The Contextual Quantization and the Principle of Complementarity of Probabilities

Andrei Khrennikov*
International Center for Mathematical Modelling in Physics and Cognitive Sciences, University of Växjö, S-35195, Sweden, e–mail: Andrei.Khrennikov@msi.vxu.se

Sergei Kozyrev†
Institute of Chemical Physics, Russian Academy of Science, Moscow, Russia, e–mail: kozyrev@mi.ras.ru

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Abstract

The contextual probabilistic quantization procedure is formulated. This approach to quantization has much broader field of applications,

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compared with the canonical quantization. The contextual probabilistic quantization procedure is based on the notions of probability context and the Principle of Complementarity of Probabilities. The general definition of probability context is given. The Principle of Complementarity of Probabilities, which combines the ideas of the Bohr complementarity principle and the technique of noncommutative probability, is introduced. The Principle of Complementarity of Probabilities is the criterion of possibility of the contextual quantization.

1 Introduction

The canonical approach to quantization (originated by Heisenberg’s ”matrix mechanics” [1]) is based, in essence, on purely algebraic ideas. But quantum mechanics is a probabilistic theory, see for discussion [2], [3], [4]. Therefore it would be natural to create a quantization procedure by starting directly with the probabilistic structure of an experiment. Of course the authors are well aware that a probabilistic structure can be added to the Heisenberg algebraic quantization through the representation of noncommutative Heisenberg variables by operators in a Hilbert space and by using Born’s probabilistic interpretation of normalized vectors. P.Dirac developed ideas of W.Heisenberg and created the modern variant of canonical quantization: mapping dynamical variables to quantum observables and Poisson bracket to commutator. Intrinsically Dirac’s canonical quantization is also an algebraic quantization. Probabilistic structure is added via the Hilbert space representation of quantum observables.

In this paper we formulate the purely probabilistic approach to quantization, based on the ideas of the Bohr complementarity principle and the methods of noncommutative probability theory, especially on the approach of contextual probability. For this reason we formulate here the new general definition of probabilistic context as a map of (in general, noncommutative) probability spaces. We find that this new approach to quantization is applicable for considerably broader class of models, compared with the canonical quantization.

Let us remind the algebraic version of canonical approach to quantization. Assume that we have a family of noncommutative involutive algebra $B(h)$
over complex numbers with the commutation relations, which depend (say polynomially) on the number parameter $h$, and for $h = 0$ the algebra $B(0)$ is commutative.

Let us have the commutative Poissonian $*$-algebra $A$ over the complex numbers, which is isomorphic to $B(h)$ as a linear space, and is isomorphic to $B(0)$ as a commutative $*$-algebra.

Therefore the mentioned isomorphism of linear spaces, which relates the algebras $A$ and $B(h)$, defines the $h$-dependent family of algebras $B(h)$ with the elements $x(h)$ depending on $h$ (where $x(0) = x \in A$).

Let the Poissonian structure, or the Poissonian bracket, on the algebra $A$ (the Lie bracket which is the differentiation of $A$), be related to the product in $B(h)$ as follows:

$$\{x, y\} = \lim_{h \to 0} \frac{i}{h} [x(h), y(h)], \quad x, y \in A, \quad x(h), y(h) \in B(h)$$

Due to polynomial dependence of the commutation relation in $B(h)$ on $h$ the RHS (right hand side) of the above is well defined.

Then we say that $B(h)$ is the quantization of $A$, and the linear map $x \mapsto x(h)$ is the quantization map. Note that the quantization map is non unique in the following sense: the different families $x(h)$ (corresponding to the different isomorphisms of linear spaces) will reproduce the same Poissonian bracket.

This is the canonical approach to quantization, which is sufficient to reproduce the standard models of quantum mechanics. But this approach, in our opinion, does not take into account the probabilistic structure of quantum mechanics and moreover, looks too restrictive.

First, it is based on purely algebraic ideas, when quantum mechanics is a probabilistic theory. In the discussed above standard approach we never used the structure of the state (and any probabilistic arguments).

