On the Large-$N_c$ Limit and Electroweak Interactions: Some Properties of the $N_c$-Extended Standard Model\textsuperscript{a}

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We discuss properties of QCD with variable and large $N_c$, taking into account electroweak interactions; i.e., we analyze the generalization of the standard model based on the gauge group $G = \text{SU}(N_c) \times \text{SU}(2)_L \times \text{U}(1)_Y$. General classes of solutions to anomaly constraints are given, and it is shown that these allow a $T_3 = \pm 1/2$ quark or $T_3 = -1/2$ lepton rather than the neutrino as an electrically neutral fermion. The issue of grand unification is addressed, and it is shown that $G$ cannot be embedded in the usual way in a simple SO($N$) or SU($N$) gauge group unless $N_c = 3$. The ratio of strengths of QCD and electroweak interactions is discussed.

1 Introduction

The large-$N$ limit has long been of use in both statistical mechanics and field theory. Early examples of its application include the exact solutions of an O($N$)-invariant spin model by Stanley\textsuperscript{1,2}, 2D U($N_c$) QCD by ’t Hooft\textsuperscript{3}, 2D models with four-fermion interactions and O($N$)-invariant scalar interactions,\textsuperscript{4} and the O($N$)-invariant nonlinear sigma model in $d = 2 + \epsilon$ dimensions.\textsuperscript{5,6}

The large-$N_c$ expansion has been of great value as an analytic approach to the nonperturbative properties of 4D QCD\textsuperscript{7,8}; work continues on properties of baryons, on finite-temperature and density behavior, and other topics (reviews include\textsuperscript{9,10}).

In applications of the large-$N_c$ expansion to QCD, a common practice has been to turn off electroweak interactions and analyze the QCD sector by itself. However, it is of considerable interest to take into account the electroweak interactions in the context of the large-$N_c$ limit and, more generally, to investigate the properties of the full $N_c$-extended standard model (SM) defined by the gauge group

$$G = \text{SU}(N_c) \times \text{SU}(2) \times \text{U}(1)_Y$$

(1)

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with fermions transforming as

\[ Q_{iL} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L : (N_c, 2, Y_{Q_L}) \] (2)

\[ u_{iR} : (N_c, 1, Y_{u_R}) \quad d_{iR} : (N_c, 1, Y_{d_R}) \] (3)

\[ L_{iL} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L : (1, 2, Y_{\nu_L}) \] (4)

\[ \nu_{jR} : (1, 1, Y_{\nu_R}) \quad e_{iR} : (1, 1, Y_{e_R}) \] (5)

where \( i \) denotes generation, \( i = 1, \ldots, N_{\text{gen}} \), with \( u_1 = u, u_2 = c, u_3 = t, d_1 = d, d_2 = s, d_3 = b \), etc. Here \( N_c \) is not necessarily large. For generality, we shall consider arbitrary \( N_{\text{gen}} \). We have analyzed this theory and report additional results here.

In this \( N_c \)-extended standard model the usual relations \( Q = T_3 + Y/2 \), \( Y_{Q_L} = q_u + q_d, q_u = q_d + 1, q_e = q_e + 1, Y_{f_R} = 2q_{f_R} \) and the vectorial property of electric charge, \( q_{f_L} = q_{f_R} = q_f \) continue to hold. Hence \( Y_{u_R} = Y_{Q_L} + 1, Y_{d_R} = Y_{Q_L} - 1, Y_{u_R} = Y_{\nu_L} + 1, \) and \( Y_{e_R} = Y_{\nu_L} - 1 \). Before imposing anomaly cancellation conditions, there are thus two independent electric charges among the fermions; we take these to be \( q_d \) and \( q_e \).

