Parikh’s theorem for infinite alphabets

Piotr Hofman
University of Warsaw

Marta Juzepczuk
University of Warsaw

Sławomir Lasota
University of Warsaw

Mohnish Pattathurajan
University of Warsaw

Abstract—We investigate commutative images of languages recognised by register automata and grammars. Semi-linear and rational sets can be naturally extended to this setting by allowing for orbit-finite unions instead of only finite ones. We prove that commutative images of languages of one-register automata are not always semi-linear, but they are always rational. We also lift the latter result to grammars: commutative images of one-register context-free languages are rational, and in consequence commutatively equivalent to register automata. We conjecture analogous results for automata and grammars with arbitrarily many registers.

I. INTRODUCTION

Register automata, introduced over 25 year ago by Francez and Kaminski [1], are nondeterministic finite-state devices equipped with a finite number of registers that can store data values from an infinite data domain. A register automaton inputs a string of data values (a data word) and compares each consecutive input data value to its registers; based on this comparison and on the current control state, it chooses a next control state and possibly stores the input value in one of its registers. The only allowed comparisons of data values considered in this paper are equality tests. An automaton can also guess a fresh data value not previously seen in the input, and store it in a register (we thus consider nondeterministic register automata with guessing [2]). Likewise one defines register context-free grammars [3], [4] Sect.5.

Register automata lack most of the good properties known from the classical theory of finite automata, like determinisation or closure properties. In particular, no satisfactory characterisation in terms of rational (regular) expressions is known. Indeed, all known generalisations of Kleene’s theorem for register automata either apply to a restricted subclass of the model, or introduce an involved syntax significantly extending the concept of rational expressions [5], [6], [7].

Register automata are expressively equivalent to orbit-finite automata [8], [9], a natural extension of finite automata where one allows for input alphabets and state spaces which are infinite, but finite up to permutation of the data domain (= orbit-finite). Along the same lines, in this paper we focus on a natural extension of rational expressions, which differ from the classical ones just by allowing for orbit-finite unions. In other words, we consider the class of rational languages, defined as the smallest class of languages closed under concatenation, star, and orbit-finite unions. In particular, the class contains the empty language, all finite and all orbit-finite languages.

Languages of register automata are not rational in general, even in case of deterministic one-register automata. Kleene theorem may be however recovered when commutative images (Parikh images) are considered: we prove that the language of every one-register automaton is Parikh-equivalent to (i.e., has the same Parikh image as) a rational language.

Example 1. Fix the data domain ATOMS = \{0, 1, 2, \ldots\} and consider the language $L_1$ consisting of all nonempty words over ATOMS \{0, 1, 2, \ldots, 9\} where every two consecutive letters are different:

$$L_1 = \{a_1 a_2 \ldots a_n \in ATOMS^* : a_1 \neq a_2 \neq \ldots \neq a_n\}.$$  

The language is recognised by a deterministic one-register automaton but it is not rational (cf. Section III). It is however Parikh-equivalent to a larger language $L_2$, where the non-equality constraint is imposed at every second position only:

$$L_2 = \{a_1 a_2 \ldots a_n \in ATOMS^* : a_1 \neq a_2, a_3 \neq a_4, \ldots\},$$

which is defined by the rational (regular) expression

$$L_2 = \left( \bigcup_{a, b \in ATOMS, a \neq b} ab \right)^* (\varepsilon \cup \bigcup_{a \in ATOMS} a)$$  \hspace{1cm} (1)

and is thus rational (the formal definition of rational languages will be given in Section III). Indeed, every $w \in L_2$ can be transformed, by swapping letters, to a word in $L_1$. Let $w = a_1 a_2 \ldots a_n$. If $a_2 = a_3$ we swap non-equal letters $a_3$ and $a_4$ thus achieving $a_1 \neq a_2 \neq a_3 \neq a_4$. Next, if $a_4 = a_5$ we swap analogously $a_5$ and $a_6$, and so on. Continuing in this way we finally arrive at a word in $L_1$.

\square

Contribution: We contribute to understanding commutative images of languages of one-register automata and grammars, by investigating sets of data vectors obtainable as Parikh images of these languages. Parikh images of rational languages we call rational as well. Here are our contributions:

(1) We show that Parikh images of languages of one-register automata are not semilinear sets of data vectors in general. By semilinear sets we naturally mean orbit-finite unions of linear sets, which in turn are determined by a base and an orbit-finite set of periods, like classically.

(2) We prove that languages of one-register automata have rational Parikh images. The crucial part of the proof resorts to a graph-theoretical characterisation of these Parikh images, and uses a necessary condition for a Hamiltonian cycle in directed graphs [9], [10].

This work was partially supported by NCN grants 2016/21/D/ST6/01368, 2017/27/B/ST6/02093 and 2019/35/B/ST6/02322.

978-1-6654-4895-6/21/$31.00 ©2021 IEEE
(3) Finally, we extend (2) to context-free grammars by showing that one-register context-free languages have rational Parikh images. The result is obtained by a novel type of transformations of derivation trees.

We conjecture that the restriction on the number of registers in (2) and (3) can be dropped; the combinatoric complexity we have encountered already in one-register case makes it however difficult to envisage a generalisation of our approach to the general case. According to (1), one-register automata and grammars fail to have semilinear Parikh images in general. However, as a direct corollary of (3) we recover an analog of the Parikh’s classical theorem [11]: one-register context-free grammars are Parikh-equivalent to register automata (but not to one-register automata).

Related research: Register automata have been intensively studied with respect to their foundational properties [11], [12], [5], [13]. Following the seminal paper of Francez and Kaminski [1], subsequent extensions of the model allow for comparing data values with respect to some fixed relations such as a total order, or introduce alternation, variations on the allowed form of nondeterminism, etc. The model is well known to satisfy almost no semantic equivalences that hold for classical finite automata. Here are few positive results: simulation of two-way nondeterministic automata by one-way alternating automata with guessing [4]; Myhill-Nerode-style characterisation of languages of deterministic automata [14], [8], [3]; and the well-behaved class of languages definable by orbit-finite monoids [15], characterised in terms of logic [16] and a syntactic subclass of deterministic automata [17]. Register automata have been also intensively studied with respect to their applications to XML databases and logics [18], [13], [19] (see [2] for a survey). Register context-free grammars are equivalent to register pushdown automata [4], [20].

Other extensions of finite-state machines to infinite alphabets include: abstract reformulation or register automata, known as orbit-finite automata, or nominal automata, or automata over atoms) [8], [3], [4]; symbolic automata [21]; pebble automata [22]; and data automata [23], [24].

II. PRELIMINARIES

Sets with atoms: Our definitions rely on basic notions and results of the theory of sets with atoms [4], also known as nominal sets [25]. This paper is a part of a uniform abstract approach to register automata in the realm of orbit-finite sets with atoms, developed in [8], [3], [4].

Fix a countably infinite set ATOMS, whose elements we call atoms. We reserve initial alphabet letters $a, b, \ldots$ to range over atoms. Informally speaking, a set with atoms is a set that can have atoms, or other sets with atoms, as elements. Formally, we define the universe of sets with atoms by a suitably adapted cumulative hierarchy of sets, by transfinite induction: the only set of rank 0 is the empty set; and for a cardinal $\gamma$, a set of rank $\gamma$ may contain, as elements, sets of rank smaller than $\gamma$ as well as atoms. In particular, nonempty subsets $X \subseteq \text{ATOMS}$ have rank 1.

Denote by PERM the group of all permutations of ATOMS. Atom permutations $\pi : \text{ATOMS} \to \text{ATOMS}$ act on sets with atoms by consistently renaming all atoms in a given set. Formally, by another transfinite induction we define $\pi(X) = \{\pi(x) : x \in X\}$. Via standard set-theoretic encodings of pairs or finite sequences we obtain, in particular, the pointwise action on pairs $\pi(x, y) = (\pi(x), \pi(y))$, and likewise on finite sequences. Relations and functions from $X$ to $Y$ are considered as subsets of $X \times Y$; for instance, in case of $f : \text{ATOMS} \to \text{ATOMS}$, we have $\pi(f)(a) = \pi(f(\pi^{-1}(a)))$.

We restrict to sets with atoms that only depend on finitely many atoms, in the following sense. A support of $x$ is any set $S \subseteq \text{ATOMS}$ such that the following implication holds for all $\pi \in \text{PERM}$: if $\pi(s) = s$ for all $s \in S$, then $\pi(x) = x$. An element (or set) $x$ is finitely supported if it has some finite support; in this case $x$ has the least support, denoted $\text{SUPP}(x)$, called the support of $x$ (cf. [4] Sect. 6). Sets supported by $\emptyset$ we call equivariant. For instance, given $a, b \in \text{ATOMS}$, the support of the set

$L_{ab} = \{a_1a_2 \ldots a_n \in \text{ATOMS}^* : n \geq 2, a_1 \neq a, a_n = b\}

$ is $\{a, b\}$; the projection function $\pi_1 : \text{ATOMS}^2 \to \text{ATOMS} : \langle a, b \rangle \mapsto a$ is equivariant; the support of a sequence $\langle a_1 \ldots a_n \rangle \in \text{ATOMS}^*$, encoded as a set in a standard way, is the set of atoms $\{a_1, \ldots, a_n\}$ appearing in it; and the support of a function $f : \text{ATOMS} \to \mathbb{N}$ such that $\text{DOM}(f) = \{a \in \text{ATOMS} : f(a) > 0\}$ is finite, is exactly $\text{DOM}(f)$.

From now on, we shall only consider sets with atoms that are hereditarily finitely supported (called briefly legal), i.e., ones that have a finite support, whose every element has some finite support, and so on.

 Orbit-finite sets: Two (elements of) sets with atoms $x, y$ are in the same orbit if $\pi(x) = y$ for some $\pi \in \text{PERM}$. This equivalence relation splits every set with atoms $X$ into equivalence classes, which we call orbits in $X$. A (legal) set is orbit-finite if it splits into finitely many orbits. Examples of orbit-finite sets are: ATOMS (1 orbit); ATOMS $\setminus \{a\}$ for some $a \in \text{ATOMS}$ (1 orbit); ATOMS$^2$ (2 orbits: diagonal and non-diagonal); ATOMS$^3$ (5 orbits, corresponding to equality types of triples); $\{1, \ldots, n\} \times \text{ATOMS}$ ($n$ orbits, as $\pi(i) = i$ for every $i \in \mathbb{N}$ and $\pi \in \text{PERM}$, according to the standard set-theoretic definition of natural numbers); the set of $n$-element subsets of atoms $\mathcal{P}_n(\text{ATOMS}) = \{X \subseteq \text{ATOMS} : |X| = n\}$ (1 orbit). Given a family $(X_i)_{i \in I}$ of sets indexed by an orbit-finite set $I$, the union $\bigcup_{i \in I} X_i$ we call orbit-finite union of sets $X_i$. (Formally, not only each set $X_i$ is assumed to be legal, but also the indexing function $i \mapsto X_i$.) As an example, consider $\bigcup_{b \in \text{ATOMS}} (L_{ab})$. The indexing function $b \mapsto L_{ab}$ is supported by $\{a\}$, and so is the union:

$\bigcup_{b \in \text{ATOMS}} L_{ab} = \{a_1a_2 \ldots a_n \in \text{ATOMS}^* : n \geq 2, a_1 \neq a\}$. Orbit-finite sets are closed under Cartesian products, subsets, and orbit-finite unions: if each of $X_i$ is orbit-finite, their union $\bigcup_{i \in I} X_i$ is orbit-finite too [4] Sect. 3].
Data words and vectors: By a finite multiset over a set \( \Sigma \) we mean any function \( v : \Sigma \to \mathbb{N} \) such that \( v(\alpha) = 0 \) for all \( \alpha \in \Sigma \) except finitely many. We define the domain of \( v \) as \( \text{dom}(v) = \{ \alpha \in \Sigma : v(\alpha) > 0 \} \), and its size as \( |v| = \sum_{\alpha \in \text{dom}(v)} v(\alpha) \) (the same notation is used for the size of a set). The Parikh image (commutative image) of a word \( w \in \Sigma^* \) is the multiset \( \text{Parikh}(w) : \Sigma \to \mathbb{N} \), where \( \text{Parikh}(w)(\alpha) \) is the number of appearances of a letter \( \alpha \in \Sigma \) in \( w \). For a language \( L \subseteq \Sigma^* \), its Parikh image is \( \text{Parikh}(L) = \{ \text{Parikh}(w) : w \in L \} \). Two languages \( L, L' \subseteq \Sigma^* \) are Parikh-equivalent if they have the same Parikh images: \( \text{Parikh}(L) = \text{Parikh}(L') \). We write \( |w| \) for the length of \( w \), hence \( |v| = |w| \) if \( v = \text{Parikh}(w) \). We order multisets pointwise: \( v \preceq v' \) if \( v(\alpha) \leq v'(\alpha) \) for all \( \alpha \in \Sigma \). The zero (empty) multiset \( 0 \) satisfies \( 0(\alpha) = 0 \) for every \( \alpha \in \Sigma \). A singleton, written \( \{ \alpha \} \), maps \( \alpha \) to 1 and all other letters to 0. Addition of multisets is pointwise: \( (v + v')(\alpha) = v(\alpha) + v'(\alpha) \) for every \( \alpha \in \Sigma \); likewise subtraction \( v - v' \), for \( v' \subseteq v \).

When \( \Sigma \) is an orbit-finite alphabet, words \( w \in \Sigma^* \) we traditionally call data words, languages \( L \subseteq \Sigma^* \) we call data languages, and finite multisets \( v : \Sigma \to \mathbb{N} \) we call data vectors. Orbit-finiteness of a set of data words (or data vectors) is equivalent to bounded length (or size) of its elements:

**Lemma 1.** A set \( X \) of data words or data vectors over an orbit-finite alphabet \( \Sigma \) is orbit-finite if, and only if, \( \{|v| : v \in X\} \subseteq \mathbb{N} \) is bounded.

