ABSTRACT

In this note, we apply a special metric ansatz to simplify the equations of motion for gravitational systems. Then we construct charged brane solutions in $D = n + p + 2$ dimensions which have spherical symmetry of $S^n$ and translational symmetry along $p$ directions. They are characterized by mass density, uniform tension and electric/magnetic charges, and nonsingular only for specific tension. In particular, we find the limits and the coordinate transformations which reduce the charged brane solutions to $M2$- and $M5$-branes. We also obtain the regularity condition for an electrically charged two-brane solution which has two tensions.

Keywords: brane · tension
1 Introduction

There is only a very limited family of asymptotically flat, stationary black hole solutions to Einstein equations in four dimensions, which has a timelike Killing vector at infinity. According to no hair theorem, every four-dimensional black hole formed in a gravitational collapse possesses the properties given only by its mass, charge and angular momentum, and event horizons of non-spherical topology are forbidden. In higher dimensions, however, there exist different kinds of black objects such as conventional black holes with hyperspherical horizons $S^n$, black strings/branes, Kaluza-Klein black holes, Kaluza-Klein bubbles and black tubes. The black branes are solutions which are extended in extra $p$ spatial dimensions and do not diverge at spatial infinity $[1, 2, 3, 4, 5, 6]$. Many brane solutions have been constructed for simple truncations of supergravity theories. A black brane may carry electric or magnetic charges and couple to dilaton fields just like the black hole in four dimensions.

Black branes in more than four dimensions are of particular interest, since they exhibit new behaviors that black holes do not show. For example, since these brane solutions have translational invariance, thermodynamics can be extended to hydrodynamics (which describes long-wavelength deviations from thermal equilibrium). So, in addition to thermodynamic properties such as temperature and entropy, black branes possess hydrodynamic characteristics of continuous fluids: viscosity, diffusion constants, etc. In $[7, 8]$, a neutral, static black string solution which is characterized by two parameters, mass density $\lambda$ and tension $\tau$, of the source was obtained. This type of solutions were extended to black branes in higher dimensions $[9, 10]$. However, we will show that those solutions are not singular only for specific values of tension.

The black branes have complicated metrics, and it is usually difficult to solve the second-order differential equations of motion. In this paper, closely following $[10]$, we will exploit certain metric ansatz to simplify the equations of motion and to derive the brane solutions with tension. These solutions can be shown to be characterized by mass density and tension after some suitable coordinate transformation, and compared with deformed brane solutions in $[11]$. Since higher-derivative corrections spoil the simplification in general, we will not take them into account in this note.

The organization of the paper is as follows. In section 2 we will gain a brane solution with tension by solving simplified equations of motion. Then,
in section 3, the result is extended to charged brane cases. And we will consider the limits and the coordinate transformations which reduce the charged brane solutions to $M^2$- and $M^5$-branes in order to show that such solutions might be considered as singular extensions of $M^2$- and $M^5$-branes. An electrically charged two-brane solution with two tensions and its regularity condition are obtained in section 4. Section 5 is reserved for the discussions.

2 Neutral Brane Solutions with Tension

Let us start with the $(n+p+2)$-dimensional Einstein-Hilbert action $(n \geq 2)$,

$$I = \frac{1}{16\pi G_{n+p+2}} \int d^{n+p+2}x \sqrt{-g} \ R.$$  (1)

We would like to recombine the metric fields. For convenience, uniformity for all the translationally symmetric directions of the solution will be assumed. Then we take the metric ansatz (similarly as in [11, 12]) as follows,

$$ds^2 = -e^{2p}(j(r)+p k(r)) dt^2 + e^{2p}(j(r)-k(r)) \sum_{i=1}^{p} dz_i^2 + e^{2(n-p)}(h(r)-j(r)) \left( e^{2f(r)} dr^2 + d\Omega^2_n \right),$$  (2)

where $d\Omega^2_n$ is the metric for $S^n$, $d\Omega^2_n = d\theta_1^2 + \sum_{j=2}^{n} \prod_{i=1}^{j-1} \sin^2 \theta_i d\theta_j^2$. This gives us the following effective Lagrangian density,

$$\mathcal{L}_{\text{eff}} = e^{h(r)-f(r)} \left[ \frac{(n-1)n}{2} e^{2f(r)} + \frac{n}{2(n-1)} \{h'(r)\}^2 - \frac{p}{2(p+1)} \{k'(r)\}^2 \right. \left. - \frac{n+p}{2(n-1)(p+1)} \{j'(r)\}^2 \right] - \left[ e^{h(r)-f(r)} \left( \frac{n}{n-1} h'(r) - \frac{1}{n-1} j'(r) \right) \right]' \right].$$  (3)

