Generalization of a Result of Sylvester
Regarding the Frobenius Coin Problem and
an Elementary Proof of Eisenstein’s Lemma
for Jacobi Symbols

Damanvir Singh Binner
Department of Mathematics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada
dbinner@sfu.ca

Abstract
In a recent work, the present author generalized a fundamental result of Gauss
related to quadratic reciprocity, and also showed that the above result of Gauss is
equivalent to a special case of a well-known result of Sylvester related to the Frobenius
coin problem. In this note, we use this equivalence to show that the above generaliza-
tion of the result of Gauss naturally leads to an interesting generalization of the result
of Sylvester. To be precise, for given positive coprime integers \(a\) and \(b\), and for a family
of values of \(k\) in the interval \(0 \leq k < (a - 1)(b - 1)\), we find the number of nonnegative
integers \(\leq k\) that can be expressed in the form \(ax + by\) for nonnegative integers \(x\) and
\(y\). We also give an elementary proof of Eisenstein’s Lemma for Jacobi symbols using
floor function sums. Our proof provides a natural straightforward generalization of the
Gauss-Eisenstein proof of the law of quadratic reciprocity for Jacobi symbols.

1 Introduction
Throughout this note, \([x]\) denotes the greatest integer less than or equal to \(x\). Recall the
following well-known results of Gauss and Sylvester.

Theorem 1 (Gauss (1808)). For distinct odd primes \(p\) and \(q\),

\[
\sum_{i=1}^{p-1} \left\lfloor \frac{iq}{p} \right\rfloor + \sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = (p - 1)(q - 1) - \frac{1}{4}.
\]

Theorem 2 (Sylvester (1882)). If \(a\) and \(b\) are coprime numbers, the number of natural
numbers that cannot be expressed in the form \(ax + by\) for nonnegative integers \(x\) and \(y\) is
equal to \((a - 1)(b - 1) - \frac{1}{2}\).
Remark 3. Theorem 1 and its proof hold verbatim for any odd positive coprime integers $a$ and $b$.

In this note, the following special case of Theorem 2 will be of particular interest.

**Theorem 4.** If $p$ and $q$ are distinct odd prime numbers, the number of natural numbers that cannot be expressed in the form $px + qy$ for nonnegative integers $x$ and $y$ is equal to $rac{(p-1)(q-1)}{2}$.

We refer to Theorem 2 as *Sylvester’s Theorem* and to Theorem 4 as *Special Case of Sylvester’s Theorem*.

Gauss [9] proved Theorem 1 in 1808, and this completed his third proof of the law of quadratic reciprocity. Eisenstein [8] gave a geometric proof of Theorem 1 in 1844. We refer the reader to Baumgart [2, pp. 15–20] for more information about these classical proofs. Sylvester [15] proved Theorem 2 in 1882. In 1883, he posed it as a recreational problem and Curran [16] published a short proof based on generating functions. Several more contemporary proofs of Sylvester’s Theorem are known [11, Section 5.1].

The present author [3, Lemma 7] generalized Theorem 1 as follows.

**Theorem 5.** Let $a$, $b$, $d$, and $K$ be positive integers such that $b < a$, $d < a$, gcd($a, b$) = 1, and $K = \left\lfloor \frac{bd}{a} \right\rfloor$. Then,

$$\sum_{i=1}^{d} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor = dK.$$  

The present author [3, Section 3] proved that the Special Case of Sylvester’s Theorem is equivalent to the reciprocity relation of Gauss in Theorem 1. As described below, the importance of this equivalence is demonstrated by the fact that we are able to generalize Sylvester’s Theorem using our generalization of the Gauss’ result (Theorem 5). Prior to that, we choose an appropriate special case of Theorem 5 that is equivalent to Sylvester’s Theorem. It turns out that the following special case of Theorem 5 is precisely what we require.

