TORSION CLASSES IN THE EQUIVARIANT CHOW GROUPS OF ALGEBRAIC TORI

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Abstract. We give an example of an algebraic torus $T$ such that the group $\text{CH}^2(BT)_{\text{tor}}$ is non-trivial. This answers a question of Blinstein and Merkurjev.

1. Introduction

Let $F$ be a field, and let $G$ be a linear algebraic group over $F$. Let $i \geq 0$ be an integer, let $V$ be a linear representation of $G$ over $F$, and assume that there exists a $G$-invariant open subscheme $U$ of $V$ such that $U$ is the total space of a $G$-torsor $U \to U/G$ and $V \setminus U$ has codimension at least $i + 1$ in $V$. Following B. Totaro [14, Definition 1.2], we define

$$\text{CH}^i(BG) := \text{CH}^i(U/G).$$

This definition does not depend on the choice of $V$ and $U$; see [14, Theorem 1.1]. The graded abelian group $\text{CH}^\ast(BG) := \bigoplus_{i \geq 0} \text{CH}^i(BG)$ has the structure of a commutative ring with identity.

If $T$ is a split $F$-torus, and $\hat{T}$ is the character lattice of $T$, then there is a canonical isomorphism $\text{Sym}(\hat{T}) \simeq \text{CH}^1(BT)$. Thus, if $T$ has rank $n$, $\text{CH}^1(BT)$ is a polynomial ring with $n$ generators in degree 1, and in particular its underlying additive group is torsion-free.

When $G$ is a finite group, a lot of work on $\text{CH}^\ast(BG)$ has been carried out by a number of authors, for example N. Yagita [19], P. Guillot [8] and Totaro. Totaro’s book [15] is devoted to the study of $\text{CH}^\ast(BG)$ and to its relation to the group cohomology of $G$.

When $G$ is a split reductive group, there is an extensive literature dealing with computations of $\text{CH}^\ast(BG)$. For instance, the ring $\text{CH}^\ast(BG)$ has been computed for $G = \text{GL}_n, \text{SL}_n, \text{Sp}_{2n}$ by Totaro [14], for $G = \text{O}_n, \text{SO}_{2n+1}$ by Totaro and R. Pandharipande [14] [11], for $G = \text{SO}_{2n}$ by R. Field [6], for $G_2$ by N. Yagita [18], for $\text{PGL}_n$ by G. Vezzosi [16], and for $\text{PGL}_p$ (additively) by A. Vistoli [17].

Let $F_s$ be a separable closure of $F$, let $\mathcal{G} := \text{Gal}(F_s/F)$ be the absolute Galois group of $F$. If $X$ is an $F$-scheme, we define $X_s := X \times_F F_s$. When $G$ is not assumed to be split, a lot less is known about $\text{CH}^\ast(BG)$. Assume that $G = T$ is an $F$-torus, not necessarily split. Then we have canonical isomorphisms

$$\text{CH}^1(BT) \simeq \text{CH}^1(BT_s)^\mathcal{G} \simeq (\hat{T}_s)^\mathcal{G}.$$
On the other hand, the natural homomorphism
\[ \text{CH}^2(BT) \to \text{CH}^2(BT_s) \]
is not surjective in general; many examples can be obtained from [1, Lemma 4.2, Theorem 4.10, Theorem 4.13].

When \( X \) is a smooth variety over \( F \), the natural map
\[ \text{CH}^2(X) \to \text{CH}^2(X_s) \]
is in general neither injective nor surjective, that is, Galois descent for codimension 2 cycles may fail. It is a difficult and interesting problem to study the kernel and cokernel of the previous map, even for special families of varieties \( X \). An extensive literature is devoted to it; here we limit ourselves to mention the works of A. Pirutka [13], J.-L. Colliot-Thélène [4], Colliot-Thélène and B. Kahn [3], and R. Parimala and V. Suresh [12].

Since \( \text{CH}^2(BT_s) \) is torsion-free, a norm argument shows that
\[ \ker(\text{CH}^2(BT) \to \text{CH}^2(BT_s)) = \text{CH}^2(BT)_{\text{tors}} \]
where \( \text{CH}^2(BT)_{\text{tors}} \) is the torsion subgroup of \( \text{CH}^2(BT) \). The group \( \text{CH}^2(BT)_{\text{tors}} \) plays a prominent role in work of S. Blinstein and A. Merkurjev, where it appears as the first term of the exact sequence of [1, Theorem B]. In [1, Theorem 4.7], Blinstein and Merkurjev showed that \( \text{CH}^2(BT)_{\text{tors}} \) is finite and \( 2 \cdot \text{CH}^2(BT)_{\text{tors}} = 0 \). They posed the following question.

**Question 1.1.** ([1, Question 4.9]) Is \( \text{CH}^2(BT)_{\text{tors}} \) trivial for every torus \( T \)?

Merkurjev studied this question further in [10]. He showed that \( \text{CH}^2(BT)_{\text{tors}} = 0 \) in many cases, for example:

- when \( BT \) is 2-retract rational, by [10, Corollary 5.5];
- when the 2-Sylow subgroups of the splitting group of \( T \) are cyclic or Klein four-groups, by [10, Proposition 2.1(2), Example 4.3, and Corollary 5.3];
- when \( \text{char } F = 2 \), by [10, Corollary 5.5];
- when \( T = R_{E/F}(G_m)/G_m \) and \( E/F \) is a finite Galois extension, by [10, Example 4.2, Corollary 5.3].

The purpose of this paper is to show that Question 1.1 has a negative answer.

**Theorem 1.2.** There exist a field \( F \) and an \( F \)-torus \( T \) such that \( \text{CH}^2(BT)_{\text{tors}} \) is not trivial.

In our example, the splitting group \( G \) of \( T \) is a 2-Sylow subgroup of the Suzuki group Sz(8), and \( F = \mathbb{Q}(V)^G \), where \( V \) is a faithful representation of \( G \) over \( \mathbb{Q} \). The group \( G \) has order 64: no counterexample with a splitting group of smaller order can be detected using our method. The torus \( T \) has dimension \( 2^{12} - 2^6 + 1 = 4033 \).

The paper is structured as follows. In Section 2, we recall a construction due to Merkurjev [10], which to every \( G \)-lattice \( L \) associates an abelian group \( \Phi(G, L) \). By a result of Merkurjev, to show that Question 1.1 has a negative answer, it suffices to exhibit \( G \) and \( L \) such that \( \Phi(G, L) \neq 0 \); see Theorem 2.3. This reduces Question 1.1 to a problem in integral representation theory. In Section 3, we associate to every finite group \( G \) a \( G \)-lattice \( M \). In Sections 4 and 5 we show that if the group cohomology of \( G \) with \( \mathbb{Z}/2 \) coefficients satisfies a certain condition, then \( \Phi(G, M) \neq 0 \); see Proposition 5.3(b). Finally, in Section 6, we show that the condition of Proposition 5.3(b) is satisfied when \( G \) is a 2-Sylow subgroup of Sz(8).
2. Merkurjev’s reformulation of Question 1.1

Let $G$ be a finite group, and let $L$ be a $G$-lattice, that is, a finitely generated free $G$-module. By definition, the second exterior power $\bigwedge^2(L)$ of $L$ is the quotient of $L \otimes L$ by the subgroup generated by all elements of the form $x \otimes x, x \in L$. We denote by $\Gamma^2(L)$ the factor group of $L \otimes L$ by the subgroup generated by $x \otimes y + y \otimes x$, $x, y \in L$. We write $x \land y$ for the coset of $x \otimes y$ in $\bigwedge^2(L)$, and $x \star y$ for the coset of $x \otimes y$ in $\Gamma^2(L)$.