In the general solution to the anomaly cancellation conditions for \( N_c \neq 3 \), the electric charges of all of the fermions will differ from their \( N_c = 3 \) values. In particular, since \( q_\nu \neq 0 \) in general, the theory should include electroweak-singlet neutrinos \( \nu_{jR} \) in order to form Dirac mass terms \([\bar{\nu}_L \nu_R]\); for \( q_\nu \neq 0 \), Majorana mass terms would violate electric charge conservation. This contrasts with the situation in the standard model where, because \( \nu_{jR} \)’s are singlets under \( G_{\text{SM}} \), they can be, and are, excluded from the fermion content. Hence, in the context of the standard model as a renormalizable QFT with only operators of dimension \( \leq 4 \), there are no neutrino mass terms. Of course, in the context of a grand unified theory (GUT) such as SO(10), one does include \( \nu_{jR} \)’s with \( j = 1, \ldots, N_{\text{gen}} \). The current evidence for neutrino masses and lepton mixing also motivates the inclusion of heavy \( \nu_{jR} \)’s to drive a seesaw mechanism and provide an explanation for small neutrino masses.

In passing, we note that a more restricted \( N_c \)-extension of the standard model avoids \( \nu_{jR} \)’s. There are then no gauge-invariant renormalizable terms in the Lagrangian that can give neutrino masses, so in the SM as a renormalizable QFT, the neutrinos are massless, a prediction disfavored by current experimental indications. In this restricted extension, one keeps \( q_\nu = 0 \) to avoid massless unconfined fermions. Here we shall consider the full generalization with \( \nu_{jR} \)’s.
The electroweak part of the gauge group $G$ is still $G_{EW} = SU(2) \times U(1)_Y$, and the issues of electroweak symmetry breaking (EWSB) are similar in the $N_c$-extended SM and the SM itself.

2 Anomaly Cancellation Conditions

The absence of anomalies is a necessary property of an acceptable quantum field theory. For the usual $d = 4$ dimensional spacetime considered here, there are three types of possible anomalies: (i) triangle anomalies in gauged currents which, if present, would spoil current conservation and renormalizability; (ii) the global $SU(2)$ Witten anomaly resulting from the nontrivial homotopy group $\pi_4(SU(2)) = Z_2$ which, if present, would render the path integral ill-defined; and (iii) mixed gauge-gravitational anomalies (if one includes gravity). We discuss here the constraints due to the absence of these anomalies. For anomalies (i), (iii), we suppress generation index since the cancellation occurs separately for each generation.

2.1 Anomalies in Gauged Currents

The $SU(N_c)^3$ anomaly vanishes automatically because of the vectorial nature of the gluon-fermion couplings and the $SU(N_c)^2 U(1)_Y$ anomalies vanish automatically because of the vectorial nature of the electromagnetic couplings. The condition for the vanishing of the $SU(2)^2 U(1)_Y$ anomaly is

$$N_c Y_{Q_L} + Y_{L_L} = 0 \text{, i.e. } N_c(2q_d + 1) + (2q_e + 1) = 0 \quad (6)$$

The condition for the vanishing of the $U(1)_Y^3$ anomaly is

$$N_c(2Y_{Q_L}^3 - Y_u^3 - Y_d^3) + (2Y_{L_L}^3 - Y_e^3 - Y_{e_R}^3) = 0 \quad (7)$$

This yields the same condition as for the vanishing of the $SU(2)^2 U(1)_Y$ anomaly. Solving the equation for the vanishing of this anomaly gives, say for $q_d$ in terms of $q_e$,

$$q_d = q_u - 1 = -\frac{1}{2} \left( 1 + \frac{Y_{L_L}}{N_c} \right) = \frac{1}{2} \left( 1 + \frac{1}{N_c}(2q_e + 1) \right) \quad (8)$$

or equivalently, taking $q_d$ as the independent variable,

$$q_e = q_\nu - 1 = -\frac{1}{2} \left( 1 + N_c Y_{L_L} \right) = -\frac{1}{2} \left( 1 + N_c(2q_d + 1) \right) \quad (9)$$
2.2 Global SU(2) Anomaly

The constraint from the Witten global SU(2) anomaly is that the number $N_{db}$ of SU(2) doublets, $N_{db} = (1 + N_c)N_{gen}$, is even. We consider, a priori the possibility of even and odd $N_{gen}$. If $N_{gen}$ is odd, the absence of the global SU(2) anomaly implies that $N_c$ is odd. In this case, there is a connection between $N_{gen}$ and $N_c$. In contrast, if $N_{gen}$ is even, there is no constraint on $N_c$. Of course, if $N_c$ were even, baryons would be bosons, and the properties of the world would be quite different than for odd $N_c$.

