Safe Learning of Linear Time-Invariant Systems

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Abstract

We consider safety in simultaneous learning and control of discrete-time linear time-invariant systems. We provide rigorous confidence bounds on the learned model of the system based on the number of utilized state measurements. These bounds are used to modify control inputs to the system via an optimization problem with potentially time-varying safety constraints. We prove that the state can only exit the safe set with small probability, provided a feasible solution to the safety-constrained optimization exists. This optimization problem is then reformulated in a more computationally-friendly format by tightening the safety constraints to account for model uncertainty during learning. The tightening decreases as the confidence in the learned model improves. We finally prove that, under persistence of excitation, the tightening becomes negligible as more measurements are gathered.

1 Introduction

Safety of a feedback control system while simultaneously learning and controlling the system is important. We do not want to potentially destroy the system that we are controlling or hurt others while learning to manoeuvre it.

Safe control where model uncertainties are unknown and learnt is an active topic of research [Taylor et al., 2020, Cheng et al., 2019, 2020, Choi et al., 2020, Jagtap et al., 2020]. For instance, the model and its uncertainties can be represented using Gaussian processes and learnt [Cheng et al., 2019, 2020, Jagtap et al., 2020]. The Gaussian process model can then be used to ensure safety while controlling the system. This allows for learning and control of large classes of systems. These studies however share a common, often implicit, assumption that the model and representation of the uncertainty are learnt prior to control or that we can alternate between learning and control [Choi et al., 2020, Jagtap et al., 2020, Cheng et al., 2020, 2019, Taylor et al., 2020]. In contrast, our main interest in this paper is to perform learning and control simultaneously while new state measurements arrive, and to prove that the system remains safe with high probability.

There are few studies that consider safety in simultaneous learning and control. The work of [Devonport et al., 2020] proposes evaluating the confidence of learned Gaussian processes when modelling potentially non-linear systems and environments. The confidence is used to make decisions regarding safety of the closed-loop system based on the number of measurements used for learning. This work follows the results of [Srinivas et al., 2012] on learning Gaussian processes. Although very powerful, the work of [Devonport et al., 2020] does not provide computationally ef-
cient methods for ensuring safety, as their framework relies on Lyapunov functions, which can be notoriously difficult to compute [Parrilo, 2000]. The other relevant study is [Farokhi et al., 2021], which proposes computationally efficient methods for projecting control signals into safe sets by computing confidence of learned Gaussian models. However, that work only considers learning stochastic disturbances caused by the environment and assumes that the underlying model of the system is known. Therefore, a computationally friendly method for ensuring safety while learning the model is missing from the literature. This is the topic of the current paper.

2 Safe Learning of Model

Consider a linear time-invariant discrete-time dynamic system of the form:

\[ x[k + 1] = Ax[k] + Bu[k] + w[k], \]  

where \( x[k] \in \mathbb{R}^n \) is the state, \( u[k] \in \mathbb{R}^m \) is the control input, and \( w[k] \in \mathbb{R}^n \) is the process noise. The process noise is a sequence of independent and identically distributed Gaussian random variables with zero mean and co-variance \( W \in \mathbb{S}^n \), where \( \mathbb{S}^n \) is the set of positive semi-definite matrices. We do not know \( A, B, \) and \( W \) and thus must learn them to control the system. The objective of this paper is to safely learn matrices \( A \) and \( B \). The safety is encoded by time-varying linear constraints:

\[ H[k]x[k] \leq h[k], \]  

where the inequality must hold entry-wise. The control signal is also constrained by

\[ u[k] \in \mathcal{U}. \]  

We make the following standing assumption about knowing an upper bound on the co-variance of the process noise.

**Assumption 1.** There exists some known \( r > 0 \) such that \( W \leq rI \), where \( \leq \) is the positive semi-definite partial ordering.

