Inversion-like and Major-like Statistics of an Ordered Partition of a Multiset

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Abstract. Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) of a positive integer \( n \), let \( \text{Tab}(\lambda, k) \) be the set of all tabloids of shape \( \lambda \) whose weights range over the set of all \( k \)-compositions of \( n \) and \( \text{OP}^k_\lambda \) the set of all ordered partitions into \( k \) blocks of the multiset \( \{1^{\lambda_1}2^{\lambda_2} \cdots l^{\lambda_l}\} \). In [2], Butler introduced an inversion-like statistic on \( \text{Tab}(\lambda, k) \) to show that the rank-selected M"{o}bius invariant arising from the subgroup lattice of a finite abelian \( p \)-group of type \( \lambda \) has nonnegative coefficients as a polynomial in \( p \). In this paper, we introduce an inversion-like statistic on the set of ordered partitions of a multiset and construct an inversion-preserving bijection between \( \text{Tab}(\lambda, k) \) and \( \text{OP}^k_\lambda \). When \( k = 2 \), we also introduce a major-like statistic on \( \text{Tab}(\lambda, 2) \) and study its connection to the inversion statistic due to Butler.

1. Ordered Partitions of a Multiset

Let \( n \) be a positive integer. An ordered partition of \( [n] := \{1, 2, \ldots, n\} \) is a disjoint union of nonempty subsets of \( [n] \), and its nonempty subsets are called blocks. Conventionally we denote by \( \pi = B_1/B_2/\cdots/B_k \) an ordered partition of \( [n] \) into \( k \) blocks, where the elements in each block are arranged in the increasing order. The set of all ordered partitions of \( [n] \) into \( k \) blocks will be denoted by \( \text{OP}^k_n \).

In the exactly same manner, one can define an ordered partition of a finite multiset. The set of all ordered partitions of a multiset \( S \) will be denoted by \( \text{OP}^k_S \). In particular, in case where \( S \) is a multiset given by

\[
\underbrace{1, \ldots, 1, 2, \ldots, 2, \ldots, l, \ldots, l}_{c_1 \text{-times} \quad c_2 \text{-times} \quad \cdots \quad c_l \text{-times}}, \quad \text{(simply denoted by } \{1^{c_1}2^{c_2} \cdots l^{c_l}\} \text{)},
\]

we write \( \text{OP}^k_{(c_1, \ldots, c_l)} \) for \( \text{OP}^k_S \). For each \( \pi = B_1/B_2/\cdots/B_k \in \text{OP}^k_S \), the type of \( \pi \) is defined by a sequence \( (b_1(\pi), b_2(\pi), \ldots, b_k(\pi)) \), where \( b_i(\pi) \) is the cardinality of

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in the present paper, a \( k \)-tuple, say \((c_1, \ldots, c_k)\), we define \( \text{Perm} \) by the total number of inversions of \( \xi \), that is, the number of \((i, j)\)'s such that \( 1 \leq i < j \leq |S| \), but \( \xi_i > \xi_j \). On the other hand, for an ordered partition

\[
\pi = \pi_{11} \pi_{12} \cdots \pi_{1b_1} / \cdots / \pi_{k1} \pi_{k2} \cdots \pi_{kb_k} \in \mathcal{OP}_n^k,
\]

we define \( \text{inv}(\pi) \) by the number of all pairs \(((i, j), (i', j'))\)'s with \( \pi_{ij} > \pi_{i'j'} \), where \( 1 \leq i < i' \leq k \), \( 1 \leq j \leq b_i \) and \( 1 \leq j' \leq b_i' \).

For each positive integer \( n \), a \( k \)-composition of \( n \) is usually defined by a \( k \)-tuple, say \((c_1, c_2, \ldots, c_k)\), of nonnegative integers whose sum equals \( n \). However, in the present paper, a \( k \)-composition of \( n \) will be used to mean by a \( k \)-tuple of positive integers and will be denoted by \((c_1, c_2, \ldots, c_k) \vdash n \). For positive integers \( n \) and \( k \) with \( k \leq n \) and for a \( k \)-composition \( c = (c_1, c_2, \ldots, c_k) \) of \( n \), we simply denote by \( \text{Perm}(n, k; c) \) the set of all permutations on \( \{1^c_1 2^c_2 \cdots k^c_k\} \) of type \( c \).

