EXISTENCE OF PERIODIC SOLUTIONS IN SHIFTS $\delta_{\pm}$ FOR NEUTRAL NONLINEAR DYNAMIC SYSTEMS

MURAT ADIVAR, H. CAN KOYUNCUĞLU, AND YOUSSEF N. RAFFOUL

Abstract. In this study, we focus on the existence of a periodic solution for the neutral nonlinear dynamic systems with delay

$$x^{\Delta}(t) = A(t)x(t) + Q^{\Delta}(t, x(\delta_{-}(s, t))) + G(t, x(t), x(\delta_{-}(s, t))).$$

We utilize the new periodicity concept in terms of shifts operators, which allows us to extend the concept of periodicity to time scales where the additivity requirement $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ and for a fixed $T > 0$, may not hold. More importantly, the new concept will easily handle time scales that are not periodic in the conventional way such as; $\mathbb{Q}^*$ and $\bigcup_{k=1}^{\infty} [3^{2^k}, 2.3^{2^k}] \cup \{0\}$. Hence, we develop a tool that enables the investigation of periodic solutions of $q$-difference systems. Since we are dealing with systems, in order to convert our equation to an integral systems, we resort to the transition matrix of the homogeneous Floquet system $y^{\Delta}(t) = A(t)y(t)$ and then make use of Krasnoselskii’s fixed point theorem to obtain a fixed point.

1. Introduction and preliminaries

In recent decades, the theory of neutral functional equations with delays have seen prominent attention due to its tremendous potential of its application in applied mathematics. There are many papers that handle neutral differential equations on regular time scales, such as discrete and continuous cases, but few that deal with general time scales. A time scale is a nonempty arbitrary closed subset of reals. Existence of periodic solutions is of importance to biologists since most models deal with certain types of populations. In the paper of Kaufmann and Raffoul [13], the authors were the first to define the notion of periodic time scales, by requiring the additivity requirement $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ and for a fixed $T > 0$, to hold. Of course, as we have mentioned above, this type of requirement leaves out many important time scales that are of interest to biologists and scientists that [13] could not handle.

To overcome such difficulties, in the famous paper of Adivar [1], the author introduced to concept of shift periodic operators which we will utilize in our work to obtain the existence of a periodic solution. For more on the existence of periodic solutions on regular time scales, we refer the readers to [12], and [16]. In addition, the papers [10], [11] and [15] study the existence of a periodic solution of system of delayed neutral functional equations by using Sadovskii and Krasnoselskii’s fixed point theorems, respectively.

Application of time scales has been extended to logistic equation modeling population growth. We refer the reader to May [14] for a detailed model construction of

$$x^{\Delta} = -a(t)x^\sigma + f(t)$$

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in the case $T = \mathbb{R}$, and to [8] for the derivation of the equivalent time scale equation

\[(1.2) \quad x^\Delta = [a(t) \odot (f(t)x)]x.\]

More interesting application, is the version of the hematopoiesis model (Weng and Liang [18]),

\[(1.3) \quad x^\Delta(t) = -a(t)x(t) + \alpha(t) \int_0^\infty B(s)e^{-x_\beta(t,s)} \Delta s,\]

where $x(t)$ is the number of red blood cells at time $t$, $\alpha, \beta, \gamma \in C(\mathbb{T}, \mathbb{R})$ are $T$-periodic, and $B$ is a non-negative and integrable function. This is an extension of the red cell system on $\mathbb{R}$ introduced by Wazewska-Czyzewska and Lasota [17].

Throughout the paper, we assume the reader is familiar with the calculus of time scales and for those who are interested in the theory of time scales, we refer them to the books [4] and [5].

Motivated by the papers [11] and [15], we consider the nonlinear neutral dynamic system with delay

\[x^\Delta(t) = A(t)x(t) + Q^\Delta(t,x(\delta_-(s,t))) + G(t,x(t),x(\delta_+(s,t))), \quad t \in \mathbb{T}\]

and by employing the results of [2] and ([16]-[19]) we invert our system and then by appealing to Krasnoselski’s fixed point theorem we will show the existence of a nonzero periodic solution by assuming suitable conditions.

We begin by stating basic results from [3] regarding shift operators and then in the last section we focus on proving the existence of a periodic solution using shift periodic operators. Hereafter, we use the notation $[a,b]_\mathbb{T}$ to indicate the set $[a,b) \cap \mathbb{T}$. The intervals $[a,b]_\mathbb{T}, (a,b)_\mathbb{T}$, and $(a,b)_\mathbb{T}$ are defined similarly.

1.1. Shift operators and periodicity. Shift operators that are periodic provide alternative tool for investigating periodicity on time scales that may not be additive. Periodicity by means of shift operators was first introduced in [1]. In this section, we aim to introduce basic definitions and properties of shift operators. The following definitions, lemmas and examples can be found in [1], and [3].

**Definition 1.** Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exists operators $\delta_{\pm} : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

1. The function $\delta_{\pm}$ are strictly increasing with respect to their second arguments, if

\[(T_0,t), (T_0,u) \in \mathcal{D}_{\pm} := \{(s,t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\pm}(s,t) \in \mathbb{T}^*\},\]

then

\[T_0 \leq t \leq u \text{ implies } \delta_{\pm}(T_0,t) \leq \delta_{\pm}(T_0,u);\]

2. If $(T_1,u), (T_2,u) \in \mathcal{D}_{-}$ with $T_1 < T_2$, then $\delta_{-}(T_1,u) > \delta_{-}(T_2,u)$ and if $(T_1,u), (T_2,u) \in \mathcal{D}_{+}$ with $T_1 < T_2$, then $\delta_{+}(T_1,u) < \delta_{+}(T_2,u)$;

3. If $t \in [t_0, \infty)_{\mathbb{T}}$, then $(t,t_0) \in \mathcal{D}_{+}$ and $\delta_{+}(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in \mathcal{D}_{+}$ and $\delta_{+}(t_0,t) = t$;

4. If $(s,t) \in \mathcal{D}_{\pm}$, then $(s,\delta_{\pm}(s,t)) \in \mathcal{D}_{\mp}$ and $\delta_{\mp}(s,\delta_{\pm}(s,t)) = t$;

5. If $(s,t) \in \mathcal{D}_{\pm}$ and $(u,\delta_{\pm}(u,t)) \in \mathcal{D}_{\mp}$, then $(s,\delta_{\mp}(u,t)) \in \mathcal{D}_{\pm}$ and $\delta_{\pm}(u,\delta_{\pm}(s,t)) = \delta_{\pm}(s,\delta_{\mp}(u,t))$.