We define

\[ \theta_i := \begin{bmatrix} A_i^T \\ B_i^T \end{bmatrix}, \]  

where \( A_i \) and \( B_i \) are, respectively, the \( i \)-th rows of matrices \( A \) and \( B \). Similarly, let \( x_i \) and \( w_i \) denote the \( i \)-th entries of vectors \( x \) and \( w \). We can rearrange the system dynamics in (1) to obtain

\[ X_i[k] = Z[k] \theta_i + W_i[k], \]  

where

\[ X_i[k] = \begin{bmatrix} x_i[1] \\ x_i[2] \\ \vdots \\ x_i[k] \end{bmatrix}, \quad W_i[k] = \begin{bmatrix} w_i[0] \\ w_i[1] \\ \vdots \\ w_i[k-1] \end{bmatrix}, \quad Z[k] = \begin{bmatrix} x[0]^T \\ x[1]^T \\ \vdots \\ x[k-1]^T \end{bmatrix}. \]

The regularized least-squares estimate of \( \theta_i \) is given by

\[ \hat{\theta}_i[k] \in \arg \min_{\theta_i \in \mathbb{R}^{n \times m}} \left\| X_i[k] - Z[k] \hat{\theta}_i[k] + \lambda \hat{\theta}_i[k] \right\|_2^2. \]  

The solution to this regularized least-squares problem is given by

\[ \hat{\theta}_i[k] := (Z[k]^T Z[k] + \lambda I)^{-1} Z[k]^T X_i[k]. \]  

The regularized estimates can be concatenated to get the learned model:

\[ [\hat{A}[k] \hat{B}[k]] := \begin{bmatrix} \hat{\theta}_1[k] \\ \vdots \\ \hat{\theta}_n[k] \end{bmatrix}. \]  

2
We may see that
\[
(\hat{A}[k], \hat{B}[k]) \in \arg\min_{(\tilde{A}, \tilde{B})} \left( \sum_{t=0}^{k-1} \| x[t+1] - (\tilde{A} x[t] + \tilde{B} u[t]) \|^2 \right) + \lambda (\|\hat{A}\|_F^2 + \|\hat{B}\|_F^2).
\] (8)

Note that the least square problems in (8) and (9) can be solved recursively to reduce the memory requirements of the presented safe learning framework. We next make the following standing assumption regarding the “size” of the model parameters.

**Assumption 2.** There exists some known \( s > 0 \) such that \( \|A - B\|_F \leq s \), where \( \| \cdot \|_F \) denotes the Frobenius norm.

Note that we can always select a large enough constant to meet this assumption.

**Theorem 1.** Let \( V[k] = Z[k]^\top Z[k] + \lambda I. \) Then, \( \mathbb{P}(\|V[k]\|^{1/2}(\hat{\theta}_t[k] - \theta_i)\|_2 \leq \beta_k(\delta/n), \forall i \geq 1 - \delta, \) where \( \beta_k(\delta) := r \sqrt{2 \log \det(V[k])^{1/2}(\delta^{1/2})} + \lambda^{3/2} s. \)

**Proof.** See Appendix A in the supplementary material.

Safe control can be achieved by modifying a nominal control input \( \bar{u}_t \) at each iteration to ensure safety. We can do this by solving:

\[
\begin{align*}
\bar{u}[k] &= \arg\min_{u \in \mathcal{U}} d(u, \bar{u}[k]), \\
\text{s.t.} \quad H[k + 1](\hat{A}[k] x[k] + \hat{B}[k] u + v + w) &\leq h[k + 1], \quad \forall w : w^\top w \leq \frac{2rn}{\delta}, \\
\forall v : v^\top v &\leq \zeta_k^2 n \beta_k^2 \left( \frac{\delta}{2n} \right),
\end{align*}
\] (9a, 9b)

where \( d(\cdot, \cdot) \) denotes a distance metric, \( \beta_k \) is defined in Theorem 1 and \( \zeta_k := \max_{u \in \mathcal{U}} \|V[k]^{1/2} [x[k]^\top u]^\top \|_2. \)

**Theorem 2.** Assume that problem (9) is feasible. Then, if we implement the control action \( u[k], x[k+1] \) is safe with probability of at least \( 1 - \delta \), i.e., \( \mathbb{P}(H[k + 1] x[k + 1] \leq h[k + 1]) \geq 1 - \delta. \)

**Proof.** See Appendix B in the supplementary material.

Note that problem (9) as written involves infinitely many constraints and is thus not easy to solve numerically. We next provide a computationally-friendly reformulation of (9).

