The Space of Local Fields as a Module over the Ring of Local Integrals of Motion

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Abstract

The chiral space of local fields in Sine-Gordon or the SU(2)-invariant Thirring model is studied as a module over the commutative algebra $D$ of local integrals of motion. Using the recent construction of form factors by means of quantum affine algebra at root of unity due to Feigin et al. we construct a $D$-free resolution of the space of local fields. In general the cohomologies of the de Rham type complex associated with the space of local fields are determined and shown to be the irreducible representations of the symplectic group $Sp(2\infty)$. Babelon-Bernard-Smirnov’s description of the space of local fields automatically incorporated in this framework.

1 Introduction

Let $U_{\sqrt{-1}}(\hat{sl}_2)$ be the quantum affine algebra at $q = \sqrt{-1}$, $x_k^\pm$, $b_n$ (a part of) Drinfeld generators, $V_{\sqrt{-1}}(\Lambda_i)$ the level one representation and $V_{\sqrt{-1}}(\Lambda_i)_{2m+i}$ the subspace of vectors whose $sl_2$ weight is $2m + i$.

In [4] Feigin et al. proved that the chiral subspace of local fields in the quantum sine-Gordon model at generic coupling constant and that in the SU(2) invariant Thirring model (ITM) are isomorphic to the space $A_{2m+i} = V_{\sqrt{-1}}(\Lambda_i)_{2m+i} / x_0^+ V_{\sqrt{-1}}(\Lambda_i)_{2m+i+2} + (x_0^+)^2 V_{\sqrt{-1}}(\Lambda_i)_{2m+i+4}$, $m \geq 0$, where $(x_0^+)^2 = (x_0^-)^2 / (q + q^{-1})$ is the divided power of $x_0^-$. The local integral of motion with spin $s$, $s$ being odd, is identified with $b_s$. They become central at $q = \sqrt{-1}$ and $A_{2m+i}$ becomes a module over the commutative ring $D = \mathbb{C}[b_{-1}, b_{-3}, \ldots]$.

To study the structure of $A_{2m+i}$ as a $D$-module is an interesting problem because it is a common problem for any integrable system. In particular it will give a suitable framework to compare structures of different integrable systems such as massive integrable field theories, conformal field theories and classical integrable systems [11 14].

The main aim of this article is to determine the $D$-module structure of $A_{2m+i}$ completely. More precisely we construct a $D$-free resolution of $A_{2m+i}$. Each term of the resolution is described in terms of the Fock space of free fermions.

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The problem to study the space of local fields as a module over the ring of local integrals of motion was initiated by Babelon-Bernard-Smirnov [1]. They studied the restricted sine-Gordon model and derived the $c=1$ character of the Virasoro algebra. The important observation in [1] is that the fermionization simplifies the description of polynomials giving rise to null vectors. The description of the space of local fields in terms of the representations of the quantum affine algebra automatically incorporates the fermions and related operators in [1]. We remark here that it is difficult to determine the $D$-module structure of $A_{2m+i}$ in the previous description by minimal form factors [10, 11, 5, 6].

The commutative ring $D$ and a module over it determines a de Rham type complex. Let $\Omega^\infty = \mathbb{C}^{dt_{-1} \wedge dt_{-3} \wedge \cdots}$, be the space of highest degree forms and $\Omega^\infty - p$ be the vector space generated by differential forms which are obtained from $dt_{-1} \wedge dt_{-3} \wedge \cdots$ by removing $p$ $dt_i$'s. Then the pair of the vector space and a differential

$$C^\infty_{2m+i} = A_{2m+i} \otimes \Omega^\infty - p,$$

$$d = \sum_{s=1}^{\infty} b_{-(2s-1)} \otimes dt_{-(2s-1)},$$

defines a complex. The highest cohomology group is isomorphic to

$$\frac{A_{2m+i}}{\sum_{s=1}^{\infty} b_{-(2s-1)} A_{2m+i}}.$$  

This space is the space of chiral local fields modulo the action of local integrals of motion and its basis gives the minimal set of generators of $A_{2m+i}$ as a $D$-module. Thus to determine it is in fact a first step to construct a free resolution of $A_{2m+i}$. In turn, using the free resolution of $A_{2m+i}$ it is possible to describe all cohomology groups in terms of the Fock space of free fermions. We show that those cohomology groups become the irreducible representations of the symplectic group $Sp(2\infty)$, which is obtained as the inductive limit of $Sp(2n)$'s. This result suggests that the cohomology groups of [1, 2] give the universal structure of the cohomology groups of affine hyperelliptic Jacobian varieties [3] [12]. Thus one can think of the result as an example of the comparison of different integrable systems mentioned above. We shall study this subject in a subsequent paper.

The present paper is organized in the following manner. After introduction the notations concerning the quantum affine algebra at $q = \sqrt{-1}$ and its level one representations are explained in section two. Then the chiral space of local fields are defined using them. In section 3 fermions are introduced and the integral expressions of $x_0$ and $(x_0^2)^{(2)}$ are given in terms of them. The free resolution of $A_{2m}$ are constructed in section 4. In section five the de Rham type cohomology groups associated with $A_{2m}$ are determined. The space $A_{2m+1}$ is studied in section 6. We prove that $A_{2m+1}$ is a free $D$-module. In section 7 the cohomology groups are shown to be the irreducible representations of the symplectic group.

### 2 The chiral space

We recall the results of [4]. Notations mainly follows that paper. Let $q$ be an indeterminate. The quantum affine algebra $U_q(\widehat{sl_2})$ is the $\mathbb{C}(q)$ Hopf algebra generated by $x_k^\pm (k \in \mathbb{Z})$, $b_n(n \in \mathbb{Z}\{0\})$, $e_1^\pm$, $C^\pm$, $D^\pm$ with the following defining relations:
Since we do not use the coproduct in this paper we omit the description it.

The Frenkel-Jing realization of the level one integrable highest weight representation

We denote the subalgebra of \( U \) to this end we need some notations. For \( x \) are given by the equations

There are two notable properties which distinguish \( U \) to a complex number one has to define the so called integral form.

Let \( A = \mathbb{C}[q, q^{-1}] \). Then the integral form \( U_A \) is defined to be the \( A \)-subalgebra of \( U_q(sl_2) \) generated by \( (x_k^\pm)^{(r)} (k \in \mathbb{Z}, r \geq 0), b_n, n \in \mathbb{Z}\setminus\{0\}, t_1^{\pm1}, C^{\pm1}, D^{\pm1}. \)

The algebra \( U_{\sqrt{-1}}(sl_2) \) is then defined by

We denote the subalgebra of \( U_A \) without \( D^{\pm1} \) by \( U'_A \) and similarly for \( U'_{\sqrt{-1}}(sl_2) \).

