Effective equation of state for dark energy: mimicking quintessence and phantom energy through a variable $\Lambda$

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Abstract. While there is mounting evidence in all fronts of experimental cosmology for a non-vanishing dark energy component in the Universe, we are still far away from understanding its ultimate nature. A fundamental cosmological constant, $\Lambda$, is the most natural candidate, but many dynamical mechanisms to generate an effective $\Lambda$ have been devised which postulate the existence of a peculiar scalar field (so-called quintessence, and generalizations thereof). These models are essentially ad hoc, but they lead to the attractive possibility of a time-evolving dark energy with a non-trivial equation of state (EOS). Most, if not all, future experimental studies on precision cosmology (e.g. the SNAP and PLANCK projects) address very carefully the determination of an EOS parametrized a la quintessence. Here we show that by fitting cosmological data to an EOS of that kind can also be interpreted as a hint of a fundamental, but time-evolving, cosmological term: $\Lambda = \Lambda(t)$. We exemplify this possibility by studying the effective EOS associated to a renormalization group (RG) model for $\Lambda$. We find that the effective EOS can correspond to both normal quintessence and phantom dark energy, depending on the value of a single parameter of the RG model. We conclude that behind a non-trivial EOS of a purported quintessence or phantom scalar field there can actually be a running cosmological term $\Lambda$ of a fundamental quantum field theory.
Introduction

During the last few years we are witnessing how Cosmology is rapidly becoming an experimental branch of physics. It is no longer a pure realm of philosophical speculation; theoretical models can be tested, and new and more accurate data in the near future will restrict our conceptions of the Universe to within few percent accuracy. Although the list of unsolved problems in Cosmology does not run short, there is a preeminent one that seems to overshoot the strict domain of Cosmology and remains boldly defiant since its first formulation by Zeldovich in 1967 [1]. We are referring to the famous cosmological constant (CC) problem [2, 3]. Its ultimate solution desperately cries out for help, hopefully to come from theoretical physics at its deepest level. The CC problem is the problem of understanding the theoretical meaning and the measured value of the cosmological term, \( \Lambda \), in Einstein’s equations. As it is well-known, the quantum field theory (QFT) contributions prove to be exceedingly large as compared to the measured \( \Lambda \) inferred from the accelerated expansion of the Universe [4], the anisotropies of the CMB [5] and the large scale structure [6].

In recent times the CC problem has become manifold and has been rephrased in a more general way, namely one interprets the observed accelerated expansion of the Universe as caused by a generic entity called the Dark Energy (DE) component, \( \rho_D \), of the total energy density \( \rho_T \). Within this new conception the DE could be related to the existence of a dynamical field that would generate an effective CC. Obviously the very notion of CC in such broader context becomes degraded, the CC could just be inexistent or simply relegated to the status of one among many other possible candidates. For example, an alternate candidate to DE that has spurred an abundant literature goes under the name of quintessence [7], meaning some scalar field \( \chi \) which generates a non-vanishing \( \rho_D \) from the sum of its potential and kinetic energy term at the present time: \( \rho_D = \{(1/2)\xi \dot{\chi}^2 + V(\chi)\}_{t=t_0} \). Here \( \xi \) is a coefficient whose sign can be of some significance, as we shall see. If the kinetic energy for \( \chi \) is small enough, it is clear that \( \rho_D \) looks as an effective cosmological constant \( \Lambda_{\text{eff}} \). The scalar field \( \chi \) is in principle unrelated to the Higgs boson or any other field of the Standard Model (SM) of particle physics, including all of its known extensions (e.g. the supersymmetric generalizations of the SM); in other words, the \( \chi \) field is an entirely ad hoc construct just introduced to mimic the cosmological term. Actually, it was long ago that it was considered the general possibility that the cosmological term could evolve with time [8, 9] or even to be a dynamical scalar field variable [10, 11], but only in more recent times this idea took the popular form of the quintessence proposal mentioned above [12, 7]. In fact, so popular that all parametrizations of the DE seem to presume it.

The reason why the quintessence idea can be useful, in principle, is because if \( \chi \) is a time-evolving field it may help to understand another aspect of the CC problem which is also rather intriguing, the so-called “coincidence problem”, to wit: why the presently measured value of the CC/DE is so close to the matter density? In other words, why the current cosmological parameters \( \Omega_\Lambda \) and \( \Omega_M \) are of the same order? Unfortunately, in spite of its virtues the quintessence idea has a big theoretical drawback: the typical mass of the quintessence field should be of the order of the

