Tensor ghosts in the inflationary cosmology

Tim Clunan\textsuperscript{1,2} and Misao Sasaki\textsuperscript{1,2}

\textsuperscript{1} Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge, CB3 0WA, UK
\textsuperscript{2} Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

E-mail: T.P.Clunan@damtp.cam.ac.uk and Misao@yukawa.kyoto-u.ac.jp

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Abstract

Theories with curvature-squared terms in the action are known to contain ghost modes in general. However, if we regard curvature-squared terms as quantum corrections to the original theory, the emergence of ghosts may be simply due to the perturbative truncation of a full non-perturbative theory. If this is the case, there should be a way to live with ghosts. In this paper, we take the Euclidean path integral approach, in which ghost degrees of freedom can be, and are integrated out in the Euclideanized spacetime. We apply this procedure to Einstein gravity with a Weyl curvature-squared correction in the inflationary background. We find that the amplitude of tensor perturbations is modified by a term of \( O(\alpha^2 H^2) \) where \( \alpha^2 \) is a coupling constant in front of the Weyl-squared term and \( H \) is the Hubble parameter during inflation.

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1. Introduction

Previously in the literature corrections to the Einstein–Hilbert action formed with terms of higher order in the Riemann tensor have been considered. In the past these have been considered in connection with the possibility of a non-local theory [1] and also renormalization [2–4]. These higher derivative theories do, however, suffer from the problem of ghosts. There are, of course, ‘good’ ghosts, such as the Faddeev–Popov ghost but higher derivative ghosts are ‘bad’ ghosts. Even if one adds ‘small’ higher derivative terms, so that the Lagrangian is almost the same as the second-order theory, the resulting theory is not well-described perturbatively since there are now extra degrees of freedom. These bring with them a number of problems [1, 5–7]. They are negative energy particles, with energies which blow-up in the limit where the higher derivative terms are removed from the Lagrangian. Ghosts also form states of negative norm and hence lead to non-unitarity.
The simple harmonic oscillator can be made to mimic the effect of these terms by adding higher derivative terms to its Lagrangian. This was considered by Hawking and Hertog [6]; they showed that despite these problems it is possible to obtain a sensible probability distribution for observations of the field. This result relied on the Euclidean path integral formulation of quantum theory. This, of course, echoes the no-boundary cosmology of Hartle and Hawking [8]. Other authors have considered the effect of extra terms in the Lagrangian. In the cosmological case the full theory is not taken seriously, and instead backsubstitution [5, 9] is typically used. Here we run contrary to this trend by extending and comparing the result of Hawking and Hertog to the cosmological case where the Lagrangian contains a Weyl tensor-squared term.

It is normally the case for ghosts that even if one could deal with the non-interacting theory, the interacting theory would have the problem of run-away production of positive energy and negative energy particles. This would require further work, but is not expected to be a problem in the Euclidean approach where a late-time boundary condition is specified [6, 10].

The very early Universe provides us with the ultimate high-energy physics experiment. Post-inflation the energies are sufficiently low that one could not rule out the occurrence of higher order curvature terms in the Lagrangian; however, during inflation these terms could be of importance. One could consider a whole series of terms of arbitrary order in the Riemann tensor, but it makes sense to start with just second order.

Having accepted that there is a possibility of higher curvature terms, the question is how one should deal with them. One can handle them simply by backsubstituting the modes from the usual equations of motion. This approach has been used in [5]. Here one finds a momentum cut-off much higher than the Plank energy.

On the other hand [11] argues that by using a perturbative approach, rather than solving the full system, one can miss essential physical features. For example, in the case of the heat equation initial data of compact support lead to a perturbative solution with the same support; however, the actual solution clearly spreads.

There is something special about the case where the only additional terms we consider are curvature-squared terms. This theory has better renormalization properties [12]. We deal with the slow roll case in four dimensions and therefore restrict our attention to the Weyl tensor-squared terms.

The structure of this paper is as follows. In section 2 we review the simple harmonic oscillator. In section 3 we find the action. In section 4 we solve the equation of motion (EOM). In section 5 we find the wavefunction. In section 6 we find the two-point function, and in section 7 we draw our conclusions.

2. The higher derivative simple harmonic oscillator

In [6] the authors consider a higher derivative simple harmonic oscillator, the extra terms being set up to emulate the situation in gravity. This has the advantage of fewer terms in the Lagrangian and doing away with issues to do with gauge, thus providing clarity in the exposition of the method. One can expect that the method works in the case of gravity, and if one is not interested in the detailed form of the resulting observables, it is more than adequate. However, as future experiments advance one may expect to be more interested in these details, which an ersatz theory cannot provide. Wishing to examine the form of the result more closely in gravity, the natural place to start is the tensor sector as it is already gauge invariant.

Hawking and Hertog find that despite the problems presented by typical higher derivative theories it is possible to take the full theory seriously (not removing any modes) and arrive at
a sensible probability distribution. Key to this method is the fact that they consider the field as a Euclidean field.

