IN Variant Tori of a Two Dimensional Periodic System With the Linear-Cubic Unperturbed Part

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Abstract. Two classes of time-periodic systems of ordinary differential equations with a small parameter $\varepsilon \geq 0$, those with "fast" and "slow" time, are studied. Right-hand sides of these systems are three times continuously differentiable with respect to phase variables and the parameter, the corresponding unperturbed systems are autonomous, conservative and have nine equilibrium points. For the perturbed systems, which do not depend on the parameter explicitly, we obtain the conditions yielding that the initial system has a certain number of two-dimensional invariant surfaces homeomorphic to a torus for each sufficiently small values of parameter $\varepsilon$ and the formulas of such surfaces. A class of systems with seven invariant surfaces enclosing different configurations of equilibrium points is studied as an example of applications of our method.

1. Introduction.

We study a periodic two-dimensional system of ODE’s with a small parameter $\varepsilon$ defined by the equations

$$\begin{align*}
\dot{x} &= (\gamma(y^3 - y) + X(t, x, y, \varepsilon))\varepsilon^{\nu}, \\
\dot{y} &= (- (x^3 - x) + Y(t, x, y, \varepsilon))\varepsilon^{\nu},
\end{align*}$$

where $\gamma \in (0, 1]$, $\nu = 0, 1$; $X, Y$ are continuous $C^3_{x,y,\varepsilon}$ functions, $T$-periodic in $t$ for $t \in \mathbb{R}$, $|x| < M_x$, $|y| < M_y$ ($M_x > \sqrt{2}$, $M_y > \sqrt{1 + \gamma^{-1/2}}$); $\varepsilon \in [0, \varepsilon_0)$.

In essence, formula (1) determines two different systems: one with $\nu = 0$, another with $\nu = 1$. Comparing these systems we can say that the system with $\nu = 1$, which is usually called the standard system, has the "fast" time, because reducing it to the system with $\nu = 0$ we obtain the period $T\varepsilon$.

It is natural to refer to the autonomous system

$$\begin{align*}
\dot{x} &= \gamma(y^3 - y)\varepsilon^{\nu}, \\
\dot{y} &= -(x^3 - x)\varepsilon^{\nu} \quad (\gamma \in (0, 1], \nu = 0, 1)
\end{align*}$$

as to the system of the first approximation or the unperturbed system with respect to (1).

System (2) is conservative. It has nine equilibrium points and its phase plane is filled, in addition to the equilibrium points, with the closed orbits and separatrices determined by the integral $(x^2 - 1)^2 + \gamma(y^2 - 1)^2 = a$.

The goal of this paper is to find, for system (1) with any sufficiently small $\varepsilon > 0$, a certain number of 2-dimensional cylindrical invariant surfaces homeomorphic to the torus, which is obtained by factoring time with respect to the period. The projections of such surfaces on the phase plane are contained in a small neighborhood of corresponding closed orbits of the unperturbed system (2).

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We explicitly write out conditions (depending on the parameter $\gamma$) on the unperturbed functions $X(t, x, y, 0)$ and $Y(t, x, y, 0)$ under which the perturbed system (1) has invariant surfaces described above and obtain asymptotic expansions in powers of $\varepsilon$ for each of them. Besides, we provide conditions that specify a class of systems with seven invariant surfaces. Phase plane projections of such surfaces encloses one, three or nine equilibrium points of the unperturbed system.

We also provide examples of systems (1) for which the three obtained bifurcational equations have admissible solutions $c_i$ $(i = 0, 1, 2)$.

Systems (1) with $\nu = 0$ and $\nu = 1$ can be studied simultaneously, because the invariant tori can be found using the method developed in [1, 2] and essentially modified in [3–7]. But significant differences arise in the process of averaging the systems obtained as the result of a special polar coordinates change.

One of the most significant differences of this paper from the previous works is the fact that equilibrium points are located not only on the abscissa axis. Due to this, a new interesting phenomenon that has been never observed before arises – it is the inability to perform the special polar coordinate change in a small neighborhood of singular points $(1, 0)$ and $(-1, 0)$ despite moving the origin to these points.

2. Parametrization of the orbits of the unperturbed system.

2.1. Construction of the Phase Portrait. Consider an autonomous conservative system with nine equilibrium points similar to system (2) defined by the equations

$$C'(\varphi) = \gamma(S^3(\varphi) - S(\varphi)), \quad S'(\varphi) = -(C^3(\varphi) - C(\varphi)) \quad (0 < \gamma \leq 1).$$

For each $a > 0$, we consider the set $\Gamma_a$ of closed orbits on the plane $(C, S)$ determined by the integrals of system (3) given by the equation

$$(C^2 - 1)^2 + \gamma(S^2 - 1)^2 = a.$$  

Obviously, systems (3) and (2) have the same closed orbits, but if $\nu = 1$ the solutions of (3) and (2) determine different trajectories.

For system (3) five of nine singular points are centers. Points $(\pm 1, 1), (\pm 1, -1)$ are solutions of (4) when $a = 0$, and $(0, 0)$ is the solution of (4) when $a = 1 + \gamma$. The other singular points are the saddle points: points $(\pm 1, 0)$ are the solutions of (4) when $a = \gamma$, and $(0, \pm 1)$ are the solutions of (4) when $a = 1$.

It is sufficient to describe orbits or parts of orbits that lie in the first quadrant, because equation (4) is invariant with respect to the change $C \to -C, S \to -S$. We denote such set of orbits by $\Gamma_a^*$.

The extremal values of the curves that belong to $\Gamma_a^*$ are:

$$r_\gamma = \sqrt{1 + \gamma^{1/2}}, \quad l_\gamma = \sqrt{1 - \gamma^{1/2}},$$

$$r_i = \sqrt{1 - (1 - \gamma)^{1/2}}, \quad r_e = \sqrt{1 + (1 - \gamma)^{1/2}}, \quad u_e = \sqrt{1 + \gamma^{-1/2}},$$

$$r_{0e}, r_1 = \sqrt{1 - (a - \gamma)^{1/2}}, \quad u_{0e}, u_1 = \sqrt{1 + (a - \gamma)^{1/2}}, \quad r_{0e}, r_1 = \sqrt{1 + a^{1/2}},$$

$$u_{0e}, u_1 = \sqrt{1 - (a - 1)/\gamma^{1/2}}, \quad u_{0e}, u_1 = \sqrt{1 + (a - 1)/\gamma^{1/2}}, \quad u_{0e}, u_1 = \sqrt{1 + (a/\gamma)^{1/2}},$$

$$r_1 = \sqrt{1 + a^{1/2}}, \quad l_1, l_2 = \sqrt{1 - a^{1/2}}, \quad l_0 = \sqrt{1 - (a/\gamma)^{1/2}},$$

where $r, l, u, lo, i, e$ means right, left, upper, lower, internal and external, respectively, the first five constants characterize $\Gamma_0^*$ and $\Gamma_1^*$, for the other constants the superscript determines.
Figure 1. Phase portrait of the unperturbed system with \( \gamma = 1/2 \) (left) and \( \gamma = 1 \) (right).

the value of the second coordinate, while the subscript determines the number of a class which we will introduce below. We first consider the "separatrix" curves determined by equation (4), namely, the curves passing through the singular points of system (3).

For \( \gamma < 1 \) the set \( \Gamma_{\gamma} (a = \gamma) \) consists of four closed curves (or two "eights"). Two of those curves contact each other at the singular point \((1,0)\) and enclose points \((1,\pm 1)\), other two contact each others at the singular point \((-1,0)\) and enclose singular points \((-1,\pm 1)\). Thus, \( \Gamma^*_{\gamma} \) is a top part of right "eight" and has the following extremal points: \((r_{\gamma},1)\), \((l_{\gamma},1)\), \((1,\sqrt{2})\), \((1,0)\).

For \( a \neq \gamma \) the set \( \Gamma_a \) consists of closed orbits of system (3). The separatrices separate the set \( \Gamma_a \) into three classes, which we denote by \([0,1], [1,2] \) and define as follows:

- For \( a > 1 \) the set \( \Gamma_a \) consists of two closed orbits: the inner orbit \( \Gamma_{ai} \) which encloses \((0,0)\) and lies inside \( \Gamma_{1i} \), and the outer orbit \( \Gamma_{ae} \) which encloses \( \Gamma_{1e} \). Then the points \((r_{0i}^0,0)\), \((0,u_{0i}^0)\) are extremal for \( \Gamma_{ai}^* \); the points \((r_{0e}^1,0)\), \((r_{0e}^1,1)\), \((1,u_{0e}^1)\), \((0,u_{0e}^0)\) are extremal for \( \Gamma_{ae}^* \). Moreover, \( r_{0i}^0 \in (0,r_i) \), \( r_{0e}^1 \in (r_e,\infty) \). Since \( \Gamma_{ai} \) degenerates into the point \((0,0)\) for \( a = 1 + \gamma \), for \( a > 1 + \gamma \), only the closed orbit \( \Gamma_{ae} \) exists.
As the result, the class $[0]$ naturally splits into two subclasses: $0_i$ for closed orbits $\Gamma_{ai}$ with $1 < a < 1 + \gamma$ and $0_e$ for closed orbits $\Gamma_{ae}$ with $a > 1$.