**Theorem 3.** The optimization problem in (9) is equivalent to

\[
\begin{align*}
u_t &= \arg\min_{u \in \mathcal{U}} d(u, \bar{u}_t), \\
\text{s.t.} \quad H[k + 1](\hat{A}[k] x[k] + \hat{B}[k] u) &\leq h[k + 1] - \bar{c}[k + 1],
\end{align*}
\] (10a, 10b)

where \( \bar{c}[k + 1] = \left( \zeta_k n \beta_k (\delta/(2n)) + \sqrt{2rn/\delta} \right) \|H_t[k + 1]\|_2. \)

**Proof.** See Appendix C in the supplementary material.

The constraint-tightening term in (10) is composed of two independent terms: one is caused by the uncertainty of the learned model and the other is caused by the process noise. We can show that the constraint-tightening term due to the uncertainty of the learned model goes to zero under persistence of excitation.

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1 The nominal control (which in general does not take into account safety) can, e.g., be generated by standard control design techniques (such as PID or model predictive control) or by a reinforcement learning algorithm.

2 For instance, \( d(u, \bar{u}) = \|u - \bar{u}\|_2. \)
Definition 1 (Persistence of Excitation [Sastry and Bodson [2011]])

The system in (1) is said to be persistently excited if there exists constants \( \gamma \geq \alpha > 0 \) and an integer \( T_0 > 0 \) such that

\[
\alpha I \leq \begin{bmatrix}
\sum_{t=k}^{k+T_0-1} x[t]x[t]^T & \sum_{t=k}^{k+T_0-1} x[t]u[t]^T \\
\sum_{t=k}^{k+T_0-1} u[t]x[t]^T & \sum_{t=k}^{k+T_0-1} u[t]u[t]^T
\end{bmatrix} \leq \gamma I, \quad \forall k.
\]

Proposition 4. If the persistence of excitation holds, then \( \lim_{k \to \infty} \zeta_k \eta_k / \beta_k (\delta/(2n)) = 0. \)

Proof. See Appendix D in the supplementary material.

Proposition 4 shows that, assuming persistence of excitation, the effect of the uncertainty caused by learning the model in the constraint tightening in Theorem 3 tends to zero as more measurements are gathered. Therefore, when \( k \) grows large, we can solve (10) with \( \bar{e}_1 \) and \( \bar{e}_2 \) only.

The remaining constraint tightening term in this optimization problem is because of the process noise. Note that, because we have not attempted at learning the statistics of the process noise, we consider the worst-case scenario in light of Assumption 1. After we have recovered the model parameters, we can then use the techniques of [Farokhi et al., 2021] to learn the statistics of the noise and also shrink this term.

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A Proof of Theorem 1

First, note that
\[ \mathbb{E}\{\exp(\mu w_i[k])\} = \exp(\mu^2 \mathbb{E}\{w_i[k]^2\}/2) \leq \exp(\mu^2 r/2), \]
where the equality follows from that \( w_i[k] \) is a zero mean Gaussian random variable and the inequality follows from Assumption 1. Further, Assumption 2 implies that \( \|\theta_i\|_2 \leq \|A B\|_F \leq s \). Now, using [Abbasi-Yadkori et al., 2011, Theorem 2], we get
\[ \mathbb{P}\left\{ \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 \leq \beta_k(\delta/n) \right\} \geq 1 - \frac{\delta}{n}. \]

Then
\[ \mathbb{P}\left\{ \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 \leq \beta_k(\delta/n), \forall i \right\} = \mathbb{P}\left\{ \bigwedge_{i=1}^n \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 \leq \beta_k(\delta/n) \right\} \]
\[ = 1 - \mathbb{P}\left\{ \bigvee_{i=1}^n \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 > \beta_k(\delta/n) \right\} \]
\[ \geq 1 - \sum_{i=1}^n \mathbb{P}\left\{ \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 > \beta_k(\delta/n) \right\} \]
\[ = 1 - \delta. \]

This concludes the proof.

