The Maxwell-Boltzmann Distribution is not the Equilibrium on a Hyperboloid

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July 14, 2009

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We give a geometric formulation of the Fokker-Planck-Kramer equations for a particle moving on a Lie algebra under the influence of a dissipative and a random force. Special cases of interest are fluid mechanics, the Stochastic Loewner Equation and the rigid body. We find that the Boltzmann distribution, although a static solution, is not normalizable when the algebra is not unimodular. This is because the invariant measure of integration in momentum space is not the standard one. We solve the special case of the upper half-plane (hyperboloid) explicitly: there is another equilibrium solution to the Fokker-Planck equation, which is integrable. It breaks rotation invariance; moreover, the most likely value for velocity is not zero.

1 Introduction

Arnold [1] showed that the Euler equations of an ideal fluid describe geodesics in the group of volume preserving co-ordinate transformations of the space filled by the fluid. The negative curvature of this geometry implies that the geodesics diverge from each other exponentially: an elegant explanation for the observed instability of fluid motion. A more realistic description would include dissipation (Navier-Stokes) as well as small random forces not included in the ideal model. Since the infinite dimensional fluid system is difficult to understand directly, we look for analogous finite dimensional examples that have (i) Negative Curvature (exponentially diverging trajectories), (ii) Dissipation and (iii) Random Forcing.

The celebrated Langevin equation for the velocity of a particle in Brownian motion includes (2) and (3); being one dimensional, it cannot have curvature.

\[ \frac{dv}{dt} = -\gamma v + \eta. \]

Here \( \gamma \) is the dissipation constant, proportional to the viscosity of the medium in which the particle is moving. Also, \( \eta \) is a random force, modelled as a Gaussian of zero mean and zero correlation time (“white noise”):
\[<\eta(t)\eta(t')>=2D\delta(t-t').\]

The constant \(D\) measures the strength of the fluctuations. This leads to the Fokker-Planck equation for the probability density of velocity

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[ D \frac{\partial P}{\partial v} + \gamma v P \right]
\]

The equilibrium (static) solution of this equation is the Boltzmann distribution

\[P(v) = \frac{1}{Z} e^{-\beta \frac{v^2}{2}}\]

with an inverse temperature given by the Einstein relation

\[\beta D = \gamma.\]

How would things change if the particle is moving in a space of negative curvature? We will study in this paper the simplest such model. We will show, by explicit solution, that the equilibrium solution is not always the Boltzmann distribution. In particular, the most likely value of velocity is not zero. The essential reason is that the invariant measure on the space of velocities is not translation invariant: the Boltzmann distribution is not integrable in this measure. There is another static solution to the Fokker-Planck which is integrable. In simple cases we can find it explicitly in terms of classical functions.

There is already an extensive literature on Brownian motion on curved manifolds. The main approach is to solve the heat equation in position space. However, this does not address the question of the equilibrium velocity distribution, which we study here.

## 2 The Fokker-Planck-Kramers Equation

We review theory of a dynamical system subject to dissipation and fluctuation, paying special attention to the role of the measure on phase space in the Fokker-Planck-Kramers equation. We base our discussion on refs. [2, 3] but use a more geometric formulation. The erudite reader might want to skip this section after a glance at the notation.

### 2.1 The Ideal System

Consider a dynamical system with equations of motion

\[
\frac{du_a}{dt} = V_a
\]

for some vector field \(V^a\). We will assume that there is a measure \(\mu\) that is invariant under this evolution,
\[ \partial^a [\mu V_a] = 0; \]  

as well as a quantity \( E(\text{energy}) \) that is conserved by it:

\[ \frac{dE}{dt} = V_a \frac{\partial E}{\partial u_a} = 0. \]

For example, if there is a sympletic form \( \omega_{ab} \) and hamiltonian \( H \) such that

\[ V_a = \{ H, u_a \} \]

for the induced Poisson bracket, the invariant measure is the Pfaffian (square root of the determinant) of \( \omega \) :

\[ \mu = \text{Pf} \, \omega. \]

Then (1) is just Liouville’s theorem. Energy will be conserved if

\[ \{ H, E \} = 0 \]

This is automatic if the hamiltonian itself is the energy; but this is not necessary.

