ON A NORM INEQUALITY FOR A POSITIVE BLOCK-MATRIX.

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ABSTRACT. For a positive semidefinite matrix $H = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$, we consider the norm inequality $||H|| \leq ||A + B||$. We show that this inequality holds under certain conditions. Some related topics are also investigated.

1. Introduction

In this paper we investigate the following problems posed by Minghua Lin [9].

Problem 1. Let $A > 0$, $B > 0$ and $X$ be matrices satisfying $H = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$, or equivalently $B \geq X^*A^{-1}X$. Under what condition can we conclude $||H|| \leq ||A + B||$?

Problem 2. Let $A > 0$ and $X$ be matrices. Under what condition can we conclude

$$||A + A^{-\frac{1}{2}}XX^*A^{-\frac{1}{2}}|| \leq ||A + X^*A^{-1}X||?$$

As explained later the problem 2 is a special case of the problem 1. It is shown in [3][10] that if $X = X^*$, then the inequality in the problem 1 holds. Hiroshima [7] showed that if we have both $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$ and $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geq 0$, then the inequality in the problem 1 is true. (See also [8] and [11] for more information on this topic.) Related to these problems, Lin conjectured the following.

Lin’s conjecture ([9] See also [4] Conjecture 2.14). If $X$ is normal and all matrices are $2 \times 2$, then the inequality in the problem 1 is true.

Lin showed that this conjecture is OK in the case that $B = X^*A^{-1}X$. More generally it is shown in [12] that the inequality in the problem 1 is true in the case that the numerical range of $X$ is a line segment. Here we should remark that in this case $X$ must be normal. On the other hand if $X$ is $2 \times 2$ and normal, then its numerical range is a line segment. Thus we know that Lin’s conjecture is true. In [5] Bourin and Mhanna generalized this theorem.

There are two main results in this paper. The first one is a partial answer to the problem 1. We show that if $||A + X^*A^{-1}X|| \geq ||A +XA^{-1}X^*||$ then the inequality $||H|| \leq ||A + B||$ is true. Moreover we show that if $||A + X^*A^{-1}X|| \leq ...
$$\|A + XA^{-1}X^*\|$$ then we have

$$\left\| \begin{bmatrix} A & X^* \\ X & C \end{bmatrix} \right\| \leq \|A + C\|$$

for any $C \geq XA^{-1}X^*$. The second main result is as follows. We expect that if the inequality in the problem 1 holds for any $A > 0$ and $B > 0$ with $B \geq X^*A^{-1}X$, then $X$ must be normal. We show that this is true if the eigenvalues of $XX^*$ are distinct.

The author wishes to express his hearty gratitude to Professor Minghua Lin for his kind explanation and advice. The author also would like to thank Professors J.-C. Bourin and Antoine Mhanna for valuable comments.

2. Main results.

Throughout this paper we consider $n \times n$-matrices acting on $\mathbb{C}^n$. We denote by $\|A\|$ the operator norm of the matrix $A$. That is, $\|A\|^2$ is the maximal eigenvalue of $A^*A$. For two vectors $\xi, \eta \in \mathbb{C}^n$ their inner product is denoted by $\langle \xi, \eta \rangle$. We define the norm of the vector $\xi \in \mathbb{C}^n$ by $\|\xi\| = \langle \xi, \xi \rangle^{\frac{1}{2}}$. The matrix $A$ is called positive semidefinite if $\langle A\xi, \xi \rangle \geq 0$ for any $\xi \in \mathbb{C}^n$ and we use the notation $A \geq 0$. We also use the notation $A > 0$ if $A \geq 0$ and $A$ is invertible. For two self-adjoint matrices $A$ and $B$ the order $A \leq B$ is defined by $B - A \geq 0$.

At first we give some remarks on the problems.