\[ \Lambda = 8\pi G \Lambda, \text{ where } G \text{ is Newton’s constant.} \]
Hubble parameter now: \( m_\chi \sim H_0 \sim 10^{-33} \text{ eV} \), meaning a particle mass 30 orders of magnitude below the very small mass scale associated to the measured value of the cosmological constant: \( m_\Lambda \equiv \Lambda_0^{1/4} \sim 10^{-33} \text{ eV} \). One may wonder if by admitting the existence of an ultralight field like \( \chi \) (totally unrelated to the rest of the particle physics world) is not just creating a problem far more worrisome than the CC problem itself! In view of these facts, it is more than advisable to seek for alternatives to quintessence which nevertheless should preserve the major virtue of that proposal, such as the possibility to have a dynamical DE that can help explaining why the CC is very small at present (comparable to the matter density) and perhaps much larger in the past. One possibility is to have a “true”, but variable, \( \Lambda \) parameter. This idea has been cherished many times in the literature, but only on purely phenomenological grounds \[8, 9, 13\]. In Ref. \[14, 15\], however, a proposal was put forward aiming at a model of variable \( \Lambda \) stemming from fundamental physics: viz. the renormalization group (RG) methods of QFT in curved space-time. The basic idea is that in QFT the CC should be treated as a running parameter, much in the same way as the electric charge in QED or the strong coupling constant in QCD\(^4\). More recently this RG cosmological model has been shown to be testable in the next generation of precision experiments \[17, 18\]. The general idea of a running CC has been further elaborated in \[19, 20, 21\], and its phenomenological consequences have been explored in great detail in \[22\] (see also the framework of \[23\]). However in practice – meaning in all future experimental projects for precision cosmology (like SNAP and PLANCK \[24\]) – the general strategy to explore the properties of the DE is to assume that there is an underlying equation of state (EOS), \( p_\chi = \omega_\chi \rho_\chi \), that describes the field \( \chi \) presumably responsible for the accelerated expansion of the universe \[25\]. If \( \omega_\chi \) lies in the interval \(-1 < \omega_\chi < -1/3\), the field \( \chi \) is a standard quintessence field; if \( \omega_\chi < -1 \), then \( \chi \) is called a “phantom field” because this possibility is non-canonical in QFT (namely it enforces \( \xi < 0 \) in its kinetic energy term) and violates the weak energy condition. Still, it cannot be discarded at present because it seems to be slightly preferred by the combined analysis of the supernovae and CMB data \[26\].\(^5\)

At variance with the idea of a canonical or non-canonical scalar field description of the DE, a fundamental CC (whether strictly constant or a variable one) can only have a “trivial” EOS: \( \omega_\Lambda = -1 \). Notwithstanding, one may describe such a variable \( \Lambda \) within the scalar field parametrization of the DE and try to uncover what is the effective EOS for the running CC term. A main result of this work is that a fundamental running \( \Lambda \) can mimic the effective vacuum energy of a dynamical field \( \chi \) both in the quintessence and phantom mode. At the same time our analysis will illustrate that an eventual determination of an EOS from experiment should not necessarily be interpreted as a sign that there is a dynamical field responsible for the DE component of the Universe.

**Running \( \Lambda \) versus quintessence**

Let us compare an scenario with a variable \( \Lambda \) with one with a DE component represented by a quintessence field \( \chi \). In the first case the full energy-momentum tensor of the cosmological perfect

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\(^4\)See Ref.\[15, 16\] for attempts to relate the running of \( \Lambda \) and of the DE to neutrino physics.

