Finiteness of central configurations in the Coulomb four-body problem

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Abstract

For the Coulomb four-body problem, we show that there are finitely many isometry classes of planar central configurations (i.e., relative equilibria).

Key Words: N-body problem; Relative equilibrium; Celestial mechanics; Finiteness.

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1 Introduction

We consider N particles with (positive or negative) masses moving in an Euclidean space \( \mathbb{R}^2 \) interacting under the law of universal gravitation (or more precisely, N charged particles moving in \( \mathbb{R}^2 \) interacting under the Coulomb law). Let the k-th particle have mass \( m_k \) and position \( r_k \in \mathbb{R}^2 \) \( (k = 1, 2, \cdots, N) \), then the equations of motion of the N-body problem are written as

\[
m_k \ddot{r}_k = \sum_{1 \leq j \leq N, j \neq k} \frac{m_k m_j (r_j - r_k)}{|r_j - r_k|^3}, \quad k = 1, 2, \cdots, N. \tag{1.1}
\]

where \( |\cdot| \) denotes the Euclidean norm in \( \mathbb{R}^2 \).

The N-body problem is an attractive and difficult topic. Central configurations are helpful for the understanding of the N-body problem. However, despite the enormous amount of effort spent to understand the central configurations in recent years, the problem of central configurations is far from being solved. Indeed, little is known about central configurations for \( N > 3 \) [6, 4, 1, etc].

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The so-called hypothesis on finiteness of relative equilibria for the $N$-body problem was proposed by Chazy \[3\] and Wintner \[7\], and was listed by Smale as the sixth problem on his list of problems for the 21-st century \[6\]: Is the number of relative equilibria finite, in the $N$-body problem of celestial mechanics, for any choice of positive real numbers $m_1, \ldots, m_N$ as the masses?

Hampton and Moeckel \[4\] answered positively the question in 2005 for $N = 4$. Albouy and Kaloshin \[1\] answered positively the question in 2012 for $N = 5$ except perhaps if the masses satisfy two given polynomial conditions.

In particular, one of the main obstacles in proving finiteness of relative equilibria for $N = 5$ is the emergence of Roberts’ continuum of relative equilibria (and/or its variant) \[5, 1\]. That is, if negative masses are allowed in the planar five-body problem, then there is a continuum of relative equilibria. It is natural to ask whether a continuum of relative equilibria is possible for $N = 4$ bodies. This is an open problem up to now \[4\]. In this paper, we prove that there is no continuum of relative equilibria in the general four-body problem.

**Theorem 1.1** Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero, then there are finitely many relative equilibria in the general four-body problem.

This work is inspired by the papers \[1, 8\]. Especially, the proof of Theorem 1.1 is motivated by the elegant method of Albouy and Kaloshin \[1\]. The principle of the method is to follow a possible continuum of relative equilibria in the complex domain and to study its possible singularities there.

We remark that

**Theorem 1.2** Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero in the general four-body problem. Let $x, y$ be the unique positive root of the two equations in \(8.82\) respectively. A complex continuum of relative equilibria exists only for the two groups of masses: $m_1 = m_2 = -m_3 = -m_4$, $m_1 = m_2, m_3 = -x^2 m_1, m_4 = -y^2 m_1$. Numerically, $x \approx 1.2407, y \approx 0.678731$.

2 Preliminaries

The notations and definitions used in this paper are almost the same as that in Section 2. structure of the proof of \[1\]. We only state some of them, please refer to \[1\] for more details.

Set $r_k = (x_k, y_k) \in \mathbb{R}^2$, $k = 1, \ldots, N$. A central configuration is a solution of the following system

\[
\begin{align*}
(x_1) & = m_2 r_{12}^{-3} (x_{21}) + m_3 r_{13}^{-3} (x_{31}) + \ldots + m_N r_{1N}^{-3} (x_{N1}) \quad (x1) \\
(y_1) & = m_2 r_{12}^{-3} (y_{21}) + m_3 r_{13}^{-3} (y_{31}) + \ldots + m_N r_{1N}^{-3} (y_{N1}) \\
(x_2) & = m_1 r_{12}^{-3} (x_{12}) + m_3 r_{23}^{-3} (x_{32}) + \ldots + m_N r_{2N}^{-3} (x_{N2}) \quad (x2) \\
(y_2) & = m_1 r_{12}^{-3} (y_{12}) + m_3 r_{23}^{-3} (y_{32}) + \ldots + m_N r_{2N}^{-3} (y_{N2}) \\
& \quad \vdots \quad (xN) \\
(y_N) & = m_1 r_{1N}^{-3} (x_{1N}) + \ldots + m_{N-1} r_{(N-1)N}^{-3} (x_{(N-1)N}) \quad (yN)
\end{align*}
\]
where \( x_{kl} = x_l - x_k, y_{kl} = y_l - y_k \) and \( r_{kl} = (x_{kl}^2 + y_{kl}^2)^{1/2} > 0 \).

Some notations will be useful. We call \( f_k \in \mathbb{R}^2, k = 1, \ldots, N \), the right-hand sides of the equations. System (2.2) is

\[
q_k = f_k, \ k = 1, \ldots, N, \text{ with } q_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} \in \mathbb{R}^2.
\]

We embed the system (2.2) above into a polynomial system in \( \mathbb{C}^{2N} \times \mathbb{C}^{N(N-1)/2} \):

\[
\begin{align*}
(x_1) &= m_2 \delta_{12}^3 \begin{pmatrix} x_{21} \\ y_{21} \end{pmatrix} + m_3 \delta_{13}^3 \begin{pmatrix} x_{31} \\ y_{31} \end{pmatrix} + \ldots \\
(x_2) &= m_1 \delta_{12}^3 \begin{pmatrix} x_{12} \\ y_{12} \end{pmatrix} + m_3 \delta_{23}^3 \begin{pmatrix} x_{32} \\ y_{32} \end{pmatrix} + \ldots \\
&\vdots \\
\delta_{12}^2 (x_{12}^2 + y_{12}^2) &= 1 \\
\delta_{13}^2 (x_{13}^2 + y_{13}^2) &= 1 \\
&\vdots \\
y_{12} &= 0.
\end{align*}
\]

**Definition 2.1 (Normalized central configuration)** A normalized central configuration is a solution of (2.3). A real normalized central configuration is a normalized central configuration such that \((x_k, y_k) \in \mathbb{R}^2 \) for any \( k = 1, 2, \ldots, N \). A positive normalized central configuration is a real normalized central configuration such that \( \delta_{jk} = \pm 1/\sqrt{x_k^2 + y_k^2} \) is positive for any \( j, k, j \neq k \).

**Definition 2.2** The following quantities are defined:

- **Total mass** \( m = \sum_{j=1}^N m_j \)
- **Potential function** \( U = \sum_{1 \leq j < k \leq N} m_j m_k \delta_{jk} \)
- **Moment of mass** \( M = \sum_{j=1}^N m_j r_j \)
- **Moment of inertia** \( I = \sum_{j=1}^N m_j |r_j|^2 = \sum_{j=1}^N m_j (x_j^2 + y_j^2)^2 \)

Then it is easy to see that

\[
mI = \sum_{1 \leq j < k \leq N} m_j m_k r_{jk}^2 \triangleq S.
\]

**Lemma 2.1 ([1])** Let \( X \) be a closed algebraic subset of \( \mathbb{C}^N \) and \( f : \mathbb{C}^N \rightarrow \mathbb{C} \) be a polynomial. Either the image \( f(X) \in \mathbb{C} \) is a finite set, or it is the complement of a finite set. In the second case one says that \( f \) is dominating.

**Lemma 2.2 ([1])** Consider the closed algebraic subset \( A \in \mathbb{C}^{2N} \times \mathbb{C}^{N(N-1)/2} \) defined by system (2.3) and the polynomial functions \( U, I \) and \( S \) on it. Then \( U(A), I(A) \) and \( S(A) \) are three finite sets.
2.1 Complex central configurations

For convenience we will use again the variables $r_{jk} = 1/\delta_{jk}$ instead of the $\delta_{jk}$’s. We still think of a closed algebraic subset in the variables $x_k, y_k$ and $\delta_{jk}$.

Set $z_k = x_k + iy_k$ and $w_k = x_k - iy_k$. In the case of a real configuration the $z_k$’s form this configuration in the complex plane, while the $w_k$’s form its conjugate. We have $x_k^2 + y_k^2 = z_k w_k$ and $x_k^2 + y_k^2 = z_j w_j$. System (2.3) becomes

\[
\begin{align*}
  z_1 &= m_2 z_{21}^{-1/2} w_{21}^{-3/2} + m_3 z_{31}^{-1/2} w_{31}^{-3/2} + \ldots \\
  w_1 &= m_2 z_{21}^{-3/2} w_{21}^{-1/2} + m_3 z_{31}^{-3/2} w_{31}^{-1/2} + \ldots \\
  \vdots \\
  z_{12} &= w_{12}.
\end{align*}
\]  

We will use the name “distance” for the $r_{jk} = \sqrt{z_{jk} w_{jk}}$. We will use the name $z$-separation (respectively $w$-separation) for the $z_j$’s (respectively the $w_j$’s) in the complex plane.

Set $Z_{jk} = z_{jk}^{-1/2} w_{jk}^{-3/2}$ and $W_{jk} = z_{jk}^{-3/2} w_{jk}^{-1/2}$, then system (2.5) becomes

\[
\begin{align*}
  z_1 &= m_2 Z_{21} + m_3 Z_{31} + \ldots + m_N Z_{N1}, \\
  w_1 &= m_2 W_{21} + m_3 W_{31} + \ldots + m_N W_{N1}, \\
  \vdots \\
  z_{12} &= w_{12}.
\end{align*}
\]

Here

\[
\begin{align*}
  r_{jk} &= r_{kj} = 1/\sqrt{Z_{jk} W_{jk}}, \\
  Z_{jk} &= z_{jk}/r_{jk}^3, & W_{jk} &= w_{jk}/r_{jk}^3, & Z_{jk} &= -Z_{kj}, & W_{jk} &= -W_{kj}.
\end{align*}
\]

Let $N = N(N+1)/2$. To a gravitational configuration

\[
\begin{align*}
  \mathcal{Q} = (z_1, z_2, \ldots, z_N, w_1, w_2, \ldots, w_N, \delta_{12}, \delta_{13}, \ldots, \delta_{(N-1)N})
\end{align*}
\]

we associate two vectors in $\mathbb{C}^N$

\[
\begin{align*}
  \mathcal{Z} = (z_1, z_2, \ldots, z_N, Z_{12}, Z_{13}, \ldots, Z_{(N-1)N}), \\
  \mathcal{W} = (w_1, w_2, \ldots, w_N, W_{12}, W_{13}, \ldots, W_{(N-1)N}).
\end{align*}
\]

A solution $\mathcal{Q}$ of (2.6) is a normalized central configuration.

Let $||\mathcal{Z}|| = \max_{j=1, 2, \ldots, N} |z_j|$ be the modulus of the maximal component of the vector $\mathcal{Z} \in \mathbb{C}^N$. Similarly, set $||\mathcal{W}|| = \max_{k=1, 2, \ldots, N} |w_k|$.