(i) The problem 2 is a special case of the problem 1. Indeed we have

$$\begin{bmatrix} A & X \\ X^* & X^*A^{-1}X \end{bmatrix} \succeq 0$$

and

$$\left\| \begin{bmatrix} A & X \\ X^* & X^*A^{-1}X \end{bmatrix} \right\| = \left\| \begin{bmatrix} A^\frac{1}{2} & \frac{A^\frac{1}{2}X}{X^*A^{-\frac{1}{2}}} \\ \frac{X^*A^{-\frac{1}{2}}}{A^\frac{1}{2}} & A^{-\frac{1}{2}} \end{bmatrix} \right\|

= \left\| \begin{bmatrix} A^\frac{1}{2} & \frac{A^\frac{1}{2}X}{X^*A^{-\frac{1}{2}}} \\ \frac{X^*A^{-\frac{1}{2}}}{A^\frac{1}{2}} & A^{-\frac{1}{2}} \end{bmatrix} \right\| = \|A + A^{-\frac{1}{2}}XX^*A^{-\frac{1}{2}}\|.$$

(ii) In general the inequality in the problem 2 does not hold. Assume that the inequality in the problem 2 is true for any matrices $A > 0$ and $X$. By this assumption we have

$$\|A + A^{-\frac{1}{2}}XX^*A^{-\frac{1}{2}}\| \leq \|A + X^*A^{-1}X\|$$

We also have

$$\|A + A^{-\frac{1}{2}}(A^\frac{1}{2}X^*A^{-\frac{1}{2}})(A^\frac{1}{2}X^*A^{-\frac{1}{2}})^{*}A^{-\frac{1}{2}}\|

\leq \|A + (A^\frac{1}{2}X^*A^{-\frac{1}{2}})^{*}A^{-1}(A^\frac{1}{2}X^*A^{-\frac{1}{2}})\|.$$
and hence
\[ \| A + X^*A^{-1}X \| \leq \| A + A^{-\frac{1}{2}}XX^*A^{-\frac{1}{2}} \|. \]

Therefore we conclude that
\[ \| A + A^{-\frac{1}{2}}XX^*A^{-\frac{1}{2}} \| = \| A + X^*A^{-1}X \|. \]

Consider the case \( X = U \) unitary. Then we have
\[ \| A + A^{-1} \| = \| A + U^*A^{-1}U \|. \]

If \( A \) is a \( 3 \times 3 \) diagonal matrix and \( U \) is a permutation matrix, then it is easy to construct a counterexample.

The following is a key lemma for our investigation.

**Lemma 2.1.** Let \( A > 0, \ B > 0 \) and \( X \) be matrices. (We don’t have to assume \( B \geq X^*A^{-1}X \).) If \( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} > \| A + B \| \), then we have
\[ \| A + XA^{-1}X^* \| \geq \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} > \| A + B \|. \]

In particular if \( B \geq X^*A^{-1}X \), then we have
\[ \| A + XA^{-1}X^* \| > \| A + X^*A^{-1}X \|. \]

After finishing this paper the author learned that there is a similar result in [6] in the case \( B = k - A \) for some positive constant \( k \).

**Proof.** We set \( H = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \) and \( \lambda = \| H \| \). By the assumption, we have \( \lambda > \| A + B \| \). We can find two vectors \( \xi \) and \( \eta \) such that \( ||\xi||^2 + ||\eta||^2 \neq 0 \) and
\[ \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \lambda \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \]

Then we get
\[ A\xi + X\eta = \lambda \xi, \quad X^*\xi + B\eta = \lambda \eta. \]

Since \( \lambda > \| A + B \| \), both \( \lambda - A \) and \( \lambda - B \) are invertible. Then we can rewrite the above relations as
\[ (\lambda - A)^{-1}X\eta = \xi, \quad (\lambda - B)^{-1}X^*\xi = \eta. \]

Therefore we get
\[ (\lambda - A)^{-1}X(\lambda - B)^{-1}X^*\xi = \xi \]

and hence
\[ X(\lambda - B)^{-1}X^*\xi = \lambda \xi - A\xi. \]

Thus we have
\[ (A + X(\lambda - B)^{-1}X^*)\xi = \lambda \xi. \]
Here we remark that $\xi \neq 0$. Indeed recall the relation $(\lambda - B)^{-1}X^*\xi = \eta$. By this equality, if $\xi = 0$, then we must have $\eta = 0$. This contradicts the fact $||\xi||^2 + ||\eta||^2 \neq 0$. So we conclude

$$||A + X(\lambda - B)^{-1}X^*|| \geq \lambda > ||A + B||.$$ 

Since $A + B \leq \lambda$, we have $(\lambda - B)^{-1} \leq A^{-1}$. Thus we get

$$||A + XA^{-1}X^*|| \geq \lambda > ||A + B||.$$

By this lemma, we have the following.

