Mean magnetic field renormalization and Kolmogorov’s energy spectrum in MHD turbulence

Mahendra K. Verma *

Department of Physics, Indian Institute of Technology, Kanpur – 208016, INDIA

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Abstract

In this paper we construct a self-consistent renormalization group procedure for MHD turbulence in which small wavenumber modes are averaged out, and effective mean magnetic field at large wavenumbers is obtained. In this scheme the mean magnetic field scales as \( k^{-1/3} \), while the energy spectrum scales as \( k^{-5/3} \) similar to that in fluid turbulence. We also deduce from the formalism that the magnitude of cascade rate decreases as the strength of the mean magnetic field is increased.

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*email: mkv@iitk.ernet.in
I. INTRODUCTION

Kolmogorov hypothesized that the energy spectrum \( E(k) \) of fluid turbulence in the inertial range is isotropic and is a power law with a spectral index of \(-5/3\), i.e.,

\[
E(k) = K_{Ko} \Pi^{2/3} k^{-5/3}
\]

where \( K_{Ko} \) is an universal constant called Kolmogorov’s constant, \( k \) is the wavenumber, and \( \Pi \) is the nonlinear energy cascade rate. Note that \( \Pi \) is equal to the dissipation rate and also the energy supply rate of the fluid. Experiments \cite{1}, simulations \cite{2}, and some of the analytical calculations based on Direct interaction approximation \cite{3,4}, renormalization group (RG) techniques \cite{5–11}, self-consistent mode coupling \cite{12} etc. are in good agreement with the above phenomenology.

In this paper we will discuss the energy spectrum in magnetohydrodynamic (MHD) turbulence. In MHD there are two fields, the velocity field \( u \) and the magnetic field \( B = B_0 + b \), where \( B_0 \) is the mean magnetic field or the magnetic field of the large eddies, and \( b \) is the magnetic field fluctuation. One usually uses Elsässer variables \( z^\pm = u \pm b \). Here the magnetic field has been written in velocity units \((b/\sqrt{4\pi \rho})\), where \( \rho \) is the density of the fluid). We also assume that the plasma is incompressible.

There are two time-scales in magnetofluid: (i) nonlinear time-scale \( 1/(kz^+_k) \) (similar to that in fluid turbulence) and (ii) Alfvén time-scale \( 1/(kB_0) \). Kraichnan \cite{13} and Dobrowolny et al. \cite{14} argued that the interacting \( z^+_k \) and \( z^-_k \) modes will get separated in one Alfvén time-scale because of the mean magnetic field. Therefore, they chose Alfvén time scale \( \tau_A = (kB_0)^{-1} \) as the relevant time-scale and found that

\[
\Pi^+ \approx \Pi^- \approx \frac{1}{B_0} E^+(k)E^-(k)k^3 = \Pi.
\]

where \( \Pi^\pm \) are the cascade rates of \( z^\pm_k \). If \( E^+(k) \approx E^-(k) \), then the above equation implies that

\[
E^+(k) \approx E^-(k) \approx (B_0 \Pi)^{1/2} k^{-3/2}
\]
In absence of mean magnetic field, the magnetic field of the largest eddy was taken as $B_0$. Kraichnan [13] also argued that the fluid and magnetic energies are equipartitioned. The above phenomenology is referred to as Dobrowolny et al.’s generalized Kraichnan (KD) phenomenology.

If the nonlinear time-scale $\tau_{NL}^\pm \approx k z_k^\mp$ is chosen as the interaction time-scales for the eddies $z_k^\pm$, we obtain

$$\Pi^\pm \approx \left( z_k^\pm \right)^2 \left( z_k^\mp \right) k,$$

which in turn leads to

$$E^\pm(k) = K^\pm (\Pi^\pm)^{4/3} (\Pi^\mp)^{-2/3} k^{-5/3},$$

where $K^\pm$ are constants, which we will refer to as Kolmogorov’s constants for MHD turbulence. Because of its similarity with Kolmogorov’s fluid turbulence phenomenology, this phenomenology is referred to as Kolmogorov-like MHD turbulence phenomenology. This phenomenology was first given by Marsch [14], Matthaeus and Zhou [16], and Zhou and Matthaeus [17] (it is a limiting case of a more generalized phenomenology constructed by Matthaeus and Zhou [16], and Zhou and Matthaeus [17]). It is implicit in these phenomenological arguments that KD phenomenology is expected to hold when $B_0 \gg \sqrt{k E^\pm(k)}$, while Kolmogorov-like phenomenology is expected to be applicable when $B_0 \ll \sqrt{k E^\pm(k)}$.