One-register automata: For defining register automata we consider input alphabets of the form \( \Sigma = H \times \text{ATOMS} \), where \( H \) is a finite set. We use three fixed variables \( x, y, x' \) to represent register values and input atoms. A nondeterministic register automaton (1-NRA) \( A \) consists of: a finite set \( H \) (finite component of the alphabet), a finite set of control locations \( Q \), subsets \( I, F \subseteq Q \) of initial resp. accepting locations, and a finite set \( \Delta \) of transition rules of the form

\[
(q(x), \langle h, y \rangle, \varphi, q'(x'))
\]

where \( q, q' \in Q \), \( h \in H \), and \( \varphi(x, y, x') \) is a Boolean combination of equalities involving the variables \( x, y, x' \), specifying relation between current register value (\( x \)), input atom (\( y \)), and next register value (\( x' \)) resulting from a transition.

A configuration \( \langle q, a \rangle \in Q \times \text{ATOMS} \) of \( A \), written \( q(a) \), consists of a control location \( q \in Q \) and a register value \( a \in \text{ATOMS} \). For all atoms \( a, b, a' \) such that \( (a, b, a') \models \varphi \), a rule \( \varphi \) induces a transition

\[
q(a) \xrightarrow{(h,b)} q'(a')
\]

from a configuration \( q(a) \) to a configuration \( q'(a') \). The semantics of 1-NRA is defined as in case of classical NFA, with configurations considered as states and \( \Sigma = H \times \text{ATOMS} \) as an alphabet. A run of \( A \) over a data word \( w = \langle h_1, b_1 \rangle \langle h_2, b_2 \rangle \ldots \langle h_n, b_n \rangle \in \Sigma^* \) is any sequence

\[
q_0(a_0) \xrightarrow{\langle h_1, b_1 \rangle} q_1(a_1) \xrightarrow{\langle h_2, b_2 \rangle} \ldots \xrightarrow{\langle h_n, b_n \rangle} q_n(a_n).
\]

Let \( L_{q(a)} q'(a')(A) \) be the set of data words admitting a run starting in \( q_0(a_0) = q(a) \) and ending in \( q_n(a_n) = q'(a') \). The language recognised by \( A \), denoted \( L(A) \), is defined as:

\[
L(A) = \bigcup_{q \in I, q' \in F, a,a' \in \text{ATOMS}} L_{q(a)} q'(a')(A).
\]

**Remark 2.** The definition allows for guessing, i.e., an automaton may nondeterministically guess, and store in its register, an atom not yet seen in the input (cf. [2]). In particular, the initial register value is guessed nondeterministically.

**Example 2.** Let \( H \) be a singleton, omitted below; we thus consider \( \text{ATOMS} \) as an alphabet. The 1-NRA consisting of \( Q = F = \{ q, p \}, I = \{ q \} \), and two transition rules:

\[
(q(x), y, y = x', p(x')) \quad (p(x), y, x \neq y = x', p(x'))
\]

recognises \( L_1 \) from Example 1 and can be drawn as:

```
  q
  q

  y = x'  q
  q
```

One-register context-free grammars: For technical convenience we restrict to production rules of arity at most 2 (higher arities can be treated similarly, but inessentially increase the combinatorial complexity of Section VII see the comment in Section VIII). Unary production rules are easily simulated using binary and nullary ones.

A context-free grammar with one register (1-CFG) \( G \) consists of: two finite sets \( H \) and \( Q \) of terminals and nonterminals, an initial nonterminal \( q_0 \in Q \), and two finite sets \( \Delta_2 \) and \( \Delta_0 \) of binary and nullary production rules, of the forms

\[
q(x, y, y = x', p(x')) \in \Delta_2, \quad q(x) \in \Delta_0,
\]

where \( q \in Q, p, p' \in Q \cup H \), and \( \varphi(x, y, y') \) is a Boolean combination of equalities involving the three (still fixed) variables. Similarly as before, a configuration \( q(a) \in Q \times \text{ATOMS} \) of \( A \) consists of a nonterminal \( q \in Q \) and a register value \( a \in \text{ATOMS} \). Elements of \( \Sigma = H \times \text{ATOMS} \) we denote either as \( h(a) \) or as \( (h, a) \). Production rules \( \varphi \) induce productions

\[
q(a) \rightarrow p(b) p'(b'), \quad q(a) \rightarrow \varepsilon,
\]

the former one under the condition \( (a, b, b') \models \varphi \). We denote by \( \Pi_2 \) and \( \Pi_0 \), respectively, the (infinite) sets of productions induced by rules from \( \Delta_2 \) and \( \Delta_0 \). The semantics of 1-CFG is defined as for classical CFG, with configurations considered as nonterminals, alphabet \( \Sigma = H \times \text{ATOMS} \), and productions \( \Pi_2 \cup \Pi_0 \). Derivation trees \( T \) of \( G \) are labeled by configurations, alphabet letters \( (h, a) = h(a) \in \Sigma \), or the empty word \( \varepsilon \), in a way consistent with productions \( \varphi \).
Complete derivation trees have all leaves labeled by elements of $\Sigma \cup \{\varepsilon\}$. We write $L_{q(a)}(G) \subseteq \Sigma^*$ for the language of yields of all complete derivation trees $T$ with root labeled by $q(a)$, as usual, where $\text{YIELD}(T) \subseteq \Sigma^*$ is obtained as concatenation of labels of the leaves of $T$. The language $L(G) \subseteq \Sigma^*$ generated by $G$ is defined as the union (as in case of 1-NRA, the initial register value is guessed nondeterministically):

$$L(G) = \bigcup_{a \in \text{ATOMS}} L_{q(a)}(G).$$

Example 3. The 1-CFG consisting of nonterminals $Q = \{l, r\}$, terminals $H = \{l, r\}$, initial nonterminal $q$, and rules

$$q(x) \quad x \neq y \rightarrow (l, y)p(y') \quad p(x) \quad x = y \rightarrow q(y)l,r \quad q(x) \rightarrow \varepsilon$$
generates palindrome-like words of the form

$$\langle l, a_1 \rangle \langle l, a_2 \rangle \ldots \langle l, a_n \rangle \langle r, a_n \rangle \ldots \langle r, a_2 \rangle \langle r, a_1 \rangle$$

where $n \geq 0$ and $a_1 \neq a_2 \neq \ldots \neq a_n$.

Remark 3. An alphabet $H \times \text{ATOMS}$ and configurations $Q \times \text{ATOMS}$ are orbit-finite. 1-NRA and 1-CFG are thus special cases of the abstract notions of orbit-finite automata and context-free grammars (cf. [4 Sect. 5]), where alphabets, state spaces and nonterminals may be arbitrary orbit-finite sets.

Normal forms: In the sequel we assume, w.l.o.g., that each constraint $\varphi$ appearing in a transition rule [2] defines a single orbit of $\text{ATOMS}^3$. In other terms, $\varphi$ contains either equality or disequality of every pair of variables. This can be easily achieved by splitting every constraint into a number of single-orbit ones. For the automaton from Example 2 we get:

There are thus just five possible constraints $\varphi$ and, correspondingly, five types of transition rules. The first two types preserve register value ($x = x'$):

1. $\varphi_1 \equiv x = x' = y$ (register value equal to input atom);
2. $\varphi_2 \equiv x = x' \neq y$ (register value different from input).

The remaining types describe an update of register value:

3. $\varphi_3 \equiv x \neq y = x'$ (register updated with input atom);
4. $\varphi_4 \equiv x = y \neq x'$ (register updated freshly);
5. $\varphi_5 \equiv x \neq y \neq x'$ (register updated freshly).

In the sequel we distinguish between register-preserving (types (1), (2)) and register-updating (types (3)–(5)) constraints $\varphi$, transition rules, and transitions.

Likewise we assume, w.l.o.g., that each constraint $\varphi$ appearing in a production rule of a 1-CFG defines a single orbit of $\text{ATOMS}^3$. The grammar in Example 2 is in normal form.

III. RATIONAL SETS

In this section we define rational sets of data words and data vectors, prove their closure under substitutions, and formulate our main results.

**Orbit-finite unions:** Consider a family of sets $\mathcal{X}$. We say that $\mathcal{X}$ is closed under orbit-finite unions if for every orbit-finite family $(X_i)_{i \in I}$ of sets $X_i \in \mathcal{X}$, the union $\bigcup_{i \in I} X_i$ belongs to $\mathcal{X}$. We instantiate below this abstract definition to families $\mathcal{X}$ of sets of data words and data vectors.

**Rational data languages:** We consider data languages over a fixed orbit-finite alphabet $\Sigma$.

As usual, we define concatenation of two data languages $LL' = \{ww' : w \in L, w' \in L'\}$, and the Kleene star (iteration): $L^* = \{w_1 \ldots w_n : n \geq 0, w_1, \ldots, w_n \in L\}$. Let rational data languages be the smallest class of data languages that contains all singleton languages $\{w\}$, for $w \in \Sigma^*$, and is closed under concatenation, iteration, and orbit-finite unions. In particular the empty language, all finite languages and all orbit-finite ones are rational. For finite $\Sigma$ we obtain the classical rational (regular) sets. As expected, without the Kleene star we obtain exactly sets of words of bounded length, or equivalently, due to Lemma 1, orbit-finite languages.

When convenient, we may speak of rational expressions, by which we mean formal derivations of rational languages according to the closure rules listed above.

Example 4. Continuing Example 1 the language $L_2$ is rational, as it can be presented by a rational expression:

$$L_2 = \bigcup_{a, b \in \text{ATOMS}, a \neq b} \{ab\}^* \left\{\varepsilon\right\} \cup \bigcup_{a \in \text{ATOMS}} \{a\}.$$

For readability we omit brackets $\{}$ in the sequel, as in [1]. On the other hand, one easy shows that the language $L_1$ is not rational.

**Rational sets of data vectors:** We consider sets of data vectors over a fixed orbit-finite alphabet $\Sigma$. Let addition of two sets $X, Y$ of data vectors be defined by Minkowski sum

$$X + Y = \{x + y : x \in X, y \in Y\},$$

and let $X^*$ contain all finite sums of elements of $X$:

$$X^* = \{x_1 + \ldots + x_n : n \geq 0, x_1, \ldots, x_n \in X\}.$$
Substitutions: Consider a language $L$ over an orbit-finite alphabet $\Sigma$ and a (legal) family of languages $K = \{K_\sigma\}_{\sigma \in \Sigma}$ over an alphabet $\Gamma$, indexed by $\Sigma$. We typically use the anonymous function notation

$$\sigma \mapsto K_\sigma.$$ 

The substitution $L(K)$ is the language over $\Gamma$ containing all words obtained from some word $\sigma_1\sigma_2\ldots\sigma_n \in L$, by replacing every letter $\sigma_i$ by some word from $K_\sigma_i$:

$$L(K) = \bigcup_{\sigma_1\sigma_2\ldots\sigma_n \in L} K_{\sigma_1}K_{\sigma_2}\ldots K_{\sigma_n}.$$ 

Example 6. As usual we use the shorthand $L^+$ as $L^+ = L^*L$. Consider the language $L_1$ from Example 1 and $\Sigma = \Gamma = \text{ATOMS}$. By the equivariant language $K_\sigma = (aa)^+$, or $\sigma \mapsto (aa)^+$, we obtain the language $L_1(K) = \bigcup_{a \in \text{ATOMS}} aa^+$ containing words, where all maximal constant infixes have even length.

**Lemma 5.** If $L$ and all languages $K_\sigma$ have rational Parikh images (resp. are rational) then the substitution $L(K)$ has also rational Parikh image (resp. is rational).

**Proof.** Intuitively speaking, it is enough to replace syntactically, in the rational expression defining $\text{PAR}(L)$, every appearance of a letter $\sigma$ by an expression defining $\text{PAR}(K_\sigma)$.

Formally, we proceed by induction on a derivation of $L$. By Claim 1 we assume, w.l.o.g., that languages $L$ and $K_\sigma$ are rational. If $L = \{\sigma\}$ is a singleton, $\sigma \in \Sigma$, then $L(K) = K_\sigma$ and hence is rational. The cases of $L = L_1L_2$, or $L = (L')^*$, are both immediate, as both the operations preserve rationality, and $L_1, L_2$ and $L'$ are rational by induction assumption. Finally, when $L = \bigcup_{x \in X} L_x$, by induction assumption we know rationality of the languages $L_x(K)$ for $x \in X$. As

$$L(K) = \bigcup_{x \in X} L_x(K)$$

and the mapping $x \mapsto L_x(K)$ is supported by the union of the supports of $x \mapsto L_x$ and $\sigma \mapsto K_\sigma$, we deduce that $L(K)$ is an orbit-finite union of rational sets and hence rational.

Main results: As our main contribution, we prove rationality of Parikh images of 1-NRA and 1-CFG:

**Theorem 6.** Parikh images of 1-NRA languages are rational.

**Theorem 7.** Parikh images of 1-CFG languages are rational.

We actually prove a refined version of Theorem 6 (needed also for proving Theorem 7) which, due to [4], implies Theorem 6.

**Lemma 8.** For every 1-NRA $A$, the languages $L_{q(a)}q'(a')(A)$ have rational Parikh images.

Before proving Theorem 7 and Lemma 8 in Sections V, VII we first demonstrate that semi-linear sets are not sufficient to capture Parikh images of 1-CFG, or even 1-NRA.

**IV. Semi-linear sets**

Analogously to rational sets, we lift semi-linear sets to orbit-finite alphabets. Consider data vectors over a fixed orbit-finite alphabet $\Sigma$. A *linear set* is then any set of the form

$$\bigcup_{i \in I} N_i,$$

for a data vector $g$ and an orbit-finite set $P$ of data vectors, and a semi-linear set is any orbit-finite union of linear sets:

$$\bigcup_{i \in I} g_i + P_i^*.$$ 

In particular, $I$ is orbit-finite, and the function $i \mapsto (g_i, P_i)$ mapping $i \mapsto (g_i, P_i)$ to a data vector $g_i$ (base) and an orbit-finite set $P_i$ of data vectors (periods) is legal. By definition, semi-linear sets are a subset of rational sets of star-height 1 (star-height is defined as usual, as the maximal nesting depth of stars).