We also let \( \text{Perm}(n, k) \) be the set of all permutations on \( \{1^c_1 2^c_2 \cdots k^c_k\} \), where \((c_1, c_2, \ldots, c_k)\) ranges over the set of all \( k \)-compositions of \( n \). Moreover, we simply denote by \( \mathcal{OP}_n^k(c) \) the set of ordered partitions of \( [n] \) into \( k \)-blocks of type \( c \). With this notation, we can establish the following lemma.

**Lemma 1.1.** Let \( n, k \) be positive integers with \( k \leq n \).

(a) Let \( c = (c_1, c_2, \ldots, c_k) \) be a \( k \)-composition of \( n \). Then there exists a bijection, say \( \phi : \text{Perm}(n, k; c) \rightarrow \mathcal{OP}_n^k(c) \), such that \( \text{inv}(\xi) = \text{inv}(\phi(\xi)) \) for all \( \xi \in \text{Perm}(n, k; c) \). Therefore

\[
\sum_{\xi \in \text{Perm}(n, k; c)} q^{\text{inv}(\xi)} = \sum_{\pi \in \mathcal{OP}_n^k(c)} q^{\text{inv}(\pi)}.
\]

(b) There exists a bijection \( \phi : \text{Perm}(n, k) \rightarrow \mathcal{OP}_n^k \) such that \( \text{inv}(\xi) = \text{inv}(\phi(\xi)) \) for all \( \xi \in \text{Perm}(n, k) \). Therefore

\[
\sum_{\xi \in \text{Perm}(n, k)} q^{\text{inv}(\xi)} = \sum_{\pi \in \mathcal{OP}_n^k} q^{\text{inv}(\pi)}.
\]

**Proof.** (a) For a permutation \( \xi = \xi_1 \xi_2 \cdots \xi_n \) on \( \{1^c_1 2^c_2 \cdots k^c_k\} \), let \( \phi(\xi) \) be the unique ordered partition whose \( r \)-th block consists of \( i \)'s with \( \xi_i = r \) for all \( 1 \leq r \leq k \). For instance, if \( \xi = 1343222 \), then \( \phi(\xi) = 1/567/24/3 \). Obviously this correspondence is bijective. One can easily verify that \( \phi \) preserves the number of total inversions since \((i, j)\) is an inversion if and only if the block containing \( j \) precedes the block containing \( i \) in \( \phi(\xi) \).

(b) The desired assertion is straightforward from (a). \( \square \)

In the following, we introduce a generalization of Lemma 1.1. For this purpose, the notion of tabloids is required since they can be viewed as a generalization of multiset permutations. Given a partition \( \lambda \) of \( n \), a tabloid \( T \) of shape \( \lambda \) is a filling of the squares of \( \lambda \) with positive integers such that the entries in each row weakly
increase from left to right. The weight of a tabloid is defined by its multiset of entries.

**Definition 1.2.** (see [2, Definition 1.3.1])

(a) If $T$ is a tabloid of any shape, then the value of an entry $x$ in $T$ is the number of smaller entries in the same column and above $x$ or in the next column to the right and below $x$. The value of $T$, denoted by $\nu(T)$, is the sum of the values of the entries in $T$.

(b) For a $k$-composition $c$ of $n$, let $\text{Tab}(\lambda, k; c)$ be the set of all tabloids of shape $\lambda$ and weight $1^{c_1}2^{c_2}\cdots k^{c_k}$. And we let $\text{Tab}(\lambda, k)$ be the set of all tabloids of shape $\lambda$ whose weights range over the set of all $k$-compositions of $n$.

