The Art of Space Filling in Penrose Tilings and Fractals

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Abstract
Incorporating designs into the tiles that form tessellations presents an interesting challenge for artists. Creating a viable MC Escher like image that works esthetically as well as functionally requires resolving incongruencies at a tile’s edge while constrained by its shape. Escher was the most well known practitioner in this style of mathematical visualization, but there are significant mathematical shapes to which he never applied his artistry. These shapes can incorporate designs that form images as appealing as those produced by Escher, and our paper explores this for traditional tessellations, Penrose Tilings, fractals, and fractal/tessellation combinations. To illustrate the versatility of tiling art, images were created with multiple figures and negative space leading to patterns distinct from the work of others.  

1 Introduction

MC Escher was the most prominent artist working with tessellations and space filling. Forty years after his death, his creations are still foremost in people’s minds in the field of tiling art. One of the reasons Escher continues to hold such a monopoly in this specialty are the unique challenges that come with creating Escher type designs inside a tessellation[1]. When an image is drawn into a tile and extends to the tile’s edge, it introduces incongruencies which are resolved by continuously aligning and refining the image. This is particularly true when the image consists of the lizards, fish, angels, etc. which populated Escher’s tilings because they do not have the quadrilateral symmetry that would make it

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possible to arbitrarily rotate the image ± 90, 180 degrees and have all the pieces fit [2]. Rather, they have bilateral symmetry which requires creating a compliment for every edge. This is true for any type of tile that incorporates such an image.

A collection of papers in honor of Escher, *MC Escher’s Legacy: A Centennial Celebration* contains a comprehensive study of his work [4]. Of the articles emphasizing art in this collection, most authors produced tile images that continued the practice of having one dominant figure in a tile that is completely filled. There are greater possibilities beyond this, and this paper contains the result of our studies.

The rules to creating tiling art are straightforward, and we describe the process and apply it to the mathematical geometries that Escher did not have a chance to include in his vast body of work. In particular, we create tiling art based on constructions that have gained prominence since Escher’s time: Penrose Tilings and fractals. In the case of the latter, we also put a tessellation inside a fractal tile to allow for growth in various directions. To give our designs a classical quality, they will consist of intertwined human figures. We also deviate from Escher and others by having negative spaces between the figures that allowed for different combinations in connecting the tiles.

Most of the visualizations related to Penrose Tilings and fractals are abstract representations purely in service of the underlying numbers [3]. This represents a missed opportunity as the geometry of tessellations and fractals is highly suited to the application of design art offering a rich and mostly unmined domain awaiting exploration. There are rewards in its provinces beyond what can be achieved otherwise. The images produced will often excite that part of the mind which responds to the inherent esthetic of mathematical shapes [5, 6]. In addition, the artist enjoys another level of discovery to his or her creation. Unlike a picture in isolation, the tiling images in this paper form patterns that only revealed themselves once all the tiles are assembled. The compelling nature of this art was best explained by Escher himself, saying, "... For once one has crossed over the threshold of the early stages this activity takes on a more worth than any other form of decorative art" [7].

2 Creating a tessellation

To illustrate the process and challenges of making tiles that contain bilateral symmetry, we begin with the simplest tessellation made of squares. Figure 1 illustrates a square with quadrilateral symmetry, and any ± 90,
180 degree rotation will give back the same image. Any side $A$ can be connected to another side $A$ and the resulting tessellation is given in Figure 2.

In contrast, when the design is of a person as in Leonardo da Vinci’s The Vitruvian Man shown in Figure 3[8], it is not possible to match every side to every other if the image is extended to the edges. The overlapping of arms and legs requires that each side is complimented by another side. To create a tessellation from Figure 3, we label sides as before, but now we have four unique sides. In the single tile given by Figure 4, side $A$ connects to side $A'$ and $B$ to $B'$. The resulting tessellation is given in Figure 5.

3 Beyond Escher

Once the simple rules of making a tessellation are understood, other possibilities exist in how tiles are connected and the types of images they contain.

