A CLASS OF ETERNAL SOLUTIONS TO THE $G_2$-LAPLACIAN FLOW

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Abstract. We explicitly describe the solution of the $G_2$-Laplacian flow starting from an extremally Ricci-pinched closed $G_2$-structure on a compact 7-manifold and we investigate its properties. In particular, we show that the solution exists for all real times and that it remains extremally Ricci-pinched. This result holds more generally on any 7-manifold whenever the intrinsic torsion of the extremally Ricci-pinched $G_2$-structure has constant norm. We also discuss various examples.

1. Introduction

A $G_2$-structure on a seven-dimensional smooth manifold $M$ is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition. Such a 3-form gives rise to a Riemannian metric $g_\varphi$ and to a volume form $dV_\varphi$ on $M$.

By [10], the holonomy of $g_\varphi$ is contained in $G_2$ if both $d\varphi$ and $d^*\varphi$ vanish, $d^*\varphi$ being the Hodge operator of $g_\varphi$. On the other hand, when a Riemannian metric $g$ has $\text{Hol}(g) \subseteq G_2$, then there exists a unique $G_2$-structure $\varphi$ satisfying $d\varphi = 0$, $d^*\varphi = 0$ and such that $g_\varphi = g$. A $G_2$-structure defined by a non-degenerate 3-form $\varphi$ which is both closed and co-closed is said to be torsion-free and the corresponding Riemannian metric $g_\varphi$ is Ricci-flat. A $G_2$-structure $\varphi$ satisfying the less restrictive condition $d\varphi = 0$ is called closed. In such a case, the intrinsic torsion can be identified with a unique 2-form $\tau$ such that $d^*\varphi = \tau \wedge \varphi$, and the scalar curvature of $g_\varphi$ is given by $-\frac{1}{2}|\tau|^2_\varphi$, where $|\cdot|_\varphi$ denotes the norm induced by $g_\varphi$ (cf. [3]).

Closed $G_2$-structures with small torsion constitute the starting point in Joyce’s construction of compact 7-manifolds with holonomy $G_2$ [19]. Besides this and the glueing constructions [7, 21, 24], in recent years a lot of effort has been made in order to understand whether it is possible to obtain metrics with holonomy $G_2$ using a geometric flow approach. So far, the main results in this direction have been obtained for the $G_2$-Laplacian flow introduced by Bryant in [3]:

$$\begin{align*}
\frac{\partial}{\partial t} \varphi(t) &= \Delta_{\varphi(t)} \varphi(t), \\
d\varphi(t) &= 0, \\
\varphi(0) &= \varphi.
\end{align*}$$

Here, $\varphi$ is a given closed $G_2$-structure and $\Delta_{\varphi(t)}$ denotes the Hodge Laplacian of $g_{\varphi(t)}$.

Short-time existence and uniqueness of the solution of the Laplacian flow on a compact manifold were proved by Bryant and Xu in [4], while the geometric and analytic properties

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of the flow were deeply investigated by Lotay and Wei in [30, 31, 32]. In particular, they proved that the solution \( \varphi(t) \) exists as long as the velocity of the flow \( |\Delta \varphi(t)\varphi(t)| \) remains bounded. It is still an open problem whether a bound on the scalar curvature is sufficient to obtain a long-time existence result (cf. [30]). Further aspects of the Laplacian flow were studied in [9, 11, 12, 18, 25, 26, 29].

By [3], on a compact 7-manifold \( M \) the Ricci tensor and the scalar curvature of the metric induced by a closed \( G_2 \)-structure \( \varphi \) must satisfy the following inequality

\[
\int_M [\text{Scal}(g_\varphi)]^2 \, dV_\varphi \leq 3 \int_M |\text{Ric}(g_\varphi)|^2 \, dV_\varphi.
\]

In particular, the metric \( g_\varphi \) is Einstein if and only if the \( G_2 \)-structure is torsion-free (see also [5]). The above inequality reduces to an equality if and only if the intrinsic torsion form \( \tau \) fulfills the equation 

\[
d\tau = \frac{1}{6} |\tau|^2 \varphi + \frac{1}{6} \ast \varphi (\tau \wedge \tau).
\]

When this happens, the closed \( G_2 \)-structure is called extremally Ricci-pinched (ERP for short).

Two examples of manifolds endowed with an ERP closed \( G_2 \)-structure were obtained by Bryant [3] and Lauret [26]. Both can be described as simply connected solvable Lie groups endowed with a left-invariant ERP closed \( G_2 \)-structure. In the first case, the Lie group is not unimodular. Nevertheless, it admits a compact quotient by a torsion-free discrete subgroup of the full automorphism group of the \( G_2 \)-structure. In the second case, the Lie group is unimodular and the existence of a compact quotient has been recently proved in [23]. In [26], Lauret proved that in both examples the ERP closed \( G_2 \)-structure \( \varphi \) is a steady Laplacian soliton, i.e., it satisfies the equation

\[
\Delta \varphi = \mathcal{L}_X \varphi + \lambda \varphi,
\]

for \( \lambda = 0 \) and for some vector field \( X \). General results on Laplacian solitons (cf. e.g. [30, Sect. 9]) allow one to conclude that the solution of the Laplacian flow starting from one of these ERP closed \( G_2 \)-structures is self-similar and eternal, i.e., it exists for all real times.

By [29], compact Laplacian solitons with \( \lambda = 0 \) are necessarily torsion-free. Thus, none of the above examples can descend to a steady Laplacian soliton on any compact quotient of the corresponding Lie group and, more generally, ERP closed \( G_2 \)-structures on compact manifolds cannot be steady Laplacian solitons.

In the present paper, we study the behaviour of the Laplacian flow starting from an ERP closed \( G_2 \)-structure in greater generality. Our main results are contained in Section 4. In Theorem 4.1 we show that the solution of the Laplacian flow starting from an ERP closed \( G_2 \)-structure \( \varphi \) whose intrinsic torsion form \( \tau \) has constant norm is given by

\[
\varphi(t) = \varphi + f(t) \, d\tau,
\]

where \( f(t) = \frac{6}{|\tau|^2} \left( \exp \left( \frac{|\tau|^2}{6} t \right) - 1 \right) \). From this expression, we easily see that the solution exists for all real times. This result holds, in particular, when the closed \( G_2 \)-structure is ERP and the manifold \( M \) is compact. To prove Theorem 4.1 we first show some useful results on ERP closed \( G_2 \)-structures in Proposition 4.2 and then we use them to show that the Laplacian flow we are considering is equivalent to a Cauchy problem for the function \( f(t) \). The properties obtained in Proposition 4.2 allow us to conclude that the solution \( \varphi(t) \) is ERP with constant velocity for all \( t \in \mathbb{R} \), and that the Ricci tensor of \( g_\varphi(t) \) is constant along the flow. Finally, by backward uniqueness and real analyticity of the solution of the
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Laplacian flow on compact 7-manifolds \[30, 32\], we conclude that a solution cannot become ERP in finite time unless it starts from an ERP closed G₂-structure.

In Section 6, we study the asymptotic behaviour of the ERP solution \(\varphi(t)\) in the compact case. In particular, we show that the volume of the manifold with respect to the Riemannian metric \(g_{\varphi(t)}\) increases without bound as \(t \to +\infty\), while it shrinks as \(t \to -\infty\).

In Section 5, we review the two examples of ERP closed G₂-structures mentioned above, and we discuss some related results. In Example 6.4, we show that Bryant’s example belongs to a one-parameter family of inequivalent solvable Lie groups admitting a left-invariant ERP closed G₂-structure, while in Theorem 6.8 we prove that a unimodular Lie group endowed with a left-invariant ERP closed G₂-structure is isomorphic to Lauret’s example.

2. Preliminaries

2.1. Stable forms in dimension seven. According to [17], a \(k\)-form on a real \(n\)-dimensional vector space \(V\) is said to be stable if its \(\text{GL}(V)\)-orbit is open in \(\Lambda^k(V^*)\).

In the present paper, we shall mainly deal with stable 3-forms in dimension seven. They can be characterized as follows.

