NONLINEAR DEGENERATE ELLIPTIC PROBLEMS
WITH $W^{1,1}_0(\Omega)$ SOLUTIONS

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Abstract. We study a nonlinear equation with an elliptic operator having degenerate coercivity. We prove the existence of a unique $W^{1,1}_0(\Omega)$ distributional solution under suitable summability assumptions on the source in Lebesgue spaces. Moreover, we prove that our problem has no solution if the source is a Radon measure concentrated on a set of zero harmonic capacity.

1. Introduction and statement of the results

In this paper we are going to study the nonlinear elliptic equation

\begin{equation}
\begin{cases}
-\text{div} \left( \frac{a(x, \nabla u)}{(1 + |u|)^\gamma} \right) + u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

under the following assumptions. The set $\Omega$ is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $\gamma > 0$, $f$ belongs to some Lebesgue space, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e., $a(\cdot, \xi)$ is measurable on $\Omega$ for every $\xi$ in $\mathbb{R}^N$, and $a(x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost every $x$ in $\Omega$) such that

\begin{align}
&a(x, \xi) \cdot \xi \geq \alpha |\xi|^2, \\
&|a(x, \xi)| \leq \beta |\xi|, \\
&[a(x, \xi) - a(x, \eta)] \cdot (\xi - \eta) > 0,
\end{align}

for almost every $x$ in $\Omega$ and for every $\xi$ and $\eta$ in $\mathbb{R}^N$, $\xi \neq \eta$, where $\alpha$ and $\beta$ are positive constants. We are going to prove that, under suitable assumptions on $\gamma$ and $f$, problem (1.1) has a unique distributional solution $u$ obtained by approximation, with $u$ belonging to the (non-reflexive) Sobolev space $W^{1,1}_0(\Omega)$. Furthermore, we are going to prove that problem (1.1) does not have a solution if $\gamma > 1$ and the datum $f$ is a bounded Radon measure concentrated on a set of zero harmonic capacity.
Problems like (1.1) have been extensively studied in the past. In [7] (see also [15], [16], [19]), existence and regularity results were proved, under the assumption that \(a(x, \xi) = A(x) \xi\), with \(A\) a uniformly elliptic bounded matrix, and \(0 < \gamma \leq 1\), for the problem

\[
\begin{align*}
-\text{div} \left( \frac{A(x) \nabla u}{(1 + |u|)^\gamma} \right) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \(f\) belongs to \(L^m(\Omega)\) for some \(m \geq 1\).

The main difficulty in dealing with problem (1.5) (or (1.1)) is that the differential operator, even if well defined between \(H^1_0(\Omega)\) and its dual \(H^{-1}(\Omega)\), is not coercive on \(H^1_0(\Omega)\) due to the fact that if \(u\) is large, \(\frac{1}{(1 + |u|)^\gamma}\) tends to zero (see [19] for an explicit example).

This lack of coercivity implies that the classical methods used in order to prove the existence of a solution for elliptic equations (see [18]) cannot be applied even if the datum \(f\) is regular. However, in [7], a whole range of existence results was proved, yielding solutions belonging to some Sobolev space \(W^{1,q}_0(\Omega)\), with \(q = q(\gamma, m) \leq 2\), if \(f\) is regular enough. Under weaker summability assumptions on \(f\), the gradient of \(u\) (and even \(u\) itself) may not be in \(L^1(\Omega)\); in this case, it is possible to give a meaning to solutions of problem (1.5), using the concept of \textit{entropy solutions} which has been introduced in [3].

If \(\gamma > 1\), a non existence result for problem (1.5) was proved in [1] (where the principal part is nonlinear with respect to the gradient), even for \(L^\infty(\Omega)\) data \(f\). Therefore, if the operator becomes “too degenerate”, existence may be lost even for data expected to give bounded solutions. However, as proved in [5], existence of solutions can be recovered by adding a lower order term of order zero. Indeed, if we consider the problem

\[
\begin{align*}
-\text{div} \left( \frac{A(x) \nabla u}{(1 + |u|)^\gamma} \right) + u &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

with \(f\) in \(L^m(\Omega)\), then the following results can be proved in the case \(\gamma > 1\) (see [5] and [11]):

i) if \(m > \gamma \frac{N}{2}\), then there exists a weak solution in \(H^1_0(\Omega) \cap L^\infty(\Omega)\);

ii) if \(m \geq \gamma + 2\), then there exists a weak solution in \(H^1_0(\Omega) \cap L^m(\Omega)\);

iii) if \(\frac{\gamma + 2}{2} < m < \gamma + 2\), then there exists a distributional solution in \(W^{1,\frac{2m}{\gamma + 2}}_0(\Omega) \cap L^m(\Omega)\).
iv) if \(1 \leq m \leq \frac{\gamma+2}{2}\), then there exists an entropy solution in \(L^m(\Omega)\) whose gradient belongs to the Marcinkiewicz space \(M^{\frac{2m}{\gamma+2}}(\Omega)\).

Note that if \(\gamma+2 \leq m < \gamma\frac{N}{2}\) and \(m\) tends to \(\gamma\frac{N}{2}\), the summability result of ii) is not “continuous” with the boundedness result of i), according to the following example (see also Example 3.3 of [5]).

**Example 1.1.** If \(\frac{2}{3} < \sigma < N - 2\), then \(u(x) = \frac{1}{|x|^{\sigma-1}}\) is a distributional solution of (1.6) with \(A(x) \equiv I\), and \(f(x) = \frac{\gamma(1+(\gamma-1))}{|x|^{\sigma}} + \frac{1}{|x|^{\sigma}} - 1\).

Due to the assumptions on \(\sigma\), both \(f\) and \(u\) belong to \(L^m(\Omega)\), with \(m < \gamma\frac{N}{2}\). If \(m\) tends to \(\gamma\frac{N}{2}\), i.e., if \(\sigma\) tends to \(\frac{2}{3}\), the solution \(u\) does not become bounded.

As stated before, this paper is concerned with two borderline cases connected with point iv) above:

A. if \(m = \frac{\gamma+2}{2}\), we will prove in Section 2 the existence of \(W^{1,1}_0(\Omega)\) distributional solutions, and in Section 3 their uniqueness;

B. if \(f\) is a bounded Radon measure concentrated on a set \(E\) of zero harmonic capacity and \(\gamma > 1\), we will prove in Section 4 non existence of solutions.

In the linear case, i.e., for the boundary value problem (1.6), a simple proof of the existence result is given in [6].

