Renormalization group approach to matrix models via noncommutative space

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Abstract

We develop a new renormalization group approach to the large-$N$ limit of matrix models. It has been proposed that a procedure, in which a matrix model of size $(N-1) \times (N-1)$ is obtained by integrating out one row and column of an $N \times N$ matrix model, can be regarded as a renormalization group and that its fixed point reveals critical behavior in the large-$N$ limit. We instead utilize the fuzzy sphere structure based on which we construct a new map (renormalization group) from $N \times N$ matrix model to that of rank $N-1$. Our renormalization group has great advantage of being a nice analog of the standard renormalization group in field theory. It is naturally endowed with the concept of high/low energy, and consequently it is in a sense local and admits derivative expansions in the space of matrices. In construction we also find that our renormalization in general generates multi-trace operators, and that nonplanar diagrams yield a nonlocal operation on a matrix, whose action is to transport the matrix to the antipode on the sphere. Furthermore the noncommutativity of the fuzzy sphere is renormalized in our formalism. We then analyze our renormalization group equation, and Gaussian and nontrivial fixed points are found. We further clarify how to read off scaling dimensions from our renormalization group equation. Finally the critical exponent of the model of two-dimensional gravity based on our formalism is examined.

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1 Introduction

There have been a lot of studies on nonperturbative aspects of string theory that have revealed that string theory would be nonperturbatively formulated in terms of a matrix model or a gauge theory in the large-\(N\) limit. In fact, it is well known that one- or two-matrix models provide nonperturbative formulations of noncritical strings [1] defined in less than one dimension [2]. Furthermore, several large-\(N\) gauge theories or matrix models are proposed as nonperturbative formulations of critical superstring theories [3].
However, the latter models are incomplete so far because we do not know precisely how to take a double scaling limit in which the gauge coupling is tuned to its critical value in a correlated way with the large-$N$ limit. In fact, in order to specify the recipe of the double scaling limit, we need information on dynamics of theory such as a critical exponent. In the case of noncritical strings, we can get it via the orthogonal polynomial method for a matrix model, or conformal field theory techniques for continuum (Liouville) theory. On the other hand, matrix models or gauge theories proposed as the models of critical string theories have more complicated actions with multiple matrices and this fact makes these models difficult to solve exactly, by traditional methods like the orthogonal polynomials, the Schwinger-Dyson equations, symmetry argument and so on. However, we here notice that in order to extract a critical exponent, we do not always have to solve a model explicitly. Indeed, we know that universal quantities like a critical exponent can be often derived by the renormalization group (RG) approach [4], where we do not have to integrate all degrees of freedom in a model. In the case of matrix models, this procedure has been materialized in a concrete way which is recapitulated in the next subsection.

1.1 A brief review of large-$N$ renormalization group

We review a good example in the case of Hermitian one-matrix model presented by Brezin and Zinn-Justin in [5]. For the purpose of examining the critical exponent in the large-$N$ limit, they first decompose $N \times N$ matrix $\phi_N$ by

$$
\phi_N = \begin{pmatrix}
\phi_{N-1} & v \\
v^\dagger & \alpha
\end{pmatrix},
$$

(1.1)

where $\phi_{N-1}$ is an $(N - 1) \times (N - 1)$ matrix, $v$ is an $(N - 1)$-component vector, and $\alpha$ is a real number. Then only $v$ and $\alpha$, namely one row and one column, are integrated out to yield a matrix model of $\phi_{N-1}$. As illustration, starting from the action

$$
S_N = N \text{tr}_N \left( \frac{1}{2} \phi_N^2 + \frac{g}{4} \phi_N^4 \right),
$$

(1.2)

then at one-loop level and to the leading order in $1/N$ expansion, we obtain an action of $(N - 1) \times (N - 1)$ matrix model,

$$
S_{N-1} = (N-1) \text{tr}_{N-1} \left( \left( \frac{1}{2} + \frac{g}{N} \right) \phi_{N-1}^2 + \frac{g'}{4} \phi_{N-1}^4 \right),
$$

(1.3)

The key observation here is that we can repeat this procedure and as such this is quite analogous to the usual RG. This motivates us to make a “wave function renormalization”

$$
\phi_{N-1} = \rho \phi'_{N-1}, \quad \rho = 1 - \frac{2g + 1}{2N} + \mathcal{O} \left( \frac{1}{N^2} \right),
$$

(1.4)

so that the action (1.3) will have again the standard kinetic term in the $(N - 1) \times (N - 1)$ matrix model

$$
S_{N-1} = (N - 1) \text{tr}_{N-1} \left( \frac{1}{2} \phi'^2_{N-1} + \frac{g'}{4} \phi'^4_{N-1} \right),
$$

(1.5)
from which we read off the change of the coupling constant as

$$g' = g - \frac{1}{N} (g + 4g^2). \quad (1.6)$$

Now it is crucial that since $N$ plays a role of the cutoff in the standard RG, we assume the Callan-Symanzik like RG equation

$$\left[ N \frac{\partial}{\partial N} - \beta(g) \frac{\partial}{\partial g} + \gamma(g) \right] F_N(g) = r(g), \quad (1.7)$$

where $F_N(g)$ is the free energy of the $N \times N$ matrix model:

$$F_N(g) = -\frac{1}{N^2} \log Z_N(g), \quad Z_N(g) = \int d\phi e^{-S_N}. \quad (1.8)$$

Namely, (1.7) prescribes the $N$-dependence of $F_N$. By plugging the known exact result, we can check that the free energy near the critical point $g^* = -1/12$ in fact satisfies (1.7) and there $\beta'(g)$ is related to the critical exponent (string susceptibility) $\gamma_1$ as

$$\gamma_1 = \frac{2}{\beta'(g^*)}. \quad (1.9)$$

We expect that at a fixed point of the above RG (1.6), critical behavior would show up and analysis of the RG near the fixed point gives us the critical exponent. This is indeed the case. From (1.6) we can read off the $\beta$-function, namely dependence of the coupling constant on the cutoff $N$ as

$$\beta(g) = -g - 4g^2, \quad (1.10)$$

and we find the nontrivial fixed point $g^* = -1/4$ and the critical exponent $\gamma_1 = 2$ according to (1.9). They relatively well approximate the exact values $g^* = -1/12$ and $\gamma_1 = 5/2$.

Hereafter we call this approach as large-$N$ renormalization group (RG).  

When we attempt to apply the large-$N$ RG to matrix models for critical strings, we immediately recognize the following issue: interpretation of matrices in these “new” matrix models is quite different from that in the “old” matrix models describing noncritical strings. Namely, in the former we interpret matrices as carrying information on the space-time itself; for example, eigenvalues as the space-time coordinates (of D-branes) and off-diagonal components as open strings connecting them. This is in contrast to the case of noncritical string where a matrix model is interpreted as a tool for describing a discretized worldsheet. Hence according to the spirit of the RG [4], it would be better to integrate out off-diagonal components far from diagonal ones, because they correspond to highly massive modes in the space-time, not just integrating one row and column as in [5]. These considerations tempt us to develop a new large-$N$ RG in which we assign the concept of high/low energy to each matrix element, and we integrate out modes with

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1More applications of the large-$N$ RG are presented in e.g. [6].
the highest energy. If it is possible, the new large-$N$ RG evidently accords with the spirit of the original RG \[4\] and is expected to have nice properties like locality. This is the motivation of the present work. We show that we can in fact define a new large-$N$ RG in a well-defined manner based on this idea. The key is utilizing the fuzzy sphere structure \[7\] by which each matrix element carries angular momentum. As a consequence, our large-$N$ RG has a nice correspondence with the usual RG in field theory. In formulating our large-$N$ RG, we will also clarify in what sense it is “local” and mention its particular features. We then analyze fixed points of our RG and propose how to read off a critical exponent.

This paper is organized as follows: in the next section we formulate our large-$N$ RG on the fuzzy sphere and derive the RG equation concretely. Fixed points of the RG equation are analyzed in section 3. It is desirable that these fixed points can be related to critical phenomena which have been observed by numerical simulations \[8, 9, 10\] and an analytical study \[11\] of fuzzy sphere $\phi^4$ theory via a matrix model. Our results will be compared to these works. Section 4 are devoted to conclusions and discussions. In appendices we enumerate useful formulas which are necessary in our calculations and show some identities satisfied by the fuzzy spherical harmonics.

## 2 Formulation of large-$N$ renormalization group on fuzzy sphere

In this section we define a filed theory on fuzzy sphere as a matrix model. Then we propose a new large-$N$ RG based on the analogy of the angular momentum realized in the space of matrices by the fuzzy sphere.

### 2.1 Definition of fuzzy sphere

We begin with constructing a map from the space of functions on the sphere to that of $N \times N$ matrices following \[7\]. A function on $S^2$ with radius $\rho$ can be expanded as
\[
\phi(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm} Y_{lm}(\theta, \varphi),
\]
where $\theta, \varphi$ are the polar coordinates, and $l, m$ correspond to the angular momentum, the magnetic quantum number (projection of the angular momentum), respectively. $Y_{lm}$ is the spherical harmonics
\[
Y_{lm}(\theta, \varphi) = \sqrt{\frac{(2l+1)(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{i m \varphi},
\]
which satisfies the orthogonality and completeness condition\[2\]
\[
\frac{1}{4\pi} \int d\Omega Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{ll'} \delta_{mm'},
\]
\[2\]For later use we change the normalization of $Y_{lm}$ by $\sqrt{4\pi}$ from the standard one.
\[
\frac{1}{4\pi} \sum_{lm} Y_{lm}(\theta, \varphi) Y_{lm}(\theta', \varphi') = \frac{\delta(\theta - \theta')\delta(\varphi - \varphi')}{\sin \theta}.
\] (2.4)

\(Y_{lm}(\theta, \varphi)\) can be also expanded as
\[
Y_{lm}(\theta, \varphi) = \rho^{-l} \sum_{i_1, \ldots, i_l} c_{i_1 \ldots i_l}^{(lm)} \hat{x}^{i_1} \cdots \hat{x}^{i_l},
\] (2.5)

where \(x^i (i = 1 \sim 3)\) are the standard flat coordinate of \(\mathbb{R}^3\), and \(c_{i_1 \ldots i_l}^{(lm)}\) is traceless and totally symmetric with respect to \(i_1, \ldots, i_l\). \(Y^*_{lm} = (-1)^m Y_{l-m}\) implies that \(c_{i_1 \ldots i_l}^{(lm)*} = (-1)^m c_{i_1 \ldots i_l}^{(l-m)}\).

In order to define an \(N \times N\) matrix corresponding to \(Y_{lm}\), we first introduce the spin \(L = (N - 1)/2\) representation of \(SU(2)\), \(L_i (i = 1 \sim 3)\) which satisfies
\[
[L_i, L_j] = i \varepsilon_{ijk} L_k,
\] (2.6)

and define
\[
\hat{x}^i = \alpha L_i, \quad \alpha = \rho \sqrt{\frac{4}{N^2 - 1}}.
\] (2.7)

Thus \(N \times N\) matrices \(\hat{x}^i\)'s satisfy an analogous equation of \(S^2\): \(\sum_{i=1}^{3} \hat{x}^{i2} = \rho^2\). We then define an \(N \times N\) matrix \(T_{lm}\) corresponding to \(Y_{lm}\) in (2.5) by
\[
T_{lm} = \rho^{-l} \sum_{i_1, \ldots, i_l} c_{i_1 \ldots i_l}^{(lm)} \hat{x}^{i_1} \cdots \hat{x}^{i_l},
\] (2.8)

which we call the fuzzy spherical harmonics hereafter and they are considered to form a noncommutative algebra of functions on the fuzzy sphere. It follows immediately from the property of \(c_{i_1 \ldots i_l}^{(lm)}\) that
\[
T_{lm}^\dagger = (-1)^m T_{l-m}.
\] (2.9)

Since \(T_{lm}\) is in the same representation as that of \(Y_{lm}\) under the rotation \(SU(2)\) of \(S^2\),
\[
U(R)T_{lm}U(R)^{-1} = \sum_{l'} T_{l'm'} R_{m'm}(R),
\] (2.10)

where \(U(R)\) and \(R^l(R)\) are the \(N\)-dimensional and \((2l + 1)\)-dimensional representation of \(R \in SU(2)\), respectively. The \(N\)-dimensional representation space of \(T_{lm}\) \((0 \leq l \leq 2L = N - 1, -l \leq m \leq l)\) is labeled by an integer \(s (-L \leq s \leq L)\). From the Wigner-Eckart theorem, (2.10) leads to
\[
\langle s | T_{lm} | s' \rangle = (-1)^{L-s} \binom{L}{-s} \binom{l}{m}^{L} \binom{l}{s} R(N, l),
\] (2.11)
where \( R(N,l) \) is independent of \( s, s' \) and depends on the choice of \( |s\rangle \). Following the convention in [12], we take

\[
\langle s| T_{lm} | s' \rangle = (-1)^{l-s} \left( \frac{L}{-s} \frac{l}{m} \frac{L}{s'} \right) \sqrt{(2l + 1)N},
\]

so that

\[
\frac{1}{N} \text{tr}_N \left( T_{lm} T_{l'm'}^\dagger \right) = \delta_{ll'} \delta_{mm'},
\]

and

\[
\frac{1}{N} \sum_{lm} \langle s_1| T_{lm} | s_2 \rangle \langle s_3| T_{lm}^\dagger | s_4 \rangle = \delta_{s_1 s_4} \delta_{s_2 s_3},
\]

which can be derived easily from (2.12) by using (A.2) and (A.3), where the trace is taken over an \( N \times N \) matrix. In particular, (2.14) leads to

\[
\frac{1}{N} \sum_{lm} \text{tr}_N \left( O_1 T_{lm} \right) \text{tr}_N \left( O_2 T_{lm}^\dagger \right) = \text{tr}_N \left( O_1 O_2 \right),
\]

\[
\frac{1}{N} \sum_{lm} \text{tr}_N \left( O_1 T_{lm} O_2 T_{lm}^\dagger \right) = \text{tr}_N O_1 \text{tr}_N O_2,
\]

for arbitrary \( N \times N \) matrices \( O_1, O_2 \). (2.13) and (2.14) show that \( T_{lm} \) form an orthogonal and complete basis of the space of \( N \times N \) matrices. Namely, any \( N \times N \) matrix \( \phi \) can be decomposed as

\[
\phi = \sum_{l=0}^{2L} \sum_{m=-l}^{l} \phi_{lm} T_{lm},
\]

Comparing (2.1) and (2.17), we find that the space of \( N \times N \) matrices can be regarded as a regularized space of functions on \( S^2 \) in which the angular momentum \( l \) is cut off at \( 2L = N - 1 \). It cannot be overemphasized that the space of functions on the fuzzy sphere has therefore completely finite dimensions, but that nevertheless they form a closed algebra and preserve the rotational symmetry \( SO(3) \) as in (2.10).

By taking the large-\( N \) limit with \( \rho \) fixed, it is clear from (2.17) that

\[
[\hat{x}^i, \hat{x}^j] = i \alpha \epsilon_{ijk} \hat{x}^k \rightarrow 0,
\]

namely the standard commutative \( S^2 \) will be recovered. More precisely, as shown in appendix [B] the structure constant of the fuzzy spherical harmonics

\[
[T_{l_1 m_1}, T_{l_2 m_2}] = \sum_{l_3 m_3} f_{l_1 m_1 l_2 m_2 l_3 m_3} T_{l_3 m_3}^\dagger,
\]

tends to that of the usual spherical harmonics in the large-\( N \) limit with \( \rho \) fixed.

