LOCAL ESTIMATES FOR POSITIVE SOLUTIONS OF POROUS MEDIUM EQUATIONS

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Abstract. We derive several new gradient estimates of Aronson-Bénilan type for positive solutions of porous medium equation and fast diffusion equation on a complete manifold that satisfies the curvature dimension condition.

1. Introduction

The porous medium equation

\[ u_t = \Delta u^m \]

where \( m \in (0, \infty) \), has been studied firstly as useful models of describing the physical processes of liquid flows passing through a concrete medium. If the physical medium is not homogenous, then one may model the physical processes in terms of porous medium equations in a manifold. The curvature effect of the manifold will have an impact on the fluid flow through the non-homogenous medium. On the other hand, the porous medium equation is an archetypical example for the study of degenerate parabolic equations, and there is a large number of papers dealing with different aspects of the porous medium equations, see for example J. L. Vázquez’s monographs [10] [11].

For a general parameter \( m > 0 \), only positive solutions to the porous medium equation are interesting, so in this paper we aim to establish a priori estimates for positive solutions. Let \( u \) be a positive solution to (1.1), which may be rewritten as the following

\[ u_t = mu^{m-1} \Delta u + m(m-1)u^{m-2}|\nabla u|^2. \]

Observe, in the non-linear case i.e. \( m \neq 1 \), the diffusion coefficient in (1.2), \( mu^{m-1} \), depends on the solution \( u \). The degeneracy of the diffusion part in (1.2) happens if \( \inf u = 0 \) when \( m > 1 \) and \( \sup u = \infty \) when \( m < 1 \). If \( u \) is bounded away from 0 and bounded from above, the differential operator \( mu^{m-1} \Delta \) is uniformly elliptic. Only in this case, the theory of fully non-linear equations applies, and the regularity of the solutions follows from Krylov-Safonov’s theory [4].

From this point of view, it is an interesting problem to derive, a priori space derivative estimates for a positive solution \( u \) in terms of its time derivative, the time parameter \( t \), and, if it is possible, the solution \( u \) only. Otherwise we wish to establish estimates depending on its time derivative, time parameter \( t \) and \( \sup u \) for the case \( m > 1 \) (resp. \( \inf u \) for the case \( m < 1 \)).

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In a seminal short note [1], Aronson and Bénilan established a powerful estimate, now under the name of gradient estimate, for a positive solution $u$ of the porous medium equation on $\mathbb{R}^n$. Explicitly, if $m > \frac{2n}{n-2}$, $v = f + \frac{m}{m-1}$, then
\begin{equation}
-\Delta f \leq \frac{N}{2t} \tag{1.3}
\end{equation}
where
\begin{equation}
f = m \frac{u^{m-1} - 1}{m - 1}, \quad \frac{2}{N} = \frac{2}{n} + (m - 1). \tag{1.4}
\end{equation}

(1.3) is equivalent to the following type gradient estimate:
\begin{equation}
\frac{|\nabla f|^2}{(m-1)f + m} - \frac{f_t}{(m-1)f + m} \leq \frac{N}{2t} \tag{1.5}
\end{equation}

The striking feature of (1.5) lies in the fact that the estimate is independent of any bounds of $u$, and therefore is useful in addressing the regularity of the solutions. This fundamental estimate was later employed in [2] for the study of existence theory, for obtaining $L^\infty_{loc}(\mathbb{R}^n)$ estimates in [5], and for obtaining regularity results for the free boundary of solutions in [3].

The story of the porous medium equations in a non-homogenous medium began with the fundamental paper [7] in 1986 by P. Li and S. T. Yau, in which they derived the celebrated gradient estimate for the linear heat equation on manifold, i.e. the case $m = 1$.

Suppose that $u$ is a positive solution to the heat equation $u_t = \Delta u$ in $(0, \infty) \times M$, where $M$ is a complete manifold of dimension $n$, $\Delta$ is the Laplace-Beltrami operator associated with a complete metric $(g_{ij})$ (which accounts for the non-homogenous continued medium). In a seminal paper [7], Li and Yau showed that, if the Ricci curvature $R_{ij}$ of $(g_{ij})$ is non-negative (for an explanation of the Ricci curvature, see below)
\begin{equation}
|\nabla f|^2 - f_t \leq \frac{n}{2t} \tag{1.6}
\end{equation}
where
\begin{equation}
f = \lim_{m \to 1} m \frac{u^{m-1} - 1}{m - 1} = \ln u
\end{equation}
is the Hopf transformation of $u$. In fact, they established, under the condition that $R_{ij} \geq -K$ for some constant $K \geq 0$, the following estimate
\begin{equation}
|\nabla f|^2 - \alpha f_t \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2K}{2(\alpha - 1)}, \quad \forall \alpha > 1. \tag{1.7}
\end{equation}

Here, the significant feature is that the quantity $|\nabla f|^2 - \alpha f_t$ is controlled by the curvature bound, but independent of any bounds of the solution $u$ itself, which is one of the special feature of the linear case. The ellipticity of the operator $\Delta$ is somehow dominated by the Ricci tensor, but independent of the solution $u$.

