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ON THE GROWTH OF THE HOMOLOGY OF A FREE LOOP SPACE II

by Yves FÉLIX, Steve HALPERIN & Jean-Claude THOMAS

Abstract. — Controlled exponential growth is a stronger version of exponential growth. We prove that the homology of the free loop space \( L_X \) has controlled exponential growth in two important situations: (1) when \( X \) is a connected sum of manifolds whose rational cohomologies are not monogenic, (2) when the rational homotopy Lie algebra \( L_X \) contains an inert element and \( \rho(L_X) < \rho(L_X/[L_X, L_X]) \), where \( \rho(V) \) denotes the radius of convergence of \( V \).

Résumé. — La croissance exponentielle controlée est une version forte de la croissance exponentielle. Nous prouvons que les nombres de Betti des lacets libres sur un espace \( X \) ont une croissance exponentielle controlée dans deux cas : lorsque \( X \) est la somme connexe de variétés dont la cohomologie n’est pas monogène, et lorsque l’algèbre de Lie \( L_X \) a une croissance exponentielle strictement plus grande que ses indécomposables.

1. Introduction

In this paper we are concerned with the growth of the homology \( H_*(X^{S^1}; \mathbb{Q}) \) of a free loop space on a simply connected space, \( X \).

A graded vector space \( V = V_{\geq 0} \) grows exponentially if there are constants \( 1 < C_1 < C_2 \) such that for some \( N \),

\[
C_1^k \leq \sum_{i \leq k} \dim V_i \leq C_2^k, \quad k \geq N.
\]

In particular, if \( X \) is a simply connected CW complex of finite type and finite Lusternik–Schnirelmann category then [3] either \( \dim \pi_*(X) \otimes \mathbb{Q} < \infty \) (\( X \) is rationally elliptic) or \( \pi_*(X) \otimes \mathbb{Q} \) grows exponentially (\( X \) is rationally hyperbolic). The first examples of elliptic spaces are given by compact homogeneous spaces, but the generic situation is given by hyperbolic spaces.

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For instance if the Euler characteristic $\chi(X) < 0$ then $X$ is hyperbolic (see [4] for other examples of elliptic or hyperbolic spaces)

In [7] Gromov conjectured that $H_\ast(XS^1; \mathbb{Q})$ grows exponentially for almost all cases when $X$ is a closed manifold. This would have an important consequence in Riemannian geometry, due to a theorem of Gromov, improved by Ballmann and Ziller:

**Theorem 1.1** ([7], [2]). — Let $N_g(t)$ denote the number of geometrically distinct closed geodesics of length $\leq t$ on a simply connected closed Riemannian manifold $(M, g)$. Then, for generic metrics $g$, there are constants $K > 0$ and $\beta > 0$ such that for $k$ sufficiently large,

$$N_g(k) \geq K \cdot \max_{t \leq \beta k} \dim H_t(MS^1; \mathbb{Q}).$$

One of the first applications of Sullivan’s minimal models $(\wedge V, d)$ of a space $X$ was the construction [16] (when $X$ is simply connected) of the minimal model $(\wedge W, d)$ of $XS^1$ where $W^k = V^k \oplus V^{k-1}$. Since $X$ is elliptic if and only if $\dim V < \infty$ it follows that in that case $H_\ast(XS^1; \mathbb{Q})$ grows at most polynomially. In [16] Vigué-Poirrier conjectures that in the hyperbolic case, $H_\ast(XS^1; \mathbb{Q})$ should grow exponentially, a conjecture which would give Gromov’s conjecture as a special case.

The Vigué-Poirrier conjecture has been proved for a finite wedge of spheres [16], for a non-trivial connected sum of closed manifolds [11] and in the case $X$ is coformal [12].

For simplicity we write $H(X)$ and $H^\ast(X)$ respectively for the rational homology and cohomology of a space $X$, and denote the free loop space of maps $S^1 \to X$ by $\mathcal{L}X$. If $X$ is simply connected and $\dim \pi_\ast(X) \otimes \mathbb{Q} < \infty$ then it is immediate from Sullivan’s model of $\mathcal{L}X$ [15] that $H(\mathcal{L}X)$ grows at most polynomially. However, even in the case when $X$ is a rationally hyperbolic finite simply connected complex it is not known if $H(\mathcal{L}X)$ grows exponentially.

Next, for a graded vector space $V$ denote by

$$V(z) := \sum_{k \geq 0} \dim V_k z^k$$

the formal Hilbert series of $V$ and denote by $\rho_V$ or $\rho(V)$ the radius of convergence of $V(z)$. If $X$ is a topological space we denote by $X(z)$ and by $\rho_X$ or by $\rho(X)$ the Hilbert series of $H(X)$ and its radius of convergence.