There are two notable properties which distinguish \( U'_{\sqrt{-1}}(sl_2) \) from the generic \( q \) case. They are given by the equations

Let \( \alpha_0, \alpha_1 \) and \( \Lambda_0, \Lambda_1 \) be the simple roots and the fundamental weights of \( sl_2 \) respectively. The Frenkel-Jing realization of the level one integrable highest weight representation \( V(\Lambda_i) \) of \( U_q(sl_2) \) determines the level one representation of \( U_{\sqrt{-1}}(sl_2) \). Define the integral form \( V_A(\Lambda_i) \) by the free \( A \)-module

\[
V_A(\Lambda_i) = \oplus_{m \in \mathbb{Z}} A[b_n | n < 0]e^{\Lambda_i + m\alpha_1}.
\]
The action of $U_A$ on this space is defined in the following way. For $Pe^\beta \in V_A(\Lambda_i)$ define the actions of $C$, $\partial$ by

$$C(PE^\beta) = q(PE^\beta),$$
$$\partial(PE^\beta) = <h_1, \beta > (PE^\beta),$$

where $< h_1, \alpha_1 > = 2$, $< h_1, \Lambda_j > = \delta_{1,j}$. For $n < 0$ $b_n$ acts by the multiplication by itself and for $n > 0$ it acts by taking the commutator $[b_n, \cdot]$. Define the grading on $V A(\Lambda_i)$ as

$$\deg b_n = n, \quad \deg e^{\Lambda_i+m\alpha_1} = -m^2 - im.$$ Let

$$x^\pm(z) = \sum_{n \in \mathbb{Z}} x^\pm_n z^{-n-1}.$$ Then the generators of $U_A$ acts on $V A(\Lambda_i)$ by the following formulas

$$x^+(z) = \exp\left(\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{b_n}{n} (qz)^{-n}\right) e^{\alpha_1 z}$$
$$x^-(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{b_{-n}}{n} (qz)^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n} z^{-n}\right) e^{-\alpha_1 z}$$

(5)

$$t_1 = q^\partial, \quad Du = q^{\deg u} u \text{ for a homogeneous } u \in V A(\Lambda_i).$$

We set

$$V A(\Lambda_i)_{2m+i} = A[b_n \mid n < 0] e^{\Lambda_i+m\alpha_1}.$$ Then the space

$$V_\sqrt{-1}(\Lambda_i) = \left(\frac{A}{(q - \sqrt{-1})A}\right) \otimes_A V A(\Lambda_i),$$

becomes a $U_\sqrt{-1}(\hat{sl}_2)$-module and

$$V_\sqrt{-1}(\Lambda_i)_{2m+i} = \left(\frac{A}{(q - \sqrt{-1})A}\right) \otimes_A V A(\Lambda_i)_{2m+i},$$

becomes its subspace.

Set

$$A_{2m+i} = \frac{V_\sqrt{-1}(\Lambda_i)_{2m+i}}{x_0 V_\sqrt{-1}(\Lambda_i)_{2m+i+2} + (x_0)^{(2)} V_\sqrt{-1}(\Lambda_i)_{2m+i+4}}, \quad i = 0, 1, \ m \geq 0.$$ Because of $\Box$ it becomes a module over the commutative algebra

$$D = \mathbb{C}[b_{-1}, b_{-3}, \ldots].$$

The chiral space of local fields in SG-model at generic coupling constant and that in SU(2)-ITM are both isomorphic to $A_{2m+i}$, $\Box \Box \Box$. In SG case the fields are highest weight vectors with the weight $2m + i$ with respect to a certain quantum group $U_p(\hat{sl}_2)$ for some $p$ and in SU(2)-ITM case the fields are highest weight vectors with weight $2m + i$ respect to $sl_2$. Local integrals of motion with spin $2s - 1$ are identified with $b_{2s-1}$. Since we consider the chiral space, only the local integrals of motion with negative spins act on it.
3 Fermions

We consider the Neveu-Schwarz (NS) and Ramond (R) fermions \( \{ \psi_{2n-1}, \psi^*_{2n-1} \mid n \in \mathbb{Z} \} \) and \( \{ \psi_{2n}, \psi^*_{2n} \mid n \in \mathbb{Z} \} \) respectively. They satisfy the canonical anti-commutation relations

\[
[\psi_m, \psi_n]_+ = [\psi^*_m, \psi^*_n]_+ = 0, \quad [\psi_m, \psi^*_n]_+ = \delta_{m,n}.
\]

In the following, objects with odd indices are for NS-fermions and those with even indices are for R-fermions. The vacuum vectors \( |m> \) and \( <m| \) are introduced by the following relations

\[
< m | \psi_n = 0 \quad \text{for} \quad n \leq m, \quad < m | \psi^*_n = 0 \quad \text{for} \quad n > m,
\]

\[
\psi_n | m > = 0 \quad \text{for} \quad n > m, \quad \psi^*_n | m > = 0 \quad \text{for} \quad n \leq m.
\]

These vacuums are related by

\[
\psi^*_m | m - 2 > = |m>, \quad < m - 2 | \psi^*_m = < m |.
\]

The Fock spaces \( H_m, H^*_m \) are constructed from \( |m> \) and \( <m| \) respectively by the same number of \( \psi_n \) and \( \psi^*_n \). The pairing between \( H_m \) and \( H^*_m \) are defined by the condition

\[
< m | m > = 1.
\]

The fermion operators are introduced as

\[
\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{2n+1} z^{-2n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi^*_{2n+1} z^{2n+1} \quad \text{for NS-fermions},
\]

\[
\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{2n} z^{-2n}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi^*_{2n} z^{2n} \quad \text{for R-fermions}.
\]

Let \( \tilde{b}_{2n} (n \in \mathbb{Z}) \) satisfy the canonical commutation relations

\[
[\tilde{b}_{2n}, \tilde{b}_{2n'}] = \delta_{n,n'}.
\]

We set

\[
H(b) = \sum_{l=1}^{\infty} \tilde{b}_{-2l} h_{2l}, \quad h_{2l} = \sum_n \psi_n \psi^*_{n-2l},
\]

where the summation in \( n \) of \( h_{2l} \) is taken over odd \( n \in \mathbb{Z} \) for NS-fermions and even \( n \) for R-fermions.

The boson-fermion correspondence gives the isomorphism of Fock spaces \[ \text{3} \]

\[
H_{2m+i} \simeq \mathbb{C}[\tilde{b}_{-2}, \tilde{b}_{-4}, \ldots] e^{\Lambda_i, + \Lambda_i},
\]

\[
a | 2m + i > \mapsto < 2m + i | e^{H(b)} a | 2m + i > .
\]

By this correspondence the fermion operators are written as \[ \text{3} \]

\[
\psi(z) = \exp(\sum_{n=1}^{\infty} \tilde{b}_{-2n} z^{2n}) \exp(-\sum_{n=1}^{\infty} \frac{\tilde{b}_{2n}}{n} z^{-2n}) e^{-\alpha z^2},
\]

\[
\psi^*(z) = \exp(-\sum_{n=1}^{\infty} \tilde{b}_{-2n} z^{2n}) \exp(\sum_{n=1}^{\infty} \frac{\tilde{b}_{2n}}{n} z^{-2n}) z^\alpha e^{\alpha}.
\]
for both NS and R.

