A GENERAL RELATION BETWEEN REAL AND IMAGINARY PARTS OF THE MAGNETIC SUSCEPTIBILITY

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Abstract—This paper is devoted to the study and the obtaining of the general relation between the real part and the imaginary part of the magnetic susceptibility function in the Laplace domain.

This new theoretical technique is general, and can be applied to any magnetic material, that can be considered like causal and Linear time invariant (LTI).

A discussion of the causality which is extensively used in Physics has been done. To obtain the relations, some important concepts like Titchmarsh’s theorem and Cauchy’s Theorem have been reviewed, which results in the integral of a analytic function, that is formed with the magnetic susceptibility used in the Laplace domain.

The Cauchy Integral expression in the Laplace domain under certain conditions leads to a general relations between real and imaginary part of the magnetic susceptibility in the complex s-plane. These new relationships allow the validation of the magnetic susceptibility functions developed by different researchers, in the Laplace domain, not just the frequency response like the well known Kramers-Kronig relations. Under certain conditions in these new general relations, the well known K-K relations can be obtained as a particular case.

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1. INTRODUCTION

It has been widely acknowledged that causality in a linear system and
the K-K relations are equivalent in a way that they are necessary and
sufficient conditions of each other, this can be seen in [1], [2], [3].

E. C. Titchmarsh (1926) has enounced a Theorem in the frequency
complex plane, which shows the equivalence between causality and K-
K relations, using Fourier Transform [4]:

If square integrable function \( G(\omega) \) fulfills one of the four conditions
below, then it fulfills all four of them.

(i) The inverse Fourier transform \( g(t) \) of \( G(\omega) \) vanishes: \( g(t) = 0 \) if
\( t < 0 \)
(ii) \( G(u) \) is for almost all \( u \), the limit as \( v \to 0^+ \) of an analitic
function \( G(u + jv) \) that is holomorfic in the upper half plane
and square integrable over any line parallel to the real axis:
\[
\int_{-\infty}^{+\infty} |G(u + jv)|^2 \, du < C \, v > 0
\]
(iii) Re(G) and Im(G) verify the first Plemelj Formula:
\[
ReG(\omega) = \frac{1}{\pi} PV \int_{-\infty}^{+\infty} \frac{Im(G(\omega'))}{\omega' - \omega} \, d\omega'
\]
(iv) Re(G) and Im(G) verify the second Plemelj Formula:
\[
ImG(\omega) = \frac{1}{\pi} PV \int_{-\infty}^{+\infty} \frac{Re(G(\omega'))}{\omega' - \omega} \, d\omega'
\]

where a function \( f(x) \) is square integrable if \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx \) is finite.

A function \( G(\omega) \) verifying one of the conditions of the Titchmarsh
Theorem (and consequently all four of them), will be called causal
transform.

In NiZn, MnZn, Ni2Y, and NiZnCu ferrites and their composites
the causality and the numerical response have been investigated, using
K-K relations [5], where is possible to measure the real component of
the complex magnetic permeability, in order to obtain the imaginary
part of the magnetic permeability.
A General Relation Between Real and Imaginary Parts of the Magnetic Susceptibility

Usually the magnetic susceptibility of the MnZn and NiZn soft ferrites have been computed in the frequency domain [6], but another more powerful way of analysis of the magnetic susceptibility function is by mean of the Laplace transform, which has been recently obtained [7]. This Transformation allows the analysis of the system like the bounded-input, bounded-output (BIBO), and the application of the Routh-Hurwitz stability criterion.

The ferrite media under study can be considered as a linear, time invariant, isotropic and homogeneous, then the magnetization vector can be expressed by mean of a convolution product [7], [8], [9], thus:

\[ \vec{M}(t) = (\chi \ast \vec{H})(t) \]  \hspace{1cm} (1)

where: \( t \) is the time, and \( \chi \) is the magnetic susceptibility.

Remembering the Laplace transform of \( f(t) \) [10]:

\[ L[f(t)] = F(s) = \int_{0}^{\infty} e^{-st} f(t) dt \]  \hspace{1cm} (2)

Applying the Laplace transform to the magnetization vector \( \vec{M}(t) \) of eqn. (1):

\[ \vec{M}(s) = \chi(s)\vec{H}(s) \]  \hspace{1cm} (3)

where \( \chi(s) = \chi(s)' - j\chi(s)'' \) is the complex magnetic susceptibility in the Laplace domain.