In [5] we introduced a much stronger version of exponential growth: $V$ has **controlled exponential growth** if $0 < \rho_V < 1$ and for each $\lambda > 1$ there
is an infinite sequence \( n_1 < n_2 < \cdots \) such that \( n_{i+1} < \lambda n_i, \ i \geq 1 \), and
\[
\lim_i \frac{\log \dim V_{n_i}}{n_i} = -\log(\rho_V).
\]

As usual, \( \Omega X \) denotes the (based) loop space on a space \( X \). We recall [14] or [4] that if \( X \) is simply connected, then \( H(\Omega X) \) is the universal enveloping algebra of the graded Lie algebra \( L_X = \pi_* (\Omega X) \otimes \mathbb{Q} \); \( L_X \) is called the homotopy Lie algebra of \( X \). According to [5, Lemma 4],
\[(1.1) \quad \rho_{\Omega X} = \rho(L_X).\]
If \( X \) has rational homology of finite type and infinite dimensional rational homotopy, then Sullivan’s model for \( L_X \) gives
\[(1.2) \quad \rho L_X \leq \rho_{\Omega X}.\]

Our objective here is to establish new classes of spaces \( X \) (Theorems 1.3 and 1.4 below) for which \( H(LX) \) has controlled exponential growth and
\[\rho L_X = \rho_{\Omega X}.\]

Our approach is by constructing maps
\[F \to X \xrightarrow{p} Y\]
in which \( F \) is the homotopy fibre of \( p \).

**Theorem 1.2.** — With the above notations if \( F \) is rationally a wedge of spheres, and if \( 0 < \rho_{\Omega F} < \rho_{\Omega Y} \) then \( H(LX) \) has controlled exponential growth and \( \rho L_X = \rho_{\Omega X} \).

**Proof.** — This follows from [5, formula (4)], together with Theorems 1.2 and 1.4. \( \square \)

One method for constructing other maps \( p : X \to Y \) is via inert elements \( \alpha \in L_X \), where \( L_X \) is the homotopy Lie algebra of \( X \). Any \( \alpha \in (L_X)_k \) corresponds up to a scalar multiple to a map \( \sigma : S^{k+1} \to X \) and \( \alpha \) is called inert if the map
\[p : X \to X \cup_\sigma D^{k+2}\]
is surjective in rational homotopy. In Lemma 2.2 we recall the proof that if \( \alpha \) is inert then the homotopy fibre of \( p \) is a wedge of spheres with homology isomorphic to \( H(\Omega(X \cup_\sigma D^{k+2})) \otimes \mathbb{Q} \alpha \). For instance the attaching map of the top cell in a simply connected manifold whose cohomology is not monogenic is inert [8]. (Recall that a graded algebra \( A = \mathbb{Q} \oplus A^{\geq 1} \) is monogenic if it is generated by a single element \( a \in A^{\geq 1} \)). Also, every nonzero element \( \alpha \) in a free Lie algebra generated by elements of even degrees is inert ([8]).
A key condition in our theorems is the hypothesis

\[ \Omega_X(\rho_{\Omega X}) := \lim_{z \to \rho_{\Omega X}} \Omega_X(z) = \infty. \]

There are no examples where this is known to fail if \( X \) is a rationally hyperbolic, finite, simply connected CW complex. In fact (Proposition 2.1) this follows from the condition

\[ \rho(L_X) < \rho \left( \frac{L_X}{[L_X, L_X]} \right), \]

which is not known to fail for such \( X \). When \( \dim L_X / [L_X, L_X] < \infty \), Proposition 2.1 follows from a result of Anick [1].

With this preamble we can state our two theorems:

**Theorem 1.3.** — Suppose \( X \) is a simply connected CW complex with rational homology of finite type. If \( L_X \) contains an inert element \( \gamma \) and if \( \rho(L_X) < \rho(L_X / [L_X, L_X]) \) then \( H(L_X) \) has controlled exponential growth and \( \rho_{L_X} = \rho_{\Omega X} \).

**Theorem 1.4.** — Suppose \( M \# N \) is the connected sum of two closed simply connected \( n \)-manifolds with \( H^*(N) \) not monogenic and \( M \) not rationally a sphere. If \( \rho_{\Omega N} \leq \rho_{\Omega M} \) and if \( \Omega_N(\rho_{\Omega N}) = \infty \) then \( H(L(M \# N)) \) has controlled exponential growth and \( \rho_{L(M \# N)} = \rho_{\Omega(M \# N)} \).

**Remarks 1.5.**

1. Theorem 1.3 is proved in [5] under the considerably stronger hypothesis that \( \dim L_X / [L_X, L_X] < \infty \).
2. If \( H^*(M) \) and \( H^*(N) \) are monogenic, but of dimension > 2 then \( M \# N \) is elliptic and so \( H(L(M \# N)) \) grows at most polynomially.
3. Theorem 1.4 strengthens a result of Lambrechts [10], which asserts that \( H(L(M \# N)) \) grows exponentially unless both \( H^*(M) \) and \( H^*(N) \) are monogenic.

