On Differential Geometric Approach to Nonlinear Systems Affine in Control

July 18, 2017


| 4 | Backstepping Design Procedure | 53 |
|---|--------------------------------|----|
| 4.1 | Introduction and Problem Statement | 53 |
| 4.2 | Review of the Backstepping Design Technique | 54 |
| 4.3 | Backstepping Design Procedures Revisited | 57 |
| 4.3.1 | Conventional Chain-by-Chain Backstepping | 57 |
| 4.3.2 | Level-by-Level Backstepping | 58 |
| 4.3.3 | Mixed Chain-by-Chain and Level-by-Level Backstepping | 63 |
| 4.4 | Summary of the Chapter | 70 |

| 5 | Semi-global Stabilization for Nonlinear Systems | 71 |
|---|-----------------------------------------------|----|
| 5.1 | Introduction and Problem Statement | 71 |
| 5.2 | Main Results | 73 |
| 5.3 | Summary of the Chapter | 80 |

| 6 | Disturbance Attenuation for Nonlinear Systems | 83 |
|---|-----------------------------------------------|----|
| 6.1 | Introduction and Problem Statement | 83 |
| 6.2 | Preliminary Results | 85 |
| 6.3 | Disturbance Attenuation and Almost Disturbance Decoupling with Stability | 87 |
| 6.4 | Summary of the Chapter | 96 |

| 7 | Summary | 99 |

| References | 101 |
1

Introduction

1.1 Literature Overview

The note is concerned with nonlinear systems affine in control described by ordinary differential equations of the following form,

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x),
\end{align*}
\]

(1.1)

where \(x\), \(u\) and \(y\) denote the system state, input and output, respectively.

The differential geometric approach to nonlinear control has been proven to be a powerful tool to deal with fundamental questions in the state space formulation of nonlinear control systems. Elliott [1], and Nijmeijer & Schaft [2] had good reviews on the development of differential geometric control theory. In the 1960’s, the popularity of Pontryagin’s Maximal Principle led to the need to understand controllability, and the researchers realized that some technical assumptions about the nonlinear systems, such as smoothness and analyticity, could lead to a general mathematical approach. Hermann [3,5] studied controllability with methods based on vector fields and differential forms, which is analogous to Kalman’s criterion for linear systems. In the early 1970’s Brockett, Boothby, Elliott, et al. were promoting the use of Lie algebra methods to study controllability. Brockett [6,7] and Willems also considered systems invariants equivalent by coordinate change and a class of feedback transformations. Isidori, Krener, Gori-Giorgi & Monaco [8], and Hirschorn [9] used the concept of controlled invariant distribution for the solving of the problem of decoupling problems. Many concepts of differential geometric control on nonlinear systems are indeed the generalization of concepts of geometric control of linear systems. Wonham and Morse [10,13] and Basile and Marro [14,15] developed a systematic geometric approach to solving the problems of pole placement, noninteracting control, disturbance decoupling, and regulation. This approach depends on global linear space structure. Isidori [16,17] generalized a local approach of this nature to nonlinear control problems. He
brought the geometry and Volterra series methods together and used them appropriately for stabilization, regulation, disturbance decoupling, noninteracting control, tracking and regulation [1].

The nonlinear analogues of linear system structural properties, such as relative degree (or infinite zero structure), zero dynamics (or finite zero structure) and invertibility properties, have played critical roles in recent literature on the analysis and control design for nonlinear systems. The normal forms that are associated with these structural properties, along with some basic tools, have enabled many major breakthroughs in nonlinear control theory.

A single input single output system has a relative degree, if the system can be reduced to the zero dynamics cascaded with a clean chain of integrators linking the input to the output. Here by clean we mean that no other signal enters the middle of the chain. This structural feature is extended to nonlinear systems with more than one input/output pair. For a square invertible nonlinear system, the notion of vector relative degree was introduced in [18, 19], and the systems can be transferred into the zero dynamics connecting to clean chains of integrators.

The clean chains of integrators are called the prime form in [10] for linear systems, and the necessary and sufficient geometric conditions for the existence of prime forms for nonlinear systems is were established [20]. The lengths of chains of integrators are the nonlinear extension of infinite zeros. However, vector relative degree is a rather restrictive structural property that not even all square invertible linear systems, with the freedom of choosing coordinates for the state, output and input spaces and state feedback, could possess.

A major generalization of the normal form representations was made in [16, 17, 21, 22], where square invertible systems are considered. With the assumption that the rank of certain matrices are constant on a sequence of nested submanifolds, or with some stronger assumptions [17, 22], the nonlinear systems can be represented by the zero dynamics cascaded with chains of integrators. Note that chains of integrators here need not to be clean. Interconnection between chains of integrators are allowed. This greatly enlarges the class of nonlinear systems that normal forms can represent. But in these normal forms, the lengths of chains of integrators are no longer the nonlinear extension of infinite zeros. The applications of these normal forms in solving the problem of asymptotic stabilization, disturbance decoupling, tracking and regulation can be found in [16, 17] and the references therein.

In the note, we make an attempt to study structural properties of affine nonlinear systems. We will develop a constructive algorithm to represent nonlinear systems in normal forms. In the special case when the system is square and invertible, our normal forms take forms similar to those in [16, 17, 22], but with an additional property that allows the normal forms to reveal the nonlinear extension of infinite zeros of linear systems. In addition, our algorithms require fewer assumptions, can apply to general nonlinear systems that are not necessarily square, and can explicitly show invertibility structures of the systems. We will also study the applications of these new normal forms
to solving the problems of global stabilization, semi-global stabilization and disturbance attenuation.

1.2 Note Outline

The note focuses on the differential geometric approach to the study of nonlinear systems that are affine in control. We first develop normal forms for nonlinear system affine in control. Based on these normal forms, we then address the problems of global stabilization, semi-global stabilization and disturbance attenuation. The results presented are based on the works [23–43].

The note can naturally be divided into three parts.

The first part is Chapter 2, which presents a brief introduction to the differential geometric concepts for use in the note. It includes the fundamental concepts of manifolds, submanifolds, tangent vectors, vector fields, and distributions.

The second part is Chapter 3. In this chapter, we propose constructive algorithms for decomposing a nonlinear system that is affine in control but otherwise general. These algorithms require modest assumptions on the system and apply to general multiple input multiple output systems that do not necessarily have the same number of inputs and outputs. They lead to various normal form representations and reveal the structure at infinity, the zero dynamics and the invertibility properties, all of which represent nonlinear extensions of relevant linear system structural properties of the system they represent.

The third part of the note consists of Chapters 4, 5 and 6. They contain some applications of the structural decomposition developed in Chapter 3. In Chapter 4, we exploit the properties of such a decomposition for the purpose of solving the stabilization problem. In particular, this structural decomposition simplifies the conventional backstepping design and motivates new backstepping design procedures that are able to stabilize some systems on which the conventional backstepping is not applicable.

In Chapter 5, we exploit the properties of such a decomposition for the purpose of solving the semi-global stabilization problem for minimum phase nonlinear systems without vector relative degrees. By taking advantage of the special structure of the decomposed system, we first apply the low gain design to the part of system that possesses linear dynamics. The low gain design results in an augmented zero dynamics that is locally stable at the origin with a domain of attraction that can be made arbitrarily large by lowering the gain. With this augmented zero dynamics, the backstepping design procedure is then applied to achieve semi-global stabilization of the overall system.

Chapter 6 considers the problems of disturbance attenuation and almost disturbance decoupling, which have played a central role in control theory. By employing the structural decomposition of multiple input multiple output nonlinear systems and the backstepping procedures that we have developed,
we show that these two problems can be solved for a larger class of nonlinear systems.

Finally, Chapter 7 is the conclusions to the note, and some topics for the future research are also mentioned.
The chapter recalls some basic concepts and facts of differential geometry that will be used in the following chapters. The detail can be found in \[2, 16, 44-47]. Differential geometry is a discipline on curves and surfaces. It studies the functions that define curves and surfaces, and the transformations between the coordinates that are used to specify curves and surfaces. It also treats the differential relations that put pieces of curves or surfaces together.

2.1 Manifolds

A manifold is a mathematical space that on a small enough scale resembles the Euclidean space of a specific dimension. A line and a circle are one-dimensional manifolds, and a plane and the surface of a ball are two-dimensional manifolds. Although manifolds resemble Euclidean spaces near each point locally, the global structure of a manifold is more complicated. A chart of a manifold is an invertible map between a subset of the manifold and the Euclidean space such that both the map and its inverse preserve the desired structure. The description of most manifolds requires more than one chart. A specific collection of charts which covers a manifold is called an atlas. Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. Given two overlapping charts, a transition map can be defined which goes from an open ball in Euclidean space to the manifold and then back to another open ball in Euclidean space.

Topological spaces are structures that define convergence, connectedness, and continuity. A topological space is a set \(X\) together with \(\mathcal{O}\), a collection of subsets of \(X\), satisfying the following axioms:

1) The empty set and \(X\) are in \(\mathcal{O}\).
2) The union of any collection of sets in \(\mathcal{O}\) is also in \(\mathcal{O}\).
3) The intersection of any finite collection of sets in \(\mathcal{O}\) is also in \(\mathcal{O}\).
The collection $\Omega$ is called a topology on $X$. The elements of $X$ are usually called points. It is customary to require that the space be Hausdorff and second countable.

A topological manifold is a topological space locally homeomorphic to a Euclidean space, which means that every point has a neighborhood for which there exists a homeomorphism (a bijective continuous function whose inverse is also continuous) mapping that neighborhood to a Euclidean space.

A differentiable manifold is a topological manifold that allow one to do differential calculus. The primary object of study in differential calculus is the derivative. We now consider the derivative of a function $f$ with domain an open subset $U$ of $\mathbb{R}^n$ and with range in $\mathbb{R}^m$. The function $f$ is differentiable at $x \in U$ if there is a linear map $A(x)$, a Jacobian matrix, from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - A(x)h|}{|h|} = 0.$$ 

Then $A(x)$ is called the derivative of $f$. A $C^k$ manifold is a differential manifold with an atlas whose transition maps are all $k$-times continuously differentiable.

A smooth manifold ( $C^\infty$ manifold ) is a differentiable manifold for which all the transition maps are smooth. That is, derivatives of all orders exist. An analytic manifold, ( $C^\omega$ manifold ) is a smooth manifold with the additional condition that each transition map is analytic: the Taylor expansion is absolutely convergent on some open ball.

Consider a topological space $(X, \Omega)$. Suppose that for any $p \in X$, there exists an open set $U \in \Omega$ with $p \in U$, and a bijection $\phi$ mapping $U$ onto an open subset of $\mathbb{R}^n$,

$$\phi : U \to \phi(U) \subset \mathbb{R}^n.$$ 

The grid defined on $\phi(U) \subset \mathbb{R}^n$ is transforms into a grid on $U$. A coordinate chart is the pair $(U, \phi)$. The map $\phi$ can be represented as a set $(\phi_1, \phi_2, \ldots, \phi_n)$ and $\phi_i : U \to \mathbb{R}$ is called the $i$-th coordinate function. The $n$-tuple of real numbers $(\phi_1(p), \phi_2(p), \ldots, \phi_n(p))$ is called the set of local coordinates of $p$ in the coordinate chart $(U, \phi)$.

For example, the helix represented by

$$z_1 = \cos x_1$$
$$z_2 = \sin x_1$$
$$z_3 = x_1$$

is a smooth path embedded in Euclidean space $\mathbb{R}^3$. It is 1-dimensional smooth manifold. The parameters $x_1$ is local coordinate, and $z_1$, $z_2$ and $z_3$ are global coordinates or ambient coordinates.

The sphere $z_1^2 + z_2^2 + z_3^2 = 1$ is a smooth surface embedded in Euclidean space $\mathbb{R}^3$. It is 2-dimensional smooth manifold. Using spherical polar coordinates, the sphere is represented by
2.1 Manifolds

\begin{align*}
    z_1 &= \sin x_1 \cos x_2 \\
    z_2 &= \sin x_1 \sin x_2 \\
    z_3 &= \cos x_1
\end{align*}

For points other than \((0, 0, \pm 1)\),

\begin{align*}
    x_1 &= \arccos z_3 \\
    x_2 &= \begin{cases} 
    \arccos \left( \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \right) & \text{if } z_2 \geq 0 \\
    2\pi - \arccos \left( \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \right) & \text{if } z_2 < 0.
    \end{cases}
\end{align*} \tag{2.1}

The chart of the sphere is the pair of functions in (2.1). The parameters \(x_1\) and \(x_2\) are called local coordinates, while \(z_1, z_2\) and \(z_3\) are called global coordinates or ambient coordinates. The ambient coordinates are superfluous data that often have nothing to do with the problem at hand. It is a tremendous advantage to be able to work with manifolds, without the excess baggage of such an ambient space.

Let \((U, \phi)\) and \((V, \varphi)\) be two coordinate charts on a manifold \(N\) with \(U \cap V \neq 0\). The coordinates transformation on \(U \cap V\)

\[ \varphi \circ \phi^{-1} : \phi(U \cap V) \to \varphi(U \cap V) \]

transfers the set of the local coordinate \((\phi_1(p), \phi_2(p), \ldots, \phi_n(p))\) to the set of the local coordinate \((\varphi_1(p), \varphi_2(p), \ldots, \varphi_n(p))\). Two coordinate charts \((U, \phi)\) and \((V, \varphi)\) are \(C^\infty\)-compatible if \(\varphi \circ \phi^{-1}\) is smooth \((C^\infty)\), i.e., \(\varphi \circ \phi^{-1}\) is a diffeomorphism.

The set \((\phi_1(p), \phi_2(p), \ldots, \phi_n(p))\) can be represented as an \(n\)-vector \(x = \text{col} \{x_1, x_2, \ldots, x_n\}\), and the set \((\varphi_1(p), \varphi_2(p), \ldots, \varphi_n(p))\) as \(y = \text{col} \{y_1, y_2, \ldots, y_n\}\). Therefore, the coordinate transformation \(\varphi \circ \phi^{-1}\) can be represented as

\[ y = \begin{pmatrix} y_1(x_1, x_2, \ldots, x_n) \\
                 y_2(x_1, x_2, \ldots, x_n) \\
                 \vdots \\
                 y_n(x_1, x_2, \ldots, x_n) \end{pmatrix} = y(x). \]

and \(\phi \circ \varphi^{-1}\) as

\[ x = \begin{pmatrix} x_1(y_1, y_2, \ldots, y_n) \\
                 x_2(y_1, y_2, \ldots, y_n) \\
                 \vdots \\
                 x_n(y_1, y_2, \ldots, y_n) \end{pmatrix} = x(y). \]

A \(C^\infty\) atlas on a manifold \(N\) is a collection \(\mathcal{A} = \{(U^i, \phi^i) : i \in I\}\) of pairwise \(C^\infty\)-compatible coordinate charts with \(\bigcup_{i \in I} U^i = N\). An atlas is complete if not properly contained in any other atlas. A smooth manifold is a manifold equipped with a complete \(C^\infty\) atlas.
Let $N$ and $M$ be manifolds of dimension $n$ and $m$, $(U, \phi)$ and $(V, \varphi)$ be coordinate charts on the manifolds $N$ and $M$, respectively. $F : N \to M$ is a mapping. The mapping

$$\tilde{F} = \phi \circ F \circ \phi^{-1}$$

is called an expression of $F$ in local coordinates.

Let $N$ and $M$ be smooth manifolds of dimension $n$. A mapping $F : N \to M$ is a smooth mapping if for each $p \in N$ there exist coordinate charts $(U, \phi)$ of $N$ and $(V, \varphi)$ of $M$, with $p \in U$ and $F(p) \in V$, such that the expression of $F$ in local coordinates is $C^\infty$.

Let $N$ and $M$ be smooth manifolds of dimension $n$. A mapping $F : N \to M$ is a diffeomorphism if $F$ is bijective and both $F$ and $F^{-1}$ are smooth mappings. Two manifolds $N$ and $M$ are diffeomorphic if there exists a diffeomorphism $F : N \to M$.

### 2.2 Submanifolds

Let $N$ be a smooth manifold of dimension $n$. A non-empty open set $V \subset N$ is itself a smooth manifold of dimension $m$ with coordinate charts obtained by restricting the coordinate charts for $N$ to $V$. $V$ is called an open submanifold of $N$.

Let $N$ be a smooth manifold of dimension $n$. A subset $N'$ of $N$ is an embedded submanifold of dimension $m < n$ if and only if for each $p \in N'$ there exists a cubic coordinate chart $(U, \phi)$ of $N$, with $p \in U$, such that $U \cap N'$ coincides with an $n$-dimensional slice of $U$ passing through $p$.

Let $F : N \to M$ be a smooth mapping of manifolds. $F$ is an immersion if $\text{rank}(F) = \text{dim}(N)$ for all $p \in N$. $F$ is an univalent immersion if $F$ is an immersion and is injective. $F$ is an embedding if $F$ is an univalent immersion and the topology induced on $F(N)$ by the one of $N$ coincides with the topology of $F(N)$ as a subset of $M$.

The image $F(N)$ of a univalent immersion is called an immersed submanifold of $M$. The image $F(N)$ of an embedding is called an embedded submanifold of $M$.

Let $F : N \to M$ be an immersion. For each $p \in N$ there exists a neighborhood $U$ of $p$ such that the restriction of $F$ to $U$ is an embedding.

For $F : N \to M$, let $M' = F(N)$ and $F' : N \to M'$. If the topology of $M'$ is the one induced by one of $N$, $F'$ is a homeomorphism. Any coordinate chart $(U, \phi)$ of $N$ induces a coordinate chart $(V, \varphi)$ of $M'$, i.e.,

$$V = F'(U), \quad \varphi = \phi \circ (F')^{-1}.$$

The smooth manifold $M'$ is diffeomorphic to the smooth manifold $N$. 

2.3 Tangent Vectors

Let $N$ be a smooth manifold of dimension $n$, and $x$ be a point in $N$. A tangent space is a real vector space that tangentially pass through the point $x$. The elements of the tangent space are called tangent vectors at $x$.

All the tangent spaces can be “glued together” to form a new differentiable manifold of twice the dimension, the tangent bundle of the manifold.

Let $N$ be a smooth manifold. A real-valued function $\lambda$ is said to be smooth in a neighborhood of $p$, if the domain of $\lambda$ includes an open set $U$ of $N$ containing $p$ and the restriction of $\lambda$ to $U$ is a smooth function. The set of all smooth functions in a neighborhood of $p$ is denoted $C^\infty(p)$. Consider $\lambda \in C^\infty(p)$, $\gamma \in C^\infty(p)$, and $a, b \in \mathbb{R}$. Define the functions $a\lambda + b\gamma$ and $\lambda \gamma$ as

$$(a\lambda + b\gamma)(q) = a\lambda(q) + b\gamma(q),$$

$$(\lambda \gamma)(q) = \lambda(q)\gamma(q),$$

for all $q$ in the neighborhood of $p$. It is obvious that $a\lambda + b\gamma \in C^\infty(p)$ and $\lambda \gamma \in C^\infty(p)$. So $C^\infty(p)$ forms a vector space over the field $\mathbb{R}$.

A tangent vector $v$ at $p$ is a map $v : C^\infty(p) \to \mathbb{R}$ with

$$v(a\lambda + b\gamma) = av(\lambda) + bv(\gamma),$$

$$v(\lambda \gamma) = \gamma(p)v(\lambda) + \lambda(p)v(\gamma),$$

for all $\lambda, \gamma \in C^\infty(p)$ and $a, b \in \mathbb{R}$.

Let $N$ be a smooth manifold. The tangent space to $N$ at $p$, denoted by $T_pN$, is the set of all tangent vectors at $p$. The set $T_pN$ forms a vector space over the field $\mathbb{R}$ under the normal rules of scalar multiplication and addition.

Let $N$ be smooth manifold of dimension $n$. Let $p$ be any point of $N$, and $(U, \phi)$ be a coordinate chart around $p$. In this coordinate, the tangent vectors $(\frac{\partial}{\partial \phi_1})_p, \cdots, (\frac{\partial}{\partial \phi_n})_p$ form a basis of $T_pN$, which is called the natural basis of $T_pN$ induced by the coordinate chart $(U, \phi)$. Let $v$ be a tangent vector at $p$, we have

$$v = \sum_{i=1}^{n} v_i \left( \frac{\partial}{\partial \phi_i} \right)_p,$$

where $v_1, \cdots, v_n$ are real numbers.

Let $(U, \phi)$ and $(V, \varphi)$ be coordinate charts around $p$. If $v$ is a tangent vector, then

$$v = \sum_{i=1}^{n} v_i \left( \frac{\partial}{\partial \phi_i} \right)_p = \sum_{i=1}^{n} w_i \left( \frac{\partial}{\partial \varphi_i} \right)_p,$$

where
\[
\begin{pmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n 
\end{pmatrix} =
\begin{bmatrix}
  \partial x_1 & \partial x_1 & \cdots & \partial x_1 \\
  \partial y_1 & \partial y_2 & \cdots & \partial y_n \\
  \partial x_2 & \partial x_2 & \cdots & \partial x_1 \\
  \partial y_1 & \partial y_2 & \cdots & \partial y_n \\
  \vdots & \vdots & \ddots & \vdots \\
  \partial x_n & \partial x_n & \cdots & \partial x_1 \\
  \partial y_1 & \partial y_2 & \cdots & \partial y_n 
\end{bmatrix}
\begin{pmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_n 
\end{pmatrix},
\]

and \( x = x(y) \) represents the coordinate transformation \( \phi \circ \varphi^{-1} \).

Let \( N \) and \( M \) be smooth manifolds. Let \( F : N \rightarrow M \) be a smooth mapping. The differential of \( F \) at \( p \in N \) is the map

\[ F_* : T_p N \rightarrow T_p \rightarrow T_{F(p)} M \]

defined as

\[ (F_*(v))(\lambda) = v(\lambda \circ F), \]

where \( v \in T_p N \) and \( \lambda \in C^\infty(F(p)) \).

Let \( (U, \phi) \) be a coordinate chart around \( p \), \( (V, \varphi) \) a coordinate chart around \( q = F(p) \). The natural basis of \( T_p N \) and \( T_q M \) are \( \left\{ \left( \frac{\partial}{\partial \varphi_1} \right)_p, \left( \frac{\partial}{\partial \varphi_2} \right)_p, \cdots, \left( \frac{\partial}{\partial \varphi_n} \right)_p \right\} \) and \( \left\{ \left( \frac{\partial}{\partial \varphi_1} \right)_q, \left( \frac{\partial}{\partial \varphi_2} \right)_q, \cdots, \left( \frac{\partial}{\partial \varphi_n} \right)_q \right\} \), respectively. Denote the mapping \( \varphi \circ F \circ \phi^{-1} \) as

\[ F(x) = F(x_1, x_2, \cdots, x_n) = \begin{pmatrix}
  F_1(x_1, x_2, \cdots, x_n) \\
  F_2(x_1, x_2, \cdots, x_n) \\
  \vdots \\
  F_m(x_1, x_2, \cdots, x_n)
\end{pmatrix}. \]

Suppose \( v \in T_p N \) and \( w = F_*(v) \in T_{F(p)} M \) are expressed as

\[ v = \sum_{i=1}^n v_i \left( \frac{\partial}{\partial \varphi_i} \right)_p, \quad w = \sum_{i=1}^m w_i \left( \frac{\partial}{\partial \varphi_i} \right)_q, \]

then

\[ \begin{pmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_m 
\end{pmatrix} = \begin{bmatrix}
  \partial F_1 & \partial F_1 & \cdots & \partial F_1 \\
  \partial F_2 & \partial F_2 & \cdots & \partial F_1 \\
  \vdots & \vdots & \ddots & \vdots \\
  \partial F_m & \partial F_m & \cdots & \partial F_m
\end{bmatrix}
\begin{pmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n
\end{pmatrix}. \]

### 2.4 Vector Fields

Consider a smooth manifold \( N \) of dimension \( n \). A vector field \( f \) on \( N \) is a mapping assigning to each point \( p \in N \) a tangent vector \( f(p) \) in \( T_p N \). A
A vector field $f$ is smooth if for each $p \in N$ there exists a coordinate chart $(U, \phi)$ about $p$ and $n$ real-valued smooth function $f_1, f_2, \ldots, f_n$ defined on $U$ such that for all $q \in U$

$$f(q) = \sum_{i=1}^{n} f_i(q) \left( \frac{\partial}{\partial \phi_i} \right)_q.$$  

In local coordinates, $f_i$ can be expressed as

$$f_k = f_i \circ \phi^{-1}.$$  

If $p$ is a point of coordinates $(x_1, x_2, \ldots, x_n)$ in the chart $(U, \phi)$, $f(p)$ is a tangent vector of coefficients $(f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n))$ in the basis $\left\{ \left( \frac{\partial}{\partial \phi_1} \right)_p, \left( \frac{\partial}{\partial \phi_2} \right)_p, \ldots, \left( \frac{\partial}{\partial \phi_n} \right)_p \right\}$ of $T_p N$. Usually, $f_i$ is used to replace $f_i \circ \phi^{-1}$, therefore, $f$ in the local coordinates is given by

$$f = \text{col} (f_1, f_2, \ldots, f_n).$$

A smooth curve $\sigma : (t_1, t_2) \rightarrow N$ is an integral curve of $f$ if

$$\sigma_* \left( \frac{d}{dt} \right)_t = f(\sigma(t))$$

for all $t \in (t_1, t_2)$. By

$$f(\sigma(t)) = \sum_{i=1}^{n} f_i(\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t)) \left( \frac{\partial}{\partial \phi_i} \right)_{\sigma(t)}$$

$$\sigma_* \left( \frac{d}{dt} \right)_t = \sum_{i=1}^{n} \frac{d\sigma_i}{dt} \left( \frac{\partial}{\partial \phi_i} \right)_{\sigma(t)}.$$  

One obtains

$$\frac{d\sigma_i}{dt} = f_i(\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t)).$$

Let $f$ be a smooth vector field on $N$ and $\lambda$ a smooth real valued function on $N$. The derivative of $\lambda$ along $f$ is a function $N \rightarrow \mathbb{R}$, defined as

$$(L_f \lambda)(p) = (f(p))(\lambda).$$

In the local coordinates,

$$(L_f)(x_1, x_2, \ldots, x_n) = \left( \frac{\partial \lambda}{\partial x_1} \frac{\partial \lambda}{\partial x_2} \ldots \frac{\partial \lambda}{\partial x_n} \right) \left( \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \right).$$

The set of smooth vector fields on a manifold $N$, denoted by $V(N)$, is a vector space over $\mathbb{R}$. The vector space $V(N)$ is a Lie algebra if a binary operation $V \times V \rightarrow V$, called a product and denoted by $[, [, ]$, is defined such that
(i) \([v, w] = -[w, v]\);
(ii) \([\alpha_1 v_1 + \alpha_2 v_2, w] = \alpha_1 [v_1, w] + \alpha_2 [v_2, w]\);
(iii) \([v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0\).

If the product \([\cdot, \cdot]\) is defined as

\[
([f, g](p))(\lambda) = (L_f L_g \lambda)(p) - (L_g L_f \lambda)(p),
\]

the set \(V(N)\) with the product forms a Lie algebra.

The product \([f, g]\) in local coordinates is given by

\[
\begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n}
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{pmatrix}
- \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}
\begin{pmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{pmatrix}
= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.
\]

The repeating product is possible. To avoid the notation of \([f, [f, \cdots [f, g] \cdots]]\) for recursive operation, define

\[
ad^k_f g(x) = [f, ad^{k-1}_f g](x)
\]

for \(k \geq 1\), where \(ad^0_f g(x) = g(x)\).

2.5 Distributions

A distribution \(D\) on a manifold \(N\) is a map which assigns to each \(p \in N\) a linear subspace \(D(p)\) of the tangent space \(T_p N\). If for each \(p \in N\) there exists a neighborhood \(U\) of \(p\) and a set of smooth vector fields \(X_i, i \in I\), such that

\[
D(q) = \text{span}\{X_i(q), i \in I\}, \quad q \in U.
\]

The dimension of a distribution \(D\) at \(p \in N\) is the dimension of the subspace \(D(p)\). A distribution is constant dimensional if the dimension of \(D(p)\) does not depend on the point \(p \in N\).

Let \(D\) be a constant dimensional distribution of dimension \(k\). Then around any \(p \in M\) there exist \(k\) independent vector fields \(X_1, X_2, \cdots, X_k\) such that

\[
D(q) = \text{span}\{X_1(q), X_2(q), \cdots, X_k(q)\}.
\]

The vector fields \(X_1, X_2, \cdots, X_k\) are called the local generators of \(D\). Every vector field \(X \in D\) can be represented by

\[
X(q) = \sum_{i=1}^{k} \alpha_i(q) X_i(q)
\]
for some smooth function $\alpha_i$, $i = 1, 2, \cdots, k$.

A distribution $D$ is involutive if

$$[X, Y] \in D$$

for all $X \in D$ and $Y \in D$.

A submanifold $P$ of $M$ is an integral manifold of a distribution $D$ on $M$ if

$$T_q P = D(q), \forall q \in P.$$ 

Let $X_1, X_2, \cdots, X_k$ be linearly independent vector fields with $[X_i, X_j] = 0$, $1 \leq i, j \leq k$. Then there exist local coordinates such that

$$X_i = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq k.$$ 

In other words, if $D$ is an involutive distribution of constant dimension $k$, then there exist local coordinates $x_1, x_2, \cdots, x_n$ such that

$$D = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_k}\}.$$
Normal Forms of Nonlinear Systems Affine in Control

The nonlinear extensions of both finite and infinite zero structures of linear systems have been well understood for single input single output systems and have found many applications in nonlinear control theory. The extensions of these notions to multiple input multiple output systems have proven to be highly sophisticated. Existing extensions either were made under restrictive assumptions that not even square invertible linear systems can satisfy or do not represent the nonlinear extensions of the related linear system notions. In this chapter, we propose constructive algorithms for decomposing a nonlinear system that is affine in control. These algorithms require modest assumptions on the system and apply to general multiple input multiple output systems that do not necessarily have the same number of inputs and outputs. They lead to various normal form representations and reveal the structure at infinity, the zero dynamics and the invertibility properties, all of which represent nonlinear extensions of relevant linear system structural properties of the system they represent.

3.1 Introduction

The nonlinear analogues of linear system structural properties, such as relative degrees (or infinite zero structure), zero dynamics (or finite zero structure) and invertibility properties, have played critical roles in recent literature on the analysis and control design for nonlinear systems (see, e.g., [2, 16, 15, 64] and the references therein for a sample of this literature). The normal forms that are associated with these structural properties, along with the basic tools like those reported in [17, 65, 68], have enabled many major breakthroughs in nonlinear control theory.

Consider a multiple input multiple output (MIMO) nonlinear system affine in control

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x),
\end{align*}
\]

(3.1)
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^p \) are the state, input and output, respectively. Let the mappings \( f \), \( g \) and \( h \) be smooth in an open set \( U \subset \mathbb{R}^n \) containing the origin \( x = 0 \), with \( f(0) = 0 \) and \( h(0) = 0 \).

A single input single output system, i.e., \( m = p = 1 \) in (3.1), has a relative degree \( r \) at \( x = 0 \) if

\[
L_g L_f^k h(x) = 0, \quad k < r - 1,
\]

in a neighborhood of \( x = 0 \), and

\[
L_g L_f^{r-1} h(0) \neq 0. \tag{3.3}
\]

If system (3.1) has a relative degree \( r \), then on an appropriate set of coordinates in a neighborhood of \( x = 0 \), it takes the following normal form (see, e.g., [58]),

\[
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi), \\
\dot{\xi}_i &= \xi_{i+1}, & i &= 1, 2, \ldots, r - 1, \\
\dot{\xi}_r &= a_1(\eta, \xi) + b_1(\eta, \xi)u, \\
y &= \xi_1,
\end{aligned}
\tag{3.4}
\]

where \( \xi = \text{col}\{\xi_1, \xi_2, \ldots, \xi_r\} \), \( b_1(0, 0) \neq 0 \), and \( \dot{\eta} = f_0(\eta, 0) \) is the zero dynamics. With a state feedback, this normal form reduces to the zero dynamics cascaded with a clean chain of integrators linking the input to the output. Here by clean we mean that no other signal enters the middle of the chain.