Frequently the Fourier Transform is used to obtain the connection between the real and imaginary part of the magnetic susceptibility [2]. In this paper the Laplace transformation is used to obtain these relations (see Appendix A).

\[ \chi''(\omega) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\chi'(\xi)}{\xi - \omega} d\xi \]

\[ \chi'(\omega) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\chi''(\xi)}{\xi - \omega} d\xi \]  \hspace{1cm} (4)

where the integral is the well known Hilbert transform [2], [5].

2. FORMULATION

A general relation between real and imaginary part of the magnetic susceptibility in the complex \( s \)-plane will be developed in this paper. This will be obtained by mean of the same integral equation used in K-K of Appendix A, with the singularity placed on the arc as is depicted in the Figure [11].
\[ \oint_c \left( \frac{\chi(s)}{s - s_0} \right) ds = 0 \]  
(5)

where \( \chi(s) \) is the magnetic susceptibility function in the Laplace domain, that is analytic in the half right of the complex \( s \)-plane.

**Figure 1.** Integration’s path ”C” in the complex \( s \)-plane

The integral can be expressed as a sum of two terms:

\[ \oint_c \left( \frac{\chi(s)}{s - s_0} \right) ds = \oint_{Ca} \left( \frac{\chi(s)}{s - s_0} \right) ds + \oint_{Cb} \left( \frac{\chi(s)}{s - s_0} \right) ds \]  
(6)

**2.1. Integral over ”Ca”**

\[ \oint_{Ca} \left( \frac{\chi(s)}{s - s_0} \right) ds \]  
(7)

**Figure 2.** Integration’s path ”Ca” in the complex \( s \)-plane
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Assuming:

\[ s_0 = \sigma_0 + j\omega_0 \]  
(8)

\[ s = j\xi \]  
(9)

\[ s - s_0 = j\xi - (\sigma_0 + j\omega_0) = -\sigma_0 + j(\xi - \omega_0) \]  
(10)

The integral can be expressed thus:

\[ \int_{-r}^{+r} \frac{\chi(s)}{s - s_0} ds \]  
(11)

where: \( s = j\xi \)

\[ \int_{-r}^{+r} \frac{\chi(j\xi)}{-\sigma_0 + j(\xi - \omega_0)} j d\xi \]  
(12)

Resolving:

\[ j \int_{-r}^{+r} \left( \frac{\chi(j\xi)}{-\sigma_0 + j(\xi - \omega_0)} \right) \left( \frac{-\sigma_0 - j(\xi - \omega_0)}{-\sigma_0 - j(\xi - \omega_0)} \right) d\xi \]  
(13)

Then:

\[ j \int_{-r}^{+r} \left( \frac{\chi(j\xi)}{-\sigma_0 + j(\xi - \omega_0)} \right) \left( -\sigma_0 - j(\xi - \omega_0) \right) d\xi \]  
(14)

Using:

\[ \chi(\xi) = \chi'(\xi) - j\chi''(\xi) \]  
(15)

Then:

\[ \int_{-r}^{+r} \left( \frac{\chi'(\xi)}{\sigma_0^2 + (\xi - \omega_0)^2} \right) \left( \xi - \omega_0 - \sigma_0 \chi''(\xi) \right) d\xi \]  
(16)

\[ -j \int_{-r}^{+r} \left( \frac{\chi'(\xi)}{\sigma_0^2 + (\xi - \omega_0)^2} \right) \left( \sigma_0 + \chi'(\xi) (\xi - \omega_0) \right) d\xi \]  
(17)
Figure 3. Integration’s path "Cb" in the complex s-plane

2.2. Integral on the path "Cb"

\[
\int_{Cb} \left( \frac{\chi(s)}{s - s_0} \right) ds
\]  

(18)

The integral can be expressed thus:

\[
\int_{C} \left( \frac{\chi(s)}{s - s_0} \right) ds = \int_{C_1} \left( \frac{\chi(s)}{s - s_0} \right) ds + \int_{C_2} \left( \frac{\chi(s)}{s - s_0} \right) ds + \int_{C_3} \left( \frac{\chi(s)}{s - s_0} \right) ds
\]  

(19)

where \( C_2 \) is the path around the singularity. \( C_1 \) and \( C_3 \) are the rest of the arc.

