Two interacting particles on the half-line

Joachim Kerner[^1] and Tobias Mühlenbruch[^2]

Department of Mathematics and Computer Science
FernUniversität in Hagen
58084 Hagen
Germany

Abstract

In the case of compact quantum graphs, many-particle models with singular two-particle interactions where introduced in [BK13a, BK13b] to provide a paradigm for further studies on many-particle quantum chaos. In this note, we discuss various aspects of such singular interactions in a two-particle system restricted to the half-line $\mathbb{R}_+$. Among others, we give a description of the spectrum of the two-particle Hamiltonian and obtain upper bounds on the number of eigenstates below the essential spectrum. We also specify conditions under which there is at most one such eigenstate. As a final result, it is shown that the ground state is non-degenerate and decays exponentially as $\sqrt{x^2 + y^2} \to \infty$.

[^1]: E-mail address: Joachim.Kerner@fernuni-hagen.de
[^2]: E-mail address: Tobias.Muehlenbruch@fernuni-hagen.de
1 Introduction

Quantum graphs have proven to be useful models in various areas of mathematical and theoretical physics. From a mathematical point of view, they combine the simplicity of a (quasi) one-dimensional system with the complexity of a graph-like structure. It is exactly this underlying complexity that turned quantum graphs into particular important models in the field of quantum chaos by showing that eigenvalue correlations in quantum graphs are generically the same as in quantum systems with chaotic classical limit [KS97, GS06]. Intuitively, the chaotic behavior stems from the scattering of the quantum particle in the vertices of the graph. Therefore, in order to provide a useful model for the investigation of many-particle quantum chaos, many-particle systems on (finite, compact) quantum graphs with localised many-particle interactions were introduced in [BK13a, BK13b]. Localised in this context means that the particles interact only in the vicinity of the vertices of the graph, i.e., one particle has to sit at a vertex whereas the other particles are close to it. The important consequence of such singular many-particle interactions is that the scattering of the particles and hence the dynamics of the system are affected in a way encoding genuine many-particle correlations.

Also, apart from the fact that the model which will be discussed in this paper originated from the field of quantum chaos, it is also an interesting model in its own right. From the point of view of applications, singular many-particle interactions on graphs were already discussed in [MP95] in order to investigate the effects of short-range many-body interactions on, e.g., the conductivity of nanoelectronic devices. From a more theoretical point of view, it is well-known that quantum many-body problems are generally hard to solve and, indeed, there are only few models which are explicitly solvable [AGHK+88]. One important model in this respect is the Lieb-Liniger model [LL63] (originally formulated for an $N$-particle system) whose Hamiltonian, in the case of two particles, is formally given by

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \alpha \delta(x - y),$$

(1.1)

where $\alpha \in \mathbb{R}$ is the interaction strength and $\delta(x)$ the Dirac delta function. Explicitly solvable for this model means that its eigenfunctions can be (if it is considered on an interval together with periodic boundary conditions) explicitly given and the spectrum can, at least in principle, be calculated. We will see below that the formal Hamiltonian of our model resembles the Hamiltonian (1.1) to some extent. The difference is that the two-particle interactions which we will consider are even more singular in the sense that they do not solely depend on the relative position of the two particles. It is important to realise that, although the Lieb-Liniger model was originally considered as being of theoretical interest only, it is nowadays recognized to describe realistic gases in one dimension very well at low temperatures [CCG+11]. Hence, in the same spirit as for the Lieb-Liniger model, we consider the model to be discussed as an interesting model for theoretical as well as for practical purposes.

This paper is organized as follows: in Section 2 we introduce our model and give a precise mathematical formulation, i.e., we establish a quadratic form which is then shown to
characterize a self-adjoint operator (the Hamiltonian of our system) uniquely. Section 3 is devoted to the spectral analysis of this operator: we will characterize its essential spectrum and prove that, in the case of mostly attractive two-particle interactions with bounded support, there always exists at least one eigenstate below the essential spectrum. We will also provide upper bounds on the number of such eigenstates. Finally, in Section 4 we will establish (pointwise) upper bounds on the ground state eigenfunction proving an exponential decay as $\sqrt{x^2 + y^2} \to \infty$.