In the solar wind, which is a good testing ground for MHD turbulence theories, Matthaeus and Goldstein [18] found that the exponent of the total energy is $1.69 \pm 0.08$, whereas the exponent of the magnetic energy is $1.73 \pm 0.08$, somewhat closer to $5/3$ than $3/2$. This is more surprising because $B_0 \gg \sqrt{k E^\pm(k)}$ for inertial range wavenumbers in the solar wind. The numerical simulations also tend to indicate that the Kolmogorov-like phenomenology, rather than KD phenomenology, is probably applicable in MHD turbulence [19]. Hence, the comparison of the solar wind observations and simulation results with the phenomenological predictions appears to show that there are some inconsistencies in the phenomenological arguments given above. To resolve these inconsistencies, we have attempted to examine the MHD equations using renormalization group analysis.
For fluid turbulence Forster et al. [5] and Yakhot and Orszag [6] have applied dynamical RG procedure in which a forcing term with a power law distribution in wavenumber space is introduced. McComb [8], McComb and Shanmugasundaram [9], McComb and Watt [10], and Zhou et al. [11] applied a self-consistent RG procedure that yields Kolmogorov’s energy spectrum. For MHD turbulence, Fournier et al. [20] and Camargo and Tasso [21] have used RG procedure similar to that of Forster et al. [5] and Yakhot and Orszag [6]. In all these schemes the averaging is done over the small scales (based on Wilson’s approach in his Fourier space RG). Till date the RG methods applied to MHD turbulence do not find direct evidence of Kolmogorov-like power law in MHD turbulence. In a more recent work, Verma and Bhattacharjee [22] have applied Kraichnan’s DIA [3,4] to MHD turbulence and obtained the Kolmogorov’s constant for MHD, but in Verma and Bhattacharjee’s work $k^{-5/3}$ energy spectra was assumed, and an artificial cutoff was introduced for the self energy integral.

In this paper we construct a self-consistent RG procedure similar to that used by McComb [8], McComb and Shanmugasundaram [9], McComb and Watt [10], and Zhou et al. [11] for fluid turbulence. However, one major difference is that we integrate the small wavenumber modes instead of large wavenumber mode integration used by earlier authors. In our procedure we obtain the effective mean magnetic field $B_0(k)$ as we go from small wavenumbers to large wavenumbers. At small wavenumbers the MHD equations are approximately linear. During the RG process, the effects of the nonlinear terms in the small wavenumber shells is translated to the modification of $B_0(k)$ at larger wavenumbers.

We postulate that the effective mean magnetic field is the magnetic field of the next-largest eddy contrary to the KD phenomenology where the effective mean magnetic at any scale is constant. To illustrate, for Alfvén waves of wavenumber $k$, the effective magnetic field $B_i(k)$ (after $i$th iteration of the RG procedure defined below) will be the magnetic field of the eddy of size $k/10$ or so. This argument is based on the physical intuition that for the scattering of the Alfvén waves at a wavenumber $k$, the effects of the magnetic field of the next-largest eddy is much more than that of the external field. The mean magnetic field at the largest scale will simply convect the waves; the local inhomogeneities contribute
to the scattering of waves which leads to turbulence (note that in WKB method, the local inhomogeneity of the medium determines the amplitude and the phase evolution). In our self consistent scheme we find that $B_0$ appearing in the Kraichnan’s or Dobrowolny et al.’s argument must be $k$ dependent. The substitution of $k$ dependent $B_0(k)$ leads to $k^{-5/3}$ energy spectra, which is consistent with the solar wind observations and the simulation results. We will describe these ideas in more detail in the following section.

The normalized cross helicity $\sigma_c$, defined as $(E^+ - E^-)/(E^+ + E^-)$, and the Alfvén ratio $r_A$, defined as the ratio of fluid energy and magnetic energy, are important factors in MHD turbulence. For simplicity of the calculation, we have taken $E^+(k) = E^-(k)$ and $r_A = 1$. These conditions are met at many places in the solar wind and in other astrophysical plasmas.