Two bosons $b_{2n}$ and $\tilde{b}_{2n}$ are related by

$$\tilde{b}_{2n} = \begin{cases} \frac{-1}{2n} b_{2n}, & n \geq 1 \\ \frac{-1}{2n} b_{2n}, & n \leq -1. \end{cases}$$

Then we have

**Proposition 1** By the boson-fermion correspondence $x_0^{-}$ and $(x_0^{-})^2$ on $V(\Lambda_i)$ are expressed as

$$x_0^{-} = \int \frac{dz}{2\pi i} e^{X(z)} \psi(z),$$

$$(x_0^{-})^2 = \frac{-i}{2} \int \int_{|z_1| > |z_2|} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \tau(z_1) e^{X(z_1)} e^{X(z_2)} \psi(z_1) \psi(z_2),$$

where

$$X(z) = \sum_{n=1}^{\infty} \frac{\tilde{b}_{-(2n-1)}}{2n-1} z^{2n-1},$$

$$\tau(z) = \sum_{n=1}^{\infty} z^{2n-1} - 2 \sum_{n=1}^{\infty} z^{2n},$$

and $\psi(z)$ is R-fermion for $i = 0$ and NS-fermion for $i = 1$.

**Remark.** The right hand sides of (7) and (8) are similar to the operators appeared in [1]. The only difference, besides the overall constant multiples, is the coefficients of even and odd powers of $z$ in $\tau(z)$.

**Proof.** The expression for $x_0^{-}$ follows from (4) and the definitions of $\psi(z)$, $X(z)$. Let us prove (8). For $n \geq 1$, $b_n$ can be written as

$$b_n = n(q^n + q^{-n}) \frac{\partial}{\partial b_{-n}}.$$ 

We substitute this into (8). In this description of $x^{-}(z)$ we have

$$(x_0^{-})^2 = \frac{1}{2} \lim_{q \to i} \frac{(x_0^{-})^2}{q - i} = \frac{1}{2} \int \int \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} d\tau(z_1) d\tau(z_2) e^{X(z_1)} e^{X(z_2)} \psi(z_1) \psi(z_2),$$

where we write the $q$-dependence of $x^{-}(z)$ explicitly and denote $i = \sqrt{-1}$. By calculation

$$\left( \frac{d}{dq} x^{-}(z; q) \right)|_{q = i}$$

$$= i \sum_{n=1}^{\infty} b_{-n} i^n z^n x^{-}(z_1; i) + x^{-}(z; q) \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1) z^{-2n-1} \frac{\partial}{\partial b_{-(2n-1)}}.$$
Then
\[
\frac{d}{dq} \left( x^-(z_1; q) x^-(z_2; q) \right) \bigg|_{q=i} =
\]
\[
i \sum_{n=1}^{\infty} b_{-n} i^n (z_1^n + z_2^n) x^-(z_1; i) x^-(z_2; i)
\]
\[
+ x^-(z_1; i) x^-(z_2; i) \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1) \frac{\partial}{\partial b_{-(2n-1)}}
\]
\[
- i \tau \left( \frac{z_2}{z_1} \right) x^-(z_1; i) x^-(z_2; i).
\]

Here we use (4) and the commutation relations of \(b_n\). Due to (3)
\[
\int \frac{dz_1}{2\pi i} \int \frac{dz_2}{2\pi i} (z_1^n + z_2^n) x^-(z_1; i) x^-(z_2; i) = 0 \quad \forall \, n \in \mathbb{Z}.
\]

\section{Resolution of \(A_{2m}\)}

In terms of the Fock space of fermions \(A_{2m}\) is written as
\[
A_{2m} \simeq \frac{\mathcal{D} \otimes H_{2m}}{x_0^{-} (\mathcal{D} \otimes H_{2m+2}) + (x_0^{-})^{(2)} (\mathcal{D} \otimes H_{2m+4})}.
\]

Since \((x_0^{-})^2 = 0\) and \([x_0^{-}, (x_0^{-})^{(2)}] = 0\) it is possible to define a complex
\[
\ldots \rightarrow \frac{\mathcal{D} \otimes H_{2m+2}}{(x_0^{-})^{(2)} (\mathcal{D} \otimes H_{2m+6})} \rightarrow \frac{\mathcal{D} \otimes H_{2m}}{(x_0^{-})^{(2)} (\mathcal{D} \otimes H_{2m+4})} \rightarrow A_{2m} \rightarrow 0,
\]

where the map
\[
\frac{\mathcal{D} \otimes H_{2n}}{(x_0^{-})^{(2)} (\mathcal{D} \otimes H_{2n+4})} \rightarrow \frac{\mathcal{D} \otimes H_{2n-2}}{(x_0^{-})^{(2)} (\mathcal{D} \otimes H_{2n+2})}
\]

is given by the left multiplication by \(x_0^{-}\).

**Proposition 2** The module
\[
\frac{\mathcal{D} \otimes H_{2n}}{(x_0^{-})^{(2)} (\mathcal{D} \otimes H_{2n+4})}
\]
is a free \(\mathcal{D}\)-module.

**Proof.** To prove the proposition we write \(x_0^{-}\) and \((x_0^{-})^{(2)}\) in the component form. Let us write
\[
e^{X(z)} = \sum_{n=0}^{\infty} T_{-n} z^n.
\]
Notice that $T_{-(2n-1)}$ is a homogeneous polynomial of $\tilde{b}_{-(2m-1)}$ with the degree $-(2n-1)$ and it has the form

$$T_{-(2n-1)} = \frac{\tilde{b}_{-(2n-1)}}{2n-1} + \cdots,$$

where $\cdots$ part does not contain $\tilde{b}_{-(2n-1)}$. In particular

$$\mathcal{D} = \mathbb{C}[T_{-1}, T_{-3}, \ldots].$$

By calculation we have

$$x_0^- = \sum_{n=1}^{\infty} T_{-(2n-1)} \psi_{2n},$$

$$\left( x_0^- \right)^{(2)} = -\frac{i}{2} \sum_{n=1}^{\infty} \left( \psi_{2(-n+1)} + \sum_{l=1}^{\infty} Q_{n,l}(T) \psi_{2(-n+1+l)} \right) \psi_{2n},$$

where

$$Q_{n,l}(T) = \sum_{n_1+n_2=l, \sigma_1 \leq n_1, \sigma_2 \leq n_2 \leq n} \left( T_{-2n_1} T_{-2n_2} - 2T_{-(2n_1+1)} T_{-(2n_2-1)} \right).$$

