Mass-Induced Crystalline Color Superconductivity

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(MIT-CTP-3224, hep-ph/0112206, December 14, 2001)

Abstract

We demonstrate that crystalline color superconductivity may arise as a result of pairing between massless quarks and quarks with nonzero mass $m_s$. Previous analyses of this phase of cold dense quark matter have all utilized a chemical potential difference $\delta \mu$ to favor crystalline color superconductivity over ordinary BCS pairing. In any context in which crystalline color superconductivity occurs in nature, however, it will be $m_s$-induced. The effect of $m_s$ is qualitatively different from that of $\delta \mu$ in one crucial respect: $m_s$ depresses the value of the BCS gap $\Delta_0$ whereas $\delta \mu$ leaves $\Delta_0$ unchanged. This effect in the BCS phase must be taken into account before $m_s$-induced and $\delta \mu$-induced crystalline color superconductivity can sensibly be compared.

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I. INTRODUCTION

Since quarks which are antisymmetric in color attract, cold dense quark matter is unstable to the formation of a condensate of Cooper pairs, making it a color superconductor [1]. At asymptotic densities, the ground state of QCD with quarks of three flavors ($u$, $d$ and $s$) with equal masses is expected to be the color-flavor locked (CFL) phase [2–4]. This phase features a condensate of Cooper pairs of quarks which includes $ud$, $us$, and $ds$ pairs. Quarks of all colors and all flavors participate in the pairing, and all excitations with quark quantum numbers are gapped. In this phase, left-flavor and right-flavor symmetries are both locked to color, breaking chiral symmetry [2]. As in any BCS state, the Cooper pairing in the CFL state pairs quarks whose momenta are equal in magnitude and opposite in direction, and pairing is strongest between pairs of quarks whose momenta are both near their respective Fermi surfaces. Pairing persists even in the face of a stress (such as a chemical potential difference or a mass difference) that seeks to push the Fermi surfaces apart, although a stress that is too strong will ultimately disrupt Cooper pairing [5,6]. Thus, the CFL phase persists for unequal quark masses, so long as the differences are not too large [5,6]. This means that the CFL phase is the ground state for real QCD, assumed to be in equilibrium with respect to the weak interactions, as long as the density is high enough.

Imagine decreasing the quark number chemical potential $\mu$ from asymptotically large values. The quark matter at first remains color-flavor locked, although the CFL condensate may rotate in flavor space as terms of order $m_s^4$ in the free energy become important [7]. Color-flavor locking is maintained until a transition to a state in which some quarks become ungapped. This “unlocking transition”, which must be first order [5,6], occurs when

$$\mu \approx \frac{m_s^2}{4\Delta_0}.$$  

In this expression, $\Delta_0$ is the BCS pairing gap, estimated in both models and asymptotic analyses to be of order tens to 100 MeV [4], and $m_s$ is the strange quark mass parameter. $m_s$ includes the contribution from any $\langle \bar{s}s \rangle$ condensate induced by the nonzero current strange quark mass, making it a density-dependent effective mass, decreasing as density increases and equaling the current strange quark mass only at asymptotically high densities. At densities which may arise at the center of compact stars, corresponding to $\mu \sim 400 – 500$ MeV, $m_s$ is certainly significantly larger than the current quark mass, and its value is not well-known. In fact, $m_s$ decreases discontinuously at the unlocking transition [10]. Thus, the criterion (1) can only be used as a rough guide to the location of the unlocking transition in nature [10]. Given this quantitative uncertainty, there remain two logical possibilities for what happens as a function of decreasing $\mu$. One possibility is a first order phase transition directly from color-flavor locked quark matter to hadronic matter, as explored in Ref. [9]. The second possibility is an unlocking transition [5,6] to quark matter in which not all quarks participate in the dominant pairing, followed only at a lower $\mu$ by a transition to hadronic matter. We assume the second possibility here, and explore its consequences.

Once CFL is disrupted, leaving some species of quarks with differing Fermi momenta and therefore unable to participate in BCS pairing, it is natural to ask whether there is some generalization of the ansatz in which pairing between two species of quarks persists even once their Fermi momenta differ. Crystalline color superconductivity is the answer to this question. The idea is that it may be favorable for quarks with differing Fermi momenta...
to form pairs whose momenta are not equal in magnitude and opposite in sign \[11,12\]. This generalization of the pairing ansatz (beyond BCS ansätze in which only quarks with momenta which add to zero pair) is favored because it gives rise to a region of phase space where both of the quarks in a pair are close to their respective Fermi surfaces, and such pairs can be created at low cost in free energy. Condensates of this sort spontaneously break translational and rotational invariance, leading to gaps which vary in a crystalline pattern. As a function of increasing depth in a compact star, \(\mu\) increases, \(m_s\) decreases, and \(\Delta_0\) changes also. If in some shell within the quark matter core of a neutron star (or within a strange quark star) the quark number densities are such that crystalline color superconductivity arises, rotational vortices may be pinned in this shell, making it a locus for glitch phenomena \[12,13\].

