Persuading Risk-Conscious Agents: A Geometric Approach

Jerry Anunrojwong
Massachusetts Institute of Technology and Chulalongkorn University, jerryanunroj@gmail.com

Krishnamurthy Iyer
Industrial and Systems Engineering, University of Minnesota, Minneapolis, MN 55455, kriyer@umn.edu

David Lingenbrink
School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853, dal299@cornell.edu

We consider a persuasion problem between a sender and a non-expected utility maximizing receiver whose utility may be nonlinear in her belief; we call such receivers risk-conscious. Such utility models arise, for example, when the receiver exhibits sensitivity to the variance of the payoff on choosing an action (e.g., uncertainty-aversion when waiting for a service). Due to this nonlinearity, the revelation principle fails and action recommendations no longer suffice for optimal persuasion. To overcome this challenge, we develop an optimization framework using the underlying geometry of the persuasion problem to pose it as a convex optimization program. We use this approach to analyze the setting of binary persuasion, where the receiver has two actions and the sender prefers one of them over the other in every state. Under a mild convexity assumption, we reduce the persuasion problem to a linear program, and establish a canonical basis for the set of signals in an optimal signaling scheme. The signals in this canonical set either reveal the state, or induce in the receiver uncertainty between two states. Finally, we apply our methods to optimally share waiting time information in a queueing system with uncertainty-averse customers.

Key words: Bayesian persuasion, information design, non-expected utility, convex optimization

1. Introduction

Given the inherent informational asymmetries in online marketplaces between the platform’s operators and its users, information design can potentially play an important role in marketplace design. Starting from the seminal papers of Rayo and Segal (2010) and Kamenica and Gentzkow (2011), the methodology of Bayesian persuasion and information design has received a lot of recent attention from the academic community. A number of papers have applied Bayesian persuasion to
various application contexts, such as engagement/misinformation in social networks (Candogan and Drakopoulos 2020), queueing systems (Lingenbrink and Iyer 2019), and online retail (Lingenbrink and Iyer 2018, Drakopoulos et al. 2018).

In most of the previous work, the standard assumption is that the agent being persuaded (the receiver), is an expected utility maximizer (EUM). Although this assumption is well-supported from a theoretical perspective via axiomatic characterizations (Savage 1954), it is empirically well-documented (Ellsberg 1961, Allais 1979, Rabin 1998, DellaVigna 2009) and conventionally accepted that human behavior is not adequately explained by the central tenets of the theory. Further, there is a long line of work in the theoretical economics literature (Kahneman and Tversky 1972, 1979, Machina 1982, Tversky and Kahneman 1992) studying the systematic biases in human behavior. Because of these shortcomings of the expected utility model, existing models of Bayesian persuasion may not satisfactorily apply to information design problems in online marketplaces and other practical settings.

Motivated by this concern, our main goal in this paper is to extend the methodology of Bayesian persuasion to settings where the receiver may not be an expected utility maximizer. Specifically, we allow for general models of the receiver’s utility under uncertainty, where the receiver’s utility is an arbitrary function of her beliefs. We call such receivers risk-conscious. The only assumption we make on the receiver’s utility is that it is continuous in her beliefs. With this assumption, we study the persuasion problem faced by a sender seeking to optimal share information about payoff-relevant uncertainty with the receiver to affect the latter’s actions.

We emphasize that the notion of risk-consciousness is different from, and much more general than, the conventional notion of risk-aversion, where utility is modeled as the expectation of a function that is concave in the payoffs. A risk-averse agent is still an expected utility maximizer, and this expected utility is necessarily linear in her belief. In particular, the level sets in the belief space of an expected utility maximizer are hyperplanes. In contrast, we allow a risk-conscious agent’s payoff to depend nonlinearly on her belief, such as the variance of her payoff distribution as is the
case when the agent is uncertainty-averse. In particular, the level sets of such an agent need not be hyperplanes (see Figure 1).

From a theoretical perspective, optimal persuasion of risk-conscious agents presents new analytical challenges. When agents are expected utility maximizers, a revelation-principle style argument is often invoked to reduce the set of possible messages the sender might send (i.e., signals) to the set of actions available to the receiver. This reduction simplifies the sender’s optimization problem substantially, and the resulting problem can be written as a linear program with one obedience constraint for each action. In contrast, with risk-conscious agents, due to the nonlinearity of the receiver’s utility, a key step of the argument fails (as we describe in Section 3), rendering the usual approach to finding the optimal signaling scheme ineffective.

Despite the analytical challenges, the setting with risk-conscious agents also yields new possibilities for persuasion. We illustrate this with the following example.

**Example 1.** Consider a persuasion setting where a sender seeks to persuade a receiver to take an action. Let there be three payoff-relevant states of the world \(\omega \in \Omega = \{0, 1, 2\}\). The sender and the receiver’s prior belief about the state is given by \(\mu^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\). For any belief \(\mu = (\mu_0, \mu_1, \mu_2)\), the receiver’s utility for taking the action equals \(\tilde{\rho}(\mu) \equiv \mu_2 + \frac{3}{4}(\mu_1 - \mu_0)^2 - \frac{2}{3}\); the receiver will take the action if and only if the utility is non-negative. It is easy to verify that \(\tilde{\rho}(\mu^*) < 0\) and hence, under the prior belief \(\mu^*\), the receiver will not take the action.

On the other hand, consider the following signaling scheme: the sender reveals a signal \(s = 0\) if \(\omega = 0\), reveals \(s = 1\) if \(\omega = 1\), and if \(\omega = 2\) reveals \(s = 0\) and \(s = 1\) with equal probability 0.5. Thus,
upon receiving the signal $s = 0$, the receiver’s belief is $\mu_0 = (\frac{1}{2}, 0, \frac{1}{2})$ and upon receiving the signal $s = 1$, the receiver’s belief is $\mu_1 = (0, \frac{1}{2}, \frac{1}{2})$. In either case, it follows that $\bar{\rho}(\mu_i) > 0$, and hence the receiver can be fully persuaded to take the action. □

In the example, the receiver is persuaded to always take an action that is not her preferred action under the prior belief; we refer to this outcome as full persuasion. This phenomenon, which arises due to the nonlinearity inherent in the receiver’s utility, cannot occur when the receiver is an expected utility maximizer.

The main contributions of our paper are two-fold: (1) we provide an optimization framework to overcome the analytical challenges in the persuasion of risk-conscious agents; and (2) we demonstrate the structural complexities of the resulting optimal persuasion mechanisms, by examining settings that impose additional regularity assumptions on the receiver’s utility.

The optimization framework we develop in Section 3 uses the underlying geometry of the sender’s persuasion problem to cast it as a convex program. The variables of this program are the joint distribution of the underlying state and the receiver’s actions, and take values in the convex hull of the set of beliefs for which a fixed action is optimal for the receiver. The optimal solution to this convex program is then decomposed to obtain the optimal persuasion mechanism. While related, our approach is different from the concavification approach (Kamenica and Gentzkow 2011), yielding a different convex program; we discuss the connection in detail in Appendix A.

Using the convex program, we then investigate the structure of the optimal persuasion mechanism in Section 4. In particular, we study a specialized setting, namely binary persuasion, in which the receiver has two actions (0 and 1), and the sender always prefers that the receiver choose action 1 over action 0. Under a mild assumption on the receiver’s utility function, we show that the convex program in fact reduces to a linear program whose solution can be efficiently computed. By analyzing this linear program, we establish a canonical basis for the set of signals in an optimal signaling scheme. In other words, we show there exists an optimal signaling scheme that always sends signals in this canonical set for any prior belief of the receiver. This canonical set of signals
consists of pure signals, which fully reveal the state to the receiver, and binary mixed signals, which induce in the receiver uncertainty between two states. Finally, we characterize the conditions under which a risk-conscious receiver can be fully persuaded to choose the sender-preferred action.

To illustrate our methodology, in Section 5 we analyze a model of a queueing system introduced in Lingenbrink and Iyer (2019), where the service provider seeks to persuade arriving customers to join an unobservable single server queue. Inspired by the literature on the psychology of waiting in queues (Maister et al. 1984), we extend their model to allow for customers who exhibit uncertainty aversion. Formally, the customers have a mean-standard deviation utility (Nikolova and Stier-Moses 2014, Cominetti and Torrico 2016, Lianeas et al. 2019), where the disutility for joining the queue is the sum of the mean waiting time and a multiple of its standard deviation. We show that our theoretical results carry over to this setting, despite the prior being endogenously determined from an equilibrium. We find that the optimal signaling scheme comprises multiple “join” signals (i.e., signals for which the customer’s optimal action is to join the queue) and a single “leave” signal. Finally, we numerically find that the scheme has a “sandwich” structure which orders the different “join” signals based on expectation of the corresponding waiting times (or, equivalently its variance). This suggests that the service provider can effectively persuade by directly communicating the expected waiting times.

In summary, our work highlights risk-consciousness as a potential explanation for the rich vocabulary of informative signals that is often observed in practice. More broadly, our work demonstrates the fruitfulness of analyzing more realistic models of human behavior. We conclude our paper in Section 6 by describing how our approach can be used to analyze two common persuasion settings, namely, (1) public persuasion, where a sender seeks to persuade a group of receivers by sending a common public signal; and (2) robust persuasion, where a receiver with an unknown type is persuaded by a sender taking a worst-case view.

1.1. Literature Review

Our work contributes to the literature on Bayesian persuasion (Rayo and Segal 2010, Kamenica and Gentzkow 2011, Bergemann and Morris 2016, 2018, Taneva 2019, Kolotilin et al. 2017, Dughmi
and Xu 2016), where a sender commits to a mechanism of sharing payoff-relevant information with a receiver in order to influence the latter’s actions. For a recent review of the literature, see Kamenica (2018). Our work particularly takes influence from Kamenica and Gentzkow (2011), who use a convex-analytic concavification approach to study the persuasion problem; we discuss the close relation to our work in Appendix A.

In the operations research literature, a number of authors have applied the methodology of Bayesian persuasion to study varied settings such as crowdsourced exploration (Papanastasiou et al. 2018), spatial resource competitions (Yang et al. 2019), engagement-misinformation trade-offs in online social networks (Candogan and Drakopoulos 2020, Candogan 2019), warning policies for disaster mitigation (Alizamir et al. 2018), throughput maximization in queues (Lingenbrink and Iyer 2019), inventory/demand signaling in retail (Lingenbrink and Iyer 2018, Drakopoulos et al. 2018), and quality of matches in matching markets (Romanyuk and Smolin 2018). Our work is inspired by this stream of work, and seeks to broaden the domain of applicability by incorporating more general utility models for the receiver.

Our other source of inspiration is the economics literature on the theory of individual preferences towards risk. Apart from the standard expected utility hypothesis, there have been a number of theoretical frameworks proposed to model preferences under uncertainty, including the widely studied prospect theory (Kahneman and Tversky 1979) and the cumulative prospect theory (Tversky and Kahneman 1992). Machina (1982, 1995) singles out the “independence axiom” in the expected utility framework which leads the utilities to be “linear in the probabilities”, and study models that relax the independence axiom. Our notion of risk-consciousness encompasses all of these non-expected utility models, as it only requires the utility to be continuous in beliefs. On the other hand, our notion does not capture some non-expected utility models, such as the maxmin expected utility with multiple priors (Gilboa and Schmeidler 1989).

The notion of risk-consciousness is related to the concept of risk measures in mathematical finance (Artzner et al. 2001, Föllmer and Schied 2011). Commonly studied risk-measures, such as...
the variance of the portfolio return (Markowitz 1952), the value-at-risk (Jorion 2006), the expected shortfall (Carlo Acerbi 2002), and the entropic value-at-risk (Ahmadi-Javid 2012) are all nonlinear functions of the distribution of the return. We note that to be a good measure of financial risk, a risk measure needs to satisfy a number of properties (e.g., coherence (Artzner et al. 2001)) that make sense in the context of portfolio management but may not be relevant for capturing aspects of human decision-making.

We end our discussion by mentioning two closely related recent works. Beauchêne et al. (2019) study a persuasion setting where a sender uses an ambiguous communication device comprising multiple ways of sending signals. The receiver is ambiguity-averse and has the maxmin expected utility (Gilboa and Schmeidler 1989) over the multiple resulting posteriors. As mentioned earlier, such maxmin-utility preferences are distinct from the class of risk-conscious receivers considered in this paper. The authors analyze optimal persuasion, and parallel to our result, note that full persuasion can be achieved through the use of ambiguous communication device. In contrast, our work demonstrates that full persuasion can be obtained from unambiguous communication, purely due to risk-conscious preferences.

Lipnowski and Mathevet (2018) consider persuasion of a receiver with similar nonlinear preferences as in this paper, but focus on the setting where the sender’s preferences are perfectly aligned with those of the receiver (for instance, the sender might be a trusted advisor). Noting the failure of the revelation principle, the authors provide a sufficient condition, namely that the receiver’s payoffs are concave in the belief for any fixed action, under which action recommendations are optimal. In our intended application, the sender is a platform or a marketplace, and hence we model the sender as an expected utility maximizer. Due to this, apart from trivial instances of our model, the sender’s preferences will not be aligned with those of the receiver.

2. Model

In the following, we present the model of Bayesian persuasion with risk-conscious agents. Our development of the model follows closely to that of the standard Bayesian persuasion setting (Kamenica and Gentzkow 2011, Kamenica 2018).
2.1. Setup

We consider a persuasion problem with one sender and one receiver. Let $X$ be a payoff-relevant random variable with support on a known set $\mathcal{X}$. We assume that neither the receiver nor the sender observes $X$. However, as we describe below, the sender has more information about $X$ than the receiver, and seeks to use this information to influence the receiver’s actions.

Formally, we assume that the distribution of $X$ depends on the state of the world $\bar{\omega}$ taking values in a finite set $\Omega$ and which is observed by the sender but not the receiver. We denote the distribution of $X$, conditional on $\bar{\omega} = \omega$, by $F_{\omega}$. The distributions $\{F_{\omega} : \omega \in \Omega\}$ are commonly known between the sender and the receiver, and both share a common prior $\mu^* \in \Delta(\Omega)$ about the state of the world $\bar{\omega}$. For each $\mu \in \Delta(\Omega)$, we let $F_{\mu}$ be the distribution of $X$ when $\bar{\omega}$ is distributed as $\mu$: we have $F_{\mu} = \sum_{\omega} \mu(\omega) F_{\omega}$. Finally, we let $X_{\mu}$ (and $X_{\omega}$) denote an independent random variable distributed as $F_{\mu}$ (resp., $F_{\omega}$).

As in the standard Bayesian persuasion setting, we assume that the receiver is Bayesian and that the sender can commit to a signaling scheme to influence receiver’s choice of an action (which is described below in detail). A signaling scheme $(S, \pi)$ consists of a signal space $S$ and a joint distribution $\pi \in \Delta(\Omega \times S)$ such that the marginal of $\pi$ over $\Omega$ equals $\mu^*$: for each $\omega \in \Omega$, $\pi(\omega, S) = \mu^*(\omega)$. Specifically, we assume that, if the realized state is $\bar{\omega} = \omega$, the sender draws a signal $\bar{s} \in S$ according to the conditional distribution $\pi(\cdot | \bar{\omega} = \omega)$, and conveys it to the receiver. For simplicity of notation, we denote a signaling scheme $(S, \pi)$ by the joint distribution $\pi$. Throughout, we assume that the sender commits to a signaling scheme prior to observing the state $\bar{\omega}$, and that the sender’s choice of the signaling scheme $\pi$ is common knowledge between the sender and the receiver.

