Sampling Algorithms and Coresets for $\ell_p$ Regression

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Abstract

The $\ell_p$ regression problem takes as input a matrix $A \in \mathbb{R}^{n \times d}$, a vector $b \in \mathbb{R}^n$, and a number $p \in [1, \infty)$, and it returns as output a number $Z$ and a vector $x_{\text{OPT}} \in \mathbb{R}^d$ such that $Z = \min_{x \in \mathbb{R}^d} \|Ax - b\|_p = \|Ax_{\text{OPT}} - b\|_p$. In this paper, we construct coresets and obtain an efficient two-stage sampling-based approximation algorithm for the very overconstrained ($n \gg d$) version of this classical problem, for all $p \in [1, \infty)$. The first stage of our algorithm non-uniformly samples $\hat{r}_1 = O(36d^{\max\{p/2+1, p\}} + 1)$ rows of $A$ and the corresponding elements of $b$, and then it solves the $\ell_p$ regression problem on the sample; we prove this is an $8$-approximation. The second stage of our algorithm uses the output of the first stage to resample $\hat{r}_1/\epsilon^2$ constraints, and then it solves the $\ell_p$ regression problem on the new sample; we prove this is a $(1+\epsilon)$-approximation. Our algorithm unifies, improves upon, and extends the existing algorithms for special cases of $\ell_p$ regression, namely $p = 1, 2$ [11, 13]. In course of proving our result, we develop two concepts—well-conditioned bases and subspace-preserving sampling—that are of independent interest.
1 Introduction
An important question in algorithmic problem solving is whether there exists a small subset of the input such that if computations are performed only on this subset, then the solution to the given problem can be approximated well. Such a subset is often known as a coreset for the problem. The concept of coresets has been extensively used in solving many problems in optimization and computational geometry; e.g., see the excellent survey by Agarwal, Har-Peled, and Varadarajan [2].

In this paper, we construct coresets and obtain efficient sampling algorithms for the classical $\ell_p$ regression problem, for all $p \in [1, \infty)$. Recall the $\ell_p$ regression problem:

**Problem 1 ($\ell_p$ regression problem).** Let $\|\cdot\|_p$ denote the $p$-norm of a vector. Given as input a matrix $A \in \mathbb{R}^{n \times m}$, a target vector $b \in \mathbb{R}^n$, and a real number $p \in [1, \infty)$, find a vector $x_{\text{OPT}}$ and a number $Z$ such that

$$
Z = \min_{x \in \mathbb{R}^m} \|Ax - b\|_p = \|Ax_{\text{OPT}} - b\|_p.
$$

In this paper, we will use the following $\ell_p$ regression coreset concept:

**Definition 2 ($\ell_p$ regression coreset).** Let $0 < \epsilon < 1$. A coreset for Problem [7] is a set of indices $I$ such that the solution $\hat{x}_{\text{OPT}}$ to $\min_{x \in \mathbb{R}^m} \|Ax - b\|_p$, where $A$ is composed of those rows of $A$ whose indices are in $I$ and $\hat{b}$ consists of the corresponding elements of $b$, satisfies $\|A\hat{x}_{\text{OPT}} - b\|_p \leq (1 + \epsilon) \min_x \|Ax - b\|_p$.

If $n \gg m$, i.e., if there are many more constraints than variables, then (1) is an overconstrained $\ell_p$ regression problem. In this case, there does not in general exist a vector $x$ such that $Ax = b$, and thus $Z > 0$. Overconstrained regression problems are fundamental in statistical data analysis and have numerous applications in applied mathematics, data mining, and machine learning [16, 10]. Even though convex programming methods can be used to solve the overconstrained regression problem in time $O((mn)^c)$, for $c > 1$, this is prohibitive if $n$ is large. This raises the natural question of developing more efficient algorithms that run in time $O(m^c n)$, for $c > 1$, while possibly relaxing the solution to Equation (1). In particular: Can we get a $\kappa$-approximation to the $\ell_p$ regression problem, i.e., a vector $\hat{x}$ such that $\|A\hat{x} - b\|_p \leq \kappa Z$, where $\kappa > 1$? Note that a coreset of small size would strongly satisfy our requirements and result in an efficiently computed solution that’s almost as good as the optimal. Thus, the question becomes: Do coresets exist for the $\ell_p$ regression problem, and if so can we compute them efficiently?

Our main result is an efficient two-stage sampling-based approximation algorithm that constructs a coreset and thus achieves a $(1 + \epsilon)$-approximation for the $\ell_p$ regression problem. The first-stage of the algorithm is sufficient to obtain a (fixed) constant factor approximation. The second-stage of the algorithm carefully uses the output of the first-stage to construct a coreset and achieve arbitrary constant factor approximation.

1.1 Our contributions

**Summary of results.** For simplicity of presentation, we summarize the results for the case of $m = d = \text{rank}(A)$. Let $k = \max\{p/2 + 1, \ p\}$ and let $\phi(r, d)$ be the time required to solve an $\ell_p$ regression problem with $r$ constraints and $d$ variables. In the first stage of the algorithm, we compute a set of sampling probabilities $p_1, \ldots, p_n$ in time $O(nd^5 \log n)$, sample $\hat{r_1} = O(36^p d^{k+1})$ rows of $A$ and the corresponding elements of $b$ according to the $p_i$’s, and solve an $\ell_p$ regression problem on the (much smaller) sample; we prove this is an $8$-approximation algorithm with a running time of $O (nd^5 \log n + \phi(\hat{r_1}, d))$. In the second stage of the algorithm, we use the residual from the first stage to compute a new set of sampling probabilities $q_1, \ldots, q_n$, sample additional $\hat{r_2} = O(\hat{r_1}/\epsilon^2)$ rows of $A$ and the corresponding elements of $b$ according to the $q_i$’s, and solve an $\ell_p$ regression problem on the (much smaller) sample; we prove this is a $(1 + \epsilon)$-approximation

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[1] For the special case of $p = 2$, vector space methods can solve the regression problem in time $O(m^2 n)$, and if $p = 1$ linear programming methods can be used.
algorithm with a total running time of \(O(nd^5 \log n + \phi(\hat{r}, d))\) (Section 4). We also show how to extend our basic algorithm to commonly encountered and more general settings of constrained, generalized, and weighted \(\ell_p\) regression problems (Section 5).

We note that the \(\ell_p\) regression problem for \(p = 1, 2\) has been studied before. For \(p = 1\), Clarkson [11] uses a subgradient based algorithm to preprocess \(A\) and \(b\) and then samples the rows of the modified problem; these elegant techniques however depend crucially on the linear structure of the \(\ell_1\) regression problem. Furthermore, this algorithm does not yield coresets. For \(p = 2\), Drineas, Mahoney, and Muthukrishnan [13] construct coresets by exploiting the singular value decomposition, a property peculiar to the \(\ell_2\) space. Thus in order to efficiently compute coresets for the \(\ell_p\) regression problem for all \(p \in [1, \infty)\), we need tools that capture the geometry of \(\ell_p\) norms. In this paper we develop the following two tools that may be of independent interest (Section 5).

1. **Well-conditioned bases.** Informally speaking, if \(U\) is a well-conditioned basis, then for all \(z \in \mathbb{R}^d\), \(\|z\|_p\) should be close to \(\|Uz\|_p\). We will formalize this by requiring that for all \(z \in \mathbb{R}^d\), \(\|z\|_q\) multiplicatively approximates \(\|Uz\|_p\) by a factor that can depend on \(d\) but is independent of \(n\) (where \(p\) and \(q\) are conjugate; i.e., \(q = p/(p - 1)\)). We show that these bases exist and can be constructed in time \(O(nd^5 \log n)\). In fact, our notion of a well-conditioned basis can be interpreted as a computational analog of the Auerbach and Lewis bases studied in functional analysis [23]. They are also related to the barycentric spanners recently introduced by Awerbuch and R. Kleinberg [5] (Section 3.1). J. Kleinberg and Sandler [17] defined the notion of an \(\ell_1\)-independent basis, and our well-conditioned basis can be used to obtain an exponentially better “condition number” than their construction. Further, Clarkson [11] defined the notion of an “\(\ell_1\)-conditioned matrix,” and he preprocessed the input matrix to an \(\ell_1\) regression problem so that it satisfies conditions similar to those satisfied by our bases.

2. **Subspace-preserving sampling.** We show that sampling rows of \(A\) according to information in the rows of a well-conditioned basis of \(A\) minimizes the sampling variance and consequently, the rank of \(A\) is not lost by sampling. This is critical for our relative-error approximation guarantees. The notion of subspace-preserving sampling was used in [13] for \(p = 2\), but we abstract and generalize this concept for all \(p \in [1, \infty)\).

We note that for \(p = 2\), our sampling complexity matches that of [13], which is \(O(d^2/\epsilon^2)\); and for \(p = 1\), it improves that of [11] from \(O(d^{3.5}(\log d)/\epsilon^2)\) to \(O(d^{2.5}/\epsilon^2)\).

**Overview of our methods.** Given an input matrix \(A\), we first construct a well-conditioned basis for \(A\) and use that to obtain bounds on a slightly non-standard notion of a \(p\)-norm condition number of a matrix. The use of this particular condition number is crucial since the variance in the subspace preserving sampling can be upper bounded in terms of it. An \(\epsilon\)-net argument then shows that the first stage sampling gives us a \(2\)-approximation. The next twist is to use the output of the first stage as a feedback to fine-tune the sampling probabilities. This is done so that the “positional information” of \(b\) with respect to \(A\) is also preserved in addition to the subspace. A more careful use of a different \(\epsilon\)-net shows that the second stage sampling achieves a \((1 + \epsilon)\)-approximation.

