Coulomb Law in the Non-Uniform Euler-Heisenberg Theory

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Abstract

We consider the non-linear classical field theory which results from adding to the Maxwell’s Lagrangian the contributions from the weak-field Euler-Heisenberg Lagrangian and a non-uniform part which involves derivatives of the electric and magnetic fields. We focus on the electrostatic case where the magnetic field is set to zero, and we derive the modified Gauss law, resulting in a higher order differential equation. This equation gives the electric field produced by stationary charges in the higher order non-linear electrodynamics. Specializing for the case of a point charge, we investigate the solutions of the modified Gauss law and calculate the correction to the Coulomb law.

1 Introduction

The search for new forces of short and long range or for new contributions to established well known results has enjoyed over the years a continued effort [1, 2, 3]. Most of the paradigms have been suggested and derived within quantum field theory and the famous ones like the Casimir or van der Waals force have changed our understanding of quantum vacuum [4].

The established Coulomb law and the Newtonian gravitational potential are, of course, known to be two important pillars of physics. The quantum corrections to these results are therefore of utmost relevance as they can have observable consequences. In the case of the Coulomb law these corrections are mostly, but as discussed below not exclusively, derived by using the Fourier transforms of Feynman amplitudes for elastic scattering giving rise to vacuum polarization corrections (the Uehling-Serber potential) [5], fine and hyperfine...
structure and finite size corrections \[6, 7\]. Its physical relevance gets revealed in
precision atomic physics and even in nuclear tunneling problems \[8\]. In the case
of gravity the procedure is similar, but uses effective field theory approach in
view of the lack of a quantum gravity theory. Here corrections proportional to \(\hbar\)
have been calculated and confirmed \[9, 10\]. They have applications in quantum
corrections to the Schwarzschild metric and cosmology \[11, 12\].

Apart from the Fourier transform of the amplitude as a tool to calculate new
potentials from quantum field theory (for yet different examples see \[13, 14\]),
there exists another source of corrections in the case of the electromagnetic inter-
action. The light-light scattering (recently observed directly \[15\]) gives rise to a
quantum corrected Lagrangian known as the Euler-Heisenberg theory (see, e.g.,
\[5\]). The resulting theory is, in nature, a non-linear modification to Maxwell’s
equations subjected to restrictions on the field strength, i.e. \(E \ll m^2/e\), (partly
to avoid pair production) and its variation, \(|\nabla E|/m \ll E\), (partly due to mainly
considering uniform fields). Since in the usual treatment derivatives of the field
are ignored, we will call this case the uniform Euler-Heisenberg theory. To this,
corrections to the electromagnetic Lagrangian involving higher derivatives have
been found in \[16, 18\] and have been used in \[19, 20, 21, 22\]. We refer to these
contributions as the non-uniform part of the theory. Apart from dynamical is-
sues \[23, 24, 25, 26, 27\], the static case of an electric potential deserves a special
attention. In the purely uniform Euler-Heisenberg theory the resulting equation
for the electric field produced by a static charge distribution reduces to a third
order polynomial \[5, 28, 29\], and can be extended to higher order polynomials
if we include higher order uniform corrections \[30\]. As shown below, adding the
non-uniform part to the theory at first order gives a non-linear differential equa-
tion which reduces to the polynomial one if higher order derivatives are switched
off. We give the solutions for the asymptotic case where the restrictions on the
theory are satisfied.

2 The Euler-Heisenberg Lagrangian and field equa-
tions

The Euler-Heisenberg theory of non-linear electrodynamics derived from quan-
tum corrections to the Maxwell theory has been with us for more than ninety
years \[31\]. Many consequences of this theory have been carefully investigated
(see for example Ref \[32\] and \[33\] for reviews and a large list of references), but
in the majority of the cases they focus in the modifications to the electromagnetic
Lagrangian given by powers of \(F_{\mu\nu}F^{\mu\nu}\) and its dual form \(F_{\mu\nu}F^{\mu\nu}\) while
the terms involving derivatives of \(F_{\mu\nu}\) are ignored. The above is justified since
in many of the cases studied the uniform field approximation holds to a good
degree (see \[23\] for a quantification of this statement). However, such derivative
terms, called here the non-uniform Euler-Heisenberg part, exist \[16, 23\], and
their consequences must be studied.