2 The model and its Hamiltonian

The model we would like to discuss in this paper consists of two (distinguishable) interacting particles where the one-particle configuration space is given by the half-line $\mathbb{R}_+ = (0, \infty)$. The two-particle interactions shall be of singular type, i.e., the formal Hamiltonian of the system reads

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + v(x, y) [\delta(x) + \delta(y)],$$

where $v(x, y) = v(y, x)$ is some symmetric interaction potential. On a physical level we can read off from (2.1) that the two particles interact only when at least one particle is situated at origin. Furthermore, choosing $v(x, y)$ such that $\text{supp } v \subset B_\varepsilon(0) \cap \mathbb{R}_+^2$, where $B_\varepsilon(0) \subset \mathbb{R}^2$ is the open ball with radius $\varepsilon > 0$ around $0 \in \mathbb{R}^2$, we see that the particles interact only whenever one particle is situated at the origin and the other particle is contained in a small neighborhood around it.

On a mathematical level, the methods employed in [BK13a] show how the Hamiltonian (2.1) can be realised via a quadratic form on $L^2(\mathbb{R}_+^2)$. Most interestingly, this quadratic form then corresponds to a variational formulation of a boundary-value problem for the two-dimensional Laplacian

$$-\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2},$$

with coordinate dependent Robin boundary conditions. Indeed, defining $\sigma(y) := -v(0, y)$, the boundary conditions read

$$\frac{\partial \varphi}{\partial n}(0, y) + \sigma(y) \varphi(0, y) = 0, \quad \text{and}$$

$$\frac{\partial \varphi}{\partial n}(y, 0) + \sigma(y) \varphi(y, 0) = 0,$$

where $\frac{\partial}{\partial n}$ denotes the inward normal derivative on $\partial \mathbb{R}_+^2$.

Remark 2.1. We require $\sigma(y)$ to be a measurable, essentially bounded function throughout this paper.
The associated quadratic form is given by

\[
q[\varphi] = \int_{\mathbb{R}^2_+} |\nabla \varphi|^2 \, dx - \int_{\partial \mathbb{R}^2_+} \sigma(y) \, |\varphi_{bv}(y)|^2 \, dy ,
\]  

(2.4)

defined on \( \mathcal{D}_q = H^1(\mathbb{R}^2_+) \). The \( \varphi_{bv}(y) \) are the boundary values of \( \varphi \in H^1(\mathbb{R}^2_+) \) which are, according to the trace theorem for Sobolev functions \([\text{Dob05}]\), well defined. It is then readily verified that the form (2.4) is meaningful and we can establish the following result.

**Theorem 2.2.** The form (2.4) is closed and semi-bounded.

**Proof.** For the proof we note that the restrictions of all functions in \( C_0^\infty(\mathbb{R}^2) \) on \( \mathbb{R}^2_+ \) are dense in \( H^1(\mathbb{R}^2_+) \) (see, e.g., \([\text{Dob05}]\)). This allows us to transfer trace estimates for bounded (Lipschitz) domains to the unbounded case, i.e., to \( \mathbb{R}^2_+ \). More explicitly, in the proof of Theorem 3.1 in \([\text{BK13a}]\) an estimate is used which extends to our case: we have

\[
\|\varphi_{bv}\|_{L^2(\partial \mathbb{R}^2_+)}^2 \leq 4 \left( \frac{2}{\delta} \|\varphi\|_{L^2(\mathbb{R}^2_+)}^2 + \delta \|\nabla \varphi\|_{L^2(\mathbb{R}^2_+)}^2 \right) ,
\]  

(2.5)

for all \( \delta > 0 \). We hence obtain

\[
\left| \int_{\partial \mathbb{R}^2_+} \sigma(y) \, |\varphi_{bv}(y)|^2 \, dy \right| \leq 4 \|\sigma(y)\|_{\infty} \left( \frac{2}{\delta} \|\varphi\|_{L^2(\mathbb{R}^2_+)}^2 + \delta \|\nabla \varphi\|_{L^2(\mathbb{R}^2_+)}^2 \right) ,
\]  

(2.6)

which implies

\[
q[\varphi] \geq (1 - 4\delta \|\sigma(y)\|_{\infty}) \|\nabla \varphi\|_{L^2(\mathbb{R}^2_+)}^2 - \frac{8\|\sigma(y)\|_{\infty}}{\delta} \|\varphi\|_{L^2(\mathbb{R}^2_+)}^2 .
\]  

(2.7)

Hence, choosing \( \delta \) small enough, we see that the form \( q[\cdot] \) is bounded from below.

Furthermore, the estimate (2.6) also implies that the form-norm is equivalent to the \( H^1 \)-norm. Since \( H^1 \) is complete, we conclude that \( q[\cdot] \) is indeed closed.