2.3 Mixed Gauge-Gravitational Anomalies

The absence of mixed gauge-gravitational anomalies does not add any further constraints; the mixed gauge-gravitational anomaly involving two graviton vertices and an $SU(N_c)$ or SU(2) gauge vertex vanishes identically since $\text{Tr}(T_a) = 0$ where $T_a$ is the generator of a nonabelian group. The anomaly involving a $U(1)_Y$ vertex is proportional to

$$N_c(2Y_Q - Y_u - Y_d) + (2Y_L - Y_e - Y_{\nu}) = 0$$

where the expression vanishes because of the vectorial nature of the electromagnetic coupling. Indeed, the two separate terms in parentheses each vanish individually: $2Y_Q - Y_u - Y_d = 0$ and $2Y_L - Y_e - Y_{\nu} = 0$, so that this anomaly does not connect quark and lepton sectors. Hence, for a given $N_c$, there is a one-parameter family of solutions for the fermion charges. The values of these charges are real, but not, in general, rational numbers, so that electric and hypercharge are not quantized, although if one is rational, then all are, as is clear from (6). In the SM extension with no $\nu_{jR}$’s and $q_\nu$ fixed at zero, one does get charge quantization: $q_e = -1$, $q_d = q_u - 1 = (1/2)(-1 + N_c^{-1})$.

3 Issue of Grand Unification

An important question is whether one can embed the $N_c$-extended standard model with gauge group $G$ in a grand unified theory (GUT) based on a simple group $G_{GUT}$. We recall that grand unification is appealing since it unifies quarks and leptons, predicts the relative sizes of gauge couplings in the SM factor groups, and quantizes electric charge, since $Q$ is a generator of $G_{GUT}$. Clearly,

$$\text{rank}(G_{GUT}) \geq \text{rank}(G) = N_c + 1$$

Following the standard procedure for constructing a GUT, one puts the fermions in complex representations to avoid bare mass terms that would produce
masses of order $M_{GUT}$ for all fermions and requires no anomalies of types (i)-(iii) in the theory. The absence of mixed gauge-gravitational anomalies is automatic, since there are no U(1) factor groups in $G_{GUT}$. Ideally, one places all fermions of a given generation in a single irreducible representation of $G_{GUT}$, although we shall also consider weakening this condition.

In order to place all fermions of each generation into one representation, a necessary condition is that $\text{Tr}(Y) = \sum_f Y_f = 0$ (for each generation) since the hypercharge $Y$ is a generator of $G_{GUT}$. This condition is satisfied:

$$\text{Tr}(Y) = 2(N_c Y_Q + Y_L) + N_c (Y_u + Y_d) + Y_e + Y_\nu = 0 \quad (12)$$

Since the exceptional groups have bounded ranks, they cannot satisfy the rank condition above for arbitrary $N_c$ and are excluded as candidates for $G_{GUT}$ for general $N_c$. To guarantee the absence of anomalies in gauged currents, one uses a “safe” group, for which the generators satisfy $A_{abc} = \text{Tr}\{T_a, T_b\} T_c = 0 \ \forall \ a, b, c$. Recall that SU($N$) is not safe for $N \geq 3$, SO($N$) has only real representations for odd $N$ and for $N \equiv 0 \mod 4$, while SO($N$) has complex representations and is safe for $N = 2 \mod 4$, except for $N = 6$ (SO(6) $\simeq$ SU(4)). This leads to the choice $G_{GUT} = \text{SO}(4k + 2)$ as the GUT group in which to embed $G$. This is the natural generalization of the SO(10) GUT in which the SM gauge group $G_{SM} = \text{SU}(3)_c \times \text{SU}(2)_L \times U(1)_Y$ is embedded. There are then no triangle anomalies in gauged currents and also no global anomaly, since $\pi_4(\text{SO}(N)) = \emptyset$ for $N \geq 6$. Now, rank(SO($2n$)) = $n$, so rank(SO($4k + 2$)) = $2k + 1$. Substituting into the inequality rank($G_{GUT}$) $\geq N_c + 1$ yields $2k \geq N_c$. Consider first the case of odd $N_{\text{gen}}$. For this case, $N_c$ must be odd for the absence of the global SU(2) anomaly in $G$, so the above inequality becomes $2k \geq N_c + 1$, and hence the minimal-rank GUT group is SO($4k + 2$) = SO($2N_c + 4$) with rank $N_c + 2$. Since $N_c + 2$ is odd, $2N_c + 4 = 2 \mod 4$, and SO($2N_c + 4$) has a complex spinor representation of dimension $2^{N_c + 1}$. For each generation, there are $N_f = 4(N_c + 1)$ Weyl fermions. The condition that these fit into a spinor is then

$$2^{N_c + 1} = 4(N_c + 1) \quad (13)$$

But this has a solution only for $N_c = 3$.