An analysis of these ideas in the context of the disruption of CFL pairing is complicated by the fact that in quark matter in which CFL pairing does not occur, up and down quarks may nevertheless continue to pair in the usual BCS fashion. In this 2SC phase, which was the earliest color superconducting phase to be studied \[1\], the attractive channel involves the formation of Cooper pairs which are antisymmetric in both color and flavor, yielding a condensate with color (Greek indices) and flavor (Latin indices) structure \(\langle q_\alpha^a q_\beta^b \rangle \sim \epsilon_{ab} \epsilon^{\alpha\beta} \).

This condensate leaves five quarks unpaired: up and down quarks of the third color, and strange quarks of all three colors. Because the BCS pairing scheme leaves ungapped quarks with differing Fermi momenta, crystalline color superconductivity may result.

To date, crystalline color superconductivity has only been studied in the simplified model context with pairing between two quark species whose Fermi momenta are pushed apart by turning on a chemical potential difference \[12,14–16\], rather than considering CFL pairing in the presence of quark mass differences. Our goal in this paper is to investigate the ways in which the response of the system to mass differences is similar to or different from the response to chemical potential differences. We can address this question within the two-flavor model by generalizing it to describe pairing between massless up quarks and strange quarks with mass \(m_s\). For completeness, we introduce

\[
\begin{align*}
\mu_a &= \mu - \delta \mu \\
\mu_s &= \mu + \delta \mu,
\end{align*}
\]

allowing us to consider the effects of \(m_s\) and \(\delta \mu\) simultaneously. We shall use this two-flavor toy model throughout, deferring an analysis of crystalline color superconductivity induced by the effects of \(m_s\) on three-flavor quark matter to future work.

A. Consequences of \(\delta \mu \neq 0\), with \(m_s=0\)

Before describing the consequences of \(m_s \neq 0\), let us review the salient facts known about the consequences of \(\delta \mu \neq 0\), upon taking \(m_s = 0\). If \(|\delta \mu|\) is nonzero but less than some \(\delta \mu_1\), the ground state in the two-flavor toy-model is precisely that obtained for \(\delta \mu = 0\) \[17,18,12\].

1In this two-flavor toy-model the diquark condensate is a flavor singlet. As the condensate breaks no flavor symmetries, there is no analogue of the rotations of the condensate in flavor space which occur within the CFL phase with nonzero \(\delta \mu\) \[19\].
In this 2SC state, red and green up and strange quarks pair, yielding four quasiparticles with superconducting gap $\Delta_0$. Furthermore, the number density of red and green up quarks is the same as that of red and green strange quarks. As long as $|\delta \mu|$ is not too large, this BCS state remains unchanged (and favored) because maintaining equal number densities, and thus coincident Fermi surfaces, maximizes the pairing and hence the interaction energy. As $|\delta \mu|$ is increased, the BCS state remains the ground state of the system only as long as its negative interaction energy offsets the large positive free energy cost associated with forcing the Fermi seas to remain coincident. In the weak coupling limit, in which $\Delta_0/\mu \ll 1$, the BCS state persists for $|\delta \mu| < \delta \mu_1 = \Delta_0/\sqrt{2}$ \cite{17,12}. For larger $\Delta_0$, the $1/\sqrt{2}$ coefficient changes in value. These conclusions are the same whether the interaction between quarks is modeled as a point-like four-fermion interaction or is approximated by single-gluon exchange \cite{4}.

The loss of BCS pairing at $|\delta \mu| = \delta \mu_1$ is the analogue in this toy model of the unlocking transition. If $|\delta \mu| > \delta \mu_1$, BCS pairing between $u$ and $s$ is not possible. However, in a range $\delta \mu_1 < |\delta \mu| < \delta \mu_2$ near the unpairing transition, it is favorable to form a crystalline color superconducting state in which the Cooper pairs have nonzero momentum. This phenomenon was first analyzed by Larkin and Ovchinnikov and Fulde and Ferrell \cite{11,12} (LOFF) in the context of pairing between electrons in which spin-up and spin-down Fermi momenta differ. It has proven difficult to find a condensed matter physics system which is well described simply as BCS pairing in the presence of a Zeeman effect: any magnetic perturbation that may induce a Zeeman effect tends to have much larger effects on the motion of the electrons, as in the Meissner effect. The QCD context of interest to us, in which the Fermi momenta being split are those of different flavors rather than of different spins, therefore turns out to be the natural arena for the phenomenon first analyzed by LOFF.