Then the operators $\delta_{\pm}$ and $\delta_{-}$ are called forward and backward shift operators associated with the initial point $t_0$ on $\mathbb{T}^*$ and the sets $\mathcal{D}_{+}$ and $\mathcal{D}_{-}$ are domain of the operators, respectively.
Example 1. The following table shows the shift operators \( \delta_\pm (s,t) \) on some time scales:

| \( T \) | \( T^* \) | \( \delta_- (s,t) \) | \( \delta_+ (s,t) \) |
|-------|--------|----------------|----------------|
| \( \mathbb{R} \) | \( \mathbb{R} \) | \( t-s \) | \( t+s \) |
| \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( t-s \) | \( t+s \) |
| \( \mathbb{Z}^{2/2} \cup \{0\} \) | \( \mathbb{Z}^{1/2} \) | \( \frac{t}{2} \) | \( st \) |

Lemma 1. Let \( \delta_\pm \) be the shift operators associated with the initial point \( t_0 \). Then we have the following:

1. \( \delta_- (t,t) = t_0 \) for all \( t \in [t_0, \infty)_T \);
2. \( \delta_- (t,t) = t \) for all \( t \in T^* \);
3. If \( (s,t) \in D_+ \), then \( \delta_+ (s,t) = u \) implies \( \delta_- (s,u) = t \) and if \( (s,u) \in D_- \), then \( \delta_- (s,u) = t \) implies \( \delta_+ (s,t) = u \);
4. \( \delta_+ (t, \delta_- (s,t_0)) = \delta_- (t,u) \) for all \( (s,t) \in D_+ \) with \( t \geq t_0 \);
5. \( \delta_+ (u,t) = \delta_- (t,u) \) for all \( (u,t) \in (\{0\}, [0, \infty)_T) \cap D_+ \);
6. \( \delta_+ (s,t) \in [t_0, \infty)_T \) for all \( (s,t) \in D_+ \) with \( t \geq t_0 \);
7. \( \delta_- (s,t) \in ([0, \infty)_T \times [0, \infty)_T) \) \cap \( D_- \);
8. If \( \delta_+ (s,) \) is \( \Delta \)-differentiable in its second variable, then \( \delta_+^{\Delta} (s,) > 0 \);
9. \( \delta_+ (\delta_- (u,v), \delta_- (u,v)) = \delta_- (u,v) \) for all \( (u,v) \in ([0, \infty)_T \times [0, \infty)_T) \) \cap \( D_- \);
10. If \( (s,t) \in D_- \) and \( \delta_- (s,t) = t_0 \), then \( s = t \).

Definition 2 (Periodicity in shifts). Let \( T \) be a time scale with the shift operators \( \delta_\pm \) associated with the initial point \( t_0 \in T^* \), then \( T \) is said to be periodic in shifts \( \delta_\pm \), if there exists a \( p \in (t_0, \infty)_T \) such that \( (p,t) \in D_\pm \) for all \( t \in T^* \). \( P \) is called the period of \( T \) if

\[ P = \inf \left\{ p \in (t_0, \infty)_T : (p,t) \in D_\pm \text{ for all } t \in T^* \right\} > t_0. \]

Observe that an additive periodic time scale must be unbounded. However, unlike additive periodic time scales, a period in shifts, may be bounded.

Example 2. The following time scales are not additive periodic but periodic in shifts \( \delta_\pm \).

1. \( T_1 = \{ \pm n^2 : n \in \mathbb{Z} \} \), \( \delta_\pm (P,t) = \begin{cases} \left( \sqrt{t} \pm \sqrt{P} \right)^2 & \text{if } t > 0 \\ \pm P & \text{if } t = 0, \ P = 1, \ t_0 = 0, \\ -\left( \sqrt{-t} \pm \sqrt{P} \right)^2 & \text{if } t < 0 \end{cases} \)
2. \( T_2 = \mathbb{Z}^2, \ \delta_\pm (P,t) = P^{\pm 1} t, \ P = q, \ t_0 = 1, \)
3. \( T_3 = \bigcup_{n \in \mathbb{Z}} [2^n, 2^{n+1}], \ \delta_\pm (P,t) = P^{\pm 1} t, \ P = 4, \ t_0 = 1, \)
4. \( T_4 = \left\{ \frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z} \right\} \cup \{0,1\}, \)

\[ \delta_\pm (P,t) = \frac{q \left( \frac{\ln(t)}{\ln(P)} \right)^{\frac{1}{\ln(q)}}}{1 + q \left( \frac{\ln(t)}{\ln(P)} \right)^{\frac{1}{\ln(q)}}}, \ P = \frac{q}{1+q}. \]

Notice that the time scale \( T_4 \) in Example 2 is bounded above and below and

\[ T_4^* = \left\{ \frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z} \right\}. \]
Corollary 1. Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period $P$. Then we have

$$\delta_{\pm}(P, \sigma(t)) = \sigma(\delta_{\pm}(P, t)) \text{ for all } t \in \mathbb{T}^*.$$  

Example 3. The time scale $\tilde{T} = (-\infty, 0] \cup [1, \infty)$ cannot be periodic in shifts $\delta_{\pm}$. Because if there existed a $p \in (t_0, \infty)_{\tilde{T}}$ such that $\delta_{\pm}(p, t) \in \tilde{T}^*$, then the point $\delta_-(p, 0)$ would be right scattered due to (1.4). However, we have $\delta_-(p, 0) < 0$ by (i) of Definition 1. This leads to a contradiction since every point less than 0 is right dense.