2.2 Dissipation

A more realistic description could include a dissipative force. We will assume that the dissipative force is a gradient of energy; i.e.,

\[ \Gamma_a = -\Gamma_{ab} \frac{\partial E}{\partial u_b} \]

for some positive \( \Gamma_{ab} \). The resulting equations

\[ \frac{du_a}{dt} = \Gamma_a + V_a \]

imply that energy decreases:

\[ \frac{dE}{dt} = -\Gamma_{ab} \frac{\partial E}{\partial u_a} \frac{\partial E}{\partial u_b} \leq 0. \]

2.3 Fluctuation

We assume a standard model of fluctuating force: a Gaussian \( \eta_a(t) \) with zero correlation time ( “white noise”) 

\[ < \eta_a(t) \eta_b(t') > = 2D_{ab} \delta(t - t'). \]

The correlation tensor \( D^{ab} \) must be positive as well. The time evolution is now a system of stochastic ordinary differential equations

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\[^1\text{That is,} \Gamma_{ab} \text{is symmetric and} \Gamma_{ab} u^a u^b \geq 0 \text{ for all} \ u_a.\]
\[ \frac{du_a}{dt} = \Gamma_a + V_a + \eta_a. \]

The probability density \( P(u,t) \) for the random variable \( u \) will then satisfy a partial differential equation: a diffusion equation with drift. The total probability

\[ \int P \mu du = 1 \]

so that \( \frac{\partial (\mu P)}{\partial t} \) must be a total derivative. This leads us to the Fokker-Planck-Kramers equation

\[ \mu \frac{\partial P}{\partial t} = \frac{\partial}{\partial u_a} \left[ \mu \left\{ D_{ab} \frac{\partial P}{\partial u_b} - (\Gamma_a + V_a) P \right\} \right]. \]

### 2.4 The Boltzmann Solution

The function

\[ P = e^{-\beta E} \]

is a solution to this equation provided that the dissipation and fluctuation tensors are proportional (the Einstein-Smoluchowski relation):

\[ \beta D_{ab} = \Gamma_{ab}. \] (3)

For, in this case

\[ D_{ab} \frac{\partial P}{\partial u_b} = \Gamma_a P. \]

Moreover,

\[ D_{ab} \frac{\partial P}{\partial u_b} = \Gamma_a P. \]

Moreover,

\[ \frac{\partial [\mu V^a P]}{\partial u_a} = \frac{\partial [\mu V^a]}{\partial u_a} P + \mu V_a \frac{\partial P}{\partial u_a} = \mu \{H, P\} = -\mu \beta E \{H, E\} = 0 \]

since the ideal evolution \( V^a \) preserves the measure \( \mu \) and conserves energy. If, in addition to (3) the convergence condition

\[ Z = \int e^{-\beta E} \mu du < \infty \] (4)

is satisfied, we can normalize the solution and get equilibrium probability distribution

\[ \frac{1}{Z} e^{-\beta E}. \]

If (3) or (4) is violated, the equilibrium is not given by the Boltzmann distribution. We will exhibit an example below where (3) holds but not (4). In
our case, there is another static, normalizable solution which we will determine explicitly.

In general, there may be no equilibrium state; or the system might reach a steady state which dissipates energy at some constant rate.

2.5 The Adjoint Equation

Suppose that the Einstein relation holds, so that $e^{-\beta E}$ is a static solution. If we make the change of variables

$$P = e^{-\beta E} Q$$

the FPK equation becomes

$$\frac{\partial Q}{\partial t} = \frac{1}{\mu} \frac{\partial}{\partial u_a} \left[ \mu D_{ab} \frac{\partial Q}{\partial u_b} \right] + \left[ \Gamma_a - V_a \right] \frac{\partial Q}{\partial u_a} \tag{5}$$

Even when the Boltzmann solution is not normalizable, this is a good starting point to search for a normalizable static solution.

