Local finiteness, distinguishing numbers and Tucker’s conjecture

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December 3, 2014

Abstract

A distinguishing colouring of a graph is a colouring of the vertex set such that no non-trivial automorphism preserves the colouring. Tucker conjectured that if every non-trivial automorphism of a locally finite graph moves infinitely many vertices, then there is a distinguishing 2-colouring.

We show that the requirement of local finiteness is necessary by giving a non-locally finite graph for which no finite number of colours suffices.

1 Introduction

A colouring of the vertices of a graph $G$ is called distinguishing if no non-trivial automorphism of $G$ preserves the colouring. This notion was first studied by Albertson and Collins [1], motivated by a recreational mathematics problem posed Rubin [7].

While a distinguishing colouring clearly exists for every graph (simply colour every vertex with a different colour), finding a distinguishing colouring with the minimum number of colours can be challenging.

For infinite graphs one of the most intriguing questions is whether or not the following conjecture of Tucker [10] is true.

Conjecture 1.1 (Tucker [10]). Let $G$ be an infinite, connected, locally finite graph with infinite motion. Then there is a distinguishing 2-colouring of $G$.

This conjecture can be viewed as a generalisation of a result on finite graphs due to Russell and Sundaram [8]. The conjecture is known to be true for many classes of infinite graphs including trees [11], tree-like graphs [2], and graphs with countable automorphism group [3]. In [9] it is shown that graphs satisfying the so-called distinct spheres condition have infinite motion as well as distinguishing number two. Examples for such graphs include leafless trees, graphs with infinite diameter and primitive automorphism group, vertex-transitive graphs of connectivity 1, and Cartesian products of graphs where at

*The author acknowledges the support of the Austrian Science Fund (FWF), project W1230-N13.
least two factors have infinite diameter. It is also known that Conjecture 1.1 is true for graphs fulfilling certain growth conditions [5]. In [6] it is shown that for locally finite graphs random colourings have a good chance of being distinguishing.

Many of the above results also hold for non-locally finite graphs which raises the question, whether the condition of local finiteness in Tucker’s conjecture can be dropped.

A first indication, that local finiteness is necessary has been given in the setting of permutation groups acting on countable sets. Here, instead of considering the automorphism group of a graph acting on the vertex set, we consider (faithful) group actions. A generalization of Conjecture 1.1 to this setting has been given by Imrich et al. [3].

Conjecture 1.2. Let $\Gamma$ be a closed, subdegree finite permutation group on a set $S$. Then there is a distinguishing 2-colouring of $S$.

For this generalization subdegree finiteness (which plays the role of local finiteness) is known to be necessary [4].

In this short note we show that local finiteness is also necessary in the graph case. More precisely we give a non-locally finite, arc transitive graph with infinite motion which does not admit a distinguishing colouring with any finite number of colours.

2 Preliminaries

Throughout this paper we will use Greek letters for group related variables while the Latin alphabet will be reserved for sets on which the group acts.

Let $S$ be a countable set and let $\Gamma$ be a group acting faithfully (i.e. the identity is the only group element which acts trivially) on $S$ from the left. The image of a point $s \in S$ under an element $\gamma \in \Gamma$ is denoted by $\gamma s$.

The stabilizer of $s$ in $\Gamma$ is defined as the subgroup $\Gamma_s = \{ \gamma \in \Gamma \mid \gamma s = s \}$ We say that $\Gamma$ is subdegree finite if for every $s \in S$ all orbits of $\Gamma_s$ are finite.

The motion of an element $\gamma \in \Gamma$ is the number (possibly infinite) of elements of $S$ which are not fixed by $\gamma$. The motion of the group $\Gamma$ is the minimal motion of a non-trivial element of $\Gamma$. Notice that the motion is not necessarily finite, in fact all groups considered in this paper have infinite motion. The motion of a graph $G$ is the motion of $\text{Aut } G$ acting on the vertex set.

