SPECTRAL GEOMETRY AND THE KAehler CONDITION FOR HERMITIAN MANIFOLDS WITH BOUNDARY

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Abstract. Let \((M, g, J)\) be a compact Hermitian manifold with a smooth boundary. Let \(\Delta_p, B\) and \(\bigwedge_p, B\) be the realizations of the real and complex Laplacians on \(p\) forms with either Dirichlet or Neumann boundary conditions. We generalize previous results in the closed setting to show that \((M, g, J)\) is Kaehler if and only if \(\text{Spec}(\Delta_p, B) = \text{Spec}(2 \bigwedge_p, B)\) for \(p = 0, 1\). We also give a characterization of manifolds with constant sectional curvature or constant Ricci tensor (in the real setting) and manifolds of constant holomorphic sectional curvature (in the complex setting) in terms of spectral geometry.

1. Introduction

The relationship between the spectrum of certain natural operators of Laplace type and the underlying geometry of a Riemannian manifold has been studied by many authors. Let \((M, g)\) be a compact Riemannian manifold with smooth boundary \(\partial M\). Let \(V\) be a smooth Hermitian vector bundle over \(M\) and let \(D\) be a formally self-adjoint operator of Laplace type acting on the space of smooth sections \(C^\infty(V)\). Let \(D_B\) denote the realization of \(D\) with respect to either the Dirichlet \((B = B_D)\) or the Neumann \((B = B_N)\) boundary operators. Then \(D_B\) is self-adjoint and has a complete discrete spectral resolution \(S(D_B) = \{\phi_n, \lambda_n\}\). The \(\phi_n \in C^\infty(V)\) form a complete orthonormal basis for \(L^2(V)\) such that \(D\phi_n = \lambda_n \phi_n\) and \(B\phi_n = 0\).

We let the spectrum \(\text{Spec}(D_B) = \{\lambda_n\}\) be the collection of eigenvalues. We repeat the eigenvalues according to multiplicity and order the eigenvalues so \(\lambda_1 \leq \lambda_2 \ldots\). For example, if \(M = [0, \pi]\) and if \(D = -\partial_x^2\) on \(C^\infty(M)\), then:

\[
S(D_{B_D}) = \left\{ \left( \frac{\sqrt{2}}{\pi} \sin(nx), n^2 \right) \right\}_{n=1}^\infty \\
\text{Spec}(D_{B_D}) = \{1, 4, 9, \ldots\} \\
S(D_{B_N}) = \left\{ \left( 1, \sqrt{\frac{1}{2}} \right) \right\} \cup \left\{ \left( \sqrt{\frac{2}{\pi}} \cos(nx), n^2 \right) \right\}_{n=1}^\infty \\
\text{Spec}(D_{B_N}) = \{0, 1, 4, 9, \ldots\}.
\]

Let \((M, g, J)\) be a Hermitian manifold of complex dimension \(\hat{m}\) and corresponding real dimension \(m = 2\hat{m}\); here \(J\) is an integrable almost complex structure which

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is unitary with respect to the Riemannian metric $g$. Let $\Lambda^n M$ be the bundle of complex $n$ forms on $M$. Let
\[
\Delta_n := d^*d + dd^* \quad \text{and} \quad \Box_n := \bar{\partial}\partial + \bar{\partial}\partial^* \quad \text{on} \quad C^\infty(\Lambda^n M)
\]
be real and complex form valued Laplacians. We further decompose
\[
\Box_n = \bigoplus_{p+q=n} \Box_{(p,q)} \quad \text{on} \quad C^\infty(\Lambda^{(p,q)} M).
\]

We introduce the associated Kaehler form $\Omega(X,Y) := g(X,JY)$. Extend the metric $g$ to be Hermitian on the complexified tangent bundle. Let $\nabla$ be the Levi-Civita connection. The following notions are equivalent and any defines the notion of a Kaehler manifold:

(1) For every $P$ in $M$, there exist local holomorphic coordinates so $dg(P) = 0$.
(2) We have $d\Omega = 0$.
(3) We have $\nabla J = 0$.