\(^5\)See e.g.\[24, 25\] for some recent literature on phantom DE.
fluid with 4-vector velocity field $U^\mu$ is given by
\[ \tilde{T}_{\mu\nu} = T_{\mu\nu} + g_{\mu\nu}\Lambda = (\Lambda - p)g_{\mu\nu} + (\rho + p)U_\mu U_\nu, \]  
(1)
where $T_{\mu\nu}$ is the ordinary matter-radiation energy-momentum tensor, $p$ is the proper isotropic pressure and $\rho$ is the proper energy density of matter-radiation. The basic cosmological equations with non-vanishing $\Lambda$ are the Friedmann equation
\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} (\rho + \Lambda) - \frac{k}{a^2}, \]  
(2)
and together with the dynamical field equation for the scale factor:
\[ \ddot{a} = -\frac{4\pi}{3} G (\rho + 3p - 2\Lambda) a. \]  
(3)
Let us first assume that $G = G(t)$ and $\Lambda = \Lambda(t)$ can be both arbitrary functions of the cosmic time. This is allowed by the Cosmological Principle embodied in the FLRW metric. Then one can check that the Bianchi identities lead to the following first integral of the previous system of differential equations
\[ \frac{d}{dt} [G(\Lambda + \rho)] + 3G H (\rho + p) = 0. \]  
(4)
Equivalently, this also follows from Eq. (1) and $\nabla^\mu \tilde{T}_{\mu\nu} = 0$. When $G$ is constant, the identity above implies that $\Lambda$ is also a constant, if and only if the ordinary energy-momentum tensor is individually conserved ($\nabla^\mu T_{\mu\nu} = 0$), i.e. $\dot{\rho} + 3 H (\rho + p) = 0$. However, a first non-trivial situation appears when $G = const$ but $\Lambda = \Lambda(t)$. Then (1) boils down to
\[ \dot{\Lambda} + \dot{\rho} + 3 H (\rho + p) = 0. \]  
(5)
This scenario exemplifies that a time-variable $\Lambda = \Lambda(t)$ cosmology may exist such that transfer of energy may occur from matter-radiation into vacuum energy, and vice versa. The solution of a generic cosmological model of this kind is contained in part in the coupled system of differential equations (2) and (5) together with the equation of state $p = p(\rho)$ for matter and radiation. However, still another equation is needed to completely solve this cosmological model in terms of the basic set of cosmological functions $(H(t), \rho(t), p(t), \Lambda(t))$. At this point one may either resort to any of the various phenomenological models available in the market [13] or use some new idea. The particular case of a continually decaying $\Lambda$ has been examined long ago [9]. In the absence of a fundamental calculation to specify how rapidly the vacuum energy decays and how it couples to non-relativistic matter and radiation, these authors decided to make some assumptions and examine the potential phenomenological consequences. Here we generalize this approach for a variable $\Lambda = \Lambda(t)$ that can either increase or decrease with time, and show that this kind of cosmological scenario could emerge from QFT. To illustrate the last possibility, we are going to make use of the renormalization group model of Ref.[17, 18, 14]6. In a few words this model is based on an RG equation for $\Lambda$ of the general form
\[ \frac{d\Lambda}{d\ln \mu} = \sum_{n=1}^{\infty} A_n \mu^{2n}. \]  
(6)
\footnote{A more general RG cosmological model with both running $G$ and running $\Lambda$ can also be constructed within QFT in curved space-time, see Ref.[20]. However, for simplicity hereafter we limit ourselves to the case $G = const.$}
Here $\mu$ is the energy scale associated to the RG running. One can argue that $\mu$ can be identified with the Hubble parameter $\mu = H$ at any given epoch [17, 18, 20, 14]. Since $H$ evolves with the cosmic time, the cosmological term $\Lambda$ inherits a time-dependence (which one may transform for convenience into redshift dependence) through its primary scale evolution with the renormalization scale $\mu$. Coefficients $A_n$ are obtained after summing over the loop contributions of fields of different masses $M_i$ and spins $\sigma_i$. The general behavior is $A_n \sim \sum M_i^{4-2n}$ [14, 19]. Therefore, for $\mu \ll M_i$, the series above is an expansion in powers of the small quantities $\mu/M_i$. Given that $A_1 \sim \sum M_i^2$, the heaviest fields give the dominant contribution. This feature (“soft-decoupling”) represents a generalization of the decoupling theorem in QFT [29]—see [14, 19, 22] for a more detailed discussion. In fact, it is characteristic of the $\Lambda$ parameter because it is the only dimension-4 parameter available in the SM, whereas quantum effects on dimensionless couplings and masses just decouple in the standard way. Now, since $\mu = H_0 \sim 10^{-33}$ eV the condition $\mu \ll M_i$ is amply met for all known particles, and the series on the r.h.s of Eq. (6) converges extremely fast. Notice that only even powers of $\mu = H$ are consistent with general covariance [17]. The $n = 0$ contribution is absent because it corresponds to terms $\propto M_i^4$ that give an extremely fast evolution. These are to be banished if we should describe a successful phenomenology; actually from the renormalization group point of view they are excluded because, as noted above, $\mu \ll M_i$ for all known masses. In practice only the first term $n = 1$ is needed, with $M_i$ of the order of the highest mass available. We may assume that the dominant masses $M_i$ are all of order of a high mass scale $M$ near the Planck mass $M_P$. Let us define (as in [17]) the ratio

$$\nu = \frac{\sigma}{12 \pi} \frac{M^2}{M_P^2}. \tag{7}$$

Here $\sigma = \pm 1$ depending on whether bosons or fermions dominate in their loop contributions to $\Lambda$. Then, to within very good approximation, the solution of the renormalization group equation (6) reads

$$\Lambda(t) = C_0 + C_1 H^2(t), \tag{8}$$

with

$$C_0 = \Lambda_0 - \frac{3 \nu}{8 \pi} M_P^2 H_0^2, \quad C_1 = \frac{3 \nu}{8 \pi} M_P^2, \tag{9}$$

where $H(t)$ is given by [2]. For $t = t_0$ we just get $\Lambda(t_0) = \Lambda_0$, the value of the CC at present. Moreover, for $t$ around $t_0$ the variation of $\Lambda$ is $\delta \Lambda(t_0) \sim \nu H_0^2 M_P^2 \sim H^2 M^2$. This is numerically in the ballpark of $\Lambda_0$ for $M \lesssim M_P$. As we see, this provides the fourth equation $\Lambda = \Lambda(t)$ needed to solve the cosmological model. It is well-behaved and it predicts a small evolution of $\Lambda$ around our time, which nevertheless may have some measurable effects [17, 22]. In the next section we will translate these effects into the language of the quintessence parametrization of the DE.