**Theorem 6.** Let $a$ and $b$ be positive coprime integers. Then,

$$\sum_{i=1}^{\left\lfloor \frac{a}{2} \right\rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\left\lfloor \frac{b}{2} \right\rfloor} \left\lfloor \frac{ia}{b} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{b}{2} \right\rfloor.$$  

In Section 2, we prove Theorem 6 and also show that it is equivalent to Theorem 2. Note that Theorem 5 is a generalization of Theorem 6. Thus, it is natural to wonder whether the equivalence leads to a generalization of Theorem 2 that is equivalent to Theorem 5. This is in fact true and leads to an interesting result (Theorem 8). We make this sequence of equivalence of results more clear in Figure 1 below.

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Theorem 4 (Special Case of Sylvester’s Theorem) \( \Leftrightarrow \) Theorem 1 (Gauss (1808))

\( \bigcap \bigcup \bigcap \bigcup \)

Theorem 2 (Sylvester’s Theorem (1882)) \( \Leftrightarrow \) Theorem 6 (Binner (2021))

\( \bigcap \bigcup \bigcap \bigcup \)

Theorem 8 (Binner (2021)) \( \Leftrightarrow \) Theorem 5 (Binner (2020))

Figure 1: By Theorem X \( \subset \) Theorem Y, we mean that Theorem X is a special case of Theorem Y, and by Theorem X \( \Leftrightarrow \) Theorem Y, we mean that Theorems X and Y are equivalent. The present author [3, Section 2.3] proved Equivalence A in 2020. We prove Equivalences B and C in Sections 2 and 3, respectively.

Sylvester’s Theorem shows that exactly half of nonnegative integers till the Frobenius number \( ab - a - b \) can be expressed in the form \( ax + by \). It is natural to ask a more general question.

**Question 7.** For given positive coprime integers \( a \) and \( b \), and given \( k \) such that \( 0 \leq k < (a - 1)(b - 1) \), find the number of nonnegative integers \( \leq k \) that can be expressed in the form \( ax + by \) for nonnegative integers \( x \) and \( y \).

We denote this number by \( N_0(a, b; k) \). For \( k < 0 \), we define \( N_0(a, b; k) = 0 \). By Sylvester’s Theorem, \( N_0(a, b; ab - a - b) = \frac{(a-1)(b-1)}{2} \). In this note, we answer Question 7 for a specific family of values of \( k \) by proving Theorem 8.

**Theorem 8.** Let \( a \) and \( b \) be positive coprime integers with \( b < a \). Further, let \( 0 < \alpha < a \) be such that \( \alpha \equiv a \pmod{2} \), and \( \beta = 2 \left\lfloor \frac{b(\alpha + a)}{2a} \right\rfloor - b \). Then,

\[
N_0 \left( a, b; \frac{b\alpha + a\beta}{2} \right) = \frac{(\alpha + 1)(\beta + 1)}{2}.
\]

Setting \( \alpha = a - 2 \) in the above theorem gives Theorem 2. Note that

\[
\beta \geq 2 \left\lfloor \frac{b}{2} \right\rfloor - b \geq -1.
\]

Thus, for \( \frac{b\alpha + a\beta}{2} < 0 \), it must be true that \( \beta = -1 \), and then by Theorem 8 \( N_0 \left( a, b; \frac{b\alpha + a\beta}{2} \right) = 0 \), which is consistent with our definition of \( N_0(a, b; k) \) for negative values of \( k \).

We demonstrate Theorem 8 for an example. Suppose that \( a = 29 \) and \( b = 23 \). Then Theorem 2 shows that \( N_0(29, 23; 615) = 308 \). However, Theorem 8 gives us \( N_0(29, 23; k) \) for a lot of values of \( k \), as described in Table 1.
| $\alpha$ | $k$ | $N_0(29, 23; k)$ |
|--------|-----|-----------------|
| 1      | -1  | 0               |
| 3      | 49  | 4               |
| 5      | 101 | 12              |
| 7      | 153 | 24              |
| 9      | 205 | 40              |
| 11     | 228 | 48              |
| 13     | 280 | 70              |
| 15     | 332 | 96              |
| 17     | 384 | 126             |
| 19     | 436 | 160             |
| 21     | 459 | 176             |
| 23     | 511 | 216             |
| 25     | 563 | 260             |
| 27     | 615 | 308             |

Table 1: The values of $N_0(29, 23; k)$ versus $k$, as $\alpha$ varies from 1 to 27 such that $\alpha$ is odd.

