VARIATIONAL PRINCIPLES OF INVARIANCE PRESSURES ON PARTITIONS

XING-FU ZHONG

School of Mathematics and Statistics, Guangdong University of Foreign Studies
Guangzhou 510006, China

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Abstract. We investigate the relations between Bowen and packing invariance pressures and measure-theoretical lower and upper invariance pressures for invariant partitions of a controlled invariant set respectively. We mainly show that Bowen and packing invariance pressures can be determined via the local lower and upper invariance pressures of probability measures, which are analogues of Billingsley’s Theorem for the Hausdorff dimension; and give variational principles between Bowen and packing invariance pressures and measure-theoretical lower and upper invariance pressures under some technical assumptions.

1. Introduction. Topological feedback entropy was first introduced by Nair et al. [14] by using invariant open covers to characterize the minimal data rate for making a subset of the state space invariant. Later, Colonius and Kawan [5] introduced invariance entropy, which is defined via spanning sets, to describe the exponential growth rate of the minimal number of different control functions sufficient for orbits to stay in a given set when starting in a subset of this set. The fact that these two entropies are equivalent was shown by Colonius, Kawan, and Nair [6]. Recently, Huang and Zhong [10] use the theory of Carathéodory-Pesin structure to obtain a dimension-like characterization for invariance entropy, which is called Bowen invariance entropy. We refer the reader to the monograph written by Kawan [11] for more about invariance entropy.

By choosing conditionally invariant measures and quasi-stationary measures, Colonius [2, 3] first introduced four notions of metric invariance entropy in analogy to the topological notion of invariance entropy of deterministic control systems. In [2], Colonius showed that the metric entropy of a given controlled invariant set coincides with the minimal entropy of coder-controllers associated with a quasi-stationary measure rendering that set invariant. Variational principle [17] for topological entropy [1] and measure theoretic entropy [12] in classical dynamical systems states that topological entropy is determined by the supremum of measure theoretic entropies. Feng and Huang gave variational principles between Bowen
and packing topological entropy and measure-theoretical lower and upper topological entropies of subsets respectively. Motivated by the works of Huang–Zhong [10], Feng–Huang [9], and Colonius [2, 3], Wang, Huang, and Sun [18] introduced packing invariance entropy and gave variational principles between Bowen and packing invariance entropies and measure-theoretical lower and upper invariance entropies in some special situations respectively.

Invariance pressure, as a generalization of invariance entropy, was first introduced by Colonius, Santana, and Cossich [8, 4]. Recently, the same authors in [7] obtain some bounds of invariance pressure and get an explicit formula for hyperbolic linear control systems. Zhong and Huang [19] introduced a version of invariance pressure in a way resembling Hausdorff dimension, which is called Bowen invariance pressure. On the other hand, Tang, Cheng, and Zhao [16] extended Feng and Huang’s result and gave variational principle between Pesin-Pitskel topological pressure (also called Bowen topological pressure) and measure-theoretic lower pressure.

Invariance feedback pressure via invariant open covers was first introduced by Colonius, Santana, and Cossich [8]. Wang, Huang, and Sun [18] introduced packing invariance pressure for invariant partitions and give corresponding variational principles which extend Theorems 6.4 and 7.2 in [18] (see Theorems 3.1 and 4.2).

The outline of this paper is as follows. In Section 2, we give the definitions and some basic properties of invariance pressures. In Sections 3 and 4, we respectively give invariance entropies and measure-theoretical lower and upper invariance entropies for invariant partitions. Encouraged by these works, we in this paper define packing invariance pressure and measure-theoretical lower and upper invariance pressures for invariant partitions and give corresponding variational principles which extend Theorems 6.4 and 7.2 in [18] (see Theorems 3.1 and 4.2). The Theorem of this paper is as follows. In Section 2, we give the definitions and some basic properties of invariance pressures. In Sections 3 and 4, we respectively give variational principles for Bowen and packing invariance pressures for some special cases.

2. Invariance pressures of invariant partitions. In this paper, we mainly consider a discrete-time control system on a metric space $X$ of the following form

$$x_{n+1} = F(x_n, u_n) = F_{u_n}(x_n), \quad n \in \mathbb{N}_0 = \{0, 1, \ldots\},$$

where $F$ is a map from $X \times U$ to $X$, $U$ is a compact set, and $F_{u}(\cdot) \equiv F(\cdot, u)$ is continuous for every $u \in U$. Given a control sequence $\omega = (\omega_0, \omega_1, \ldots)$ in $U$, the solution of (1) can be written as

$$\phi(k, x, \omega) = F_{\omega_{k-1}} \circ \cdots \circ F_{\omega_0}(x).$$

For convenience, we denote system (1) by $\Sigma = (\mathbb{N}_0, X, U, \mathcal{W}, \phi)$, where $\mathcal{W} \subset U^{\mathbb{N}_0}$ is a nonempty set such that

(i). if $\omega = (\omega_0, \omega_1, \ldots) \in \mathcal{W}$ and $m \in \mathbb{N}$ then $(\omega_m, \omega_{m+1}, \ldots) \in \mathcal{W}$;

(ii). if $m, n \in \mathbb{N}$ and

$$\omega^1 = (\omega^1_0, \omega^1_1, \ldots, \omega^1_{m-1}, \ldots), \omega^2 = (\omega^2_0, \omega^2_1, \ldots, \omega^2_{n-1}, \ldots) \in \mathcal{W},$$

then there exists $\omega^3 \in \mathcal{W}$ such that

$$\omega^3_{0,m+n-1} = (\omega^1_0, \omega^1_1, \ldots, \omega^1_{m-1}, \omega^2_0, \omega^2_1, \ldots, \omega^2_{n-1}).$$

Suppose $\Sigma = (\mathbb{N}_0, X, U, \mathcal{W}, \phi)$ is a system. A set $Q \subset X$ is said to be controlled invariant if for every $x \in Q$ there exists $\omega_x \in \mathcal{W}$ such that $\phi(\mathbb{N}_0, x, \omega_x) \subset Q$. In the rest of this paper, we always assume that $Q$ is a compact subset of $X$.

**Definition 2.1.** A triple $\mathcal{C} = (\mathcal{A}, \tau, v)$ is said to be an invariant partition of $Q$ if (i) $\tau \in \mathbb{N}$; (ii) $\mathcal{A} = \{A_1, \ldots, A_q\}$ is a Borel measurable partition of $Q$; (iii) $v$ is a map from $\mathcal{A}$ to $U^+$ such that $\phi(A_i, [0, \tau], v(A_i)) \subset Q$ for all $i = 1, \ldots, q$. 




Given $a \in \{1, \ldots, q\}$, let $\omega_a$ denote the control function $v(A_a)$. Put $S_{\omega} = \{1, \ldots, q\}$. Given $s = (s_0, s_1, \ldots, s_{N-1}) \in S_{\omega}^N$, $N \in \mathbb{N}$, we define

$$\omega_s = \omega_{s_0} \omega_{s_1} \cdots \omega_{s_{N-1}}.$$ 

Let

$$C_s(Q) = \{x \in Q : \phi(j, x, \omega_{s_0}, \ldots, s_{N-1}) \in A_{s_j}, \text{ for } j = 0, 1, \ldots, N - 1\}.$$ 

If $C_s(Q) \neq \emptyset$, then we say that $s$ is an admissible word of length $N$. Let $l(s)$ denote the length of $s$ and let $\mathcal{L}^j(C)$ denote the set of all admissible words of length $j$, $j \in \mathbb{N}$. Put $S(C) = \cup_{j=1}^{\infty} \mathcal{L}^j(C)$. Let $K \subset Q$. A subset $\mathcal{G} \subset S(C)$ is called a cover of $K$ with respect to $C$ provided that $K \subset \cup_{\mathcal{G}} C_s(Q)$.

Since $C = (\mathcal{A}, \tau, v)$ is an invariant partition, for each $x \in Q$ there exists a unique sequence in $S_{\omega}^N$ denoted by $C(x)$ such that

$$\phi(j, x, \omega_{c(j)}) \in A_{C_i(x)}, \forall j \in \mathbb{N}_0.$$

For $m \leq n$, we denote $C_m(x)C_{m+1}(x) \cdots C_n(x)$ by $C_{[m, n]}(x)$. Given $n \in \mathbb{N}$ and $x \in Q$, let

$$[C_n(x)] = \{y \in Q : C_i(y) = C_i(x), 0 \leq i < n\}.$$ 

It is obvious that $[C_n(x)]$ is measurable and $[C_n(x)] = C_{[n]}(Q)$, where $s_{[n]} = C_{[0, n]}(x)$. We call $[C_n(x)]$ the cylinder of $x$ of length $n$ with respect to $C$. If $x \neq y$ then either $[C_n(x)] = [C_n(y)]$ or $[C_n(x)] \cap [C_n(y)] = \emptyset$.

