Effect of thermal radiation and slip on unsteady 3D MHD nanofluid flow over a non-linear stretching sheet in a porous medium with convective boundary condition

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Abstract. The main aim of the article is to analyze the non-linear thermal radiation and velocity slip effects on three-dimensional MHD convective flow of nanofluid over a non-linear stretching sheet in a porous medium with convective boundary condition. Using suitable transformations, the governing equations are transformed into ordinary differential equations and are solved by using homotopy analysis method (HAM). The momentum boundary layer thickness along x- and y-directions and skin friction declines with raise in velocity slip. The swelling of thermal boundary layer thickness and Nusselt number occurs while boosting the thermal radiation.

Key Words: Nanofluid; slip; non-linear thermal radiation; porous medium; MHD.

1. Introduction

The importance of convective flow and heat transfer of nanofluid is discussed in [1]-[3]. Eswaramoorthi et al. [4] studied about the Newtonian heating and partial slip effects of Carreau liquid in MHD flow. Sivasankaran et al. [5] analyzed about the the radiation, chemical reaction and slip effects on mixed convective MHD stagnation point flow in a porous medium. The effect of non-linear thermal radiation and slip velocity on 3D nanofluid MHD flow were examined by Hayat et al. [6]. Unsteady MHD flow of nanofluid towards a slendering stretching surface with slip effects were demonstrated by Reddy et al. [7]. Bhattacharyya et al. [8] examined about the slip effects on an unsteady stagnation-point boundary layer flow towards a stretching sheet. Nazar et al. [9] investigate about the unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium The effect of thermal radiation and MHD on heat unsteady fluid flow over a stretching surface were studied by Abdel-Rahman [10]. The slip and convective boundary condition and thermal radiation effects on unsteady Casson nanofluid flow over a stretching sheet were examined by Oyelakin et al. [11]. The unsteady 3D MHD flow and mass transfer in a porous space were deliberated by Hayat et al. [12].

In this paper the investigation is made on three-dimensional unsteady MHD convective flow of
nanofluid over a non-linear stretching sheet in a porous medium in the presence of non-linear thermal radiation, slip effect and convective boundary condition.

2. Mathematical Formulation

The 3D unsteady electrically conducting flow of nanofluid towards a stretching sheet in the presence of a constant applied magnetic field of strength \( B_0 \) is considered. At \( t = 0 \), the velocity components \( l_1 \) and \( l_2 \) are taken as \( l_1 = a_1 x \) and \( l_2 = b_1 y \) along \( x \)- and \( y \)-axis and \( a_1 \) and \( b_1 \) are positive constants. The radiative heat flux is taken as \( q_r = \frac{4\pi}{3} \frac{\partial T_e}{\partial z} \). The surface temperature and concentration vary from \((T_e)_\infty\) to \((T_e)_w\) and \((Cn)_\infty\) to \((Cn)_w\) where \((T_e)_\infty\) and \((Cn)_\infty\) are constant temperature and concentration. The governing equations for the analysis can be expressed as follows

\[
\frac{\partial l_1}{\partial t} + \frac{\partial l_2}{\partial y} + \frac{\partial l_3}{\partial z} = 0, \tag{1}
\]
\[
\frac{\partial l_1}{\partial t} + l_1 \frac{\partial l_1}{\partial x} + l_2 \frac{\partial l_1}{\partial y} + l_3 \frac{\partial l_1}{\partial z} = \nu l_1 - \frac{\sigma B_0^2}{\rho} l_1 - \frac{\nu \varphi}{k_1} l_1, \tag{2}
\]
\[
\frac{\partial l_2}{\partial t} + l_1 \frac{\partial l_2}{\partial x} + l_2 \frac{\partial l_2}{\partial y} + l_3 \frac{\partial l_2}{\partial z} = \nu l_2 - \frac{\sigma B_0^2}{\rho} l_2 - \frac{\nu \varphi}{k_1} l_2, \tag{3}
\]
\[
\frac{\partial (T_e)}{\partial t} + l_1 \frac{\partial (T_e)}{\partial x} + l_2 \frac{\partial (T_e)}{\partial y} + l_3 \frac{\partial (T_e)}{\partial z} = \frac{k}{\rho c_p} \frac{\partial^2 (T_e)}{\partial z^2} + \tau D_B \left( \frac{\partial (T_e)}{\partial z} \frac{\partial (Cn)}{\partial z} \right) + \frac{\tau D_{T_e}}{(T_e)_\infty} \left( \frac{\partial (T_e)}{\partial z} \right)^2 - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial z}, \tag{4}
\]
\[
\frac{\partial (Cn)}{\partial t} + l_1 \frac{\partial (Cn)}{\partial x} + l_2 \frac{\partial (Cn)}{\partial y} + l_3 \frac{\partial (Cn)}{\partial z} = D_B \frac{\partial^2 (Cn)}{\partial z^2} + \frac{D_{T_e}}{(T_e)_\infty} \frac{\partial^2 (T_e)}{\partial z^2}, \tag{5}
\]

