ON TROPICAL DUALITIES IN CLUSTER ALGEBRAS

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Abstract. We study two families of integer vectors playing a crucial part in the structural theory of cluster algebras: the $g$-vectors parameterizing cluster variables, and the $c$-vectors parameterizing the coefficients. We prove two identities relating these vectors to each other. The proofs depend on the sign-coherence assumption for $c$-vectors that still remains unproved in full generality.

1. Introduction

In the theory of cluster algebras, the central role is played by two families of elements obtained by certain algebraic recurrences: cluster variables and coefficients (V. Fock and A. Goncharov refer to them as $a$-coordinates, and $x$-coordinates, respectively). In this paper we study certain tropical limits of these elements: two families of integer vectors ($g$-vectors and $c$-vectors, respectively) introduced in [5]. Despite the fact that these two families are given by very different piecewise-polynomial recurrences, it turns out that they are closely related to each other. We refer to the identities involving $g$-vectors and $c$-vectors as tropical dualities. The main result of this paper is Theorem 1.2 containing two such dualities.

We start by briefly recalling the formalism of cluster algebras (more details can be found in [5]). In the heart of this formalism there are several discrete dynamical systems on a $n$-regular tree given by birational recurrences and their “tropical” versions. More precisely, we fix a positive integer $n$, and denote by $T_n$ an $n$-regular tree whose edges are labeled by the numbers $1, \ldots, n$, so that the $n$ edges emanating from each vertex receive different labels. We write $t \rightarrow t'$ to indicate that vertices $t, t' \in T_n$ are joined by an edge labeled by $k$.

We need a little algebraic preparation. We call a semifield an abelian multiplicative group $P$, supplied with the addition operation $\otimes$, which is commutative, associative, and distributive with respect to multiplication. Let $\mathbb{Z}P$ denote the integer group ring of (the multiplicative group of) $P$ (note that its definition ignores the addition $\oplus$). It is easy to show that (the multiplicative group of) $P$ is torsion-free (see the remark before Definition 5.3 in [4]), hence $\mathbb{Z}P$ is a domain, hence it has the field of fractions $\mathbb{Q}(P)$.

We call an ambient field a field $F$ isomorphic to the field of rational functions in $n$ indeterminates with the coefficients from $\mathbb{Q}(P)$.

Fix a semifield $P$ and an ambient field $F$. Following [5], we call a (labeled) seed a triple $(x, y, B)$, where
• $B = (b_{ij})$ is an $n \times n$ integer matrix which is skew-symmetrizable, that is, its transpose $B^T$ is equal to $-DBD^{-1}$ for some diagonal matrix $D$ with positive integer diagonal entries $d_1, \ldots, d_n$;
• $y = (y_1, \ldots, y_n)$ is an $n$-tuple of elements of $\mathbb{P}$, and
• $x = (x_1, \ldots, x_n)$ is an $n$-tuple of elements of $\mathcal{F}$ forming a free generating set, that is, being algebraically independent and such that $\mathcal{F} = \mathbb{Q}(\mathbb{P})(x_1, \ldots, x_n)$.

We refer to $B$ as the exchange matrix of a seed, $y$ as the coefficient tuple, and $x$ as the cluster.

Throughout the paper we use the notation $[b]_+ = \max(b, 0)$. For $k = 1, \ldots, n$, the seed mutation $\mu_k$ transforms $(x, y, B)$ into the labeled seed $(x', y', B')$ defined as follows:

• The entries of the exchange matrix $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+ & \text{otherwise.} \end{cases}$$

• The coefficient tuple $y' = (y'_1, \ldots, y'_n)$ is given by

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k. \end{cases}$$

• The cluster $x' = (x'_1, \ldots, x'_n)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange relation

$$x'_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{-[b_{ik}]_+}}{(y_k \oplus 1)x_k}.$$

It is easy to see that $B'$ is skew-symmetrizable (with the same choice of $D$), implying that $(x', y', B')$ is indeed a seed. Furthermore, the seed mutation $\mu_k$ is involutive, that is, it transforms $(x', y', B')$ into the original seed $(x, y, B)$. We shall also use the notation $B' = \mu_k(B)$ (resp. $(y', B') = \mu_k(y, B)$) and call the transformation $B \mapsto B'$ the matrix mutation (resp. $(y, B) \mapsto (y', B')$ the $Y$-seed mutation).

