SYMMETRIES AND SOLUTIONS OF A THIRD ORDER EQUATION

JÚLIO CESAR SANTOS Sampaio
Instituto de Matemática, Estatística e Computação Científica
IMECC – UNICAMP
Sérgio Buarque de Holanda, 651, 13083 – 859
Campinas, SP, Brasil

IGOR LEITE FREIRE
Centro de Matemática, Computação e Cognição
Universidade Federal do ABC – UFABC
Rua Santa Adélia, 166, Bairro Bangu, 09210 – 170
Santo André, SP, Brasil

Abstract. In this paper we study a new third order evolution equation discovered a couple of years ago using a genetic programming. We show that the Lie symmetries of this equation corresponds to space and time translations, as well as a dilation on the space of independent variables and another one with respect to the depend variable. From its symmetries, explicit solutions of the equation are obtained, some of them expressed in terms of the solutions of the Airy equation and Abel equation of the second kind. Additionally, by using the direct method we establish three conservation laws for the equation, one of them new.

1. Introduction. This work corresponds to a talk given by the second author in the session SS 129: Qualitative and Quantitative Techniques for Differential Equations arising in Economics, Finance and Natural Sciences, during The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, which took place in Madrid, Spain, from July 07 – July 11, 2014. We would like to thank the organizers of the session and conference for the opportunity to discuss our results in that nice event.

Among other results we discussed about the Lie symmetries and some solutions obtained for the equation
\[ u_t + \frac{2\alpha}{u} u_x u_{xx} = \epsilon au_{xxx}, \tag{1} \]
which was introduced in [11]. Taking \( \epsilon = 1 \) and the dilation \((x,t) \mapsto (\sqrt[3]{\alpha}x,t)\), equation (1) is reduced to
\[ u_t + \frac{2}{u} u_x u_{xx} = u_{xxx}. \tag{2} \]
In this paper we shall consider equation (1) with the restriction \( \epsilon \alpha \neq 0 \).

According to [11], (2) was discovered by using a genetic programming. Such program was employed for deducing model equations once a solution had been given. Then, (2) was obtained when it was given the function
\[ u(x,t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{\epsilon}}{2} (x - ct - x_0) \right), \tag{3} \]

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where \( c > 0 \) and \( x_0 \in \mathbb{R} \). Function (3) is an well known solution of the celebrated KdV equation.

Additionally, in [11] the authors showed that equation (1) can be transformed in a linear one when \( \epsilon = -2/3 \). In fact, for this value of \( \epsilon \), equation (1) can be rewritten as the following conservation law

\[
\frac{\partial}{\partial t} \left( \frac{u^2}{2} \right) + \frac{\partial}{\partial x} \left( \frac{a}{3} \epsilon^2 u^2 \right) = 0.
\]

Therefore, under the change \((x, t) \mapsto (\sqrt[3]{\alpha} x, t)\) and introducing \( \rho := u^2 \), the last equation is equivalent to the Airy’s equation

\[
\rho_t + \frac{2}{3} \rho_{xxx} = 0,
\]

which is an approximation of the KdV equation without nonlinear effects.

Furthermore, letting \( w = w(x, t) \) be a function such that \( u = e^w \), equation (1) becomes

\[
w_t + a[(2 - \epsilon)w^2_x + (2 - 3\epsilon)w_{xxx}]w_x = \epsilon aw_{xxx}.
\]

Differentiating (6) with respect to \( x \), defining \( v := w_x \), making \((x, t) \mapsto (\sqrt[3]{\alpha} x, 3t/2)\) and letting \( \epsilon = 2/3 \), one obtains the modified KdV equation

\[
v_t = v_{xxx} - 6v^2v_x.
\]

In regard to solutions, in [11] the authors found some similarity solutions of (1). An interesting case arise at the dispersionless limit \( \epsilon \to 0 \). Using the similarity variable \( z = x^3/54t \), two solutions of (1) were obtained, given by

\[
u_1(x, t) = \Gamma \left( \frac{2}{3} \right) \left( \frac{x^3}{54t} \right)^{\frac{1}{3}} I_{\frac{1}{3}} \left( 2 \sqrt{\frac{x^3}{54t}} \right),
\]

\[
u_2(x, t) = \Gamma \left( \frac{4}{3} \right) \left( \frac{x^3}{54t} \right)^{\frac{1}{3}} I_{\frac{1}{3}} \left( 2 \sqrt{\frac{x^3}{54t}} \right),
\]

where \( I_\alpha(z) \) is the modified Bessel function of the first kind. We would like to emphasise that these solutions where obtained when \( \epsilon = 0 \) in (1). Further solutions of (1), with \( \epsilon \neq 0 \), were also discovered in [11].