### 1.2 Related work

As mentioned earlier, in course of providing a sampling-based approximation algorithm for \(\ell_1\) regression, Clarkson [11] shows that coresets exist and can be computed efficiently for a controlled \(\ell_1\) regression problem. Clarkson first preprocesses the input matrix \(A\) to make it well-conditioned with respect to the \(\ell_1\) norm then applies a subgradient-descent-based approximation algorithm to guarantee that the \(\ell_1\) norm of the target vector is conveniently bounded. Coresets of size \(O(d^{3.5} \log d/\epsilon^2)\) are thereupon exhibited for

\(^2\)Two ingredients of [11] use the linear structure: the subgradient based preprocessing itself, and the counting argument for the concentration bound.
this modified regression problem. For the $\ell_2$ case, Drineas, Mahoney and Muthukrishnan [13] designed sampling strategies to preserve the subspace information of $A$ and proved the existence of a coreset of rows of size $O(d^2/\varepsilon^2)$—for the original $\ell_2$ regression problem; this leads to a $(1+\varepsilon)$-approximation algorithm. While their algorithm used $O(nd^2)$ time to construct the coreset and solve the $\ell_2$ regression problem—which is sufficient time to solve the regression problem—in a subsequent work, Sarlós [19] improved the running time for solving the regression problem to $O(nd)$ by using random sketches based on the Fast Johnson–Lindenstrauss transform of Ailon and Chazelle [3].

More generally, embedding $d$-dimensional subspaces of $L_p$ into $\ell_p^{(d)}$ using coordinate restrictions has been extensively studied [20, 8, 22, 23, 21]. Using well-conditioned bases, one can provide a constructive analog of Schechtman’s existential $L_1$ embedding result [20] (see also [8]), that any $d$-dimensional subspace of $L_1[0,1]$ can be embedded in $\ell_1^d$ with distortion $(1+\varepsilon)$ with $r = O(d^2/\varepsilon^2)$, albeit with an extra factor of $\sqrt{d}$ in the sampling complexity. Coresets have been analyzed by the computation geometry community as a tool for efficiently approximating various extent measures [1, 2]; see also [15, 6, 14] for applications of coresets in combinatorial optimization. An important difference is that most of the coreset constructions are exponential in the dimension, and thus applicable only to low-dimensional problems, whereas our coresets are polynomial in the dimension, and thus applicable to high-dimensional problems.

2 Preliminaries

Given a vector $x \in \mathbb{R}^m$, its $p$-norm is $\|x\|_p = \sum_{i=1}^{m} |x_i|^p$, and the dual norm of $\|\cdot\|_p$ is denoted $\|\cdot\|_p'$, where $1/p + 1/q = 1$. Given a matrix $A \in \mathbb{R}^{n \times m}$, its generalized $p$-norm is $\|A\|_p = (\sum_{i=1}^{n} \sum_{j=1}^{m} |A_{ij}|^p)^{1/p}$. This is a submultiplicative matrix norm that generalizes the Frobenius norm from $p = 2$ to all $p \in [1, \infty)$, but it is not a vector-induced matrix norm. The $j$-th column of $A$ is denoted $A_{\cdot j}$, and the $i$-th row is denoted $A_{i \cdot}$. In this notation, $\|A\|_p = (\sum_{j} \|A_{\cdot j}\|_p^p)^{1/p} = (\sum_{i} \|A_{i \cdot}\|_p^p)^{1/p}$. For $x, x', x'' \in \mathbb{R}^m$, it can be shown using Hölder’s inequality that $\|x - x''\|^p \leq 2^{p-1} \left(\|x - x'\|^p + \|x'' - x'\|^p\right)$.

Two crucial ingredients in our proofs are $\varepsilon$-nets and tail-inequalities. A subset $\mathcal{N}(D)$ of a set $D$ is called an $\varepsilon$-net in $D$ for some $\varepsilon > 0$ if for every $x \in D$, there is a $y \in \mathcal{N}(D)$ with $\|x - y\| \leq \varepsilon$. In order to construct an $\varepsilon$-net for $D$ it is enough to choose $\mathcal{N}(D)$ to be the maximal set of points that are pairwise $\varepsilon$ apart. It is well known that the unit ball of a $d$-dimensional space has an $\varepsilon$-net of size at most $(3/\varepsilon)^d$ [8].

Finally, throughout this paper, we will use the following sampling matrix formalism to represent our sampling operations. Given a set of $n$ probabilities, $p_i \in (0, 1]$, for $i = 1, \ldots, n$, let $S$ be an $n \times n$ diagonal sampling matrix such that $S_{ii}$ is set to $1/p_i^{1/p}$ with probability $p_i$ and to zero otherwise. Clearly, premultiplying $A$ or $b$ by $S$ determines whether the $i$-th row of $A$ and the corresponding element of $b$ will be included in the sample, and the expected number of rows/elements selected is $r' = \sum_{i=1}^{n} p_i$. (In what follows, we will abuse notation slightly by ignoring zeroed out rows and regarding $S$ as an $r' \times n$ matrix and thus $SA$ as an $r' \times m$ matrix.) Thus, e.g., sampling constraints from Equation (1) and solving the induced subproblem may be represented as solving

$$
\hat{Z} = \min_{\hat{x} \in \mathbb{R}^m} \|SA\hat{x} - Sb\|_p.
$$

A vector $\hat{x}$ is said to be a $\kappa$-approximation to the $\ell_p$ regression problem of Equation (1), for $\kappa \geq 1$, if $\|A\hat{x} - b\|_p \leq \kappa \hat{Z}$. Finally, the Appendix contains all the missing proofs.

3 Main technical ingredients

3.1 Well-conditioned bases

We introduce the following notion of a “well-conditioned” basis.
Definition 3 (Well-conditioned basis). Let $A$ be an $n \times m$ matrix of rank $d$, let $p \in [1, \infty)$, and let $q$ be its dual norm. Then an $n \times d$ matrix $U$ is an $(\alpha, \beta, p)$-well-conditioned basis for the column space of $A$ if (1) $\|U\|_p \leq \alpha$, and (2) for all $z \in \mathbb{R}^d$, $\|z\|_q \leq \beta \|Uz\|_p$. We will say that $U$ is a $p$-well-conditioned basis for the column space of $A$ if $\alpha$ and $\beta$ are $dO(1)$, independent of $m$ and $n$.

Recall that any orthonormal basis $U$ for $\text{span}(A)$ satisfies both $\|U\|_2 = \|U\|_F = \sqrt{d}$ and also $\|z\|_2 = \|Uz\|_2$ for all $z \in \mathbb{R}^d$, and thus is a $(\sqrt{d}, 1, 2)$-well-conditioned basis. Thus, Definition 3 generalizes to an arbitrary $p$-norm, for $p \in [1, \infty)$, the notion that an orthogonal matrix is well-conditioned with respect to the 2-norm. Note also that duality is incorporated into Definition 3 since it relates the $p$-norm of the vector $z \in \mathbb{R}^d$ to the $q$-norm of the vector $Uz \in \mathbb{R}^n$, where $p$ and $q$ are dual.

The existence and efficient construction of these bases is given by the following.

Theorem 4. Let $A$ be an $n \times m$ matrix of rank $d$, let $p \in [1, \infty)$, and let $q$ be its dual norm. Then there exists an $(\alpha, \beta, p)$-well-conditioned basis $U$ for the column space of $A$ such that: if $p < 2$, then $\alpha = d^{\frac{1}{p} + \frac{1}{2}}$ and $\beta = 1$, if $p = 2$, then $\alpha = d^{\frac{1}{2}}$ and $\beta = 1$, and if $p > 2$, then $\alpha = d^{\frac{1}{p} + \frac{1}{2}}$ and $\beta = d^{\frac{1}{2} - \frac{1}{p}}$. Moreover, $U$ can be computed in $O(nmd + nd^6 \log n)$ time (or in just $O(ndm)$ time if $p = 2$).

Proof. Let $A = QR$, where $Q$ is any $n \times d$ matrix that is an orthonormal basis for $\text{span}(A)$ and $R$ is a $d \times m$ matrix. If $p = 2$, then $Q$ is the desired basis $U$; from the discussion following Definition 3, $\alpha = \sqrt{d}$ and $\beta = 1$, and computing it requires $O(ndm)$ time. Otherwise, fix $Q$ and $p$ and define the norm, $\|z\|_{Q,p} \triangleq \|Qz\|_p$. A quick check shows that $\|\cdot\|_{Q,p}$ is indeed a norm. $(\|z\|_{Q,p} = 0$ if and only if $z = 0$ since $Q$ has full column rank; $\|\gamma z\|_{Q,p} = |\gamma| \|Qz\|_p = |\gamma| \|z\|_{Q,p}$; and $\|z + z'\|_{Q,p} \leq \|Qz\|_p + \|Qz'\|_p = \|z\|_{Q,p} + \|z'\|_{Q,p})$

Consider the set $C = \{z \in \mathbb{R}^d : \|z\|_{Q,p} \leq 1\}$, which is the unit ball of the norm $\|\cdot\|_{Q,p}$. In addition, define the $d \times d$ matrix $F$ such that $E_{1,1} = \{z \in \mathbb{R}^d : z^T F z \leq 1\}$ is the Löwner–John ellipsoid of $C$. Since $C$ is symmetric about the origin, $(1/\sqrt{d})E_{1,1} \subseteq C \subseteq E_{1,1}$; thus, for all $z \in \mathbb{R}^d$,

$$\|z\|_{E_{1,1}} \leq \|z\|_{Q,p} \leq \sqrt{d} \|z\|_{E_{1,1}},$$

where $\|z\|_{E_{1,1}}^2 = z^T F z$ (see, e.g. [9, pp. 413–4]). Since the matrix $F$ is symmetric positive definite, we can express it as $F = G^T G$, where $G$ is full rank and upper triangular. Since $Q$ is an orthogonal basis for $\text{span}(A)$ and $G$ is a $d \times d$ matrix of full rank, it follows that $U = QG^{-1}$ is an $n \times d$ matrix that spans the column space of $A$. We claim that $U \triangleq QG^{-1}$ is the desired $p$-well-conditioned basis.

