SEMISIMPLICITY OF THE LYAPUNOV SPECTRUM FOR
IRREDUCIBLE COCYCLES

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ABSTRACT. Let $G$ be a semisimple Lie group acting on a space $X$, let $\mu$ be a
compactly supported measure on $G$, and let $A$ be a strongly irreducible linear
cocycle over the action of $G$. We then have a random walk on $X$, and let $T$ be the
associated shift map. We show that the cocycle $A$ over the action of $T$ is conjugate
to a block conformal cocycle.

This statement is used in the recent paper by Eskin-Mirzakhani on the classifica-
tion of invariant measures for the $SL(2, \mathbb{R})$ action on moduli space. The ingredients
of the proof are essentially contained in the papers of Guivarch and Raugi and also
Goldsheid and Margulis.

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1. Introduction

1.1. Statement of the main results. Let $G$ be a locally compact second countable
group. Denote by $\mu$ a symmetric compactly supported probability measure on $G$.

Let $X$ be a space where $G$ acts and denote by $\nu$ a $\mu$-stationary measure (that is,
$\mu \ast \nu = \nu$ where $\mu \ast \nu := \int_G g_*\nu \mu(g)$). We assume that $\nu$ is $\mu$-ergodic.

Consider $L$ a real finite-dimensional vector space and $A : G \times X \to SL(L)$ a (linear)
cocycle. Since it is sufficient for our purposes, we will assume that $A(g, x)$ is bounded
for $g$ in the support of $\mu$. Denote by $H$ the algebraic hull of $A(\cdot, \cdot)$ in Zimmer’s

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sense, that is, the smallest linear $\mathbb{R}$-algebraic subgroup $\mathbf{H}$ such that there exists a measurable map $C : X \to SL(L)$ with $C(g(x))A(g, x)C(x)^{-1} \in \mathbf{H}$ for $\mu$-almost all $g \in G$ and $\nu$-almost all $x \in X$. In what follows, we will assume that $\mathbf{H}$ is a $\mathbb{R}$-simple Lie group with finite center, and a basis of $L$ is (measurably) chosen at each $x \in X$ so that the cocycle $A(\cdot, \cdot)$ takes its values in $\mathbf{H}$.

**Definition 1.1.** We say that the cocycle $A(\cdot, \cdot)$ has an *invariant system of subspaces* if there exists a measurable family $W(x)$ of subspaces of $L$ such that $A(g, x)W(x) \subset W(g(x))$ for $\mu$-almost every $g \in G$ and $\nu$-almost every $x \in X$.

**Definition 1.2** (Strong irreducibility). We say that $A(\cdot, \cdot)$ is *strongly irreducible* if the natural cocycles induced by $A(\cdot, \cdot)$ on finite covers of $X$ do not have non-trivial and proper invariant systems of subspaces.

We will be interested in the behavior of a strongly irreducible cocycle $A(\cdot, \cdot)$ on the Lyapunov subspaces obtained after multiplying the matrices $A(g, x)$ while following a random walk on $G$. For this reason, let us introduce the following objects.

Let $\Omega = G^\mathbb{N}$. Denote by $T : \Omega \times X \to \Omega \times X$ the natural forward shift map on $\Omega \times X$:

$$T(u, x) = (\sigma(u), u_1(x))$$

where $\sigma(u) = (u_2, \ldots)$ for $u = (u_1, u_2, \ldots) \in \Omega$. Denoting by $\beta = \mu^\mathbb{N}$ the probability measure on $\Omega$ naturally induced by $\mu$, it follows from the fact that $\nu$ is $\mu$-stationary that the probability measure $\beta \times \nu$ is $T$-invariant.

As we already mentioned above, from now on, we will assume that the stationary measure $\nu$ is $\mu$-ergodic, that is, $\beta \times \nu$ is $T$-ergodic.

In this language, we can study the products of matrices of the cocycle $A(\cdot, \cdot)$ along random walks with the aid of the cocycle dynamics $F_A : \Omega \times X \times H \to \Omega \times X \times H$ naturally associated to $A(\cdot, \cdot)$:

$$F_A(u, x, h) = (T(u, x), A(u_1, x)h)$$

Actually, for our purposes, the “fiber dynamics” of $F_A$ will be more important than the base dynamics $T$. For this reason, given $u \in \Omega$ and $x \in X$, let us denote by $A^n(u, x)$ the matrix given by the formula:

$$F^n_A(u, x, \text{Id}) = (T^n(u, x), A(u_n, u_{n-1} \ldots u_1(x)) \ldots A(u_1, x)) =: (T^n(u, x), A^n(u, x))$$

In this context, the multiplicative ergodic theorem of V. Oseledets [Os] says that, if $\int \log^+ \|A(g, x)\| d\mu(g) d\nu(x) < \infty$, then there is a collection of numbers $\lambda_1 > \cdots > \lambda_k$ with multiplicities $m_1, \ldots, m_k$ called *Lyapunov exponents* and, at $\beta \times \nu$-almost every point $(u, x) \in \Omega \times X$, we have a *Lyapunov flag*

$$\{0\} = V_{k+1}^+ \subset V_k^+(u, x) \subset \cdots \subset V_1^+(u, x) = L$$

\[1\] Recall that the algebraic hull is unique up to conjugation (cf. Zimmer’s book [Zi]).
such that $V^+_i(u, x)$ has dimension $m_i + \cdots + m_k$ and
\[
\lim_{n \to \infty} \frac{1}{n} \log \|A^n(u, x)\vec{p}\| = \lambda_i
\]
whenever $\vec{p} \in V^+_i(u, x) \setminus V^+_{i+1}(u, x)$.

In this paper, we will study the consequences of the strong irreducibility of a cocycle for its Lyapunov spectrum (i.e., collection of Lyapunov exponents and flags). In particular, we will focus on the following property:

**Definition 1.3.** We say that $F_A$ or simply $A(\cdot, \cdot)$ has semisimple Lyapunov spectrum if its algebraic hull $H$ is block-conformal in the sense that, for each $i = 1, \ldots, k$, $V^+_i(u, x)/V^+_{i+1}(u, x)$ possesses an invariant splitting,

\[
V^+_i(u, x)/V^+_{i+1}(u, x) = \bigoplus_{j=1}^{n_i} E_{ij}(u, x),
\]

and on each $E_{ij}(u, x)$ there exists a (non-degenerate) quadratic form $\langle \cdot, \cdot \rangle_{ij,u,x}$ such that, for all $\vec{p}, \vec{q} \in E_{ij}(u, x)$ and for all $n \in \mathbb{N},$

\[
\langle A^n(u, x)\vec{p}, A^n(u, x)\vec{q} \rangle_{ij,T^n(u,x)} = e^{\lambda_{ij}(u,x,n)} \langle \vec{p}, \vec{q} \rangle_{ij,u,x}
\]

for some cocycle $\lambda_{ij} : \Omega \times X \times \mathbb{N} \to \mathbb{R}$.

The main result of this paper is the following theorem:

**Theorem 1.4.** If $A(\cdot, \cdot)$ is strongly irreducible, then it has semisimple Lyapunov spectrum.

Furthermore, the top Lyapunov exponent corresponds to a single conformal block, that is, for $\beta \times \nu$-a.e. $(u, x)$ there are a (non-degenerate) quadratic form $\langle \cdot, \cdot \rangle_{u,x}$ and a cocycle $\lambda : \Omega \times X \times \mathbb{N} \to \mathbb{R}$ such that

\[
\langle A^n(u, x)\vec{p}, A^n(u, x)\vec{q} \rangle_{T^n(u,x)} = e^{\lambda(u,x,n)} \langle \vec{p}, \vec{q} \rangle_{u,x}
\]

for all $\vec{p}, \vec{q} \in V^+_1(u, x)/V^+_2(u, x)$.

In fact, the ingredients of the proof of this result are essentially contained in the articles of Goldshheid-Margulis [GM] and Guivarc’h-Raugi [GR1], [GR2]. In particular, the fact that such a result holds is no surprise to the experts.

Nevertheless, we decided to write down a proof of this theorem here mainly for two reasons: firstly, this precise statement is hard to locate in these references, and, secondly, this result is relevant in the recent paper [EMi] where a Ratner-type theorem is shown for the action of $SL(2, \mathbb{R})$ on moduli spaces of Abelian differentials.

1.2. The backwards cocycle. As it turns out, for the application in Eskin-Mirzakhani paper [EMi], one needs the analog of Theorem 1.4 for the backward shift.

More precisely, let $\Omega^- = G^{\mathbb{Z}^-} \times \Omega$ and $\tilde{\Omega} = \Omega^- \times \Omega$. Denote by $T^- : \Omega^- \times X \to \Omega^- \times X$ the natural backward shift map on $\Omega^- \times X$:

\[
T^-(v, y) = (\sigma^-(v), v_0^{-1}(y))
\]

That is, $\lambda_{ij}(u, x, m + 1) = \lambda_{ij}(T^{m-1}(u, x), 1) + \lambda_{ij}(u, x, m)$. 

where $\sigma^-(v) = (\ldots, v_{-1})$ for $v = (\ldots, v_0) \in \Sigma^-$. Similarly, denote by $\hat{T}: \hat{\Omega} \times X \to \hat{\Omega} \times X$ the natural forward shift map on $\hat{\Omega} \times X$:

$$\hat{T}(v, u, x) = (\hat{\sigma}(v, u), u_1(x))$$

where $\hat{\sigma}(v, u) = (c_i)_{i \in \mathbb{Z}}$ for $(v, u) = (c_i)_{i \in \mathbb{Z}}$.