We first deal with the case where $\lambda = 1^n$. For each $k$-composition $c$ of $n$, consider the function

$$\psi : \text{Tab}(1^n, k; c) \rightarrow \text{Perm}(n, k; c), \quad T \mapsto \psi(T),$$

where $\psi(T)$ is the permutation obtained by reading entries of $T$ from bottom to top. Alternatively, $\psi(T)$ is obtained by rotating $T$ by $90^\circ$ clockwise. Obviously $\psi$ is a bijection. Moreover, from Definition it follows that $\psi$ is an inversion-preserving map, that is, $\nu(T) = \text{inv}(\psi(T))$ for all $T \in \text{Tab}(1^n, k; c)$. In view of Lemma, one can deduce that

$$\sum_{T \in \text{Tab}(1^n, k; c)} q^{\nu(T)} = \sum_{\pi \in O^k_p(c)} q^{\text{inv}(\pi)}.$$  

(1.1)

by composing $\phi$ with $\psi$. In fact, $\phi \circ \psi(T)$ is the ordered partition into $k$ blocks whose $r$-th block consists of $i$'s with $T(i, 1) = r$ for each $1 \leq r \leq k$. Since $c$ ranges over the set of all compositions of $n$, one can finally obtain the following identity:

$$\sum_{T \in \text{Tab}(1^n, k)} q^{\nu(T)} = \sum_{\pi \in O^k_p} q^{\text{inv}(\pi)}.$$  

(1.2)

Before introducing a generalization of Eq.(1.1) and Eq.(1.2), let us explain the motivation briefly. Recall that $|O^k_p| = k!S(n, k)$, where $S(n, k)$ is a Stirling number of the second kind. There is a natural one to one correspondence between $O^k_p$ and the set of $k$-chains in the Boolean algebra, $B_n = [0, 1]^n$. (For the precise information, see [8].)

On the other hand, it was shown in [2] that there exists a close relationship between the subgroup lattice of a finite abelian $p$-group $G$ of type $\lambda = (\lambda_1, \ldots, \lambda_l)$ and the lattice $[0, \lambda_1] \times \cdots \times [0, \lambda_l]$. Here the *type of $G$* denotes the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$ when $G$ is isomorphic to $\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \mathbb{Z}/p^{\lambda_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_l}\mathbb{Z}$. For instance, under this relationship, the case $\lambda = 1^n$ corresponds to $B_n = [0, 1]^n$. When $G$ is a finite abelian $p$-group of type $\lambda$, define $\alpha_\lambda(S; p)$ to be the total number of chains with rank set $S$ in the subgroup lattice of $G$.

**Lemma 1.3.** (see [2, Proposition 1.3.2]) If $S = \{a_1, a_2, \cdots, a_{k-1}\} \subseteq |\lambda| - 1$, then

$$\alpha_\lambda(S; p) = \sum_{T \in \text{Tab}(\lambda, k; c)} p^{\nu(T)},$$  

(1.3)
where \( c = (a_1, a_2 - a_1, \ldots, |\lambda|-a_{k-1}) \).

Recall that, given a tabloid \( T \), the value \( v(T) \) of \( T \) is defined by

\[
\sum_{(i,j) \in T} v(T; (i,j)),
\]

where \( v(T; (i,j)) \) is the sum of

\[
\sharp\{(i', j) \in T : 1 \leq i' < i \text{ and } T(i', j) < T(i, j)\}
\]

and

\[
\sharp\{(i', j + 1) \in T : i' > i \text{ and } T(i', j + 1) < T(i, j)\}.
\]

Here the notation \( \sharp \) is used to denote the cardinality of a set. Now we define the inversion of an ordered partition of \( n \) into \( k \)-blocks as follows.

**Definition 1.4.** Given a \( k \)-composition \( c \) of \( n \), let

\[
\pi = \pi_{11} \pi_{12} \cdots \pi_{1c_1} / \cdots / \pi_{k1} \pi_{k2} \cdots \pi_{kc_k}
\]

be an ordered partition into \( k \) blocks of type \( c \).

(a) Given \((i, j)\), we define \( \text{rep}(\pi; (i, j)) \) by

\[
\sharp\{(i', j') : (i) 1 \leq i' < i \text{ and } \pi_{i'j'} = \pi_{ij}, \text{ or } (ii) i' = i \text{ and } \pi_{ij} = \pi_{i'j'}\}.
\]

(b) Given \((i, j)\), \( \text{inv}(\pi; (i, j)) \) is defined by the sum of

\[
\sharp\{(i', j') : (i) 1 \leq i' < i, (ii) \pi_{ij} < \pi_{i'j'}, \text{ and } (iii) \text{rep}(\pi; (i', j')) = \text{rep}(\pi; (i, j))\}
\]

and

\[
\sharp\{(i'', j'') : (i) 1 \leq i'' < i, (ii) \pi_{ij} < \pi_{i''j''}, \text{ and } (iii) \text{rep}(\pi; (i'', j'')) = \text{rep}(\pi; (i, j)) + 1\}.
\]