From the definition of \( T_{lm} \) (2.8), we find that

\[
[L_i, [L_i, T_{lm}]] = l(l + 1)T_{lm},
\]
which implies that \([L_i, [L_i, \cdot]]\) corresponds to the Laplacian on the unit \(S^2\). In fact, the operator on the unit \(S^2\) \(L_i = -i\epsilon_{ijk} x_j \partial_k\) satisfies
\[
L_i^2 = -\Delta_\Omega, \\
\Delta_\Omega = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.
\]  
(2.21)

Hence \(L_i\) essentially corresponds to the adjoint action of \(L_i\).

In summary of this subsection, we have constructed the mapping rules:

1. function \(\rightarrow\) matrix:
   \[
   \phi(\theta, \varphi) = \sum_{l=0}^{2L} \sum_{m=-l}^{l} \phi_{lm} Y_{lm}(\theta, \varphi) \rightarrow \phi = \sum_{l=0}^{2L} \sum_{m=-l}^{l} \phi_{lm} T_{lm}.
   \]  
   (2.22)

2. integration \(\rightarrow\) trace:
   \[
   \int \frac{d\Omega}{4\pi} \phi(\theta, \varphi) = \frac{1}{N} \text{tr}_N \phi.
   \]  
   (2.23)

   Notice that this holds as equality.

3. Laplacian \(\rightarrow\) double adjoint action:
   \[
   -\Delta_\Omega \phi(\theta, \varphi) \rightarrow [L_i, [L_i, \phi]].
   \]  
   (2.24)

### 2.2 Field theory on fuzzy sphere

In the previous subsection we have seen that functions on the fuzzy sphere are defined as \(N \times N\) matrices. Hence considering a field theory on the fuzzy sphere amounts to constructing a corresponding matrix model. We have identified \(N^2\) bases \(\{T_{lm}\}\) in the space of \(N \times N\) matrices, and it is remarkable that they enjoy the concept of the angular momentum \((l, m)\) based on which we will formulate a new RG in the next subsection.

In the following we restrict our interest to a Hermitian matrix, which corresponds to a real function on \(S^2\). \(\Box\) leads to
\[
\phi = \phi^\dagger \leftrightarrow \phi^*_{lm} = (-1)^m \phi_{l-m},
\]  
(2.25)

for \(\Box\). As an illustration we consider\(^3\)
\[
S_N = \frac{\rho^2}{N} \text{tr}_N \left( -\frac{1}{2\rho^2} [L_i, \phi]^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4} \phi^4 \right),
\]  
(2.26)

which is a natural regularization of the scalar field theory on \(S^2\) with radius \(\rho\)
\[
S = \int \frac{\rho^2 d\Omega}{4\pi} \left( -\frac{1}{2\rho^2} (\mathcal{L}_i \phi(\theta, \varphi))^2 + \frac{m^2}{2} \phi(\theta, \varphi)^2 + \frac{g}{4} \phi(\theta, \varphi)^4 \right).
\]  
(2.27)

\(^3\)A coupling constant \(m^2\) in front of \(\text{tr}_N \phi^2\) should not be confused with the magnetic quantum number \(m\). Later, the mass \(m^2\) is always accompanied by a subscript \(N\), and may not be confusing.
Note that this action is classically equivalent to \((2.26)\) by \((2.23)\) and \((2.24)\). Expanding \(\phi\) as in \((2.17)\) and using the trace formula \((B.2)\) given in appendix B, \((2.26)\) can be written in the momentum space as

\[
S_N = \sum_{lm} \frac{1}{2}(l(l + 1) + \rho^2 m^2)\phi^*_{lm}\phi_{lm} \\
+ \frac{N\rho^2 g}{4} \sum_{l_0 m_r} \prod_{r=1}^{4} \left( (2l_r + 1)\frac{1}{2} \phi_{l_r m_r}\right) \\
\times \sum_{lm} (-1)^{-m}(2l + 1) \left( \begin{array}{ccc} l_1 & l_4 & l \\ m_1 & m_4 & m \end{array} \right) \left( \begin{array}{ccc} l & l_2 \\ -m & m_2 \end{array} \right) \left\{ \begin{array}{ccc} l_1 & l_4 & l \\ L & L & L \end{array} \right\} \left\{ \begin{array}{ccc} l & l_3 & l_2 \\ L & L & L \end{array} \right\}, \tag{2.28}
\]

where the summations effectively run in regions constrained by the triangular conditions (mentioned in \((A.1)\)) imposed by the 3\(j\)- and 6\(j\)-symbols. Hereafter a sum over \(l, m\) should be understood likewise and we will not show their range explicitly.

### 2.3 Large-\(N\) renormalization group on fuzzy sphere

In this subsection we demonstrate a new large-\(N\) renormalization group (RG) by taking advantage of the fuzzy sphere. As mentioned in the introduction, the main motivation to develop the RG on the fuzzy sphere is that there we can introduce the concept of the angular momentum in the space of \(N \times N\) matrices. This contrasts with the original large-\(N\) RG proposed in [5], where there is no clear notion of momentum. Thus the fuzzy sphere enables us to perform a new RG transformation lowering \(N\): \(N \rightarrow N - 1\) by integrating out matrix elements with the maximal angular momentum. The difference between our RG and that in [5] is just the choice of the basis in the space of \(N \times N\) matrices based on which the large-\(N\) RG is formulated. There would then be no gauge-invariant notion of matrix elements of high/low momenta since they are mixed up under \(U(N)\) rotations. However, introducing the kinetic term breaks \(U(N)\) and enables us to define high/low momentum with respect to the choice of the kinetic term action. Our formulation then manifestly accords with the spirit of the RG [4]. The resulting RG is expected to be a local and well-defined transformation and to give rise to a local Lagrangian with nice behavior in UV. Notice that for the purpose of the large-\(N\) RG we have to formulate the RG for finite \(N\). The fuzzy sphere which can be formulated with finite \(N\) fits in our purpose, while the noncommutative plane based on the Heisenberg algebra does not, at least in a straightforward manner.

Now we formulate our large-\(N\) RG. We start from the \(N \times N\) matrix model as in \((2.26)\)

\[
S_N = \frac{\rho^2}{N} \text{tr}_N \left( -\frac{1}{2\rho^2}[L_i, \phi_N]^2 + \frac{m^2}{2}\phi^2 + \frac{g_N}{4}\phi^4 \right), \tag{2.29}
\]

\(^4\)Strictly speaking, we are considering an analogy of the angular momentum space, which will be called the momentum space for simplicity.

\(^5\)We will mention a serious problem in use of the fuzzy torus in section.
where we have put the subscript \( N \) on the parameters in order to keep track of their RG flow. The RG transformation is carried out through the following two procedures: the first is coarse-graining, and the second is rescaling. For the first, we integrate out modes with the maximal angular momentum, i.e. \( \phi_{lm} \) with \( l = 2L \) and \( -2L \leq m \leq 2L \) in the expansion (2.17). Then the number of remaining modes will be \( N^2 - (4L + 1) = (N - 1)^2 \), which is exactly the number of components of \((N - 1) \times (N - 1)\) matrix. Thus we would reinterpret the resulting action as that of \((N - 1) \times (N - 1)\) matrix. Namely we construct a new map from a matrix model with rank \( N \) to that with rank \( N - 1 \). Next we make a scale transformation \( \rho_N \rightarrow \rho_{N-1} \) in order to recover the original scale, and finally read off the change of parameters \( m^2_N \) and \( g_N \) to \( m^2_{N-1} \) and \( g_{N-1} \). This is our new RG by utilizing the fuzzy sphere.\(^6\) In the following, we will present the details of each step in order.

### 2.3.1 Coarse-graining

For notational simplicity, let us divide the space of the angular momentum \( \Lambda \) as

\[
\Lambda = \{(l, m) \mid 0 \leq l \leq 2L, -l \leq m \leq l\} \subset \mathbb{Z}^2,
\]

\[
\Lambda_{\text{out}} = \{(l, m) \mid l = 2L, -2L \leq m \leq 2L\} \subset \Lambda,
\]

\[
\Lambda_{\text{in}} = \Lambda \setminus \Lambda_{\text{out}} = \{(l, m) \mid 0 \leq l \leq 2L - 1, -l \leq m \leq l\},
\]

and define correspondingly

\[
\phi_{lm} = \begin{cases} 
\phi_{2Lm}^{\text{out}} & (l, m) \in \Lambda_{\text{out}} \\
\phi_{lm}^{\text{in}} & (l, m) \in \Lambda_{\text{in}}
\end{cases}.
\]

Thus we formulate the coarse-graining procedure as

\[
S_{N-1}(m^2_{N-1}, g_{N-1}) = -\log \int \prod_{m=-2L}^{2L} d\phi_{2Lm}^{\text{out}} e^{-S_N(m^2_N, g_N)},
\]

where \( S_{N-1}(m^2_{N-1}, g_{N-1}) \) is an action of \((N - 1) \times (N - 1)\) matrix\(^7\). It is important that our large-\( N \) RG formalism respects the rotational symmetry \( SO(3) \), because it integrates out the whole components of one irreducible representation of \( SO(3) \) which does not mix modes with different \( l \) as in (2.10).

In order to make calculation of (2.32) tractable, we first note that the kinetic term in (2.29) can be decomposed as

\[
S_N^{(\text{kin})} = \frac{\rho_N^2}{N} \text{tr}_N \left( -\frac{1}{2}\rho_N^2 [L_i, \phi]^2 + \frac{1}{2}m_N^2 \phi^2 \right) = S_N^{(\text{kin})}_{\text{in}} + S_N^{(\text{kin})}_{\text{out}},
\]

\(^6\) Another RG approach to matrix models based on the configuration space is proposed in [13].

\(^7\) A similar RG has been proposed in [14] for the purpose of analyzing the noncommutative field theory. However, it should be noticed that in our large-\( N \) RG we construct an \((N - 1) \times (N - 1)\) matrix model, not just integrating the out-modes. As a result, we have to take account of renormalization of noncommutativity as discussed in present subsection, which is in contrast to [14], where the noncommutativity is fixed.
\[ S_N^{(\text{kin.\,in})} = \sum_{(l,m) \in \Lambda_{\text{in}}} \frac{1}{2}(l(l+1) + \rho_N^2 m_N^2) \phi_{lm}^{\text{in}} \phi_{lm}^{* \text{in}}, \]

\[ S_N^{(\text{kin.\,out})} = \sum_{m=-2L}^{2L} \frac{1}{2}(N(N-1) + \rho_N^2 m_N^2) \phi_{2Lm}^{\text{out}} \phi_{2Lm}^{* \text{out}}, \]  

(2.33)

and define

\[ Z_0 = \int \prod_{m=-2L}^{2L} d\phi_{2Lm}^{\text{out}} e^{-S_N^{(\text{kin.\,out})}}, \quad \langle \mathcal{O} \rangle_0 = \frac{1}{Z_0} \int \prod_{m=-2L}^{2L} d\phi_{2Lm}^{\text{out}} \mathcal{O} e^{-S_N^{(\text{kin.\,out})}}, \]  

(2.34)

then our RG equation (2.32) becomes

\[ S_{N-1}(m_{N-1}^2, g_{N-1}) = -\log Z_0 + S_N^{(\text{kin.\,in})} - \log \langle e^{-S_N^{(\text{pot.})}} \rangle_0, \]

\[ S_N^{(\text{pot.})} = \frac{\rho_N^2 g_N}{4N} \text{tr}_N \phi^4. \]  

(2.35)

Thus the calculation of \( S_{N-1}(m_{N-1}^2, g_{N-1}) \) amounts to evaluating \( \langle e^{-S_N^{(\text{pot.})}} \rangle_0 \). For this purpose, we classify the potential term by the number of out-modes as

\[ S_N^{(\text{pot.})} = \frac{\rho_N^2 g_N}{N} \sum_{i=0}^{4} V_i, \]

\[ V_0 = \frac{1}{4} \text{tr}_N \phi^{\text{in}4}, \]

\[ V_1 = \text{tr}_N \left( \phi^{\text{in}3} \phi^{\text{out}2} \right) = \sum_{m=-2L}^{2L} \phi_{2Lm}^{\text{out}} \text{tr}_N \left( \phi_{2Lm}^{\text{in}3} T_{2Lm} \right), \]

\[ V_2 = V_2^P + V_2^{NP}, \]

\[ V_2^P = \text{tr}_N \left( \phi^{\text{in}2} \phi^{\text{out}2} \right) = \sum_{m,m'==-2L}^{2L} \phi_{2Lm}^{\text{out}} \phi_{2Lm'}^{\text{out}} \text{tr}_N \left( \phi^{\text{in}2} T_{2Lm} T_{2Lm'} \right), \]

\[ V_2^{NP} = \frac{1}{2} \text{tr}_N \left( \phi^{\text{in}} \phi^{\text{out}} \phi^{\text{in}} \phi^{\text{out}} \right) = \frac{1}{2} \sum_{m,m'==-2L}^{2L} \phi_{2Lm}^{\text{out}} \phi_{2Lm'}^{\text{out}} \text{tr}_N \left( \phi^{\text{in}} T_{2Lm} \phi^{\text{in}} T_{2Lm'} \right), \]

\[ V_3 = \text{tr}_N \left( \phi^{\text{in}} \phi^{\text{out}3} \right) = \sum_{m_1,m_2,m_3=-2L}^{2L} \phi_{2Lm_1}^{\text{out}} \phi_{2Lm_2}^{\text{out}} \phi_{2Lm_3}^{\text{out}} \text{tr}_N \left( \phi^{\text{in}} T_{2Lm_1} T_{2Lm_2} T_{2Lm_3} \right), \]

\[ V_4 = \frac{1}{4} \text{tr}_N \phi^{\text{out}4} = \sum_{m_1=m_4=-2L}^{2L} \prod_{i=1}^{4} \phi_{2Lm_i}^{\text{out}} \text{tr}_N \left( \prod_{i=1}^{4} T_{2Lm_i} \right), \]  

(2.36)

where \( \phi^{\text{in}} \) and \( \phi^{\text{out}} \) denote a matrix only with in-modes and out-modes, respectively:

\[ \phi^{\text{in}} = \sum_{(l,m) \in \Lambda_{\text{in}}} \phi_{lm}^{\text{in}} T_{lm}, \quad \phi^{\text{out}} = \sum_{m=-2L}^{2L} \phi_{2Lm}^{\text{out}} T_{2Lm}. \]  

(2.37)
The expectation value can be evaluated by using the propagator of the out-modes

\[ \langle \phi_{2L,m}^{\text{out}} \phi_{2L,m'}^{\text{out}} \rangle_0 = \delta_{m+m'}(-1)^m P_N, \]  

(2.38)

where \( P_N \) does not depend on \( m, m' \) as

\[ P_N = \frac{1}{N(N-1) + \rho_N^2 m_N^2}. \]  

(2.39)

By using this, we can perturbatively integrate out \( \phi^{\text{out}} \) as

\[ -\log \left\langle e^{-S^{(\text{pot.})}_N}} \right\rangle_0 = \frac{\rho_N^2 g_N}{N} \sum_i \langle V_i \rangle_0 - \frac{1}{2} \left( \frac{\rho_N^2 g_N}{N} \right)^2 \sum_{ij} \langle V_i V_j \rangle_c + \mathcal{O}(g_N^3), \]  

(2.40)

where the subscript \( c \) means taking the connected part. In the rest of the paper, we will demonstrate the RG transformation to the second order in \( g_N \). Noting the \( \mathbb{Z}_2 \)-symmetry of the original action (2.29), we find that

\[ S_{N-1}(m_{N-1}^2, g_{N-1}) = S^{(\text{kin.})}_{N-1} + \frac{\rho_N^2 g_N}{N} (\langle V_0 + \langle V_2 \rangle_0) - \frac{1}{2} \left( \frac{\rho_N^2 g_N}{N} \right)^2 \left( \langle V_1^2 \rangle_c + \langle V_2^2 \rangle_c + \langle V_3^2 \rangle_c + 2 \langle V_1 V_3 \rangle_c + 2 \langle V_2 V_4 \rangle_c \right) + \mathcal{O}(g_N^3) + \text{const.}, \]  

(2.41)

where we do not include the expectation values containing only \( V_4 \), since they do not depend on \( \phi^{\text{in}} \) and then will not affect the renormalization of the parameters \( m_N^2, g_N \). We postpone carrying out the actual calculations and first explain the rest of the large-\( N \) RG procedure.