In the hypothetical case that $N_{\text{gen}}$ is even, then the global SU(2) anomaly condition allows $N_c$ to be either even or odd. The case of odd $N_c$ has been covered. For even $N_c$, the minimum-rank $G_{GUT}$ is SO($2N_c + 2$). Since $N_c$ is even, $2N_c + 2 = 2 \mod 4$, and SO($2N_c + 2$) has a complex spinor of dimension $2^{N_c + 1}$, so that one is again led to the same condition (13) and conclusion. This is an important result since it shows that one can grand-unify the $N_f$-extended standard model as discussed above only for the single case $N_c = 3$.  

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One could also attempt a less complete type of grand unification, in which one assigns fermions of a given generation to more than one representation of $G_{\text{GUT}}$. Since $\text{SU}(N_{\text{GUT}})$ is not a safe group for $N_{\text{GUT}} \geq 3$, one must arrange for the triangle anomaly in gauged currents to cancel between representations. Since fermions are not placed in a single representation, the conditions $\text{Tr}(Y) = 0$ and $\text{Tr}(T_3) = 0$ or $\text{Tr}(Q) = 0$ are also not automatically satisfied for each representation, as they would be if all the fermions of a given generation are placed in a single representation of $G_{\text{GUT}}$.

As a generalization of the Georgi-Glashow SU(5) GUT \cite{14}, we fix $q_\nu = 0$ so that one can exclude $n_{jR}$’s, and try to fit the remaining $4(N_c + 1) - 1 = 4N_c + 3$ Weyl fermions of each generation in some conjugate fundamental representations $(\psi_{k,L}^c)_\alpha$ and an antisymmetric second-rank tensor rep. $\psi_L^{[\alpha\beta]}$ of SU($N_{\text{GUT}}$), with $\alpha, \beta = 1, \ldots, N_{\text{GUT}}$. Since the contribution to the triangle anomaly in gauged currents from $\psi_L^{[\alpha\beta]}$ is $(N_{\text{GUT}} - 4)$ times that from the fundamental representation, $(\psi_L^c)_\alpha$, we use $N_{\text{GUT}} - 4$ copies of $(\psi_{k,L}^c)_\alpha$, with $k = 1, 2, \ldots, N_{\text{GUT}} - 4$. The condition that the fermions of each generation fit in these representations is then

$$4N_c + 3 = (N_{\text{GUT}} - 4)N_{\text{GUT}} + \frac{N_{\text{GUT}}(N_{\text{GUT}} - 1)}{2} = \frac{3N_{\text{GUT}}(N_{\text{GUT}} - 3)}{2}$$  \hspace{1cm} (14)

The solution is

$$N_{\text{GUT}} = \frac{1}{2}\left[3 + \frac{1}{3}\sqrt{3(51 + 32N_c)}\right]$$  \hspace{1cm} (15)

for integer $N_{\text{GUT}}$. As before, one also requires

$$\text{rank}(G_{\text{GUT}}) \geq \text{rank}(G) = N_c + 1$$  \hspace{1cm} (16)

i.e.,

$$\Delta_{\text{rank}} = \text{rank}(G_{\text{GUT}}) - \text{rank}(G) = N_{\text{GUT}} - N_c - 2 \geq 0$$  \hspace{1cm} (17)