The action for a Euclidean higher derivative simple harmonic oscillator is

\[ I = \int \mathrm{d} \tau \left( \frac{\alpha^2}{2} \phi_{\tau\tau}^2 + \frac{1}{2} \phi_{\tau}^2 + \frac{1}{2} m^2 \phi^2 \right), \]

(1)

and, although this theory does not contain any interaction terms, the authors argue that these would not cause further problems in the case where we prescribe the field value on a late-time surface. For real \( \phi \) and real \( \tau \) this action is clearly positive semi-definite and hence results in a convergent path integral. The resulting EOM

\[ \alpha^2 \phi_{\tau\tau\tau\tau} - \phi_{\tau\tau} + m^2 \phi = 0 \]

(2)

has a general solution which is easily seen to be

\[ \phi(\tau) = A \sinh(\lambda_+ \tau) + B \cosh(\lambda_+ \tau) + C \sinh(\lambda_- \tau) + D \cosh(\lambda_- \tau), \]

(3)

where

\[ \lambda_{\pm} = \frac{1}{\sqrt{2\alpha}} \sqrt{1 \pm \sqrt{1 - 4m^2\alpha^2}} \approx \frac{m}{\alpha} \]

(4)

for small alpha (which is our area of interest since this results in a theory which has ‘small’ higher derivative terms). The ground state wavefunction \( \Psi_{0}\tau = 0 \) is defined in a similar way, but integration goes from \( \tau = 0 \) to \( \infty \); it is the analogue of complex conjugating the Lorentzian wavefunction. The associated probability is thus

\[ P(\phi_0, \phi_{0,\tau}) = N \exp \left( -2F \left( \phi_0^2 + \frac{m}{\alpha} \phi_{0,\tau}^2 \right) \right), \]

(5)

where

\[ F = \frac{1 - 4m^2\alpha^2}{2\alpha^2(\lambda_+ + \lambda_-)(\lambda_- - \lambda_+)^2}. \]

(6)

The Euclidean conjugate ground state wavefunction \( \Psi^* \) is defined in a similar way, but integration goes from \( \tau = 0 \) to \( \infty \); it is the analogue of complex conjugating the Lorentzian wavefunction. The associated probability is thus

\[ P(\phi_0, \phi_{0,\tau}) = \Psi^* \Psi = N^2 \exp \left( -2F \left( \phi_0^2 + \frac{m}{\alpha} \phi_{0,\tau}^2 \right) \right). \]

(7)

and we see that we may integrate over the unobserved \( \phi_{0,\tau} \). If we had been working with \( \phi_{0,\tau} \), where \( \tau \) is the Lorentzian time, the sign in front of the \( \phi_{0,\tau}^2 \) term would have been wrong and integration not possible: the possibility is afforded by working in the Euclidean formalism. This results in a probability

\[ P(\phi_0) = \sqrt{\frac{2Fm}{\pi \alpha}} \exp \left( -\frac{2mF}{\alpha} \phi_0^2 \right) \]

(8)

\[ \approx \sqrt{\frac{m}{\pi}} \left( 1 + \frac{m\alpha}{2} \right) \exp \left( -m(1 + m\alpha)\phi_0^2 \right), \]

(9)

the approximation holding for small \( \alpha \). So we see that the second-order theory is corrected by terms in \( \alpha \), rather than \( \alpha^2 \) as we would see had we backsubstituted the modes from the second-order theory into the fourth-order theory.
3. Preliminaries

We are interested in actions with terms up to second order in the curvature tensor. Since we are working in four dimensions the Gauss–Bonnet term is a total derivative and so we may replace \( R_{\mu\nu} R^{\mu\nu} \) with a combination of \( R^2 \) and \( C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \). The \( R^2 \) term leads to an additional massive scalar field which does not cause any problems\(^3\). We are interested in the negative-energy ghost component, which Boulware [13] has shown to be contained in the tensor part of the Weyl-squared term. Thus, we focus on this.

The York–Gibbons–Hawking boundary term \( B_{\text{YGH}} \) is usually added to the Einstein–Hilbert action in order to make the variational problem (where one chooses the metric on a spacelike surface) well posed [14, 15]. This comes down to the fact that if we specify ‘\( q \)’ at initial and final times we do not also wish to specify ‘\( \dot{q} \)’. In the context of the higher derivative theory we consider here that we will specify ‘\( q \)’ and ‘\( \dot{q} \)’, so at first glance it might seem we do not need \( B_{\text{YGH}} \); however, since this term also makes the Euclidean action for the inhomogeneous modes positive definite [16] it transpires that we will need it.