In this work we present, in section 2, the Lie point symmetries of (1) using the classical approach [1, 7, 9, 10, 12]. Then, our next step is to use the generators of point symmetries for constructing solutions of (1), which is done in section 3. Particularly, we obtain solutions of (1), which were not found in [11]. In section 4 we use differential invariants to reduce the order of certain ODEs obtained in section 3. Additionally, we employ the direct method to construct conservation laws for (1) in section 5. Consequently, this enables us to find more conserved currents for (1) than those obtained in [11]. Finally, conclusions are presented at the end.

2. Lie symmetries. Here we present a very short recall on Lie symmetries. For further details, see [1, 7, 9, 10, 12].

Let \( x = (x^1, \cdots, x^n) \in \mathbb{R}^n \) and \( u = u(x) \in \mathbb{R} \) be, respectively, \( n \) independent and a dependent variable. The set of all \( j \)th derivatives of \( u \) is denoted by \( u_{(j)} \). We also assume the summation over the repeated indices and all functions in this paper are assumed to be smooth.

The operator

\[
D_i := \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 0, \cdots, n
\]

is called total derivative and, in particular, given a function \( u \), we have \( u_{i_1 \cdots i_j} = D_{i_1} \cdots D_{i_j} (u) \).

We denote by \( \mathcal{A} \) the set of all locally analytic functions of a finite number of the variables \( x, u \) and \( u_{(j)} \) and, given a \( F \in \mathcal{A} \), we consider the equation

\[
F(x, u, u_{(1)}, \cdots, u_{(k)}) = 0.
\]
An operator

\[ X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \]

(9)
is called Lie point symmetry generator of the equation (8) if

\[ X^{(k)} F = \lambda F, \]

(10)
for a certain function \( \lambda \in \mathcal{A} \). The operator \( X \) is the generator of a one-parameter transformation \( T_x : (x, u) \mapsto (\tilde{x}, \tilde{u}) \) acting on a subset of \( \mathbb{R}^{n+1} \), where \( x = (x^1, \cdots, x^n) \) and

\[
\begin{align*}
\tilde{x}^i & = x^i + \varepsilon X x^i + \frac{\varepsilon^2}{2!} X(X x^i) + \cdots + \frac{\varepsilon^n}{n!} X(X^{n-1} x^i) + \cdots = x^i + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} X^j(x^i), \\
\tilde{u} & = u + \varepsilon Xu + \frac{\varepsilon^2}{2!} X(X u) + \cdots + \frac{\varepsilon^n}{n!} X(X^{n-1} u) + \cdots = u + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} X^j(u).
\end{align*}
\]

The set of these transformations is an additive group when endowed with the composition of functions, since \( T_x \circ T_{\xi} = T_{x+\xi} \), whose identity is given by the identity map \( I(x, u) = (x, u) \).

The operator \( X^{(k)} \) in (10) is given by

\[ X^{(k)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{ij} \frac{\partial}{\partial u_{ij}} + \cdots + \zeta_{i_1 \cdots i_k} \frac{\partial}{\partial u_{i_1 \cdots i_k}}, \]

(11)
where \( \zeta_i = D_i \eta - (D_i \xi^j) u_j, \cdots, \zeta_{i_1 \cdots i_k} = D_{i_1} \cdots D_{i_k} \eta - (D_{i_k} \xi^j) u_{i_1 \cdots i_{k-1} j} \), is the \( k \)-th prolongation of the vector field \( X \). In this case, we say that \( (x, u) \mapsto (\tilde{x}, \tilde{u}) \) is a Lie point symmetry of (8) and (10) is called invariance condition.

Given a differential equation, from the invariance condition we obtain an overdetermined, linear system of partial differential equations for the coefficients of the Lie point symmetry generator. The solution of such system gives the set of all Lie point symmetry generators of the equation.