Assuming:

\[
s_0 = re^{j\phi_0}
\]  

(20)

\[
s = re^{j\phi}
\]  

(21)

Then:

\[
s - s_0 = re^{j\phi} - re^{j\phi_0}
\]  

(22)

\[
ds = re^{j\phi} d\phi
\]  

(23)

The integral is:
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\[ \int_{C_1} \left( \frac{\chi(s)}{re^{j\varphi} - re^{j\varphi_0}} \right) re^{j\varphi} d\varphi - j\pi \left( \chi(s_0) \right) \]

+ \int_{C_3} \left( \frac{\chi(s)}{re^{j\varphi} - re^{j\varphi_0}} \right) re^{j\varphi} d\varphi \quad (24)

Solving:

\[ \int_{C_1} \left( \frac{\chi(s)}{1 - e^{j(\varphi_0 - \varphi)}} \right) d\varphi - j\pi \left( \chi(s_0) \right) \]

+ \int_{C_3} \left( \frac{\chi(s)}{1 - e^{j(\varphi_0 - \varphi)}} \right) d\varphi \quad (25)

Using:

\[ 1 - e^{j(\varphi_0 - \varphi)} = 1 - \cos(\varphi_0 - \varphi) - j\sin(\varphi_0 - \varphi) \quad (26) \]

\[ \frac{1}{e^{j(\varphi_0 - \varphi)}} = \frac{1 - \cos(\varphi_0 - \varphi) + j\sin(\varphi_0 - \varphi)}{(1 - \cos(\varphi_0 - \varphi))^2 + \sin^2(\varphi_0 - \varphi)} \quad (27) \]

where the real and imaginary parts:

\[ \frac{1}{e^{j(\varphi_0 - \varphi)}} = \frac{1 - \cos(\varphi_0 - \varphi)}{(1 - \cos(\varphi_0 - \varphi))^2 + \sin^2(\varphi_0 - \varphi)} \quad (28) \]

assuming:

\[ \frac{1}{e^{j(\varphi_0 - \varphi)}} = K_1(\varphi_0, \varphi) + jK_2(\varphi_0, \varphi) \quad (29) \]

\[ \int_{\varphi_0}^{\varphi_1} \left( \chi'(s) - j\chi''(s) \right) (K_1(\varphi_0, \varphi) + jK_2(\varphi_0, \varphi)) d\varphi 

- j\pi \left( \chi(s_0) \right) + j\pi \left( \chi(s_0) \right) \quad (30) \]

If the radius \( \rho \) of the path around the singularity \( s_0, \rho \to 0 \):

\[ PV \int_{0}^{\pi} \left( \chi'(s) - j\chi''(s) \right) (K_1(\varphi_0, \varphi) + jK_2(\varphi_0, \varphi)) d\varphi 

- j\pi \left( \chi(s_0) - j\chi''(s_0) \right) \quad (31) \]

With the real and imaginary parts:
\[\pi \left(-\chi''(s_0)\right) + \text{PV} \int_0^\pi \left(\chi'(s)\right)\left(K_1(\phi_0, \phi) + (\chi''(s))K_2(\phi_0, \phi)\right)d\phi \]
\[-j\pi \left(\chi'(s_0)\right) + j\text{PV} \int_0^\pi \left(\chi'(s)\right)\left(K_2(\theta, \phi) - (\chi''(s))K_1(\theta, \phi)\right)d\phi \]

Rewriting the Cauchy theorem:

\[
\oint_c \left(\chi'(s)\right)\left(s - s_0\right)ds = \oint_{c_a} \left(\chi(s)\right)\left(s - s_0\right)ds + \oint_{c_b} \left(\chi(s)\right)\left(s - s_0\right)ds = 0 + j0 \quad (33)
\]