### 2.3.2 Rescaling

The second procedure of the RG transformation is the rescaling. We will see that it also involves the renormalization of noncommutativity. After coarse-graining, we should have a matrix model of size \((N-1) \times (N-1)\), say \( \tilde{\phi} \), instead of \( N \times N \) matrix \( \phi^{\text{in}} \). To do it, we relate \( \phi^{\text{in}} \) and \( \tilde{\phi} \) by

\[ \frac{1}{N} \text{tr}_N \left( -\frac{1}{2} [L_i, \phi^{\text{in}}]^2 \right) = \frac{1}{N-1} \text{tr}_{N-1} \left( -\frac{1}{2} [\tilde{L}_i, \tilde{\phi}]^2 \right), \]  

(2.42)

which implies that

\[ \phi^{\text{in}}_{lm} = \tilde{\phi}_{lm}, \]  

(2.43)

with \( \tilde{\phi} = \sum_{(l,m) \in \Lambda_{\text{in}}} \tilde{\phi}_{lm} \tilde{T}_{lm} \), where \( \tilde{L}_i \) is the \( SU(2) \) generator (2.6) of spin \( L - 1/2 = (N-2)/2 \) representation, and \( \tilde{T} \) is the fuzzy spherical harmonics with rank \( N-1 \). Namely,
we fix the wave function renormalization in such a way that the kinetic term is canonically normalized. Then it immediately follows that

$$\frac{1}{N} \text{tr}_N \left( \phi^{in2} \right) = \frac{1}{N-1} \text{tr}_{N-1} \left( \tilde{\phi}^2 \right). \tag{2.44}$$

On the other hand, it is nontrivial how the $\text{tr}_N \left( \phi^{in4} \right)$ vertex becomes in terms of $\tilde{\phi}$. For this purpose it is evident that we have to shift all $L$’s appearing in the $6j$-symbols in (2.28) by $-1/2$ (recall $L = (N - 1)/2$). Here the key recursion relation of the $6j$-symbol is [15]

$$\begin{align*}
\{a \ b \ l \ L \ L \ L\} &= \frac{1}{\sqrt{(2L + a + 1)(2L - a)(2L + b + 1)(2L - b)}} \\
&\times \left\{ -2L\sqrt{(2L + l + 1)(2L - l)} \right\} \left\{ a \ b \ l \ L - \frac{1}{2} \ L - \frac{1}{2} \ L - \frac{1}{2} \right\} + \sqrt{(a + 1)a(b + 1)b} \left\{ a \ b \ l \ L \ L \ L \right\}.
\end{align*} \tag{2.45}$$

We find that the first term indeed provides the necessary factor for the $\text{tr}_{N-1} \left( \tilde{\phi}^4 \right)$, while the second term does not seem to correspond to natural operation of an $(N - 1) \times (N - 1)$ matrix. However, as in the usual RG, we are interested in the low energy regime $l_i \ll L$ ($i = 1 \sim 4$) in $\text{tr}_N(T_{i_1 m_1}T_{i_2 m_2}T_{i_3 m_3}T_{i_4 m_4})$ and $a, b$ in (2.45) applied to (2.28) correspond to low energy modes. We then find that the second term is of $O(1/N^2)$ compared to the first term, which can be therefore neglected in the large-$N$ RG discussing corrections of $O(1/N)$ when $N \to N - 1$. Substituting (2.45) into an explicit form (B.3), a little calculation shows that

$$\text{tr}_N \left( \phi^4 \right) = \left( 1 + \frac{1}{N} \right) \text{tr}_{N-1} \left( \tilde{\phi}^4 \right) + O \left( \frac{1}{N^2} \right). \tag{2.46}$$

We also rescale $\rho^2_N$ in our RG transformation. In the ordinary RG on the two-dimensional square lattice, the lattice spacing will be multiplied by $\sqrt{2}$ after the block spin transformation, and we rescale the length scale by $1/\sqrt{2}$ in order to recover the original lattice spacing. In the case of the RG of field theory, after integrating out high energy modes $\Lambda/b \leq p \leq \Lambda$ (for some $b > 1$) with $\Lambda$ the momentum cutoff, we make a scale transformation as $p \to bp$ in order to retrieve the original momentum space. Since in the case of the fuzzy sphere the maximal momentum is given by the Casimir of the mode with $2L$, we rescale $\rho_N$ as

$$\frac{2L(2L+1)}{\rho^2_N} = \frac{2L(2L-1)}{\rho^2_{N-1}}, \tag{2.47}$$

namely, in such a way that the maximal momentum will be fixed in a rotational invariant manner. Here we define a ratio

$$b^2_N \equiv \frac{\rho^2_N}{\rho^2_{N-1}}. \tag{2.48}$$
Thus \( b^2_N \) is given by

\[
 b^2_N = \frac{N}{N - 2},
\]

(2.49)

by which we make a scale transformation in order to recover the original scale as in the usual RG of field theory. Notice that the degrees of freedom in a unit volume would be then multiplied by \( N/(N - 1) \), which is consistent with the picture that the fuzzy sphere defined by \( N \times N \) matrices describes \( N \) quanta with volume given by \( \alpha^2_N \) as in (2.7)

\[
 \alpha^2_N = \frac{4}{N^2 - 1} \rho^2_N.
\]

(2.50)

Correspondingly, it should be noticed that under the scale transformation (2.47), \( \alpha_N \) is invariant:

\[
 \frac{\alpha^2_N}{\alpha^2_{N-1}} = 1 + \mathcal{O} \left( \frac{1}{N^2} \right).
\]

(2.51)

In other words, in the case of fuzzy sphere we take the large-\( N \) limit with the physical length scale \( \alpha^2 \) fixed, as the continuum limit in the lattice QCD with the physical pion mass fixed. Since \( \alpha \) is the physical scale characterizing the fuzzy sphere, this limit is natural, and accordingly we examine it via the large-\( N \) RG which respects \( \alpha \).

On the other hand, it is known that by taking another large-\( N \) limit, we can obtain the noncommutative field theory (NCFT) \([17, 18]\) on the flat two-dimensional plane (see e.g. \([12, 19]\)). In this case, the large-\( N \) limit is taken with the noncommutativity \( \theta \) in the NCFT fixed as \([19]\).

\[
 N \to \infty \quad \text{with} \quad \theta = \frac{2\rho^2}{N} : \text{fixed}.
\]

(2.52)

Hence in order to examine the NCFT via our large-\( N \) RG, we should fix \( \theta_N \). This implies that

\[
 \frac{\theta^2_N}{N} = \frac{\theta_{N-1}}{2} = \frac{\theta_{N-1}}{N - 1}.
\]

(2.53)

Namely, in this case we have instead of (2.49)

\[
 b^2_N = \frac{N}{N - 1}.
\]

(2.54)

### 2.3.3 Change of the parameters

After evaluating (2.41) and performing the scale transformation, the renormalized action can schematically be written as

\[
 S_{N-1} = \frac{\rho^2_{N-1}}{N - 1} \text{tr}_{N-1} \left[ -\frac{1}{2\rho^2_{N-1}} \left( 1 + K_N(N, m^2_N, g_N) \right) [\tilde{L}, \tilde{\phi}]^2 + \frac{b^2_N}{2} \left( m^2_N + M_N(N, m^2_N, g_N) \right) \tilde{\phi}^2 \right]
\]

For more details, see e.g. \([16]\).
\[
+ \frac{b_N^2}{4} \left( g_N + G_N(N, m_N^2, g_N) \right) \phi^4 \right] \]
\[ + \text{(others)}, \] (2.55)

where we have taken account of (2.44), (2.46) and (2.48). In general, \( K(N, m_N^2, g_N) \neq 0 \), and a further rescaling of \( \tilde{\phi} \) has to be done in order to make the kinetic term canonical. The resulting action should be compared with

\[
S_{N-1} = \frac{\rho_{N-1}^2}{N-1} \text{tr}_{N-1} \left[ \frac{1}{2\rho_{N-1}^2} [\tilde{L}_i, \tilde{\phi}]^2 + \frac{m_{N-1}^2}{2} \tilde{\phi}^2 + \frac{g_{N-1}}{4} \tilde{\phi}^4 \right].
\] (2.56)

Now we read off the change of parameters as

\[
m_{N-1}^2 = b_N^2 m_N^2 + M_N(N, m_N^2, g_N),
\]
\[
g_{N-1} = b_N^2 \frac{g_N + G_N(N, m_N^2, g_N)}{(1 + K(N, m_N^2, g_N))^2}.
\] (2.57)

These are RG flow equations for \( m^2 \) and \( g \), as \( N \to N-1 \).

In the last line of (2.55), \( \text{(others)} \) includes terms of the form which did not appear in the original action, induced through the coarse-graining procedure. It contains higher order interaction terms \( \text{tr}_{N-1} \tilde{\phi}^{2n} \ (n \geq 3) \) and multi-trace deformations. They would induce the RG flow in an enlarged space of coupling constants. However, to the order of the perturbation in the present paper, they are not relevant. It also contains various “derivative expansions” of kinetic and potential terms, and highly “nonlocal” terms which are related to nonplanar diagrams in perturbation theory. In the following subsections, we will see how they appear and how to deal with them.

In following subsections we will concretely evaluate the terms in (2.41) and determine the corrections shown in (2.55). There we will find several quite interesting and characteristic aspects of our large-\( N \) RG, namely, appearance of “antipode” transformed fields and multi-trace operators, and “derivative” expansions in the space of matrices.

### 2.4 \( \mathcal{O}(g_N) \) and appearance of antipode transformation

As a demonstration let us calculate contributions of \( \mathcal{O}(g_N) \) in (2.41). From (2.36) and (2.38), we obtain

\[
\langle V_{2}^{P} \rangle_0 = \sum_{m, m'} \langle \phi_{2Lm}^{\text{out}} \phi_{2Lm'}^{\text{out}} \rangle_0 \text{tr}_N \left( \phi_{2Lm}^{\text{in}} T_{2L m} T_{2L m'} \right)
\]
\[
= P_N \sum_{m} (-1)^m \text{tr}_N \left( \phi_{2Lm}^{\text{in}} T_{2L m} T_{2L -m} \right).
\] (2.58)

Here we note the formula given in (B.13) in appendix B

\[
\sum_{m=-2L}^{2L} (-1)^m T_{2Lm} T_{2L-m} = (2N-1) 1_N,
\] (2.59)
and get

\[ \langle V_2^P \rangle_0 = (2N - 1) P_N \text{tr}_N \phi^\text{in}^2, \]  

which therefore contributes to the mass correction. This contribution comes from a planar diagram given in [a] in Fig. 1.

On the other hand, we also have a contribution from the nonplanar diagram shown in [b] in Fig. 1

\[ \langle V_2^{NP} \rangle_0 = \frac{1}{2} \sum_{m,m} \langle \phi_{2Lm}^\text{out} \phi_{2Lm'}^\text{out} \rangle_0 \text{tr}_N \left( \phi_{2Lm}^\text{in} T_{2Lm} \phi_{2Lm'}^\text{in} T_{2Lm'} \right) \]

\[ = \frac{P_N}{2} \sum_m (-1)^m \text{tr}_N \left( \phi_{2Lm}^\text{in} T_{2Lm} \phi_{2Lm}^\text{in} T_{2L-m} \right). \]  

Hence the nonplanar diagram generates a new operator. In order to elaborate on it, we apply the formula in (B.12) in appendix B as

\[ \sum_m (-1)^m \text{tr}_N \left( \phi_{2Lm}^\text{in} T_{2Lm} \phi_{2Lm}^\text{in} T_{2L-m} \right) = N(2N - 1) \sum_{(l,m) \in \Lambda_m} \left\{ \frac{L}{2L} \right\} \phi_{lm}^\text{in} \text{tr}_N \left( \phi_{lm}^\text{in} (-1)^l T_{lm} \right). \]  

It is then natural to define an operation on \( N \times N \) matrices as

\[ T_{lm} \quad \mapsto \quad T_{lm}^A \equiv (-1)^l T_{lm}, \]  

thus

\[ \phi^\text{in} = \sum_{(l,m) \in \Lambda_m} \phi_{lm}^\text{in} T_{lm} \quad \mapsto \quad \phi^\text{in}^A = \sum_{(l,m) \in \Lambda_m} (-1)^l \phi_{lm}^\text{in} T_{lm}. \]  

Although this transformation looks strange in the space of matrices, it has a natural interpretation on the fuzzy sphere. In fact, it corresponds to a discrete transformation on \( S^2 \)

\[ Y_{lm}(\theta, \varphi) \quad \mapsto \quad (-1)^l Y_{lm}(\theta, \varphi) = Y_{lm}(\pi - \theta, \varphi + \pi), \]
namely, it transports a field to the antipode point on \( S^2 \). (2.64) is therefore a natural counterpart on the fuzzy sphere of this transformation and hereafter we call it antipode transformation. Using this, (2.61) becomes

\[
\langle V_{NP}^2 \rangle_0 = \frac{P_N}{2} N (2N - 1) \sum_{(l,m) \in \Lambda_{in}} \left\{ \begin{array}{c} L \ L \ l \\ L \ L \ 2L \end{array} \right\} \phi_{lm}^{in} \text{tr}_N (\phi_{lm}^{in} T^{A}_{lm}) .
\] (2.66)

Furthermore, we can make a sort of “derivative expansion” of this expression. In fact, from (A.9) the 6\textit{j}-symbol in (2.66) has a large-\( L \) (thus large-\( N \)) expansion\(^9\)

\[
\left\{ \begin{array}{c} L \ L \ l \\ L \ L \ 2L \end{array} \right\} \simeq \frac{(-1)^l}{2L + 1} P_l \left( -1 + \frac{1}{L + 1} \right)
\]

\[= \frac{1}{2L + 1} \left( 1 - \frac{l(l + 1)}{2(L + 1)} + \mathcal{O} \left( \frac{1}{L^2} \right) \right) ,
\] (2.67)

to give

\[
\langle V_{NP}^2 \rangle_0 = \frac{P_N}{2} (2N - 1) \left[ \text{tr}_N \left( \phi^{in} \phi^{in A} \right) - \frac{1}{N} \text{tr}_N \left( \phi^{in} [L_i, [L_i, \phi^{in A}]] \right) \right] + \mathcal{O} \left( \frac{1}{N^2} \right) ,
\] (2.68)

where we recall \( L = (N - 1)/2 \) and (2.20). It is quite interesting that owing to the fuzzy sphere structure, the space of \( N \times N \) matrices admits the derivative expansion of an operator. Here higher derivative terms come from the \( l \)-dependent terms of the expansion of the Legendre polynomial in (2.67) in terms of \( 1/L \) and are suppressed by \( 1/N \) as such. Notice that this expansion should be in terms of the Laplacian (2.20), because the fuzzy sphere preserves the rotational symmetry \( SO(3) \), which is respected in our large-\( N \) RG formalism. Thus we find that the operator on the right-hand side in (2.66) contains \( \text{tr}_N \left( \phi^{in} \phi^{in A} \right) \) and its derivatives\(^10\). We point out here that new operators that are not contained in the original Lagrangian (2.29) are thus generated in our RG, but that they may be regarded as small corrections. For example, the first term in (2.68) agrees with the overlap between a wave function and its antipode

\[
\frac{1}{N} \text{tr}_N \left( \phi^{in} \phi^{in A} \right) = \int \frac{d\Omega}{4 \pi} \phi^{in}(\theta, \varphi) \phi^{in}(\pi - \theta, \varphi + \pi) = \sum_{(l,m) \in \Lambda_{in}} (-1)^l |\phi_{lm}^{in}|^2 ,
\] (2.69)

which is always smaller than \( \frac{1}{N} \text{tr}_N \left( \phi^{in} \phi^{in} \right) = \sum_{(l,m) \in \Lambda_{in}} |\phi_{lm}^{in}|^2 \) in (2.60) coming from the planar diagram. In fact, it is easy to check that for the Gaussian wave function (under the stereographic projection) the former is exponentially smaller than the latter with respect

\(^9\)Throughout this paper, we concentrate on the case with the “in” fields carrying low enough momenta. Therefore in the expression of the perturbation theory, we write errors simply as \( \mathcal{O}(1/L^2) \) or \( \mathcal{O}(1/N^2) \), though it could also depend on the momenta of associated “in” fields.