**Example 7.** Parikh image of $L_1$ (cf. Example 5 $\Sigma = \text{ATOMS}$) is semi-linear: (with all sets $P_i$ equal):

$$I = (0 \cup \bigcup_{a \in \text{ATOMS}} a) g_i = i \quad P_i = \bigcup_{a,b \in \text{ATOMS}, a \neq b} a + b.$$ 

**Proposition 9.** Semi-linear sets of data vectors are exactly rational sets of star-height at most 1.

**Semi-linear sets are not sufficient:** We demonstrate that Parikh images of 1-NRA languages are not semi-linear in general. As a counterexample we take the following language $L_3 \subseteq \text{ATOMS}$. For $a \in \text{ATOMS}$, let

$$K_a = \bigcup_{b \in \text{ATOMS} - \{a\}} b; \quad L_a = aa\left(aK_a\right)^*.$$ 

Let $L_3$ be the language obtained from $L_1$ by the substitution: $a \mapsto L_a$. The language is clearly rational, and recognised by a (deterministic) one-register automaton ($H$ is omitted):

As we show, its Parikh image is not semi-linear, which motivates consideration of rational sets in forthcoming sections.

**Lemma 10.** $\text{PAR}(L_3)$ is not semi-linear.

Let $\text{SING}(v) = |\{a \in \text{ATOMS} : v(a) = 1\}|$ denote the number of atoms appearing exactly once in $v$. The argument relies on a careful analysis of the limit value of the *singularity ratio* $\frac{\text{SING}(v)}{|v|}$ for $v \in \text{PAR}(L_3)$, when $|v|$ tends to infinity.

**V. Proof of Theorem 7**

Consider a fixed 1-CFG $G = (H, Q, q_0, \Delta_2, \Delta_0)$.
Proof strategy: We proceed in three steps. First, by a Ramsey’s argument, we prove that a sufficiently large set of productions contains a compatible pair (Lemma 11). Then we define width of derivation trees and show that for a sufficiently large \( n \in \mathbb{N} \), every derivation tree can be transformed into a tree of width at most \( n \) while preserving the Parikh image of its yield (Lemma 15). The cut-and-paste transformation relies on compatibility of productions in a tree. Finally, we argue that Parikh image of the set of words generated by derivation trees of width bounded by \( n \) is rational, for every fixed \( n \in \mathbb{N} \) (Lemma 15). Lemmas 13 and 15 imply Theorem 7.

Compatibility: The equality type of a tuple \( (a_1, \ldots, a_k) \in \text{ATOMS}^k \) is defined as the set \( \{(i, j) : 1 \leq i < j \leq k, \ a_i = a_j\} \). Intuitively speaking, tuples of the same equality type admit the same equalities between their coordinates. Two tuples \( \alpha = (a_1, \ldots, a_k) \) and \( \beta = (b_1, \ldots, b_k) \) we call compatible if they have the same equality type, and for every coordinate \( i \in \{1, \ldots, k\} \) one of two conditions holds: either (1) \( a_i = b_i \); or (2) \( a_i \neq b_i \) and both \( a_i \) and \( b_i \) do not appear in the other tuple: \( a_i \notin \{b_1, \ldots, b_k\} \). In particular, two equal \( k \)-tuples are always compatible.

Lemma 11. For every \( k \in \mathbb{N} \) there is some \( l = f(k) \in \mathbb{N} \) such that every finite multiset of \( k \)-tuples of atoms \( A : \text{ATOMS}^k \rightarrow \mathbb{N} \) of size at least \( l \) contains two compatible \( k \)-tuples.

Proof. Let \( k \in \mathbb{N} \) be fixed. If \( A \) contains two equal tuples, they are compatible. Thus we can assume \( A \) to be a set. We take \( l = f(k) \) large enough to satisfy the constraint (9) below.

The number of different equality types \( E_k \) is finite and equal to the number of partitions of the coordinates set \( \{1, \ldots, k\} \) (the \( k \)th Bell number). By the pigeonhole principle, for \( l = |A| \) large enough, there is a subset \( A' \subseteq A \) of size \( l' = |A'| = \frac{l}{E_k} \) whose elements have all the same equality type.

We now consider an undirected clique of size \( l' \) with vertices\( A' \), where the edge between vertices \( \alpha = (a_1, \ldots, a_k) \) and \( \beta = (b_1, \ldots, b_k) \) is labeled (coloured) by the set \( D_{\alpha, \beta} = \{i \in \{1, \ldots, k\} : a_i \neq b_i\} \). Intuitively, the colour describes the coordinates on which \( \alpha \) and \( \beta \) disagree. The number of colours is at most \( C = 2^k \). By Ramsey’s theorem, for \( l' \) large enough the graph contains a monochromatic clique \( A'' \) of size \( l'' = k^2 + 1 \); indeed, it suffices to take

\[
   l' \geq R(l'', l'', \ldots, l''). \tag{9}
\]

Thus every two elements of \( A'' \) disagree on the same coordinates \( D \subseteq \{1, \ldots, k\} \), and hence also agree on the same coordinates \( \{1, \ldots, k\} - D \).

Take any \( \alpha = (a_1, \ldots, a_k) \in A'' \). For every coordinate \( i \in D \), all tuples \( \beta \in A'' \) are pairwise different on that coordinate. Therefore, at most \( k \) tuples \( \beta = (b_1, \ldots, b_k) \in A'' \) may satisfy

\[
   b_i \in \{a_1, \ldots, a_k\}, \tag{10}
\]

i.e., \( b_i \) appears in \( \alpha \). As \( |D| \leq k \), at most \( k^2 \) tuples (including \( \alpha \) itself) may satisfy the condition (10) for some coordinate \( i \in D \). Therefore taking any of the remaining tuples, say \( \beta \), we obtain a compatible pair \( \alpha, \beta \).

Traversals and side-effects: The number of children of a node \( x \) in a derivation tree \( T \) we call arity of \( x \) (leaves are nodes of arity 0). Let \( \preceq \) denote the tree order \( (x \preceq y \text{ if } x \text{ is an ancestor of } y) \). A path from a node \( x \) to a node \( y \), assuming \( x \preceq y \), is the set \( \{z \in T : x \preceq z \preceq y\} \) of all nodes \( z \) appearing between the nodes \( x \) and \( y \), including \( x \) and \( y \).

Consider an arbitrary derivation tree \( T \) of \( G \). We distinguish two ways of traversing a production \( q(a) \rightarrow p(b) \ p'(b') \in \Pi_2 \) appearing in \( T \) by a path, namely left and right traversal:

Once left or right traversal is chosen, say the right one, a production \( q(a) \rightarrow p(b) \ p'(b') \in \Pi_2 \) resembles a transition of 1-NRA (over the extended input alphabet \( \Gamma = (Q \cup H) \times \text{ATOMS} \)) from \( q(a) \) to \( p'(b') \) which inputs the label of the remaining node, namely \( p(b) \). We call the pair \( p(b) \in \Gamma \) the side-effect of the right traversal; symmetrically we call \( p'(b') \) the side-effect of the left traversal. For two configurations \( q(a) \) and \( p(b) \) of \( G \), we denote by \( S_{q(a)}(p(b) \subseteq \Gamma^* \) the set of all sequences of side-effects that may appear along a path from a node labeled by \( q(a) \) to a node labeled by \( p(b) \) in a derivation tree of \( G \). As a corollary of Lemma 8 we get:

Lemma 12. Languages \( S_{q(a)}(p(b) \) have rational Parikh images.

Proof. Indeed, the claim follows immediately by Lemma 8 if production traversals are considered as transitions of a 1-NRA over the input alphabet \( \Gamma \), and the side-effect of a traversal is considered as input of a transition.

Height, width, and rank: Recall the normal form of constraints (1)–(5) as defined in Section II. Similarly as in case of 1-NRA, the right traversal of a production \( q(a) \rightarrow p(b) \ p'(b') \in \Pi_2 \) is called register-preserving if \( a = b' \), and register-updating if \( a \neq b' \); likewise for the left traversal.

We define the length of a path in a derivation tree \( T \) as the number of register-updating production traversals along the path, and the height of a node \( x \) in \( T \) as the maximal length of a path from \( x \) to a leaf. A cut in \( T \) is a set of nodes which are pairwise incomparable with respect to the tree ordering. A cut is called \( n \)-cut if its size is at least \( n \) and the height of every node in the cut is at least \( n \). The width of a derivation tree \( T \) is the maximal \( n \) for which \( T \) contains some \( n \)-cut.

The rank of a derivation tree is defined as the multiset of lengths of all paths from the root to some leaf. For a finite multiset \( r : \mathbb{N} \rightarrow \mathbb{N} \) of natural numbers, let the diagram of \( r \) be the unique non-increasing sequence \( w \in \mathbb{N}^* \) such that \( \text{PAR}(w) = r \). We define the order on ranks as follows: \( r \leq r' \) if
the diagram of \( r \) is lexicographically smaller than the diagram of \( r' \). For instance, \( \{7,5,2,2\} < \{7,7,3\} \).

We call two derivation trees \( T, T' \) Parikh-equivalent if \( \text{Par}(\text{Yield}(T)) = \text{Par}(\text{Yield}(T')) \).

**Lemma 13.** For a sufficiently large \( n \), every derivation tree is Parikh-equivalent to a derivation tree of width at most \( n \).

**Proof.** Let \( m = |\Delta_2| \). Fix an arbitrary \( n \geq f(6) \cdot 2m \), for \( f \) given by Lemma 11. We show:

**Claim 2.** Every derivation tree \( T \) of \( G \) of width \( \geq n \) can be transformed, by cutting and pasting of some parts, into a Parikh-equivalent derivation tree \( T' \) of rank strictly larger than \( T \), but of the same size (= the number of nodes) as \( T \).

The claim is sufficient for proving Lemma 13. Indeed, as the transformation preserves the size, the rank can increase only finitely many times. Therefore, by iterating the transformation we ultimately arrive at a derivation tree \( T' \) whose rank can not be further increased. By Claim 2, the width of \( T' \) is forcedly at most \( n - 1 \), as required.

From now on we concentrate on proving Claim 2. Let \( T \) be a derivation tree of width \( \geq n \). Consider some fixed \( n \)-cut \( \{x_1, \ldots, x_n\} \) and disjoint paths \( \pi_1, \ldots, \pi_n \) in \( T \) of length \( \geq n \), each path \( \pi_i \) going from \( x_i \) to some leaf.

Consider a fixed path \( \pi_i \). It contains \( \geq n \) register-updating production traversals, and therefore by the pigeonhole principle the same production rule \( q \xrightarrow{\Delta_2} p p' \in \Delta_2 \) and the same (say left) register-updating traversal repeats at least \( n' = \frac{n}{2m} \) times along \( \pi_i \). We apply Lemma 11 for \( k = 3 \) to deduce that, as \( n' \geq f(6) \geq f(3) \), some two of these traversals are compatible, by which we mean that their underlying 3-tuples \( \langle a, b, b' \rangle \) and \( \langle c, d, d' \rangle \) are so. Thus each path \( \pi_i \) traverses a pair of compatible productions \( \delta_i, \sigma_i \) which agree on the production rule and (left or right) traversal.

We now repeat a similar argument for paths. As before, by the pigeonhole principle in at least \( n' \) paths \( \pi_i \), the same production rule and the same traversal was used in productions \( \delta_i \) and \( \sigma_i \), derived in the above reasoning. We now apply Lemma 11 for \( k = 6 \) to deal with pairs \( (\delta_i, \sigma_i) \) of productions, where a pair \( (\delta_i, \sigma_i) \) induces a 6-tuple obtained by concatenating two underlying 3-tuples of \( \delta_i \) and \( \sigma_i \). Since \( n' \geq f(6) \), according to the lemma some two of these pairs, say \( (\delta_i, \sigma_i) \) and \( (\delta_j, \sigma_j) \), are compatible (by which we mean that the two induced 6-tuples are so).

We have thus four productions \( \delta, \sigma, \bar{\delta}, \bar{\sigma} \), traversed by two disjoint paths in \( T \) (we do not depict nonterminals as all the four productions are induced by the same rule):

---

**Claim 3.** The four underlying triples \( \langle a, b, b' \rangle, \langle c, d, d' \rangle, \langle \bar{a}, \bar{b}, b' \rangle \) and \( \langle \bar{c}, d, d' \rangle \) are pairwise compatible.

**Proof of Claim 3** By the construction we have compatibility of triples \( \langle a, b, b' \rangle \) and \( \langle c, d, d' \rangle \), and of triples \( \langle \bar{a}, \bar{b}, b' \rangle \) and \( \langle \bar{c}, d, d' \rangle \). Furthermore, we have also compatibility of 6-tuples \( \langle a, b, b', c, d, d' \rangle \) and \( \langle \bar{a}, \bar{b}, b', c, d, d' \rangle \), which implies compatibility of triples \( \langle a, b, b' \rangle \) and \( \langle \bar{a}, \bar{b}, b' \rangle \), and of \( \langle c, d, d' \rangle \) and \( \langle \bar{c}, d, d' \rangle \). Therefore, it only remains to prove compatibility of \( \langle a, b, b' \rangle \) and \( \langle \bar{c}, d, d' \rangle \), and of \( \langle c, d, d' \rangle \) and \( \langle \bar{a}, \bar{b}, b' \rangle \). We concentrate of the former pair, as the other one is dealt with similarly.

The equality types of triples \( \langle a, b, b' \rangle \) and \( \langle \bar{c}, d, d' \rangle \) are the same, since so are the equality types of \( \langle a, b, b' \rangle \) and \( \langle c, d, d' \rangle \), and of \( \langle c, d, d' \rangle \) and \( \langle \bar{c}, d, d' \rangle \). We concern the first coordinate of the triples. Supposing \( a \neq \bar{c} \), we derive \( a \notin \{\bar{c}, d, d'\} \): if \( a \neq \bar{c} \) then this follows due to compatibility of the two 6-tuples, and if \( a = \bar{c} \) then this follows due to compatibility of \( \langle \bar{a}, \bar{b}, b' \rangle \) and \( \langle \bar{c}, d, d' \rangle \); symmetrically we derive \( \bar{c} \notin \{a, b, b' \} \). The two remaining coordinates are dealt with similarly.