Such a nice feature is extended to nonlinear systems with more than one input output pairs. That is, a special class of square invertible nonlinear system with \( m = p > 1 \) can be transformed into the zero dynamics cascaded with \( m \) clean chains of integrators. To do this, the notion of vector relative degree was introduced in [18, 19]. System (3.1) with \( m = p > 1 \) has a vector relative degree \( \{r_1, r_2, \ldots, r_m\} \) at \( x = 0 \) if

\[
L_g L_f^k h_i(x) = 0, \quad 0 \leq k < r_i - 1, \quad 1 \leq i, j \leq m, \tag{3.5}
\]

in a neighborhood of \( x = 0 \), and

\[
\det\{L_g L_f^{r_i-1} h_i(0)\}_{m \times m} \neq 0. \tag{3.6}
\]

If system (3.1) has a vector relative degree \( \{r_1, r_2, \ldots, r_m\} \) at \( x = 0 \), then with an appropriate change of coordinates, it can be described by

\[
\begin{aligned}
\dot{\eta} &= f_0(x) + g_0(x)u, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1}, & j &= 1, 2, \ldots, r_i - 1, \\
\dot{\xi}_{i,r_i} &= a_i(x) + b_i(x)u, \\
y_i &= \xi_{i,1}, & i &= 1, 2, \ldots, m,
\end{aligned}
\tag{3.7}
\]

which contains \( m \) clean chains of integrators. Moreover, if the distribution spanned by the column vectors of \( g(x) \) is involutive in a neighborhood of \( x = 0 \), a set of local coordinates can be selected such that \( g_0(x) = 0 \). The clean
chains of integrators are called a prime form in [10] for linear systems, and the necessary and sufficient geometric conditions for the existence of prime forms for nonlinear systems is developed in [20]. There is a large body of nonlinear systems and control literature based on the form (3.7) (see e.g., [69, 74], for a small sample).

The conditions for the existence of a vector relative degree, (3.5) and (3.6), though similar to (3.2) and (3.3) in form, are not easy to be satisfied. Simple change of coordinates in the output space could alter the property (3.5). That is, (3.5) is satisfied only under certain output coordinates. Consider the linear system \((A, B, C)\) from [22],

\[
A = \begin{bmatrix}
1 & 1 & -2 & 0 & 0 \\
0 & 5 & -4 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 \\
-2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
1 & 1 \\
0 & 0 \\
-1 & 1 \\
0 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1 & -2 & 0 & 0 \\
0 & 1 & -2 & 0 & 1
\end{bmatrix}.
\]

(3.8)

As shown in [22], the system does not possess a vector relative degree. If we apply an output transformation

\[
T_o = \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix},
\]

it can be verified that \((A, B, T_o C)\) has a vector relative degree \(\{1, 2\}\). In other words, the system (3.1) meets the vector relative degree conditions only under appropriate coordinates of the output space.

In general, the vector relative degree is a rather restrictive structural property that not even all square invertible linear systems, with the freedom of choosing coordinates for the state, output and input spaces and state feedback, could possess. A square invertible linear system with \(m = p > 1\) in general can only be transformed into the zero dynamics cascaded with \(m\) chains of integrators, with all but one chains containing output injection terms (see [75]). That is, there are interconnections between these chains. For example, consider a linear system \((A, B, C)\) with

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\alpha & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \alpha \neq 0.
\]

(3.9)

The system contains two chains of integrators of lengths 1 and 3. The parameter \(\alpha\) represents an output injection term, which in turn represents the interconnections between the two chains. Such an interconnection cannot be removed through coordinate transformations and state feedback, and thus system (3.9) cannot be represented by two clean chains of integrators. In other
words, even with the freedom of choosing coordinates and static state feedback, \((3.9)\) does not have a vector relative degree. To see this, suppose that there exist nonsingular coordinate transformations \(T_s, T_i\) and \(T_o\) such that

\[
\tilde{A} = T_s^{-1}AT_s = \begin{bmatrix}
* & * & * & * \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
* & * & * & * 
\end{bmatrix}, \quad T_s^{-1}BT_i = B, \quad T_o^{-1}CT_s = C,
\]

which indicates that the system can be decoupled into two clean chains of integrators. Denote \(T_s = \{t_{i,j}\}_{4 \times 4}\). By \(T_sB = BT_i\) and \(CT_s = T_oC\), we obtain

\[
t_{1,3} = t_{1,4} = t_{2,1} = t_{2,3} = t_{3,1} = t_{3,4} = 0. \quad \text{The (2, 1) entry of } AT_s - T_s\tilde{A}
\]
is \(\alpha t_{1,1} = 0\). So, \(t_{1,1} = 0\), consequently, \(T_s\) is singular. This is a contradiction. Similarly, \(A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
\alpha & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad \alpha \neq 0.
\]

The system in \((3.10)\) does not have a vector relative degree even with the freedom of choosing coordinates and static state feedback. The parameter \(\alpha\) here represents an input coupling term between the two chains.

A major generalization of the form \((3.7)\) was made in \([16, 17, 21, 22]\), where MIMO square invertible systems are considered. In \([16]\), with Zero Dynamics Algorithm, a sequence of nested submanifolds \(M_0 \supset M_1 \supset \cdots \supset M_k \supset \cdots = Z^*\) are defined, and system \((3.1)\) is transformed into the form,

\[
\begin{align*}
\dot{\eta} &= f_0(x) + g_0(x)u, \\
\dot{x}_{i,j} &= x_{i,j+1} + \sum_{i=1}^{n_i-1} \delta_{i,j,l}(x)v_l + \sigma_{i,j}(x)u, \quad j = 1, 2, \cdots, n_i - 1, \\
\dot{x}_{i,n_i} &= v_i, \\
y_i &= x_{i,1}, \quad i = 1, 2, \cdots, m,
\end{align*}
\]

where \(\sigma_{i,j}(x) = 0, j = 1, 2, \cdots, n_i - 1,\) and \(\sigma_{i,j}(x) = 0, i > 1, j = 1, 2, \cdots, n_i - 1,\) in \(Z^*\), and the static state feedback is given by \(v_i = a_i(x) + b_i(x)u, i = 1, 2, \cdots, m,\) with the matrix \(\text{col}\{b_1(x), b_2(x), \cdots, b_m(x)\}\) being smooth and nonsingular. In the algorithm, the rank of certain matrices are assumed to be constant on these nested submanifolds. With some stronger assumptions imposed in the algorithm \([17, 22]\), i.e., the rank of certain matrices were assumed to be constant for all \(x \in U\) (not just in these submanifolds), one can have all \(\sigma_{i,j}(x) = 0\). Moreover, if certain vector fields commute, one can select coordinates such that \(g_0(x) = 0\). Thus, system \((3.11)\) becomes...
The applications of the form (3.12) in solving the problem of asymptotic stabilization, disturbance decoupling, tracking and regulation can be found in [17] and the references therein.

The infinite zeros of a linear system can be defined either through the root locus theory or as the Smith-McMillan zeros of the transfer function at infinity [76, 77]. They can also be characterized in state-space [10, 11]. On the other hand, the structure at infinity was introduced for a certain class of nonlinear systems in [78], and was further developed for smooth systems or analytic systems in [79] and for meromorphic systems in [80–82].

In [79] and [2] (Chapter 9), formal zeros at infinity are defined in terms of a set of geometric conditions. In particular, for system (3.1) defined on a smooth manifold $M$, a sequence of the locally controlled invariant distributions $D_0 \supset D_1 \supset \cdots \supset D_n$ in $\ker dh(x)$ are defined, where

$$
\begin{align*}
\mathcal{D}_0 &= TM, \\
\mathcal{D}_{i+1} &= \{ X \in V(M) | [f, X] \in \mathcal{D}_i + G, [g_j, X] \in \mathcal{D}_i + G, \ j = 1, 2, \cdots, m \} \\
&\cap \ker dh, \quad i = 1, 2, \cdots, n - 1,
\end{align*}
$$

and where $G = \text{span} \{g_1, g_2, \cdots, g_m\}$, $V(M)$ denotes the set of smooth vector fields on a smooth manifold $M$, and $TM$ denotes the tangent bundle of $M$. Under the assumption of the distributions $D_i$ and $D_i \cap G$ on $M$ having constant dimensions, the formal zeros at infinity can be defined. Formal zeros at infinity plays an important role in the input output decoupling problem by static state feedback, in which after possible relabeling of inputs, the control $u_i$ does not influence the output $y_j$, $j \neq i$. But this structure information does not show in a normal form in the references [2, 79].

In [50, 82], a linear-algebraic strategy is developed based on the use of vector spaces over the field of meromorphic functions. As a counterpart to the above differential-geometric approach, the algebraic approach considers system (3.1) with $f, g$ and $h$ being meromorphic. Except for some singular points, the two approaches lead to the same results as in [83], in particular, the same notions of rank and structure at infinity. The structure at infinity is related to a chain of subspaces $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n$, where

$$
\mathcal{E}_0 = \text{span}_K \{dx\},
$$
\[ \mathcal{E}_i = \text{span}_K \{dx, dy, \cdots, dy^{(i)}\}, \quad i = 1, 2, \cdots, n, \]

and where \( K \) are meromorphic functions. The structure at infinity is then determined by

\[ \sigma_k = \dim_K \frac{\mathcal{E}_k}{\mathcal{E}_{k-1}}. \]

With a generalized state space transformation, a regular generalized state feedback and a universal, additive output injection, system (3.1) can be transformed into a canonical form, which contains time derivatives of inputs and shows the structure at infinity explicitly.

As pointed out in [16], if all \( \delta_{i,j,l}(x) = 0 \), the set of integers \( \{n_1, n_2, \cdots, n_m\} \) in (3.12) corresponds to the vector relative degree, which in this case, represents the infinite zero structure if the system is linear. These integers however are not related to the infinite zero structure of linear systems when \( \delta_{i,j,l}(x) \neq 0 \), and thus cannot be defined as the nonlinear extension of and expected to play a similar role as infinite zeros. To see this, consider the following linear system \((A,B,C)\),

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
\alpha & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}, \\
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}, \quad \alpha \neq 0,
\]

which is in the form of (3.12) with \( n_1 = 2 \) and \( n_2 = 3 \). However, by using the toolkit [25, 84, 85], we can find state, input and output transformations \( T_s \), \( T_i \) and \( T_o \) such that

\[
T_s^{-1} A T_s = \begin{bmatrix}
1 & 0 & 0 & -\alpha & 0 \\
1/\alpha & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad T_s^{-1} B T_i = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}, \\
T_o^{-1} C T_s = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

Thus, according to [10, 75], the system is invertible with two infinite zeros \( \{1, 4\} \). Therefore, the integers \( n_1 \) and \( n_2 \) in the form (3.12) does not generalize the notion of infinite zero structure of linear systems.
Invertibility of linear systems was first studied in [86–88]. In these references, inversion algorithms and invertibility criteria are given. Invertibility of nonlinear control systems was considered in [53–63], which generalized the structure algorithm for linear systems [88]. Refs. [83, 89, 90] carry out a systematic study of invertibility of general nonlinear systems that are not necessarily affine in control. The authors gave a list of equivalent conditions for right and left invertibility for linear systems, and examined when and how these conditions can be generalized to nonlinear systems. Based on [63], ref. [59] explicitly constructs the left inverse of an affine output-input stable system.

Invertibility of nonlinear systems can also be determined by using the structure algorithm in [50, 81–83]. In particular, with a generalized state space transformation, a regular generalized state feedback and a universal, additive output injection, system (3.1) can be transformed into a canonical form, which contains time derivatives of inputs and shows the structure at infinity and invertibility structures explicitly.

A key feature of the normal forms is that they represent a system in several interconnected subsystems. These subsystems, along with the interconnections that exist among them, lead us to a deeper insight into how control would take effect on the system, and thus to the construction of control laws that meet our design specifications. The structure of a linear system, characterized by a matrix triple \((A, B, C)\), has been studied in great depth. In 1973, Morse [10] showed that, under a group of state, input and output transformations, state feedback and output injection, any matrix triple \((A, B, C)\) is uniquely characterized by three lists of positive integers and a list of monic polynomials. By identifying state variables in the structure algorithm in [88], Sannuti and Saberi [75] explicitly constructed state, input and output transformations that transform a general MIMO system, not necessarily square, into a so-called special coordinate basis form, which displays all structural properties of the system, including the finite and infinite zero structures and invertibility properties.

Motivated by the many efforts reported in the nonlinear control literature and a complete understanding and numerous applications of the structural decomposition of linear systems, we make an attempt to study structural properties of affine nonlinear systems beyond the case of square invertible systems. For a general nonlinear system (3.1) in the absence of the vector relative degree assumption, we develop an algorithm, which is referred to as the infinite zero structure algorithm and, under certain constant rank assumptions over \(U\), results in diffeomorphic state, input and output transformations and state feedback laws under which the system can be represented in normal forms. In the special case when the system is square and invertible, our normal forms take a form similar to those in [17, 22], but with an additional property that allows the normal forms to reveal the nonlinear extension of infinite zeros of linear systems. In addition, our development enhances the existing results in some other ways. First, fewer assumptions are required. Second, the resulting
normal forms explicitly show invertibility structures and nonlinear extension of invariant zeros. Third, our development applies to general MIMO nonlinear systems that are not necessarily square.

The infinite zero structure algorithm will also be adapted to develop normal forms that reveal system structural properties when the output is restricted to zero. The adapted algorithm will be referred to as the zero output structure algorithm. The assumptions required will also be in the form of constant ranks, but in a sequence of nested subsets, rather than the more stringent constant ranks on $U$ as required by the infinite zero structure algorithm. Our results on zero output normal forms inherit the features pertaining to the infinite zero structure algorithm and thus enhance the existing results on the zero output normal forms in similar ways as the normal forms resulting from the infinite zero structure algorithm.

These normal forms include the ones identified in [16, 17, 21, 22] for square invertible systems as special cases. In particular, Under the milder assumptions on nonlinear systems, and by carefully selecting new coordinates, simpler normal forms can be derived. These normal forms not only reveal the infinite zero structure and zero dynamics of the system, but also provide explicit information on the system invertibility properties. So far, the structure at infinity is related only to input-output decoupleable nonlinear systems. In the chapter, we try to extend the concept of structure at infinity to input-output coupling nonlinear In doing so, we introduce the notions of infinite zero of nonlinear systems. The systems are not necessarily square. We also explore the structural properties of nonlinear systems along the trajectory in which the output is and introduce the notions of zero-structure at infinity, zero-invertibility of nonlinear systems at an equilibrium point $x = 0$.

The remainder of this chapter is organized as follows. The infinite zero structure algorithm and the resulting normal forms are presented in Section 3.2. The zero output structure algorithm and the resulting normal forms are given in Section 3.3. Section 3.4 contains a few examples that illustrate the main results of the chapter. A brief conclusion to the chapter is drawn in Section 3.5. For clarity in the presentation, all proofs are given in the appendices.

### 3.2 Normal Forms and Structure Properties of Nonlinear Systems

In this section, we will find diffeomorphic state, input and output transformations and static state feedback laws under which system (3.1) can be represented in normal forms and discuss about the intrinsic structural properties these normal forms reveal. Similarly to many existing results (see, e.g., [17, 22]), we rely on constant rank assumptions over $U$. However, as will become clear, our development here enhances the existing results in several ways. First, weaker assumptions are required. Second, normal forms with simpler structure are resulted in, based on which nonlinear extension of infinite zeros
can be defined. Third, the resulting normal forms explicitly show invertibility structures and nonlinear extension of invariant zeros. Finally, our normal form development applies to nonlinear systems that are not necessarily square.

In particular, we first separate from the overall system dynamics the dynamics associated with the infinite zeros, and then carry out some further decomposition of the zero dynamics and the remaining dynamics.

### 3.2.1 The Infinite Zero Structure Algorithm

Both our algorithm and the algorithm in [17, 22] involve repetitive differentiations of the output and, under certain constant rank assumptions, identification of functions to serve as new state variables. What distinguishes our algorithm is how we identify the new state variables. In each step of our algorithm, we identify not only \( \Theta_k(x) \), from which new state variables will be selected, but also \( \Omega_k(x) \), which contains \( \Omega_{k-1}(x) \) and part of \( \Theta_{k-1}(x) \), in such a way that \( L_g \Omega_k(x) \) is of full row rank and

\[
\text{rank}(L_g \Omega_k(x)) = \text{rank}(L_g \{ \Theta_0(x), \Theta_1(x), \ldots, \Theta_{k-1}(x) \}).
\]

More specifically, we first identify \( \Omega_k(x) \), then define \( \Theta_k(x) \) to depend only on \( \Theta_{k-1}(x) \) and \( \Omega_k(x) \), rather than on \( \Theta_i(x) \), \( i = 1, 2, \ldots, k-1 \). Such an approach will be helpful in selecting state variables that render the more informative normal forms.

Moreover, by choosing the function \( \Theta_k(x) \) in such a way, we will be able to carry out the algorithm with fewer constant rank assumptions than the algorithm in [17, 22], and more importantly, allow the algorithm to be applicable to square but non-invertible systems and non-square systems.

We also will device criteria for the above repetitive procedure to stop. The times the derivatives are taken on each output variable and which stopping criterion is met determine the structure at infinity and the invertibility properties, respectively.

**Initial Step.** Let \( \Theta_0(x) = h(x) \), \( \Omega_0(x) = \emptyset \), \( \rho_0 = 0 \) and \( k = 1 \).

**Step k.** We start with \( \Theta_{k-1}(x) : U \to \mathbb{R}^{p-k-1} \), \( \Omega_{k-1}(x) : U \to \mathbb{R}^{\rho_k-1} \), where the matrix \( L_g \Omega_{k-1}(x) \) has full row rank \( \rho_k \). Suppose that the following assumption holds.

**Assumption \( A_k \):** The matrix \( \begin{bmatrix} L_g \Omega_{k-1}(x) \\ L_g \Theta_{k-1}(x) \end{bmatrix} \) has constant rank \( \rho_k \) for \( x \in U \), and there exists an \( R_k \in \mathbb{R}^{(p_k-\rho_k-1) \times (p-\rho_k-1)} \) such that the matrix \( \begin{bmatrix} L_g \Omega_{k-1}(x) \\ L_g R_k \Theta_{k-1}(x) \end{bmatrix} \) is of full row rank \( \rho_k \) for \( x \in U \).

Let \( S_k \in \mathbb{R}^{(p-\rho_k) \times (p-\rho_k-1)} \) be such that

\[
\det \left( \begin{bmatrix} R_k \\ S_k \end{bmatrix} \right) \neq 0. \tag{3.14}
\]

Denote
\[ \Omega_k(x) = \text{col} \{ R_1 \Theta_0(x), R_2 \Theta_1(x), \ldots, R_k \Theta_{k-1}(x) \}. \]  

(3.15)

The matrix \( L_g \Omega_k(x) \) has full row rank \( \rho_k \) and

\[
\text{rank} \left( \begin{bmatrix} L_g \Omega_k(x) \\ L_g S_k \Theta_{k-1}(x) \end{bmatrix} \right) = \rho_k.
\]

Thus, there exist unique smooth functions

\[ P_{k,l}(x) : U \rightarrow \mathbb{R}^{(p-\rho_k) \times (m-\rho_{l-1})}, \quad l = 1, 2, \ldots, k, \]

such that

\[
L_g S_k \Theta_{k-1}(x) - \sum_{l=1}^{k} P_{k,l}(x) L_g R_l \Theta_{l-1}(x) = 0.
\]

(3.16)

Define

\[
\Theta_k(x) = L_f S_k \Theta_{k-1}(x) - \sum_{l=1}^{k} P_{k,l}(x) L_f R_l \Theta_{l-1}(x).
\]

(3.17)

If \( k + \sum_{j=1}^{k} j (\rho_j - \rho_{j-1}) < n \) and \( \rho_k < \min\{p, m\} \), then increase \( k \) by 1 and repeat the above step. Otherwise, go to Final Step.

**Final Step.** Let \( k^* = k \), we have

\[ k^* + \sum_{j=1}^{k^*} j (\rho_j - \rho_{j-1}) = n \quad \text{or} \quad \rho_{k^*} = \min\{p, m\}. \]  

(3.18)

Let \( m_d = \rho_{k^*} \) and

\[ n_d = \sum_{j=1}^{k^*} j (\rho_j - \rho_{j-1}). \]

Denote the set \( \rho = \{\rho_1, \rho_2, \ldots, \rho_{k^*}\} \). Define a set of integers \( 0 < q_1 \leq q_2 \leq \ldots \leq q_{m_d} \) as

\[ q = \{q_1, q_2, \ldots, q_{m_d}\} = \{1, \ldots, 1, 2, \ldots, 2, \ldots, k^*, \ldots, k^*\}. \]

**End.**

**Definition 3.2.1** System (3.1) is said to be regular, if Assumption \( A_k, k = 1, 2, \ldots, k^* \), are satisfied.

**3.2.2 Normal Forms**

We will base on the infinite zero structure algorithm to derive normal forms of system (3.1). Denote
\[ v_k = L_j R_k \Theta_{k-1}(x) + L_y R_y \Theta_{k-1}(x) u. \] (3.19)

By (3.16) and (3.17),
\[ \Theta_j(x) = \frac{d}{dt} S_j \Theta_{j-1}(x) - \sum_{i=1}^{j} P_{j,i}(x)v_i, \quad j = 1, 2, \ldots, k^*. \] (3.20)

For the notational brevity, denote \( S_i S_{i-1} \cdots S_{j+1} S_j \) as \( S_{i+j} \), with \( S_{i+j} = 1 \) for \( j > i \). We first define the new states representing the dynamics of \( i (\rho_i - \rho_{i-1}) \) integrators, which connect the input \( L_y R_y \Theta_{i-1}(x) u \) to the output \( R_i S_{i-1+1} y \),
\[ \zeta_{i,j} = R_i S_{i-1+j} \Theta_{j-1}(x), \quad j = 1, 2, \ldots, i, \]
\[ \zeta_i = \text{col} \{ \zeta_{i,1}, \zeta_{i,2}, \ldots, \zeta_{i,i} \}, \]
\[ = \text{col} \{ R_i S_{i-1+j} \Theta_0(x), R_i S_{i+2+j} \Theta_1(x), \ldots, R_i \Theta_{i-1}(x) \}, \quad i = 1, 2, \ldots, k^*. \]

Note that \( \zeta_{i,j} : U \rightarrow \mathbb{R}^{\rho_i - \rho_{i-1}} \). In view of (3.19) and (3.20), we have,
\[ \dot{\zeta}_{i,j} = \zeta_{i,j+1} + R_i S_{i-1+j+1} \sum_{l=1}^{j} P_{j,l}(x)v_l, \quad j = 1, 2, \ldots, i - 1, \]
\[ \dot{\zeta}_{i,j} = v_i, \quad i = 1, 2, \ldots, k^*. \] (3.21)

Let
\[ \Phi_d(x) = \text{col} \{ \zeta_1, \zeta_2, \ldots, \zeta_k \}, \]
\[ \Gamma_{id}(x) = \text{col} \{ L_y R_k \Theta_0(x), L_y R_k \Theta_1(x), \ldots, L_y R_k \Theta_{k-1}(x) \} \]
\[ = L_y \Omega_k \Theta_k(x), \]
\[ \Gamma_{od}(x) = \text{col} \{ R_1, R_2 S_1, \ldots, R_k, S_{k-1+1} \} . \] (3.23) (3.24)

It is obvious that \( \Phi_d(x) : U \rightarrow \mathbb{R}^{n_x}, \Gamma_{id}(x) : U \rightarrow \mathbb{R}^{m_i \times m} \) and \( \Gamma_{od} \in \mathbb{R}^{m_d \times p} \).

To construct a new set of coordinates, we need the following assumption.

**Assumption B:** The matrix \( d \Phi_d(x) \) is of full row rank for \( x \in U \).

Note that Assumption B is automatically satisfied if \( P_{k,i}(x), \quad l = 1, 2, \ldots, k, \)
\( k = 1, 2, \ldots, k^* \), in the infinite zero structure algorithm are independent of \( x \).

**Lemma 3.1.** Suppose that system \( (3.7) \) is regular, and that \( P_{k,i}(x), \quad l = 1, 2, \ldots, k, \quad k = 1, 2, \ldots, k^* \), in the infinite zero structure algorithm are constant matrices. Then, \( d \Phi_d(x) \) is of full row rank for \( x \in U \).

**Proof:** See Appendices [3.6.1](#). 

By the infinite zero structure algorithm, we know that \( \Gamma_{id}(x) = L_y \Omega_k \Theta_k(x) \) is of full row rank. Note that \( R_i S_{i-1+1}, \quad i = 1, 2, \ldots, k^* \), are the coefficients in \( \zeta_{i,1} = R_i S_{i-1+1} \Theta_0(x) \). Under Assumption B, \( \Gamma_{od} \) is of full row rank. In what follows, we augment the state variables \( \zeta_{i,j} \)'s with \( n - n_d \) additional state variables to form a full set of state variables for the system. Similarly, we also need to augment the input variables \( v_i \)'s and the output variables \( \zeta_{i,1} \)'s with...
\[ m - m_d \text{ additional input variables and } p - m_d \text{ output variables to form a full input vector and output vector, respectively.} \]

Note that \( \zeta_{i,j} \text{ contains } \rho_i - \rho_{i-1} \text{ states, and thus } \zeta_{i,j}, j = 1, 2, \ldots, i, \text{ define } \rho_i - \rho_{i-1} \text{ chains containing a total of } i(\rho_i - \rho_{i-1}) \text{ integrators. If } \rho_i - \rho_{i-1} > 1, \text{ we introduce the permutation matrix } \Xi(\rho_i - \rho_{i-1}, i) \text{ to reorder the states such that each chain contains } (\rho_i - \rho_{i-1}) \text{ integrators and corresponds to only one input and one output, where } \Xi(s, t) \in \mathbb{R}^{i \times st} \text{ with} \]

\[
\Xi(s, t) = \begin{bmatrix}
  e_1 & e_{t+1} & \cdots & e_{(s-1)t+1} & e_2 & e_{t+2} & \cdots & e_{(s-1)t+2} \\
  & \cdots & & \cdots & & \cdots & & \cdots \\
  & & e_{s+t+s} & \cdots & e_{st} 
\end{bmatrix}^T
\]

and \( e_i \) being the \( i \)-th column of the identity matrix \( I_{st} \). Define

\[
\xi = \Phi_d(x) = Y\Phi_d(x) = \text{col} \{ \Xi(\rho_1 - \rho_0, 1), \Xi(\rho_2 - \rho_1, 2), \cdots, \Xi(\rho_k - \rho_{k-1}, k^*) \}, \quad (3.25)
\]

where \( Y = \text{blkdiag} \{ \Xi(\rho_1 - \rho_0, 1), \Xi(\rho_2 - \rho_1, 2), \cdots, \Xi(\rho_k - \rho_{k-1}, k^*) \} \).

Note that if \( \rho_i - \rho_{i-1} \leq 1 \) for \( 0 \leq i \leq k^* \), then \( Y = I. \) Define a new set of coordinates,

\[
\begin{pmatrix}
  \eta \\
  \xi 
\end{pmatrix}
= \begin{pmatrix}
  \Phi_e(x) \\
  \Phi_d(x)
\end{pmatrix}
= \Phi(x), \quad \begin{pmatrix}
  u_c \\
  u_d
\end{pmatrix}
= \Gamma_1(x)u = \begin{bmatrix}
  \Gamma_{ie}(x) \\
  \Gamma_{id}(x)
\end{bmatrix}
\]

\[
\begin{pmatrix}
  y_c \\
  y_d
\end{pmatrix}
= \Gamma_2 y = \begin{bmatrix}
  \Gamma_{oe} \\
  \Gamma_{od}
\end{bmatrix}
y, \quad (3.26)
\]

where \( \Phi_e(x) : U \rightarrow \mathbb{R}^{n-m_d} \) is smooth and such that \( \Phi(x) \) is a diffeomorphism on \( x \in U, \Gamma_{ie}(x) : U \rightarrow \mathbb{R}^{(m-m_d) \times m} \) is smooth and such that the matrix \( \Gamma_1(x) \) is nonsingular, and \( \Gamma_{oe} \in \mathbb{R}^{(p-m_d) \times p} \) is such that the constant matrix \( \Gamma_o \) is nonsingular.

The variables \( \xi, u_d \) and \( y_d \) correspond to the structure at infinity, and the variables \( \eta, u_c \) and \( y_c \) represent the additional state, input and output variables, respectively, to form complete sets of state, input and output variables.

Denote

\[
\text{col} \{ a_1(x), a_2(x), \cdots, a_{m_d}(x) \} = L_f \Omega_k(x), \quad \text{col} \{ b_1(x), b_2(x), \cdots, b_{m_d}(x) \} = L_g \Omega_k(x).
\]

Let \( \delta_{i,1,1}(x) \) be smooth functions and

\[
\delta_i(x) = \begin{bmatrix}
  \delta_{i,1,1}(x) & \delta_{i,1,2}(x) & \cdots & \delta_{i,1,i-1}(x) \\
  \delta_{i,2,1}(x) & \delta_{i,2,2}(x) & \cdots & \delta_{i,2,i-1}(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  \delta_{i,q-1,1}(x) & \delta_{i,q-1,2}(x) & \cdots & \delta_{i,q-1,i-1}(x) \\
  0 & 0 & \cdots & 0
\end{bmatrix}, \quad i = 1, 2, \cdots, m_d.
\]
Define
\[
\text{col} \{ [\delta_1(x) \ 0], [\delta_2(x) \ 0], \ldots, [\delta_{\mu_1(x)}(x)] \} = T \text{col} \{ [\mu_1(x) \ 0], [\mu_2(x) \ 0], \ldots, [\mu_{\mu_k}(x)] \},
\] (3.27)
where
\[
\mu_i(x) = \text{blkdiag} \{ R_{iS_{i-1}+2}, R_{iS_{i-1}+3}, \ldots, R_i, I_{\rho_i-\rho_{i-1}} \}
\]
\[
\begin{bmatrix}
P_{1,1}(x) & 0 & \cdots & 0 \\
0 & P_{2,2}(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P_{i-1,i-1}(x) \\
\end{bmatrix}.
\]

We have the following result.

**Theorem 3.2.1** Suppose that system (3.1) is regular, and that Assumption \( B \) holds. Let \( \text{col} \{ b_1(x), b_2(x), \ldots, b_{\mu_1}(x) \} \) be as obtained in the infinite zero structure algorithm. Then there exist a set of coordinates in \( U \), i.e., diffeomorphic state, input and output transformations, such that the system takes the following form,

\[
\begin{cases}
\dot{\eta} = f_e(x) + g_e(x)u_e + \sum_{l=1}^{\mu_1(x)} \phi_l(x)v_{d,l}, \\
\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{d,l}, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} = v_{d,i}, \\
y_e = h_e(x), \\
y_{d,i} = \xi_{i,1}, \quad i = 1, 2, \ldots, m_d,
\end{cases}
\]
(3.28)

where \( v_{d,i} = a_i(x) + b_i(x)u_i \), \( i = 1, 2, \ldots, m_d \), with \( \text{col} \{ b_1(x), b_2(x), \ldots, b_{\mu_1}(x) \} \) being nonsingular for \( x \in U \), and

\[
\delta_{i,j,l}(x) = 0, \quad \text{for} \ j < q_i, \ i = 1, 2, \ldots, m_d.
\] (3.29)

By (3.21), the dynamics \( \dot{\xi}_{i,j} \) does not relate to the state feedbacks \( v_l \) with \( l > j \). Inequality (3.29) follows from this fact. Indeed, (3.29) can be combined into the from (3.28) by replacing \( \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{d,l} \) with \( \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{d,l} \).

