Note on the properties of exact solutions in Lovelock gravity

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We study the properties of cosmological solution for a flat multidimensional anisotropic Universe in Lovelock gravity. A particular attention is paid to some features of solutions in a general Lovelock gravity which have no their counterparts in analogous solutions of General Relativity. We consider exponential and so called generalized Milne solutions and discuss reason for these solutions to exist in Lovelock gravity and do not exist in General Relativity.

PACS numbers: 04.20.Jb, 04.50.-h, 04.50.Kd, 98.80.-k

I. INTRODUCTION

The Lovelock gravity being a natural generalization of General Relativity (GR) with equations of motion of the same order as in GR became recently a matter of intense investigation mainly due to popularity of higher-dimensional paradigm. For any fixed number of spatial dimensions the Lovelock Lagrangian includes finite number of terms (in contrast to string gravity where we have an infinite raw of curvature corrections). The Einstein-Gilbert Lagrangian is the first term in this theory, and this term is the only one in (3+1) dimensions. Higher order terms can be important in multidimensional scenarios. The second term is the famous Gauss-Bonnet (GB) combination, and in (4+1) and (5+1) dimension there are no other terms.

The system containing Einstein-Gilbert and Gauss-Bonnet terms have been studied in many papers during at least last 30 years (see e.g. [2–6]). In particular, many works have been devoted to cosmological dynamics near a cosmological singularity where quadratic in curvature contribution from Gauss-Bonnet term should be important. Moreover, it is possible to assume that in high-curvature regime the contribution from Einstein-Gilbert term is negligible, and to consider a pure Gauss-Bonnet gravity. Equations of motion simplify considerably under this assumption, and it is possible to find a number of analytical solutions. Additionally, if we are looking for solution in the power-law form (and we are), it is impossible to construct it with a mixture of different Lovelock
contribution in the time-independent form (since different Lovelock contributions scale differently in time: Einstein-Hilbert are $\propto t^{-2}$, Gauss-Bonnet are $\propto t^{-4}$ and so on).

The equations of motion as well as the resulting dynamics near a singularity for Gauss-Bonnet in (5+1) in a flat anisotropic Universe have been obtained previously [4, 5, 7] and generalized in [8] for a general Lovelock gravity in various dimensions. Further studies including presence of matter or/and a subdominant Einstein-Gilbert term have been done in [9–14].

The equations of motion for the Gauss-Bonnet gravity take the form

$$\sum_{i>j>k>l} D_{ijkl} H_i H_j H_k H_l = 0 \quad \text{(constraint equation);}$$

$$\sum_{j \neq i} \left( H_j + H_j^2 \right) \sum_{\{k>l\} \neq \{i,j\}} H_k H_l + 3 \sum_{\{a>b>c>d\} \neq i} H_a H_b H_c H_d = 0 \quad \text{(i-th dynamical equation),}$$

where $H_i \equiv \dot{a}_i / a_i$ is the Hubble parameter and $a_i \equiv a_i(t)$ is the scale factor corresponding to the $i$th coordinate.

For the power-law ansatz $(a_i(t) = a_0^i t^{p_i}, \quad \dot{H}_i(t) = \dot{a}_i(t)/a_i(t) = p_i/t, \quad \ddot{H}_i(t) = -p_i/t^2)$ equations (1) and (2) take the form:

$$\sum_{i>j>k>l} p_i p_j p_k p_l = 0 \quad \text{(constraint equation);}$$

$$\sum_{j \neq i} (p_j^2 - p_j) \sum_{\{k>l\} \neq \{i,j\}} p_k p_l + 3 \sum_{\{a>b>c>d\} \neq i} p_a p_b p_c p_d = 0 \quad \text{(i-th dynamical equation).}$$

In Gauss-Bonnet gravity one of two conditions should be applied to the power indices:

$$\sum_{i=1}^D p_i = 3 \quad \text{or} \quad \sum_{j>k>l} p_j p_k p_l = 0,$$

where we call the first of them as the generalized Kasner solution and the second one as the generalized Milne solution for brevity. The power indices for the generalized Milne solution can be expressed in the form $(a, b, 0, 0, 0)$ where $a$ and $b$ are arbitrary numbers. This fact for (5+1)-dimensional case was demonstrated in [11] and later in [8] was generalized for general Lovelock
case. There are no analogs of such solution in GR. On the other hand, generalized Kasner solution looks similar to the power-law flat anisotropic solution in GR (“classical” Kasner multidimensional solution).