In Section 3, we prove Theorem 8 and also show that it is equivalent to Theorem 5 (Equivalence C in Figure 1). For other values of $k$ not covered by Theorem 8, we describe an easy method to calculate $N_0(a, b; k)$ in Section 4.

We conclude our discussion on Question 7 by mentioning that for given coprime natural numbers $a$ and $b$, the study of properties of numbers that cannot be expressed in the form $ax + by$ (also called nonrepresentable numbers), where $x$ and $y$ are nonnegative integers, continues to be an active area of research. Let $NR(a, b)$ denote the set of nonnegative integers $n$ that cannot be expressed in the form $ax + by$. Then, by Sylvester’s Theorem, $|NR(a, b)| = \frac{(a-1)(b-1)}{2}$. Brown and Shiue [4] discovered the sum $S(a, b)$ of nonrepresentable numbers. They proved that

$$S(a, b) := \sum_{n \in NR(a, b)} n = \frac{1}{12}(a - 1)(b - 1)(2ab - a - b - 1).$$

Rødseth considered a generalization of this sum

$$S_m(a, b) = \sum_{n \in NR(a, b)} n^m.$$

These sums $S_m(a, b)$ are commonly known as the Sylvester sums. Rødseth [13] discovered a formula for these sums in terms of Bernoulli numbers and found that

$$S_2(a, b) = \sum_{n \in NR(a, b)} n^2 = \frac{1}{12}(a - 1)(b - 1)ab(ab - a - b).$$

Recently, Komatsu and Zhang [10] considered the weighted Sylvester sums

$$S_m^{(\lambda)} = \sum_{n \in NR(a, b)} \lambda^{n-1} n^m.$$
They gave explicit expressions for these sums in terms of the Apostol-Bernoulli numbers.

However, in Question 7 we consider the number of nonrepresentable numbers below a given number \( k \), instead of considering all the nonrepresentable numbers.

In Section 5, we describe an application of floor function summation results to Jacobi symbols. Our approach leads to a natural straightforward generalization of the Gauss-Eisenstein proof of the law of quadratic reciprocity for Jacobi symbols. We recall some main steps in the Gauss-Eisenstein proof of the law of quadratic reciprocity for Legendre symbols. More details about these results can be found in [12, Chapter 3].

**Definition 1.** Let \( a \) and \( m \) be integers such that \( \gcd(a,m) = 1 \). Then \( a \) is called a quadratic residue modulo \( m \) if the congruence \( x^2 \equiv a \pmod{m} \) has a solution. If the congruence has no solution, then \( a \) is called a quadratic nonresidue modulo \( m \).

**Definition 2.** Let \( p \) denote an odd prime. The Legendre symbol \((a/p)\) is defined to be 1 if \( a \) is a quadratic residue modulo \( p \), \(-1\) if \( a \) is a quadratic nonresidue modulo \( p \), and \(0\) if \( p \) divides \( a \).

**Theorem 9** (Gauss’ Lemma for Legendre symbols). Suppose \( p \) is an odd prime and \( \gcd(a,p) = 1 \). Consider the integers \( a, 2a, 3a, \ldots, (p-1)/2 \) \( a \) and their least positive residues modulo \( p \). If \( n \) denotes the number of these residues that exceed \( p/2 \), then the Legendre symbol \((a/p)\) satisfies \( (a/p) = (-1)^n \).