1] $\gamma < a < 1$. The set $\Gamma_a$ consists of two closed orbits. The top half of the right orbit $\Gamma^*_a$ is located between $\Gamma^*_r$ and $\Gamma^*_l$ and has the following extremal points: $(r^0_1, 0), (r^1_1, 1), (1, u^1_1)$ $(l^1_1, 1), (l^0_1, 0)$, where $r^0_1 \in (1, r_e)$.

2] $0 < a < \gamma$. The set $\Gamma_a$ consists of four closed orbits. The orbit $\Gamma^*_a$ encloses $(1, 1)$ and lies inside of $\Gamma^*_r$, its extremal points are $(r^2_1, 1), (1, u^2_1), (l^1_2, 1), (1, lo^2_2)$, where $r^2_1 \in (1, r_\gamma)$.

2.2. Parametrization of the closed orbits. The parameter $a$ does not determine a specific closed orbit from $\Gamma_a$, therefore it cannot be used to parametrize the orbits. To define the parametrization we use the extremal points: $(r^0_0, 0), (r^0_0, 0)$ in class $0], (\pm r^1_1, 0)$ in class $1], (\pm r^2_1, 1)$ and $(\pm r^2_1, -1)$ in class $2]$. Each such point defines the corresponding closed orbit and the parameter $a$ from (4) can be explicitly expressed through $r$ mentioned above.

Thus, an arbitrary closed orbit of system (3) is parameterized by the functions $C(\varphi), S(\varphi)$. These functions are solutions of the initial value problem

$$
\begin{align*}
C(0) = b_{kl}, & \quad S(0) = l \quad (k, l = 0, \pm 1, (k, l) \neq (0, \pm 1)); \\
\begin{cases}
 b_{00}^0 = r^0_0, & r^0_0 \in (0, r_0) \\
 b_{0e}^0 = r^0_0, & r^0_0 \in (r_e, r_M), \\
 b_{00}^0 = r^0_0, & r^0_0 \in (r_e, r_M), \\
 b_{0e}^0 = r^0_0, & r^0_0 \in (r_e, r_M),
\end{cases} \\
\begin{cases}
 b_{00} = 1 - |l|, & b_{0e} = 1 - |l|, \\
 b_{1+1} = -r^0_1, & b_{-1+1} = -r^0_1, \\
 r_M = \sqrt{1 + (M - \gamma)^2},
\end{cases}
\end{align*}
$$

where $k$ determines the "shift" along the abscissa axis, $l$ determines the "shift" along the ordinate axis and $|k| + |l|$ determines the class number related to the parameterized closed orbit, and the constants $r$ are introduced in (4).

Let us note that the restriction $b_{00}^0 < r_M$ is introduced according to Remark 1 so we can work with system (3).

We will denote the real analytic $\omega_{b_{kl}}$-periodic solution of the initial value problem with the initial conditions (6) by $(C(\varphi, b_{kl}), S(\varphi, b_{kl}))$.

2.3. Calculation of periods. Let us introduce the auxiliary functions

$$
\begin{align*}
S^\pm(C^2(\varphi)) = \sqrt{1 \pm \gamma^{-1/2}(a_{b_{kl}} - (C^2(\varphi) - 1)^2)^{1/2}}.
\end{align*}
$$

Then, in the first integral $S = \pm S^-(C^2)$ or $S = \pm S^+(C^2)$, and the signs are chosen according to the location of points from the closed orbit parameterized by the solution $(C(\varphi), S(\varphi))$ with respect to the lines $S = 0, \pm 1$.

We can write down the first equation of system (3) as

$$
\begin{align*}
d\varphi = (\gamma(S^3(\varphi) - S(\varphi)))^{-1}dC(\varphi).
\end{align*}
$$

**Proposition 1.** The following equalities hold:

$$
\begin{align*}
\omega_{b_{00}}^+ = 4\varphi^-_l, & \quad \omega_{b_{00}}^- = 4(\varphi^+_e + \varphi^-_e), \\
\omega_{b_{0e}} = 2(\varphi^-_l + \varphi^+_u + \varphi^-_r), & \quad \omega_{b_{1+1}} = \omega_{b_{-1+1}} = \varphi^-_2 + \varphi^+_2.
\end{align*}
$$
where \( \varphi_i^- = \int_{b_{10}}^{0} \zeta^- dC; \quad \varphi_e^+ = \int_{0}^{r_{1e}} \zeta^+ dC, \quad \varphi_e^- = \int_{r_{1e}}^{b_{10}} \zeta^- dC; \quad \varphi_l^- = \int_{l_{1}}^{r_{1l}} \zeta^- dC, \quad \varphi_u^+ = \int_{r_{1l}}^{r_{1l}^1} \zeta^+ dC, \quad \varphi_r^- = \int_{r_{1l}^1}^{b_{10}} \zeta^- dC; \quad \varphi_2 = \int_{r_{1l}^1}^{l_{1}} \zeta^- dC, \quad \varphi_2^+ = \int_{l_{1}}^{b_{10}} \zeta^+ dC, \) the limits of integration are defined in \((5)\), with \( a = a_{b_{kl}} \) from \((6)\), and \( \zeta^\pm(C^2) = (g(S^\pm(C^2) - S(\pm(C^2)))^{-1}.

**Proof.** Let us calculate the period of solution \((C(\varphi_{b_{kl}}), S(\varphi_{b_{kl}}))\) [this solution parameterizes the closed orbit from the class 1] when \( b_{10} = r_{1l}^0 \in (1, r_e) \) and \( C(0) = r_{1l}^0, S(0) = 0 \).

There exists a \( \varphi_* \) such that \( C(\varphi_*) = l_{1l}^0, S(\varphi_*) = 0 \) \((0 < \varphi_* < \omega_{b_{10}})\). The constants \( l_{1l}^0, l_{1l}^1, r_{1l}^1 \) from \((5)\) are abscissas of the left intersection point of the closed orbit and the lines \( S = 0 \) and \( S = 1 \), and \( S'(0) = r_{1l}^0 - (r_{1l}^0)^3 < 0 \). Therefore, integrating \((8)\) from \( \varphi_* \) to \( \omega_{b_{10}} \), we get

\[
\omega_{b_{10}} - \varphi_* = \varphi_l^- + \varphi_u^+ + \varphi_r^-.
\]

This calculated value is the "period of time," when the orbit is located in the first quadrant: below and above the line \( S = 1 \). Similarly, integrating formula \((8)\) from 0 to \( \varphi_* \) and taking into account that \( l_{1l}^1, r_{1l}^1 \) are also the abscissas of the intersection points of the closed orbits and the line \( S = -1 \), we get \( \varphi_* = \varphi_r^- + \varphi_u^+ + \varphi_l^- \). Hence, \( \varphi_* = \omega_{b_{10}}/2 \) and \( \omega_{b_{10}} = 2(\varphi_l^- + \varphi_u^+ + \varphi_r^-) \);

\( \omega_{b_{10}} = \omega_{b_{10}} \) due to the symmetry.

Computations are similar for classes \([0], [0_0], \text{ and } 2\]. Moreover, \( C(\omega_{b_{10}}/4) = 0, S(\omega_{b_{10}})/4 = u_0^0 \) for the class \([0]\). \( \square \)

### 3. Dynamics in a neighborhood of a closed orbit.

#### 3.1. Monotonicity indicator of the angular variable.

For each \( b_{kl} \) from \((6)\) we consider the function defined on the periodic solution \((C(\varphi_{b_{kl}}), S(\varphi_{b_{kl}}))\) of system \((3)\) by the formula

\[
\alpha_{kl}(\varphi) = C'(\varphi)(S(\varphi) - l) - (C(\varphi) - k)S'(\varphi).
\]

Using formula \((4)\) the function can be written in a simpler form as

\[
\alpha_{kl} = a_{b_{kl}} - 1 - \gamma + C^2 + \gamma S^2 - k(C^3 - C) - \gamma l(S^3 - S).
\]

Let us study how the function \( \alpha_{kl}(\varphi) \) changes its sign along the orbits which go counterclockwise when \( \varphi \) increases for the class \([0] \), and go clockwise when \( \varphi \) increases for the other classes.

The passage to a neighborhood of an arbitrary closed orbit is possible only if \( \alpha_{kl} \) is a function of a fixed sign, because geometrically the sign of \( \alpha_{kl} \) reflects the monotonicity of the change of the angular variable assuming we observe the movement along the closed orbit from the point \((k, l)\).