Recall that $\Omega$ is equipped with the probability measure $\beta = \mu^N$, so that $\beta \times \nu$ is a $T$-invariant probability measure on $\Omega \times X$. Note that Borel measures on $\Omega$ and $\hat{\Omega}$ are uniquely determined by their values on cylinders. In particular, the natural projection $\pi_+: \hat{\Omega} \times X \to \Omega \times X$ induces a bijection $(\pi_+)_*$ between the spaces of $\hat{T}$-invariant and $T$-invariant Borel probability measures, and, a fortiori, there exists an unique probability measure $\hat{\beta} \times \nu$ on $\hat{\Omega} \times X$ projecting to $\beta \times \nu$ under $(\pi_+)_*$. In this context, the natural $T^-$-invariant probability measure $\hat{\beta}^X$ constructed in Lemma 3.1 of Benoist and Quint [BQ] is $\hat{\beta}^X = (\pi_-)_* \circ (\pi_+)_*^{-1}(\beta \times \nu) := (\hat{\beta} \times \nu)^{-}$, where $\pi_-: \Omega \times X \to \Omega^- \times X$ is the natural projection.

Similarly to the previous subsection, we can study the products of matrices of the cocycle $A(\cdot, \cdot)$ along backward random walks with the aid of the dynamical system $F^-_A: \Omega^- \times X \times H \to \Omega^- \times X \times H$ given by

$$F^-_A(v, y, h) = (T^-(v, y), A(v_0, v_0^{-1}(y))^{-1}h)$$

naturally associated to $A$, or, equivalently, the “fiber” dynamics $A^{-n}(v, y)$ given by the formula:

$$(F^-_A)^n(v, y, \text{Id}) = ((T^-)^n(v, y), A(v_{-n-1}, v_{-n-1}^{-1}\ldots v_0^{-1}(y))^{-1}\ldots A(v_0, v_0^{-1}(y))^{-1})$$

$$= ((T^-)^n(v, y), A^{-n}(v, y))$$

By Oseledec's multiplicative ergodic theorem, if $\int \log^+ \|A(g, x)^{\pm 1}\|d\mu(g)d\nu(x) < \infty$, then we have a Lyapunov flag

$$\{0\} = V_0^- \subset V_1^- (v, y) \subset \cdots \subset V_k^- (v, y) = L$$

such that $V_j^- (v, y)$ has dimension $m_1 + \cdots + m_j$ and $\lim_{n \to \infty} \frac{1}{n} \log \|A^{-n}(v, y)\overline{q}\| = -\lambda_j$ for $\overline{q} \in V_j^- (v, y) \setminus V_{j-1}^- (v, y)$, where $\lambda_i$ are the Lyapunov exponents of $F_A$ and $m_i$ are their multiplicities.

In this setting, we will show the following:

**Theorem 1.5.** Suppose that $A(\cdot, \cdot)$ is strongly irreducible. Then, $F^-_A$ has semisimple Lyapunov spectrum.

Furthermore, the largest Lyapunov exponent corresponds to a single conformal block, i.e., for $\beta^X = (\hat{\beta} \times \nu)^{-}$-a.e. $(v, y)$ there are a (non-degenerate) quadratic form $\langle \cdot, \cdot \rangle_{v,y}$ and a cocycle $\lambda: \Omega^- \times X \times \mathbb{R}$ such that

$$\langle A^{-n}(v, y)\overline{p}, A^{-n}(v, y)\overline{q} \rangle_{(T^-)^n(v, y)} = e^{\lambda(v, y, n)} \langle \overline{p}, \overline{q} \rangle_{v,y}$$

for all $\overline{p}, \overline{q} \in V_1^- (v, y)$. 

1.3. The invertible cocycle. Both Theorem 1.4 and Theorem 1.5 are derived as a consequence of a theorem about the two-sided walk. By Oseledets theorem applied to \( \hat{T} \), the flags (1.1) and (1.3) exist for \( \beta \times \nu \)-a.e. \( (v, u, x) \in \hat{\Omega} \times X \) (and, moreover, \( V_i^+(v, u, x) = V_i^+(u, x) \) and \( V_j^-(v, u, x) = V_j^-(v, x) \)).

Then for \( \beta \times \nu \)-a.e. \( (v, u, x) \in \hat{\Omega} \times X \), let us define

\[
V_\ell(v, u, x) = V_\ell^+(u, x) \cap V_\ell^-(v, x)
\]

for every \( 1 \leq \ell \leq k \). By [GM, Lemma 1.5], \( V_\ell(v, u, x) \) has dimension \( m_\ell \) and

\[
V_i^+(u, x) = \bigoplus_{\ell=i}^k V_\ell(v, u, x) \quad \text{and} \quad V_j^-(u, x) = \bigoplus_{\ell=1}^j V_\ell(v, u, x)
\]

In particular, for \( \beta \times \nu \)-a.e. \( (v, u, x) \),

\[
V_j^+(u, x)/V_{j+1}^+(u, x) \simeq V_j(v, u, x) / V_{j+1}^-(v, x)
\]

Therefore, the statements in Theorem 1.4 about the action of \( F_A^- \) on \( V_j^- (v, x)/V_{j-1}^- (v, x) \) and the corresponding statements in Theorem 1.5 about the action of \( F_A \) on \( V_j^+(u, x)/V_{j+1}^+(u, x) \) follow from the corresponding result for the two-sided walk:

**Theorem 1.6.** If \( A(,\ldots,) \) is strongly irreducible, then it has semisimple Lyapunov spectrum, in the sense that the restriction of \( A^n(v, u, x) \) to each \( V_i(v, u, x) \) is block-conformal.

Furthermore, the top Lyapunov exponent corresponds to a single conformal block, that is, for \( \beta \times \nu \)-a.e. \( (v, u, x) \) there are a (non-degenerate) quadratic form \( \langle \cdot, \cdot \rangle_{v,u,x} \) and a cocycle \( \lambda : \hat{\Omega} \times X \times \mathbb{N} \to \mathbb{R} \) such that

\[
\langle A^n(v, u, x)\vec{p}, A^n(v, u, x)\vec{q} \rangle_{T^n(v,u,x)} = e^{\lambda(v,u,x,n)} \langle \vec{p}, \vec{q} \rangle_{v,u,x}
\]

for all \( \vec{p}, \vec{q} \in V_1(v, u, x) \).

**Remark 1.7.** It is shown in [EM1, Appendix C] that if the algebraic hull \( H \) is the whole group \( SL(L) \), then all Lyapunov exponents are associated to single conformal blocks, i.e., for \( \beta \times \nu \)-a.e. \( (v, u, x) \in \Omega \times X \) and for each \( 1 \leq i \leq k \), there are a (non-degenerate) quadratic form \( \langle \cdot, \cdot \rangle_{i,v,u,x} \) and a cocycle \( \lambda_i : \Omega \times X \times \mathbb{N} \to \mathbb{R} \) such that

\[
\langle A^n(v, u, x)\vec{p}, A^n(v, u, x)\vec{q} \rangle_{i,T^n(v,u,x)} = e^{\lambda_i(v,u,x,n)} \langle \vec{p}, \vec{q} \rangle_{i,v,u,x}
\]

for all \( \vec{p}, \vec{q} \in V_i(v, u, x) \). Furthermore, analogous statements hold for the forward and backward walks.

The remainder of this paper is devoted to the proof of Theorem 1.6.
2. $\mathbb{R}$-simple Lie groups

Let $H$ be a $\mathbb{R}$-simple Lie group. We will always assume that $H$ is a linear algebraic group with finite center. Let $\theta$ denote a Cartan involution of $H$, and let $K$ denote the set of fixed points of $\theta$. Then, $K$ is a maximal compact subgroup of $H$.

Let $A$ denote a maximal $\mathbb{R}$-split torus of $H$ such that $\theta(A) = A$, and let $\Sigma$ denote the associated root system. Let $\Sigma^+$ denote the set of positive roots, and let $\Delta$ denote the set of simple roots. Let $B$ denote the Borel subgroup of $H$ corresponding to $\Sigma^+$.

Let $W$ denote the Weyl group of $(H, A)$. Let $A_+ := \{a \in A : \alpha(\log a) \geq 0 \text{ for all } \alpha \in \Sigma^+\}$.

We have the decomposition

$$H = K A_+ K.$$  \hfill (2.1)

If $g \in H$ is written as $g = k_1 a k_2$ where $k_1, k_2 \in K$ and $a \in A_+$, we write for $\alpha \in \Sigma^+$

$$\alpha(g) = \alpha(\log a).$$  \hfill (2.2)

We also have the Bruhat decomposition

$$H = \bigsqcup_{w \in W} B w B.$$  

Let $w_0 \in W$ be the longest root. Then, $B w_0 B$ is open and dense in $H$. Let

$$J \subset H/B$$ denote the complement of $B w_0 B / B$ in $H/B$. \hfill (2.3)

Given a subset $I \subset \Delta$, let $P_I$ denote the parabolic subgroup of $H$ associated to $I$. We have the Langlands decomposition

$$P_I = M_I A_I N_I,$$

where

$$A_I = \{a \in A : \alpha(\log a) = 0 \text{ for all } \alpha \in I\}.$$  

The group $M_I$ is semisimple, and commutes with $A_I$. The group $N_I$ is unipotent, and is normalized by $A_I N_I$.

For later use, we denote $N_I = w_0 N_I w_0^{-1}$ and let $J_I$ be the complement of $(B w_0 P_I) / P_I$ in $H / P_I$.

We will use the rest of this section to deduce some general properties of the actions of elements of $H$ on $H / P_I$. In particular, even though these properties help in the proof of Theorem 1.6, we decided to present them in their own section because they have nothing to do with the cocycle $A$ but only with the group $H$. 
2.1. A lemma of Furstenberg.