Let $I := \text{int}(\Omega)$. Let $\delta'$ be the formal adjoint of $\partial$ and $\delta''$ be the formal adjoint of $\bar{\partial}$. For a Kaehler manifold, one has the following relationships:

(1) $\partial I - I \bar{\partial} = \sqrt{-1} \delta'$.
(2) $\bar{\partial}\delta' + \delta\bar{\partial} = 0$ and $\partial\delta'' + \delta'' = 0$.
(3) $d^*d + dd^* = \delta' + \delta'' + \bar{\delta}' + \bar{\delta}''$.
(4) $\partial\delta I - \delta I \bar{\partial} = \sqrt{-1}(\delta' and \bar{\partial}\delta I - I \bar{\partial} = \sqrt{-1}\delta''$.
(5) $\sqrt{-1}(\delta' + \delta'' = \partial\delta I - \delta I \bar{\partial} + \bar{\partial}\delta I - I \bar{\partial}\delta$.
(6) $\partial\delta' + \delta\bar{\partial} = \bar{\delta}'' + \bar{\delta}''$.

The following well known result is now immediate.

**Theorem 1.1.** Let $(M,g,J)$ be a compact Kaehler manifold without boundary of complex dimension $m$. Then $\Delta = 2\Box$ and so $\text{Spec}(\Delta_p) = \text{Spec}(2\Box_p)$ for all $p$.

Conversely, one has the following result to T. Tsujishita (reported by Gilkey [3]):

**Theorem 1.2.** Let $(M,g,J)$ be a compact Hermitian manifold without boundary. If $\text{Spec}(\Delta_0) = \text{Spec}(2\Box_0)$ and if $\text{Spec}(\Delta_1) = \text{Spec}(2\Box_1)$, then $(M,g,J)$ is Kaehler.

Donnelly [2] established a similar characterization of the Kaehler property using the reduced complex Laplacian. Pak [12] extended these results to the context of almost isospectral manifolds.

Theorem 1.2 is sharp. We refer to Gilkey [7] for the proof of the following result:

**Theorem 1.3.** Let $ds^2 = dz_1 \odot d\bar{z}_1 + e^{\psi(z_1)}dz_2 \odot d\bar{z}_2 + e^{-\psi(z_1)}dz_3 \odot d\bar{z}_3$ be a Hermitian metric on the torus $M_3$ where $\psi(z_1)$ is an arbitrary smooth real valued function. Then $\Delta_0 = 2\Box_0$ but the metric is not Kaehler.

A Riemannian manifold of constant sectional curvature $c$ is said to be a space form; a Kaehler manifold of constant holomorphic sectional curvature $c$ is said to be a complex space form. Modulo rescaling, any space form is locally isometric to the unit sphere, to flat space, or to hyperbolic space. Similarly, modulo rescaling, any complex space form is locally isometric to complex projective space, to flat space, or to the negative curvature dual. Thus the geometries are very rigid in this context.

Patodi [11] established the following spectral characterization of space forms:

**Theorem 1.4.** Let $(M_1, g_1)$ be compact Riemannian manifolds without boundary. Assume that $\text{Spec}(\Delta_p, M_1) = \text{Spec}(\Delta_p, M_2)$ for $0 \leq p \leq 2$. Then:
The manifold $M_1$ has constant scalar curvature $c$ if and only if the manifold $M_2$ has constant scalar curvature $c$.

(2) The manifold $M_1$ is Einstein if and only if the manifold $M_2$ is Einstein.

(3) The manifold $M_1$ has constant sectional curvature $c$ if and only if the manifold $M_2$ has constant sectional curvature $c$.

Donnelly [3] and Gilkey and Sacks [9] extended Theorem 1.4 to the complex setting – see also related work by Friedland [5, 6,], C.C. Hsuing et. al. [10], and Pak [13].