But before doing that, let us recall how the cosmological picture becomes modified when one trades the CC for a dynamical scalar field $\chi$, with an EOS of the general form $p_\chi = \omega_\chi \rho_\chi$. Consider the present time where $\rho \simeq \rho_M$ and $p \simeq 0$. Then equations (2) and (3) become

$$H^2 = \frac{8 \pi G}{3} (\rho_M + \rho_\chi) - \frac{k}{a^2}, \tag{10}$$
and

$$\ddot{a} = -\frac{4\pi G}{3} \left[ \rho_M + (1 + 3\omega_\chi) \rho_\chi \right] a.$$  \hfill (11)

From the last equation it is clear that for $\rho_M \rightarrow 0$ the expansion will accelerate if $\omega_\chi < -1/3$. However, for $\chi$ to mimic a positive CC one needs $\omega_\chi \gtrsim -1$ (quintessence). If $\omega_\chi < -1$ the Universe will still accelerate, but the $\chi$ field is non-canonical (phantom) because it should have a small, and negative, kinetic term at present:

$$\omega_\chi \equiv \frac{p_\chi}{\rho_\chi} = \left\{ \begin{array}{ll}
\frac{1}{2} \xi \chi^2 - V(\chi) & t = t_0, \\
\frac{1}{2} \xi \chi^2 + V(\chi) & \end{array} \right. \lesssim -1 \quad \text{if} \quad |\xi| \chi^2 \ll V(\chi) \quad \text{and} \quad \xi < 0. \hfill (12)
$$

Here we assumed a positive potential for $\chi$, the simplest possibility being $V(\chi) = (1/2) m_\chi^2 \chi^2$. The field $\chi$ is usually thought of as a high energy field (unrelated to SM physics), i.e. $\chi \simeq M_X$ where $M_X$ is some high energy scale typically around $M_P$. Neglecting the contribution from the kinetic term at the present time, such scalar field model would produce an effective cosmological constant of the order of the measured one, $\Lambda_{\text{eff}} \simeq V(\chi) \simeq \Lambda_0$, provided the mass of that (high-energy) field is $m_\chi \sim H_0 \sim 10^{-33} \text{eV}$, which looks rather contrived – to say the least. Even if (by some unknown mechanism) $\chi$ would be related to the electroweak scale (say $\chi \simeq G_F^{-1/2} \simeq 300 \text{GeV}$, where $G_F$ is Fermi’s constant in electroweak theory) the previous condition would imply $m_\chi \sim 10^{-12} \text{eV}$. This mass scale is 21 orders of magnitude larger than before, but still one billion times smaller than the tiny mass scale associated to the measured value of the cosmological constant: $\Lambda_0^{1/4} \sim 10^{-3} \text{eV}$. It is very difficult to understand the mass $m_\chi$ in particle physics, and this is of course a serious problem underlying the quintessence models.

The corresponding full energy-momentum tensor replacing (II) in this case is $\tilde{T}_{\mu\nu} = T_{\mu\nu} + T^\chi_{\mu\nu}$, where one assumes that the two components are of perfect fluid form and are conserved separately. For the $\chi$ part, $\nabla^\mu T^\chi_{\mu\nu} = 0$ leads to

$$\dot{\rho}_\chi + 3(1 + \omega_\chi) H \rho_\chi = 0,$$

instead of (8). We can easily convert this into a redshift equation using the correspondence between time derivatives and redshift derivatives: $d/dt = -(1 + z) H d/dz$. Then integrating (13) we have

$$\rho_\chi(z) = \rho_\chi(0) \zeta(z) \quad \text{where} \quad \zeta(z) = \exp\left\{ 3 \int_0^z dz' \frac{1 + \omega_\chi(z')}{1 + z'} \right\}. \hfill (14)$$

If we plug this equation into (II) we may write the Hubble expansion rate as a function of the redshift and the unknown (z-dependent) barotropic index $\omega_\chi = \omega_\chi(z)$ as follows:

$$H^2(z) = H_0^2 \left[ \tilde{\Omega}_M^0 (1 + z)^3 + \tilde{\Omega}_K^0 (1 + z)^2 + \tilde{\Omega}_\chi^0 \zeta(z) \right]. \hfill (15)$$

If one expands

$$\omega_\chi(z) = \omega_0 + \omega_1 z + ...$$

then for small redshifts one can replace $\zeta(z)$ in (15) with

$$\zeta(z) \simeq e^{3\omega_1 z} (1 + z)^3 (1 + \omega_0 - \omega_1), \hfill (17)$$
where one expects $\omega_0 \simeq -1$ and $| \omega_1 | \ll 1$ in order that $\chi$ can mimic a slowly varying CC. In Eq. (15) we have defined the cosmological parameters $\tilde{\Omega}_M$ and $\tilde{\Omega}_K$ in the usual way. The tilde indicates that they are presumably determined from a fit to experimental data assuming a true quintessence model. This notation will help to distinguish them from the cosmological parameters associated to the aforementioned RG model (more on this in the next section). Finally, we have defined $\tilde{\Omega}_0^\chi$ in (15) as the value of $\rho\chi(0) = \{(1/2)\xi \dot{\chi}^2 + V(\chi)\}_{z=0}$ in units of the critical density at present.

