Partition regularity of affine algebraic plane curves

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Abstract

We characterise all systems of Diophantine equations in two variables admitting infinitely many monochromatic solutions in N. Extending Rado’s Theorem, we prove that a system of polynomial equations in two variables admits infinitely many monochromatic solutions if and only if all polynomials are multiple of $x - y$. Working with algebraically closed fields, we provide two different proofs: the first uses nonstandard analysis, ultrafilters, and basic algebraic geometry; and the second explores a known result about colorings related to functions with no fixed points. The second method also provides a bound on the number of colors needed to show that a system of Diophantine equations in two variables does not admit infinitely many monochromatic solutions. This provides one of the first examples of a complete characterization of a large class of nonlinear systems of Diophantine equations admitting infinitely many monochromatic solutions.

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1 Introduction

This paper deals with the study of the so-called partition regularity of equations.

Definition 1.1. Given polynomials $P_1, \ldots, P_n \in \mathbb{Z}[x_1, \ldots, x_n]$, let

$$\sigma(x_1, \ldots, x_n) = \begin{cases} P_1(x_1, \ldots, x_n), \\ \vdots \\ P_m(x_1, \ldots, x_m). \end{cases}$$

We say that the system $\sigma(x_1, \ldots, x_n) = 0$ is partition regular over $\mathbb{N}$ if for all $r \in \mathbb{N}$ and all partitions $A_1, \ldots, A_r$ of $\mathbb{N}$, there are $i \in \{1, \ldots, r\}$ and $a_1, \ldots, a_n \in A_i$ such that $\sigma(a_1, \ldots, a_r) = 0$.

Historically, the first result of this kind is Schur’s Theorem which dates back to 1916 and states that the equation $x + y - z = 0$ is partition regular over $\mathbb{N}$ [1]. In 1933, R. Rado [2] completely characterized partition regularity of systems of linear Diophantine equations over $\mathbb{N}$; for a single equation, his characterization (that we call Rado condition (R) below) reads as follows (see sections 3.2 and 3.3 of [3] for a full treatment of Rado’s Theorem).

Theorem 1.2 (Rado’s Theorem). A linear Diophantine equation with no constant term $c_1x_1 + \cdots + c_nx_n = 0$ is partition regular over $\mathbb{N}$ if and only if

(R) There exists a nonempty set $J \subseteq \{1, \ldots, n\}$ such that $\sum_{j \in J} c_j = 0$.

Whilst there are plenty of results regarding various aspects of partition regularity of finite and infinite systems of linear equations in the literature, progress on the nonlinear case has been scarce. We refer to the introductory section of [4] for a complete list of nonlinear results proven until 2018, and to [5, 6, 7, 8, 9] for the latest advancements we are aware of. Some of these advancements have been produced through a nonstandard characterization of ultrafilters that first appeared in [10, 11]; these methods have proven to be particularly effective in producing necessary conditions for the partition regularity of equations in several classes. However, at the best of our knowledge, the only nonlinear classes whose partition regularity over $\mathbb{N}$ has been completely characterized are$^1$:

$^1$The situation is slightly different when it comes to the partition regularity over $\mathbb{R}$, $\mathbb{C}$; see e.g. [12] for a discussion of this fact.
1. equations of the form \( \sum_{i=1}^{n} c_i x_i = P(y) \), for \( P(y) \) nonlinear polynomial with no constant term (see [4, Corollary 3.14]); these equations are partition regular if and only if the linear part satisfies Rado’s condition, namely if and only if there exists a nonempty \( I \subseteq \{1, \ldots, n\} \) such that \( \sum_{i \in I} c_i = 0 \);

2. equation of the form \( \sum_{i=1}^{h(n)} c_i x_i^n = 0 \) for \( h(n) \) large enough (namely, \( h(n) \geq (1 + o(n)) n \log n \), see [13] for details), which are partition regular over \( \mathbb{N} \) if and only if there exists a nonempty \( I \subseteq \{1, \ldots, n\} \) such that \( \sum_{i \in I} c_i = 0 \);

3. equations of the form \( \sum_{i=1}^{s} a_i x_i^2 = \sum_{j=1}^{t} b_j y_j \), where the \( a_i, b_j \) are non-zero integers, with the exception of those equations of the form \( a_1 (x_1^2 - x_2^2) = a_2 x_3^2 + b_1 y_1 \), for which the partition regularity is still unknown (see [9, Theorem 1.10]). These equations are partition regular if and only if there exists \( I \neq \emptyset \) such that \( \sum_{i \in I} a_i = 0 \) or \( \sum_{j \in I} b_j = 0 \).