The full quantum mechanical corrections to the Maxwell Lagrangian are
given to the lowest order by

\[ L = L_0 + L_{EH} + L_{int} + b \left\{ (\partial_\alpha F^\alpha_\beta) (\partial_\nu F^{\nu\beta}) + F_{\alpha\beta} \left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) F^{\alpha\beta} \right\}, \quad (1) \]

where

\[ L_0 = \frac{E^2 - B^2}{8\pi} \quad (2) \]

is the Maxwell Lagrangian,

\[ L_{EH} = a \left[ (E^2 - B^2)^2 + 7(E \cdot B)^2 \right] \quad (3) \]

is the Euler-Kochel Lagrangian [17] (to which we will refer in the text as the uniform Euler-Hesienberg theory) and

\[ L_{int} = A^\mu j_\mu \quad (4) \]

is the matter-field interaction term. For the constants we have \( a = e^4/(360\pi^2 m^4) \) and \( b = e^2/360\pi m^2 \) (\( m \) being the electron mass and \( e \) the charge). The non-uniform part seems to be less suppressed with a lower power of \( e/m \) which makes it interesting to search for phenomenological consequences. In Eq. (1) the uniform and the non-uniform contributions to the effective Lagrangian are taken at the lowest possible order. Higher order terms exist [18] and, in principle, could also be considered. However, such terms are suppressed by coefficients of higher powers in both \( \alpha \) and \( 1/m \). Our objective in this work is to derive the leading order asymptotic corrections to the Coulomb law. We will argue later that for this purpose the Lagrangian (1) is sufficient, hence, we will not take terms of higher order into account.

The equations of motion which come from the variation of Lagrangian (1) give corrections to the classical Gauss and Ampere-Maxwell Laws. The Magnetic Gauss and Faraday’s laws do not proceed from any Lagrangian, therefore, they remain the same in any non-linear version of electrodynamics. As a consequence, the standard relation between fields and potentials remains the same, a fact that will be used in the rest of this work.

Specializing for the purely electrostatic case (\( B = 0 \) and time independent potentials), we have \( L_0 = \frac{1}{8\pi} E^2, L_{EH} = aE^4, L_{int} = -\rho\phi \) and

\[ (\partial_\alpha F^\alpha_\beta) (\partial_\nu F^{\nu\beta}) = (\nabla \cdot E)^2, \quad (5) \]

\[ -F_{\alpha\beta} \nabla^2 F^{\alpha\beta} = 2E \cdot (\nabla^2 E). \quad (6) \]

Therefore, the Lagrangian (1) simplifies to

\[ L = \frac{1}{8\pi} E^2 + aE^4 + b \left\{ (\nabla \cdot E)^2 + 2E \cdot (\nabla^2 E) \right\} - \rho\phi. \quad (7) \]

Since in the electrostatic case we have \( E = -\nabla \phi \), the Lagrangian under discussion contains higher order derivatives of \( \phi \). To obtain the equations of
motion we cannot simply take the standard Euler-Lagrange equations to first order. The classical field theory of Lagrangians with higher derivatives can be found in [34, 35]. Using the abbreviations \( \partial_{ij} = \partial_i \partial_k \) and \( \partial_{ijk} = \partial_i \partial_j \partial_k \) the version of the Euler-Lagrange equations we will need for our purpose is

\[
\frac{\partial L}{\partial \phi} - \partial_i \left( \frac{\partial L}{\partial (\partial_i \phi)} \right) + \partial_{ij} \left( \frac{\partial L}{\partial (\partial_i \partial_j \phi)} \right) - \partial_{ijk} \left( \frac{\partial L}{\partial (\partial_i \partial_j \partial_k \phi)} \right) = 0, \tag{8}
\]

where we have used the Einstein’s summation convention and the indices \( i, j \) and \( k \) run from 1 to 3. An important assumption we need for the formula (8) to be valid is that \( \phi \) has continuous derivatives at least up to third order so that the Clairut theorem holds and the order of the derivatives can be interchanged.