**Remark 1.2.** Let \(a(x,\xi) = A(x)\xi\), with \(A\) a bounded and measurable uniformly elliptic matrix, and let \(u \geq 0\) be a solution of

\[
-\text{div} \left( \frac{A(x)\nabla u}{(1+u)^\gamma} \right) + u = f,
\]

with \(\gamma > 1\) and \(f \geq 0\). If we define

\[
z = \frac{1}{\gamma - 1} \left( 1 - \frac{1}{(1+u)^{\gamma-1}} \right),
\]

then \(z\) is a solution of

\[
-\text{div}(A(x)\nabla z) + \left( \frac{1}{1-(\gamma - 1)z} \right)^{\frac{\gamma-1}{\gamma}} - 1 = f,
\]

which is an equation whose lower order term becomes singular as \(z\) tends to the value \(\frac{1}{\gamma - 1}\). For a study of these problems, see [3] and [14].

**Remark 1.3.** We explicitly state that our existence results can be generalized to equations with differential operators defined on \(W^{1,p}_0(\Omega)\), with \(p > 1\): if \(\gamma \geq \frac{(p-2)+}{p-1}\) and if \(m = \frac{\gamma(p-1)+2}{p}\), then it is possible to
prove the existence of a distributional solution $u$ in $W^{1,1}_0(\Omega) \cap L^m(\Omega)$ of the boundary value problem

$$
(1.7) \begin{cases}
-\text{div} \left( \frac{a(x, \nabla u)}{(1 + |u|)^{(p-1)\gamma}} \right) + u = f & \text{in } \Omega,
\quad \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $a(x, \xi)$ satisfies (1.2), (1.3) and (1.4) with $p$ instead of 2 (in (1.3), $a$ grows as $|\xi|^{p-1}$).

2. Existence of a $W^{1,1}_0(\Omega)$ solution

In this section we prove the existence of a $W^{1,1}_0(\Omega)$ solution to problem (1.1). Our result is the following.

**Theorem 2.1.** Let $\gamma > 0$, and let $f$ be a function in $L^\gamma(\Omega) + 2$. Then there exists a distributional solution $u$ in $W^{1,1}_0(\Omega) \cap L^\gamma(\Omega)$ of (1.1), that is,

$$
(2.1) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi + \int_{\Omega} u \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in W^{1,\infty}_0(\Omega).
$$

**Remark 2.2.** The previous result gives existence of a solution $u$ in $W^{1,1}_0(\Omega)$ to (1.6) for every $\gamma > 0$. If $0 < \gamma \leq 1$ existence results for (1.1) can also be proved by the same techniques of [7]. More precisely, if $f$ belongs to $L^m(\Omega)$ with $m > \frac{N}{N(1-\gamma)+1+\gamma}$ then (1.1) has a solution in $W^{1,q}_0(\Omega)$, with $q = \frac{Nm(1-\gamma)}{N-m(1+\gamma)}$. Note that when $m$ tends to $\frac{N}{N(1-\gamma)+1+\gamma}$, then $q$ tends to 1. We have now two cases: if $\frac{\gamma+2}{2} > \frac{N}{N(1-\gamma)+1+\gamma}$, that is, if $0 < \gamma < \frac{2}{2N-1}$, our result is weaker than the one in [7]. On the other hand, if $\frac{2}{N-1} \leq \gamma \leq 1$, then our result, which strongly uses the lower order term of order zero, is better.

**Remark 2.3.** The same existence result, with the same proof, holds for the following boundary value problem

$$
\begin{cases}
-\text{div}(b(x, u, \nabla u)) + u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

with $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function such that

$$
\frac{\alpha |\xi|^2}{(1 + |s|)^\gamma} \leq b(x, s, \xi) \cdot \xi \leq \beta |\xi|^2,
$$

where $\alpha, \beta, \gamma$ are positive constants.
To prove Theorem 2.1 we will work by approximation. First of all, let \( g \) be a function in \( L^\infty(\Omega) \). Then, by the results of [5], there exists a solution \( v \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) of

\[
\begin{cases}
  -\text{div} \left( \frac{a(x, \nabla v)}{(1 + |v|)^\gamma} \right) + v = g & \text{in } \Omega, \\
  v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.2)

In order for this paper to be self contained, we give here the easy proof of this fact. Let \( M = \|g\|_{L^\infty(\Omega)} + 1 \), and consider the problem

\[
\begin{cases}
  -\text{div} \left( \frac{a(x, \nabla v)}{(1 + |T_M(v)|)^\gamma} \right) + v = g & \text{in } \Omega, \\
  v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.3)

Here and in the following we define \( T_k(s) = \max(-k, \min(s, k)) \) for \( k \geq 0 \) and \( s \in \mathbb{R} \). Since the differential operator is pseudomonotone and coercive thanks to the assumptions on \( a \) and to the truncature, by the results of [18] there exists a weak solution \( v \) in \( H^1_0(\Omega) \) of (2.3).

Choosing \((|v| - \|g\|_{L^\infty(\Omega)}) + \text{sgn}(v)\) as a test function we obtain, dropping the nonnegative first term,

\[
\int_\Omega |v| (|v| - \|g\|_{L^\infty(\Omega)}) + \leq \int_\Omega \|g\|_{L^\infty(\Omega)} (|v| - \|g\|_{L^\infty(\Omega)})^+.
\]

Thus,

\[
\int_\Omega (|v| - \|g\|_{L^\infty(\Omega)}) (|v| - \|g\|_{L^\infty(\Omega)})^+ \leq 0,
\]

so that \(|v| \leq \|g\|_{L^\infty(\Omega)} < M\). Therefore, \( T_M(v) = v \), and \( v \) is a bounded weak solution of (2.2).

Let now \( f_n \) be a sequence of \( L^\infty(\Omega) \) functions which converges to \( f \) in \( L^{\frac{2+2}{2}}(\Omega) \), and such that \(|f_n| \leq |f|\) almost everywhere in \( \Omega \), and consider the approximating problems

\[
\begin{cases}
  -\text{div} \left( \frac{a(x, \nabla u_n)}{(1 + |u_n|)^\gamma} \right) + u_n = f_n & \text{in } \Omega, \\
  u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.4)

A solution \( u_n \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \) exists choosing \( g = f_n \) in (2.2). We begin with some a priori estimates on the sequence \( \{u_n\} \).
Lemma 2.4. If $u_n$ is a solution to problem (2.4), then, for every $k \geq 0$,

\begin{equation}
\int_{\{|u_n| \geq k\}} |u_n|^{\frac{2+2}{2}} \leq \int_{\{|u_n| \geq k\}} |f|^{\frac{2+2}{2}}; \tag{2.5}
\end{equation}

\begin{equation}
\int_{\{|u_n| \geq k\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{2+2}{2}}} \leq C \left[ \int_{\{|u_n| \geq k\}} |f|^{\frac{2+2}{2}} \right]^{\frac{1}{2+2}}; \tag{2.6}
\end{equation}

\begin{equation}
\int_{\{|u_n| \geq k\}} |\nabla u_n| \leq C \left[ \int_{\{|u_n| \geq k\}} |f|^{\frac{2+2}{2}} \right]^{\frac{1}{2+2}}; \tag{2.7}
\end{equation}

\begin{equation}
\alpha \int_{\Omega} |\nabla T_k(u_n)|^2 \leq k(1 + k)^{\gamma} \int_{\Omega} |f|. \tag{2.8}
\end{equation}

Here, and in the following, $C$ denotes a positive constant depending on $\alpha$, $\gamma$, $\text{meas}(\Omega)$, and the norm of $f$ in $L^{2+2}$. 

