Identifying Powers of Half-Twists and Computing its Root

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Abstract

In this paper we give an algorithm for solving a main case of the conjugacy problem in the braid groups. We also prove that half-twists satisfy a special root property which allows us to reduce the solution for the conjugacy problem in half-twists into the free group. Using this algorithm one is able to check conjugacy of a given braid to one of E. Artin’s generators in any power, and compute its root. Moreover, the braid element which conjugates a given half-twist to one of E. Artin’s generators in any power can be restored. The result is applicable to calculations of braid monodromy of branch curves and verification of Hurwitz equivalence of braid monodromy factorizations, which are essential in order to determine braid monodromy type of algebraic surfaces and symplectic 4-manifolds.

Introduction

During past decades braid groups have become important in many fields. Hence, a practical solution for its conjugacy problem has become extremely important. Although the groups conjugacy problem was first solved by Garside [4] (1969) and was addressed many times in the past (i.e., [5], [2]), still a practical polynomial algorithm for its solution is unknown.

This has lead to the research for the solution of partial problems such as identification of special conjugacy classes. In [6], a random algorithm for the identifying half-twists in any power was given, and the aim of this paper is to give a deterministic algorithm for the problem, which although exponential is of interest because of the simplicity of the proofs involved, and the combination of techniques used in order to solve the problem.

The algorithm presented here enables to test factors of braid monodromy type of surfaces, and is a partial solution for the conjugacy problem in the braid groups. Moreover, it enables to solve specific cases of quasi-positivity.

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We begin in Section 1, by giving some basic definitions related with braid
groups. Then, in Section 2, we prove a simple manner of the root property con-
jectured and tested by a computer in [6] for half-twists. This means that if we
know that two identical powers of half-twists are conjugated to one another by a
positive braid word \( w \), then the two half-twists are conjugated by the same word
\( w \). In section 3 we give some properties of powers of half-twists which will be used
in Section 4 to present the algorithm for identifying the powers of half-twists in
the braid group and to compute an element which conjugates them in any power
to one given Artin generator.

1 Braid group preliminaries

1.1 Artin’s braid group

In this section we will give the definition and some properties of the braid group.

**Definition 1.1.** Artin’s braid group \( B_n \) is the group generated by \( \{\sigma_1, ..., \sigma_{n-1}\} \)
submitted to the relations

1. \( \sigma_i \sigma_j = \sigma_j \sigma_i \) where \( |i - j| \geq 2 \)

2. \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for all \( i = 1, ..., n-2 \)

This algebraic definition can be looked from a geometric point of view, by
associating to every generator of the braid group \( \sigma_i \) a tie between \( n \) strings going
monotonically from top to bottom, such that we switch by a positive rotation
between the two adjacent pair of strings \( i \) and \( i+1 \). The operation for the geometric
group is the concatenation of two geometric sets of strings.

![Figure 1: The geometrical braid associated with \( \sigma_i \).](image)

**Definition 1.2.** There is a natural homomorphism, denoted by \( \pi \) from the braid
group to the symmetric group, sending each braid to the permutation induced by
the braid strings. \( \pi \) is defined by \( \pi(\sigma_i) = (i, i + 1) \)
We need some definitions:

**Definition 1.3.** Let \( w \in B_n \) be a braid. Then it is clear that \( w = \sigma_{i_1}^{e_1} \cdot \ldots \cdot \sigma_{i_l}^{e_l} \) for some sequence of generators, where \( i_1, \ldots, i_l \in \{1, \ldots, n-1\} \) and \( e_1, \ldots, e_l \in \{1, -1\} \). We will call such presentation of \( w \) a braid word, and \( \sigma_{i_k}^{e_k} \) will be called the \( k^{th} \) letter of the word \( w \). \( l \) is the length of the braid word, and we define \( \exp(w) = \sum_{i=1}^{l} e_i \), which is the sum of exponents of the letters of \( w \), and known to be invariant in the conjugacy class of \( w \).

We will distinguish between two relations on the braid words.