This is the reason why the same function \( \alpha_{00} \) cannot be used for passing to orbits from classes \([1] \) and \([2] \). Assuming we observe the movement from the origin the polar angle is not changing monotonically. The subtraction of constants \( k \) and \( l \) in formula \((10)\) means the the origin shifts to the point \((k, l)\) in system \((3)\).

#### 3.2. Monotonicity of the angular variable in class \([0] \).

Let us show that \( \alpha_{00} = a_{b_{00}} - 1 - \gamma + C^2(\varphi) + \gamma S^2(\varphi) \) is a function of a fixed sign.

We have: \( \alpha_{00}' = 2C(\varphi)C(\varphi)' + 2\gamma S(\varphi)S'(\varphi) = 2\gamma C(\varphi) S(\varphi)(S^2(\varphi) - C^2(\varphi)). \)
For the class 0\textsuperscript{th} ($a = a_{00} \in (1, 1 + \gamma)$), orbits are located in the first quadrant when $\varphi \in [0, \omega / 4]$ and pass through the points $(r_{0e}^0, 0)$ when $\varphi = 0$ and $(0, u_{0e}^0)$ when $\varphi = \omega / 4$. Constants $r_{0e}^0$ and $u_{0e}^0$ are given in \[(5)\].

Note that $\alpha_{00}(\varphi) < 0$ when $C_0(\varphi) > S_0(\varphi)$ and $\alpha_{00}(\varphi) > 0$ when $C_0(\varphi) < S_0(\varphi)$. Therefore, the function $\alpha_{00}(\varphi)$ takes maximum values at the endpoints $[0, \omega / 4]$. However, $\alpha_{00}(0) = a - 1 - \gamma + (r_{0e}^0)^2 = (a - \gamma)^{1/2}((a - \gamma)^{1/2} - 1) < 0$, $\alpha_{00}(\omega / 4) = a - 1 - \gamma + (u_{0e}^0)^2 = (a - 1)^{1/2}((a - 1)^{1/2} - \sqrt{\gamma}) < 0$.

Thus, due to the symmetry, we conclude that $\alpha_{00}(\varphi) < 0$ for any $\varphi$.

For the class 0\textsuperscript{th} ($a = a_{00} > 1$), the orbits are located in the first quadrant when $\varphi \in [3\omega / 4, \omega]$ and pass through the point $(0, u_{0e}^0)$ when $\varphi = 3\omega / 4$ and the point $(r_{0e}^0, 0)$ when $\varphi = \omega$. The constants $r_{0e}^0 = b_{0e}^0$ and $u_{0e}^0$ are given in \[(5)\].

We observe also that $\alpha_{00}(\varphi) > 0$ when $C_0(\varphi) < S_0(\varphi)$ and $\alpha_{00}(\varphi) < 0$ when $C_0(\varphi) > S_0(\varphi)$.

Therefore, the function $\alpha_{00}(\varphi)$ takes minimum values at the endpoints $[3\omega / 4, \omega]$.

However, $\alpha_{00}(3\omega / 4) = a - 1 - \gamma + (u_{0e}^0)^2 > 0$ and $\alpha_{00}(0) = a - 1 - \gamma + (r_{0e}^0)^2 > 0$. Thus, for each $\varphi$, we conclude that $\alpha_{00}(\varphi) > 0$ due to the symmetry.

### 3.3. Monotonicity of the angular variable for the class 1]

We first prove that for $b_{k0}$ from \[(6)\] close to $k$ the function $\alpha_{k0} = \alpha_{k0}(\varphi, b_{k0}) \; (k = \pm 1)$ from \[(10)\] alternates its sign.

Let $k = 1$. Then, for $b_{10} \rightarrow 1_{+0}$, the orbits of system \[(3)\] converge to the right ”eight” from the outside. Let us show that, for each $\gamma \in (0, 1)$, $\alpha_{10}(\varphi, b_{k0})$ takes both positive and negative values assuming we move along the right ”eight”.

For $b_{10} = 1$, in formula \[(5)\] $a_{10} = \gamma$, therefore, function $\alpha_{10}(\varphi, 1) = \gamma S^2(\varphi, 1) - C^3(\varphi, 1) + C^2(\varphi, 1) - C(\varphi, 1) - 1$. It is possible to find such $\varphi_*$, that $C(\varphi_*, 1) = 1$, $S(\varphi_*, 1) = \sqrt{2}$, because $(1, \sqrt{2})$ is the apex point of $\Gamma_*$. Then $\alpha_{10}(\varphi_*, 1) = 2\gamma > 0$.

For each $\varphi$, such that $C(\varphi, 1) \in ((1 - \gamma)^{1/2}, 1)$, $|S(\varphi, 1)| \in (0, 1)$, in integral \[(4)\] $\gamma(S^2 - 1)^2 = \gamma - (C^2 - 1)^2 (\geq 0)$ or $S^2(\varphi, 1) = 1 - (1 - \gamma^{-1}(C^2(\varphi, 1) - 1)^2)^{1/2}$. Omitting the arguments $\varphi, 1$ we obtain $\alpha_{10} = \gamma - \gamma^{1/2}(\gamma - (C^2 - 1)^2)^{1/2} - (C - 1)^2(C + 1)$. Then $\alpha_{10} < 0 \Leftrightarrow (\gamma - (C - 1)^2(C + 1))^2 < \gamma(\gamma - (C^2 - 1))^2 \Leftrightarrow C(C^2 + \gamma - 1) < 0$, which is the correct statement.

Now, continuity of $\alpha_{10}$ as a function of $b_{10}$ implies that for $b_{10}$ close enough to zero, the function $\alpha_{10}(\varphi, b_{10})$ also alternates its sign. It changes its sign on any closed orbit from some neighborhood of the right ”eight,” to be more specific, on the part of the orbit located to the left of the singular point $(1, 0)$ of system \[(3)\].

The statement that $\alpha_{-1,0}(\varphi, b_{-1,0})$ alternates its sign can be proved similarly.

Thus, the class 1] is different from the other classes in the sense that the algorithm used in the paper requiring the function $\alpha_{k0}$ to be of a fixed sign can be used only for $\gamma$ which are separated from 1 and on a smaller interval of $b_{k0}$.

Thereby, we introduce the following restriction for the class 1]:

\begin{equation}
0 < \gamma \leq \gamma_* = 0.806, \quad kb_{k0} \in (b^-, b^+) \; (k = \pm 1),
\end{equation}

where $b^+ = \{ \sqrt{1 + (1 - \gamma)^{1/2}} \text{ for } \gamma \in (0, 0.5], \sqrt{1 + (3.25 - \gamma - 3\gamma^2)^{1/2}} / 2 \text{ for } \gamma \in [0.5, \gamma_*] \}$,

$\quad b^- = \{1.01 \text{ for } \gamma \in (0, 0.2], 0.20 + 0.97 \text{ for } \gamma \in [0.2, 0.75], 0.60 + 0.67 \text{ for } \gamma \in [0.75, \gamma_*] \}$.

In particular, $(b^-, b^+) \subset (1, r_e)$ and $1.154 < b_{-0}(\gamma_*) < b_{+0}(\gamma_*) < 1.162$. 
Lemma 1. Let \( b^-_k = \{ 1 \text{ for } \gamma \in (0, 0.2), \ 0.2\gamma + 0.96 \text{ for } \gamma \in [0.2, 0.75], \ 0.6\gamma + 0.66 \text{ for } \gamma \in [0.75, \gamma_* + 10^{-3}] \} \), \( b^+_k = \{ r_e \text{ for } \gamma \in (0, 0.5), \ \sqrt{1 + (3.5 - \gamma - \gamma^2)^{1/2}/2 + 0.02(\gamma - 0.5)} \text{ for } \gamma \in [0.5, \gamma_* + 10^{-3}] \} \).

Then, \( \)

1) for all \( \gamma \in (0, \gamma_*), \) and for all \( b_{k0}: \) \( kb_{k0} \in (b^-(\gamma), b^+(\gamma)) \Rightarrow \alpha_{k0}(\varphi, b_{k0}) > 0; \)

2) for all \( \gamma \in (\gamma_* + 10^{-3}), \) and for all \( b_{k0}: \) \( kb_{k0} \in (0, r_e) \Rightarrow \alpha_{k0}(\varphi, b_{k0}) \) alternates its sign;

3) for all \( \gamma \in (0, \gamma_* + 10^{-3}), \) for all \( b_{k0}: \) \( kb_{k0} \in (1, b^-(\gamma)) \cup (b^+(\gamma), r_e) \Rightarrow \alpha_{k0}(\varphi, b_{k0}) \) alternates its sign (\( k = \pm 1 \)).

Proof. Functions and constants used in Lemma 1 were calculated approximately using MAPLE by tracking the sign of \( \alpha_{k0} \) while changing parameters \( \gamma \) and \( b_{k0}(\gamma) \) with the step \( 10^{-3} \).