Due to the representation theorem of quadratic forms (see, e.g., \([\text{BHE08}]\)), Theorem 2.2 implies the existence of a unique self-adjoint operator being associated to the form \( q[\cdot] \). This operator is the Hamiltonian of our system which we denote by \(-\Delta_\sigma\).

### 3 Spectral properties of \(-\Delta_\sigma\)

In this section we characterize the spectrum of \(-\Delta_\sigma\), i.e., we describe the essential as well as the discrete part of the spectrum. We will prove the existence of at least one eigenstate below the essential spectrum in the case of mostly attractive interactions (as defined in eq. (3.4)) with bounded support. Furthermore, we provide upper bounds on the number of eigenstates below the essential spectrum and specify conditions on \( \sigma(y) \) for which there at most one such eigenstate.
Proposition 3.1. For $\sigma \equiv 0$, the operator $-\Delta_\sigma$ has purely essential spectrum with
\[
\sigma_{\text{ess}}(-\Delta_\sigma) = [0, \infty) .
\] (3.1)

Proof. The proof employs Weyl’s characterization of the essential spectrum in the sense of quadratic forms [Sto01]. This means that, in order to prove that $\lambda \geq 0$ is in the essential spectrum, we construct a sequence $(\varphi_n)_{n\in \mathbb{N}} \subset H^1(\mathbb{R}^2_+)$ such that $\|\varphi_n\| = 1$, $\varphi_n \rightharpoonup 0$ and
\[
\sup \left\{ |s(\varphi_n, u) - \lambda \langle \varphi_n, u \rangle_{L^2(\mathbb{R}^2_+)}| \, \bigg| u \in H^1(\mathbb{R}^2_+), \|u\|_{H^1(\mathbb{R}^2_+)} \leq 1 \right\} \to 0 ,
\] (3.2)
as $n \to \infty$ and where $s(\cdot, \cdot)$ is the sesquilinear form associated with (2.4). To construct such a sequence we consider the rectangle $R_n := [0, L_n] \times [y_n, y_n + B_n]$ with some constants $L_n, B_n, y_n > 0$ and choose $\varphi_n$ to be the normalized ground state eigenfunction of the two-dimensional Laplacian (with Dirichlet boundary conditions) on $R_n$, extended by zero on all of $\mathbb{R}^2_+$. A direct calculation then gives
\[
s(\varphi_n, u) = \left( \frac{\pi^2}{L_n^2} + \frac{\pi^2}{B_n^2} \right) \langle \varphi_n, u \rangle_{L^2(\mathbb{R}^2_+)} .
\] (3.3)
Hence, letting $L_n \to \infty$, $\frac{\pi^2}{B_n^2} \to \lambda$ and $y_n \to \infty$ fast enough, we see that $(\varphi_n)_{n\in \mathbb{N}}$ is an appropriate sequence showing that $\sigma_{\text{ess}}(-\Delta_\sigma) = [0, \infty)$. Finally, taking into account that $-\Delta_\sigma$ is positive, the claim follows.

Remark 3.2. We note that the case $\sigma \equiv 0$ corresponds to the so called Neumann-Laplacian on $\mathbb{R}^2_+$ being self-adjoint on the domain $\mathcal{D}_N := \{ \varphi \in H^2(\mathbb{R}^2_+) \mid \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \mathbb{R}^2_+ \}$.

In a second step we prove the existence of an eigenstate at negative energy whenever the two-particle interactions are mostly attractive and have bounded support. Mostly attractive shall mean
\[
\int_0^\infty \sigma(y) \, dy > \frac{3\pi}{16}
\] (3.4)
for $\sigma(y) \in L^1(\mathbb{R}_+)$. Note that the constant on the right hand-side of (3.4) follows from a test-function argument below and is not assumed to be of particular significance. For us, it will only be important to establish the existence of an eigenstate at negative energy based on the integral $\int_0^\infty \sigma(y) \, dy$.

From a physical point of view, condition (3.4) implies that the potential $\sigma(y)$ can exhibit some oscillatory behavior while $-\Delta_\sigma$ still maintains an eigenstate at negative energy. This, on the other hand, allows to consider “more natural” interactions since, from a classical perspective, repulsive interactions are always expected if the particles are too close to each other (hardcore repulsion).