**Proposition 2.1 ([17]).** Let \(V\) be a seven-dimensional real vector space. Consider a 3-form \(\phi \in \Lambda^3(V^*)\) and the symmetric bilinear map

\[
b_{\phi} : V \times V \to \Lambda^7(V^*), \quad b_{\phi}(v, w) = \frac{1}{6} \iota_v \phi \wedge \iota_w \phi \wedge \phi.
\]

Then, \(\phi\) is stable if and only if \(\det(b_{\phi})^{1/9} \in \Lambda^7(V^*)\) is not zero.

Given a stable 3-form \(\phi\), the symmetric bilinear map

\[
g_{\phi} := \det(b_{\phi})^{-1/9} b_{\phi} : V \times V \to \mathbb{R}
\]

is either positive definite or it has signature \((3, 4)\). These conditions characterize the only two open \(\text{GL}(V)\)-orbits contained in \(\Lambda^3(V^*)\).

We denote the open orbit of stable 3-forms for which (2.1) is positive definite by \(\Lambda^3(V^*)^+\). It is well-known that the \(\text{GL}^+(V)\)-stabilizer of a 3-form \(\phi \in \Lambda^3(V^*)^+\) is isomorphic to the exceptional Lie group \(G_2\), and that there exists a basis \((e^1, \ldots, e^7)\) of \(V^*\) for which

\[
\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},
\]

\(e^{ijk}\) being a shorthand for \(e^i \wedge e^j \wedge e^k\).

2.2. Closed \(G_2\)-structures. Let \(M\) be a seven-dimensional smooth manifold and let \(\Lambda_+^3(T^*M)\) denote the open subbundle of \(\Lambda^3(T^*M)\) whose fibre over each point \(x \in M\) is given by \(\Lambda^3_+(T^*_xM)\).

A \(G_2\)-structure on \(M\), namely a \(G_2\)-reduction of the frame bundle \(FM \to M\), is characterized by the existence of a stable 3-form \(\varphi \in \Omega^3_+(M) := \Gamma(\Lambda_+^3(T^*M))\). This 3-form gives rise to a Riemannian metric \(g_{\varphi}\) with volume form \(dV_{\varphi}\) via the identity

\[
g_{\varphi}(X, Y) dV_{\varphi} = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,
\]

for all vector fields \(X, Y \in \mathfrak{X}(M)\). We denote by \(\ast_{\varphi}\) the Hodge operator determined by \(g_{\varphi}\), and by \(| \cdot |_{\varphi}\) the pointwise norm induced by \(g_{\varphi}\).
It follows from the discussion in Section 2.1 that at each point $x$ of $M$ there exists a basis $B^* = (e^1, \ldots, e^7)$ of the cotangent space $T^*_x M$ for which $\varphi|_x$ can be written as in (2.2). We shall call $B^*$ an adapted basis for the $G_2$-structure $\varphi$.

The intrinsic torsion of a $G_2$-structure $\varphi$ can be identified with the covariant derivative of $\varphi$ with respect to the Levi Civita connection $\nabla^\varphi$ of $g_\varphi$. By [10], a $G_2$-structure $\varphi$ is torsion-free, i.e., $\nabla^\varphi \varphi \equiv 0$, if and only if the 3-form $\varphi$ is both closed and coclosed.

A $G_2$-structure is said to be closed if the defining 3-form $\varphi$ satisfies the equation $d \varphi = 0$. When this happens, the intrinsic torsion can be identified with a unique 2-form $\tau \in \Omega^2_1(M) := \{\kappa \in \Omega^2(M) | \kappa \wedge *\varphi = - * \varphi \kappa \}$ such that $d * \varphi = \tau \wedge \varphi$.

Clearly, the intrinsic torsion form $\tau$ vanishes identically if and only if the $G_2$-structure is torsion-free. Notice that $\tau = d^* \varphi = - * \varphi d^* \varphi$, thus it is coclosed and its exterior derivative coincides with the Hodge Laplacian $\Delta \varphi \varphi = (dd^* + d^* d) \varphi = - d^* * \varphi d^* \varphi$ of $\varphi$. Properties of closed $G_2$-structures were investigated in [3, sect. 4.6] and in [5].

By [15], the closed 3-form $\varphi$ defines a calibration on $M$. An oriented three-dimensional submanifold of $M$ is called associative if it is calibrated by $\varphi$, while an oriented four-dimensional submanifold $N$ is called coassociative if $\varphi|_N \equiv 0$ (see [15, Sect. IV] and [20, Ch. 12] for more details).

By [3], the Ricci tensor and the scalar curvature of the Riemannian metric $g_\varphi$ induced by a $G_2$-structure $\varphi$ can be expressed in terms of the intrinsic torsion. In particular, when $\varphi$ is closed the Ricci tensor has the following expression,

$$\text{Ric}(g_\varphi) = \frac{1}{4} |\tau|_\varphi^2 g_\varphi - \frac{1}{4} j_\varphi \left( d\tau - \frac{1}{2} * \varphi (\tau \wedge \tau) \right),$$

where the map $j_\varphi : \Omega^3(M) \to S^2(M)$ is defined as follows

$$j_\varphi(\beta)(X,Y) = * \varphi (\iota_X \varphi \wedge \iota_Y \varphi \wedge \beta),$$

and the scalar curvature is given by

$$\text{Scal}(g_\varphi) = - \frac{1}{2} |\tau|_\varphi^2.$$

2.3. The $G_2$-Laplacian flow. Consider a 7-manifold $M$ endowed with a closed $G_2$-structure $\varphi$. The Laplacian flow starting from $\varphi$ is the initial value problem

$$\begin{cases}
\frac{\partial}{\partial t} \varphi(t) = \Delta_\varphi(t) \varphi(t), \\
d \varphi(t) = 0, \\
\varphi(0) = \varphi.
\end{cases}$$

This flow was introduced by Bryant in [3] to study seven-dimensional manifolds admitting closed $G_2$-structures. Short-time existence and uniqueness of the solution of (2.3) when $M$ is compact were proved in [4].

**Theorem 2.2 ([4]).** Assume that $M$ is compact. Then, the Laplacian flow (2.3) has a unique solution defined for a short time $t \in [0, \varepsilon)$, with $\varepsilon$ depending on $\varphi$. 
Remark 2.3. The condition \( d\varphi(t) = 0 \) implies that the solution of \((2.3)\) must belong to the open set

\[ [\varphi]_+ := [\varphi] \cap \Omega^3_+(M) \]

in the de Rham cohomology class of \( \varphi \) as long as it exists.

By [30, Thm. 1.6], the solution \( \varphi(t) \) exists as long as the velocity of the flow \( |\Delta\varphi(t)\varphi(t)|_{\varphi(t)} \) remains bounded. Moreover, if \( \varphi(t) \) is defined on some interval \([0, T]\), then for each fixed time \( t \in (0, T) \), \((M, \varphi(t), g_{\varphi(t)})\) is real analytic [32].

Using general results on flows of \( G_2 \)-structures (see e.g. [3, 22]), it is possible to check that the evolution equation of the Riemannian metric \( g_{\varphi(t)} \) induced by a \( G_2 \)-structure \( \varphi(t) \) evolving under the Laplacian flow is given by

\[
\frac{\partial}{\partial t} g_{\varphi(t)} = -2 \text{Ric}(g_{\varphi(t)}) + \frac{|\tau(t)|^2_{\varphi(t)}}{6} g_{\varphi(t)} + \frac{1}{4} I_{\varphi(t)}(\ast_{\varphi(t)}(\tau(t) \wedge \tau(t))) ,
\]

(see also [30]), and the corresponding volume form \( dV_{\varphi(t)} \) evolves as follows

\[
\frac{\partial}{\partial t} dV_{\varphi(t)} = \frac{|\tau(t)|^2_{\varphi(t)}}{3} dV_{\varphi(t)}.
\]

In particular, \( dV_{\varphi(t)} \) is pointwise non-decreasing.