Consider a sequence $\mathcal{Q}^{(n)}$, $n = 1, 2, \ldots$, of normalized central configurations. Extract a sub-sequence such that the maximal component of $\mathcal{Z}^{(n)}$ is always the same, i.e., $||\mathcal{Z}^{(n)}|| = |z_j^{(n)}|$ for a $j \in \{1, 2, \ldots, N\}$ that does not depend on $n$. Extract again in such a way that the vector sequence $\mathcal{Z}^{(n)}/||\mathcal{Z}^{(n)}||$ converges. Extract again in such a way
that there is similarly an integer $k \in \{1, 2, \cdots, N\}$ such that $\|W^{(n)}\| = |W_k^{(n)}|$ for all $n$. Extract a last time in such a way that the vector sequence $W^{(n)}/\|W^{(n)}\|$ converges. If the initial sequence is such that $Z^{(n)}$ or $W^{(n)}$ is unbounded, so is the extracted sequence. Note that $\|Z^{(n)}\|$ and $\|W^{(n)}\|$ are bounded away from zero: if the first $N$ components of the vector $Z^{(n)}$ or $W^{(n)}$ all go to zero, then the denominator $z_{12} = w_{12}$ of the component $Z_{12} = W_{12}$ go to zero and $Z^{(n)}$ and $W^{(n)}$ are unbounded. There are two possibilities for the extracted sub-sequences above:

- $Z^{(n)}$ and $W^{(n)}$ are bounded,
- at least one of $Z^{(n)}$ and $W^{(n)}$ is unbounded.

**Definition 2.3 (Singular sequence)** Consider a sequence of normalized central configurations. A sub-sequence extracted by the above process, in the unbounded case, is called a singular sequence.

### 3 Rules of colored diagram

We extend the results in the section 3. tools in to classify the singular sequences of $\mathbb{P}$ to the general $N$-body problem.

**Definition 3.1 (Notation of asymptotic estimates)** $a \sim b$ means $a/b \to 1$

- $a \prec b$ means $a/b \to 0$
- $a \preceq b$ means $a/b$ is bounded
- $a \approx b$ means $a \preceq b$ and $a \succeq b$

**Definition 3.2 (Strokes and circles.)** We pick a singular sequence. We write the indices of the bodies in a figure and use two colors for edges and vertices.

The first color, the $z$-color, is used to mark the maximal order components of

$$Z = (z_1, z_2, \cdots, z_N, Z_{12}, Z_{13}, \cdots, Z_{(N-1)N}).$$

They correspond to the components of the converging vector sequence $Z^{(n)}/\|Z^{(n)}\|$ that do not tend to zero. We draw a circle around the name of vertex $j$ if the term $z_j^{(n)}$ is of maximal order among all the components of $Z^{(n)}$. We draw a stroke between the names $k$ and $l$ if the term $Z_{kl}^{(n)}$ is of maximal order among all the components of $Z^{(n)}$.

The following rules mainly concern $z$-diagram, but they apply as well to the $w$-diagram.

If there is a maximal order term in an equation, there should be another one. This gives immediately the following Rule 1:
Rule I  There is something at each end of any z-stroke: another z-stroke or/and a z-circle drawn around the name of the body. A z-circle cannot be isolated; there must be a z-stroke emanating from it. There is at least one z-stroke in the z-diagram.

Definition 3.3 (z-close) Consider a singular sequence. We say that bodies $k$ and $l$ are close in z-coordinate, or z-close, or that $z_k$ and $z_l$ are close, if $z_{kl}^{(n)} < \|Z^{(n)}\|$.

The following statement is obvious.

Rule II  If bodies $k$ and $l$ are z-close, they are both z-circled or both not z-circled.

Definition 3.4 (Isolated component) An isolated component of the z-diagram is a subset of vertices such that no z-stroke is joining a vertex of this subset to a vertex of the complement.

Rule III  The moment of mass of a set of bodies forming an isolated component of the z-diagram is z-close to the origin.

Rule IV  Consider the z-diagram or an isolated component of it. If there is a z-circled body, there is another one. The z-circled bodies can not all be z-close together except that the total masses of these bodies is zero.

Definition 3.5 (Maximal z-stroke) Consider a z-stroke from vertex $k$ to vertex $l$. We say it is a maximal z-stroke if $k$ and $l$ are not z-close.

Rule V  At least one of the two ends of any maximal z-stroke is z-circled.

On the same diagram we also draw w-strokes and w-circles. Graphically we use another color. The previous rules and definitions apply to w-strokes and w-circles. What we will call simply the diagram is the superposition of the z-diagram and the w-diagram. We will, for example, adapt Definition 3.4 of an isolated component: a subset of bodies forms an isolated component of the diagram if and only if it forms an isolated component of the z-diagram and an isolated component of the w-diagram.

Definition 3.6 (Edges and strokes) There is an edge between vertex $k$ and vertex $l$ if there is either a z-stroke, or a w-stroke, or both. There are three types of edges, z-edges, w-edges and zw-edges, and only two types of strokes, represented with two different colors.

![Figure 1: A z-stroke, a z-stroke plus a z-stroke, a w-stroke, forming respectively a z-edge, a zw-edge, a w-edge.](image-url)
3.1 New normalization. Main estimates.

One does not change a central configuration by multiplying the $z$ coordinates by $a \in \mathbb{C}\{0\}$ and the $w$ coordinates by $a^{-1}$. Our diagram is invariant by such an operation, as it considers the $z$-coordinates and the $z$-coordinates separately.

We used the normalization $z_{12} = w_{12}$ in the previous considerations. In the following we will normalize instead with $\|Z\| = \|W\|$. We start with a central configuration normalized with the condition $z_{12} = w_{12}$, then multiply the $z$-coordinates by $a > 0$, the $w$-coordinates by $a^{-1}$, in such a way that the maximal component of $Z$ and the maximal component of $W$ have the same modulus, i.e., $\|Z\| = \|W\|$.

A singular sequence was defined by the condition either $\|Z^{(n)}\| \to \infty$ or $\|W^{(n)}\| \to \infty$. We also remarked that both $\|Z^{(n)}\|$ and $\|W^{(n)}\|$ were bounded away from zero. With the new normalization, a singular sequence is simply characterized by $\|Z^{(n)}\| = \|W^{(n)}\| \to \infty$. From now on we only discuss singular sequences.

Set $\|Z^{(n)}\| = \|W^{(n)}\| = 1/\epsilon^2$, then $\epsilon \to 0$.

**Proposition 3.1 (Estimate 1)** For any $(k, l)$, $1 \leq k < l \leq N$, we have $\epsilon^2 \leq z_{kl} \leq \epsilon^{-2}$, $\epsilon^2 \leq w_{kl} \leq \epsilon^{-2}$ and $\epsilon \leq r_{kl} \leq \epsilon^{-2}$. There is a $zw$-edge between $k$ and $l$ if and only if $r_{kl} \approx \epsilon$. There is a maximal $z$-edge between $k$ and $l$ if and only if $w_{kl} \approx \epsilon^2$.

**Proposition 3.2 (Estimate 2)** We assume that there is a $z$-stroke between $k$ and $l$. Then

$$\epsilon \leq r_{kl} \leq 1, \quad \epsilon \leq z_{kl} \leq \epsilon^{-2}, \quad \epsilon \geq w_{kl} \geq \epsilon^2.$$  

Under the same hypothesis the “equality case” are characterized as follows:

- Left : $r_{kl} \approx \epsilon \iff z_{kl} \approx \epsilon \iff w_{kl} \approx \epsilon \iff zw$-edge between $k$ and $l$,
- Right : $r_{kl} \approx 1 \iff z_{kl} \approx \epsilon^{-2} \iff w_{kl} \approx \epsilon^2 \iff$ maximal $z$-edge between $k$ and $l$.

\[ \square \]

**Remark 3.1** By the estimates above, the strokes in a $zw$-edge are not maximal. A maximal $z$-stroke is exactly a maximal $z$-edge.

**Rule VI** Two consecutive $zw$-edges. If there are two consecutive $zw$-edges, there is a third $zw$-edge closing the triangle.

Clusters. At the limit when following a singular sequence, the $z_k$’s form clusters. If, for example, bodies 1, 2 and 3 are such that $z_{12} \prec z_{23}$, we say that 1 clusters with 2 in $z$-coordinate, relatively to the subset of bodies 1,2,3. We may then consider a fourth body, which may form a sub-cluster, e.g., together with body 2. Altogether this means $z_{24} \prec z_{12} \prec z_{23}$.

We will often write a clustering scheme in each coordinate. In the latter situation we would write simple $z : 421...3$, three dots being the largest separation within the group, one dot the intermediate separation, no dot the smallest separation.

In the rule below we discuss clustering relation inside a sub-system of three bodies. Nothing forbids that these three bodies for, e.g., in $z$-coordinate, a cluster relatively to the whole configuration.
Rule VII  *Skew clustering.* Consider two consecutive edges that are not part of a triangle, e.g., an edge from vertex 1 to vertex 2, and an edge from vertex 2 to vertex 3. Then the clustering schemes are $z: 1.2...3, w: 1...2.3$, or $z: 1...2.3, w: 1.2...3$. We say there is “skew clustering”.

**Corollary 3.1** *Two consecutive z-edges cannot be maximal if they are not part of a triangle of edges.*

Rule VIII  *Cycles.* Consider a cycle of edges, the list of z-separations corresponding to the edges, and the maximal order of the z-separations within this list. Two or more of the z-separations are of this order. The corresponding edges have the same type. If there are only two, the corresponding separations are not only of the same order, but equivalent.

Rule IX  *Triangles.* Consider a triangle of edges in the diagram. Then the edges have the same type (all z-edges or all w-edges or all zw-edges), all the z-separations are of the same order, all the w-separations are of the same order.

**Corollary 3.2** *Consider three vertices. There are 6, 3, 2, 1, or 0 strokes joining them. If there are three forming a triangle, they are of the same color.*

Rule XI  *Fully edged sub-diagram.* Consider in the diagram: a triangle of edges, plus a fourth vertex attached to the triangle by at least two edges, plus a fifth vertex attached to the four previous vertices by at least two edges, and so on up to a $p$-th vertex, $p \geq 3$. Then there is indeed an edge between any pair of the $p$ vertices, the edges are of the same type, all the z-separations are of the same order, all the w-separations are of the same order.

Rule XII  If four edges form a quadrilateral, then the opposite edges are of the same type.

Rule XIII  *Bounded potential.* Consider a singular sequence. Pick bodies $k_0$ and $l_0$ such that $r_{k_0l_0} \preceq r_{kl}$ for any $k, l, 1 \leq k < l \leq n$. If $r_{k_0l_0} \to 0$, then there is another pair of bodies $(k_1, l_1)$ such that $r_{k_0l_0} \approx r_{k_1l_1}$.

**Corollary 3.3** *If there is a zw-edge in the diagram, there is another one.*

## 4  Exclusion of 4-body diagrams

We call a bicolored vertex of the diagram a vertex which connects at least a stroke of z-color with at least a stroke of w-color. The number of edges from a bicolored vertex is at least 1 and at most 3. The number of strokes from a bicolored vertex is at least 2 and at most 6. Given a diagram, we define $C$ as the maximal number of strokes from a bicolored vertex. We use this number to classify all possible diagrams.

Recall that the z-diagram indicates the maximal terms among a finite set of terms. It is nonempty. If there is a circle, there is an edge of the same color emanating from it. So there is at least a z-stroke, and similarly, at least a w-stroke.
4.1 No bicolored vertex

If there is no bicolored vertex, then there are at most two strokes and they are “parallel”. Thus the only possible diagram is the following Figure 2.

![Figure 2: No bicolored vertex](image)

4.2 $C = 2$

There are two cases: a $zw$-edge exists or not.

If it is present, it should be isolated. On the other hand, there should be another $zw$-edge by Corollary 3.3. Then the only possible diagram is the one in Figure 3. By Rule IV, we have $m_1 + m_2 = 0$ since body 1 and 2 are both $z$-circled and $z$-close. Similarly, we get $m_3 + m_4 = 0$, then $\sum_{i=1}^{4} m_i = 0$.