3.4. Monotonicity of the angular variable in the class 2]. Let us now show that \( \alpha_{kl}(\varphi, b_{kl}) (k, l = \pm 1) \) from \([10]\) is positive.

We have that \( \alpha_{kl}'(\varphi, b_{kl}) = \gamma(C - k)(S - l)(kC - lS)(3klCS + kC + lS + 1). \)

Let \( kl = 1. \) Since \( IS > 0 > -(C + k)/(3C + k), \) by multiplying the inequality above by \( k(3C + k)(> 0) \) we get that \( 3klCS + kC + lS + 1 > 0. \)

Considering that the movement along the orbit is clockwise the analysis of the sign of \( \alpha_{kl}'s \) shows that the function \( \alpha_{kl}(\varphi, b_{kl}) \) has local minimums at the points \( \varphi_1, \varphi_2, \varphi_3 \) determined by the equations \( C(\varphi_1) = k, S(\varphi_1) = l \cdot lo^1_2; \ S(\varphi_2) = l, C(\varphi_2) = kl^1_2; \ kC(\varphi_3) = lS(\varphi_3) > 1. \)

According to formulas \([5]\) \( l^2_2 = \sqrt{1 - a^2/2}, \ lo^2_2 = \sqrt{1 - (a/\gamma)^2}; \) hence, \( a = \gamma(1 - (lo^2_2)^2) = (1 - (l^2_2)^2); \) then, \( \alpha_{kl}(\varphi_1) = a + \gamma((lo^2_2)^2 - 1)(1 - lo^2_2) = \gamma lo^2_2(lo^2_2 - 1)^2(l^2_2 + 1) > 0. \) Similarly, \( \alpha_{kl}(\varphi_2) = a + ((l^2_2)^2 - 1)(1 - l^2_2) = l^2_2(l^2_2 - 1)^2(l^2_2 + 1) > 0. \) Finally, \( \alpha_{kl}(\varphi_3) = (1 + \gamma)(S^2 - 1)S(S - l) > 0. \)

The property that \( \alpha_{kl} > 0 \) when \( kl = -1 \) can be proved analogously.

3.5. Special polar coordinates. We introduce the notations:

\[
\begin{align*}
\mu_{kl} &= ((b_{kl} + k)^2 - 1)^2, \quad C_k(\varphi) = C(\varphi) - k, \quad S_k(\varphi) = S(\varphi) - l; \\
R_{kl}(t, \varphi, r, \varepsilon) &= C_k'(\tilde{\varphi}_{kl}) - S_k'(\tilde{\varphi}_{kl}), \quad (\tilde{\varphi}_{kl}) = (t, C + C_k r, S + S_k r, \varepsilon), \\
\Phi_{kl}(t, \varphi, r, \varepsilon) &= \alpha_{kl}^{-1}(1 + r)^{-1}(S_k'X(\tilde{\varphi}_{kl}) - C_k'Y(\tilde{\varphi}_{kl})); \\
R^{o}_{kl} &= R_{kl}(t, \varphi, 0, 0) = C_k'(t, C, S, 0) - S_k'X(t, C, S, 0), \\
\Phi^{o}_{kl} &= \Phi_{kl}(t, \varphi, 0, 0) = \alpha_{kl}^{-1}(S_k'X(t, C, S, 0) - C_k'Y(t, C, S, 0)).
\end{align*}
\]
Then $R_{kl} = R_{kl}^0 + (R_{kl}')^o r + (R_{kl}')^c \varepsilon + O(|r| + \varepsilon)^2$, $\Phi_{kl} = \Phi_{kl}^0 + O(|r| + \varepsilon)$, where $(R_{kl}')^o = C'(C_k Y_x^t(t, C, S, 0) + S_l Y_y^t(t, C, S, 0)) - S'(C_k X_x^t(t, C, S, 0) + S_l X_y^t(t, C, S, 0))$, $(R_{kl}')^c = C'_0 Y_x^t(t, C, S, 0) - S'_0 X_y^t(t, C, S, 0)$.

Let us perform the special affine-polar coordinate change in system (1) setting

$$x = C(\varphi) + C_k(\varphi) r, \quad y = S(\varphi) + S_l(\varphi) r \quad (|r| < r_0 \leq 1, \ k, l = 0, \pm 1),$$

where $(C(\varphi), S(\varphi))$ are $\omega_{b_{kl}}$-periodic solution of system (3) with the initial values $C(0) = b_{kl}$, $S(0) = l$ from (6) and restriction (11).

Let us differentiate (13) with respect to $t$. Using (10) we solve the obtained equalities with respect to $\dot{r}$ and $\dot{\varphi}$. As the result we get

$$\alpha_{kl} \dot{r} = C'(\varphi) \dot{y} - S'(\varphi) \dot{x}, \quad (r + 1) \alpha_{kl}(\varphi) \dot{\varphi} = S_l(\varphi) \dot{x} - C_k(\varphi) \dot{y}.$$  

Substituting the right-hand sides of system (1) into the formulas given above and using notations (12) we obtain the system

$$\varepsilon^{-\nu} \alpha_{kl} \dot{r} = -\alpha_{kl}^o r - p_{kl} r^2 + O(|r|^3) + (R_{kl}'^o + (R_{kl}')^c \varepsilon + O(|r| + \varepsilon)^2) \varepsilon,$$

$$\varepsilon^{-\nu} \dot{\varphi} = 1 + \alpha_{kl} q_{kl} r + O(|r|^2) + (\Phi_{kl}^c + O(|r| + \varepsilon)) \varepsilon,$$

where $p_{kl}(\varphi) = 3\gamma C S((S^2 - 1)(C - k)^2 - (C^2 - 1)(S - l)^2)$ and $q_{kl}(\varphi) = \alpha_{kl}^{-2}((C - k)^3(2C + k) + \gamma(S - l)^3(2S + l))$.

4. Radial averaging and the determining equation.

4.1. Primary radial averaging. Let us introduce the functions

$$\beta_{kl}(\varphi) = \int_0^\varphi \xi_{kl}(s) ds, \quad \xi_{kl}(\varphi) = \alpha_{kl}^{-1}(\varphi)(\alpha_{kl}'(\varphi) q_{kl}(\varphi) - \alpha_{kl}^{-1}(\varphi) p_{kl}(\varphi)).$$

Lemma 2. The function $\beta_{kl}(\varphi, b_{kl})$ is $\omega_{b_{kl}}$-periodic in $\varphi$.

Proof. We will use formulas (14), (10), (10) and (12).

For the class 0] $(k, l = 0)$ we have: $p_{00} = 3\alpha_{00}'/2$, $q_{00} = 4\alpha_{00}^{-1} + 2\mu_{00}\alpha_{00}^{-2}$. Thus, $\xi_{00} = \alpha_{00}^{-2}(5/2 + 2\mu_{00}\alpha_{00}^{-1})$, $\beta_{00} = \int_0^{\varphi} (5\alpha_{00}^{-2}(s)/2 + 2\mu_{00}\alpha_{00}^{-3}(s)) d\alpha_{00}(s) = 5(\alpha_{00}^{-1}(0) - \alpha_{00}^{-1}(\varphi)) / 2 + \mu_{00}(\alpha_{00}^{-2}(0) - \alpha_{00}^{-2}(\varphi)) = \beta_{00}(000, 000).$

For the class 1] $(k = \pm 1, l = 0)$, we have: $p_{k0} = \alpha_{k0}' - \gamma C_k S_l(C_k^2 + 2 S_l^2 + 3(2 - k)C_k S_l^2)$, $q_{k0} = 5\alpha_{k0}^{-1} + \alpha_{k0}^{-2}(2C_k^2 - \gamma S_l^2 - 3\mu_{k0})$. Therefore $\xi_{k0} = \alpha_{k0}^{-2}(2C_k^2 - \gamma S_l^2 - 3\mu_{k0}) - \gamma \alpha_{k0}^{-2}C_k S_l(C_k^2 + 2 S_l^2 + 3(2 - k)C_k S_l^2)$.

For the class 2] $(k, l = 1)$, we have: $p_{kl} = \alpha_{kl}' + \gamma C_k S_l(kk - lS_l)(3kkC_k S_l + 2kkC_k + 2l S_l)$, $q_{kl} = 5\alpha_{kl}^{-1} + \alpha_{kl}^{-2}(2C_k^2 + 2\gamma S_l^2 - 3\mu_{kl})$, then $\xi_{kl} = \alpha_{kl}^{-2}(2C_k^2 + 2\gamma S_l^2 - 3\mu_{kl}) - \gamma \alpha_{kl}^{-2}C_k S_l(kk - lS_l)(2kkC_k + 3kkC_k S_l + 2l S_l)$.