If only the highest order Lovelock term is considered, these results can be generalized [8]. It is interesting that though the “classical” Kasner solution can be considered as a particular case in this scheme, it has some properties which separate it from any other Lovelock cases. In the present paper we discuss several features which are specific for the Kasner solution and do not shared by its analogs in Lovelock gravity.

The structure of the manuscript is as follows: in the Section 2 we prove the noncompactness of the power indices space in GB and link it with the existence of the exponential solutions. In Section 3 we discuss the properties of the generalized Milne solution. Finally in the Conclusions we outlook briefly our results.

II. NONCOMPACTNESS OF THE POWER INDICES SPACE IN THE GENERAL LOVELOCK CASE

Regarding generalized Kasner solution it is possible to note that despite first Kasner relation is not altered seriously (the sum of power indices is equal to some odd number depending on the order of the Lovelock term), the simple form of the second relation (the sum of indices squared is equal to unity) does not survive in higher order Lovelock theories [4, 5, 7]. The Kasner sphere of Einstein theory is replaced by some less simple surface which appears to be noncompact.

First we proof the noncompactness of Kasner solutions space for the case with even number of spatial dimensions. The simplest case is (4+1)-dimensional Gauss-Bonnet; the Kasner conditions are governed by two equations: \( \sum_{i=1}^{4} p_i = 3 \) and \( p_1 p_2 p_3 p_4 = 0 \). From the second of them it is clear that one of \( p_i \) should be zero while others compose a plane that obey \( \sum_{j\neq i; j=1}^{4} p_j = 3 \). The complete set joins four planes according to four different \( p_i \) which can be equal to zero.

The general Lovelock case is similar to the Gauss-Bonnet one; the Kasner conditions are [8]: \( \sum_{i=1}^{2n} p_i = (2n - 1) \) and \( p_1 p_2 \ldots p_{2n} = 0 \). Similarly, from the second of them one of \( p_i \) should be zero while others compose a hyper-plane that obey \( \sum_{j\neq i; j=1}^{2n} p_j = (2n - 1) \). And again, the complete set joins 2n hyper-planes according to 2n choices of \( p_i \).

The proof for the case of the odd number of spatial dimensions is a bit more complicated; again, first we give a proof for the simplest ((5+1)-dimensional Gauss-Bonnet) case and then extend it onto a general case.
In (5+1)-dimensional Gauss-Bonnet universe the generalized Kasner conditions are \( \sum_{i=1}^{5} p_i = 3 \) and \( \sum_{i>j>k>l} p_i p_j p_k p_l = 0 \). Let us express one of \( p_i \) from the first equation (without loss of generality let it be \( p_5 \)), substitute it into the second equation and resolve the result with respect to another \( p_i \) (let it be \( p_4 \)):

\[
p_1^2 (p_1 p_2 + p_1 p_3 + p_2 p_3) - p_4 ((3 - p_1 - p_2 - p_3)(p_1 p_2 + p_1 p_3 + p_2 p_3)) - p_1 p_2 p_3 (3 - p_1 - p_2 - p_3). \tag{6}
\]

We need to show that there exist a solution of this equation when at least one of the power indices from remaining three is arbitrary large. Let us denote one of \( \{p_1, p_2, p_3\} \) as \( A \) (nonzero and arbitrary large) (let it be \( p_3 \)); then the discriminant \( D \) of (6) in leading on \( A \) order takes a form

\[
D = A^4 (p_1 + p_2)^2 + O(A^3).
\]

It is obvious that if \( p_1 \neq -p_2 \) then \( D > 0 \) so the solution always exists (we do not consider the \( p_1 = -p_2 \) case since we are only interested in noncompactness and not in some exact solutions and with \( p_1 \neq -p_2 \) noncompactness is already proven).