![Figure 3: $C = 2$, $zw$-edge appears](image)

If it is not present, there are adjacent $z$-edges and $w$-edges. From any such adjacency there is no other edge. By trying to continue it, we see that the only diagram is the following Figure 4.

![Figure 4: $C = 2$, no $zw$-edge](image)
4.3 \( C = 3 \)

Consider a bicolored vertex with three strokes. There are two cases: a \( zw \)-edge exists or not.

If it is not present, it is Y-shaped. Suppose that vertex 1 connects with vertex 2 and vertex 3 by \( z \)-edges, and connects with vertex 4 by a \( w \)-edge. By Rule I, vertex 1 is \( w \)-circled, then vertex 2 and vertex 3 are also \( w \)-circled by Estimate 2. By Rule I again, there is \( w \)-stroke emanating from vertex 2 and vertex 3, which leads to a triangle with edges of different types. We exclude this by Rule IX.

If it is present, let vertex 2 be bicolored vertex with three strokes. Suppose it connects with vertex 1 by a \( zw \)-edge, with vertex 3 by a \( z \)-edge. By circling method, we circle the three vertices by \( w \)-color. Then there is \( w \)-stroke from vertex 3, which can not connect to 1 by Rule IX. Then there are two cases: an edge between vertex 1 and vertex 4 or not. If it is not present, then we continue it to the first one of Figure 5. If it is present, then they form a quadrilateral. By Rule XII, we see the types of edges. There is no diagonal edges by Rule IX. Then circling method gives the next three diagrams in Figure 5. Consider the isolated components of the \( w \)-diagram. We see \( m_1 + m_2 = 0 \) and \( m_3 + m_4 = 0 \) by Rule IV, for all of Figure 5.

![Figure 5: \( C = 3 \)](image)

4.4 \( C = 4 \)

Consider a bicolored vertex (let us say, 1) with four strokes.

In the first case, vertex 1 has two adjacent \( zw \)-edges connected vertex 2 and vertex 3 separately. A third \( zw \)-edge closes the triangles by Rule VI. As \( C = 4 \) there is no any stroke connected vertex 4, thus vertex 4 is neither \( z \)-circled nor \( w \)-circled. Hence the only possible diagrams are that in the following Figure 6.

![Figure 6: \( C = 4 \), three \( zw \)-edges](image)
In the second case, vertex 1 has one adjacent zw-edge connected with vertex 2, a 
$z$-edges connected with vertex 3 and a w-edges connected with vertex 4. Any other 
edge in this diagram would close a triangle, which would contradicts Rule IX. Hence, 
1 is z and w-circled, and so is vertex 3 by Rule I and Rule II. Then there is w-edge 
emanating from 3 by Rule I. Contradiction.

In the third case, vertex 1 has one adjacent zw-edge connected with vertex 2, two 
z-edges connected with vertex 3 and 4 separately. If there are more strokes, it should 
be a z-stroke between vertex 3 and 4 by Rule IX. However, 2 is z and w-circled, and 
so is vertex 1 by Rule II. Then both of vertex 3 and 4 are w-circled by Rule II. Then 
there should be w-stroke emanating from 3. Contradiction.

4.5 $C = 5$

Consider a bicolored vertex (let us say, 1) with five strokes.

Suppose vertex 1 has one adjacent w-edge connected vertex 4, and two adjacent 
zw-edges connected vertex 2 and vertex 3 separately. Then vertexes 1,2,3 formed 
a fully zw-edged triangle by Rule VI. Rule VI also implies that there are no more 
edges. By Rule I and IV, all vertices are w-circled. There is no z-circle, otherwise all 
vertices are z-circled by Rule IV and II. Then there is z-edge emanating from vertex 4. 
Contradiction.

4.6 $C = 6$

Consider a bicolored vertex with six strokes. Then vertexes 1,2,3,4 form a fully 
zw-edged diagram by Rule VI. According to if there are z-circle or w-circle at vertexes, 
the only possible diagrams are that in the following Figure 8.
5 Problematic diagrams with nonvanishing total mass

Notations. From now on, $x^{1/n}$ (or $\sqrt[n]{x}$) is understood as one appropriate value of the $n$-th root of $x$. We denote $\pm 1$ by $\sigma_i$, and $\sqrt{|m_i|}$ by $\mu_i$, $i = 1, 2, 3, 4$. We denote the set $f = 0$ by $\mathcal{V}_f$, and $f = 0, g = 0$ by $\mathcal{V}_f \cap \mathcal{V}_g$ or simply $\mathcal{V}_f \mathcal{V}_g$; the set $f = 0$ or $g = 0$ (i.e., $fg = 0$) by $\mathcal{V}_f \cup \mathcal{V}_g$. Recall that $m$ is the total mass.

First it is easy to see that all of diagrams in Figure 3 and Figure 5 and the last two diagrams in Figure 8 must deduce that $m = 0$.

Next we prove that the second diagram in Figure 6 is impossible.

First, we have

$$m_1 + m_2 + m_3 = 0, \quad (5.7)$$

and

$$z_1 \sim z_2 \sim z_3 \sim a\varepsilon^{-2}.$$  

Thus $Z_{13} \sim Z_{12} + Z_{23}$. Set

$$Z_{12} \sim x_1\varepsilon^{-2}, \; Z_{23} \sim x_2\varepsilon^{-2}.$$  

Then $Z_{13} \sim (x_1 + x_2)\varepsilon^{-2}$ and

$$m_1x_1 - m_3x_2 = a. \quad (5.8)$$

By

$$w_k = \sum_{j \neq k} m_j W_{jk}, \quad k = 1, 2, 3,$$

we arrive at the estimation

$$\frac{W_{12}}{m_3} \sim \frac{W_{23}}{m_1} \sim \frac{W_{31}}{m_1} \sim b\varepsilon^{-2}.$$  

We have

$$z_{12}^4 = \frac{Z_{12}}{W_{12}^3} \sim \frac{x_1\varepsilon^4}{(bm_3)^3}, \; z_{23}^4 = \frac{Z_{23}}{W_{23}^3} \sim \frac{x_2\varepsilon^4}{(bm_1)^3}, \; z_{13}^4 = \frac{Z_{13}}{W_{13}^3} \sim \frac{-(x_1 + x_2)\varepsilon^4}{(bm_2)^3}, \quad (5.9)$$

or

$$z_{12} \sim \frac{x_1}{(bm_3)^3}\varepsilon, \; z_{23} \sim \frac{x_2}{(bm_1)^3}\varepsilon, \; z_{13} \sim \frac{-(x_1 + x_2)}{(bm_2)^3}\varepsilon.$$
So
\[
\left(\frac{x_1}{m_3}\right)^4 + \left(\frac{x_2}{m_1}\right)^4 - \left(-\frac{x_1 + x_2}{m_3}\right)^4 = 0. \tag{5.10}
\]
Similarly
\[
\left(\frac{m_3}{x_1}\right)^4 + \left(\frac{m_1}{x_2}\right)^4 - \left(-\frac{m_2}{(x_1 + x_2)^3}\right)^4 = 0. \tag{5.11}
\]
We have
\[
r_{12} \sim \frac{\epsilon}{(bx_1m_3)^4}, \quad r_{23} \sim \frac{\epsilon}{(bx_2m_1)^4}, \quad r_{13} \sim \frac{\epsilon}{(-b(x_1 + x_2)m_2)^4}.
\]
By \(U\) is bounded, we have
\[
m_1m_2(x_1m_3)^4 + m_2m_3(x_2m_1)^4 + m_1m_3(-(x_1 + x_2)m_2)^4 = 0. \tag{5.12}
\]
A straightforward computation shows that there is no solution for equations (5.7), (5.10), (5.11) and (5.12).

We could not eliminate the diagrams in Figure 9. Some singular sequence could still exist and approach any of these diagrams.

5.1 Diagram I

Following the argument from [I], we obtain the equation
\[
\frac{m_1m_3}{(m_2m_4)^4} + \frac{m_2m_3}{(-m_1m_4)^4} + \frac{m_1m_4}{(-m_2m_3)^4} + \frac{m_2m_4}{(m_1m_3)^4} = 0,
\]
or equivalently,

\[ m_1 m_3 \sqrt{m_1 m_3} + \sigma_1 m_2 m_3 \sqrt{-m_2 m_3} + \sigma_2 m_1 m_4 \sqrt{-m_1 m_4} + \sigma_3 m_2 m_4 \sqrt{m_2 m_4} = 0, \] (5.13)

where \( \sqrt{x} (x \in \mathbb{R}) \) is understood as the square root of \( x \) on the positive real axis if \( x > 0 \), and on the positive imaginary axis if \( x < 0 \).

However, we claim that

\[ (m_1 + m_2)(m_3 + m_4) \neq 0. \] (5.14)

This is a consequence of the fact

\[ w_{12} = (m_1 + m_2)W_{12} + m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}), \]
\[ z_{34} = (m_3 + m_4)Z_{34} + m_1(Z_{14} - Z_{13}) + m_4(Z_{24} - Z_{23}) \] (5.15)

It also follows that

\[ r_{12}^3 \sim m_1 + m_2, r_{34}^3 \sim m_3 + m_4. \] (5.16)

On the other hand, it is easy to see that

\[ r_{jk} \approx \epsilon^{-2}, j = 1, 2, k = 3, 4. \] (5.17)

We will simplify the algebraic equation (5.13) in all possible cases on the signs of the masses. Equation (5.13) is symmetric under the transformations

\[ (m_1, m_2, m_3, m_4) \mapsto k(m_1, m_2, m_3, m_4), \ k \in \mathbb{R} \setminus \{0\}, \]
\[ (m_1, m_2, m_3, m_4) \mapsto (m_2, m_1, m_4, m_3). \] (5.18)

The two symmetries reduce the discussions on the signs of the masses.

Assume that the signs of the masses are the same. It suffices to consider only the case \((+, +, +, +)\). Then equation (5.13) becomes

\[ m_1 m_3 \sqrt{m_1 m_3} + \sigma_3 m_2 m_4 \sqrt{m_2 m_4} + i(\sigma_1 m_2 m_3 \sqrt{m_2 m_3} + \sigma_2 m_1 m_4 \sqrt{m_1 m_4}) = 0, \]

which is \( m_1^2 m_3^3 = m_2^2 m_4^3 \) and \( m_2^2 m_3^3 = m_1^2 m_4^3 \). Then the necessary relations on the masses are

\[ m_1 = m_2, \ m_3 = m_4, \ m_1 m_3 > 0. \]

Assume that only one of the signs of the masses is different from the others. It suffices to consider only \((+, +, +, -)\) and \((+, -, +, +)\). In the first subcase, equation (5.13) is equivalent to

\[
\begin{cases}
    m_1 m_3 \sqrt{m_1 m_3} + \sigma_2 m_1 m_4 \sqrt{-m_1 m_4} = 0, \\
    m_2 m_3 \sqrt{-m_2 m_3} + \sigma_3 m_2 m_4 \sqrt{m_2 m_4} = 0,
\end{cases}
\]

which holds only if \( m_3 = -m_4 \). We exclude this subcase according to (5.21). Similarly, we exclude the second subcase.
Assume that only two of the signs of the masses is different from the others. It suffices to consider only $(+, +, -, -)$, $(+, -, +, -)$ and $(+, -, -, +)$. In the first subcase, equation (5.13) is equivalent to

$$\begin{cases}
m_1 m_3 \sqrt{m_1 m_3} + \sigma_3 m_2 m_4 \sqrt{m_2 m_4} = 0, \\
m_2 m_3 \sqrt{-m_2 m_3} + \sigma_2 m_1 m_4 \sqrt{-m_1 m_4} = 0,
\end{cases}$$

which is equivalent to

$$m_1 = m_2, \ m_3 = m_4, \ m_1 m_3 < 0.$$  

In the rest subcases, equation (5.13) can not be simplified. Then equation (5.13) becomes

$$\mu_1^3 \mu_3^3 + \sigma_1 \mu_2^3 \mu_3^3 + \sigma_2 \mu_2^3 \mu_4^3 + \sigma_3 \mu_2^3 \mu_4^3 = 0, \ m_1 m_2 < 0, \ m_3 m_4 < 0.$$  

To summarize, the masses of central configurations corresponding to the first diagram belong to one of the following two sets of constraints

$$\mathcal{V}_{IA}[12, 34] : \ m_1 = m_2, \ m_3 = m_4; \ \ \ \ \ \mathcal{V}_{IB}[12, 34] : \ mu_1^3 \mu_3^3 + \sigma_1 \mu_2^3 \mu_3^3 + \sigma_2 \mu_2^3 \mu_4^3 + \sigma_3 \mu_2^3 \mu_4^3 = 0, \ m_1 m_2 < 0, \ m_3 m_4 < 0.$$  

Furthermore, we claim that if $I = 0$, then the masses belong to the set

$$\mathcal{J}_I[12, 34] : \ \frac{m_1 m_2}{\sqrt{m_1 + m_2}} + \frac{m_3 m_4}{\sqrt{m_3 + m_4}} = 0,$$  

(5.19)

this is a consequence of the fact that $U = I = 0$.