We are now prepared to cutting and pasting in \( T \). For convenience we use below atoms \( a, b \), etc. to identify respective nodes (keeping in mind potential equalities between these atoms). Recall that all the four traversals are register-updating, and hence \( a \neq b \), and likewise for other tuples. We distinguish three cases, depending on the relation of \( b' \) to \( a \) and \( b \):

**Case 1** \( b' = a \): Define the relevance of a node \( x \) in \( T \) as the maximal length of a path from the root of \( T \) to a leaf that traverses \( x \). By symmetry assume, w.l.o.g., that the relevance \( \bar{r} \) of the node \( \bar{b} \) is larger or equal to the relevance \( r \) of the node \( b \). We cut the segment of \( T \) starting from the edge \( a \to b \) and ending with the edge \( c \to d \), and paste this segment between the nodes \( \bar{a} \) and \( \bar{b} \) as depicted in the figure:

By Claim 3 the tree \( T' \) so obtained is still a derivation tree:
Indeed, \( d \neq a \) (because either \( d = b \) or \( d \) does not appear elsewhere) and hence \( q(a) \rightarrow p(d) \) \( p'(b') \) \( \in \Pi_2 \) is a production; likewise for the two remaining productions above.

Furthermore, we claim that rank of \( T' \) is strictly larger than rank of \( T \). To this aim we analyse the effect of cut and paste on the lengths of the paths from the root to a leaf in \( T \). First, all paths not traversing \( b \) or \( b' \) remain untouched. Furthermore, the lengths of all paths traversing \( b \) strictly increase. Thus some path of length \( r \) in \( T \) gets strictly prolonged, and all other affected paths in \( T \) have lengths at most \( r \leq \tilde{r} \). As before, these two properties ensure that the rank of \( T' \) is strictly larger than the rank of \( T \).

**Case 2** \( b' = b \): By symmetry assume, w.l.o.g., that the relevance \( \tilde{r} \) of the node \( a \) is larger or equal to the relevance \( r \) of the node \( a \). We cut the segment of \( T \) starting from the edge \( a \rightarrow b \) and ending with edges \( c \rightarrow d \) and \( c \rightarrow d' \), and paste this segment between the node \( a \) and the nodes \( b, b' \), and moreover cut the subtree rooted in \( b' \) and paste it in place of the subtree rooted in \( b' \), as depicted in the figure:

By Claim 3 the tree \( T' \) obtained is a derivation tree, as before:

Similarly as before, we claim that the rank of \( T' \) is strictly larger than the rank of \( T \). First, all paths from the root to a leaf in \( T \) not traversing \( a \) or \( a \) remain untouched. Furthermore, the lengths of all paths from the root to a leaf that traverse \( a \) strictly increase. Thus some path of length \( \tilde{r} \) in \( T \) gets strictly prolonged, and all other affected paths in \( T \) have lengths at most \( r \leq \tilde{r} \). As before, these two properties ensure that the rank of \( T' \) is strictly larger than the rank of \( T \).

**Case 3** \( b' \notin \{a, b\} \): In this case one can use any of the two cut-and-paste schemes described above.

The proof of Claim 2 is thus completed, and hence so is the proof of Lemma 15.

We denote by \( H_{q(a),n} \subseteq L_q(G) \) the subset of words generated by a derivation tree of height at most \( n \), and by \( W_n \) the subset of \( L(G) \) of words generated by a derivation tree of width at most \( n \). We now prove, for every \( n \in \mathbb{N} \), rationality of the languages \( H_{q(a),n} \), and then use it to derive rationality of the Parikh image of the language \( W_n \).

**Lemma 14.** For every \( n \in \mathbb{N} \), the languages \( H_{q(a),n} \) have rational Parikh images.

**Lemma 15.** For every \( n \in \mathbb{N} \), the language \( W_n \) has rational Parikh image.

**Proof.** For a fixed \( n \in \mathbb{N} \), consider an arbitrary derivation tree \( T \) of width at most \( n \), and the subset \( \mathcal{H} \subseteq T \) of those nodes which have height at least \( n + 1 \). The set \( \mathcal{H} \) is closed under ancestors and is thus itself a tree; contrarily to \( T \) whose non-leaf nodes have arity 2, the tree \( \mathcal{H} \) may contain nodes of arity 1. Notably, as a special case \( \mathcal{H} \) may be empty.

By assumption, width of \( T \) is at most \( n \), and hence it may contain \( n \)-cuts but no \( (n+1) \)-cuts. This implies that the largest cut in \( \mathcal{H} \) has size \( n \). In consequence:

**Claim 4.** \( \mathcal{H} \) has at most \( n \) leaves, and hence at most \( n + 1 \) nodes of arity 2.

Let \( \mathcal{L} \) denote the finite multiset (of size at most \( n \)) of configurations \( q(a) \) labelling leaves of \( \mathcal{H} \).

Any maximal path consisting of nodes of arity 1 we call a segment. Thus \( \mathcal{H} \) decomposes uniquely into leaves, nodes of arity 2, and segments. An example tree on the right has \( n = 4 \) leaves, 3 nodes of arity 2 and 4 segments (depicted by blue areas) of size 3, 2, 2 and 1, respectively. Using Claim 4 we deduce:

**Claim 5.** \( \mathcal{H} \) contains at most \( 2n – 1 \) segments.

Let \( \mathcal{S} \) denote the finite multiset (of size at most \( 2n – 1 \)) of pairs of configurations \( \langle q(a), p(b) \rangle \) labelling ends of segments. Let \( \overline{S}_{q(a), p(b)} \) be obtained from the side-effect language \( S_{q(a), p(b)} \) by the equivariant substitution (for \( q' \in Q \)):

\[
q'(c) \mapsto H_{q'(c),n-1};
\]

by Lemmas 14 and 12 languages \( \overline{S}_{q(a), p(b)} \) have thus rational Parikh images. Let’s define (\( \prod \) denotes concatenation)

\[
L_{\mathcal{L},\mathcal{S}} = \left( \prod_{q(a) \in \mathcal{L}} H_{q(a),n} \right) \left( \prod_{\langle q(a), p(b) \rangle \in \mathcal{S}} \overline{S}_{q(a), p(b)} \right)
\]
as the concatenation of two concatenations, one of them ranging over \( \mathcal{L} \) and the other one over \( \mathcal{S} \). By the very definition of the language \( L_{\mathcal{L},\mathcal{S}} \) we have

Claim 6. \( \text{Par} \langle \text{yield} (T) \rangle \in \text{Par}(L_{\mathcal{L},\mathcal{S}}) \).

Claim 7. The languages \( W_n \) and \( K = \bigcup_{\mathcal{L},\mathcal{S}} L_{\mathcal{L},\mathcal{S}} \) are Parikh-equivalent, where \( \mathcal{L}, \mathcal{S} \) range over all possible sets arising from all derivation trees \( T \) of \( \mathcal{G} \) of width at most \( n \).

Proof. The inclusion \( \text{Par}(W_n) \subseteq \text{Par}(K) \) we deduce by Claim 6. For the converse inclusion \( \text{Par}(K) \subseteq \text{Par}(W_n) \) we should prove: for every \( \mathcal{L}, \mathcal{S} \) arising from some derivation tree \( T \) of width at most \( n \), the language \( L_{\mathcal{L},\mathcal{S}} \) is included in \( W_n \). Indeed, given \( T \) and \( \mathcal{H} \) used to derive sets \( \mathcal{L}, \mathcal{S} \), we observe that every word \( w \in L_{\mathcal{L},\mathcal{S}} \) is Parikh-equivalent to the yield of a derivation tree \( T' \) of width at most \( n \), obtained from \( \mathcal{H} \) by replacing each leaf labelled by \( q(a) \) with a tree of height \( \leq n \) with root labelled by \( q(a) \), and replacing each segment with a sequence of productions, where every side-effect \( q(a) \) is replaced by a tree of height at most \( n - 1 \) with root labelled by \( q(a) \). Thus \( \text{Par}(w) \in \text{Par}(W_n) \).

Finally, we derive rationality of \( \text{Par}(K) \). By Lemmas 14 and 12 the languages \( L_{\mathcal{L},\mathcal{S}} \) have rational Parikh images. Due to the bounds on the size of \( \mathcal{L} \) and \( \mathcal{S} \) (cf. Claims 4 and 5), by Lemma 1 the set of all possible pairs \( \mathcal{L}, \mathcal{S} \) is orbit-finite. Therefore \( K \), as an orbit-finite union of languages with rational Parikh images, has a rational Parikh image too.

VI. PROOF OF LEMMA 8

Consider a fixed 1-NRA \( \mathcal{A} = (H, Q, I, F, \Delta) \).

Proof strategy: The proof proceeds by a sequence of simplifying steps, as stated in consecutive Lemmas [17–18] in this section and in Lemmas [19–23] in the next one. Instead of only considering Parikh images of input words, in the proof we investigate Parikh images of runs, mostly concentrating on alterings of register value along a run. This leads us to consider, besides languages over the alphabet \( H \times \text{ATOMS} \) of a 1-NRA, also languages over richer alphabets:

- languages of altering paths over the alphabet \( (Q \times \text{ATOMS} \times Q) \cup (H \times \text{ATOMS}) \) in Lemma 17;
- languages of altering loops over the alphabet \( \text{ATOMS}^2 \times \text{ATOMS} \) in Lemma 18;
- languages of anti-paths and anti-cycles over \( \text{ATOMS} \times \text{ATOMS}^2 \) in Lemmas 19–23.

The intuitive idea underlying the final, most technical steps (Lemmas 19–23) is, roughly speaking, that the set of words

\[ \langle a_1, b_1 \rangle \langle a_2, b_2 \rangle \ldots \langle a_n, b_n \rangle \]

over \( \text{ATOMS}^2 \), satisfying \( b_i \neq a_{i+1} \) for all \( i = 1, \ldots, n - 1 \), has rational Parikh image. Notably, this is not true for paths, where one requires \( b_i = a_{i+1} \) instead.

A. Proof of Lemma 8

For locations \( q, p \in Q \) of \( \mathcal{A} \) and \( a \in \text{ATOMS} \), let \( L_{\text{gap}} \) be the language of all data words read by a run from configuration \( q(a) \) to \( p(a) \) that use register-preserving transitions only (thus the register stores \( a \) along the whole run).

Lemma 16. The languages \( L_{\text{gap}} \) are rational.

Proof. We only need to consider register-preserving transitions. Define the finite alphabet \( \Delta = H \times \{ \varphi_1, \varphi_2 \} \) and consider every transition rule \((q(x), \langle h, y \rangle, \varphi, q'(x'))\) to be labeled by \((h, \varphi) \in \Delta\).

Fix \( q, a \) and \( p \) and let \( E_{\text{gap}} \) be the classical regular expression over \( \Delta \) defining the labels of all those runs from \( q \) to \( p \) that only use transitions of types \((1)\) and \((2)\). Then the language \( L_{\text{gap}} \) is defined by the expression \( E_{\text{gap}} \) obtained from \( E_{\text{gap}} \) by replacing \((h, \varphi_1)\) with \( \langle h, a \rangle \) and replacing \((h, \varphi_2)\) with

\[ L_{\text{gap}} = \bigcup_{b \in \text{ATOMS} - \{a\}} \langle h, b \rangle. \]

Thus \( L_{\text{gap}} \) is a rational data language.

We now state the central lemma that generalises Example 11.

Define the language \( P \) over the alphabet \((Q \times \text{ATOMS} \times Q) \cup (H \times \text{ATOMS})\) containing words of the form \((n \geq 1)\):

\[ \langle q_1, a_1, p_1 \rangle \langle h_1, b_1 \rangle \langle q_2, a_2, p_2 \rangle \langle h_2, b_2 \rangle \ldots \langle q_n, a_n, p_n \rangle \]

such that \( p_i(a_i) \rightarrow (h_i, b_i) \rightarrow q_{i+1}(a_{i+1}) \) is a register-updating transition for \( i = 1, \ldots, n - 1 \) (in particular \( a_i \neq a_{i+1} \) for \( i = 1, \ldots, n - 1 \)). Words in \( P \) we call altering paths. Furthermore, define the subsets \( P_{q(a)} q'(a') \subseteq P \) of those altering paths as in (11) where \( q(a) = q_1(a_1) \) and \( q'(a') = p_n(a_n) \).

Lemma 17. Altering path languages \( P_{q(a)} q'(a') \) have rational Parikh images.

Before proving the lemma we use it to complete the proof of Lemma 8. Indeed, \( L_{q(a)} q'(a') (\mathcal{A}) \) is obtained from the altering path language \( P_{q(a)} q'(a') \) using the equivariant substitution \((q, p) \) range over locations and \( a, b \) over \( \text{ATOMS} \):

\[ (q, p) \rightarrow L_{\text{gap}} \quad (h, b) \rightarrow (h, b). \]

As a substitution by languages with rational Parikh images preserves rationality of Parikh image (cf. Lemma 5), by Lemmas 16 and 17 we deduce that the languages \( L_{q(a)} q'(a') (\mathcal{A}) \) have rational Parikh images, as required.

B. Proof of Lemma 17

We define, for a register-updating transition constraint \( \varphi \in \{ \varphi_4, \varphi_5 \} \) and (not necessarily distinct) atoms \( a', b, a \in \text{ATOMS} \), the language \( L_{(a',b),\varphi a} \) over the alphabet \( \text{ATOMS}^2 \times \text{ATOMS} \) as follows: let \( L_{(a',b),\varphi a} \) contain all (possibly empty) words of the form

\[ \langle a_1, a'_1 \rangle b_1 \langle a_2, a'_2 \rangle b_2 \ldots \langle a_n, a'_n \rangle b_n \]

such that \( (a'_i, b_i, a_{i+1}) \models \varphi \) for \( i = 1, \ldots, n \). We omit the case \( \varphi = \varphi_3 \) as it is can be treated symmetrically to the case \( \varphi = \varphi_4 \).
Words in $L_{(a_i', b_i)}\varphi_{a_i+1}$ we call altering loops. Intuitively, a letter $\langle (d', d''), e \rangle \in ATOMS^2 \times ATOMS$ represents (cf. the substitution (13) below), for some locations $p', h$ and $h \in H$, an altering path from $p(d)$ to $p'(d'')$ followed by a register-updating transition that inputs $(h, e)$. We derive Lemma 17 from the following one (proved itself in Section VI-C below):

Lemma 18. Altering loop languages $L_{(a', b)\varphi a}$ have rational Parikh images.