**Definition 1.4.** Let \( w_1 \) and \( w_2 \) be two braid words. We will say that \( w_1 = w_2 \) if they represent the same element of the braid group.

**Definition 1.5.** Let \( w_1 \) and \( w_2 \) be two braid words. We will say that \( w_1 \equiv w_2 \) if \( w_1 \) and \( w_2 \) are identical letter by letter.

**Definition 1.6.** A positive braid is an element of \( B_n \) which can be written as a word in positive powers of the generators \( \{\sigma_i\} \), without the use of the inverse elements \( \sigma_i^{-1} \). We denote this subsemigroup \( B_n^+ \).

**Definition 1.7.** Let \( \beta \in B_n \), consider two different strings, the \( i \)-th and the \( j \)-th \( i \neq j \). In a general diagram of \( \beta \) these two strings “intersect” at several points. Let \( p(i,j) \) (and \( n(i,j) \)) denote the number of times the \( i \)-th string crosses under the \( j \)-th string starting from the left (from the right). Then the number \( p(i,j) - n(i,j) \) is called the crossing index from the \( i \)-th string to the \( j \)-th string denoted by \( cr_\beta(i,j) \).

The following Lemma is well known:

**Lemma 1.8.**
1. The crossing index of two strings is an invariant of the braid.
2. If \( \pi(\beta) = Id \) then \( cr_\beta(i,j) = cr_\beta(j,i) \) \( \forall i, j \).
3. \( \Sigma_{i\neq j} cr_\beta(i,j) = \exp(\beta) \).

1.2 The half-twists

We are going to describe a specific conjugacy class in the braid groups which we will call its elements Half-twists.

**Definition 1.9.** Let \( H \) be the conjugacy class of \( \sigma_1 \), (i.e. \( H = \{q^{-1}\sigma_1 q : q \in B_n \} \)). We call \( H \) the set of half-twists in \( B_n \), and we call an element \( \beta \in H \) a half-twist.
Lemma 1.10. If $b \in H$ is a half-twist then there exists $1 \leq i, j \leq n$, such that $\pi(b) = (i, j)$.

Proof: Trivial, since $\pi(P \sigma_i P^{-1}) = \pi(P)^{-1}(i, i + 1)\pi(P)$ which is also a transposition.

It is proven in [7] that the half-twists occupy a full conjugacy class in the braid group. Moreover, it is proven in [7] that all the generators of the braid group are conjugated to each other.

1.3 Half-twists and paths

Another way of looking at the braid group on $n$ generators is by the group of isotopy classes of orientation preserving automorphisms of an $n$ punctured disk that fix the set of punctures and is the identity on the boundary of the disk. It is proven in [7] that the all half-twists can be regarded as an automorphism that is described by a path connecting two punctures which does not self intersect nor intersect the punctures aside for its beginning and ending points. The automorphism induced by this path actually exchange the two punctures clockwise in a small neighborhood of the it, which does not contain any of the other punctures.

The geometric braid can be determined by taking the trace of the punctures under the isotopy as can be seen in the figure below:

![Figure 2: The geometric braid and its defining path.](image)

Remark 1.11. This actually means that for half-twist braids, we have two strings which interact along a small neighborhood of a surface in $\mathbb{R}^3$, while the other strings
go simply straight from top to bottom. The surface is what is found underneath the path.

2 The root property of half-twists

In [6], we conjectured after testing with a computer, that a braid word \( w \) is conjugated to the generator \( \sigma_i \) if and only if \( w^2 \) is conjugated to \( \sigma_i^2 \).

The aim of this section is to give a proof for theorem 2.12 which is a simple version of the above conjecture, and says that if \( W\sigma_i^m = \sigma_j^m W \) for some \( W \in B_n^+ \), \( m \in \mathbb{N} \). Then \( W\sigma_i = \sigma_j W \). However, before we can go to the proof we need some properties of positive braids.

2.1 Properties of positive braids

In this section we will recall two properties of positive braids. The first is concerning with the equivalence of two positive braid words, and the second is concerning the ability to write every conjugated element in the braid group using a positive braid word as its conjugator.