At zeroth order in slow roll the spacetime during inflation reduces to de Sitter space. We use the metric

\[
d s^2 = e^{2\rho} (-d\eta^2 + (\delta_{ij} + \gamma_{ij})\,dx^i\,dx^j),
\]

where \( \gamma_{ij} \) is a transverse traceless perturbation and we can conveniently take \( e^{2\rho} \) to be 1 or \( 1/(H\eta)^2 \) according to whether we wish to discuss flat or de Sitter space. From Boulware [13] we have

\[
C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} = 8C^{0k0l} C_{0k0l} + 4C^{0kln} C_{0kln}.
\]

This leads to

\[
C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} = \frac{1}{2} e^{-4\rho} ((\gamma_{ij}'' + \gamma_{ij,nn})(\gamma_{ij}'' + \gamma_{ij,mm}) - 2(\gamma_{ij,k} - \gamma_{ik,j})(\gamma_{ij,k} - \gamma_{ik,j}))
\]

\[
= \frac{1}{2} e^{-4\rho}(\gamma_{ij}'' \gamma_{ij}'' + 2\gamma_{ij,nm} \gamma_{ij}'' + 4\gamma_{ij}'' \gamma_{ij,nn} + \gamma_{ij,nn} \gamma_{ij,mm}),
\]

at second order, where spatial boundary terms have been dropped\(^4\). Here and in what follows the repeated spatial indices \( i, j, k, \ldots \) are to be summed over. Since the metric is conformally flat the Weyl tensor is vanishing at zeroth order and hence it is only necessary to calculate it to first order if one is finding the \( C^2 \) term to second order.

Considerations so far lead us to start with the action

\[
S = \int \sqrt{g} \, d^4x \left( \frac{1}{2} R - \Lambda - a^2 C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} \right) + B_{\text{YGH}},
\]

where

\[
B_{\text{YGH}} = \int d^3x \sqrt{h} K;
\]

however, we will need an additional boundary term relating to the \( C^2 \) term.

\(^3\) One might expect a problem with the \( R^2 \) term; however, the only metric component having a non-degenerate higher derivative term due to this is the gravitational potential energy in the usual theory. The extra modes due to the higher derivatives therefore have positive energy.

\(^4\) We are only ever concerned with the fields on future and past boundaries of a spacetime volume in the definition and propagation of the wavefunction.
3.1. Action for flat space

If \( \Lambda = 0 \), the Minkowski spacetime is a solution of the theory. On this background action (14) becomes

\[
S = \int d\eta d^3x \left[ \frac{1}{8} (\gamma''_{ij}\gamma''_{ij} - \gamma_{ij,k}\gamma_{ij,k}) - \frac{\alpha^2}{2} (\gamma''_{ij}\gamma''_{ij} + 2\gamma_{ij,nn}\gamma''_{ij} - 4\gamma_{ij,k}\gamma_{ij,k} + \gamma_{ij,nn}\gamma_{ij,mm}) \right]
\]  
(16)

at second order in the gravitational wave perturbation. We have dropped spatial boundary terms as these vanish for localized fields. We only consider the transverse traceless perturbations, and at second order these decouple from the scalar and vector perturbations. This gives us the ghost content of the theory. Considering now \( iS \) (the argument of the path integral defining the wavefunction) rotated into the Euclidean with \( \tau = i\eta \) being the Euclidean ‘time’ we see that the only term preventing the Euclidean action \( I = -iS \) being positive definite on real Euclidean fields is

\[
-\int d\eta d^3x \alpha^2 \gamma_{ij,nn}\gamma''_{ij}.
\]  
(17)

This leads us to the addition of a boundary term on the initial and final boundary surfaces. In this way we can do an integration by parts in the time coordinate and have a positive definite Euclidean action, so the path integral will be well defined. This boundary term is the same in the de Sitter case.

3.2. Action for de Sitter space

The action in the case of metric (10) is

\[
S = \int d\eta d^3x \left[ -3 e^{2\rho} \rho'' - e^{4\rho} \Lambda + \frac{1}{8} e^{2\rho} (\gamma''_{ij}\gamma''_{ij} - \gamma_{ij,k}\gamma_{ij,k}) \right.
\]

\[
-\frac{\alpha^2}{2} (\gamma''_{ij}\gamma''_{ij} + 2\gamma_{ij,nn}\gamma''_{ij} - 4\gamma_{ij,k}\gamma_{ij,k} + \gamma_{ij,nn}\gamma_{ij,mm}) \right],
\]  
(18)

where the background EOM holds. The EOM for the tensor fluctuations is thus

\[
\frac{1}{4} ((e^{2\rho} \gamma''_{ij})' - e^{2\rho} \gamma_{ij,kk}) + \alpha^2 (\gamma''_{ij} - 2\gamma_{ij,kk} + \gamma_{ij,mm}) = 0.
\]  
(19)

So expanding \( \gamma_{ij} \) as

\[
\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} e^s(k) \gamma_s^k(\eta) e^{i\vec{k} \cdot \vec{x}},
\]  
(20)

where \( e^s(k) = 0 = k' e^s(i\vec{k}) \), \( e^s(-\vec{k}) = e^s(i\vec{k}) e^s(i\vec{k}) = 2\delta_{ss} \), we find

\[
\frac{1}{4} ((e^{2\rho} \gamma''_k) + k^2 e^{2\rho} \gamma_k^2) + \alpha^2 (\gamma''_k + 2k^2 \gamma''_k + k^4 \gamma_k^2) = 0.
\]  
(21)