Since there is no derivative term of $f(r)$ in the effective Lagrangian, it is not a dynamical variable, but simply gives us a constraint. Thus, we can fix the gauge $f(r) = h(r)$ after varying the Lagrangian. First varying the effective Lagrangian with respect to $f(r)$, $h(r)$, $j(r)$ and $k(r)$, we can easily derive the equations of motion. Then, after choosing the gauge $f(r) = h(r)$ they are expressed in very simple forms and their solutions can be readily obtained,

$$h''(r) = (n-1)^2 e^{2h(r)} \Rightarrow e^{-h(r)} = -\frac{n-1}{\kappa} \sinh(\kappa r + c_2),$$  (4)

$$j''(r) = 0 \Rightarrow j(r) = -\mu r + c_3,$$  (5)

$$k''(r) = 0 \Rightarrow k(r) = -\nu r + c_4.$$  (6)
with

\[ \frac{n}{n-1} \kappa^2 = \frac{n+p}{(n-1)(p+1)} \mu^2 + \frac{p}{p+1} \nu^2, \quad (7) \]

where \( \kappa, \mu, \nu, c_2, c_3 \) and \( c_4 \) are the integration constants. Using the definition of \( r \) coordinate and the symmetries of \( t \) and \( z_i \) rescalings we can set \( c_2, c_3 \) and \( c_4 \) to vanish. Then, we have a two-parameter family of solutions. After recombining the two parameters and making the coordinate transformation in a suitable way given by

\[ \rho^{n-1} = \bar{M} \coth[(n-1)\bar{M}r], \quad \bar{M} = \frac{4\pi G n+p+2M}{\text{Vol}(S^n)} \sqrt{\frac{(n-1)(n \rho a^2 - 2pa + n + p - 1)}{n(n+p)}}, \quad (8) \]

\[ \kappa = 2(n-1)\bar{M}, \quad \mu = -\frac{2b(n-1)^2(pa+1)}{pa - n - p + 1} \bar{M}, \quad \nu = \frac{2b(n-1)(n+p)(a-1)}{pa - n - p + 1} \bar{M}, \]

\[ b^2 = \frac{n(pa - n - p + 1)^2}{(n+p)(n-1)(n \rho a^2 - 2pa + n + p - 1)}. \]

where \( a \) is the ratio of mass density to tension, we can identify the parameters as black brane mass density and tension. The resultant metric is

\[ ds^2 = -F(\rho) dt^2 + G(\rho) d\rho^2 + \rho^2 G(\rho) d\Omega_n^2 + H(\rho) \sum_{i=1}^{p} dz_i^2, \quad (9) \]

where

\[ F(\rho) = \Lambda(\rho)^{2b}, \quad G(\rho) = (1 - \frac{\bar{M}}{\rho^{2(n-1)}})^{\frac{2}{a}}, \quad \Lambda(\rho) \frac{2(pa+1)}{pa-n-p+1} b, \]

\[ H(\rho) = \Lambda(\rho) - \frac{2(na-1)}{na-n+1} b, \quad \Lambda(\rho) = \rho^{n-1} - \bar{M}. \quad (10) \]

We can check explicitly that the above solution is identified as the vacuum \( p \)-brane with trans-spherical symmetry of \([9]\) by equalizing all the \( p \) directions of tensions, and reduces to the boosted black brane in the literature \([7]\) by an appropriate coordinate transformation, \( \Lambda(\rho) = e^{-K \Delta x} \), and fixing the symmetries of \( t \) and \( z_i \) rescalings and the overall scaling of the metric,

\[ ds^2 = -e^{K \delta x} dt^2 + e^{-2\beta K x} \sum_{i=1}^{p} dz_i^2 + e^{\frac{uK}{(n-1)\gamma}}(1+2\gamma(\beta+\frac{1}{4\gamma}))x \]

\[ \cdot \sinh^{-\frac{2u}{n-1}}(K \Delta x)[dx^2 + (\frac{n-1}{K^2 \Delta^2}) \sinh^2(K \Delta x)d\Omega_n^2], \quad (11) \]
where

$$\Delta^2 = \frac{\delta}{4} \left( \delta - \frac{1}{\gamma} \right) - \frac{1 + \beta}{4 \gamma}, \quad \beta = -\frac{1}{n + p}, \quad \gamma = -\frac{n(n + p)}{4p}, \quad \delta = -\frac{2(pa - n - p + 1)}{(n + p)(na - 1)}. \quad (12)$$