A solution of the Laplacian flow is said to be self-similar if it is of the form

\[
\varphi(t) = \varrho(t) F_t^* \varphi,
\]

where \( F_t \in \text{Diff}(M) \) and \( \varrho(t) \in \mathbb{R} \smallsetminus \{0\} \) is a scaling factor. A standard argument allows one to show that the solution of the Laplacian flow is self-similar if and only if the initial datum \( \varphi \) satisfies the equation

\[
\Delta \varphi = \mathcal{L}_X \varphi + \lambda \varphi,
\]

for some vector field \( X \) on \( M \) and some \( \lambda \in \mathbb{R} \) (see e.g. [29, 30]). In such a case, \( \varrho(t) = (1 + \frac{2}{3} \lambda t)^{3/2} \). A closed \( G_2 \)-structure for which \((2.5)\) holds is called a Laplacian soliton. Depending on the sign of \( \lambda \), a Laplacian soliton is said to be shrinking (\( \lambda < 0 \)), steady (\( \lambda = 0 \)), or expanding (\( \lambda > 0 \)), and the corresponding self-similar solution exists on the maximal time interval \((-\infty, -\frac{3}{\lambda}), (-\infty, +\infty), (-\frac{3}{\lambda}, +\infty)\), respectively.

3. Extremally Ricci-pinched closed \( G_2 \)-structures

Let \( M \) be a compact 7-manifold endowed with a closed \( G_2 \)-structure \( \varphi \). It was proved independently in [3] and [5] that the Riemannian metric \( g_\varphi \) cannot be Einstein unless \( \varphi \) is torsion-free. Moreover, by [3] the Ricci tensor \( \text{Ric}(g_\varphi) \) and the scalar curvature \( \text{Scal}(g_\varphi) \) of \( g_\varphi \) must satisfy the integral inequality

\[
\int_M [\text{Scal}(g_\varphi)]^2 dV_\varphi \leq 3 \int_M |\text{Ric}(g_\varphi)|^2 dV_\varphi ,
\]

(3.1)

and (3.1) reduces to an equality if and only if the intrinsic torsion form \( \tau \) fulfills

\[
d\tau = \frac{1}{6} |\tau|^2_{\varphi} \varphi + \frac{1}{6} \ast_{\varphi}(\tau \wedge \tau). 
\]

(3.2)

This motivates the following.
Definition 3.1 (3). A closed $G_2$-structure $\varphi$ whose intrinsic torsion form $\tau$ satisfies (3.2) is said to be extremally Ricci-pinched (ERP for short).

Useful properties of ERP closed $G_2$-structures can be derived starting from (3.2) (cf. 3, Sect. 4.6). We summarize some of them in the next proposition.

Proposition 3.2 (3). Let $M$ be a 7-manifold endowed with an ERP closed $G_2$-structure $\varphi$ with intrinsic torsion form $\tau \in \Omega^2_{14}(M)$ not identically vanishing. If $M$ is compact, then $\tau$ has constant (non-zero) norm. More generally, if $|\tau|_\varphi$ is constant, then the following results hold

1) $\tau \wedge \tau \wedge \tau = 0$;
2) $\tau \wedge \tau$ is a non-zero closed simple 4-form of constant norm;
3) $^\ast_\varphi(\tau \wedge \tau)$ is a non-zero closed simple 3-form of constant norm;
4) by points 2) and 3), the tangent bundle of $M$ splits into the orthogonal direct sum of two integrable subbundles $TM = P \oplus Q$ with

$$P := \{X \in TM \mid i_X (\tau \wedge \tau) = 0\}, \quad Q := \{X \in TM \mid i_X {^\ast_\varphi}(\tau \wedge \tau) = 0\}.$$

Moreover, the $P$-leaves are associative submanifolds calibrated by $- \frac{1}{6} |\tau|_\varphi^4$, while the $Q$-leaves are coassociative submanifolds calibrated by $- \frac{1}{6} |\tau|_\varphi^2$.

4. The Laplacian flow starting from an ERP closed $G_2$-structure

In this section, we prove the following result.

Theorem 4.1. Let $M$ be a seven-dimensional manifold endowed with an ERP closed $G_2$-structure $\varphi$ whose intrinsic torsion form $\tau$ has constant non-zero norm. Then, the solution of the Laplacian flow starting from $\varphi$ at $t = 0$ is

$$\varphi(t) = \varphi + f(t) \, d\tau,$$

where

$$f(t) = 6 \frac{|\tau|_\varphi^2}{|\tau|_\varphi^4} \left( \exp \left( \frac{|\tau|_\varphi^2}{6} t \right) - 1 \right).$$
In particular, \( \varphi(t) \) is defined for all real times, it is ERP, and the corresponding intrinsic torsion form is given by

\[
\tau(t) = \exp \left( \frac{|\tau|^2}{6} t \right) \tau.
\]

Finally, the Ricci tensor of the Riemannian metric \( g_{\varphi(t)} \) induced by \( \varphi(t) \) is constant along the flow, i.e., \( \text{Ric}(g_{\varphi(t)}) = \text{Ric}(g_{\varphi}) \), and \( g_{\varphi(t)} \) evolves as follows

\[
\frac{\partial}{\partial t} g_{\varphi(t)} = \frac{|\tau|^2}{6} g_{\varphi(t)} \mid_Q.
\]

The proof of the first assertion in Theorem 4.1 consists in showing that the closed 3-form \( \varphi(t) \) given by (4.1) defines a G\( _2 \)-structure and that the G\( _2 \)-Laplacian flow (2.3) for \( \varphi(t) \) is equivalent to a Cauchy problem for the function \( f(t) \). To this aim, it is useful to investigate the properties of the 3-form

\[
(4.2) \quad \bar{\varphi} := \varphi + a \, d\tau = \left( 1 + \frac{1}{6} |\tau|^2 a \right) \varphi + \frac{a}{6} \ast_{\varphi} (\tau \wedge \tau),
\]

where \( \varphi \) is an ERP closed G\( _2 \)-structure with intrinsic torsion form \( \tau \) of constant norm, and \( a \) is a real number. We collect them in the next result.

**Proposition 4.2.** Let \( \varphi \) be an ERP closed G\( _2 \)-structure, assume that its intrinsic torsion form \( \tau \) has constant norm, and consider the closed 3-form \( \bar{\varphi} \) given by (4.2). Then, \( \bar{\varphi} \) defines a closed G\( _2 \)-structure for all \( a > -6|\tau|^2 \). Whenever this happens, the following hold

1) the \( \varphi \)-orthogonal decomposition \( TM = P \oplus Q \) given in point iv) of Proposition 3.2 is also \( \bar{\varphi} \)-orthogonal. Moreover, \( g_{\varphi} \mid_P = g_{\bar{\varphi}} \mid_P \), and \( g_{\varphi} \mid_Q = \left( 1 + \frac{1}{6} |\tau|^2 a \right) g_{\varphi} \mid_Q \);
2) the volume form induced by \( \bar{\varphi} \) is \( dV_{\bar{\varphi}} = (1 + \frac{1}{6} |\tau|^2 a) \frac{1}{2} dV_{\varphi} \);
3) the Hodge dual of \( \bar{\varphi} \) is \( \ast_{\bar{\varphi}} \bar{\varphi} = (1 + \frac{1}{6} |\tau|^2 a) \left( \ast_{\varphi} \varphi - \frac{a}{6} \tau \wedge \tau \right) \);
4) the intrinsic torsion form of \( \bar{\varphi} \) is given by \( \bar{\tau} = (1 + \frac{1}{6} |\tau|^2 a) \tau \), it satisfies the identity \( \ast_{\bar{\varphi}} (\bar{\tau} \wedge \bar{\tau}) = \ast_{\varphi} (\tau \wedge \tau) \), and its \( \bar{\varphi} \)-norm coincides with the \( \varphi \)-norm of \( \tau \), i.e., \( |\bar{\tau}|_{\bar{\varphi}} = |\tau|_{\varphi} \).
Consequently, \( \text{Scal}(g_{\bar{\varphi}}) = \text{Scal}(g_{\varphi}) \);
5) the closed G\( _2 \)-structure \( \bar{\varphi} \) is ERP and \( \text{Ric}(g_{\bar{\varphi}}) = \text{Ric}(g_{\varphi}) \).

