ON REGULARITY AND STABILITY FOR A CLASS OF NONLOCAL EVOLUTION EQUATIONS WITH NONLINEAR PERTURBATIONS

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Abstract. We study a class of nonlocal partial differential equations with nonlinear perturbations, which is a general model for some equations arose from fluid dynamics. Our aim is to analyze some sufficient conditions ensuring the global solvability, regularity and stability of solutions. Our analysis is based on the theory of completely positive kernel functions, local estimates and a new Gronwall type inequality.

1. Introduction. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$. Consider the following problem

$$
\partial_t u + \partial_t (m \ast (-\Delta)^\gamma u) - \Delta u = f(u) \quad \text{in } \Omega, t > 0,
$$

(1.1)

$$
u = 0 \quad \text{on } \partial \Omega, \quad t \geq 0,
$$

(1.2)

$$
u(\cdot, 0) = \xi \quad \text{in } \Omega,
$$

(1.3)

where $\partial_t = \frac{\partial}{\partial t}$, $m \in L^1_{\text{loc}}(\mathbb{R}^+)$ is a nonnegative function, $\gamma \in [0, 1]$, and the notation $\ast$ stands for the Laplace convolution with respect to the time $t$, i.e.,

$$(m \ast v)(t) = \int_0^t m(t - s)v(s)ds.
$$

In our model, $(-\Delta)^\gamma$ denotes the fractional power of the Laplacian, $f$ is a nonlinear function and $\xi \in L^2(\Omega)$ is given.

We first mention some special cases of (1.1). If $\gamma = 1$ and $m$ is a nonnegative constant then (1.1) is the classical reaction-diffusion equation with nonlinear sources. In the case $\gamma = 1$ and $m(t) = m_0 g_{1-\alpha}(t) = \frac{m_0 t^{-\alpha}}{\Gamma(1-\alpha)}$, $m_0 > 0$ and $\alpha \in (0, 1)$, our equation reads

$$
\partial_t u - (1 + m_0 \partial_t^\alpha) \Delta u = f(u),
$$

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which is the generalized Rayleigh-Stokes equation, here $\partial^\alpha_\gamma$ denotes the fractional derivative of order $\alpha$ in the sense of Riemann-Liouville. This equation is employed to describe the behavior of non-Newtonian fluids and has been a subject of numerous studies (see, e.g. [2, 13, 14, 19, 21, 24, 25]). In the case $\gamma = 0$ and $m(t) = m_0\theta_{1-\alpha}(t)$, we get
\[
\partial_t u + m_0\partial^\alpha_\gamma u - \Delta u = f(u),
\]
that is the Basset equation mentioned in [1, 15, 23].

In addition, if $\gamma = 1$ and $m$ is a regular function, e.g. $m \in C^1(\mathbb{R}^+)$, then (1.1) is a nonclassical diffusion equation, namely
\[
\partial_t u - m_0\Delta u - \int_0^t m_1(t-s)\Delta u(s)ds = f(u),
\]
where $m_0 = 1 + m(0), m_1(t) = m'(t)$. This class of equations has been a topic of an extensive study, see e.g., [3, 4, 6, 7, 9, 17, 18].

In this work, we are interested in the case $m$ is possibly singular (e.g., the case of Rayleigh-Stokes or Basset equation). Up to our knowledge, the regularity and stability analysis for (1.1)-(1.3) have not been investigated in literature, and we aim at addressing fundamental questions. Based on the relaxation integral equation with completely positive kernel, we give a representation of the resolvent operator, which enjoys some properties such as smoothness, decaying estimate, etc. In addition, a new Gronwall type inequality will be established and utilized in stability analysis. This inequality is also employed to prove the convergence of solution to equilibrium point, i.e. the solution of the elliptic problem
\[
-\Delta v = f(v) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega.
\]
The construction of the resolvent family $\{S(t)\}_{t \geq 0}$ and its properties are represented in the next section. It should be noted that, in our case the regularity of $S(\cdot)$ cannot be obtained by using the resolvent theory given by Prüss [20] as in the recent work [11]. More precisely, it is impossible to transform our system to the integral equation of the form
\[
u(t) + l * Au(t) = g(t),
\]
as in [20] since (1.1) involves two different operators, i.e. $-\Delta$ and its fractional power one.

We will prove the differentiability of $S(\cdot)$ on $(0, \infty)$ in order to prove the regularity of solutions. Section 3 is devoted to the results on global solvability and regularity. In the case when $f$ is locally Lipschitzian, we show that the mild solution to (1.1)-(1.3) is classical, provided that $m$ is nonincreasing. In Section 4, we prove some results on stability of solution to (1.1)-(1.3) such as the dissipativity, the asymptotic stability and the convergence to equilibrium. It should be mentioned that, the obtained results can be applied to the problems governed by nonclassical diffusion equation, the semilinear Rayleigh-Stokes equation and the Basset equation. In particular, our results extend those given in [12]. Moreover, in comparison with the studies in [11, 22] for subdiffusion equations, the decay rate in our stability results established for (1.1) is more explicit and faster (see Remark 2).

2. Preliminaries. We first find a representation for solution of the linear problem
\[
\partial_t u + \partial_t(m * (-\Delta)^\gamma u) - \Delta u = F \text{ in } \Omega, t \in (0, T],
\]
\[
u = 0 \text{ on } \partial\Omega, \ t \in [0, T],
\]
\[
u(\cdot, 0) = \xi \text{ in } \Omega,
\]
where $F \in C([0, T]; L^2(\Omega))$. Assume that
\[ u(\cdot, t) = \sum_{n=1}^{\infty} u_n(t) \varphi_n, \quad F(\cdot, t) = \sum_{n=1}^{\infty} F_n(t) \varphi_n, \quad (2.4) \]
where $\{\varphi_n\}$ is the sequence of eigenfunctions of the Laplacian $-\Delta$ subject to homogeneous Dirichlet boundary condition. Let $\lambda_n$ be the eigenvalue corresponding to $\varphi_n$. Then the fractional power operator $(-\Delta)^{\beta}$ is defined as follows
\[ (-\Delta)^{\beta} v = \sum_{n=1}^{\infty} \lambda_n^{\beta}(v, \varphi_n) \varphi_n, \]
\[ D((-\Delta)^{\beta}) = \{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\beta}(v, \varphi_n)^2 < \infty \}, \]
here the notation $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.
Substituting (2.4) into (2.1), we get
\[ u'_n(t) + \lambda_n u_n(t) + \lambda_n^2 (m * u_n)'(t) = F_n(t), \quad (2.5) \]
\[ u_n(0) = \xi_n := (\xi, \varphi_n). \quad (2.6) \]
In order to find a representation of $u_n$, we consider the relaxation equation
\[ \omega'(t) + \mu \omega(t) + \nu (m * \omega)'(t) = 0, \quad \text{for } t > 0, \quad \omega(0) = 1, \]
where $\mu$ and $\nu$ are positive numbers. This equation can be rewritten as
\[ \omega(t) + \mu (1 + \nu \mu^{-1} m) * \omega(t) = 1. \quad (2.7) \]
We make the following standing assumption:
\[ (M) \quad m \in L^1_{\text{loc}}(\mathbb{R}^+) \text{ is a nonnegative function such that } a_\eta(t) := 1 + \eta m(t) \text{ is completely positive for any } \eta > 0. \]
Recall that, the complete positivity of a function $\ell \in L^1_{\text{loc}}(\mathbb{R}^+)$ means that the solutions of the following equations
\[ s(t) + \theta(\ell \ast s)(t) = 1, \quad (2.8) \]
\[ r(t) + \theta(\ell \ast r)(t) = \ell(t), \quad (2.9) \]
take nonnegative values for all $\theta > 0$. This is equivalent to that, there exist $\varepsilon \geq 0$ and a nonnegative and nonincreasing function $k \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $\varepsilon \ell + \ell \ast k = 1$ on $(0, \infty)$ (see [16]). It was shown in [5] that, if $\varepsilon = 0$, i.e. $\ell \ast k = 1$, then
\[ \int_0^t r(\tau)d\tau = \theta^{-1}(1 - s(t)), \quad \forall t \geq 0. \quad (2.10) \]
Nevertheless, one can get (2.10) even in the case $\varepsilon > 0$. Indeed, it follows from (2.8)-(2.9) that
\[ (\ell \ast s)(t) = \theta^{-1}(1 - s(t)), \]
\[ (\ell \ast s)(t) + \theta(\ell \ast (\ell \ast s))(t) = (1 \ast \ell)(t), \]
\[ (1 \ast r)(t) + \theta(\ell \ast (1 \ast r))(t) = (1 \ast \ell)(t), \quad \forall t \geq 0, \]
which imply $1 \ast r = \ell \ast s$, and then (2.10) takes place.
Following [16], if $a_\eta$ is completely monotonic, i.e. $(-1)^n a_\eta^{(n)}(t) \geq 0$ for all $n \in \mathbb{N}$ and $t \in (0, \infty)$, then it is completely positive. Noting that, if $m$ is completely monotonic, so is $a_\eta$. A weaker condition ensuring the complete positivity of $a_\eta$ is
that, \( a_\eta \in C^1(0, \infty) \) is positive and \( \log a_\eta \) is a convex function (log-convex). It should be mentioned that, if \( m \in C^1(0, \infty) \) is positive and log-convex, so is \( a_\eta \). Indeed, if \( m \) is log-convex then it is convex. That means \( \frac{m'}{m} \) and \( m' \) are increasing simultaneously, which implies that \( \frac{d}{dt} \log a_\eta = \frac{m'}{1 + \eta m} \) is increasing as well. That is, \( a_\eta \) is log-convex for any \( \eta > 0 \).

