ARTIN-SCHREIER CURVES GIVEN BY $\mathbb{F}_q$-LINEARIZED POLYNOMIALS

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Abstract. Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is a power of an odd prime $p$. In this paper we associate circulant matrices and quadratic forms with the Artin-Schreier curve $y^q - y = x \cdot F(x) - \lambda$, where $F(x)$ is a $\mathbb{F}_q$-linearized polynomial and $\lambda \in \mathbb{F}_q$. Our results provide a characterization of the number of affine rational points of this curve in the extension $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$, for $\gcd(q,r) = 1$. In the case $F(x) = x^{q^r} - x$ we give a complete description of the number of affine rational points in terms of Legendre symbols and quadratic characters.

1. Introduction

Information about the number of affine rational points of algebraic curves over finite fields has many applications in coding theory, cryptography, communications and related areas, e.g., [8, 14]. In this paper we investigate the number of affine rational points of plane curves given by

$$C_g : y^q - y = g(x)$$  \hspace{1cm} (1)

in extensions $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$ where $q = p^e$, $p$ is an odd prime, $r \in \mathbb{N}$ and $g(x) \in \mathbb{F}_q[x]$. These curves are called Artin-Schreier curves and have been extensively studied in several contexts, e.g., [4, 5, 11, 15].

For a polynomial $g(x) \in \mathbb{F}_q[x]$ and $\mathbb{F}_{q^r}$ a finite extension of $\mathbb{F}_q$ we can associate the map

$$Q_g : \mathbb{F}_{q^r} \rightarrow \mathbb{F}_q$$

$$\alpha \mapsto \text{Tr}(g(\alpha)),$$  \hspace{1cm} (2)

where $\text{Tr} : \mathbb{F}_{q^r} \rightarrow \mathbb{F}_q$ denotes the trace function. Let $N_r(Q_g)$ denote the number of zeroes of $Q_g$ in $\mathbb{F}_{q^r}$ and $N_r(C_g)$ the number of affine rational points of $C_g$ over $\mathbb{F}_{q^r}$. From Hilbert’s Theorem 90 we have

$$N_r(C_g) = qN_r(Q_g).$$

It follows that the determination of $N_r(C_g)$ is equivalent to the determination of $N_r(Q_g)$. Details can be found in [1, 2, 12].

In [16], Wolfmann determined the number of rational points of the algebraic plane curve defined over $\mathbb{F}_{q^k}$ by the equation $y^q - y = ax^k + b$ where $a \in \mathbb{F}_{q^k}^*$ and $b \in \mathbb{F}_{q^k}$, for even $k$ and special integers $s$. In [4], Coulter determined the number of $\mathbb{F}_{q^r}$-rational points of the curve $y^{q^r} - y = ax^{q^{r-1}} + L(x)$, where $a \in \mathbb{F}_{q^r}^*$, $t = \gcd(n,e)$ divides $d = (\alpha,e)$ and $L(x) \in \mathbb{F}_q[x]$ is a $\mathbb{F}_q$-linearized polynomial. In this paper, we determine $N_r(C_g)$ for some families of Artin-Schreier curves given by specific polynomials $g(x) \in \mathbb{F}_q[x]$.

The first aim of this paper is to find $N_r(C_g)$ when $g(x) = xF(x) - \lambda$, where $F(x)$ is a $\mathbb{F}_q$-linearized polynomial, $\lambda \in \mathbb{F}_q$ and we assume $\gcd(r,p) = 1$. In this case, we denote $C_g$ by $C_{F,\lambda}$ for $F(x)$ and $\lambda$ fixed. In order to do so, we prove that $Q_{g+\lambda}$ defines

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a quadratic form and use this form to find a connection between the number of affine rational points with the rank of an appropriate circulant matrix. Theorem 3.2 provides an explicit formula for $N_r(C_{F,\lambda})$ in this case.

A second point of this paper, assuming the hypothesis of Theorem 3.2, is to study the case $F(x) = x^q - x$ when $i$ is a positive integer, i.e., $C_{F,\lambda}$ defined by

$$y^i - y = x^{i+1} - x^2 - \lambda.$$  

In Theorem 4.10 we find an expression of $N_r(C_{x^q-x,\lambda})$ in terms of Legendre symbols and $p$-adic valuations.

When $i = 1$ and $\lambda = 0$ we obtain the curve $y^q - y = x^{i+1} - x^2$, which is associated to the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree $r$ with prescribed first two coefficients, i.e., when we fix the coefficients of $x^{r-1}$ and $x^{r-2}$. To see this, consider the map

$$T_2 : \mathbb{F}_{q^r} \to \mathbb{F}_q$$

$$\alpha \mapsto \sum_{0 \leq i < j \leq r-1} \alpha^i \alpha^j.$$  

$\text{Tr}(\alpha)$ and $T_2(\alpha)$ determine, respectively, the coefficients of $x^{r-1}$ and $x^{r-2}$ of the characteristic polynomial of $\alpha \in \mathbb{F}_{q^r}$ over $\mathbb{F}_q$. A straightforward calculation shows that $T_2(\beta^j - \beta) = \text{Tr}(\beta^{j+1} - \beta^j)$ for all $\beta \in \mathbb{F}_{q^r}$. By Hilbert's Theorem 90 we have that $\text{Tr}(\beta) = 0$ if and only if there exists $\alpha \in \mathbb{F}_{q^r}$ such that $\beta = \alpha^q - \alpha$ and therefore $\text{Tr}(\beta) = 0$ and $T_2(\beta) = 0$ if and only if $0 = T_2(\beta) = T_2(\alpha^q - \alpha) = \text{Tr}(\alpha^{q+1} - \alpha^2)$. Consequently, the number of irreducible polynomials of degree $r$ with first two coefficients being zero can be related to the number of affine rational points of the curve $y^q - y = x^{i+1} - x^2$.

For more details see [3, 10].

This paper is organized as follows. Section 2 provides background material and preliminary results. In Section 3 we discuss the case $g(x) = xF(x) - \lambda$ where $F(x)$ is a $\mathbb{F}_q$-linearized polynomial and $\lambda \in \mathbb{F}_q$. In Section 4 we give an explicit formula for $N_r(C_{x^q-x,\lambda})$ when $\gcd(r, p) = 1$.

2. Preliminary results

Throughout this paper, $\mathbb{F}_q$ denotes a finite field with $q$ elements, where $q$ is a power of an odd prime $p$ and $r$ is a fixed positive integer. We define the trace function

$$\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q} : \mathbb{F}_{q^r} \to \mathbb{F}_q$$

$$\alpha \mapsto \alpha + \alpha^q + \cdots + \alpha^{q^{r-1}}$$

and, for simplicity, we denote the trace function $\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ by $\text{Tr}$. A polynomial $F(x) \in \mathbb{F}_q[x]$ is called $\mathbb{F}_q$-linearized if is of the form $a_0x + a_1x^2 + a_2x^3 + \cdots + a_lx^l$, where $a_j \in \mathbb{F}_q$ for all $0 \leq j \leq l$. The polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_lx^l$ is called associated polynomial of $F(x)$.

In what follows, for $F(x)$ a $\mathbb{F}_q$-linearized and $\lambda \in \mathbb{F}_q$, $C_{F,\lambda}$ denotes the curve determined by the equation

$$y^q - y = xF(x) - \lambda$$

and $Q_{xF(x)} : \mathbb{F}_{q^r} \to \mathbb{F}_{q^r}$ is the quadratic form given by $Q_{xF(x)}(\alpha) = \text{Tr}(\alpha F(\alpha))$. We observe that if $(\alpha, \beta) \in \mathbb{F}_{q^2}^2$, is a point of $C_{F,\lambda}$, i.e., $\beta^q - \beta = \alpha F(\alpha) - \lambda$ then

$$0 = \text{Tr}(\beta^q - \beta) = \text{Tr}(\alpha F(\alpha) - \lambda) = \text{Tr}(\alpha F(\alpha)) - r\lambda.$$  

Reciprocally, if $\alpha \in \mathbb{F}_{q^r}$ satisfies the equation $\text{Tr}(\alpha F(\alpha)) = r\lambda$, then $\text{Tr}(\alpha F(\alpha) - \lambda) = 0$ and by Hilbert’s Theorem 90 there exists $\beta \in \mathbb{F}_{q^r}$ such that $\beta^q - \beta = \alpha F(\alpha) - \lambda$. In
addition, any other solution is of the form $\beta + c$ for $c \in \mathbb{F}_q$ and it follows that

$$N_r(C_{F, \lambda}) = qN_r(Q_{xF(x) - \lambda}).$$

(5)

We now recall the following standard definitions.

**Definition 2.1.** Let $L$ be a field and $\mathbb{F}$ a finite extension of $\mathbb{L}$, where $\text{char}(\mathbb{L}) \neq 2$. For a quadratic form $\Phi : L \to L$, we define the symmetric bilinear form $\varphi : \mathbb{F} \times \mathbb{F} \to L$ associated to $\Phi$ by $\varphi(\alpha, \beta) = \frac{1}{2}(\Phi(\alpha + \beta) - \Phi(\alpha) - \Phi(\beta))$. The radical of symmetric bilinear form $\Phi : \mathbb{F} \to L$ is:

$$\text{rad}(\Phi) = \{ \alpha \in \mathbb{F} : \varphi(\alpha, \beta) = 0 \text{ for all } \beta \in \mathbb{F} \}.$$

If $\text{rad}(\Phi) = \{0\}$, $\Phi$ is non-degenerate.

Let $B = \{v_1, \ldots, v_r\}$ be a basis of $\mathbb{F}$ over $\mathbb{L}$. The $r \times r$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} \Phi(v_i), & \text{if } i = j; \\ \frac{1}{2}(\Phi(v_i + v_j) - \Phi(v_i) - \Phi(v_j)), & \text{if } i \neq j. \end{cases}$$

is the associated matrix of the quadratic form $\Phi$ in the basis $B$. In particular, the dimension of $\text{rad}(\Phi)$ is equal to $r - \text{rank}(A)$.