**Proof.** We begin showing that \( \bar{\varphi} \) is stable for all \( a \neq -6|\tau|^2 \) and that it defines a G\( _2 \)-structure for all \( a > -6|\tau|^2 \). What we need to prove is that \( \bar{\varphi} \mid_x \in \Lambda^3_x(T_xM) \) for all \( x \in M \). Let us fix a basis \( (e_1, \ldots, e_7) \) of \( T_xM \) whose dual basis \( (e^1, \ldots, e^7) \) of \( T^*_xM \) is an adapted basis for the closed G\( _2 \)-structure \( \varphi \). Then, we have

\[
\varphi \mid_x = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},
\]

and \( g_{\varphi} \mid_x = \sum_{i=1}^7 (e^i)^2 \). By point iv) of Proposition 3.2 we know that \( T_xM = P_x \oplus Q_x \). Without loss of generality, we may assume that \( P_x = \langle e_1, e_2, e_3 \rangle \) and \( Q_x = \langle e_4, e_5, e_6, e_7 \rangle \). Since \( P = \ker(\tau \wedge \tau) \), the simple 4-form \( \tau \wedge \tau \mid_x \) must be proportional to \( e^{4567} \). This implies that \( \tau \mid_x \in \Lambda^2(Q^*_x) \). Consequently, as \( \tau \wedge \ast_{\varphi} \varphi = 0 \), there exist some real numbers \( c_1, c_2, c_3 \) for which

\[
\tau \mid_x = c_1 (e^{45} - e^{67}) + c_2 (e^{46} + e^{57}) + c_3 (e^{47} - e^{56}),
\]
In particular, \((\tau \wedge \tau)|_x = -|\tau|^2|_\varphi e^{4567}\) and \(*_\varphi(\tau \wedge \tau)|_x = -|\tau|^2|_\varphi e^{123}\).

We can now compute the symmetric bilinear map \(b_{\varphi}|_x : T_xM \times T_xM \rightarrow \Lambda^7(T_x^*M)\) and see that

\[
\det(b_{\varphi}|_x)^\frac{1}{6} = \left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right)^2 \det(b_{\varphi}|_x)^\frac{1}{6}.
\]

Thus, the 3-form \(\tilde{\varphi}_|_x\) is stable if and only if \(a \neq -6|\tau|^2\). Moreover, we have that

\[
g_{\varphi}|_x = \det(b_{\varphi}|_x)^\frac{1}{6} b_{\varphi}|_x = \sum_{i=1}^3 (e_i)^2 + \left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right) \sum_{i=4}^7 (e_i)^2.
\]

Hence, \(\varphi|_x \in \Lambda^3(T_x^*M)\) if and only if \(a > -6|\tau|^2\), as we claimed.

Now, assertions \((1)\) and \((2)\) follow immediately from the above discussion, while assertion \((3)\) can be checked pointwise using the adapted basis for \(\varphi\) we are considering.

As for \((4)\), we have

\[
d *_{\tilde{\varphi}} \tilde{\varphi} = \left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right) \left(d *_{\varphi} \varphi - \frac{a}{6} d(\tau \wedge \tau)\right) = \left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right) \tau \wedge \tilde{\varphi},
\]

since \(\tau \wedge \tau\) is closed. Thus, by the uniqueness of the 2-form \(\tilde{\tau}\) for which \(d *_{\tilde{\varphi}} \tilde{\varphi} = \tilde{\tau} \wedge \tilde{\varphi}\), we obtain \(\tilde{\tau} = (1 + \frac{1}{6}|\tau|^2|_\varphi a\) \tau. We can compute the Hodge dual of \(\tilde{\tau}\) as follows

\[
*_{\tilde{\varphi}} \tilde{\tau} = -\tilde{\tau} \wedge \tilde{\varphi} = -\left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right) \tau \wedge \varphi = \left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right) *_{\varphi} \tau.
\]

Consequently, we get

\[
|\tau|^2|_\varphi dV_{\varphi} = \tilde{\tau} \wedge *_{\tilde{\varphi}} \tilde{\tau} = \left(1 + \frac{1}{6}|\tau|^2|_\varphi a\right) \left|\tau|^2|_\varphi dV_{\varphi} = |\tau|^2|_\varphi dV_{\varphi}.
\]

Finally, a pointwise computation as in \((3)\) allows us to show that \(*_{\tilde{\varphi}}(\tilde{\tau} \wedge \tilde{\varphi}) = *_{\varphi}(\tau \wedge \tau)\). Using this, it is straightforward to check that \(\tilde{\varphi}\) is ERP. Now, the integrable subbundles \(\tilde{P}\) and \(\tilde{Q}\) determined by the ERP closed \(G_2\)-structure \(\tilde{\varphi}\) coincide with those determined by \(\varphi\). Consequently, by point \((\mathfrak{v})\) of Proposition \((3.2)\) and point \((\mathfrak{i})\) above, we have

\[
Ric(g_{\varphi}) = -\frac{1}{6}|\tau|^2|_\varphi g_{\varphi}|_\tilde{P} = -\frac{1}{6}|\tau|^2|_\varphi g_{\varphi}|_P = -\frac{1}{6}|\tau|^2|_\varphi g_{\varphi}|_P = Ric(g_{\varphi}).
\]

We are now ready to prove the main theorem of this section.

**Proof of Theorem \((4.1)\).** Consider the closed 3-form \(\varphi(t) = \varphi + f(t)d\tau\), where \(f\) is a real valued smooth function such that \(f(0) = 0\). Let \(t\) be small enough so that \(f(t) > -6|\tau|^2\).

Then, by Proposition \((4.2)\) the 3-form \(\varphi(t)\) defines a closed \(G_2\)-structure with intrinsic torsion form \(\tau(t) = (1 + \frac{1}{6}|\tau|^2|_\varphi f(t)) \tau\). Now, the Laplacian flow equation \(\frac{\partial}{\partial t} \varphi(t) = \Delta_{\varphi(t)} \varphi(t)\) reads

\[
\frac{d}{dt} f(t) d\tau = \left(1 + \frac{1}{6}|\tau|^2|_\varphi f(t)\right) d\tau.
\]
Consequently, $\varphi(t)$ is the solution of the Laplacian flow starting from $\varphi$ at $t = 0$ if and only if $f(t)$ solves the Cauchy problem

$$\begin{cases}
\frac{df}{dt} = 1 + \frac{1}{6}\left|\tau\right|^2 f(t), \\
f(0) = 0.
\end{cases}$$

Thus, we have

$$f(t) = \frac{6}{\left|\tau\right|^2} \left( \exp \left( \frac{\left|\tau\right|^2}{6} t \right) - 1 \right),$$

whence we see that $f(t)$ is defined for all $t \in \mathbb{R}$ and that it satisfies the condition $f(t) > -6\left|\tau\right|^2$. The second part of the theorem follows immediately from points 4) and 5) of Proposition 4.2. In particular, the evolution equation of the metric $g_{\varphi(t)}$ can be obtained starting from equation (2.4) and using the identity $\frac{1}{4} j_{\varphi}(\ast\varphi(\tau \wedge \tau)) = 3 \text{Ric}(g_{\varphi})$ (cf. point v) of Proposition 3.2).

When $M$ is compact, backward uniqueness and real analyticity of the solution of the Laplacian flow (cf. [30, Thm. 1.4] and [32]) together with Theorem 4.1 imply the following.

**Corollary 4.3.** Let $\varphi(t)$ be the solution of the Laplacian flow (2.3) on a compact 7-manifold $M$ and assume that $\varphi(0)$ is not ERP. Then, $\varphi(t)$ cannot become ERP in finite time.

Furthermore, from point 4) of Proposition 4.2 and Remark 3.3 we see that the velocity of the flow is constant for all $t \in \mathbb{R}$:

$$\left|\Delta_{\varphi(t)}\varphi(t)\right|_{\varphi(t)} = \left|d\tau(t)\right|_{\varphi(t)} = \frac{1}{\sqrt{6}}\left|\tau\right|^2_{\varphi}.$$

Since the solution $\varphi(t)$ of the Laplacian flow starting from an ERP closed $G_2$-structure $\varphi$ exists for all real times, and since the Ricci tensor of the corresponding metric $g_{\varphi(t)}$ does not evolve along the flow, it is natural to ask whether $\varphi(t)$ is self-similar or, equivalently, if $\varphi$ is a steady Laplacian soliton. It follows from [29, Cor. 1] that the answer is negative when the manifold $M$ is compact (see also [30, Prop. 9.5]). In detail:

**Theorem 4.4** ([29]). Let $M$ be a compact 7-manifold. Then, the only steady Laplacian solitons on $M$ are given by torsion-free $G_2$-structures.