There are attempts to extend the Aronson-Bénilan estimate to solutions of porous medium equations on manifolds, or equivalently extend Li-Yau’s estimate for non-linear parabolic equations in the presence of curvature.

In [8], P. Lu, L. Ni, J. L. Vázquez and C. Villani studied the porous medium equation on a complete manifold, and established a gradient estimate in terms of the supremum of the solution for the case that $m > 1$. Explicitly, let $B(O, 2R)$ denote a geodesic ball with centre $O$ and radius $2R > 0$. Assume that $u$ is a positive
solution to \([11]\) on \(B(O, 2R) \times [0, T]\) and that the Ricci curvature \(R_{ij} \geq -K^2\) on \(B(O, 2R)\) for some \(K \geq 0\). Let \(v = \frac{m}{m-1} u^{m-1}\). If \(m > 1\) and \(\beta > 1\), it holds on \(B(O, R) \times [0, T]\) that
\[
\frac{\nabla v}{v} - \beta \frac{v_t}{v} \leq \alpha \beta^2 \left( \frac{1}{t} + C_2 R^2 v_{\max}^{2R,T} \right) + \alpha \beta^2 \frac{v_{\max}^{2R,T}}{R^2} C_1
\]
where
\[
v_{\max}^{2R,T} = \max_{B(O, 2R) \times [0, T]} v.
\]
If \(m \in (1 - \frac{2}{n}, 1)\), they proved that on \(B(O, R) \times [0, T]\), for any \(\gamma \in (0, 1)\),
\[
\frac{\nabla v}{v} - \gamma \frac{v_t}{v} \geq \frac{\alpha \gamma^2}{C_3} \left( \frac{1}{t} + C_4 \sqrt{C_5} K^2 v_{\max}^{2R,T} \right) + \frac{\alpha \gamma^2 \bar{v}_{\max}^{2R,T}}{C_3} \frac{v_{\max}^{2R,T}}{R^2} C_5
\]
with
\[
\bar{v}_{\max}^{2R,T} = \max_{B(O, 2R) \times [0, T]} (-v).
\]
In \([6]\), G. Huang, Z. Huang and H. Li obtained similar estimates. Note that these gradient bounds depend on the bounds of the solution \(u\), and therefore are not local.

S.T. Yau \([12]\) established a similar type of gradient bounds depending on derivatives of initial data for degenerate parabolic equations of the form
\[
u_t = \Delta (F(u))
\]
with \(F \in C^2 (0, \infty)\) and \(F' > 0\). In particular, as explained in \([9]\), Yau’s result implies that for any function \(c(t) \in C^1 (0, \infty)\) satisfying
\[
\begin{align*}
c(t) &\leq 0 \\
c'(t) &\geq 0 \\
|\nabla v|^2 - 2v_t + 2m \left( \frac{m-1}{m} v \right)^{m-2} &\leq c(t) \quad \text{at } t = 0
\end{align*}
\]
it holds for all \(t > 0\) that
\[
|\nabla v|^2 - 2v_t + 2m \left( \frac{m-1}{m} v \right)^{m-2} \leq c(t).
\]
The gradient estimate of Yau therefore depends on the initial data \(u\).

In this paper, we consider a positive solution \(u\) of the equation
\[
u_t = Lu^m \quad \text{in } (0, \infty) \times M
\]
where \(m > 0\) is a real exponent, and \(L = \Delta + V\) with some smooth vector field \(V\). The Bakry-merg curvature-dimension condition \(CD(n, -K)\) is assumed to hold for \(L\), where \(K \geq 0\) and \(m \neq 1\). (See below for an explanation of \(CD(n, -K)\).)

One of the contributions in the present paper is the gradient estimate for porous medium equation on a complete manifold, which is independent of the bounds of \(u\), for the case \(m < 1\). Our estimate is a generalization of Aronson-Bénilan’s estimate and Li-Yau’s estimate.

**Theorem 1.** Let \(M\) be a complete manifold of dimension \(n\) and the curvature-dimension condition \(CD(n, -K)\) hold for some constant \(K \geq 0\). Let \(u\) be a positive solution to the porous medium equation \((1.8)\) in \((0, \infty) \times M\). Let \(N > 0\) be defined by \(\tfrac{N}{m} = \tfrac{2}{n} + m - 1\).
1) If \( m > \max \left\{ \frac{1}{2}, 1 - \frac{2}{n} \right\} \) and \( m < 1 \), then
\[
(1.9) \quad m \frac{\nabla u}{u^{3-m}} - \frac{u_t}{2t} \leq \frac{N}{2t} + \frac{2Km}{(1-m)(2m-1)} u^{m-1}.
\]

2) If \( K = 0 \), \( m > 1 - \frac{2}{n} \), and if \( c \in [0, \infty) \) such that \( -\frac{m}{m-1} Lu^{m-1}(0, \cdot) \leq c \), then
\[
(1.10) \quad m \frac{\nabla u}{u^{3-m}} - \frac{u_t}{2t} \leq \frac{c}{2t} + 1.
\]

If \( K \neq 0 \), and \( m > 1 \), we are unable to derive an estimate with the following form
\[
|\nabla u|^2 \leq Q(t, u_t, u)
\]
and we do not believe it is possible for the reason which will become clear, which makes striking difference from the linear case. In this aspect, we therefore seek for best possible estimates for \( |\nabla u|^2 \) in terms of \( t, u_t, u \) and \( \sup u \) as well. To state our results, let us introduce several notations.