Notice that all these characterizations use some Rado condition on the coefficients of the equation. Recently, a generalization of the Rado condition was proposed in [6].

**Conjecture 1.1.** [6, Conjecture 2.20] Let \( P \in \mathbb{Z}[x_1, \ldots, x_n] \) such that \( \tilde{P}(w) = P(w, \ldots, w) \) is a nonzero linear and homogeneous polynomial. Suppose that \( P(x_1, \ldots, x_n) = 0 \) has infinitely many solutions in \( \mathbb{N} \); then the following are equivalent

1. the equation \( P(x_1, \ldots, x_n) = 0 \) is partition regular; and

2. \( P \) satisfies the maximal and the minimal Rado condition (see Definitions 2.16 and 2.17 of [6]).

In this paper, we restrict ourselves to the study of infinite partition regularity of systems of polynomial equations in two variables.

**Definition 1.3.** Under the assumptions and notations of Definition 1.1, we say that the system of equations \( \sigma(x_1, \ldots, x_n) = 0 \) is infinitely partition regular over \( \mathbb{N} \) if for all \( r \in \mathbb{N} \) and all partition \( A_1, \ldots, A_r \) of \( \mathbb{N} \), there are \( i \in \{1, \ldots, r\} \) and infinitely many \( n \)-tuples \( (a_1, \ldots, a_n) \in A_i^n \) such that \( \sigma(a_1, \ldots, a_n) = 0 \).

The number of different results that appeared in the past few years shows that, whilst several techniques from different areas of mathematics can be used to prove the (non) partition regularity of a specific
family of systems of non-linear equations, the general problem of characterizing all possible partition regular systems of non-linear equations is, currently, too difficult. For this reason, it is important to produce new examples of systems of non-linear equations whose partition regularity can be characterized in simple terms, such as Rado’s Theorem. In this paper, we provide a characterization of all possible systems of equations in two variables that are infinitely partition regular. Our main result reads as follows:

**Theorem 1.4.** Let \( P_1, \ldots, P_m \in \mathbb{Z}[x, y] \) and

\[
\sigma(x, y) = \begin{cases} 
  P_1(x, y), \\
  \vdots \\
  P_m(x, y).
\end{cases}
\]

The following facts are equivalent:

1. The system \( \sigma(x, y) = 0 \) is infinitely partition regular over \( \mathbb{N} \);
2. \( x - y \) divides \( P_i(x, y) \) in \( \mathbb{C}[x, y] \) for all \( i \leq n \).

Rephrasing, Theorem 1.4 shows that the equality is the only polynomial relation among pairs of numbers that is infinitely partition regular. In particular, Theorem 1.4 shows that there are no injectively infinite partition regular\(^2\) systems of polynomial equations in two variables, i.e. that admits infinitely many monochromatic solutions \( x \) and \( y \) such that \( x \neq y \).

At the best of our knowledge, the only known result regarding the infinite partition regularity of two-variables nonlinear equations are consequences of [6, Theorem 2.10], namely:

- for \( P(x, y) = x^2 - xy + ax + by = 0 \) or \( P(x, y) = x^2 - y^2 + ax + by = 0 \) with \( a, b \in \mathbb{Z} \), the equation \( P(x, y) = 0 \) is partition regular over \( \mathbb{N} \) if and only if \( a = -b \) (this is [6, Theorem 2.10] for \( c = 0 \));
- for \( P(x, y) = x^2 + y^2 - xy - ax - by + ab \) or \( P(x, y) = P(x, y) = x^2 - y^2 + xy - ax - by + ab \) with \( a, b \in \mathbb{Z} \), the equation \( P(x, y) = 0 \) is never infinitely partition regular over \( \mathbb{N} \) (these are [6, Example 3.4, Example 3.5]).