2.6 Fast Dynamics

If the dynamics $V^a$ is much faster than the dissipation and fluctuation effects, the details of the vector field do not matter: the system will wander around in phase space and fill it. A vestige of the ideal dynamics survives: the invariant measure $\mu$. (A kind of micro-canonical ensemble.) In this limit we get the Fokker-Planck equation

$$\mu \frac{\partial P}{\partial t} = \partial_a \left[ \mu \{ D^{ab} \partial_b P - \Gamma^a P \} \right].$$

Again, if the Einstein relation holds, we have the adjoint equation,

$$\frac{\partial Q}{\partial t} = \frac{1}{\mu} \frac{\partial}{\partial u_a} \left[ \mu D_{ab} \frac{\partial Q}{\partial u_b} \right] + \Gamma_a \frac{\partial Q}{\partial u_a}.$$

3 Stochastic Geodesic Motion on Groups

There are several examples in physics where we are interested in geodesic motion on Lie groups. Arnold’s observation that the Euler equations of an ideal fluid are geodesic equations on the group of incompressible diffeomorphisms is the most important example. Navier-Stokes equations follow from adding a dissipation and many standard discussions of turbulence involve adding a Gaussian random force\[4\]. Self-erasing random walks can reformulated as random walks on the diffeomorphism group of the circle, the Schramm-Loewner Equation\[5\]. This dynamics takes place on a generalization of the upper half-plane, the space of univalent functions. A much more elementary example would be rigid body motion; random motion of rigid bodies have been used to model dust grains.
in astronomy[6]. We will now give a general framework for this theory. The ingredients are the structure constants $f_{ab}^c$ of the Lie algebra and three symmetric positive tensors $G^{ab}$, $\Gamma^{ab}$, $D^{ab}$, characterizing the energy, dissipation and fluctuations respectively. The main result is that the Boltzmann distribution is the equilibrium distribution only for unimodular Lie algebras: the trace of the matrices in the adjoint representation must vanish.

We will study in detail the simplest example that is not unimodular. There is an equilibrium distribution, but it is not the Boltzmann distribution.

3.1 Geodesic Motion as Hamiltonian Dynamics

Geodesics on a Riemann manifold form a hamiltonian system[7]. If the manifold is a Lie group, and the metric is invariant under the left action of the group on itself, the evolution of the tangent vector ("momentum") is independent of the position and can be studied separately[8, 9]. The Poisson Brackets of velocity components are just the commutation relations of the Lie algebra in some basis:

$$\{v_a, v_b\} = f_{ab}^c v_c.$$

A left-invariant metric on the group is the same thing as a quadratic form on the Lie algebra ("energy")

$$E = \frac{1}{2} G^{ab} v_a v_b.$$

The geodesic equations follow by the usual rules of hamiltonian mechanics:

$$\frac{dv_a}{dt} = \{v_a, E\} = f_{ab}^c G^{bd} v_c v_d \equiv V_a.$$

In the special case of the rigid body, the Lie algebra is $SO(3)$; the tensor $G^{ab}$ is the inverse of moment of inertia and the geodesic equations are the Euler equations of the rigid body. If the metric on the group is also right invariant, the tensor $G^{ab}$ is isotropic

$$f_{ab}^c G^{bd} + f_{ab}^d G^{bc} = 0.$$

and the momentum is a constant. This is true of the isotropic rigid body (equal moment of inertia in all directions) but is not usually the interesting case.