Let \( N > 0 \) and \( R \geq 0 \). Define
\[
(1.11) \quad w(t, y) = \frac{4t}{N} \sqrt{NR} \sqrt{y + \frac{NR}{4}}
\]
and \( Q(t, y) = C(t, y) + y \), where \( C \) is the solution to the differential equation
\[
\frac{d}{dt} C + \frac{2}{N} C^2 - 2R(C + y) = 0, \quad C(0) = \infty.
\]
That is
\[
(1.12) \quad Q(t, y) = \frac{NR}{2} + y + \sqrt{NR} \sqrt{y + \frac{NR}{4}} \coth \frac{w(t, y)}{2}.
\]

We are now in a position to state our result for the case that \( K > 0 \) and \( m > 1 \).

**Theorem 2.** Let \( M \) be a complete manifold \( M \) of dimension \( n \) and the curvature-dimension condition \( CD(n, -K) \) hold for some constant \( K \geq 0 \). Let \( u \) be a positive solution to the porous medium equation \( (1.8) \) in \( (0, \infty) \times M \). Let \( f = m (u^{m-1} - 1) / (m - 1) \) and \( U = (m - 1) f + m \). Let \( R = \sup \{ KU \} \). Then
\[
\frac{|\nabla f|}{U}^2 \leq Q \left( t, \frac{f_t}{U} \right) \text{ on } \frac{f_t}{U} > -\frac{NR}{4}
\]
and
\[
\frac{|\nabla f|}{U}^2 - \frac{f_t}{U} \leq \frac{N}{2t} + \frac{NR}{2} \text{ on } \frac{f_t}{U} \leq -\frac{NR}{4}.
\]

Note that if \( m = 1 \) then \( U = 1 \), so \( R \) reduces in the linear case to \( K \). If \( K = 0 \) then \( R = 0 \), thus our estimate reduces to Aronson-Bénilan’s estimate.

**2. Notations and geometric assumptions**

Let us begin with a description of our precise work setting. Let \( M \) be a complete manifold of dimension \( n \), and the complete metric \( ds^2 = \sum_{i,j=1}^d g_{ij} dx^i dx^j \) in a local coordinate system. Thus \( (g_{ij}) \) is a symmetric, positive definite type \((0, 2)\) smooth tensor field on \( M \). Let \( \Delta \) denote the Laplace-Beltrami operator on \( M \) given in a local coordinate system by
\[
(2.1) \quad \Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} g^{ij} \sqrt{g} \frac{\partial}{\partial x^j}.
\]
which is an elliptic differential operator of second order, where \((g^{ij})\) is the inverse matrix of \((g_{ij})\), and \(g = \det (g_{ij})\). Let \(V\) be a smooth vector field, and let \(L = \Delta + V\).

The geometric condition we are going to use, which is actually the only place where the geometry of the complete manifold \((M, g_{ij})\) enters into our study, is the curvature-dimension condition, a concept that is closely related to Ricci curvature lower bound.

The metric \((g_{ij})\) may be recovered from the elliptic operator by means of

\[
\Gamma (u, v) = \frac{1}{2} (L(uv) - uLv - vLu)
\]

\[= \sum_{i,j=1}^{d} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} = \langle \nabla u, \nabla v \rangle ,
\]

then the curvature operator \(\Gamma_2 (u, v)\) is defined by iterating \(\Gamma\) and given by

\[\Gamma_2 (u, v) = \frac{1}{2} (L\Gamma (u, v) - \Gamma (u, Lv) - \Gamma (v, Lu)) .\]

The Bochner identity then gives the explicit formula in terms of the Ricci curvature tensor \((R_{ij})\) associated with the Riemannian metric \((g_{ij})\) and the symmetric part of the total co-variant derivative \((V_{ij})\) of the vector field \(\nabla V\), in a local coordinate system,

\[\Gamma_2 (u, u) = \| \nabla^2 u \|^2 + \sum_{i,j=1}^{d} (R_{ij} - V_{ij}) \nabla^i f \nabla^j f
\]

where \(V_{ij} = \frac{1}{2} (\nabla_i V_j + \nabla_j V_i)\), and \(V = \sum_{i=1}^{d} V^i \frac{\partial}{\partial x^i}\) is the vector field represented in terms of coordinate partial differentiation. It is elementary that

\[\| \nabla^2 u \|^2 \geq \frac{1}{n} (\Delta u)^2 .\]

Therefore, in the case that \(V = 0\), we have the following curvature-dimension inequality

\[\Gamma_2 (u, u) \geq \frac{1}{n} (\Delta u)^2 - K |\nabla u|^2 .\]

Hence, for a general vector field \(V\), the geometric condition on \(L\) should be the following curvature-dimension inequality \(CD (n, -K)\)

\[\Gamma_2 (u, u) \geq \frac{1}{n} (Lu)^2 - K |\nabla u|^2
\]

for some \(n > 0\) and a constant \(K\).