Lemma 3.3. Assume that $\text{supp } \sigma(y) \subset [0, L]$ for some $L > 0$ and $\int_0^\infty \sigma(y) \, dy > \frac{3\pi}{16}$. Then
\[
\inf \sigma_{\text{ess}}(-\Delta_\sigma) \geq 0
\] (3.5)
and there exists an eigenstate state below the essential spectrum.
Proof. We split \( \mathbb{R}_+^2 \) into two disjoint subsets, i.e., we write \( \mathbb{R}_+^2 = B_{2L}(0)|_{\mathbb{R}_+^2} \cup \Omega_- \) where \( \Omega_- := \mathbb{R}_+^2 \setminus B_{2L}(0) \). As a comparison operator we consider \( -\Delta_{2L}^N \oplus -\Delta_{\Omega_-}^N \) where the index \( N \) refers to (additional) Neumann boundary conditions along the boundary of \( B_{2L}(0)|_{\mathbb{R}_+^2} \). Note that both operators, \( -\Delta_{2L}^N \) as well as \( -\Delta_{\Omega_-}^N \), can be defined via their associated quadratic forms which are similar to (2.4). More precisely, \( -\Delta_{2L}^N \) is the unique self-adjoint operator being associated with

\[
q_1[\varphi] := \int_{B_{2L}(0)|_{\mathbb{R}_+^2}} |\nabla \varphi|^2 \, dx - \int_{\partial \mathbb{R}_+^2 \cap B_{2L}(0)} \sigma(y) |\varphi_{\nu}(y)|^2 \, dy ,
\]

defined on \( H^1(B_{2L}(0)|_{\mathbb{R}_+^2}) \). Furthermore, \( -\Delta_{\Omega_-}^N \) is the unique self-adjoint operator being associated with

\[
q_2[\varphi] := \int_{\Omega_-} |\nabla \varphi|^2 \, dx ,
\]

defined on \( H^1(\Omega_-) \).

A standard bracketing argument of operators [BHE08] then implies

\[
\inf \sigma_{ess}(-\Delta_{2L}^N \oplus -\Delta_{\Omega_-}^N) \leq \inf \sigma_{ess}(-\Delta_{\sigma}) .
\]

Since \( \sigma_{ess}(-\Delta_{2L}^N) = \emptyset \) (note that \( -\Delta_{2L}^N \) is defined on a bounded Lipschitz domain and hence has only discrete spectrum [Dob05]) we have

\[
\inf \sigma_{ess}(-\Delta_{\Omega_-}^N) \leq \inf \sigma_{ess}(-\Delta_{\sigma}) .
\]

On the other hand (using the same arguments as in the proof of Proposition 3.1) we see that \( \sigma_{ess}(-\Delta_{\Omega_-}^N) = [0, \infty) \) and hence the first claim follows.

To prove the second claim we use a test-function argument. For \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( R_0 > L \), consider the (unnormalised) function

\[
\tilde{\varphi}(r, \theta) := \begin{cases} e^{-\left(\frac{r}{R_0}-1\right)} & \text{for } r > R_0 , \\ 1 & \text{for } r \leq R_0 , 
\end{cases}
\]

where \( f(r) := e^{-\left(\frac{r}{R_0}-1\right)} \). Then

\[
q[\tilde{\varphi}] = \frac{\pi}{2} \int_{R_0}^\infty |f'(r)|^2 \, r \, dr - 2 \int_0^L \sigma(y) \, dy ,
\]

\[
= \frac{3\pi}{8} - 2 \int_0^L \sigma(y) \, dy .
\]

Hence, due to (3.4) we have \( q[\tilde{\varphi}] < 0 \). This proves the statement. \( \square \)

Using the same methods as in the proof of Lemma 3.3 and of Proposition 3.1, we can actually determine the essential spectrum of \( -\Delta_{\sigma} \) completely.
Remark 3.5. Using the Weyl sequence as constructed in the proof of Proposition 3.1 it is readily verified that, for \( \sigma(y) \leq 0 \) (i.e., repulsive two-particle interactions), the spectrum is purely essential with \( \sigma_{\text{ess}}(-\Delta_{\sigma}) = [0, \infty) \).

In a next step, assuming that \( \text{supp} \sigma(y) \subset [0, L] \), we estimate the number of eigenstates of \(-\Delta_{\sigma}\) below the essential spectrum. As in the proof of Lemma 3.3 we use a comparison argument between \(-\Delta_{\sigma}\) and a direct sum of Laplacians \(-\Delta_{\Omega_1} \oplus -\Delta_{\Omega_2}\) where \( \mathbb{R}_+ = \Omega_1 \cup \Omega_2 \) and \(\Omega_1\) is the triangle which is obtained by connecting the three points \((0,0),(L,0)\) and \((0,L)\). On an operator level, we impose Neumann boundary conditions along the line connecting \((L,0)\) and \((0,L)\) as well as on \(\partial \Omega_2\). Furthermore, Robin boundary conditions as in (2.3) with constant \(\tilde{\sigma} := \|\sigma(y)\|_{\infty}\) are imposed along the other sides of the triangle.

