Edwards-Wilkinson Fluctuations in the Howitt-Warren Flows

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December 15, 2014

Abstract

We study current fluctuations in a one-dimensional interacting particle system known as the dual smoothing process that is dual to random motions in a Howitt-Warren flow. The Howitt-Warren flow can be regarded as the transition kernels of a random motion in a continuous space-time random environment. It turns out that the current fluctuations of the dual smoothing process fall in the Edwards-Wilkinson universality class, where the fluctuations occur on the scale $t^{1/4}$ and the limit is a universal Gaussian process. Along the way, we prove a quenched invariance principle for a random motion in the Howitt-Warren flow. Meanwhile, the centered quenched mean process of the random motion also converges on the scale $t^{1/4}$, where the limit is another universal Gaussian process.

AMS 2000 subject classification. 60K35, 60K37, 60F17

1 Introduction

1.1 Overview

In the review article [Sep10], Seppäläinen discussed the processes of particle currents in several dynamical stochastic systems of particles on the one-dimensional integer lattice. It turns out that for independent random walks, independent random walks in an i.i.d space random environment, and the random average process (RAP), there is a universal limit for the current fluctuations on the scale $n^{1/4}$, which is a certain family of self-similar Gaussian processes. These three models all belong to the so-called Edwards-Wilkinson (EW) universality class. In EW class the limiting current fluctuations are described by the linear stochastic heat equation $Z_t = \nu Z_{xx} + \dot{W}$ where $\dot{W}$ is space-time white noise and $\nu$ is a non-zero parameter. In contrast, asymmetric simple exclusion process and a class of totally asymmetric zero range processes have nontrivial current fluctuations on the scale $n^{1/3}$, and the Tracy-Widom distributions are the universal limits. These two models belong to the Kardar-Parisi-Zhang (KPZ) universality class. More discussions about EW and KPZ universality classes and their relations can be found in [Sep10] and [Cor12]. However, all the models that were shown to be in the EW universality up to now are discrete models defined on $\mathbb{Z}$. The motivation of this paper is to present a model in continuous space and time that also falls in the EW class.

Recently, in [LJR04] Le Jan and Raimond introduced the so-called stochastic flow of kernels, which is a collection of random probability kernels. Heuristically, a stochastic flow of kernels can be interpreted as the transition kernels of a Markov process in a space-time random environment, where restrictions of the environment to disjoint space-time regions are independent and the law

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of the environment satisfies translation-invariance in space and time. Given the environment, one can sample $n$ independent translation-invariant Markov processes (random motions) and then average over the environment. This leads to a Markov process known as the $n$-point motion of the flow and their joint law satisfies a natural consistency condition: the marginal distribution of any $k$ components of an $n$-point motion is necessarily a $k$-point motion. The main result of Le Jan and Raimond [LJR04] is that any family of Feller processes that is consistent in this way gives rise to a unique stochastic flow of kernels. Later, Howitt and Warren, using martingale problems, constructed a class of consistent Feller processes on $\mathbb{R}$ which are Brownian motions with sticky interaction when they meet, and thus by the fundamental result of Le Jan and Raimond, it determines the unique stochastic flow of kernels which is now called the Howitt-Warren flow. In [SSS14], Schertzer, Sun and Swart showed that the Howitt-Warren flows can be realized as the transition kernels of a random motion in a space-time environment, constructed explicitly from the Brownian web and Brownian net. Thus the heuristic interpretation above naturally becomes rigorous.

Dual smoothing process dual to Howitt-Warren flows, which is a function-valued process, was also introduced in [SSS14]. As a continuous space-time analogue of RAP, dual smoothing process can be thought of as the evolution of the interface height function in a growth model as well. In one dimension, conservative interacting particle systems can always be equivalently formulated as interface models. Here the connection goes by regarding the gradient of the interface height function as a measure governing the distribution of the particles. The movement of particle currents can then be viewed as deposition or removal of particles from the growing interface. With such an equivalent formulation, the current process maps directly to the height function. We will show that on the scale $t^{1/4}$, the fluctuations of the height function (dual smoothing process), which is the first continuum space-time model shown to be in the EW class, converges weakly to a universal Gaussian process. Along the way, we will show that for the random motions in the Howitt-Warren flows, the process of the centered quenched mean, indexed by space and time, converges to a Gaussian process after rescaling by $t^{-1/4}$. Moreover, we will prove a quenched invariance principle for the random motion in the Howitt-Warren flows, which is of independent interest.

1.2 Stochastic flows of kernels and Howitt-Warren flows

In this subsection, we recall the notion of a stochastic flow of kernels and the characterization of Howitt-Warren flows. We then state a quenched invariance principle for the random motion in a Howitt-Warren flow, as well as the first convergence theorem.

We first give the definition of stochastic flows of kernels as introduced in [LJR04]. Given a Polish space $E$, let $\mathcal{B}(E)$ be the Borel $\sigma$-field of $E$ and $\mathcal{M}_1(E)$ be the space of probability measures on $E$ equipped with the topology of weak convergence and the associated Borel $\sigma$-field. A random probability kernel on $E$ is a measurable function $K : \Omega \times E \to \mathcal{M}_1(E)$, where $(\Omega, \mathcal{F}, P)$ is the underlying probability space. We say that two random probability kernels $K, K'$ are equal in finite dimensional distributions if for any $n$ and $x_1, \ldots, x_n \in E$, the distributions of the $n$-tuple of random probability measures $\{K(x_1, \cdot), \ldots, K(x_n, \cdot)\}$ and $\{K'(x_1, \cdot), \ldots, K'(x_n, \cdot)\}$ are equal. We say that two or more random probability kernels are independent if their finite dimensional distributions are independent. Under these notations, Le Jan and Raimond (see [LJR04] Definition 1.6) defines:

**Definition 1.1. (Stochastic flow of kernels)** A stochastic flow of kernels is a collection $(K_{s,t})_{s\leq t}$ of random probability kernels on the Polish space $E$ such that

(i) For every $s \leq t \leq u$ and $x \in E$, almost surely, $K_{s,s}(x, dz) = \delta_x(dz)$ and

\[
\int K_{s,t}(x, dy)K_{t,u}(y, dz) = K_{s,u}(x, dz)
\]

(ii) For every $s \leq t$ and $u \in \mathbb{R}$, $K_{s,t}$ and $K_{s+u,s+t+u}$ are equal in finite dimensional distributions.

(iii) For any $t_0 < \cdots < t_n$, the random probability kernels $\{K_{t_{i-1}, t_i}\}_{i=1}^n$ are independent.
Remark. In general, it is not known whether condition (i) can be strengthened to
(i’) A.s., \( K_{s,t}(x, dz) = \delta_x(dz) \) and \( \int K_{s,t}(x, dy)K_{t,u}(y, dz) = K_{s,u}(x, dz) \) for all \( x \in E \)
and \( s \leq t \leq u \),
so that \( (K_{s,t})_{s \leq t} \) is a bona fide family of transition kernels of a random motion in a random space-
time environment and the random motion satisfies the Chapman-Kolmogorov equation. However, for Howitt-Warren flows, this has been shown to be possible.

Given a stochastic flow of kernels \( (K_{s,t})_{s \leq t} \), if we set
\[
P^{(n)}_{t-s}(\vec{x}, \vec{y}) := \mathbb{E}[K_{s,t}(x_1, dy_1), \ldots, K_{s,t}(x_n, dy_n)] \quad (\vec{x}, \vec{y} \in E^n, s \leq t),
\]
then it defines a family of Markov transition probability kernels on \( E^n \). We call the Markov process
with these transition probabilities the \( n \)-point motion associated with the stochastic flow of kernels
\( (K_{s,t})_{s \leq t} \) and a natural consistent condition is satisfied. Conversely, a fundamental theorem of
Le Jan and Raimond [LJR04, Theorem 2.1] shows that any consistent family of Feller processes
on a locally compact space \( E \) gives rise to a stochastic flow of kernels on \( E \) and it is unique in
the sense that any two versions of such stochastic flows of kernels are equal in finite dimensional distributions.

Howitt and Warren constructed in [HW09] a consistent family of Feller processes on \( \mathbb{R} \) via a well
posed martingale problems, which are Brownian motions with drift \( \beta \in \mathbb{R} \) and sticky interactions
that can be characterized by a finite measure \( \mu \) on \([0, 1] \). The associated stochastic flow of kernels
is now called the Howitt-Warren flow. The associated \( n \)-point motion evolves as \( n \) independent
Brownian motions with the same drift when they do not coincide, but it is possible that two or
more Brownian motions may meet at the same location because of the stickiness which makes the
\( n \)-point motion spend positive Lebesgue time together. In [SSS14, Proposition 2.3], it was shown
that for the Howitt-Warren flows, the random set of probability 1 where Definition 1.1 (i) holds
\( \{X^1(t), X^2(t)\}_{t \geq 0} \) is the well-known sticky Brownian motion which
\( \nu \)-coupled Brownian motions as in [SSS14], Proposition 2.3], it was shown
that for the Howitt-Warren flows, the random set of probability 1 where Definition 1.1 (i) holds
\( \{X^1(t), X^2(t)\}_{t \geq 0} \) is the well-known sticky Brownian motion which
that can be characterized by a finite measure \( \mu \) on \([0, 1] \). Moreover, \( X^1(t) - X^2(t) \) is the well-known sticky Brownian motion which
because of time-changing a standard Brownian motion in such a way that its local time
at the origin becomes \( 1/2\nu \) times the real time, and it behaves as a standard Brownian motion on
\( \mathbb{R} \setminus \{0\} \). In [HW09], such \( X \) is called \( \theta \)-coupled Brownian motions with \( \theta = 1/2\nu \).
For the \( n \)-point motion, we will give a SDE representation in the next section.
Given a realization of the Howitt-Warren flow \((K^ω_{t,s})_{s,t \leq 1}\), one can sample a set of independent random motions \((X^1_t, \ldots, X^n_t)\). We let \(P\) (resp. \(E\)) denote the probability (resp. expectation) for the environment \(\omega\), let \(P^ω\) (resp. \(E^ω\)) denote the quenched law (resp. quenched expectation) for the random motions given the environment \(\omega\), and let \(P := EP^ω(\cdot)\) (resp. \(E\)) denote the averaged law (or annealed law) (resp. averaged expectation) of the random motions by integrating out the environment. Under this notation, for example, two random motions \((X^1_t, X^2_t)\) independent under the law \(P^ω\) are in fact a 2-point motion under the averaged law \(P\).

If we consider a random motion \((X_t)_{t \geq 0}\) starting from the origin in the Howitt-Warren flow \((K^ω_{s,t})_{s,t \leq 1}\) with drift \(\beta\) and characteristic measure \(\mu\) (so that the stickiness parameter for the 2-point motion is \(\nu = 1/4\mu([0,1])\)), then our first result is an almost sure quenched invariance principle for \((X_t)_{t \geq 0}\), which is analogous to the one for the random walk in i.i.d space-time random environment \([\text{RAS05}, \text{Theorem 1}]\).

**Theorem 1.1.** Let \(Y_t := X_t - \beta t\), then for \(\mathbb{P}\)-a.e. \(\omega\), the process \(\left(\frac{Y_{nt}}{\sqrt{n}}\right)_{t \geq 0}\) converges weakly to a standard Brownian motion as \(n \to \infty\). Moreover, for \(\mathbb{P}\)-a.e. \(\omega\), \(n^{-1/2}\max_{s \leq nt} |E^ω X_s - \beta s|\) converges to 0, and therefore the same quenched invariance principle also holds for the process \(\tilde{Y}_t := X_t - E^ω[X_t]\).

Since \(\mathbb{P}\) is invariant w.r.t. the space-time shift of the environment \(\omega\), this invariant principle holds for the random motion starting from any space-time point.

If we use the superscript to represent the starting point of the random motion, i.e., \(X^{x_0,t_0}_t\) is a random motion starting from the space-time point \((x_0, t_0)\), then we can state our second result:

**Theorem 1.2.** For every \((t, r) \in \mathbb{R}^+ \times \mathbb{R}\), define two rescaled centered quenched means as follows:

\[
a_n(t, r) := n^{-1/4} \left( E^ω \left[ X^r_{0} \sqrt{n} - \beta nt, -nt \right] - r \sqrt{n} \right),
\]

\[
b_n(t, r) := n^{-1/4} \left( E^ω \left[ X^r_{nt} \sqrt{n} - \beta nt \right] - r \sqrt{n} - \beta nt \right),
\]

then the finite dimensional distributions of the processes \(\{a_n(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\}\) and \(\{b_n(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\}\) converges to those of the Gaussian processes \(\{a(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\}\) and \(\{b(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\}\) with covariance functions given by \(\Gamma((t, r), (s, q))\) and \(\Gamma((t+s, r), (t+s, q))\) respectively, where

\[
\Gamma((t, r), (s, q)) := \nu \int_{|t-s|}^{t+s} \frac{1}{\sqrt{\pi u}} e^{-\frac{(u-t-s)^2}{4u}} du.
\]

**Remark.** We will only give the proof of the convergence of \((a_n(t, r))\), since the argument for \((b_n(t, r))\) is almost the same. For any \((t, r) \in \mathbb{R}^+ \times \mathbb{R}\), \(a_n(t, r)\) is a random variable of the environment between time \(-nt\) and 0. The variance of the quenched mean process is of order \(n^{1/2}\), and this leads to the choice of the scale \(n^{-1/4}\) in \(a_n(t, r)\) and \(b_n(t, r)\).

### 1.3 Dual smoothing process

Given a Howitt-Warren flow \((K^ω_{t,s})_{s,t \leq 1}\), a Howitt-Warren process, which is a measure-valued Markov process, is defined by

\[
\rho_t(dy) := \int \rho_0(dx)K^ω_{0,t}(x, dy) \quad (t \geq 0),
\]

where \(\rho_0\) a finite measure on \(\mathbb{R}\). A function-valued dual smoothing process is defined by

\[
\zeta_t(x) := \int K^ω_{t,0}(x, dy)\zeta_0(y) \quad (x \in \mathbb{R}, \ t \geq 0),
\]
where \( \zeta_0 \in D_v(\mathbb{R}) \), the space of bounded cádlág function on \( \mathbb{R} \). These two processes are shown to be dual to each other in \[SSS14\] Lemma 11.1. Indeed, from \((K_{\tau t}^x)_{x \leq t}^1\) one can define a dual Howitt-Warren process \((\tilde{\rho}_t)_{t \geq 0}\), and regard \( \zeta_t \) as its height function at time \( t \). To see this fact at a heuristic level, we begin with the description of the discrete Howitt-Warren flow.

Let \( \mathbb{Z}_{even}^2 = \{ (x, t) : x, t \in \mathbb{Z}, x+t \text{ is even} \} \), where the first and second coordinates are interpreted as space and time. Let \( \omega := (\omega_z)_{z \in \mathbb{Z}_{even}^2} \) be i.i.d. \([0,1]\)-valued random variables with common distribution \( \mathbb{Q} \). We view \( \omega \) as a random space-time environment for a random walk. That is, conditional on \( \omega \), if a random walk is at time \( t \) at the position \( x \), then in the next unit time step the walk jumps to \( x \pm 1 \) with probability \( \omega_{(x,t)} \) and to \( x-1 \) with the remaining probability \( 1-\omega_{(x,t)} \).

If we use \( P^\omega_{(s,x)} \) to denote the quenched law of a random walk \( X := (X_t)_{t \geq s} \) starting from the space-time point \((x,s)\), then setting \( K^x_{\tau t}(x,y) := P^\omega_{(s,x)}(X(t) = y) \) defines the discrete Howitt-Warren flow \((K^x_{\tau t})_{s \leq t} \). It is shown in \[SSS14\] that with suitable assumption the discrete flow under diffusive scaling converges to the Howitt-Warren flow. A natural corollary of this graphical construction is that in Definition 1.1, condition (i) can be strengthened to (i)' for Howitt-Warren flows.