Effective equation of state for $\Lambda$

Let us now come back to the RG cosmological model. Solving the system (2),(5) and (8) one finds [17] $\rho = \rho(z; \nu)$ and $\Lambda = \Lambda(z; \nu)$ as explicit functions of the redshift and depending on the single additional parameter $\nu$, Eq. (7). These functions can be substituted back into Eq.(2) to obtain the expansion parameter as a function of the redshift:

$$H^2(z; \nu) = H_0^2 \left\{ 1 + \Omega_0^\Lambda \frac{(1+z)^3(1-\nu)}{1-\nu} - 1 + \frac{\Omega_0^\Lambda}{1-3\nu} \left[ (1+z)^2 - 1 - 2\nu \frac{(1+z)^3(1-\nu)}{1-\nu} - 1 \right] \right\}.$$ (18)

For $\nu = 0$ we recover the standard form corresponding to strictly constant $\Lambda$. Here the cosmological parameters are denoted without tilde because they need not to be the same ones as in (15). In fact, in Ref.[22] it has been shown how to fit the high-z supernovae data using this RG model. The fit crucially depends on the luminosity distance function, which is determined by the explicit structure of (18), so that the fitting parameters $\Omega_0^\Lambda, \Omega_0^K$ can be different from those obtained by substituting the alternate function (15) in the luminosity distance function. The potential differences between these parameters,

$$\Delta \Omega_M = \Omega_0^\Lambda - \tilde{\Omega}_M, \quad \Delta \Omega_\Lambda = \Omega_0^\Lambda - \tilde{\Omega}_\Lambda, \quad \Delta \Omega_K = \Omega_0^K - \tilde{\Omega}_K$$ (19)

can play a role in our discussion, but the main effect under consideration would be there even if these differences would exactly be zero. What we are really searching for is an effective dark energy EOS

$$p_D = \omega_{\text{eff}} \rho_D$$ (20)

associated to the running $\Lambda$ model that gives rise to the expansion rate (18). This means the following. In practice we would have experimental data, and we would usually fit it to a quintessence-like DE model in order to determine its EOS. But suppose that the RG model described above should be the correct one and that the experimental data would follow the Hubble function (18) for some value of $\nu$. In that case the data would actually adapt perfectly well to a fundamental running $\Lambda$. But of course it could be that we just ignore this fact, and insist in fitting the data to a quintessence-like model (15) with $\omega_{\chi}$ replaced by an effective $\omega_{\text{eff}}$. Then the natural questions that emerge are the following: i) what would be the effective barotropic index, $\omega_{\text{eff}}$, for the EOS of this model? ii) would it appear as a normal quintessence model ($\omega_{\text{eff}} \gtrsim -1$)?, iii) could it effectively behave as a phantom model ($\omega_{\text{eff}} < -1$) for some values of $\nu$ and/or in some range of redshift?; iv) what is the impact on these questions if we have non-vanishing parameter differences (19) in the
two independent fits of the same data? To answer these points we have to solve for the barotropic index function \( \omega_{\text{eff}} = \omega_{\text{eff}}(z) \) obtained after equating (15) and (18). Since \( \omega_{\text{eff}}(z) \) appears in the integral at the exponent of (14), the procedure can be simplified as follows. We first note from this equation that

\[
\omega_{\text{eff}}(z) = -1 + \frac{1}{3} (1 + z) \frac{1}{\zeta} \frac{d\zeta}{dz}. \tag{21}
\]

Next we compute the redshift derivative of (15) and arrive at

\[
\bar{\Omega}_K^0 \frac{d\zeta}{dz} = \frac{d}{dz} \left( \frac{H^2}{H_0^2} \right) - 2 \bar{\Omega}_K^0 (1 + z) - 3 \bar{\Omega}_M^0 (1 + z)^2. \tag{22}
\]

The pending derivative on the r.h.s. of this equation can be computed from (18). Finally we insert the result for \( d\zeta/dz \) in (21). In doing this we keep non-vanishing parameter differences (\( \Delta \Omega \neq 0 \)) in (19). The final result is obtained after a straightforward calculation, but in the non-flat case (\( \Omega_0^K \neq \bar{\Omega}_K^0 = 0 \)) the result is a bit too cumbersome and will not be quoted here. Let us quote here only the result for the flat-space case (\( \Omega_0^K = \bar{\Omega}_K^0 = 0 \)). This should be enough to illustrate the basic facts, and moreover it is the most realistic situation in the light of the present data. One finds the following barotropic index function for the effective EOS of the running \( \Lambda \) model:

\[
\omega_{\text{eff}}(z) \mid_{\Delta \Omega \neq 0} = -1 + (1 - \nu) \frac{\Omega_M^0 (1 + z)^{3(1-\nu)} - \bar{\Omega}_M^0 (1 + z)^3}{\Omega_M^0 [(1 + z)^{3(1-\nu)} - 1] - (1 - \nu) \left[ \Omega_M^0 (1 + z)^3 - 1 \right]} . \tag{23}
\]