On a form level, the quadratic form associated with \(-\Delta_{\Omega_1}\) is given by

\[
q_3[\varphi] := \int_{\Omega_1} |\nabla \varphi|^2 \, dx - \tilde{\sigma} \int_{\partial \Omega_2 \cap \partial \Omega_1} |\varphi_{\text{bc}}(y)|^2 \, dy
\]

and it is defined on \(H^1(\Omega_1)\). Furthermore, the quadratic form associated with \(-\Delta_{\Omega_2}\) is given by

\[
q_4[\varphi] := \int_{\Omega_2} |\nabla \varphi|^2 \, dx,
\]

being defined on \(H^1(\Omega_2)\). As a consequence, the operator \(-\Delta_{\Omega_1} \oplus -\Delta_{\Omega_2}\) is (in the sense of an operator bracketing) smaller than \(-\Delta_{\sigma}\). Hence, denoting by \(N_-(A)\) the number of negative (discrete) eigenvalues of a self-adjoint operator \(A\), we obtain the following result.

Lemma 3.6. Let \(-\Delta_{\sigma}\) be given with \(\text{supp} \sigma(y) \subset [0, L]\) and corresponding \(-\Delta_{\Omega_1}\) as constructed above. Then

\[
N_-(\Delta_{\sigma}) \leq N_-(\Delta_{\Omega_1}).
\]

Proof. As mentioned before, in the sense of an operator bracketing we have \(-\Delta_{\Omega_1} \oplus -\Delta_{\Omega_2} \leq -\Delta_{\sigma}\). A direct application of the minimax principle (see, e.g., Corollary 12.3 in [Sch12]) hence implies \(N_-(\Delta_{\sigma}) \leq N_-(\Delta_{\Omega_1} \oplus -\Delta_{\Omega_2})\). The statement then follows by taking into account that \(N_-(\Delta_{\Omega_1} \oplus -\Delta_{\Omega_2}) = N_-(\Delta_{\Omega_1})\). \(\square\)

There is now an interesting way to estimate \(N_-(\Delta_{\Omega_1})\) further by reducing the two-particle (or two-dimensional) problem to a one-particle and hence one-dimensional problem. For this, let \(-\bar{\Delta}_\sigma\) denote the one-dimensional Laplacian \(-\frac{d^2}{dx^2}\) being defined on

\[
\mathcal{D}(-\bar{\Delta}_\sigma) = \{ \varphi \in H^2(0,L) \mid \varphi'(0) + \bar{\sigma} \varphi(0) = 0 \text{ and } -\varphi'(L) + \bar{\sigma} \varphi(L) = 0 \},
\]

with \(\bar{\sigma} > 0\). It is well-known that \((-\bar{\Delta}_\sigma, \mathcal{D}(-\bar{\Delta}_\sigma))\) is self-adjoint and has at most two eigenstates at negative energy (see, e.g., [BL10]). Furthermore, the ground state eigenvalue \(\varepsilon_0 := -\kappa^2\) corresponds to the solution of

\[
\kappa \tanh \left( \frac{\kappa L}{2} \right) = \bar{\sigma},
\]

\[
\sigma_{\text{ess}}(-\Delta_{\sigma}) = [0, \infty) .
\]
with $\kappa > \hat{\sigma} > 0$ \cite{BE09}.

The next step is then to extend the triangle $\Omega_1$ on which the operator $-\Delta_{\Omega_1}$ is defined in a suitable way: i.e., $\Omega_1$ is extended by reflection across the line connecting $(L, 0)$ and $(0, L)$ to a square $\bar{\Omega}_1$ of side length $L$. By construction, Robin-boundary conditions with constant $\hat{\sigma}$ are imposed along the sides of $\bar{\Omega}_1$. However, the Neumann boundary conditions which were present along line connecting $(L, 0)$ and $(0, L)$ are now implicitly implemented through requiring symmetry of the functions across this line. To repeat and to be more precise, we define the operator

$$-\hat{\Delta}_{\hat{\sigma}} \otimes 1 + 1 \otimes -\hat{\Delta}_{\hat{\sigma}},$$

(3.18)
on the Hilbert space $L^2_s(\Omega)$ where $\Omega = [0, L] \times [0, L]$ and where the index $s$ refers to the fact that only functions which are symmetric across line connecting $(L, 0)$ and $(0, L)$ are considered, i.e., $f(x, y) = f(L - y, L - x)$. By symmetry it is readily verified that the normal derivative of any (differentiable) function $f$ vanishes along the diagonal $y = L - x$.