and the boundary conditions are

\[
l_1 = 0, \quad l_2 = 0, \quad l_3 = 0, \quad T_e = (T_e)_\infty, \quad Cn = (Cn)_\infty; \quad t < 0.
\]
\[
l_1 = l_1,w = a_1 x + A_1 \frac{\partial l_1}{\partial z}, \quad l_2 = l_2,w = b_1 y + A_2 \frac{\partial l_2}{\partial z}, \quad l_3 = 0,
\]
\[
-k \frac{\partial (T_e)}{\partial z} = h \left[ (T_e)_w - T_e \right], \quad Cn = (Cn)_w, \quad z = 0; \quad t \geq 0.
\]
\[
l_1 \to 0, \quad l_2 \to 0, \quad T_e \to (T_e)_\infty, \quad Cn \to (Cn)_\infty, \quad as \ z \to \infty; \quad t \geq 0.
\]

where \( l_3 \) is velocity component along \( z \)-axis, \( \mu, \rho, \nu, \sigma, \varphi, k_1, k, C, D_B, D_T, A_1 \) and \( A_2 \) are dynamic viscosity, density, kinematic viscosity, specific heat, electrical conductivity, porosity, porous medium permeability, thermal conductivity, species concentration, the Brownian motion, thermophoresis coefficient and velocity slip factor along \( x \)- and \( y \)-directions respectively. The similarity transformations are

\[
\omega = z \sqrt{\frac{a_1}{\nu \zeta}}, \quad \zeta = 1 - e^{-\tau}, \quad \tau = a_1 t, \quad l_1 = a_1 x m', \quad l_2 = a_1 y n',
\]
\[
l_3 = -\sqrt{a_1 \nu} \zeta (m + n), \quad \theta (\omega) = \frac{T_e - (T_e)_\infty}{(T_e)_w - (T_e)_\infty}, \quad \phi (\omega) = \frac{Cn - (Cn)_\infty}{(Cn)_w - (Cn)_\infty}. \tag{7}
\]
where prime denotes the derivative with respect of $\omega$.

Substituting (7) in (1), (1) is satisfied identically. Substituting (7) in (2) to (6), we get

\begin{equation}
m'' + (\zeta - 1) \left[ \zeta \frac{\partial m'}{\partial \zeta} - \frac{\omega}{2} m'' \right] - \frac{\omega}{2} \left[ m'' - m'' (m + n) \right] - \zeta \left[ m' (\lambda + M^2) \right] = 0, \tag{8}
\end{equation}

\begin{equation}
n'' + (\zeta - 1) \left[ \zeta \frac{\partial n'}{\partial \zeta} - \frac{\omega}{2} n'' \right] - \frac{\omega}{2} \left[ n'' - n'' (m + n) \right] - \zeta \left[ n' (\lambda + M^2) \right] = 0, \tag{9}
\end{equation}

\begin{equation}
\left( 1 + \frac{4}{3}Rd \right) \theta'' + \frac{4}{3}Rd \left[ (\theta_w - 1) \theta^2 (3\theta^2 + \theta \theta'') + 3(\theta_w - 1)^2 \theta (2\theta^2 + \theta \theta'') \right.
\end{equation}

\begin{equation}
+ 3 (\theta_w - 1) (\theta^2 + \theta \theta'') \bigg] + PrNb \theta' \phi' + PrNt \theta'^2 + Pr (\zeta - 1) \left[ \frac{\zeta}{2} \frac{\partial \theta}{\partial \zeta} - \frac{\omega}{2} \theta' \right]
\end{equation}