**Remark 3.2.1** The results of Theorem 3.2.1 are applicable to general MIMO systems that are not necessarily square. For square and invertible systems, normal form (3.28) is in the same form as the one derived in [17, 22], where no vector relative degree assumption is required either. However, normal form (3.28) possesses an extra property (3.29) (see Example 3.9). As will be seen, such a property plays a key role in defining the nonlinear extension of the infinite zeros of linear systems.
In what follows, we further simplify the normal form in Theorem 3.2.1.

Assumption $C$: There exists a $\Gamma_i(x)$ in (3.26) such that the distribution spanned by the column vectors of $g^d(x) = g(x)\Gamma_i^{-1}(x) \begin{bmatrix} 0 \\ I_{m_d} \end{bmatrix}$ is involutive.

**Theorem 3.2.2** Suppose that the conditions in Theorem 3.2.1 and Assumption $C$ are satisfied. Then there exists a set of coordinates in $\mathcal{U}$ such that the system takes the form of Theorem 3.2.1 with

$$\varphi_l(x) = 0, \quad l = 1, 2, \ldots, m_d.$$  \hfill (3.30)

**Proof:** See Appendices 3.6.2 \hfill \Box

Let us apply the infinite zero structure algorithm to a linear system $(A, B, C)$, i.e., system (3.1) with $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$.

It is obvious that Assumptions $A_k$, $k = 1, 2, \ldots, k^*$, $B$ and $C$ automatically hold.

**Theorem 3.2.3** Consider a linear system $(A, B, C)$. There exist nonsingular state, input and output transformations, and a state feedback, such that the system takes the form

$$\begin{aligned}
\dot{\eta} &= A_{11}\eta + A_{12}\text{col}\{\xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m_d,1}\} + B_1 u_e, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{q_i-1} \delta_{i,j,l} v_{d,l}, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_{d,i}, \\
y_e &= C_1 \eta, \\
y_{d,i} &= \xi_{i,1}, \quad i = 1, 2, \ldots, m_d, 
\end{aligned} \quad \hfill (3.31)$$

where $(A_{11}, B_1, C_1)$ does not contain dynamics that is simultaneously controllable and observable, $v_{d,i} = a_i\text{col}\{\eta, \xi\} + b_i u, \quad i = 1, 2, \ldots, m_d$, with $\text{col}\{b_1, b_2, \ldots, b_{m_d}\}$ being nonsingular, and $\delta_{i,j,l} = 0$, for $j < q_i, \quad i = 1, 2, \ldots, m_d$. \hfill \Box

The form (3.31) can be achieved by some additional state transformation on the linear counterpart of (3.28). In (3.31), the dynamics of $\eta$ depends only on $\eta$ and $\xi_{i,1}, \quad i = 1, 2, \ldots, m_d$, and $y_e$ only depends on $\eta$. It can be verified that the finite zeros are given by the simultaneously uncontrollable and unobservable dynamics of $(A_{11}, B_1, C_1)$. The infinite zeros are $\{q_1, q_2, \ldots, q_{m_d}\}$. The system is left invertible if $u_e$ is absent, right invertible if $y_e$ is absent, invertible if both $u_e$ and $y_e$ are absent, and degenerate if both $u_e$ and $y_e$ are present.

Some remarks on the infinite zero structure algorithm and normal forms are given as follows.
Remark 3.2.2 In the infinite zero structure algorithm, there is only one constant rank assumption in each step, while in the constrained dynamics algorithm [2] and the zero dynamics algorithm [16,17,22], each step involves two constant rank assumptions. However, in the infinite structure algorithm, to construct a new set of coordinates, Assumption $B$ is needed. Assumption $B$ automatically holds if certain matrices are constant (see Lemma 3.1).

Remark 3.2.3 In the structure algorithm, the smooth matrix valued functions $P_{k,l}(x), l = 1, 2, \ldots, k$, can be found as follows. By (3.16),

$$L_gS_k \Theta_{k-1}(x)(L_g \Omega_k(x))^T = [P_{k,1}(x) P_{k,2}(x) \cdots P_{k,k}(x)] L_g \Omega_k(x)(L_g \Omega_k(x))^T.$$ 

The matrix $L_g \Omega_k(x)$ is of full row rank, thus

$$\det (L_g \Omega_k(x)(L_g \Omega_k(x))^T) \neq 0.$$ 

Therefore,

$$[P_{k,1}(x) P_{k,2}(x) \cdots P_{k,k}(x)] = [L_gS_k \Theta_{k-1}(x)][L_g \Omega_k(x)]^T[L_g \Omega_k(x)]^{-1}.$$ 

Remark 3.2.4 Suppose $dh(x)$ is of full row rank and $g(x)$ is of full column rank in $U$, then we can stop repeat Step $k$ in the infinite zero structure algorithm and go to Final Step if

$$[\max \{m,p\} - \rho_k - 1] + k + \sum_{j=1}^{k} j(\rho_j - \rho_{j-1}) = n,$$

rather than

$$k + \sum_{j=1}^{k} j(\rho_j - \rho_{j-1}) = n.$$ 

This will lead to fewer steps in the algorithm.

Remark 3.2.5 Consider $m = p$. If we further assume $P_{k,j}(x), j = 1, 2, \ldots, k, k = 1, 2, \ldots, k^*$ are independent of $x$, we obtain the structure algorithm of Chapter 5 in [16].

Remark 3.2.6 By Lemma 3.1, we do not request that the matrices $d\Theta_i(x), i = 1, 2, \ldots, k^*$ or their combinations have constant rank in $U$. By the infinite zero structure algorithm, we always can find $d\Phi_d(x)$, i.e., linear combinations of $d\Theta_i(x), i = 1, 2, \ldots, k^*$, has constant rank. And thus $d\Phi_d(x)$ can be used as part of the new state coordinate.
Remark 3.2.7 The infinite zero structure algorithm stops at Step $k^*$ when (3.18) is satisfied. Carrying on the algorithm further would not increase $\rho_k$. That is, $\rho_k = \rho_{k^*}$, for $k > k^*$. This can be seen in two cases. Case 1: $\rho_{k^*} = \min\{p, m\}$. Suppose there exists a $k^0 > k^*$ such that $\rho_{k^0} > \rho_{k^*} = \min\{p, m\}$. Then, by the algorithm, $L_\varnothing \Omega_{k^*}(x)$ is a $\rho_{k^0} \times m$ full row rank matrix and $p > \rho_{k^0}$. This is a contradiction. Case 2: $k^0 + \sum_{j=1}^{k^0} j(\rho_j - \rho_{j-1}) = n$. Suppose there exists a $k^0 > k^*$ such that $\rho_{k^0} > \rho_{k^*}$. Then, $\sum_{j=1}^{k^0} j(\rho_j - \rho_{j-1}) > k^*(\rho_{k^0} - \rho_{k^*}) + \sum_{j=1}^{k^0} j(\rho_j - \rho_{j-1}) \geq n$. However, it can be easily verified that $\text{col}\{d\zeta_1, d\zeta_2, \ldots, d\zeta_{k^0}\}_{x=0}$ is a $(\sum_{j=1}^{k^0} j(\rho_j - \rho_{j-1})) \times n$ matrix with a full row rank. This is also a contradiction.

Remark 3.2.8 As observed in [16], it is in general difficult to construct a set of coordinates such that (3.30) is satisfied. It entails the solution of a system of $n - n_d$ partial differential equations. However, in the special case that $\varphi_l(x)$, $l = 1, 2, \ldots, m_d$, in (3.28) are independent of $x$, i.e., $\varphi_l(x) = \varphi_l$ is a constant, by renaming the state variable

$$\bar{\eta} = \eta - \sum_{l=1}^{m_d} \varphi_l \xi_{l,q_l},$$

the term $\sum_{l=1}^{m_d} \varphi_l(x) v_{d,l}$ in (3.28) disappears under the new set of coordinates.

Remark 3.2.9 The variables $\zeta_{i,j}$, $j = 1, 2, \ldots, i$, $i = 1, 2, \ldots, k^*$, constitute all the states associated with the structure at infinity. Note that for some $i$ with $\rho_i = \rho_{i-1}$, $\zeta_{i,j}$ is not defined. For each $i = 1, 2, \ldots, k^*$, the states $\zeta_{i,j}$, $j = 1, 2, \ldots, i$, form $\rho_i - \rho_{i-1}$ chains of integrators, and each chain contains $i$ integrators. However, except for the smallest $i = i_0$ such that $\rho_{i_0} > 0$, in which $\zeta_{i_0,j}$ form $\rho_{i_0}$ clean chains of integrators that link the transformed inputs to the transformed outputs $\zeta_{i_0,1}$, for each remaining $i$ with $\rho_i \neq \rho_{i-1}$, the equations governing the states $\zeta_{i,j}$ represent chains of $i$ integrators with the previous transformed inputs $v_l$ ($l < i$) injected into the integrators with $j \geq q_i$.

3.2.3 Infinite Zeros

We now extend the linear system notion of infinite zeros to nonlinear systems. Consider the normal form in Theorem 3.2.3. The set $q = \{q_1, q_2, \ldots, q_{m_d}\}$ as obtained in the infinite zero structure algorithm coincides with the infinite zeros of this linear system as defined in [10, 75]. This motivates the following definition.

Definition 3.2.2 Suppose that the nonlinear system (3.1) is regular. The infinite zeros of the system are the set of integers $q = \{q_1, q_2, \ldots, q_{m_d}\}$ as identified in the infinite zero structure algorithm.
Roughly speaking, each integer $q_i$ in the set $q$ represents a chain of integrators of length $q_i$ connecting an input and output pair. We will further justify Definition 3.2.2 as an extension of the linear system notion of infinite zeros to nonlinear systems by showing that the set $q$ is invariant under diffeomorphic state, input and output transformations, static state feedback and output injection.

Consider a diffeomorphic state transformation $z = \Phi(x)$ in $U$, we have
\[
\begin{align*}
\dot{z} &= \hat{f}(z) + \hat{g}(z)u, \\
y &= \hat{h}(z),
\end{align*}
\]
where
\[
\begin{align*}
\hat{f}(z) &= \left( \frac{\partial \Phi}{\partial x} f(x) \right)_{x = \Phi^{-1}(z)}, \\
\hat{g}(z) &= \left( \frac{\partial \Phi}{\partial x} g(x) \right)_{x = \Phi^{-1}(z)}, \\
\hat{h}(z) &= [h(x)]_{x = \Phi^{-1}(z)}.
\end{align*}
\]
Following the infinite zero structure algorithm, it is easy to verify the following result.

**Lemma 3.2.** If system (3.1) is regular, then system (3.32) is regular too. Moreover, both systems have the same infinite zeros.

We also have the following result.

**Lemma 3.3.** The infinite zeros of system (3.1) are invariant under
1. input transformation $\tilde{u} = \Gamma(x)u$ with $\Gamma(x) : U \rightarrow \mathbb{R}^{m \times m}$ being smooth and nonsingular;
2. output transformation $\tilde{y} = \Gamma_o y$ with $\Gamma_o \in \mathbb{R}^{p \times p}$ being nonsingular;
3. static state feedback $\tilde{u} = u - K(x)$ with $K(x) : U \rightarrow \mathbb{R}^m$ being smooth; and
4. output injection, i.e.,
\[
\begin{align*}
\dot{x} &= f(x) + F(x)h(x) + g(x)u, \\
y &= h(x),
\end{align*}
\]
where $F(x) : U \rightarrow \mathbb{R}^{n \times p}$ is smooth.

**Proof:** See Appendices 3.6.3.

### 3.2.4 Invertibility and Zero Dynamics

Equations (3.18) in the infinite zero structure algorithm indicates the invertibility property of the system.

**Lemma 3.4.** System (3.1) is left invertible if $\rho_{k^*} = m < p$, right invertible if $\rho_{k^*} = p < m$, invertible if $\rho_{k^*} = m = p$, and degenerate if $\rho_{k^*} < \min\{m, p\}$.
Equivalently, the system in Theorem 3.2.1 is left invertible if \( u_e \) is absent, right invertible if \( y_e \) is absent, invertible if both \( u_e \) and \( y_e \) are absent, and degenerate if both \( u_e \) and \( y_e \) are present.

In [10], the zero dynamics of a nonlinear system is defined for a square invertible nonlinear system. Let \( M \) be a smooth connected submanifold of \( U \). The manifold \( M \) is said to be locally controlled invariant at \( x = 0 \) if there exist a smooth mapping \( u : M \to \mathbb{R}^m \) and a neighborhood \( U^* \) of \( x = 0 \) such that \( M \) is locally invariant under the vector field \( f(x) + g(x)u(x) \). A zero output submanifold in a neighborhood of \( x = 0 \) for the nonlinear system (3.1) is a smooth connected submanifold \( M \), which is locally controlled invariant at \( x = 0 \) and for each \( x \in M \), \( h(x) = 0 \). Suppose \( Z^* \) is the locally maximal zero output submanifold with span \( \{g(0)\} \cap T_{x=0}Z^* = 0 \), where \( T_{x=0}Z^* \) represents the tangent space to \( Z^* \) at \( x = 0 \). Then, there exists a unique smooth mapping \( u^* : Z^* \to \mathbb{R}^m \) such that the vector field \( f^*(x) = f(x) + g(x)u^*(x) \) is tangent to \( Z^* \). The pair \( (Z^*, f^*) \) is called the zero dynamic of (3.1).

The global version of \( Z^* \) for a square invertible nonlinear system is defined in [17] [22] as a controlled invariant smooth embedded submanifold of \( \mathbb{R}^n \).

Here, we want to use the form (3.28) to derive the zero dynamics of general nonlinear system in \( U \). In particular, let \( y_0 = 0 \) in (3.28). It then follows from the dynamic equations that \( \xi = 0 \) and \( v_{d,i} = 0 \), \( i = 1, 2, \ldots, m_d \). Consequently, the remaining dynamics reduces to

\[
\begin{aligned}
\dot{\eta} &= f_e(\eta, 0) + g_e(\eta, 0)u_e, \\
y_e &= h_e(\eta, 0).
\end{aligned}
\]

Let \( C_0 \) be the smallest distribution that is invariant for (3.34) and contains the distribution spanned by the column vectors of \( g_e(\eta, 0) \), and \( d\mathcal{O} \) be the smallest codistribution that is invariant for (3.34) and contains the codistribution spanned by the row vectors of \( dh_e(\eta, 0) \). Note that the distribution \( C_0 \) characterizes local strong accessibility and the codistribution \( d\mathcal{O} \) characterizes local observability. The subsystem (3.34) does not contain any subspace that is both strong locally accessible (by \( u_e \)) and locally observable (through \( y_e \)). Otherwise, the infinite zeros are no longer \( q = \{q_1, q_2, \ldots, q_{m_d}\} \). Thus by [2] [10], we have the following result.

**Lemma 3.5.** Consider system (3.34). Assume that the distributions \( C_0 \), \( \ker d\mathcal{O} \) and \( C_0 + \ker d\mathcal{O} \) of (3.34) each has a constant dimension. Then there exist a set of coordinates \( \hat{z} = \text{col} \{z_a, z_b, z_c\} \) such that (3.34) takes the form

\[
\begin{aligned}
\dot{z}_a &= f_a(z_a, z_b), \\
\dot{z}_b &= f_b(z_b), \\
\dot{z}_c &= f_c(z_a, z_b, z_c) + g_{ce}(z_a, z_b, z_c)u_e, \\
y_e &= h_{ce}(z_b),
\end{aligned}
\]

with \( C_0 = \text{span} \{\frac{\partial}{\partial z_c}\} \) and \( \ker d\mathcal{O} = \text{span} \{\frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_b}\} \). \( \square \)
3.2 Normal Forms and Structure Properties of Nonlinear Systems

The decomposition (3.35) allows us to decompose normal form (3.28) into four distinct subsystems (see Example 3.10) as we can do in a linear system (10, 75). In a generalization to the notion of invariant zero of linear systems [75], the dynamics \( \dot{z}_a = f_a(z_a, 0) \) is referred to as the zero dynamics of system (3.1). The case of \( m = p = m_d = \rho_k \) has been studied in [16, 22]. In this case, \( y_e \) and \( u_e \) are absent from (3.28), and \( \dot{\eta} = f_e(\eta, 0) \) is directly obtained as the zero dynamics of system (3.1).

3.2.5 Normal Forms of Square Invertible Systems

We now consider the normal forms of system (3.1) with \( m = p = \rho_k = m_d \), i.e., a square invertible system, which has been considered in [16, 17, 22]. In this case, \( y_e \) and \( u_e \) do not exist and we have the following result, as a corollary to Theorems 3.2.1 and 3.2.2.

**Corollary 3.2.1** Suppose that a square invertible system (3.1) has infinite zeros \( q = \{q_1, q_2, \ldots, q_m\} \), Assumption B holds, and the distribution spanned by the column vectors of \( g(x) \) is involutive. Under a new set of coordinates, the system takes the form,

\[
\begin{align*}
\dot{\eta} &= f_e(x) + \sum_{l=1}^{m} \varphi_l(x)v_l, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_l, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
\]  

where \( v_i = a_i(x) + b_i(x)u \) with \( \text{col}\{b_1(x), b_2(x), \ldots, b_m(x)\} \) being nonsingular, and

\[
\delta_{i,j,l}(x) = 0, \quad \text{for} \quad j < q_i, \quad i = 1, 2, \ldots, m.
\]

If, in addition, the distribution spanned by the column vectors of \( g(x) \) is involutive for \( x \in U \), then there exist a set of coordinates such that

\[
\varphi_l(x) = 0, \quad l = 1, 2, \ldots, m.
\]

Note that the form given in Corollary 3.2.1 is the same as (3.12) except for the additional structural property (3.37). The \( \dot{\xi}_{i,j} \) equation in (3.36) displays a triangular structure of the control inputs that enter the system. Property (3.37) imposes an additional structure within each chain of integrators on how control inputs enter the system. With this additional structural property, the set \( q = \{q_1, q_2, \ldots, q_m\} \) represents infinite zeros when the system is linear. Note that the property (3.37) which can be deduced from (3.21), is a key feature which the form (3.11) resulting from the algorithm in [16, 17, 21, 22]
does not possess. To see the significance of property (3.37), we transfer system (3.13) into the normal form (3.36),

\[
\dot{\tilde{x}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix} \tilde{x} + \begin{bmatrix}
1 & 0 \\
0 & 0 \\
1/\alpha & 0 \\
0 & 0 \\
0 & 0 
\end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},
\]

\[
\tilde{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{x},
\]

by using the following state and output transformations and state feedback,

\[
\tilde{x} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/\alpha & 0 \\
0 & 0 & 0 & 0 & -1/\alpha 
\end{bmatrix} x,
\quad \tilde{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} y,
\]

\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\alpha \tilde{x}_4 + \alpha u_1 \\ u_2 \end{bmatrix}.
\]

It is obvious that \( q_1 = 1 \) and \( q_2 = 4 \), which coincide with the infinite zeros of this linear system (see, e.g., \([10, 75]\)).

**Remark 3.2.10** The normal form in Corollary 3.2.1 can be further simplified by using the method in \([17, 22]\). Define the vector fields \( Y_{k,j}(x), \ 1 \leq j \leq m, \ 1 \leq k \leq q_j \). If these vector fields commute, then there exist a set of coordinates such that the dynamics of \( \eta \) in Corollary 3.2.1 simplifies to

\[
\dot{\eta} = f(\eta, \xi_1, \xi_2, \ldots, \xi_m),
\]

Next, we follow the method in \([17, 22]\) to further simplify the dynamic of \( \eta \) in Corollary 3.2.1 Define

\[
\dot{f}(x) = f(x) - g(x) \text{col} \{ b_1(x), b_2(x), \ldots, b_m(x) \}^{-1} \text{col} \{ a_1(x), a_2(x), \ldots, a_m(x) \},
\]

\[
\dot{g}(x) = g(x) \text{col} \{ b_1(x), b_2(x), \ldots, b_m(x) \}^{-1},
\]

and let

\[
Y_{m,k}^k(x) = (-1)^{k-1} \text{ad}_f^{k-1} \tilde{g}_m(x), \ 1 \leq k \leq q_m,
\]

and for \( 1 \leq j \leq m - 1, \ 1 < k \leq q_j \),

\[
Y_{j,1}^1(x) = \tilde{g}_j(x) - \sum_{i=j+1}^{m} \sum_{\ell=2}^{q_i} \delta_{i,q_i-\ell+1,j}(x) Y_{i,\ell}^\ell(x),
\]

\[
Y_{j,k}^k(x) = (-1)^{k-1} \text{ad}_f^{k-1} Y_{j,1}^1(x).
\]

**Assumption \( \mathcal{D} \) :** The vector fields \( Y_{j,k}^k(x), \ 1 \leq j \leq m, \ 1 \leq k \leq q_j \), commute, i.e.,
\[ [Y^s_i, Y^k_j] = \frac{\partial Y^k_j}{\partial x} Y^s_i - \frac{\partial Y^s_i}{\partial x} Y^k_j = 0, \quad 1 \leq i, j \leq m, \quad 1 \leq s \leq q_i, \quad 1 \leq k \leq q_j. \]

The following result is immediate from [17].

**Theorem 3.2.4** Suppose that the conditions of Corollary 3.2.1 and Assumption D hold, then the dynamics of \( \eta \) in Corollary 3.2.1 can be simplified to
\[ \dot{\eta} = f_c(\eta, \xi_1, \xi_2, \cdots, \xi_m). \]

### 3.3 Normal Forms of Nonlinear Systems Relating to the Zero Output

In determining the zero dynamics, a normal form representation of the nonlinear system is also given in Chapter 6 of [16], which displays structure information along one special output trajectory, the zero output. Two constant rank assumptions are made in the nested submanifolds \( M_k, k = 1, 2, \cdots, k^* \).

Here, we will show that the infinite zero structure algorithm can be adapted for the same problem. In particular, for system (3.1), we will introduce Assumption \( A_k, k = 1, 2, \cdots, k^* \), in the nested subsets \( M_k, k = 1, 2, \cdots, k^* \), rather than for all \( x \in U \). Because the nested subsets \( M_k, k = 1, 2, \cdots, k^* \), are related to the zero output, we refer to the resulting algorithm as the zero output structure algorithm.

**Zero Output Structure Algorithm**

**Initial Step.** Let \( \Theta_0(x) = h(x), \quad O_0(x) = \emptyset, \quad \rho_0 = 0 \) and \( k = 1 \).

**Step \( k \).** We start with \( \Theta_{k-1}(x) : U \rightarrow \mathbb{R}^{p-\rho_k-1} \) and \( \Omega_{k-1}(x) : U \rightarrow \mathbb{R}^{\rho_k-1} \), where the matrix \( L_\Theta\Omega_{k-1}(x) \) has full row rank \( \rho_k-1 \) in \( M_k \cap O_k \), with \( M_k = \{ x : \Theta_i(x) = 0, \quad i = 0, 1, \cdots, k-1 \} \) and \( O_k \) being a neighborhood of \( x = 0 \).

**Assumption \( \tilde{A}_k \):** The matrix
\[
\begin{bmatrix}
L_\Theta \Omega_{k-1}(x) \\
L_\Theta \Theta_{k-1}(x)
\end{bmatrix}
\]
has a constant rank \( \rho_k \) in \( M_k \cap O_k \), and there exists an \( R_k \in \mathbb{R}^{(\rho_k-\rho_{k-1}) \times (p-\rho_k-1)} \) such that
\[
\text{rank} \left( \begin{bmatrix}
L_\Theta \Omega_{k-1}(x) \\
L_\Theta \Theta_{k-1}(x)
\end{bmatrix} \right) = \rho_k, \quad \forall \ x \in (M_k \cap O_k)^c, \tag{3.38}
\]
where \( (M_k \cap O_k)^c \) is the connected component of \( M_k \cap O_k \) containing \( x = 0 \).

Suppose that Assumption \( \tilde{A}_k \) is satisfied. Let \( S_k \) and \( \Omega_k(x) \) be as in (3.14) and (3.15). Thus, the matrix \( L_\Theta \Omega_k(x) \) has full row rank \( \rho_k \) for \( x \in (M_k \cap O_k)^c \), and
\[
\text{rank} \left( \begin{bmatrix}
L_\Theta \Omega_k(x) \\
L_\Theta S_k \Theta_{k-1}(x)
\end{bmatrix} \right) = \rho_k, \quad \forall \ x \in (M_k \cap O_k)^c.
\]
Therefore, there exist smooth functions \( P_{k,l}(x) \in \mathbb{R}^{(p-\rho_k) \times (\rho_l-\rho_{l-1})}, \ l = 1, 2, \cdots, k \), such that
\[ L_g S_k \Theta_{k-1}(x) - \sum_{l=1}^{k} P_{k,l}(x) L_g R_l \Theta_{l-1}(x) - W_k(x) = 0, \quad (3.39) \]

where \( W_k(x) \) is a matrix valued smooth function with \( W_k(x) = 0 \) in \( (M_k \cap O_k)^c \). Denote
\[ v_k = L_f R_k \Theta_{k-1}(x) + L_g R_k \Theta_{k-1}(x) u, \quad (3.40) \]
and define
\[ \Theta_k(x) = \frac{d}{dt} S_k \Theta_{k-1}(x) - \sum_{l=1}^{k} P_{k,l}(x) v_l - W_k(x) u \]
\[ = L_f S_k \Theta_{k-1}(x) - \sum_{l=1}^{k} P_{k,l}(x) L_f R_l \Theta_{l-1}(x). \quad (3.41) \]

If \( k + \sum_{j=1}^{k} j (\rho_j - \rho_j - 1) < n \) and \( \rho_k < \min\{p, m\} \), then increase \( k \) by 1 and repeat the above step. Otherwise, go to Final Step.

**Final Step.** The same as the final step in the infinite zeros structure algorithm in Section 3.2.1.

**End.**

**Definition 3.3.1** The point \( x = 0 \) is said to be a regular point of system (3.1) if Assumption \( \bar{A}_k, k = 1, 2, \cdots, k^* \), in the zero output structure algorithm are satisfied.

Note that in Step \( k \), the choice of the matrices \( R_k \) and \( S_k \), which satisfy (3.38) and (3.14), are not unique.

**Lemma 3.6.** Suppose that \( \dot{R}_i \) and \( \dot{S}_i, i = 1, 2, \cdots, k^* \), are different choices yielding \( \dot{\Theta}_i(x), \dot{\Omega}_i(x), \dot{M}_i \) and \( \dot{O}_i \). Then,
\[ \dot{M}_i = M_i, \quad \dot{\Theta}_i(x) = \sum_{l=1}^{i-1} Q_{i,l}(x) \Theta_l(x) + T_i(x) \Theta_i(x) + V_i(x), \quad (3.42) \]

where \( T_i(x) \) is a nonsingular matrix valued smooth function, and \( V_i(x) \) is smooth with \( V_i(x) = 0 \) in \( M_i \cap \dot{O}_i \cap O_i \).

**Proof:** See Appendices 3.6.4. \( \square \)

The following result follows directly from Lemma 3.6.

**Lemma 3.7.** The set \( \rho \), and hence the set \( q \), as identified in the zero output structure algorithm are invariant with respect to the choice of matrices \( R_i \) and \( S_i, i = 1, 2, \cdots, k^* \).

Define \( \dot{\Phi}_d(x), \Gamma_1(x) \) and \( \Gamma_o \) as in (3.22)-(3.24). We have the following crucial result.
Lemma 3.8. Let \( x = 0 \) be a regular point of system (3.1). Then, \( d\Phi_d(0) \), \( \Gamma_d(0) \) and \( \Gamma_{o,d} \) are of full row rank.

Proof: See Appendices 3.6.5.

Theorem 3.3.1 Consider system (3.1). Suppose that \( x = 0 \) is a regular point. Let \( q = \{ q_1, q_2, \ldots, q_m \} \) be as obtained in the zero output structure algorithm. There exist a set of coordinates, i.e., diffeomorphic state, input and output transformations, such that the system assumes the following form,

\[
\begin{align*}
\dot{y} &= f_e(x) + g_e(x)u_e + \sum_{i=1}^{m_d} \phi_i(x)v_{d,i}, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{i=1}^{i-1} \delta_{i,j,i}(x)v_{d,i} + \sigma_{i,j}(x)u, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_{d,i}, \\
y_e &= h_e(x),
\end{align*}
\]

(3.43)

where \( \sigma_{1,j}(x) = 0, \quad j = 1, 2, \ldots, n_1 - 1, \) and \( \sigma_{i,j}(x) = 0, \quad i > 1, \quad j = 1, 2, \ldots, n_i - 1, \) in \( (M_j \cap O_j)^c, \) \( v_{d,i} = a_i(x) + b_i(x)u, \) \( i = 1, 2, \ldots, m_d, \) with \( \text{col} \{ b_1(x), b_2(x), \ldots, b_{m_d}(x) \} \) being nonsingular, and

\[
\delta_{i,j,i}(x) = 0, \quad \text{for} \quad j < q_i, \quad i = 1, 2, \ldots, m_d.
\]

(3.44)

We next consider system (3.1) with \( m = p = m_d = \rho_k \). In this case, \( y_e \) and \( u_e \) do not exist and we have the following result.

Corollary 3.3.1 Suppose that the conditions in Theorem 3.3.1 hold with \( m = p = m_d = \rho_k \). Then, there exist a set of local coordinates such that the system takes the form,

\[
\begin{align*}
\dot{y} &= f_e(x) + \sum_{i=1}^{m} \phi_i(x)v_i, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{i=1}^{i-1} \delta_{i,j,i}(x)v_i + \sigma_{i,j}(x)u, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

(3.45)

where \( \sigma_{1,j}(x) = 0, \quad j = 1, 2, \ldots, n_1 - 1, \) and \( \sigma_{i,j}(x) = 0, \quad i > 1, \quad j = 1, 2, \ldots, n_i - 1, \) in \( (M_j \cap O_j)^c, \) \( v_i = a_i(x) + b_i(x)u \) with \( \text{col} \{ b_1(x), b_2(x), \ldots, b_m(x) \} \) being nonsingular, and

\[
\delta_{i,j,i}(x) = 0, \quad \text{for} \quad j < q_i, \quad i = 1, 2, \ldots, m.
\]

(3.46)

The submanifold \( Z^* \) is given as \( Z^* = \{ x \in U : \xi_{i,j}(x) = 0, \quad j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, m. \} \).
Remark 3.3.1 Corollary 3.3.1 is the same as the result in Chapter 6 of [16], except that there is property (3.46) here.

Remark 3.3.2 The zero output structure algorithm requires milder regularity assumptions than the infinite zero structure algorithm. For example, consider
\[
\begin{align*}
  f(x) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
  g(x) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
  h(x) &= \begin{bmatrix} x_1 \\ x_1 x_2 \end{bmatrix}.
\end{align*}
\]
By the infinite zero structure algorithm, the system is not regular, since
\[
L_g h(x) = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ x_1 \end{bmatrix}
\]
does not have a constant rank in a neighborhood of \( x = 0 \). However, by the zero output structure algorithm, \( x = 0 \) is a regular point with \( \rho = \{1, 1\} \), and the locally maximal zero output submanifold is \( Z^* = \{0\} \).

Remark 3.3.3 We have similar results as in Lemmas 3.2, 3.3 and 3.5 for zero output structure algorithm. If the point \( x = 0 \) of system (3.1) is regular, then the point \( \Phi(0) \) of system (3.32) is regular too. The set of the integers \( q \) as identified in the zero output structure algorithm are invariant under the state, input and output transformations, state feedback and output injection as defined in Lemma 3.3. The zero dynamics can be computed similarly as in Lemma 3.3.

3.4 Examples

Examples 3.9 and 3.10 illustrate the infinite zero structure algorithm, and Example 3.11 illustrates the zero output structure algorithm. and Example 3.12 is an application of our results to a practical system.

