r}\) is equivalent to showing that \((\phi_{U,r})_{red}\) vanishes on the kernel of \(q_U\).

Let \(V = \ker\{q_U\}\) and let \(\tau^* : S^*(\oplus_{\ell=0}^{r-1}(u^{\#(r)}[2p^r-\ell-1])) \rightarrow S^*(\oplus_{\ell=0}^{r-1}(gl_N^{\#(r)}[2p^r-\ell-1]))\) be induced by the \(T\)-splitting of \(u \rightarrow b \rightarrow g \rightarrow gl_N\) as in Proposition 1.6. Then the commutativity of (1.7.1) and the equality \(i_{\psi_U}^* \circ \tau_U^* = 1\) implies that \((i_{\phi_{GLN,r}}^* \circ \phi_{GLN,r} \circ q_{GLN}) (V) = 0\). Appealing once again to the commutativity of (1.7.1), we conclude that

\[(F_r^\tau \circ q_U)(V) = (\psi_{U,r} \circ \phi_{U,r})(V) = 0.\]

Since \((\psi_{U,r})_{red}\) is injective by Theorem 1.2 we conclude that \((\phi_{U,r})_{red}(V) = 0\) as required. \(\square\)

**Example 1.8.** We produce many examples of unipotent groups \(U\) satisfying the conditions of Theorem 1.7 as follows. As stated in 21.1.8, the classical simple algebraic groups \(Sp_{2n}\), \(SO_n\) and \(SL_n\) admit embeddings of exponential type. Namely, one considers a vector space (of dimension \(2n\) for \(Sp_{2n}\), of dimension \(n\) for \(On\)) equipped with a non-degenerate bilinear form and one takes the embedding given by considering those linear isomorphisms preserving the form.

Let \(G\) be a simple algebraic group of adjoint type and \(P_J \subset G\) a parabolic subgroup corresponding to a choice of \(J \subset \Pi\) (see Proposition 21. If \(G\) admits an embedding of exponential type \(G \subset GL_N\), then the restriction of the exponentiation \(E : N_G(Lie(P_J)) \times G_a \rightarrow P_J\) to some term \(U\) of the lower central series of \(U_J\) determines

\(E_U : u = Lie(U) \times G_a \rightarrow U;\)

thus, the composition \(U \subset U_J \subset P_J \subset G \subset GL_N\) is also an embedding of exponential type.
Furthermore, each of these embeddings \( U \to G \subset GL_N \) is an embedding of exponential type defined over \( \mathbb{F}_p \).

In the very special case of the Heisenberg group \( U_3 \subset GL_3 \), the following proposition makes \( \phi_{U_3,r} \) more explicit.

**Proposition 1.9.** As always, we assume \( p > 2 \) and let \( \Gamma_2 \subset U_3 \) denote the commutator subgroup of \( U_3 \subset GL_3 \). The coordinate algebra \( k[V_r(U_3)] \) admits a natural tensor product decomposition

\[
k[V_r(U_3)] = k[Y_r(U_3/\Gamma_2)] \otimes S^*(\{X^{1,3}(\ell), 0 \leq \ell < r\}), \quad V_r(U_3) \simeq Y_r(U_3/\Gamma_2) \times A^r.
\]

Moreover, the natural map \( k[Y_r(U_3/\Gamma_2)] \to k[V_r(U_3)] \) factors as a surjection \( k[Y_r(U_3/\Gamma_2)] \to k[Y_r(U_3)] \) followed by the split inclusion \( k[Y_r(U_3)] \to k[V_r(U_3)] \).

Furthermore, \( k[Y_r(U_3)/\Gamma_2] \) is an integral domain smooth outside of the origin with field of fractions a purely transcendental extension of \( k \) transcendence degree \( r + 1 \). Consequently, \( k[V_r(U_3)] \) is an integrally closed domain of dimension \( 2r + 1 \). In particular, \( k[V_r(U_3)] = k[V_r(U_3/\Gamma_2)]_{\text{red}} \).

**Proof.** We use the presentation of \( k[V_r(U_3)] \) derived from that given for \( V_r(GL_N) \) in Theorem 1.1 as justified by Proposition 1.6. The coordinate algebra \( k[Y_r(U_3)] \) is generated by \( \{X^{s,s+1}(\ell), 1 \leq s < 3, 0 \leq \ell < r\} \), and subject to the relations

\[
\{X^{1,2}(\ell) \cdot X^{2,3}(\ell') - X^{1,2}(\ell') \cdot X^{2,3}(\ell); 0 \leq \ell < \ell' < r\}
\]

(for example, see Theorem 1.1). The tensor product decomposition is immediate from the observation that the relations for \( k[V_r(U_3)] \) do not involve \( X^{1,3}(\ell) \). The factorization follows from the observation that \( k[Y_r(U_3/\Gamma_2)] \) can be identified with the polynomial algebra on \( \{X^{s,s+1}(\ell), 1 \leq s < 3, 0 \leq \ell < r\} \).

If \( r = 1 \), then \( V_r(U_3) = u_3 \) so that \( k[Y_1(U_3/\Gamma_2)] \) can be identified with the polynomial algebra \( S^*(\{X^{1,3}, X^{2,3}\}) \). For the remainder of the proof, we assume \( r > 1 \).

For any \( \ell_1, 0 \leq \ell_1 < r \), the algebra \( k[Y_r(U_3/\Gamma_2)]/[X^{2,3}(\ell_1)]^{-1} \) is isomorphic to

\[
k[X^{2,3}(\ell), 0 \leq \ell < r; X^{1,2}(\ell)]/[X^{2,3}(\ell_1)]^{-1},
\]

since \( X^{1,2}(\ell) = X^{1,2}(\ell_1)X^{2,3}(\ell_1)^{-1}X^{2,3}(\ell) \); similarly, for any \( \ell_0, 0 \leq \ell_0 < r \), the algebra \( k[Y_r(U_3/\Gamma_2)]/[X^{1,2}(\ell_0)]^{-1} \) is isomorphic to

\[
k[X^{1,2}(\ell), 0 \leq \ell < r; X^{2,3}(\ell_0)]/[X^{1,2}(\ell_0)]^{-1}.
\]

This verifies the computation of the field of fractions of \( k[Y_r(U_3/\Gamma_2)] \) and shows that \( k[Y_r(U_3/\Gamma_2)] \) is smooth outside the common zeros of \( \{X^{1,2}(\ell_0), X^{2,3}(\ell_1); 0 \leq \ell_0, \ell_1 < r\} \); namely, the origin. A theorem of Serre (see [10, Thm 39]) tells us that \( k[Y_r(U_3/\Gamma_2)] \) is an integrally closed domain since the codimension of this zero locus is at least 2. \qed

We conclude this section with the following observation that \( \phi_{U,r} \) is a graded map, an easy consequence of Theorem 1.1.

**Corollary 1.10.** Let \( U \) be a unipotent algebraic group admitting an embedding \( U \subset G \subset GL_N \) satisfying the hypotheses of Theorem 1.7, and let \( U(r) \subset U \) be its \( r \)-th Frobenius kernel. Then \( (\phi_{U,r})_{\text{red}} : k[V_r(U)]_{\text{red}} \to H^\bullet(U(r), k)_{\text{red}} \) multiplies degrees by 2, where the grading is that of [21] (see Remark 1.3).
Proof. First, observe that if $A^* = \oplus_{n \geq 0} A^n$ is a graded commutative algebra, then $(A^*)_{red} = \oplus_{n \geq 0} \{A^n \rightarrow (A^*)_{red}\}$. Namely, if $a \in A^m, b \in A^n$ satisfy the condition that the image of $a - b$ in $(A^*)_{red}$ is 0, then for $s \gg 0$ we have that $a^s - b^s = (a - b)^s = 0$. Thus, either both $a, b$ are nilpotent, or $p^s \cdot m = p^s \cdot n$ for $s \gg 0$ so that $m$ must equal $n$.

Thus, $k[V_r(U)]_{red}$ inherits from $H^*(U_{(r)}, k)_{red}$, a grading concentrated in even degrees; we give $k[V_r(U)]_{red}$ this grading with degrees divided by 2. Since $F^r : k[V_r(U)]_{red} \rightarrow k[V_r(U)]_{red}$ is injective and multiplies degrees by $p^r$, we conclude that the grading just defined on $k[V_r(U)]_{red}$ and that of Remark 1.3 agree upon applying $F^r$ and hence must agree. □

2. THE MAP $\eta_{U_{J_3}} : T(U_{J_3}, r) \rightarrow H^*(U_{J_3}/\Gamma_3, k)$

In this section, we construct $\eta_{U_{J_3}} : S^v(U_3, r) \rightarrow H^*((U_3)_3, k)$, a more natural formulation of $\eta_{U_{J_3}}$ of Section 11 this map induces $\eta_{U_{J_3}}$. With the much more general context of Section 3 in mind, we consider unipotent groups more general than $U_3$; namely, groups of the form $U_J / \Gamma_3$, where $\Gamma_3 \equiv \Gamma_3(U_J)$ is the third stage of the descending central series of the unipotent radical of a parabolic subgroup $P_J \subset G$ of a simple algebraic group. Proposition 2.4 identifies $\eta_{U_{J_3}}$ with $\phi_{U_{J_3}}$. Much of this section utilizes the construction of $\eta_{U_{J_3}}$ to show that $\eta_{U_{J_3}}$ is an “good model” for $H^*((U_3)_3, k)$, culminating in Theorem 2.17.

Our primary tool is the Lyndon-Hochshield-Serre spectral sequence for a central extension (see Proposition 2.4) together with the action of the mod-$p$ Steenrod algebra. At a key point (Proposition 2.7), we use the Andersen-Jantzen spectral sequence recalled in Proposition 2.6.

We recall from 3 the description due to H. Azad, M. Barry, and G. Seitz of the terms of the descending central series of the unipotent radical $U_J$ of $P_J \subset G$ for some $J \subset \Pi$. We fix an ordering of $\Pi$ which respects addition. For a positive root $\beta \in \Sigma^+ - \Sigma^+_J$ (where $\Sigma^+_J$ is the set of positive roots for the root system of $G \supset B \supset T$ and $\Sigma^+_J$ is the set of positive roots for the root system of $L_J = P_J / U_J \supset T_J$), we adopt the terminology of 3: write $\beta = \beta_J + \beta_J'$ where $\beta_J$ is a sum $\sum_i c_i \alpha_i$ with each $\alpha_i \in J$ and $\beta_J'$ is a sum $\sum_j d_j \alpha_j$ with each $\alpha_j \in \Pi - J$; then the height of $\beta$ is defined to be $\sum_i c_i + \sum_j d_j$, the level of $\beta$ is defined to be $\sum_j d_j$ and the shape of $\beta$ is defined to be $\beta_J'$.

Proposition 2.1. (summary of §2 of 3) Let $G$ be a simple algebraic group of adjoint type, and $P = P_J \subset G$ a parabolic subgroup, $L_J$ its Levi factor, and $U_J$ its unipotent radical for some subset $J \subset \Pi$. As usual, assume $p > 2$; for $G$ of type $G_2$, assume $p > 3$. Consider the descending central series for $U_J$:

$$\cdots \subset \Gamma_{v+1} = [U_J, \Gamma_v] \subset \cdots \subset \Gamma_2 = [U_J, U_J] \subset \Gamma_1 = U_J.$$  

For any $v > 1$, we have the central extension with a natural action of $L_J$:

$$(2.1.1) \quad 1 \rightarrow \Gamma_v/\Gamma_{v+1} \rightarrow U_J/\Gamma_{v+1} \rightarrow U_J/\Gamma_v \rightarrow 1.$$  

The commutative group $\Gamma_v/\Gamma_{v+1}$ is a direct product of irreducible $L_J$-modules $V_S$ indexed by shapes $S$ of level $v$; each $V_S$ is $T$ isomorphic to a product of $U_- \beta$ indexed by $\beta \in \Sigma^+ - \Sigma^+_J$ of shape $S$ and level $v$, where $U_- \beta$ is the root subgroup with $T$-weight $-\beta$; $V_S$ is a high weight $L_J$-module with highest weight $-\beta_S$, where $\beta_S$ is the unique root of minimal height and shape $S$. 

In particular, if \( P_f \) is the minimal parabolic (i.e., equal to the given Borel subgroup \( B \subset G \), corresponding to \( \Pi = \emptyset \)), then \( \Gamma_v/\Gamma_{v+1} \) is \( T \)-isomorphic to \( \prod U_{-\beta} \) where the product is indexed by \( \beta \in \Sigma^+ \) of level \( v \).

In what follows, we shall denote \( \text{Lie}(\Gamma_v(U_J)) \) by \( \gamma_v \). As in Section 2.1, \( r \) will denote a fixed (but arbitrary) positive integer.

In order to fix notation and \( T \)-weights for Frobenius twists, we recall the known computation of \( H^*(G_a, k) \) and \( H^*(G_a(r), k) \) (see, for example, [6]):

\[
H^*(G_a, k) \simeq S^*(V(1)[2]) \otimes \Lambda^*(V[1]),
\]

where \( V = H^1(G_a, k) \) is a countable \( k \)-vector space spanned by \( y_1, y_2, \ldots, y_s \) with Frobenius action \( F^*(y_a) = y_{a+1} \), \( \Lambda^*(V([1])) \) is the exterior algebra on \( V \) placed in degree 1, and \( S^*(V(1)[2]) \) is the polynomial algebra on the Frobenius twist \( V^* \) of \( V \) placed in degree 2 spanned by \( x_1 = \beta(y_1), x_2 = \beta(y_2), \ldots, x_s = \beta(y_s) \) \( \ldots \) where \( \beta : H^1(G_a, k) \to H^2(G_a, k) \) is the \((\mathbb{P}_p)\)-linear Bockstein homomorphism. The action of multiplication by \( c \in k \) on \( G_a \) induces an action on \( H^*(G_a, k) \) given by \( c^s(y_i) = c^{s-1}y_i, \quad c^r(x_i) = c^{r-i}x_i. \) This indexing is that of [6] and [22, Thm 1.3]. We recall that the cohomology algebra \( H^*(G_a(r), k) \) of the \( r \)-th Frobenius kernel \( G_a(r) \) of \( G_a \) can be identified with the quotient of \( H^*(G_a, k) \) obtained by setting \( y_s = 0 = x_s \) for \( s > r \). This finitely generated cohomology algebra admits the natural action of the mod-p Steenrod algebra \( \mathcal{A}_p \) (see [22, 1.7] for an explicit description of the action of the generators \( \mathcal{P}^r, \beta \mathcal{P}^s \) of \( \mathcal{A}_p \) on \( H^*(G_a, k) \)).

Observe that \( U_J/\Gamma_2 \) is a product \( G_a^{\times s} \) of copies of \( G_a \), so that its cohomology and that of \((G_a^{\times s})(r)\) are determined by the above computation and the Künneth theorem. In particular, we conclude that there is a natural map

\[
\eta_{U_J/\Gamma_2, r} : S^*((u_J/\gamma_{2})^\#(\ell+1)[2]) \to H^*((U_J/\Gamma_2)(r), k)
\]

which is injective and is surjective modulo nilpotents.