**Proposition 2.1.** If \( \alpha = \beta \in B_n^+ \), then the length \( |\alpha| \) of \( \alpha \) is equal to the length \( |\beta| \) of \( \beta \).

**Proof:** This follows immediately from the relations of the braid group since they do not change the number of generators.

**Definition 2.2.** Let \( \alpha, \beta \in B_n^+ \) be two positive braid words. We call \( \alpha \) and \( \beta \) positively equivalent if there is a sequence of positive words \( \alpha \equiv w_0 = w_1 = ... = w_k \equiv \beta \), such that \( w_{i+1} \) is obtained from \( w_i \) by a single activation of a relations of \( B_n \).

**Proposition 2.3.** Let \( \alpha, \beta \in B_n^+ \) be two braid words such that \( \alpha = \beta \) in \( B_n \). Then \( \alpha \) and \( \beta \) are positively equivalent.

**Proof.** See [4]

**Proposition 2.4.** Let \( \alpha = \beta \in B_n^+ \), such that \( \alpha \equiv \sigma_i w \) for some generator \( \sigma_i \) and a positive braid word \( w \). Then, there is an algorithm that writes \( \beta \equiv \sigma_i w' \) for some positive braid word \( w' \) such that \( |w| = |w'| \).

**Proof:** The proof can be found in [5].

If we examine this algorithm in detail we result with the following corollary:
Corollary 2.5. Let $\alpha \equiv \sigma_i \sigma_j w \in B_n^+$ for some generators $\sigma_i, \sigma_j$ $i \neq j$ and a positive braid word $w$, and suppose that $\alpha = \sigma_j w'$ for some $w' \in B_n^+$. Then, one of the following must happen:

1. If $|i - j| > 1$ we can write $\alpha = \sigma_j \sigma_i w$.
2. If $|i - j| = 1$ we can write $w = \sigma_i w''$

Proof: Although the proof is obvious from the proof given in [5] for the correctness of the algorithm in Proposition 2.4 that extract a letter to the left of a braid word, we will give the basic idea of it. In the first case simply $\sigma_i$ and $\sigma_j$ commute, so write $w = w'$ and we have finished. In the second case the only way to transform to the left the letter $\sigma_j$ is to use the second braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. Now since it is given that $\alpha = \sigma_j w'$ for some $w' \in B_n^+$, we must be able to activate the braid relations in order make this switch, but this means that we can write $w = \sigma_i w''$ for some $w'' \in B_n^+$.

Corollary 2.6. Let $\alpha \equiv \sigma_i^k \sigma_j w \in B_n^+$ for some generators $\sigma_i, \sigma_j$ $i \neq j$ and a positive braid word $w$, and suppose that $\alpha = \sigma_j w'$ for some $w' \in B_n^+$. Then, one of the following must happen:

1. If $|i - j| > 1$ we can write $\alpha = \sigma_j \sigma_i^k w$.
2. If $|i - j| = 1$ we can write $w = \sigma_i w''$

Proof: The proof is clear from the proof of Corollary 2.5 and the algorithms in [5].

Corollary 2.7. If we use the second case in the corollary above then $|w''| < |w|$.

The following theorem will be of great importance in what follows.

Theorem 2.8. Let $\alpha, \beta \in B_n$ (not necessarily positive) two conjugated braids. Then, there exist a positive braid $w \in B_n^+$ such that $w^{-1} \alpha w = \beta$.

Proof. See [3]

2.2 Proof of The Theorem

Before we start with the theorem and its proof we want to point out that the unique root property, that we are going to prove for the conjugacy class of the half-twists, does not hold for the entire braid group elements. For example:
Example 2.9. Consider the two braids in $B_4$:

\[ B = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \]
\[ \Delta = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \]

Proposition 2.10. The two braids described above are different.

Proof: If $B = \Delta$ then,

\[ B = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 = \Delta \]
\[ \sigma_3 = \sigma_1 \]

a contradiction. \qed

But, it is a common knowledge that the square of both $B$ and $\Delta$ equals $\Delta^2$ which is the generator of the cyclic center of the braid group.