We mimic the standard proof of Kleene theorem, exploiting altering loops to capture all iterations along loops in $A$. We proceed by induction on the number of register-updating transition rules in $A$. If there is no such transition rules, we have trivial (and obviously rational) altering path languages

$$P_{\varphi_4} = \{ \langle q, a, q' \rangle \} \quad \text{if } a = a', \quad \text{otherwise.}$$

Otherwise, remove an arbitrary register-updating transition rule $t = \langle q(x), (h, y), \varphi, q'(a') \rangle$ from $A$ (if $\varphi = \varphi_4$ consider the inverse of $A$ and $\varphi = \varphi_4$ instead), and use the induction assumption for the so obtained automaton $A'$ to get altering path languages $K_{\varphi_4}(q(a'))$ for every locations $q, q'$ and atoms $a, a'$, with rational Parikh images. Let $L_{(c', b)\varphi c}$ be the language obtained from the altering loops $L_{(c', b)\varphi c}$ by the equivalence sublanguage $(d, d', e \text{ range over ATOMS})$

$$\langle (d, d'), e \rangle \rightarrow K_{p(d)p'(d')} \langle h, e \rangle. \quad (13)$$

Rationality of the Parikh images of the altering path languages $P_{\varphi_4}(q'(a'))$ of $A$ follows by the fact that $P_{\varphi_4}(q'(a'))$ is equal to the union of $K_{\varphi_4}(q'(a'))$ and the following set

$$\bigcup_{c', b, c \in ATOMS} K_{\varphi_4}(q'(c')) \langle h, b \rangle L_{(c', b)\varphi c} K_{\varphi_4}(q'(a')). \quad (14)$$

To show the equality, we observe that $K_{\varphi_4}(a'(q'))$ contains all altering paths in $A$ that do not use $t$, and claim that the set (14) contains those altering paths in $A$ that do use $t$. Specifically, as $e \in L_{(c', b)\varphi c}$, we obtain altering paths using $t$ exactly once (dotted arrow depict altering paths in $A'$):

$$K_{\varphi_4}(q'(c')) \langle h, b \rangle (c', b, c) = \varphi \rightarrow K_{\varphi_4}(q'(a')) \langle h, c \rangle (c', b, c) = \varphi$$

or more than once (for instance twice, as shown in the figure):

In general, a word in (14) factorises into a prefix before the first use of $t$ (an altering path from $q(a)$ to $p'(c')$), the suffix after the last use of $t$ (an altering path from $p(c)$ to $q'(a')$), and the infix leading from $p'(c')$ to $p(c)$. The infix starts with the letter $(h, b)$ input by the first traversal of $t$, and then contains alternately altering paths that do not use $t$ (from $p(d)to p'(d')$, for some $d, d' \in ATOMS$) and traversals of $t$ (a letter $(h, e)$ for some $e \in ATOMS$), cf. the substitution (13). Therefore, by the definition (12) of altering loops $L_{(a', b)\varphi a}$, the set (14) contains exactly those altering paths in $A$ that do use $t$, as claimed.

C. Proof of Lemma 18

We concentrate on the hardest case $\varphi = \varphi_5$ (all the three atoms involved in $\varphi$ are pairwise distinct). The remaining case $\varphi = \varphi_4$ is obtained then using the substitution

$$\langle (a, a'), b \rangle \rightarrow \langle (a, a'), a' \rangle.$$

We need to show that the altering loop languages $L_{(c, b)\varphi c}$ have rational Parikh images. Recall that $L_{(c, b)\varphi c}$ contains all words over ATOMS$^2 \times ATOMS$ of the form

$$\langle (a_1, c_1), b_1 \rangle \langle (a_2, c_2), b_2 \rangle \ldots \langle (a_n, c_n), b_n \rangle \quad (15)$$

such that $c_i, b_i, a_{i+1}$ are pairwise different for $i = 0, \ldots, n$.

We reduce Lemma 18 to Lemma 19 (which constitutes the technical core of the proof of Lemma 8). Relying on the observation that $b_i$ and $c_i$ play entirely symmetric roles in (15) and are forcedly distinct, we rearrange words (15) into words over the alphabet $\Gamma = ATOMS \times P_2(ATOMS)$ as follows:

$$\langle a_1, \{b_1, c_1\} \rangle \langle a_2, \{b_2, c_2\} \rangle \ldots \langle a_n, \{b_n, c_n\} \rangle. \quad (16)$$

Let $P_{\{b_i, c_i\}a_{i+1}} \subseteq \Gamma^*$ denote the language of all nonempty words of the form (16) subject to the same constraints as in (15), namely $a_{i+1} \notin \{b_i, c_i\}$ for $i = 0, \ldots, n$; these words we call anti-paths in the sequel. Note that $a_i \in \{b_i, c_i\}$ is allowed. We observe that $L_{(c, b)\varphi c}$ is obtained from $P_{\{b, c\}a}$ using the equivariant substitution

$$\langle d, \{e, f\} \rangle \rightarrow \langle d, e \rangle f \bigcup \langle d, f \rangle e,$$

and adding the empty word. Therefore the language $L_{(c, b)\varphi c}$ has rational Parikh image assuming $P_{\{b, c\}a}$ has so, and Lemma 18 is implied by the following core technical result:

Lemma 19. The anti-path languages $P_{\{b, c\}a} \subseteq \Gamma^*$ have rational Parikh images.

The proof of Theorem 6 is thus completed once we prove Lemma 19. The whole next section is devoted to this task.

VII. Anti-paths: Proof of Lemma 19

For a letter $\alpha = \langle a, \{b, c\} \rangle \in \Gamma$ we call the atom $a$ its source, and the two-element set $\{b, c\}$ its target, denoted $a = SRC(\alpha)$ and $\{b, c\} = TRG(\alpha)$, respectively. For a word $w = \alpha_1 \ldots \alpha_n \in \Gamma^*$ we denote by $SRC(w) = SRC(\alpha_1)$ the first source, and by $TRG(w) = TRG(\alpha_n)$ the last target.
Anti-cycles: An anti-path \( w \) is called an anti-cycle if \( \text{SRC}(w) \notin \text{TRG}(w) \) (the first source does not belong to the last target). Anti-cycles are closed under cyclic shifts, and hence we use the cyclic order when speaking about precedence of letters in anti-cycles. Denote the set of all anti-cycles by \( C \). We build on a simple but crucial observation: anti-paths \( P_{\{a,b\}} \) are exactly those words \( w \in \Gamma^* \) which, prolonged with a single letter \( w \langle a, \{b, e\} \rangle \in C \), form an anti-cycle:

Claim 8. \( P_{\{b,c\}} = \{ w \in \Gamma^* : w \langle a, \{b, e\} \rangle \in C \} \).

Lemma 20. If \( L \subseteq \Gamma^* \) is rational and \( \alpha \in \Gamma \) then the language \( L \prec \alpha = \{ w \in \Gamma^* : w \alpha \in L \} \) is rational too.

Claim 8 and Lemma 20 prove Lemma 19 once we have:

Lemma 21. \( C \) has rational Parikh image.

Indeed, let \( L \subseteq \Gamma^* \) be rational and Parikh-equivalent to \( C \). By Claim 8, \( P_{\{b,c\}} \) is Parikh-equivalent to \( L \prec \alpha \langle a, \{b, e\} \rangle \), which is rational by Lemma 20. Thus it suffices to prove Lemma 21.

We mostly focus on a special but central case of Lemma 21, namely we restrict to the sub-alphabet

\[ \Sigma = \{ \alpha \in \Gamma : \text{SRC}(\alpha) \notin \{b, e\} \} \subseteq \Gamma. \]

Lemma 22. If the language \( D = C \cap \Sigma^* \) has rational Parikh image then \( C \) has rational Parikh image too.

Lemma 23. The language \( D \) has rational Parikh image.

A. Proof of Lemma 23

The number of different sources of letters appearing in a data vector \( v : \Sigma \to \mathbb{N} \), i.e., the size of the set

\[ V_v = \{ \text{SRC}(\alpha) : \alpha \in \text{DOM}(v) \}, \quad (17) \]

we denote by \( \text{ORD}(v) \) and call the order of \( v \) (clearly, an atom can be the source of more than one letter in \( \text{DOM}(v) \)).

The order of a data word \( w \in \Sigma^* \) is defined naturally as \( \text{ORD}(w) = \text{ORD}(|\text{PAR}(w)|) \). We write \( D^{\leq n} \) (resp. \( D^{>n} \)) for the subsets of \( D \) containing anti-cycles of order smaller than \( n \) (resp. at least \( n \)). Anti-cycles of bounded order can be easily dealt with separately:

Lemma 24. For every \( n \in \mathbb{N} \), the language \( D^{\leq n} \) is rational.

Therefore, in the rest of the proof we concentrate on anti-cycles or order at least \( n \), for a sufficiently large \( n \in \mathbb{N} \).

Source graphs: In the sequel we consider directed graphs without self-loops or parallel edges, but possibly containing tight two-vertex cycles.

Let \( v : \Sigma \to \mathbb{N} \) be a fixed data vector. Guided by the crucial property of anti-paths that the source of every letter does not belong to the target of the preceding letter, we define the directed graph \( G_v = (V_v, E_v) \), called source graph induced by \( v \); let the vertices \( V_v \) of \( G_v \) be the sources of all letters appearing in \( v \), as defined in (17), and let \( (d,e) \in E_v \) be an edge if, and only if

\[ \exists \alpha \in \text{DOM}(v) : d = \text{SRC}(\alpha), e \notin \text{TRG}(\alpha). \]

Whenever \( (d,e) \notin E_v \), for distinct atoms \( d \neq e \), we say that \( d \) excludes \( e \) (or call \( (d,e) \) an excluded edge); equivalently, \( e \) belongs to the target of every letter in \( \text{DOM}(v) \) with source \( d \):

\[ \forall \alpha \in \text{DOM}(v) : d = \text{SRC}(\alpha) \implies e \in \text{TRG}(\alpha). \]

Note that an atom never excludes itself, due to restriction to \( \Sigma \), and that \( G_v \) depends only on the set \( \text{DOM}(v) \subseteq \Sigma \) of letters appearing in \( v \), and not on cardinalities of letters in \( v \).

Let \( \text{IN}(e) = \{ d \in V_v : (d,e) \in E_v \} \) denote the set of in-neighbours of a vertex \( e \), and let \( \text{IN-DEG}(e) = |\text{IN}(e)| \) denote the in-degree of \( e \). Symmetrically we define out-neighbours \( \text{OUT}(e) \) and out-degree \( \text{OUT-DEG}(e) \). Clearly, an atom may exclude at most two other atoms, and hence (let \( n = \text{ORD}(v) \)):

Claim 9. \( \text{OUT-DEG}(d) \geq n - 3 \) for every vertex \( d \in V_v \).

Corollary 25. There are at most \( 2n \) excluded edges.

In the sequel we rely on Claim 9 and Corollary 25 according to which \( G_v \) is not much different from the full directed clique.

For \( A \subseteq \text{DOM}(v) \), let \( v_{|A} \) denote the restriction of \( v \) to \( A \):

\[ v_{|A}(\alpha) = v(\alpha) \text{ if } \alpha \in A, \text{ and } v_{|A}(\alpha) = 0 \text{ otherwise}. \]

In the proof of Lemma 29 we transform cycles in \( G_v \) into anti-cycles, using the following lemma:

Lemma 26. For every simple cycle \( \pi = a_1 a_2 \ldots a_n \) in \( G_v \), there exists an anti-cycle \( w \) with \( \text{PAR}(w) = v_{|A} \) where \( A = \{ \alpha \in \text{DOM}(v) : \text{SRC}(\alpha) \in \{a_1, a_2, \ldots , a_n\} \} \).

Proof. We arrange the letters into an anti-path \( w \) by taking first all \( a_1 \)-sourced letters in a consecutive block, then all \( a_2 \)-sourced ones in a consecutive block, etc. The order of \( a_1 \)-sourced letters inside a block (including repetitions of equal letters) is irrelevant as long as the last one, say \( \alpha \), satisfies \( a_{n+1} \notin \text{TRG}(\alpha) \) (where \( n + 1 \) is identified cyclicly with 1).

In the proof of Lemma 29 we also use a sufficient condition for a directed graph to admit a Hamiltonian cycle:

Theorem 27 \([9]\), cf. also Thm. 1 in \([10]\). Let \( G \) be a strongly connected directed graph with \( n \) vertices such that for every two vertices \( d, d' \), \( \text{IN-DEG}(d) + \text{OUT-DEG}(d') \geq n \). Then \( G \) contains a Hamiltonian cycle.

The tool will be applicable due to the following observation:

Lemma 28. For sufficiently large \( n \), a directed graph with \( n \) vertices such that \( \text{IN-DEG}(d) \geq 3 \) and \( \text{OUT-DEG}(d) \geq n - 3 \) for every vertex \( d \), is necessarily strongly connected.

Proof. Consider the decomposition of the graph into strongly connected components. As the first step we observe that there may be no singleton components \( \{d\} \). Indeed, by the assumption we have \( \text{IN-DEG}(d) + \text{OUT-DEG}(d) \geq n \), and hence \( d \) forms a tight 2-vertex cycle with some other vertex \( d' \).
In the sequel we use Corollary 25. As the second step we argue that for sufficiently large $n$, a component $\{d, e\}$ of size 2 is impossible (and, in consequence, a component of size $n - 2$ is impossible too). Towards contradiction, suppose $\{d, e\}$ is a strongly connected component (hence the two vertices form a tight cycle). In consequence, (a) the sets $V_d = \text{OUT}(d) - \{e\}$ and $V_e = \text{IN}(e) - \{d\}$ are disjoint, and (b) there is no edge from $V_d$ to $V_e \cup \{d, e\}$. As $\text{OUT-DEG}(d) \geq 3$ and $\text{IN-DEG}(e) \geq 3$, we have $|V_d| \geq n - 4$ and $|V_e| \geq 2$. By (a) we deduce $|V_d| = n - 4$ and $|V_e| = 2$. By (b), all $(4 - n)$ edges from $V_d$ to $V_e \cup \{d, e\}$ are excluded. This is impossible as long as $4(n - 4) > 2n$.