In this section we shall investigate \( H^*((U_J/\Gamma_3)(r), k) \), whereas in the following section we consider the cohomology of the general quotient \((\Gamma_i/\Gamma_v)(r)\).

We designate \( T \)-eigenvector generators for

\[
H^*(U_J/\Gamma_2, k) \simeq S^*(\oplus_{\ell=0}^{\infty} (u_J/\gamma_2)^\#(\ell+1)[2]) \otimes \Lambda^*(\oplus_{\ell=0}^{\infty} (u_J/\gamma_2)^\#(\ell)[1])
\]

by \( x_\alpha^{(\ell)} \) of cohomological degree 2 and \( T \)-weight \( p^{\ell+1}\alpha \) and \( y_{\beta}^{(\ell)} \) of cohomological degree 1 and \( T \)-weight \( p^\ell \alpha \); here, \( \alpha \) ranges over roots of \( U_J \) of level 1 and \( \ell \) is a non-negative integer satisfying \( 0 \leq \ell \). Similarly, we designate generators for

\[
H^*(\Gamma_i/\Gamma_{i+1}, k) \simeq S^*(\oplus_{\ell=0}^{\infty} (\gamma_i/\gamma_{i+1})^{\#(\ell+1)}[2]) \otimes \Lambda^*(\oplus_{\ell=0}^{\infty} (\gamma_i/\gamma_{i+1})^{\#(\ell)}[1])
\]

by \( x_{\beta}^{(\ell)} \) of cohomological degree 2 and \( y_{\beta}^{(\ell)} \) of cohomological degree 1, with \( 0 \leq \ell \) and with \( \beta \) ranging over \( T \)-weights of \( U_J \) of level 1.

The indexing we adopt (for example, in Proposition 2.2) relates to the indexing of [6] as follows for cohomology classes of \( G_a \): \( y_i \) corresponds to \( y_{\ell}^{(\ell)} \) and \( x_i \) corresponds to \( x_{\ell}^{(\ell)} \) with \( \ell = i - 1 \).

**Proposition 2.2.** Retain the notation and hypotheses of Proposition 2.1. Consider the \( T \)-equivariant Lyndon-Hochshild-Serre spectral sequence [12] for the extension

\[
1 \to \Gamma_2/\Gamma_3 \to U_J/\Gamma_3 \to U_J/\Gamma_2 \to 1;
\]

(2.2.1) \( E_2^{a,b}(U_J/\Gamma_3) = H^a(U_J/\Gamma_2, k) \otimes H^b(\Gamma_2/\Gamma_3, k) \Rightarrow H^{a+b}(U_J/\Gamma_3, k). \)

For any \( \ell \geq 0, j \geq 0, \beta \) a weight of level 2:
(1) \[ d_2^{0,1}(y^{(f)}_{\beta}) = \sum_{\alpha + \alpha' = \beta, \alpha < \alpha'} y^{(f)}_{\alpha} \wedge y^{(f)}_{\alpha'} \in H^2(U_J/\Gamma_2, k), \]

where the sum is indexed by pairs \( \alpha, \alpha' \) of weights in \( (u/\gamma_2)^{\#} \) (i.e., hence, roots in \( \Sigma^+ - \Sigma^{+J} \)) such that \( \beta = \alpha + \alpha' \) and \( \alpha < \alpha' \).

(2) \[ d_2^{0,2p+1}((x^{(f)}_{\beta})^{p'}) = \sum_{\alpha + \alpha' = \beta, \alpha < \alpha'} \{(x^{(f)}_{\alpha})^{p'} \otimes y^{(f+1+j)}_{\alpha'} - (x^{(f)}_{\alpha'})^{p'} \otimes y^{(f+1+j)}_{\alpha}\} \]

is non-zero in \( H^{2p+1+1}(U_J/\Gamma_2, k) \). Thus, \( (x^{(f)}_{\beta})^{p'} \) does not lie in the image of \( H^*(U_J/\Gamma_3, k) \).

(3) \[ \beta P^0 (\sum_{\alpha + \alpha' = \beta, \alpha < \alpha'} \{(x^{(f)}_{\alpha})^{p'} \otimes y^{(f+1+j)}_{\alpha'} - (x^{(f)}_{\alpha'})^{p'} \otimes y^{(f+1+j)}_{\alpha}\}) \]

is non-zero in \( H^{2p+1+2}(U_J/\Gamma_2, k) \). The expression \( (x^{(f)}_{\beta})^{p'} \) maps to 0 in \( H^{2p+1+2}(U_J/\Gamma_3, k) \).

Proof. If \( U_J = U_3 \), the unipotent radical of a maximal Borel of \( SL_3 \), then the computation of \( d_2^{0,1}(y^{(f)}_{\beta}) \) follows from the well known identification of the extension class of each

\[ (G_a)_{(i)/(G_a)_{(i-1)} \rightarrow (U_3)_{(i)/(U_3)_{(i-1)} \rightarrow (G_a)^2)_{(i)}/(G_a^2)_{(i-1)}}. \]

More generally, the root subgroup \( U_\beta \subset U_J/\Gamma_3 \) maps injectively to a product of root subgroups under a quotient map \( U_J/\Gamma_3 \rightarrow U_\beta \approx U_3 \). Thus, functoriality tells us that \( d_2^{0,1}(y^{(f)}_{\beta}) \) has the asserted value plus other terms with \( T \)-weights not equal to \( \beta \). Since the differential \( d_2^{0,1} \) respects \( T \)-weights,

For \( j = 0 \), (2) follows from the equality \( x^{(f)}_{\beta} = (\beta P^0)(y^{(f)}_{\beta}) \), the fact that \( \beta P^0 \) commutes with transgression (11), and the Cartan formula telling us the

\[ \beta P^0 (y^{(f)}_{\alpha} \wedge y^{(f)}_{\alpha'}) = \beta P^0 (y^{(f)}_{\alpha} \otimes y^{(f)}_{\alpha'}) - \beta P^0 (y^{(f)}_{\alpha} \otimes y^{(f)}_{\alpha'}) = x^{(f)}_{\alpha} \otimes y^{(f+1)}_{\alpha'} - x^{(f)}_{\alpha'} \otimes y^{(f+1)}_{\alpha}. \]

To prove (2) for \( j > 0 \), we recall that \( P^{p'} \) applied to \( (x^{(f)}_{\beta})^{p'} \) equals \( (x^{(f)}_{\beta})^{p'+1} \). Using the fact that Steenrod action commute with differential in the spectral sequence and repeated applications of the Cartan formula, we verify (2) by computing \( d_2^{0,2p'}((x^{(f)}_{\beta})^{p'}) \), the result of applying \( d_2^{0,2p'} \) to \( (P^{p'} \circ \cdots P^1 \circ \beta P^0)(y^{(f)}_{\beta}) \). The fact that \( d_2^{0,2p'}((x^{(f)}_{\beta})^{p'}) \neq 0 \) follows from the explicit computation of \( H^*(U_J/\Gamma_2, k) \).

Because some differential in the spectral sequence is non-vanishing on \( (x^{(f)}_{\beta})^{p'} \), it does not lie in the image of \( H^*(U_J/\Gamma_3, k) \).

The computation of assertion (3) follows from the Cartan formula for \( \beta P^{p'} \) and the detailed description of \( P^1 \) and \( \beta P^1 \) given in [22, 1.7]. The non-vanishing of \( (x^{(f)}_{\beta})^{p'} \) follows once again from the explicit computation of \( H^*(U_J/\Gamma_2, k) \).

Remark 2.3. In all examples we have considered, the sum \( \sum_{\alpha' = \beta, \alpha < \alpha'} \) has only one summand: namely, for each \( \beta \), there exist unique \( \alpha < \alpha' \) such that \( \beta = \alpha + \alpha' \).

The restriction map for the embedding \( (U_J/\Gamma_3)_{(r)} \rightarrow U_J/\Gamma_3 \) determines a map from the spectral sequence (2.21) to the spectral sequence (2.22) considered in the next proposition. On \( E_2 \)-terms, this map sends \( y^{(f)}_{\beta}, x^{(f)}_{\beta}, y^{(f)}_{\alpha}, x^{(f)}_{\alpha} \) to 0 for \( \ell \geq r \).
Proposition 2.4. Retain the notation and hypotheses of Proposition 2.1 and consider the central extension

\[(2.4.1) \quad 1 \to (\Gamma_2/\Gamma_3)_{(r)} \to (U_J/\Gamma_3)_{(r)} \to (U_J/\Gamma_2)_{(r)} \to 0.\]

The L-H-S spectral sequence for (2.4.1) has the form

\[(2.4.2) \quad E_2^{a,b}((U_J/\Gamma_3)_{(r)}) = H^a((U_J/\Gamma_2)_{(r)}, k) \otimes H^b((\Gamma_2/\Gamma_3)_{(r)}, k) \Rightarrow H^{a+b}((U_J/\Gamma_3)_{(r)}, k).\]

Using Proposition 2.2 we conclude

\[(2.4.3) \quad \sum_{\alpha + \alpha' = \beta, \alpha < \alpha'} \{(x^{(\ell)}_\alpha)^{p^{\ell+1}} \otimes x^{(\ell+1)j}_{\alpha'} - (x^{(\ell)}_{\alpha'})^{p^{\ell+1}} \otimes x^{(\ell+1)j}_\alpha \} \in S^{p^{\ell+1}+1}(\oplus_{k=0}^{r-1}(U_J/\gamma_2)^{\#(\ell+1)}[2])\]

lies in the kernel of the restriction map \(H^*(U_J/\Gamma_2)_{(r)}, k) \to H^*(U_J/\Gamma_3)_{(r)}, k);\)

in other words, gives the relation

\[(2.4.4) \quad \sum_{\alpha + \alpha' = \beta, \alpha < \alpha'} \{(x^{(\ell)}_\alpha)^{p^{\ell+1}} \otimes x^{(\ell+1)j}_{\alpha'} - (x^{(\ell)}_{\alpha'})^{p^{\ell+1}} \otimes x^{(\ell+1)j}_\alpha \} = 0 \in H^{2p^{\ell+1}+2}((U_J/\Gamma_3)_{(r)}, k).\]

Proof. The vanishing of \(y^{(\ell)}_\alpha, \ell \geq r\) together with Proposition 2.2 immediately implies that \((x^{(\ell)}_\beta)^{p^{\ell}}\) is a permanent cycle if \(\ell + 1 + j \geq r\). Conversely, if \(\ell + 1 + j < r\), then Proposition 2.2 tells us that \(d^{p^{\ell+1}+2}_{2p^{\ell}}\) does not vanish on \((x^{(\ell)}_\beta)^{p^{\ell}}\).

Assertion (2) follows from Proposition 2.2(2),(3), since

\[\sum_{\alpha + \alpha' = \beta, \alpha < \alpha'} \{(x^{(\ell)}_\alpha)^{p^{\ell+1}} \otimes y^{(\ell+1)j}_{\alpha'} - (x^{(\ell)}_{\alpha'})^{p^{\ell+1}} \otimes y^{(\ell+1)j}_\alpha \} \]

is a boundary and the restriction map commutes with the Bockstein.

The fact that the \(p^{r-\ell-j-1}\)-st power of (2.4.4) equals the relation

\[-X^{1,2}(\ell) \cdot X^{2,3}(\ell') + X^{2,3}(\ell) \cdot X^{1,2}(\ell') \]

of Theorem 1.1 is immediate from the identification of \(X^{1,2}(\ell)\) with \((x^{(\ell)}_\alpha)^{p^{r-\ell-1}}\) and \(X^{2,3}(\ell)\) with \((x^{(\ell)}_\alpha)^{p^{r-\ell-1}}\).

Corollary 2.5. If \(x^{(\ell)}_\alpha \in S^*(\oplus_{\ell=0}^{r-1}(\gamma_2/\gamma_3)^{\#(\ell+1)}[2])\) does not lie in the subspace \(S^*(\oplus_{\ell=0}^{r-1}(\gamma_2/\gamma_3)^{\#(\ell+1)}[2p^{r-\ell-1}])\), then there exists some differential of (2.4.4) which is non-zero on \(x^{(\ell)}_\alpha\).

Proof. We employ the fact that the differentials in the spectral sequence are \(k\)-linear derivatives. Consider a monomial \(w = \prod_{\ell=0}^{r-1}(x^{(\ell)}_\beta)^{n_\ell}\) with some \(n_\ell\) not divisible by \(p^{r-\ell-1}\). Let \(p^j\) be the smallest power of \(p\) such that there exists some \(\ell\) with \(j < r - \ell - 1\). \(p^j\) divides \(n_\ell\), and \(p^{j+1}\) does not divide \(n_\ell\). Then \(d^{p_{j+1}}_{2p^{j+1}}(w)\) is a sum of non-zero terms indexed by those \(\ell\) with \(p^j\) but not \(p^{j+1}\) dividing \(n_\ell\).

Moreover, each non-zero summand of \(d^{p_{j+1}}_{2p^{j+1}}(w)\) is a monomial of total degree 1 less than that of \(w\) of a form given by Proposition 2.2(2), and thus uniquely associated to the monomial \(w\). In other words, there is no cancellation occurring when one applies \(d^{2p^j}_{2p^j}\) to a sum of monomials of the form \(w\). Alternatively, one can
observe that the $T$-weight spaces of $S^*\left(\bigoplus_{i=0}^{r-1}(2p^i)[2]\right)$ in a given degree are 1-dimensional and that differential preserve the weights. This implies the corollary \[\square\]

We next proceed to “lift” the permanent cycles $(x^{(\ell)}_\beta)p^\ell$, $\ell + j \geq r$, to elements of $H^*\left((U_j/\Gamma_3)(r), k\right)$, determining the map

$$
\eta_{U_j/\Gamma_3,r}: S^*\left(\bigoplus_{i=0}^{r-1}(u_j/\gamma_3)[2]\right) \otimes S^*\left(\bigoplus_{i=0}^{r-1}(\gamma_2/\gamma_3)[2p^r-1]\right) \to H^*\left((U_j/\Gamma_3)(r), k\right).
$$

For this, we use a weight computation in the Andersen-Janzten spectral sequence of $k$ coordinate algebra of the vector group scheme $gr$. We employ the indexing of $[14]$.

**Proposition 2.6.** $[14]$, $[59]$ Let $H$ be an irreducible affine group scheme (over $k$) and let $I_1 \subset k[H]$ denote the maximal ideal at the identity of $H$. The filtration of $k[H]$ by powers of $I_1$ leads to an associated graded Hopf algebra which is the coordinate algebra of the vector group scheme $gr(H)$. For any rational $H$-module $M$, there is a naturally associated convergent spectral sequence

$$(2.6.1) \quad A^jE^{i,j}_1(H) = H^{i+j}(gr(H), k)_i \otimes M \Rightarrow H^{i+j}(H, M),$$

where $H^*(gr(H), k)_i$ is the cohomology algebra of the $i$th graded summand of the Hochschild complex of $gr(H)$.