More generally, a function $H(\rho)$ where $\rho^2 = G^{ab} v_a v_b$ can be chosen as the hamiltonian. The equations of motion become

$$\frac{dv_a}{dt} = \{v_a, H\} = \frac{H'(\rho)}{\rho} f_{ab}^c G^{bd} v_c v_d \equiv V_a.$$

The solutions are still geodesics, differing only by a constant reparametrization: the time variable gets multiplied by a constant (which can depend on energy). We will see that an unconventional choice $H(\rho) = -\frac{\rho}{\rho}$ will simplify the FPK equation in the special case we study in detail below.
3.1.1 Invariant Measure

The obvious measure of integration $dv = dv_1 \wedge dv_2 \cdots$ is only invariant if

$$\frac{\partial V_a}{\partial v_a} = 0.$$ 

For a Hamiltonian $H$,

$$V_a = \{v_a, H\} = f_{abc} v_c \frac{\partial H}{\partial v_b},$$

so that

$$\frac{\partial V_a}{\partial v_a} = f_{abc} \frac{\partial H}{\partial v_b}.$$ 

A Lie algebra is said to be unimodular if the trace of the structure constants is zero:

$$f_{ab} = 0.$$ 

This is the condition for the measure $dv$ on the Lie algebra to be invariant under the adjoint action. The Haar measure on the corresponding Lie group will be both left and right invariant. Any semi-simple Lie algebra is unimodular; such as the rotations $SO(n)$, linear canonical transformations $Sp(n)$, unitary transformations $U(n)$ or any products of these groups.

An example that is not unimodular is the Lie algebra of the affine group: the only non-abelian Lie algebra in two dimensions. Typical examples are solvable and nilpotent algebras or Lie algebras containing them as subalgebras. When the algebra is not unimodular, there is still a measure $\mu dv$ that is invariant under the adjoint action: it is not anymore just $dv$.

3.2 Dissipation

Adding a fluctuation that is a gradient of energy leads to an Ohmic force

$$\Gamma_a = \Gamma_{ad} G^{db} v_b \equiv \Gamma^b_a v_b$$

and

$$\frac{dv_a}{dt} = -\Gamma^b_a v_b + f_{abc} G^{bd} v_c v_d.$$ 

3.2.1 The Navier-Stokes Equation

A special case is the Navier-Stokes equation. The divergence free vector fields form a Lie algebra under the usual commutator. The energy is just the $L^2$-norm

$$E = \frac{1}{2} \int v_i v_i dx = \frac{1}{2} G^{ab} v_a v_b.$$
The tensor $G^{ab}$ is just the Dirac delta function.

The dissipation tensor, thought of as a quadratic form on vector fields is the $H_1$ norm:

$$\Gamma^{ab} v_a v_b = \int \partial_i v_j \partial_i v_j \, dx$$

This leads (after some calculations [9]) to the Navier-Stokes equations

$$\frac{\partial v_i}{\partial t} = \partial^2 v_i - v_j \partial_j v_i - \partial_i p$$

where the pressure $p$ is determined from the constraint

$$\partial_i v_i = 0.$$ 

Replacing the space of incompressible vector fields by a finite dimensional Lie algebra allows to study simpler models of this important physical system.

### 3.3 Fluctuation

Adding a random force leads to a Langevin equation

$$\frac{dv_a}{dt} = -\Gamma^{ab} v_b + f^{c}_{ab} G^{cd} v^d + \eta_a.$$ 

The Fokker-Planck-Kramers equation becomes

$$\frac{\partial P}{\partial t} = \frac{1}{\mu} \frac{\partial}{\partial u_a} \left[ \mu \left( D_{ab} \frac{\partial P}{\partial u_b} + (\Gamma^{ab} v_b - f^{c}_{ab} G^{cd} v^d) P \right) \right]$$

There is a static solution

$$P_B(v) = e^{-\beta \frac{v^2}{2}}$$

if

$$\beta D_{ab} = \Gamma_{ab}.$$ 

This can be interpreted as a probability distribution if

$$\int \mu e^{-\beta \frac{v^2}{2}} \, dv$$

converges.
3.3.1 The Randomly Forced Navier-Stokes Equation

In fluid mechanics, the Navier-Stokes equation with random forcing

\[
\frac{\partial v_i}{\partial t} = \frac{\partial^2 v_i}{\partial x^2} - v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \eta_i(x, t)
\]

is often used to model turbulence. The correlation tensor of fluctuation is chosen to be translation invariant

\[
D_{ij}(x, y) = \int \left[ \delta_{ij} - \frac{k_i k_j}{k^2} \right] \tilde{D}(p) dp.
\]