The crystalline color superconducting phase (also called the LOFF phase) has been described in Ref. \cite{12} (following Refs. \cite{11}) upon making the simplifying assumption that quarks interact via a four-fermion interaction with the quantum numbers of single gluon exchange. In the LOFF state, each Cooper pair carries momentum $2q$ with $|q| \approx 1.2\delta \mu$. The condensate and gap parameter vary in space with wavelength $\pi/|q|$. Although the magnitude $|q|$ is determined energetically, the direction $\hat{q}$ is chosen spontaneously. The LOFF state is characterized by a gap parameter $\Delta$ and a diquark condensate, but not by an energy gap in the dispersion relation: the quasiparticle dispersion relations vary with the direction of the momentum, yielding gaps that vary from zero up to a maximum of $\Delta$. The condensate is dominated by those regions in momentum space in which a quark pair with total momentum $2q$ has both members of the pair within $\sim \Delta$ of their respective Fermi surfaces. These regions form circular bands on the two Fermi surfaces. Making the ansatz that all Cooper pairs make the same choice of direction $\hat{q}$ corresponds to choosing a single circular band on each Fermi surface. In position space, it corresponds to a condensate which varies in space like

$$\langle \psi(x)\psi(x) \rangle \propto e^{2iq \cdot x}. \quad (3)$$

This ansatz is certainly not the best choice. If a single plane wave is favored, why not two? That is, if one choice of $\hat{q}$ is favored, why not add a second $q$, with the same $|q|$ but a different $\hat{q}$? If two are favored, why not three? This question, namely, the determination of the favored crystal structure of the crystalline color superconductor phase, is unresolved but
is under investigation. Note, however, that if we find a region $\delta \mu_1 < |\delta \mu| < \delta \mu_2$ in which the simple LOFF ansatz with a single $\hat{q}$ is favored over the BCS state and over no pairing, then the LOFF state with whatever crystal structure turns out to be optimal must be favored in *at least* this region. Note also that even the single $\hat{q}$ ansatz, which we use henceforth, breaks translational and rotational invariance spontaneously. The resulting phonon has been analyzed in Ref. [15].

Crystalline color superconductivity is favored within a window $\delta \mu_1 < |\delta \mu| < \delta \mu_2$. As $|\delta \mu|$ increases from 0, one finds a first order phase transition from the ordinary BCS phase to the crystalline color superconducting phase at $|\delta \mu| = \delta \mu_1$ and then a second order phase transition at $|\delta \mu| = \delta \mu_2$ at which $\Delta$ decreases to zero. Because the condensation energy in the LOFF phase is much smaller than that of the BCS condensate at $\delta \mu = 0$, the value of $\delta \mu_1$ is almost identical to that at which the naive unpairing transition from the BCS state to the state with no pairing would occur if one ignored the possibility of a LOFF phase, namely $\delta \mu_1 = \Delta_0 / \sqrt{2}$. For all practical purposes, therefore, the LOFF gap equation is not required in order to determine $\delta \mu_1$. The LOFF gap equation is used to determine $\delta \mu_2$ and the properties of the crystalline color superconducting phase [12]. In the limit of a weak four-fermion interaction, the crystalline color superconductivity window is bounded by $\delta \mu_1 = \Delta_0 / \sqrt{2}$ and $\delta \mu_2 = 0.754 \Delta_0$, as first demonstrated in Refs. [11]. These results have been extended beyond the weak four-fermion interaction limit in Ref. [12].

We now know that the use of the simplified point-like interaction significantly underestimates the width of the LOFF window: assuming instead that quarks interact by exchanging medium-modified gluons yields a much larger value of $\delta \mu_2$ [16]. This can be understood upon noting that quark-quark interaction by gluon exchange is dominated by forward scattering. In most scatterings, the angular positions on their respective Fermi surfaces do not change much. In the LOFF state, small-angle scattering is advantageous because it cannot scatter a pair of quarks out of the region of momentum space in which both members of the pair are in their respective circular bands, where pairing is favored. This means that it is natural that a forward-scattering dominated interaction like single-gluon exchange is much more favorable for crystalline color superconductivity that a point-like interaction, which yields $s$-wave scattering. Thus, although for the present we shall use the point-like interaction in our analysis of $m_s$-induced crystalline color superconductivity, it is worth remembering that this is very conservative.