If $G$ is a linear algebraic group and $p \neq 2$. Then $A^jE^{i,j}_1(G)$ can be identified with the direct sum of tensor products of the form

$$(2.6.2) \quad S^{a_1}(g\#(1)[2]) \otimes S^{a_2}(g\#(2)[2]) \otimes \cdots \otimes \Lambda^{b_1}(g\#(1)[1]) \otimes \cdots$$

where the sum is over all sequences $\{a_n\}, \{b_n\}$ with each $a_n \geq 0$, each $b_n \geq 0$ and

$$i = \sum_{n \geq 1} (a_n p^n + b_n p^{n-1}), \quad i + j = \sum (2a_n + b_n).$$

Moreover, for any $r \geq 1$, $A^jE^{i,j}_1(G_{(r)})$ can be identified with the direct sum of those tensor products of the form $(2.6.2)$ with $a_n = b_n = 0$, $n > r$.

We shall apply Proposition 2.6 to Frobenius kernels $U_{(r)}$ of a closed subgroup of a simple algebraic group $G$ stable under the (adjoint) action of a maximal torus $T$ of $G$. In this case, the spectral sequence $\{A^jE^{i,j}_1(U_{(r)}); s \geq 1\}$ admits a natural action of $T$ whose $T$-weights are identified using $(2.6.2)$.

The uniqueness given in the following proposition enables us to specify the map $\eta_{U_j/\Gamma_3,r}$. We presume that the somewhat strange condition on roots $\beta$ of level 2 holds for all $U_j$, but this condition becomes non-vacuous in the more general context of Section 3.

**Proposition 2.7.** Retain the notation and hypotheses of Propositions 2.6. Assume for each root $\beta$ of $U_j$ of level 2 there do not exist $2p$ distinct roots $\alpha_1, \ldots, \alpha_{2p}$ of $U_j$ of level 1 such that $\beta = \alpha_{2i-1} + \alpha_{2i}$ for each $i$, $1 \leq i \leq p$. Then there exists a unique $T$-equivariant $k$-linear map

$$\eta: (u_j/\gamma_3)^{(1)}[2p^{r-1}] \to H^{2p^{r-1}}((U_j/\Gamma_3)(r), k).$$

which fits in the following commutative diagram
\[(2.7.1)\]
\[
\begin{array}{c}
(u_j/\gamma_2)^{#(r)(2p^{-1})} \\
\eta \\
H^{2p^{-1}}((U_j/\Gamma_2)_{(r)}, k) \\
\end{array}
\begin{array}{c}
(u_j/\gamma_3)^{#(r)(2p^{-1})} \\
\rightarrow \\
H^{2p^{-1}}((U_j/\Gamma_3)_{(r)}, k) \\
\rightarrow \\
H^{2p^{-1}}((\Gamma_2/\Gamma_3)_{(r)}, k).
\end{array}
\]

Here, the left and right vertical maps are given by the inclusions $S^*((u_j/\gamma_2)^{(1)}[2]) \rightarrow H^*((U_j/\Gamma_2)_{(r)}, k)$ and $S^*((\gamma_2/\gamma_3)^{(1)}[2]) \rightarrow H^*((\Gamma_2/\Gamma_3)_{(r)}, k)$, the upper horizontal maps are the evident ones, the lower horizontal maps are those given by functoriality.

Proof. The existence of some $\eta$ fitting in diagram \[(2.7.1)\] is implied by Proposition \[(2.3.1)\]. Thus, to prove the proposition it suffices to verify for each root $\beta$ of $U_j$ of level 2 and that the $T$-weight space of $H^{2p^{-1}}((U_j/\Gamma_3)_{(r)}, k)$ of weight $p^r\beta$ is 1-dimensional. This would imply the uniqueness of the choice of class $\eta((x^{(0)}_\beta p^{-1}) \in H^{2p^{-1}}((U_j/\Gamma_3)_{(r)}, k)$ mapping to $(x^{(0)}_\beta p^{-1} \in H^{2p^{-1}}((\Gamma_2/\Gamma_3)_{(r)}, k)$.

We search in $AJ E^*_1$ as given in \[(2.6.2)\] for $T$-weight vectors with $T$-weight $p^r\beta$ and cohomology degree $2p^{-1}$ other than $(x^{(0)}_\beta p^{-1} \in A J E^{0,2p^{-1}}_1$. Consider a simple tensor of the specified weight and degree, in other words a monomial in $x$’s and $y$’s. Because the degree is even, there must be a wedge of an even number of $y$’s in the monomial. If some $y^{(0)}$ is a factor, then we would need $y_{a_1}^{(1)} \otimes y_{a_2}^{(1)} \otimes \ldots \otimes y_{a_{2p-1}}^{(1)} \otimes y_{a_{2p}}^{(1)}$ to divide our simple tensor with each $a_{2i-1} + a_{2i} = \beta$ in order for the weight to be divisible by $p$. Our hypothesis excludes this possibility, so that no $y^{(0)}$ is a factor.

None of the factors of this monomial can be of the form $x^{(l)}$ or $y^{(l)}$ for $l > 1$ because such a factor would increase the weight too “fast” with respect to increase of the resulting degree by either 2 or 1. The only remaining possibility is to “replace” factors $x^{(0)}_\beta$ by $y_{a_{2i-1}}^{(1)} \otimes y_{a_{2i}}^{(1)}$ with $\beta = a_{2i-1} + a_{2i}$. Since

\[
(\text{AJ} d^1_{1,0})^{2p^{-1}}(x^{(0)}_\beta) = 0 = \text{AJ} d^{1,0}_1(\alpha^{(1)}_i),
\]

the value of the derivation \[(\text{AJ} d^1_{1,0})^{2p^{-1}+n-1,2p^{-1}} \otimes y^{(1)} \wedge y_{a_1}^{(1)} \otimes \ldots \otimes y_{a_{2p}}^{(1)}\) applied to \[(x^{(0)}_\beta)^{2p^{-1}-n} \otimes y^{(1)} \wedge y_{a_1}^{(1)} \otimes \ldots \otimes y_{a_{2p}}^{(1)}\] equals \[(x^{(0)}_\beta)^{2p^{-1}-n} \otimes y^{(1)} \wedge y_{a_1}^{(1)} \otimes \ldots \otimes y_{a_{2p}}^{(1)}\] plus other terms in different tensor powers because the differentials preserve $T$-weight and increase cohomological degree by 1. Thus, the latter is not a permanent cycle in $\text{AJ} E^*_1$ so that the $p^r\beta$ weight space of $H^{2p^{-1}}((U_j/\Gamma_3)_{(r)}, k)$ is 1-dimensional. \hfill \Box

Definition 2.8. Retain the notation and hypotheses of Propositions \[(2.1)\]. We define $S^*((U_j/\Gamma_v)_{(r)})$ to be
\[(2.8.1)\]
\[
S^*((\otimes_{\ell=0}^{r-1}(u_j/\gamma_2)^{#(r-1)[2]})) \otimes S^*((\otimes_{\ell=0}^{r-1}(u_j/\gamma_2)^{#(r)[2p^{-1}-\ell-1]})) S^*((\otimes_{\ell=0}^{r-1}(u_j/\gamma_v)^{#(r)[2p^{-1}-\ell-1]})
\]

In other words, $S^*((U_j/\Gamma_v)_{(r)})$ is the coproduct in the category of commutative $k$ algebras of $S^*((\otimes_{\ell=0}^{r-1}(u_j/\gamma_2)^{#(r-1)[2]}))$ and $S^*((\otimes_{\ell=0}^{r-1}(u_j/\gamma_v)^{#(r)[2p^{-1}-\ell-1]})$ over $S^*((\otimes_{\ell=0}^{r-1}(u_j/\gamma_2)^{#(r)[2p^{-1}-\ell-1]})).
We define $Q((U_J/Γ_3)_{(r)})$ to be the quotient of $S^*(⊕_{ℓ=0}^{r-1}(u_j/γ_2)^{(ℓ-1)}[2])$ by the ideal generated by the elements of $S^*(H_q^l)$, which we denote by $J_2$:

$$Q((U_J/Γ_2)_{(r)}) \equiv S^*(⊕_{ℓ=0}^{r-1}(u_j/γ_2)^{(ℓ-1)}[2])/J_2.$$

We define $\overline{S}^*((U_J/Γ_3)_{(r)})$ to be the tensor product of $S^*((U_J/Γ_3)_{(r)})$ and $Q((U_J/Γ_2)_{(r)})$ over $S^*(⊕_{ℓ=0}^{r-1}(u_j/γ_2)^{(ℓ-1)}[2])$

We observe that there is a $T$-equivariant splitting

$$S^*(U_J/Γ_3) \simeq S^*(⊕_{ℓ=0}^{r-1}(u_j/γ_2)^{(ℓ+1)}[2]) \otimes S^*(⊕_{ℓ=0}^{r-1}(γ_2/γ_3)^{(r)}[2p^r-ℓ-1])$$

given by the $T$-equivariant splitting $u_j/γ_2 \simeq (u_j/γ_2) \oplus (γ_2/γ_3)$ which gives the $T$-equivariant splitting

$$\overline{S}^*((U_J/Γ_3)_{(r)}) \simeq Q((U_J/Γ_2)_{(r)}) \otimes S^*(⊕_{ℓ=0}^{r-1}(γ_2/γ_3)^{(r)}[2p^r-ℓ-1]).$$

We view $\overline{S}^*((U_J/Γ_3)_{(r)})$ as

$$\overline{S}^*((U_J/Γ_3)_{(r)}) \simeq S^*((U_J/Γ_3)_{(r)})/I_3$$

where $I_3 \subseteq S^*((U_J/Γ_3)_{(r)})$ is the ideal generated by $J_2 \otimes S^*(⊕_{ℓ=0}^{r-1}(γ_2/γ_3)^{(r)}[2p^r-ℓ-1])$.

**Example 2.9.** $\overline{S}^*((U_J)_{(r)})$ is generated by elements $(x_j^{(ℓ)})^{r-ℓ-1} \in (γ_2)^{(r)}[2p^r-ℓ-1]$ and $x^{(ℓ)}_{α^r} \in (u_3/γ_2)^{(r+2)}[2]$ with $0 \leq ℓ < r$. A set of relations for $\overline{S}^*((U_J)_{(r)})$ is given by

$$(x^{(ℓ)}_{α^r})^p \otimes x^{(ℓ+1)}_{α^r} - (x^{(ℓ)}_{α^r})^p \otimes x^{(ℓ+1)}_{α^r}, \quad 0 \leq j < r - ℓ - 1.$$  

Similarly, $Q((U_J/Γ_2)_{(r)})$ is generated by $x^{(ℓ)}_{α^r}, x^{(ℓ)}_{α^r} \in (u_3/γ_2)^{(r+2)}[2]$ for $0 \leq ℓ < r$ with the same set of relations (2.9.1).

**Definition 2.10.** Adopt the hypotheses and notation of Proposition 2.7. We define

$$(2.10.1) \quad η_{U_J/Γ_3,r} : S^*((U_J/Γ_3)_{(r)}) \to H^*((U_J/Γ_3)_{(r)}, k)$$

as follows. Restricted to $S^*(⊕_{ℓ=0}^{r-1}(u_j/γ_2)^{(ℓ-1)}[2])$, $η_{U_J/Γ_3}$ is defined as the composition of the natural map $S^*(⊕_{ℓ=0}^{r-1}(u_j/γ_2)^{(ℓ-1)}[2]) \to H^*((U_J/Γ_2)_{(r)}, k)$ and the restriction map $H^*((U_J/Γ_2)_{(r)}, k) \to H^*((U_J/Γ_3)_{(r)}, k)$. On $(γ_2/γ_3)^{(r)}[2p^r-ℓ-1]$, $η_{U_J/Γ_3,r}$ is defined to be $ℓ$-th Frobenius twist of the map of Proposition 2.7 $η : (u_j/γ_3)^{(r-ℓ)}[2p^r-ℓ-1] \to H^2\overline{S}^*((U_J/Γ_3)_{(r-ℓ)}, k)$; in other words, the pull-back along $(U_J/Γ_3)_{(r)} \to (U_J/Γ_3)_{(r-ℓ)}$ of $η$.

Justified by Proposition 2.4(2), we define

$$(2.10.2) \quad η_{U_J/Γ_3,r} : S^*((U_J/Γ_3)_{(r)}) \to H^*((U_J/Γ_3)_{(r)}, k)$$

to be the map whose composition with the quotient $q_{U_J/Γ_3,r} : S^*(U_J/Γ_3) \to S^*((U_J/Γ_3)_{(r)})$ equals $η_{U_J/Γ_3,r}$.

At the moment, the only unipotent group $U$ for which we have defined both $φ_{U,r}$ and $η_{U,r}$ is $U = U_3$, the Heisenberg group. We verify that our two definitions agree. Unlike the definition of $φ_{U,r}$, the definition of $η_{U,r}$ is intrinsic, without reference to an embedding of $U_3 \subset GL_N$. 

Proposition 2.11. The following maps are equal,
\[ \phi_{U_3, r} = \eta_{U_3, r} : S^r((\oplus_{i=0}^{r-1} (U_3)^{\#(r)} [2p^{r-\ell-1}] )) \to H^\bullet((U_3)(r), k), \]
where \( \phi_{U_3, r} \) is constructed in Proposition 1.3 and \( \eta_{U_3, r} \) in Definition 2.10 (restricted to \( S^r((\oplus_{i=0}^{r-1} (U_3)^{\#(r)} [2p^{r-\ell-1}] )) \)).

Proof. The uniqueness argument of Proposition 1.3 applies (in a simplified form) to roots of level 1 as well as roots of level 2 of \( U_j/\Gamma_3 \). Working in the special case \( U_j = U_3, \ \Gamma_3 = 1 \), we apply this uniqueness to any weight \( \beta \) of \( U_3 \). Observe that \( \phi_{U_3, r}(x_\beta^{i \ell}) \) has cohomological degree \( 2p^{r-\ell-1} \) and \( T \)-weight \( p^r \cdot \beta \), and lies in the image of \( (F^\ell)^* \), as does \( \eta_{U_3, r}(x_\beta^{i \ell}) \). Thus, the uniqueness property established in the proof of Proposition 1.3 implies the asserted equality. \( \square \)

We next verify the compatibility of \( \eta_{U_3, r} \) with \( \phi_{GL_3, r} \).

Proposition 2.12. There is a naturally constructed injective map
\[ (2.12.1) \quad \theta_{U_3, r} : k[V_r(U_3)] \to \overline{S^\bullet}((U_3)(r)). \]
with the property that the \( p^{r-1} \)-st power of an element in \( \overline{S^\bullet}((U_3)(r)) \) lies in the image of \( \theta_{U_3, r} \).