(c) The inversion of \( \pi \), denoted by \( \text{inv}(\pi) \), is defined by

\[
\sum_{\pi_{ij} \in \pi} \text{inv}(\pi; (i, j)).
\]

**Example 1.5.** Let \( \pi = 444/555/5/2/1/3 \). It is easy to show that the inversion table \( (\text{inv}(\pi; (i, j)))_{i,j} \) is given by \( 000/110/0/2/3/2 \). For example, \( \text{inv}(\pi; (2, 1)) \) and \( \text{inv}(\pi; (4, 1)) \) can be illustrated well in the following diagram:

```
        4 4 4 / 5 5 5 / 5 / 2 / 1 / 3
         ^  ^  ^
        4 4 4 / 5 5 5 / 5 / 2 / 1 / 3
```
Let \( C_{\lambda}(k; p) \) denote the total number of \( k \)-chains in the subgroup lattice of the finite abelian \( p \)-group \( G \) of type \( \lambda \). The following theorem is the main result of this section.

**Theorem 1.6.** For positive integers \( n, k \) with \( k \leq n \), let \( \lambda \) be a partition of \( n \) with \( \ell(\lambda) = l \).

(a) Given a \( k \)-composition \( c \) of \( n \), there exists a bijection

\[
\eta : \text{Tab}(\lambda, k; c) \longrightarrow \mathcal{O} \mathcal{P}^{\lambda, k}_\text{rev}(c)
\]

such that \( v(T) = \text{inv}(\phi(T)) \) for all \( T \in \text{Tab}(\lambda, k; c) \). Here \( \lambda_{\text{rev}} \) denotes the partition \((\lambda_1, \ldots, \lambda_1)\). Therefore

\[
\sum_{T \in \text{Tab}(\lambda, k; c)} q^{v(T)} = \sum_{\pi \in \mathcal{O} \mathcal{P}^{\lambda}_{\text{rev}}(c)} q^{\text{inv}(\pi)}.
\]

(b) For a prime \( p \), we have

\[
C_{\lambda}(k; p) = \sum_{T \in \text{Tab}(\lambda, k)} p^{v(T)} = \sum_{\pi \in \mathcal{O} \mathcal{P}^{\lambda}_{\text{rev}}} p^{\text{inv}(\pi)}.
\]

**Proof.** (a) Given \( T \in \text{Tab}(\lambda, k) \), let us obtain a word, say \( w(T) = w_1w_2 \cdots w_n \), by the reading which proceeds up columns from bottom to top and from left to right. Now define \( \eta(T) \) in the following steps:

- **step 1:** Each \( T(i, j) \) is placed in the \( T(i, j) \)-th block as the value \( l - i + 1 \).
- **step 2:** Assume that \( T(i, j) \) appears in the \( k \)-th place among \( w_i \)'s with the same value with \( T(i, j) \). Then \( l - i + 1 \) appears in the \( k \)-th place of the \( T(i, j) \)-th block. Since \( \eta(T) \) is an ordered partition with length \( k \) and weight \( \text{sh}(T) \), \( \phi \) is well-defined. Furthermore, it is a bijection since there is the inverse map which can be defined by reversing the above process.

In the following, let us show that \( \eta \) is weight preserving. Fix \((i, j) \in T \). Note that, if \( \eta(T)_{ij} \) corresponds to the \( T(a, b) \) under \( \eta \), then \( \text{rep}(\eta(T); (i, j)) \) is nothing other than \( b \), the column of \( T(a, b) \). Now we claim that there is a bijection between

\[
\{(i', j) \in T : 1 \leq i' < a \text{ and } T(i', b) < T(a, b)\}
\]

and

\[
\{(i', j') : \ (i) \ 1 \leq i' < i, \ (ii) \ \eta(T)_{ij} < \eta(T)_{i'j'}, \ (iii) \ \text{rep}(\eta(T); (i', j')) = \text{rep}(\eta(T); (i, j))\}.
\]