**Theorem 2.2.** Let $A > 0$ and $X$ be matrices. We set

$$\alpha = ||A + X^*A^{-1}X||, \quad \beta = ||A + XA^{-1}X^*||$$

Then we have the following.

(i) If $\alpha > \beta$, then for any $B \geq X^*A^{-1}X$ we have

$$\left\|\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right\| \leq ||A + B||.$$  \hspace{1cm} (1)

(ii) If $\alpha < \beta$, then for any $C \geq XA^{-1}X^*$ we have

$$\left\|\begin{bmatrix} A & X^* \\ X & C \end{bmatrix}\right\| \leq ||A + C||.$$  \hspace{1cm} (2)

(iii) If $\alpha = \beta$, then for any $B \geq X^*A^{-1}X$ and $C \geq XA^{-1}X^*$ we have

$$\left\|\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right\| \leq ||A + B||, \quad \left\|\begin{bmatrix} A & X^* \\ X & C \end{bmatrix}\right\| \leq ||A + C||.$$ 

In particular either the inequality (1) or (2) is always true.

**Proof.** This immediately follows from the previous lemma. Indeed if $B \geq X^*A^{-1}X$ does not satisfy the inequality (1), then by the lemma we have $\alpha < \beta$. Similarly if $C \geq XA^{-1}X^*$ does not satisfy the inequality (2), then by the lemma we have $\alpha > \beta$. So we have shown both (i) and (ii). The statement (iii) is also obvious. \hspace{1cm} □

Next we want to consider a special case in which $X$ is unitary.

**Proposition 2.3.** For any positive invertible matrix $A$ and any unitary $U$, we have

$$||A + A^{-1}|| \leq ||A + U^*A^{-1}U||.$$ 

That is, the inequality in the problem 2 is true if $X$ is unitary.
Proof. Let $\lambda_{\text{min}}$ be the minimal inequality of $A$. Consider the function $f(t) = t + t^{-1}$. Then since $f'(t) = \frac{t^2 - 1}{t^2}$, the maximum of $f(t)$ on the interval $0 < a \leq t \leq b$ is given by $\max\{a + a^{-1}, b + b^{-1}\}$. Therefore we have

$$||A + A^{-1}|| = \max\{\lambda_{\text{min}} + \lambda_{\text{min}}^{-1}, ||A|| + ||A||^{-1}\}.$$ 

We may assume that

$$||A + A^{-1}|| = ||A|| + ||A||^{-1}.$$ 

Indeed, by setting $B = A^{-1}$, we see that $||A + A^{-1}|| = ||B + B^{-1}||$ and $||A + U^*A^{-1}U|| = ||B + UB^{-1}U^*||$. Moreover the spectrum of $B$ is located in the interval $||A||^{-1} \leq t \leq \lambda_{\text{min}}^{-1} = ||B||$. Therefore if $||A + A^{-1}|| = \lambda_{\text{min}} + \lambda_{\text{min}}^{-1}$, then we have

$$||B + B^{-1}|| = ||A + A^{-1}|| = \lambda_{\text{min}} + \lambda_{\text{min}}^{-1} = ||B|| + ||B||^{-1}.$$ 

Now we have only to show

$$||A|| + ||A||^{-1} \leq ||A + U^*A^{-1}U||.$$ 

Since $A \leq ||A||$, we have $U^*A^{-1}U \geq ||A||^{-1}$. Thus we get

$$||A + U^*A^{-1}U|| \geq ||A + ||A||^{-1}|| = ||A|| + ||A||^{-1}.$$ 

\[\square\]

In the case that $X$ is a unitary $U$, we can rewrite the problem 1 as follows.