**Theorem 1.5.** Let $(M_1, g_1, J_1)$ be compact Kaehler manifolds without boundary. Assume that $\text{Spec}(\Delta_{0, B}, M_1) = \text{Spec}(\Delta_{0, B}, M_2)$ for $0 \leq p \leq 2$ and $0 \leq q \leq 2$. Then the manifold $M_1$ has constant holomorphic sectional curvature $c$ if and only if the manifold $M_2$ has constant holomorphic sectional curvature $c$.

We can extend Theorem 1.2 to manifolds with boundary:

**Theorem 1.6.** Let $B$ denote either Dirichlet or Neumann boundary conditions. Let $(M, g, J)$ be a compact Hermitian manifold with smooth boundary $\partial M$. Assume that $\text{Spec}(\Delta_{0, B}) = \text{Spec}(2 \Delta_{0, B})$ and that $\text{Spec}(\Delta_{1, B}) = \text{Spec}(2 \Delta_{1, B})$. Then $(M, g, J)$ is Kaehler.

We can also generalize Theorems 1.4 and 1.5 to the category of manifolds with boundary under the additional technical hypothesis that the manifolds in question have constant scalar curvature.

**Theorem 1.7.** Let $B$ denote either Dirichlet or Neumann boundary conditions. Let $(M_i, g_i, J_i)$ be compact Riemannian manifolds with smooth boundaries $\partial M_i$ which have constant scalar curvatures $\tau_i$. Assume that $\text{Spec}(\Delta_{p, B}, M_1) = \text{Spec}(\Delta_{p, B}, M_2)$ for $0 \leq p \leq 2$. Then:

(1) $\tau_1 = \tau_2$.

(2) The manifold $M_1$ is Einstein if and only if the manifold $M_2$ is Einstein.

(3) The manifold $M_1$ has constant sectional curvature $c$ if and only if the manifold $M_2$ has constant sectional curvature $c$.

**Theorem 1.8.** Let $B$ denote either Dirichlet or Neumann boundary conditions. Let $(M_i, g_i, J_i)$ be compact Kaehler manifolds with smooth boundaries $\partial M_i$ which have constant scalar curvatures $\tau_i$. Assume that $\text{Spec}(\Box_{p, q}, M_1) = \text{Spec}(\Box_{p, q}, M_2)$ for $0 \leq p \leq 2, 0 \leq q \leq 2$. Then the manifold $M_1$ has constant holomorphic sectional curvature $c$ if and only if the manifold $M_2$ has constant holomorphic sectional curvature $c$.

Here is a brief outline to the remainder of this paper. In Section 2 we review some previous results concerning the heat trace asymptotics. In Section 3 we complete the proof of Theorem 1.6 and in Section 4 we complete the proof of Theorems 1.7 and 1.8. In Remark 4.1 we extend Theorems 1.7 and 1.8 to more general boundary conditions of Robin type where the auxiliary endomorphism $S$ is ‘universal’ in a certain sense.

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2. Heat trace asymptotics

Let $M$ be a compact Riemannian manifold of real dimension $m$ with smooth boundary $\partial M$ and let $D_B$ be the realization of an operator of Laplace type on $M$ with respect to either Dirichlet or Neumann boundary conditions. Let $e^{-tD_B}$ be the fundamental solution of the heat equation. This operator is of trace class and as $t \downarrow 0$ there is a complete asymptotic expansion with locally computable coefficients in the form:

$$\text{Tr} e^{-tD_B} \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(D, B).$$

To study the heat trace coefficients $a_n(D, B)$, we must introduce a bit of additional notation. There is a canonically defined connection $\nabla = \nabla(D)$ and a canonically defined endomorphism $E = E(D)$ so that

$$D = -(\text{Tr}(\nabla^2) + E).$$

Let indices $i, j, k$ range from 1 to $m$ and index a local orthonormal frame \{\(e_1, \ldots, e_m\)\} for $TM$. Let indices $a, b, c$ range from 1 to $m - 1$ and index a local orthonormal frame \{\(e_1, \ldots, e_{m-1}\)\} for $T \partial M$; on $\partial M$, we let $e_m$ be the inward unit normal vector field. Let $\Omega$ be the curvature of $\nabla$, let $\tau := R_{ijkl}$ be the normalized scalar curvature, let $\rho_{ij} := R_{ikkj}$ be the Ricci tensor, and let $L_{ab}$ be the second fundamental form. We adopt the Einstein convention and sum over repeated indices. Let ‘;’ denote multiple covariant differentiation. We refer to [1] for the proof of the following result:

**Theorem 2.1.** Let $D$ be an operator of Laplace type on the space of sections $C^\infty(V)$ to a vector bundle $V$ over a compact manifold $M$ with smooth boundary $\partial M$. Let $I$ be the identity endomorphism of $V$. With Dirichlet boundary conditions, we have:

1. $a_0(D, B_D) = (4\pi)^{-m/2} \int_M \text{Tr}\{I\}$.
2. $a_1(D, B_D) = -(4\pi)^{-(m-1)/2} \frac{4}{\pi} \int_{\partial M} \text{Tr}\{I\}$.
3. $a_2(D, B_D) = (4\pi)^{-m/2} \frac{1}{6} \int_M \text{Tr}\{6E + \tau I\} + (4\pi)^{-m/2} \frac{1}{6} \int_{\partial M} \text{Tr}\{2L_{aaI}\}$.
4. $a_3(D, B_D) = \frac{1}{816} (4\pi)^{-(m-1)/2} \int_M \text{Tr}\{96E + (16\tau + 8R_{amam} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab})I\}$.
5. $a_4(D, B_D) = (4\pi)^{-m/2} \frac{1}{2430} \int_M \text{Tr}\{60E_{kk} + 60\tau E + 180E^2 + 30\tau^2 + (12\tau_{kk} + 5\tau^2 - 2|\rho|^2 + 2|\nabla R|^2)I + (4\pi)^{-m/2} \frac{1}{360} \int_{\partial M} \text{Tr}\{-120E_{mm} + 120EL_{aa} + (-18\tau_{mm} + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambn}L_{ab} + 4R_{abcb}L_{ac} + \frac{40}{27} L_{aa}L_{bb}L_{cc} - \frac{88}{27} L_{ab}L_{ab}L_{cc} + \frac{320}{27} L_{ab}L_{bc}L_{ac})I\}}.$

Let $\nabla_m$ denote the covariant derivative with respect to $e_m$ on $\partial M$. Let $S$ be an auxiliary endomorphism of $V|_{\partial M}$. The Robin boundary operator is then given by:

$$B_S := (\nabla_m + S \phi)|_{\partial M}.$$ We take $S = 0$ to define Neumann boundary conditions. Again, we refer to [1] for the proof of the following result:

**Theorem 2.2.** With Robin boundary conditions, we have:

1. $a_0(D, B_S) = (4\pi)^{-m/2} \int_M \text{Tr}\{I\}$.
2. $a_1(D, B_S) = (4\pi)^{(1-m)/2} \frac{4}{\pi} \int_{\partial M} \text{Tr}\{I\}$.
3. $a_2(D, B_S) = (4\pi)^{-m/2} \frac{1}{6} \int_M \text{Tr}\{6E + \tau I\} + (4\pi)^{-m/2} \frac{1}{6} \int_{\partial M} \text{Tr}\{2L_{aaI} + 12S\}.$
Since \( a \) (3.2) Equations (3.1) and (3.2) then imply Dirichlet boundary conditions. By Theorems 2.1 and 2.2, we use this relation and the relation \( a \)  