Classically \( \gamma_{ij} \) is taken as a real field. Then the second-order action may be written in terms of the Fourier components as

\[
S_k = \sum_{s=\pm} \int d\eta \left[ \frac{1}{4} e^{2\rho} (|\gamma'_k|^2 - k^2 |\gamma_k^2|^2) - \alpha^2 (|\gamma'_k|^2 - 2k^2 |\gamma_k^2|^2 + k^4 |\gamma_k^2|^2) \right],
\]  
(22)

which we regard as an action for the real and imaginary parts of \( \gamma_k^s \) as independent fields.
3.3. Canonical formalism

In [6] it is argued that whilst most authors considering the canonical formalism for fourth-order theories would take ‘q’ and ‘q̄’ to be the canonical ‘position’ coordinates, in the path integral formalism one should describe a state by ‘q’ and ‘q̄’ at initial and final times. With this in mind we consider \( Q_{q'r} = γ_k^{r'} \) and \( Q_{q''r} = γ_k^{r''} \), these have conjugate momenta

\[
P_{q'r} = 2\left( \frac{1}{2} e^{-\frac{e^2}{4} γ_k^{r'} + α^2 γ_k^{r''} + α^2 k^2 γ_k^{r'}} \right),
\]

and similarly for the imaginary components of the fields. Thus we have a Hamiltonian \( \mathcal{H}_k = -\alpha^2 |γ_k^{r'}|^2 + 2α^2 \Re( γ_k^{r''} γ_k^{r'}) + \frac{e^{2α^2}}{4} |γ_k^{r'}|^2 + 2α^2 k^2 |γ_k^{r'}|^2 + \frac{e^{2α^2} k^2}{4} |γ_k^{r'}|^2 + α^2 k^4 |γ_k^{r'}|^2 \)

\[
= \Re(P_{q'r} Q_{q''r}) - \frac{1}{4α^2} |P_{q''r}|^2 - \frac{e^{2α^2}}{4} |Q_{q'r}|^2 + \frac{e^{2α^2} k^2}{4} |Q_{q'r}|^2 - 2α^2 k^2 |Q_{q'r}|^2 + α^2 k^4 |Q_{q'r}|^2,
\]

where again we take the independent variables to be the real and imaginary components of each field.

Taking the classical Hamiltonian (26) one sees the ghost instability is present because the only occurrence of \( P_{q'} \) is in the \( P_{q'} Q_{q''} \) term and hence the Hamiltonian can be made arbitrarily negative by fixing \( Q_{q''} \neq 0 \) and taking \( P_{q'} \) appropriately large positive or negative. Unlike the case of a particle orbiting in a central potential\(^5\), this problem is present for a large volume of phase space, and it is this which makes the difference [17]. Quantizing the higher derivative theory will not result in a lower bound on the energy of states.

3.4. Flat space wavefunction

In flat space the EOM (21) becomes

\[
0 = \frac{1}{2} γ_k^{r''} + \frac{1}{2} k^2 γ_k^{r'} + α^2 (γ_k^{r''} + 2k^2 γ_k^{r'} + k^4 γ_k^{r'}),
\]

which is easy to solve in terms of exponentials. It factorizes to give solutions with \( k_+ = k \) and \( k_- = \sqrt{k^2 + 1/(4α^2)} \).

The Lorentzian flat space wavefunction for the mode \( γ_k^{r'} \) is

\[
\Psi_k^r(Q_{q'r}, Q_{q''r}) = N(η) \exp\left( -\alpha^2 k_+(k_+ + k_-)|Q_{q''r}|^2 \right)
- iα^2 k_- k_+(Q_{q'r}^* Q_{q'r} + Q_{q''r} Q_{q''r}) + α^2 (k_+ + k_-)|Q_{q''r}|^2),
\]

which may be obtained through the Euclidean path integral prescription, followed by rotation back to Minkowski time, and satisfies the Wheeler–de-Witt equation,

\[
\mathcal{H}_k^r \Psi_k^r = i \partial_η \Psi_k^r
\]

\(^5\) Of course, it is also the case that the classical hydrogen atom Hamiltonian can be made arbitrarily negative by taking the electron to have small momentum and be close enough to the nucleus. This problem with a small region of phase space suggests that the hydrogen atom has states of arbitrarily negative energy. Heisenberg’s uncertainty principle stops a state being localized on this problematic region, and so there is a ground state.
with
\[ \mathcal{H}_k^\gamma = \Re(P_{ij}^* Q_{ij}^\gamma) - \frac{1}{4 \alpha^2} |P_{ij}^\gamma|^2 - \frac{1}{4} |Q_{ij}^\gamma|^2 + \frac{k^2}{4} |Q_{ij}^\gamma|^2 - 2\alpha^2 k^2 |Q_{ij}^\gamma|^2 + \alpha^2 k^4 |Q_{ij}^\gamma|^2, \]
(30)
where, as usual, \( P = -i \partial Q \). With the boundary term chosen to make the path integral defining the wavefunction well defined we will see that this is also true in the de Sitter case.