A seed pattern is an assignment of a seed $(x_t, y_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \to t'$ are obtained from each other by the seed mutation $\mu_k$. We write:

$$x_t = (x_{1:t}, \ldots, x_{n:t}), \quad y_t = (y_{1:t}, \ldots, y_{n:t}), \quad B_t = (b_{ij:t}).$$

We also refer to a family $(y_t, B_t)|_{t \in \mathbb{T}_n}$ as a $Y$-seed pattern, and a family $(B_t)|_{t \in \mathbb{T}_n}$ as an exchange matrix pattern.

It is convenient to fix a vertex (root) $t_0 \in \mathbb{T}_n$. Then a seed pattern is uniquely determined by a seed at $t_0$, which can be chosen arbitrarily. We will just write this initial seed as $(x, y, B)$, with

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad B = (b_{ij}).$$

The cluster algebra associated with a seed $(x, y, B)$ (or rather with the corresponding seed pattern) is the $\mathbb{Z}\mathbb{P}$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster
variables $x_{j:t}$. One of the central problems in the theory of cluster algebras is to find explicit expressions for all $x_{j:t}$ and $y_{j:t}$ as rational functions in $x_1, \ldots, x_n; y_1, \ldots, y_n$. A step in this direction was made in [5], where the following was proved.

**Proposition 1.1.** [5] Proposition 3.13, Corollary 6.3] Every pair $(B; t_0)$ gives rise to a family of polynomials $F_{j:t} = F_{j:t}^{B; t_0} \in \mathbb{Z}[u_1, \ldots, u_n]$ and two families of integer vectors $c_{j:t} = c_{j:t}^{B; t_0} = (c_{1j:t}, \ldots, c_{nj:t}) \in \mathbb{Z}^n$ and $g_{j:t} = g_{j:t}^{B; t_0} = (g_{1j:t}, \ldots, g_{nj:t}) \in \mathbb{Z}^n$ (where $j \in \{1, \ldots, n\}$ and $t \in \mathbb{T}_n$) with the following properties:

1. Each $F_{j:t}$ is not divisible by any $u_i$, and can be expressed as a ratio of two polynomials in $u_1, \ldots, u_n$ with positive integer coefficients, thus can be evaluated in every semifield $\mathbb{P}$.
2. For any $j$ and $t$, we have
   \begin{equation}
   y_{j:t} = y_{1j:t}^{c_{1j:t}} \cdots y_{nj:t}^{c_{nj:t}} \prod_i F_{i:t} | \mathbb{P}(y_1, \ldots, y_n)^{b_{i:j}}. \tag{1.6}
   \end{equation}
3. For any $j$ and $t$, we have
   \begin{equation}
   x_{j:t} = x_{1j:t}^{a_{1j:t}} \cdots x_{nj:t}^{a_{nj:t}} \frac{F_{j:t} | \mathbb{P}(y_1, \ldots, y_n)}{F_{j:t} | \mathbb{P}(y_1, \ldots, y_n)}, \tag{1.7}
   \end{equation}
   where the elements $\hat{y}_j$ are given by
   \[ \hat{y}_j = y_j \prod_i x_i^{b_{i:j}}. \]

Following [5], we refer to $g_{j:t}$ as the g-vector of a cluster variable $x_{j:t}$. It is instructive to view the g-vectors as some kind of discrete (or tropical) limits of cluster variables. In fact, as explained in [5] Remark 7.15], these vectors are expected to provide a parametrization of cluster variables (and more generally, cluster monomials) by “Langlands dual tropical Y-seed patterns” in accordance with a conjecture by V. Fock and A. Goncharov [3, Conjecture 5.1]. Some properties of g-vectors and F-polynomials were established in [5] but several basic properties still remain conjectural.

Comparing (1.6) and (1.7), it is natural to view the c-vector $c_{j:t}$ as a tropical limit of a coefficient $y_{j:t}$. These vectors also made their appearance in [5], although not under this name, and only as a tool for studying g-vectors and F-polynomials. In this note we focus on the properties of c-vectors and their relationships with g-vectors.