**Theorem 2.1.** The Lie point symmetry generators of equation (1) is a linear combination of the operators

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad X_4 = u \frac{\partial}{\partial u}. \]

(12)

**Proof.** Let

\[ X = \xi (x, t, u) \frac{\partial}{\partial x} + \tau (x, t, u) \frac{\partial}{\partial t} + \eta (x, t, u) \frac{\partial}{\partial u}, \]

(13)
be a Lie point symmetry generator of (1). Using the results of Chapter 4 of [1], we can affirm that the coefficients \( \xi \) and \( \tau \) in (13) do not have any dependence with respect to \( u \). Now, define

\[ F = u_t + \frac{2a}{u} u_x u_{xx} - \epsilon a u_{xxx}. \]

Then we have

\[
X^{(3)} F = -\frac{2a}{u^2} \eta u_x u_{xx} + \eta_t + (\eta_x - \tau_t) u_t - \zeta_t u_x + \frac{2a}{u} u_{xx} (\eta_x + (\eta_x - \zeta_x) u_x - \tau_x u_t)
\]

\[ + \frac{2a}{u} u_x (\eta_{xx} + (2 \eta_{xx} - \zeta_{xx}) u_x - \tau_{xx} u_t - 2 \tau_x u_{xt} + \eta_x u_x^2 + (\eta_x - 2 \zeta_x) u_{xx})
\]

\[ - \epsilon a (\eta_{xxx} + (3 \eta_{xxx} - \zeta_{xxx}) u_x + 3 \eta_{xx} u_x^2 + (3 \eta_{xx} - 3 \zeta_{xx}) u_{xx} + 3 \eta_{uu} u_x u_{xx}
\]

\[ + (\eta_x - 3 \zeta_x) u_{xxx} - \tau_{xxx} u_t - 3 \tau_x u_{xt}). \]
Condition (10) can therefore be written as the following
\[
-2a \frac{u^2}{u_x} u_{xx} + \eta_t + (\eta_x - \epsilon) u_t - \zeta_t u_x + 2a \frac{u}{u_x} (\eta_x + (\eta_u - \zeta_x) u_x - \tau_x u_t)
\]
\[
+ 2a \frac{u}{u_x} (\eta_{xx} + (2\eta_{xx} - \zeta_x) u_x - \tau_x u_t - 2\tau_x u_{xx} + \eta_{uu} u_x^2 + (\eta_u - 2\zeta_x) u_{xx} - \epsilon)(\eta_{xx} + (3\eta_{xx} - \zeta_x) u_x + 3\eta_{uu} u_x^2 + (3\eta_{uu} - 3\zeta_x) u_{xx} + 3\eta_{uu} u_x u_{xx} + (\eta_u - 3\zeta_x) u_{xxx} - 3\tau_{xx} u_t - 3\tau_{xx} u_{xx})) = \lambda (u_t + 2a \frac{u}{u_x} u_x u_{xx} - \epsilon u u_{xxx}).
\] (14)

The coefficients of the terms \(u_t, u_x, u_{xxx}, \ldots\), give
\[
\eta_t - \epsilon a \tau_{xxx} = \lambda,
\] (15)
\[
\frac{2a}{u} \left( - \frac{1}{u} \eta + 2\eta_u - 3\zeta_x \right) - 3ae \eta_{uu} = \frac{2a}{u} \lambda, \quad (16)
\]
\[
- \epsilon a (\eta_u - 3\zeta_x) = - \epsilon a \lambda,
\] (17)
\[
\eta_t - \epsilon a \eta_{xxx} = 0,
\] (18)
\[
- \zeta_t + \frac{2a}{u} \eta_{xx} - 3e \eta_{uu} + \epsilon a \zeta_{xxx} = 0,
\] (19)
\[
\frac{2a}{u} \eta_x - 3e \eta_{uu} + 3e a \zeta_{xx} = 0,
\] (20)
\[
\frac{2a}{u} \eta_{uu} = 0, \quad - \frac{2a}{u} \tau_x = 0, \quad - \frac{2a}{u} \tau_{xx} = 0, \quad - \frac{4a}{u} \tau_x = 0, \quad - 3\tau_{xx} = 0, \quad - 5\tau_x = 0.
\] (21)

System (15) – (21) can easily be solved and its solution is given by
\[
\xi = \frac{c_1}{3} x + c_2, \quad \tau = c_1 t + c_3, \quad \eta = c_4 u.
\] (22)

Substituting (22) into (13) we obtain (12).