Given a realization of the environment, one can sample coalescing random walks starting from every point in \( \mathbb{Z}_{even}^2 \), which is called a (random) discrete web. Moreover, one can couple a dual discrete web in the following way. Consider the coalescing random walks running backwards in time, starting from every point in \( \mathbb{Z}_{odd}^2 := \mathbb{Z}^2 \setminus \mathbb{Z}_{even}^2 \). For each \((x,t+1) \in \mathbb{Z}_{odd}^2 \), the backward walk at time \( t+1 \) at the position \( x \) jumps to \((x-1,t)\) if the forward walk in the discrete web jumps from \((x,t)\) to \((x+1,t+1)\), and otherwise the backward walk jumps to \((x+1,t)\). It is then easy to see that the dual discrete web determines the dual Howitt-Warren flow \((\tilde{K}_{\tau t})_{s \leq t} \), which is equal in distribution to the Howitt-Warren flow. Furthermore, noting the non-crossing property (i.e., in the coupling, random walks in the discrete web do not cross any random walk in the dual web), we have the following relationship:

\[
K_{s,t}(x, [y, \infty)) = \tilde{K}_{t,s}(y, (-\infty, x)) \quad (x, y \in \mathbb{R}, s \leq t).
\]

(1.8)

In other words, if we sample a forward random motion \( X := (X_u)_{u \geq s} \) from the space-time point \((x,s) \in (K_{\tau t})_{s \leq t} \) and a backward random motion \( \tilde{X} := (\tilde{X}_u)_{u \leq t} \) from \((y,t) \in (\tilde{K}_{\tau t})_{s \leq t} \), and if \( s \leq t \) and \( X_t \geq y \), then since the paths of \( X \) and \( \tilde{X} \) do not cross each other, we must have \( \tilde{X}_s \leq x \). Noting that for deterministic \( y \), the probability that \( X_t = y \) is zero, in \[1.8\] we can change the closed interval to open interval.

Now if we consider the dual Howitt-Warren process with finite initial measure \( \rho_0 \),

\[
\hat{\rho}_t(dy) := \int \rho_0(dx) K_{0,-t}(x,dx),
\]

(1.9)

then setting \( \zeta_0 \) in \[1.7\] as the height function of \( \hat{\rho}_0 \)

\[
\zeta_0(x) = \int_{(\infty,x]} \hat{\rho}_0(dy) \quad (x \in \mathbb{R}),
\]

(1.10)

we have that \( \zeta_t \) is the height function of \( \hat{\rho}_t \), and by \[1.8\] the current of \( \hat{\rho} \) over the line segment from \((0,x)\) to \((-t,y)\) is:

\[
\int_{(x,\infty)} \hat{\rho}_0(dz) K_{0,-t}(z,(-\infty,y)) - \int_{(-\infty,x]} \hat{\rho}_0(dz) K_{0,-t}(z,(y,\infty))
\]

\[
= \int_{(x,\infty)} \left\{ \int_{z}^{\infty} \rho_{t,0}(y, dw) \right\} \hat{\rho}_0(dz) - \int_{(-\infty,x]} \left\{ \int_{-\infty}^{z} \rho_{t,0}(y, dw) \right\} \hat{\rho}_0(dz)
\]

\[
= \int \left\{ \int_{-\infty}^{x} \hat{\rho}_0(dz) \right\} \rho_{t,0}(y, dw)
\]

\[
= \zeta_t(y) - \zeta_0(x).
\]

(1.11)

5
As a result, considering the fluctuations of the dual Howitt-Warren process is equivalent to considering the current fluctuations of the dual smoothing process.

Now we consider the fluctuations of a class of generalized dual smoothing processes. For any deterministic point \( x_0 \in \mathbb{R} \), we look at the fluctuation rescaled by \( n^{-1/4} \) along the characteristic line \( x(t) = x_0 - \beta t \), namely the quantity

\[
z_n(t, r) := n^{-1/4} \left\{ \zeta^{(n)}(nx_0 + r\sqrt{n} - \beta nt) - \zeta^{(n)}_0(nx_0 + r\sqrt{n}) \right\}.
\]

(1.12)

For our purpose, we assume the following initial condition:

**Assumption I.** For each \( n \in \mathbb{N} \), define an initial condition \( \zeta_0^{(n)}(x) = f^{(n)}(x) + W(x) \), where \( (W(x))_{x \in \mathbb{R}} \) is a two-sided Brownian motion, independent of the Howitt-Warren flow, with \( W(0) = 0 \), and \( f^{(n)}(x) := nf(x) \), where \( f \) is a \( C^1 \) function such that \( f(0) = 0 \), \( f' \) is bounded and satisfies the following H"older continuity condition: there exist constants \( C > 0 \) and \( \gamma > 1/2 \) such that

\[
|f'(x) - f'(y)| < C |x - y|^{\gamma} \quad (x, y \in \mathbb{R}).
\]

(1.13)

It turns out that under Assumption I, as \( n \) tends to \( \infty \), \( \{z_n(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\} \) converges to a Gaussian process \( \{z(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\} \) in finite dimensional distributions. So we next describe the limit process.

Define a covariance function \( \Gamma_0 \) on \( (\mathbb{R}^+ \times \mathbb{R}) \times (\mathbb{R}^+ \times \mathbb{R}) \),

\[
\Gamma_0((t, r), (s, q)) := \int_{r \vee q}^{\infty} P(W(t) > z - r) P(W(s) > z - q) \, dz
\]

\[
-1_{\{r > q\}} \int_{q}^{r} P(W(t) < z - r) P(W(s) > z - q) \, dz
\]

\[
-1_{\{r < q\}} \int_{r}^{q} P(W(t) > z - r) P(W(s) < z - q) \, dz
\]

\[
+ \int_{-\infty}^{r \wedge q} P(W(t) < z - r) P(W(s) < z - q) \, dz
\]

(1.14)

where \( W \) is the standard Brownian motion, and recall the covariance function \( \Gamma \) defined in last subsection:

\[
\Gamma((t, r), (s, q)) := \nu \int_{|t-s|}^{t+s} \frac{1}{\sqrt{2\pi u}} e^{-\frac{(u-s)^2}{2u}} \, du.
\]

(1.15)

Then \( \{z(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\} \) is a mean zero Gaussian process with covariance given by

\[
Ez(t, r)z(s, q) = f^2(x_0)\Gamma((t, r), (s, q)) + \Gamma_0((t, r), (s, q)).
\]

(1.16)

In fact, the first term in the right-hand side of (1.16) comes from the fluctuation caused by the dynamics, while the second is from the initial noise \( \zeta_0 \).

**Theorem 1.3.** Under Assumption I, the finite dimensional distribution of the current fluctuations \( \{z_n(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\} \) defined in (1.14) converges weakly to that of the mean zero Gaussian process \( \{z(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\} \) with covariance function (1.16).

The rest of the paper is organized as follows. Section 2 provides the SDEs for the 2-point motion and establishes some properties of their collision local time. Section 3 proves the quenched invariance principle for the random motion in Howitt-Warren flows. Section 4 and Section 5 prove Theorem 1.2 and Theorem 1.3 respectively. In Appendix A, we provide a proof of the quenched invariance principle of the random motion in an i.i.d. space-time random environment via the second moment method, as a result of independent interest.
2 Howitt-Warren 2-point motion preliminaries

In this section we first give a set of SDEs that characterizes the Howitt-Warren 2-point motion, which are Brownian motions with sticky interactions when they meet. We then derive two useful lemmas about the difference process of the 2-point motion, namely the sticky Brownian motion.

2.1 SDEs for the Howitt-Warren 2-point motion

We first recall the concept of local time of a continuous semimartingale, and then give a set of SDEs which has a unique weak solution. We will show that a Howitt-Warren 2-point motion can be represented by the weak solution of the SDEs. Moreover, with this representation, the invariance principle for Howitt-Warren flows will be proved using the second moment method in next section.

In Section 3.7 of [KS91], the local time \( \Lambda_{x_0}(t,x) \) for a continuous semimartingale \( (X_t)_{t \geq 0} \) is discussed. It is a generalized concept of the local time of Brownian motion, first introduced by P. Lévy, which is used to measure the time that a Brownian motion spends in the vicinity of a deterministic point. Now we list some of the properties of \( \Lambda_{x_0}(t,x) \) as a proposition. For further theory of local time, we refer to [KS91] Chapter 3.

Proposition 2.1. Let \( X_t = x_0 + M_t + V_t \) be a continuous semimartingale on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( X_0 = x_0 \in \mathbb{R} \), \( (M_t)_{t \geq 0} \) with \( M_0 = 0 \) is a continuous local martingale adapted to a filtration \( (\mathcal{F}_t)_{t \geq 0} \) and \( (V_t)_{t \geq 0} \) is the difference of continuous, nondecreasing, adapted processes with \( V_0 = 0 \) a.s.. Then there exists an a.s. unique process \( \Lambda_{x_0}(t,x) \), which is called the semimartingale local time, defined on \( \mathbb{R}^+ \times \mathbb{R} \times \Omega \), such that the following holds:

(i) For all \( (t,x,\omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \), \( \Lambda_{x_0}(t,x) \) is nonnegative.

(ii) For every fixed \( x \in \mathbb{R} \), \( \Lambda_{x_0}(0,x) = 0 \), \( \Lambda_{x_0}(t,x) \) is continuous and nondecreasing in \( t \), and

\[
\int_0^\infty 1_{\mathbb{R} \setminus \{x\}}(X_t)\Lambda_{x_0}(dt,x) = 0, \quad \text{for } \mathbb{P}-\text{a.e. } \omega \in \Omega. \tag{2.1}
\]

(iii) (Tanaka-Meyer formula) For every fixed \( x \in \mathbb{R} \),

\[
|X_t - x| = |X_0 - x| + \int_0^t \text{sgn}(X_s - x) \, dM_s + \int_0^t \text{sgn}(X_s - x) \, dV_s + 2\Lambda_{x_0}(t,x), \tag{2.2}
\]

where \( \text{sgn}(x) \) is the sign function.

(iv) If \( X_t \) is a Brownian motion \( B_t \) with \( B_0 = x_0 \) and \( \mathbb{E}[B_t^2] = \sigma^2 t \) (usually we use the notation \( L_{x_0}(t,x) \) instead of \( \Lambda_{x_0}(t,x) \) in this case), then for every measurable function \( f : \mathbb{R} \to [0, \infty) \), we have a.s.,

\[
\sigma^2 \int_0^t f(B_s) \, ds = 2 \int_{-\infty}^\infty f(x) L_{x_0}(t,x) \, dx,
\]

and \( L_{x_0}(t,x) \) is continuous in \( (t,x) \).

Later, when the starting point of \( X_t \) is clear, we will abbreviate the notation \( \Lambda_{x_0}(t,x) \) by \( \Lambda(t,x) \).

Now we consider the following SDEs:

\[
\begin{align*}
\text{d}X_1^1 &= 1_{\{X_1^1 \neq X_2^1\}} \, dB_1^1 + 1_{\{X_1^1 = X_2^1\}} \, dB_3^1 + \beta \, dt, \\
\text{d}X_2^1 &= 1_{\{X_1^1 \neq X_2^1\}} \, dB_2^1 + 1_{\{X_1^1 = X_2^1\}} \, dB_3^1 + \beta \, dt, \\
1_{\{X_1^1 = X_2^1\}} \, dt &= 2\nu \Lambda(dt,0),
\end{align*}
\]

with initial condition \( X_1^1_0 = x_1 \) and \( X_2^1_0 = x_2 \). Here \( \{B_i^1 \mid i = 1,2,3\} \) are independent standard Brownian motions starting form the origin, \( \nu \) is a constant parameter (later we will see that \( \nu \)
 coincides with the stickiness parameter given in Definition 1.2, and $\Lambda(t, x)$ is the local time for the difference process $X_t^1 - X_t^2$. Note that from the first two equations, the processes $X_t^1$ and $X_t^2$ must be semimartingales, which leads to the existence of $\Lambda(t, x)$ by Proposition 2.1. In particular, $\Lambda(t, 0)$ is continuous and nondecreasing in $t$ for a.e. $\omega$. Consequently, it induces a measure on $\mathbb{R}^+$ and the third equation of (2.3) is meaningful.

The SDEs (2.3) gives a representation of the Howitt-Warren flows as stated in the following theorem.

**Theorem 2.2.** The SDEs (2.3) is well posed, i.e., for every initial condition $(x_1, x_2) \in \mathbb{R}^2$, (2.3) admits a weak solution which is unique in law. Furthermore, any Howitt-Warren 2-point motion $(X_t^1, X_t^2)$ is a solution of (2.3).

We prove this theorem by the following lemmas.

**Lemma 2.3.** Given initial condition $X_0^1 = x_1$, $X_0^2 = x_2$ for any $x_1, x_2 \in \mathbb{R}$, the SDEs (2.3) has a weak solution, that is, there is a quintuple $(X_t^1, X_t^2, B_t^1, B_t^2, B_t^3)$ and a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that the quintuple is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, $B_t^1, B_t^2, B_t^3$ are independent Brownian motions and $(X_t^1, X_t^2)$ satisfies (2.3) in Itô-integral form.

**Proof.** Let $\{\hat{B}_i, \hat{B}_i^1; i = 1, 2, 3\}$ be independent standard Brownian motions starting from 0 and $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by these Brownian motions. Define $W_t := x_1 + B_t^1 - x_2 - B_t^2$, then $W_t$ is a Brownian motion and let $L(t, x)$ denote the local time for $W_t$.

Set $A_t := t + 2\nu L(t, 0)$, then $A_t$ is a strictly increasing and continuous function and $A_t \geq t$. Therefore, we can define the inverse function of $A_t$ by $T_t := A_t^{-1}$ and define also $S_t := t - T_t$. Now let

$$X_t^1 := x_1 + \hat{B}_{T_t}^1 + \hat{B}_{S_t}^3 + \beta t, \quad (2.4)$$

$$X_t^2 := x_2 + \hat{B}_{T_t}^2 + \hat{B}_{S_t}^3 + \beta t. \quad (2.5)$$

Define then

$$B_t^1 := \hat{B}_{T_t}^1 + \int_0^{T_t} 1_{\{X_s^1 = X_s^2\}} d\hat{B}_s^1,$$

$$B_t^2 := \hat{B}_{T_t}^2 + \int_0^{T_t} 1_{\{X_s^1 = X_s^2\}} d\hat{B}_s^2,$$

$$B_t^3 := \hat{B}_{S_t}^3 + \int_0^{S_t} 1_{\{X_s^1 \neq X_s^2\}} d\hat{B}_s^3. \quad (2.6)$$

We claim that the quintuple $(X_t^1, X_t^2, B_t^1, B_t^2, B_t^3)$ together with the filtration $(\mathcal{F}_t)_{t \geq 0}$ is a weak solution of (2.3).

To see this, first we note that the quintuple is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. Next we are going to prove that $B_t^1, B_t^2, B_t^3$ are independent Brownian motions by Lévy’s characterization of Brownian motion. Since

$$\mathbb{E} \left[ \left( \int_0^{T_t} 1_{\{W_s = 0\}} d\hat{B}_s^3 \right)^2 \right] = \mathbb{E} \left[ \int_0^{T_t} 1_{\{W_s = 0\}} ds \right] = 0,$$

so $\int_0^{T_t} 1_{\{W_s = 0\}} d\hat{B}_s^3 = 0$ a.s.. Combine this with the fact that $W_{T_t} = 0$ if and only if $X_t^1 = X_t^2$, we have a.s.,

$$\hat{B}_{T_t}^1 = \int_0^{T_t} 1_{\{W_s \neq 0\}} d\hat{B}_s^1 = \int_0^{t} 1_{\{W_s \neq 0\}} d\hat{B}_s^1 = \int_0^{t} 1_{\{X_s^1 \neq X_s^2\}} d\hat{B}_s^1, \quad (2.7)$$

and the quadratic variation

$$\langle \hat{B}_{T_t}^1 \rangle_t = T_t = \int_0^{T_t} 1_{\{W_s \neq 0\}} ds = \int_0^{T_t} 1_{\{W_s \neq 0\}} (ds + 2\nu L(ds, 0))$$

$$= \int_0^{T_t} 1_{\{W_s \neq 0\}} dA_s = \int_0^{t} 1_{\{W_s \neq 0\}} ds = \int_0^{t} 1_{\{X_s^1 \neq X_s^2\}} ds. \quad (2.8)$$

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The third equality holds because of Proposition 2.1 (ii). Hence, the quadratic variation of $B^1_t$ is given by:

$$
(B^1)_t = T_t + \int_0^t 1_{\{X^1_t = X^2_t\}} ds = \int_0^t 1 ds = t.
$$

(2.9)

Notice that $B^1_t$ is a continuous martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$, so by Lévy’s characterization $B^1_t$ is a Brownian motion. Similarly, $B^2_t$ is also a Brownian motion. As to $B^3_t$, we have

$$
(B^3)_t = S_t + \int_0^t 1_{\{X^1_t \neq X^2_t\}} ds = S_t + T_t = t,
$$

(2.10)

which means that $B^3_t$ is also a Brownian motion. And it is not difficult to see the independence of $B^1_t$, $B^2_t$ and $B^3_t$ since the covariation process of each two is zero.