**2. Proposition 2.1 and Theorem 1.3**

Suppose \( A = \mathbb{Q}1 \oplus A_{\geq 1} \) is a finitely generated graded algebra satisfying \( \rho_A < 1 \). Then it follows from a result of Anick [1] that

\[ A(\rho_A) = \infty. \]

We generalize this with
Proposition 2.1. — Let $L = L_{\geq 1}$ be a graded Lie algebra of finite type such that $0 < \rho_{UL} < 1$. If $L$ is generated by a subspace $V$ with $\rho_{UL} < \rho_V$ then $UL(\rho_{UL}) = \infty$.

Proof. — We assume $UL(\rho_{UL}) < \infty$, and deduce a contradiction. By Anick's result we have $\dim V = \infty$. Choose some $\sigma$ with $\rho_{UL} < \sigma < \rho_V$. Then $V(\sigma) < \infty$ and so $V_{\geq r}(\sigma) \to 0$ as $r \to \infty$. In particular, we may choose $r$ so that $UL(\rho_{UL}) \cdot V_{\geq r}(\sigma) < 1$.

Now let $E$ be the sub Lie algebra generated by $V_{< r}$ and note that by Anick's result, $E \neq L$. In particular, $UE(\rho_{UL}) < UL(\rho_{UL})$. Clearly $\rho_{UE} \geq \rho_{UL}$. If $\rho_{UE} = \rho_{UL}$, then $0 < \rho_{UE} < 1$. Then by Anick's result $UE(\rho_{UE}) = \infty$, and $UL(\rho_{UL}) = \infty$. It follows that $\rho_{UE} > \rho_{UL}$. Thus for some $\tau$ with $\rho_{UL} < \tau < \rho_{UE}$ we have $UE(\tau) < UL(\rho_{UL})$.

Choose $\rho$ so that $\rho_{UL} < \rho < \tau$ and $\rho < \sigma$. Then $UE(\rho) \cdot V_{\geq r}(\rho) < UE(\tau) \cdot V_{\geq r}(\tau) < UL(\rho_{UL}) \cdot V_{\geq r}(\sigma) < 1$.

Now let $W = UE \circ V_{\geq r}$ where “$\circ$” denotes the adjoint action and note that $W(\rho) < 1$. Then, let $I$ be the sub Lie algebra generated by $W$. The inclusion of $W$ in $I$ extends to a surjection $TW \to UI$. Since $(TW)(\rho) = \frac{1}{1-W(\rho)} < \infty$, it follows that $\rho_{UI} \geq \rho_{TW} \geq \rho > \rho_{UL}$.

On the other hand, since $W \supset V_{\geq r}$ and $[E, W] \subset W$, it follows that $I$ is an ideal in $L$. The surjection $L \to L/I$ kills $V_{\geq r}$, and so it restricts to a surjection $E \to L/I$. Thus $\rho_{U(L/I)} \geq \rho_{UE} > \rho_{UL}$. But as graded vector spaces $UL \cong UI \otimes U(L/I)$ and so $\rho_{UL} = \min\{\rho_{UI}, \rho_{U(L/I)}\}$.

This is the desired contradiction because $\rho_{UL} < \rho_{UI}$ and $\rho_{UL} < \rho_{U(L/I)}$. \hfill \Box

We also require the following lemma announced in the Introduction, and which is essentially proved, if not stated, in [8].

Lemma 2.2. — Let $X$ be a simply connected CW complex that is not rationally a sphere. If $\alpha \in (L_X)_k$ is an inert element corresponding to $\sigma : S^{k+1} \to X$, then

1. The homotopy fibre $i : F \to X$ of $p : X \to X \cup_{\sigma} D^{k+2} = Y$ is rationally a wedge of spheres.

2. $H(\Omega i)$ restricts to an isomorphism $L_F \cong I$, where $I \subset L_X$ is the ideal generated by $\alpha$. 

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(3) $I$ is a free Lie algebra and $I/[I, I] \cong U(L_X/I) \otimes \mathbb{Q}\alpha$.

(4) $H_\ast(\Omega p)$ induces an isomorphism $U(L_X/I) \cong H_\ast(\Omega Y)$.