Let $C(U, \mathbb{R})$ denote the collection of all continuous functions from $U$ to $\mathbb{R}$. Given $f \in C(U, \mathbb{R})$ and $n \in \mathbb{N}$, let

$$S_n f(w) := \sum_{i=0}^{n-1} f(w_i), w \in U^n.$$ 

2.1. Bowen invariance pressure. Let $C = (\mathcal{A}, \tau, v)$ be an invariant partition of $Q$ and $f \in C(U, \mathbb{R})$. For any $K \subset Q$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, let

$$M_C(N, \alpha, K, Q, f) = \inf_{\mathcal{G} \subset S(C)} \left\{ \sum_{s \in \mathcal{G}} e^{-\alpha l(s) + S_l(s), f(\omega_s)} \right\},$$

where the infimum is taken over all finite or countable covers $\mathcal{G} \subset S(C)$ of $K$ with $l(s) \geq N$ for any $s \in S$. Since $M_C(N, \alpha, K, Q, f)$ is monotonically increasing with $N$, the following limit exists

$$m_C(\alpha, K, Q, f) := \lim_{N \to \infty} M_C(N, \alpha, K, Q, f).$$

It is routine to check that $m_C(\alpha, K, Q, f)$ as a function of $\alpha$ has a critical point $\alpha' \in [-\infty, +\infty]$ denoted by $P_C(K, Q, f)$ such that

$$m_C(\alpha, K, Q, f) = 0, \alpha > \alpha' \text{ and } m_C(\alpha, K, Q, f) = \infty, \alpha < \alpha'.$$

**Definition 2.2.** The quantity $P_C(K, Q, f)$ is called Bowen invariance pressure of $K$ with respect to $Q, C$, and $f$.

2.2. Lower and upper capacity invariance pressures. Let $C = (\mathcal{A}, \tau, v)$ be an invariant partition of $Q$ and $f \in C(U, \mathbb{R})$. For any $K \subset Q$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, we define

$$R_C(N, \alpha, K, Q, f) = \inf_{\mathcal{G} \subset \mathcal{L}^N(C)} \left\{ \sum_{s \in \mathcal{G}} e^{-\alpha^N + S_N, f(\omega_s)} \right\},$$

where the infimum is taken over all finite or countable covers $\mathcal{G} \subset S(C)$ of $K$ with $l(s) \geq N$ for any $s \in S$. Since $R_C(N, \alpha, K, Q, f)$ is monotonically increasing with $N$, the following limit exists

$$r_C(\alpha, K, Q, f) := \lim_{N \to \infty} R_C(N, \alpha, K, Q, f).$$

It is routine to check that $r_C(\alpha, K, Q, f)$ as a function of $\alpha$ has a critical point $\alpha'' \in [-\infty, +\infty]$ denoted by $P_C(K, Q, f)$ such that

$$r_C(\alpha, K, Q, f) = 0, \alpha > \alpha'' \text{ and } r_C(\alpha, K, Q, f) = \infty, \alpha < \alpha''.$$

**Definition 2.2.** The quantity $P_C(K, Q, f)$ is called Bowen invariance pressure of $K$ with respect to $Q, C$, and $f$. 

where the infimum is taken over all finite covers $G \subset L$ of $K$. Set

$$R_C(N, \alpha, K, Q, f) = \liminf_{N \to \infty} R_C(N, \alpha, K, Q, f),$$

$$\tau_C(N, \alpha, K, Q, f) = \limsup_{N \to \infty} R_C(N, \alpha, K, Q, f).$$

Define the “jump-up” points of $R_C(\alpha, K, Q, f)$ and $\tau_C(\alpha, K, Q, f)$ as

$$\mathcal{C}P_C(K, Q, f) = \inf\{a : \mathcal{R}_C(\alpha, K, Q, f) = 0\} = \sup\{a : \mathcal{R}_C(\alpha, K, Q, f) = \infty\},$$

$$\mathcal{C}P_C(K, Q, f) = \inf\{a : \mathcal{R}_C(\alpha, K, Q, f) = 0\} = \sup\{a : \mathcal{R}_C(\alpha, K, Q, f) = \infty\},$$

respectively.

**Definition 2.3.** We call the quantities $\mathcal{C}P_C(K, Q, f)$ and $\mathcal{C}P_C(K, Q, f)$ the lower and upper capacity invariance pressures of $K$ with respect to $Q, C$, and $f$, respectively.

2.3. **Packing invariance pressure.** Given $s, s' \in S(C)$, we say that $s$ is equal to $s'$ if $l(s) = l(s')$ and $s_i = s'_i$ for every $0 \leq i \leq l(s) - 1$. Given $G \subset S(C)$, we say that $G$ is pairwise disjoint if $C_s(Q) \cap C_{s'}(Q) = \emptyset$ for any $s, s' \in G$ with $s \neq s'$.

Let $C = (\omega, \tau, \nu)$ be an invariant partition of $Q$ and $f \in C(U, R)$. For any $K \subset Q, \alpha \in R$, and $N \in N$, define

$$P_C(N, \alpha, K, Q, f) = \sup_{\mathcal{G} \subset S(C)} \left\{ \sum_{s \in \mathcal{G}} e^{-a_l(s)\tau + S(s)} \right\},$$

where the supremum is taken over all finite or countable pairwise disjoint collections $\mathcal{G}$ such that $C_s(Q) \cap K \neq \emptyset$ and $l(s) \geq N$ for every $s \in \mathcal{G}$. The quantity $P_C(N, \alpha, K, Q, f)$ decreases as $N$ increases. Therefore the following limit exists:

$$P_C(\alpha, K, Q, f) = \lim_{N \to \infty} P_C(N, \alpha, K, Q, f).$$

Define

$$P_C(\alpha, K, Q, f) = \inf\{\sum_{i=1}^{\infty} P_C(\alpha, K_i, Q, f) : K \subset \cup_{i=1}^{\infty} K_i\}.$$

There clearly exists a unique point denoted by $P_C^P(K, Q, f)$ such that

$$\left\{\begin{array}{ll}
P_C(\alpha, K, Q, f) = \infty, & \alpha < P_C^P(K, Q, f); \\
P_C(\alpha, K, Q, f) = 0, & \alpha > P_C^P(K, Q, f).
\end{array}\right.$$ 

**Definition 2.4.** We call the quantity $P_C^P(K, Q, f)$ packing invariance pressure of $K$ with respect to $Q, C$, and $f$.

2.4. **Properties of invariance pressures.**

**Proposition 1.** Let $Q$ be controlled invariant and $C$ be an invariant partition of $Q$. For any $f \in C(U, R)$, the following assertions hold:

(i) If $K_1 \subset K_2 \subset Q$, then $\mathcal{P}(K_1, Q, f) \leq \mathcal{P}(K_2, Q, f)$ (monotonicity), where $\mathcal{P}$ denotes either $P_C$ or $\mathcal{C}P_C$ or $\mathcal{C}P_C^P$.

(ii) If $K_1 \subset Q, i \geq 1$, then

$$m_C(\alpha, \cup_{i=1}^{\infty} K_i, Q, f) \leq \sum_{i=1}^{\infty} m_C(\alpha, K_i, Q, f),$$

$$P_C(\alpha, \cup_{i=1}^{\infty} K_i, Q, f) \leq \sum_{i=1}^{\infty} P_C(\alpha, K_i, Q, f).$$
(iii). If \( K_i \subset Q, \ i \geq 1 \), then
\[
P_C(\cup_i K_i, Q, f) = \sup_i P_C(K_i, Q, \psi) \text{(countable stability)},
\]
\[
P'_C(\cup_i K_i, Q, f) = \sup_i P'_C(K_i, Q, \psi) \text{(countable stability)},
\]
\[
CP_C(\cup_i K_i, Q, f) \geq \sup_i CP_C(K_i, Q, \psi),
\]
\[
CP_C(\cup_i K_i, Q, f) \geq \sup_i CP'_C(K_i, Q, \psi).
\]

(iv). Assume \( K_i \subset Q \), where \( i = 1, \ldots, n \). Then
\[
CP_C(\cup_{i=1}^n K_i, Q, f) = \max_{1 \leq i \leq n} CP_C(K_i, Q, f) \text{(finite stability)}.
\]

(v). \( P_C(K, Q, f) \leq CP_C(K, Q, f) \leq CP'_C(K, Q, f) \) for any \( K \subset Q \).