Having the Lagrangian and the Euler-Lagrange equation to be used, we now proceed to compute the equation for the electric field. We already know that

\[
\frac{\partial L_{\text{int}}}{\partial \phi} = -\rho, \tag{9}
\]

\[
\partial_i \left( \frac{\partial (L_0 + L_{EH})}{\partial (\partial_i \phi)} \right) = -\nabla \cdot \left( \frac{1}{4\pi} E + 4\alpha E^2 \right). \tag{10}
\]

The term \( (\nabla \cdot E)^2 \) contains second derivatives of the potential while \( E \cdot (\nabla^2 E) \) has first and third derivatives. Plugging \( (\nabla \cdot E)^2 \) into the Euler-Lagrange equation (8) we obtain the following

\[
\partial_{ij} \left( \frac{\partial L}{\partial (\partial_i \partial_j \phi)} \right) = -\partial_{ij} \left( \frac{\partial L}{\partial (\partial_i E_j)} \right) = -\partial_{ij} \left( \frac{\partial (\nabla \cdot E)^2}{\partial (\partial_i E_j)} \right) = -2\nabla^2 (\nabla \cdot E). \tag{11}
\]

On the other hand, the first derivative variation of \( E \cdot (\nabla^2 E) \) gives

\[
\partial_i \left( 2 \frac{\partial [E \cdot (\nabla^2 E)]}{\partial (\partial_i \phi)} \right) = -\partial_i \left( 2 \frac{\partial [E \cdot (\nabla^2 E)]}{\partial (\partial_i E_i)} \right) = -2\nabla \cdot (\nabla^2 E). \tag{12}
\]

The third derivative variation of \( E \cdot (\nabla^2 E) \) can be restated as

\[
\partial_{ijk} \left( 2 \frac{\partial [E \cdot (\nabla^2 E)]}{\partial (\partial_i \partial_j \partial_k \phi)} \right) = -\partial_{ijk} \left( 2 \frac{\partial [E \cdot (\nabla^2 E)]}{\partial (\partial_i \partial_j E_k)} \right). \tag{13}
\]

\(^1\text{We point out that references [20, 21, 22] use the standard Euler-Lagrange equations, } \frac{\partial L}{\partial \phi} - \partial_i \left( \frac{\partial L}{\partial (\partial_i \phi)} \right) = 0, \text{ to derive their correction to the Maxwell equations. This is an incorrect procedure since we are dealing here with a Lagrangian with higher derivatives of the potentials.}\)
Now, recalling that \( \mathbf{E} \cdot (\nabla^2 \mathbf{E}) = E_k \partial_i \partial_j E_k \), we obtain
\[
\partial_{ijk} \left( 2 \frac{\partial (\mathbf{E} \cdot (\nabla^2 \mathbf{E}))}{\partial (\partial_i \partial_j E_k)} \right) = -2 \nabla^2 (\nabla \cdot \mathbf{E}).
\] (14)
Collecting the results we arrive at the modified Gauss law
\[
b' \nabla^2 (\nabla \cdot \mathbf{E}) + \nabla \cdot (\mathbf{E} + a' E^2 \mathbf{E}) = 4\pi \rho,
\] (15)
where \( a' = 16\pi a \) and \( b' = 8\pi b \).
Equation (15) gives the electric field produced by any arbitrary charge distribution once the quantum corrections from the effective Lagrangian are taken into account. Since this equation contains higher derivatives, special attention should be paid to the boundary conditions. Fortunately, the equation can be simplified by noticing that the source \( \rho \) is the same for both the Maxwell and the Euler-Heisenberg theories. Hence, we can replace \( 4\pi \rho \) by
\[
4\pi \rho = \nabla \cdot \mathbf{E}_c,
\] (16)
where \( \mathbf{E}_c \) would be the field produced by the charge distribution due to the classical Gauss law. Thus, we can write (15) as
\[
\nabla \cdot (b' \nabla^2 \mathbf{E}) + \nabla \cdot (a' E^2 \mathbf{E} + \mathbf{E}) = \nabla \cdot \mathbf{E}_c,
\] (17)
where we have used the fact that the potential is continuous up to third order to interchange the order of the derivatives.
As we are in the electrostatic case, we have that the electric field must obey \( \nabla \times \mathbf{E} = 0 \). Thus, we can cancel the divergence operator on both sides of (17) to obtain the simplified equation
\[
b' \nabla^2 \mathbf{E} + a' E^2 \mathbf{E} + \mathbf{E} = \mathbf{E}_c,
\] (18)
which is of second order.
For the spherically symmetric case of a point charge we have
\[
\mathbf{E}_c = \frac{e}{r^2} \hat{r},
\] (19)
\[
\mathbf{E} = E \hat{r},
\] (20)
\[
E = E(r).
\] (21)
In spherical coordinates, and taking into account the conditions (20) and (21), the vector Laplacian \( \nabla^2 \mathbf{E} \) reduces to
\[
\nabla^2 \mathbf{E} = \left( \nabla^2 E - \frac{2E}{r^2} \right) \hat{r}
\] 
\[
= \left[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dE}{dr} \right) - \frac{2E}{r^2} \right] \hat{r}.
\] (22)
Care has to be taken since the Laplacian acting on a vector differs from its counterpart acting on a scalar. This well-founded result goes back to the analogy between differential operators in curvilinear coordinates and the covariant derivatives in differential geometry where the action of the covariant derivatives depends on the the tensor type on which it acts \[37\].