Proof. Let $k \geq 0$, $h > 0$, and let $\psi_{h,k}(s)$ be the function defined by

\[
\psi_{h,k}(s) = \begin{cases} 
0 & \text{if } 0 \leq s \leq k, \\
h(s - k) & \text{if } k < s \leq k + \frac{1}{h}, \\
1 & \text{if } s > k + \frac{1}{h}, \\
\psi_{h,k}(s) = -\psi_{h,k}(-s) & \text{if } s < 0.
\end{cases}
\]

Note that

\[
\lim_{h \to +\infty} \psi_{h,k}(s) = \begin{cases} 
1 & \text{if } s > k, \\
0 & \text{if } |s| \leq k, \\
-1 & \text{if } s < -k.
\end{cases}
\]

Let $\varepsilon > 0$, and choose $(\varepsilon + |u_n|)^{\frac{2}{2}} \psi_{h,k}(u_n)$ as a test function in (2.4); such a test function is admissible since $u_n$ belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$.
and \( \psi_{h,k}(0) = 0 \). We obtain

\[
\frac{\gamma}{2} \int_{\Omega} a(x, \nabla u_n) \cdot \nabla u_n \frac{(\varepsilon + |u_n|)^{\gamma - 1}}{(1 + |u_n|)^{\gamma}} |\psi_{h,k}(u_n)| \\
+ \int_{\Omega} a(x, \nabla u_n) \cdot \nabla u_n \psi'_{h,k}(u_n) (\varepsilon + |u_n|)^{\frac{\gamma}{2}} \\
+ \int_{\Omega} u_n (\varepsilon + |u_n|)^{\frac{\gamma}{2}} \psi_{h,k}(u_n) \\
= \int_{\Omega} f_n (\varepsilon + |u_n|)^{\frac{\gamma}{2}} \psi_{h,k}(u_n). \tag{2.9}
\]

By (1.2), and since \( \psi'_{h,k}(s) \geq 0 \), the first two terms are nonnegative, so that we obtain, recalling that \(|f_n| \leq |f|\),

\[
\int_{\Omega} u_n (\varepsilon + |u_n|)^{\frac{\gamma}{2}} \psi_{h,k}(u_n) \leq \int_{\Omega} |f| (\varepsilon + |u_n|)^{\frac{\gamma}{2}} |\psi_{h,k}(u_n)|.
\]

Letting \( \varepsilon \to 0 \) and \( h \to \infty \), we obtain, by Fatou’s lemma (on the left hand side) and by Lebesgue’s theorem (on the right hand side, recall that \( u_n \) belongs to \( L^\infty(\Omega) \)),

\[
\int_{\{u_n \geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{u_n \geq k\}} |f| |u_n|^{\frac{\gamma}{2}}.
\]

Using Hölder’s inequality on the right hand side we obtain

\[
\int_{\{u_n \geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \left[ \int_{\{u_n \geq k\}} |f|^{\frac{\gamma+2}{2}} \right]^{\frac{\gamma}{\gamma+2}} \left[ \int_{\{u_n \geq k\}} |u_n|^{\frac{\gamma+2}{2}} \right]^{\frac{2}{\gamma+2}}.
\]

Simplifying equal terms we thus have

\[
\int_{\{u_n \geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{u_n \geq k\}} |f|^{\frac{\gamma+2}{2}},
\]

which is (2.5). Note that from (2.5), written for \( k = 0 \), it follows

\[
\int_{\Omega} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\Omega} |f|^{\frac{\gamma+2}{2}} = \|f\|^{\frac{\gamma+2}{2}}_{L^{\frac{\gamma+2}{2}}(\Omega)}. \tag{2.10}
\]

Now we consider (2.9) written for \( \varepsilon = 1 \). Dropping the nonnegative second and third terms, and using that \(|f_n| \leq |f|\), we have

\[
\frac{\gamma}{2} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} |\psi_{h,k}(u_n)| \leq \int_{\Omega} |f| (1 + |u_n|)^{\frac{\gamma+2}{2}} |\psi_{h,k}(u_n)|.
\]

Using (1.2), and letting \( h \to \infty \), we get (using again Fatou’s lemma and Lebesgue’s theorem)

\[
\frac{\alpha \gamma}{2} \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} \leq \int_{\{u_n \geq k\}} |f| (1 + |u_n|)^{\frac{\gamma}{2}}.
\]
Hölder’s inequality on the right hand side then gives
\[
\alpha \gamma \int_{\{|u_n| \geq k\}} \frac{\|\nabla u_n\|^2}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} \leq \left( \int_{\{|u_n| \geq k\}} |f|^\frac{\gamma+2}{\gamma} \right)^{\frac{2}{\gamma}} \left( \int_{\{|u_n| \geq k\}} (1 + |u_n|)^{\frac{\gamma+2}{2}} \right)^{\frac{2}{\gamma+2}},
\]
so that, by (2.10),
\[
\alpha \gamma \int_{\{|u_n| \geq k\}} \frac{\|\nabla u_n\|^2}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} \leq C \left( \int_{\{|u_n| \geq k\}} |f|^\frac{\gamma+2}{\gamma} \right)^{\frac{2}{\gamma}},
\]
which is (2.6).

Then, again by Hölder’s inequality, and by (2.6) and (2.10),
\[
\int_{\{|u_n| \geq k\}} |\nabla u_n| = \int_{\{|u_n| \geq k\}} \frac{\|\nabla u_n\|^2}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} (1 + |u_n|)^{\frac{\gamma+2}{2}} \leq \left( \int_{\{|u_n| \geq k\}} |\nabla u_n|^2 \right)^{\frac{1}{2}} \left( \int_{\{|u_n| \geq k\}} (1 + |u_n|)^{\frac{\gamma+2}{2}} \right)^{\frac{1}{2}} \leq C \left( \int_{\{|u_n| \geq k\}} |f|^\frac{\gamma+2}{\gamma} \right)^{\frac{1}{2}} \left( \int_{\Omega} (1 + |u_n|)^{\frac{\gamma+2}{2}} \right)^{\frac{1}{2}} \leq C \left( \int_{\{|u_n| \geq k\}} |f|^\frac{\gamma+2}{\gamma} \right)^{\frac{1}{\gamma+2}},
\]
so that (2.7) is proved.