As mentioned above, we assume that the receiver is Bayesian. Given the sender’s signaling scheme $\pi$, upon observing the signal $\bar{s} = s$, the receiver updates her belief about the state using Bayes’ rule from her prior $\mu^*$ to the posterior $\mu^*_s \in \Delta(\Omega)$. In particular, we have for all $\omega \in \Omega$,

$$
\mu^*_s(\omega) = \frac{\pi(\omega, s)}{\sum_{\omega' \in \Omega} \pi(\omega', s)},
$$

(1)
whenever the denominator on the right-hand side is positive. (We let \( \mu_s \in \Delta(\Omega) \) be arbitrary if the denominator is zero.) This implies that upon receiving the signal \( \bar{s} = s \), the receiver believes that the payoff-relevant variable \( X \) is distributed as \( F_{\mu_s} \).

### 2.2. Actions, strategy and utility

Upon observing the signal \( \bar{s} \), the receiver chooses an action \( a \in A \), where the set of actions \( A \) is assumed to be finite. Given a signaling scheme \( \pi \), the receiver’s strategy \( a(\cdot) \) specifies an action \( a(s) \) for each realization \( s \in S \) of the signal \( \bar{s} \).

We let \( v(\omega, a) \) denote the sender’s utility in state \( \bar{\omega} = \omega \) when the receiver chooses the action \( a \in A \). Furthermore, we assume that the sender is an expected utility maximizer. (One can equivalently represent the utility function \( v(\omega, a) \) as an expectation of a utility function \( \hat{v}(X, a) \) over the payoff-relevant variable \( X \) and the action \( a \), conditional on \( \bar{\omega} = \omega \); we suppress the details for brevity.)

Our point of departure from the standard persuasion framework is in the definition of the receiver’s utility. Specifically, we relax the assumption that the receiver is an expected utility maximizer; as we describe next, our setup allows for more general models of the receiver’s utility over the uncertain outcome \( X \). We refer to such receivers as being **risk-conscious**.

Formally, for any belief \( \mu \in \Delta \) of the receiver, we assume that the receiver’s utility upon taking an action \( a \in A \) is given by \( \hat{\rho}(F_\mu, a) \in \mathbb{R} \). For notational simplicity, we define the utility function \( \rho : \Delta(\Omega) \times A \rightarrow \mathbb{R} \) as \( \rho(\mu, a) \equiv \hat{\rho}(F_\mu, a) \). Given a belief \( \mu \in \Delta(\Omega) \), we assume that the receiver chooses an action \( a \in A \) that achieves the highest utility \( \rho(\mu, a) \).

Observe that a receiver is an expected utility maximizer if and only if, for each \( a \in A \), the utility function \( \rho(\mu, a) \) is linear in \( \mu \). In particular, there exists a function \( u : \mathcal{X} \times \Omega \times A \rightarrow \mathbb{R} \) such that \( \rho(\mu, a) = \mathbb{E}_{\mu}[u(X, \bar{\omega}, a)] = \sum_{\omega \in \Omega} \mu(\omega) \mathbb{E}[u(X_\omega, \omega, a)] \) for all \( \mu \in \Delta(\Omega) \) and \( a \in A \), if and only if the receiver is an expected utility maximizer with von Neumann-Morgenstern (vNM) utility function \( u \). As illustrated in Figure 1, this in turn implies that the level sets of \( \rho(\cdot, a) \) in the belief space are hyperplanes. Our setup therefore includes as a special case the standard Bayesian persuasion
framework with an expected utility maximizing receiver. However, the generality of our setting allows us to capture a much wider range of receiver behavior.

As an illustration, consider the setting of a customer deciding whether or not to wait for service in a queue. The receiver’s utility depends on her unknown waiting time $X$, and suppose the queue-operator observes some correlated feature $\omega$ (queue-length, congestion, server availability, etc). A natural risk-conscious customer model posits that the customer only waits for service if, given her beliefs, the mean of her waiting time plus a multiple of its standard deviation is below a threshold (Nikolova and Stier-Moses 2014, Cominetti and Torrico 2016, Lianeas et al. 2019). Such a behavioral model may arise from the customer’s requirement for service reliability, or from an aversion to uncertainty due to a desire to plan her day subsequent to service completion. This model can be captured in our setting by letting $A = \{\text{wait, leave}\}$ and assuming, for example, that $\rho(\mu, \text{wait}) = \tau - \left( E[X_\mu] + \beta \sqrt{\text{Var}(X_\mu)} \right)$ for some $\beta, \tau > 0$, and $\rho(\mu, \text{leave}) = 0$. It is straightforward to check that $\rho(\mu, \text{wait})$ is not linear in $\mu$. (We explore this example in more detail in Section 5.)

Throughout this paper, we make the following assumption:

**Assumption 1.** For each $a \in A$, the utility function $\rho(\cdot, a)$ is continuous.$^4$

### 2.3. Persuasion game

We are now ready to describe the sender’s persuasion problem. First, we require that for any choice of the signaling scheme $\pi$, the receiver’s strategy maximizes her utility with respect to her posterior beliefs: for each $s \in S$, we have

$$a(s) \in \arg \max_{a \in A} \rho(\mu_s, a). \quad (2)$$

We call any strategy that satisfies (2) an optimal strategy for the receiver. Given an optimal strategy $a(\cdot)$, the sender’s expected utility for choosing a signaling scheme $\pi$ is given by

$$E_\pi [v(\tilde{\omega}, a(\tilde{s}))],$$

where $E_\pi$ denotes the expectation$^5$ over $(\tilde{\omega}, \tilde{s})$ with respect to $\pi$. The sender seeks to choose a signaling scheme $\pi$ that maximizes her expected utility, assuming that the receiver responds with
an optimal strategy. (When the receiver has multiple optimal strategies, we assume that the sender chooses her most preferred one; the literature refers to this as the sender-preferred subgame-perfect equilibrium (Kamenica and Gentzkow 2011).) Thus, the sender’s persuasion problem can be posed as

$$\max_{\pi \in \Delta(\Omega \times S)} E_{\pi}[v(\omega, a(s))]$$

subject to, $$a(s) \in \arg \max_{a \in A} \rho(\mu_s, a),$$ for all $$s \in S,$$

$$\pi(\omega, S) = \mu^*(\omega),$$ for all $$\omega \in \Omega.$$ (3)

Our main goal in this paper is to find and characterize the sender’s optimal signaling scheme to the persuasion problem (3). Note that the problem as posed, is challenging, as it requires first choosing an optimal set of signals $$S$$ and then a joint distribution $$\pi$$ over $$\Omega \times S.$$ Without an explicit bound on the cardinality of set $$S,$$ the persuasion problem seems intractable. In the next section, we reframe the problem to obtain a tractable formulation.

3. Towards a tractable formulation

When the receiver is treated as an expected utility maximizer, a revelation-principle style argument is typically invoked (Bergemann and Morris 2016) to restrict attention to signaling schemes where the signal space $$S$$ satisfies $$|S| = |A|.$$ Before we discuss our approach for general risk-conscious agents, we provide a more detailed discussion of this argument, and discuss why it fails in our setting.

The revelation-principle style argument rests on the following observation: when the receiver is an expected utility maximizer, if two signals $$s_1$$ and $$s_2$$ both lead to the same optimal action $$a(s_1) = a(s_2) = a,$$ then $$a$$ is still an optimal action for the receiver if the signaling scheme reveals only that $$\bar{s} \in \{s_1, s_2\}$$ whenever it was supposed to reveal $$s_1$$ or $$s_2.$$ This property is straightforward to show using the linearity of the utility function $$\rho(\mu, a)$$ in $$\mu$$ for an expected utility maximizer.

One can then use this property to coalesce all signals that lead to the same optimal action for the receiver into a single signal. Such a coalesced signaling scheme has at most one signal per action, which after identifying the signal with the corresponding action, can be turned into an action.
recommendation. Moreover, for such a signaling scheme, the agent’s optimal strategy is obedient, i.e., it is optimal for the agent to follow the action recommendation.

However, when the receiver is risk-conscious, the preceding argument may no longer hold. This is because, when signals with the same optimal action are coalesced, it may alter the posterior of the receiver on the coalesced signal, and without linearity of \( \rho(\cdot,a) \), the receiver’s optimal action may change. (To see this, consider Example 1 in reverse: under signals \( s_1 \) and \( s_2 \), it is optimal for the receiver to take the action. However, coalescing the two signals is equivalent to providing no information, and not taking the action is uniquely optimal for the receiver under her prior belief.) Thus, it no longer suffices to consider only those signaling schemes with action recommendations.

Despite this difficulty, a version of the preceding argument, which we term coalescence, continues to hold with a risk-conscious receiver. To see this, observe that if two signals \( \bar{s} = s_1 \) and \( \bar{s} = s_2 \) lead to the same posterior \( \mu \in \Delta(\Omega) \) for the receiver, then the receiver’s posterior is still \( \mu \) if the signaling scheme reveals only that \( \bar{s} \in \{ s_1, s_2 \} \) whenever it was supposed to reveal \( s_1 \) or \( s_2 \). This coalescence property follows immediately from the fact that the receiver’s posterior beliefs are expectations and expectations are linear. Thus, using the same argument as before, the coalescence property allows us to coalesce all signals that lead to the same posterior belief of the receiver into a belief recommendation. In such a coalesced signaling scheme, we can take the signal space \( S \) to be \( \Delta(\Omega) \), the set of posteriors. Furthermore, in such a scheme, a property akin to obedience holds: if the receiver is recommended a belief \( \mu \), her posterior belief is indeed \( \mu \).

Summarizing the preceding discussion, we can write the sender’s persuasion problem (3) as

\[
\max_{\pi \in \Delta(\Omega \times \Delta(\Omega))} E_{\pi} [v(\omega, a(s))] \\
\text{subject to, } \quad a(s) \in \arg \max_{a \in A} \rho(\mu_s, a), \quad \text{for all } s \in \Delta(\Omega), \\
\pi(\omega, \Delta(\Omega)) = \mu^*(\omega), \quad \text{for each } \omega \in \Omega, \\
\mu_s = s, \quad \text{for all } s \in \Delta(\Omega).
\]

(4)

Although we have characterized the set of signals, this is still a challenging problem because of the complexity of the set \( \Delta(\Omega \times \Delta(\Omega)) \). To make further progress, we state the following lemma (Aumann...
and Maschler 1995, Kamenica and Gentzkow 2011), which gives an equivalent formulation using the notion of Bayes-plausibility.

**Lemma 1 (Aumann and Maschler (1995), Kamenica and Gentzkow (2011)).** A signaling scheme \( \pi \in \Delta(\Omega \times \Delta(\Omega)) \) satisfies the condition \( \mu_s = s \) for almost all \( s \in \Delta(\Omega) \), only if the measure \( \eta(\cdot) \equiv \pi(\Omega, \cdot) \in \Delta(\Delta(\Omega)) \) is Bayes-plausible, i.e., only if the following condition holds:

\[
E_{\eta}[\bar{s}(\omega)] = \mu^*(\omega), \quad \text{for each } \omega \in \Omega.
\]

Here \( E_{\eta} \) denotes the expectation with respect to \( \eta \). Conversely, for any Bayes-plausible measure \( \eta \), the signaling scheme defined as \( \pi(\omega, ds) = s(\omega)\eta(ds) \) satisfies \( \mu_s = s \) for all \( s \in \Delta(\Omega) \).

The preceding lemma allows us to optimize over the space of Bayes-plausible measures \( \eta \in \Delta(\Delta(\Omega)) \), which are probability measures over the set of posteriors \( \Delta(\Omega) \) with the property that the mean of the distribution of the induced posteriors equals the prior. To see this, observe that for any Bayes-plausible measure \( \eta \), the sender’s expected utility under the corresponding signaling scheme \( \pi(\omega, ds) = s(\omega)\eta(ds) \) can be written as

\[
E_{\pi}[v(\bar{\omega}, a(\bar{s}))] = E_{\pi}\left[ E_{\pi}\left[ v(\bar{\omega}, a(\bar{s})) | \bar{s} \right] \right] = E_{\pi}\left[ \sum_{\omega \in \Omega} \mu_s(\omega)v(\omega, a(\bar{s})) \right] = E_{\pi}\left[ \sum_{\omega \in \Omega} \bar{s}(\omega)v(\omega, a(\bar{s})) \right] = E_{\eta}\left[ \sum_{\omega \in \Omega} \bar{s}(\omega)v(\omega, a(\bar{s})) \right],
\]

where the third equality follows from the fact that \( \mu_s = s \) for almost all \( s \in \Delta(\Omega) \), and the final equality follows from the definition of \( \eta \). Since the sender’s expected utility can be written as a function of the receiver’s strategy and the probability measure \( \eta \), we obtain the following intermediate reformulation of the sender’s problem:

\[
\max_{\eta \in \Delta(\Delta(\Omega))} E_{\eta}\left[ \sum_{\omega \in \Omega} \bar{s}(\omega)v(\omega, a(\bar{s})) \right]
\]

subject to, \( a(s) \in \arg\max_{a \in A} \rho(s, a) \), for all \( s \in \Delta(\Omega) \),

\[
E_{\eta}[\bar{s}(\omega)] = \mu^*(\omega), \quad \text{for each } \omega \in \Omega.
\]

(5)

Electronic copy available at: https://ssrn.com/abstract=3386273
This reformulation is still challenging, as it involves optimizing over distributions over beliefs, i.e., over the set $\Delta(\Delta(\Omega))$. Our first result allows us to overcome this difficulty, by establishing that one can instead optimize over a much simpler space. To state our result, we need a definition and some notation. Define the following sets:

$$P_a \triangleq \left\{ \mu \in \Delta(\Omega) : a \in \arg \max_{a' \in A} \rho(\mu, a) \right\}, \text{ for each } a \in A.$$  

(6)

The set $P_a$ denotes the set of posterior beliefs for which the action $a \in A$ is optimal for the receiver. By the continuity of $\rho(\cdot, a)$, the set $P_a$ is closed (and hence, compact) for each $a \in A$. Moreover, we have $\cup_{a \in A} P_a = \Delta(\Omega)$ and if $\mu \in P_a \cup P_a'$, then both actions $a$ and $a'$ are optimal for the receiver with posterior $\mu$. Next, for any set $A \in \mathbb{R}^m$, let $\text{Conv}(A)$ denote the convex hull of $A$, defined as:

$$\text{Conv}(A) = \left\{ y : y = \sum_{i=1}^{j} \lambda_i x_i, \text{ for some } j \geq 1, \lambda_i \geq 0, x_i \in A \text{ for all } 1 \leq i \leq j \text{ and } \sum_{i=1}^{j} \lambda_i = 1 \right\}.$$  

In words, $\text{Conv}(A)$ is the set of all finite convex combinations of elements in $A$.

