Dixmier trace and the DOS of magnetic operators

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Abstract
The main goal of this work is to provide two new formulas for the computation of the trace per unit volume, and consequently the integrated density of states (IDOS), for magnetic operators. These formulas also allow the use of the Dixmier trace in the spectral analysis of magnetic operators. The second of these formulas, named energy shell formula, allows us to approximate the IDOS by a finite sum of averaged expectation values of the spectral projections.

Keywords Landau Hamiltonian · IDOS and DOS · Dixmier trace · Trace per unit volume

Mathematics Subject Classification Primary: 81R15 · Secondary: 81V70 · 58B34 · 81R60

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1 Introduction

The main goal of this work is to provide two new formulas for the computation of the trace per unit volume, and consequently the density of states (DOS), for magnetic operators. These formulas also allow us to use the Dixmier trace in the spectral analysis of these operators. In order to describe the new results, we first need to introduce some definitions and notations. We will use [7], and references therein, as the source for this background material.

The (dual) magnetic translations\(^1\) on \(L^2(\mathbb{R}^2)\) are the unitary operators defined by

\[
(V(a)\psi)(x) = e^{i \frac{x \wedge a}{2\ell^2}} \psi(x - a), \quad a \in \mathbb{R}^2, \quad \psi \in L^2(\mathbb{R}^2) \quad (1.1)
\]

where \(x \wedge a := x_1 a_2 - x_2 a_1\) for all \(x = (x_1, x_2)\) and \(a = (a_1, a_2)\). A direct computation shows that

\[
V(a)V(b) = e^{i \frac{b \wedge a}{2\ell^2}} V(a + b) , \quad a, b \in \mathbb{R}^2.
\]

The parameter \(\ell > 0\) is known as magnetic length. From a physical point of view, it is proportional to \(\beta^{-\frac{1}{2}}\) where \(\beta > 0\) is the strength of a constant magnetic field perpendicular to the plane \(\mathbb{R}^2\). Therefore, \(\ell \to \infty\) represents the “singular” limit of a vanishing magnetic field [7, Remark 2.2]. Let \(\mathcal{V} := C^*(V(a), a \in \mathbb{R}^2)\) be the \(C^*\)-algebra generated by the unitaries \(V(a)\). The magnetic von Neumann algebra \(\mathcal{M}\) (or magnetic algebra, for short) is by definition the commutant of \(\mathcal{V}\) [7, Proposition 2.18] i.e.,

\[
\mathcal{M} := \mathcal{V}' . \quad (1.2)
\]

The name magnetic algebra is justified by the fact the Landau Hamiltonian

\[
H_L := \frac{1}{2} \left( -i \ell \frac{\partial}{\partial x_1} - \frac{1}{2\ell} x_2 \right)^2 + \frac{1}{2} \left( -i \ell \frac{\partial}{\partial x_2} + \frac{1}{2\ell} x_1 \right)^2 , \quad (1.3)
\]

is affiliated to \(\mathcal{M}\) [7, Section 2.3].

The magnetic algebra \(\mathcal{M}\) admits a canonical, faithful, semi-finite and normal (FSN) trace \(\tau\), defined on the two-sided self-adjoint ideal \(\mathcal{I}_\tau \subset \mathcal{M}\) [7, Proposition 2.21]. We will provide a precise definition for \(\mathcal{I}_\tau\) and \(\tau\) in Sect. 2. For the purpose of this introduction, it is important to point out that \(\tau\) can be realized as the trace per unit volume. For that, let \(\Lambda_n \subseteq \mathbb{R}^2\) be an increasing sequence of compact subsets such that \(\Lambda_n \nearrow \mathbb{R}^2\) and which satisfies the Følner condition (see [13] for more details). Let \(\chi_{\Lambda_n}\) be the projection defined as the multiplication operator by the characteristic function of \(\Lambda_n\). A bounded operator \(S\) admits the trace per unit volume (with respect to the Følner sequence \(\Lambda_n\)) if the limit

\[
\mathcal{I}_{u,v}(S) := \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\Lambda_n} S \chi_{\Lambda_n})
\]

\(^1\) The name magnetic translations is common in the condensed matter community since the works of Zak [23, 24]. Mathematically, they are known as Weyl systems.
exists. It follows that every $S \in \mathcal{S}_\tau$ admits the trace per unit volume, and

$$\tau(S) = \frac{\Omega_\ell}{2} T_{u.v.}(S), \quad \Omega_\ell := \pi (2\ell)^2$$

(1.4)

independently of the election of any particular Følner sequence [7, Lemma 2.23 and Remark 2.24]. It is interesting to notice that the constant $\Omega_\ell$ has the physical meaning of the area of the magnetic disk of radius $2\ell$. For the aims of this introduction Eq. (1.4) can be used as the definition of $\tau$, although a more direct and intrinsic definition will be provided in Sect. 2. It is worth to point out that a crucial aspect for the existence of the thermodynamic limit defining $T_{u.v.}$ is that the elements of $\mathcal{S}_\tau$ are left invariant by the action of $\mathbb{R}^2$ implemented by the magnetic translations (1.1).

The trace per unit volume plays a crucial role for the study of the spectral and thermodynamic properties of quantum systems. An important example is given by the DOS of a Hamiltonian as discussed in Sect. 5. For this reason, it is important to have formulas that allow to calculate, or approximate, the trace per unit volume. One of the main contributions of this work is to provide two new formulas which allow us to compute $\tau$. The first of these formulas relates the computation of the trace to the estimate of a residue. To state this result we need to introduce the family of operators

$$Q_{\lambda}^{−s} := (Q + \lambda 1)^{−s}$$

(1.5)

with $s > 0$ and $\lambda > −1$ where

$$Q := −\ell^2 \Delta + \frac{1}{4\ell^2} |x|^2$$

(1.6)

is the two-dimensional isotropic harmonic oscillator given by the sum of the Laplacian $\Delta := \partial^2_{x_1} + \partial^2_{x_2}$ and the harmonic potential $|x|^2 := x_1^2 + x_2^2$. Since the spectrum of $Q$ is $\sigma(Q) = \mathbb{N} = \{1, 2, 3, \ldots\}$, it follows that $Q + \lambda 1$ is invertible for every $\lambda > −1$.

**Theorem 1.1** (Residue formula) *For every $S \in \mathcal{S}_\tau$ it holds true that*

$$\tau(S) = \lim_{x \to 0^+} x \, \text{Tr} \left( Q_{\lambda}^{−(1+s)} S \right).$$

The Proof of Theorem 1.1 requires several intermediate steps and will be presented in Sect. 3.

For the second formula let us recall that the harmonic oscillator $Q$ is diagonalized by the Laguerre basis $\psi_{n,m}$ defined in (2.1). More precisely, one has that

$$Q \psi_{n,m} = (n + m + 1) \psi_{n,m}, \quad \forall \ (n, m) \in \mathbb{N}_0^2,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As a consequence the eigenspace of $Q$ associated to the eigenvalue $j \in \mathbb{N}$, also called $j$th energy shell, is spanned by the $\psi_{n,m}$ such that
n + m = j − 1 and has dimension (or degeneracy) \( j \). Therefore, the quantity

\[
w_j(S) := \frac{1}{j} \sum_{n+m=j-1} \langle \psi_{n,m}, S\psi_{n,m} \rangle_{L^2}, \quad j \in \mathbb{N}
\]  

(1.7)

represents the averaged expectation value of the operator \( S \) on the \( j \)th energy shell of \( Q \).

**Theorem 1.2** (Energy shell formula) For every \( S \in \mathcal{I}_\tau \) it holds true that

\[
\tau(S) = \lim_{N \to +\infty} \left( \frac{1}{\log(N)} \sum_{j=1}^{N} w_j(S) \right)
\]

where \( w_j \) is as defined by (1.7).

The Proof of Theorem 1.2 is postponed to Sect. 4.

**Remark 1.3** (Equivalence of traces) The use of the Laguerre basis induces the unitary isomorphism of Hilbert spaces \( L^2(\mathbb{R}^2) \cong \bigoplus_{m \in \mathbb{N}_0} \mathcal{H}_m \) where \( \mathcal{H}_m \cong \ell^2(\mathbb{N}_0) \) is the Hilbert space generated by the vectors \( \{\psi_{n,m} \mid n \in \mathbb{N}_0\} \). Under this isomorphism one can prove that \( \mathcal{M} \) is unitarily equivalent to \( \bigoplus_{m \in \mathbb{N}_0} \mathcal{B}(\mathcal{H}_m) \) [7,Proposition 2.11 & Proposition 2.20]. The restriction to one of the summands \( \mathcal{H}_m \) (for instance \( m = 0 \)) induces the non-unitary \(*\)-isomorphism \( \rho_0 : \mathcal{M} \to \mathcal{B}(\ell^2(\mathbb{N}_0)) \). Since the trace \( \tau \) is faithful, semi-finite and normal and in view of the fact that every faithful, semi-finite and normal trace on \( \mathcal{B}(\ell^2(\mathbb{N}_0)) \) is (a constant multiple of) the standard trace on \( \ell^2(\mathbb{N}_0) \) one gets that

\[
\tau = \text{Tr}_{\ell^2(\mathbb{N}_0)} \circ \rho_0.
\]  

(1.8)

The constant of proportionality (which is 1) can be fixed by observing that the isomorphism \( \rho_0 \) sends the Landau projection \( \Pi_j \) (see Sect. 2) to the standard rank one projection \( |j\rangle \langle j| \) on \( \ell^2(\mathbb{N}_0) \). Equation (1.8) can be used to provide a different (but equivalent) Proof of Theorem 1.2 but it cannot be used in the Proof of Theorem 1.1 since the operator \( Q_{-t}^{-\lambda} \) does not factor through the isomorphism \( \rho_0 \). Indeed Eq. (1.8) works since the algebra \( \mathcal{M} \) is, in a certain sense, “small”. If one thinks in terms of trace per unit volume, one realizes that \( \mathcal{T}_{u.v.} \) is defined on a bigger algebra of operators on \( \mathcal{B}(L^2(\mathbb{R}^2)) \). For instance the trace per unit volume makes sense for magnetic operators perturbed by periodic or covariant potentials (see Sect. 6) while the Eq. (1.8) makes no sense for this type of operator. However, it is aspected that Theorem 1.2 should continue to work, in the form as it is stated, also in presence of perturbations by potential.

