On the relative quantum entanglement
with respect to tensor product structure

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Mathematical foundation of the novel concept of quantum tensor product by Zanardi et al is rigorously established. The concept of relative quantum entanglement is naturally introduced and its meaning is made clear both mathematically and physically. For a finite or an infinite dimensional vector space $W$ the so called tensor product partition (TPP) is introduced on $\text{End}(W)$, the set of endmorphisms of $W$, and a natural correspondence is constructed between the set of TPP’s of $\text{End}(W)$ and the set of tensor product structures (TPS’s) of $W$. As a byproduct, it is shown that an arbitrarily given wave function belonging to an $n$-dimensional Hilbert space, $n$ being not a prime number, can be interpreted as a separable state with respect to some man-made TPS, and thus a quantum entangled state of a many-body system with respect to the “God-given” TPS can be regarded as a quantum state without entanglement in some sense. The concept of standard set of observables is also introduced to probe the underlying structure of the object TPP and to establish its connection with practical physical measurement.

I. INTRODUCTION

Quantum entanglement is a fundamental concept of quantum mechanics and plays a central role in quantum information processing \cite{1}. It has also motivated many investigations in mathematical physics \cite{2, 3}. What is less obvious is the fact that quantum entanglement is not an intrinsically defined concept. For example, the state of a bi-particle system described by the separable wave function with respect to the two position coordinates is generally an entangled one with respect to the center of mass and relative coordinates. It seems that this point has been ignored for a long time by physicists. But recently Zanardi et.al have brought this problem to our attention in the context of quantum information. They explicitly point out that whether a state is an entangled one or not depends on the tensor product structure (TPS) of the state space \cite{4, 5} and they argue that quantum system can be partitioned into the so called virtual subsystems according to a man-made TPS selected by a set of observables operationally relevant in the sense of interactions and measurements. Accordingly, quantum entanglement is observable induced and hence relative. As a matter of fact, we have in some sense considered the relativity of quantum entanglement.
in the adiabatic separation of the fast and slow variables of a composite system by means of the Born-Oppenheimer approximation \[6, 7\].

In this paper, to characterize the above mentioned man-made TPS and the related quantum entanglement we develop a general algebraic approach from the viewpoint of observable algebra. Along this line, we manage to show that a quantum state can justifiably be called entangled or unentangled with respect to a particular partition of the observable algebra. And among others, we can show that in an \(n\)-dimensional Hilbert space \(W\) where the dimension \(n\) is not a prime, an arbitrarily given quantum state \(|s\rangle \in W\) is a separable one with respect to some man-made TPS of \(W = V_{k'} \otimes V_{l'}\) where \(V_{k'}, V_{l'}\) are respectively \(k'\) and \(l'\) dimensional subspaces of \(W\). Particularly, an entangled state in \(W\) with respect to the natural (or “a priori God-given”) TPS \(W = V_k \otimes V_l\) with \(n = kl\), can always be decoded as a separable state with respect to some artificially introduced TPS. But this is only one side of the coin. It is in fact equally true that when the dimension of \(W\) is not a prime an arbitrarily given quantum state \(|s\rangle \in W\) is an entangled one with respect to some man-made TPS. In short, we have made it mathematically clear that it is impossible to make a clear cut between entanglement and unentanglement as expected. To emphasize the physical aspect of the TPS of state space we also show how it is related to the so called complete set of observables.

It should be pointed out that the main idea of this paper originates from the remarkable observations implied in the interesting paper by Zanardi et al.\[4, 5\]. But in this paper we prove the uniqueness theorem concerning the TPS related to a particular partition of the observable algebra while only the existence theorem is proved in the original paper. Thanks to the uniqueness theorem we can characterize entanglement from the viewpoint of partition of observable algebra. Moreover, the whole theory is here developed within strict mathematical framework and thus some vague points have been clarified. Especially, we have weakened the conditions for the partition of observable algebra and developed a method which is free of the restriction from dimension. Indeed, most of the results obtained in this paper are valid in the infinite dimensional case as well as in the finite dimensional case. As we do not wish to restrict ourselves to the finite dimensional case from the very beginning some proofs become inevitably more complicated. But our effort is rewarded: we finally clarify the arguments about the entanglement of identical particle system. It is found that different definitions of entanglement for distinguishable-particle tacitly presuppose different TPS’s corresponding to the measurement of different observables, and from our approach a separability criterion for two-identical particle system can be correctly given without any contradiction\[8, 9\].

The rest parts of this paper is organized as follows. In section II we present some basic
knowledge of module theory and multilinear algebra for later use and fix the notation. The materials are standard and can be found in any relevant text books (for example see [10]). The reader who is familiar with these topics can safely skip this section. In section III, we introduce and investigate the concept of tensor product partition (TPP) of the set of linear operators $\text{End}(W)$ on a finite or an infinite dimensional vector space $W$, which turns out to have a close connection with the TPS of $W$. In section IV we introduce the concept of complete set of operators to probe the underlying structure of the TPP of $\text{End}(W)$. With these preparations we study the interrelationship between the TPP of $\text{End}(W)$ and the TPS of $W$ in section V. Some major propositions are proved in this section concerning the correspondence between these two objects. For the application in quantum mechanics, we take into accounts the inner product structure of $W$ in section VI and discuss the inner product compatible TPS after introducing a natural compatibility condition. In section VII, we explore the relationship among the TPP, the TPS and the product vector set, and the relativity of entanglement then becomes clear. Finally, three examples are analyzed in section VIII as an illustration of the theory and some concluding remarks are made in section IX.

II. PRELIMINARIES

In this paper all the algebras and vector spaces dealt with are over the complex number field and of countable dimension. Moreover, we will not consider the topology of vector space at all. So infinite summation does not mean any limit process. Rather, its meaning will be specified in the context.

First let us review an elementary part of the module theory. Let $A$ be an associative algebra, $V$ an $A$ module. $V$ is called a irreducible module if it has no nontrivial submodule. If $V$ can be decomposed into a direct sum of irreducible modules, then it is called decomposable.

**Theorem 2.1** Let $V$ be an $A$ module, then (1) $V$ is decomposable if and only if every submodule of $V$ is decomposable; (2) $V$ is decomposable only if for any submodule $V_1$ of $V$ there is a complementary submodule $V_2$: $V = V_1 \oplus V_2$.

**Remark** If every submodule of $V$ is finitely generated, then the converse of (2) in the above theorem is true. That is, if for any submodule of $V$ there exists a complementary submodule, then $V$ is decomposable.

**Theorem 2.2** Let $V$ be a decomposable $A$ module, $V = \sum_i \oplus V_i$ the decomposition of $V$ into a direct sum of irreducible submodules. If $U$ is a irreducible submodule of $V$, then $U$ is isomorphic to some $V_i$. 
Let $V$ and $W$ be $A$ modules. A linear map from $V$ to $W$ is called a *module homomorphism* if it commutes with the action of $A$. An injective and surjective module homomorphism is called a module isomorphism. The following result concerning module homomorphism is well known.

**Theorem 2.3** Let $V$ and $W$ be irreducible $A$ modules, $f$ a homomorphism from $V$ to $W$. Then $f$ is either a zero map or an isomorphism.

According to this theorem, given two irreducible modules, to prove that they are isomorphic we only need to show that there exists a non zero homomorphism between them. This fact will be used in the next section.

**Definition 2.1** An irreducible $A$ module $V$ will be called a normal module if the following condition is satisfied: every homomorphism $f$ from $V$ to itself is equal to the identity map up to a scalar multiple.

**Remark** A finite dimensional irreducible $A$ module is necessarily a normal module by Schur’s Lemma. But a normal module is not necessarily finite dimensional. So normality in the sense of Definition 2.1 does not characterize finite dimensionality completely. In this paper we will assume the normality, instead of the finite dimensionality, of certain modules, and hence our discussion is applicable to some interesting infinite dimensional cases.

Next let us recall some basic knowledge concerning the concept of tensor product.

**Definition 2.2** Let $V_1, V_2$ and $W$ be vector spaces, $f$ a bilinear map from $(V_1,V_2)$ to $W$. If for any vector space $U$ and any bilinear map $g$ from $(V_1,V_2)$ to $U$ there exists a unique linear map $h$ from $W$ to $U$ such that $g = h \circ f$, then $(V_1,V_2,f)$ is called a TPS of $W$ and $W$ is called a tensor product of $V_1$ and $V_2$ with respect to $f$, or simply a tensor product of $V_1$ and $V_2$ if no confusion will arise.

Conventionally, $f$ is denoted by the symbol $\otimes$ and $W$ is written as $W = V_1 \otimes V_2$. Here is a major fact about tensor product: If $\{x_i\}$ and $\{y_j\}$ are two bases of $V_1$ and $V_2$ respectively, then $W = V_1 \otimes V_2$ if and only if $\{x_i \otimes y_j\}$ is a basis of $W$. Especially in the finite dimensional case, $(V_1,V_2,f)$ is a TPS of $W$ if and only if the image of $f$ spans $W$ and $\dim W = \dim V_1 \cdot \dim V_2$.

**Definition 2.3** Let $(V_1,V_2,\otimes)$ be a TPS of $W$. A vector $w \in W$ is called decomposable if it is of the form $x \otimes y$ where $x \in V_1$ and $y \in V_2$.

Obviously, any vector $w \in W$ can be written as a sum of decomposable vectors. We call such a sum an expression of $w$ in terms of decomposable vectors or just an expression of $w$ for short. Notice that expressions of $w$ may not be unique. The length of an expression of $w$ is defined to be the number of nonzero decomposable vectors it contains and the rank of $w$ is defined to be the length of the shortest expressions of $w$. Then by definition the rank of a decomposable vector is 1.
The following result about the rank of a vector is useful.

**Proposition 2.1** The expression \( w = \sum_{i,j} u_i \otimes v_j \) is a shortest one if and only if \( \{u_i\} \) and \( \{v_j\} \) are linearly independent in \( V_1 \) and \( V_2 \) respectively.

Finally we consider \( \text{End}(W) \), the set of endmorphisms of \( W \). Take \( a \in \text{End}(V_1) \) and \( b \in \text{End}(V_2) \). Then we can define a bilinear map \( g \) from \( (V_1, V_2) \) to \( W \) such that \( g(u, v) = au \otimes bv \forall u \in V_1, v \in V_2 \). So there is a unique endmorphism \( h \) of \( W \) such that \( h(u \otimes v) = au \otimes bv \). By convention such an \( h \) will be denoted by \( a \otimes b \) from now on. Thus we have \( (a \otimes b)(u \otimes v) = au \otimes bv \). Denote by \( S \) the linear subspace of \( \text{End}(W) \) spanned by the endmorphisms of the form \( a \otimes b \). Then \( (\text{End}(V_1), \text{End}(V_2), \otimes) \) is a TPS of \( S \). Here, \( \otimes \) stands for the bilinear map satisfying \( \otimes(a, b) = a \otimes b \) as the symbol itself suggests. In the finite dimensional case we have \( S = \text{End}(W) \), so \( \text{End}(W) = \text{End}(V_1) \otimes \text{End}(V_2) \). But when \( W \) is of infinite dimension, this is no longer true. We will investigate this problem in more detail in the next sections.

### III. Tensor Product Partition

In this section and the next one, we introduce the concept of TPP for the endmorphisms of the finite or infinite dimensional vector space \( W \). In this section, \( A \) always stands for \( \text{End}(W) \). The concept of TPP is at the core of this paper. It turns out to be useful in understanding relativity of quantum entanglement.

**Definition 3.1** (a) For \( a_i \in A \), the summation \( \sum_i a_i \) is called well defined if \( (\sum_i a_i) w \) is a well defined vector of \( W \). (b) A subset \( B \) of \( A \) is called an extended subalgebra if it is a subalgebra in the usual sense and is closed under well defined summation.

Let \( A_1 \) and \( A_2 \) be two extended subalgebras of \( A \) such that \( [A_1, A_2] = 0 \), namely \( [a, b] = 0 \) for \( a \in A_1 \) and \( b \in A_2 \). We denote by \( A_1 \vee A_2 \) the associative algebra generated by \( A_1 \) and \( A_2 \). By definition \( A_1 \vee A_2 \subseteq A \) and an element \( c \) of \( A \) belongs to \( A_1 \vee A_2 \) if and only if \( c \) is of the well defined summation form \( \sum_i a_i b_i \) where \( a_i \in A_1 \) and \( b_i \in A_2 \). Notice that the sum \( \sum_i a_i b_i \) may contain infinitely many terms but \( (\sum_i a_i b_i) w \) contains only finite terms for each \( w \in W \).

**Definition 3.2** The ordered pair \( (A_1, A_2) \) is called a pre-tensor-product partition of \( A \) if the following two conditions are satisfied: (1) \( [A_1, A_2] = 0 \) and \( A = A_1 \vee A_2 \); (2) \( W \) is a decomposable \( A_1 \) and \( A_2 \) modules respectively.

For arbitrary extended subalgebras \( A_1 \) and \( A_2 \) of \( A \), \( W \) becomes \( A_1 \) and \( A_2 \) modules in the natural way. We find that when \( (A_1, A_2) \) is a pre-tensor-product partition of \( A \) the modules enjoy a very nice property.
Lemma 3.1 If \((A_1, A_2)\) is a pre-tensor-product partition of \(A\), then all irreducible \(A_1\) \(A_2\) submodules of \(W\) are isomorphic.

Proof. Since \(W\) is a decomposable \(A_1\) module, we have the decomposition of \(W\) into a direct sum of irreducible \(A_1\) modules:

\[ W = \sum_i \oplus M_i. \]

It follows that each irreducible \(A_1\) submodule is isomorphic to some \(M_i\). We need to show that all \(M_i\)'s are isomorphic to one another. For different indices \(i, j\), we choose \(c \in A\) such that \(cM_i \subseteq M_j\) and \(c|_{M_i}\), the restriction of \(c\) to \(M_i\), is nonzero. As \(A = A_1 \lor A_2\) we can write \(c = \sum_k a_kb_k\) where \(a_k \in A_1\) and \(b_k \in A_2\). Denote by \(p_l\) the projection onto \(M_l\). Obviously, \(p_l\) is an \(A_1\) module homomorphism and we have \(\sum_l p_l = 1\), where 1 stands for the identity map. Now \(c\) can be rewritten as

\[ c = \sum_{k,l} a_kb_k. \]

Notice that \(p_l b_k M_i \subseteq M_l\) and \(a_k p_l b_k M_i \subseteq M_l\) for each \(l\). So it follows from \(cM_i \subseteq M_j\) that

\[ c|_{M_i} = \sum_k a_k p_j b_k. \]

But \(c|_{M_i} \neq 0\), thus there exists a \(k\) such that \(p_j b_k \neq 0\). Finally \(b_k \in A_2\) implies that \(b_k\) is an \(A_1\) module homomorphism. Therefore \(p_j b_k\) is a nonzero \(A_1\) module homomorphism from \(M_i\) to \(M_j\), and \(M_i\) and \(M_j\) are isomorphic according to Theorem 2.3. This proves the lemma for \(A_1\). The parallel result for \(A_2\) can be proved in the same way.

\[ \square \]

For a pre-tensor-product partition \((A_1, A_2)\), by definition we have the decompositions

\[ W = \sum_i \oplus M_i = \sum_j \oplus N_j, \]

where \(M_i\) and \(N_j\) are irreducible \(A_1\) and \(A_2\) modules respectively. According to Lemma 2.1, all \(M_i\)'s and all \(N_j\)'s are isomorphic. This allows us to denote them by \(M\) and \(N\) respectively. For convenience, \(M, N\) will be called characteristic modules, and \(\{M_i\}, \{N_j\}\) irreducible component sets, of the partition \((A_1, A_2)\).