**Theorem 10** (Eisenstein’s Lemma for Legendre symbols). If \( p \) is an odd prime and \( a \) is any odd number not divisible by \( p \), then \( (a/p) = (-1)^t \) where

\[
t = \sum_{i=1}^{p-1} \left\lfloor \frac{i a}{p} \right\rfloor.
\]

Theorems 9 and 10 immediately lead to the law of quadratic reciprocity which is as follows:

**Theorem 11** (Law of quadratic reciprocity for Legendre symbols). For distinct odd primes \( p \) and \( q \),

\[
\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{(p-1)(q-1)/4}.
\]

The theory of quadratic residues can be extended further by generalizing Legendre symbols to Jacobi symbols.

**Definition 3.** Let \( b \) be an odd number and \( b = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \) be its prime factorization. Then the Jacobi symbol \((a/b)\) is defined as

\[
\left( \frac{a}{b} \right) := \left( \frac{a}{p_1} \right)^{r_1} \left( \frac{a}{p_2} \right)^{r_2} \cdots \left( \frac{a}{p_k} \right)^{r_k},
\]

where \( \left( \frac{a}{p_i} \right) \) is the Legendre symbol.
Schering [14] generalized Gauss’ Lemma to the Jacobi symbol. However, a direct proof of the Gauss-Schering Lemma [14, 6, 11] seems to be quite technical. Zolotarev [18] observed that Legendre and Jacobi symbols are connected to signatures of naturally associated permutations. Using this approach, there are some other proofs [7, 5] showing that Gauss’ Lemma can be generalized to the Jacobi symbol. These methods give direct proofs of the law of quadratic reciprocity for Jacobi symbols at the cost of introducing some auxiliary concepts of an abstract algebraic nature. In the present note, we provide an elementary proof using floor function sums, showing that Eisenstein’s Lemma also holds for Jacobi symbols.

Theorem 12 (Eisenstein’s Lemma for Jacobi symbols). For odd positive coprime integers $a$ and $b$, the Jacobi symbol $\left(\frac{a}{b}\right)$ is given as

\[
\left(\frac{a}{b}\right) = (-1)^{\frac{b-1}{2} \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor}.
\]

To prove Theorem 12 using Eisenstein’s Lemma for Legendre symbols (Theorem 10), it suffices to prove the following result.

Lemma 13. For odd positive integers $a$, $b$ and $c$ such that $b$ and $c$ are coprime with $a$,

\[
\sum_{i=1}^{\frac{bc-1}{2}} \left\lfloor \frac{ia}{bc} \right\rfloor \equiv \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor + \sum_{i=1}^{\frac{c-1}{2}} \left\lfloor \frac{ia}{c} \right\rfloor \quad \text{(mod 2)}.
\]

Using Theorem 1 and Remark 3 (or Theorem 6 instead), we can easily express all of the sums above in terms of summations of fractions having denominator $a$ and summation index $\frac{a-1}{2}$. From there, Lemma 13 reduces to proving the following result.

Lemma 14. For odd positive integers $a$, $b$ and $c$ such that $b$ and $c$ are coprime with $a$,

\[
\sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ibc}{a} \right\rfloor \equiv \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ic}{a} \right\rfloor \quad \text{(mod 2)}.
\]

We prove Lemma 14 in Section 5. The law of quadratic reciprocity for Jacobi symbols (Theorem 15 below) is then immediately obtained using Eisenstein’s Lemma for Jacobi symbols (Theorem 12), and Theorem 1 and Remark 3 (or Theorem 6 instead).

Theorem 15 (Law of quadratic reciprocity for Jacobi symbols). If $a$ and $b$ are positive odd coprime integers, then

\[
\left(\frac{a}{b}\right) \left(\frac{b}{a}\right) = (-1)^{\frac{(a-1)(b-1)}{4}}.
\]

Remark 16. Using the standard techniques in the proof of Theorem 10, 12, Theorem 3.3], it is straightforward to deduce Gauss’ Lemma for Jacobi symbols from Eisenstein’s Lemma for Jacobi symbols.
2 Equivalence between Theorems 2 and 6

In this section, we prove Theorem 6 and show that it is equivalent to Theorem 2 (Equivalence B in Figure 1).