**Definition 2.1** \((\epsilon, \delta)\)-regular. Suppose \(\epsilon > 0\) and \(\delta > 0\) are fixed. A measure \(\eta\) on \(H/B\) is \((\epsilon, \delta)\)-regular if for any \(g \in H\),

\[ \eta(\text{Nbd}_\epsilon(gJ)) < \delta, \]

where \(J\) is as in (2.3). A measure \(\eta_I\) on \(H/P_I\) is \((\epsilon, \delta)\)-regular if for any \(g \in H\),

\[ \eta_I(\text{Nbd}_\epsilon(gJ_I)) < \delta, \]

where \(J_I\) is the complement of \((Bw_0P_I)/P_I\) in \(H/P_I\).

**Lemma 2.2.** Suppose \(I \subset \Delta\), \(g_n \in H\) is a sequence, and \(\eta_n\) is a sequence of uniformly \((\epsilon, \delta)\)-regular measures on \(H/P_I\). Suppose \(\delta \ll 1\). Write

\[ g_n = k_n a_n k'_n, \]

where \(k_n \in K\), \(k'_n \in K\) and \(a_n \in A_+\).

(a) Suppose \(I \subset \Delta\) is such that for all \(\alpha \in \Delta \setminus I\),

\[ \alpha(a_n) \to \infty. \]

Then, for any subsequential limit \(\lambda\) of \(g_n \eta_n\), we have

\[ k_n P_I \to k_\infty P_I \quad \text{and} \quad \lambda(\{k_\infty P_I\}) \geq 1 - \delta \]

for some element \(k_\infty \in K\).

(b) Suppose \(g_n \eta_n \to \lambda\) where \(\lambda\) is some measure on \(H/P_I\). Suppose also that there exists an element \(k_\infty\) such that \(\lambda(\{k_\infty P_I\}) > 5\delta\). Then, as \(n \to \infty\), (2.7) holds for all \(\alpha \in \Delta \setminus I\). As a consequence, by part (a), (2.5) holds and \(\lambda(\{k_\infty P_I\}) \geq 1 - \delta\).

**Proof of (a).** Without loss of generality, \(k'_n\) is the identity (or else we replace \(\eta_n\) by \(k'_n \eta_n\)).

Let \(\tilde{N}_I = w_0 N_I w_0^{-1}\). By our assumptions, for \(\tilde{n} \in \tilde{N}_I\),

\[ a_n \tilde{n} P_I = (a_n \tilde{n} a_n^{-1}) P_I \to P_I \quad \text{in} \ H/P_I. \]

For any \(z \in H/P_I\) such that \(z \not\in J_I\), we may write \(z = \tilde{n} P_I\) for some \(\tilde{n} \in \tilde{N}_I\). Therefore, \(d(g_n z, k_n P_I) \to 0\), where \(d(\cdot, \cdot)\) denotes some distance on \(H/P_I\). It then follows from the \((\epsilon, \delta)\)-regularity of \(\eta_n\) that (2.5) holds, and any limit of \(g_n \eta_n\) must give weight at least \(1 - \delta\) to \(k_\infty P_I\) (where \(k_n \to k_\infty\)).

**Proof of (b).** This is similar to [GM, Lemma 3.9]. There is a subsequence of the \(g_n\) (which we again denote by \(g_n = k_n a_n k'_n\)) such that for all \(\gamma \in \Delta\), either \(\gamma(a_n) \to \infty\) or \(\gamma(a_n)\) is bounded. After passing again to a subsequence, we may assume that \(k_n \to k_\infty\). Also, without loss of generality, we may assume that \(k'_n\) is the identity (or else we replace \(\eta_n\) by \(k'_n \eta_n\)).
Suppose there exists $\alpha \in \Delta \setminus I$ such that \([2.4]\) fails. Let $I' \subset \Delta$ denote the set of $\gamma \in \Delta$ such that, for $\gamma \in \Delta \setminus I'$, $\gamma(a_n) \to \infty$. Since we are assuming that $\alpha \in \Delta \setminus I$ and $\alpha \not\in \Delta \setminus I'$, we have $\Delta \setminus I \not\subset \Delta \setminus I'$, and thus $I' \not\subset I$.

Let $N_\alpha \subset N$ denote the subgroup obtained by exponentiating the root subspace $-\alpha$. We may write $N_I = N_\alpha N'$ for some subgroup $N'$ of $N$. Note that the action by left multiplication by $g_n$ on $H/P_I$ does not shrink the direction $N_\alpha$.

Write $k_\infty = n_\alpha \bar{n}$, where $n_\alpha \in N_\alpha$, $\bar{n}' \in N'$. Then, for $z \in H/P_I$, $g_n z$ does not converge to $k_\infty P_I$ unless if $z \in \bar{n}_\alpha N'P_I$ or $z \in J_I$. In particular, since $\bar{n}_\alpha N'P_I \subset \bar{n}_\alpha J_I$ (because $w_0 N' w_0^{-1} \in B w_0 w_0 B$), we obtain that if $g_n z$ converges to $k_\infty P_I$ then $z \in J_I \cup \bar{n}_\alpha J_I$.

On the other hand, since $\eta_n$ is $(\epsilon, \delta)$-regular,
\[ \eta_n(\text{Nbh}_{\delta}(J_I \cup \bar{n}_\alpha J_I)) < 2 \delta. \]

Therefore $\lambda(k_\infty P_I) < 3 \delta$ which is a contradiction. Thus $\alpha(g_n) \to \infty$ for all $\alpha \in \Delta \setminus I$. Now, by part (a), \([2.5]\) holds, and $\lambda(k_\infty P_I) \geq 1 - \delta$. \hfill \Box

2.2. The functions $\xi_\alpha(\cdot, \cdot)$ and $\hat{\sigma}_\alpha(\cdot, \cdot)$. Let $\omega_\alpha$ be the fundamental weight corresponding to $\alpha$, i.e. for $\gamma \in \Delta$,
\[ \langle \omega_\alpha, \gamma \rangle = \begin{cases} 1 & \text{if } \alpha = \gamma \\ 0 & \text{if } \gamma \in \Delta \setminus \{\alpha\}. \end{cases} \]

Then,
\[ \alpha = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \omega_\gamma. \quad (2.6) \]

We write
\[ \omega_\alpha(g) = \omega_\alpha(\log a), \quad \text{where } g = k_1 a k_2, \quad k_1, k_2 \in K, \quad a \in A_+. \quad (2.7) \]

Note that for all $\alpha \in \Delta$ and all $g \in H$,
\[ \alpha(g) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \omega_\gamma(g). \quad (2.8) \]

**Lemma 2.3.** For all $g_1 \in H$, $g_2 \in H$, and for all $\alpha \in \Delta$,

\[ \omega_\alpha(g_1 g_2) \leq \omega_\alpha(g_1) + \omega_\alpha(g_2). \quad (2.9) \]

and
\[ \omega_\alpha(g_1 g_2) \geq \omega_\alpha(g_1) - \omega_\alpha(g_2^{-1}). \quad (2.10) \]

**Proof.** There exists a representation $\rho_\alpha : H \to GL(V)$ such that its highest weight is $\omega_\alpha$. Let $\| \cdot \|$ be any $K$-invariant norm on $V$. Then, since $\omega_\alpha$ is the highest weight,
\[ \| \rho_\alpha(g) \| = \sup_{v \in V \setminus \{0\}} \frac{\| \rho_\alpha(g) v \|}{\| v \|} = e^{\omega_\alpha(g)}. \]
Since $\|\rho_\alpha(g_1 g_2)\| \leq \|\rho_\alpha(g_1)\|\|\rho_\alpha(g_2)\|$, (2.9) follows.

Now write $g_1 = h_1 h_2$, $g_2 = h_2^{-1}$, so that $g_1 g_2 = h_1$. Substituting into (2.9), we get

$$\omega_\alpha(h_1) \leq \omega_\alpha(h_1 h_2) + \omega_\alpha(h_2^{-1})$$

which immediately implies (2.10).

Let $P_\alpha$ be the parabolic subgroup corresponding to the subset $\Delta \setminus \{\alpha\} \subset \Delta$. We can write

$$P_\alpha = M_\alpha A_\alpha N_\alpha,$$

where

$$A_\alpha = \{a \in A : \gamma(\log a) = 0 \text{ for all } \gamma \in \Delta \setminus \{\alpha\}\}.$$

Note that $A_\alpha$ is one dimensional, and that $M_\alpha$ commutes with $A_\alpha$. We have the Iwasawa decomposition

$$H = K P_\alpha = K M_\alpha A_\alpha N_\alpha.$$

Let $P_\alpha^0 = M_\alpha N_\alpha$. If we decompose $g \in H$ as $g = k_m a_n n_\alpha$ with $k_m \in K$, $m_\alpha \in M_\alpha$, $a_\alpha \in A_\alpha$ and $n_\alpha \in N_\alpha$, then the decomposition is unique up to the transformation $k_m \to k_m m_1$, $m_\alpha \to m_\alpha^{-1} m_\alpha$ for $m_1 \in K \cap M_\alpha$. We can thus define the function

$$\xi_\alpha : H/P_\alpha^0 \to \mathbb{R}$$

by

$$\xi_\alpha(g) = \omega_\alpha(\log a), \quad \text{where } g = k m a n, k \in K, m \in M_\alpha, a \in A_\alpha \text{ and } n \in N_\alpha.$$

By definition, we have for $a \in A_\alpha$,

$$\xi_\alpha(ga) = \xi_\alpha(g) + \xi_\alpha(a). \quad (2.11)$$

We now define for $g \in H$, $z \in H/P_\alpha^0$,

$$\xi_\alpha(g, z) = \xi_\alpha(gz) - \xi_\alpha(z).$$

Then, in view of (2.11), for $a \in A_\alpha$, $\xi_\alpha(g, za) = \xi_\alpha(g, z)$. Thus, we may consider $\xi_\alpha(\cdot, \cdot)$ to be a function from $H \times H/P_\alpha$ to $\mathbb{R}$.