Under this curvature-dimension condition, we will study the porous medium equation

\[u_t = Lu^m\ in\ [0, \infty) \times M ,
\]

where \(m > 0\) is a real exponent.

Throughout the paper, we will use \(t\) to denote the time parameter and \(\partial_t\) to denote the partial derivative with respect to \(t\). For simplicity, if no confusion may arise, then \(h_t\) denotes \(\partial_t h\).
3. FUNDAMENTAL DIFFERENTIAL INEQUALITIES

Given a positive solution $u$ of equation (2.7), consider the modified Hopf transformation

$$f = m \frac{u^{m-1} - 1}{m - 1}$$

which recovers to the logarithm transform in the sense that

$$\lim_{m \to 1} m \frac{u^{m-1} - 1}{m - 1} = \log u.$$

It is clear that $(m - 1) f + m = mu^{m-1}$, which is positive. This quantity will play an important role below.

The porous medium equation is then transformed into a parabolic equation for $f$:

$$[(m - 1) f + m] L - \partial_t f = - |\nabla f|^2.$$

The parabolic equation (3.2) for $f$ has the same form as the linear case. Although the elliptic operator $A$ involved is much more complicated than the linear case, it however suggests the possibility for obtaining gradient estimates that are similar to Aronson-Bénilan [1] and Li-Yau [7] type. In fact our arguments from now on only rely on the fact that $f$ is a solution to (3.2) and the fact (or assumption) that $(m - 1) f + m$ is positive.

The parabolic equation (3.2) may be written as

$$Lf = Y - X$$

where

$$X = \frac{|\nabla f|^2}{(m - 1) f + m} \quad \text{and} \quad Y = \frac{f_t}{(m - 1) f + m}.$$

In general, a gradient estimate will take the form $-L f \leq c(t, f, f_t)$. Due to a reason which will become clear only later on we wish to point out that we will seek for a gradient estimate in terms of $X$ and $Y$ with the following form

$$X - Y \leq B(t, U, Y)$$

for some function $B$ which can be either computed explicitly or can be determined by simple ordinary differential equations (see below for details), where $U = (m - 1) f + m$. As long as the problem of determining $B(t, U, Y)$ has been settled, the idea then is to apply the maximum principle to test function

$$G = t [X - Y - B(t, U, Y)]$$

to conclude that $G \leq 0$, which in turn yields the desired estimate (3.5). To this end we will consider the heat operator $(A - \partial_t)$ applying to $G$:

$$\frac{1}{t} G = - \frac{1}{t} G + t (A - \partial_t) (X - Y) - t (A - \partial_t) B(t, U, Y)$$

where $A$ is an elliptic differential operator, yet need to be determined as well. So our first task is to derive parabolic equations that $X$ and $Y$ must satisfy.

Let us start with the parabolic equation (3.2) for $f$, which may be written as

$$f_t = ((m - 1) f + m) (Lf + X).$$
Therefore we seek for gradient estimates. In the case for linear equations, so that
\[ (3.14) \]
By using (3.2) we obtain
\[ A \]
Therefore, (3.12) is the right choice of
\[ (3.13) \]
Similarly, we have
\[ (3.11) \]
By plugging these equations into (3.9) we obtain
\[ \partial_t Y = ((m - 1) f + m)\nabla Y + 2m \langle \nabla f, \nabla Y \rangle + (m - 1) Y^2. \]
Define an elliptic differential operator of second order as
\[ (A - \partial_t) Y = -(m - 1) Y^2. \]
Therefore, (3.12) is the right choice of \( A \) we have been looking for.
Next, let us consider \((A - \partial_t)X\). Again we begin with the time derivative of \(X\):
\[ \partial_t X = 2 ((m - 1) f + m)^{-1} \nabla f, \nabla f_t - (m - 1) Y X. \]
By using (3.2) we obtain
\[ (3.14) \]
and
\[ LX = L \left[ ((m - 1) f + m)^{-1} |\nabla f|^2 \right] \]
so that
\[ AX = 2 \langle \nabla f, \nabla X \rangle + L |\nabla f|^2 - (m - 1) X (Y - X). \]
Therefore
\[ (3.15) \]
It is also convenient to consider \(U = (m - 1) f + m\) as a basic function when we seek for gradient estimates. In the case for linear equations, \(U\) reduces to a
constant which explains why the solution does not appear in Li-Yau type gradient estimates. It follows from (3.16) the evolution equation for $U$ which is given by

$$(A - \partial_t)U = (2m - 1)m(m - 1)UX.$$ 

The two functions related to the space and time derivatives of $f$, $X$ and $Y$, may be rewritten as

$$(3.17) \quad X = \frac{\|\nabla f\|^2}{U} \quad \text{and} \quad Y = \frac{f}{U}.$$ 