Under the assumption \((M)\), the solution of \((2.7)\) is positive on \( \mathbb{R}^+ \), since \( \omega \) is the solution of \((2.8)\) with \( \theta = \mu \) and \( \ell = 1 + \mu^{-1}m \). It should be noted that, the positivity of \( \omega \) plays an important role in our analysis. We have the following additional properties of \( \omega \).

**Proposition 1.** Let \( \omega(\cdot, \mu, \nu) \) be the solution of \((2.7)\). Then
(a) The function \( t \mapsto \omega(t, \mu, \nu) \) is nonincreasing on \( \mathbb{R}^+ \) and obeys the inequality
\[
0 < \omega(t, \mu, \nu) \leq \frac{1}{1 + \mu \int_0^t (1 + \nu \mu^{-1}m(\tau))d\tau}, \quad \forall t \geq 0.
\]
Consequently, \( \lim_{t \to \infty} \omega(t, \mu, \nu) = 0 \).
(b) We have
\[
\int_0^t \omega(\tau, \mu, \nu)d\tau \leq \mu^{-1}(1 - \omega(t, \mu, \nu)), \quad \forall t \geq 0.
\]
(c) The function \( \mu \mapsto \omega(t, \mu, \mu^\gamma) \) is nonincreasing on \((0, \infty)\) for each \( t \geq 0 \).

**Proof.** (a) As pointed out in [5], the solution of \((2.8)\) is nonincreasing, so is \( \omega(\cdot, \mu) \). Then it follows that
\[
1 = \omega(t, \mu, \nu) + \mu \int_0^t (1 + \nu \mu^{-1}m(t - \tau)) \omega(\tau, \mu)d\tau \\
\geq \omega(t, \mu, \nu) + \mu \omega(t, \mu, \nu) \int_0^t (1 + \nu \mu^{-1}m(t - \tau))d\tau,
\]
which implies
\[
\omega(t, \mu, \nu) \leq \frac{1}{1 + \mu \int_0^t (1 + \nu \mu^{-1}m(\tau))d\tau}, \quad \forall t \geq 0.
\]
(b) Thanks to the fact that \( m \) is nonnegative, we have
\[
1 = \omega(t, \mu, \nu) + \mu \int_0^t (1 + \nu \mu^{-1}m(t - \tau)) \omega(\tau, \mu, \nu)d\tau \\
\geq \omega(t, \mu, \nu) + \mu \int_0^t \omega(\tau, \mu, \nu)d\tau,
\]
which ensures that
\[
\int_0^t \omega(\tau, \mu, \nu)d\tau \leq \mu^{-1}(1 - \omega(t, \mu, \nu)).
\]
(c) Let \( \nu = \mu^\gamma \). Taking differentiation of \((2.7)\) in \( \mu \), one gets
\[
\partial_\mu \omega + \mu(1 + \mu^{-1}m) * \partial_\mu \omega = -(1 + \gamma \mu^{-1}m) * \omega.
\]

Let \( \hat{\ell} = 1 + \mu^{-1}m \), then by \((M)\), \( \hat{\ell} \) is completely positive and there exist \( \varepsilon \geq 0 \) and a function \( \hat{k} \in L_{loc}^1(\mathbb{R}^+) \) which is nonnegative and nonincreasing such that
\[
\varepsilon \hat{\ell} + \hat{\ell} * \hat{k} = 1.
\]
Noting that
\[
1 + \gamma \mu^{-1}m = \gamma(1 + \mu^{-1}m) + 1 - \gamma
\]
The solution of 

\[ γ\hat{\ell} + (1 - γ)(\hat{z} \ell + \hat{\ell} \ast \hat{k}) \]

\[ = (γ + (1 - γ)ε)\hat{\ell} + (1 - γ)\hat{\ell} \ast \hat{k}, \]

we can rewrite (2.11) as

\[ \partial_μω + μ\hat{\ell} \ast \partial_μω = -[(γ + (1 - γ)ε)ω + ω \ast \hat{k}] \ast \hat{\ell}. \] (2.12)

Let \( \hat{r} \) be the solution of \( \hat{r} + μ\hat{\ell} \ast \hat{r} = \hat{\ell} \). Then we see that, the solution of (2.12) is given by

\[ \partial_μω = -[(γ + (1 - γ)ε)ω + ω \ast \hat{k}] \ast \hat{r}. \]

The last relation guarantees that \( \partial_μω(t, μ, μ) \leq 0 \) for each \( t \geq 0 \) and \( μ > 0 \). The proof is complete.

We are now in a position to consider the inhomogeneous equation

\[ v'(t) + μv(t) + ν(m \ast v)'(t) = g(t), \quad \text{for } t > 0, \quad v(0) = v_0, \] (2.13)

where \( g \in C(\mathbb{R}^+) \) is a given function, \( μ \) and \( ν \) are positive numbers.

**Proposition 2.** The solution of (2.13) is unique and given by

\[ v(t) = ω(t, μ, ν)v_0 + \int_0^t ω(t - τ, μ, ν)g(τ)dτ, \quad t \geq 0. \] (2.14)

**Proof.** Put \( L[y] = y' + μy + ν(m \ast y)', \ y \in C^1(\mathbb{R}^+) \). Then we have \( L[ω] = 0 \). In addition, we see that

\[ L[v] = L[ω]v_0 + L[ω \ast g] = L[ω \ast g]. \]

We will prove that \( L[ω \ast g] = g \). Indeed, one gets

\[(ω \ast g)' + μω \ast g + ν(m \ast ω \ast g)' = g + ω' \ast g + μω \ast g + ν(m \ast ω)' \ast g = g + [ω' + μω + ν(m \ast ω)'] \ast g = g + L[ω] \ast g = g.\]

Conversely, if \( v \) is a solution of (2.13), we have

\[ z\hat{v}(z) + μ\hat{v}(z) + νz\hat{m}(z)\hat{v}(z) = v_0 + \hat{g}(z), \]

where \( \hat{v} \) is the Laplace transform of \( v \). Then

\[ \hat{v}(z) = (z + μ + νz\hat{m}(z))^{-1}v_0 + (z + μ + νz\hat{m}(z))^{-1}\hat{g}(z) = \hat{ω}(z)v_0 + \hat{ω}(z)\hat{g}(z), \]

where \( \hat{ω} \) is the Laplace transform of \( ω \) with respect to \( t \). Taking the inverse Laplace transform yields \( v = ωv_0 + ω \ast g \), which is (2.14). The proof is complete.