To see this, let \( \eta(T)_{i'j'} \) correspond to the \( T(a', b') \) under \( \eta \). Then the third condition \( \text{rep}(\eta(T); (i', j')) = \text{rep}(\eta(T); (i, j)) \) implies \( b' = b \). The first condition \( i \leq i' < i \) implies that \( a' < a \). Finally, the second condition \( \eta(T)_{ij} < \eta(T)_{i'j'} \) is equivalent to the condition \( T(a', b') < T(a, b) \). In the same fashion, we can show that there is a bijection between

\[
\{(i', j + 1) \in T : i' > i \text{ and } T(i', j + 1) < T(i, j)\}
\]
and
\[
\{(i'', j''): \ (i) \ 1 \leq i'' < i, \ (ii) \ \pi_{i''j''} < \pi_{ij}, \\
(iii) \ \text{rep}(\pi; (i'', j'')) = \text{rep}(\pi; (i, j)) + 1\}.
\]

(b) First, note that there is a one to one correspondence between \(\{S \subset [n - 1] : |S| = k - 1\}\) and the set all \(k\)-compositions of \(n\). Thus, in view of Lemma 1.3, (b) can be derived from (a) by specializing \(q\) into \(p\).

\[\square\]

Remark 1.7. In the proof of Theorem 1.6, we can derive a bijection
\[
\text{reading} : \text{Tab}(\lambda, k; c) \to \mathcal{O}^d_c(\lambda^{\text{rev}})
\]
by reading the entries of a tabloid. So, by composing \(\eta^{-1}\) with \text{reading}, we obtain a bijection
\[
\theta : \mathcal{O}^d_c(\lambda^{\text{rev}}) \to \mathcal{O}^k_{\lambda^{\text{rev}}}(c).
\]
For instance, let \(T\) be the tabloid given by
\[
\begin{array}{cccc}
2 & 2 & 2 & 3 \\
1 & 1 & 1 & \\
6 & & & \\
4 & & & \\
5 & & & \\
\end{array}
\]
then \text{reading}(T) = 5/4/6/111/2223 and \(\eta(T) = 444/555/5/2/1/3\). Thus we have
\[
\theta(5/4/6/111/2223) = 444/555/5/2/1/3.
\]

2. A Major-like Statistics on the Set of Tabloids

In 1913, P. MacMahon introduced the major index statistic on the set of multiset permutations, which is defined by the sum of the descents, and this statistic is the same as the distribution of inversions. That is,
\[
(2.1) \quad \sum_{\sigma \in \text{Perm}(n,k;c)} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \text{Perm}(n,k;c)} q^{\text{inv}(\sigma)} = \left[\begin{array}{c} n \\ c_1, \ldots, c_k \end{array}\right]_q.
\]
In 1968, Foata constructed a map which assigns a permutation with a given major index to another with the same inversion index (see [4]).

After that, there were many results on these two statistics. An extension of MacMahon’s result has been given by Kasraoui in [5] and many major-like statistics on the ordered partitions have been found by Kasraoui and Zeng in [6].

The purpose of this section is to introduce a major-like statistic, \(m\), on the set \(\text{Tab}(\lambda, 2)\) of tabloids such that
\[
\sum_{T \in \text{Tab}(\lambda, 2)} p^{v(T)} = \sum_{T \in \text{Tab}(\lambda, 2)} p^{m(T)}.
\]
Definition 2.1. If $T$ is a tabloid with shape $\lambda$ and weight $\{c_1, c_2\}$, then the array of $T$, denoted $A_T$, is obtained from $T$ as follows.

(i) The last column of $A_T$ is equal to that of $T$.

(ii) For every $1 \leq j \leq \lambda_1 - 1$, the $j$th column of $A_T$ is defined by the following conditions:

- If there is an entry smaller than $T(i, j)$ in the $(j+1)$th column and below the $i$th row in $T$, then switch $T(i, j)$ with $T(i+1, j)$. Denote the result by $\hat{T}$.
- Continue the above process until no entries in $\hat{T}$ satisfy the conditions in the above step.