Proof of Theorem 6. Without loss of generality, suppose that $b < a$. By setting the index of summation equal to $\left\lfloor \frac{a}{2} \right\rfloor$ in Theorem 5, we get

$$\sum_{i=1}^{\left\lfloor \frac{a}{2} \right\rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor = K \left\lfloor \frac{a}{2} \right\rfloor,$$

where

$$K = \left\lfloor \frac{\frac{a}{2} b}{a} \right\rfloor.$$

We split the calculation into three cases based on the parity of $a$ and $b$.

Case 1: Suppose $a$ is even, then $K = \left\lfloor \frac{b}{2} \right\rfloor$, and we are done.

Case 2: Suppose $a$ and $b$ are both odd, then

$$K = \left\lfloor \frac{(a-1)b}{2a} \right\rfloor = \left\lfloor \frac{b-1}{2} + \frac{a-b}{2a} \right\rfloor = \frac{b-1}{2} = \left\lfloor \frac{b}{2} \right\rfloor.$$

Case 3: Suppose $a$ is odd and $b$ is even, then

$$K = \left\lfloor \frac{(a-1)b}{2a} \right\rfloor = \left\lfloor \frac{b}{2} - \frac{b}{2a} \right\rfloor = \frac{b}{2} - 1 = \left\lfloor \frac{b}{2} \right\rfloor - 1,$$

and thus (1) becomes

$$\sum_{i=1}^{\left\lfloor \frac{a}{2} \right\rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\left\lfloor \frac{b}{2} \right\rfloor - 1} \left\lfloor \frac{ia}{b} \right\rfloor = \left( \left\lfloor \frac{b}{2} \right\rfloor - 1 \right) \left\lfloor \frac{a}{2} \right\rfloor.$$

From (2), the theorem easily follows in this case.

We establish the equivalence between Theorems 2 and 6 (Equivalence B in Figure 1). Since the proofs of Lemmas 17, 18, and 19 below are easy generalizations of our proofs in [3, Lemma 10, Lemma 12, and Lemma 13] respectively, we skip the details here. For the remainder of this section, suppose $a$ and $b$ are positive coprime integers.

Lemma 17. The number of nonnegative integer solutions $(x, y, z)$ of the equation $ax + by + z = b\left\lfloor \frac{a}{2} \right\rfloor$ is given by

$$\left\lfloor \frac{a}{2} \right\rfloor + 1 + \sum_{i=1}^{\left\lfloor \frac{a}{2} \right\rfloor} \left\lfloor \frac{ib}{a} \right\rfloor.$$
Lemma 18. The number of nonnegative integer solutions \((x, y, z)\) of the equation

\[ ax + by + z = a \left\lfloor \frac{b}{2} \right\rfloor + b \left\lfloor \frac{a}{2} \right\rfloor \]

is equal to

\[ a \left\lfloor \frac{b}{2} \right\rfloor + b \left\lfloor \frac{a}{2} \right\rfloor + 1 - N_0, \]

where \(N_0\) is the number of natural numbers which cannot be expressed as \(ax + by\) for any nonnegative integers \(x\) and \(y\).

Lemma 19. The number of nonnegative integer solutions \((x, y, z)\) of the equation

\[ ax + by + z = a \left\lfloor \frac{b}{2} \right\rfloor + b \left\lfloor \frac{a}{2} \right\rfloor \]

is equal to

\[ 2 \left( \sum_{i=1}^{\frac{a}{2}} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\frac{b}{2}} \left\lfloor \frac{ia}{b} \right\rfloor \right) + \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor + 1. \]