(3.1) Let \( \kappa := \tau \) be the geodesic curvature of the boundary. We may then use Theorem 2.1 to extend results of Gilkey [7] to see: 

\[
\begin{align*}
\phi_2(2 \odot_0) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 2\kappa + \int_M (\tau + 3K_2 + 3K_3) \right\} \\
\phi_2(\Delta_0) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 2\kappa + \int_M \tau \right\} \\
\phi_2(2 \odot_1,0) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 2\hat{m}\kappa + (\hat{m} - 3) \int_M (\tau + 3K_2 + 3K_3) \right. \\
&\quad + \left. \int_M (-6K_1 + 6K_2 + 3K_3) \right\} \\
\phi_2(2 \odot_0,1) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 2\hat{m}\kappa + (\hat{m} - 3) \int_M (\tau + 3K_2 + 3K_3) \right. \\
&\quad + \left. \int_M (6K_1 + 6K_2 + 3K_3) \right\} \\
\phi_2(\Delta_1) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 4\hat{m}\kappa + 2(\hat{m} - 3) \int_M \tau \right. \\
\end{align*}
\]

Since \( \phi_2(\Delta_0) = \phi_2(2 \odot_0) \), we have

\[
\int_M (3K_2 + 3K_3) = 0.
\]

We use this relation and the relation \( \phi_2(\Delta_1) = \phi_2(2 \odot_1) \) to see

\[
\int_M (6K_2 + 3K_3) = 0.
\]

Equations 3.1 and 3.2 then imply \( \int_M K_2 = 0 \) and hence \( M \) is Kaehler.

If \( \hat{m} = 2 \), we have the formulae:

\[
\begin{align*}
\phi_2(2 \odot_0) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 2\kappa + \int_M (\tau + 3K_2) \right\} \\
\phi_2(\Delta_0) &= (4\pi)^{-\hat{m}} \left\{ \int_{\partial M} 2\kappa + \int_M \tau \right\}.
\end{align*}
\]

Since \( \phi_2(\Delta_0) = \phi_2(2 \odot_0) \), \( \int_M K_2 = 0 \) and \( M \) is Kaehler; the condition relating \( \Delta_1 \) and \( 2 \odot_1 \) is not necessary in this instance. Finally, if \( \hat{m} = 1 \), then \( M \) is automatically Kaehler. \( \square \)

4. PROOF OF THEOREMS 1.7 AND 1.8

We set \( S = 0 \). Let \( \varepsilon := \frac{1}{4} \) for Neumann boundary conditions and \( \varepsilon := -\frac{1}{4} \) for Dirichlet boundary conditions. By Theorems 2.1 and 2.2,

\[
\text{Tr}_{L^2}(e^{-t\Delta_0,\eta}) = (4\pi t)^{-m/2} \{ \text{Vol}(M) + \varepsilon \sqrt{t} \text{Vol}(\partial M) + O(t) \}
\]

so

\[
\begin{align*}
\text{Vol}(M_1) &= \text{Vol}(M_2), \quad \text{Vol}(\partial M_1) = \text{Vol}(\partial M_2), \quad \text{and} \\
\dim_{\mathbb{R}}(M_1) &= \dim_{\mathbb{R}}(M_2).
\end{align*}
\]
We set \( m := \dim_{\mathbb{R}}(M) \) to this common value and compute:

\[
a_2(\Delta_{0,B}, M_i) = (4\pi)^{-m/2} \frac{1}{\beta}(\int_M \tau_i + \int_{\partial M_i} 2\kappa_i) \\
a_2(\Delta_{1,B}, M_i) = (4\pi)^{-m/2} \frac{1}{\beta}(\int_M (m-6)\tau_i + \int_{\partial M_i} 2m\kappa_i).
\]

We may then establish assertion (1) by computing:

\[
\tau_1 = (4\pi)^{m/2} \text{Vol}(M_1)^{-1}\{ma_2(\Delta_{0,B}, M_1) - a_2(\Delta_{1,B}, M_1)\} \\
= (4\pi)^{m/2} \text{Vol}(M_2)^{-1}\{ma_2(\Delta_{0,B}, M_2) - a_2(\Delta_{1,B}, M_2)\} \\
= \tau_2.
\]

For subsequent use, we compute similarly that:

\[
\int_{\partial M_i} \kappa_1 = 3(4\pi)^{m/2}a_2(\Delta_{0,B}, M_1) - \frac{1}{2} \int_M \tau_1 \\
= 3(4\pi)^{m/2}a_2(\Delta_{0,B}, M_2) - \frac{1}{2} \int_M \tau_2 \\
= \int_{\partial M_2} \kappa_2.
\]