To establish this claim, let $z' = G z$. Thus, $\|z\|_{E_{1,1}}^2 = z^T F z = z^T G^T G z = (G z)^T G z = z'^T z' = \|z'\|_2^2$. Furthermore, since $G$ is invertible, $z = G^{-1} z'$, and thus $\|z\|_{Q,p} = \|Qz\|_p = \|QG^{-1} z'\|_p$. By combining these expression with (3), it follows that for all $z' \in \mathbb{R}^d$,

$$\|z'\|_2 \leq \|Uz'\|_p \leq \sqrt{d} \|z'\|_2.$$ 

Since $\|U\|_p^2 = \sum_j |U_{j,j}|_p^2 = \sum_j |U_{j,j}|_p^2 \leq \sum_j d^\frac{1}{p} \|e_j\|_2^p = d^{\frac{1}{p} + 1}$, where the inequality follows from the upper bound in (4), it follows that $\alpha = d^{\frac{1}{p} + \frac{1}{2}}$. If $p < 2$, then $q > 2$ and $\|z\|_q \leq \|z\|_2$ for all $z \in \mathbb{R}^d$; by combining this with (4), it follows that $\beta = 1$. On the other hand, if $p > 2$, then $q < 2$ and $\|z\|_q \leq d^{\frac{1}{2} - \frac{1}{p}} \|z\|_2$; by combining this with (4), it follows that $\beta = d^{\frac{1}{2} - \frac{1}{p}}$.

Footnote: For $p = 2$, Drineas, Mahoney, and Muthukrishnan used this basis, i.e., an orthonormal matrix, to construct probabilities to sample the original matrix. For $p = 1$, Clarkson used a procedure similar to the one we describe in the proof of Theorem 4 to preprocess $A$ such that the 1-norm of $z$ is a $d\sqrt{d}$ factor away from the 1-norm of $Az$. 4.
In order to construct $U$, we need to compute $Q$ and $G$ and then invert $G$. Our matrix $A$ can be decomposed into $QR$ using the compact $QR$ decomposition in $O(nd)$ time. The matrix $F$ describing the Löwner–John ellipsoid of the unit ball of $\|\cdot\|_{Q,p}$ can be computed in $O(nd^5 \log n)$ time. Finally, computing $G$ from $F$ takes $O(d^3)$ time, and inverting $G$ takes $O(d^3)$ time.

**Connection to barycentric spanners.** A point set $K = \{K_1, \ldots, K_d\} \subseteq D \subseteq \mathbb{R}^d$ is a barycentric spanner for the set $D$ if every $z \in D$ can be expressed as a linear combination of elements of $K$ using coefficients in $[-C,C]$, for $C = 1$. When $C > 1$, $K$ is called a $C$-approximate barycentric spanner. Barycentric spanners were introduced by Awerbuch and R. Kleinberg in [5]. They showed that if a set is compact, then it has a barycentric spanner. Our proof shows that if $A$ is an $n \times d$ matrix, then $\tau^{-1} = R^{-1}G^{-1} \in \mathbb{R}^{d \times d}$ is a $\sqrt{d}$-approximate barycentric spanner for $D = \{z \in \mathbb{R}^d : \|Az\|_p \leq 1\}$. To see this, first note that each $\tau^{-1}_j$ belongs to $D$ since $\|A\tau^{-1}_j\|_p = \|Ue_j\|_p \leq \|e_j\|_2 = 1$, where the inequality is obtained from Equation (4).

Moreover, since $\tau^{-1}$ spans $\mathbb{R}^d$, we can write any $z \in D$ as $z = \tau^{-1} \nu$. Hence,

$$\frac{\|\nu\|_{\infty}}{\sqrt{d}} \leq \frac{\|\nu\|_2}{\sqrt{d}} \leq \frac{\|U\nu\|_p}{\sqrt{d}} = \frac{\|A\tau^{-1} \nu\|_p}{\sqrt{d}} = \frac{\|Az\|_p}{\sqrt{d}} \leq 1,$$

where the second inequality is also obtained from Equation (4). This shows that our basis has the added property that every element $z \in D$ can be expressed as a linear combination of elements (or columns) of $\tau^{-1}$ using coefficients whose $\ell_2$ norm is bounded by $\sqrt{d}$.

**Connection to Auerbach bases.** An Auerbach basis $U = \{U_{s,j}\}_{j=1}^d$ for a $d$-dimensional normed space $A$ is a basis such that $\|U_{s,j}\|_p = 1$ for all $j$ and such that whenever $y = \sum_j \nu_j U_{s,j}$ is in the unit ball of $A$ then $|\nu_j| \leq 1$. The existence of such a basis for every finite dimensional normed space was first proved by Herman Auerbach [4] (see also [12][24]). It can easily be shown that an Auerbach basis is an $(\alpha, \beta, p)$-well-conditioned basis, with $\alpha = d$ and $\beta = 1$ for all $p$. Further, suppose $U$ is an Auerbach basis for $\text{span}(A)$, where $A$ is an $n \times d$ matrix of rank $d$. Writing $A = U\tau$, it follows that $\tau^{-1}$ is an exact barycentric spanner for $D = \{z \in \mathbb{R}^d : \|Az\|_p \leq 1\}$. Specifically, each $\tau^{-1}_j \in D$ since $\|A\tau^{-1}_j\|_p = \|U_{s,j}\|_p = 1$. Now write $z \in D$ as $z = \tau^{-1} \nu$. Since the vector $y = Az = U \nu$ is in the unit ball of $\text{span}(A)$, we have $|\nu_j| \leq 1$ for all $1 \leq j \leq d$. Therefore, computing a barycentric spanner for the compact set $D$—which is the pre-image of the unit ball of $\text{span}(A)$—is equivalent (up to polynomial factors) to computing an Auerbach basis for $\text{span}(A)$.

### 3.2 Subspace-preserving sampling

In the previous subsection (and in the notation of the proof of Theorem 4), we saw that given $p \in [1, \infty)$, any $n \times m$ matrix $A$ of rank $d$ can be decomposed as

$$A = QR = QG^{-1}GR = U\tau,$$

where $U = QG^{-1}$ is a $p$-well-conditioned basis for $\text{span}(A)$ and $\tau = GR$. The significance of a $p$-well-conditioned basis is that we are able to minimize the variance in our sampling process by randomly sampling rows of the matrix $A$ and elements of the vector $b$ according to a probability distribution that depends on norms of the rows of the matrix $U$. This will allow us to preserve the subspace structure of $\text{span}(A)$ and thus to achieve relative-error approximation guarantees.

More precisely, given $p \in [1, \infty)$ and any $n \times m$ matrix $A$ of rank $d$ decomposed as $A = U\tau$, where $U$ is an $(\alpha, \beta, p)$-well-conditioned basis for $\text{span}(A)$, consider any set of sampling probabilities $p_i$ for $i = 1, \ldots, n$, that satisfy:

$$p_i \geq \min \left\{1, \frac{\|U_{i,*}\|_p^p}{\|U\|_p^p} \right\}, \quad (5)$$
where \( r = r(\alpha, \beta, p, d, \epsilon) \) to be determined below. Let us randomly sample the \( i^{th} \) row of \( A \) with probability \( p_i \), for all \( i = 1, \ldots, n \). Recall that we can construct a diagonal sampling matrix \( S \), where each \( S_{ii} = 1/p_i^{1/p} \) with probability \( p_i \) and 0 otherwise, in which case we can represent the sampling operation as \( SA \).

The following theorem is our main result regarding this subspace-preserving sampling procedure.