B Proof of Theorem 2

We have
\[ x[k+1] = \begin{bmatrix} \hat{\theta}_1[k]^T \\ \vdots \\ \hat{\theta}_n[k]^T \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} + v[k] + w[k], \]
where
\[ v[k] = \begin{bmatrix} \theta_1[k]^T - \hat{\theta}_1[k]^T \\ \vdots \\ \theta_n[k]^T - \hat{\theta}_n[k]^T \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}. \]

In what follows, we bound the perturbation terms \( v[k] \) and \( w[k] \) with high probability. Theorem 1 implies that
\[ \mathbb{P}\left\{ \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 \leq \beta_k(\delta/(2n)), \forall i \right\} \geq 1 - \frac{\delta}{2}, \]
and, as a result,
\[ \mathbb{P}\left\{ \sum_{i=1}^n \|V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i)\|_2 \leq n\beta_k(\delta/(2n)) \right\} \geq 1 - \frac{\delta}{2}. \]
Now note that
\[
\left\| \begin{bmatrix} \hat{\theta}_1[k] - \theta_1[k] \\ \vdots \\ \hat{\theta}_n[k] - \theta_n[k] \end{bmatrix} V[k]^{1/2} \right\|_2 \leq \sum_{i=1}^n \left\| V[k]^{1/2}(\hat{\theta}_i[k] - \theta_i[k]) \right\|_2,
\]
where the first inequality follows from [Gentle, 2007, Eq. (3.241)] and the second inequality follows from Lemma 5. Therefore,
\[
P \left\{ \left\| \begin{bmatrix} \hat{\theta}_1[k] - \theta_1[k] \\ \vdots \\ \hat{\theta}_n[k] - \theta_n[k] \end{bmatrix} V[k]^{1/2} \right\|_2 \leq n \beta_k \left( \frac{\delta}{2n} \right) \right\} \geq 1 - \frac{\delta}{2}.
\]
Further, we have
\[
\|v[k]\|_2 = \left\| \begin{bmatrix} \hat{\theta}_1[k] - \theta_1[k] \\ \vdots \\ \hat{\theta}_n[k] - \theta_n[k] \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} x[k] \\ \vdots \\ u[k] \end{bmatrix} \right\|_2 V[k]^{1/2} V[k]^{-1/2} \left\| \begin{bmatrix} x[k] \\ \vdots \\ u[k] \end{bmatrix} \right\|_2 \leq \zeta_k,
\]
and thus
\[
P \left\{ \|v[k]\|_2 \leq \zeta_k \beta_k \left( \frac{\delta}{2n} \right) \right\} \geq 1 - \frac{\delta}{2}.
\]
For the process noise, we have
\[
P \{w[k]^{\top}(rI)^{-1}w[k] \leq \varepsilon\} \geq P \{w[k]^{\top}W^{-1}w[k] \leq \varepsilon\} \geq 1 - \frac{n}{\varepsilon},
\]
where the last inequality follows by a similar calculation as in [Farokhi et al., 2021, proof of Theorem III.1]. Selecting \(\varepsilon = (2n)/\delta\) gives
\[
P \{w[k]^{\top}(rI)^{-1}w[k] \leq (2n)/\delta\} \geq 1 - \delta.
\]
Finally, we note that
\[
P \left\{ w[k]^{\top}w[k] \leq \frac{2rn}{\delta} \wedge \|v[k]\|_2 \leq \zeta_k \beta_k \left( \frac{\delta}{2n} \right) \right\} = 1 - P \left\{ w[k]^{\top}w[k] > \frac{2rn}{\delta} \wedge \|v[k]\|_2 > \zeta_k \beta_k \left( \frac{\delta}{2n} \right) \right\} = 1 - P \left\{ w[k]^{\top}w[k] > \frac{2rn}{\delta} \right\} - P \left\{ \|v[k]\|_2 > \zeta_k \beta_k \left( \frac{\delta}{2n} \right) \right\} = 1 - \delta.
\]
This concludes the proof.
C Proof of Theorem 3

Lemma 6 can be used to eliminate $v$ in (9) to get

\[ u[k] = \arg\min_{u \in \mathcal{U}} d(u, \hat{u}[k]), \]

s.t. \[ H[k + 1](\hat{A}[k]x[k] + \hat{B}[k]u + w) \leq h[k + 1] - e[k + 1], \forall w : w^\top w \leq \frac{2\kappa n}{\delta}, \]

where

\[ e_i[k + 1] = \zeta_k n \beta_k \left( \frac{\delta}{2n} \right) \| H_i[k + 1] \|_2. \]

An additional application of Lemma 7 to eliminate $w$ concludes the proof.

