Semiclassical trajectory-coherent approximation in quantum mechanics: II. High order corrections to the Dirac operators in external electromagnetic field

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Abstract

High approximations of semiclassical trajectory-coherent states (TCS) and of semiclassical Green function (in the class of semiclassically concentrated states) for the Dirac operator with anomalous Pauli interaction are obtained. For Schrödinger and Dirac operators trajectory-coherent representations are constructed up to any precision with respect to $\hbar$, $\hbar \to 0$.

Introduction

This paper is the second part of our work \[1\] and deals with higher approximations of semiclassical trajectory-coherent states (TCS) for the Dirac equation with anomalous Pauli interaction in an external electromagnetic field. The approach given in \[3\] is based on the concept that classical equations of motion can be considered as the limiting (as $\hbar \to 0$) equations of motion for averaged values of corresponding quantum mechanical magnitudes. In physical literature this passage to the limit is justified by describing the dynamical states of a quantum system as wave packets localized near the position of a classical particle. For more detailed references, see paper \[1\] and reviews \[5, 6\]. Such states can be obtained up to any power of $\hbar^{1/2}$, $\hbar \to 0$, by the complex WKB-Maslov method called the complex germ theory \[4\].

In \[1, 7–9\], complete orthonormalized sets of asymptotic (as $\hbar \to 0$) semiclassical solutions to wave equations (Schrödinger, Klein–Gordon, pseudodifferential equation of the Schrödinger type for a relativistic particle) were constructed for a scalar quantum particle in an arbitrary external field. These sets satisfy the coherence condition: the quantum mechanical values (averaged with respect to the functions from these sets) for the operators of coordinates and momenta are (as $\hbar \to 0$) solutions of classical equations of motion (Newton and Lorentz respectively).

Such solutions, called semiclassical trajectory-coherent states (TCS), generalize the well-known coherent \[10–14\] and correlated coherent states of nonrelativistic quadratic systems \[13\] to the case of an arbitrary external field and scalar relativistic equations.

For a relativistic quantum particle in an arbitrary external field, which is described by the Dirac equation, the semiclassical TCS were constructed in \[16–18\]. They are characterized by the fact that, beside the coherence condition with respect to orbital variables, they must also satisfy the following condition of spin coherence: in the limit as $\hbar \to 0$, the quantum mechanical averages of spin operator, i.e., of the Bargman polarization vector $\hat{S}^\mu$ \[19\], are solutions to the classical relativistic equations of spin motion, which are the Bargmann–Michel–Telegdi equations \[20\].

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Note that the construction of semiclassical TC-states for quantum systems is based on the semiclassical trajectory coherent representation of the corresponding Hamiltonians. For scalar quantum mechanical equations, this representation was implicitly used in essence in [3]. For the Dirac operator, it was introduced first in [14, 17]. From the viewpoint of semiclassical asymptotics, the constructed TC-states give the principal term of asymptotic expansion in the powers of \( h^{1/2} \), \( h \to 0 \). They satisfy the corresponding evolutionary quantum mechanical equations up to the functions, whose \( L_2 \)-norm is of the order \( O(h^{3/2}) \) as \( h \to 0 \) (mod \( O(h^{3/2}) \)). In this paper we construct the higher approximations of the trajectory-coherent states of the Dirac operator in an arbitrary magnetic field, i.e., the asymptotic solutions of these equations up to \( O(h^{(N+1)/2}) \), where \( N \) is arbitrary integer independent of \( h \); \( N = 3, 4, \ldots \).

In the general situation the scheme for construction of corrections to the principal term of a semiclassical asymptotic solution is well known [3]. Nevertheless, its realization for particular quantum equations with arbitrary electromagnetic potentials requires a large amount of nontrivial calculations. In particular, we succeeded in constructing a unitary (up to any order in \( h^{1/2} \)) operator in the form of an asymptotic series in powers of \( h^{1/2} \), \( h \to 0 \), which defines the passage to the semiclassical trajectory-coherent representation.