In particular, the first result above can be rephrased by saying that equations \( P(x, y) = x^2 - xy + ax + by = 0 \) or \( P(x, y) = x^2 - y^2 + ax +

\(^2\)This is often used as the strengthening of the notion of partition regularity needed to avoid constant solutions [4, Definition 1.7].
by = 0 are infinitely partition regular only when \( P(x, y) \) is divisible by \( x - y \).

Notice that, by Bézout’s Theorem, it had to be expected that the infinite partition regularity of \( \sigma(x, y) = 0 \) had to be related with some property of a common factor of the \( P_i(x, y) \)’s, as without such a common factor this system could have only a finite amount of solutions. Under this intuition, to prove Theorem 1.4, we extend the problem of partition regularity of polynomial equations to algebraically closed fields of characteristic 0 in which basic tools from algebraic geometry, namely the Bézout’s Theorem and the Newton-Puiseux Theorem, can be applied. As such, we are able to settle Theorem 1.4 as a consequence of the following seemingly weaker statement:

**Theorem 1.5.** Let \( F \) be an algebraically closed field of characteristic 0, let \( P(x, y) \in F[x, y] \) and \( S \subseteq F \) be any infinite subset of \( F \). Then the following are equivalent:

1. \( P(x, y) \) is infinitely partition regular over \( S \);
2. \( x - y \) divides \( P(x, y) \) in \( F[x, y] \).

As a direct corollary of Theorem 1.5, we have that

**Corollary 1.6.** Let \( F \) be an algebraically closed field of characteristic 0, \( P_1, \ldots, P_m \in F[x, y] \) and \( S \subseteq F \) an infinite set. If

\[
\sigma(x, y) = \begin{cases} 
P_1(x, y), \\
\vdots \\
P_m(x, y). 
\end{cases}
\]

then following facts are equivalent:

1. The system \( \sigma(x, y) = 0 \) is infinitely partition regular over \( S \);
2. \( x - y \) divides \( P_i(x, y) \) in \( F[x, y] \) for all \( i \leq n \).

As a straightforward consequence of Corollary 1.6, when \( F = \mathbb{C} \), we deduce the Theorem 1.4, i.e. a complete characterization of infinitely partition regular systems of polynomial equations in two variables over \( \mathbb{N} \). Notice that, in particular, no equation in two variables containing a nonzero constant term is infinitely partition regular over \( \mathbb{N} \) in contrast with what happens with 3 or more variables even in the linear case (see e.g. [6, Proposition 2.3] and [11, Theorem 3.5.3]).

We will provide two different proofs of Theorem 1.5. The first will be given for \( F = \mathbb{C} \), but its method that could be (and has been in
adapted to study the partition regularity of other classes of equations. This proof uses a nonstandard characterization of ultrafilters mixed with Newton-Puiseux and Bézout’s Theorems; at the best of our knowledge, this is the first proof of the partition regularity of a class of nonlinear equations that uses methods coming from algebraic geometry. The second one is a standard proof that works for any field \( F \) of characteristics 0, which gives also an upper bound on the number of colors that are sufficient to disprove the infinite partition regularity when \( P(x, y) \) is not divided by \( x - y \). Whilst more direct than the previous one, this second proof cannot be as easily generalized to study other classes of equations.

The paper is structured as follows: in Section 2 we recall the basic facts regarding the interplay between ultrafilters, partition regularity of equations and nonstandard methods that we need in order to prove Theorem 1.5. In Section 3 we present the two proofs our main result.

2 Facts about ultrafilters

In this Section, we recall the basics of the relationships between ultrafilters, nonstandard analysis, and partition regularity. We refer to the monograph [14] for applications of ultrafilter theory in Ramsey theory and combinatorics, and the book [15] for a gentle introduction to nonstandard analysis and its applications in Ramsey theory and combinatorics.

