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Huppert’s conjecture for character codegrees

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Abstract

Huppert’s $\rho$-$\sigma$ conjecture is one of the main open problems on character degrees of finite groups. A number of $\rho$-$\sigma$ problems have been studied. For instance, T. Keller and J. Zhang considered the $\rho$-$\sigma$ problem for element orders in the 1990s. Recently, a lot of research is being done on character codegrees. Y. Yang and G. Qian studied the $\rho$-$\sigma$ problem for character codegrees in 2017. In this note, we obtain a sharp bound for groups with trivial solvable radical. As a consequence, we improve the general bound of Yang and Qian. For solvable groups, we notice that the $\rho$-$\sigma$ problem for character codegrees is very closely related to the $\rho$-$\sigma$ problem for element orders. In particular, we give a partial negative answer to a speculation by Yang and Qian on the exact bound for the $\rho$-$\sigma$ problem for character codegrees.

Key words

character codegrees, character degrees, element orders, Huppert’s $\rho$-$\sigma$ conjecture

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1 | INTRODUCTION

One of the main problems on character degrees of finite groups is Huppert’s $\rho$-$\sigma$ conjecture. Given an integer $n$, $\pi(n)$ is the set of prime divisors of $n$. If $G$ is a group, then $\pi(G)$ stands for the set of prime divisors of $|G|$, $\rho(G)$ is the set of primes that divide the degree of some irreducible character of $G$ and

$$\sigma(G) = \max\{|\pi(\chi(1))| \mid \chi \in \text{Irr}(G)\}.$$ 

The $\rho$-$\sigma$ conjecture asserts that $|\rho(G)|$ is bounded in terms of $\sigma(G)$ and that, if $G$ is solvable, then $|\rho(G)| \leq 2\sigma(G)$. The first bound for the general $\rho$-$\sigma$ conjecture was obtained in [11]. The bound in [11] was improved by C. Casolo and S. Dolfi in [1] to a linear bound, namely,

$$|\rho(G)| \leq 7\sigma(G).$$

It is expected that $|\rho(G)| \leq 3\sigma(G)$ holds, but this is still unproven. For solvable groups, the first bound was obtained by M. Isaacs in [4]. Isaacs’ bound was subsequently improved in a number of papers. The best known bound, due to O. Manz

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and T. Wolf [9], is

$$|\rho(G)| \leq 3\sigma(G) + 2.$$  

Several particular cases of the $\rho$-$\sigma$ conjecture are known to be true, for instance when $\sigma(G) \leq 2$, but finding the exact bounds for any group $G$ or when $G$ is solvable seems a very tough problem.

There have been a number of variations on Huppert’s $\rho$-$\sigma$ conjecture. One of them will be very relevant in this note: the $\rho$-$\sigma$ problem for element orders, especially for solvable groups. If

$$\sigma_e(G) = \max \{|\pi(o(g))| \mid g \in G\},$$  

it was proved by J. Zhang [15] that there exists a quadratic bound for $|\pi(G)|$ in terms of $\sigma_e(G)$. This was improved to a linear bound by T. Keller in [8]. He proved that

$$|\pi(G)| \leq C(\sigma_e(G))\sigma_e(G),$$  

where $C(n)$ is a function that decreases to 4 when $n \to \infty$. It is believed that, perhaps, $|\pi(G)| \leq 3\sigma_e(G)$ holds.

Recently, a variation on character degrees is gaining interest: character codegrees. This concept was defined in [13] as follows. If $\chi \in \text{Irr}(G)$, the codegree of $\chi$ is

$$\chi^c(1) = \frac{|G : \text{Ker}\chi|}{\chi(1)}.$$  

The $\rho$-$\sigma$ problem for character codegrees was studied by Y. Yang and G. Qian in [14]. If

$$\sigma^c(G) = \max \{|\pi(\chi^c(1))| \mid \chi \in \text{Irr}(G)\},$$  

they proved that

$$|\pi(|G|)| \leq (K + 3)\sigma^c(G),$$  

where $K$ is an upper bound for Keller’s function $C(n)$. This answered the first part of Question A of [13]. The connection between character codegrees and element orders comes from a beautiful theorem of Isaacs [6]: if $G$ is a group and $g \in G$ then there exists $\chi \in \text{Irr}(G)$ such that $\pi(o(g)) \subseteq \pi(\chi^c(1))$. (The case of solvable groups had been first proved by Qian [12].)