Finally, choosing \(T_k(u_n)\) as a test function in (2.4) we get, dropping the nonnegative linear term, and using (1.2),
\[
\alpha \int_{\Omega} |\nabla T_k(u_n)|^2 \leq k(1 + k)\gamma \int_{\Omega} |f|,
\]
which is (2.8).

**Lemma 2.5.** If \(\{u_n\}\) is the sequence of solutions to (2.4), there exists a subsequence, still denoted by \(\{u_n\}\), and a function \(u\in L^{\frac{\gamma+2}{\gamma}}(\Omega)\), with \(T_k(u)\) belonging to \(H^1_0(\Omega)\) for every \(k > 0\), such that \(u_n\) almost everywhere converges to \(u\) in \(\Omega\), and \(T_k(u_n)\) weakly converges to \(T_k(u)\) in \(H^1_0(\Omega)\).

**Proof.** Consider (2.6) written for \(k = 0\):
\[
\int_{\Omega} \frac{\|\nabla u_n\|^2}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} \leq C \|f\|_{L^{\frac{\gamma+2}{\gamma}}(\Omega)}.
\]
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Since (if $\gamma \neq 2$)
\[
\frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{2}{2-\gamma}}} = \frac{16}{(2 - \gamma)^2} |\nabla [(1 + |u_n|)^{\frac{2}{2-\gamma}} - 1]|^2,
\]
the sequence $v_n = \frac{4}{2-\gamma} [(1 + |u_n|)^{\frac{2}{2-\gamma}} - 1] \text{sgn}(u_n)$ is bounded in $H^1_0(\Omega)$ by (2.12). If $\gamma = 2$ we have
\[
\frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{2}{2}}} = |\nabla \log(1 + |u_n|)|^2,
\]
so that $v_n = [\log(1 + |u_n|)] \text{sgn}(u_n)$ is bounded in $H^1_0(\Omega)$. In both cases, up to a subsequence still denoted by $v_n$, $v_n$ converges to some function $v$ weakly in $H^1_0(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in $\Omega$. If $\gamma < 2$, define
\[
u(x) = \left[ \left(\frac{2-\gamma}{4} |v(x)| + 1 \right)^{\frac{4}{2-\gamma}} - 1 \right] \text{sgn}(v(x)),
\]
if $\gamma > 2$ define
\[
u(x) = \begin{cases} 
\left[ \left(\frac{2-\gamma}{4} |v(x)| + 1 \right)^{\frac{4}{2-\gamma}} - 1 \right] \text{sgn}(v(x)) & \text{if } |v(x)| < \frac{4}{\gamma-2}, \\
+\infty & \text{if } v(x) = \frac{4}{\gamma-2}, \\
-\infty & \text{if } v(x) = -\frac{4}{\gamma-2},
\end{cases}
\]
while if $\gamma = 2$, define
\[
u(x) = [e^{\text{sgn}(v(x)} - 1] \text{sgn}(v(x)).
\]
Thus, $u_n$ almost everywhere converges, up to a subsequence still denoted by $u_n$, to $u$. From now on, we will consider this particular subsequence, for which it holds that $u_n$ almost everywhere converges to $u$.

We use now (2.5) written for $k = 0$:
\[
\int_{\Omega} |u_n|^{\frac{\gamma+2}{\gamma}} \leq \int_{\Omega} |f|^{\frac{\gamma+2}{\gamma}} \leq C.
\]
Since $u_n$ almost everywhere converges to $u$, we have from Fatou’s lemma that
\[
\int_{\Omega} |u|^{\frac{\gamma+2}{\gamma}} \leq C.
\]
Hence $u$ belongs to $L^{\frac{\gamma+2}{\gamma}}(\Omega)$, which implies that $u$ is almost everywhere finite (note that if $\gamma > 2$ this fact did not follow from the definition of $u$, since $|v|$ could have assumed the value $\frac{4}{\gamma-2}$ on a set of positive measure).
Let now $k > 0$; since from (2.8) it follows that the sequence $\{T_k(u_n)\}$ is bounded in $H^1_0(\Omega)$, there exists a subsequence $T_k(u_{n_j})$ which weakly converges to some function $v_k$ in $H^1_0(\Omega)$. Using the almost everywhere convergence of $u_n$ to $u$, we have that $v_k = T_k(u)$. Since the limit is independent on the subsequence, then the whole sequence $\{T_k(u_n)\}$ weakly converges to $T_k(u)$, for every $k > 0$. □

Remark 2.6. Using the fact that $T_k(u)$ is in $H^1_0(\Omega)$ for every $k > 0$, and the results of [3], we have that there exists a unique measurable function $v_k$ with values in $\mathbb{R}^N$, such that

$$\nabla T_k(u) = v_k \chi_{\{|u| \leq k\}}$$

almost everywhere in $\Omega$, for every $k > 0$.

Following again [3], we will define $\nabla u = v$, the approximate gradient of $u$.

Remark 2.7. We emphasize that if $\gamma = 2$, then (2.11), written for $k = 0$, becomes

$$\int_{\Omega} |\nabla u_n|^2 \leq \left[ \int_{\Omega} \frac{|\nabla u_n|^2}{(1 + |u_n|)^2} \right]^\frac{1}{2} \left[ \int_{\Omega} (1 + |u_n|)^2 \right]^{\frac{1}{2}}.$$

Since

$$\frac{|\nabla u_n|^2}{(1 + |u_n|)^2} = |\nabla \log(1 + |u_n|)|^2,$$

a nonlinear interpolation result follows: let $A$ be in $\mathbb{R}^+$ and let $v$ in $L^2(\Omega)$ be such that $\log(A + |v|)$ belongs to $H^1_0(\Omega)$. Then $v$ belongs to $W^{1,1}_0(\Omega)$, and

$$\int_{\Omega} |\nabla| \leq \| \log(A + |v|) \|_{H^1_0(\Omega)} \left[ \int_{\Omega} (A + |v|) \right]^\frac{1}{2}.$$

Our next result deals with the strong convergence of $T_k(u_n)$ in $H^1_0(\Omega)$.

Proposition 2.8. Let $u_n$ and $u$ be the sequence of solutions to problems (2.4) and the function in $L^{\frac{2\gamma}{\gamma+2}}(\Omega)$ given by Lemma 2.5. Then, for every fixed $k > 0$, $T_k(u_n)$ strongly converges to $T_k(u)$ in $H^1_0(\Omega)$, as $n$ tends to infinity.