Then the real part and the imaginary part are:

\[
-\pi \chi''(s_0) + \text{PV} \int_0^\pi \left(\chi'(s)\right)K_1(\phi_0, \phi) + \chi''(s)K_2(\phi_0, \phi)d\phi \\
+ \int_{-r}^{+r} \frac{\chi'(\xi)(\xi - \omega_0) - \sigma_0 \chi(\xi)}{\sigma_0^2 + (\xi - \omega_0)^2}d\xi = 0 \quad (34)
\]

\[
-j\pi \chi'(s_0) + j\text{PV} \int_0^\pi \left(\chi'(s)\right)K_2(\phi_0, \phi) - \chi''(s)K_1(\phi_0, \phi)d\phi \\
- j \int_{-r}^{+r} \frac{\chi'(\xi)\sigma_0 + \chi(\xi)(\xi - \omega_0)}{\sigma_0^2 + (\xi - \omega_0)^2}d\xi = 0 \quad (35)
\]

Finally the real part:

\[
\text{PV} \int_0^\pi \chi'(s)K_1(\phi_0, \phi)d\phi + \int_{-r}^{+r} \frac{\chi'(\xi)(\xi - \omega_0)}{\sigma_0^2 + (\xi - \omega_0)^2}d\xi \\
= \pi \chi''(s_0) - \text{PV} \int_0^\pi \chi''(s)K_2(\phi_0, \phi)d\phi + \int_{-r}^{+r} \frac{\sigma_0 \chi''(\xi)}{\sigma_0^2 + (\xi - \omega_0)^2}d\xi \quad (36)
\]

and the imaginary part:

\[
- \text{PV} \int_0^\pi \chi''(s)K_1(\phi_0, \phi)d\phi - \int_{-r}^{+r} \frac{\chi''(\xi)(\xi - \omega_0)}{\sigma_0^2 + (\xi - \omega_0)^2}d\xi \\
= \pi \chi'(s_0) - \text{PV} \int_0^\pi \chi'(s)K_2(\phi_0, \phi)d\phi + \int_{-r}^{+r} \frac{\chi(\xi)\sigma_0}{\sigma_0^2 + (\xi - \omega_0)^2}d\xi \quad (37)
\]

### 3. RESULTS AND DISCUSSION

#### 3.1. Solving the equations

The new general relations obtained, shows a connection between the real and imaginary part of the magnetic susceptibility. Rewriting the eqn. (36) that corresponds to the imaginary part:
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\[ PV \int_0^\pi \chi'(s)K_1(\phi_0, \phi)d\phi + \int_{-r}^{+r} \frac{\chi'(\xi)(\xi-\omega_0)}{\sigma_0^2 + (\xi-\omega_0)^2}d\xi = \pi \chi''(s_0) \]  

(38)

Neglecting the integrals of the right side, the imaginary part of the magnetic susceptibility \( \chi'' \) can be calculated, as a first step.

\[ PV \int_0^\pi \chi'(s)K_1(\phi_0, \phi)d\phi + \int_{-r}^{+r} \frac{\chi'(\xi)(\xi-\omega_0)}{\sigma_0^2 + (\xi-\omega_0)^2}d\xi = \pi \chi''(s_0) \]  

(39)

After this, an iterative method can be used with the complete eqn. (38), but it is not employed in this paper.

### 3.2. Obtaining the K-K relations from the general equations

Another interesting test can be performed from the equations (36) and (37).

\[ PV \int_0^\pi \chi'(s)K_1(\phi_0, \phi)d\phi + \int_{-r}^{+r} \frac{\chi'(\xi)(\xi-\omega_0)}{\sigma_0^2 + (\xi-\omega_0)^2}d\xi = \pi \chi''(s_0) \]  

(40)

If the radius of the arc are decresing, up to a coincidence between both path in the Figure 1, then: \( \sigma_0 = 0 \). In this situation \( \frac{\partial}{\partial \phi} = 0 \). Then the equations (40) and (41) can be written thus:

\[ \chi''(\omega) = \frac{1}{\pi} PV \int_{-r}^{+r} \frac{\chi'(\xi)}{\xi-\omega}d\xi \]  

