Entanglement criterion for pure $M \otimes N$ bipartite quantum states

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We propose a entanglement measure for pure $M \otimes N$ bipartite quantum states. We obtain the measure by generalizing the equivalent measure for a $2 \otimes 2$ system, via a $2 \otimes 3$ system, to the general bipartite case. The measure emphasizes the role Bell states have, both for forming the measure, and for experimentally measuring the entanglement. The form of the measure is similar to generalized concurrence. In the case of $2 \otimes 3$ systems, we prove that our measure, that is directly measurable, equals the concurrence. It is also shown that in order to measure the entanglement, it is sufficient to measure the projections of the state onto a maximum of $M(M - 1)N(N - 1)/2$ Bell states.

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I. INTRODUCTION

The concept of quantum entanglement is not new, it goes back to the early days of quantum theory where it was initiated by Einstein, Podolsky, and Rosen [1], and Schrödinger [2]. Many years has passed since the dawn of quantum mechanics, but we have still not been able to solve the enigma of entanglement, e.g., finding a complete mathematical model to describe and quantify this interesting feature of quantum mechanical systems, and in the same time reveal the physical implications of this feature. During recent years, separability and entanglement has been a vital research field. Peres pioneered quantification of entanglement by showing that a necessary criterion for separability was positivity of the density matrix upon partial transposition [3]. Soon thereafter, Horodeckis proved that the criterion was also sufficient [4].

Recently, several quantitative measures of entanglement for nonseparable states, such as entanglement of formation [5], distillable entanglement [6], relative entropy of entanglement [7], concurrence [8], or concurrence related measures [9, 10, 11, 12, 13] have been suggested. In particular, the definition of concurrence is based on the spin-flip operation, and Rungta et al. [10] have generalized this operator and defined I-concurrence for pure bipartite state in any dimension. For the mixed $2 \otimes M$ bipartite state we have a separability criterion given in [14] and a lower bound of concurrence of mixed such quantum states [15]. Entanglement witnesses is another method of detecting entanglement [16].

In this paper, we develop a measure for general pure $N \otimes M$ bipartite quantum states, inspired by the measure of entanglement proposed in [15,18]. It is based on bipartite phase sums and differences, and we conjecture that if it is properly normalized, it equals the I-concurrence for all pure bipartite states [10]. In Sec. [14] we briefly discuss the physical grounds for our measure for $2 \otimes 2$ systems (two qubits). In Sec. [11] we argue what a similar measure should be for a $2 \otimes 3$. In Sec. [14] we derive a measure using a relative-phase positive operator valued measure (POVM) as the entanglement quantifier. In Sec. [11] we compare our measure to existing measures. In Sec. [11] we extent the measure to encompass bipartite systems of any dimension. In Sec. [11] we apply our measure to a few sample states and discuss its normalization and the role of the Bell states. Finally, in Sec. [11] we summarize our findings. The main novelty of our paper is not the measure itself, since it is essentially proportional to I-concurrence, but rather the way the measure is derived. We have based our derivation on physical argumentation in contrast to earlier derivations that are primarily based on mathematical arguments.

II. ENTANGLEMENT BETWEEN TWO QUBITS, A BRIEF REVIEW

In an earlier paper we investigated the entanglement properties of a $2 \otimes 2$ bipartite state [19]. The starting point of that investigation was the assumption that the entanglement properties are, or can be, expressed in the bipartite state’s joint phase properties. On basis of this assumption, we found that the state’s non-local properties are found in two of the state’s off-diagonal density matrix coefficients (when the density matrix was expressed in the standard basis $|11\rangle$, $|12\rangle$, $|21\rangle$, and $|22\rangle$). Examining the reasons for this, a rather simple physical reasoning shows why this is the case. Tensor multiplying two qubit density matrices, $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$, where each factor $\hat{\rho}_i$, $i=A,B$, is of the form

$$\hat{\rho}_i = \left( \begin{array}{cc} \rho_{11} & \rho_{12} e^{i\varphi_i} \\ \rho_{12} e^{-i\varphi_i} & 1 - \rho_{11} \end{array} \right),$$

and subdividing the ensuing $4 \times 4$ matrix into four $2 \times 2$ quadrants, it is clear that the upper left quadrant does not contain any information about qubit A’s phase $\varphi_A$. The lower right quadrant lacks this information, too. Due