In what follows, in the case \( ν = μ^γ \), we write \( ω(t, μ) \) instead of \( ω(t, μ, μ^γ) \). Using the last proposition, we have the following formula for the solution of (2.5)-(2.6):

\[ u_n(t) = ω(t, λ_n)ξ_n + \int_0^t ω(t - τ, λ_n)F_n(τ)dτ. \]

Thus one has the following representation for the solution of (2.1)-(2.3):

\[ u(\cdot, t) = \sum_{n=1}^{∞} ω(t, λ_n)ξ_nφ_n + \sum_{n=1}^{∞} \int_0^t ω(t - τ, λ_n)F_n(τ)dτφ_n \]

\[ = S(t)ξ + \int_0^t S(t - τ)F(\cdot, τ)dτ, \] (2.15)
where $S(t)$ is defined by
\[ S(t)\xi = \sum_{n=1}^{\infty} \omega(t, \lambda_n)\xi_n \varphi_n, \quad \xi \in L^2(\Omega). \tag{2.16} \]

Hereafter, the notation $\| \cdot \|$ stands for the standard norm in $L^2(\Omega)$ and $\| \cdot \|_\mathcal{L}$ represents the operator norm of bounded linear operator acting on $L^2(\Omega)$.

**Lemma 2.1.** Let $\{S(t)\}_{t\geq 0}$ be the resolvent operators defined by (2.16), $\xi \in L^2(\Omega)$ and $T > 0$. Then

(a) $S(\cdot)\xi \in C([0, T]; L^2(\Omega))$ and $\|S(t)\|_\mathcal{L} \leq \omega(t, \lambda_1)$ for all $t \geq 0$.

(b) For $g \in C([0, T]; L^2(\Omega))$, $(-\Delta)^{\frac{1}{2}} S * g \in C([0, T]; L^2(\Omega))$ and it holds that
\[ \|(-\Delta)^{\frac{1}{2}} S * g(t)\| \leq \left( \int_{0}^{t} \omega(t - \tau, \lambda_1)\|g(\tau)\|^2 d\tau \right)^{\frac{1}{2}}, \quad \forall t \geq 0. \tag{2.17} \]

(c) If $m$ is nonincreasing, then $S(\cdot)\xi \in C^1([0, T]; L^2(\Omega))$ and it holds that
\[ \|S'(t)\|_\mathcal{L} \leq t^{-1} \quad \text{for all } t > 0. \]

(d) $\Delta S(\cdot)\xi \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; L^2(\Omega))$ and we have the estimates
\[ \|\Delta S(t)\xi\| \leq t^{-1}\|\xi\|, \quad \text{for all } t > 0, \]
\[ \|\int_{0}^{t} \Delta S(\tau)\xi d\tau\| \leq \|\xi\|, \quad \text{for all } t \geq 0. \]

**Proof.** (a) It follows from (2.16) that
\[ \|S(t)\xi\|^2 = \sum_{n=1}^{\infty} \omega(t, \lambda_n)^2 \xi_n^2 \leq \omega(t, \lambda_1)^2 \sum_{n=1}^{\infty} \xi_n^2 = \omega(t, \lambda_1)^2 \|\xi\|^2, \quad \xi_n = (\xi, \varphi_n), \]

thanks to Proposition 1(c), which implies the uniform convergence of series (2.16) on $[0, T]$ and the estimate $\|S(t)\|_\mathcal{L} \leq \omega(t, \lambda_1)$ for all $t \geq 0$.

(b) We observe that
\[ (-\Delta)^{\frac{1}{2}} S * g(t) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \int_{0}^{t} \omega(t - \tau, \lambda_n)g_n(\tau) d\tau \varphi_n, \tag{2.18} \]

where $g_n(t) = (g(t), \varphi_n)$. Using the Hölder inequality, we get
\[
\lambda_n \left( \int_{0}^{t} \omega(t - \tau, \lambda_n)g_n(\tau) d\tau \right)^2 \leq \lambda_n \int_{0}^{t} \omega(t - \tau, \lambda_n) d\tau \int_{0}^{t} \omega(t - \tau, \lambda_n) |g_n(\tau)|^2 d\tau
\leq (1 - \omega(t, \lambda_n)) \int_{0}^{t} \omega(t - \tau, \lambda_n) |g_n(\tau)|^2 d\tau
\leq \int_{0}^{t} \omega(t - \tau, \lambda_1) |g_n(\tau)|^2 d\tau,
\]

thanks to Proposition 1(b). So
\[
\|(-\Delta)^{\frac{1}{2}} S * g(t)\|^2 = \sum_{n=1}^{\infty} \lambda_n \left( \int_{0}^{t} \omega(t - \tau, \lambda_n)g_n(\tau) d\tau \right)^2
\leq \sum_{n=1}^{\infty} \int_{0}^{t} \omega(t - \tau, \lambda_1) |g_n(\tau)|^2 d\tau = \int_{0}^{t} \omega(t - \tau, \lambda_1) \|g(\tau)\|^2 d\tau,
\]
which implies (2.17). In order to show \((-\Delta)^{\frac{3}{2}} S \ast g \in C([0, T]; L^2(\Omega))\), it suffices to check that series (2.18) is uniformly convergent on \([0, T]\). Since \(g\) is continuous, the series \(\sum_{n=1}^{\infty} |g_n(\tau)|^2\) is uniformly convergent on \([0, T]\). That means, for any \(\epsilon > 0\), there exists \(N_\epsilon \in \mathbb{N}\) such that \(\sum_{n=N_\epsilon}^{N+p} |g_n(\tau)|^2 < \epsilon\) for all \(N \geq N_\epsilon, p \in \mathbb{N}\) and \(\tau \in [0, T]\). It follows that
\[
\sum_{n=N}^{N+p} \lambda_n \left( \int_0^t \omega(t-\tau, \lambda_n)g_n(\tau)d\tau \right)^2 \leq \sum_{n=N}^{N+p} \int_0^t \omega(t-\tau, \lambda_1)|g_n(\tau)|^2d\tau
\]
\[
= \int_0^t \omega(t-\tau, \lambda_1) \sum_{n=N}^{N+p} |g_n(\tau)|^2d\tau \leq \lambda_1^{-1}\epsilon,
\]
for all \(t \in [0, T]\), which guarantees the uniform convergence of (2.18) on \([0, T]\).

(c) Let \(r(\cdot, \lambda)\) be the solution of (2.9) with \(\theta = \lambda\) and \(\ell(t) = 1 + \lambda^{-1}m(t)\). Then, due to the assumption that \(m\) is nonincreasing, we have
\[
(2.19)
\]
\[
In addition, it follows from (2.10) that
\[
\int_0^t r(\tau, \lambda)d\tau = \lambda^{-1}(1 - \omega(t, \lambda)) \geq \frac{t + \lambda^{-1}(1 * m)(t)}{1 + \lambda t + \lambda^2(1 * m)(t)},
\]
in accordance with Proposition 1(a). Hence
\[
(2.19)
\]
\[
ru(t, \lambda) \leq [1 + \lambda^{-1}m(t)] \left[ 1 - \frac{\lambda t + \lambda^2(1 * m)(t)}{1 + \lambda t + \lambda^2(1 * m)(t)} \right] = \frac{1 + \lambda^{-1}m(t) + \lambda t + \lambda^2(1 * m)(t)}{1 + \lambda t + \lambda^2(1 * m)(t)}. \tag{2.19}
\]
Considering the series
\[
\sum_{n=1}^{\infty} \omega'(t, \lambda_n)\xi_n\varphi_n, \quad t > 0, \xi_n = (\xi, \varphi_n), \quad \xi \in L^2(\Omega), \tag{2.20}
\]
we see that
\[
|\omega'(t, \lambda_n)| = \lambda_n ru(t, \lambda_n) \leq \frac{\lambda_n + \lambda_n^2m(t)}{1 + \lambda_n t + \lambda_n^2(1 * m)(t)} \leq \frac{\lambda_n + \lambda_n^2m(t)}{\lambda_n t + \lambda_n^2tm(t)} = t^{-1}, \quad \forall t > 0.
\]
thanks to (2.19) and the fact that \(1 * m(t) \geq tm(t)\) for \(t > 0\). This ensures the uniform convergence of series (2.20) on \([\epsilon, T]\) and it holds that
\[
S'(t)\xi = \sum_{n=1}^{\infty} \omega'(t, \lambda_n)\xi_n\varphi_n, \quad ||S'(t)\xi|| \leq t^{-1}||\xi||, \quad \forall t > 0.
\]
(d) It follows from Proposition 1 that \(\lambda_n\omega(t, \lambda_n) \leq t^{-1}\) for all \(t > 0\). Then
\[
||\Delta S(t)\xi||^2 = \sum_{n=1}^{\infty} [\lambda_n\omega(t, \lambda_n)]^2\xi_n^2 \leq t^{-2}||\xi||^2,
\]
which infers that the series
\[
\Delta S(t)\xi = \sum_{n=1}^{\infty} \lambda_n\omega(t, \lambda_n)\xi_n\varphi_n
\]
is uniformly convergent on \([\epsilon, T]\) for any \(\epsilon \in (0, T)\). Thus \(\Delta S(\cdot)\xi \in C((0, T]; L^2(\Omega))\).
On the other hand, we see that
\[
\int_0^t \Delta S(\tau) \xi d\tau = -\sum_{n=1}^{\infty} \int_0^t \lambda_n \omega(\tau, \lambda_n) \xi_n d\tau \varphi_n. \tag{2.21}
\]