Definition 3.3 Let \((A_1, A_2)\) be a pre-tensor-product partition of \(A\). It is called a TPP if its characteristic modules are normal modules.

Remark In the finite dimensional case the concepts of pre-tensor-product partition and TPP are equivalent. But in the infinite dimensional cases a pre-tensor-product partition of \(A\) may not be a TPP of \(A\). This is because Schur’s Lemma may be false in the infinite dimensional case.
Lemma 3.2 If \((A_1, A_2)\) is a TPP, then \(A_1|_{M_i} = \text{End}(M_i)\) and \(A_2|_{N_j} = \text{End}(N_j)\), where \(\{M_i\}\) (\(\{N_j\}\)) is an irreducible component set of \(A_1\) (\(A_2\)).

Proof. The notation is the same as in the proof of Lemma 3.1, unless explicitly pointed out. We use the contradiction method. If \(A_1|_{M_i} \subsetneq \text{End}(M_i)\), then there exists an element \(c_i \in \text{End}(M_i)\) which cannot be written as \(a|_{M_i}\) with \(a \in A_1\). Take \(c \in A\) such that \(c|_{M_i} = c_i\). We can write
\[
c = \sum_k a_kb_k = \sum_{k,l} a_k p_l b_k.
\]
It then follows that
\[
c|_{M_i} = \sum_k a_k p_l b_k.
\]
On the other hand, \(p_l b_k|_{M_i}\) is an \(A_1\) module homomorphism from \(M_i\) to \(M_i\), so there is a constant \(\alpha_k\) such that \(p_l b_k|_{M_i} = \alpha_k \cdot 1\) because \(M_i\) is a normal module by assumption. Thus we can write
\[
c|_{M_i} = \left(\sum_k \alpha_k a_k\right)|_{M_i}.
\]
Now we define \(a = \sum_k \alpha_k a_k\). We claim that \(a \in A_1\). To prove this point, it suffices to show that \(a\) is well defined. In other words, we only need to show that \(aw\) contains only finite terms for each \(w \in W\). Note that \(a|_{M_i} = c|_{M_i}\) is well defined and \(W = \sum_i \oplus M_i\). The well definedness of \(a\) then follows directly from the fact that all \(M_i\)'s are isomorphic \(A_1\) modules. As \(a|_{M_i} = c_i\) we are led to a contradiction. This proves \(A_1|_{M_i} = \text{End}(M_i)\). The other conclusion can be proved in the same way.

\[
\square
\]

Let \((A_1, A_2)\) be a TPP, \(\{M_i\}\) the irreducible component set for \(A_1\). According to Lemma 3.1, for each \(i\) we can choose an ordered basis \(\{x_{ji}| j = 1, 2, \cdots\}\) of \(M_i\) such that \(A_1\) has the same matrix representation with respect to these bases. In other words, for each \(a \in A_1\) there exists a complex number set \(\{a_{kl}| k, l = 1, 2, \cdots\}\), which is independent of the index \(i\), such that
\[
a x_{ji} = \sum_k x_{ki} a_{kj}, \forall i.
\]
Obviously, \(\bigcup_i \{x_{ji}| j = 1, 2, \cdots\}\) is a basis of \(W\). We call such an ordered basis a synchronic basis with respect to the irreducible component set \(\{M_i\}\). Actually, \(\bigcup_i \{x_{ji}| j = 1, 2, \cdots\}\) is a synchronic basis with respect to the irreducible component set \(\{M_i\}\) if and only if the linear map \(f_i\) from \(M_1\) to \(M_i\) that sends \(x_{j1}\) to \(x_{ji}\) is a module isomorphism.
Now for each \( j \) define \( N_j \) to be the vector space spanned by \( \{x_{ji} | i = 1, 2, \cdots \} \). Then we have the following result.

**Lemma 3.3** (1) \( \{N_j\} \) is the irreducible component set for \( A_2 \); (2) \( \bigcup_j \{x_{ji} | i = 1, 2, \cdots \} \) is a synchronic basis with respect to \( \{N_j\} \).

**Proof.** (1) Take a set \( \{\lambda_j | j = 1, 2, \cdots \} \) consisting of distinct complex numbers and define \( r_i \in \text{End}(M_i) \) such that \( r_i x_{ji} = \lambda_j x_{ji} \). Then according to Lemma 3.2, there is \( r \in A_1 \) such that \( r|_{M_i} = r_i \).

As \( \bigcup_i \{x_{ji} | j = 1, 2, \cdots \} \) is a synchronic basis we have \( rx_{ji} = \lambda_j x_{ji}, \forall i \). It follows that \( N_j \) is none other than the eigenspace of \( r \) corresponding to the eigenvalue \( \lambda_j \). But \([r, A_2] = 0\) by definition, so \( N_j \) is stable under the action of \( A_2 \), that is, \( N_j \) is an \( A_2 \) module. Clearly we have

\[
W = \bigoplus_i M_i = \bigoplus_j N_j.
\]

Now it remains to show that \( N_j \) is an irreducible module.

If \( N_j \) is not irreducible, then by Theorem 1.1 it can be decomposed into a direct sum of at least two irreducible modules:

\[
N_j = \bigoplus_k N_{jk}.
\]

Consider \( c \in A = \text{End}(W) \) such that \( cN_{j1} \subseteq N_{j2} \) and \( c|_{N_{j1}} \neq 0 \). We claim that \( c \notin A_1 \lor A_2 \). This contradicts the condition \( A_1 \lor A_2 = \text{End}(W) \). Hence, it is sufficient to prove the claim.

Denote by \( p_j \) the projection to \( N_j \). We first prove that for each \( a \in A_1 \) and \( z \in N_j \) there exists a complex number \( \alpha \) such that \( p_j a z = \alpha z \). In fact, for \( z \in N_j \) we can write

\[
z = \sum_i \alpha_i x_{ji}, \alpha_i \in \mathbb{C}.
\]

Because \( \bigcup_i \{x_{ji} | j = 1, 2, \cdots \} \) is a synchronic basis with respect to \( \{M_i\} \), the action of \( a \) is of the form

\[
a x_{ji} = \sum_k x_{ki} a_{kj}, \forall i,
\]

where \( a_{kj} \) is independent of \( i \). Thus

\[
a z = \sum_i \sum_k \alpha_i x_{ki} a_{kj},
\]

and

\[
p_j a z = a_{jj} \sum_i \alpha_i x_{ji} = a_{jj} z.
\]
Now we prove \( c \notin A_1 \lor A_2 \). If \( c \in A_1 \lor A_2 \), then there exist \( a_k \in A_1, b_k \in A_2 \) such that

\[
c = \sum_k a_k b_k.
\]

By assumption \( cN_{j_1} \subseteq N_{j_2} \not\subseteq N_j \), so for \( z \in N_{j_1} \) we have

\[
cz = p_j cz = \sum_k p_j a_k b_k z.
\]

Define \( z_k = b_k z \). Obviously, \( z_k \in N_{j_1} \not\subseteq N_j \). Then by the above argument for each \( k \) there exists a complex number \( \alpha_k \) such that \( p_j a_k z_k = \alpha_k z_k \). It now follows that

\[
cz = \sum_k p_j a_k z_k = \sum_k \alpha_k z_k \in N_{j_1}.
\]

But \( cz \in N_{j_2} \) by definition, so \( cz = 0, \forall z \in N_{j_1} \). This is a contradiction. The proof for the first part of the lemma is thus completed.

(2) As \( N_j \) is an \( A_2 \) module, for \( b \in A_2 \) most generally the action on the basis element \( x_{ji} \) can be written as

\[
b x_{ji} = \sum_k x_{jk} b_{ki}^j,
\]

where \( b_{ki}^j \in \mathbb{C} \) depends on the index \( j \). Suppose that \( \bigcup_j \{ x_{ji} | i = 1, 2, \cdots \} \) is not a synchronic basis with respect to \( \{ N_j \} \). Then there exist an element \( b \in A_2 \) and indices \( j_1, j_2, k, i \) such that \( b_{k_1}^{j_1} \neq b_{k_2}^{j_2} \).

Since \( A_1 |_{M_i} = \text{End}(M_i) \), there exists an element \( a \in A_1 \) satisfying \( ax_{ji} = x_{j2i} \). It then follows that \( ax_{j1k} = x_{j2k}, \forall k \) because \( \bigcup_i \{ x_{ji} | j = 1, 2, \cdots \} \) is a synchronic basis with respect to \( \{ M_i \} \). Now consider \( ab \) and \( ba \). We have

\[
ab x_{ji} = \sum_k ax_{j1k} b_{k1}^{j1} = \sum_k x_{j2k} b_{k1}^{j1},
\]

\[
ba x_{ji} = bx_{j2i} = \sum_k x_{j2k} b_{k2}^{j2}.
\]

So we come to the conclusion that \( ab x_{ji} \neq ba x_{ji} \). But this is impossible since \( [A_1, A_2] = 0 \) by definition. The proof is thus completed.

\[\square\]

**Corollary 3.1.** Let \( (A_1, A_2) \) be a TPP of \( \text{End}(W) \), then there exist an irreducible component set \( \{ M_i \} \) for \( A_1 \), an irreducible component set \( \{ N_j \} \) for \( A_2 \), and a basis \( \{ x_{ji} \} \) such that \( \bigcup_i \{ x_{ji} | j = 1, 2, \cdots \} \) is a synchronic basis with respect to \( \{ M_i \} \) and \( \bigcup_j \{ x_{ji} | i = 1, 2, \cdots \} \) a synchronic basis with respect to \( \{ N_j \} \).
This corollary follows directly from the above lemma. A basis \( \{x_{ji}\} \) with the property specified in the corollary will be called a standard basis associated with the irreducible component sets \( \{M_i\} \) and \( \{N_j\} \).

**Corollary 3.2.** Let \((A_1, A_2)\) be a TPP of \(\text{End}(W)\), \(\{\lambda_j\}\) and \(\{\mu_i\}\) two sets of distinct complex numbers. Then there exist \(r \in A_1\), \(t \in A_2\), and a decompositions of \(W\) into direct sum of vector spaces:

\[
W = \bigoplus_i M_i = \bigoplus_j N_j
\]

such that

\[
\forall j, r|_{N_j} = \lambda_j \cdot 1, t|_{M_i} = \mu_i \cdot 1
\]

and the standard module decomposition

\[
M_i = \bigoplus_j M^\lambda_{ij}, \quad N_j = \bigoplus_i N^\mu_{ij}
\]

where the summations range over all \(\lambda_j\)'s and all \(\mu_i\)'s respectively, and \(M^\lambda_{ij}\) and \(N^\mu_{ij}\) are one dimensional. Here \(M^\lambda_{ij}\) stands for the eigenspace of \(r\) in \(M_i\) corresponding to the eigenvalue \(\lambda_j\) and \(N^\mu_{ij}\) the eigenspace of \(t\) in \(N_j\) corresponding to the eigenvalue \(\mu_i\).

**Proof.** According to Corollary 3.1, there exist an irreducible component set \(\{M_i\}\) for \(A_1\), an irreducible component set \(\{N_j\}\) for \(A_2\), and a basis \(\{x_{ji}\}\) such that \(\bigcup_i \{x_{ji}|j = 1, 2, \cdots\}\) is a synchronic basis with respect to \(\{M_i\}\) and \(\bigcup_j \{x_{ji}|i = 1, 2, \cdots\}\) a synchronic basis with respect to \(\{N_j\}\). Define \(r, t \in A\) such that

\[
rx_{ji} = \lambda_j x_{ji}, \quad tx_{ji} = \mu_i x_{ji}.
\]

It is easy to verify that \(\{M_i\}, \{N_j\}, r, t\) meet the requirement of Corollary 3.2.

**IV. STANDARD COMPLETE SET OF OPERATORS**

In the last section, some fine properties for TPP have been proved to prepare for the introduction of the TPS. In this section we proceed along to probe the underlying structure of TPP, aiming at describing the TPS of a finite or infinite dimensional vector space \(W\) in a constructive way. We will show how the TPP of \(\text{End}(W)\) is determined by particular sets of operators, the so called standard complete sets of operators, contained in \(\text{End}(W)\).
Definition 4.1 If \( r, t \in A \) satisfy the conditions specified in Corollary 3.2, then \((r, t)\) is called a standard complete set of operators of \( A \), and \( \{M_i\}, \{N_j\} \) are called characteristic sets of \( r \) and \( t \) respectively.

Definition 4.2 If \((r, t)\) is a standard complete set of operators of \( A \) and \((A_1, A_2)\) is a TPP of \( A \) such that \( r \in A_1 \) and \( t \in A_2 \), then \((A_1, A_2)\) is called a TPP containing \((r, t)\).

It is obvious that, if \((r, t)\) is a standard complete set of operators, then necessarily \([r, t] = 0\) according to the above definitions.

We have seen that a TPP contains standard complete sets of operators. Now it is natural to ask how to determine a TPP from a standard complete set of operators. The remaining part of this section is devoted to this problem.

Proposition 4.1 If \((r, t)\) is a standard complete set of operators of \( A \), then there exists a TPP \((A_1, A_2)\) containing \((r, t)\).

Proof. By the definition of complete set of operators, there are two sets of distinct complex numbers \( \{\lambda_j\} \) and \( \{\mu_i\} \) and two decompositions of \( W \) into direct sum of subspaces

\[
W = \bigoplus_i M_i = \bigoplus_j N_j
\]

such that

\[
M_i = \bigoplus_j \mathbb{C}x_{ji}, \quad N_j = \bigoplus_i \mathbb{C}x_{ji}
\]

where \( x_{ji} \) is the common eigenvector of \( r \) and \( t \):

\[
rx_{ji} = \lambda_j x_{ji}, \quad tx_{ji} = \mu_i x_{ji}.
\]

It then follows that we can define two extended subalgebras \( A_1, A_2 \subseteq A \) such that the following two conditions are satisfied: (1) \( A_1|_{M_i} = \text{End}(M_i) \) and \( A_2|_{N_j} = \text{End}(N_j) \); (2) \( \bigcup_i \{x_{ji}: j = 1, 2, \ldots\} \) becomes a synchronic basis with respect to \( \{M_i\} \) and \( \bigcup_j \{x_{ji}: i = 1, 2, \ldots\} \) a synchronic basis with respect to \( \{N_j\} \). We claim that \((A_1, A_2)\) is a TPP and \( r \in A_1, t \in A_2 \). The claim is almost immediate from the definition. In fact, the first condition guarantees that \( W \) are decomposable \( A_1 \) and \( A_2 \) modules, and \( M_i, N_j \) are respectively irreducible normal \( A_1 \) and \( A_2 \) modules, while the second condition leads to the commutation relation \([A_1, A_2] = 0\). The fact that \( r \in A_1 \) and \( t \in A_2 \) is also a direct consequence of the two conditions. Now it remains to show that \( A_1 \lor A_2 = A \). This point is proved as follows.

Let \( a \in A \) be an arbitrary element. It suffices to prove that \( a \in A_1 \lor A_2 \). Define \( a_{k,l}, b_{k,l} \in A \) such that

\[
a_{k,l}x_{ji} = \delta_{kj}x_{li}, \quad b_{k,l}x_{ji} = \delta_{ki}x_{jl}.
\]
It is readily verified that \( \{a_{k,l}\} \) and \( \{b_{k,l}\} \) are bases of \( A_1 \) and \( A_2 \) respectively. As \( \{x_{ji}\} \) is a basis of \( W \), \( a \) is determined by its action on each \( x_{ji} \). Generally we can write

\[
a x_{ji} = \sum_{k,l} x_{kl} a_{kl,ji}, \quad a_{kl,ji} \in \mathbb{C}.
\]

Notice that for each \( x_{ji} \), there are only finite nonzero coefficients \( a_{kl,ji} \). One can now easily convince oneself that the expression \( \sum_j \sum_l a_{kl,ji} x_{ji} \) is a well defined summation and equal to the given \( a \). Obviously this is an element of \( A_1 \lor A_2 \). The proposition is thus proved.

\[\square\]

Notice that Proposition 4.1 solves the problem of existence of TPP containing a given complete set of operators. To probe the problem of uniqueness in some sense, we need to make some more preparation. The next proposition is also interesting in its own right.