4. Solving the equation in the de Sitter case

The EOM (21) for \( \gamma_k^\gamma \) with \( e^{2\rho} = \frac{H}{\beta} \) has the two convenient factorizations:
\[ 0 = \left( \frac{d^2}{dz^2} + 2 \frac{d}{z} + 1 - \frac{1}{4\beta z^2} \right) \left( \frac{d^2}{dz^2} - \frac{2}{z} \frac{d}{dz} + 1 \right) \gamma_k^\gamma, \]
(31)
\[ 0 = \left( \frac{1}{z^2} \frac{d^2}{dz^2} - \frac{2}{z^3} \frac{d}{dz} + \frac{1}{z^2} \right) \left( z^2 \frac{d^2}{dz^2} - 2z \frac{d}{dz} + 2 + \frac{1}{4\beta} \right) \gamma_k^\gamma, \]
(32)
where \( z = -k \eta \) and \( \beta = -H^2 \alpha^2 \). Hence, the solutions vanishing in the upper half \( \eta \) plane are
\[ (1 + iz) e^{-iz}, \]
(33)
\[ z^{3/2} (J_{\frac{1}{2} \sqrt{4\beta z}}(z) - i Y_{\frac{1}{2} \sqrt{4\beta z}}(z)); \]
(34)
these each also solve the first factor in each of (31) and (32). Whilst mode (33) is conveniently the same as in the Einstein–Hilbert case, we will see that it differs in normalization. One can also see that this mode is an eigenfunction of the first factor of (32) with non-zero eigenvalue and thus obviously solves (32). One can see from the second-order equations these satisfy that they both also obey an equation of the form
\[ \frac{1}{z^2} f^* \frac{d}{dz} f - f \frac{d}{dz} f^* = \text{const.} \]
(35)
This and similar equations are useful in the normalization of the modes.

4.1. Normalizing the modes

With the canonical ‘position’ coordinates \( Q_{ij} \) and \( Q_{ij}' \) the conjugate momenta are
\[ P_{ij} = \frac{1}{4H^2 \eta^2} \gamma_{ij}' + \alpha^2 \gamma_{ij}''' - 2\alpha^2 \gamma_{ij,kk}', \]
(36)
\[ P_{ij}' = -\alpha^2 \gamma_{ij}'. \]
(37)
For the position space Hamiltonian these give rise to correctly generate the canonical Hamiltonian evolution equations, and thus these put the Poisson brackets in the canonical form. The operator versions of these satisfy the equal time commutation relations
\[ [Q_{ij}(\eta, \bar{x}), P_{ij}(\eta, \bar{y})] = 2\delta^{(3)}(\bar{x} - \bar{y}), \]
(38)
\[ [Q_{ij}'(\eta, \bar{x}), P_{ij}'(\eta, \bar{y})] = 2\delta^{(3)}(\bar{x} - \bar{y}), \]
(39)
where the factor of 2 occurs because summing over \(i\), \(j\) sums over the ‘plus’ and ‘cross’ modes. If we use

\[
\gamma'_k = u_k a_k^+ + u_k^* a_k^- + v_k b_k^+ + v_k^* b_k^-
\]

in (20) the normalized modes

\[
u_k = \frac{H}{\sqrt{k^3(1 - 8\beta)}} \left(1 - ik\eta\right) e^{ik\eta},
\]

\[
v_k = \frac{H}{\sqrt{k^3(1 - 8\beta)}} \sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{4} \sqrt{1 + \frac{1}{\beta} \left(-k\eta\right)^2}}
\]

result in

\[
\left[a_k^+, a_k^-ight] = (2\pi)^3 \delta_{ij} \delta^{(3)}(\vec{k} - \vec{k}),
\]

\[
\left[b_k^+, b_k^-ight] = -(2\pi)^3 \delta_{ij} \delta^{(3)}(\vec{k} - \vec{k}).
\]

Thus, the states created by \(b_k^\dagger\) are of negative norm, as expected for ghosts.

5. The wavefunction for a de Sitter background

In the case of de Sitter space the definition of the wavefunction in the Euclidean path integral formalism would normally be done in global coordinates with a foliation by equal time slices which are copies of \(S^3\). This is the Hartle–Hawking wavefunction [8]; however, there is a simplification in our case. We know [18] that for wavelengths passing through the horizon sufficiently late\(^6\), it suffices to use the coordinates of (10) which cover only half the spacetime, resulting in a wavefunction similar to, and extending, that in [19]. So, the observables closely approximate those found in the calculation with \(S^3\) hypersurfaces, and for all but the \(\ell \lesssim 20\) on the \(S^2\) of last scattering there is no loss in not doing the calculation in global coordinates. At late times the curvature terms do not affect the background equations of motion and we may use the coordinates with hypersurfaces which are copies of \(\mathbb{R}^3\); this is where we use the Euclidean formalism to integrate out over the unobserved variable \(Q_{\gamma'}\) before rotating to Lorentzian time. The Wheeler–de-Witt equation, being based on a Hamiltonian formulation, is Lorentzian. It is generally convenient to give our expressions in a Lorentzian form and point out the changes in the Euclidean form where relevant.