Then
\[
\| \int_0^t \Delta S(\tau) \xi d\tau \|^2 = \sum_{n=1}^{\infty} \left( \int_0^t \lambda_n \omega(\tau, \lambda_n) d\tau \right)^2 \xi_n^2 \leq \sum_{n=1}^{\infty} (1 - \omega(t, \lambda_n))^2 \xi_n^2 \leq \| \xi \|^2,
\]
due to Proposition 1(b), which implies that \( \Delta S(\cdot) \xi \in L^1(0, T; L^2(\Omega)) \). The proof is complete. \( \square \)

We end this section by proving a Gronwall type inequality, which will be used in our stability analysis.

**Proposition 3.** Let \( z \) be a nonnegative function obeying the inequality
\[
z(t) \leq \omega(t, \mu, \nu) z_0 + \int_0^t \omega(t - \tau, \mu, \nu) [az(\tau) + b(\tau)] d\tau, \quad t \geq 0, \tag{2.22}
\]
where \( a \in [0, \mu), \nu > 0, b \in L_{loc}(\mathbb{R}^+) \). Then
\[
z(t) \leq \omega(t, \mu - a, \nu) z_0 + \int_0^t \omega(t - \tau, \mu - a, \nu) b(\tau) d\tau.
\]

**Proof.** Let \( y(t) \) be the right hand side of (2.22). Then \( z(t) \leq y(t) \) and \( y \) solves the equation
\[
y'(t) + \mu y(t) + \nu (m * y)'(t) = az(t) + b(t), \quad t > 0, y(0) = z_0,
\]
as stated by Proposition 2. It follows that
\[
y'(t) + (\mu - a) y(t) + \nu (m * y)'(t) = a[z(t) - y(t)] + b(t), \quad t > 0, y(0) = z_0,
\]
and then \( y \) admits the representation
\[
y(t) = \omega(t, \mu - a, \nu) z_0 + \int_0^t \omega(t - \tau, \mu - a, \nu) (a[z(\tau) - y(\tau)] + b(\tau)) d\tau
\]
\[
\leq \omega(t, \mu - a, \nu) z_0 + \int_0^t \omega(t - \tau, \mu - a, \nu) b(\tau) d\tau,
\]
thanks to the positivity of \( \omega \) and the fact that \( z(t) - y(\tau) \leq 0 \) for \( t \geq 0 \). So we get the conclusion as desired. \( \square \)

3. **Solvability and regularity.** Based on representation (2.15), we give the following definition of mild solution for (1.1)-(1.3).

**Definition 3.1.** A function \( u \in C([0, T]; L^2(\Omega)) \) is said to be a mild solution to the problem (1.1)-(1.3) on \([0, T]\) iff
\[
u(\cdot, t) = S(t) \xi + \int_0^t S(t - \tau) f(u(\cdot, \tau)) d\tau \text{ for any } t \in [0, T].
\]

We first prove a global solvability result for (1.1)-(1.3).

**Theorem 3.2.** Let (M) hold. Assume that
The function $f : L^2(\Omega) \to L^2(\Omega)$ satisfies $f(0) = 0$ and is locally Lipschitzian, i.e.
\[ \| f(v_1) - f(v_2) \| \leq \kappa(r) \| v_1 - v_2 \|, \quad \forall v_1, v_2 \in B_r, \]
where $B_r$ is the closed ball in $L^2(\Omega)$ with radius $r$ and center at origin, $\kappa(\cdot)$ is a nonnegative function such that $\limsup_{r \to 0} \kappa(r) = \ell \in [0, \lambda_1)$.

Then there exists $\delta > 0$ such that the problem (1.1)-(1.3) has a unique mild solution on $[0, T]$, provided $\| \xi \| \leq \delta$.

**Proof.** Let $\Phi : C([0, T]; L^2(\Omega)) \to C([0, T]; L^2(\Omega))$ be the mapping defined by
\[ \Phi(u)(t) = S(t)\xi + \int_0^t S(t - \tau) f(u(\cdot, \tau))d\tau \quad \text{for} \quad t \in [0, T]. \]
We first look for $\rho > 0$ such that $\Phi(B_\rho) \subset B_\rho$, where $B_\rho$ is the closed ball in $C([0, T]; L^2(\Omega))$ centered at origin with radius $\rho$. Taking $\epsilon \in (0, \lambda_1 - \ell)$, we can find $\rho > 0$ such that $\kappa(r) \leq \ell + \epsilon$ for any $r \leq \rho$. Considering $\Phi : B_\rho \to C([0, T]; L^2(\Omega))$, we have
\[ \| \Phi(u)(\cdot, t) \| \leq \| S(t)\xi \| + \int_0^t \| S(t - \tau)\| \| f(u(\cdot, \tau)) \| d\tau \]
\[ \leq \omega(t, \lambda_1) \| \xi \| + \int_0^t \omega(t - \tau, \lambda_1) \kappa(\rho) \| u(\cdot, \tau) \| d\tau \]
\[ \leq \omega(t, \lambda_1) \| \xi \| + (\ell + \epsilon) \rho \int_0^t \omega(t - \tau, \lambda_1)d\tau \]
\[ \leq \omega(t, \lambda_1) \| \xi \| + (\ell + \epsilon) \rho \lambda_1^{-1} (1 - \omega(t, \lambda_1)) \]
\[ = \omega(t, \lambda_1) \| \xi \| - (\ell + \epsilon) \rho \lambda_1^{-1} + (\ell + \epsilon) \rho \lambda_1^{-1}, \quad \forall u \in B_\rho, t \in [0, T], \]
here we used Lemma 2.1(a) and Proposition 1(b). Choosing $\| \xi \| \leq \delta := \ell \rho \lambda_1^{-1}$, we see that
\[ \| \Phi(u)(\cdot, t) \| \leq (\ell + \epsilon) \rho \lambda_1^{-1} \leq \rho, \quad \forall u \in B_\rho, t \in [0, T], \]
which implies $\Phi(B_\rho) \subset B_\rho$. We now prove that $\Phi$ is a contraction mapping on $B_\rho$. For $u_1, u_2 \in B_\rho$, one gets
\[ \| \Phi(u_1)(\cdot, t) - \Phi(u_2)(\cdot, t) \| \leq \int_0^t \omega(t - \tau, \lambda_1) \| f(u_1(\cdot, \tau)) - f(u_2(\cdot, \tau)) \| d\tau \]
\[ \leq \kappa(\rho) \int_0^t \omega(t - \tau, \lambda_1) \| u_1(\cdot, \tau) - u_2(\cdot, \tau) \| d\tau \]
\[ \leq (\ell + \epsilon) \| u_1 - u_2 \|_\infty \int_0^t \omega(t - \tau, \lambda_1)d\tau \]
\[ \leq (\ell + \epsilon) \lambda_1^{-1} (1 - \omega(t, \lambda_1)) \| u_1 - u_2 \|_\infty, \quad \forall t \in [0, T], \]
which ensures that
\[ \| \Phi(u_1) - \Phi(u_2) \|_\infty \leq (\ell + \epsilon) \lambda_1^{-1} \| u_1 - u_2 \|_\infty. \]
Hence $\Phi$ is a contraction mapping and it admits a fixed point in $B_\rho$, which is a mild solution to (1.1)-(1.3). In order to testify the uniqueness, we observe that, if $u, v \in C([0, T]; L^2(\Omega))$ are solutions of (1.1)-(1.3), then one can assume that $u, v \in B_R$ for some $R > 0$. So
\[ \| u(\cdot, t) - v(\cdot, t) \| \leq \int_0^t \omega(t - \tau, \lambda_1) \kappa(R) \| u(\cdot, \tau) - v(\cdot, \tau) \| d\tau \]
\[ \leq \kappa(R) \int_0^t \|u(\cdot, \tau) - v(\cdot, \tau)\| d\tau, \forall t \in [0, T], \]

according to the fact that \( \omega(t, \lambda_1) \leq 1 \) for all \( t \geq 0 \). By using the Gronwall inequality, we get \( \|u(\cdot, t) - v(\cdot, t)\| = 0 \) for all \( t \in [0, T] \), which implies \( u = v \). The proof is complete. \( \square \)