\(^10\)At a glance, the antipode transformation seems to correspond to exchange of two matrices. However, as shown in appendix D it is related to the reverse of the ordering of all matrices inside a trace.
to the inverse square of the width. However, since in the RG we are interested in low energy modes, it is not obvious that the overlap is small enough to ignore. Thus in the fixed point analysis given in section 3 we will concentrate on the flows of $m_N^2$ and $g_N$ up to $\mathcal{O}(g_N^2)$ and simply assume that to this order, interactions involving $\phi^{inA}$ would not affect them considerably, namely, would not modify $\langle V_2^2 \rangle_c$ so much. Here it is important to note that as we see by comparing the coefficients in (2.57) and (2.61), the nonplanar contribution is not suppressed by $\mathcal{O}(1/N)$ in itself compared to the planar one, but the momentum dependence (in the present case $(-1)^l$ in the antipode transformation) can make it suppressed. Recall that this is also the case with the NCFT, where nonplanar diagrams are in general dropped in a high energy regime by the oscillating phase depending on momenta.

### 2.5 $\mathcal{O}(g_N^2)$ and appearance of double trace operator

#### 2.5.1 Vertex correction

Among contributions of $\mathcal{O}(g_N^2)$ shown in the second line in (2.41), $\phi^4$ vertex correction originates only from $\langle V_2^2 \rangle_c$. Thus in this subsection, we focus on this part and present only the results of the calculation of the other contributions. The details are given in appendix C.

Since $V_2 = V_2^P + V_2^{NP}$ as in (2.56), let us begin with $\langle V_2^{P2} \rangle_c$. From (2.36),

$$
\langle V_2^{P2} \rangle_c = \sum_{m_1,m_2,m_1',m_2'} \left( \left( \phi^{out}_{2Lm_1}\phi^{out}_{2Lm_2} \right)_0 \left( \phi^{out}_{2Lm_1}\phi^{out}_{2Lm_2} \right)_0 + \left( \phi^{out}_{2Lm_1}\phi^{out}_{2Lm_2} \right)_0 \left( \phi^{out}_{2Lm_1}\phi^{out}_{2Lm_2} \right)_0 \right)
\times \text{tr}_N \left( \phi^{in2}_{T2Lm_1T2Lm_2} \right) \text{tr}_N \left( \phi^{in2}_{T2Lm_1'T2Lm_2'} \right)
= P_N^2 \sum_{m_1,m_2} (-1)^{m_1+m_2} \left( \text{tr}_N \left( \phi^{in2}_{T2Lm_1T2Lm_2} \right) \text{tr}_N \left( \phi^{in2}_{T2L-m_2T2L-m_1} \right) \right)
+ \text{tr}_N \left( \phi^{in2}_{T2Lm_1T2Lm_2} \right) \text{tr}_N \left( \phi^{in2}_{T2L-m_1T2L-m_2} \right).
$$

(2.70)

The first and second term correspond to the planar and nonplanar diagram shown in Fig.2[a] and [b] respectively. Important observation here is that double trace terms as above are in general generated in the large-$N$ RG. At first sight, they seem to be new interactions arising in our RG and do not play any role in renormalization of parameters

---

[1] Here we should notice that it is likely that an interaction with $\phi^{inA}$ originating from a nonplanar diagram is not irrelevant in itself near a fixed point. In fact, it is shown in [19] that a contribution from the 1PI nonplanar two-point function $\Gamma^{(2)}_{\text{nonplanar}}$ (corresponding to $\langle V_2^{NP} \rangle$ with all modes running the loop integrated) gives rise to the IR singularity in the large-$N$ NCFT limit (2.52) and hence becomes the origin of the UV/IR mixing. Since the large-$N$ limit should be captured by a fixed point in the large-$N$ RG, this suggests that if an interaction involving $\phi^{inA}$ has something to do with $\Gamma^{(2)}_{\text{nonplanar}}$ which becomes quite large in the IR in the large-$N$ limit, it would be rather relevant around a corresponding fixed point. We will further make a few comments on such an interesting aspect in section 4.
\[ \rho_N, \; m^2_N, \; g_N \] in the original action (2.29). But this is not the case. Recall that in the usual RG of the \( \phi^4 \) scalar field theory we also get (in the configuration space) from the diagram in \( (a) \) in Fig. 2

\[ \frac{g^2}{8} \int dxdy \Delta(x - y)^2 \phi(x)^2 \phi(y)^2, \quad (2.71) \]

where \( \Delta(x - y) = \langle \phi(x)\phi(y) \rangle \) is the propagator of the scalar field. Namely, bi-local interactions are generated in general in the RG corresponding to the double trace term in (2.70). However, we know that if the propagator \( \Delta(x - y) \) is for a highly massive mode of \( \phi \), it would rapidly damp as \( x \) and \( y \) are separated. In such a case the derivative expansion as

\[ \frac{g^2}{8} \int dx \left[ \left( \int dz \Delta(z)^2 \right) \phi(x)^4 + \left( \int dz \frac{z^2}{2} \Delta(z)^2 \right) \phi(x)^2 \partial_x^2 \phi(x)^2 + \cdots \right], \quad (2.72) \]

is valid at least for a low energy mode of \( \phi \) of interest in the RG. Namely, bi-local interactions generated by the RG can be decomposed as a sum of local interactions and higher derivative interactions are suppressed if we integrated out sufficiently high energy modes. In other words, the RG by integrating only high energy modes only generates local interactions. Notice that the first term in (2.72) contributes to \( \phi^4 \) vertex correction.

Hence in our large-\( N \) RG we also have to make a “derivative” expansion as in (2.72) in the space of matrices and to read off what kind of “local” interactions arise as a result of the RG. Interestingly enough, it is indeed possible in our formalism because our RG is based on the fuzzy sphere structure and we in fact integrate out modes with the maximal momentum. As illustration let us consider the planar contribution of \( \langle V_2^{P^2} \rangle_c \) given by the first term in (2.70). By using the fusion coefficient (B.4) with (B.5) in appendix B

\[ \langle V_2^{P^2} \rangle_c \bigg|_{\text{planar}} = P_N^2 (4L + 1)^2 N \sum_{m_1, m_2} (-1)^{m_1 + m_2} \sum_{(l, m), (l', m')} \sqrt{(2l + 1)(2l' + 1)}(-1)^{l + m + l' + m'} \]
\[
\times \begin{pmatrix} 2L & 2L & l \\ m_1 & m_2 & -m \end{pmatrix} \begin{pmatrix} 2L & 2L & l' \\ -m_2 & -m_1 & -m' \end{pmatrix} \begin{pmatrix} 2L & 2L & l \\ L & L & L \end{pmatrix} \begin{pmatrix} 2L & 2L & l' \\ L & L & L \end{pmatrix} \\
\times \text{tr}_N(\phi^{in}T_{lm}) \text{tr}_N(\phi^{in}T_{lm}^T). \tag{2.73}
\]

Then we can take the summation over \(m_1, m_2\) first by the formula (A.2) in appendix A to get

\[
\left\langle V_2 P^2 \right\rangle_{c, \text{planar}} = P_N^2 (2N - 1)^2 N \sum_{lm} \begin{pmatrix} 2L & 2L & l \\ L & L & L \end{pmatrix}^2 \text{tr}_N(\phi^{in}T_{lm}) \text{tr}_N(\phi^{in}T_{lm}^T). \tag{2.74}
\]

Now we make a “derivative” expansion. In the large-\(N\) RG we are interested in sufficiently low energy modes \(\phi^{in}\) with \(l \ll L\) as in the usual RG in field theory, and then (2.13) tells us that the internal momentum \(l\) in (2.74) should be also sufficiently smaller than \(L\) so that the traces will not vanish. Thus as far as \(\phi^{in}\) only with low energy modes is concerned, we can evaluate the 6\(j\)-symbol in (2.74) under \(l \ll N\) by using (A.9) as

\[
\begin{pmatrix} 2L & 2L & l \\ L & L & L \end{pmatrix} \approx \frac{(-1)^l}{\sqrt{(4L + 1)(2L + 1)}} P_l \left( \frac{2L + 1}{2L + 2} \right) = \frac{(-1)^l}{\sqrt{(2N - 1)N}} \left( 1 - \frac{l(l + 1)}{4N} + O \left( \frac{1}{N^2} \right) \right). \tag{2.75}
\]

Plugging this into (2.74), we obtain

\[
\left\langle V_2 P^2 \right\rangle_{c, \text{planar}} = P_N^2 (2N - 1) N \sum_{lm} \left( \text{tr}_N(\phi^{in}T_{lm}) \text{tr}_N(\phi^{in}T_{lm}^T) - \frac{1}{2N} \text{tr}_N(\phi^{in}[L_i, [L_i, T_{lm}]] \text{tr}_N(\phi^{in}T_{lm}^T) + O \left( \frac{1}{N^2} \right) \right). \tag{2.76}
\]

Then we see that the first term in fact becomes the single trace by virtue of the completeness of \(T_{lm}\) (2.15). It is rather technical but interesting that in the second term we can also take the summation over \(l, m\) by applying (2.15), because by the “partial integration”

\[
\text{tr}_N(\phi^{in}T_{lm}) \text{tr}_N(\phi^{in}T_{lm}^T) = \text{tr}_N \left( [L_i, [L_i, \phi^{in}]] T_{lm} \right) \text{tr}_N(\phi^{in}T_{lm}^T). \tag{2.77}
\]

As a consequence we have

\[
\left\langle V_2 P^2 \right\rangle_{c, \text{planar}} = P_N^2 (2N - 1) N \left[ \text{tr}_N(\phi^{in^2}) - \frac{1}{2N} \text{tr}_N(\phi^{in^2} + O \left( \frac{1}{N^2} \right) \right]. \tag{2.78}
\]
Namely, we have the vertex correction of $O(g_N^2)$ from the planar diagram. Detailed calculations presented in appendix C show that the other contributions in $\langle V_2^2 \rangle_c$ always generate $\phi^4$ vertex corrections with both $\phi^{\text{in}}$ and $\phi^{\text{inA}}$:

\[
\left. \langle V_2^{P2} \rangle \right|_{\text{nonplanar}} = P_N^2 (2N - 1) N \left[ \text{tr}_N \left( \phi^{\text{in2}} \phi^{\text{inA2}} \right) - \frac{1}{2N} \text{tr}_N \left( \left[ L_i, \left[ L_i, \phi^{\text{in2}} \right] \right] \phi^{\text{inA2}} \right) + O \left( \frac{1}{N^2} \right) \right],
\]

\[
\langle V_2^P V_2^{NP} \rangle_c = P_N^2 (2N - 1) N \left[ \text{tr}_N \left( \phi^{\text{in3}} \phi^{\text{inA}} \right) - \frac{1}{2N} \text{tr}_N \left( \left[ L_i, \left[ L_i, \phi^{\text{in}} \right] \right] \phi^{\text{in2}} \phi^{\text{inA}} \right) - \frac{1}{2N} \text{tr}_N \left( \left[ L_i, \left[ L_i, \phi^{\text{inA}} \right] \right] \phi^{\text{in}} \right) + O \left( \frac{1}{N^2} \right) \right],
\]

\[
\langle V_2^{NP2} \rangle_c = \frac{1}{2} P_N^2 (2N - 1) N \left[ \text{tr}_N \left( \phi^{\text{inA}} \phi^{\text{inA}} \phi^{\text{in}} \phi^{\text{in}} \right) - \frac{2}{N} \text{tr}_N \left( \left[ L_i, \left[ L_i, \phi^{\text{in}} \right] \right] \phi^{\text{inA}} \phi^{\text{inA}} \phi^{\text{in}} \phi^{\text{in}} \right) - \frac{1}{2N} \text{tr}_N \left( \left[ L_i, \left[ L_i, \phi^{\text{inA}} \phi^{\text{in}} \phi^{\text{inA}} \phi^{\text{in}} \right] \phi^{\text{inA}} \phi^{\text{in}} \phi^{\text{inA}} \phi^{\text{in}} \right) + O \left( \frac{1}{N^2} \right) \right],
\]

where we notice that all derivative corrections come in as the Laplacian which reflects the fact that our large-$N$ RG as well as the fuzzy sphere respects the $SO(3)$ symmetry, as anticipated below (2.68). Therefore the RG again generates new interactions containing both $\phi^{\text{in}}$ and $\phi^{\text{inA}}$ that are not present in the original action. For the same reason as in (2.68) we regard them as small corrections and negligible in discussion of the RG of the parameters $\rho_N^2, m_N^2$ and $g_N$ in the original Lagrangian (2.29).

### 2.5.2 Mass correction

Let us examine the other terms than $\langle V_2^2 \rangle_c$ in the second line in (2.41). It is obvious that they contribute to correction in the quadratic order of $\phi^{\text{in}}$. The results are summarized as:

1. $\langle V_2^2 \rangle_c$ vanishes in sufficiently low energy regime where the momentum $l$ of $\phi^{\text{in}}$ is much smaller than the cutoff $2L$.

2. $\langle V_3^2 \rangle_c$ is exponentially suppressed by $L$ when $L \gg 1$.

3. $\langle V_1 V_3 \rangle_c = 0$ by the momentum conservation.

4. $\langle V_2^P V_4 \rangle_c \approx P_N^3 (2N - 1)^2 N \text{tr}_N \left( \phi^{\text{in2}} \right)$, which arises from the planar diagram.

5. $\langle V_2^{NP} V_4 \rangle_c$ again provides derivative expansions of terms with both the usual and the antipode fields.
Therefore only $\langle V^2_P V_4 \rangle_c$ part contributes to the mass correction. The details of the calculations are shown in appendix C.