Our goal in this note is threefold. First, we give a negative answer to the second part of Question A of [13], where it was asked whether the number of prime divisors of a group with $\sigma^c(G) = 2$ was at most 4.

**Theorem 1.1.** There exist solvable groups $G$ such that

1. $|G|$ is divisible by 5 different primes and $\sigma^c(G) = 2$.
2. $|G|$ is divisible by 8 different primes and $\sigma^c(G) = 3$.
3. $|G|$ is divisible by 12 different primes and $\sigma^c(G) = 4$.

Theorem 1.1 also gives a partial negative answer to a speculation in [14], where it was mentioned that it would be interesting to find the best possible constant $k$ such that $|\pi(|G|)| \leq k\sigma^c(G)$ and it was guessed by the authors that $k = 2$ or 3 (see p. 219). Theorem 1.1(3) shows that there is not any bound better than $|\pi(|G|)| \leq 3\sigma^c(G)$. The examples used to prove Theorem 1.1 are not original. They were built by Zhang (in the case of part (1)) and by Keller (parts (2) and (3)) as examples of groups with at most 2, 3 and 4 prime divisors of element orders, respectively, and many primes dividing the order of the group.

Our second goal in this paper is to remark that, as already indicated by the examples in Theorem 1.1, the connection between element orders and character codegrees in solvable groups is perhaps even stronger than expected. We propose the following question, which is the converse of the above mentioned theorem of Isaacs and Qian for solvable groups.

**Question 1.2.** Let $G$ be a solvable group and $\chi \in \text{Irr}(G)$. Does there exist $g \in G$ such that $\pi(\chi^c(1)) \subseteq \pi(o(g))$?
At first, we were convinced that there would be plenty of counterexamples. However, we have been unable to find any such example and it seems that, if they exist, they are rare. This suggests that the problem of finding sharp bounds for the $\rho$-$\sigma$ problem for character codegrees of solvable groups is very closely related to the $\rho$-$\sigma$ problem for element orders.

As has always been the case with all the variations of the $\rho$-$\sigma$ problems for arbitrary groups, the proof of Yang and Qian in [14] is divided into two parts. First, they find a bound for groups with trivial solvable radical and then they find a bound for solvable groups (which is an immediate consequence of Keller’s bound for the problem for element orders in this case). “Gluing” both parts they get the bound for arbitrary groups. They proved that $|\pi(G)| \leq 3\sigma^c(G)$ for groups with trivial solvable radical. Our third goal is to obtain a sharp bound in this case.

**Theorem 1.3.** Let $G$ be a group with trivial solvable radical. Then

$$|\pi(G)| \leq \frac{3}{2}\sigma^c(G).$$

For simple groups, we obtain a much better bound than Theorem 1.3 provides: with 3 sporadic groups as exceptions, if $G$ is simple then $|\pi(G)| \leq \sigma^c(G) + 1$, i.e., there exists an irreducible character whose codegree if divisible by all the prime divisors of $|G|$ except for, perhaps, one. It is interesting to observe that, unlike for solvable groups, character codegrees of simple groups are divisible by many more primes than element orders. In particular, we will see that any nonabelian simple group would be a counterexample to Question 1.2 if we removed the solvability hypothesis.

Using Theorem 1.3, we deduce the following improvement on the main theorem of [14].

**Corollary 1.4.** Let $G$ be a finite group. Then

$$|\pi(G)| \leq \left(K + \frac{3}{2}\right)\sigma^c(G),$$

where $K$ is an upper bound for Keller’s function.

## 2 PROOFS

We start with the proof of Theorem 1.3. We consider first the case of simple groups.

**Lemma 2.1.** Let $G$ be a nonabelian simple group. Then the following hold:

1. If $G \notin \{J_4, F_{24}^t, M\}$ there exists $\chi \in \text{Irr}(G)$ that extends to $\text{Aut}(G)$ such that such that $|\pi(\chi^c(1))| \geq |\pi(G)| - 1$.
2. If $G = J_4$ or $F_{24}^t$ then there exists $\chi \in \text{Irr}(G)$ that extends to $\text{Aut}(G)$ such that such that $|\pi(\chi^c(1))| = |\pi(G)| - 2$.
3. If $G = M$ then there exists $\chi \in \text{Irr}(G)$ (that extends to $\text{Aut}(G) = G$) such that such that $|\pi(\chi^c(1))| = |\pi(G)| - 3$.