We omit the tensor symbol $\otimes$ in writing elements of $\mathcal{D} \otimes H_{2n}$ etc. for the sake of simplicity. Using this expression we make a base change of $\mathcal{D} \otimes H_{2n}$. Let us define $\{\tilde{\psi}^*_{2n}, \tilde{\psi}^*_{-2n}\}$ in the following way. First we set

$$\tilde{\psi}^*_{2n} = \psi_{2n} \quad \text{for} \quad n \geq 1,$$

$$\tilde{\psi}^*_{-2n} = \psi_{-2n} + \sum_{l=1}^{\infty} Q_{n,l}(T) \psi_{2(-n+l)} \quad \text{for} \quad n \geq 0,$$

and write it as

$$\tilde{\psi}^*_i = \sum_{j} B_{ij} \psi^*_j,$$

where $i, j$ are even integers. Then the matrix $B = (B_{ij})$ is a triangular matrix such that its diagonal entries are all 1 and $B_{ij} = \delta_{ij}$ for $i \geq 2$. Thus $B^{-1}$ exists. Define the matrix $C$ by

$$C = (c_{ij}) = t(B^{-1})$$

and set

$$\tilde{\psi}^*_i = \sum_{j} c_{ij} \psi^*_j.$$

Then $\{\tilde{\psi}_k, \tilde{\psi}^*_l\}$ satisfy the canonical anti-commutation relations

$$[\tilde{\psi}_k, \tilde{\psi}^*_l]_+ = [\tilde{\psi}^*_k, \tilde{\psi}^*_l]_+ = 0, \quad [\tilde{\psi}_k, \tilde{\psi}^*_l]_+ = \delta_{k,l}.$$

Moreover the vacuum is the same for $\{\tilde{\psi}_k, \tilde{\psi}^*_l\}$, that is, the following relations hold

$$\tilde{\psi}^*_{2n}|2m > = 0 \quad \text{for} \quad n > m, \quad \tilde{\psi}^*_{-2n}|2m > = 0 \quad \text{for} \quad n \leq m,$$

$$< 2m|\tilde{\psi}_{2n} = 0 \quad \text{for} \quad n \leq m, \quad < 2m|\tilde{\psi}^*_{2n} = 0 \quad \text{for} \quad n > m.$$
Therefore we denote the vacuum for \( \{ \tilde{\psi}_k, \tilde{\psi}^*_l \} \) by the same symbol as for \( \{ \psi_k, \psi^*_l \} \). We denote the Fock space of \( \{ \tilde{\psi}_k, \tilde{\psi}^*_l \} \) by \( \tilde{H}_m \). Then we have the isomorphism of \( D \) modules

\[
D \otimes H_2m \simeq D \otimes \tilde{H}_2m.
\]

In terms of \( \{ \tilde{\psi}_k, \tilde{\psi}^*_l \} \), \( x^-_0 \) and \( (x^-_0)^{(2)} \) take simple forms

\[
x^-_0 = \sum_{n=1}^{\infty} T_{-(2n-1)}\tilde{\psi}_{2n},
\]

\[
(x^-_0)^{(2)} = \frac{i}{2} \sum_{n=1}^{\infty} \tilde{\psi}_{-2(2n-1)}\tilde{\psi}_{2n}.
\]

In particular \( (x^-_0)^{(2)} \) defines a map

\[
(x^-_0)^{(2)} : \tilde{H}_2n \rightarrow \tilde{H}_{2n-4}.
\]  

Thus we have the isomorphism

\[
\frac{D \otimes H_2n}{(x^-_0)^{(2)}(D \otimes H_{2n+4})} \simeq \frac{\tilde{H}_2n}{(x^-_0)^{(2)}\tilde{H}_{2n+4}}.
\]

This proves the proposition. \( \blacksquare \)

**Remark.** The argument used in the proof of the proposition is essentially due to Babelon-Bernard-Smirnov \[1\].

We set

\[
W_{2n} = \frac{\tilde{H}_{2n}}{(x^-_0)^{(2)}\tilde{H}_{2n+4}}.
\]

**Theorem 1** The complex \([4]\)

\[
\cdots \rightarrow D \otimes W_{2m+2} \rightarrow D \otimes W_{2m} \rightarrow A_{2m} \rightarrow 0,
\]

is a \( D \)-free resolution of \( A_{2m} \).

The following lemmas are sufficient to prove the theorem.

**Lemma 1** The complex

\[
\cdots \rightarrow D \otimes \tilde{H}_4 \rightarrow D \otimes \tilde{H}_2 \rightarrow D \otimes \tilde{H}_0 \rightarrow 0
\]

is exact at \( \tilde{H}_{2n} \) \( n \geq 1 \), where the maps are defined by the multiplication by \( x^-_0 \).

**Lemma 2** The map \([10]\) is injective for \( n \geq 1 \).

Assuming these lemmas we give a proof of the theorem first.

**Proof of Theorem** \([7]\)

It is sufficient to prove the exactness at \( D \otimes W_{2n}, n > m \). Suppose that \( v \in \tilde{H}_{2n} \) satisfies

\[
x^-_0 v = (x^-_0)^{(2)} w
\]
for some \( w \in \tilde{H}_{2n+2} \). Then

\[
(x_0^-)^{(2)}(x_0^- w) = 0.
\]

Since \((x_0^-)^{(2)}\) is injective by Lemma 2, \( x_0^- w = 0 \). By Lemma 1, \( w = x_0^- u \) for some \( u \in \tilde{H}_{2n+4} \).

Then

\[
(x_0^-)(v - (x_0^-)^{(2)} u) = 0,
\]

and again by Lemma 1

\[
v - (x_0^-)^{(2)} u = x_0^- y
\]

for some \( y \in \tilde{H}_{n+2} \). This proves the theorem.

**Proof of Lemma 1**

Notice that

\[
\tilde{H}_2 = \sum C \tilde{\psi}^*_{2r_1} \cdots \tilde{\psi}^*_{2r_k} \tilde{\psi}^{2s_1} \cdots \tilde{\psi}^{2s_k} |0>.
\]

where the summation is taken for all

\[
0 < r_1 < \cdots < r_k+m, \quad s_1 < \cdots < s_k \leq 0. \tag{11}
\]

We set

\[
d = x_0^- = \sum_{s=1}^{\infty} T_{-(2s-1)}\psi_{2s},
\]

for the sake of simplicity. For each \((s_1, \ldots, s_k)\) satisfying (11), we set

\[
\tilde{H}_{2m}(s_1, \ldots, s_k) = \sum_{0 < r_1 < \cdots < r_k+m} C \tilde{\psi}^*_{2r_1} \cdots \tilde{\psi}^*_{2r_k+m} \tilde{\psi}^{2s_1} \cdots \tilde{\psi}^{2s_k} |0>.
\]

Then \( \tilde{H}_{2m} \) is a direct sum of \( \tilde{H}_{2m}(s_1, \ldots, s_k) \)'s and

\[
d(\tilde{H}_{2m}(s_1, \ldots, s_k)) \subset \tilde{H}_{2m-2}(s_1, \ldots, s_k).
\]

Thus it is sufficient to prove the exactness of the complex

\[
(D \otimes \tilde{H}_{2m}(s_1, \ldots, s_k), d)
\]

at \( m \geq 1 \) for each \((s_1, \ldots, s_k)\). Set, for \( 0 < r_1 < \cdots < r_k+m \),

\[
e_{r_1 \cdots r_k+m} = \tilde{\psi}^*_{2r_1} \cdots \tilde{\psi}^*_{2r_k+m} \tilde{\psi}^{2s_1} \cdots \tilde{\psi}^{2s_k} |0>.
\]