We have a short exact sequence

$$0 \to L/2 \to \Gamma^2(L) \to \bigwedge^2(L) \to 0,$$

where $i(x + 2L) = x \star x$, and $\pi(x \star y) = x \land y$. We write

$$\alpha_L : H^1(G, \bigwedge^2(L)) \to H^2(G, L/2)$$

for the connecting homomorphism for (2.1). Recall that a $G$-lattice is called a permutation lattice if it admits a permutation basis, i.e., a $\mathbb{Z}$-basis stable under the $G$-action. A $G$-lattice $L'$ is said to be stably equivalent to $L$ if there exist permutation $G$-lattices $P$ and $P'$ such that $L \oplus P \simeq L' \oplus P'$.

Lemma 2.1. (a) Assume that $L$ is a permutation $G$-lattice, and let $x_1, \ldots, x_n$ be a permutation basis of $L$. Then the homomorphism

$$\Gamma^2(L) \to L/2, \quad x_i \star x_j \mapsto 0 \ (i \neq j), \quad x_i \star x_i \mapsto x_i + 2L,$$

defines a splitting of (2.1). Moreover, the homomorphism

$$\bigwedge^2(L) \to \Gamma^2(L), \quad x_i \land x_j \mapsto x_i \star x_j \ (i < j)$$

is a section of $\pi$.

(b) Let $L'$ be a $G$-lattice stably equivalent to $L$. Then $\text{Im}(\alpha_L) \simeq \text{Im}(\alpha_{L'})$.

Proof. This is contained in [10, §2].

Remark 2.2. In Lemma 2.1(a), the homomorphism $\Gamma^2(L) \to L/2$ is clearly independent of the ordering of the $x_i$. Since $x \land y = -y \land x$ and $x \star y = -y \star x$ for every $x, y \in L$, the homomorphism $\bigwedge^2(L) \to \Gamma^2(L)$ sends $x_i \land x_j$ to $x_i \star x_j$ for every $i \neq j$. In particular, the homomorphism $\bigwedge^2(L) \to \Gamma^2(L)$ is also independent of the choice of the ordering of the $x_i$.

Recall that a coflasque resolution of $L$ is a short exact sequence of $G$-lattices

$$0 \to L'' \to L' \to L \to 0$$

such that $L'$ is a permutation $G$-lattice and $L''$ is coflasque, i.e., $H^1(H, L'') = 0$ for every subgroup $H$ of $G$. By [5, Lemme 3], coflasque resolutions always exist. Let (2.2) be a coflasque resolution of $L$, and define

$$\Phi(G, L) := \text{Im}(\alpha_{L''}).$$

By Lemma 2.1(b) and [5, Lemme 5], this definition does not depend on the choice of a coflasque resolution of $L$.

The following is a reformulation of Question 1.1 purely in terms of integral representation theory, and is the starting point for the present work. It is due to Merkurjev [10], and builds upon the results of Blinstein-Merkurjev [1].

Theorem 2.3. Let $F$ be a field, let $T$ be an $F$-torus with character lattice $\hat{T}$, minimal splitting field $E$ and splitting group $G = \text{Gal}(E/F)$. 

(a) The group $\text{CH}^2(BT)_{\text{tors}}$ is a quotient of $\Phi(G, \hat{T})$.

(b) Let $V$ be a faithful representation of $G$ over $\mathbb{Q}$, and assume that $E = \mathbb{Q}(V)$ and $F = \mathbb{Q}(V)^G$. Then $\Phi(G, \hat{T}) \simeq \text{CH}^2(BT)_{\text{tors}}$.

Proof. (a) This is [10, Corollary 5.2].

(b) Let $T^\circ$ be the dual torus of $T$. For every field extension $K/F$, there is a pairing

$$
H^1(K, T^\circ) \times H^1(K, T) \xrightarrow{\cup} H^3(K, \mathbb{Q}/\mathbb{Z}(2)), \quad (\alpha, \beta) \mapsto \alpha \cup \beta;
$$

see [1, (4-5)]. By definition, the left kernel of (2.3) is the subgroup of all $\alpha \in H^1(F, T^\circ)$ such that for every field extension $K/F$ and every $\beta \in H^1(K, T)$ we have $\alpha_K \cup \beta = 0$.

By [10, Proposition 5.6], the left kernel of the pairing (2.3) is isomorphic to $\Phi(G, \hat{T})$. By [1, Theorem B], the left kernel of (2.3) is isomorphic to $\text{CH}^2(BT)_{\text{tors}}$.

\[ \square \]

3. The Example

Let $G$ be a finite group. In this section we define a $G$-lattice $M$. In Proposition 5.3, we will show that $\Phi(G, M) \neq 0$ when the group cohomology of $G$ with $\mathbb{Z}/2$ coefficients satisfies a certain condition.

The $G$-lattice $\mathbb{Z}[G \times G]$ is a free $\mathbb{Z}[G]$-module, and has a canonical permutation basis $\{(g, g') : g, g' \in G\}$. The free $G$-lattice $\mathbb{Z}[G] \oplus \mathbb{Z}[G]$ has a canonical basis $\{(g, 0), (0, g) : g \in G\}$. Let $M := \sum_{g \in G} g\mathbb{Z}[G]$. Consider the homomorphism

$$
\rho : \mathbb{Z}[G]^{|G|^2} \rightarrow \mathbb{Z}[G \times G], \quad (g, 0) \mapsto \sum_{g' \in G} (g, g'), \quad (0, g) \mapsto \sum_{g' \in G} (g', g).
$$

Then $\text{Ker} \rho = \langle (\gamma, -\gamma) \rangle$ has rank 1 and trivial $G$-action. Define

$$
M := \text{Coker} \rho,
$$

and let $\pi : \mathbb{Z}[G \times G] \rightarrow M$ be the natural projection.

Lemma 3.1. The $G$-module $M$ is $\mathbb{Z}$-free.

Proof. Let

$$
x = \sum_{g, g' \in G} a_{g, g'}(g, g') \in \mathbb{Z}[G \times G].
$$

It is easy to show that $x \in \text{Im} \rho$ if and only if there exist integers $b_g, c_g$ (where $g \in G$) such that $a_{g, g'} = b_g + c_{g'}$ for every $g, g' \in G$.