Proof. (i) and (v) follow directly from the definitions of invariance pressures. The first inequality in (ii) comes from (3) of Proposition 1.1 in [15] and we shall prove the second one. Given \( \delta > 0 \), for every \( i \in \mathbb{N} \), since
\[
P_C(\alpha, K_i, Q, f) = \inf \left\{ \sum_{j=1}^{\infty} P_C(\alpha, K_j, Q, f) : K_i \subset \bigcup_{j=1}^{\infty} K_j \right\},
\]
we can choose a sequence of subsets \( \{K_j\}_{j \in \mathbb{N}} \) (denoted by \( \{K_{ij}\}_{j \in \mathbb{N}} \)) such that \( K_i \subset \bigcup_{j=1}^{\infty} K_{ij} \) and
\[
\sum_{j=1}^{\infty} P_C(\alpha, K_{ij}, Q, f) < P_C(\alpha, K_i, Q, f) + \frac{\delta}{2^i}.
\]
It follows from that
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_C(\alpha, K_{ij}, Q, f) \leq \sum_{i=1}^{\infty} P_C(\alpha, K_i, Q, f) + \delta.
\]
Let
\[
K = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{K_{ij}\}.
\]
Let us rearrange the elements in \( K \), say,
\[
K = \{K_{11}, K_{21}, K_{12}, K_{31}, K_{32}, K_{33}, \ldots\} = \{L_1, L_2, L_3, L_4, L_5, L_6, \ldots\}.
\]
Hence \( K \) is at most countable, and \( K \subset \bigcup_{i=1}^{\infty} L_i \). This yields that
\[
P_C(\alpha, K, Q, f) \leq \sum_{i=1}^{\infty} P_C(\alpha, K_i, Q, f) + \delta.
\]
Since \( \delta \) is arbitrarily small, we get the desired result. The third and fourth inequalities in (iii) follow from monotonicity. We now show the other equalities in (iii). If \( s > \mathcal{P}(K_i, Q, f) \) for all \( i \in \mathbb{N} \), where \( \mathcal{P} \) denotes either \( P_C \) or \( P'_C \), then \( \mathcal{P}'(s, K_i, Q, f) = 0 \), where \( \mathcal{P}' \) denotes either \( m_C \) or \( P_C \). This implies \( \mathcal{P}'(s, \cup_i K_i, Q, f) = 0 \) by (ii). Hence \( \mathcal{P}(\cup_i K_i, Q, f) \leq s \). The opposite inequalities follow from monotonicity. (iv) comes from Theorem 2.4 in [15].
Given \( n \in \mathbb{N} \), \( f \in C(U, \mathbb{R}) \), and \( K \subset Q \), let
\[
\Lambda(C(n, K, Q, f)) = \inf_{G \subset L^n(C)} \left\{ \sum_{s \in G} e^{S_{n\tau}f(\omega_s)} \right\},
\]
where the infimum is taken over all finite covers \( G \subset L^n(C) \) of \( K \). From Theorem 2.2 in [15], we have

**Proposition 2.** For each \( K \subset Q \), \( f \in C(U, \mathbb{R}) \),
\[
\text{CP}_C(K, Q, f) = \liminf_{n \to \infty} \frac{1}{n\tau} \log \Lambda(C(n, K, Q, f)),
\]
\[
\text{CP}_C(K, Q, f) = \limsup_{n \to \infty} \frac{1}{n\tau} \log \Lambda(C(n, K, Q, f)).
\]

**Theorem 2.5.** Let \( Q \) be controlled invariant, \( C = (\mathcal{A}, \tau, \nu) \) be an invariant partition of \( Q \), and \( K \subset Q \). Then for any \( f \in C(U, \mathbb{R}) \) we have
\[
P_C(K, Q, f) \leq P^P_C(K, Q, f).
\]
If furthermore \( f \) is a constant function, then
\[
P^P_C(K, Q, f) \leq \text{CP}_C(K, Q, f).
\]

**Proof.** We shall prove the first inequality. Without loss of generality, we can assume that \( P_C(K, Q, f) > -\infty \). Let \( \alpha \in (-\infty, P_C(K, Q, f)) \). For any \( n \in \mathbb{N} \) and \( K' \subset Q \), since \( L^n(C) \) is finite and \( K' \subset Q = \bigcup_{s \in L^n(C)} C_s(Q) \), we can find a disjoint collection \( G \subset L^n(C) \) such that \( K' \subset \bigcup_{s \in G} C_s(Q) \). This yields that
\[
M_C(n, \alpha, K', Q, f) \leq \sum_{s \in G} e^{-n\tau\alpha + S_{n\tau}f(\omega_s)} \leq P_C(n, \alpha, K', Q, f).
\]
By taking \( n \to \infty \), we have
\[
m_C(\alpha, K', Q, f) \leq P_C(\alpha, K', Q, f).
\]
If \( K \subset \bigcup_{i=1}^\infty K_i \), then we see from (ii) of Proposition 1 that
\[
+\infty = m_C(\alpha, K, Q, f) \leq \sum_{i=1}^\infty m_C(\alpha, K_i, Q, f) \leq \sum_{i=1}^\infty P_C(\alpha, K_i, Q, f).
\]
Hence \( P_C(\alpha, K, Q, f) = +\infty \) and \( P^P_C(K, Q, f) \geq \alpha \), which implies the desired inequality.

We now prove the second inequality. Suppose
\[
-\infty < \beta < \alpha < P^P_C(K, Q, f),
\]
Then for any \( N \in \mathbb{N} \) we have
\[
P_C(N, \alpha, K, Q, f) = \infty.
\]
This implies that there exists a disjoint collection \( G \subset S(C) \) such that \( C_s(Q) \cap K \neq \emptyset \), \( l(s) \geq N \) for every \( s \in G \), and
\[
\sum_{s \in G} e^{-\alpha l(s)\tau + S_{l(s)\tau}f(\omega_s)} = \sum_{s \in G} e^{-\alpha l(s)\tau + l(s)c\tau} > 1,
\]
where \( c := f(u), u \in U \). For each \( m \), let \( l_m \) be the number of \( s \) such that \( l(s) = m \). Thus
\[
\sum_{m=N}^\infty l_m e^{-\alpha m\tau + mct} > 1.
\]
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There must be some $m \geq N$ with $l_m > e^{m(\beta-c)\tau}(1 - e^{(\beta-c)\tau})$, otherwise
\[
\sum_{m=N}^{\infty} l_m e^{-m\alpha\tau + mc\tau} \leq \sum_{m=1}^{\infty} e^{m(\beta-c)\tau}(1 - e^{(\beta-c)\tau}) e^{-m\alpha\tau + mc\tau} = (1 - e^{\tau(\beta-c)}) \sum_{m=1}^{\infty} e^{\tau(\beta-c)m} < 1.
\]
Therefore, we have
\[
\Lambda_c(m, K, Q, f) \geq e^{m(\beta-c)\tau}(1 - e^{(\beta-c)\tau}) e^{mc\tau} = e^{m\beta\tau}(1 - e^{(\beta-c)\tau})
\]
By Proposition 2, we obtain $\overline{CP}_c(K, Q, f) \geq \beta$. \\
Using Theorem 1 in [19], we have

**Theorem 2.6.** Let $Q$ be controlled invariant and $f \in C(U, \mathbb{R})$. Then
1. $\overline{CP}_c(Q, Q, f) = \overline{CP}_c(Q, Q, f)$.
2. If the infimum in Eq. (2) is taken over all finite collections, then
\[
P_c(Q, Q, f) = \overline{CP}_c(Q, Q, f) = \overline{CP}_c(Q, Q, f).
\]

**Corollary 1.** Under the assumptions of Theorem 2.6, if moreover $f$ is a constant function, then
\[
P_c(Q, Q, f) = \overline{CP}_c(Q, Q, f) = P_c^p(Q, Q, f) = \overline{CP}_c(Q, Q, f).
\]

Using Proposition 5 in [19], we have

**Proposition 3** (Time discretization). Suppose $f \in C(U, \mathbb{R})$. Then for any $m \in \mathbb{N}$ we have
\[
\overline{CP}_c(K, Q, f) = \limsup_{N \to \infty} \frac{\log \Lambda_c(mN, K, Q, f)}{mN},
\]
\[
P_c(K, Q, f) = \liminf_{N \to \infty} \frac{\log \Lambda_c(mN, K, Q, f)}{mN}.
\]

2.5. **Measure-theoretic invariance pressures.** Let $\mathcal{M}(Q)$ denote the set of all Borel probability measures on $Q$.