Then, we have the following lemma which states that corresponding to each Bayes-plausible measure $\eta$, there exists $\{b_a \geq 0\}_{a \in A}$ and $\{m_a \in \Delta(\Omega)\}_{a \in A}$ such that the sender’s utility under $\eta$ can be written as a bilinear function of $m_a$ and $b_a$. Thus, the lemma allows us to directly optimize over $m_a$ and $b_a$, instead of over Bayes-plausible measures $\eta$.

**Lemma 2.** For any Bayes-plausible measure $\eta \in \Delta(\Delta(\Omega))$ and optimal receiver strategy $a(\cdot)$, there exists $\{(b_a, m_a)\}_{a \in A}$, with $b_a \geq 0$ and $m_a \in \text{Conv}(P_a)$ for each $a \in A$, such that

$$\sum_{a \in A} b_a m_a = \mu^*,$$  

(7)

$$E_{\eta}[\bar{s}(\omega)v(\omega, a(\bar{s}))] = \sum_{a \in A} b_a m_a(\omega)v(\omega, a) \text{ for each } \omega \in \Omega.$$  

(8)

Conversely, for any $\{(b_a, m_a) : b_a \geq 0 \text{ and } m_a \in \text{Conv}(P_a) \text{ for each } a \in A\}$ satisfying (7), there exists a Bayes-plausible measure $\eta$ and an optimal receiver strategy $a(\cdot)$ such that (8) holds.

The full proof of Lemma 2 is given in Appendix B. Here, we provide some interpretation for the quantities $m_a$ and $b_a$. For each $a$, the quantity $b_a$ denotes the probability that the receiver plays
Figure 2  In the first figure, $\eta \in \Delta(\Delta(\Omega))$ has support in the green region. Since $\eta$ is Bayes-plausible, $E_{\eta}[\bar{s}(\omega)] = \mu^*(\omega)$ for each $\omega \in \Omega$. In the second figure, we separate $\eta$ into $\eta_0$ and $\eta_1$, where $\eta_a$ has support in $P_a$. In the third figure, we depict $m_0$ and $m_1$, where $E_{\eta_a}[\bar{s}(\omega)] = m^*_a(\omega)$ and $m_a \in \text{Conv}(P_a)$. Finally, $b_a$ is defined as $P_{\eta}(a(\bar{s}) = a)$.

action $a$ under the optimal strategy $a(\cdot)$, when the sender uses the Bayes-plausible measure $\eta$; in other words, $b_a = P_{\eta}(a(\bar{s}) = a)$. Similarly, $m_a$ denotes the distribution of the state $\bar{\omega}$, conditioned on the receiver choosing action $a$. By iterated expectation, we obtain $m_a(\omega) = E_{\eta}[\bar{s}(\omega)|a(\bar{s}) = a]$. Thus, for any $a \in A$, the quantity $m_a$ denotes the mean of all posterior beliefs the receiver holds, conditioned on her choosing action $a$. Due to this reason, we refer to $m_a$ as the mean-posterior of the receiver corresponding to action $a \in A$. Note that $m_a$ may not correspond to any actual posterior that the receiver holds about the state when she chooses action $a \in A$; in fact, the mean-posterior $m_a \in \text{Conv}(P_a)$ may not even lie in the set $P_a$. Figure 2 gives some geometric intuition for these quantities, as well as for the distributions $\eta_a$ introduced in the proof.

Using the preceding lemma, we can now reframe the sender’s persuasion problem (5) as

$$\max_{\{b_a, m_a : a \in A\}} \sum_{\omega \in \Omega} \sum_{a \in A} b_a m_a(\omega) v(\omega, a)$$

subject to,

$$\sum_{a \in A} b_a m_a = \mu^*,$$

$$m_a \in \text{Conv}(P_a), \quad b_a \in [0, 1] \quad \text{for each } a \in A. \quad (9)$$

Note that the preceding problem is extremely simple when compared to (5); the optimization is over $|A|(1 + |\Omega|)$ real variables, and the variables $m_a$ belong to a convex set in $\mathbb{R}^{\Omega}$. Although this
is not yet a convex program because of the bilinearity in the objective and the constraints, we can convert it into one by letting 
\[ t_a(\omega) = b_a m_a(\omega), \]
and noticing that \( b_a \in [0, 1] \) and \( m_a \in \text{Conv}(\mathcal{P}_a) \) if and only if \( t_a \in \text{Conv}(\mathcal{P}_a \cup \{0\}) \) where \( 0 \in \mathbb{R}^{[\Omega]} \) is the vector of all zeros. Using these expressions, we obtain our main theorem:

**Theorem 1.** The sender’s persuasion problem (3) can be optimized by solving the following convex optimization problem:

\[
\begin{align*}
\max_{\{t_a; a \in A\}} & \quad \sum_{\omega \in \Omega} \sum_{a \in A} t_a(\omega)v(\omega, a) \\
\text{subject to} & \quad \sum_{a \in A} t_a(\omega) = \mu^*(\omega), \quad \text{for each } \omega \in \Omega, \\
& \quad t_a \in \text{Conv}(\mathcal{P}_a \cup \{0\}), \quad \text{for each } a \in A.
\end{align*}
\]

**Proof.** Note that for any Bayes-plausible \( \eta \in \Delta(\Delta(\Omega)) \), Lemma 2 guarantees a corresponding \( t_a = b_a m_a \) that is feasible for (10), and for which the objective equals the sender’s utility. Conversely, any feasible \( t_a \in \text{Conv}(\mathcal{P}_a \cup \{0\}) \) can be written as \( t_a = b_a m_a + (1 - b_a)0 \) for some \( b_a \in [0, 1] \) and \( m_a \in \text{Conv}(\mathcal{P}_a) \). Then, from Lemma 2, we obtain a corresponding Bayes-plausible \( \eta \) and an optimal strategy for the receiver \( a(s) \) for which the sender’s utility equals the objective of (10). □

Recall that the variable \( m_a \) denotes the mean posterior of the receiver conditional on her choosing action \( a \). Furthermore, \( b_a \) denotes the probability that the receiver chooses action \( a \). Together this implies that \( t_a(\omega) = b_a m_a(\omega) \) denotes the joint probability that the receiver takes action \( a \) and the realized state is \( \bar{\omega} = \omega \). Thus, the reformulated problem (10) directly optimizes the joint probability distribution of the state and the receiver’s actions.

Next, we describe how to get an optimal signaling scheme \( \pi \) from the optimal solution \( t_a \) to the problem (10). As in the proof of Theorem 1, for each \( a \in A \), let \( m_a \in \text{Conv}(\mathcal{P}_a) \) be such that \( t_a = b_a m_a \) for some \( b_a \in [0, 1] \). (Note that for \( t_a \neq 0 \), the corresponding \( m_a \) is uniquely defined.) Since \( m_a \in \text{Conv}(\mathcal{P}_a) \), there exists a convex decomposition of \( m_a \) in terms of the elements of \( \mathcal{P}_a \). That is, there exists \( \{\mu_i^a : i = 1, \cdots, j_a\} \) and \( \{\lambda_i^0 : i = 1, \cdots, j_a\} \) for some \( j_a \geq 1 \), such that \( \mu_i^a \in \mathcal{P}_a \), \( \lambda_i^0 \geq 0 \), \( \sum_{a \in A} \lambda_i^0 = 1 \), and \( m_a = \sum_{i=1}^{j_a} \lambda_i^0 \mu_i^a \). An optimal signaling scheme \( \pi \) is then given by the (discrete)
distribution over $\Delta(\Omega \times \Delta(\Omega))$ that chooses $(\bar{\omega}, \bar{s}) = (\omega, \mu^a_i)$ with probability $b_a \lambda^a_i \mu^a_i(\omega)$. Observe that conditional on $\bar{\omega} = \omega$, the optimal signaling scheme $\pi$ makes the belief recommendation $\mu^a_i$ to the receiver with probability

$$\pi(\bar{s} = \mu^a_i | \bar{\omega} = \omega) = \frac{b_a \lambda^a_i \mu^a_i(\omega)}{\sum_{a' \in A} b_{a'} \lambda^{a'} \mu^{a'}(\omega)}.$$  \hfill (11)

We note that in general, the representation of $m_a$ as a convex combination of $\mu^a_i \in \mathcal{P}_a$ need not be unique. Since the preceding construction works for any convex decomposition of $m_a$, we conclude that there may exist multiple optimal signaling schemes for the receiver.

The preceding characterization also allows us to bound the size of the set of signals the sender needs to use to optimally persuade the receiver. For any set $A \in \mathbb{R}^{|\Omega|}$, let $\text{Cara}(A)$ denote the minimum value of $j$ such that any point $x \in \text{Conv}(A)$ can be written as a convex combination of at most $j$ points in $A$. Using Caratheodory’s theorem (Bárány and Onn 1995), we obtain the following bound:

**Proposition 1.** There exists an optimal signaling scheme $\pi \in \Delta(\Omega \times S)$, where the set $S$ satisfies $|S| \leq \sum_{a \in A} \text{Cara}(\mathcal{P}_a)$. Specifically, for any $a \in A$, the signaling scheme sends at most $\text{Cara}(\mathcal{P}_a) \leq |\Omega|$ signals for which the receiver’s optimal action is $a$.

**Proof.** Since the optimal $m_a \in \text{Conv}(\mathcal{P}_a)$, it follows that $m_a$ can be written as a convex combination of at most $\text{Cara}(\mathcal{P}_a)$ points in $\mathcal{P}_a$. As detailed in the discussion preceding the proposition statement, one can then construct an optimal signaling scheme using such a convex combination that sends, for each $a \in A$, at most $\text{Cara}(\mathcal{P}_a)$ signals for which the receiver’s optimal action is $a$. Hence, the total number of signals is at most $\sum_{a \in A} \text{Cara}(\mathcal{P}_a)$.

Using Caratheodory’s theorem, we have $\text{Cara}(\mathcal{P}_a) \leq \dim(\mathcal{P}_a) + 1$, where $\dim(A)$ is the dimension of the smallest affine space containing $A$. Since the set $\mathcal{P}_a \subseteq \Delta(\Omega)$ lies in an affine space of dimension $\mathbb{R}^{|\Omega| - 1}$, we obtain $\text{Cara}(\mathcal{P}_a) \leq |\Omega|$. \qed

The bound we obtain here is related to the bound obtained by Kamenica and Gentzkow (2011), who also use convex analytic arguments to show that at most $|\Omega|$ signals suffice for an optimal
signaling scheme. In Appendix A, we discuss this connection in detail, and use their bound to provide an alternative approach to arrive at the convex problem (10). Finally, note that \( \text{Cara}(A) = 1 \) for any convex set \( A \). Thus, in the case of expected utility maximizing agents, where each set \( \mathcal{P}_a \) is convex, the preceding proposition states that at most \( |A| \) signals suffice for an optimal signaling scheme, matching that obtained from the revelation-principle argument.

To conclude the section, we briefly remark on the complexity of finding the optimal solution to the problem (10). Since the optimization problem is convex in the variables \( \{t_a\}_{a \in A} \), the complexity rests on whether there exists an efficient characterization of the set \( \text{Conv}(\mathcal{P}_a \cup \{0\}) \) for each \( a \in A \). Observe that this set is fully determined by the model primitives, namely the utility functions \( \rho(\cdot, a) \) for each \( a \in A \). Thus, whether an efficient characterization of the set \( \text{Conv}(\mathcal{P}_a \cup \{0\}) \) exists depends solely on the properties of the receiver’s utility functions \( \rho(\cdot, a) \) for each \( a \). In the next section, we show that under some natural convexity properties, one can replace the sets \( \text{Conv}(\mathcal{P}_a \cup \{0\}) \) by the convex hull of a finite number of pre-specified points. Using this, the sender’s persuasion problem can be reduced to a linear program.

4. Binary persuasion

We now focus on a specific setting of practical importance that we refer to as binary persuasion. In this setting, the receiver’s actions are binary, i.e., \( A = \{0, 1\} \), and the sender’s utility is always weakly higher when the receiver takes action 1, i.e., \( v(\omega, 1) \geq v(\omega, 0) \) for all \( \omega \in \Omega \). This model matches settings where, independent of the state, the sender seeks to persuade the receiver to take an action, such as engage with social media platforms (Candogan and Drakopoulos 2020), wait in a queue (Lingenbrink and Iyer 2019), or purchase a product (Lingenbrink and Iyer 2018, Drakopoulos et al. 2018).

To aid our discussion, we define the receiver’s differential utility \( \bar{\rho}(\cdot) \) function as the difference in the utility between choosing action 1 and action 0: \( \bar{\rho}(\mu) \triangleq \rho(\mu, 1) - \rho(\mu, 0) \) for all \( \mu \in \Delta(\Omega) \). Note that action \( a = 1 \) is optimal for the receiver at belief \( \mu \) if and only if \( \bar{\rho}(\mu) \geq 0 \).
(a) The set $K_0 = \Omega \cap \mathcal{P}_0$.  (b) The set $K_1 = \Omega \cap \mathcal{P}_1$.  (c) The set $K_{01}$ as defined in (12).

**Figure 3**  Geometry of $K_0, K_1$ and $K_{01}$.

### 4.1. Geometry of the convex program

The convex program (10) obtained in the previous section has variables defined over the domain $\text{Conv}(\mathcal{P}_a \cup \{0\})$ for each action $a$. In this section, we show that the domain can be further simplified under the following assumption:

**Assumption 2.** The set $\mathcal{P}_1^c = \{\mu \in \Delta(\Omega) : \bar{\rho}(\mu) < 0\}$ is convex.

Intuitively, the assumption implies that the receiver is averse to uncertainty when choosing action 1: if action 1 is not optimal under beliefs $\mu$ and $\mu'$, then it cannot be optimal under a belief $\gamma\mu + (1 - \gamma)\mu'$ that is obtained by inducing uncertainty between $\mu$ and $\mu'$. Furthermore, the assumption holds for a wide class of utility functions, as the following lemma establishes.

**Lemma 3.** Suppose the differential utility $\bar{\rho}(\cdot)$ is quasiconvex. Then Assumption 2 holds.

Specifically, Assumption 2 holds when $\rho(\mu, 1)$ is convex and $\rho(\mu, 0)$ is concave. At the same time, note that Assumption 2 is substantially weaker than requiring quasiconvexity of $\bar{\rho}$, which implies that every level set $\{\mu : \bar{\rho}(\mu) < a\}$ is convex.

Using Assumption 2, we now show that the convex program (10) can be simplified to a linear program. In the following, to improve readability, we slightly abuse notation and let $\omega \in \Omega$ also denote the belief in $\Delta(\Omega)$ that assigns all the weight to $\omega$. 

Electronic copy available at: https://ssrn.com/abstract=3386273
Let \( K_1 \triangleq \mathcal{P}_1 \cap \Omega \) denote the set of states where action 1 is optimal for the receiver under full-information. Similarly, let \( K_0 \triangleq \mathcal{P}_0 \cap \Omega \) be the set of states where action 0 is optimal for the receiver under full-information. Note, \( K_0 \cap K_1 \) may be non-empty if the receiver finds both actions optimal at some state. We let \( L_0 \triangleq K_0 \cap K_1^c \) denote the set of states for which the action 0 is uniquely optimal for the receiver.