According to their evolution equations (3.13) and (3.15), it is a good idea to introduce a new variable $Z = X - Y = -Lf$, whose evolution equation can be obtained by taking the difference of (3.13) and (3.15) as follows

$$(3.18) \quad (A - \partial_t)Z = 2\Gamma_2 (f, f) + \left( m - 1 \right) Z^2.$$ 

Suppose that the following curvature-dimension condition $CD(n, -K)$ holds

$$\Gamma_2 (f, f) \geq \frac{1}{n} (Lf)^2 - K \|\nabla f\|^2 = \frac{1}{n} Z^2 - KU (Z + Y)^2.$$ 

It follows therefore from (3.18) that

$$(3.19) \quad (A - \partial_t) Z \geq \frac{2}{N} Z^2 - 2KU (Z + Y)^2$$

where

$$\frac{2}{N} = \frac{2}{n} + m - 1.$$ 

(3.19) is a fundamental inequality, which, together with the evolution equations for $U$ and $Y$, will lead to different kinds of gradient estimates in the sequel.

4. LOCAL GRADIENT ESTIMATES

It seems a long and uninitiated procedure to derive various differential inequalities, and up to now we have not derived any useful estimates. From now on however we are going to deliver useful estimates for $u$ (in terms of its Hopf transformation $f$), based solely on the following differential inequalities

$$\begin{align*}
(A - \partial_t)U &= (2m - 1)(m - 1)U(Y + Z), \\
(A - \partial_t)Y &= -(m - 1)Y^2, \\
(A - \partial_t)Z &\geq \frac{2}{N} Z^2 - 2KU(Y + Z)
\end{align*}$$

where $Z = X - Y$, $U = (m - 1)f + m$. We remark the facts that $U \geq 0$ and $Z + Y \geq 0$.

Proof of (1.10) in Theorem 1. When $K = 0$, the third differential inequality in (4.1) reduces to

$$(4.2) \quad (A - \partial_t)Z \geq \frac{2}{N} Z^2.$$ 

Assume at time 0

$$Z = X - Y \leq c.$$ 

We apply maximum principle to

$$J = \left( \frac{2t}{N} + \frac{1}{c} \right) Z.$$
On the one hand, by (4.2) we have

\[(A - \partial_t) J = \left(\frac{2t}{N} + \frac{1}{c}\right) (A - \partial_t) Z - \frac{2}{N} Z \geq \left(\frac{2t}{N} + \frac{1}{c}\right) \frac{2}{N} Z^2 - \frac{2}{N} Z \]

\[= \frac{2}{N} Z (J - 1),\]

which ensures that the maximum of \(J\) over \((0, \infty) \times M\), if exists, can not be greater than 1. On the other hand, at time 0, since \(Z \leq C\), we have \(J \leq 1\). Therefore, we conclude that

\[X - Y \leq \left(\frac{2t}{N} + \frac{1}{c}\right)^{-1},\]

which is (1.10) in Theorem 1.

In order to derive (1.9) in Theorem 1 and Theorem 2, we need several facts, which we are going to establish now.

When \(K > 0\), the differential inequality for \(Z\) involves \(Y\) and \(U\). Hence we look for gradient estimates in the form

\[Z \leq B(t, U, Y),\]

and we wish to prove this estimate by applying the maximum principle to the test function \(G = t (Z - B(t, U, Y))\). The idea is to apply the heat operator \((A - \partial_t)\) to \(G\). Here we recall that

\[A = UL + 2m\nabla f \nabla\]

is an elliptic operator of second order.

**Lemma 3.** Assume that \(B(t, U, Y)\) is smooth in \((t, U, Y)\) and is convex in \((U, Y)\). Let \(G = t (Z - B(t, U, Y))\). Then it holds that

\[(A - \partial_t) G \geq \frac{2}{Nt} G^2 + \left\{ \frac{4}{N} B - \frac{1}{t} - (2K + (m - 1) (2m - 1) B_U) U \right\} G + \left\{ B_t + \frac{2}{N} B^2 - (2K + (m - 1) (2m - 1) B_U) U (Y + B) \right\} t + (m - 1) t Y^2 B_Y.\]

**Proof.** Define \(F = Z - B(t, U, Y)\). By product rule

\[(A - \partial_t) F = (A - \partial_t) Z - (A - \partial_t) B(t, U, Y) = (A - \partial_t) Z + B_t - B_Y (A - \partial_t) Y - B_U (A - \partial_t) U - U \left\{ B_Y |\nabla Y|^2 + 2B_Y \langle \nabla U, \nabla Y \rangle + B_U |\nabla U|^2 \right\} .\]

For smooth function \(B(t, U, Y)\) which is convex in \((U, Y)\),

\[B_Y |\nabla Y|^2 + 2B_Y \langle \nabla U, \nabla Y \rangle + B_U |\nabla U|^2 \leq 0.\]

