Hyperbolicity of Semigroup Algebras II*

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Abstract

In 1996 Jespers and Wang classified finite semigroups whose integral semigroup ring has finitely many units. In a recent paper, Iwaki-Juriaans-Souza Filho continued this line of research by partially classifying the finite semigroups whose rational semigroup algebra contains a Z-order with hyperbolic unit group. In this paper we complete this classification by handling the case in which the semigroup is semi-simple.

1 Introduction

In this paper we continue the investigations on the hyperbolic property. Recall ([3]) that a unital Q-algebra $A$ is said to have the hyperbolic property if the unit group $\mathcal{U}(\Gamma)$ of every Z-order $\Gamma$ in $A$ does not contain a free abelian subgroup $\mathbb{Z}^2$ of rank 2. If $A$ has finite Q-dimension, then having the hyperbolic property is equivalent to the unit group of Z-orders being hyperbolic groups in the sense of Gromov [2] and it is sufficient to verify the required condition on only one Z-order in $A$.

The finite groups $G$ for which its rational group algebra $\mathbb{Q}G$ is hyperbolic have been characterized in [4]. In [6] this problem has been dealt with for rational group algebras of arbitrary groups and in [7] one deals with groups algebras of finite groups over quadratic extensions of the rationals. A natural question is to extend these results to the much wider class of (unital) rational semigroup algebras of finite semigroups. A first step in this direction is given in [5] where the finite semigroups $S$ are classified whose integral semigroup ring $\mathbb{Z}S$ has trivial units. In a recent paper ([3]), Iwaki-Juriaans-Souza Filho classified the finite semigroups $S$ such that $\mathbb{Q}S$ has the hyperbolic property, this provided that either $\mathbb{Q}S$ is semi-simple or $S$ is not a semi-simple semigroup. See also [8] and [12] for results in the same direction.

In this paper we complete the classification started in [3]. In particular, we need to handle the case in which $S$ is semi-simple, that is, every principal factor of $S$ is simple or 0-simple and $\mathbb{Q}S$ is not semi-simple. The reader is referred to [11] [10] for background on semigroups and semigroup algebras. For convenience sake, we recall that the contracted semigroup algebra of a semigroup with zero element $\theta$ is denoted by $\mathbb{Q}_0S$ and defined as $\mathbb{Q}S/\mathbb{Q}\theta$. Since any semigroup algebra can be considered as a contracted semigroup algebra, we will state our results in this more general context. Without specifically stating this, throughout the paper we assume that $\mathbb{Q}_0S$ is unital. Also recall that a finite 0-simple semigroup is isomorphic with a Rees matrix semigroup $\mathcal{M}^0(G^{\theta}; m, n; P)$.

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where \( n \) and \( m \) are positive integers and \( G \) is a finite group. This is by definition the set of all \( m \times n \)-matrices over \( G^\theta \), the group \( G \) with a zero \( \theta \) adjoined, with at all entries in \( G^\theta \) and at most one entry different from \( \theta \). The matrix \( P \) is an \( n \times m \)-matrix with entries in \( G^\theta \) and it is regular (i.e., each row and column contains at least one non-zero element). The product of \( A, B \in \mathcal{M}^0(G^\theta, m, n; P) \) is \( A \circ P \circ B \), where \( \circ \) denotes the usual matrix product. The Jacobson radical of a ring \( R \) we denote by \( J(R) \). An example that will be of importance in our classification is the Rees matrix semigroup \( T = \mathcal{M}^0(\{1\}, 1, 2, P) \) with \( P = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Clearly, \( T = \{e, f, \theta\} \) with \( ef = f, e^2 = e, f^2 = f, fe = e, \theta^2 = \theta, \) and \( e\theta = \theta e = f\theta = \theta f = \theta \). By \( T^1 \) we denote the smallest monoid containing \( T \), thus \( T^1 = T \cup \{1\} \), with \( t1 = 1t \) for all \( t \in T \). Clearly, \( J(Q_3T^1) = Q(e - f) \) and \( Q_3T^1 = Q(e - f) + Q(1 - e) + Q(e) \cong T_2(Q) \), the upper triangular \( 2 \times 2 \)-matrices over \( Q \). So the semigroup algebra \( Q_3T^1 \) has the hyperbolic property. We will show, in some sense, that this is precisely the only case not covered by the main result of \( [3] \).

## 2 Hyperbolic semigroup algebras

For the convenience of the reader we recall the main result in \( [3] \) on the structure of finite dimensional algebras with the hyperbolic property.