For the cases $k \neq 0$ let us describe the movement along the closed orbit when $\varphi$ changes from zero to $\omega_{b_{kl}}$. To this end, we express $S_l(\varphi)$ as a composite function using formula (1) for the integral, notations (12), constants from (5) and the fact that $S'(0) > 0$ for system.
Thus, we obtain:

\[ S_0(C_k(\varphi)) = \{ S_0^- \text{ for } kC_k \setminus \omega_{l_1-k}^{r_{l_1-k}}, S_0^+ \text{ for } kC_k \setminus \omega_{l_2-k}^{r_{l_2-k}}, \}
\]

where

\[ S_1(C_k(\varphi)) = \{ S_1^+ \text{ for } kC_k \setminus \omega_{l_1-k}^{b_{l_1-k}}}, S_1^- \text{ for } kC_k \setminus \omega_{l_2-k}^{b_{l_2-k}} \} \]

Thus, we obtain:

\[ S_1(C_k(\varphi)) = \{ S_1^+ \text{ for } kC_k \setminus \omega_{l_1-k}^{b_{l_1-k}}}, S_1^- \text{ for } kC_k \setminus \omega_{l_2-k}^{b_{l_2-k}} \} \]

Therefore, \( \xi_{kl} = \xi_{kl}(C_k(\varphi)) \), because \( \alpha_{kl}(\varphi) = \alpha_{kl}(C(\varphi)) = \alpha_{kl}(C_k(\varphi) + k) \) in \( \theta_{b_{l_1}} \). Thus, taking into account \( \theta_{b_{l_1}} \), \( \beta_{kl} = \beta_{kl}(C_k(\varphi)) \). 

We now show that the \( \omega_{b_{l_1}} \)-periodic averaging change

\[ r = \alpha_{kl}^{-1}(\varphi, b_{kl})(z + \beta_{kl}(\varphi, b_{kl})z^2) \quad (k, l = 0, \pm 1), \]

transforms system \( \theta_{l_1} \) into the system

\[ \begin{align*}
\epsilon^{-\nu} \dot{z} &= O(|z|^3) + (R_{Z}^o + Z_{kl}z + (R_{kl}^o)^\varphi + O((|z|^2))^\varphi)\varphi \\
\epsilon^{-\nu} \dot{\varphi} &= 1 + q_{kl}z + O(|z|^2) + (\Phi_{kl}^o + O(|z|^2))\varphi
\end{align*} \]

where \( Z_{kl}(t, \varphi) = \alpha_{kl}^{-1}(R_{kl}^o)^\varphi - 2\beta_{kl}R_{kl}^o + \alpha_{kl}^{-1} \alpha_{kl}^\prime \Phi_{kl}^o \).

Differentiating \( \theta_{l_1} \) with respect to system \( \theta_{l_1} \) and \( \theta_{l_1} \) we obtain

\[ -\alpha_{kl}^{-1} \alpha_{kl}^\prime (z + \beta_{kl}z^2) - \alpha_{kl}^{-2} p_{kl}z^2 + O(|z|^3) + (R_{kl}^o + \alpha_{kl}^{-1}(R_{kl}^o)^\varphi z + (R_{kl}^o)^\varphi + O((|z|^2))^\varphi)\varphi = (1 + 2\beta_{kl}z)(R_{kl}^o + Z_{kl}z + (R_{kl}^o)^\varphi + (\beta_{kl}z^2 - \alpha_{kl}^{-1} \alpha_{kl}^\prime z - \alpha_{kl}^{-1} \alpha_{kl}^\prime (\beta_{kl}z^2)))(1 + q_{kl}z + \Phi_{kl}^o)\varphi. \]

Equating the coefficients of \( z \) and \( z^2 \) we get the formula for \( Z_{kl} \) and the equation \( \beta_{kl}^\prime = \alpha_{kl}^{-1}(\alpha_{kl}^\prime q_{kl} - \alpha_{kl}^{-1} p_{kl}) \). Solving this equation we find \( \beta_{kl}(\varphi, b_{kl}) \).

### 4.2. Decompositions of two-periodic functions, the Siegel condition.

We continue the search for two-dimensional invariant surfaces of system \( \theta_{l_1} \). For this purpose we average the functions \( R_{kl}^o, Z_{kl} \) and \( (R_{kl}^o)^\varphi \) in system \( \theta_{l_1} \). However for this system \( \dot{\varphi} = 1 + \ldots \) when \( \nu = 0, \dot{\varphi} = \varepsilon + \ldots \) for \( \nu = 1 \). Therefore the following averaging changes and their existence conditions will be different for each \( \nu \). To distinguish the cases the functions, constants and formulas that depend on \( \nu \), will have the superscript 0 or 1.

For continuous, \( T \)-periodic in \( t \), real analytic and \( \omega \)-periodic in \( \varphi \) functions \( v = v^\nu(t, \varphi) \) we use the following decomposition depending on the value of parameter \( \nu \):

\[ v^\nu(t, \varphi) = \overline{v^\nu} + \tilde{v}^\nu(\varphi) + \hat{v}^\nu(t, \varphi) \quad (\nu = 0, 1), \]

where \( \overline{v^\nu} = \frac{1}{\omega T} \int_0^\omega \int_0^T v^\nu(t, \varphi) \, dt \, d\varphi \) is the average value of the function \( v^\nu \), \( \hat{v}^0 = 0, \quad \hat{v}^1 = \frac{1}{T} \int_0^T v^1(t, \varphi) \, dt \). Then, the function \( \tilde{v}^\nu(t, \varphi) \) has zero average value with respect to \( t \), which implies periodicity of the function \( \int_0^t \tilde{v}^\nu(\tau, \varphi) \, d\tau \), which also has zero average value by the virtue of the choice of the constant \( t_0 \in [0, T] \).
Proposition 2. Assume \( \nu = 0 \) and the periods \( T \) and \( \omega \) of continuous, \( T \)-periodic in \( t \), real analytic and \( \omega \)-periodic in \( \varphi \) function \( \tilde{v}(t, \varphi) \) satisfy the Siegel condition
\[
|pT - q\omega| > K(p + q)^{-\tau} \quad (K > 0, \quad \tau \geq 1, \quad p, q \text{ – positive integers}).
\]
Then the equation
\[
\hat{\zeta}'(t, \varphi) + \hat{\zeta}(t, \varphi) = \tilde{v}(t, \varphi)
\]
has the unique solution \( \hat{\zeta}(t, \varphi) \), which has the same properties as the function \( \tilde{v}(t, \varphi) \).

Here and in what follows we will denote the derivative with respect to \( t \) of any function which has \( t \) and \( \varphi \) as its arguments by a dot, and the derivative with respect to \( \varphi \) by a prime.

4.3. The choice of generating orbits. The function
\[
R_{kl}^0(t, \varphi) = C'(\varphi)Y(t, C(\varphi), S(\varphi), 0) - S'(\varphi)X(t, C(\varphi), S(\varphi), 0),
\]
introduced in (12) will play the key role in our analysis. In (18) \((C(\varphi), S(\varphi))\) is a real analytic \( \omega_{b_{kl}} \)-periodic solution of the initial value problem for system \((3)\) with the initial values \( C(0) = b_{kl}, S(0) = l \), the parameter \( b_{kl} \) is any number from (6) and the period \( \omega_{b_{kl}} \) is calculated in (9).

Using the mentioned above decomposition, we write out \( R_{kl}^\nu(t, \varphi) \) as the following sum:
\[
R_{kl}^\nu = R_{kl}^0 + \tilde{R}_{kl}(\varphi) + \tilde{R}_{kl}(t, \varphi),
\]
where \( R_{kl}^\nu(b_{kl}) = \frac{1}{T\omega} \int_0^\omega \int_0^T R_{kl}^\nu(t, \varphi)dt d\varphi \) is the average value of \( R_{kl}^\nu \) \((\omega = \omega_{b_{kl}})\).

For each \( k, l \) \((k, l = 0, \pm 1, \; (k, l) \neq (0, \pm 1))\), we also introduce the generating equation
\[
\hat{\g}_{kl}^0(t, \varphi) + \hat{\g}_{kl}^0(t, \varphi) = \tilde{R}_{kl}(t, \varphi)\hat{\g}_{kl}^0 + \hat{\g}_{kl}^0 = R_{kl}(t, \varphi) \quad (\tilde{R}_{kl} = 0).
\]

Moreover, if for \( \nu = 0 \) the periods \( T \) and \( \omega_{kl} \) satisfy Siegel condition \((17)\), then by Proposition 2 the equation
\[
\hat{\g}_{kl}(t, \varphi) + \hat{\g}_{kl}(t, \varphi) = \tilde{R}_{kl}(t, \varphi)\hat{\g}_{kl} + \hat{\g}_{kl} = R_{kl}(t, \varphi) \quad (\tilde{R}_{kl} = 0).
\]

This solution has the same properties as the function \( \tilde{R}_{kl}(t, \varphi) \).