Moreover, by the construction of $X^1_t$, $X^2_t$, $B^1_t$, $B^2_t$ and $B^3_t$,

$$
X^1_t = x_1 + \int_0^t 1_{\{X^1_t \neq X^2_t\}} dB^1_t + \int_0^t 1_{\{X^1_t = X^2_t\}} dB^3_t + \beta t = x_1 + \int_0^t 1_{\{X^1_t \neq X^2_t\}} dB^1_t + \int_0^t 1_{\{X^1_t = X^2_t\}} dB^3_t + \beta t,
$$

$$
X^2_t = x_2 + \int_0^t 1_{\{X^1_t \neq X^2_t\}} dB^2_t + \int_0^t 1_{\{X^1_t = X^2_t\}} dB^3_t + \beta t = x_2 + \int_0^t 1_{\{X^1_t \neq X^2_t\}} dB^2_t + \int_0^t 1_{\{X^1_t = X^2_t\}} dB^3_t + \beta t.
$$

We can see that $X^1_t$ and $X^2_t$ solve the first two equations of (2.3).

It remains to show that $(X^1_t, X^2_t)$ satisfies the third equation in (2.3). Since $X^1_t - X^2_t = W_{T_t}$, we have $\Lambda(t, x) = L(T_t, x)$. Observe that

$$
\int_0^t 1_{\{X^1_t = X^2_t\}} ds = \int_0^t 1_{\{W_{T_t} = 0\}} ds = \int_0^{T_t} 1_{\{W_s = 0\}} dB^3_s
$$

$$
= \int_0^{T_t} 1_{\{W_s = 0\}} \left(ds + 2\nu L(ds, 0)\right)
$$

$$
= 2\nu \int_0^{T_t} 1_{\{W_s = 0\}} L(ds, 0)
$$

$$
= 2\nu L(T_t, 0) = 2\nu \Lambda(t, 0).
$$

(2.11)

Thus, $(X^1_t, X^2_t, B^1_t, B^2_t, B^3_t)$ solves the SDEs in (2.3).

\[ \square \]

In fact, the Howitt-Warren 2-point motion is equal in distribution to the solution of the SDEs as shown in the following lemma.

**Lemma 2.4.** Given initial condition $X^1_0 = x_1$, $X^2_0 = x_2$ for any $x_1, x_2 \in \mathbb{R}$, the solution $(X^1_t, X^2_t)$ (in weak sense) solves the martingale problem for the Howitt-Warren 2-point motion as defined in Definition 1.2.

**Proof.** Suppose that $(X^1_t, X^2_t)$ is a solution of (2.3). Then

$$
X^1_t - x_1 - \beta t = \int_0^t 1_{\{X^1_t \neq X^2_t\}} dB^1_s + \int_0^t 1_{\{X^1_t = X^2_t\}} dB^3_s,
$$

where $B^1_t$ and $B^3_t$ are independent Brownian motion. It is easy to see that $X^1_t - \beta t$ is a continuous martingale and the quadratic variation is $t$. By Lévy’s characterization $X^1_t - \beta t$ is a Brownian motion, and so is $X^2_t - \beta t$.

Furthermore, applying Tanaka-Meyer formula to the martingale $X^1_t - X^2_t$, we have

$$
|X^1_t - X^2_t| = |x_1 - x_2| + \int_0^t \text{sgn}(X^1_t - X^2_t) d(X^1_t - X^2_t) + 2\Lambda(t, 0).
$$

(2.12)
Since $\int_0^t \text{sgn}(X_1^t - X_2^t)d(X_1^t - X_2^t)$ is a martingale and $2\Lambda(t,0) = \frac{1}{\nu} \int_0^t 1_{\{X_1^s = X_2^s\}} ds$ by (2.3), we conclude that $\nu[X_1^t - X_2^t] - \int_0^t 1_{\{X_1^s = X_2^s\}} ds$ is a martingale.

Therefore, the solution $(X_1^t, X_2^t)$ solves the martingale problem for the Howitt-Warren 2-point motion.

Lastly, the uniqueness of the SDEs (2.3) follows from the uniqueness of the 2-point motion, where the latter one is a special case of the uniqueness of the martingale problem shown in [HW09]. As a result, we have proved Theorem 2.2.

From now on, we will identify the 2-point motion and the solution of (2.3) since we can couple three independent Brownian motion and a Howitt-Warren 2-point motion by (2.3).

2.2 Local time preliminaries

In this subsection, we always let $(X_1^t, X_2^t)$ be a Howitt-Warren 2-point motion starting from $(x_1,x_2)$, and $\nu$ be the stickiness parameter. We will derive the first moment of the local time $L$.

Lemma 2.5. For the local time $\Lambda_{x_1-x_2}(t,0)$ we have $E[\Lambda_{x_1-x_2}(t,0)] = O(t^{1/2})$ for all $x_1,x_2 \in \mathbb{R}$, where $f(t) = O(g(t))$ denotes that there exists a constant $C > 0$ such that $f(t) \leq C g(t)$ as $t \rightarrow \infty$. Moreover, if $x_1 - x_2 = 0$, then

$$E[\Lambda_0(t,0)] = \sqrt{\frac{2}{\pi}} t^{1/2} + \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right) \nu, \quad (2.13)$$

where $\Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$.

Proof. We follow the notations in Lemma 2.5 and first derive the distribution of the local time $\Lambda(t,0)$. In the proof of Lemma 2.5, we have the relation $X_1^t - X_2^t = W_{T_1}$, where $W_t$ is a Brownian motion (not standard, but $E[W_t^2] = 2t$) starting from $x_1 - x_2$ and $T_1$ is the time change defined in Lemma 2.5. For the local times $X_1 - X_2$ and $W$, we have $\Lambda(t,x) = L(T_1, x)$ and in particular $\Lambda(t,0) = L(T_1, 0)$. Temporarily we use the notation $\Lambda(t)$ (resp. $L(t)$) to denote $\Lambda(t,0)$ (resp. $L(t,0)$) in the left of this paragraph, and define the left inverse $L^{-1}(u) := \inf\{t : L(t) > u\}$ (same for $\Lambda^{-1}(u)$). Then $L(L^{-1}(u)) = u$ and $L(t) \leq u$ if and only if $L^{-1}(u) \geq t$. Since $\Lambda(t)$ is $L(T_1)$ and $T_1 = T^{-1}$ is a continuous and strictly increasing function where $A_t = t + 2\nu L(t)$, we have

$$\Lambda^{-1}(u) = T^{-1}(L^{-1}(u)) = A(L^{-1}(u)) = L^{-1}(u) + 2\nu L(L^{-1}(u)) = L^{-1}(u) + 2\nu u. \quad (2.14)$$

Hereby we get

$$P(\Lambda(t) \leq u) = P(\Lambda^{-1}(u) \geq t) = P(L^{-1}(u) \geq t - 2\nu u). \quad (2.15)$$

Then

$$P(\Lambda(t,0) > u) = \begin{cases} P(L(t - 2\nu u, 0) > u) & u \leq \frac{t}{2\nu}, \\ 0 & u > \frac{t}{2\nu}. \end{cases} \quad (2.16)$$

So it suffices to derive the distribution of $L(t,0)$ in order to see the one of $\Lambda(t,0)$. Apply the Tanaka-Meyer formula (Proposition 2.1 (iii)) to $W_t$,

$$|W_t| = |x_1 - x_2| + \int_0^t \text{sgn}(W_s) dW_s + 2L(t,0). \quad (2.17)$$
By Lévy’s characterization, \( B(t) := \frac{1}{\sqrt{2\pi}} \int_{0}^{t} \text{sgn}(W_{s}) dW_{s} \) is a standard Brownian motion starting from 0. Since \( L(0,0) = 0 \) and \( \int_{0}^{\infty} 1_{\{W_{t} > \xi \}} L(ds,0) = 0 \), a.s. the equation (2.17) is a Skorohod equation (see Lemma 6.14 in [KS91, Chapter 3]) with the unique solution \( 2L(t,0) \) given by

\[
2L(t,0) = \max \left\{ 0, \max_{0 \leq s \leq t} \left\{ -\left( |x_{1} - x_{2}| + 2B(s) \right) \right\} \right\}, \quad 0 \leq t \leq \infty.
\]

(2.18)

By the reflection principle,

\[
\mathbb{P}(2L(t,0) = 0) = \mathbb{P}\left( \min_{0 \leq s \leq t} B(s) \geq -\frac{|x_{1} - x_{2}|}{2} \right) = 1 - 2\mathbb{P} \left( B(t) > \frac{|x_{1} - x_{2}|}{2} \right) = 2\Phi \left( \frac{|x_{1} - x_{2}|}{2\sqrt{t}} \right) - 1,
\]

and for all \( u \geq 0 \),

\[
\mathbb{P}(2L(t,0) > u) = \mathbb{P} \left( -\min_{0 \leq s \leq t} 2B(s) - |x_{1} - x_{2}| > u \right) = 2\mathbb{P}(2B(t) > |x_{1} - x_{2}| + u) = 2 - 2\Phi \left( \frac{|x_{1} - x_{2}| + u}{2\sqrt{t}} \right).
\]

(2.20)

In particular, when \( x_{1} - x_{2} = 0 \), for \( t > 0 \),

\[
\mathbb{P}(\Lambda_{0}(t,0) = 0) = \mathbb{P}(L_{0}(t,0) = 0) = 0 \quad (2.21)
\]

\[
\mathbb{P}(\Lambda_{0}(t,0) > u) = \begin{cases} \mathbb{P}(L_{0}(t - 2\nu u,0) > u) = 2 - 2\Phi \left( \frac{u}{\sqrt{t - 2\nu u}} \right), & t > 2\nu u > 0; \\ \mathbb{P}(L_{0}(t - 2\nu u,0) = 0) = 2, & 2\nu u \geq t > 0. \end{cases} \quad (2.22)
\]

For \( x_{1} - x_{2} \neq 0 \), it is easy to see that for all \( u \geq 0 \),

\[
\mathbb{P}(\Lambda_{x_{1} - x_{2}}(t,0) > u) \leq \mathbb{P}(\Lambda_{0}(t,0) > u).
\]

(2.23)

We then consider the quantity \( \mathbb{E}[\Lambda_{0}(t,0)] \).

\[
\mathbb{E}[\Lambda_{0}(t,0)] = \int_{0}^{\infty} \mathbb{P}(\Lambda_{0}(t,0) > u) du = 2 \int_{0}^{\infty} \int_{\frac{u}{\sqrt{t}}}^{\infty} \phi(y) y du dy, \quad (2.24)
\]

where \( \phi(y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{y^{2}}{2}} \) is the Gaussian density. For the right-hand side of (2.24), change the order of the integrals, apply the integration by parts and the substitution \( z = \sqrt{\nu^{2}y^{2} + t/\nu} \),

\[
\mathbb{E}[\Lambda_{0}(t,0)] = 2 \int_{0}^{\infty} \left( y \sqrt{\nu^{2}y^{2} + t - \nu y^{2}} \right) \phi(y) dy
\]

\[
= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \int_{\frac{u}{\sqrt{t}}}^{\infty} \nu z e^{-\frac{z^{2}}{4\nu} + \frac{z^{2}}{2\nu} - 2\nu} \int_{0}^{\infty} y^{2} \phi(y) dy
\]

\[
= \sqrt{\frac{2}{\pi}} e^{\frac{t}{\nu}} + \int_{0}^{\infty} \sqrt{\frac{2}{\pi}} e^{\nu \sqrt{t/\nu}} e^{-\frac{z^{2}}{4\nu} + \frac{z^{2}}{2\nu} - 2\nu} dz - \nu
\]

\[
= \sqrt{\frac{2}{\pi}} e^{t/2\nu^{2}} \left[ 1 - \Phi \left( \sqrt{\frac{t}{\nu}} \right) \right] - 1 \nu.
\]

(2.25)

Note that \( e^{x^{2}/2} \left( 1 - \Phi(x) \right) \) is a decreasing function when \( x \geq 0 \), so the sup norm of the term in the bracket of the last line in (2.25) is 1. Consequently, the order of \( \mathbb{E}[\Lambda_{0}(t,0)] \) is \( t^{1/2} \).

Generally when \( x_{1} - x_{2} \neq 0 \), according to (2.23),

\[
\mathbb{E}[\Lambda_{x_{1} - x_{2}}(t,0)] \leq \mathbb{E}[\Lambda_{0}(t,0)] = O(t^{1/2}).
\]

(2.26)

This completes the proof.
Lemma 2.6. Let \( x_1 - x_2 = 0 \) and \( \tilde{W}_t := X_1^t - X_2^t \). For a fixed \( t \), let \( B_k := \{ \tilde{W}_s = 0 : \text{for some} \ s \in (kt, (k+1)t] \} \). Then for any \( \alpha > 0 \),
\[
\sum_{k=0}^{n} \mathbb{P}(B_k) = o(n^{1/2+\alpha}),
\]
where \( f(n) = o(g(n)) \) denotes that \( f(n)/g(n) \to 0 \) as \( n \to \infty \).