They form a basis of \( \tilde{H}_{2m}(s_1, \ldots, s_k) \). Now suppose that

\[
v \in D \otimes \tilde{H}_{2m}(s_1, \ldots, s_k), \quad dv = 0,
\]

and write

\[
v = \sum_{0 < r_1 \cdots r_k+m} P_{r_1 \cdots r_k+m} e_{r_1 \cdots r_k+m}, \quad P_{r_1 \cdots r_k+m} \in D.
\]
If we set $t_s = T_{-(2s-1)}$ we have
\[
d(e_{r_1...r_{k+m}}) = \sum_{i=1}^{k+m} (-1)^{i-1} t_s e_{r_1...r_{i-1}r_{i+1}...r_{k+m}}.
\]

Let $N_1$ be the maximum among $r_1, ..., r_{k+m}$ such that $P_{r_1...r_{k+m}} \neq 0$, $N$ an integer such that $N > N_1$ and $P_{r_1...r_{k+m}} \in \mathbb{C}[t_1, \ldots, t_N]$ for all $P_{r_1...r_{k+m}} \neq 0$. We set
\[
B = \mathbb{C}[t_1, \ldots, t_N], \quad K_p = \sum_{1 \leq i_1 < \cdots < i_p \leq N} B e_{i_1...i_p}.
\]

Then
\[
v \in K_{m+k}
\]

and the complex
\[
0 \rightarrow K_N \xrightarrow{d} K_{N-1} \xrightarrow{d} \cdots \xrightarrow{d} K_0 \rightarrow 0
\]
is the Koszul complex associated with the regular sequence $(t_1, \cdots, t_N)$ of $B$. Thus the complex $(K,d)$ is exact at $K_n$, $n \geq 1$. Since $m \geq 1$, this proves the lemma.

Proof of Lemma

Let us set
\[
\omega = 2i(x_0^-)^{(2)} = -\sum_{n=1}^{\infty} \tilde{\psi}_{-2(n-1)} \tilde{\psi}_{2n}.
\]

Suppose that $v \in \tilde{H}_{2m}$ satisfies
\[
\omega v = 0.
\]

Let us write
\[
v = \sum c(r_1, \ldots, r_k | s_1, \ldots, s_{k+m}) \tilde{\psi}_{2r_1} \cdots \tilde{\psi}_{2r_k} \tilde{\psi}_{2s_1} \cdots \tilde{\psi}_{2s_{k+m}} | 0 >,
\]

where
\[
r_1 < \cdots < r_k \leq 0 < s_1 < \cdots < s_{k+m}.
\]

Let $N$ be the maximum number among $-r_1, \ldots, -r_k, s_1, \ldots, s_{k+m}$ such that $c(r_1, \ldots, r_k | s_1, \ldots, s_{k+m}) \neq 0$. Then
\[
\omega v = \omega_N v, \quad \omega_N = -\sum_{n=1}^{N} \tilde{\psi}_{-2(n-1)} \tilde{\psi}_{2n}.
\]

Define
\[
\eta_N = \sum_{n=1}^{N} \tilde{\psi}_{-2(n-1)} \tilde{\psi}_{2n}, \quad \xi_N = -\sum_{n=1}^{N} \tilde{\psi}_{-2(n-1)} \tilde{\psi}_{2n} + \sum_{n=1}^{N} \tilde{\psi}_{2n} \tilde{\psi}_{2n}.
\]
They act on the direct sum of the spaces

\[ \hat{H}_{2m}^{(N)} = \sum_{-N+1 < r_1 < \cdots < r_k \leq 0 < s_1 < \cdots < s_{k+m} \leq N} \mathbb{C}\bar{\psi}_{2r_1} \cdots \bar{\psi}_{2r_k} \bar{\psi}_{2s_1} \cdots \bar{\psi}_{2s_{k+m}} |0\rangle, \]

and satisfy the relations

\[ [\eta_N, \omega_N] = \xi_N, \quad [\xi_N, \eta_N] = 2\eta_N, \quad [\xi_N, \omega_N] = -2\omega_N. \]

Thus \( \omega_N, \eta_N \) and \( \xi_N \) determine the action of \( sl_2 \) on \( \oplus_{m \in \mathbb{Z}} \hat{H}_{2m}^{(N)} \) by the correspondence

\[ e = \eta_N, \quad f = \omega_N, \quad h = \xi_N. \]

We have \( h = m \) on \( \hat{H}_{2m}^{(N)} \). Thus by the representation theory of \( sl_2 \), \( \omega_N \) is injective at \( \hat{H}_{2m}^{(N)} \), \( m \geq 1 \). Since \( v \in \hat{H}_{2m}^{(N)}, v = 0 \).

Let us write \( D = q^d \in U_q(sl_2) \). Then \( d \) can be considered as a degree operator of \( V(\Lambda_i) \). The grading of \( V(\Lambda_i) \) induces those of \( V_{\sqrt{-1}}(\Lambda_i) \) and \( H_{2m} \). Since \( x_0^- \) and \( (x_0^-)^{(2)} \) are homogeneous, the quotient space \( A_{2m} \) is also graded. In general, for a graded vector space \( V \) by the degree operator \( d \) such that each homogeneous subspace is finite dimensional, we define its character by

\[ \text{ch} V = \text{tr}_V (p^{-d}). \]

We remark that, with this definition of the degree, we have

\[ \text{deg} \psi_{2n} = 2n - 1, \quad \text{deg} \bar{\psi}_{2n}^* = -(2n - 1), \quad \text{deg} |2m\rangle = -m^2. \]

Then

\[ \text{deg} \bar{\psi}_{2n} = 2n - 1, \quad \text{deg} \bar{\psi}_{2n}^* = -(2n - 1), \]

and

\[ \text{ch} \hat{H}_{2m} = \text{ch} H_{2m}. \quad (12) \]

**Corollary 1**

\[ \text{ch} A_{2m} = \frac{p^{m^2}(1 - p^{2m+1})}{(p : p)_{\infty}}. \]

**Proof.** Since

\[ H_{2m} \simeq \mathbb{C}[b_{-2}, b_{-4}, \ldots] e^{\Lambda_0 + m\alpha_1}, \]

we have, by (12),

\[ \text{ch} \hat{H}_{2m} = \frac{p^{m^2}}{(p^2 : p^2)_{\infty}}. \]

By Lemma 2 \( \omega \) is injective. Thus

\[ \text{ch} W_{2m} = \text{ch} \hat{H}_{2m} - \text{ch} \hat{H}_{2m+4}, \]

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and
\[ \text{ch } A_{2m} = \sum_{n=m}^{\infty} \left(-1\right)^{n-m} \text{ch} \left(D \otimes W_{2m}\right) = \text{ch} D(\text{ch } \hat{H}_{2m} - \text{ch } \hat{H}_{2m+2}). \]

This proves the corollary. ■

**Remark.** The character of \( A_{2m} \) is calculated in \([10, 4]\) in a so-called fermionic form. Combining the identity of \( q \)-series \([7]\) with it the character formula in the corollary is proved. Here we have shown that the character formula in this bosonic form can be derived from the resolution of \( A_{2m} \) independently to other results.