### 2.6 Flow equations

So far we see that when $\phi^{\text{in}}$ is in a sufficiently low energy regime and $L \gg 1$, (2.41) becomes

$$S_{N-1} = \text{tr}_N \left[ -\frac{1}{2N} [L_i, \phi^{\text{in}}]^2 + \frac{\rho^2_N m^2_N}{2N} \phi^{\text{in}}^2 + \frac{\rho^2_N g_N}{N} \left( \frac{1}{4} \phi^{\text{in}}^4 + P_N (2N-1) \phi^{\text{in}}^2 \right) 
- \frac{1}{2} \left( \frac{\rho^2_N g_N}{N} \right)^2 \left( P^2_N (2N-1) N \phi^{\text{in}}^4 + 4 P^3_N (2N-1)^2 N \phi^{\text{in}}^2 \right) \right] + \cdots$$

$$= \frac{\rho^2_N}{N} \text{tr}_N \left[ -\frac{1}{2 \rho^2_N} [L_i, \phi^{\text{in}}]^2 + \frac{1}{2} \left( m^2_N + g_N B_1(N, m^2_N) - \rho^2_N g^2_N B_1(N, m^2_N) B_2(N, m^2_N) \right) \phi^{\text{in}}^2 
+ \frac{1}{4} \left( g_N - \rho^2_N g^2_N B_2(N, m^2_N) \right) \phi^{\text{in}}^4 \right] + \cdots,$$

where

$$B_1(N, m^2_N) = 2(2N-1) P_N,$$
$$B_2(N, m^2_N) = 2(2N-1) P^2_N,$$

(2.81)

with $P_N$ given in (2.39). In the above, $\cdots$ denotes the terms including $\phi^{\text{in}}$ independent constants, $O(g^3_N)$, antipode fields, and $1/N$ suppressed derivative expansions. They are all omitted in the following RG analysis. Now the RG flow equations (2.57) become

$$m^2_{N-1} = b^2_{N} \left( m^2_N + g_N B_1(N, m^2_N) - \rho^2_N g^2_N B_1(N, m^2_N) B_2(N, m^2_N) \right),$$
$$g_{N-1} = b^2_{N} \left( g_N - \rho^2_N g^2_N B_2(N, m^2_N) \right),$$

(2.82)

with keeping $O(g^3_N)$ terms.

In the next section we analyze the RG transformation (2.82). Before closing this section we comment on a couple of features of our large-$N$ RG equation in order.

1. The mass corrections of $O(g_N)$, $O(g^2_N)$ and the vertex correction of $O(g^3_N)$ arise from the planar diagrams in $\langle V^2_P \rangle_0$, $\langle V_2 V_4 \rangle_c$, and $\langle V^2_P \rangle_c$, respectively. Nonplanar diagrams generate terms with the antipode fields, which are neglected in the current analysis.

2. No kinetic term correction appears. This reflects the fact that the result of $\langle V^2_P \rangle_0$ in (2.60) is exact and does not have any derivative corrections.

3. Since our large-$N$ RG is constructed by integrating out the modes with the highest energy, it should be in a sense local as in the case of the usual RG. In fact, all
corrections carry the propagator $P_N$ given in (2.38), which is highly suppressed for large-$N$. Furthermore, our RG admits the “derivative” expansion as mentioned in section 2.5 by which the double trace corrections can be written as a sum of the single traces. This fact also reflects locality of our RG, because the reason why we can utilize the asymptotic formula of $6j$-symbol as in (2.75) is that we have integrated the modes with $l = 2L$ and hence its upper entries become the cutoff $2L$ itself. Accordingly, we have observed that the “derivative” corrections are suppressed by $1/N$ as in (2.78).

3 Fixed point analysis

In this section we examine Gaussian and nontrivial fixed points of our RG transformation (2.82) and discuss their properties. We also determine the scaling dimensions of small perturbations around these fixed points from (2.82). First we discuss fixed points corresponding to field theories, and then turn to the two-dimensional gravity originally considered in [5]. Results in this section are compared to preceding works [8][9][10][11], regarding nonperturbative studies of $\phi^4$ theory on the fuzzy sphere via the matrix model by different methods.

3.1 Gaussian fixed point

It is evident that $m_* = g_* = 0$ is indeed the fixed point (Gaussian fixed point) of the RG transformation (2.82). Near the Gaussian fixed point, we linearize (2.82) with respect to small perturbations $m^2 \ll 1$ and $g \ll 1$ as

$$
m_{N-1}^2 = b_N^2 (m_N^2 + g_N B_1(N)),
g_{N-1} = b_N^2 g_N.
$$

(3.1)

Here we expanded $B_1(N, m_N^2)$ and $B_2(N, m_N^2)$ as

$$
B_1(N, m_N^2) = B_1(N) + B_{1,1}(N) m_N^2 + \cdots, \quad B_2(N, m_N^2) = B_2(N) + B_{2,1}(N) m_N^2 + \cdots,
$$

(3.2)

where

$$
B_1(N) = \frac{2(2N - 1)}{N(N - 1)}, \quad B_2(N) = \frac{2(2N - 1)}{N^2(N - 1)^2}.
$$

(3.3)

In order to diagonalize the RG transformation near the Gaussian fixed point, we assume the eigenvector as $m_N^2 - \bar{m}^2 (g_N)$, where $\bar{m}^2 (g_N) = A_N g_N + B_N g_N^2 + \cdots$. We then need to solve\footnote{As the readers can see, the coefficient $A_N$ in $\bar{m}^2 (g)$ depends on the step of RG transformation, namely varies as $N \to N - n$. Therefore $\bar{m}^2 (g)$ is slightly different from the critical line of the RG flow. The leading behavior of $A_N$ shown here approximates the slope of critical line in the vicinity of the fixed point.}

$$
m_{N-1}^2 - m_{N-1}^2 (g_{N-1}) = b_N^2 (m_N^2 - \bar{m}^2 (g_N)),
$$

(3.4)
\[ g_{N-1} = b_N^2 g_N. \] 

These equations determine \( \bar{m}^2(g_N) \) as

\[ \bar{m}^2(g_N) = -(4 \log N) g_N + O \left( \frac{1}{N} \right). \] 

(3.5)

Thus \( \bar{m}^2 \) approximately give a critical line of the RG flow, and then we find that the critical line is vertical in \( m^2 - g \) plane in large-\( N \) limit. This feature originates from the fact that \( m_N^2 \) and \( g_N \) have the same scaling dimension and thus it is peculiar to two dimensions. In fact, this is also the case with the RG of the ordinary two-dimensional field theory where the critical line becomes vertical. This vertical line can be derived in a different manner. In order to regularize the slope of the vertical line, we introduce an extra tiny \( N \)-dependence \( c(N) \sim N^{-\varepsilon} \) to \( g_N \) in (2.29)

\[ S_N = \frac{\rho_N^2}{N} \text{tr}_N \left( -\frac{1}{2\rho_N^2}[L_i,\phi_N]^2 + \frac{m_N^2}{2} \phi^2 + \frac{c(N)g_N}{4} \phi^4 \right), \] 

(3.6)

and repeat our large-\( N \) RG to obtain the RG equation

\[ m_{N-1}^2 = b_N^2 \left( m_N^2 + c(N)g_NB_1(N,m_N^2) - c(N)^2 \rho_N^2 g_N^2 B_1(N,m_N^2)B_2(N,m_N^2) \right), \] 

(3.7)

\[ g_{N-1} = \frac{c(N)}{c(N-1)} b_N^2 \left( g_N - \rho_N^2 c(N)g_N^2 B_2(N,m_N^2) \right). \] 

(3.8)

Now critical line \( m_c^2(g) \) near the Gaussian fixed point can be determined as

\[ m_c^2(g) = -\frac{12(2N-1)}{\varepsilon (N-1)} g \rightarrow -\frac{4}{\varepsilon} g \quad (N \rightarrow \infty). \] 

(3.9)

Thus we also find the vertical critical line. Compared to the case of the ordinary field theory, we see that the introduction of \( c(N) \sim N^{-\varepsilon} \) is an analog of considering \((2 + \varepsilon)\)-dimension. Furthermore, (3.7) and (3.8) tell us that the eigenvalues of the RG transformations are \( b_N^2 \) for \( m_N^2 \) and \( \frac{c(N)}{c(N-1)} b_N^2 \) for \( g_N \). Now we propose how we read off the scaling dimension in our large-\( N \) RG. Recall that we make the scale transformation by \( b_N^2 = \rho_N^2/\rho_{N-1}^2 \) in order to compensate the maximal momentum integrated out in the large-\( N \) RG. Therefore, the scaling dimension should be given by how many powers of \( b_N \) an eigenvalue of the RG transformation has, as in the standard way to extract the scaling dimension in the usual RG. In other words, since \( b_N = 1 + O(1/N) \), we can read off the scaling dimension by observing deviation of \( O(1/N) \) from 1 in an eigenvalue of the linearized RG transformation. Applying this method to (3.7) and (3.8), we conclude that our Gaussian fixed point is an UV fixed point and the scaling dimensions of \( m^2 \) and \( g \) are both 2 for infinitesimal \( \varepsilon \). It is in fact consistent with the result from the mean-field, or the Landau theory, as it should be near the Gaussian fixed point.

---

\(^{13}\)Since we have just changed \( N \) by one in the large-\( N \) RG, the eigenvalue must be 1 in the leading order in the \( 1/N \) expansion.
The critical line found in present analysis also have been seen in numerical works [8, 9, 10]. The uniform-disordered phase transitions are reported on this critical line. Our results shows that the slope of the critical line diverges in large-$N$ limit, while [9] reports weak $N$-dependence of the critical line. This is because our slope of the critical line is logarithmically divergent, and thus it is hard to see the vertical of the critical line if $N$ is not so large.

It is worth noticing that (3.7) and (3.8) also imply the Gaussian fixed point in the NCFT, where the scaling dimensions of $m_N^2$ and $g_N$ are again 2, because the difference between the field theory on the fuzzy sphere and the NCFT is just in the $N$-dependence of $b_N$ itself as in (2.49) and (2.54), namely in a quantity which should be fixed in the large-$N$ limit. It would be intriguing aspect of our large-$N$ RG formalism that we can discuss a field theory on the fuzzy sphere and on the noncommutative plane in a unified way.

3.2 Nontrivial fixed points

Next we search for nontrivial fixed points. If there is a nontrivial fixed point, we would have nontrivial wave function renormalization there. In such a case if we normalize the kinetic term canonically as in (2.42), it would give rise to nontrivial $N$-dependence of $g_N$ as in (3.6). Here we assume general leading $N$-dependence of $c(N)$ in (3.6) as $c(N) = cN^a$ and fix the value of $a$ in such a way that there exists a fixed point of $\mathcal{O}(N^0)$. Near such a fixed point, if any, the perturbative expansion in terms of $g_N$ can be in danger due to higher order interactions. We observe that since $\rho_N^2c(N)g_NP_N^2N$ is the loop expansion parameter, $\rho_N^2c(N)g_NP_N^2N \ll 1$ is required for the loop expansion to be a good approximation. With this condition the third term in the left hand side of (3.7) is subleading. Therefore up to $\mathcal{O}(g_N^2)$ the RG equations (3.7) and (3.8) can be recast into

$$m_{N-1}^2 = b_N^2 \left( m_N^2 + c(N)g_NB_1(N, m_N^2) \right), \quad (3.10)$$

$$\frac{1}{g_{N-1}} = \frac{1}{c(N)b_N^2} \left( \frac{1}{g_N} + c(N)\rho_N^2B_2(N, m_N^2) \right). \quad (3.11)$$

We find a fixed point of (3.10) and (3.11) analogous to the Wilson-Fisher fixed point

$$m^2_* = -\frac{N(N - 1)}{\rho_N^2} \frac{b_N^2}{c(N)b_N^4 - 1}, \quad g_* = \frac{N^2(N - 1)^2}{2(2N - 1)} \frac{c(N)}{c(N-1)} \left( \frac{b_N^2 - 1}{c(N)b_N^4 - 1} \right)^2. \quad (3.12)$$

We then confirm that the loop expansion parameter $\rho_N^2c(N)g_NP_N^2N$ is small around this fixed point as a consistency check. Using these fixed point values, we evaluate

$$\rho_N^2c(N)g_*P_N^2N = \frac{N}{2(2N - 1)} \left( \frac{c(N)}{c(N-1)} b_N^2 - 1 \right). \quad (3.13)$$

\[^{14}\text{Here we do not take further possible suppression factors due to } 6j\text{-symbols into account.}\]
For arbitrary $a$, this is of $O(N^{-1})$ and negligible in the large-$N$ limit. Therefore our fixed point (3.12) is stable against higher order loop corrections, and this also suggests that the fixed point will not receive much effects in general from higher order interactions induced by the RG transformation. Such feature can be understood as the locality of our RG transformation itself.

We linearize (3.10) and (3.11) around the fixed point. The linear perturbations obey

$$
\delta m^2_{N-1} = b^2_N \left( 2 - \frac{c(N)}{c(N-1)} b^2_N \right) \delta m^2_N + \left( 1 - b^2_N \right) \frac{m_*^2}{g_*} \delta g_N,
$$

(3.14)

$$
\delta g_{N-1} = -\frac{1}{c(N-1) b^2_N} \delta g_N + \left( 1 - \frac{1}{c(N-1) b^2_N} \right) \frac{1 - b^2_N}{b^2_N} \frac{m_*^2}{(2N-1) c(N)} \delta m^2_N.
$$

(3.15)

The scaling dimensions are determined from the eigenvalues of these linear perturbation equations. Note that off-diagonal terms in these equations are of $O(N-1)$. Thus to the leading order, the eigenvalues for $\delta m^2$ and $\delta g$ are

$$
\delta m^2 : b^2_N \left( 2 - \frac{c(N)}{c(N-1)} b^2_N \right), \quad \delta g : \left( \frac{c(N)}{c(N-1)} b^2_N \right)^{-1}.
$$

(3.16)

with $O(N^{-2})$ corrections. As concrete examples, we discuss scaling dimensions of these linear perturbations in both cases of fuzzy sphere and NCFT, separately.

**fuzzy sphere case:** Substituting $\rho^2_N \simeq N^2 \alpha^2 / 4$ where $\alpha^2 = \alpha^2_N$ is fixed in the RG, we find

$$
m_*^2 \to -\frac{4}{\alpha^2} \frac{a + 2}{a + 4}, \quad g_* \to \frac{4}{\alpha \alpha^2} \frac{a + 2}{(a + 4)^2} N^{-a} \quad (N \to \infty),
$$

(3.17)

where we have used (2.49). We choose $a = 0$ to have $g_*$ of $O(1)$, then $(m_*^2, g_*) \to (-\frac{4}{\alpha^2}, \frac{1}{2\alpha^2})$. Since $b_N \simeq 1 + 1/N$ from (2.49), the scaling dimensions of the linear perturbations are obtained from coefficients of $1/N$ terms of the eigenvalues (3.16). They are 0 for $\delta m^2$ and $-2$ for $\delta g$. Namely there are one marginal and one irrelevant perturbations to this order in this large-$N$ limit.