**Theorem 2.1** \([3, \text{Theorem 3.1}]\) A finite dimensional \( Q \)-algebra \( A \) has the hyperbolic property if and only if one of the following holds:

1. \( J(A) = \{0\} \) and \( A = \oplus_i D_i \oplus B \),
2. \( \dim_Q J(A) = 1 \) and \( A/J(A) = \oplus_i D_i \),

where each \( D_i \) is either \( Q \), a quadratic imaginary extension of \( Q \) or is a totally definite quaternion algebra over the rationals and where either \( B \in \{\{0\}, M_2(Q)\} \) or \( B \) has a \( \mathbb{Z} \)-order whose unit group is non-torsion and hyperbolic.

In case \( A \) has the hyperbolic property with \( J(A) \neq \{0\} \) then either \( J(A) \) is central in \( A \) or \( A \) is a direct product of division algebras and \( T_2(Q) \), the \( 2 \times 2 \)-upper triangular matrices over \( Q \).

Note that required conditions on the division algebras \( D_i \) say that the unit group of an order in \( D_i \) has to be finite.

An obvious consequence of the Theorem is that ideals \( I \) of finite dimensional hyperbolic rational algebras \( A \) also have an algebraic structure as described in 1. or 2. of Theorem 2.1. Here we consider \( I \) as a \( K \)-algebra, but it does not necessarily contain an identity. Abusing terminology, we also will say that such algebras are hyperbolic. Since these facts are crucial for the proof of our description of hyperbolic rational semigroup algebras we formulate them in a corollary.

**Corollary 2.2** Let \( A \) be a finite dimensional rational algebra. If \( A \) is hyperbolic then epimorphic images and ideals of \( A \) are hyperbolic. In particular, if \( J(I) = J(A) \cap I = \{0\} \) then \( I \) has an identity and thus \( A \cong I \times (A/I) \), a direct product of algebras.

**Lemma 2.3** Let \( G \) be a finite group and \( S \) a Rees semigroup \( S = \mathcal{M}^0(G; m, n; P) \) with regular sandwich matrix \( P = (p_{i,j}) \). If \( Q_0 S \) has the hyperbolic property then \( nm = 1, 2 \) or 4. Furthermore, if \( nm \neq 1 \) then \( G = \{1\} \) and \( nm = 2 \) or \( n = m = 2 \) and \( P \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). If \( nm = 2 \) then \( J(Q_0 S) \neq \{0\} \) and \( Q_0 S/J(Q_0 S) \cong Q \), and if \( n = m = 2 \) then \( Q_0 S \cong M_2(Q) \).
Proof. Normalizing $P$, we may without loss of generality assume that $p_{1,1} = 1$. There is a natural epimorphism $S = M^n(G; m, n; P) \to \overline{S} = M^n((1); m, n; \overline{P})$, where $\overline{P}$ is the matrix obtained from $P$ by replacing an entry $g \in G$ by 1. We thus obtain a natural $Q$-algebra epimorphism of contracted semigroup algebras $Q_0S \to Q_0\overline{S}$. Because of Corollary 2.2, also the algebra $Q_0\overline{S}$ has the hyperbolic property. Recall the well-known fact that $Q_0M^n((1); m, n; \overline{P}) \cong M(Q, n, m; \overline{P})$, a Munn algebra over the rationals. Hence (see [9, Theorem 3.6] or [10, Lemma 5.21]), $Q_0\overline{S}/J(Q_0\overline{S}) \cong M_2(Q, 1)$, where $t$ is the rank of $\overline{P}$. Because of Theorem 2.1, we then also know that $t = 1$ if the radical is non-trivial and $t = 2$ otherwise. As also dim$_Q J(Q_0\overline{S}) \leq 2$ we thus obtain that $mn - 1 = 1$ or $nm = 4$ respectively. The former yields $mn = 2$.

Now, as usual, we denote a non-zero matrix in $S$ as $(g, i, j)$, where $g \in G$ is the only non-zero entry in position $(i, j)$. All matrices of the type $(\theta, i, j)$ are identified with the zero element (also denoted by $\theta$) of $S$. Hence the product in $S$ is as follows: $(g, i, j)(h, k, l) = (gp_{j,k}h, i, l)$.

We now exclude the case that $m = 1$ and $n = 4$ (and similarly, $m = 4$ and $n = 1$ is excluded). Suppose the contrary, then $(1, 1, 1) - (1, 1, 2)$ and $(1, 1, 1) - (1, 1, 3)$ are linearly independent nilpotent elements of the hyperbolic algebra $Q_0\overline{S}$. However, this is in contradiction with the dimension of the Jacobson radical being at most one.