On the other hand, it was recently shown in [27] that any left-invariant ERP closed $G_2$-structure on a non-compact Lie group is a steady Laplacian soliton. The converse of this result does not hold, as there are examples of left-invariant steady Laplacian solitons that are not ERP, see [14].

5. **Asymptotic behaviour of the ERP solution**

In this section, we assume that the 7-manifold $M$ is compact and we investigate the behaviour of the ERP solution when $t \to \pm \infty$.

Using point 2) of Proposition 4.2 we obtain the following expression for the volume form induced by $\varphi(t)$

$$dV_{\varphi(t)} = \exp \left( \frac{\left|\tau\right|^2}{3} t \right) dV_{\varphi}.$$
Consequently, the total volume of the compact 7-manifold $M$ with respect to the metric $g_{\varphi(t)}$ is
\[
\text{Vol}_{g_{\varphi(t)}}(M) = \int_M dV_{\varphi(t)} = \exp\left(\frac{|\tau|^2_{\varphi} t}{3}\right) \text{Vol}_{g_{\varphi}}(M),
\]
whence we easily see that it increases without bound as $t$ goes to $+\infty$, while it shrinks as $t$ goes to $-\infty$:
\[
\lim_{t \to +\infty} \text{Vol}_{g_{\varphi(t)}}(M) = +\infty, \quad \lim_{t \to -\infty} \text{Vol}_{g_{\varphi(t)}}(M) = 0.
\]

We now study the behaviour of the associative $P$-leaves and coassociative $Q$-leaves along the flow. As the integrable subbundles $P(t)$ and $Q(t)$ determined by the ERP closed $G_2$-structure $\varphi(t)$ coincide with the subbundles $P$ and $Q$ determined by $\varphi = \varphi(0)$, at each time $t \in \mathbb{R}$ we can endow the $P$-leaves and the $Q$-leaves with the Riemannian metric induced by $g_{\varphi(t)}$. We denote the corresponding Riemannian volume forms by $dV_P(t)$ and $dV_Q(t)$, respectively, and we let $dV_P := dV_P(0)$, $dV_Q := dV_Q(0)$.

Consider an oriented $P$-leaf $L_P \hookrightarrow M$. By point iv) of Proposition 3.2, we know that the volume form $dV_P(t)$ on $L_P$ coincides with the restriction of the closed 3-form $-|\tau(t)|^2_{\varphi(t)} \ast \varphi(t) (\tau(t) \wedge \tau(t))$. Such a volume form is constant along the flow, as
\[
dV_P = -|\tau|^2_{\varphi} \ast \varphi (\tau \wedge \tau)\big|_{L_P} = dV_P(t),
\]
by point 4) of Proposition 4.2. Using the same result, we see that the volume form $dV_Q(t)$ of an oriented $Q$-leaf $L_Q \hookrightarrow M$ is given by
\[
dV_Q(t) = -|\tau(t)|^2_{\varphi(t)} (\tau(t) \wedge \tau(t))\big|_{L_Q} = -|\tau|^2_{\varphi} \exp\left(\frac{|\tau|^2_{\varphi} t}{3}\right) (\tau \wedge \tau)\big|_{L_Q}
\]
\[
= \exp\left(\frac{|\tau|^2_{\varphi} t}{3}\right) dV_Q.
\]

It is now immediate to prove the following.

**Proposition 5.1.** Let $\varphi(t)$ be the solution of the Laplacian flow starting from an ERP closed $G_2$-structure on a compact 7-manifold $M$. Then, the volume of the $P$-leaves is constant along the flow. Moreover:

1. when $t \to +\infty$, the volume of the $P$-leaves goes to zero relative to the volume of the manifold, while the volume of the $Q$-leaves and the volume of the manifold $M$ grow at the same rate;
2. when $t \to -\infty$, the volume of the $Q$-leaves and the volume of the manifold $M$ tend to zero at the same rate.

**Remark 5.2.** Rescaling the metric $g_{\varphi(t)}$ as $\exp\left(-\frac{|\tau|^2_{\varphi} t}{6}\right) g_{\varphi(t)}$ shows that the volume of the $P$-leaves goes to zero as $t \to +\infty$, that is, the $P$-leaves collapse as $t \to +\infty$. 

6. Examples

In this section, we review two examples of 7-manifolds endowed with an ERP closed $G_2$-structure obtained in [3, 26], and we discuss some related results. Further examples are considered in [2, 27, 28].

We begin with the example obtained by Bryant in [3, Ex. 1]. It consists of an invariant ERP closed $G_2$-structure on the non-compact homogeneous space $SL(2, \mathbb{C}) \rtimes \mathbb{C}^2 / SU(2)$. For the sake of convenience, we give the following alternative description (cf. [6, Sect. 6.3] and [26, Ex. 4.13]).

**Example 6.1.** Let $\mathfrak{r} = \langle e_1, e_2, e_3 \rangle$ be the three-dimensional non-unimodular solvable Lie algebra with non-zero Lie brackets

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3.$$

Consider the abelian Lie algebra $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7 \rangle$, and the Lie algebra homomorphism $\mu : \mathfrak{r} \to \text{Der}(\mathbb{R}^4) \cong gl(4, \mathbb{R})$ defined as follows

$$\mu(e_1) = \text{diag} \left( -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \quad \mu(e_2) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right), \quad \mu(e_3) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right).$$

The semidirect product $\mathfrak{s} := \mathfrak{r} \ltimes \mu \mathbb{R}^4$ is a seven-dimensional non-unimodular completely solvable Lie algebra. Denoted by $(e^1, \ldots, e^7)$ the dual basis of $(e_1, \ldots, e_7)$, the structure equations $(de^i) = (0, -e^{12}, -e^{13}, \frac{1}{2} e^{14}, \frac{1}{2} e^{15}, -\frac{1}{2} e^{16} + e^{25} + e^{34}, -\frac{1}{2} e^{17} + e^{24} - e^{35})$.

It is now straightforward to check that the 3-form $\varphi$ given by (2.2) defines an ERP closed $G_2$-structure on $\mathfrak{s}$ with intrinsic torsion form

$$\tau = 3 e^{45} - 3 e^{67}.$$ 

From this we see that $|\tau|^2 = 18$, $P = \mathfrak{r}$, and $Q = \mathbb{R}^4$. Notice that $\varphi$ is exact:

$$\varphi = d \left( -\frac{1}{2} e^{23} + e^{45} - e^{67} \right).$$

Left-multiplication allows to extend the 3-form $\varphi$ to a left-invariant one, say $\check{\varphi}$, on the simply connected solvable Lie group $S$ with Lie algebra $\mathfrak{s}$. Bryant’s example is then described by the pair $(S, \check{\varphi})$.

**Remark 6.2.** Albeit the Lie algebra $\mathfrak{s}$ is not unimodular, the corresponding simply connected solvable Lie group $S$ admits a compact quotient, as it is acted on by a torsion-free discrete subgroup $\Gamma \subset \text{Aut}(S, \check{\varphi})$ (cf. [3, Ex. 1] and [26, Remark 4.14]). This gives rise to a compact locally homogeneous example of ERP closed $G_2$-structure on $\Gamma \backslash S$.

As observed by Cleyton and Ivanov [6], in the above example the intrinsic torsion form $\tau$ is parallel with respect to the canonical $G_2$-connection $\nabla$. More generally, the following holds.
Let $M$ be a 7-manifold endowed with a closed $G_2$-structure $\varphi$ and let $\nabla$ be the corresponding canonical $G_2$-connection. If the intrinsic torsion form $\tau$ of $\varphi$ is parallel with respect to $\nabla$, then $\varphi$ is ERP and $(M, g_\varphi)$ is locally isometric to Bryant’s example.

We now describe a one-parameter family of pairwise non-isomorphic solvable Lie algebras admitting an ERP closed $G_2$-structure and including the above example.