II. CALCULATION

The MHD equation in the Fourier space is

$$(-i\omega \mp i (B_0 \cdot k)) z_\pm^i(k,\omega) = -iM_{ijm}(k) \int dp d\omega' z_\mp^j(p,\omega') z_\pm^m(k-p,\omega-\omega')$$

where

$$M_{ijm}(k) = k_j P_{im}(k); \quad P_{im}(k) = \delta_{im} - \frac{k_i k_m}{k^2},$$

Here we have ignored the viscous terms. The above equation will, in principle, yield an anisotropic energy spectra (different spectra along and perpendicular to $B_0$). Solving anisotropic equations is quite complicated. Therefore, we modify the above equation to the following form to preserve isotropy:

$$(-i\omega \mp i (B_0 k)) z_\pm^i(k,\omega) = -iM_{ijm}(k) \int dp d\omega' z_\mp^j(p,\omega') z_\pm^m(k-p,\omega-\omega')$$

This equation can be thought of as an effective MHD equation in an isotropic random mean magnetic field.
In our RG procedure the wavenumber range \(( k_0 .. k_N )\) is divided logarithmically into \(N\) shells. The \(n\)th shell is \(( k_{n-1} .. k_n )\) where \( k_n = s^n k_o (s > 1) \). In the following discussion, firstly we carry out the elimination of the first shell \(( k_0 .. k_1 )\) and obtain the modified MHD equation. We then proceed iteratively to eliminate higher shells and get a general expression for the modified MHD equation after elimination of \(n\)th shell. The details of the renormalization group operation is as follows:

**A. RG Procedure**

1. Decompose the modes into the modes to be eliminated \(( k^< )\) and the modes to be retained \(( k^> )\). In the first iteration \(( k_0 .. k_1 ) = k^< \) and \(( k_1 .. k_N ) = k^> \). Note that \( B_0(k) \) is the mean magnetic field before the elimination of the first shell.

2. We rewrite the Eq. (8) for \( k^< \) and \( k^> \). The equation for \( z_i^{>\pm}(k, t) \) modes is

\[
(-i\omega \mp i (B_0 k)) z_i^{>\pm}(k, \omega) = -iM_{ijm}(k) \int dp d\omega' \left[z_j^{>\pm}(p, \omega') z_m^{>\pm}(k - p, \omega - \omega') + z_j^{>\pm}(p, \omega') z_m^{>\pm}(k - p, \omega - \omega') + z_j^{<\pm}(p, \omega') z_m^{<\pm}(k - p, \omega - \omega')\right]
\]

while the equation for \( z_i^{<\pm}(k, t) \) modes can be obtained by interchanging \(<\) and \(>\) in the above equation.

3. The terms given in the second and third brackets in the RHS of Eq. (8) is calculated perturbatively. We perform ensemble average over the first shell which is to be eliminated. We assume that \( z_i^{<\pm}(k, t) \) has a gaussian distribution with zero mean. Hence,

\[
\left\langle z_i^{<\pm}(k, t) \right\rangle = 0
\]

\[
\left\langle z_i^{>\pm}(k, t) \right\rangle = z_i^{>\pm}(k, \omega)
\]

and
\[ \langle z^{a<}_s(p, \omega')z^{b<}_m(q, \omega'') \rangle = P_{sm}(p)C^{ab}(p, \omega')\delta(p + q)\delta(\omega' + \omega'') \]  

(11) 

where \(a, b = \pm\). Also, the triple order correlations \( \langle z^{\pm<}_s(k, \omega)z^{\pm<}_m(p, \omega')z^{\pm<}_t(q, \omega'') \rangle \) are zero. We keep only the nonvanishing terms to first order. For the relevant Feynmann diagrams, refer to Zhou et al. [11]. Taking \( r_A = 1 \) and \( E^+(k) = E^-(k) \), the Eq. (3) becomes 

\[
(-i\omega + i(B_0 k))z^{+>}_i(k, \omega) = -iM_{ijm}(k) \int d\mathbf{p}d\omega' \left[ z^{+>}_j(p, \omega')z^{+>}_m(p - k, \omega - \omega') \right] + \\
(-i)^2 M_{ijm}(k) \int d\mathbf{q}d\omega'M_{mst}(p)P_{js}(q)G^{++}(p, \omega')C^{++<}(q, \omega - \omega')z^{+>}_i(k, \omega) + \\
(-i)^2 M_{ijm}(k) \int d\mathbf{q}d\omega'M_{mst}(p)P_{js}(q)G^{+-}(p, \omega')C^{+-<}(q, \omega - \omega')z^{+>}_i(k, \omega)
\]  