**Proof.** \( \tilde{W}_t \) can be obtained by time-changing a Brownian motion \( W_t \) as in Lemma 2.5. Following the notations there we write \( A_t = t + 2\nu L_0(t, 0) \), where \( L_0(t, x) \) is the local time of \( W_t \), and \( T_t = A_t^{-1} \) so that \( \tilde{W}_t = W_{T_t} \). Now we define a measure \( m(dx) := dx + 2\nu1_{(0)}(x) \) (in some references this measure is called speed measure, such as [Fre71]), then by Proposition 2.1 (ii), (iv) and \( \mathbb{E}[W_t^2] = 2t \), for any Borel measurable function \( f \), we have a.s.
\[
\int_0^t f(W_{T_t})ds = \int_0^{T_t} f(W_s)dA_s
\]
\[
= \int_0^{T_t} f(W_s)ds + 2\nu \int_0^{T_t} f(W_s)L(ds, 0)
\]
\[
= \int_{-\infty}^{\infty} f(x)L(T_t,x)dx + 2\nu f(0)L_0(T_t,0)
\]
\[
= \int_{-\infty}^{\infty} f(x)L_0(T_t,x)m(dx)
\]
Since \( L_0(T_t, x) = \Lambda_0(t, x) \), where \( \Lambda(t, x) \) is the local time of \( \tilde{W} \), this equality is equivalent to
\[
\int_0^t f(\tilde{W}_s)ds = \int_{-\infty}^{\infty} f(x)\Lambda_0(t, x)m(dx)
\]
Taking differentiation of both sides with respect to \( t \) and then taking the expectation gives us the probability density \( p_t(x) \) of \( \tilde{W}_t \) with respect to \( m(dx) \),
\[
p_t(x) = \frac{\partial \mathbb{E}\Lambda_0(t, x)}{\partial t}.
\]
From the expression (2.26), \( p_t(0) \leq Ct^{-1/2} \) for some constant \( C \) when \( t \) is large enough (notice that \( 1 - \Phi(x) \approx x^{-1}e^{-x^2/2} \) for large \( x \)). For \( x \neq 0 \), it is also true that \( p_t(x) \leq Ct^{-1/2} \) because \( \tilde{W} \) behaves as a Brownian motion when it is not at 0. So for any \( K > 0 \),
\[
\mathbb{P}(|\tilde{W}_t| \leq K) = \int_{-K}^{K} p_t(x)m(dx) \leq (K + 2\nu)Ct^{-1/2}
\]
For the event \( B_k \), for any \( \alpha > 0 \),
\[
\mathbb{P}(B_k) \leq \mathbb{P}(B_k \cap \{|\tilde{W}_kt| > (kt)^{\alpha}\}) + \mathbb{P}(|\tilde{W}_kt| \leq (kt)^{\alpha}).
\]
Conditional on \( \tilde{W}_kt = x > 0 \), the probability of \( \tilde{W}_s \) hitting zero is same as the one of Brownian motion starting from \( x \) hitting zero, and it decreases as \( |x| \) increases. Therefore, recall the meaning of \( B_k \),
\[
\mathbb{P}(B_k \cap \{|\tilde{W}_kt| > (kt)^{\alpha}\}) \leq 2 \left(1 - \Phi\left(\frac{(kt)^{\alpha}}{\sqrt{2t}}\right)\right)
\]
where the right-hand side is the probability of a Brownian motion starting at \((kt)^\alpha\) hitting zero before time \(t\). Since \(1 - \Phi(x)\) has an exponential decay, as \(k\) large enough,

\[
\mathbb{P}(B_k \cap \{|\tilde{W}_{kt}| > (kt)^\alpha\}) \leq Ck^{-1/2+\alpha}
\]  
(2.34)

Combine this with (2.31),

\[
\mathbb{P}(B_k) = o(k^{-1/2+\alpha}).
\]  
(2.35)

Taking summation of (2.35) from 1 to \(n\) gives the desired result.

\[\square\]

3 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on a second moment method, which is inspired by [BCDG13]. The idea is that the averaged law of a random motion \((X_t)_{t \geq 0}\) in the Howitt-Warren flow converges to the law of a drifted Brownian motion, and the quenched law satisfies a law of large numbers by variance calculations so that it also converges to the same limit. In the proof, we consider a class of test functions applied to \(Y^{(n)} := (Y_{nt}/\sqrt{n})_{0 \leq t \leq T}\) for some fixed \(T > 0\), where \(Y_t := X_t - \beta t\), and bound the variance of their quenched mean \(E^\omega[f(Y^{(n)})]\) in such a way that we can apply the Borel-Cantelli lemma to get an almost sure convergence for a subsequence of the quenched mean. Then with some modification we will reach our goal. This method can also be applied to show the quenched invariance principle for the random walk in an i.i.d. space-time random environment (see Appendix A). We begin with two lemmas.

Lemma 3.1. Let \(T > 0\) and \(C[0,T]\) be the space of continuous function on \([0,T]\) equipped with the sup norm. If for any bounded Lipschitz function \(f : C[0,T] \to \mathbb{R}\), \(E^\omega[f(Y^{(n)})]\) converges to \(E[f(B)]\) a.s. as \(n\) tends to \(\infty\), where \(B := (B_t)_{0 \leq t \leq T}\) is a standard Brownian motion, then for \(\mathbb{P} - \text{a.e. } \omega\), \(Y^{(n)}\) converges weakly to \(B\).

Proof. It suffices to show that there exists a convergence determining class for \(C[0,T]\) that consists of countably many bounded Lipschitz functions. However, the proof of Proposition 3.17 in [Res87] shows how to find such a convergence determining class for general Polish space. As a particular case Lemma 3.1 holds.

To check the almost sure convergence for bounded Lipschitz function \(f\), we first consider its variance.

Lemma 3.2. For any bounded Lipschitz function \(f\) and a random motion \((X_t)_{t \geq 0}\) with \(X_0 = 0\) in the Howitt-Warren flow, there exists a constant \(C_{f,T,\nu} > 0\), depending only on \(f\), \(T\), and the stickiness parameter \(\nu\) of the Howitt-Warren 2-point motion, such that

\[
E\left[ (E^\omega f(Y^{(n)}) - E f(B))^2 \right] \leq C_{f,T,\nu} n^{-1/4}.
\]  
(3.1)

Proof. Let \(X^1_t, X^2_t\) with \(X^1_0 = X^2_0 = 0\) be two independent random motions in the same environment, i.e., for a fixed realization of the Howitt-Warren flow. Then under the averaged law \(P\), \((X^1_t, X^2_t)\) is a 2-point motion. Moreover, by Theorem 2.2, we have a coupling \((X^1_t, X^2_t, B^1_t, B^2_t, B^3_t)\) as in (2.2), where \(B^1_t, B^2_t\) and \(B^3_t\) are independent Brownian motions. Let \(Y^i_t := X^i_t - \beta t\), then \(Y^{(n)} := (Y^i_{nt}/\sqrt{n})_{0 \leq t \leq T}\) for \(i=1,2\), and \(B^{(n)} := (B^i_{nt}/\sqrt{n})_{0 \leq t \leq T}\) for \(j = 1, 2, 3\), and

\[
E\left[ \| Y^{(n)} - B \| \right] = n^{-\frac{1}{2}} E\left[ \sup_{0 \leq t \leq T} \int_0^{nt} 1_{\{X^i_t = X^j_t\}} dB^3_t - \int_0^{nt} 1_{\{X^i_t = X^j_t\}} dB^1_t \right] \\
\leq 4n^{-\frac{1}{2}} \left( E\left[ \left( \int_0^{nt} 1_{\{X^i_t = X^j_t\}} dB^3_t \right)^2 \right] \right)^{\frac{1}{2}} + 4n^{-\frac{1}{2}} \left( E\left[ \left( \int_0^{nt} 1_{\{X^i_t = X^j_t\}} dB^1_t \right)^2 \right] \right)^{\frac{1}{2}} \\
= 8n^{-\frac{1}{2}} \left( E\left[ \int_0^{nt} 1_{\{X^i_t = X^j_t\}} ds \right] \right)^{\frac{1}{2}} + 16n^{-\frac{1}{2}} \nu \left( E[\Lambda_0(nT,0)] \right)^{\frac{1}{2}} \\
\leq C_{T,\nu} n^{-\frac{1}{2}}.
\]
where the first equality holds because of the coupling (23), the second step is by Doob’s maximal inequality, and the last two steps are by (24) and (2.26). Since X^1_t, X^2_t are independent in a same environment \( \omega \) and behave as a 2-point motion under the averaged law \( P \), we have

\[
\begin{align*}
\mathbb{E}[ (E^\omega f(Y^{(n)})) - Ef(B) ]^2 &= E[ f(Y^{(n)}) f(Y^{(n)}) - f(B^{(n)}) f(B^{(n)}) ] \\
&\leq \| f \|_{\infty} C_f \left( E[ \| Y^{(n)} - B^{(n)} \| ] + E[ \| Y^{(n)} - B^{(n)} \| ] \right) \\
&\leq C_{f,T,\nu} n^{-1/4}.
\end{align*}
\]

(3.2)

Now we can finish the proof of Theorem [11].

Proof of Theorem [11]. For the first statement, we only need to prove that for all \( T > 0, (\frac{Y_t}{\sqrt{n}})_{0 \leq t \leq T} \) converges weakly to \((B_t)_{0 \leq t \leq T} \) in \( C[0,T] \) equipped with the sup norm. We follow the notations given in Lemma 3.1 and Lemma 3.2. By Lemma 3.1 it only remains to show that for each bounded Lipschitz function \( f \), a.s., \( E^\omega f(Y^{(n)}) \rightarrow Ef(B) \). Without loss of generality, we assume \( Y_0 = 0 \).

First, observe that along a subsequence \( n_k = n^3 \), the almost sure convergence holds. Indeed, for any \( \epsilon > 0 \), by Lemma 3.2 and Markov inequality,

\[
\mathbb{P} \left( \left| E^\omega f(Y^{(n)}) - Ef(B) \right| \geq \epsilon \right) \leq C_{f,T,\nu} \epsilon^{-2} n^{-5/4}.
\]

(3.3)

It is summable, hence by Borel-Cantelli lemma \( E^\omega f(Y^{(n^3)}) \rightarrow Ef(B) \) a.s..

For general \( m \in [n^3, (n+1)^3] \), we control the maximum of the difference between \( E^\omega f(Y^{(m)}) \) and \( E^\omega f(Y^{(n^3)}) \). Since \((n+1)^3 - n^3 \leq 6n^4 \) for large \( n \),

\[
\begin{align*}
&\max_{n^3 \leq m < (n+1)^3} \left| E^\omega f(Y^{(m)}) - E^\omega f(Y^{(n^3)}) \right| \\
&\leq \max_{n^3 \leq m < (n+1)^3} C_f \left\{ \left( \sqrt{\frac{m}{n^3}} - 1 \right) E^\omega \| Y^{(m)} \| + E^\omega \left[ \max_{0 \leq t \leq T} |Y_{mt} - Y_{n^3t}| \right] \right\} \\
&\leq C_f n^{-1} \max_{n^3 \leq m < (n+1)^3} E^\omega \| Y^{(m)} \| + C_f n^{-1/2} \max_{0 \leq m < 6n^4} E^\omega \| \tilde{Y}^{(m)} \|,
\end{align*}
\]

(3.4)

where \( \tilde{Y}^{(m)} := m^{-1/2} (Y_{m+n^3} - Y_{n^3}) \) is equal to \( Y^{(m)} \) in distribution. Since under the averaged law \( P, Y_t = X_t - \beta t \) is a standard Brownian motion starting form 0, the \( k \)-th moment of the maximum process of \( |Y_t| \) is of order \( t^{k/2} \). Therefore, for any \( \delta > 0 \),

\[
\mathbb{P} \left( n^{-1} \max_{n^3 \leq m < (n+1)^3} E^\omega \| Y^{(m)} \| \geq \delta \right) \leq \delta^{-2} n^{-7} E \left[ \max_{0 \leq m < (n+1)^3} |Y_t|^2 \right] = O(n^{-2}),
\]

(3.5)

\[
\mathbb{P} \left( n^{-1/2} \max_{0 \leq m < 6n^4} E^\omega \| \tilde{Y}^{(m)} \| \geq \delta \right) \leq \delta^{-4} n^{-10} E \left[ \max_{0 \leq m < 6n^4} |Y_t|^4 \right] = O(n^{-2}).
\]

(3.6)

Both (3.5) and (3.6) are summable, which means that for \( \mathbb{P} \)-a.e. \( \omega \), as \( n \to \infty \),

\[
\max_{n^3 \leq m < (n+1)^3} \left| E^\omega f(Y^{(m)}) - Ef(Y^{(n^3)}) \right| \to 0.
\]

(3.7)

Therefore, for each bounded Lipschitz function \( f \), \( \mathbb{P} \)-a.s., \( E^\omega [ f(Y^n) ] \to Ef([B]) \), which finishes the proof of the first part.
To show that for $\mathbb{P}$-a.e. $\omega$, $Z_n := n^{-1/2} \max_{s \leq nt} |E^\omega X_s - \beta s|$ converges to 0, we again consider the second moment. By Doob’s maximal inequality and Markov inequality,
\[ E \left( n^{-1/2} \max_{s \leq nt} |E^\omega X_s - \beta s| \right)^2 \leq 4n^{-1} E[Y_{nt}^1 Y_{nt}^2] = 4n^{-1} \int_0^{nt} 1_{\{X_t^1 = X_t^2\}} ds, \] (3.8)
where $E[Y_{nt}^1 Y_{nt}^2]$ is the covariance of the 2-point motion and from Definition 1.2 we get the equality. In Lemma 1.2 we have computed (the calculation there is independent of this section, hence applicable here) that the expectation in the right-hand side of (3.8) is of order $n^{1/2}$. As a result, the order of the second moment of $Z_n$ is $n^{-1/2}$. So along the subsequence \{3n\}, $Z_{3n} \to 0$ by Borel-Cantelli lemma. As for $n^3 \leq k < (n + 1)^3$,
\[ |Z_k - Z_{n^3}| \leq C_1 n^{-5/2} \max_{s \leq nt} |E^\omega X_s - \beta s| + C_2 n^{-3/2} \max_{n^3 \leq s \leq (n+1) n^3} |E^\omega [X_s - X_{n^3} - \beta (s - n^3 t)]|. \] (3.9)
Similar arguments as (3.5) and (3.6) can be applied here, and therefore we can show that for P-a.e. $\omega$, \( \max_{n^3 + k < (n+1)^3} |Z_k - Z_{n^3}| \) converges to 0. Thus for $\mathbb{P}$-a.e. $\omega$, $Z_n$ converges to 0.

4 Proof of Theorem 1.2

In this section we will only focus on $a_n(t, r)$, since the proof for $b_n(t, r)$ can be easily obtained by a change of time interval from $[-nt, 0]$ to $[0, nt]$.

The strategy is essentially the same as in [BRAS06]. We proceed in two steps. First it is shown that for a fixed time $t$, the distribution of $(a_n(t, r_1), \ldots, a_n(t, r_k))$ for any given integer $k > 0$ converges weakly to $(a(t, r_1), \ldots, a(t, r_k))$ in Lemma 1.2.

In the second step, observe that by decomposing $X_0^r \sqrt{\pi} \beta nt - nt$ in terms of its increments on $[-nt, -n(t-s)]$ and $[-n(t-s), 0]$,\[ a_n(t, r) = n^{-1/4} \left( E^\omega X_0^r \sqrt{\pi} \beta nt - nt - r \sqrt{n} \right) \]
\[ = n^{-1/4} \left( E^\omega X_0^r \sqrt{\pi} \beta nt - nt - E^\omega X_0^r \sqrt{\pi} \beta nt - nt - r \sqrt{n} \right) + n^{-1/4} \left( E^\omega X_0^r \sqrt{\pi} \beta nt - nt - r \sqrt{n} \right) \]
\[ = n^{-1/4} \int E^\omega X_0^r \sqrt{\pi} \beta nt - nt - r \sqrt{n} \]
\[ = \int a_n(s, \frac{z}{\sqrt{n}}) P^\omega \left( X_0^r \sqrt{\pi} \beta nt - nt + \beta ns \right) ds + a_n(t-s, r) \circ T_{-ns, -\beta ns}, \] (4.1)
where $T_{t, x}$ denotes the transition of the random environment that makes $(x, t)$ the new space-time origin. Here we note that in the decomposition as functions of the random environment in disjoint time intervals, $a_n(s, \frac{z}{\sqrt{n}})$ and $a_n(t-s, r) \circ T_{-ns, -\beta ns}$ are independent, while the random measure on line (4.1) converge to Gaussian distribution by Theorem 1.1.

Meanwhile, for the limiting Gaussian process, [BRAS06] Lemma 3.1 shows:

**Proposition 4.1.** There is a version of the Gaussian process $\{a(t, r) : (t, r) \in \mathbb{R}^+ \times \mathbb{R}\}$ that is continuous in $(t, r)$. Moreover, given $0 = t_0 < t_1 < \cdots < t_n$, let $\{\tilde{a}(t_i - t_{i-1}, \cdot) : 1 \leq i \leq n\}$ be independent random functions such that $\tilde{a}(t_i - t_{i-1}, \cdot)$ has the distribution of $a(t_i - t_{i-1}, \cdot)$ for all $i$. Define $a^*(t_i, r) := \tilde{a}(t_i, r)$ for $r \in \mathbb{R}$ and inductively for $i = 2, \ldots, n$ and $r \in \mathbb{R}$,
\[ a^*(t_i, r) := \int a^*(t_{i-1}, r + z) \phi\left( \frac{z}{\sqrt{t_i - t_{i-1}}} \right) dz + \tilde{a}(t_i - t_{i-1}, r), \] (4.2)
where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then the joint distribution of the random functions $\{a^*(t_i, \cdot) : 1 \leq i \leq n\}$ is the same as that of $\{a(t_i, \cdot) : 1 \leq i \leq n\}$.
We see that the decompositions (4.1) and (4.2) have the same structure. In order to show the convergence of the finite dimensional distributions of \( a_n(t, r) \), we only need to take advantage of this structure, and apply the induction to \( a_n(t, r) \) based on Lemma 4.2.