Note that the above brane solution covers only exterior region of the outer
horizon, which is located at \( r = \infty \). To avoid a conical singularity at the
event horizon imposes another condition on the parameters,

$$n(p + 1)\kappa = (n + p)\mu + p(n - 1)\nu. \quad (13)$$

Then (7) and (13) imply \( \mu = \nu \). Thus, the tension parameter must have a
definite value for the solution to be regular on the horizon.

### 3 Dyonic Solutions with Tension

The extension of the previous result to the charged solutions can be easily
done. The result may be compared with the known metric for M-branes in
eleven dimensions. Let us consider the \((n + p + 2)\)-dimensional Einstein-
Hilbert-Maxwell action \((n \geq 2)\),

$$S = \int \frac{\eta^{n+p+2}x \sqrt{-g}}{16\pi G_{n+p+2}} \left[ \mathcal{R} - \frac{F_{A\mu_1\cdots\mu_{p+2}} F_{A}^{\mu_1\cdots\mu_{p+2}}}{2(p+2)!} - \frac{\mathcal{F}^{B\nu_1\cdots\nu_n} \mathcal{F}^{B}_{\nu_1\cdots\nu_n}}{2n!}\right], \quad (14)$$

where \(A\) runs over \( A = 1, \cdots, N_e\) and \(B\) over \( B = 1, \cdots, N_m\). \(N_e\) is the
number of electric fields and \(N_m\) is that of magnetic ones. As in the previous
case, we recombine the metric fields and take the same ansatz as (2). We
want to solve the Maxwell equations first, and the Einstein equations next.
The Maxwell equations are given by

$$\partial_{\mu_1} \left[ \sqrt{-g} F^{A}_{\mu_1\cdots\mu_{p+2}} \right] = 0, \quad \partial_{\nu_1} \left[ \sqrt{-g} \mathcal{F}^{B}_{\nu_1\cdots\nu_n} \right] = 0, \quad (15)$$

and they can be easily solved,

$$F^{A}_{\tau_1\cdots z_p} = Q^A e^{f(r) - h(r) + 2j(r)}, \quad \mathcal{F}^{B}_{\theta_1\cdots \theta_n} = P^B \omega_n, \quad (16)$$

where \(Q^A\) and \(P^B\) are electric and magnetic charges, respectively and \(\omega_n\)
stands for the volume form of \(S^n\). The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} + \frac{1}{4} g_{\mu\nu} \left( \frac{1}{(p+2)!} F_{A\mu_1\cdots\mu_{p+2}} F_{A}^{\mu_1\cdots\mu_{p+2}} + \frac{1}{n!} \mathcal{F}^{B\nu_1\cdots\nu_n} \mathcal{F}^{B}_{\nu_1\cdots\nu_n} \right)$$

$$- \frac{1}{2(p+1)!} F_{\mu_1\cdots\nu_{p+2}}^{A} F_{A}^{\mu_1\cdots\mu_{p+2}} = \frac{1}{2(n-1)!} \mathcal{F}^{B}_{\mu_2\cdots\nu_n} \mathcal{F}^{B}_{\nu_2\cdots\nu_n} = 0. \quad (17)$$
Since the action does not have a \( f'(r) \) term, it is easily found that after plugging the results (16) into the Einstein equations (17) and appropriately recombining them one equation is merely a constraint which is given by

\[
(n - 1)ne^{2f(r)} - \frac{n}{n - 1}\{h'(r)\}^2 + \frac{n + p}{(n - 1)(p + 1)}\{j'(r)\}^2
- \frac{(n - 1)(n + p)}{p + 1}q^2e^{2j(r)} + \frac{p}{p + 1}\{k'(r)\}^2 = 0
\]

(18)

with

\[
q = \sqrt{\frac{p + 1}{2(n - 1)(n + p)}\left[\sum_{A=1}^{N_a}(Q^A)^2 + \sum_{B=1}^{N_m}(P^B)^2\right].}
\]

