Floating functions *

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Abstract

We introduce floating bodies for convex, not necessarily bounded subsets of $\mathbb{R}^n$. This allows us to define floating functions for convex and log concave functions and log concave measures. We establish the asymptotic behavior of the integral difference of a log concave function and its floating function. This gives rise to a new affine invariant which bears striking similarities to the Euclidean affine surface area.

1 Introduction

Two important closely related notions in affine convex geometry are the floating body and the affine surface area of a convex body. The floating body of a convex body is obtained by cutting off caps of volume less or equal to a fixed positive constant $\delta$. Taking the right-derivative of the volume of the floating body gives rise to the affine surface area. This was established for all convex bodies in all dimensions by Schütt and Werner in [54].

The affine surface area was introduced by Blaschke in 1923 [12]. Due to its important properties, which make it an effective and powerful tool, it is omnipresent in geometry. The affine surface area and its generalizations in the rapidly developing $L_p$ and Orlicz Brunn–Minkowski theory are the focus of intensive investigations (see e.g., [25, 37, 57, 63]) and have proven to be a valuable tool in a variety of settings, among them solutions for the affine Bernstein and Plateau problems by Trudinger and Wang [60, 61, 62]. Totally new connections opened up to e.g., PDEs and ODEs (see e.g., the papers [15, 30, 39, 40], lattice polytopes [16] and to concentration of volume, e.g., [21, 35].

A first characterization of affine surface area was achieved by Ludwig and Reitzner [36] and had a profound impact on valuation theory of convex bodies. That started a line of research (see e.g., [27, 34, 40, 52]) leading up to the recent characterization of all centro-affine valuations by Haberl and Parapatits [28].

There is a natural inequality associated with affine surface area, the affine isoperimetric inequality, which states that among all convex bodies, with fixed volume, affine surface area is maximized for ellipsoids. This inequality has sparked interest into affine isoperimetric inequalities with a multitude of results and proved to be the key ingredient in many problems (see e.g., [29, 41, 42, 65, 66]). In particular, it was used to

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*Keywords: 2010 Mathematics Subject Classification: 52A20, 53A15

†Partially supported by NSF grant DMS-1504701
show uniqueness of self-similar solutions of the affine curvature flow and to study its asymptotic behavior by Sapiro & Tannenbaum [50] and by Andrews [2, 3].

There are numerous other applications for affine surface area, such as, the approximation theory of convex bodies by polytopes [14, 20, 26, 47, 56], affine curvature flows [2, 32, 31, 58, 59], information theory [18, 19, 20, 44, 64] and partial differential equations [38]. Very recent developments are the introduction of floating bodies in spherical and hyperbolic space [9, 10]. This has already led to applications in approximation of convex bodies by polytopes [11].

The study of log-concave functions is a natural extension of convexity theory. One of the most important discoveries in recent investigations in this direction is the functional version of the famous Blaschke-Santaló inequality [4, 5, 7, 22, 33].

Here we introduce floating bodies for unbounded convex sets in the same way as for bounded sets by cutting off sets of fixed volume $\delta$. We apply that to the epigraph of a convex function, which is an unbounded convex set. This allows us to define floating functions $\psi_\delta$ for convex functions $\psi$, and consequently for log concave functions $f$ which are of the form $f = e^{-\psi}$, where $\psi$ is convex. Namely we put $f_\delta = e^{-\psi_\delta}$.

Taking the right-derivative of the integral of the floating function $f_\delta$ (see below for the definition) of a log concave function $f = e^{-\psi}$ gives rise to a new affine invariant for log concave functions. This is the content of our main result which reads as follows. There, $\nabla^2 \psi$ denotes the Hessian of the convex function $\psi$.

**Theorem 1.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} dx < \infty$. Let $c_{n+1} = \frac{1}{4} \left( \frac{n+2}{\text{vol}_n(\Omega_1^2)} \right)^{\frac{2}{n+2}}$. Then

$$\lim_{\delta \to 0} \frac{\int_{\mathbb{R}^n} (e^{-\psi(x)} - e^{-\psi_\delta(x)})}{\delta^{2/(n+2)}} dx = c_{n+1} \int_{\mathbb{R}^n} (\det (\nabla^2 \psi(x)))^{\frac{1}{n+2}} e^{-\psi(x)} dx.$$ 

The comparison with convex bodies leads us to call the right hand side of this theorem the affine surface area of the log concave function $f$,

$$\text{as}(f) = \int_{\mathbb{R}^n} (\det (\nabla^2 \psi(x)))^{\frac{1}{n+2}} e^{-\psi(x)} dx.$$ 

This is further justified as the expression shares many properties of the affine surface area for convex bodies. It is invariant under affine transformations with determinant 1 and it is a valuation (see below). A slightly different definition of affine surface area for log concave functions has been suggested in [13]. Both coincide in many cases. We compare the two definitions in section 3.

We lay the foundation for further investigations of floating functions and the affine surface area of log concave functions. The authors believe that both notions are of interest in its own right and will in particular be useful for applications, such as, isoperimetric inequalities.

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2 Floating functions and floating sets

Throughout the paper we will use the following notation. We denote by $B_n^2(x, r)$ the $n$-dimensional closed Euclidean ball centered at $x$ with radius $r$. We write in short $B_n^2 = B_n^2(0, 1)$ for the Euclidean unit ball centered at 0. A convex body $K$ in $\mathbb{R}^n$ is a convex compact subset of $\mathbb{R}^n$ with non-empty interior. $\partial K$ denotes the boundary of $K$ and $S^{n-1} = \partial B_n^2$. Finally, $c, c_0, c_1$ denote absolute constants that may change from line to line.

For background on convex bodies we refer to the books [24, 51] and for background on convex function to [48, 49].

We first recall the definition of the floating body [8, 54].

Let $H$ be a hyperplane. Then there is $u \in S^{n-1}$ and $a \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n : \langle x, u \rangle = a\}$. $H^+ = \{x \in \mathbb{R}^n : \langle x, u \rangle \geq a\}$ and $H^- = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq a\}$ are the two closed half spaces determined by $H$. The hyperplane passing through the point $x$ and being orthogonal to the vector $\xi$ is denoted by $H(x, \xi)$.

Let $K$ be a convex body in $\mathbb{R}^n$ and let $\delta > 0$. Then the (convex) floating body $K_\delta$ of $K$ is the intersection of all halfspaces $H^+$ whose defining hyperplanes $H$ cut off sets of volume at most $\delta$ from $K$,

$$K_\delta = \bigcap_{\{H: \text{vol}_n(H^{-} \cap K) \leq \delta\}} H^+. \quad (1)$$

The floating body exists, i.e., is non-empty, if $\delta$ is small enough and clearly, $K_0 = K$ and $K_\delta \subseteq K$, for all $\delta \geq 0$.

Similarly, we now introduce the analogue notion for convex, closed, not necessarily bounded sets $C$.

**Definition 1 (Floating Set).** Let $C$ be a closed convex subset of $\mathbb{R}^n$ with non-empty interior. For $\delta > 0$, we define the floating set $C_\delta$ by

$$C_\delta = \bigcap_{H: \text{vol}_n(H^{-} \cap C) \leq \delta} \{H^+: \text{vol}_n(H^{-} \cap C) \leq \delta\},$$

where $H$ denotes a hyperplane.

It is clear that $C_\delta$ is a closed convex subset of $C$. While for a convex body $K$, $K_\delta$ is a proper subset of $K$ if $\delta > 0$, it is now possible that $C_\delta = C$ for $\delta > 0$, e.g., when $C$ is a halfspace.

The next proposition states a property of the floating set $C_\delta$ which we will need later. The proof is similar to the one given for the floating body in [55] and we omit it.

**Proposition 1.** Let $C$ be a closed convex subset of $\mathbb{R}^n$. For all $\delta$ such that $C_\delta \neq \emptyset$ and all $x_\delta \in \partial(C_\delta) \cap \text{int}(C)$ there exists a support hyperplane $H$ at $x_\delta$ to $C_\delta$ such that $\delta = \text{vol}_n(C \cap H^-)$. 

3
2.1 Log concave functions

Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a convex function. We always consider in this paper convex functions \( \psi \) such that \( 0 < \int_{\mathbb{R}^n} e^{-\psi(x)} dx < \infty \).

In the general case, when \( \psi \) is neither smooth nor strictly convex, the gradient of \( \psi \), denoted by \( \nabla \psi \), exists almost everywhere by Rademacher’s theorem (see, e.g., [13]), and a theorem of Alexandrov [1] and Busemann and Feller [17] guarantees the existence of the (generalized) Hessian, denoted by \( \nabla^2 \psi \), almost everywhere in \( \mathbb{R}^n \). The Hessian is a quadratic form on \( \mathbb{R}^n \), and if \( \psi \) is a convex function, for almost every \( x \in \mathbb{R}^n \) one has, when \( y \to 0 \), that

\[
\psi(x + y) = \psi(x) + \langle \nabla \psi(x), y \rangle + \frac{1}{2} \langle \nabla^2 \psi(x)(y), y \rangle + o(\|y\|^2).
\]

A function \( f : \mathbb{R}^n \to \mathbb{R}_+ \) is log concave, if it is of the form \( f = \exp(-\psi) \) where \( \psi : \mathbb{R}^n \to \mathbb{R} \) is convex. Recall also that a measure \( \nu \) with density \( e^{-\psi} \) with respect to the Lebesgue measure is called log-concave if \( \psi : \mathbb{R}^n \to \mathbb{R} \) is a convex function.

Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a convex function and let \( \text{epi}(\psi) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \geq \psi(x)\} \) be the epigraph of \( \psi \). Then \( \text{epi}(\psi) \) is a closed convex set in \( \mathbb{R}^{n+1} \) and for sufficiently small \( \delta \) its floating sets \( \text{epi}(\psi)_\delta \) are, by above,

\[
\text{epi}(\psi)_\delta = \bigcap_{H : \text{vol}_{n+1}(H \cap \text{epi}(\psi) \leq \delta)} H^+.
\]  

(2)

It is easy to see that there exists a unique convex function \( \psi_\delta : \mathbb{R}^n \to \mathbb{R} \) such that \( \text{epi}(\psi)_\delta = \text{epi}(\psi_\delta) \). This leads to the definitions of a floating function for convex and log concave functions.

**Definition 2.** Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a convex function. Let \( \text{epi}(\psi) \) be its epigraph. Let \( \delta > 0 \).

(i) The floating function of \( \psi \) is defined to be this function \( \psi_\delta \) such that

\[
\text{epi}(\psi)_\delta = \text{epi}(\psi_\delta).
\]  

(3)

(ii) Let \( f(x) = \exp(-\psi(x)) \) be a log concave function. The floating function \( f_\delta \) of \( f \) is defined as

\[
f_\delta(x) = \exp(-\psi_\delta(x)).
\]  

(4)

Note that when \( \psi \) is affine, \( \psi_\delta = \psi \) and, for \( f = e^{-\psi} \), \( f_\delta = f \).

3 Main Theorem and consequences

Let \( C \) be a closed convex set in \( \mathbb{R}^n \) and let \( z \in \partial C \) be such that \( N_C(z) \), the outer normal vector, is unique. The following notion was introduced for convex bodies in [54]. We
define it in the same way for closed convex sets: We put $r_C(z)$ to be the radius of the biggest Euclidean ball contained in $C$ that touches $C$ in $z$,

$$r_C(z) = \max\{\rho : B_2^\rho(z - \rho N_C(z), \rho) \subset C\}.$$  

(5)

$r_C$ is called the rolling function of $C$. If $N_C(z)$ is not unique, $r_C(z) = 0$. If $C = \text{epi} (\psi)$, we will use, from now on, the notation

$$r_\psi (x) = r_{\text{epi}(\psi)}((x, \psi(x))).$$  

(6)

Since $\psi$ is continuous, the epigraph of $\psi$ is a closed set. For functions $\psi$ such that $e^{-\psi}$ is integrable, we have that $r_\psi(z)$ is bounded and for almost every $x \in \mathbb{R}^n$, $(x, \psi(x))$ is an element of a Euclidean ball contained in the epigraph of $\psi$.

For the remainder of the paper, $c_{n+1}$ will always be

$$c_{n+1} = \frac{1}{2} \left( \frac{n + 2}{\text{vol}_n(B_2^1)} \right)^{\frac{1}{n+1}}.$$  

(7)

**Theorem 1.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} dx < \infty$. Then with $c_{n+1}$ as in (7),

$$\lim_{\delta \to 0} \int_{\mathbb{R}^n} \frac{(e^{-\psi(x)} - e^{-\psi_\delta(x)})}{\delta^{2/(n+2)}} \, dx = c_{n+1} \int_{\mathbb{R}^n} \left( \frac{\det (\nabla^2 \psi(x))}{(1 + \|
abla \psi(x)\|^2)^{\frac{n+2}{2}}} \right) e^{-\psi(x)} \, dx.$$  

(8)

We note that under the assumptions of the theorem, the expression on the right hand side of the theorem is finite. Indeed, for a convex function $\psi$ the following formulas hold for the Gaussian curvature $\kappa_\psi(z)$ and the outer unit normal $N_\psi(z)$ in $z = (x, \psi(x)) \in \partial \text{epi}(\psi)$ (see, e.g., [LS]),

$$\kappa_\psi(z) = \frac{\det (\nabla^2 \psi(x))}{(1 + \|
abla \psi(x)\|^2)^{\frac{n+2}{2}}}.$$  

(9)

and

$$\langle N_\psi(z), e_{n+1} \rangle = \frac{1}{(1 + \|
abla \psi(x)\|^2)^\frac{n}{2}}.$$  

(10)

As $\kappa_\psi(z) = \prod_{i=1}^n \frac{1}{\rho_i^\psi(z)}$, where $\rho_i^\psi(z)$, $1 \leq i \leq n$, are the principal radii of curvature, we have for almost all $x \in \Omega_\psi$ that $r_\psi(x) \leq \frac{1}{(\kappa_\psi(z))^{\frac{1}{n}}}$. With (9) we thus get

$$r_\psi(x) \leq \frac{1}{(\kappa_\psi(z))^{\frac{1}{n}}} = \frac{(1 + \|
abla \psi(x)\|^2)^{\frac{n+2}{2}}}{(\det \nabla^2 \psi(x))^{\frac{1}{n}}}.$$  

Therefore

$$\int_{\mathbb{R}^n} \left( \frac{\det (\nabla^2 (\psi(x)))}{(1 + \|
abla \psi(x)\|^2)^{\frac{n+2}{2}}} \right) e^{-\psi(x)} \, dx \leq \int_{\mathbb{R}^n} \frac{(1 + \|
abla \psi(x)\|^2)^\frac{n}{2}}{r_\psi(x)^{\frac{n+2}{2}}} e^{-\psi(x)} \, dx$$

and we prove in Lemma [S] that the last integral is finite.

If the determinant of the Hessian of $\psi$ is 0 almost everywhere, the right hand term of the theorem is 0. This is in particular the case when $\psi$ is piecewise affine. As noted above, the left hand side of the theorem and the proposition will then also be 0.
We postpone the proof of the theorem and discuss some consequences first. The next Proposition will follow from the lemmas needed for the proof of Theorem 1.

**Proposition 2.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx < \infty$. Then with $c_{n+1}$ as given by (7),

$$
\lim_{\delta \to 0} \frac{\int_{\mathbb{R}^n} |\psi_\delta(x) - \psi(x)| \, e^{-\psi} \, dx}{\delta^{2/(n+2)}} = c_{n+1} \int_{\mathbb{R}^n} \left( \det (\nabla^2 \psi(x)) \right)^{1/(n+2)} \, e^{-\psi(x)} \, dx.
$$

We call the quantity $\int_{\mathbb{R}^n} \left( \det (\nabla^2 \psi(x)) \right)^{1/(n+2)} \, e^{-\psi(x)} \, d(x)$ the **affine surface area** of the log concave function $f = e^{-\psi}$ or, equivalently, the affine surface area of the log concave function $\psi$ and its floating function $\psi_\delta$ and call the expression on the right hand side of the theorem and the corollary the affine surface area $as(\psi)$ of the convex function $\psi$.

We now give reasons why we call (11) affine surface area. We first recall the definition of the $L_p$-affine surface areas $as_p(K)$ for convex bodies $K$. For $-\infty \leq p \leq \infty$, $p \neq -n$, they are defined as 12, 57, 57

$$
as_p(K) = \int_{\partial K} \frac{\kappa_K(z)^{\frac{n}{p+1}}}{\langle z, N_K(z) \rangle^{\frac{n(p+1)}{n+1}}} \, d\mu_K(z). \tag{12}
$$

Here, $N_K(z)$ is the outer normal at $z \in \partial K$, $\mu_K$ is the usual surface area measure on $\partial K$ and $\kappa_K(z)$ is the Gauss curvature at $x$. In particular, for $p = 1$ we get the (usual) affine surface area of $K$,

$$
as(K) = \int_{\partial K} (\kappa_K(z))^{\frac{1}{n+1}} \, d\mu_K(z). \tag{13}
$$

We pass from integration over $\mathbb{R}^n$ in (11) to integration over $\partial \text{epi}(\psi)$ with the change of variable formula $\left( 1 + \|\nabla \psi(x)\|^2 \right)^{\frac{1}{2}} \, dx = d\mu_{\text{epi}(\psi)}$. With (9) we get

$$
as(f) = \int_{\partial \text{epi}(\psi)} (\kappa_{\text{epi}(\psi)}(z))^{\frac{n}{n+1}} \, e^{-\langle z, e_{n+1} \rangle} \, d\mu_{\text{epi}(\psi)}(z). \tag{14}
$$

Thus the expression (11) coincides (for the unbounded convex set $\text{epi}(\psi)$) with the one for the affine surface area of a convex body in $\mathbb{R}^{n+1}$, given in (13). This is one reason to call the quantity the affine surface area of $f$.

Moreover, $as(f)$ has similar properties as $as_1(K)$. Firstly, an affine invariance property holds (with the same degree of homogeneity as the affine surface area for convex bodies in $\mathbb{R}^{n+1}$): For all affine transformations $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det A \neq 0$

$$
as(f \circ A) = |\det A|^{-\frac{n}{n+1}} \, as(f).$$
This identity is easily checked using $\nabla^2_x(\psi \circ A) = A^t \nabla^2 A \psi A$.

Secondly, as for convex bodies, a valuation property holds for $as(f)$, i.e., we have for log concave functions $f_1 = e^{-\psi_1}$ and $f_2 = e^{-\psi_2}$ that

$$as(f_1) + as(f_2) = as(\max(f_1, f_2)) + as(\min(f_1, f_2)),$$

provided $\min(\psi_1, \psi_2)$ is convex.