**Corollary 1.** Transformations

\[(x, t, u) \mapsto (x + \varepsilon, t, u), \quad (x, t, u) \mapsto (x, t + \varepsilon, u), \quad (x, t, u) \mapsto (x, t + \varepsilon, u), \quad (x, t, u) \mapsto (x, t + \varepsilon, u), \quad (x, t, u) \mapsto (x, t + \varepsilon, u), \quad \varepsilon \in \mathbb{R}.
\]

are the one-parameter symmetry groups of equation (1).

3. Invariant solutions. In this section we construct invariant solutions of (1). We assume that \(\varepsilon < 2\).

We say that a function \(\varphi = \varphi(x, t)\) is an invariant solution of (1) with respect to the symmetry generated by \(X\) if
\[
X(u - \varphi(x, t))|_{u=\varphi(x, t)} = 0
\] (23)
and \(u = \varphi(x, t)\) is a solution of (1).

Condition (23) can be rewritten in the characteristic form
\[
\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta},
\] (24)
where it is assumed that the generator in (23) is given by (13).

Let us firstly consider a linear combination of the generators \(X_1, X_2\) and \(X_4\), that is
\[
X = \epsilon \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \gamma u \frac{\partial}{\partial u},
\] (25)
where $c$ and $\gamma$ are constants. From (24) we have
\[
\frac{dx}{c} = \frac{dt}{1} = \frac{du}{\gamma u}.
\]
Consequently, we can look for a solution of the form $u = e^{\gamma t} \phi(x - ct)$. Substituting this function into (1), we obtain
\[
\gamma \phi - c \phi' + \frac{2a}{\phi} \phi' \phi'' - \epsilon a \phi''' = 0. \tag{26}
\]
We do the ansatz $\phi = Ae^{m(x-ct)}$, with $A \neq 0$. Substituting $\phi = Ae^{m(x-ct)}$ into (26), we obtain the following algebraic equation
\[
a(2 - \epsilon)m^3 - cm + \gamma = 0. \tag{27}
\]
Since $\epsilon < 2$, we can write (27) as
\[
m^3 - \frac{c}{a(2 - \epsilon)} m + \frac{\gamma}{a(2 - \epsilon)} = 0. \tag{28}
\]
In particular, when $\gamma = 0$, the roots of (28) are $0$ and
\[
\pm \sqrt{\frac{c}{a(2 - \epsilon)}}.
\]
The first root corresponds to a constant solution. The last two cases lead us to the solutions
\[
u_\pm(x,t) = Ae^{\pm\sqrt{\frac{c}{a(2 - \epsilon)}}(x-ct)}, \tag{29}
\]
which are traveling wave solutions.

In the general case, we have three different solutions
\[
u_i(x,t) = e^{m_i x + (\gamma - m_i c)t}, \quad i = 1, 2, 3, \tag{30}
\]
each one corresponding to the different roots of the cubic equation (28).

Now we consider the generator
\[
X_4 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}. \tag{31}
\]
Again, from (24) we obtain
\[
\frac{dx}{x} = \frac{dt}{3t} = \frac{du}{0},
\]
which enables us to seek for a solution of the type $u = \phi(z)$, where $z = xt^{-\frac{2}{3}}$. Then, substituting this $u$ into (1), we have
\[
\frac{z}{3} \phi' - \frac{2a}{\phi} \phi' \phi'' + \epsilon a \phi''' = 0. \tag{32}
\]
Multiplying (32) by $\phi$ and taking the expression $\phi \phi'' = (\phi \phi')'' - 3\phi' \phi''$ into account, we can write
\[
\frac{z}{3a\epsilon} \phi \phi' - \left(\frac{2}{\epsilon} + 3\right) \phi' \phi'' + (\phi \phi')'' = 0. \tag{33}
\]
If $\epsilon = -\frac{2}{3}$ we obtain the Airy’s equation
\[
y'' = \frac{z}{2a} y, \tag{34}
\]
where $y = \phi \phi'$. The solution of (34) is given by
\[
Y(z) = \frac{a_0}{2a} \left(1 + \sum_{n=1}^{\infty} \frac{z^{3n}}{2.3\ldots(3n-1).(3n)}\right) + \frac{a_1}{2a} \left(z + \sum_{n=1}^{\infty} \frac{z^{3n+1}}{3.4\ldots(3n).(3n+1)}\right). \tag{35}
\]
where $a_0$ and $a_1$ are constants. Therefore, for $\epsilon = -\frac{2}{3}$ we have a solution of (1) given by
\[
u(x, t) = \phi(x t^{-\frac{2}{3}}) = \pm \sqrt{2H(x t^{-\frac{2}{3}}) + k}, \tag{36}\]
where $k = \text{const.}$, $H' = Y$ and $Y$ is given by (35).