**NCFT case:** $\rho^2_N = N \theta / 2$ from (2.53) yields

$$
m_*^2 \to -\frac{2N a + 1}{\theta} \frac{a + 2}{a + 2}, \quad g_* \to \frac{N^{1-a}}{2c \theta} \frac{a + 1}{(a + 2)^2} \quad (N \to \infty),
$$

(3.18)

where we have utilized (2.54). We set $a = 1$ to have $g_*$ of $O(1)$, then $(m_*^2, g_*) \to (-\frac{2N}{\theta}, \frac{1}{2c \theta})$. Although the position of $m_*^2$ goes to negative infinity as $N$ goes large, the condition for higher loop corrections to be suppressed is still satisfied. Note that now the rescaling factor is given as $b^2_N = 1 + \frac{1}{N}$. Thus the scaling dimensions of linear perturbations can be read off as $-2$ for $\delta m^2$ and $-4$ for $\delta g$. This IR fixed point is stable in this large-$N$ limit.
large \( \rho_N^2 \) limit: When \( \rho_N^2 m_N^2 \gg N^2 \), the factor from the propagator \( P_N \) in \( B_1(N, m_N^2) \) and \( B_2(N, m_N^2) \) can be approximated by \( \rho_N^{-2} m_N \). Then we easily observe from (3.11) how the position of the fixed point depends on \( N \) as (for simplicity, we choose \( c(N) = 1 \))

\[
m_4^* = \frac{2(2N - 1) g_*}{b_N^2 - 1} \rho_N^2.
\] (3.19)

This suggests a phase transition which occurs on such points. Indeed, the so-called disordered–matrix phase transition is observed on such points in numerical simulations \[8, 9, 10\]. In \[11\] this phase transition is explained by a topology change of matrix eigenvalue distribution. In these previous works, the matrix model action is given as

\[
S = \frac{4\pi}{N} \text{tr}(\phi[L_i, [L_i, \phi]] + r R^2 \phi^2 + \lambda R^2 \phi^4). \tag{3.20}
\]

The relation between \((R, r, \lambda)\) and our \((\rho_N, m_N^2, g_N)\) is determined as \( \rho_N = R, \ m_N^2 = r, \ g_N = \frac{\lambda}{4\pi} \). In \[11\], noncommutativity is fixed as \( R^2 = \frac{N\theta}{2} \), thus we adopt \( b_N^2 = \frac{N}{N-1} \) as in \[2.54\] for a comparison. Then the equation (3.19) becomes

\[
\frac{r}{N} = \pm \frac{\sqrt{\lambda}}{\sqrt{\pi}R} = \pm 0.564 \frac{\sqrt{\lambda}}{R} \quad (N \to \infty). \tag{3.21}
\]

On the other hand, the position of the critical point on \( \lambda = 1 \) line is calculated numerically in \[8\], and also estimated by an analytical method in \[11\]. Their results are

- observed in \[8\] : \( \frac{r}{N} = -\frac{0.56}{R} \),
- estimated in \[11\] : \( \frac{r}{N} = \pm \frac{3}{2\sqrt{\pi}} \frac{1}{R} = \pm \frac{0.846}{R} \).

Our result (3.21) have a good agreement with the numerical simulation\[15\]. It strongly suggests that our formula (3.19) describes the disordered–matrix phase transition points. Analysis of critical behavior of various order parameters will provide further evidence.

And the phase structures of this matrix model are further investigated numerically in \[10\]. It would be interesting to understand these structures by our RG methods.

### 3.3 Two-dimensional gravity

As a final application of our large-\( N \) RG, in this section let us consider the two-dimensional gravity. For comparison we start from the same action as in the original large-\( N \) RG in \[5\] and apply ours, namely, perform the large-\( N \) RG by a different basis of \( N \times N \) matrices from that in \[5\]. Then we compare our results with those in \[5\].

We begin with the one-matrix model (1.2) defining the two-dimensional quantum gravity

\[
S_N = N \text{tr}_N \left( \frac{1}{2} \phi_N^2 + \frac{g}{4} \phi_N^4 \right), \tag{3.22}
\]

---

\[15\]If we adopt \( b_N^2 = \frac{N}{N-1} \) as in (2.49), the number in front of \( \frac{1}{R} \) in (3.21) differs by \( 2^{-1/2} \).
here dropping the kinetic term $\frac{1}{2} [L_i, [L_i, \phi_N]]^2$ would be essential for the two-dimensional gravity, because by doing so we have the $U(N)$ gauge symmetry in (3.22) which would be somehow related to the diffeomorphism invariance. Then it is straightforward to repeat our formalism. Two important differences from before are that now we fix the wave function renormalization as

$$N \text{tr}_N \left( \phi^{\text{in}2} \right) = (N - 1) \text{tr}_{N-1} \left( \tilde{\phi}^2 \right),$$

instead of (2.42), and that the propagator becomes

$$P_N = \frac{1}{N^2}.$$ (3.24)

With these in mind, it is easy to derive the RG equation for $g_N$ again as

$$g_{N-1} = g_N \left( 1 - \frac{2}{N} (1 + 6g_N) \right) + O(g_N^2),$$

from which we obtain as in (1.10)

$$\beta(g) = -2g - 12g^2.$$ (3.26)

Therefore we again find a nontrivial fixed point $g_s = -1/6$ and a critical exponent there $\gamma_1 = 2/\beta'(g_s) = 1$. Although $\gamma_1$ is worse than that obtained in [5], they are still comparable with the exact results $g_s = -1/12$ and $\gamma_1 = 5/2$. There may be several reasons why our RG becomes worse than in [5] for the two-dimensional gravity. First we notice that our formalism respects the rotational symmetry $SO(3)$. It is evident that this property fits in field theory on the (fuzzy) sphere, but not in the two-dimensional gravity describing random surfaces. Another reason could be that in our formalism we have dropped contributions from the antipode matrix, while in [5] all $O(g)$ contributions in the systematic $1/N$ expansion are taken into account.

4 Conclusions and discussions

In this paper we propose a new large-$N$ RG based on the fuzzy sphere. It has a nice analogy with the usual RG in some aspects such as the locality of the RG transformation and the derivative expansion. It also reveals the interesting features in matrix models as the appearance of multi-trace operators, the antipode transformation, and renormalization of noncommutativity. Fixed point analysis gives consistent results for the Gaussian fixed point of field theory on the fuzzy sphere, or the NCFT on the two-dimensional plane. A nontrivial fixed point is also found and their properties are discussed. There is a critical line if the radius of the sphere is large. Our RG equation provides an expression for the critical line which well agrees with the numerical simulation. A comparable value of the critical exponent in the two-dimensional gravity can be also obtained.

Since it seems straightforward to apply our formalism to multi-matrix models, it is desirable to apply it to the large-$N$ limit of more interesting models, related to string
theory, like the Chern-Simons type matrix models as in e.g. [12, 20, 21]. Our formalism is expected to reveal some aspects of them, in particular, on their universal properties. It is also anticipated that our large-\(N\) RG also sheds light on the large-\(N\) limit of supersymmetric gauge theories on higher dimensional sphere that have attracted much attention recently. It would be likely that in order to clarify vast universality class of large-\(N\) gauge theories suggested by the large-\(N\) reduced models [22], we would need formalism based on the RG like ours. We mention that it would be also interesting to examine a relation between our large-\(N\) RG and a master field approach on the fuzzy sphere [23].

Before closing the paper, we discuss two issues which would be considerable for further understanding of the matrix RG method in the present paper.

**Fuzzy torus**

One may think that our large-\(N\) RG is also available on the fuzzy torus, where the algebra of functions is more tractable. In fact, an \(N \times N\) matrix \(\phi_N\) can be expanded analogously to the Fourier expansion as (assuming odd \(N\))

\[
\phi_N = \frac{1}{(2\pi)^2} \sum_{n_1,n_2=-N-1/2}^{N-1} \phi_{n_1,n_2} \omega^{2n_1n_2} U^{n_1} V^{n_2},
\]

(4.1)

where \(U\) and \(V\) are the standard 't Hooft matrices with rank \(N\) satisfying \(UV = \omega^{-1}VU\) with \(\omega = \exp(2\pi i/N)\). Then each vertex of a field theory on the fuzzy torus has a factor imposing the momentum conservation modulo \(N\): \(\delta(N)(\sum_r n_1^{(r)})\delta(N)(\sum_r n_2^{(r)})\). In our large-\(N\) RG, it is natural to integrate first over \((4N-4)\) modes with the maximal momentum of \(n_1\) or \(n_2\), namely, over \(\phi_{n_1,n_2}\) with \(|n_1|\) or \(|n_2| = (N-1)/2\), and then to rewrite the resulting action as an \((N-2) \times (N-2)\) matrix model. However, in the \((N-2) \times (N-2)\) matrix model the momentum conservation should hold in each vertex modulo \(N-2\). For example, if a vertex consists only of in-modes with \(|n_1|, |n_2| < (N-1)/2\), it remains intact in the RG, and hence there the momentum is conserved modulo \(N\), not \(N-2\). This serious problem would be bypassed by considering the large-\(N\) RG not by \(N \to N-2\), but by \(N \to N/2\) as the block spin transformation. We will report our study in this direction elsewhere.

**Antipode, UV/IR mixing, and noncommutative anomaly**

Another interesting aspect in our formalism is the appearance of the antipode transformation. Here we point out an interesting connection between the antipode matrix and the nonplanar diagram. If we draw nonplanar diagrams by the standard double line notation, we find that they resemble annulus diagrams appearing at one-loop of open string theory. They have two edges from which external lines emanate. Our intriguing observation up to \(O(g_N^2)\) is that if we identify matrices brought by external lines from one edge as “usual” ones, external lines from another edge provide the antipode matrices. Namely, there is a connection between the Feynman diagram and the positions on the (fuzzy) sphere. We
hope that further analysis of nonplanar diagrams makes clear the origin of the antipode transformation.

Related to this, in \[14\] it is claimed that such antipode degrees of freedom trigger the UV/IR mixing \[17\]. After \[14\], it is rigorously argued in \[19\] that there is no UV/IR mixing on the fuzzy sphere for finite \(N\) based on the one-loop effective action including a nonplanar diagram, and the IR singularity seen in field theory on the noncommutative plane is reproduced as the large-\(N\) limit of a term arising from a nonplanar diagram, that the authors call noncommutative anomaly (it is regular for finite \(N\)). As seen below in detail, the appearance of the antipode matrix reported in this paper can be understood as a part of such a noncommutative anomaly for finite \(N\).

Here we make a detailed comparison of \(\langle V_{2NP}^2 \rangle\) between that in this paper and in \[19\]. The authors of \[19\] use the following formula in calculating \(\langle V_{2NP}^2 \rangle\)

\[
\left\{ \begin{array}{c} L \ L \ l \\ L \ L \ J \end{array} \right\} \simeq \frac{(-1)^{2L+l+J}}{2L} P_l \left( 1 - \frac{J^2}{2L^2} \right).
\]

The noncommutative anomaly for finite \(N\) is obtained as the difference between \(\langle V_{2NP}^2 \rangle\) and \(\langle V_{2P}^2 \rangle\) with the internal mode integrated over the whole region \(0 \leq J \leq 2L\) by use of this formula. On the other hand, our antipode matrix originates in \(\langle V_{2NP}^2 \rangle_0\) with only the \(J = 2L\) mode integrated and with \(2.67\) applied. The term with \(J = 2L\) in \(4.2\) indeed reproduces the first term \[16\] in the formula \(2.67\). Therefore, the antipode effect of \(\langle V_{2NP}^2 \rangle_0\) in this paper can be regarded as the maximum momentum part of the noncommutative anomaly discussed in \[19\]. It is then preferable that the other appearances of the antipode matrix can also be interpreted in a similar way. If we can repeatedly integrate out the highest modes, the total antipode effect will eventually, in principle, reproduce the noncommutative anomaly for finite \(N\). Our RG approach would be helpful toward such a problem, as discussed next.

Since as shown in \[19\] the noncommutative anomaly develops the IR singularity in the large-\(N\) NCFT limit \(2.52\) and triggers the UV/IR mixing, the above observation implies that around a fixed point corresponding to the large-\(N\) limit in our RG, interactions involving the antipode matrix would be relevant and that analysis of them there would reveal universal nature of the noncommutative anomaly and the UV/IR mixing. More generally, an important question from our study is on existence of nontrivial theory with the antipode matrix. According to the spirit of the RG, we are led to consider a new matrix model like

\[
S_N = N\text{tr}_N \left( -\frac{1}{2} [L_i, \phi]^2 - \alpha [L_i, \phi][L_i, \phi^A] + \frac{\rho_N^2 m^2}{2} \phi^2 + \rho_N^2 \tilde{m}^2 \phi \phi^A + \cdots \right),
\]

and look for a nontrivial fixed point of our large-\(N\) RG. For example, the fixed point analysis of \(\tilde{m}^2\) is expected to be useful for understanding the UV/IR mixing.

\[16\]Note that \(4.2\) and \(2.67\) are different approximation formulas even when \(J = 2L\), and the “derivative corrections” in \(2.67\) are not visible by use of \(4.2\).
Finally, we comment on the observations made in [17, 24]. In these papers the UV/IR mixing is understood as the exchanges of light degrees of freedom which are called closed string modes. This nicely fits with stringy interpretation of a nonplanar diagram. Recall that the 1-loop correction of the scalar field 2-point function is UV divergent in the 2-dimensional continuum theory and that if we try to regularize it via the noncommutativity as in the fuzzy sphere or in the noncommutative plane, a nonplanar diagram in general yields the noncommutative anomaly, or the UV/IR mixing, in the large-$N$ limit. Then the IR singularity there can be regarded as the effect of the propagation of the closed string mode. Therefore it is natural to expect that even for finite $N$ a nonplanar diagram of the 2-point function in the matrix model on fuzzy sphere has a counterpart of such a closed string mode. In fact, the formula (2.68) can be written as the exchange of particles with propagator

$$\Delta(p) = \int \frac{d\mu^2}{2\rho^2} \frac{\lambda(\mu^2)}{p \circ p/\rho^2 + \mu^2}, \quad \lambda(\mu^2) = \delta\left(\mu^2 - \frac{1}{N^2}\right),$$

following the expression in [24]. Here identification is made as

$$l(l + 1) = p^2 \rho^2, \quad \frac{\rho^4}{N^3} p^2 = p \circ p, \quad (4.5)$$

and $\rho$ is the radius of sphere. The spectral function $\lambda(\mu^2)$ in [24] has a uniform distribution which vanishes below $1/(\rho \Lambda)^2$, where $\Lambda$ is a UV cutoff of Wilsonian RG in NCFT and $\rho$ is replaced by a parameter with dimensions of squared length in the reference. On the other hand, the spectral function in (4.4) is the delta function. This is because we calculated a one-step RG flow $N \to N - 1$. It is shown that matrix models on the fuzzy sphere are free from the IR divergence for finite $N$ [19]. In fact, this feature can be also seen in (4.4). The spectral function does not contain massless modes unless $N$ is infinity. An IR regularization of the closed string picture is realized in this way.

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**A Useful formulas of $3nj$-symbols**

In this appendix we enumerate useful formulas for our study from [15]. The Latin and corresponding Greek indices denote angular momenta and magnetic quantum numbers (projection of angular momentum), respectively. For example, the magnetic quantum
number \( \alpha \) runs \(-a \leq \alpha \leq a\) for the angular momentum \( a \). In the following we define

\[
\{abc\} = \begin{cases} 
1 & |a - b| \leq c \leq a + b \\
0 & \text{otherwise}
\end{cases}, \quad \text{(A.1)}
\]

namely, \( \{abc\} \) does not vanish only if \( a, b, \) and \( c \) satisfy the triangular conditions which are symmetric under the interchange of variables.