### 4.1 A deterministic time-level

In this subsection we prove the weak convergence of the finite dimensional distributions for a fixed time \( t \) as the following lemma:

**Lemma 4.2.** For any fixed \( t > 0 \) and \( N \in \mathbb{N} \), suppose \( r_1 < \cdots < r_N \) to be any \( N \) different points on the real line. Then the \( \mathbb{R}^N \)-valued random vector \( (a_n(t, r_1), \ldots, a_n(t, r_N)) \) converges weakly to the mean zero Gaussian vector \( (a(t, r_1), \ldots, a(t, r_N)) \) with covariance matrix \( (\Gamma((t, r_i), (t, r_j)))_{1 \leq i, j \leq N} \), where \( \Gamma((t, r_i), (t, r_j)) = \nu \int_0^{2t} \frac{1}{\sqrt{2\pi u}} e^{-u^2/2} du \).

**Proof.** Only need to show that for each \( \bar{\theta} \in \mathbb{R}^N \), \( \sum_{i=1}^N \theta_i a_n(t, r_i) \) converges weakly to \( \sum_{i=1}^N \theta_i a(t, r_i) \) by Cramér-Wold device. Abbreviate \( X_i^{n, \pi} \) and \( X_i^{n, 1} \) by \( X_i^{n, \pi} \). Since for \( i \leq 1 \) we have the decomposition

\[
\sum_{i=1}^N \theta_i a_n(t, r_i) = n^{-1/4} \sum_{i=1}^N \theta_i \int E^{\omega} \left[ X_{kt}^{n,i} - X_{(k-1)t}^{n,i} \right] \]

\[
= \sum_{k=1}^n \left( n^{-1/4} \sum_{i=1}^N \theta_i E^{\omega} \left[ X_{kt}^{n,i} - X_{(k-1)t}^{n,i} \right] \right),
\]

(4.3)

if we define \( z_{n,k} := n^{-1/4} \sum_{i=1}^N \theta_i E^{\omega} \left[ X_{kt}^{n,i} - X_{(k-1)t}^{n,i} \right] \) and the filtration \( \mathcal{F}_{n,k} := \sigma (K_{u,v} : -nt \leq u \leq v \leq kt - nt) \) \( k = 0, 1, 2, \cdots \), then \( \sum_{i=1}^N \theta_i a_n(t, r_i) = \sum_{k=1}^n z_{n,k} \), and the process \( (\sum_{k=1}^n z_{n,k})_{j \in \mathbb{N}} \) is adapted to \( \{ \mathcal{F}_{n,j} \}_{j \in \mathbb{N}} \), and \( \mathbb{E} [z_{n,k} | \mathcal{F}_{n,k-1}] = 0 \). Therefore, \( (\sum_{k=1}^n z_{n,k})_{j=0}^n \) is a martingale with respect to the filtration \( \{ \mathcal{F}_{n,j} \}_{j=1}^n \), where \( z_{n,k} \) is the martingale difference.

Recall that our goal is to prove \( \sum_{k=1}^n z_{n,k} \) converges in distribution to a Gaussian. By the martingale central limit theorem, it suffices to check that for any \( \epsilon > 0 \), when \( n \to \infty \),

\[ (i) \quad n \sum_{k=1}^n \mathbb{E} [z_{n,k}^2 | \mathcal{F}_{n,k-1}] \overset{p}{\to} \sum_{1 \leq i, j \leq N} \theta_i \theta_j \Gamma((t, r_i), (t, r_j)), \]

\[ (ii) \quad n \sum_{k=1}^n \mathbb{E} [z_{n,k}^2 \mathbb{1}_{|z_{n,k}| > \epsilon} | \mathcal{F}_{n,k-1}] \overset{p}{\to} 0. \]

(4.4)

(4.5)

Note that to show Lindeberg’s condition (4.5), it suffices to show Lyapunov’s condition:

\[ n \sum_{k=1}^n \mathbb{E} [z_{n,k}^6 | \mathcal{F}_{n,k-1}] \overset{p}{\to} 0 \quad \text{as} \ n \to \infty, \]

(4.6)

which is implied by the following estimate:

\[ n \sum_{k=1}^n \mathbb{E} [z_{n,k}^6] = n^{-3/2} \sum_{k=1}^n \mathbb{E} \left[ \left( \sum_{i=1}^N \theta_i E^{\omega} \left[ X_{kt}^{n,i} - X_{(k-1)t}^{n,i} \right] \right)^6 \right] \]

\[ \leq n^{-3/2} \sum_{i=1}^N N^5 \theta_i^6 \sum_{k=1}^n \mathbb{E} \left[ \left( X_{kt}^{n,i} - X_{(k-1)t}^{n,i} \right)^6 \right] \]

\[ = C(N, t, \bar{\theta}) n^{-1/2}, \]

(4.7)
where in the last equality we used that $X_{nt}^{n,i} - X_{(k-1)t}^{n,i}$ is a drifted Brownian motion under the averaged law and therefore the 6th-moment is a constant depending on $t$.

It only remains to check condition (3.12). First we observe that

$$\sum_{k=1}^{n} \mathbb{E}[z_{n,k}^2] = n^{-1/2} \sum_{1 \leq i,j \leq N} \theta_{ij} \sum_{k=1}^{n} \mathbb{E}[\sum_{k=1}^{n} \mathbb{E}[X_{kt}^{n,i} - X_{(k-1)t}^{n,i}] \mathbb{E}[X_{kt}^{n,j} - X_{(k-1)t}^{n,j}]]$$

$$= n^{-1/2} \sum_{1 \leq i,j \leq N} \theta_{ij} \sum_{k=1}^{n} \mathbb{E}[X_{kt}^{n,i} - X_{(k-1)t}^{n,i}] \mathbb{E}[X_{kt}^{n,j} - X_{(k-1)t}^{n,j}]$$

$$= \sum_{1 \leq i,j \leq N} \theta_{ij} n^{-1/2} \sum_{k=1}^{n} \mathbb{E}[(X_{kt}^{n,i} - X_{(k-1)t}^{n,i}) (X_{kt}^{n,j} - X_{(k-1)t}^{n,j})]$$

$$= \sum_{1 \leq i,j \leq N} \theta_{ij} n^{-1/2} E \int_{0}^{nt} 1_{\{X_{t}^{n,i} = X_{t}^{n,j}\}} ds$$

$$= \sum_{1 \leq i,j \leq N} \theta_{ij} \nu n^{-1/2} \left[ E[X_{nt}^{n,i} - X_{nt}^{n,j}] - |r_{i} \sqrt{n} - r_{j} \sqrt{n}| \right] \quad (4.8)$$

where in the second equality, given the environment $\omega$ we take independent copies $X_{t}^{n,i}, X_{n}^{2,n,i}$ of $X_{n,i}$, and then under the averaged law $P$, $(X_{t}^{n,i}, X_{n}^{2,n,i})$ is a 2-point motion; the forth and fifth equality holds because the covariation process of $(X_{t}^{n,i}, X_{n}^{2,n,i})$ is given by the integral and

$$\nu \int_{0}^{1} \{X_{t}^{n,i} = X_{n}^{2,n,j}\} ds$$

is a martingale as stated in Definition 1.2.

Notice that $X_{t}^{n,i} - X_{n}^{2,n,j}$ is a sticky Brownian motion with sticky interaction at the origin starting form $r_{i} \sqrt{n} - r_{j} \sqrt{n}$, then it behaves as a Brownian motion before time $\tau$, where $\tau$ is the stopping time of the sticky Brownian motion first hitting the origin. Let $\sigma$ be the stopping time of the Brownian motion $W_{t}^{[x]}$ first hitting the origin, where the quadratic variation of $W_{t}^{[x]}$ is $2t$ and $|x| = |r_{i} \sqrt{n} - r_{j} \sqrt{n}|$, then $\tau$ is equal in distribution to $\sigma$. Thus by reflection principle we have

$$P(\tau > s) = \mathbb{P}(\inf_{0 \leq u \leq s} W_{u}^{[x]} > 0) = 2 - 2 \Phi \left( \frac{|x|}{\sqrt{2s}} \right) = 2 - 2 \Phi \left( \frac{|r_{i} - r_{j}| \sqrt{n}}{\sqrt{2s}} \right), \quad (4.9)$$

and

$$E \left[ X_{t}^{n,i} - X_{n}^{2,n,j} \right] \mathbb{1}_{\{\tau > s\}} = E \left[ W_{s}^{[x]} \right] \mathbb{1}_{\{\tau > s\}} = E \left[ W_{s}^{[x]} \right] - E \left[ W_{s}^{[x]} \right] \mathbb{1}_{\{\sigma \leq s\}}$$

$$= |x| - E \left[ W_{s}^{[x]} \right] \mathbb{1}_{\{\sigma \leq s\}} = |x| = |r_{i} \sqrt{n} - r_{j} \sqrt{n}|. \quad (4.10)$$

The last calculation shows that

$$E \left[ X_{nt}^{n,i} - X_{n}^{2,n,j} \right] = |r_{i} \sqrt{n} - r_{j} \sqrt{n}| = E \left[ X_{nt}^{n,i} - X_{n}^{2,n,j} \right] \mathbb{1}_{\{\tau \leq nt\}}. \quad (4.11)$$

Conditional on the stopping time $\tau$, $X_{t}^{n,i} - X_{n}^{2,n,j}$ is a sticky Brownian motion starting from 0. Consider the first moment of a sticky Brownian motion $\tilde{W}_{s}^{0}$ starting form the origin with stickiness also at origin. By Tanaka-Meyer formula and Lemma 2.3

$$E[\tilde{W}_{s}^{0}] = 2E[A_{0}(s,0)] = 2 \sqrt{\frac{2}{\pi}} \frac{s^{1/2}}{2} + 2\nu \left( \sqrt{\frac{2}{\pi}} \frac{s^{1/2}}{2} \Phi \left( \frac{\sqrt{n}}{\sqrt{2s}} \right) - 1 \right) \quad (4.12)$$

If we condition the difference process $X_{t}^{n,i} - X_{n}^{2,n,j}$ on $\tau$ and use the Markov property, then by
(4.9), (4.11) and (4.12), the last line of (4.8) equals
\[
\sum_{1 \leq i,j \leq N} \theta_i \theta_j \nu n^{-1/2} \int_0^{nt} \left[ 2 \sqrt{\frac{2(nl - s)}{\pi}} + 2\nu \left( \sqrt{\frac{2}{\pi}} e^{(nt-s)/2\nu^2} \left[ 1 - \Phi \left( \frac{\sqrt{n}l - s}{\nu} \right) \right] - 1 \right) \right] P(\tau \in ds)
= \sum_{1 \leq i,j \leq N} \theta_i \theta_j \nu n^{-1/2} \left[ \sqrt{2} \int_0^{nt} e^{2nt - n(r_i - r_j)^2s} e^{-1/2s} ds \right] + 2\nu \int_0^{nt} \left\{ \sqrt{\frac{2}{\pi}} e^{(nt-s)/2\nu^2} \left[ 1 - \Phi \left( \frac{\sqrt{n}l - s}{\nu} \right) \right] - 1 \right\} P(\tau \in ds).
\] (4.13)

Define
\[
G(x,t) := \sqrt{2} \int_0^{2t/x^2} \frac{e^{2lt - x^2s}}{\pi s^{3/2}} e^{-1/2s} ds,
\] (4.15)
\[
H(x,t) := 2\nu \int_0^t \left\{ \sqrt{\frac{2}{\pi}} e^{(t-s)/2\nu^2} \left[ 1 - \Phi \left( \frac{\sqrt{n}l - s}{\nu} \right) \right] - 1 \right\} P(\tau \in ds),
\] (4.16)
then
\[
G(\sqrt{n}l, nt) = \sqrt{n}G(x,t), \quad G(0,t) = \sqrt{t},
\] (4.17)
\[
\left| \frac{\partial G}{\partial x} (x,t) \right| = \sqrt{2} \int_0^{2t/x^2} \frac{-x^2s}{\pi s^{3/2} \sqrt{2t - x^2s}} e^{-1/2s} ds \leq 2 \nu \int_0^{2t/x^2} \frac{|x|}{\sqrt{s} 2t - x^2s} ds \leq 2,
\] (4.18)
and since that $e^{x^2/2} (1 - \Phi(x))$ is a decreasing function when $x \geq 0$ bounded by 1,
\[
|H(x,t)| \leq 2\nu \int_0^t (1 + 1) P(\tau \in ds) \leq 4\nu.
\] (4.19)

Furthermore, we have a non-trivial equality
\[
\Gamma((t, r), (t, q)) = \nu G(r - q, t),
\] (4.20)
This can be obtained via comparing the second order partial derivative with respect to $x$ of these two quantities, and we also provide a probabilistic method to prove this identity in Appendix B.

Applying (4.13)–(4.20) to (4.8), we have
\[
\sum_{k=1}^n \mathbb{E} \left[ z_{n,k}^2 \right] = \sum_{1 \leq i,j \leq N} \theta_i \theta_j \nu n^{-1/2} \left( G(\sqrt{n}(r_i - r_j), nt) + n^{-1/2}H(\sqrt{n}(r_i - r_j), nt) \right)
= \sum_{1 \leq i,j \leq N} \theta_i \theta_j \nu \left( G((r_i - r_j), t) + n^{-1/2}H(\sqrt{n}(r_i - r_j), nt) \right)
\] (4.21)
\[
\xrightarrow{n \to \infty} \sum_{1 \leq i,j \leq N} \theta_i \theta_j \Gamma((t, r_i), (t, r_j)).
\] (4.22)

Hence, to check condition (4.4), it suffices to show that as $n$ tends to $\infty$,
\[
\sum_{k=1}^n \left( \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n, k-1} \right] - \mathbb{E} \left[ z_{n,k}^2 \right] \right) \xrightarrow{P} 0.
\] (4.23)
Actually, we will show the convergence in $L^2$. Rewrite the left-hand side of (4.23) as

$$
\sum_{k=1}^{n} \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,k-1} \right] - \sum_{k=1}^{n} \mathbb{E} \left[ z_{n,k}^2 \right]
$$

$$
= \sum_{k=1}^{n} \left( \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l} \right] - \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l-1} \right] \right)
$$

$$
= \sum_{l=1}^{n-1} \sum_{k=l+1}^{n} \left( \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l} \right] - \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l-1} \right] \right)
$$

$$
= \sum_{l=1}^{n-1} R_l,
$$

(4.24)

where $R_l := \sum_{k=l+1}^{n} \left( \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l} \right] - \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l-1} \right] \right)$ is a random variable determined by the environment up to time $t_l$. Consequently, for $l > l'$, $\mathbb{E} \left[ R_l R_{l'} \right] = \mathbb{E} \left[ \mathbb{E} \left[ R_l \mathcal{F}_{n,l-1} \right] R_{l'} \right] = 0$. As a result, the $L^2$ norm of (4.23) is $\sum_{l=1}^{n} E \left[ R_l^2 \right] = \sum_{l=1}^{n} \mathbb{E} \left[ \mathbb{E} \left[ R_l \mathcal{F}_{n,l-1} \right] R_{l'} \right]$. Next we are going to bound this second moment. Mimic the derivation of (4.8), setting $I_t(x) := G(x, (n - l)t) + H(x, (n - l)t)$ and $E_{n,l} := E[\mathcal{F}_{n,l}]$, it is not hard to see that

$$
\sum_{k=l+1}^{n} \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l} \right] = n^{-1/2} \sum_{1 \leq i,j \leq N} \theta_i \theta_j \mathbb{E}_{n,l} \left[ E \omega \left[ X_{n,i}^1 - X_{n,j}^1 \right] E \omega \left[ X_{n,j}^2 - X_{n,j}^2 \right] \right]
$$

$$
= n^{-1/2} \sum_{1 \leq i,j \leq N} \theta_i \theta_j \mathbb{E} \omega \left[ G(X_{n,i}^1 - X_{n,j}^2, (n - l)t) + H(X_{n,i}^1 - X_{n,j}^2, (n - l)t) \right]
$$

$$
= n^{-1/2} \sum_{1 \leq i,j \leq N} \theta_i \theta_j \mathbb{E} \omega \left[ I_t(X_{n,i}^1 - X_{n,j}^2) \right],
$$