If $n = m = 2$ (and thus $J(Q_0\overline{S}) = \{0\}$) then the mapping $Q_0\overline{S} \cong M(Q, 2, 2; \overline{P}) \to M_2(Q) : A \mapsto A \circ \overline{P}$ is an isomorphism (note that its kernel consists of nilpotent elements). Hence $P \in GL_2(Q)$. Therefore the regular matrix can not be equal to $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Hence $\overline{P}$ (and thus also $P$) is an upper or lower triangular matrix. So $P$ is invertible over $QG$. Consequently the mapping $Q_0S \cong M(QG, 2, 2; P) \to M_2(QG) : A \mapsto A \circ \overline{P}$ is an isomorphism. Theorem 2.1 then yields that $|G| = 1$.

Finally, suppose that $nm = 2$ and thus $t = 1$. So all nilpotent elements belong $J(Q_0S)$. By symmetry, we then may assume that $n \geq 2$. Let $g \in G$, $a = (gp_{2,1,1,1})$ and $b = (g, 1, 2)$. Then, because $p_{1,1} = 1$, $(a - b)^2 = (gp_{2,1,1,1}) + (g, 1, 2) - (gp_{2,1,1,1}) - (g, 1, 2) = 0$. Consequently, $a - b \in J(Q_0(S))$, for all $g \in G$. Since dim$_Q J(Q_0(S)) \leq 1$ this implies that $|G| = 1$. This finishes the proof.

Recall that a finite group is said to be a Higman group if it is either abelian of exponent dividing 4 or 6 or a Hamiltonian 2-group. These are precisely the finite groups $G$ so that $U(ZG)$ is finite (see for example [11]). By $C_n$ we denote the cyclic group of order $n$. Put $S_3$ the symmetric group of degree 3, $D_4$ the dihedral group of order 8, $Q_8$ the quaternion group of order 12 and $C_4 \times C_4$ the semi-direct product where $C_4$ acts non-trivially. These four groups are precisely those non-abelian finite groups $G$ so that $QG$ has the hyperbolic property. It turns out that $U(ZG)$ has a free subgroup of rank 2 that is of finite index 2 ([3]). From Dirichlet’s Unit it follows that the only finite abelian groups $H$ with $QH$ hyperbolic are $C_5$, $C_6$ and $C_{12}$. These are the non-trivial finite abelian groups $H$ with $U(ZH)$ having an infinite cyclic subgroup of finite index. Also recall that a null semigroup is a non-trivial semigroup with zero in which the product of any two elements is the zero element $\theta$. Of course such a semigroup is 0-simple if it has only one non-zero element. Finally, if a principal factor of a finite semigroup $S$ is of the form $G^0$, for some group $G$, then, we simply say that this principal factor is isomorphic with the group $G$.

We are now ready to prove our main result which, together with [3, Theorem 4.8], completes the classification of the finite semigroups $S$ whose contracted semigroup algebra $Q_0S$ has the hyperbolic property. As said in the introduction, only the case of simple-semi simple semigroups has to be dealt with. However, because of Lemma 2.3, the proof works for all finite semigroups. So for this reason and
because of completeness’ sake, we give a complete proof.

**Theorem 2.4** Let $S$ be a finite semigroup (with zero). The contracted semigroup algebra $Q_0 S$ has the hyperbolic property if and only if all principal factors of $S$, except possibly one, say $K$, are Higman groups, and $K$ is isomorphic to one of the following simple semigroups:

1. a null semigroup.
2. $\mathcal{M}^0(\{1\},1,2,P)$, with $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
3. $\mathcal{M}^0(\{1\},2,2,P)$ with $P \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$.
4. $C_5$, $C_8$, or $C_{12}$
5. $S_3$, $D_4$, $Q_{12}$, or $C_4 \times C_4$.

**Proof.** Let $S_n = \{\emptyset\} \subset S_{n-1} \subset \cdots \subset S_1 = S$ be a principal series of $S$. So, the Rees factors $S_i/S_{i+1}$, with $1 \leq i \leq n-1$, are the non-trivial principal factors of $S$. Each of these either is a null semigroup or a Rees Matrix semigroup $\mathcal{M}^0(G_i,m_i,n_i;P_i)$, with $G_i$ a finite group.