We firstly extend the definition of (infinite) partition regularity to integral domain in the obvious way:

**Definition 2.1.** Let \( D \) be an integral domain and \( S \subseteq D \); given polynomials \( P_1, \ldots, P_n \in D[x_1, \ldots, x_n] \), let

\[
\sigma(x_1, \ldots, x_n) = \begin{cases} 
P_1(x_1, \ldots, x_n), \\
\vdots \\
P_m(x_1, \ldots, x_m).
\end{cases}
\]

We say that the system \( \sigma(x_1, \ldots, x_n) = 0 \) is (ininitely) partition regular over \( S \) if for all \( r \in \mathbb{N} \) and all partitions \( A_1, \ldots, A_r \) of \( \mathbb{N} \), there is an \( i \in \{1, \ldots, r\} \) and (infinitely many) \( (a_1, \ldots, a_n) \in A_i^n \) such that \( \sigma(a_1, \ldots, a_r) = 0 \).

Partition regularity problems can be easily translated to existence problems for ultrafilters as follows:
Theorem 2.2. [14, Theorem 3.11] Under the assumptions and notations of Definition 2.1, the system \( \sigma(x_1, \ldots, x_n) = 0 \) is partition regular over \( S \) if and only if there is an ultrafilter \( U \in \beta S \) such that for all \( A \in U \) there are \( a_1, \ldots, a_n \in A \) such that \( \sigma(a_1, \ldots, a_n) = 0 \).

Definition 2.3. Under the conditions of Theorem 2.2, we say that \( U \in \beta S \) is a witness of the partition regularity over \( S \) of the system \( \sigma(x_1, \ldots, x_n) = 0 \).

Specialized to infinite partition regularity, Theorem 2.2 can be reformulated as follows.

Theorem 2.4. Under the assumptions and notations of Definition 2.1, the system \( \sigma(x_1, \ldots, x_n) = 0 \) is infinitely partition regular over \( S \) if and only if there is a free ultrafilter \( U \in \beta S \) such that for all \( A \in U \) there are \( a_1, \ldots, a_n \in A \) such that \( \sigma(a_1, \ldots, a_n) = 0 \).

In recent years, a nonstandard take on ultrafilters has been used in various ways to provide necessary, and in a few cases also sufficient, conditions for the partition regularity of several classes of nonlinear equations, see e.g. [6, 10, 4, 11, 16]. This translation can be obtained as follows. Let \( \mathbb{U} \) be a mathematical universe with \( D \in \mathbb{U} \) and let \( \langle \ast, \mathbb{U}, \mathbb{U}' \rangle \) be an enlarging in the sense of nonstandard analysis.

Definition 2.5. Let \( S \subseteq D \) and \( U \in \beta S \). The set

\[
\mu(U) := \bigcap_{A \in U} \ast A
\]

is called the monad of \( U \); any \( \alpha \in \mu(U) \) is called a generator of \( U \).

Conversely, given any \( \alpha \in \ast S \), we let the ultrafilter generated by \( \alpha \) to be

\[
U_\alpha = \{ A \subseteq S \mid \alpha \in \ast A \}.
\]

Given any \( \beta \in \ast S \), we say that \( \alpha \) and \( \beta \) are \( u \)-equivalent (denoted by \( \alpha \sim_u \beta \)) if \( U_\alpha = U_\beta \).

For a detailed study of monads and their properties, we refer to [16]. Here, we simply observe that \( \sim_u \) is an equivalence relation on \( \ast S \); notice also that, by definition, \( \alpha \sim_u \beta \) if and only if \( \alpha, \beta \) cannot be distinguished by any hyper-extension, namely if for every \( A \subseteq S \) one has either \( \alpha, \beta \in \ast A \) or \( \alpha, \beta \notin \ast A \).

\( \footnote{This hypothesis is needed to ensure that \( \mu(U) \neq \emptyset \) for all \( U \in \beta S \), see e.g. [15, Proposition 3.6].} \)
For our purposes, Theorem 2.2 can be translated into nonstandard terms as follows (this is a particular case of [16, Theorem 5.4]).