This maps fits in the following commutative diagram, strengthening diagram (1.7.1) for \( U = U_3 \):
\[ (2.12.2) \]
\[ \begin{array}{ccc}
S^r((\oplus_{i=0}^{r-1} (U_3)^{\#(r)} [2p^{r-\ell-1}] )) & \xrightarrow{q_{GL_3}} & k[V_r(GL_3)] \\
\phi_{GL_3, r} & \phi_{GL_3, r} & \phi_{GL_3, r} \\
\eta_{U_3, r} & \eta_{U_3, r} & \eta_{U_3, r} \\
S^\bullet((U_3)(r)) & \xrightarrow{\pi_{U_3, r} \circ \theta_{U_3, r}} & k[V_r(U_3)] \\
S^\bullet((U_3)(r)) & \xrightarrow{\pi_{U_3, r} \circ \theta_{U_3, r}} & k[V_r(U_3)] \\
\end{array} \]

Proof. Because \( V_r(U_3) \subset \mathfrak{u}_3^{x^r} \) is the pull-back along \( \mathfrak{u}_3^{x^r} \subset \mathfrak{gl}_3^{x^r} \) of \( V_r(GL_3) \subset \mathfrak{gl}_3^{x^r} \), the upper left square is cocartesian. Thus, to define \( \theta_{U_3, r} \) it suffices to define \( \theta_{U_3, r} \circ i_{V_r} \) which is exhibited using the fact that the defining relations for \( q_{GL_3} \) (given in Theorem 1.3) are mapped to the \((p^{r-\ell-1})\)-st power of the relation \((x_\alpha^{i \ell})^{p^{r-\ell-1}} \times x_{\alpha'}^{(\ell+1+j)} - (x_{\alpha'}^{i \ell})^{p^{r-\ell-1}} \times x_\alpha^{(\ell+1+j)}) \) of (2.4.4). (Here, \( \alpha, \alpha' \) are weights of level 1.) By construction, the lower left square commutes as does the lower middle triangle.

The injectivity of \( \theta_{U_3, r} \) is shown by checking that the intersection in \( S^\bullet((U_3)(r)) \) of \( S^\bullet((X^{1.2}(\ell), X^{2.3}(\ell), X^{1.3}(\ell); 0 \leq \ell < r)) \) with the ideal defining \( \overline{S^\bullet}((U_3)(r)) \) is precisely the ideal defining \( k[V_r(U_3)] \). The generators \( x_\alpha^{i \ell}, x_{\alpha'}^{i \ell} \) have \( p^{r-\ell-1} \)-st power in the image of \( \theta_{U_3, r} \), whereas the generators \( (x_\beta^{i \ell})^{p^{r-\ell-1}} \) are in the image of \( \pi_{U_3, r} \); thus, the \( p^{r-1} \)-st power of any element in \( \overline{S^\bullet}((U_3)(r)) \) lies in the image of \( \theta_{U_3, r} \).
The commutativity of the left square of (2.12.2) is that of Proposition 1.6. The commutativity of the middle square (2.12.2) follows from the commutativity of the rectangle composed of the left and middle squares given by Proposition 2.11, the surjectivity of $q_{GL_3}$, and the commutativity of the left square. The commutativity of the right square follows from the naturality of $\psi$. □

**Proposition 2.13.** Adopt the hypotheses and notation of Proposition 2.7. The map $\eta_{U_J/\Gamma_3}: S^1((U_J/\Gamma_3)(r)) \to H^1(Q((U_J/\Gamma_3)(r)), k)$ restricts to $f_Q: Q((U_J/\Gamma_2)(r)) \to E^0((U_J/\Gamma_3)(r))$ and projects to the natural inclusion

$g_Q: S^1(\gamma_2^\#(r)[2p^r-t-1]) \to E^0((U_J/\Gamma_3)(r))$, determining commutative diagrams of algebras

(2.13.1)

$$S^1(\oplus_{s=0}^{r-1}(\gamma_1/\gamma_2)^{(r+1)[2]}) \xrightarrow{f_Q} Q((U_J/\Gamma_2)(r)) \xrightarrow{\eta_{U_J/\Gamma_3}} S^1((U_J/\Gamma_3)(r))$$

(2.13.2)



Moreover, for any element $z \in E^0((U_J/\Gamma_3)(r))$, there exists some $\tilde{z} \in Q((U_J/\Gamma_2)(r))$ such that $z - f_Q(\tilde{z})$ has square 0; similarly, for any element $w \in E^0((U_J/\Gamma_3)(r))$, there exists some $\tilde{w} \in S^1(\oplus_{s=0}^{r-1}(\gamma_2/\gamma_3)^{(r)}[2p^r-t-1])$ such that $w - g_Q(\tilde{w})$ has square 0.

**Proof.** By Proposition 2.4(2) and the definition of $Q((U_J/\Gamma_2)(r))$, the left square of (2.13.1) commutes. By definition of $\eta_{U_J/\Gamma_3}$ in Definition 2.10 the right square of (2.13.1) also commutes.

The commutativity of the outer square of (2.13.2) arises from the naturality of the restriction maps for $\Gamma_2/\Gamma_3 \to U_3/\Gamma_3$. The commutativity of the two squares of (2.13.2) thus follows from the fact that $S^1(\oplus_{s=0}^{r-1}(\gamma_2/\gamma_3)^{(r)}[2p^r-t-1])$ is the image of the upper composition of (2.13.2) and the fact that $E^0((U_J/\Gamma_3)(r))$ is the image of the restriction map $H^1((U_J/\Gamma_3)(r), k) \to H^1((U_J/\Gamma_2)(r), k)$ by a standard property of Grothendieck spectral sequences.

The surjectivity statement for $f_Q$ follows from the surjectivity modulo squares of the left vertical arrow of diagram (2.13.1). The surjectivity statement for $g_Q$ follows from Corollary 2.5 and the fact that the lower right map of (2.13.2) is an isomorphism modulo squares. □

We give $S^1((U_J/\Gamma_3)(r)) = Q^1((U_J/\Gamma_2)(r)) \otimes S^1(\oplus_{s=0}^{r-1}(\gamma_2/\gamma_3)^{(r)}[2p^r-t-1])$ the filtration associated with the degree of $Q^1((U_J/\Gamma_2)(r))$, so that $F^d(S^1((U_J/\Gamma_3)(r)))$ equals $\oplus_{l \geq 0}Q^d((U_J/\Gamma_2)(r)) \otimes S^1(\oplus_{s=0}^{r-1}(\gamma_2/\gamma_3)^{(r)}[2p^r-t-1])$. We give $H^1((U_J/\Gamma_3)(r), k)$ the filtration associated to the spectral sequence (2.4.1).

**Proposition 2.14.** Adopt the hypotheses and notation of Proposition 2.7.
(1) We may identify $\pi_{U_j/G_3,r}$ with the tensor product of maps

\[(2.14.1) \quad \pi_{U_j/G_3,r} = f_Q \otimes g_Q : S^*((U_j/G_3)_r) \to H^*((U_j/G_3)_r,k).\]

Here, $f_Q : Q^*((U_j/G_2)_r) \to E^*_\infty((U_j/G_3)_r) \subset H^*((U_j/G_3)_r,k)$ and $g_Q$ is the restriction of $\pi_{U_j/G_3,r}$ to $S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}])$.

(2) $\pi_{U_j/G_3,r}$ is a map of filtered algebras (doubling cohomological and filtration degree) with associated graded map

\[gr(\pi_{U_j/G_3,r}) = f_Q \otimes g_Q : S^*((U_j/G_3)_r) \to E^*_\infty((U_j/G_3)_r).\]

(3) $gr(\pi_{U_j/G_3,r})$ is surjective modulo squares: for each element $z \in E^*_\infty((U_j/G_3)_r)$ there exists some element $w \in S^*((U_j/G_3)_r)$ such that $z - gr(\pi_{U_j/G_3,r})(w) \in E^*_\infty((U_j/G_3)_r)$ has square 0.

Proof. The splitting $S^*((U_j/G_3)_r) \simeq Q^*((U_j/G_3)_r) \otimes S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}]$ of (2.13.2), the fact that $\pi_{U_j/G_3,r}$ is a map of algebras, and Proposition 2.13 imply the identification in (2.14.1).

The map $\pi_{U_j/G_3} : Q^*((U_j/G_2)_r) \otimes S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}]) \to H^*((U_j/G_3)_r,k)$ restricted to $Q^*((U_j/G_2)_r) \otimes 1$ is induced by $U_j/G_3 \to U_j/G_2$ and thus is filtration preserving. Since $1 \otimes S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}] \subset S^*((U_j/G_3)_r)$ has filtration degree 0, the commutativity of the left square of (2.13.2) implies that $\pi_{U_j/G_3,r}$ restricted to $1 \otimes S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}]$ is also filtration preserving. The multiplicativity properties of these filtrations thus imply that $\pi_{U_j/G_3,r}$ itself is a map of filtered algebras.

By Proposition 2.13, the multiplicative map $gr(\pi_{U_j/G_3,r})$ equals the composition of

\[f_Q \otimes g_Q : Q^*((U_j/G_2)_r) \otimes S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}] \to E^*_\infty \otimes E^*_\infty\]

with the natural map $E^*_\infty \otimes E^*_\infty \to E^*_\infty$ arising from the multiplicative structure. Because $g_Q$ is a lifting of $S^*((U_j/G_3)_r) \to S^*((\oplus_{\ell=0}^{\infty}(\gamma_2/\gamma_3)^{[\ell]}[2p^{\ell-1}] \to E^*_\infty$, we conclude that $gr(\pi_{U_j/G_3,r}) = f_Q \otimes g_Q : S^*((U_j/G_3)_r) \to E^*_\infty((U_j/G_3)_r)$. To prove (3), we first observe that $S^*((\oplus_{\ell=0}^{\infty}(\eta_j/\gamma_3)^{[\ell+1]}[2]) \to E^*_2$ is surjective modulo squares. Consider the following commutative diagram

\[\begin{array}{ccc} S^*((U_j/G_3)_r) & \longrightarrow & S^*((\oplus_{\ell=0}^{\infty}(\eta_j/\gamma_3)^{[\ell+1]}[2]) \\ & E^*_\infty \longleftarrow & Z^*_\infty \longleftarrow E^*_2 \end{array}\]

where $Z^*_\infty \subset E^*_2$ is the (bigraded) subalgebra of permanent cycles mapping subjectively to $E^*_2$. By Corollary 2.5, the right square is a pull-back of $(k)$-vector spaces. Given a bighomogeneous class $z \in E^*_2$, consider a lifting $\tilde{z} \in Z^*_2$. There exists some $w \in S^*((\eta_j/\gamma_3)^{[\ell+1]}[2])$ whose image in $E^*_2$ equals $\tilde{z} + t$, where $t \in E^*_2$ satisfies $t^2 = 0$. Since $d^*_2(t^2) = 0$, $d^*_2(w) = d^*_2(t)$ and thus $d^*_2(t^2) = d^*_2(t^2) = 0$ for any $s \geq 2$; thus, $w$ lies in $S^*((U_j/G_3)_r)$ by Corollary 2.5. Since $z - gr(\pi_{U_j/G_3,r})(w)$ in $E^*_2$ has square 0, assertion (3) follows. \qed
We can not adapt the surjectivity argument of Proposition 2.13 for the AJ filtration in large part because we have not established Steenrod operations in the Andersen-Jantzen spectral sequence. In particular, we have not shown that \((x_j^{(0)})^{p_j}\) is a permanent cycle in that spectral sequence if \(\ell > 0\). We do provide some information about this filtration in the following proposition.

**Proposition 2.15.** With respect to the Andersen-Jantzen filtrations of Proposition 2.6, the map \(\eta_{UJ/\Gamma_3} : S^*(\langle U_J/\Gamma_3 \rangle (r)) \to H^*(\langle U_J/\Gamma_3 \rangle (r), k)\) preserves filtrations.

Moreover, \(A^Jgr(\eta_{UJ/\Gamma_3}) : S^*(\langle U_J/\Gamma_3 \rangle (r)) \to A^J E^{r,*}_\infty\) factors through the map \(f_Q \otimes g_Q\) of Proposition 2.14.

**Proof.** To prove that \(\eta_{UJ/\Gamma_3,r}\) preserves filtrations, it suffices to observe that \(\eta_{UJ/\Gamma_3,r}\) restricted to \(S^*(\oplus_{j=0}^{r-1}(u_j/\gamma_2)^{\#(\ell+1)}[2])\) preserves filtrations by functoriality and that \(\overline{\eta}_{UJ/\Gamma_3,r}\) restricted to \(S^*(\oplus_{j=0}^{r-1}(\gamma_2/\gamma_3)^{\#(\ell+1)}[2])\) (using the \(T\)-equivariant splitting of \(\gamma_2/\gamma_3 \to u_j/\gamma_3\)) preserves filtrations thanks to the definition given in Definition 2.10 based upon the construction given in the proof of Proposition 2.7.

To show that \(A^Jgr(\eta_{UJ/\Gamma_3,r})\) factors through \(f_Q \otimes g_Q\) of Proposition 2.14 we first observe that functoriality implies that the restriction to \(S^*(\oplus_{j=0}^{r-1}(u_j/\gamma_2)^{\#(\ell+1)}[2])\) of \(\overline{\eta}_{UJ/\Gamma_3,r}\) is given by \(f_Q\). Since the Andersen-Jantzen filtrations on \(Q\) and on \(S^*(\oplus_{j=0}^{r-1}(u_j/\gamma_2)^{\#(\ell+1)}[2])\) both split, we conclude that \(A^Jgr(\eta_{UJ/\Gamma_3,r})\) restricted \(S^*(\oplus_{j=0}^{r-1}(u_j/\gamma_2)^{\#(\ell+1)}[2])\) equals \(f_Q\).

The restriction of \(\eta_{UJ/\Gamma_3,r}\) to \(S^*(\oplus_{j=0}^{r-1}(\gamma_2/\gamma_3)^{\#(\ell+1)}[2])\) is specified by the construction of Proposition 2.7 (in the statement of Proposition 2.14 this is \(\overline{\eta}Q\) whose associated graded map with respect to \(\text{L-H-S}\) filtrations is \(g_Q\)). Yet this construction is formulated using \(A^J E^{r,*}_\infty\), so that the associated graded map with respect to AJ filtrations is also \(g_Q\).

We provide further information about the maps \(\theta_{U3,r} : k[V_r(U3)] \to S^*(\langle U3 \rangle (r))\) and \(\overline{\theta}_{U3,r} : k[Y_r(U3)] \to Q^*(\langle U3/\Gamma_2 \rangle (r))\), including the observation that \(Q^*(\langle U3/\Gamma_2 \rangle (r))\) and thus \(S^*(\langle U3 \rangle (r))\) are domains.

**Proposition 2.16.** The map \(\theta_{U3,r} : k[V_r(U3)] \to S^*(\langle U3 \rangle (r))\) of Proposition 2.14 can be identified as the tensor product map

\[
\overline{\theta}_{U3,r} \otimes 1 : k[V_r(U3)] = k[Y_r(U3)] \otimes S^*(\{X^{1,3}(\ell), 0 \leq \ell < r\}) \to Q^*(\langle U3/\Gamma_2 \rangle (r)) \otimes S^*(\{\gamma_2^{\#(\ell)}|2p^{r-\ell-1}\}) = S^*(\langle U3 \rangle (r))
\]

The map \(\overline{\theta}_{U3,r} : k[Y_r(U3)] \to Q^*(\langle U3/\Gamma_2 \rangle (r))\) is a finite map of integral domains of degree \(p^{(r+2)r-1}/2\) obtained by taking \(p^{r-\ell-1}\)-st roots of \(X^{1,2}(\ell), X^{2,3}(\ell)\) for each \(\ell, 0 \leq \ell < r\). Consequently, \(S^*(\langle U3 \rangle (r))\) is also an integral domain.