### B. Consequences of $m_s \neq 0$

In the absence of any interaction, and thus in the absence of pairing, the effect of a strange quark mass is to shift the Fermi momenta to

\[
\begin{align*}
    p_s^\mu_F &= \mu - \delta \mu \\
    p_s^s &= \sqrt{(\mu + \delta \mu)^2 - m_s^2}.
\end{align*}
\]

Assuming both $|\delta \mu|/\mu$ and $m_s/\mu$ are small, the separation between the two Fermi momenta is $\approx 2|\delta \mu - m_s^2/2\mu|$. This suggests the conjecture that even when $m_s \neq 0$ the description given in the previous subsection continues to be valid upon replacing $|\delta \mu|$ by $|\delta \mu - m_s^2/4\mu|$. We shall see in Section 3 that this conjecture is *incorrect* in one key respect: whereas if
$m_s = 0$ a $|\delta \mu|$ which is nonzero but smaller than $\delta \mu_{1}$ has no effect on the BCS state, the BCS gap $\Delta_{0}$ decreases with increasing $m_s^2$, as we discuss further in an appendix. We show that for small $m_s^2$, $\Delta_{0}(m_s)/\Delta_{0}(0)$ decreases linearly with $m_s^2$. Because $\Delta_{0}$ occurs in the free energy in a term of order $\Delta_{0}^2\mu^2$, the $m_s$-dependence of $\Delta_{0}$ corrects the free energy by of order $\Delta_{0}(0)^2m_s^2$. As $\delta \mu$ has no analogous effect, we conclude that $m_s^2/4\mu$ and $\delta \mu$ have qualitatively different effects on the paired state.

At another level, however, the story is quite similar to that for $m_s = 0$: if $|\delta \mu - m_s^2/4\mu|$ is small enough, we find the BCS state; if $|\delta \mu - m_s^2/4\mu|$ lies within an intermediate window, we find LOFF pairing; if $|\delta \mu - m_s^2/4\mu|$ is large enough, no pairing is possible. The boundaries between the phases, however, are related to a $\Delta_{0}(m_s)$, rather than simply to a constant $\Delta_{0}$. That is, the definitions of "small enough" and "large enough" are $m_s$-dependent. We map the $(m_s, \delta \mu)$ plane in this paper. In Section 2, we derive a gap equation which we then use in Section 3 to analyze the BCS phase, finding the region of the $(m_s, \delta \mu)$ plane in which BCS pairing is favored, and again use in Section 4 to analyze crystalline color superconductivity, finding windows in the $(m_s, \delta \mu)$ plane in which LOFF pairing is favored.

II. THE GAP EQUATION

In this Section, we sketch the derivation of the gap equation which describes either the BCS or the LOFF states in a two-flavor theory with $m_s \neq 0$ and $\delta \mu \neq 0$. In the crystalline color superconducting phase, the condensate contains pairs of $u$ and $s$ quarks with momenta such that the total momentum of each Cooper pair is given by $2q$, with the direction of $q$ chosen spontaneously. As noted above, wherever there is an instability towards (6), we expect the true ground state to be a crystalline condensate which varies in space like a sum of several such plane waves with the same $|q|$. (To recover the BCS gap equation from that which we now derive, simply set $|q| = 0$.)

In order to describe pairing between $u$ quarks with momentum $p + q$ and $s$ quarks with momentum $-p + q$, we use a modified Nambu-Gorkov propagator:

$$\Psi(p, q) = \begin{pmatrix} \psi_u(p + q) \\ \psi_s(p - q) \\ \bar{\psi}_s^T(-p + q) \\ \bar{\psi}_u^T(-p - q) \end{pmatrix}.$$  (5)

Note that by $q$ we mean the four-vector $(0, q)$. The Cooper pairs have nonzero total momentum but the ground state condensate (6) is static. The momentum dependence of (5) is motivated by the fact that in the presence of a crystalline color superconducting condensate, anomalous propagation does not only mean picking up or losing two quarks from the condensate. It also means picking up or losing momentum $2q$. The basis (5) has been chosen so that the inverse fermion propagator in the crystalline color superconducting phase is diagonal in $p$-space and is given by

