PRIMES IN THE FORM $[\alpha p + \beta]$  

HONGZE LI AND HAO PAN

Abstract. Let $\beta$ be a real number. Then for almost all irrational $\alpha > 0$ (in the sense of Lebesgue measure)
\[
\limsup_{x \to \infty} \pi_{\alpha, \beta}^*(x) (\log x)^2 / x \geq 1,
\]
where
\[
\pi_{\alpha, \beta}^*(x) = \{ p \leq x : \text{both } p \text{ and } [\alpha p + \beta] \text{ are primes} \}.
\]

Recently Jia [4] solved a conjecture of Long and showed that for any irrational number $\alpha > 0$, there exist infinitely many primes not in the form $2n + 2[\alpha n] + 1$, where $[x]$ denotes the largest integer not exceeding $x$. Subsequently, in [2] Banks and Shparlinski investigated the distribution of primes in the Beatty sequence $\{[\alpha n + \beta] : n \geq 1 \}$. Motivated by the binary Goldbach conjecture and the twin primes conjecture, we have the following conjecture:

**Conjecture 1.** Let $\alpha > 0$ be an irrational number and $\beta$ be a real number. Then there exist infinitely many primes $p$ such that $[\alpha p + \beta]$ is also prime.

On the other hand, Deshouillers [3] proved that for almost all (in the sense of Lebesgue measure) $\gamma > 1$ there exist infinitely many primes $p$ in the form $[n^\gamma]$. Furthermore, Balog [1] showed that for almost all $\gamma > 1$
\[
\limsup_{x \to \infty} \frac{|\{ p \leq x : \text{both } p \text{ and } [p^\gamma] \text{ are primes} \}|}{x / (\log x)^2} \geq \gamma.
\]

In this note we shall show that Conjecture holds for almost all $\alpha$. Define
\[
\pi_{\alpha, \beta}^*(x) = \{ p \leq x : \text{both } p \text{ and } [\alpha p + \beta] \text{ are primes} \}.
\]

**Theorem 1.** Let $\beta$ be a real number. Then
\[
\limsup_{x \to \infty} \pi_{\alpha, \beta}^*(x) (\log x)^2 / x \geq 1 (1)
\]
for almost all irrational $\alpha > 0$.

For a set $X \subseteq \mathbb{R}$, let $\text{mes}(X)$ denote its Lebesgue measure. Without the additional mentions, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ will be always absolute.

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**Lemma 1.** Let \( I \subseteq [0, 1) \) be an interval. Suppose that \( b, l > 0 \). Then
\[
\text{mes}(\{\alpha \in (0, b) : \lfloor \alpha/l \rfloor \in I\}) = O(b + l)\text{mes}(I),
\]
where \( \{x\} = x - \lfloor x \rfloor \).

**Proof.** Without loss of generality, we may assume that \( I = (c_1, c_2) \) with \( 0 \leq c_1 < c_2 \leq 1 \). Let \( J = \{\alpha \in (0, b) : \lfloor \alpha/l \rfloor \in I\} \). Clearly
\[
J \subseteq \bigcup_{0 \leq j \leq b/l} ((j + c_1)l, \min\{b, (j + c_2)l\}).
\]
If \( l \leq b \), then
\[
\text{mes}(J) \leq (\lfloor b/l \rfloor + 1)(c_2 - c_1)l = O(b)\text{mes}(I).
\]
And if \( l > b \), then
\[
\text{mes}(J) = \text{mes}((c_1l, \min\{b, c_2l\})) \leq (c_2 - c_1)l.
\]
\( \square \)

**Lemma 2.** Suppose that \( b_2 > b_1 > 0 \) and \( \beta \) are arbitrarily real numbers. Let \( \epsilon > 0 \) be a small number and \( x \) be a sufficiently large (depending on \( b_1, b_2, \beta \) and \( \epsilon \)) integer. Then there exists an exceptional set \( J_E \subseteq (b_1, b_2) \) with \( \text{mes}(J_E) = O(x^{-\epsilon}) \) such that for any square-free \( d \leq x^{1/3-2\epsilon} \) and irrational \( \alpha \in (b_1, b_2) \setminus J_E \),
\[
|\{1 \leq n \leq x : n[\alpha n + \beta] \equiv 0 \pmod{d}\}| = \frac{x}{d} \prod_{p|d} \left( 1 - \frac{1}{p} \right) + O(x^{1-\epsilon}/d). \tag{2}
\]