In the semiclassical description of the behavior of a quantum particle with its spin properties taken into account, this allows to construct the two-component theory, i.e., to use the space of positive-frequency solutions to the Dirac equations, to exclude explicitly the Hamiltonian which is a relativistic generalization of the Pauli operator [3]. The scalar part of this Hamiltonian describes quantum fluctuations of the wave packet near the particle position on a classical trajectory, and its “vector” part describes the interaction between the particle spin and the external field and the quantum fluctuations with respect to orbital variables.

Note that in order to study certain specific physical problems, e.g., to consider with a given precision as \( h \to 0 \) [22, 23] the quantum effects which appear when a charge radiates spontaneously in an arbitrary external field, we must take into account the higher (as \( h \to 0 \)) corrections to the principal term of TCS. In this case, to obtain the first (up to \( O(h^2) \)) quantum corrections to the power of radiation (which are uniform with respect to relativism, i.e., they can be used in any region of the particle energy), it is necessary to take the semiclassical TC-states satisfying the Dirac equation up to \( O(h^{5/2}) \).

The influence of corrections to the principal term of asymptotics is essential when nonadiabatic phases in quantum mechanics are considered [24], as well as when equations of motion of quantum averages are derived [1, 25–27].

This paper is organized as follows. Section 1 contains some relevant facts about the semiclassical trajectory-coherent representation (TC-representation) for the Schrödinger equation. In Section 2 we give the definition and investigate the simple properties of the semiclassically concentrated states of the Dirac equation. Higher approximations of semiclassical trajectory-coherent states \( \Psi^{(N)}(\vec{x}, t, h) \) and of semiclassical Green function \( G^{(N)}(\vec{x}, \vec{p}, t) \) (in the class of semiclassically concentrated states) for the Dirac equation are constructed in Sections 3 and 4. In Section 5 we show that in the class of positive-frequency (negative-frequency) \( \sqrt{h} \), \( h \to 0 \), semiclassically concentrated states it can be gone to the one particle two-component theory whose Hamiltonian \( \hat{H}^{(N)}_D(t) \) is a self-adjoint operator for any order of \( h \to 0 \). Portion of necessary material is taken into appendices. Our paper is essentially based on the results and notation of Part 1 [1], some of them are not defined in this text, but only referred to as, e.g., (I.2.3), which means formula (2.3) in the first part [1].

1 Semiclassical trajectory-coherent representation for the Schrödinger equation

As is known, physical results in quantum theory are independent of the choice of the representation for its principal dynamical variables. A good choice of one or another representation often allows to simplify the problem or to solve it completely. To solve the problem of passing in the limit from quantum to classical mechanics, a new semiclassical trajectory-coherent (TC) representation was constructed in [16] up to mod \( O(h^{1/2}) \), \( h \to 0 \). In this representation, in the limit as \( h \to 0 \), the quantum averages for an arbitrary observable \( \hat{A}_0(t) \) with classical analog \( \hat{A}_0(t) = \hat{A}(\vec{p}, \vec{x}, t) \) go over into the classical observable \( A(\vec{p}, \vec{x}, t) \) translated along the trajectories of the corresponding Hamilton system (I.1.1). In concrete problems, from the viewpoint of applications, it is important to know how the quantum-mechanical averages tend to their classical limit, i.e., the corresponding asymptotic expansions must be obtained up to any precision with respect to \( h \), \( h \to 0 \). To solve this problem, we shall construct a semiclassical trajectory-coherent representation for the Schrödinger equation (I.0.1) up to \( O(h^{(N+1)/2}) \), \( N = 1, 2, \ldots \). In Section 4, in
order to construct a semiclassical TC-representation of the Dirac operator in an arbitrary external field, we shall use the construction of a unitary operator (with a given precision in $\hbar \to 0$), which defines the passage to such representations.