Finally, collecting everything together, we have the following equation in spherical coordinates for the electric field produced by a point charge

\[ b' \left[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dE}{dr} \right) - \frac{2E}{r^2} \right] + E + a'E^3 = \frac{e}{r^2}. \]  

(23)

3 The asymptotic solution

In this section we investigate the behavior of the solutions to Eq.(23). We focus in the asymptotic case of large \( r \) since near the point charge we can expect the field to be too strong and too rapidly varying so \( E \ll m^2/e \) and \( |\nabla E|/m \ll E \) most likely break down (though the inclusion of Euler-Heisenberg terms has the effect of taming the resulting produced field \[24\]). For completeness, we give a short range solution to Eq. (23) in an appendix at the end of the article.

To write an ansatz for an asymptotic solution \((mr \gg 1)\) we notice that our differential equation contains the small parameters \( a' \) and \( b' \). Therefore, it is clear that a solution can also be constructed by using these parameters. Secondly, the asymptotic solution will have a form \( E = e/r^2(1 + \xi) \) where dimensionless \( \xi \) represents the quantum corrections of the form \( c_n/(rm)^n \). Since \( a' \) and \( b' \) contain inverse even powers of the mass \( m \), only even exponents \( n \) are allowed. A quick dimensional analysis reveals that, for \( n = 4 \) the coefficient \( c_n \) must be proportional to \((b')^2 \) or \( a' \), for \( n = 6 \) we would expect \( a'b' \) or \((b')^3 \) and finally for \( n = 8 \) the coefficient is expected to be proportional to one of the terms of the form: \((a')^2, a'(b')^2, (b')^4\). Therefore, the most general possible ansatz reads

\[
E(r) = \frac{e}{r^2} + a'f_1(r) + b'f_2(r) + (a')^2f_3(r) + (b')^2f_4(r) + a'b'f_5(r) + \\
+ (b')^3f_6(r) + a'(b')^2f_7(r) + (b')^4f_8(r) \cdots ,
\]  

(24)

where \( r > 0 \) and the \( f_i \)‘s are differentiable functions in the radial variable \( r \) yet to be determined. Substituting (24) into the differential equation (23), combining like terms, i.e. powers of the parameters \( a' \) and \( b' \), and imposing that (24) is a solution of (23) lead to the following system of ODEs for the unknown functions
Solving recursively the above system of equations leads to the following asymptotic expression for the field produced by a point charge

\[
E(r) = \frac{e}{r^2} + \frac{e^3}{r^6} a' + \frac{3e^5}{r^{10}} (a')^2 + \frac{28e^3}{r^8} a' b' - \frac{1512e^3}{r^{10}} a'(b')^2 + \cdots .
\]