Now we can formulate the dissipativity condition as
\[
L_{kl}^\nu \neq 0 \quad (\nu = 0, 1),
\]
where \( L_{kl}^0 = Z_{kl} - \hat{q}_{kl}^0 \), \( L_{kl}^1 = Z_{kl} - \tilde{R}_{kl}^0 q_{kl} \); and \( q_{kl}(\varphi) \) is from (14), \( Z_{kl}(t, \varphi) \) is from (16).

Definition 1. The parameter \( b_{kl}^* \) from (6) is called admissible for (1) and is denoted by \( b_{kl}^* \), if it is a solution of generating equation (19), condition (11) holds for \( |k| + |l| = 1 \), the periods \( T \) and \( \omega^* = \omega_{b_{kl}} \) satisfy Siegel condition (17) for \( \nu = 0 \) and the dissipativity condition (21) holds.

Remark 2. We will be interested in systems (1) with a nonempty set of admissible values. For each \( b_{kl}^* \), it will be proved that the perturbed system (1) retains a two-periodic invariant surface homeomorphic to a torus, which is obtained by factoring the time with respect to the period in a small neighborhood of the cylindric surface, whose generatrix is a closed orbit of unperturbed system (2) that passes through the point \((b_{kl}^*, l)\) for each sufficiently small \( \varepsilon \).

\(^1\)See Lemma B.5 in [9, p. 17].
From now on we fix an admissible parameter $b_{kl}^*$ of system (1) which in turn fixes initial values for the $\omega^*$-periodic initial value problem solution $(C(\varphi),S(\varphi))$ of system (3), which parameterizes a specific closed orbit related to the class $|k| + |l|$

All subsequent changes and introduced functions are fixed by the choice of $b_{kl}^*$. In particular, conditions (11), (17), (21) hold by the definition and $R_{kl}^o$, which is a part of systems (14) and (16), has zero average value, that is,

$$R_{kl}^o(t,\varphi) = \tilde{R}_{kl}(t,\varphi).$$

5. The construction of invariant surfaces. The results.

5.1. Secondary radial averaging. We introduce the functions:

$$g_{kl}^0 = \frac{\bar{g}_{kl}^0}{\hat{g}_{kl}^0}, \quad \hat{g}_{kl}^0 \text{ from } (20), \quad g_{kl}^0 = (\hat{g}_{kl}^0 \Phi_{kl}^o - (R_{kl}^o \nu) - \hat{g}_{kl}^0 (Z_{kl} - g_{kl}^0 q_{kl})) / L_{kl}^0,$$

(22) \hspace{1cm} \hat{g}_{kl}^0 = \hat{R}_{kl}^o, \quad \bar{g}_{kl} = (\hat{R}_{kl}^o \Phi_{kl}^o - (R_{kl}^o \nu) - \hat{g}_{kl}^0 (Z_{kl} - \hat{R}_{kl}^o q_{kl})) / L_{kl}^1.

Let us show that the coordinate change

(23) \hspace{1cm} z = u + G_{kl}(t,\varphi,\varepsilon) \varepsilon + H_{kl}(t,\varphi,\varepsilon) u \varepsilon + F_{kl}(t,\varphi,\varepsilon) \varepsilon^2 \quad (\nu = 0, 1),

where $G_{kl}^0 = g_{kl}^0 (t,\varphi), \quad H_{kl}^0 = \bar{h}_{kl}^0 (t,\varphi), \quad F_{kl}^0 = \bar{f}_{kl}^0 (t,\varphi,\varepsilon), \quad G_{kl}^1 = \bar{g}_{kl}^1 (t,\varphi,\varepsilon), \quad H_{kl}^1 = \bar{h}_{kl}^1 (t,\varphi,\varepsilon), \quad F_{kl}^1 = \bar{f}_{kl}^1 (t,\varphi,\varepsilon)$, transforms system (16) into the system

(24) \hspace{1cm} \dot{u} = (L_{kl}^0 u \varepsilon + O((|u| + |\varepsilon|^3)) \varepsilon^\nu, \quad \dot{\varphi} = (1 + \Theta_{kl}^\nu \varepsilon + q_{kl} u + O(|u| + |\varepsilon|^2)) \varepsilon^\nu,

where, obviously, $\Theta_{kl}^0 (t,\varphi) = \Phi_{kl}^o + q_{kl} g_{kl}^0, \quad \Theta_{kl}^1 (t,\varphi) = \Phi_{kl}^o + q_{kl} (\bar{g}_{kl}^0 + \bar{g}_{kl}^1)$.

We achieve this by differentiating coordinate change (23) with respect to systems (16) and (24) and reducing the result by $\varepsilon^\nu$. The calculations yield the identity

$$(R_{kl}^o + (u + G_{kl}^0 \varepsilon) Z_{kl} + (R_{kl}^o \nu) \varepsilon + O((|u| + |\varepsilon|^3)) = L_{kl}^0 u \varepsilon + G_{kl}^0 \varepsilon (1 + \Theta_{kl}^\nu \varepsilon + q_{kl} u) + H_{kl}^0 u \varepsilon + F_{kl}^0 \varepsilon^2 + (\hat{G}_{kl}^o + \hat{H}_{kl}^o u + \hat{F}_{kl}^o \varepsilon) \varepsilon^{1-\nu}.$$

Let $\nu$ be equal to 0. Then, the coefficients of $\varepsilon$ constitute the equation (20), which has a single two-periodic solution $\bar{g}_{kl}^0$.

The coefficients of $u \varepsilon$ constitute the equation $\dot{h}_{kl}^0 + \dot{h}_{kl}^0 = Z_{kl} - g_{kl}^0 q_{kl} - L_{kl}^0$, which, by Proposition 2, has a single two-periodic solution $\bar{h}_{kl}^0 (t,\varphi)$, since the right-hand side of the equation has the zero average value by the virtue of choice of constant $L_{kl}^0$ in (21).

The equation $g_{kl}^0 Z_{kl} + (R_{kl}^o \nu) = g_{kl}^0 ((\Phi_{kl}^o + q_{kl} g_{kl}^0) + f_{kl}^0 + \hat{f}_{kl}^0)$ is constituted by the coefficients of $\varepsilon^2$. Substituting into this equation $Z_{kl}$ from the previous equation we get $\dot{g}_{kl}^0 + \dot{g}_{kl}^0 = (\hat{g}_{kl}^0 + \dot{g}_{kl}^0) (L_{kl}^0 + \hat{h}_{kl}^0 + \hat{h}_{kl}^0) + (R_{kl}^o \nu) - g_{kl}^0 \Phi_{kl}^o$. The two-periodic function $\hat{f}_{kl}^0 (t,\varphi)$ can be found explicitly by the virtue of the choice of constant $\dot{g}_{kl}^0$ from (22).

Now, consider the case $\nu = 1$. Let us substitute decompositions of functions $G^1, H^1$ and $F^1$, introduced in (23) into the identity.

For $\varepsilon$, we obtain the equation $R_{kl}^o = \bar{g}_{kl}^1 + \bar{g}_{kl}^1$, which can be explicitly solved by splitting it into two equations: $\hat{g}_{kl}^1 = \hat{R}_{kl}^o$ and $\bar{g}_{kl}^1 = \hat{R}_{kl}^o$.

For $u \varepsilon$, we have $Z_{kl} = L_{kl}^1 + \hat{g}_{kl}^1 q_{kl} + \hat{h}_{kl}^1 + \hat{h}_{kl}^1$ with $L_{kl}^1 = Z_{kl} - \hat{g}_{kl}^1 q_{kl}$ from (21). We express $\hat{h}_{kl}^1, \hat{h}_{kl}^1$ from equations $\hat{h}_{kl}^1 = \hat{Z}_{kl} - \hat{g}_{kl}^1 q_{kl}$ and $\hat{h}_{kl}^1 = \hat{Z}_{kl}$ ($\hat{g}_{kl}^1 = 0$).
For $\varepsilon^2$, we have \( (\dot{g}_{kl} + \dot{g}_{kl}^1)Z_{kl} + (R_{kl})^{(\varphi)} = \ddot{g}_{kl} + \delta_{kl}^{(\varphi)}(\dot{\Phi}_{kl}^\varphi + (\dot{g}_{kl} + \dot{g}_{kl}^1)q_{kl}) + \dot{f}_{kl} + \dot{f}_{kl}^1 \). Similarly, to the case $\nu = 0$, we obtain the equation
\[
\dot{g}_{kl}L_{kl}^1 = \ddot{g}_{kl}^\nu - (R_{kl})^{(\varphi)} - \dot{g}_{kl}^1(\dot{h}_{kl}^\nu + \dot{h}_{kl}^0).
\]
This equation allows to fix the value of $\ddot{g}_{kl}^1$ introduced in (22).