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to hermiticity, the upper right and the lower left quadrant contain the same information. Therefore it suffices to consider, e.g., the upper right quadrant. Of its four coefficients, its diagonal terms (that is, the coefficients 13 and 24 of the joint system density matrix) do not contain information about qubit B’s phase \( \varphi_B \). Hence, all the joint phase information is collected in the off-diagonal coefficients 14 and 23, and we found that a relevant measure of entanglement for a two qubit system was

\[
\Gamma_{\text{sup}} = \text{Sup}(2|\rho_{14} - |\rho_{23}|),
\]

where Sup refers to the supremum of the function with respect to any local unitary transformation(s).

One then notes that by a local phase rotations, it is always possible to make \( \rho_{14} \) and \( \rho_{23} \) simultaneously real. In this case

\[
\text{Re}(\rho_{14}) = |\rho_{14}| = \langle \Psi_+ | \hat{\rho} | \Psi_+ \rangle - \langle \Psi_- | \hat{\rho} | \Psi_- \rangle,
\]

or

\[
\text{Re}(\rho_{23}) = |\rho_{23}| = \langle \Phi_+ | \hat{\rho} | \Phi_+ \rangle - \langle \Phi_- | \hat{\rho} | \Phi_- \rangle,
\]

where \( |\psi_\pm\rangle = (|11\rangle \pm |22\rangle)/\sqrt{2}, |\phi_\pm\rangle = (|12\rangle \pm |21\rangle)/\sqrt{2}, \) and \( \text{Re} \) denotes the real part. That is, the entanglement is simply the maximum of the difference between the state’s projections onto the Bell states. Hence, it can be measured as the Bell-analyzer visibility \( V \).

III. AN ENTANGLEMENT MEASURE FOR A 2 \times 3 BIPARTITE STATE

Now we apply a physical reasoning similar to that in Sec. II to a bipartite 2 \times 3 system (a qubit and a qutrit). We start by subdividing the system density operator (a 6 \times 6 matrix) into four 3 \times 3 matrix quadrants. It is clear that the upper left and lower right quadrants do not contain information about the qubit phase \( \varphi_A \). The remaining two quadrants contain the same information so let us focus on, e.g., the upper right quadrant. Disregarding the diagonal coefficients in this quadrant (they contain information about \( \varphi_A \) but not about the qutrit phases), we see that the joint phase properties of the state lies in the density matrix coefficients 15, 16, 24, 26, 34, and 35. These coefficients correspond to the projectors \(|11\rangle\langle22|, |11\rangle\langle23|, |12\rangle\langle21|, |12\rangle\langle23|, |13\rangle\langle21|, \) and \(|13\rangle\langle22| \), respectively. Consider now the two possible complete Bell bases:

\[
|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle), \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |22\rangle),
\]

\[
|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |23\rangle), \quad |\Psi_4\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |23\rangle),
\]

and

\[
|\Psi_5\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |21\rangle), \quad |\Psi_6\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |21\rangle),
\]

or

\[
|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |23\rangle), \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |23\rangle),
\]

\[
|\Phi_3\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \quad |\Phi_4\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle),
\]

and

\[
|\Phi_5\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |22\rangle), \quad |\Phi_6\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |22\rangle).
\]