(19)

Then fixing the gauge \( f(r) = h(r) \), the remaining recombined Einstein equations are given as follows,

\[
h''(r) = (n - 1)^2e^{2h(r)}, \quad j''(r) = (n - 1)^2q^2e^{2j(r)} \quad \text{and} \quad k''(r) = 0.
\]

(20)

As in the previous section all the Einstein equations could have been alternatively obtained from variation of the effective Lagrangian,

\[
\mathcal{L}_{\text{eff}} = e^{h(r) - f(r)}\left[\frac{(n - 1)n}{2}e^{2f(r)} + \frac{n}{2(n - 1)}\{h'(r)\}^2 - \frac{p}{2(p + 1)}\{k'(r)\}^2
- \frac{n + p}{2(n - 1)(p + 1)}\{j'(r)\}^2 - \frac{(n + p)(n - 1)}{2(p + 1)}q^2e^{2f(r) - h(r) + j(r)}\right]
- \left[e^{h(r) - f(r)}\left(\frac{n}{n - 1}h'(r) - \frac{1}{n - 1}j'(r)\right)\right]'.
\]

(21)

where the sign of the \( Q^2 \)-term is flipped [11]. The solutions of the equations of motion are

\[
\{h'(r)\}^2 = (n - 1)^2e^{2h(r)} + \kappa^2 \quad \Rightarrow \quad e^{-h(r)} = \frac{n - 1}{\kappa}\sinh(\kappa r + c_2),
\]

(22)

\[
\{j'(r)\}^2 = (n - 1)^2q^2e^{2j(r)} + \mu^2 \quad \Rightarrow \quad e^{-j(r)} = \frac{(n - 1)q}{\mu}\sinh(\mu r + c_3),
\]

(23)

\[
k'(r) + \nu = 0 \quad \Rightarrow \quad k(r) = -\nu r + c_4,
\]

(24)

where \( \kappa, \mu, \nu, c_2, c_3 \) and \( c_4 \) are the integration constants and we used the fact that \( \kappa^2 \) is positive, \( \frac{n}{n - 1}\kappa^2 = \frac{n + p}{(n - 1)(p + 1)}\mu^2 + \frac{p}{p + 1}\nu^2 \) from the constraint (18).
Here $c_2$ and $c_4$ can be set to zero, and $c_3$ to $\text{arcsinh}\left(\frac{\mu}{(n-1)q}\right)$ by the definition of $r$ coordinate and the symmetries of $t$ and $z_i$ coordinate rescalings. Then, we are left with a $(2 + N_e + N_m)$-parameter family of solutions. More generally, unless we assume the uniformity, a solution with $p$ nonuniform tensions has $1 + p + N_e + N_m$ parameters.

Now let us take the electrically charged brane with $p = 2$ and $n = 7$ into account. This solution may be considered as a singular extension of the $M2$-brane in that setting $\mu = \nu$, we can find a coordinate transformation from our electrically charged brane with $p = 2$ and $n = 7$ to the nonextremal $M2$-brane as follows,

$$e^{-2\kappa r} = 1 - \frac{\kappa}{3\rho^6}$$

where $\kappa = \mu = \nu$ and $\rho$ is the radial coordinate. Similarly, we would like to consider the magnetically charged brane with $p = 5$ and $n = 4$. This case could be considered as a singular generalization of the $M5$-brane in that for the case $\mu = \nu$, there is a coordinate transformation from the magnetically charged brane with $p = 5$ and $n = 4$ to the nonextremal $M5$-brane as follows,

$$e^{-2\kappa r} = 1 - \frac{2\kappa}{3\rho^3}.$$  

More generically, $M2(M5)$-brane solutions can be extended to have two(five) tension parameters along the extended spatial directions as in [9] in the same method.