If the parameter differences (19) vanish, this yields

\[
\omega_{\text{eff}}(z) \mid_{\Delta \Omega = 0} = -1 + (1 - \nu) \frac{\Omega_M^0 (1 + z)^3 [(1 + z)^{-3\nu} - 1]}{1 - \nu - \Omega_M^0 + \Omega_M^0 (1 + z)^3 [(1 + z)^{-3\nu} - 1 + \nu]]. \tag{24}
\]

In the next section we analyze some phenomenological consequences and perform a detailed numerical analysis of these formulae.

**Effective quintessence and phantom behavior**

Before embarking on an exact numerical analysis of the formulae for \( \omega_{\text{eff}} = \omega_{\text{eff}}(z) \) found in the previous section, we can identify some interesting features from simple analytical methods. Let us concentrate on Eq. (24). First of all, as it could be expected, for \( \nu = 0 \) one retrieves the pure CC behavior \( \omega_{\text{eff}} = -1 \) at all redshifts. On the other hand, for non-vanishing \( \nu \) and \( z \to \infty \) we get \( \omega_{\text{eff}} \to 0 \) (for \( \nu > 0 \)) and \( \omega_{\text{eff}} \to -\nu \) (for \( \nu < 0 \)). And in the infinite future (\( z \to -1 \)) the EOS recovers again a pure CC behavior \( \omega_{\text{eff}} = -1 \) for any \( \nu < 1 \). We have seen that \( \nu \) is a naturally small parameter. For example, if \( M = M_P \) in (7) then \( \nu = \nu_0 \), where

\[
\nu_0 \equiv \frac{1}{12\pi} \simeq 0.026. \tag{25}
\]

In general we expect \( |\nu| \leq \nu_0 \) because from the effective field theory point of view we should have \( M \leq M_P \). This is also suggested from the bounds on \( \nu \) obtained from nucleosynthesis [17 22] and also from the CMB, although in this latter case the preferred values for \( \nu \) are smaller [30].
Figure 1: (a) Numerical analysis of $\omega_{\text{eff}}$, Eq. (23), as a function of the redshift for fixed $\nu = \nu_0 > 0$, Eq. (7), and for various values of $\Delta \Omega$ in (19). The Universe is assumed to be spatially flat ($\Omega_0^M = 0$) with the standard parameter choice $\Omega_0^M = 0.3$, $\Omega_0^\Lambda = 0.7$; (b) Extended $z$ range of the plot (a).

Therefore it is natural to expand the previous results for small $\nu \ll 1$. Again we take the simplest case (24) and we find, in linear approximation in $\nu$ (and for not very large values of the redshift):

$$\omega_{\text{eff}}(z) = -1 - 3 \nu \frac{\Omega_0^M}{\Omega_0^\Lambda} (1 + z)^3 \ln(1 + z).$$

(26)

This result is simple and interesting, and contains the basic qualitative features of our analysis. Of course it boils down to $\omega_{\text{eff}} = -1$ for $\nu = 0$. But for $\nu > 0$ it shows that we can get an (effective) phantom-like behavior ($\omega_{\text{eff}} < -1$)! The cubic enhancement with redshift indicates that a significant effective phantom phase can actually be reached already for redshifts of order 1 corresponding to our “recent” Universe. For example, for the standard flat-space choice ($\Omega_0^M$, $\Omega_0^\Lambda$) = (0.3, 0.7), and a typical value of $\nu$ as in (25), we get $\omega_{\text{eff}} \simeq (-1.2, -1.5)$ for $z = (1, 1.5)$ respectively. Even for $\nu > 0$ ten times smaller ($\nu = 0.1 \nu_0$) we get a non-negligible phantom-like behavior $\omega_{\text{eff}} \simeq -1.1$ near $z = 2$. These results are approximate, but the exact numerical analysis of equations (23)–(24) is shown in Figures 1b where we have also included the possibility of having non-vanishing parameter differences $\Delta \Omega$ in (19). For $\nu \simeq \nu_0$ at $z = 1$, the differences between the exact result and the approximate one (26) are of order of a few percent, and for $z = 1.5$ there is a difference of 10%; in this last case the more accurate value reads $\omega_{\text{eff}}(z = 1.5) = -1.67$.