**Proof.** The fact that \(\theta_{U3,r}\) is a tensor product of the form \(\overline{\theta}_{U3,r} \otimes 1\) arises from the fact that the tensor decomposition of \(k[V_r(U3)]\) in Proposition 2.9 and that of \(\overline{\theta}^*\) of \(\langle U3 \rangle (r)\) in 2.8,2 both arise because the relations do not involve weights of level 2.

Arguing as in the proof of Proposition 2.9 we verify that \(Q^*(\langle U3/\Gamma_2 \rangle (r))(x_\alpha^{(0)\ell})^{-1}\) is the localization of the polynomial algebra on generators \(x_\alpha^{(0)\ell}, 0 \leq \ell < r; Y_1^{1,2}(0)\) with \(x_\alpha^{(0)}\) inverted. Thus, to show that \(Q^*(\langle U3/\Gamma_2 \rangle (r))\) is a domain it suffices to
show that the localization map $Q^*((U_3/T_2)_{(r)}) \to Q^*((U_3/T_2)_{(r)}))[(x_0^{(0)})^{-1}]$ is injective. This is verified by examining the relations (2.18.3) to show that $x_0^{(0)} \in Q^*((U_3/T_2)_{(r)})$ is not a zero-divisor.

Because $k[V_r(U_3)]$ is a domain, $F^r = \psi_{U_3,r} \circ \theta_{U_3,r} : k[V_r(U_3)] \to k[V_r(U)]$ is injective and thus $\theta_{U_3,r}$ is injective. Since $S^*((\oplus_{\ell=0}^{r-1} (U_3)^{\#(r)}[2p^{r-\ell-1}]) \to S^*((U_3)_{(r)})$ is obtained by taking $p^{r-\ell-1}$-st roots of $(x_0^{(0)})^{pr-\ell-1}$, $(\alpha^{(0)})^{pr-\ell-1}$ for each $0 \leq \ell < r$, we conclude that $Q^*((U_3/T_2)_{(r)})$ is similarly obtained from $k[V_r(U_3)]$.

To compute the degree of $\overline{\eta}_{U_3,r}$, we consider the map $k[V_r(U_3)][X^{1,2}] \to Q^*((U_3/T_2)_{(r)}))[(x_0^{(0)})^{-1}]$ and utilize the facts that $x_0((t)$ is the $p^{r-\ell-1}$-st root of the image of $X^{2,3}(t)$ and that $x_0^{(0)}$ is the $p^{r-1}$-st root of the image of $X^{1,2}(0)$ (using the notation of Proposition 1.19).

The following theorem is a culmination of previous propositions.

**Theorem 2.17. Retain the notation of Proposition 2.14.** Then

1. $\overline{\eta}_{U_3,r} : \overline{\eta}^* ((U_3)_{(r)}) \to H^*((U_3)_{(r)}, k)$ is injective.
2. $gr(\overline{\eta}_{U_3,r}) : \overline{\eta}^* ((U_3)_{(r)}) \to gr(H^*((U_3)_{(r)}, k)$ is also injective.
3. $gr(\overline{\eta}_{U_3,r})$ is surjective modulo squares (i.e., every element of $\xi \in gr(H^*((U_3)_{(r)}, k)$ there exists some $z \in \overline{\eta}((U_3)_{(r)})$ such that $\xi - gr(\overline{\eta}_{U_3,r})(z)$ has square 0).

**Proof.** Observe that $\overline{\phi}_{U_3,r} : k[V_r(U_3)] \to H^*((U_3)_{(r)}, k)$ is injective because the composition with $\psi_{U_3}$ equals $F^r$ which is an injective endomorphism of the $k[V_r(U_3)]$ (which is an integral domain by Proposition 1.10). Since the $p^{r}$-th power of any non-zero element of $\overline{\eta}^* ((U_3)_{(r)})$ is a non-zero element of $k[V_r(U_3)]$ by Proposition 2.16 we conclude that $\overline{\eta}_{U_3,r}$ must also be injective since $\overline{\eta}_{U_3,r}$ extends $\phi_{U_3,r}$.

The surjectivity of $gr(\overline{\eta}_{U_3,r})$ can be seen by inspection using the equality $gr(\overline{\eta}_{U_3,r}) = f_Q \otimes g_Q$ of Proposition 2.14.

The statement of the surjectivity modulo squares of $gr(\overline{\eta}_{U_3,r})$ is verified in Propositions 2.18.

Theorem 2.17 easily implies the following result for $H^*((U_3)_{(r)}, k)$.

**Proposition 2.18.** The restriction of $\overline{\eta}_{U_3,r} : \overline{\eta}^* ((U_3)_{(r)}) \to H^*((U_3)_{(r)}, k)$ to $(T_3)_{(r)}$-invariants equals the map

$$\overline{\phi}_{B_3,r} : k[V_r(U_3)] \to H^*((B_3)_{(r)}, k).$$

This is an injective map of $k$-algebras which is surjective onto $p^r$-th powers.

More generally, the restriction of $\overline{\eta}_{U_3/T_3,r} : \overline{\eta}^* ((U_3/T_3)_{(r)}) \to H^*((U_3/T_3)_{(r)}, k)$ to $(T_3)_{(r)}$-invariants equals the map

$$\overline{\phi}_{T \cdot U_3/T_3,r} : k[V_r((T \cdot U_3)_{(r)}, k) \to H^*((T \cdot U_3)_{(r)}, k).$$

**Proof.** We verify by inspection the equality

$$H^0((T_3)_{(r)}, \overline{\eta}^* ((U_3)_{(r)})) = S^*\left(\oplus_{\ell=0}^{r-1} (U_3)^{\#(r)}[2p^{r-\ell-1}\right] = k[V_r(U_3)].$$

Moreover, the spectral sequence for the extension

$$1 \to (U_3)_{(r)} \to (B_3)_{(r)} \to (T_3)_{(r)} \to 1$$

and the semi-simplicity of $(T_3)_{(r)}$ imply the equality

$$H^0((T_3)_{(r)}, H^*((U_3)_{(r)}, k) = H^*((B_3)_{(r)}, k).$$
The fact that the restriction of $\pi_{U_j,r}$ to these $(T_3)_{(r)}$-invariants equals $\overline{\phi}_{B_3,r}$ follows from (2.4.3) and the fact that $\pi_{U_3,r}$ restricted to $S^*((\oplus_{\ell=0}^{r-1}u_3)\#(r)[2p^r-\ell-1])$ equals $\overline{\phi}_{U_3,r}$ by Proposition 2.11.

The injectivity of $\overline{\phi}_{B_3,r}$ follows from the injectivity of $\overline{\phi}_{U_3,r}$. By Theorem 2.17 the $p^r$-th element of any $\zeta \in H^*(((B_3)_{(r)}, k)$ lies in $S^*((U_3)_{(r)})$; such an element must lie in $k[V_r(U_3)]$ because it is $(T_3)_{(r)}$-invariant.

Consider the short exact sequence

$$1 \to U_j \to \Gamma_3 \to (U_j/\Gamma_3) \to 1,$$

of algebraic groups restricting to the short exact sequence

$$1 \to (U_j/\Gamma_3)_{(r)} \to (U_j/\Gamma_3) \to T_{(r)} \to 1.$$  

The argument of the first paragraph applies to this short exact sequence, implying that $H^0(T_{(r)}, \pi_{U_j/\Gamma_3,r})$ equals (2.18.2).

We briefly consider the variety $X_r(U_j/\Gamma_3)$, equal to $V_r(U_3)$ in the special case $U_j = U_3$. As we verify below, $X_r(U_4/\Gamma_3)$ is not irreducible.

**Proposition 2.19.** Assume $p > 2$ and $r > 1$. Define $X_r(U_j/\Gamma_3)$ to be the prime ideal spectral of the quotient of $S^*((\oplus_{\ell=0}^{r-1}u_j)\#(r)[2p^r-\ell-1])$ by the ideal generated by elements of (2.4.3).

Then $X_r(U_4/\Gamma_3)$ is isomorphic to the product $\overline{X_r(U_4/\Gamma_3) \times A^{2r}}$, where $Y_r(U_4/\Gamma_3) = V_1 \cup V_2$ is a non-trivial union of two irreducible affine subvarieties. The first component, $V_1$, is isomorphic to $A^{4r}$ and therefore is smooth. The second, $V_2$, has dimension $3r + 2$, is smooth outside of $V_1 \cap V_2$; moreover, $V_1 \cap V_2 \subset V_2$ is the singular locus of $V_2$ and has codimension 1 in $V_2$.

**Proof.** Assume $r \geq 2$. As in Proposition 13, we use a presentation of $k[X_r(U_4/\Gamma_3)]$ based on the presentation of $V_r(GL_N)$ in Theorem 1.1. Observe that $X_r(U_4/\Gamma_3)$ is a product of the affine space $A^{2r}$ with coordinate algebra $S^*((X^{1,3}(\ell), X^{2,4}(\ell); 0 \leq \ell < r])$ and a variety $Y_r(U_4/\Gamma_3)$ given as the spectrum of the quotient of the $k$-algebra $k[X^{s,s+1}(\ell); 1 \leq s < 4; 0 \leq \ell < r]$ by the ideal generated by the relations $(X^{s,s+1}(\ell)) \cdot (X^{s,s+1}(\ell') \cdot X^{s+1,s+2}(\ell')) = 0$.

We proceed to analyze the irreducible component structure of $Y_r(U_4/\Gamma_3)$ (and thus $X_r(U_4/\Gamma_3)$). Consider the map $f_{2,3} : Y_r(U_4/\Gamma_3) \to A^r$ sending an element of $A \in Y_r(U_4/\Gamma_3)$ (represented by an $r$-tuple of strictly upper triangular $4 \times 4$ matrices) to its entries $A(0)_{2,3}, \ldots, A(r-1)_{2,3}$. Let $Y_1$ denote the zero set of $f_{2,3}(0)$: the entries $A(\ell)_{1,2}, A(\ell')_{3,4}$ of an element of $Y_1$ can be arbitrary, so that $Y_1 \simeq A^{2r}$. The fiber above some $0 \neq (a^{0}_{2,3}, \ldots, a^{r-1}_{2,3})$ has dimension 2: if for example $a^{0}_{2,3} \neq 0$, then a point in the fiber above $(a^{0}_{2,3}, \ldots, a^{r-1}_{2,3})$ is given by an arbitrary choice of $A(0)_{1,2}, A(0)_{3,4}$. Let $Y_2$ denote the closure in $Y_r(U_4/\Gamma_3)$ of $f_{2,3}^{-1}(A^r - \{0\})$, so that $Y_2$ has dimension $r + 2$. Thus, $Y_r(U_4/\Gamma_3)$ is the non-trivial union of the two irreducible closed subsets $Y_1$, $Y_2$.

Observe that $Y_1$ is the zero locus of the equations $\{X^{2,3}(\ell); 0 \leq \ell < r\}$ whereas $Y_2$ is the zero locus of the equations $\{X^{1,2}(\ell) \cdot X^{3,4}(\ell') - X^{1,2}(\ell') \cdot X^{3,4}(\ell)\}$. An open subset of the intersection $Y_1 \cap Y_2$ is given by setting $X^{1,2}(0) \neq 0$: projecting this open subset to $A^r$ by taking the (1,2)-coordinate, we see that a point in the fiber above some $(a^{0}_{1,2}, \ldots, a^{r-1}_{1,2})$ with $a^{0}_{1,2} \neq 0$ is specified by an arbitrary choice of $A(0)_{1,2}$. Consequently, $Y_1 \cap Y_2$ has codimension 1 in $Y_2$. 


Finally, $Y_2 - Y_1 \cap Y_2 = f_{2,3}^{-1}(A^r - \{0\})$ is smooth, with fibers of $f_{2,3}$ above a point of $A^r - \{0\}$ isomorphic to $A^2$, whereas the points of $Y_1 \cap Y_2$ are singular in $Y_2$: we require all of the equations $X_{2,3}(\ell)$, $0 \leq \ell < r$ to carve out this intersection. □

We suggest a pattern for the irreducible components of $X_r(U_N/\Gamma_3)$ extending Proposition 2.19 and Proposition 2.30.

Remark 2.20. Assume $p > 2$. Then for $N \geq 3$, the variety $X_r(U_N/\Gamma_3) \simeq Y_r(U_N/\Gamma_3) \times A^{(N-3)r}$ is irreducible for $r = 1$ or $N = 3$. For $N > 3$ and $r > 1$, this variety is reducible.

The irreducible components of $Y_r(U_N/\Gamma_3)$ appear to be indexed by the set $S$ consisting of sub-diagrams of the Dynkin diagram for $A_{N-1}$ obtained recursively as follows: the full Dynkin diagram is in $S$, and recursively a sub-diagram $D'$ is in $S$ if it can be obtained by removing a node of a sub-diagram $D$ in $S$ in such a way that $D'$ has 1 more component than $D$. The irreducible component $V_D \subseteq Y_r(U_N/\Gamma_3)$ corresponding to some $D \in S$ appears to be a rational variety of dimension equal the sum of the number of nodes of $D$ plus $(r - 1)$ times the number of components of $D$. For any $D$, the complement of $\cup_{D' \subseteq D} V_{D'}$ in $V_D$ appears to be smooth.

In particular, it appears that the Frobenius map $F : V_r(U_N/\Gamma_3) \to V_r(U_N/\Gamma_3)$ is injective.

3. The map $\eta_{U_J/\Gamma_{v+1}} : \overline{S}^{\star}((U/\Gamma_{v+1})_{(r)}) \to H^{\star}((U/\Gamma_{v+1})_{(r)}, k)$

We proceed by ascending induction on $v$ to define $\eta_{U_J/\Gamma_{v+1,r}} : S^{\star}((\sum_{\ell=0}^{r-1} (u_J/\gamma_{v+1})^{\#(r)[2p^{r-\ell-1}]})) \to H^{\star}((U/\Gamma_{v+1})_{(r)}, k)$ extending $\eta_{U_J/\Gamma_{3,r}}$ of Section 2 (in the special case $v = 1$). Despite some cumbersome notation and inductive arguments, the reader will find that the constructions and results of this section are natural extensions of those of Section 2. In order to execute our inductive arguments, we consider terms $\Gamma_i = \Gamma_i(U_J)$ in the descending central series of $U_J$ and various quotients $\Gamma_i/\Gamma_{v+1}$. Theorems 3.9 and 3.10 establish various good properties of $\eta_{U_J/\Gamma_{v+1,r}}$ and its induced map $\overline{\eta}_{U_J/\Gamma_{v+1,r}}$, extending many of the results of Section 2.

We begin our somewhat lengthy inductive argument with the following consequence for central extension $1 \to \Gamma_v/\Gamma_{v+1} \to U_J/\Gamma_{v+1} \to U_J/\Gamma_v \to 1$. The statement of Proposition 3.1 is very close to that of Proposition 2.2 except that we use the inductively defined map $\eta_{U_J/\Gamma_{v,r}}$ to designate values of differentials in the spectral sequence.