It should be noted that, from Definition 2.1, $T$ can be recovered from $A_T$ in a unique way. For all $1 \leq j \leq \lambda_1$, its $j$th column of $A_T$ is contained in $\text{Perm}(\lambda'_j, 2; c)$, where $c = (\lambda'_j - \nu'_j, \nu'_j)$. Define the inversion, $\text{inv}$, of $A_T$ by

$$\text{inv}(A_T) = \sum_{1 \leq i \leq \lambda_1} \text{inv}(\text{the } i\text{th column of } A_T).$$

Then, by Definition 2.1, one has that $\text{inv}(A_T) = v(T)$.

Example 2.2. Let $T$ be the tabloid as below. Then we have $v(T)$.

| $T$       | $v(T)$ | $A_T$ | $\text{inv}(A_T)$ |
|----------|--------|-------|-------------------|
| 2 2 2    | 1 1 0  | 1 1 2 | 1 1 0             |
| 1 1 1    |        | 2 1   |                   |
| 1 2      |        |       |                   |
| 2        |        | 2     |                   |
| 1        |        | 1     |                   |

Given a tabloid $T$ with weight $(c_1, c_2)$, we denote by $[k]T(j)$ the number of occurrences of $k$, $(k = 1, 2)$ in the $j$th column of $T$. Similarly, we denote by $[k]A_T(j)$ the number of occurrences of $k$ $(k = 1, 2)$ in the $j$th column of $A_T$. Note that, for all $j$, we have $[k]T(j) = [k]A_T(j)$.

Before introducing a major-like statistic on the tabloid $T$, let us define the set of descents of $A_T$, denoted $\text{DES}(A_T)$, by

$$\{(i, j) \in A_T : 1 \leq j \leq \lambda_1 - 1, [1]T^{(j+1)} + 1 < i \leq \lambda'_j, A_T(i, j) > A_T(i - 1, j)\} \cup \{(i, \lambda_1) \in A_T : 2 \leq i \leq \lambda'_j, A_T(i, \lambda_1) > A_T(i - 1, \lambda_1)\}.$$ 

Now, define a major-like statistic on $A_T$, denoted $\text{maj}$, by

$$\text{maj}(A_T) = \sum_{(i, j) \in \text{DES}(A_T)} \lambda'_j - i + 1.$$
Definition 2.3. With the above notation, for a tabloid $T$ of weight $(c_1, c_2)$, the value of $T$, denoted by $m(T)$, is defined to be
\[
\text{maj}(A_T) + \sum_{1 \leq j \leq \lambda_1 - 1} [2]T^{(j)} \cdot [1]T^{(j+1)}.
\]

Example 2.4. Let $T$ be the tabloid as below. Then we have
\[
\begin{array}{cccc}
2 & 2 & 2 \\
1 & 1 & 1 \\
1 & 2 \\
2 \\
1
\end{array}
\rightarrow
\begin{array}{cccc}
1 & 1 & 2 \\
2 & 2 & 1 \\
1 & 2 \\
2 \\
1
\end{array}
\text{maj}(A_T) = 2.
\]

In this case, $m(T)$ equals 4. Let $\alpha_\lambda(\mu; p)$ denote the number of subgroups of type $\mu$ in the finite abelian $p$-group of type $\lambda$.

Lemma 2.5. ([2, Lemma 1.4.1], [3]) For any partition $\mu$ with $\emptyset \neq \nu \subseteq \lambda$,
\[
(2.2) \quad \alpha_\lambda(\nu; p) = \prod_{j \geq 1} p^{\nu'_j-\nu'_j+1}(\lambda'_j - \nu'_j+1) \cdot (\lambda'_j - \nu'_j+1) \cdot (\lambda'_j - \nu'_j+1) \cdot (\lambda'_j - \nu'_j+1)
\]
where $\lambda'$ is the conjugate of $\lambda$ and
\[
\binom{n}{k}_p = \frac{(1 - p^n)(1 - p^{n-1}) \cdots (1 - p^{n-k+1})}{(1 - p)(1 - p^2) \cdots (1 - p^k)}.
\]

Lemma 2.6. Let $\lambda, \nu$ be partitions with $\nu \subseteq \lambda$. Then we have
\[
\alpha_\lambda(\nu; p) = \sum_{T \in \text{Tab}(\lambda, 2; (1^{|\nu|}, 2^{(\lambda - |\nu|)}))} p^{m(T)}.
\]