Example 3.9. Consider system (3.1) with
\[
\begin{align*}
  f(x) &= \begin{bmatrix} x_3 \\ x_5 \\ x_1 \\ x_1 x_2 \\ x_4 \end{bmatrix}, \\
  g(x) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & x_3 \\ 0 & 1 \\ x_4 & x_3 x_4 \end{bmatrix}, \\
  h(x) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
  U &= \{ x : x_1 < 1 \}.
\end{align*}
\]
We carry out the infinite zero structure algorithm as follows.

Initial Step. Let \( \Theta_0(x) = h(x), \Omega_0(x) = \emptyset, \rho_0 = 0 \) and \( k = 1 \).

Step 1.
\[
L_f \Theta_0(x) = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}, \\
L_g \Theta_0(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Hence, $\rho_1 = 0$. Let
\[ R_1 = \emptyset, \quad S_1 = I_2. \]
Thus,
\[ v_1 = \emptyset, \quad \Omega_1(x) = \emptyset, \quad P_{1,1}(x) = \emptyset, \quad \Theta_1(x) = \text{col}\{x_3, x_5\}. \]

**Step 2.**
\[ L_f \Theta_1(x) = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}, \quad L_g \Theta_1(x) = \begin{bmatrix} 1 & x_3 \\ x_4 & x_3 x_4 \end{bmatrix}. \]
Hence, $\rho_2 = 1$. Let
\[ R_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]
Thus,
\[ v_2 = x_1 + [1 \quad x_3] u, \quad \Omega_2(x) = x_3, \quad P_{2,1}(x) = \emptyset, \quad P_{2,2}(x) = x_4, \quad \Theta_2(x) = x_4 - x_1 x_4. \]

**Step 3.**
\[ L_f \Theta_2(x) = x_1 x_2 - x_3 x_4 - x_1^2 x_2, \quad L_g \Theta_2(x) = \begin{bmatrix} 0 & 1 - x_1 \end{bmatrix}. \]
Hence, $\rho_3 = 2$. Let
\[ R_3 = 1, \quad S_3 = \emptyset. \quad v_3 = x_1 x_2 - x_3 x_4 - x_1^2 x_2 + [0 \quad 1 - x_1] u. \]

**Final Step.** $k^* = 3, m_d = 2, n_d = 5, \rho = \{0, 1, 2\}, q = \{2, 3\}$.

Let
\[ \text{col}\{\xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2}, \xi_{2,3}\} = \text{col}\{x_1, x_3, x_2, x_5, x_4 - x_1 x_4\}, \]
\[ y_{d,1} = y_1, \quad y_{d,2} = y_2. \]
\[ v_{d,1} = x_1 + [1 \quad x_3] u, \quad v_{d,2} = x_1 x_2 - x_3 x_4 - x_1^2 x_2 + [0 \quad 1 - x_1] u. \]

The form (3.28) is given by
\[
\begin{cases}
\dot{\xi}_{1,1} = \xi_{1,2}, \\
\dot{\xi}_{1,2} = v_{d,1}, \\
\dot{\xi}_{2,1} = \xi_{2,2}, \\
\dot{\xi}_{2,2} = \xi_{2,3} + \frac{\xi_{2,3}}{1 - \xi_{1,1}} v_{d,1}, \\
\dot{\xi}_{2,3} = v_{d,2}, \\
y_{d,1} = \xi_{1,1}, \\
y_{d,2} = \xi_{2,2}.
\end{cases}
\]

The system is invertible with two infinite zeros of order 2 and 3. The zero dynamic degenerates to the single point $x = 0$. Note that $\delta_{2,1,1}(x) = 0$, i.e., the term of $v_{d,1}$ does not appear in the dynamic equation of $\dot{\xi}_{2,1}$.
Example 3.10. Consider system (3.1) with
\[
\begin{bmatrix}
-x_1 + x_3 \\
x_2 x_4 \\
-x_2 x_4 - x_2 x_4^2 \\
-x_4
\end{bmatrix}, \quad \begin{bmatrix}
x_2 \\
0 \\
0 \\
1
\end{bmatrix}, \quad h(x) = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}.
\]

The system is defined globally, i.e., \( U = \mathbb{R}^4 \). We apply the infinite zero structure algorithm.

**Initial Step.** Let \( \Theta_0(x) = h(x) \), \( \Omega_0(x) = \emptyset \), \( \rho_0 = 0 \) and \( k = 1 \).

**Step 1.**
\[
L_f \Theta_0(x) = \begin{bmatrix} x_2 x_4 \\ -x_4 \end{bmatrix}, \quad L_g \Theta_0(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
\]
So, \( \rho_1 = 1 \). Let \( R_1 = [0 \ 1] \), \( S_1 = [1 \ 0] \).

Thus,
\[
v_1 = -x_4 + u_1, \quad \Omega_1(x) = x_4, \quad P_{1,1}(x) = 0, \quad \Theta_1(x) = x_2 x_4.
\]

**Step 2.**
\[
L_f \Theta_1(x) = -x_2 x_4 + x_2 x_4^2, \quad L_g \Theta_1(x) = [x_2 \ 0].
\]
So, \( \rho_2 = 1 \). Let \( R_2 = \emptyset \), \( S_2 = 1 \). Thus,
\[
v_2 = \emptyset, \quad \Omega_2(x) = x_4, \quad P_{2,1}(x) = x_2, \quad P_{2,2}(x) = \emptyset, \quad \Theta_2(x) = x_2 x_4^2.
\]

**Step 3.**
\[
L_f \Theta_2(x) = -2x_2 x_4^2 + x_2 x_4^3, \quad L_g \Theta_2(x) = [2x_2 x_4 \ 0].
\]
So, \( \rho_3 = 1 \). Let \( R_3 = \emptyset \), \( S_3 = 1 \), \( v_3 = \emptyset \), \( \Omega_3(x) = x_4 \),
\[
P_{3,1}(x) = 2x_2 x_4, \quad P_{3,2}(x) = \emptyset, \quad P_{3,3}(x) = \emptyset, \quad \Theta_3(x) = x_2 x_4^3.
\]

**Final Step.** \( k^* = 3, \ n_d = 1, \ n_d = 1, \ \rho = \{1,1,1\} \) and \( q = \{1\} \). It is obvious that \( \Phi_d(x) = x_4 \), \( \Gamma_d(x) = [1 \ 0] \), \( \Gamma_{od}(x) = [0 \ 1] \). Let
\[
\Gamma_1(x) = \begin{bmatrix} 0 & e^{-x_4} \\ 1 & 0 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Thus, \( g_d(x) = \text{col} \{x_2,0,0,1\} \). Find \( \Phi_e \) such that \( d\Phi_e g_d(x) = 0 \), i.e.,
\[
\frac{\partial \Phi_e}{\partial x_2} + \frac{\partial \Phi_e}{\partial x_4} = 0.
\]
We obtain $\Phi_{\varepsilon}(x) = \text{col}\{x_1 - x_2x_4, x_2, x_3\}$. Let $\text{col}\{\eta_1, \eta_2, \eta_3, \xi\} = \text{col}\{x_1 - x_2x_4, x_2, x_3, x_4\}$. Thus,

$$
\begin{align*}
\dot{\eta}_1 &= -\eta_1 + \eta_3 - \eta_2\xi^2 + u_e, \\
\dot{\eta}_2 &= \eta_2\xi, \\
\dot{\eta}_3 &= -\eta_2\xi - \eta_2\xi^2 + u_e, \\
\dot{\xi} &= v_d, \\
y_c &= \eta_2, \\
y_d &= \xi,
\end{align*}
$$

with $u_e = e^{-x_4u_2}$, and $v_d = -x_4 + u_1$. We take the following further transformation on $\eta$. Let $z_a = \eta_1 - \eta_3$, $z_b = \eta_2$, and $z_c = \eta_2 + \eta_3$. Then, the system takes the following form

$$
\begin{align*}
\dot{z}_a &= -z_a + z_b\xi, \\
\dot{z}_b &= z_b\xi, \\
\dot{z}_c &= -z_b\xi^2 + u_e, \\
\dot{\xi} &= v_d, \\
y_c &= z_b, \\
y_d &= \xi,
\end{align*}
$$

with $\text{col}\{z_a, z_b, z_c, \xi\} = \text{col}\{x_1 - x_3 - x_2x_4, x_2, x_2 + x_3, x_4\}$. The zero dynamics is $\dot{z}_a = -z_a$. It is also clear from the normal form above that the system has an infinite zero of order 1 and is not invertible.

**Example 3.11.** Consider system (3.1) with

$$
\begin{align*}
f(x) &= \begin{pmatrix} x_3 \\ x_4 \\ x_3x_4 \\ x_1x_3x_4 \end{pmatrix}, \\
g(x) &= \begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \\ x_2 & -x_3 \\ x_3 & 1 \end{bmatrix}, \\
h(x) &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{align*}
$$

It is obvious that $f(0) = 0$ and $h(0) = 0$. We carry out the zero output structure algorithm as follows.

**Initial Step.** Let $\Theta_0(x) = h(x)$, $\Omega_0(x) = \emptyset$, $\rho_0 = 0$ and $k = 1$.

**Step 1.** Let $M_1 = \{x : x_1 = x_2 = 0\}$,

$$
L_f\Theta_0(x) = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \\
L_g\Theta_0(x) = \begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix}.
$$

Hence, $\rho_1 = 1$. Let $R_1 = [1 \ 0], S_1 = [0 \ 1]$. Thus,

$$
\begin{align*}
v_1 &= x_3 + u_1 + x_1u_2, \\
\Omega_1(x) &= x_1, \\
P_{1,1}(x) &= x_1,
\end{align*}
$$

$$
W_1(x) = \begin{bmatrix} 0 & x_2 - x_2^2 \end{bmatrix}, \\
\Theta_1(x) &= x_4 - x_1x_3.
$$

**Step 2.** Let $M_2 = \{x : x_1 = x_2 = x_4 = 0\}$.

$$
L_f\Theta_1(x) = -x_3^2, \\
L_g\Theta_1(x) = [-x_1x_2 \ 1].
$$
42 3 Normal Forms of Nonlinear Systems Affine in Control

Hence, \( \rho_2 = 2 \). Let \( R_3 = 1, S_2 = \emptyset \). Thus,

\[
v_2 = -x_3^2 - x_1x_2u_1 + u_2, \quad \Omega_2(x) = \text{col} \{x_1, x_4 - x_1x_3\}.
\]

**Final Step.** \( k^* = 2, m_4 = 2, n_4 = 3, \rho = \{1, 2\} \) and \( q = \{1, 2\} \).

The distribution spanned by the column vectors of \( g(x) \) is not involutive. Define

\[
\text{col} \{\eta, \xi_{1,1}, \xi_{2,1}, \xi_{2,2}\} = \text{col} \{x_3, x_1, x_2, x_4 - x_1x_3\},
\]

\[
v_{d,1} = x_3 + u_1 + x_1u_2
\]

and

\[
v_{d,2} = -x_3^2 - x_1x_2u_1 + u_2.
\]

In the region \( \{x : x_1^2 + x_2^2 < 1\} \),

\[
\dot{\eta} = x_3x_4 + [x_2 - x_3]u = x_3x_4 + \frac{-x_2x_3 - x_3^3}{1 + x_1^2x_2} + \frac{1}{1 + x_1^2x_2} [x_2 - x_1x_2x_3 - x_1x_2 - x_3] (\begin{bmatrix} v_{d,1} \\ v_{d,2} \end{bmatrix}),
\]

and thus the form \([3.43]\) is given by

\[
\begin{align*}
\dot{\eta} &= (\xi_{2,2} + \xi_{1,1})\eta + \frac{-\xi_{2,2}\eta - \eta^3}{\tau + \xi_{1,1}^2}, \\
\dot{\xi}_{1,1} &= v_{d,1}, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + \xi_{1,1}v_{d,1} - \left[0 \quad \xi_{2,1} - \xi_{1,1}^2 \right] u, \\
\dot{\xi}_{2,2} &= v_{d,2}, \\
y_1 &= \xi_{1,1}, \\
y_2 &= \xi_{2,1}.
\end{align*}
\]

By letting \( y = 0 \), we obtain the zero dynamics

\[
\dot{\eta} = -\eta^3.
\]

**Example 3.12.** Consider the dynamics of an underactuated vehicle \([52]\). The model is given by

\[
M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + g(\mu) = \begin{bmatrix} \tau \\ 0 \end{bmatrix}, \quad \dot{\mu} = J(\mu)\nu, \quad y = \nu,
\]

where \( \mu \in \mathbb{R}^{n_1} \) denotes the position and orientation, \( \nu \in \mathbb{R}^p \) denotes velocities to be controlled and \( \tau \in \mathbb{R}^m \) denotes control forces and moments with \( p > m \), \( g(\mu) \) is the gravitation and buoyancy vector, and the inertia matrix \( M \) is constant, symmetric, nonsingular and positive definite. Denote \( \text{col} \{\nu_1, \nu_2\} = M\nu \), with \( \nu_1 \in \mathbb{R}^m \). The model is given by
where \( \nu \) is evaluated by \( M^{-1} \text{col} \{ \nu_1, \nu_2 \} \). Note that (3.47) is already in the form of (3.28), with \( \eta = \text{col} \{ \mu, \nu_2 \} \), \( \xi = \nu_1 \), \( \nu_d = \tau - [I_m 0] C(\nu) \nu - [I_m 0] D(\nu) \nu - [I_m 0] g(\mu) \), and \( u_e \) nonexistent. Thus, the system is left invertible and has \( m \) infinite zeros of order 1. To determine its zero dynamics, we consider the subsystem (3.34) as follows,

\[
\begin{dcases}
\dot{\mu} = 0, \\
\dot{\nu}_2 = -[0 I_{p-m}] g(\mu), \\
y_e = \nu_2, \\
y_d = \nu_1,
\end{dcases}
\]

The zero dynamics depends heavily on \([0 I_{p-m}] g(\mu)\). If \( \mu \) is observed through \( \nu_2 \), there is no zero dynamics. Otherwise, if \( g(\mu) = 0 \), then zero dynamics is given by \( \dot{\mu} = 0 \). It is interesting to note that it has been shown in [61] that \( g(\mu) \) is important for the stabilizability of underactuated vehicles. This difficulty can also be seen in the form (3.47). In the absence of \( g(\mu) \), neither \( y_e \) nor \( y_d \) contains any information of the state \( \mu \).

### 3.5 Summary of the Chapter

We have presented constructive algorithms for decomposing an affine nonlinear system into its normal form representations. Such algorithms generalize the existing results in several ways. They require less restrictive assumptions on the system and apply to general MIMO systems that do not necessarily have the same number of inputs and outputs. The resulting normal forms reveal various nonlinear extensions of linear system structural properties. These algorithms and the resulting normal forms are thus expected to facilitate the solution of several nonlinear control problems.

### 3.6 Proofs

#### 3.6.1 Proof of Lemma 3.1

Denote

\[
ad_J g = [ad_J g_1 \quad ad_J g_2 \quad \cdots \quad ad_J g_m],
\]

and

\[
\langle d\Theta, g \rangle = [\langle d\Theta, g_1 \rangle \quad \langle d\Theta, g_2 \rangle \quad \cdots \quad \langle d\Theta, g_m \rangle].
\]
We want to show that all row vectors in the following list are linearly independent:

\[
d\zeta_1 : \quad dR_1\theta_0 \\
d\zeta_2 : \quad dR_2S_1\theta_0 \quad dR_2\theta_1 \\
\vdots \quad \vdots \quad \vdots \\
d\zeta_k : \quad dR_kS_{k-1+1}\theta_0 \quad dR_kS_{k-1+2}\theta_1 \quad \cdots \quad dR_k\theta_{k-1} \\
d\zeta_{k+1} : \quad dR_{k+1}S_{k+1}\theta_0 \quad dR_{k+1}S_{k+2}\theta_1 \quad \cdots \quad dR_{k+1}S_k\theta_{k-1} \quad dR_{k+1}\theta_k.
\]

The rows of \(dR_1\theta_0\) are linearly independent since the matrix \(L_gR_1\theta_0\) is of full row rank. Next, we show that the rows of \(dR_1\theta_0, dR_2S_1\theta_0\) and \(dR_2\theta_1\) are linearly independent. To do this, consider

\[
\begin{bmatrix}
  dR_1\theta_0 \\
dR_2S_1\theta_0 \\
dR_2\theta_1
\end{bmatrix}
\begin{bmatrix}
g \\
ad_f g \\
\ast
\end{bmatrix} =
\begin{bmatrix}
L_gR_1\theta_0 & \langle dR_1\theta_0, \ ad_f g \rangle \\
L_gR_2S_1\theta_0 & \langle dR_2S_1\theta_0, \ ad_f g \rangle - R_2 P_{1,1} \langle dR_1\theta_0, \ ad_f g \rangle \\
L_gR_2\theta_1 & \ast
\end{bmatrix}.
\]

By row operation, the right hand side of (3.48) can be transformed to

\[
\begin{bmatrix}
L_gR_1\theta_0 \\
\langle dR_1\theta_0, \ ad_f g \rangle \\
L_gR_2\theta_1
\end{bmatrix}.
\]

Considering \(\langle d\phi, \ ad_f g \rangle = L_f \langle d\phi, g \rangle - \langle dL_f \phi, g \rangle\), we have

\[
\langle dR_2S_1\theta_0, \ ad_f g \rangle - R_2 P_{1,1} \langle dR_1\theta_0, \ ad_f g \rangle \\
= L_f \langle dR_2S_1\theta_0, g \rangle - \langle dL_f R_2S_1\theta_0, g \rangle - R_2 P_{1,1} L_f \langle dR_1\theta_0, g \rangle \\
= L_g R_2 \theta_1.
\]

Therefore, (3.48) is of full row rank for \(x \in U\). Hence the row vectors \(dR_1\theta_0, dR_2S_1\theta_0\), and \(dR_2\theta_1\) are linearly independent.

Similarly, the row vectors of \(\text{col} \{d\zeta_1(x), d\zeta_2(x) \cdots, d\zeta_{k+1}(x)\}\) are linearly independent.

### 3.6.2 Proof of Theorem 3.2.2

Note that

\[
d\Omega_k(x) g_4(x) = \Gamma_{1d}(x) \Gamma_1(x)^{-1} \begin{bmatrix} 0 \\ I_{m_d} \end{bmatrix} = I_{m_d}.
\]

Thus, the column vectors of \(g_4(x)\) are linearly independent for \(x \in U\). By Frobenius’ Theorem, there exist \(n-m_d\) real-valued functions \(\lambda_1(x), \lambda_2(x), \cdots, \lambda_{n-m_d}\) such that the rows of \(d\lambda_1(x), d\lambda_2(x), \cdots, d\lambda_{n-m_d}\) are linearly independent and
In view of (3.49) and the fact that \( \lambda_1 \) and \( \lambda_2 \) are obvious from the infinite zero structure algorithm.

3.6.3 Proof of Lemma 3.3

Let \( \tilde{\theta}_0(x) \) and \( \tilde{\theta}_0(x) \) be

\[
\begin{align*}
d\tilde{\theta}_0(x) &= L_{(f+gK)}\tilde{\theta}_0(x) + L_g\tilde{\theta}_0(x)u = L_f\tilde{\theta}_0(x) + [L_g\tilde{\theta}_0(x)]K(x) + L_g\tilde{\theta}_0(x)u. \\
\end{align*}
\]

Letting \( \tilde{R}_1 = R_1 \) and \( \tilde{S}_1 = S_1 \), we have \( -P_{1,1}(x)\tilde{R}_1 + \tilde{S}_1L_g\tilde{\theta}_0(x) = 0 \). By (3.16) and (3.17),

\[
\begin{align*}
\hat{\theta}_1(x) &= \left[ -P_{1,1}(x)\tilde{R}_1 + \tilde{S}_1L_{(f+gK)}\tilde{\theta}_0(x) \right] = \Theta_1(x). \\
\end{align*}
\]

Similarly, letting \( \tilde{R}_k = R_k \) and \( \tilde{S}_k = S_k \), we obtain \( \hat{\theta}_k(x) = \Theta_k(x) \). Thus, \( \hat{\rho}_k = \rho_k \).

4) Let \( \hat{\theta}_0(x) = h(x) = \Theta_0(x) \). Then,

\[
\begin{align*}
d\hat{\theta}_0(x) &= L_{(f+Fh)}\hat{\theta}_0(x) + L_g\hat{\theta}_0(x)u \\
&= L_f\hat{\theta}_0(x) + [d\hat{\theta}_0(x)]F(x)h(x) + L_g\hat{\theta}_0(x)u. \\
\end{align*}
\]
Let $\tilde{R}_1 = R_1$ and $\tilde{S}_1 = S_1$. We have
\[ [-P_{1,1}(x)\tilde{R}_1 + \tilde{S}_1]L_2\Theta_0(x) = 0. \]
By (3.16) and (3.17),
\[ \Theta_1(x) = [-P_{1,1}(x)\tilde{R}_1 + \tilde{S}_1]L_{f+fb}\Theta_0(x) = \Theta_1(x). \]
Similarly, letting $\tilde{R}_k = R_k$ and $\tilde{S}_k = S_k$, we obtain $\tilde{\Theta}_k(x) = \Theta_k(x)$, Thus, $\tilde{\rho}_k = \rho_k$.

3.6.4 Proof of Lemma 3.6

We first establish the following result.

Lemma 3.13. Let $G_i(x) = \text{col}\{\Theta_0(x), \Theta_1(x), \ldots, \Theta_{i-1}(x)\}$, $i = 1, 2, \ldots, k^*$. We have
\[ C_i(x)L_2G_i(x) = \text{col}\{W_1(x), W_2(x), \ldots, W_i(x)\}, \quad (3.51) \]
\[ C_i(x)L_2G_i(x) = \text{col}\{\Theta_1(x), \Theta_2(x), \ldots, \Theta_i(x)\}, \quad (3.52) \]
where $C_1(x) = -P_{1,1}(x)R_1 + S_1$,
\[ C_1(x) = \begin{bmatrix} C_{i-1}(x) & 0 \\ -P_{1,c}(x)\text{blkdiag}\{R_1, R_2, \ldots, R_{i-1}\} & -P_{1,i}(x)R_i + S_i \end{bmatrix}, \]
and
\[ P_{i,0}(x) = [P_{1,1}(x) \quad P_{1,2}(x) \quad \cdots \quad P_{1,i-1}(x)]. \]
Moreover, the rows of $C_i(x)$ form a basis of the solution space of the homogeneous linear equation $\gamma L_2G_i(x) = 0$ in $(M_1 \cap O_i)^c$.

Proof: We carry out the proof by induction. By Assumption $A_1$, the matrix $L_2\Theta_0(x)$ has a constant rank $\rho_1$ in $(M_1 \cap O_1)^c$. By (3.39), we have $C_1(x)L_2\Theta_0(x) = W_1(x)$. Since $\text{col}\{R_1, S_1\}$ is nonsingular, $C_1(x)$ has full row rank $p - \rho_1$, and hence its rows form a basis of the solution space of $\gamma L_2\Theta_0(x) = 0$ in $(M_1 \cap O_1)^c$. By (3.41), we have $\Theta_1(x) = C_1(x)L_2\Theta_0(x)$.

Assume that
\[ C_{j-1}(x)L_2G_{j-1}(x) = \text{col}\{W_1(x), W_2(x), \ldots, W_{j-1}(x)\}, \]
\[ C_{j-1}(x)L_2G_{j-1}(x) = \text{col}\{\Theta_1(x), \Theta_2(x), \ldots, \Theta_{j-1}(x)\}, \]
and the rows of $C_{j-1}(x)$ form a basis of the solution space of $\gamma L_2G_{j-1}(x) = 0$ in $(M_{j-1} \cap O_{j-1})^c$. By (3.39) and (3.41), we have
\[ \begin{bmatrix} -P_{j,0}(x) & -P_{j,1}(x)R_j + S_j \end{bmatrix} \begin{bmatrix} L_2\Theta_{j-1}(x) \\ L_2\Theta_{j-1}(x) \end{bmatrix} = W_j(x), \]
Thus, the matrix 

\[
\begin{bmatrix}
-P_{j,:}(x) & -P_{j,j}(x)R_j + S_j
\end{bmatrix}
\begin{bmatrix}
L_f \Omega_{j-1}(x) \\
L_f \Theta_{j-1}(x)
\end{bmatrix} = \Theta_j(x).
\]

Hence,

\[
\begin{bmatrix}
-P_{j,:}(x) & -P_{j,j}(x)R_j + S_j
\end{bmatrix}
bldg \{ R_1, R_2, \ldots, R_{j-1} \} = W_j(x),
\]

\[
\begin{bmatrix}
L_g G_{j-1}(x) \\
L_g \Theta_{j-1}(x)
\end{bmatrix} = \Theta_j(x).
\]

Thus,

\[
C_j(x)L_g G_j(x) = W_j(x),
\]

\[
C_j(x)L_f G_j(x) = \col \{ \Theta_1(x), \Theta_2(x), \ldots, \Theta_j(x) \}.
\]

The matrix \(C_j(x)\) is of full row rank, since \(-P_{j,j}(x)R_j + S_j\) is of full row rank. The matrices \(C_j(x)\) and \(L_g G_j(x)\) have \(\sum_{\ell=1}^j (p - \rho_\ell)\) and \(p + \sum_{\ell=1}^{j-1} (p - \rho_\ell)\) rows, respectively, and the rank of \(L_g G_j(x)\) is \(\rho_j\). Thus, the rows of \(C_j(x)\) form a basis of the solution space of \(\gamma L_g G_j(x) = 0\) in \((M_j \cap \hat{O}_j)^c\).

Now we are ready to prove Lemma 3.6. We do it by induction. Consider \(i = 1\). According to the algorithm, \(\Theta_0(x) = h(x) = \Theta_0(x)\), thus, \(M_1 = M_1\). The rows of \(C_1(x)\) form a basis of the solution space of the homogeneous linear equation \(\gamma L_g \Theta_0(x) = 0\) in \((M_1 \cap \hat{O}_1)^c\). Similarly, the rows of \(\hat{C}_1(x) = -\hat{P}_{1,1}(x)\hat{R}_1 + \hat{S}1\) span the same solution space in \((M_1 \cap \hat{O}_1)^c\). Therefore,

\[
\hat{C}_1(x) = T_1(x)C_1(x) + \varpi_1(x),
\]

where the matrix \(T_1(x) : (O_k \cap \hat{O}_k) \to \mathbb{R}^{(p-\rho_1) \times (p-\rho_1)}\) is a nonsingular and smooth, and \(\varpi_1(x)\) is smooth with \(\varpi_1(x) = 0\) in \(M_1 \cap O_1 \cap \hat{O}_1\). Thus, by Lemma 3.4.1,

\[
\hat{\Theta}_1(x) = \hat{C}_1(x)L_f \Theta_0(x) = T_1(x)\Theta_1(x) + V_1(x),
\]

where \(V_1(x) = \varpi_1(x)L_f \Theta_0(x) = 0\) in \(M_1 \cap O_1 \cap \hat{O}_1\).

Assume that, for \(i = 1, 2, \ldots, j-1\), equations in (3.42) are satisfied. That is,

\[
M_i = M_i, \quad \hat{\Theta}_i(x) = \sum_{l=1}^{i-1} Q_{i,l}(x)\theta_l(x) + T_i(x)\theta_i(x) + V_i(x), \quad i = 1, 2, \ldots, j-1,
\]

where \(T_i(x)\) is nonsingular and \(V_i(x)\) is smooth with \(V_i(x) = 0\) in \(M_i \cap O_i \cap \hat{O}_i\). Thus,

\[
\hat{G}_j(x) = E_j(x)G_j(x) + \col \{ 0, V_1(x), \ldots, V_{j-1}(x) \}, \quad (3.53)
\]

with \(\col \{ 0, V_1(x), \ldots, V_{j-1}(x) \} = 0\) in \(M_{j-1} \cap O_{j-1} \cap \hat{O}_{j-1}\), and \(E_j(x)\) being nonsingular, where
\[ E_j(x) = \begin{bmatrix} I_p & 0 & 0 & \cdots & 0 \\ 0 & T_1(x) & 0 & \cdots & 0 \\ 0 & Q_{2,1}(x) & T_2(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & Q_{j-1,1}(x) & Q_{j-1,2}(x) & \cdots & T_{j-1}(x) \end{bmatrix}. \]

By (3.53), we know that \( G_j(x) = 0 \) is equivalent to \( \tilde{G}_j(x) = 0 \). Thus, \( \tilde{M}_j = M_j \). We also have

\[
d\tilde{G}_j(x) = E_j(x)dG_j(x) + D_j(x), \tag{3.54}
\]

where

\[
D_j(x) = \sum_{l=1}^{\varsigma} G_{j,l} \frac{\partial E_{j,1}}{\partial x} + \frac{\partial}{\partial x} \text{col}\{0, V_1(x), \cdots, V_{j-1}(x)\},
\]

with \( \text{col}\{G_{j,1}, G_{j,2}, \cdots, G_{j,\varsigma}\} = G_j(x) \), \( [E_{j,1}, E_{j,2}, \cdots, E_{j,\varsigma}] = E_j(x) \), and \( \varsigma = \sum_{l=1}^{p-\rho_j}(p-\rho_j) \). It is obvious that \( D_j(x) = 0 \) in \( M_j \cap O_j \cap \tilde{O}_j \).

By (3.54), \( C_j(x)L_yG_j(x) = 0 \) and \( \tilde{C}_j(x)L_y\tilde{G}_j(x) = 0 \) in \( M_j \cap O_j \cap \tilde{O}_j \). The rows of \( C_j(x) \) and \( \tilde{C}_j(x) \) span the solution spaces of homogeneous linear equations \( \gamma L_yG_j(x) = 0 \) and \( \gamma L_y\tilde{G}_j(x) = 0 \) in \( M_j \cap O_j \cap \tilde{O}_j \), respectively. By (3.54),

\[
L_y\tilde{C}_j(x) = E_j(x)L_yG_j(x) + D_j(x)g(x),
\]

and thus,

\[
\tilde{C}_j(x)E_j(x) = F_j(x)C_j(x) + \varpi_{j+1}(x), \tag{3.55}
\]

where \( F_j(x) \) is a nonsingular matrix valued smooth function, and \( \varpi_j(x) = 0 \) in \( M_j \cap O_j \cap \tilde{O}_j \).

Denote

\[
F_j(x) = \begin{bmatrix} Y_j(x) & \tilde{Y}_j(x) \\ Q_j(x) & T_j(x) \end{bmatrix}, \quad \varpi_{j+1}(x) = \begin{bmatrix} \mu_j(x) \\ \rho_j(x) \end{bmatrix},
\]

where \( T_j(x) \) is a \((p-\rho_j) \times (p-\rho_j)\) matrix, and \( \mu_j(x) \) is smooth with \( \mu_j(x) = 0 \) in \( M_j \cap O_j \cap \tilde{O}_j \). Due to the structure of \( C_j(x) \) and \( \tilde{C}_j(x) \), we know that \( \tilde{Y}_j(x) = 0 \) in \( M_j \cap O_j \cap \tilde{O}_j \). Thus, \( T_j(x) \) is nonsingular in a neighborhood of \( x = 0 \), which contains \( M_j \cap O_j \cap \tilde{O}_j \). By (3.55),

\[
[-\tilde{P}_{j,\varsigma}(x)\text{blkdiag}\{\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_{j-1}\}] - \tilde{P}_{j,j}(x)\tilde{R}_j + \tilde{S}_j \right] E_j(x) \\
= Q_j(x) [C_j(x) \ 0] + T_j(x)[-\tilde{P}_{j,\varsigma}(x)\text{blkdiag}\{R_1, R_2, \cdots, R_{j-1}\}] - P_{j,j}(x)R_j + S_j \right] U_j(x). \tag{3.56}
\]

And by (3.41), we have

\[
\Theta_j(x) = [-\tilde{P}_{j,\varsigma}(x)\text{blkdiag}\{R_1, R_2, \cdots, R_{j-1}\}] - P_{j,j}(x)R_j + S_j \right] L_fG_j(x),
\]

\[
\tilde{\Theta}_j(x) = [-\tilde{P}_{j,\varsigma}(x)\text{blkdiag}\{\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_{j-1}\}] - \tilde{P}_{j,j}(x)\tilde{R}_j + \tilde{S}_j \right] L_f\tilde{G}_j(x).
\]
Thus, multiplying \([3.56]\) to the right by \(L_f G_j(x)\) and using \([3.52]\) and \([3.54]\), we have

\[
\hat{\Theta}_j(x) = Q_j(x)C_j(x)L_f G_{j-1}(x) + T_j(x)\Theta_j(x) + V_j(x)
\]

\[
= \sum_{i=1}^{j-1} Q_{j,i}(x)\Theta_i(x) + T_j(x)\Theta_j(x) + V_j(x),
\]

where \([ Q_{j,1}(x) \ Q_{j,1}(x) \cdots Q_{j,j-1}(x) ] = Q_j(x)\) and

\[V_j(x) = \mu_j(x)L_f G_j+ \]

\[
[-P_{j,i}(\cdot)\blkdiag \{R_1, R_2, \cdots, R_{j-1}\} \ -P_{j,j}(\cdot)R_j + S_j] D_j(x)f(x).
\]

Therefore, \(V_j(x) = 0\) in \(M_j \cap O_j \cap \hat{O}_j\).