Definition 3 (Periodic function in shifts $\delta_{\pm}$). Let $\mathbb{T}$ be a time scale $P$-periodic in shifts. We say that a real valued function $f$ defined on $\mathbb{T}^*$ is periodic in shifts $\delta_{\pm}$ if there exists a $T \in [P, \infty)_{\mathbb{T}}$ such that

$$f(\delta_{\pm}(T, t)) = f(t) \text{ for all } t \in \mathbb{T}^*,$$

where $\delta_{\pm}(T, t) = \delta_{\pm}(T, t)$. $T$ is called period of $f$, if it is the smallest number satisfying (1.5).

Example 4. Let $\mathbb{T} = \mathbb{R}$ with initial point $t_0 = 1$, the function

$$f(t) = \sin \left( \frac{\ln |t|}{\ln (1/2)} \pi \right), \ t \in \mathbb{R}^* := \mathbb{R} - \{0\}$$

is 4-periodic in shifts $\delta_{\pm}$ since

$$f(\delta_{\pm}(4, t)) = \begin{cases} f(t^4 \pm 1) & \text{if } t \geq 0 \\ f(t^{4 \pm 1}) & \text{if } t < 0 \end{cases}$$

$$= \sin \left( \frac{\ln |t| \pm 2 \ln (1/2) \pi}{\ln (1/2)} \right)$$

$$= \sin \left( \frac{\ln |t|}{\ln (1/2)} \pi \pm 2 \pi \right)$$

$$= \sin \left( \frac{\ln |t|}{\ln (1/2)} \pi \right)$$

$$= f(t).$$

Definition 4 ($\Delta$-periodic function in shifts $\delta_{\pm}$). Let $\mathbb{T}$ be a time scale $P$-periodic in shifts. A real valued function $f$ defined on $\mathbb{T}^*$ is $\Delta$-periodic function in shifts if there exists a $T \in [P, \infty)_{\mathbb{T}}$, such that

$$f(\delta_{\pm}(T, t)) = f(t), \ \forall t \in \mathbb{T}^*$$

the shifts $\delta_{\pm}^T$ are $\Delta$-differentiable with rd-continuous derivatives and for all $t \in \mathbb{T}^*$, where $\delta_{\pm}^T(t) = \delta_{\pm}(T, t)$. The smallest number $T$ satisfying (1.6) is called period of $f$.

Example 5. The function $f(t) = 1/t$ is $\Delta$-periodic function on $q^\mathbb{Z}$ with the period $T = q$. 

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**Theorem 1.** Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period $P \in (t_0, \infty)_{\mathbb{T}}$, and $f$ a $\Delta$-periodic function in shifts $\delta_{\pm}$ with period $T \in [P, \infty)_{\mathbb{T}}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then

$$
\int_{t_0}^{t} f(s) \Delta s = \int_{\delta_{\pm}(t_0)}^{\delta_{\pm}(t)} f(s) \Delta s.
$$

1.2. **Unified Floquet theory with respect to new periodicity concept.** In this section, we list some results of [2] for further use.

1.2.1. **Homogeneous case.** Consider the regressive time varying linear dynamic initial value problem

(1.9) 
$$
x^\Delta(t) = A(t) x(t), \quad x(t_0) = x_0,
$$

where $A : \mathbb{T}^r \rightarrow \mathbb{R}^{n \times n}$ is $\Delta$-periodic in shifts with period $T$. Notice that if the time scale is additive periodic, then $\delta_{\pm}(T,t) = 1$ and $\Delta$-periodicity in shifts becomes the same as the periodicity in shifts. Hence, the homogeneous system we consider in this section is more general than the systems handled in literature.

In [9], the solution of the system (1.9) (for an arbitrary matrix $A$) is expressed by the equality

$$
x(t) = \Phi_A(t,t_0) x_0,
$$

where $\Phi_A(t,t_0)$, called the transition matrix for the system (1.9), is given by

$$
\Phi_A(t,t_0) = I + \int_{t_0}^{t} A(\tau_1) \Delta \tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_2} A(\tau_2) \Delta \tau_2 \Delta \tau_1 + \ldots
$$

$$
+ \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{t} A(\tau_2) \ldots \int_{t_0}^{t} A(\tau_i) \Delta \tau_i \ldots \Delta \tau_1 + \ldots.
$$

(1.10)

As mentioned in [6] the matrix exponential $e_A(t,t_0)$ is not always identical to $\Phi_A(t,t_0)$ since

$$
A(t) e_A(t,t_0) = e_A(t,t_0) A(t)
$$

is always true but the equality

$$
A(t) \Phi_A(t,t_0) = \Phi_A(t,t_0) A(t)
$$

is not. It can be seen from (1.10) that one has $e_A(t,t_0) \equiv \Phi_A(t,t_0)$ only if the matrix $A$ satisfies

$$
A(t) \int_{s}^{t} A(\tau) \Delta \tau = \int_{s}^{t} A(\tau) \Delta \tau A(t).
$$

In preparation for the next result we define the set

(1.11) 
$$
P(t_0) := \left\{ \delta_{\pm}^{(k)}(T,t_0), \ k = 0, 1, 2, \ldots \right\}
$$

and the function

(1.12) 
$$
\Theta(t) := \sum_{j=1}^{m(t)} \delta_{-} \left( \delta_{\pm}^{(j-1)}(T,t_0), \delta_{\pm}^{(j)}(T,t_0) \right) + G(t),
$$

where $G(t)$ is a function.
where
\begin{equation}
m(t) := \min \left\{ k \in \mathbb{N} : \delta^{(k)}_+(T, t_0) \geq t \right\}
\end{equation}
and
\begin{equation}
G(t) := \begin{cases} 
0 & \text{if } t \in P(t_0) \\
-\delta_\sigma \left( t, \delta^{(m(t))}_+ (T, t_0) \right) & \text{if } t \notin P(t_0) 
\end{cases}
\end{equation}

**Remark 1.** For an additive periodic time scale we always have \( \Theta(t) = t - t_0 \).

Following theorem constructs the matrix \( R \) as a solution of matrix exponential equation.

**Theorem 2** ([2]). For a nonsingular, \( n \times n \) constant matrix \( M \) a solution \( R : T \rightarrow \mathbb{C}^{n \times n} \) of matrix exponential equation
\begin{equation}
e_R \left( \delta^T_+ (t_0), t_0 \right) = M
\end{equation}
can be given by
\begin{equation}
R(t) = \lim_{s \rightarrow t} \frac{M^{|\Theta(\sigma(t))-\Theta(s)|} - I}{\sigma(t) - s},
\end{equation}
where \( I \) is the \( n \times n \) identity matrix and \( \Theta \) is as in (1.12).