**Definition 2.7.** Let $\mu \in \mathcal{M}(Q)$. Given a controlled invariant set $Q$, an invariant partition $\mathcal{C}$ of $Q$, and $f \in C(U, \mathbb{R})$, the measure-theoretical lower and upper invariance pressures of $\mu$ for $Q$ with respect to $\mathcal{C}$ are defined respectively by
\[
h_{\mu, \mathcal{C}}(Q, f) = \int_Q h_{\mu, \mathcal{C}}(x, f) \, d\mu(x), \quad \overline{h}_{\mu, \mathcal{C}}(Q, f) = \int_Q \overline{h}_{\mu, \mathcal{C}}(x, f) \, d\mu(x),
\]
where
\[
h_{\mu, \mathcal{C}}(x, f) = \liminf_{n \to \infty} \frac{1}{n\tau} \left[ -\log \mu([\mathcal{C}_n(x)]) + S_{n\tau} f(\omega_{[0,n]}(x)) \right],
\]
\[
\overline{h}_{\mu, \mathcal{C}}(x, f) = \limsup_{n \to \infty} \frac{1}{n\tau} \left[ -\log \mu([\mathcal{C}_n(x)]) + S_{n\tau} f(\omega_{[0,n]}(x)) \right].
\]
Remark 1. It is easy to check that $h_{\mu,C}(x, f)$ and $\mu_{\mu,C}(x, f)$ are measurable by using the fact that $[C_n(x)] = [C_n(y)]$ for any $y \in [C_n(x)]$. Let $m = \inf_{u \in U} f(u)$. Then

$$m \leq \frac{1}{n^T} \left[ -\log \mu([C_n(x)]) + S_{nT} f(\omega_{x,[0,n)}(x)) \right].$$

This implies that the negative parts of $h_{\mu,C}(x, f)$ and $\mu_{\mu,C}(x, f)$ are finite. Hence $h_{\mu,C}(Q, f)$ and $\mu_{\mu,C}(Q, f)$ are well defined.

Remark 2. If $U = \{ u \}$ is a singleton, then $\Sigma = (N_0, X, U, \mathcal{U}, \phi)$ is a classical dynamical system, where $N_0 = \{ 0, 1, 2, \ldots \}$, and we write $\Sigma$ as $(X, T)$, where $T = F_0$. If furthermore $\mu$ is a $T$-ergodic Borel probability measure, then by Shannon-McMillan-Breiman theorem we see

$$\lim_{n \to \infty} \frac{-\log \mu([C_n(x)])}{n} = h_\mu(T, \mathcal{A})$$

for a.e. $x \in Q$, where $\mathcal{A}$ is the partition in $\mathcal{C}$ and $h_\mu(T, \mathcal{A})$ is the entropy of $T$ with respect to $\mathcal{A}$ (see Definition 4.9 in [17]). Thus

$$h_{\mu,C}(Q, f) = \mu_{\mu,C}(Q, f) = h_\mu(T, \mathcal{A}) + f(u).$$

Ma and Wen showed that Bowen entropy can be determined via the local entropies of measures in [13, Theorem 1], which is an analogue of Billingsley’s Theorem for the Hausdorff dimension. Next, we shall give analogous results for Bowen and packing invariance pressures. The proof is adapted from [13, Theorem 1] and [18, Theorems 5.2 and 5.3].

Theorem 2.8. Let $\mathcal{C} = (\mathcal{A}, \tau, v)$ be an invariant partition of $Q$, $\mu \in \mathcal{M}(Q)$, and $f$ be a continuous function on $U$. For $\beta \in \mathbb{R}$, the following assertions hold:

1. If $h_{\mu,C}(x, f) \leq \beta$ for every $x \in K$, then $P_C(K, Q, f) \leq \beta$;
2. If $h_{\mu,C}(x, f) \geq \beta$ for every $x \in K$ and $\mu(K) > 0$, then $P_C(K, Q, f) \geq \beta$;
3. If $\mu_{\mu,C}(x, f) \leq \beta$ for every $x \in K$, then $P_C^p(K, Q, f) \leq \beta$;
4. If $\mu_{\mu,C}(x, f) \geq \beta$ for every $x \in K$ and $\mu(K) > 0$, then $P_C^p(K, Q, f) \geq \beta$.

Proof. 1. Given $r > 0$, for any $x \in K$, there exists a strictly increasing sequence $(n_j(x))_{j=1}^\infty$ such that

$$\mu([C_{n_j(x)}(x)]) \geq e^{-n_j(x)\tau(\beta+r)+S_{n_j(x)} f(\omega_{x,[0,n_j(x)]}(x))}.$$

For any $N \geq 1$, let

$$\mathcal{F}_N = \{ [C_{n_j(x)}(x)] : x \in K, n_j(x) \geq N \}.$$

Then $K$ is contained in the union of the sets in $\mathcal{F}_N$. Since $\mathcal{F}_N \subset \{ \mathcal{C}_s(Q) : s \in \mathcal{S}(\mathcal{C}) \}$ and $\mathcal{S}(\mathcal{C})$ is at most countable, $\mathcal{F}_N$ is at most countable. It is clear that $\mathcal{C}_s(Q) \neq \mathcal{C}_{s'}(Q)$ if $s \neq s'$ with $l(s) = l(s')$. Hence, by induction, we can pick a finite or countable pairwise disjoint subfamily $\mathcal{G} \subset \mathcal{S}(\mathcal{C})$ such that

$$K \subset \bigcup_{s \in \mathcal{G}} \mathcal{C}_s(Q), \text{ and } \mathcal{C}_s(Q) \in \mathcal{F}_N, \forall \ s \in \mathcal{G}.$$

This implies that

$$M_C(N, \beta + r, K, Q, f) \leq \sum_{s \in \mathcal{G}} e^{-l(s) \tau(\beta+r)+S_{l(s)} f(\omega_s)} \leq \sum_{s \in \mathcal{G}} \mu(\mathcal{C}_s(Q)) \leq 1.$$

It follows that

$$m_C(\beta + r, K, Q, f) \leq 1 \text{ and } P_C(K, Q, f) \leq \beta + r.$$
Since \( r \) is arbitrary, we get \( P_c(K, Q, f) \leq \beta \).

2. Fix \( r > 0 \). Given \( N \in \mathbb{N} \), put

\[
K_N = \{ x \in K : \frac{1}{n^r} \left[ -\log \mu([C_n(x)]) + S_n f(\omega^{[0,n]}(x)) \right] > \beta - r, \ \forall \ n \geq N \}.
\]

Since the sequence \( \{ K_N \}_{N=1}^{\infty} \) increases to \( K \), \( \lim_{N \to \infty} \mu(K_N) = \mu(K) \). Then we can pick \( M \geq 1 \) such that \( \mu(K_M) > \frac{1}{2} \mu(K) \). For any \( x \in K_M \), we have

\[
\mu([C_n(x)]) \leq e^{-n^r(\beta - r) + S_n f(\omega^{[0,n]}(x))}, \ \forall \ n \geq M.
\]

Similar to the proof for 1, we have

\[
P_c(K_M, Q, f) \geq \beta - r,
\]

which implies that, by Proposition 1 and arbitrariness of \( r \),

\[
P_c(K, Q, f) \geq \beta.
\]

3. For any \( \alpha > \beta \) and \( N \in \mathbb{N} \), let \( r = \frac{\alpha - \beta}{2} \) and

\[
K_N = \{ x \in K : \mu([C_n(x)]) \geq e^{-n^r(\alpha - r) + S_n f(\omega^{[0,n]}(x))}, \ \forall \ n \geq N \}.
\]

Thus \( K = \bigcup_{N=1}^{\infty} K_N \). Fix \( M \geq N \). Let \( \mathcal{G} \subset \mathcal{S}(\mathcal{C}) \) be a finite or countable pairwise disjoint collection such that \( C_s(Q) \cap K_N \neq \emptyset \) and \( l(s) \geq M \). Pick \( x_s \in C_s(Q) \cap K_N \) for \( s \in \mathcal{G} \). Then

\[
\sum_{s \in \mathcal{G}} e^{-l(s)\tau(\alpha) + S_l f(\omega_s)} = \sum_{s \in \mathcal{G}} e^{-l(s)\tau(\alpha - r) - l(s)\tau + S_l f(\omega^{[0,l]}(x_s))}
\]

\[
\leq e^{-Mr^\tau} \sum_{s \in \mathcal{G}} e^{-l(s)\tau(\alpha - r) + S_l f(\omega^{[0,l]}(x_s))}
\]

\[
\leq e^{-Mr^\tau} \mu([C_l(x_s)])
\]

\[
\leq e^{-Mr^\tau}.
\]

This shows that \( P(M, \alpha, K_N, Q, f) \leq e^{-Mr^\tau} \). Thus \( P(\alpha, K_N, Q, f) = 0 \) and

\[
P(\alpha, K, Q, f) \leq \sum_{N=1}^{\infty} P(\alpha, K_N, Q, f) = 0.
\]

It follows that \( P^P_c(K, Q, f) \leq \alpha \). Therefore, we get the desired result.