Proposition 3.1. Retain the notation and hypotheses of Proposition 2.1. For some $v \geq 2$, consider the $T$-equivariant Lyndon-Hochschild-Serre spectral sequence for the central extension

\begin{equation}
1 \to \Gamma_v/\Gamma_{v+1} \to U_J/\Gamma_{v+1} \to 1 \to U_J/\Gamma_v \to 1
\end{equation}

which takes the form

\begin{equation}
E_2^{a,b}(U_J/\Gamma_{v+1}) = H^a(U_J/\Gamma_v, k) \otimes H^b(\Gamma_v/\Gamma_{v+1}, k) \Rightarrow H^{a+b}(U_J/\Gamma_{v+1}, k).
\end{equation}

For any $\ell \geq 0$, $j \geq 0$ and $\beta$ a root of $U_J$ of level $v$:
The formula (3.1.3) is computed exactly as for Proposition 2.2(2). The vanishing of the $U$-structure and the restriction of $R$ follows from the fact that $\beta = \alpha + \alpha'$ and $\alpha < \alpha'$.

Proof. Given $\alpha < \alpha'$ with $\alpha + \alpha' = \beta$, let $R_{\alpha, \alpha'} \subset U_J/\Gamma_v + 1$ be the subgroup generated by the root subgroups $U_{\alpha}, U_{\alpha'}$ and $U_{\beta}$. Then $R_{\alpha, \alpha'} \simeq U_3$ so that Proposition 2.2(1) applies to $R_{\alpha, \alpha'}$. Using the naturality of the LHS spectral sequence we conclude that the restriction of $d_{2, \alpha}^{\beta}((x_{\beta}^{(l)})^{p^l})$ to $H^2(R_{\alpha, \alpha'}, k)$ equals $y_{\alpha}^{(l)} \wedge y_{\alpha'}^{(l)}$. A weight argument now implies statement (1).

To verify statement (2), we see that the restriction of $d_{2, \alpha}^{\beta}((x_{\beta}^{(l)})^{p^l})$ to $H^2(R_{\alpha, \alpha'}, k)$ is non-zero so that $d_{2, \alpha}^{\beta}((x_{\beta}^{(l)})^{p^l})$ itself is also nonzero. The formula (3.1.3) is computed exactly as for Proposition 2.2(2). The vanishing of the restriction to $H^2(J/\Gamma_v + 1, ((U_J/\Gamma_v)_r, k))$ if $\ell + 1 + j \geq r$ follows from the fact that $y_{\alpha}^{(l)} = 0$ whenever $\ell + 1 + j \geq r$.

The fact that the image of (3.1.3) equals $\beta P^{p^l}(d_{2, \alpha}^{\beta}((x_{\beta}^{(l)})^{p^l}))$ follows as argued for Proposition 2.2(3).

The final statement follows from the observation that the restriction map $H^{2p^l + 1, 2}((U_J/\Gamma_v)_r, k) \to H^{2p^l + 1, 2}((U_J/\Gamma_v + 1)_r, k)$ commutes with $\beta P^{p^l}$ in view of the naturality of Steenrod operations, and that $d_{2, \alpha}^{\beta}((x_{\beta}^{(l)})^{p^l})$ maps to zero in $H^{2p^l + 1, 2}((U_J/\Gamma_v + 1)_r, k)$ because it is a boundary in the spectral sequence.

Corollary 3.2. Retain the notation and hypotheses of Proposition 2.1. For some $v \geq 2$, consider the Lyndon-Hochschild-Serre spectral sequence 3.2.3 for the central extension

$$\alpha, \alpha$$}

which takes the form

$$E_2^{a, b}((U_J/\Gamma_v + 1)_r, k) \Rightarrow H^a((U_J/\Gamma_v)_r, k) \otimes H^b((\Gamma_v/\Gamma_v + 1)_r, k) \Rightarrow H^{a+b}((U_J/\Gamma_v + 1)_r, k).$$
Then for any root $\beta$ of $U_J$ of level $v$, $(x^{(0)}_{\beta})^{p^{\ell}} \in H^{\bullet}(\Gamma_v/\Gamma_{v+1}(r), k)$ is a permanent cycle if and only if $\ell + j \geq r - 1$.

Proof. If $\ell + j \geq r - 1$ so that $y_{\alpha^{\ell+1+j}} = y_{\alpha^{\ell+1+j}}$, then (3.1.3) implies that $(x^{(0)}_{\beta})^{p^{\ell}}$ is a permanent cycle.

Let $R_{\alpha, \alpha'}$ denote the quotient of $R_{\alpha, \alpha'}$ by its center, $U_{\beta} \simeq \mathbb{G}_a$. If $\ell + j < r - 1$, then (3.1.3) restricted to $H^{2p^{\ell}}((R_{\alpha, \alpha'})(r), k)$ is non-zero so that $(x^{(0)}_{\beta})^{p^{\ell}}$ is not a permanent cycle. \hfill $\square$

The next proposition extends Proposition 2.7 leading to the definition of $\eta^r_{i, v + 1}$ in Definition 3.3.

**Proposition 3.3.** Retain the notation and hypotheses of Propositions 2.7 and consider $\Gamma_i \equiv \Gamma_i(U_J)$ for some $J \subset \Pi$, some $i > 2$. For any $v \geq i$, there exists a unique $T$-equivariant $k$-linear map

$$\eta_{i, v + 1} : (\gamma_i/\gamma_{i+1})^{\#(r)[2p^{r-1}]} \rightarrow H^{2p^{r-1}}((\Gamma_i/\Gamma_{v+1})(r), k)$$

which fits in the following commutative diagram

(3.3.1)

$$\xymatrix{ (\gamma_i/\gamma_{i+1})^{\#(r)[2p^{r-1}]} \ar[r]_-{\eta_{i, v}} & (\gamma_i/\gamma_{i+1})^{\#(r)[2p^{r-1}]} \ar[r]_-{\eta_{i, v+1}} & (\gamma_i/\gamma_{i+1})^{\#(r)[2p^{r-1}]} \ar[d]^-{\eta_{i, v+1}} \ar[d]^-{\eta_{i, v+1}} \\
H^{2p^{r-1}}((\Gamma_i/\Gamma_v)(r), k) \ar[r] & H^{2p^{r-1}}((\Gamma_i/\Gamma_{v+1})(r), k) \ar[r] & H^{2p^{r-1}}((\Gamma_i/\Gamma_{v+1})(r), k). \ar[d]^-{\eta_{i, v+1}} \ar[d]^-{\eta_{i, v+1}}}

Here, the left vertical map is that defined recursively and the right vertical map is obtained after identifying $\Gamma_v/\Gamma_{v+1}$ with products of $\mathbb{G}_a$’s; the upper horizontal maps are the evident ones, the lower horizontal maps are those given by functoriality.

Let $N$ be some positive integer such that $\Gamma_N(U_J) = 1$ and denote by $\eta_i$ the map $\eta_{i, N}$ constructed above. Then $\eta_i$ and $\eta_{i+1}$ fit in a commutative diagram

(3.3.2)

$$\xymatrix{ (\gamma_i/\gamma_{i+1})^{\#(r)[2p^{r-1}]} \ar[r]_-{\eta_{i, v}} \ar[d]^-{\eta_{i+1}} & (\gamma_i)^{\#(r)[2p^{r-1}]} \ar[r]_-{\eta_{i+1}} \ar[d]^-{\eta_{i+1}} & (\gamma_{i+1})^{\#(r)[2p^{r-1}]} \ar[d]^-{\eta_{i+1}} \\
H^{2p^{r-1}}((\Gamma_i/\Gamma_{i+1})(r), k) \ar[r] & H^{2p^{r-1}}((\Gamma_i)(r), k) \ar[r] & H^{2p^{r-1}}((\Gamma_{i+1})(r), k). \ar[d]^-{\eta_{i, v+1}} \ar[d]^-{\eta_{i, v+1}}}

whose left vertical map is obtained after identifying $\Gamma_i/\Gamma_{i+1}$ with a product of $\mathbb{G}_a$’s

Proof. In the special case $\eta_{i, i+1}$, $\Gamma_i/\Gamma_{i+1}$ is a product of $\mathbb{G}_a$’s; in this case, $\eta_{i, i+1}$ is the $p^{r-1}$-st power of the map $(\gamma_i/\gamma_{i+1})^{(1)[2]} \rightarrow H^2((\Gamma_i/\Gamma_{i+1})(r), k)$ uniquely determined by weight considerations in view of (2.1.2).

We replace $U_J$ in the proof of Proposition 2.7 by $\Gamma_i$ and we consider $\Gamma_{v+1} \subset \Gamma_v$ in place of $\Gamma_3 \subset \Gamma_2$ in that proof. We proceed by ascending induction on $v$, assuming that $\eta_{i, v} : (\gamma_i/\gamma_{i+1})^{\#(r)[2p^{r-1}]} \rightarrow H^{2p^{r-1}}((\Gamma_i/\Gamma_{v})(r), k)$ has been constructed.

The argument in the proof given in Proposition 2.7 applies to construct $\eta_{i, v+1}$ fitting in (3.3.1). Namely, we consider some root $\beta$ of $\Gamma_i$ of level $v$ for $U_J$ and verify that any two $T$-eigenvectors $[\beta]$ in $A^J E_{1+}^\ast((\Gamma_i/\Gamma_{v+1})$ mapping to $(x^{(0)}_{\beta})^{p^{r-1}} \in H^{2p^{r-1}}((\Gamma_i/\Gamma_{v+1})(r), k)$ have difference in the image of $A^J d_{1+}^\ast$. Moreover, the $T$-eigenvector $(x^{(0)}_{\beta})^{p^{r-1}}$ itself is a permanent cycle by Corollary 3.2.
To prove the commutativity of \[(3.3.2),\] we first observe that the left square can be obtained by iterating the left square of \[(3.3.1)\] until \(v + 1\) equals \(N\). To prove the commutativity of the right square of \[(3.3.2)\], we utilize the \(T\)-splitting \(\gamma_i \cong \gamma_{i+1} \oplus (\gamma_i / \gamma_{i+1})\). Then \(\eta_i\) lifts \(\eta_{i+1}\) since the unique choice (up to boundaries) of \(T\)-eigenvector \([\beta]\) in \(A^T\mathcal{E}_1^*(\Gamma_i/\Gamma_{v+1})\) lifts the unique choice of \(T\)-eigenvector \([\beta]\) in \(A^T\mathcal{E}_1^*(\Gamma_{i+1}/\Gamma_{v+1})\). Thus, the right hand square of \[(3.3.2)\] commutes when restricted to \((\gamma_{i+1})^{#(r)}\), whereas the two compositions of \[(3.3.2)\] from the upper middle to the lower right are 0 on \((\gamma_i / \gamma_{i+1})^{#(r)}\).

**Definition 3.4.** Retain the notation and hypotheses of Propositions 2.7 and consider integers \(i < v \geq 2\). We define \(S^*((\Gamma_i/\Gamma_{v+1})_{(r)}) \to H^*((\Gamma_i/\Gamma_{v+1})_{(r)}, k)\) as follows. Restricted to \((\oplus_{\ell=0}^{r-1} (\gamma_i / \gamma_{i+1})^{#(\ell-1)(2)})\), \(\eta_{\Gamma_i/\Gamma_{v+1},r}\) is defined as the composition of \[(3.3.1)\] and the restriction map \(H^*((\Gamma_i/\Gamma_{v+1})_{(r)}, k) \to H^*((\Gamma_i/\Gamma_{v+1})_{(r)}, k)\). Restricted to \((\gamma_i / \gamma_{i+1})^{#(\ell-r)}[2p^{\ell-r-1}]\), \(\eta_{\Gamma_i/\Gamma_{v+1},r}\) is defined to be \(\ell\)-th Frobenius twist of the map \(\eta_{(\gamma_i / \gamma_{i+1})^{#(\ell-r)}[2p^{\ell-r-1}] \to H^{2p^{\ell-r-1}}((\Gamma_i/\Gamma_{v+1})_{(r-\ell)}, k)\) of Proposition 3.3, replacing \(r\) in that proposition by \(r - \ell\). (These two definitions agree when restricted to \(S^*((\oplus_{\ell=0}^{r-1} (\gamma_i / \gamma_{i+1})^{#(\ell)}[2p^{\ell-r-1}])\).)

Taking \(i = 1\), we have defined

\[(3.4.3)\] \(\eta_{U_j,\Gamma_{v+1},r} : S^*(U_j/\Gamma_{v+1})_{(r)} \to H^*((U_j/\Gamma_{v+1})_{(r)}, k)\).

Using Frobenius twists of the diagram \[(3.3.1)\] of Proposition 3.3 and extending the vertical maps of that diagram multiplicatively, we make explicit how we have recursively constructed \(\eta_{\Gamma_i/\Gamma_{v+1},r}\).

**Corollary 3.5.** The maps \(\eta_{\Gamma_i,\Gamma_{v+1}}\) and \(\eta_{\Gamma_i,\Gamma_{v+1},r}\) fit in commutative diagrams of \(k\)-algebras

\[(3.5.1)\]

\[S^*((\Gamma_i/\Gamma_v)_{(r)}) \xrightarrow{\eta_{\Gamma_i/\Gamma_v,r}} S^*((\Gamma_i/\Gamma_{v+1})_{(r)}) \xrightarrow{\eta_{\Gamma_i/\Gamma_{v+1},r}} H^*((\Gamma_i/\Gamma_{v+1})_{(r)}, k) \]

where \(\phi_{U,v}\) equals \(\eta_{U,v}\) (restricted to \(S^*((\oplus_{\ell=0}^{r-1} (\gamma_i)_{#(\ell)(2p^{\ell-r-1})})\) for all \(U = \Gamma_i(U_j)\).

**Proposition 3.6.** Retain the notation and hypotheses of Propositions 2.7 and consider \(U_j\) for some \(J \subset \Pi\). For any \(i \geq 1\), we have equality of maps

\[\phi_{\Gamma_i,r} = \eta_{\Gamma_i,r} : S^*((\oplus_{\ell=0}^{r-1} (\gamma_i)_{#(\ell)(2p^{\ell-r-1})}) \to H^*((\Gamma_i)_{(r)}, k)\]

where \(\phi_{\Gamma_i,r}\) is constructed in Proposition 3.4 and \(\eta_{\Gamma_i,r}\) is (the restriction of) the map \(\eta_{\Gamma_i/\Gamma_{v+1},r}\) of Definition 3.4 for any \(v \gg 0\).
Proof. As constructed in Proposition 1.3, the map $\phi_{r_i,r}$ is determined by its restrictions $(\gamma_i)^{(r-\ell-1)}[2^p-\ell-1] \to H^2p^{-\ell-1}((\Gamma_i)_{(r)},k)$; since these are obtained using the Frobenius twists $(e_{r_i}^{(\ell)}) : (gN)^{(r)}[2^p-\ell-1] \to H^2p^{-\ell-1}((GL_N)_{(r)},k)$, these maps arise as the $\ell$-th Frobenius twists of maps $(\gamma_i)^{(r-\ell)}[2^p-\ell-1] \to H^2p^{-\ell-1}((\Gamma_i)_{(r-\ell)},k)$. By Definition 3.3, the map $\eta_{r_i,r}$ is similarly determined by $\ell$-th Frobenius twists of maps $(\gamma_i)^{(r-\ell)}[2^p-\ell-1] \to H^2p^{-\ell-1}((\Gamma_i)_{(r-\ell)},k)$.