Likewise one argues that there may be no component of size strictly between 2 and $n - 2$. Indeed, supposing there is a component $C$ of size $k$, for $2 < k < n - 2$, no vertex in $C$ may form a tight cycle with other vertex outside of $C$, and hence at least $k(n - k)$ edges are excluded. This is impossible as long as $k(n - k) > 2n$. As $k(n - k)$ reaches its minimum for $k = 3$ or $k = n - 3$, there may be no component of size strictly between 2 and $n - 2$ as long as $3(n - 3) > 2n$.

Non-degeneracy: Let $\text{PRE}(d) = \{\alpha \in \text{DOM}(v) : d \not\in \text{SRC}(\alpha), d \not\in \text{TRG}(\alpha)\}$ denote the set of letters that can precede a $d$-sourced letter and have themselves source different than $d$. A data vector $v : \Sigma \rightarrow \mathbb{N}$ is called non-degenerate if the following conditions hold:

1. $\text{IN}(d) \not= \emptyset$ for every $d \in V_v$,
2. $\text{IN}(d) \cup \text{IN}(e) \not\subseteq \{d, e\}$ for every non-equal $d, e \in V_v$,
3. $|\text{IN}(d) \cup \text{IN}(e)| \geq 2$ for every non-equal $d, e \in V_v$.

(1) excludes vertices of in-degree 0, (2) excludes pairs of vertices $d, e$ with $\text{IN}(d) = \{e\}$ and $\text{IN}(e) = \{d\}$, (3) excludes the case when there is only one letter $\alpha \in \text{DOM}(v)$ that can precede $d$- or $e$-sourced letters, and moreover $\sigma(\alpha) = 1$.

**Lemma 29.** For data vectors $v : \Sigma \rightarrow \mathbb{N}$ of sufficiently large order, $v \in \text{PAR}(D)$ if, and only if $v$ is non-degenerate.

**Proof.** Let $G_v = (V_v, E_v)$ be the source graph and $n = \text{ORD}(v)$.

The ‘only if’ implication is immediate for data vectors of order at least 3. Indeed, suppose $v = \text{PAR}(w)$ for an anti-cycle $w \in D$. By the definition of anti-cycles, $\text{IN}(d) \not= \emptyset$ for every $d \in V_v$ and hence (1) forcedly holds. The other two conditions are easily shown by contradiction. Indeed, if (2) fails for some $d, e \in V_v$ then every $d$- or $e$-sourced letter would be preceded in $w$ by a $d$- or $e$-sourced one, which is impossible as long as $\text{ORD}(v) \geq 3$. Finally, if (3) fails then the same letter $\alpha$ would have to precede two different letters in $w$.

For the ‘if’ implication, we assume that $v$ is non-degenerate ((1)–(3) hold) and prove that $v = \text{PAR}(w)$ for some $w \in D$.

Let $k = 9$. Due to Corollary 25 we can assume $n$ to be large enough so that:

**Claim 10.** At most two atoms in $V_v$ have in-degree $< k$.

In other words, this means that there are no 3 atoms excluded by at least $n - k$ vertices. Therefore, relying on Corollary 25 it is enough to assume $3(n - k) > 2n$, i.e., $n > 3k$.

Let $a_1, a_2 \in V$ be the vertices with the smallest in-degrees. By assumption, $\text{IN-DEG}(a_1) \geq 1, \text{IN-DEG}(a_2) \geq 1$, and by Claim 10 we have:

**Claim 11.** Every $d \in V_v - \{a_1, a_2\}$ satisfies $\text{IN-DEG}(d) \geq k$.

We construct a cycle $\pi$ in $G_v$, such that (o) its first vertex $d$, as well as vertices $d$ not contained in $\pi$, satisfy $\text{IN-DEG}(d) \geq k$. Due to (1), it suffices to consider the following cases:

**Case 1.** $|\text{IN}(a_1) \cup \text{IN}(a_2)| \geq 2$: Relying on (1), choose in $V_v - \{a_1, a_2\}$ two distinct atoms $d \neq d'$ with $d \in \text{IN}(a_1)$ and $d' \in \text{IN}(a_2)$. Due to (2) the atoms can be chosen so that $d \neq a_2$ or $d' \neq a_1$. By symmetry we assume w.l.o.g. that $d \neq a_2$. If $d' = a_1$ we take the following simple path $\pi$ in $G_v$ satisfying (o):

$$
\begin{align*}
&d \longrightarrow a_1 \longrightarrow a_2
\end{align*}
$$

Otherwise, suppose $d' \neq a_1$ either. By Claim 11 $\text{IN-DEG}(d) \geq k$ and $\text{IN-DEG}(d') \geq k$. Choose in $V_v - \{a_1, a_2\}$ any atom $e$ with $e \in \text{IN}(d') \cap \text{OUT}(a_1)$ (since $\text{IN-DEG}(d') \geq k$, such $e$ exists as $a_1$ excludes at most two atoms, as long as $k \geq 7$). This yields the following simple path $\pi$ in $G_v$ satisfying (o):

$$
\begin{align*}
&d \longrightarrow a_1 \longrightarrow e \longrightarrow d' \longrightarrow a_2
\end{align*}
$$

**Case 2.** $\text{IN}(a_1) = \text{IN}(a_2) = \{d\}$ for some $d \in V_v - \{a_1, a_2\}$: Take some two letters $a_1, a_2$ appearing in $v$ such that $a_1 \notin \text{TRG}(a_1)$ and $a_2 \notin \text{TRG}(a_2)$. Due to (3) we can assume that either $a_1 \neq a_2$, or $a_1 = a_2$ but $\sigma(a_1) \geq 2$ (their cardinality in $v$ is at least 2). Note that $\text{SRC}(a_1) = \text{SRC}(a_2) = d$, and by Claim 11 $\text{IN-DEG}(d) \geq k$. Choose in $V_v - \{d, a_1, a_2\}$ any atom $e$ with $e \in \text{IN}(d) \cap \text{OUT}(a_1)$ (similarly as before, such $e$ exists as long as $k \geq 6$). This yields the non-simple path $\pi$ in $G_v$ satisfying (o):

$$
\begin{align*}
&d \longrightarrow a_1 \longrightarrow e \longrightarrow d \longrightarrow a_2
\end{align*}
$$

We have thus constructed a path $\pi$ in $G_v$ from $a_1$ to $a_2$. If $a_2 \not\in \text{IN}(d)$, append at the end of $\pi$ any vertex $c$ such that $c \in \text{OUT}(a_2) \cap \text{IN}(d)$. As before, such a vertex exists since $a_2$ excludes at most 2 atoms and $\text{IN-DEG}(d) \geq k$, as long as $k \geq 8$. Therefore the last vertex $c$ of $\pi$ satisfies $c \in \text{IN}(d)$, which means that $\pi$ is a cycle as required.

In Case 1 we transform $\pi$, using Lemma 26 into an anti-cycle $\bar{w}$. In Case 2 we proceed similarly, except that the vertex $d$ appears twice in $\pi$; this exception is treated by splitting all $d$-sourced letters into two disjoint blocks (cf. the proof of Lemma 26), containing $a_1$ and $a_2$, respectively.

We now remove, intuitively speaking, the anti-cycle $\bar{w}$ from $v$ thus obtaining a smaller data vector $v'$ to which we apply Theorem 27 and Lemma 26. We remove from $v$ all letters appearing in $\bar{w}$, and add a single letter $\beta = \langle \text{SRC}(\bar{w}), \text{TRG}(\bar{w}) \rangle \in \Sigma$. This yields a data vector $v'$. As the length of $\pi$ is at most 6, the in-degree of a node $e$ in the graph $G_v'$ may be smaller by at most 6 than in the graph $G_v$. Thus IN-DEG($e$) $\geq 3$ in $G_v'$, as $k \geq 9$. Moreover OUT-DEG($e$) $\geq n' - 3$ in $G_v'$, where $n'$ is the number of nodes of $G_v'$, by Corollary 25. Therefore
the graph $G_{v'}$, assuming $n$ to be sufficiently large, satisfies assumptions of Lemma 29 by which $G_{v'}$ is strongly connected. In consequence, $G_{v'}$ satisfies assumptions of Theorem 27 by which we derive a Hamiltonian cycle $C$ in $G$. The Hamiltonian cycle is turned, using Lemma 26, into an anti-cycle in $w$ with $\text{Par}(w) = v'$. Finally, replacing the letter $\beta$ in $w'$ by $\bar{w}$, yields an anti-cycle $w$ with $\text{Par}(w) = v$, as required.

Let $N$ denote the set of all non-degenerate data vectors, and $N_{\geq n} = \{v \in N : \text{ORD}(v) \geq n\}$. In these terms, Lemma 29 claims $N_{\geq n} = \text{Par}(D_{\geq n})$ for sufficiently large $n$.

Lemma 30. $N_{\geq n}$ is rational, for sufficiently large $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ sufficiently large, for Lemmas 29 and 30 to hold, we decompose the Parikh image of anti-cycles into $\text{Par}(D) = \text{Par}(D_{< n}) \cup N_{\geq n}$, both of them rational by Lemmas 24 and Lemma 30 respectively. Lemma 23 is thus proved.

VIII. Final remarks

We have shown that Parikh images of languages of one-register automata are not semi-linear in general, but are rational; and likewise for one-register context-free languages. As a corollary of Theorem 7 we obtain an analog of Parikh’s theorem mentioned in the introduction: one-register context-free grammars are Parikh-equivalent to register automata (but not to one-register ones). Indeed, every rational set of data vectors is the Parikh image of some register automaton.

We conjecture that the restriction to one register can be dropped, and that general register context-free grammars have Parikh images and are Parikh-equivalent to register automata; our present proof techniques do not allow however to tackle the general case. On the other hand our proof method routinely (but tediously) adapts to 1-CFG of any arity, but at the price of considering anti-paths over a larger alphabet $\text{ATOMS} \times P_n(\text{ATOMS})$, where $n$ is the largest arity of a 1-CFG.

Besides dropping one-register restriction, we envisage several potential directions of generalisation: richer input alphabets, more structured atoms, etc. As future work we leave also investigation of algorithmic problems on rational sets, like testing equality of such sets. Finally, we hope to develop a general theory of rational sets of data vectors, e.g., study closure properties, strictness of the star-height hierarchy, or logical characterisations.

Acknowledgment

The authors would like to thank the anonymous reviewers for helpful remarks and suggestions.

References

[1] N. Francez and M. Kaminski, “Finite-memory automata,” Theor. Comput. Sci., vol. 134, no. 2, pp. 329–363, 1994.
[2] L. Segoufin, “Automata and logics for words over an infinite alphabet,” in Proc. CSL 2006, ser. Lecture Notes in Computer Science, vol. 4207. Springer, 2006, pp. 41–57.
[3] M. Bojańczyk, B. Klin, and S. Lasota, “Automata theory in nominal sets,” Log. Methods Comput. Sci., vol. 10, no. 3, 2014.
[4] M. Bojańczyk, “Slightly infinite sets,” a draft of a book. [Online]. Available: https://www.mimuw.edu.pl/~bojan/paper/atom-book
[5] M. Kaminski and T. Tan, “Regular expressions for languages over infinite alphabets,” Fundam. Informaticae, vol. 69, no. 3, pp. 301–318, 2006.
[6] L. Libkin, T. Tan, and D. Vrgoc, “Regular expressions for data words,” J. Comput. Syst. Sci., vol. 81, no. 7, pp. 1278–1297, 2015.
[7] A. Kurz, T. Suzuki, and E. Tuosto, “On nominal regular languages with binders,” in Proc. FOSSACS 2012, ser. Lecture Notes in Computer Science, L. Birkedal, Ed., vol. 7213. Springer, 2012, pp. 255–269.
[8] M. Bojańczyk, B. Klin, and S. Lasota, “Automata with group actions,” in Proc. LICS 2011, 2011, pp. 355–364.
[9] A. Ghouila-Houri, “Une condition suffisante d’existance d’un circuit hamiltonien,” C. R. Acad. Sci. Paris, vol. 25, pp. 495–497, 1960.
[10] D. Kühn and D. Oshius, “A survey on hamilton cycles in directed graphs,” European Journal of Combinatorics, vol. 33, no. 5, pp. 750–766, 2012.
[11] R. Parikh, “On context-free languages,” J. ACM, vol. 13, no. 4, pp. 570–581, 1966.
[12] H. Sakamoto and D. Ikeda, “Intractability of decision problems for finite-memory automata,” Theor. Comput. Sci., vol. 231, no. 2, pp. 297–308, 2000.
[13] F. Neven, T. Schwentick, and V. Vianu, “Finite state machines for strings over infinite alphabets,” ACM Trans. Comput. Log., vol. 5, no. 3, pp. 403–435, 2004.
[14] N. Francez and M. Kaminski, “An algebraic characterization of deterministic regular languages over infinite alphabets,” Theor. Comput. Sci., vol. 306, no. 1-3, pp. 155–175, 2003.
[15] M. Bojańczyk, “Data monoids,” in Proc. STACS 2011, ser. LIPIcs, vol. 9, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2011, pp. 105–116.
[16] T. Colcombet, C. Ley, and G. Puppis, “Logics with rigidly guarded data tests,” Log. Methods Comput. Sci., vol. 11, no. 3, 2015. [Online]. Available: https://doi.org/10.2168/LMCS-11(3:10)2015
[17] M. Bojańczyk and R. Szefińska, “Single-use automata and transducers for infinite alphabites,” in Proc. ICALP 2020, ser. LIPIcs, vol. 168. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, pp. 113:1–113:14.
[18] S. Demri and R. Lazic, “LTL with the freeze quantifier and register automata,” ACM Trans. Comput. Log., vol. 10, no. 3, pp. 16:1–16:30, 2009.
[19] T. Colcombet and A. Manuel, “Generalized data automata and fixpoint logic,” in Proc. FSTTCS 2014, ser. LIPIcs, vol. 29. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014, pp. 267–278.
[20] L. Clemente and S. Lasota, “Reachability analysis of first-order definable pushdown systems,” in Proc. CSL 2015, ser. LIPIcs, S. Kreutzer, Ed., vol. 41. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2015, pp. 244–259.
[21] L. D’Antoni and M. Veanes, “Minimization of symbolic automata,” in Proc. POPL ’14. ACM, 2014, pp. 541–554.
[22] T. Milo, D. Sucić, and V. Vianu, “Typechecking for XML transformers,” J. Comput. Syst. Sci., vol. 66, no. 1, pp. 66–97, 2003.
[23] M. Bojańczyk, C. David, A. Muscholl, T. Schwentick, and L. Segoufin, “Two-variable logic on data words,” ACM Trans. Comput. Log., vol. 12, no. 4, pp. 27:1–27:26, 2011.
[24] M. Bojańczyk and S. Lasota, “An extension of data automata that captures XPath,” Log. Methods Comput. Sci., vol. 8, no. 1, 2012.
[25] A. M. Pitts, Nominal Sets: Names and Symmetry in Computer Science, ser. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2013, vol. 57.
[26] S. Eilenberg, Automata, languages, and machines. A, ser. Pure and applied mathematics. Academic Press, 1974. [Online]. Available: https://www.worldcat.org/oclc/310535248
[27] M. Juzepczuk, “Zbiory semielinowe nad nieskończonym alfabetem (in Polish),” Master’s thesis, University of Warsaw, 2013.
APPENDIX