The result can also be given in terms of an expansion in \( \alpha = e^2 \) as

\[
E(r) = \frac{e}{r^2} \left( 1 - 1.4 \times 10^{-2} \frac{\alpha^3}{(rm)^4} + 8.8 \times 10^{-3} \frac{\alpha^4}{(rm)^6} + 1.0 \times 10^{-2} \frac{\alpha^5}{(rm)^8} + \cdots \right).
\]

Equation (34) represents the Euler-Heisenberg contribution to the Coulomb law of an electron at large distances including contribution from the non-uniform terms.

Clearly, equation (34) reduces to Coulomb law when \( b' = a' = 0 \), as it should be. Additionally, we can check that equation (34) reproduces the known result from the uniform Euler-Heisenberg theory by noticing that in the case \( b' = 0 \) the differential equation (23) reduces to a cubic equation whose only real solution is

\[
E_c(r) = \frac{2f(r)}{3a'r} - \frac{r}{2f(r)}, \quad f(r) = \frac{3}{16} \frac{(a')^2r}{16} \left( 27e + 3\sqrt{12r^4 + 81a' e^2} a' \right),
\]

which expanded around \( a' = 0 \) reproduces (34) with \( b' = 0 \). Hence, the new term proportional \( a'b' r^{-8} \) interpolates between \( r^{-6} \) and \( r^{-10} \) obtained from the Euler-Heisenberg theory without derivative corrections.
A second check consists in putting $a' = 0$ which sheds some light on the reason for why we do not encounter terms proportional only to $b'\,n$ in equation (34). In the case $b' > 0$ and $a' = 0$, two linearly independent solutions to equation (23) are found to be

$$y_\pm(r) = \frac{e^{\pm i \beta}}{r^2} \sqrt{\mp i(r \pm i\beta)^2}, \quad \beta = \sqrt{b'}.$$  \hspace{1cm} (37)

Before taking the square roots appearing in the above expression, we need to fix a branch cut. In particular, we choose the principal value in such a way that the branch cut lies on the negative $r$-axis. Then,

$$\sqrt{\pm i} = e^{\pm \frac{i}{2}}, \quad \sqrt{(r \pm i\beta)^2} = \sqrt{r^2 + \beta^2 e^{\pm i\vartheta}}, \quad \vartheta = \arctan \left( \frac{\beta}{r} \right)$$  \hspace{1cm} (38)

and the particular solutions to (23) can be cast into the compact form

$$y_\pm(r) = \frac{\sqrt{r^2 + \beta^2}}{r^2} e^{\pm i\left( \frac{\beta}{r} \mp \vartheta \right)}.$$  \hspace{1cm} (39)

At this point, it is trivial to construct two linearly independent real solutions to equation (23) by means of (39), and therefore, we limit us to state the general real and exact solution to the non-homogeneous ODE (23) in the case $a' = 0$, namely

$$E(r) = \frac{e}{r^2} + \frac{\sqrt{r^2 + \beta^2}}{r^2} \left[ c_1 \cos \left( \frac{r}{\beta} - \frac{\pi}{4} + \vartheta \right) + c_2 \sin \left( \frac{r}{\beta} - \frac{\pi}{4} + \vartheta \right) \right],$$  \hspace{1cm} (40)

where $\vartheta$ is a function of the radial variable given in (38). Equation (40) is not a physical result, but serves us to explain certain curious features. For large $r$ one would not even recover the correct Coulomb law from (40) which actually means that the Coulomb law comes from the uniform Euler-Heisenberg part of the theory. Due to the presence of the term $1/\beta$ in (40), we expect that in the case $b' \ll 1$ and $a' \neq 0$ the corresponding perturbative solution will not possess terms exhibiting powers of $b'$ only as we discovered already in (34).