The function \( \tilde{D}(p) = p^2 \) would satisfy the condition \( |p|^{2\alpha} \). But it is not the one usually used in the literature on turbulence. Generally speaking, dissipation is important at short distance scales (viscous force is proportional to the gradient of velocity) while fluctuations are believed to pump energy into the system at large distance scales. Thus \( \tilde{D}(p) = |p|^{-\alpha} \), with a negative power of momentum is more reasonable. So an equilibrium of the Boltzmann type is unlikely to exist in fluid mechanics: instead, a steady state solution in which energy is dissipated at a constant rate is more likely to be the correct answer. The theory of Kolmogorov that leads to scale invariant velocity correlations, in one such example.

4 The Langevin Equation on a Hyperboloid

It is often useful to replace an infinite dimensional physical system by a finite dimensional toy model, which still has some of the basic structure intact. In our case, it is important that the underlying Lie algebra be non-abelian. Otherwise it simply reduces to the standard Brownian motion in Euclidean space. Thus to get something non-trivial, the Lie algebra must be at least two dimensional.

The only non-abelian Lie algebra in two dimensions is

\[
\{ v_0, v_1 \} = v_1.
\] (6)

The corresponding Lie group is the set of triangular matrices

\[
\begin{pmatrix}
  a_0 & a_1 \\
  0 & 1
\end{pmatrix}
\]

with \( a_0 > 0 \). The upper half-plane

\[
\mathcal{U} = \{ (a_0, a_1) | a_0 > 0 \}
\]

parametrizes such matrices. The group multiplication law is

\[
(a_0, a_1)(a_0', a_1') = (a_0 a_0', a_0 a_1' + a_1).
\]

This group is often called the affine group, as it acts on the real line by translations and scaling (affine transformation).
\[(a_0, a_1) = a_0 t + a_1.\]

The natural geometry on the upper half plane is the Poincare’ metric, which has constant negative curvature: the simplest geometry with negative curvature. There is a one-one correspondence of the hyperboloid with the upper half-plane; the induced metric on the hyperboloid is just the the Poincare’ metric on the upper half-plane. A particle constrained to move on the hyperboloid, but free of other forces, will move along geodesics.

The Poincare’ metric is invariant under the left action (but not the right action) of the affine group. Because it has negative curvature, the geodesics do not have constant tangent vectors: only the length is preserved, while the direction changes through parallel transport. It is straightforward to derive the geodesic equations \[8, 9\]

\[
\frac{dv_0}{dt} = -v_1^2
\]

\[
\frac{dv_1}{dt} = v_0 v_1
\]

where \(t\) is the arc-length. Thee can be thought of as Hamilton’s equations implied by the Poisson brackets (6) and the Hamiltonian

\[
E = \frac{1}{2} [v_0^2 + v_1^2]
\]

which is just the kinetic energy.

It is convenient to use a kind of hyperbolic analogue of the polar co-ordinate system in velocity space

\[
v_0 = -\rho \tanh \theta, \ v_1 = \rho \text{sech} \theta
\]

in terms of which

\[
\frac{d\theta}{dt} = \rho
\]

\[
\frac{d\rho}{dt} = 0.
\]

The “angular” variable \(\theta\) has the range \(-\infty < \theta < \infty\). It is canonically conjugate to the radial (or “action”) variable \(\rho\)

\[
\{\rho, \theta\} = 1
\]

under the above Poisson bracket (6). The discrete variable \(\epsilon = \pm 1\) is needed in addition to cover both signs of \(v_1\). The solutions are semi-circles in the \((v_0, v_1)\) plane:

\[
v_0 = -\rho \tanh \rho t
\]
\[ v_1 = \rho \text{sech} \, \rho t. \]

Such a geodesic motion on a space of negative curvature models the Euler equation of an ideal fluid, which are geodesic equations on the group of volume preserving diffeomorphisms. Indeed, the affine Lie algebra is a subalgebra of the incompressible vector fields. In a plane for example, the pair of vector fields
\[ -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad y \frac{\partial}{\partial x} \]
span such a subalgebra.