Again, the only solution is $N_c = 3$, $N_{\text{GUT}} = 5$, i.e., SU(5), with $\Delta_{\text{rank}} = 0$. The next two solutions with integer values of $N_{\text{GUT}}$ are

$$(N_c, N_{\text{GUT}}, \Delta_{\text{rank}}) = (6, 6, -2), (48, 13, -37)$$  \hspace{1cm} (18)

and both are excluded by the negative $\Delta_{\text{rank}}$, which becomes increasingly negative for larger $N_c$. Thus, one reaches a negative conclusion even before addressing whether the above trace conditions could be satisfied. So even if one attempts this less ambitious type of grand unification of the group \cite{14} with general $N_c$, it is only possible for the special case $N_c = 3$. Of course these results do not reduce the usefulness of the large-$N_c$ expansion in pure QCD. However, they do show how special the value $N_c = 3$ is from the point of view of grand unification.
Table 1: Classes of solutions for quark charges

| case  | $q_d$ | $(q_u, q_d)$ | $Y_{QL}$ | $Y_{LL}$     |
|-------|-------|-------------|----------|-------------|
| $C_1q$| $> 0$ | $(+,-)$     | $> 1$    | $< -N_c$    |
| $C_2q$| $-1 < q_d < 0$ | $(+,-)$    | $-1 < Y_{QL} < 1$ | $-N_c < Y_{LL} < N_c$ |
| $C_2q_{sym}$| $-1/2$ | $(1/2, -1/2)$ | $0$     | $0$         |
| $C_3q$| $< -1$ | $(-,-)$     | $< -1$   | $> N_c$     |
| $C_4q$| $0$   | $(1,0)$     | $1$      | $-N_c$      |
| $C_5q$| $-1$  | $(0,-1)$    | $-1$     | $N_c$       |

4 Classification of Solutions for Fermion Charges

In the general $N_c$-extended standard model, there are several generic and special classes of solutions for the fermion charges. The classes, denoted $C_nq$, for the quark charges are given in the table. Here classes $C_1q$, $C_2q$, and $C_3q$ are generic, while $C_2q_{sym}$ is a symmetric subcase of $C_2$ and $C_4q$ and $C_5q$ are other special solutions. The charges $q_u$ and $q_d$ are monotonically increasing functions of $N_c$ if $q_e < -1/2$ and monotonically increasing functions of $N_c$ if $q_e > -1/2$. In the borderline case $q_e = -q_\nu = -1/2$ (whence $Y_{LL} = 0$), $q_u$ and $q_d$ are independent of $N_c$ and have the values in $C_2q_{sym}$, $q_d = -q_u = -1/2$ (whence $Y_{QL} = 0$), so that the anomalies of type (i) cancel separately in the quark and lepton sectors.

One can also work out analogous classes of solutions $C_1\ell$ to $C_5\ell$ for lepton charges satisfying the anomaly constraints. These results show that the solutions to the anomaly cancellation conditions allow a $T_3 = \pm 1/2$ quark or $T_3 = -1/2$ lepton rather than the neutrino as an electrically neutral fermion.

5 Relative Strengths of Color and Electroweak Interactions

As the above solutions show, if $Y_{QL} \neq 0$, then the lepton charges $q_e$ and $q_\nu$ will diverge as $N_c \to \infty$: $q_e, q_\nu \sim -Y_{QL}N_c/2$. A necessary condition for the lepton charges to remain finite as $N_c \to \infty$ is that $\lim_{N_c \to \infty} Y_{QL} = 0$. In contrast, for any fixed finite value of $q_e$, the quark charges $q_u$ and $q_d$ have finite limits as $N_c \to \infty$: $\lim_{N_c \to \infty} q_d = -1/2$, $\lim_{N_c \to \infty} q_u = 1/2$.

Even if all of the fermion charges remain finite as $N_c \to \infty$, electroweak effects that involve quarks in loops can still lead to divergent behavior in this
limit. Some examples are provided by quark loop corrections to the \( W \) and \( Z \) propagators, which are \( \propto N_c \). To keep these finite in the \( N_c \to \infty \) limit, it is sufficient to require that the SU(2) and U(1)\(_Y\) gauge couplings \( g \) and \( g' = (3/5)^{1/2}g \) satisfy \( g^2 N_c = c_2 \) and \( g'^2 N_c = c_Y \) analogous to the condition on the color SU(3)\(_c\) coupling \( (g_s)^2 N_c = c_3 \). From the relation \( e = gg' / \sqrt{g^2 + g'^2} \), it follows that \( e^2 N_c = \text{const.} \), which keeps the quark-loop contributions to the photon propagator finite. The electromagnetic corrections to the baryon mass arising from one-photon exchange between quark lines then go like \( (\delta m_B)_{\text{em}} \propto e^2 N_c \), which is the same as the behavior \( m_B \propto N_c \) in the QCD sector.