Assume that $nx \in \text{Im} \rho$. Thus, for every $g, g' \in G$, we can find integers $p_g, q_{g'}$ such that $na_{g, g'} = p_g + q_{g'}$. We may write $p_g = nb_g + p'_{g'}$, $q_{g'} = nc_{g'} - q'_{g'}$, where $0 \leq p'_g < n$ and $0 \leq q'_{g'} < n$ for all $g, g' \in G$. Then $n(a_{g, g'} - b_g - c_{g'}) = p'_g - q'_{g'}$. In particular, $n$ divides $p'_g - q'_{g'}$. On the other hand, we have $|p'_g - q'_{g'}| < n$, hence $p'_g = q'_{g'}$. We conclude that $a_{g, g'} = b_g + c_{g'}$ for every $g, g' \in G$, hence $x \in \text{Im} \rho$. This shows that $M$ is torsion-free, as desired.

By construction, we have a long exact sequence

$$
0 \rightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}[G]^{|G|^2} \xrightarrow{\rho} \mathbb{Z}[G \times G] \xrightarrow{\pi} M \rightarrow 0,
$$

(3.1)
where \( i(1) := (\gamma, -\gamma) \). We set \( N := \text{Im} \rho \), and we split (3.1) into the two short exact sequences

\[
0 \to N \to \mathbb{Z}[G \times G] \xrightarrow{\pi} M \to 0
\]

(3.2)

and

\[
0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}[G]^{\oplus 2} \to N \to 0.
\]

(3.3)

Let

\[
0 \to R \to P \xrightarrow{\pi'} M \to 0
\]

(3.4)

be a coflasque resolution of \( M \). We define a \( G \)-lattice \( Q \) and homomorphisms \( \iota_1 : Q \to \mathbb{Z}[G \times G] \) and \( \iota_2 : Q \to P \) by the short exact sequence

\[
0 \to Q \xrightarrow{(\iota_1, -\iota_2)} \mathbb{Z}[G \times G] \oplus P \xrightarrow{\pi \pm \pi'} M \to 0.
\]

(3.5)

Thus we have a commutative diagram

\[
\begin{array}{cccccc}
0 & 0 & & & & \\
\uparrow & & & & & \\
0 & N & \mathbb{Z}[G \times G] & \xrightarrow{\pi} M & 0 & \\
\uparrow \iota_1 & \uparrow & \uparrow \pi' & & & \\
0 & N & Q & \xrightarrow{\iota_2} P & 0 & \\
\uparrow & & \uparrow & & \uparrow & \\
R & R & & R & & \\
0 & 0 & & & & \\
\end{array}
\]

where the rows and columns are exact, and the top right square is a pullback square. For every subgroup \( H \) of \( G \) we have \( H^1(H, \mathbb{Z}[G \times G]) = H^1(H, R) = 0 \), hence from the diagram \( H^1(H, Q) = 0 \), that is, \( Q \) is coflasque. It follows that (3.5) is a coflasque resolution of \( M \).

4. The map \( \alpha_N \)

We maintain the notation of the previous section. The purpose of the next two sections is the proof of Proposition 5.3, which gives a sufficient condition for \( \Phi(G, M) \neq 0 \). In order to prove Proposition 5.3, we must understand the map \( \alpha_Q \).

In this section we study \( \alpha_N \), and in the next section we use the acquired knowledge to derive information on \( \alpha_Q \).

**Lemma 4.1.** Let

\[
0 \to \mathbb{Z} \xrightarrow{\psi} X \xrightarrow{\psi} Y \to 0
\]

be a short exact sequence of \( G \)-lattices, and let \( \sigma := \varphi(1) \). Then we have a short exact sequence of \( G \)-lattices

\[
0 \to Y \xrightarrow{\eta} \Lambda^2(X) \xrightarrow{\Lambda^2 \psi} \Lambda^2(Y) \to 0.
\]

Here, for every \( y \in Y \), we set \( \eta(y) := x \wedge \sigma \), where \( x \in X \) is any element satisfying \( \psi(x) = y \).
It is clear that $\eta$ is a homomorphism of $G$-lattices, that $\wedge^2 \psi$ is surjective, and that $(\wedge^2 \psi) \circ \eta = 0$.

Since $Y$ is torsion-free, we may complete $\sigma$ to a $\mathbb{Z}$-basis $x_1, \ldots, x_n, \sigma$ of $X$. Then $\psi(x_1), \ldots, \psi(x_n)$ is a basis of $Y$. An arbitrary element $z$ of $\wedge^2(X)$ may be uniquely written as

$$z = \sum_{1 \leq i < j \leq n} a_{ij} x_i \wedge x_j + \sum_{i=1}^{n} b_i x_i \wedge \sigma$$

for suitable integers $a_{ij}, b_i$. In particular, $x_1 \wedge \sigma, \ldots, x_n \wedge \sigma$ are linearly independent. Since $\eta(\psi(x_i)) = x_i \wedge \sigma$ for every $i$, we deduce that $\eta$ is injective. Moreover, the free $\mathbb{Z}$-module $\wedge^2(Y)$ has a basis consisting of $\psi(x_i) \wedge \psi(x_j)$, for $0 \leq i < j \leq n$. We conclude that $z$ belongs to $\text{Ker} \wedge^2 \psi$ if and only if $a_{ij} = 0$ for every $i, j$, that is, if and only if $z$ belongs to $\text{Im} \eta$. \hfill $\square$

Let $C_2 = \langle \tau \rangle$ be the cyclic group of order 2, and let $\mathbb{Z}^-$ be the $C_2$-lattice of rank 1 on which $\tau$ acts by $-\text{Id}$. If $g \in G$ is an element of order 2, we denote by $\text{Ind}_G^G(Z^-)$ the $G$-lattice induced by the $(g)$-lattice $\mathbb{Z}^-$. 

**Lemma 4.2.** Let $G$ be a finite group of order $n$, let $g_1, \ldots, g_n$ be the elements of $G$, and let $\gamma := \sum_{i=1}^{n} g_i \in \mathbb{Z}[G]$. If $m \in \mathbb{Z}[G]^{\oplus 2}$, we let $\overline{m} \in N$ be its image under the homomorphism $\mathbb{Z}[G]^{\oplus 2} \to N$ of (3.3).

(a) We have a short exact sequence

$$0 \to N \xrightarrow{\eta} \wedge^2(\mathbb{Z}[G]^{\oplus 2}) \to \wedge^2 N \to 0,$$

where for every $m \in \mathbb{Z}[G]^{\oplus 2}$ we have $\eta(\overline{m}) = m \wedge (\gamma, -\gamma)$.

(b) We have

$$\wedge^2(\mathbb{Z}[G]) \simeq \mathbb{Z}[G]^{\oplus r} \oplus \bigoplus_{g^2 = e, g \neq e} \text{Ind}_G^G(\mathbb{Z}^-)$$

for some $r \geq 0$.

(c) For every $i \geq 1$, we have

$$H^i(G, \wedge^2(\mathbb{Z}[G]^{\oplus 2})) \simeq \bigoplus_{g^2 = e, g \neq e} H^{i+1}(\langle g \rangle, \mathbb{Z})^{\oplus 2}.$$

(d) The coboundary $\partial : H^1(G, \wedge^2 N) \to H^2(G, N)$ associated to (4.1) is surjective.