The trace per unit volume, or equivalently the trace \( \tau \) in view of (1.4), is the central object for the construction of the integrated density of states (IDOS) \( N_H \) associated to the self-adjoint operator \( H \) affiliated with \( \mathcal{M} \). Albeit the precise description of \( N_H \) (which requires also the assumption of the spectral regularity for \( H \)) will be provided in Definition 5.1, we can anticipate that \( N_H \) is a positive, non-decreasing and right-continuous function defined on \( \mathbb{R} \) (Lemma 5.3). According to Definition 5.4, the DOS...
of $H$ is the Lebesgue–Stieltjes measure associated with $N_H$ and will be denoted with $\mu_H$. Let $\epsilon_\infty := \sup \sigma(H)$ the supremum of the spectrum of $H$ (with the convention that $\epsilon_\infty = +\infty$ when $H$ is unbounded from above) and $C_c((-\infty, \epsilon_\infty))$ the space of compactly supported continuous functions on the open interval $(-\infty, \epsilon_\infty)$. Then, the following spectral formula

$$\tau(f(H)) = \frac{\Omega_\ell}{2} \int_{\mathbb{R}} d\mu_H(\epsilon) f(\epsilon)$$

(1.9)

holds true for every $f \in C_c((-\infty, \epsilon_\infty))$ (Proposition 5.6). The combination of formula (1.9) with Theorem 1.2 provides the following result:

**Theorem 1.4** (Spectral energy shell formula) Let $H$ be a self-adjoint and spectrally regular operator affiliated to the magnetic algebra $\mathcal{M}$. For every $f \in C_c((-\infty, \epsilon_\infty))$ the following asymptotic formula holds true

$$\lim_{N \to +\infty} \left( \frac{1}{\log(N)} \sum_{j=1}^{N} w_j(f(H)) \right) = \frac{\Omega_\ell}{2} \int_{\mathbb{R}} d\mu_H(\epsilon) f(\epsilon),$$

where $w_j$ is as defined by (1.7).

In the same vein, by combining Theorem 1.2 with the definition of the IDOS $N_H$ (Definition 5.4) one obtains the following approximated formula

$$N_H(\epsilon) = \lim_{N \to +\infty} \frac{2}{\Omega_\ell \log(N)} \sum_{j=1}^{N} w_j(P_H(\epsilon)),$$

as $N \to +\infty$ (1.10)

where $P_H(\epsilon)$ is the spectral projection of $H$ for the spectral interval $(-\infty, \epsilon]$. Formula (1.10) seems to be new in the literature and can be useful for numerical computations. To some extent this formula plays for magnetic (continuous) operators a similar role that the local DOS and the windowed DOS play for tight-binding models [15].

An important result obtained in [7] ensures that on a certain $\ast$-subalgebra $\mathcal{L}^1$ of $\mathcal{S}$ (see Sect. 2 for the precise definition) the trace $\tau$ can be expressed in terms of the Dixmier trace weighed$^2$ by $Q^{-1}_\lambda$. We will assume here some familiarity of the reader with the theory of the Dixmier trace and we provide in Appendix 1 a short summary of the main properties of the Dixmier trace along with some useful references. For the moment let us recall that the natural domain of the Dixmier trace is an ideal of the compact operators denoted with $\mathcal{S}_1^+$ and called the Dixmier ideal. The construction of the Dixmier trace requires the choice of a generalized scale-invariant limit $\omega$. However, there exists a relevant subset $\mathcal{S}_m^+ \subset \mathcal{S}_1^+$ where the computation of the Dixmier trace is independent of the election of $\omega$. The elements of $\mathcal{S}_m^+$ are called

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$^2$ It is worth pointing out that the use of the “weight” $Q^{-1}_\lambda$ in (1.11) has a precise justification in terms of the noncommutative geometry of the magnetic spectral triple for the magnetic algebra constructed in [7]. In such a geometric context it is more appropriate to refer to $Q^{-1}_\lambda$ as the “magnetic volume form”. These aspects have been discussed in detail in [7], and we will not dwell further on this point.
measurable operators. The action of any Dixmier trace on $\mathfrak{S}^+_m$ will be denoted simply with $\TrDix: \mathfrak{S}^+_m \to \mathbb{C}$. In [7,Proposition 2.27] it has been proved that when $S \in \mathcal{L}^1$ then the equality

$$\tau(S) = \TrDix(T_S)$$  \hfill (1.11)

holds true for every

$$T_S \in \left\{ Q^{-1}_\lambda S, SQ^{-1}_\lambda, Q^{-\frac{1}{2}}_\lambda SQ^{-\frac{1}{2}}_\lambda \right\} \subset \mathfrak{S}^+_m. \hfill (1.12)$$

independently of $\lambda, \lambda' > -1$. In this case the spectral formula (1.9) can be reinterpreted as follows:

**Theorem 1.5** Let $H$ be a self-adjoint and $\mathcal{L}^1$-spectrally regular operator affiliated to the magnetic algebra $\mathcal{M}$. For every $f \in C_c((-%\infty, %\infty))$ let

$$T_f(H) \in \left\{ Q^{-1}_\lambda f(H), f(H)Q^{-1}_\lambda, Q^{-\frac{1}{2}}_\lambda f(H)Q^{-\frac{1}{2}}_\lambda \right\}.$$  

Then $T_f(H) \in \mathfrak{S}^+_m$ and it holds true that

$$\TrDix(T_f(H)) = \frac{\Omega_\ell}{2} \int_{\mathbb{R}} d\mu_H(\epsilon) f(\epsilon)$$

independently of $\lambda, \lambda' > -1$.

The Proof of Theorem 1.5 follows directly form formula (1.11) and Proposition 5.6 applied to $\mathcal{L}^1$-spectrally regular operators. The latter property (described in Definition 5.1) and Lemma 2.1 ensure that $f(H) \in \mathcal{L}^1$, and in turn the applicability of the Eq. (1.11). The restriction imposed by the $\mathcal{L}^1$-spectral regularity could be removed if one could prove that equality (1.11) holds for every $S \in \mathcal{H}$. This aspect will be discussed in Sect. 6.

Theorem 1.5 represents the “magnetic version” of [2,Theorem 1.1] in the special case $d = 2$. Aside from the obvious similarity, there are few differences we would like to point out. Theorem 1.1 in [2] works for Schrödinger operators of type $-\Delta + V$ with real-valued potentials $V \in L^\infty(\mathbb{R})$, while Theorem 1.5 is valid for every ($\mathcal{L}^1$-spectrally regular) magnetic operator affiliated with the magnetic algebra $\mathcal{M}$ but without potentials. Perturbations by potentials will be considered in a future work along the lines anticipated in Sect. 6. Anyway, it is suggestive to compare the case of the *free Laplacian* [2,Example 1.2] with the case of the *free Landau operator* in Example 5.8. There is also a second relevant difference. The “weight” introduced in [2,Theorem 1.1] is the multiplication operator by the function $(1 + |x|^2)^{-1}$ which is evidently not compact. On the contrary, the “weight” $Q^{-1}_\lambda$ used in Theorem 1.5 is a compact operator which is diagonalized on the basis of the generalized Laguerre functions $\psi_{n,m}$ defined in (2.1). The latter fact provides a significant computational advantage. In fact it is the hidden reason behind the energy shell formulas of Theorems 1.2 and 1.4.
Structure of the paper. In Sect. 2 we will introduce the background material about magnetic operators necessary as the starting point for the formulation of our main results. Section 3 contains the proof of the residue-type formula anticipated in Theorem 1.1 and Sect. 4 contains the proof of the energy shell formula provided in Theorem 1.2. Section 5 concerns with the construction of the IDOS and of the DOS and contains the proof of the spectral formula (1.9), and consequently of Theorem 1.5. Possible generalizations of our main results along with the related open problems are briefly discussed in Sect. 6. Appendix 1 contains some useful results about the elements of the magnetic algebra \( \mathcal{M} \) which are needed for the Proof of Lemma 2.1. Finally, Appendix 1 contains a very short introduction to the Dixmier trace in order to make this work self-contained.

2 Relevant aspects about magnetic operators

The material presented in this preliminary section is borrowed from [6, 7]. Consider the Hilbert space \( L^2(\mathbb{R}^2) \) and let \( \{\psi_{n,m}\} \subset L^2(\mathbb{R}^2) \), with \( n, m \in \mathbb{N}_0 \), be the orthonormal Laguerre basis defined by

\[
\psi_{n,m}(x) := \psi_0(x) \sqrt{\frac{n!}{m!}} \left( \frac{x_1 + i x_2}{\ell \sqrt{2}} \right)^{m-n} L_n^{(m-n)} \left( \frac{|x|^2}{2\ell^2} \right), \tag{2.1}
\]

where

\[
L_n^{(\alpha)}(\zeta) := \sum_{j=0}^{n} \frac{(\alpha+n)(\alpha+n-1)\ldots(\alpha+j+1)}{j!(n-j)!} (-\zeta)^j, \quad \alpha, \zeta \in \mathbb{R}
\]

are the generalized Laguerre polynomial of degree \( m \) (with the usual convention \( 0! = 1 \)) and

\[
\psi_0(x) := \frac{1}{\sqrt{2\pi \ell}} e^{-\frac{|x|^2}{4\ell^2}}. \tag{2.2}
\]

From the definition it follows that \( \psi_{0,0} = \psi_0 \).

Let us introduce the family \( \{\Upsilon_{j \to k} \mid (j, k) \in \mathbb{N}_0^2\} \) of transition operators on \( L^2(\mathbb{R}^2) \) defined by

\[
\Upsilon_{j \to k} \psi_{n,m} := \delta_{j,n} \psi_{k,m}, \quad k, j, n, m \in \mathbb{N}_0. \tag{2.3}
\]

A direct computation shows that [7,Proposition 2.10]

\[
(\Upsilon_{j \to k})^* = \Upsilon_{k \to j}, \quad \Upsilon_{j \to k} \Upsilon_{m \to n} = \delta_{j,n} \Upsilon_{m \to k} \tag{2.4}
\]

for every \( j, k, n, m \in \mathbb{N}_0^2 \). The relations (2.4) allow to introduce the \( C^* \)-algebra

\[
\mathcal{C} = C^*(\Upsilon_{j \to k}, k, j \in \mathbb{N}_0) \tag{2.5}
\]

generated inside the algebra of bounded operators \( \mathcal{B}(L^2(\mathbb{R}^2)) \) by the norm closure of polynomials in the generators \( \Upsilon_{j \to k} \). It turns out that \( \mathcal{C} \) is non-unital. We will refer
to $\mathcal{C}$ as the magnetic $C^*$-algebra. Such a name is justified by the fact that the Landau Hamiltonian $H_L$ defined in (1.3) is affiliated with $\mathcal{C}$. More precisely, it turns out that the Landau projections $\Pi_j := \Upsilon_{j \rightarrow j}$ are the spectral projections of $H_L$ which provide the spectral representation

$$H_L = \sum_{j \in \mathbb{N}_0} \lambda_j \Pi_j, \quad \lambda_j := \left( j + \frac{1}{2} \right).$$

(2.6)

The eigenvalue $\lambda_j$ is known as the $j$th Landau level.