It is worth to note that in the GR case the second bracket in the second term of the Eq. (6) is absent, and \( p_3 \) enters linearly in the second term while still quadratically in the third term, so the asymptotic of the discriminant in the limit \( p_3 \to \infty \) is no longer valid.

Now we prove our statement for a general case with odd number of spatial dimensions. The generalized Kasner conditions for a general case are [8]: \( \sum_{i=1}^{2n+1} p_i = (2n - 1) \) and \( \sum_{i_1 > i_2 > \cdots > i_{2n}} p_{i_1} p_{i_2} \cdots p_{i_{2n}} = 0 \). Similarly to (5+1) case we express \( p_{2n+1} \) from first equation, substitute it into the second equation, solve the result with respect to \( p_{2n} \) (one can verify that it is still a quadratic equation), denote \( p_{2n-1} \) as \( A \) and write down the discriminant in leading order in \( A \):

\[
D = A^4 \left( \sum_{i_1 > i_2 > \cdots > i_{2n-3}} p_{i_1} p_{i_2} \cdots p_{i_{2n-3}} \right)^2 + O(A^3). \tag{7}
\]

Similarly to (5+1) case, if the multiplier at \( A^4 \) is nonzero, then the solution always exists, and for the same reason we do not consider the case when the multiplier is equal to zero.

It is interesting that power-law ansatz is not a unique possibility for a flat Universe in Lovelock gravity. Substituting \( H_i = \text{const}; \dot{H}_i = 0 \) it is possible to get solutions corresponding to exponentially increasing or decreasing scale factors [15, 16]. Such solutions are absent in GR – formal substitution of this ansatz only lead to trivial solution with all \( H_i \equiv 0 \). One can note an interesting link: in GR we have compact power indices space (\( \sum p_i^2 = 1 \)) and no exponential solutions. On the
contrary, in GB case we have noncompact power indices space and there are exponential solutions. It is reasonable to think that existence of exponential solutions is linked with noncompactness of the power indices space. Indeed, from the definition of $p_i = -H_i^2/\dot{H}_i$ (where $p_i \neq 0$) one can see that the compactness of $p_i$ space ($|p_i| \leq 1$) in GR means that $|\dot{H}_i| \geq H_i^2$. This assures that $\dot{H}_i$ cannot be nullified without putting to zero the corresponding $H_i$, making exponential solution non-existent. However, in GB case we have noncompact $p_i$ space and $\dot{H}_i$ can be arbitrary small in modulus. Formally, from the definition $p_i = -H_i^2/\dot{H}_i$ one can see that zeroth $\dot{H}_i$ (with nonzero $H_i$) corresponds to infinite $p_i$, which could be achieved only if the space of possible $p_i$ is noncompact.

III. PROPERTIES OF THE GENERALIZED MILNE SOLUTION

We remind a reader that the generalized Milne solution in Gauss-Bonnet gravity is a power-law solution with indices $(a, b, 0\ldots0)$ with only two non-zero indices which can be absolutely arbitrary. To shed a light on this rather particular solution we go back to the $(H_i, \dot{H}_i)$ coordinates (this can be done from the beginning, however, historically this solution have been obtained in power-law ansatz which obscure its real meaning). Hubble parameters, associated with zero power indices are zeros, while those associated with arbitrary power indices appears to be arbitrary functions. All equations of motion nullify\(^1\), making Hubble parameters completely unconstrained. This situation have not been remarked previously, because the Milne solution have been usually obtained after imposing the power-law ansatz, and in its power-law form the values of indices $a$ and $b$ can be fixed by initial conditions. However, we can see easily that setting three Hubble functions to zero in enough for satisfying the equations of motion – two remaining Hubble functions remain unconstrained. We treat this situation as an unphysical one, and we find the meaning behind it as follows: Milne solution is some kind of “artifact” which remains in the system if we neglect lower-order contribution: dealing with pure Gauss-Bonnet gravity we indeed neglect lower-order Einstein-Hilbert contribution. As on the solution under consideration all terms originating from ”dominating” Gauss-Bonnet combination vanish, neglecting Einstein-Gilbert term is obviously incorrect.