\begin{equation}
+ Pr \zeta \theta' (m + n) = 0, \tag{10}
\end{equation}

\begin{equation}
\phi'' + Sc \left[ (\zeta - 1) \left[ \zeta \frac{\partial \phi}{\partial \zeta} - \frac{\omega}{2} \phi' \right] + \zeta \phi' (m + n) \right] + \frac{Nt}{Nb} \theta'' = 0, \tag{11}
\end{equation}

\begin{equation}
m(\zeta, \omega) = 0, n(\zeta, \omega) = 0, m'(\zeta, \omega) = 1 + \epsilon_1 m'' (\zeta, \omega), n'(\zeta, \omega) = c + \epsilon_2 n'' (\zeta, \omega), \tag{12}
\end{equation}

\begin{equation}
\theta'(\zeta, \omega) = -\alpha [1 - \theta(\zeta, \omega)], \phi' (\zeta, \omega) = 1 \text{ as } \omega \rightarrow 0.
\end{equation}

\begin{equation}
m'(\zeta, \omega) = 0, n'(\zeta, \omega) = 0, \theta (\zeta, \omega) = 0, \phi (\zeta, \omega) = 0 \text{ as } \omega \rightarrow \infty.
\end{equation}

where $\alpha, Pr, M, Sc, c, \lambda, Nb, Nt, Rd, \theta_w, \epsilon_1$ and $\epsilon_2$ are thermal Biot number, Prandtl number, local Hartman number, Schmidt number, stretching ratio, local porosity, Brownian motion, thermophoresis, non-linear thermal radiation, temperature ratio, and velocity slip parameters along $x$- and $y$-axis are defined as

\begin{equation}
\alpha = \frac{h \sqrt{\nu \zeta / a_1}}{k}, \quad Pr = \frac{\mu_c}{k}, \quad M^2 = \frac{\sigma B_0^2}{a_1 \rho}, \quad Sc = \frac{\nu}{D}, \quad c = \frac{b_1}{a_1}, \quad \lambda = \frac{\nu \varphi}{a_1 k_1},
\end{equation}

\begin{equation}
Nb = \tau D_B \left[ (Cn)_w - (Cn)_\infty \right], \quad Nt = \tau D_B \left[ (Te)_w - (Te)_\infty \right], \quad Rd = \frac{4 \sigma * (Te)_\infty^3}{kk^*},
\end{equation}

\begin{equation}
\theta_w = \frac{(Te)_w}{(Te)_\infty}, \quad \epsilon_1 = A_1 \sqrt{\frac{a_1}{\nu c}}, \quad \epsilon_2 = A_2 \sqrt{\frac{a_1}{\nu c}}.
\end{equation}

The local skin-friction coefficient and local Nusselt number are defined as follows.

\begin{equation}
\zeta^2 Re_x^{1/2} C_{m_x} = -m'' (\zeta, 0),
\end{equation}

\begin{equation}
\zeta^2 Re_y^{1/2} C_{n_y} = -n'' (\zeta, 0),
\end{equation}

\begin{equation}
\zeta^2 Re_x^{-1/2} Nu = - \left( 1 + \frac{4}{3} Rd (\theta_w)^3 \right) \theta' (\zeta, 0)
\end{equation}

where $Re_x = \frac{\tau_{1,w} x}{\nu}$ and $Re_y = \frac{\tau_{2,w} y}{\nu}$ are local Reynolds numbers.
The Hartmann number $M$, porosity parameter $\lambda$, and thermal Biot number $\alpha$ diminishes with increase in velocity slip parameter ($\epsilon_1$ and $\epsilon_2$) along $x-$ and $y-$axis. And also it declines while boosting the porosity parameter $\lambda$ and Hartmann number $M$, non-linear thermal radiation parameter $Rd$ and thermal Biot number $\alpha$.