The two bases can be obtained from each other by permutation of any two of the qutrit states. However, such a permutation changes all six states, that is, one whole basis is transformed into the other. We note that

\[
\rho_{15} + \rho_{51} = 2\text{Re}(\rho_{15}) = \langle \Psi_1 | \hat{\rho} | \Psi_1 \rangle - \langle \Psi_2 | \hat{\rho} | \Psi_2 \rangle,
\]

and

\[
\rho_{16} + \rho_{61} = 2\text{Re}(\rho_{16}) = \langle \Phi_1 | \hat{\rho} | \Phi_1 \rangle + \langle \Phi_2 | \hat{\rho} | \Phi_2 \rangle,
\]

etc. By applying phase shifts, local to the qutrit B, it is always possible to make all three coefficients \( \rho_{15}, \rho_{26}, \) and \( \rho_{34} \) real, simultaneously. Hence, the absolute values of density matrix coefficients 15, 26, and 34 are associated with, and can be obtained from, a Bell-state analysis using the complete basis set \( \{|\Psi_i\rangle\} \), while, in a similar manner, coefficients 16, 24, and 35 are associated with the complete and noncompatible set \( \{|\Phi_i\rangle\} \). Intuitively, one would expect the entanglement to be greatest if only the states in one of the sets were excited. Moreover, the entanglement should be maximized if only one of the states in one of the sets were excited. A reasonable measure of entanglement would therefore be

\[
\Gamma = (N_2(|\rho_{15}|^2 + |\rho_{24}|^2 + |\rho_{26}|^2 - |\rho_{15}|^2 - |\rho_{24}|^2 - |\rho_{26}|^2)^{\frac{1}{2}},
\]

where, again, \( \Gamma \) is not invariant to transformations local to qubit A and qutrit B, respectively, so the state’s entanglement is understood to be given by \( \Gamma_{\text{sup}} \), the supremum of \( \Gamma \) taken over all possible local unitary transformations. \( N_2 \) is a normalization factor that, if taken to be 2, makes \( 0 \leq \Gamma \leq 1 \). Eq. (6) is our central result for 2 \times 3 systems. Due to the absolute signs, the expression is symmetric with respect to the two Bell basis sets. It is also clear that for any separable state, \( \hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B \), one gets

\[
\Gamma = 0
\]

since \( \rho_{15} = \rho_{1A2\beta_{B12}}, \rho_{24} = \rho_{1A2\beta_{B12}}, \rho_{26} = \rho_{1A2\beta_{B23}}, \rho_{35} = \rho_{1A2\beta_{B23}}, \) and \( \rho_{16} = \rho_{2A3\beta_{B13}}, \rho_{34} = \rho_{2A3\beta_{B13}} \). These relations explain why we have chosen the specific pairing between the coefficients in sets \( \{|\rho_{15}, \rho_{26}, \rho_{34}\} \) and \( \{|\rho_{16}, \rho_{24}, \rho_{35}\} \) associated with the respective Bell basis.
IV. A MATHEMATICAL DERIVATION

In order to eventually generalize the results, we need to be on a little bit firmer mathematical ground. We start by introducing the basis vectors $|1\rangle_A, |2\rangle_A$, and $|1\rangle_B, |2\rangle_B, |3\rangle_B$ for subsystems A and B. Subsequently we form the Hermitian operator

$$ \hat{\Delta}_A(\varphi_{A;12}) = \frac{1}{2\pi} (\hat{1} + e^{i\varphi_{A;12}} |1\rangle_A \langle 2|_A + \text{h.c.}), $$

where h.c. denotes the hermitian conjugate. In the same manner we define

$$ \hat{\Delta}_B(\varphi_{B;12}, \varphi_{B;13}, \varphi_{B;23}) = \frac{1}{2\pi} (\hat{1} + e^{i\varphi_{B;12}} |1\rangle_B \langle 2|_B + e^{i\varphi_{B;13}} |1\rangle_B \langle 3|_B + e^{i\varphi_{B;23}} |2\rangle_B \langle 3|_B + \text{h.c.}). $$

The bipartite system's phase properties are described by the operator

$$ \hat{\Delta}(\varphi_{A;12}, \varphi_{B;12}, \varphi_{B;13}, \varphi_{B;23}) = \hat{\Delta}_A \otimes \hat{\Delta}_B. $$