### A.1 Identities

\[
\sum_{\psi \kappa} (-1)^{p+q-\kappa} \begin{pmatrix} a & p & q \\ -\psi & \kappa \end{pmatrix} \begin{pmatrix} p & q & a' \\ -\psi & -\kappa & \alpha' \end{pmatrix} = \frac{(-1)^{a+\alpha}}{2a + 1} \{apq\} \delta_{aa'} \delta_{\alpha \alpha'}. \quad \text{(A.2)}
\]

\[
\sum_{q \kappa} (-1)^{q-\kappa} (2q + 1) \begin{pmatrix} a & b & q \\ -\alpha & -\beta & \kappa \end{pmatrix} \begin{pmatrix} q & a & b \\ -\kappa & \alpha' & \beta' \end{pmatrix} = (-1)^{a+\alpha+b+\beta} \delta_{aa'} \delta_{\beta \beta'}. \quad \text{(A.3)}
\]

\[
\sum_{\kappa \psi \rho} (-1)^{p-\psi+q-r-\rho} \begin{pmatrix} p & a & q \\ \psi & \alpha & -\kappa \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa & \beta & -\rho \end{pmatrix} \begin{pmatrix} r & c & p \\ \rho & \gamma & -\psi \end{pmatrix} = \begin{pmatrix} a & b & c \\ -\alpha & -\beta & -\gamma \end{pmatrix} \begin{pmatrix} a & b & c \\ r & p & q \end{pmatrix}. \quad \text{(A.4)}
\]

\[
\sum_{\psi \kappa \rho \sigma} (-1)^{p-\psi+q-r+s-r} \begin{pmatrix} p & a & q \\ \psi & \alpha & -\kappa \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa & \beta & -\rho \end{pmatrix} \begin{pmatrix} r & c & s \\ \rho & \gamma & -\sigma \end{pmatrix} \begin{pmatrix} s & d & p \\ \sigma & \delta & -\psi \end{pmatrix} = (-1)^{s-\alpha-d-a} \sum_{x \xi} (-1)^{x-\xi} (2x + 1) \begin{pmatrix} a & x & d \\ \alpha & -\xi & \delta \end{pmatrix} \begin{pmatrix} b & x & c \\ \beta & \xi & \gamma \end{pmatrix} \begin{pmatrix} a & x & d \\ s & p & q \end{pmatrix} \begin{pmatrix} b & x & c \\ s & r & q \end{pmatrix}. \quad \text{(A.5)}
\]

\[
\sum_{x \xi} (-1)^{x-\xi} (2x + 1) \begin{pmatrix} a & x & c \\ \alpha & -\xi & \gamma \end{pmatrix} \begin{pmatrix} b & x & d \\ \beta & \xi & \delta \end{pmatrix} \begin{pmatrix} a & p & q \\ x & d & b \end{pmatrix} \begin{pmatrix} c & a & q \\ x & d & b \end{pmatrix} = \begin{pmatrix} a & f & r \\ d & q & e \end{pmatrix} \begin{pmatrix} a & b & r \\ p & c & b \end{pmatrix}. \quad \text{(A.6)}
\]

\[
\sum_{X} (-1)^{p+q+X} (2X + 1) \begin{pmatrix} a & b & X \\ c & d & p \end{pmatrix} \begin{pmatrix} c & d & X \\ b & a & q \end{pmatrix} = \begin{pmatrix} c & a & q \\ d & b & p \end{pmatrix}. \quad \text{(A.7)}
\]

\[
\sum_{X} (-1)^{2X} (2X + 1) \begin{pmatrix} a & b & X \\ c & d & p \end{pmatrix} \begin{pmatrix} c & d & X \\ e & f & q \end{pmatrix} \begin{pmatrix} e & f & X \\ a & b & r \end{pmatrix} = \begin{pmatrix} a & f & r \\ d & q & e \end{pmatrix} \begin{pmatrix} a & b & r \\ p & c & b \end{pmatrix}. \quad \text{(A.8)}
\]
A.2 Asymptotic formulas of $6j$-symbol

If $a, b, c \gg f$ and $f$ is an arbitrary integer,

$$\begin{bmatrix} a & b & c \\ b & a & f \end{bmatrix} \simeq (-1)^{a+b+c+f} \frac{P_f(\cos \theta)}{\sqrt{(2a+1)(2b+1)}} \sqrt{(2a+1)(2b+1)} P_f(\cos \theta),$$  \hspace{1cm} (A.9)

with

$$\cos \theta = \frac{a(a+1) + b(b+1) - c(c+1)}{2a(a+1)b(b+1)}.$$  \hspace{1cm} (A.10)

If $R \gg 1$ and $a, b, c$ are arbitrary,

$$(-1)^{2R} \begin{bmatrix} a & b & c \\ d & R & e + R & f + R \end{bmatrix} \simeq (-1)^{c+d+e} \frac{C_{a+e-d+R}^{c+R}}{\sqrt{2R(2c+1)}} C_{a-e}^c R(\cos \theta),$$  \hspace{1cm} (A.11)

where $C_{a+e-d+R}^{c+R}$ is the Clebsch-Gordan coefficient.\(^{17}\)

To investigate the asymptotic behavior of the $6j$-symbols, it is sometimes useful to use the explicit expression due to Racah,

$$\begin{bmatrix} A & B & C \\ a & b & c \end{bmatrix} = \sqrt{\Delta(A, B, C)\Delta(a, b, c)\Delta(a, b, C)\Delta(a, C, b)} \sum_t (-1)^t \frac{(t+1)!}{f(t)} \Delta(x, y, z) = \frac{(x+y-z)!}{(x+y+z+1)!},$$  \hspace{1cm} (A.12)

where the sum of $t$ runs in the region where no argument of the factorials takes negative values, namely,

$$\max(n_1, n_2, n_3, n_4) \leq t \leq \min(m_1, m_2, m_3).$$

Several asymptotic formulas are obtained by evaluating the factorials by the use of the Stirling’s formula.

B Useful formulas of the fuzzy spherical harmonics

In this appendix useful formulas of the $N \times N$ matrix $T_{lm}$ defined in (2.12) are presented. We set $L = (N - 1)/2$ as in the main text. (A.4), (A.5) and (A.6) give formulas of the trace of $T_{lm}$'s

$$\text{tr}_N (T_{l_1 m_1} T_{l_2 m_2} T_{l_3 m_3}) = N^2 \sum_{i=1}^3 (2l_i + 1)\frac{1}{2} (-1)^{2L+\sum_{i=1}^3 l_i} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ L & L & L \end{pmatrix},$$

\(^{17}\)We have supplied the formula presented in [15] with a necessary phase which was also added in [19].
Notice that (B.2) can be also derived easily from (B.1) and (2.14), and that $F(B.4)$ and (B.5) reproduces the fusion of the usual spherical harmonics [15] also coincides with that of the usual spherical harmonics in the large-$N$ limit.
Furthermore, by using the fusion (B.4) and (B.5) repeatedly, we find

\[
T \equiv T_{2Lm}T_{1,m_1}T_{2L,-m} = N(4L+1) \sum_{l'\, m'} \sqrt{(2l_1+1)(2l''+1)} (-1)^{l_1+l''+m'+m''} \times \left( \frac{l_1}{m_1 -m'} \right) \left( \frac{l''}{m'' -m} \right) \left\{ \frac{2l_1 \; l'}{L \; L \; L} \right\} \left\{ \frac{l'' \; l''}{L \; L \; L} \right\} T_{l',m'}. 
\]  

(B.10)

Taking the sum \( \sum_{m=-2L}^{2L} (-1)^m \) of both sides of this equation and applying (A.2), we have

\[
\sum_{m=-2L}^{2L} (-1)^m T_{2Lm}T_{1,m_1}T_{2L,-m} = (-1)^{2L+l_1} N(2N-1)T_{1,m_1} \sum_{l'} (-1)^{l'} (2l'+1) \left\{ \frac{2l_1 \; l'}{L \; L \; L} \right\}^2 . 
\]  

(B.11)

where the summation over \( l' \) can be also taken by utilizing (A.7) to yield

\[
\sum_{m=-2L}^{2L} (-1)^m T_{2Lm}T_{1,m_1}T_{2L,-m} = N(2N-1) \left\{ \frac{L \; L \; l_1}{L \; L \; 2L} \right\} (-1)^{l_1} T_{1,m_1}. 
\]  

(B.12)

This equation tells us an interesting fact that making the similarity transformation with respect to \( T_{2Lm} \) and taking the summation over \( m \) is essentially equivalent to performing the “antipode” transformation \( T_{lm} \rightarrow (-1)^l T_{lm} \), namely transporting a field to the antipode on the sphere. As a corollary, we find that

\[
\sum_{m=-2L}^{2L} (-1)^m T_{2Lm}T_{2L,-m} = (2N-1) 1_N. 
\]  

(B.13)

where we have used \( T_{00} = 1_N \) and

\[
\left\{ \frac{L \; L \; 0}{L \; L \; 2L} \right\} = \frac{1}{N}. 
\]  

(B.14)

(B.12) and (B.13) are of great help in calculating our RG.

### C Calculations of \( \mathcal{O}(g_N^2) \)

In this appendix we evaluate contributions of \( \mathcal{O}(g_N^2) \) in the large-\( N \) RG shown in the second line in (2.41). First we consider \( \langle V_2^2 \rangle_c \) that contributes to vertex correction and then turn to the other terms giving rise to mass correction.
C.1 Vertex correction

\[ V_2 = V_2^P + V_2^{NP} \text{ as in (2.36)} \] and the only possible planar diagram in \( \langle V_2^2 \rangle_c \) arises in \( \langle V_2^{P^2} \rangle_c \) calculated in (2.74). Furthermore, in the low energy regime where the momentum \( l \) of \( \phi_{lm}^\text{in} \) is much smaller than the cutoff \( 2L \), we can perform “derivative” expansion of it as in (2.76) to get (2.78). Similarly we can evaluate the nonplanar diagram in \( \langle V_2^{P^2} \rangle_c \) given as the second term in (2.70) and obtain a similar result to (2.74)

\[
\left\langle V_2^{P^2} \right\rangle_c \mid_{\text{nonplanar}} = P_N^2 (2N - 1)^2 N \sum_{lm} \left\{ \frac{2L}{L} \right\}^2 (-1)^l \text{tr}_N \left( \phi_{lm}^\text{in} T_{lm} \right) \text{tr}_N \left( \phi_{lm}^\text{in} T_{lm}^\dagger \right). 
\]

(C.1)

Here we note that

\[
(-1)^l \text{tr}_N \left( \phi_{lm}^\text{in} T_{lm} \right) = \text{tr}_N \left( \phi_{lm}^\text{in} A_{lm} T_{lm} \right),
\]

which can be readily shown directly from the expression of the trace of three \( T_{lm} \)'s given in (B.1) by using the well-known property of the 3\( j \)-symbol, or from the proposition proved in the appendix D. Then in the low energy regime the same derivation as in (2.75) leads to

\[
\left\langle V_2^{P^2} \right\rangle_c \mid_{\text{nonplanar}} = P_N^2 (2N - 1) N \left[ \text{tr}_N \left( \phi_{lm}^\text{in} A_{lm} \phi_{lm}^\text{in} A_{lm} \right) - \frac{1}{2N} \text{tr}_N \left( \left[ L_i, \phi_{lm}^\text{in} A_{lm} \phi_{lm}^\text{in} A_{lm} \right] \phi_{lm}^\text{in} A_{lm} \right) + O \left( \frac{1}{N^2} \right) \right].
\]

(C.3)

On the other hand, from (2.36) we obtain

\[
\langle V_2^{P^2} V_2^{NP} \rangle_c = \frac{1}{2} \sum_{m_1, m_2, m_1', m_2'} \left( \phi_{2Lm_1}^\text{out} \phi_{2Lm_1'}^\text{out} \phi_{2Lm_2}^\text{out} \phi_{2Lm_2'}^\text{out} \right)_0 \left( \phi_{2Lm_1}^\text{out} \phi_{2Lm_1'}^\text{out} \phi_{2Lm_2}^\text{out} \phi_{2Lm_2'}^\text{out} \right)_0 \frac{2L}{L} \times \text{tr}_N \left( \phi_{lm}^\text{in} T_{2Lm_1} T_{2Lm_2} \right) \text{tr}_N \left( \phi_{lm}^\text{in} T_{2Lm_1} T_{2Lm_2'} \right) \text{tr}_N \left( \phi_{lm}^\text{in} T_{2Lm_2} T_{2Lm_2'} \right) + O \left( \frac{1}{N^2} \right)
\]

\[
= P_N^2 \sum_{m, m'} (-1)^{m + m'} \text{tr}_N \left( \phi_{lm}^\text{in} T_{2Lm} T_{2Lm'} \right) \text{tr}_N \left( \phi_{lm}^\text{in} T_{2Lm'} \phi_{lm}^\text{in} T_{2Lm} \right),
\]

(C.4)

where the two kinds of the contractions yielding nonplanar diagrams turn out to be equivalent from the cyclicity of the trace. By expanding \( \phi_{lm}^\text{in} \) as \( \phi_{lm}^\text{in} = \sum_{lm} \phi_{lm}^\text{in (2)} T_{lm} \) and using the trace formula in (B.1) and in (B.2), we have

\[
\langle V_2^{P^2} V_2^{NP} \rangle_c = P_N^2 (2N - 1)^2 N^2 \sum_{m, m'} (-1)^{m + m'} \sum_{(l_r, m_r) \in \Lambda_m} \phi_{l_1 m_1}^\text{in (2)} \phi_{l_2 m_2}^\text{in} \phi_{l_3 m_3}^\text{in} \]

\[= P_N^2 (2N - 1)^2 N^2 \sum_{m, m'} (-1)^{m + m'} \sum_{(l_r, m_r) \in \Lambda_m} \phi_{l_1 m_1}^\text{in (2)} \phi_{l_2 m_2}^\text{in} \phi_{l_3 m_3}^\text{in} \]

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\[
\times (-1)^{2L+l} \prod_{r=1}^{3} (2l_r + 1)^{\frac{1}{2}} \left\{ \begin{array}{ccc} l_1 & 2L & 2L \\ m_1 & m & m' \end{array} \right\} \left\{ \begin{array}{ccc} l_1 & 2L & 2L \\ L & L & L \end{array} \right\} \\
\times \sum_{l''_{m''}} (-1)^{m''} (2l'' + 1) \left\{ \begin{array}{ccc} l_2 & 2L & l'' \\ m_2 & -m & -m'' \end{array} \right\} \left\{ \begin{array}{ccc} l'' & l_3 & 2L \\ m'' & m_3 & m' \end{array} \right\} \left\{ \begin{array}{ccc} l_2 & 2L & l'' \\ L & L & L \end{array} \right\} \left\{ \begin{array}{ccc} l'' & l_3 & 2L \\ L & L & L \end{array} \right\} .
\]
\]

(C.5)

Here we can take the summations over \(m, m', \) and \(m''\) according to (A.4), while the sum on \(l''\) can be also taken by (A.8). We thus get

\[
\langle V_2^PV_2^NP \rangle_c = P_N^2 (2N - 1)^2 N^2 \sum_{(r, r') \in \Lambda_m} \phi_{1,m_1}^{in} \phi_{2,m_2}^{in} \phi_{3,m_3}^{in} \prod_{i=1}^{3} (2l_i + 1)^{\frac{1}{2}} (-1)^{l_1} \\
\times \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right\} \left\{ \begin{array}{ccc} l_1 & 2L & 2L \\ L & L & L \end{array} \right\} \left\{ \begin{array}{ccc} l_2 & l_3 & l_1 \\ L & L & 2L \end{array} \right\} .
\]
\]

(C.6)