Consequently, it suffices to consider the case $\ell = 0$. We proceed by induction on $i$. We use the commutativity of (3.3.2) as well as the commutativity of the square implied by the commutativity of (1.5.1). To show that the liftings $\eta_i$ and $\phi_{r_i,r}$ (restricted to $(\gamma_i)^{(r-\ell)}[2^p-\ell-1]$) of $\eta_{r+1} = \phi_{r+1,r}$ are equal, we use the uniqueness of liftings verified in the proof of Proposition 1.3.

The map $\eta_{U_J/\Gamma_{v+1},r}$ sends the relations (3.1.4) to 0 (as well as the relations (2.2.3)), leading us to consider the quotient $\overline{S}^*((U_J/\Gamma_{v+1})_{(r)})$ of $S^*((U_J/\Gamma_{v+1})_{(r)})$ by these relations.

Definition 3.7. Retain the notation and hypotheses of Propositions 2.1 and consider $v \geq 2$. We define $I_{v+1} \subset S^*((U_J/\Gamma_{v+1})_{(r)})$ to be the ideal generated by elements

\[
\sum_{\alpha+\alpha' = \beta, \alpha < \alpha'} \{(x_{\alpha}^{(\ell)})^{p^{-\ell-1}} \otimes (x_{\alpha'}^{(\ell')})^{p^{-\ell'-1}} - (x_{\alpha}^{(\ell')})^{p^{-\ell-1}} \otimes (x_{\alpha'}^{(\ell)})^{p^{-\ell'-1}} \}
\]

with $0 \leq \ell < \ell' < r$ and level($\beta$) $\leq v$ (see 3.1.4) together with the additional elements

\[
\sum_{\alpha+\alpha' = \beta, \alpha < \alpha'} \{(x_{\alpha}^{(\ell)})^{p^{\ell+1}} \otimes x_{\alpha'}^{(\ell+1+j)} - (x_{\alpha'}^{(\ell+1+j)})^{p^{\ell+1}} \otimes x_{\alpha}^{(\ell)}) \} \in S^{p^{\ell+1}+1}((U_J)_{(r)}^{(\ell)}((U_{\gamma_2})_{(r)})^{(\ell+1)}[2])
\]

of (2.4.3) for $\beta$ of level 2.

We define

\[
S^*((U_J/\Gamma_{v+1})) \to \overline{S}^*((U_J/\Gamma_{v+1}))
\]

to be the quotient of of $S^*((U_J/\Gamma_{v+1})_{(r)})$ by the ideal $I_{v+1}$. We define

\[
Q((U_J/\Gamma_{v})_{(r)}) \equiv \text{im}\{S^*((U_J/\Gamma_{v})_{(r)}) \to \overline{S}^*((U_J/\Gamma_{v+1})_{(r)})\}.
\]

Lemma 3.8. Restriction induces the natural commutative square

\[
\begin{array}{ccc}
S^*((U_J/\Gamma_{v})_{(r)}) & \longrightarrow & \overline{S}^*((U_J/\Gamma_{v})_{(r)}) \\
\downarrow & & \downarrow \\
S^*((U_J/\Gamma_{v+1})_{(r)}) & \longrightarrow & \overline{S}^*((U_J/\Gamma_{v+1})_{(r)})
\end{array}
\]

inducing the natural map $Q((U_J/\Gamma_{v-1})_{(r)}) \to Q((U_J/\Gamma_{v})_{(r)})$. 

Moreover, the defining embedding \( Q((U_J/\Gamma_v)_r) \subset \mathcal{S}^*((U_J/\Gamma_{v+1})_r) \) splits, with splitting given by the identification

\[
\mathcal{S}^*((U_J/\Gamma_{v+1})_r) \cong Q((U_J/\Gamma_v)_r) \otimes S^*(\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_r [2p^{r-\ell}-1]).
\]

**Proof.** The commutativity of (3.8.1) is an immediate consequence of the fact that the restriction map \( S^*((U_J/\Gamma_v)_r) \rightarrow S^*((U_J/\Gamma_{v+1})_r) \) sends \( I_v \) to \( I_v \).

The identification

\[
\mathcal{S}^*((U_J/\Gamma_{v+1})_r) \cong Q((U_J/\Gamma_v)_r) \otimes S^*(\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_r [2p^{r-\ell}-1])
\]

follows from the identification

\[
S^*((U_J/\Gamma_{v+1})_r) \cong S^*((U_J/\Gamma_v)_r) \otimes S^*(\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_r [2p^{r-\ell}-1])
\]

and the fact that the generators of the ideal \( I_v \) do not involve elements of \( S^*(\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_r [2p^{r-\ell}-1]) \). \( \square \)

We summarize the results of the preceding propositions concerning \( \eta_{U_J/\Gamma_{v+1},r} \) in the following theorem. It is interesting to observe that our construction is independent of the earlier work of [21], [22].

**Theorem 3.9.** Retain the notation and hypotheses of Proposition 2.7. Consider the map

\[
\eta_{U_J/\Gamma_{v+1},r} : S^*((U_J/\Gamma_{v+1})_r) \rightarrow H^*((U_J/\Gamma_{v+1})(r), k)
\]
of (3.9).

1. In the special case in which \( U_J/\Gamma_{v+1} \) equals \( U_3 \), \( \eta_{U_J/\Gamma_{v+1},r} \) agrees with the map \( \eta_{U_3,r} \) of Definition 2.10.
2. If \( \Gamma_{v+1} = 1 \) so that \( U_J/\Gamma_{v+1} = U_J \), then

\[
\eta_{U_J,r} = \phi_{U_J,r} : S^*(\oplus_{\ell=0}^{r-1}(u_J)_r [2p^{r-\ell}-1]) \rightarrow H^*((U_J)_r, k),
\]

where \( \phi_{U_J,r} \) is the map of Theorem 1.7.
3. \( \eta_{U_J/\Gamma_{v+1},r} \) factors through the quotient \( S^*((U_J/\Gamma_{v+1})_r) \rightarrow \mathcal{S}^*((U_J/\Gamma_{v+1})_r) \) by the ideal \( I_v \), determining

\[
\eta_{U_J/\Gamma_{v+1},r} : \mathcal{S}^*((U_J/\Gamma_{v+1})_r) \rightarrow H^*((U_J/\Gamma_{v+1})(r), k).
\]

4. Restricted to \( S^*(\oplus_{\ell=0}^{r-1}(u_J/\gamma_2)_r [2^{(\ell+1)}]) \rightarrow \mathcal{S}^*((U_J/\Gamma_{v+1})_r) \), \( \eta_{U_J/\Gamma_{v+1},r} \) is given by the natural embedding \( S^*(\oplus_{\ell=0}^{r-1}(u_J/\gamma_2)_r [2^{(\ell+1)}]) \subset H^*((U_J/\Gamma_2)(r), k) \) and the restriction map induced by \( U_J/\Gamma_{v+1} \rightarrow U_J/\Gamma_2 \).

5. \( \eta_{U_J/\Gamma_{v+1},r} \) fits is the commutative square

\[
\begin{array}{ccc}
S^*((U_J/\Gamma_{v+1})_r) & \rightarrow & \mathcal{S}^*((U_J/\Gamma_{v+1})_r) \\
\eta_{U_J/\Gamma_{v+1},r} & \downarrow & \eta_{U_J/\Gamma_{v+1},r} \\
S^*((U_J/\Gamma_{v})_r) & \rightarrow & \mathcal{S}^*((U_J/\Gamma_{v})_r)
\end{array}
\]

\[
\begin{array}{ccc}
& & \rightarrow \\
\mathcal{S}^*((U_J/\Gamma_{v+1})_r) & \rightarrow & H^*((U_J/\Gamma_{v+1})(r), k) \\
\eta_{U_J/\Gamma_{v+1},r} & \downarrow & \eta_{U_J/\Gamma_{v+1},r} \\
& & \rightarrow \\
S^*((U_J/\Gamma_{v})_r) & \rightarrow & H^*((U_J/\Gamma_{v})(r), k)
\end{array}
\]

(3.9.1)

whose left square is the square of Lemma 3.8 and whose right vertical arrow is the restriction map.
Proof. In the special case \( \Gamma_{v+1} \) equals \( U_3 \), this is precisely the construction of Definition 2.10 as asserted in (1). Statement (2) is the content of Proposition 3.6.

The fact that \( \eta_{U,J}/\Gamma_{v+1, r} \) factors through \( \eta_{U,J}/\Gamma_{v+1, r} \) as asserted in (3) follows from the fact that \( \eta_{U,J}/\Gamma_{v+1, r} \) so defined sends \( I_v \) to 0 (see Definition 3.7). Statement (4) now follows from Definition 2.10 since the recursive construction of \( \eta_{U,J}/\Gamma_{v+1, r} \) implies that \( \eta_{U,J}/\Gamma_{v+1, r} \) “extends” \( \eta_{U,J}/\Gamma_{v, r} \).

The commutativity of the left square of (3.9.1) is given by Lemma 3.8. Since the left horizontal maps of (3.9.1) are surjective, to prove Statement (5) it suffices to prove the commutativity of the outer square of (3.10.2). This follows from the above construction of \( \eta_{U,J}/\Gamma_{v+1, r} \) as a lifting of \( \eta_{U,J}/\Gamma_{v, r} \).

The following theorem extends Proposition 2.12. This theorem relates our construction of \( \eta_{U,J,r} \) to the results of [21] for the reductive group \( GL_N \).

**Theorem 3.10.** Retain the notation and hypotheses of Proposition 2.12. Assume that \( (-)^{[p]} : u_J \rightarrow u_J \) is the zero map and that \( G \) admits an embedding \( G \subset GL_N \) of exponential type. Denote the inclusion \( U_J \subset G \subset GL_N \) by \( i : U_N \rightarrow GL_N \). Then there is a naturally constructed injective map

\[
\theta_{U_J,r} : k[V_r(U_J)] \rightarrow \tilde{S}((U_J)_{(r)})
\]

fitting in the commutative diagram (3.10.2)

Consequently, the \( p^r \)-th power of each element of \( \ker\{\overline{\theta}_{U_J,r}\} \) equals 0.

Furthermore, the \( p^{-1} \)-th power of each element of \( \tilde{S}((U_J)_{(r)}) \) lies in the image of \( \theta_{U_J,r} \), so that every element in the kernel of \( \eta_{U_J,r} \) has \( p^{2r-1} \)-st power equal to 0.

Proof. The proof of Proposition 2.12 applies to prove this theorem once one replaces \( U_3 \) by \( U_J \). We point out that to verify that \( \theta_{U_J,r} \circ i_r \) is well defined, one uses the observation that the relations \( \{R_{i,J,t,E}\} \) of Theorem 1.1 map to the \( p^{-\ell-1} \)-st power of the relations \( \{\tilde{S}_{i,j,t}\} \) and \( \{\tilde{S}_{i,j,t}\} \). One also uses the observation that the relations \( \{S_{i,j,t}\} \) of Theorem 1.1 map to 0 in \( \tilde{S}((U_J)_{(r)}) \) because \( (-)^{[p]} : u_J \rightarrow u_J \) (the restriction of the \( p \)-th iterate of multiplication in \( gl_N \)) is the zero map.

4. Stabilization with Respect to \( r \)

Much of the author’s motivation for considering the cohomology algebra \( H^*(U_{(r)}, k) \) has been the hope that some form of “continuous cohomology” for the linear algebraic group \( U \) would prove useful in the study of the (rational) representations.
of \( U \). This requires understanding the limiting behavior of \( H^*(U(r), k) \) as \( r \) increases. Earlier computational information for \( H^*(U(r), k) \) (especially in \([21],[22]\)) shed little if any light on this limiting behavior.

In order to investigate how the map \( \phi_{GL,N,r} \) in \((1.0.1)\) behaves as \( r \) increases, we introduce in the next proposition the map \((-)^p\).

**Proposition 4.1.** Define the map

\[
(-)^p : S^*(\oplus_{\ell=0}^{r-1} \mathfrak{gl}_N^{(r-\ell)}[2p^{r-\ell-1}]) \to S^*(\oplus_{\ell=0}^{r-2} \mathfrak{gl}_N^{(r-1-\ell)}[2p^{r-\ell-2}])
\]

by sending \( X^{s,t}(\ell) \in \mathfrak{gl}_N^{(r-\ell)}[2p^{r-\ell-1}] \) to the \( p \)-th power \( (X^{s,t}(\ell))^p \in S^p(\mathfrak{gl}_N^{(r-1-\ell)}[2p^{r-\ell-2}]) \) if \( \ell < r - 1 \) and to 0 if \( \ell = r - 1 \). Then \((-)^p\) fits in the \( GL_N \)-equivariant commutative square

\[
\begin{array}{ccc}
S^*(\oplus_{\ell=0}^{r-1} \mathfrak{gl}_N^{(r-\ell)}[2p^{r-\ell-1}]) & \xrightarrow{\phi_{GL,N,r}} & H^*(GL_N(r), k) \\
(-)^p & & \text{res} \\
S^*(\oplus_{\ell=0}^{r-2} \mathfrak{gl}_N^{(r-1-\ell)}[2p^{r-\ell-2}]) & \xrightarrow{\phi_{GL,N,r-1}} & H^*(GL_N(r-1), k).
\end{array}
\]

**Proof.** We first show that \( \phi_{GL,N,r}(X^{s,t}(0)) \in H^{2p^{r-1}}(GL_N(r), k) \) restricts to the \( p \)-th power of \( \phi_{GL,N,r-1}(X^{s,t}(0)) \in H^{2p^{r-2}}(GL_N(r-1), k) \). By \([21] 3.4\), both of these classes restrict to the \( p \)-th power of the image of \( \phi_{GL,N,1}(X^{s,t}(0)) \) in \( H^2(GL_{N-1}(1), k) \). Consequently, the outer square of the following diagram of \( GL_N \)-modules commutes:

\[
\begin{array}{ccc}
\mathfrak{gl}_N^{(r)}[2p^{r-1}] & \xrightarrow{e_r} & H^{2p^{r-1}}(GL_N(r), k) \\
(-)^p & & \text{res} \\
S^p(\mathfrak{gl}_N^{(r-1)}[2p^{r-2}]) & \xrightarrow{e_{r-1}} & H^{2p^{r-2}}(GL_N(r-1), k) \\
(-)^{p-2} & & \text{res} \\
S^{p-2}(\mathfrak{gl}_N^{(1)}[2]) & \xrightarrow{e_1} & H^{2p^{r-2}}(GL_N(1), k)
\end{array}
\]

The images in \( H^{2p^{r-1}}(GL_N(r-1), k) \) of the two compositions in the upper square of \((4.1.3)\) are each irreducible \( GL_N \)-modules (copies of \( \mathfrak{gl}_N^{(r)}[2p^{r-1}]/(k \cdot T^r) \)) which restrict non-trivially to \( H^{2p^{r-1}}(GL_N(1), k) \). Using the form of the \( E_1 \)-term of the A-J spectral sequence \((2.6.1)\) and the behavior of this \( E_1 \)-term upon restriction along \( GL_N(1) \to GL_N(r) \), we conclude that the upper square of \((4.1.3)\) also commutes. Namely, there is a unique copy of \( \mathfrak{gl}_N^{(r)}[2p^{r-1}]/(k \cdot T^r) \) in \( A^J_{1}(GL_N(r-1)) \) of homological degree \( 2p^{r-1} \).