In fact, it is easy to see that if $I = 0$, then the masses belong to the set

$$\mathcal{V}_{I0}[12, 34] \triangleq \mathcal{V}_{IB}[12, 34] \mathcal{J}_I[12, 34].$$  

(5.20)

### 5.2 Diagram II

Following the argument from [11], the masses of central configurations corresponding to the second diagram belong to the following set

$$\mathcal{V}_{II}[13, 24] : \ m_1 m_3 = m_2 m_4.$$  

Furthermore, we claim that

$$(m_1 + m_2)(m_2 + m_3)(m_3 + m_4)(m_1 + m_4) \neq 0.$$  

(5.21)

This is a consequence of the fact $z_{12} = (m_1 + m_2) z_{12} + m_3 (Z_{32} - Z_{31}) + m_4 (Z_{42} - Z_{41})$ and so on. This also deduces

$$r_{12}^3 \sim (m_1 + m_2), \ r_{23}^3 \sim (m_2 + m_3), \ r_{34}^3 \sim (m_3 + m_4) r_{14}^3 \sim (m_1 + m_4).$$  

(5.22)

Note that

$$r_{13} \approx \epsilon^{-2}, \ r_{24} \approx \epsilon^{-2}.$$  

(5.23)

And if $I = 0$, then the masses further belong to the set

$$\mathcal{J}_{II}[13, 24] : \ \frac{m_1 m_2}{\sqrt{m_1 + m_2}} + \frac{m_2 m_3}{\sqrt{m_2 + m_3}} + \frac{m_3 m_4}{\sqrt{m_3 + m_4}} + \frac{m_1 m_4}{\sqrt{m_1 + m_4}} = 0,$$  

(5.24)

that is, if $I = 0$, then the masses belong to the set

$$\mathcal{V}_{I0}[13, 24] \triangleq \mathcal{V}_{II}[13, 24] \mathcal{J}_{II}[13, 24].$$  

(5.25)

This is a consequence of the fact that $U = I = 0$. 

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5.3 Diagram III

Following the argument from [11], we arrive at the estimation

\[ \frac{Z_{12}}{m_3} \sim \frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_1}, \quad \frac{W_{12}}{m_3} \sim \frac{W_{23}}{m_1} \sim \frac{W_{31}}{m_1}. \]

Since \( Z_{12}, W_{12} \) are both of the maximal order, we assume the first set is asymptotic to \( ac^{-2} \) and the second asymptotic to \( be^{-2} \). Multiplying the two, by \( Z_{kl}W_{kl} = r_{kl}^{-4} \), we obtain

\[ \frac{1}{m_3} \sim \frac{1}{m_1^2r_{23}^4} \sim \frac{1}{m_2^2r_{31}^4} \sim abc^{-4}. \]

Then we have

\[ r_{12} \sim \frac{\epsilon}{\sqrt{|m_3|}}c, \quad r_{23} \sim \frac{\epsilon}{\sqrt{|m_1|}}c, \quad r_{31} \sim \frac{\epsilon}{\sqrt{|m_2|}}c, \]

where \( \epsilon_1^4 = \epsilon_2^4 = \epsilon_3^4 = 1 \) and \( c \) is some nonzero constant. Then by \( z_{kl} = r_{kl}^3 Z_{kl} \), we have

\[ z_{12} \sim \frac{\epsilon}{\sqrt{|m_3|}}d, \quad z_{23} \sim \frac{\epsilon}{\sqrt{|m_1|}}d, \quad z_{31} \sim \frac{\epsilon}{\sqrt{|m_2|}}d, \]

with \( d \) being some nonzero constant. Then the identity \( z_{12} + z_{23} + z_{31} = 0 \) implies that the masses further belong to the set

\[ V_{III}[123] : \frac{1}{|m_3|^{\frac{1}{2}}} + \frac{1}{|m_1|^{\frac{1}{2}}} + \frac{1}{|m_2|^{\frac{1}{2}}} = 0. \]

Thus the masses of central configurations corresponding to the third diagram satisfy

\[ \mu_2\mu_3 + \sigma_1\mu_1\mu_3 + \sigma_2\mu_1\mu_2 = 0. \]

It is easy to see that

\[ r_{12}, r_{23}, r_{13} \approx \epsilon, r_{14}, r_{24}, r_{34} \succ \epsilon. \]

Without loss of generality, suppose \( z_{14} \lesssim w_{14} \). Then \( \epsilon \lesssim w_{14} \sim w_{24} \sim w_{34} \).

If \( z_{14} \lesssim \epsilon \). Then \( z_{24} \approx z_{34} \approx \epsilon \) and \( r_{14} \approx r_{24} \approx r_{34} \). By \( w_4 = \sum_{j=1}^{3} m_j Z_{j4} \) and \( z_4 = \sum_{j=1}^{3} m_j Z_{j4} \) it follows that \( w_4 \approx m_1 W_{14}, z_4 \approx m_1 Z_{14} \). But \( \sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4} \leq w_{14} \) and \( \sum_{j=1}^{4} m_j z_4 = \sum_{j=1}^{3} m_j z_{j4} \leq \epsilon \), therefore,

\[ r_{14} \geq 1, z_4 \lesssim z_{14} \lesssim \epsilon. \]

Thus \( z_4 \lesssim z_1 \lesssim \epsilon \approx z_2 \approx z_3 \). It is easy to see that \( z_{12} \approx z_2 \approx -m_2 c \epsilon, z_{13} \approx z_3 \approx m_2 c \epsilon \) and \( z_{23} = z_3 - z_2 \approx (m_2 + m_3) c \epsilon \). It follows that

\[ \frac{m_2}{m_3} + \frac{11^4 m_3}{m_2} = 0, \quad \frac{m_1 m_2}{(-m_3)^{\frac{1}{2}}} + \frac{m_2 m_3}{(m_2 + m_3)^{\frac{1}{2}}} + \frac{m_1 m_3}{m_2^{\frac{1}{2}}} = 0. \]

There is no solution.

So \( z_{14} \succ \epsilon \) or \( z_{14} \approx z_{24} \approx z_{34} \approx \epsilon \).
5.3.1 \[ \sum_{j=1}^{3} m_j \neq 0 \]

By \( \sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4} \) it follows that
\[ w_4 \sim \frac{\sum_{j=1}^{3} m_j}{\sum_{j=1}^{4} m_j} w_{14} \geq \epsilon. \quad (5.29) \]

By \( w_4 = \sum_{j=1}^{3} m_j W_{j4} \) and \( z_4 = \sum_{j=1}^{3} m_j Z_{j4} \), it follows that
Case 1). If \( z_{14} \gg \epsilon \). Then \( \epsilon \prec z_{14} \sim z_{24} \sim z_{34} \) and \( r_{14} \sim \pm r_{24} \sim \pm r_{34} \).

\[ r_{14} \sim \pm r_{24} \sim \pm r_{34} \approx 1. \quad (5.30) \]

Case 2). If \( z_{14} \approx z_{24} \approx z_{34} \approx \epsilon \), then \( w_{14} \leq \epsilon^{-1} \) and \( r_{14} \approx r_{24} \approx r_{34} \leq 1. \)

5.3.2 \[ \sum_{j=1}^{3} m_j = 0 \]

By \( \sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4} \) it follows that
\[ w_4 \prec w_{14}. \quad (5.31) \]

Case 1). If \( z_{14} \gg \epsilon \). Then \( \epsilon \prec z_{14} \sim z_{24} \sim z_{34} \) and
\[ 1 \prec r_{14} \sim \pm r_{24} \sim \pm r_{34} \prec \epsilon^{-2}. \quad (5.32) \]

Case 2). If \( z_{14} \approx z_{24} \approx z_{34} \approx \epsilon \), then \( 1 \prec r_{14} \approx r_{24} \approx r_{34} \prec \sqrt{\epsilon^{-1}}. \)

5.3.3 \( I = 0 \)

Case 1). If \( z_{14} \gg \epsilon \). Then \( \epsilon \prec z_{14} \sim z_{24} \sim z_{34} \) and
\[ 1 \preceq r_{14}^2 \sim r_{24}^2 \sim r_{34}^2 \preceq \epsilon^{-4}. \]

It follows that
\[ \sum_{j=1}^{3} m_j = 0, \quad z_4 \prec z_{14}, \quad w_4 \prec w_{14}. \quad (5.33) \]

So
\[ 1 \prec r_{14} \sim \pm r_{24} \sim \pm r_{34} \prec \epsilon^{-2}. \quad (5.34) \]

Case 2). If \( z_{14} \approx z_{24} \approx z_{34} \approx \epsilon \), then \( r_{14} \approx r_{24} \approx r_{34} \prec \sqrt{\epsilon^{-1}}. \)

5.4 Diagram IV

The masses belong to the set
\[ V_{IV}[123] : \quad \sum_{j=1}^{3} m_j = 0, \]
\[ r_{12}, r_{23}, r_{13} \approx \epsilon, r_{14} \sim \pm r_{24} \sim \pm r_{34} \approx \epsilon^{-2}. \quad (5.35) \]
5.5 Diagram V