(4.25)

where $X_{n,i}^1, X_{n,j}^2$ are independent copies of $X_{n,i}^1$ under the quenched law $P^\omega$ as before, and the conditional expectation of the environment in the first line means integrating out the environment after time $t_l$ and conditional on the environment before this time. Similarly we have

$$
\sum_{k=l+1}^{n} \mathbb{E} \left[ z_{n,k}^2 | \mathcal{F}_{n,l-1} \right] = n^{-1/2} \sum_{1 \leq i,j \leq N} \theta_i \theta_j \mathbb{E}_{n,l-1} \left\{ E \omega \left[ I_t(X_{n,i}^1 - X_{n,j}^2) \right] \right\},
$$

(4.26)

Now let $(Y_{t,l}^1, Y_{t,l}^2)$ be an independent copy of $(X_{n,i}^1, X_{n,j}^2)$ under the quenched law $P^\omega$, then $(X_{n,i}^1, X_{n,j}^2, Y_{t,l}^1, Y_{t,l}^2)$ is a 4-point motion under the averaged law $\bar{P}$. Abbreviate $I_t(X_{n,i}^1 - X_{n,j}^2)$ by $I(\bar{X}_t)$ for convenience (similar for $I(\bar{Y}_t)$), then

$$
\mathbb{E} \left[ E \omega \left[ I(\bar{X}_t) \right] - \mathbb{E}_{n,l-1} E \omega \left[ I(\bar{X}_t) \right] \right]^2
$$

$$
= \mathbb{E} \left[ I(\bar{X}_t) I(\bar{Y}_t) \right] - \mathbb{E} \left[ \left( \mathbb{E}_{n,l-1} E \omega \left[ I(\bar{X}_t) \right] \right) \left( \mathbb{E}_{n,l-1} E \omega \left[ I(\bar{Y}_t) \right] \right) \right].
$$

(4.27)

The expectations in the second term of (4.27) equals $\bar{E} \left[ I(\bar{X}_t) I(\bar{Y}_t) \right]$, where the law $\bar{P}$ is the one such that $(X_{n,i}^1, X_{n,j}^2, Y_{t,l}^1, Y_{t,l}^2)$ is the 4-point motion on time interval $[0, (l - 1)t)$, then $(X_{n,i}^1, X_{n,j}^2)$ and $(Y_{t,l}^1, Y_{t,l}^2)$ behave as two sets of independent 2-point motions between time $(l - 1)t$ and $lt$. Let $A_t$ denote the event that for some $s \in [(l - 1)t, lt]$ \{ $X_{n,i}^1, X_{n,j}^2$ \}$ \cap \{ Y_{t,l}^1, Y_{t,l}^2 \} \neq \phi$. Notice that on the event $A_t$, the 4-point motion behaves the same as two sets of independent 2-point motions (since no interaction). That is, the averaged law $\bar{P}$ and the
law $\tilde{P}$ are equal on the event $A'_1$. Therefore, the left-hand side of (4.27) is equal to
\[
E \left[ 1_{A_1} I(\tilde{X}_i) I(\tilde{Y}_i) \right] - \tilde{E} \left[ 1_{A_1} I(\tilde{X}_i) I(\tilde{Y}_i) \right] = \ E \left[ 1_{A_1} (I(\tilde{X}_i) - I(\tilde{X}_{i-1})) (I(\tilde{Y}_i) - I(\tilde{Y}_{i-1})) \right] - \tilde{E} \left[ 1_{A_1} (I(\tilde{X}_i) - I(\tilde{X}_{i-1})) (I(\tilde{Y}_i) - I(\tilde{Y}_{i-1})) \right]
\leq \ E \left[ 1_{A_1} (I(\tilde{X}_i) - I(\tilde{X}_{i-1})) \right]^2 + \tilde{E} \left[ 1_{A_1} (I(\tilde{Y}_i) - I(\tilde{Y}_{i-1})) \right]^2 + \tilde{E} \left[ 1_{A_1} (I(\tilde{X}_i) - I(\tilde{X}_{i-1})) \right]^2
\]
where by definition $I(\tilde{X}_{i-1}) = I_t(X_{(i-1)t}^1 - X_{(i-1)t}^2)$, and the equality holds because before time $(l - 1)t$, $P$ and $\tilde{P}$ are equal.

Recall that $I_t(x) := G(x, (n-l)t) + H(x, (n-l)t)$. Since $\frac{\alpha C}{\nu}$ and $H$ are uniformly bounded by 2 and $4\nu$, respectively, by Jensen's inequality we have
\[
(I(\tilde{X}_i) - I(\tilde{X}_{i-1}))^2 \leq 12|X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}|^2 + 12|X_{it}^{2,n,i_2} - X_{(l-1)t}^{2,n,i_2}|^2 + C\nu. \tag{4.29}
\]
Since marginally both $X^{n,i_1}$ and $X^{n,i_2}$ are Brownian motions with drift $\beta$ under law $P$ (and $\tilde{P}$),
\[
E \left[ 1_{A_1} (I(\tilde{X}_i) - I(\tilde{X}_{i-1})) \right]^2 \leq 12E \left[ 1_{A_1} |X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}| \right]^2 \leq 12E \left[ 1_{A_1} |X_{it}^{2,n,i_2} - X_{(l-1)t}^{2,n,i_2}| \right]^2 + C\nu P(A_1)
\leq 24E \left[ X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1} \right]^2 |X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}| > n^{1/6}
\]
where in the second inequality we decomposed the integral to two parts: $|X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}| > n^{1/6}$ and $|X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}| \leq n^{1/6}$, and in the first part we used $1_{A_1} \leq 1$, and in the second part we controlled the random motion directly by $n^{1/6}$. Since $X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}$ has an exponential decay under law $P$, when $n$ is large enough,
\[
E \left[ (X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1})^2 |X_{it}^{1,n,i_1} - X_{(l-1)t}^{1,n,i_1}| > n^{1/6} \right] < n^{-1}.
\]
So we have a bound
\[
E \left[ 1_{A_1} (I(\tilde{X}_i) - I(\tilde{X}_{i-1})) \right]^2 \leq Cn^{-1} + Cn^{1/3} P(A_1). \tag{4.31}
\]
With similar argument, this estimate also holds for the other three terms in the right-hand side of (4.28). By Lemma 2.6, $\sum_{i=1}^{n-1} P(A_i) = o(n^{1/2 + \alpha})$ for any $\alpha > 0$. Here we take $\alpha = 1/6$ and then as $n \to \infty$, by (4.27), (4.28) and (4.31),
\[
\sum_{i=1}^{n-1} E \left[ R_i^2 \right] = n^{-1} \sum_{i_1,i_2,i_3,i_4} \theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4} E \left[ \left( E^\nu[I(\tilde{X}_{i})] - E_n t_1 E^\nu[I(\tilde{X}_{i})] \right)^2 \right] \leq C' n^{-2} + C'' n^{-2/3} \sum_{i=1}^{n-1} P(A_i) = o(1) \tag{4.32}
\]
This checks the condition (1.3), and therefore completes the proof of Lemma 4.2 \qed

4.2 Multiple time-levels

In this subsection we finish the second step stated at the beginning of this section, and thus finish the proof of Theorem 1.2.

20
Proof of Theorem 1.2. In this step we use induction argument to show the convergence of the finite dimensional distributions.

We assume that for some $M \in \mathbb{N}^+$,

$$(a_n(t_i, r_j) : 1 \leq i \leq M, 1 \leq j \leq N) \rightarrow (a(t_i, r_j) : 1 \leq i \leq M, 1 \leq j \leq N)$$

weakly on $\mathbb{R}^{NM}$ for any finite $N$, $0 \leq t_1 < \cdots < t_M$ and $r_1 < \cdots < r_N$.

Then when $M = 1$, it is just Lemma 4.2. It remains to deal with the case $M + 1$. Let $0 \leq t_1 < \cdots < t_M < t_{M+1}$, by the Cramér-Wold device, it suffices to prove that for any $(M+1)N$ vector $(\theta_{i,j})$, the linear combination

$$\sum_{1 \leq i \leq M+1} \sum_{1 \leq j \leq N} \theta_{i,j} a_n(t_i, r_j)$$

of $(a_n(t_i, r_j))$ converges weakly to that of $(a(t_i, r_j))$.

For Borel sets $B \in \mathcal{B}$, denote the probability measures

$$p_{n,j}(B) := P^n \left( X^n_{\lfloor n(M+1-t) \rfloor} + \beta ns \in B \right),$$

and let $s := t_{M+1} - t_M$.

By the decomposition (4.1),

$$a_n(t_{M+1}, r_j) = \int a_n(t_M, \frac{z}{\sqrt{n}}) p_{n,j}(dz) + \tilde{a}_n(s, r_j).$$

(4.35)

In order to apply Lemma 4.2, we need to discretize the integral in (4.35). Given $A > 0$, define a partition of $[-A, A]$ by

$$-A = u_0 < u_1 < \cdots < u_L = A.$$  

(4.36)

with mesh size $\Delta = \max_{1 \leq i \leq L} \{u_i - u_{i-1}\}$. For $z \in (-A, A]$, let $u(z)$ be the value $u_i$ where $u_{i-1}\sqrt{n} < z \leq u_i\sqrt{n}$. Then

$$a_n(t_{M+1}, r_j) = \sum_{i=1}^L a_n(t_M, u_i) p_{n,j}((u_{i-1}\sqrt{n}, u_i\sqrt{n})) + \tilde{a}_n(s, r_j) + R_{n,j}(A),$$

(4.37)

where the error term $R_{n,j}(A)$ is given by

$$R_{n,j}(A) = \sum_{i=1}^L \int_{u_{i-1}\sqrt{n}}^{u_i\sqrt{n}} \left( a_n(t_M, \frac{z}{\sqrt{n}}) - a_n(t_M, u_i) \right) p_{n,j}(dz)$$

(4.38)

$$+ \int_{-\infty}^{u_{i-1}\sqrt{n}} a_n(t_M, \frac{z}{\sqrt{n}}) p_{n,j}(dz)$$

(4.39)

Let $R_n = \sum_j \theta_{M+1,j} R_{n,j}(A)$, then we can rewrite

$$\sum_{1 \leq i \leq M+1} \sum_{1 \leq j \leq N} \theta_{i,j} a_n(t_i, r_j)$$

$$= \sum_{1 \leq i \leq M} \sum_{1 \leq k \leq K} \rho_{n,i,k} t_i, q_k + \sum_{1 \leq j \leq N} \theta_{M+1,j} \tilde{a}_n(s, r_j) + R_n(A).$$

(4.40)
In the above the spatial points \( \{ q_k \} \) are a relabeling of \( \{ r_j, u_l \} \), and the \( \omega \)-dependent coefficients \( \rho^{n, i, k}_{n, j} \) consist of constants \( \theta_{i, j} \), zeros and probabilities \( p_{n, j}^{(\omega)}(u_{t-1}\sqrt{m}, u_{t}\sqrt{m}) \). By the quenched invariance principle Theorem 1.1, the constant limits \( \rho^{n, i, k}_{n, j} \to \rho_{i, k} \) exist \( \mathbb{P} \)-a.s. as \( n \to \infty \).

On a probability space where the limit process \( (a(t, r)) \) has been defined, let \( \tilde{a}(s, \cdot) \) be a random function which equals \( a(s, \cdot) \) in distribution but independent of \( (a(t, r)) \).

Showing the weak convergence of the linear combination in (4.40) is equivalent to showing that for any bounded Lipschitz function \( f \) on \( \mathbb{R} \), (4.41) below vanishes as \( n \) tends to \( \infty \). Note that

\[
\mathbb{E} f \left( \sum_{1 \leq i \leq M+1} \sum_{1 \leq j \leq N} \theta_{i, j} a_n(t, r_j) \right) - \mathbb{E} f \left( \sum_{1 \leq i \leq M+1} \sum_{1 \leq j \leq N} \theta_{i, j} a(t, r_j) \right) = \left\{ \begin{array}{l} \mathbb{E} f \left( \sum_{1 \leq i \leq M+1} \sum_{1 \leq j \leq N} \theta_{i, j} a_n(t, r_j) \right) \\ - \mathbb{E} f \left( \sum_{1 \leq i \leq M+1} \sum_{1 \leq j \leq N} \theta_{i, j} a(t, r_j) \right) \end{array} \right\} (4.41)
\]

The rest is to show that these three differences all converge to zero.

By the Lipschitz continuity of \( f \) and the decomposition (4.40), the difference (4.42) is bounded by

\[
C_f \mathbb{E} |R_n(A)|,
\]

where \( C_f \) is the Lipschitz constant of \( f \). To bound \( R_n(A) \), it suffices to bound each \( R_{n,j}(A) \), for which we will deal with the term (4.38) and (4.39) separately.

First by (4.21), the covariance of \( (a_n(t, r)) \) is given by

\[
\mathbb{E} [a(t, r)a_n(t, q)] = n^{-1/2} \nu \left( G(\sqrt{n}(r - q), nt) + H(\sqrt{n}(r - q), nt) \right),
\]

from which together with the fact that \(|\frac{d}{dy}G(x, st)| \leq 2\) and \( H(x, t) \leq 4\nu \) uniformly in \((x, t)\), we can get

\[
\mathbb{E} \left( (a(t, r) - a_n(t, q))^2 \right) = 2n^{-1/2} \nu \left( G(0, nt) - G(\sqrt{n}(r - q), nt) + 8\nu \right) \leq C_1 |r - q| + C_2 n^{-1/2}.
\]

Noting the independence of \( a_n(t_M, r) \) and \( p_{n, j}^{(\omega)} \) on line (4.38), the \( L^1 \) norm of (4.38) can be controlled
by

\[
\mathbb{E} \left| \sum_{l=1}^{L} \int_{[u_{i-1}, u_{i}]} \left( a_n(t_M, \frac{z}{\sqrt{n}}) - a_n(t_M, u_l) \right) \rho_{n,j}^\omega (dz) \right|
\leq \sum_{l=1}^{L} \int_{[u_{i-1}, u_{i}]} \left( \mathbb{E} \left[ \left( a_n(t_M, \frac{z}{\sqrt{n}}) - a_n(t_M, u_l) \right)^2 \right] \right)^{1/2} \mathbb{E} [\rho_{n,j}^\omega (dz)]
\leq C_1 \sqrt{\Delta} + C_2 n^{-1/4},
\tag{4.47}
\]

where \( \Delta \) is the mesh size of the partition.