The rest of this section is dedicated to the proof of the following theorem:

Theorem 2.11. Let $H_1, H_2$ be two half-twists such that $H_1^m = H_2^m$. Then, $H_1 = H_2$.

This means that inside the conjugacy class of the half-twists the braid group has the unique root property.

First, we want to reduce the problem into an equivalent one. Since we know that $H_1$ and $H_2$ are two half-twists we know that there are two positive braid words $w_1, w_2 \in B_n^+$ such that $H_1 = w_1^{-1} \sigma_i w_1$ and $H_2 = w_2 \sigma_j w_2^{-1}$. Therefore, since $H_1^m = H_2^m$ we have that $w_1^{-1} \sigma_i^m w_1 = w_2 \sigma_j^m w_2^{-1}$. But this means that $w_2^{-1} \sigma_i^m w_1 w_2 = \sigma_j^m$. If we can prove that in this case $w_2^{-1} \sigma_i w_1 w_2 = \sigma_j$ we result immediately with what we want. Therefore, we can reduce the theorem into the following theorem:

Theorem 2.12. Let $W \sigma_i^m = \sigma_j^m W$ for some $W \in B_n^+$, $m \in \mathbb{N}$. Then $W \sigma_i = \sigma_j W$.

Proof: We will proof this theorem by induction on the length $|W|$ of $W$.

The base:

1. If $|W| = 0$ it means that $\sigma_i^m = \sigma_j^m$. Therefore $i = j$ and we have that $\sigma_i = \sigma_j$ as we need.
2. If $|W| = 1$ we have $\sigma_k \sigma_i^m = \sigma_j^m \sigma_k$.

If $|k - i| > 1$ or $|k - j| > 1$ or $k = i$ or $k = j$ then, at least one of $\sigma_i$ or $\sigma_j$ commutes with $\sigma_k$, and therefore we have (without loss of generality) $\sigma_k \sigma_i^m = \sigma_k \sigma_j^m$, which means that $\sigma_i^m = \sigma_j^m$.

If $|i - k| = 1$, we have 2 different cases:

(a) If $k = i + 1$ and $k = j + 1$, or $k = i - 1$ and $k = j - 1$ we have $i = j$, but then when checking the linking numbers associated with, $\sigma_k \sigma_i^m \sigma_k^{-1}$ and $\sigma_j^m = \sigma_i^m$ we find a contradiction, which means that this case can not happen.

(b) If $k = i + 1$ and $k = j - 1$, or $k = i - 1$ and $k = j + 1$ we have again that the linking numbers associated with the braids $\sigma_k \sigma_i^m \sigma_k^{-1}$ and $\sigma_j^m$ give us a contradiction, which means that this case can not happen.

Now, suppose that for all $W$ such that $|W| < n$ the following is true $W \sigma_i^m = \sigma_j^m W \Rightarrow W \sigma_i = \sigma_j W$. We need to prove that if $|W| = n$ then, $W \sigma_i^m = \sigma_j^m W \Rightarrow W \sigma_i = \sigma_j W$.

Now lets look at $W$. We divide the proof into some cases:

1. We can write $W = \sigma_k W'$ where $W' \in B_n^+$ and $|k - j| > 1$ or $k = j$.

Then,

$$
\sigma_k W' \sigma_i^m = W \sigma_i^m = \sigma_j^m W = \sigma_j^m \sigma_k W' = \sigma_k \sigma_j^m W' \\
\downarrow \\
W' \sigma_i^m = \sigma_j^m W'
$$

and we finished by induction since $|W'| < |W|$.

2. We can write $W = W' \sigma_k$ where $W' \in B_n^+$ and $|k - i| > 1$ or $k = i$. Then,

$$
W' \sigma_i^m \sigma_j = W' \sigma_k \sigma_i^m = W \sigma_i^m = \sigma_j^m W = \sigma_j^m W' \sigma_k \\
\downarrow \\
W' \sigma_i^m = \sigma_j^m W'
$$

and we finished by induction since $|W'| < |W|$.