Second, it is obvious that in this approach one can obtain only the commutation relations for the algebras which become commutative for $h = 0$, such as the Heisenberg algebra. It is not possible to reproduce in the canonical approach the relations of the quantum Boltzmann algebra (or the algebra of the free creation and annihilation operators) of the form

$$A_i A_j^\dagger = \delta_{ij}$$
which, for example, describes the statistics of collective excitations in the quantum electrodynamics (and many other interacting theories) [5]. The relations above has no contradiction with the celebrated spin–statistics theorem, since they are valid for the collective interacting degrees of freedom. This relations are derived by the time averaging procedure (called the stochastic limit), and in [5] the phenomenon of arising of the quantum Boltzmann relations in the stochastic limit was called the third quantization\(^1\), see Appendix for details.

It would be worth to include the mentioned above probabilistic arguments into more general frameworks of quantization. In the present paper we investigate the quantization procedure based on the contextual approach to probability. We call this procedure the contextual quantization. This quantization procedure will be relevant to a wide class of noncommutative probability spaces with the statistics different from the Bose or Fermi statistics, for instance for the quantum Boltzmann statistics.

In papers [6]–[9] there was developed a contextual probabilistic approach to the statistical theory of measurements over quantum as well as classical physical systems. It was demonstrated that by taking into account dependence of probabilities on complexes of experimental physical conditions, physical contexts, we can derive quantum interference for probabilities of alternatives. Such a contextual derivation is not directly related to special quantum (e.g. superposition) features of physical systems.

In the present paper we are performing the next step in developing of the contextual probabilistic approach — the step from the contextual interpretation of noncommutative probability to the contextual quantization procedure. We show that the contextual probabilistic approach gives us not only interpretation of quantum mechanics, but, after corresponding generalization, allows to formulate a new quantization procedure, which is applicable, in principle, not only to models of standard quantum mechanics, but to a wide class of physical phenomena, which may be described by models of noncommutative probability theory, see discussion of [10], [11], [12].

To formulate the contextual quantization procedure we introduce the following notions.

First, we introduce the general definition of probabilistic context as a de-

\(^1\)The terminology third quantization is quite divertive. Some authors use it in a totally different framework.
formation of embedding of probability spaces: the probability space $A$ has context $f$ in the probability space $B$, if $f$ is the map $f : A \rightarrow B$ which is a deformation of embedding of probability spaces, see Section 3 for details. This definition gives a contextual probabilistic formulation of the correspondence principle in quantum mechanics, but our aim is to apply it to more general situation. So the aim of this paper is not just a new reformulation of the conventional quantum formalism, but extension of quantum ideology to new domains.

Second, we propose a probabilistic version of the Bohr complementarity principle, which we call the Principle of Complementarity of Probabilities (the PCP) and show, that this principle is the criterion of possibility of the contextual quantization. The Principle of Complementarity of Probabilities describes possibility of unification of probability contexts of several probability spaces in the frameworks of a new larger probability space.

We discuss the relation of the introduced contextual quantization procedure and the noncommutative replica procedure, introduced in [10], [11] and show, that the noncommutative replica procedure is the example of the contextual quantization. This shows that the contextual quantization is applicable in the multiplicity of the fields which is much wider than quantum mechanics. This in some sense reflects the original ideas by Bohr.

Then we discuss applications of the contextual quantization to collective interacting commutation relations of the stochastic limit, see also [5], [12] for details.

For the review of quantum probability see [2]. Quantization of thermodynamics was discussed in [13]. Preliminary discussion of the mathematical framework for the contextual approach was given in [14]. Discussion of the Chameleon point of view on quantum measurements which is similar to contextual approach was given in [15], [16]. Contextual approach to Kolmogorov probability with relations to quantum mechanics was discussed in [7].

We also underline that our Principle of Complementarity of Probabilities can be coupled to investigations on quantum entropy and information dynamics, see [17] – [19]. It might be that our principle can be reformulated by using the language of quantum entropy and information.

The structure of the present paper is as follows.

In Section 2 we introduce the Principle of Complementarity of Probabilities.

In Section 3 we introduce the definition of probability context as (defor-
mation of) embeddings of probability spaces.

In Section 4 we introduce the contextual quantization procedure based on the notions of probability context and the Principle of Complementarity of Probabilities.

In Section 5 we discuss the relation of the contextual quantization and the noncommutative replica procedure of [10].

In Section 6 we discuss the stochastic limit approach, in which commutation relations for collective operators, which can not be described by canonical quantization, but can be described by the contextual quantization, were obtained.

2 The Principle of Complementarity of Probabilities

In the present Section we discuss the meaning of the complementarity principle in quantum mechanics from the point of view of the probabilistic interpretation of quantum mechanics and introduce the Principle of Complementarity of Probabilities.