For the term (4.39), observe that

\[
\mathbb{E} [a_n^2(t, r)] = n^{-1/2} \nu \left( G(0, nt) + H(0, nt) \right)
\leq n^{-1/2} \nu (\sqrt{nt} + C) = O(1).
\tag{4.48}
\]

So

\[
\mathbb{E} \left| \left( \int_{-\infty, -A\sqrt{n}} + \int_{A\sqrt{n}, \infty} \right) a_n(t_M, \frac{z}{\sqrt{n}}) \rho_{n,j}^\omega (dz) \right|
\leq \left( \int_{-\infty, -A\sqrt{n}} + \int_{A\sqrt{n}, \infty} \right) \left( \mathbb{E} [a_n^2(t_M, \frac{z}{\sqrt{n}})] \right)^{1/2} \mathbb{E} [\rho_{n,j}^\omega (dz)]
\leq CP \left( \left| X_0^{v-\beta nt_M, -nt_M} \right| > A\sqrt{n} \right).
\tag{4.49}
\]

As a result, for any given \( \epsilon > 0 \), we can first choose \( A \) large enough and then \( \Delta \) small enough so that the term (4.42) satisfies

\[
\limsup_{n \to \infty} (4.42) < \epsilon.
\tag{4.50}
\]

For the difference (4.43), we cite Lemma 5.3 in [BRAS09], which states as follows:

**Lemma 4.3.** For any \( k \in \mathbb{N}^+ \), for each \( n \), let \( V_n = (V_n^1, \ldots, V_n^k) \), \( X_n = (X_n^1, \ldots, X_n^k) \) and \( Y_n \) be random variables in some probability space. If for each \( n \), \( X_n \) and \( Y_n \) are independent, and marginally the weak convergence \( V_n \to v \), \( X_n \to X \) and \( Y_n \to Y \) hold, where \( v \) is a constant \( k \)-vector, \( X \) a random \( k \)-vector and \( Y \) a random variable, then the weak convergence \( V_n X_n + Y_n \to vX + Y \) holds, where \( X \) and \( Y \) are independent.

Now note that in (4.43) \( \rho_{n,k}^\omega \to \rho_{-k} \) \( \mathbb{P} \)-a.s., hence in distribution. By the induction assumption, \( \{a_n(t_i, q_k) : 1 \leq i \leq M, 1 \leq k \leq K\} \) converges weakly to \( \{a(t_i, q_k) : 1 \leq i \leq M, 1 \leq k \leq K\} \), and by Lemma 4.2 \( \{a_n(s, r_j) : 1 \leq j \leq N\} \) converges weakly to \( \{a(s, r_j) : 1 \leq j \leq N\} \). Moreover, for each \( n \), \( a(t_i, q_k) \) is independent of \( a(s, r_j) \). This implies that

\[
\lim_{n \to \infty} \left[ (4.43) \right] = 0.
\tag{4.51}
\]

For the last difference (4.44), the method is the same as for (4.42). By Proposition 4.1 there is a representation (equal in finite dimensional distributions) of \( a(t_{M+1}, r_j) \) given by

\[
a(t_{M+1}, r_j) := \tilde{a}(s, r_j) + \int a(t_M, r + z) \phi_{t_{M+1}} dz.
\tag{4.52}
\]
Substitute it into the second term of the difference (4.41) and compare it with the first term (4.42), check the coefficients \( \rho_{i,k} \), then we have a similar error term as \( R_n(A) \) under the same partition. Since the covariance function \( \Gamma((t, r), (q, t)) = \nu G(r - q, t) \), we still have

\[
E \left[ (a(t, r) - \bar{a}(t, q))^2 \right] \leq C|r - q|, \tag{4.53}
\]

\[
E \left[ a^2(t, r) \right] \leq C, \tag{4.54}
\]

which allows us bound the error term with the same method, and we omit the detail here. After all, if we take large enough \( A \) of the partition and then make them mesh \( \Delta \) small enough, then

\[
\lim_{n \to \infty} \sup_{n} \left| E \left[ \sum_{1 \leq i \leq M_1 + 1} \sum_{1 \leq j \leq N} \theta_{i,j} a_n(t_i, r_j) \right] - E \left[ \sum_{1 \leq i \leq M_1 + 1} \sum_{1 \leq j \leq N} \theta_{i,j} a(t_i, r_j) \right] \right| < \epsilon. \tag{4.55}
\]

In sum, given any bounded and Lipschitz continuous function \( f \) and \( \epsilon > 0 \), by taking the partition \( \{ u_l \} \) good enough,

\[
\lim_{n \to \infty} \sup_{n} \left| E \left[ \sum_{1 \leq i \leq M_1 + 1} \sum_{1 \leq j \leq N} \theta_{i,j} a_n(t_i, r_j) \right] - E \left[ \sum_{1 \leq i \leq M_1 + 1} \sum_{1 \leq j \leq N} \theta_{i,j} a(t_i, r_j) \right] \right| < 2 \epsilon. \tag{4.56}
\]

This complete the proof of Theorem 1.2. \( \square \)

5 Proof of Theorem 1.3

Before going into the proof, we do some analysis to \( z_n(t, r) \). Denote \( x(n, r) := nx_0 + r\sqrt{n} \), then

\[
z_n(t, r) = n^{-1/4} \left\{ \int_{s(n, r)}^{g} f(\tilde{z})d\tilde{z} \right\} K^\omega_{\gamma n t,0}(x(n, r) - \beta nt, dy)
\]

\[
+ n^{-1/4} \left\{ \int_{s(n, r)}^{g} W(y)K^\omega_{\gamma n t,0}(x(n, r) - \beta nt, dy) - W(x(n, r)) \right\}
\]

\[
=: H_n(t, r) + I_n(t, r). \tag{5.3}
\]

We will consider the processes \( H_n \) and \( I_n \) separately.

For any \( T, Q > 0 \) and \( (t, r) \in [0, T] \times [-Q, Q] \), to analyze \( H_n(t, r) \), we break the domain of integration in line (5.1) into two parts: \(-\infty, x(n, r)) \) and \([x(n, r), \infty) \). Recall that \( f^{(n)}(x) = nf(\frac{x}{n}) \), then the second part of the integral is equal to

\[
\int_{x(n, r)}^{g} f'(\frac{z}{n})K^\omega_{\gamma n t,0}(x(n, r) - \beta nt, dy)
\]

\[
= \int_{x(n, r)}^{g} f'(\frac{z}{n})P^{\omega} \left[ X^{\gamma x(n, r) - \beta nt, -nt} > z \right] dz
\]

\[
= \int_{x(n, r)}^{g} f'(x_0)P^{\omega} \left[ X^{\gamma x(n, r) - \beta nt, -nt} > z \right] dz + R_n
\]

\[
= f'(x_0)E^{\omega} \left[ \left( X^{\gamma x(n, r) - \beta nt, -nt} - x(n, r) \right)^+ \right] + R_n, \tag{5.4}
\]

where \( X^{\gamma x(n, r) - \beta nt, -nt} \) is the random motion in the Howitt-Warren flow, and \( R_n \) is the remainder.
dominated by
\[ R_n = \int_{x(n,r)}^\infty \left( f'(\frac{z}{n}) - f'(x_0) \right) P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt > z \right) dz \]
\leq \int_{x(n,r)+n^{1/2+\delta}}^{x(n,r)+n^{1/2+\delta}} \left| f'(\frac{z}{n}) - f'(x_0) \right| dz + C \int_{x(n,r)+n^{1/2+\delta}}^\infty P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt > z \right) dz
= R_{n1} + R_{n2}, \quad (5.5)\]
where in the inequality we bounded the quenched probability by 1 for the first term, and by Assumption I we take a bound C/2 of f’ for the second. For R_{n1}, by the Hölder continuity of f’,
\[ R_{n1} \leq Cn^{1/2+\delta} \frac{r\sqrt{n} + n^{1/2+\delta}}{n} = o(n^{1/4}), \quad (5.6)\]
where we take 0 < \delta < (2\gamma - 1)/(4\gamma + 4). As to R_{n2}, for any \epsilon > 0,
\[ \mathbb{P} \left[ \int_{x(n,r)+n^{1/2+\delta}}^\infty P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt > z \right) dz > \epsilon \right] \leq \epsilon^{-1} \int_{x(n,r)+n^{1/2+\delta}}^\infty P \left( X_0^{x(n,r)} - \beta nt, -nt > z \right) dz \leq \epsilon^{-1} \int_{\sqrt{n} + n^{1/2+\delta}}^\infty \frac{1}{\sqrt{2\pi nt}} \exp\left\{-\frac{z^2}{2nt}\right\} dz \leq \epsilon \frac{C_{n, t}}{\sqrt{2\pi nt}} \frac{\exp\{-n^{2\delta}/(2t)\}}{(n^{1/2+\delta})^2}. \quad (5.7)\]
Since (5.7) is summable with respect to n uniformly for t \in [0, T] and r \in [-Q, Q], by Borel-Cantelli lemma, for \mathbb{P}\text{-a.e. } \omega,
\[ \sup_{(t, r)} \int_{x(n,r)+n^{1/2+\delta}}^\infty P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt > z \right) dz \longrightarrow 0 \quad (5.8)\]
as \n \to 0. Applying the same argument to the integral in line \[5.1\] on the domain (-\infty, x(n,r)), we can get a similar result as \[5.4\]. Therefore the following lemma holds:

**Lemma 5.1.** For any T, Q > 0 denote \( A = [0, T] \times [-Q, Q] \), then \mathbb{P}\text{-a.s.,}
\[ \lim_{n \to \infty} \sup_{(t, r) \in A} n^{-1/4} \left| H_n(t, r) - f'(x_0) E^\omega \left[ X_0^{x(n,r)} - \beta nt, -nt x(n,r) \right] \right| = 0. \quad (5.9)\]
As to the process \( I_n(t, r) \), we have a representation:
\[ \int W(y) K_{nt, 0}(x(n, r) - \beta nt, dy) - W(x(n, r)) \]
\[ = \int_{x(n,r)}^\infty \left\{ \int_{x(n,r)}^\infty 1_{(x(n,r), y)}(s) dW_s \right\} P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt \in dy \right) \]
\[ = \int_{x(n,r)}^\infty \left\{ \int_{-\infty}^y 1_{(y, x(n,r))}(s) dW_s \right\} P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt \in dy \right) \]
\[ = \int_{x(n,r)}^\infty P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt > s \right) dW_s \quad (5.10)\]
\[ - \int_{-\infty}^{x(n,r)} P^\omega \left( X_0^{x(n,r)} - \beta nt, -nt < s \right) dW_s, \quad (5.11)\]
where in the last equality we interchanged the Lebesgue integral and Itô integral because of the following lemma and the fact that $E^\omega |X_{0}^{(n,r)} - \beta nt, -nt|$ exists for $\mathbb{P}$-a.e. $\omega$.

**Lemma 5.2.** Let $X$ be a random variable on the probability space $(\Omega, \mathcal{F}, P)$ with $E|X| < \infty$ and independent of the Brownian motion $W_t$. Then for any constant $a$, the following interchange of Lebesgue integral and Itô integral is permissible:

$$\int_{a}^{\infty} \int_{a}^{\infty} 1_{(a,y)}(s) dW_s P(X \in dy) = \int_{a}^{\infty} \int_{a}^{\infty} 1_{(a,y)}(s) P(X \in dy) dW_s. \quad (5.12)$$

**Proof.** Without loss of the generality, we assume $a = 0$. The left side of (5.12) can be approximated a.s. by

$$\sum_{k=0}^{\infty} P\left(\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right) \int_{0}^{\infty} 1_{(0,2^n)}(s) dW_s, \quad (5.13)$$

since a.s., Brownian motion is $\gamma$-Hölder continuous for all $\gamma < 1/2$ and $P$ is a probability measure. On the other hand, since

$$\left|\int_{0}^{\infty} 1_{(0,y)}(s) P(X \in dy) - \sum_{k=0}^{\infty} P\left(\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right) 1_{(0,2^n)}(s)\right| \leq \sum_{k=0}^{\infty} P\left(\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right) 1_{(0,2^n)}(s), \quad (5.14)$$

the second moment of the difference of (5.13) and the right-hand side of (5.12) is controlled by

$$\int_{0}^{\infty} \sum_{k=0}^{\infty} P^2\left(\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right) 1_{(0,2^n)}(s) ds, \quad (5.15)$$

which is dominated by $\int_{0}^{\infty} P(X > s/2) ds \leq 2E|X| < \infty$. Therefore, by dominated convergence theorem (5.13) converges to the right-hand side of (5.12) in $L^2$ as $n \to \infty$, and this shows that the interchange is permissible. \hfill \Box

With the form of (5.10)-(5.11), it is clear that for $\mathbb{P}$-a.e. $\omega$, $I_n^\omega(t,r)$ is a Gaussian process. The quenched covariance of the process $I_n^\omega(t,r)$ is given by

$$n^{-1/2}E^\omega \left[ I_n^\omega(t,r) I_n^\omega(s,q) \right] = \int_{x(n,r) \cap x(n,q)} P^n\left(X_{0}^{(n,r)} - \beta nt, -nt > z\right) P^n\left(X_{0}^{(n,q)} - \beta ns, -ns > z\right) dz$$

$$- \int_{x(n,r) \cap x(n,q)} P^n\left(X_{0}^{(n,r)} - \beta nt, -nt < z\right) P^n\left(X_{0}^{(n,q)} - \beta ns, -ns > z\right) dz$$

$$- \int_{x(n,r) \cap x(n,q)} P^n\left(X_{0}^{(n,r)} - \beta nt, -nt > z\right) P^n\left(X_{0}^{(n,q)} - \beta ns, -ns < z\right) dz$$

$$+ \int_{-\infty}^{x(n,r) \cap x(n,q)} P^n\left(X_{0}^{(n,r)} - \beta nt, -nt < z\right) P^n\left(X_{0}^{(n,q)} - \beta ns, -ns < z\right) dz. \quad (5.16)$$

Since when $n$ tends to $\infty$, for $\mathbb{P}$-a.e. $\omega$, by the quenched invariance principle Theorem 1.1

$$P^n\left(X_{0}^{(n,r)} - \beta nt, -nt > nx_0 + \sqrt{nt}\right) \to P(W(t) > z - r),$$

$$P^n\left(X_{0}^{(n,q)} - \beta ns, -ns > nx_0 + \sqrt{nt}\right) \to P(W(s) > z - q),$$

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where $W$ is a standard Brownian motion, we get for $\mathbb{P}$-a.e. $\omega$,
\[
E^\omega [I_n^\omega(t, r)I_n^\omega(s, q)] \longrightarrow \Gamma_0((t, r), (s, q))
\]  
(5.17)
as $n$ tends to $\infty$.

With these useful observations, we take advantage of the following lemma cited from [BRAS06] Lemma 7.1 to show Theorem 1.3.

**Lemma 5.3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Xi, \mathcal{G}, \mathbb{P})$ be two probability spaces. On the product space $(\Omega \times \Xi, \mathcal{F} \times \mathcal{G}, \mathbb{P} \times \mathbb{P})$, define two sequences of $\mathbb{R}^N$-valued random vectors $H_n(\omega)$ and $I_n(\omega, \xi)$, where $H_n$ depends only on $\omega$. Denote the conditional probability measure $P^{\omega} = \delta_\omega \times \mathbb{P}$ given $\omega$. If $H_n$ and $I_n$ satisfy the following two conditions:

(i) As $n$ tends to $\infty$, the random vector $H_n$ converges weakly to an $\mathbb{R}^N$-valued random vector $H$;

(ii) There exists an $\mathbb{R}^N$-valued random vector $I$ such that for all $\lambda \in \mathbb{R}$,
\[
E^\omega [e^{i\lambda I_n}] \longrightarrow E [e^{i\lambda I}]
\]  
(5.18)in $\mathbb{P}$-probability as $n$ tends to $\infty$;

then $H_n + I_n$ converges weakly to $H + I$, where $H$ and $I$ are independent.

In the lemma, the limit of $H_n + I_n$ exists and consists of two independent parts. This is because for any $\omega$, there is a common limit of $I_n$ which has nothing to do with $\omega$, while $H_n$ converges and only depends on $\omega$. The proof of the lemma is also straightforward, we only need to show that the difference of $\mathbb{E}E^\omega [e^{ib H_n + i\lambda I_n}]$ and $E [e^{i\lambda H}] E [e^{i\lambda I}]$ for arbitrary $\theta, \lambda \in \mathbb{R}^N$ vanishes when $n$ tends to $\infty$.