**Theorem 2.6.** Let $D$ be an integral domain. Let $P(x_1, \ldots, x_n) \in D[x_1, \ldots, x_n]$ and $S \subseteq D$. The following facts are equivalent:

1. the equation $P(x_1, \ldots, x_n) = 0$ is infinitely partition regular over $S$;
2. there are $u$-equivalent $\alpha_1, \ldots, \alpha_n \in *S \setminus S$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$.

As an example of application of Theorem 2.6 we show that, when trying to identify conditions for the partition regularity of equations, we can always restrict to irreducible polynomials.

**Theorem 2.7.** Let $P(x_1, \ldots, x_n) \in D[x_1, \ldots, x_n]$ and let

$$P(x_1, \ldots, x_n) = \prod_{j=1}^{k} Q_j(x_1, \ldots, x_n)$$

be its factorization as a product of irreducible polynomials. Let $S \subseteq D$. Then $P(x_1, \ldots, x_n) = 0$ is infinitely partition regular over $S$ if and only if there exists $j \leq k$ such that $Q_j(x_1, \ldots, x_n) = 0$ is infinitely partition regular over $S$.

**Proof.** If $P(x_1, \ldots, x_n) = 0$ is infinitely partition regular, by Theorem 2.6 there are $u$-equivalent $\alpha_1, \ldots, \alpha_n \in *S \setminus S$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$. But then there is an $i \leq k$ such that $Q_i(\alpha_1, \ldots, \alpha_n) = 0$ which shows, again by Theorem 2.6, that $Q_i(x_1, \ldots, x_n) = 0$ is infinitely partition regular. The converse is immediate by Definition 2.1.

From now on in this Section, we work with the field of the complex numbers $\mathbb{C}$ and its usual topology. If $\alpha \in {}^*\mathbb{C}$, we let the standard part of $\alpha$ (notation $st(\alpha)$) to be

- the unique $c \in \mathbb{C}$ such that $\alpha \in {}^*I$ for all $I$ open neighborhood of $c$, if such a $c$ exists; in this case, we will say that $\alpha$ is nearstandard;
- $\infty$, otherwise.

By the definition and the nonstandard characterization of compactness (see e.g. [17, Theorem 3.5.1]), it follows that $\alpha \in {}^*\mathbb{C}$ is nearstandard if and only if $\|\alpha\|$ is finite, where $\|\cdot\|$ denotes the nonstandard extension of the usual complex absolute value $\|\cdot\| : \mathbb{C} \to \mathbb{R}$. 
Definition 2.8. We say that $U \in \beta C$ is a nearprincipal ultrafilter if $\mu(U)$ consists solely of nearstandard points; equivalently, $U$ is nearprincipal if there exists $c \in C$ such that $st(\alpha) = c$ for all $\alpha \in \mu(U)$. In this case, we set $c = st(U)$.

Notice that the well-posedness of Definition 2.8 follows as the topology on $C$ is Hausdorff and by the fact that, for all $\alpha \sim_u \beta \in \ast C$, for all $c \in C$ and $I$ open neighborhood of $c$, $\alpha \in \ast I$ if and only if $\beta \in \ast I$. Moreover, if $st(\alpha) = \infty$ then $\alpha^{-1}$ is nearprincipal and $st(\alpha^{-1}) = 0$.

3 Partition regularity of affine algebraic plane curves

Recently, in [18] J. Byszewski and E. Krawczyk extended Rado’s characterization to infinite integral domains and reduced noetherian rings; for our purposes, we recall their characterization of the partition regularity of a single linear equation, which is a particular case of [18, Theorem 3.7], that characterizes in general finite systems of linear equations and thus extending Rado condition (R).

Theorem 3.1. [18, Theorem 3.7] Let $D$ be an infinite integral domain and $c_1, \ldots, c_n \in D \setminus \{0\}$. The equation $\sum_{i=1}^{n} c_i x_i = 0$ is partition regular over $D \setminus \{0\}$ if and only if there is a non-empty $I \subseteq \{1, \ldots, n\}$ such $\sum_{i \in I} c_i = 0$.

To proceed with the proof of the Theorem 1.5, we remark that the implication $(2) \Rightarrow (1)$ follows by Theorem 2.7, as $x - y$ is infinitely partition regular by Theorem 3.1. Therefore, the only implication to be proven is $(1) \Rightarrow (2)$.