This wavefunction describing fluctuations about a de Sitter background may be found by assessing the action on a solution of the EOM. which has prescribed the values of \(Q_{\gamma}, Q_{\gamma'}\) (on a late-time surface at \(\eta_0\)) and vanishes in the upper half \(\eta\) plane. This last condition restricts us to the modes in (33) and (34); thus, the \(\gamma'_k\) in (20) is

\[
\gamma'_k = \frac{\left(Q_{\gamma'}(u(\eta_0)v(\eta) - v(\eta_0)u(\eta)) - Q_{\gamma'}(u'(\eta_0)v(\eta) - v'(\eta_0)u(\eta))\right)}{u(\eta_0)v'(\eta_0) - v(\eta_0)u'(\eta_0)}.
\]

From this it is clear that any normalization of the modes will cancel out and thus does not concern us here. Reality of the field and its time derivative at \(\eta_0\) require \(Q_{\gamma'} = (Q_{\gamma'})^\dagger\) and \(Q_{\gamma''} = (Q_{\gamma''})^\dagger\). We assess the action on a solution of the EOM and hence we deal with a boundary term on the future boundary.\(^7\)

\(^6\) That is, sufficiently late so that they do not see the curvature of the Universe at horizon exit.

\(^7\) It is noted in [20] that the pre-exponential term in the wavefunction in the quadratic case is independent of the canonical coordinates; this is borne out by our result.
\[ t S |_{\eta_0} = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{k^3}{H^2} \left[ -\frac{1}{4z^2} \gamma_1^s \gamma_2^s + \beta \left( \gamma_3^s \gamma_2^s \gamma_1^s \gamma_4^s - \gamma_3^s \gamma_2^s \gamma_1^s \gamma_4^s + 2\gamma_3^s \gamma_2^s \right) \right] |_{\eta_0} \]  

(46)

\[ = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{k^3}{H^2} \left[ |Q_{\gamma s}^i| \bar{A} - \frac{1}{2k} (Q_{\gamma s}^i Q_{\gamma s}^* + Q_{\gamma s}^* Q_{\gamma s}^i) B + \frac{1}{k^2} |Q_{\gamma s}^i|^2 \bar{C} \right], \]  

(47)

where the coefficients

\[ \bar{A} = \beta \left( \frac{u_{zz}v_z - u_{zz}u_z}{uv_z - vu_z} \right) \mid_{\eta_0}, \]  

(48)

\[ \bar{B} = \left( -\frac{1}{4z^2} + \beta \frac{uv_{zz} - vu_{zz} + 2uv_z - 2uv_z + u_z v_{zz} - v_z u_{zz}}{uv_z - vu_z} \right) \mid_{\eta_0}, \]  

(49)

\[ \bar{C} = \beta \left( \frac{uv_{zz} - vu_{zz}}{uv_z - vu_z} \right) \mid_{\eta_0}, \]  

(50)

are obtained by the insertion of (45) into (46). This leaves us with a wavefunction

\[ \Psi(Q_{\gamma s}^i, Q_{\gamma s}^i) = N(\eta_0) \exp\left(i S |_{\eta_0}\right) \]

\[ = N(\eta_0) \exp\left[ i \frac{k^3}{H^2} \left( |Q_{\gamma s}^i|^2 \bar{A} - \frac{1}{2k} (Q_{\gamma s}^i Q_{\gamma s}^* + Q_{\gamma s}^* Q_{\gamma s}^i) B + \frac{1}{k^2} |Q_{\gamma s}^i|^2 \bar{C} \right) \right]. \]

(51)

The Wheeler–de-Witt equation, constructed from (26), \( \hat{\mathcal{H}} \Psi = i \partial_{\eta} \Psi \) and \( P = -i \partial_Q \), is then equivalent to

\[ \frac{1}{4z^2} + \frac{B^2}{4\beta} - \beta = \bar{A}, \]  

(52)

\[ -2\bar{A} + \frac{B\bar{C}}{\beta} = \bar{B}, \]  

(53)

\[ -\bar{B} - \frac{1}{4z^2} + \frac{C^2}{\beta} + 2\beta = \bar{C}, \]  

(54)

which are satisfied by (48), (49) and (50).

5.1. The small \( \beta \) approximation

We need to calculate \( \bar{A}, \bar{B}, \bar{C} \) of (48), (49) and (50), though little useful progress can be made employing the full form of the mode (42). This is facilitated by only considering the physical case where \( \beta \approx 0 \) (since \( H \approx 0 \) and one can expect \( \alpha \) is of order 1).