If we consider now the impact of the parameter differences (19), we see that in the case $\nu = \nu_0$ the phantom effect can either be more dramatic (if $\Delta \Omega < 0$) or it can be smoothed out, and even disappear, for small $z$ when $\Delta \Omega > 0$. In the last case the phantom behavior is nevertheless retrieved at larger redshifts, see Fig. 1b. In the same figure we show the behavior of the $\nu > 0$ models for an extended redshift range up to $z = 10$. Of course this behavior cannot be described with the approximate expression (26), only with the full equation (23). At very large $z$ one attains very slowly the asymptotic limit $\omega_{\text{eff}} \to 0$ (cf. Fig. 1b). But well before reaching this limit one can appreciate a kind of divergent behavior, e.g. around $z \gtrsim 2$ for the $\Delta \Omega = 0$ case. It is due to the denominator of Eq. (23) which vanishes at that point. This can only happen for $\nu > 0$. Of course
there is nothing odd going on here because the presumed fundamental RG model is well-behaved for all values of $z$ — cf. Eq. (18). It is only the effective EOS description that displays this fake singularity, which is nothing but an artifact of the EOS parametrization of a true $\Lambda$ model. If we would discover a sort of anomaly like this when fitting the data we could suspect that there is no fundamental dynamical field behind the EOS but something else, like e.g. the RG model under discussion.

On the other hand, there is the class of models with $\nu < 0$, with an entirely different qualitative behavior. Here we have normal quintessence ($\omega_{\text{eff}} \gtrsim -1$) for $z > 0$ whenever $\Delta \Omega \geq 0$. This is obvious from Eq. (26). For example, if we fix $\nu = -\nu_0$, then for $z = (1, 1.5)$ we find $\omega_{\text{eff}} \simeq (-0.82, -0.62)$ respectively using the exact formula. For $\nu$ ten times smaller ($\nu = -0.1\nu_0$), we have $\omega_{\text{eff}} \simeq (-0.98 - 0.95)$ at the respective redshift values. Moreover, from Fig. 2 (which displays the exact numerical analysis of the case $\nu < 0$) it is apparent that this model can easily accommodate the possibility of a relatively recent EOS transition from a quintessence phase into a phantom phase. This would indeed happen for $\Delta \Omega < 0$ in Eq. (19). If, instead, $\Delta \Omega > 0$, then at small $z$ the index $\omega_{\text{eff}}$ increases with redshift faster than for $\Delta \Omega \leq 0$. However, in all cases with negative $\nu$ the effective barotropic index climbs fast with $z$ up to positive values before reaching the asymptotically small value $\omega_{\text{eff}} \rightarrow -\nu > 0$ (cf. Fig. 2a). For example, for $\nu = -\nu_0$ one achieves $\omega_{\text{eff}} \simeq +0.2$ around $z = 5$. This positive behavior of $\omega_{\text{eff}}$ effectively looks as additional radiation, and it is sustained for a long redshift interval. Finally, in Fig. 3a and 3b we plot $\omega_{\text{eff}}$ in detail for various values of $\nu$ and both signs, but for vanishing parameter differences $\Delta \Omega = 0$ in (19). It is patent that the effects (both normal quintessence and phantom-like behavior) should be visible even for $|\nu| \lesssim 0.1 \nu_0$, i.e. for $\nu$ of order of a few per mil.

It is interesting to compare the previous result for the effective EOS with usual expansions like (16), (17). One could naively think that the parameter $\omega_1$ is the direct analog of $\nu$ for the RG model. In fact it is, but only in part. Already from the approximate formula (26) it is patent that the first two terms in the expansion (16) describe very poorly the redshift behavior of the RG model. This is because the coefficient $\nu$ is highly enhanced by the cubic powers of $1 + z$, whereas $\omega_1$ is just the coefficient of the linear term in $z$. It means that if one would enforce the
data fit to be of the linear form \((16)\) the quality of the EOS could be rather bad – e.g. if the data would hypothetically adapt perfectly well to the RG model under discussion. There are alternative parametrizations of the EOS that may overcome some of these difficulties \([25]\), but the example \([26]\) shows that the effective EOS of variable \(\Lambda\) models can have a much stronger redshift dependence than usually assumed for scalar field models of the DE. This issue can be further illustrated using e.g. the (model-independent) analysis of the SNe(Gold)+CMB data \([4, 5]\) performed in Ref. \([31]\). In this analysis a polynomial fit to the expansion parameter and EOS of the DE is made as a function of \(z\). The results show that the fitted function \(\omega_{\text{eff}} = \omega_{\text{eff}}(z)\) in the redshift range \(0 \leq z \lesssim 1.7\) does uphold the possibility of a slowly varying \(\omega_{\text{eff}}(z)\) which is monotonically increasing with \(z\) from \(\omega_{\text{eff}}(0) < -1\) (today) and then reaching a long period \(\omega_{\text{eff}}(z) > -1\) at higher redshifts, with a crossing of the CC threshold \(\omega_{\text{eff}} = -1\) at some intermediate redshift in that interval. In other words, these model-independent fits of the data show that the effective dynamical evolution of the DE can be assimilated to a phantom-like behavior near our time preceded by a long quintessence-like regime. This is exactly the kind of behavior that the effective EOS of our RG model predicts for \(\nu < 0\) and \(\Delta \Omega < 0\) (as can be seen in Fig. \([2]\)).