**Lemma 2.4.** We have for all $\alpha \in \Delta$:

(a) For all $g_1, g_2 \in H$,

$$\xi_\alpha(g_1 g_2, z) = \xi_\alpha(g_1, g_2 z) + \xi_\alpha(g_2, z).$$

(b) For all $g \in H$ and all $z \in H/P_\alpha$,

$$\xi_\alpha(g, z) \leq \omega_\alpha(g),$$

where $\omega_\alpha(g)$ is as defined in (2.7).

(c) For all $\epsilon > 0$ there exists $C = C(\epsilon) > 0$ such that for all $k_2 \in K$, for all $g \in K A^+ k_2$ and all $z \in H/P_\alpha$ with $d(k_2 z, (\bar{N}_\alpha P_\alpha)^c) > \epsilon$,

$$\xi_\alpha(g, z) \geq \omega_\alpha(g) - C.$$
Proof. Part (a) is clear from the definition of $\xi_\alpha(\cdot, \cdot)$. To show part (b), note that there exists a representation $\rho_\alpha : H \to GL(V)$ with highest weight $\omega_\alpha$. Let $\| \cdot \|$ be any $K$-invariant norm on $V$. Let $v_\alpha$ be the highest weight vector. Then $P_\alpha^0$ is the stabilizer of $v_\alpha$, and for all $g \in H$, 

$$\xi_\alpha(g) = \log \frac{\|\rho_\alpha(g)v_\alpha\|}{\|v_\alpha\|}. $$

As in the proof of Lemma 2.3, 

$$\sup_{v \in V \setminus \{0\}} \log \frac{\|\rho_\alpha(g)v\|}{\|v\|} = \omega_\alpha(g).$$

Then, part (b) of Lemma 2.4 follows.

To show part (c), write $g = k_1ak_2$, $k_1, k_2 \in K$, $a \in A_+$. Note that if $d(k_2z, (\alpha_\alpha(a_\alpha)\alpha_\alpha) > \epsilon$, then we can write 

$$k_2z = \alpha_\alpha a_\alpha,$$

with $d(\alpha_\alpha, \epsilon) \leq C(\epsilon)$. Then, $|\omega_\alpha(\alpha_\alpha)| < C(\epsilon)$. We have 

$$\xi_\alpha(g, z) = \xi_\alpha(k_1ak_2, z)$$

$$= \xi_\alpha(k_1a, k_2z)$$

$$= \xi_\alpha(k_1a, \alpha_\alpha)$$

$$= \xi_\alpha(a, \alpha_\alpha) - \xi_\alpha(\alpha_\alpha, \alpha_\alpha)$$

$$\geq \xi_\alpha(a, \alpha_\alpha) - C(\epsilon)$$

$$= \xi_\alpha(a) - \omega_\alpha(a) - C(\epsilon)$$

$$\geq \omega_\alpha(a) - 2C(\epsilon)$$

For $\alpha \in \Delta$, $g \in H/(MN)$, let 

$$\hat{\alpha}(g) = \alpha(a), \text{ where } g = kman, k \in K, m \in M, a \in A \text{ and } n \in N.$$

By definition, we have for $a \in A$, 

(2.12) 

$$\hat{\alpha}(ga) = \hat{\alpha}(g) + \hat{\alpha}(a).$$

We now define for $g \in H$, $z \in H/(MN)$, 

$$\hat{\alpha}(g, z) = \hat{\alpha}(gz) - \hat{\alpha}(z).$$

Then, in view of (2.12), for $a \in A$, $\hat{\alpha}(g, za) = \hat{\alpha}(g, z)$. Thus, we may consider $\hat{\alpha}(\cdot, \cdot)$ to be a function $H \times H/B \to \mathbb{R}$.

Lemma 2.5. We have for all $\alpha \in \Delta$: 
(a) For all \( g_1, g_2 \in H \) and \( z \in H/B \),
\[
\hat{\sigma}_\alpha(g_1g_2, z) = \hat{\sigma}_\alpha(g_1, g_2z) + \hat{\sigma}_\alpha(g_2, z).
\]

(b) For all \( \varepsilon > 0 \) there exists \( C = C(\varepsilon) > 0 \) such that for all \( k_2 \in K \), for all \( g \in KA, k_2 \) and all \( z \in H/B \) with \( d(k_2z, (\bar{N}_\alpha P_\alpha)^c) > \varepsilon \),
\[
\hat{\sigma}_\alpha(g, z) \geq \alpha(g) - C,
\]
where \( \alpha(g) \) is as defined in (2.2).

**Proof.** The natural map \( H/P_\alpha \rightarrow H/B \) allows us to consider the functions \( \xi_\alpha(\cdot, \cdot) \) to be functions \( H \times H/B \rightarrow \mathbb{R} \). Then, in view of (2.6), we have
\[
\hat{\sigma}_\alpha(g, z) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \xi_\alpha(g, z).
\]

Then (a) immediately follows from (a) of Lemma 2.4. Also,
\[
\hat{\sigma}_\alpha(g, z) = \langle \alpha, \alpha \rangle \xi_\alpha(g, z) + \sum_{\gamma \neq \alpha} \langle \alpha, \gamma \rangle \xi_\alpha(g, z)
\]
\[
\geq \langle \alpha, \alpha \rangle \xi_\alpha(g, z) + \sum_{\gamma \neq \alpha} \langle \alpha, \gamma \rangle \omega_\gamma(g) \quad \text{by Lemma 2.4(b) and since } \langle \alpha, \gamma \rangle \leq 0
\]
\[
\geq \langle \alpha, \alpha \rangle \omega_\alpha(g) - C(\varepsilon) + \sum_{\gamma \neq \alpha} \langle \alpha, \gamma \rangle \omega_\gamma(g) \quad \text{by Lemma 2.4(c)}
\]
\[
= \alpha(g) - C(\varepsilon).
\]
This completes the proof of (b). \( \square \)

### 3. Cocycles with values in \( \mathbb{R} \)-simple Lie groups

Let \( A : G \times X \rightarrow SL(L) \) be a linear cocycle satisfying the properties described in §1 above. In particular, we will assume that \( A(\cdot, \cdot) \) takes values in its algebraic hull \( H \). Furthermore, we will suppose that \( H \) is a \( \mathbb{R} \)-simple Lie group with finite center.

For \( \alpha \in \Delta \), let
\[
\lambda_\alpha \equiv \limsup_{n \to +\infty} \frac{1}{n} \alpha(A^n(u, x))
\]

By (2.8) and Lemma 2.3 the map
\[
(u, x, n) \rightarrow \lambda_\alpha(A^n(u, x))
\]
is a linear combination of subadditive cocycles.

Therefore, by the subadditive ergodic theorem, the limsup is actually a limit. Also, by the ergodicity of \( T \), \( \lambda_\alpha \) is constant a.e. on \( \Omega \times X \).
From now on, let us fix \( I \subset \Delta \) minimal such that for a \( \beta \times \nu \)-positive measure set of \((u, x) \in \Omega \times X\), we have

\[
I = \{ \alpha \in \Delta : \lambda_\alpha = 0 \}.
\]

Thus, for all \( \alpha \in \Delta \setminus I \), \( \lambda_\alpha > 0 \).

We will deduce Theorem 1.6 from the following:

**Theorem 3.1.** Let \( I \subset \Delta \) be as in (3.2). Then, for almost all \((v, u, x) \in \hat{\Omega} \times X\) there exists \( C(v, u, x) \in H \) such that almost all \((v, u, x)\) we may write

\[
C(T^n(v, u, x))^{-1}A^n(v, u, x)C(v, u, x) = k_n(v, u, x)a_n(u, v, x),
\]

where \( k_n(v, u, x) \in K \cap M_I \) and \( a_n(v, u, x) \in A_I \), and for all \( \alpha \in \Delta \setminus I \),

\[
\lim_{|n| \to \infty} \frac{1}{n} \alpha(\log a_n(v, u, x)) = \lambda_\alpha > 0
\]

where \( \lambda_\alpha \) is as in (3.1).

Let \( w_0 \in W \) be the longest root. Let \( I' \subset \Delta \) be defined by:

\[
I' = \{-w_0\alpha w_0^{-1} : \alpha \in I\}.
\]

Theorem 3.1 will be deduced from the following results.

**Proposition 3.2.** Let \( I' \subset \Delta \) be as in (3.3). Then,

(a) For almost all \((u, x) \in \Omega \times X\) there exists \( C^+(u, x) \in H \) such that for all \( n \) and almost all \((u, x)\),

\[
C^+(T^n(u, x))^{-1}A^n(u, x)C^+(u, x) \in P_I.
\]

(b) For almost all \((v, x) \in \Omega^+ \times X\) there exists \( C^-(v, x) \in H \) such that for all \( n \) and almost all \((v, x)\),

\[
C^-(T^{-n}(v, x))^{-1}A^{-n}(v, x)C^-(v, x) \in P_I.
\]

(c) For almost all \((v, u, x) \in \hat{\Omega} \times X\),

\[
C^+(u, x)^{-1}C^-(v, x) \in P_I w_0 P_I.
\]

We note that Proposition 3.2 is essentially a restatement of the Osceledets multiplicative ergodic theorem in this context. The standard proof (see e.g. [GM]) is based on the subadditive ergodic theorem. We give a proof in [4] below based on the martingale convergence theorem. Parts of this proof will be used again in the proof of Theorem 3.1 in [5].
4. Proof of Proposition 3.2

4.1. A Zero One Law. Let \( \nu \) be an ergodic stationary measure on \( X \). Let \( \hat{X} = X \times H/B \). We then have an action of \( G \) on \( \hat{X} \), by

\[
g \cdot (x, z) = (gx, A(g, x)z).
\]

Let \( \hat{\nu} \) be a \( \mu \)-stationary measure on \( \hat{X} \) which projects to \( \nu \) under the natural map \( \hat{X} \to X \). (At the moment we do not assume that \( \hat{\nu} \) is \( \mu \)-ergodic). We may write

\[
d\hat{\nu}(x, z) = d\nu(x)\, d\eta_x(z),
\]

where \( \eta_x \) is a measure on \( H/B \).

