OBSERVABLE PROPERTIES OF $q$-DEFORMED PHYSICAL SYSTEMS

R. J. Finkelstein

Department of Physics and Astronomy
University of California, Los Angeles, CA 90095-1547

To Moshé Flato

Abstract. Examples are given of $q$-deformed systems that may be interpreted by the standard rules of quantum theory in terms of new degrees of freedom and supplementary quantum numbers.
1. Introduction.

In standard quantum mechanical theories the $S$-matrix encodes the physically observable properties: transition probabilities between scattering-states and energy levels of bound states. In this note we are similarly concerned with energy levels and transition probabilities of physical systems which have been $q$-deformed. The examples we consider are speculative deformations of real physical systems which have been extensively discussed in diverse ways. Because of the consequent arbitrariness and ambiguities in these different approaches it is perhaps useful to attempt a more uniform interpretation of a few typical cases.

2. The $q$-Oscillator.

In the Fock representation the $q$-oscillator is defined by the Hamiltonian

$$H = \frac{1}{2}(a\bar{a} + \bar{a}a)\hbar\omega$$

where $\bar{a}$ and $a$ satisfy

$$(a, \bar{a})_q = a\bar{a} - q\bar{a}a = 1.$$  

The time dependence of these operators is given by the $q$-Dirac rule:

$$i\hbar\dot{a} = (a, H)_q \rightarrow a \sim e^{-i\omega t}$$

Operating on eigenstates of $H$ the $a$ and $\bar{a}$ satisfy

$$a|n\rangle = \langle n\rangle^{1/2}|n - 1\rangle$$

$$\bar{a}|n\rangle = \langle n + 1\rangle^{1/2}|n + 1\rangle$$

and

$$H|n\rangle = \frac{1}{2}[(\langle n\rangle + \langle n + 1\rangle)|n + 1\rangle\hbar\omega$$

where $\langle n\rangle$ is the basic number

$$\langle n\rangle = \frac{q^n - 1}{q - 1}.$$  

Denote the ground state by $|0\rangle$. Then the $n^{th}$ state is

$$|n\rangle \sim \bar{a}^n|0\rangle.$$  

Set

$$A = \left( \begin{array}{c} a \\ \bar{a} \end{array} \right).$$

Then (2.2) may be rewritten as

$$A^t \epsilon A = q^{-1/2}$$
where
\[
\epsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}.
\] (2.10)

One may transform to other representations by the following canonical transformation, \( T \), belonging to \( SU_q(2) \)
\[
A = TX , \quad T \in SU_q(2)
\] (2.11)

where \( T \) satisfies
\[
T^t \epsilon T = T \epsilon T^t = \epsilon .
\] (2.12)

Then
\[
X^t \epsilon X = q^{-1/2}
\] (2.13)

and if one sets
\[
X = \left( \frac{\hat{p}}{x} \right)
\] (2.14)

then
\[
qx - px = i\hbar .
\] (2.15)

These relations may be satisfied in configuration space by the pair \((-i\hbar D^q_{\bar{x}}, x)\) or in momentum space by the pair \((p, -iq_1 D^q_{\bar{p}})\) where the \( D \) are the following difference operators:
\[
D^q_{\bar{x}} \psi(x) = \frac{\psi(qx) - \psi(x)}{qx - x}
\] (2.16a)
\[
D^{q_1}_{\bar{p}} \varphi(p) = \frac{\varphi(q_1 p) - \varphi(p)}{q_1 p - p}
\] (2.16b)

and
\[
q_1 = q^{-1}.
\] (2.17)

Set
\[
T = \begin{pmatrix} \alpha & \beta \\ -q_1 \bar{\beta} & \bar{\alpha} \end{pmatrix}.
\] (2.18)

Then (2.12) implies
\[
\alpha \beta = q \beta \alpha \quad (a)
\]
\[
\alpha \bar{\alpha} + \beta \bar{\beta} = 1 \quad (d)
\]
\[
\alpha \bar{\beta} = q \bar{\beta} \alpha \quad (b)
\]
\[
\bar{\alpha} \alpha + q_1^2 \bar{\beta} \beta = 1 \quad (e)
\] (2.19)

By (2.11)
\[
a = \alpha \left( \frac{i}{\hbar} \hat{p} \right) + \beta x
\] (2.20)
\[
\bar{a} = -q_1 \bar{\beta} \left( \frac{i}{\hbar} \hat{p} \right) + \bar{\alpha} x.
\]