Proof. We follow the proof of [17].

Let $h > k$ and choose $T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]$ as a test function in (2.4). We have

$$\int_{\Omega} a(x, \nabla u_n) \cdot \nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]$$

$$= -\int_{\Omega} (u_n - f_n) T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)].$$

(2.13)
We observe that the right hand side converges to zero as first $n$ and then $h$ tend to infinity, since $u_n$ converges to $u$ almost everywhere in $\Omega$ and $u_n$ and $f_n$ are bounded in $L^{\frac{2}{1+\gamma}}(\Omega)$. Thus, if we define $\varepsilon(n, h)$ as any quantity such that

$$\lim_{h \to +\infty} \lim_{n \to +\infty} \varepsilon(n, h) = 0,$$

then

$$\int_\Omega \varepsilon(n, h) = 0.$$

Let $M = 4k + h$. Observing that $\nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)] = 0$ if $|u_n| \geq M$, by (2.13) we have

$$\varepsilon(n, h) = \int_{\{|u_n| < k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla [u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1 + |u_n|)^\gamma} + \int_{\{|u_n| \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla [u_n - T_h(u_n)]}{(1 + |u_n|)^\gamma}.$$

Since $u_n - T_h(u_n) = 0$ in $\{|u_n| < k\}$ and $\nabla T_k(u_n) = 0$ in $\{|u_n| \geq k\}$, we have, using that $a(x, 0) = 0$,

$$\varepsilon(n, h) = \int_\Omega \frac{a(x, \nabla T_k(u_n)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + |u_n|)^\gamma}.$$

The second term of the right hand side is positive, so that

$$\varepsilon(n, h) \geq \int_\Omega \frac{a(x, \nabla T_k(u_n) - a(x, \nabla T_k(u)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + k)^\gamma} + \int_\Omega \frac{a(x, \nabla T_k(u)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + |u_n|)^\gamma} - \int_{\{|u_n| \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)}{(1 + |u_n|)^\gamma} = I_n + J_n - K_n.$$

The last two terms tend to zero as $n$ tends to infinity. Indeed

$$\lim_{n \to +\infty} J_n = \lim_{n \to +\infty} \int_\Omega \frac{a(x, \nabla T_k(u)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + |u_n|)^\gamma} = 0,$$

since $T_k(u_n)$ converges to $T_k(u)$ weakly in $H^1_0(\Omega)$ and $\frac{a(x, \nabla T_k(u))}{(1 + |u_n|)^\gamma}$ is strongly compact in $(L^2(\Omega))^N$ by the growth assumption (1.3) on $a$. 

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The last term can be rewritten as
\[ K_n = \int_{\Omega} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u) \chi_{\{|u_n| \geq k\}}}{(1 + |u_n|)^\gamma} \cdot \nabla T_k(u) \chi_{\{|u_n| \geq k\}} \cdot \nabla T_k(u) \chi_{\{|u_n| \geq k\}} \cdot \nabla T_k(u) \chi_{\{|u_n| \geq k\}} \cdot (1 + |u_n|)^\gamma. \]

Since \( M \) is fixed with respect to \( n \), then the sequence \( \{a(x, \nabla T_M(u_n))\} \) is bounded in \((L^2(\Omega))^N\). Hence, there exists \( \sigma \) in \((L^2(\Omega))^N\), and a subsequence \( \{a(x, \nabla T_M(u_{n_j}))\} \), such that
\[ \lim_{j \to +\infty} a(x, \nabla T_M(u_{n_j})) = \sigma, \]
weakly in \((L^2(\Omega))^N\). On the other hand,
\[ \lim_{n \to +\infty} \nabla T_k(u) \chi_{\{|u_n| \leq k\}} \cdot (1 + |u_n|)^\gamma = \nabla T_k(u) \chi_{\{|u_n| \leq k\}} \cdot (1 + |u_n|)^\gamma = 0, \]
strongly in \((L^2(\Omega))^N\), and so
\[ \lim_{j \to +\infty} K_{n_j} = \lim_{j \to +\infty} \int_{\{|u_n| \geq k\}} \frac{a(x, \nabla T_M(u_{n_j})) \cdot \nabla T_k(u)}{(1 + |u_{n_j}|)^\gamma} = 0. \]
Since the limit does not depend on the subsequence, we have
\[ \lim_{n \to +\infty} K_n = \lim_{n \to +\infty} \int_{\{|u_n| \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)}{(1 + |u_n|)^\gamma} = 0, \]
as desired. Therefore,
\[ \varepsilon(n, h) \geq I_n = \int_{\Omega} \frac{[a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + k)^\gamma}, \]
so that, thanks to (1.4),
\[ \lim_{n \to +\infty} \int_{\Omega} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla [T_k(u_n) - T_k(u)] = 0. \]
Using this formula, (1.4) and the results of [10] and [8], we conclude that \( T_k(u_n) \) strongly converges to \( T_k(u) \) in \( H_0^1(\Omega) \), as desired. □

**Corollary 2.9.** Let \( u_n \) and \( u \) be as in Proposition 2.8. Then \( \nabla u_n \) converges to \( \nabla u \) almost everywhere in \( \Omega \), where \( \nabla u \) has been defined in Remark 2.6.

**Lemma 2.10.** Let \( u_n \) and \( u \) be as in Proposition 2.8. Then \( \nabla u_n \) strongly converges to \( \nabla u \) in \((L^1(\Omega))^N\). Moreover \( u_n \) strongly converges to \( u \) in \( L^{1/2+2}(\Omega) \).

**Proof.** We begin by proving the convergence of \( \nabla u_n \) to \( \nabla u \). Let \( \varepsilon > 0 \), and let \( k > 0 \) be sufficiently large such that
\[ (2.14) \quad \left[ \int_{\{|u_n| \geq k\}} |f|^{\frac{4+2}{2}} \right]^{\frac{2}{4+2}} < \varepsilon, \]
uniformly with respect to \( n \). This can be done thanks to (2.10) and to the absolute continuity of the integral. Let \( E \) be a measurable set. Writing
\[
\int_E |\nabla u_n| = \int_E |\nabla T_k(u_n)| + \int_{E \cap \{|u_n| \geq k\}} |\nabla u_n|
\]
we have, by (2.7), and by (2.14),
\[
\int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + C \varepsilon.
\]
Using Hölder’s inequality and (2.8), we obtain
\[
\int_E |\nabla u_n| \leq C \text{meas}(E)^{\frac{1}{2}} k^{\frac{1}{2}} (1 + k)^{\frac{1}{2}} \left( \int_{\Omega} |f| \right)^{\frac{1}{2}} + C \varepsilon.
\]
Choosing \( \text{meas}(E) \) small enough (recall that \( k \) is now fixed) we have
\[
\int_E |\nabla u_n| \leq C \varepsilon,
\]
uniformly with respect to \( n \), where \( C \) does not depend on \( n \) or \( \varepsilon \). Since \( \nabla u_n \) almost everywhere converges to \( \nabla u \) by Corollary 2.9, we can apply Vitali’s theorem to obtain the strong convergence of \( \nabla u_n \) to \( \nabla u \) in \( (L^1(\Omega))^N \).