**Proof.** (a) This follows from Lemma 4.1, applied to (3.3).

(b) Recall that $\{g_i \wedge g_j : i < j\}$ is a basis of $\wedge^2(\mathbb{Z}[G])$. It follows that a subset of $\{g \wedge h : g, h \in G, g \neq h\}$ is linearly dependent if and only if it contains a subset of the form $\{g \wedge h, g \wedge h \wedge g\}$ for some distinct elements $g, h$ of $G$.

Let $e \in G$ be the identity element. We may write

$$G = \{e\} \amalg S_1 \amalg S_2 \amalg S_2^{-1},$$
where $S_1 = \{ g \in G \setminus \{ e \} : g^2 = e \}$, and for every $g \in G$ such that $g^2 \neq e$, exactly one of \( g, g^{-1} \) belongs to $S_2$. For every $g \in G \setminus \{ e \}$, let $M_g$ be the $G$-sublattice of $\wedge^2(\mathbb{Z}[G])$ generated by $g \wedge e$, that is, the $\mathbb{Z}$-submodule generated by $\{hg \cdot h : h \in G\}$.

Assume first that $g \in S_2$. We have a $G$-homomorphism $f_g : \mathbb{Z}[G] \to M_g$ given by sending $e \mapsto g \wedge e$. The homomorphism $f_g$ is surjective because $M_g$ is generated by $g \wedge e$ as a $G$-lattice. The $G$-orbit of $g \wedge e$ has $n$ elements, and since $g^2 \neq e$, it does not contain $e \wedge g$, and so it is a linear independent set of $n$ elements. It follows that $M_g \cong \mathbb{Z}[G]$ if $g \in S_2$.

Assume now that $g \in S_1$ (so $n$ is even). Then
\[
g(g \wedge e) = g^2 \wedge g = e \wedge g = -g \wedge e.
\]
Let $\text{Res}^G_{\langle g \rangle} (M_g)$ be the restriction of $M_g$ to $\langle g \rangle$. The previous calculation shows that sending $1 \in \mathbb{Z}^\times$ to $g \wedge e$ gives a well-defined homomorphism of $\langle g \rangle$-modules $\mathbb{Z}^\times \to \text{Res}^G_{\langle g \rangle} (M_g)$. By definition, we have the identifications $\text{Ind}^G_{\langle g \rangle} (\mathbb{Z}^\times) \cong \mathbb{Z}^\times \otimes_{\mathbb{Z}[\langle g \rangle]} \mathbb{Z}[G]$ and $\text{Res}^G_{\langle g \rangle} (M_g) \cong \text{Hom}_{\langle g \rangle} (\mathbb{Z}[G], M_g)$, hence the tensor-hom adjunction yields a canonical isomorphism
\[
\text{Hom}_G (\text{Ind}^G_{\langle g \rangle} (\mathbb{Z}^\times), M_g) \cong \text{Hom}_{\langle g \rangle} (\mathbb{Z}^\times, \text{Res}^G_{\langle g \rangle} (M_g)).
\]
We obtain an induced $G$-homomorphism $f_g : \text{Ind}^G_{\langle g \rangle} (\mathbb{Z}^\times) \to M_g$. As $M_g$ is generated by $g \wedge e$ as a $G$-lattice, the homomorphism $f_g$ is surjective. It is clear that $\text{Ind}^G_{\langle g \rangle} (\mathbb{Z}^\times)$ has rank $n/2$. The orbit of $g \wedge e$ has $n$ elements, and by the above calculation it contains $e \wedge g$, hence it is closed under taking the opposite. Thus $M_g$ also has rank $n/2$, hence $M_g \cong \text{Ind}^G_{\langle g \rangle} (\mathbb{Z}^\times)$ if $g \in S_1$.

To prove (b), it is thus enough to establish a $G$-equivariant direct sum decomposition
\[
\wedge^2(\mathbb{Z}[G]) = \oplus_{g \in S_1 \cup S_2} M_g.
\]
As an abelian group, $\wedge^2(\mathbb{Z}[G])$ is generated by $\{ g' \wedge g : g, g' \in G, g \neq g' \}$. Since $g^{-1}(g' \wedge g) = g^{-1}g' \wedge e$, we see that $\wedge^2(\mathbb{Z}[G])$ is generated by $\{ g \wedge e : g \in G \setminus \{ e \} \}$ as a $G$-lattice. If $g \in S_2^{-1}$, then $g^{-1} \in S_2$ and $g^{-1}(g \wedge e) = e \wedge g^{-1} = -g^{-1} \wedge e$, hence $\wedge^2(\mathbb{Z}[G])$ is generated by $\{ g \wedge e : g \in S_1 \cup S_2 \}$ as a $G$-lattice. In other words, $\wedge^2(\mathbb{Z}[G])$ is the sum of the $M_g$ for $g \in S_1 \cup S_2$.

We conclude by a rank computation. Note that $|S_1| + 2|S_2| = n - 1$. We know that $M_g$ has rank $n/2$ if $g \in S_1$, and rank $n$ if $g \in S_2$. Therefore, the right hand side of (4.2) has rank $|S_1| \cdot (n/2) + |S_2| \cdot n = n(n - 1)/2$. This is equal to the rank of $\wedge^2(\mathbb{Z}[G])$, hence (4.2) holds and the proof of (b) is complete.

(c) We have a short exact sequence of $C_2$-lattices
\[
0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \to \mathbb{Z}^\times \to 0.
\]
We have $H^i(C_2, \mathbb{Z}[C_2]) = 0$ for every $i \geq 1$, hence the cohomology long exact sequence associated to (4.3) gives $H^i(C_2, \mathbb{Z}^\times) \cong H^{i+1}(C_2, \mathbb{Z})$ for every $i \geq 1$. We have a $G$-equivariant decomposition
\[
\wedge^2(\mathbb{Z}[G]) \cong \wedge^2(\mathbb{Z}[G]) \oplus \wedge^2(\mathbb{Z}[G]) \oplus (\mathbb{Z}[G] \otimes \mathbb{Z}[G]).
\]
The conclusion now follows from (b) and the fact that $H^i(G, \mathbb{Z}[G]) = 0$ for every $i \geq 1$. 

\[
(4.3)
0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \to \mathbb{Z}^\times \to 0.
\]
(d) The group $H^3(C_2, \mathbb{Z})$ is trivial. By (c), we deduce that $H^2(G, \wedge^2(\mathbb{Z}[G]^{\oplus 2}))$ is also trivial. The cohomology long exact sequence associated to (4.1) gives

$$H^1(G, \wedge^2(N)) \xrightarrow{\partial} H^2(G, N) \rightarrow H^2(G, \wedge^2(\mathbb{Z}[G]^{\oplus 2})) = 0,$$

proving the surjectivity of $\partial$. \hfill \Box

**Lemma 4.3.** There exists a homomorphism $h : H^1(G, \wedge^2(N)) \rightarrow H^1(G, \wedge^2(N))$ making the following diagram commute:

$$
\begin{array}{ccc}
H^1(G, \wedge^2(N)) & \xrightarrow{\partial} & H^2(G, N) \\
\downarrow h & & \downarrow \pi_2 \\
H^1(G, \wedge^2(N)) & \xrightarrow{\alpha_N} & H^2(G, N/2).
\end{array}
$$

Here $\partial$ is the connecting homomorphism of (4.1), and $\alpha_N$ is the connecting homomorphism of (2.1) with $L = N$.