Now we are ready to give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We only needs to show that for any $N$ space-time points $(t_1, r_1), \ldots, (t_N, r_N)$ in $\mathbb{R}^+ \times \mathbb{R}, (z_n(t_1, r_1), \ldots, z_n(t_N, r_N))$ converges weakly to $(z(t_1, r_1), \ldots, z(t_N, r_N))$. According to the decomposition (5.3), Lemma 5.1 and Theorem 1.2, we can see that $(H_n(t_1, r_1), \ldots, H_n(t_N, r_N))$ depends only on $\omega$ and converges weakly to a random vector $(H(t_1, r_1), \ldots, H(t_N, r_N))$. On the other hand, for $\mathbb{P}$-a.e. $\omega$, $(I_n^\omega(t_1, r_1), \ldots, I_n^\omega(t_N, r_N))$ is Gaussian and thereby
\[
E^\omega [e^{i\lambda I_n^\omega}] = \exp\left\{-\frac{1}{2} \lambda^\top \Sigma^\omega \lambda\right\},
\]  
(5.19)in which $\Sigma^\omega$ is the covariance matrix $(E^\omega [I_n^\omega(t_i, r_i)I_n^\omega(t_j, r_j)])_{i,j}$ given in (5.10). By (5.17), in the limit the matrix becomes $\Sigma = (\Gamma_0((t_i, r_i), (t_j, r_j)))_{i,j}$, and therefore $(I_n^\omega(t_1, r_1), \ldots, I_n^\omega(t_N, r_N))$ converges to a Gaussian vector $(I(t_1, r_1), \ldots, I(t_N, r_N))$ satisfying the condition (ii) in Lemma 5.3. Hence there exists a mean zero Gaussian weak limit of $(z_n(t_1, r_1), \ldots, z_n(t_N, r_N))$ in the form of Lemma 5.3. From the covariances, the limit is $(z(t_1, r_1), \ldots, z(t_N, r_N))$. 

**Appendix A** Quenched invariance principle for random walk in an i.i.d space-time random environment

The quenched invariance principle for one-dimensional random walk in an i.i.d space-time random environment has previously been established by Rassoul-Agha and Seppäläinen in [RAS05]. They proved the result based on the view of the particle and martingale techniques. Now we apply the second moment method, similar to Section 3, to give an alternative proof of this result, since the
method in this case is concise and self-contained. Actually, the second moment method was first used by Bolthausen and Sznitman in [BS02], and also by Comets and Yoshida in [CY06].

In this basic model, an environment is a collection of transition probabilities \( \omega = (\pi_{xy})_{x,y \in \mathbb{Z}} \in \Omega \) where \( \Omega = \{(p_y)_{y \in \mathbb{Z}} \in [0,1]^\mathbb{Z} : \sum_y p_y = 1 \} \). The space \( \Omega \) is equipped with the canonical product \( \sigma \)-algebra \( \mathcal{F} \) and given an i.i.d probability measure \( \mathbb{P} \). Here we say \( \mathbb{P} \) is i.i.d in the sense that the distribution of the random probability vectors \( (\pi_{xy})_{y \in \mathbb{Z}} \) are i.i.d over distinct sites \( x \) and \( \mathbb{P} \) is their product measure.

Once the environment \( \omega \) is chosen from the distribution \( \mathbb{P} \), we fix it and sample a Markov process \( X = (X_n)_{n \geq 0} \) with the state space \( \mathbb{Z} \), starting from the site \( z \), with the transition probability given by:

\[
P^\omega(x,y) = \pi^\omega_{xy}.
\]

We call \( X \) the one-dimensional random walk in an i.i.d space-time random environment. \( P^\omega \) denotes the quenched law and we denote the averaged law by \( P(\cdot) := \mathbb{E}P^\omega(\cdot) \). Under the averaged law, the averaged walk \( X \) is just a random walk with transition probability \( p(x,x+y) = p(0,y) = \mathbb{E}[\pi^\omega_{0y}] \). Let \( \mu = \sum_z z \pi(0,z) \) and \( \sigma^2 = \sum_z z^2 \pi(0,z) \) be the mean and the variance of the averaged walk. For \( t \geq 0 \), define the linear interpolation of \( X \) by \( X_t := X_n + (t - [t])(X_{[t]+1} - X_{[t]}) \) where \( [x] = \max\{n \in \mathbb{Z} : n \leq x\} \). Let \( B_n(t) = \frac{X_{nt} - nt\mu}{\sqrt{\sigma^2 n t}} \) and \( \bar{B}_n(t) = \frac{X_{nt} - nt\mu}{\sqrt{\pi^\omega_{0y}}} \) be random variables in \( C[0,\infty) \), then we have the following theorem:

**Theorem A.1.** With the notations introduced above, if \( \sigma^2 < \infty \) and \( \mathbb{P}(\sup_{y \in \mathbb{Z}} \pi^\omega_{xy} < 1) > 0 \), then for \( \mathbb{P}-a.e. \omega \), \( B_n(t) \) converges weakly to a standard Brownian motion \( B(t) \). Moreover, for \( \mathbb{P}-a.e. \omega \), \( n^{-1/2} \max_{1 \leq n} |\mathbb{E}X_{kt} - kt\mu| \) converges to 0, and therefore the same a.s. invariance principle also holds for \( \bar{B}_n \).

**Proof.** The argument is similar to the one for Theorem 1.1. First we choose a proper coupling. Given the environment \( \omega \), sample two independent random walk \( (X^1_n)_{n \geq 0} \) and \( (X^2_n)_{n \geq 0} \), and assume \( X^1_0 = X^2_0 = 0 \) without loss of generality. Then under the averaged law \( P \), \( (X^1, X^2) \) is a Markov process with transition probability:

\[
P^\omega((X^1_{n+1}, X^2_{n+1})) = (y_1,y_2) | (X^1_n, X^2_n) = (x_1, x_2)) = p(x_1, y_1)p(x_2, y_2), \quad x_1 \neq x_2;
\]

\[
P^\omega((X^1_{n+1}, X^2_{n+1})) = (y_1,y_2) | (X^1_n, X^2_n) = (x, x)) = \mathbb{E}[\pi^\omega_{xy_1} \pi^\omega_{xy_2}].
\]

Define a sequence of stopping times:

\[
\tau_0 := 0; \quad \tau_{n+1} := \min\{k > \tau_n : X^1_k = X^2_k\} \quad (n \geq 0),
\]

and two sequences of i.i.d random variables in the same probability space \( (\xi^1_n)_{n \geq 0} \) and \( (\xi^2_n)_{n \geq 0} \), which are independent of \( (X^1, X^2) \) and each other, with

\[
P(\xi^1_n = z) = P(\xi^2_n = z) = p(0,z) \quad (z \in \mathbb{Z}).
\]

Now couple a Markov process \( (Y^1, Y^2) \) as follows:

\[
(Y^1_0, Y^2_0) = (0,0),
\]

for \( \tau_k < n \leq \tau_{k+1} \),

\[
Y^j_n = X^j_n - X^j_{\tau_{j+1}} + \sum_{i=1}^{\tau_k} (X^j_i - X^j_{\tau_{i-1}+1}) + \sum_{i=0}^{\tau_k} \xi^j_i \quad (j = 1, 2).
\]

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In words, the way \((Y_1^n, Y_2^n)\) jumping is exactly the same as that of \((X_1^n, X_2^n)\) when \(X_1^n \neq X_2^n\). But if \(X_1^n = X_2^n = x_n\), then \(Y_1^n\) and \(Y_2^n\) still jump independently with transition probability \(p(x_n, \cdot)\). Therefore, \(Y^1\) and \(Y^2\) are two independent random walks with transition kernels given by the function \(p\). We also do linear interpolation to \((X_1^n)_{n \geq 0}\) and \((Y_2^n)_{n \geq 0}\) for \(j = 1, 2\), and denote them by \((X_1^j)_{j \geq 0}\) and \((Y_2^j)_{j \geq 0}\).

Moreover, by Donsker’s theorem \(Ef\), we have

\[
\lim_{n \to \infty} P(X_1^n = x) = P(X_1^j = x) = \frac{n}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}}.
\]

and

\[
\lim_{n \to \infty} P(Y_2^n = x) = P(Y_2^j = x) = \frac{n}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}}.
\]

These results are due to [FF98, Lemma 3.3]. Again, following the proof of theorem 1.1 gives us that for \(P\)

\[
P(\Delta X_1^j \leq \epsilon) = O(n^{1/2}),
\]

where in the last inequality we applied the independence of \((X_1^j, X_2^j)\) and \((\Delta X_1^j, \zeta^j_1)\), and the inequality \(E|\Delta X_1^j - \zeta^j_1| \leq (E[(\Delta X_1^j - \zeta^j_1)^2])^{1/2} \leq 2\sigma\); the last step is due to [BRAS06, Lemma 4.1]. Denote \(X_1^j = \frac{X_1^j}{\sqrt{n\sigma}}\) and \(Y_2^j = \frac{Y_2^j}{\sqrt{n\sigma}}\) \((j=1,2)\) by \(B_1^n(t)\) and \(W_2^n(t)\), then with the same argument as in the proof of Lemma 3.2 we have

\[
E \left[ (\frac{\sigma_1^j}{\sqrt{n\sigma}})^2 \right] \leq Cn^{-1/2},
\]

Hence following the proof of theorem 1.1 gives us that for \(P\)-a.e. \(\omega\), \(Ef(B_1^n) - Ef(W_2^n) \to 0\). Moreover, by Donsker’s theorem \(Ef(W_2^n(t)) \to Ef(B(t))\), so for \(P\)-a.e. \(\omega\), \(B_n(t)\) converges weakly to \(B(t)\).

As to the second part of the theorem, let \(\sigma_0^2 = \sum_{y_1,y_2 \in \mathbb{Z}} (y_1 - \mu)(y_2 - \mu)E[\pi_0y_1\pi_0y_2]\), then

\[
E \left[ (\frac{\sigma_1^j}{\sqrt{n\sigma}})^2 \right] = \left[ \frac{\sigma_1^j}{\sqrt{n\sigma}} \right]^2 = \sum_{k=0}^{[nt]-1} \sum_{l=0}^{[nt]-1} \sum_{k=0}^{[nt]-1} E \left[ (X_{n+1}^j - X_{n}^j - \mu) (X_{n+1}^j - X_{n}^j - \mu) \right] = O(n^{1/2}),
\]

where in the second line the summands vanish unless \(k = j\) and \(X_{n+1}^j = X_{n}^j\) because \(X_{n}^j\) and \(X_{n}^k\) are independent when they do not meet; the last estimate is due to [PT98, Lemma 3.3]. Again, with the same argument as in the proof of the Theorem 1.1, we have \(n^{-1/2} \max_{k \leq n} |E_{\omega}X_{kt} - kt\mu| \to 0\) for \(P\)-a.e. \(\omega\).

**Appendix B  An integral identity**

This section gives a probabilistic method to show the identity \(4.20\).

**Lemma B.1.** For all \((t, x) \in \mathbb{R}^+ \times \mathbb{R},

\[
\int_0^t \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds = \int_0^t \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds.
\]

Therefore \(4.20\) holds.
Proof. To see the equality, we in turn consider the following probability question: For a standard Brownian motion $B_t$ starting from the point $x$, what is the expectation of the local time at the origin $E[L_x(t,0)]$ up to time $t$. We compute this quantity in two ways.

First we use the same method as in Section 2.2. Decompose it according to the first hitting time to 0 of $B_t$. Since by Lemma 2.5 we have $E[L_0(t,0)] = \sqrt{\frac{t}{2\pi}}$, we have

$$E[L_x(t,0)] = \int_0^t E[L_0(t-s,0)] \frac{|x|}{\sqrt{2\pi s^{3/2}}} e^{-\frac{x^2}{2s}} ds = \int_0^t \sqrt{\frac{t-s}{2\pi s^{3/2}}} |x| e^{-\frac{x^2}{2s}} ds. \tag{B.2}$$

On the other hand, by Proposition 2.1 (iv), we have

$$2E \int_{-\infty}^{\infty} f(y)L_x(t,y)dy = \mathbb{E} \int_0^t f(B_s)ds = \int_{-\infty}^{\infty} \int_0^t f(y) \frac{1}{\sqrt{2\pi s}} e^{-\frac{(y-x)^2}{2s}} dsdy, \tag{B.3}$$

for all measurable functions $f : \mathbb{R} \rightarrow [0,\infty)$. Therefore,

$$E[L_x(t,0)] = \int_0^t \frac{1}{2\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds \tag{B.4}$$

From (B.2) and (B.4) we get the result. \hfill \Box

Acknowledgement

I am deeply indebted to my advisor Associate Professor Rongfeng Sun, who introduced me to this topic, and read carefully an earlier version of this paper and provided many corrections and suggestions.

References

[Ami91] M. Amir. Sticky Brownian motion as the strong limit of a sequence of random walks. Stochastic Process. Appl., 39(2):221–237, 1991.

[BˇCDG13] M. Birkner, J. Černý, A. Depperschmidt, and N. Gantert. Directed random walk on the backbone of an oriented percolation cluster. Electron. J. Probab., 18:no. 80, 35, 2013.

[BRAS06] M. Balázs, F. Rassoul-Agha, and T. Seppäläinen. The random average process and random walk in a space-time random environment in one dimension. Comm. Math. Phys., 266(2):499–545, 2006.

[BS02] E. Bolthausen and A.-S. Sznitman. On the static and dynamic points of view for certain random walks in random environment. Methods Appl. Anal., 9(3):345–375, 2002. Special issue dedicated to Daniel W. Stroock and Srinivasa S. R. Varadhan on the occasion of their 60th birthday.

[Cor12] I. Corwin. The Kardar-Parisi-Zhang equation and universality class. Random Matrices Theory Appl., 1(1):1130001, 76, 2012.

[CY06] F. Comets and N. Yoshida. Directed polymers in random environment are diffusive at weak disorder. Ann. Probab., 34(5):1746–1770, 2006.

[DL01] Y. Derriennic and M. Lin. Fractional Poisson equations and ergodic theorems for fractional coboundaries. Israel J. Math., 123:93–130, 2001.
[DL03] Y. Derriennic and M. Lin. The central limit theorem for Markov chains started at a point. *Probab. Theory Related Fields*, 125(1):73–76, 2003.

[Dur10] R. Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[FF98] P. A. Ferrari and L. R. G. Fontes. Fluctuations of a surface submitted to a random average process. *Electron. J. Probab.*, 3:no. 6, 34 pp. (electronic), 1998.

[FINR04] L. R. G. Fontes, M. Isopi, C. M. Newman, and K. Ravishankar. The Brownian web: characterization and convergence. *Ann. Probab.*, 32(4):2857–2883, 2004.

[Fre71] D. Freedman. *Brownian motion and diffusion*. Holden-Day, San Francisco, Calif.-Cambridge-Amsterdam, 1971.

[HW09] C. Howitt and J. Warren. Consistent families of Brownian motions and stochastic flows of kernels. *Ann. Probab.*, 37(4):1237–1272, 2009.

[KS91] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[LJR04] Y. Le Jan and O. Raimond. Flows, coalescence and noise. *Ann. Probab.*, 32(2):1247–1315, 2004.

[MW00] M. Maxwell and M. Woodroofe. Central limit theorems for additive functionals of Markov chains. *Ann. Probab.*, 28(2):713–724, 2000.

[RAS05] F. Rassoul-Agha and T. Seppäläinen. An almost sure invariance principle for random walks in a space-time random environment. *Probab. Theory Related Fields*, 133(3):299–314, 2005.

[Reb80] R. Rebolledo. Central limit theorems for local martingales. *Z. Wahrsch. Verw. Gebiete*, 51(3):269–286, 1980.

[Res87] S. I. Resnick. *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust*. Springer-Verlag, New York, 1987.

[Sep10] T. Seppäläinen. *Current fluctuations for stochastic particle systems with drift in one spatial dimension*, volume 18 of *Ensaios Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2010.

[SS08] R. Sun and J. M. Swart. The Brownian net. *Ann. Probab.*, 36(3):1153–1208, 2008.

[SSS14] E. Schertzer, R. Sun, and J. M. Swart. Stochastic flows in the Brownian web and net. *Mem. Amer. Math. Soc.*, 227(1065):vi+160, 2014.