$$S^{-1}(p, q) = \begin{pmatrix} \hat{p} + \hat{q} + \mu_u \gamma_0 & 0 & -\bar{\Delta}(p, -q) & 0 \\ 0 & \hat{p} - \hat{q} + \mu_s \gamma_0 - m_s & 0 & \bar{\Delta}(p, q) \\ -\bar{\Delta}(p, -q) & 0 & (\hat{p} - \hat{q} - \mu_s \gamma_0 + m_s)^T & 0 \\ 0 & \Delta(p, q) & 0 & (\hat{p} + \hat{q} - \mu_u \gamma_0)^T \end{pmatrix}.$$  (6)
where $\bar{\Delta} = \gamma_0 \Delta^+ \gamma_0$ and the matrix $\Delta$ is given by $\Delta = \Delta C \gamma_5 \epsilon^{\alpha \beta \gamma \delta}$. Note that the condensate is explicitly antisymmetric in flavor. $2p$ is the relative momentum of the quarks in a given pair and is different for different pairs. In the gap equation below, we shall integrate over $p_0$ and $p$. As desired, the off-diagonal blocks describe anomalous propagation in the presence of a condensate of diquarks with momentum $2q$.

We obtain the gap equation by solving the one-loop Schwinger-Dyson equation given by

$$S^{-1}(k, q) - S_0^{-1}(k, q) = -g^2 \int \frac{d^4p}{(2\pi)^4} \Gamma^A \mu \pi \Gamma^B \mu \nu D_{AB}^{\mu \nu}(k - p),$$

(7)

where $D_{AB}^{\mu \nu} = D^{\mu \nu} \delta_{AB}$ is the gluon propagator, $S$ is the full quark propagator, whose inverse is given by (6), and $S_0$ is the fermion propagator in the absence of interaction, given by $S$ with $\Delta = 0$. The vertex is defined as

$$\Gamma^A \mu = \begin{pmatrix} \gamma^\mu \lambda^A / 2 & 0 & 0 & 0 \\ 0 & \gamma^\mu \lambda^A / 2 & 0 & 0 \\ 0 & 0 & -(\gamma^\mu \lambda^A / 2)^T & 0 \\ 0 & 0 & 0 & -(\gamma^\mu \lambda^A / 2)^T \end{pmatrix}.$$  

(8)

As in Refs. [12,14], we simplify the interaction between quarks by making it point-like. That is, we replace $g^2 D^{\mu \nu}$ by $3G g^{\mu \nu}$. Our model Hamiltonian has two parameters, the four-fermion coupling $G$ and a cutoff $\Lambda$ on the three-momentum $p$. The dependence on the cutoff has been studied in Ref. [12]: upon varying $\Lambda$ while at the same time varying the coupling $G$ such that $\Delta_0$ is kept fixed, the relation between other physical quantities and $\Delta_0$ is reasonably insensitive to variation of $\Lambda$. We often use the value of the BCS gap $\Delta_0$ obtained at $m_s = \delta \mu = 0$ to parametrize the strength of the interaction. We can do this because $\Delta_0$ increases monotonically with increasing $G$. Doing so proves convenient because both $\delta \mu_1$ and $\delta \mu_2$ are given simply when written in terms of the physical quantity $\Delta_0$, whereas writing them in terms of the model-dependent parameters $G$ and $\Lambda$ requires unwieldy expressions.

After some algebra (essentially the determination of $S$ given $S^{-1}$ specified above), and upon suitable projection, the Schwinger-Dyson equation (7) reduces to a gap equation for the gap parameter $\Delta$ given (in Euclidean space) by

$$\Delta = \int \frac{d^4p}{(2\pi)^4 \Delta^4 + 2\Delta^2 w^2 + [(p_0 + i \mu + i \delta \mu)^2 - |p - q|^2 - m_s^2]} ((p_0 - i \mu + i \delta \mu)^2 - |p + q|^2),$$

(9)

where

$$w^2 \equiv p^2 + \mu^2 - q^2 + (p_0 + i \delta \mu)^2.$$  

(10)

Eq. (9) reduces to the LOFF gap equation of Ref. [14] when $m_s = 0$. The gap equation (9) includes the contributions of pairing involving antiparticles, obviously far from the Fermi surfaces. These contributions can be removed using the methods of Ref. [14] (and if one does so and then sets $m_s = 0$ one obtains the gap equation of Ref. [12]). However, as the antiparticle contributions are suppressed by $\Delta / \mu$ and as removing them leads to rather unwieldy expressions, we shall simply use eq. (9).
III. THE BCS GAP AND THE BCS REGION

We expect BCS pairing to be possible when the $u$ and $s$ Fermi momenta in noninteracting quark matter are similar in value. From (4) we see that $p_u^F = p_s^F$ in noninteracting quark matter when

$$\delta \mu = \frac{m_s^2}{4 \mu}.$$  \hspace{1cm} (11)

Note that we have not assumed that $\delta \mu$ or $m_s$ is small compared to $\mu$ in this expression.