Let $\Phi : \mathbb{F}_q^m \to \mathbb{F}_q$ and $\Psi : \mathbb{F}_q^n \to \mathbb{F}_q$ be quadratic forms where $m \geq n$. Let $A$ and $B$ be associated matrix of $\Phi$ and $\Psi$, respectively. We say that $\Phi$ is equivalent to $\Psi$ if there exists $M \in \text{GL}_m(\mathbb{F}_q)$ such that

$$M^TAM = \left( \begin{array}{cc} B & 0 \\ 0 & 0 \end{array} \right) \in \text{M}_m(\mathbb{F}_q),$$

where $\text{GL}_m(\mathbb{F}_q)$ denotes the $m \times m$ invertible matrices over $\mathbb{F}_q$, and $\text{M}_m(\mathbb{F}_q)$ denotes the $m \times m$ matrices over $\mathbb{F}_q$. Furthermore, $\Psi$ is called a reduced form of $\Phi$ if $\text{rad}(\Psi) = \{0\}$.

**Remark 2.2.** Let $F(x)$ be $\mathbb{F}_q$-linearized and $r$ a positive integer. It can be easily seen that the map $\Phi : \mathbb{F}_q^r \to \mathbb{F}_q$ given by $\alpha \mapsto \alpha F(\alpha)$ is a quadratic form. Furthermore, since $\text{Tr}$ is a $\mathbb{F}_q$-linear form, we have that the map $\Phi : \mathbb{F}_q^r \to \mathbb{F}_q$ given by $\Phi(\alpha) = \text{Tr}(\Phi(\alpha))$ is also a quadratic form. In fact, for all $c \in \mathbb{F}_q$ and $\beta \in \mathbb{F}_q$, we have

$$\text{Tr}(c\beta F(c\beta)) = \text{Tr}(c^2\beta F(\beta)) = c^2\text{Tr}(\beta F(\beta)).$$

hence $\text{Tr}((x+y)F(x+y)) - \text{Tr}(xF(x)) - \text{Tr}(yF(y))$ defines a symmetric bilinear form.

The following theorem is a well known result about the number of the solutions of a special equation over finite fields.

**Theorem 2.3 ([7], Theorems 6.26 and 6.27).** Let $\Phi$ be a quadratic form over $\mathbb{F}_q$. Let $\varphi$ be the bilinear symmetric form associated to $\Phi$, $v = \dim(\text{rad}(\Phi))$ and $\Psi$ a reduced non-degenerate quadratic form equivalent to $\Phi$. We define $S_\lambda = \{x \in \mathbb{F}_q | \Phi(x) = \lambda \}$ and denote by $\Delta$ the determinant of the matrix associated to $\Psi$ and $\chi$ the canonical quadratic character of $\mathbb{F}_q$.

(i) If $r + v$ is even, then

$$S_\lambda = \begin{cases} q^{r-1} + Dq^{(r+v-2)/2}(q-1) & \text{if } \lambda = 0; \\ q^{r-1} - Dq^{(r+v-2)/2} & \text{if } \lambda \neq 0, \end{cases}$$

(6)

where $D = \chi((-1)^{(r-v)/2}\Delta)$.

(ii) If $r + v$ is odd, then

$$S_\lambda = \begin{cases} q^{r-1} & \text{if } \lambda = 0; \\ q^{r-1} + Dq^{(r+v-1)/2} & \text{if } \lambda \neq 0, \end{cases}$$

(7)

where $D = \chi((-1)^{(r-v-1)/2}\lambda\Delta)$. 

Clearly, in order to determine $S_{\lambda}$ we need to calculate the dimension of the radical of $\Phi$ and the determinant of a reduced matrix associate to $\Phi$. For this, important tools for our work are complete homogeneous symmetric polynomials and circulant matrices, which we define below.

**Definition 2.4.** a) The complete homogeneous symmetric polynomials of degree $k$ is defined by

$$h_k(x_1, \ldots, x_r) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq r} x_{i_1} \cdots x_{i_k}.$$ 

We denote this polynomial by $h_k(r)$.

b) Let $a_0, a_1, \ldots, a_{r-1}$ be elements of a field $L$. The $r \times r$ circulant matrix $C = (c_{ij})$ associated to the $r$-tuple $(a_0, a_1, \ldots, a_{r-1})$ is given by $c_{ij} = a_k$ for each pair $(i, j)$ such that $j - i \equiv k \pmod{r}$. We denote by $C(a_0, a_1, \ldots, a_{r-1})$ this circulant matrix, and the vector $(a_0, a_1, \ldots, a_{r-1})$ is called the generator vector of the matrix $C$.

c) The associated polynomial of the circulant matrix $C(a_0, a_1, \ldots, a_{r-1})$ is $f(x) = \sum_{i=0}^{r-1} a_i x^i$.

We will show that, with some additional hypotheses (see Proposition 3.1), there exists a base of $F_{q^e}$ over $F_q$ such that the associated matrix of the quadratic form $Tr(Q_{f+\lambda})$ in this base is circulant.

The following theorem, that can be found as an exercise in [13], describes another representation of the polynomial $h_k(r)$.

**Theorem 2.5** ([13], Ex. 7.4). The polynomial $h_k(r)$ can be expressed as

$$h_k(r) = \sum_{l=1}^{r} x_l^{r+k-1} \prod_{m=1}^{r} (x_l - x_m).$$

These polynomials will be useful to determine the rank of some circulant matrices. In the case when $r$ is relatively prime with the characteristic of the field $F_q$, it is well known (e.g. [6]) that the determinant of any circulant matrix $C = C(a_0, a_1, \ldots, a_{r-1})$ satisfies the relation

$$\det C = \prod_{i=1}^{r} (a_0 + a_1 \omega_i + \cdots + a_{r-1} \omega_i^{r-1}),$$

where $\omega_1, \ldots, \omega_r$ are the $r$-th roots of unity in some extension of $F_q$. We will use this fact in order to determine the rank of $C$. More precisely, the rank of $C$ is equal to the number of common roots of $f(x)$ and $x^r - 1$, as we prove in Theorem 2.12. Before we prove this we need the following definition.

**Definition 2.6.** For each $0 < j \leq k$ integers, $A_k$ and $A_{k,j}$ denote the polynomials

1) $A_k(x_1, \ldots, x_k) = \prod_{1 \leq t < s \leq k} (x_s - x_t)$, for all $k \geq 2$.

2) $A_{k,j}(x_1, \ldots, x_k) = (-1)^{j+1} \prod_{1 \leq t < s \leq k} (x_s - x_t)$, for all $k \geq 3$.

The following lemmas show some relations between the complete homogeneous symmetric polynomials and the polynomials $A_k$ and $A_{k,j}$.

**Lemma 2.7.** Let $k$ be a positive integer and for each $1 \leq j \leq k$, $h_{r,j}(k)$ denotes the polynomial $h_r(x_1, \ldots, \hat{x}_j, \ldots, x_k)$, where $\hat{x}_j$ means that the variable $x_j$ is omitted. Then

$$\sum_{j=1}^{k} x_j^{r-1} A_{k,j} h_{r-k+1,j}(k) = 0, \text{ for all } k \geq 3.$$
Proof. Let us denote $\epsilon_{l,j} = \begin{cases} 1 & \text{if } l > j \\ -1 & \text{if } l < j \\ 0 & \text{if } l = j \end{cases}$. By Theorem 2.5 it follows that

$$\sum_{j=1}^{k} x_j^{r-1} A_{k,j} h_{r-k+1,j}(k) = \sum_{j=1}^{k} x_j^{r-1} (-1)^{j+1} \prod_{1 \leq l < s \leq k, l \neq j} (x_s - x_l) \sum_{l=1}^{k} \left( \prod_{m=1}^{k} (x_l - x_m) \right)$$

$$= \sum_{j=1}^{k} x_j^{r-1} (-1)^{j+1} \sum_{l=1}^{k} x_l^{r-1} \prod_{1 \leq r < s \leq k, r \neq j} (x_s - x_r) (-1)^{k-l+1} \epsilon_{l,j}$$

$$= \sum_{j=1}^{k} \sum_{l=1}^{k} x_j^{r-1} x_l^{r-1} \prod_{1 \leq r < s \leq j} (x_s - x_r) (-1)^{k+j-l+1} \epsilon_{l,j}.$$

For each $l$ and $j$ fixed, the sum runs over the term $(x_l x_j)^{r-1} = (x_l x_j)^{r-1}$ twice and then

$$x_j^{r-1} x_l^{r-1} \left( \prod_{1 \leq r < s \leq k, r \neq j} (x_s - x_r) (-1)^{k+j-l+1} (\epsilon_{l,j} + \epsilon_{j,l}) \right) = 0.$$

$$\square$$

Lemma 2.8. Let $k \geq 2$ be an integer. Then

$$A_{k+1} = \sum_{j=1}^{k+1} \frac{x_1 \cdots x_{k+1}}{x_j} A_{k+1,j}.$$

In addition,

$$\sum_{j=1}^{k} \frac{x_1 \cdots x_k}{x_j} \frac{1}{\prod_{r \neq j} (x_r - x_j)} = \frac{F(x_1, \ldots, x_{k+1})}{A_{k+1}} = 1.$$