Probabilistic interpretation of quantum mechanics is based on the notion of noncommutative (or quantum) probability space. A noncommutative probability space is a pair $\mathcal{B} = (\mathcal{B}, \psi)$, where $\mathcal{B}$, called the algebra of observables, is the involutive algebra with unit over the complex numbers, and $\psi$ is a state (positive normed linear functional) on $\mathcal{B}$.

The commutative, or Kolmogorovian, probability space is a particular variant of noncommutative probability space for the case when the algebra is commutative. Such a probability space is called classical.

The state of a quantum system is described by density matrix — positive functional on noncommutative algebra of observables. Non compatible observables correspond to noncommuting operators, which we may consider belonging to different classical (commutative) probability subspaces in the full noncommutative probability space. In particular, in the ordinary quantum mechanics (where $\mathcal{B}$ is the Heisenberg algebra) the position and momentum observables generate commutative subalgebras of the Heisenberg algebra — algebras of functions of the position and momentum observables, respectively. Measuring the observables from the classical subalgebra we build the (classi-
cal) state on the classical subalgebra. Therefore, after the observation of the full set of incompatible observables, we obtain the set of classical states on noncommuting classical subalgebras, or the set of classical probability spaces.

The following formulation presented in Discussion with Einstein on Epistemological Problems in Atomic Physics (see [20], vol. 2, p. 40), perhaps is Bohr's most refined formulation of what he means by the complementary situations of measurements:

Evidence obtained under different experimental conditions [e.g. those of the position vs. the momentum measurement] cannot be comprehended within a single picture, but must be regarded as [mutually exclusive and] complementary in the sense that only the totality of the [observable] phenomena exhausts the possible information about the [quantum] objects [themselves].

We propose the following probabilistic version of the complementarity principle. From the beginning we consider very general model, which is essentially wider than the conventional quantum model. Thus our aim is not only a probabilistic reformulation of Bohr's complementarity principle, but the extension of this principle to more general situation.

\(^2\)N.Bohr discussed at many occasions the possibility to use the principle of complementarity outside of the quantum domain [20], see [21], [22], [23], [4], [24]. However, his proposals were presented on merely philosophical level and, as a consequence, we did not see any fruitful applications of the principle of complementarity in other domains. Another reason for the absence of such applications of the principle of complementarity was Bohr's attitude to present this principle as a kind of no go principle. For N.Bohr the main consequence of the principle of complementarity was that in quantum mechanics, we are not dealing with an arbitrary renunciation of a more detailed analysis of atomic phenomena, but with a recognition that such an analysis is in principle excluded [20] (Bohr's emphasis). In the opposite, we shall present the principle of complementarity in a constructive form by concentrating on the possibility of unification of statistical data obtained in incompatible experiments into a single quantum probabilistic model. Here we are coming to the crucial difference between Bohr's view and our view to complementarity. Bohr's complementarity was merely individual complementarity and our complementarity is probabilistic complementarity. It is well known that historically N.Bohr came to the principle of complementarity through discussion with W.Heisenberg on his uncertainty principle. The original source of all those considerations was the idea that for a single quantum system the position and momentum could not be simultaneously measured. Our main idea is that statistical data for incompatible observables (e.g. the position and momentum) can not be obtained in a single experiment. But, nevertheless, such data, obtained in different experiments can be unified in a single quantum probabilistic model. Thus, in the opposite to N.Bohr and W.Heisenberg, we present a constructive program of the unification of statistical data and not a no go program. Such a constructive approach gives
Assume we have noncommutative involutive algebra of observables $B$, and the set $B_i$ of subalgebras in $B$ with the states $\psi_i$ defined on the corresponding classical subalgebras $B_i$ (thus $B_i = (B_i, \psi_i)$ are probability spaces).

**Definition 1** We say that the states $\psi_i$ on the subalgebras $B_i$ of the algebra $B$ of observables, corresponding to measurements of physically incompatible observables, satisfy the Principle of Complementarity of Probabilities (or the PCP), if they may be unified into the state $\psi$ on the full algebra of observables $B$.

In short the PCP may be formulated as follows: probability spaces, corresponding to measurements of physically incompatible observables, may be unified into a larger probability space.