In particular, for this character $\chi$, $|\pi(\chi^c(1))| \geq 2|\pi(G)|/3$.

**Proof.** For the sporadic groups, this can be checked in the Atlas [2]. For the groups of Lie type, consider the Steinberg character. (It was shown in [3] that it extends to $\text{Aut}(G)$.)

Assume now that $G = A_n$, with $n > 6$ ($A_5$ and $A_6$ are groups of Lie type), and consider the irreducible character $\chi \in \text{Irr}(G)$ of degree $n - 1$. If $n - 1$ is a prime power, the result is clear. Assume now that $n - 1$ is not a prime power. Let $p > 2$ be a prime divisor of $n - 1$. Then $n - 1 \geq 2(n - 1)_{p^2}$, so

$$(n/2)_p \geq (n - 1)^2_{p^2}.$$ 

This implies that $p$ divides the codegree of the irreducible character of $G$ of degree $n - 1$. Notice also that $(n - 1)_2 < |G|_2$, so 2 also divides the codegree of the irreducible character of $G$ of degree $n - 1$. Hence $\chi^c(1)$ is divisible by all the prime divisors of $|G|$ and $|\pi(\chi^c(1))| = |\pi(G)|$ in this case. \qed
Notice that these bounds are best possible, as shown by $A_5$. One can check that if $G$ is any simple group then there exists $\chi \in \text{Irr}(G)$ such that for every $g \in G$, $\pi(\chi(1)) \notin \pi(o(g))$, so any simple group would be a counterexample to Question 1.2 if we removed the solvability hypothesis.

Now, we complete the proof of Theorem 1.3. Recall that if $G$ is a finite group with trivial solvable radical then the generalized Fitting subgroup $F^*(G)$ is the direct product of the minimal normal subgroups of $G$, and $G$ is isomorphic to a subgroup of $\text{Aut}(F^*(G))$.

**Proof of Theorem 1.3.** Write $F = F^*(G) = N_1 \times \cdots \times N_t$ as a direct product of the minimal normal subgroups of $G$. Each $N_i$ is a direct product of $n_i$ copies of a nonabelian simple group $U_i$. Let $\alpha_i \in \text{Irr}(U_i)$ be the irreducible character whose existence is guaranteed by the previous lemma and let $\beta_i = \alpha_i \times \cdots \times \alpha_i \in \text{Irr}(N_i)$. Put $\delta = \beta_1 \times \cdots \times \beta_t \in \text{Irr}(F)$. We claim that $\delta$ extends to $G$.

In order to see this, notice that $G$ is isomorphic to a subgroup of the direct product $\Gamma = \text{Aut}(U_i) \wr S_{n_i}$, where $S_{n_i}$ is the symmetric group on $n_i$ letters. We will see that $\delta$ extends to $\Gamma$. Equivalently, it suffices to see that $\beta_i$ extends to $\tilde{\beta}_i = \tilde{\alpha}_i \times \cdots \times \tilde{\alpha}_i \in \text{Irr}(B_i)$, where $B_i$ is the base group of $\Gamma$. Now, the claim follows from Lemma 1.3 of [10].

Let $\chi \in \text{Irr}(G)$ be an extension of $\delta$. Since $\chi_{N_i}$ is a multiple of $\beta_i$, $N_i \notin \text{Ker} \chi$ for every $i$. This implies that $\chi$ is faithful. Hence,

$$\chi^c(1) = \frac{|G|}{\chi(1)} = \frac{|F|}{|\delta(1)|}.$$  \hspace{1cm} (2.1)

Note that

$$\pi \left( \frac{|F|}{\delta(1)} \right) = \pi \left( \frac{|U_1|}{\alpha_1(1)} \right) \cup \cdots \cup \pi \left( \frac{|U_t|}{\alpha_t(1)} \right) = \pi \left( \alpha_1^c(1) \cdots \alpha_t^c(1) \right).$$