The following (unfortunately, still conjectural) sign-coherence property of c-vectors is crucial for our analysis:

1. Each vector $c_{j:t}$ has either all entries nonnegative or all entries nonpositive.

As shown in [5] Proposition 5.6], (1.8) is equivalent to the following conjecture made in [5] Conjecture 5.4]:

1. Each polynomial $F_{j:t}(u_1, \ldots, u_n)$ has constant term 1.

This conjecture (hence the property (1.8)) was proved in [2] for the case of skew-symmetric exchange matrices, using quivers with potentials and their representations (two different proofs were recently given in [6, 8]).
We denote by $C_t^{B \cdot t_0}$ (resp. $G_t^{B \cdot t_0}$) the integer matrix with columns $c_{1; \ell}, \ldots, c_{n; \ell}$ (resp. with columns $g_{1; \ell}, \ldots, g_{n; \ell}$), where $B$ is the exchange matrix at $t_0$. In particular, we have

\begin{equation} \label{eq:identity}
C_t^{B \cdot t_0} = G_t^{B \cdot t_0} = I \quad \text{(the identity matrix)}.
\end{equation}

The main results of this paper are the following two identities.

**Theorem 1.2.** Under the assumption \((1.8)\), for any skew-symmetrizable exchange matrix $B$, and any $t_0, t \in \mathbb{T}_n$, we have

\begin{equation} \label{eq:relation}
(G_t^{B \cdot t_0})^T = (C_t^{B^T \cdot t_0})^{-1},
\end{equation}

and

\begin{equation} \label{eq:relation2}
C_t^{B \cdot t_0} = (C_{t_0}^{B_{t}^T})^{-1},
\end{equation}

where $t \mapsto B_t$ is the exchange matrix pattern on $\mathbb{T}_n$ such that $B_{t_0} = B$, and $B^T$ stands for the transpose matrix of $B$.

Note that in the case when $B$ is skew-symmetric, \((1.11)\) takes the form $(G_t^{B \cdot t_0})^T = (C_t^{B \cdot t_0})^{-1}$. In this case it was proved in \cite{2} Proposition 4.1.

One can show that identities \((1.11)\) and \((1.12)\) imply most of the conjectures made in \cite{5} and recast in \cite{2}, namely Conjectures 1.1 - 1.4 and 1.6 from \cite{2}. For instance, by combining them, we obtain the equality

\begin{equation} \label{eq:relation3}
(G_t^{B \cdot t_0})^T = C_{t_0}^{B^T \cdot t},
\end{equation}

which can be shown to imply \cite{2} Conjecture 1.6. More details will be given in Section 4.

In contrast with \cite{2}, the proofs of \((1.11)\) and \((1.12)\) given below do not use any categorical interpretation of $g$- or $c$-vectors but are based on the analysis of recurrences satisfied by them. To describe these recurrences we need to develop some matrix formalism.

We extend the notation $[b]_+ = \max(0, b)$ to matrices, writing $[B]_+$ for the matrix obtained from $B$ by applying the operation $b \mapsto [b]_+$ to all entries of $B$. For a matrix index $k$, we denote by $B^{\bullet k}$ the matrix obtained from $B$ by replacing all entries outside of the $k$-th column with zeros; the matrix $B^{k \bullet}$ is defined similarly using the $k$-th row instead of the column. Note that the operations $B \mapsto [B]_+$ and $B \mapsto B^{\bullet k}$ commute with each other, making the notation $[B]_+^{\bullet k}$ (and $[B]^{k \bullet}_+$) unambiguous.

Using this formalism, we can rephrase the sign-coherence condition \((1.8)\) for a matrix $C = C_t^{B \cdot t_0}$ as follows:

\begin{equation} \label{eq:condition}
\text{For every } j, \text{ there exists the sign } \varepsilon_j(C) = \pm 1 \text{ such that } [-\varepsilon_j(C)C]_+^{\bullet j} = 0.
\end{equation}

Let $J_k$ denote the diagonal matrix obtained from the identity matrix by replacing the $(k, k)$-entry with $-1$. We deduce \((1.11)\) from the following proposition.

**Proposition 1.3.** Suppose $t \underset{\ell}{\longrightarrow} t'$ in $\mathbb{T}_n$, and let $C = C_t^{B \cdot t_0}$, $G = G_t^{B \cdot t_0}$, $C' = C_{t'}^{B \cdot t_0}$, and $G' = G_{t'}^{B \cdot t_0}$. Then, under the assumption that $C$ satisfies \((1.14)\), we have

\begin{equation} \label{eq:prop}
C' = C(J_\ell + [\varepsilon_\ell(C)B_{t_0}]_+^{\bullet}), \quad G' = G(J_\ell + [-\varepsilon_\ell(C)B_{t_0}]_+^{\bullet}).
\end{equation}
As for (1.12), we prove it together with the following proposition that provides a recurrence relation for the matrices $C_{t}^{B_{t}B_{t0}}$ “at the opposite end.” We still assume that every matrix of the form $C = C_{t}^{B_{t}B_{t0}}$ satisfies (1.8) or equivalently (1.14).