It is evident from figures 4a and 4b that the skin friction in both direction declines with raise in velocity slip parameter along $x-$ and $y-$axis, but, it enhances with raise in the porosity parameter $\lambda$ and Hartmann number $M$ (see figures 5a, 5b, 5c and 5d). The local Nusselt

3. Method of the solution

The equations (8) to (12) are solved using HAM by choosing the initial approximation and auxiliary linear operators as

$$m_0(\zeta, \omega) = \left(\frac{2*\epsilon_1}{1+\epsilon_1}\right) exp(-\omega) + \omega*exp(-\omega) - \left(\frac{2*\epsilon_1}{1+\epsilon_1}\right),$$

$$n_0(\zeta, \omega) = c * \left(\frac{2*\epsilon_2}{1+\epsilon_2}\right) exp(-\omega) + \omega*exp(-\omega) - \left(\frac{2*\epsilon_2}{1+\epsilon_2}\right),$$

$$\theta_0(\zeta, \omega) = \frac{\alpha}{1+\alpha} exp(-\omega),$$

$$L_m(m) = \frac{d^3m}{d\omega^3} - \frac{dm}{d\omega}, L_n(n) = \frac{d^3n}{d\omega^3} - \frac{dn}{d\omega}, L_\theta(\theta) = \frac{d^2\theta}{d\omega^2} - \theta.$$  

which satisfies the property

$$L_m[A_1 + A_2 exp(-\omega) + A_3 exp(\omega)] = 0,$$

$$L_n[A_4 + A_5 exp(-\omega) + A_6 exp(\omega)] = 0,$$

$$L_\theta[A_7 exp(-\omega) + A_8 exp(\omega)] = 0.$$  

where $A_1, A_2, ..., A_8$ are the arbitrary constants.

The resulting equations contains the auxiliary parameters $h_m, h_n$ and $h_\theta$. The h-curve is plotted for $c = 0.5$, $\lambda = 0.5$, $M = 1.0$, $Nb = 0.2$, $Nt = 0.2$, $Pr = 1.5$, $Rd = 0.3$, $Sc = 1.0$, $\theta_w = 0.1$ and $\zeta = 1$. From figure 1, it is clear that the admissible range of $h_m, h_n$ and $h_\theta$ are $-1.0 \leq h_m \leq -0.15$, $-1.0 \leq h_n \leq -0.1$ and $-1.3 \leq h_\theta \leq 0.0$.

4. Result and Discussion

The discussions are made for different combinations of the pertinent parameters involved in the study. From figures 2c and 2f, it is clear that the velocity profile along $x-$ and $y-$directions diminishes with increase in velocity slip parameter ($\epsilon_1$ and $\epsilon_2$) along $x-$ and $y-$axis. And also it declines while boosting the porosity parameter $\lambda$ and Hartmann number $M$, see figures 2a, 2b, 2d and 2e. From figures 3a,3b,3c and 3d, the temperature profile enhances while boosting the Hartmann number $M$, porosity parameter $\lambda$, non-linear thermal radiation parameter $Rd$ and thermal Biot number $\alpha$.
number rises while raising the non-linear thermal radiation parameter $Rd$ and thermal Biot number $\alpha$ (see figures 6a and 6b).

5. Conclusion
The study of 3D unsteady MHD convective nanofluid flow over a non-linear stretching sheet with non-linear thermal radiation, slip and convective boundary condition in a porous medium is examined. The thermal boundary layer thickness enhances while magnifying the non-linear thermal radiation and thermal Biot number which results in enhancement on heat transfer rate. Along $x$– and $y$–directions while increasing porosity and Hartmann number, the skin friction rises but momentum boundary layer thickness declines . The thickness of momentum boundary layer and skin friction in both the directions declines while magnifying the velocity slip.

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Figure 2: Influence of $\lambda$ and $M$ on $m'(\zeta, \omega)$ and $n'(\zeta, \omega)$, $\epsilon_1$ on $m'(\zeta, \omega)$ and $\epsilon_2$ on $n'(\zeta, \omega)$. 
Figure 3: Influence of $\lambda$, $M$, $Rd$ and $\alpha$ on $\theta(\zeta, \omega)$.

Figure 4: Influence of $\epsilon_1$ on $\zeta^2 \text{Re}^{1/2} C_{\text{mx}}$ and $\epsilon_2$ on $\zeta^2 \text{Re}^{1/2} C_{\text{my}}$. 

**Figure 5:** Influence of $\lambda$ and $M$ on $\frac{1}{2} Re_x^{1/2} C_{m_x}$ and $\frac{1}{2} Re_y^{1/2} C_{n_y}$.

**Figure 6:** Influence of Rd and $\alpha$ on $\frac{1}{2} Re_x^{-1/2} Nu$. 