**Proof.** As in Lemma 4.2, if $m$ is an element of $\mathbb{Z}[G]^{\oplus 2}$, we denote by $m \in N$ its image under the homomorphism $\mathbb{Z}[G]^{\oplus 2} \rightarrow N$ of (3.3). Let $n$ be the order of $G$, let $g_1, \ldots, g_n$ be the elements of $G$, and denote by $(g_1, 0), \ldots, (g_n, 0), (0, g_1), \ldots, (0, g_n)$ the canonical permutation basis of $\mathbb{Z}[G]^{\oplus 2}$. By Lemma 2.1(a), this choice of basis yields a section $s : \wedge^2(\mathbb{Z}[G]^{\oplus 2}) \rightarrow \Gamma^2(\mathbb{Z}[G]^{\oplus 2})$

$$s : \wedge^2(\mathbb{Z}[G]^{\oplus 2}) \xrightarrow{s} \Gamma^2(\mathbb{Z}[G]^{\oplus 2})$$

$$(g_i, 0) \wedge (g_j, 0) \mapsto (g_i, 0) \star (g_j, 0), \quad i < j,$$

$$(g_i, 0) \wedge (0, g_j) \mapsto (g_i, 0) \star (0, g_j), \quad i \neq j,$$

$$(0, g_i) \wedge (0, g_j) \mapsto (0, g_i) \star (0, g_j), \quad i < j.$$ We let

$$f : \wedge^2(\mathbb{Z}[G]^{\oplus 2}) \xrightarrow{f} \Gamma^2(\mathbb{Z}[G]^{\oplus 2}) \rightarrow \Gamma^2(N),$$

where the second homomorphism is induced by (3.3). Let $\eta : N \rightarrow \wedge^2(\mathbb{Z}[G]^{\oplus 2})$ be the injection of Lemma 4.2(a), and let $s : N/2 \rightarrow \Gamma^2(N)$ be the injective homomorphism of (2.1) with $L = N$. We claim that the square

$$
\begin{array}{ccc}
N & \xrightarrow{\eta} & \wedge^2(\mathbb{Z}[G]^{\oplus 2}) \\
\downarrow \pi_2 & & \downarrow f \\
N/2 & \xrightarrow{s} & \Gamma^2(N)
\end{array}
$$

commutes. It is enough to verify the commutativity on the $(g_i, 0)$ and the $(0, g_i)$. By Lemma 4.2(a), we have

$$\eta((g_i, 0)) = (g_i, 0) \wedge (\gamma, -\gamma) = \sum_{j \neq i} (g_i, 0) \wedge (g_j, 0) - \sum_{j=1}^n (g_i, 0) \wedge (0, g_j).$$
From the definition of $s$ and Remark 2.2, we see that
\[
s(\eta(g_1, 0)) = \sum_{j \neq i}^n (g_i, 0) \ast (g_j, 0) - \sum_{j=1}^n (g_i, 0) \ast (0, g_j)
\]
\[= (g_i, 0) \ast (\gamma, -\gamma) - (g_i, 0) \ast (g_i, 0)
\]
\[= (g_i, 0) \ast (\gamma, -\gamma) + (g_i, 0) \ast (g_i, 0).
\]
Since $(\gamma, -\gamma) = 0$, we have
\[
f(\eta((g_i, 0))) = (g_i, 0) \ast (\gamma, -\gamma) + (g_i, 0) \ast (g_i, 0) = (g_i, 0) \ast (g_i, 0) = \iota(\pi_2((g_i, 0))).
\]
The proof of the commutativity of (4.5) on the $(0, g_i)$ follows by symmetry. Therefore (4.5) commutes, and so there exists $\overline{T} : \Lambda^2(N) \to \Lambda^2(N)$ making the diagram
\[
\begin{array}{ccc}
0 & \rightarrow & N \\
\downarrow \pi_2 & & \downarrow f \\
0 & \rightarrow & N/2 \rightarrow \Gamma^2(N) \rightarrow \Lambda^2(N) \rightarrow 0.
\end{array}
\]

5. REDUCTION TO GROUP COHOMOLOGY WITH CONSTANT COEFFICIENTS

Recall that if $S$ is a permutation lattice, by Lemma 2.1(a) the choice of a permutation basis $x_1, \ldots, x_n$ for $S$ defines a splitting of the sequence
\[
0 \to S/2 \to \Gamma^2(S) \to \Lambda^2(S) \to 0.
\]
More precisely, we get a section $\Lambda^2(S) \to \Gamma^2(S)$ by sending $x_i \wedge x_j \mapsto x_i \ast x_j$ for every $i < j$, and a retraction $\Gamma^2(S) \to S/2$ by sending $x_i \ast x_j \mapsto 0$ if $i < j$, and $x_i \ast x_i \mapsto x_i + 2S$.

Applying this to $S = \mathbb{Z}[G \times G]$, with the canonical basis $\{(g_i, g_j)\}_{i,j}$, we obtain a homomorphism
\[
\varphi_N : \Gamma^2(N) \to \Gamma^2(\mathbb{Z}[G \times G]) \to \mathbb{Z}[G \times G]/2,
\]
where the map on the left is induced by (3.2).

We now fix a permutation basis $\{x_k\}$ of the $G$-lattice $P$ of (3.4). Using the canonical basis $\{(g_i, g_j)\}_{i,j}$ of $\mathbb{Z}[G \times G]$, this extends to a permutation basis of $\mathbb{Z}[G \times G] \oplus P$. We obtain a homomorphism
\[
\varphi_Q : \Gamma^2(\mathbb{Z}[G \times G] \oplus P) \to (\mathbb{Z}[G \times G] \oplus P)/2,
\]
where $Q$ was defined in (3.5), and where the homomorphism on the left is induced by the map $(\iota_1, \iota_2)$ of (3.5). We stress that $\varphi_N$ and $\varphi_Q$ depend on the choice of a permutation basis for $\mathbb{Z}[G \times G]$ and $P$. It may be helpful for the reader to note that $\varphi_N$ and $\varphi_Q$ do not depend on the choice of ordering of the bases; see Remark 2.2.
Lemma 5.1. We have a commutative diagram with exact rows

\begin{align*}
0 & \longrightarrow N/2 \longrightarrow \Gamma^2(N) \longrightarrow \Lambda^2(N) \longrightarrow 0 \\
0 & \longrightarrow Q/2 \longrightarrow \Gamma^2(Q) \longrightarrow \Lambda^2(Q) \longrightarrow 0 \\
0 & \longrightarrow Q/2 \longrightarrow (\mathbb{Z}[G \times G] \oplus P)/2 \longrightarrow M/2 \longrightarrow 0 \\
0 & \longrightarrow N/2 \longrightarrow \mathbb{Z}[G \times G]/2 \longrightarrow M/2 \longrightarrow 0.
\end{align*}

Here, the first two rows are (2.1) for \( L = N, Q \). The third and fourth rows are the reductions modulo 2 of (3.5) and (3.2), respectively.