**Proposition 4.2** If \( (A_1, A_2) \) is a TPP of \( A \), then \( A'_1 = A_2, A'_2 = A_1 \). Here \( A'_i \) \((i = 1, 2)\) stands for the commutator of \( A_i \) in \( A \): \( A'_i = \{a \in A | [a, A_i] = 0\} \).

**Proof.** Let \( \{M_i\} \) and \( \{N_j\} \) be irreducible component sets of \( (A_1, A_2) \), \( \{x_{ji}\} \) a synchronic basis associated with \( \{M_i\} \) and \( \{N_j\} \). Define \( a_{k,l}, b_{k,l} \in A \) in the same way as in the proof of Proposition 4.1. Now we prove the proposition in three steps as follows.

1. If \( \sum_l a_{l,k,l} b_{l,k,l} = 0 \), then \( \sum_l a_{l,k,l} b_{l,k,l} = 0 \) for each \( l \), where \( a_{l,k,l} \in A_1 \). In fact, we have

\[
\left( \sum_l a_{l,k,l} b_{l,k,l} \right) W \subseteq M_i, \forall l,
\]

so the conclusion directly follows from the decomposition \( W = \sum_i \oplus M_i \).

2. If \( \sum_k a_{k,l} b_{k,l} = 0 \), then \( a_{k,l} = 0 \) for each \( k \), where \( a_{k,l} \in A_1 \). If, on the contrary, there is some \( a_i \neq 0 \), then \( a_i \) can be written as

\[
a_i = \sum_{k,l} \alpha_{k,l} a_{k,l},
\]

where there is at least a nonzero coefficient. Suppose that \( \alpha_{m,n} \neq 0 \). Then it is readily check that \( (\sum_k a_{k,l} b_{k,l}) x_{m,l} \neq 0 \). This contradicts the condition \( \sum_k a_{k,l} b_{k,l} = 0 \). The statement is thus proved.

3. \( A_2 = A'_1, A_1 = A'_2 \). By definition \( A_2 \subseteq A'_1 \). So to prove \( A_2 = A'_1 \), we only need to show that \( A'_1 \subseteq A_2 \). Let \( a \in A'_1 \subseteq A \). As \( A_1 \lor A_2 = A \) we can express \( a \) in the form

\[
a = \sum_l \sum_k a_{l,k} b_{l,k,l}.
\]
where $a(l_k,l) \in A_1$. To prove that $a \in A_2$ it is sufficient to show that $a(l_k,l)$ is equal to the identity map up to a scalar multiple. In fact, if this is not the case, then there exist an $a(l_k,l)$ and some $c \in A_1$ such that $[c, a(l_k,l)] \neq 0$ because $A_1|_{M_i} = \text{End}(M_i)$ for every $i$. On the other hand, we have

$$0 = [c, a] = \sum_l \sum_{k} [c, a(l_k,l)] b_{l_k,l}.$$ 

It then follows from (1) and (2) that $[c, a(l_k,l)] = 0$, for every $l$ and $k$. This contradiction proves that $A_2 = A'_2$. Similarly, we can prove that $A_1 = A'_2$.

\[\square\]

**Corollary 4.1** If $(A_1, A_2)$ is a TPP of $A$, then $A_1 \cap A'_1 = A_2 \cap A'_2 = C1$.

**Proof.** First we notice that it is a direct consequence of Proposition 3.2 that $A_1 \cap A'_1 = A_2 \cap A'_2$. Let $\{M_i\}$ and $\{N_j\}$ be irreducible component sets for $A_1$ and $A_2$ respectively, and $\{x_{ji}\}$ a synchronic basis associated with them. By definition

$$M_i = \sum_j \oplus \mathbb{C} x_{ji}, \quad N_j = \sum_i \oplus \mathbb{C} x_{ji}.$$ 

As $A_1|_{M_i} = \text{End}(M_i)$ and $A_2|_{N_j} = \text{End}(N_j)$ it is clear that $C1 \subseteq A_1 \cap A_2 = A_1 \cap A'_1$. For the same reason, if $a \in A_1 \cap A'_1 = A_2 \cap A'_2$, then there exist constant sets $\{\alpha_i\}, \{\beta_j\} \subseteq \mathbb{C}$ such that $a|_{M_i} = \alpha_i \cdot 1$, $a|_{N_j} = \beta_j \cdot 1$. It then follows that all these constants are identical. Thus $a \in C1$, that is, $A_1 \cap A'_1 = A_2 \cap A'_2 \subseteq C1$. This completes the proof.

\[\square\]

**Remark** In the finite dimensional case, from this corollary we conclude that if $(A_1, A_2)$ is a TPP, then both $A_1$ and $A_2$ are the so called factors.

**Lemma 4.1** Let $W$ be a vector space, $q$ a linear transformation of $W$. If $q$ is diagonalizable and all of its eigenvalues are distinct, then $q$ is diagonalizable in any $q$ invariant subspace of $W$.

Before proving this lemma we remark that the conclusion is obvious if $W$ is finite dimensional, but if this is not the case the lemma seems to need a proof. Certainly, we present the lemma and its proof here not to claim the originality. Rather, we do so just for completeness.

**Proof.** Let $\{\lambda_j\}$ be the set of eigenvalues of $q$. Then we have the decomposition

$$W = \sum_j \mathbb{C} x_j,$$ 

where $x_j$ is an eigenvector of $q$ corresponding to the eigenvalue $\lambda_j$: $q x_j = \lambda_j x_j$. Suppose that $W_1 \subseteq W$ is a $q$ invariant subspace, namely, $qW_1 \subseteq W_1$. For an arbitrary $y \in W_1$, we can write

$$y = \sum_j \alpha_j(y) x_j, \quad \alpha_j(y) \in \mathbb{C}.$$
If there exists \( y \in W_1 \) such that \( \alpha_j(y) \neq 0 \), then we call \( \lambda_j \) an eigenvalue related to the subspace \( W_1 \). We claim that

\[
W_1 = \sum_{\lambda_j} \oplus W^{\lambda_j}
\]

where \( W^{\lambda_j} = \mathbb{C}x_j \) and the summation ranges over all the eigenvalues that are related to \( W_1 \). Clearly we have

\[
W_1 \subseteq \sum_{\lambda_j} \oplus W^{\lambda_j}.
\]

So to prove the claim it is sufficient to show that for each eigenvalue \( \lambda_j \) that is related to \( W_1 \) we have \( x_j \in W_1 \). In fact, if \( \lambda_{j_0} \) is related to \( W_1 \), then there exists \( y \in W_1 \) such that

\[
y = \sum_{j \in I} \alpha_j(y) x_j
\]

where \( I \) is a finite set containing \( j_0 \) and \( \alpha_j(y) \neq 0, \forall j \in I \). Suppose that \( I \) contains \( n \) elements. Then we have the following system of linear equations:

\[
q^i y = \sum_{j \in I} \alpha_j(y) \lambda_j^i x_j, \ i = 1, 2, \ldots, n.
\]

As \( qW_1 \subseteq W_1 \), we have \( q^i y \in W_1, \forall i \in I \). On the other hand, the determinant of the coefficient matrix is nonzero since all the \( \lambda_j \)'s are distinct. Therefore, we have \( x_j \in W_1, \forall j \in I \), especially, \( x_{j_0} \in W_1 \). This proves the claim, and hence the lemma.

\[\Box\]

**Lemma 4.2** Let \((r, t)\) be a standard complete set of operators of \( A (= \text{End}(W)) \), \( \{M_i\} \), \( \{N_j\} \) the characteristic sets of \( r \) and \( t \) respectively. If \((A_1, A_2)\) is a TPP containing \((r, t)\), then \( \{M_i\} \), \( \{N_j\} \) are irreducible component sets for \( A_1 \) and \( A_2 \) respectively.

**Proof.** By definition we have

\[
W = \sum_i \oplus M_i = \sum_j \oplus N_j,
\]

and

\[
r|_{N_j} = \lambda_j \cdot 1, \quad t|_{M_i} = \mu_i \cdot 1.
\]

Let us focus on \( \{M_i\} \). Notice that \( M_i \) is none other than the eigenspace of \( t \) corresponding to the eigenvalue \( \mu_i \). As \( t \in A_2 \), we have \([t, A_1] = 0\). It then follows that \( A_1 M_i \subseteq M_i \), that is, \( M_i \) is an \( A_1 \)
module. We observe that proving that \( \{ M_i \} \) is a irreducible component set for \( A_1 \) boils down to proving that \( M_i \) is irreducible as \( A_1 \) module. The proof is as follows.

Suppose that \( M_i \) is not irreducible. Then \( M_i \) can be decomposed into a direct sum of nonzero irreducible modules:

\[
M_i = \sum_k \oplus M_{ik}.
\]

By Lemma 4.1 all \( M_{ik} \)'s are isomorphic. On the other hand, according to Lemma 4.1, \( r \) is diagonalizable in each \( M_{ik} \). Note that \( r \in A_1 \). Thus \( r \) has the same eigenvalues in different \( M_{ik} \)'s. But this is impossible because by definition all the eigenvalues of \( r \) in \( M_i \) have the multiplicity 1. In the same way we can prove that \( N_j \) is a irreducible \( A_2 \) module.

□

Now we are prepared to prove the following result concerning the uniqueness of TPP containing a given complete set of operators.

**Proposition 4.3** Let \((r,t)\) be a standard complete set of operators of \( A (= \text{End}(W)) \), \((A_1, A_2)\) and \((B_1, B_2)\) two tensor product partitions containing \((r,t)\). Then there exists an isomorphism \( \varphi \in \text{End}(W) \), diagonal with respect to the basis consisting of common eigenvectors of \( r \) and \( t \), such that \( B_1 = \varphi \cdot A_1 \cdot \varphi^{-1} \) and \( B_2 = \varphi \cdot A_2 \cdot \varphi^{-1} \).

**Proof.** Keep the same notation as in the proof of Lemma 4.2. According to Lemma 4.2, \( \{ M_i \} \) is a irreducible component set for both \( A_1 \) and \( B_1 \). Fix an index \( i_0 \) and choose a basis \( \{ x_{ji} \} \) of \( M_{i_0} \) such that \( rx_{ji} = \lambda_j x_{ji} \). Obviously we can extend this basis to a synchronic basis \( \bigcup_i \{ x_{ji} | j = 1, 2, \cdots \} \) with respect to \( \{ M_i \} \) as irreducible component set for \( A_1 \) and a synchronic basis \( \bigcup_i \{ y_{ji} | j = 1, 2, \cdots \} \) with respect to \( \{ M_i \} \) as irreducible component set for \( B_1 \). Since \( r \in A_1 \), \( B_1 \), we have

\[
rx_{ji} = \lambda_j x_{ji}, \quad ry_{ji} = \lambda_j y_{ji}, \quad \forall i.
\]

It then follows that for each pair of index \((i,j)\) there exists a complex number \( \alpha_{ji} \) such that \( y_{ji} = \alpha_{ji} x_{ji} \). This is because that all the eigenvalues of \( r \) in \( M_i \) are of multiplicity 1. Now define \( \varphi \in \text{End}(W) \) such that \( \varphi x_{ji} = y_{ji} \). Then \( \varphi \) is an isomorphism diagonal with respect to the basis \( \{ x_{ji} \} \). It is clear that \( B_1 = \varphi \cdot A_1 \cdot \varphi^{-1} \). Indeed, this relation follows directly from the fact that \( A_1 |_{M_i} = B_1 |_{M_i} = \text{End}(M_i) \). Finally, we consider the set \( \varphi \cdot A_2 \cdot \varphi^{-1} \). We have \([ B_1, \varphi \cdot A_2 \cdot \varphi^{-1} ] = 0\), so by Proposition 4.2 \( \varphi \cdot A_2 \cdot \varphi^{-1} \subseteq B_2 \). Similarly, we can prove \( \varphi^{-1} \cdot B_2 \cdot \varphi \subseteq A_2 \). Thus \( B_2 = \varphi \cdot A_2 \cdot \varphi^{-1} \).

This completes the proof.
We have seen that a TPP is determined up to an isomorphism by a standard complete set of operators contained in it. Now, in the remaining part of this section, we study how to determine a TPP completely by some standard complete sets of operators satisfying certain conditions. For convenience, we first introduce a new concept as follows. For \( p, q \in \text{End}(W) \), we denote by \( S_{p,q} \) the extended subalgebra of \( \text{End}(W) \) generated by them. Let \((r, t), (r', t')\) be standard complete sets of operators with the characteristic sets \( \{M_i\}, \{N_j\}\) and \( \{M'_i\}, \{N'_j\}\) respectively.

**Definition 4.3** \((r, t)\) and \((r', t')\) are called complementary if (1) \( M_i = M'_i \) and all \( M_i \)'s are isomorphic normal \( S_{r, \tilde{r}} \) modules or (2) \( N_j = N'_j \) and all \( N_j \)'s are isomorphic normal \( S_{t, \tilde{t}} \) modules.

**Remark** Both (1) and (2) cannot be satisfied unless \( M_i \) and \( N_j \) are both of one dimension.

Next we prove the following results on the construction of TPP.

**Proposition 4.4** A TPP contains complementary standard complete sets of operators.

**Proof.** Let \((A_1, A_2)\) be a TPP with the irreducible component sets \( \{M_i\}, \{N_j\}\). Take a synchronic basis \( \{x_{ji}\} \) associated with \( \{M_i\}, \{N_j\}\). Then there exists a standard complete set of operators \((r, t)\) with the characteristic sets \( \{M_i\}, \{N_j\}\): \( rx_{ji} = \lambda_j x_{ji}, tx_{ji} = \mu_i x_{ji} \). Now define \( \tilde{r}, \tilde{t} \in \text{End}(W) \) such that:

\[
\tilde{t} = t, \\
\tilde{r} x_{1i} = \lambda_1 x_{1i}, \quad \tilde{r} (x_{ji} + x_{j+1} i) = \lambda_{j+1} (x_{ji} + x_{j+1} i).
\]

It is readily check that \((\tilde{r}, \tilde{t})\) is a standard complete set of operators contained in \((A_1, A_2)\). Obviously all \( M_i \)'s are isomorphic \( S_{r, \tilde{r}} \) modules by definition. Now to prove the proposition it suffices to show that \( M_i \) is a normal \( S_{r, \tilde{r}} \) module. Let \( f : M_i \to M_i \) be an \( S_{r, \tilde{r}} \) module homomorphism. Then we have

\[
rf(x_{ji}) = \lambda_j f(x_{ji}), \\
\tilde{r} f(x_{1i}) = \lambda_1 f(x_{1i}), \tilde{r} f(x_{ji} + x_{j+1} i) = \lambda_{j+1} f(x_{ji} + x_{j+1} i).
\]

It follows that there are \( \alpha_j, \beta_j \in \mathbb{C} \) such that

\[
f(x_{ji}) = \alpha_j x_{ji}, \quad f(x_{ji} + x_{j+1} i) = \beta_{j+1} (x_{ji} + x_{j+1} i).
\]

We thus conclude that all \( \alpha_j \)'s must be identical, that is, \( f = \alpha \cdot 1 \) for some \( \alpha \in \mathbb{C} \). Hence, \( M_i \) is a normal \( S_{r, \tilde{r}} \) module.

\[\square\]
Proposition 4.5 If \((r, t)\) and \((\tilde{r}, \tilde{t})\) are complementary standard complete sets of operators, then there exists a unique TPP \((A_1, A_2)\) such that \(r, \tilde{r} \in A_1\) and \(t, \tilde{t} \in A_2\).