A few words of interpretation are in order. There is no one-to-one correspondence between the Lagrangian (1) in coordinate space and the matrix elements in momentum space. Indeed, the term proportional to $a$ would result in momentum space in a point-like vertex of four photons and would not reflect the full light-light scattering. Similarly, the term proportional to $b$ in (1) would correspond to corrections to the two-point function (vacuum polarization) and we should not expect to re-derive the Uehling-Serber potential. Indeed, Eq. (40) shows that this is impossible and we should interpret the derivative terms as a non-uniform correction to the Euler-Heisenberg Lagrangian which taken by itself does not lead to physically viable results in the situation considered in this paper. Such an interpretation is reaffirmed by Eq. (34) where $b$ appears only in combination with $a$. Notice that the Uehling-Serber potential does not have an inverse power law expansion for the case $mr \gg 1$. 8
We can ask what is the effect of including higher order derivative corrections in the Lagrangian. The next derivative term would be proportional to $c = \frac{e^4}{4m^4}$. As explained above, we expect the effect of such a term to appear only in conjunction with $a$, the proportionality constant of the Euler-Heisenberg Lagrangian. Hence, by dimensional analysis, we can expect the contribution of the next higher derivative term to be of the order of $ea'b'r^{-10}$. Therefore, such a new term would only modify the term $r^{-10}$ in (34), the first contribution from the derivative Lagrangian, $28ea'b'r^{-8}$, remains untouched. Similar lines of arguments apply to other higher order terms appearing in [18].

Finally, we comment that we have limited ourselves to the 1-loop corrections to the Lagrangian. Incorporating a 2-loop correction to the first non-linear term in the Lagrangian [38, 39] would modify our $a$ by the addition of a term that is approximately $\alpha$ smaller. All of our results remain the same but with a small modification in $a$. This, and the effect of incorporating higher non-linear terms in the electric field produced by point (and an extended) charge, is discussed in [30].

4 Conclusions

Calculating quantum corrections to the Coulomb law has been an active field over decades. The interest goes back to the fact that the Coulomb law governs many phenomena in classical and quantum physics. Therefore, knowing its exact quantum correction is of some importance. Any new interaction or particle can be subjected to the test of reproducing well established results and as a consequence can, at the most, add corrections to these results. Similarly, any new higher order Lagrangian established within the Standard Model should be examined carefully for its consequences.

The Euler-Heisenberg theory presents such an example. However, most of its effects have been derived using only the uniform part neglecting hereby the non-uniform contribution with higher order derivatives. The dynamical and static properties of the latter deserve a closer inspection. A first step towards such a program has been done in this paper: we have derived a non-linear equation for the static electric field within the full Euler-Heisenberg theory (in the leading order). We have solved this equation asymptotically and found terms which add to the standard solution derived using only the uniform part. Although we have worked with a point charge, the same methods also hold for the case of extended charge distribution just by taking the corresponding classical field in the right hand side of Eq. (23) in a manner similar to the procedure given in [30].

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A Appendix

The uniform Euler-Heisenberg Lagrangian contains higher powers of the electric field $E$ (we restrict ourselves again to the static case with $B = 0$) which can be used to derive higher order polynomials for $E$ [30]. To the lowest order a polynomial of the third order emerges which we discussed in the main text. Solving these higher order polynomials displays a self-regulating effect: the field strength of a proton can be brought below $m^2/e$ and the short distance solution can become viable. For the non-uniform part of the Euler-Heisenberg theory such a procedure does not exist, but could become available in future. Therefore, we conclude the paper by deriving a power series solution of (23) around the point $r = 0$. We warn the reader that although the following solution is mathematically sound, we do not make much emphasis in its physical relevance due to the limitations in the range of validity of the Euler-Heisenberg theory.