**Remark 1.** In the case \( f \) is globally Lipschitzian, i.e.

\[ \|f(v_1) - f(v_2)\| \leq \kappa_0\|v_1 - v_2\|, \forall v_1, v_2 \in L^2(\Omega), \]

for some \( \kappa_0 > 0 \), one can prove the existence and uniqueness of mild solution to (1.1)-(1.3) by using the contraction mapping principle, regardless the assumption \( f(0) = 0 \) and the smallness of initial data (see, e.g. [11, Theorem 4.2]).

We are in a position to show the first result on regularity of mild solution to (1.1)-(1.3).

**Theorem 3.3.** Assume that \( m \) satisfies (M) with an additional condition that \( m \) is nonincreasing. If \( f : L^2(\Omega) \to L^2(\Omega) \) is locally Lipschitzian, i.e.

\[ \|f(v_1) - f(v_2)\| \leq \kappa(r)\|v_1 - v_2\|, \forall v_1, v_2 \in B_r, \]

where \( \kappa(\cdot) \) is a nonnegative function, and \( u \) is a mild solution to (1.1)-(1.3) on \([0, T]\), then \( u \in C^1([0, T]; L^2(\Omega)) \).

**Proof.** The proof consists of two steps.

**Step 1.** We show that \( u \) is Hölder continuous on \((0, T]\). By assumption, the resolvent family \( S(\cdot) \) is continuously differentiable on \((0, \infty)\) and

\[ \|S'(t)\|_C \leq t^{-1}, \text{ for all } t > 0, \]

thanks to Lemma 2.1.

For \( t \in (0, T] \) and \( h \in (0, T - t) \), we have

\[
\begin{align*}
\|u(\cdot, t+h) - u(\cdot, t)\| & \leq \|[S(t+h) - S(t)]\| \xi + \int_t^{t+h} \|S(t+h-\tau)\| f(u(\cdot, \tau)) d\tau \\
& \quad + \int_0^t \|[S(t+h-\tau) - S(t-\tau)]f(u(\cdot, \tau))\| d\tau \\
& = E_1(t) + E_2(t) + E_3(t).
\end{align*}
\]

Using the mean value formula, we have

\[ [S(t+h) - S(t)]\xi = h \int_0^1 S'(t + \zeta h)\xi d\zeta. \]

Then

\[ E_1(t) = \|[S(t+h) - S(t)]\| \leq h\|\xi\| \int_0^1 \frac{d\zeta}{t + \zeta h} \]

\[ = M\|\xi\| \ln \left(1 + \frac{h}{t}\right) \leq \|\xi\| \beta^{-1} \left(\frac{h}{t}\right)\beta, \text{ for any } \beta \in (0, 1), \]

where we used the inequality \( \ln(1+r) \leq \frac{r^\beta}{\beta} \) for all \( r > 0, \beta \in (0, 1) \).

Dealing with \( E_2(t) \), we put \( R = \|u\|_\infty \) and make use of the inequality

\[ \|f(u(\cdot, t))\| \leq \kappa(R)\|u(\cdot, t)\| + \|f(0)\| \leq \kappa(R)R + \|f(0)\|. \]
So
\[ E_2(t) \leq \int_t^{t+h} \|f(u(\cdot, \tau))\| d\tau \leq [\kappa(R)R + \|f(0)\|]h \leq [\kappa(R)R + \|f(0)\|]T^{1-\beta}h^\beta. \]

Regarding \( E_3(t) \), we note that
\[ [S(t + h - \tau) - S(t - \tau)]f(u(\cdot, \tau)) = h \int_0^1 S'(t - \tau + \zeta h) f(u(\cdot, \tau)) d\zeta. \]

Then by the same argument used to estimate \( E_1(t) \), we obtain
\[ ||[S(t + h - \tau) - S(t - \tau)]f(u(\cdot, \tau))|| \leq [\kappa(R)R + \|f(0)\|]T^{1-\beta}h^\beta. \]

Therefore,
\[ E_3(t) \leq [\kappa(R)R + \|f(0)\|]T^{1-\beta}h^\beta. \]

Summing up, we get
\[ E_1(t) + E_2(t) + E_3(t) \leq (C_1t^{-\beta} + C_2)h^\beta, \]
where
\[ C_1 = \|\xi\|\beta^{-1}, \]
\[ C_2 = [\kappa(R)R + \|f(0)\|]T^{1-\beta}(1 + \beta^{-1}(1 - \beta)^{-1}). \]

**Step 2.** We now prove that \( u \) is continuously differentiable on \( (0, T] \). Writing
\[ u(\cdot, t) = S(t)\xi + \int_0^t S(t - \tau)f(u(\cdot, \tau))d\tau = u_1(t) + u_2(t), \]
we see that \( u_1 = S(\cdot)\xi \in C^1((0, T]; L^2(\Omega)) \) as stated in Lemma 2.1(c). In addition, we have
\[ \partial_t u_2(t) = f(u(\cdot, t)) + \int_0^t S'(t - \tau)f(u(\cdot, \tau))d\tau, \]
where the second term makes sense by the following reasoning
\[ \| \int_0^t S'(t - \tau)f(u(\cdot, \tau))d\tau \|
\leq \| \int_0^t S'(t - \tau)[f(u(\cdot, \tau)) - f(u(\cdot, t))]d\tau \| + \| \int_0^t S'(t - \tau)f(u(\cdot, t))d\tau \|
\leq \| f(u(\cdot, \tau)) - f(u(\cdot, t)) \| \| S'(t - \tau) \| d\tau + \| (S(t) - I)f(u(\cdot, t)) \|
\leq \int_0^t (t - \tau)^{-1}(C_1\tau^{-\beta} + C_2)(t - \tau)^\beta d\tau + \| (S(t) - I)f(u(\cdot, t)) \|
\leq \int_0^t (t - \tau)^{\beta-1}(C_1\tau^{-\beta} + C_2)d\tau + \| (S(t) - I)f(u(\cdot, t)) \|
< \infty, \text{ for all } t \geq 0, \]
thanks to the result of Step 1. It remains to show that the mapping
\[ F(t) = \int_0^t S'(t - \tau)f(u(\cdot, \tau))d\tau \]
is continuous on \((0, T]\). Let \(t > 0\) and \(h \in (0, T - t)\), then

\[
\|F(t + h) - F(t)\| \leq \int_0^{t+h} |S'(t + h - \tau)| f(u(\cdot, \tau)) d\tau
\]

\[
+ \| \int_0^t [S'(t + h - \tau) - S'(t - \tau)] f(u(\cdot, \tau)) d\tau \|
\]

\[
+ \| \int_0^t [S'(t + h - \tau) - S'(t - \tau)] [f(u(\cdot, \tau)) - f(u(\cdot, t))] d\tau \|
\]

\[
= F_1(t) + F_2(t) + F_3(t).
\]

Obviously, \(F_1(t) \to 0\) as \(h \to 0\) due to the fact that \(\tau \mapsto S'(t - \tau) f(u(\cdot, \tau))\) is integrable on \((0, t)\) for any \(t \in (0, T)\). Regarding \(F_2(t)\), we see that

\[
F_2(t) = \| [S(t + h) - S(t)] f(u(\cdot, t)) \| \to 0 \text{ as } h \to 0.
\]