Proof. — Since $\alpha$ is inert $\pi_\ast(p) \otimes \mathbb{Q}$ is surjective. Thus $\pi_\ast(\Omega p) \otimes \mathbb{Q}$ is surjective and

$$
\pi_\ast(\Omega i) \otimes \mathbb{Q} : L_F = \pi_\ast(\Omega F) \otimes \mathbb{Q} \cong \ker \pi_\ast(\Omega p) \otimes \mathbb{Q}.
$$

Moreover, it follows from [8, Theorem 1.1], that $L_F = I$, and so $H_\ast(\Omega p) \otimes \mathbb{Q}$ induces an isomorphism $U(L_X/I) \cong H_\ast(\Omega Y)$. Theorem 1.1 of [8] also asserts that $I$ is a free Lie algebra, and that

$$I/[I, I] \cong U(L_X/I) \otimes \mathbb{Q}\alpha.$$

It remains to show that $F$ is rationally a wedge of spheres. Let $\sigma_i : S^{n_i} \to F$ corresponding to elements $\alpha_i \in L_F$ which represent a basis of $I/[I, I]$. Then the map

$$
\varphi = \vee_i \sigma_i : \vee S^{n_i} \to F
$$

induces a map $\Omega \varphi : \Omega(\vee S^{n_i}) \to \Omega F$ and $\pi_\ast(\Omega \varphi) \otimes \mathbb{Q}$ is a morphism between free Lie algebras inducing an isomorphism $I/[I, I] \cong L_F/[L_F, L_F]$. Thus $\pi_\ast(\Omega \varphi) \otimes \mathbb{Q}$ is an isomorphism and $\varphi$ is a rational homotopy equivalence. □

Proof of Theorem 1.3. — Denote $L_X$ simply by $L$, let $\alpha \in L_k$ be the inert element corresponding to $\sigma : S^{k+1} \to X$, and let $p : X \to X \cup_\sigma D^{k+2}$ be the map considered in Lemma 1. Then by Lemma 1, with $I$ the ideal generated by $\alpha$ and $V = I/[I, I]$, we have isomorphisms

$$H_\ast(\Omega F) \cong UI \cong TV \quad \text{and} \quad H(\Omega(X \cup_\sigma D^{k+2})) \cong U(L/I).$$

Thus, as observed in the Introduction, Theorem 1.3 will be established once we prove

$$\rho_{UI} < \rho_{U(L/I)}.$$  

Clearly $\rho_{UL} \leq \rho_{U(L/I)}$ and if $\rho_{UL} < \rho_{U(L/I)}$ then $\rho_{UI} < \rho_{U(L/I)}$ since $UL \cong UI \otimes U(L/I)$. It remains to consider the case that $\rho_{UL} = \rho_{U(L/I)}$. Since $UI \cong TV$ and since $\dim V \geq 2$ it follows that $\rho_{UL} \leq \rho_{UI} < 1$. Since $L/[L, L]$ maps surjectively to $(L/I)/[L/I, L/I]$, we obtain

$$\rho_{U(L/I)} = \rho_{UL} < \rho_{L/[L, L]} \leq \rho_{(L/I)/[L/I, L/I]}.$$ 

Thus by Proposition 2.1,

$$U(L/I)(\rho_{U(L/I)}) = \infty.$$ 

On the other hand, $UI \cong TV$ with $V \cong U(L/I) \otimes \mathbb{Q}\alpha$. Thus

$$UI(z) = \frac{1}{1 - z^k U(L/I)(z)}.$$
Since $\lim_{z \to \rho(U(L/I))} U(L/I)(z) = \infty$, it follows that $r^k U(L/I)(r) = 1$ for some $r < \rho U(L/I)$. But then $r = \rho U I$ and so again $\rho U I < \rho(U(L/I))$. \(\square\)

3. Connected sums

The objective of this section is to prove Theorem 1.4, and we shall frequently rely on the acyclic closure [6] of a cdga, $(A, d)$ in which $A^0 = \mathbb{Q}$ and $H^1(A) = 0$. This is a cdga of the form $(A \otimes \wedge U, d)$ containing $(A, d)$ as a sub cdga, where the quotient $(\wedge U, d)$ is a minimal Sullivan algebra, and such that $H(A \otimes \wedge U, d) = \mathbb{Q}$. The acyclic closure is determined up to isomorphism ([6, Theorem 3.2]).

For the proof of Theorem 1.4 we establish a preliminary proposition to deal with the case that $H^\ast(M)$ is monogenic and $H^\ast(N)$ is not. Recall that a model for a space $X$ is a connected commutative graded differential algebra whose minimal Sullivan model is also a minimal Sullivan model for the rational polynomial differential forms on $X$ ([15], [4]).

Let $(A, d)$ and $(B, d)$ be finite dimensional models for the closed $n$-manifolds $M$ and $N$ of Theorem 1.4. We may suppose $A^0 = B^0 = \mathbb{Q}$, $A^1 = B^1 = 0$, $A^{>n} = B^{>n} = 0$, $A^n = \mathbb{Q} \alpha$ and $B^n = \mathbb{Q} \beta$.

**Lemma 3.1.** — A model for the connected sum $M \# N$ is given the cdga

$$(A \oplus \mathbb{Q} B) \oplus \mathbb{Q} w, d)$$

with $dw = \alpha - \beta$ and $w \cdot A^+ = w \cdot B^+ = 0$.