The magnetic algebra $\mathcal{M}$ defined by (1.2) coincides with the enveloping von Neumann algebra of $\mathcal{C}$ i.e., $\mathcal{M} = \mathcal{C}''$ where on the right-hand side one has the bicommutant of $\mathcal{C}$ [7,Proposition 2.18]. Besides $\mathcal{C}$, there are other interesting subspaces contained in $\mathcal{M}$. Let us introduce the following family of spaces

$$L^p := \left\{ A := \sum_{(j,k) \in \mathbb{N}_0^2} a_{j,k} \Upsilon_{j \rightarrow k} \mid \{a_{j,k}\} \in \ell^p(\mathbb{N}_0^2) \right\},$$

(2.7)

where $\ell^p(\mathbb{N}_0^2)$ are the usual spaces of $p$-summable sequences on $\mathbb{N}_0^2$. Every $L^p$, obtained as the closure of the polynomials in the generators $\Upsilon_{j \rightarrow k}$ with respect the associated $\ell^p$-norms, turns out to be a Banach space. One has that [7,Proposition 2.17]

$$L^1 \subset \mathcal{I}_\tau \subset L^2 \subset \mathcal{C} \subset \mathcal{M},$$

where

$$\mathcal{I}_\tau := \left\{ S = AB \mid A, B \in L^2 \right\} \equiv \left( L^2 \right)^2.$$

All these subspaces are dense in $\mathcal{C}$ with respect to the operator norm, and in $\mathcal{M}$ with respect to the weak or strong topology. Both $L^2$, and consequently $\mathcal{I}_\tau$, are self-adjoint two-sided ideals of $\mathcal{M}$ [7,Proposition 2.18].

The space $L^2$ admits a special characterization in terms of integral kernel operators [7,Section 2.4]. Indeed, it turns out that $A \in L^2$ if and only if there is a $f_A \in L^2(\mathbb{R}^2)$ such that

$$(A\varphi)(x) = \frac{1}{2\pi \ell^2} \int_{\mathbb{R}^2} dy \ f_A(y - x) \ e^{i\frac{x\cdot y}{2\ell^2}} \varphi(y), \quad \forall \ \varphi \in L^2(\mathbb{R}^2).$$

(2.8)

The relation between the integral kernel $f_A$ and the sequence $\{a_{j,k}\} \in \ell^2(\mathbb{N}_0^2)$ which identifies the expansion of $A$ in the basis $\Upsilon_{j \rightarrow k}$ is given by

$$f_A = \sqrt{2\pi \ell} \sum_{(j,k) \in \mathbb{N}_0^2} (-1)^{j-k} a_{j,k} \psi_{k,j}$$

(2.9)
and the norm bound \( \sqrt{2\pi \ell} \| A \| \leq \| f_A \|_{L^2} \) holds true.

The space \( \mathcal{L}^1 \) is a Banach \( \ast \)-algebra with respect to the \( \ell^1 \)-norm [7,Lemma B.1]. It is not an ideal of \( \mathcal{M} \) but fulfills the following “absorption” property:

**Lemma 2.1** Let \( A_1, A_2 \in \mathcal{L}^1 \) and \( T \in \mathcal{M} \). Then \( A_1 T A_2 \in \mathcal{L}^1 \).

The proof of this result is postponed to Appendix 1.

As discussed in [7,Section 2.6], the von Neumann algebra \( \mathcal{M} \) admits a canonical, faithful, semi-finite and normal (FSN) trace \( \tau \) defined on the ideal \( \mathcal{I}_\tau \), and uniquely specified by the prescription

\[
\tau(A^*B) := \frac{1}{2\pi \ell} \langle f_A, f_B \rangle_{L^2}, \quad \forall \, A, B \in \mathcal{L}^2
\]

where \( \langle \cdot, \cdot \rangle_{L^2} \) is the usual scalar product in \( L^2(\mathbb{R}^2) \) and \( f_A, f_B \in L^2(\mathbb{R}^2) \) are the integral kernels of \( A \) and \( B \) respectively, as given by (2.9). By using the orthonormality of the Laguerre functions and the expansions of \( A \) and \( B \) in terms of the operators \( \Upsilon_{j \rightarrow k} \) and the \( \ell^2 \)-sequences \( \{a_{j,k}\} \) and \( \{b_{j,k}\} \) respectively, one gets from (2.10) the following formula

\[
\tau(A^*B) := \sum_{(j,k) \in \mathbb{N}_0^2} \overline{a_{j,k}} b_{j,k}.
\]

Let \( S := A^*B \) be a generic element of \( \mathcal{I}_\tau \). Then, a direct computation shows that \( S \) can be expanded as

\[
S = \sum_{(j,k) \in \mathbb{N}_0^2} s_{j,k} \Upsilon_{j \rightarrow k} \quad \text{with} \quad s_{j,k} := \sum_{n \in \mathbb{N}_0} \overline{a_{k,n}} b_{j,n},
\]

and a comparison with (2.10) provides

\[
\tau(S) = \sum_{k \in \mathbb{N}_0} s_{k,k} := \lim_{N \rightarrow +\infty} \left( \sum_{k=0}^{N} s_{k,k} \right), \quad \forall \, S \in \mathcal{I}_\tau.
\]

Finally, let us recall that every \( S \in \mathcal{I}_\tau \) has an integral kernel of type (2.9) which satisfies \( f_S \in L^2(\mathbb{R}^2) \cap C_0(\mathbb{R}^2) \), where \( C_0(\mathbb{R}^2) \) is the space of continuous functions which vanish at infinity. Therefore, it makes sense to evaluate \( f_S \) pointwise and one can prove that \( \tau(S) = f_S(0) \) [7,Corollary 2.22]

The harmonic oscillator \( Q \) defined by (1.6) admits the spectral decomposition

\[
Q = \sum_{(n,m) \in \mathbb{N}_0^2} (n + m + 1) \Pi_n P_m
\]

which involves the Landau projections \( \Pi_n \) and the *transverse projections* \( P_m \) defined by

\[
P_m \psi_{j,k} := \delta_{m,k} \psi_{j,k}, \quad k, j, m \in \mathbb{N}_0.
\]
It is important to notice that \( \Pi_n P_m = P_m \Pi_n \) for every \((n, m) \in \mathbb{N}_0^2\). It turns out that
\[
Q^{-s}_\lambda = \sum_{(n, m) \in \mathbb{N}_0^2} \frac{1}{(n + m + 1 + \lambda)^s} \Pi_n P_m
\]
provides the spectral representation of the operator defined by (1.5). In particular the last equation shows that \( Q^{-s}_\lambda \) is a well-defined compact operator for every \( s > 0 \) and \( \lambda > -1 \).

The relation between the family \( Q^{-s}_\lambda \) and the Dixmier trace has been investigated in [6, Lemma B.4 & Lemma B.5] and in [7, Corollary 2.26 & Lemma 3.10]. Let \( \mathcal{S}^p \) be the \( p \)th Schatten ideal, \( \mathcal{S}^p_+ \) the Mačaev ideal of order \( p \) and \( \mathcal{S}^1_+ \subset \mathcal{S}^1_\infty \) the space of measurable operators (cf. Appendix 1). It results that \( Q^{-s}_\lambda \in \mathcal{S}^1 \) for every \( s > 2 \) and \( \lambda > -1 \). Moreover, \( Q^{-2}_\lambda \in \mathcal{S}^1_+ \) and
\[
\text{Tr}_{\text{Dix}}(Q^{-2}_\lambda) = \frac{1}{2}
\]
independently of \( \lambda > -1 \). When \( Q^{-1}_\lambda \) is multiplied by an element in \( \mathcal{S}^1 \) one obtains the equality (1.11) i.e.,
\[
\text{Tr}_{\text{Dix}}(T_S) = \sum_{k \in \mathbb{N}_0} s_{k,k}
\]
where \( T_S \) is one of the operators in (1.12) and \( \{s_{j,k}\} \) are the coefficients of \( S \in \mathcal{S}^1 \).

3 The residue formula

In this section, we will show that the trace \( \tau \) on \( \mathcal{S}_\tau \) can be computed via the residue-type formula anticipated in Theorem 1.1.

Let us start by observing that the Laguerre basis induces the isomorphism of Hilbert spaces \( L^2(\mathbb{R}^2) \cong \ell^2(\mathbb{N}_0^2) \) via the unitary map \( \mathcal{U} : \psi_{n,m} \mapsto |n, m\rangle \). Here, the Dirac notation\(^3\) \( |n, m\rangle \) for the canonical basis of \( \ell^2(\mathbb{N}_0^2) \) is used. Every operator \( T \) acting on \( \ell^2(\mathbb{N}_0^2) \) can be represented as
\[
T = \sum_{(n', m') \in \mathbb{N}_0^2} \sum_{(n, m) \in \mathbb{N}_0^2} \kappa_T[(n', m'), (n, m)] |n', m'\rangle \langle n, m|
\]
for a given function \( \kappa_T : \mathbb{N}_0^2 \times \mathbb{N}_0^2 \rightarrow \mathbb{C} \) defined through the matrix elements
\[
\kappa_T[(n', m'), (n, m)] := \langle n', m'|T|n, m\rangle.
\]
\(^3\) We will assume a certain familiarity of the reader with the Dirac notation. Let us just recall that \(|n', m'\rangle \langle n, m|\) denotes the rank-one operator with initial space spanned by \(|n, m\rangle\) and target space spanned by \(|n', m'\rangle\).