Proof. In view of Eq. (2.2) and Definition 2.3, it suffices to show the equality
\[
(2.3) \quad \prod_{j \geq 1} p^{\nu'_j+1}(\lambda'_j - \nu'_j) \cdot (\lambda'_j - \nu'_j+1) \cdot (\lambda'_j - \nu'_j+1) \cdot (\lambda'_j - \nu'_j+1)
\]

The left hand side of the above equation can be rewritten as
\[
(2.4) \quad \prod_{j \geq 1} p^{\nu'_j+1}(\lambda'_j - \nu'_j) \left( \sum_{\xi \in \text{Perm}(\lambda'_j - \nu'_j+1, 2^{(\lambda'_j - \nu'_j+1)})} p^{\text{maj}(\xi)} \right)
\]

(2.5) \quad \sum_{\xi = (\xi^{(1)}, \ldots, \xi^{(\lambda_1)})} p^{\text{maj}(\xi)} p^{\nu'_j+1}(\lambda'_j - \nu'_j),
where the sum ranges over the set
\[ P := \prod_{j \geq 1} \text{Perm}(\lambda_j' - \nu_j' + 1; (1^{\lambda_j' - \nu_j'} \cdot 2^{\nu_j' - \nu_j' + 1})) \]
and \( \text{maj}(\xi) = \sum_{1 \leq i \leq \lambda} \text{maj}(\xi^{(i)}) \).

Given a sequence of permutations \( \xi = (\xi^{(1)}, \ldots, \xi^{(\lambda)}) \) in \( P \), we let \( A_\xi \) be the array with shape \( \lambda \) defined by
\[ A_\xi(i, j) = \begin{cases} 1 & \text{if } i \leq \lambda_j' - \nu_j', \\ \xi(j)_{\lambda_j' - i + 1} & \text{if } \lambda_j' - \nu_j' < i \leq \lambda_j. \end{cases} \]

If we denote by \( A_j \xi \) the \( j \)th column of \( A_\xi \), then Eq. (2.5) can be rewritten as
\[ \sum_{A_\xi} p[2; A_\xi^{(j)}, 1; A_\xi^{(j+1)}] p[\text{maj}(A_\xi)], \]
where the sum runs over all \( A_\xi \)'s contained in
\[ \{ \text{sh}(A_\xi) = \lambda : [1]A_\xi = |\nu|, [2]A_\xi = |\lambda| - |\nu| \text{ and } A_\xi \text{ satisfies Eq. (2.6)} \}. \]

Also, recall that \( \text{maj}(A_\xi) = \sum_{1 \leq j \leq \lambda} \text{maj}(\xi^{(j)}) \). Finally, by Definition, we can derive the equality
\[ \sum_{T \in \text{Tab}(\lambda, 2; (1^{|\nu|}, 2^{|\lambda| - |\nu|}))} p[2; T^{(j)}, 1; T^{(j+1)}] p[\text{maj}(A_T)] = \sum_{T \in \text{Tab}(\lambda, 2; (1^{|\nu|}, 2^{|\lambda| - |\nu|}))} p[\text{maj}(T)], \]

The main theorem of this section is stated as follows.

**Theorem 2.7.** Let \( \lambda \) be a nonempty partition. Then
\[ C_{\lambda}(2; p) = \sum_{T \in \text{Tab}(\lambda, 2)} p[\text{maj}(T)]. \]

**Proof.** The desired result is almost straightforward since
\[ C_{\lambda}(2; p) = \sum_{\emptyset \subseteq \nu \subseteq \lambda} \alpha_{\lambda}(\nu; p) \]
\[ = \sum_{\emptyset \subseteq \nu \subseteq \lambda} \left( \sum_{T \in \text{Tab}(\lambda, 2; (1^{|\nu|}, 2^{|\lambda| - |\nu|}))} p[\text{maj}(T)] \right) \]
\[ = \sum_{T \in \text{Tab}(\lambda, 2)} p[\text{maj}(T)]. \]

According to [7], let us define
\[ \psi : \text{Perm}(n, k; c) \rightarrow \text{Perm}(n, k; c), \sigma \mapsto \beta^{(n-1)} \]
in the following steps.
Proof. More precisely, for any \( \{ \text{maj} \} \), there is a bijective map from Theorem 2.8.