5 Dissipation

We now modify the above system to include a dissipation, to model the Navier-Stokes equation
\[
\frac{dv_0}{dt} = -\gamma v_0 - v_1^2 \\
\frac{dv_1}{dt} = -\gamma v_1 + v_0 v_1
\]

In terms of hyperbolic polar co-ordinates \( \rho, \theta \) as above(7),
\[
\frac{d\rho}{dt} = -\gamma \rho, \quad \frac{d\theta}{dt} = \rho.
\]
The energy \( E(\rho) = \frac{1}{2} \rho^2 \) is then monotonically decreasing:
\[
\frac{dE}{dt} = -2\gamma E
\]

But the equations can still be solved analytically. The solution that passes through the point with hyperbolic polar co-ordinates \( (\rho_1, \theta_1) \) at time \( t_1 \) is,
\[
v_0 = -\rho_1 e^{-\gamma(t-t_1)} \text{tanh} \left[ \theta_1 + \rho_1 \frac{1 - e^{-\gamma(t-t_1)}}{\gamma} \right] \\
v_1 = \rho_1 e^{-\gamma(t-t_1)} \text{sech} \left[ \theta_1 + \rho_1 \frac{1 - e^{-\gamma(t-t_1)}}{\gamma} \right]
\]

As \( t \to \infty \), the velocities tend to zero; as \( t \to -\infty \), \( v_1(t) \to 0 \) and \( v_0(t) \to -\infty \). For intermediate values it roughly follows the semi-circle.

If we choose as Hamiltonian \( H(\rho) = -\frac{k}{\rho} \), we would get instead
\[
\frac{d\rho}{dt} = -\gamma \rho, \quad \frac{d\theta}{dt} = H'(\rho) = \frac{k}{\rho^2}
\]
which is just as easily solvable.
6 Random Forcing

Adding a random force with

\[ < \eta_a(t) > = 0, \quad < \eta_a(t) \eta_b(t') > = 2D \delta_{ab} \delta(t - t'). \]

we get the Langevin equation on a hyperboloid

\[
\frac{dv_0}{dt} = -\gamma v_0 - v_1^2 + \eta_1 \tag{8}
\]

\[
\frac{dv_1}{dt} = -\gamma v_1 + v_0 v_1 + \eta_2
\]

As noted earlier, the affine Lie algebra is not unimodular so the invariant measure is not \( dv \). It is instead \( d\rho d\theta = \mu dv_0 dv_1 \). The vector field \( V_a \) arising from a hamiltonian \( H \) and Poisson Brackets (6) preserve this measure:

\[
V_0 = -v_1 \frac{\partial H}{\partial v_1}, \quad V_1 = v_1 \frac{\partial H}{\partial v_0} \Rightarrow \frac{\partial}{\partial v^a} [\mu V_a] = 0.
\]

The above (8) corresponds to \( H = \frac{1}{2} v^2 \). The choice \( H = \frac{k}{v} \) yields equivalent geodesic equations, and is more convenient for solving the FKP equation.

We are led to the Fokker-Plank equation

\[
\frac{\partial P}{\partial t} = \frac{1}{\mu} \frac{\partial}{\partial v^a} [\mu \{ D \partial_a P + (\gamma v_a - V_a) P \}].
\]

The function \( e^{-\beta \frac{v^2}{2}} \) is a static solution if

\[ \beta D = \gamma. \]

But the integral

\[
\int e^{-\beta \frac{v^2 + v_1^2}{2}} \frac{d^2v}{v^1}
\]

is logarithmically divergent near \( v^1 = 0 \). In canonical co-ordinates the measure is just the Liouville measure

\[ \mu d^2v = d\rho d\theta \]

In this language, we have a linear divergence in the angular co-ordinate:

\[
\int e^{-\frac{1}{2} \beta \rho^2} \frac{dp}{\rho} \int d\theta
\]