3. If none of the above cases is true, it means that we can’t write $W$ beginning or ending with a letter that commutes with $\sigma_i$ or with $\sigma_j$. So we know that in every way of writing $W = \sigma_k W' \sigma_l$ where $W' \in B_n^+$ and $|k - j| = 1$ and $|l - i| = 1$. Now we have to split again into cases:
(a) Suppose that $W = \sigma_{j+1}W'\sigma_{i+1}$ Then by what is given we know that

$$W\sigma_i^m = \sigma_{j+1}W'\sigma_{i+1}\sigma_i^m = \sigma_j^m \sigma_{j+1} W'\sigma_{i+1} = \sigma_j^m W$$

(1)

Therefore, by 2.6 when we take $\alpha = \sigma_j^m \sigma_{j+1} W'\sigma_{i+1}$, and $w = W'\sigma_{i+1}$ we have that $w = W'\sigma_{i+1} = \sigma_j W''$.

Substituting this in (1) we get

$$\sigma_{j+1} \sigma_j W'' \sigma_i^m = \sigma_j^m \sigma_{j+1} \sigma_j W'' = \sigma_j^{m-1} \sigma_{j+1} \sigma_j W'' = \cdots = \sigma_{j+1} \sigma_j \sigma_{j+1} W''$$

$\Downarrow$

$$W'' \sigma_i^m = \sigma_j^{m-1} W''$$

Because of 2.7 we know that $|W''| < |W|$ so, by induction we have:

$$W'' \sigma_i = \sigma_{j+1} W''$$

Now, by multiplying this equation by $\sigma_{j+1} \sigma_j$ on the left we get:

$$\sigma_{j+1} \sigma_j W'' \sigma_i = \sigma_{j+1} \sigma_j \sigma_{j+1} W''$$

$\Downarrow$

$$\sigma_{j+1} \sigma_j W'' \sigma_i = \sigma_j \sigma_{j+1} \sigma_j W''$$

$\Downarrow$

$$W \sigma_i = \sigma_{j+1} W' \sigma_{i+1} \sigma_i = \sigma_j \sigma_{j+1} W' \sigma_{i+1} = \sigma_j W$$

(b) A similar argument implies that When $W = \sigma_{j-1} W' \sigma_{i-1}$ the theorem holds.

(c) Suppose that $W = \sigma_{j-1} W' \sigma_{i+1}$ Then by what is given we know that

$$W\sigma_i^m = \sigma_{j-1} W'\sigma_{i+1}\sigma_i^m = \sigma_j^m \sigma_{j-1} W'\sigma_{i+1} = \sigma_j^m W$$

(2)

Therefore, by 2.6 when we take $\alpha = \sigma_j^m \sigma_{j-1} W'\sigma_{i+1}$ and $w = W'\sigma_{i+1}$ we must be able to write $w = W'\sigma_{i+1} = \sigma_j W''$.  

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Substituting this in (2) we get
\[ \sigma_{j-1} \sigma_j W'' \sigma_i^m = \sigma_j^m \sigma_{j-1} \sigma_j W'' \]

However, \( \sigma_j^m \sigma_{j-1} \sigma_j W'' = \sigma_j^{m-1} \sigma_{j-1} \sigma_j \sigma_{j-1} W'' = \cdots = \sigma_{j-1} \sigma_j \sigma_j^m W'' \), hence we have:

\[ \sigma_{j-1} \sigma_j W'' \sigma_i^m = \sigma_{j-1} \sigma_j \sigma_j^m W'' \]

\[ \Downarrow \]

\[ W'' \sigma_i^m = \sigma_j^m W'' \]

Because of 2.7 we know that \( |W''| < |W| \) so, by induction we have:

\[ W'' \sigma_i = \sigma_{j-1} W'' \]