A. Missing items in Section [7]

Proof of Lemma [1] Fix an orbit-finite set $\Sigma$. The 'only if' implication is immediate, as the length (or size) is invariant inside an orbit. Towards the 'if' implication for data languages, we observe that the set $\Sigma^n$ of words of length $n$ is orbit-finite, for every $n \in \mathbb{N}$, as Cartesian products preserve orbit-finiteness. Therefore a language $X \subseteq \Sigma^*$ satisfying $|v| \leq n$ for $v \in X$, is a subset of a finite union of orbit-finite sets and hence orbit-finite itself. In consequence, $\text{PAR}(X)$ is also orbit-finite, as the image of $X$ under an equivariant function, which proves the claim for sets of data vectors.

B. Missing items in Section [11]

Lemma 31. The language $L_1$ from Example [7] is not rational.

Proof. Indeed, towards contradiction suppose $L_1$ is rational, and hence generated by a rational expression $R$. Consider the sublanguage $L \subseteq L_1$ containing words in which all atoms are different. The language $L$ is orbit-infinite and hence it cannot be generated without star; indeed, concatenation and orbit-finite sums preserve orbit-finiteness of languages. Therefore, there must be a star subexpression $R'$ of $R$ such that the number of iterations of $R'$ is unbounded in generation of words in $L$. In other words, for every $n \in \mathbb{N}$ there is a word $w \in L$ whose some infix $u$ is generated by at least $n$ iterations of $R'$. Thus $w = w''w''$, the infix $u$ splits into $u = u_1 \ldots u_n$, and each of factors $u_i$ is generated by $R'$. Choose $n$ sufficiently large, namely $n > 2 \cdot |\text{SUPP}(R')|$ (considering union operations as atom-binding constructs, the support of a rational expression $R'$ consists of those atoms appearing in $R'$ which are not bounded by any union). As no atom repeats twice in words in $L$, some of words $u_i$ is fresh for $R'$, i.e., $\text{SUPP}(u) \cap \text{SUPP}(R') = \emptyset$, and $u_i$ is either preceded or succeeded in $w$ by an atom $a \notin \text{SUPP}(R')$. Consider w.l.o.g. the first case, and let $a'$ be the first atom in $u_i$. Necessarily $a \neq a'$. As $R'$ is invariant under the swap $a' \leftrightarrow a$, applying this swap to $u_i$ yields a word $w'$ still generated by $R'$. Replacing $u_i$ by $w'$ in $w$ yields a word $w'$ still generated by $R$, but $w' \notin L_1$ as it contains two consecutive atoms $a$. The contradiction completes the proof.

C. Missing items in Section [14]

Proof of Proposition [9] Every semilinear set is, by definition, a rational set of star-height at most 1. For the converse inclusion we use a distributive law of addition over orbit-finite unions:

$$\bigcup_{i \in I} L_i + \bigcup_{j \in J} K_j = \bigcup_{(i,j) \in I \times J} L_i + K_j.$$

Note that the Cartesian product $I \times J$ of orbit-finite sets $I$ and $J$ is necessarily orbit-finite (cf. [4] Sect. 3).

Consider a rational set $X$ of data vectors of star-height $h \leq 1$. If $h = 0$, by the distributive law the set $X$ is orbit-finite and hence vacuously semi-linear. If $h = 1$, by the distributive law we similarly deduce that, for every star subexpression $Y^*$, the set $Y$ is orbit-finite; and moreover, the set $X$ is an orbit-finite union

$$X = \bigcup_{i \in I} X_i,$$

where each $X_i$ is a sum of star subexpressions $Y^*$ and orbit-finite sets. As addition is commutative, preserves orbit-finiteness, and admits merging of stars:

$$Y^* + Z^* = (Y \cup Z)^*,$$

each of sets $X_i$ is of the form

$$Z + Y^*$$

where $Y$ and $Z$ are both orbit-finite. Therefore each $X_i$ is semi-linear, and hence the orbit-finite union is semi-linear too.

Proof of Lemma 10. The proof is an adaptation of the argument from [27]. Towards contradiction, suppose $\text{PAR}(L_3)$ is orbit-finite:

$$\text{PAR}(L_3) = \bigcup_{i \in I} g_i + P_i^*.$$

For a data vector $p : \text{ATOMS} \to \mathbb{N}$, let $\text{SING}(p) = \{|a \in \text{ATOMS} : p(a) = 1\}$ denote the number of atoms appearing exactly once in $p$. Our argument relies on a careful analysis of the limit value of the singularity ratio $\frac{\text{SING}(p)}{|p|}$, for $p \in \text{PAR}(L_3)$, when $|p|$ tends to infinity.

By induction on the length of $w \in L_3$ one easily proves:

Claim 12. Every $p \in \text{PAR}(L_3)$ satisfies $\text{SING}(p) < \frac{1}{2}|p|$.

Claim 13. Every $p \in \text{PAR}(L_3)$ satisfies $\text{DOM}(p) \leq \frac{1}{2}|p|$.

For a data vector $p : \text{ATOMS} \to \mathbb{N}$ and $S \subseteq \text{ATOMS}$, we denote by $p - S$ the data vector obtained from $p$ by removing all occurrences of atoms from $S$:

$$(p - S)(a) = \begin{cases} p(a) & \text{if } a \notin S \\ 0 & \text{otherwise.} \end{cases}$$

Let $S_l = \text{SUPP}(P_l)$. As a consequence of Claim 12 we get:

Claim 14. Every $p \in P_l$ (i.e. $l$) satisfies $\text{SING}(p - S_l) \leq \frac{1}{2}|p|$.

Proof. Towards contradiction, suppose $\text{SING}(p - S_l) > \frac{1}{2}|p|$ for some $p \in P_l$. Let $S' = \text{DOM}(p - S_l)$. For an arbitrary permutation of atoms $\pi \in \text{PERM}$ such that $\pi(a) = a$ for all $a \in S_l$, we have $\pi(p) \in P_l$. Consider such permutations $\pi_1, \ldots, \pi_n \in \text{PERM}$, such that $\pi_k(S')$ and $\pi_l(S')$ are disjoint for $k \neq l$. As $|g_i|$ is fixed, for sufficiently large $n$ the vector $g_i + \pi_1(p) + \ldots + \pi_n(p) \in \text{PAR}(L_3)$ contradicts Claim 12. This completes the proof.

We call a data vector $p : \text{ATOMS} \to \mathbb{N}$ non-singular if $p(a) > 1$ for some $a \in \text{ATOMS}$. Claim 14 can be strengthened as long as non-singular data vectors are considered:
Claim 15. Every $p \in P_i$ ($i \in I$) such that $p - S_i$ is non-singular, satisfies $\text{SING}(p - S_i) < \frac{1}{2} |p|$.

Proof. Indeed, suppose $\text{SING}(p - S_i) \geq \frac{1}{2} |p|$ for some $p \in P_i$, and hence $\text{DOM}(p - S_i) > \frac{1}{2} |p|$ due to non-singularity of $p - S_i$. Considering similar permutations of $p$ as in the argument for Claim 14, we contradict Claim 15.

Let $k = \max \{|S_i| : i \in I\}$ be the maximal size of the support of $P_i$; note that $k$ is well defined as the family of sets $\{P_i\}_{i \in I}$ is orbit-finite, and the size of the support is invariant in an orbit. Likewise, let $t = \max \{|g_i| : i \in I\}$ be the maximal size of a base and let $s = \max \{|p| : p \in \bigcup_i P_i\}$ be the maximal size of a period.

Let $Z = \{a_0, \ldots, a_k\} \subseteq \text{ATOMS}$ be some fixed $k + 1$ atoms. A word $v \in L_a$ (cf. (8)) we call varied if all atoms different than $a$ appear at most once in $v$. For every $m \in \mathbb{N}$ choose some arbitrary but fixed word $w_m \in L_3$ of the form:

$$w_m = v_0 v_1 \ldots v_k \in L_{a_0} L_{a_1} \ldots L_{a_k},$$

where each $v_i \in L_{a_i}$ is a varied word of length $2m$ and no atom appears in two distinct words $v_i, v_j$, for $i \neq j$. Let $q_m = \text{PAR}(w_m)$. Hence $|w_m| = |q_m| = 2m(k + 1)$. As $\text{SING}(q_m) = (m - 1)(k + 1)$, in the limit we have:

$$\lim_{m \to \infty} \frac{\text{SING}(q_m)}{|q_m|} = \lim_{m \to \infty} \frac{(m - 1)(k + 1)}{2m(k + 1)} = \frac{1}{2}$$

irrespective of the choice of the words $w_m$. Let $g_m + P_{m,*}$ $(m \in \mathbb{N})$ be a linear set to which $q_m$ belongs. Thus $q_m = g_m + p_m$, for $p_m \in P_{m,*}$. Recalling (19), choose $a_m \in Z$ so that $a_m \notin S_{m,*}$ (such $a_m$ exists as $|S_m| \leq k$). We split $g_m$:

$$q_m = (g_m + p_{m,0}) + p_{m,1} + p_{m,>1},$$

where $p_{m,1}$ is a sum of vectors from $P_{m,1}$ that contain exactly one appearance of $a_m$; $p_{m,>1}$ is a sum of vectors from $P_{m,>1}$ that contain more than one appearance of $a_m$, and $p_{m,0}$ is a sum of vectors from $P_{m,0}$ that contain no appearance of $a_m$ at all. Applying Claim 12 to $g_m + p_{m,0} \in \text{PAR}(L_3)$, we obtain:

$$\limsup_{m \to \infty} \frac{\text{SING}(g_m + p_{m,0})}{|g_m + p_{m,0}|} \leq \frac{1}{2}.$$

Observe that the size of the sum of the last two data vectors in (21) constitutes, up to a constant $t$, at least $\frac{1}{2(k + 1)}|p_m|$ (recall that $|g_m|$ is bounded by $t$):

$$\frac{1}{2(k + 1)}|p_m| - |g_m| < |p_{m,1} + p_{m,>1}|$$

as it includes all $\frac{1}{2(k + 1)}|p_m| + 1$ appearances of $a_m$ in $g_m$, except for at most $|g_m|$ many of them, possibly appearing in $g_i$. We are going to prove the following strict inequality:

$$\limsup_{m \to \infty} \frac{\text{SING}(p_{m,1} + p_{m,>1})}{|p_{m,1} + p_{m,>1}|} < \frac{1}{2}$$

which, together with inequalities (22) and (23), implies

$$\limsup_{m \to \infty} \frac{\text{SING}(q_m)}{|q_m|} < \frac{1}{2}$$

and thus contradicts the equality (20). Call $p_{m,1}$ non-trivial if it is a sum of at least two vectors from $P_{m,1}$. When $p_{m,1}$ is trivial, $|p_{m,1}| \leq s$ is bounded and hence $p_{m,1}$ can be ignored in (24). We split the inequality (24) into two separate ones

$$\limsup_{m \to \infty} \frac{\text{SING}(p_{m,1})}{|p_{m,1}|} < \frac{1}{2}, \quad \limsup_{m \to \infty} \frac{\text{SING}(p_{m,>1})}{|p_{m,>1}|} < \frac{1}{2}$$

and prove the first one assuming that $p_{m,1}$ is non-trivial, and the second one unconditionally. This is enough to derive (24).

Concerning the first inequality, we observe that the atom $a_m \notin S_{m,*}$ is counted in $\text{SING}(v - S_{m,*})$ for every data vector $v$ contributing to the sum $p_{m,1}$, but if there are more than one of these vectors $v$, then the atom $a_m$ is no more counted in $\text{SING}(p_{m,1} - S_{m,*})$. Thus $\text{SING}(p_{m,1} - S_{m,*})$ loses, intuitively speaking, at least the $\frac{1}{2}$ fraction of the maximal possible value $\frac{1}{2}|p_{m,1}|$ according to Claim 14. This allows us to deduce:

$$\frac{\text{SING}(p_{m,1} - S_{m,*})}{|p_{m,1}|} \leq \frac{1}{2} \left(1 - \frac{1}{s}\right) < \frac{1}{2}$$

which implies, in the limit, the first inequality in (25), as $|S_{m,*}|$ is bounded (by $k$).