By (2.7) and (2.20) the eigenstates may be expressed in either the \( x \) and \( p \) representation and are the products of \( q \)-Gaussian and \( q \)-Hermite polynomials.\(^1\) These expressions will
depend upon basic numbers and some version of basic Hermite functions in either $x$ or $p$
and will agree with the usual expressions in the $q = 1$ limit. The $x$ and $p$ amplitudes are
related by $q$-Fourier transforms:

$$
\psi_n(x) = \int_{-\infty}^{\infty} \mathcal{E}_q(ipx) \varphi_n(p) dp
$$

$$
\varphi_n(x) = \int_{-\infty}^{\infty} \mathcal{E}_{q_1}(-iqpx) \psi_n(x) dx
$$

where $\mathcal{E}_q(ipx)$ is an eigenstate of $p$:

$$
\mathcal{E}_q(ipx) = \sum \frac{(ipx)^n}{\langle n \rangle !}
$$

and the $q$-integrals are Jackson integrals (inverse of $q$-differentiation).

However, all of these functions lie in the subalgebra generated by $\alpha$ and $\beta$. That
basic numbers and basic hyper-geometric functions replace the usual expressions does not
concern us. On the other hand, that the wave functions lie in an algebra and are not
numerically valued requires attention.

If the operator associated with a transition between two states, $m$ and $n$, is $A$, then
the usual transition amplitude is $\langle n | A | m \rangle$ and the corresponding transition probability is
$|\langle n | A | m \rangle|^2$; but if the states as well as the operator lie in the algebra, $|\langle n | A | m \rangle|^2$ does
not have its usual interpretation since it is not numerically valued. There are two natural
procedures for dealing with the problem.

The first way to attach a numerical measure to $|\langle n | A | m \rangle|^2$ is to form the Woronowicz
measure as follows:

$$
h|\langle n | A | m \rangle|^2
$$

where $h$ stands for the Woronowicz integral over the algebra. It may be evaluated term
by term with the aid of the following result:

$$
h[\alpha^s \beta^n \bar{\beta}^m] = \delta^{s\bar{s}} \delta^{mn} q^n / [m + 1]_q
$$

where

$$
[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.
$$

On top of this integration there is the usual $x$ or $p$ integration.

In an alternative procedure, which is also natural since it is also based on the $q$-algebra,
we construct a Hilbert space as follows:

Because $\beta$ and $\bar{\beta}$ commute they have common eigenstates. Let $|0\rangle$, the groundstate
of this Hilbert space, be a common eigenstate of $\beta$ and $\bar{\beta}$. Then

$$
\beta|0\rangle = b|0\rangle
\bar{\beta}|0\rangle = \bar{b}|0\rangle
$$

$$
\alpha|0\rangle = 0
$$
By (2.19e) it follows that $|b|^2 = q^2$.

To see that $\bar{\alpha}$ and $\alpha$ are consistent creation and destruction equations define

$$|n\rangle \sim \alpha^n |0\rangle .$$

(2.25)

By iteration of (2.19a)

$$\beta |n\rangle = (q^n b) |n\rangle .$$

(2.26)

Then $|n\rangle$ is an eigenstate of $\beta$ and $\bar{\beta}$ and

$$\bar{\alpha} |n\rangle = \lambda_n |n + 1\rangle$$

$$\alpha |n\rangle = \mu_n |n - 1\rangle$$

(2.27) (2.28)

where

$$\lambda_n = (1 - |b|^2 q^{2n})^{1/2}$$

$$\mu_n = \lambda_{n-1} .$$

(2.29) (2.30)

If $\alpha$ and $\bar{\alpha}$ are adjoint and the Hilbert space has a positive definite scalar product, then $\langle n|\alpha \bar{\alpha}|n\rangle$ and $\langle n|\bar{\alpha} \alpha|n\rangle$ must remain positive, and therefore $q^2 \leq 1$ by Eqs. (2.19d) and (2.19e). Continuing this procedure one now replaces $\langle N|A|M\rangle$ by $\langle nN|A|mM\rangle$ where $n$ and $m$ refer to states of the associated Hilbert space. Now a state of the $q$-oscillator carries two quantum numbers, $n$ and $N$, and the matrix elements are numerically valued. Attaching a new quantum number, $n$, is analogous to the procedure in Pauli spin theory where one multiplies an atomic wave function by an independent spin function. The $q$-oscillator interpreted in this way has internal degrees of freedom.