**Corollary 2** We have the isomorphism
\[ \frac{A_{2m}}{\sum_{s=1}^{\infty} b_{-(2s-1)} A_{2m}} \simeq W_{2m}. \]

5 De Rham type cohomologies

Let us set
\[ v_\phi = dt_{-1} \wedge dt_{-3} \wedge dt_{-5} \wedge \cdots. \]

For \( 0 > i_1 > \cdots > i_l \), \( i_k \)'s being odd, we define \( v_{i_1, \ldots, i_l} \) to be the formal infinite wedges obtained from \( v_\phi \) by removing \( dt_{i_1}, \ldots, dt_{i_l} \). We define \( \Omega^\infty_-^p, p \geq 0 \)
by
\[ \Omega^\infty_-^p = \mathbb{C} v_\phi, \]
\[ \Omega^\infty_-^{p-p} = \sum_{0 > i_1 > \cdots > i_p, \text{odd}} \mathbb{C} v_{i_1, \ldots, i_p} \quad \text{for } p \geq 1 \]

and set
\[ C_{2m}^\infty_-^p = A_{2m} \otimes \Omega^\infty_-^p. \]

Then by defining
\[ d = \sum_{s=1}^{\infty} b_{-(2s-1)} \otimes dt_{-(2s-1)}, \]

to be a differential \((C_{2m}^\infty_-^p, d)\) becomes a complex. We denote its cohomology group at \( C_{2m}^\infty_-^p \) by \( H_{2m}^\infty_-^p \). Notice that
\[ H_{2m}^\infty_-^p \simeq \frac{A_{2m}}{\sum_{s=1}^{\infty} b_{-(2s-1)} A_{2m}}, \]

which is described in Corollary 2. Other cohomology groups are described by

**Proposition 3**
\[ H_{2m}^\infty_-^p \simeq W_{2(m+p)} \quad \text{for } p \geq 0. \]
As a consequence of this proposition we have
\[
\text{ch } H^{\infty-2}_{2m} = \frac{p^{m+1} (1 - p^{4(m+p+1)})}{(p^2 : p^2)_{\infty}}.
\]

Proof of Proposition 3
Consider two chain complexes of $D$-modules
\[
K_p = D \otimes \Omega^{\infty-p}, \quad dK = \sum_{n=1}^\infty b_{-(2n-1)} \otimes dt_{-(2n-1)},
\]
\[
L_p = D \otimes W_{2(m+p)}, \quad dL = x_0.
\]
We set $K_p = L_p = 0$ for $p < 0$. Let
\[
M_{p,q} = K_p \otimes D L_q
\]
be the double complex obtained from $K$ and $L$. Here $p$ denotes the row index and $q$ denotes the column index. We denote by $M = (M_n)$ the total complex of $(M_{p,q})$. The bordered chain complexes of $(M_{p,q})$ in the vertical and horizontal directions are
\[
H_0(M^{\cdot,-}) \simeq C_{2m}^{-p}, \quad (13)
\]
\[
H_0(M^{-\cdot}) \simeq W_{2(m+q)}, \quad (14)
\]
respectively, where the symbol $H_0$ denotes the 0-th homology group of a complex. Notice that all maps in the complex $\text{(14)}$ are zero. Therefore $q$-th homology group of $\text{(14)}$ are given by
\[
H_q(W_{2(m+q)}) = W_{2(m+q)}.
\]
By the definition $p$-th homology group of the complex $\text{(14)}$ is
\[
H_p(C_{2m}^{-p}) = H^{\infty-p}_{2m}.
\]
Since two complexes
\[
\cdots \rightarrow M_{1,q} \rightarrow M_{0,q} \rightarrow W_{2(m+q)} \rightarrow 0,
\]
\[
\cdots \rightarrow M_{p,1} \rightarrow M_{p,0} \rightarrow C_{2m}^{-p} \rightarrow 0,
\]
are exact for $p, q \geq 0$, we have
\[
H_n(W_{2(m+q)}) \simeq H_n(M) \simeq H_n(C_{2m}^{-p}).
\]

6 The structure of $A_{2m+1}$
The strategy is similar to the case of $A_{2m}$, that is, we make a base change of fermions and describe $A_{2m+1}$ using the Fock space of new fermions. The structure of $A_{2m+1}$ is simpler than that of $A_{2m}$.
The component forms of $x_0^-$ and $(x_0^-)^{(2)}$ are given by

$$x_0^- = \sum_{n=0}^{\infty} T_{-2n} \psi_{2n+1},$$

$$(x_0^-)^{(2)} = -\frac{i}{2} \sum_{n=1}^{\infty} \left( -2\psi_{-2n+1} + \sum_{l=1}^{\infty} Q_{n,l}(T)\psi_{2(-n+l)+1} \right) \psi_{2n+1},$$

where

$$Q_{n,l}(T) = \sum_{n_1+n_2=l,0\leq n_1,0\leq n_2\leq n} (-2T_{-2n_1}T_{-2n_2} + T_{-(2n_1-1)}T_{-(2n_2+1)}).$$

We set

$$\tilde{\psi}_{2n+1} = \psi_{2n+1} \quad \text{for } n \geq 0,$$

$$\tilde{\psi}_{-(2n+1)} = -2\psi_{-(2n+1)} + \sum_{l=1}^{\infty} Q_{n,l}(T)\psi_{2(-n+l)+1} \quad \text{for } n \geq 0.$$  

Then we have

$$x_0^- = \sum_{n=0}^{\infty} T_{-2n} \tilde{\psi}_{2n+1},$$

$$(x_0^-)^{(2)} = -\frac{i}{2} \sum_{n=1}^{\infty} \tilde{\psi}_{-(2n-1)} \tilde{\psi}_{2n+1}.$$  

We define $\tilde{\psi}_{2n+1}$, $n \in \mathbb{Z}$ similarly to the case of $A_{2m}$. Again the vacuums are invariant by this change. Thus the Fock space $\tilde{H}_{2m+1}$ is defined over the same vacuum as fermions $\psi_{2n+1}, \psi_{2n+1}^*.$  

Let us define the subspace of $\tilde{H}_{2m+1}$ by

$$\tilde{H}_{2m+1,0} = \sum_{r_1 < \cdots < r_k < 0 < s_1 < \cdots < s_{k+m}} C_{2r_1+1} \cdots \tilde{\psi}_{2r_k+1} \psi_{2s_1+1}^* \cdots \psi_{2s_{k+m}+1}^* |1 \rangle,$$

where $r_1 < \cdots < r_k < 0 < s_1 < \cdots < s_{k+m}$. Notice that $\tilde{\psi}_1$ is absent in $\tilde{H}_{2m+1,0}$. Since $(x_0^-)^{(2)}$ does not contain $\tilde{\psi}_1$,

$$(x_0^-)^{(2)} \tilde{H}_{2m+1,0} \subset \tilde{H}_{2m-3,0}.$$  

We set

$$W_{2n+1,0} = \frac{\tilde{H}_{2n+1,0}}{(x_0^-)^{(2)} \tilde{H}_{2n+5,0}}.$$  

The degree induced from $V_{\sqrt{-1}}(\Lambda_1)$ is given by

$$\deg \tilde{\psi}_{2n+1} = 2n, \quad \deg \tilde{\psi}_{2n+1}^* = -2n, \quad \deg |2m+1 \rangle = -m^2 - m.$$  