Proof. By the definition of \( \varphi_N \) and \( \varphi_Q \), we have a commutative diagram

\begin{align*}
\Gamma^2(N) & \longrightarrow \Gamma^2(\mathbb{Z}[G \times G]) \longrightarrow \mathbb{Z}[G \times G]/2 \\
\Gamma^2(Q) & \longrightarrow \Gamma^2(\mathbb{Z}[G \times G] \oplus P) \longrightarrow (\mathbb{Z}[G \times G] \oplus P)/2,
\end{align*}

where the composition of the top horizontal homomorphisms is \( \varphi_N \), and the composition of the bottom horizontal homomorphisms is \( \varphi_Q \). The commutativity of the square on the left follows from the functoriality of \( \Gamma^2(-) \). The square on the right commutes because the choice of permutation basis that we have made respects the decomposition \( \mathbb{Z}[G \times G] \oplus P \). We deduce that the diagram

\begin{align*}
\Gamma^2(N) & \xrightarrow{\varphi_N} \mathbb{Z}[G \times G]/2 \\
\Gamma^2(Q) & \xrightarrow{\varphi_Q} (\mathbb{Z}[G \times G] \oplus P)/2
\end{align*}

is commutative. We now show that the four squares on the left side of the diagram commute. The commutativity of

\begin{align*}
N/2 & \longrightarrow \Gamma^2(N) \longrightarrow \mathbb{Z}[G \times G]/2 \\
Q/2 & \longrightarrow \Gamma^2(Q) \longrightarrow (\mathbb{Z}[G \times G] \oplus P)/2
\end{align*}

is clear. We have a commutative diagram

\begin{align*}
Q/2 & \longrightarrow \Gamma^2(Q) \\
(\mathbb{Z}[G \times G] \oplus P)/2 & \longrightarrow \Gamma^2(\mathbb{Z}[G \times G] \oplus P) \longrightarrow (\mathbb{Z}[G \times G] \oplus P)/2,
\end{align*}

where the square comes from the functoriality of (2.1), and the triangle from the definition of \( \varphi_Q \). By construction, the composition of the horizontal homomorphisms
at the bottom is the identity. Therefore, the square
\[ \begin{array}{ccc}
Q/2 & \longrightarrow & \Gamma^2(Q) \\
\downarrow & & \downarrow_{\rho_{Q}} \\
Q/2 & \hookrightarrow & (\mathbb{Z}[G \times G] \oplus P)/2
\end{array} \]
appearing in the diagram of the lemma commutes. A similar reasoning yields the commutativity of
\[ \begin{array}{ccc}
N/2 & \longrightarrow & \Gamma^2(N) \\
\downarrow & & \downarrow_{\rho_{N}} \\
N/2 & \hookrightarrow & \mathbb{Z}[G \times G]/2.
\end{array} \]
The existence and commutativity of the right side of the diagram of the lemma now follow from the universal property of cokernels. \qed

For every \( i \geq 0 \), let \( \pi_2 : H^i(G, \mathbb{Z}) \rightarrow H^i(G, \mathbb{Z}/2) \) be the homomorphism of reduction modulo 2, and let \( \text{Sq}^1 : H^i(G, \mathbb{Z}/2) \rightarrow H^{i+1}(G, \mathbb{Z}/2) \) be the first Steenrod square, that is, the Bockstein homomorphism for the sequence \( 0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0 \). We also denote by \( \beta : H^i(G, \mathbb{Z}/2) \rightarrow H^{i+1}(G, \mathbb{Z}) \) the Bockstein homomorphism for \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0 \). It is well known and easy to show that \( \text{Sq}^1 = \pi_2 \circ \beta \).

For every subgroup \( H \), let \( \text{Cor}_H^G : H^*(H, \mathbb{Z}/2) \rightarrow H^*(G, \mathbb{Z}/2) \) denote the corestriction homomorphism, and let \( \tau_H \) be the composition
(5.1) \[ \tau_H : H^1(H, \mathbb{Z}/2) \otimes H^2(H, \mathbb{Z}) \xrightarrow{\text{Id} \otimes \pi_2} H^1(H, \mathbb{Z}/2) \otimes H^2(H, \mathbb{Z}/2) \xrightarrow{\subseteq} H^3(H, \mathbb{Z}/2) \xrightarrow{\text{Cor}_H^G} H^3(G, \mathbb{Z}/2). \]

**Definition 5.2.** Let \( G \) be a finite group. We define \( V_G \) as the subgroup of \( H^3(G, \mathbb{Z}/2) \) generated by the union of the \( \text{Im} \tau_H \), where \( H \) varies among all subgroups of \( G \).

**Proposition 5.3.**

(a) Assume that \( \text{Im}(\pi_2 : H^3(G, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}/2)) \) is not contained in \( V_G \). Then \( \Phi(G, M) \neq 0 \).

(b) Assume that \( \text{Im}(\text{Sq}^1 : H^2(G, \mathbb{Z}/2) \rightarrow H^3(G, \mathbb{Z}/2)) \) is not contained in \( V_G \). Then \( \Phi(G, M) \neq 0 \).

**Proof.** Following the notation of Merkurjev [10, §3], using (3.5) we define
\[ H^1(G, M/2) := \text{Im}(H^1(G, (\mathbb{Z}[G \times G] \oplus P)/2) \rightarrow H^1(G, M/2)) = \ker(H^1(G, M/2) \rightarrow H^2(G, Q/2)). \]

By [10, Lemma 3.1], \( H^1(G, M/2) \) is generated by the images of the compositions
\[ \sigma_H : H^1(H, \mathbb{Z}/2) \otimes M^H \xrightarrow{\subseteq} H^1(H, M/2) \xrightarrow{\text{Cor}_H^G} H^1(G, M/2), \]
where \( H \) ranges among all subgroups of \( G \). For every subgroup \( H \) of \( G \) and every \( i \geq 0 \), we denote by
\[ d_1 : H^i(H, M) \rightarrow H^{i+1}(H, N), \quad d_2 : H^i(H, N) \rightarrow H^{i+1}(H, Z), \]
\[ \delta_1 : H^i(H, M/2) \rightarrow H^{i+1}(H, N/2), \quad \delta_2 : H^i(H, N/2) \rightarrow H^{i+1}(H, Z/2). \]
the connecting homomorphisms associated to (3.2), (3.3) and their reduction modulo 2. Recall that free $\mathbb{Z}[G]$-modules and free $(\mathbb{Z}/2)[G]$-modules have trivial cohomology in positive degrees. It follows that $d_1, d_2, \delta_1, \delta_2$ are surjective in all non-negative degrees, and isomorphisms in positive degrees.