We are now ready to show the equivalence between Theorems 2 and 6 (Equivalence B in Figure 1). Upon comparing the number of nonnegative integer solutions of the equation \(ax + by + z = a \left\lfloor \frac{b}{2} \right\rfloor + b \left\lfloor \frac{a}{2} \right\rfloor\) obtained in Lemmas 18 and 19 and then using Lemma 17, we find

\[ N_0 + 2 \left( \sum_{i=1}^{\frac{a}{2}} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\frac{b}{2}} \left\lfloor \frac{ia}{b} \right\rfloor \right) = (a - 1) \left\lfloor \frac{b}{2} \right\rfloor + (b - 1) \left\lfloor \frac{a}{2} \right\rfloor. \] (3)

By taking three cases based on the parity of \(a\) and \(b\), it can be easily verified that

\[ (a - 1) \left\lfloor \frac{b}{2} \right\rfloor + (b - 1) \left\lfloor \frac{a}{2} \right\rfloor = \frac{(a - 1)(b - 1)}{2} + 2 \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{b}{2} \right\rfloor. \] (4)

From (3) and (4), we get that

\[ N_0 + 2 \left( \sum_{i=1}^{\frac{a}{2}} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\frac{b}{2}} \left\lfloor \frac{ia}{b} \right\rfloor \right) = \frac{(a - 1)(b - 1)}{2} + 2 \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{b}{2} \right\rfloor. \]

The equivalence between Theorems 2 and 6 (Equivalence B in Figure 1) now readily follows.
3 Proof of Theorem 8

In this section, we prove Theorem 8 and the Equivalence C in Figure 1. Recall that for given positive coprime integers \(a\) and \(b\), and given \(k\) such that \(0 \leq k < (a - 1)(b - 1)\), the symbol \(N_0(a, b; k)\) denotes the number of natural numbers \(\leq k\) that can be expressed in the form \(ax + by\) for nonnegative integers \(x\) and \(y\).

For positive coprime integers \(a\) and \(b\) and any natural number \(n\), let \(N(a, b; n)\) denote the number of nonnegative integer solutions of \(ax + by = n\). An exact formula for \(N(a, b; n)\) [17] is known. Further, it is well-known that \(N(a, b; n + ab) = N(a, b; n) + 1\) (see [17, Lemma 1]). Using this fact while generalizing the proof of Lemma 18, we easily get the following result.

**Lemma 20.** Let \(a, b, d,\) and \(K\) be positive integers such that \(b < a\), \(\frac{a}{2} < d < a\), \(\gcd(a, b) = 1\), and \(K = \left\lfloor \frac{bd}{a} \right\rfloor\). The number of nonnegative integer solutions of the equation

\[
ax + by + z = bd + aK
\]

is equal to

\[
bd + aK + 1 - \frac{(a - 1)(b - 1)}{2} + N_0(a, b; bd + aK - ab).
\]

Generalizing the proof of Lemma 19 and then using Theorem 5, we get the following result.

**Lemma 21.** Let \(a, b, d,\) and \(K\) be positive integers such that \(b < a\), \(d < a\), \(\gcd(a, b) = 1\), and \(K = \left\lfloor \frac{bd}{a} \right\rfloor\). The number of nonnegative integer solutions \((x, y, z)\) of the equation

\[
ax + by + z = bd + aK
\]

is equal to

\[
2 \left( \sum_{i=1}^{d} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor \right) + d + K + 1.
\]

The following lemma immediately follows from Lemmas 20 and 21.

**Lemma 22.** Let \(a, b, d,\) and \(K\) be positive integers such that \(b < a\), \(\frac{a}{2} < d < a\), \(\gcd(a, b) = 1\), and \(K = \left\lfloor \frac{bd}{a} \right\rfloor\). Then

\[
N_0(a, b; bd + aK - ab) = 2 \left( \sum_{i=1}^{d} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor \right) - dK + \frac{(2d - a + 1)(2K - b + 1)}{2}.
\]

Using Lemma 22, it is clear that Theorem 5 is equivalent to the following theorem.