As for the second convergence, by (2.8) we have
\[
\int_E |u_n|^\gamma \leq \int_{E \cap \{|u_n| \leq k\}} |u_n|^\gamma + \int_{E \cap \{|u_n| \geq k\}} |u_n|^\gamma
\]
\[
\leq k^{\frac{\gamma+2}{2}} \text{meas}(E) + \int_{\{|u_n| \geq k\}} |f|^\gamma.
\]
As before, we first choose \( k \) such that the second integral is small, uniformly with respect to \( n \), and then the measure of \( E \) small enough such that the first term is small. The almost everywhere convergence of \( u_n \) to \( u \), and Vitali’s theorem, then imply that \( u_n \) strongly converges to \( u \) in \( L^{\frac{\gamma+2}{2}}(\Omega) \).

**Remark 2.11.** Since we have proved that \( \nabla u_n \) strongly converges to \( \nabla u \) in \( (L^1(\Omega))^N \), so that \( u \) belongs to \( W^{1,1}_0(\Omega) \), then the approximate gradient \( \nabla u \) of \( u \) is nothing but the distributional gradient of \( u \) (see [3]).

**Proof of Theorem 2.7.** Using the previous results, we pass to the limit, as \( n \) tends to infinity, in the weak formulation of (2.4). Starting from
\[
\int_{\Omega} a(x, \nabla u_n) \cdot \nabla \varphi + \int_{\Omega} u_n \varphi = \int_{\Omega} f_n \varphi, \quad \varphi \in W^{1,\infty}_0(\Omega),
\]
the limit of the second and the last integral is easy to compute; indeed, recall that by Lemma \ref{lemma:convergence} and by definition of \( f_n \), the sequences \( \{u_n\} \) and \( \{f_n\} \) strongly converge to \( u \) and \( f \) respectively in \( L^{\frac{\gamma+2}{2}}(\Omega) \), hence in \( L^1(\Omega) \). For the first integral, we have that \( a(x, \nabla u_n) \) converges almost everywhere in \( \Omega \) to \( a(x, \nabla u) \) thanks to Corollary \ref{corollary:conv_lem} and the continuity assumption on \( a(x, \cdot) \); furthermore, \ref{eq:beta} implies that

\[
|a(x, \nabla u_n)| \leq \beta |\nabla u_n|,
\]

and the right hand side is compact in \( L^1(\Omega) \) by Lemma \ref{lemma:convergence}. Thus, by Vitali’s theorem \( a(x, \nabla u_n) \) strongly converges to \( a(x, \nabla u) \) in \( (L^1(\Omega))^N \), so that

\[
\lim_{n \to +\infty} \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla \varphi}{(1 + |u_n|)^\gamma} = \int_\Omega \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^\gamma},
\]

where we have also used that \( u_n \) almost everywhere converges to \( u \), and Lebesgue’s theorem. Thus, we have that

\[
\int_\Omega \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^\gamma} + \int_\Omega u \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in W^{1,\infty}(\Omega),
\]

i.e., \( u \) satisfies \ref{eq:2.1}. \qed

3. **Uniqueness of the solution obtained by approximation**

Let \( f \in L^{\frac{\gamma+2}{2}}(\Omega) \), let \( f_n \) be a sequence of \( L^\infty(\Omega) \) functions converging to \( f \) in \( L^{\frac{\gamma+2}{2}}(\Omega) \), with \( |f_n| \leq |f| \), and let \( u_n \) be a solution of \ref{eq:2.4}. In Section 2 we proved the existence of a distributional solution \( u \) in \( W^{1,\infty}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega) \) to \ref{eq:1.1}, such that, up to a subsequence,

\[
\lim_{n \to +\infty} \|u_n - u\|_{W^{1,\infty}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)} = 0.
\]

Now, let \( g \in L^{\frac{\gamma+2}{2}}(\Omega) \), let \( g_n \) be a sequence of \( L^\infty(\Omega) \) functions converging to \( g \) in \( L^{\frac{\gamma+2}{2}}(\Omega) \), with \( |g_n| \leq |g| \), and let \( z_n \) in \( H^{1}(\Omega) \cap L^\infty(\Omega) \) be a weak solution of

\[
\begin{cases}
-\text{div} \left( \frac{a(x, \nabla z_n)}{(1 + |z_n|)^\gamma} \right) + z_n = g_n & \text{in } \Omega, \\
z_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Then, up to a subsequence, we can assume that

\[
\lim_{n \to +\infty} \|z_n - z\|_{W^{1,\infty}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)} = 0,
\]

and
where $z$ in $W^{1,1}_0(\Omega) \cap L^{\frac{2^*}{2}}(\Omega)$ is a distributional solution of

$$(3.4) \begin{cases} -\text{div} \left( \frac{a(x, \nabla z)}{(1 + |z|^\gamma)} \right) + z = f & \text{in } \Omega, \\ z = 0 & \text{on } \partial \Omega. \end{cases}$$

Our result, which will imply the uniqueness of the solution by approximation (see [12]) of (1.1), is the following.

**Theorem 3.1.** Assume that $u_n$ and $z_n$ are solutions of (2.4) and (3.2) respectively, and that (3.1) and (3.3) hold true, with $u$ and $z$ solutions of (1.1) and (3.4) respectively. Then

$$(3.5) \int_\Omega |u - z| \leq \int_\Omega |f - g|.$$  

Moreover,

$$(3.6) \quad f \leq g \text{ a.e. in } \Omega \quad \text{implies} \quad u \leq z \text{ a.e. in } \Omega.$$  

**Proof.** Substracting (3.2) from (2.4) we obtain

$$-\text{div} \left( \left[ \frac{a(x, \nabla u_n)}{(1 + |u_n|^\gamma)} - \frac{a(x, \nabla z_n)}{(1 + |z_n|^\gamma)} \right] \right) + u_n - z_n = f_n - g_n.$$  

Choosing $T_h(u_n - z_n)$ as a test function we have

$$\int_\Omega \left[ \frac{a(x, \nabla u_n)}{(1 + |u_n|^\gamma)} - \frac{a(x, \nabla z_n)}{(1 + |z_n|^\gamma)} \right] \cdot \nabla T_h(u_n - z_n) + \int_\Omega (u_n - z_n) T_h(u_n - z_n) = \int_\Omega (f_n - g_n) T_h(u_n - z_n).$$