Following the argument from [1], we arrive at the estimation
\[
\frac{Z_{12}}{m_3} \sim \frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_1} \sim a\epsilon^{-2}, \quad \text{and} \quad z_{12} = m_3, \quad z_{31} = m_2,
\]
with \(a\) being some nonzero constant. By \(Z_{kl} = r_{kl}^{-3}z_{kl}\), we see
\[
\frac{m_1}{r_{12}^3} \sim -\frac{(m_2 + m_3)}{r_{23}^3} \sim \frac{m_1}{r_{31}^3} \sim am_1\epsilon^{-2}.
\]
Then we have
\[
r_{12} \sim \omega_1\epsilon^{\frac{2}{3}}\sqrt[3]{m_1c}, \quad r_{23} \sim \omega_2\epsilon^{\frac{2}{3}}\sqrt[3]{-(m_2 + m_3)c}, \quad r_{31} \sim \omega_3\epsilon^{\frac{2}{3}}\sqrt[3]{m_1c}
\]
where \(\omega_1 = \omega_2 = \omega_3 = 1\) and \(\sqrt[3]{x} (x \in \mathbb{R})\) is understood as the real cubic root of \(x\), and \(c\) is some nonzero constant. Then the asymptotic relation \(\frac{1}{m_1r_{23}} + \frac{1}{m_2r_{31}} + \frac{1}{m_3r_{12}} \sim 0\) implies
\[
\frac{1}{m_1\sqrt[3]{-(m_2 + m_3)}} + \frac{\omega_2}{m_2\sqrt[3]{m_1}} + \frac{\omega_3}{m_3\sqrt[3]{m_1}} = 0.
\]
If \(\omega_2\) is not real, then \(\omega_3\) should be its conjugate. Then
\[
m_2 = m_3, \quad \frac{1}{m_1\sqrt[3]{-(2m_2)}} + \frac{-1}{m_2\sqrt[3]{m_1}} = 0.
\]
The sign of the two terms are the same, so it is a contradiction. Then
\[
\omega_2 = \omega_3 = 1, \quad \frac{1}{m_1\sqrt[3]{-(m_2 + m_3)}} + \frac{1}{m_2\sqrt[3]{m_1}} + \frac{1}{m_3\sqrt[3]{m_1}} = 0,
\]
which reduces to the set
\[
V_V[1, 23] : \quad m_2^2(m_2 + m_3)^4 = m_3^2m_3^3,
\]
which holds only if \(m_2m_3 > 0\). Thus, the masses of central configurations corresponding to the Diagram V satisfy
\[
\mu_1^2(\mu_2^2 + \mu_3^2)^2 = \mu_2^3\mu_3^3, \quad m_2m_3 > 0.
\]  

(5.36)

We have
\[
\sum_{j=1}^3 m_j \neq 0, r_{12}, r_{23}, r_{13} \approx \epsilon, r_{14} \approx 1, r_{24} \approx r_{34} \approx \sqrt[3]{\epsilon^{-1}}.
\]  

(5.37)
5.6 Diagram VI

We will reach our result without discussing the mass polynomial of this diagram. Here is one remark about this diagram. Note that the momentum of inertia $I$ tends to zero by estimation 2. By Lemma 2.2, the momentum of inertia is constant on a continuum of central configurations, so such a singular sequence exists only on the subset $I = 0$.

Proposition 5.1 Consider singular sequences corresponding to Diagram III, V, and VI. For Diagram V and VI, the product $r_{jk}r_{lm}$ of any two nonadjacent distances tends to zero and the product $r_{jk}^2r_{lm} \leq 1$. The same estimate holds for Diagram III, except for the case $\sum_{j=1}^{3} m_j = 0$ and $1 < r_{14} \sim \pm r_{24} \sim \pm r_{34} < \epsilon^{-2}$, which are possible only for six groups of masses.

6 Finiteness of Central configurations with $m \neq 0$ and $\prod_{1 \leq k \leq 4}(m - m_k) \neq 0$

Theorem 6.1 Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero, if $\sum_{i=1}^{4} m_i \neq 0$ and $\prod_{1 \leq j < k \leq 4}(m_j + m_k + m_l) \neq 0$, then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.

In this case only Diagram I, Diagram II, Diagram III, Diagram V and Diagram VI are possible.

The proof of Theorem 6.1 reduces to the following Theorem 6.2 and Theorem 6.3.

6.1 Finiteness with $m \neq 0$, $\prod_{1 \leq k \leq 4}(m - m_k) \neq 0$ and $I \neq 0$

Theorem 6.2 Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero, if $\sum_{i=1}^{4} m_i \neq 0$, $\prod_{1 \leq j < k \leq 4}(m_j + m_k + m_l) \neq 0$ and $I \neq 0$, then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.

In this case only Diagram I, Diagram II, Diagram III and Diagram V are possible.

Proof of Theorem 6.2: It is easy to see that giving five of $r_{kl}^2$'s, $1 \leq k < l \leq 4$, determines only finitely many geometrical configurations up to rotation. Suppose that there are infinitely many solutions of system (4) in the complex domain. Then at least two of $\{r_{kl}^2\}$’s must take infinitely many values and thus are dominating by Lemma 2.1. Suppose that $r_{kl}^2$ is dominating for some $1 \leq k < l \leq 4$. There must exist a singular sequence of central configurations with $r_{kl}^2 \to 0$, which happens only in the Diagram III or Diagram V. In either case, by Proposition 5.1,

$$r_{ki}^2r_{ij}^2 \to 0, \quad r_{kl}^2r_{jl}^2 \to 0, \quad r_{ki}^2r_{kj}^2 \to 0,$$

along the singular sequence. Then all the three polynomials

$$r_{13}^2r_{24}^2, \quad r_{14}^2r_{23}^2, \quad r_{12}^2r_{34}^2,$$
are dominating. It is easy to see that there exist \( \Delta_{ijk} \to 0 \) (say \( \Delta_{123} \to 0 \)), so \( r_{12}, r_{23} \) and \( r_{13} \) are dominating.

Then there also exist singular sequences with \( r_{12}^2 r_{34}^2 \to \infty \), singular sequences with \( r_{12}^2 r_{24}^2 \to \infty \) and singular sequences with \( r_{12}^2 r_{23}^2 \to \infty \). These sequences must correspond to Diagram I or Diagram II.

Consider a singular sequence with \( r_{12}^2 r_{23}^2 \to \infty \), thus, the masses must be belong to

\[
\mathcal{V}_{IA}[12, 34] \cup \mathcal{V}_{IA}[14, 23] \cup \mathcal{V}_{IB}[12, 34] \cup \mathcal{V}_{IB}[14, 23] \cup \mathcal{V}_{II}[13, 24].
\]  
(6.38)

Note that \( \mathcal{V}_{IA}[12, 34] \cup \mathcal{V}_{IA}[14, 23] \subset \mathcal{V}_{II}[13, 24] \). Thus we classify the cases above into two classes:

1). \( \mathcal{V}_{II}[13, 24] \);
2). \( \mathcal{V}_{IB}[12, 34] \cup \mathcal{V}_{IB}[14, 23] \).  
(6.39)

Repeat the argument with \( r_{12}^2 r_{23}^2 \). Then the masses must satisfy

1). \( \mathcal{V}_{II}[12, 34] \);
2). \( \mathcal{V}_{IB}[12, 34] \cup \mathcal{V}_{IB}[13, 24] \).  
(6.40)

Repeat the argument with \( r_{12}^2 r_{34}^2 \). Then the masses must satisfy

1). \( \mathcal{V}_{II}[12, 34] \);
2). \( \mathcal{V}_{IB}[13, 24] \cup \mathcal{V}_{IB}[14, 23] \).  
(6.41)

Our strategy is to show that the three constraints (6.43), (6.44) and (6.45), together with other available constraints, can not be satisfied simultaneously.

**Case 1: Three of 1) are satisfied.** That is, \( m_1 m_3 = m_2 m_4, m_1 m_4 = m_3 m_2, m_1 m_2 = m_3 m_4 \). Then \( m_1 = m_2 = m_3 = m_4 \), or \( \pm m_1 = \pm m_2 = \pm m_3 = m_4 \) with two minus.

However, recall that \( \Delta_{123} \to 0 \), one checks that conditions (5.26) and (5.36) corresponding to the remaining two diagrams are impossible for \( m_1 = m_2 = m_3 = m_4 \), or \( \pm m_1 = \pm m_2 = \pm m_3 = m_4 \) with two minus.

**Case 2: Two of 1) are satisfied.** Without lose of generality, assume that the masses belong to the set \( \mathcal{V}_{II}[13, 24] \cup \mathcal{V}_{II}[14, 23] \), i.e., \( m_1 m_3 = m_2 m_4, m_1 m_4 = m_2 m_3 \). Then \( m_1 = \sigma m_2, m_3 = \sigma m_4 \) (\( \sigma = \pm 1 \)), and the masses also belong to the set \( \mathcal{V}_{IB}[13, 24] \cup \mathcal{V}_{IB}[14, 23] \).

However, recall that \( \Delta_{123} \to 0 \), one checks that conditions (5.26) and (5.36) corresponding to the remaining two diagrams are impossible.

**Case 3: One of 1) is satisfied.** Without lose of generality, assume that the masses belong to the set \( \mathcal{V}_{II}[13, 24] \), i.e., \( m_1 m_3 = m_2 m_4 \). Then the masses also belong to the set \( \mathcal{V}_{IB}[13, 24] \cup \mathcal{V}_{IB}[14, 23] \).

\[
(V_{IB}[12, 34] \cup V_{IB}[13, 24]) (V_{IB}[13, 24] \cup V_{IB}[14, 23]) = (V_{IB}[12, 34] V_{IB}[14, 23]) \cup V_{IB}[13, 24].
\]
Subcase 1: $V_{IB}[12, 34]V_{IB}[14, 23]$

However, recall that $\Delta_{123} \to 0$, one checks that all of conditions are impossible.

Subcase 2: $V_{IB}[13, 24]$

Except that $\Delta_{123} \to 0$, note that in this case one of the two diagrams in Figure 10 occur.

Thus we can consider $r_{14} \to 0$ and $r_{34} \to 0$, we have $\Delta_{134} \to 0$ or at least two of $\Delta_{124} \to 0$, $\Delta_{134} \to 0$ and $\Delta_{234} \to 0$ occur.

Similarly, one checks that all of conditions are impossible.

Case 4: No 1) is satisfied. Then the masses belong to the set

$$
(V_{IB}[12, 34] \cup V_{IB}[14, 23])(V_{IB}[12, 34] \cup V_{IB}[13, 24])(V_{IB}[13, 24] \cup V_{IB}[14, 23])
= (V_{IB}[12, 34] \cup V_{IB}[14, 23]) \cup (V_{IB}[12, 34] \cup V_{IB}[13, 24]) \cup (V_{IB}[13, 24] \cup V_{IB}[14, 23]).
$$

Without lose of generality, assume that the masses belong to the set $V_{IB}[12, 34]V_{IB}[13, 24]$.

Except that $\Delta_{123} \to 0$, note that:

By $V_{IB}[12, 34]$ we consider $r_{14}^2 \to 0$ and $r_{24}^2 \to 0$, then $\Delta_{124} \to 0$ or at least two of $\Delta_{124} \to 0$, $\Delta_{134} \to 0$ and $\Delta_{234} \to 0$ occur.

By $V_{IB}[12, 34]$ we consider $r_{14}^2 \to 0$ and $r_{34}^2 \to 0$, then $\Delta_{134} \to 0$ or at least two of $\Delta_{124} \to 0$, $\Delta_{134} \to 0$ and $\Delta_{234} \to 0$ occur.

In short, at least two of $\Delta_{124} \to 0$, $\Delta_{134} \to 0$ and $\Delta_{234} \to 0$ occur.

Subcase 1: $\Delta_{124} \to 0$ and $\Delta_{134} \to 0$:

However, one checks that all of conditions are impossible.

Subcase 2: $\Delta_{124} \to 0$ and $\Delta_{234} \to 0$:

Similarly, one checks that all of conditions are impossible.

Subcase 3: $\Delta_{134} \to 0$ and $\Delta_{234} \to 0$:

Similarly, one checks that all of conditions are impossible.

To summarize, we proved that the system (2.3) possesses finitely many solutions, if

$$
\sum_{i=1}^{4} m_i \neq 0, \quad \prod_{1 \leq j < k \leq 4} (m_j + m_k + m_l) \neq 0 \quad \text{and} \quad I \neq 0.
$$

\[\square\]

6.2 Finiteness with $m \neq 0$, $\prod_{1 \leq k \leq 4} (m - m_k) \neq 0$ and $I = 0$

Theorem 6.3 Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero, if $\sum_{i=1}^{4} m_i \neq 0$, $\prod_{1 \leq j < k \leq 4} (m_j + m_k + m_l) \neq 0$ and $I = 0$, then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.
In this case only Diagram I, Diagram II, Diagram III, Diagram V and Diagram VI are possible.