If $\sigma(-\Delta_{\hat{\sigma}}) = \{\varepsilon_n \mid n \in \mathbb{N}_0\}$ denotes the spectrum of $-\Delta_{\hat{\sigma}}$ and $\{\varepsilon_n\}_{n \in \mathbb{N}_0}$ are the corresponding eigenvalues in increasing order and counted with multiplicity, then

$$\sigma(-\hat{\Delta}_{\hat{\sigma}} \otimes 1 + 1 \otimes -\hat{\Delta}_{\hat{\sigma}}) = \{\varepsilon_n + \varepsilon_m \mid n, m \in \mathbb{N}_0 \text{ and } n \geq m\}$$

(3.19)
and $\{\varepsilon_n + \varepsilon_m \mid n, m \in \mathbb{N}_0 \text{ and } n \geq m\}$ are the eigenvalues of $-\hat{\Delta}_{\hat{\sigma}} \otimes 1 + 1 \otimes -\hat{\Delta}_{\hat{\sigma}}$. Note that requiring $n \geq m$ is due to the imposed symmetry.

Since $\varepsilon_0 = -\kappa^2$ we obtain, assuming that $(-\hat{\Delta}_{\hat{\sigma}}, D(-\hat{\Delta}_{\hat{\sigma}}))$ has exactly one eigenstate at negative energy,

$$N_-(\Delta_{\Omega_1}) \leq \#\{n \in \mathbb{N}_0 \mid \varepsilon_n < \kappa^2\}.$$  
(3.20)

Furthermore, employing the results obtained in \cite{BL10}, we see that $(-\hat{\Delta}_{\hat{\sigma}}, D(-\hat{\Delta}_{\hat{\sigma}}))$ has only one eigenstate at negative energy if and only if

$$\hat{\sigma} \leq \frac{2}{L}.$$  
(3.21)

Hence, summarizing the conclusions from above we have established the subsequent statement.

**Theorem 3.7.** Let $-\Delta_{\hat{\sigma}}$ be given with $\text{supp} \sigma(y) \subset [0, L]$ and $\hat{\sigma} = \|\sigma(y)\|_{\infty} \leq \frac{2}{L}$. If $-\hat{\Delta}_{\hat{\sigma}}$ denotes the one-dimensional Laplacian as constructed above with eigenvalues $\{\varepsilon_n\}_{n \in \mathbb{N}_0}$, one has

$$N_-(\Delta_{\Omega_1}) \leq \#\{n \in \mathbb{N}_0 \mid \varepsilon_n < |\varepsilon_0|\}.$$  
(3.22)

As a consequence, combining equations (3.17) and (3.23) and using Theorem 3.7 we arrive at the following.
Proposition 3.8. For any fixed value \( L > 0 \) there exists \( \sigma_0 > 0 \) such that \(-\Delta_{\sigma}\) has at most one eigenstate at negative energy for potentials \( \sigma(y) \) with \( \text{supp} \sigma(y) \subset [0, L] \) and \( \|\sigma(y)\|_{\infty} \leq \sigma_0 \).

In a final result of this section we want to estimate the ground state energy of \(-\Delta_{\sigma}\) in the case where an eigenstate exists below the essential spectrum.

Lemma 3.9. Under the assumptions of Lemma 3.3, let \( E_{\sigma} \) denote the ground state energy of \(-\Delta_{\sigma}\) and \( \hat{\sigma} = \|\sigma(y)\|_{\infty} \). Then

\[
-2\hat{\sigma}^2 \leq E_{\sigma} \leq -2\hat{\sigma}^2 + 8\hat{\sigma}^2 \int_{0}^{\infty} [\hat{\sigma} - \sigma(y)] e^{-2\hat{\sigma}y} \, dy. \tag{3.24}
\]