Let us now analyze the BCS gap equation, keeping this in mind. Upon setting $q = 0$, Eq. (9) reduces to the BCS gap equation of Ref. [8]:

$$\Delta_0 = \int \frac{d^4p}{(2\pi)^4} \frac{8G\Delta_0 (\Delta_0^2 + w_0^2)}{\Delta_0^2 + 2\Delta_0^2 w_0^2 + [(p_0 + i \mu + i \delta \mu)^2 - p^2 - m_s^2] [(p_0 - i \mu + i \delta \mu)^2 - p^2]}$$ \hspace{1cm} (12)

where

$$w_0^2 = p^2 + \mu^2 + (p_0 + i \delta \mu)^2$$

and where we use the notation $\Delta_0$ for the BCS gap, to keep track of the fact that here $q = 0$. Eq. (12) involves an integration over $p_0$ from $-\infty$ to $+\infty$, which can be thought of as a contour in the complex $p_0$-plane, which we choose to close in the upper half-plane. With $q = 0$, the poles in the integrand of Eq. (12) are easily analyzed. There are four poles in the $p_0$-plane. Of these, two correspond to the contribution of antiparticle pairing and have residues which are suppressed by $\Delta_0/\mu$ relative to the residues of the two remaining poles, which correspond to particle pairing. The residues of the two particle poles are equal in magnitude and opposite in sign. As a result, the existence of a nonzero $\Delta_0$ which solves (12) requires that one and only one of the particle poles lies in the upper half-plane. If we choose $\delta \mu$ according to (11), we do in fact find one particle pole in the upper half-plane and thus a nonzero, $m_s$-dependent, solution $\Delta_0$ as shown in Fig. 1. $\Delta_0$ decreases with $m_s^2$, linearly for small $m_s^2$. The $m_s$-dependence of $\Delta_0$ can be further understood, but as this would divert us we defer doing so to an appendix.

What is the effect of changing $\delta \mu$ away from the choice (11)? This is simply understood upon noticing that $p_0$ and $\delta \mu$ enter the gap equation only in the combination $p_0 + i \delta \mu$. This means that the only effect of $\delta \mu$ is to shift the location of the poles. Whereas $m_s$ affects the value of the residues, $\delta \mu$ does not. As long as $|\delta \mu - m_s^2/4\mu| \lesssim \Delta_0$, there is one particle pole in the upper half-plane and $\Delta_0$ is identical to that found when $\delta \mu = m_s^2/4\mu$, as in Fig. 1. Once $|\delta \mu - m_s^2/4\mu| \gtrsim \Delta_0$, there are either zero or two particle poles in the upper half-plane and therefore no such solutions $\Delta_0$ exist.

As long as $|\delta \mu - m_s^2/4\mu| \lesssim \Delta_0(m_s)$, we have found a $\delta \mu$-independent, $m_s$-dependent solution $\Delta_0(m_s)$ (shown in Fig. 1) that describes a BCS state which is a local minimum of the free energy $\Omega$ at a given $m_s$ and $\delta \mu$. However, this BCS state need not be global minimum.

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2The criterion is $|\delta \mu - m_s^2/4\mu| < \Delta_0$ in the limit in which all these quantities are small compared to $\mu$. As we shall see momentarily, this is not the most important criterion anyway, and we therefore do not specify it precisely.
FIG. 1. BCS gap $\Delta_0$ vs. $m_s^2/4\mu$, with $\delta\mu$ set equal to $m_s^2/4\mu$ to ensure that in the absence of any pairing, the $u$ and $s$ Fermi momenta would be equal. In this and in all subsequent figures, we set $\mu = 400$ MeV and choose a cutoff $\Lambda = 1000$ MeV. We have chosen three different values of the coupling constant $G$, yielding $\Delta_0 = 10, 40, 80$ MeV at $m_s = 0$. $\Delta_0$ decreases with increasing $m_s^2$, linearly for small $m_s^2$.

of the free energy. To determine the region in the $(m_s^2/4\mu, \delta\mu)$ plane in which BCS pairing is favored over unpaired quark matter, we must evaluate the free energy difference between the BCS state and the unpaired state. This is given by integrating the gap equation over $\Delta$ up to the equation’s solution, $\Delta_0(m_s)$.