Hence we have

\[(A - \partial_t) F \geq (A - \partial_t) Z + B_t - B_Y (A - \partial_t) Y - B_U (A - \partial_t) U\]
which yields that

\[(A - \partial_t) G = \frac{1}{t} G + t (A - \partial_t) F \geq \frac{1}{t} G + t (A - \partial_t) Z + tB_t \]

\[(A - \partial_t) G = -tB_Y (A - \partial_t) Y - tB_U (A - \partial_t) U.\]  

By applying the fundamental differential inequalities (4.1) we obtain

\[(A - \partial_t) G \geq -\frac{1}{t} G + t \left\{ 2N \left[ \frac{1}{t} G + B \right]^2 - 2KU(Y + Z) \right\} + tB_t \]

\[+ (m-1) t \left\{ Y^2 B_Y - (2m-1) U(Y + Z) B_U \right\}. \]

Now replace \( Z \) by \( \frac{1}{t} G + B = Z \) to obtain

\[(A - \partial_t) G \geq -\frac{1}{t} G + t \left\{ 2N \left[ \frac{1}{t} G + B \right]^2 - 2KU(Y + \frac{1}{t} G + B) \right\} \]

\[+ tB_t + (m-1) t \left\{ Y^2 B_Y - (2m-1) U(\frac{1}{t} G + B) \right\}. \]

By rearranging the terms we deduce that

\[(A - \partial_t) G \geq \frac{2}{Nt} G^2 + \left\{ \frac{4}{N} B - \frac{1}{t} - (2K + (m-1) (2m-1) B_U) U \right\} G \]

\[+ \left\{ B_t + \frac{2}{N} B^2 - (2K + (m-1) (2m-1) B_U) U(Y + B) \right\} t \]

\[+ (m-1) t Y^2 B_Y. \]

With this result at hand, we are able to test whether a function \( B(t, U, Y) \) convex in \( (U, Y) \) is an upper bound on \( Z \).

Let us consider the linear form

\[B(t, U, Y) = \theta + aU + bY\]

where \( \theta, a \) and \( b \) are functions in \( t \). Then by the lemma above,

\[(A - \partial_t) G \geq \frac{2}{Nt} G^2 + \left\{ \frac{4}{N} \theta - \frac{1}{t} \right\} G \]

\[+ 2 \left( \frac{2b}{N} Y - \lambda_1 U \right) G + E_1 t\]  

where

\[E_1 = B_t + \frac{2}{N} B^2 - \lambda_2 ((1 + b) U + aU^2 + \theta U) + (m-1) b Y^2 \]

\[= B_t + \frac{2}{N} \theta^2 + \left( \frac{4a}{N} - \lambda_2 \right) \theta U + \left( \frac{2}{N} a - \lambda_2 \right) a U^2 \]

\[+ \frac{4b}{N} \theta Y + \left( \frac{2}{N} b^2 + (m-1) b \right) Y^2 + \left( \frac{4}{N} ab - \lambda_2 (1 + b) \right) U Y, \]
and

$$\lambda_1 = \left( K + \left[ \frac{(m-1)(2m-3)}{2} - \frac{2}{n} a \right] \right),$$

$$\lambda_2 = (2K + (m-1)(2m-1)a).$$

Introduce a function $\psi$ and rearrange the differential inequality as

$$\begin{align*}
(A - \partial_t) G & \geq \frac{2}{Nt} \left[ G + \frac{Nt}{2} \left( \frac{2b}{N} Y - \lambda_1 U + \psi \right) \right]^2 \\
& \quad + \left( \frac{4}{N} \theta - \frac{1}{t} - 2\psi \right) G + E_2 t \tag{4.8}
\end{align*}$$

where

$$E_2 = E_1 - \frac{N}{2} \left( \frac{2b}{N} Y - \lambda_1 U + \psi \right)^2$$

$$= E_1 - \frac{2b^2}{N} Y^2 - \frac{N}{2} (\lambda_1 U - \psi)^2 + 2b(\lambda_1 U - \psi) Y$$

$$= B_t + \frac{2}{N} \theta^2 + \left( \frac{4a}{N} - \lambda_2 \right) \theta U + \left( \frac{2}{N} a^2 - \lambda_2 a - \frac{N}{2} \lambda_1^2 \right) U^2$$

$$+ \frac{4b}{N} \theta Y + (m - 1)bY^2 + \left( \frac{4}{N} ab - \lambda_2 (1 + b) + 2\lambda_1 b \right) UY$$

$$+ N\lambda_1 U \psi - 2b\psi Y - \frac{N}{2} \psi^2.$$

Insert the values of $\lambda_1$ and $\lambda_2$ to obtain

$$E_2 = B_t + \frac{2}{N} \theta^2 + \frac{4b}{N} \theta Y + (m - 1)bY^2$$

$$+ \left( \frac{4}{n} a - (m - 1)(2m - 3)a - 2K \right) \theta U$$

$$+ \left( \frac{2}{N} a^2 - (m - 1)(2m - 1)a^2 - 2Ka - \frac{N}{2} \lambda_1^2 \right) U^2$$