We start by rewriting (23) as follows

$$b' r^2 \frac{d^2 E}{dr^2} + 2b' r \frac{dE}{dr} - e + (r^2 - 2b'r)E + a'r^2E^3 = 0. \quad (41)$$

The above equation is a special case of the generalized second order Riccati equation treated in [40], namely

$$(A_0 + B_0 y) \frac{d^2 y}{dx^2} + (C_0 + D_0 y) \frac{dy}{dx} - 2B_0 \left( \frac{dy}{dx} \right)^2 + E_0 + F_0 y + G_0 y^2 + H_0 y^3 = 0, \quad (42)$$

where each coefficient can be Taylor expanded about $x = 0$, more precisely

$$A_0 = x^2 \sum_{k=0}^{\infty} a_k x^k, \quad B_0 = x^2 \sum_{k=0}^{\infty} b_k x^k, \quad C_0 = x \sum_{k=0}^{\infty} c_k x^k, \quad (43)$$

$$D_0 = x \sum_{k=0}^{\infty} d_k x^k, \quad E_0 = \sum_{k=0}^{\infty} e_k x^k, \quad F_0 = \sum_{k=0}^{\infty} f_k x^k, \quad (44)$$

$$G_0 = x \sum_{k=0}^{\infty} g_k x^k, \quad H_0 = x \sum_{k=0}^{\infty} h_k x^k, \quad (45)$$

with

$$e_0 \neq 0 \neq f_0, \quad (46)$$

and

$$A_0(0) = B_0(0) = C_0(0) = D_0(0) = G_0(0) = H_0(0) = 0. \quad (47)$$

In our case, both conditions (46) and (47) are satisfied provided that $b' \neq 0 \neq e$. Moreover, we have

$$A_0 = b'^2, \quad C_0 = 2b'r, \quad E_0 = -e, \quad F_0 = -2b' + r^2, \quad H_0 = a'r^2. \quad (48)$$

As in [40], we are interested in a solution to (41) around $r = 0$ admitting a power series of the form

$$E(r) = a_0 + \sum_{k=1}^{\infty} \beta_k r^k. \quad (49)$$
The coefficients in (49) do not have in general the properties

\[ \Delta_p = \det \begin{bmatrix} \alpha_0 & \beta_1 & \cdots & \beta_p \\ \beta_1 & \beta_2 & \cdots & \beta_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_p & \beta_{p+1} & \cdots & \beta_{2p} \end{bmatrix} \neq 0 \quad \forall p = 0, 1, 2, \ldots \quad (50) \]

and

\[ \Gamma_{2p+1} = \det \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{p+1} \\ \beta_2 & \beta_3 & \cdots & \beta_{p+2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{p+1} & \beta_{p+2} & \cdots & \beta_{2p+1} \end{bmatrix} \neq 0 \quad \forall p = 0, 1, 2, \ldots \quad (51) \]

because as we will see soon in our case it turns out that \( \Gamma_1 = \beta_1 = 0 \). Hence, according to [41], the solution to equation (41) does not admit a representation in terms of a continued fraction of the form

\[ E(r) = \frac{\alpha_0}{1 + \frac{\alpha_1 r}{1 + \frac{\alpha_2 r}{\ddots}}} \quad (52) \]

Finally, replacing (49) into (41) yields the following expansion for the electric field in a neighbourhood of \( r = 0 \), namely

\[ E(r) = -\frac{e}{2b'} + \beta_2 r^2 + \beta_4 r^4 + \beta_6 r^6 + \beta_8 r^8 + \mathcal{O}(r^{10}), \quad (53) \]

with

\[ \beta_2 = \frac{e[a' e^2 + 4(b')^2]}{512(b')^4}, \quad (54) \]

\[ \beta_4 = -\frac{e[3(a')^2 e^4 + 16a'(b')^2 e^2 + 16(b')^4]}{147456(b')^7}, \quad (55) \]

\[ \beta_6 = \frac{e[45(a')^3 e^6 + 336(a')^2(b')^2 e^4 + 656a'(b')^4 e^2 + 128(b')^6]}{188743680(b')^{10}}, \quad (56) \]

and

\[ \beta_8 = -\frac{e[585(a')^4 e^8 + 5628(a')^3(b')^2 e^6 + 17232(a')^2(b')^4 e^4 + 16448a'(b')^6 e^2 + 512(b')^8]}{211392921600(b')^{13}}, \quad (57) \]

It is peculiar to see that the electric field at the origin given by Eq. (53) has a negative sign. Near a positive charge Eq. (53) gives a field that points toward the charge.