(42)

\[ \chi'(\omega) = -\frac{1}{\pi} PV \int_{-r}^{+r} \frac{\chi'(\xi)}{\xi-\omega}d\xi \]  

If \( r \to \infty \) in order to cover the half right \( s \)-complex plane. The well known K-K equations explained in [A13] are obtained:

\[ \chi''(\omega) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\chi'(\xi)}{\xi-\omega}d\xi \]  

(43)

\[ \chi'(\omega) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\chi'(\xi)}{\xi-\omega}d\xi \]

The K-K relations are a particular case of these new general relations written in eqns. (36) and (37)
4. CONCLUSIONS

In this paper the magnetic susceptibility has been considered as a linear, time invariant, isotropic and homogeneous. The study of a new relation between the real and the imaginary part of the magnetic susceptibility function in the Laplace domain, in the $s$-complex plane have been realized.

A discussion of the causality and another concepts like Titchmarsh’s theorem and Cauchy’s Theorem have been done, considering a analytic function, that is formed with the magnetic susceptibility in the Laplace domain.

By mean of the Cauchy theorem in the Laplace domain under certain conditions leads to a general relations between real and imaginary part of the magnetic susceptibility in the complex $s$-plane.

The K-K relations can be obtained reducing these new general relations under certain assumptions, because K-K realltions are a particular case of these relations.

These new relationships allow the validation of the magnetic susceptibility functions developed by different researchers, in the Laplace domain, not just the frequency response like the well known Kramers-Kronig relations.

These new general relations could be applied to the dielectric materials as well.

APPENDIX A. K-K RELATIONS WITH LAPLACE TRANSFORMATION

Using the expresion of the magnetic susceptibility in the Laplace domain [7], the well known K-K equation can be obtained from the integral over the path “c” on the complex $s$-plane of the following function (see Figure A1):

$$\oint_c (\chi(s) s - s_0) ds = 0$$  \hspace{1cm} (A1)

where $\chi(s)$ is the magnetic susceptibility in the Laplace domain.

The function $\left(\frac{\chi(s)}{s - s_0}\right)$ on the path c and inside the domain doesn’t has any poles, and is analytic, then the Cauchy theorem can be used [10]:

$$\oint_c \left(\frac{\chi(s)}{s - s_0}\right) ds = 0$$  \hspace{1cm} (A2)
Using the path of the Figure 1, the integral of eqn. (A2) can be expressed like a sum of two terms: one term is the circumference arc (Ca) and the other term is the imaginary axe ω (Cb).

A.1. Integral in (Ca)

As the radio of the half circumference tends to infinity, then:
\[ \lim_{r \to \infty} \left( \frac{\chi(s)}{s - s_0} \right) = 0 \quad (A3) \]

Results:

\[ \int_{ca} \left( \frac{\chi(s)}{s - s_0} \right) ds = 0 \quad (A4) \]

### A.2. Integral in Cb

#### Figure A3. Integration's path "Cb" in the complex s-plane of the imaginary axis

The integral on the path Cb can be expressed in three terms, this can be observed in the Figure [A3]

\[ \oint_{Cb} \left( \frac{\chi(s)}{s - s_0} \right) ds = \int_{\rho-s_0}^{\rho+s_0} \left( \frac{\chi(s)}{s - s_0} \right) ds + \int_{\rho-s_0}^{\rho+s_0} \left( \frac{\chi(s)}{s - s_0} \right) ds + \int_{-\infty}^{\rho-s_0} \left( \frac{\chi(s)}{s - s_0} \right) ds \quad (A5) \]

If \( \rho \to 0 \):

\[ \oint_{Cb} \left( \frac{\chi(s)}{s - s_0} \right) ds = \lim_{\rho \to 0} \left( \int_{\rho-s_0}^{\rho+s_0} \left( \frac{\chi(s)}{s - s_0} \right) ds \right) + PV \int_{-\infty}^{\infty} \left( \frac{\chi(s)}{s - s_0} \right) ds \quad (A6) \]