Let $H$ be a subgroup of $G$. By a double application of [7, Proposition 3.4.8], first to (3.2) and then to (3.3), the diagram

\[
\begin{array}{ccc}
H^1(H, \mathbb{Z}/2) \otimes M^H & \xrightarrow{\cup} & H^1(H, M/2) \\
\downarrow \text{Id} \otimes (d_2 \circ d_1) & & \downarrow \delta_2 \circ \delta_1 \\
H^1(H, \mathbb{Z}/2) \otimes H^2(H, \mathbb{Z}) & \xrightarrow{\cup} & H^3(H, \mathbb{Z}/2)
\end{array}
\]

commutes. Here, the horizontal homomorphism at the bottom is (5.1). Since connecting homomorphisms commute with corestrictions, for every subgroup $H$ of $G$ we obtain the following commutative square:

\[
\begin{array}{ccc}
H^1(H, \mathbb{Z}/2) \otimes M^H & \xrightarrow{\sigma_H} & H^1(G, M/2) \\
\downarrow \text{Id} \otimes (d_2 \circ d_1) & & \downarrow \delta_2 \circ \delta_1 \\
H^1(H, \mathbb{Z}/2) \otimes H^2(H, \mathbb{Z}) & \xrightarrow{\tau_H} & H^3(G, \mathbb{Z}/2).
\end{array}
\]

We conclude that

(5.2) \quad (\delta_2 \circ \delta_1)(H^1(G, M/2)^{(1)}) = V_G.

(a) Consider the commutative diagram

\[
\begin{array}{ccc}
H^1(G, \bigwedge^2(N)) & & \\
H^1(G, M) \xrightarrow{\delta_1} H^2(G, N) \xrightarrow{\delta_2} H^3(G, \mathbb{Z}) \\
\downarrow \pi_2 & & \downarrow \pi_2 & \Delta H^1(G, M/2) \xrightarrow{\delta_1} H^2(G, N/2) \xrightarrow{\delta_2} H^3(G, \mathbb{Z}/2)
\end{array}
\]

where the triangle in the middle comes from Lemma 4.3, the horizontal maps are isomorphisms and $\delta$ is the surjective homomorphism of Lemma 4.2(d). We claim that there exists $x \in H^1(G, \bigwedge^2(N))$ such that

(5.3) \quad \delta_1^{-1}(\alpha_N(x)) \in H^1(G, M/2) \setminus H^1(G, M/2)^{(1)}.

By assumption, there exists $a \in H^3(G, \mathbb{Z})$ such that \(\pi_2(a) \in H^3(G, \mathbb{Z}/2) \setminus V_G\). We let $b := (d_2 \circ d_1)^{-1}(a) \in H^1(G, M)$. By (5.2), $\pi_2(b) \in H^1(G, M/2) \setminus H^1(G, M/2)^{(1)}$. Since $\delta$ is surjective, there exists $c \in H^1(G, \bigwedge^2(N))$ such that $\delta(c) = d_1(b)$. If we let $x := h(c) \in H^1(G, \bigwedge^2(N))$, then

\[
\delta_1^{-1}(\alpha_N(x)) = \delta_1^{-1}(\pi_2(\delta(c))) = \pi_2(d_1^{-1}(\delta(c))) = \pi_2(b),
\]

hence $x$ satisfies (5.3).
Passing to group cohomology in Lemma 5.1, we obtain a commutative diagram

\[
\begin{array}{ccc}
H^1(G, \Lambda^2(N)) & \xrightarrow{\alpha_N} & H^1(G, \Lambda^2(Q)) \\
\downarrow & & \downarrow \\
H^2(G, N/2) & \xrightarrow{\alpha_Q} & H^2(G, Q/2)
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
& k^* & \\
\downarrow & & \downarrow \\
& H^1(G, M/2) & \xrightarrow{\delta_1} & H^2(G, N/2).
\end{array}
\]

Let now \( x \in H^1(G, \Lambda^2(N)) \) be such that (5.3) holds for \( x \), and let \( y \in H^1(G, \Lambda^2(Q)) \) be the image of \( x \). From the commutativity of the outer square of the diagram above, we see that the composition of the homomorphisms of the top row is \( \delta_1^{-1} \circ \alpha_N \). Therefore

\[
k^*(y) = \delta_1^{-1}(\alpha_N(x)) \in H^1(G, M/2) \setminus H^1(G, M/2)^{(1)}.
\]

By definition of \( H^1(G, M/2)^{(1)} \), this means that \( k^*(y) \) does not map to zero in \( H^2(G, Q/2) \). It follows from the commutativity of the middle square of the diagram that \( \alpha_Q(y) \) is a non-zero element of \( H^2(G, Q/2) \). This shows that \( \text{Im} \alpha_Q \neq 0 \), that is, \( \Phi(G, M) \neq 0 \).

(b) Since \( \text{Sq}^1 = \pi_2 \circ \beta \), this follows immediately from (a). \qedhere

**Remark 5.4.** The hypotheses of (a) and (b) are equivalent if \( H^3(G, \mathbb{Z}) \) is 2-torsion. In general we only have that the hypothesis of (b) implies that of (a). However, the hypothesis of (b) is easier to check: if \( H^3(G, \mathbb{Z}) \) has exponent \( 2^m \), to check the hypothesis of (a) one needs to know the degree 3 differentials of the first \( m \) pages of the Bockstein spectral sequence associated to the short exact sequence of \( G \)-modules \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \).

If \( G \) is abelian or has order \( \leq 32 \), we have checked that the hypotheses of (a) and (b) are not satisfied.

### 6. Proof of Theorem 1.2

Let \( G \) be a 2-Sylow subgroup of the Suzuki group \( \text{Sz}(8) \). We start by giving an explicit description of \( G \). Let \( \mathbb{F}_8 \) be the field of 8 elements, let \( \mathbb{F}_8^+ \) be its underlying additive group, and let \( \theta : \mathbb{F}_8 \to \mathbb{F}_8^+ \) be the field automorphism given by \( \theta(a) = a^4 \), so that \( \theta(\theta(a)) = a^2 \). For a pair of elements \( a, b \in \mathbb{F}_8 \), let \( S(a, b) \) be the following lower-triangular matrix with entries in \( \mathbb{F}_8 \):

\[
S(a, b) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
a & 1 & 0 & 0 \\
b & \theta(a) & 1 & 0 \\
a^2 \theta(a) + ab + \theta(b) & a \theta(a) + b & a & 1
\end{pmatrix}
\]

Then \( G \) can be identified with the set of matrices \( S(a, b) \); see [9, Chapter XI, §3]. A matrix calculation shows that

\[
S(a, b)S(c, d) = S(a + c, \theta(a)c + b + d).
\]