Another reason comes from the next observation which shows that the definition for affine surface area for a function agrees with the definition for convex bodies if the function is the gauge function $\| \cdot \|_K$ of a convex body $K$ with $0$ in its interior,

$$\|x\|_K = \min\{\alpha \geq 0 : x \in \alpha K\} = \max_{y \in K^o} \langle x, y \rangle = h_{K^o}(x).$$

If $\psi(x) = \frac{\|x\|^2}{2}$, then

$$as\left(\frac{\| \cdot \|_K^2}{2}\right) = \frac{(2\pi)^{\frac{n}{2}}}{\text{vol}_n(B^n_2)} as_{\frac{n}{2}}(K).$$

This was already observed in [18]. There, a slightly different definition of affine surface area for log concave functions $f = e^{-\psi}$ was introduced, namely

$$\int_{\mathbb{R}^n} e^{-\frac{n}{n+2} \psi(x) + \frac{1}{n+2} \langle x, \nabla \psi(x) \rangle} \left(\det \nabla^2 \psi(x)\right)^\frac{n}{n+2} dx. \quad (16)$$

Both definitions coincide for 2-homogeneous functions $\psi$. From an affine isoperimetric inequality proved in [18] for the expression (16) we can deduce the following corollary.

**Corollary 1.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a 2-homogeneous, convex function such that $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} dx < \infty$. Then

$$as(f) \leq (2\pi)^{\frac{n}{n+2}} \left(\int_{\mathbb{R}^n} e^{-\psi(x)} dx\right)^{\frac{n}{n+2}},$$

with equality if and only if there are $a \in \mathbb{R}$ and a positive definite matrix $A$ such that for all $x \in \mathbb{R}^n$

$$\psi(x) = \langle Ax, x \rangle + a.$$

We conjecture that this inequality holds for general, convex functions.

We include the argument for [15] for completeness.

We integrate in polar coordinates with respect to the normalized cone measure $\sigma_K$ of $K$. Thus, if we write $x = r\theta$, with $\theta \in \partial K$, then $dx = n \text{vol}_n(K) r^{n-1} dr d\sigma_K(\theta)$ and we get

$$as\left(\frac{\| \cdot \|^2_K}{2}\right) = \int_{\mathbb{R}^n} \det \left(\nabla^2(\psi(x))\right)^\frac{1}{n+2} e^{-\psi(x)} dx$$

$$= n \text{vol}_n(K) \int_0^{\infty} r^{n-1} e^{-\frac{r^2}{2}} dr \int_{\partial K} \left(\det \nabla^2 \psi(\theta)\right)^\frac{1}{n+2} d\sigma_K(\theta)$$

$$= \frac{(2\pi)^{\frac{n}{2}}}{\text{vol}_n(B^n_2)} \int_{\partial K} \left(\det \nabla^2 \psi(\theta)\right)^\frac{1}{n+2} d\sigma_K(\theta).$$

The relation between the normalized cone measure $\sigma_K$ and the Hausdorff measure $\mu_K$ on $\partial K$ is given by

$$d\sigma_K(x) = \frac{\langle \theta, N_K(\theta) \rangle d\mu_K(\theta)}{n \text{vol}_n(K)}.$$
E.g., Lemma 1 of [18] (and its proof) show that \( \det \nabla^2 \psi(\theta) = \frac{\kappa_{K(\theta)}}{\langle \theta, N_K(\theta) \rangle^{n+1}} \). Thus
\[
\text{as} \left( \frac{\| \cdot \|^2}{2} \right) = \frac{(2\pi)^{\frac{n}{2}}}{n \text{vol}(B_n^2)} \int_{\partial K} \left( \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^{n+1}} \right)^{\frac{n+1}{n+2}} \langle x, N_K(x) \rangle d\mu_K(x)
= \frac{(2\pi)^{\frac{n}{2}}}{n \text{vol}(B_n^2)} \text{as}_{\frac{n}{n+2}}(K).
\]

Finally, the most compelling reason to call the quantity \( \text{as}(f) \) affine surface area is the following theorem proved in [54] in the case of convex bodies in \( \mathbb{R}^n \).
\[
\lim_{\delta \to 0} \frac{\text{vol}_n(K) - \text{vol}_n(K_\delta)}{\delta^{\frac{n}{n+2}}} = c_n \int_{\partial K} (\kappa_K(z))^{\frac{n}{n+2}} d\mu_K(z),
\]
where \( c_n = \frac{1}{2} \left( \frac{n+1}{\text{vol}_{n-1}(B_n^{n-1})} \right)^{\frac{n}{n+2}} \). Theorem [11] is its analogue for log concave functions. Thus this theorem provides a geometric description of affine surface area for such functions.

### 4 Proof of Theorem [11]

We need several lemmas. The first lemma is standard and well known (see, e.g., [56]).

**Lemma 1.** Let
\[
\mathcal{E} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \left| \frac{x_i}{a_i} \right|^2 \leq 1 \right\},
\]
and let \( H_h = H((a_n - h)e_n, e_n) \). Then for all \( h \leq a_n \)
\[
h^{\frac{n+1}{2}} \left( 1 - \frac{h}{2a_n} \right)^{\frac{n+1}{2}} \leq \frac{(n+1) a_n^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}} \text{vol}_{n-1}(B_n^{n-1})} \prod_{i=1}^{n-1} a_i \leq h^{\frac{n+1}{2}}.
\]
In particular, if \( \mathcal{E} = rB_n^2 \) is a Euclidean ball with radius \( r \) in \( \mathbb{R}^n \), then for all \( u \in S^{n-1} \), for \( h \leq r \) and \( H_h = H((r - h)u, u) \),
\[
h^{\frac{n+1}{2}} \left( 1 - \frac{h}{2r} \right)^{\frac{n+1}{2}} \leq \frac{(n+1) \text{vol}_n (rB_n^2 \cap H_h^n)}{2^{\frac{n+1}{2}} \text{vol}_{n-1}(B_n^{n-1})} \leq h^{\frac{n+1}{2}}.
\]

The next lemma is well known. We refer to e.g., [49].

**Lemma 2.** Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a convex function. Then \( \int e^{-\psi(x)} dx \leq \infty \) if and only if for some \( \gamma > 0 \) there exists \( \beta \in (-\infty, \infty) \) such that for all \( x \in \mathbb{R}^n \),
\[
\psi(x) \geq \gamma \| x \| + \beta.
\] (17)
In particular it follows from (17) that
\[
\lim_{\| x \| \to \infty} \psi(x) = \infty.
\] (18)
This property is sometimes called coercivity (see, e.g., [49]).
Lemma 3. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If $0 < \int_{\mathbb{R}^n} e^{-\psi(x)} dx < \infty$, then
\[ \int_{\mathbb{R}^n} e^{-\psi(x)} (1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}} dx < \infty. \]

Proof.

\[ \int_{\mathbb{R}^n} e^{-\psi(x)} (1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}} dx \leq \int_{\mathbb{R}^n} e^{-\psi(x)} (1 + \|\nabla \psi(x)\|) dx \]
\[ = \int_{\mathbb{R}^n} e^{-\psi(x)} dx + \int_{\mathbb{R}^n} e^{-\psi(x)} \left( \sum_{i=1}^{n} \left| \frac{\partial \psi}{\partial x_i}(x) \right|^2 \right)^{\frac{1}{2}} dx \]

The first integral is finite by assumption. We consider the second integral.

\[ \int_{\mathbb{R}^n} e^{-\psi(x)} \left( \sum_{i=1}^{n} \left| \frac{\partial \psi}{\partial x_i}(x) \right|^2 \right)^{\frac{1}{2}} dx \leq \int_{\mathbb{R}^n} e^{-\psi(x)} \sum_{i=1}^{n} \left| \frac{\partial \psi}{\partial x_i}(x) \right| dx \]
\[ = \sum_{i=1}^{n} \int_{\mathbb{R}^n} e^{-\psi(x)} \left| \frac{\partial \psi}{\partial x_i}(x) \right| dx. \]

Let $y = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ and let $m(y) \in \mathbb{R}$ satisfy
\[ \psi((m(y), y) = \min_{x_1 \in \mathbb{R}} \psi(x_1, y). \]

By Lemma 2, $\psi$ satisfies (18). This means that $\psi$ has a global minimum (which needs not be unique, unless $\psi$ is strictly convex) and therefore, $m(y)$ exists. Then

\[ \int_{\mathbb{R}^n} e^{-\psi(x)} \left| \frac{\partial \psi}{\partial x_1}(x) \right| dx_1 \cdots dx_n \]
\[ = \int_{\mathbb{R}^{n-1}} \left( \int_{m(y)}^{\infty} e^{-\psi(x_1, y)} \frac{\partial \psi}{\partial x_1}(x_1, y) dx_1 \right) dy - \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{m(y)} e^{-\psi(x_1, y)} \frac{\partial \psi}{\partial x_1}(x_1, y) dx_1 \right) dy \]
\[ = \int_{\mathbb{R}^{n-1}} \left( \int_{m(y)}^{\infty} - \frac{\partial}{\partial x_1} \left( e^{-\psi(x_1, y)} \right) dx_1 \right) dy + \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{m(y)} \frac{\partial}{\partial x_1} \left( e^{-\psi(x_1, y)} \right) dx_1 \right) dy \]
\[ = 2 \int_{\mathbb{R}^{n-1}} e^{-\psi(m(y), y)} dy. \]

By Lemma 2, one has for all $x_1 \in \mathbb{R}$ that $\psi(x_1, y) \geq \gamma \|(x_1, y)\| + \beta$ for some $\gamma > 0$ and some $\beta$. It follows that $\psi(m(y), y) \geq \gamma \|(m(y), y)\| + \beta \geq \gamma \|y\| + \beta$. It follows that this term is finite.