**Lemma 2** ([2]). Let \( T \) be a time scale and \( P \in \mathcal{R}(\mathbb{R}^*, \mathbb{R}^{n \times n}) \) be a \( \Delta \)-periodic matrix valued function in shifts with period \( T \), i.e.
\begin{equation}
P(t) = P\left( \delta^T_+ (t) \right) \delta_+ \Delta^T (t)
\end{equation}
Then the solution of the dynamic matrix initial value problem
\begin{equation}
Y^\Delta (t) = P(t) Y(t), \ Y(t_0) = Y_0,
\end{equation}
is unique up to a period \( T \) in shifts. That is
\begin{equation}
\Phi_P(t, t_0) = \Phi_P\left( \delta^T_+ (t), \delta^T_+ (t_0) \right)
\end{equation}
for all \( t \in \mathbb{T}^* \).

**Corollary 2** ([2]). Let \( T \) be a time scale and \( P \in \mathcal{R}(\mathbb{R}^*, \mathbb{R}^{n \times n}) \) be a \( \Delta \)-periodic matrix valued function in shifts, i.e.
\begin{equation}
P(t) = P\left( \delta^T_+ (t) \right) \delta_+ \Delta^T (t)
\end{equation}
Then
\begin{equation}
e_P(t, t_0) = e_P \left( \delta^T_+ (t), \delta^T_+ (t_0) \right).
\end{equation}

**Theorem 3** ([2], Floquet decomposition). Let \( A \) be a matrix valued function that is \( \Delta \)-periodic in shifts with period \( T \). The transition matrix for \( A \) can be given in the form
\begin{equation}
\Phi_A(t, \tau) = L(t) e_R(t, \tau) L^{-1}(\tau), \ \text{for all } t, \tau \in \mathbb{T}^*,
\end{equation}
where \( R : \mathbb{T} \rightarrow \mathbb{C}^{n \times n} \) and \( L(t) \in C^1_r(\mathbb{T}^*, \mathbb{R}^{n \times n}) \) are both periodic in shifts with period \( T \) and invertible.

**Theorem 4** ([2]). There exists an initial state \( x(t_0) = x_0 \neq 0 \) such that the solution of (1.9) is \( T \)-periodic in shifts if and only if one of the eigenvalues of the matrix
\begin{equation}
e_R \left( \delta^T_+ (t_0), t_0 \right) = \Phi_A \left( \delta^T_+ (t_0), t_0 \right)
\end{equation}
is 1.
1.2.2. Nonhomogeneous case. Let us consider the nonhomogeneous regressive nonautonomous linear dynamic initial value problem

\[ x^\Delta (t) = A(t) x(t) + F(t), \quad x(t_0) = x_0, \]

where \( A : \mathbb{T}^* \rightarrow \mathbb{R}^{n \times n}, \) \( F \in C_{rd}(\mathbb{T}^*, \mathbb{R}^n) \cap \mathcal{R}(\mathbb{T}^*, \mathbb{R}^n). \) Hereafter, we suppose both \( A \) and \( F \) are \( \Delta \)-periodic in shifts with the period \( T. \)

**Theorem 5** \([2]\). For any initial point \( t_0 \in \mathbb{T}^* \) and for any function \( F \) that is \( \Delta \)-periodic in shifts with period \( T, \) there exists an initial state \( x(t_0) = x_0 \) such that the solution of (1.20) is \( T \)-periodic in shifts if and only if there does not exist a nonzero \( z(t_0) = z_0 \) and \( t_0 \in \mathbb{T}^* \) such that the \( T \)-periodic homogeneous initial value problem

\[ z^\Delta (t) = A(t) z(t), \quad z(t_0) = z_0, \]

has a solution that is \( T \)-periodic in shifts.

For details about Floquet theory based on new periodicity concept on time scales, we refer readers \([2]\).

2. Existence of periodic solutions

For \( T > 0, \) let \( P_T \) be the set of all \( n \)-vector functions \( x(t), \) periodic in shifts with period \( T. \) Then \((P_T, \|\|)\) is a Banach space endowed with the norm

\[ \|x\| = \max_{t \in [t_0, \delta^T_{\pm}(t_0)]_{\mathbb{T}}} |x(t)|. \]

Also for an \( n \times n \) matrix valued function \( A, \) given by \( A(t) := [a_{ij}(t)], \) we define the norm \( \|A\| \) by

\[ \|A\| = \sup_{t \in [t_0, \infty)_{\mathbb{T}}} |A(t)|, \]

where

\[ |A(t)| := \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}(t)|. \]

Now assume that \( \mathbb{T} \) is a time scale that is \( T \)-periodic in shifts and consider the delay dynamic system

\[ x^\Delta(t) = A(t)x(t) + Q^\Delta \left( t, x(\delta_{-}(s, t)) \right) + G \left( t, x(t), x(\delta_{-}(s, t)) \right), \]

where \( A \in C_{rd}(\mathbb{T}^*, \mathbb{R}^{n \times n}), \) \( Q \in C_{rd}(\mathbb{T}^* \times \mathbb{T}^* \times \mathbb{T}^* \times \mathbb{R}^n) \) and \( G \in C_{rd}(\mathbb{T}^* \times \mathbb{T}^* \times \mathbb{T}^* \times \mathbb{R}^n). \) Since we focus on the existence of a periodic solution of (2.1), we have the following periodicity assumptions:

\[ A \text{ is } \Delta \text{-periodic in shifts, i.e. } A(\delta_{\pm}(t)) = A(t) \text{ for all } t \in \mathbb{T}^*, \]

\[ Q(\delta^T_{\pm}(t), x(\delta_{-}(s, \delta^T_{\pm}(t)))) = Q(t, x(\delta_{-}(s, t))) \text{ for all } t \in \mathbb{T}^* \text{ and } x \in P_T, \]

and

\[ G(\delta^T_{\pm}(t), x(\delta^T_{\pm}(t)), x(\delta_{-}(s, \delta^T_{\pm}(t)))) \delta_{\pm}^\Delta(t) = G(t, x(t), x(\delta_{-}(s, t))) \text{ for all } t \in \mathbb{T}^* \text{ and } x \in P_T. \]

In order to prove the existence of a nonzero periodic solution in shifts \( \delta_{\pm} \) for system (2.1), we assume that

\[ Q^\Delta(0, 0) + G(0, 0, 0) \neq 0. \]
for some $t \in T^*$.