D Proof of Proposition 4

Let us define

\[ \Xi_{T_0}(k) := \begin{bmatrix} \sum_{t=0}^{k} x[t] x[t]^\top & \sum_{t=0}^{k} x[t] u[t]^\top \\ \sum_{t=0}^{k} u[t] x[t]^\top & \sum_{t=0}^{k} u[t] u[t]^\top \end{bmatrix} \]

Under persistence of excitation, $\alpha I \leq \Xi_{T_0}(k) \leq \gamma I$ for all $k$. Note that

\[ V[k] = \lambda I + \Xi_{T_0}(0) + \Xi_{T_0}(T_0) + \cdots + \Xi_{T_0}([k/T_0]T_0 - 1) \]

The persistence of excitation therefore implies that

\[ ([k/T_0] \alpha + \lambda) I \leq V[k] \leq ([k/T_0] + 1) \gamma + \lambda) I. \]

Hence, it must be that $\det(V[k]) = \mathcal{O}(k^n)$ and $\sigma_{\min}(V[k]^{-1/2}) = \mathcal{O}(1/\sqrt{k})$. Recalling the definitions of $\beta_k$ and $\zeta_k$, we obtain

\[ \zeta_k n \beta_k \left( \frac{\delta}{2n} \right) = \mathcal{O} \left( \sqrt{\log(\det(V[k])^{-1/2})} \sigma_{\min}(V[k]^{-1/2}) \right) = \mathcal{O} \left( \sqrt{\log(k)/\sqrt{k}} \right). \]

This concludes the proof.

E Useful Lemmas

Lemma 5. For $x_i > 0, \forall i$,

\[ \sqrt{x_1 + \cdots + x_n} \leq \sqrt{x_1} + \cdots + \sqrt{x_n}. \tag{11} \]

Proof. For $x_i > 0, i \in \{1, \ldots, n\}$, we have \[ \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i + 2 \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} \sqrt{x_i x_j} \]

Let \[ X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \]

\(\text{\Box}\)

Lemma 6. Let
Then, for any positive semi-definite matrix $Y$,

$$\|XY\|_F \leq \sum_{i=1}^{n} \|Y(X_i)^T\|_2.$$ 

**Proof.** The definition of Frobenius norm results in $\|X\|_F^2 = \sum_{i=1}^{n} \|X_i\|_2^2$. Furthermore,

$$XY = \begin{bmatrix} X_1Y \\ \vdots \\ X_nY \end{bmatrix}.$$ 

Therefore,

$$\|XY\|_F^2 = \sum_{i=1}^{n} \|(X_i)^T\|_2^2 = \sum_{i=1}^{n} \|Y(X_i)^T\|_2^2.$$ 

Finally, using Lemma 5 we get

$$\|XY\|_F = \sqrt{\sum_{i=1}^{n} \|Y(X_i)^T\|_2^2} \leq \sum_{i=1}^{n} \sqrt{\|Y(X_i)^T\|_2^2} = \sum_{i=1}^{n} \|Y(X_i)^T\|_2.$$ 

This concludes the proof. 

**Lemma 7.** For $W \succeq 0$ and $d \geq 0$, $\{u| a^T u + b^T w \leq c, \forall w : w^T W w \leq d\} = \{u| a^T u \leq c - \sqrt{d}\|W^{-1/2}b\|_2\}$. 

**Proof.** With the change of variables $\tilde{w} = W^{1/2} w$ and $\tilde{b} = W^{-1/2} b$, we have $\{u| a^T u + b^T \tilde{w} \leq c, \forall \tilde{w} : \tilde{w}^T W \tilde{w} \leq d\} = \{u| a^T u + \tilde{b}^T \tilde{w} \leq c, \forall \tilde{w} : \tilde{w}^T \tilde{w} \leq d\}$. Then, following the approach of [Ben-Tal et al., 2009, Example 1.3.3], we can obtain $\{u| a^T u + \tilde{b}^T \tilde{w} \leq c, \forall \tilde{w} : \tilde{w}^T \tilde{w} \leq d\} = \{u| \sqrt{d}\|\tilde{b}\|_2 \leq c - a^T u\}$. 