Next, for \( \omega_0 \in K_0 \) and \( \omega_1 \in K_1 \), consider the set of beliefs obtained as the convex combination of \( \omega_0 \) and \( \omega_1 \). In this set, we let \( \chi(\omega_0, \omega_1) \) denote the belief that puts the largest weight on \( \omega_0 \) while still preserving the optimality of action 1 for the receiver. (Since \( \mathcal{P}_1 \) is closed, such a maximal convex combination exists.) Formally, for \( \omega_0 \in K_0 \) and \( \omega_1 \in K_1 \), we define \( \gamma(\omega_0, \omega_1) = \sup_{\gamma \in [0,1]} \{ \gamma : \gamma \omega_0 + (1 - \gamma) \omega_1 \in \mathcal{P}_1 \} \), and let

\[
\chi(\omega_0, \omega_1) = \gamma(\omega_0, \omega_1) \omega_0 + (1 - \gamma(\omega_0, \omega_1)) \omega_1.
\]

Since \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) are closed, we obtain that \( \chi(\omega_0, \omega_1) \in \mathcal{P}_0 \cap \mathcal{P}_1 \), and hence the receiver is indifferent between action 0 and 1 at belief \( \chi(\omega_0, \omega_1) \). Finally, we define the set \( K_{01} \) to be a subset of such maximal convex combinations \( \chi(\omega_0, \omega_1) \):

\[
K_{01} = \{ \chi(\omega_0, \omega_1) \in \Delta(\Omega) \text{ for some } \omega_0 \in L_0 \text{ and } \omega_1 \in K_1 \}.
\] (12)

We note while \( K_{01} \cap K_0 \) must be empty, the set \( K_{01} \cap K_1 \) can be non-empty.

With these definitions, we can state the main theorem of this section.

**Theorem 2.** Under Assumption 2, the sender’s persuasion problem (10) can be optimized by solving the following linear program:

\[
\max_{t_0, t_1} \sum_{\omega \in \Omega} v(\omega, 1)t_1(\omega) + v(\omega, 0)t_0(\omega)
\]

subject to, \( t_1 \in \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \),

\( t_0 \in \text{Conv}(L_0 \cup \{0\}) \),

\( t_0(\omega) + t_1(\omega) = \mu^*(\omega) \text{ for each } \omega \in \Omega. \)

(13)
The full proof is provided in Appendix C. The proof intuition is as follows. First, under Assumption 2, we prove in Lemma 5 that the set $\text{Conv}(\mathcal{P}_1)$ is a convex polytope with extreme points in $K_0 \cup K_{01}$, and hence the constraint $t_1 \in \text{Conv}(\mathcal{P}_1 \cup \{0\})$ in (10) is equivalent to the set of linear constraints $t_1 \in \text{Conv}(K_1 \cup K_{01} \cup \{0\})$. Second, we establish that any $t_0 \in \text{Conv}(\mathcal{P}_0 \cup \{0\})$ that is not in $\text{Conv}(L_0 \cup \{0\})$ cannot be part of an optimal solution, by improving on any such feasible solution. See Fig. 4 (and Fig. 8 in Appendix C) for some geometric intuition.

4.2. Structural Characterizations

The preceding theorem provides several implications about the structure of the optimal signaling scheme, that we discuss next.

First, Theorem 2 establishes a canonical basis of signals for an optimal signaling scheme, namely $K_0 \cup K_1 \cup K_{01}$. In other words, for a given binary persuasion setting, irrespective of the prior belief $\mu^*$, it suffices to only send signals in the set $K_0 \cup K_1 \cup K_{01}$. This follows from the fact that Theorem 2 implies that the mean-posteriors $m_0$ and $m_1$ satisfy $m_0 \in \text{Conv}(L_0)$ and $m_1 \in \text{Conv}(K_1 \cup K_{01})$. Hence, $m_0$ can be expressed as a convex combination of beliefs in $L_0$, and $m_1$ can be expressed as a convex combination of beliefs in $K_1 \cup K_{01}$. Thus, it follows that inducing beliefs in the set $L_0 \cup K_1 \cup K_{01} = K_0 \cup K_1 \cup K_{01}$ suffices for optimal persuasion.

Second, observe that the canonical basis set consists of pure signals $\omega \in K_0 \cup K_1$ which fully reveal the state $\omega$, and binary mixed signals $\chi(\omega_0, \omega_1) \in K_{01}$ which induce a belief over two states $\omega_0 \in K_0$ and $\omega_1 \in K_1$. Thus, the optimal persuasion can always be achieved by either fully revealing the state, or by making the receiver uncertain about two states. Of course, due to the convexity of

Figure 4  Geometry of $\text{Conv}(K_0 \cup K_{01})$ and $\text{Conv}(K_1 \cup K_{01})$. 
Figure 5  Full persuasion of a receiver.

$P^c_1$, there also exists an optimal signaling scheme where all posteriors that lead to a receiver taking action 0 are replaced with a single action recommendation “take action 0”.

Finally, Theorem 2 provides a characterization of when full persuasion, as described in the introduction, is possible in the binary persuasion setting. To build intuition, consider Figure 5 where we have $\mu^* \in P^c_1$, implying that the receiver strictly prefers choosing action 0 under her prior. However, we can find beliefs $s_1, s_2 \in P_1$ such that $\gamma s_1 + (1 - \gamma) s_2 = \mu^*$. The signaling scheme that induces posterior $s_1$ with probability $\gamma$ and $s_2$ with probability $1 - \gamma$ is Bayes-plausible. Since the receiver strictly prefers $a = 1$ at both $s_1$ and $s_2$, the sender can fully persuade the receiver. The above intuition is formalized in the following proposition, whose proof is in Appendix C:

**Proposition 2.** The receiver can be fully persuaded if and only if $\mu^* \in \text{Conv}(P_1) \cap P^c_1$. Under Assumption 2, the latter condition is equivalent to $\mu^* \in \text{Conv}(K_1 \cup K_{01}) \cap P^c_1$.

Since it is straightforward to check whether $\mu^* \in \text{Conv}(K_1 \cup K_{01})$, the preceding proposition provides a simple criterion to determine whether a prior belief $\mu^* \in P^c_1$ allows for full persuasion. The proposition also highlights the impossibility of full persuasion when the receiver is an expected utility maximizer: in this case, $P_1$ is convex, and hence $\text{Conv}(P_1) \cap P^c_1 = P_1 \cap P^c_1 = \emptyset$.

5. Application: Signaling in Unobservable Queues

As an illustration of our methodology, in this section we apply it to study information sharing in an service system where arriving customers must choose whether or not to join an unobservable
queue to obtain service. The model is based on that of Lingenbrink and Iyer (2019), with the difference being that here customers are not expected utility maximizers. In particular, we model the customers as being averse to the uncertainty in their waiting times (Maister et al. 1984). Moreover, the setting has a key difference from the model in Section 2: the customers’ (receiver’s) prior belief is endogenously determined from the queue-dynamics.

5.1. Model

We consider a service system modeled as an $M/M/1/C$ FIFO queue, i.e. a single-server queue with Poisson arrivals (with rate $\lambda$), independent exponential service times (with unit mean), and queue capacity $C > 0$. Upon arrival, each customer chooses whether to join the queue to receive service ($a = 1$) or leave without obtaining service ($a = 0$); a customer cannot join the queue if there are $C$ customers already in queue. We assume that customers are averse to waiting, but cannot observe the queue-length before deciding whether to join or leave. Instead, the service provider can observe the queue-length and communicate this information to arriving customers. The service provider aims to maximize the throughput of the queue; if service is offered at a fixed price, this translates to maximizing the rate of revenue.

The state $\omega$, as observed by the service provider, is the queue-length upon the arrival of a customer. Since the customer cannot join the queue if the queue-length is already $C$, we let $\Omega = \{0, 1, \ldots, C - 1\}$ denote the relevant state space. Since we focus on throughput-maximization, we set $v(\omega, a) = a$ for $\omega \in \Omega$ and $a \in \{0, 1\}$.

For an arriving customer, the payoff-relevant variable $X$ is their waiting time until service completion. When $\omega$ equals $n \in \Omega$, the waiting time $X$ is distributed as the sum of $n + 1$ independent unit exponentials (the waiting times for $n$ customers in the queue plus the customer’s own service time). If a customer has belief $\mu \in \Delta(\Omega)$ about the queue-length $\omega$, her differential utility for joining the queue is given by $\bar{\rho}(\mu)$. To capture uncertainty aversion on the part of the customers (Maister et al. 1984), we focus on the following the differential utility function:

$$\bar{\rho}(\mu) \equiv \tau - \left( E_{\mu}[X] + \beta \sqrt{Var_{\mu}[X]} \right), \quad (14)$$
where \( \tau > 0 \) captures the value of service, and \( \beta \geq 0 \) captures the degree of risk-consciousness. It is straightforward to verify that \( \bar{\rho}(\cdot) \) is convex in \( \mu \), and hence satisfies Assumption 2. (See Appendix D for the details.)

Using the results from Section 4 and the same approach as in Lingenbrink and Iyer (2019) to handle endogenous priors, we obtain that the service provider’s signaling problem can be optimized by solving the following linear program:

\[
\begin{align*}
\max_{t_0, t_1} & \sum_{\omega \in \Omega} t_1(\omega) \\
\text{subject to,} & \quad t_1 \in \text{Conv}(K_1 \cup K_{01} \cup \{0\}), \\
& \quad t_0 \in \text{Conv}(L_0 \cup \{0\}), \\
& \quad \sum_{\omega \in \Omega} t_0(\omega) + \sum_{\omega \in \Omega} t_1(\omega) + \lambda t_1(C - 1) = 1, \\
& \quad t_0(\omega + 1) + t_1(\omega + 1) = \lambda t_1(\omega), \quad \text{for each } \omega < C - 1.
\end{align*}
\]

(15a)

(15b)

(15c)

(15d)

Here, the constraint (15d) captures the detailed-balance conditions on the steady-state distribution of the queue, whereas the constraint (15c) is the normalization condition for the steady-state distribution (with \( \lambda_1 t_1(C - 1) \) being the probability that the queue is at capacity). Similar to Lingenbrink and Iyer (2019) we find below that the optimal signaling scheme induces a simple structure on the customers’ actions. The proof is available in Appendix D.1.

**Proposition 3.** The optimal signaling scheme induces a threshold structure in the customers’ actions: there exists an \( m \in \Omega \) and \( q \in [0, 1] \) such that each arriving customer joins the queue whenever the queue-length upon arrival is strictly less than \( m \), leaves whenever the queue-length is strictly greater than \( m \), and joins with probability \( q \) if the queue-length equals \( m \).

We note that the proof of the preceding result, apart from using Assumption 2, relies only on the following property satisfied by \( \bar{\rho}(\cdot) \), namely that \( \bar{\rho}(\gamma m + (1 - \gamma)n) \) is strictly decreasing in \( n > m \) for all \( \gamma \in [0, 1] \). Thus, the proposition holds for a much wider class of differential utility functions. Finally, we emphasize that it is the induced actions of the customers that has a threshold structure; the optimal signaling scheme has a more intricate structure that we explore numerically next.
Figure 6  Optimal signaling schemes

5.2. Numerical Computations

In Figure 6, we display the optimal signaling schemes for a range of values of the model parameters \((\lambda, \beta, \tau)\). Throughout our numerical computations, we set the queue capacity \(C = 100\).

To interpret these plots, consider Fig. 6a which shows the optimal signaling scheme for \((\lambda, \beta, \tau) = (0.95, 2.5, 7.5)\). For these parameter values, the optimal signaling scheme uses 5 different signals which we label as Join_1, Join_2, Join_3, Join_4 and Leave. In particular, the customer’s optimal action under signals Join_k \((k = 1, 2, 3, 4)\) is to join the queue, whereas under Leave, the optimal action is to
not join the queue. In Fig. 6a, we plot the probabilities with which the different signals are sent for each value of the queue-length upon a customers’ arrival. For instance, if the queue-length is \( \omega = 3 \) upon the arrival of a customer, the signal \( \text{Join}_4 \) is sent with probability 0.08, the signal \( \text{Join}_3 \) is sent with probability 0.63, and the signal \( \text{Join}_2 \) is sent with the remaining probability 0.29. Similarly, if the queue length is \( \omega = 4 \) upon the arrival of a customer, the signal \( \text{Join}_1 \) is sent with probability 0.25 and the signal \( \text{Leave} \) is sent with probability 0.75. As established in Proposition 3, these plots reveal the threshold structure induced on the customers’ actions.

From the customers’ perspective, we note that each signal induces a posterior belief about the queue-length, and hence about the waiting time upon joining. The computations reveal an interesting “sandwich” structure in the beliefs induced by the \( \text{Join}_k \) signals. For instance, in Fig. 6a, the posterior belief upon receiving the \( \text{Join}_1 \) signal has support \( \omega \in \{0, 4\} \), has support \( \omega \in \{0, 3\} \) for \( \text{Join}_2 \), has support \( \omega \in \{1, 3\} \) for \( \text{Join}_3 \) and has support \( \omega \in \{2, 3\} \) for \( \text{Join}_4 \). More generally, the computations show that the signals \( \{\text{Join}_j : j \geq 1\} \) possess a sandwich structure, where for each \( j \), the posterior under \( \text{Join}_j \) is supported on at most two queue-lengths \( a_j \) and \( b_j \) with \( a_j \leq b_j \), such that \( a_i \leq a_j \) and \( b_i \geq b_j \) for \( i \leq j \).

The aforementioned sandwich structure also leads to an ordering on the signals based on the degree of “uncertainty” in the induced posteriors. In particular, for all \( i < j \) we find \( \text{Var}(X | \text{Join}_i) > \text{Var}(X | \text{Join}_j) \) and \( \mathbb{E}[X | \text{Join}_i] < \mathbb{E}[X | \text{Join}_j] \). Thus, although the expected waiting time under the signal \( \text{Join}_1 \) is the lowest, it also conveys the highest variance in the waiting time. Note that this ordering provides a convenient vocabulary for the service provider to communicate with the customers: rather than sending an abstract signal such as \( \text{Join}_k \), the service provider can directly convey the induced expected waiting times \( \mathbb{E}[X | \text{Join}_k] \) (or the induced uncertainty \( \text{Var}(X | \text{Join}_k) \)).

To complete the discussion, in Figure 7, we compare the throughput under the optimal signaling scheme with that under two extreme benchmarks: the full-information mechanism, where the service provider reveals the queue-length to each arriving customer, and the no-information mechanism, where she reveals nothing to each arriving customer. From the plots, we observe that the optimal
signaling scheme provides the most increase in throughput over the full-information and the no-information mechanisms for moderate values of the degree of risk-consciousness $\beta$ and for values of the arrival rate $\lambda$ close to the service rate.

We can see in Fig. 7a that under all mechanisms, as $\beta$ increases, the throughput decreases: when customers are more risk-conscious, they are less likely to join the queue. Up to a point, persuasion can increase throughput beyond what can be achieved under fully-revealing or no-information mechanisms, but when $\beta$ is high enough, no customer joins as she finds the uncertainty in just the service time to be unacceptable.

6. Discussion

We conclude with a brief discussion of our modeling choices and the applicability of our approach to broader contexts.