4. Reductions of order by differential invariants. In this section we use differential invariants to reduce the order of equations (26) and (32).

To begin with, we recall that if $G$ is a local group of point transformations acting on $M \subseteq \mathbb{R}^2$, a differential invariant of order $n$ of $G$ is an invariant function of the prolonged action of the group.

Consider the generator
\[
X = \phi \frac{\partial}{\partial \phi}, \tag{37}
\]
which generates the transformation $(z, \phi) \mapsto (z, \lambda \phi)$, $\lambda > 0$, on the plane $(z, \phi)$.

The first order prolongation of (37) is
\[
X^{(1)} = \phi \frac{\partial}{\partial \phi} + \phi' \frac{\partial}{\partial \phi'}. \quad \text{(38)}
\]

Let
\[
u = z, \quad v = \phi'/\phi. \tag{39}
\]

They provide a set of first order differential invariants under the scaling transformation $(z, \phi) \mapsto (z, \lambda \phi)$, $\lambda > 0$. We observe that this last transformation is a point symmetry of equation (26).

From (38), we have
\[
\frac{dv}{du} = \frac{\phi''}{\phi} - v^2, \quad \frac{d^2v}{du^2} = \frac{\phi'''}{\phi} - 3\frac{dv}{du} - v^3. \tag{40}
\]

From (39) we can obtain $\phi'$, $\phi''$ and $\phi'''$ in terms of $v$ and its derivatives. Next, substituting $\phi'$, $\phi''$ and $\phi'''$ into (26) and after reckoning we obtain the following second order ODE:
\[
\frac{d^2v}{du^2} = \left(\frac{2}{\epsilon} - 3\right) \frac{dv}{du} + \left(\frac{2}{\epsilon} - 1\right) v^3 - \frac{c}{\epsilon a} v + \frac{\gamma}{\epsilon a}. \tag{41}
\]

We now observe that equation (40) is invariant under the translation $(u, v) \mapsto (u + \epsilon, v)$. Then, let $w = w(v)$ be a function such that
\[
\frac{dv}{du} = w. \tag{42}
\]

Let
\[
w' := \frac{dw}{dv}. \tag{43}
\]

Then equation (40) is reduced to the first order ODE
\[
w w' = \left(\frac{2}{\epsilon} - 3\right) w v + \left(\frac{2}{\epsilon} - 1\right) v^3 - \frac{c}{\epsilon a} v + \frac{\gamma}{\epsilon a}, \tag{44}
\]
which is an Abel differential equation of the second kind.

Remark 1. Solution (30) can now be obtained in a different way. Assume that $w = 0$. On one hand, (41) implies that $v$ is a constant. On the other hand, from (42) and the fact that $w = 0$, we conclude that $v$ is a solution of the algebraic equation (28). Then, using the transformations (38) and the fact that $\nu(x, t) = e^{\gamma t} \phi(x - ct)$, we arrived again at the solution (30).
Remark 2. Solutions of equation (42) can be found using different techniques such as those suggested in [13], even though it is not a trivial task. As example, consider the case $\epsilon = 2/3$, $\gamma = 0$ and $3c = 2a$. For this case we obtain $w_\pm = \pm \sqrt{v^4 - v^2}$. Considering the positive root, from (41) we have

$$\frac{dv}{\sqrt{v^4 - v^2}} = du,$$

which has the solution\(^1\) $v = -\cossec(u + 2B)$, where $B$ is an arbitrary constant. Then, from (38) we have

$$\frac{\phi'}{\phi} = -\cossec(z + 2B),$$

whose solution is

$$\phi = A \cot \left( \frac{z}{2} + B \right),$$

where $A$ is another arbitrary constant. Therefore, one has the solution

$$u(x, t) = A \cot \left( \frac{x - ct}{2} + B \right)$$

of (1), provided that $3c = 2a$ and $\epsilon = 2/3$.