Lemma 4.1 (cf. [GM, Lemma 4.2], cf. [EMi, Lemma C.10], cf. [GR1, Théorème 2.6]). For almost all \( x \in X \) and any \( g \in H \),

\[
\eta_x(gJ) = 0,
\]

where \( J \) is defined in (2.3).

Proof. Let \( d \) be the smallest number such that there exists a subset \( E \subset X \) with \( \nu(E) > 0 \) and for all \( x \in E \) an irreducible algebraic subvariety \( J_x \subset H/B \) of dimension \( d \) with \( \eta_x(J_x) > 0 \). For \( x \in E \), let \( S(x) \) denote the set of irreducible algebraic subvarieties of \( H/B \) of dimension \( d \) such that for \( Q \in S(x) \), \( \eta_x(Q) > 0 \).

Note that for a.e. \( x \in X \), for any \( Q_1 \in S(x) \), \( Q_2 \in S(x) \) with \( Q_2 \neq Q_1 \),

\[
\eta_x(Q_1 \cap Q_2) = 0.
\]

(since \( Q_1 \cap Q_2 \) is an algebraic subvariety of dimension lower than \( d \)). Thus

\[
\sum_{Q \in S(x)} \eta_x(Q) \leq 1.
\]

Therefore \( S(x) \) is at most countable. Moreover, by setting

\[
(4.1) \quad f(x) = \max_{Q \in S(x)} \eta_x(Q)
\]

and \( S_{\max}(x) := \{Q \in S(x) : \eta_x(Q) = f(x)\} \), we see that \( S_{\max}(x) \) is finite.

Consider the measurable subset \( S_{\max} = \{(x, z) \in \hat{X} : z \in S_{\max}(x)\} \subset \hat{X} = X \times H/B \). By definition, for each \( x \in X \), the fiber \( \{z \in H/B : (x, z) \in E\} := S_{\max}(x) \) of \( S_{\max} \) at \( x \) is a finite set, and, in particular, \( S_{\max}(x) \) is a countable union of compact sets. By a result of Kallman (see, e.g., the statement of Theorem A.5 in Appendix A of Zimmer’s book [Zi]), we can find a Borel measurable section for the restriction to \( S_{\max} \) of the natural projection \( \pi : \hat{X} \to X \). In other words, one has a Borel measurable map \( X \ni x \mapsto Q_x^{(1)} \in S_{\max}(x) \) whose graph \( E_1 := \{(x, Q_x^{(1)}) \in \hat{X} : x \in X\} \) is a measurable subset of \( S_{\max} \). If \( E_1 = S_{\max} \), we are done. Otherwise, we apply once more Kallman’s theorem to \( S_{\max} - E_1 \) in order to obtain a measurable subset \( E_2 \) of \( S_{\max} - E_1 \) given by the graph of a Borel measurable map \( \pi(E_2) := \{y \in X : \).
\#S_{\text{max}}(y) \geq 2 \} \ni x \mapsto Q^{(2)}(x) \in S_{\text{max}}(x) \) that we extend (in a measurable way) to \( X \) by setting \( Q^{(2)}(x) := Q^{(1)}(x) \) whenever \( \#S_{\text{max}}(x) = 1 \). Since the fibers \( S_{\text{max}}(x) \) of \( S_{\text{max}} \) are finite sets, by iterating this procedure at most countably many times, we obtain a non-empty subset \( Z \subset \mathbb{N} \) and, for each \( m \in Z \), a Borel measurable map

\[ X \ni x \mapsto Q^{(m)}(x) \in S_{\text{max}}(x) \]

such that \( S_{\text{max}}(x) = \{ Q^{(1)}(x), \ldots, Q^{(\#S_{\text{max}}(x))}(x) \} \) for almost every \( x \in X \).

Fix \( m \in Z \). Since \( \hat{\nu} \) is \( \mu \)-stationary, we have \( \mu * \hat{\nu}(\text{graph}(Q^{(m)})) = \hat{\nu}(\text{graph}(Q^{(m)})) \), that is,

\[
\int_X \eta_x(Q^{(m)}(x)) d\nu(x) = \hat{\nu}(\text{graph}(Q^{(m)})) = \mu * \hat{\nu}(\text{graph}(Q^{(m)}))
\]

\[
= \int_G \int_X \chi_{\text{graph}(Q^{(m)})}(g, A(g, x)z) d\eta_x(z) d\nu(x) d\mu(g)
\]

\[
= \int_G \int_X \eta_x(A(g, x)^{-1}Q^{(m)}(gx)) d\nu(x) d\mu(g)
\]

\[
= \int_X \left( \int_G \eta_x(A(g, x)^{-1}Q^{(m)}(gx)) d\mu(g) \right) d\nu(x).
\]

On the other hand, since \( \eta_x(Q^{(m)}(x)) = f(x) = \max_{Q \in S(x)} \eta_x(Q) \), we see that

\[
\int_G \eta_x(A(g, x)^{-1}Q^{(m)}(gx)) d\mu(g) \leq f(x) = \eta_x(Q^{(m)}(x))
\]

By combining (1.2) and (1.3), we deduce that

\[
f(x) = \eta_x(Q^{(m)}(x)) = \eta_x(A(g, x)^{-1}Q^{(m)}(gx)),
\]

i.e., \( A(g, x)^{-1}Q^{(m)}(gx) \in S_{\text{max}}(x) \) for \( \mu \)-almost every \( g \) and \( \nu \)-almost every \( x \). In other terms, for all \( m \in Z \), \( \mu \)-almost every \( g \) and \( \nu \)-almost every \( x \), one has \( Q^{(m)}(gx) \in A(g, x)S_{\text{max}}(x) \). By putting together this inclusion with the facts that \( S_{\text{max}}(y) = \{ Q^{(1)}(y), \ldots, Q^{(\#S_{\text{max}}(y))}(y) \} \) for \( \nu \)-almost every \( y \) and \( \nu \) is \( \mu \)-stationary, one has that \( S_{\text{max}}(gx) \subset A(g, x)S_{\text{max}}(x) \) for \( \mu \)-almost every \( g \) and \( \nu \)-almost \( x \).

Now, let \( n_0 \in Z \) the smallest integer in \( Z \) such that \( \{ x \in X : \#S_{\text{max}}(x) \leq n_0 \} \) has positive \( \nu \)-measure. Because \( S_{\text{max}}(gx) \subset A(g, x)S_{\text{max}}(x) \) (for \( \mu \times \nu \)-almost every \( (g, x) \)), the set \( \{ x \in X : S_{\text{max}}(x) \leq n_0 \} \) is essentially invariant. Thus, from the \( \mu \)-ergodicity of \( \nu \) and our choice of \( n_0 \), we conclude that \( \{ x \in X : \#S_{\text{max}}(x) = n_0 \} \) has full \( \nu \)-measure. Hence, from the \( \mu \)-stationarity of \( \nu \), we obtain that \( \#S_{\text{max}}(gx) = \#S_{\text{max}}(x) = n_0 \) for \( (\mu \times \nu) \)-almost every \( (g, x) \). In particular, the inclusion \( S_{\text{max}}(gx) \subset A(g, x)S_{\text{max}}(x) \) is actually an equality \( S_{\text{max}}(gx) = A(g, x)S_{\text{max}}(x) \) for \( (\mu \times \nu) \)-almost every \( (g, x) \), that is, the cocycle \( A \) permutes the finite sets \( S_{\text{max}}(x) \), as desired.

Therefore the same is true for the algebraic hull \( H \). But this is impossible since \( H \) acts transitively on \( H/B \). \qed
4.2. **Another lemma of Furstenberg.** Let $\hat{X} = X \times H/B$. The group $G$ acts on the space $\hat{X}$ is by

$$g \cdot (x, z) = (gx, A(g, x)z).$$

(4.4)

Let $\hat{\nu}$ be an ergodic $\mu$-stationary measure on $\hat{X}$ which projects to $\nu$ under the natural map $\hat{X} \to X$. Note there is always at least one such: one chooses $\hat{\nu}$ to be an extreme point among the measures which project to $\nu$. If $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$ where the $\hat{\nu}_i$ are $\mu$-stationary measures then $\nu = \pi_*(\hat{\nu}) = \pi_*(\hat{\nu}_1) + \pi_*(\hat{\nu}_2)$. Since $\nu$ is $\mu$-ergodic, this implies that $\pi_*(\hat{\nu}_1) = \nu$ or $\pi_*(\hat{\nu}_2) = \nu$, hence the measure $\hat{\nu}_1$ or $\hat{\nu}_2$ also projects to $\nu$. Since $\hat{\nu}$ is an extreme point among such measures, we must have $\hat{\nu}_1 = \nu$ or $\hat{\nu}_2 = \nu$. This $\hat{\nu}$ is $\mu$-ergodic.

As above, we write

$$d\hat{\nu}(x, z) = d\nu(x) d\eta_x(z).$$

Note that Lemma 4.1 applies to the measures $\eta_x$ on $H/B$.