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Every vector $\phi \in \ell^2(\mathbb{N}_0^2)$ can be expanded as $\phi := \sum_{(n,m) \in \mathbb{N}_0^2} \phi(n,m) |n,m\rangle$ and the explicit computation

$$(T\phi)(n,m) = \sum_{(n',m') \in \mathbb{N}_0^2} \kappa_T[(n,m), (n',m')] \phi(n',m') ,$$

shows that $\kappa_T$ is the integral kernel of $T$.

The integral kernel associate to the transition operator $\Upsilon_{j\rightarrow k}$ is given by

$$\kappa_{\Upsilon_{j\rightarrow k}}[(n,m), (n',m')] := \langle n, m | \Upsilon \Upsilon_{j\rightarrow k} \Upsilon^{-1} | n', m' \rangle = \delta_{n,k} \delta_{n',j} \delta_{m,m'} .$$

More in general, let $A \in \mathcal{L}^p$ with associated coefficients $\{a_{n,n'}\} \in \ell^p(\mathbb{N}_0^2)$ according to (2.7). Then, the associated integral kernel is given by

$$\kappa_A[(n,m), (n',m')] := \langle n, m | \Upsilon A \Upsilon^{-1} | n', m' \rangle = a_{n',n} \delta_{m,m'} . \quad (3.2)$$

In the same way the integral kernel associate to $Q^{-s}_\lambda$ is given by

$$\kappa_{Q^{-s}_\lambda}[(n,m), (n',m')] := \langle n, m | \Upsilon Q^{-s} \Upsilon^{-1} | n', m' \rangle = \frac{1}{(n + m + 1 + \lambda)^s} \delta_{n,n'} \delta_{m,m'} . \quad (3.3)$$

We will need the family of auxiliary operators

$$M^r := \sum_{m \in \mathbb{N}_0} (m + 1)^r P_m \quad (3.4)$$

where $r \in \mathbb{R}$ is a real parameter and $P_m$ are the transverse projections (2.15). For $r \leq 0$ Eq. (3.4) defines a bounded operator, while for $r > 0$ one has an unbounded operator densely defined on the finite linear combination of the Laguerre functions. From its very definition it is immediate to recognize that $M^r$ commutes with $Q^{-s}_\lambda$ since they share the same family of spectral projections. On the other hand, $M^r$ also commutes with every element of the magnetic algebra $\mathcal{A}$ since it belongs to (if $r \leq 0$), or is affiliated with (if $r > 0$) the commutant $\mathcal{A}'$. The integral kernel of $M^r$ is given by

$$\kappa_{M^r}[(n,m), (n',m')] := \langle n, m | \Upsilon M^r \Upsilon^{-1} | n', m' \rangle = (m + 1)^r \delta_{n,n'} \delta_{m,m'} . \quad (3.5)$$

Let $\mathcal{S}^2$ be the ideal of Hilbert-Schmidt operators (the 2-nd Schatten ideal) on $L^2(\mathbb{R}^2)$. We are now in position to prove some preliminary results.

---

4 In the unbounded case $r > 0$ the resolvents of $M^r$ are elements of $\mathcal{A}'$ and thus commute with $\mathcal{A}$. This is enough to ensure that every $A \in \mathcal{A}$ sends the vectors in the domain of $M^r$ into the domain of $M^r$ itself, and the equation $AM^r - M^r A = 0$ is initially well defined on the domain of $M^r$ (cf. the proof of [7, Lemma 2.19]). By a standard continuity argument one then extends the commutation relation to the full Hilbert space.
Lemma 3.1 Let $A \in \mathcal{L}^2$. Then

$$M^{-r}A = AM^{-r} \in \mathcal{S}^2$$

for every $r > \frac{1}{2}$.

**Proof** First of all a bounded operator $B$ on $L^2(\mathbb{R}^2)$ is Hilbert-Schmidt if and only if $\|BU^{-1}\|$ is Hilbert-Schmidt as operator on $\ell^2(\mathbb{N}_0^2)$. Secondly, the property of being Hilbert-Schmidt on $\ell^2(\mathbb{N}_0^2)$ can be checked by showing that the associated integral kernel is in $\ell^2(\mathbb{N}_0^2 \times \mathbb{N}_0^2)$ [20, Theorem VI.23]. By using formulas (3.2) and (3.5) on can compute explicitly the integral kernel of $\|M^{-r}AU^{-1}\|$ which is given by

$$\kappa_{M^{-r}A}(n,m), (n',m') := a_{n',n} (m+1)^{-r} \delta_{m,m'}.$$

Therefore

$$\|\kappa_{M^{-r}A}\|^2_{\ell^2} = \sum_{(n,n',m) \in \mathbb{N}_0^3} |a_{n',n}|^2 (m+1)^{-2r}$$

$$= \frac{\|f_A\|^2_{L^2}}{2\pi \ell^2} \sum_{m \in \mathbb{N}_0} (m+1)^{-2r}$$

and $f_A \in L^2(\mathbb{R}^2)$ is the convolution kernel of $A$ as given by (2.8) and (2.9). It is evident that the sum on the right-hand side converges whenever $r > \frac{1}{2}$. ☐

Lemma 3.2 Let $A \in \mathcal{L}^2$. Then

$$M^r Q^{-s}_\lambda A = Q^{-s}_\lambda M^r A = Q^{-s}_\lambda AM^r \in \mathcal{S}^2$$

for every $r \geq 0$ and $s > 0$ such that $s - r > \frac{1}{2}$.

**Proof** We will use the same strategy of the Proof of Lemma 3.1. The integral kernel of $\|M^r Q^{-s}_\lambda AU^{-1}\|$ can be computed from (3.2), (3.3) and (3.5) and is given by

$$\kappa_{M^r Q^{-s}_\lambda A}(n,m), (n',m') := \frac{(m+1)^r}{(n+m+1+\lambda)^s} a_{n',n} \delta_{m,m'}.$$

Therefore

$$\|\kappa_{M^r Q^{-s}_\lambda A}\|^2_{\ell^2} = \sum_{(n,n',m) \in \mathbb{N}_0^3} \frac{(m+1)^2r}{(n+m+1+\lambda)^{2s}} |a_{n',n}|^2$$

$$\leq \sum_{(n,n',m) \in \mathbb{N}_0^3} \frac{(m+1)^2r}{(m+1+\lambda)^{2s}} |a_{n',n}|^2$$

$$= \frac{\|f_A\|^2_{L^2}}{2\pi \ell^2} \sum_{m \in \mathbb{N}_0} \frac{(m+1)^2r}{(m+1+\lambda)^{2s}}$$
and the last series converges whenever \(2(s - r) > 1\). \(\square\)

Let \(\mathcal{S}^1\) be the ideal of trace-class operators (the 1-st Schatten ideal) on \(L^2(\mathbb{R}^2)\). It is known that for a given pair \(A, B \in \mathcal{S}^2\) of Hilbert-Schmidt operators, the product \(AB \in \mathcal{S}^1\) is trace-class.

**Corollary 3.3** Let \(A, B \in \mathcal{L}^2\). Then

\[
\{ Q_{s}^{-r}AB, BQ_{s}^{-r}A, ABQ_{s}^{-r} \} \subset \mathcal{S}^1
\]

for every \(s > 1\). Moreover,

\[
\text{Tr}(Q_{s}^{-r}AB) = \text{Tr}(BQ_{s}^{-r}A) = \text{Tr}(ABQ_{s}^{-r}).
\]

**Proof** Let \(\frac{1}{2} < r < \frac{1}{2} + (s - 1)\). The identity

\[
Q_{s}^{-r}AB = (Q_{s}^{-r}AM^r)(M^{-r}B)
\]

along with Lemmas 3.1 and 3.2 shows that \(Q_{s}^{-r}AB\) is the product of two Hilbert-Schmidt operators, hence it is trace-class. The remaining cases can be proved in a similar way. The equality of the traces is guaranteed by [21,Corollary 3.8]. \(\square\)

**Corollary 3.4** Let \(S \in \mathcal{I}_r\). Then

\[
Q_{s}^{-\frac{s}{2}}SQ_{s}^{-\frac{s}{2}} \in \mathcal{S}^1
\]

for every \(s > 1\) and

\[
\text{Tr}\left( Q_{s}^{-\frac{s}{2}}SQ_{s}^{-\frac{s}{2}} \right) = \text{Tr}(Q_{s}^{-s}S) = \text{Tr}(SQ_{s}^{-s}).
\]

**Proof** Let \(S = AB\) with \(A, B \in \mathcal{L}^2\). Since Lemma 3.2 ensures that \(Q_{s}^{-\frac{s}{2}}A\) and \(BQ_{s}^{-\frac{s}{2}}\) are both Hilbert-Schmidt, their product is trace-class. The equality of the traces is guaranteed by [21,Corollary 3.8]. \(\square\)

Corollaries 3.3 and 3.4 establish that for every \(S \in \mathcal{I}_r\) the products \(Q_{s}^{-s}S, SQ_{s}^{-s}\) and \(Q_{s}^{-\frac{s}{2}}SQ_{s}^{-\frac{s}{2}}\) are trace-class operators, whenever \(s > 1\). This allows to associate to every \(S \in \mathcal{I}_r\) the function \(\theta_S : (0, +\infty) \to \mathbb{C}\) defined equivalently by one of the following equalities:

\[
\theta_S(x) := \text{Tr} \left( Q_{s}^{-\frac{s(1+x)}{2}}S \right) = \text{Tr}(SQ_{s}^{-\frac{s(1+x)}{2}}) = \text{Tr} \left( Q_{s}^{-\frac{s}{2}}SQ_{s}^{-\frac{s}{2}} \right). \tag{3.7}
\]
Proof of Theorem 1.1 It is sufficient to prove the claim for positive elements in $\mathcal{I}_\tau$. In fact, every $S \in \mathcal{I}_\tau$ is a linear combination of at most four non-negative elements and the functions $S \mapsto \theta_S(x)$ and $S \mapsto \tau(S)$ are linear. Then, let us assume $S \geq 0$. This implies that $s_{n,n} \geq 0$ for all $n \in \mathbb{N}_0$ where $\{s_{n,m}\} \subset \ell^2(\mathbb{N}_0^2)$ are the coefficients associated with the expansion of $S$ as in (2.7). The computation of the trace of $Q^{-1}_\lambda S$ can be performed by integrating “along the diagonal” the integral kernel of $\mathcal{U} Q^{-1}_\lambda (1+x) S \mathcal{U}^{-1}$. This kernel can be computed by putting $r = 0$ in Eq. (3.6) and one gets

$$\kappa_{Q^{-1}_\lambda S}((n, m), (n', m')) := \frac{1}{(n + m + 1 + \lambda)^{1+x} s_{n', n} \delta_{m,m'}}. \tag{3.8}$$