It follows that we have a short exact sequence

\[
1 \to \mathbb{Z} \to G \xrightarrow{\pi} \mathbb{C} \to 1,
\]

where \( \mathbb{C} := \mathbb{F}_8^+ \simeq (\mathbb{Z}/2)^3 \), \( \pi(S(a, b)) := a \), and \( \mathbb{Z} \simeq (\mathbb{Z}/2)^3 \) is the center of \( G \). The subgroup \( Z \) coincides with the derived subgroup of \( G \), and so \( \mathbb{C} \) is the abelianization of \( G \); see [9, Chapter 11, Lemma 3.1].
If $R$ is a ring, $A^* = \oplus_{n \geq 0} A^n$ is a graded $R$-algebra, and $i$ is a non-negative integer, we denote by $A^{\leq i}$ the quotient of $A^*$ by the ideal $\oplus_{n > i} A^n$. The ring $H^*(G, \mathbb{Z}/2)$ is extremely complicated. It may be found, together with restriction and corestriction homomorphisms, in the book [2, #153(64) p. 566]. It can also be obtained using a computer algebra software such as Magma or GAP, where $G$ is SmallGroup(64,82). For our purposes, we are only interested in degrees $\leq 3$. We have:

$$H^{\leq 3}(G, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)[z_1, y_1, x_1, w_2, v_2, u_3, t_3, s_3, r_3, q_3, p_3]^{\leq 3}/I.$$  

Here the indices denote the degrees of the variables, and will be dropped in the future. The ideal $I$ is defined by

$$I := (z^2 + yx, zy + zx + x^2, zx + y^2 + yx, zx^2, yx^2, x^3, zw + xv, vz + yw, yv + xw + xv).$$

**Lemma 6.1.** Let $V_G$ be as in Definition 5.2. The class $Sq^1(w)$ does not belong to $V_G$.

**Proof.** Let

$$W_G := \text{Im}(H^1(G, \mathbb{Z}/2) \otimes \mathbb{Z}/2, -1 \rightarrow H^3(G, \mathbb{Z}/2))$$

be the image of the triple cup product map, and let $V_G$ be as in Definition 5.2. We claim that $V_G \subseteq W_G$. Since $H^1(G, \mathbb{Z}) = 0$ and

$$H^2(G, \mathbb{Z}) \simeq H^1(G, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hom}(\overline{G}, \mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Z}/2)^3$$

is 2-torsion, the Bockstein homomorphism $\beta : H^1(G, \mathbb{Z}/2) \rightarrow H^2(G, \mathbb{Z})$ is an isomorphism. It follows that $H^2(G, \mathbb{Z})$ is generated by $\beta(z), \beta(y), \beta(x)$, and so the image of $\pi_2 : H^2(G, \mathbb{Z}) \rightarrow H^2(G, \mathbb{Z}/2)$ is generated by $z^2, y^2, x^2$ (recall that $Sq^1 = \pi_2 \circ \beta$, and that $Sq^1(c) = c^2$ when $c$ has degree 1). It follows that $\text{Im}(\tau_G) \subseteq W_G$, where $\tau_G$ is from (5.1).

If $H$ is a subgroup of $G$ and $K$ is a subgroup of $H$, then $\text{Cor}^G_K = \text{Cor}^H_K \circ \text{Cor}^H_K$, hence to prove that $V_G \subseteq W_G$ it is enough to show that $\text{Im} \text{Cor}^G_K \subseteq W_G$ for every maximal proper subgroup $H$ of $G$. The ideal of $H^*(G, \mathbb{Z}/2)$ generated by the images of the $\text{Cor}^G_K$, where $H$ ranges over all maximal proper subgroups $H$ of $G$, can be read from [2, #153(64) ImTrans, p.569]. In degree $\leq 3$, it coincides with the ideal generated by $zx, yx, x^2$. Therefore, in degree 3 we have

$$\text{Im} \text{Cor}^G_K \subseteq \langle zxa, yxb, x^2c : a, b, c \in H^1(G, \mathbb{Z}/2) \rangle \subseteq W_G,$$

hence $V_G \subseteq W_G$, as claimed.

To finish the proof, it suffices to show that $Sq^1(w)$ does not belong to $W_G$. Let $G_1$ be the first maximal subgroup of $G$ in the list of [2, p. 569] (that is, #1 in MaxRes). Its cohomology ring can be found in [2, #18(32), p. 367]:

$$H^*(G_1, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)[z'_1, y'_1, x'_2, w'_2, v'_2, u'_2, t'_2]/I',$$

for some ideal $I'$, a precise description of which we will not need. By [2, MaxRes, p. 569], the restriction $\text{Res}^G_{G_1}$ sends

$$z \mapsto 0, \quad y \mapsto y', \quad x \mapsto x', \quad w \mapsto w'.$$

Let $G_2$ be the first maximal subgroup of $G_1$ in the list of [2, MaxRes, p. 368] (we have $G_2 \simeq \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$). The cohomology ring of $G_2$ can be read from [2, #2(16), p. 349]:

$$H^*(G_2, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)[z''_1, y''_1, x''_2, w''_2]/((z''_1)^2).$$
By [2, MaxRes #1, p. 368], \( \text{Res}_{G_2}^G \) sends
\[ z' \mapsto z'', \quad y' \mapsto 0, \quad v' \mapsto z''x''. \]
It follows that \( \text{Res}_{G_2}^G = \text{Res}_{G_2}^G \circ \text{Res}_{G_1}^G \) sends
\[ z \mapsto 0, \quad y \mapsto 0, \quad x \mapsto z'', \quad w \mapsto z''x''. \]

Let \( \alpha := z'^1 y'^m x'^n \in H^3(G, \mathbb{Z}/2) \), where \( l, m, n \geq 0 \) and \( l + m + n = 3 \). If \( l \geq 1 \) or \( m \geq 1 \), then clearly \( \text{Res}_{G_2}^G(\alpha) = 0 \). If \( l = m = 0 \), then \( \alpha = x^3 \), and so \( \text{Res}_{G_2}^G(\alpha) = (z'')^3 = 0 \). It follows that \( \text{Res}_{G_2}^G(W_G) = \{0\} \). On the other hand, since Steenrod operations commute with restriction homomorphisms, we have
\[ \text{Res}_{G_2}^G(\text{Sq}^1(w)) = \text{Sq}^1(z''x'') = (z'')^2x'' + z''(x'')^2 = z''(x'')^2 \neq 0. \]
This shows that \( \text{Res}_{G_2}^G(\text{Sq}^1(w)) \) does not belong to \( \text{Res}_{G_2}^G(W_G) = \{0\} \). We conclude that \( \text{Sq}^1(w) \) does not belong to \( W_G \), as desired.

**Proof of Theorem 1.2.** Let \( G \) be a 2-Sylow subgroup of the Suzuki group \( \text{Sz}(8) \). By Lemma 6.1, the image of \( \text{Sq}^1 : H^2(G, \mathbb{Z}/2) \to H^3(G, \mathbb{Z}/2) \) is not contained in \( V_G \). Let \( M \) be the \( G \)-lattice of (3.1). Then by Proposition 5.3(b) we have \( \Phi(G, M) \neq 0 \). Let \( V \) be a faithful representation of \( G \) over \( \mathbb{Q} \), and set \( E := \mathbb{Q}(V) \) and \( F := \mathbb{Q}(V)^G \). The extension \( E/F \) is Galois, with Galois group \( G \). Let \( T \) be an \( F \)-torus split by \( E \) and whose character lattice \( \hat{T} \) is isomorphic to \( M \). By Theorem 2.3(b), we have \( \text{CH}^2(BT)_{\text{tors}} \neq 0 \). \( \square \)

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