Dealing with \(F_3(t)\), we get

\[
F_3(t) \leq \kappa(R) \int_0^t \|S'(t + h - \tau) - S'(t - \tau)\| \|u(\cdot, t) - u(\cdot, \tau)\| d\tau
\]

\[
\leq \kappa(R) \int_0^t \|S'(t + h - \tau) - S'(t - \tau)\| L(C_1 \tau^{-\beta} + C_2)(t - \tau)^\beta d\tau.
\]

Let us denote \(Q_h(\tau) = \|S'(t + h - \tau) - S'(t - \tau)\| L(C_1 \tau^{-\beta} + C_2)(t - \tau)^\beta\), then \(\lim_{h \to 0} Q_h(\tau) = 0\) for any \(\tau \in (0, t)\). Moreover, we observe that

\[
Q_h(\tau) \leq \|S'(t + h - \tau)\| L + \|S'(t - \tau)\| L(C_1 \tau^{-\beta} + C_2)(t - \tau)^\beta
\]

\[
\leq [(t + h - \tau)^{-1} + (t - \tau)^{-1}] (C_1 \tau^{-\beta} + C_2)(t - \tau)^\beta
\]

\[
\leq 2(t - \tau)^{-1}(C_1 \tau^{-\beta} + C_2)(t - \tau)^\beta
\]

\[
= 2(t - \tau)^{\beta - 1}(C_1 \tau^{-\beta} + C_2) = Q^\star(\tau).
\]

Since \(Q^\star \in L^1(0, t)\), one can utilize the Lebesgue dominated convergence theorem to claim that

\[
F_3(t) \leq \kappa(R) \int_0^t Q_h(\tau) d\tau \to 0 \text{ as } h \to 0.
\]

Thus \(F\) is continuous on \((0, T]\) and the theorem is proved.

**Theorem 3.4.** Let the assumptions of Theorem 3.3 hold. Then the mild solution of (1.1)-(1.3) on \([0, T]\) satisfies \(\Delta u \in C((0, T]; L^2(\Omega))\).

**Proof.** Let \(u\) be a mild solution of (1.1)-(1.3), then

\[
u(\cdot, t) = S(t) \xi + \int_0^t S(t - \tau) f(u(\cdot, \tau)) d\tau = u_1(t) + u_2(t), \forall t \in [0, T].
\]

By Lemma 2.1(d), we get \(\Delta u_1 \in C((0, T]; L^2(\Omega))\). In addition, let \(R = \|u\|_{\infty}\) then

\[
\|u_2(t)\| = \| \int_0^t \Delta S(t - \tau) f(u(\cdot, \tau)) d\tau \|
\]

\[
\leq \| \int_0^t \Delta S(t - \tau)[f(u(\cdot, \tau)) - f(u(\cdot, t))] d\tau \| + \| \int_0^t \Delta S(t - \tau) f(u(\cdot, t)) d\tau \|
\]

\[
\leq \kappa(R) \int_0^t (t - \tau)^{-1} \|u(\cdot, \tau) - u(\cdot, t)\| + \| \int_0^t \Delta S(\tau) f(u(\cdot, t)) d\tau \|
\]

\[
= F_1(t) + F_2(t) + F_3(t).
\]
\[
\leq \kappa(R) \int_0^t (t - \tau)^{\beta - 1} (C_1 \tau^{-\beta} + C_2) d\tau + \|f(u(\cdot, t))\|,
\]
here we employed the Lipschitz condition on \( f \), the Hölder continuity shown in the proof of Theorem 3.3, and the estimates stated in Lemma 2.1(d). This means, \( \Delta u_2 \) makes sense. It remains to check that the mapping \( t \mapsto \Delta u_2(t) \) is continuous on \( (0, T] \). Let \( t > 0 \) and \( h \in (0, T - t] \), then
\[
\|\Delta u_2(t + h) - \Delta u_2(t)\| \leq \| \int_t^{t+h} \Delta S(t+h-\tau)f(u(\cdot, \tau)) d\tau \|
+ \| \int_0^t [\Delta S(t+h-\tau) - \Delta S(t-\tau)] f(u(\cdot, t)) d\tau \|
+ \| \int_0^t [\Delta S(t+h-\tau) - \Delta S(t-\tau)] [f(u(\cdot, \tau) - f(u(\cdot, t))] d\tau \|
= G_1(t) + G_2(t) + G_3(t).
\]
Since the function \( \tau \mapsto \Delta S(t+h-\tau)f(u(\cdot, \tau)) \) is integrable, we see that \( G_1(t) \to 0 \) as \( h \to 0 \).

For \( G_2(t) \), we have
\[
G_2(t) = \int_t^{t+h} \Delta S(\tau)f(u(\cdot, \tau)) d\tau - \int_0^t \Delta S(\tau)f(u(\cdot, t)) d\tau
\leq \int_t^{t+h} \Delta S(\tau)f(u(\cdot, t)) d\tau + \int_0^h \Delta S(\tau)f(u(\cdot, t)) d\tau
\to 0 \quad \text{as} \quad h \to 0,
\]
according to the fact that \( \tau \mapsto \Delta S(\tau)\xi \) is integrable for any \( \xi \in L^2(\Omega) \), as stated in Lemma 2.1(d).

Concerning \( G_3(t) \), we observe that
\[
G_h(\tau) := \| [\Delta S(t+h-\tau) - \Delta S(t-\tau)] [f(u(\cdot, \tau) - f(u(\cdot, t))] \| \to 0 \quad \text{as} \quad h \to 0,
\]
for all \( \tau \in (0, t) \), thanks to the fact that \( \Delta S(\cdot)\xi \in C((0, T]; L^2(\Omega)) \) for any \( \xi \in L^2(\Omega) \). Moreover, we get
\[
G_h(\tau) \leq [ (t+h-\tau)^{-1} + (t-\tau)^{-1} ] \kappa(R) \| u(\cdot, t) - u(\cdot, \tau) \|
\leq 2\kappa(R)(t-\tau)^{\beta - 1} (C_1 \tau^{-\beta} + C_2),
\]
according to the Hölder continuity of \( u \) and the estimate of \( \Delta S(t) \) given by Lemma 2.1(d). Thus
\[
G_3(t) \leq \int_0^t G_h(\tau) d\tau \to 0 \quad \text{as} \quad h \to 0,
\]
due to the Lebesgue dominated convergence theorem. This completes the proof. \( \square \)

**Theorem 3.5.** Assume that the assumptions of Theorem 3.3 are satisfied. Then the mild solution of (1.1)-(1.3) on \( [0, T] \) obeys \( m * (-\Delta)^{\gamma} u \in C^1((0, T]; L^2(\Omega)) \). Consequently, this solution is classical.

**Proof.** Noting that
\[
u(\cdot, t) = \sum_{n=1}^{\infty} [\omega(t, \lambda_n)\xi_n + \omega(\cdot, \lambda_n) * F_n(t)]\varphi_n = \sum_{n=1}^{\infty} u_n(t)\varphi_n,
\]
where \( F_n(t) = (f(u(\cdot, t)), \varphi_n) \), we have
\[
\sigma \star (-\Delta)^y u(\cdot, t) = \sum_{n=1}^{\infty} \lambda_n \varphi_n.
\]
So it suffices to show that, the series \( \sum_{n=1}^{\infty} \lambda_n \varphi_n \) is uniformly convergent on \([\epsilon, T]\) for any \( \epsilon \in (0, T) \). We first see that \( \lambda_n \varphi_n \) is uniformly convergent on \([\epsilon, T]\) thanks to the relaxation equation, then
\[
\lambda_n \varphi_n = \lambda_n \varphi_n = \lambda_n \varphi_n - \lambda_n \varphi_n - \lambda_n \varphi_n - \lambda_n \varphi_n.
\]
One can write
\[
\sum_{n=1}^{\infty} \lambda_n \varphi_n = -(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4),
\]
where
- \( \sigma_1 = \sum_{n=1}^{\infty} \omega'(t, \lambda_n) \xi_n \varphi_n \) is uniformly convergent on \([\epsilon, T]\) since \( \sigma_1 = S'(t) \xi \) and \( S(t) \xi \in C^1([0, T]; L^2(\Omega)) \).
- \( \sigma_2 = \sum_{n=1}^{\infty} \lambda_n \omega(t, \lambda_n) \xi_n \varphi_n \) is uniformly convergent on \([\epsilon, T]\) to \(-\Delta S(t) \xi\) due to that \( \Delta S(t) \xi \in C([0, T]; L^2(\Omega)) \).
- \( \sigma_3 = \sum_{n=1}^{\infty} \omega'(t, \lambda_n) \varphi_n \) is uniformly convergent on \([\epsilon, T]\) to \(-\Delta S(t) \xi\) due to that \( \Delta S(t) \xi \in C([0, T]; L^2(\Omega)) \).
- \( \sigma_4 = \sum_{n=1}^{\infty} \lambda_n \omega(t, \lambda_n) \varphi_n \) is uniformly convergent on \([\epsilon, T]\) to \(-\Delta S(t) \xi\) due to that \( \Delta S(t) \xi \in C([0, T]; L^2(\Omega)) \).