Let us define the Hilbert space of functions depending on $\vec{x}$ and $\hbar$ with the following scalar product of two functions $\varphi_1(\vec{x}, t, \hbar)$ and $\varphi_2(\vec{x}, t, \hbar)$:

$$
(\varphi_1 | \varphi_2)_{L^2_h} = \int_{\mathbb{R}^3} \varphi_1^*(\vec{x}, t, \hbar) \varphi_2(\vec{x}, t, \hbar) \rho_{\hbar}^0(\vec{x}, t) \, d^3x,
$$

where the normalized measure density $\rho_{\hbar}^0(\vec{x}, t)$ is equal to

$$
\rho_{\hbar}^0(\vec{x}, t) = N_{\hbar}^2 |J(t, z_0)|^{-1} \exp \left( \frac{2}{\hbar} \text{Im} S(\vec{x}, t) \right).
$$

Here the complex action $S(\vec{x}, t)$ and the normalizing factor $N_{\hbar}$ are defined in (I.11.11). It is natural to consider the space $L^1_h$ as the space of states of quantum system (I.0.2) which, as $\hbar \to 0$, are localized in a neighborhood of the position $z(t, z_0)$ of a classical particle on the phase trajectory, since $\rho_{\hbar}^0(\vec{x}, t) \to \delta(\vec{x} - \vec{x}(t))$ as $\hbar \to 0$ and, in the $p$-representation, we have $\hat{F}_{\vec{x} \to p} \rho_{\hbar}^0(\vec{x}, t) \to \delta(\vec{p} - \vec{p}(t))$, where $\hat{F}_{\vec{x} \to p}$ is the $\hbar^{-1}$-Fourier transform [28]. The measure depends on a small parameter which implies that one can approximate (up to a given precision $\hbar \to 0$) the smooth functions $\varphi(\vec{x}, t)$ by partial sums of the Taylor series in powers of $\Delta \vec{x}$:

$$
\varphi(\vec{x}, t) = \sum_{k=0}^{N} \frac{1}{k!} d^k\varphi(t) + R_N(\vec{x}, t),
$$

where $d^k\varphi(t)$ is the $k$-th term of the expansion of $\varphi(\vec{x}(t, z_0) + \Delta \vec{x}, t)$ into a Taylor series in powers of $\Delta \vec{x}$, and for $R_N(\vec{x}, t)$ we have the estimate: $\|R_N(\vec{x}, t)\|_{L^2_h} = O(\hbar^{(N+1)/2})$. This estimate follows from the relation $\int \rho_{\hbar}^0(\vec{x}, t) \langle \vec{a}, \Delta \vec{x} \rangle^{N+1} d\vec{x} = O(\hbar^{(N+1)/2})$ which holds for any constant vector $\vec{a}$ (comp. with (1.4.8)). On the other hand, the space $L^1_h$ contains, as its elements, the functions $\varphi(\vec{x}, t, \hbar)$ which depend on $\hbar \to 0$ in a singular way and are of the order $O(1)$ as $\hbar \to 0$ in the norm of $L^1_h$. For example,

$$
\varphi(\vec{x}, t, \hbar) = \left( \frac{\langle \vec{a}(t), \Delta \vec{x} \rangle}{\sqrt{\hbar}} \right)^k.
$$

Denote by $\mathcal{P}_h$ the set of polynomials in powers of $\hbar^{-1/2} \Delta \vec{x}$ with coefficients depending on $t$. The set $\mathcal{P}_h$ is everywhere dense in $L^1_h$ and is natural domain of definition for differential operators acting in $L^1_h$.

By $\hat{O}(\hbar^N)$ we shall denote the operator $\hat{F}(\hbar) : L^1_h \to L^1_h$ for which on the set $\mathcal{P}_h$ the estimate holds: $\|\hat{F}(\hbar)\varphi\|_{L^2_h} = O(\hbar^N)$ as $\hbar \to 0$ uniformly in $t \in [0, T]$. Note that in this sense the operators

$$
\sqrt{\hbar} \nabla, \quad \langle \Delta \vec{x}, \nabla \rangle, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \langle \dot{\vec{x}}(t, z_0), \nabla \rangle
$$

are the order $O(1)$ as $\hbar \to 0$.