(12) 

where \( G \) is the Green’s function obtained from the equation 

\[
G^{-1}(k, \omega) = \begin{pmatrix} -i\omega - ikB_0^{++}(k) & -ikB_0^{+-}(k) \\ ikB_0^{++}(k) & -i\omega + ikB_0^{+-}(k) \end{pmatrix}.
\]  

(13) 

In deriving Eq. (12) we have neglected the contribution of the triple nonlinearity \( z^{+>}_s(k, \omega)z^{+>}_m(p, \omega')z^{+>}_t(q, \omega'') \). McComb, McComb and Shanmugsundaram, and Mc-Comb and Watt [8-10] have also ignored the triple nonlinearity for fluid turbulence.

4. Since \( r_A = 1 \) and \( E^+(k) = E^-(k) \), we find that \( B_0^{++}(k) = B_0^{+-}(k) \). We also assume that the correlation functions \( C^{\pm\pm} \) have the same frequency dependence as \( G^{\pm\pm} \), i.e., 

\[
C^{\pm\pm}(k, \omega^\pm) = \frac{C^{\pm\pm}(k)}{-i\omega^\pm \mp ikB_0^{\pm\pm}(k)}
\]  

(14) 

Note that \( C^{\pm\pm}(k) = E^{\pm\pm}(k)/(4\pi k^2) \) in three dimensions. From dynamical scaling arguments 

\[
\omega^\pm = \mp kB_0^{\pm\pm}(k)
\]  

(15) 

After some manipulations the Eq. (12) becomes
\[ (-i\omega + i [B_0(k) + \delta B^\pm_0(k)] k) z_i^{\pm}(k, t) = i\delta B^\pm_0(k) z_i^{\mp}(k, t) \]
\[ = M_{ijm}(k) \int dp \left[ z_j^{\mp>(p, t)} z_m^{\pm>(k - p, t)} \right] \]

where

\[ \delta B^\pm_0(k) = -k \int p+q=k d\mathbf{q} \left( \frac{E(q)}{4\pi q^2} \times \right. \]
\[ \left. \frac{a_2(k,p,q)(X_0^{\pm}(p)+B_0^{\pm}(p))-a_4(k,p,q)B_0^-(p)}{2X_0(p)(k^2(p)+pX_0^{\pm}(p)-qX_0^{\pm}(q))} \right] \]

and

\[ \delta B^\mp_0(k) = -k \int p+q=k d\mathbf{q} \left( \frac{E(q)}{4\pi q^2} \times \right. \]
\[ \left. \frac{a_3(k,p,q)B_0^{\mp}(p)-a_3(k,p,q)(X_0^{\pm}(p)+B_0^{\pm}(p))}{2X_0^{\pm}(p)(k^2(p)+pX_0^{\pm}(p)-qX_0^{\pm}(q))} \right] \]

where \(2k^2 a_i(k, p, q) = A_i(k, p, q)\) and \(X_0^{\pm}(k) = \sqrt{(B_0^{\pm}(k))^2 - (B_0^{\pm2}(k))^2}\). The terms \(A_i(k, p, q)\) are given in the Appendix of Leslie [4] as \(B_i(k, p, q)\). Since, \(E^+ = E^-\) and \(r_A = 1\), it is clear that \(\delta B^+_0(k) = \delta B^-_0(k)\). Therefore, \(B^{++}_0(k) = B^{+-}_0(k) = B_0(k)\) and \(X_0^{++}(k) = X_0^{+-}(k) = X_0(k)\).

Let us denote \(B_1(k)\) as the effective mean magnetic field after the elimination of the first shell.

\[ B_1(k) = B_0(k) + \delta B_0(k) \]

Similarly,

\[ B_1^{+-}(k) = B_0^{+-}(k) + \delta B_0^{+-}(k) \]

5. We keep eliminating the shells one after the other by the above procedure. After \(n+1\) iterations we obtain

\[ B^{ab}_{n+1}(k) = B^{ab}_n(k) + \delta B^{ab}_n(k) \]

where the equations for \(\delta B^\pm_0(k)\) and \(\delta B^\pm_0(k)\) are the same as the equations (17,18) except that the terms \(B^{ab}_0(k)\) and \(X^{ab}_0(k)\) are to be replaced by \(B^{ab}_n(k)\) and \(X^{ab}_n(k)\)
respectively. Clearly $B_{n+1}(k)$ is the effective mean magnetic field after the elimination of the $(n + 1)$th shell.