In our model formulation, we have taken the receiver’s utility $\rho(\cdot)$ as the primitive. However, this is a modeling choice, and our approach could equally assume the sets $\{P_a : a \in A\}$ to be the modeling primitives. Namely, rather than modeling receiver utilities directly, we could model the receiver as being characterized by the sets of beliefs for which she chooses each particular action. As we discuss next, this perspective allows us to apply our methods to broader settings beyond the context of risk-conscious receivers.

First, our methods can be used to study public persuasion (Arieli and Babichenko 2019) of a group of interacting (expected utility maximizing) agents, where the sender shares information...
publicly with all the agents. For any public signal, the agents share a common posterior and subsequently play an equilibrium of an incomplete information game. The sender seeks to publicly share payoff-relevant information to influence the agents’ choice of the equilibrium. To apply our methods, we view the group of agents as a single risk-conscious receiver, and the equilibrium profile as the action chosen by the receiver. Then, for any equilibrium \( a \), the set \( \mathcal{P}_a \) describes the set of common posteriors for which the agents play the same equilibrium profile \( a \). With this mapping, our results can be used to find the optimal public signaling scheme, as long as the set of equilibria (over all common posteriors) is finite.

Second, our methods can be applied to study robust persuasion (Hu and Weng 2018), where a sender persuades a single (expected utility maximizing) receiver with a private type \( \theta \in \Theta \). The sender takes a worst-case view, and seeks to persuade the receiver irrespective of her type. Formally, suppose the utility of the receiver with type \( \theta \) for action \( a \) is given by \( \rho_\theta(\mu, a) \), and let \( a_\theta(\mu) \in \arg\max_{a} \rho_\theta(\mu, a) \) denote the optimal action chosen by the receiver with belief \( \mu \in \Delta(\Omega) \). Let \( v(a) \) denote the sender’s utility when the receiver chooses action \( a \). Since the receiver’s type \( \theta \) is unknown to the sender, she maximizes the (expectation) of the minimum of her utility across all receiver types: \( E[\min_{\theta} v(a_\theta(\mu))] \). Such a setting of robust persuasion maps to our model, with the sets \( \mathcal{P}_a \) given by \( \mathcal{P}_a = \{ \mu \in \Delta(\Omega) : v(a) = \min_{\theta} v(a_\theta(\mu)) \} \) for each \( a \in A \). Thus, our results yield a robust signaling scheme through a convex program.

Finally, while our analysis considers receivers who may not be expected utility maximizers, we always assume the sender to be one. An interesting direction for further research would be explore what occurs when this modeling assumption is relaxed.

Endnotes

1. Throughout, for any set \( S \), we let \( \Delta(S) \) denote the set of probability distributions over \( S \). When \( S \) is finite, we consider \( \Delta(S) \) as a subset of \( \mathbb{R}^{|S|} \), endowed with the Euclidean topology.

2. Although our definition implies a pure strategy, we can easily incorporate mixed strategies where the receiver chooses an action at random. We suppress this technicality for the sake of readability.
3. Note that if \( \rho(\mu, a) \) is a utility function of an agent, then so is \( g(\rho(\mu, a)) \) for any increasing function \( g: \mathbb{R} \to \mathbb{R} \). Thus, this linearity holds only up to an increasing transformation. We suppress such (irrelevant) transformations for the sake of clarity.

4. Here, continuity is with respect to the Euclidean topology on \( \Delta(\Omega) \).

5. Throughout, we use the notation \( E_\lambda[\cdot] \) (and \( P_\lambda(\cdot) \)) to denote expectation (resp., probability) with respect to a distribution \( \lambda \).

6. We also note that all the \( \text{Join}_k \) signals result in the same differential utility of 0 for the parameter values shown.

7. We discuss the equilibrium under these mechanisms in Appendix D.2.

Appendix A: Relation to the Concavification Approach

In their seminal work, Kamenica and Gentzkow (2011) (hereafter [KG11]) present a convex analytic concavification approach to find the optimal signaling scheme in persuasion problems. Furthermore, in Proposition 4 of the online appendix of [KG11], the authors establish that the number of signals in an optimal signaling scheme can be upper-bounded by the cardinality \( |\Omega| \) of the state space. While the focus in [KG11] is on the case of an expected utility maximizing receiver, the approach presented therein also applies to the case of risk-conscious receiver. In this appendix, we discuss how the concavification approach relates to ours, and show how their results provide an alternative path to arrive at the convex programming formulation (10).

A.1. Convex analytic argument of Kamenica and Gentzkow (2011)

To describe their approach in detail, define \( \hat{v}(s) \triangleq \sum_{\omega \in \Omega} s(\omega)v(\omega, a(s)) \) for \( s \in \Delta(\Omega) \), where \( a(s) \in \arg \max_{a \in A} \rho(s, a) \) denotes the receiver’s optimal action under belief \( s \in \Delta(\Omega) \). Using this definition, the optimization problem (5) can be written as the following problem:

\[
\max_{\eta \in \Delta(\Delta(\Omega))} E_\eta[\hat{v}(\bar{s})] \quad \text{subject to} \quad E_\eta[\bar{s}] = \mu^*,
\]

where \( E_\eta \) denotes expectation with respect to \( \eta \). Thus, letting \( V(\mu) \triangleq \sup_{\eta \in \Delta(\Delta(\Omega))} \{ E_\eta[\hat{v}(\bar{s})] : E_\eta[\bar{s}] = \mu \} \), the sender’s largest payoff from persuasion is given by \( V(\mu^*) \).

The main result of [KG11] is that \( V(\cdot) \) is the smallest concave function that dominates \( \hat{v} \). In particular, the authors establish that, for all \( \mu \in \Delta(\Omega) \),

\[
V(\mu) = \sup\{ x : (x, \mu) \in \text{Conv}(\text{hyp}(\hat{v})) \},
\]

where \( \hat{v} \) denotes the hypograph of \( \hat{v} \) defined as \( \text{hyp}(\hat{v}) \triangleq \{ (x, s) : x \in \mathbb{R}, s \in \Delta(\Omega), x \leq \hat{v}(s) \} \). Furthermore, the authors show that any representation of \( (\mu^*, V(\mu^*)) \) as a convex combination of elements
of $\text{hyp}(\hat{v})$ yields an optimal signaling scheme. Note that (16) yields a convex program to compute $V(\mu^*)$.

To obtain the bound on the number of signals in the optimal signaling scheme, the authors apply the Fenchel-Bunt theorem (see the online appendix of [KG11] for the detailed argument) to $\text{hyp}(\hat{v})$ to show that any $(x,\mu) \in \text{Conv}(\text{hyp}(\hat{v}))$ can be written as a convex combination of at most $|\Omega|$ elements of $\text{hyp}(\hat{v})$. From this, the authors deduce the existence of an optimal signaling scheme with at most $|\Omega|$ signals.

Our approach has many parallels with the preceding approach. First, our approach requires us to compute the set $\mathcal{P}_a$ for each $a \in A$, and compute the convex hull $\text{Conv}(\mathcal{P}_a \cup \{0\})$. In contrast, the preceding approach requires computing the set $\text{Conv}(\text{hyp}(\hat{v}))$. Second, we formulate the convex optimization problem (10) with variables $t_a \in \text{Conv}(\mathcal{P}_a \cup \{0\})$ for each $a \in A$. In contrast, the characterization (16) of $V(\mu^*)$ suggests a convex optimization formulation with variables $(x,\mu) \in \text{Conv}(\text{hyp}(\hat{v}))$. Finally, we use the Caratheodory theorem to split each optimal $t_a$ into at most $\text{Cara}(\mathcal{P}_a) \leq |\Omega|$ signals (see Proposition 1). In contrast, the preceding approach directly splits the point $(\mu^*,V(\mu^*))$ into at most $|\Omega|$ signals using the Fenchel-Bunt theorem.

We note that since our splitting argument applies to each $t_a$ separately, our argument furnishes an upper-bound of at most $\text{Cara}(\mathcal{P}_a) \leq |\Omega|$ signals per action, and $\sum_a \text{Cara}(\mathcal{P}_a) \leq |\Omega| \cdot |A|$ signals in total, in the optimal signaling scheme. In comparison, the argument of [KG11] provides an upper-bound of at most $|\Omega|$ signals in total in the optimal signaling scheme. Thus, when the state space is large, our bound might be stronger, whereas the bound of $|\Omega|$ might be better if the action space is large. We further note that the advantage of our approach is that in some cases, it provides a canonical basis for the set of signals, as we show in the case of binary persuasion in Section 4.

A.2. Alternative argument yielding convex formulation (10)

In this section, we present an alternative argument to arrive at the convex programming formulation (10) using the upper-bound result in [KG11]. In fact, we show that this the formulation can be arrived at using any finite bound larger than $|\Omega|$ on the number of signals in the optimal signaling scheme.

To begin, let $B \geq |\Omega|$ be a bound on the number of signals required in the optimal signaling scheme. Hence, for any receiver action $a \in A$, there are at most $B$ signals that induce it under the optimal signaling scheme. Thus, it suffices to consider a signaling scheme, with signals $S = \{(a,i) : a \in A, 1 \leq i \leq B\}$, where each signal $(a,i) \in S$ induces the receiver to take action $a \in A$. Thus, the sender’s optimization problem (3) can be written as

$$\max_{\pi} \sum_{\omega \in \Omega} \sum_{(a,i) \in S} \pi(\omega,(a,i))v(\omega,a)$$

subject to, $a \in \arg \max_{a' \in A} \rho(\mu_{(a,i)},a')$, for all $(a,i) \in S$,

$$\pi(\omega,S) = \mu^*(\omega), \text{ for all } \omega \in \Omega.$$
where \( \mu_{(a,i)}(\omega) = \frac{\pi(\omega, (a,i))}{\sum_{a'} \pi(\omega', (a,i))} \) denotes the receiver’s belief after receiving the signal \((a,i) \in S\). Note that the constraint \( a \in \arg \max_{a' \in A} \rho(\mu_{(a,i)}, a') \) is essentially an obedience constraint, which requires that the receiver, upon receiving the signal \((a,i)\), finds it optimal to choose action \(a \in A\). Using the definition of \( P_a \), this constraint can be equivalently written as \(\mu_{(a,i)} \in P_a\). Letting \( t_a(\omega) = \sum_{i=1}^{B} \pi(\omega, (a,i)) \) for each \(a \in A\), the problem can be reformulated as

\[
\max_{\pi} \sum_{\omega \in \Omega} \sum_{a \in A} t_a(\omega) v(\omega, a)
\]

subject to, \(\mu_{(a,i)} \in P_a\), for all \((a,i) \in S\),

\[
\sum_{a \in A} t_a(\omega) = \mu^*(\omega), \quad \text{for all } \omega \in \Omega.
\]

\[
t_a(\omega) = \sum_{i=1}^{B} \pi(\omega, (a,i)), \quad \text{for all } \omega \in \Omega \text{ and } a \in A.
\]

Let \(k(a,i) = \sum_{\omega' \in \Omega} \pi(\omega', (a,i))\). Note that \(k(a,i) \geq 0\), and \(\sum_{i=1}^{B} k(a,i) \leq 1\). Since \(\mu_{(a,i)}(\omega) = \frac{\pi(\omega, (a,i))}{\sum_{a'} \pi(\omega', (a,i))} = \frac{\pi(\omega, (a,i))}{k(a,i)}\), we obtain for all \(a \in A\) and \(\omega \in \Omega\),

\[
t_a(\omega) = \sum_{i=1}^{B} k(a,i) \mu_{(a,i)}(\omega) = \sum_{i=1}^{B} k(a,i) \mu_{(a,i)}(\omega) + \left(1 - \sum_{i=1}^{B} k(a,i)\right) \mathbf{0}.
\]

Since \(\mu_{(a,i)} \in P_a\) for all \(1 \leq i \leq B\), we deduce that \(t_a \in \text{Conv}(P_a \cup \{\mathbf{0}\})\). Hence, the problem can be written as

\[
\max_{\pi} \sum_{\omega \in \Omega} \sum_{a \in A} t_a(\omega) v(\omega, a)
\]

subject to, \(t_a \in \text{Conv}(P_a \cup \{\mathbf{0}\})\), for all \(a \in A\),

\[
\sum_{a \in A} t_a(\omega) = \mu^*(\omega), \quad \text{for all } \omega \in \Omega.
\]

\[
\mu_{(a,i)} \in P_a, \quad \text{for all } (a,i) \in S,
\]

\[
t_a(\omega) = \sum_{i=1}^{B} \pi(\omega, (a,i)), \quad \text{for all } \omega \in \Omega \text{ and } a \in A.
\]

The formulation (10) then follows from an application of the Caratheodory’s theorem, which implies that in the preceding optimization problem, the last two constraints are redundant (implied by the first constraint) as long as \(B \geq |\Omega|\), and the optimization can be done directly over \(\{t_a\}\) instead of \(\pi\). The optimal signaling scheme \(\pi\) (along with the beliefs \(\mu_{(a,i)}\)) can then be obtained from the optimal \(t_a\) using (11), as described in the discussion following Theorem 1.

**Appendix B: Proofs from Section 3**

In this section, we provide the missing proofs from Section 3.
Proof of Lemma 1. Consider a signaling scheme \( \pi \in \Delta(\Omega \times \Delta(\Omega)) \) satisfying \( \mu_s = s \) for almost all \( s \in \Delta(\Omega) \). Let \( \eta(\cdot) \triangleq \pi(\Omega, \cdot) \in \Delta(\Delta(\Omega)) \). By definition, we have for any \( \omega \in \Omega \),

\[
\mu^*(\omega) = \pi(\omega, \Delta(\Omega)) = \mathbb{E}_\pi[\mathbb{I}\{\bar{\omega} = \omega\}] = \mathbb{E}_\pi[\mathbb{E}_\omega[\mathbb{I}\{\bar{\omega} = \omega\}|\bar{s}] = \mathbb{E}_\omega[\mu_s(\omega)] = \mathbb{E}_\omega[\bar{s}(\omega)] = \mathbb{E}_\eta[\bar{s}(\omega)].
\]

Thus, \( \eta \) is Bayes-plausible.