Let us now reduce the order of equation (32). We firstly observe that it admits the scaling $(z, \phi) \mapsto (z, \lambda \phi)$, $\lambda > 0$, as a symmetry. This means that (38) are differential invariants of this equation and then, substituting (39) into (32) we arrive at

$$\frac{d^2 v}{du^2} + \left( \frac{3\epsilon - 2}{\epsilon} \right) v \frac{dv}{du} + \frac{\epsilon - 2}{\epsilon} v^3 + \frac{u}{3c a} v = 0. \quad (43)$$

5. Conservation laws. We recall that a conservation law for an equation of the type (1) is a vanishing expression

$$D_t C^0 + D_x C^1 = 0$$

on the solutions $u = u(x, t)$ of (1).

The vector field $C = (C^0, C^1)$ is called conserved vector, while component $C^0$ is the conserved density and the component $C^1$ is the conserved flux. Defining

$$Q(t) := \int_{-\infty}^{\infty} C^0 dx,$$

from the divergence theorem we conclude that\(^2\)

$$Q'(t) = \int_{-\infty}^{\infty} D_t C^0 dx = \int_{-\infty}^{\infty} D_x C^1 dx = - C^1|_{-\infty}^{\infty}.$$  

Assuming that $C^1 \to 0$ when $x \to \pm \infty$, we conclude that $Q'(t) \equiv 0$, which provides a time conserved quantity.

Here we establish conservation laws for equation (1) using the direct method [8]. For further details, see also [3, 4, 5, 6].

Let $\mu = \mu(u)$ be a function such that

$$\mu \left( u_t + \frac{2a}{u} u_x u_{xx} - c a u_{xxx} \right) = D_t C^0 + D_x C^1. \quad (44)$$

Such function is called multiplier, see [3, 4, 5, 6, 8].

Equation (44) implies that on the solutions of (1), the vector field $C = (C^0, C^1)$ is a vanishing divergence, that is, it is a conservation law for (1).

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\(^1\)It was obtained with the aid of Wolfram, see http://www.wolframalpha.com/

\(^2\)Formally speaking, $Q$ is a functional of $u$, that is, $Q = Q[u]$.  

On one hand, since
\[ \frac{\delta}{\delta u} D_t = 0, \]
where
\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=0}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}} \]
is the Euler-Lagrange operator, from equation (44) we have
\[ \frac{\delta}{\delta u} \left[ \mu \left( u_t + \frac{2a}{u} u_x u_{xx} - \epsilon au_{xxx} \right) \right] \equiv 0. \tag{45} \]

On the other hand, after reckoning, we have
\[ \frac{\delta}{\delta u} \left[ \mu \left( u_t + \frac{2a}{u} u_x u_{xx} - \epsilon au_{xxx} \right) \right] = a \left[ 2 \left( \frac{\mu''}{u} - 2 \frac{\mu'}{u^2} + 2 \frac{\mu}{u^3} \right) + \epsilon \mu'' \right] u_x^3 \]
\[ + 3a \left[ 2 \left( \frac{\mu'}{u} - \frac{\mu}{u^2} \right) + \epsilon \mu' \right] u_x u_{xx}. \tag{46} \]

Thus, substituting (46) into (45) and equating the coefficients of \( u_x^3 \) and \( u_x u_{xx} \) to 0, one concludes that
\[ \epsilon \mu'' + 2 \left( \frac{\mu''}{u} - 2 \frac{\mu'}{u^2} + 2 \frac{\mu}{u^3} \right) = 0 \tag{47} \]
and
\[ \epsilon \mu'' + 2 \left( \frac{\mu'}{u} - \frac{\mu}{u^2} \right) = 0. \tag{48} \]

Equation (47) is a consequence of (48) and it is then enough to solve this last one, which gives \( \mu = c_1 u + c_2 u^{-\frac{7}{2}} \), if \( \epsilon \neq -2 \), and \( \mu = c_1 u + c_2 \ln u \), in the case \( \epsilon = -2 \).