Example 6.4. Let us consider the one-parameter family of three-dimensional non-unimodular solvable Lie algebras $\mathfrak{t}_\eta = \langle e_1, e_2, e_3 \rangle$ with non-zero Lie brackets

$$[e_1, e_2] = e_2 + \eta e_3, \quad [e_1, e_3] = -\eta e_2 + e_3, \quad \eta \in \mathbb{R}.$$ 

Notice that $\mathfrak{t}_0 = \mathfrak{t}$, while the number $1 + \eta^2$ provides a complete isomorphism invariant for $\mathfrak{t}_\eta$ when $\eta \neq 0$ (see [[33] Lemma 4.10]). Moreover, it is possible to check that the Ricci endomorphism of the inner product $g = (e^1)^2 + (e^2)^2 + (e^3)^2$ on $\mathfrak{t}_\eta$ is diagonal with eigenvalue $-2$ of multiplicity three (cf. [[33] Thm. 4.11]).

We let $\mathfrak{s}_\eta := \mathfrak{t}_\eta \ltimes \mu_\eta : \mathbb{R}^4$, where $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7 \rangle$ is the abelian Lie algebra, and the Lie algebra homomorphism $\mu_\eta : \mathfrak{t}_\eta \to \text{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ is defined as follows

$$\mu_\eta(e_1) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \eta \\ 0 & 0 & -\eta & \frac{1}{2} \end{pmatrix}, \quad \mu_\eta(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu_\eta(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

When $\eta \neq 0$, the seven-dimensional non-unimodular Lie algebra $\mathfrak{s}_\eta$ is solvable but not completely solvable, as the complex numbers $1 \pm i\eta, \frac{1}{2} \pm i\eta$ are eigenvalues of $\text{ad}_{e_1}$. Furthermore, $\mathfrak{s}_0$ coincides with the Lie algebra $\mathfrak{s}$ considered in Example 6.1.

Now, the 3-form $\varphi$ given by $\varphi = (2.2)$ is exact, and it defines an ERP closed $G_2$-structure on $\mathfrak{s}_\eta$ with intrinsic torsion form $\tau = 3 e^{45} - 3 e^{67}$. By left-multiplication, we can extend $\varphi$ to a left-invariant ERP closed $G_2$-structure $\widetilde{\varphi}_\eta$ on the simply connected solvable Lie group $S_\eta$ with Lie algebra $\mathfrak{s}_\eta$. Clearly, the Lie groups $S_\eta$ are pairwise non-isomorphic for all $\eta \geq 0$. However, it is possible to check that the intrinsic torsion form of the ERP closed $G_2$-structure on $S_\eta$ is parallel with respect to the canonical $G_2$-connection. Hence, by the result of Cleyn and Ivanov recalled above, $(S_\eta, g_{\varphi_\eta})$ is locally isometric to Bryant’s example.

Remark 6.5. Although the simply connected solvable Lie groups $\eta$ are pairwise non-isomorphic for different values of $\eta \geq 0$, it is possible to show that the $G_2$-structures $(S_\eta, \varphi_\eta)$ are pairwise equivalent. This has been proved in the recent work [[27]].

The next example was obtained by Lauret in [[26]]. As shown in the same paper, it is not equivalent to Bryant’s example (cf. [[26] Rem. 4.15]).

Example 6.6. Consider the abelian Lie algebras $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$ and $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7 \rangle$, and define the seven-dimensional unimodular solvable Lie algebra $\mathfrak{u}$ as the semidirect product $\mathbb{R}^3 \ltimes \mu \mathbb{R}^4$, with $\mu : \mathbb{R}^3 \to \text{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ given by

$$\mu(e_1) = \text{diag}(1, 1, -1, -1), \quad \mu(e_2) = \text{diag}(1, -1, 1, -1), \quad \mu(e_3) = \text{diag}(1, -1, -1, 1).$$

The structure equations of $\mathfrak{u}$ can be written with respect to the dual basis $(e^1, \ldots, e^7)$ of $(e_1, \ldots, e_7)$ as follows

$$\begin{pmatrix} 0, 0, 0, -e^{14} - e^{24} - e^{34}, -e^{15} + e^{25} + e^{35}, e^{16} - e^{26} + e^{36}, e^{17} + e^{27} - e^{37} \end{pmatrix}.$$
The 3-form $\varphi$ given by (2.2) defines an ERP closed $G_2$-structure on $u$ with intrinsic torsion form

$$\tau = -2e^{45} + 2e^{67} - 2e^{46} - 2e^{57} + 2e^{47} - 2e^{56}.$$ 

Notice that $|\tau|^2_\varphi = 24$, and that $P$ and $Q$ coincide with $R^3$ and $R^4$, respectively.

**Remark 6.7.** As shown in [26], in both Examples 6.1 and 6.6 the left-invariant ERP closed $G_2$-structure is a steady Laplacian soliton on the corresponding non-compact simply connected solvable Lie group. By [29, Cor. 1], it cannot descend to an invariant steady Laplacian soliton on any compact quotient.

In the next theorem, we show that the Lie algebra $u$ described in the previous example is the unique unimodular Lie algebra admitting ERP closed $G_2$-structures up to isomorphism.

**Theorem 6.8.** Let $G$ be a unimodular Lie group endowed with a left-invariant ERP closed $G_2$-structure. Then, its Lie algebra is isomorphic to the Lie algebra $u$ described in Example 6.6.

**Proof.** Consider the Lie algebra $g = \text{Lie}(G)$ endowed with the ERP closed $G_2$-structure $\varphi$ corresponding to the left-invariant one on $G$, and let $g = P \oplus Q$ be the induced $g_\varphi$-orthogonal decomposition of $g \cong T_{10}G$. Since $g$ is unimodular and the Ricci tensor of $g_\varphi$ is non-positive, the nilradical $n$ of $g$ is abelian by [8, Cor. 1]. Moreover, by [8, Lemma 1] we have $\text{Ric}(g_\varphi)|_n = 0$. Consequently, point (v) of Proposition 3.2 implies that the nilradical $n$ is contained in $Q$.

To prove the assertion, we consider all possible seven-dimensional unimodular Lie algebras with abelian nilradical of dimension at most four, and we show that $u$ is the only one admitting an ERP closed $G_2$-structure up to isomorphism. We shall deal with the cases $g$ solvable and $g$ non-solvable separately.

If $g$ is solvable, by [36, Thm. 1]

$$\dim(g) = \dim(n) + k,$$

with $k \leq \dim(n) - \dim([n, n])$. As $n$ is abelian, we have

$$7 = \dim(n) + k, \quad k \leq \dim(n) \leq \dim(Q) = 4,$$

whence $n = Q$. Thus, the nilradical of $g$ is four-dimensional and abelian. Moreover, since $g$ is solvable, the dimension of its center $z(g)$ must satisfy $\dim(z(g)) \leq 2 \dim(n) - \dim(g) = 1$ (see e.g. [34]). We now claim that $g$ is not decomposable. Indeed, otherwise we could write $g = s_1 \oplus s_2$ for some solvable unimodular ideals $s_1, s_2 \subseteq g$. If $\dim(s_1) = 2$, then necessarily $s_1 \cong \mathbb{R}^2$ and $z(g)$ would have dimension at least two, a contradiction. If $\dim(s_1) = 3$, then $s_1$ is isomorphic either to the Lie algebra $\mathfrak{e}(1,1)$ of the group of rigid motions of the Minkowski 2-space or to the Lie algebra $\mathfrak{e}(2)$ of the group of rigid motions of the Euclidean 2-space (see, for instance, [11, Thm. 1.1]). Since both these Lie algebras have two-dimensional abelian nilradical, it follows that the unimodular solvable Lie algebra $s_2$ must be four-dimensional with two-dimensional abelian nilradical. However, there are no four-dimensional Lie algebras satisfying these properties by [11, Thm. 1.5]. Finally, if $\dim(s_1) = 1$, then $s_1 \cong \mathbb{R}$ and $s_2$ must be six-dimensional with three-dimensional nilradical. Any such $s_2$ must be decomposable by [35, Thm. 3]. We can then conclude as in the previous cases. Therefore, $g$ is indecomposable.
Indecomposable seven-dimensional solvable Lie algebras with an abelian nilradical of dimension four were classified in [16]. A scan of all possibilities allows to conclude that only three unimodular Lie algebras of this type occur up to isomorphism. One is isomorphic to the Lie algebra \( u \) described in Example [6.6] thus it admits an ERP closed \( G_2 \)-structure. The remaining ones correspond to the fourth and the fifth pencil in [16] Prop. 5.4. Their structure equations with respect to a suitable basis \((e_1, \ldots, e_7)\) are the following:

\[
\begin{align*}
\begin{cases}
    de^i &= 0, & i = 1, 2, 3, \\
    de^4 &= \alpha e^{14} - e^{15} + \beta e^{24}, \\
    de^5 &= e^{14} + \alpha e^{15} + \gamma e^{25}, \\
    de^6 &= -\alpha e^{16} - \beta e^{17} - \gamma e^{26} - \rho e^{27} - \sigma e^{37}, \\
    de^7 &= \beta e^{16} - \alpha e^{17} + \rho e^{26} - \gamma e^{27} + \sigma e^{36},
\end{cases}
\end{align*}
\]