**Theorem 23.** Let \(a, b, d,\) and \(K\) be positive integers such that \(b < a\), \(\frac{a}{2} < d < a\), \(\gcd(a, b) = 1\), and \(K = \left\lfloor \frac{bd}{a} \right\rfloor\). Then

\[
N_0(a, b; bd + aK - ab) = \frac{(2d - a + 1)(2K - b + 1)}{2}.
\]

Observe that Theorem 8 is just another version of Theorem 23 obtained by setting \(\alpha = 2d - a\) and \(\beta = 2K - b\) in Theorem 23. This completes the proof of Theorem 8 and its equivalence with Theorem 5 (Equivalence C in Figure 1).
4 \( N_0(a, b; k) \) for other values of \( k \)

In this section, we describe an easy method to calculate \( N_0(a, b; k) \) for other values of \( k \) not covered by Theorem 8. It is well-known that for \( l < ab \), the equation \( ax + by = l \) has at most one solution [17, Lemma 2 and Lemma 4]. Using this fact, it is easy to see that for \( k < ab \), \( N_0(a, b; k) \) is equal to the number of nonnegative integer solutions of \( ax + by + z = k \), which can be easily calculated using the algorithm described in [3, Section 2.3]. For example, suppose we want to calculate \( N_0(29, 23; 257) \). Then, by [3, Theorem 5], we get that

\[
N_0(29, 23; 257) = 15 + \sum_{i=1}^{8} \left\lfloor \frac{23i}{29} \right\rfloor + \sum_{i=1}^{18} \left\lfloor \frac{4i}{23} \right\rfloor.
\]

By repeated applications of Theorem 5 and the division algorithm, as described in [3, Section 2.3], we easily get that \( \sum_{i=1}^{8} \left\lfloor \frac{23i}{29} \right\rfloor = 24 \) and \( \sum_{i=1}^{18} \left\lfloor \frac{4i}{23} \right\rfloor = 21 \), and thus \( N_0(29, 23; 257) = 60 \).

5 Proof of Lemma 14

Proof of Lemma 14. For brevity of notation, let \( r(m) \) denote the remainder when \( m \) is divided by \( a \). Since \( a, b \) and \( c \) are odd, we get

\[
\sum_{i=1}^{a-1} \left\lfloor \frac{ibc}{a} \right\rfloor \equiv \sum_{i=1}^{a-1} a \left\lfloor \frac{ibc}{a} \right\rfloor \pmod{2}
\]

\[
= \sum_{i=1}^{a-1} ibc - r(ibc)
\]

\[
\equiv \sum_{i=1}^{a-1} i + \sum_{i=1}^{a-1} r(ibc) \pmod{2}. \tag{5}
\]

Next, we study the latter sum. Let \( c_i \) denote the remainder when \( ic \) is divided by \( a \). Note that

\[
c_i = ic - a \left\lfloor \frac{ic}{a} \right\rfloor
\]

\[
\equiv i - \left\lfloor \frac{ic}{a} \right\rfloor \pmod{2}.
\]

Therefore,

\[
r(ibc) = r(bc_i)
\]

\[
= bc_i - a \left\lfloor \frac{bc_i}{a} \right\rfloor
\]

\[
\equiv c_i - \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod{2}
\]

\[
\equiv i - \left\lfloor \frac{ic}{a} \right\rfloor - \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod{2}. \tag{6}
\]
Thus, from (5) and (6), we get that
\[ \sum_{i=1}^{a-1} \left\lfloor \frac{ibc_i}{a} \right\rfloor \equiv \sum_{i=1}^{a-1} \left\lfloor \frac{ic_i}{a} \right\rfloor + \sum_{i=1}^{a-1} \left\lfloor \frac{bci_i}{a} \right\rfloor \pmod{2}. \]

Therefore, to complete the proof of the lemma, it suffices to show that
\[ \sum_{i=1}^{a-1} \left\lfloor \frac{bc_i}{a} \right\rfloor \equiv \sum_{i=1}^{a-1} \left\lfloor \frac{ib}{a} \right\rfloor \pmod{2} \tag{7} \]