The contribution of the first integral is \(-j\pi\chi(s_0)\) [2]:

\[ \oint_{Cb} \left( \frac{\chi(s)}{s - s_0} \right) ds = -j\pi\chi(s_0) + PV \int_{-\infty}^{\infty} \left( \frac{\chi(s)}{s - s_0} \right) ds \quad (A7) \]

Using:
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\[ \oint (c_a + c_b) \left( \frac{\chi(s)}{s - s_0} \right) ds = 0 \]  
(A8)

Considering (A7), (A4), and (A2):

\[ -j\pi \chi(s_0) + PV \int_{-\infty}^{\infty} \left( \frac{\chi(s)}{s - s_0} \right) ds = 0 \]  
(A9)

Then:

\[ PV \int_{-\infty}^{\infty} \left( \frac{\chi(s)}{s - s_0} \right) ds = j\pi \left( \chi(s_0) \right) \]  
(A10)

Splitting real and imaginary parts:

\[ PV \int_{-\infty}^{\infty} \left( \frac{\chi^\prime(x)}{x - s_0} \right) dx = \pi \left( \chi^\prime(s_0) \right) \]

\[ -jPV \int_{-\infty}^{\infty} \left( \frac{\chi^\omega(x)}{x - s_0} \right) dx = j\pi \left( \chi^\omega(s_0) \right) \]  
(A11)

Results:

\[ \chi^\omega(s_0) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \left( \frac{\chi^\prime(x)}{x - s_0} \right) dx \]

\[ \chi^\prime(s_0) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \left( \frac{\chi^\omega(x)}{x - s_0} \right) dx \]  
(A12)

If \( s_0 = j\omega, x = j\xi \), and changing the integral limits the K-K relations can be obtained:

\[ \chi^\omega(\omega) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \left( \frac{\chi^\prime(\xi)}{\xi - \omega} \right) d\xi \]

\[ \chi^\prime(\omega) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \left( \frac{\chi^\omega(\xi)}{\xi - \omega} \right) d\xi \]  
(A13)

where the integral of the eqn. [A13] is the well known Hilbert transform

**APPENDIX B. SUSCEPTIBILITY OF FERRITES**

As a reference the magnetic susceptibility function that can be used in the \( s \)-complex plane \( \chi(s) \) of the MnZn and NiZn soft ferrite can be computed with the following function [7]:

\[ \chi(s) = \frac{s^3a + s^2b + cs + d}{(s^2 + \beta_1 s + \omega_d^2)(s^2 + s \frac{2\omega_a}{1+\alpha^2} + \frac{\omega^2}{1+\alpha^2})} \]  
(B1)

where:
\[ a = \frac{(\alpha \omega \chi_0)}{1 + \alpha^2} \]
\[ b = \frac{(1 + \alpha^2) \omega^2 \chi_{d0} + \chi_s \omega^2 \chi_{s0} + \alpha \omega \chi_0 \beta_1}{1 + \alpha^2} \]
\[ c = \frac{2 \omega \beta_1 \chi_{d0} + \chi_s \omega \beta_1 + \alpha \omega \chi_0 \omega}{(1 + \alpha^2)} \]
\[ d = \frac{\omega^2 \chi_{d0} + \chi_s \omega \beta_1}{(1 + \alpha^2)} \]

(B2)

The expressions of the magnetic susceptibility \( \chi(s) \) for MnZn and NiZn ferrites have been obtained by Fano et al [7]:

\[
\chi(s) = \frac{s^3 4.3951 \cdot 10^9 + s^2 8.3019 \cdot 10^{16} + 3.5404 \cdot 10^{23} s + 3.0857 \cdot 10^{29}}{(s^2 + 9.3 \cdot 10^6 s + 6.25 \cdot 10^{12})(s^2 + 3.0086 \cdot 10^{13} + 5.9706 \cdot 10^{26})}
\]

(B3)

\[
\chi(s) = \frac{s^3 7.7202 \cdot 10^9 + s^2 8.3570 \cdot 10^{16} + 2.9710 \cdot 10^{23} s + 1.7749 \cdot 10^{29}}{(s^2 + 3.5 \cdot 10^6 s + 7.84 \cdot 10^{12})(s^2 + 1.5030 \cdot 10^{16} + 5.6481 \cdot 10^{31})}
\]

(B4)

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