The set of RG equations to be solved are (17,18) with $B_0$ replaced by $B_n$, and (21).

**B. Solution of RG equations**

To solve the Eqs. (17,18) with $B_n$ and (21), we substitute the following forms for $E(k)$ and $B_n(k)$ in the modified equations (17,18)

\[
E(k) = K \pi^{2/3} k^{-5/3},
\]
\[
B_n^{ab}(k_nk') = K^{1/2} \pi^{1/3} k_n^{-1/3} B_n^{*ab}(k')
\]

with $k = k_{n+1}k'$ ( $k' > 1$). We expect that $B_n^{*ab}(k')$ is an universal function for large $n$. We use $\Pi^+ = \Pi^- = \Pi$ due to symmetry. After the substitution we obtain the equations for $B_n^{*ab}(k')$ that are

\[
\delta B_n^*(k') = - \int_{p' + q' = k'} dq' \left( \frac{E(q')}{4\pi q'^2} \right) \times \frac{a_2(k,p,q)(X_n(sp) + B_n(sp)) - a_4(k,p,q)B_n^{*+}(sp)}{2X_n(sp')(k_n B_n(sk') + p' X_n(sp') - q' X_n(sq'))}
\]
\[
\delta B_n^{**+}(k') = - \int_{p' + q' = k'} dq' \left( \frac{E(q')}{4\pi q'^2} \right) \times \frac{a_3(k,p,q)B_n^{**+}(sp') - a_4(k,p,q)(X_n(sp') + B_n(sp'))}{2X_n(sp')(k_n B_n(sk') + p' X_n(sp') - q' X_n(sq'))}
\]
\[
B_{n+1}^{*ab}(k) = s^{1/3} B_n^{*ab}(k) + s^{-1/3} \delta B_n^{*ab}(k)
\]

Now we need to solve these three equations self consistently. The integrals in the Eqs. (23,24) is performed over a region $1/s \leq p', q' \leq 1$ with the constraint that $p' + q' = k'$. We use Monte Carlo technique to solve the integral. Since the integrals are identically zero for $k' > 2$, the initial $B_0^*(k'_i) = B_0^{initial}$ for $k'_i < 2$ and $B_0^*(k'_i) = B_0^{initial} \ast (k'_i/2)^{-1/3}$ for $k' > 2$. We take $B_0^{**} = 0$. The Eqs. (23,24) are solved iteratively. We continue iterating the equations till $B_{n+1}^*(k') \approx B_n^*(k')$, that is, till the solution converges. For $B_0^{initial} = 1.0$, the $B'_n$s for
various $n$ ranging from 0...3 is shown in Figure 1. Here the convergence is very fast, and after $n = 3 - 4$ iterations $B_n^*(k)$ converges to an universal function

$$f(k') = 1.24 * k'^{-0.32}. $$

From the above arguments, we have shown that $B_n^*(k')$ is approximately proportional to $k'^{-1/3}$. The other parameter $B_{n+}^*(k')$ remains close to zero.

We infer from the above analysis that the mean magnetic field scales as $k^{-1/3}$, and the energy spectra scales as $k^{-5/3}$. Essentially, the scaling of $B_0$ leads to $k^{-5/3}$ energy (Kolmogorov-like) spectra in our scheme. We have calculated $B_n^*(k')$ for $B_0^{initial} = 1, 2, 10$ and found that for large $n$, $B_n^*(k') \approx 1.25 B_0^{initial} k'^{-1/3}$ or

$$B_n(k) = 1.25 B_0^{initial} K^{1/2} \Pi^{1/3} k^{-1/3}. \tag{26}$$

C. Calculation of $K$

We can calculate the Kolmogorov’s constant for MHD turbulence $K$ by calculating the cascade rate $\Pi$. In MHD the cascade rates are