An example of a ratio of electroweak cross sections that involves \( N_c \) is

\[
\sigma(e^+e^- \to \mu^+\mu^-)/\sigma(e^+e^- \to \text{hadrons})
\]

at a center-of-mass energy \( \sqrt{s} \gg \text{hadron masses} \) (i.e. quite high, given that \( m_B \sim N_c \)) and away from the positions of narrow meson states, so that scaling can work:

\[
\frac{\sigma(e^+e^- \to \mu^+\mu^-)}{\sigma(e^+e^- \to \text{hadrons})} \sim \frac{1}{N_c}
\]  

A different issue concerning the strength of electroweak interactions arises for fixed but arbitrary \( N_c \): even if the gauge couplings \( g, g' \) are small, the one-parameter solutions for the fermion charges allow arbitrarily large values of these charges and similarly for hypercharges. Hence, the actual strengths of the U(1)\(_Y\) and electromagnetic interactions, \( |g'Y_f| \) and \( |eq_f| \) can be arbitrarily large. One can avoid this problem by setting some finite (not necessarily zero) input value for \( q_\nu \).

Part of the evidence for grand unification in our actual world is that, even before calculating precision evolution of the gauge couplings, one has the requisite ordering: at a given scale \( \mu \), of the 3! = 6 possible relative orderings in size of the SU(3), SU(2), and U(1) couplings \( g_2^2, g_1^2, g_1^2 \), one has \( g_2^2 > g_1^2 > g_1^2 \). For \( N_c \neq 3 \), since we do not have a usual GUT, different orderings might, \textit{a priori}, occur, and could produce quite different theories. For example, for fixed (moderate) \( N_c \), consider the possibility that \( g^2 \gg g_2^2, g_1^2 \). In this case, for sufficiently large \( g^2/(4\pi) \sim O(1) \), the SU(2) gauge interaction could produce the following SU(2)-invariant condensates breaking color- and/or charge as well as baryon and/or lepton number:

\[
\langle \epsilon_{ab}Q^{\alpha\alpha T}CQ^{\beta\beta T}_{\beta\beta L} \rangle = \langle u_{i\alpha L}^T C q_{j\beta L} - d_{i\alpha L}^T Cu_{j\beta L} \rangle
\]

\[
\langle \epsilon_{ab}Q^{\alpha\alpha T}CL^{\beta\beta T}_{\beta\beta L} \rangle = \langle u_{i\alpha L}^T C e_{j\beta L} - d_{i\alpha L}^T Cu_{j\beta L} \rangle
\]

\[
\langle \epsilon_{ab}L^{\alpha\alpha T}CL^{\beta\beta T}_{\beta\beta L} \rangle = \langle \nu_{i\alpha L}^T C e_{j\beta L} - e_{i\alpha L}C\nu_{j\beta L} \rangle
\]

where \( a, \alpha \), and \( (i, j) \) are SU(2), SU(\( N_c \)), and generation indices, respectively. This shows that even though QCD and electromagnetism are vectorial gauge
symmetries, they could be broken because of the SU(2)$_L$ interaction. This is prevented if $g^2$ is sufficiently small.

6 Some Properties of Bound States

First, consider baryons composed of $r$ up-type and $N_c - r$ down-type quarks; these have electric charge $q[B(r,N_c-r)] = r + N_c q_d$. The charge difference $q[B(r,N_c-r)] - q[B(N_c-r,r)] = 2r - N_c$. Now assume that $N_c$ is odd and that $m_u, m_d << \Lambda_{QCD}$, as in the physical world. A strong–isospin mirror pair which constitutes a kind of generalization of the proton and neutron is

$$\mathcal{P} = B\left(\frac{N_c+1}{2}, \frac{N_c-1}{2}\right), \quad \mathcal{N} = B\left(\frac{N_c-1}{2}, \frac{N_c+1}{2}\right) \quad (23)$$

These baryons have charges $q_\mathcal{P} = q_N + 1$ satisfying $q_\mathcal{P} = -q_e$ and $q_\mathcal{N} = -q_\nu$.