This equality can be written in an equivalent way as

$$\int_\Omega \left[ \frac{a(x, \nabla u_n) - a(x, \nabla z_n)}{(1 + |u_n|^\gamma)} \cdot \nabla T_h(u_n - z_n) + \int_\Omega (u_n - z_n) T_h(u_n - z_n) = \int_\Omega (f_n - g_n) T_h(u_n - z_n) \right. \left. - \int_\Omega \left[ \frac{1}{(1 + |u_n|^\gamma)} - \frac{1}{(1 + |z_n|^\gamma)} \right] a(x, \nabla z_n) \cdot \nabla T_h(u_n - z_n). \right.$$  

By (1.4), the first term of the left hand side is nonnegative, so that it can be dropped; using Lagrange’s theorem on the last term of the right hand side, we therefore have, since the absolute value of the derivative
of the function $s \mapsto \frac{1}{(1+|s|)^\gamma}$ is bounded by $\gamma$,

\[ \int_{\Omega} (u_n - z_n)T_h(u_n - z_n) \leq \int_{\Omega} (f_n - g_n)T_h(u_n - z_n) + \gamma h \int_{\Omega} \|a(x, \nabla z_n)\| \nabla T_h(u_n - z_n). \]

Dividing by $h$ we obtain

\[ \int_{\Omega} \frac{(u_n - z_n)T_h(u_n - z_n)}{h} \leq \int_{\Omega} \frac{|f_n - g_n| T_h(u_n - z_n)}{h} + \gamma \int_{\Omega} \|a(x, \nabla z_n)\| \nabla T_h(u_n - z_n). \]

Since, for every fixed $n$, $u_n$ and $z_n$ belong to $H^1_0(\Omega)$, and $a(x, \xi)$ satisfies (1.3), the limit as $h$ tends to zero gives

\[ (3.7) \quad \int_{\Omega} |u_n - z_n| \leq \int_{\Omega} |f_n - g_n|, \]

which then yields (3.5) passing to the limit and using the second part of Lemma 2.10.

The use of $T_h(u_n - z_n)^+$ as a test function and the same technique as above imply that

\[ \int_{\Omega} (u_n - z_n)^+ \leq \int_{\{u_n \geq z_n\}} (f_n - g_n). \]

Hence, passing to the limit as $n$ tends to infinity, we obtain, if we suppose that $f \leq g$ almost everywhere in $\Omega$,

\[ \int_{\Omega} (u - z)^+ \leq \int_{\{u \geq z\}} (f - g) \leq 0, \]

so that (3.6) is proved.

Thanks to (3.5), we can prove that problem (1.1) has a unique solution obtained by approximation.

**Corollary 3.2.** There exists a unique solution obtained by approximation of (1.1), in the sense that the solution $u$ in $W^{1,1}_0(\Omega) \cap L^{\frac{\gamma+2}{\gamma}}(\Omega)$ obtained as limit of the sequence $u_n$ of solutions of (2.4) does not depend on the sequence $f_n$ chosen to approximate the datum $f$ in $L^{\frac{\gamma+2}{\gamma}}(\Omega)$.

**Remark 3.3.** Note that (3.7) implies the uniqueness of the solution of (2.2), while (3.6) implies that if $f \geq 0$, then the solution $u$ of (1.1) is nonnegative.
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Remark 3.4. Corollary 3.2, together with estimates (3.5) and (2.5), implies that the map $S$ from $L^{\frac{\gamma+2}{2}}(\Omega)$ into itself defined by $S(f) = u$, where $u$ is the solution of (1.1) with datum $f$, is well defined and satisfies

$$\|S(f) - S(g)\|_{L^1(\Omega)} \leq \|f - g\|_{L^1(\Omega)}, \quad \|S(f)\|_{L^{\frac{\gamma+2}{2}}(\Omega)} \leq \|f\|_{L^{\frac{\gamma+2}{2}}(\Omega)}.$$ 

4. A NON EXISTENCE RESULT

As stated in the Introduction, we prove here a non existence result for solutions of (1.1) if the datum is a bounded Radon measure concentrated on a set $E$ of zero harmonic capacity.

Theorem 4.1. Assume that $\gamma > 1$, and let $\mu$ be a nonnegative Radon measure, concentrated on a set $E$ of zero harmonic capacity. Then there is no solution to

$$\begin{cases}
-\text{div} \left( \frac{a(x, \nabla u)}{(1 + u)\gamma} \right) + u = \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

More precisely, if $\{f_n\}$ is a sequence of nonnegative $L^\infty(\Omega)$ functions which converges to $\mu$ in the tight sense of measures, and if $u_n$ is the sequence of solutions to (2.4), then $u_n$ tends to zero almost everywhere in $\Omega$ and

$$\lim_{n \to +\infty} \int_\Omega u_n \varphi = \int_\Omega \varphi \, d\mu \quad \forall \varphi \in W^{1,\infty}_0(\Omega).$$

Remark 4.2. A similar non existence result for the case $\gamma \leq 1$ is much more complicated to obtain. Indeed, if for example $a(x, \xi) = \xi$, and $\gamma = 1$, the change of variables $v = \log(1 + u)$ yields that $v$ is a solution to

$$\begin{cases}
-\Delta v + e^v - 1 = \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Existence and non existence of solutions for such a problem has been studied in [9] (where the concept of “good measure” was introduced) and in [21] (if $N = 2$) and [2] (if $N \geq 3$).

Proof. Let $\mu$ be as in the statement. Then (see [13]) for every $\delta > 0$ there exists a function $\psi_\delta$ in $C^\infty_0(\Omega)$ such that

$$0 \leq \psi_\delta \leq 1, \quad \int_\Omega |\nabla \psi_\delta|^2 \leq \delta, \quad \int_\Omega (1 - \psi_\delta) \, d\mu \leq \delta.$$
Note that, as a consequence of the estimate on $\psi_\delta$ in $H^1_0(\Omega)$, and of the fact that $0 \leq \psi_\delta \leq 1$, $\psi_\delta$ tends to zero in the weak* topology of $L^\infty(\Omega)$ as $\delta$ tends to zero.