3. The $q$-Coulomb Problem (The $q$-Hydrogen Atom).²

The Schrödinger differential equation in configuration space becomes an integral equation in momentum space. To emphasize the $O(3)$ symmetry of the Coulomb problem one may map momentum space onto the group space of $O(3)$ according to

$$\frac{p_o - i \vec{p} \vec{\sigma}}{p_o + i \vec{p} \vec{\sigma}} = e^{\frac{1}{2} i \vec{\sigma} \vec{w}}$$

(3.1)

where $p_o$ and $\vec{p}$ refer to the energy and momentum while $\vec{w}$ fixes the magnitude and axis of rotation. Since the problem is non-relativistic the relativistic bound state energy $E$ is

$$E = -\frac{p_o^2}{2m} .$$

(3.2)

It is then possible to cast this integral equation into the following form:

$$\int K(\vec{p},\vec{p'})\Phi(\vec{p'})d\tau(\vec{p'}) = C p_o \Phi(\vec{p})$$

(3.3)
where
\[ C = \frac{\hbar}{me^2} \]
and
\[ K(\vec{p}, \vec{p}') = \sum_{jm\mu} \bar{D}_{jm\mu}(\vec{p})D_{jm\mu}(\vec{p}') . \] (3.4)

Here \( K \) is a rescaled Fourier transform of the potential and \( \Phi \) is the rescaled wave function. The \( D_{jm\mu}(\vec{p}') \) are Wigner functions (matrix elements of irreducible representations of the rotation group). These functions satisfy the orthogonality relations
\[ \int D_{jm\mu}(\vec{p})D_{j'm'\mu'}^*(\vec{p})d\tau(\vec{p}) = \delta_{jj'}\delta_{mm'}\delta_{\mu\mu'}/2j + 1. \] (3.5)

By (3.3), (3.4), and (3.5) one sees that the eigenfunctions of (3.3) are
\[ \Phi_{jm\mu}(\vec{p}) = D_{jm\mu}(\vec{p}) \] (3.6)
and the eigenvalues are
\[ Cp_o = \frac{1}{2j + 1} \] (3.7)
and by (3.2) one finds the Balmer formula for the energy
\[ E = \frac{1}{2m} \frac{1}{e^2} \frac{1}{(2j + 1)^2} = \frac{1}{2}mc^2\left(\frac{e^2}{\hbar c}\right)^2 \frac{1}{N^2} \] (3.8)
where the principal quantum number is
\[ N = 2j + 1. \] (3.9)

The other indices \( m \) and \( \mu \) labelling \( \Phi_{jm\mu} \) refer to the \( z \)-components of the Lenz and angular momentum vectors. These are the conserved integrals of the motion. The \( O(4) \) algebra of this system is reflected in the \( O(3) \times O(3) \) symmetry of the left and right parameter groups.

The amplitude in momentum space \( \varphi(\vec{p}) \) is related to \( \Phi(\vec{p}) \) by a rescaling as follows:
\[ \varphi_{mm'}^j(\vec{p}) = G^2(p)D_{mm'}^j(\vec{p}) . \] (3.10)

Here \( G(p) \) is the “propagator”:
\[ G(p) = \frac{p_o}{p^2 + p_o^2} . \] (3.11)

We shall now define the \( q \)-Coulomb problem by deforming the integral equation (3.3). To formulate this new equation it is necessary to define integration over the “\( q \)-group space”. This may be done by again making use of the Haar measure introduced by Woronowicz who established orthogonality relations that may be expressed as follows:
\[ h(\bar{D}_{jm\mu}^jD_{m'm'}^{j'}) = \delta^{jj'}\delta_{mm'}\delta_{\mu\mu'}q^{2m}/[2j + 1]_q \] (3.12)
where $h$ is the Haar measure and $D$ a unitary representation. In our notation the preceding equation is

$$\int_W \bar{D}_{m\mu}^j(\alpha|q) D_{m'\mu'}^{j'}(\alpha|q) d\tau(\alpha) = \delta^{jj'} \delta_{mm'} \delta_{\mu\mu'} q^{2m} / [2j + 1]_q$$  \hspace{1cm} (3.13)

where $f(\alpha)$ lies in the algebra generated by $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$, $\int_W$ means a Woronowicz integral, and $D_{m\mu}^j(\alpha|q)$ is a matrix element of an irreducible representation of the “$q$-group”. The deformed integral equation corresponding to (3.3) is

$$\int_W K_q(\alpha, \alpha') \Phi(\alpha', q) d\tau(\alpha') = C \Phi(\alpha|q)$$ \hspace{1cm} (3.14)

where the kernel is

$$K_q(\alpha, \alpha') = \sum D_{m\mu}^j(\alpha|q) \bar{D}_{m'\mu'}^{j'}(\alpha'|q) .$$  \hspace{1cm} (3.15)