Once derivatives are neglected we are only dealing with expressions which are homogeneous of degree zero in \( u \) and \( v \), so we can drop overall factors and use \( u \) and \( v \) in the form (33) and (34). This can be further simplified by using

\[ H^{(2)}_\lambda(z) = \frac{J_{-\lambda}(z) - e^{\pi \lambda} J_{\lambda}(z)}{-i \sin(\lambda \pi)}, \]  

(55)

where \( \lambda = \sqrt{1+1/\beta}/2 \), and dropping the factor of \( i/\sin(\lambda \pi) \). With real \( z \) and imaginary \( \lambda \), the complex conjugate of \( J_{-\lambda}(z) \) is \( J_{\lambda}(z) \). Since \( \beta \) is negative the \( e^{\pi \lambda} \) suppresses the second
term in (55) more than any power of $\beta$. The late-time (small $z$) expansion for the Bessel functions
\[ J_\lambda(z) = \frac{z^\lambda}{2^{\lambda}\Gamma(1+\lambda)} \left(1 - \frac{z^2}{4(1+\lambda)} + \frac{z^4}{32(1+\lambda)(2+\lambda)} + \cdots\right) \] (56)
shows us that we can drop another factor, and we are left with the important part of the ghost mode being
\[ z^{1/2} \left(1 - \frac{z^2}{4(1-\lambda)} + \frac{z^4}{32(1-\lambda)(2-\lambda)} + \cdots\right). \] (57)
The first few terms of the resulting late time, small $\beta$ series are
\[
\bar{A} = -\frac{1}{4z} + \frac{i}{4}(1 - 2\beta) + \frac{1}{4} \left(1 + 2\beta \left(-1 + \sqrt{1 + \frac{1}{\beta}} \right)\right) z + \cdots, \] (58)
\[
\bar{B} = -\beta \left(1 + \sqrt{1 + \frac{1}{\beta}} \right) + \frac{i\beta}{2} \left(3 + \sqrt{1 + \frac{1}{\beta}} \right) z + \cdots, \] (59)
\[
\bar{C} = \frac{\sqrt{1 + \frac{1}{\beta}} - 1}{2z} \beta \left(3 + \sqrt{1 + \frac{1}{\beta}} \right) z + \cdots. \] (60)

6. Observables from tensor perturbations

6.1. The usual gravity

Here we make a few notes about gravity where the Lagrangian consists only of the Einstein–Hilbert term. From equation (18) the second-order part of the action is
\[
S = \int d\eta d^3x \frac{1}{8} e^{2\rho} (\gamma'_{ij} \gamma'_{ij} - \gamma_{ij,\lambda} \gamma_{ij,\lambda}), \] (61)
with Fourier components as in equation (22) given by
\[
S_{\vec{k}} = \sum_{s=\pm} \int d\eta \frac{1}{4} e^{2\rho} (|\gamma'_{s\vec{k}}|^2 - k^2 |\gamma'_{s\vec{k}}|^2). \] (62)
In this case, of course, we have canonical coordinates $Q_{\gamma'_{s\vec{k}}} = \gamma'_{s\vec{k}}$ and $P_{\gamma'_{s\vec{k}}} = \frac{1}{2} e^{2\rho} \gamma'_{s\vec{k}}$, and the Hamiltonian is given by
\[
\mathcal{H}_{\vec{k}} = \frac{e^{2\rho}}{4} |\gamma'_{\vec{k}}|^2 + \frac{e^{2\rho} k^2}{4} |\gamma'_{\vec{k}}|^2 = e^{-2\rho} |P_{\gamma'_{s\vec{k}}}|^2 + \frac{e^{2\rho} k^2}{4} |Q_{\gamma'_{s\vec{k}}}|^2. \] (63)
The wavefunction is constructed in section 5, where equation (45) takes the simplified form
\[
\gamma'_{s\vec{k}} = \frac{Q_{\gamma'_{s\vec{k}}} u(\eta)}{u(\eta_0)}, \] (64)
and is plugged into a simplified form of equation (46),
\[
\imath S_{\eta_0} = -\frac{1}{4} \sum_{s=\pm} \int d^3k \frac{k^3}{(2\pi)^3} 4H^2 z^2 \mathcal{H}_{\vec{k}} \gamma'_{s\vec{k}}. \] (65)
to give a wavefunction
\[ \Psi_{\gamma}^{\prime}(Q_{\gamma}^{\prime}) = N \exp \left( -\frac{k^3}{4H^2} \left( \frac{1}{1+z^2} + \frac{t}{z(1+z^2)} \right) |Q_{\gamma}^{\prime}|^2 \right). \]  
(66)
From this we see that the probability distribution for \( Q_{\gamma}^{\prime} \),
\[ P(Q_{\gamma}^{\prime}) = |\Psi_{\gamma}^{\prime}|^2 = |N|^2 \exp \left( -\frac{k^3}{2H^2(1+z^2)} |Q_{\gamma}^{\prime}|^2 \right), \]  
(67)
freezes out at late times (\( z \rightarrow 0 \)); hence, observables involving \( Q_{\gamma}^{\prime} \) all freeze out. We note that in order for the various correlation functions of \( \gamma \) to freeze out only the real component of the argument of wavefunction (66) needs to freeze out. This is also observed in [19]. We will see that a similar situation arises for the two-point function when we include the Weyl-squared term\(^8\). We also note that the above result is valid for a quasi-de Sitter background as long as \( |H|/H^2 \ll 1 \), provided that \( H \) in the above formula is regarded as the value of the Hubble parameter at the time of horizon crossing, \( z = -kH = 1 \).