4. Given \( \alpha < \beta \), we shall show \( P^P_c(K, Q, f) \geq \alpha \). To see this, we only need to show that \( P(\alpha, K', Q, f) = \infty \) for any \( K' \subset K \) with \( \mu(K') > 0 \). Then \( r = \frac{\beta - \alpha}{2} \). Then there exists a strictly increasing sequence \( \{ n_j(x) \}_{j=1}^{\infty} \) such that

\[
\mu([C_{n_j}(x)]) \leq e^{-n_j(x)^r(\alpha + r) + S_{n_j}(x) f(\omega^{[0,n_j]}(x))}.
\]

For any \( N \geq 1 \), let

\[
K_N = \{ [C_{n_j}(x)] : x \in K', \ n_j(x) \geq N \}.
\]

Then \( K' \) is contained in the union of the sets in \( K_N \). Similar to the proof of 2, we can find a finite or countable pairwise disjoint subfamily \( \mathcal{G} \subset \mathcal{S}(\mathcal{C}) \) such that

\[
K' \subset \bigcup_{s \in \mathcal{G}} C_s(Q), \ \text{and} \ C_s(Q) \subset K_N, \ \forall \ s \in \mathcal{G}.
\]
This yields that
\[ P_C(N, \alpha, K', Q, f) \geq \sum_{s \in G} e^{-l(s)\tau(\alpha+\tau r)S_l(s)}f(\omega_s) \]
\[ \geq e^{N\tau r} \sum_{s \in G} \mu(s) \]
\[ \geq e^{N\tau r} \sum_{s \in G} \mu(C_s(Q)) \geq e^{N\tau r} \mu(K'). \]

Taking \( N \to \infty \), we get \( P(\alpha, K', Q, f) = \infty \). \( \square \)

3. Variational principle for Bowen invariance pressure. Let \( C = (\mathcal{A}, \tau, v) \) be an invariant partition. We say that \( C \) is a clopen invariant partition if every element in \( \mathcal{A} \) is closed and open. The main theorem in this section is

**Theorem 3.1.** Let \( C = (\mathcal{A}, \tau, v) \) be a clopen invariant partition of \( Q \), where \( Q \) is a controlled invariant compact set, and \( f \) be a continuous function on \( U \). If \( K \subset Q \) is non-empty and compact, then
\[ P_C(K, Q, f) = \sup \{ h_{\mu, C}(Q, f) : \mu \in \mathcal{M}(Q), \mu(K) = 1 \}. \]

**Remark 3.** If \( f \) is null, then Theorem 6.4 in [18] is a special case of Theorem 3.1.

3.1. Weighted invariance pressure. Let \( Q \) be a controlled invariant set, \( C = (\mathcal{A}, \tau, v) \) be an invariant partition. For any \( \alpha \in \mathbb{R}, N \in \mathbb{N}, f \in C(U, \mathbb{R}) \), and bounded function \( g : Q \to \mathbb{R} \), define
\[ W_C(N, \alpha, g, Q, f) = \inf \{ \sum_s \gamma_s e^{-\alpha l(s)\tau + S_l(s)} f(\omega_s) \}, \]
where the infimum is taken over all finite or countable families of \( \{(s, \gamma_s)\} \) such that \( 0 < \gamma_s < \infty, s \in S(C), l(s) \geq N \) for all \( s \), and
\[ \sum_s \gamma_s \chi_{C_s(Q)} \geq g, \]
where \( \chi_B \) denotes the characteristic function of \( B \), i.e., \( \chi_B(x) = 1 \) if \( x \in B \) and 0 if \( x \in X \setminus B \). For \( K \subset Q \) and \( g = \chi_K \), we set
\[ W_C(N, \alpha, K, Q, f) = W_C(N, \alpha, K, Q, f). \]
Since the quantity \( W_C(N, \alpha, K, Q, f) \) increases as \( N \) increases, the following limit exists:
\[ W_C(\alpha, K, Q, f) = \lim_{N \to \infty} W_C(N, \alpha, K, Q, f). \]

Obviously, there exists a critical value denoted by \( P_C^W(K, Q, f) \) such that
\[ \begin{cases} W_C(\alpha, K, Q, f) = \infty, & \alpha < P_C^W(K, Q, f); \\ W_C(\alpha, K, Q, f) = 0, & \alpha > P_C^W(K, Q, f). \end{cases} \]

**Definition 3.2.** We call \( P_C^W(K, Q, f) \) weighted invariance pressure of \( K \) with respect to \( Q, C \), and \( f \).

**Remark 4.** If \( f = 0 \) then weighted invariance pressure is the weighted \( C\)-P dimension [18].
Proposition 4. Let $Q$ be a controlled invariant set and $C = \{s', \tau, v\}$ be an invariant partition of $Q$. For any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

\[ M_C(N, \alpha + \varepsilon, K, Q, f) \leq W_C(N, \alpha, K, Q, f) \leq M_C(N, \alpha, K, Q, f) \]

for any $n \geq N$. Consequently, we obtain $P_C(K, Q, f) = P^W_C(K, Q, f)$.

Proof. Let $K \subset Q$, $\alpha \in \mathbb{R}$. Taking $f = \chi_K$ and $\gamma_s = 1$ in the definition (4), we see that the second inequality holds. In the following, we prove the first inequality holds when $N$ is large enough.

Given $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, choose $N \in \mathbb{N}$ such that $n^2 e^{-n \varepsilon \tau} \leq 1$ for any $n \geq N$. Let $\{(s, \gamma_s)\}_{s \in \mathcal{G}}$ be a family such that $\mathcal{G}$ is at most countable, $s \in \mathcal{S}(\mathcal{C})$, $0 < \gamma_s < \infty$, $l(s) \geq N$, and

\[ \sum_s \gamma_s \chi_{C_s(Q)} \geq \chi_K. \]

We show below that

\[ M_C(N, \alpha + \varepsilon, K, Q, f) \leq \sum_{s \in \mathcal{G}} \gamma_s e^{-\alpha l(s) \tau + S(l(s), f(\omega_s))}, \]

which implies

\[ M_C(N, \alpha + \varepsilon, K, Q, f) \leq W_C(N, \alpha, K, Q, f). \]

For any $n \geq N$, let $\mathcal{G}_n = \{s \in \mathcal{G} : l(s) = n\}$. Without loss of generality, we can assume that $C_s(Q) \cap C_{s'}(Q) = \emptyset$ for $s, s' \in \mathcal{G}_n$ with $s \neq s'$ (otherwise we replace $(C_s(Q), \gamma_s)$ and $(C_{s'}(Q), \gamma_{s'})$ by $(C_s(Q), \gamma_s + \gamma_{s'})$). For $t > 0$, put

\[ K_{n,t} = \{x \in K : \sum_{s \in \mathcal{G}_n} \gamma_s \chi_{C_s(Q)}(x) > t\}, \]

and

\[ \mathcal{G}_{n,t} = \{s \in \mathcal{G}_n : K_{n,t} \cap C_s(Q) \neq \emptyset\}. \]

Since $K_{n,t} \subset \bigcup_{s \in \mathcal{G}_{n,t}} C_s(Q)$, we have

\[ M_C(N, \alpha + \varepsilon, K_{n,t}, Q, f) \leq \sum_{s \in \mathcal{G}_{n,t}} e^{-(\alpha + \varepsilon)n \tau + S_{n,t}f(\omega_s)} \]

\[ \leq \sum_{s \in \mathcal{G}_{n,t}} \frac{\gamma_s}{t} e^{-(\alpha + \varepsilon)n \tau + S_{n,t}f(\omega_s)} \]

\[ = \frac{1}{n^2 t} \sum_{s \in \mathcal{G}_{n,t}} \gamma_s e^{-\alpha \tau + S_{n,t}f(\omega_s)} \]

\[ \leq \frac{1}{n^2 t} \sum_{s \in \mathcal{G}_{n,t}} \gamma_s e^{-\alpha \tau + S_{n,t}f(\omega_s)}. \]

Noting that $\sum_{n=N}^{\infty} n^{-2} < 1$, we claim that $K = \bigcup_{n=N}^{\infty} K_{n,n-2t}$. If

\[ x \in K \setminus \bigcup_{n=N}^{\infty} K_{n,n-2t}, \]

then

\[ 1 \leq \sum_{s \in \mathcal{G}} \gamma_s \chi_{C_s(Q)}(x) \leq \sum_{n=N}^{\infty} \sum_{s \in \mathcal{G}_n} \gamma_s \chi_{C_s(Q)}(x) < 1, \]
which is a contradiction. Hence
\[
M_C(N, \alpha + \varepsilon, K, Q, f) \leq \sum_{n=N}^{\infty} M_C(N, \alpha + \varepsilon, K, Q, f)
\]
\[
\leq \sum_{n=N}^{\infty} \frac{1}{t} \sum_{s \in g_n} \gamma_s e^{-\alpha n \tau + S_n \tau f(\omega_s)}
\]
\[
= \frac{1}{t} \sum_{s \in g} \gamma_s e^{-\alpha n \tau + S(\varepsilon) \tau f(\omega_s)}.
\]
We finish the proof by letting \( t \to 1 \). \( \square \)

3.2. A dynamical Frostman’s lemma and the proof of Theorem 3.1. To prove the first equality in Theorem 3.1, we need the following dynamical Frostman’s lemma, which is an analogue of the classical Frostman’s lemma for compact metric space. Our proof is adapted from [9, Lemma 3.4] and [18, Lemma 6.3].