The other coordinates are treated similarly. This finishes the proof of the lemma. \qed

Lemma 4. (i) Let $x \in \mathbb{R}^n$ be such that the Hessian $\nabla^2 \psi$ at $x$ is positive definite. Then there are constants $\beta_1$ and $\beta_2$ such that for all $\varepsilon > 0$ there is $\delta_0 = \delta_0(x, \varepsilon)$ such that for all $\delta \leq \delta_0$,

\[ (1 - \beta_2 \varepsilon) c_{n+1} \left( \det \left( \nabla^2 \psi(x) \right) \right)^{\frac{1}{2}} \leq \frac{\psi_{\delta}(x) - \psi(x)}{\delta^{n+\frac{1}{2}}} \leq (1 + \beta_1 \varepsilon) c_{n+1} \left( \det \left( \nabla^2 \psi(x) \right) \right)^{\frac{1}{2}}. \]
where \(c_{n+1}\) is given by [7]. Consequently, for \(f = e^{-\psi}\) we get with (new) constants \(\beta_1\) and \(\beta_2\)

\[
(1 - \beta_2 \varepsilon) f(x) \, c_{n+1} \left( \det \left( \nabla^2 \psi(x) \right) \right)^{\frac{1}{n+2}} \leq \frac{f(x) - f_\delta(x)}{\delta^{\frac{1}{n+2}}} \leq (1 + \beta_1 \varepsilon) f(x) \, c_{n+1} \left( \det \left( \nabla^2 \psi(x) \right) \right)^{\frac{1}{n+2}}.
\]

(ii) Let \(x \in \mathbb{R}^n\) be such that \(\det \left( \nabla^2 \psi(x) \right) = 0\). Then for all \(\varepsilon > 0\) there is \(\delta_0 = \delta_0(x, \varepsilon)\) such that for all \(\delta \leq \delta_0\),

\[
0 \leq \frac{\psi_\delta(x) - \psi(x)}{\delta^{\frac{1}{n+2}}} \leq \varepsilon.
\]

Consequently, for \(f = e^{-\psi}\) we get for all \(\varepsilon > 0\) that there is \(\delta_0 = \delta_0(x, \varepsilon)\) such that for all \(\delta \leq \delta_0\),

\[
0 \leq \frac{f(x) - f_\delta(x)}{\delta^{\frac{1}{n+2}}} \leq \varepsilon.
\]

Proof. Let \(0 < \varepsilon < \frac{1}{n+2}\) be given and let \(x_0 \in \mathbb{R}^n\). We put \(z_{x_0} = (x_0, \psi(x_0))\). Denote by \(N_\psi(z_{x_0})\) the outer unit normal in \(z_{x_0}\) to the surface described by \(\psi\). As recalled above, \(N_\psi(z_{x_0})\) exists uniquely for almost all \(x_0\).

(i) We assume that \(x_0\) is such that the Hessian \(\nabla^2 \psi(x_0)\) is positive definite. Then, locally around \(z_{x_0}\), the graph of \(\psi\) can be approximated by an ellipsoid \(E\). We make this precise:

Let \(E\) be such that the lengths of its principal axes are \(a_1, \ldots, a_{n+1}\) and such that its center is at \(z_{x_0} - a_{n+1} N_\psi(z_{x_0})\). Let \(E(\varepsilon^-)\) be the ellipsoid centered at \(z_{x_0} - a_{n+1} N_\psi(z_{x_0})\) whose principal axes coincide with the ones of \(E\), but have lengths \((1 - \varepsilon)a_1, \ldots, (1 - \varepsilon)a_n, a_{n+1}\). Similarly, let \(E(\varepsilon^+)\) be the ellipsoid centered at \(z_{x_0} - a_{n+1} N_\psi(z_{x_0})\), with the same principal axes as \(E\), but with lengths \((1 + \varepsilon)a_1, \ldots, (1 + \varepsilon)a_n, a_{n+1}\). Then

\[z_{x_0} \in \partial E\] and \(N_\psi(z_{x_0}) = N_\psi(z_{x_0}),\)

and (see, e.g., [60]) there exists a \(\Delta_\varepsilon > 0\) such that the hyperplane \(H\left(z_{x_0} - \Delta_\varepsilon N_\psi(z_{x_0}), N_\psi(z_{x_0})\right)\) such that

\[
H^- \left(z_{x_0} - \Delta_\varepsilon N_\psi(z_{x_0}), N_\psi(z_{x_0})\right) \cap E(\varepsilon^-) \subseteq H^- \left(z_{x_0} - \Delta_\varepsilon N_\psi(z_{x_0}), N_\psi(z_{x_0})\right) \cap \{(x, y) : y \geq \psi(x)\} \subseteq H^- \left(z_{x_0} - \Delta_\varepsilon N_\psi(z_{x_0}), N_\psi(z_{x_0})\right) \cap E(\varepsilon^+). \tag{19}
\]

For \(\delta \geq 0\), let \(z_\delta = (x_0, \psi_\delta(x_0))\). We choose \(\delta\) so small that for all support hyperplanes \(H(z_\delta)\) to \(\mathrm{epi}(\psi)_\delta\) through \(z_\delta\) we have

\[
H(z_\delta)^- \cap E(\varepsilon^-) \subseteq H^- \left(z_{x_0} - \Delta_\varepsilon N_\psi(z_{x_0}), N_\psi(z_{x_0})\right) \cap E(\varepsilon^-).
\]

Let \(\Delta_\delta\) be such that

\[
H(z_{x_0} - \Delta_\delta N_\psi(z_{x_0}), N_\psi(z_{x_0})) \tag{20}
\]
is a supporting hyperplane to $\text{epi}(\psi_\delta)$. Moreover, we choose $\delta$ so small that $\Delta_\delta \leq \Delta_\varepsilon$ of (19). As $\partial \text{epi}(\psi)$ is approximated by an ellipsoid in $z_{x_0}$, we have that $z_\delta \in \text{int}(\text{epi}(\psi))$. Thus we get by definition of $\text{epi}(\psi)$, respectively $\psi_\delta$, by Proposition 1 and Lemma 1

$$
\delta \leq \text{vol}_{n+1} \left( H \left( z_{x_0} - \Delta_\delta N_{\psi}(z_{x_0}), N_{\psi}(z_{x_0}) \right) \cap \text{epi}(\psi) \right)
$$

$$
\leq \text{vol}_{n+1} \left( H \left( z_{x_0} - \Delta_\delta N_{\psi}(z_{x_0}), N_{\psi}(z_{x_0}) \right) \cap \mathcal{E}(\varepsilon^+) \right)
$$

$$
\leq \left( 1 + \varepsilon \right)^n \frac{a^n + 2}{n + 2} \prod_{i=1}^{n} \frac{a_i}{\sqrt[4]{\Delta_\delta}}.
$$

As $\kappa_\psi(z_{x_0}) = \prod_{i=1}^{n} \frac{a_i}{\sqrt[4]{\Delta_\delta}}$ (see, e.g., [56]), (11) yields

$$
\Delta_\delta \geq \frac{c_{n+1}}{(1 + \varepsilon)^{\frac{n+2}{4}}} \frac{\left( \text{det} \nabla^2 \psi(z_{x_0}) \right)^{\frac{1}{n+2}}}{\left( 1 + \| \nabla \psi(z_{x_0}) \|^2 \right)^{\frac{1}{2}}},
$$

(21)

where $c_{n+1}$ is as given by (7). By (20)

$$
\Delta_\delta \leq \left< N_\psi(z), c_{n+1} \right> \left( \psi_\delta(x_0) - \psi(x_0) \right).
$$

Therefore, with (10),

$$
\Delta_\delta \leq \frac{\psi_\delta(x_0) - \psi(x_0)}{(1 + \| \nabla \psi(x_0) \|^2)^{\frac{1}{2}}},
$$

and thus with (21) that

$$
\psi_\delta(x_0) - \psi(x_0) \geq \frac{c_{n+1}}{(1 + \varepsilon)^{\frac{n+2}{4}}} \left( \text{det} \nabla^2 \psi(z_{x_0}) \right)^{\frac{1}{n+2}} \delta^{\frac{n+2}{4}}.
$$

(22)

Now we estimate $\delta$ from below. By Proposition 1 there exists a hyperplane $H_\delta$ such that $\delta = \text{vol}_{n+1} \left( H_\delta \cap G_\psi \right)$. By (19),

$$
\delta \geq \text{vol}_{n+1} \left( H_\delta \cap \mathcal{E}(\varepsilon^-) \right).
$$

(23)

The expression $\text{vol}_{n+1} \left( H_\delta \cap \mathcal{E}(\varepsilon^-) \right)$ is invariant under affine transformations with determinant 1. We apply an affine transformation that maps $\mathcal{E}(\varepsilon^-)$ into a Euclidean ball with radius

$$
r = (1 - \varepsilon) \left( \frac{1}{\kappa_\psi(z_{x_0})} \right)^{\frac{1}{n+2}}.
$$

(24)