**Proposition 1.4.** Suppose $t_{0} \frac{k}{2} t_{1}$ in $\mathbb{T}_{n}$, and let $B_{1} = \mu_{k}(B)$. For $t \in \mathbb{T}_{n}$, abbreviate $C_{t} = C_{t}^{B_{t}B_{t0}}$, and $C_{t}' = C_{t}^{B_{t1}B_{t1}}$. Then we have

$$
C_{t}' = (J_{k} + [-\varepsilon_{k}(C_{t}^{B_{t1}B_{t1}})B]_{+}^{\epsilon})C_{t}.
$$

Proposition 1.3 and the identity (1.11) will be proved in Section 2, while Proposition 1.4 and (1.12) will be proved in Section 3. In the last section we give some corollaries.

## 2. Proofs of Proposition 1.3 and Identity (1.11)

**Proof of Proposition 1.3.** We start with the following identity involving the function $[b]_{+} = \max(b, 0)$: for any real numbers $b$ and $c$, and a sign $\varepsilon = \pm 1$, we have

$$
[c]_{+}[b]_{+} - [-c]_{+}[-b]_{+} = c[\varepsilon b]_{+} + b[-\varepsilon c]_{+}.
$$

This is a consequence of an obvious identity

$$
[b]_{+} - [-b]_{+} = b.
$$

Indeed, we have

$$
c[b]_{+} + b[-c]_{+} = ([c]_{+} - [-c]_{+})[b]_{+} + ([b]_{+} - [-b]_{+})[-c]_{+} = [c][b]_{+} - [-c][b]_{+},
$$

proving (2.1) for $\varepsilon = 1$; the case $\varepsilon = -1$ is proved similarly.

The first equality in (1.15) was essentially shown in [11 (3.2)] (in a different notation). For the convenience of the reader, we reproduce the argument here. Recall from [5, Remarks 3.2, 3.14] that the relationship between $C = C_{t}^{B_{t}B_{t0}}$ and $C' = C_{t'}^{B_{t}B_{t0}}$ can be described as follows: if $\tilde{B}_{t}$ is a $2n \times n$ integer matrix with the top $n \times n$ block $B_{t}$ and the bottom block $C_{t}^{B_{t0}}$, then $\tilde{B}_{t'}$ is obtained from $\tilde{B}_{t}$ by the matrix mutation $\mu_{\ell}$ given by the same formula as in (1.11). Thus, we get

$$
c_{ij}' = \begin{cases} 
- c_{ij} & \text{if } j = \ell; \\
 c_{ij} + [c_{ij}]_{+}[b_{ij}]_{+} - [-c_{ij}]_{+}[-b_{ij}]_{+} & \text{otherwise.}
\end{cases}
$$

Using (2.1) and rewriting the resulting formula in the matrix form, we get

$$
C' = C(J_{\ell} + [\varepsilon B_{t}]_{+}^{\epsilon}) + [-\varepsilon C]_{+}^{\epsilon}B_{t}
$$

for any choice of the sign $\varepsilon$. Setting $\varepsilon = \varepsilon_{t}(C)$ and remembering (1.14), we see that the second term in (2.4) disappears, implying the desired equality.

To prove the second equality in (1.15), we rewrite [5, (6.12)-(6.13)] in the matrix form as follows:

$$
G' = G(J_{\ell} + [-\varepsilon B_{t}]_{+}^{\epsilon}) - B[-\varepsilon C]_{+}^{\epsilon}
$$

(again for any choice of the sign $\varepsilon$). Again setting $\varepsilon = \varepsilon_{t}(C)$ makes the second term in (2.5) disappear, completing the proof of Proposition 1.3. □
Proof of identity (1.11). We prove (1.11) by induction on the distance between \( t \) and \( t_0 \) in the tree \( T_n \). For \( t = t_0 \), the claim follows from (1.10). It remains to show that if (1.11) holds for some \( t \in T_n \) then it also holds for \( t' \in T_n \) such that \( t \rightarrow t' \). This implication is a consequence of (1.15) combined with the following three simple observations:

- For every integer square matrix \( B \) and an index \( \ell \) such that \( b_{\ell t} = 0 \), we have
  \[
  (J_{\ell} + B^{\bullet \bullet})^T = J_{\ell} + (B^T)^{\bullet \bullet}, \quad (J_{\ell} + B^{\bullet \bullet})^{-1} = J_{\ell} + B^{\bullet \bullet}.
  \]

- The replacement of the initial exchange matrix \( B_{t_0} = B \) with \(-B^T\) leads to the replacement of each \( B_t \) with \(-B_t^T\), and we have
  \[
  C_{t}^{-B^T; t_0} = DC_{t}^{B; t_0} D^{-1},
  \]
  where \( D \) is a diagonal \( n \times n \) matrix with positive integer entries such that \(-B^T = DBD^{-1}\).

- In particular, we have \( \varepsilon_{\ell}(C_{t}^{-B^T; t_0}) = \varepsilon_{\ell}(C_{t}^{B; t_0}) \) for every matrix index \( \ell \).

\[\Box\]

Remark 2.1. As in Proposition 1.3 let us abbreviate \( C = C_{t}^{B; t_0} \), and \( G = G_{t}^{B; t_0} \). In the equality (2.5), the fact that two choices of the sign give the same answer implies the following identity for \( G \) (shown already in [5, (6.14)]):

\[
(2.8) \quad GB_t = BC.
\]

Combining this identity with (1.11) and (2.7), we obtain the following:

\[
(2.9) \quad DB_t = C^T \cdot DB \cdot C,
\]

where the diagonal matrix \( D \) is as in (2.7). This shows that the exchange matrix \( B_t \) at every vertex \( t \in T_n \) (that is, the top part of the \( 2n \times n \) matrix \( B_t \)) is determined by the initial exchange matrix \( B \) and the matrix \( C \) which is the bottom part of \( B_t \).

3. Proofs of Proposition 1.4 and identity (1.12)

We prove (1.16) and (1.12) by a simultaneous induction. To prepare the ground for it, we note that, under the notation and assumptions of Proposition 1.4, we have \( B_t^{\bullet \bullet} = -B_t^{\bullet \bullet} \) and \( (C_{t_1}^{-B_t; t})^{\bullet k} = -(C_{t_0}^{-B_t; t})^{\bullet k} \); hence \( \varepsilon_{k}(C_{t_1}^{-B_t; t}) = -\varepsilon_{k}(C_{t_0}^{-B_t; t}) \). In view of the second equality in (2.6), this allows us to interchange \( t_0 \) and \( t_1 \) in (1.16). Thus, it suffices to prove (1.16) under the assumption that

\[
(3.1) \quad t_0 \text{ belongs to the (unique) path between } t \text{ and } t_1 \text{ in } T_n.
\]

In other words, (3.1) means that \( d(t, t_1) = d(t, t_0) + 1 \), where \( d(t, t_0) \) denotes the distance between \( t \) and \( t_0 \) in \( T_n \).

For \( d \geq 0 \), we denote by \((I_{d})\) and \((II_{d})\) the following two statements:

(I_{d}) The equality (1.12) holds for \( d(t, t_0) = d \).

(II_{d}) The equality (1.16) holds under the assumption (3.1) whenever \( d(t, t_0) = d \).
We will prove both \((I_d)\) and \((II_d)\) simultaneously by induction on \(d\).

For \(d = 0\) we have \(t = t_0\), and both \((1.12)\) and \((1.16)\) are immediate from the definitions.

It remains to prove the implications \((I_d) \& (II_d) \implies (I_{d+1})\) and \((I_d) \& (II_d) \implies (II_{d+1})\).