Proof. The lower bound follows from a comparison argument. Indeed, we see that the quadratic form

\[
\tilde{q}[\varphi] := \int_{\mathbb{R}_+^2} |\nabla \varphi|^2 \, dx - \hat{\sigma} \int_{\partial \mathbb{R}_+^2} |\varphi_{by}(y)|^2 \, dy, \tag{3.25}
\]

is smaller (in the sense of a bracketing of forms) than \( q[\cdot] \) when defined on \( H^1(\mathbb{R}_+^2) \). On the other hand, the form \( \tilde{q}[\cdot] \) corresponds to the Laplacian

\[-\Delta_{\hat{\sigma}} \otimes 1 + 1 \otimes -\Delta_{\hat{\sigma}} \tag{3.26}\]

where \(-\Delta_{\hat{\sigma}}\) is the (self-adjoint) one-dimensional Laplacian \(-\frac{d^2}{dx^2}\) defined on \( D_{\hat{\sigma}} = \{ \varphi \in H^2(\mathbb{R}_+) \mid \varphi'(0) + \hat{\sigma} \varphi(0) = 0 \} \). Also, it is well-known that \(-\Delta_{\hat{\sigma}}\) has exactly one eigenstate of negative energy at \(-\hat{\sigma}^2\) [BL10]. Hence the lower bound follows.

To obtain the upper bound we use the (normalised) test-function \( \varphi(x, y) = 2\hat{\sigma} e^{-\hat{\sigma}(x+y)} \).

We calculate

\[
q[\varphi] = 2\hat{\sigma}^2 - 8\hat{\sigma}^2 \int_{0}^{\infty} \sigma(y) e^{-2\hat{\sigma}y} \, dy
= -2\hat{\sigma}^2 + 8\hat{\sigma}^2 \int_{0}^{\infty} [\hat{\sigma} - \sigma(y)] e^{-2\hat{\sigma}y} \, dy. \tag{3.27}
\]

This proves the claim. \( \square \)

We can illustrate the result of Lemma 3.9 by the following example. We choose the step-potential

\[
\sigma(y) := \begin{cases} \sigma & \text{if } 0 \leq y \leq L \\ 0 & \text{if } y > L \end{cases} \tag{3.28}
\]

where \( \sigma > 0 \) is the depth of the potential and \( L > 0 \) the range of it. In this case, Lemma 3.9 implies

\[
-2\sigma^2 \leq E_{\sigma} \leq -2\sigma^2 + 4\sigma^2 e^{-2\sigma L}. \tag{3.29}
\]
4 Asymptotics of the ground state

In this final section we discuss certain properties of the ground state \( \varphi_0(x) \) (where \( x \in \mathbb{R}^2_+ \)) of the system, which exists according to Lemma 3.3 for mostly attractive interactions with bounded support. In particular, we want to describe its asymptotic behavior as \(|x| \to \infty|\).

Adopting standard methods available in the theory of Schrödinger operators and, in particular, the methods used in the proof of Theorem 11.8 of [LL01], we directly arrive at the following result.

**Lemma 4.1.** Under the assumptions of Lemma 3.3, let \( \varphi_0(x) \) denote the ground state of \(-\Delta_\sigma\) with eigenvalue \(-|E_\sigma|<0\). Then \( \varphi_0(x) \) is non-degenerate and can be chosen to be real-valued and strictly positive in \( \mathbb{R}^2_+ \).

As a final result we derive (pointwise) upper bounds on \( \varphi_0(x) \) that describe its asymptotic behavior as \(|x| \to \infty|\). The key ingredient is an approach based on the concept of (positive) supersolutions as described in [Agm85]. More precisely, if \( E_\sigma < 0 \) denotes the ground state energy, the supersolution in our case will be the function

\[
\begin{align*}
  u(x) = & 
  e^{-\frac{\sqrt{|E_\sigma||x|}}{|x|}} \\
  & \text{for } x \in \Omega_R := \{x \in \mathbb{R}^2_+ : |x| > R \} \text{ for some } R > 0 \text{ large enough.} 
\end{align*}
\] (4.1)

considered on \( \Omega_R \). It is called a supersolution since it fulfills \((-\Delta + |E_\sigma|)u(x) \geq 0 \) on \( \Omega_R \). In the proof we will also use the fact that the eigenfunction \( \varphi_0 \) can be extended (by reflection across the coordinate axes) to all of \( \Omega_R \) and assumed to be continuous. Indeed, for \( R \) large enough, the boundary conditions on \( \partial \mathbb{R}^2_+ \cap \Omega_R \) are Neumann boundary conditions only. Hence, standard regularity theory (see, e.g., [GT83]) implies that \( \varphi_0|_{\partial \mathbb{R}^2_+ \cap \Omega_R} \) is in \( H^1(\mathbb{R}^2_+ \cap \Omega_R) \). After having extended \( \varphi_0 \) to \( \Omega_R \) by reflection and then, using a cut-off function, to all of \( \mathbb{R}^2 \), standard Sobolev embedding theorems imply continuity (see, e.g., [HS96]).