Now we use Lemma 11 of [54]. Please note that $z_{x_0}$ corresponds to 0 of Lemma 11, that $z_\delta$ corresponds to $z$ and that $N_\psi(z_{x_0})$ corresponds to $N(0) = (0, \cdots, 0, -1)$. We choose $\varepsilon < \varepsilon_0$, where $\varepsilon_0$ is given by Lemma 11, and we choose $\delta$ so small that $\psi_\delta(x_0) - \psi(x_0) = \| z_\delta - z(x_0) \| \leq \varepsilon < \varepsilon_0$. By Lemma 11 (iii) of [54],

$$
\text{vol}_{n+1} \left( H_\delta \cap \mathcal{E}(\varepsilon^-) \right) = \text{vol}_{n+1} \left( H_\delta \cap B^{n+1}_{C} \left( z_{x_0} - r \ N_\psi(z_{x_0}) \right) \right)
$$

$$
\geq \eta(\gamma)^{-n} \text{vol}_{n+1} \left( C \left( r, d_0 \left( 1 - c(\eta(\gamma) - 1) \right) \right) \right),
$$

where $c$ is an absolute constant, $C(r, d_0(1 - c(\eta(\gamma) - 1)))$ is the cap of the $(n + 1)$-dimensional Euclidean ball $B^{n+1}_{C} \left( z_{x_0} - r \ N_\psi(z_{x_0}) \right)$ of height $d_0(1 - c(\eta(\gamma) - 1))$ and $d_0$ is the distance from $z_\delta$ to the boundary of $B^{n+1}_{C} \left( z_{x_0} - r \ N_\psi(z_{x_0}) \right)$. $\gamma = 4\sqrt{2r}/d_0$ and $\eta$ is a monotone function on $\mathbb{R}^+$ such that $\lim_{t \to 0} \eta(t) = 1$. Thus, by Lemma 1

$$
\delta \geq \frac{\frac{a^n + 2}{n + 2} \text{vol}(B^n_2)}{\eta(\gamma)^n} \left( d_0 \left( 1 - c(\eta(\gamma) - 1) \right) \right)^{\frac{n+2}{2}} \left( \frac{1 - d_0(1 - c(\eta(\gamma) - 1))}{2r} \right)^{\frac{n+2}{2}}.
$$

(25)
We apply Lemma 11 (ii) of [54] next. Please note that \( z_n \) of Lemma 11 corresponds to \( z_n = \langle c_{n+1}, N_\psi(z_{x_0}) \rangle (\psi_0(x_0) - \psi(x_0)) \) in our case and \( \frac{C}{|r|} = c_{n+1} \). Then by Lemma 11 (ii),
\[
d_0 \leq \langle c_{n+1}, N_\psi(z_{x_0}) \rangle (\psi_0(x_0) - \psi(x_0)) \leq d_0 + \frac{2d_0^2}{r^2 |\langle c_{n+1}, N_\psi(z_{x_0}) \rangle|^2}. \tag{26}
\]
Thus we get for sufficiently small \( \delta \), with an absolute constant \( \beta_1 \), that
\[
\eta(\gamma) = \eta(4\sqrt{2r}d_0) \leq 1 + \beta_1 \epsilon \tag{27}
\]
and hence
\[
1 - c(\eta(\gamma) - 1) \geq 1 - \beta_2 \epsilon, \tag{28}
\]
with an absolute constant \( \beta_2 \). It follows from (26) that
\[
d_0 \geq \langle c_{n+1}, N_\psi(z_{x_0}) \rangle (\psi_0(x_0) - \psi(x_0)) \left( 1 - \frac{2}{r^2} \frac{\psi_0(x_0) - \psi(x_0)}{|\langle c_{n+1}, N_\psi(z_{x_0}) \rangle|} \right).
\]
We conclude with (10), (24), (26), (27) and (28) that with \( c_{n+1} = \frac{1}{4} \left( \frac{n+2}{\text{vol}(B_2)} \right) \frac{n+2}{n+4} \) and (new) absolute constants \( \beta_1, \beta_2 \),
\[
\delta \frac{n+2}{n+4} \geq 1 - \beta_2 \epsilon \frac{r^{n+2}}{(1 + \epsilon)^{n+2}} c_{n+1} \frac{(1 + \|\nabla\psi(x_0)\|)^2}{2} \left( 1 - \frac{2}{r^2} \frac{\psi_0(x_0) - \psi(x_0)}{(1 + \|\nabla\psi(x_0)\|)^2} \right) \frac{n+2}{n+4} \tag{29}
\]
For \( \delta \) small enough, (9) and (24) give with (new) absolute constants \( \beta_1, \beta_2 \),
\[
\psi_0(x_0) - \psi(x_0) \leq \frac{(1 + \beta_1 \epsilon)^{n+2}}{(1 - \beta_2 \epsilon)^{n+4}} c_{n+1} \left( \text{det} \nabla^2 \psi(x_0) \right)^{\frac{n+2}{n+4}} \delta^{\frac{n+4}{n+2}}.
\]
(ii) Now we assume that \( \text{det} (\nabla^2 \psi(x_0)) = 0 \). Suppose first that there is \( \delta_0 \) such that \( z_{\delta_0} \in \partial \text{epi}(\psi) \). Then \( z_{\delta} \in \partial \text{epi}(\psi) \) for all \( \delta \leq \delta_0 \). As \( z_{\delta} = (x_0, \psi_0(x_0)) \) and \( z_{x_0} = (x_0, \psi(x_0)) \), we thus get that \( \psi_0(x_0) = \psi(x_0) \) for all \( \delta \leq \delta_0 \), and hence \( \psi_0(x_0) - \psi(x_0) = 0 \).

Suppose next that for all \( \delta > 0 \), \( z_{\delta} \in \text{int}(\text{epi}(\psi)) \). As \( \text{det} (\nabla^2 \psi(x_0)) = 0 \), the indicatrix of Dupin at \( z_{x_0} \) is an elliptic cylinder and we may assume that the first \( k \) axes have infinite lengths and the others not. Then, (see e.g., [57], Lemma 23, proof), for all \( \epsilon > 0 \) there is an ellipsoid \( E \) and \( \Delta_\epsilon > 0 \) such that for all \( \Delta \leq \Delta_\epsilon \)
\[
E \cap H^-(z_{x_0} - \Delta N_\psi(z_{x_0}), N_\psi(z_{x_0})) \subset \text{epi}(\psi) \cap H^- (z_{x_0} - \Delta N_\psi(z_{x_0}), N_\psi(z_{x_0}))
\]
and such that the lengths of the \( k \) first principal axes of \( E \) are larger than \( \frac{1}{\epsilon} \). By Proposition [1] there is a hyperplane \( H_\delta \) such that \( z_{\delta} \in H_\delta \) and such that \( \delta = \text{vol}_{n+1} (\text{epi}(\psi) \cap \overline{H_\delta^-} ) \). We choose \( \delta \) so small that
\[
E \cap H_\delta^- \subset E \cap H^- (z_{x_0} - \Delta N_\psi(z_{x_0}), N_\psi(z_{x_0})).
\]
We have
\[
\delta = \text{vol}_{n+1} (\text{epi}(\psi) \cap \overline{H_\delta^-} ) \geq \text{vol}_{n+1} (E \cap \overline{H_\delta^-}).
\]
Now we continue as in [23] and after. We arrive at

\[
\frac{\psi_{\delta}(x_0) - \psi(x_0)}{\delta^{\frac{n}{n+2}}} \leq \frac{(1 + \beta_1 \varepsilon)^{\frac{2n}{n+2}}}{(1 - \beta_2 \varepsilon)^{\frac{2n}{n+2}}} \frac{c_{n+1}}{n+1} \left( \prod_{i=1}^{n} \frac{a_i}{\sqrt{d_{n+1}}} \right)^{\frac{2}{n+1}}
\]

\[
\leq \frac{(1 + \beta_1 \varepsilon)^{\frac{2n}{n+2}}}{(1 - \beta_2 \varepsilon)^{\frac{2n}{n+2}}} \frac{c_{n+1}}{n+1} \left( \prod_{i=k+1}^{n} \frac{a_i}{\sqrt{d_{n+1}}} \right)^{\frac{2}{n+1}} \varepsilon^{\frac{2k+2}{n+1}},
\]

where in the last inequality we have used that for all \(1 \leq i \leq k, a_i = \frac{1}{2}, \)

We require a uniform bound in \(\delta\) for the quantity \(\frac{x_i(x) - \psi(x)}{\delta^{\frac{n}{n+2}}}\) so that we can apply the Dominated Convergence theorem. This is achieved in the next lemma.

**Lemma 5.** Let \(\psi : \mathbb{R}^n \to \mathbb{R}\) be a convex function such that \(0 < \int_{\mathbb{R}^n} e^{-\psi(x)}dx < \infty\). Then there exists \(\delta_0\) such that for all \(\delta < \delta_0\), for all \(x \in \mathbb{R}^n\),

\[
0 \leq \frac{\psi_{\delta}(x) - \psi(x)}{\delta^{\frac{n}{n+2}}} \leq \frac{2n+4}{n+2} \left( \frac{n+2}{\text{vol}_n(B_2^n)} \right)^{\frac{2}{n+2}} \left( 1 + \frac{\|\nabla \psi(x)\|^2}{r_{\psi}(x)^{\frac{n}{n+2}}} \right)^{\frac{1}{2}}.
\]

where \(r_{\psi}(x)\) is as in [23]. Consequently we have for all \(\delta < \delta_0\), for all \(x \in \Omega_{\psi}\),

\[
0 \leq f(x) - f_\delta(x) \leq \frac{2n+4}{n+2} \left( \frac{n+2}{\text{vol}_n(B_2^n)} \right)^{\frac{2}{n+2}} \left( 1 + \frac{\|\nabla \psi(x)\|^2}{r_{\psi}(x)^{\frac{n}{n+2}}} \right)^{\frac{1}{2}} f(x).
\]

**Proof.** Let \(z_x = (x, \psi(x)) \in \partial (\text{epi}(\psi))\) and let \(z_\delta = (x, \psi_\delta(x))\). Let \(r_{\psi}(x) = r_{\text{epi}(\psi)}(z_x)\) be as in [23]. If \(N_{\psi}(z_x)\) is not unique, then \(r_{\text{epi}(\psi)}(z_x) = 0\) and the inequality holds trivially. Moreover, if \(\psi_\delta(x) = \psi(x)\), then \(\psi_\delta(x) - \psi(x) = 0\) and again, the inequality holds trivially.