### 3.6.5 Proof of Lemma 3.8

By the infinite zero structure algorithm, we know that \(\Gamma_{i0}(x) = L_{i0}Q_{k*}(x)\) is of full row rank. Note that \(R_i S_{i-1+j,1}, i = 1, 2, \cdots, k^*,\) are the coefficients in \(\zeta_{i,1}\). So if \(d\hat{\Phi}_d(0)\) is of full row rank, \(\Gamma_{i0}\) is of full row rank. Thus, we only need to prove that \(d\hat{\Phi}_d(0)\) is of full row rank. We prove it by induction.

Recall that \(d\hat{\Phi}_d(0) = \text{col}\{d\zeta_1, d\zeta_2, \cdots, d\zeta_k\}\). We first prove that the row vectors of \(d\zeta_1(0), dR_i\Theta_0(0)\), are linearly independent. It follows directly from the fact that \(L_g R_i \Theta_0(x)\) has full row rank.

Assume that the rows of \(\text{col}\{d\zeta_1(0), d\zeta_2(0) \cdots, d\zeta_k(0)\}\) are linearly independent. We want to prove that the rows of \(\text{col}\{d\zeta_1(0), d\zeta_2(0) \cdots, d\zeta_{k+1}(0)\}\) are linearly independent.

Let \(w_{i,j}(x) : U \rightarrow \mathbb{R}^1 \times (\rho_{i-1} - \rho_{i}), j = 0, 1, \cdots, i - 1,\) and \(i = 1, 2, \cdots, k + 1\).

Define

\[
\beta(x) = \sum_{i=1}^{k+1} w_{i,j-1}(x)dR_i\Theta_{i-1}(x) + \sum_{j=1}^{k+1} \sum_{i=j+1}^{k+1} w_{i,j-1}(x)dR_i S_{i-1+j} \Theta_{j-1}(x).
\]

(3.57)

By \([3.39]\),

\[
\beta(x)g(x) = \sum_{i=1}^{k+1} w_{i,j-1}(x)L_g R_i \Theta_{i-1}(x)
\]

\[
+ \sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(x)R_i S_{i-1+j} L_g S_j \Theta_{j-1}(x)
\]

\[
= \sum_{i=1}^{k+1} w_{i,j-1}(x)L_g R_i \Theta_{i-1}(x) + \sum_{j=1}^{k} \sum_{i=j+1}^{k+1} \left( w_{i,j-1}(x)R_i S_{i-1+j} + \sum_{l=1}^{j} P_{j,l}(x)L_g R_l \Theta_{l-1}(x) + W_j(x) \right)
\]
where

$$
\psi_l(x) = w_{l,l-1}(x) + \sum_{j=l}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(x) R_i S_{i-1+j+1} P_{j,l}(x), \quad l = 1, 2, \cdots, k.
$$

By (3.41),

$$
\beta(x)f(x) = \sum_{l=1}^{k+1} w_{l,l-1}(x) L_f R_l \Theta_{l-1}(x) + \sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(x) R_i S_{i-1+j+1} L_f S_j \Theta_{j-1}(x)
$$

$$
= \sum_{l=1}^{k+1} w_{l,l-1}(x) L_f R_l \Theta_{l-1}(x) + \sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(x) R_i S_{i-1+j+1} \left[ \Theta_j(x) + \sum_{l=1}^{j} P_{j,l}(x) L_f R_l \Theta_{l-1}(x) \right]
$$

$$
= w_{k+1,k}(x) L_f R_{k+1} \Theta_k(x) + \sum_{l=1}^{k} \psi_l(x) L_f R_l \Theta_{l-1}(x)
$$

$$
+ \sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(x) R_i S_{i-1+j+1} \Theta_j(x). \tag{3.58}
$$

Let

$$
\beta(x) = 0. \tag{3.59}
$$

Thus, $\beta(x)g(x) = 0$. Since the matrix

$$
\text{col} \{ L_g R_1 \Theta_0(x), L_g R_2 \Theta_1(x), \cdots, L_g R_{k+1} \Theta_k(x) \}
$$

is of full row rank in $(M_k \cap O_k)^c$, we have

$$
w_{k+1,k}(x) = 0, \quad \psi_l(x) = 0, \quad l = 1, 2, \cdots, k, \tag{3.60}
$$

in $(M_k \cap O_k)^c$. Thus, by (3.59) and (3.60),

$$
\sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(0) dR_i S_{i-1+j} \Theta_{j-1}(0) = 0. \tag{3.61}
$$

By (3.58),

$$
d \left[ \beta(x)f(x) - \sum_{l=1}^{k+1} \psi_l(x) L_f R_l \Theta_{l-1}(x) \right]
$$

$$
= \sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(x) dR_i S_{i-1+j+1} \Theta_j(x) + \sum_{l=1}^{k} \Theta_l(x) \Pi_l(x). \tag{3.62}
$$
where $\Pi_l(x)$, $l = 1, 2, \cdots, k$, are matrix valued functions of $w_{i,j}(x)$. Consider $\Theta_i(0) = 0$. We have

$$
\sum_{j=1}^{k} \sum_{i=j+1}^{k+1} w_{i,j-1}(0) dR_i S_{i-1+j+1} \Theta_j(0) = 0. \quad (3.63)
$$

By (3.61) and (3.63), we have $w_{i,j}(0) = 0$, for $j = 0, 1, \cdots, i - 1$ and $i = 1, 2, \cdots, k + 1$. In conclusion, the row vectors of col \{\{d\zeta_1(0), d\zeta_2(0), \cdots, d\zeta_{k+1}(0)\} \} are linearly independent.
Backstepping Design Procedure

In Chapter 3, we developed a structural decomposition for multiple input multiple output nonlinear systems that are affine in control but otherwise general. In this chapter, we exploit the properties of such a decomposition for the purpose of solving the stabilization problem. In particular, this decomposition simplifies the conventional backstepping design and motivates a new backstepping design procedure that is able to stabilize some systems on which the conventional backstepping is not applicable. An numerical example also shows that different backstepping procedure lead to different control performance.

4.1 Introduction and Problem Statement

Consider a nonlinear system of the form

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x),
\end{align*}
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^p \) are the state, input, and output, respectively, and the mappings \( f \), \( g \) and \( h \) are smooth with \( f(0) = 0 \) and \( h(0) = 0 \).

In Chapter 3, we study the structural properties of affine-in-control nonlinear systems beyond the case of square invertible systems. We propose an algorithm that identifies a set of integers that are equivalent to the infinite zero structure of linear systems and leads to a normal form representation that corresponds to these integers as well as to the system invertibility structure. This new normal form representation takes the following form

\[
\begin{align*}
\dot{\eta} &= f_e(\eta, \xi) + g_e(\eta, \xi)u_e, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{d,l}, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_{d,i}, \\
y_e &= h_e(\eta, \xi), \\
y_{d,i} &= \xi_{i,1}, \quad i = 1, 2, \ldots, m_d,
\end{align*}
\]

(4.2)
54  4 Backstepping Design Procedure

where $q_1 \leq q_2 \leq \cdots \leq q_{m_d}$, $\xi_i = \{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\}$, $i = 1, 2, \cdots, m_d$, $\xi = \{\xi_1, \xi_2, \cdots, \xi_{m_d}\}$, $v_{d,i} = a_i(x) + b_i(x)u$, with the matrix $\text{col}\{b_1(x), b_2(x), \cdots, b_{m_d}(x)\}$ being of full row rank and smooth, and $\delta_{i,j,l}(x) = 0$, for $j < q_i$, $i = 1, 2, \cdots, m_d$.

We note here that $m_d$ is the largest integer for which the system assumes the above form. The system is left invertible if $u_e$ is non-existent, right invertible if $y_e$ is non-existent, and invertible if both are non-existent. In the case that the system is square and invertible, i.e., the system that was considered in [16, 17], $m = p = m_d$ and the parts containing $y_e$ and $u_e$ drop off. Thus, the normal form (4.2) simplifies to

$$\begin{cases}
\dot{\eta} = f_e(\eta, \xi), \\
\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{q_i-1} \delta_{i,j,l}(x)v_l, \quad j = 1, 2, \cdots, q_i - 1, \\
\xi_{i,q_i} = v_i, \\
y_i = \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{cases}$$

(4.3)

where $q_1 \leq q_2 \leq \cdots \leq q_{m_d}$, and

$$\delta_{i,j,l}(x) = 0, \quad \text{for } j < q_i, \ i = 1, 2, \cdots, m. \quad (4.4)$$

We note that if $q_1 = q_2 = \cdots = q_{m_d}$ in (4.3), then by the property (4.4), the system turns out to have uniform relative degrees $q_1$. The $\dot{\xi}_{i,j}$ equation in (4.3) displays a triangular structure of the control inputs that enter the system. The property (4.4) imposes additional structure within each chain of integrators on how control inputs enter the system. With this additional structural property, the set of integers $\{q_1, q_2, \cdots, q_{m_d}\}$ indeed represent infinite zero structure when the system is linear.

Control design techniques and structural decompositions of nonlinear systems have been developed interweavingly. The discovery of structural properties and the corresponding normal form representation of the system motivates new control designs. On the other hand, the desire for achieving more stringent closed-loop performances for a larger class of systems entails the exploitation of more intricate structural properties. For example, various stabilization results have been obtained in this process. In this chapter, we would like to revisit the problem of stabilization. We will show how the property (4.4) simplifies the conventional backstepping design and motivates a new backstepping design technique that is able to stabilize some systems that cannot be stabilized by the conventional backstepping technique.

4.2 Review of the Backstepping Design Technique

In the section, we recall some results on the backstepping design methodology [16, 17, 66]. We first recall the integrator backstepping, on which the recursive backstepping procedure is develop.
Lemma 4.1. \[\text{Consider} \]

\[
\begin{align*}
\dot{\eta} &= f(\eta, \xi), \\
\dot{\xi} &= u,
\end{align*}
\]

(4.5)

where \((\eta, \xi) \in \mathbb{R}^n \times \mathbb{R}\) and \(f(0,0) = 0\). Suppose there exists a smooth real-valued function \(\xi = v^*(\eta)\), with \(v^*(0) = 0\), and a smooth real-valued function \(V(\eta)\), which is positive definite and proper, such that

\[
\frac{\partial V}{\partial \eta} f(\eta, v^*(\eta)) < 0, \quad \forall \eta \neq 0.
\]

Then, there exists a smooth static feedback law \(u = u(\eta, \xi)\) with \(u(0,0) = 0\), and a smooth real-valued function \(W(\eta, \xi)\), which is positive definite and proper, such that

\[
\frac{\partial W}{\partial \eta} f(\eta, \xi) + \frac{\partial W}{\partial \xi} u(\eta, \xi) < 0, \quad \forall (\eta, \xi) \neq 0.
\]

(4.6)

That is, \(u = u(\eta, \xi)\) globally asymptotically stabilizes (4.5) at its equilibrium \((\eta, \xi) = 0\).

The negative definiteness property in (4.6) can be replaced with a negative semi-definiteness property along with a LaSalle’s Invariance argument.

The backstepping design method is readily applicable to systems that have vector relative degrees and are represented in the following form,

\[
\begin{align*}
\dot{\eta} &= f_0(x) + g_0(x)u, \\
\dot{\xi}_{i,j} &= \xi_{i,j+1}, \quad j = 1, 2, \cdots, r_i - 1, \\
\dot{\xi}_{i,r_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{align*}
\]

which contains \(m\) clean chains of integrators. Each of these chains is independently controlled by a separate input. Let us consider the following assumption.

Assumption 1 The dynamics \(\eta\) is driven only by \(\xi_{i,1}\), \(i = 1, 2, \cdots, m\), i.e.,

\[
\dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}),
\]

(4.7)

and there exist smooth functions \(\phi_{i,1}(\eta)\), with \(\phi_{i,1}(0) = 0\), \(i = 1, 2, \cdots, m\), such that \(\dot{\eta} = f_0(\eta, \phi_{1,1}(\eta), \phi_{2,1}(\eta), \cdots, \phi_{m,1}(\eta))\) is globally asymptotically stable at its equilibrium \(\eta = 0\).

Suppose that Assumption 1 holds, then for the systems with vector relative degree, it is straightforward to design a globally asymptotically stabilizing feedback law recursively, by viewing the next integrators as a new virtual input. Such a design procedure is thus referred to as “backstepping.”
The technique of backstepping, however, cannot as easily be implemented if the system does not have a vector relative degree. An additional assumption, which requires the coefficient functions $\delta_{i,j,l}$ in the following normal form

\[
\begin{aligned}
\dot{\eta} &= f_0(x), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_l, & j = 1, 2, \ldots, n_i - 1, \\
\dot{\xi}_{i,n_i} &= v_i, & i = 1, 2, \ldots, m.
\end{aligned}
\]

(4.8)

(4.8)

\[
\begin{aligned}
\dot{\eta} &= f_0(x), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_l, & j = 1, 2, \ldots, n_i - 1, \\
\dot{\xi}_{i,n_i} &= v_i, & i = 1, 2, \ldots, m.
\end{aligned}
\]

to display a certain “triangular” dependency on the state variables, is needed \[16, 17\]. In what follows, we recall from \[17\] such an additional assumption and the backstepping design procedure that is implemented under these assumptions.

**Assumption 2** The functions $\delta_{i,j,l}$ depend only on variable $\xi_{\ell_p,\ell_b}$, with

1. $1 \leq \ell_p \leq m$ and $\ell_b = 1$; or,
2. $\ell_p \leq i - 1$; or,
3. $\ell_p = i$ and $\ell_b \leq j$.

Under Assumptions 1 and 2, a feedback law

\[
v_i = v_i(\eta; \xi_1, \xi_2, \ldots, \xi_i), \quad i = 1, 2, \ldots, m,
\]

that globally stabilizes the whole system can be constructed from $\phi_{i,1}(\eta)$, $i = 1, 2, \ldots, m$, through a backstepping procedure \[17\]. The procedure commences with the subsystem \[4.7\], and is followed by backstepping $n_1$ times through the variables in first chain of integrators to obtain

\[
v_1 = v_1(\eta; \xi_1, \phi_{2,1}(\eta), \phi_{3,1}(\eta), \ldots, \phi_{m,1}(\eta)),
\]

and backstepping $n_2$ times through the variables in the second chain of integrators to obtain the feedback law

\[
v_2 = v_2(\eta; \xi_1, \phi_{2,1}(\eta), \phi_{3,1}(\eta), \phi_{4,1}(\eta), \ldots, \phi_{m,1}(\eta)).
\]

This procedure is continued chain by chain for $i = 1$ through $m$, each backstepping $n_i$ times through $i$-th chain of integrators to discover the feedback law

\[
v_i = v_i(\eta; \xi_1, \phi_{i+1,1}(\eta), \phi_{i+2,1}(\eta), \ldots, \phi_{m,1}(\eta)).
\]

As the backstepping is implemented on the integrators chain by chain, we will refer to the above backstepping procedure as the chain-by-chain backstepping.
4.3 Backstepping Design Procedures Revisited

In this section we focus on systems that are square invertible and discuss about their stabilization by the backstepping technique. We will first show that the conventional chain-by-chain backstepping design technique as described in [17] and recalled in Section 4.2 is applicable to our new normal form (4.3)-(4.4), and its implementation on this new normal form is simpler than on the earlier normal form (4.8). We the propose a new backstepping procedure which we refer to as the level-by-level backstepping. In the level-by-level backstepping design procedure, the backstepping is first implemented on the first integrators of all chains and then on the second integrators of all chains, and so on. We will show that the level-by-level backstepping will allow the backstepping to be implemented on some systems for which the chain-by-chain backstepping procedure is not applicable. We will also show that the chain-by-chain backstepping and the level-by-level backstepping can be mixed and implemented on a same system to allow stabilization of a larger class of systems.

4.3.1 Conventional Chain-by-Chain Backstepping

Since the normal form (4.3)-(4.4) is a special case of the normal form (4.8), backstepping is applicable to it. As explained in [17], the chain-by-chain backstepping requires the system (4.3) to satisfy Assumptions 1 and 2. Under these two assumptions, the normal form (4.3)-(4.4) is much simpler than the normal form (4.8). This simpler form makes the implementation of the chain-by-chain backstepping procedure to be easier.

Example 4.2. A three input three output system in the form (3.12) with three chains of integrators of lengths \{2, 4, 4\} and satisfying Assumption 2 will take the form (see Fig. 4.1),

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= \nu_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + \delta_{2,1,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1})\nu_1, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1})\nu_1, \\
\dot{\xi}_{2,3} &= \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_{1,1}, \xi_{2,3}, \xi_{3,1})\nu_1, \\
\dot{\xi}_{2,4} &= \nu_2, \\
\dot{\xi}_{3,1} &= \xi_{3,2} + \delta_{3,1,1}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1})\nu_1 + \delta_{3,1,2}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1})\nu_2, \\
\dot{\xi}_{3,2} &= \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_{1,1}, \xi_{2,3}, \xi_{3,1}, \xi_{3,2})\nu_1 + \delta_{3,2,2}(\eta, \xi_{1,1}, \xi_{2,3}, \xi_{3,1}, \xi_{3,2})\nu_2, \\
\dot{\xi}_{3,3} &= \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_{1,1}, \xi_{2,3}, \xi_{3,2}, \xi_{3,3})\nu_1 \\
&\quad + \delta_{3,3,2}(\eta, \xi_{1,1}, \xi_{2,3}, \xi_{3,2}, \xi_{3,3})\nu_2, \\
\dot{\xi}_{3,4} &= \nu_3.
\end{align*}
\]
Fig. 4.1. Nonlinear system in the form (4.8) with three chains of integrators of lengths \{2, 4, 4\}.

On the other hand, under the same assumption, the normal form (4.3)-(4.4) would take the following simpler form

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{3,1})v_1, \\
\dot{\xi}_{2,3} &= \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{2,3}, \xi_{3,1})v_1, \\
\dot{\xi}_{2,4} &= v_2, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_1, \xi_{2,1}, \xi_{3,1}, \xi_{3,2})v_1, \\
\dot{\xi}_{3,3} &= \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_1, \xi_{2,1}, \xi_{3,1}, \xi_{3,2}, \xi_{3,3})v_1, \\
\dot{\xi}_{3,4} &= v_3.
\end{align*}
\]  

(4.10)

Suppose that Assumption 1 is satisfied. The form (4.10) makes the implementation of the chain-by-chain backstepping simpler, due to the simpler structure (see Fig. 4.2).

4.3.2 Level-by-Level Backstepping

Let us call all \(\xi_{i,1}\), i.e., the “leading” variables in each chain of integrators which connect an input to an output, the first level integrators, and call all \(\xi_{i,2}\) the second level integrators, and so on. As an alternative to the chain-by-chain backstepping, we here propose to carry out the backstepping on all first level integrators, and then repeat the procedure on all second level integrators until we reach to last level of integrators. We will refer to such a backstepping procedure as the level-by-level backstepping, in contrast with the chain-by-chain backstepping procedure.
4.3 Backstepping Design Procedures Revisited

Fig. 4.2. Nonlinear system in the form (4.3)-(4.4) with three chains of integrators of lengths {2, 4, 4}.

To make the level-by-level backstepping possible, the coefficients $\delta_{i,j,l}$ in the normal form (4.8) with the property (4.4) should satisfy the following assumption:

**Assumption 3** The functions $\delta_{i,j,l}$ depend only on variable $\xi_{\ell_p, \ell_b}$, with

1. $1 \leq \ell_p \leq m$ and $\ell_b = 1$; or,
2. $\ell_b \leq j - 1$; or
3. $\ell_b = j$ and $\ell_p \leq i$.

We will say that the coefficients $\delta_{i,j,l}$ in the normal form (4.8) have the chain-by-chain triangular dependency on state variables if they satisfy Assumption 2. The coefficients $\delta_{i,j,l}$ in the normal form (4.3) - (4.4) have the level-by-level triangular dependency on state variables if they satisfy Assumption 3.

Under Assumptions 1 and 3, the level-by-level backstepping procedure for the normal form (4.3) with the property (4.4) can be described as follows. We will start with

$$\dot{\eta} = f_0(\eta, \phi_{1,1}(\eta), \phi_{2,1}(\eta), \cdots, \phi_{m,1}(\eta)).$$

After the first-level backstepping, we obtain the feedback laws

$$v_i = v_i(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}), \quad i = 1, 2, \cdots, \alpha_1,$$

where $\alpha_1$ is the number of chains that contain exactly one integrator, i.e., $n_1 = n_2 = \cdots = n_{\alpha_1} = 1$. For chains that contain more than one integrator, $\xi_{i,2}$ are viewed as virtual inputs, and the desired $\xi_{i,2}$ are defined as

$$\xi_{i,2}^* = \phi_{i,2}(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}), \quad i = \alpha_1 + 1, \alpha_1 + 2, \cdots, m.$$  

We next proceed with backstepping on the second level integrators. After the second level backstepping, we obtain the feedback laws

$$v_i = v_i(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{\alpha_1+1,2}, \xi_{\alpha_1+2,2}, \cdots, \xi_{i,2}), \quad i = \alpha_1 + 1, \alpha_1 + 2, \cdots, \alpha_2,$$
where \( \alpha_2 - \alpha_1 \) is the number of chains that contain exactly two integrators, i.e., \( n_{\alpha_1+1} = n_{\alpha_1+2} = \cdots = n_{\alpha_2} = 2 \). For chains with lengths greater than 2, the variables \( \xi_{i,3} \) are viewed as virtual inputs, and the desired \( \xi_{i,3} \) are defined as

\[
\xi_{i,3} = \phi_{i,3}(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{\alpha_1+1,2}, \xi_{\alpha_1+2,2}, \cdots, \xi_{i,2}),
\]

\[i = \alpha_2 + 1, \alpha_2 + 2, \cdots, m.\]

Continuing in this way, we finally obtain

\[
v_i = v_i(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{\alpha_1+1,2}, \xi_{\alpha_1+2,2}, \cdots, \xi_{m,2}; \cdots; \xi_{\alpha_{m-1}+1,n_m}, \xi_{\alpha_{m-1}+2,n_m}, \cdots, \xi_{i,n_m}),
\]

for chains that contain \( n_m \) integrators.

**Example 4.3.** Consider a system in the normal form \([4.3], [4.4]\) with three chains of integrators of lengths \{2, 4, 4\}. See Fig. 4.2.

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2})v_1, \\
\dot{\xi}_{2,3} &= \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2}, \xi_{3,2})v_1, \\
\dot{\xi}_{2,4} &= v_2, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2}, \xi_{3,2})v_1, \\
\dot{\xi}_{3,3} &= \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2}, \xi_{3,2}, \xi_{3,3})v_1, \\
\dot{\xi}_{3,4} &= v_3.
\end{align*}
\]

Clearly, Assumption 3 is satisfied, but Assumption 2 is not. Consequently, the chain-by-chain backstepping cannot be implemented on this system. In what follows, we will illustrate how to implement the level-by-level backstepping on this system.

Let Assumption 1 be satisfied, i.e., there exist smooth functions \( \phi_{i,1}(\eta) \), with \( \phi_{i,1}(0) = 0, i = 1, 2, 3 \), such that the equilibrium \( \eta = 0 \) of the subsystem

\[
\dot{\eta} = f_0(\eta, \phi_{1,1}(\eta), \phi_{2,1}(\eta), \phi_{3,1}(\eta))
\]

is globally asymptotically stable. The backstepping procedure starts with the subsystem \([4.12]\). Now we consider the backstepping on the first level variables. The variable \( \xi_{1,1} \) can be viewed as the virtual input of the subsystem

\[
\dot{\eta} = f_0(\eta, \xi_{1,1}, \phi_{2,1}(\eta), \phi_{3,1}(\eta)),
\]

and the desired input is given by \( \xi_{1,1} = \phi_{1,1}(\eta) \).

To carry out the backstepping from \( \xi_{1,1} \) to \( \xi_{1,2} \), we consider the subsystem

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,1}, \phi_{2,1}(\eta), \phi_{3,1}(\eta)), \\
\dot{\xi}_{1,1} &= \xi_{1,2}.
\end{align*}
\]

60 4 Backstepping Design Procedure
with $\xi_{1,2}$ as the virtual input. By Lemma 4.1, this subsystem can be globally asymptotically stabilized by a control of the form

$$\xi_{1,2}^* = \phi_{1,2}(\eta, \xi_{1,1}, \phi_{2,1}(\eta), \phi_{3,1}(\eta)). \quad (4.13)$$

The variable $\xi_{2,1}$ can be viewed as the virtual input of the subsystem

$$\begin{cases} 
\dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \phi_{3,1}(\eta)), \\
\dot{\xi}_{1,1} = \xi_{1,2}, \\
\dot{\xi}_{2,1} = \xi_{2,2}, \\
\dot{\xi}_{3,1} = \xi_{3,2}, 
\end{cases} \quad (4.14)$$

and $\xi_{2,1}^* = \phi_{2,1}(\eta)$ globally asymptotically stabilizes its equilibrium $\col{\eta, \xi_{1,1}} = 0$. To carry out the backstepping from $\xi_{2,1}$ to $\xi_{2,2}$, we next look at the subsystem

$$\begin{cases} 
\dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \phi_{3,1}(\eta)), \\
\dot{\xi}_{1,1} = \xi_{1,2}, \\
\dot{\xi}_{2,1} = \xi_{2,2}, \\
\dot{\xi}_{3,1} = \xi_{3,2}, 
\end{cases} \quad (4.15)$$

Thus, after the first level backstepping, the subsystem

$$\begin{cases} 
\dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\
\dot{\xi}_{1,1} = \xi_{1,2}, \\
\dot{\xi}_{2,1} = \xi_{2,2}, \\
\dot{\xi}_{3,1} = \xi_{3,2}, 
\end{cases} \quad (4.16)$$

can be written as

$$\dot{\eta}_I = f_I(\eta_I, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \quad (4.16)$$

where $\eta_I = \col{\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}}$, and $\xi_{1,2}$, $\xi_{2,2}$ and $\xi_{3,2}$ are the virtual inputs. The equilibrium $\eta_I = 0$ of this subsystem (4.16) is globally asymptotically stabilized by the virtual inputs $\xi_{1,2}$, $\xi_{2,2}$ and $\xi_{3,2}$ as given by (4.13), (4.14) and (4.15), respectively.
For the second level backstepping, consider
\[
\begin{align*}
\dot{\eta} &= f_1(\eta, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \\
\dot{\xi}_{1,2} &= v_1, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2})v_1, \\
\dot{\xi}_{3,2} &= \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2}, \xi_{3,2})v_1,
\end{align*}
\]
(4.17)
and view \(\xi_{2,3}\) and \(\xi_{3,3}\) as its virtual inputs. Following the same procedure as in the first level backstepping, we find the controls of the form
\[
\begin{align*}
\begin{cases}
v_1 = v_1(\eta, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \\
\xi_{2,3}^* = \phi_{2,3}(\eta, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \\
\xi_{3,3}^* = \phi_{3,3}(\eta, \xi_{1,2}, \xi_{2,2}, \xi_{3,2})
\end{cases}
\end{align*}
\]
(4.18)
that globally asymptotically stabilize the equilibrium
\[
\eta_m = \text{col} \{\eta_1, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}\} = 0
\]
of the subsystem (4.17). In other word, the subsystem (4.17) can be written as
\[
\dot{\eta}_m = f_m(\eta_m, v_1, \xi_{2,3}, \xi_{3,3}),
\]
whose equilibrium \(\eta_m = 0\) is globally asymptotically stabilized by the input \(v_1\), and virtual inputs \(\xi_{2,3}\) and \(\xi_{3,3}\) given by (4.18).

Define
\[
\begin{align*}
\dot{\eta}_m &= f_m(\eta_m, v_1, \xi_{2,3}, \xi_{3,3}), \\
\dot{\xi}_{2,3} &= \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2}, \xi_{3,2}, \xi_{3,3})v_1, \\
\dot{\xi}_{3,3} &= \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{2,2}, \xi_{3,2}, \xi_{3,3})v_1,
\end{align*}
\]
(4.19)
on which we carry out the third level of backstepping to obtain
\[
\begin{align*}
\begin{cases}
\xi_{2,4}^* = \phi_{2,4}(\eta_m, \xi_{2,3}, \xi_{3,3}), \\
\xi_{3,4}^* = \phi_{3,4}(\eta_m, \xi_{2,3}, \xi_{3,3})
\end{cases}
\end{align*}
\]
(4.20)
The subsystem (4.19) can be defined as
\[
\dot{\eta}_m = f_m(\eta_m, v_1, \xi_{2,4}, \xi_{3,4}),
\]
whose equilibrium \(\eta_m = \text{col} \{\eta_1, \xi_{2,3}, \xi_{3,3}\} = 0\) is globally asymptotically stabilized by the virtual inputs \(\xi_{2,4}\) and \(\xi_{3,4}\) given by (4.20).

Finally, for the fourth level backstepping, we define
\[
\begin{align*}
\dot{\eta}_m &= f_m(\eta_m, v_2, \xi_{2,4}, \xi_{3,4}), \\
\dot{\xi}_{2,4} &= v_2, \\
\dot{\xi}_{3,4} &= v_3,
\end{align*}
\]
on which we carry out the last level of backstepping to obtain
4.3 Backstepping Design Procedures Revisited

\[
\begin{align*}
    v_2 &= v_2(\eta_{III}, \xi_{2,4}, \xi_{4,4}), \\
    v_3 &= v_3(\eta_{III}, \xi_{2,4}, \xi_{3,4}).
\end{align*}
\]

The inputs \(v_1, v_2\) and \(v_3\) globally asymptotically stabilize the origin of the system (4.11).

**Remark 4.3.1** The structural property (4.4) makes the level-by-level backstepping possible. It is not possible to implement the level-by-level backstepping technique on the normal form (4.8). For example, consider a system in the form (4.8) with two chains of integrators of lengths \(\{2, 3\}\), backstepping the virtual input from \(\xi_{2,1} = \phi_{2,1}(\eta)\) to \(\xi_{2,2}\) by the dynamical equation \(\dot{\xi}_{2,1} = \xi_{2,2} + \delta_{2,1,1}(\eta, \xi) v_1\) is infeasible. At this stage, \(v_1\) is not yet available.

**4.3.3 Mixed Chain-by-Chain and Level-by-Level Backstepping**

A system with a vector relative degree is a special case of the systems (4.8) with all \(\delta_{i,j,l} = 0\). Thus, both chain-by-chain backstepping and level-by-level backstepping can be implemented on it. Furthermore, backstepping can be switched across chains and levels as long as a variable of lower level in a chain is backstepped earlier than variables of higher levels in the same chain.