**Proof of Theorem 6.3.** It is easy to see that giving five of \( r_{kl}^2 \)'s, \( 1 \leq k < l \leq 4 \), determines only finitely many geometrical configurations up to rotation. Suppose that there are infinitely many solutions of system (4) in the complex domain. Then at least two of \( \{ r_{kl}^2 \} \)’s must take infinitely many values and thus are dominating by Lemma 2.1. Suppose that \( r_{kl}^2 \) is dominating for some \( 1 \leq k < l \leq 4 \). There must exist a singular sequence of central configurations with \( r_{kl}^2 \to 0 \), which happens only in the Diagram III or Diagram V. In either case, by Proposition 5.1,

\[
r_{kl}^2 r_{ij}^2 \to 0, \quad r_{kl}^2 r_{jl}^2 \to 0, \quad r_{il}^2 r_{kj}^2 \to 0,
\]

along the singular sequence. Then all the three polynomials

\[
r_{13}^2 r_{24}^2, \quad r_{14}^2 r_{23}^2, \quad r_{12}^2 r_{34}^2,
\]

are dominating. It is easy to see that there exist \( \Delta_{ijk} \to 0 \) (say \( \Delta_{123} \to 0 \)), so \( r_{12}, r_{23} \) and \( r_{13} \) are dominating.

Then there also exist singular sequences with \( r_{12}^2 r_{34}^2 \to \infty \), singular sequences with \( r_{13}^2 r_{24}^2 \to \infty \) and singular sequences with \( r_{14}^2 r_{23}^2 \to \infty \). These sequences must correspond to Diagram I or Diagram II.

Consider a singular sequence with \( r_{13}^2 r_{24}^2 \to \infty \), thus, the masses must belong to the set

\[
V_{I0}[12, 34] \cup V_{I0}[14, 23] \cup V_{II0}[13, 24].
\]

(6.42)

We classify the cases above into two classes:

1). \( V_{II0}[13, 24] \);
2). \( V_{I0}[12, 34] \cup V_{I0}[14, 23] \). (6.43)

Repeat the argument with \( r_{14}^2 r_{23}^2 \). Then the masses must belong to one of the sets

1). \( V_{II0}[14, 23] \);
2). \( V_{I0}[12, 34] \cup V_{I0}[13, 24] \). (6.44)

Repeat the argument with \( r_{12}^2 r_{34}^2 \). Then the masses belong to one of the sets

1). \( V_{II0}[12, 34] \);
2). \( V_{I0}[13, 24] \cup V_{I0}[14, 23] \). (6.45)

Our strategy is to show that the three constraints (6.43), (6.44) and (6.45) are incompatible.

**Case 1: Three of 1) are satisfied.** A straightforward computation shows that all the conditions are impossible.
Case 2: Two of 1) are satisfied. Without lose of generality, assume that the masses belong to the set

\[ V_{II0}[13, 24]V_{I0}[14, 23](V_{I0}[13, 24] \cup V_{I0}[14, 23]). \]

However, one checks that the set is empty.

Case 3: One of 1) is satisfied. Without lose of generality, assume that the masses belong to the set

\[ V_{II0}[13, 24](V_{I0}[12, 34] \cup V_{I0}[13, 24]) \cup V_{II0}[14, 23](V_{I0}[13, 24] \cup V_{I0}[14, 23]). \]

However, a straightforward computation shows that that the set is empty.

Case 4: No 1) is satisfied. Then the masses belong to the set

\[ (V_{I0}[12, 34] \cup V_{I0}[14, 23])(V_{I0}[12, 34] \cup V_{I0}[13, 24]) \cup V_{I0}[14, 23](V_{I0}[13, 24] \cup V_{I0}[14, 23]). \]

Without lose of generality, assume that the masses belong to the set \( V_{I0}[12, 34]V_{I0}[14, 23] \). A straightforward computation shows that the set is empty.

To summarize, we proved that the system (2.3) possesses finitely many solutions, if

\[ \sum_{i=1}^{4} m_i \neq 0, \prod_{1 \leq j < k < l \leq 4} (m_j + m_k + m_l) \neq 0 \text{ and } I = 0. \]

\[ \Box \]

7 Finiteness of Central configurations with \( m \neq 0 \) and \( \prod_{1 \leq k \leq 4} (m - m_k) = 0 \)

Theorem 7.1 Suppose that \( m_1, m_2, m_3, m_4 \) are real and nonzero, if \( \sum_{i=1}^{4} m_i \neq 0 \) and \( \prod_{1 \leq j < k < l \leq 4} (m_j + m_k + m_l) = 0 \), then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.

In this case Diagram I, Diagram II, Diagram III, Diagram IV, Diagram V and Diagram VI are all possible.

The proof of Theorem 7.1 reduces to the following Theorem 7.3 and Theorem 7.4.

First, we establish the following result:

Lemma 7.2 If some product \( r_{jk}^4 r_{lm}^2 \) is not dominating on the closed algebraic subset \( A \). Then system (2.3) possesses finitely many solutions.

Proof of Lemma 7.2

If system (2.3) possesses infinitely many solutions, without lose of generality, assume that \( r_{12}^4 r_{34}^2 \) is not dominating on the closed algebraic subset \( A \). Then some level set \( r_{12}^4 r_{34}^2 \equiv \text{const} \neq 0 \) also includes infinitely many solutions of system (2.3).
Figure 11: Complete diagrams in the level set $r_{12}^4 r_{34}^2$
In this level set the complete diagrams may be approached by some singular sequence
are the Figure 11.

Similar as previous discussion, at least two of \( \{r_{kl}^2\} \)'s must take infinitely many values
and thus are dominating. Suppose that \( r_{kl}^2 \) is dominating for some \( 1 \leq k < l \leq 4 \). There
must exist a singular sequence of central configurations with \( r_{kl}^2 \to 0 \), which happens
only in the Diagram III, Diagram IV or Diagram V. In each case, note that \( r_{12}r_{34} \approx \sqrt{\epsilon} \),
by Proposition 5.1
\[
r_{ki}^2r_{lj}^2 \to 0, \quad r_{ik}^2r_{jl}^2 \to 0, \quad r_{il}^2r_{kj}^2 \to 0,
\]
along the singular sequence. Then all the three polynomials
\[
r_{13}^2r_{24}^2, \quad r_{14}^2r_{23}^2, \quad r_{12}^2r_{34}^2,
\]
are dominating.

Consider a singular sequence with \( r_{12}^2r_{34}^2 \to \infty \), then it is easy to see that this
happens only in the Diagram IV1 or IV2. Without lose of generality, assume that
Diagram IV1 occur. Then
\[
m_1 + m_2 + m_3 = 0. \tag{7.46}
\]

Note that the relations of masses in Diagram III1 or III2 are not consistent with
that in Diagram IV1, and the relations of masses in Diagram V1-4 are not consistent
with that in Diagram IV1 and IV2.

Thus, when we consider \( r_{12}^2 \to \infty \), we are in Diagram V1-4. So when we consider
\( r_{14}^2 \to 0 \), we are in Diagram V1 or V2, and the masses must satisfy
\[
(m_3^2(m_1 + m_4)^4 - m_1^3m_4^3)(m_2^2(m_3 + m_1)^4 - m_3^3m_1^3) = 0. \tag{7.47}
\]

Similarly, consider \( r_{24}^2 \to 0 \), we are in Diagram V3 or V4, and the masses must satisfy
\[
(m_2^2(m_2 + m_4)^4 - m_2^3m_4^3)(m_3^2(m_3 + m_2)^4 - m_3^3m_2^3) = 0. \tag{7.48}
\]

A straightforward computation shows that the only solution of equations (7.46),
(7.47) and (7.48) is \( m_2 = -m_1, m_3 = m_4 = 0 \), this is a contradiction.

\[
\square
\]

So we consider all of \( r_{jkl}^2r_{lm}^2 \) are dominating on the closed algebraic subset \( \mathcal{A} \) from
now on.

If Diagram IV does not occur, then it reduces to previous section. Thus, without
lose of generality, we assume that there always exists a singular sequence of central
configurations approaching Diagram IV such that \( \Delta_{124} \to 0 \) below. Then
\[
m_1 + m_2 + m_3 = 0 \tag{7.49}
\]
and it is easy to see that all of \( r_{kl}^2 \)'s, \( 1 \leq k < l \leq 4 \), and \( r_{13}^2r_{24}^2, \quad r_{14}^2r_{23}^2, \quad r_{12}^2r_{34}^2 \)
are dominating.
7.1 Finiteness with \( m \neq 0 \), \( \prod_{1 \leq k \leq 4} (m - m_k) = 0 \) and \( I \neq 0 \)

**Theorem 7.3** Suppose that \( m_1, m_2, m_3, m_4 \) are real and nonzero, if \( \sum_{i=1}^{4} m_i \neq 0 \), \( \prod_{1 \leq j < k < l \leq 4} (m_j + m_k + m_l) = 0 \) and \( I \neq 0 \), then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.

In this case only Diagram I, Diagram II, Diagram III, Diagram IV and Diagram V are possible.

**Proof of Theorem 7.3**

We consider \( r_{14}^2 \to 0, r_{24}^2 \to 0 \) and \( r_{34}^2 \to 0 \), then it is easy to see that at least two of \( \Delta_{124} \to 0, \Delta_{134} \to 0 \) and \( \Delta_{234} \to 0 \) occur. Without lose of generality, assume that \( \Delta_{124} \to 0 \) and \( \Delta_{134} \to 0 \) occur.

Corresponding to \( \Delta_{124} \to 0 \) we have

\[
\left( \frac{1}{|m_1|^2} + \frac{1}{|m_2|^2} + \frac{1}{|m_4|^2} \right) (m_1 + m_2 + m_4)(m_1^2(m_2 + m_4)^4 - m_2^3 m_4^3) \\
(m_2^2(m_1 + m_4)^4 - m_1^3 m_4^3)(m_4^2(m_2 + m_1)^4 - m_2^3 m_1^3) = 0 \quad (7.50)
\]

Corresponding to \( \Delta_{134} \to 0 \) we have

\[
\left( \frac{1}{|m_1|^2} + \frac{1}{|m_3|^2} + \frac{1}{|m_4|^2} \right) (m_1 + m_3 + m_4)(m_1^2(m_3 + m_4)^4 - m_3^3 m_4^3) \\
(m_3^2(m_1 + m_4)^4 - m_1^3 m_4^3)(m_4^2(m_3 + m_1)^4 - m_2^3 m_1^3) = 0 \quad (7.51)
\]

Let us further consider \( r_{12}^2 r_{34}^2 \to \infty \), then we are in Diagram I, Diagram II, Diagram III or Diagram IV.

**Case 1: If we are in Diagram I.** That is, the masses belong to the set

\[
\mathcal{V}_{IA}[13, 24] \cup \mathcal{V}_{IA}[14, 23] \cup \mathcal{V}_{IB}[13, 24] \cup \mathcal{V}_{IB}[14, 23]. \quad (7.52)
\]

However, a straightforward computation shows that all of these masses are not consistent with equations (7.49), (7.50) and (7.51).

**Case 2: If we are in Diagram II.** That is, we have the masses belong to the set

\[
\mathcal{V}_{II}[12, 34]. \quad (7.53)
\]

However, a straightforward computation shows that all of these masses are not consistent with equations (7.49), (7.50) and (7.51).