Proof. We now let

$$F_{k+1} = \sum_{j=1}^{k+1} \frac{x_1 \cdots x_{k+1}}{x_j} A_{k+1,j}.$$

We will prove that $A_{k+1} = F_{k+1}$ by induction on the number of variables. For $k = 2$ we have

$$F_3 = \sum_{j=1}^{3} \frac{x_1 x_2 x_3}{x_j} (-1)^{j+1} \prod_{1 \leq l < s \leq 3, l \neq j} (x_l - x_s)$$

$$= x_2 x_3 (x_3 - x_2) - x_1 x_2 x_3 (x_3 - x_1) + x_1 x_3 (x_3 - x_1)$$

$$= (x_2 - x_1) (x_3 - x_2) + (x_1 x_2 - x_2 x_3) = A_3.$$

Now suppose that the result is true for some $k \geq 2$. The polynomial $F_{k+1}$ has degree $\binom{k+1}{2}$ and if $x_i = x_j$ for any $1 \leq i < j \leq k+1$, it follows that $F_{k+1} = 0$, which implies that $A_{k+1}$ divides $F_{k+1}$. Since $A_{k+1}$ has the same degree of $F_{k+1}$ we obtain that $F_{k+1} = c A_{k+1}$ for some $c \in \mathbb{F}_q$. From the fact that $A_{k+1}$ is a monic polynomial in respect to the variable.
Since this relation proves the case and we have the case

and this means that $F_k = cA_k$. By the induction hypothesis, it follows that $c = 1$. \hfill \Box

**Lemma 2.9.** Let $C$ be a circulant matrix over $\mathbb{F}_q$ with generator vector $(a_0, a_1, \ldots, a_{r-1})$ and $f(x)$ be the associated polynomial to the matrix $C$. Let $g(x) = \gcd(f(x), x^r - 1)$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ the roots of $g(x)$. If $g(x)$ has only simple roots, then for each positive integer $j \leq m$ the relation

$$(a_0, a_1, \ldots, a_{r-j}) \cdot (1, h_1(\alpha_1, \ldots, \alpha_j), \ldots, h_{r-j}(\alpha_1, \ldots, \alpha_j)) = 0$$

is satisfied, where $\cdot$ denotes the inner product.

**Proof.** We set $\bar{a}_k = (\alpha_1, \ldots, \alpha_{k+1})$. We proceed by induction on the number of roots of $g$. For any $\alpha$ root of $g$ we have

$$a_0 + a_1\alpha + \cdots + a_{r-1}\alpha^{r-1} = 0.$$ 

Then $(a_0, a_1, \ldots, a_{r-1}) \cdot (1, \alpha, \ldots, \alpha^{r-1}) = 0$ and this relation is equivalent to the first case of the induction. If $j = 2$, for each pair of roots $\alpha_1$ and $\alpha_2$, we have the relations

$$\begin{cases}
 a_0\alpha_1 + a_1\alpha_1^2 + \cdots + a_{r-1}\alpha_1^{r-1} = 0 \\
 a_0\alpha_2 + a_1\alpha_2^2 + \cdots + a_{r-1}\alpha_2^{r-1} = 0.
\end{cases}$$

Subtracting, we get

$$a_0(\alpha_2 - \alpha_1) + a_1(\alpha_2^2 - \alpha_1^2) + \cdots + a_{r-2}(\alpha_2^{r-1} - \alpha_1^{r-1}) = 0.$$ 

Since $A_2 = \alpha_2 - \alpha_1 \neq 0$ it follows that

$$0 = (a_0, a_1, \ldots, a_{r-2}) \cdot (\alpha_2 - \alpha_1, \alpha_2^2 - \alpha_1^2, \ldots, \alpha_2^{r-1} - \alpha_1^{r-1})$$

$$= (a_0, a_1, \ldots, a_{r-2}) \cdot A_2 (1, h_1(\alpha_1, \alpha_2), \ldots, h_{r-2}(\alpha_1, \alpha_2)).$$

and this relation proves the case and we have the case $j = 2$. Let us suppose now that (8) is true for any choice of $k$ different roots of $g$ and let $\alpha_1, \ldots, \alpha_{k+1}$ be $k + 1$ roots of $g$. By the induction hypothesis, we have $k + 1$ equations of the form (8), where for each one we do not consider one of the roots, i.e., the $j$-th equation is given by

$$(a_0, \ldots, a_{r-k}) \cdot (1, h_{1,j}(\bar{a}_{k+1}), \ldots, h_{r-k,j}(\bar{a}_{k+1})) = 0. \quad (9)$$

Multiplying the vector $(1, h_{1,j}(\bar{a}_{k+1}), \ldots, h_{r-k,j}(\bar{a}_{k+1}))$ for $\alpha_j^{-1}A_{k+1,j}$ and adding these vectors we obtain the vector

$$\bar{u} = \sum_{j=1}^{k+1} \alpha_j^{-1} (A_{k+1,j}, A_{k+1,j}h_{1,j}(\bar{a}_{k+1}), \ldots, A_{k+1,j}h_{r-k,j}(\bar{a}_{k+1})).$$

By Lemma 2.7, the last coordinate of $\bar{u}$ is

$$\sum_{j=1}^{k+1} \alpha_j^{-1}A_{k+1,j}h_{r-k,j}(\bar{a}_{k+1}) = 0. \quad (10)$$
Let us put $\alpha = \alpha_1 \cdots \alpha_{k+1}$. The first coordinate of $\vec{u}$ is
\[
\sum_{j=1}^{k+1} \alpha_j^{-1} A_{k+1,j} = \sum_{j=1}^{k+1} \alpha_j^{-1} \left( (-1)^{j+1} \prod_{i<l \leq k+1} (\alpha_l - \alpha_s) \right)
\]
\[
= \sum_{j=1}^{k+1} \left( \frac{\alpha_j}{\alpha} \prod_{i<l \leq k+1} (\alpha_l - \alpha_s) \right) = \frac{A}{\alpha} A_{k+1}, \quad \text{(11)}
\]

where in the last equality we use Lemma 2.8 and the fact that $\alpha_j$'s are $r$-th roots of unity. For $2 \leq l \leq r - k - 1$, the $l$-th coordinate of $\vec{u}$ is equal to
\[
\sum_{j=1}^{k+1} \alpha_j^{-1} A_{k+1,j} h_{i,j}(\vec{a}_{k+1}) = \sum_{j=1}^{k+1} \alpha_j^{-1} \left( (-1)^{j+1} \prod_{i<l \leq k+1} (\alpha_l - \alpha_s) \right)
\]
\[
= \frac{\alpha}{\alpha} \sum_{j=1}^{k+1} \alpha_j^{-1} \left( (-1)^{j+1} \prod_{i<l \leq k+1} (\alpha_l - \alpha_s) \right) \prod_{i<j \leq k+1} (\alpha_i - \alpha_s \epsilon_{i,j}). \quad \text{(12)}
\]

Let $G_{k+1}$ denote the polynomial
\[
G_{k+1} = \sum_{i=1}^{k+1} \prod_{m \neq i} (x_i - x_m) \prod_{j \neq i} (x_j - x_i) \prod_{r \neq i,j} (x_r - x_j).
\]
We observe that for $x_i = x_j$, $i \neq j$, we have $G_{k+1} = 0$ and therefore $(x_i - x_j)$ divides $G_{k+1}$ for all $i \neq j$. We conclude that $A_{k+1}$ divides $G_{k+1}$, we can write
\[
\frac{G_{k+1}}{A_{k+1}} = \sum_{i=1}^{k+1} \prod_{m \neq i} (x_i - x_m) \sum_{j \neq i} \prod_{r \neq i,j} (x_r - x_j)
\]
\[
= \sum_{i=1}^{k+1} \prod_{m \neq i} (x_i - x_m) \sum_{j \neq i} \prod_{r \neq i,j} (x_r - x_j) = \frac{1}{x_i x_j} \prod_{r \neq i,j} (x_r - x_j).
\]

Fixing $i$, it follows from Lemma 2.8 that
\[
\sum_{j \neq i} \prod_{m \neq i} x_j^{1+k} = 1. \quad \text{Therefore}
\]
\[
G_{k+1} = A_{k+1} \sum_{i=1}^{k+1} \prod_{m \neq i} (x_i - x_m) = A_{k+1} h_{k+1}. \quad \text{(13)}
\]

By (12) we have
\[
\sum_{j=1}^{k+1} \alpha_j^{-1} A_{k+1,j} h_{i,j}(\vec{a}_{k+1}) = \frac{\alpha}{\alpha} A_{k+1} h_{i,k+1}.
\]
From (10), (11) and (13) we conclude that
\[(a_0, \ldots, a_{r-1}) \cdot (1, h_1(k + 1), \ldots, h_{n-k-1}(k + 1)) = 0.\]
\[\square\]

**Remark 2.10.**

(1) Let \(\lambda\) be a root of \(g(x)\). Multiplying \(f(\lambda)\) by \(\lambda^i\) we obtain \(a_{r-i} + a_{r-i+1}\lambda + \cdots + a_{r-1}\lambda^{i-1} = 0\) and therefore Lemma 2.9 is true for any shift of the coefficients \(a_0, a_1, \ldots, a_{r-1}\).

(2) In particular, Lemma 2.9 is true if \(\gcd(r, q) = 1\), since in this case \(g(x)\) has only simple roots.

For what follows, we will need the following definition.

**Definition 2.11.** Let \(f(x) \in \mathbb{F}_q[x]\) be a monic polynomial of degree \(n\) such that \(f(0) \neq 0\). The reciprocal polynomial \(f^*\) of the polynomial \(f\) is defined by \(f^*(x) = \frac{1}{f(1)} x^n f(\frac{1}{x})\). The polynomial \(f\) is self-reciprocal if \(f = f^*\).

Now, we show how to find the rank and an equivalent reduced matrix to the circulant matrix \(C\).