Now, multiplying this equation by \( \sigma_{j-1} \sigma_j \) on the left we get:

\[ \sigma_{j-1} \sigma_j W'' \sigma_i = \sigma_{j-1} \sigma_j \sigma_{j-1} W'' \]

\[ \Downarrow \]

\[ \sigma_{j-1} \sigma_j W'' \sigma_i = \sigma_j \sigma_{j-1} \sigma_j W'' \]

\[ \Downarrow \]

\[ W \sigma_i = \sigma_{j-1} W' \sigma_{i+1} \sigma_i = \sigma_j \sigma_{j-1} W' \sigma_{i+1} = \sigma_j W \]

(d) A similar argument shows that the theorem holds when \( W = \sigma_{j+1} W' \sigma_{i-1} \)

\[ \square \]

### 3 Properties of half-twists powers

In the following section we introduce some properties of half-twists and power of half-twists. In particular, we prove that the problem of determining if a braid is an event power of a half-twist is equivalent to the conjugacy problem in the free group.
**Definition 3.1.** A braid which is equivalent to a braid that can be drawn in a way, where strings 2, ..., n are parallel and straight, is called a combed braid, the process of transforming a given combed braid into it’s equivalent representation with strings 2, ..., n straight is called combing.

**Lemma 3.2.**
1. The set of all combed braids is a subgroup of \( B_n \) denoted by \( A_n \).
2. \( A_n \) is the free group generated by \( \{a_1, ..., a_{n-1}\} \), where,
   \[
   a_i = \sigma_1 ... \sigma_{i-1}\sigma_i^2\sigma_{i-1}^{-1}...\sigma_1^{-1}
   \]

**Proof:** [1].

The algorithm for getting the presentation of a combed braid in terms of the \( \{a_1, ..., a_{n-1}\} \) generators is well known. We recall here its steps, for clarification.

**Algorithm 3.3.**

**Step 1:** 'Comb' the 2, ..., n strings of the braid so they are all parallel, and only the 1st string is moving in between them.

**Step 2:** Moving along the first string, every time, the string does not match to the \( A_n \) presentation, 'pull' the string at this point to the left side of the braid. An example of Step 2, is shown in figure 3. In the example we show how does the combed braid \( \sigma_1\sigma_2^{-1}\sigma_3^2\sigma_2^2\sigma_2^{-1}\sigma_1 \) can be written in \( A_n \) presentation as \( a_2^{-1}a_3^{-1}a_2a_3a_1 \).

It is known that the complexity of Algorithm 3.3 is exponential.

**Lemma 3.4.** Let \( \beta \) be a half-twist such that \( \pi(\beta) = (i, j) \) and \( k > 0 \) then

\[
cr_{\beta^k}(s, t) = \begin{cases} 
  k & \text{if } \{s, t\} = \{i, j\} \\
  0 & \text{other}
\end{cases}
\]

**Proof:** Since \( \pi(\beta^2) = Id \), and by Lemma 1.8, it is clear that

\[
cr_{\beta^k}(i, j) = k \cdot cr_{\beta^2}(i, j)
\]

Therefore, it is enough to prove the lemma for \( k = 1 \).
Figure 3: Changing to $A_n$ presentation by pulling back all non-matching points.

Observing $\beta^2$ as a trace of $n$ punctures as shown in Figure 2, we notice the following 3 cases:

**Case 1:** The $s$-th and the $t$-th strings where ${s, t} \cap {i, j} = \emptyset$ are parallel, and therefore, $cr_{\beta^2}(s, t) = 0$.

**Case 2:** Assume that $s = i$ and $t \neq j$ the $i$-th string crosses the $t$-th string and then moves back in the same path and we get that $p(s, i) = n(i, s)$, therefore $cr_{\beta^2}(i, s) = 0$.

**Case 3:** If ${s, t} \cap {i, j} = \{i, j\}$, since, $exp(\beta^2)$ is 2 and from Lemma 1.8,

$$\Sigma_{s \neq t} cr_{\beta^2}(s, t) = 2 \quad (3)$$

which is, from the above, equal to $cr_{\beta^2}(i, j) + cr_{\beta^2}(j, i)$.