**Lemma 3.3.** Let \( C = (\mathcal{A}, \tau, v) \) be a clopen invariant partition of \( Q \), where \( Q \) is a controlled invariant compact set, and \( K \) be a non-empty compact subset of \( Q \). Let \( \alpha \in \mathbb{R}, N \in \mathbb{N} \), and \( f \in C(U, \mathbb{R}) \). Suppose that \( \gamma := W_C(N, \alpha, K, Q, f) > 0 \). Then there exists a Borel probability measure \( \mu \) on \( Q \) such that \( \mu(K) = 1 \) and
\[
\mu([C_n(x)]) \leq \frac{1}{\gamma} e^{-n \alpha + S_n \tau f(\omega_{c_{0,n}(x)})} := \beta_{x,n}, \quad \forall \ x \in Q, n \geq N.
\]

**Proof.** By definition, we have \( c < \infty \). We define a function \( p \) on the space \( C(Q) \) of continuous real-valued functions on \( Q \) by
\[
p(g) = \frac{1}{\gamma} W_C(N, \alpha, \chi_K \cdot g, Q, f).
\]
Let \( 1 \) denote the constant function \( 1(x) \equiv 1 \). It is easy to check that
1. \( p(h + g) \leq p(h) + p(g) \) for any \( h, g \in C(Q) \).
2. \( p(tg) = tp(g) \) for any \( t \geq 0 \) and \( g \in C(Q) \).
3. \( p(1) = 1, 0 \leq p(g) \leq \|g\|_{\infty} \) for any \( g \in C(Q) \), and \( p(g) = 0 \) for \( g \in C(Q) \) with \( g \leq 0 \).

By the Hahn-Banach theorem, we can extend the linear functional \( t \mapsto tp(1), t \in \mathbb{R} \), from the subspace of the constant functions to a linear functional \( L : C(Q) \to \mathbb{R} \) satisfying
\[
L(1) = p(1) = 1 \quad \text{and} \quad -p(-g) \leq L(g) \leq p(g) \quad \text{for any} \ g \in C(Q).
\]
If \( g \in C(Q) \) with \( g \geq 0 \), then \( p(-g) = 0 \) and so \( L(g) \geq 0 \). Hence combining the fact that \( L(1) = 1 \), we can use the Riesz representation theorem to find a Borel probability measure \( \mu \) on \( Q \) such that \( L(g) = \int_Q g \, d\mu \) for \( g \in C(Q) \).

If \( K = Q \) then \( \mu(K) = 1 \); otherwise, for any compact set \( E \subset Q \setminus K \), by the Urysohn lemma there exists \( g \in C(Q) \) such that \( 0 \leq g \leq 1 \), \( g(x) = 1 \) for \( x \in E \) and \( g(x) = 0 \) for \( x \in K \). It follows that \( g \cdot \chi_K \equiv 0 \) and thus \( p(g) = 0 \). This yields that \( \mu(E) \leq L(g) \leq p(g) = 0 \). Since \( \mu \) is regular, we have \( \mu(Q \setminus K) = 0 \). This means that \( \mu(K) = 1 \).

We now show \( \mu([C_n(x)]) \leq \beta_{x,n} \) for any \( x \in Q \) and \( n \geq N \). For any compact set \( E \subset [C_n(x)] \), by the Urysohn lemma, there exists \( g \in C(Q) \) such that \( 0 \leq g \leq 1 \), \( g(y) = 1 \) for \( y \in E \) and \( g(y) = 1 \) for \( y \in Q \setminus [C_n(x)] \). Then \( \mu(E) \leq L(g) \leq p(g) \) and \( g \cdot \chi_K \leq \chi_{C_n(x)} \). Let \( s = C_{[0,n]}(x) \) and \( \gamma_s = 1 \). Then we get \( W_C(N, \alpha, g \cdot \chi_K, Q, f) \leq \beta_{x,n} \).
\[ \gamma \beta_{x,n}, \text{ which implies that } p(g) \leq \beta_{x,n} \text{ and } \mu(E) \leq \beta_{x,n}. \text{ Using the regularity of } \mu, \] we obtain \[ \mu([C_n(x)]) \leq \beta_{x,n}. \]

**Remark 5.** Since we use Riesz representation theorem and Urysohn lemma in this proof, we need that \( Q \) is compact and \( Q \setminus C_n(x) \) is closed. That is why we suppose \( Q \) is a controlled invariant compact set and \( C \) is a clopen invariant partition.

**Proof of Theorem 3.1.** We first show that \( P_C(K, Q, f) \geq h_{\mu,C}(Q, f) \) for any \( \mu \in \mathcal{M}(Q) \) with \( \mu(K) = 1 \). Fix \( l \in \mathbb{N} \). Let

\[ K_l = \{ x \in K : h_{\mu,C}(x, f) \geq h_{\mu,C}(Q, f) - \frac{1}{l} \}. \]

Thus \( \mu(K_l) > 0 \). By Theorem 2.8, we have

\[ P_C(K_l, Q, f) \geq h_{\mu,C}(Q, f) - \frac{1}{l}. \]

Thus

\[ P_C(K, Q, f) \geq h_{\mu,C}(Q, f) - \frac{1}{l}. \]

Letting \( l \to \infty \), we have the desired inequality.

We now show the converse inequality. We can assume that \( P_C(K, Q, f) > -\infty \), otherwise we have nothing to prove. By Proposition 4, we see that \( P_C(K, Q, f) = P_C^W(K, Q, f) \). Then for any \(-\infty < \alpha < P_C(K, Q, f)\), there exists \( N \in \mathbb{N} \) such that

\[ \gamma := W_C(N, \alpha, K, Q, f) > 0. \]

By Lemma 3.3, there exists \( \mu \in \mathcal{M}(Q) \) with \( \mu(K) = 1 \) such that

\[ \mu(C_n(x)) \leq \frac{1}{\gamma} e^{-n\tau + S_{l(s)} f(\omega_c^{[0,n]}(s))} \]

for any \( x \in Q \) and \( n \geq N \). By a direct computation we obtain

\[ h_{\mu,C}(x, f) \geq \alpha. \]

Thus \( h_{\mu,C}(Q, f) \geq \alpha \) and we get the desired inequality. \( \square \)

**4. Variational principle for packing invariance pressure.**

**Lemma 4.1.** Let \( C = (\alpha, \tau, v) \) be an invariant partition of \( Q \), \( K \subset Q \), \( f \in C(U, \mathbb{R}) \) with \( \| f \|_\infty \leq 0 \), where \( \| f \|_\infty = \sup_{u \in U} f(u) \). If \( P_C(\alpha, K, Q, f) = \infty \), then for any given finite interval \( (a, b) \subset \mathbb{R} \) with \( a \geq 0 \) and any \( N \in \mathbb{N} \), there exists a finite disjoint collection \( G \) such that \( C_s(Q) \cap K \neq \emptyset \), \( l(s) \geq N \) for every \( s \in G \), and

\[ \sum_{s \in G} e^{-l(s)\alpha + S_{l(s)} f(\omega_c^{[0,n]}(s))} \in (a, b). \]

**Proof.** Since \( \| f \|_\infty \leq 0 \),

\[ \lim_{l(s) \to \infty} e^{-l(s)\alpha + S_{l(s)} f(\omega_c)} = 0. \]

Choose \( N \) large enough such that \( e^{-l(s)\alpha + S_{l(s)} f(\omega_c)} < b - a \) when \( l(s) \geq N \). Since \( P_C(\alpha, K, Q, f) = \infty \), we have \( P_C(N, \alpha, K, Q, f) = \infty \). Thus there exists a finite disjoint collection \( G' \) with \( C_s(Q) \cap K \neq \emptyset \), \( l(s) \geq N \) for every \( s \in G \), and

\[ \sum_{s \in G'} e^{-l(s)\alpha + S_{l(s)} f(\omega_c)} > b. \]

Since \( e^{-l(s)\alpha + S_{l(s)} f(\omega_c)} < b - a \), we can discard elements in \( G' \) one by one until we get a desired collection. \( \square \)
Theorem 4.2. Let $C = (\mathcal{A}, \tau, v)$ be a clopen invariant partition of $Q$, where $Q$ is a controlled invariant compact set, and $f$ be a non-positive continuous function on $U$. If $K \subset Q$ is non-empty and compact, then

$$P_C^P(K, Q, f) = \sup \{ \mathcal{T}_{\mu, C}(Q, f) : \mu \in \mathcal{M}(Q), \mu(K) = 1 \}.$$ 

Remark 6. If $f$ is null, then Theorem 7.2 in [18] is a special case of Theorem 4.2.