In summary, we get
\[
\partial_t(m \star (-\Delta)^y u(t)) = -(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)
\]
\[
= -S'(t) \xi + \Delta S(t) \xi - \int_0^t S'(t - \tau) f(u(\cdot, \tau)) d\tau + \int_0^t \Delta S(t - \tau) f(u(\cdot, \tau)) d\tau
\]
\[
= \Delta u(\cdot, t) - \partial_t u(\cdot, t) + f(u(\cdot, t)), \ \forall t \in (0, T],
\]
which ensures that \( u \) satisfies (1.1). The theorem is proved.

### 4. Stability results.
In this section, we show some results on the asymptotic stability, dissipativity and the convergence to the solution of the elliptic equation \(-\Delta v = f(v)\).

**Theorem 4.1.** Let the hypotheses of Theorem 3.2 hold. Then the zero solution to (1.1) is asymptotically stable.

**Proof.** Taking \( \rho, \delta \) and \( \epsilon \in (0, \lambda_1 - \ell) \) from the proof of Theorem 3.2, we see that for any initial data \( \|\xi\| \leq \delta \), there exists a unique solution \( u \) of (1.1)-(1.3) such that \( \|u(\cdot, t)\| \leq \rho \) and \( \|f(u(\cdot, t))\| \leq (\ell + \epsilon)\|u(\cdot, t)\| \) for any \( t \geq 0 \). Then
\[
\|u(\cdot, t)\| \leq \|S(t)\| + \int_0^t \|S(t - \tau)\| \|f(u(\cdot, \tau))\| d\tau
\]
\[
\leq \omega(t, \lambda_1, \lambda_1^\gamma) \|\xi\| + \int_0^t \omega(t - \tau, \lambda_1, \lambda_1^\gamma)(\ell + \epsilon)\|u(\cdot, \tau)\| d\tau,
\]
thanks to Proposition 1. Applying the Gronwall type inequality given by Proposition 3, we get
\[ \|u(\cdot, t)\| \leq \omega(t, \lambda_1 - \ell - \epsilon, \lambda_1)\|\xi\|, \quad \forall t \geq 0, \]
which implies that the zero solution of (1.1) is asymptotically stable, according to the fact that \( \lambda_1 - \ell - \epsilon > 0 \) and \( \omega(t, \mu) \to 0 \) as \( t \to \infty \) for any \( \mu > 0 \).

If the nonlinearity function \( f \) is globally Lipschitzian, we have a stronger result, whose proof is simply utilizing Proposition 3.

**Theorem 4.2.** Let (M) hold. If \( f \) satisfies the Lipschitz condition
\[ \|f(v_1) - f(v_2)\| \leq \kappa_0\|v_1 - v_2\|, \quad \forall v_1, v_2 \in L^2(\Omega), \]
where \( \kappa_0 \in [0, \lambda_1) \), then every solution of (1.1)-(1.2) is asymptotically stable.

**Remark 2.** The decay rate in the stability results given by Theorem 4.1 and 4.2 is governed by the relaxation function \( \omega \). Due to Proposition 1(a), if the kernel function \( m \) is nonincreasing then
\[ \omega(t, \lambda, \nu) \leq \frac{1}{1 + \lambda t(1 + \nu \lambda^{-1} m(t))}, \quad t > 0, \lambda > 0, \nu > 0. \]
Thus the decay rate is of \( O(t^{-1}) \) as \( t \to \infty \). In comparison with the decay rate for solutions of subdiffusion equations studied in [11, 22], this one is faster.

In the next theorem, we show a result on dissipativity of (1.1)-(1.3).

**Theorem 4.3.** Let (M) hold. If \( f \) satisfies the condition
\[ \|f(v)\| \leq a\|v\| + b, \quad \forall v \in L^2(\Omega), \]
where \( a \in [0, \lambda_1) \), and \( b \) is a nonnegative constant. Then there exists an absorbing set of mild solutions to (1.1)-(1.3) for any initial data.

**Proof.** Let \( u \) be a mild solution of (1.1)-(1.3). Then
\[ \|u(\cdot, t)\| \leq \omega(t, \lambda_1)\|\xi\| + \int_0^t \omega(t - \tau, \lambda_1)(a\|u(\cdot, \tau)\| + b)d\tau. \]
By using Proposition 3 again, we obtain
\[ \|u(\cdot, t)\| \leq \omega(t, \lambda_1 - a, \lambda_1)\|\xi\| + b \int_0^t \omega(t - \tau, \lambda_1 - a, \lambda_1) d\tau \]
\[ \leq \omega(t, \lambda_1 - a, \lambda_1)\|\xi\| + b \int_0^t \frac{1}{\lambda_1 - a} (1 - \omega(t, \lambda_1 - a, \lambda_1)) d\tau \]
\[ \leq \omega(t, \lambda_1 - a, \lambda_1)\|\xi\| + b \frac{\lambda_1 - a}{\lambda_1 - a}, \]
thanks to Proposition 1(b). It follows that the closed ball \( B_R \subset L^2(\Omega) \) with \( R = \frac{b}{\lambda_1 - a} + 1 \) is an absorbing set of solutions to (1.1)-(1.3) for any initial data. The proof is complete.

We are in a position to show a result on the convergence of solution of (1.1)-(1.3) to equilibrium point.
Let (M) hold. Assume that \( \partial \Omega \subset C^{2} \) and \( f : L^{2}(\Omega) \to L^{2}(\Omega) \) is continuous and locally bounded. If a mild solution of (1.1)-(1.3) obeys \( \lim_{t \to \infty} u(\cdot, t) = u^{\ast} \) in \( L^{2}(\Omega) \), then \( u^{\ast} \) is a strong solution of the elliptic problem

\[
-\Delta v = f(v) \quad \text{in} \quad \Omega, \quad (4.1)
\]

\[
v = 0 \quad \text{on} \quad \partial \Omega, \quad (4.2)
\]

provided that \( z \hat{m}(z) \to 0 \) as \( z \to 0 \) in \( \mathbb{C} \), where \( \hat{m} \) is the Laplace transform of \( m \). Conversely, if \( f \) satisfies the Lipschitz condition

\[
\| f(v_{1}) - f(v_{2}) \| \leq \kappa_{0} \| v_{1} - v_{2} \|, \quad \forall v_{1}, v_{2} \in L^{2}(\Omega), \quad (4.3)
\]

where \( \kappa_{0} \in [0, \lambda_{1}) \), then the solution of (1.1)-(1.3) converges to the unique strong solution of (4.1)-(4.2), provided that \( m \) is nonincreasing and \( \lim_{t \to \infty} m(t) = 0 \).