**Lemma 4.2 (Furstenberg).** For $\alpha \in \Delta$, let $\bar{\sigma}_\alpha : G \times \hat{X} \to \mathbb{R}$ be given by

$$\bar{\sigma}_\alpha(g, x, z) = \hat{\sigma}_\alpha(A(g, x)z)$$

with $\hat{\sigma}_\alpha(\cdot, \cdot)$ as in §2.2. Then, we have

$$\lambda_\alpha = \int_G \int_{\hat{X}} \bar{\sigma}_\alpha(g, x, z) d\hat{\nu}(x, z) d\mu(g).$$

where $\lambda_\alpha$ is as in (3.7).

**Proof.** This is similar to the proof of [GM, Lemma 5.2]. Note that

$$\xi_\alpha(g, z) = \sum_{\gamma \in \Delta} \langle \omega_\alpha, \omega_\gamma \rangle \hat{\sigma}_\alpha(g, z)$$

where $\xi_\alpha(\cdot, \cdot)$ and $\hat{\sigma}_\alpha(\cdot, \cdot)$ as in §2.2. Therefore, it is enough to show that for all $\alpha \in \Delta$,

(4.5)

$$\int_G \int_{\hat{X}} \xi_\alpha(A(g, x)z) d\hat{\nu}(x, z) d\mu(g) = \sum_{\gamma \in \Delta} \langle \omega_\alpha, \omega_\gamma \rangle \lambda_\gamma.$$
Write $A^n(g, x) = \bar{k}_n(g, x)\bar{a}_n(g, x)\bar{k}'_n(g, x)$, where $\bar{k}_n(g, x), \bar{k}'_n(g, x) \in K$, $\bar{a}_n(g, x) \in A_+$, and fix $\epsilon > 0$. Then, by Lemma 2.4 (a) for all $z \in H/P_\alpha$ with $d(\bar{k}'_n(g, x)z, (N_\alpha P_\alpha)\epsilon) > \epsilon$, we have

$$\omega_\alpha(A^n(g, x)) \geq \xi_\alpha(A^n(g, x), z) \geq \omega_\alpha(A^n(g, x)) - C'(\epsilon).$$

Hence, by Lemma 1.1

$$\sum_{\gamma \in \Delta} \langle \omega_\alpha, \omega_\gamma \rangle \lambda_\gamma = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times X} \xi_\alpha(A^n(g, x), z) d\beta(g) d\hat{\nu}(x, z).$$

By Lemma 2.4 (a),

$$\xi_\alpha(A^n(g, x), z) = \sum_{k=1}^n \xi_\alpha(A(g_k, g_{k-1} \ldots g_1 x), A(g_{k-1} \ldots g_1, x) z),$$

Since $\hat{\nu}$ is stationary, each of the terms in the sum has the same integral over $\Omega \times \hat{X}$ (with respect to $\beta \times \hat{\nu}$). Therefore

$$\frac{1}{n} \int_{\Omega \times X} \xi_\alpha(A^n(g, x), z) d\beta(g) d\hat{\nu}(x, z) = \int_{\Omega} \int_{\hat{X}} \xi_\alpha(A(g, x), z) d\hat{\nu}(x, z) d\mu(g),$$

which completes the proof of (4.5). \hfill \Box

4.3. Proof of Proposition 3.2 (a). For $u \in \Omega$, let the measures $\nu_u, \hat{\nu}_u$ be essentially\(^3\) as defined in [BQ] Lemma 3.2, i.e.

$$\nu_u = \lim_{n \to \infty} (u_n \ldots u_1)^{-1} \nu$$

$$\hat{\nu}_u = \lim_{n \to \infty} (u_n \ldots u_1)^{-1} \hat{\nu}.$$ 

The limits exist by the martingale convergence theorem. We disintegrate

$$d\hat{\nu}(x, z) = d\nu(x) d\eta(x(z)), \quad d\hat{\nu}_u(x, z) = d\nu_u(x) d\eta_{u,x}(z).$$

Then each $\eta_x$ is a measure on $H/B$.

We have for $\beta$-a.e. $u \in \Omega$,

$$\hat{\nu}_u = \lim_{n \to \infty} (u_n \ldots u_1)^{-1} \hat{\nu}.$$ 

Therefore, on a set of $\beta \times \nu$ full measure,

$$\lim_{n \to \infty} (u_n \ldots u_1)^{-1} \eta_{u_n \ldots u_1 x} = \eta_{u,x}.$$

Using (4.4), this means

$$\lim_{n \to \infty} A((u_n \ldots u_1)^{-1}, u_n \ldots u_1 x) \eta_{u_n \ldots u_1 x} = \eta_{u,x}.$$ 

\(^3\)It is shown in [BQ] Lemma 3.2 that $(u_1 \ldots u_n) \nu$ and $(u_1 \ldots u_n) \hat{\nu}$ converge for $\beta$-almost every $u = (u_1, u_2, \ldots) \in \Omega$. In our setting, this implies the convergence of $(u_n \ldots u_1)^{-1} \nu$ and $(u_n \ldots u_1)^{-1} \hat{\nu}$ for $\beta$-almost $u \in \Omega$ because $(u_n \ldots u_1)^{-1} = u_1^{-1} \ldots u_n^{-1}$, $\beta = \mu^N$, and $\mu$ is symmetric.
Note that, by the cocycle relation $A(g^{-1}, gx) = A(g, x)^{-1}$, one has
\[ A((u_n \ldots u_1)^{-1}, u_n \ldots u_1 x) = A(u_n \ldots u_1, x)^{-1}. \]

Hence, on a set of $\beta \times \nu$-full measure,
\[ \lim_{n \to \infty} A(u_n \ldots u_1, x)^{-1} \eta_{u_n \ldots u_1 x} = \eta_{u, x}. \tag{4.6} \]

In view of Lemma 4.1 (see also the proof of [EM1, Lemma 14.4]), there exists $\epsilon > 0$ and a compact $K_\delta \subset X$ with $\nu(K_\delta) > 1 - \delta$ such that the family of measures $\{\eta_x\}_{x \in K_\delta}$ is uniformly $(\epsilon, \delta/5)$-regular. Let
\[ N_\delta(u, x) = \{n \in \mathbb{N} : u_n \ldots u_1 x \in K_\delta\}. \]

Write
\[ A(u_n \ldots u_1, x)^{-1} = k_n(u, x) a_n(u, x) k'_n(u, x) \tag{4.7} \]
where $k_n(u, x) \in K$, $k'_n(u, x) \in K$ and $a_n(u, x) \in A_+$. We also write
\[ A(u_n \ldots u_1, x) = \bar{k}_n(u, x) \bar{a}_n(u, x) \bar{k}'_n(u, x) \tag{4.8} \]
where $\bar{k}_n$ and $\bar{k}'_n$ are elements of $K$, and $\bar{a}_n \in A_+$. Then, $\bar{a}_n(u, x) = w_0 a_n(u, x)^{-1} w_0^{-1}$ and thus,
\[ \alpha'(a_n(u, x)) = \alpha(\bar{a}_n(u, x)), \]
\[ \bar{k}_n(u, x) = k'_n(u, x)^{-1} w_0^{-1}, \quad \bar{k}'_n(u, x) = w_0 k_n(u, x)^{-1}, \]
where $w_0$ is longest element of the Weyl group, and $\alpha' = -w_0 \alpha w_0^{-1}$.

Let $\pi_{I'} : H/B \to H/P_{I'}$ be the natural map. Let $\eta_x' = (\pi_{I'})_* \eta_x$ and $\eta_{u, x}' = (\pi_{I'})_* \eta_{u, x}$. Then, $\eta_x'$ and $\eta_{u, x}'$ are measures on $H/P_{I'}$.

Suppose $\alpha \in \Delta \setminus I$. Then, $\lambda_\alpha > 0$ and, a fortiori,
\[ \lim_{n \to \infty} \alpha(\bar{a}_n(u, x)) \to \infty. \]

Thus,
\[ \lim_{n \to \infty} \alpha'(a_n(u, x)) \to \infty \]
for each $\alpha' \in \Delta \setminus I'$.

Applying Lemma 2.2(a) to $g_n = A(u_n \ldots u_1, x)^{-1}$ for $n \in N_\delta(u, x)$ and the $(\epsilon, \delta)$-regular measures $\eta_n = \eta_{u_0 \ldots u_1 x}$ we get that there exists $\bar{k} = \bar{k}(I', u, x) \in K$ such that, for $n \in N_\delta(u, x)$, one has $k_n(u, x) P_{I'} \to \bar{k} P_{I'}$ and
\[ \eta_{u, x}'(\bar{k} P_{I'}) \geq 1 - \delta. \]

Since $\delta > 0$ is arbitrary, we get that for almost all $(u, x)$, $\eta_{u, x}'$ is supported on one point of $H/P_{I'}$. Now choose $C^+(u, x) \in H/B$ so that $\pi_{I'}(C^+(u, x)) = \bar{k}(I', u, x) P_{I'}$. The desired property about $C^+(u, x)$ follows from the stationarity of $\hat{\nu}$. \(\square\)
4.4. Proof of Proposition 3.2 (b), (c). The proof of Proposition 3.2 (b) is virtually identical to the proof of Proposition 3.2 (a), and so we omit the details. Part (c) of Proposition 3.2 is also a classical fact, cf. [GM, Lemma 1.5]. We give an outline of a geometric argument as follows.

Let \( H/K \) be the symmetric space associated to \( H \), and let \( e = K \) denote the origin of \( H/K \). We say that two geodesic rays (parametrized by arc length) are equivalent if they stay a bounded distance apart.