**Proof of the implication \((I_d) \& (II_d) \implies (I_{d+1})\).** We need to show that if \((1.12)\) and \((1.16)\) hold for some \(t, t_0,\) and \(t_1\) satisfying \((3.1)\), then \((1.12)\) also holds if \(t_0\) is replaced with \(t_1\), and \(B\) is replaced by \(B_1 = \mu_k(B)\) (note that this replacement leaves \(B_t\) unchanged by the definition of an exchange matrix pattern). Let us abbreviate \(C = C_t^{B; t_0}\). To prove the desired equality
\[
C_t^{B; t_1} = (C_t^{B; t_1})^{-1}
\]
we use \((1.16)\), the second equality in \((2.6)\), the inductive assumption \(C^{-1} = C_{t_0}^{-B; t_1}\), and the first equality in \((1.15)\) to get
\[
(C_t^{B; t_1})^{-1} = ((J_k + [-
\varepsilon_k(C_{t_0}^{-B; t_1})B]_t^{1\bullet}) C)^{-1}
= C^{-1}(J_k + [-
\varepsilon_k(C_{t_0}^{-B; t_1})B]_t^{1\bullet})
= C_{t_0}^{-B; t_1}(J_k + [\varepsilon_k(C_{t_0}^{-B; t_1}) \cdot (-B)]_t^{1\bullet})
= C_t^{B; t_1}
\]
(to see that the last equality is indeed an instance of \((1.15)\), note that the matrix mutation commutes with the operation \(B \mapsto -B\), thus the exchange matrix pattern that assigns \(-B_t\) to \(t\), also assigns \(-B\) to \(t_0\).)

**Proof of the implication \((I_d) \& (II_d) \implies (II_{d+1})\).** This proof is more involved than the previous one. Let us first summarize our assumptions, and what has to be proven.

Suppose \(d(t_0, t) = d\) in \(\mathbb{T}_n\), and let \(t_1, t' \in \mathbb{T}_n\) be such that \(t_0 \xrightarrow{k} t_1, t \xrightarrow{\ell} t'\), and \(t\) and \(t_0\) belong to the unique path between \(t'\) and \(t_1\). Thus we have \(d(t_1, t) = d(t_0, t') = d + 1,\) and \(d(t_1, t') = d + 2.\) The assumption \((I_d)\) is just the equality \((1.12)\).

Since we have already shown that \((I_d)\) and \((II_d)\) imply \((I_{d+1})\), we can also assume the equalities
\[
(3.2) \quad C_t^{B; t_1} = (C_{t_1}^{-B; t})^{-1}, \quad C_t^{B; t_0} = (C_{t_0}^{-B; t'}^{-1})^{-1}.
\]
Now the assumption \((II_d)\) gives us the equalities
\[
(3.3) \quad C_t^{B; t_1} = (J_k + [-\varepsilon_k(C_{t_0}^{-B; t_1})B]_t^{1\bullet})C_t^{B; t_0},
C_{t_0}^{-B; t'}^{-1} = (J_\ell + [-\varepsilon_\ell(C_{t_0}^{B; t_0})B]_t^{1\bullet})^{-1}C_t^{B; t_1},
\]
while our goal is to prove that
\[
(3.4) \quad C_{t'}^{B; t_0} = (J_k + [-\varepsilon_k(C_{t_0}^{-B; t'})B]_t^{1\bullet})C_{t'}^{B; t_0}.
\]
To prove \((3.4)\), we invoke the following equalities which are instances of \((1.15)\):
\[
(3.5) \quad C_t^{B; t_0} = C_t^{B; t_0}(J_\ell + [\varepsilon_\ell(C_t^{B; t_0})B]_t^{1\bullet}), \quad C_t^{B; t_1} = C_t^{B; t_1}(J_k + [\varepsilon_k(C_t^{B; t_1})B]_t^{1\bullet}).
\]
The following observation is immediate from the first equality in (3.3):

(3.6) The transformation \( C_{t}^{B; t_0} \rightarrow C_{t}^{B; t_1} \) affects only the entries in the \( k \)-th row.

Now we consider separately the following two mutually exclusive cases:

**Case 1:** the matrix \( C_{t}^{B; t_0} \) has a non-zero entry \((i, \ell)\) for some \(i \neq k\).

**Case 2:** the only non-zero entry in the \( \ell \)-th column of \( C_{t}^{B; t_0} \) is the \((k, \ell)\) entry; equivalently, the \( \ell \)-th column of \( C_{t}^{B; t_0} \) is of the form \( \varepsilon e_k \), where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{Z}^n \). Note that in this case the coefficient \( \varepsilon \) is equal to \( \pm 1 \), since both \( C_{t}^{B; t_0} \) and its inverse \( C_{t_0}^{-B; t} \) are integer matrices.