We can re-express this operator in terms of the system's sum and difference phases

$$ \varphi_{p,q}^{k,l} = \varphi_{A;kl} \pm \varphi_{B;pq}. $$

The linear dependence between $\varphi_{A;kl}$, $\varphi_{B;pq}$ and $\varphi_{p,q}^{k,l}$, $\varphi_{p,q}^{k,l}$ allows us to express the operator in Eq. (10) as a function of the sum and difference phases. Hence we can write $\hat{\Delta}(\varphi_{1,2+}, \varphi_{1,3-}, \varphi_{1,3+}, \varphi_{1,3-}, \varphi_{2,3+}, \varphi_{2,3-})$. Next we compute to what extent the density operator depends on these phase sums and differences. Since the sum and difference phase POVM is a periodic function of the phases, we can compute the Fourier components of the POVM's expectation value. We define, e.g.,

$$ \Gamma_{1,2+} = \frac{1}{2\pi} \int d\varphi_{1,2+} e^{i\varphi_{1,2+}} Tr(\hat{\rho} \hat{\Delta}(\varphi_{1,2+}, \varphi_{1,3-}, \varphi_{1,3+}, \varphi_{2,3+}, \varphi_{2,3-})$$

$$ + \hat{\Delta}(\varphi_{1,2+} + \varphi_{1,3-}, \varphi_{1,3+}, \varphi_{1,3-}, \varphi_{2,3+}, \varphi_{2,3-}) + \hat{\Delta}(\varphi_{1,2+} + \varphi_{1,3-}, \varphi_{1,3+}, \varphi_{1,3-}, \varphi_{2,3+}, \varphi_{2,3-})$$

$$ + \hat{\Delta}(\varphi_{1,2+} + \varphi_{1,3-}, \varphi_{1,3+}, \varphi_{1,3-}, \varphi_{2,3+}, \varphi_{2,3-})].$$

The addition of all the $\pi$ terms in the right-most term inside the trace operation above makes the functions $\Gamma_{p,q}^{k,l}$ above insensitive to the diagonal coefficients of the density matrix, so only the joint (separable) properties are probed by $\Gamma_{p,q}^{k,l}$.

Finally, we compute

$$ \Gamma = (2\pi \mathcal{N}_2)[||\Gamma_{1,2+}^{1,2+}||^2 + ||\Gamma_{1,3+}^{1,2+}||^2 + ||\Gamma_{1,3-}^{1,2+}||^2$$

$$ + ||\Gamma_{2,3+}^{1,2+}||^2 + ||\Gamma_{2,3-}^{1,2+}||^2]^{1/2}$$

$$ = (2\pi \mathcal{N}_2)[||\rho_{15}||^2 + ||\rho_{24}||^2 + ||\rho_{26}||^2 + ||\rho_{35}||^2 + ||\rho_{34}||^2]^{1/2}. $$

The result is identical to the one we obtained by reasoning in Eq. (6), above. The derivation may seem rather ad hoc, but follows our derivation for the similar entanglement measure for 2 $\otimes$ 2 systems. The various steps are motivated in [12].

V. A COMPARISON TO EXISTING MEASURES

In this section we compare our degree of entanglement with concurrence for pure bipartite quantum systems. There are several attempts to generalize concurrence for pure bipartite quantum systems. One of these generalizations is called I-concurrence and is defined in terms of a super operator that is a generalization of a spin 1/2 flip-operation to higher Hilbert-space dimensions. For a pure state $|\Phi\rangle$, of dimension $N \otimes M$, I-concurrence is defined [10] as

$$ C_I = \sqrt{2(1 - Tr(\rho^2_A))} = \sqrt{2(1 - Tr(\rho^2_B))}, $$

where $\rho_A = Tr_B(|\Phi\rangle \langle \Phi|)$ and $\rho_B = Tr_A(|\Phi\rangle \langle \Phi|)$. Let $|\Psi\rangle$ be a pure state for a $2 \otimes 3$ bipartite quantum systems given by

$$ |\Psi\rangle = \sum_{k=1}^{2} \sum_{l=1}^{3} \alpha_{kl} |k\rangle \otimes |l\rangle = \sum_{k=1}^{2} \sum_{l=1}^{3} \alpha_{kl} |kl\rangle $$

where $\{ |k\rangle \}$ and $\{ |l\rangle \}$ are two complete orthonormal basis vector sets spanning a $M$- and $N$-dimensional Hilbert space, respectively, and where $\sum_{k,l} \sum_{l,k} |\alpha_{kl}|^2 = 1$. For such a state, the I-concurrence $C_I$ coincides with the concurrence $C$ for the state $|12\rangle$, and it can be rather simply expressed in the state's probability amplitudes as

$$ C = (2(|\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}|^2 + |\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}|^2$$

$$ + |\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22}|^2)^{1/2}. $$

(16)