Therefore one obtains

$$\theta_S(x) = \sum_{(n,m) \in \mathbb{N}_0^2} \kappa_{Q^{-1}_\lambda S}((n, m), (n, m))$$

$$= \sum_{(n,m) \in \mathbb{N}_0^2} \frac{s_{n,n}}{(n + m + 1 + \lambda)^{1+x}}.$$  

The latter is a convergent series with positive terms (hence, absolutely convergent) and therefore it can be rearranged as

$$\theta_S(x) = \sum_{n \in \mathbb{N}_0} s_{n,n} \zeta(1 + x, n + 1 + \lambda) \tag{3.9}$$

where

$$\zeta(t, q) := \sum_{m \in \mathbb{N}_0} \frac{1}{(m + q)^t}, \quad t > 1, \quad q > 0$$

is the Hurwitz zeta function [18, Chapter 64]. It is known that $\zeta(t, q)$ has a simple pole with residue 1 in $t = 1$, i.e.

$$\lim_{t \to 1^+} g_q(t) = 1$$

where $g_q(t) := (t - 1)\zeta(t, q)$. Moreover, one has that

$$\lim_{t \to +\infty} g_q(t) = 0, \quad \forall \ q > 1.$$

Then, $g_q$ is a bounded continuous function on $(1, +\infty)$ and

$$\|g_q\|_\infty := \sup_{t \in (1, +\infty)} |g_q(t)| < +\infty, \quad \forall \ q > 1.$$
From $\zeta(t, q) < \zeta(t, q + \delta)$ when $\delta > 0$, one infers that

$$\|g_{q+\delta}\|_\infty \leq \|g_q\|_\infty.$$  

We can use the expression (3.9) to write

$$x \theta_S(x) = s_{0,0} g_{1+\lambda} (1 + x) + \left( \sum_{n \in \mathbb{N}} s_{n,n} g_{n+1+\lambda} (1 + x) \right)$$

where term of order zero has been isolated to consider possible $\lambda > -1$. The series in the right is uniformly convergent since

$$\left\| \sum_{n \in \mathbb{N}} s_{n,n} g_{n+1+\lambda} (1 + x) \right\|_\infty \leq \|g_{2+\lambda}\|_\infty \sum_{n \in \mathbb{N}} s_{n,n} = \|g_{2+\lambda}\|_\infty (\tau(S) - s_{0,0})$$

in view of the relation (2.13). Therefore, one gets

$$\lim_{x \to 0^+} x \theta_S(x) = \sum_{n \in \mathbb{N}_0} s_{n,n} \lim_{t \to 1^+} (t - 1) \zeta(t, n + 1 + \lambda) = \sum_{n \in \mathbb{N}_0} s_{n,n} = \tau(S)$$

where the last equality is provided again by (2.13). \hfill \Box

The next formula may be useful in some application.

**Corollary 3.5** Let $A, B \in \mathcal{L}^2$. Then

$$\lim_{x \to 0^+} x \operatorname{Tr} \left( B \mathcal{Q}_\lambda^{-1} A^* \right) = \frac{1}{2\pi \ell^2} \langle f_A, f_B \rangle_{\mathcal{L}^2}$$

where $f_A, f_B$ are the kernels associated to $A$ and $B$ respectively by (2.9).

**Proof** Let $S := A^* B \in \mathcal{F}_\tau$. By combining Corollary 3.3 with the definition (3.7) one gets

$$\operatorname{Tr} \left( B \mathcal{Q}_\lambda^{-1} A^* \right) = \theta_S(x).$$

Therefore, in view of Theorem 1.1, the limit in the claim equals $\tau(S) = \tau(A^* B)$. Finally, it is enough to apply Eq. (2.10). \hfill \Box

### 4 The energy shell formula

In this section we will build the Proof of Theorem 1.2. We will start with a preliminary result valid for elements in $\mathcal{L}^2$. 

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Lemma 4.1 Let

\[ S = \sum_{(j,k) \in \mathbb{N}_0^2} s_{j,k} \Upsilon_{j \rightarrow k} \in \mathcal{L}^2. \]

Then, it holds true that

\[
\lim_{N \rightarrow +\infty} \left( \frac{1}{\log(N)} \sum_{j=1}^{N} w_j(S) \right) = \lim_{N \rightarrow +\infty} \left( \sum_{n=0}^{N} s_{n,n} \right) \tag{4.1}
\]

where \( w_j(S) \) is defined by (1.7).

Proof A direct computation based on (2.3), and the orthonormality of the Laguerre basis provide

\[ \langle \psi_n, m, S \psi_n, m \rangle_{L^2} = s_{n,n}, \quad \forall (n,m) \in \mathbb{N}_0^2. \]

Therefore, in view of (1.7) one gets that

\[ w_j(S) = \frac{1}{j} \sum_{(n,m) \in \mathbb{N}_0^2} s_{n,n} = \frac{1}{j} \sum_{n=0}^{j-1} s_{n,n}. \]

A rearrangement of the double finite sum provides

\[
\sum_{j=1}^{N} w_j(S) = s_{0,0} + \frac{1}{2}(s_{0,0} + s_{1,1}) + \ldots + \frac{1}{N} \sum_{n=0}^{N-1} s_{n,n} \\
= s_{0,0} h_N + s_{1,1}(h_N - 1) + \ldots + \frac{1}{N} s_{N-1,N-1} \\
= \sum_{n=0}^{N-1} (h_N - h_n)s_{n,n}
\]

where \( h_n := \sum_{k=1}^{n} k^{-1} \) is the \( k \)th harmonic number with the convention \( h_0 = 0 \). Let us consider the following sequence in \( l^\infty(\mathbb{N}) \).

\[ b_n^N = \begin{cases} 
\frac{h_n}{\log(N)}, & n < N, \\
1, & n \geq N.
\end{cases} \]
Notice that \( b^N \) is uniformly bounded and \( \lim_{N \to \infty} b^N_n = 0 \), for all \( n \in \mathbb{N} \). Thus, the dominated convergence theorem implies that \( \lim_{N \to \infty} \sum_{n=0}^{\infty} b^N_n s_{n,n} = 0 \). Moreover,

\[
\frac{1}{\log(N)} \sum_{n=0}^{N-1} \eta_{n,n} = \sum_{n=0}^{\infty} b^N_n s_{n,n} - \sum_{n=N}^{\infty} s_{n,n}.
\]

The latter implies that

\[
\lim_{N \to +\infty} \left( \frac{1}{\log(N)} \sum_{n=0}^{N-1} \eta_{n,n} \right) = 0
\]

and in turn

\[
\lim_{N \to +\infty} \left( \frac{1}{\log(N)} \sum_{j=1}^{N} w_j(S) \right) = \lim_{N \to +\infty} \left( \frac{\eta_N}{\log(N)} \sum_{n=0}^{N-1} s_{n,n} \right).
\]

Since \( \eta_N \sim \log(N) \) when \( N \to +\infty \) one gets the desired result. \( \square \)

Equality (4.1) must be understood in the sense that the two limits have the same behavior. In particular, it says that one of the two limits converges to a finite value if and only if the other also converges to the same value. However, the two limits can be both divergent or indefinite. For instance, as long as \( S \) belongs to \( \mathcal{L}^2 \) there is no guarantee that the series on the right-hand side of (4.1) converges. On the other hand, the convergence is evidently guaranteed if one restricts to elements in \( \mathcal{L}^1 \). However, there is a less restrictive condition which guarantee the convergence. Let \( A, B \in \mathcal{L}^2 \) and consider \( S = A^* B \). Then, by using (2.12) one gets

\[
\left| \sum_{n \in \mathbb{N}_0} s_{n,n} \right|^2 = \left| \sum_{(n,k) \in \mathbb{N}_0^2} \overline{a_{n,k}} b_{n,k} \right|^2 \leq \|a_{i,j}\|^2_{\ell^2} \|b_{i,j}\|^2_{\ell^2} < +\infty
\]

in view of the Cauchy-Schwarz inequality. At this point we have all the ingredients to prove the energy shell formula.

**Proof of Theorem 1.2** Let \( S \in \mathcal{S}_\tau \). Then, according to (2.13) one has that

\[
\lim_{N \to +\infty} \left( \sum_{n=0}^{N} s_{n,n} \right) = \sum_{n \in \mathbb{N}_0} s_{n,n} = \tau(S)
\]

and this concludes the proof. \( \square \)

In the case \( S \in \mathcal{L}^1 \) we can identify the trace \( \tau \) with the Dixmier trace according to (1.11). In such a case one has the following result.
Corollary 4.2 Let $S \in \mathcal{L}^1$ and $T_S$ one of the elements of the set (1.12). Then, it holds true that

$$\text{Tr}_{\text{Dix}}(T_S) = \lim_{N \to +\infty} \left( \frac{1}{\log(N)} \sum_{j=1}^{N} w_j(S) \right),$$

independently of $\lambda, \lambda' > -1$.

Corollary 4.2 is reminiscent of the formula for the computation of the Dixmier trace of modulated operators [16, Section 11.2]. This aspect will be explored a little more in Sect. 6.

5 IDOS and DOS

In this section we will introduce the notions of integrated density of states (IDOS) and density of states (DOS) from a slightly generalized point of view, and we will provide the relation between the DOS and the Dixmier trace for magnetic operators. For a general exposition on in or of the theory of the DOS we will refer to the modern monograph [22], as well as to the classical textbooks [3] or [19].

In this section $\mathcal{M}$ will denote a von Neumann algebra of bounded operators acting on the Hilbert space $\mathcal{H}$ and endowed with a normal, faithful and semi-finite trace $\tau$ defined on the ideal $\mathcal{I}_\tau$. When necessary, $\mathcal{M}$ will be interpreted as the magnetic algebra introduced in Sect. 2. A (non-necessary bounded) self-adjoint operator $H$ acting on $\mathcal{H}$ is affiliated to $\mathcal{M}$ if for every borelian subset $\Sigma \subseteq \sigma(H)$ of the spectrum of $H$ the associated spectral projection $\chi_{\Sigma}(H)$ belongs to $\mathcal{M}$. For every given “energy” $\epsilon \in \mathbb{R}$ let $P_H(\epsilon) := \chi_{(-\infty, \epsilon]}(H)$ be the spectral projection associated with the set $(-\infty, \epsilon]$. Let $\epsilon_\infty := \sup \sigma(H)$ be the supremum of the spectrum of $H$ with the convention that $\epsilon_\infty = +\infty$ when $H$ is unbounded from above.