Thus we can assume that \(N_{\psi}(z_x)\) is unique and \(\psi_\delta(x) > \psi(x)\). By Proposition [1] there is a hyperplane \(H_\delta\) such that \(z_x \in H_\delta\) and

\[
\delta = \text{vol}_{n+1}(H_\delta \cap \text{epi}(\psi)) \geq \text{vol}_{n+1} \left( H_\delta^- \cap B_2^{n+1}(z_x - r_{\psi}(x)N_{\psi}(z_x), r_{\psi}(x)) \right).
\]

We will estimate \(\text{vol}_{n+1} \left( H_\delta^- \cap B_2^{n+1}(z_x - r_{\psi}(x)N_{\psi}(z_x), r_{\psi}(x)) \right)\). We choose \(\delta_0\) so small that for all \(\delta \leq \delta_0, z_x \in H_\delta^-\).

We treat first the case

\[
\psi_\delta(x) - \psi(x) \geq r(x) \langle e_{n+1}, N_{\psi}(z_x) \rangle.
\]

In this case we have for all hyperplanes \(H(z_\delta)\) through \(z_\delta\) and such that \(z_x \in H^{-}(z_\delta)\),

\[
\text{vol}_{n+1} \left( H^{-}(z_\delta) \cap B_2^{n+1}(z_x - r_{\psi}(x)N_{\psi}(z_x), r_{\psi}(x)) \right) \geq
\text{vol}_{n+1} \left( H_0^{-}(z_\delta) \cap B_2^{n+1}(z_x - r_{\psi}(x)N_{\psi}(z_x), r_{\psi}(x)) \right),
\]

where \(H_0(z_\delta)\) is this hyperplane orthogonal to \(x\) and such that both, \(z_x\) and \(z_\delta\) are in \(H_0(z_\delta)\). We can estimate the latter from below by the cone with base \(\frac{1}{2} (\psi_\delta(x) - \psi(x)) B_2^n\)
and height $h \geq (\langle e_{n+1}, N_\psi(z_x) \rangle)^2 \frac{r_\psi(x)}{2}$. Hence, by (31),
\[
\text{vol}_{n+1} \left( H^- (z_\delta) \cap B_2^{n+1} (z_x - r_\psi(x)N_\psi(z_x), r_\psi(x)) \right) \geq
\]
\[
\frac{\text{vol}_n (B_2^n)}{2^{n+1}(n+1)} \left( \langle e_{n+1}, N_\psi(z_x) \rangle \right)^2 r_\psi(x) (\psi_\delta(x) - \psi(x))^n \geq
\]
\[
\frac{\text{vol}_n (B_2^n)}{2^{n+1}(n+1)} \left( \langle e_{n+1}, N_\psi(z_x) \rangle \right)^2 \frac{n+2}{2} r_\psi(x) (\psi_\delta(x) - \psi(x))^\frac{n+2}{2} \geq
\]
\[
\frac{\text{vol}_n (B_2^n)}{2^{n+1}(n+1)} \left( \langle e_{n+1}, N_\psi(z_x) \rangle \right)^\frac{n+2}{2} r_\psi(x)^\frac{n+2}{2} (\psi_\delta(x) - \psi(x))^\frac{n+2}{2}.
\]
Since $\langle e_{n+1}, N_\psi(z_x) \rangle = \frac{1}{(1 + \|\nabla \psi_x\|^2)^{\frac{1}{2}}}$, we get
\[
\frac{\psi_\delta(x) - \psi(x)}{\delta^{\frac{n+2}{2}}} \leq \left( \frac{2^{n+1}(n+1)}{\text{vol}_n (B_2^n)} \right)^\frac{n+2}{2} \frac{r_\psi(x)^\frac{n+2}{2}}{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}. \tag{32}
\]
Now we treat the case
\[
0 < \psi_\delta(x) - \psi(x) < r_\psi(x) \langle e_{n+1}, N_\psi(z_x) \rangle.
\]
For all hyperplanes $H^- (z_\delta)$ through $z_\delta$ such that $z_x \in H^- (z_\delta),$
\[
\text{vol}_{n+1} \left( H^- (z_\delta) \cap B_2^{n+1} (z_x - r_\psi(x)N_\psi(z_x), r_\psi(x)) \right)
\]
is minimal if the line segment $[z_\delta, z_x - r_\psi(x)N_\psi(z_x)]$ is orthogonal to the hyperplane $H^- (z_\delta).$
Then $H^- (z_\delta) \cap B_2^{n+1} (z_x - r(x)N_\psi(z_x), r_\psi(x))$ is a cap of $B_2^{n+1} (z_x - r(x)N_\psi(z_x), r_\psi(x))$ of height $d,$ where $d = \text{dist} \left( z_\delta, \partial B_2^{n+1} (z_x - r_\psi(x)N_\psi(z_x), r_\psi(x)) \right).$ Let $h$ be the height of the cap $H^- (z_\delta) \cap B_2^{n+1} (z_x - r(x)N_\psi(z_x), r_\psi(x))$ and let $\beta$ be the angle between the normal to $H_0$ and the line segment $[z_\delta, z_x - r_\psi(x)N_\psi(z_x)].$

If $\beta = 0,$ then $d = h = \psi_\delta(x) - \psi(x)$ and
\[
\delta \geq \frac{\text{vol}_n (B_2^n)}{2^{n+1}(n+2)} (\psi_\delta(x) - \psi(x))^\frac{n+2}{2} r_\psi(x)^\frac{n+2}{2}
\]
and thus
\[
\frac{\psi_\delta(x) - \psi(x)}{\delta^{\frac{n+2}{2}}} \leq 2 \frac{n+2}{\text{vol}_n (B_2^n)} (\psi_\delta(x) - \psi(x))^\frac{n+2}{2} r_\psi(x)^{-\frac{n+2}{2}}.
\]
Assume now $\beta > 0.$ We first consider the case $h < r_\psi(x).$ Then
\[
\cos \beta = \frac{r_\psi(x) - h}{r_\psi(x) - d} \quad \text{and} \quad \sin \beta = \frac{r_\psi(x) \langle e_{n+1}, N_\psi(z_x) \rangle - (\psi_\delta(x) - \psi(x))}{r_\psi(x) - d}.
\]
From this we get that
\[
d = r_\psi(x) - \left( (r_\psi(x) - h)^2 + (r_\psi(x) \langle e_{n+1}, N_\psi(z_x) \rangle - (\psi_\delta(x) - \psi(x))^2) \right)^{\frac{1}{2}}
\geq r_\psi(x) \left( 1 - \left( 1 + \frac{(\psi_\delta(x) - \psi(x))^2}{r_\psi(x)^2} - 2 \langle e_{n+1}, N_\psi(z_x) \rangle \frac{(\psi_\delta(x) - \psi(x))}{r_\psi(x)} \right) \right)^{\frac{1}{2}}
\geq \langle e_{n+1}, N_\psi(z_x) \rangle (\psi_\delta(x) - \psi(x)) \left( 1 - \frac{\psi_\delta(x) - \psi(x)}{2 r(x) \langle e_{n+1}, N_\psi(z_x) \rangle} \right)
\geq \frac{1}{2} \langle e_{n+1}, N_\psi(z_x) \rangle (\psi_\delta(x) - \psi(x)),$
The latter inequality holds as \( \psi_\alpha(x) - \psi(x) < r_\psi(x) \langle e_{n+1}, N_\psi(z_x) \rangle \).

Thus we get with Lemma 6
\[
\delta \geq \frac{\text{vol}_n(B^2_{\alpha})}{2^{\frac{n+2}{2}}} d^{n+2} r(x)^{\frac{n+2}{2}} \geq \frac{\text{vol}_n(B^2_{\alpha})}{2^{n+1}(n+2)} \langle e_{n+1}, N_\psi(z_x) \rangle^{\frac{n+2}{2}} (\psi_\alpha(x) - \psi(x))^{\frac{n+2}{2}} r_\psi(x)^{\frac{n+2}{2}},
\]
which implies that
\[
\frac{\psi_\alpha(x) - \psi(x)}{\delta^{\frac{1}{n+2}}} \leq 2^{\frac{n+2}{2}} \left( \frac{n+2}{\text{vol}_n(B^2_{\alpha})} \right)^{\frac{1}{n+2}} \frac{(1 + \|\nabla\psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^{\frac{1}{n+2}}}.
\]

If \( h > r_\psi(x) \), then \( \sin \beta \) is as above and \( \cos \beta = \frac{h - r_\psi(x)}{r_\psi(x)^{\frac{n}{2}}} \). We continue as above.