**Theorem 5.** Let \( A \) be an \( n \times m \) matrix of rank \( d \), and let \( p \in [1, \infty) \). Let \( U \) be an \((\alpha, \beta, p)\)-well-conditioned basis for \( \text{span}(A) \), and let us randomly sample rows of \( A \) according to the procedure described above using the probability distribution given by Equation (5), where \( r \geq 32^p(\alpha\beta)^p(\log(12) + \log(2))/(p^2\epsilon^2) \). Then, with probability \( 1 - \delta \), the following holds for all \( x \in \mathbb{R}^m \):

\[
| \| SAx \|_p - \| Ax \|_p | \leq \epsilon \| Ax \|_p .
\]

Several things should be noted about this result. First, it implies that \( \text{rank}(SA) = \text{rank}(A) \), since otherwise we could choose a vector \( x \in \text{null}(SA) \) and violate the theorem. In this sense, this theorem generalizes the subspace-preservation result of Lemma 4.1 of [13] to all \( p \in [1, \infty) \). Second, regarding sampling complexity: if \( p < 2 \) the sampling complexity is \( O(d^{p+2}) \), if \( p = 2 \) it is \( O(d^2) \), and if \( p > 2 \) it is \( O(d\frac{p+1}{p}d^{\frac{1}{p}} \frac{1}{d-\frac{1}{2}} \frac{1}{d-\frac{1}{2}}) = O(d^{p+1}) \). Finally, note that this theorem is analogous to the main result of Schechtman [20], which uses the notion of Auerbach bases.

## 4 The sampling algorithm

### 4.1 Statement of our main algorithm and theorem

Our main sampling algorithm for approximating the solution to the \( \ell_p \) regression problem is presented in Figure 4. The algorithm takes as input an \( n \times m \) matrix \( A \) of rank \( d \), a vector \( b \in \mathbb{R}^n \), and a number \( p \in [1, \infty) \). It is a two-stage algorithm that returns as output a vector \( \hat{x}_{\text{OPT}} \in \mathbb{R}^m \) (or a vector \( \hat{x}_{\text{c}} \in \mathbb{R}^m \) if only the first stage is run). In either case, the output is the solution to the induced \( \ell_p \) regression subproblem constructed on the randomly sampled constraints.

The algorithm first computes a \( p \)-well-conditioned basis \( U \) for \( \text{span}(A) \), as described in the proof of Theorem 4. Then, in the first stage, the algorithm uses information from the norms of the rows of \( U \) to sample constraints from the input \( \ell_p \) regression problem. In particular, roughly \( O(d^{p+1}) \) rows of \( A \), and the corresponding elements of \( b \), are randomly sampled according to the probability distribution given by

\[
p_i = \min \left\{ 1, \frac{\| Ux \|_p}{\| U \|_p^2 r_1} \right\}, \quad \text{where} \quad r_1 = 8^2 \cdot 36^p \cdot d^k \left( d\ln(8 \cdot 36) + \ln(200) \right). \tag{6}
\]

implicitly represented by a diagonal sampling matrix \( S \), where each \( S_{ii} = 1/p_i^{1/p} \). For the remainder of the paper, we will use \( S \) to denote the sampling matrix for the first-stage sampling probabilities. The algorithm then solves, using any \( \ell_p \) solver of one’s choice, the smaller subproblem. If the solution to the induced subproblem is denoted \( \hat{x}_{\text{c}} \), then, as we will see in Theorem 6, this is an \( 8 \)-approximation to the original problem.\footnote{It has been brought to our attention by an anonymous reviewer that one of the main results of this section can be obtained with a simpler analysis. In particular, one can show that one can obtain a relative error (as opposed to a constant factor) approximation in one stage, if the sampling probabilities are constructed from subspace information in the augmented matrix \( [Ab] \) (as opposed to using just subspace information from the matrix \( A \)), i.e., by using information in both the data matrix \( A \) and the target vector \( b \).}

\footnote{For \( p = 2 \), Drineas, Mahoney, and Muthukrishnan show that this first stage actually leads to a \((1 + \epsilon)\)-approximation. For \( p = 1 \), Clarkson develops a subgradient-based algorithm and runs it, after preprocessing the input, on all the input constraints to obtain a constant-factor approximation in a stage analogous to our first stage. Here, however, we solve an \( \ell_p \) regression problem on a small subset of the constraints to obtain the constant-factor approximation. Moreover, our procedure works for all \( p \in [1, \infty) \).}
Input: An $n \times m$ matrix $A$ of rank $d$, a vector $b \in \mathbb{R}^n$, and $p \in [1, \infty)$.

Let $0 < \epsilon < 1/7$, and define $k = \max\{p/2 + 1, p\}$.

- Find a $p$-well-conditioned basis $U \in \mathbb{R}^{n \times d}$ for $\text{span}(A)$ (as in the proof of Theorem 4).

- Stage 1: Define $p_i = \min\left\{1, \frac{\|U_i\|_p}{\|U\|_p} r_1 \right\}$ where $r_1 = 8^2 \cdot 36^p d^k \left(d \ln(8 \cdot 36) + \ln(200)\right)$.
  
  - Generate (implicitly) $S$ where $S_{ii} = 1/p_i^{1/p}$ with probability $p_i$ and 0 otherwise.
  
  - Let $\hat{x}_c$ be the solution to $\min_{x \in \mathbb{R}^m} \|S(Ax - b)\|_p$.

- Stage 2: Let $\hat{\rho} = A\hat{x}_c - b$, and unless $\hat{\rho} = 0$ define $q_i = \min\left\{1, \max\left(\frac{p_i}{\|\hat{\rho}\|_p} r_2, 1\right)\right\}$ with $r_2 = \frac{36^p d^k}{\epsilon^2} \left(d \ln\left(\frac{36}{\epsilon}\right) + \ln(200)\right)$.
  
  - Generate (implicitly, a new) $T$ where $T_{ii} = 1/q_i^{1/p}$ with probability $q_i$ and 0 otherwise.
  
  - Let $\hat{x}_\text{OPT}$ be the solution to $\min_{x \in \mathbb{R}^m} \|T(Ax - b)\|_p$.

Output: $\hat{x}_\text{OPT}$ (or $\hat{x}_c$ if only the first stage is run).

Figure 1: Sampling algorithm for $\ell_p$ regression.

In the second stage, the algorithm uses information from the residual of the $8$-approximation computed in the first stage to refine the sampling probabilities. Define the residual $\hat{\rho} = A\hat{x}_c - b$ (and note that $\|\hat{\rho}\|_p \leq 8 Z$). Then, roughly $O(d^{p+1}/\epsilon^2)$ rows of $A$, and the corresponding elements of $b$, are randomly sampled according to the probability distribution

$$q_i = \min\left\{1, \max\left(\frac{p_i}{\|\hat{\rho}\|_p} r_2, 1\right)\right\},$$

where $r_2 = \frac{36^p d^k}{\epsilon^2} \left(d \ln\left(\frac{36}{\epsilon}\right) + \ln(200)\right)$.

As before, this can be represented as a diagonal sampling matrix $T$, where each $T_{ii} = 1/q_i^{1/p}$ with probability $q_i$ and 0 otherwise. For the remainder of the paper, we will use $T$ to denote the sampling matrix for the second-stage sampling probabilities. Again, the algorithm solves, using any $\ell_p$ solver of one’s choice, the smaller subproblem. If the solution to the induced subproblem at the second stage is denoted $\hat{x}_\text{OPT}$, then, as we will see in Theorem 5, this is a $(1 + \epsilon)$-approximation to the original problem.

The following is our main theorem for the $\ell_p$ regression algorithm presented in Figure 1.

**Theorem 6.** Let $A$ be an $n \times m$ matrix of rank $d$, let $b \in \mathbb{R}^n$, and let $p \in [1, \infty)$. Recall that $r_1 = 8^2 \cdot 36^p d^k \left(d \ln(8 \cdot 36) + \ln(200)\right)$ and $r_2 = \frac{36^p d^k}{\epsilon^2} \left(d \ln\left(\frac{36}{\epsilon}\right) + \ln(200)\right)$. Then,

- **Constant-factor approximation.** If only the first stage of the algorithm in Figure 7 is run, then with probability at least $0.6$, the solution $\hat{x}_c$ to the sampled problem based on the $p_i$’s of Equation 5 is an $8$-approximation to the $\ell_p$ regression problem;

---

6The subspace-based sampling probabilities are similar to those used by Drineas, Mahoney, and Muthukrishnan, while the residual-based sampling probabilities are similar to those used by Clarkson.
• **Relative-error approximation.** If both stages of the algorithm are run, then with probability at least 0.5, the solution \( \hat{x}_{\text{OPT}} \) to the sampled problem based on the \( q_i \)'s of Equation (7) is a \((1 + \epsilon)\)-approximation to the \( \ell_p \) regression problem;

• **Running time.** The \( s \)th stage of the algorithm runs in time \( O(nmd + nd^2 \log n + \phi(20ir_1, m)) \), where \( \phi(s, t) \) is the time taken to solve the regression problem \( \min_{x \in \mathbb{R}^t} \| A'x - b' \|_p \), where \( A' \in \mathbb{R}^{s \times t} \) is of rank \( d \) and \( b' \in \mathbb{R}^s \).

Note that since the algorithm of Figure 1 constructs the \((\alpha, \beta, p)\)-well-conditioned basis \( U \) using the procedure in the proof of Theorem 4, our sampling complexity depends on \( \alpha \) and \( \beta \). In particular, it will be \( O(d^{(\alpha \beta)p}) \). Thus, if \( p < 2 \) our sampling complexity is \( O(d \cdot d^{(\alpha \beta)^2}) = O(d^{(\alpha \beta)^2 + 1}) \); if \( p > 2 \) it is \( O(d(d^{-\frac{1}{p}} + \frac{1}{d} d^{-\frac{1}{p}})^p) = O(d^{p+1}) \); and (although not explicitly stated, our proof will make it clear that) if \( p = 2 \) it is \( O(d^2) \). Note also that we have stated the claims of the theorem as holding with constant probability, but they can be shown to hold with probability at least \( 1 - \delta \) by using standard amplification techniques.