## 4 Charged Two-Brane Solution with Two Tensions

In this section we will repeat the previous calculations in a similar way to derive an electrically charged two-brane solution with two tensions. Consider the following $(n + 4)$-dimensional Einstein-Hilbert-Maxwell action ($n \geq 2$),

$$I = \frac{1}{16\pi G_{n+4}} \int d^{n+4}x \sqrt{-g} \left[ \mathcal{R} - \frac{F^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}}{48} \right].$$

On the contrary to the previous cases, we will not assume uniformity for all the translationally symmetric directions of the solution. Then we take the
similar, but slightly different metric ansatz,
\[ ds^2 = -e^{\frac{2}{3}(j(r)+2k(r))} dt^2 + e^{\frac{2}{3}(j(r)-k(r))} \left( e^{l(r)} dz_1^2 + e^{-l(r)} dz_2^2 \right) + e^{\frac{2}{3-1}h(r)-j(r)} \left( e^{2f(r)} dr^2 + d\Omega^2_2 \right). \] (28)

We can solve the Maxwell and the Einstein equations in a similar way to the previous section. Alternatively, we may first plug the solution of the Maxwell equations into the action with flipping the sign of the Maxwell term to find the following effective Lagrangian density,
\[ \mathcal{L}_{\text{eff}} = e^{h(r)-f(r)} \left[ \frac{(n-1)n}{2} e^{2f(r)} + \frac{n}{2(n-1)} \{ h'(r) \}^2 - \frac{1}{3} \{ k'(r) \}^2 - \frac{n+2}{6(n-1)} \{ j'(r) \}^2 - \frac{1}{4} \{ l'(r) \}^2 - \frac{(n+2)(n-1)}{6} q^2 e^{2(f(r)-h(r)+j(r))} \right] \nonumber \\
- \left[ e^{h(r)-f(r)} \left( \frac{n}{n-1} h'(r) - \frac{1}{n-1} j'(r) \right) \right]' \right], \] (29)

where \( q = \sqrt{\frac{3}{2(n-1)(n+2)}} Q^2 \) and \( Q \) is the electric charge, and then solve the equations derived from the variation. The solutions of the equations of motion with the gauge \( f(r) = h(r) \) are
\[ e^{-h(r)} = \frac{n-1}{\kappa} \sinh(\kappa r + c_2), \] (30)
\[ e^{-j(r)} = \frac{(n-1)q}{\mu} \sinh(\mu r + c_3), \] (31)
\[ k(r) = -\nu r + c_4, \] (32)
\[ l(r) = -\sigma r + c_5 \] (33)

with the constraint,
\[ \frac{n}{n-1} \kappa^2 - \frac{n+2}{3(n-1)} \mu^2 - \frac{2}{3} \nu^2 - \frac{1}{2} \sigma^2 = 0, \] (34)

where \( \kappa, \mu, \nu, \sigma, c_2, c_3, c_4 \) and \( c_5 \) are the integration constants. Using the definition of \( r \) coordinate and the symmetries of \( t, z_1 \) and \( z_2 \) rescalings we can set \( c_2, c_4 \) and \( c_5 \) to vanish, and \( c_3 \) to \( \arcsinh(\frac{\mu}{(n-1)q}) \). Then, we get a four-parameter family of solutions. These four parameters are some combinations of mass density, charge and two tensions of the solution. It
is straightforward to find generalized $M5$-brane solutions with five different
tensions by the similar process.

For the above solution, the event horizon is located at $r = \infty$. The
regularity condition requires

$$3n\kappa = (n + 2)\mu + 2(n - 1)\nu.$$  \hfill (35)

Then (34) and (35) give us $\mu = \nu$ and $\sigma = 0$. Thus, for the solution to have
a regular horizon the tension parameters cannot be arbitrary.

5 Discussions

In this paper, we have obtained dyonic brane solutions with tension. The
parameters are mass density and tension, and electric/magnetic charges. We
also have gotten an electrically charged two-brane solution with two arbitrary
tensions, which, particularly, could be considered as a singular extension of
the nonextremal $M2$-brane. It is straightforward to generalize the charged
brane solutions further to have $p$ different tensions along $p$ directions with
translational symmetry and to find a generalized $M5$-brane solution with five
different tensions. The brane solutions with uniform tension are the deformed
ones which were already found in [11], where the first-order formalism [12]
was employed. It would be interesting to derive solutions which have $p$
different tensions along $p$ extended directions. More importantly, however,
for most values of tensions, these solutions represent just naked singularities
[9, 13, 14, 15]. In other words, the regularity conditions narrow meaningful
solutions and require definite values of tension parameters. Therefore, it
would be interesting to extend the charged brane solution to have $p$
different arbitrary tensions along $p$ translationally symmetric directions and to seek
the role of the singular black brane solutions. We leave it for future study.

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