Let us recall that the RG model underlying the effective EOS under consideration predicts a redshift evolution of the cosmological constant. An approximate formula for the relative variation of \(\Lambda\) (valid for small \(\nu\) and not very high redshift \(z\)) reads \([17]\)

\[
\delta_\Lambda \equiv \frac{\Lambda(z; \nu) - \Lambda_0}{\Lambda_0} = \nu \frac{\Omega_M^0}{\Omega_\Lambda^0} \left[(1 + z)^3 - 1\right].
\] (27)

Again taking the flat-space case with \(\Omega_M^0 = 0.3, \Omega_\Lambda^0 = 0.7, \) and \(\nu = \nu_0,\) one obtains \(\delta_\Lambda = 16.3\%\) for \(z = 1.5\) (reachable by SNAP \([24]\)). This effect is big enough to be measurable in the next generation of high precision cosmological experiments. At the end of the day we see that, either by direct measurement of the evolution of the cosmological constant, or indirectly through the rich class of qualitatively different behaviors of its effective EOS, it should be possible to get a handle on the underlying RG cosmological model. Finally, let us clarify that in the non-flat case \((\Omega_K \neq 0)\) we have checked that the numerical results are not significantly different from those presented here.

Figure 3: As in Fig. \([1]\) but assuming \(\Delta \Omega = 0\) in \((19)\): (a) for three values \(\nu > 0;\) (b) for three values \(\nu < 0.\)
for the flat Universe. A more complete numerical analysis of these effective EOS models, including the possibility of a running Newton’s constant, will be presented elsewhere.

Conclusions

We have illustrated the possibility that a “true” or fundamental cosmological term Λ can mimic the behavior expected for quintessence-like representations of the dark energy. Specifically, we have shown that a running cosmological constant based on the principles of quantum field theory – more concretely on the renormalization group (RG) – can achieve this goal. This suggests that the usual description of the dark energy in terms of a dynamical field should be cautiously interpreted more as a general parametrization rather than as a fundamental one. That is to say, the fact that the cosmological precision data may turn out to be adjustable to an equation of state (EOS) of a dynamical field does not necessarily mean that in such case we would have proven that there is such a field there. It could be an effective description of fundamental physics going on at higher energy scales, for example near the Planck scale. This physics could be based on just the cosmological constant, Λ, as the ultimate explanation for the dark energy, except that Λ should then be a running parameter, Λ = Λ(µ), namely one that evolves with an energy scale µ characteristic of the cosmological system. A picture of Λ like this is not essentially different from the quantum field theoretical running of, say, the electromagnetic charge, e = e(µ), in QED. While in the latter case µ should be in the ballpark of the collider energy, e.g. µ ≃ √s in a e+e− interaction at LEP, in cosmology the scale µ should be suitably identified from some testable ansatz. In previous work[14] the appropriate running scale µ for the cosmological context was identified with H(t), because the expansion rate gives the typical energy of the cosmological gravitons. Indeed H is of the order of the square root of the 4-curvature scalar of the FLRW metric. From this ansatz the primary renormalization group running of the cosmological term with µ, i.e. Λ = Λ(µ), can be easily converted into time-evolution, or alternatively into redshift dependence Λ = Λ(z). And this redshift dependence can then be matched to the general quintessence-like behavior, leading to an effective EOS for the DE, ρ_D = ω_eff ρ_D, where ω_eff = ω_eff(z) is a non-trivial function of the redshift precisely determined by the RG model. Remarkably enough it turns out that this effective EOS for Λ can be both of normal quintessence and of phantom type, depending on the value and sign of a single parameter, ν, in the RG cosmological model. In this respect we should recall that the present data suggest some tilt of the dark energy EOS into the phantom phase. Further remarkable is that the effective EOS of our RG model follows, with striking resemblance, the qualitative behavior derived in some model-independent fits to the most recent data[31]. These fits suggest that ω_eff > −1 for a long (quintessence-like) period in the past, and at the same time they suggest that the universe has just entered a phantom phase (ω_eff < −1) near our present. Irrespective of the credit we may wish to give to this possibility at present, our analysis shows how to possibly account for anomalies of this sort without resorting to a true phantom scalar field. Finally, we have shown that the effects resulting from the effective EOS are quite sizeable even for the relatively close redshift range z = 1 – 2 and for values of the ν parameter of order of a few per mil. This should be welcome because the next generation of supernovae experiments, such as SNAP, is going to scan intensively that particular redshift range. The net outcome of our analysis
is that an experimental determination, even with high precision, of a non-trivial EOS for the dark energy must be interpreted with great care, whether it results into normal quintessence or into phantom energy. A running cosmological constant, based on the standard principles of quantum field theory, could still be responsible for the observed dark energy of the universe.

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