The following lemmas were proved in [54].

**Lemma 6.** [54] Let \( K \) be a convex body in \( \mathbb{R}^n \). Then we have for all \( 0 \leq \alpha < 1 \),
\[
\int_{\partial K} \frac{d\mu_K(x)}{r_K(x)^\alpha} < \infty,
\]
where \( r_K \) is as in (2).

**Lemma 7.** [54] Let \( K \) be a convex body in \( \mathbb{R}^n \) that contains \( B^2_2 \). Then we have for all \( t \) with \( 0 < t \leq 1 \) that \( \{x \in \partial K | r_K(x) \geq t\} \) is a closed set and
\[
(1 - t)^{n-1} \text{vol}_{n-1}(\partial K) \leq \mathcal{H}_{n-1}(\{x \in \partial K : r_K(x) \geq t\}),
\]
where \( \mathcal{H}_{n-1} \) is the \( (n-1) \)-dimensional Hausdorff measure.

When \( K \) contains a Euclidean ball with radius \( \lambda \) we get for all \( 0 < s \leq \lambda \),
\[
\left( 1 - \frac{s}{\lambda} \right)^{n-1} \text{vol}_{n-1}(\partial K) \leq \mathcal{H}_{n-1}(\{x \in \partial K : r_K(x) \geq s\}). \tag{33}
\]
It follows for all \( \alpha \) with \( 0 < \alpha < 1 \),
\[
\int_{\partial K} r_K^{-\alpha} d\mu_K = \int_0^1 \mathcal{H}_{n-1}(\{x \in \partial K | r_K^{-\alpha}(x) > s\}) ds
= \int_0^1 \mathcal{H}_{n-1}(\{x \in \partial K | r_K^{-\alpha}(x) > s\}) ds + \int_1^\infty \mathcal{H}_{n-1}(\{x \in \partial K | r_K^{-\alpha}(x) > s\}) ds
\leq \text{vol}_{n-1}(\partial K) + \text{vol}_{n-1}(\partial K) \int_1^\infty (n-1)s^{\frac{n}{2} - 1} ds
= \text{vol}_{n-1}(\partial K) \left( 1 + \frac{n-1}{\lambda} \left( \frac{1}{\alpha} - 1 \right) \right). \tag{34}
\]

The next lemma is the analogue of Lemma 6 in the present context. The definition of \( r_\psi(x) \) is given in (6).
Lemma 8. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $e^{-\psi}$ is integrable. Then we have for all $0 \leq \alpha < 1$,

$$\int_{\mathbb{R}^n} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx < \infty. \quad (35)$$

In particular, this holds for $\alpha = \frac{1}{n+2}$.

Proof. By Lemma 2, there are $\gamma$ and $\beta$ such that for all $x \in \mathbb{R}^n$

$$\psi(x) \geq \gamma \|x\| + \beta. \quad (36)$$

We can assume that $\alpha > 0$. The case $\alpha = 0$ follows by Lemma 3.

We first assume that $\psi(x) < \infty$ for all $x \in \mathbb{R}^n$. For $k \in \mathbb{N}$, let

$$E_k(\psi) = \{x \in \Omega_\psi : \psi(x) \leq 2^k\}.$$ 

As $\psi$ is convex on $\mathbb{R}^n$, it is continuous on $\mathbb{R}^n$. Therefore the sets $E_k(\psi)$ are convex and closed. By Lemma 2 the sets $E_k(\psi)$ are bounded. For $k \in \mathbb{N}$ we put

$$K_k = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \psi(x) \leq y \leq 2^k\}$$

and

$$A_k = \text{epi}(\psi) \cap \{(x, t) \in \mathbb{R}^{n+1} : 2^k \leq t \leq 2^{k+1}\}. \quad (37)$$

Let $k_0 \in \mathbb{N}$ be the smallest number such that $K_{k_0}$ contains an $(n + 1)$-dimensional Euclidean ball and such that

$$\|\beta\| \leq 2^{k_0-1}. \quad (38)$$

We denote its radius by $\lambda$. We may assume that $\lambda \leq 1$. Thus, by convexity, for $k \geq k_0$ the sets $A_k$ contain an $(n + 1)$-dimensional Euclidean ball with radius $\lambda$. Then

$$\int_{\mathbb{R}^n} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx \leq \int_{E_{k_0}(\psi)} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx + \int_{(E_{k_0}(\psi))^c} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx.$$

By (34), (35) and as $r_{K_{k_0}}(x, \psi(x)) \leq r_\psi(x)$ for all $x \in E_{k_0}$

$$\int_{E_{k_0}(\psi)} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx \leq e^{-\beta} \int_{\partial K_{k_0}} \frac{d\mu_{K_{k_0}}(z)}{r_{K_{k_0}}(z)^\alpha} \leq e^{-\beta} \text{vol}(\partial K_{k_0}) \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right) \leq 2e^{-\beta} \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right) \text{vol}(\{(x, \gamma \|x\| + \beta) : \gamma \|x\| + \beta \leq 2^{k_0}\}).$$

The convex set $\{(x, \gamma \|x\| + \beta) : \gamma \|x\| + \beta \leq 2^{k_0}\}$ is contained in the convex cylinder of height $2^{k_0}$ and radius $\frac{2^{k_0} - \beta}{\gamma}$. By (38), $\frac{2^{k_0} - \beta}{\gamma} \geq 2^{k_0}$. Therefore, the surface area of the
first set is smaller than the surface area of this cylinder. Thus

\[
\int_{E_{t_0}(\psi)} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx \leq 4e^{-\beta} \left(1 + \frac{n}{\alpha} \left(\frac{1}{\alpha} - 1\right)\right) \left(\left(\frac{2k_0 - \beta}{\gamma}\right)^n \text{vol}_n(B^n_2) + 2^{k_0} \left(\frac{2k_0 - \beta}{\gamma}\right)^{n-1} \text{vol}_{n-1}(\partial B^n_2)\right),
\]

which is finite. Let \(G(\psi) = \{(x, \psi(x)) : x \in \mathbb{R}^n\}\) be the graph of \(\psi\) and let

\[
\Gamma_k = G(\psi) \cap \{y \in \mathbb{R}^{n+1} : 2^k \leq y_{n+1} \leq 2^{k+1}\},
\]

We denote by \(P\) the orthogonal projection onto \(\mathbb{R}^n\). Then

\[
\int_{(E_{t_0}(\psi))_C} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} e^{-\psi(x)} \, dx \leq \sum_{k = k_0}^{\infty} e^{-2^k} \int_{P(\Gamma_k)} \frac{(1 + \|\nabla \psi(x)\|^2)^{\frac{1}{2}}}{r_\psi(x)^\alpha} \, dx
\]

\[
\leq \sum_{k = k_0}^{\infty} e^{-2^k} \int_{\Gamma_k} \frac{d\mu_{\Gamma_k}(z)}{r_{\Gamma_k}(z)^\alpha} \leq \sum_{k = k_0}^{\infty} e^{-2^k} \int_{\partial A_k} \frac{d\mu_{A_k}(z)}{r_{A_k}(z)^\alpha}
\]

\[
\leq \sum_{k = k_0}^{\infty} e^{-2^k} \text{vol}_n(\partial A_k) \left(1 + \frac{n}{\alpha} \left(\frac{1}{\alpha} - 1\right)\right). \tag{41}
\]

The last inequality follows by (34), as \(A_k\) contains a ball of radius \(\lambda\).

Recall that \(H(x, \xi)\) denotes the hyperplane through \(x\) and orthogonal to \(\xi\). Then

\[
\partial A_k = \Gamma_k \cup (\text{epi}(\psi) \cap H(2^k e_{n+1}, e_{n+1})) \cup (\text{epi}(\psi) \cap H(2^{k+1} e_{n+1}, e_{n+1})). \tag{42}
\]

We shall show that there is a constant \(\alpha_n\) such that for all \(k \geq k_0,

\[
\text{vol}_n(\partial A_k) \leq \alpha_n \text{vol}_n(\Gamma_k). \tag{43}
\]

As above

\[
\text{vol}_n(\text{epi}(\psi) \cap H(2^k e_{n+1}, e_{n+1})) \leq \left(\frac{2k - \beta}{\gamma}\right)^n \text{vol}_n(B^n_2) \tag{44}
\]

\[
\text{vol}_n(\text{epi}(\psi) \cap H(2^{k+1} e_{n+1}, e_{n+1})) \leq \left(\frac{2^{k+1} - \beta}{\gamma}\right)^n \text{vol}_n(B^n_2). \tag{45}
\]

To show (43), it is enough to show

\[
\text{vol}_n(\text{epi}(\psi) \cap H(2^k e_{n+1}, e_{n+1})) \leq \alpha_n \text{vol}_n(\Gamma_k) \tag{45}
\]

\[
\text{vol}_n(\text{epi}(\psi) \cap H(2^{k+1} e_{n+1}, e_{n+1})) \leq \alpha_n \text{vol}_n(\Gamma_k). \tag{45}
\]

To do so, we apply the Schwarz symmetrization (see e.g. [24, 51]) with axis \(e_{n+1}\) to \(A_k\). Then there is a rotationally invariant function \(\tilde{\psi} : \mathbb{R}^n \to \mathbb{R}\) such that

\[
\text{Schw}(A_k) = \text{epi}(\tilde{\psi}) \cap \{(x, t) : x \in \mathbb{R}^n, 2^k \leq t \leq 2^{k+1}\} \tag{46}
\]