In particular, the initial probability spaces $B_i$ can be classical (commutative) as in the above considerations on classical probability spaces generated by incompatible observables. Note that in this definition we did not claim that the initial probability spaces are necessarily commutative and their unification is noncommutative and our definition is more general. Therefore the PCP is not only a reformulation of the Bohr complementarity principle in the probabilistic language, but it also is an extension the field of applicability of the ideas of complementarity.

One can see, that the PCP is not always trivially satisfied. For example, on the algebra with the relation

$$[a, a^*] = -1$$

there is no positive faithful state. Therefore, classical states on the subalgebras, generated by $a + a^*$ and $i(a - a^*)$ can not be unified into the faithful state on the full algebra of observables. Therefore, this algebra satisfies the original Bohr complementarity principle but can not satisfy the Principle of Complementarity of Probabilities.

the possibility to extend essentially the domain of applications of the modified principle of complementarity.

At the same time we see the real bounds of the extension of quantum ideology: models in which statistical data, obtained in experiments with incompatible observables, can not be even in principle unified into a single quantum–like probability model, see the example at the end of this section.
3 Probabilistic contexts

In the present Section we discuss the contextual approach to noncommutative probability and introduce the new general definition of probabilistic context.

We remind that a morphism of probability spaces is a $\ast$–homomorphism of algebras which conserves all the correlation functions, i.e. for the morphism $f: A = (A, \phi) \to B = (B, \psi)$, one has $\psi(f(a)) = \phi(a)$ for any $a \in A$. A subspace of probability space is defined by a subalgebra of algebra of observables and the restriction of the state on the algebra of observables on this subalgebra. A morphism is an embedding, if it is an injection as a $\ast$–homomorphism.

The contextual approach in probability theory, see [6]–[9], discusses the definition of probability with respect to the complex of physically relevant conditions, or the context. An attempt to give a formal definition of a context was done in [14]. Let us formulate the new general definition of a context in noncommutative probability:

**Definition 2** The contextual representation (or simply context) of the probability space $A = (A, \phi)$ in the probability space $B = (B, \psi)$ is the injective $\ast$–homomorphism $f: A \to B$, which satisfies the correspondence principle for the states $\phi$ and $\psi$.

In the following we, if no confusion is possible, will use the term context also for the image of $f$ in $B$.

Two contexts are incompatible, if their images can not be considered in the frameworks of commutative (or classical) probability space.

Now we define (in philosofical sense) the correspondence principle, see also [8] for the discussion. Identifying $A$ with its image in $B$, we formulate the following:

**Definition 3** The states $\phi$ and $\psi$ on the involutive algebra $A$ satisfy the correspondence principle, if $\psi$ is a deformation of $\phi$.

Of course, the definition above does not make sense, if we do not formulate the definition of deformation of the states. For different algebras $A$ and different contexts we may have different definitions of deformation. One of the natural examples is the following.
Definition 4 We say that the state $\psi$ on algebra $A$ is the deformation of the state $\phi$, if the state $\psi = \psi_h$ depends on the (real) parameter $h$, $\psi_0 = \phi$, and for any $a \in A$ we have $\lim_{h \to 0} \psi_h(a) = \phi(a)$.

For the prototypical example the correspondent states mean simply equal states, and the context in Definition 2 will be simply an embedding of probability spaces. But this simple case does not cover the case of standard quantum mechanics, although may be useful for some other applications of the contextual quantization.

Discuss the important examples of probability contexts, describing the well known two slit experiment, in which we observe quantum interference of a particle, passing through two slits to a screen. In quantum mechanics it is naturally to define a context by fixing of classical probability subspace (classical subalgebra with the restriction of the state). Note that in the definition of context we noted that we will identify the contextual map with it’s image, if no confusion is possible.

In the two slit experiment we have two important probabilistic contexts:

1) The context of measurements, in which we perform observations of particles. This context, as a probability space, is generated by projections onto the basis of measurements, and in the considered case is given by probability space in which the operator of coordinate along the screen is diagonal.

2) The context of dynamics — probability space, in which density matrix of the particle is diagonal.

Since these two contexts are incompatible (i.e. density matrix of the particle is non diagonal in the basis of measurements), we observe quantum interference.