### 4.2 Proof for first-stage sampling – constant-factor approximation

To prove the claims of Theorem 6 having to do with the output of the algorithm after the first stage of sampling, we begin with two lemmas. First note that, because of our choice of \( r_1 \), we can use the subspace preserving Theorem 5 with only a constant distortion, i.e., for all \( x \), we have

\[
\frac{7}{8} \| Ax \|_p \leq \| SAx \|_p \leq \frac{9}{8} \| Ax \|_p
\]

with probability at least 0.99. The first lemma below now states that the optimal solution to the original problem provides a small (constant-factor) residual when evaluated in the sampled problem.

**Lemma 7.** \( \| S(Ax_{\text{OPT}} - b) \| \leq 3Z \), with probability at least \( 1 - 1/3^p \).

The next lemma states that if the solution to the sampled problem provides a constant-factor approximation (when evaluated in the sampled problem), then when this solution is evaluated in the original regression problem we get a (slightly weaker) constant-factor approximation.

**Lemma 8.** If \( \| S(A\hat{x}_c - b) \| \leq 3Z \), then \( \| A\hat{x}_c - b \| \leq 8Z \).

Clearly, \( \| S(A\hat{x}_c - b) \| \leq \| S(Ax_{\text{OPT}} - b) \| \) (since \( \hat{x}_c \) is an optimum for the sampled \( \ell_p \) regression problem). Combining this with Lemmas 7 and 8 it follows that the solution \( \hat{x}_c \) to the the sampled problem based on the \( p_i \)'s of Equation (5) satisfies \( \| A\hat{x}_c - b \| \leq 8Z \), i.e., \( \hat{x}_c \) is an \( 8 \)-approximation to the original \( Z \).

To conclude the proof of the claims for the first stage of sampling, note that by our choice of \( r_1 \), Theorem 5 fails to hold for our first stage sampling with probability no greater than \( 1/100 \). In addition, Lemma 7 fails to hold with probability no greater than \( 1/3^p \), which is no greater than \( 1/3 \) for all \( p \in [1, \infty) \). Finally, let \( \hat{r}_1 \) be a random variable representing the number of rows actually chosen by our sampling schema, and note that \( E[\hat{r}_1] \leq r_1 \). By Markov’s inequality, it follows that \( \hat{r}_1 > 20r_1 \) with probability less than \( 1/20 \). Thus, the first stage of our algorithm fails to give an \( 8 \)-approximation in the specified running time with a probability bounded by \( 1/3 + 1/20 + 1/100 < 2/5 \).

### 4.3 Proof for second-stage sampling – relative-error approximation

The proof of the claims of Theorem 6 having to do with the output of the algorithm after the second stage of sampling will parallel that for the first stage, but it will have several technical complexities that arise since the first triangle inequality approximation in the proof of Lemma 8 is too coarse for relative-error approximation.
approximation. By our choice of \( r_2 \) again, we have a finer result for subspace preservation. Thus, with probability 0.99, the following holds for all \( x \)

\[
(1 - \epsilon) \|Ax\|_p \leq \|S Ax\|_p \leq (1 + \epsilon) \|Ax\|_p
\]

As before, we start with a lemma that states that the optimal solution to the original problem provides a small (now a relative-error) residual when evaluated in the sampled problem. This is the analog of Lemma 7. An important difference is that the second stage sampling probabilities significantly enhance the probability of success.

**Lemma 9.** \( \|T(Ax_{\text{OPT}} - b)\| \leq (1 + \epsilon)Z, \) with probability at least 0.99.

Next we show that if the solution to the sampled problem provides a relative-error approximation (when evaluated in the sampled problem), then when this solution is evaluated in the original regression problem we get a (slightly weaker) relative-error approximation. We first establish two technical lemmas.

The following lemma says that for all optimal solutions \( \hat{x}_{\text{OPT}} \) to the second-stage sampled problem, \( A\hat{x}_{\text{OPT}} \) is not too far from \( Ax_c \), where \( x_c \) is the optimal solution from the first stage, in a \( p \)-norm sense. Hence, the lemma will allow us to restrict our calculations in Lemmas 11 and 12 to the ball of radius 12 \( Z \) centered at \( A\hat{x}_c \).

**Lemma 10.** \( \|A\hat{x}_{\text{OPT}} - Ax_c\| \leq 12 Z. \)

Thus, if we define the affine ball of radius 12 \( Z \) that is centered at \( A\hat{x}_c \) and that lies in \( \text{span}(A) \),

\[
B = \{ y \in \mathbb{R}^n : y = Ax, x \in \mathbb{R}^m, \|Ax_c - y\| \leq 12 Z \} , \tag{8}
\]

then Lemma 10 states that \( A\hat{x}_{\text{OPT}} \in B \), for all optimal solutions \( \hat{x}_{\text{OPT}} \) to the sampled problem. Let us consider an \( \epsilon \)-net, call it \( B_\epsilon \), with \( \epsilon = \epsilon Z \), for this ball \( B \). Using standard arguments, the size of the \( \epsilon \)-net is \( (\frac{36\sqrt{2}}{\epsilon Z})^d = (\frac{36}{\epsilon})^d \). The next lemma states that for all points in the \( \epsilon \)-net, if that point provides a relative-error approximation (when evaluated in the sampled problem), then when this point is evaluated in the original regression problem we get a (slightly weaker) relative-error approximation.

**Lemma 11.** For all points \( Ax_\epsilon \) in the \( \epsilon \)-net, \( B_\epsilon \), if \( \|T(Ax_\epsilon - b)\| \leq (1 + 3\epsilon)Z \), then \( \|Ax_\epsilon - b\| \leq (1 + 6\epsilon)Z \), with probability 0.99.

Finally, the next lemma states that if the solution to the sampled problem (in the second stage of sampling) provides a relative-error approximation (when evaluated in the sampled problem), then when this solution is evaluated in the original regression problem we get a (slightly weaker) relative-error approximation. This is the analog of Lemma 8 and its proof will use Lemma 11.

**Lemma 12.** If \( \|T(A\hat{x}_{\text{OPT}} - b)\| \leq (1 + \epsilon)Z \), then \( \|A\hat{x}_{\text{OPT}} - b\| \leq (1 + 7\epsilon)Z \).

Clearly, \( \|T(A\hat{x}_{\text{OPT}} - b)\| \leq \|T(Ax_{\text{OPT}} - b)\| \), since \( \hat{x}_{\text{OPT}} \) is an optimum for the sampled \( \ell_p \) regression problem. Combining this with Lemmas 9 and 12 it follows that the solution \( \hat{x}_{\text{OPT}} \) to the sampled problem based on the \( q_i \)'s of Equation 7 satisfies \( \|A\hat{x}_{\text{OPT}} - b\| \leq (1 + \epsilon)Z \), i.e., \( \hat{x}_{\text{OPT}} \) is a \( (1 + \epsilon) \)-approximation to the original \( Z \).

To conclude the proof of the claims for the second stage of sampling, recall that the first stage failed with probability no greater than \( 2/5 \). Note also that by our choice of \( r_2 \), Theorem 5 fails to hold for our second stage sampling with probability no greater than \( 1/100 \). In addition, Lemma 9 and Lemma 11 each
fails to hold with probability no greater than 1/100. Finally, let \( \hat{r}_2 \) be a random variable representing the number of rows actually chosen by our sampling schema in the second stage, and note that \( E[\hat{r}_2] \leq 2r_2 \). By Markov’s inequality, it follows that \( \hat{r}_2 > 40r_2 \) with probability less than 1/20. Thus, the second stage of our algorithm fails with probability less than 1/20 + 1/100 + 1/100 + 1/100 < 1/10. By combining both stages, our algorithm fails to give a \((1 + \epsilon)\)-approximation in the specified running time with a probability bounded from above by 2/5 + 1/10 = 1/2.

5 Extensions

In this section we outline several immediate extensions of our main algorithmic result.

Constrained \( \ell_p \) regression. Our sampling strategies are transparent to constraints placed on \( x \). In particular, suppose we constrain the output of our algorithm to lie within a convex set \( C \subseteq \mathbb{R}^m \). If there is an algorithm to solve the constrained \( \ell_p \) regression problem \( \min_{z \in C} \| A'x - b' \| \), where \( A' \in \mathbb{R}^{s \times m} \) is of rank \( d \) and \( b' \in \mathbb{R}^s \), in time \( \phi(s, m) \), then by modifying our main algorithm in a straightforward manner, we can obtain an algorithm that gives a \((1 + \epsilon)\)-approximation to the constrained \( \ell_p \) regression problem in time \( O(nmd + nd^5 \log n + \phi(40r_2, m)) \).

Generalized \( \ell_p \) regression. Our sampling strategies extend to the case of generalized \( \ell_p \) regression: given as input a matrix \( A \in \mathbb{R}^{n \times m} \) of rank \( d \), a target matrix \( B \in \mathbb{R}^{n \times p} \), and a real number \( p \in [1, \infty) \), find a matrix \( X \in \mathbb{R}^{m \times p} \) such that \( \| AX - B \|_p \) is minimized. To do so, we generalize our sampling strategies in a straightforward manner. The probabilities \( p_i \) for the first stage of sampling are the same as before. Then, if \( \hat{X} \) is the solution to the first-stage sampled problem, we can define the \( n \times p \) matrix \( \hat{\rho} = AX - B \), and define the second stage sampling probabilities to be \( q_i = \min(1, \max\{p_i, r_2\|\hat{\rho}_i\|_p/\|\hat{\rho}\|_p\}) \). Then, we can show that the \( \hat{X}_{\text{OPT}} \) computed from the second-stage sampled problem satisfies \( \| AX_{\text{OPT}} - B \|_p \leq (1 + \epsilon) \min_{X \in \mathbb{R}^{m \times p}} \| AX - B \|_p \), with probability at least 1/2.