Let

\[
\tilde{\Gamma}_k = G(\tilde{\psi}) \cap \{(x, t) : x \in \mathbb{R}^{n+1}, 2^k \leq t \leq 2^{k+1}\}.
\]
Observe that
\[ \partial \text{Schw}(A_k) = \tilde{\Gamma}_k \cup (\text{epi}(\tilde{\psi}) \cap H(2^k e_{n+1}, e_{n+1})) \cup (\text{epi}(\tilde{\psi}) \cap H(2^{k+1} e_{n+1}, e_{n+1})). \] (47)

There are radii \( r_k \) and \( R_k \) with
\[ \text{epi}(\tilde{\psi}) \cap H(2^k e_{n+1}, e_{n+1}) = B^n_2(2^k e_{n+1}, r_k) \cap H(2^k e_{n+1}, e_{n+1}) \]
\[ \text{epi}(\tilde{\psi}) \cap H(2^{k+1} e_{n+1}, e_{n+1}) = B^n_2(2^{k+1} e_{n+1}, R_k) \cap H(2^{k+1} e_{n+1}, e_{n+1}). \]

We have
\[ \text{vol}_n(\text{epi}(\psi) \cap H(2^k e_{n+1}, e_{n+1})) = \text{vol}_n(\text{epi}(\tilde{\psi}) \cap H(2^k e_{n+1}, e_{n+1})) = r_k^n \text{vol}_n(B^n_2) \] (48)

and
\[ \text{vol}_n(\text{epi}(\psi) \cap H(2^{k+1} e_{n+1}, e_{n+1})) = \text{vol}_n(\text{epi}(\tilde{\psi}) \cap H(2^{k+1} e_{n+1}, e_{n+1})) = R_k^n \text{vol}_n(B^n_2). \] (49)

From the above considerations it follows that
\[ \text{vol}_n(\tilde{\Gamma}_k) \leq \text{vol}_n(\Gamma_k). \] (50)

Indeed, since a Schwarz symmetrization reduces the surface area of a convex body
\[ \text{vol}_n(\partial \text{Schw}(A_k)) \leq \text{vol}_n(\partial(A_k)) \]
and thus by (12) and (17) and as the unions are disjoint
\[ \text{vol}_n(\tilde{\Gamma}_k) + \text{vol}_n(\text{epi}(\tilde{\psi}) \cap H(2^k e_{n+1}, e_{n+1}))) + \text{vol}_n(\text{epi}(\tilde{\psi}) \cap H(2^{k+1} e_{n+1}, e_{n+1})) \]
\[ \leq \text{vol}_n(\Gamma_k) + \text{vol}_n(\text{epi}(\psi) \cap H(2^k e_{n+1}, e_{n+1}))) + \text{vol}_n(\text{epi}(\psi) \cap H(2^{k+1} e_{n+1}, e_{n+1}))). \]

By (48) and (49), the inequality (50) follows.

We show now that for some constant \( b_n \)
\[ b_n \frac{R^n_k}{2^n} \text{vol}_n(B^n_2) \leq \text{vol}_n(\tilde{\Gamma}_k). \] (51)

For this we show
\[ (R^n_k - r^n_k) \text{vol}_n(B^n_2) \leq \text{vol}_n(\tilde{\Gamma}_k) \] (52)

and
\[ r^{n-1}_k \text{vol}_{n-1}(\partial B^n_2)2^k \leq \text{vol}_n(\tilde{\Gamma}_k). \] (53)

To prove (52) we observe
\[ \text{vol}_n(\tilde{\Gamma}_k) \geq \text{vol}_n(\{x : r_k \leq x \leq R_k\}). \]

To prove (53) we observe that the surface area of the cylinder (without bottom and top)
\[ \{(x, t) \in \mathbb{R}^{n+1} : 2^k \leq t \leq 2^{k+1}, \|x\|_2 = r_k\} \]
is less than the surface area of $\tilde{\Gamma}_k$. In order to prove (51) we consider two cases. The first case is $r_k \leq \frac{R_n}{2}$. By (52),
\[
\left(1 - \frac{1}{2^n}\right) R_n^k \text{vol}_n(B_2^n) \leq \text{vol}_n(\tilde{\Gamma}_k).
\]
The second case is $r_k \geq \frac{R_n}{2}$. By (53),
\[
\text{vol}_n(\tilde{\Gamma}_k) \geq r_k^{n-1} \text{vol}_{n-1}(\partial B_2^n) 2^k \geq \frac{R_n^{n-1}}{2^{n-1}} \text{vol}_{n-1}(\partial B_2^n) 2^k.
\]
By (41) and (49)
\[
\left(\frac{2^{k+1} - \beta}{\gamma}\right)^n \text{vol}_n(B_2^n) \geq \text{vol}_n((\text{epi}(\psi) \cap H(2^{k+1}e_{n+1}, e_{n+1})) = R_k^n \text{vol}_n(B_2^n).
\]
Therefore
\[
\frac{2^{k+1} - \beta}{\gamma} \geq R_k
\]
and consequently, by (38)
\[
\text{vol}_n(\partial(A_k)) = \text{vol}_n(\tilde{\Gamma}_k) + \text{vol}_n((\text{epi}(\psi) \cap H(2^k e_{n+1}, e_{n+1})) + \text{vol}_n((\text{epi}(\psi) \cap H(2^{k+1} e_{n+1}, e_{n+1})).
\]
By (48), (49), (50) and (51)
\[
\text{vol}_n(\partial(A_k)) \leq \text{vol}_n(\Gamma_k) + r_k^n \text{vol}_n(B_2^n) + R_k^n \text{vol}_n(B_2^n) \leq \text{vol}_n(\Gamma_k) + 2^n b_n \text{vol}_n(\Gamma_k),
\]
which shows (43). By (41) and (43)
\[
\int_{(E_k \cap \psi) \cap \epsilon} \frac{1 + \|\nabla \psi(x)\|^2}{{r}_{\psi(x)}^\alpha} e^{-\psi(x)} dx \leq \alpha_n \sum_{k=k_0}^{\infty} e^{-2^k} \text{vol}_n(\Gamma_k) \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right)
\]
\[
= \alpha_n \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right) \sum_{k=k_0}^{\infty} e^{-2^k} \int_{E_{k+1} \setminus E_k} (1 + \|\nabla \psi(x)\|^2)\frac{1}{2^k} dx.
\]
Since $2^k \leq \psi(x) \leq 2^{k+1}$ on $E_{k+1} \setminus E_k$, we have $\psi(x) \leq 2^k$. Therefore
\[
\int_{(E_k \cap \psi) \cap \epsilon} \frac{1 + \|\nabla \psi(x)\|^2}{{r}_{\psi(x)}^\alpha} e^{-\psi(x)} dx
\]
\[
\leq \alpha_n \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right) \sum_{k=k_0}^{\infty} \int_{E_{k+1} \setminus E_k} (1 + \|\nabla \psi(x)\|^2)\frac{1}{2^k} e^{-\frac{\psi(x)}{2^{k+1}}} dx
\]
\[
= \alpha_n \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right) \int_{(E_k \cap \psi) \cap \epsilon} (1 + \|\nabla \psi(x)\|^2)\frac{1}{2^k} e^{-\frac{\psi(x)}{2^k}} dx
\]
\[
\leq \alpha_n \left(1 + \frac{n}{\lambda} \left(\frac{1}{\alpha} - 1\right)\right) \int_{\mathbb{R}^{n}} (1 + \|\nabla \psi(x)\|^2)\frac{1}{2^k} e^{-\frac{\psi(x)}{2^k}} dx.
\]
As in the proof of Lemma 3, we get
\[
\int_{(E_{k_0}(\psi))^c} \left(1 + \frac{\|\nabla \psi(x)\|^2}{r_\psi(x)^\alpha}\right)^{\frac{1}{2}} e^{-\psi(x)} \, dx \\
\leq \alpha_n \left(1 + \frac{n}{\alpha} \left(\frac{1}{\alpha} - 1\right)\right) \left(\int_{\mathbb{R}^n} e^{-\frac{\psi(x)}{\alpha}} \, dx + 4e^{-\frac{\beta}{2}} \text{vol}_{n-2}(\partial B_{\beta}^{n-1}) \left(\frac{2}{\gamma}\right)^{n-1} \Gamma(n-1)\right)
\]
and thus
\[
\int_{\mathbb{R}^n} \left(1 + \frac{\|\nabla \psi(x)\|^2}{r_\psi(x)^\alpha}\right)^{\frac{1}{2}} e^{-\psi(x)} \, dx \leq c(\beta, \gamma, n, k_0).
\]
(54)

The proof of Theorem 1 and Corollary 2 follows immediately from these lemmas.

**Proof of Theorem 1.** By the assumptions of the theorem, Lemmas 5, 8 and Lebesgue’s Dominated Convergence Theorem we get that
\[
\lim_{\delta \to 0} \frac{\int (f(x) - f_\delta(x)) \, dx}{\delta^{2/(n+2)}} = \int \frac{\lim_{\delta \to 0} (f(x) - f_\delta(x)) \, dx}{\delta^{2/(n+2)}}
\]
Lemma 4 finishes the proof.

**Proof of Corollary 2.** The proof is done in the same way using Lemmas 4, 5, 8 and Lebesgue’s Dominated Convergence Theorem.

**Acknowledgement**
This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester.
We want to thank the referee for the careful reading and suggestions for improvement.

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