(6.1)

where \( \alpha, \beta, \rho \in \mathbb{R} \) and \( \gamma, \sigma \in \mathbb{R} \setminus \{0\} \), and

\[
\begin{align*}
\begin{cases}
    de^i &= 0, & i = 1, 2, 3, \\
    de^4 &= \frac{1}{2}\alpha e^{14} - e^{15} + \frac{1}{2}\beta e^{24}, \\
    de^5 &= e^{14} + \frac{1}{2}\alpha e^{15} + \frac{1}{2}\beta e^{25}, \\
    de^6 &= -e^{36}, \\
    de^7 &= -\alpha e^{17} - \beta e^{27} + e^{37},
\end{cases}
\end{align*}
\]

(6.2)

where \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \setminus \{0\} \). We now show that none of these Lie algebras admits closed \( G_2 \)-structures. The generic closed 3-form \( \phi \) on the Lie algebra with structure equations (6.1) is given by

\[
\begin{align*}
\phi &= \phi_{123} e^{123} + \phi_{124} e^{124} + \phi_{125} e^{125} + \phi_{135} e^{135} + \phi_{136} e^{136} + \phi_{137} e^{137} + \phi_{147} e^{147} + \phi_{157} e^{157} \\
&\quad + \frac{\rho}{\sigma} \phi_{357} e^{257} + \frac{\rho}{\gamma} \phi_{347} e^{247} + \frac{\alpha}{\gamma} \phi_{245} e^{145} + (\sigma \phi_{147} - \beta \phi_{347}) e^{356} + (\beta \phi_{357} - \sigma \phi_{157}) e^{346} \\
&\quad + (\alpha \phi_{235} - \gamma \phi_{135}) e^{234} + \rho \left( \phi_{147} - \beta \phi_{347} \right) e^{256} + \rho \left( \frac{\beta}{\sigma} \phi_{357} - \phi_{157} \right) e^{246} + \phi_{245} e^{245} \\
&\quad + \frac{\alpha^2 \phi_{235} - \alpha \gamma \phi_{135} + \phi_{235}}{\gamma} e^{134} + \phi_{235} e^{235} + \phi_{236} e^{236} + \phi_{237} e^{237} + \phi_{347} e^{347} + \phi_{357} e^{357} \\
&\quad + \frac{\beta^2 \phi_{357} - \beta \sigma \phi_{157} - \phi_{357}}{\sigma} e^{146} - \frac{\beta^2 \phi_{347} - \beta \sigma \phi_{147} - \phi_{347}}{\sigma} e^{156} + \frac{\alpha}{\gamma} \phi_{267} e^{167} + \phi_{267} e^{267} \\
&\quad + \frac{\alpha \phi_{236} - \beta \phi_{237} - \gamma \phi_{136} + \rho \phi_{137}}{\sigma} e^{127} - \frac{\alpha \phi_{237} + \beta \phi_{236} - \gamma \phi_{137} - \rho \phi_{136}}{\sigma} e^{126},
\end{align*}
\]

where \( \phi_{ijk} \in \mathbb{R} \). A simple computation shows that

\[
b_{\phi}(e_6, e_6) = -\phi_{267} (\phi_{147} \phi_{357} - \phi_{157} \phi_{347}) e^{1234567} = -b_{\phi}(e_7, e_7),
\]

thus \( \phi \) cannot define a \( G_2 \)-structure.
As for the Lie algebra with structure equations (6.2), the generic closed 3-form has the following expression

\[ \phi = \phi_{123}e^{123} + \phi_{124}e^{124} + \phi_{125}e^{125} + \phi_{135}e^{135} + \phi_{136}e^{136} + \phi_{137}e^{137} + \phi_{235}e^{235} + \phi_{236}e^{236} + \phi_{237}e^{237} + \phi_{245}e^{245} + \phi_{267}e^{267} + \phi_{346}e^{346} + \phi_{347}e^{347} + \phi_{356}e^{356} + \phi_{357}e^{357} + \frac{1}{2}(\alpha\phi_{235} - \beta\phi_{135})e^{234} + \left(\phi_{346} - \frac{\alpha}{2}\phi_{356}\right)e^{156} - \left(\phi_{347} + \frac{\alpha}{2}\phi_{357}\right)e^{157} + \alpha\phi_{267}e^{167} - \left(\phi_{356} + \frac{\alpha}{2}\phi_{346}\right)e^{146} + \left(\phi_{357} - \frac{\alpha}{2}\phi_{347}\right)e^{147} + (\alpha\phi_{237} - \beta\phi_{137})e^{127} + \alpha\phi_{245}e^{145} - \frac{\beta}{2}(\phi_{347}e^{247} + \phi_{346}e^{246} + \phi_{357}e^{257} + \phi_{356}e^{256}) + \left(\frac{\alpha^2}{2}\phi_{235} - \frac{\alpha}{2}\phi_{135} + 2\phi_{233}\right)e^{134}, \]

with \( \phi_{ijk} \in \mathbb{R} \), and we immediately see that also in this case \( b_\phi(e_6, e_6) = -b_\phi(e_7, e_7) \).

We now focus on the case when \( g \) is unimodular and non-solvable. By the classification in (13), \( g \) is isomorphic to one of the following

\[
\begin{align*}
&(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, \frac{1}{2}e^{46} - e^{47}, \frac{1}{2}e^{47}); \\
&(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, -\alpha e^{46}, (1 + \alpha) e^{47}), \quad -1 < \alpha \leq -\frac{1}{2}; \\
&(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, \frac{\alpha}{2} e^{46} - e^{47}, e^{46} + \frac{\alpha}{2} e^{47}), \quad \alpha > 0; \\
&(-e^{23}, -2e^{12}, 2e^{13}, -e^{14} - e^{25} - e^{47}, e^{15} - e^{34} - e^{57}, 2e^{67}, 0).
\end{align*}
\]