**Theorem 2.12.** Let \(A = A(a_0, a_1, \ldots, a_{r-1})\) be a circulant matrix over \(\mathbb{F}_q\) and assume \(\gcd(r, p) = 1\). If \(f(x)\) is the polynomial associated to \(A\) is such that \(g(x) = \gcd(f(x), x^r - 1)\) is a self-reciprocal polynomial with \(\deg g(x) = m\) then \(\text{rank}(A) = l = r - m\) and there exists \(M \in GL_r(\mathbb{F}_q)\) such that \(M A M^T = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 0 \end{array}\right)\), where \(R = (r_{ij})\) denotes the \(l \times l\) matrix defined by \(r_{ij} = a_{i-j}\) for \(0 \leq i, j \leq l\) and \(M^T\) is the transpose matrix of \(M\).

**Proof.** Let \(\alpha_1, \ldots, \alpha_m\) be the roots of \(g(x)\). Let \(B_i\) be the matrices obtained from the identity matrix changing the entries of the \(r-i+1\)-th row by
\[ (1, h_1(\alpha_1, \ldots, \alpha_1), \ldots, h_{r-i}(\alpha_1, \ldots, \alpha_i), 0, \ldots, 0). \]
Observe that
\[ B_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \\ 1 & \alpha_1 & \cdots & \alpha_1^{r-1} \alpha_1^{-1} & \cdots & \alpha_1^{-1} \end{pmatrix}, \]
and since \(\alpha_1\) and \(\alpha_1^{-1}\) are roots of \(g\), the product \(B_1 A B_1^T\) has last row and column with null entries. From Lemma 2.9 and Remark 2.10 it follows that \(M A M^T = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 0 \end{array}\right)\), where \(M = B_m B_{m-1} \cdots B_2 B_1\) and \(R\) is the matrix \(A\) reduced to its first \(l\) rows and \(l\) columns.

\[\square\]

**Example 2.13.** Let \(q = 27\), \(r = 7\) and \(\Omega_r\) denote the \(r\)-th cyclotomic polynomial. Since \(\text{ord}_q q = 2\), \(\Omega_r\) splits into three monic irreducible polynomials over \(\mathbb{F}_q[x]\) of degree 2. Let \(\langle a \rangle = \mathbb{F}_2^{27}\), where we can choose \(a\) with minimal polynomial \(x^3 + 2x + 1\). Then \(\Omega_r(x) = (x^2 + 2a^2x + 1)(x^2 + (2a^2 + a + 2)x + 1)(x^2 + (2a^2 + 2a + 2)x + 1)\).

Let us define
\[ f(x) = (x^2 + 2a^2x + 1)(x^2 + (2a^2 + a + 2)x + 1)(x - a) = x^3 + x^2(a^2 + 2) + x^2(a^2 + a + 1) + x^2(2a + 1) + x(a^2 + 2a) + 2a \]
and therefore the circulant matrix associated to the polynomial \( f(x) \) is
\[
A = \begin{pmatrix}
2a & a^2 + 2a & 2a^2 + 1 & 0 \\
0 & 2a & a^2 + 2a & 2a^2 + 1 \\
1 & 0 & 2a & a^2 + 2a \\
a^2 + 2 & 1 & 0 & 2a \\
a^2 + a + 1 & a^2 + 2 & 2a + 1 & a^2 + a + 1 \\
2a + 1 & a^2 + a + 1 & a^2 + 2 & 1 \\
a^2 + 2a & 2a + 1 & a^2 + a + 1 & a^2 + 2 \\
a^2 + 2a & 2a + 1 & a^2 + a + 1 & a^2 + 2
\end{pmatrix}.
\]

Since
\[
g(x) = \gcd(f(x), x^r - 1) = (x^2 + 2a^2x + 1)(x^2 + (2a^2 + a + 2)x + 1)
= x^4 + x^3(a^2 + a + 2) + x^2(2a^2 + a) + x(a^2 + a + 2) + 1
\]
is a self-reciprocal polynomial, it follows from Theorem 2.12 that \( \text{rank}(A) \) is 3 and the reduced matrix associated to \( A \) is \( A' = \begin{pmatrix} 2a & a^2 + 2a & 2a^2 + 1 \\
0 & 2a & a^2 + 2a \\
1 & 0 & 2a \\
a^2 + 2a & 2a + 1 & a^2 + a + 1 \\
2a + 1 & a^2 + a + 1 & a^2 + 2 \\
a^2 + 2a & 2a + 1 & a^2 + a + 1 \\
a^2 + 2a & 2a + 1 & a^2 + a + 1 \end{pmatrix} \). In addition, \( \det A' = a^4 + 12a^3 + a = a^2 \neq 0 \).

3. The number of affine rational points of \( y^q - y = xF(x) - \lambda \)

In this section, in order to find the number of affine rational points of the curve \( y^q - y = xF(x) - \lambda \), where \( F(x) \) is a \( \mathbb{F}_q \)-linearized, we determine the number of solutions of the equation \( \text{Tr}(xF(x)) = \lambda \) in \( \mathbb{F}_q^r \). In fact, by (5)
\[
y^q - y = xF(x) - \lambda. \tag{14}
\]
We recall that we have
\[
N_r(C_{F, \lambda}) = qS_\lambda,
\]
where \( S_\lambda = |\{x \in \mathbb{F}_q^r \mid \text{Tr}(xF(x)) = r\lambda\}| \).

In that follows, \( P \) denotes the \( r \times r \) cyclic permutation matrix
\[
P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 
\end{pmatrix}. \tag{15}
\]
The following proposition associates the \( \mathbb{F}_q \)-linearized polynomial \( F(x) \) with an appropriated circulant matrix.

**Proposition 3.1.** Let \( F(x) = \sum_{i=0}^l a_i x^i \) be \( \mathbb{F}_q \)-linearized. For \( \lambda \in \mathbb{F}_q \), the number of solutions of \( \text{Tr}(xF(x)) = r\lambda \) in \( \mathbb{F}_q^r \) is equal to the number of solutions \( \bar{z} = (z_1, z_2, \ldots, z_n)^T \in \mathbb{F}_q^r \) of the quadratic form
\[
\bar{z}^T A \bar{z} = r\lambda
\]
where \( A = \frac{1}{2} \sum_{i=0}^l a_i (P^i + (P^i)^T) \).

**Proof.** Let \( \Gamma = \{\beta_1, \ldots, \beta_r\} \) be a basis of \( \mathbb{F}_q^r \) over \( \mathbb{F}_q \) and
\[
N_\Gamma = \begin{pmatrix}
\beta_1 & \beta_1^q & \ldots & \beta_1^{q^{r-1}} \\
\beta_2 & \beta_2^q & \ldots & \beta_2^{q^{r-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_r & \beta_r^q & \ldots & \beta_r^{q^{r-1}}
\end{pmatrix}.
\]
Then \( N_\Gamma \) is an invertible matrix and for \( x \in \mathbb{F}_q^r \) we can write \( x = \sum_{j=1}^r \beta_j x_j \), where \( x_1, \ldots, x_r \in \mathbb{F}_q \). The equation \( \text{Tr}(xF(x)) = r\lambda \) is equivalent to
\[
\sum_{j=0}^{r-1} a_j q^j F(x) y^j = r\lambda, \tag{16}
\]
Since $F(x)$ is a $\mathbb{F}_q$-linearized and the trace is $\mathbb{F}_q$-linear, we need to express the monomials of the form $x \cdot x^{q^r}$ in terms of the basis $\Gamma$. We have

$$\text{Tr}(x^{q^r+1}) = \sum_{j=0}^{r-1} x^{q^j} \cdot (x^{q^r})^{q^j} = \sum_{j=0}^{r-1} \left( \sum_{s=1}^{r} \beta_s x_s \right)^{q^j} \left( \sum_{k=1}^{r} \beta_k x_k \right)^{q^j+1} = \sum_{s,k=1}^{r} \left( \sum_{j=0}^{r-1} \beta_s^{q^j} \beta_k^{q^j+1} \right) x_s x_k.$$

Consequently $\text{Tr}(x^{q^r+1})$ has the following symmetric representation

$$(x_1, x_2, \ldots, x_r)B_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix},$$

where $B_1 = \frac{1}{2}N_r(P^l + (P^l)^T)N_{2r}^T$.

Making the change of variables $(z_1, z_2, \ldots, z_r) = (x_1, x_2, \ldots, x_r)N_r$ we get

$$(z_1, z_2, \ldots, z_r) \begin{pmatrix} \frac{1}{2}(P^l + (P^l)^T) \\ \vdots \\ \frac{1}{2}(P^l + (P^l)^T) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{pmatrix}$$

which has the same number of solutions of $\text{Tr}(x^{q^r+1})$. Using equation (16) and the definition of $A$, the result follows.

The following theorem is straightforward consequence of Theorems 2.12 and 2.3 and Proposition 3.1.

**Theorem 3.2.** Let $F(x) = \sum_{i=0}^{l} a_i x^{q^i}$ be $\mathbb{F}_q$-linearized and $f(x) = \sum_{i=0}^{l} a_i x^i$ its associated polynomial. We assume $\gcd(r, p) = 1$ and that $g(x) = \gcd(f(x), x^r - 1)$ is a self-reciprocal polynomial of degree $m$. Let also $R$ be the matrix defined as in Theorem 2.12 and $a = \det R$. Then for each $\lambda \in \mathbb{F}_q$, the number of affine rational points in $\mathbb{F}_q^2$ of the curve $y^q - y = xF(x) - \lambda$ is

$$N_r(C_{F, \lambda}) = \begin{cases} q^r - q^{r+m-2/2} \chi((-1)^{(r-m)/2}a), & \text{if } r + m \text{ is even and } \lambda \neq 0; \\
q^r + (q-1)q^{r+m-2/2} \chi((-1)^{(r-m)/2}a), & \text{if } r + m \text{ is even and } \lambda = 0; \\
q^r + q^{r+m-1/2} \chi((-1)^{(r-m-1)/2}2r\lambda a), & \text{if } r + m \text{ is odd.} \end{cases}$$

**Corollary 3.3.** Let $F$, $f$ and $g$ be polynomials which satisfy the conditions of Theorem 3.2. Then

$$|N_r(C_{F, \lambda}) - q^r| \leq (q-1)q^{\frac{r+m-2}{2}}.$$

In addition, the upper (resp. lower) bound is attained if and only if $r + m$ is even, $\lambda = 0$ and $(-1)^{\frac{r+m}{2}}a$ is (resp. not) a square in $\mathbb{F}_q$.