Let $C$ be a (usually finite) set. A $C$-colouring of $S$ is a map $c : S \to C$. Given a colouring $c$ and $\gamma \in \Gamma$ we say that $\gamma$ preserves $c$ if $c(\gamma s) = c(s)$ for every $s \in S$. Call a colouring distinguishing if no non-trivial group element preserves the colouring.

3 The example

The construction that we use relies on the following result from [4] which also shows that there are permutation groups on a countable sets whose distinguishing number is infinite. The proof uses a standard back-and-forth argument and is included for the convenience of the reader.
Theorem 3.1 (Laflamme et al. [4]). Let $\Gamma$ be the group of all bijective, order preserving functions $\gamma: \mathbb{Q} \to \mathbb{Q}$. Then $\Gamma$ has infinite motion but no distinguishing colouring with finitely many colours.

Proof. Suppose we have a colouring of $\mathbb{Q}$ with finitely many colours. We show that this colouring is not distinguishing by constructing a non-trivial order and colour preserving function $\gamma: \mathbb{Q} \to \mathbb{Q}$.

It is not assumed that all the colours are dense but by considering an open interval $I$ we may assume that all the colours that occur on that interval are dense in the interval.

Now we use induction to construct a non-identity bijective, order and colour preserving, function $\gamma: \mathbb{Q} \to \mathbb{Q}$. First, if $x$ is not on the open interval $I$ with end points $a$ and $b$ then we set $\gamma(x) = x$. Find two enumerations $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ of the interval $I$, and take care that $p_1$ and $q_1$ are different but have the same colour. Start with setting $\gamma(p_1) = q_1$. The next step is to find the element $p_i$ with the lowest number $i$ such that $p_i$ has the same colour as $q_2$ and $p_i$ is in the same relationship to $p_1$ as $q_2$ is to $q_1$, i.e. if $q_2 > q_1$ then we want $p_i > p_1$ and if $q_2 < q_1$ then we want $p_i < p_1$. Then set $\gamma(p_i) = q_2$.

At step $n + 1$ in the induction we have already defined $\gamma$ on a set $I_n$ that contains $n$ of the $p_i$'s.

If $n + 1$ is an odd number, find the lowest number $i$ such that $p_i$ is not in $I_n$, then find the lowest number $j$ such that $q_j$ is not in $\gamma(I_n)$ and $q_j$ has the same colour as $p_i$ and with respect to the order $p_i$ and $q_j$ are in the same relationship to the points in $I_n$ and $\gamma(I_n)$, respectively. Such a vertex exists because each colour is supposed to be dense in $I$. Then set $\gamma(p_i) = q_j$.

If $n + 1$ is even then we let $j$ be the lowest number such that $q_j$ is not in $\gamma(I_n)$. Then let $i$ be the lowest number such that $p_i$ is not in $I_n$ and $p_i$ has the same colour as $q_j$ and with respect to the order $p_i$ and $q_j$ are in the same relationship to the points in $I_n$ and $\gamma(I_n)$, respectively. Then set $\gamma(p_i) = q_j$.

Proceeding in this way it is clear that in the end we will have defined $\gamma$ on the whole of $I$ and that $\gamma$ is both injective and surjective. It is also clear that at each step we have a map $I_n \to \gamma(I_n)$ that preserves the colouring and the ordering.

Thus we have defined a function $\gamma: \mathbb{Q} \to \mathbb{Q}$ that preserves the colouring and the ordering and $\gamma$ is non-trivial because $\gamma(p_1) = q_1 \neq p_1$. 

Clearly the group $\Gamma$ of the above theorem is the full automorphism of a directed graph. Simply draw an arrow from $q$ to $r$ if $q \leq r$. The underlying undirected graph is the complete countable graph which also has infinite distinguishing number but finite motion.

Theorem 3.2. There is a countable, connected, arc transitive graph with infinite motion which has no distinguishing colouring with a finite number of colours.