**Problem 3.** For any $A > 0$, any $C \geq A^{-1}$ and any unitary $U$, under what condition can we conclude

$$\left\| \begin{bmatrix} A & 1 \\ 1 & C \end{bmatrix} \right\| \leq ||A + U^*CU||?$$ 

Indeed if $X$ is a unitary $U$ in problem 1, we see that

$$\left\| \begin{bmatrix} A & U \\ U^* & B \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} A & 1 \\ 1 & UB \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} \right\| = \left\| \begin{bmatrix} A & 1 \\ 1 & UB \end{bmatrix} \right\|$$

and $||A + B|| = ||A + U^*(UBU^*)U||$. Thus by letting $C = UBU^*$ we obtain the problem 3.

In the previous proposition we have shown that the inequality in the problem 3 is true in the case $C = A^{-1}$. In the same way we can also show that the inequality in the problem 3 is true in the case $C = \alpha A^{-1}$ for any scalar $\alpha \geq 1$. These facts might suggest that the inequality in the problem 3 is true when $AC = CA$. However we can construct a counter example as follows.

Set

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$
Here we remark that
\[ A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \leq C. \]

We observe
\[ \| A + U^* C U \| = \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \right\| = \frac{7}{2} \]

Next we compute the norm \( \| A \oplus I \| C \). We observe that
\[ \| a 1 c \| = a + c + \sqrt{(a - c)^2 + 4} \]
for any positive numbers \( a \) and \( c \). Then we see that
\[ \| A 1 C \| \geq \| 3 1 2 \| = \frac{5 + \sqrt{5}}{2} > \frac{7}{2} = \| A + U^* C U \|. \]

Recall the following theorem due to Ando.

**Theorem 2.4** (Ando [1]). *The matrix \( B \) is fixed.*

If the implication
\[ \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \implies \| A C \| \geq \| B \| \]
is true for any \( A \geq 0, C \geq 0 \), then we have \( \| B \| = r(B) \), where \( r(B) \) is the spectral radius of \( B \). (\( B \) is normaloid.)

Inspired by this theorem, we want to ask the following.

**Problem 4.** If the inequality in the problem 1 is true for any \( A > 0 \) and \( B \geq X^* A^{-1} X \), what can we say about \( X \)? Can we conclude that \( X \) is normal?

Next we will make some observation for the problem 4. We can rewrite the problem 4 as follows.

**Problem 5.** Let \( D \) be positive and let \( U \) be unitary. If \( \| \begin{bmatrix} A & D \\ D & C \end{bmatrix} \| \leq \| A + U^* C U \| \) for any \( A > 0 \) and \( C \geq DA^{-1} D \), can we conclude \( UD = DU \)?

Indeed, take a polar decomposition \( X = DU \) and set \( C = UBU^* \). Then we see that
\[ \| A \oplus X \| = \| A \oplus D U \| = \| A \oplus D C \| \]
and \( \| A + B \| = \| A + U^* C U \| \). On the other hand we observe that the inequality \( B \geq X^* A^{-1} X = U^* DA^{-1} DU \) is equivalent to \( C = UBU^* \geq DA^{-1} D \). Therefore we conclude that the problem 4 is equivalent to the problem 5. Here we remark that \( D = (XX^*)^\frac{1}{2} \).
Lemma 2.5. Under the assumption in the problem 5, we have
\[ ||D + U^*DU|| = 2||D||. \]

Proof. Set \( A = C = D \). Here we remark that \( C = D = DA^{-1}D \). By the assumption we have
\[ \left\| \begin{bmatrix} D & D \\ D & D \end{bmatrix} \right\| \leq ||D + U^*DU||. \]

Then since \( \left\| \begin{bmatrix} D & D \\ D & D \end{bmatrix} \right\| = 2||D|| \), we see that
\[ 2||D|| = \left\| \begin{bmatrix} D & D \\ D & D \end{bmatrix} \right\| \leq ||D + U^*DU|| \leq 2||D||. \]

So we are done. \( \square \)

Lemma 2.6. Under the assumption in the problem 5, we can find a unit vector \( \xi \) satisfying both \( D\xi = ||D||\xi \) and \( DU\xi = ||D||U\xi \).