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Theorem 2  (i) The space $A_{2m+1}$ becomes a free $\mathcal{D}$-module as

$$A_{2m+1} \simeq \mathcal{D} \otimes W_{2m+1,0}.$$ 

(ii) 

$$ch A_{2m+1} = \frac{p^{m(m+1)}(1 - p^{2m+2})}{(p : p)_{\infty}}.$$ 

Corollary 3 

$$\sum_{s=1}^{\infty} b_{-(2s-1)} A_{2m+1} \simeq W_{2m+1,0}.$$ 

Proof of Theorem 2

Since

$$x_0 = \tilde{\psi}_1 + T_2 \tilde{\psi}_3 + T_4 \tilde{\psi}_5 + \cdots,$$

we have the isomorphism of $\mathcal{D}$-modules

$$\mathcal{D} \otimes \tilde{H}_{2m+1,0} \simeq \frac{\mathcal{D} \otimes \tilde{H}_{2m+1}}{x_0 (\mathcal{D} \otimes \tilde{H}_{2m+1})}.$$ 

Thus the natural map

$$\mathcal{D} \otimes \tilde{H}_{2m+1,0} \rightarrow A_{2m+1}^{(0)}$$

is surjective and its kernel is given by

$$\frac{x_0 (\mathcal{D} \otimes \tilde{H}_{2m+1}) + (x_0^{(2)} (\mathcal{D} \otimes \tilde{H}_{2m+5}))}{x_0 (\mathcal{D} \otimes \tilde{H}_{2m+1})} \simeq (x_0^{(2)} (\mathcal{D} \otimes \tilde{H}_{2m+5,0}),$$

which proves (i).

Next let us prove (ii). Set

$$\omega_N = - \sum_{n=1}^{N} \tilde{\psi}_{-(2n-1)} \tilde{\psi}_{2n+1}^{*},$$

$$\eta_N = \sum_{n=1}^{N} \tilde{\psi}_{-(2n-1)}^{*} \tilde{\psi}_{2n+1}^{*},$$

$$\xi_N = - \sum_{n=1}^{N} \tilde{\psi}_{-(2n-1)} \tilde{\psi}_{-(2n-1)}^{*} + \sum_{n=1}^{N} \tilde{\psi}_{2n+1} \tilde{\psi}_{2n+1}^{*}.$$ 

Then they satisfy the relations of $\mathfrak{sl}_2$ as in the case of $A_{2m}$. It follows that $(x_0^{-})^{(2)}$ is injective on $\tilde{H}_{2m+1,0}$, $m \geq 1$. Using the arguments of [1] we have

$$ch W_{2m+1,0} = ch \tilde{H}_{2m+1,0} - ch \tilde{H}_{2m+5,0} = (1 - p^{2m+2}) ch \tilde{H}_{2m+1}.$$
Thus (ii) follows from
\[
\text{ch } \hat{H}_{2m+1} = \frac{p^{m(m+1)}}{(p^2 : p^2)_\infty}.
\]

We define de Rham type complex \((C_{2m+1}^\infty, d)\) and its cohomology \(H_{2m+1}^\infty\) for \(A_{2m+1}\) in a similar manner to the case of \(A_{2m}\). Then

**Proposition 4**

\[
H_{2m+1}^\infty \cong \begin{cases} W_{2m+1}^\infty, & \text{for } p = 0 \\ 0, & \text{for } p \geq 1. \end{cases}
\]

The proof of this proposition is similar to that of Proposition 3 and we leave it to the reader.

7 **Action of symplectic group on cohomologies**

Let \(Sp(2n) \subset GL(2n, \mathbb{C})\) be the symplectic group. For \(n < n'\) we have the embedding

\[
GL(2n, \mathbb{C}) \subset GL(2n', \mathbb{C}),
\]

\[
A \mapsto \begin{bmatrix} 1_{n'-n} & A \\ & 1_{n'-n} \end{bmatrix},
\]

where \(1_r\) denotes the \(r\) by \(r\) unit matrix. We define the group \(Sp(2\infty)\) as the inductive limit of \(Sp(2n)\) with respect to this embedding. In this section we shall show that \(W_{2m}\) and \(W_{2m+1,0}\) are irreducible representations of \(Sp(2\infty)\).

Let us consider \(W_{2m}\) first. Set

\[
\alpha_n = \tilde{\psi}_{-2(n-1)}, \quad \beta_n = \tilde{\psi}_{2n}, \quad \alpha_n^* = \tilde{\psi}_n, \quad \beta_n^* = \tilde{\psi}_n, \quad n \geq 1.
\]

Then

\[
\omega = - \sum_{n=1}^\infty \alpha_n^* \beta_n^*, \quad \eta = \sum_{n=1}^\infty \alpha_n \beta_n. \quad (15)
\]

Let

\[
V_N = \bigoplus_{i=1}^N \mathbb{C} \alpha_i \oplus \bigoplus_{i=1}^N \mathbb{C} \beta_i. \quad (16)
\]

We consider \(V_N\) as the vector representation of \(Sp(2N)\). Then for \(k \geq 1\)

\[
M_k^{(N)} := \frac{\wedge^k V_N}{\eta_N \wedge^{k-2} V_N}. \quad (17)
\]

is isomorphic to the \(k\)-th fundamental representation of \(Sp(2N)\), where

\[
\eta_N = \sum_{i=1}^N \alpha_i \wedge \beta_i.
\]
To each $m \geq 0$ we associate $M_{N-m}^{(N)}$, $N \geq m$. For $N < N'$ we define a map
\[
\wedge^{N-m}V_N \to \wedge^{N'-m}V_{N'},
\]
\[
v \mapsto v \wedge \alpha_{N+1} \wedge \cdots \wedge \alpha_{N'}.
\] (18)

It induces a map
\[
M_{N-m}^{(N)} \to M_{N'-m}^{(N')}.
\] (19)

**Lemma 3** The map (19) is injective.

**Proof.** It is sufficient to prove the lemma for $N' = N + 1$. Suppose that $v \in \wedge^{N-m}V_N$ satisfy
\[
v \wedge \alpha_{N+1} = \eta_{N+1} \wedge w, \quad w \in \wedge^{N-m-1}V_{N+1}.
\] (20)

Let us write
\[
w = w' + w'' \wedge \alpha_{N+1}, \quad w' \in \wedge^{N-m-1}V_{N+1}, \quad w'' \in \wedge^{N-m-2}V_{N+1},
\]
where $w'$ and $w''$ do not contain $\alpha_{N+1}$. Then
\[
\eta_{N+1} \wedge w = \eta_{N} \wedge w' + (-w' \wedge \beta_{N+1} + \eta_{N} \wedge w'') \wedge \alpha_{N+1}.
\]
By (20) we have
\[
\eta_{N} \wedge w' = 0.
\]

By the representation theory of $\mathfrak{sl}_2$ $\eta_N$ is injective on $\wedge^{N-m-1}V_N \oplus \beta_{N+1} \wedge^{N-m-2}V_{N+1}$. Thus
\[
w' = 0 \text{ and we have } v \wedge \alpha_{N+1} = \eta_{N} \wedge w'' \wedge \alpha_{N+1}.
\]

This shows $v = \eta_n \wedge w''$. [Q.E.D.]