Weighted \( \ell_p \) regression. Our sampling strategies also generalize to the case of \( \ell_p \) regression involving weighted \( p \)-norms: if \( w_1, \ldots, w_m \) are a set of non-negative weights then the weighted \( p \)-norm of a vector \( x \in \mathbb{R}^m \) may be defined as \( \| x \|_{p, w} = \left( \sum_{i=1}^{m} w_i |x_i|^p \right)^{1/p} \), and the weighted analog of the matrix \( p \)-norm \( \| \cdot \|_p \) may be defined as \( \| U \|_{p, w} = \left( \sum_{j=1}^{d} \| U_{\cdot j} \|_{p, w} \right)^{1/p} \). Our sampling schema proceeds as before. First, we compute a “well-conditioned” basis \( U \) for \( \text{span}(A) \) with respect to this weighted \( p \)-norm. The sampling probabilities \( p_i \) for the first stage of the algorithm are then \( p_i = \min(1, r_1 w_i\|U_{i \cdot}\|_p^p/\|U\|_{p, w}^p) \), and the sampling probabilities \( q_i \) for the second stage are \( q_i = \min(1, \max\{p_i, r_2 w_i\|\hat{\rho}_i\|_p/\|\hat{\rho}\|_{p, w}^p\}) \), where \( \hat{\rho} \) is the residual from the first stage.

General sampling probabilities. More generally, consider any sampling probabilities of the form: \( p_i \geq \min\left\{ 1, \max\left\{ \frac{\|U_{i \cdot}\|_p^p}{\|U\|_{p, w}^p}, \frac{\|\rho_{\text{opt}}\|_p^p}{2^p}\right\} r \right\} \), where \( \rho_{\text{opt}} = AX_{\text{opt}} - b \) and \( r \geq \frac{36^p d^p}{\epsilon^2} \left( d \ln\left(\frac{36}{\epsilon}\right) + \ln(200)\right) \) and where we adopt the convention that \( 0^0 = 0 \). Then, by an analysis similar to that presented for our two stage algorithm, we can show that, by picking \( O(36^p d^p + 1/\epsilon^2) \) rows of \( A \) and the corresponding elements of \( b \) (in a single stage of sampling) according to these probabilities, the solution \( \hat{x}_{\text{OPT}} \) to the sampled \( \ell_p \) regression problem is a \((1 + \epsilon)\)-approximation to the original problem, with probability at least 1/2. (Note that these sampling probabilities, if an equality is used in this expression, depend on the entries of the vector \( \rho_{\text{opt}} = AX_{\text{opt}} - b \); in particular, they require the solution of the original problem. This is reminiscent of the results of [13]. Our main two-stage algorithm shows that by solving a problem in the first stage based on coarse probabilities, we can refine our probabilities to approximate these probabilities and thus obtain an \((1 + \epsilon)\)-approximation to the \( \ell_p \) regression problem more efficiently.)
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A Tail inequalities

With respect to tail inequalities, we will use the following version of the Bernstein’s inequality.

**Theorem 13** ([18] [7]). Let \( \{X_i\}_{i=1}^n \) be independent random variables with \( E[X_i^2] < \infty \) and \( X_i \geq 0 \). Set \( Y = \sum X_i \) and let \( \gamma > 0 \). Then

\[
\Pr [Y \leq E[Y] - \gamma] \leq \exp \left( \frac{-\gamma^2}{2 \sum_i E[X_i^2]} \right) .
\]

(9)

If \( X_i - E[X_i] \leq \Delta \) for all \( i \), then with \( \sigma_i^2 = E[X_i^2] - E[X_i]^2 \) we have

\[
\Pr [Y \geq E[Y] + \gamma] \leq \exp \left( \frac{-\gamma^2}{2 \sum_i \sigma_i^2 + 2\gamma \Delta / 3} \right) .
\]

(10)

B Proofs for Section 3

B.1 Proof of Theorem 5

**Proof.** For simplicity of presentation, in this proof we will generally drop the subscript from our matrix and vector \( p \)-norms; i.e., unsubscripted norms will be \( p \)-norms. Note that it suffices to prove that, for all \( x \in \mathbb{R}^m \),

\[
(1 - \epsilon)^p \|Ax\|^p \leq \|SAx\|^p \leq (1 + \epsilon)^p \|Ax\|^p ,
\]

(11)

with probability \( 1 - \delta \). To this end, fix a vector \( x \in \mathbb{R}^m \), define the random variable \( X_i = (S_{ii} | A_{ix} | x)^p \), and recall that \( A_{ix} = U_{ix} \tau \) since \( A = U \tau \). Clearly, \( \sum_{i=1}^n X_i = \|SAx\|^p \). In addition, since \( E[X_i] = \|A_{ix}x\|^p \), it follows that \( \sum_{i=1}^n E[X_i] = \|Ax\|^p \). To bound Equation (11), first note that

\[
\sum_{i=1}^n (X_i - E[X_i]) = \sum_{i:p_i < 1} (X_i - E[X_i]) .
\]

(12)

Equation (12) follows since, according to the definition of \( p_i \) in Equation (5), \( p_i \) may equal 1 for some rows, and since these rows are always included in the random sample, \( X_i = E[X_i] \) for these rows. To bound the right hand side of Equation (12) note that for all \( i \) such that \( p_i < 1 \),

\[
\frac{|A_{ix}x|^p}{p_i} \leq \frac{\|U_{ix}\|^p \|\tau x\|^p}{p_i} \leq \frac{\|U\|^p \|\tau x\|^p}{r} \quad \text{(by H"{o}lders inequality)}
\]

\[
\leq \frac{(\alpha \beta)^p \|Ax\|^p}{r} \quad \text{(by Equation (5))}
\]

(13)

From Equation (13), if follows that for each \( i \) such that \( p_i < 1 \),

\[
X_i - E[X_i] \leq |A_{ix}x|^p / p_i \leq (\alpha \beta)^p \|Ax\|^p / r .
\]

Thus, we may define \( \Delta = (\alpha \beta)^p \|Ax\|^p / r \). In addition, it also follows from Equation (13) that

\[
\sum_{i:p_i < 1} E[X_i^2] = \sum_{i:p_i < 1} \frac{|A_{ix}x|^p}{p_i} \frac{|A_{ix}x|^p}{p_i} \leq \frac{(\alpha \beta)^p \|Ax\|^p}{r} \sum_{i:p_i < 1} |A_{ix}x|^p \quad \text{(by Equation (13))}
\]

\[
\leq (\alpha \beta)^p \|Ax\|^{2p} / r ,
\]

13
from which it follows that \( \sum_{i:p_i < 1} \sigma_i^2 \leq \sum_{i:p_i < 1} E[X_i^2] \leq (\alpha \beta)^p \|Ax\|^{2p}/r. \)

To apply the upper tail bound in Theorem 13, define \( \gamma = ((1 + \epsilon/4)^p - 1) \|Ax\|^p \). It follows that \( \gamma^2 \geq (p \epsilon/4)^2 \|Ax\|^{2p} \) and also that

\[
2 \sum_{i:p_i < 1} \sigma_i^2 + 2\gamma \Delta /3 \leq 2(\alpha \beta)^p \|Ax\|^{2p}/r + 2((1 + \epsilon/4)^p - 1)(\alpha \beta)^p \|Ax\|^{2p}/3r
\]

\[
\leq 32p(\alpha \beta)^p \|Ax\|^{2p}/r,
\]

where the second inequality follows by standard manipulations since \( \epsilon \leq 1 \) and since \( p \geq 1 \). Thus, by Equation (10) of Theorem 13, it follows that

\[
\text{Pr} [\|SAx\|^p > \|Ax\|^p + \gamma] = \text{Pr} \left[ \sum_{i:p_i < 1} X_i > E \left[ \sum_{i:p_i < 1} X_i \right] + \gamma \right]
\]

\[
\leq \exp \left( \frac{-\gamma^2}{2 \sum_{i:p_i < 1} \sigma_i^2 + 2\gamma \Delta /3} \right)
\]

\[
\leq \exp \left( -\epsilon^2 p^2 r/(\alpha \beta)^p 32p \right).
\]

Similarly, to apply the lower tail bound of Equation (9) of Theorem 13, define \( \gamma = (1 - (1 - \epsilon/4)^p) \|Ax\|^p \). Since \( \gamma \geq \epsilon \|Ax\|^p / 4 \), we can follow a similar line of reasoning to show that

\[
\text{Pr} [\|SAx\|^p < \|Ax\|^p - \gamma] \leq \exp \left( -\epsilon^2 p^2 r/(\alpha \beta)^p 32 \right).
\]

Choosing \( r \geq 32p(\alpha \beta)^p (d \ln(\frac{12}{\epsilon}) + \ln(\frac{1}{\delta}))/(p^2 \epsilon^2) \), we get that for every fixed \( x \), the following is true with probability at least \( 1 - (\frac{7}{12})^d \delta \):

\[
(1 - \epsilon/4)^p \|Ax\|^p \leq \|SAx\|^p \leq (1 + \epsilon/4)^p \|Ax\|^p.
\]

Now, consider the ball \( B = \{ y \in \mathbb{R}^n : y = Ax, \|y\| \leq 1 \} \) and consider an \( \epsilon \)-net for \( B \), with \( \epsilon = \epsilon/4 \). The number of points in the \( \epsilon \)-net is \( (\frac{12}{\epsilon})^d \). Thus, by the union bound, with probability \( 1 - \delta \), Equation (11) holds for all \( \epsilon \)-net. Now, to show that with the same probability Equation (11) holds for all points \( y \in B \), let \( y^* \in B \) be such that \( \|Sy^*\| - \|y^*\| \) is maximized, and let \( \eta = \sup \{\|Sy\| - \|y\| : y \in B\} \).