First we deal with Case 1. In view of (3.6), we have \( \varepsilon_{\ell}(C_{t}^{B; t_1}) = \varepsilon_{\ell}(C_{t}^{B; t_0}) \) (here we use the assumption (1.14)). Combining (3.5) and (3.3), we see that the desired equality (3.4) follows from the equality

\[
\varepsilon_k(C_{t_0}^{-B; t}) = \varepsilon_k(C_{t}^{B; t_0}).
\]

Applying the same argument as above to the transformation \( C_{t_0}^{-B; t} \rightarrow C_{t_0}^{-B; t'} \) given by the second equality in (3.3), we see that it is enough to show that \( C_{t_0}^{-B; t} \) satisfies the analog of Case 1, namely has a non-zero entry \((i, k)\) for some \(i \neq \ell\). Now recall that by our assumption \((I_d)\), the matrices \( C_{t}^{B; t_0} \) and \( C_{t_0}^{-B; t} \) are inverses of each other. Thus, it is enough to show the following statement from linear algebra:

(3.8) For any indices \( k \) and \( \ell \), an invertible matrix \( C \) has a non-zero entry

\((i, \ell)\) for some \(i \neq k\) if and only if \( C^{-1} \) has a non-zero entry \((i, k)\) for some \(i \neq \ell\).

The easiest way to prove (3.8) is to observe that it becomes obvious after replacing the equivalent statements in question by their negations:

(3.9) For any \( k \) and \( \ell \), the \( \ell \)-th column of \( C \) is of the form \( \varepsilon e_k \)

with \( \varepsilon = \pm 1 \) if and only if the \( k \)-th column of \( C^{-1} \) is \( \varepsilon e_\ell \).

This completes the proof of (3.4) in Case 1.

Now assume that we are in Case 2, that is, in view of (3.9), the \( \ell \)-th column of \( C_{t}^{B; t_0} \) is \( \varepsilon e_k \), while the \( k \)-th column of \( (C_{t}^{B; t_0})^{-1} = C_{t_0}^{-B; t} \) is \( \varepsilon e_\ell \) for some \( \varepsilon = \pm 1 \). In particular, we have

\[
\varepsilon_{\ell}(C_{t}^{B; t_0}) = \varepsilon_k(C_{t_0}^{-B; t}) = \varepsilon.
\]

Using the two equalities in (3.3), we conclude that the \( \ell \)-th column of \( C_{t}^{B; t_1} \) is \(-\varepsilon e_k\), while the \( k \)-th column of \( C_{t_0}^{-B; t'} \) is \(-\varepsilon e_\ell\), hence

\[
\varepsilon_{\ell}(C_{t}^{B; t_1}) = \varepsilon_k(C_{t_0}^{-B; t'}) = -\varepsilon.
\]

Combining (3.5) and (3.3), we can rewrite the desired equality (3.4) as follows:

\[
(J_k + [-\varepsilon B^k]_+)C_{t}^{B; t_0}(J_\ell + [-\varepsilon B_\ell]_+) = (J_k + [\varepsilon B^k]_+)C_{t}^{B; t_0}(J_\ell + [\varepsilon B_\ell]_+),
\]

or equivalently (in view of (2.6)) as

\[
(J_k + [\varepsilon B^k]_+)(J_\ell + [-\varepsilon B^\ell]_+)C_{t}^{B; t_0} = C_{t}^{B; t_0}(J_\ell + [\varepsilon B_\ell]_+)(J_k + [-\varepsilon B^k]_+).
\]

Performing the matrix multiplication and remembering (2.2), we can simplify the last equality to

\[
(I + \varepsilon B^k)C_{t}^{B; t_0} = C_{t}^{B; t_0}(I + \varepsilon B^\ell).
\]
or even further to
\[ B^{k\bullet}C^{B; t_0} = C^{B; t_0}B^i\bullet. \]

To prove (3.10) we recall the identity (1.11), and in particular the matrix $G^{B; t_0}$ that appears there. To simplify the notation, we abbreviate $G^{B; t_0} = G$, $C^{B; t_0} = C$; as shown in the proof of (1.11) given above, it can be rewritten as follows:
\[ G = (C^{-B^T; t_0})^{-1} = DC^{-1}D^{-1}, \]
where $D$ is the diagonal matrix with positive diagonal entries such that $-B^T = DBD^{-1}$. In particular, the $k$th column of $G$ is equal to $d^{-1}k \varepsilon e^T$.