When $P(x, y)$ is homogeneous, this implication is trivial; in fact, as $F$ is an algebraically closed field of characteristic 0, any homogeneous two variables polynomial $P(x, y)$ can be decomposed as a product of linear factors, say $P(x, y) = \prod_{i=1}^{k} (u_i x + v_i y)$; by Theorem 2.7, one of this factors has to be partition regular, and we conclude by Theorem 3.1.

When $P(x, y)$ is not homogeneous, we cannot factorize it globally as a product of linear factors. As said in the previous sections, to overcome this problem we can proceed in (at least) two different ways. The Subsection 3.1 deals with the proof of the particular case where $F = \mathbb{C}$; for this proof, we mimic the homogeneous case locally with the help of the Newton-Puisseux Theorem.
In the Subsection 3.2, we work with an arbitrary algebraically closed field $F$ of characteristic 0 and produce a bound on the number of color needed to disproof the infinitely partition regular of a system of polynomial equations whose irreducible factors do not include $x - y$; while the first proof was carried with nonstandard techniques, this proof will be carried in the standard universe.

In both proofs, the following result will be applied:

**Lemma 3.2.** [15, Theorem 2.83] Let $S$ be a set and let $f : S \to S$ be a function such that $f(x) \neq x$ for all $x \in S$. Then there exists a 3-coloring $\Xi : S \to \{1, 2, 3\}$ such that $\Xi(x) \neq \Xi(f(x))$ for all $x \in S$.

### 3.1 Nonstandard Ultrafilters Proof

We restrict to the case $F = \mathbb{C}$, and discuss how the ultrafilters proof could be adapted to more general fields at the end of this Section. In this case, Newton-Puisseux Theorem allows to adapt the proof of the homogeneous case to the general one. For a clear exposition of Puiseux series and the Newton-Puisseux Theorem, we refer to [19]; for our purposes, it is sufficient to consider the following much weaker consequence of Newton-Puisseux Theorem, which is a rephrasing of [19, Proposition 2.3.1] where we emphasize that the explicit nature of the functions $f$ will not play any role in our proof.

**Theorem 3.3.** [Newton-Puisseux] Let $P \in \mathbb{C}[x, y]$. For every $(x_0, y_0) \in \mathbb{C}^2$ such that $P(x_0, y_0) = 0$ there exists an open neighborhood $U_{(x_0, y_0)}$ of $(x_0, y_0)$ and a finite set $F_{(x_0, y_0)}$ of functions such that for all $x, y \in U_{(x_0, y_0)}$ one has that $P(x, y) = 0$ if and only if there exists $f \in F_{(x_0, y_0)}$ such that $x = f(y)$ or $y = f(x)$.

**Proof.** By Theorem 2.6, there exist $\alpha, \beta \in \ast \mathbb{C} \setminus \mathbb{C}$ such that $\alpha \sim_u \beta$ and $P(\alpha, \beta) = 0$. We divide the proof into the two cases below:

**Case 1:** $\alpha, \beta$ nearstandard.

Let $c = \text{st}(\alpha) = \text{st}(\beta)$. As $P(\alpha, \beta) = 0$, by continuity $\text{st} \left( P(c, c) \right) = \text{st} \left( P(\alpha, \beta) \right) = 0$. Let $U$ neighbourhood of $c$ and functions $f_1, \ldots, f_k : U \to \mathbb{C}$ be given by Theorem 3.3. We extend the $f_i$’s to the whole of $\mathbb{C}$ arbitrarily, keeping the same notation. $U$ is a neighborhood of $c = \text{st}(\alpha)$, so $\alpha, \beta \in \ast \mathbb{C}$. Hence, there exists $i \in \{1, \ldots, k\}$ such that $\alpha = \ast f_i(\beta)$ (the case where $\beta = \ast f_i(\alpha)$ is similar). In particular,
\( \beta \sim_u \ast f_i(\beta) \) and, as \( f_i \) is standard, the trivial nonstandard translation\(^4\) of Lemma 3.2 forces \( \ast f_i(\beta) = \beta \), so \( \alpha = \beta \). Hence

\[
(\alpha, \beta) \in \ast \{(x, y) \in \mathbb{C}^2 \mid x = y \text{ and } P(x, y) = 0\}
\]

which, as \( \alpha, \beta \in \ast \mathbb{C} \setminus \mathbb{C} \), shows that the intersection between the curves \( P(x, y) = 0 \) and \( x = y \) is infinite. By Bézout’s Theorem, this can happen only when \( P(x, y) \) is divisible by \( x - y \).