Proof. Using an analogous method in the proof of the first part of Theorem 3.1, we can show that

$$P_C^P(K, Q, f) \geq \sup \{ \mathcal{T}_{\mu, C}(Q, f) : \mu \in \mathcal{M}(Q), \mu(K) = 1 \}.$$ 

We now prove that the converse inequality holds. To see this, we employ the approach given by Wang, Huang and Sun [18], which is adapted from the method used by Feng and Huang [9].

Without losing generality, we can assume that $P_C^P(K, Q, f) > -\infty$. Let $-\infty < \alpha < \beta < P_C^P(K, Q, f)$. We are going to construct inductively a sequence of finite sets $(K_i)_{i=1}^{\infty}$ and a sequence of finite measures $(\mu_i)_{i=1}^{\infty}$ so that $K_i \subset K$ and $\mu_i$ is supported on $K_i$ for each $i$. Together with these two sequences, we construct a sequence of integer-valued functions $(m_i : K_i \to \mathbb{N})$ and a sequence of positive numbers $(M_i)$. The construction is divided into three steps:

**Step 1.** Construct $K_1$, $\mu_1$, $m_1(\cdot)$, and $M_1$.

Since $-\infty < \alpha < \beta < P_C^P(K, Q, f)$, we have $\mathcal{P}_C(\beta, K, Q, f) = \infty$. Let

$$H = \bigcup \{ G \subset Q : G \text{ is open}, \mathcal{P}_C(\beta, K \cap G, Q, f) = 0 \}.$$ 

Then $\mathcal{P}_C(\beta, K \cap H, Q, f) = 0$ by the separability of $Q$. Let $K' = K \setminus H = K \cap (Q \setminus H)$. We claim that for any open set $G \subset Q$, either $K' \cap G = \emptyset$ or $\mathcal{P}_C(\beta, K' \cap G, Q, f) > 0$.

To see this, assume $\mathcal{P}_C(\beta, K' \cap G, Q, f) = 0$ for an open set $G \subset Q$. Since $K = K' \cup (K \cap H)$, we have

$$\mathcal{P}_C(\beta, K \cap G, Q, f) \leq \mathcal{P}_C(\beta, K' \cap G, Q, f) + \mathcal{P}_C(\beta, K \cap H, Q, f) = 0,$$

which implies that $G \subset H$. Hence $K' \cap G = \emptyset$.

Since

$$\mathcal{P}_C(\beta, K, Q, f) \leq \mathcal{P}_C(\beta, K \cap H, Q, f) + \mathcal{P}_C(\beta, K', Q, f)$$

and $\mathcal{P}_C(\beta, K \cap H, Q, f) = 0$, we have

$$\mathcal{P}_C(\beta, K', Q, f) = \mathcal{P}_C(\beta, K, Q, f) = \infty.$$ 

It follows that

$$\mathcal{P}_C(\alpha, K', Q, f) = \infty.$$ 

By Lemma 4.1, we can find a finite disjoint collection $\mathcal{G}_1$ such that $C_s(Q) \cap K' \neq \emptyset$, $l(s) \geq N$ for every $s \in \mathcal{G}$, and

$$\sum_{s \in \mathcal{G}_1} e^{-l(s)\tau_\alpha + S_{l(s)} f(\omega_s)} \in (1, 2).$$

Choose $x_s \in C_s(Q) \cap K'$ for $s \in \mathcal{G}_1$. Let $m_1(x_s) = l(s)$ for $s \in \mathcal{G}_1$. Put $K_1 = \{ x_s : s \in \mathcal{G}_1 \}$. Then the collection $\{ [C_{m_1(x_s)}(x)] \}_{x \in K_1}$ is disjoint and

$$\sum_{x \in K_1} e^{-m_1(x)\tau_\alpha + S_{m_1(x)} f(\omega_{[0,m_1(x)]}(x))} \in (1, 2).$$
Define
\[ \mu_1 = \sum_{x \in K_1} e^{-m_1(x)\tau + S_{m_1(x)} + f(\omega_{z_0,m_1(x)}(x))} \delta_x, \]
where \( \delta_x \) denotes the Dirac measure at \( x \). Let \( M_1 = \max\{m_1(x) : x \in K_1\} \). Then for any \( z_x \in [C_{M_1}(x)] \subset [C_{m_1(x)}(x)] \), we have
\[ \left([C_{M_1}(z_x)] \cup [C_{m_1(x)}(x)]\right) \cap \left( \bigcup_{y \in K_1 \setminus \{x\}} [C_{M_1}(z_y)] \cup [C_{m_1(y)}(y)] \right) = \emptyset. \tag{5} \]
Since \( x \in K_1 \subset K' \), \( x \in K' \cap [C_{M_1}(x)] \). This implies that
\[ P_C(\beta, K \cap [C_{M_1}(x)], Q, f) \geq P_C(\beta, K' \cap [C_{M_1}(x)], Q, f) > 0. \]

**Step 2.** Construct \( K_2, \mu_2, m_2(\cdot) \), and \( M_2 \).
For each \( x \in K_1 \), since \( P_C(\beta, K \cap [C_{M_1}(x)], Q, f) > 0 \), we can construct, as in Step 1, a finite set \( E_2(x) \subset K \cap [C_{M_1}(x)] \) and an integer-valued function
\[ m_2 : E_2(x) \to \mathbb{N} \cap [M_1, \infty), \]
such that
\begin{align*}
(2-a) & P_C(\beta, K \cap G, Q, f) > 0 \text{ for each open set } G \subset Q \text{ with } G \cap E_2(x) \neq \emptyset; \\
(2-b) & \text{ the elements in } \{[C_{m_2(y)}(y)] \} \text{ are disjoint, and } \\
& \mu_1(\{x\}) < \sum_{y \in E_2(x)} e^{-m_2(y)\tau + S_{m_2(y)} + f(\omega_{z_0,m_2(y)}(y))} \leq (1 + 2^{-2})\mu_1(\{x\}).
\end{align*}
To see it, we fix \( x \in K_1 \). Denote \( F = K \cap [C_{M_1}(x)] \). Let
\[ H_x = \bigcup\{G \subset Q : G \text{ is open, } P_C(\beta, F \cap G, Q, f) = 0\}. \]
Set \( F' = F \setminus H_x \). Then as in Step 1, we can show that
\[ P_C(\beta, F', Q, f) = P_C(\beta, F, Q, f) > 0 \]
and
\[ P_C(\beta, F' \cap G, Q, f) > 0 \]
for any open set \( G \subset Q \) with \( G \cap F' \neq \emptyset \). Note that
\[ P_C(\alpha, F' \cap G, Q, f) = \infty \] (since \( \alpha < \beta \)).
Using Lemma 4.1 again, we can find a finite set \( E_2(x) \subset F' \) and an integer-valued function \( m_2 : E_2(x) \to \mathbb{N} \cap [M_1, \infty) \) so that (2-b) holds. If \( G \subset \) is an open set with \( G \cap E_2(x) \neq \emptyset \), then \( G \cap F' \neq \emptyset \). Thus
\[ P_C(\beta, K \cap G, Q, f) \geq P_C(\beta, F' \cap G, Q, f) > 0. \]
Since the family \( \{[C_{M_1}(x)] \} \) is disjoint, \( E_2(x) \cap E_2(x') = \emptyset \) for different \( x, x' \in K_1 \). Define \( K_2 = \cup_{x \in K_1} E_2(x) \) and
\[ \mu_2 = \sum_{y \in K_2} e^{-m_2(y)\tau + S_{m_2(y)} + f(\omega_{z_0,m_2(y)}(y))} \delta_y. \]
By Eq. 5 and (2-b), the elements in \( \{[C_{m_2(y)}(y)] \} \) are pairwise disjoint. Let \( M_2 = \max\{m_2(x) : x \in K_2\} \). Then for any \( z_x \in [C_{M_2}(x)] \subset [C_{m_2(x)}(x)] \), we have
\[ \left([C_{M_2}(z_x)] \cup [C_{m_2(x)}(x)]\right) \cap \left( \bigcup_{y \in K_2 \setminus \{x\}} [C_{M_2}(z_y)] \cup [C_{m_2(y)}(y)] \right) = \emptyset. \tag{6} \]
Step 3. Assume that there exists \( x \) such that \( y \in K_1 \) such that \( x \in E_2(y) \). Thus \( C_{M_2}(x) \cap E_2(y) \neq \emptyset \). By (2-a), we have
\[
P_{\mathcal{C}}(\beta, K \cap [C_{M_2}(x)], Q, f) > 0.
\]