Proof. By the previous lemma, we can take a unit vector \( \xi \) such that
\[ (D + U^*DU)\xi = 2||D||\xi. \]

Then since
\[ 2||D|| = \left\| (D + U^*DU)\xi \right\| \leq ||D\xi|| + ||U^*DU\xi|| \leq 2||D||, \]
we have both \( ||D\xi|| = ||D|| \) and \( ||U^*DU\xi|| = ||D|| \). Since
\[ \left\| (||D||^2 - D^2)^{1/2} \xi \right\|^2 = \left\langle (||D||^2 - D^2)\xi, \xi \right\rangle = ||D||^2 - ||D\xi||^2 = 0, \]
we have \( D\xi = ||D||\xi \). Similarly we get \( U^*DU\xi = ||D||\xi \). \( \square \)

We have the partial answer to the problem 5 as follows.

Theorem 2.7. Under the assumption in the problem 5, if the \( n \times n \)-matrix \( D \) has \( n \) distinct eigenvalues, then we have \( UD = DU \). That is, the problem 5 is true in this case.

Here recall that \( D = (XX^*)^{1/2} \) and that the problem 4 is equivalent to the problem 5. These mean that the problem 4 is true if \((XX^*)^{1/2}\) has \( n \) distinct eigenvalues.

For the proof we need some preparation.

Lemma 2.8. [2] Lemma 2.1 For two positive operators \( A \) and \( B \), if they satisfy
\[ ||A + B|| = ||A|| + ||B||, \]
then we have \( ||\alpha A + \beta B|| = \alpha||A|| + \beta||B|| \) for any \( \alpha \geq 0 \) and \( \beta \geq 0 \).
Proof. We would like to include its proof for completeness. Without loss of generality we may assume that \( \alpha \geq \beta \geq 0 \). We see that
\[
||\alpha A + \beta B|| = ||\alpha (A + B) - (\alpha - \beta) B|| \geq \alpha||A + B|| - (\alpha - \beta)||B||
\]
\[
= \alpha(||A|| + ||B||) - (\alpha - \beta)||B|| = \alpha||A|| + \beta||B||.
\]
The reverse inequality follows from the triangle inequality. \( \square \)

Lemma 2.9. Consider the matrices as in the problem 5. Let \( q \) be a projection with \( Dq = qD \) and \( Uq = qU \) and we set \( p = 1 - q \). Then we have
\[
||Dp + U^* DpU|| = 2||Dp||.
\]
Proof. We set
\[
A = kDp + q
\]
where \( k \) is a positive constant. Later we will take \( k \) large enough. By the assumption we have
\[
\left\| \begin{bmatrix} A & D \\ D & DA^{-1}D \end{bmatrix} \right\| = \left\| A + A^{-\frac{1}{2}} D^2 A^{-\frac{1}{2}} \right\| \leq \|A + U^* DA^{-1}DU\|.
\]
We see that
\[
||A + A^{-\frac{1}{2}} D^2 A^{-\frac{1}{2}}|| \geq \|(A + A^{-\frac{1}{2}} D^2 A^{-\frac{1}{2}})p|| = \left( k + \frac{1}{k} \right)||Dp||.
\]
Thus we conclude that
\[
\left( k + \frac{1}{k} \right)||Dp|| \leq \|A + U^* DA^{-1}DU\|.
\]
On the other hand we observe
\[
||A + U^* DA^{-1}DU|| = ||kDp + \frac{1}{k} U^* DpU + (1 + U^* D^2 U)q||
\]
\[
= \max\{||kDp + \frac{1}{k} U^* DpU||, \ ||(1 + U^* D^2 U)q||\}
\]
because the operator \( kDp + \frac{1}{k} U^* DpU \) is orthogonal to \( (1 + U^* D^2 U)q \). (Recall that both \( p \) and \( q \) commute with \( D \) and \( U \).) If \( Dp = 0 \), we have nothing to do. If \( Dp \neq 0 \), we can take the constant \( k > 0 \) large enough such that
\[
||kDp + \frac{1}{k} U^* DpU|| \geq k||Dp|| - \frac{1}{k}||U^* DpU|| \geq ||(1 + U^* D^2 U)q||
\]
and hence
\[
||A + U^* DA^{-1}DU|| = ||kDp + \frac{1}{k} U^* DpU||.
\]
Then we have
\[
\left( k + \frac{1}{k} \right)||Dp|| \leq ||kDp + \frac{1}{k} U^* DpU|| \leq ||kDp|| + ||\frac{1}{k} U^* DpU|| = (k + \frac{1}{k})||Dp||.
\]
That is, we get
\[ ||kDp + \frac{1}{k}U^*DpU|| = ||kDp|| + ||\frac{1}{k}U^*DpU||.\]