**Proof.** Let \( u \) be a mild solution to (1.1)-(1.3) such that \( \lim_{t \to \infty} u(\cdot, t) = u^{\ast} \) in \( L^{2}(\Omega) \). Then

\[
u(\cdot, t) = S(t) \xi + \int_{0}^{t} S(t - \tau) [f(u(\cdot, \tau)) - f(u^{\ast})] d\tau + \int_{0}^{t} S(t - \tau) f(u^{\ast}) d\tau.
\]

Obviously, \( \| S(t) \xi \| \leq \omega(t, \lambda_{1}) \| \xi \| \to 0 \) as \( t \to \infty \). On the other hand, since \( f(u(\cdot, t)) \to f(u^{\ast}) \) as \( t \to \infty \) in \( L^{2}(\Omega) \), there exists \( T > 0 \) such that \( \| f(u(\cdot, t)) - f(u^{\ast}) \| < \varepsilon \) for all \( t \geq T \), here \( \varepsilon > 0 \) is given. So for any \( t \geq T \), we get

\[
\| \int_{0}^{t} S(t - \tau) [f(u(\cdot, \tau)) - f(u^{\ast})] d\tau \|
\]

\[
\leq \int_{0}^{T} \omega(t - \tau, \lambda_{1}) \| f(u(\cdot, \tau)) - f(u^{\ast}) \| d\tau + \int_{T}^{t} \omega(t - \tau, \lambda_{1}) \| f(u(\cdot, \tau)) - f(u^{\ast}) \| d\tau
\]

\[
\leq \int_{0}^{T} \omega(t - \tau, \lambda_{1}) \| f(u(\cdot, \tau)) - f(u^{\ast}) \| d\tau + \varepsilon \int_{T}^{t} \omega(t - \tau, \lambda_{1}) d\tau
\]

\[
\leq C \int_{0}^{T} \omega(t - \tau, \lambda_{1}) d\tau + \varepsilon \int_{T}^{t} \omega(t - \tau, \lambda_{1}) d\tau,
\]

for some \( C > 0 \), thanks to the local boundedness of \( f \). Since \( \omega(\cdot, \lambda_{1}) \in L^{1}(\mathbb{R}^{+}) \), we have

\[
C \int_{0}^{T} \omega(t - \tau, \lambda_{1}) d\tau = C \int_{-T}^{t} \omega(\tau, \lambda_{1}) d\tau < C \varepsilon,
\]

for \( t \) large enough. In addition, we see that

\[
\varepsilon \int_{T}^{t} \omega(t - \tau, \lambda_{1}) d\tau = \varepsilon \int_{0}^{t - T} \omega(\tau, \lambda_{1}) d\tau < \varepsilon \lambda_{1}^{-1}.
\]

Thus

\[
\int_{0}^{t} S(t - \tau) [f(u(\cdot, \tau)) - f(u^{\ast})] d\tau \to 0 \quad \text{as} \quad t \to \infty.
\]

It follows that

\[
\lim_{t \to \infty} u(\cdot, t) = \lim_{t \to \infty} \int_{0}^{t} S(t - \tau) f(u^{\ast}) d\tau = \int_{0}^{\infty} S(\tau) f(u^{\ast}) d\tau.
\]

Equivalently, one has

\[
u^{\ast} = \hat{S}(0) f(u^{\ast}), \quad (4.4)
\]
where $\hat{S}$ is the Laplace transform of the resolvent family $S(\cdot)$, i.e.

$$\hat{S}(z)\xi = \int_0^\infty e^{-zt}S(t)\xi dt, \xi \in L^2(\Omega).$$

Since $S(t)$ is the resolvent operator for the Cauchy problem

$$\partial_t u + \partial_t (m * (-\Delta)^\gamma u) - \Delta u = 0 \text{ in } \Omega, t > 0,$$

$$u = 0 \text{ on } \partial\Omega, t \geq 0,$$

$$u(\cdot, 0) = \xi \text{ in } \Omega,$$

we get $u(\cdot, t) = S(t)\xi$ and then $\hat{S}(z)\xi = \hat{u}(\cdot, z)$. Taking the Laplace transform of the last system, we have

$$z\hat{u} + z\hat{m}(z)(-\Delta)^\gamma \hat{u} - \Delta \hat{u} = \xi.$$  

Thus $\hat{S}(z) = (zI + z\hat{m}(z)(-\Delta)^\gamma - \Delta)^{-1}$ and $\hat{S}(0) = (-\Delta)^{-1}$. Plugging into (4.4) yields $u^* = (-\Delta)^{-1}f(u^*)$. Employing the regularity result in [10, Sect. 6.3.2] with the assumption $\partial\Omega \in C^2$, we get $u^* \in H^2(\Omega)$. Thus $u^*$ is a strong solution of (4.1)-(4.2).

We now prove the converse statement. Assume that (4.3) holds, $m$ is nonincreasing and $m(t) \to 0$ as $t \to \infty$. By Remark 1 and Theorem 3.5, the problem (1.1)-(1.3) has a unique classical solution. On the other hand, due to [8, Theorem 7.4.1], the problem (4.1)-(4.2) has a unique weak solution $u^* \in H^1(\Omega)$ if the Lipschitz constant $\kappa_0$ satisfies $\kappa_0 < C^{-2}$, where $C_e$ is the constant of embedding $H^1(\Omega) \subset C^2$. By the smoothness of $\partial\Omega$, we have

$$C^{-2}_e = \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2} = \lambda_1.$$  

Observe that $f(u^*) \in L^2(\Omega)$. Then using the regularity result in [10, Sect. 6.3.2] again, we obtain $u^* \in H^2(\Omega)$ and hence $u^*$ is a unique strong solution of (4.1)-(4.2).

We now combine (1.1)-(1.3) with (4.1)-(4.2) to obtain

$$\partial_t (u - u^*) + \partial_t [m * (-\Delta)^\gamma (u - u^*)] - \Delta (u - u^*) = f(u) - f(u^*) - m(-\Delta)^\gamma u^*$$

in $\Omega$, with the boundary condition

$$u - u^* = 0 \text{ on } \partial\Omega,$$

and the initial condition

$$u(\cdot, 0) - u^* = \xi - u^* \text{ in } \Omega.$$  

Then $u - u^*$ admits the following representation

$$u(\cdot, t) - u^* = S(t)(\xi - u^*) + \int_0^t S(t - \tau)[f(\cdot, \tau) - f(u^*) - m(\tau)(-\Delta)^\gamma u^*]d\tau,$$

and hence

$$\|u(\cdot, t) - u^*\| \leq \omega(t, \lambda_1)\|\xi - u^*\|$$

$$+ \int_0^t \omega(t - \tau, \lambda_1)\|u(\cdot, \tau) - u^*\| + m(\tau)(-\Delta)^\gamma u^*\|d\tau.$$  

Utilizing the Gronwall type inequality in Proposition 3, we get

$$\|u(\cdot, t) - u^*\| \leq \omega(t, \lambda_1 - \kappa_0, \lambda_1^\gamma)\|\xi - u^*\| + \|(-\Delta)^\gamma u^*\| \int_0^t \omega(t - \tau, \lambda_1 - \kappa_0, \lambda_1^\gamma) m(\tau)d\tau.$$
Since $\omega(t, \lambda_1 - \kappa_0, \lambda_1^2) \to 0$ as $t \to \infty$, it is sufficient to testify that
\[
M(t) = \int_0^t \omega(t, \lambda_1 - \kappa_0, \lambda_1^2) m(\tau) d\tau \to 0 \text{ as } t \to \infty.
\]
By assumption, for given $\varepsilon > 0$, there is $T > 0$ such that $m(t) < \varepsilon$ for all $t \geq T$. Then for any $t \geq T$, one gets
\[
M(t) \leq \int_0^T \omega(t, \lambda_1 - \kappa_0, \lambda_1^2) m(\tau) d\tau + \varepsilon \int_T^t \omega(t, \lambda_1 - \kappa_0, \lambda_1^2) d\tau
\]
\[
\leq \omega(t - T, \lambda_1 - \kappa_0, \lambda_1^2) \int_0^T m(\tau) d\tau + \varepsilon \int_0^{t-T} \omega(\tau, \lambda_1 - \kappa_0, \lambda_1^2) d\tau
\]
\[
\leq \omega(t - T, \lambda_1 - \kappa_0, \lambda_1^2) \int_0^T m(\tau) d\tau + \varepsilon (\lambda_1 - \kappa_0)^{-1},
\]
thanks to Proposition 1(b). This implies
\[
M(t) \leq [1 + (\lambda_1 - \kappa_0)^{-1}] \varepsilon,
\]
provided $t$ chosen such that
\[
\omega(t - T, \lambda_1 - \kappa_0, \lambda_1^2) \int_0^T m(\tau) d\tau < \varepsilon,
\]
which is possible since $\omega(t - T, \lambda_1 - \kappa_0, \lambda_1^2) \to 0$ as $t \to \infty$. The proof is complete. 

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