Also, let \( y^*_\epsilon \in B \) be the point in the \( \epsilon \)-net that is closest to \( y^* \). By the triangle inequality,

\[
\eta = \|Sy^*\| - \|y^*\| = \|Sy^*_\epsilon + S(y^* - y^*_\epsilon)\| - \|y^*_\epsilon + (y^* - y^*_\epsilon)\|
\]

\[
\leq \|Sy^*_\epsilon\| + \|S(y^* - y^*_\epsilon)\| - \|y^*_\epsilon\| + 2 \|y^* - y^*_\epsilon\| - \|y^* - y^*_\epsilon\|
\]

\[
\leq \|Sy^*_\epsilon\| - \|y^*_\epsilon\| + \|S(y^* - y^*_\epsilon)\| - \|y^* - y^*_\epsilon\| + 2 \|y^* - y^*_\epsilon\|
\]

\[
\leq \epsilon/4 \|y^*_\epsilon\| + \epsilon \eta/4 + \epsilon/2 ,
\]

where the last inequality follows since \( \|y^* - y^*_\epsilon\| \leq \epsilon, \epsilon \leq 1/7 \). Thus, Equation (11) holds for all points \( y \in B \), with probability at least \( 1 - \delta \). Similarly, it holds for any \( y \in \mathbb{R}^n \) such that \( y = Ax \), since \( y/\|y\| \in B \) and since \( \|S(y/\|y\|) - y/\|y\|\| \leq \epsilon \) implies that \( \|Sy - y\| \leq \epsilon \|y\| \), which completes the proof of the theorem.
C Proofs for Section 4

As in the proof of Theorem 5, unsubscripted norms will be $p$-norms.

C.1 Proof of Lemma 7

Proof. Define $X_i = (S_i |A_i x_{OPT} - b_i|)^p$. Thus, $\sum_i X_i = \|S(A x_{OPT} - b)\|^p$, and the first moment is $E[\sum_i X_i] = \|A x_{OPT} - b\|^p = Z$. The lemma follows since, by Markov’s inequality,

$$\Pr \left[ \sum_i X_i > 3^p E \left[ \sum_i X_i \right] \right] \leq \frac{1}{3^p},$$

i.e., $\|S(A x_{OPT} - b)\|^p > 3^p \|A x_{OPT} - b\|^p$, with probability no more than $1/3^p$. \hfill \qed

C.2 Proof of Lemma 8

Proof. We will prove the contrapositive: If $\|A \hat{x}_c - b\| > 8Z$, then $\|S(A \hat{x}_c - b)\| > 3Z$. To do so, note that, by Theorem 5 and the choice of $r_1$, we have that

$$\frac{7}{8} \|A\|_p \leq \|SA\|_p \leq \frac{9}{8} \|A\|_p.$$

Using this,

\[
\begin{align*}
\|S(A \hat{x}_c - b)\| &\geq \|SA\hat{x}_c - x_{OPT}\| - \|S(A x_{OPT} - b)\| \\
&\geq \frac{7}{8} \|A \hat{x}_c - x_{OPT}\| - 3Z \\
&\geq \frac{7}{8} (\|A \hat{x}_c - b\| - \|A x_{OPT} - b\|) - 3Z \\
&> \frac{7}{8} (8Z - 3Z) - 3Z \\
&> 3Z,
\end{align*}
\]

which establishes the lemma. \hfill \qed

C.3 Proof of Lemma 9

Proof. Define the random variable $X_i = (T_{ii} |A_i x_{OPT} - b_i|)^p$, and recall that $A_i = U_i \tau$ since $A = U \tau$. Clearly, $\sum_{i=1}^n X_i = \|T(A x_{OPT} - b)\|^p$. In addition, since $E[X_i] = |A_i x_{OPT} - b_i|^p$, it follows that $\sum_{i=1}^n E[X_i] = \|A x_{OPT} - b\|^p$. We will use Equation (10) of Theorem 13 to provide a bound for $\sum_i (X_i - E[X_i]) = \|T(A x_{OPT} - b)\|^p - \|A x_{OPT} - b\|^p$.

From the definition of $q_i$ in Equation (7), it follows that for some of the rows, $q_i$ may equal 1 (just as in the proof of Theorem 5). Since $X_i = E[X_i]$ for these rows, $\sum_i (X_i - E[X_i]) = \sum_{i:q_i < 1} (X_i - E[X_i])$, and thus we will bound this latter quantity with Equation (10). To do so, we must first provide a bound for
where the final inequality follows from the definition of $Z$ and the results from the first stage of sampling. Next, note that from the conditions on the probabilities $q_i$ in Equation (7), as well as by Definition 3 and the output of the first-stage of sampling, it follows that

$$X_i \leq E[X_i] \leq X_i$$

and for $\sum_{i: q_i < 1} \sigma_i^2 \leq \sum_i E[X_i^2]$. To that end, note that:

$$|A_{i*}(x_{OPT} - \hat{x}_c)| \leq \|U_{i*}\|_p \|\tau(x_{OPT} - \hat{x}_c)\|_q \leq \|U_{i*}\|_p \beta \|\tau(x_{OPT} - \hat{x}_c)\|_p \leq \|U_{i*}\|_p \beta (\|Ax_{OPT} - b\| + \|A\hat{x}_c - b\|) \leq \|U_{i*}\|_p \beta 9 Z,$$  \hspace{1cm} (14)

where $\gamma$ is set such that $\gamma = \|x_{OPT} - \hat{x}_c\|$, and since $\gamma \leq \|x_{OPT} - \hat{x}_c\|$, it follows that for all $i$ such that $q_i < 1$,

$$|\hat{\rho}| \leq \|\hat{\rho}\|_r \leq \frac{8 \|Z\|}{r_2}$$

and

$$\frac{\|U_{i*}\|}{r_2} \leq \|U\|_p \leq \frac{\alpha^p}{r_2},$$  \hspace{1cm} (15)

for all $i$ such that $q_i < 1$.

Thus, since $X_i \leq E[X_i] \leq A_{i*}(x_{OPT} - b_i)^p/q_i$, it follows that for all $i$ such that $q_i < 1$,

$$X_i - E[X_i] \leq \frac{2^{p-1}}{q_i} \left( |A_{i*}(x_{OPT} - \hat{x}_c)|^p + |\hat{\rho}|^p \right) \leq \frac{2^{p-1}}{q_i} \left( \frac{\|U_{i*}\|_p^p \beta^p 9^p Z^p}{q_i} + |\hat{\rho}|^p \right) \leq \frac{2^{p-1}}{q_i} \left( \frac{\|\hat{\rho}\|_r^p \alpha^p Z^p + 8^p Z^p}{r_2} \right) \leq \frac{c_p (\alpha \beta)^p Z^p}{r_2},$$  \hspace{1cm} (16)

where we set $c_p = 2^{p-1}(9^p + 8^p)$. Thus, we may define $\Delta = c_p (\alpha \beta)^p Z^p / r_2$. In addition, it follows that

$$\sum_{i: q_i < 1} E[X_i^2] = \sum_{i: q_i < 1} |A_{i*} x_{OPT} - b_i|^p \frac{|A_{i*} x_{OPT} - b_i|^p}{q_i} \leq \Delta \sum_i |A_{i*} x_{OPT} - b_i|^p \leq c_p (\alpha \beta)^p Z^{2p} / r_2.$$  \hspace{1cm} (17)

To apply the upper tail bound of Equation (10) of Theorem 13 define $\gamma = ((1 + \epsilon)^p - 1) Z^p$. We have $\gamma \geq p \epsilon Z^p$, and since $\epsilon \leq 1/7$, we also have $\gamma \leq \left( \left( \frac{8}{7} \right)^p - 1 \right) Z^p$. Hence, by Equation (10) of Theorem 13, it follows that

$$\ln Pr \left[ \|T(A_{OPT} - b)\|^p > \|A_{OPT} - b\|^p + \gamma \right] \leq \frac{-\gamma^2}{2 \sum_{i: q_i < 1} \sigma_i^2 + 2 \gamma \Delta / 3} \leq \frac{-\gamma^2}{36^p (\alpha \beta)^p}.$$  \hspace{1cm} (18)

Thus, $Pr \left[ \|T(A_{OPT} - b)\|^p > (1 + \epsilon) Z \right] \leq \exp \left( \frac{-p^2 \epsilon^2 r_2}{36^p (\alpha \beta)^p} \right)$, from which the lemma follows by our choice of $r_2$. \hspace{1cm} \square
C.4 Proof of Lemma 10