By the previous lemma we have the desired statement. \(\square\)

**Proof of Theorem 2.7.** Since each eigenvalue of \(D\) has multiplicity 1, by lemma 2.6 there exists a rank one projection \(q_1\) such that \(Dq_1 = q_1D = ||D||q_1\) and \(Uq_1 = q_1U\). We set \(p_1 = 1 - q_1\). Then applying lemma 2.9 to \(q_1\) and \(p_1\), we obtain
\[ ||Dp_1 + U^*Dp_1U|| = 2||Dp_1||.\]

Then by the proof of lemma 2.6, we can find a unit vector \(\xi = p_1\xi\) such that \(D\xi = ||Dp_1||\xi\) and \(DU\xi = ||Dp_1||U\xi\). Since the eigenvalue \(||Dp_1||\) of \(D\) has multiplicity 1, we can find a rank 1 projection \(q_2 \leq p_1\) such that \(Dq_2 = q_2D = ||Dp_1||q_2\) and \(Uq_2 = q_2U\). We set \(p_2 = 1 - (q_1 + q_2)\). By applying lemma 2.9 again, we get
\[ ||Dp_2 + U^*Dp_2U|| = 2||Dp_2||.\]

By continuing this procedure, we can construct mutually orthogonal rank 1 projections \(q_1, q_2, \ldots, q_n\) such that \(Uq_j = q_jU\) and \(D = \lambda_1q_1 + \cdots + \lambda_nq_n\), \((\lambda_1 > \lambda_2 > \cdots > \lambda_n)\). Then we conclude that \(UD = DU\). \(\square\)

**References**

[1] T. Ando, *Geometric mean and norm Schwarz inequality*. Ann. Funct. Anal. 7 (2016), no. 1, 1–8.

[2] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *The Daugavet equation in uniformly convex Banach spaces*. J. Funct. Anal. 97 (1991), no. 1, 215–230.

[3] J-C. Bourin and E-Y. Lee, *Decomposition and partial trace of positive matrices with Hermitian blocks*. Internat. J. Math. 24 (2013), no.1, 1350010, 13 pp.

[4] J-C. Bourin, E-Y. Lee and M. Lin, *On a decomposition lemma for positive semi-definite block-matrices*. Linear Algebra Appl. 437 (2012), no. 7, 1906–1912.

[5] J-C. Bourin and A. Mhanna, *Positive block matrices and numerical ranges*. C. R. Math. Acad. Sci. Paris 355 (2017), no.10, 1077–1081.

[6] M. Gumus, J. Liu, S. Raouafi and T-Y. Tam, *Positive semi-definite 2 × 2 block matrices and norm inequalities*. Linear Algebra Appl. 551 (2018), 83-91.

[7] T. Hiroshima, *Majorization criterion for distillability of a bipartite quantum state*. Phys. Rev. Lett. 91 (2003) no. 5 057902, 4 pp.

[8] M Lin, *Some applications of a majorization inequality due to Bapat and Sunder*. Linear Algebra Appl. 469 (2015), 510–517.

[9] , private communication.

[10] M. Lin and H. Wołkowicz, *An eigenvalue majorization inequality for positive semidefinite block matrices*. Linear Multilinear Algebra 60 (2012), no. 11-12, 1365–1368.

[11] , *Hiroshima’s theorem and matrix norm inequalities*. Acta Sci. Math. (Szeged) 81 (2015), no. 1-2, 45–53.

[12] A. Mhanna, *On symmetric norm inequalities and positive definite block-matrices*. Math. Inequal. Appl. 21 (2018), no. 1, 133–138.
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