By the geometric version of the multiplicative ergodic theorem [KM], [Ka], for almost all \((u, x) \in \Omega \times X\) there exists a geodesic ray \( \gamma^+: [0, \infty) \rightarrow H/K \) with \( \gamma^+(0) = e \) such that

\[
\lim_{n \to \infty} \frac{1}{n} d(A^n(u, x)^{-1}K, \gamma^+(n)) = 0.
\]

Similarly, by applying the same argument to the backwards walk, we get that for almost all \((v, x) \in \Omega^- \times X\) there exists a geodesic ray \( \gamma^-: [0, \infty) \rightarrow H/K \) such that

\[
\lim_{n \to \infty} \frac{1}{n} d(A^{-n}(v, x)^{-1}K, \gamma^-(n)) = 0.
\]

Let \( F = F(v, u, x) \) be a flat in \( H/K \) which contains rays \( \dot{\gamma}^+ \) and \( \dot{\gamma}^- \) equivalent to \( \gamma^+ \) and \( \gamma^- \) respectively. Then, we have

\[
\lim_{n \to \infty} \frac{1}{n} d(A^n(v, u, x)^{-1}K, \dot{\gamma}^+(n)) = 0.
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} d(A^{-n}(v, u, x)^{-1}K, \dot{\gamma}^-(n)) = 0.
\]

Therefore, for every \( \delta > 0 \) there exists a set \( K_\delta \subset \Omega \times X \) with \( \beta \times \nu(K_\delta) > 1 - \delta \) and \( N > 0 \) such that for \((v, u, x) \in K_\delta \) and \( n > N \),

\[
d(A^n(v, u, x)^{-1}K, \dot{\gamma}^+(n)) \leq \delta n, \quad \text{and} \quad d(A^{-n}(v, u, x)^{-1}K, \dot{\gamma}^-(n)) < \delta n.
\]

Let \( X_n = A^n(v, u, x)^{-1}K \), and let \( \hat{X}_n \) be the closest point to \( X_n \) on \( \dot{\gamma}^+_n \). Then, by (4.12), for \((v, u, x) \in K_\delta \) and \( n > N \),

\[
d(X_n, \hat{X}_n) \leq \delta n.
\]

Let \( \hat{\gamma}^-_n(t) \) be unique geodesic ray equivalent to \( \dot{\gamma}^- \) such that \( \hat{\gamma}^-_n(0) = \hat{X}_n \). Then, as long as \( T^n(v, u, x) \in K_\delta \), and \( m > N \), by (4.14) and (15), we have

\[
d(A^{-m}(v, u, x)^{-1}X_n, \hat{\gamma}^-_n(m)) \leq \delta n + \delta m.
\]

Since \( A^n(v, u, x) \) and \( A^{-n}(T^n(v, u, x)) \) are inverses, we have

\[
d(\dot{\gamma}^-_n(n), e) \leq 2\delta n.
\]
Note that $\hat{X}_n$, $\hat{\gamma}^+$, $\hat{\gamma}^-$ all lie in $F$. However in that case, \(^4.10\) (for sufficiently small $\delta$ and large enough $n$) implies that
\[(4.17) \quad \hat{\gamma}^+ \text{ and } \hat{\gamma}^- \text{ belong to the closures of opposite Weyl chambers in } F.\]

We now interpret \(^4.17\) in terms of $C^+(u, x)$ and $C^-(v, x)$. We can write
\[\gamma^+(t) = k(u, x)\hat{\Lambda}^t K,\]
where $k(u, x) \in K$ and $\hat{\Lambda}^t \in A_+$. Then, by comparing \(^4.10\) with \(^4.7\), we get
\[k(u, x)P_I = C^+(u, x)P_I,\]
where $C^+(u, x)$ is as in Proposition \(^3.2\) (a), and $I'$ is as in \(^3.5\). Similarly, if we may write
\[\gamma^-(t) = \bar{k}(v, x)\Lambda^t K,\]
where $\bar{k}(u, x) \in K$ and $\Lambda^t \in A_+$. Then, by comparing \(^4.11\) with \(^4.8\), we get
\[\bar{k}(u, x)P_I = C^-(u, x)P_I,\]
where $C^-(u, x)$ is as in Proposition \(^3.2\) (b), and $I$ is as in \(^3.2\). Then, \(^4.17\) implies \(^3.6\). \(\square\)

5. Proof of Theorem \(^3.1\)

5.1. Conformal blocks and Schmidt’s criterion. We will use the following criterion of K. Schmidt [Sch] for the detection of conformal blocks.

**Definition 5.1** (cf. Definition 4.6 in [Sch]). We say that a cocycle $A : G \times X \to H$ is Schmidt-bounded if, for every $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset H$ such that
\[\overline{\beta \times \nu \left\{ ((v, u), x) \in \hat{\Omega} \times X : A^n(v, u, x) \not\in K(\varepsilon) \right\}} < \varepsilon\]
for all $n \in \mathbb{N}$.

The importance of this notion in the search of conformal blocks becomes apparent in view of the next result, which follows from [Sch Theorem 4.7].

**Theorem 5.2** (Schmidt). $A(,\cdot)$ is Schmidt-bounded if and only if there exists a measurable map $C : X \to H$ such that the cocycle $C(g(x))A(g, x)C(x)^{-1}$ takes its values in a compact subgroup of $H$. 
5.2. Proof of Theorem 3.1. We use the notation from §4.3

Lemma 5.3. For any α ∈ I, let α′ = −w_0αν_0^{-1} (so that α′ ∈ I'). Then, β × ν-almost all (u, x) ∈ Ω × X, the measure η_{α,x} has no atoms; i.e. for any element k_{u,x} ∈ K, we have η_{α,x}(k_{u,x}P_{α'}) = 0.

Proof. Suppose there exists δ > 0 so that, for some α′ ∈ I' and for a set (u, x) of positive measure, there exists k_{u,x} ∈ K with η_{α,x}(k_{u,x}P_{α'}) > δ. Then this happens for a subset of full measure by ergodicity. Note that (4.6) holds.

Then, by Lemma 2.2 (b), for β × ν almost all (u, x), η_{α,x}(k_{u,x}P_{α'}) ≥ 1 − δ (and thus k_{u,x}P_{α'} is unique) and, as n → ∞ along N_δ(α, x), we have:

\[ \alpha'(a_n(u, x)) \to \infty, \]

and

\[ k_n(u, x)P_{α'} \to k_{u,x}P_{α'}. \]

Then, by (4.9),

\[ \alpha(\bar{a}_n(u, x)) \to \infty, \]

and

\[ \bar{k}'_n(u, x)^{-1}w_0P_{α'} \to \bar{k}_{u,x}P_{α'}. \]

Therefore for any ε_1 > 0 there exists a subset H_{ε_1} ⊂ Ω × X of β × ν-measure at least 1 − ε_1 such that the convergence in (5.2) and (5.1) is uniform over (u, x) ∈ H_{ε_1}. Hence there exists M > 0 such that for any (u, x) ∈ H_{ε_1} and any n ∈ N_δ(α, x) with n > M,

\[ \bar{k}'_n(u, x)^{-1}w_0P_{α'} \in \text{Nbd}_ε(\bar{k}_{u,x}P_{α'}). \]

By Lemma 4.1 (see also the proof of [EMR] Lemma 14.4]) there exists a subset H''_{ε_1} ⊂ X with ν(H''_{ε_1}) > 1 − ε_2(ε_1) with ε_2(ε_1) → 0 as ε_1 → 0 such that for all x ∈ H''_{ε_1}, and any g ∈ H,

\[ η_x(\text{Nbd}_δ(gJ)) < c_3(ε_1), \]

where c_3(ε_1) → 0 as ε_1 → 0. Let

\[ H'_{ε_1} = \{(u, x, z) : (u, x) ∈ H_{ε_1}, \ x ∈ H''_{ε_1} \ \text{and} \ d(z, \bar{k}_{u,x}J) > 2ε_1\}. \]

Then, \( (β \times \hat{ν})(H'_{ε_1}) > 1 − ε_1 − c_2(ε_1) − c_3(ε_1) \), hence \( (β \times \hat{ν})(H'_{ε_1}) \to 1 \) as \( ε_1 \to 0 \).

We now claim that for \( (u, x, z) ∈ H'_{ε_1} \) and \( n ∈ N_δ(α, x) \), we have

\[ d(\bar{k}'_n(u, x)z, (\bar{N}_αP_α)^c) > ε_1. \]

Suppose not, then there exist (u, x, z) ∈ H'_{ε_1} and \( n ∈ N_δ(α, x) \) such that

\[ d(\bar{k}'_n(u, x)z, (\bar{N}_αP_α)^c) ≤ ε_1. \]

Let \( W_α ⊂ W \) denote the subgroup of the Weyl group which fixes \( M_α \). Then, \( d(\bar{k}'_n(u, x)z, w_0 \bigsqcup_{w ∈ W_α w_0^{-1}W_α} BwB) ≤ ε_1. \)
Hence,
\[
d(z, k'_n(u, x)^{-1}w_0 \bigsqcup_{w \notin W_\alpha w_0^{-1}W_\alpha} BwB) \leq \epsilon_1.
\]
(5.6)

Note that
\[
P_\alpha' \bigsqcup_{w \notin W_\alpha w_0^{-1}W_\alpha} BwB = \bigsqcup_{w \notin W_\alpha w_0^{-1}W_\alpha} BwB.
\]
By (5.3) and (5.6), this implies that
\[
d(z, \tilde{k}_{u,x} \bigsqcup_{w \notin W_\alpha w_0^{-1}W_\alpha} BwB) \leq 2\epsilon_1,
\]
contradicting (5.4). This completes the proof of (5.5).

Therefore, in view of Lemma 2.5, there exists
\lim_{n \to \infty} \lambda_\alpha' C_1(u, x, z) = \lambda_\alpha'(u, x, z) \in \hat{X}.