**Proof.** — By [4, §12], the cdga $A \oplus \mathbb{Q} B$ is a model for the wedge $M \lor N$. Denote by $p : M \# N \to M \lor N$ the pinch map and $(\wedge X, d)$ a Sullivan minimal model for $M \lor N$. Since $H^{<n}(p)$ is an isomorphism and $H^n(p)$ simply identifies the classes $\alpha$ and $\beta$, a model of $p$ is given by the inclusion $(\wedge X, d) \to (\wedge X \otimes \wedge u \otimes \wedge Z, d)$ where $du = \alpha - \beta$ with $[\alpha]$ and $[\beta]$ the fundamental classes of $M$ and $N$, and where $Z = \mathbb{Z}^{<n-1}$ is introduced to kill recursively all new cohomology classes. We then have clearly a commutative diagram, where the vertical maps are quasi-isomorphisms

$$
\begin{array}{ccc}
(\wedge X, d) & \xrightarrow{\varphi} & (\wedge Y, d) \\
\downarrow \cong & & \downarrow \cong \\
A \oplus \mathbb{Q} B & \longrightarrow & (A \oplus \mathbb{Q} B) \oplus \mathbb{Q} w, d).
\end{array}
$$

Now consider the case that $H^\ast(M)$ is monogenic. Then $H^\ast(M) = \wedge a/a^{n+1}$, where $\deg a = 2p$, $n = 2pk$, and $k \geq 2$ because $M$ is not rationally
a sphere. In this case \((\wedge a / a^{n+1}, 0)\) is a model for \(M\) and we choose as model 
\((B, d)\) for \(N\) a quotient of the minimal Sullivan model such that \(B^{> n} = 0\) and \(B^n = \mathbb{Q}\beta\). Then \(a\) represents a cohomology class in \(H^{2p}(M \# N)\) and hence determines a map \(p : M \# N \to K(2p, \mathbb{Q})\) with homotopy fibre \(F\).

**Proposition 3.2.** — The homotopy fibre \(F\) has a model of the form 
\[
(C, d) = (B / \beta, d) \oplus (B^{\geq 1}, d) \otimes \mathbb{Q}a
\]
where \(\deg a = 2p - 1\), \((B / \beta, d)\) is the quotient cdga of \((B, d)\) acting by multiplication on the left on \((B^{\geq 1}, d) \otimes \mathbb{Q}a\), and \((B^{\geq 1} \otimes \mathbb{Q}a)(B^{\geq 1} \otimes \mathbb{Q}a) = 0\).

**Proof.** — As observed above, a model for \(M \# N\) is given by \(((\wedge a / a^{k+1} \times \mathbb{Q}B) \oplus \mathbb{Q}w, d)\) with \(dw = a^k - \beta\). Now a quasi-isomorphism
\[
((\wedge a \otimes \wedge w) \times \mathbb{Q}B, d) \xrightarrow{\sim} (\wedge a / a^{k+1} \times \mathbb{Q}B) \oplus \mathbb{Q}w
\]
is given by dividing by the elements \(a^q\) and \(a^r w\), \(q \geq k + 1\) and \(r \geq 1\); here on the left \(dw = a^k - \beta\). (This follows by filtering by the degree in \(B\).)

Thus it follows from Theorem 15.3 in [4] or Theorem 5.1 in [6] that the Sullivan fibre of the morphism \(\wedge a \to ((\wedge a \otimes \wedge w) \times \mathbb{Q}B)\) is a model for \(F\). Let \((\wedge a \otimes \wedge \bar{a}, d \bar{a} = a)\) be the acyclic closure of \((\wedge a, 0)\). Then this Sullivan fibre is given by \(((\wedge a \otimes \wedge w) \times \mathbb{Q}B) \otimes_{\wedge a} (\wedge a \otimes \wedge \bar{a})\). Hence
\[
(\wedge a \otimes \wedge w \otimes \wedge \bar{a}) \oplus (B^{\geq 1} \otimes \wedge \bar{a}) = (\wedge a \otimes \wedge w \otimes \wedge \bar{a}) \times_{\wedge \bar{a}} (B \otimes \wedge \bar{a})
\]
is also a model for \(F\).