3.1 Other Images and Connecting Rules

Instead of limiting a tile to one prominent entity as typically seen in most work emulating Escher, Figure 6 contains multiple subjects and its rectangular tile leads to the tessellation of Figure 7.
Figure 2: Tessellation of Figure 1
Figure 3: Leonardo da Vinci’s *The Vitruvian Man*

Figure 4: A single tile of *The Vitruvian Man* for tessellation
Figure 5: Tessellation of *The Vitruvian Man*
Figure 6: A tile made up of multiple figures
Figure 7: Tessellation of Figure 6
The next two figures show other matching rules that square tiles can have. As seen in Figure 9 formed by the tiles of Figure 8, by making the complimentary sides adjacent, the final image forms an interesting pattern of swirling figures rather than a simple shifting of the tile vertically or horizontally. In Figure 10, the same adjacent compliments exist, but with each side complimenting two sides instead of one. This leads to the image seen in Figure 11 which has the same turning of the figures, but also allows for non periodic arrangements. Once one row of tiles is set, the next row has two possible arrangements depending on how the first tile of the new row is placed. This is true for every subsequent row.

3.2 Penrose Tilings

Though Escher was friends with Roger Penrose, he died before applying his space filling work to Penrose’s aperiodic tilings[9, 10]. These types of tilings are even more challenging to incorporating designs because they are made of two tiles which, depending on the arrangement, may not form a repeating pattern. Fortunately, the more interesting and non-repeating (hence aperiodic) arrangements are created by following side matching rules in the same manner as discussed before[11]. The Kite and Dart tiles, given in Figure 12, show how every side is connected to create the aperiodic tiling of 13. Using these same rules, the artistic tiles in Figure 14 lead to 15.
Figure 9: Tessellation of Figure 8
Figure 10: Tile having two adjacent complimentary sides
Figure 11: Tessellation of Figure 10
Figure 12: Kite and Dart Penrose Tilings
Figure 13: Tessellation of Figure 12
Figure 14: Space filled kite and dart Penrose Tiles
Figure 15: Tessellation of Figure 14
The other Penrose Tiling made of fat and thin rhombi has similar rules for creating an aperiodic pattern. Figure 16 shows how the sides match to create 17. The individual tiles containing a design and the resulting image are given in Figures 18 and 19 respectively.

3.3 Fractal Tilings

For fractals, Escher’s Circle Limit series invokes the concept but it can also be interpreted as a tessellation of the hyperbolic plane[12]. To make a fractal tiling that purely manifests the notion of self similarity, many of the previous techniques are applied again with a few modifications. Starting with a single tile, a matching rule is created for how other tiles will be connected to its sides. An additional constraint is that the added tiles are decreased in size by an amount that allows for growth of the fractal without the branches intersecting. Figure 20 shows one rectangular tile where subsequent additions decrease in size by 1/2. So the next tile is half of the original, and its side A will connect with A’ and A”.

To improve the esthetics, the two connecting sides are made to alternate such that side A” connects to the next tile after it is rotated 90 degrees clockwise while side A’ connects to the mirror image of this tile and rotated 90 degrees counter-clockwise. Figure 21 shows the result.
Figure 17: Tessellation of Figure 16
Figure 18:  Space filled rhombus Penrose Tiles
Figure 19: Tessellation of Figure 18
Figure 20: Fractal tile
Figure 21: Final image using the tile of Figure 20
For the next fractal tile made of an isosceles triangle, the same mirroring connection on side $A'$ is made. In addition, the principals of tessellation are used so that the fractal can progress in other directions. As seen in Figure 22, the area around the centre of the triangle is a region of symmetry for the design which is repeated every 120 degrees. This is delineated by 3 sub-triangles which form the tessellation inside the isosceles. From here, the lower triangle can be swapped with either of the other two leading to the fractal of Figure 23 which has branches in the downward direction. In this fractal, the decrease of each tile was only one third rather than one half which illustrates the problem of branches colliding. Despite this, the final result is very appealing.

4 Concluding Remarks

Escher’s work introduced the world to the beauty of geometrical art. But non-mathematician artists tended not to follow his example, and so a wealth of trigonometric shapes only exists as blank tiles waiting to be filled. By describing the process of incorporating tessellations and fractals into art,
Figure 23: Final image using the tile of Figure 22
we hope to show that the challenges are artistic rather than mathematical.

In the same way that mathematical analysis can give a deeper appreciation of art, Escher used art as a means of gaining insight into the mathematics of geometry. Continuing this tradition, our paper demonstrates that the connections contained within tessellations and fractals are best seen when combined with tiling art. With the limitless possibilities as to what can be put inside a tile, artists are well suited to find the undiscovered pattern contained in each.

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