The analog of Milne solution in GR is Taub\(^{17}\) solution, it imply $(p_1, p_2, p_3) = (1, 0, 0)$, but it is not a full analog of the Milne solution. One can easily see that the Taub solution is a particular case of the Kasner solution – formally it follows $\sum p_i = \sum p_i^2 = 1$. A ”true” GB analog of generalized Milne solution in GR would be $(p_1, p_2, p_3) = (a, 0, 0)$, but the equations of motion require $a \equiv 1$.

\(^1\) actually the statement is even stronger – “all individual terms in all the equations of motion nullify”
The formal reason is that in the GR case the combination \( (\dot{H}_1 + H_1^2) \) is not a multiplier, so if even all other Hubble vanish, this combination requires \( p_1 = 1 \).

If we consider not only the Gauss-Bonnet contribution, but also the Einstein-Hilbert part, and impose generalized Milne conditions, the Gauss-Bonnet contribution will vanish and the constraint equation for Einstein-Hilbert contribution takes the form \( H_1 H_2 = 0 \) (all other Hubble parameters are already set to zero) which imply one of \( \{H_1, H_2\} \) (say, \( H_2 \)) is also always equal to zero. After that the only term that remains in the dynamical equations is \( (\dot{H}_1 + H_1^2) \) which is also should be equal to zero, which leads to \( p_1 = 1 \) – the Taub solution, mentioned above.

The structure of this reductions is similar for higher-order corrections as well – imposing generalized Milne of the order \( n \) we nullify the \( n \)th order contribution and leaving \( 2n - 2 \) nonzero power indices \( [8] \). If there is only one additional next lower order Lovelock correction, then one additional Hubble parameter is also set to zero (as it happened with the Einstein-Gauss-Bonnet case), however, this is not enough to nullify this next order Lovelock contribution, which gives us well defined equations of motion. If there are more than one additional next lower order Lovelock corrections, then no additional nullification of Hubble parameters occur. For example, in \((7+1)\) with cubic and quadric (Gauss-Bonnet) terms only by imposing third order generalized Milne we set three Hubble parameters to zero, which nullify cubic Lovelock contribution. Remaining four Hubble parameters act as effective \((4+1)\) Gauss-Bonnet model and we have one additional Hubble parameter nullified (the generalized Kasner solution requires one of remaining Hubble parameters to vanish in this case, see above). However, if we have originally in \((7+1)\) the linear Lovelock contribution as well (the Einstein-Hilbert action) then the remaining four Hubble parameters effectively form \((4+1)\) Einstein-Gauss-Bonnet model and no additional nullification occurs. In both cases we obtain well-defined solutions with no arbitrary functions.

IV. CONCLUSIONS

In this paper we considered the dynamics of a flat anisotropic Universe in Lovelock gravity and described two situation in which corresponding behavior in GR is different from any theory with higher order Lovelock terms.

First, we demonstrate that unlike \( N \)-dimensional GR where the space of possible Kasner indices represented by \((N - 2)\)-dimensional sphere, in the general Lovelock theory it is noncompact. It is interesting to link this fact with existence of exponential solutions in the theory under consideration.
and absence of those in GR.

Secondly, we discuss the nature of exceptional solution (which we call as generalized Milne solution here), which does not exist in GR where two Kasner conditions are the only conditions for a power-law solution. On the contrary, in $n$th order Lovelock theory setting large enough number of Hubble parameters to zero results in vanishing of all terms in equations of motion identically, leaving the rest of Hubble functions absolutely unconstrained (we even need not to impose a power-law ansatz here!). It is reasonable to treat this situation as an artifact of neglecting lower-order Lovelock contribution. As such solution does not exist in GR, retaining Einstein-Gilbert contribution destroys it.

V. ACKNOWLEDGMENTS

This work was supported via RFBR grant No. 11-02-00643. S.A.P. was supported by FONDECYT under Project No. 3130599.

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