Throughout the paper, we assume that the homogeneous system

\[ (2.6) \]

\[ z^A(t) = A(t) z(t), \ z(t_0) = z_0 \]

is non-critical. That is, (2.6) has no periodic nonzero solution in shifts $\delta_\pm$.

**Lemma 3.** Suppose that (2.2-2.5) hold. If $x(t) \in P_T$, then $x(t)$ is a solution of equation (2.1) satisfying $x(t_0) = x_0$ if and only if

\[ x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta_+(t, t_0), t_0) - I \right)^{-1} \times \]

\[ \int_t^{\delta_+(t)} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u. \]

**Proof.** Let $x(t) \in P_T$ be a solution of (2.1) satisfying $x(t_0) = x_0$ and $\Phi_A(t, t_0)$ be the transition matrix of system (2.6). The necessity part of the proof is straightforward. For the sufficiency part we employ (2.1) to get

\[ [x(t) - Q(t, x(\delta_-(s, t)))] = A(t) [x(t) - Q(t, x(\delta_-(s, t))) + A(t)Q(t, x(\delta_-(s, t))) + G(t, x(t), x(\delta_-(s, t)))]. \]

Since $\Phi_A(t, t_0)\Phi_A^{-1}(t, t_0) = I$, we have

\[ 0 = (\Phi_A(t, t_0)\Phi_A^{-1}(t, t_0))^\Delta A(t) \]

\[ = \Phi_A^2(t, t_0)\Phi_A^{-1}(t, t_0) + \Phi_A(\sigma(t), t_0) \left( \Phi_A^{-1}(t, t_0) \right)^\Delta A(t) \]

\[ = (A(t)\Phi_A(t, t_0)) \Phi_A^{-1}(t, t_0) + \Phi_A(\sigma(t), t_0) \left( \Phi_A^{-1}(t, t_0) \right)^\Delta A(t) \]

\[ = A(t) + \Phi_A(\sigma(t), t_0) \left( \Phi_A^{-1}(t, t_0) \right)^\Delta A(t). \]

That is,

\[ (\Phi_A^{-1}(t, t_0))^\Delta = -\Phi_A^{-1}(\sigma(t), t_0)A(t). \]

If $x(t)$ is a solution of (2.1) satisfying $x(t_0) = x_0$, then

\[ \{ \Phi_A^{-1}(t, t_0) (x(t) - Q(t, x(\delta_-(s, t)))) \}^\Delta = (\Phi_A^{-1}(t, t_0))^\Delta (x(t) - Q(t, x(\delta_-(s, t)))) \]

\[ + \Phi_A^{-1}(\sigma(t), t_0) (x(t) - Q(t, x(\delta_-(s, t))))^\Delta A(t) \]

\[ = -\Phi_A^{-1}(\sigma(t), t_0)A(t) (x(t) - Q(t, x(\delta_-(s, t)))) \]

\[ + \Phi_A^{-1}(\sigma(t), t_0) [A(t) (x(t) - Q(t, x(\delta_-(s, t)))) + G(t, x(t), x(\delta_-(s, t)))] \]

\[ = \Phi_A^{-1}(\sigma(t), t_0) [A(t)Q(t, x(\delta_-(s, t))) + G(t, x(t), x(\delta_-(s, t)))] \Delta u. \]

Integrating the last equality from $t_0$ to $t$, we arrive at

\[ x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) (x_0 - Q(t_0, x(\delta_-(s, t_0)))) \]

\[ + \Phi_A(t, t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u. \]
Since \( x(\delta^T_+(t_0)) = x(t_0) = x_0 \), (2.9) implies

\[
x(t_0) - Q(t_0, x(\delta_-(s, t_0))) = \Phi_A(\delta^T_+(t_0), t_0) (x_0 - Q(t_0, x(\delta_-(s, t_0))))
\]

(2.10)

\[
+ \Phi_A(\delta^T_+(t_0), t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u.
\]

Substituting (2.10) into (2.9) yields

\[
x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) (I - \Phi_A(\delta^T_+(t_0), t_0))^{-1} \Phi_A(\delta^T_+(t_0), t_0)
\]

\[
\times \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u
\]

(2.11)

\[
+ \Phi_A(t, t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u.
\]

In order to show that (2.11) is equivalent to (2.7) we use

\[
(I - \Phi_A(\delta^T_+(t_0), t_0))^{-1} = (\Phi_A(\delta^T_+(t_0), t_0) (\Phi_A^{-1}(\delta^T_+(t_0), t_0) - I))^{-1}
\]

\[
= (\Phi_A^{-1}(\delta^T_+(t_0), t_0) - I)^{-1} \Phi_A^{-1}(\delta^T_+(t_0), t_0).
\]

to get

\[
x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) (\Phi_A^{-1}(\delta^T_+(t_0), t_0) - I)^{-1} \Phi_A^{-1}(\delta^T_+(t_0), t_0) \Phi_A(\delta^T_+(t_0), t_0)
\]

\[
\times \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u
\]

\[
+ \Phi_A(t, t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u,
\]
and we have the following equality:

\[
x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta_T^+(t_0), t_0) - I \right)^{-1} \
\times \left[ \frac{\delta_T^+(t_0)}{t} \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) \left[ A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u))) \right] \Delta u \
+ \Phi_A^{-1}(\delta_T^+(t_0), t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) \left[ A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u))) \right] \Delta u \
- \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) \left[ A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u))) \right] \Delta u \right].
\]

Thus, \( x(t) \) can be stated as follows

\[
x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta_T^+(t_0), t_0) - I \right)^{-1} \
\times \left[ \frac{\delta_T^+(t_0)}{t} \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) \left[ A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u))) \right] \Delta u \
+ \Phi_A^{-1}(\delta_T^+(t_0), t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) \left[ A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u))) \right] \Delta u \right].
\]