4 The contextual quantization

The introduced in Section 2 Principle of Complementarity of Probabilities is a necessary condition for the existence of noncommutative probability space. In the present Section we use the Principle of Complementarity of Probabilities to define the quantization procedure based on probabilistic arguments.
Assume we have the family of probability spaces \( A_i = (\mathcal{A}_i, \phi_i) \), and the family \( f_{ij} : \mathcal{A}_i \to \mathcal{B}_j \) of \(*\)-homomorphisms of \( \mathcal{A}_i \) onto the subalgebras \( \mathcal{B}_j \) of \(*\)-algebra \( \mathcal{B} \), which define the states \( \psi_j \) on \( \mathcal{B}_j \):

\[
\psi_j(f_{ij}(a)) = \phi_i(a)
\]

and therefore make these subalgebras the probability spaces \( \mathcal{B}_j = (\mathcal{B}_j, \psi_j) \).

We propose the following definition of the contextual quantization.

**Definition 5** Let the images \( \mathcal{B}_j \) of algebras \( \mathcal{A}_i \) generate the \(*\)-algebra \( \mathcal{B} \), and moreover the states \( \psi_j \) on the subalgebras \( \mathcal{B}_j \in \mathcal{B} \) satisfy the Principle of Complementarity of Probabilities and therefore there exists the state \( \psi \) on the whole \(*\)-algebra \( \mathcal{B} \).

In this case we will say that the probability space \( \mathcal{B} = (\mathcal{B}, \psi) \) is the contextual quantization of the family of probability spaces \( \mathcal{A}_i \) with respect to the contexts \( \mathcal{B}_j = f_{ij}(\mathcal{A}_i) \).

**Remark 6** We see that in the definition above \( \mathcal{B}_j \) will be the contexts of \( \mathcal{A}_i \), and in the contextual approach the quantization of the probability space is defined by the family of contexts with the images satisfying the Principle of Complementarity of Probabilities. Hence we will also call this principle the Principle of Complementarity of Probabilistic Contexts.

**Remark 7** Note that the context map \( f \) in Definition 2 is a homomorphism, while in the canonical quantization we usually can not use homomorphisms to map classical objects to quantum. This is related to the fact that the context of classical probability space in noncommutative probability space is a classical probability subspace (and classical subalgebra (say of coordinates) in quantum algebra may be related to classical algebra by a homomorphism). The contextual quantization is based on totally different idea, compared to canonical quantization: instead of looking for classical constructions, which will be classical traces of quantum phenomena (as in the canonical quantization), we unify several (not necessarily classical) statistics into more general noncommutative statistics.

Since in a general situation the morphisms of probability spaces (and moreover their deformations) are highly non unique, the contextual quantization is applicable in the situations which are beyond of the frameworks of
the canonical quantization. Moreover, the contextual quantization in principle may be applied to some special classical systems.

Consider the following examples. The first example describes contextual quantization of harmonic oscillator.

**Example 1.** Consider the classical probability space \( (A, \phi) \) described by the real valued classical random variable \( x \) with the Gaussian mean zero state \( \phi \), which is the Gibbs state of a classical harmonic oscillator.

Consider the noncommutative probability space \( (B, \psi) \), where \( B \) is the Heisenberg algebra with the (selfadjoint) generators \( q_i, p_i, i = 1, \ldots, d \) and the relations

\[
[p_i, q_j] = -i\hbar \delta_{ij}
\]

and the state \( \psi \) which is the Gibbs state for the harmonic oscillator with \( d \) degrees of freedom:

\[
\psi(X) = \text{tr} \ e^{-\beta H} X, \quad H = \frac{1}{2} \sum_{i=1}^{d} \left(p_i^2 + q_i^2\right)
\]

Then the set of \( 2d \) injections

\[
f_i : x \mapsto q_i, \quad g_j : x \mapsto p_j, \quad i, j = 1, \ldots, d
\]  

of probability spaces \( A \to B \) satisfies the conditions of the Definition 5 and defines the contextual quantization, which transforms the classical mean zero Gaussian real valued random variable into the quantum probability space describing the quantum harmonic oscillator.

The correspondence principle in this example relates the Gibbs states of classical and quantum harmonic oscillators.

Note that in the described contextual quantization, unlike in the canonical quantization, different degrees of freedom are described by different maps \( f_i, g_j \) (the initial classical probability space \( A \) has one degree of freedom). In principle we may distinguish different degrees of freedom from the beginning and start contextual quantization from a family of classical probability spaces.