Next, let \( \eta \in \Delta(\Delta(\Omega)) \) be Bayes-plausible, and let \( \pi(\omega, ds) \triangleq s(\omega)\eta(ds) \) for all \( \omega \in \Omega \) and \( s \in \Delta(\Omega) \). Then, using Bayes’ rule, for any \( s \in \Delta(\Omega) \) (\( \eta \)-almost surely), we have \( \mu_s(\omega) = \mathbb{P}_s(\bar{\omega} = \omega|\bar{s} = s) = \pi(\omega, ds)/\pi(\Omega, ds) = s(\omega) \) for all \( \omega \in \Omega \). Thus, under \( \pi \), the receiver’s belief \( \mu_s \) upon receiving signal \( s \in \Delta(\Omega) \) satisfies \( \mu_s = s \) almost surely. \( \square \)

Proof of Lemma 2. Fix any Bayes-plausible measure \( \eta \) and an optimal receiver strategy \( a(\cdot) \). For each \( a \in A \), define \( b_a \triangleq \mathbb{P}_\eta(a(\bar{s}) = a) \in [0, 1] \) to be the probability that the receiver chooses action \( a \) under the corresponding signaling scheme. For each \( a \in A \), if \( b_a = 0 \), then let \( \eta_a \) be any probability measure with support on \( \mathcal{P}_a \). Otherwise, define \( \eta_a \) to the measure obtained by conditioning \( \eta \) on the event \( a(\bar{s}) = a \). More precisely, we have \( \eta_a(ds) \triangleq \frac{1}{b_a} \mathbb{I}\{a(\bar{s}) = a\}\eta(ds) \) if \( b_a > 0 \). Note that, by the definition of the sets \( \mathcal{P}_a \), the support of \( \eta_a \) is \( \mathcal{P}_a \) for each \( a \in A \). The following equations are immediate from the definitions:

\[
\sum_{a \in A} b_a \eta_a = \eta, \quad \sum_{a \in A} b_a = 1.
\]

We let \( m_a(\omega) \triangleq \mathbb{E}_{\eta_a}[\bar{s}(\omega)] \) for each \( \omega \in \Omega \). Note that if \( b_a > 0 \), then \( m_a(\omega) = \mathbb{E}_\eta[\bar{s}(\omega)|a(\bar{s}) = a] \). Thus, \( m_a(\omega) \) is the mean-posterior belief of the receiver that the state is \( \omega \), given that she chooses the action \( a \). From the Bayes-plausibility of \( \eta \), we obtain for each \( \omega \in \Omega \):

\[
\sum_{a \in A} b_a m_a(\omega) = \sum_{a \in A} b_a \mathbb{E}_{\eta_a}[\bar{s}(\omega)] = \mathbb{E}_\eta[\bar{s}(\omega)] = \mu^*(\omega),
\]

where the second equality follows from the fact that \( \sum_{a \in A} b_a \eta_a = \eta \). Moreover, it is straightforward to verify that \( m_a \in \text{Conv}(\mathcal{P}_a) \), since \( m_a \) is the mean of the posterior distribution \( \eta_a \) with support on \( \mathcal{P}_a \), and \( \mathcal{P}_a \) is closed and compact. Finally, note that for each \( \omega \in \Omega \),

\[
\sum_{a \in A} b_a m_a(\omega)v(\omega, a) = \sum_{a \in A} b_a \mathbb{E}_{\eta_a}[\bar{s}(\omega)]v(\omega, a)
\]

\[
= \sum_{a \in A} b_a \mathbb{E}_{\eta_a}[\bar{s}(\omega)v(\omega, a(\bar{s}))]
\]

\[
= \mathbb{E}_\eta[\bar{s}(\omega)v(\omega, a(\bar{s}))],
\]

where the first equality follows from the fact that \( a(\bar{s}) = a \) when \( \bar{s} \sim \eta_a \), and the second equality follows from the fact that \( \sum_{a \in A} b_a \eta_a = \eta \).
Conversely, suppose we have \(\{(b_a, m_a)\}_{a \in A}\) with \(b_a \in [0, 1]\) and \(m_a \in \text{Conv}(\mathcal{P}_a)\) with \(\sum_{a \in A} b_a m_a = \mu^*\). By the definition of the convex hull, \(m_a \in \text{Conv}(\mathcal{P}_a)\) implies the existence of \(\{(\mu^a_i, \lambda^a_i) : i = 1, \ldots, j_a\}\) such that \(\mu^a_i \in \mathcal{P}_a\) and \(\lambda^a_i \geq 0\) for each \(i \leq j_a\) with \(\sum_{i=1}^{j_a} \lambda^a_i = 1\) and \(m_a = \sum_{i=1}^{j_a} \lambda^a_i \mu^a_i\). Define \(\eta \in \Delta(\Delta(\Omega))\) to be the discrete distribution that selects the posterior \(\mu^a_i\) with probability \(b_a \lambda^a_i\). Then, we have for all \(\omega \in \Omega\),

\[
E_{\eta}[\bar{s}(\omega)] = \sum_{a \in A} \sum_{i=1}^{j_a} b_a \lambda^a_i \mu^a_i(\omega) = \sum_{a \in A} b_a \left( \sum_{i=1}^{j_a} \lambda^a_i \mu^a_i(\omega) \right) = \sum_{a \in A} b_a m_a(\omega) = \mu^*(\omega).
\]

This proves the Bayes-plausibility of \(\eta\). Finally, define the strategy \(a(\cdot) : \Delta(\Omega) \rightarrow A\) so that \(a(\mu^a_i) = a\) for each \(i \leq j_a\) and \(a \in A\), and for other values of \(\mu\), let \(a(\mu)\) be an arbitrary element in \(\arg \max_{a \in A} \rho(\mu, a)\). Since \(\mu^a_i \in \mathcal{P}_a\), it is straightforward to verify that the strategy \(a(\cdot)\) is optimal. Finally, we have for each \(\omega \in \Omega\),

\[
\sum_{a \in A} b_a m_a(\omega) v(\omega, a) = \sum_{a \in A} b_a \sum_{i=1}^{j_a} \lambda^a_i \mu^a_i(\omega) v(\omega, a) = \sum_{a \in A} b_a \sum_{i=1}^{j_a} (b_a \lambda^a_i) \cdot \mu^a_i(\omega) \cdot v(\omega, a(\mu^a_i)) = \sum_{a \in A} \sum_{i=1}^{j_a} \eta(\mu^a_i) \cdot \mu^a_i(\omega) \cdot v(\omega, a(\mu^a_i)) = E_{\eta}[\bar{s}(\omega) v(\omega, \bar{s})].
\]

Here, the first equation follows from the fact that \(m_a = \sum_{i=1}^{j_a} \mu^a_i \lambda^a_i\), the second equation follows from the fact that \(a(\mu^a_i) = a\), and the third equation follows from the definition of \(\eta\). This completes the proof of the lemma. \(\square\)

**Appendix C: Proofs from Section 4**

In this section, we provide the proofs of the results in Section 4. We begin with the following simple argument showing that the quasiconvexity of \(\bar{\rho}(\cdot)\) implies Assumption 2.

**Proof of Lemma 3.** The proof follows immediately from the fact that if \(\bar{\rho}\) is quasiconvex, then \(\bar{\rho}(\gamma \mu + (1 - \gamma) \mu') \leq \max\{\bar{\rho}(\mu), \bar{\rho}(\mu')\}\) for \(\gamma \in [0, 1]\) and \(\mu, \mu' \in \Delta(\Omega)\). \(\square\)

We next focus on the proof of Theorem 2. The proof of the theorem rests on two helper lemmas that characterize the geometry of the sets \(\Delta(\Omega)\) and \(\text{Conv}(\mathcal{P}_1)\). The first lemma shows that the set \(\Delta(\Omega)\) can be viewed as the union of two regions, each of which is the convex hull of a finite set of points. Fig. 4 and Fig. 8 illustrate the geometric intuition behind this lemma. Recall that \(L_0 \triangleq K_0 \cap K_1^\circ\) denotes the set of states for which action 0 is uniquely optimal for the receiver.

**Lemma 4.** \(\Delta(\Omega) = \text{Conv}(L_0 \cup K_{(1)}) \cup \text{Conv}(K_1 \cup K_{(1)})\).
Lemma 5. Under Assumption 2, \( \text{Conv} (K_0) \) is not a hyperplane.

Proof. Let \( \mu \in \Delta(\Omega) \). Since \( \Omega = L_0 \cup K_1 \), we have \( \mu \in \text{Conv}(L_0 \cup K_1) = \text{Conv}(L_0 \cup K_1 \cup K_{01}) \).

Consider any convex decomposition of \( \mu = \sum_{\omega \in L_0 \cup K_1 \cup K_{01}} \alpha_\omega \phi \). If the convex decomposition does not place positive weights either on the elements of \( L_0 \) or on those of \( K_1 \), then we are done. Otherwise, let \( \omega_0 \in L_0 \) and \( \omega \in K_1 \) be such that \( \alpha_\omega > 0 \) for \( \phi \in \{ \omega_0, \omega_1 \} \). Let \( \mu' \triangleq \frac{1}{\alpha_\omega_0 + \alpha_\omega_1} (\alpha_\omega_0 \omega_0 + \alpha_\omega \omega_1) \), and note that \( \mu = (\alpha_\omega_0 + \alpha_\omega_1) \mu' + \sum_{\phi \in L_0 \setminus \{\omega_0\}} \alpha_\phi \phi + \sum_{\phi \in K_1 \setminus \{\omega_1\}} \alpha_\phi \phi + \sum_{\phi \in K_{01}} \alpha_\phi \phi \). Since \( \mu' \in \text{Conv}(\{\omega_0, \omega_1\}) \), one can write \( \mu' \) as either a convex combination of \( \omega_0 \) and \( \chi(\omega_0, \omega_1) \) or a convex combination of \( \omega_1 \) and \( \chi(\omega_0, \omega_1) \). In either scenario, using this decomposition for \( \mu' \), we obtain a convex decomposition of \( \mu \) that places positive weight on at least one fewer element of \( L_0 \cup K_1 \). Continuing with this process, we obtain a convex decomposition of \( \mu \) that places no positive weight either on elements on \( L_0 \) or on elements of \( K_1 \), yielding the lemma statement. \( \square \)

The second lemma uses this result to establish that \( \text{Conv}(P_1) \) is a convex polytope with extreme points in the set \( K_1 \cup K_{01} \).

Lemma 5. Under Assumption 2, \( \text{Conv}(P_1) = \text{Conv}(K_1 \cup K_{01}) \).

Proof. We have \( K_1 \subseteq P_1 \), and furthermore, as \( P_1 \) is closed, we have \( K_{01} \subseteq P_1 \). Hence, we obtain that \( \text{Conv}(K_1 \cup K_{01}) \subseteq \text{Conv}(P_1) \). Thus, to prove the lemma statement, we must show \( \text{Conv}(P_1) \subseteq \text{Conv}(K_1 \cup K_{01}) \). Moreover, since \( \text{Conv}(P_1) \) is the smallest convex set containing \( P_1 \), it suffices to show \( P_1 \subseteq \text{Conv}(K_1 \cup K_{01}) \).

Let \( \mu \in P_1 \). By Lemma 4, \( \mu \in \text{Conv}(L_0 \cup K_{01}) \cup \text{Conv}(K_1 \cup K_{01}) \). Suppose for the sake of contradiction, \( \mu \notin \text{Conv}(K_1 \cup K_{01}) \). Then, it follows that \( \mu \in \text{Conv}(L_0 \cup K_{01}) \) and \( \mu \notin \text{Conv}(K_0) \). Consider any convex decomposition of \( \mu \) as follows:

\[
\mu = \sum_{\omega \in L_0} \alpha^0_\omega \omega + \sum_{\omega \in K_0 \cap K_1} \alpha^0_\omega \omega + \sum_{\phi \in K_{01} \cap K_1^2} \alpha^0_\phi \phi.
\]

Define \( H^0_0 = \{ \omega \in L_0 : \alpha_\omega > 0 \} \), \( H^0_1 = \{ \omega \in K_0 \cap K_1 : \alpha^0_\omega > 0 \} \), and \( H^3_0 = \{ \phi \in K_{01} \cap K_1^2 : \alpha^0_\phi > 0 \} \). Since \( \mu \notin \text{Conv}(K_{01}) \), it follows that \( \sum_{\omega \in L_0} \alpha_\omega > 0 \). Thus, we obtain that \( H^0_0 \) is non-empty.

In the following, we inductively define sets \( \{ H^i_n : i = 0, 1, 2 \} \) for \( n \geq 1 \) as long as both \( H^0_{n-1} \) and \( H^1_{n-1} \) are non-empty, such that the following four properties hold: (1) \( H^0_n \subseteq L_0 \); (2) \( H^1_n \subseteq K_{01} \cap K_1 \); (3) \( H^2_n \subseteq K_0 \cap K_1 \); (4) \( H^3_n \subseteq K_0 \cap K_1^2 \).
for any \( \chi(\omega_0, \omega_1) \in H_n^2 \subseteq K_{01} \), either \( \chi(\omega_0, \omega_1) \in K_{01} \cap K_{11}^c \) or \( \omega_0 \in H_n^0 \); and (4) \( \mu \) is a strict convex combination of elements in \( H_n^0 \cup H_n^1 \cup H_n^2 \). Towards that end, first note that the sets \( \{ H_n^i : i = 0, 1, 2 \} \) satisfy the aforementioned four properties.

Next, suppose for some \( n \geq 1 \) the sets \( \{ H_{n-1}^i : i = 0, 1, 2 \} \) satisfy the four properties, with both \( H_{n-1}^0 \) and \( H_{n-1}^1 \) being non-empty. Consider a strict convex decomposition of \( \mu \) in terms of elements in \( \cup_{i=0}^2 H_{n-1}^i \) with coefficients \( \alpha_n^{\mu} > 0 \) for \( \phi \in \cup_{i=0}^2 H_{n-1}^i \). Choose some \( \omega_0 \in H_{n-1}^0 \) and \( \omega_1 \in H_{n-1}^1 \), and let \( \beta_{n-1} = \alpha_n^{\omega_0} / (\alpha_n^{\omega_0} + \alpha_n^{\omega_1}) \). If \( \beta_{n-1} \omega_0 + (1 - \beta_{n-1}) \omega_1 \) equals \( \chi(\omega_0, \omega_1) \), let \( H_n^i = H_{n-1}^i \setminus \{ \omega_i \} \) for \( i \in \{0, 1\} \) and \( H_n^2 = H_{n-1}^2 \cup \{ \chi(\omega_0, \omega_1) \} \). Since \( \alpha_n^{\omega_i} > 0 \) for \( i \in \{0, 1\} \), we have \( \beta_{n-1} \in (0, 1) \) and hence \( \chi(\omega_0, \omega_1) \in K_{01} \cap K_{11}^c \). Finally, letting \( \alpha_n^{\mu} = \alpha_n^{\omega_0} \) for all \( \phi \in \cup_{i=0}^n H_n^i \setminus \{ \chi(\omega_0, \omega_1) \} \) and \( \alpha_n^{\omega_0} = (\alpha_n^{\omega_0} + \alpha_n^{\omega_1}) > 0 \) for \( \phi = \chi(\omega_0, \omega_1) \), we obtain a strict convex combination of \( \mu \) in terms of elements of \( \cup_{i=0}^2 H_n^i \). Thus, the four properties continue to hold for \( \{ H_n^i : i = 0, 1, 2 \} \).

If \( \beta_{n-1} \omega_0 + (1 - \beta_{n-1}) \omega_1 \) does not equal \( \chi(\omega_0, \omega_1) \), it can be written either as a strict convex combination (1) of \( \omega_0 \) and \( \chi(\omega_0, \omega_1) \) or (2) of \( \chi(\omega_0, \omega_1) \) and \( \omega_1 \).