However, we know that both $G$ and its inverse $C^{-B^T; t_0}$ are integer matrices. Therefore, $d_k = d_k$, and we have
\[ \text{the } k\text{th row of } G \text{ is equal to } \varepsilon e^T. \]

Now we recall the equality (2.8). Computing the entries in the $k$th row on both sides of (2.8) and using (3.12), we get, for any $j = 1, \ldots, n$:
\[ \varepsilon b_{ij} = \sum_p b_{kp}c_{pj}. \]
Rewriting this in the matrix form yields (3.10), thus completing the proofs of Proposition 1.4 and identity (1.12).

4. Some corollaries

In this section we show that the assumption (1.8) and the identities (1.11) and (1.12) imply most of the conjectures made in [5] and recast in [2], namely Conjectures 1.1 - 1.4 and 1.6 from [2]. For the convenience of the reader we reproduce their statements.

**Conjecture 4.1.**

(i) Each polynomial $F^{B; t_0}_{j; t}$ has constant term 1.

(ii) Each polynomial $F^{B; t_0}_{j; t}$ has a unique monomial of maximal degree. Furthermore, this monomial has coefficient 1, and it is divisible by all the other occurring monomials.

(iii) For every $t \in T_n$, the vectors $g_{1; t}^{B; t_0}, \ldots, g_{n; t}^{B; t_0}$ are sign-coherent, i.e., for any $i = 1, \ldots, n$, the $i$-th components of all these vectors are either all nonnegative, or all nonpositive.

(iv) For every $t \in T_n$, the vectors $g_{1; t}^{B; t_0}, \ldots, g_{n; t}^{B; t_0}$ form a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$.

(v) Let $t_0 \xrightarrow{k} t_1$ be two adjacent vertices in $T_n$, and let $B' = \mu_k(B)$. Then, for any $t \in T_n$ and $j = 1, \ldots, n$, the $g$-vectors $g_{j; t_0}^{B; t_0} = (g_1, \ldots, g_n)$ and $g_{j; t_1}^{B'; t_1} = (g'_1, \ldots, g'_n)$ are related as follows:
\[ g'_i = \begin{cases} -g_k & \text{if } i = k; \\ g_i + [b_{ik}] + b_k \min(g_k, 0) & \text{if } i \neq k. \end{cases} \]
Proposition 4.2. The assumption (1.8) implies all the statements in Conjecture 4.1.

Proof. The equivalence of (i) and (ii) was shown in [5, Section 5] (see Conjectures 5.4, 5.5 and Proposition 5.3 there). The fact that (1.8) implies conjectures (i) and (ii) was shown in [5, Proposition 5.6] (note that (1.8) appears as condition (ii') in the proof).

Part (iii) is immediate from (1.8) and (1.13).

Part (iv) can be restated by saying that the matrix $G_{t_0}^{B^t}$ with columns $g_{j,t_0}$ is invertible over $\mathbb{Z}$. But this is immediate from (1.11).

It remains to prove (v). Replacing $\min(g_k, 0)$ with $-\lfloor g_k \rfloor_+$, and using (2.1), we rewrite the desired equality (4.1) in the matrix form as follows:

\[(4.2) G_{t_1}^{B^t,t_1} = (J_k + [\varepsilon B]_+^k)G_{t_0}^{B^t} + B[-\varepsilon G_{t_0}^{B^t}]_+^k, \]

for any choice of sign $\varepsilon = \pm 1$. Taking transpose matrices on both sides of (4.2) and using (1.11), we can rewrite (4.2) as

\[(4.3) C_{t_1}^{B^t,t_1} = C_{t_0}^{B^t,t_1}(J_k + [\varepsilon B^T]_+^k) + [-\varepsilon C_{t_0}^{B^t,t_1}]_+^k B^T . \]

In view of (1.14), choosing $\varepsilon = \varepsilon_k(C_{t_0}^{B^t,t_1})$, we see that (1.3) takes the form

\[(4.4) C_{t_1}^{B^t,t_1} = C_{t_0}^{B^t,t_1}(J_k + [\varepsilon_k(C_{t_0}^{B^t,t_1}) B^T]_+^k), \]

which is just the first equality in (1.15) (with $(t, t')$ replaced by $(t_0, t_1)$, and $(B; t_0)$ replaced by $(B^t; t)$). This proves (4.2) and completes the proof of Proposition 4.2. \[\square\]

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