The interior integrands defining \( a_4(\Delta_p) \) have been determined by Patodi. Motivated by his work, we introduce constants:

\[
c_{m,p}^1 = \frac{1}{12} \frac{m!}{p!(m-p)!} - \frac{1}{6} \frac{(m-2)!}{(p-1)!(m-p-1)!}, \\
c_{m,p}^2 = -\frac{1}{180} \frac{m!}{p!(m-p)!} + \frac{1}{2} \frac{(m-2)!}{(p-1)!(m-p-1)!}, \\
c_{m,p}^3 = -\frac{1}{180} \frac{m!}{p!(m-p)!} - \frac{1}{12} \frac{(m-2)!}{(p-1)!(m-p-1)!}, \\
c_{m,p}^4 = \frac{1}{30} \frac{m!}{p!(m-p)!} - \frac{1}{6} \frac{(m-2)!}{(p-1)!(m-p-1)!}.
\]

The work of Patodi then shows if \( \partial M \) is empty that:

\[
a_4(\Delta_p) = (4\pi)^{-m/2} \int_M \{c_{m,p}^1 \tau^2 + c_{m,p}^2 |\rho|^2 + c_{m,p}^3 |R|^2 + c_{m,p}^4 \tau_{ii}\}.
\]

To simplify the notation, we introduce reduced invariants

\[
\tilde{a}_n(\Delta_p, B) := a_n(\Delta_p, B) - \frac{m!}{p!(m-p)!} a_n(\Delta_0, B).
\]

The terms

\[
\{R_{ab}L_{bc}, R_{ab}L_{bc}, R_{abc}L_{ac}, L_{ab}L_{bc}, L_{ab}L_{bc}, L_{ab}L_{bc}L_{ac}\}
\]

appearing in Theorem 2.1 are all multiplied by \( \text{Tr}(I_{A'}) = \frac{m!}{p!(m-p)!} \). Thus they do not appear in \( \tilde{a}_4(\Delta_p, B) \); only the terms involving \( E(\Delta_p) \) survive in the boundary contributions. One can use the Weitzenböck formula to see that

\[
\text{Tr}(E(\Delta_p)) = -\frac{m!}{p!(m-p)!} \frac{(m-2)!}{(p-1)!(m-p-1)!} \tau.
\]

Using equation (4.3), we see there exist universal constants so

\[
\tilde{a}_4(\Delta_p, B) = (4\pi)^{-m/2} \int_M \{c_{m,p}^1 \tau^2 + c_{m,p}^2 |\rho|^2 + c_{m,p}^3 |R|^2 + c_{m,p}^4 \tau_{kk}\} + (4\pi)^{-m/2} \int_{\partial M} \{c_{m,p}^5 \tau_{\kappa} + \tau_{m}^6 \tau_{m}\},
\]
where, by Patodi’s result, we have:

\begin{align}
\hat{c}^1_{m,p} &:= c^1_{m,p} - \frac{m!}{p!(m-p)!} c_{m,0} = - \frac{1}{6} \left( \frac{m-2}{(m-1)!} \right) + \frac{1}{2} \left( \frac{m-4}{(m-2)!} \right), \\
\hat{c}^2_{m,p} &:= c^2_{m,p} - \frac{m!}{p!(m-p)!} c_{m,0} = \frac{1}{2} \left( \frac{m-2}{(m-1)!} \right) - \frac{1}{2} \left( \frac{m-4}{(m-2)!} \right), \\
\hat{c}^3_{m,p} &:= c^3_{m,p} - \frac{m!}{p!(m-p)!} c_{m,0} = - \frac{1}{12} \left( \frac{m-2}{(m-1)!} \right) + \frac{1}{2} \left( \frac{m-4}{(m-2)!} \right), \\
\hat{c}^4_{m,p} &:= c^4_{m,p} - \frac{m!}{p!(m-p)!} c_{m,0} = - \frac{1}{6} \left( \frac{m-2}{(m-1)!} \right).
\end{align}