(Very similar expressions have been also derived in [11] and in [13]). This expression should be compared to our entanglement measure

$$ \Gamma_{\text{sup}} = \text{Sup} [\mathcal{N}_2(||\alpha_{11}\alpha_{22}^* - |\alpha_{12}\alpha_{21}|^2$$

$$ + |\alpha_{11}\alpha_{23}^*| + |\alpha_{13}\alpha_{21}|^2$$

$$ + |\alpha_{12}\alpha_{23}^* - |\alpha_{13}\alpha_{22}|^2])^{1/2}. $$

It is seen that for $\mathcal{N}_2 = 2$, we get $\Gamma_{\text{sup}} \leq C$. Suppose we rotate this state with local unitary operations $\hat{U}_A \otimes \hat{U}_B$ such that

$$ \hat{U}_A \otimes \hat{U}_B |\psi\rangle = \sum_{k=1}^{2} \sum_{l=1}^{3} \alpha_{kl} |kl\rangle. $$

The state's concurrence is invariant with respect of the operation. Hence, if it is always possible to make, e.g. $\alpha_{11} = \alpha_{23} = 0$, then the two expressions above coincide to yield $(2(|\alpha_{12}\alpha_{21}|^2 + |\alpha_{11}\alpha_{22}|^2 + |\alpha_{13}\alpha_{22}|^2)^{1/2}$, and such a transformation maximizes $\Gamma$. We shall now prove that this is always possible. Consider the local unitary transformations

$$ \hat{U}_A(\theta, \phi) = \begin{pmatrix} \cos \theta & i e^{i\phi} \sin \theta \\ i \sin \theta & e^{i\phi} \cos \theta \end{pmatrix}. $$

1. \footnote{Very similar expressions have been also derived in [11] and in [13].}

2. \footnote{It is seen that for $\mathcal{N}_2 = 2$, we get $\Gamma_{\text{sup}} \leq C$. Suppose we rotate this state with local unitary operations $\hat{U}_A \otimes \hat{U}_B$ such that $\hat{U}_A \otimes \hat{U}_B |\psi\rangle = \sum_{k=1}^{2} \sum_{l=1}^{3} \alpha_{kl} |kl\rangle$. The state's concurrence is invariant with respect of the operation. Hence, if it is always possible to make, e.g. $\alpha_{11} = \alpha_{23} = 0$, then the two expressions above coincide to yield $(2(|\alpha_{12}\alpha_{21}|^2 + |\alpha_{11}\alpha_{22}|^2 + |\alpha_{13}\alpha_{22}|^2)^{1/2}$, and such a transformation maximizes $\Gamma$. We shall now prove that this is always possible. Consider the local unitary transformations $\hat{U}_A(\theta, \phi) = \begin{pmatrix} \cos \theta & i e^{i\phi} \sin \theta \\ i \sin \theta & e^{i\phi} \cos \theta \end{pmatrix}$.
and

$$\hat{U}_B(\vartheta, \varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & i e^{i \varphi} \sin \vartheta \\ 0 & i \sin \vartheta & e^{i \varphi} \cos \vartheta \end{pmatrix}.$$ 

From these transformations we get

$$\alpha'_{11} = \alpha_{11} \cos \theta + i \alpha_{21} e^{i \phi} \sin \theta.$$ 

(19)

It is seen that it is always possible to make $\alpha'_{11} = 0$ by an appropriate choice of $\theta$ and $\phi$. We also get

$$\alpha'_{23} = i \sin \vartheta (i \alpha_{12} \sin \theta + \alpha_{22} e^{i \phi} \cos \theta) + e^{i \varphi} \cos \vartheta (i \alpha_{13} \sin \theta + \alpha_{32} e^{i \phi} \cos \theta).$$

(20)

For a given choice of $\theta$ and $\phi$, the expressions in the parenthesis on the right hand side of the equation above are two fixed complex numbers. Therefore, it is always simultaneously possible to assure that $\alpha'_{23} = 0$ by an appropriate choice of the parameters $\vartheta$ and $\varphi$. Hence, the proof that our measure coincides with the concurrence for pure states is completed.