First assume that $Q_0(S)$ is semisimple. It is well known (see for example [10, Theorem 14.24]) that this occurs precisely when $n_i = m_i$ and $P_i$ is invertible in $M_{n_i}(QG)$, for each $1 \leq i \leq n-1$ and it follows that $Q_0 S \cong \bigoplus_{i=1}^{n-1} Q_0(S_i/S_{i+1})$. So, $Q_0 S$ is hyperbolic if and only if each $Q_0(S_i/S_{i+1})$ has the hyperbolic property. From Theorem 2.1 and Lemma 2.3 we obtain that all $n_i \leq 2$ and at most one of them, say $n_i_0$, equals 2. Furthermore, if the latter occurs then $S_{i_0}/S_{i_0+1}$ is of the type described in 3. For all other indices $i$, we have that $n_i = 1$ and $ZG_i$ must have finitely many units. Hence $G_i$ is a Higman group. If, on the other hand, all $n_i = 1$ then all $ZG_i$ have finite unit group except possibly one which (by the comments before the Theorem) is then of the type given in 4. or 5. In the former case $Q S_0$ is a sum of division rings. In the latter case there is a simple component of the type $M_2(Q)$. This finishes the proof if $Q_0 S$ is semisimple.

Second, assume $Q_0 S$ has the hyperbolic property but not semi-simple. Choose the smallest index $i_0$ such that the ideal $Q_0 S_{i_0+1}$ of $Q_0 S$ has no nilpotent elements. Note that $i_0 > 1$. As $Q_0 S_{i_0}$ is an ideal of $Q_0 S$ with trivial Jacobson radical, Corollary 2.2 yields that $Q_0 S \cong Q_0 S_{i_0} \oplus Q_0(S/S_{i_0})$. Again using Theorem 2.1 we obtain that $Q_0 S_{i_0}$ is hyperbolic and a direct sum of division algebras of which the unit group of an order is finite. So, as above, the principal factors of $S_{i_0}$ are all Higman groups. So $Q_0(S/S_{i_0})$ has the hyperbolic property and is not semi-simple. Hence, to prove the necessity of the conditions, we may assume that $i_0 = n-1$.

Suppose $S_{n-1}$ is a null semigroup then $Q_0 S_{n-1} \subseteq J(Q_0 S)$ and thus, because of the dimension of the radical, we have equality. Again, by Theorem 2.1 we then also have that $Q_0 S/Q_0 S_{n-1} \cong Q_0 S/S_{n-1}$ is semisimple and hyperbolic with finitely many units in an order. So, all principal factors $S_i/S_{i+1}$, with $1 \geq i > n-1$ are Higman groups, as desired.

Next, suppose $S_{n-1}$ is not a null semigroup. Because of Lemma 2.3 we get that $S_{n-1}$ is as described in 2. So $Q_0 S_{n-1}/J(Q_0 S_{n-1}) \cong Q$ and $Q_0 S/J(Q_0 S) \cong Q \oplus Q_0 S/S_{n-1}$. Theorem 2.1 then yields that $Q_0(S/S_{n-1})$ is a sum of division rings with finitely many units in an order. So, all principal factors $S_i/S_{i+1}$, with $1 \leq i \leq n-2$, are Higman groups. This finishes the proof of the necessity.

Conversely, suppose that the factors of the principal series $S_n = \{\emptyset\} \subset S_{n-1} \subset \cdots \subset S_1 = S$ of $S$ are of the types described and that $S_{i_0}/S_{i_0+1}$ is the exceptional factor. If this principal factor is
of type 3. then \(Q_0 S\) is semisimple and the result follows from the first part of the proof. So, assume \(S_i/S_i + 1\) is of the type 1 or 2. Now, \(S_i + 1\) is a union of groups and therefore \(Q_0 S_i + 1\) is semisimple and thus has an identity. Moreover, \(Q_0 S \cong Q_0 S_i + 1 \oplus Q_0 (S/S_i + 1)\) and, by the semisimple part of the proof, an order in \(Q_0 S_i + 1\) has finitely many units. So, we only need to verify that \(Q_0 (S/S_i + 1)\) is hyperbolic. By the assumptions and Corollary 2.3, we know that \(J(Q_0 (S_i/S_i + 1))\) is of dimension one and \(Q_0 (S/S_i + 1)/J(Q_0 (S_i/S_i + 1)) \cong \mathbb{Q}\). The assumptions also imply that \(Q_0 (S/S_i)\) is a direct sum of division rings. Hence, \(Q_0 (S/S_i + 1)/J(Q_0 (S_i/S_i + 1)) = \mathbb{Q} \oplus Q_0 (S/S_i + 1)\) is a direct sum of division rings with only finitely many units in an order. Therefore, \(Q_0 S\) has the hyperbolic property.

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