**Theorem 4.2.** Under the assumptions of Lemma 3.3, let \( \varphi_0 \) denote the ground state of \(-\Delta_\sigma\). Then

\[
|\varphi_0(x)| \leq c e^{-\frac{\sqrt{|E_\sigma||x|}}{|x|}} , \quad \forall x \in \mathbb{R}^2_+ : |x| > R ,
\] (4.2)

where \( R, c > 0 \) are some constants and \( E_\sigma \) is the ground state energy.

**Proof.** We only sketch the proof. For more details see Theorem 2.7 and Theorem 3.2 in [Agm85]. Also, as described above, we will assume \( \varphi_0(x) \) to be extended and continuous on all of \( \Omega_R \). Choosing \( R \) large enough and defining \( R_0 := R + 1 \) we choose \( c > 0 \) such that

\[
  cu(x) - \varphi_0(x) > 0 , \quad \forall x : |x| = R_0 .
\] (4.3)

Note that (4.3) is meaningful due to continuity of both functions \( u(x) \) and \( \varphi_0(x) \) on \( \{x : |x| = R_0\} \). The key idea now is to prove that the function

\[
  u_0(x) := [\varphi_0(x) - cu(x)]_+
\] (4.4)
vanishes on all of $\Omega_{R_0}$. Note that $\cdot_+$ denotes the positive part of a function.

From continuity we conclude that there exists $\delta > 0$ such that $u_0(x) = 0$ for all $x$ s.t. $R_0 < |x| < R_0 + \delta$. Furthermore, we note that $\varphi_0(x) - cu(x)$ is a non-negative subsolution, i.e., $(-\Delta + |E_\sigma|)(\varphi_0(x) - cu(x)) \leq 0$ on $\Omega_{R_0}$. Lemma 2.9 of [Agm85] then shows that $u_0(x)$ is also a non-negative subsolution (in the weak sense). Hence,

$$
\int_{\Omega_{R_0}} \nabla u_0 \nabla (\zeta^2 u_0) \, dx + |E_\sigma| \int_{\Omega_{R_0}} \zeta^2 u_0^2 \, dx \leq 0 ,
$$

(4.5)

for any real function $\zeta \in C_0^\infty(\Omega_{R_0})$. Using the identity (see equation (2.25) in [Agm85])

$$
\nabla u_0 \nabla (\zeta^2 u_0) = |\nabla (\zeta u_0)|^2 - u_0^2 |\nabla \zeta|^2
$$

(4.6)

we obtain

$$
\int_{\Omega_{R_0}} \left( |\nabla (\zeta u_0)|^2 + |E_\sigma|^2 u_0^2 \right) \, dx \leq \int_{\Omega_{R_0}} u_0^2 |\nabla \zeta|^2 \, dx .
$$

(4.7)

Now, using (4.7) and the identity (see equation (2.6) in [Agm85])

$$
\nabla u \nabla \left( \frac{\zeta^2 u_0}{u} \right) = |\nabla (\zeta u_0)|^2 - u^2 \left| \nabla \left( \frac{\zeta u_0}{u} \right) \right|^2 ,
$$

(4.8)

we obtain

$$
\int_{\Omega_{R_0}} u^2 \left| \nabla \left( \frac{\zeta u_0}{u} \right) \right|^2 \, dx \leq \int_{\Omega_{R_0}} u_0^2 |\nabla \zeta|^2 \, dx ,
$$

(4.9)

in analogy to equation (2.28) in [Agm85].

We then choose a suitable sequence $(\chi_n)_{n \in \mathbb{N}}$ with $\chi_n(x) := \chi(\frac{x}{n})$, $\chi \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \chi \leq 1$ and $\chi(x) = 1$ for $|x| \leq 1$. Employing the Lemma of Fatou as well as the estimate (4.9) we obtain

$$
\int_{\Omega_{R_0}} u^2 \left| \nabla \left( \frac{\chi_n}{u} \right) \right|^2 \, dx \leq \liminf_{n \to \infty} \int_{\Omega_{R_0}} u_0^2 |\nabla \chi_n|^2 \, dx
$$

$$
= 0 .
$$

(4.10)

As a consequence, $u_0(x) = \lambda u(x)$ with some constant $\lambda$. Since $u > 0$ in $\Omega_{R_0}$ while $u_0$ vanishes in some strip around $R_0$ it follows that $\lambda = 0$. Thus $u_0(x) = 0$ and the theorem is proved.

\[\square\]

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