Proof. Let us consider the case where \(r, \tilde{r}\) have the same characteristic set \(\{M_i\}\) and all \(M_i\)'s are isomorphic normal \(S_{r, \tilde{r}}\) modules. The other case can be discussed in the same way.

Let \(f_i : M_1 \rightarrow M_i\) be an \(S_{r, \tilde{r}}\) module isomorphism. By the definition of standard complete set of operators, there exist bases \(\{x_{j1}\}, \{\tilde{x}_{j1}\}\) of \(M_1\) and sets \(\{\lambda_j\}, \{\tilde{\lambda}_j\}\) of distinct complex numbers such that

\[
x_{j1} = \lambda_j x_{j1}, \quad \tilde{r}\tilde{x}_{j1} = \tilde{\lambda}_j \tilde{x}_{j1}.
\]

Let \(x_{ji} = f_i x_{j1}, \tilde{x}_{ji} = f_i \tilde{x}_{j1}\). Then according to the proof of Proposition 3.1, there are TPP's \((A_1, A_2)\) and \((\tilde{A}_1, \tilde{A}_2)\) such that (1) \(r \in A_1, \tilde{r} \in \tilde{A}_1, t \in A_2, \tilde{t} \in \tilde{A}_2\); (2) \(\cup_i \{x_{ji}| j = 1, 2, \cdots \}\) and \(\cup_i \{\tilde{x}_{ji}| j = 1, 2, \cdots \}\) are synchronic bases with respect to \(\{M_i\}\) as irreducible component sets for \(A_1\) and \(\tilde{A}_1\) respectively. Since \(f_i\) is an \(S_{r, \tilde{r}}\) module isomorphism, \(\cup_i \{\tilde{x}_{ji}| j = 1, 2, \cdots \}\) is also a synchronic basis with respect to \(\{M_i\}\) as irreducible component sets for \(A_1\). It then follows that \(A_1 = \tilde{A}_1\) and hence that \(A_2 = \tilde{A}_2\) as \(A_2 = A_1'\) and \(\tilde{A}_2 = \tilde{A}_1'\). Thus \((A_1, A_2)\) is a TPP meeting the requirement. This proves the existence.

Now let \((B_1, B_2)\) be an arbitrary TPP satisfying the condition. To prove the uniqueness we only need to show that \((B_1, B_2) = (A_1, A_2)\), which is defined above. According to Lemma 4.2, \(\{M_i\}\) is an irreducible component set for \(B_1\). Then there exists a synchronic basis \(\cup_i \{y_{ji}| j = 1, 2, \cdots \}\) with respect to \(\{M_i\}\) such that \(y_{j1} = x_{j1}, \quad j = 1, 2, \cdots \). As \(r, \tilde{r} \in B_1\) the linear map \(g_i : M_1 \rightarrow M_i\) that sends \(y_{j1}\) to \(y_{ji}\) is an \(S_{r, \tilde{r}}\) module isomorphism. But \(M_1\) is a normal \(S_{r, \tilde{r}}\) module, so there exists \(\alpha_i \in \mathbb{C}\) such that \(f_i^{-1} \cdot g_i = \alpha_i \cdot 1\), and we have \(y_{ji} = \alpha_i x_{ji}\). It then follows that \(\cup_i \{x_{ji}| j = 1, 2, \cdots \}\) is also a synchronic basis with respect to \(\{M_i\}\) as irreducible component set for \(B_1\). Consequently, we have \(A_1 = B_1\) and hence \(A_2 = B_2\). The uniqueness is thus proved.

\[\qed\]

V. TENSOR PRODUCT STRUCTURE

With the above preparation in concepts we are now in a position to focus on the TPS of a vector space \(W\), one of the mainstay of this paper. In this section we will establish a correspondence between the set of TPS of \(W\) and the set of TPP of \(End(W)\), revealing the close relation between these two objects. In this section we denote \(End(W)\) by \(A\).

Definition 5.1 Let \((W_1, W_2, \otimes)\) be a TPS of \(W\), \((A_1, A_2)\) a TPP of \(A (= End(W))\), where
$W_1, W_2$ are subspaces of $W$. $(W_1, W_2, \otimes)$ is called a TPS associated with $(A_1, A_2)$ if the following condition is satisfied:

$$a(u \otimes v) = (au) \otimes v, \ b(u \otimes v) = u \otimes (bv),$$

$$\forall a \in A_1, b \in A_2, u \in W_1, v \in W_2.$$

According to this definition, if $(W_1, W_2, \otimes)$ is a TPS associated with $(A_1, A_2)$, then $W_1$ and $W_2$ are necessarily $A_1$ and $A_2$ modules respectively. Furthermore, we have the following result.

**Lemma 5.1** if $(W_1, W_2, \otimes)$ is a TPS associated with $(A_1, A_2)$, then $W_1$ and $W_2$ are irreducible $A_1$ and $A_2$ modules respectively.

*Proof.* Suppose, on the contrary, that $W_1$ is not irreducible. Then there exist nonzero $A_1$ modules $W_1^\alpha$ and $W_1^\beta$ such that $W_1 = W_1^\alpha \oplus W_1^\beta$. Take two nonzero elements $x_1 \in W_1^\alpha$, $x_2 \in W_1^\beta$ and an element $a \in A$ such that $ax_1 = x_2$. It is clear that $a \notin A_1 \lor A_2$. This contradicts the condition that $A_1 \lor A_2 = A$. That $W_2$ is irreducible can be proved in the same way.

$\square$

**Proposition 5.1** Let $(W_1, W_2, \otimes)$ be a TPS of $W$, where $W_1, W_2$ are subspaces of $W$. Define $A_1 = \text{End}(W_1) \otimes 1 \triangleright \{a \otimes 1 | a \in \text{End}(W_1)\}$ and $A_2 = 1 \otimes \text{End}(W_2) \triangleright \{1 \otimes b | b \in \text{End}(W_2)\}$. Then $(A_1, A_2)$ is a TPP of $A$ and $(W_1, W_2, \otimes)$ is a TPS associated with $(A_1, A_2)$. Conversely, if $(A_1, A_2)$ is a TPP of $A$ and $(W_1, W_2, \otimes)$ a TPS associated with it, then we have $A_1 = \{a \otimes 1 | a \in \text{End}(W_1)\}$ and $A_2 = \{1 \otimes b | b \in \text{End}(W_2)\}$.

*Proof.* The proof of the first part is immediate, and we would rather omit it. For the second part, just notice that if $\{x_j\}$ and $\{y_i\}$ are respective bases of $W_1$ and $W_2$, then $\{W_1 \otimes y_i\}$, $\{x_j \otimes W_2\}$ are irreducible component sets for $A_1$ and $A_2$ respectively and $\{x_j \otimes y_i\}$ is a standard basis associated with them. The conclusion then follows. Here $W_1 \otimes y_i = \{u \otimes y_i | u \in W_1\}$ as the symbol suggests, and $x_j \otimes W_2$ is understood similarly.

$\square$

This proposition tells us that each TPS of the form $(W_1, W_2, \otimes)$ with $W_1, W_2 \subseteq W$ is associated with some TPP determined by it. Naturally we want to ask whether a TPP can determine a TPS associated with it. The answer is positive.

**Theorem 5.1** Each TPP of $A$ determines a TPS associated with it.

*Proof.* Let $(A_1, A_2)$ be a TPP of $A$. Then there are irreducible component sets $\{M_i\}$ and $\{N_j\}$ for $A_1$ and $A_2$ respectively. By Corollary 3.1 to Lemma 3.3 we can choose a synchronic basis $\{x_{ji}\}$ associated with $\{M_i\}$ and $\{N_j\}$. Now fix a pair of index $(i_0, j_0)$ and take $W_1 = M_{i_0}$, $W_2 =$
By definition \( \{ x_{ji0} | j = 1, 2, \cdots \} \) and \( \{ x_{joi} | i = 1, 2, \cdots \} \) are bases of \( W_1 \) and \( W_2 \) respectively. Thus we can define a bilinear map \( \otimes \) from \( W_1 \times W_2 \) to \( W \) such that \( x_{ji0} \otimes x_{joi} = x_{ji} \). We claim that \( (W_1, W_2, \otimes) \) is a TPS of \( W \) associated with \( (A_1, A_2) \). As \( \{ x_{ji0} \otimes x_{joi} \} = \{ x_{ji} \} \) is a basis of \( W \), \( (W_1, W_2, \otimes) \) is obviously a TPS of \( W \). According to the definition, to prove that it is a TPS associated with \( (A_1, A_2) \) we need to show that

\[
a(x_{ji0} \otimes x_{joi}) = (ax_{ji0}) \otimes x_{joi}, \quad \forall a \in A_1, \\
b(x_{ji0} \otimes x_{joi}) = x_{ji0} \otimes (bx_{joi}), \quad \forall b \in A_2.
\]

In fact, if

\[
a x_{ji0} = \sum k x_{ki0} a_{kj}, \quad a_{kj} \in \mathbb{C},
\]

then

\[
a x_{ji} = \sum k x_{ki} a_{kj}, \quad \forall i
\]

since \( \{ x_{ji} \} \) is a standard basis. It then follows that

\[
a(x_{ji0} \otimes x_{joi}) = ax_{ji} = \sum k x_{ki} a_{kj}
\]

and

\[
(ax_{ji0}) \otimes x_{joi} = (\sum k x_{ki0} a_{kj}) \otimes x_{joi} = \sum k x_{ki} a_{kj}.
\]

This proves that \( a(x_{ji0} \otimes x_{joi}) = (ax_{ji0}) \otimes x_{joi} \). The other equation can be proved in the same way.

We observe that in the finite dimensional case, if \( (A_1, A_2) \) is a TPP of \( A \), then we have \( A = A_1 \otimes A_2 \) as a direct consequence of the above theorem. This justifies calling \( (A_1, A_2) \) a TPP of \( A \).

\[\square\]

Remark Theorem 4.1, together with the second half of Proposition 4.1, provides a simple proof for Proposition 3.2.

Now we consider to what extent a given TPP determines the TPS associated with it.

**Definition 5.2.** Two tensor product structures \( (U_1, U_2, \otimes_1) \) and \( (W_1, W_2, \otimes_2) \) of \( W \) are called equivalent if at least one of the following two conditions is satisfied: (1) There are vector space isomorphisms \( \varphi_1: U_1 \rightarrow W_1, \varphi_2: U_2 \rightarrow W_2 \), and a complex number \( \alpha \) such that

\[
u_1 \otimes_1 u_2 = \alpha (\varphi_1 u_1 \otimes_2 \varphi_2 u_2), \quad \forall u_1 \in U_1, \ u_2 \in U_2;
\]
(2) There are vector space isomorphisms $\varphi_1: U_1 \to W_2$, $\varphi_2: U_2 \to W_1$, and a complex number $\alpha$ such that

$$u_1 \otimes_1 u_2 = \alpha (\varphi_2 u_2 \otimes_2 \varphi_1 u_1), \forall u_1 \in U_1, u_2 \in U_2.$$ 

The equivalent class of the TPS $(W_1, W_2, \otimes)$ is denoted by $(W_1, W_2, \otimes)$, and the set of equivalent classes of tensor product structures of $W$ is denoted by $T(W)$.

**Proposition 5.2** If two tensor product structures are associated with the same TPP, then they are equivalent.

**Proof.** Let $(W_1, W_2, \otimes)$ be the TPS defined in the proof of Theorem 3.1. It is then sufficient to show that an arbitrary TPS $(U_1, U_2, \otimes_1)$ associated with the TPP $(A_1, A_2)$ is equivalent to $(W_1, W_2, \otimes)$.

By Lemma 5.1, $U_1, U_2$ are irreducible modules. So $U_1, U_2$ are isomorphic to $W_1, W_2$ as $A_1$ and $A_2$ modules respectively. It then follows that there exist isomorphisms $\varphi_1: U_1 \to W_1$, $\varphi_2: U_2 \to W_2$, such that $a \cdot \varphi_1 = a \cdot \varphi_1$, $b \cdot \varphi_2 = b \cdot \varphi_2$, $\forall a \in A_1$, $b \in A_2$.

Now fix a standard complete set of operators $\{r, s\}$ such that $rx_{ji} = \lambda_j x_{ji}$, $sx_{ji} = \mu_i x_{ji}$. By definition, $W_1 = M_{i_0}$, $W_2 = N_{j_0}$ and $x_{ji_0} \otimes x_{j_0i} = x_{ji}$. As $r \in A_1$, $s \in A_2$, we then have

$$r (\varphi_1 x_{ji_0} \otimes_1 \varphi_2 x_{j_0i}) = \lambda_j (\varphi_1 x_{ji_0} \otimes_1 \varphi_2 x_{j_0i}),$$

$$s (\varphi_1 x_{ji_0} \otimes_1 \varphi_2 x_{j_0i}) = \mu_i (\varphi_1 x_{ji_0} \otimes_1 \varphi_2 x_{j_0i}),$$

namely, $(\varphi_1 x_{ji_0} \otimes \varphi_2 x_{j_0i})$ belongs to the same joint eigenspace of $\{r, s\}$ as $x_{ji}$. But the joint eigenspaces of $\{r, s\}$ are all one dimensional, so we conclude that for each pair of index $(j, i)$ there exists a complex number $\alpha_{ji}$ such that

$$x_{ji_0} \otimes x_{j_0i} = \alpha_{ji} (\varphi_1 x_{ji_0} \otimes_1 \varphi_2 x_{j_0i}).$$

To prove the proposition we have to show that $\alpha_{ji}$ is independent of $(j, i)$.

For different indice $j_1, j_2$, take $a \in A_1$ such that $ax_{j_1i_0} = x_{j_1i_0} + x_{j_2i_0}$. Note that the existence of such $a$ is guaranteed by the fact that $A_1 | M_{i_0} = \text{End}(M_{i_0})$. We then have

$$a (x_{j_1i_0} \otimes x_{j_0i}) = \alpha_{j_1i} a (\varphi_1 x_{j_1i_0} \otimes_1 \varphi_2 x_{j_0i}),$$

$$(ax_{j_1i_0}) \otimes x_{j_0i} = \alpha_{j_1i} (a \varphi_1 x_{j_1i_0}) \otimes_1 (\varphi_2 x_{j_0i}) = \alpha_{j_1i} (\varphi_1 ax_{j_1i_0} \otimes_1 (\varphi_2 x_{j_0i})),$$

$$(x_{j_1i_0} + x_{j_2i_0}) \otimes x_{j_0i} = \alpha_{j_1i} (\varphi_1 (x_{j_1i_0} + x_{j_2i_0})) \otimes_1 (\varphi_2 x_{j_0i}).$$
Therefore,
\[
\alpha_{j_1i} (\varphi_1 x_{j_1i_0} \otimes_1 \varphi_2 x_{j_1i_0}) + \alpha_{j_2i} (\varphi_1 x_{j_2i_0} \otimes_1 \varphi_2 x_{j_2i_0}) = \alpha_{j_1i} (\varphi_1 x_{j_1i_0} \otimes_1 \varphi_2 x_{j_1i_0}) + \alpha_{j_1i} (\varphi_1 x_{j_2i_0} \otimes_1 \varphi_2 x_{j_2i_0}),
\]
\[
\alpha_{j_2i} (\varphi_1 x_{j_2i_0} \otimes_1 \varphi_2 x_{j_2i_0}) = \alpha_{j_1i} (\varphi_1 x_{j_2i_0} \otimes_1 \varphi_2 x_{j_2i_0}).
\]

It follows directly that \(\alpha_{j_1i} = \alpha_{j_2i}\). In the same way, we can prove that \(\alpha_{ji_1} = \alpha_{ji_2}\) for different indices \(i_1, i_2\). Consequently, all \(\alpha_{ji}\)'s are equal. The proposition is thus proved.

\[
\square
\]

**Definition 5.3** Two tensor product partitions \((A_1, A_2)\) and \((B_1, B_2)\) are called equivalent if \((A_1, A_2) = (B_1, B_2)\) or \((A_1, A_2) = (B_2, B_1)\). The equivalent class of \((A_1, A_2)\) is denoted by \([A_1, A_2]\), and the set of equivalent classes of tensor product partitions of \(\text{End}(W)\) is denoted by \(P(W)\).