In the absence of a vector relative degree, the normal form (4.3) with the property (4.4) contains coefficient functions \(\delta_{i,j,l}\). The implementation of both chain-by-chain and level-by-level backstepping require structural dependency on state variables of \(\delta_{i,j,l}\). Such structural dependency constraint can be weakened by utilizing mixed chain-by-chain and level-by-level backstepping.

**Example 4.4.** Consider a system with three chains of integrators of lengths \(\{2, 4, 4\}\).

\[
\begin{align*}
    \dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\
    \dot{\xi}_{1,1} &= \xi_{1,2}, \\
    \dot{\xi}_{1,2} &= v_1, \\
    \dot{\xi}_{2,1} &= \xi_{2,2}, \\
    \dot{\xi}_{2,2} &= \xi_{2,3}, \\
    \dot{\xi}_{2,3} &= \xi_{2,4} + \xi_{3,2} v_1, \\
    \dot{\xi}_{2,4} &= v_2, \\
    \dot{\xi}_{3,1} &= \xi_{3,2}, \\
    \dot{\xi}_{3,2} &= \xi_{3,3}, \\
    \dot{\xi}_{3,3} &= \xi_{3,4} + \xi_{2,4} v_1, \\
    \dot{\xi}_{3,4} &= v_3.
\end{align*}
\] (4.21)

Let Assumption 1 be satisfied. Due to the term \(\xi_{3,2} v_1\), Assumption 2 is not satisfied. Similarly, because of the term \(\xi_{2,4} v_1\), Assumption 3 is not met. As a result, neither the chain-by-chain nor the level-by-level backstepping can be implemented on this system. However, a mixed chain-by-chain and level-by-level backstepping will successfully stabilize this system.
In particular, by Lemma 4.1, we can carry out backstepping in the order of \(\{\xi_{1,1}, \xi_{2,1}, \xi_{3,1}, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}, \xi_{2,3}, \xi_{3,3}, \xi_{3,4}\}\) to obtain
\[
\begin{align*}
v_1 &= v_1(\eta, \xi_1), \\
v_2 &= v_2(\eta, \xi_1, \xi_2, \xi_3, \xi_3), \\
v_3 &= v_3(\eta, \xi_1, \xi_2, \xi_3).
\end{align*}
\]

Shown in Fig. 4.3 are backstepping procedures for the systems in the normal form (4.3)-(4.4) with three chains of integrators of lengths \(\{2, 4, 4\}\).

Motivated by the mixed chain-by-chain and level-by-level backstepping, we give the following result, which includes all the above backstepping procedures as special cases.

**Theorem 4.3.1** Consider a system in the form
\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{i=1}^{j-1} \delta_{i,j,i}(\eta, \xi) v_i, \quad j = 1, 2, \cdots, n_i - 1, \\
\dot{\xi}_{i,n_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{align*}
\]  
(4.22)

where \(\xi = \text{col}\{\xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}\}\), \(\xi_i = \text{col}\{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,n_i}\}\), 
\(v = \text{col}\{v_1, v_2, \cdots, v_m\}\), \(n_1 \leq n_2 \leq \cdots \leq n_m\), and all functions \(\delta_{i,j,l}\) and \(f_0\) are smooth. Then, there exists a feedback \(v = v(\eta, \xi)\) that globally asymptotically stabilizes the system at \((\eta, \xi) = 0\) if

1. there exist \(\phi_{i,1}(\eta)\), \(i = 1, 2, \cdots, m\), such that
   \[\dot{\eta} = f_0(\eta, \phi_{1,1}(\eta), \phi_{2,1}(\eta), \cdots, \phi_{m,1}(\eta))\] is globally asymptotically stable at its equilibrium \(\eta = 0\); and

2. there exists an ordered list \(\kappa\) containing all variables of \(\xi\) such that, for 
   
   \(j = 1, 2, \cdots, n_i - 1, \ l = 1, 2, \cdots, i - 1, \ i = 1, 2, \cdots, m, \)
   
   a) \(\xi_{i,j}\) appears earlier than \(\xi_{i,j+1}\) in \(\kappa\); 
   b) for \(\delta_{i,j,l} \neq 0\), the variables \(\xi_l\) appear earlier than \(\xi_{i,j}\) in \(\kappa\); 
   c) \(\delta_{i,j,l}\) depends only on \(\eta, \xi_{\ell,1}, \ell = 1, 2, \cdots, m, \) and the variables that appear no later than \(\xi_{i,j}\) in \(\kappa\).

The backstepping procedures can be carried out according to the order of \(\kappa\). In some cases, there exist more than one \(\kappa\). Backstepping in different orders lead to different dependency of controls on state variables, which can be exploited to meet certain constraints or performance requirement.

**Example 4.5.** Consider a system in the form of (4.22),
\[
\begin{align*}
\dot{\eta} &= \eta + \xi_{1,1} + \xi_{2,1}, \\
\dot{\xi}_{1,1} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + \xi_{3,2} v_1, \\
\dot{\xi}_{2,2} &= v_2, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi_{3,3}, \\
\dot{\xi}_{3,3} &= \xi_{3,4} + \xi_{2,2} v_1, \\
\dot{\xi}_{3,4} &= v_3.
\end{align*}
\]
The zero dynamics $\dot{\eta} = \eta$ can be stabilized by

$$\xi_{1,1}^* = 0, \quad \xi_{2,1}^* = -2\eta, \quad \xi_{3,1}^* = 0.$$

Neither Assumption 2 nor Assumption 3 is satisfied. So neither the chain-by-chain nor the level-by-level backstepping can be carried out. However, the system satisfies the conditions of Theorem 4.3.1 with the ordered list
The Lyapunov function of proving stability of the closed-loop system is

\[ V = \{\xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}\}. \]

By backstepping in the order of \( \kappa \), the stabilizing controller is given as

\[
\begin{cases}
v_1 = -\eta, \\
v_2 = -6\eta - 5\xi_{1,1} - 6\xi_{2,1} - 3\xi_{2,2} + \eta(-\xi_{3,1} + 3\xi_{3,2}) + \xi_{3,2}(\xi_{1,1} + \xi_{2,1}), \\
v_3 = -\xi_{3,1} - 2\xi_{3,2} - 5\xi_{3,3} - 3\xi_{3,4} + (\eta + \xi_{1,1} + \xi_{2,1})(\beta - \eta\xi_{3,2}) \\
+ \eta(\beta + \eta + 3\xi_{1,1} + 3\xi_{2,1} + 4\xi_{2,2} - 2\eta\xi_{3,2} - \eta\xi_{3,3} + 2v_2),
\end{cases}
\]

where \( \beta = 3\eta + 2\xi_{1,1} + 2\xi_{2,1} + 2\xi_{2,2} - \eta\xi_{3,2} \), the Lyapunov function

\[
V = [\eta^2 + \xi_{1,1}^2 + \xi_{1,2}^2 + (\xi_{2,1} + 2\eta)^2 + \beta^2 + (\xi_{3,3} + \xi_{3,1} + \xi_{3,2})^2 + (\xi_{3,4} + \beta)^2] / 2,
\]

and its derivative

\[
\dot{V} = -\eta^2 - \xi_{3,2}^2 - (\xi_{2,1} + 2\eta)^2 - \beta^2 - (\xi_{3,3} + \xi_{3,1} + \xi_{3,2})^2 - (\xi_{3,4} + \beta)^2 \leq 0.
\]

It can be verified that no solution other than \((\eta(t), \xi(t)) = 0\) can stay forever in \(\{(\eta, \xi) \in \mathbb{R}^5 : \dot{V}(\eta, \xi) = 0\}\). Thus, by LaSalle theorem, the closed-loop system is globally asymptotically stable at the origin. Shown in Fig. 4.4 are some state trajectories of the closed-loop system with different initial conditions.

Next, we explain how different backstepping procedures leading to different control performance by a simple linear numerical example.

**Example 4.6.** Consider the system \([3.7]\) with a vector relative degree \{2, 2\},

\[
\begin{cases}
\dot{\eta} = \eta + \xi_{1,1} + \xi_{2,1}, \\
\dot{\xi}_{1,1} = \xi_{1,2}, \\
\dot{\xi}_{1,2} = v_1, \\
\dot{\xi}_{2,1} = \xi_{2,2}, \\
\dot{\xi}_{2,2} = v_2,
\end{cases}
\]

(4.23)

The zero dynamics here is linear for the convenience of backstepping. We certainly can use linear system tools to design controllers.

Here, in each step, integrator backstepping with \(c = 1\) is implemented (see Lemma 2.8 in [66] for detail). The backstepping starts with

\[
\xi_{1,1}^* = -\eta, \quad \xi_{2,1}^* = -\eta.
\]

We first carry out chain-by-chain backstepping in the order of \(\{\xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2}\}\), and obtain

\[
\begin{cases}
v_1 = -3\eta - 5\xi_{1,1} - 3\xi_{1,2}, \\
v_2 = -11\eta - 4\xi_{1,1} - 6\xi_{1,2} - 11\xi_{2,1} - 3\xi_{2,2}.
\end{cases}
\]

(4.24)

The Lyapunov function of proving stability of the closed-loop system is
Fig. 4.4. State trajectories with two sets of initial conditions.

\[
V = \eta^2 + (\xi_{1,1} + \eta)^2 + (\xi_{1,2} + 2\eta + 2\xi_{1,1})^2 + (\xi_{2,1} + \eta)^2 + (\xi_{2,2} + 2\eta + 2\xi_{1,1} + 2\xi_{1,2} + 2\xi_{2,1})^2/2
\]

and \(\dot{V} = -2V\).

The level-by-level backstepping can be implemented in the order of \(\{\xi_{1,1}, \xi_{2,1}, \xi_{1,2}, \xi_{2,2}\}\) to arrive at

\[
\begin{align*}
\begin{cases} 
\dot{v}_1 &= -5\eta - 5\xi_{1,1} - 3\xi_{1,2} - 2\xi_{2,1}, \\
\dot{v}_2 &= -9\eta - 6\xi_{1,1} - 2\xi_{1,2} - 6\xi_{2,1} - 2\xi_{2,2},
\end{cases}
\end{align*}
\] (4.25)
\[ V = \left[ \eta^2 + (\xi_{1,1} + \eta)^2 + (\xi_{2,1} + \eta)^2 + (\xi_{1,2} + 2\eta + 2\xi_{1,1})^2 + (\xi_{2,2} + 4\eta + 2\xi_{1,1} + \xi_{2,1})^2 \right]/2 \]

and \( \dot{V} = -2V \).

Comparing the four feedback gains in (4.24) and (4.25), \( v_1 \) obtained in the chain-by-chain backstepping is the smallest, and \( v_2 \) obtained in the chain-by-chain backstepping is the largest, while \( v_1 \) and \( v_2 \) for level-by-level backstepping have average gains. The input \( v_1 \) obtained in the chain-by-chain backstepping does not depend on \( \xi_{2,1} \), while \( v_1 \) obtained in the level-by-level backstepping does. If there is disturbance in \( \xi_{2,1} \), both inputs of level-by-level backstepping can take account of it more directly.

Run simulation 1000 times with the initial conditions being uniformly distributed pseudo-random numbers in the interval \([-1, 1]\), Figure 4.5 shows
4.3 Backstepping Design Procedures Revisited

 outlines of each state trajectory, and outlines of absolute values of each input. It indicates that state trajectories of level-by-level backstepping converge faster.

Consider adding the following disturbances to each state equation,

\[ w_i(t) = \text{rand}_{i,j}, \quad t \in [0.01j, 0.01(j + 1)], \quad j \geq 0, \quad 1 \leq i \leq 5, \]

where \( \text{rand}_{i,j} \) is an uniformly distributed pseudo-random number in the interval \([0, 1]\). Run simulation 1000 times again with disturbance. Figure 4.6 shows outlines of each state and absolute values of each input. It indicates that the feedback law from level-by-level backstepping has a better disturbance rejection capability.

This example shows that backstepping in different orders lead to different dependency of controls on state variables and different performance, which
can be exploited to meet certain constraints or performance requirement. In general, we could select a backstepping order with each input depend on more state variables. In this case, each input would take full advantage of the state information, and thus the closed-loop systems tend to have a better performance.

4.4 Summary of the Chapter

We exploited the properties of a recently developed structural decomposition for the stabilization of multiple input and multiple output systems, and showed that this decomposition simplifies the conventional chain-by-chain backstepping design and motivates a new level-by-level backstepping design procedure that is able to stabilize some systems for which the conventional backstepping procedure is not applicable. The chain-by-chain backstepping and level-by-level backstepping can be combined to form a mixed backstepping design technique. The enlarged class of systems that can be stabilized by this mixed backstepping design procedure is characterized in the form of a theorem.
Semi-global Stabilization for Nonlinear Systems

In Chapter 3, we developed a structural decomposition for multiple input multiple output nonlinear systems that are affine in control but otherwise general. Chapter 4 shows that this structural decomposition simplifies the conventional backstepping design and allows a new backstepping design procedure that is able to stabilize some systems on which the conventional backstepping is not applicable. In this chapter we further exploit the properties of such a decomposition for the purpose of solving the semi-global stabilization problem for minimum phase nonlinear systems without vector relative degrees. By taking advantage of special structure of the decomposed system, we first apply the low gain design to the part of system that possesses a linear dynamics. The low gain design results in an augmented zero dynamics that is locally stable at the origin with a domain of attraction that can be made arbitrarily large by lowering the gain. With this augmented zero dynamics, backstepping design is then apply to achieve semi-global stabilization of the overall system.

5.1 Introduction and Problem Statement

Consider the problem of semi-globally stabilizing a nonlinear system of the affine-in-control form
\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x),
\end{align*}
\] (5.1)
where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^p\) are the state, input and output, respectively, and the mappings \(f\), \(g\) and \(h\) are smooth with \(f(0) = 0\) and \(h(0) = 0\).

In a semi-global stabilization problem, we are to construct, for any given, arbitrarily large, bounded set of the state space \(X_0\), a smooth feedback law, say \(u = v_{X_0}(x)\), with \(v(0) = 0\), such that the closed-loop system is asymptotically stable at the origin with \(X_0\) contained in the domain of attraction.

The non-local stabilization of nonlinear systems of the form (5.1) has been made possible by the structural decomposition, in the form of various normal forms, of these systems.
In Chapter 3, we propose an algorithm that identifies a set of integers that are equivalent to the infinite zero structure of linear systems and leads to a normal form representation that corresponds to these integers as well as to the system invertibility structure. In the case that the system is square and invertible, i.e., the system that was considered in [16, 17, 22], \( m = p = m_d \), the normal form simplifies to

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x) v_l, \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
\]  

(5.2)

where \( q_1 \leq q_2 \leq \ldots \leq q_m, \xi_i = \text{col}\{\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,q_i}\}, \) \( \xi = \text{col}\{\xi_1, \xi_2, \ldots, \xi_m\} \), and

\[
\delta_{i,j,l}(x) = 0, \quad \text{for} \quad j < q_l, \quad i = 1, 2, \ldots, m.
\]  

(5.3)

In this chapter, we would like to explore the application of the normal form (5.2)-(5.3) in solving the problem of semi-global stabilization for nonlinear systems (5.1). The normal form (5.2)-(5.3) does not require a vector relative degree. The problem of semi-global stabilization of system (5.1) with a vector relative degree has been well-studied in the literature. For example, the work of [18, 91] solved the semi-global stabilization problem for nonlinear systems with vector relative degrees, i.e., in the form of (3.7), but the globally asymptotically stable zero dynamics

\[
\dot{\eta} = f_0(\eta, \xi)
\]

is driven only by \( \xi_{i,1}, \quad i = 1, 2, \ldots, m, \) the states at the top of the \( m \) chains of integrators. The system with globally asymptotically stable zero dynamics is said to be of minimum phase. The works of [92, 93] generalized this result of [18, 91] by allowing \( f_0 \) to be dependent on any one state of each of the \( m \) chains of integrators. More specifically, the system considered in [92, 93] can be represented as follows,

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{i,\ell_i}, \xi_{i,\ell_i+1}, \ldots, \xi_{i,m}), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1}, \quad j = 1, 2, \ldots, r_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
\]  

(5.4)

where \( 1 \leq \ell_i \leq r_i + 1, \quad i = 1, 2, \ldots, m, \) and \( \xi_{i,q_i+1} \equiv v_i \). The peaking phenomenon, which was identified in [91] as a main obstacle to semi-global stabilization, in such systems is eliminated by stabilizing part of linear subsystem with a high-gain linear control and the remaining part of the linear subsystem with a small, bound nonlinear control [93]. The reference [92] shows that the
same problem can be solved by linear state feedback laws, as those in \cite{93},
depend only on the linear states. The fundamental issue in the design of such
linear state feedback laws is to induce a specific time-scale structure in the
linear part of the closed-loop system. This time-scale structure consists of a
very slow and a very fast time scale, which are the results of a linear state
feedback of the high-and-low-gain nature.

Note that in \cite{18, 91–93}, the system is considered to be minimum phase,
which means that its zero dynamic \( \dot{\eta} = f(\eta,0,\cdots,0) \) have a globally asymp-
totically stable equilibrium at the origin.

In this chapter, we consider the semi-global stabilization problem for the
following minimum phase nonlinear system,

\[
\begin{align*}
\dot{\eta} &= f_0(\eta,\xi_1,\xi_2,\cdots,\xi_m), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{q_i-1} \delta_{i,j,l}(\eta,\xi)v_l, \quad j = 1, 2, \cdots, q_i-1, \\
\dot{\xi}_{i,q_i} &= v_i, \\
y_{d,i} &= \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{align*}
\]

(5.5)

where \( q_1 \leq q_2 \leq \cdots \leq q_m, \xi = \text{col}\{\xi_1,\xi_2,\cdots,\xi_m\}, \)
\( \xi = \text{col}\{\xi_1,\xi_2,\cdots,\xi_m\}, \) and

\[
\delta_{i,j,l}(\eta,\xi) = 0, \quad \text{for} \quad j < q_l, \quad i = 1, 2, \cdots, m.
\]

(5.6)

\[
\ell_1 \leq q_1 + 1, \quad \ell_i \leq q_1, \quad i = 2, 3, \cdots, m,
\]

(5.7)

with \( \xi_{1,q_1+1} \equiv v_1. \) Note that the zero dynamics of (5.5) is given by

\[
\dot{\eta} = f_0(\eta,0,\cdots,0).
\]

As explained earlier, no vector relative degree is required for systems to
be decomposed into the above normal form.

Note that in \cite{92, 93}, \( \delta_{i,j,l} = 0. \) That is, the systems considered in \cite{92, 93}
are a cascade of a linear subsystem with the zero dynamic, which is the only
source of nonlinearity.

The remainder of this chapter is organized as follows. Section 5.2 presents
our solution to the semi-global stabilization problem for nonlinear systems
without vector relative degrees. Design examples are presented to illustrate
how the proposed design approach works. A brief conclusion to the chapter is
drawn in Section 5.3.

5.2 Main Results

Definition 5.2.1 The system (5.5) is semi-globally stabilizable by state feed-
back if, for any compact set of initial conditions \( \mathcal{X}_0 \) of the state space, there
exists a smooth state feedback
such that the equilibrium \((0,0)\) of the closed-loop system \((5.7)\) and \((5.8)\) is locally asymptotically stable and \(X_0\) is contained in its domain of attraction.

Theorem 4.3.1 can be modified to deal with the semi-global stabilization problem.

**Theorem 5.2.1** Consider a system in the form

\[
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}), \\
\dot{\xi}_{i,j} &= \delta_{i,j,l}(\eta, \xi_{\ell}, \xi_{i,0}), \quad j = 1, 2, \cdots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{aligned}
\]  

(5.9)

where \(q_1 \leq q_2 \leq \cdots \leq q_m\), \(\xi = \text{col}\{\xi_1, \xi_2, \cdots, \xi_m\}\), \(\xi_i = \text{col}\{\xi_{i,1}, \cdots, \xi_{i,q_i}\}\),

\[
\delta_{i,j,l} = 0, \quad \text{for} \quad j < q_l, \quad i = 1, 2, \cdots, m.
\]  

(5.10)

and all functions \(\delta_{i,j,l}\) and \(f_0\) are smooth. Then the system is semi-globally stabilizable if

1. for any compact set \(Z \subset \mathbb{R}^{n_0}\), there exist \(c, \phi_{i,1}(\eta), i = 1, 2, \cdots, m\), and a smooth, positive definite Lyapunov function \(W(\eta)\), such that

\[
Z \subset \{\eta : W(\eta) \leq c\},
\]

\[
\dot{W} = \frac{\partial W}{\partial \eta} f_0(\eta, \phi_{1,1}(\eta), \phi_{2,1}(\eta), \cdots, \phi_{m,1}(\eta)) < 0,
\]

\(\forall \eta \in \{\eta : W(\eta) \leq c\} \setminus \{0\};

2. there exists an ordered list \(\kappa\) containing all variables of \(\xi\) such that, for \(j = 1, 2, \cdots, q_i - 1, \ell = 1, 2, \cdots, i - 1, i = 1, 2, \cdots, m,\)

a) \(\xi_{i,j}\) appears earlier than \(\xi_{i,j+1}\) in \(\kappa\);

b) for \(\delta_{i,j,l} \neq 0\), the variables \(\xi_{\ell}\) appear earlier than \(\xi_{i,j}\) in \(\kappa\);

c) \(\delta_{i,j,l}\) depends only on \(\eta, \xi_{i,1}, \ell = 1, 2, \cdots, m,\) and the variables that appear no later than \(\xi_{i,j}\) in \(\kappa\).

In what follows, we will present an algorithm for constructing a family of feedback laws that semi-globally stabilize the system \((5.5)\). This algorithm consists of two steps.

We first find positive constants \(c_{i,k}\), such that the polynomials

\[
p_i(s) = s^{\ell_i - 1} + c_{i,\ell_i - 2} s^{\ell_i - 2} + \cdots + c_{i,1} s + c_{i,0}, \quad i = 1, 2, \cdots, m,
\]

have all roots with negative real parts. Define
\[ \xi_{i,\ell_i} = -\epsilon^{\ell_i - 1} c_{i,0} \xi_{i,1} - \epsilon^{\ell_i - 2} c_{i,1} \xi_{i,2} - \cdots - \epsilon c_{i,\ell_i - 1} \xi_{i,\ell_i - 1}, \]
\[ i = 1, 2, \cdots, m. \quad (5.11) \]

where \( \epsilon > 0 \). Consider
\[
\begin{cases}
\dot{\eta} = f_0(\eta, \xi_{1,\ell_1}, \xi_{2,\ell_2}, \cdots, \xi_{m,\ell_m}), \\
\dot{\xi}_{i,j} = \xi_{i,j+1}, & j = 1, 2, \cdots, \ell_i - 2, \\
\dot{\xi}_{i,\ell_i - 1} = \xi_{i,\ell_i}, & i = 1, 2, \cdots, m.
\end{cases} \quad (5.12)
\]

Denote \( z = \text{col} \{ \xi_{1,1}, \xi_{1,2}, \cdots, \xi_{1,\ell_1 - 1}, \cdots, \xi_{m,1}, \xi_{m,2}, \cdots, \xi_{m,\ell_m - 1} \} \).

Following \[92\, 93\], the dynamics of (5.12) with \( \xi_{i,\ell_i} \) given by (5.11) has a locally asymptotically stable equilibrium at the origin of \((\eta, z)\). Moreover, the domain of attraction of this equilibrium can be made arbitrarily large by decreasing the value of the low gain parameter \( \epsilon \).

**Lemma 5.1.** Consider the system (5.12) with \( \xi_{i,\ell_i} \) given by (5.11). Suppose that its zero dynamics \( \dot{\eta} = f_0(\eta, 0, \cdots, 0) \) has a globally asymptotically stable equilibrium at the origin. Then for any compact set \( Y \), there exist \( \epsilon, c \) and a smooth, positive definite, Lyapunov function \( W(\eta, z) \), such that
\[ Y \subset \{ (\eta, z) : W(\eta, z) \leq c \}, \]
and
\[ \dot{W} < 0, \quad \forall (\eta, z) \in \{ (\eta, z) : W(\eta, z) \leq c \} \setminus \{0\}. \]

Once the virtual inputs \( \xi_{i,\ell_i}, i = 1, 2, \cdots, m \), have been obtained, both \[93\] and \[92\] design the overall controller by using linear high-gain state feedback. This is possible because the systems considered there are linear except the zero dynamics. In our situation, the system is in the form of (5.5). Because of the nonlinearities \( \delta_{i,j,\ell}(\eta, \xi) \), we have to resort to backstepping procedure as described in Theorem 5.2.1, where a special case of (5.5), i.e., \( \ell_i = 1, i = 1, 2, \cdots, m \), is considered.

Consider the dynamics of (5.12) with \( \xi_{i,\ell_i} \) given by (5.11), as the zero dynamics of the system (5.9), by Theorem 5.2.1 and Lemma 5.1, we have

**Theorem 5.2.2** Consider the system (5.9) with (5.6) and (5.7). Assume that all functions \( \delta_{i,j,\ell} \) and \( f_0 \) are smooth. If

1. the zero dynamics \( \dot{\eta} = f_0(\eta, 0, 0, \cdots, 0) \) is globally asymptotically stable at the equilibrium \( \eta = 0 \);
2. there exists an ordered list \( \kappa \) containing all variables of \( \xi \) with \( \xi_{s,p}, p = 1, 2, \cdots, \ell_s - 1, s = 1, 2, \cdots, m \), being its first \( \sum_{s=1}^{m} (\ell_s - 1) \) variables, such that, for \( j = 1, 2, \cdots, q_i - 1, l = 1, 2, \cdots, i - 1, i = 1, 2, \cdots, m \),
   a) \( \xi_{i,j} \) appears earlier than \( \xi_{i,j+1} \) in \( \kappa \);
   b) the variables \( \xi_{i,j} \) appear earlier than \( \xi_{i,j} \) in \( \kappa \) if \( \delta_{i,j,\ell} \neq 0 \);
   c) \( \delta_{i,j,\ell} \) depends only on \( \eta, \xi_{\ell,1}, \ell = 1, 2, \cdots, m \), and the variables that appear no later than \( \xi_{i,j} \) in \( \kappa \).
Then the system is semi-globally stabilizable. That is, for any compact set $X_0$ of the state space of $(\eta, \xi)$, there exists a state feedback $v$ that locally asymptotically stabilizes the system with $X_0$ contained in the domain of attraction.

Example 5.2. Consider a three inputs three outputs system in the form of (5.5) with three chains of integrators of lengths \{3, 4, 4\}, and $\ell_1 = 2$, $\ell_2 = 1$, $\ell_3 = 3$.

Suppose its zero dynamics $\dot{\eta} = f_0(\eta, \xi_1, \xi_2, \xi_3)$ with

$$
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_1, \xi_2, \xi_3), \\
\dot{\xi}_1 &= \xi_1 + 1, \\
\dot{\xi}_2 &= \xi_2 + 1, \\
\dot{\xi}_3 &= \xi_3 + 1, \\
\xi_1 &= \xi_1 - 1, \\
\xi_2 &= \xi_2 - 1, \\
\xi_3 &= \xi_3 - 1.
\end{align*}
$$

Consider the subsystem

$$
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_1^*, \xi_2, \xi_3), \\
\dot{\xi}_1 &= \xi_1^* + 1, \\
\dot{\xi}_2 &= \xi_2 + 1, \\
\dot{\xi}_3 &= \xi_3 + 1
\end{align*}
$$

with $\xi_1^* = -\varepsilon \xi_{1.1}$, $\xi_2^* = 0$, $\xi_3^* = -\varepsilon^2 \xi_{3.1} - \varepsilon \xi_{3.2}$. It is semi-globally asymptotically stable.

In what follows, we will illustrate how to implement the level-by-level backstepping to find $v_1$, $v_2$ and $v_3$. To carry out the backstepping on the first level variable $\xi_1$ to $\xi_2$, we consider the following subsystem,

$$
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_1^*, \xi_2, \xi_3), \\
\dot{\xi}_1 &= \xi_1^* + 1, \\
\dot{\xi}_2 &= \xi_2 + 1, \\
\dot{\xi}_3 &= \xi_3 + 1
\end{align*}
$$

with $\xi_2$ as the virtual input. By backstepping, the desired input is given as

$$
\xi_2^* = \phi_2(\eta, \xi_1, \xi_2, \xi_3).
$$

Now consider backstepping from the second level variables. To backstep from $\xi_{1.2}$ to $\xi_{1.3}$, we consider
5.2 Main Results

\[\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,2}, \xi_{2,1}, \xi_{3,1}), \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= \xi_{1,3}, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi_{3,3},
\end{align*}\]

with \(\xi_{1,3}\) as the virtual input. The desired input is given as

\[\xi^*_{1,3} = \phi_{1,3}(\eta, \xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{3,1}, \xi_{3,2}).\]

To backstep from \(\xi_{2,2}\) to \(\xi_{2,3}\), we view \(\xi_{2,3}\) as the virtual input of

\[\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,2}, \xi_{2,1}, \xi^*_{3,3}), \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= \xi_{1,3}, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= \xi_{2,3}, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi_{3,3},
\end{align*}\]

And the desired input is given as

\[\xi^*_{2,3} = \phi_{2,3}(\eta, \xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}).\]

Next consider backstepping from the third level variables. To backstep from \(\xi_{1,3}\) to \(v_1\) in the subsystem,

\[\begin{align*}
\dot{\eta} &= f_0(\eta, \xi_{1,2}, \xi_{2,1}, \xi_{3,1}), \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= \xi_{1,3}, \\
\dot{\xi}_{1,3} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= \xi_{2,3}, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi^*_{3,3},
\end{align*}\]

we get

\[v_1 = v_1(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}).\]

To backstep from \(\xi_{2,3}\) to \(\xi_{2,4}\), we consider
Finally, backstepping from the fourth level variables, we obtain
\[ \xi^* = \phi_2(\eta, \xi_1, \xi_2, \xi_3, \xi_4). \]
Similarly, to backstep from \( \xi_{3,3} \) to \( \xi_{3,4} \), we obtain
\[ \xi^*_{3,4} = \phi_3(\eta, \xi_1, \xi_2, \xi_3, \xi_4). \]
Finally, backstepping from the fourth level variables, we obtain
\[ v_2 = v_2(\eta, \xi_1, \xi_2, \xi_3), \]
\[ v_3 = v_3(\eta, \xi_1, \xi_2). \]
The inputs \( v_1, v_2 \) and \( v_3 \) semi-globally asymptotically stabilize the origin of the system \((5.13)\).

In what follows, we give an example which requires the mixed chain-by-chain and level-by-level backstepping design procedure.