Utilizing the orthogonality relations as before one finds that

$$\Phi(\alpha|q) = D_{m\mu}^j(\alpha|q)$$ \hspace{1cm} (3.16)

and

$$E(N, \mu) = -\frac{1}{2} m c^2 \left(\frac{e^2}{\hbar c}\right)^2 q^{4\mu} [N]^2$$  \hspace{1cm} (3.17)

where $N(= 2j + 1)$ is again the principal quantum number. Here

$$[N] = \frac{q^N - q^{-N}}{q - q^{-1}} ,$$

inherited from (3.12).

In the $q = 1$ limit one recovers the usual results. If $q \neq 1$, the expression for the energy is modified, similar to the way it was for the oscillator, by the substitution of the basic number $[N]$ for $N$.

The factor $q^{4\mu}$ reveals that the Coulomb degeneracy is lifted and that the $q$-system no longer has spherical symmetry.

Since the wavefunctions $D_{m\mu}^j(\alpha|q)$ lie in the algebra, one has the same problem in computing transition probabilities that one has for the oscillator or for any other problem obtained by $q$-deformation; and one can handle the problem in the ways suggested for the oscillator. Probably the simpler procedure is to evaluate these probabilities as squares of matrix elements between states of the Hilbert space defined by the algebra. In doing this we are explicitly recognizing that the $q$-system has new degrees of freedom not belonging to the $q = 1$ system but we are not discarding the usual quantum mechanical rules of interpretation.

This Hilbert space is intrinsic to the formalism and in fact is very similar to the usual Fock space. The suggested procedure is analogous to the evaluation of operator fields in Fock space, i.e., the $q$-deformed problem is more like a quantum field problem than a one-particle quantum mechanical problem.
The $q = 1$ oscillator as well as the $q = 1$ hydrogen atom are themselves idealizations of real physical systems. Whether these new $q \neq 1$ systems are also useful idealizations of more complex physical systems is not known.

4. $q$-Field Theory.

An arbitrary field may be expanded as follows:

$$\psi_{\mu}(x) = \sum_{\rho} [a(\rho)f_{\mu}(\rho, x) + \bar{b}(\rho)g_{\mu}(\rho, x)]$$ (4.1)

where $\mu$ is a generic tensor index and

$$\sum_{\rho} = \sum_{r} \int d\vec{p} \rho = (\vec{p}, r)$$ (4.2a)

$$f_{\mu}(\rho, x) \sim \frac{u_{\mu}(\vec{p}, r)}{(2p_{0})^{1/2}} e^{-ipx}$$ (4.2b)

$$g_{\mu}(\rho, x) \sim \frac{v_{\mu}(\vec{p}, r)}{(2p_{0})^{1/2}} e^{ipx}.$$ (4.2c)

Here $a(\bar{a})$ and $b(\bar{b})$ are absorption (emission) operators of particles and antiparticles respectively. The $\rho$ sum is an integration over momentum and a sum over spin states ($r$). The particle and antiparticle parts of the sum are related by complex conjugation of the exponentials and by charge conjugation of the spin dependent functions.

Let us now impose the $q$-commutators

$$(a(p), \bar{a}(p'))_q = \delta(p, p')$$ (4.3a)

$$(b(p), \bar{b}(p'))_q = \delta(p, p')$$ (4.3b)

$$(a(p), b(p'))_q = 0$$ (4.3c)

$$(\bar{b}(p), \bar{a}(p'))_q = 0$$ (4.3d)

In standard theory $q = 1$ ($-1$) describes the (boson) fermion fields.