**Lemma 5.2** TPS’s associated with equivalent TPP’s are equivalent.

**Proof.** Let \((A_1, A_2)\) and \((B_1, B_2)\) be equivalent TPP’s. If \((A_1, A_2) = (B_1, B_2)\), then the assertion is just what Proposition 4.2 says. Now suppose that \((A_1, A_2) = (B_2, B_1)\). Let \((U_1, U_2, \otimes_1)\) and \((W_1, W_2, \otimes_2)\) be TPS’s associated with \((A_1, A_2)\) and \((B_1, B_2)\) respectively. We define a bilinear map \(\otimes: U_2 \times U_1 \rightarrow W\) such that \(u_2 \otimes u_1 = u_1 \otimes_1 u_2, \forall u_1 \in U_1, u_2 \in U_2\). It is readily verified that \((U_2, U_1, \otimes)\) is a TPS associated with \((B_1, B_2)\). It then follows from Proposition 4.2 that there are vector space isomorphisms \(\varphi_1: W_1 \rightarrow U_2, \varphi_2: W_2 \rightarrow U_1\), and a complex number \(\alpha\) such that
\[
W_1 \otimes_2 W_2 = \alpha (\varphi_1 w_1 \otimes \varphi_2 w_2) = \alpha (\varphi_2 w_2 \otimes_1 \varphi_1 w_1), \forall w_1 \in W_1, w_2 \in W_2.
\]

This means, according to Definition 4.2, that \((U_1, U_2, \otimes_1)\) and \((W_1, W_2, \otimes_2)\) are equivalent.

\[
\square
\]

**Lemma 5.3** An arbitrary TPS of \(W\) is equivalent to a TPS of the form \((W_1, W_2, \otimes)\) with \(W_1, W_2 \subseteq W\).

**Proof.** Let \((V_1, V_2, \otimes_1)\) be an arbitrary TPS of \(W\), and \(\{x_i\}, \{y_j\}\) be respective bases of \(V_1, V_2\). Define two subspaces \(W_1, W_2\) of \(W\) as
\[
W_1 = V_1 \otimes_1 y_1 \triangleq \{u \otimes_1 y_1 | u \in V_1\},
\]
\[
W_2 = x_1 \otimes_1 V_2 \triangleq \{x_1 \otimes_1 v | v \in V_2\},
\]
and a bilinear map \(\otimes: W_1 \times W_2 \rightarrow W\) such that
\[
(x_i \otimes_1 y_1) \otimes (x_j \otimes_1 y_j) = x_i \otimes_1 y_j.
\]

It is then straightforward to check that \((W_1, W_2, \otimes)\) meets the requirement.
Let $\tau$ denote the map from the set of TPP’s of $\text{End}(W)$ to the set of TPS’s of $W$ under which a TPP is sent to a TPS associated it. By Lemma 5.2 $\tau$ induces a map $\overline{\tau}$ from $\mathcal{P}(W)$ to $\mathcal{T}(W)$:

$$\overline{\tau}(A_1, A_2) = \tau(A_1, A_2).$$

It follows from Proposition 5.1 and Lemma 5.3 that $\overline{\tau}$ is surjective. Later we will prove that it is also injective.

VIII. INNER PRODUCT COMPATIBLE TENSOR PRODUCT STRUCTURE

Up to now we have not taken into accounts the inner product structure of $W$. But in quantum mechanics, a physical space of quantum states should be endowed with a reasonable inner product so that the probability explanation of wave function could make sense. In this section we proceed along to study the TPS in connection with the inner product structure. We will introduce a natural compatibility condition between these two structures as the starting point. Further study will still be developed in the context of TPP. Throughout this section $W$ stands for a vector space with the inner product $\langle , \rangle$, and $W_1, W_2$ stand for subspaces of $W$.

**Definition 6.1.** A TPS $(W_1, W_2, \otimes)$ of $W$ is called compatible with the inner product $\langle , \rangle$ if

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle$$

for all $u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$.

**Definition 6.2.** Let $V_1, V_2$ be two modules with the inner products $\langle , \rangle_1$ and $\langle , \rangle_2$ respectively. $V_1, V_2$ are called $U$–homomorphic (isomorphic) if there exists an inner product preserving module homomorphism (isomorphism) $f$ from $V_1$ to $V_2$ : $\langle fv_1, fv_2 \rangle_2 = \langle v_1, v_2 \rangle_1$, for all $v_1 \in V_1, v_2 \in V_2$. Such a $f$ will be called a $U$–homomorphism (isomorphism) of module.

**Definition 6.3** A TPP $(A_1, A_2)$ of $\text{End}(W)$ is called compatible with the inner product $\langle , \rangle$ if there exist irreducible component sets of $(A_1, A_2)$ such that (1) different $M_i$’s ($N_j$’s) are orthogonal to one another with respect to $\langle , \rangle$; (2) all $M_i$’s ($N_j$’s) are $U$–isomorphic $A_1(A_2)$ modules.

**Lemma 6.1.** A TPP $(A_1, A_2)$ of $\text{End}(W)$ is compatible with the inner product $\langle , \rangle$ if and only if there exist irreducible component sets of $(A_1, A_2)$ which have an orthonormal standard basis with respect to $\langle , \rangle$.

**Proof.** If $(A_1, A_2)$ is compatible with the inner product, then there exist irreducible component sets $\{M_i\}$ for $A_1$, all $M_i$’s being $U$–isomorphic. Let $f_i$ be the inner product preserving isomorphism from $M_1$ to $M_i$. Take an orthonormal basis $\{x_{j1}\}$ of $M_1$, and define $x_{ji} = f_ix_{j1}$. It is evident that
$\cup_i \{x_{ji}| j = 1, 2, \cdots \}$ is an orthonormal synchronic basis with respect to $\{M_i\}$. Now define $N_j$ to be the vector space spanned by $\{x_{ji}| i = 1, 2, \cdots \}$, it then follows from Lemma 2.3 that $\{x_{ji}\}$ is an orthonormal standard basis associated with $\{M_i\}, \{N_j\}$. This proves the necessity. For the sufficiency, just observe that if $\{x_{ji}\}$ is an orthonormal standard basis associated with $\{M_i\}, \{N_j\}$, then the linear maps $f_i : M_1 \rightarrow M_i$ which sends $x_{j1}$ to $x_{ji}$ and $g_j : N_1 \rightarrow N_j$ which sends $x_{1i}$ to $x_{ji}$ are $U$–isomorphisms of module.

\[\square\]

Remark One easily sees from the proof of Lemma 5.1 that in Definition 5.3 the two conditions for $\{M_i\}$ and the two conditions for $\{N_j\}$ are not independent. They actually imply each other.

We now probe the relation between the inner product compatible TPS of $W$ and the inner product compatible TPP of $\text{End}(W)$.

**Proposition 6.1** Let $(A_1, A_2)$ be a TPS of $\text{End}(W)$, $(W_1, W_2, \otimes)$ a TPS of $W$ associated with $(A_1, A_2)$. If $(W_1, W_2, \otimes)$ is compatible with the inner product, then so is $(A_1, A_2)$.

**Proof.** Let $\{x_j\}, \{y_i\}$ be orthonormal bases of $W_1$ and $W_2$ respectively and define $M_i = W_1 \otimes y_i$, $N_j = x_j \otimes W_2$. It follows that $\{M_i\}, \{N_j\}$ are irreducible component sets of $(A_1, A_2)$ and $\{x_j \otimes y_i\}$ is a standard basis associated with them. If $(W_1, W_2, \otimes)$ is compatible with the inner product, then $\{x_j \otimes y_i\}$ is an orthonormal basis of $W$ and hence an orthonormal standard basis associated with $\{M_i\}, \{N_j\}$. The proposition thus follows from **Lemma 6.1**.

Conversely, we have the following result.

**Proposition 6.2** If $(A_1, A_2)$ is an inner product compatible TPP of $\text{End}(W)$, then there exists an inner product compatible TPS of $W$ associated with it.

**Proof.** According to **Lemma 5.1**, we can choose irreducible component sets $\{M_i\}, \{N_j\}$ of $W$ and an orthonormal standard basis $\{x_{ji}\}$ associated with them. Fix a pair of index $(i_0, j_0)$, take $W_1 = M_{i_0}, W_2 = N_{j_0}$ and define a bilinear map $\otimes$ from $W_1 \times W_2$ to $W$ such that $x_{j_1i_0} \otimes x_{j_2i_0} = x_{ji}$, as in the proof of **Theorem 3.1**. We can then check that $(W_1, W_2, \otimes)$ is a desired TPS of $W$. That $(W_1, W_2, \otimes)$ is a TPS associated with $(A_1, A_2)$ has been proved in **Theorem 3.1**. It remains to show that for all $u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$,

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle.$$

But this is an immediate consequence of the sequi-linearity of the inner product $<, >$ and the relation:

$$\langle x_{j_1i_0} \otimes x_{j_2i_0}, x_{j_1i_0} \otimes x_{j_2i_0} \rangle = \langle x_{j_1i_0}, x_{j_2i_0} \rangle \langle x_{j_1i_0}, x_{j_2i_0} \rangle,$$
which follows from the assumption that \( \{x_{ji}\} \) is an orthonormal basis of \( W \).

Next we turn to the problem of characterizing inner product compatible TPP.

**Definition 6.4** Let \( B \) be an extended subalgebra of \( \text{End}(W) \). If for every element \( b \in B \) whose adjoint operator exists we have \( b^* \in B \), where as usual \( b^* \) stands for the adjoint operator of \( b \), then \( B \) is called a quasi-star extended subalgebra.

**Proposition 6.3** Let \((A_1, A_2)\) be a TPP of \( \text{End}(W) \). If it is compatible with the inner product, then both \( A_1 \) and \( A_2 \) are quasi-star extended subalgebras of \( \text{End}(W) \). Conversely, if \( A_1 \) (or \( A_2 \)) is a quasi-star extended subalgebra and there exists an irreducible component set \( \{M_i\} \) (or \( N_j \)) for \( A_1 \) (or \( A_2 \)) such that different \( M_i \)'s (or \( N_j \)'s) are orthogonal to one another, then \((A_1, A_2)\) is compatible with the inner product.

**Proof.** If \((A_1, A_2)\) is compatible with the inner product, according to Lemma 5.1, we can choose an orthonormal synchronic basis \( \{x_{ji}\} \) with respect to some irreducible component set \( \{M_i\} \) for \( A_1 \). Let \( a \in A_1 \). We can write

\[
ax_{ji} = \sum_k x_{ki} a_{kj},
\]

where \( a_{kj} \) is a complex number independent of the index \( i \). We observe that the adjoint operator of \( a \) exists if and only if \( \{j \mid a_{kj} \neq 0\} \) is a finite set for each \( k \), and in that case we have

\[
a^* x_{ji} = \sum_k x_{ki} \overline{a_{jk}},
\]

where \( \overline{a_{jk}} \) stands for the complex-conjugate number of \( a_{jk} \). The fact that \( \{x_{ji}\} \) is a synchronic basis with respect to \( \{M_i\} \) and \( A_1|_{M_i} = \text{End}(W) \) then implies that \( a^* \) is an element of \( A_1 \). Consequently, \( A_1 \) is a quasi-star extended subalgebra. The same conclusion for \( A_2 \) can be proved similarly.

To prove the second half of the proposition, let \( A_1 \) be a quasi-star extended subalgebra, \( \{M_i\} \) an irreducible component set for \( A_1 \) such that elements from different \( M_i \)'s are orthogonal to one another. When \( A_2 \) is a quasi-star extended subalgebra, the argument is similar. Now from the proof of Lemma 5.1, we notice that for our purpose it suffices to show that there exists an orthonormal synchronic basis with respect to \( \{M_i\} \). The existence of such a basis is proved as follows.

Let \( \{x_{j1}\} \) be an orthonormal basis of \( M_1 \) and \( \bigcup_i \{x_{ji} \mid j = 1, 2, \cdots \} \) a synchronic basis with respect to \( \{M_i\} \). If \( a \in A_1 \) and \( a^* \) exists, then we have \( a^* \in A_1 \). As a result, we can write

\[
a x_{ji} = \sum_k x_{ki} a_{kj}, a^* x_{ji} = \sum_k x_{ki} b_{kj},
\]

where \( a_{kj}, b_{kj} \) are complex numbers independent of the index \( i \). But \( \{x_{j1}\} \) is an orthonormal basis
of \( M_1 \), so we have \( b_{kj} = \overline{a_{jk}} \). Now by the definition of adjoint operator we have
\[
\langle a^* x_{ji}, x_{li} \rangle = \langle x_{ji}, a x_{li} \rangle.
\]
It then follows that
\[
\left\langle \sum_k x_{ki} a_{jk}, x_{li} \right\rangle = \left\langle x_{ji}, \sum_k x_{ki} a_{kl} \right\rangle.
\]
Now if we choose \( a \in A_1 \) such that \( a_{mn} = \delta_{ml} \delta_{nl} \), from this equation we obtain
\[
\langle x_{ji}, x_{li} \rangle = 0, \text{ for } l \neq j;
\]
and if we choose \( a \in A_1 \) such that \( a_{mn} = \delta_{mj} \delta_{nl} \), we obtain
\[
\langle x_{ji}, x_{ji} \rangle = \langle x_{li}, x_{li} \rangle.
\]
Let \( \langle x_{ji}, x_{ji} \rangle = \alpha_i^2 \) (\( \alpha_1 = 1 \) by definition), \( y_{ji} = \frac{x_{ji}}{\alpha_i} \), it is then check that \( \bigcup \{ y_{ji} | j = 1, 2, \cdots \} \) is an orthonormal synchronic basis with respect to \( \{ M_i \} \).

Before going on to another topic, let us pause to investigate the finite dimensional case. In this case, we have the following much better result.

**Theorem 6.1** When \( W \) is finite dimensional, \( (A_1, A_2) \) is a TPP of \( \text{End}(W) \) compatible with the inner product if and only if (1) \( A_1, A_2 \) are star subalgebras of \( \text{End}(W) \); (2) \( [A_1, A_2] = 0 \) and \( A = A_1 \lor A_2 \).

**Proof.** The necessity follows from the first half of Proposition 6.3 directly. For the sufficiency, according to the second half of Proposition 6.3, we need only to show that as \( A_1(A_2) \) module \( W \) possesses a decomposition into an orthogonal sum of irreducible \( A_1(A_2) \) submodules. But this is a direct consequence of the condition that \( A_1, A_2 \) are star subalgebras of \( \text{End}(W) \). Indeed, if \( V \subseteq W \) is an \( A_1(A_2) \) submodule, then the orthogonal complement \( V^\perp \) of \( V \) is also an \( A_1(A_2) \) submodule and we have the orthogonal decomposition: \( W = V + V^\perp \). Repeating this procedure, we will obtain the desired decomposition after finite steps because \( W \) is finite dimensional.

Finally, for the completeness, we now investigate the concept of standard complete set of observables, the counterpart of standard complete set of operators.