In the first case, let \( H_n^0 = H_{n-1}^0 \setminus \{ \omega_0 \} \), \( H_n^1 = H_{n-1}^1 \setminus \{ \omega_1 \} \) and \( H_n^2 = H_{n-1}^2 \cup \{ \chi(\omega_0, \omega_1) \} \). Note that properties (1) and (2) hold trivially, and since \( \omega_0 \in H_n^0 \), property (3) continues to hold. Using the strict convex combination of \( \beta_{n-1} \omega_0 + (1 - \beta_{n-1}) \omega_1 \) in terms of \( \omega_0 \) and \( \chi(\omega_0, \omega_1) \), we obtain a strict convex combination of \( \mu \) in terms of elements of \( \cup_{i=0}^2 H_n^i \), and hence property (4) also holds.

In the second case, let \( H_n^0 = H_{n-1}^0 \setminus \{ \omega_0 \} \), \( H_n^1 = H_{n-1}^1 \) and \( H_n^2 = H_{n-1}^2 \cup \{ \chi(\omega_0, \omega_1) \} \). Again, properties (1) and (2) hold trivially. Since \( \beta_{n-1} \omega_0 + (1 - \beta_{n-1}) \omega_1 \) with \( \beta_{n-1} \in (0, 1) \) is a strict convex combination of \( \chi(\omega_0, \omega_1) \) and \( \omega_1 \), it follows that \( \chi(\omega_0, \omega_1) \neq \omega_1 \) and hence \( \chi(\omega_0, \omega_1) \in K_{01} \cap K_{11}^c \). Thus property (3) holds. Finally, using the strict convex combination of \( \beta_{n-1} \omega_0 + (1 - \beta_{n-1}) \omega_1 \) in terms of \( \chi(\omega_0, \omega_1) \) and \( \omega_1 \), we obtain a strict convex combination of \( \mu \) in terms of elements of \( \cup_{i=0}^2 H_n^i \), and hence property (4) also holds.

Note that in all three cases, we have \( |H_n^0| + |H_n^1| < |H_{n-1}^0| + |H_{n-1}^1| \). Thus, this inductive process stops with sets \( H_n^i, i = 0, 1, 2 \) for some \( n \geq 0 \) with either \( H_n^0 \) or \( H_n^1 \) empty. If \( H_n^0 \) is empty, then \( \mu \in \text{Conv}(H_n^1 \cup H_n^2) \subseteq \text{Conv}(K_{01}) \), contradicting the assumption that \( \mu \notin \text{Conv}(K_{01} \cup K_1) \). Thus, it must be that \( H_n^1 \) is empty. Furthermore, consider any \( \phi = \chi(\omega_0, \omega_1) \in H_n^2 \) for which \( \omega_0 \notin H_n^0 \). By property (3), it must be that \( \chi(\omega_0, \omega_1) \in K_{01} \cap K_{11}^c \). Choose any \( \omega' \in H_n^0 \), and define \( \beta = \alpha_{n'}^{\omega'} / (\alpha_{n'}^{\omega'} + \alpha_{n}^{\mu}) \). Since \( \chi(\omega_0, \omega_1) \in K_{01} \cap K_{11}^c \) and \( \omega' \in H_n^0 \), it is straightforward to verify that \( \beta \omega' + (1 - \beta) \chi(\omega_0, \omega_1) \) can be written as a strict convex combination of \( \omega_0, \omega', \chi(\omega_0, \omega_1) \) and \( \chi(\omega', \omega_1) \). Using such a strict convex combination and adding \( \omega_0 \in L_0 \) to the set \( H_n^0 \), we can without loss of generality assume that for any \( \phi = \chi(\omega_0, \omega_1) \in H_n^2 \) we have \( \omega_0 \in H_n^0 \). We thus obtain the following strict convex decomposition:

\[
\mu = \sum_{\omega \in H_n^0} \gamma_\omega \omega + \sum_{\phi \in H_n^2} \gamma_\phi \phi,
\]
where $\sum_{\omega \in H^0_n} \gamma_\omega > 0$, and for which if $\chi(\omega_0, \omega_1) \in H^2_n$ then $\omega_0 \in H^0_n$.

This can be further rewritten as the following convex combination

$$
\mu = \sum_{\omega \in H^0_n} \frac{\gamma_\omega}{\gamma_\omega + \gamma_{\omega_0, \omega_1}} \omega + \frac{\gamma_{\omega_0, \omega_1}}{\gamma_\omega + \gamma_{\omega_0, \omega_1}} \chi(\omega_0, \omega_1),
$$

where if $\gamma_{\omega_0, \omega_1} > 0$, then $\gamma_{\omega_0} > 0$. Note that for any such $(\omega_0, \omega_1)$, by definition of $\chi(\omega_0, \omega_1)$, the belief $\xi(\omega_0, \omega_1) \triangleq \frac{1}{\gamma_\omega + \gamma_{\omega_0, \omega_1}} (\gamma_{\omega_0} \omega_0 + \gamma_{\omega_0, \omega_1} \chi(\omega_0, \omega_1))$ lies in the set $P^c_1$. Thus, $\mu$ is a convex combination of elements in $H^0_n \subseteq L_0$ and the elements $\{\xi(\omega_0, \omega_1)\}$, all of which belong to $P^c_1$.

From Assumption 2, we then obtain that $\mu$ itself is an element of $P^c_1$, contradicting the fact that $\mu \in P_1$. This proves that our initial assumption that $\mu \notin \operatorname{Conv}(K_1 \cup K_{01})$ must be false, and hence $\mu \in \operatorname{Conv}(K_1 \cup K_{01})$.

Thus, $\mu \in P_1$ implies $\mu \in \operatorname{Conv}(K_1 \cup K_{01})$, and hence $P_1 \subseteq \operatorname{Conv}(K_1 \cup K_{01})$. Thus, we conclude that $\operatorname{Conv}(P_1) = \operatorname{Conv}(K_1 \cup K_{01})$. □

With these two helper lemmas in place, we are now ready to prove Theorem 2.

Proof of Theorem 2. Since the objectives of the programs (10) and (13) are identical, to prove the result it suffices to show that optimal solution of each program is a feasible for the other.

First, consider any optimal solution to (13). Since $\operatorname{Conv}(L_0 \cup \{0\}) \subseteq \operatorname{Conv}(P_0 \cup \{0\})$ and $\operatorname{Conv}(K_1 \cup K_{01} \cup \{0\}) \subseteq \operatorname{Conv}(P_1 \cup \{0\})$, it is a feasible solution to (10).

Next, consider any optimal solution $t = (t_0, t_1)$ to (10) with $t_1 \in \operatorname{Conv}(P_1 \cup \{0\})$ and $t_0 \in \operatorname{Conv}(P_0 \cup \{0\})$. By Lemma 5, it follows that $t_1 \in \operatorname{Conv}(K_1 \cup K_{01} \cup \{0\})$. If $t_0 = 0$, then $t_0 \in \operatorname{Conv}(L_0 \cup \{0\})$ and we are done. Instead, suppose $t_0 \neq 0$. Let $t_0 = b_0 m_0$ where $b_0 \in (0, 1]$ and $m_0 \in \operatorname{Conv}(P_0)$. By Lemma 4, we obtain $m_0 \in \operatorname{Conv}(L_0 \cup K_{01}) \cup \operatorname{Conv}(K_1 \cup K_{01})$. If $m_0 \notin \operatorname{Conv}(L_0)$, then $m_0 = (1 - \alpha) v_0 + \alpha v_1$ with $v_0 \in \operatorname{Conv}(L_0)$ and $v_1 \in \operatorname{Conv}(K_{01})$ and $\alpha > 0$. However, we then have that $\tilde{t} = (\tilde{t}_0, \tilde{t}_1)$ with $\tilde{t}_0 = 0$ and $\tilde{t}_1 = t_1 + t_0$ is feasible for (10) and achieves larger utility for the sender than $(t_0, t_1)$, contradicting the latter’s optimality. Hence, $m_0 \in \operatorname{Conv}(L_0 \cup K_{01})$. If $m_0 \notin \operatorname{Conv}(L_0)$, then $m_0 = (1 - \alpha) v_0 + \alpha v_1$ with $v_0 \in \operatorname{Conv}(L_0)$ and $v_1 \in \operatorname{Conv}(K_{01})$ and $\alpha > 0$. However, we then have that $\tilde{t} = (\tilde{t}_0, \tilde{t}_1)$ with $\tilde{t}_0 = b_0 (1 - \alpha) v_0$ and $\tilde{t}_1 = t_1 + b_0 \alpha v_1$ is feasible for (10) and has larger utility for the sender than $(t_0, t_1)$, once again contradicting the latter’s optimality. Thus, we must have $m_0 \in \operatorname{Conv}(L_0)$ and hence $t_0 \in \operatorname{Conv}(L_0 \cup \{0\})$. Taken together, we obtain that $(t_0, t_1)$ is feasible for (13). □

Finally, we conclude this section with the proof of Proposition 2.

Proof of Proposition 2. Clearly, action 0 is uniquely optimal for the receiver under her prior belief if and only if $\mu^* \in P^c_1$. On the other hand, action 1 can be made optimal with probability one if and only if $\mu^*$ can be expressed as a convex combination of beliefs in $P_1$, i.e., $\mu^* \in \operatorname{Conv}(P_1)$. Thus, the receiver can be fully persuaded if and only if $\mu^* \in \operatorname{Conv}(P_1) \cap P^c_1$. Finally, using Assumption 2, we establish in Lemma 5 that $\operatorname{Conv}(P_1) = \operatorname{Conv}(K_1 \cup K_{01})$. □
Appendix D: Proofs from Section 5

The following result implies that differential utility function \( \hat{\rho}(\cdot) \) defined in (14) satisfies Assumption 2. We use this implication in the proof of Proposition 3.

**Lemma 6.** The differential utility function \( \hat{\rho}(\mu) = \tau - \left( E_\mu[X] + \beta \sqrt{\text{Var}_\mu[X]} \right) \) is convex in \( \mu \).

**Proof.** From the fact that expectation is linear in the belief, it is straightforward to show that \( \text{Var}_\mu[X] = E_\mu(X^2) - (E_\mu(X))^2 \) is concave in the belief \( \mu \). Since \( \sqrt{x} \) is concave and strictly increasing in \( x \), using Jensen’s inequality, we obtain that \( \sqrt{\text{Var}_\mu[X]} \) is also concave in \( \mu \). From these facts, we conclude that \( \hat{\rho}(\mu) = \tau - \left( E_\mu[X] + \beta \sqrt{\text{Var}_\mu[X]} \right) \) is convex in \( \mu \). \( \square \)

**D.1. Proof of Proposition 3**

Our proof follows a similar approach as in Lingenbrink and Iyer (2019): we show that any feasible solution where the customers’ actions do not have a threshold structure can be perturbed to obtain another feasible solution corresponding to a signaling scheme with higher throughput.

To begin, let \( \Omega = \{0, 1, \cdots, C - 1\} \) denote the set of queue-lengths for which an arriving customer could possibly join the queue. (We assume that immediately upon arrival a customer is informed whether the queue-length equals \( C \) or not.) For \( n \in \Omega \), let \( e_n \) denote the belief that puts all its weight on \( n \). In other words, under the belief \( e_n \), an arriving customer knows that there are exactly \( n \) people in queue already. We have \( K_1 = \{ n \in \Omega : \hat{\rho}(e_n) \geq 0 \} \), and \( L_0 = \{ n \in \Omega : \hat{\rho}(e_n) < 0 \} \). Since \( \hat{\rho}(e_n) \) is strictly decreasing in \( n \), we obtain \( K_1 = \{0, 1, \cdots, M\} \) for some \( M < C \), and \( L_0 = \{ n \in \Omega : n > M \} \).

Finally, for \( m \in L_0 \) and \( n \in K_1 \), let \( \chi(m, n) \in K_{01} \) be given by the following convex combination:

\[
\chi(m,n) = \gamma(m,n)e_m + (1 - \gamma(m,n))e_n, \quad \text{where} \quad \gamma(m,n) = \sup\{ \gamma \in [0,1] : \hat{\rho}(\gamma e_m + (1 - \gamma)e_n) \geq 0 \}.
\]

Recall that under the belief \( \chi(m, n) \), the customer is indifferent between joining and leaving the queue.

**Proof of Proposition 3.** First, note that if \( K_1 = \Omega \). i.e., \( M = C - 1 \), then joining the queue is always optimal if feasible. In this case, the full-information mechanism (a mechanism inducing a threshold structure on the actions) has the highest throughput. Hence, hereafter we focus on the case where \( K_1 \) is a strict subset of \( \Omega \), i.e., \( M < C - 1 \).

Consider a feasible solution \( t = (t_1, t_0) \) to the linear program (15). Since \( t_0 \in \text{Conv}(L_0 \cup \{0\}) \) and \( L_0 = \{M + 1, \cdots, C - 1\} \), we have \( t_0(k) = 0 \) for all \( k \leq M \). Then the constraint (15d) implies that \( t_1(k + 1) = \lambda t_1(k) \) for all \( k \leq M \), and hence \( t_1(k) = \lambda^k t_1(0) \) for all \( k \leq M + 1 \).

Let \( N \) be the largest value of \( n \) such that \( t_1(k) = \lambda^k t_1(0) \) for all \( k \leq n \). (Note the preceding argument implies \( N \geq M + 1 \).) If either \( N \geq C - 2 \) or \( t_1(N + 2) = 0 \), then we obtain that \( t \) induces a threshold structure in the customers’ actions. Hence, for the rest of the proof assume that \( N < C - 2 \) and \( t_1(N + 2) > 0 \). Using (15d), we then obtain \( 0 < t_1(N + 1) < \lambda t_1(N) \), \( t_0(N + 1) > 0 \) and \( t_0(n) = 0 \).
for \( n \leq N \). We now show that \( t = (t_1, t_0) \) cannot be an optimal solution to (15). We do this by constructing another feasible solution \( \bar{t} = (\bar{t}_1, \bar{t}_0) \) that has a larger objective value than \( t \).

For some small \( \epsilon > 0 \) and \( \beta > 0 \) to be chosen later, consider \( t = (t_1, t_0) \) with

\[
\bar{t}_1 = \frac{1}{Z} \left( t_1 - \beta \sum_{m=N+2}^{C-1} t_1(m)e_m + (\beta + \epsilon) \sum_{m=N+2}^{C-1} t_1(m)e_{N+1} \right)
\]

and \( \bar{t}_0(0) = 0, \bar{t}_0(n) = \lambda t_1(n - 1) - \bar{t}_1(n) \) for all \( n > 0 \), where \( Z \) is chosen to satisfy (15c); it is straightforward to verify that

\[
Z = 1 + \lambda \epsilon \sum_{m=N+2}^{C-1} t_1(m).
\]

In particular, for \( \epsilon > 0 \), we obtain that \( Z > 1 \).

We begin by showing that \( \bar{t} \) is feasible for (15) for all small enough \( \epsilon, \beta > 0 \). Note that \( \bar{t} \) satisfies (15d) and (15c) by definition. Thus, we verify the conditions (15a) and (15b).

