NONCLASSICAL EVOLUTION OF A FREE PARTICLE

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Abstract

A conditional kinetic energy is defined in terms of the Wigner distribution. It is shown that this kinetic energy may become negative for negative Wigner distributions. A free particle wave packet with negative kinetic energy will spread in a nonclassical manner.

1 The Free Particle

In classical statistical mechanics, a particle is described in terms of a nonnegative phase space distribution

\[ P(x, p; t) \geq 0. \]  

(1)

This is the probability of the particle having position \( x \) and momentum \( p \) at the time \( t \). The time evolution of the phase space distribution for a free particle is

\[ \frac{\partial P}{\partial t} = -\frac{p}{m} \frac{\partial P}{\partial x}. \]  

(2)

That quantum mechanics can be formulated in a similar manner was first discovered by Wigner \([1]\). However, the Wigner distribution \( W(x, p; t) \) has the nonclassical feature of being negative for certain quantum states. In fact, among pure states only the gaussian states have nonnegative Wigner distributions \([2]\). However, even if a free particle is in a state with negative Wigner distribution, it will not necessarily behave exhibit different dynamics than an ensemble of classical free particles \([3]\). For instance, the Wigner distribution obeys the same equation of motion (2) as a classical phase space distribution.

Bracken and Melloy \([4]\) recently found that the probability of observing a free particle in a certain region of space may increase even though there is zero probability for the particle to have momenta pointing towards this region. They explained this effect in terms of negative probability.

Here we shall study another dynamical effect which defies explanation in terms of the classical model (1) and (2). To this end, we consider the modulus of \( x \),

\[ \langle |x| \rangle = \int dp \int dx \ |x| W(x, p; t). \]  

(3)

We now find that

\[ \frac{d\langle |x| \rangle}{dt} = \frac{1}{m} \int dp \int dx \ \text{sign} \ x \ W(x, p; t), \]  

(4)

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and
\[
\frac{d^2 \langle |x| \rangle}{dt^2} = \frac{2}{m^2} \int dp \, p^2 W(0, p; t).
\] (5)

For an ensemble of classical, free particles we have the condition
\[
\frac{d^2 \langle |x| \rangle}{dt^2} \geq 0,
\] (6)
since the phase space distribution must be nonnegative. \( \langle |x| \rangle \) is a measure of the uncertainty in position provided that \( \langle x \rangle = 0 \). It is often called the absolute deviation. According to Eq. (6), the curvature with respect to time of the absolute deviation must be nonnegative in classical theory. This inequality therefore sets a constraint on the dynamics of the spreading of a wave packet in classical mechanics.

What is the physical significance of the r.h.s. of Eq. (5)? We introduce the moments \( \pi_n \)
defined by
\[
\pi_n(x; t) = \int dp \, p^n W(x, p; t).
\] (7)
\( \pi_n(x; t)/\pi_0(x; t) \) can be interpreted classically as the average of \( p^n \) given \( x \). Violation of (6) therefore can be interpreted in classical terms as due to negative kinetic energy given \( x \).

It may seem surprising that a negative kinetic energy can be observed. Indeed, the operator \( \hat{p}^2 \) has only nonnegative eigenvalues, and the expectation of this operator therefore is always nonnegative. But this does not imply that also conditional kinetic energy must be nonnegative. Indeed, in tunneling, it seems as if the particle traversing the tunneling region has negative kinetic energy, since the total energy is lower than the energy of the potential barrier. But also here one always finds a nonnegative kinetic energy. However, Aharonov et al. [6] have shown that if the kinetic energy is first measured followed by a position measurement, the subensemble of particles found in the tunneling region may display negative kinetic energy.

Finally, let’s study a system with negative kinetic energy. To this end, consider the (unnormalized) state
\[
|\psi\rangle = |\alpha\rangle + |\alpha\rangle.
\] (8)
This is a superposition of two coherent states 180° out of phase with respect to each other. With a choice of units so that \( h = 1 \), it has a Wigner distribution
\[
W(x, p; 0) = \frac{1}{\pi} \left[ e^{-(p-p_0)^2-(x-x_0)^2} + e^{-(p+p_0)^2-(x+x_0)^2} + 2e^{-x^2-p^2} \cos 2(p_0 x - px_0) \right].
\] (9)
This distribution has negative regions. For the choice \( x_0 = 0 \), these regions are centered along the line \( p = 0 \), and for \( p_0 = 0 \) they are centered along \( x = 0 \). As seen from Eq. (5), violation of inequality (6) for \( t = 0 \) requires that the Wigner distribution has negative regions for \( x = 0 \). We therefore use the parameter choice \( p_0 = 0 \).
FIG. 1. Contourplot of the Wigner distribution for an even coherent state where $x_0 = \sqrt{2}$ and $p_0 = 0$. Note the negative regions along $x = 0$.