**Definition 6.5** A standard complete set of operators \( (r, t) \) of \( \text{End}(W) \) is called a standard complete set of observables if (1) both \( r \) and \( t \) are self-adjoint operators; (2) there exists a characteristic set \( \{ M_i \} \) (or \( \{ N_j \} \)) of \( r \) (or \( t \)) consisting of subspaces orthogonal to one another.

**Remark** If \( W \) is finite dimensional, then the condition (1) implies the condition (2).
The results listed below are about the relation between standard complete set of observables and inner product compatible TPP. Proposition 6.5 is just a special instance of Proposition 4.3, and the other propositions can be proved by similar argument as presented in Section 2. Here we would rather omit the proof to avoid redundancy.

**Proposition 6.4** If \((r,t)\) is a standard complete set of observables of \(\text{End}(W)\), then there exists an inner product compatible TPP \((A_1,A_2)\) of \(\text{End}(W)\) containing \((r,t)\).

**Proposition 6.5** Let \((r,t)\) be a standard complete set of observables of \(\text{End}(W)\), \((A_1,A_2)\) and \((B_1,B_2)\) two inner product compatible tensor product partitions containing \((r,t)\). Then there exists an isomorphism \(\varphi \in \text{End}(W)\), diagonal with respect to the basis consisting of common eigenvectors of \(r\) and \(t\), such that \(B_1 = \varphi \cdot A_1 \cdot \varphi^{-1}\) and \(B_2 = \varphi \cdot A_2 \cdot \varphi^{-1}\).

**Definition 6.6** The complete sets of observables \((r,t)\) and \((r',t')\) are called complementary if (1) \(M_i = M'_i\) and all \(M_i\)'s are \(U\)-isomorphic normal \(S_{r,r'}\) modules or (2) \(N_j = N'_j\) and all \(N_j\)'s are \(U\)-isomorphic normal \(S_{t,t'}\) modules.

**Proposition 6.6** An inner product compatible TPP contains complementary standard complete sets of observables.

**Proposition 6.7** If \((r,t)\) and \((\tilde{r},\tilde{t})\) are complementary standard complete sets of observables, then there exists a unique inner product compatible TPP \((A_1,A_2)\) such that \(r,\tilde{r} \in A_1\) and \(t,\tilde{t} \in A_2\).

**VII. PRODUCT VECTOR SET AND RELATIVITY OF QUANTUM ENTANGLEMENT**

Now we turn to consider the set of decomposable vectors related to a TPS. Decomposable and indecomposable vectors correspond respectively to product and entangled states in physics. So from physical point of view, it is meaningful to study this topic.

**Definition 7.1** Let \((V_1,V_2,\otimes)\) be a TPS of \(W\). Then \((V_1,V_2,\otimes)\) is called nontrivial if \(\dim V_1, \dim V_2 > 1\) and the subset \(\{u \otimes v|u \in V_1,v \in V_2\}\) of \(W\) is called the decomposable vector set related to it.

**Definition 7.2** A nonempty subset \(S\) of \(W\) is called a product vector set if it is a decomposable vector set related to some TPS \((V_1,V_2,\otimes)\) of \(W\), and a nontrivial one if \((V_1,V_2,\otimes)\) is nontrivial.

**Remark** A product vector set of \(W\) is a nontrivial one if and only if it is a proper subset of \(W\).

**Definition 7.3** Let \(S\) be a product vector set, \(w\) an element of \(W\). If \(w\) belongs to \(S\), \(w\) is called a product vector. Otherwise, it is called an entangled vector.

**Remark** According to the definition, when we call an element of \(W\) a product vector or an entangled one we should have in mind a TPS. Rigorously speaking, in the above definition, \(w\)
should be called a product vector or an entangled one with respect to the TPS related to which $S$ is a decomposable vector set. Indeed, whether an element is a product vector or not strongly depends on what TPS is considered. This point will become clear as we proceed.

Obviously, every product vector set of $W$ contains a basis of $W$. Conversely, we have the following result.

**Proposition 7.1** Every basis of $W$ can be extended to a nontrivial product vector set if $\dim W$ is not a prime number.

**Proof.** Let $\{w_k\}$ be a basis of $W$. $\dim W$ being not a prime number, we can choose subspaces $W_1, W_2$ of $W$ with $\dim W_1, \dim W_2 > 1$ and the respective bases $\{x_j\}, \{y_i\}$ such that there exists a bijective map $f : \{x_j\} \times \{y_i\} \rightarrow \{w_k\}$. Extend $f$ bilinearly to a map $\otimes : W_1 \times W_2 \rightarrow W$. It is then readily check that $(W_1, W_2, \otimes)$ is a TPS of $W$ with respect to which $w_k$ is a product vector for each $k$. This proves the proposition.

**Corollary 7.1** An arbitrary element of $W$ is a product vector with respect to some TPS, which can be chosen to be nontrivial if $\dim W$ is not a prime number.

**Corollary 7.2** An arbitrary element of $W$ is a entangled vector with respect to some (nontrivial) TPS if $\dim W$ is not a prime number.

**Proof.** Keep the same notation as in the proof of Proposition 3.3. Let $w \in W$. Choose a basis $\{w_k\}$ of $W$ such that $w = w_1 + w_2$ and choose a map $f$ such that

$$f \left( x_1, y_2 \right) = w_1, \quad f \left( x_2, y_1 \right) = w_2.$$

Then $w = x_1 \otimes y_2 + x_2 \otimes y_1$ is an entangled vector with respect to the TPS $(W_1, W_2, \otimes)$.

Similarly, we can prove the following results.

**Proposition 7.1’** Let $W$ a space with inner product. Every orthonormal basis of $W$ can be extended to a nontrivial decomposable vector set related to an inner product compatible TPS if $\dim W$ is not a prime number.

**Corollary 7.1’** An arbitrary element of $W$ is a decomposable vector with respect to some inner product compatible TPS, which can be chosen to be nontrivial if $\dim W$ is not a prime number.

**Corollary 7.2’** An arbitrary element of $W$ is a entangled vector with respect to some (nontrivial) inner product compatible TPS if $\dim W$ is not a prime number.

We denote by $D(W)$ the set of product vector sets of $W$, and denote by $\sigma$ the map that sends each TPS of $W$ to the decomposable vector set related to it. We observe that equivalent TPP’s have the same decomposable vector set. So $\sigma$ naturally induces a map $\overline{\sigma}$ from $\mathcal{T}(W)$ to $D(W)$:

$$\overline{\sigma}(V_1, V_2, \otimes) = \sigma(V_1, V_2, \otimes).$$
By definition $\sigma$ is a surjective map. Thus $\sigma \cdot \tau$ is a surjective map from $\mathcal{P}(W)$ to $\mathcal{D}(W)$. A proof of the bijectivity of this map is now in order.

**Remark** Lemma 5.3 tells us that as far as product vector set is concerned considering only the TPS of the form $(W_1, W_2, \otimes)$ with $W_1, W_2 \subseteq W$ does not cause any loss of generality.

**Lemma 7.1** If $(A_1, A_2)$ and $(B_1, B_2)$ are two tensor product partitions such that $\sigma \tau(A_1, A_2) = \sigma \tau(B_1, B_2)$, then $(A_1, A_2) = (B_1, B_2)$.

**Proof.** Let $(W_1, W_2, \otimes), (W_1', W_2', \otimes')$ be tensor product structures of $W$ associated with $(A_1, A_2)$ and $(B_1, B_2)$ respectively. Denote by $D$ and $D'$ the decomposable vector sets related to $(W_1, W_2, \otimes)$ and $(W_1', W_2', \otimes')$ respectively. Suppose that $D = D'$. We then have to show that either $A_1 = B_1$ and $A_2 = B_2$ or $A_1 = B_2$ and $A_2 = B_1$. Take a basis $\{x_j\}$ of $W_1$ and a basis $\{y_i\}$ of $W_2$. We observe that when $W_1$ or $W_2$ is of one dimension the proof is trivial. So we exclude this case in the following argument. Now let us proceed in steps as follows.

1. $W_1 \otimes y_1 \subseteq W_1' \otimes' y_1'$ for some $y_1' \in W_2'$ or $W_1 \otimes y_1 \subseteq x_1' \otimes W_2'$ for some $x_1' \in W_1'$. Consider the elements $x_1 \otimes y_1$ and $x_2 \otimes y_1$ of $D$. Since $D = D'$ there exist $x_1', x_2' \in W_1'$ and $y_1', y_1'' \in W_2'$ such that

   $$x_1 \otimes y_1 = x_1' \otimes' y_1', \quad x_2 \otimes y_1 = x_2' \otimes' y_1''.$$

On the other hand, we have $x_1 \otimes y_1 + x_2 \otimes y_1 = (x_1 + x_2) \otimes y_1 \in D$. This implies that $x_1' \otimes' y_1' + x_2' \otimes' y_1'' \in D'$, that is, $x_1' \otimes' y_1' + x_2' \otimes' y_1''$ is of rank one with respect to $(W_1', W_2', \otimes')$. It then follows from Proposition 1.1 that either $\{x_1', x_2'\}$ or $\{y_1', y_1''\}$ is linearly dependent. Notice that both $\{x_1', x_2'\}$ and $\{y_1', y_1''\}$ cannot be linearly dependent since $\{x_1' \otimes y_1', x_2' \otimes y_1''\}$ is linearly independent. We thus conclude that either there exist a linearly independent set $\{x_1', x_2'\} \subseteq W_1'$ and a nonzero element $y_1'' \in W_2'$ such that

   $$x_1 \otimes y_1 = x_1' \otimes' y_1', \quad x_2 \otimes y_1 = x_2' \otimes' y_1'',$$

   or there exist a linearly independent set $\{y_1', y_1''\} \subseteq W_2'$ and a nonzero element $x_1' \in W_1'$ such that

   $$x_1 \otimes y_1 = x_1' \otimes' y_1', \quad x_2 \otimes y_1 = x_1' \otimes' y_1''.$$

If the first case happens, we assert, that $W_1 \otimes y_1 \subseteq W_1' \otimes' y_1'$ for some $y_1' \in W_2'$, and if the second case happens, $W_1 \otimes y_1 \subseteq x_1' \otimes W_2'$ for some $x_1' \in W_1'$. In fact, if $\dim W_1 = 2$ the assertion is evidently true. If $\dim W_1 > 2$ consider the element $x_j \otimes y_1, j \neq 1, 2$. Let $x_j \otimes y_1 = x_j' \otimes' y_1''$. As argued above either $\{x_1', x_2'\}$ or $\{y_1', y_1''\}$ is linearly dependent, but not both. Now the assertion clearly reduces to the claim that $\{y_1', y_1''\}$ and $\{x_1', x_2'\}$ are respectively linearly dependent in the
first and the second cases. Suppose, on the contrary, that \( \{ x'_1, x'_2 \} \) is linearly dependent in the first case. Then there exists a complex number \( \alpha \) such that \( x_j \otimes y_1 = \alpha x'_1 \otimes y'_1 \). It then follows that
\[
(x_2 + x_j) \otimes y_1 = x'_2 \otimes y'_1 + \alpha x'_1 \otimes y''_1.
\]
Since \( \{ x'_1, x'_2 \} \) and \( \{ y'_1, y''_1 \} \) are both linearly independent, the right hand side is an element of rank 2 with respect to \((W'_1, W'_2, \otimes')\) according to Proposition 1.1. But the left hand side is an element of rank 1 with respect to \((W_1, W_2, \otimes)\). This contradicts the assumption that \( D = D' \). The second case can be dealt with in the same way.

In the following discussion, we assume that \( W_1 \otimes y_1 \subseteq W'_1 \otimes' y'_1 \) for some \( y'_1 \in W'_2 \). The subsequent argument then proves that \( A_1 = B_1 \) and \( A_2 = B_2 \). If the other case happens, then a similar argument will prove that \( A_1 = B_2 \) and \( A_2 = B_1 \).

(2) \( W_1 \otimes y_1 = W'_1 \otimes' y'_1 \) for some \( y'_1 \in W'_2 \). Let \( x' \) be an arbitrary element of \( W'_1 \). It suffices to show that \( x' \otimes' y'_1 \in W_1 \otimes y_1 \). Actually, we have
\[
x'_1 \otimes' y'_1 = x_1 \otimes y_1, \quad x'_2 \otimes' y'_1 = x_2 \otimes y_1.
\]
Then by a similar argument as presented above we can prove that \( x' \otimes' y'_1 = x \otimes y_1 \) for some \( x \in W_1 \), that is \( x' \otimes' y'_1 \in W_1 \otimes y_1 \).

(3) There is a basis \( \{ x'_j \} \) of \( W_1 \) and a nonzero element \( y'_1 \) of \( W'_2 \) such that \( x_j \otimes y_1 = x'_j \otimes' y'_1 \).

This is a direct consequence of (1) and (2).

(4) There exists a linearly independent subset \( \{ y'_i \} \) of \( W'_2 \) such that \( x_1 \otimes y_i = x'_1 \otimes' y'_i \) for each \( i \).

When \( i = 1 \) the conclusion has been proved. When \( i \neq 1 \) consider the element \( x_1 \otimes y_i + x_1 \otimes y_1 \) of \( D \). Let \( x_1 \otimes y_i = x''_1 \otimes' y'_i \). Then \( x_1 \otimes y_i + x_1 \otimes y_1 = x''_1 \otimes' y'_i + x'_1 \otimes' y'_1 \) is an element of \( D' \). It follows that exactly one of the two sets \( \{ x'_1, x''_1 \} \) and \( \{ y'_1, y'_i \} \) is linearly dependent. Clearly, what we need to show is that \( \{ x'_1, x''_1 \} \) is linearly dependent. If, on the contrary, \( \{ y'_1, y'_i \} \) is linearly dependent, then we have \( y'_i = \alpha_i y'_1 \) for some \( \alpha_i \in \mathbb{C} \). As a result,
\[
x_1 \otimes y_i + x_2 \otimes y_1 = \alpha_i x''_1 \otimes' y'_1 + x'_2 \otimes' y'_1 = (\alpha_i x''_1 + x'_2) \otimes' y'_1.
\]
This is a contradiction since the right hand side is an element of \( D' \) but the left hand side is not an element of \( D \). Finally, it is evident that \( \{ y'_i \} \) is linearly independent since \( \{ x_1 \otimes y_i \} \) is linearly independent.

(5) \( x_j \otimes y_i = x''_j \otimes' y'_i \) for each \( i \). We need only to consider the case where \( i, j \neq 1 \). Let \( x_j \otimes y_i = x''_j \otimes' y''_i \). We have to show that \( \{ y'_i, y''_i \} \) is linearly dependent. If this is not the case,
by considering the rank of the element \( x_1 \otimes y_i + x_j \otimes y_i \) we can prove that \( \{ x'_1, x''_j \} \) is linearly dependent. It then follows that \( x_j \otimes y_i + x_1 \otimes y_i \) belongs to \( D' \). But when \( i, j \neq 1 \), the element \( x_j \otimes y_i + x_1 \otimes y_i \) does not lie in \( D \). This contradicts the assumption that \( D = D' \).

(6) \( x_j \otimes y_i = x'_j \otimes y'_i \) for each \( i \). By (5) we have \( x_j \otimes y_i = x''_j \otimes y''_i \). Then by a similar argument as presented in the proof of (4) we can show that \( \{ x'_j, x''_i \} \) is linearly dependent. So we can write \( x_j \otimes y_i = \alpha x'_j \otimes y'_i, \alpha \in \mathbb{C} \). Let us prove that \( \alpha = 1 \). If \( i = 1 \) or \( j = 1 \), there is nothing to prove. When \( i, j \neq 1 \), consider the element \( (x_1 + x_j) \otimes (y_1 + y_i) \) of \( D \). We have

\[
(x_1 + x_j) \otimes (y_1 + y_i) = (x'_1 + \alpha x'_j) \otimes y'_i + (x'_1 + x'_j) \otimes y'_i.
\]

As \( \{ y'_1, y'_i \} \) is linearly independent, it then follows that \( \{ x'_1 + \alpha x'_j, x'_1 + x'_j \} \) is linearly dependent. Consequently, \( \alpha = 1 \). This proves the original assertion.