After that we write out and explicitly solve the equations for $\dot{f}_{kl}^1$ and $\dot{f}_{kl}$.

5.2. **Angular averaging and Hale’s Lemma.** Let us average $\Theta_{kl}^\nu(t, \varphi)$ in system (24) performing the $T$- and $\omega^*$-periodic change of the angular variable
\[
\varphi = \psi + \Delta_{kl}^\nu(t, \psi, \varepsilon)\varepsilon,
\]
where $\Delta_{kl}^0 = \delta_{kl}^0(t, \psi)$, $\Delta_{kl}^1 = \delta_{kl}^1(\psi) + \delta_{kl}^1(t, \psi)\varepsilon$. This change transforms (24) into the system
\[
\ddot{u} = (L_{kl}^\nu u + O((|u| + \varepsilon^3))\varepsilon^\nu, \quad \ddot{\psi} = (1 + \Theta_{kl}^\nu\varepsilon + q_{kl}(\psi)u + O((|u| + \varepsilon^2))\varepsilon^\nu.
\]

It is obvious that the function $\Delta_{kl}^\nu$ from (25) is uniquely determined by the equations
\[
\delta_{kl}^0 + \delta_{kl}^0 = \Theta_{kl}^0 - \Omega_{kl}^0, \quad \delta_{kl}^1 = \tilde{\Theta}_{kl}^1, \quad \tilde{\delta}_{kl}^1 = \tilde{\Theta}_{kl}^1.
\]

It is easy to see that the change inverse to (25) can be written in the form
\[
\varphi = \psi + \Omega_{kl}^\nu(t, \varphi, \varepsilon)\varepsilon,
\]
where $\Omega_{kl}^0 = -\delta_{kl}^0(t, \varphi) + \delta_{kl}^0(t, \varphi)\delta_{kl}^1(t, \varphi)\varepsilon + O(\varepsilon^2)$, $\Omega_{kl}^1 = -\delta_{kl}^1(t, \varphi) + (\delta_{kl}^1(t, \varphi)\delta_{kl}^1(t, \varphi) - \delta_{kl}^1(t, \varphi))\varepsilon + O(\varepsilon^2)$ are $T$-periodic in $t$ and real analytic $\omega^*$-periodic in $\varphi$.

We make the scaling change
\[
u = \nu^*3/2,
\]
which transforms (26) into the system
\[
\dot{u} = (L_{kl}^\nu u + V_{kl}^\nu(t, \psi, \varepsilon)3/2)\varepsilon^\nu, \quad \dot{\psi} = (1 + \Psi_{kl}^\nu\varepsilon + \Psi_{kl}(t, \psi, \varepsilon)3/2)\varepsilon^\nu,
\]
where $V_{kl}^\nu, \Psi_{kl}^\nu$ are continuous functions of their arguments in a small neighborhood of $v$ and $\varepsilon$, which are continuously differentiable with respect to $v$ and $\psi$, $T$-periodic in $t$ and $\omega^*$-periodic in $\psi$.

Indeed, $V_{kl}^\nu(t, \psi, \varepsilon) = O((|v|3/2 + \varepsilon)^3)\varepsilon^{-3}, \Psi_{kl}^\nu(t, \psi, \varepsilon) = q_{kl}v + O((|v|3/2 + \varepsilon^2)\varepsilon^{-3}, \varepsilon)$, and the functions $O(\ldots)$ are real analytic for each $\psi$ and three times continuously differentiable in a small neighborhood of the point $v = \varepsilon = 0$. Therefore $V_{kl}^\nu, \Psi_{kl}^\nu$ are continuously differentiable with respect to $v$ at this point.

System (28) satisfies conditions of Hale’s lemmas 2.1 and 2.2 in [8], hence for all sufficiently small $\varepsilon$, it has an invariant surface of the form
\[
\nu = \Gamma_{kl}^\nu(t, \psi, \varepsilon)3/2,\]
where $\Gamma_{kl}^\nu$ is continuous continuously differentiable, $T$-periodic in $t$, and $\omega^*$-periodic in $\psi$.

5.3. **Summary of the results.** In the following statements we summarize the obtained results.

**Lemma 3.** For each admissible $b_{kl}$, for each sufficiently small $\varepsilon > 0$, system (16) has the continuous continuously differentiable, $T$-periodic in $t$, and $\omega^*$-periodic in $\varphi$ invariant surface
\[
\nu = Q_{kl}^\nu(t, \varphi, \varepsilon)3/2, \quad (\nu = 0, 1)
\]
where $Q_{kl}^\nu = G_{kl}^\nu(t, \varphi, \varepsilon) + (F_{kl}^\nu(t, \varphi, \varepsilon) + \Gamma_{kl}^\nu(t, \varphi, \varepsilon))3/2, H_{kl}^\nu(t, \varphi, \varepsilon))3/2$. 
The surface \( \Sigma_{k,l} \) is obtained by substituting the invariant surface \( \Sigma_{k,l}^0 \) into the composition of changes (23), (27) and the change inverse to (25).

**Corollary 1.** For each sufficiently small \( \varepsilon > 0 \), system (14) has the continuous continuously differentiable \( T \)-periodic in \( t \), and \( \omega^* \)-periodic in \( \varphi \) invariant surface

\[
(31) \quad r = \Upsilon_{k,l}^\varepsilon(t, \varphi, \varepsilon)
\]

where \( \Upsilon_{k,l}^\varepsilon = \alpha_{k,l}^{-1}(\varphi)(Q_{k,l}^0(t, \varphi, \varepsilon) + \beta_{k,l}(\varphi)(Q_{k,l}^\varepsilon(t, \varphi, \varepsilon))^2) \).

The surface (31) is obtained by substituting the invariant surface (30) into the change (33).

**Corollary 2.** The invariant surface (31) has the following asymptotic expansion:

\[
\Upsilon_{k,l}^0 = \alpha_{k,l}^{-1}(\varphi)(g_{k,l}^0(t, \varphi, \varepsilon) + (\tilde{g}_{k,l}^0(t, \varphi) + \Gamma_{k,l}(\varphi)g_{k,l}^0(t, \varphi, \varepsilon^2) + O(\varepsilon^3),
\]

\[
\Upsilon_{k,l}^1 = \alpha_{k,l}^{-1}(\varphi)\left((\tilde{g}_{k,l}^1(\varphi) + \hat{g}_{k,l}^1(\varphi)\varepsilon) + (\tilde{f}_{k,l}^1(\varphi) + \Gamma_{k,l}(\varphi)\hat{g}_{k,l}^1(\varphi) + \hat{g}_{k,l}^1(\varphi)\varepsilon^2) + O(\varepsilon^3).\right)
\]

**Theorem 1.** For each \( b_{k,l}^* \), for each sufficiently small \( \varepsilon > 0 \), system (1) has the continuous continuously differentiable, \( T \)-periodic in \( t \), and \( \omega^* \)-periodic in \( \varphi \) invariant surface

\[
(32) \quad x = C(\varphi) + \Upsilon_{k,l}^\varepsilon(t, \varphi, \varepsilon)(C(\varphi) - k), \quad y = S(\varphi) + \Upsilon_{k,l}^\varepsilon(t, \varphi, \varepsilon)(S(\varphi) - l).
\]

When \( \varphi = 0 \), this surface passes through a small neighborhood of the point \( (b_{k,l}^*, 0) \) and is homeomorphic to a two-dimensional torus, provided that the time is factored by the period. Here \( b_{k,l}^* \) is an admissible parameter from Definition 1, surface \( \Upsilon_{k,l}^\varepsilon \) from (31), \( (C(\varphi), S(\varphi)) \) is an \( \omega^* = \omega b_{k,l}^* \)-periodic solution of system (3) with the initial values \( C(0) = b_{k,l}^*, S(0) = l \).

The surface (32) is obtained by substituting the invariant surface (31) into the change (13).

6. The application of the obtained results.

6.1. The analysis of the analytic generating equation. Suppose that for system (1)

\[
(33) \quad X(t, x, y, 0) = \sum_{m,n=0}^{\infty} X^{(m,n)}(t)x^m y^n, \quad Y(t, x, y, 0) = \sum_{m,n=0}^{\infty} Y^{(m,n)}x^m y^n,
\]

where the power series are absolutely convergent uniformly in \( t \) in an open set \( G^0 = \{(t, x, y) | t \in \mathbb{R}, x < x_*, y < y_* \} \). Assume also that the coefficients are real, continuous, and \( T \)-periodic in \( t \) coefficients.