**Proof.** For an irrational \( \alpha \in (b_1, b_2) \), let
\[
\mathcal{A}(x; \alpha) = \{n[\alpha n + \beta] : 1 \leq n \leq x\}
\]
and
\[
\mathcal{A}_d(x; \alpha) = \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}.
\]
For a square-free \( d \), we have
\[
|\mathcal{A}_d(x; \alpha)| = \sum_{s|d} |\{1 \leq n \leq x/s : [\alpha sn + \beta] \equiv 0 \pmod{d/s}, (n, d/s) = 1\}| \leq \sum_{s|d} \sum_{t|d/s} \mu(t) |\{1 \leq n \leq x/s : [\alpha sn + \beta] \equiv 0 \pmod{d/s}, t|n\}|
\leq \sum_{s|d, t|s} \mu(t) |\{1 \leq n \leq x/s : [\alpha sn + \beta] \equiv 0 \pmod{dt/s}\}|.
\]
Clearly
\[
[\alpha sn + \beta] \equiv 0 \pmod{td/s} \iff \{\alpha sn^2/td + \beta s/td\} \in [0, s/td).
\]
Let \( \alpha' = \alpha s^2/td, \beta' = \beta s/td, d' = td/s \) and \( y = x/s \). Clearly \( y \geq x^{2/3+2\epsilon} \) and \( d' \leq d \). Let
\[
I_{a,q} = \{\theta \in [0, 1) : |\theta q - a| \leq x^{2\epsilon}/y\}.
\]
Suppose that \(d' x^{2\epsilon} \leq q \leq y/x^{2\epsilon}\) and \(1 \leq a \leq q\) with \((a, q) = 1\). If \(\{\alpha'\} \in I_{a, q}\),

\[
\{|1 \leq n \leq y : \{an/q + \beta'\} \in [1/q, 1/d' - 1/q]\| \\
\leq|\{1 \leq n \leq y : \{a'n + \beta'\} \in [0, 1/d']\| \\
\leq|\{1 \leq n \leq y : \{an/q + \beta'\} \in [0, 1/d' + 1/q) \cup [1 - 1/q, 1]\|.
\]

Hence

\[
|\{1 \leq n \leq y : \{a'n + \beta'\} \in [0, 1/d']\| = \frac{y}{d'} + O(q/d') + O(y/q).
\]

Let

\[
I_{d'} = \bigcup_{1 \leq a \leq d' x^{2\epsilon} \atop (a, q) = 1} I_{a, q}
\]

Clearly

\[
\text{mes}(I_{d'}) \leq \sum_{1 \leq a \leq d' x^{2\epsilon} \atop (a, q) = 1} \text{mes}(I_{a, q}) \ll \frac{d' x^{4\epsilon}}{y} = \frac{td x^{4\epsilon} - 1}{y}.
\]

If \(\alpha s^2/td \notin I_{td/s}\) for each \(s, t\) with \(s \mid d, t \mid s\), then

\[
|\mathcal{A}_d(x; \alpha)| = \sum_{s \mid d, t \mid s} \mu(t) x/td (1 + O(x^{-2\epsilon})) = \frac{x}{d} \prod_{p \mid d} (2 - 1/p) + O(x^{1-\epsilon}/d).
\]

Let

\[
\mathcal{J}_d = \{\alpha \in (0, b) : \{\alpha s^2/td\} \in I_{td/s}\ for some \ s, t \ with \ s \mid d, t \mid s\}.
\]

Applying Lemma 1,\[\text{mes}(\mathcal{J}_d) \ll b_2 \sum_{s \mid d, t \mid s \atop b_2 \geq td/s^2} \text{mes}(I_{td/s}) + \frac{td}{s^2} \sum_{s \mid d, t \mid s \atop b_2 < td/s^2} \text{mes}(I_{td/s}) = O(x^{-1/3+\epsilon}).\]

Finally, Let

\[
J_E = \bigcup_{d \leq x^{1/3-2\epsilon}} \mathcal{J}_d.
\]

Clearly we have \(\text{mes}(J_E) = O(x^{-\epsilon})\). \[\square\]