**Case 2:** \( \alpha, \beta \) not nearstandard.

In this case, \( \alpha^{-1} \sim_u \beta^{-1} \) are nearstandard. In particular, \( \alpha^{-1}, \beta^{-1} \) solve the functional equation \( P\left(\frac{1}{x}, \frac{1}{y}\right) = 0 \). Let \( d_1, d_2 \in \mathbb{N}, d_1, d_2 \geq 1 \) be such that \( x^{d_1}y^{d_2}P\left(\frac{1}{x}, \frac{1}{y}\right) \in \mathbb{C}[x, y] \). By what we proved in Case 1, as \( x^{d_1}y^{d_2}P\left(\frac{1}{x}, \frac{1}{y}\right) \) is infinitely partition regular, there exists \( R(x, y) \) such that

\[
(x - y)R(x, y) = x^{d_1}y^{d_2}P\left(\frac{1}{x}, \frac{1}{y}\right).
\]

But then

\[
\left(\frac{1}{x} - \frac{1}{y}\right)R\left(\frac{1}{x}, \frac{1}{y}\right) = x^{-d_1}y^{-d_2}P(x, y),
\]

namely

\[
x^{d_1-1}y^{d_2-1}(y - x)R\left(\frac{1}{x}, \frac{1}{y}\right) = P(x, y).
\]

In particular, this forces \( x^{d_1-1}y^{d_2-1}R\left(\frac{1}{x}, \frac{1}{y}\right) \in \mathbb{C}[x, y] \), hence \( P(x, y) \) is a multiple of \( x - y \).

\[\square\]

The above proof could be generalized in at least two different directions. On one hand, it could be adapted to work for different fields by observing that it uses three properties of \( F \):

1. the use of nearstandard generators, that requires to work with topological fields with no isolated points;

\(^4\)For all \( \alpha \in \ast S \), for all \( f : S \rightarrow S \) if \( \alpha \sim_u \ast f(\alpha) \) then \( \alpha = \ast f(\alpha) \); see e.g. [15, Exercise 3.5(3)].
2. the fact that the inverse of a non nearstandard number is near-
standard, which holds, e.g., in valued fields;

3. the local property given by Puiseux-Newton Theorem, that holds
similarly in more general fields (for example, a form of this result
holds for Hahn fields, see [20]).

The second direction regards the study of the localization of in-
finite partition regularity over subsets of $\mathbb{C}$ of other polynomial equa-
tions using the nonstandard characterization of ultrafilters. Some simi-
lar ideas have been used in [12] to prove the infinite partition regularity
near 0, in $\mathbb{R}$, of several classes of nonlinear equations.

3.2 Standard proof

In this Section, we let $F$ be an algebraically closed field of charac-
teristic 0 and we actually prove a stronger fact, that has Theorem 1.5
as a simple consequence. First, we fix the following definitio

Definition 3.4. Let $P(x, y) \in F[x, y]$. We let

$$d_P(x) = \max \{ n \in \mathbb{N} \mid x^n \text{ divides a monomial in } P(x, y) \}$$

and define $d_P(y)$ similarly.

Theorem 3.5. Let $P(x, y) \in F[x, y]$. Let $n = \min \{ d_P(x), d_P(y) \}$ and
$m = \max \{ d_P(x), d_P(y) \}$. If $x - y$ does not divide $P(x, y)$ in $F[x, y]$,
there is a $1 \leq k \leq m$ and a $(3^n + k)$-coloring of $F$ that disproves the
infinite partition regularity of $P(x, y) = 0$.