By (5.2) and (4.8), this implies that for \((u, x, z) \in H'_\epsilon\),
(5.8)
\[
\hat{\sigma}_\alpha(A(u_n \ldots u_1, x, z) \to \infty \text{ as } n \to \infty \text{ along } N_\delta(u, x).
\]

Since \((\beta \times \hat{\nu})(H'_\epsilon) \to 1 \text{ as } \epsilon_1 \to 0 \text{, (5.8) holds for } \beta \times \hat{\nu} \text{ almost all } (u, x, z) \in \Omega \times \hat{X}.

For \(1 \leq s \leq m\), let \(\sigma_\alpha : \Omega \times \hat{X} \to \mathbb{R}\) be defined by \(\sigma_\alpha(u, x, z) = \tilde{\sigma}_\alpha(u_1, x, z)\), where \(\tilde{\sigma}_\alpha\) is as in Lemma 4.2. Then, the left hand side of (5.8) is exactly
\[
\sum_{j=0}^{n-1} \sigma_\alpha(T^j(u, x, z)).
\]
Also, we have \(n \in N_\delta(u, x)\) if and only if \(T^n(u, x) \in \Omega \times K_\delta\). Then, by [EMI, Lemma C.6],
\[
\int_{\Omega \times \hat{X}} \sigma_\alpha(q) d(\beta \times \hat{\nu})(q) > 0.
\]
By Lemma 4.2 (Furstenberg’s formula), the above expression is \(\lambda_\alpha\). Thus \(\lambda_\alpha > 0\), contradicting our assumption that \(\alpha \in I\). This completes the proof of the lemma. □

Proof of Theorem 3.1. Let \(C^+(u, x) \in H\) and \(C^-(v, x) \in H\) be as in Proposition 3.2. By Proposition 3.2(c), for a.e. \((v, u, x),\)
\[
C^+(u, x)^{-1}C^-(v, x) = p_{I'}(v, u, x)w_0p_I(v, u, x) \quad \text{where } p_I(v, u, x) \in P_I, p_{I'}(v, u, x) \in P_{I'}.
\]
Let
\[
C_1(v, u, x) = C^+(u, x)p_{I'}(v, u, x) = C^-(v, x)p_I(v, u, x)^{-1}w_0^{-1}.
\]
Then, by Proposition 3.2(a) and (b),
\[
C_1(\hat{T}^n(v, u, x))^{-1}A^n(v, u, x)C_1(v, u, x) \in P_{I'} \cap w_0P_Iw_0^{-1} = M_{I'}A_{I'}.
\]
Let
\[ A^n_I(v, u, x) := C_1(\hat{T}^n(v, u, x))^{-1}A^n(v, u, x)C_1(v, u, x). \]
Suppose \( \delta > 0 \). Then there exist compact sets \( K_2(\delta) \subset \Omega \times X \) with \( \hat{\beta} \times \nu(K_2(\delta)) > 1 - \delta \) and \( K_3(\delta) \subset \mathcal{H} \) such that for \((v, u, x) \in K_2(\delta), C_1(v, u, x) \in K_3(\delta) \).

Therefore, by (2.8) and Lemma 2.3, there exists \( c_1(\delta) \in \mathbb{R}_+ \) such that for all \((v, u, x) \in K_2(\delta) \) and all \( n \in \mathbb{N} \) with \( T^n(v, u, x) \in K_2(\delta) \), we have, for all \( \alpha \in \Delta, \)
\[ |\alpha(A^n_I(v, u, x)) - \alpha(A^n(v, u, x))| \leq c_1(\delta). \]

Let now \( \epsilon = \epsilon(\delta) \) and \( \mathcal{K}(\delta) \subset X \) be as in the proof of Proposition 3.2(a), so that for \( x \in \mathcal{K}(\delta) \), the measure \( \eta_x \) is \((\epsilon, \delta)\)-regular.

By Lemma 5.3 for all \( \alpha' \in I' \), the measures \( \eta^\alpha_{u,x} \) are non-atomic. Therefore, we can find \( \epsilon' = \epsilon'(\delta) \) and \( \mathcal{K}'(\delta) \subset \Omega \times X \) such that for \((u, x) \in \mathcal{K}'(\delta) \), and all \( \alpha' \in I' \), for any \( z \in \mathcal{H}/\mathcal{P}_\alpha' \),
\[ \eta^\alpha_{u,x}(\text{Bhd}_{\epsilon'}(z)) < \epsilon. \]

Let
\[ \mathcal{K}_1(\delta) = \{(v, u, x) \in \Omega \times X : (v, u, x) \in \mathcal{K}(\delta) \} \]
Then, by (4.6), (4.7) and Lemma 2.2(a), there exists \( c_2 = c_2(\delta) \in \mathbb{R}^+ \) such that for all \((v, u, x) \in \mathcal{K}_1(\delta) \), all \( n \) with \( T^n(v, u, x) \in \mathcal{K}_1(\delta) \), and any \( \alpha' \in I' \),
\[ \alpha(A_n(u_n \ldots u_1, x)) < c_2(\delta), \text{ where } \alpha = -w_0\alpha'w_0^{-1}. \]
Thus, by (4.8) and (4.9), for all \((v, u, x) \in \mathcal{K}_1(\delta) \) and all \( n \in \mathbb{N} \) such that \( T^n(v, u, x) \in \mathcal{K}_1(\delta) \) and all \( \alpha' \in I' \),
\[ \alpha'(A_n(u_n \ldots u_1, x)) < c_2(\delta). \]
Let \( \tilde{A}_n^I(v, u, x) \) denote the part of \( A_n^I(v, u, x) \) which lies in \( M_I \). Then,
\[ \alpha'(\tilde{A}_n^I(v, u, x)) = \alpha'(A_n^I(v, u, x)) \text{ for } \alpha' \in I'. \]
Let \( \mathcal{K}'_1(\delta) = \mathcal{K}_2(\delta) \cap \{(v, u, x) : (u, x) \in \mathcal{K}_1(\delta) \} \). It follows from (5.9), (5.10) and (5.11), that for all \( \alpha' \in I' \), all \((v, u, x) \in \mathcal{K}'_1(\delta) \) and all \( n \in \mathbb{N} \) with \( T^n(v, u, x) \in \mathcal{K}'_1(\delta) \),
\[ \alpha'(A_n^I(v, u, x)) \leq c_3(\delta). \]
Note that \( \hat{\beta} \times \nu(\mathcal{K}_1'(\delta)) > 1 - 4\delta \). Since \( \delta > 0 \) is arbitrary, it follows that \( \tilde{A}_n^I \) is Schmidt-bounded (see Definition 5.1). Therefore, by Theorem 5.2 there exists \( \hat{C} : \Omega \times X \rightarrow \mathcal{M} \) such that \( \hat{C}(T^n(v, u, x))^{-1}\tilde{A}_n^I(v, u, x)\hat{C}(v, u, x) \in \mathcal{K} \cap \mathcal{M}_I \). Let
\[ C(v, u, x) = C_1(v, u, x)\hat{C}(v, u, x)w_0. \]
Then,
\[ C(T^n(v, u, x))^{-1}A^n(v, u, x)C(v, u, x) \in w_0^{-1}(\mathcal{M}_I \cap \mathcal{K})A_Iw_0 = (\mathcal{M}_I \cap \mathcal{K})A_I. \]
Thus, (3.3) holds.

Finally, it is easy to see that (3.4) follows from (5.9) and the definition of the \( \lambda_\alpha \) (cf. the argument in the proof of [GM Lemma 1.5]). \( \square \)
6. Proof of Theorem 1.6

Let $L$ be a vector space, and suppose $H$ is a subgroup of $SL(L)$. We may assume that the action of $H$ on $L$ is irreducible, in the sense that no non-trivial proper subspace of $L$ is fixed by $H$.

Let $K, I, A_I$ and $M_I$ be as in Theorem 3.1. By Theorem 3.1 we may assume that the cocycle $A(\cdot, \cdot)$ takes values in $(K \cap M_I)A_I$. We choose an inner product on $L$ which is preserved by $K$.

Then, the block conformality of Theorem 1.6 follows from the corresponding statement in Theorem 3.1.

We note that, by Theorem 3.1, there exists $a_*$ in the interior of $A_I$ such that the Lyapunov exponents of $A(\cdot, \cdot)$ are given by expressions of the form $\omega(\log a_*)$, where $\omega$ is a weight of the action of $H$ on $L$.

Let $V'_\omega \subset L$ be the subspace corresponding to the weight $\omega$; then for $a \in A$, and $v \in V'_\omega$,

$$a \cdot v = \omega(\log a)v.$$

Let $\omega_0$ be the highest weight. (It exists and has multiplicity 1 because the action of $H$ on $L$ is irreducible). Then, the top Lyapunov exponent $\lambda_1$ of $A(\cdot, \cdot)$ is $\omega_0(\log a_*)$. Then, since the action of $H$ on $L$ is irreducible, the Lyapunov subspace $V_1$ of $A(\cdot, \cdot)$ corresponding to the Lyapunov exponent $\lambda_1$ is given by

$$V_1 = \bigoplus_{\omega \in S_0} V'_{\omega},$$

where $S_0$ consists of weights of the form

$$\omega_0 - \sum_{\alpha \in I} c_\alpha \alpha.$$

Recall that for $\alpha \in I$, $\alpha(A_I) = 0$. Therefore, for $a \in A_I$ and $v \in V_1$,

$$a \cdot v = \omega_0(\log a)v.$$

Then, for $k \in (K \cap M_I)$, $a \in A_I$,

$$(ka) \cdot v = \omega_0(\log a)k \cdot v.$$

Since $A(\cdot, \cdot)$ takes values in $(K \cap M_I)A_I$, (1.5) follows from (6.1). \qed

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