**Remark 3.4.** The curve $C_{F, \lambda}$, where $F(x) = \sum_{i=0}^{l} a_i x^{q^i}, a_i \in \mathbb{F}_q$ and $0 \leq l < r$, has genus $g = \frac{(q-1)q^l}{2}$. The Hasse-Weil bound for $C_{F, \lambda}$ is given by

$$|N_r(C_{F, \lambda}) - q^r| \leq (q-1)q^{\frac{2l+r}{2}}.$$

Consequently, this curve is not maximal (resp. minimal) with respect to this bound.
Example 3.5. Let \( q = 27 \), \( r = 7 \) and \( f(x), g(x) \) be the polynomials of Example 2.13. The polynomial \( F(x) = x^q + (a^2 + 2)x^{q^2} + (a^2 + a + 1)x^q + (2a + 1)x^7 + (a^2 + 2a)x^9 + 2ax \) is the \( F \)-linearized of \( f(x) \). Since \( r - m \) is odd and \( \det C' = a^2 \), we get from Theorem 3.2

\[
N_r(C_{F,\lambda}) = q^7 + q^6 \chi(\lambda) = \begin{cases} 
q^7 + q^6, & \text{if } \lambda \text{ is a square in } \mathbb{F}_q^*; \\
q^7 - q^6, & \text{if } \lambda \text{ is not a square in } \mathbb{F}_q^*; \\
q^7, & \text{if } \lambda = 0.
\end{cases}
\]

In the following section we use Theorem 2.3 to compute \( D \) for some some special polynomials \( F(x) \).

4. The number of affine rational points of \( y^q - y = x \cdot (x^q - x) - \lambda \)

Throughout this section, for any prime \( t \) and a positive integer \( n \), we denote by \( (\frac{a}{t}) \) the Legendre symbol and by \( v_t(n) \) the \( t \)-adic valuation of \( n \), i.e., the maximum power of \( t \) that divides \( n \). The objective of this section is to find an expression for the number of affine rational points of the curve \( y^q - y = x^{q^2+1} - x^2 - \lambda \) in \( \mathbb{F}_q^2 \) in terms of valuation functions and Legendre symbols.

In the previous section, we used the fact that the number \( N_r(C_{F,\lambda}) \), for \( F(x) \) a \( F_q \)-linearized polynomial and \( \lambda \in \mathbb{F}_q \), is \( q \) times the number of elements \( x \in \mathbb{F}_q \) such that \( \text{Tr}(xF(x)) = r \lambda \).

In order to determine the number of solutions of \( \text{Tr}(xF(x)) = r \lambda \), it is necessary to establish the dimension of the symmetric bilinear form associated to this quadratic form, which is the content of the next proposition.

**Proposition 4.1.** Let \( 0 < i < r \) be integers and \( F(x) = \sum_{j=0}^i a_j x^j \) a \( \mathbb{F}_q \)-linearized. Let \( \Phi_F(x) = \text{Tr}(xF(x)) \). If \( a_0 \neq 0 \), then

\[
\dim \text{rad}(\varphi_F) = \deg \left( \gcd \left( \sum_{j=0}^i a_j (x^j + x^{r-j}), x^r - 1 \right) \right). \tag{17}
\]

**Proof.** In order to determine the dimension of the radical of \( \varphi_F \) it is sufficient to compute the dimension of the radical

\[
\text{dim}_{\mathbb{F}_q} \{ y \in \mathbb{F}_q | \varphi_F(x, y) = 0 \text{ for all } y \in \mathbb{F}_q \}.
\]

In fact

\[
\varphi(x, y) = \text{Tr} \left( \sum_{j=0}^i a_j (x + y)^{qj + 1} - \sum_{j=0}^i a_j x^{qj + 1} - \sum_{j=0}^i a_j y^{qj + 1} \right)
\]

\[
= \sum_{j=0}^{r-1} \left( \sum_{i=0}^i a_j (x + y)^{q^i i + q} - \sum_{j=0}^i a_j x^{q^i i + q} - \sum_{j=0}^i a_j y^{q^i i + q} \right)
\]

\[
= \sum_{j=0}^i a_j \left( \sum_{l=0}^{r-1} x^{q^i l + y^{q^i l}} + x^{q^i} y^{q^i} \right)
\]

\[
= \sum_{j=0}^i a_j \left( \sum_{j=0}^{r-1} (x^{q^i} + x^{q^i - j}) y^{q^i} \right)
\]

\[
= \sum_{j=0}^i a_j \text{Tr}( (x^{q^i} + x^{q^i - j}) y^{q^i} ) = \text{Tr} \left( \sum_{j=0}^i a_j (x^{q^i} + x^{q^i - j}) y^{q^i} \right) \tag{18}
\]
It follows that $\varphi_F(x, y) = 0$ for all $y \in \mathbb{F}_q^r$ is equivalent to
\[
\sum_{j=0}^{i} a_j(x^{q^j} + x^{q^{r-j}}) = 0. \tag{19}
\]
The $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^r$ determined by (19) is the set of roots of
\[
g(x) = \gcd(H(x), x^{q^r} - x), \quad \text{where } H(x) = \sum_{j=0}^{i} a_j(x^{q^j} + x^{q^{r-j}}).
\]
Since $g$ is a $\mathbb{F}_q$-linearized, the degree of the associated polynomial gives us the dimension of radical of $\varphi$, which is the degree of $\gcd(x^r - 1, \sum_{j=0}^{i} a_j(x^j + x^{r-j}))$. This finishes the proof. \hfill \square

4.1. The special case $F(x) = x^{q^r} - x$. For this case we explicitly determine the dimension of the quadratic form $\text{Tr}(xF(x))$. Besides that, we will use this information to compute $\mathcal{N}_{r}(\mathcal{C}_{F, \lambda})$, that is given in Theorem 4.10. In order to simplify the notations, we define the following quadratic form.

**Definition 4.2.** Let $i, r$ be integers such that $0 < i < r$. We define
\[
\Phi_i : \mathbb{F}_q^{r} \to \mathbb{F}_q,
\]
\[
x \mapsto \text{Tr}(x^{q^{i+1}} - x^2).
\]

The following corollary is consequence of Proposition 4.1.

**Corollary 4.3.** Let $i, r$ be integers such that $0 < i < r$ and $\varphi_i(x, y)$ the associated symmetric bilinear form of $\Phi_i$. Let $r = p^u\bar{r}$ and $i = p^s\bar{i}$, where $u, s$ are non-negative integers such that $\gcd(p, \bar{r}) = \gcd(p, \bar{i}) = 1$. Then
\[
\dim \text{rad } (\varphi_i) = \gcd(r, i) \min(p^u, 2p^s). \tag{20}
\]

**Proof.** By Proposition 4.1 it is enough to find the dimension of the linear space determined by the roots of
\[
H(x) = \gcd(x^{q^i} + x^{q^{r-i}} - 2x, x^{q^r} - x). \tag{21}
\]
Since $r = p^u\bar{r}$, $i = p^s\bar{i}$, the associated polynomial to the $\mathbb{F}_q$-linearized polynomial $H(x)$ is
\[
h(x) = \gcd((x^i - 2 + x^{q^{r-i}}, x^r - 1) = \gcd((x^{2i} - 2x^i + x^r, x^r - 1) = \gcd((x^{2i} - 1)^{2p^s}, (x^r - 1)^{p^u}) = (x^{\gcd(r, i)} - 1)^{\min(p^u, 2p^s)}.
\]
Since the degree of $h(x)$ is equal to the dimension of the radical, we conclude that
\[
\dim \text{rad } (\varphi_i) = \gcd(r, i) \min(p^u, 2p^s). \hfill \square
\]

Using Theorem 2.3 and the previous corollary we can determine the number of solutions of $\text{Tr}(x^{q^{i+1}} - x^2) = r\lambda$ in $\mathbb{F}_q^r$, which will give us a complete description of $\mathcal{N}_{r}(\mathcal{C}_{F, \lambda})$.

**Lemma 4.4.** Let $i, r$ be integers such that $0 < i < r$ and $\gcd(r, 2p) = 1$. Let $v$ be the dimension of the radical of the associated bilinear symmetric form $\Phi_i$. Let $i = p^s\bar{i}$, where $s$ is a non-negative integer and $\gcd(i, p) = 1$. Then $r + v$ is even and, for $\lambda \in \mathbb{F}_q^*$, the constant $D$ of Theorem 2.3 is given by
\[
D = \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max(0, v_{p_j}(r) - v_{p_j}(i))}.
\]
where $r = p_1^{a_1} \cdots p_u^{a_u}$ is the prime factorization of $r$.

Proof. Let $M_\lambda$ be the set of solutions of $\text{Tr}(x^{d+1} - x^2) = \lambda$ in $\mathbb{F}_q$. Then $S_\lambda = |M_\lambda|$ is given by Equation (6). For each $\lambda \in \mathbb{F}_q^*$, if $\text{Tr}(x^{d+1} - x^2) = \lambda$ we have

$$\text{Tr}((x^{q^j})^{q+1} - (x^{q^j})^2) = \text{Tr}((x^{q^j+1} - x^2)^q) = \text{Tr}(x^{d+1} - x^2) = \lambda,$$

(22)

for all $0 \leq j \leq r - 1$.