**Example 5.3.** Consider a system in the form of \((5.5)\) with three chains of integrators of lengths \{2, 4, 4\},
\[
\dot{\eta} = f_0(\eta, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}),
\]
\[
\begin{align*}
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1})v_1, \\
\dot{\xi}_{2,3} &= \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,2}, \xi_{3,1}, \xi_{3,2})v_1, \\
\dot{\xi}_{2,4} &= v_2, \\
\dot{\xi}_{3,1} &= \xi_{3,2}, \\
\dot{\xi}_{3,2} &= \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2})v_1, \\
\dot{\xi}_{3,3} &= \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_{1,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}, \xi_{3,3})v_1, \\
\dot{\xi}_{3,4} &= v_3,
\end{align*}
\]
where \( \ell_1 = \ell_2 = \ell_3 = 2 \). Suppose its zero dynamics \( \dot{\eta} = f_0(\eta, 0, 0, 0) \) is globally asymptotically stable at the origin. It is obvious that the system satisfies the conditions in Theorem \((5.2, 2)\) with
We first find the low-gain feedback, 
\[ \xi^*_{1,2} = -\varepsilon \xi_{1,1}, \quad \xi^*_{2,2} = -\varepsilon \xi_{2,1}, \quad \xi^*_{3,2} = -\varepsilon \xi_{3,1}. \]

Then we carry out a mixed chain-by-chain and level-by-level backstepping in the order of \( \xi_{1,2}, \xi_{2,2}, \xi_{3,2}, \xi_{2,3}, \xi_{2,4}, \xi_{3,3}, \xi_{3,4} \) to obtain 
\[
\begin{align*}
v_1 &= v_1(\eta, \xi_1, \xi_{2,1}, \xi_{3,1}), \\
v_2 &= v_2(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}), \\
v_3 &= v_3(\eta, \xi_1, \xi_2, \xi_3).
\end{align*}
\]

**Example 5.4.** Consider
\[
\begin{align*}
\dot{\eta} &= -\eta + (v_1 + \xi_{2,2}) \sin \eta, \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \eta v_1, \\
\dot{\xi}_{2,3} &= v_2.
\end{align*}
\]

Obviously the system satisfies the conditions in Theorem 5.2.2 with
\[ q_1 = 2, \quad q_2 = 3, \quad \ell_1 = 3, \quad \ell_2 = 2. \]

Choosing all poles of the linear slow subsystems to be \(-\varepsilon\), we obtain
\[
\begin{align*}
v_1 &= -\varepsilon^2 \xi_{1,1} - 2\varepsilon \xi_{1,2}, \\
\xi^*_{2,2} &= -\varepsilon \xi_{2,1}.
\end{align*}
\]

Next, we view \( \xi_{2,3} \) as a virtual input. By backstepping, the desired input is given as follows,
\[ \xi^*_{2,3} = -(\varepsilon + 1) \xi_{2,1} - (\varepsilon + 1) \xi_{2,2} - \eta (\sin \eta - \varepsilon^2 \xi_{1,1} - 2\varepsilon \xi_{1,2}). \]

Finally, backstepping one more time, we get
\[
\begin{align*}
v_2 &= -(2\varepsilon + 1) \xi_{2,1} - (2\varepsilon + 3) \xi_{2,2} - (\varepsilon + 2) \xi_{2,3} - \eta (\sin \eta + 2v_1 - \varepsilon v_1 - \varepsilon^2 \xi_{1,2}) \\
&\quad - [\eta + (v_1 + \xi_{2,2}) \sin \eta] (\sin \eta + \eta \cos \eta + v_1).
\end{align*}
\]

The Lyapunov function is given as
\[
V = \left\{ \eta^2 + \varepsilon^2 \xi_{1,1}^2 + (\varepsilon \xi_{1,1} + \xi_{1,2})^2 + \xi_{2,1}^2 + (\xi_{2,2} + \varepsilon \xi_{2,1})^2 \\
+ [\xi_{2,3} + (\varepsilon + 1) \xi_{2,1} + (\varepsilon + 1) \xi_{2,2} + \eta (\sin \eta - \varepsilon^2 \xi_{1,1} - 2\varepsilon \xi_{1,2})]^2 \right\}/2,
\]
and

\[
\dot{V} = \eta \left[ -\eta + ( -\varepsilon^2 \xi_{1,1} - 2\varepsilon \xi_{1,2} - \varepsilon \xi_{2,1}) \sin \eta \right] - \varepsilon^3 \xi_{1,1}^2 \\
- \varepsilon^2 \xi_{1,1} \xi_{1,2} - \varepsilon \xi_{1,2}^2 - \varepsilon \xi_{2,1}^2 - (\xi_{2,2} + \varepsilon \xi_{2,1})^2 \\
- [\xi_{2,3} + (\varepsilon + 1)\xi_{2,1} + (\varepsilon + 1)\xi_{2,2} + \eta (\sin \eta - \varepsilon^2 \xi_{1,1} - 2\varepsilon \xi_{1,2})]^2.
\]

Shown in Figs. 5.1 and 5.2 are state trajectories of the closed-loop system with different initial conditions.

### 5.3 Summary of the Chapter

In this chapter, we showed how the structural decomposition in Chapter 3 can be used to solve the semi-global stabilization of a class of multiple input multiple output systems without vector relative degrees. The design procedure involved several existing design techniques in nonlinear stabilization, including low gain feedback and different forms of backstepping design procedures in Chapter 4.
5.3 Summary of the Chapter

Fig. 5.1. State trajectories with the initial condition $(5, 0.5, 0.5, 0.5, 5, 5)$ and $\varepsilon = 0.5$.

Fig. 5.2. State trajectories with the initial condition $(-20, -2, -2, -2, -20, -20)$ and $\varepsilon = 0.15$. 
Disturbance Attenuation for Nonlinear Systems

The problems of disturbance attenuation and almost disturbance decoupling play a central role in control theory. In this chapter, by employing the structural decomposition of multiple input multiple output nonlinear systems in Chapter 3 and the backstepping procedures in Chapter 4, we show that these two problems can be solved for a larger class of nonlinear systems.

6.1 Introduction and Problem Statement

Consider the problems of disturbance attenuation and almost disturbance decoupling with internal stability for nonlinear systems affine in control,

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u + p(x, w), \\
y &= h(x),
\end{align*}
\]

(6.1)

where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\), \(y \in \mathbb{R}^m\) and \(w \in \mathbb{R}\) are the state, input, output and disturbance, respectively, and the mappings \(f\), \(g\), \(p\) and \(h\) are smooth with \(f(0) = 0\) and \(h(0) = 0\). The problem of almost disturbance decoupling was originally formulated and solved in [94] for linear systems and was later extended to single input single output (SISO) minimum phase nonlinear systems in [95–97]. It was further extended to SISO non-minimum phase systems in [98, 99].

The problem of almost disturbance decoupling is, for any \textit{a priori} given arbitrarily small scalar \(\gamma > 0\), to find a feedback law such that the \(L_2\) gain from the disturbance to the output is less than or equal to \(\gamma\). A practical solution to the almost disturbance decoupling problem would require the resulting closed-loop system to be globally or locally asymptotically stable as well. Here in this chapter, we will focus on the requirement of global asymptotic stability. The problem of disturbance attenuation is a less stringent one in that it does not require the bound on the resulting \(L_2\) to be arbitrarily small. The problem of almost disturbance decoupling is a special case of disturbance attenuation.
The problem of disturbance attenuation (or almost disturbance decoupling) with stability can be solved by establishing the dissipativity of the system \[17\]. That is, the problem of disturbance attenuation with stability (or almost disturbance decoupling) for a given system is, for a given (arbitrarily small) scalar \( \gamma > 0 \), to find a feedback law \( u = u(x) \) such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \( q(w,y) = \gamma^2 w^2 - y^2 \), which is equivalent to finding a feedback law \( u = u(x) \) such that, for some smooth, positive definite and proper function \( V(x) \), the dissipation inequality
\[
\frac{\partial V}{\partial x} (f(x)+g(x)u(x)+p(x,w)) \leq -\alpha(\|x\|) + \gamma^2 w^2 - h^2(x),
\]
holds for some class \( K_\infty \) function \( \alpha \).

The inequality (6.2) guarantees that the response of the closed-loop system in the absence of disturbance is globally asymptotically stable and, with \( x(0) = 0 \),
\[
\int_0^\infty y^2(t)dt \leq \gamma^2 \int_0^\infty w^2(t)dt,
\]
for every \( L_2 \) disturbance \( w \).

The solution to the problem of disturbance attenuation and almost disturbance decoupling usually resorts to transforming the nonlinear systems into certain structural normal forms. For example, in [95–97], the problem of almost disturbance decoupling problem with stability was solved for systems in the following normal form
\[
\begin{align*}
\dot{z} &= f_0(z,\xi_1) + p_0(z,\xi_1)w, \\
\dot{\xi}_i &= \xi_{i+1} + p_i(z,\xi_1,\xi_2,\cdots,\xi_i)w, \quad i = 1, 2, \cdots, r - 1, \\
\dot{\xi}_r &= u + p_r(z,\xi_1,\xi_2,\cdots,\xi_r)w, \\
y &= \xi_1,
\end{align*}
\]
(6.3)
where \( f_0(0,0) = 0 \). A critical assumption made there is that the system is of minimum phase, that is, the zero dynamics \( \dot{z} = f_0(z,0) \) is globally asymptotically stable.

The work [98] relaxes the minimum phase assumption by allowing part of the zero dynamics to be unstable as long as it is unaffected by the disturbance and is stabilizable though the output of the system. That is, it is assumed that the dynamic of \( z \) in (6.3) takes the following form,
\[
\begin{align*}
\dot{z}_1 &= f_1(z_1, z_2, \xi_1) + p_0(z_1, z_2, \xi_1)w, \\
\dot{z}_2 &= f_2(z_2, \xi_1),
\end{align*}
\]
(6.4)
where the \( z_1 \) subsystem is globally asymptotically stable at \( z_1 = 0 \) and there exists some smooth \( v_2(z) \) such that \( \dot{z} = f_2(z_2, v(z_2)) \) is globally asymptotically stable at \( z_2 = 0 \). In a further note [99], it was pointed out that, under
some further structural assumption on the $z_2$ subsystem, the problem of almost disturbance decoupling with stability can still be solved even if the $z_2$ subsystem is affected by the disturbance.

In an effort to solve the problem of disturbance attenuation for multiple input multiple output nonlinear systems, a normal form for square invertible systems was developed in [16, 17, 22].

In Chapter 3, we studied the structural properties of affine-in-control nonlinear systems beyond the case of square invertible systems. We proposed an algorithm that identifies a set of integers that are equivalent to the infinite zero structure of linear systems and leads to a normal form representation that corresponds to these integers as well as to the system invertibility structure.

This new normal form facilitates the control design. As shown in Chapter 4, it allows the development of some new backstepping design procedures, the level-by-level backstepping and the mixed chain-by-chain and level-by-level backstepping. These new backstepping procedures lead to the stabilization of a larger class of systems that the conventional chain-by-chain backstepping design procedure cannot stabilize. The objective of this chapter is to show that the backstepping design procedures of Chapter 4 can also be utilized to solve the problems of disturbance attention and almost disturbance decoupling for a larger class of multiple input multiple output systems.

The remainder of this chapter is organized as follows. In Section 6.2, we recall some results on the problems of disturbance attenuation and almost disturbance decoupling for SISO systems. We will also describe the level-by-level and the mixed chain-by-chain and level-by-level backstepping design procedures. Section 6.3 presents our solutions to the problems of disturbance attenuation and almost disturbance decoupling. A brief conclusion to the chapter is drawn in Section 6.4.

### 6.2 Preliminary Results

We first recall the follow result on disturbance attenuation with stability from [17]. This result will serve as a building block in our design procedures.

**Lemma 6.1.** Consider a system described by

\[
\begin{align*}
\dot{z} &= f_0(z, \xi) + p_0(z,w), \\
\dot{\xi} &= u + f_1(z,\xi) + p_1(z,\xi,w), \\
y &= h(z,\xi),
\end{align*}
\]

(6.5)

where $(z,\xi) \in \mathbb{R}^n \times \mathbb{R}$, $f_0(0,0) = 0$ and $f_1(0,0) = 0$. Assume that

\[
\begin{align*}
\|p_0(z,w)\| &\leq R_0(|z| |w|), \ \forall z,w, \\
|p_1(z,\xi,w)| &\leq R_1(|z,\xi| |w|), \ \forall z,\xi,w,
\end{align*}
\]

for some smooth real-valued functions $R_0(z)$ and $R_1(z,\xi)$. Suppose that there exist a number $\gamma > 0$, a smooth real-valued function $v(z)$ with $v(0) = 0$, a
smooth positive definite and radially unbounded function \( V(z) \), and a class \( K_\infty \) function \( \alpha_0(\cdot) \) such that
\[
\frac{\partial V}{\partial z}[f_0(z, v(z)) + p_0(z, w)] \leq -\alpha_0(||z||) + \gamma^2 w^2 - h^2(z, v(z)), \quad \forall z, \xi, w, \tag{6.6}
\]
that is, there exists a smooth \( v(z) \) such that the subsystem
\[
\begin{cases}
\dot{z} = f_0(z, v(z)) + p_0(z, w), \\
y = h(z, v(z)),
\end{cases}
\]
is strictly dissipative with respect to the supply rate \( q(w, y) = \gamma^2 w^2 - y^2 \).

Then, for every \( \epsilon > 0 \), there exist a smooth feedback law \( u = u(z, \xi) \), a smooth positive definite and radially unbounded function \( W(z, \xi) \), and a class \( K_\infty \) function \( \alpha(\cdot) \) such that
\[
\frac{\partial W}{\partial z}(f_0(z, \xi) + p_0(z, w)) + \frac{\partial W}{\partial \xi}(u(z, \xi) + f_1(z, \xi) + p_1(z, \xi, w)) \\
\leq -\alpha(||\text{col}\{z, \xi\}||) + (\gamma + \epsilon)^2 w^2 - h^2(z, \xi), \quad \forall z, \xi, w,
\]
or equivalently, there exist a smooth feedback law \( u = u(z, \xi) \) such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \( q(w, y) = (\gamma + \epsilon)^2 w^2 - y^2 \).

In [17], the possibility of fulfilling the main condition (6.6) in Lemma 6.1 is discussed. In the context of the almost disturbance decoupling problem, suppose that the \( z \)-subsystem
\[
\dot{z} = f(z, \xi_1, w) \tag{6.7}
\]
can be decomposed as
\[
\begin{cases}
\dot{z}_1 = f_1(z_1, z_2, \xi_1, w), \\
z_2 = f_2(z_2, \xi_1),
\end{cases} \tag{6.8}
\]
where \( z_1 \) represents “stable component” and \( z_2 \) represents “unstable but stabilizable component.”

**Lemma 6.2.** Consider system (6.7) which can be decomposed as (6.8). Suppose that

1. there exists a smooth positive definite and radially unbounded function \( V_1(z_1) \) such that
\[
\frac{\partial V_1}{\partial z_1}f_1(z_1, z_2, \xi_1, w) \leq -\alpha_1(||z_1||) + \gamma^2_0 w^2 + \gamma^2_0 ||z_2||^2 + \gamma^2_0 \xi_1^2,
\]
for some \( K_\infty \) function \( \alpha_1 \) and some \( \gamma_0 > 0 \),
2. there exist a smooth real-valued function \( v_2(z_2) \) with \( v_2(0) = 0 \), and a smooth positive definite and radially unbounded function \( V_2(z_2) \) such that
\[
\frac{\partial V_2}{\partial z_2} f_2(z_2, v_2(z_2)) + v_2^2(z_2) \leq -\alpha_2(\|z_2\|),
\]
for some \( K_\infty \) function \( \alpha_2 \).

Then, for every \( \gamma > 0 \), there exist a smooth \( v(z) \) with \( v(0) = 0 \), and a smooth positive definite and radially unbounded function \( V(z) \) such that
\[
\frac{\partial V}{\partial z} f(z, v(z), w) \leq -\alpha(\|z\|) + \gamma^2 w^2 - v^2(z),
\]
for some \( K_\infty \) function \( \alpha(\cdot) \).

### 6.3 Disturbance Attenuation and Almost Disturbance Decoupling with Stability

Suppose that, by the algorithm in [30], system (6.1) is transferred into the following form,

\[
\begin{align*}
\dot{z} &= f_0(z, \xi) + p_0(z, w), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, \xi)v_l + p_{i,j}(z, \xi, w), \quad j = 1, 2, \ldots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i + p_{i,q_i}(z, \xi)w, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

(6.9)

where \( \xi = \text{col} \{ \xi_1, \xi_2, \ldots, \xi_m \} \), \( \xi_i = \text{col} \{ \xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,q_i} \} \), \( q_1 \leq q_2 \leq \cdots \leq q_m \), and functions \( f_0, p_0 \) and \( p_{i,j}, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, m \) are smooth with \( f_0(0, 0, \ldots, 0) = 0 \). Moreover,

\[ \delta_{i,j,l} = 0, \quad \text{for} \quad j < q_i, \quad i = 1, 2, \ldots, m. \]

As in the literature on the problems of disturbance attention and almost disturbance decoupling for SISO systems, we assume that the zero dynamics is driven only by the states on the top of the \( m \) chains of integrators, \( \xi_{i,1}, \quad i = 1, 2, \ldots, m \). That is \( \dot{z} = f_0(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}) + p_0(z, w) \).

To apply the level-by-level backstepping, we also assume that the coefficients \( \delta_{i,j,l}, p_{i,j}(z, \xi, w), j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, m \), satisfy the level-by-level triangular dependency on state variables. We have following result on the problem disturbance attenuation with stability.

**Theorem 6.3.1** Consider a system given by
Then, for every $\epsilon > 0$ Disturbance Attenuation for Nonlinear Systems

\begin{align*}
\dot{z} &= \dot{f_0}(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}) + p_0(z, w), \\
\dot{\xi}_{i,j} &= \dot{\xi}_{i,j+1} + \sum_{l=1}^{i-1} \dot{\delta}_{i,j,l}(z, \xi)v_l + p_{i,j}(z, \xi, w), \quad j = 1, \cdots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i + p_{i,q_i}(z, \xi)w, \\
y_i &= \xi_{i,1}, \quad i = 1, \cdots, m,
\end{align*}

(6.10)

where $\xi = \text{col} \{\xi_1, \xi_2, \cdots, \xi_m\}$, $\xi_i = \text{col} \{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\}$, $q_1 \leq q_2 \leq \cdots \leq q_m$, and functions $f_0, p_0, \dot{\delta}_{i,j,l}$ and $p_{i,j}$, $j = 1, 2, \cdots, q_i$, $i = 1, 2, \cdots, m$, are smooth with $f_0(0, \cdots, 0) = 0$. Assume that

1) there exist a number $\gamma > 0$, smooth function $\phi_{i,1}(z)$, with $\phi_{i,1}(0) = 0$, $i = 1, 2, \cdots, m$, a smooth positive definite and radially unbounded function $V(z)$, and a class $K_\infty$ function $\alpha_0(\cdot)$ such that

\begin{align*}
\frac{\partial V}{\partial z}[f_0(z, \phi_{1,1}(z), \phi_{2,1}(z), \cdots, \phi_{m,1}(z)) + p_0(z, w)] \\
&\leq -\alpha_0(\|z\|) + \gamma^2 \|w\|^2 - \|\text{col} \{\phi_{1,1}(z), \phi_{2,1}(z), \cdots, \phi_{m,1}(z)\}\|^2,
\end{align*}

for all $z$ and $w$.

2) the functions $\dot{\delta}_{i,j,l}(z, \xi)$ and $p_{i,j}(z, \xi, \cdot)$ depend only on variables $z$ and $\xi_{\ell_p, \ell_b}$, with

a) $1 \leq \ell_p \leq m$ and $\ell_b = 1$; or,

b) $\ell_b \leq j - 1$; or

c) $\ell_b = j$ and $\ell_p \leq i$.

Then, for every $\epsilon > 0$, there exist smooth feedback laws $v_i = v_i(z, \xi)$, $i = 1, 2, \cdots, m$, such that the resulting closed-loop system is strictly dissipative with respect to the supply rate $q(w, y) = (\gamma + \epsilon)^2 \|w\|^2 - \|y\|^2$, where $y = \text{col} \{y_1, y_2, \cdots, y_m\}$.

**Proof.** The theorem can be proven by using the level-by-level backstepping design procedure [30]. In each step of the procedure, we use Lemma 6.1. Let $n_d = \sum_{i=1}^{m} q_i$.

We start the backstepping with

\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \phi_{2,1}(z), \phi_{3,1}(z), \cdots, \phi_{m,1}(z)) + p_0(z, w), \\
\dot{\xi}_{1,1} &= \xi_{1,2} + p_{1,1}(z, \xi_{1,1}, \phi_{2,1}(z), \phi_{3,1}(z), \cdots, \phi_{m,1}(z), w), \\
y_1 &= \xi_{1,1}, \\
y_i &= \phi_{i,1}(z), \quad i = 2, 3, \cdots, m.
\end{align*}

(6.11)
Here, $\xi_{1,2}$ is viewed as a virtual input. By Lemma 6.1 for every $\epsilon > 0$, there exist a smooth feedback law $\xi_{1,2} = \phi_{1,2}(z, \xi_{1,1})$, a smooth positive definite and radially unbounded function $W_{1,1}(z, \xi_{1,1})$, and a class $\mathcal{K}_\infty$ function $\alpha_{1,1}(\cdot)$ such that

$$
\frac{\partial W_{1,1}}{\partial z}[f_0(z, \xi_{1,1}, \phi_{2,1}(z), \cdots, \phi_{m,1}(z)) + p_0(z, w)]
+ \frac{\partial W_{1,1}}{\partial \xi_{1,1}}[\phi_{1,2}(z, \xi_{1,1}) + p_{1,1}(z, \xi_{1,1}, w)]
\leq -\alpha_{1,1}(\|\text{col} \{z, \xi_{1,1}\}\|) + (\gamma + \epsilon/n_d)^2\|w\|^2
- \|\text{col} \{\xi_{1,1}, \phi_{2,1}(z), \cdots, \phi_{m,1}(z)\}\|^2,
$$

(6.12)

for all $z, \xi_{1,1}$ and $w$. That is, subsystem (6.11) with the feedback $\xi_{1,2} = \phi_{1,2}(z, \xi_{1,1})$ is strictly dissipative with respect to the supply rate $q(w, y) = (\gamma + \epsilon/n_d)^2\|w\|^2 - \|y\|^2$.

Next, consider the subsystem

$$
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \phi_{3,1}(z), \cdots, \phi_{m,1}(z)) + p_0(z, w), \\
\dot{\xi}_{1,1} &= \phi_{1,2}(z, \xi_{1,1}) + p_{1,1}(z, \xi_{1,1}, \xi_{2,1}, \phi_{3,1}(z), \cdots, \phi_{m,1}(z), w), \\
\dot{\xi}_{2,1} &= \phi_{2,2} + p_{2,1}(z, \xi_{1,1}, \xi_{2,1}, \phi_{3,1}(z), \cdots, \phi_{m,1}(z), w), \\
y_1 &= \xi_{1,1}, \\
y_2 &= \xi_{2,1}, \\
y_i &= \phi_{i,1}(z), \quad i = 3, \cdots, m,
\end{align*}
$$

(6.13)

where $\xi_{2,2}$ is viewed as a virtual input. By Lemma 6.1 and in view of (6.12), there exist a smooth feedback law $\xi_{2,2} = \phi_{2,2}(z, \xi_{1,1}, \xi_{2,1})$ such that the resulting closed-loop system is strictly dissipative with respect to the supply rate $q(w, y) = (\gamma + 2\epsilon/n_d)^2\|w\|^2 - \|y\|^2$.

Similarly, we step back from the remain states in the first-level, and obtain

$$
\begin{align*}
v_i &= v_i(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}), \quad i = 1, 2, \cdots, b_1,
\end{align*}
$$

where $b_1$ is the number of chains that contain exactly one integrator, i.e., $q_1 = q_2 = \cdots = q_{b_1} = 1$.

For chains that contain more than one integrator, we have

$$
\begin{align*}
\xi_{i,2} &= \phi_{i,2}(z; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}), \\
i &= b_1 + 1, b_1 + 2, \cdots, m.
\end{align*}
$$

Thus, the following subsystem

$$
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}) + p_0(z, w), \\
\dot{\xi}_{i,1} &= v_i + p_{i,1}(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, w), \quad i = 1, 2, \cdots, b_1, \\
\dot{\xi}_{i,1} &= \phi_{i,2}(z; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}) + p_{i,1}(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, w), \quad i = b_1 + 1, b_1 + 2, \cdots, m,
\end{align*}
$$

(6.14)

is strictly dissipative with respect to the supply rate $q(w, y) = (\gamma + m\epsilon/n_d)^2\|w\|^2 - \|y\|^2$. 
To proceed backstepping on the first state in the second-level, we view \( \xi_{b_1+1,2} \) as a virtual input of the following subsystem, which consists of (6.14) and the dynamics

\[
\dot{\xi}_{b_1+1,2} = \xi_{b_1+1,3} + p_{b_1+1,2}(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, \xi_{b_1+1,2}, w).
\]

Again, by Lemma 6.1 there exists a smooth feedback law

\[
\xi_{b_1+1,3} = \phi_{b_1+1,3}(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, \xi_{b_1+1,2}),
\]

such that the resulting closed-loop subsystem is strictly dissipative with respect to the supply rate \( q(w, y) = [\gamma + (m+1)\epsilon/n_a]w^2 - \|y\|^2 \).

Continuing in this way, we finally obtain

\[
v_i = v_i(z; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2}, \cdots, \xi_{m,2}; \cdots; \xi_{1,q_m-1}, \xi_{2,q_m-1}, \cdots, \xi_{m,q_m-1}; \xi_{i,q_m}),
\]

for chains that contain \( q_m \) integrators, such that such that the closed-loop system is strictly dissipative with respect to the supply rate \( q(w, y) = (\gamma + \epsilon)^2\|w\|^2 - \|y\|^2 \). \( \square \)

The level-by-level backstepping procedure enlarges the class of systems for which the disturbance attainment problem can be solved. The triangular dependency requirement in Theorem 6.3.1 can be further weakened if we mix the chain-by-chain backstepping and the level-by-level backstepping and implement it on a same system. The following result includes the chain-by-chain backstepping and level-by-level as special cases.

**Theorem 6.3.2** Consider a system in the form

\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}) + p_0(z, w), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, \xi)v_l + p_{i,j}(z, \xi, w), \quad j = 1, 2, \cdots, q_i - 1, \\
\dot{\xi}_{i,q_i} &= v_i + p_{i,q_i}(z, \xi, w), \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{align*}
\]

(6.15)

where \( \xi = \text{col}\{\xi_1, \xi_2, \cdots, \xi_m\} \), \( \xi_i = \text{col}\{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\} \), \( q_1 \leq q_2 \leq \cdots \leq q_m \), and functions \( f_0, p_0, \delta_{i,j,l} \) and \( p_{i,j} \), \( i = 1, 2, \cdots, m \) are smooth. Assume that

\[
\|p_0(z, w)\| \leq R_0(z)\|w\|, \quad \forall z, w,
\]

\[
|p_{i,j}(z, \xi, w)| \leq R_{i,j}(z, \xi)\|w\|, \quad \forall z, \xi, w, \quad j = 1, 2, \cdots, q_i, \quad i = 1, 2, \cdots, m,
\]

for some smooth functions \( R_0(z) \) and \( R_{i,j}(z, \xi) \), \( j = 1, 2, \cdots, q_i, \quad i = 1, 2, \cdots, m \). Suppose that

1) Condition 1) in Theorem 6.3.1 holds,
6.3 Disturbance Attenuation and Almost Disturbance Decoupling with Stability

2) there exists an ordered list $\kappa$ containing all variables of $\xi$ such that, for $j = 1, 2, \ldots, q_1 - 1, l = 1, 2, \ldots, i - 1, i = 1, 2, \ldots, m$,
   a) $\xi_{i,j}$ appears earlier than $\xi_{i,j+1}$ in $\kappa$;
   b) for $\delta_{i,j,l} \neq 0$, the variables $\xi_l$ appear earlier than $\xi_{i,j}$ in $\kappa$;
   c) $\delta_{i,j,l}(z, \xi)$ and $p_{i,j}(z, \xi, \cdot)$ depend only on $z$, $\xi_{\ell,1}$, $\ell = 1, 2, \ldots, m$, and the variables that appear no later than $\xi_{i,j}$ in $\kappa$.

Then, for every $\epsilon > 0$, there exist smooth feedback laws $v_i = v_i(z, \xi)$, $i = 1, 2, \ldots, m$, such that the resulting closed-loop system is strictly dissipative with respect to the supply rate $q(w, y) = (\gamma + e)^2\|w\|_2^2 - \|y\|_2^2$, where $y = \text{col}\{y_1, y_2, \ldots, y_m\}$.

**Proof:** The backstepping can be carried out one state by one state in the order of the list $\kappa$. Suppose that, after backstepping $\ell$ state variables, we want to backstep from $\xi_{i,j}$, the $\ell + 1$-th element in the list $\kappa$, to $\xi_{i,j+1}$. Let $n_d = \sum_{i=1}^m q_i$. Denote all the state variables that come before $\xi_{i,j}$ in the list $\kappa$ as $Z$. By Condition 2), we can describe the subsystem of $Z$ and $\xi_{i,j}$ as

$$
\begin{align*}
\dot{Z} &= F_0(Z, \xi_{i,j}) + P_0(Z, w), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(Z, \xi_{i,j})v_l(Z) + P_{i,j}(Z, \xi_{i,j}, w), \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
$$

(6.16)

where $\xi_{i,j+1}$ is viewed as a virtual input. By Lemma 6.1 there exists a smooth function

$$\xi_{i,j+1} = \phi_{i,j+1}(Z, \xi_{i,j})$$

such that the resulting closed-loop subsystem is strictly dissipative with respect to the supply rate $q(w, y) = [\gamma + (\ell + 1)e/n_d]^2\|w\|_2^2 - \|y\|_2^2$. By backstepping through all the state variables in the list $\kappa$, we can find the desired feedback laws $v_i = v_i(z, \xi)$, $i = 1, 2, \ldots, m$. \hfill \Box

We illustrate the backstepping procedure of Theorem 6.3.2 by the following example.

**Example 6.3.** Consider a system in the form of (6.15), with $q_1 = 2$ and $q_2 = 3$,

$$
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}) + R_0(z)w, \\
\dot{\xi}_{1,1} &= \xi_{1,2} + \xi_{2,1}w, \\
\dot{\xi}_{1,2} &= v_1 + \xi_{1,2}w, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + z \sin w, \\
\dot{\xi}_{2,2} &= \xi_{2,3} + \xi_{2,1}v_1 + zw, \\
\dot{\xi}_{2,3} &= v_2, \\
y_1 &= \xi_{1,1}, \\
y_2 &= \xi_{2,1}.
\end{align*}
$$

(6.17)

We first note that the triangular dependency condition (2), needed for the conventional backstepping, does not hold for this system. We will thus resort...
to Theorem \ref{Theorem6.3.2}. Obviously, Condition 2) in Theorem \ref{Theorem6.3.2} holds with the
ordered list \( \kappa = \{ \xi_{2,1}, \xi_{1,1}, \xi_{1,2}, \xi_{2,2}, \xi_{2,3} \} \). Suppose that there exist a number \( \gamma > 0 \), smooth functions \( \phi_{i,1}(z) \), with \( \phi_{i,1}(0) = 0 \), \( i = 1, 2 \), a smooth positive
definite and radially unbounded function \( V(z) \), and a class \( K_\infty \) function \( \alpha_0(\cdot) \) such that
\[
\frac{\partial V}{\partial z} \left[ f_0(z, \phi_{1,1}(z), \phi_{2,1}(z)) + R_0(z)w \right]
\leq -\alpha_0(\|z\|) + \gamma^2\|w\|^2 - \| \text{col} \{ \phi_{1,1}(z), \phi_{2,1}(z) \} \|^2,
\]
for all \( z \) and \( w \).

Consider the following subsystem,
\[
\begin{align*}
\dot{z} &= f_0(z, \phi_{1,1}(z), \xi_{2,1}) + R_0(z)w, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + z \sin w, \\
y_1 &= \phi_{1,1}(z), \\
y_2 &= \xi_{2,1}.
\end{align*}
\]

(6.18)

View \( \xi_{2,2} \) as a virtual input. By Lemma \ref{Lemma6.1}, for every \( \epsilon > 0 \), there exists
a smooth feedback \( \xi_{2,2} = \phi_{2,2}(z, \xi_{2,1}) \) such that subsystem (6.18) is strictly dissi-
pative with respect to the supply rate \( q(w, y) = (\gamma + \epsilon/5)\|w\|^2 - \|y\|^2 \).

Next consider
\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}) + R_0(z)w, \\
\dot{\xi}_{1,1} &= \xi_{1,2} + \xi_{2,1}w, \\
\dot{\xi}_{2,1} &= \phi_{2,2}(z, \xi_{2,1}) + z \sin w, \\
y_1 &= \xi_{1,1}, \\
y_2 &= \xi_{2,1}.
\end{align*}
\]

(6.19)

and view \( \xi_{1,2} \) as a virtual input. Again, by Lemma \ref{Lemma6.1}, there exists a smooth
feedback \( \xi_{1,2} = \phi_{1,2}(z, \xi_{2,1}, \xi_{1,1}) \) such that subsystem (6.19) is strictly dissi-
pative with respect to the supply rate \( q(w, y) = (\gamma + 2\epsilon/5)\|w\|^2 - \|y\|^2 \).