One may also define the $q$ time-ordered product

$$T_q(\psi(x), \psi(x')) = \psi(x)\psi(x') \quad t > t'$$

$$= q\psi(x')\psi(x) \quad t < t'$$ (4.4)

again correct for both bosonic and fermionic fields. Finally introduce the $q$-modified $S$ matrix:

$$S^{(q)} = T_q \left( \exp(i \int L(x) d^4x) \right).$$ (4.5)

In the standard way, we take the transition probability between an incoming state $A$ and an outgoing state $B$ to be $|\langle B| S^{(q)} |A \rangle|^2$ where $A^+$ and $B^+$ are products of creation operators.
that represent different collections of particles. Here

\[ |A\rangle = A^+|0\rangle \quad \text{(4.6a)} \]
\[ \langle B| = \langle 0|B \quad \text{(4.6b)} \]
\[ \langle B|S|A\rangle = \langle 0|BSA^+|0\rangle \quad \text{(4.6c)} \]

Again following the usual procedures \( \langle B|S(q)|A\rangle \) is expanded and evaluated with the aid of Wick’s theorem for normal products. In putting a string of absorption and emission operators in normal form one picks up a power of \( q \) instead of a power of -1. Continuing this procedure one arrives at a set of Feynman rules differing from the standard rules by the substitution of \( q \)-dependent internal propagator for the usual internal propagators.

One finds for the vector fields

\[ D_{\mu\lambda}(x) = \left(g_{\mu\lambda} - \frac{\partial_{\mu}\partial_{\lambda}}{m^2}\right) \left(\frac{1}{2\pi}\right)^4 \frac{1 + q}{2} \int e^{-ikx} \left(\frac{1}{k^2 - m^2} + \frac{1 - q k_\omega}{1 + q \omega}\right) d^4k \quad \text{(4.7)} \]

and for spinor fields

\[ S_{\alpha\beta}(x) = (\bar{\phi} + m)_{\alpha\beta} \left(\frac{1}{2\pi}\right)^4 \frac{1 - q}{2} \int e^{-ikx} \left[\frac{1}{k^2 - m^2} + \frac{1 + q k_\omega}{1 - q \omega}\right] d^4k \quad \text{(4.8)} \]

where

\[ \omega = (\vec{k}^2 + m^2)^{1/2}. \quad \text{(4.9)} \]

For the standard vector (spinor) field \( q = 1(-1) \) these propagators reduce to their standard form.

One may test these expressions by computing, for example, electron-electron scattering or electron-positron annihilation. One obtains the standard (Möller) expression for electron-electron scattering and the standard (Bhabha) expression for electron-positron annihilation but both expressions are multiplied by frame dependent factors.

In this way one sees that \( q \)-quantization imposed in the minimal way we have described breaks the Lorentz symmetry. A somewhat similar result was obtained in discussing the \( q \)-Coulomb problem where the \( q \)-quantization (there differently implemented) breaks the rotational symmetry. The propagators (4.7) and (4.8) also illustrate the Pauli result that the Lorentz group requires \( q = 1 \) for bosons and \( q = -1 \) for fermions.

The preceding result suggests that the argument be reversed by postulating a \( q \)-symmetry group at the outset. One may therefore introduce the \( q \)-Lorentz group by defining its spin representation \( L_q \) as follows:

\[ \epsilon_q \det_q L_q = L_q^T \epsilon_q L_q = L_q \epsilon_q L_q^T, \quad \det_q L_q = 1 \quad \text{(4.10)} \]

Here \( \epsilon_q \) is the \( q \)-deformed Levi-Civita matrix as before and Eq. (4.10) is also the definition of the \( q \)-determinant of \( L_q \). If \( q = 1 \), Eq. (4.10) exactly defines the spin representation of the Lorentz group.

To see how the postulate of \( q \)-Lorentz invariance might work out, one may focus on the correlation functions (vacuum expectation values of products of interacting fields) since
these determine the structure of the theory. Without going into details, the essential new element now introduced by the $q$-symmetry is that the interacting fields lie in a $q$-algebra and, just as for the simple systems already discussed, there are new degrees of freedom that give rise to new quantum numbers determined by the Hilbert space associated with the algebra.

In a $q$-Yang-Mills theory the connection field would lie in a $q$-algebra rather than a Lie algebra. The associated Fock space would then be the direct product of the standard Fock space generated by $\bar{a}$ and the $q$-Fock space generated by $\bar{\alpha}$. If Yang-Mills theories describe point particles, then $q$-Yang-Mills theories describe particles with internal degrees of freedom.

I should like to thank Professors Fronsdal and Varadarajan for comments.

References.

1. R. Finkelstein and E. Marcus, J. Math. Phys. 36, 2652 (1995).
2. R. Finkelstein, J. Math. Phys. 37, 2628 (1996).
3. R. Finkelstein, J. Math. Phys. 37, 983 (1996).