Let us define the operator $\hat{K}_S^{(N)}(t, \hbar) : L^1_h \to L^2_2(\mathbb{R}^3)$, which defines (up to $\text{mod}\hat{O}(\hbar^{(N+1)/2})$) the passage to the semiclassical TC-representation by the formula:

$$
\hat{K}_S^{(N)}(t, \hbar) \varphi = \sum_{n=0}^{N} \left[ - \frac{i}{\hbar} \hat{K}_1(t) \right]^n \hat{K}_S^{(0)}(t, \hbar) \varphi,
$$

$$
\hat{K}_S^{(0)}(t, \hbar) = \frac{N_{\hbar}}{\sqrt{J(t, z_0)}} \exp \left[ \frac{i}{\hbar} S(\vec{x}, t) \right],
$$

$$
\hat{K}_1(t) \varphi(t) = \sum_{|\nu| = 0}^{\infty} \int_0^t \int d\tau d\nu \langle \tau, \nu \rangle |\hat{H}_1(t)|^2 \langle \varphi(\tau) \rangle,
$$

where the functions $|\nu, t\rangle$, $S(\vec{x}, t)$, $J(\vec{x}, t)$ and the constant $N_{\hbar}$ are defined in (I.11.11)–(I.11.15), and the operator $\hat{H}_1$ is defined in (I.4.7).

The operator $\hat{K}_S^{(N)}(t, \hbar)$ maps the space $L^1_h$ into $L^2_2$ unitarily up to $O(\hbar^{(N+1)/2})$ which means that

$$
(\hat{K}_S^{(N)}(t, \hbar) \varphi_1 | \hat{K}_S^{(N)}(t, \hbar) \varphi_2)_{L^2_2} = (\varphi_1 | \varphi_2)_{L^1_h} + O(\hbar^{(N+1)/2}).
$$

(1.5)
The Schrödinger equation (1.0.1) in the semiclassical TC-representation given the operator \( \hat{K}_S^{(N)}(t, h) \) (1.4) has the form

\[
[\hat{K}_S^{(N)}(t, \hbar)]^{-1} \{ - i \hbar \partial_t + \hat{H} \} \hat{K}_S^{(N)}(t, \hbar) \varphi = \hat{\pi}_0 \varphi + O(\hbar^{(N+3)/2}),
\]

(1.6)

\[
\hat{\pi}_0 = (-i\hbar) \left\{ \frac{d}{dt} + \langle \Delta \xi, [H_{zp}(t) + Q(t)H_{pp}(t)]\nabla \rangle - \frac{\hbar}{2i} \langle \nabla, H_{pp}(t)\nabla \rangle \right\},
\]

(1.7)

and \( Q(t) \) is defined in (1.1.9). Thus with precision up to functions of the order \( O(\hbar^{(N+3)/2}) \), \( N = 0, 1, 2, \ldots \), in the norm of the space \( L_k \), the Schrödinger equation in the semiclassical TC-representation is equivalent to the equation

\[
\hat{\pi}_0 \varphi = 0, \quad \varphi \in L_k.
\]

Equation (1.8) can be easily integrated if we note that the operator \( \hat{\pi}_0 \) admits the complete set of dynamical symmetries. It is easy to verify that the operators

\[
\hat{\Lambda}_j = \frac{1}{\sqrt{2\hbar \Im b_j}} \langle (\tilde{\mathbf{Z}}_j(t), \hat{p}) - \langle (\tilde{\mathbf{W}}_j^* - Q(t)\tilde{\mathbf{Z}}_j(t), \Delta \xi) \rangle, \quad j = 1, n,
\]

(1.9)

commute with the operator \( \hat{\pi}_0 \) and satisfy the Bose commutation relations [3]:

\[
[\hat{\Lambda}_k, \hat{\Lambda}_j^+] = \delta_{kj}, \quad [\hat{\Lambda}_k, \hat{\Lambda}_j] = [\hat{\Lambda}_k^+, \hat{\Lambda}_j^+] = 0.
\]

Hence the functions \( H_\nu = |H_\nu\rangle = \prod_{j=1}^{\infty} (\nu_j!)^{-1/2}(\hat{\Lambda}_j^+)\nu_j |1\rangle \) form in \( L_k \) the complete orthonormalized set of solutions of equation (1.8). Applying the operator \( \hat{K}_S^{(N)}(t, \hbar) \) to the functions \( |H_\nu\rangle \), we obtain the higher approximations for semiclassical TCS (I.4.14) of the Schrödinger equation.