Concerning the second inequality, let’s put

$$r = \max \left\{\frac{\text{SING}(v - S_i)}{|v|} : v \in P_i, v - S_i \text{ is non-singular} \right\}$$

As before, $r$ is well defined due to orbit-finiteness of all $P_i$ and $(P_i)_{i \in I}$, and moreover $r < \frac{1}{2}$ by Claim 15. A crucial observation is that

$$\frac{\text{SING}(p_{m,>1} - S_{m,*})}{|p_{m,>1}|} \leq \frac{\text{SING}(v - S_{m,*})}{|v|}$$

for some $v \in P_m$ that contributes to the sum $p_{m,>1}$, and hence

$$\frac{\text{SING}(p_{m,>1} - S_{m,*})}{|p_{m,>1}|} \leq r < \frac{1}{2}$$

which implies, in the limit, the second inequality in (25), as $|S_{m,*}|$ is bounded. The inequalities (25) are thus proved. □

D. Missing items in Section V.

Proof of Lemma 14. For a nonterminal $q \in Q$ and an atom $a \in \text{ATOMS}$, consider the set of derivation trees of $G$ with root labeled by $q(a)$, which use only productions with the left-hand side in $Q \times \{a\}$ (thus every non-leaf in such a tree belongs to $Q \times \{a\}$, and where every leaf belongs either to $H \times \text{ATOMS}$ or to $Q \times (\text{ATOMS} - \{a\})$. Intuitively, we stop derivation at a terminal, or at a configuration with register value different than $a$ (i.e., at first register update along every path). The language $L_{q(a)}$ generated by such trees is obtained by applying a substitution to a classical context-free language (with the finite set $Q \times \{a\}$ of nonterminals), and thus has rational Parikh image.

The proof is by induction on $n$. In case $n = 0$, we observe that $H_{q(a),0}$ is the restriction of $L_{q(a)}$ to terminals $H \times \{a\}$:

$$H_{q(a),0} = L_{q(a)} \cap (H \times \{a\})^*$$
and thus is itself a classical context-free language (with the finite set \( Q \times \{a\} \) of nonterminals and the finite set \( H \times \{a\} \) of terminals); in consequence, it has rational Parikh image.

For the induction step we assume rationality of languages \( H_q(a,n) \), and observe that \( H_q(a,n+1) \) is obtained by applying to the language \( L_q(a) \) the substitution:

\[
p(b) \mapsto H_{p(b)}, \quad (h,b) \mapsto (h,b),
\]

where \( p \in Q, \ h \in H, \) and \( b \in \text{ATOMS} \). Indeed, intuitively speaking, \( L_q(a) \) allows for exactly one register update, while \( H_{p(b)}, \) allows for \( \leq n \) additional register updates along every path. Therefore \( H_q(a),n+1 \) has rational Parikh image, as required.

### E. Missing items in Section [VII]

#### Proof of Lemma 20

We transform a rational expression \( E \) defining a language \( L \subseteq \Gamma^* \) into a rational expression \( \tilde{E} \) defining \( L \circ \alpha \). We proceed by structural induction on \( E \). In case of orbit-finite union the transformation is distributive:

\[
\bigcup_{i \in I} E_i := \bigcup_{i \in I} \tilde{E}_i.
\]

In case of sum, the transformation is applied to one of summands:

\[
\tilde{E}_1 + \tilde{E}_2 := (\tilde{E}_1 + \tilde{E}_2) \cup (\tilde{E}_1 + \tilde{E}_2).
\]

In case of iteration, the transformation is applied to a single iteration (which forces at least one iteration and hence rules out the vacuous generation of the zero vector \( 0 \) due to 0 iterations):

\[
\tilde{E}^* := \tilde{E} E^*.
\]

Finally, the induction base, for a singleton \( \{\beta\} \), is given by:

\[
\{\beta\} := \begin{cases} \emptyset & \text{if } \beta = \alpha \\ \{0\} & \text{otherwise}. \end{cases}
\]

For a word \( w = \alpha_1 \ldots \alpha_n \in \Gamma^* \) we denote by \( \text{SRCES}(w) \) the sequence \( \text{SRC}(\alpha_1) \ldots \text{SRC}(\alpha_n) \) of sources. For a finite subset \( X \subseteq \text{ATOMS} \) and a regular language \( K \subseteq X^* \) we define:

\[
C_a^K = \{ w \in \Gamma^* : \text{SRCES}(w) \in K, a \notin \text{TRG}(w) \}.
\]

#### Lemma 32

For every finite set \( X \subseteq \text{ATOMS} \) and regular language \( K \subseteq X^* \), the languages \( C_a^K \) are rational.

**Proof.** Consider the finite set \( \Delta = (X \cup \{a\})^2 \) as an alphabet, and the regular language \( P \subseteq \Delta^* \) of all \( K,a \)-paths, i.e., all nonempty sequences

\[
\langle d_1, d_2 \rangle \langle d_2, d_3 \rangle \ldots \langle d_n, d_{n+1} \rangle \in \Delta^*
\]

such that \( d_1 d_2 \ldots d_n \in K \) and \( d_{n+1} = a \). The language \( C_a^K \) is obtained from \( P \) by the substitution

\[
\langle d, e \rangle \mapsto \bigcup_{\{e', e''\} \in P_2(\text{ATOMS} - \{e\})} \langle d, \{e', e''\} \rangle
\]

and is thus rational.

#### Proof of Lemma 22

We show that rationality of \( \text{PAR}(D) \) implies rationality of \( \text{PAR}(C) \). To this aim we define, for distinct atoms \( b, c \in \text{ATOMS} \), the language

\[
K_{bc} := C_{b}^{(b,c)^*} \langle b, \{b, c\} \rangle \cup C_{c}^{(b,c)^*} \langle c, \{b, c\} \rangle
\]

of all anti-paths where the last target is \( \{b, c\} \), all sources are in \( \{b, c\} \), and the first one is \( b \). Languages \( K_{bc} \) are rational, due to Lemma 32. Further, for pairwise distinct atoms \( a, b, c \) we define the following rational language

\[
K_{a\{b, c\}} := \langle a, \{b, c\} \rangle \cup \bigcup_{b', c' \in \text{ATOMS} - \{b\}} \langle a, \{b', c'\} \rangle K_{bc}.
\]

Note that the source of the first letter in every word in \( K_{a\{b, c\}} \) is \( a \), and the target of the last letter is \( \{b, c\} \). Lemma 22 follows once we show the following claim:

**Claim 16.** \( \text{PAR}(C) \) is obtained from \( \text{PAR}(D) \) by applying twice the substitution

\[
\langle d, \{e, f\} \rangle \mapsto K_{d\{e, f\}}.
\]

(We consider Parikh images of \( C \) and \( D \), instead of the languages themselves, only because we reason below up to cyclic shifts.) From now on we concentrate on the proof of the claim. Let \( \tilde{D} \) denote the set of data vectors obtained from \( \text{PAR}(D) \) by applying twice the above-defined substitution. By the very definition, \( \tilde{D} \subseteq \text{PAR}(C) \). For the converse inclusion, we prove that every data vector \( v \in \text{PAR}(C) \) belongs to \( \tilde{D} \).

If \( v \) contains no unwanted letters from \( \Gamma - \Sigma \) then \( v \in \text{PAR}(D) \), and the claim follows due to \( \text{PAR}(D) \subseteq \tilde{D} \).

Otherwise, choose an anti-cycle \( w \in C \) with \( v = \text{PAR}(w) \) and consider the last appearance of an unwanted letter \( (b, \{b, c\}) \in \Gamma - \Sigma \) in \( w \). Applying a cyclic shift \( (\rightarrow) \) we can assume, w.l.o.g., that the letter is the last one in \( w \). Let \( u \) be the maximal suffix of \( w \) that belongs to \( K_{bc} \) (or, symmetrically, to \( K_{cb} \)).

\[
w = u' w.
\]

We observe that \( w' \neq \varepsilon \); indeed, as \( \text{SRC}(u) = b \in \text{TRG}(u) = \{b, c\} \), the word \( u \) itself is not an anti-cycle.

Let \( w'' = w''(a, \{b', c'\}) \); since \( w \) is an anti-chain we have \( b \notin \{b', c'\} \), and by maximality of \( u \) we have \( a \notin \{b, c\} \). Then \( u'' = \langle a, \{b', c'\} \rangle u \in K_{a\{b, c\}} \). Replace the suffix \( u'' \) by \( \langle a, \{b', c'\} \rangle \), thus obtaining a data word \( \tilde{w} = w''(a, \{b, c\}) \) with smaller number of occurrences of unwanted letters. We continue in the same way with \( w' \) until all occurrences of letters from \( \Gamma - \Sigma \) are eliminated. A crucial observation is that during elimination of all letters, except for possibly the very last one, the total sum of cyclic shifts \( (\rightarrow) \) performed does not exceed the full cyclic shift of \( w \). Therefore, Parikh image of the word obtained by elimination of all unwanted letters except for the last one, belongs to the result of application the substitution once to \( D \). In consequence, the final word belongs to the result of applying the substitution twice, as required.
Proof of Lemma 24. For a finite subset $X \subseteq \text{ATOMS}$ the language

$$C_X = C \cap \{ \alpha \in \Gamma : \text{src} (\alpha) \in X \}^*.$$  

is rational, due to Lemma 32 as it equals

$$\bigcup_{\alpha \in X} C^\alpha X^*,$$

and hence so is its restriction $D_X = C_X \cap \Sigma^*.$ The language $D_X^{<n}$, being the union of all the rational languages $D_X$ for subsets $X \subseteq \text{ATOMS}$ of cardinality $< n$, is thus rational as well.

Proof of Lemma 30. Fix $n \geq 6.$ We define the kernel of a data vector $v : \Sigma \rightarrow \mathbb{N}$ as the intersection of all targets in $v$:

$$\text{ker}(v) = \bigcap_{\alpha \in \text{dom}(v)} \text{TRG}(\alpha).$$

The size of the kernel is 0, 1 or 2. For $X \subseteq \text{ATOMS}$ of size at most 2, let

$$\mathbb{N}^{X, \geq n} = \{ v \in \mathbb{N}^{\geq n} : \text{ker}(v) = X \}.$$  

As $\mathbb{N}^{\geq n} = \bigcup_{X} \mathbb{N}^{X, \geq n}$, it is enough to show that the sets $\mathbb{N}^{X, \geq n}$ are rational. This, in turn, is implied by the following decomposition property of sets $\mathbb{N}^{X, \geq n}$:

$$\mathbb{N}^{X, \geq n} = \mathbb{N}^{X, n} + \text{par}(\Sigma_X^*),$$

where $\mathbb{N}^{X, n} = \{ v \in \mathbb{N}^{X, \geq n} : \text{ord}(v) = n \}$ and $\Sigma_X = \{ \alpha \in \Sigma : X \subseteq \text{TRG}(\alpha) \}$. Towards showing the decomposition (26) we prove that kernel-preserving extensions by one letter $\alpha \in \Sigma$ preserve membership in $\mathbb{N}$:

$$v \in \mathbb{N}, \text{ker}(v) = \text{ker}(v + \alpha) \implies v + \alpha \in \mathbb{N};$$

and also that there always exists a letter $\alpha$ that one can remove from a vector in $\mathbb{N}^{\geq n+1}$, preserving kernel and membership in $\mathbb{N}$:

$$v \in \mathbb{N}^{\geq n+1} \implies \exists \alpha \in \text{dom}(v) : \text{ker}(v) = \text{ker}(v - \alpha),$$

$$v - \alpha \in \mathbb{N}.$$

Concerning the first property, suppose $v \in \mathbb{N}$ and $\text{ker}(v) = \text{ker}(v + \alpha).$ We thus know that $v$ satisfies conditions (1)–(3) and that $d = \text{src}(\alpha) \notin \text{ker}(v)$ since $d \notin \text{TRG}(\alpha).$ This implies that $v + \alpha$ satisfies (1). For conditions (2)–(3) we consider two separate cases. If $d \in V_v$ then adding $\alpha$ may only increase in-neighbour sets $\text{IN}(_\alpha)$ and preceding-letter sets $\text{PRE}(_\alpha)$, and hence $v + \alpha$ satisfies (2)–(3). Otherwise, suppose $d \notin V_v$ is a fresh source. We reason by contradiction. If $v + \alpha$ violates (2) for $d$ and some $e \in V_v$, then $v$ necessarily violates (1) due to $\text{IN}(e) = \emptyset$. If $v + \alpha$ violates (3) for $d$ and some $e \in V_v$, then all $\beta \in \text{dom}(v)$, except for exactly one, satisfy $\text{TRG}(\beta) = \{d, e\}$ and hence forcedly $\text{src}(\beta) \neq e.$ Therefore there is exactly one $e$-sourced letter in $v$ and $\text{IN}(e) = \emptyset$, and hence $v$ violates (1) again.

We now concentrate on the second property. Removal of a letter from $v$ may only increase (inclusion-wise) the kernel, say from $X$ to $X'$, but this only happens if $v(\alpha) = 1$, $X' \subseteq \text{TRG}(\alpha)$, and $X' \subseteq \text{TRG}(\beta)$ for all $\beta \in \text{dom}(v) - \{\alpha\}$. By inspection of possible sizes 1, 2 of $X'$, one deduces that $v$ may contain at most two such kernel-increasing letters. This eliminates at most 2 potential sources $\text{src}(\alpha)$.

Non-degeneracy can be only violated by vertices in the source graph of in-degree below 2. Therefore non-degeneracy of $v - \alpha$ is guaranteed if removal of $\alpha$ does not decrease in-degree of any vertex below 2, i.e., $\text{src}(\alpha)$ does not belong to $\text{IN}(d)$ for $d \in V_v$ of in-degree $\text{IN-DEG}(d) \leq 2$. For sufficiently large $n$, similarly as in Claim 11 there are at most 2 such vertices $d$ in $V'_v$. This eliminates at most 4 potential sources $\text{src}(\alpha)$.

In total, at most 6 potential sources $\text{src}(\alpha)$ are eliminated. Therefore, as long as $\text{ord}(v) > 6$, there is $\alpha \in \text{dom}(v)$ such that $\text{ker}(v) = \text{ker}(v - \alpha)$ and $v - \alpha \in \mathbb{N}.$