Proof. By two applications of the triangle inequality, it follows that
\[
\|\hat{A}^*_x - A\hat{x}_c\| \leq \|\hat{A}^*_x - A\hat{x}_c\| + \|A\hat{x}_c - b\| + \|A\hat{x}_c - b\| \\
\leq \|\hat{A}^*_x - A\hat{x}_c\| + 9Z ,
\]
where the second inequality follows since \(\|A\hat{x}_c - b\| \leq 8Z\) from the first stage of sampling and since \(Z = \|A\hat{x}_c - b\|\). In addition, we have that
\[
\|A\hat{x}_c - b\| \leq \frac{1}{(1 - \epsilon)} \|T(A\hat{x}_c - b)\| \quad \text{(by Theorem 5)}
\]
\[
\leq (1 + \epsilon) \|T(A\hat{x}_c - b)\| + \|T(A\hat{x}_c - b)\| \quad \text{(by the triangle inequality)}
\]
\[
\leq 2(1 + \epsilon) \|T(A\hat{x}_c - b)\| \\
\leq 2(1 + \epsilon)^2 \|A\hat{x}_c - b\| \quad \text{(by Lemma 9)},
\]
where the third inequality follows since \(\hat{x}_c\) is optimal for the sampled problem. The lemma follows since \(\epsilon \leq 1/7\). \(\square\)

C.5 Proof of Lemma 11

Proof. Fix a given point \(y^*_\epsilon = A\hat{x}_c^* \in B_\epsilon\). We will prove the contrapositive for this point, i.e., we will prove that if \(\|A\hat{x}_c^* - b\| > (1 + 6\epsilon)Z\), then \(\|T(A\hat{x}_c^* - b)\| > (1 + 3\epsilon)Z\), with probability at least \(1 - \frac{1}{100} (\frac{1}{20})^d\).

The lemma will then follow from the union bound.

To this end, define the random variable \(X_i = (T_\tau A_i \epsilon \hat{x}_c - b_i)^p\), and recall that \(A_i = U_i \tau\) since \(A = U \tau\). Clearly, \(\sum_{i=1}^n X_i = \|T(A\hat{x}_c - b)\|^p\). In addition, since \(E[X_i] = A_i \epsilon \hat{x}_c - b_i)^p\), it follows that \(\sum_{i=1}^n E[X_i] = \|A\hat{x}_c - b\|^p\). We will use Equation (9) of Theorem 13 to provide an upper bound for the event that \(\|T(A\hat{x}_c - b)\|^p \leq \|A\hat{x}_c - b\|^p - \gamma\), where \(\gamma = \|A\hat{x}_c - b\|^p - (1 + 3\epsilon)Z\), under the assumption that \(\|A\hat{x}_c - b\| > (1 + 6\epsilon)Z\).

From the definition of \(q_i\) in Equation (7), it follows that for some of the rows, \(q_i\) may equal 1 (just as in the proof of Theorem 5). Since \(X_i = E[X_i]\) for these rows, \(\sum_i (X_i - E[X_i]) = \sum_{i:p_i < 1} (X_i - E[X_i])\), and thus we will bound this latter quantity with Equation (9). To do so, we must first provide a bound for \(\sum_{i:p_i < 1} E[X_i]\). To that end, note that:
\[
|A_i \epsilon \hat{x}_c - \hat{x}_c| \leq \|U_i\|_p \|\tau(\hat{x}_c^* - \hat{x}_c)|_q \quad \text{(by H"older’s inequality)}
\]
\[
\leq \|U_i\|_p \|\tau(\hat{x}_c^* - \hat{x}_c)||_p \quad \text{(by Definition 3 and Theorem 4)}
\]
\[
\leq \|U_i\|_p \|\beta 12Z ,
\]
where the final inequality follows from the radius of the high-dimensional ball in which the \(\epsilon\)-net resides. From this, we can show that
\[
\frac{|A_i \epsilon \hat{x}_c^* - b|}{q_i} \leq 2^{p-1} \frac{p}{q_i} \left(\|A_i \epsilon \hat{x}_c^* - A_i \epsilon \hat{x}_c\|^p + \|\hat{\rho}\|^p\right) \quad \text{(since } \hat{\rho} = A\hat{x}_c - b \text{)}
\]
\[
\leq 2^{p-1} \frac{p}{q_i} \left(\|U_i\|_p 12^p \beta^p Z^p + \|\hat{\rho}\|^p\right) \quad \text{(by Equation (19))}
\]
\[
\leq 2^{p-1} \left(\alpha^p 12^p \beta^p Z^p + 8^p Z^p\right) / r_2 \quad \text{(by Equation (15))}
\]
\[
\leq 4^p (\alpha \beta)^p Z^p / r_2 .
\]
Therefore, we have that
\[
\sum_{i: q_i < 1} E\left[X_i^2\right] = \sum_{i: q_i < 1} |A_{i*} x^*_\varepsilon - b_i|^p |A_{i*} x^*_\varepsilon - b_i|^p q_i
\]
\[
\leq \frac{24^p (\alpha \beta)^p Z^p}{r_2} \sum_i |A_{i*} x^*_\varepsilon - b_i|^p
\]
\[
\leq 24^p (\alpha \beta)^p \|A x^*_\varepsilon - b\|^{2p} / r_2.
\] (21)

To apply the lower tail bound of Equation (9) of Theorem 13 define \(\gamma = \|A x^*_\varepsilon - b\|^p - (1 + 3\varepsilon)^p Z^p\). Thus, by Equation (21) and by Equation (9) of Theorem 13 it follows that
\[
\ln \Pr \left[\|T(A x^*_\varepsilon - b)\|^p \leq (1 + 3\varepsilon)^p Z^p\right] \leq -r_2 (\|A x^*_\varepsilon - b\|^p - (1 + 3\varepsilon)^p Z^p)^2
\]
\[
\leq \frac{-r_2}{24^p (\alpha \beta)^p} \left(1 - \frac{(1 + 3\varepsilon)^p Z^p}{\|A x^*_\varepsilon - b\|^p}\right)^2
\]
\[
< \frac{-r_2}{24^p (\alpha \beta)^p} \left(1 - \frac{(1 + 3\varepsilon)^p Z^p}{(1 + 6\varepsilon)^p Z^p}\right)^2 \quad \text{(by the premise)}
\]
\[
\leq \frac{-r_2 \varepsilon^2}{24^p (\alpha \beta)^p} \quad \text{(since } \varepsilon \leq 1/3\text{)}.
\]

Since \(r_2 \geq 24^p (\alpha \beta)^p (d \ln(\frac{36}{\varepsilon}) + \ln(200)) / \varepsilon^2\), it follows that \(\|T(A x^*_\varepsilon - b)\| \leq (1 + 3\varepsilon) Z\), with probability no greater than \(\frac{1}{200} \left(\frac{\varepsilon}{36}\right)^d\). Since there are no more than \(\left(\frac{36}{\varepsilon}\right)^d\) such points in the \(\varepsilon\)-net, the lemma follows by the union bound.

C.6 Proof of Lemma 12

Proof. We will prove the contrapositive: If \(\|\hat{A} \hat{x}_{\text{OPT}} - b\| > (1 + 7\varepsilon) Z\) then \(\|T(\hat{A} \hat{x}_{\text{OPT}} - b)\| > (1 + \varepsilon) Z\). Since \(\hat{A} \hat{x}_{\text{OPT}}\) lies in the ball \(B\) defined by Equation (8) and since the \(\varepsilon\)-net is constructed in this ball, there exists a point \(y_\varepsilon = Ax^*_\varepsilon\), call it \(Ax^*_\varepsilon\), such that \(Ax^*_\varepsilon - Ax^*_\varepsilon\varepsilon\) \(\leq \varepsilon Z\). Thus,
\[
\|Ax^*_\varepsilon - b\| \geq \|\hat{A} \hat{x}_{\text{OPT}} - b\| - \|Ax^*_\varepsilon - \hat{A} \hat{x}_{\text{OPT}}\| \quad \text{(by the triangle inequality)}
\]
\[
\geq (1 + 7\varepsilon) Z - \varepsilon Z \quad \text{(by assumption and the definition of } Ax^*_\varepsilon\text{)}
\]
\[
= (1 + 6\varepsilon) Z.
\]

Next, since Lemma 11 holds for all points \(Ax^*_\varepsilon\) in the \(\varepsilon\)-net, it follows that
\[
\|T(A x^*_\varepsilon - b)\| > (1 + 3\varepsilon) Z. \quad \text{(22)}
\]

Finally, note that
\[
\|T(\hat{A} \hat{x}_{\text{OPT}} - b)\| \geq \|T(A x^*_\varepsilon - b)\| - \|T(A x^*_\varepsilon - \hat{x}_{\text{OPT}})\| \quad \text{(by the triangle inequality)}
\]
\[
> (1 + 3\varepsilon) Z - (1 + \varepsilon) \|A(x^*_\varepsilon - \hat{x}_{\text{OPT}})\| \quad \text{(by Equation (22) and Theorem 5)}
\]
\[
> (1 + 3\varepsilon) Z - (1 + \varepsilon) \varepsilon Z \quad \text{(by the definition of } Ax^*_\varepsilon)\}
\]
\[
> (1 + \varepsilon) Z,
\]
which establishes the lemma. \(\square\)