Definition 5.1 (IDOS and spectral regularity) Let $\mathcal{Y} \subseteq \mathcal{I}_\tau$ be a $*$-subalgebra with the following absorption property

$$Y_1 T Y_2 \in \mathcal{Y} \quad (5.1)$$

for every $Y_1, Y_2 \in \mathcal{Y}$ and every $T \in \mathcal{M}$. Let $H$ be a self-adjoint operator affiliated with $\mathcal{M}$. We will say that $H$ is $\mathcal{Y}$-spectrally regular\(^5\) if $P_H(\epsilon) \in \mathcal{Y}$ for every $\epsilon < \epsilon_\infty$. In this case the function

$$N_H(\epsilon) := C \tau (P_H(\epsilon))$$

will be called the integrated density of states (IDOS) of $H$. The positive constant $C > 0$ plays the role of a “scale factor” for the IDOS.

\(^5\) When $\mathcal{Y} = \mathcal{I}_\tau$ we will refer simply to spectral regularity instead of $\mathcal{I}_\tau$-spectral regularity.
Remark 5.2 Definition 5.1 is mainly meant for operators bounded from below. For instance, in the special case of the magnetic algebra, the operator $-H_L$, where $H_L$ denotes the Landau Hamiltonian defined in (1.3), cannot be spectrally regular. The strict inequality in the condition $\epsilon < \epsilon_\infty$ serves to exclude the identity $1 = P_H(\epsilon_\infty)$ which cannot be in the ideal of definition of the semi-finite trace $\tau$. When $Y$ is a subideal of $\mathcal{I}_\tau$ then condition (5.1) is automatically satisfied. In the specific case of the magnetic algebra $\mathcal{M}$ introduced in Sect. 2 the scale factor for the IDOS is $C = (2\pi \ell^2)^{-1}$.

The main properties of $N_H$ are described below.

Lemma 5.3 For every self-adjoint and spectrally regular operator $H$ affiliated to $\mathcal{M}$ the function $N_H : (-\infty, \epsilon_\infty) \to \mathbb{R}$ is positive, non-decreasing and right-continuous. Moreover

\[ N_H(-\infty) := \lim_{\epsilon \to -\infty} N_H(\epsilon) = 0, \]
\[ N_H(+\infty) := \lim_{\epsilon \to \epsilon_\infty} N_H(\epsilon) = +\infty. \]

Proof The positivity follows from the positivity of $\tau$. Let $\epsilon' \leq \epsilon < \epsilon_\infty$ and consider the identity $P_H(\epsilon) = P_H(\epsilon') + \Delta_H(\epsilon', \epsilon)$ with $\Delta_H(\epsilon', \epsilon) := \chi(\epsilon', \epsilon)(H)$. Therefore, the linearity and the positivity of $\tau$ imply

\[ N_H(\epsilon) = N_H(\epsilon') + C \tau(\Delta_H(\epsilon', \epsilon)) \geq N_H(\epsilon') \]

proving that $N_H$ is non-decreasing. Let $\delta > 0$ (sufficiently small) and consider $P_H(\epsilon + \delta) - P_H(\epsilon) = \Delta_H(\epsilon, \epsilon + \delta) = \chi(\epsilon, \epsilon + \delta)(H)$. Observe that $\lim_{\delta \to 0^+} \chi(\epsilon, \epsilon + \delta)(x) = 0$ for every $x \in \mathbb{R}$, namely the characteristic function $\chi(\epsilon, \epsilon + \delta)$ converges point-wise to 0. Therefore, in view of the Borel functional calculus [20, Theorem VIII.5 (d)] one gets that

\[ s - \lim_{\delta \to 0^+} P_H(\epsilon + \delta) = P_H(\epsilon), \]

namely the family of projections $P_H(\epsilon)$ is strongly right-continuous. The trace $\tau$ is normal, meaning that it is ultra-weakly continuous. Therefore, in view of [8, Part I, Chap. 3, Theorem 1 (ii)] $\tau$ is equivalently ultra-strongly continuous and strongly continuous when restricted to the ball of operators with unitary norm. As a consequence the map $\epsilon \mapsto \tau(P_H(\epsilon))$, and in turn the function $N_H$, are right-continuous on $\mathbb{R}$. Since $\lim_{\epsilon \to -\infty} P_H(\epsilon) = 0$ strongly, one obtains from the right-continuity that $N_H(-\infty) = 0$. On the other hand, by observing that $\lim_{\epsilon \to \epsilon_\infty} P_H(\epsilon) = 1$ strongly, and $1 \in \mathcal{M} \setminus \mathcal{J}_\tau$ in view of the fact that $\tau$ is semi-finite, one infers that $N_H(+\infty) = +\infty$. □

The passage from the IDOS to the DOS requires the use of the Lebesgue–Stieltjes measure (cf. [14, Sect. 15] or [9, Sect. 1.5]).

Definition 5.4 (Density of states) Let $H$ be a self-adjoint and spectrally regular operator affiliated to $\mathcal{M}$. Then, the density of states (DOS) of $H$ is by definition the Lebesgue–Stieltjes measure $\mu_H$ on $(-\infty, \epsilon_\infty)$ induced by $N_H$. 

\[ \square \]
Remark 5.5 By definition $\mu_H$ is the unique Borel measure on $\mathbb{R}$ defined by

$$\mu_H((\epsilon_1, \epsilon_2]) := N_H(\epsilon_2) - N_H(\epsilon_1)$$

via the Carathéodory’s extension. It follows from its very definition that

$$N_H(\epsilon) = \int_{-\infty}^{\epsilon} d\mu_H(\epsilon') = \int_{-\infty}^{\epsilon} d\epsilon' \rho_H(\epsilon')$$

where the second equality makes sense when $\mu_H$ is absolutely continuous with respect to the Lebesgue measure and $\rho_H$ is its Radon–Nikodym derivative. Sometimes $\rho_H$ is referred as the density of states of $H$ in the physical literature. □

Let $C_c((-\infty, \epsilon_\infty))$ be the space of compactly supported continuous functions on the open interval $(-\infty, \epsilon_\infty)$. By functional calculus $f(H) \in \mathcal{M}$. We are now in position to prove the following relevant result:

Proposition 5.6 (Spectral formula) Let $H$ be a self-adjoint and $\mathcal{Y}$-spectrally regular operator affiliated to $\mathcal{M}$. Let $f \in C_c((-\infty, \epsilon_\infty))$. Then $f(H) \in \mathcal{Y}$ and

$$\tau(f(H)) = \frac{1}{C} \int_{-\infty}^{\epsilon_\infty} d\mu_H(\epsilon') \, f(\epsilon').$$

Proof Since $f$ is compactly supported there are $\epsilon_m < \epsilon_M < \epsilon_\infty$ such that the support of $f$ is contained into $[\epsilon_m, \epsilon_M]$. Therefore $f = \chi_{(-\infty, \epsilon_M)} f \chi_{(-\infty, \epsilon_M)}$ and by functional calculus this implies $f(H) = P_H(\epsilon_M) f(H) P_H(\epsilon_M)$ with $P_H(\epsilon_M) \in \mathcal{Y}$ by hypothesis. Since $\mathcal{Y}$ meets the property (5.1) one has that $f(H) \in \mathcal{Y}$. The function $f$ can be approximated point-wise (indeed uniformly) by a sequence of simple functions $f_n$ [9,Theorem 2.10] such that $|f_n| \leq |f| \leq f_{\max} \chi_{(-\infty, \epsilon_M)}$ where $f_{\max} := \|f\|_{\infty}$. Since $f$ is continuous and compactly supported, hence uniformly continuous, it turns out that the approximants $f_n$ can be constructed as Riemann partitions. For every $n \in N$ there is a $\delta_n$ such that $|f(\epsilon) - f(\epsilon')| < 2^{-n}$ whenever $\epsilon, \epsilon' \in [\epsilon_m, \epsilon_M]$ and $|\epsilon - \epsilon'| < \delta_n$. Fix a partition $\epsilon_m =: \epsilon_0 < \ldots < \epsilon_k < \epsilon_{k+1} < \ldots < \epsilon_{N(n)} := \epsilon_M$ such that $\epsilon_{k+1} - \epsilon_k < \delta_n$ and consider the step function

$$f_n(\epsilon) = \sum_{k=0}^{N(n)-1} f_n^k \chi_{(\epsilon_k, \epsilon_{k+1}]}(\epsilon), \quad -\infty < \epsilon < \epsilon_\infty$$

where $f_n^k := f(\epsilon_{k+1})$. Let $\epsilon \in (\epsilon_k, \epsilon_{k+1}]$. Then

$$|f(\epsilon) - f_n(\epsilon)| = |f(\epsilon) - f_n^k| = |f(\epsilon) - f(\epsilon_{k+1})| < 2^{-n}$$
and this shows that \( f_n \to f \) point-wise. By observing that \( \chi(\epsilon_k, \epsilon_k+1)(H) = P_H(\epsilon_{k+1}) - P_H(\epsilon_k) \) and using the linearity of \( \tau \) and the definitions of \( N_H \) and \( \mu_H \) one gets that

\[
C \tau \left( \chi(\epsilon_k, \epsilon_{k+1})(H) \right) = N_H(\epsilon_{k+1}) - N_H(\epsilon_k) = \int_{-\infty}^{\epsilon_{k+1}} d\mu_H(\epsilon') \chi(\epsilon_{k}, \epsilon_{k+1})(\epsilon')
\]

and in turn

\[
\tau(f_n(H)) = \frac{1}{C} \sum_{k=0}^{n} f_k^k \int_{-\infty}^{\epsilon_{k+1}} d\mu_H(\epsilon') \chi(\epsilon_{k}, \epsilon_{k+1})(\epsilon') .
\]