One can also consider atoms. For all cases except $C4_\ell$ ($q_e = 0$), there exists a neutral Coulomb bound state of the generalized proton $\mathcal{P}$ and electron, $(\mathcal{P}e)$, which is the $N_c$-extended generalization of the hydrogen atom. For all cases except $C5_\ell$ ($q_\nu = 0$), there also exists a second neutral Coulomb bound state, which has no analogue in the usual $N_c = 3$ standard model, namely, $(\mathcal{N}\nu_1)$, where $\nu_1$ denotes the lightest neutrino mass eigenstate. For our discussion of the $(\mathcal{P}e)$ and $(\mathcal{N}\nu)$ atoms, we assume, respectively, that $q_e \neq 0$ and $q_\nu \neq 0$, so that these atoms exist, and we suppress the mass eigenstate index in $(\mathcal{N}\nu_1)$.

The $(\mathcal{P}e)$ and $(\mathcal{N}\nu)$ atoms are nonrelativistic bound states iff $|q_e|\alpha_{em} << 1$ and $|q_\nu|\alpha_{em} << 1$ (equivalent conditions). Then the binding energy in the ground state of the $(\mathcal{P}e)$ atom is $E_{(\mathcal{P}e)} = -(q_e\alpha)^2\frac{e^2}{2}m_e$ and the Bohr radius is $a_0 = 1/(q_e^2\alpha m_e)$. These formulas apply to $(\mathcal{N}\nu)$ with the replacements $\mathcal{P} \rightarrow \mathcal{N}$ and $e \rightarrow \nu$. For large $N_c$, $e^2N_c = \text{const.}$, and fixed $q_e$, the binding is very weak, $E_{(\mathcal{P}e)} \propto N_c^{-2}$ and $a_0 \propto N_c$. Since the size of the nucleon $r_N \sim O(1)$ as $N_c \rightarrow \infty$, the electron and neutrino clouds in these respective atoms extend over much larger distances than $r_N$. Indeed, since $a_0$ diverges as $N_c \rightarrow \infty$, the notion of an individual atom reasonably well separated from other atoms requires that interatomic separations grow at least like $N_c$.

For large $N_c$ the strong interactions of mesons become weak and these mesons are long-lived. Hence, one could also consider meson-lepton Coulomb bound states. Consider, for example, the $J = 0$ $ud$ meson $\pi^+$ (or the $J = 1$ $ud\bar{d}$ meson $\rho^+$). Possible bound states of $\pi^+$ with leptons include (i) $(\pi^+ e)$, if $q_e < 0$ as in cases $C2_\ell$ and $C3_\ell$; this would have charge $q = 1 + q_e$ and hence would be neutral for the special case $C5_\ell$; (ii) $(\pi^+ \bar{e})$ if $q_e > 0$ as in case $C1_\ell$; this would have charge $q = 1 - q_e$; (iii) $(\pi^+ \nu)$ if $q_\nu < 0$ (case $C3_\ell$) with charge $q = 1 + q_\nu$; and (iv) $(\pi^+ \bar{\nu})$ if $q_\nu > 0$ (cases $C1_\ell, C2_\ell$) with charge $1 - q_\nu$, hence neutral for case $C4_\ell$. 

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There might also be a stable purely leptonic Coulombic bound state with lepton number \( L = 2 \), \((e\nu)\). This requires that \( q_e \neq 0 \), \( q_\nu \neq 0 \), and \( \text{sgn}(q_e) = -\text{sgn}(q_\nu) \), which happens in case \( C_2e \). Unlike the neutral atoms \((P e)\) and \((N \nu)\), this possible leptonic state would, in general, be charged:

In the usual world, molecules are stable because the Coulomb repulsion of the nuclei is counterbalanced by the Coulombic attraction between nuclei and the total set of electrons, yielding various bonds (ionic, covalent). For large \( N_c \) this balancing could still occur, yielding molecules and normal crystalline matter, but on an interatomic length scale \( r_{\text{i.a.}} \sim N_c \).

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