By Lemma 2.1 $|\pi(\alpha_i^c(1))| \geq \frac{2}{3}|\pi(|U_i|)|$, so

$$\pi \left( \frac{|F|}{\delta(1)} \right) \geq \frac{2}{3}|\pi(|F|)|.$$

By (2.1),

$$|\pi(\chi^c(1))| \geq \frac{2}{3}|\pi(|G|)|.$$

Therefore

$$|\pi(|G|)| \leq \frac{3}{2}\sigma(G),$$

as desired. \hfill \Box

**Proof of Corollary 1.4.** Let $R$ be the solvable radical of $G$. By Theorem 1.3,

$$|\pi(G/R)| \leq \frac{3}{2}\sigma(G/R) \leq \frac{3}{2}\sigma(G).$$

On the other hand, by [8] and [6] (or [12]),

$$|\pi(R)| \leq K\sigma_e(R) \leq K\sigma_e(R) \leq K\sigma^e(G).$$

The result follows from the fact that $|\pi(G)| \leq |\pi(G/R)| + |\pi(R)|$. \hfill \Box

As the proof above illustrates, this “gluing” process is an obstacle to obtaining best possible bounds in any variation of the $\rho$-$\sigma$ conjecture for arbitrary groups. It would be interesting to find some new idea that avoids this argument.

Now, we sketch the proof of Theorem 1.1. The group that shows (1) appears in the example on page 43 of [15]. The groups that show (2) and (3) appear on p. 632 and p. 631 of [7], respectively. The proof is a tedious analysis of the kernels.
and degrees of all the irreducible characters of each of these groups. For this reason, we will just give some details using Zhang’s example with Keller’s notation, and leave the remaining cases to the interested reader. We prove the following.

Theorem 2.2. There exists a solvable group $G$ with $\sigma^c(G) = 2$ and $|\pi(G)| = 5$.

Proof. Let $G$ be the example on p. 43 of [15]. Write $V_1 = GF(13^4)$ and $V_2$ be the cyclic group of order 29. Let $N = R_1 \times R_2 = C_5 \times C_7$ be the cyclic subgroup of order 35 of the multiplicative group $GF(13^4)^\times$ and let $Q_1$ be a field automorphism of order 2. We have that $R_2$ and $Q_1$ act fixed point freely on $V_2$. Note that $G = Q_1 R_1 R_2 V_1 V_2$.

First, we consider a nonprincipal character $\lambda_2 \in \text{Irr}(V_2)$ and want to see that if $\chi \in \text{Irr}(G|\lambda_2)$, then $\chi^c(1)$ is divisible by at most 2 primes. Notice that $I_G(\lambda_2) = R_1 V_1 V_2$ and $\lambda_2$ extends to $I_G(\lambda_2)$. By Clifford’s correspondence and Gallagher’s theorem (Corollary 6.17 of [5]),

$$\text{Irr}(G|\lambda_2) = \left\{ (\lambda_2 \varphi)^G \mid \varphi \in \text{Irr}(I_G(\lambda_2)/V_2) \right\}.$$  

Notice also that $I_G(\lambda_2)/V_2$ is a Frobenius group with kernel $V_1$ and complement $R_1$.

If $\varphi$ is a linear character, then $V_1 \leq \text{Ker}(\lambda_2 \varphi)^G$. This implies that 13 does not divide the codegree of this character and since it induces irreducibly from $R_1 V_1 V_2$, 2 and 7 do not divide this codegree either. It follows that there are at most 2 prime divisors for the codegree of this character. If $\varphi$ is not linear, then $2 \cdot 5 \cdot 7$ divides $(\lambda_2 \varphi)^G(1)$ and again the codegree of this irreducible character is divisible by at most 2 primes.

Now, it suffices to consider the irreducible characters whose kernels contain $V_2$. Let $\lambda_1 \in \text{Irr}(V_1 V_2/V_2)$ be nonprincipal and consider $\chi \in \text{Irr}(G/V_2|\lambda_1)$. By the construction of $G$, $V_1 V_2 \leq I_G(\lambda_1) \leq Q_1 V_1 V_2$. If $I_G(\lambda_1) = V_1 V_2$ then $2 \cdot 5 \cdot 7$ divides the degree of the irreducible characters of $G$ lying over $\lambda_1$ so there are at most 2 primes dividing the codegree of such characters. Hence, we may assume that $I_G(\lambda_1) = Q_1 V_1 V_2$. In this case, 5 and 7 divide the degree of the characters of $G$ lying over $\lambda_2$ and 29 divides the size of the kernel of such characters. It follows that at most 13 and 2 divide the codegree, as desired.

Finally, it remains to consider the characters in $\text{Irr}(G/V_1 V_2)$. In this case it is immediate to see that at most 2 primes divide the codegrees.

\[ \square \]

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