Therefore, passing to the limit \( n \to +\infty \) the right-hand side converges to the Riemann–Lebesgue–Stieltjes of \( f \) with respect to the measure \( \mu_H \). On the other hand the sequence \( f_n(H) \) is equibounded by \( f_{\text{max}} \) and converges strongly to \( f(H) \) in view of the Borel functional calculus [20, Theorem VIII.5 (d)]. Since on bounded sequences strong convergence implies ultra-weak convergence, and recalling that \( \tau \) is normal, hence ultra-weakly continuous, one obtains that

\[
\tau(f(H)) = \lim_{n \to +\infty} \tau(f_n(\epsilon)) = \frac{1}{C} \lim_{n \to +\infty} \sum_{k=0}^{n} f_k^k \int_{-\infty}^{\epsilon_{k+1}} d\mu_H(\epsilon') \chi(\epsilon_{k}, \epsilon_{k+1})(\epsilon') = \frac{1}{C} \int_{-\infty}^{\epsilon_{k+1}} d\mu_H(\epsilon') f(\epsilon') .
\]

This concludes the proof. \( \square \)

**Remark 5.7** Equation (1.9) follows from Proposition 5.6 applied to the case of the magnetic algebra in which \( C = \frac{2\Omega}{\ell^2} \). Let us point out that in (1.9) we used the usual convention of thinking to the spectral measure \( \mu_H \) as (trivially) extended on the complete real axis by the prescription \( \mu_H(\mathbb{R} \setminus (-\infty, \epsilon_{\infty}]) = 0. \)

**Example 5.8 (The Landau Hamiltonian)** It is worth applying the results of this section to the Landau Hamiltonian \( H_L \) defined by (1.3). From the spectral representation of the Landau Hamiltonian (2.6) one infers that \( P_{H_L}(\epsilon) = \sum_{j \in \mathbb{N}_0} \Theta(\epsilon - \lambda_j) \Pi_j \), where \( \Theta \) is the Heaviside step function, \( \Pi_j \) is the \( j \)th Landau projection and \( \lambda_j := j + \frac{1}{2} \) is the \( j \)th Landau level (2.6). Since \( \Pi_j \in \mathcal{L}^1 \) for every \( j \in \mathbb{N}_0 \), one gets that \( P_{H_L}(\epsilon) \in \mathcal{L}^1 \) for every \( \epsilon \in \mathbb{R} \). Therefore, \( H_L \) is \( \mathcal{L}^1 \)-spectrally regular. From \( \tau(\Pi_j) = 1 \) [7, eq. (2.22)] one recovers the well-known formula for the IDOS of the Landau Hamiltonian [17, Appendix B]

\[
N_{H_L}(\epsilon) = \frac{1}{2\pi \ell^2} \sum_{j \in \mathbb{N}_0} \Theta(\epsilon - \lambda_j) .
\]
The associated DOS can be represented by
\[\mu_{H_L}(\epsilon) := \frac{d\epsilon}{2\pi \ell^2} \sum_{j \in \mathbb{N}_0} \delta(\epsilon - \lambda_j)\]
as a sum of Dirac measures concentrated at the Landau levels. The application of Theorem 1.5 provides
\[\text{Tr}_{\text{Dix}}(Q^{-1}_\lambda f(H_L)) = \sum_{j \in \mathbb{N}_0} f(\lambda_j)\]
for every \(f \in C_c(\mathbb{R})\).

6 Some open questions

There are two directions in which the main results presented in Sect. 1 can be generalized.

The first generalization concerns the fact that our Proof of Theorem 1.5 requires the assumption of \(L^1\)-spectral regularity. This is a consequence of the fact that formula (1.11) has been established only for elements \(S \in L^1\) [7,Proposition 2.27], or on a slightly bigger domain according to [7,Theorem 2.28]. However, it is our belief that formula (1.11) should work for every element in the ideal \(\mathcal{I}_\tau\) which is the natural domain of definition of the trace \(\tau\) [7,Remark 2.29]. This point is still an open conjecture which, once confirmed, would allow to extend Theorem 1.5 to every spectrally regular Hamiltonian \(H\). In this work we have not been able to prove this conjecture. However, the residue formula in Theorem 1.1 provides a small step toward the solution of this problem. In fact our residue formula is very reminiscent of the Tauberian criterion (Theorem B.1)
\[\lim_{x \to 0^+} x \text{ Tr} \left( T^{1+x} \right) = \text{Tr}_{\text{Dix}}(T) \quad (6.1)\]
which allows to compute the Dixmier trace of certain \(T \in \mathcal{S}^{1^+}\) as a residue. If one can prove the equality of the two residues in Theorem 1.1 and in (6.1) with \(T = Q^{-1}_\lambda S\), then one would immediately obtain the proof of our conjecture. Also the energy shell formula in Theorem 1.2 is reminiscent of the fact that \(\tau\) should be interpreted as a Dixmier trace on all its domain of definition. In fact, by denoting with \(\phi_r\) the orthonormal basis of eigenvectors of \(Q^{-s}_0\) \((s > 1)\) ordered according to the decreasing sequence of the related eigenvalues (counting the multiplicity) one has that
\[\sum_{j=1}^N w_j(S) = \sum_{r=0}^{d_N} \langle \phi_r, T_S \phi_r \rangle_{L^2}\]
where \( d_N = \frac{1}{2}N(N - 1) - 1 \) and \( T_S \) is one of the operators in (1.12) with \( \lambda = 0 \). For every \( N \in \mathbb{N} \) let \( M(N) := \lfloor \sqrt{2(N + 1)} \rfloor \) where the \( \lfloor \cdot \rfloor \) denotes the integer part. From one hand one has that

\[
M(N)^2 - M(N) \leq M(N)^2 = \left\lfloor \sqrt{2(N + 1)} \right\rfloor^2 \leq 2(N + 1)
\]

which implies \( d_{M(N)} \leq N \). On the other hand

\[
(M(N) + 1)^2 + (M(N) + 1) \geq (M(N) + 1)^2 = \left( \left\lfloor \sqrt{2(N + 1)} \right\rfloor + 1 \right)^2 \geq 2(N + 1)
\]

which implies \( N \leq d_{M(N) + 2} \). Then

\[
\sum_{j=1}^{M(N)} w_j(S) \leq \sum_{r=0}^{N} \langle \phi_r, T_S \phi_r \rangle_{L^2} \leq \sum_{j=1}^{M(N) + 2} w_j(S)
\]

and in turn

\[
\frac{a_N}{\log(M(N))} \sum_{j=1}^{M(N)} w_j(S) \leq \frac{1}{\log(N + 1)} \sum_{r=0}^{N} \langle \phi_r, T_S \phi_r \rangle_{L^2} \leq \frac{b_N}{\log(M(N) + 2)} \sum_{j=1}^{M(N) + 2} w_j(S)
\]

where

\[
a_N := \frac{\log(M(N))}{\log(N + 1)} , \quad b_N := \frac{\log(M(N) + 2)}{\log(N + 1)} .
\]

By observing that

\[
\lim_{N \to +\infty} a_N = \lim_{N \to +\infty} b_N = \frac{1}{2}
\]

and using Theorem 1.2 one gets

\[
\lim_{N \to +\infty} \left( \frac{1}{\log(N + 1)} \sum_{r=0}^{N} \langle \phi_r, T_S \phi_r \rangle_{L^2} \right) = \frac{1}{2} \tau(S) . \quad (6.2)
\]

The series on the left-hand side is very reminiscent of the formula for the computation of the Dixmier trace of modulated operators [16,Corollary 11.2.4 (c)]. However, although this is a strong indication, we have not (yet) been able to adapt the theory of modulated operators [16,Section 11.2] to operators \( T_S \) with \( S \in \mathcal{F}_\tau \).
The second type of generalization concerns the introduction of perturbations by (random) electrostatic potentials. From a mathematical point of view this consists in replacing the magnetic $C^*$-algebra $\mathcal{C}$, which is the twisted group $C^*$-algebra of $\mathbb{R}^2$, with the twisted $C^*$-crossed product generated by the action $t$ of $\mathbb{R}^2$ on the hull of the potentials $\Omega$ (which is a compact and Hausdorff space). In a more concrete way this $C^*$-crossed product can be thought of as the collection of $C^*$-algebras $\mathcal{C}_\omega \subset \mathcal{B}(L^2(\mathbb{R}^2))$ parametrized by $\omega \in \Omega$, where every $\mathcal{C}_\omega$ is polynomially generated by elementary operators of the type $M_g(\omega) \Upsilon_{j\rightarrow k}$. Here $\Upsilon_{j\rightarrow k}$ are the transition operators defined in (2.3), $g \in C(\Omega)$ and $M_g(\omega)$ is the multiplication operator defined by

$$
(M_g(\omega)\phi)(x) = g(t_x(\omega)) \phi(x), \quad x \in \mathbb{R}^2
$$

for every $\phi \in L^2(\mathbb{R}^2)$. It is reasonable to expect that the content of Theorems 1.5 and 1.4 could be extended to the case of the $C^*$-crossed product $\{\mathcal{C}_\omega\}_{\omega \in \Omega}$ provided that a certain averaging procedure with respect to the ergodic probability measure $\mathbb{P}$ on $\Omega$ is introduced. This problem is the object of current investigations.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A: Some properties of the magnetic algebra

In [7,Theorem A.1] it has been proved that every element $T \in \mathcal{M}$ of the magnetic algebra acts on $L^2(\mathbb{R}^2)$ as a twisted convolution operator with an integral kernel given by a suitable tempered distribution $\Psi_T \in S'(\mathbb{R}^2)$. As proved in [10] (just after the proof of Theorem 6) every tempered distribution admits an expansion in terms of the Laguerre basis $\psi_{k,j}$. Since every transition operator $\Upsilon_{j\rightarrow k}$ is nothing more than the twisted convolution with integral kernel $\psi_{k,j}$ (see the proof of [7,Proposition 2.10]) it follows that every $T \in \mathcal{M}$ admits a series representation of the form

$$
T = \sum_{(j,k) \in \mathbb{N}_0^2} t_{j,k} \Upsilon_{j\rightarrow k} \tag{A.1}
$$

where the convergence of the series is meant in the strong (eq. weak) topology. The characterization of the behavior of the series of the coefficients $t_{j,k}$ is generally a difficult task and large part of the papers [10, 11] is devoted to this question. Here we will provide only a quite weak property.