**Lemma 3.** Suppose that \(b_2 > b_1 > 0\), \(\epsilon > 0\) and \(\beta\) are arbitrarily real numbers. Then there exists an exceptional set \(J_E \subseteq (b_1, b_2)\) with \(\text{mes}(J_E) = O(x^{-\epsilon})\) such that for any irrational \(\alpha \in (b_1, b_2) \setminus J_E\),

\[
|\{1 \leq p \leq x : \text{both } p \text{ and } \lfloor \alpha p + \beta \rfloor \text{ are primes}\| \ll \frac{x}{(\log x)^2} \tag{3}
\]

for sufficiently large (depending on \(b_1, b_2, \beta\) and \(\epsilon\)) \(x\).
Proof. Let $z = x^{1/8}$. Define

$$P(z) = \prod_{p < z \text{ prime}} p$$

and

$$\mathcal{S}(A, z) = \{a \in A; (a, P(z)) = 1\}.$$ 

Let $\mathcal{A}(\alpha) = \{n[\alpha n + \beta] : 1 \leq n \leq x\}$. Clearly

$$\{p|\alpha + \beta : z + \alpha^{-1}(z + 1 - \beta) \leq p \leq x, \text{ both } p \text{ and } |\alpha + \beta| \text{ are primes}\}$$

is a subset of $\mathcal{S}(\mathcal{A}(\alpha), z)$. Furthermore, by Lemma 2, we know that there exists a set $J_E \subseteq (b_1, b_2)$ with mes$(J_E) = O(x^{-\epsilon})$ such that for any square-free $1 \leq d \leq x^{1/3 - 2\epsilon}$ and irrational $\alpha \in (b_1, b_2) \setminus J_E$,

$$|\mathcal{A}(\alpha)| = \frac{x}{d} \prod_{p|d} \left(2 - \frac{1}{p}\right) + O(x^{1-\epsilon/d}),$$

where $\mathcal{A}(\alpha) = \{y \in \mathcal{A}(\alpha) : d | y\}$. Let $g(m)$ be the completely multiplicative function such that $g(p) = 2^p - 1$ for each prime $p$. Define $G(z) = \sum_{m < z} g(m)$.

By Selberg’s sieve method,

$$|\mathcal{S}(\mathcal{A}(\alpha), z)| \leq \frac{|\mathcal{A}(\alpha)|}{G(z)} + O\left(\sum_{d < z^2} 3^{\omega(d)} x^{1-\epsilon}/d\right),$$

where $\omega(d)$ denotes the number of distinct prime divisors of $d$. Since $3^{\omega(d)} \ll d^\epsilon$,

$$\sum_{d < z^2} \frac{3^{\omega(d)}}{d} \ll z^{2\epsilon}.$$ 

So it suffices to show $G(z) \gg (\log z)^2$. By Theorem 7.14 in [5], we know

$$G(z) = \sum_{m < z} g(m) \gg \prod_{p < z} (1 - g(p))^{-1} = \prod_{p < z} (1 - 2/p + 1/p^2)^{-1} \gg (\log z)^2.$$ 

\[\square\]

Proof of Theorem 1. Suppose that $b_2 > b_1 > 0$. Let

$$\mathcal{F} = \{\alpha \in (b_1, b_2) : \limsup_{x \to \infty} \pi_{\alpha, \beta}^*(x)(\log x)^2/x < 1\}$$

and

$$\mathcal{F}_n = \{\alpha \in (b_1, b_2) : \limsup_{x \to \infty} \pi_{\alpha, \beta}^*(x)(\log x)^2/x \leq 1 - 1/n\}.$$ 

Clearly $\mathcal{F} = \bigcup_{n > 1} \mathcal{F}_n$. So it suffices to show that mes$(\mathcal{F}_n) = 0$ for every $n > 1$. (The measurability of $\mathcal{F}_n$ will be proven later.)
Assume on the contrary that there exists \( n > 1 \) such that \( \text{mes}(F_n) > 0 \). Let \( I = (c_1, c_2) \) be an arbitrary sub-interval of \((b_1, b_2)\). Clearly

\[
\int_{c_1}^{c_2} \pi_{\alpha,\beta}^*(x) \, d\alpha = \int_{c_1}^{c_2} \left( \sum_{p \leq x} \sum_{\alpha p + \beta - 1 < q \leq \alpha p + \beta} 1 \right) \, d\alpha
\]

\[
= \sum_{p \leq x} \sum_{q \text{ prime}} \text{mes}((q - \beta)/p, (q + 1 - \beta)/p) \cap [c_1, c_2])
\]

\[
\geq \sum_{p \leq x} \sum_{q \text{ prime}} \frac{1}{q p} = \frac{x}{\log p} \left( 1 + O\left( \frac{1}{\log(c_1 p)} \right) \right)
\]

\[
\geq (c_2 - c_1) \frac{x}{(\log x)^2} \left( 1 + O\left( \frac{1}{\log x} \right) \right), \quad (4)
\]

provided that \( x \) is sufficiently large (depending on \( b_1 \) and \( b_2 \)). Suppose that \( C > 1 \) is the implied constant in Lemma 3. Let \( L_I = F_n \cap I \) and

\[
L_{I,\delta}(x) = \{\alpha \in I : \pi_{\alpha,\beta}^*(x) \leq (1 - \delta)x/(\log x)^2\}.
\]