If we let \( u = \delta_T(\hat{u}), \) we get

\[
x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta_T^+(t_0), t_0) - I \right)^{-1} \
\times \left[ \frac{\delta_T^+(t_0)}{t} \int_{t_0}^{t} \Phi_A^{-1}(\sigma(u), t_0) \left[ A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u))) \right] \Delta u \
+ \Phi_A^{-1}(\delta_T^+(t_0), t_0) \int_{t_0}^{t} \Phi_A^{-1}(\sigma(\delta_T(\hat{u})), t_0) \left[ A(\delta_T^+(\hat{u}))Q(\delta_T^+(\hat{u}), x(\delta_-(s, \delta_T(\hat{u})))) \right] \Delta u \
+ \left[ G(\delta_T^+(\hat{u}), x(\delta_T^+(\hat{u})), x(\delta_-(s, \delta_T(\hat{u})))) \right] \delta_T^+(\hat{u}) \Delta u \right].
\]
and
\[ x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right)^{-1} \]
\[ \times \left[ \int_t^{\delta^T_+(t_0)} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u \right. \]
\[ + \Phi_A^{-1}(\delta^T_+(t_0), t_0) \left. \int_{\delta^T_+(t_0)}^{\delta^T_+(t_0)} \Phi_A^{-1}(\sigma(\delta^T_+(\hat{u})), t_0) [A(\hat{u})Q(\hat{u}, x(\delta_-(s, \hat{u}))) + G(\hat{u}, x(\hat{u}), x(\delta_-(s, \hat{u})))] \Delta u \right]. \]

Since
\[ \Phi_A^{-1}(\delta^T_+(t_0), t_0) \Phi_A^{-1}(\sigma(\delta^T_+(\hat{u})), t_0) = \Phi_A^{-1}(\delta^T_+(t_0), t_0) \Phi_A^{-1}(\delta^T_-(\hat{u})), t_0), \]
we have
\[ \Phi_A(t_0, \delta^T_+(t_0)) \Phi_A(t_0, \delta^T_-(\hat{u})) = \Phi_A(t_0, \delta^T_+(t_0)) \Phi_A(\delta^T_+(t_0), \sigma(\hat{u})). \]

Applying the last equality to the preceding one, we obtain
\[ x(t) = Q(t, x(\delta_-(s, t))) + \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right)^{-1} \]
\[ \times \left[ \int_t^{\delta^T_+(t_0)} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, x(\delta_-(s, u))) + G(u, x(u), x(\delta_-(s, u)))] \Delta u \right], \]
as desired. \qed

Next we state Krasnoselkii's fixed point theorem which we employ for showing existence of a periodic solution.

**Theorem 6** (Krasnoselkii). Let \( \mathbb{M} \) be a closed convex nonempty subset of a Banach space \((\mathbb{B}, \|\cdot\|)\). Suppose that \( B \) and \( C \) maps \( \mathbb{M} \) into \( \mathbb{B} \) such that

(i) \( x, y \in \mathbb{M} \), implies \( Bx + Cy \in \mathbb{M} \),
(ii) \( C \) is compact and continuous,
(iii) \( B \) is a contraction mapping.

Then there exists \( z \in \mathbb{M} \) with \( z = Bz + Cz \).

In preparation for the next result define the mapping \( H \) by
\[ (H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t), \]
where
\[ (B\varphi)(t) := Q(t, \varphi(\delta_-(s, t))) \]
and
\[ (C\varphi)(t) := \Phi_A(t, t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right)^{-1} \]
\[ \times \int_t^{\delta^T_+(t_0)} \Phi_A^{-1}(\sigma(u), t_0) [A(u)Q(u, \varphi(\delta_-(s, u))) + G(u, \varphi(u), \varphi(\delta_-(s, u)))] \Delta u. \]
Lemma 4. Suppose (2.2)-(2.3) hold. Let C be defined by (2.14). If there exist positive constants $E_1, E_2, E_3$, and $N$ such that

\begin{align}
|Q(t,x) - Q(t,y)| &\leq E_1 \|x-y\|, \\
|G(t,x,y) - G(t,z,w)| &\leq E_2 \|x-z\| + E_3 \|y-w\|, \\
\end{align}

and

\begin{equation}
\|C\varphi(\cdot)\| \leq r (\delta^T_+(t_0) - t_0) \|A(.)Q(\cdot, \varphi(\delta_-(s, \cdot))) + G(\cdot, \varphi(\cdot), \varphi(\delta_-(s, \cdot)))\|,
\end{equation}

where

\begin{equation}
r = \max_{t \in [t_0, \delta^T_+(t_0) \cap T]} \left( \max_{u \in [t, \delta^T_+(t) \cap T]} \left| \left[ \Phi_A(\sigma(u), t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right) \Phi_A^{-1}(t, t_0) \right]^{-1} \right| \right).
\end{equation}

(ii) C is continuous and compact.

Proof. Let C be defined as in (2.14). Then it can be written as in the following form

\begin{equation}
(C\varphi)(t) = \int_{t}^{\delta^T_+(t)} \left[ \Phi_A(\sigma(u), t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right) \Phi_A^{-1}(t, t_0) \right]^{-1} A(u)Q(u, \varphi(\delta_-(s, u))) + G(u, \varphi(u), \varphi(\delta_-(s, u))) \Delta u.
\end{equation}

Since $(C\varphi)(t) \in P_T$, we have

\begin{align*}
\|C\varphi(\cdot)\| &= \max_{t \in [t_0, \delta^T_+(t_0) \cap T]} \left| \int_{t}^{\delta^T_+(t)} \left[ \Phi_A(\sigma(u), t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right) \Phi_A^{-1}(t, t_0) \right]^{-1} A(u)Q(u, \varphi(\delta_-(s, u))) + G(u, \varphi(u), \varphi(\delta_-(s, u))) \Delta u \right| \\
&\leq \max_{t \in [t_0, \delta^T_+(t_0) \cap T]} \left( \max_{u \in [t, \delta^T_+(t) \cap T]} \left| \left[ \Phi_A(\sigma(u), t_0) \left( \Phi_A^{-1}(\delta^T_+(t_0), t_0) - I \right) \Phi_A^{-1}(t, t_0) \right]^{-1} \right| \right) \\
&\quad \times \max_{t \in [t_0, \delta^T_+(t_0) \cap T]} \left| \int_{t_0}^{t} A(u)Q(u, \varphi(\delta_-(s, u))) + G(u, \varphi(u), \varphi(\delta_-(s, u))) \Delta u \right| \\
&\leq r (\delta^T_+(t_0) - t_0) \|A(.)Q(\cdot, \varphi(\delta_-(s, \cdot))) + G(\cdot, \varphi(\cdot), \varphi(\delta_-(s, \cdot)))\|.
\end{align*}