**Case 3: If we are in Diagram III.** That is, we have

\[
\begin{cases} 
\frac{1}{|m_1|^2} + \frac{1}{|m_2|^2} + \frac{1}{|m_4|^2} = 0 \\
m_1 + m_3 + m_4 = 0 
\end{cases} \quad (7.54)
\]

or

\[
\begin{cases} 
\frac{1}{|m_2|^2} + \frac{1}{|m_3|^2} + \frac{1}{|m_4|^2} = 0 \\
m_2 + m_3 + m_4 = 0 
\end{cases} \quad (7.55)
\]


However, a straightforward computation shows that all of these relations are not consistent with equations (7.49), (7.50) and (7.51).

Case 4: If we are in Diagram IV. That is, we are in Diagram IV3 or Diagram IV4.

Subcase 1: If we are in Diagram IV3: Then
\[ m_2 + m_3 + m_4 = 0. \] (7.56)
A straightforward computation shows that this relation is not consistent with equations (7.49), (7.50) and (7.51).

Subcase 2: If we are in Diagram IV4: Then
\[ m_1 + m_3 + m_4 = 0. \] (7.57)
In this case, the relation (7.57) is consistent with equations (7.49), (7.50) and (7.51).

Let us further consider \( r_{12}^2 r_{34}^2 \rightarrow 0 \), then we are in Diagram III or Diagram V, indeed, we have \( \triangle_{234} \rightarrow 0 \). Thus
\[
\left( \frac{1}{|m_2|^2} + \frac{1}{|m_3|^2} + \frac{1}{|m_4|^2} \right) (m_2^2 (m_2 + m_4)^4 - m_3^2 m_4^3) - (m_2^2 (m_3 + m_4)^4 - m_3^2 m_3^3) (m_2^2 (m_2 + m_3)^4 - m_2^2 m_3^3) = 0
\] (7.58)
However, a straightforward computation shows that there is no solution for equations (7.49), (7.50), (7.51), (7.57) and (7.58).

To summarize, we proved that the system (2.3) possesses finitely many solutions, if \( \sum_{i=1}^4 m_i \neq 0 \), \( \prod_{1 \leq j < k \leq 4} (m_j + m_k + m_l) = 0 \) and \( I \neq 0 \).

\[ \square \]

7.2 Finiteness with \( m \neq 0 \), \( \prod_{1 \leq k \leq 4} (m - m_k) = 0 \) and \( I = 0 \)

Theorem 7.4 Suppose that \( m_1, m_2, m_3, m_4 \) are real and nonzero, if \( \sum_{i=1}^4 m_i \neq 0 \), \( \prod_{1 \leq j < k \leq 4} (m_j + m_k + m_l) \neq 0 \) and \( I = 0 \), then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions.

In this case all of Diagram I, Diagram II, Diagram III, Diagram IV, Diagram V and Diagram VI are possible. However we remark that Diagram I, Diagram II or Diagram III should not occur twice or more times.

Proof of Theorem 7.4
Let us first consider \( r_{12}^2 r_{34}^2 \rightarrow \infty \), then we are in Diagram I, Diagram II, Diagram III or Diagram IV.
Then the masses belong to the set
\[
\mathcal{V}_{I0}[13, 24] \cup \mathcal{V}_{I0}[14, 23] \cup \mathcal{V}_{II0}[12, 34] \\
\cup (\mathcal{V}_{III}[134] \cup \mathcal{V}_{IV}[134]) \cup (\mathcal{V}_{III}[234] \cup \mathcal{V}_{IV}[234]) \cup \mathcal{V}_{IV}[134] \cup \mathcal{V}_{IV}[234] \triangleq \mathcal{V}[12, 34].
\] (7.59)
Similarly, when considering \( r_{13}^4 r_{24}^2 \to \infty \) the masses belong to the set

\[
\bigcup (\mathcal{V}_{I0}[12, 34] \cup \mathcal{V}_{I0}[14, 23] \cup \mathcal{V}_{II0}[13, 24] \
\cup (\mathcal{V}_{III}[124] \cup \mathcal{V}_{IV}[124]) \cup (\mathcal{V}_{IIII}[124] \cup \mathcal{V}_{III}[124] \cup \mathcal{V}_{IV}[124] \cup \mathcal{V}_{IV}[234]) \equiv \mathcal{V}[13, 24];
\]

(7.60)

when considering \( r_{23}^4 r_{14}^2 \to \infty \) the masses belong to the set

\[
\bigcup (\mathcal{V}_{I0}[13, 24] \cup \mathcal{V}_{I0}[12, 34] \cup \mathcal{V}_{II0}[14, 23] \
\cup (\mathcal{V}_{IIII}[124] \cup \mathcal{V}_{IV}[124]) \cup (\mathcal{V}_{IIII}[134] \cup \mathcal{V}_{IV}[134]) \cup \mathcal{V}_{IV}[124] \cup \mathcal{V}_{IV}[134] \equiv \mathcal{V}[23, 14].
\]

(7.61)

Or

\[
\begin{align*}
V[12, 34] &= (\mathcal{V}_{I0}[13, 24] \cup (\mathcal{V}_{I0}[14, 23]) \cup (\mathcal{V}_{II0}[12, 34]) \cup \mathcal{V}_{IV}[134] \cup \mathcal{V}_{IV}[234]; \\
V[13, 24] &= (\mathcal{V}_{I0}[12, 34]) \cup (\mathcal{V}_{I0}[14, 23]) \cup (\mathcal{V}_{II0}[13, 24]) \cup \mathcal{V}_{IV}[124] \cup \mathcal{V}_{IV}[234]; \\
V[23, 14] &= (\mathcal{V}_{I0}[13, 24]) \cup (\mathcal{V}_{I0}[12, 34]) \cup (\mathcal{V}_{II0}[14, 23]) \cup \mathcal{V}_{IV}[124] \cup \mathcal{V}_{IV}[134].
\end{align*}
\]

(7.62)

Case 1: If Diagram I occurs. Without lose of generality, assume that the masses belong to the set \( \mathcal{V}_{I0}[13, 24] \), then we claim that

\[
(\mathcal{V}_{I0}[13, 24]) \cap V[13, 24] \cap \mathcal{V}_{IV}[123] = \emptyset.
\]

(7.63)

A straightforward computation shows the claim.

Case 2: If Diagram II occurs. Without lose of generality, assume that the masses belong to the set \( \mathcal{V}_{II0}[12, 34] \), then we claim that

\[
(\mathcal{V}_{II0}[12, 34]) \cap V[13, 24] \cap \mathcal{V}_{IV}[123] \cap V[14, 23] = \emptyset.
\]

(7.64)

A straightforward computation shows the claim.

We can simplify \( V[12, 34], V[13, 24] \) and \( V[14, 23] \) as follows now.

\[
\begin{align*}
V[12, 34] &= \mathcal{V}_{IV}[134] \cup \mathcal{V}_{IV}[234]; \\
V[13, 24] &= \mathcal{V}_{IV}[124] \cup \mathcal{V}_{IV}[234]; \\
V[23, 14] &= \mathcal{V}_{IV}[124] \cup \mathcal{V}_{IV}[134].
\end{align*}
\]

(7.65)

Then

\[
\begin{align*}
V[12, 34]V[13, 24]V[14, 23]\mathcal{V}_{IV}[123] &= \mathcal{V}_{IV}[123]\mathcal{V}_{IV}[134]\cup \mathcal{V}_{IV}[124]\mathcal{V}_{IV}[234]\cup \mathcal{V}_{IV}[124]\mathcal{V}_{IV}[234].
\end{align*}
\]

(7.66)

A straightforward computation shows that only the point

\[
m_2 = m_3 = m_4 = -\frac{1}{7}m_1
\]

belongs them.

It is easy to see that this point is not consistent with the constraints corresponding to Diagram I, Diagram II, Diagram III or Diagram V, therefore, only Diagram IV and Diagram VI are possible now.
We consider the function \( r_{12}^2r_{23}^2r_{34}^2r_{14}^2r_{24}^2r_{34}^2 \), then it is easy to see that it is dominating. When considering \( r_{12}^2r_{23}^2r_{34}^2r_{14}^2r_{24}^2r_{34}^2 \rightarrow 0 \), we are in Diagram VI, thus the function \( r_{12}^2r_{13}^2r_{14}^2 \rightarrow \infty \), we are in Diagram IV with \( \Delta_{234} \rightarrow 0 \), as a result, we have \( m_2 + m_3 + m_4 = 0 \), this is a contradiction.

To summarize, we proved that the system (2.3) possesses finitely many solutions, if \( \sum_{j=1}^{4} m_j \neq 0 \), \( \prod_{1 \leq j < k < l \leq 4} (m_j + m_k + m_l) = 0 \) and \( I = 0 \).

\[\blacksquare\]

8 Finiteness of Central configurations with \( m = 0 \)

8.1 Problematic diagrams with vanishing total mass

We prove that the first and third diagrams in Figure 5 and all of diagrams in 6 and 7 are impossible. Case 1: First and third diagrams in Figure 5. First, it is easy to see that

\[
\begin{align*}
m_1 + m_2 &= 0, \\
m_3 + m_4 &= 0, \\
w_1 \sim w_2 \sim w_3 \sim w_4 \sim a\epsilon^{-2}, \\
z_{12} \approx z_{34} \approx w_{12} \approx w_{34} \approx \epsilon.
\end{align*}
\]

(8.67)

It follows that

\[
w_{12} = m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}),
\]

we claim that \( z_1 \sim z_2 \sim z_3 \sim z_4 \) for the first three diagrams in Figure 5. Otherwise, we have \( z_{jk} \approx \epsilon^{-2} \) and \( W_{jk} = \frac{1}{z_{jk}w_{jk}} \leq \epsilon^2 \) for \( j = 3, 4, k = 1, 2 \), this contradicts \( w_{12} \approx \epsilon \).

Thus \( z_1 \sim z_2 \sim z_3 \sim z_4 \sim b\epsilon^{-2} \) holds for the first two diagrams and the third diagram does not exists in Figure 5.

For the first diagram we have

\[
\begin{align*}
z_1 &\sim m_2 Z_{21}, \\
z_2 &\sim m_1 Z_{12} + m_3 Z_{32},
\end{align*}
\]

(8.69)

this is a contradiction. Thus the first diagram does not exists.

Case 2: Diagrams in Figure 6. First, it is easy to see that the second and third diagrams are impossible.

For the first diagram, without loss of generality, assume that \( w_{14} > \epsilon \), then we have \( w_{24} \sim w_{34} \sim w_{14} \sim \epsilon \). By

\[
0 = \sum_{j=1}^{4} m_j w_4 = \sum_{j=1}^{3} m_j w_{j4}
\]

it follows that \( \sum_{j=1}^{3} m_j = 0 \), this is a contradiction.

Case 3: Diagram in Figure 7. First, it is easy to see that

\[
\begin{align*}
w_1 \sim w_2 \sim w_3 \sim w_4, \\
w_{14} \sim w_{24} \sim w_{34} \sim \epsilon.
\end{align*}
\]

(8.70)
By

\[ 0 = \sum_{j=1}^{4} m_jw_4 = \sum_{j=1}^{3} m_jw_j, \]

it follows that \( \sum_{j=1}^{3} m_j = 0 \), this is a contradiction.

We could not eliminate the diagrams in Figure 12. Some singular sequence could still exist and approach any of these diagrams.

8.1.1 Diagram I

Following the argument from [1], we obtain the equation

\[
\frac{m_1m_3}{(m_2m_4)^\frac{1}{2}} + \frac{m_2m_3}{(-m_1m_4)^\frac{1}{2}} + \frac{m_1m_4}{(-m_2m_3)^\frac{1}{2}} + \frac{m_2m_4}{(m_1m_3)^\frac{1}{2}} = 0,
\]

or equivalently,

\[
m_1m_3\sqrt{m_1m_3} + \sigma_1m_2m_3\sqrt{-m_2m_3} + \sigma_2m_1m_4\sqrt{-m_1m_4} + \sigma_3m_2m_4\sqrt{m_2m_4} = 0, \quad (8.71)
\]

where \( \sqrt{x} \ (x \in \mathbb{R}) \) is understood as the square root of \( x \) on the positive real axis if \( x > 0 \), and on the positive imaginary axis if \( x < 0 \).