All of these Lie algebras have a three-dimensional abelian nilradical \( n \). Since the 3-form defining an ERP closed \( G_2 \)-structure vanishes on \( Q \), it must vanish on \( n \). The first three Lie algebras in the above list do not admit any stable closed 3-form satisfying this condition. Indeed, in each case the abelian nilradical is given by \( n = \langle e_5, e_6, e_7 \rangle \) and imposing that the generic closed 3-form \( \phi \) satisfies \( \phi(e_5, e_6, e_7) = 0 \) gives \( b_\phi(e_i, e_i) = 0 \), for \( i = 5, 6, 7 \). We are then left with the last Lie algebra, whose nilradical is \( n = \langle e_4, e_5, e_6 \rangle \). Let us consider the generic closed 3-form

\[ \phi = \phi_{123}e^{123} - 3\phi_{247}e^{124} - \phi_{234}e^{125} + \phi_{267}e^{126} + \phi_{127}e^{127} - \phi_{235}e^{134} + 3\phi_{357}e^{135} - \phi_{367}e^{136} + \phi_{137}e^{137} + \phi_{467}e^{146} + (\phi_{257} - \phi_{234})e^{147} - \phi_{567}e^{156} - (\phi_{235} + \phi_{347})e^{157} + 2\phi_{236}e^{167} + \phi_{234}e^{234} + \phi_{235}e^{235} + \phi_{236}e^{236} + \phi_{237}e^{237} + \phi_{247}e^{247} + \phi_{467}e^{256} + \phi_{257}e^{257} + \phi_{267}e^{267} + \phi_{567}e^{346} + \phi_{347}e^{347} + \phi_{357}e^{357} + \phi_{367}e^{367} + \phi_{456}e^{456} + \phi_{457}e^{457} + \phi_{467}e^{467} + \phi_{567}e^{567}, \]

where \( \phi_{ijk} \in \mathbb{R} \). The condition \( \phi(e_4, e_5, e_6) = 0 \) gives \( \phi_{456} = 0 \). Up to a basis change, we may assume that \( Q = \langle e_4, e_5, e_6, v \rangle \) with \( v = v_1e_1 + v_2e_2 + v_3e_3 + v_7e_7 \) for some real numbers \( v_1, v_2, v_3, v_7 \). Now, if we consider the equation \( 0 = \phi(e_4, e_5, v) = \phi_{457}v_7 \), we see that necessarily \( v_7 = 0 \), otherwise \( b_\phi(e_4, e_4) = 0 \) and \( \phi \) would not be stable. An inspection of the equations \( \phi(e_4, e_6, v) = 0 = \phi(e_5, e_6, v) \) when \( \phi \) is stable gives the following possibilities

- \( \phi_{467} \neq 0, \phi_{567} \neq 0, \) and \( Q = \langle e_4, e_5, e_6, e_1 + \frac{\phi_{467}}{\phi_{567}}e_2 - \frac{\phi_{467}}{\phi_{567}}e_3 \rangle; \)
- \( \phi_{467} = 0, \phi_{567} \neq 0, \) and \( Q = \langle e_4, e_5, e_6, e_2 \rangle; \)
- \( \phi_{467} \neq 0, \phi_{567} = 0, \) and \( Q = \langle e_4, e_5, e_6, e_3 \rangle. \)
Now, if $\tau$ is the intrinsic torsion form of an ERP closed $G_2$-structure, then the 4-form $\tau \wedge \tau \in \Lambda^4(Q^*)$ must be closed and simple. However, in none of the above cases there exist closed simple 4-forms on $Q$. □

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**References**

[1] A. Andrada, M. L. Barberis, I. G. Dotti, and G. P. Ovando. Product structures on four dimensional solvable Lie algebras. *Homology Homotopy Appl.* 7(1), 9–37, 2005.
[2] G. Ball. Seven Dimensional Geometries with Special Torsion. Ph.D. Thesis, 2019.
[3] R. L. Bryant. Some remarks on $G_2$-structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pages 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006.
[4] R. L. Bryant and F. Xu. Laplacian flow for closed $G_2$-structures: Short time behavior. [arXiv:1101.2004](arXiv:1101.2004)
[5] R. Cleyton and S. Ivanov. On the geometry of closed $G_2$-structures. *Comm. Math. Phys.* 270 (1), 53–67, 2007.
[6] R. Cleyton and S. Ivanov. Curvature decomposition of $G_2$-manifolds. *J. Geom. Phys.* 58 (10), 1429–1449, 2008.
[7] A. Corti, M. Haskins, J. Nordström, and T. Pacini. $G_2$-manifolds and associative submanifolds via semi-Fano 3-folds. *Duke Math. J.* 164 (10), 1971–2092, 2015.
[8] I. Dotti Miatello. Metrics with nonpositive Ricci curvature on semidirect products. *Q. J. Math.* 37 (147), 309–314, 1986.
[9] M. Fernández, A. Fino, and V. Manero. Laplacian flow of closed $G_2$-structures inducing nilsolitons. *J. Geom. Anal.* 26 (3), 1808–1837, 2016.
[10] M. Fernández and A. Gray. Riemannian manifolds with structure group $G_2$. *Ann. Mat. Pura Appl.* 132, 19–45, 1982.
[11] J. Fine and C. Yao. Hypersymplectic 4-manifolds, the $G_2$-Laplacian flow and extension assuming bounded scalar curvature. *Duke Math. J.* 167, 3533–3589, 2018.
[12] A. Fino and A. Raffero. Closed warped $G_2$-structures evolving under the Laplacian flow. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 20 (1), 315–348, 2020.
[13] A. Fino and A. Raffero. Closed $G_2$-structures on non-solvable Lie groups. *Rev. Mat. Complut.* 32 (3), 837-851, 2019.
[14] A. Fino and A. Raffero. Remarks on homogeneous solitons of the $G_2$-Laplacian flow. To appear in *C. R. Math. Acad. Sci. Paris*.
[15] R. Harvey and H. B. Lawson, Jr. Calibrated geometries. *Acta Math.*, 148, 47–157, 1982.
[16] F. Hindeleh and G. Thompson. Seven dimensional Lie algebras with a four-dimensional nilradical. *Algebras Groups Geom.* 25 (3), 243–265, 2008.
[17] N. Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, volume 288 of *Contemp. Math.*, pp. 70–89. Amer. Math. Soc., Providence, RI, 2001.
[18] H. Huang, Y. Wang and C. Yao. Cohomogeneity-one $G_2$-Laplacian flow on the 7-torus. *J. London Math. Soc.* 98 (2), 349–368, 2018.
[19] D. D. Joyce. Compact Riemannian 7-manifolds with holonomy $G_2$, I, II. *J. Differential Geom.* 43 (2), 291–328, 329–375, 1996.
[20] D. D. Joyce. *Riemannian holonomy groups and calibrated geometry*, vol. 12 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2007.
[21] D. D. Joyce, S. Karigiannis. A new construction of compact torsion-free $G_2$-manifolds by gluing families of Eguchi-Hanson spaces. To appear in *J. Differential Geom.*

[22] S. Karigiannis. Flows of $G_2$-structures. *I. Q. J. Math.* 60 (4), 487–522, 2009.

[23] I. Kath, J. Lauret. A new example of a compact ERP $G_2$-structure. [arXiv:2005.02462](https://arxiv.org/abs/2005.02462)

[24] A. Kovalev. Twisted connected sums and special Riemannian holonomy. *J. Reine Angew. Math.*, 565, 125–160, 2003.

[25] J. Lauret. Laplacian flow of homogeneous $G_2$-structures and its solitons. *Proc. Lond. Math. Soc.* 114 (3), 527–560, 2017.

[26] J. Lauret. Laplacian solitons: questions and homogeneous examples. *Differential Geom. Appl.* 54 (B), 345–360, 2017.

[27] J. Lauret, M. Nicolini. Extremally Ricci pinched $G_2$-structures on Lie groups. [arXiv:1902.06375](https://arxiv.org/abs/1902.06375)

[28] J. Lauret, M. Nicolini. The classification of ERP $G_2$-structures on Lie groups. To appear in *Ann. Mat. Pura Appl.*

[29] C. Lin. Laplacian solitons and symmetry in $G_2$-geometry. *J. Geom. Phys.* 64, 111–119, 2013.

[30] J. D. Lotay and Y. Wei. Laplacian flow for closed $G_2$ structures: Shi-type estimates, uniqueness and compactness. *Geom. Funct. Anal.* 27 (1), 165–233, 2017.

[31] J. D. Lotay and Y. Wei. Stability of torsion-free $G_2$ structures along the Laplacian flow. *J. Differential Geom.* 111 (3), 495–526, 2019.

[32] J. D. Lotay and Y. Wei. Laplacian flow for closed $G_2$ structures: real analyticity. *Comm. Anal. Geom.* 27, 73–109, 2019.

[33] J. Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.* 21 (3), 293–329, 1976.

[34] G. M. Mubarakzjanov. On solvable Lie algebras. *Izv. Vyssh. Učebn. Zaved. Matematika* 32 (1), 114–123, 1963.

[35] J. C. Ndogmo and P. Winternitz. Solvable Lie algebras with abelian nilradicals. *J. Phys. A* 27 (2), 405–423, 1994.

[36] L. Šnobl. On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras. *J. Phys. A* 43 (50), 505202, 17, 2010.

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