First, note that for \( \ell \leq N \), we have \( \bar{t}_1(\ell) = \frac{t_1(\ell)}{Z} \). On the other hand, we have \( \bar{t}_1(N+1) = \frac{1}{Z}(t_1(N+1) + (\beta + \epsilon) \sum_{m=N+2}^{C-1} t_1(m)) \) and \( \bar{t}_1(\ell) = \frac{1}{Z}t_1(\ell) \) for \( \ell > N + 1 \). Using this and the definition of \( \bar{t}_0 \), it is straightforward to verify that for small enough \( \epsilon, \beta > 0 \), we have \( \bar{t}_0 \geq 0 \) and \( \bar{t}_0(0) = 0 \) for all \( \ell \leq N \). Further using the fact that \( \sum_{\ell} \bar{t}_0(\ell) \leq 1 \), we obtain \( \bar{t}_0 \in \text{Conv}(L_0 \cup \{0\}) \) for all small enough \( \epsilon, \beta > 0 \), and hence \( \bar{t} \) satisfies (15a).

Next, since \( t_1 \in \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \), we can write the decomposition

\[
t_1 = \sum_{j=0}^{M} \alpha(j)e_j + \sum_{j=0}^{M} \sum_{i=M+1}^{C-1} \delta(i,j)\chi(i,j),
\]

where \( \alpha(j) \geq 0 \) for \( j \leq M \), \( \delta(i,j) \geq 0 \) for \( j \leq M < i \), and \( \sum_{j=0}^{M} \alpha(j) + \sum_{j=0}^{M} \sum_{i=M+1}^{C-1} \delta(i,j) \leq 1 \). Thus, we have

\[
Z\bar{t}_1 = t_1 - \beta \sum_{m=N+2}^{C-1} t_1(m)(e_m - e_{N+1}) + \epsilon \sum_{m=N+2}^{C-1} t_1(m)e_{N+1}
\]

\[
= t_1 - \beta \sum_{m=N+2}^{C-1} \sum_{j=0}^{M} \delta(m,j)\gamma(m,j)(e_m - e_{N+1}) + \epsilon \sum_{m=N+2}^{C-1} \sum_{j=0}^{M} \delta(m,j)\gamma(m,j)e_{N+1}.
\]

Now, it can be readily verified that for \( 0 \leq j \leq M \) and \( m > N + 1 \),

\[
\gamma(m,j)(e_m - e_{N+1}) = \chi(m,j) - \left( \frac{\gamma(m,j)}{\gamma(N+1,j)}\chi(N+1,j) + \left( 1 - \frac{\gamma(m,j)}{\gamma(N+1,j)} \right) e_j \right).
\]
Thus, after some algebra, we obtain
\[
Z\tilde{t}_1 = t_1 - \beta \sum_{m=N+2}^{C-1} \sum_{j=0}^{M} \delta(m,j)\chi(m,j) + \beta \sum_{m=N+2}^{C-1} \sum_{j=0}^{M} \delta(m,j) \left( \frac{\gamma(m,j)}{\gamma(N+1,j)} \chi(N+1,j) + \left( 1 - \frac{\gamma(m,j)}{\gamma(N+1,j)} \right) e_j \right) + \epsilon \sum_{m=N+2}^{C-1} \sum_{j=0}^{M} \delta(m,j)\gamma(m,j)e_{N+1}.
\]

Now, since \( t_1 \in \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \), we have the first term \( t_1 - \beta \sum_{m=N+2}^{C-1} \sum_{j=0}^{M} \delta(m,j)\chi(m,j) \in \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \) for small enough \( \beta > 0 \). Moreover, for any \( j \leq M \) and \( m > N + 1 \), it is straightforward to verify that \( \bar{\rho} \gamma(m,j)e_{N+1} + (1 - \gamma(m,j))e_j > \bar{\rho} \gamma(m,j)e_m + (1 - \gamma(m,j))e_j = 0 \), and hence \( \gamma(N+1,j) > \gamma(m,j) \). Hence, for all small enough \( \beta > 0 \), we obtain that second term lies in the \( \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \), but not in \( \text{Conv}(K_{01} \cup \{0\}) \). This in turn implies that the second and third term together lie in \( \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \) for small enough \( \beta, \epsilon > 0 \). Taken together, this implies that \( Z\tilde{t}_1 \) (and hence \( \tilde{t}_1 \)) lies in the cone generated by \( \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \). Since \( \sum_{\omega} \bar{t}_1(\omega) \leq 1 \), we conclude that \( \bar{t}_1 \in \text{Conv}(K_1 \cup K_{01} \cup \{0\}) \), and hence \( \bar{t} \) satisfies (15b).

Having shown the feasibility of \( \bar{t} \), we now verify that the objective value under \( \bar{t} \) is greater than that under \( t \). Using (15c) and (15d), we obtain
\[
\sum_{\ell=0}^{C-1} \bar{t}_1(\ell) = \frac{1}{\lambda} (1 - \bar{t}_1(0)) > \frac{1}{\lambda} (1 - t_1(0)) = \sum_{\ell=0}^{C-1} t_1(\ell).
\]

Here, the inequality follows from the fact that \( \bar{t}_1(0) = \frac{1}{2} t_1(0) < t_1(0) \) since \( Z > 1 \). Thus, the objective of (15) for \( \bar{t} \) is strictly greater than that for \( t \). Thus, it follows that \( t \) cannot be optimal for (15).

To conclude the proof, we obtain from the preceding argument that for any optimal \( t = (t_1, t_0) \), if \( N \) denotes the largest value of \( n \) such that \( t_1(k) = \lambda^k t_1(0) \) for all \( k \leq n \), then either \( N \geq C - 2 \) or \( t_1(N + 2) = 0 \). In either case, we obtain that \( t \) induces a threshold structure in the customers’ actions. \( \square \)

**D.2. Full-information and no-information mechanisms**

Recall that if the queue length is \( \omega \), the waiting time \( X \) is a sum of \( \omega + 1 \) independent random variables, each exponentially distributed with rate 1 (the waiting times for \( \omega \) people in the queue plus the customer’s own service time). Therefore, the distribution of \( X \) is given by a Gamma distribution with shape \( \omega + 1 \) and scale 1; we denote the cdf of this distribution by \( F_\omega \). In particular, we obtain \( E_\omega(X) = \omega + 1 \) and \( \text{Var}_\omega(X) = \omega + 1 \).

If \( 1 + \beta > \tau \), then under any signaling scheme, no customer joins the queue as even if the queue is empty and the customer is served immediately upon arrival, her utility is \( \bar{\rho}(e_0) = \tau - (1 + \beta) < 0 \). Therefore, the throughput is 0. Given this result, we assume that \( 1 + \beta \leq \tau \).
Under the fully-revealing scheme, the customer observes the queue length $\omega \in \Omega$ and joins if and only if $\omega < C$ and $\bar{\rho}(e_\omega) = \tau - (E_\omega(X) + \beta \sqrt{\text{Var}_\omega(X)}) = \tau - (\omega + 1 + \beta \sqrt{\omega + 1}) \geq 0$. Let $\bar{\omega}$ be the largest value of $\omega \in \Omega$ for which $\bar{\rho}(e_\omega) \geq 0$, and let $\omega^* = \min\{\bar{\omega}, C - 1\}$. Then, it is straightforward to show that the equilibrium throughput under the full-revealing scheme equals $\sum_{k=1}^{\omega^*+1} \lambda^k / \sum_{k=0}^{\omega^*+1} \lambda^k$.

Under the no-information scheme, suppose the customers’ strategy is to join the queue with probability $q \in [0, 1]$. The steady state distribution $\pi^q$ under this strategy profile is then given by
\[
\pi^q_n = q^n \lambda^n \left( \frac{1 - q \lambda}{1 - (q \lambda)^{C+1}} \right), \quad \text{for } n \in \Omega.
\]
The waiting time $X$ under the steady state then has the cdf $\sum_{n \in \Omega} \pi^q_n F_n$. Using this cdf, we can compute the differential utility for joining as $\bar{\rho}(\pi^q) = \tau - E_{\pi^q}(X) - \beta \sqrt{\text{Var}_{\pi^q}(X)}$. We omit the details for brevity.

In equilibrium, $q = 0$ if and only if $\bar{\rho}(\pi^0) = \bar{\rho}(e_0) = \tau - 1 - \beta \leq 0$. When $1 + \beta > \tau$, we have $q = 1$ to be the equilibrium if and only if $\bar{\rho}(\pi^1) \geq 0$. Otherwise, there exists a $q \in (0, 1)$ with $\bar{\rho}(\pi^q) = 0$. The equilibrium under no-information mechanism is then the largest value of such $q$.

**Acknowledgments**

The second and the third authors gratefully acknowledge support from the National Science Foundation under grants CMMI-1633920 and CMMI-2002156. Portions of this work were completed when the second author was with Cornell University. A preliminary version of this work appeared as a one-page abstract at WINE 2019, and as a poster at the Workshop on Behavioral EC (2019); we thank the anonymous reviewers at these conferences for their feedback.

**References**

Ahmadi-Javid A (2012) Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization Theory and Applications* 155(3):1105–1123.

Alizamir S, de Vericourt F, Wang S (2018) Warning against recurring risks: An information design approach. *Working paper*.

Allais M (1979) The so-called allais paradox and rational decisions under uncertainty. Allais M, Hagen O, eds., *Expected Utility Hypotheses and the Allais Paradox: Contemporary Discussions of the Decisions under Uncertainty with Allais’ Rejoinder*, 437–681 (Springer Netherlands, Dordrecht), URL [http://dx.doi.org/10.1007/978-94-015-7629-1_17](http://dx.doi.org/10.1007/978-94-015-7629-1_17).
Arieli I, Babichenko Y (2019) Private bayesian persuasion. *Journal of Economic Theory* 182:185 – 217, URL http://dx.doi.org/10.1016/j.jet.2019.04.008.

Artzner P, Delbaen F, Eber J, Heath D (2001) Coherent measures of risk. *Mathematical Finance* 9:203–228, URL http://dx.doi.org/10.1111/1467-9965.00068.

Aumann RJ, Maschler M (1995) *Repeated Games with Incomplete Information* (MIT press).

Bárány I, Onn S (1995) Carathéodory’s theorem, colourful and applicable. *Intuitive Geometry* 6:11–21.

Beauchêne D, Li J, Li M (2019) Ambiguous persuasion. *Journal of Economic Theory* 179:312–365, URL http://dx.doi.org/10.1016/j.jet.2018.10.008.

Bergemann D, Morris S (2016) Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics* 11(2):487–522.

Bergemann D, Morris S (2018) Information design: A unified perspective. *Journal of Economic Literature* Forthcoming.

Candogan O (2019) Persuasion in networks: Public signals and k-cores. *Proceedings of the 2019 ACM Conference on Economics and Computation*, 133–134, EC ’19 (Association for Computing Machinery, New York, NY, USA), URL http://dx.doi.org/10.1145/3328526.3329618.

Candogan O, Drakopoulos K (2020) Optimal signaling of content accuracy: Engagement vs. misinformation. *Operations Research* 68(2):497–515, URL http://dx.doi.org/10.1287/opre.2019.1897.

Carlo Acerbi DT (2002) Expected shortfall: a natural coherent alternative to value at risk. *Economic Notes* 31(2):379–388.

Cominetti R, Torrico A (2016) Additive consistency of risk measures and its application to risk-averse routing in networks. *Mathematics of Operations Research* 41(4):1510–1521.

DellaVigna S (2009) Psychology and economics: Evidence from the field. *Journal of Economic Literature* 47(2):315–72, URL http://dx.doi.org/10.1257/jel.47.2.315.

Drakopoulos K, Jain S, Randhawa RS (2018) Persuading customers to buy early: The value of personalized information provisioning. *SSRN Electronic Journal* URL http://dx.doi.org/10.2139/ssrn.3191629.
Dughmi S, Xu H (2016) Algorithmic bayesian persuasion. *Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing*, 412–425, STOC ’16 (Association for Computing Machinery), URL http://dx.doi.org/10.1145/2897518.2897583.

Ellsberg D (1961) Risk, ambiguity, and the savage axioms. *The Quarterly Journal of Economics* 75(4):643–669.

Föllmer H, Schied A (2011) *Stochastic finance: an introduction in discrete time* (Walter de Gruyter).

Gilboa I, Schmeidler D (1989) Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics* 18:141–153, URL http://dx.doi.org/10.1016/0304-4068(89)90018-9.

Hu J, Weng X (2018) Robust persuasion of a privately informed receiver. SSRN 3134179.

Jorion P (2006) *Value at Risk: The New Benchmark for Managing Financial Risk* (McGraw-Hill, New York).

Kahneman D, Tversky A (1972) Subjective probability: A judgment of representativeness. *Cognitive Psychology* 3(3):430–454.

Kahneman D, Tversky A (1979) Prospect theory: An analysis of decision under risk. *Econometrica* 47(2):263–292.

Kamenica E (2018) Bayesian persuasion and information design. *Annual Review of Economics* Forthcoming.

Kamenica E, Gentzkow M (2011) Bayesian persuasion. *American Economic Review* 101(6):2590–2615.

Kolotilin A, Mylovanov T, Zapechelnyuk A, Li M (2017) Persuasion of a privately informed receiver. *Econometrica* 85(6):1949–1964.

Lianeas T, Nikolova E, Stier-Moses NE (2019) Risk-averse selfish routing. *Mathematics of Operations Research* 44(1):38–57, URL http://dx.doi.org/10.1287/moor.2017.0913.

Lingenbrink D, Iyer K (2018) Signaling in online retail: Efficacy of public signals. *SSRN Electronic Journal* URL http://dx.doi.org/10.2139/ssrn.3179262.

Lingenbrink D, Iyer K (2019) Optimal signaling mechanisms in unobservable queues. *Operations Research* 67(5):1397–1416, URL http://dx.doi.org/10.1287/opre.2018.1819.

Lipnowski E, Mathevet L (2018) Disclosure to a psychological audience. *American Economic Journal: Microeconomics* 10(4):67–93, URL http://dx.doi.org/10.1257/mic.20160247.

Machina MJ (1982) “Expected utility” analysis without the independence axiom. *Econometrica* 50(2):277–323.
Machina MJ (1995) Non-expected utility and the robustness of the classical insurance paradigm. *The Geneva Papers on Risk and Insurance Theory* 20(1):9–50.

Maister DH, et al. (1984) *The psychology of waiting lines* (Harvard Business School, Boston).

Markowitz HM (1952) Portfolio selection. *Journal of Finance* 7(1):77–91.

Nikolova E, Stier-Moses NE (2014) A mean-risk model for the traffic assignment problem with stochastic travel times. *Operations Research* 62(2):366–382.

Papanastasiou Y, Binpikis K, Savva N (2018) Crowdsourcing exploration. *Management Science* 64(4):1727–1746, URL http://dx.doi.org/10.1287/mnsc.2016.2697.

Rabin M (1998) Psychology and economics. *Journal of Economic Literature* 36(1):11–46.

Rayo L, Segal I (2010) Optimal information disclosure. *Journal of Political Economy* 118(5):949–987.

Romanyuk G, Smolin A (2018) Cream skimming and information design in matching markets. *American Economic Journal: Microeconomics* Forthcoming.

Savage LJ (1954) *The Foundations of Statistics* (John Wiley & Sons, New York).

Taneva I (2019) Information design. *American Economic Journal: Microeconomics* Forthcoming.

Tversky A, Kahneman D (1992) Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5(4):297–323.

Yang P, Iyer K, Frazier P (2019) Information design in spatial resource competition. *The 15th Conference on Web and Internet Economics*, WINE 2019, URL arXiv:1909.12723.