If $f_n$ is a sequence of nonnegative functions which converges to $\mu$ in the tight convergence of measures, that is, if

$$
\lim_{n \to +\infty} \int_\Omega f_n \varphi = \int_\Omega \varphi \, d\mu, \quad \forall \varphi \in C^0(\overline{\Omega}),
$$

then

$$
(4.1) \quad 0 \leq \lim_{n \to +\infty} \int_\Omega f_n(1 - \psi_\delta) = \int_\Omega (1 - \psi_\delta) \, d\mu \leq \delta.
$$

Let $u_n$ be the nonnegative solution to the approximating problem (2.4). If we choose $1 - (1 + u_n)^{1-\gamma}$ as a test function in (2.4), we have, by (1.2), and dropping the nonnegative lower order term,

$$
\alpha(\gamma - 1) \int_\Omega \left| \frac{\nabla u_n}{(1 + u_n)^\gamma} \right|^2 \leq (\gamma - 1) \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1 + u_n)^{2\gamma}} \leq \int_\Omega f_n.
$$

Recalling (1.3), we thus have

$$
\int_\Omega \left| \frac{a(x, \nabla u_n)}{(1 + u_n)^\gamma} \right|^2 \leq \beta \int_\Omega \left| \frac{\nabla u_n}{(1 + u_n)^\gamma} \right|^2 \leq C \int_\Omega f_n,
$$

with $C$ depending on $\alpha$, $\beta$ and $\gamma$. Therefore, up to a subsequence, there exist $\sigma$ in $(L^2(\Omega))^N$ and $\rho$ in $L^2(\Omega)$ such that

$$
(4.2) \quad \lim_{n \to +\infty} \frac{a(x, \nabla u_n)}{(1 + u_n)^\gamma} = \sigma, \quad \lim_{n \to +\infty} \left| \frac{\nabla u_n}{(1 + u_n)^\gamma} \right| = \rho,
$$

weakly in $(L^2(\Omega))^N$ and $L^2(\Omega)$ respectively.

The choice of $[1 - (1 + u_n)^{1-\gamma}] (1 - \psi_\delta)$ as a test function in (2.4) gives

$$
(\gamma - 1) \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1 + u_n)^{2\gamma}} (1 - \psi_\delta)
$$

$$
+ \int_\Omega u_n [1 - (1 + u_n)^{1-\gamma}] (1 - \psi_\delta)
$$

$$
= \int_\Omega f_n [1 - (1 + u_n)^{1-\gamma}] (1 - \psi_\delta)
$$

$$
+ \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla \psi_\delta}{(1 + u_n)^\gamma} [1 - (1 + u_n)^{1-\gamma}]
$$

$$
\leq \int_\Omega f_n (1 - \psi_\delta)
$$

$$
+ \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla \psi_\delta}{(1 + u_n)^\gamma} [1 - (1 + u_n)^{1-\gamma}].
$$

(4.3)
We study the right hand side. For the first term, (4.1) implies that
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\Omega} f_n (1 - \psi_\delta) = 0 ,
\]
while for the second one, we have, using (4.2), and the boundedness of 
\([1 - (1 + u_n)^{1-\gamma}]\),
\[
\lim_{n \to +\infty} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla \psi_\delta}{(1 + u_n)^\gamma} [1 - (1 + u_n)^{1-\gamma}] = \int_{\Omega} \sigma \cdot \nabla \psi_\delta [1 - (1 + u)^{1-\gamma}] .
\]
Recalling that \(\sigma\) is in \((L^2(\Omega))^N\), that \(\psi_\delta\) tends to zero in \(H^1_0(\Omega)\), and using the boundedness \([1 - (1 + u_n)^{1-\gamma}]\), we have
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla \psi_\delta}{(1 + u_n)^\gamma} [1 - (1 + u_n)^{1-\gamma}] = 0 .
\]
Therefore, since both terms of the left hand side of (4.3) are nonnegative, we obtain
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1 + u_n)^{2\gamma}} (1 - \psi_\delta) = 0 .
\]
Assumption (1.2) then gives
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \alpha \int_{\Omega} \left( \frac{\nabla u_n}{(1 + u_n)^\gamma} \right)^2 (1 - \psi_\delta) 
\leq \lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1 + u_n)^{2\gamma}} (1 - \psi_\delta) = 0 .
\]
Since the functional 
\[
v \in L^2(\Omega) \mapsto \int_{\Omega} |v|^2 (1 - \psi_\delta)
\]
is weakly lower semicontinuous on \(L^2(\Omega)\), we have
\[
\int_{\Omega} |\rho|^2 = \lim_{\delta \to 0^+} \int_{\Omega} |\rho|^2 (1 - \psi_\delta) \leq \lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\Omega} \left( \frac{\nabla u_n}{(1 + u_n)^\gamma} \right)^2 (1 - \psi_\delta) = 0 ,
\]
which implies that \(\rho = 0\). Thus, since
\[
\frac{\nabla u_n}{(1 + u_n)^\gamma} = \frac{1}{\gamma - 1} \nabla \left( 1 - (1 + u_n)^{1-\gamma} \right) ,
\]
by the second limit of (4.2) the sequence \(1 - (1 + u_n)^{1-\gamma}\) weakly converges to zero in \(H^1_0(\Omega)\), and so (up to subsequences) it strongly converges to zero in \(L^2(\Omega)\). Therefore \(u_n\) (up to subsequences) tends to zero almost everywhere in \(\Omega\). Since the limit does not depend on the subsequence, the whole sequence \(u_n\) tends to zero almost everywhere in \(\Omega\).
We now have, for $\Phi$ in $(L^2(\Omega))^N$, and by (1.3),
\[
\left| \int_{\Omega} a(x, \nabla u_n) \cdot \Phi \right| \leq \int_{\Omega} \frac{|a(x, \nabla u_n)|}{(1 + |u_n|)^\gamma} |\Phi| \leq \beta \int_{\Omega} \frac{|\nabla u_n|}{(1 + |u_n|)^\gamma} |\Phi|.
\]
Thus, by (1.2),
\[
\left| \int_{\Omega} \sigma \cdot \Phi \right| = \lim_{n \to +\infty} \left| \int_{\Omega} \frac{a(x, \nabla u_n)}{(1 + |u_n|)^\gamma} \cdot \Phi \right| \leq \beta \int_{\Omega} \rho |\Phi| = 0,
\]
which implies that $\sigma = 0$. Therefore, passing to the limit in (2.4), that is,
\[
\int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla \varphi}{(1 + u_n)^\gamma} + \int_{\Omega} u_n \varphi = \int_{\Omega} f_n \varphi, \quad \varphi \in W_0^{1,\infty}(\Omega),
\]
we get, since the first term tends to zero,
\[
\lim_{n \to +\infty} \int_{\Omega} u_n \varphi = \int_{\Omega} \varphi d\mu,
\]
for every $\varphi \in W_0^{1,\infty}(\Omega)$, as desired. \qed

Remark 4.3. With minor technical changes (see [13]) one can prove the same result if $\mu$ is a signed Radon measure concentrated on a set $E$ of zero harmonic capacity.

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