(7) \( A_1 = B_1 \) and \( A_2 = B_2 \). Let \( M_i = W_1 \otimes y_i, M'_i = W'_1 \otimes y'_i, x_{ji} = x_j \otimes y_i, x'_{ji} = x'_j \otimes y'_i \). Then \( \{ M_i \} \) is an irreducible component set of \( A_1 \) and \( \bigcup_i \{ x_{ji} | j = 1, 2, \ldots \} \) is a synchronic basis with respect to \( \{ M_i \} \). Indeed, we have obviously the direct sum decomposition \( W = \sum_i M_i \). Moreover, since \( aM_i = (aW_1) \otimes y_i, \forall a \in A_1 \), all \( M_i \)'s are irreducible \( A_1 \) modules isomorphic to \( W_1 \) and \( A_1 \) has the same matrix representation with respect to the bases \( \{ x_{ji} | j = 1, 2, \cdots \}, i = 1, 2, \cdots \). For the same reason, \( \{ M'_i \} \) is an irreducible component set for \( B_1 \) and \( \bigcup_i \{ x'_{ji} | j = 1, 2, \cdots \} \) is a synchronic basis with respect to \( \{ M'_i \} \). But it follows from (3) that \( M_i = M'_i \) and by (6) we have \( x_{ji} = x'_{ji} \). Considering \( A_1 | M_i = B_1 | M_i = \text{End}(M_i) \), we then conclude that \( A_1 = B_1 \), and hence \( A_2 = B_2 \) since \( A_2 = A'_1, B_2 = B'_1 \).

We are now in a position to present the following result on the relation among the TPP, the TPS and the product vector set.

**Theorem 7.1** \( \varphi \) and \( \varphi^* \) are both bijective.

**Proof.** By Lemma 7.1, \( \varphi \cdot \varphi^* \) is injective. It follows that \( \varphi \) is injective. But we have proved that \( \varphi \) is surjective, it is therefore bijective. \( \varphi^* \) is surjective by definition. For the bijectivity, just notice that \( \varphi^* = (\varphi \cdot \varphi^*) \cdot \varphi^{-1} \). The claim then follows directly.

**Remark** This theorem shows that each product vector set is characterized by an unordered pair \( \{ A_1, A_2 \} \) where \( (A_1, A_2) \) is a TPP.

In the remaining part of this section, we keep on studying the property of product vector set.

**Proposition 7.2** Let \( D \) be a subset of \( W \). Then \( D \) is a product vector set if and only if there exist a TPP \( (A_1, A_2) \) of \( \text{End}(W) \) and a nonzero element \( w \) of \( W \) such that \( D = A_1 A_2 w \) and \( (A_1 w, A_2 w, \otimes_D) \) is a TPS of \( W \), where \( \otimes_D \) is defined as \( aw \otimes_D bw = abw, \forall a \in A_1, b \in A_2 \).

**Proof.** If \( D \) is a product vector set, then there exist a TPP \( (A_1, A_2) \) and a TPS \( (W_1, W_2, \otimes) \)
associated with it such that

\[ D = \{ u \otimes v | u \in W_1, v \in W_2 \}. \]

Take nonzero elements \( u \in W_1, v \in W_2 \) and let \( w = u \otimes v \). By Lemma 3.1 \( W_1, W_2 \) are respectively irreducible \( A_1, A_2 \) modules. So \( A_1 u = W_1, A_2 v = W_2 \) and hence \( D = A_1 A_2 w \). On the other hand, we have by definition \( aw \otimes_D bw = au \otimes bv \). It then follows directly that \( (A_1 w, A_2 w, \otimes_D) \) is a TPS of \( W \). The necessity is thus proved. For the sufficiency, just observe that \( D \) is exactly the decomposable vector set related to the TPS \( (A_1 w, A_2 w, \otimes_D) \).

Let \( \varphi \) be an automorphism of \( W \) and \((A_1, A_2)\) a TPP. Then \( (\varphi \cdot A_1 \cdot \varphi^{-1}, \varphi \cdot A_2 \cdot \varphi^{-1}) \) is also a TPP. Moreover \( \varphi \) induces a map \( \varphi \) from \( \mathcal{P}(W) \) to \( \mathcal{P}(W) : \varphi(A_1, A_2) = (\varphi \cdot A_1 \cdot \varphi^{-1}, \varphi \cdot A_2 \cdot \varphi^{-1}) \).

**Lemma 7.2** If \( \varphi \) is an automorphism of \( W \), then \( \varphi \cdot \varphi = \varphi \cdot \varphi \).

**Proof.** Let \((A_1, A_2)\) be a TPP and \((W_1, W_2, \otimes)\) a TPS associated with it. We assert that \((\varphi W_1, \varphi W_2, \otimes_\varphi)\) is a TPS associated with \((\varphi \cdot A_1 \cdot \varphi^{-1}, \varphi \cdot A_2 \cdot \varphi^{-1})\), where \( \otimes_\varphi = \varphi \cdot \otimes \cdot (\varphi^{-1} \times \varphi^{-1}) : \)

\[ (\varphi w_1) \otimes_\varphi (\varphi w_2) = \otimes_\varphi (\varphi w_1, \varphi w_2) = \varphi (w_1 \otimes w_2), \forall w_1 \in W_1, w_2 \in W_2. \]

\((\varphi W_1, \varphi W_2, \otimes_\varphi)\) is obviously a TPS. On the other hand, for all \( w_1 \in W_1, w_2 \in W_2, \) we have

\[
(\varphi a \varphi^{-1}) ((\varphi w_1) \otimes_\varphi (\varphi w_2)) = \varphi a (w_1 \otimes w_2) = \varphi (aw_1 \otimes w_2)
\]

\[
= (\varphi aw_1) \otimes_\varphi (\varphi w_2) = ((\varphi a \varphi^{-1}) (\varphi w_1)) \otimes_\varphi (\varphi w_2), \forall a \in A_1,
\]

and similarly

\[
(\varphi b \varphi^{-1}) ((\varphi w_1) \otimes_\varphi (\varphi w_2)) = (\varphi w_1) \otimes_\varphi ((\varphi b \varphi^{-1}) (\varphi w_2)), \forall b \in A_2.
\]

The assertion then follows. Now it is clearly seen that \( \mathcal{P}(\mathcal{P}(A_1, A_2)) = \mathcal{P}(\mathcal{P}(A_1, A_2)) \) is a direct consequence of the definition of \( \otimes_\varphi \). This proves the lemma.

**Lemma 7.3** Let \((r, t)\) be a standard complete set of operators of \( A, \{M_i\}, \{N_j\}\) the characteristic sets of \( r \) and \( t \) respectively. If \((A_1, A_2)\) is a TPP containing \((r, t)\), then we have \( M_i, N_j \subseteq \mathcal{P}(A_1, A_2) \).

**Proof.** According to Lemma 4.2, \( \{M_i\} \) and \( \{N_j\} \) are irreducible component sets for \( A_1 \) and \( A_2 \) respectively. Let \( \{x_{ji}\} \) be a synchronic basis associated with \( \{M_i\} \) and \( \{N_j\} \). Then it follows from Theorem 5.1 that there exists a TPS \((W_1, W_2, \otimes)\) associated with \((A_1, A_2)\) such that for some \( i_0, j_0 \)

\[
M_i = W_1 \otimes x_{j_0i}, N_j = x_{j_0i} \otimes W_2.
\]
This implies that $M_i, N_j \subseteq \sigma\tau(A_1, A_2)$.

Now we conclude this section by the following results about the construction of product vector set.

**Proposition 7.3** Let $(r, t)$ be a standard complete set of operators of $A (= End(W))$, $\{M_i\}$, $\{N_j\}$ the characteristic sets of $r$ and $t$ respectively, then there exists a product vector set $D$ containing $M_i$ and $N_j$ as subsets. Moreover, if we require, in addition, that for $u \otimes v \in D$

$$r (u \otimes v) = (ru) \otimes v, \quad t (u \otimes v) = u \otimes (tv),$$

then such product vector set is unique up to an automorphism of $W$, which is diagonal with respect to the basis consisting of common eigenvectors of $r$ and $t$.

**Proof.** Take a TPP $(A_1, A_2)$ containing $(r, t)$ and define $D = \sigma\tau(A_1, A_2)$. Then $D$ meets the requirement according to Lemma 7.3. This proves the first half of the proposition.

To prove the second half of the proposition, let $(A_1, A_2)$ be a TPP and $(W_1, W_2, \otimes)$ a TPS associated with it such that $M_i, N_j \subseteq \sigma\tau(A_1, A_2) = \sigma(W_1, W_2, \otimes)$. We need to show that $(A_1, A_2)$ is determined uniquely up to an automorphism of $W$.

First we recall that by the definition of standard complete set of operators, there are two sets of distinct complex numbers $\{\lambda_j\}$ and $\{\mu_i\}$ and two decompositions of $W$ into direct sum of subspaces

$$W = \sum_i \oplus M_i = \sum_j \oplus N_j$$

such that

$$M_i = \sum_j \oplus Cx_{ji}, \quad N_j = \sum_i \oplus Cx_{ji}$$

where $x_{ji}$ is the common eigenvector of $r$ and $t$:

$$rx_{ji} = \lambda_j x_{ji}, \quad tx_{ji} = \mu_i x_{ji}.$$ 

Now, let us proceed in steps.

1. There exist a basis $\{u_j\}$ of $W_1$ and a basis $\{v_i\}$ of $W_2$ such that $M_i = W_1 \otimes v_i$ and $N_j = u_j \otimes W_2$. Since $M_i \subseteq \sigma(W_1, W_2, \otimes)$, there exist $u_j \in W_1$ and $v_i \in W_2$ such that $x_{ji} = u_j \otimes v_i$.

Then by the condition (*) we have

$$\lambda_j u_j \otimes v_i = rx_{ji} = (ru_j) \otimes v_i,$$

$$\mu_i u_j \otimes v_i = tx_{ji} = u_j \otimes (tv_i).$$
and hence

\[ ru_j = \lambda_j u_j, \quad tv_{ij} = \mu_i v_{ij}. \]

Now using the fact that \( M_i \) is a vector space one can easily prove that \( v_{i_1}, v_{i_2} \) are linearly dependent for different \( j_1, j_2 \). Let \( W'_1 \) be the subspace of \( W_1 \) spanned by \( \{ u_j \} \), it then follows that there exists \( v_i \in W_2 \) such that \( tv_i = \mu_i v_i \) and \( M_i = W'_1 \otimes v_i \). Notice that \( W_1 \otimes v_i \) is included in the eigenspace of \( t \) corresponding to the eigenvalue \( \mu_i \), which is just \( M_i \). Thus we have

\[ M_i = W'_1 \otimes v_i \subseteq W_1 \otimes v_i \subseteq M_i, \]

hence \( M_i = W_1 \otimes v_i \). Similarly, there exists \( u_j \in W_1 \) such that \( N_j = u_j \otimes W_2 \). Finally, it is easy to check that \( \{ u_j \}, \{ v_i \} \) are bases of \( W_1 \) and \( W_2 \) respectively.

(2) \((A_1, A_2)\) is a TPP containing \((r,t)\). According to (1), \( \{M_i\}, \{N_j\} \) are irreducible component sets of \((A_1, A_2)\), and \( \{u_j \otimes v_i\} \) is a standard basis associated with them. The assertion then follows.

Finally, by Proposition 4.3, we conclude from (2) that \((A_1, A_2)\) is determined uniquely up to an automorphism of \( W \). The conclusion then follows from Lemma 7.2.

**Proposition 7.4** Let \((r,t)\) be a standard complete set of observables of \( A(=End(W)) \), \( \{M_i\}, \{N_j\} \) the characteristic sets of \( r \) and \( t \) respectively, then there exists an inner product compatible TPS whose decomposable vector set \( D \) contains \( M_i \) and \( N_j \) as subsets. Moreover, if we require, in addition, that for \( u \otimes v \in D \)

\[ r(u \otimes v) = (ru) \otimes v, \quad t(u \otimes v) = u \otimes (tv), \]

then such decomposable vector set is unique up to an automorphism of \( W \), which is diagonal with respect to the basis consisting of common eigenvectors of \( r \) and \( t \).

The proof of this proposition is similar to that of Proposition 7.3. We would rather omit it.

**VIII. EXAMPLES FOR RELATIVITY OF QUANTUM ENTANGLEMENT**

In this section, we will analyze three examples as an illustration of the theory developed above. The first example concerns the so called Bell states, the second one deals with entanglement in Bargmann space, and the third one is about entanglement with respect to the coordinate of mass of center mentioned in the introduction. In the subsequent discussion, following the physical convention, we will sometimes call a vector a state.
A. Entanglement of Bell States

Now let us study the first example that has been considered extensively, but not rigorously from mathematical point of view, by Zanardi et al.[4].

Consider the system $S_{AB}$ consisting of two spin $\frac{1}{2}$ particles labelled by $A$ and $B$ respectively. We will not be interested in the dependence of the wave functions on the coordinates. For a spin $\frac{1}{2}$ particle, the spin operator $\vec{S}$ takes the form ($\hbar = 1$)

$$\vec{S} = \frac{1}{2} \vec{\sigma} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. Conventionally, the two eigenvectors of $S_z$ are denoted by $|\uparrow\rangle, |\downarrow\rangle$, which belong to the eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. To distinguish different particles, for operators we introduce an upper script and for states we introduce a lower script. For example, $\sigma_z^A$ denotes the spin operator for particle $A$ and $|\uparrow\rangle^B$ denotes the eigenvector of $\sigma_z^B$.

Let $V_1$ be the vector space spanned by $\{|\uparrow\rangle_A, |\downarrow\rangle_A\}$ and $V_2$ the vector space spanned by $\{|\uparrow\rangle_B, |\downarrow\rangle_B\}$. Then the space of states of the system $S_{AB}$, which we denote by $W$, has a God given inner product compatible TPS ($V_1, V_2, \otimes_0$) : $W$ is taken or defined to be the vector space spanned by the linearly independent set

$$\{|\uparrow\rangle_A \otimes_0 |\uparrow\rangle_B, |\uparrow\rangle_A \otimes_0 |\downarrow\rangle_B, |\downarrow\rangle_A \otimes_0 |\uparrow\rangle_B, |\downarrow\rangle_A \otimes_0 |\downarrow\rangle_B\},$$

which is usually written as

$$\{|\uparrow\rangle_A |\uparrow\rangle_B, |\uparrow\rangle_A |\downarrow\rangle_B, |\downarrow\rangle_A |\uparrow\rangle_B, |\downarrow\rangle_A |\downarrow\rangle_B\}.$$ 

We call this TPS God given because we are not able to define the above four vectors definitely as elements of $W$. As a matter of fact, they are tacitly understood as the common eigenstates of $\sigma_z^A$ and $\sigma_z^B$. But the problem is still there: the phase is not and cannot be determined. We have seen that from mathematical point of view, the bilinear map $\otimes_0$ is not well defined. But as far as physics is concerned, we have no choice but take it for granted and make it the starting point of our discussions in this paper.
The so-called Bell states are defined as follows:

\[ |\psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B \pm |\downarrow\rangle_A |\uparrow\rangle_B), \]

\[ |\phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\uparrow\rangle_B \pm |\downarrow\rangle_A |\downarrow\rangle_B). \]  

Obviously, they are maximally entangled states with respect to the TPS \((W, states form an orthonormal basis of such tensor product structures.