**Example 2.** Consider the contextual quantization which generates noncommutative probability space \( (B, \psi) \), where \( B \) is the quantum Boltzmann algebra, generated by the quantum Boltzmann annihilations \( A_i \) and
creations $A_i^\dagger$, $i = 1, \ldots, d$ with the relations

$$A_iA_j^\dagger = \delta_{ij}$$

(2)
in the Fock state $\psi(X) = \langle \Omega, X\Omega \rangle$, where the vacuum $\Omega$ is annihilated by all annihilations $A_i$.

Consider the probability space $\mathbf{A} = (\mathcal{A}, \phi)$, where $\mathcal{A}$ is the quantum Boltzmann algebra with one degree of freedom, i.e. the algebra generated by operators $A, A^\dagger$ with the relation

$$AA^\dagger = 1$$

and $\phi$ is the Fock state $\phi(X) = \langle \Omega, X\Omega \rangle$. Note that the algebra $\mathcal{A}$ is non-commutative.

Consider the set of $d$ embeddings of probability spaces $\mathbf{A} \rightarrow \mathbf{B}$

$$A \mapsto A_i, \quad A^\dagger \mapsto A_i^\dagger, \quad i = 1, \ldots, d$$

(3)

This set of injections satisfies the conditions of Definition 5 (where the deformation in the correspondence principle is an identity, i.e. the correspondent states are identically equal) and therefore the probability space $\mathbf{B}$ (quantum Boltzmann for $d$ degrees of freedom in the Fock state) is the contextual quantization of the probability space $\mathbf{A}$ (the same for one degree of freedom).

**Example 3.** In the Example 2 we quantized the noncommutative probability space $\mathbf{A}$ and obtained the more complicated noncommutative probability space $\mathbf{B}$. Note that if we consider in the noncommutative probability space $\mathbf{A}$ the commutative subspace $\mathbf{A}_0$ with the algebra of observables generated by $X = A + A^\dagger$ and consider the restriction of the quantization procedure on $\mathbf{A}_0$: we consider the embeddings $\mathbf{A}_0 \rightarrow \mathbf{B}_0$,

$$X \mapsto X_i = A_i + A_i^\dagger, \quad i = 1, \ldots, d$$

(4)

where $\mathbf{B}_0$ is the probability space with the algebra of observables generated by $X_i$ in the Fock state, then the contextual quantization $\mathbf{B}_0$ of noncommuting contexts of commutative probability space $\mathbf{A}_0$ will be a noncommutative probability space.

For further discussion of relations used in Examples 2 and 3 see Appendix.
Example 4. Example 1 can be generalized to quantization in superspace, see [25]. We would not like to go into details (since it needs a few new definitions which are not directly relevant to contextual quantization). We just mention that there is considered a quantization of supercommutative superalgebras. Thus initial classical algebras are not commutative, but supercommutative.

The next example describes that one can call a cognitive quantization.

Example 5. The next example has no direct relation to quantum mechanics, but we hope, it will find applications in the future. Assume we have several persons which have different points of view on some complex problem. Each point of view is described by a classical probability space, which describes the distribution of possible opinions. If the problem under consideration is complex enough, it may be possible that different points of view can not be unified within the frameworks of a single point of view, or in our description, within the frameworks of a single commutative probability space. Instead, it may be possible, that different points of view (or corresponding probability spaces) are complementary and may be unified within the frameworks of noncommutative probability space, which gives a relevant complete description of the considered complex problem, while it is not possible to give such a description using classical probability space (or single point of view). The Principle of Complementarity of Probabilities here guarantees, that there is no contradiction between different points of view and these points of view are complementary but not contradictory.

In the next Section we discuss the contextual quantization in relation to the noncommutative replica approach, introduced in [10], [11], which was one of the motivations for the definitions of the present paper.

5 The contextual quantization and replicas

In the present Section we show that the noncommutative replica procedure for disordered systems is the example of the contextual quantization and the contexts describe the way of averaging of quenched disorder.

In papers [10], [11] the noncommutative replica procedure for disordered systems was introduced. The noncommutative replica procedure has the form
of the embedding of probability spaces. It was mentioned that this embedding is non unique and moreover the combination of different morphisms can not be considered in a commutative probability space, see [10], [12].