It is well described in Chapter 10 of [7] that \( \psi \) is a bijective map such that \( \text{inv}(\sigma) = \text{maj}(\psi(\sigma)) \).

**Theorem 2.8.** There is a bijective map from \( \text{Tab}(\lambda, 2) \) into itself such that

\[
\sum_{T \in \text{Tab}(\lambda, 2)} p^v(T) = \sum_{T \in \text{Tab}(\lambda, 2)} p^m(T),
\]

where the sum runs over all tabloids in \( \text{Tab}(\lambda, 2) \).

**Proof.** More precisely, for any \( \emptyset \subseteq \nu \subseteq \lambda \), we will construct a bijection \( \Psi \) from \( \{ T \in \text{Tab}(\lambda, 2; (1^{r_1}, 2^{\lambda-|\nu|}) \} \) into itself such that \( v(T) = m(\Psi(T)) \).

Given \( T \in \{ T \in \text{Tab}(\lambda, 2; (1^{r_1}, 2^{\lambda-|\nu|}) \} \), let us obtain \( A_T \) following Definition 2.5. Set \( \beta(1) = \sigma \), where \( \sigma \in \text{Perm}(n, k; c) \).

1. Begin by setting \( \beta(1) = \sigma \), where \( \sigma \in \text{Perm}(n, k; c) \).
2. If \( \beta(1)_n \geq \beta(1)_1 \), draw a bar before each number in \( \beta(1) \) which is less than or equal to \( \beta(1)_n \), and also before \( \beta(1)_n \).
3. If \( \beta(1)_n < \beta(1)_1 \), draw a bar before each number in \( \beta(1) \) which is greater than \( \beta(1)_n \), and also before \( \beta(1)_n \).
4. Move each number at the beginning of a block to the end of the block. Then the last letter of \( \beta(1) \) is fixed.
5. Set \( \beta(2) = \beta(1) \), and compare the \( (n-1) \)st letter with the first, creating blocks and draw an extra bar before the \( (n-1) \)st letter as above. Repeat the step (4).

It is well described in Chapter 10 of [7] that \( \psi \) is a bijective map such that \( \text{inv}(\sigma) = \text{maj}(\psi(\sigma)) \).

Next, let us recover \( \hat{T} \) from \( A_T \) and then define \( \Psi(T) \) by \( \hat{T} \).

The bijectivity of \( \Psi \) follows from the uniqueness of \( \psi \). Moreover, we have

\[
v(T) = \sum_{1 \leq j \leq A_T(j)} \text{inv}(A_T(j))
= \sum_{1 \leq j \leq A_T(j)} [2]A_T(j) \cdot [1]A_T(j+1) + \text{inv}(\xi(j))
= \sum_{1 \leq j \leq A_T(j)} [2]A_T(j) \cdot [1]A_T(j+1) + \text{maj}(\psi(\xi(j)))
= \sum_{1 \leq j \leq A_T(j)} [2]A_T(j) \cdot [1]A_T(j+1) + \text{maj}(\tau(j))
= m(\Psi(T)).
\]

This completes the proof. \( \square \)
Remark 2.9. Specially, if \( \lambda = \frac{1}{n} \), then the set \( \text{Perm}(\lambda, 2) \) is related to ordered partitions of the set \( \{1, \ldots, n\} \). In view of Eq. (1.2), we can easily define a major-like statistic on \( \mathcal{O} \), that is,

\[
\sum_{T \in \text{Tab}(1^n, k)} q^{m(T)} = \sum_{\pi \in \mathcal{O}} q^{\text{maj}(\pi)}.
\]

Example 2.10. Suppose that \( \lambda = (2, 2, 2, 2, 1) \) and \( \nu = (2, 2, 1) \). Then we get the following result.

\[
\alpha_\lambda(\nu; p) = p^{2(5-3)} \binom{5-2}{3-2} p^0 \left( \frac{4}{2} \right) = p^4 \left( \frac{3}{1} \right) p \left( \frac{4}{2} \right).
\]

The next figure illustrates \( \Psi \) well.

To be precise, we have the following correspondence.
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