$$- ((m - 1)(2m - 1)a + 2K) UY$$

$$+ N\lambda_1 U \psi - 2b\psi Y - \frac{N}{2} \psi^2. \tag{4.9}$$

Let us consider the coefficient in front of $U^2$ in (4.9), which is

$$\lambda_3 = \frac{2}{N} a^2 - (m - 1)(2m - 1)a^2 - 2Ka - \frac{N}{2} \lambda_1^2$$

$$= - \left[ (m - 1)(2m - 1)a + 2K \right]^2 \frac{N}{8}. $$
Therefore

\[ E_2 = B_t + \frac{2}{N} \theta^2 + \frac{4b}{N} \theta Y + (m - 1)bY^2 \]
\[ + \left( K + \frac{m - 1}{2} \frac{a}{N} \right) \psi U - 2b\psi Y - \frac{N}{2} \psi^2 \]
\[ + \left( \frac{4}{n} a - (m - 1) (2m - 3)a - 2K \right) \theta U \]
\[ - \left[ (m - 1) (2m - 1) a + 2K \right]^2 \frac{N}{8} U^2 \]
\[ + \left[ (m - 1) (2m - 1) a + 2K \right] UY. \]

(4.10)

Inserting \( B_t = \theta' + a'U + b'Y \) into it yields

\[ E_2 = \theta' + \frac{2}{N} \theta^2 + \left( \frac{4b}{N} \theta + b' \right) Y + (m - 1)bY^2 \]
\[ + \left( K + \frac{m - 1}{2} \frac{a}{N} \right) \psi U - 2b\psi Y - \frac{N}{2} \psi^2 \]
\[ + \left( \frac{4}{n} a - (m - 1) (2m - 3)a - 2K \right) \theta U \]
\[ - \left[ (m - 1) (2m - 1) a + 2K \right]^2 \frac{N}{8} U^2 \]
\[ + \left[ (m - 1) (2m - 1) a + 2K \right] UY. \]

(4.11)

Assume that \( N = 2 ((\frac{n}{m} + m - 1)^{-1} > 0 \). According to (4.8), in order to be able to conclude that \( G \leq 0 \) by using the maximum principle, we need to choose \( \psi, a, b \) and \( \theta \) such that

\[ 4\frac{N}{\theta} - 2\psi - \frac{1}{U} \geq 0, \text{ and } E_2 \geq 0. \]

(4.12)

We are now in a position to prove (1.9) in Theorem 4.

**Proof of (1.9) in Theorem 4** By the expression (4.11) for \( E_2 \), the best we can do is to choose \( a \) such that

\( (m - 1) (2m - 1) a + 2K = 0 \)

otherwise we can not control \( U^2 \) for whatever the choice of \( b \), as long as \( m \neq 1 \). (if \( m = 1 \) then \( U = 1 \), the story would be different then). This is possible only if \((m - 1) (2m - 1) \neq 0\), so we are working with this constraint.

With this particular choice of \( a \), we have

\[ E_2 = \theta' + \frac{2}{N} \theta^2 + \left( b' + 4b \frac{\theta}{N} \right) Y + (m - 1)bY^2 \]
\[ - \frac{2}{(m - 1) (2m - 1)} \left( \frac{4}{N} \theta - 2\psi \right) KU - 2b\psi Y - \frac{N}{2} \psi^2. \]

(4.13)

Recall that we do not wish to use any upper bound on \( U \), and the only assumption we imposed on \( U \) is its positivity. Therefore, to control \( E_2 \), we need

\[ - \frac{2}{(m - 1) (2m - 1)} \left( \frac{4}{N} \theta - 2\psi \right) \geq 0. \]

(4.14)
However, the first inequality in condition (4.12) implies that $\frac{4}{N}\theta - 2\psi$ needs to be positive. Hence to ensure (4.14) to hold, we need $-\frac{2K}{(m - 1)(2m - 1)} \geq 0$, which means $m \in \left(\frac{1}{2}, 1\right)$. On the other hand, in order to control the term $(m - 1)bY^2$ in the expression (4.13) for $E_2$, we need $(m - 1)b \geq 0$. Therefore, we have to choose $b = 0$.

It only remains to determine $\theta$. By substituting $b = 0$ into (4.13) we have

$$E_2 = \theta' + \frac{2}{N} \theta^2 - \frac{N}{2} \psi^2 - \frac{2}{(m - 1)(2m - 1)} \left(\frac{4}{N}\theta - 2\psi\right) KU.$$ 

So condition (4.12) will be satisfied when

$$\frac{4}{N}\theta - 2\psi - \frac{1}{t} \geq 0, \text{ and } \theta' + \frac{2}{N} \theta^2 - \frac{N}{2} \psi^2 \geq 0.$$ 

By testing the coefficients of terms with the lowest degree in $t$, we conclude that the best choice of $(\theta, \psi)$ is $\theta = \frac{N}{2t}$ and $\psi = 0$. \hfill \Box