Next note that \(I = (\wedge^{\geq 2} a \oplus \wedge^{\geq 1} a \cdot \bar{a}) \otimes \wedge w \subset (\wedge a \otimes \wedge w \otimes \wedge \bar{a}) \oplus (B^{\geq 1} \otimes \wedge \bar{a})\) is an ideal preserved by \(d\), and that \(H(I, d) = 0\). Thus division by \(I\) produces another model for \(F\), given explicitly by
\[
(\mathbb{Q}(1 \oplus a + \bar{a}) \otimes \wedge w) \oplus (B^{\geq 1} \otimes \wedge \bar{a})
\]
with \(a^2 = a\bar{a} = \bar{a}^2 = 0\), \(d \bar{a} = a\) and, since \(k \geq 2\), \(dw = -\beta\). In this cdga, \(d(\bar{a}w) = aw + \bar{a}\beta\). Moreover, the subspace spanned by \(\bar{a}w\) and \(aw + \bar{a}\beta\) is an ideal. Thus a quasi-isomorphism
\[
(\mathbb{Q}(1 \oplus a + \bar{a}) \otimes \wedge w) \oplus (B^{\geq 1} \otimes \wedge \bar{a}) \to \mathbb{Q}(1 \oplus a \oplus \bar{a} \oplus w) \oplus (B^{\geq 1} \otimes \wedge \bar{a})
\]
is given by \(\bar{a}w \mapsto 0\) and \(aw \mapsto -\bar{a}\beta\).

Now the inclusion \(\mathbb{Q} \oplus \mathbb{Q}w \oplus (B^{\geq 1} \otimes \wedge \bar{a})\) in \(\mathbb{Q}(1 \oplus a \oplus \bar{a} \oplus w) \oplus (B^{\geq 1} \otimes \wedge \bar{a})\) is clearly a quasi-isomorphism. Since \(dw = -\beta\), division by \(w\) and \(\beta\) then gives a quasi-isomorphism
\[
\mathbb{Q} \oplus \mathbb{Q}w \oplus (B^{\geq 1} \otimes \wedge \bar{a}) \xrightarrow{\sim} B / \beta \oplus (B^{\geq 1} \otimes \mathbb{Q}a).
\]
(Note that in the left hand cdga $\beta \otimes \overline{\alpha}$ is not the product of $\beta$ and $\overline{\alpha}$, since $\overline{\alpha}$ is not an element in the cdga!).

Proof. — We consider separately the cases that $H^*(M)$ is monogenic and $H^*(N)$ is not, and that neither $H^*(M)$ nor $H^*(N)$ are monogenic. Note that since $M$ and $N$ are simply connected, and $N$ is not a rational sphere, $n \geq 4$.

Case 1: $H^*(M)$ is monogenic. — We adopt the notation of Proposition 3.2, and for simplicity denote $- \otimes \mathbb{Q}\overline{\alpha}$ simply by $- \otimes \overline{\alpha}$. It is immediate from Theorem 3 and (4) in [5] that it is sufficient to prove that $H(LF)$ has controlled exponential growth and that $\rho_{LF} = \rho_{\Omega F}$. Let $(\Lambda W, d) \to (B/\beta, d)$ be a minimal Sullivan model, and extend this to a Sullivan model $(\Lambda W \otimes \Lambda Z, d) \to (C, d)$. By Proposition 3.2, $(\Lambda W \otimes \Lambda Z, d)$ is a Sullivan model for $F$. Now, letting $(\Lambda W \otimes \Lambda U, d)$ be the acyclic closure of $(\Lambda W, d)$, we have for the Sullivan fibre $(\Lambda Z, d)$ that

\[(\Lambda Z, d) \simeq (\Lambda W \otimes \Lambda Z \otimes \Lambda W \otimes \Lambda U, d) = (\Lambda W \otimes \Lambda Z \otimes \Lambda U, d) \to (B/\beta \oplus (B^{\geq 1} \otimes \overline{\alpha}) \otimes \Lambda U, d) \to \mathbb{Q} \oplus (B^{\geq 1} \otimes \overline{\alpha} \otimes \Lambda U, d).
\]

Since products in $(B^{\geq 1} \otimes \overline{\alpha})$ are zero it follows that $(\Lambda Z, d)$ is the minimal Sullivan model of a wedge of spheres with cohomology $\mathbb{Q} \oplus H(B^{\geq 1} \otimes \overline{\alpha} \otimes \Lambda U, d)$.

Thus in this case Theorem 1.4 will follow from the Sullivan model version of Theorem 3 and (4) in [5] once we show that the Sullivan acyclic closure $(\Lambda Z \otimes \Lambda S, d)$ of $(\Lambda Z, d)$ satisfies

\[(3.1) \quad \rho_{\Lambda S} < \rho_{\Lambda U}.
\]

Denote $H(B^{\geq 1} \otimes \overline{\alpha} \otimes \Lambda U)$ simply by $H$. Since $(\Lambda Z, d)$ is the model of a wedge of spheres, it follows that $\Lambda S$ is the dual of a tensor algebra $TE$ with $E_i \simeq H^{i+1}$. Thus

\[(3.2) \quad \Lambda S(z) = \frac{1}{1 - E(z)} = \frac{1}{1 - \frac{1}{z}H(z)}.
\]

It remains to estimate $H(z)$.