For any two primes \( p \) and \( q \), clearly

\[
J_{p,q} := \{\alpha \in I : \lfloor \alpha p + \beta \rfloor = q\}
\]

is an interval or empty set. Hence

\[
L_{I,\delta}(x) = I \setminus \left( \bigcup_{k > (1 - \delta)x/(\log x)^2} \bigcap_{j=1}^{k} J_{p_j, q_j} \right)
\]

is measurable in the sense of Lebesgue measure. Let \( \epsilon > 0 \) be a very small number. By Lemma 3,

\[
\int_{c_1}^{c_2} \pi_{\alpha,\beta}^*(x) \, d\alpha \leq O(x^{1-\epsilon}) + 
\]

\[
+ \text{mes}(L_{I,\delta}(x))\frac{(1 - \delta)x}{(\log x)^2} + (c_2 - c_1 - \text{mes}(L_{I,\delta}(x))) \frac{Cx}{(\log x)^2} \quad (5)
\]

provided that \( x \) is sufficiently large. Combining (4) and (5), we have

\[
\text{mes}(L_{I,\delta}(x)) \leq \frac{C - 1}{C - 1 + \delta/2} \text{mes}(I). \quad (6)
\]
We claim that
\[ L_I = \bigcap_{m>n} \bigcup_{y \geq 1} \bigcap_{x \geq y} L_{1/n-1/m}(x). \]  
(7)

In fact, for any \( m>n \), if
\[ \limsup_{x \to \infty} \frac{\pi_{a,b}(x)}{x/(\log x)^2} < 1 - \frac{1}{n} + \frac{1}{m}, \]
then there exists \( y_0 \) such that for any \( x \geq y_0 \)
\[ \pi_{a,b}(x) \leq \left( 1 - \frac{1}{n} + \frac{1}{m} \right) \frac{x}{(\log x)^2}. \]

On the other hand, if \( \alpha \in \bigcup_{y} \bigcap_{x \geq y} L_{1/n-1/m}(x) \), clearly we have
\[ \limsup_{x \to \infty} \frac{\pi_{a,b}(x)}{x/(\log x)^2} \leq 1 - \frac{1}{n} + \frac{1}{m}. \]

By (6) and (7), we get
\[ \text{mes}(L_I) \leq \limsup_{x \to \infty} L_{1/n-2/3n}(x) \leq \frac{C - 1}{C - 1 + 1/3n} \text{mes}(I). \]

Since \( \text{mes}(\mathcal{F}_n) > 0 \), there exist open intervals \( I_1, I_2, \ldots \subseteq (b_1, b_2) \) such that
\[ \mathcal{F}_n \subseteq \bigcup_{k=1}^{\infty} I_k \]
and
\[ \sum_{k=1}^{\infty} \text{mes}(I_k) \leq \frac{C - 1 + 1/4n}{C - 1} \text{mes}(\mathcal{F}_n). \]

By (6),
\[ \text{mes}(\mathcal{F}_n) = \sum_{k=1}^{\infty} \text{mes}(L_k) \leq \frac{C - 1}{C - 1 + 1/n} \sum_{k=1}^{\infty} \text{mes}(I_k) \leq \frac{C - 1 + 1/4n}{C - 1 + 1/3n} \text{mes}(\mathcal{F}_n). \]

This evidently leads to a contradiction. \( \square \)

Remark. In [6] and [8], Harman proved that for almost all real \( \alpha > 0 \) there are infinitely many pairs of \((p, q)\) satisfying
\[ |\alpha p - q| < \psi(p), \quad p, q \text{ are primes}, \]
provided that \( \psi \) is a non-increasing positive function and
\[ \sum_{2 \leq p \leq \infty} \frac{\psi(p)}{\log p} \]
(8)
diverges. (In fact, in [8] Harman established a quantitative version of the above result, on condition that \( \psi(n) \in (0, 1/2) \) for each \( n \).) As an immediate consequence,
for almost all $\alpha > 0$, there exists infinitely many pair of primes $(p,q)$ such that $\lfloor \alpha p \rfloor = q$, where $\lfloor x \rfloor$ is the nearest integer to $x$. For more related results, the readers may refer to [7, Chapter 6].

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E-mail address: lihz@sjtu.edu.cn

E-mail address: haopan79@yahoo.com.cn

Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, People’s Republic of China