By definition of \( e_{r-\ell} \) as the pull-back via \( F^\ell : GL_N \to GL_N \), we have the commutativity of the following square

\[
\begin{array}{ccc}
\mathfrak{gl}_N^{(r-\ell)}[2p^{r-\ell-1}] & \xrightarrow{e_{r-\ell}} & H^{2p^{r-\ell-1}}(GL_N(r-\ell), k) \\
(-)^{(r)} & & \text{res} \\
\mathfrak{gl}_N^{(r)}[2p^{r-1}] & \xrightarrow{e_{r}} & H^{2p^{r-1}}(GL_N(r), k)
\end{array}
\]
Consequently, pulling back via $F^\ell$ the commutative upper square of (4.1.3) with $r$ replaced by $r - \ell$ determines the following commutative square for each $\ell, 0 \leq \ell < r$:

$$
\begin{array}{ccc}
\mathfrak{gl}_N^{(r)} & [2p^{r-\ell-1}] & H^{2p^{r-\ell-1}}(GL_N(r), k) \\
\mathfrak{gl}_N^{(r-1)} & [2p^{r-\ell-2}] & H^{2p^{r-\ell-1}}(GL_{N(r-1)}, k).
\end{array}
$$

(4.1.5)

The proposition now follows since the maps of (4.1.2) are maps of $k$-algebras and the commutativity of (4.1.5) implies the commutativity of (4.1.2) on generators. □

We extend Proposition 4.1 to the unipotent groups considered in Section 3.

**Definition 4.2.** As in Definition 3.4 we retain the notation and hypotheses of Proposition 2.1 (so that $\Gamma_i$ denotes the $i$-th term of the descending central series for $U_j, J \subset H$) and consider integers $i < v \geq 2$. We define

$$(-)[p]: S^*((\Gamma_i/\Gamma_{v+1})(r)) \to S^*((\Gamma_i/\Gamma_{v+1})(r-1))$$

to be the coproduct of the projection $S^*((\oplus_{\ell=0}^{r-1}(\gamma_i/\gamma_2)^{(\ell-1)}[2])) \to S^*(\oplus_{\ell=0}^{2}(\gamma_i/\gamma_2)^{(\ell-1)}[2])$ and the map $S^*(\oplus_{\ell=0}^{r-1}(\gamma_i/\gamma_{v+1})^{(\ell-1)}[2p^{r-\ell-1}]) \to S^*(\oplus_{\ell=0}^{v-2}(\gamma_i/\gamma_{v+1})^{(\ell-1)}[2p^{r-\ell-2}])$ obtained by extending multiplicatively for each $\ell, 0 \leq \ell < r-1$ the maps

$$(\gamma_i/\gamma_{v+1})^{(r)}[2p^{r-\ell-1}] \to S^*((\gamma_i/\gamma_{v+1})^{(r)}[2p^{r-\ell-2}])$$
given by the $p$-th power map.

**Proposition 4.3.** Retain the hypotheses and notation of Definition 4.2. Then $(-)[p]$ fits in the $P_i$-equivariant commutative square for each $i < v \geq 2$

$$
\begin{array}{ccc}
S^*((\Gamma_i/\Gamma_{v+1})(r)) & H^*((\Gamma_i/\Gamma_{v+1})(r), k) \\
(-)[p] & & res \\
S^*((\Gamma_i/\Gamma_{v+1})(r-1)) & H^*((\Gamma_i/\Gamma_{v+1})(r-1), k),
\end{array}
$$

(4.3.1)

where $\eta_{P_i/\Gamma_{v+1}, r}$ is the map constructed in Definition 3.4.

Moreover, restriction along $\Gamma_i/\Gamma_{v+1} \to \Gamma_i/\Gamma_v$ determines a commutative cube from the square of the form (4.3.1) associated to $\Gamma_i/\Gamma_v$ to (4.3.1) associated to $\Gamma_i/\Gamma_{v+1}$.

**Proof.** We proceed by (ascending) induction on both $v$ and $r$ to prove the commutativity of the “cube” (diagram involving eight groups) mapping square of the form (4.3.1) associated to $\Gamma_i/\Gamma_v$ to the square square of the form (4.3.1) associated to $\Gamma_i/\Gamma_{v+1}$. This “cube” consists of an outer square of the form (4.3.1) associated to $\Gamma_i/\Gamma_v$ which commutes by induction, four “intermediate” squares discussed below, and an innermost square which is (4.3.1).
The “upper intermediate square”

$$S^*((\Gamma_i/\Gamma_v)(r)) \xrightarrow{\eta_{v/r}} H^*((\Gamma_i/\Gamma_{v+1})(r), k) \xrightarrow{\text{res}} S^*((\Gamma_i/\Gamma_{v+1+1})(r), k),$$

(4.3.2)

commutes by the recursive construction of $\eta_{v/r}$; the same argument verifies the commutativity of the “lower intermediate square”

$$S^*((\Gamma_i/\Gamma_{v+1})(r-1)) \xrightarrow{\eta_{v/r}} H^*((\Gamma_i/\Gamma_{v+1+1})(r-1), k) \xrightarrow{\text{res}} S^*((\Gamma_i/\Gamma_{v+1+1+1})(r-1), k),$$

(4.3.3)

The “left intermediate square”

$$S^*((\Gamma_i/\Gamma_v)(r)) \xrightarrow{} S^*((\Gamma_i/\Gamma_{v+1})(r))$$

(4.3.4)

is shown to commute by a quick examination of maps of coproducts (as in Definition 4.2), recognizing that the vertical maps are obtained by dropping the summands indexed by $\ell - 1$. The commutativity of the “right intermediate square”

$$H^*((\Gamma_i/\Gamma_v)(r), k) \xrightarrow{} H^*((\Gamma_i/\Gamma_{v+1})(r), k)$$

(4.3.5)

is a consequence of functoriality of cohomology.

An easy diagram chase around the “cube” using the commutativity just proved of the “outer square” and four “intermediate squares” together with the fact that $S^*((\Gamma_i/\Gamma_{v+1})(r))$ is the coproduct of $S^*((\Gamma_i/\Gamma_v)(r))$ and $S^*((\bigoplus_{t=0}^{r-1}(\gamma_{v}/\gamma_{v+1})^{\#(r)}[2p^{-\ell-1}])$ implies that to prove the commutativity of the innermost square 4.3.3 it suffices to verify its commutativity restricted to each $(\gamma_{v}/\gamma_{v+1})^{\#(r)}[2p^{-\ell-1}])$. This becomes the statement that for every weight $\beta$ of level $v$ of $\Gamma_i$ the value of $\eta_{v/r}$ on $((x_{\beta})^{#(t)})p^{-\ell-1} \in H^{2p^{-\ell-1}}((\Gamma_i/\Gamma_{v+1})(r), k)$ when restricted to $H^{2p^{-\ell-1}}((\Gamma_i/\Gamma_{v+1})(r-1), k)$ equals the $p$-th power of the value of $\eta_{v/r}$ on $((x_{\beta})^{#(t)})p^{-\ell-2} \in H^{2p^{-\ell-2}}((\Gamma_i/\Gamma_{v+1})(r-1), k)$.

This follows from the construction of $\eta_{v+1}$ given in Proposition 4.3 which gives the value in $H^{2p^{-\ell-1}}((\Gamma_i/\Gamma_{v+1})(r), k)$ in terms of the unique cohomology class determined by $(x_{\beta}^{(0)})^{-p^{-\ell-1}} \in \Lambda^J E^{*,*}_1((\Gamma_i/\Gamma_{v+1})(r), k)$. \[\square\]

We obtain the following somewhat surprising result concerning inverse limits whose connecting maps are given by $(-)^{[p]}$.

**Proposition 4.4.** As in Definition 3.4 we retain the notation and hypotheses of Proposition 2.4 (so that $\Gamma_i$ denotes the $i$-th term of the descending central series.
for $U, J \subset \Pi$) and consider integers $i < v \geq 2$. Then the natural map between limits with respect to $r$ (using (4.3.4))

$$
\lim_{s \geq r} S^\ast((\Gamma_i/\Gamma_v)_{(s)}) \to \lim_{s \geq r} S^\ast((\Gamma_i/\Gamma_{v+1})_{(s)})
$$

is an isomorphism.

Proof. Observe that

$$
\lim_{s \geq r} S^\ast((\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_{\ast}(2p^{r-\ell-1})) = 0,
$$

where $S^\ast$ denotes the ideal of positive degree elements of the symmetric algebra $S^\ast$. Namely, in the notation of Proposition 4.3 any element homogeneous of degree $d, 0 < d < p^r$ in $S^\ast((\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_{\ast}(2p^{r-\ell-1})) \subset S^\ast((U/\Gamma_{v+1})_{(r)})$ does not lie the image of $(-)^s : S^\ast((U/\Gamma_{v+1})_{(r,s)}) \to S^\ast((U/\Gamma_{v+1})_{(r)})$.

By taking the limit (in each cohomological degree) with respect to $r$ of the $T$-equivariant isomorphisms

$$
S^\ast((\Gamma_i/\Gamma_v)_{(r)}) \simeq S^\ast((\Gamma_i/\Gamma_v)_{(r)}) \otimes S^\ast((\oplus_{\ell=0}^{r-1}(\gamma_v/\gamma_{v+1})_{\ast}(2p^{r-\ell-1}))
$$

of (3.4.1), we conclude the isomorphism (4.4.1).

Proposition 4.4 enables the following description of the image of the intersection (for varying $s$) of the images of $S^\ast((\Gamma_i/\Gamma_{v+1})_{(s)})_{m_i/\Gamma_{v+1}^\ast} \to H^\ast((\Gamma_i/\Gamma_{v+1})_{(r)}, k)$.

Corollary 4.5. Adopt the hypotheses of and notation of Proposition 4.4; in particular, $\Gamma_i = \Gamma_i(U, J)$ for some $J \subset \Pi$ and $i < v$. Then

$$
im\left(\lim_{s \geq r} S^\ast((\Gamma_i/\Gamma_v)_{(s)})_{m_i/\Gamma_{v+1}^\ast} \to H^\ast((\Gamma_i/\Gamma_{v+1})_{(r)}, k)\right)
$$

equals

$$
im\{S^\ast((\Gamma_i/\Gamma_v)_{(s)})_{m_i/\Gamma_{v+1}^\ast} \to H^\ast((\Gamma_i/\Gamma_{v+1})_{(r)}, k)\}.
$$

We conclude with a brief discussion of $H^\ast(U, k)$. Although Theorem 4.6 tells us that $H^\ast(U, k) \to \lim_{r \to \infty} H^\ast(U_{(r)}, k)$ is injective, Question 4.8 emphasizes our lack of understanding of nilpotence in $H^\ast(U, k)$.

The unipotent algebraic groups $V$ considered in the following theorem are more general than the class of unipotent groups we have been considering in Section 3 and earlier in this section.

Theorem 4.6. Let $G$ be a simple algebraic group provided with a choice of Borel subgroup $B \subset G$ with maximal torus $T$ and unipotent radical $U$. Let $U_1 \subset U$ be a $T$-stable closed subgroup, $U_2 \subset U_1$ a $T$-stable, normal closed subgroup of $U_1$, and consider $V \equiv U_1/U_2$. The natural map

$$
H^\ast(V, k) \to \lim_{r \to \infty} H^\ast(V_{(r)}, k)
$$

is an isomorphism.

Proof. Let $0 \neq \xi \in H^d(V, k)$ be a $T$-eigenvector of weight $\omega = \sum_{m=1}^{\ell} w_m \alpha_m, w_m \geq 0$, an element in the positive cone of the root lattice for $V = \text{Lie}(V)$; here, $\alpha_1, \ldots, \alpha_\ell$ are the simple roots determined by $B \subset G$. One verifies by inspection that the restriction map $A^d \text{E}_{1^*}^\ast(V) \to A^d \text{E}_{1^*}^\ast(V_{(r)})$ is an isomorphism of $\omega$-weight spaces $(A^d \text{E}_{1^*}^\ast(V))_\omega \to (A^d \text{E}_{1^*}^\ast(V_{(r)}))_\omega$ provided that $p^{r-1}$ is greater than any of
Under what conditions is the image of the restriction map \( H^* \) which we now state. Let 

\[ \text{Theorem 4.6 suggests several questions concerning the relationship} \]

\[ \text{Question 4.8.} \]

is a weight of \( \Box \)

\[ \text{V between the multiplicative structure of} H \]

The action of \( H \)

\[ \text{Theorem 4.6 for A-J spectral sequences} \]

\[ \text{Proof.} \]

\[ \text{We repeat the argument of the proof of Theorem 4.6 for A-J spectral sequences} \]

\[ \text{Thus, the restriction map} (H^* (V, k))_\omega \rightarrow \lim_{r \to \infty} (H^* (V_r, k))_\omega \]

is an isomorphism for all weights \( \omega \) in the positive cone of the root lattice for \( u \). So that \( H^* (V, k) \rightarrow \lim_{r \to \infty} H^* (V_r, k) \) is also an isomorphism.

\[ \text{Corollary 4.7. Retain the notation of Theorem 4.6 and let} M \]

\[ \text{is injective. Moreover, if} \zeta_r \in H^d (V_r, M) \]

\[ \text{for all} s \geq r, \]

\[ \text{there exists some} \zeta \in H^d (V, M) \]

\[ \text{which restricts to} \zeta_r. \]

\[ \text{Proof.} \]

\[ \text{We repeat the argument of the proof of Theorem 4.6 for A-J spectral sequences} \]

\[ \text{The action of} V \text{ on the associated graded group of} M \]

\[ \text{These spectral sequences now split as a direct sum of spectral sequences indexed} \]

\[ \text{by weights given as the sum of a weight of} M \]

\[ \text{and an element in the positive cone of the root lattice for} u. \]

\[ \text{As in the proof of Theorem 4.6 we conclude that} \]

\[ \text{the restriction map} \]

\[ \text{is an isomorphism of} \omega \text{-weight spaces} \]

\[ \text{provided that} p^{r-1} \text{ does not divide} \]

\[ \text{is a weight of} M. \]

\[ \text{The remainder of the proof is a repetition of that of Theorem 4.6} \]

\[ \text{Question 4.8. Theorem 4.6 suggests several questions concerning the relationship} \]

\[ \text{between the multiplicative structure of} H^* (V, k) \]

\[ \text{of} H^* (V_r, k), \text{two of} \]

\[ \text{which we now state. Let} i_r : V_r \rightarrow V \text{ be the natural embedding.} \]

\[ \text{(1) For} V \text{ as in Theorem 4.6 does there exist a non-nilpotent cohomology class} \]

\[ \text{each of whose restrictions} i_r^* (\alpha) \in H^* (V_r, k) \]

\[ \text{is nilpotent?} \]

\[ \text{(2) Under what conditions is the image of the restriction map} H^* (V, k) \rightarrow \]

\[ H^* (V_r, k) \text{ finitely generated?} \]

\[ \text{References} \]

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