$$\Omega_{\text{BCS}} - \Omega_{\text{unpaired}} = \int_0^{\Delta_0(m_s)} d\Delta \left( \frac{\Delta}{G} - \int d^4p \frac{\text{integrand}}{(2\pi)^4} \right), \quad (13)$$

where “integrand” refers to that of the right-hand side of Eq. (12). The curves in the $(m_s^2/4\mu, \delta\mu)$ plane along which the above expression vanishes determine the region in which the BCS phase is favored. These curves are plotted in Fig. 2. At any given $m_s$, BCS pairing is favored within a range of values of $\delta\mu$ around $m_s^2/4\mu$. Everywhere within this range, $\Delta_0$ takes on the same ($m_s$-dependent; $\delta\mu$-independent) value. The boundaries of the BCS region are given approximately by

$$\left| \delta\mu - \frac{m_s^2}{4\mu} \right| < \frac{\Delta_0(m_s)}{\sqrt{2}}, \quad (14)$$

derived in Ref. [8] and valid if $\Delta_0(m_s)$, $m_s$ and $\delta\mu$ are all small compared to $\mu$. We see that as $m_s$ increases and $\Delta_0(m_s)$ decreases, the BCS region narrows.
FIG. 2. The region in the \((m_s^2/4\mu, \delta\mu)\) plane where the BCS state is favored over unpaired quark matter. We have chosen a single value of the coupling, corresponding to \(\Delta_0(0) = 40\) MeV. At each value of \(m_s^2/4\mu\), \(\Delta_0(m_s)\) is independent of \(\delta\mu\) within the shaded region and zero outside it. \(\Delta_0(m_s)\) decreases with \(m_s^2/4\mu\) as shown in Fig. 1.

IV. THE CRYSTALLINE COLOR SUPERCONDUCTIVITY WINDOWS

We now determine the regions of the \((m_s^2/4\mu, \delta\mu)\) plane where the crystalline color superconducting phase, with \(|q| \neq 0\), is favored. These regions will be strips just above and just below the BCS region of Fig. 2. Let us denote the upper boundary of the BCS region \(\delta\mu_1(m_s)\). We now show that at a given \(m_s\), the crystalline color superconducting state is favored in some region \(\delta\mu_1(m_s) < \delta\mu < \delta\mu_2(m_s)\). There is also a crystalline color superconductivity window of \(m_s\)-dependent width just below the BCS region.

At \(\delta\mu = \delta\mu_2\), the gap \(\Delta\) in the LOFF state goes continuously to zero. This means that we can determine \(\delta\mu_2\) as follows. We set \(\Delta = 0\) in the gap equation (9), obtaining a “zero-gap relation” among \(\delta\mu, m_s\), and \(|q|\). We choose a value of \(m_s\), and find the largest value of \(\delta\mu\) at which the zero-gap relation can be satisfied. This is \(\delta\mu_2\). (The smallest value of \(\delta\mu\) at which the zero-gap relation can be satisfied is the lower boundary of the crystalline color superconductivity window below the BCS region of Fig. 2.) In Fig. 3 we plot \([\delta\mu_2(m_s) - \delta\mu_1(m_s)]/\Delta_0(m_s)\) vs. \(m_s^2/4\mu\) for four different values of the coupling, corresponding to \(\Delta_0(0) = 10, 40, 80, 100\) MeV. At \(m_s = 0\) and in the weak coupling limit in which \(\Delta_0(0) \rightarrow 0\), \([\delta\mu_2(0) - \delta\mu_1(0)]/\Delta_0(0) \rightarrow [0.754 - 0.707] = 0.047\). The results at \(m_s = 0\) with \(\Delta_0 \neq 0\) which we find in Fig. 3 agree with those derived in Ref. [12].

These results are symmetric about \(\delta\mu = 0\). For \(m_s \neq 0\), we now see that when it is scaled by \(\Delta_0(m_s)\), the width of the crystalline color superconductivity window changes little, for

\[\delta\mu_1(m_s)\] was determined in the previous section by comparing the free energy of the BCS and unpaired states. Here, we should in principle compare BCS and LOFF states to determine \(\delta\mu_1(m_s)\), but this makes very little difference [12].
FIG. 3. Upper panel: width of the crystalline color superconductivity window above the BCS region. At a given $m_s$, crystalline color superconductivity occurs for $\delta \mu_1 (m_s) < \delta \mu < \delta \mu_2 (m_s)$. Here, we show the width of the window $\delta \mu_2 (m_s) - \delta \mu_1 (m_s)$ relative to $\Delta_0 (m_s)$ for four values of the coupling. The four curves from top to bottom correspond to couplings for which $\Delta_0 (0) = 10, 40, 80, 100$ MeV. Lower panel: width of the crystalline color superconductivity window below the BCS region. The curves from top to bottom correspond to increasing couplings, as in the upper panel.

smaller values of $\Delta_0 (0)$. For larger values of $\Delta_0 (0)$, the width of the window increases as $m_s$ increases from zero. Indeed, we see that for $\Delta_0 (0) = 100$ MeV, the window has closed at $m_s = 0$ whereas crystalline color superconductivity continues to occur at large enough $m_s$.