---

6 This result is similar to the $N$-representation theorem [20,Theorem V.14] for $S'(\mathbb{R}^2)$ with the only difference that the Hermite basis is replaced by the Laguerre basis.
Lemma A.1 Let $T \in \mathcal{M}$ and $t_{j,k}$ the coefficients in the expansion (A.1). Then, $|t_{n,k}| \leq \|T\|$ for every $n, m \in \mathbb{N}_0$.

**Proof** Since the elements of the Laguerre basis are normalized, one has that $\|T\psi_{n,m}\|_{L^2} \leq \|T\|$ for every $n, m \in \mathbb{N}_0$. By using the series representation (A.1) and the relation (2.3) one gets

$$T\psi_{n,m} = \sum_{(j,k) \in \mathbb{N}_0^2} t_{j,k}(\delta_{j,n}\psi_{k,m}) = \sum_{k \in \mathbb{N}_0} t_{n,k}\psi_{k,m}$$

and in turn

$$\|T\psi_{n,m}\|_{L^2}^2 = \sum_{k \in \mathbb{N}_0} |t_{n,k}|^2 \leq \|T\|^2$$

for every $n \in \mathbb{N}_0$. As a consequence none of the coefficients $|t_{n,k}|$ can exceed $\|T\|$.

Lemma A.1 enters in the Proof of Lemma 2.1.

**Proof of Lemma 2.1** Let

$$A_i = \sum_{(j,k) \in \mathbb{N}_0^2} a_{j,k}^{(i)} \Upsilon_{j\rightarrow k} , \quad i = 1, 2$$

be two elements in $\mathcal{L}^1$ and $T \in \mathcal{M}$. By using the series representation (A.1) one gets

$$A_1TA_2 = \sum_{(r,s) \in \mathbb{N}_0^2} \sum_{(j,k) \in \mathbb{N}_0^2} \sum_{(p,q) \in \mathbb{N}_0^2} a_{r,s}^{(1)}t_{j,k}a_{p,q}^{(2)}(\Upsilon_{r\rightarrow s}\Upsilon_{j\rightarrow k}\Upsilon_{p\rightarrow q})$$

and since

$$\Upsilon_{r\rightarrow s}\Upsilon_{j\rightarrow k}\Upsilon_{p\rightarrow q} = \delta_{j,q}\delta_{r,k}\Upsilon_{p\rightarrow s}$$

one ends with

$$A_1TA_2 = \sum_{(p,q) \in \mathbb{N}_0^2} \kappa_{p,s}\Upsilon_{p\rightarrow s}$$

where

$$\kappa_{p,s} := \sum_{(r,q) \in \mathbb{N}_0^2} a_{r,s}^{(1)}t_{q,r}a_{p,q}^{(2)}.$$
By invoking Lemma A.1 one has that

\[ \sum_{(p,s) \in \mathbb{N}_0^2} |\kappa_{p,s}| \leq \sum_{(p,s) \in \mathbb{N}_0^2} \sum_{(r,q) \in \mathbb{N}_0^2} |a_{r,s}^{(1)}| |a_{p,q}^{(2)}| \]
\[ \leq \|T\| \sum_{(p,s) \in \mathbb{N}_0^2} \sum_{(r,q) \in \mathbb{N}_0^2} |a_{r,s}^{(1)}| |a_{p,q}^{(2)}| \]
\[ = \|T\| \|a_{r,s}^{(1)}\|_{\ell^1} \|a_{p,q}^{(2)}\|_{\ell^1} \leq +\infty . \]

Therefore, the coefficients \( \kappa_{p,s} \) are in \( \ell^1(\mathbb{N}_0^2) \) and as a consequence \( A_1 T A_2 \in \mathcal{L}^1 \).

\[ \square \]

Appendix B: A primer on the Dixmier trace

There are several standard references for the theory of the Dixmier trace, see e.g. [5, Chap. 4, Sect. 2], [4, Appendix A], [12, Sect. 7.5 & App. 7.C], [16, Chapter 6], [1]. Here, we will recall only the basic facts concerning the Dixmier trace. Let \( \mathcal{H} \) be a separable Hilbert space. The singular values \( \mu_n(T) \) of the compact operator \( T \in \mathcal{K}(\mathcal{H}) \) are, by definition, the eigenvalues of \( |T| := \sqrt{T^* T} \). By convention the singular values will be listed in decreasing order, repeated according to their multiplicity, i.e.

\[ \mu_0(T) \geq \mu_1(T) \geq \ldots \geq \mu_n(T) \geq \mu_{n+1}(T) \geq \ldots \geq 0 . \]

In particular \( \mu_0(T) = \| |T| \| = \|T\| \). Let

\[ \sigma_N^p(T) := \sum_{n=0}^{N-1} \mu_n(T)^p , \quad p \in [1, +\infty) . \quad (B.1) \]

A compact operator \( T \) is in the \( p \)th Schatten ideal \( \mathcal{S}^p \), if and only if,

\[ \|T\|_p^p := \lim_{N \to \infty} \sigma_N^p(T) < +\infty . \]

Accordingly, \( \mathcal{S}^1 \) is the ideal of trace-class operators. Let

\[ \gamma_N(T) := \frac{\sigma_N^1(T)}{\log(N)} = \frac{1}{\log(N)} \sum_{n=0}^{N-1} \mu_n(T) , \quad N > 1 . \quad (B.2) \]

A compact operator \( T \) is in the Dixmier ideal \( \mathcal{S}^{1+} \) if its (Calderón) norm

\[ \|T\|_{1+} := \sup_{N > 1} \gamma_N(T) < +\infty \quad (B.3) \]
is finite. It turns out that \( S^1 \) is a two-sided self-adjoint ideal which is closed with respect to the norm (B.3) (but not with respect to the operator norm). The set of operators such that \( \lim_{N \to \infty} \gamma_N(T) = 0 \) forms an ideal inside \( S^1 \) denoted with \( S^1_0 \). The latter coincides with the closure for the norm (B.3) of the ideal of finite-rank operators. The chain of (proper) inclusions \( S^1 \subset S^1_0 \subset S^1 \) holds true for every \( \epsilon > 0 \). To define a trace functional with domain the Dixmier ideal \( S^1 \) we need to fix a generalized scale-invariant limit\(^7\) \( \omega : \ell^\infty(\mathbb{N}) \to \mathbb{C} \). The \( \omega \)-Dixmier trace of a positive element of the Dixmier ideal is defined as

\[
\text{Tr}_{\text{Dix}, \omega}(T) := \omega(\{\gamma_N(T)\}_N), \quad T \in S^1_0, \quad T \geq 0.
\]

The definition of \( \text{Tr}_{\text{Dix}, \omega} \) extends to non-positive elements of \( S^1 \) by linearity. The \( \omega \)-Dixmier trace provides an example of a singular (hence non-normal) trace and it is continuous with respect to the norm (B.3) i.e., \( |\text{Tr}_{\text{Dix}, \omega}(T)| \leq \|T\|_{1^+} \). Every Dixmier trace fulfills the cyclicity property

\[
\text{Tr}_{\text{Dix}, \omega}(TA) = \text{Tr}_{\text{Dix}, \omega}(AT), \quad \forall \ T \in S^1_0, \ A \in \mathcal{B}(\mathcal{H})
\]

the Hölder inequalities\(^{123}\) which, in the special case \( p = 1 \) and \( q = +\infty \), provide

\[
\text{Tr}_{\text{Dix}, \omega}(|ATB|) \leq \|A\|\|B\| \text{Tr}_{\text{Dix}, \omega}(|T|), \quad \forall \ T \in S^1_0, \ A, B \in \mathcal{B}(\mathcal{H}).
\]

An element \( T \in S^1 \) is called measurable if the value of \( \omega(\{\gamma_N(T)\}_N) \) is independent of the choice of the generalized scale-invariant limit \( \omega \). For a positive element \( T \geq 0 \) this is equivalent to the convergence of a certain Cesàro mean of \( \gamma_N(T) \). Moreover, the set of measurable operators \( S^1 \) is a closed subspace of \( S^1 \) (but not an ideal) which is invariant under conjugation by bounded invertible operators [5, Chap. 4, Sect. 2, Proposition 6]. Evidently, \( S^1_0 \subset S^1 \). A compact operator \( T \) is called Tauberian [16, Definition 9.7.1] if the limit

\[
\lim_{N \to \infty} \left( \frac{1}{\log(N)} \sum_{n=0}^{N-1} \lambda_n(T) \right) = L,
\]

exists. Here, \( \lambda_n(T) \) denotes an eigenvalue sequence [16, Definition 1.1.10] of \( T \). A non-negative operator \( T \geq 0 \) is Tauberian if and only if \( T \in S^1 \) is measurable [16, Theorem 9.3.1], and in that case

\[
\text{Tr}_{\text{Dix}}(T) := \lim_{N \to \infty} \left( \frac{1}{\log(N)} \sum_{n=0}^{N-1} \mu_n(T) \right)
\]

\(^7\) A generalized scale-invariant limit is a continuous positive linear functional \( \omega : \ell^\infty(\mathbb{N}) \to \mathbb{C} \) which coincides with the ordinary limit on the subspace of convergent sequences and is invariant under “dilations” of the sequences of the type \( \{a_1, a_2, a_3, \ldots\} \mapsto \{a_1, a_1, a_2, a_2, a_3, a_3, \ldots\} \).
where the equality $\lambda_n(T) = \mu_n(T)$ has been used. However, not every element in $\mathcal{S}_m^{1+}$ is Tauberian as shown in [16,Example 9.7.6].

The following result provides a useful criterion to determine whether $\text{Tr}_{\text{Dix}}(T)$ is independent of $\omega$ and to compute its value.

**Theorem B.1** (The Tauberian criterion) Let $T \geq 0$ be a non-negative compact operator such that $T^{1+x} \in \mathcal{S}^1$ for every $x > 0$, and define the zeta function

$$
\zeta_T(x) := \text{Tr} \left( (T^{1+x}) \right).
$$

Then, the residue condition

$$
\lim_{x \to 0^+} x \, \zeta_T(x) = L
$$

implies that $T \in \mathcal{S}_m^{1+}$ and

$$
\text{Tr}_{\text{Dix}}(T) = \lim_{N \to \infty} \frac{1}{\log(N)} \sum_{n=0}^{N-1} \mu_n(T) = L,
$$

independently of the choice of $\omega$.

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