Then, for any value of the parameter \( b_{k,l} \) from (6), that satisfies additional conditions (11), (17) and (21), the function \( R_{k,l}^\omega \) from (18) takes the form

\[
R_{k,l}^\omega(t, \varphi) = \sum_{m,n=0}^{\infty} \left( X^{(m,n)}(t)C'(\varphi) - X^{(m,n)}(t)S'(\varphi) \right) C^m(\varphi)S^n(\varphi).
\]

Therefore, the left-hand side of generating equation (19) with \( \omega = \omega b_{k,l} \) takes the form

\[
\bar{R}_{k,l}^\omega(b_{k,l}) = \frac{1}{T \omega} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_0^\omega C^m S^n C' \, d\varphi - \int_0^\omega C^m S^n S' \, d\varphi.
\]

When \( n = 0 \), the first integral is equal to zero and, when \( m = 0 \), the second one is equal to zero.
Integrating the identity \((C^m S^{n+1})' = m C^{m-1} S^{n+1} C' + (n + 1) C^m S^n S'\) with respect to the period and collecting coefficients, we get
\[
\overline{R}_{kl}(b_{kl}) = \frac{1}{T \omega} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m + 1}{n} \overline{X}(m+1,n-1) + \overline{Y}(m,n) \right) I_{kl}^{m, n},
\]
where \(I_{kl}^{m, n} = \int_{0}^{\omega} C^m(\varphi) S^n(\varphi) C'(\varphi) \, d\varphi\). Notice, that for each of the three classes, this formula can be simplified as follows:

\[
(34) \quad \overline{R}_{00}(b_{00}) = \frac{4}{T \omega b_{00}} \sum_{m,n=0}^{\infty} P(2m+1,2n+1) J_{00}^{m, n},
\]
\[
\overline{R}_{k0}(b_{00}) = \frac{2}{T \omega b_{00}} \sum_{m,n=0}^{\infty} P(m,2n+1) J_{k0}^{m, n}, \overline{R}_{kl}(b_{kl}) = \frac{1}{T \omega b_{kl}} \sum_{m,n=0}^{\infty} P(m,n+1) J_{kl}^{m, n} \quad (k, l = \pm 1),
\]
where \(P(m,n) = m \frac{1 + 1}{n} \overline{X}(m+1,n-1) + \overline{Y}(m,n)\) end \(J_{00}^{m, n} = \left\{ \int_{0}^{r_{1c}} \eta^{2m}(S^+(\eta^2))^{2n+1} d\eta + \int_{r_{0c}}^{b_{0c}} \eta^{2m}(S^-(\eta^2))^{2n+1} d\eta \right\} \quad \text{for} \ b_{0c} \in (r_e, r_M), \int_{0}^{0} \eta^{2m}(S^-) \eta^{2n+1} d\eta \quad \text{for} \ b_{00} \in (0, r_i) \right\},
\]
\[
k_{k0}^{m, n} = \left\{ \int_{b_{0c}}^{kr_{0c}} \eta^m(S^+(\eta^2))^{2n+1} d\eta + \int_{kr_{1c}}^{b_{0c}} \eta^m(S^-(\eta^2))^{2n+1} d\eta + \int_{kr_{1c}}^{b_{0c}} \eta^m(S^-(\eta^2))^{2n+1} d\eta, \quad k, l = 0, 1.
\]

For example, let us deduce the first formula from (34) taking into account that the constants are introduced in (5).

For the class 0, when \(b_{00} \in (r_e, r_M)\), the movement along a closed orbit is clockwise and the function \(S^2(\varphi) - 1\) changes its sign in each quadrant when \(|C(\varphi)| = r_{0c}^1\). Therefore, \(I_{00}^{m, n} = 4I_e\), when \(m\) is even and \(n\) is odd. In the other cases \(I_{00}^{m, n} = 0\).

For the class 0, when \(b_{00} \in (0, r_i)\), the movement along a closed orbit is counterclockwise. Therefore,
\[
I_{00}^{m, n} = I_i + \int_{0}^{b_{00}} \eta^m(S^-(\eta^2))^{n} d\eta + \int_{0}^{b_{00}} \eta^m(S^-(\eta^2))^{n} d\eta + \int_{0}^{b_{00}} \eta^m(S^-(\eta^2))^{n} d\eta,
\]
where \(I_i = \int_{0}^{0} \eta^m(S^-(\eta^2))^{n} d\eta\). Then, \(I_{00}^{m, n} = 4I_i\), if \(m\) is even and \(n\) is odd. In the other cases \(I_{00}^{m, n} = 0\).
6.2. The practical application of the obtained results. As an example, we investigate the $T$-periodic in $t$ system (32) with $\gamma = 0.5$, $M_x, M_y \geq 2$. The functions $X(t, x, y, 0)$ and $Y(t, x, y, 0)$ take the form (33) and

$$\forall q, s \in \mathbb{Z}_+: \quad \begin{array}{l}
X^{(q,s)} = 0, \\
Y^{(q,s)} = 0,
\end{array}
$$

except $Y^{(0,1)} = 4.57$, $Y^{(0,3)} = -1.66$, $Y^{(1,1)} = -0.855$, $Y^{(0,2)} = 0.513$.

By (5) for this system $r_i = 2^{-1/4}(\sqrt{2} - 1) \approx 0.348$, $r_e = r_v = 2^{-1/4}(\sqrt{2} + 1) \approx 1.306$; according to (6) $r_M = \sqrt{3} > 1.732$ (since the constant is from Remark 1 $M = 4.5$). Therefore, by (5) for the class 0 $b_{00}^i \in (0, 0.348)$, $b_{00}^e \in (1.306, 1.732)$, for class 1, $kb_{k0} \in (1.07, 1.306)$. Moreover, the lower bound is given taking into account conditions (11), for the class 2 $kb_{kl} \in (1, 1.306)$ (here $k, l \in \{-1, 1\}$).

In turn, formulas (34) take the form:

$$T \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} = 4(y^{(0,1)}J^{00}_{k0} + y^{(0,3)}J^{01}_{k0}),$$

$$T \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} = 2(y^{(0,1)}J^{00}_{k0} + y^{(0,3)}J^{01}_{k0} + y^{(1,1)}J^{11}_{k0}),$$

$$T \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} = y^{(0,1)}J^{00}_{k0} + y^{(0,3)}J^{01}_{k0} + y^{(1,1)}J^{11}_{k0} + y^{(0,2)}J^{01}_{k0}.$$

Now, for each class we change the initial value of $b$ in the given limits by the step $10^{-2}$, calculate the values of the function $T_\omega b_{kl} \frac{P^{(0)}_{k0}}{(b_{00})}$ $(k, l \in \{0, \pm 1\})$ at each step value and observe its sign. In case we find intervals with endpoints of a different sign, it is meaningful to reiterate the process but with a smaller step value.

For the given system we obtain:

$$T \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} > 10^{-5}, \quad T \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} < -10^{-5} \text{ with } b_{00}^{<} = 1.5, \quad b_{00}^{>} = 1.501;$$

$$kT \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} < -10^{-5}, \quad kT \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} > 10^{-5} \text{ for } b_{k0}^{<} = 1.2, \quad b_{k0}^{>} = 1.202;$$

$$kT \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} < -10^{-5}, \quad kT \omega_{b_{k0}} \frac{P^{(0)}_{k0}}{(b_{00})} > 10^{-5} \text{ for } b_{kl}^{<} = 1.2, \quad b_{kl}^{>} = 1.202.$$

Therefore, we find seven values of the parameter $b$: $b_{00}^{0} \in (1.5, 1.501), \quad kb_{k0}^{0}, \quad kb_{k1}^{0}, \quad kb_{k2}^{0} \in (1.2, 1.202)$ $(k = \pm 1)$. These values are the zeros of generating equation (19) and belong to the corresponding intervals from (6) and (11). By Definition 1 the mentioned values are admissible if the Siegel and dissipativity conditions hold.

In such case by Theorem 1 under conditions (35), for each sufficiently small $\varepsilon > 0$ system (1) has seven two-dimensional invariant surfaces (32). When $t = 0$, four of them generate closed orbits of class 2], which enclose one of singular points $(1, \pm 1), (-1, \pm 1)$ of the unperturbed system [2]; another two surfaces are closed orbits of class 1], which enclose one of separatrix ”eight” and three singular points: $(\pm 1, 1), (\pm 1, 0), (\pm 1, -1)$. Finally, the seventh surface is of the class $0^e$ and encloses all nine singular points.

We have not found the eighth invariant surface generated by a cycle of the class $0^e$ which encloses only the point $(0, 0)$; since for each $b_{00} \in (0, 0.348)$ we have $\frac{P^{(0)}_{k0}}{(b_{00})} < 0$.

Calculations, that prove that all seven constants $L$ from dissipativity condition (21) are not equal to zero, are not provided due to their cumbersome nature. To perform these calculations for each initial value $b_{00}$, we interpolated solutions of the initial value problems $C(\phi), S(\phi)$ for system (3) with the fifth degree polynomials depending on $\phi$.

Now, the only thing left to do is to notice that the Siegel condition, required for the case $\nu = 0$, holds for almost all vectors with integer components with respect to the Lebesgue measure.
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