We first consider the case $r = p_1^l$, where $p_1$ is an odd prime and $\gcd(p_1, p) = 1$. By Equation (22), for each $\alpha \in M_\lambda$ we can associate another $d - 1$ elements of $M_\lambda$, where $d$ is the smallest positive divisor of $r = p_1^l$ such that $\alpha^{q^d} = \alpha$. For each $\alpha \in M_\lambda$ we have $d > 1$, because otherwise $\alpha^q = \alpha$, which implies $\alpha^{q^d+1} - \alpha^2 = \alpha^2 - \alpha^2 = 0$, which is a contradiction because $\lambda \neq 0$. Then $d$ is a multiple of $p_1$ and Equation (6) of Theorem 2.3 can be rewritten modulo $p_1$ as

$$q^{r-1} - D q^{(r+v-2)/2} \equiv 0 \pmod{p_1},$$

which is equivalent to

$$D \equiv q^{(r+v-2)/2} \equiv q^{(r+v-2)/2} \pmod{p_1},$$

where in the last congruence we use the fact that $D = \pm 1$. By Lemma 4.3 we obtain

$$D \equiv q^{p_1^{a_1} + p_1^{\min(a,v p_1(1))} + 1}/2 - 1 \pmod{p_1}$$

$$\equiv q^{p_1^{\min(a,v p_1(1))} - 1}/2 \pmod{p_1}$$

$$\equiv q^{p_1^{\max(0,a,v p_1(1))} - 1}/2 \pmod{p_1}$$

$$\equiv \left( \frac{q}{p_1} \right)^{\max(0,a,v p_1(1))} \pmod{p_1}$$

Since $\left( \frac{q}{p_1} \right)$ assumes only the values $\{-1, 1\}$ and $\frac{p_1^l - 1}{p_1 - 1} \equiv l \pmod{2}$, we conclude that

$$D = \left( \frac{q}{p_1} \right)^{\max(0,a,v p_1(1))}.$$

Now we consider the general case $r = p_1^{a_1} \cdots p_u^{a_u}$, with $u \geq 1$. We will prove the result by induction on $u$. We already proved the case when $u = 1$. Now suppose $u \geq 2$. It follows from Lemma 4.3 that the dimension of the radical of the bilinear symmetric form associated to $\Phi(x) = v = \gcd(p_1^{a_1} \cdots p_u^{a_u}, i)$ and therefore $v$ divides $p_1^{a_1} \cdots p_u^{a_u}$ and $r + v$ is even. Using Theorem 2.3, for $\lambda \in \mathbb{F}_q^*$, we obtain that

$$S_\lambda = q^{r-1} - D q^{(r+v-2)/2}.$$

Now let $\lambda \in \mathbb{F}_q^*$ and $r = \tilde{r} p_u^{a_u}$ where $\tilde{r} = p_1^{a_1} \cdots p_{u-1}^{a_{u-1}}$. We now consider the subfield $\mathbb{F}_{q^\tilde{r}} \subset \mathbb{F}_{q^r}$. The induction hypothesis, the number of solutions of $\text{Tr}_{\mathbb{F}_{q^{\tilde{r}}} / \mathbb{F}_{q}}(x^{d+1} - x^2) = \lambda$ is

$$S_{\lambda, \tilde{r}} = q^{\tilde{r}-1} - \prod_{j=1}^{\tilde{r}-1} \left( \frac{q}{p_j} \right)^{\max(0,v p_j(r) - v p_j(1))} q^{(\tilde{r}+v-2)/2},$$

where $v_1$ is the dimension of radical of the bilinear symmetric form associated to $\text{Tr}_{\mathbb{F}_{q^{\tilde{r}}} / \mathbb{F}_{q}}(x^{d+1} - x^2)$. From Lemma 4.3 we know that $v_1 = \gcd(\tilde{r}, i)$ which implies $v = v_1 \cdot \gcd(p_u^{a_u}, i)$. Since
we have that
\[
\left\lfloor \frac{\alpha^{q^u} - 1}{\alpha^{q^u} - 1} \right\rfloor = \frac{\alpha^{q^u} - 1}{\alpha^{q^u} - 1} \equiv q^{v_1 - 1} - \prod_{j=1}^{u-1} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} q^{(r + v_1 - 1)/2} \pmod{p_u}.
\]

Since \( q^{u} \equiv q \pmod{p_u} \), the previous equation is equivalent to
\[
Dq^{(r + v_1 - 1)/2} = \prod_{j=1}^{u-1} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} q^{(r + v_1 - 1)/2} \pmod{p_u}.
\]

Now let \( v_2 = \gcd(p_u^{\nu_u}, i) \). We observe that \( q^{(u^{\nu_u} - 1)/2} \equiv \left( \frac{q}{p_u} \right)^{a_u} \pmod{p_u} \) from which it follows that
\[
q^{(r + v_1 - r^{\nu_u} - v_1)/2} \equiv q^{-r} \left( \frac{\alpha^{q^u} - 1}{\alpha^{q^u} - 1} \right) q^{\frac{u^{\nu_u} - 1}{2}} \pmod{p_u}
\]
\[
\equiv \left( \frac{q}{p_u} \right)^{a_u} q^{\frac{u^{\nu_u} - 1}{2}} \pmod{p_u}
\]
\[
\equiv \left( \frac{q}{p_u} \right)^{a_u} q^{\frac{u^{\nu_u} - 1}{2}} \pmod{p_u}
\]
\[
\equiv \left( \frac{q}{p_u} \right)^{a_u} q^{-v_1 \frac{\min(a_u, \nu_{p_u}(i)) - 1}{2}} \pmod{p_u}.
\]

Equations (23) and (24) allow us to conclude that
\[
D \equiv \prod_{j=1}^{u-1} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} q^{(r + v_1 - r^{\nu_u} - v_1)/2} \pmod{p_u}
\]
\[
\equiv \prod_{j=1}^{u-1} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} \left( \frac{q}{p_u} \right)^{a_u} q^{-v_1 \frac{\min(a_u, \nu_{p_u}(i)) - 1}{2}} \pmod{p_u}
\]
\[
\equiv \prod_{j=1}^{u-1} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} \left( \frac{q}{p_u} \right)^{a_u} \left( \frac{q}{p_u} \right)^{-\min(a_u, \nu_{p_u}(i))} \pmod{p_u}
\]
\[
\equiv \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} \pmod{p_u}
\]

and consequently \( D = \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))} \).

\[\square\]

Remark 4.5. From Theorem 2.3 we have that \( D \) independent of the value of \( \lambda \in \mathbb{F}_{q^r} \). Then, by Lemma 4.4, for \( \lambda = 0 \) we have that the value of \( D \) in Equation (6) is
\[
\prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max(0, \nu_{p_j}(r) - \nu_{p_j}(i))}.
\]

For extensions of degree power of 2 we have the following result.
Lemma 4.6. Let $b, i, r$ be integers such that $0 < i < r$ and $r = 2^b$. Let $v$ be the dimension of the radical of the associate bilinear symmetric form of $\Phi_i$. For any $\lambda \in \mathbb{F}_q$ the value of $D$ as defined in Theorem 2.3, is given by

$$
D = \begin{cases}
\chi(-\lambda) & \text{if } b = 1; \\
(-1)^{(q-1)(2b-v)}/4 & \text{if } b \geq 2 \text{ and } r + v \text{ is even}; \\
(-1)^{(q+1)/2} \cdot \chi\left(-\frac{\lambda}{2^{r-1}}\right) & \text{if } b \geq 2 \text{ and } r + v \text{ is odd}.
\end{cases}
$$

Proof. When $r = 2$ it follows that $i = 1$ and $\text{Tr}(\alpha^{q+1} - \alpha^2) = \lambda$ is equivalent to

$$
\alpha^{q^2+q} - \alpha^{2q} + \alpha^{q+1} - \alpha^2 = \lambda,
$$

which can be written as $(\alpha^q - \alpha)^2 = -\lambda$. If $\lambda = 0$ that relation is equivalent to $\alpha^q - \alpha = 0$, and then $\alpha \in \mathbb{F}_q$, in which case we have $q$ solutions. For $\lambda \in \mathbb{F}_q$, let us consider the following maps:

$$
\psi : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2} \quad \text{and} \quad \tau : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2},
$$

$$
x \mapsto x^2 - x \quad \text{and} \quad x \mapsto x^2.
$$

In order to determine the number of solutions of Equation (25) it is enough to fix $a \in \mathbb{F}_{q^2}$ such that $\tau(\psi(a)) = -\lambda$. Let $\{1, \alpha\}$ be a basis of $\mathbb{F}_{q^2}/\mathbb{F}_q$. The image of $\{1, \alpha\}$ by $\psi$ is $\{0, \beta\}$, where $\beta = \alpha^q - \alpha$. Since $\ker(\psi) = \mathbb{F}_q$, the image of $\psi$ is generated by $\beta$. Therefore it is sufficient to consider the elements of the form $c\alpha$, with $c \in \mathbb{F}_q$, i.e.,

$$
\tau(\psi(\alpha c)) = \tau(c\beta) = c^2\beta^2.
$$

We now claim that $\beta \notin \mathbb{F}_q$. For that, suppose by contradiction that $\beta^q = \beta$. Then

$$
\alpha^{q^2} - \alpha^q - \alpha^q + \alpha = 0
$$

which only happens if $-2(\alpha^q - \alpha) = 0$. But this is not possible because $\alpha \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ and $p \neq 2$. Consequently $\tau(\psi(\alpha c)) = -\lambda$ if and only if $c^2\beta^2 = -\lambda$, and since $\beta \notin \mathbb{F}_q$, this equation has solutions if and only if $-\lambda$ is not a square in $\mathbb{F}_q$. In this case, Equation (25) has $2q$ solutions.