The solution of Eq. (2) is $W(x, p; t) = W(x - pt/m, p; 0)$. Using this and Eq. (9), we get from Eq. (7) by integration

$$
\pi_2(0; t) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{x_0^2}{1 + t^2} \right) \frac{1 - x_0^2 + t^2 + x_0^2t^2}{(1 + t^2)^{5/2}},
$$

(10)

where we have assumed that $p_0 = 0$. We see that $\pi_2$ is negative if

$$
t < \frac{x_0^2 - 1}{x_0^2 + 1}.
$$

(11)

Thus $\pi_2$ may become negative provided that $x_0 > 1$, and it has a relative minimum for $t = 0$ and $x_0 = \sqrt{2}$.

FIG. 2. Plot of $\pi_2(0; t)$ for a free particle in an even coherent state, where $x_0 = \sqrt{2}$ and $p_0 = 0$. It is negative for $t < 1/\sqrt{3}$. This is impossible for an ensemble of classical, free particles, since it implies negative kinetic energy. According to Eq. (5), $\pi_2(0; t)$ is also proportional to the curvature of the expected absolute value of position.
2 Quantum Optics

We have found that violation of inequality (6) can only be explained in terms of a negative Wigner distribution. Let’s now consider a simple realization of a similar scheme in quantum optics.

Consider the rotated quadrature variable

\[ x_\theta = x \cos \theta + p \sin \theta. \]  \hspace{1cm} (12)

This observable can be measured in homodyne detection, if the radiation mode described by \( W(x, p) \) is mixed with a strong local oscillator with phase \( \theta \) \cite{7, 8}. We may “simulate” free particle evolution by introducing the variable \( \chi_\tau \)

\[
\chi_\tau = \frac{x_\theta}{\cos \theta} = x + p\tau,
\]  \hspace{1cm} (13)

where

\[
\tau = \tan \theta.
\]  \hspace{1cm} (14)

We thus have

\[
\langle |\chi_\tau| \rangle = \int dp \int dx \ | x + p\tau | W(x, p).
\]  \hspace{1cm} (15)

We substitute \( x' = x + p\tau \), so that

\[
\langle |\chi_\tau| \rangle = \int dp \int dx' \ | x' | W(x' - p\tau, p).
\]  \hspace{1cm} (16)

This clarifies that there is no ordering problem associated with Eq. (15) \cite{5}. We may now proceed to demonstrate that

\[
\frac{d^2 \langle |\chi_\tau| \rangle}{dt^2} = 2 \int dp \ p^2 \ W(0, p).
\]  \hspace{1cm} (17)

In analogy with the free particle case, we therefore see that

\[
\frac{d^2 \langle |\chi_\tau| \rangle}{d\tau^2} \geq 0
\]  \hspace{1cm} (18)

for nonnegative Wigner distributions. Violation of this inequality indicates that the Wigner distribution has negative regions along \( x = 0 \). It can be tested in homodyne detection.

Conclusion

We have seen that it is possible to observe negative kinetic energy for free particles. It was shown that this leads to nonclassical evolution of the position absolute deviation. The scheme was also applied to quantum optics, where a simple experiment was proposed to detect negative Wigner distributions.

Depending on the operator ordering we assign to a scalar product of \( x \) and \( p \), we get a different quasi phase space distribution \cite{11}. Thus, since there is no unique phase space distribution in quantum mechanics, there is no unique conditional kinetic energy either. It turns out, e.g., that the state examined by Aharonov et al. \cite{9} does not give a negative Wigner kinetic energy. An
essential part of the analysis of Aharonov et al. was an inherent uncertainty in the pointer position of their measurement apparatus.

When the classical model expressed by Eqs. (1) and (2) breaks down, one may abandon either assumption (1) or (2). By using the Wigner distribution, we abandon (1) and the concept of nonnegative probability. Using other distributions, one might instead abandon (2) [3]. This amounts to abandoning the idea that a point in phase space moves with constant velocity.

The analysis done here can be generalized to particles in arbitrary potentials. Also in this case it can be shown that negative kinetic energy leads to nonclassical evolution of the position absolute value.

Acknowledgments

I would like to thank Howard M. Wiseman and Stefan Weigert for drawing my attention to the paper by Bracken and Melloy [4]. I would also like to thank Ulf Leonhardt for useful comments.

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