6.2. With the Weyl-squared term

The probability distribution the wavefunction of equation (51) leads to is
\[ P(Q_{\gamma}^{\prime}, Q_{\gamma}^{\prime}) = N(\eta_0)^2 \exp \left( tS_0 (\eta_0) + (tS_0 (\eta_0))^* \right) \]
\[ = N(\eta_0)^2 \exp \left( \frac{k^3}{H^2} \left( |Q_{\gamma}^{\prime}|^2 \tilde{\Lambda} - \frac{1}{2k} (Q_{\gamma}^{\prime} Q_{\gamma}^{\prime*} + Q_{\gamma}^{\prime*} Q_{\gamma}^{\prime}) \tilde{B} + \frac{1}{k^2} |Q_{\gamma}^{\prime}|^2 \tilde{C} \right) \right) \]
\[ = N(\eta_0)^2 \exp \left( \frac{k^3}{H^2} \left( \left( \tilde{\Lambda} - \frac{\tilde{B}^2}{4\tilde{C}} \right) |Q_{\gamma}^{\prime}|^2 + \frac{\tilde{C}}{k^2} |Q_{\gamma}^{\prime} - \frac{k\tilde{B}}{2\tilde{C}} Q_{\gamma}^{\prime}|^2 \right) \right), \]  
(68)
where \( \tilde{\Lambda} = 2\Re(t\tilde{\Lambda}) \), etc.

As before (section 5.1) we may do a late-time expansion of the ghost mode to select the dominant terms when \( \beta \) is small and negative:
\[ P(Q_{\gamma}^{\prime}, Q_{\gamma}^{\prime}) = |N(\eta_0)|^2 \exp \left( \frac{k^3}{H^2} \left( \left( -\frac{1}{2} + 4\beta + \frac{1}{2} - 4\beta \right) z^2 + \cdots \right) |Q_{\gamma}^{\prime}|^2 \right) \]
\[ + \frac{1}{k^2} \left( \frac{1}{\beta} + \frac{1}{\beta} \frac{1}{z} + \frac{10i}{\beta} \left( \frac{1}{\beta} + \frac{1}{\beta} \beta z^2 \right) \right) \left( -1 + 3\beta \right) \frac{2\beta(1+10\beta)z^2}{\left( -1 + 8\beta \right)} + \cdots \]
\[ \times |Q_{\gamma}^{\prime} + k \left( z - \frac{3z^2}{\sqrt{1 + \frac{1}{\beta}}} + \cdots \right) Q_{\gamma}^{\prime}|^2 \right). \]
Since \(-1 < \beta < 0 \) and \( z > 0 \) it looks like that the dependence on \( Q_{\gamma}^{\prime} \) is of the wrong sign to be integrated out. This problem does not emerge in the Euclidean formalism (as noted in [6]). We can rotate into the Euclidean and thus get an extra minus sign, which allows us to integrate over the unobserved variable. This leaves us with
\[ P(Q_{\gamma}) = |N(\eta_0)|^2 \exp \left[ \frac{k^3}{H^2} \left( \left( -\frac{1}{2} - 4H^2\alpha^2 + \frac{1}{2} + 4H^2\alpha^2 \right) z^2 + \cdots \right) |Q_{\gamma}|^2 \right]. \]  
(69)
The simple \( k \) dependence here is due to the fact that the \( k \) dependence can be removed from the EOM that the modes satisfy; in the case of the flat space wavefunction the \( k \) dependence

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\( ^8 \) Linear combinations of the classical mode solutions in equations (33) and (34) freeze out since they start with a constant and have no term of order one in \( z \); thus, we must have freeze-out of observables involving \( \gamma \).
is not an overall factor. So observables freeze out at late times, with the two-point function having a leading contribution for a large $k$ of

$$\langle |\gamma_s|^2 \rangle = \frac{H^2}{k^3} (1 - 8H^2a^2 + \cdots)$$

(70)

for the tensor perturbations $\gamma_s$. We note again that $H$ in this formula should be regarded as the value of the Hubble parameter at horizon crossing, if the background is not pure de Sitter.

7. Conclusions

We have shown that in higher derivative theories of gravity it is possible to live with ghosts, so there is no need for backsubstitution. So the result of taking the theory with the Weyl-squared term seriously is not that we see corrections at order $O(H)$, as we might have expected given the results for a higher derivative simple harmonic oscillator; instead, as in the case of backsubstitution we get corrections at order $O(H^2)$. Thus, if we only consider the addition of a Weyl-squared term to the action we see no qualitative difference between backsubstitution and taking the full theory seriously, as we do here. This similarity may not extend further: further work is required to determine whether the unusually high cut-off found by backsubstitution [5] would be removed in our approach.

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