Furthermore, for any \( x, z \) such that \( x \in M_p \), \( y \in K_p \), \( y \neq x \), and \( \emptyset \neq [C_{M_p}(x)] \), we have, for each \( x \in K_p \),
\[
([C_{M_p}(x)] \cup [C_{M_p}(y)]) \cap \left( \bigcup_{y \in K_p \setminus \{x\}} [C_{M_p}(y)] \cup [C_{M_p}(z)] \right) = \emptyset. \tag{7}
\]
and \( P_{\mathcal{C}}(\beta, K \cap [C_{M_p}(x)], Q, f) > 0 \). We construct below each term of them for \( i = p + 1 \) in a way similar to Step 2.

Since for each \( x \in K_p \) we have \( P_{\mathcal{C}}(\beta, K \cap [C_{M_p}(x)], Q, f) > 0 \), we can find, as in Step 2, a finite set \( E_{p+1}(x) \subset K \cap [C_{M_p}(x)] \) and an integer-valued function
\[
m_{p+1} : E_{p+1}(x) \to \mathbb{N} \cap [M_p, \infty)
\]
such that
(3-a) \( P_{\mathcal{C}}(\beta, K \cap G, Q, f) > 0 \) for each open set \( G \subset Q \) with \( G \cap E_{p+1}(x) \neq \emptyset \);
(3-b) the elements in \( \{C_{m_{p+1}}(y)\}_{y \in E_{p+1}(x)} \) are disjoint, and
\[
\mu_p(\{x\}) < \sum_{y \in E_{p+1}(x)} e^{-m_{p+1}(y)\tau + S_{m_{p+1}(y)}f(\omega_{0,m_{p+1}(y)(y)})} \leq (1 + 2^{-p-1})\mu_p(\{x\}).
\]

Since the elements in \( \{C_{M_p}(x)\}_{x \in K_p} \) are pairwise disjoint, we have
\[
E_{p+1}(x) \cap E_{p+1}(y) = \emptyset
\]
for different \( x, y \in K_p \). Define \( K_{p+1} = \bigcup_{x \in K_p} E_{p+1}(x) \) and
\[
\mu_{p+1} = \sum_{y \in K_{p+1}} e^{-m_{p+1}(y)\tau + S_{m_{p+1}(y)}f(\omega_{0,m_{p+1}(y)(y)})} \delta_y.
\]
It follows from (7) and (3-b) that the elements in \( \{C_{m_{p+1}}(y)\}_{y \in K_{p+1}} \) are pairwise disjoint. Let \( M_{p+1} = \max\{m_{p+1}(x) : x \in K_{p+1}\} \). Then for any \( x \in C_{M_{p+1}}(x), x \in K_{p+1} \), we have, for each \( x \in K_{p+1} \),
\[
([C_{M_{p+1}}(x)] \cup [C_{m_{p+1}(x)}(z)]) \cap \left( \bigcup_{y \in K_{p+1} \setminus \{x\}} [C_{M_{p+1}}(y)] \cup [C_{m_{p+1}}(y)(y)] \right) = \emptyset. \tag{8}
\]
Furthermore, for any \( x \in K_{p+1} \), there exists \( y \in K_p \) such that \( C_{m_{p+1}}(x) \cap E_{p+1}(y) \neq \emptyset \). Then, by (3-a),
\[
P_{\mathcal{C}}(\beta, K \cap [C_{M_{p+1}}(x)], Q, f) > 0.
\]

As in the above steps, we can construct by induction the sequences \( \{K_i\}, \{\mu_i\}, \{m_i(\cdot)\}, \) and \( \{M_i\} \). We summarize some of their basic properties as follows:
(a) For each \( i \), the family \( \mathcal{F}_i := \{C_{M_i}(x) : x \in K_i\} \) is disjoint. For any \( x \in K_{i+1} \), there exists \( y \in K_i \) such that \( C_{M_{i+1}}(x) \subset C_{M_i}(y) \).
(b) For each \( x \in K_i \) and \( z \in C_{M_i}(x) \),
\[
C_{m_i(x)}(z) \cap \bigcup_{y \in K_i \setminus \{x\}} C_{M_i}(y) = \emptyset.
\]
and

\[ \mu_i(\mathcal{C}_i(x)) = e^{-m_i(x)\tau + S_{m_i(x)} \cdot f(\omega_{[0,m_i(x)]}(x))} \]

\[ \leq \sum_{y \in E_{i+1}(x)} e^{-m_{i+1}(y)\tau + S_{m_{i+1}(y)} \cdot f(\omega_{[0,m_{i+1}(y)]}(y))} \]

\[ \leq (1 + 2^{-i-1})\mu_i(\mathcal{C}_i(x)), \]

where \( E_{i+1}(x) = \mathcal{C}_i(x) \cap K_{i+1} \). The second part in (b) implies

\[ \mu_i(F_i) \leq \mu_{i+1}(F_i) = \sum_{F \in \mathcal{F}_{i+1}: F \subset \mathcal{F}_i} \mu_i(F) \]

\[ \leq \sum_{F \in \mathcal{F}_{i+1}: F \subset \mathcal{F}_i} (1 + 2^{-i-1})\mu_i(F) \]

\[ = (1 + 2^{-i-1}) \sum_{F \in \mathcal{F}_{i+1}: F \subset \mathcal{F}_i} \mu_i(F) \]

\[ \leq (1 + 2^{-i-1})\mu_i(F_i), \quad F_i \in \mathcal{F}_i. \]

Using the above inequalities repeatedly, we have for any \( j > i \),

\[ \mu_i(F_i) \leq \mu_j(F_i) \leq \prod_{n=i+1}^{j} (1 + 2^{-n})\mu_i(F_i) \leq C\mu_i(F_i), \quad \forall F_i \in \mathcal{F}_i, \quad (9) \]

where \( C := \prod_{n=1}^{\infty} (1 + 2^{-n}) \).

Let \( \hat{\mu} \) be a limit point of \( \{\mu_i\} \) in the weak-star topology. Let

\[ K^* = \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} K_i = \lim_{n \to \infty} \bigcup_{i \geq n} K_i. \]

Then \( \hat{\mu} \) is supported on \( K^* \) and \( K^* \subset K \).

The second part in (b) implies that \( \hat{\mu}(K^*) \geq 1 \). From (a), we see that \( K^* \subset \bigcup_{x \in K_i} [\mathcal{C}_M(x)] \) for any \( i \in \mathbb{N} \). This implies that

\[ \hat{\mu}K^* \leq \hat{\mu} \left( \bigcup_{x \in K_i} [\mathcal{C}_M(x)] \right) \leq \sum_{x \in K_i} C\mu_1([\mathcal{C}_M(x)]) \leq 2C. \]

By the first part of (b), for each \( x \in K_i \) and \( z \in \mathcal{C}_M(x) \),

\[ \hat{\mu}(\mathcal{C}_M(x)(z)) \leq \hat{\mu}(\mathcal{C}_M(x)) \leq Ce^{-m_i(x)\tau + S_{m_i(x)} \cdot f(\omega_{[0,m_i(x)]}(x))}. \]

Since for each \( z \in K^* \) and \( i \in \mathbb{N} \), there exists \( x \in K_i \) such that \( z \in \mathcal{C}_M(x) \). Hence

\[ \hat{\mu}(\mathcal{C}_M(x)(z)) \leq Ce^{-m_i(x)\tau + S_{m_i(x)} \cdot f(\omega_{[0,m_i(x)]}(x))}. \]
Define $\mu = \hat{\mu}/\hat{\mu}(K^*)$. Then $\mu \in \mathcal{M}(Q)$ and $\mu(K^*) = 1$. For each $z \in K^*$, there exists an increasing sequence $\{k_i\}_{i \geq 1}$ such that

$$
\mu(C_{k_i}(z)) \leq Ce^{-k_i \tau \alpha + S_{k_i} f(\omega C_{[0,k_i]}(z))} \hat{\mu}(K^*).
$$

It follows that $h_{\mu,C}(Q,f) \geq \alpha$.

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E-mail address: xfzhong@gdufs.edu.cn