5. **Non-local estimates**

In this part we prove Theorem 2. We have seen from the last section that when $K > 0$ and $m > 1$, there is no way to control the term containing $KU$ in $E_2$ (4.13). To deal with this problem, the idea is to replace $KU$ by its upper bound

$$R = \sup KU$$

and then seek for a bound on $Z$ involving constant $R$ rather than $U$. Therefore, we choose $a = 0$, and consider gradient bounds in the form $B(t, U, Y) = \theta + bY$. By Lemma 3

$$(A - \partial_t) G \geq \frac{2}{Nt} G^2 + \left(\frac{4}{N}\theta - \frac{1}{t}\right) G$$

$$-2KUt \left(\frac{G}{t} + \theta + (1 + b)Y\right)$$

$$+ \frac{4b}{N} YG + \frac{2}{N} (\theta + bY)^2 t + tB_t$$

$$+ t(m - 1)bY^2.$$ 

Since

$$\frac{G}{t} + \theta + (1 + b)Y = X \geq 0,$$

we have

$$(A - \partial_t) G \geq \frac{2}{Nt} G^2 + \left(\frac{4}{N}\theta - \frac{1}{t}\right) G$$

$$-2Rt \left(\frac{G}{t} + \theta + (1 + b)Y\right)$$

$$+ \frac{4b}{N} YG + \frac{2}{N} (\theta + bY)^2 t + tB_t$$

$$+ t(m - 1)bY^2.$$
Introduce a function $y$ and rearrange the terms as

$$(A - \partial_t) G \geq \frac{2}{Nt} [G + b(Y - y)t]^2$$

$$+ \left( \frac{4}{N} (\theta + by) - 2R - \frac{1}{t} \right) G + E_3 t$$

where

$$E_3 = (m - 1) bY^2$$

$$+ \left( \frac{4}{N} b(\theta + by) - 2R(1 + b) + b' \right) (Y - y)$$

$$(5.2)$$

$$+ \theta' + b' y + \frac{2}{N} (\theta + by)^2 - 2R(\theta + by + y).$$

From this differential inequality, to conclude $G \leq 0$ by maximum principle, we need

$$(5.3)$$

$$\frac{4}{N} (\theta + by) - 2R - \frac{1}{t} \geq 0, \quad \text{and} \quad E_3 \geq 0.$$

Suppose that $(m - 1) b \geq 0$. To ensure that $E_3$ is non-negative, let us solve

$$(5.4)$$

$$\begin{cases}
\theta' + b' y + \frac{2}{N} (\theta + by)^2 - 2R(\theta + by + y) = 0 \\
\frac{4}{N} b(\theta + by) - 2R(1 + b) + b' = 0 \\
\theta(0) = \infty \\
b(0) = 0.
\end{cases}$$

Define $C = \theta + by$. It follows from $(5.4)$ that

$$\frac{d}{dt} C + \frac{2}{N} C^2 - 2R (C + y) = 0, \quad C(0) = \infty.$$

Assume that $y$ is a constant such that $y \geq -\frac{NR}{4}$. Then we have

$$(5.5)$$

$$C = C(t, y) = \frac{NR}{2} + \sqrt{NR} \sqrt{y + \frac{NR}{4}} \coth \frac{w(t, y)}{2}$$

where

$$w(t, y) = \frac{4t}{\sqrt{NR} \sqrt{y + \frac{NR}{4}}}.$$

By substituting it into the second equation in $(5.3)$, we can solve $b$ as

$$(5.6)$$

$$b = \partial_y C(t, y) = 2t R \frac{e^{2w(t, y)} - 1 - 2e^{w(t, y)} w(t, y)}{(e^{w(t, y)} - 1)^2 w(t, y)}.$$

One can verify that $b \geq 0$. So we have found $(\theta, b, y)$ such that $E_3 \geq 0$ when $m > 1$. Moreover, $\theta + by = C$ satisfies the first inequality in $(5.3)$. Therefore, we have obtained a family of bound on $X - Y$ with index $y \geq -\frac{NR}{4}$ as

$$B = \theta + bY = C(t, y) + \partial_y C(t, y) (Y - y).$$

This may be organized as the following lemma.

**Lemma 4.** Let $X, Y, R, C$ and $\partial_y C$ be defined according to $(5.4)$, $(5.7)$, $(5.5)$ and $(5.6)$. When $m > 1$, for any $y \geq -\frac{NR}{4}$, we have

$$(5.7)$$

$$X - Y \leq C(t, y) + \partial_y C(t, y) (Y - y).$$
To derive Theorem 2 from this lemma, notice that when \( Y \geq -\frac{NR}{4} \), we can let \( y = Y \) in (5.7), hence obtaining the first inequality in Theorem 2. As for the second inequality in Theorem 2, it follows from (5.5) and (5.6) that
\[
\lim_{y \to -\frac{NR}{4}} C(t, y) = \frac{NR}{2} + \frac{N}{2t} \text{ and } \lim_{y \to -\frac{NR}{4}} \partial_Y C(t, y) = \frac{2R}{3}.
\]
Therefore, by letting \( y \to -\frac{NR}{4} \) in (5.7) we arrived at the desiring result.

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