Proof. Let $\mathbb{Q}^+$ and $\mathbb{Q}^-$ be two disjoint copies of $\mathbb{Q}$. Denote the elements corresponding to $q \in \mathbb{Q}$ in these copies by $q^+$ and $q^−$ respectively. Consider the (undirected) graph $G = (V, E)$ where $V = \mathbb{Q}^+ \cup \mathbb{Q}^−$ and $q^+ r^- \in E$ whenever $q < r$. Figure [4] shows a small
Figure 1: An induced subgraph of the graph in Theorem 3.2. Note that edges only go from top left to bottom right. By the definition of the graph all such edges are present and every edge is of this type.

subgraph of this graph to give an idea of what it looks like. Clearly $G$ is countable and connected.

Next, note that if $\gamma$ is an order automorphism of $\mathbb{Q}$, then the maps $\gamma^\uparrow$ and $\gamma^\downarrow$ where $\gamma^\uparrow(q^+) = (\gamma(q))^+$, $\gamma^\uparrow(q^-) = (\gamma(q))^-$, $\gamma^\downarrow(q^+) = (-\gamma(q))^+$, and $\gamma^\downarrow(q^-) = (-\gamma(q))^-$ are automorphisms of $G$.

To see that $G$ is arc transitive notice that the arc $0^+1^-$ can be mapped to any arc of the form $q^+r^-$ by the automorphism $\gamma^\uparrow$ where

$$\gamma(x) = q + (r - q)x.$$ 

In this case $\gamma$ is an order automorphism of $\mathbb{Q}$ since $q^+r^- \in E$ implies that $q < r$. By analogous arguments, the arc $0^+1^-$ can be mapped to any arc of the form $q^-r^+$ by the automorphism $\gamma^\downarrow$ where

$$\gamma(x) = -q + (q - r)x.$$ 

To show that $G$ has infinite motion, we prove that the automorphisms of the form $\gamma^\uparrow$ and $\gamma^\downarrow$ as defined above are the only automorphisms of $G$.

Note that $G$ is connected and bipartite with bipartition $\mathbb{Q}^+ \cup \mathbb{Q}^-$. Hence every automorphism $\gamma$ of $G$ either fixes $\mathbb{Q}^+$ and $\mathbb{Q}^-$ set-wise, or swaps the two sets. Furthermore if $\gamma q^+ = r^+$ then $\gamma q^- = r^-$ because $q^-$ is the unique vertex with the property $N(q^-) = \bigcap_{v \sim q^+} N(v) \setminus \{q^+\}$. A similar argument shows that if $\gamma q^+ = r^-$ then $\gamma q^- = r^+$. So the action on $\mathbb{Q}^+$ uniquely determines an automorphism of $G$.

It is not hard to see that $q \leq r$ if and only if $N(q^+) \subseteq N(r^+)$. This implies that $N(\phi(q^+)) \subseteq N(\phi(r^+))$ for every automorphism $\phi$ of $G$. If $\phi$ fixes $\mathbb{Q}^+$ set-wise we conclude that $\phi$ preserves the order on $\mathbb{Q}^+$, hence it is equal to $\gamma^\uparrow$ for a suitable order automorphism $\gamma$. An analogous argument shows that if $\phi$ swaps $\mathbb{Q}^+$ and $\mathbb{Q}^-$, then $\phi = \gamma^\downarrow$ for an order automorphism $\gamma$ of $\mathbb{Q}$.
Every map of the type $\gamma_\uparrow$ and $\gamma_\downarrow$ moves infinitely many vertices, hence $G$ has infinite motion.

Finally, assume that there is a distinguishing colouring $c$ of $G$ with $n \in \mathbb{N}$ colours. In particular this colouring would break every automorphism of the form $\gamma_\uparrow$. Hence the map $q \mapsto (c(q^+), c(q^-))$ would be a distinguishing colouring of $Q$ with $n^2 < \infty$ colours, a contradiction to Theorem 3.1.

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