Since $\pi(\beta^2) = Id$, and from (3) and Lemma 1.8 (2), $cr_{\beta^2}(i, j) = cr_{\beta^2}(j, i) = 1$.

**Lemma 3.5.** Let $b$ be a half-twist such that $\pi(b) = (1, n)$, then, $b^2$ is combed braid conjugated in $A_n$ to $a_{n-1}$

**Proof:** Let $b$ be a half-twist with $\pi(b) = (1, n)$. By definition 1.9, $b$ can be written as $P^{-1}\sigma_1 P$, for some $P \in B_n$.

By Remark 1.11, the 1-st and the $n$-th strings can be embedded in a small neighborhood of a surface $F$, which does not intersect any other string.

$b^2 = P^{-1}\sigma_1^2 P$ and therefore, by changing $\sigma_1$ into $\sigma_1^2$ we see that $b^2$ can be embedded...
in the same neighborhood of the surface $F$ (See Figure 2).
In $b^2$, strings 1 and $n$ move along the surface, go around each other and move back along the surface to their original point. Now, since $F$ does not intersect any other string, we may straight the $n$th string along the surface $F$, and make the $n$th string parallel to the other $n - 2$ parallel strings. Hence, we get that $b^2$ is equivalent to a combed braid (See Figure 4).
By the algorithm 3.3, getting the $A_n$ generators of the combed braid $b^2$, the braid still remain symmetric with respect to $\sigma_{n-1}^k$. Therefore, $b^2$ is conjugated to $a_{n-1}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{$b^2$ is a combed braid}
\end{figure}

**Theorem 3.6.** Let $b \in B_n$ with $\pi(b) = (1, n)$, then, $b$ is a half-twist in power of $2k$ if and only if $b$ is a combed braid conjugated to $a_{n-1}^k$ in $A_n$.

**Proof:** Let $b = h^{2k}$, where $h$ is a half-twist. By Lemma 3.5, $h^2 = P^{-1}a_{n-1}P$ for some $P \in A_n$ and therefore,

$$b = h^{2k} = P^{-1}a_{n-1}^kP.$$  

The second direction of the theorem is trivial since,

$$b = P^{-1}(\sigma_1\ldots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}^{-1}\ldots \sigma_1^{-1})^kP = (P^{-1}\sigma_1\ldots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\ldots \sigma_1^{-1}P)^{2k}$$

which is a half-twist in the power of $2k$.

\end{proof}

4 The algorithm

We are now ready to present the algorithm. Given a braid $b$, the algorithm will determine if $b$ is a power of a half-twist. If the answer is positive, and $b$ is half-twist in the power of $k, k \in \mathbb{N}$, the algorithm computes a $P \in B_n$ such that, $b = P^{-1}\sigma_1^kP$. 

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Algorithm 4.1.

**Step 1:** Compute $\exp(b)$. It is clear that if $b$ is a half-twist in power $k$ then $k = \exp(b)$.

**Step 2:**

**Case 1:** If $\exp(b)$ is odd, compute $\pi(b)$. If $\pi(b)$ is not a transposition, then, by Lemma 1.10 $b$ is not an odd power of a half-twist. Therefore return false.

**Case 2:** If $\exp(b)$ is even, compute $\pi(b)$. If $\pi(b) \neq Id$, then, by Lemma 1.10 $b$ is not an even power of a half-twist. Therefore, return false. If $\pi(b) = Id$ compute $cr_k(i, j) \forall i, j$ and find $i$ and $j$ such that $cr_k(i, j) = cr_k(j, i) = \exp(b)/2$ all other $cr_k(k, l)$ must be zero. By Lemma 3.4 if such $i$ and $j$ do not exist, or $cr_k(k, l) \neq 0$ where $\{i, j\} \neq \{k, l\}$ then $b$ is not a power of a half-twist, and we can return false.

At this point we know, that if $b$ is a power of a half-twist than $b$ is equivalent to a braid where all strings other than $i$ and $j$ are parallel. We call the strings $i$ and $j$ the switching strings.