For \( p = 1, 2 \), we have, by assumption, that:

\begin{align}
\hat{a}_4(\Delta_{p,B}, M_1) &= a_4(\Delta_{p,B}, M_1) - \frac{m!}{p!(m-p)!} a_4(\Delta_{0,B}, M_1) \\
&= \hat{a}_4(\Delta_{p,B}, M_2) = a_4(\Delta_{p,B}, M_2) - \frac{m!}{p!(m-p)!} a_4(\Delta_{0,B}, M_2).
\end{align}

By equation (4.4), \( \int_{\partial M_1} \kappa_1 = \int_{\partial M_2} \kappa_2 \). By assertion (1), \( \tau_1 = \tau_2 \). Since the scalar curvature is constant, \( \tau_m = 0 \). Thus since Vol(\( \partial M_1 \)) = Vol(\( \partial M_2 \)), the boundary integrals are equal. Furthermore, since Vol(\( M_1 \)) = Vol(\( M_2 \)), the interior integrals of \( \tau^2 \) are equal. Since \( \tau_{ii} = 0 \), we have

\begin{align}
\int_{M_1} (\hat{c}^2_{m,p} |p_1|^2 + \hat{c}^3_{m,p} |R_1|^2) &= \int_{M_2} (\hat{c}^2_{m,p} |p_2|^2 + \hat{c}^3_{m,p} |R_2|^2)
\end{align}

for \( n = 1, 2 \); these two equations are independent since, by display (4.1),

\begin{align}
\det \begin{pmatrix} \hat{c}^2_{m,1} & \hat{c}^3_{m,1} \\ \hat{c}^2_{m,2} & \hat{c}^3_{m,2} \end{pmatrix} = \det \left( \frac{1}{2} - \frac{1}{12} \right) = \frac{1}{4} - \frac{1}{6} \neq 0.
\end{align}

Consequently

\begin{align}
\int_{M_1} |p_1|^2 &= \int_{M_2} |p_2|^2, \quad \text{and} \quad \int_{M_1} |R_1|^2 = \int_{M_2} |R_2|^2.
\end{align}

A manifold \( M \) has constant sectional curvature \( c \) if and only if

\[ 0 = \int_M |R_{ijkl} - c(\delta_i \delta_j - \delta_i \delta_k)|^2 = \int_M (|R|^2 - 4c\tau + c^2 \varepsilon_m) \]

where \( \varepsilon_m := |\delta_i \delta_j - \delta_i \delta_k|^2 \) is polynomial in \( m \). We use equation (4.1), equation (4.7), and assertion (1) to complete the proof of Theorem 1.7 (3) by computing:

\[ \int_{M_1} (|R_1|^2 - 4c\tau_1 + c^2 \varepsilon_m) = \int_{M_2} (|R_2|^2 - 4c\tau_2 + c^2 \varepsilon_m). \]

Note that \( M \) is Einstein if and only if there is a constant \( c \) so

\[ 0 = |p_{ij} - c\delta_{ij}|^2 = |p|^2 - 2c\tau + mc^2. \]

Thus Theorem 1.7 (2) can be established by verifying that:

\[ 0 = \int_{M_1} (|p_1|^2 - 2c\tau_1 + mc^2) = \int_{M_2} (|p_2|^2 - 2c\tau_2 + mc^2). \]

As a similar argument based on the results of 3.9 establishes Theorem 1.8, we shall omit the details of the proof of Theorem 1.8 in the interests of brevity.

**Remark 4.1.** We can generalize Theorems 1.7 and 1.8 to the context of Robin boundary conditions as follows. One could take \( S = c_1 + c_2 \kappa \); the same cancellation argument as that given above to establish equation (4.6) shows the additional boundary terms cancel off for the reduced invariant. What is crucial is that the boundary condition be natural and universal in the context in which we are working.
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