This proves the following result.

**Theorem 5.1.** Let \( \mu = \mu(u) \) be a multiplier of equation (1). Then, for any value of \( \epsilon \), \( \mu = u \) is a multiplier. For \( \epsilon \neq -2 \), equation (1) also admits the multiplier \( \mu = u^{-\frac{7}{2}} \). In the particular case when \( \epsilon = -2 \), the additional multiplier is \( \mu = u \ln u \).

Our next results are the conservation laws of (1) obtained via multipliers.

**Theorem 5.2.** Conservation laws of equation (1) are given by
\[ D_t \left( \frac{u^2}{2} \right) + D_x \left( \frac{2 + \epsilon}{2} au_x^2 - \epsilon au_{xx} \right) = 0, \tag{49} \]
\[ D_t \left( u^{-\frac{7}{2}} \right) + D_x \left( (2 - \epsilon) au^{-\frac{7}{2}} u_{xx} \right) = 0, \tag{50} \]
\[ D_t \left( \frac{2a^2 \ln u - u^2}{4} \right) + D_x \left( 2au \ln u u_{xx} - au_x^2 \right) = 0. \tag{51} \]

**Proof.** The proof follows from the fact that
\[ u \left( u_t + \frac{2a}{u} u_x u_{xx} - \epsilon au_{xxx} \right) = D_t \left( \frac{u^2}{2} \right) + D_x \left( \frac{2a^2 x}{2} au_x^2 - \epsilon au_{xx} \right), \]
\[ u^{-\frac{7}{2}} \left( u_t + \frac{2a}{u} u_x u_{xx} - \epsilon au_{xxx} \right) = D_t \left( u^{-\frac{7}{2}} \right) + D_x \left( (2 - \epsilon) au^{-\frac{7}{2}} u_{xx} \right) \tag{52} \]
and
\[ u \ln u \left( u_t + \frac{2a}{u} u_x u_{xx} + 2au_{xx} \right) = D_t \left( \frac{2a^2 \ln u - u^2}{4} \right) + D_x \left( 2au \ln u u_{xx} - au_x^2 \right). \tag{53} \]

\( \square \)
6. Conclusions. In this communication we found the Lie point symmetry generators of equation (1), given in the Theorem 2.1. From these generators, we obtained some explicit, exact solutions for the considered equation.

We would like to call attention to the fact that in [11] the authors found some solutions. In particular, they sought by solutions of the type \( u(x,t) = f(z) \), where \( z = x^3/54t \). A solution of this type is an invariant solution under the one-parameter group of symmetries \((x,t,u) \mapsto (e^\varepsilon x, e^{3\varepsilon}t, u)\), which is a solution obtained using the generator (31). However, in [11] the authors had obtained two different solutions in terms of the confluent hypergeometric function, but only to the dispersionless limit \( \varepsilon = 0 \). In our case, the invariance under the scaling \((x,t,u) \mapsto (e^{\varepsilon}x, e^{3\varepsilon}t, u)\) led us, when \( \varepsilon = -2/3 \), to the solution (36), where the function \( Y(z) \) is given by the serie (35).

On the other hand, although in [11] the authors had looked for travelling wave solutions, they only obtained implicit ones. We, however, assuming the ansatz \( u = e^{\gamma t}\phi(x-ct) \), found two explicit solutions given by (29), as well as other three (not exactly travelling waves, of course), given by (30), where the constants \( m_1, m_2, m_3 \) are the roots of the equation (28). Additionally, from the results of section 4 we can obtain solutions of (1) in terms of the solutions of the Abel equation of the second kind (42) or the solutions of the second order ODE (43). Then our results complement those found in the interesting paper [11] with respect to the solutions of (1).

In regard to conservation laws, in [11] the authors established (52). Therefore, the conservation law (53) is new, even though it is restricted to the case \( \varepsilon = -2 \).

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Received September 2014; revised January 2015.
E-mail address: juliocesar.santossampaio@gmail.com
E-mail address: igor.freire@ufabc.edu.br and igor.leite.freire@gmail.com