**Step 3:** To apply Theorem 3.6 we need that the two switching strings found in step 2, $i$ and $j$ will be 1, $n$. Assuming that $i < j$, by conjugating $\sigma_{i-1}...\sigma_2\sigma_1$ to $b$ and then conjugating $\sigma_{j+1}...\sigma_{n-1}$ to the result, we get a braid $b'$ with this property. Since the half-twists occupy a full conjugacy class $b'$ is a power of a half-twist if and only if $b$ is a power of half-twist. If we manage to find $P_1 \in B_n$ such that, $b' = P_1^{-1}\sigma_1P_1$ computing $P$ is trivial. At this point of the proof we may assume that the two switching strings are 1, $n$.

**Step 4:** If $\exp(b)$ is odd, Compute $b^2$. If $\exp(b)$ is even, $b$ remain the same. At this point we know that, if $b$ is a power of a half-twist, the result of this step is an even power of a half-twist, and therefore a combed braid.

**Step 5:** Operate Algorithm 3.3 on the result of the previous step. If the algorithm fails (since the braid is not a combed braid), By Theorem 3.6, $b$ is not a power of a half-twist, and we return false. On the contrary, if the algorithm succeeded, the result of this step is the presentation of $b$ (or $b^2$, depending on the previous step) in $A_n$.  

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Step 6: Check if the result is conjugated to $a^k_{n-1}$ ($a^k_{n-1}$ in the case of $b^2$) in $A_n$. The conjugacy problem in $A_n$ is easy to compute since $A_n$ is a free group (Lemma 3.2). By Theorem 3.6, the answer is positive if and only if $b$ (or $b^2$) is a power of a half-twist. So, if the answer is false, the algorithm returns false.

Step 7: In case that the result of step 6 is true, we compute the k-th ($2k$-th) root of the result: change back to $B_n$ presentation, and since the braid is conjugated to $a^k_{n-1}$ ($a^k_{n-1}$ in the case of $b^2$) we get the form:

$$b = Q^{-1}(\sigma_1...\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\sigma_1^{-1})^kQ$$

and the k-th root is:

$$r = Q^{-1}(\sigma_1...\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\sigma_1^{-1})Q$$

which is obviously a half-twist.

In case of $b^2$ we apply the same procedure. If $exp(b)$ is even, the algorithm is finished. If $exp(b)$ is odd, at this point we know that $b^2$ is a half-twist of the power $2k$, and we computed its $2k$-th root, $r$. Since the $2k$-th root is unique (Theorem 2.11), if $b$ is a half-twist in the power of $k$, $r$ must be it’s k-th root. Therefore, in order to check if $b$ is a power of a half-twist it is enough to check whether, $b$ is equivalent to $r^k$ in $B_n$. There are many solutions for the word problem in the braid group, one can be found for example in, [2].

This completes the algorithm.

The complexity of Algorithm 4.1 is exponential, since in Step 5 we comb the braid which might take exponential time as described after Algorithm 3.3. All other steps are bounded by this exponential function and hence it gives the upper bound for the number of operations which the algorithm performs.

References

[1] Artin, E., Theory of braids, Ann. Math. 48 (1947), 101-126.

[2] Birman, J.S., Ko, K.H. and Lee, S.J., A new approach to the word and conjugacy problems in the braid groups, Adv. Math. 139 (1998), 322-353.
[3] Elrifai, E.A. and Morton, H.R., *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) 45 (1994), 479-497.

[4] Garside, F.A., *The braid group and other groups*, Quart. J. Math. Oxford Ser. (2) 78 (1969), 235-254.

[5] Jacquemard, A., *About the effective classification of conjugacy classes of braids*, J. Pure. Appl. Alg. 63 (1990), 161-169.

[6] Kaplan, S. and Teicher, M., *Identifying Half-Twists Using Randomized Algorithm Methods*, preprint.

[7] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry I, Line arrangements in \( \mathbb{CP}^2 \)*, Contemporary Math. 78 (1988), 425-555.