Thus the Boltzmann distribution is not the correct equilibrium solution.
6.1 The Adjoint Equation

After the change of variables
\[ P = e^{-\beta \frac{v^2}{2}} Q \]

the FPK equation becomes its adjoint
\[ \frac{\partial Q}{\partial t} = D \mu \frac{\partial}{\partial v} \left[ \mu \frac{\partial Q}{\partial v_a} \right] + [-\gamma v_a - V_a] \frac{\partial Q}{\partial v_a} \quad (9) \]

Notice that the first term on the r.h.s. is a kind of Laplacian, but is not the Laplace-Beltrami operator: the symplectic volume element implied by the Poisson brackets appears in place of the Riemannian volume element. In polar co-ordinates
\[ \frac{\partial Q}{\partial t} = D \frac{\partial^2 Q}{\partial \rho^2} - \gamma \frac{\partial Q}{\partial \rho} + \frac{D}{\rho^2} \frac{\partial}{\partial \theta} \left[ \cosh^2 \theta \frac{\partial Q}{\partial \theta} \right] - H'(\rho) \frac{\partial Q}{\partial \theta} \]

We want a solution for which
\[ \int_{0}^{\infty} d\rho e^{-\frac{\beta}{2} \rho^2} \int_{-\infty}^{\infty} d\theta Q(\rho, \theta, t) \]

converges. In particular, it cannot be a constant in the \( \theta \) variable.

If the Hamiltonian is \( H = -\frac{k}{\rho} \) the last term would be \( \frac{k}{\rho^2} \frac{\partial Q}{\partial \theta} \) and the equation would be solvable by separation of variables:

\[ Q(\rho, \theta, t) = Q_1(\theta)Q_2(\rho)e^{-D/\beta_2 t} \]

\[ \frac{\partial}{\partial \theta} \left[ \cosh^2 \theta \frac{\partial Q_1}{\partial \theta} \right] + a \frac{\partial Q_1}{\partial \theta} + \beta_1 Q_1 = 0 \]

\[ \frac{\partial^2 Q_2}{\partial \rho^2} - \beta \frac{\partial Q_2}{\partial \rho} - \beta_1 \frac{\partial Q_2}{\partial \theta} + \beta_2 Q_2 = 0 \]

where \( a = \frac{k}{D} \). The separation constants \( \beta_1, \beta_2 \) are eigenvalues of these differential operators; they depend on \( a \) as well as on discrete 'quantum numbers' labelling eigenfunctions. For a static solution \( \beta_2 = 0 \).

6.2 Angular Equation

Change to
\[ u = \tanh \theta \]

\[ \frac{\partial^2 Q_1}{\partial u^2} + 2a \frac{\partial Q_1}{\partial u} + \frac{\beta_1}{1-u^2} Q_1 = 0 \]
\[ Q_1 = e^{-au} \sqrt{1-u^2} \phi(u) \]

\[(1-u^2)\phi'' - 2u \phi' + \left[ \beta_1 - a^2(1-u^2) - \frac{1}{1-u^2} \right] \phi = 0 \]

This is the Prolate Angular Spheroidal Wave Equation [10].

A simple solution is \( \phi(u) = \sqrt{1-u^2} \) when \( a = 0, \beta_1 = 2 \). Then \( Q_1[u] = (1-u^2) = \text{sech}^2 \theta \) is integrable. This belongs to a different branch from the non-integrable constant solution, for \( a = 0 \). When \( a \neq 0 \), this solution continues to a solution \( S_{11}(\theta) \) integrable in \( \theta \), with \( \beta_1 > 2 \).

### 6.3 Radial Wave Function

For a static solution,

\[
\frac{\partial^2 Q_2}{\partial \rho^2} - \beta_1 \frac{\partial Q_2}{\partial \rho} - \frac{\beta_1}{\rho^2} Q_2 = 0.
\]

The solution is a confluent hypergeometric function. The solution is

\[ Q_2 = \rho^{1+\sqrt{1+\beta_1}} F_1^1 \left( 1 + \sqrt{1+\beta_1}, 1 + \frac{1}{2} \sqrt{1+4\beta_1}, \frac{\sqrt{\beta_1} \rho}{2} \right) \]

If \( \beta_1 = 0 \), a solution is constant, which is not normalizable in the angular co-ordinate.