What do we conclude from our results? First, it is clear that the appropriate variable to use to describe the width of the crystalline color superconductivity window is $\delta \mu / \Delta_0 (m_s)$, as opposed to $\delta \mu / \Delta_0 (0)$. When plotted as in Fig. 3 the width of the window is almost independent of $m_s$ at weak coupling, indicating that we have found the correct scaling variable. This means that if we were to plot the crystalline color superconducting windows as strips above and below the BCS region of Fig. 2, the “horizontal width” of the window...
on the $\delta\mu = 0$ axis, expressed as a window in $m_s^2/[4\mu\Delta_0(m_s)]$, is the same as the “vertical width” of the windows in $\delta\mu/\Delta_0(0)$ on the $m_s = 0$ axis. Thus, to the extent that the curves in Fig. 3 are horizontal we conclude that $m_s$-induced and $\delta\mu$-induced crystalline color superconductivity are equally robust. However, the curves in Fig. 3 are not quite horizontal. At all but the weakest of couplings, the width of the crystalline color superconductivity window increases with $m_s$, meaning that crystalline color superconductivity is somewhat more robust if it is $m_s$-induced than if it is $\delta\mu$-induced. Indeed, we find that at the moderate coupling corresponding to $\Delta_0(0) = 100$ MeV, $m_s$-induced crystalline color superconductivity occurs whereas $\delta\mu$-induced crystalline color superconductivity does not.

ACKNOWLEDGMENTS

We acknowledge helpful conversations with M. Alford, J. Bowers, B. Fore, A. Leibovich and E. Shuster.

Research supported in part by the DOE under cooperative research agreement DE-FC02-94ER40818. The work of JK is supported in part by a DOD National Defense Science and Engineering Graduate Fellowship.
FIG. 4. BCS gap $\Delta_0$ vs. $m_s^2/4\mu(m_s)$, with $\mu(m_s)$ taken to increase with increasing $m_s$ in just such a way that, in the absence of interaction, the Fermi momenta of both the $u$ and $s$ quarks are $m_s$-independent. We plot the ratio $\Delta_0(m_s)/\Delta_0(0)$, for couplings corresponding to $\Delta_0(0) = 10, 40, 80$ MeV, from bottom to top. The curves are essentially independent of the coupling when plotted in this way.

APPENDIX A: THE $M_S$-DEPENDENCE OF $\Delta_0$

In this appendix, we seek to better understand the $m_s$-dependence of the BCS gap $\Delta_0$, shown in Fig. 1. To the extent that the coupling can be taken as weak when $\Delta_0 = 80$ MeV, one might expect that when each of these three curves are scaled by their value of $\Delta_0(0)$, they should fall on top of one another. This does not occur. The reason for this can be understood upon first setting the interaction to zero. In the absence of pairing, increasing $m_s$ while setting $\delta\mu = m_s^2/4\mu$ to ensure that $p^u_F = p^s_F$ has two effects: the value of $p_F$ decreases, and the strange quark Fermi velocity decreases. Since two different effects are at play, it is not surprising that the curves in Fig. 1 do not scale trivially. To disentangle these effects, we have redone the analysis of Fig. 1 as follows. As we increase $m_s$, we now increase $\mu$, while keeping $\delta\mu = m_s^2/4\mu(m_s)$. As before, the latter condition ensures that, in the absence of interaction, $p^u_F = p^s_F$. Here, we choose $\mu(m_s)$ in just such a way that both $p^u_F$ and $p^s_F$ are $m_s$-independent. Any remaining $m_s$-dependence of $\Delta_0$ can now be seen as arising from the variation of the difference between the Fermi velocities in the unpaired state. The plot analogous to Fig. 1 is given in Fig. 4, and the scaling is now manifest: the curves are almost independent of coupling. Note also that $\Delta_0$ decreases linearly with $m_s^2$ as before, but the slope in Fig. 4 is much less than that in Fig. 1. This indicates that the $m_s$-dependence in Fig. 4 can be attributed more to the variation of $p_F$ in the unpaired state than to the variation of $v_F$ in the unpaired state.
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