The noncommutative replica procedure is defined (by S. Kozyrev [10], [11]) as follows. Consider the system of random Gaussian $N \times N$ matrices $J_{ij}$ with independent matrix elements with the zero mean and unit dispersion of each of the matrix element. Consider disordered system with the Hamiltonian $H[\sigma, J]$, where $\sigma$ enumerates the states of the system and the disorder $J$ is the mentioned above random matrix.

Introduce the commutative probability space where random variable corresponds to the random matrix $J$ and the correlations are defined as by the correlator

$$
\langle J^k \rangle = Z^{-1} \frac{1}{N^k} E \left( \sum_{\sigma} e^{-\beta H[\sigma, J]} \operatorname{tr} J^k \right)
$$

(5)

where $E$ is the expectation of the matrix elements and

$$
Z = E \left( \sum_{\sigma} e^{-\beta H[\sigma, J]} \right)
$$

This probability space describes annealed disordered system and the averaging over $J$ describes the averaging of the disorder. For $\beta = 0$ (5) reduces to the Gaussian state $E$ on random matrices.

We consider the family of contexts of the annealed probability space in the larger replica probability space generated by the replicas $J^{(a)}_{ij}$ of the random matrix $J_{ij}$, see [10] for details. These contexts we call the noncommutative replica procedures and they have the form

$$
\Delta : J_{ij} \mapsto \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} c_a J^{(a)}_{ij} ;
$$

(6)

This embedding of probability spaces describes the way of selfaveraging of the quenched disorder.

The context $\Delta$ maps the matrix element $J_{ij}$ into the linear combination of independent replicas $J^{(a)}_{ij}$, enumerated by the replica index $a$. Here $c_a$ are complex coefficients, which should satisfy the condition

$$
\sum_{a=0}^{p-1} |c_a|^2 = p
$$
Varying coefficients $c_a$ we will obtain different morphisms $\Delta$ of probability spaces.

After the morphism $\Delta$ the probability space is described by the correlation functions (5) with $J$ replaced by $\Delta J$ and mathematical expectation $E$ replaced by the analogous expectation for the set of independent random matrices (replicas of $J$).

Discuss the large $N$ limit of the probability space (5) for the free case $\beta = 0$.

In the large $N$ limit, by the Wigner theorem, see [26]–[28], the system of $p$ random matrices with independent variables will give rise to the quantum Boltzmann algebra with $p$ degrees of freedom with the generators $A_a, A_a^\dagger$, $a = 0, \ldots, p - 1$ and the relations

$$A_a A_b^\dagger = \delta_{ab}$$

The Gaussian state on large random matrices in the $N \to \infty$ limit becomes the Fock, or vacuum, state on the quantum Boltzmann algebra: the Fock state is generated by the expectation $\langle \Omega, X \Omega \rangle$ where $\Omega$ is the vacuum: $A_a \Omega = 0, \forall a$.

The operators $A_a$ are the limits of the large random matrices

$$\lim_{N \to \infty} \frac{1}{N} J^{(a)}_{ij} = Q_a = A_a + A_a^\dagger$$

where the convergence is understood in the sense of correlators (as in the central limit theorem).

Then in the thermodynamic limit $N \to \infty$ the map (6) will take the form of the following map (for which we use the same notation) of the quantum Boltzmann algebra with one degree of freedom into quantum Boltzmann algebra with $p$ degrees of freedom:

$$\Delta : A \mapsto \frac{1}{\sqrt{p}} \sum_{a=0} c_a A_a$$

and correspondingly

$$\Delta : Q \mapsto \frac{1}{\sqrt{p}} \sum_{a=0} c_a Q_a$$

We see that the set of contexts $\Delta$ with the different coefficients $c_a$ defines the contextual quantization of the commutative probability space, described in (4).
6 Appendix: the Stochastic Limit

In the present Section we briefly discuss the stochastic limit approach, in which deformations of quantum Boltzmann commutation relations were obtained, see [5]. In this approach we consider the quantum system with the Hamiltonian in the form

$$H = H_0 + \lambda H_I$$

where $H_0$ is called the free Hamiltonian, $H_I$ is called the interaction Hamiltonian, and $\lambda \in \mathbb{R}$ is the coupling constant.