By backstepping in a similar way through \( \xi_{1,2}, \xi_{2,2}, \xi_{2,3} \), we obtain the smooth feedback laws
\[
\begin{align*}
v_1 &= v_1(z, \xi_{2,2}, \xi_{1,1}, \xi_{1,2}), \\
v_2 &= v_2(z, \xi_{2,2}, \xi_{1,1}, \xi_{2,2}, \xi_{2,3}),
\end{align*}
\]
such that system (6.17) is strictly dissipative with respect to the supply rate
\( q(w, y) = (\gamma + \epsilon)^2\|w\|^2 - \|y\|^2 \).

As the problem of almost disturbance decoupling is a special case of the
problem of disturbance attenuation, the following result on almost disturbance
decoupling with stability is a corollary to Theorem \ref{Theorem6.3.2}.
Corollary 6.3.1  Consider a system in the form

\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, w), \\
\dot{\zeta}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{q_i-1} \delta_{i,j,l}(z, \xi) v_l + p_{i,j}(z, \xi, w), \quad j = 1, 2, \cdots, q_i - 1, \\
y_i &= \xi_{i,1}, \quad i = 1, 2, \cdots, m,
\end{align*}
\]

where \( \xi = \text{col} \{ \xi_1, \xi_2, \cdots, \xi_m \}, \xi_i = \text{col} \{ \xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i} \}, q_1 \leq q_2 \leq \cdots \leq q_m \), and functions \( f_0, \delta_{i,j,l} \) and \( p_{i,j}, i = 1, 2, \cdots, m \), are smooth. Assume that

\[
|p_{i,j}(z, \xi, w)| \leq R_{i,j}(z, \xi) \| w \|, \quad \forall z, \xi, w, \quad j = 1, 2, \cdots, q_i, \quad i = 1, 2, \cdots, m,
\]

for some smooth functions \( R_{i,j}(z, \xi), j = 1, 2, \cdots, q_i, \quad i = 1, 2, \cdots, m \). Suppose that

1) for every \( \gamma_0 > 0 \), there exist smooth \( \phi_{i,1}(z) \) with \( \phi_{i,1}(0) = 0, \quad i = 1, 2, \cdots, m \), and a smooth positive definite and radially unbounded function \( V(z) \) such that

\[
\frac{\partial V}{\partial z} f_0(z, \phi_{1,1}(z), \phi_{2,1}(z), \cdots, \phi_{m,1}(z), w) \\
\leq -\alpha(\| z \|) + \gamma_0^2 \| w \|^2 - \| \text{col} \{ \phi_{1,1}(z), \phi_{2,1}(z), \cdots, \phi_{m,1}(z) \} \|^2, \quad \forall z, w,
\]

for some \( K_\infty \) function \( \alpha(\cdot) \).

2) Condition 2) in Theorem 6.3.2 holds.

Then, for every \( \gamma > 0 \), there exist smooth feedback laws \( v_i = v_i(z, \xi), \quad i = 1, 2, \cdots, m \), such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \( q(w, y) = \gamma^2 \| w \|^2 - \| y \|^2 \), where \( y = \text{col} \{ y_1, y_2, \cdots, y_m \} \).

We next consider further the fulfillment of Condition 1) in Corollary 6.3.1. It is a generalization of Lemma 6.2 to multiple input multiple output systems. Suppose that the \( z \)-subsystem

\[
\dot{z} = f_0(z, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, w)
\]

can be decomposed as

\[
\begin{align*}
\dot{z}_1 &= f_1(z_1, z_2, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, w), \\
\dot{z}_2 &= f_2(z_2, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}),
\end{align*}
\]

where \( z_1 \) represents “stable component” and \( z_2 \) represents “unstable but stabilizable component.” We have the following result.
Corollary 6.3.2 Consider system (6.21) which can be decomposed as (6.22). Suppose that

1) there exists a smooth positive definite and radially unbounded function \(V_1(z_1)\) such that

\[
\frac{\partial V_1}{\partial z_1} f_1(z_1, z_2, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}, w)
\leq -\alpha_1(\|z_1\|) + \gamma_0^2 \|z_2\|^2 + \gamma_0^2 \|w\|^2 + \gamma_0^2 \|\text{col}\{\xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}\}\|^2,
\]

for some \(\mathcal{K}_\infty\) function \(\alpha_1\) and some \(\gamma_0 > 0\), and

2) there exist smooth functions \(\bar{v}_i(z_2)\) with \(\bar{v}_i(0) = 0\) for \(i = 1, 2, \cdots, m\), and a smooth positive definite and radially unbounded function \(V_2(z_2)\) such that

\[
\frac{\partial V_2}{\partial z_2} f_2(z_2, \bar{v}_1(z_2), \bar{v}_2(z_2), \cdots, \bar{v}_m(z_2)) + \|\text{col}\{\bar{v}_1(z_2), \bar{v}_2(z_2), \cdots, \bar{v}_m(z_2)\}\|^2
\leq -\alpha_2(\|z_2\|)
\]

for some \(\mathcal{K}_\infty\) function \(\alpha_2\).

Then, for every \(\gamma > 0\), there exist smooth \(v_i(z)\) with \(v_i(0) = 0\), \(i = 1, 2, \cdots, m\), and a smooth positive definite and radially unbounded function \(V(z)\) such that

\[
\frac{\partial V}{\partial z} f_0(z, v_1(z), v_2(z), \cdots, v_m(z), w) \leq -\alpha(\|z\|) + \gamma^2 \|w\|^2 - \|\text{col}\{v_1(z), v_2(z), \cdots, v_m(z)\}\|^2, \forall z, w,
\]

for some \(\mathcal{K}_\infty\) function \(\alpha(\cdot)\).

Corollary 6.3.2 provides, for the system in Corollary 6.3.1, a starting point from which the backstepping can be carried out. We next use a numerical example with unstable zero dynamics to illustrate Corollary 6.3.2.

Example 6.4. Consider a system in the form of (6.20) with \(q_1 = 1\) and \(q_2 = 2\),

\[
\begin{cases}
\dot{z} = z + \xi_{1,1} + \xi_{2,1}, \\
\xi_{1,1} = v_1 + \xi_{2,1} w, \\
\xi_{2,1} = \xi_{2,2} + z w, \\
\xi_{2,2} = v_2 + (\cos \xi_{1,1}) \sin w, \\
y_1 = \xi_{1,1}, \\
y_2 = \xi_{2,1}.
\end{cases}
\tag{6.23}
\]

Note that the dependency requirement in Corollary 6.3.1 holds and zero dynamics satisfies the conditions in Corollary 6.3.2. The zero dynamics \(\dot{z} = z\) is unstable. View \(\xi_{1,1}\) and \(\xi_{2,1}\) as virtual input of...
\[ \dot{z} = z + \xi_{1,1} + \xi_{2,1}. \] (6.24)

Then \( \xi_{1,1} = \phi_{1,1}(z) = -2z \) and \( \xi_{2,1} = 0 \) stabilizes (6.24) with Lyapunov function \( V_0 = z^2/2 \).

Condition 2) of Corollary 6.3.1 holds with \( \kappa = \{ \xi_{2,1}, \xi_{1,1}, \xi_{2,2} \} \).

To begin the mixed chain-by-chain and level-by-level backstepping procedure, we consider the subsystem
\[
\begin{align*}
\dot{z} &= -z + \xi_{2,1}, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + zw,
\end{align*}
\] (6.25)
and view \( \xi_{2,2} \) as a virtual input. Consider the Lyapunov function \( V_1 = V_0 + \xi_{2,1}^2/2 = z^2/2 + \xi_{2,1}^2/2 \). Its time derivative is given by
\[
\dot{V}_1 = \dot{V}_0 + \xi_{2,1} \dot{\xi}_{2,1} + zw.
\]
Let
\[
\xi_{2,2} = \phi_{2,2}(z, \xi_{2,1}) = -z - \xi_{2,1} - \frac{3}{4\gamma^2} \xi_{2,1}(1 + z^2),
\]
which renders
\[
\dot{V}_1 \leq -\xi_{2,1}^2 + 2\gamma^2 w^2 - z^2.
\]

We next consider
\[
\begin{align*}
\dot{z} &= z + \xi_{1,1} + \xi_{2,1}, \\
\dot{\xi}_{2,1} &= -z - \xi_{2,1} - \frac{3}{4\gamma^2} \xi_{2,1}(1 + z^2) + zw, \\
\dot{\xi}_{1,1} &= v_1 + \xi_{2,1}w.
\end{align*}
\] (6.26)
Letting \( V_2 = V_1 + (\xi_{1,1} + 2z)^2/2 \), we have
\[
\dot{V}_2 \leq \dot{V}_1 + (\xi_{1,1} + 2z)(v_1 + 3z + \xi_{1,1} + \xi_{2,1} + \xi_{2,1}w).
\]
Let
\[
v_1 = -\frac{11}{3} z - \frac{4}{3} \xi_{1,1} - \xi_{2,1} - \frac{3}{4\gamma^2} (\xi_{1,1} + 2z)(1 + \xi_{2,1}^2).
\]
We have
\[
\dot{V}_2 \leq -\xi_{2,1}^2 + \frac{2\gamma^2}{3} w^2 - z^2 - (\xi_{1,1} + 2z)^2/3 
\leq -\xi_{2,1}^2 + \frac{2\gamma^2}{3} w^2 - 4z^2 - \xi_{1,1}^2.
\]
Finally, consider
\[
\begin{align*}
\dot{z} &= z + \xi_{1,1} + \xi_{2,1}, \\
\dot{\xi}_{1,1} &= v_1 + \xi_{2,1}w, \\
\dot{\xi}_{2,1} &= \xi_{2,2} + zw, \\
\dot{\xi}_{2,2} &= v_2 + (\cos \xi_{1,1}) \sin w,
\end{align*}
\] (6.27)
for which we let \( V_3 = V_2 + (\xi_{2,2} - \phi_{2,2})^2/2 \). Thus,

\[
\dot{V}_3 = \dot{V}_2 + (\xi_{2,2} - \phi_{2,2}) \left( \xi_{2,1} + v_2 + (\cos \xi_{1,1}) \sin w - \phi_{2,2} \right)
= \dot{V}_2 + (\xi_{2,2} - \phi_{2,2}) (v_2 + \Psi + \Phi w + (\cos \xi_{1,1}) \sin w),
\]

where

\[
\Phi = z + \frac{3}{4\gamma^2} (z + z^3),
\]
\[
\Psi = z + \xi_{1,1} + 2\xi_{2,1} + \xi_{2,2} + \frac{3}{4\gamma^2} \xi_{2,2} (1 + z^2) + \frac{3}{2\gamma^2} \xi_{2,1} z (z + \xi_{1,1} + \xi_{2,1}).
\]

Let

\[
v_2 = -\xi_{2,2} + \phi_{2,2} - \Psi - \frac{3}{4\gamma^2} (\xi_{2,2} - \phi_{2,2}) \left( 1 + (|\Phi| + |\cos \xi_{1,1}|)^2 \right).
\]

We have

\[
\dot{V}_3 \leq - (\xi_{2,2} - \phi_{2,2})^2 - \xi_{2,1}^2 + \gamma^2 w^2 - 4z^2 - \xi_{1,1}^2
\leq -\xi_{2,1}^2 - \xi_{1,1}^2 + \gamma^2 w^2,
\]

from which we have

\[
\int_0^t \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t w(\tau)^2 d\tau,
\]
in the absence of initial condition. In the presence of initial condition \( x(0) \), we have

\[
\int_0^t \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t w(\tau)^2 d\tau + V_3(x(0)).
\]

Shown in Fig. 6.1 are some state trajectories of the closed-loop system with \( x(0) = 0 \) and \( w(t) = 1(t) - 1(t-2) \). Shown in Fig. 6.2 are some state trajectories with \( x(0) = [1 1 1 1]^T \) and \( w(t) = (1(t) - 1(t-2)) \times 10 \).

### 6.4 Summary of the Chapter

In this chapter, we have revisited the problems disturbance attenuation and almost disturbance decoupling for nonlinear systems and showed how a recently developed structural decomposition of multiple input multiple output systems and the new backstepping design procedure it motivates can lead to the solution of these two problems for a larger class of systems.
Fig. 6.1. State trajectories with \( x(0) = 0 \) and \( w(t) = 1(t) - 1(t - 2) \).
Fig. 6.2. State trajectories with $x(0) = [1 \ 1 \ 1]^T$ and $w(t) = (1(t) - 1(t - 2)) \times 10$. 
In this note, we have obtained a few further results in differential geometric nonlinear control theory. We first developed normal forms for nonlinear system affine in control. Then, based on these normal forms, we revisited stabilization, semi-global stabilization and disturbance attenuation.

We presented constructive algorithms for decomposing an affine nonlinear system into its normal form representations. Such algorithms generalize the existing results in several ways. They require fewer restrictive assumptions on the system and apply to general multiple input multiple output nonlinear systems that do not necessarily have the same number of inputs and outputs. The resulting normal forms reveal various nonlinear extensions of linear system structural properties. These algorithms and the resulting normal forms are thus expected to facilitate the solution of several nonlinear control problems.

We exploited the properties of the structural decomposition for the stabilization of multiple input and multiple output systems, and showed that this decomposition simplifies the conventional chain-by-chain backstepping design procedure and motivates a new level-by-level backstepping design procedure that is able to stabilize some systems for which the conventional backstepping procedure is not applicable. The chain-by-chain and level-by-level backstepping procedures can be combined to form a mixed backstepping design technique. The enlarged class of systems that can be stabilized by this mixed backstepping design procedure is characterized in the form of a theorem.

We then showed how the structural decomposition can be used to solve the problem of semi-global stabilization for a class of multiple input multiple output systems without vector relative degrees. The design procedure involved several existing design techniques in nonlinear stabilization, including low gain feedback and different forms of backstepping design procedures.

We also revisited the problems of disturbance attenuation and almost disturbance decoupling for nonlinear systems and showed how the structural decomposition of nonlinear systems and the new backstepping design proce-
dures it motivates can lead to the solution of these two problems for a larger class of systems.

For the future research, we are interested in the problems of non-interacting control, tracking and regulation of nonlinear systems. These control problems can be dealt with based on the normal forms in Chapter 3.

Output feedback control is a more challenging problem. The normal forms proposed in the note, which reveals system structure at infinity, will also facilitate the construction of high gain observers, which will result in output feedback laws.

The structural algorithms can be applied to general nonlinear systems that are not necessarily square invertible. We have only considered their application to square invertible nonlinear systems in the note. We will utilize these normal forms to study control problems for non-invertible nonlinear systems, in particular, underactuated nonlinear systems.
References

[1] D. L. Elliott, “Book Reviews of Nonlinear Control Systems, Alberto Isidori, 1995,” *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 1043–1044, 1997.

[2] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. Springer, 1990.

[3] R. Hermann, “On the Accessibility Problem in Control Theory,” in *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics: proceedings*. Academic Press, 1963, p. 325.

[4] ———, *Differential geometry and the calculus of variations*. Academic Press, 1968.

[5] R. Hermann and A. J. Krener, “Nonlinear controllability and observability,” *IEEE Transactions on Automatic Control*, vol. 22, no. 5, pp. 728–740, 1977.

[6] R. W. Brockett, “Feedback invariants for nonlinear systems,” *A link between science and applications of automatic control*, pp. 1115–1120, 1979.

[7] ———, “Asymptotic stability and feedback stabilization,” in *Differential Geometric Control Theory*. Birkhäuser, Dec. 1983, pp. 181–191.

[8] A. Isidori, A. Krener, C. Gori-Giorgi, and S. Monaco, “Nonlinear decoupling via feedback: a differential geometric approach,” *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 331–345, 1981.

[9] R. Hirschorn, “(A, B)-invariant distributions and disturbance decoupling of nonlinear systems,” *SIAM Journal on Control and Optimization*, vol. 19, p. 1, 1981.

[10] A. S. Morse, “Structural invariants of linear multivariable systems,” *SIAM Journal on Control*, vol. 11, pp. 446–465, 1973.

[11] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag, 1979.

[12] W. M. Wonham and A. S. Morse, “Decoupling and pole assignment in linear multivariable systems: A geometric approach,” *SIAM Journal on Control*, vol. 8, no. 1, pp. 1–18, 1970.
[13] ——, “Feedback invariants of linear multivariable systems,” *Automatica*, vol. 8, no. 1, pp. 93–100, 1972.

[14] G. Basile and G. Marro, “Controlled and conditioned invariant subspaces in linear system theory,” *Journal of Optimization Theory and Applications*, vol. 3, no. 5, pp. 306–315, 1969.

[15] ——, *Controlled and conditioned invariants in linear system theory*. Prentice Hall Englewood Cliffs, New Jersey, 1992.

[16] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Springer, 1995.

[17] ——, *Nonlinear Control Systems II*. Springer, 1999.

[18] C. I. Byrnes and A. Isidori, “Asymptotic stabilization of minimum phase nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 36, pp. 1122–1137, 1991.

[19] S. S. Sastry and A. Isidori, “Adaptive control of linearizable systems,” *IEEE Transactions on Automatic Control*, vol. 34, no. 11, pp. 1123–1131, 1989.

[20] R. Marino, W. Respondek, and A. J. van der Schaft, “Equivalence of nonlinear systems to input-output prime forms,” *SIAM Journal of Control and Optimization*, vol. 32, p. 387, 1994.

[21] C. I. Byrnes and A. Isidori, “Local stabilization of minimum-phase nonlinear systems,” *Systems & Control Letters*, vol. 11, no. 1, pp. 9–17, 1988.

[22] B. Schwartz, A. Isidori, and T. J. Tarn, “Global normal forms for mimo nonlinear systems, with application to stabilization and disturbance attenuation,” *Mathematics of Control, Signals and Systems*, vol. 12, pp. 121–142, 1999.

[23] D. Chu, X. Liu, and R. C. Tan, “On the numerical computation of a structural decomposition in systems and control,” *IEEE Transactions on Automatic Control*, vol. 47, no. 11, pp. 1786–1799, 2002.

[24] X. Liu, B. M. Chen, and Z. Lin, “On the problem of general structural assignments of linear systems through sensor/actuator selection,” *Automatica*, vol. 39, no. 2, pp. 233–241, 2003.

[25] ——, “Linear systems toolkit in matlab: structural decompositions and their applications,” *Journal of Control Theory and Applications*, vol. 3, no. 3, pp. 287–294, 2005.

[26] X. Liu, Z. Lin, and B. Chen, “Symbolic realization of asymptotic time-scale and eigenstructure assignment design method in multivariable control,” *International Journal of Control*, vol. 79, no. 11, pp. 1471–1484, 2006.

[27] X. Liu, Z. Lin, and B. M. Chen, “Further results on structural assignment of linear systems via sensor selection,” *Automatica*, vol. 43, no. 9, pp. 1631–1639, 2007.

[28] D. Chu, X. Liu, and V. Mehrmann, “A numerical method for computing the hamiltonian schur form,” *Numerische Mathematik*, vol. 105, no. 3, pp. 375–412, 2007.
[29] B. M. Chen, X. Liu, and Z. Lin, “Interconnection of kronecker canonical form and special coordinate basis of multivariable linear systems,” *Systems & Control Letters*, vol. 57, no. 1, pp. 28–33, 2008.

[30] X. Liu and Z. Lin, “On stabilization of nonlinear systems affine in control,” in *Proc. 2008 American Control Conference*, Seattle, WA, Jun. 2008, pp. 4123–4128.

[31] X. Liu, Z. Lin, and B. M. Chen, “Assignment of complete structural properties of linear systems via sensor selection,” *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2072–2086, 2009.

[32] X. Liu and Z. Lin, “On semi-global stabilization of minimum phase nonlinear systems without vector relative degrees,” *Science in China. Series F, Information sciences*, vol. 52, no. 11, pp. 2153–2162, 2009.

[33] ——, “Further results on disturbance attenuation for multiple input multiple output nonlinear systems,” in *Proc. 2010 American Control Conference*, Baltimore, MD, Jun. 2010.

[34] ——, “On normal forms of nonlinear systems affine in control,” *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 239–253, 2011.

[35] ——, “On the backstepping design procedure for multiple input nonlinear systems,” *International Journal of Robust and Nonlinear Control*, vol. 22, no. 8, pp. 918–932, 2012.

[36] X. Liu, Z. Lin, and S. Acton, “A grid-based bayesian approach to robust visual tracking,” *Digital Signal Processing*, vol. 22, no. 1, pp. 54–65, 2012.

[37] X. Liu, T. Iwasaki, and F. Fish, “Dynamic modeling and gait analysis of batoid swimming,” in *American Control Conference, 2013. ACC’13*. IEEE, 2013.

[38] A. H. Belcher, X. Liu, Z. Grelewicz, E. Pearson, and R. D. Wiersma, “Development of a 6dof robotic motion phantom for radiation therapy,” *Medical physics*, vol. 41, no. 12, p. 121704, 2014.

[39] X. Liu, A. H. Belcher, Z. Grelewicz, and R. D. Wiersma, “Robotic real-time translational and rotational head motion correction during frameless stereotactic radiosurgery,” *Medical Physics*, vol. 42, no. 6, pp. 2757–2763, 2015.

[40] A. H. Belcher, X. Liu, Z. Grelewicz, and R. D. Wiersma, “Spatial and rotational quality assurance of 6dof patient tracking systems,” *Medical Physics*, vol. 43, no. 6, pp. 2785–2793, 2016.

[41] X. Liu, F. Fish, R. S. Russo, S. S. Blemker, and T. Iwasaki, “Modeling and optimality analysis of pectoral fin locomotion,” in *Neuromechanical Modeling of Posture and Locomotion*. Springer, 2016, pp. 309–332.

[42] X. Liu and T. Iwasaki, “Design of coupled harmonic oscillators for synchronization and coordination,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, 2017.

[43] X. Liu, C. Pelizzari, A. H. Belcher, Z. Grelewicz, and R. D. Wiersma, “Use of proximal operator graph solver for radiation therapy inverse treatment planning,” *Medical Physics*, vol. 44, no. 4, pp. 1246–1256, 2017.
[44] A. A. Agrachev and Y. L. Sachkov, Control Theory from the Geometric Viewpoint. Springer Heidelberg, 2004.
[45] F. Bullo and A. Lewis, Geometric control of mechanical systems. Springer Berlin, 2005.
[46] B. F. Doolin and C. Martin, Introduction to differential geometry for engineers. Marcel Dekker Inc, 1990.
[47] S. Waner and G. C. Levine, Introduction to Differential Geometry and General Relativity. Hofstra University, 2005.
[48] C. I. Byrnes and A. Isidori, “A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control,” in Proc. 23rdIEEE Conference on Decision and Control, Dec. 1984, pp. 1569–1573.
[49] D. Cheng and L. Zhang, “Generalized normal form and stabilization of non-linear systems,” International Journal of Control, vol. 76, no. 2, pp. 116–128, 2003.
[50] G. Conte, C. H. Moog, and A. M. Perdon, Nonlinear Control Systems: an Algebraic Setting. Springer, 1999.
[51] Z. Ding, “Asymptotic rejection of asymmetric periodic disturbances in output-feedback nonlinear systems,” Automatica, vol. 43, no. 3, pp. 555–561, 2007.
[52] T. I. Fossen, Guidance and Control of Ocean Vehicles. Chichester, 1994.
[53] R. M. Hirschorn, “Invertibility of multivariable nonlinear control systems,” IEEE Transactions on Automatic Control, vol. 24, pp. 855–865, 1979.
[54] A. Isidori, A. J. Krener, C. Gori-Giorgi, and S. Monaco, “Nonlinear decoupling via feedback: a differential geometric approach,” IEEE Transactions on Automatic Control, vol. 26, pp. 331–345, 1981.
[55] Z. P. Jiang, I. Mareels, D. J. Hill, and J. Huang, “A unifying framework for global regulation via nonlinear output feedback: from ISS to iISS,” IEEE Transactions on Automatic Control, vol. 49, no. 4, pp. 549–562, 2004.
[56] G. Kaliora, A. Astolfi, and L. Praly, “Norm estimators and global output feedback stabilization of nonlinear systems With ISS inverse dynamics,” IEEE Transactions on Automatic Control, vol. 51, no. 3, pp. 493–498, 2006.
[57] D. Karagiannis, Z. P. Jiang, R. Ortega, and A. Astolfi, “Output-feedback stabilization of a class of uncertain non-minimum-phase nonlinear systems,” Automatica, vol. 41, no. 9, pp. 1609–1615, 2005.
[58] H. K. Khalil, Nonlinear Systems, 2nd ed. Prentice Hall, 2002.
[59] D. Liberzon, “Output–input stability implies feedback stabilization,” Systems & Control Letters, vol. 53, no. 3-4, pp. 237–248, 2004.
[60] R. Marino and P. Tomei, Nonlinear Control Design: Geometric, Adaptive and Robust. London: Prentice-Hall, 1996.
[61] K. Y. Pettersen and O. Egeland, “Exponential stabilization of an underactuated surface vessel,” in *Proc. 35th IEEE Conference on Decision and Control*, vol. 1, 1996, pp. 967–972.

[62] R. Sepulchre, M. Arcak, and A. R. Teel, “Trading the stability of finite zeros for global stabilization of nonlinear cascade systems,” *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 521–525, 2002.

[63] S. N. Singh, “A modified algorithm for invertibility in nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 26, pp. 595–598, 1981.

[64] A. R. Teel and L. Praly, “Global stabilizability and observability imply semi-global stabilizability by output feedback,” *Systems & Control Letters*, vol. 22, no. 5, pp. 313–325, 1994.

[65] F. Esfandiari and H. K. Khalil, “Output feedback stabilization of fully linearizable systems,” *International Journal of Control*, vol. 56, no. 5, pp. 1007–1037, 1992.

[66] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York, 1995.

[67] A. Saberi, P. V. Kokotović, and H. J. Sussmann, “Global stabilization of partially linear composite systems,” *SIAM Journal of Control and Optimization*, vol. 28, no. 6, pp. 1491–1503, 1990.

[68] A. R. Teel and L. Praly, “Tools for semiglobal stabilization by partial state and output feedback,” *SIAM Journal of Control and Optimization*, vol. 33, pp. 1443–1488, 1995.

[69] A. N. Atassi and H. K. Khalil, “A separation principle for the stabilization of a class of nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 44, no. 9, 1999.

[70] J. Huang, “On the solvability of the regulator equations for a class of nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 48, no. 5, pp. 880–885, 2003.

[71] L. R. Hunt and G. Meyer, “Stable inversion for nonlinear systems,” *Automatica*, vol. 33, no. 8, pp. 1549–1554, 1997.

[72] Z. F. Jiang and L. P. Pražak, “Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties,” *Automatica*, vol. 34, no. 7, pp. 825–840, 1998.

[73] D. Liberzon, A. S. Morse, and E. D. Sontag, “Output-input stability and minimum-phase nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 422–436, 2002.

[74] R. Ortega, L. Hsu, and A. Astolfi, “Immersion and invariance adaptive control of linear multivariable systems,” *Systems & Control Letters*, vol. 49, no. 1, pp. 37–47, 2003.

[75] P. Sannuti and A. Saberi, “A special coordinate basis of multivariable linear systems – finite and infinite zero structure, squaring down and decoupling,” *International Journal of Control*, vol. 45, pp. 1655–1704, 1987.

[76] H. H. Rosenbrock, *State Space and Multivariable Theory*. New York: John-Wiley, 1970.
[77] M. L. J. Hautus, “The formal Laplace transform for smooth linear systems,” in Mathematical Systems Theory, Lect. Notes Econ. Math. Syst., vol. 131, 1976, pp. 29–47.

[78] A. Isidori, “Nonlinear feedback, structure at infinity and the input-output linearization problem,” in Mathematical theory of networks and systems. Springer, 1983.

[79] H. Nijmeijer and J. Schumacher, “Zeros at infinity for affine nonlinear control systems,” IEEE Transactions on Automatic Control, vol. 30, no. 6, pp. 566–573, 1985.

[80] M. Fliess, “A new approach to the structure at infinity of nonlinear systems,” Systems & Control Letters, vol. 7, no. 5, pp. 419–421, 1986.

[81] C. H. Moog, “Nonlinear decoupling and structure at infinity,” Mathematics of Control, Signals and Systems, vol. 1, no. 3, pp. 257–268, 1988.

[82] M. D. D. Benedetto, J. W. Grizzle, and C. H. Moog, “Rank invariants of nonlinear systems,” SIAM Journal of Control and Optimization, vol. 27, pp. 658–672, 1989.

[83] W. Respondek, “Right and left invertibility of nonlinear control systems,” in Nonlinear Controllability and Optimal Control, New York and Basel, 1990, pp. 133–176.

[84] B. M. Chen, Z. Lin, and Y. Shamash, Linear Systems Theory: A Structural Decomposition Approach. Boston: Birkhäuser, 2004.

[85] Z. Lin, B. M. Chen, and X. Liu, Linear Systems Toolkit, 2004. [Online]. Available: http://linearsystemskit.net.

[86] R. W. Brockett and M. R. Mesarović, “The reproducibility of multivariable systems,” Journal of Mathematics Analysis Application, vol. 11, pp. 548–563, 1965.

[87] M. K. Sain and J. L. Massey, “Invertibility of linear time-invariant dynamical systems,” IEEE Transactions on Automatic Control, vol. 14, pp. 141–149, 1969.

[88] L. M. Silverman, “Inversion of multivariable linear systems,” IEEE Transactions on Automatic Control, vol. 14, pp. 270–276, 1969.

[89] H. Nijmeijer and W. Respondek, “Dynamic input-output decoupling of nonlinear control systems,” IEEE Transactions on Automatic Control, vol. 33, no. 11, pp. 1065–1070, 1988.

[90] W. Respondek and H. Nijmeijer, “On local right-invertibility of nonlinear control systems,” Control Theory and Advanced Technology, vol. 4, pp. 325–348, 1988.

[91] H. J. Sussmann and P. Kokotović, “The peaking phenomenon and the global stabilization of nonlinear systems,” IEEE Transactions on Automatic Control, vol. 36, no. 4, pp. 424–440, 1991.

[92] Z. Lin and A. Saberi, “Semi-global stabilization of minimum phase nonlinear systems in special normal form via linear high-and-low-gain state feedback,” International Journal of Robust and Nonlinear Control, vol. 4, pp. 353–362, 1994.
[93] A. R. Teel, “Semi-global stabilization of minimum phase nonlinear systems in special normal forms,” *Systems & Control Letters*, vol. 19, no. 3, pp. 187–192, 1992.

[94] J. Willems, “Almost invariant subspaces: An approach to high gain feedback design–Part I: Almost controlled invariant subspaces,” *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 235–252, 1981.

[95] A. Isidori, “A note on almost disturbance decoupling for nonlinear minimum phase systems,” *Systems & Control Letters*, vol. 27, no. 3, pp. 191–194, 1996.

[96] R. Marino, W. Respondek, and A. J. Van der Schaft, “Almost disturbance decoupling for single-input single-output nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 34, no. 9, pp. 1013–1017, 1989.

[97] R. Marino, W. Respondek, A. J. Van der Schaft, and P. Tomei, “Nonlinear $H_{\infty}$ almost disturbance decoupling,” *Systems & Control Letters*, vol. 23, no. 3, pp. 159–68, 1994.

[98] A. Isidori, “Global almost disturbance decoupling with stability for non minimum-phase single-input single-output nonlinear systems,” *Systems & Control Letters*, vol. 28, no. 2, pp. 115–122, 1996.

[99] Z. Lin, “Almost disturbance decoupling with global asymptotic stability for nonlinear systems with disturbance-affected unstable zero dynamics,” *Systems & Control Letters*, vol. 33, no. 3, pp. 163–169, 1998.