From (17) above, we see that it is formally also possible to prove that the measures coincide if, e.g., $\alpha'_{11} = \alpha'_{21} = 0$, or if $\alpha_{11} = \alpha_{22} = 0$, $\alpha'_{11} = \alpha'_{22} = 0$, or $\alpha_{11} = \alpha_{13} = 0$, where we have looked at the possibilities where $\alpha'_{11} = 0$. (The other possibilities lead to equivalent conclusions.) The first case was already proven, above. In the second case, a proof outlined like the one pertaining to the first case, shown above, can be used. In the third case we can prove the assertion as follows: The pure state can be written $|\psi\rangle = |1\rangle \otimes (\alpha_{11}|1\rangle + \alpha_{12}|2\rangle + \alpha_{13}|3\rangle) + ...$. We see that in order to prove our assertions, it only makes sense to consider rotations in subspace B. However, in order to make $\alpha'_{11} = \alpha'_{12} = 0$ we need to find three new basis vectors in the subspace that are mutually orthogonal, and that makes $\alpha_{11}|1\rangle + \alpha_{12}|2\rangle + \alpha_{13}|3\rangle \rightarrow \alpha'_{11}|3\rangle$. It is quite obvious that the choice $|3\rangle' = (\alpha_{11}|1\rangle + \alpha_{13}|3\rangle)/\sqrt{N}$, where $N$ is a normalization factor, and where $|1\rangle'$ and $|2\rangle'$ can be chosen arbitrary, as long as they are orthogonal to $|3\rangle'$ and to each other, satisfies our requirement. In the fourth case we have $|\psi\rangle = (\alpha_{11}|1\rangle + \alpha_{21}|2\rangle) \otimes |1\rangle + ...$. We see that in order to get $\alpha'_{11} = \alpha'_{21} = 0$, we need to find a unitary transformation in space A rendering $\alpha_{11}|1\rangle + \alpha_{21}|2\rangle \rightarrow 0|1\rangle' + |2\rangle'$. In is obvious that if either $\alpha_{11}$ or $\alpha_{21}$ are nonzero, no such transformation can be found.

As a consequence of the conclusions above, we can deduce that in order to measure the entanglement of a pure state, one can do Bell measurements, but it is necessary to use not one, but a minimum of two different Bell bases. Suppose, e.g., that we have found a basis such that $\alpha'_{11} = \alpha'_{21} = 0$. We then need to find $|\rho_{24}\rangle$, $|\rho_{35}\rangle$, and $|\rho_{54}\rangle$ to get $\Gamma_{\text{sup}}$. The first of the density matrix coefficients can be found by projection on the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ subtracting one of the outcome probabilities from the other, taking the absolute of the difference, and dividing by two. To obtain $\rho_{35}$, we can likewise project on the states $|\Phi_5\rangle$ and $|\Phi_6\rangle$ (orthogonal to the first pair of projectors). Finally, to get $\rho_{54}$, we need to project on the states $|\Phi_5\rangle$ and $|\Phi_6\rangle$. These two states, however, are not orthogonal to the any of the previous four Bell states.