This completes the proof of part (i).
To see that $C$ is continuous, suppose $\varphi$ and $\psi$ belong to $P_T$. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\varphi - \psi\| < \delta$ implies

$$
\|((C\varphi)(\cdot) - (C\psi)(\cdot))\| \leq r \int_{t_0}^{t_0} \|[A\| E_1 \|\varphi - \psi\| + (E_2 + E_3) \|\varphi - \psi\|]\Delta u
$$

By choosing $\delta = \varepsilon/N$, we prove that $C$ is continuous.

In order to show that $C$ is compact, we consider the set $D := \{\varphi \in P_T : \|\varphi\| \leq R\}$ for a positive fixed constant $R$. Consider a sequence of $T$-periodic functions in shifts, $\{\varphi_n\}$, and assume that $\{\varphi_n\} \in D$. Moreover, from (2.15) and (2.16) we get

$$(2.19)$$

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$$(2.19)$$
Substituting
\[ \Phi_A^{-1}(\sigma(T_+\delta(t)) + \theta) - \Phi_A^{-1}(\sigma(t) + \theta) = (\Phi_A^{-1}(\delta(T_+\delta(t), t_0) - I) \Phi_A^{-1}(\sigma(t) + \theta) \]
\[ \text{in (2.19), we obtain} \]
\[ (C\varphi_n)^{(T_+\delta(t))} = A\varphi_n(T_+\delta(t)) + A(t)Q(t, \varphi_n(\delta_-(s, t))) + G(t, \varphi_n(t), \varphi_n(\delta_-(s, t))). \]

Thus, \((C\varphi_n)^{(T_+\delta(t))}\) is bounded. This means that \((C\varphi_n)\) is uniformly bounded and equicontinuous. Hence by Arzela-Ascoli theorem \(C(D)\) is compact.

**Lemma 5.** Let \(B\) be given by (2.13). If (2.15) holds with \(E_1 < \zeta < 1\), then \(B\) is a contraction.

**Proof.** Let \(B\) be defined by (2.13). Then for \(\varphi, \psi \in P_T\), we have
\[ \|B\varphi(\cdot) - (B\psi)(\cdot)\| = \max_{t \in [t_0, T_+\delta(t_0)]} |(B\varphi)(t) - (B\psi)(t)| \]
\[ = \max_{t \in [t_0, T_+\delta(t_0)]} |Q(t, \varphi(\delta_-(s, t))) - Q(t, \psi(\delta_-(s, t)))| \]
\[ \leq E_1 \|\varphi - \psi\| \]
\[ \leq \zeta \|\varphi - \psi\|. \]

This shows \(B\) is a contraction mapping with contraction constant \(\zeta\).

**Theorem 7.** Assume that all hypothesis of Lemma 4 are satisfied. Let \(r\) be given by (2.18), \(\alpha := \|Q(t, 0)\|\) and \(\beta := \|G(t, 0, 0)\|\). Let \(J\) be a positive constant satisfying the inequality
\[ E_1J + \alpha + r (\delta_T(t_0) - t_0) [\|A\| (\alpha + E_1J) + (E_2 + E_3)J + \beta] \leq J. \]

Then the equation (2.1) has a solution in \(M := \{\varphi \in P_T : \|\varphi\| \leq J\}\).

**Proof.** By Lemma 4, \(C\) is continuous and compact. Also \(B\) is a contraction of \(P_T\). Now, we have to show that \(\|B\psi + C\varphi\| \leq J\) for \(\varphi, \psi \in M\). Take \(\varphi\) and \(\psi\) from \(M\) then
\[ \|B\psi(\cdot) + C\varphi(\cdot)\| \leq E_1 \|\psi\| + \alpha + r \int_{t_0}^{\delta_T(t_0)} [\|A\| (\alpha + E_1 \|\varphi\|) + (E_2 + E_3) \|\varphi\| + \beta] \Delta u \]
\[ \leq E_1J + \alpha + r (\delta_T(t_0) - t_0) [\|A\| (\alpha + E_1J) + (E_2 + E_3)J + \beta] \]
\[ \leq J. \]

By Krasnoselskii’s theorem, there exists a fixed point \(z \in M\) such that \(z = Bz + Cz\). This fixed point is also a \(T\)-periodic solution of (2.1) in shifts \(\delta_T\). The proof is complete.

**Theorem 8.** In addition to all hypothesis of Lemma 4 suppose also that
\[ E_1 + r (\delta_T(t_0) - t_0) (\|A\| E_1 + E_2 + E_3) < 1. \]

Then the equation (2.1) has a unique solution which is \(T\)-periodic in shifts.

**Proof.** Let the mapping \(H\) be defined by (2.12) and \(\varphi, \psi \in P_T\). Since
\[ \|(H\varphi)(\cdot) - (H\psi)(\cdot)\| \leq [E_1 + r (\delta_T(t_0) - t_0) (\|A\| E_1 + E_2 + E_3)] \|\varphi - \psi\| \]
the proof follows from contraction mapping principle. The proof is complete.