Now we consider the case when $r = 2^b$ with $b > 1$. Let $i = p^s\bar{t}$, where $\gcd(p, \bar{t}) = 1$. From Lemma 4.3 we know that $v = \gcd(2^b, \bar{t}) \min(1, 2^s) = \gcd(2^b, \bar{t})$, which implies that $v$ is of the form $2^w$ with $0 \leq c \leq b$. Let $M_{\lambda}$ be the set of solutions of $\text{Tr}(x^{q^i+1} - x^2) = \lambda$ in $\mathbb{F}_{q^2}$. We now consider two cases.

1. $r + v$ is even.

In this case $v$ and $i$ are even. The number of solutions of $\text{Tr}(x^{q^i+1} - x^2) = \lambda$ is given by Equation (6). If $\text{Tr}(\alpha^{q^i+1} - \alpha^2) = \lambda$ for some $\alpha \in \mathbb{F}_{q^2}$, we have

$$
\text{Tr}((\alpha^{q^i})^{q^j+1} - (\alpha^{q^i})^2) = \text{Tr}(\alpha^{q^{i+1}} - \alpha^2)^q = \text{Tr}(\alpha^{q^{i+1}} - \alpha^2) = \lambda,
$$

for each $0 \leq j \leq r - 1$. Since $r = 2^b \geq 4$ and by Equation (26), for each $\alpha \in M_{\lambda}$ we can associate another $d - 1$ elements of $M_{\lambda}$, where $d$ is the smallest positive divisor of $r = 2^b$ such that $\alpha^{q^i} = \alpha$. We claim that $d > 2$. In fact, if $d = 1$ we have $(\alpha^{q^i+1} - \alpha^2)^q = \alpha^2 - \alpha^2 = 0$, and this implies $\lambda = 0$, which is a contradiction. In the case $d = 2$, for $\alpha \in \mathbb{F}_{q^2}$ we have $(\alpha^{q^i+1} - \alpha^2)^q = \alpha^{q^i+1} - \alpha^2 = \alpha^2 - \alpha^2 = 0$, since $i$ is even. This relation also implies $\lambda = 0$, which is also a contradiction. Then Equation (26) does not have solutions in these cases. Consequently $d > 2$, which implies that 4 divides $d$ for any $\alpha \in M_{\lambda}$ and seen Equation (6) of Theorem 2.3, we obtain the relation

$$
q^{2^b-1} - Dq^{(2^b+2)/2} \equiv 0 \pmod{4},
$$

where $D = \text{gcd}(2^b, \bar{t}) \min(1, 2^s)$. This completes the proof.
which is equivalent to
\[ D \equiv q^{2^k - 1 - (2^k + v - 2)/2} \equiv q^{(2^k - v)/2} \pmod{4}. \]

We conclude that
\[ D = (-1)^{(q-1)/(2^k - v)/4}. \]

(2) $r + v$ is odd.

In this case $v$ is an odd divisor of $2^k$ and therefore $v = 1$. By the same argument used in the previous case the number of solutions of $\text{Tr}(\alpha q^{i+1} - \alpha^2) = \lambda$ is given by Equation (7). Furthermore, it follows that for $\lambda \in \mathbb{F}_q^*$ we have $\text{Tr}(\alpha q^{i+1} - \alpha^2) = \lambda$ if and only if
\[ \text{Tr}((\alpha q^i)^{q^i+1} - (\alpha q^i)^2) = \text{Tr}((\alpha q^{i+1} - \alpha^2)q^i) = \text{Tr}(\alpha q^{i+1} - \alpha^2) = \lambda, \]
for all $0 \leq j \leq r - 1$. Therefore for each $\alpha \in M_{\lambda}$ we can associate another $d - 1$ elements of $S_{\lambda}$, where $d$ is the smallest divisor of $r = 2^k \geq 4$ such that $\alpha d^i = \alpha$. The case $d = 1$ does not happen, otherwise we would have $\lambda = 0$.

Suppose now that $\alpha \in \mathbb{F}_{q^2} \cap M_{\lambda} \subseteq \mathbb{F}_q$. We then have
\[ \lambda = \text{Tr}(\alpha q^{i+1} - \alpha^2) = 2^{b-1} \cdot ((\alpha q^{i+1} + q - \alpha 2q) + \alpha q^{i+1} - \alpha^2). \]
As in the previous case, Equation (28) does not have solution in $\mathbb{F}_q$. Therefore $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and the equation
\[ \lambda = 2^{b-1} \cdot (\alpha q^{i+1} + \alpha q^{i+1} - \alpha 2q - \alpha^2) \]
can be written as $(\alpha^q - \alpha)^2 = \gamma$, where $\gamma = -\frac{\lambda}{2^b - 1}$. Using the same argument of item (i) of Lemma 4.6 it follows that $(\alpha^q - \alpha)^2 = \gamma$ has solutions in $\mathbb{F}_{q^2}$ if and only if $\gamma$ is not a square in $\mathbb{F}_q$, which is equivalent to $-\frac{\lambda}{2^b - 1}$ not being a square in $\mathbb{F}_q$, and in this case we have $2^b$ solutions for Equation (25) in $\mathbb{F}_{q^2}$. Consequently the number of solutions of Equation (28) in $\mathbb{F}_{q^2}$ is $(1 - \chi(-\frac{\lambda}{2^b - 1})) q$ and then
\[ S_{\lambda} = \left(1 - \chi \left(-\frac{\lambda}{2^b - 1}\right)\right) q \equiv 0 \pmod{4}. \]

By Theorem 2.3, it follows that
\[ q^{2^k - 1} + D q^{(2^k + v - 1)/2} \equiv \left(1 - \chi \left(-\frac{\lambda}{2^b - 1}\right)\right) q \pmod{4}, \]
i.e.,
\[ D \equiv \left(1 - \chi \left(-\frac{\lambda}{2^b - 1}\right)\right) q^{1 - (2^k + v - 1)/2} - q^{(2^k - v - 1)/2} \pmod{4}. \]
Therefore
\[ D \equiv \left(1 - \chi \left(-\frac{\lambda}{2^b - 1}\right)\right) q^{2^k - 2^k - 1} \equiv q^{(2^k - v - 1)/2} \pmod{4}. \]
Since $v = 1$ and $q^2 \equiv 1 \pmod{4}$ we conclude that
\[ D \equiv -q^{2^k - 1} \cdot \chi \left(-\frac{\lambda}{2^b - 1}\right) \equiv -q \cdot \chi \left(-\frac{\lambda}{2^b - 1}\right) \pmod{4}. \]
and consequently
\[ D = (-1)^{(q+1)/2} \cdot \lambda \left( -\frac{\lambda}{2^b-1} \right). \]

The case \( \lambda = 0 \) follows from Theorem 2.3, which tells us that \( D \) is the same for any \( \lambda \in \mathbb{F}_q \), if \( r + v \) is even. In the case when \( r + v \) is odd, we have \( D = 0 \). \( \square \)

The following definitions are helpful to allow us to rewrite the expressions of \( D \) in a more simpler way.

**Definition 4.7.** For each \( \lambda \in \mathbb{F}_q \) we define
\[ \varepsilon_\lambda = \begin{cases} q - 1 & \text{if } \lambda = 0 \\ -1 & \text{otherwise.} \end{cases} \quad \text{and} \quad \varepsilon'_\lambda = \begin{cases} 0 & \text{if } \lambda = 0 \\ -1 & \text{otherwise.} \end{cases} \]

In the following theorem we use Theorem 2.3 and Lemma 4.6, to determine the value of \( S_\lambda \).

**Theorem 4.8.** Let \( b,i,r \) be integers such that \( 0 < i < r \) and \( r = 2^b \). For \( \lambda \in \mathbb{F}_q \), the number of solutions \( S_\lambda \) of \( \Phi_i(x) = \lambda \) in \( \mathbb{F}_{q^r} \) is given by
\[
S_\lambda = \begin{cases} (1 - \chi(\lambda))q & \text{if } b = 1; \\
q^{2^b-1} + (-1)(q-1)(2^{b-v}/4q^{2^{b+v-2}})\varepsilon_\lambda & \text{if } b \geq 2 \text{ and } r + v \text{ is even}; \\
q^{2^b-1} + (-1)(q-1)/2 : \chi \left( -\frac{\lambda}{2^b-1} \right) q^{(2^b+v-1)/2}\varepsilon'_\lambda & \text{if } b \geq 2 \text{ and } r + v \text{ is odd},
\end{cases}
\]
where \( v = \gcd(2^b,i) \) is the dimension of the radical of the bilinear symmetric form associated to \( \Phi_i \).

The results obtained in Lemmas 4.4 and 4.8 can be used inductively to obtain the following result for extensions of degree \( r \) satisfying \( \gcd(r,p) = 1 \).

**Theorem 4.9.** Let \( b,i,r \) be integers such that \( 0 < i < r \), \( r = 2^b \tilde{r} \), \( \tilde{r} = p_1^{\nu_1} \cdots p_n^{\nu_n} \) the prime factorization of \( \tilde{r} \) and \( \gcd(\tilde{r},2p) = 1 \). For \( \lambda \in \mathbb{F}_q \), the number of solutions of \( \Phi_i(x) = \lambda \) in \( \mathbb{F}_{q^r} \) is
\[
S_\lambda = \begin{cases} q^{r-1} + \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max(0,\nu_{p_j}(r) - \nu_{p_j}(i))} q^{\frac{(r+1-\nu_0-\nu_1)}{2}} \varepsilon_\lambda & \text{if } i \text{ is even and } b = 1; \\
q^{r-1} + (-1)(q-1)(2^{b-v_1}/4q^{(r-v_1-2)/2})\varepsilon_\lambda & \text{if } i \text{ is even and } b \geq 2; \\
q^{r-1} + \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max(0,\nu_{p_j}(r) - \nu_{p_j}(i))} q^{(r+2\nu_0-2)/2}\varepsilon'_\lambda & \text{if } i \text{ is odd and } b \geq 2,
\end{cases}
\]
where \( \nu_0 = \gcd(\tilde{r},i) \) and \( v_1 = \gcd(2^b,i) \).