We denote $M_{\infty-m}$ the inductive limit of $M_{N-m}^{(N)}$,
\[
M_{\infty-m} = \lim_{\to} M_{N-m}^{(N)}.
\]

For $N < N'$ the subgroup $Sp(2N)$ in $Sp(2N')$ fixes $\alpha_{N+1}, \ldots, \alpha_{N'}$, $\beta_{N+1}, \ldots, \beta_{N'}$. Therefore $Sp(2\infty)$ acts on $M_{\infty-m}$. It is straightforward to check that this representation is irreducible.

**Proposition 5**
\[
W_{2m} \simeq M_{\infty-m}.
\]

**Proof.** By the representation theory of $\mathfrak{sl}_2$ we have the isomorphisms,
\[
\omega^m : \tilde{H}_{2m} \simeq \tilde{H}_{-2m},
\]
\[
\omega^{m+1} \tilde{H}_{2m+4} \simeq \eta \tilde{H}_{-2m-4}.
\]

Thus
\[
W_{2m} = \frac{\tilde{H}_{2m}}{\omega \tilde{H}_{2m+4}} \simeq \frac{\tilde{H}_{-2m}}{\eta \tilde{H}_{-2m-4}}.
\]
We shall show that $W_{2m}$ is an inductive limit of the subspaces isomorphic to $M_{N-m}^{(N)}$.

We set, for $N \geq m$,

$$
\tilde{H}_{-2m}(N) = \sum_{k=0}^{N-m} \sum C_{\beta_1 \cdots \beta_k \alpha_{j_1, \cdots, j_{N-m-k}}} \mid 2N >,
$$

where the second summation is taken for all

$$
1 \leq i_1 < \cdots < i_k \leq N, \quad 1 \leq j_1 < \cdots < j_{N-m-k} \leq N.
$$

For $N < N'$ we have the inclusion

$$
\tilde{H}_{-2m}(N) \subset \tilde{H}_{-2m}(N'),
$$

$$
x \mid 2N > = x_{\alpha_{N+1} \cdots \alpha_{N'}} \mid 2N >.
$$

Thus $\{\tilde{H}_{-2m}(N)\}$ defines an increasing filtration and satisfy

$$
\tilde{H}_{-2m} = \cup_{N=m}^{\infty} \tilde{H}_{-2m}(N).
$$

There is an isomorphism,

$$
\tilde{H}_{-2m}(N) \simeq \wedge^{N-m} V_N,
$$

$$
\beta_{i_1} \cdots \beta_{i_k} \alpha_{j_1} \cdots \alpha_{j_{N-m-k}} \mid 2N > \mapsto \beta_{i_1} \wedge \cdots \wedge \alpha_{j_{N-m-k}}.
$$

It induces the isomorphism

$$
\frac{\tilde{H}_{-2m}(N)}{\eta N \tilde{H}_{-2m-4}(N)} \simeq \frac{\wedge^{N-m} V_N}{\eta N \wedge^{N-m-2} V_N} = M_{N-m}^{(N)}.
$$

By the isomorphism (22) the inclusion (21) is transformed to the map (18). Thus we have

$$
\lim_{\rightarrow} \frac{\tilde{H}_{-2m}}{\eta \tilde{H}_{-2m-4}} = \frac{\tilde{H}_{-2m}(N)}{\eta N \tilde{H}_{-2m-4}(N)} \simeq M_{\infty-m}.
$$

As to $W_{2m+1,0}$ we have

**Proposition 6**

$$
W_{2m+1,0} \simeq M_{\infty-m-1}.
$$

**Proof.** In this case we set

$$
\alpha_n = \tilde{\psi}_{-(2n-1)}, \quad \beta_n = \tilde{\psi}_{2n+1}, \quad \alpha_n^* = \tilde{\psi}_{-(2n-1)}, \quad \beta_n^* = \tilde{\psi}_{2n+1}, \quad n \geq 1.
$$

Then $\omega$ and $\eta$ are given by (13). Define $V_N$ and $M_{k}^{(N)}$ by (16) and (17) respectively. As in the case of $H_{2m}$ we have the isomorphisms

$$
\omega^{m+1} : \tilde{H}_{2m+1,0} \simeq \tilde{H}_{-2m-1,0},
$$

$$
\omega^{m+2} \tilde{H}_{2m+5,0} \simeq \eta \tilde{H}_{-2m-5,0}.
$$
and consequently
\[ W_{2m+1,0} \simeq \frac{\tilde{H}_{-2m-1,0}}{\eta \tilde{H}_{-2m-5,0}}. \]

By the definition
\[ \tilde{H}_{-2m-1,0} = \sum_{k=m+1}^{\infty} \sum_{i_1, \ldots, i_{k-m-1}, j_1, \ldots, j_k} \sum_{C} \eta \beta_{i_1} \cdots \beta_{i_{k-m-1}} \alpha_{j_1}^* \cdots \alpha_{j_k}^* |1>, \]
where the second summation is taken for all
\[ 1 \leq i_1 < \cdots < i_{k-m-1}, \ 1 \leq j_1 < \cdots < j_k. \]

We identify \( \tilde{H}_{-2m-1,0} \) as a subspace of \( \tilde{H}_{-2m-3} \) by
\[ x|1> = x\tilde{\psi}_1^*|1> \mapsto x|1>, \quad (23) \]
that is, by removing \( \tilde{\psi}_1^* \). Since \( x \) does not contain \( \tilde{\psi}_1 \), this map is well-defined. Moreover the map (23) preserves the degree because \( \deg \tilde{\psi}_1^* = 0 \). We set
\[ \tilde{H}_{-2m-1,0}(N) = \sum_{k=0}^{N-m-1} \sum_{i_1, \ldots, i_k, j_1, \ldots, j_{N-m-1-k}} \sum_{C} \eta \beta_{i_1} \cdots \beta_{i_k} \alpha_{j_1} \cdots \alpha_{j_{N-m-1-k}} |1>, \]
where the second summation is taken for all
\[ 1 \leq i_1 < \cdots < i_k \leq N, \ 1 \leq j_1 < \cdots < j_{N-m-1-k} \leq N. \]

Then
\[ \tilde{H}_{-2m-1,0}(N) \simeq \wedge^{N-m-1} V_N. \quad (24) \]

For \( N < N' \) we have the natural inclusion map
\[ \tilde{H}_{-2m-1,0}(N) \to \tilde{H}_{-2m-1,0}(N'), \]
\[ x|1> - (2N+1) \mapsto x\alpha_{N+1} \cdots \alpha_{N'} |1> - (2N'+1) >. \]

This map is transplanted to [18] by the isomorphism (24). Thus we have
\[ \frac{\tilde{H}_{-2m-1,0}}{\eta \tilde{H}_{-2m-5,0}} \simeq \lim_{\eta \to \infty} \frac{\tilde{H}_{-2m-1,0}(N)}{\eta \tilde{H}_{-2m-5,0}(N)} \simeq M_{\infty-m-1}. \]

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