We investigate the dynamics of the system in the new slow time scale of the stochastic limit, taking the van Hove time rescaling [29]

$$t \mapsto t/\lambda^2$$

and considering the limit $\lambda \to 0$. In this limit the free evolutions of the suitable collective operators

$$A(t, k) = e^{itH_0} A(k) e^{-itH_0}$$

will become quantum white noises:

$$\lim_{\lambda \to 0} \frac{1}{\lambda} A \left( \frac{t}{\lambda^2}, k \right) = b(t, k)$$

The convergence is understood in the sense of correlators. The $\lambda \to 0$ limit describes the time averaging over infinitesimal intervals of time and allows to investigate the dynamics on large time scale, where the effects of interaction with the small coupling constant $\lambda$ are important. For the details of the procedure see [5].

The collective operators describe joint excitations of different degrees of freedom in systems with interaction, and may have the form of polynomials over creations and annihilation of the field, or may look like combinations of the field and particles operators etc.

For example, for nonrelativistic quantum electrodynamics without the dipole approximation the collective operator is

$$A_j(k) = e^{ikq} a_j(k)$$  \hspace{1cm} (7)

where $a_j(k)$ is the annihilation of the electromagnetic (Bose) field with wave vector $k$ and polarization $j$, $q = (q_1, q_2, q_3)$ is the position operator of quantum particle (say electron), $qk = \sum_i q_i k_i$. 
The nontrivial fact is that, after the $\lambda \to 0$ limit, depending on the form of the collective operator, the statistics of the noise $b(t,k)$ depends on the form of the collective operator and may be nontrivial.

Consider the following examples.

1) We may have the following possibility

$$[b_i(t,k), b_j^\dagger(t',k')] = 2\pi\delta_{ij}\delta(t - t')\delta(k - k')\delta(\omega(k) - \omega_0)$$

which corresponds to the quantum electrodynamics in the dipole approximation, describing the interaction of the electromagnetic field with two level atom with the level spacing (energy difference of the levels) equal to $\omega_0$. Here $\omega(k)$ is the dispersion of quantum field.

In this case the quantum noise will have the Bose statistics, and different annihilations of the noise will commute

$$[b_i(t,k), b_j(t',k')] = 0$$

2) The another possibility is the relation

$$b_i(t,k)b_j^\dagger(t',k') = 2\pi\delta_{ij}\delta(t - t')\delta(k - k')\delta(\omega(k) + \varepsilon(p) - \varepsilon(p + k))$$

which corresponds to the quantum electrodynamics without the dipole approximation with interacting operator (7). Here $\omega(k)$ and $\varepsilon(p)$ are dispersion functions of the field and of the particle correspondingly.

In this case the quantum noise will have the quantum Boltzmann statistics [5], [30], and different annihilations of the noise will not commute

$$b_i(t,k)b_j(t',k') \neq b_j(t',k')b_i(t,k)$$

The commutation relations of the types (8), (9) are universal in the stochastic limit approach (a lot of systems will have similar relations in the stochastic limit $\lambda \to 0$).

7 Appendix: Copenhagen and Växjö complementarities

In the first version of this paper, see [31], we used the following version of the complementarity principle (Växjö complementarity):
To obtain the full information about the state of a quantum system, one has to perform measurements of the set of physically incompatible (noncommuting) observables, say the momentum and the coordinate. Measuring only the momentum or only the coordinate we will not obtain the full information about the state of the quantum system.

In his Email to one of the authors (A. Yu. Khrennikov) A. Plotnitsky remarked:

“First of all, this formulation does not appear to me to correspond to Bohr’s view to complementarity and indeed implies a conflict between the respective interpretations of quantum mechanics, Bohr’s and your own\(^3\), or any interpretation consistent with your formulation...”

We agree with A. Plotnitsky and in this paper we use the original Bohr’s formulation. The difference between two formulations, “Copenhagen and Växjö complementarities”, is that N. Bohr considered possible information and we considered full information. To consider “possible information” one need not to use a realists interpretation and to consider “full information” (about something) we need to use a realists interpretation, e.g., the Växjö interpretation.

However, in this paper we do not try to connect our mathematical formulation of an extended principle of complementarity to any fixed interpretation of quantum mechanics. This is just a formal mathematically formalized principle. This principle can be considered as the formalization of either the Bohr’s principle or Växjö principle.

Finally, we remark that problems discussed in this paper are closely related to the problem of understanding of information in quantum theory, see [34]–[37], [21].

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\(^3\)A. Plotnitsky mentioned the so called Växjö interpretation of quantum mechanics, see [32], [33], cf. [34], [35].
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