Now suppose \( a = 0, \beta_1 = 2 \), for which the angular integral converges. Then there is a solution that is normalizable in the radial variable as well.

\[ Q_2 = 1 - \frac{1}{\sqrt{\beta_1} \rho} e^{-\frac{\beta_1}{4} \rho^2} \text{erf} \left( \frac{\sqrt{\beta_1} \rho}{2} \right) \]

Thus, although the Boltzmann distribution is not a normalizable solution, there is another solution that is static and normalizable. But it breaks rotation invariance spontaneously! This solution can be continued to \( \beta_1 > 2 \), and remains integrable, when the hamiltonian is not zero. Thus rotation invariance is spontaneously broken and moreover, the most likely value of energy is not zero.

### 6.4 Equilibrium Solution

We have putting all of the above together, in the limit of \( k = 0 \) when the hamiltonian has a small effect,

\[
P(\rho, \theta) d\rho d\theta = e^{-\frac{1}{2} \rho^2} \left[ 1 - \frac{1}{\sqrt{\beta_1} \rho} e^{-\frac{\beta_1}{4} \rho^2} \text{erf} \left( \frac{\sqrt{\beta_1} \rho}{2} \right) \right] \text{sech}^2 \theta d\rho d\theta
\]

or in the original co-ordinates
When $a \neq 0$, we still have an explicit solution

\[
P(v_0, v_1) \frac{dv_0 dv_1}{v_1} = e^{-\frac{a v_0^2}{\rho}} \left[ 1 - \frac{1}{\sqrt{\beta}} e^{-\frac{\beta v_1^2}{2}} \right] dv_0 dv_1
\]

which has a similar shape. It is easy to check that this is normalizable.

7 Conclusion

There is a large literature on Brownian motion in curved manifolds. Mostly, this amounts to a study of the heat kernel of the Laplace-Beltrami operator. This is justified in the over-damped limit, where velocity (rather than acceleration) is proportional to the force. Our work shows that there are unexpected subtleties
in the more general case. The velocities do not tend to the expected Maxwell-Boltzmann distribution asymptotically, although for short times they might appear to do so.

For fluid mechanics, the phase space is infinite dimensional. It is possible to embed the phase space into an infinite dimensional analogue of the upper half plane (the space of complex symmetric matrices with positive imaginary part), which has a very similar behavior for the velocities. We plan to return to this case, which requires much deeper mathematics than used in this paper, in a future publication. It is hoped that the equilibrium velocity distribution of fluids under random forces is experimentally accessible and gives useful some information about strongly turbulent flows.

8 Appendix: Formulas For Change Of Variables

\[ v_0 = -\rho \tanh \theta, \quad v_1 = \rho \text{sech} \theta \]
\[ dv_0 = -d\rho \tanh \theta - \rho \text{sech}^2 \theta d\theta \]
\[ dv_1 = d\rho \text{sech} \theta - \rho \text{sech} \theta \tanh \theta d\theta \]
\[ dv_0^2 + dv_1^2 = d\rho^2 + \rho^2 \text{sech}^2 \theta d\theta^2 \]

\[ g_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & \rho^2 \text{sech}^2 \theta \end{bmatrix} \]
\[ g^{ab} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\rho^2} \cosh^2 \theta \end{bmatrix} \]
\[ dv_0 \wedge dv_1 = \rho \text{sech} \theta d\rho \wedge d\theta \]
\[ \mu = \frac{1}{\rho \text{sech} \theta} \]
\[ \mu dv_0 \wedge dv_1 = d\rho \wedge d\theta \]

Thus these co-ordinates are canonically conjugate:

\[ \{ \rho, \theta \} = 1 \]

9 Acknowledgement

This work was supported in part by a grant from the US Department of Energy under contract DE-FG02-91ER40685.
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