VI. GENERALIZING TO $M \otimes N$ BIPARTITE QUANTUM STATES

The derivation of the entanglement measure for a $2 \otimes 3$ system made in Sec. [14] can be extended to an arbitrary bipartite system. We label the basis vectors $|1\rangle_A, ..., |M\rangle_A$, and $|1\rangle_B, ..., |N\rangle_B$. Subsequently we form the operator

$$\hat{\Delta}_A = \frac{1}{2\pi} (\hat{I} + \sum_{k=1}^{M} \sum_{l=k+1}^{M} e^{i \phi_{A,k,l}} |k\rangle_A \langle l| A + \text{h.c.}),$$

where h.c. denotes the hermitian conjugate. In the same manner we define $\hat{\Delta}_B$. Then we form the tensor product operator

$$\hat{\Delta} = \hat{\Delta}_A \otimes \hat{\Delta}_B.$$ 

(22)

We re-express this operator in terms of the system’s sum and difference phases, given in Eq. [14] above, as

$$\hat{\Delta}(\varphi_{1,1,2}, \varphi_{1,2,-}, ..., \varphi_{1,1,0}, ..., \varphi_{N-1,N-1,1,1}, ..., \varphi_{N-1,N-1,1,1})$$

(23)

This operator depends on $M(N - 1)/2$ phase sums or differences. We define, e.g.,

$$\Gamma_{p,k,l}^{k,l} = \frac{1}{2\pi} \int_{2\pi} d\varphi_{p,k,l} e^{i \varphi_{p,k,l}}$$

(24)

$$\text{Tr}(|\Delta_{p,k,l}|^{2} + \varphi_{p,k,l} + \pi, ..., \varphi_{N-1,N-1,1,1})\text{]}.$$ 

Our final result, general to any bipartite system, is

$$\Gamma = (2\pi)$$

(25)

$$\mathcal{N}_2 \sum_{k=1}^{M} \sum_{l=k+1}^{M} \sum_{p=1}^{N} \sum_{q=p+1}^{N} \left| |\Gamma_{k,l}^{k,l}| - |\Gamma_{p,q}^{p,q}| \right|^2 1/2$$

$$= (\mathcal{N}_2 \sum_{k=1}^{M} \sum_{l=k+1}^{M} \sum_{p=1}^{N} \sum_{q=p+1}^{N} \left| |\hat{\rho}_{k,l}^{(1)} N + p, (l-1) N + q \right|^2 1/2.$$ 

(26)

This is our main result, where the entanglement $\Gamma_{\text{sup}}$ of the state $\hat{\rho}$ is understood to be the supremum of the equation (26) under all local unitary transformations. For a separable state $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$, the expression (26), above, simplifies to

$$\Gamma = (\mathcal{N}_2 \sum_{k=1}^{M} \sum_{l=k+1}^{M} \sum_{p=1}^{N} \sum_{q=p+1}^{N} \left| |\hat{\rho}_{A,kl}^{k,l} \hat{\rho}_{B,qp}^{p,q}| - |\hat{\rho}_{A,kl}^{k,l} \hat{\rho}_{B,qp}^{p,q}| \right|^2 1/2 = 0.$$ 

(27)
That is, our measure is identically zero for any separable state. Hence, the expression can be used as a separability criteria for any state, not only pure states. For a pure state where
\[ |\Psi\rangle = \sum_{k=1}^{M} \sum_{l=1}^{N} \alpha_{kl} |kl\rangle, \]
we have
\[ \Gamma = \left( \mathcal{N}_2 \sum_{k=1}^{M-1} \sum_{l=k+1}^{M} \sum_{p=1}^{N-1} \sum_{q=p+1}^{N} |\alpha_{kp}\alpha_{lq}^* - |\alpha_{kq}\alpha_{lp}^*| |^2 \right)^{1/2}. \]
We can compare this expression with the concurrence (or similar measures such as the concurrence vector) for pure \( M \otimes N \) bipartite states \( \{9, 10, 11, 12\} \)
\[ C \propto \left( \sum_{k=1}^{M-1} \sum_{l=k+1}^{M} \sum_{p=1}^{N-1} \sum_{q=p+1}^{N} |\alpha_{kp}\alpha_{lq}^* - |\alpha_{kq}\alpha_{lp}^*| |^2 \right)^{1/2}. \]
We conjecture that, properly normalized, \( \Gamma_{\text{sup}} = C \) for any state. An extensive numerical testing has always confirmed the hypothesis. We are presently working on a strict mathematical proof of the conjecture.