Let \( d_i = \min(c_i, a - c_i) \). Note that
\[ \left\lfloor \frac{b(a - c_i)}{a} \right\rfloor = b - 1 - \left\lfloor \frac{bc_i}{a} \right\rfloor \equiv \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod{2}. \]

Therefore, for each \( i \),
\[ \left\lfloor \frac{bd_i}{a} \right\rfloor \equiv \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod{2}. \]

Moreover, note that as \( i \) varies from 1 to \( \frac{a-1}{2} \), so does \( d_i \). Therefore,
\[ \sum_{i=1}^{a-1} \left\lfloor \frac{bc_i}{a} \right\rfloor \equiv \sum_{i=1}^{a-1} \left\lfloor \frac{bd_i}{a} \right\rfloor \pmod{2} \]
\[ = \sum_{i=1}^{a-1} \left\lfloor \frac{ib}{a} \right\rfloor \pmod{2}, \]
completing the proof of (7), and thus of Lemma 14.

\[ \square \]

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References

[1] J. L. Ramírez Alfonsín, The Diophantine Frobenius Problem, volume 30 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2005.

[2] O. Baumgart, The Quadratic Reciprocity Law: A Collection of Classical Proofs, Springer International Publishing, 2015.
[3] D. S. Binner, The number of solutions to $ax + by + cz = n$ and its relation to quadratic residues, *J. Integer Seq.* 23 (2020), Article 20.6.5.

[4] T. C. Brown and P. J. Shiue, A remark related to the Frobenius problem, *Fibonacci Quart.* 31, 32–36, 1993.

[5] A. Brunyate and P. L. Clark, Extending the Zolotarev–Frobenius approach to quadratic reciprocity, *Ramanujan J.* 37, 25–50, 2015.

[6] P. Cartier, Sur une généralisation des symboles de Legendre-Jacobi, *Enseign. Math.* 16, 31–48, 1970.

[7] W. Duke and K. Hopkins, Quadratic reciprocity in a finite group, *Amer. Math. Monthly* 112, 251–256, 2005.

[8] G. Eisenstein. Geometrischer Beweis des Fundamentaltheorems für die quadratischen Reste, *J. Reine Angew. Math.* 28, 246–248, 1844.

[9] C. F. Gauss, Theorematis arithmeticici demonstratio nova, *Comment. Soc. regiae sci. Göttingen XVI*, 69, 1808.

[10] T. Komatsu and Y. Zhang, Weighted Sylvester sums on the Frobenius set, arxiv preprint, 2021. Available at [https://arxiv.org/pdf/2105.08274.pdf](https://arxiv.org/pdf/2105.08274.pdf).

[11] A. Kuroki and S. I. Katayama, A variation of Takagi’s proof for quadratic reciprocity laws of Jacobi symbols, *J. Math. Tokushima Univ.* 43, 9–23, 2009.

[12] I. Niven, H. S. Zuckerman and H. L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley and Sons Inc, New York, 2004.

[13] Ø. J. Rødseth, A note on Brown and Shiue’s paper on a remark related to the Frobenius problem, *Fibonacci Quart.* 32, 407–408, 1994.

[14] E. Schering, Zur Theorie der quadratischen Reste, *Acta Math.* 1, (1882), 153–170; Werke II, 69–86.

[15] J. J. Sylvester, On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order, *Amer. J. Math.* 5 (1882), 79–136.

[16] J. J. Sylvester, Problem 7382, *Mathematical Questions, with their Solutions, from the Educational Times* 41 (1884), 21.

[17] A. Tripathi, The number of solutions to $ax + by = n$, *Fibonacci Quart.* 38, 290–293, 2000.

[18] G. Zolotarev, Nouvelle démonstration de la loi de réciprocité de Legendre. *Nouvelles Annales de Mathématiques* 11, 354–362, 1872.
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