Proof. Without loss of generality, let us assume that $n = d_P(y)$ and
$m = d_P(x)$. For each $s \in F$, let $P_s(y) \in F[y]$ be given by $P_s(y) =
P(s, y)$; define $A$ as the set of all $s \in F$ such that $P_s(y)$ is the zero
polynomial and, for each $s \in F \setminus A$, let $F_s = \{ t \in F : P_s(t) = 0 \}$. If
we let

$$P(x, y) = \sum_{j=0}^{n} P_j(x)y^j,$$

we have that $s \in A$ if and only if $P_i(s) = 0$ for all $i \leq n$. As $d_P(x) = m$,

$$|\{ s \in F : \forall i \leq n (P_i(s) = 0) \}| \leq m,$$

hence $|A| = k \leq m$ and, for each $s \in F \setminus A$, $|F_s| \leq n$. 

12
By the well-ordering principle, there is a linear order on $F$ so that every finite and non-empty subset of $F$ admits a minimum. Let us fix such an order, and let us define $f_1, \ldots, f_n : F \to F$ recursively as follows: fix an $s_0 \in F \setminus A$ and define $f_1 : F \to F$ by

$$f_1(s) = \begin{cases} 
\min F, & \text{if } s \notin A; \\
s_0, & \text{otherwise.}
\end{cases}$$

Suppose that for some $2 \leq i \leq n$ the functions $f_1, \ldots, f_{i-1}$ are already defined; then, define

$$f_i(s) = \begin{cases} 
\min(F \setminus \{f_j(s) \mid j < i\}), & \text{if } s \notin A \text{ and } F \setminus \{f_j(s) \mid j < i\} \neq \emptyset; \\
\min(F \setminus \{f_j(s) \mid j < i\}), & \text{if } s \notin A \text{ and } F \setminus \{f_j(s) \mid j < i\} = \emptyset; \\
s_{i-1}, & \text{otherwise.}
\end{cases}$$

By construction, given $s \in F \setminus A$ and $i \leq n$, $f_i(s) = s$ implies that $P(s, s) = 0$. Hence, the fixed points of the functions $f_1, \ldots, f_n$ lay in the set

$$B = \{s \in F \mid P(s, s) = 0\}.$$

If $x - y$ does not divide $P(x, y)$ in $F[x, y]$, then Bézout’s Theorem proves that $B$ is a finite set. For each $i \leq n$, let $g_i : F \to F$ be a function that agrees with $f_i$ on $F \setminus B$ and has no fixed points in $B$; in particular, $g_i$ has no fixed points in $F$. So Theorem 3.2 gives a $3$–coloring $\Xi_i$ of $F$ such that for all $s \in F$ $\Xi_i(s) \neq \Xi_i(g_i(s))$. Enumerate $A = \{s_1, \ldots, s_k\}$ and define a $(3^n + k)$-coloring $\Xi$ of $F$ as follows:

$$\Xi(s) = \begin{cases} 
(\Xi_1(s), \ldots, \Xi_n(s)), & \text{if } s \in F \setminus A; \\
i, & \text{if } s = s_i \in A.
\end{cases}$$

We claim that the set of all $\Xi$-monochromatic solutions of $P(x, y) = 0$ in $F$ is finite. In fact, given $(s, t) \in F \times F$ such that $P(s, t) = 0$,

- if $s = s_i \in A$, then, by construction, $\Xi(s) = \Xi(t)$ if and only if $s = t$;
- if $s \in B \setminus A$, then we must have that $t \in F_s$;
- if $s \notin A \cup B$, then there is $i \leq n$ such that $t = f_i(s) = g_i(s)$ and thus $\Xi(s) \neq \Xi(t)$ by construction.
Hence, the set of all $\Xi$-monochromatic pairs $(s, t) \in F \times F$ such that $P(x, y) = 0$ in $F$ is a subset of

$$M = \{(s, s) \mid s \in A\} \cup \left( (B \setminus A) \times \bigcup_{s \in B \setminus A} F_s \right).$$

As both $A$ and $B$ are finite, $M$ is a finite subset of $F$. \qed

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14
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