VII. BIPARTITE ENTANGLEMENT, THE BELL STATES, AND NORMALIZATION

From the previous section we conjecture that by measuring certain of the density matrix coefficients, a pure state’s entanglement can be quantified. All the needed coefficients can be obtained by projections on Bell states. E.g., the coefficient \( \hat{p}_{(k-1)N+p,(l-1)N+q} \) can be obtained by projection onto the states \( (|kp\rangle \pm |lq\rangle)/\sqrt{2} \). We note that for an \( M \otimes N \) bipartite system, there exist \( M(M-1)N(N-1) \) Bell states. However, not all of those states are needed to measure the bipartite state’s entanglement. In analogy with our results for \( 2 \otimes 3 \) systems, we conjecture that at most \( M(M-1)N(N-1)/2 \) of the Bell states are needed, due to the degrees of freedom local unitary transformations give us. For \( M, N \geq 2 \) this implies that more than one Bell basis is needed, in general, to measure the state’s entanglement (we can note that when the product \( MN \) is odd, complete Bell bases do not exist). At any rate, our results demonstrate that for bipartite states, the Bell states play a fundamental role in defining entanglement properties.

At first sight, the central role of the Bell states in entanglement classification may seem obvious. However, perhaps surprisingly, these states do not have the largest entanglement of the bipartite states. For a Bell state, \( \Gamma_{\text{sup}} = (\mathcal{N}_2/2)^{1/2} \). However, the upper limit for \( \Gamma_{\text{sup}} \) is reached for, e.g., the state
\[ \frac{1}{\sqrt{K}} \sum_{k=1}^{K} |kk\rangle, \]
where \( K \) is the smaller of \( M \) and \( N \). For this state we get \( \Gamma_{\text{sup}} = \mathcal{N}_2^{1/2} \). When \( M = N = K \), such states are, e.g., manifested as the eigenstates of the relative-phase operator \( \{13\} \), and they have been experimentally demonstrated \( \{20\} \). The states also have a role in quantum polarimetry \( \{21\} \), and in quantum cryptography \( \{22\} \).

In spite of this, they seem to have no direct role as quantifying states of entanglement.

The observation above raises an important question: How do we normalize \( \Gamma_{\text{sup}} \) (or equivalently, the concurrence)? If we set \( \mathcal{N}_2 = 1 \) we get \( 0 \leq \Gamma_{\text{sup}} \leq 1 \). However, a Bell state in this Hilbert space will then have \( \Gamma_{\text{sup}} = (1/2)^{1/2} < 1 \). Another possibility is to set \( \mathcal{N}_2 = 2 \) rendering \( \Gamma_{\text{sup}} = 1 \) for a Bell state and \( 0 \leq \Gamma_{\text{sup}} \leq (2)^{1/2} \) in general. Ideally, one would like to find a way to quantify entanglement so that one gets a quantitative measure that corresponds to the state’s utility in quantum information tasks. Such a general quantification hierarchy lies beyond the scope of this paper. Investigations of minimum reversible entanglement generating sets \( \{6, 23, 24\} \) have shown that even for relatively simple Hilbert spaces, such as those housing tripartite qubit systems, there exist two classes of states, W-states and GHZ-states, that cannot be transformed into the other class by local operations \( \{25, 26, 27\} \). Hence, it is not clear to us how an entanglement hierarchy should be defined.

VIII. DISCUSSION AND CONCLUSION

In conclusion we have derived a quantitative measure of entanglement for pure \( M \otimes N \) bipartite quantum states, based, essentially, on simple physical considerations. The measure is positive, bounded, invariant to local unitary operations, and it is shown that it equals zero for all separable states. Our measure is less than, or equal to generalized concurrence for pure bipartite state. We conjecture that it is actually always proportional to the I-concurrence. In contrast to the latter, our measure can be obtained by measurement in a direct fashion. To this end, one projects the state onto Bell states and maximizes the the sum of certain differences with respect to local unitary transformations. The fact only projections onto Bell states are needed suggests that these states are of particular significance for all bipartite systems.

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