Obviously, they are maximally entangled states with respect to the TPS \((V_1, V_2, \otimes_0)\). The Bell states form an orthonormal basis of \(W\), so according to Proposition 7.1', there exists an inner product compatible TPS with respect to which they are product states. Let us explicitly construct such tensor product structures.

Let \(R_x(\pi)\) be the two bi-particle rotation operator through the angle \(\pi\) about the \(x\)-axis. We have

\[ R_x(\pi) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes_0 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \]

Notice that \(R_x(\pi)\) is unitary and self-adjoint as well. It is easy to check that

\[ R_x(\pi) |\psi^\pm\rangle_{AB} = \mp |\psi^\pm\rangle_{AB}, \quad R_x(\pi) |\phi^\pm\rangle_{AB} = \mp |\phi^\pm\rangle_{AB}. \]

Denote by \(M_1, M_2\) the subspaces spanned by \(|\psi^\pm\rangle_{AB}\) and \(|\phi^\pm\rangle_{AB}\) respectively, and by \(N_1, N_2\) the subspaces spanned by \(|\psi^\pm\rangle_{AB}, |\phi^\pm\rangle_{AB}\) and \(|\psi^-\rangle_{AB}, |\phi^-\rangle_{AB}\) respectively. Then we have the orthogonal decomposition

\[ W = M_1 \oplus M_2 = N_1 \oplus N_2. \]

Now let \(S_z = S_z^A + S_z^B\). Clearly we have

\[ S_z^2 |\psi^\pm\rangle_{AB} = 0, \quad S_z^2 |\phi^\pm\rangle_{AB} = |\phi^\pm\rangle_{AB}. \]

It then follows that \((R_x(\pi), S_z^2)\) is a standard complete set of observables, and \(\{M_1, M_2\}, \{N_1, N_2\}\) are characteristic sets of \(R_x(\pi)\) and \(S_z^2\) respectively. According to Proposition 6.4, there exists an inner product compatible TPS whose decomposable vector set contains \(M_1, M_2, N_1, N_2\).

Let us first construct an inner product compatible TPP contains \((R_x(\pi), S_z^2)\). Using the previous notation, let \(x_{11} = |\psi^+\rangle_{AB}, \ x_{21} = |\psi^-\rangle_{AB}, \ x_{12} = |\phi^+\rangle_{AB}, \ x_{22} = |\phi^-\rangle_{AB}\). Define the subalgebras \(A_1, A_2 \subseteq End(W)\) as follows. An element \(a\) belongs to \(A_1\) if and only if

\[ a(x_{11}, x_{21}) = (x_{11}, x_{21}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \]

\[ a(x_{12}, x_{22}) = (x_{12}, x_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \]
and an element \( b \) belongs to \( A_2 \) if and only if

\[
\begin{align*}
b(x_{11}, x_{12}) &= (x_{11}, x_{12}) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \\
b(x_{21}, x_{22}) &= (x_{21}, x_{22}) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},
\end{align*}
\] (9)

where \( a_{ij}, b_{ij} \) are arbitrary complex numbers. It is straightforward to check that \( (A_1, A_2) \) is the desired TPP and \( \{M_1, M_2\}, \{N_1, N_2\} \) are irreducible component sets of \( (A_1, A_2) \).

Next we construct an inner product compatible TPS associated with \( (A_1, A_2) \). Take \( W_1 = M_1, W_2 = N_2 \) and define a bilinear map \( \otimes \) from \( W_1 \times W_2 \) to \( W \) such that

\[
\begin{align*}
x_{11} \otimes x_{21} &= x_{11}, & x_{11} \otimes x_{22} &= x_{12}, \\
x_{21} \otimes x_{21} &= x_{21}, & x_{21} \otimes x_{22} &= x_{22}.
\end{align*}
\] (11)

Then \( (W_1, W_2, \otimes) \) is an inner product compatible TPS. Notice that

\[
\begin{align*}
M_1 &= W_1 \otimes x_{21}, & M_2 &= W_1 \otimes x_{22}, \\
N_1 &= x_{11} \otimes W_2, & N_2 &= x_{21} \otimes W_2.
\end{align*}
\] (13)

Hence, \( M_1, M_2, N_1, N_2 \) are included in the decomposable vector set related to \( (W_1, W_2, \otimes) \). The construction is thus completed.

Before leaving this example, we would like to point out that \( \{\sigma_x^A \sigma_x^B, \sigma_z^A \sigma_z^B\} \) is also a standard complete set of observables. In fact, we have

\[
\begin{align*}
\sigma_x^A \sigma_x^B &|\psi^\pm\rangle_{AB} = \pm |\psi^\pm\rangle_{AB}, & \sigma_x^A \sigma_x^B &|\phi^\pm\rangle_{AB} = \pm |\phi^\pm\rangle_{AB}, \\
\sigma_z^A \sigma_z^B &|\psi^\pm\rangle_{AB} = - |\psi^\pm\rangle_{AB}, & \sigma_z^A \sigma_z^B &|\phi^\pm\rangle_{AB} = + |\phi^\pm\rangle_{AB}.
\end{align*}
\]

It follows that \( \{\sigma_x^A \sigma_x^B, \sigma_z^A \sigma_z^B\} \) is a standard complete set of observables with the same characteristic sets \( \{M_1, M_2\}, \{N_1, N_2\} \) as defined above and the above constructed \( (A_1, A_2) \) is also an inner product compatible TPP containing \( \{\sigma_x^A \sigma_x^B, \sigma_z^A \sigma_z^B\} \). In this sense, we may well call \( (R_x(\pi), S_z^2) \) and \( (\sigma_x^A \sigma_x^B, \sigma_z^A \sigma_z^B) \) equivalent.

B. Entanglement in Bargmann Space

Let \( W \) be the space \( \mathbb{C}[x_1, x_2] \) of two variable polynomial functions that span a Bargmann space of rank 2 [11]. Notice that here an element of \( \mathbb{C}[x_1, x_2] \) is regarded as a function rather
than a polynomial in the indeterminates $x_1$ and $x_2$. In studying this example, we have in mind the composite system of two one dimensional subsystems. Indeed, $W$ can be viewed in some way as a subspace of the space of states of such a system, $x_1$ and $x_2$ understood as coordinates of the two subsystems. This point of view was most recently casted on the narrowing effects of wave packets of two free particles due to their relative entanglement.

Obviously, $\{ x_1^j x_2^i | j, i = 0, 1, \cdots \}$ is a basis of $W$. Take $W_1 = \mathbb{C}[x_1]$, $W_2 = \mathbb{C}[x_2]$ and define the bilinear map $\otimes_1$ from $W_1 \times W_2$ to $W : \ x_1^i \otimes_1 x_2^j = x_1^i x_2^j$. Then $(W_1, W_2, \otimes_1)$ is a TPS of $W$, and actually this TPS is taken for granted. But we notice that if we define a bilinear map $\otimes_1'$: $W_1 \times W_2 \rightarrow W$ such that $x_1^j \otimes_1' x_2^i = \alpha_{ji} x_1^j x_2^i$ where $\alpha_{ji}$ is a nonzero complex number, then $(W_1, W_2, \otimes_1')$ is also a TPS. The state $x_1^j x_2^i$ is a product state with respect to both of the two tensor product structures. Nevertheless, the decomposable vector states related to the two tensor product structures are not identical when all $\alpha_{ji}$’s are not identical. For example, if $\alpha_{11} = \alpha_{12} = \alpha_{21} = 1$, but $\alpha_{22} = 2$, then

$$
x_1 x_2 + x_1^2 x_2 + x_1^2 x_2 + x_1^2 x_2 = (x_1 + x_1^2) \otimes_1 (x_2 + x_2^2),
$$

$$
x_1 x_2 + x_1^2 x_2 + x_1^2 x_2 + x_1^2 x_2 = x_1 \otimes_1' (x_2 + x_2^2) + x_1^2 \otimes_1' (x_2 + \frac{1}{2} x_2^2).
$$

So this is a product state with respect to $(W_1, W_2, \otimes_1)$ but an entangled state with respect to $(W_1, W_2, \otimes_1')$. This result is no surprise. In fact, when all $\alpha_{ji}$’s are not identical, generally speaking, $(W_1, W_2, \otimes_1)$ and $(W_1, W_2, \otimes_1')$ are associated with inequivalent tensor product partitions. This being true, the result is then implied by Lemma 7.1. This point can be argued as follows.

For simplicity, we suppose that $\alpha_{j0} = \alpha_{0i} = 1$ for all $j, i$. Let $M_i$ be the space spanned by $\{ x_1^j x_2^i | j = 0, 1, \cdots \}$, $N_j$ the space spanned by $\{ x_1^j x_2^i | i = 0, 1, \cdots \}$. For $\{ \alpha_{ji} \} \subseteq \mathbb{C}$, define a TPP $(A_1 (\alpha), A_2 (\alpha))$ such that (1) $\{ M_i \}, \{ N_j \}$ are irreducible component sets for $A_1 (\alpha)$ and $A_2 (\alpha)$ respectively; (2) $\{ \alpha_{ji} x_1^j x_2^i \}$ is a standard basis associated with $\{ M_i \}, \{ N_j \}$. Now one can check that $(W_1, W_2, \otimes_1)$ is associated with $(A_1 (\alpha), A_2 (\alpha))$ and

$$
A_1 (\alpha) = \varphi \cdot A_1 (1) \cdot \varphi^{-1}, \ A_2 (\alpha) = \varphi \cdot A_2 (1) \cdot \varphi^{-1}
$$

where $\varphi$ is an automorphism of $W$ such that

$$
\varphi (x_1^j x_2^i) = \alpha_{ji} x_1^j x_2^i.
$$

When all $\alpha_{ji}$’s are not identical, $\varphi$ is not an identity map. So it is highly possible that $(A_1 (\alpha), A_2 (\alpha))$ and $(A_1 (1), A_2 (1))$ are not equivalent.
C. Entanglement with respect to Mass of Center Coordinate

Now let us study another TPS of $W$ as a special case of the above example. The discussion is motivated by the simple consideration that a separable wave function of bi-particle system with respect to the two position coordinates can be regarded as an entangling one with respect to the center of mass and relative coordinates.

Let $X$ be the “coordinate of center of mass”, $x$ the “relative coordinate”:

$$X = \frac{1}{2}(x_1 + x_2), \; x = x_1 - x_2.$$  \hspace{1cm} (16)

One can check that $\{X^j x^i | j, i = 0, 1, \cdots \}$ is also a basis of $W$. Consider the operators $X \cdot \frac{\partial}{\partial X}$ and $x \cdot \frac{\partial}{\partial x}$, where $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial x}$ are derivatives with respect to $X$ and $x$ respectively. By definition we have

$$\left( X \cdot \frac{\partial}{\partial X} \right) X^j x^i = j X^j x^i, \quad (17)$$
$$\left( x \cdot \frac{\partial}{\partial x} \right) X^j x^i = i X^j x^i. $$

Let $M_i, N_j$ be the space spanned by $\{X^j x^i | j = 0, 1, \cdots \}$ and $\{X^j x^i | i = 0, 1, \cdots \}$ respectively. Then it follows that $(X \cdot \frac{\partial}{\partial X}, x \cdot \frac{\partial}{\partial x})$ is a standard complete set of operators, and $\{M_i\}, \{N_j\}$ are the characteristic sets of $X \cdot \frac{\partial}{\partial X}$ and $x \cdot \frac{\partial}{\partial x}$ respectively.

According to Proposition 4.1, there is a TPP containing $(X \cdot \frac{\partial}{\partial X}, x \cdot \frac{\partial}{\partial x})$. Let us explicitly construct such a TPP. We define the extended subalgebras $A_1, A_2 \subseteq \text{End}(W)$ as follows. An element $a$ belongs to $A_1$ if and only if

$$a X^j x^i = \sum_k X^k x^i a_{kj},$$  \hspace{1cm} (18)

and an element $b$ belongs to $A_2$ if and only if

$$b X^j x^i = \sum_k X^j x^k b_{ki},$$  \hspace{1cm} (19)

where $a_{kj} \in \mathbb{C}$ is independent of $i$, $b_{ki} \in \mathbb{C}$ is independent of $j$, and $\{k | a_{kj} \neq 0\}, \{k | b_{ki} \neq 0\}$ are both finite sets for each $j$ and each $i$ respectively. According to the proof of Proposition 4.1, $(A_1, A_2)$ is a TPP containing $(X \cdot \frac{\partial}{\partial X}, x \cdot \frac{\partial}{\partial x})$.

Now take $W'_1$ and $W'_2$ to be the subspaces spanned by $\{X^j\}$ and $\{x^i\}$ respectively, and define a bilinear map $\otimes_2 : W'_1 \times W'_2 \rightarrow W$ such that $X^j \otimes_2 x^i = X^j x^i$. It is then readily check that $(W'_1, W'_2, \otimes_2)$ is a TPS associated with $(A_1, A_2)$. 
Finally we point out that the decomposable vector set related to \( (W_1', W_2', \otimes_2) \) and that related to \( (W_1, W_2, \otimes_1) \) are different. For example, we consider the state \( x_1x_2 \). We have

\[
x_1x_2 = \frac{2X + x}{2} \frac{2X - x}{2} = X^2 \otimes_2 1 - \frac{1}{4} (1 \otimes_2 x^2).
\]

(20)

So it is an entangled state with respect to \( (W_1', W_2', \otimes_2) \). But it is a product state with respect to \( (W_1, W_2, \otimes_1) \).

The above argument demonstrate a simple but profound physical fact about the relativity of entanglement: generally speaking, the factorized wave function \( \Psi(x_1, x_2) = \Psi_1(x_1)\Psi_2(x_2) \) is entangled with respect to the "coordinate of center of mass" \( (X, x) \) since generally there are no functions \( \Phi_1, \Phi_2 \) such that

\[
\Psi(x_1, x_2) = \Phi_1(X)\Phi_2(x)
\]

(21)

though we do have

\[
\Psi(x_1, x_2) = \Psi_1(X + \frac{x}{2})\Psi_2(X - \frac{x}{2}).
\]

Remark: If we interprets \( X, x \) as creation operators and \( \frac{\partial}{\partial X}, \frac{\partial}{\partial x} \) as annihilation operators respectively, then the above discussion is applicable to settling down the issue of entanglement of Fock states, as promised in the introduction.

**IX. CONCLUDING REMARKS**

We have presented a rigorous algebraic description for the relativity of quantum entanglement due to the non-uniqueness of TPS of a vector space. Physically, there are many ways to subdivide the Hilbert space of a large system according to various physical purposes. In practice, different partitions correspond to different choices of observables in the measurement. According to the above discussion, this means that the notion of entanglement depends on the definition of tensor product in association with the subsystem partition. This reveals the seemingly exotic fact that multi-particle states that are entangled with respect to some subsystem partition may be separable with respect to different observations in the measurement. For example, a symmetrized state for two boson system is obviously an entangled state in the coordinate representation, but it is a tensor product of two number state with respect to some TPS. We think that it is safe to say
that the present paper has made clear the cloudy physical concept—quantum entanglement in a mathematical way.

Finally, we would like to remark that in this paper we have made efforts not to leave out the infinite dimensional case, avoiding the argument’s being too restrictive and excluding many physically interesting examples. But on the other hand, we have completely sacrificed topology for mathematical simplicity. So from mathematical point, especially analytical point of view, the present paper has left much to be desired. It seems desirable to study the inner product compatible TPS of an infinite dimensional Hilbert space $H$. Then we will have to consider the topology of $L(H)$, the set of linear operators on $H$, and to study some kind of partition of $L(H)$ related to the inner product compatible TPS of $H$ we might need to enter the field of operator algebra.

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