$$\Pi^+(k) = \Pi^-(k) = - \int_0^k dk' T(k') \tag{27}$$

The numerical solution of the cascade rate integral yields

$$\frac{1.24 B_0^{initial}}{K^{3/2}} = 3.85 \tag{28}$$

From the above equation it is evident that the Kolmogorov’s constant $K$ is dependent on the mean magnetic field $B_0^{initial}$, in fact, $K \propto (B_0^{initial})^{2/3}$. Clearly, an increase in the mean magnetic field leads to an increase in the Kolmogorov constant, which in turn will lead to a decrease in the cascade rate (cf. Eq. (5)). This result is consistent with the simulation results of Oughton. However, a cautious remark is necessary here. We have considered the mean magnetic field to be isotropic; this isotropy assumption needs to be relaxed for studies of realistic situations.
III. CONCLUSIONS

We obtain Kolmogorov-like energy spectrum in MHD turbulence in presence of arbitrary $B_0$ by postulating that the effective $B_0$ is scale dependent. In our renormalization group scheme we find that the self consistent $B_n(k)$ is proportional to $k^{-1/3}$ and $E(k)$ is proportional to $k^{-5/3}$. This analysis has been worked out when $E^+ = E^-$ and $r_A = 1$. The generalization to arbitrary parameters is planned for future studies.

In our methodology, the averaging has been performed for small wavenumbers in contrast to earlier RG analysis of turbulence in which the higher wavenumbers were averaged out. Our scheme yields a power law solution for large wavenumber, and is independent of the small wavenumber forcing states. This is in agreement with the Kolmogorov’s hypothesis which states that the energy spectrum of the intermediate scale is independent of the large-scale forcing. Any extension of our scheme to fluid turbulence in presence of large-scale shear etc. will yield interesting insights into the connection of energy spectrum with large-scale forcing.

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REFERENCES

[1] H. L. Grant, R. W. Stewart, and A. Molliet, J. Fluid Mech. 12, 241 (1962).

[2] U. Frisch and P. L. Sulem, Phys. Fluids 27, 1921 (1984).

[3] R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959).

[4] D. C. Leslie, Development in the Theory of Turbulence (Claredon, Oxford, 1973).

[5] D. Forster, D. R. Nelson, and M. J. stephen, Phys. Rev. A 16, 732 (1977).

[6] V. Yakhot and S. A. Orszag, J. Sci. Comput. 1, 3 (1986).

[7] D. Ronis, Phys. Rev. A 36, 3322 (1987).

[8] W. D. McComb, Phys. Rev. A 26, 1078 (1982).

[9] W. D. McComb and V. Shanmugasundaram, Phys. Rev. A 28, 2588 (1983).

[10] W. D. McComb and A. G. Watt, Phys. Rev. A 46, 4797 (1992).

[11] Y. Zhou, G. Vahala, and M. Hussain, Phys. Rev. A 37, 2590 (1988).

[12] J. K. Bhattacharjee, Phys. Fluids A 3, 879 (1991).

[13] R. H. Kraichnan, Phys. Fluids 8, 1385 (1965).

[14] M. Dobrowhny, A. Mangeney, and P. Veltri, Phys. Rev. Lett. 45, 144 (1980).

[15] E. Marsch, in Reviews in Modern Astronomy, edited by G. Klare (Springer-Verlog, 1990), p. 43.

[16] W. H. Matthaeus and Y. Zhou, Phys. Fluids B 1, 1929 (1989).

[17] Y. Zhou and W. H. Matthaeus, J. Geophys. Res. 95, 10291 (1990).

[18] W. H. Matthaeus and M. L. Goldstein, J. Geophys. Res. 87, 6011 (1982).

[19] M. K. Verma et al., J. Geophys. Res. 101, 21619 (1996).
[20] J. D. Fournier, P.-L. Sulem, and A. Pouquet, J. Phys. A 15, 1393 (1982).

[21] S. J. Camargo and H. Tasso, Phys. Fluids B 4, 1199 (1992).

[22] M. K. Verma and J. K. Bhattacharjee, Europhys. Lett. 31, 195 (1995).

[23] S. Oughton, E. R. Priest, and W. H. Matthaeus, J. Fluid Mech. 280, 95 (1994).
FIGURES

Figure 1: $B_n^*(k')$ for $n = 0 \ldots 3$. The line of best fit $f(k')$ to $B_3^*(k')$ overlaps with $B_3^*$. 