For this recall that the morphism $B \to B/\beta$ corresponds to the inclusion

\[N - D^n \to (N - D^n) \cup_{S^{n-1}} D^n,
\]

where $S^{n-1}$ is the boundary of a small disk $D^n \subset N$. Since $H(N)$ is not monogenic Theorem 5.1 of [8] asserts that the sphere $S^{n-1}$ corresponds
to an inert element in the homotopy Lie algebra of $N - D^n$. Thus by [8, Theorem 1.1],

$$H(\Omega(N - D^n)) \cong TV \otimes H(\Omega N)$$

where $V \cong H(\Omega N) \otimes v$ and $\deg v = n - 2$. Since $V(z) = z^{n-2}\Omega N(z)$ it follows that $\rho_V = \rho_{\Omega N}$ and that $V(\rho_V) = \infty$. Since

$$TV(z) = \frac{1}{1 - V(z)}$$

it follows that $\rho_{TV} < \rho_V$ and that $TV(\rho_{TV}) = \infty$.

Moreover, the minimal Sullivan model $(\wedge W, d)$ of $B/\beta$ has the form $(\wedge W_N \otimes \wedge P, d)$ in which $\wedge W_N$ is the minimal Sullivan model of $N$. Thus the acyclic closure $(\wedge W \otimes \wedge U, d)$ has the form

$$(\wedge W_N \otimes \wedge U_N \otimes \wedge P \otimes \wedge U_P, d)$$

in which $(\wedge W_N \otimes \wedge U_N, d)$ is the acyclic closure of $(\wedge W_N, d)$. In particular, $\wedge U \cong \wedge U_N \otimes \wedge U_P$, and there are linear isomorphisms

$$(3.3) \quad \wedge U_N \cong H^*(\Omega N) \quad \text{and} \quad \wedge U_P \cong TV^#,$$

$V^#$ denoting the dual of $V$. Thus

$$\rho_{\wedge U_P} = \rho_{TV} < \rho_V = \rho_{\wedge U_N}.$$ 

Since $\wedge U = \wedge U_N \otimes \wedge U_P$, it follows that

$$\rho_{\wedge U} = \rho(\wedge U_N \otimes \wedge U_P) = \rho_{\wedge U_P},$$

and that $\wedge U(\rho_{\wedge U}) = \infty$.

Now consider the short exact sequence

$$0 \rightarrow (B^{\geq 1} \otimes \bar{a} \otimes \wedge U, d) \rightarrow (B \otimes \bar{a} \otimes \wedge U, d) \rightarrow (\bar{a} \otimes \wedge U, 0) \rightarrow 0.$$ 

Since $(B \otimes \bar{a} \otimes \wedge U, d) = (B \otimes \bar{a} \otimes \wedge U_N \otimes \wedge U_P, d)$ it follows that

$$H = H(B \otimes \bar{a} \otimes \wedge U, d) \cong \bar{a} \otimes \wedge U_P.$$ 

It follows that $H(B^{\geq 1} \otimes \bar{a} \otimes \wedge U, d)$ contains a subspace $T$ with

$$T^{i + \deg \bar{a} + 1} \cong (\wedge^{\geq 1} U_N \otimes \wedge U_P)^i.$$ 

In particular, with $\gg$ denoting coefficient-wise inequality, we have

$$E(z) \gg z^{\deg \bar{a}} \cdot (\wedge^{\geq 1} U_N)(z) \cdot (\wedge U_P)(z).$$ 

Thus $\rho_E \leq \rho_{\wedge U}$ and if $\rho_E = \rho_{\wedge U}$, then $E(\rho_E) = \infty$. Since

$$\wedge S(z) = \frac{1}{1 - E(z)}$$

it follows in either case that $\rho_{\wedge S} < \rho_{\wedge U}$, which completes the proof of Theorem 1.4 in this case.
Case 2: Neither $H(M)$ nor $H(N)$ is monogenic. — In this case Theorem 5.4 of [8] asserts that the collar sphere $S^{n-1}$ joining $M - \{pt\}$ to $N - \{pt\}$ represents an inert element in $L_{M \# N}$. Attaching a disk to this sphere gives $M \vee N$ and thus by Theorem 1.1 in [8] the homotopy fibre $F$ of the map $p : M \# N \to M \vee N$ is rationally a wedge of spheres with

$$H_i(F) \cong H_{i-n+2}(\Omega(M \vee N)).$$

Thus

$$H(\Omega F) = TV \quad \text{and} \quad V_i \cong H_{i-n+2}(\Omega(M \vee N)),$$

and so

$$\Omega F(z) = \frac{1}{1 - z^{n-2}(\Omega(M \vee N))(z)}.$$

On the other hand it is a classical fact that the homotopy fibre $G$ of the map $q : M \vee N \to M \times N$ is the join $\Omega M * \Omega N$, (we sketch the proof in Lemma 3.3 below). Thus $G$ is the suspension of $\Omega M \wedge \Omega N$ and therefore rationally a wedge of spheres. Since $\pi_*(q)$ is trivially surjective. It follows that

$$H(\Omega G) = TW \quad \text{with} \quad W_i \cong H_{i-1}(\Omega M * \Omega N).$$