However, we claim that

\[
(m_1 + m_2)(m_3 + m_4) \neq 0. \quad (8.72)
\]
This is a consequence of the fact
\[
\begin{align*}
    w_{12} &= (m_1 + m_2)W_{12} + m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}), \\
    z_{34} &= (m_3 + m_4)Z_{34} + m_1(Z_{14} - Z_{13}) + m_4(Z_{24} - Z_{23})
\end{align*}
\]  
(8.73)

It also follows that
\[
    r_{12}^3 \sim m_1 + m_2, \quad r_{34}^3 \sim m_3 + m_4. \quad (8.74)
\]

On the other hand, it is easy to see that
\[
    r_{jk} \approx \epsilon^{-2}, \quad j = 1, 2, k = 3, 4. \quad (8.75)
\]

To summarize, the masses of central configurations corresponding to the first diagram satisfy one of the following two sets of constraints
\[
\begin{align*}
    \text{Set } & \mathcal{V}_{IA}[12, 34], \quad m_1 = m_2, \quad m_3 = m_4, \quad m_1 + m_3 = 0; \\
    \text{Set } & \mathcal{V}_{IB}[12, 34], \quad \mu_2 \mu_3^3 + \sigma_1 \mu_2 \mu_4^3 + \sigma_2 \mu_1 \mu_4^3 + \sigma_3 \mu_2 \mu_4^3 = 0, \quad m_1 m_2 < 0, \quad m_3 m_4 < 0.
\end{align*}
\]

### 8.1.2 Diagram II

Following the argument from [II], the masses of central configurations corresponding to the second diagram satisfy the following equation
\[
    m_1 m_3 = m_2 m_4.
\]

Furthermore, we claim that
\[
    m_1 + m_2 = m_2 + m_3 = m_3 + m_4 = m_1 + m_4 = 0. \quad (8.76)
\]

Indeed, if \( m_1 + m_2 \neq 0 \), then \( z_1 \sim z_4, z_2 \sim z_3, z_1 \sim z_2 \), thus
\[
    z_{12} \approx \epsilon^{-2}, \quad z_{13} \sim z_{12}, z_{14} \sim \epsilon.
\]

By
\[
    0 = \sum_{j=1}^{4} m_j z_1 = \sum_{j=2}^{4} m_j z_1
\]

it follows that \( m_2 + m_3 = 0, \) then \( m_1 + m_4 = 0. \) this is a contradiction.

By the fact \( w_{14} = (m_1 + m_4)W_{14} + m_2(W_{24} - W_{21}) + m_3(W_{34} - W_{31}), \) we can get a contradiction.

Note that
\[
    r_{12}, r_{23}, r_{34}, r_{14} \ll 1, \\
    r_{13} \ll \epsilon^{-2}, \quad r_{24} \ll \epsilon^{-2}. \quad (8.77)
\]
8.1.3 Diagrams with two $zw$-edges

We claim that

$$m_1 + m_2 = m_3 + m_4 = 0, \quad m_1^2 = m_3^2.$$  \hspace{1cm} (8.78)

Indeed, by

$$w_{12} = (m_1 + m_2)W_{12} + m_3(W_{32} - W_{31}) + m_4(W_{42} - W_{41}),$$
$$z_{34} = (m_3 + m_4)Z_{34} + m_1(Z_{14} - Z_{13}) + m_4(Z_{24} - Z_{23})$$  \hspace{1cm} (8.79)

and so on it is easy to see that $z_1 \sim z_2 \sim z_3 \sim z_4, w_1 \sim w_2 \sim w_3 \sim w_4$.  \hspace{1cm} (8.80)

Thus $m_1 W_{12} \sim m_3 W_{34}$ and $w_{12} \sim -w_{34}$. So

$$\frac{m_1}{r_{12}} \sim -\frac{m_3}{r_{34}}.$$

By $r_{jk} > \epsilon, j = 1, 2, k = 3, 4$ and $U$ is bounded, it follows that

$$\frac{m_1 m_2}{r_{12}} \sim -\frac{m_3 m_4}{r_{34}}.$$

Thus $m_1^2 = m_3^2$.

8.1.4 Fully edged diagrams

We will reach our result without discussing the mass polynomial of this diagram.

8.2 Finiteness of Central configurations with $m = 0$

**Theorem 8.1** Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero with $\sum_{i=1}^4 m_i = 0$, then system (2.3), which defines the normalized central configurations in the complex domain, possesses finitely many solutions except perhaps for (up to renumeration of the bodies) $m_1 = m_2 = -m_3 = -m_4$ or $m_1 = m_2, m_3 = -x^2 m_1, m_4 = -y^2 m_1$, here $x, y$ are respectively the unique positive root of the two equations of (8.82). Numerically, $x \approx 1.2407, y \approx 0.678731$.

Note that if any of Diagram I with $\mathcal{V}_{I A_0}$ or Diagrams with two $zw$-edges occur, the masses must satisfy $\pm m_1 = \pm m_2 = \pm m_3 = \pm m_4$. Thus we assume that Diagram I with $\mathcal{V}_{I A_0}$ and Diagrams with two $zw$-edges do not occur in the following discussion.

**Proof of Theorem 8.1** It is easy to see that giving five of $r_{kl}^2$'s, $1 \leq k < l \leq 4$, determines only finitely many geometrical configurations up to rotation. Suppose that there are infinitely many solutions of system (4) in the complex domain. Then at least two of $\{r_{kl}^2\}$'s must take infinitely many values and thus are dominating by Lemma 2.1. Suppose that $r_{kl}^2$ is dominating for some $1 \leq k < l \leq 4$. There must exist a singular sequence of central configurations with $r_{kl}^2 \to 0$, which happens only in the
fully edged diagrams. In each case, it is easy to see that all of $r_{jk}^2$'s, $1 \leq k < l \leq 4$, and $r_{13}^2 r_{24}^2$, $r_{14}^2 r_{23}^2$, $r_{12}^2 r_{34}^2$ are dominating.

We consider $r_{12}^2 \to \infty$, $r_{13}^2 \to \infty$ and $r_{14}^2 \to \infty$, then it is easy to see that at least two of $V_{IB}[12, 24]$, $V_{IB}[13, 24]$ and $V_{IB}[14, 23]$ occur. Without lose of generality, assume that $V_{IB}[13, 24]$ and $V_{IB}[14, 23]$ occur.

We claim that

$$V_{IB}[13, 24] \bigcap V_{IB}[14, 23] \bigcap V_{m0} = \{m_2 = m_1, m_3 = -x^2m_1, m_4 = -y^2m_1 \}
\bigcup \{m_1 = -x^2m_4, m_2 = -y^2m_4, m_3 = m_4 \}
\bigcup \{m_1 = -y^2m_4, m_2 = -x^2m_4, m_3 = m_4 \},$$

where $V_{m0}$ denotes the set $\sum_{j=1}^4 m_j = 0$, and $x, y$ are respectively the unique positive root of the following equations:

$$x^{12} - 6x^{10} + 2x^9 + 12x^8 - 12x^7 - 6x^6 + 24x^5 - 6x^4 - 18x^3 + 12x^2 - 7 = 0,$$

$$y^{12} - 6y^{10} - 2y^9 + 12y^8 + 12y^7 - 6y^6 - 24y^5 - 6y^4 + 18y^3 + 12y^2 - 7 = 0;$$

where $x \approx 1.2407, y \approx 0.678731$.

Indeed, first it is easy to see that negative and positive of $m_1, m_2$ and $m_3, m_4$ are same respectively. Without loss of generality, assume that $m_1 = \mu_2^2, m_2 = \mu_3^2, m_3 = -\mu_3^2, m_4 = -\mu_4^2$, then a straightforward computation shows the claim.

\[\square\]

### 8.3 Finiteness with $m_1 = m_2 = -m_3 = -m_4$ or $m_1 = m_2, m_3 = -x^2m_1, m_4 = -y^2m_1$

Recall that solving system (2.2) is equivalent to finding critical points of the potential restricted on the set $I = c$. For these two groups of masses, following Celli [2], we use $r_{jk}$ as coordinates. For system with zero total mass, there are extra constraints linear in the squared distances, namely

$$\sum_{j \neq 1} m_j r_{j1}^2 = \sum_{j \neq 2} m_j r_{j2}^2 = \sum_{j \neq 3} m_j r_{j3}^2 = \sum_{j \neq 4} m_j r_{j4}^2 = c_0.$$

Then system (2.2) is equivalent to the following equations

$$m_j m_k + 2(\lambda_j m_k + \lambda_k m_j) r_{jk}^3 = 0, \quad 1 \leq j < k \leq 4$$

$$\sum_{j \neq k} m_j r_{jk}^2 = c_0, \quad 1 \leq k \leq 4.$$  

(8.83)

Here $\lambda_k (k = 1, 2, 3, 4)$ are four Lagrange multipliers.

**Theorem 8.2** Suppose that $m_1, m_2, m_3, m_4$ are real and nonzero, if $m_1 = m_2 = -m_3 = -m_4$ or $m_1 = m_2, m_3 = -x^2m_1, m_4 = -y^2m_1$, then system (2.2) possesses finitely many solutions.

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Proof of Theorem 8.2

Consider the equations (8.83).

**Case 1:** $m_1 = m_2 = -m_3 = -m_4$. In this case, it is easy to see that

$$r_{13} = r_{14} = r_{23} = r_{24} = tr_{34},$$
$$r_{12} = \sqrt{4t^2 - Tr_{34}},$$

where $t$ is one of two positive root of the following equation:

$$(4t^2 - 1)^3 (2 - t^3)^2 - t^6 = 0 \quad (8.84)$$
such that $4t^2 - 1 > 0$ and $2 - t^3 > 0$, here $t \approx 0.547752, 1.22442$.

**Case 2:** $m_1 = m_2, m_3 = -x^2m_1, m_4 = -y^2m_1$. In this case, after eliminating the $\lambda_j$ and $I$ we have the relations:

$$\frac{1}{r_{12}} + \frac{1}{r_{34}} = \frac{1}{r_{13}} + \frac{1}{r_{24}} = \frac{1}{r_{14}} + \frac{1}{r_{23}},$$
$$x^2 (r_{13}^2 - r_{23}) = y^2 (r_{24}^2 - r_{14}^2). \quad (8.85)$$
It follows that

$$r_{13} - r_{23} = r_{14} - r_{24} = 0, \quad (8.86)$$
then by imposing the normalization $r_{34} = 1 \quad (8.83)$ reduces to

$$\frac{1}{r_{12}^2} = \frac{1}{r_{13}^2} + \frac{1}{r_{14}^2} - 1,$$
$$r_{12}^2 - (r_{13}^2 - 1) x^2 - r_{14}^2 (y^2 + 2) = 0,$$
$$2r_{13}^2 - 2r_{14}^2 + x^2 - y^2 = 0. \quad (8.87)$$
By substituting $r_{13}^2 = r_{14}^2 - \frac{x^2}{2} + \frac{y^2}{2}$ and $r_{12}^2 = r_{14}^2 (x^2 + y^2 + 2) - \frac{1}{2} x^2 (x^2 - y^2 + 2)$
into the first equation in (8.87), rationalizing it and eliminating denominator, we get a polynomial of degree 36 in $r_{14}$, thus solutions (8.87) are finite.

$\square$
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