**Proof.** By Lemma 4.3 it follows that \( v = \gcd(r,i) \). We split the proof in cases.

(i) **Case \( i \) even and \( b = 1 \).** Using the transitivity of the trace function, we obtain
\[
\lambda = \Tr(\alpha^{q^{r+1}+1} - \alpha^2) = \Tr_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\Tr_{\mathbb{F}_{q^r}/\mathbb{F}_{q^2}}(\alpha^{q^{r+1}+1} - \alpha^2)),
\]
(29)
where \( \alpha \) is such that \( \Phi_i(\alpha) = \lambda \). Let \( \mu = \Tr_{\mathbb{F}_{q^r}/\mathbb{F}_{q^2}}(\alpha^{q^{r+1}+1} - \alpha^2) \in \mathbb{F}_{q^2} \). Then Equation (29) is equivalent to the system
\[
\begin{cases}
\Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\mu) = \lambda; \\
\Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\alpha^{q^{r+1}+1} - \alpha^2) = \mu.
\end{cases}
\]
Let us define $Q = q^2$, then $Q^\tilde{r} = q^{\tilde{r}}$. Since $b = 1$, by Lemma 4.4 it follows that the number of solutions of $\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(x^{q^r+1} - x^2)) = \mu$ is

$$Q^{\tilde{r}-1} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(\tilde{r}) - \nu_{p_j}(i)\}} Q^{\frac{r+q-2}{2}}.$$ 

The dimension of the radical of $\text{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x^{q^2+1} - x^2))$ is $v_0 = \text{gcd}(\tilde{r}, i)$. The number of solutions of $\text{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\mu) = \lambda$ is $q$ and therefore

$$S_\lambda = q \left( Q^{\tilde{r}-1} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(\tilde{r}) - \nu_{p_j}(i)\}} Q^{\frac{r+q-2}{2}} \right) = q \left( q^{2\tilde{r}-2} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(\tilde{r}) - \nu_{p_j}(i)\}} q^{\frac{2r+2q-4}{2}} \right) = q^{\tilde{r}-1} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(\tilde{r}) - \nu_{p_j}(i)\}} q^{\frac{r+2q-2}{2}}.$$

(ii) Case $i$ even and $b \geq 2$. As above, we have

$$\lambda = \text{Tr}(\alpha^{q^b+1} - \alpha^2) = \text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\alpha^{q^b+1} - \alpha^2)),$$

where $\alpha$ is such that $\Phi_i(\alpha) = \lambda$. Finding solutions of the Equation (30) is equivalent to finding solutions of the system

$$\begin{cases} 
\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\mu) = \lambda; \\
\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\alpha^{q^b+1} - \alpha^2) = \mu.
\end{cases}$$

Putting $Q = q^\tilde{r}$ we have that $Q^{2\tilde{r}} = q^r$. Since $b \geq 2$, it follows from Lemma 4.8(ii) that the number of solutions of $\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(x^{q^b+1} - x^2)) = \mu$ is

$$S_\mu = Q^{2\tilde{r}-1} - (-1)^{Q-1} \left( \frac{\tilde{r}-1}{2} \right) \frac{Q^{(\tilde{r}-1)(\tilde{r}-2)} - 1}{2},$$

where $v_1 = \text{gcd}(2b, i)$ is the dimension of the radical of $\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(x^{q^b+1} - x^2))$. Besides that, the number of solutions of $\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\mu) = \lambda$ is $q^{\tilde{r}-1}$ and the number of solutions of $\Phi_i(x) = \lambda$ is

$$q^{\tilde{r}-1}(Q^{2\tilde{r}-1} - (-1)^{Q-1} \left( \frac{\tilde{r}-1}{2} \right) \frac{Q^{(\tilde{r}-1)(\tilde{r}-2)} - 1}{2}) = q^{\tilde{r}-1}(q^{r-\tilde{r}} - (-1)^{Q-1} \left( \frac{\tilde{r}-1}{2} \right) \frac{Q^{(\tilde{r}-1)(\tilde{r}-2)} - 1}{2}) \frac{1}{q^{r-\tilde{r}-2}} = q^{\tilde{r}-1} - (-1)^{Q-1} \left( \frac{\tilde{r}-1}{2} \right) \frac{Q^{(\tilde{r}-1)(\tilde{r}-2)} - 1}{2}.$$

(iii) Case $i$ odd. We define $Q = q^b$; then $Q^\tilde{r} = q^{\tilde{r}}$ and

$$\lambda = \text{Tr}(\alpha^{q^b+1} - \alpha^2) = \text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\alpha^{q^b+1} - \alpha^2)),$$

where $\alpha$ is such that $\Phi_i(\alpha) = \lambda$. It follows that the number of solutions of (31) is equal to the number of solutions of the system

$$\begin{cases} 
\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\mu) = \lambda; \\
\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(\alpha^{q^b+1} - \alpha^2)) = \mu.
\end{cases}$$
By Corollary 4.4 we have that the number of solutions of \( T_{\mathbb{F}_q^r} \cap \mathbb{F}_q \left(x^{q^r+1} - x^2\right) = \mu \) with \( \mu \neq 0 \) is
\[
Q^{\bar{r}-1} \cdot \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(r) - \nu_{p_j}(i)} Q^{(r + q - 2)},
\]
where \( \nu_0 = \gcd(\bar{r}, i) \) is the dimension of the radical of \( T_{\mathbb{F}_q^r} \cap \mathbb{F}_q \left(x^{q^r+1} - x^2\right) \). Since the number of solutions of \( T_{\mathbb{F}_q^r} \cap \mathbb{F}_q \left(x^{q^r+1} - x^2\right) \) is \( q^{4^r-1} \), we conclude that the number of solutions of \( \Phi_i(x) = \lambda \) is
\[
S_\lambda = q^{2^r-1} \left(Q^{\bar{r}-1} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(r) - \nu_{p_j}(i)} Q^{(r + q - 2)}\right)
= q^{2^r-1} \left(q^{r-2^r} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(r) - \nu_{p_j}(i)} Q^{(r + q - 2)}\right)
= q^{r-2^r} - \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(r) - \nu_{p_j}(i)} Q^{(r + q - 2)}\right),
\]
The case \( \lambda = 0 \) follows using the same ideas and analogous formulas for \( S_0 \), for extensions of \( \mathbb{F}_q \) of degree power of 2 and odd.

Using Lemma 4.4 and Theorem 4.9 we can determine the number of affine rational points of the curve \( y^q - y = x^{q^r+1} - x^2 - \lambda \), that we show as in the following theorem.

**Theorem 4.10.** Let \( i, r \) be integers such that \( 0 < i < r \). Let \( \bar{r} \) be an integer such that \( r = 2^\bar{r} \), \( \gcd(\bar{r}, 2p) = 1 \) and \( \bar{r} = p_1^{a_1} \cdots p_u^{a_u} \) the prime factorization. For \( F(x) = x^q - x \) and \( \lambda \in \mathbb{F}_q \) we have
\[
N_r(C_{F, \lambda}) = q^r + Dq^{(r + E)/2}
\]
where
\[
\begin{align*}
(i) \quad & D = \left(1 - \chi(-\lambda)\right)q^{\bar{r}+1} \epsilon_{x, \lambda} L = -\bar{r} \gcd(2^\bar{r}, i) \text{ if } b = 1 \text{ and } i \text{ is even}; \\
(ii) \quad & D = \left(-1\right)^{(q-1)(2^\bar{r} - \gcd(2^\bar{r}, i))}/4 \epsilon_{x, \lambda} L = -\bar{r} \gcd(2^\bar{r}, i) \text{ if } b \geq 2 \text{ and } i \text{ is even}; \\
(iii) \quad & D = \prod_{j=1}^{u} \left( \frac{q}{p_j} \right)^{\max\{0, \nu_{p_j}(r) - \nu_{p_j}(i)} \epsilon_{x, \lambda} L = 2^b \gcd(\bar{r}, i) \text{ if } b = 0 \text{ or } b \geq 1 \text{ and } i \text{ is odd}.
\end{align*}
\]

5. Some open problems

We finish this paper enumerating some open problems. We note that in Theorem 3.2 we determined the number of affine rational points in \( \mathbb{F}_q^2 \) of \( C : y^q - y = xF(x) - \lambda \) when \( F(x) \) is a \( \mathbb{F}_q \) linearized and such that \( g(x) = \gcd(f(x), x^r - 1) \) is a self-reciprocal, where \( f(x) \) is the associated to \( F(x) \). We then have two problems:

**Problem 1.** Determine \( N_r(C_{F, \lambda}) \) when \( F(x) \) is not a \( \mathbb{F}_q \)-linearized.

**Problem 2.** Determine \( N_r(C_{F, \lambda}) \) when \( g(x) \) is not a self-reciprocal.

In Section 4, we only considered extensions of degree \( r \) such that \( \gcd(r, p) = 1 \). This is the content of the next problem.

**Problem 3.** Determine explicitly \( N_r(C_{F(x), \lambda}) \), when \( \lambda \in \mathbb{F}_q \) and \( F(x) \) is a \( \mathbb{F}_q \) linearized and such that \( \gcd(r, p) = p \).
In [9], the authors show that the number of monic irreducible polynomials in \( \mathbb{F}_q[x] \) of degree \( r \) and with the first and third coefficients prescribed is related to the curve
\[
y^q - y = x^{2q+1} - x^{q+2}.
\]
Then, we have the following problem.

**Problem 4.** Determine \( N_r(C_{FG}) \) when \( F(x), G(x) \in \mathbb{F}_q[x] \) are \( \mathbb{F}_q \)-linearized.

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