We next use the cyclic symmetry of the 9j-symbol and utilize (A.8) once again. Then the 9j-symbol can be rewritten as

\[
\left\{ \begin{array}{ccc} l_2 & l_3 & l_1 \\ L & L & 2L \end{array} \right\} = \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ 2L & L & L \end{array} \right\} = \sum_x (-1)^{2x} (2x + 1) \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ L & L & x \end{array} \right\} \left\{ \begin{array}{ccc} l_2 & l_3 & l_1 \\ L & L & 2L \end{array} \right\} \left\{ \begin{array}{ccc} 2L & L & L \\ x & l_1 & 2L \end{array} \right\} .
\]
\]

(C.7)

Note that in contrast to \(l''\) in (C.5), here the selection rules of the 6j-symbols impose \(L \leq x \leq L + \text{min}(l_1, l_2)\) and \(x \in \mathbb{Z} \). Thus setting \(x = L + m\), we have

\[
(-1)^{2L} \sum_{m=0}^{\text{min}(l_1, l_2)} (N + 2m) \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ L & L & L + m \end{array} \right\} \left\{ \begin{array}{ccc} 2L & L & L \\ l_2 & L + m & L \end{array} \right\} \left\{ \begin{array}{ccc} 2L & L & L \\ L + m & l_1 & 2L \end{array} \right\} .
\]
\]

(C.8)

Three 6j-symbols above can be explicitly evaluated via the Racah formula (A.12) and it can be further approximated by the Stirling formula in the low energy regime \(l_r \ll L\) \((r = 1 \sim 3)\). For instance,

\[
\left\{ \begin{array}{ccc} 2L & L & L \\ l & L + m & L \end{array} \right\} = (-1)^m \sqrt{m!} \sqrt{\frac{(l + m)!}{(l - m)!}} \sqrt{\frac{(2L)!}{(4L + 1)!}} \sqrt{\frac{(2L - m)!}{(2L + l + 1)!}} \sqrt{\frac{(2L + m - l)!}{(2L + m + l + 1)!}} \\
= (-1)^m \sqrt{m!} \sqrt{\frac{(l + m)!}{(l - m)!}} \frac{1}{2L^{\frac{m}{2} + 1}} \left( 1 - \frac{8l(l + 1) - m(m - 5) + 8}{16L} + O \left( \frac{1}{L^2} \right) \right),
\]
\]

(C.9)

where we note that terms with higher \(m\) are suppressed as \(L^{-1 - \frac{m}{2}}\) when \(L \gg 1\). We can evaluate the last 6j-symbol in (C.8) in a similar manner,

\[
\left\{ \begin{array}{ccc} 2L & L & L \\ L + m & l & 2L \end{array} \right\} = (-1)^l \sqrt{m!} \sqrt{\frac{(l + m)!}{(l - m)!}} \sqrt{\frac{2}{(4L)^{\frac{m}{2} + 1}}} \left( 1 - \frac{2l(l + 1) + m(5m + 7) + 6}{16L} + O \left( \frac{1}{L^2} \right) \right),
\]

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and this is also suppressed as $L^{-1 - \frac{m}{2}}$. Therefore, in the $1/L$ expansion, it is sufficient to consider $m = 0$ and $1$ in the summation. As for the remaining $6j$-symbol, we can relate $m = 1$ one with $m = 0$ case for $l_i \ll L \ (i = 1 \sim 3)$

$$\begin{align*}
\left\{ \begin{array}{c}
l_1 \ l_2 \ l_3 \\
L \ L \ L + 1
\end{array} \right\} & \simeq \frac{l_3(l_3 + 1) - l_1(l_1 + 1) - l_2(l_2 + 1)}{2\sqrt{l_1(l_1 + 1)l_2(l_2 + 1)}} \left\{ \begin{array}{c}
l_1 \ l_2 \ l_3 \\
L \ L \ L
\end{array} \right\} ,
\end{align*}$$

(C.10)

by use of the asymptotic formula (A.11) and a recursion relation for the Clebsch-Gordan coefficient

$$C_{a_{1b-1}}^0 = C_{a_{0b0}}^0 \frac{c(c + 1) - a(a + 1) - b(b + 1)}{2\sqrt{a(a + 1)b(b + 1)}} .$$

(C.11)

Combining these, we obtain

$$\min(l_1,l_2) \sum_{m=0}^{(N + 2m)} \left\{ \begin{array}{c}
l_1 \ l_2 \ l_3 \\
L \ L \ L + m
\end{array} \right\} \left\{ \begin{array}{c}
2L \ L \ L \\
L + m \ L
\end{array} \right\} \left\{ \begin{array}{c}
2L \ L \ L \\
L + m \ l_1 \ 2L
\end{array} \right\} = \frac{(-1)^{l_1}}{\sqrt{2N}} \left\{ \begin{array}{c}
l_1 \ l_2 \ l_3 \\
L \ L \ L
\end{array} \right\} \left[ 1 - \frac{1}{N} \left( -\frac{1}{4}l_1(l_1 + 1) + \frac{1}{2}l_2(l_2 + 1) + \frac{1}{2}l_3(l_3 + 1) - \frac{1}{4} \right) + O \left( \frac{1}{N^2} \right) \right] .$$

(C.12)

Plugging (C.12) into (C.6), together with (2.75), we see that the remaining factors are exactly of the form of (B.11) and finally obtain

$$\langle V_2^P V_2^{NP} \rangle_c = P_N^2(2N - 1)N \text{tr}_N \left( \phi^{in3} \phi^{inA} \right) - \frac{1}{2N} \text{tr}_N \left( [L_i, [L_i, \phi^{in}]] \phi^{in2} \phi^{inA} \right)
- \frac{1}{2N} \text{tr}_N \left( [L_i, [L_i, \phi^{inA}]] \phi^{in3} \right) + O \left( \frac{1}{N^2} \right) .$$

(C.13)

The calculation of $\langle V_2^{NP2} \rangle_c$ is similar. The two ways of contractions again give the same result as

$$\langle V_2^{NP2} \rangle_c = \frac{P_N^2}{2} \sum_{m_3,m_4} (-1)^{m_3 + m_4} \text{tr}_N \left( \phi^{in} T_{2L,m_3} \phi^{in} T_{2L,m_4} \right) \text{tr}_N \left( \phi^{in} T_{2L-m_4} \phi^{in} T_{2L-m_3} \right)
- \frac{P_N^2}{2} \langle 2N - 1 \rangle^2 N^4 \sum_{(l_{r,m_r}, l'_{r,m'_r}) \in \Delta_{2L} \ r=1} \prod_{r=1}^{2} \left( 2l_{r} + 1 \right)^{\frac{3}{2}} \phi^{in}_{l_{r,m_r}} \phi^{in}_{l'_{r,m'_r}} \left( -1 \right)^{l_2 + l'_{2}} \times \sum_{l,m} (-1)^{-m} \left( 2l + 1 \right) \left\{ l_1 \ l_2 \ l \right\} \left\{ m_1 \ m_2 \ -m \right\} \left\{ \begin{array}{c}
l_1 \ l_2 \ l \\
m_1 \ m_2 \ m
\end{array} \right\} R_{L \ L \ L \ 2L \ L \ L} ,$$

(C.14)
where we have used the formula for the trace of four $T_{lm}$’s in terms of 9$j$-symbols \([B.3]\) instead of 6$j$-symbols \([B.2]\). Then we apply a similar evaluation from \((C.7)\) to \((C.12)\) to two 9$j$-symbols in this expression. A lengthy calculation as above results in the last equation in \((2.79)\).

### C.2 Mass correction

\(\langle V_1^2 \rangle_c\), would give rise to a \(\phi^6\) vertex. But we shall see that it is negligible in the low energy regime. In fact, from \((2.36)\), \(\langle V_1^2 \rangle_c\) becomes

\[
\langle V_1^2 \rangle_c = \sum_{m,m'} \langle \phi_{2Lm}^{out} \phi_{2Lm'}^{out} \rangle_0 \mathrm{tr}_N \left( \phi^{in} T_{2Lm} \right) \mathrm{tr}_N \left( \phi^{in} T_{2Lm'} \right)
= P_N \sum_m (-1)^m \mathrm{tr}_N \left( \phi^{in} T_{2Lm} \right) \mathrm{tr}_N \left( \phi^{in} T_{2L-m} \right).
\]

\(\text{(C.15)}\)

However, when the momentum \(l\) of \(\phi^{in}\) is much smaller than \(2L\), that of \(\phi^{in}\) cannot be equal to \(2L\) and the traces above vanish by \((2.13)\). Hence in the low energy regime of interest in the RG we can neglect this contribution.

The other remaining terms of \(O(g_N^2)\) will contribute to quadratic terms of \(\phi^{in}\). First let us consider \(\langle V_3^2 \rangle_c\). Using the trace formula \((B.2)\), we have after contractions

\[
\langle V_3^2 \rangle_c = P_N^3 (2N-1)^3 N^4 \sum_{(l_1,m_1),(l'_1,m'_1)\in\Lambda_{\text{in}}} \phi^{in}_{l_1m_1} \phi^{in}_{l'_1m'_1} \sqrt{(2l_1+1)(2l'_1+1)}
\times \sum_{l,l'} (2l+1)(2l'+1) \left\{ \begin{array}{ccc} l_1 & 2L & l \\ L & L & L \end{array} \right\} \left\{ \begin{array}{ccc} l'_1 & 2L & l' \\ L & L & L \end{array} \right\} \left\{ \begin{array}{ccc} l & 2L & 2L \\ L & L & L \end{array} \right\} \left\{ \begin{array}{ccc} l' & 2L & 2L \\ L & L & L \end{array} \right\}
\times \sum_{m_1,-m_3,m_4,m_3,m_4} (-1)^{m_1+m_2+m_3-m-m'} \left( \frac{l_1}{m_1} 2L l \quad \frac{l'_1}{m'_1} 2L l' \right)
\times \left( \frac{l}{m} 2L l \quad \frac{l'}{m'} 2L l' \right)
\times \left( \frac{2L}{m} 2L l \quad \frac{l'}{m'} 2L l' \right)
+ (\text{permutations of } m_2 \sim m_4).
\]

\(\text{(C.16)}\)

When \(l_1, l'_1 \ll L\), the first two 6$j$-symbols impose \(l = 2L - m, l' = 2L - n\) with \(m \leq l_1, n \leq l'_1\). Then the Racah formula \((A.12)\) and the Stirling’s formula show that the large-\(L\) behavior of the third 6$j$-symbol as

\[
\left\{ \begin{array}{ccc} 2L-m & 2L & 2L \\ L & L & L \end{array} \right\} \sim \frac{(-1)^{2L-m} 3^2 (2\pi)^{\frac{1}{4}}}{8 \sqrt{m!}} L^{\frac{3L-3}{4}} \left( \frac{3}{4} \right)^{\frac{3L-3}{4}},
\]

\(\text{(C.17)}\)

namely, it is exponentially suppressed for \(L \gg 1\). Since it is easy to see that other factors in the above equation are bounded at least by polynomials of \(L\), we conclude that \(\langle V_3^2 \rangle_c\) only gives exponentially small contribution in the low energy regime. Furthermore, it is easy to show that \(\langle V_1 V_3 \rangle_c = 0\) by the momentum conservation. Thus only \(\langle V_2 V_4 \rangle_c\) provides a nonzero contribution even at low energy. From \((2.36)\) we first find that

\[
\langle V_2^P V_4 \rangle_c = P_N^3 \sum_{m_1,\ldots,m_3} (-1)^{m_1+m_2+m_3} \mathrm{tr}_N \left( \phi^{in} T_{2Lm_1} T_{2Lm_2} \right)
\]

\(38\)
\[ \times \left( \text{tr}_N \left( T_{2L-m_2} T_{2L_m} T_{2L - m} \right) + \text{tr}_N \left( T_{2L - m_1} T_{2L m_2} T_{2L m} T_{2L - m} \right) \right) + \text{tr}_N \left( T_{2L - m_2} T_{2L_m} T_{2L m_1} T_{2L - m} \right) \right) ; \quad (C.18) \]

where the first two terms correspond to planar diagrams, while the last term to a nonplanar one. The formulas \((B.12)\) and \((B.13)\) enable us to rewrite this as

\[ \langle V^P V^4 \rangle_c = P^3 \frac{(2N - 1)^2 N \left( 2 + (-1)^{2L} \left\{ L L \frac{2L}{L L} L L \right\} \right)}{2N} \text{tr}_N \left( \phi^{in^2} \right) . \quad (C.19) \]

In the second term that comes from the nonplanar diagram, the 6\(j\)-symbol is again exponentially small as

\[ \left\{ L L \frac{2L}{L L} L L \right\} \simeq \sqrt{2\pi L} \frac{L^2}{L^2 - 4L - 2} , \quad (C.20) \]

for \( L \gg 1 \). Therefore for \( L \gg 1 \) the planar diagram gives

\[ \langle V^P V^4 \rangle_c \simeq P^3 (2N - 1)^2 N \text{tr}_N \left( \phi^{in^2} \right) . \quad (C.21) \]

Finally, we evaluate \( \langle V^{NP} V^4 \rangle_c \). This can be carried out by using the technique explained so far. We quote only the result,

\[ \langle V^{NP} V^4 \rangle_c = N (2N - 1)^2 P^3 \text{tr}_N \left( \phi^{in^2} \phi^{inA} - \frac{1}{N} \phi^{in} \left[ L_i, \left[ L_i, \phi^{inA} \right] \right] + \cdots \right) . \quad (C.22) \]

Therefore, this term involves the antipode fields.

### D Antipode transformation and ordering reverse

In this appendix we prove a proposition that reveals an interesting connection between the ordering of matrices inside the trace and the antipode transformation.

**Proposition:**

\[ \text{tr}_N \left( \prod_{i=1}^n \phi_i^A \right) = \text{tr}_N \left( \prod_{i=1}^n \phi_{n+1-i}^A \right) . \quad (D.1) \]

**Proof:**

The \( n = 1 \) case is trivial. The \( n = 2 \) and \( n = 3 \) cases are also obvious because of the orthogonality \((2.13)\) and the explicit expression of the trace of three generators \((B.1)\) with the symmetry property of the 3\(j\)-symbol. Assuming \((D.1)\) for \( n = 1, \cdots, k \), from \((2.15),\)

\[ \text{tr}_N \left( \prod_{i=1}^{k+1} \phi_i^A \right) = \frac{1}{N} \sum_{lm} \text{tr}_N \left( \prod_{i=1}^{k-1} \phi_i^A T^l_{lm} \right) \text{tr}_N \left( \phi_k^A \phi_{k+1}^A T^l_{lm} \right) \]
$$\frac{1}{N} \sum_{lm} \text{tr}_N \left( (-1)^l T_{lm} \prod_{i=1}^{k-1} \phi_{k-i} \right) \text{tr}_N \left( (-1)^l T_{lm}^\dagger \phi_{k+1} \phi_k \right)$$

$$= \frac{1}{N} \sum_{lm} \text{tr}_N \left( \prod_{i=1}^{k-1} \phi_{k-i} T_{lm} \right) \text{tr}_N \left( \phi_{k+1} \phi_k T_{lm}^\dagger \right)$$

$$= \text{tr}_N \left( \phi_{k+1} \phi_k \prod_{i=1}^{k-1} \phi_{k-i} \right) = \text{tr}_N \left( \prod_{i=1}^{k+1} \phi_{k+2-i} \right), \quad (D.2)$$

thus (D.1) holds for $n = k + 1$. This completes the induction.

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