The following result is a generalization of [11] Corollary 2.7 and [15] Corollary 2.8.
Corollary 3. Assume that (2.2), (2.3) hold. Let $\alpha$ and $\beta$ be the constants as in Theorem 4. Suppose that there exist positive constants $E_1^*, E_2^*$ and $E_3^*$ such that

\begin{align}
|Q(t, x) - Q(t, y)| &\leq E_1^* \|x - y\|, \\
|G(t, x, y) - G(t, z, w)| &\leq E_2^* \|x - z\| + E_3^* \|y - w\|
\end{align}

and

\begin{equation}
E_1^* J + \alpha + r \left( \delta_1^T(t_0) - t_0 \right) \left\| \alpha + (E_1^* J + (E_2^* + E_3^*) J + \beta \right\| \leq J
\end{equation}

holds for all $x, y, z,$ and $w \in \mathcal{M}$. Then (2.1) has a solution in $\mathcal{M}$. Moreover, if

\begin{equation}
E_1^* + r \left( \delta_1^T(t_0) - t_0 \right) \left( \|A\| E_1^* + E_2^* + E_3^* \right) < 1,
\end{equation}

then the solution in $\mathcal{M}$ is unique.

The following example illustrates our existence results.

Example 6. Consider the time scale $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$, which is 2-periodic in shifts $\delta_{\pm}(s, t) = s^\pm t$ associated with the initial point $t_0 = 1$. For a positive constant $J$ define the set $\mathcal{M}_J$ by

$\mathcal{M}_J := \{\varphi \in P_2 : \|\varphi\| \leq J\},$

where $P_2$ is the set of 2-periodic functions in shifts $\delta_{\pm}(s, t) = s^\pm t$. Substituting

\begin{equation}
A(t) = \begin{bmatrix}
\frac{1}{7} & 0 \\
0 & \frac{1}{7}
\end{bmatrix},
\end{equation}

\begin{equation}
Q(t, u) = \begin{bmatrix}
\frac{1}{7} \left( (-1) \frac{\ln t}{\sqrt{2}} + u \right) \\
0
\end{bmatrix},
\end{equation}

and

\begin{equation}
G(t, u, v) = \begin{bmatrix}
\frac{1}{\sqrt{8}} \sin \left( \frac{\ln t}{\ln \sqrt{2}} \pi \right) u \\
0
\end{bmatrix},
\end{equation}

into (2.1) we obtain the dynamic system

\begin{equation}
\begin{bmatrix}
\frac{1}{7} & 0 \\
0 & \frac{1}{7}
\end{bmatrix} x(t) + \begin{bmatrix}
\frac{1}{7} \left( (-1) \frac{\ln t}{\sqrt{2}} + x(\delta_-(s, t)) \right) \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{1}{\sqrt{8}} \sin \left( \frac{\ln t}{\ln \sqrt{2}} \pi \right) x(t)
\end{bmatrix}.
\end{equation}

One may easily verify that (2.2)–(2.5) hold for all $x \in P_2$ and all $t \in \mathbb{T}^r = \{2^n : n \in \mathbb{Z}\}$. Similar to [2, Example 6], one may conclude that

\begin{equation}
\Phi_A \left( \delta_+^2(1), 1 \right) = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix},
\end{equation}

which along with Theorem 4 shows that the homogeneous system

\begin{equation}
\begin{bmatrix}
\frac{1}{7} & 0 \\
0 & \frac{1}{7}
\end{bmatrix} x(t)
\end{equation}

has no periodic solution in shifts. For $\varphi, \psi \in \mathcal{M}$, we have

\begin{equation}
|Q(t, \varphi(\delta_-(s, t))) - Q(t, \psi(\delta_-(s, t)))| \leq \frac{1}{8} \max_{t \in [1, 4]t} |(\varphi(\delta_-(s, t)) - \psi(\delta_-(s, t)))| \\
\leq \frac{1}{8} \|\varphi - \psi\|
\end{equation}
which shows that (2.20) holds for \( E_1^* = \frac{1}{8} \). For \( \varphi, \psi \in \mathcal{M} \) we also have
\[
|G(t, \varphi(t), \varphi(\delta_-(s, t))) - G(t, \psi(t), \psi(\delta_-(s, t)))| \leq \frac{1}{8} \max_{t \in [1,2]} \left| \frac{1}{t} \sin \left( \frac{\ln t}{\ln \sqrt{2}} \right) (\varphi(t) - \psi(t)) \right|
\]
\[
\leq \frac{1}{8} \|\varphi - \psi\|
\]
Thus, (2.21) holds for \( E_2^* = \frac{1}{8} \) and \( E_3^* = 0 \). Hence, for such \( E_1^* \), \( E_2^* \), and \( E_3^* \) the inequality (2.22) turns into
\[
(2.24) \quad \frac{2}{5} \leq J
\]
since \( \alpha = \frac{1}{8}, \beta = 0, \|A\| = 1, \) and
\[
\begin{align*}
    r &= \max_{t \in [1,2]} \left( \max_{u \in [0,1]} \left| \Phi_A(\sigma(u), t_0) \left( \Phi_A^{-1}(\delta^T_+(1, 1) - I) \Phi_A^{-1}(t, t_0) \right)^{-1} \right| \right) \\
    &= \max_{t \in [1,2]} \left( \max_{u \in [0,1]} \left| \left[ -2 \frac{\ln u}{\ln 2} \Phi_A^{-1}(t, t_0) \right]^{-1} \right| \right) \\
    &\leq \max_{t \in [1,2]} \left( \left| 2^{-\frac{\ln 4}{\ln 2}} \Phi_A(t, t_0) \right| \right) \\
    &= \max_{t \in [1,2]} \left( \left| 2^{-\frac{\ln 4}{\ln 2}} t I \right| \right) \\
    &= 1.
\end{align*}
\]
By Corollary 3 the system (2.23) has a 2-periodic solution in shifts. Moreover, since
\[
E_1^* + r \left( \delta^T_+(t_0) - t_0 \right) (\|A\| E_1^* + E_2^* + E_3^*) = \frac{3}{8} < 1
\]
the periodic solution of the system (2.23) is unique.

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(M. Adıvar) Izmir University of Economics, Department of Mathematics, 35330, Izmir, Turkey
E-mail address, M. Adıvar: murat.adivar@ieu.edu.tr

(H. C. Koyuncuoğlu) Izmir University of Economics, Department of Mathematics, 35330, Izmir, Turkey
E-mail address, H. C. Koyuncuoğlu: can.koyuncuoglu@ieu.edu.tr

(Y. N. Raffoul) University of Dayton, Department of Mathematics, Dayton, OH 45469-2316, USA
E-mail address: yraffoul1@udayton.edu