By hypothesis, $\rho_{\Omega N} \leq \rho_{\Omega M}$ and $\rho_{\Omega N}(\Omega N) = \infty$. In particular, $W(\rho_{\Omega N}) = \infty$ and, since $\Omega G(z) = \frac{1}{1 - W(z)}$, it follows that the radius of convergence, $\rho$, of $\Omega G(z)$ satisfies

$$\rho < \rho_{\Omega N} \leq \rho_{\Omega M} \quad \text{and} \quad W(\rho) = 1.$$

Moreover, since $\pi_*(q)$ is surjective,

$$H(\Omega(M \vee N)) = H(\Omega G) \otimes H(\Omega M) \otimes H(\Omega N)$$

and so $\rho$ is also the radius of convergence of $\Omega(M \vee N)(z)$ and

$$\Omega(M \vee N)(\rho) = \infty.$$

Finally, since

$$\Omega F(z) = \frac{1}{1 - z^{n-2}(\Omega(M \vee N))(z)}$$

it follows that $\rho_{\Omega F} < \rho = \rho_{\Omega(M \vee N)}$ and Theorem 1.4 follows from Theorem 1, Theorem 3 and (4) in [5].

Lemma 3.3. — The homotopy fiber $G$ of the injection $q : M \vee N \to M \times N$ has the homotopy type of $\Omega M * \Omega N$.

Proof. — Recall the Cube Lemma ([13]): In a homotopy commutative cube, if the vertical faces are homotopy pullbacks and the lower face an homotopy push-out, then the upper face is also an homotopy push-out.
Let $j : G \to M \vee N$ be the homotopy fibre of the inclusion $q$. Then we form the following cube by taking the pullbacks of $j$ along the injections $M \to M \vee N$ and $N \to M \vee N$.

\[
\begin{array}{ccc}
\Omega N \times \Omega M & \to & \Omega M \\
\downarrow & & \downarrow \\
\Omega N & \to & G \\
\downarrow & & \downarrow \\
\{\ast\} & \to & N \\
\downarrow & & \downarrow \\
M & \to & M \vee N \\
\end{array}
\]

This shows that $G \cong \Omega M \ast \Omega N$. □

BIBLIOGRAPHY

[1] D. J. Anick, “The smallest singularity of a Hilbert series”, *Math. Scand* 51 (1982), p. 35-44.

[2] W. Ballmann & W. Ziller, “On the number of closed geodesics on a compact riemannian manifold”, *Duke Math. J.* 49 (1982), p. 629-632.

[3] Y. Félix, S. Halperin & T. Jean-Claude, “The homotopy Lie algebra for finite complexes”, *Publ. Math., Inst. Hautes Étud. Sci.* 56 (1983), p. 387-410.

[4] ———, *Rational Homotopy Theory*, Graduate Texts in Mathematics, vol. 205, Springer, 2001, xxxii+535 pages.

[5] ———, “On the growth of the homology of a free loop space”, *Pure Appl. Math. Q.* 9 (2013), no. 1, p. 167-187.

[6] ———, *Rational Homotopy II*, World Scientific, 2015, xxxvi+412 pages.

[7] M. Gromov, “Homotopical effects of dilatations”, *J. Differ. Geom.* 13 (1978), p. 303-310.

[8] S. Halperin & L. Jean-Michel, “Suites inertes dans les algèbres de Lie graduées”, *Math. Scand.* 61 (1987), p. 39-67.

[9] S. Halperin & G. Levin, “High skeleta of CW complexes”, in *Algebra, Algebraic Topology and their Interactions (Stockholm 1983)*, Lecture Notes in Math., vol. 1183, Springer, 1986, p. 211-217.

[10] P. Lambrechts, “Analytic properties of Poincaré series of spaces”, *Topology* 37 (1998), p. 1363-1370.

[11] ———, “The Betti numbers of the free loop space of a connected sum”, *J. Lond. Math. Soc.* 64 (2001), p. 205-228.

[12] ———, “On the Betti numbers of the free loop space of a coformal space”, *J. Pure Appl. Algebra* 161 (2001), p. 177-192.

[13] M. Mather, “Pull-backs in homotopy theory”, *Can. J. Math.* 28 (1976), p. 225-263.

[14] J. W. Milnor & J. Moore, “On the structure of Hopf algebras”, *Ann. Math.* 81 (1965), p. 211-264.
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[15] D. Sullivan, “Infinitesimal Computations in Topology”, Publ. Math., Inst. Hautes Étud. Sci. 47 (1977), p. 269-331.

[16] M. Vigué-Poirrier, “Homotopie rationnelle et nombre de géodésiques fermées”, Ann. Sci. Éc. Norm. Supér. 17 (1984), p. 413-431.

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