If follows from (1.6), (1.9) that the operators

\[
[\hat{\Lambda}_j^{(N)}], \quad \langle [\hat{\Lambda}_j^{(N)}]^{-1} [\hat{\Lambda}_j^{(N)}]^{-1} \rangle = \hat{O}(\hbar^{(N+3)/2})
\]

and satisfy the following commutation relations:

\[
\hat{\Lambda}_j^{(N)} [\hat{\Lambda}_k^{(N)}] = \delta_{jk} + \hat{\Lambda}_k^{(N)} [\hat{\Lambda}_j^{(N)}] = \hat{\pi}_0 [\hat{\Lambda}_j^{(N)}] = \hat{\pi}_0 [\hat{\Lambda}_k^{(N)}].
\]

For \( N = 2 \) we calculate the operator \( \hat{K}_S^{(2)}(t, \hbar) \) explicitly. By (1.4) and the nonrelativistic Hamiltonian function

\[
\hat{H} = \frac{1}{2m} \hat{\mathbf{p}}^2 + e\Phi, \quad \hat{\mathbf{p}} = \hat{\mathbf{p}} - e\frac{\mathbf{A}}{c},
\]

we have

\[
\hat{K}_S^{(2)}(t, \hbar) = \hat{K}_S^{(0)}(t, \hbar) \left[ 1 - i\sqrt{\hbar} \pi_1 - i\hbar \pi_2 - \hbar^2 \pi_2 \right], \quad (1.10)
\]

where

\[
\frac{1}{\hbar^{j/2}} \hat{\pi}_j \varphi(t) = \frac{1}{(j + 2)!} \sum_{|\nu| = 0}^{\infty} |H_\nu\rangle \int_0^t d\tau \langle H_\nu | \frac{1}{R} \hat{D}^{j+2} \hat{H}(\tau) | \varphi(\tau) \rangle; \quad j = 1, 2,
\]

\[
\frac{1}{3!} \hat{D}^3 \hat{H}(t) = - \frac{e}{4mc} \langle (\hat{\mathbf{p}}_1, d^2 \hat{\mathbf{A}}) + \langle d^2 \mathbf{A}, \hat{\mathbf{p}}_1 \rangle \rangle + \frac{e}{3!} (d^3 \Phi - \langle \vec{\mathbf{p}}, \mathbf{A} \rangle),
\]

\[
\frac{1}{4!} \hat{D}^4 \hat{H}(t) = \frac{e}{4!} (d^4 \Phi - \langle \vec{\mathbf{p}}, d^4 \hat{\mathbf{A}} \rangle) + \frac{e^2}{8mc^2} (d^2 \hat{\mathbf{A}}, d^2 \hat{\mathbf{A}}) - \frac{e}{2mc} \frac{1}{3!} \langle (\hat{\mathbf{p}}_1, d^2 \hat{\mathbf{A}}) + \langle d^2 \hat{\mathbf{A}}, \hat{\mathbf{p}}_1 \rangle \rangle,
\]

(1.11)
\[ \hat{\mathcal{P}}_1^t = -i\hbar \nabla + Q(t)\Delta \vec{x} - \frac{e}{c} d\vec{A}(t), \quad \vec{\beta} = \frac{1}{c} \vec{v}(t, z_0), \]

and \( d^k A(t) \) denotes the function \( d^k A(t) = (\langle \Delta \vec{x}, \partial / \partial \vec{g} \rangle)^k A(\vec{g}, t) \big|_{\vec{g} = \vec{x}(t, z_0)} \). Note that since the operators \( \hat{\mathcal{P}}_1^t \) and \( \Delta \vec{x} \) are self-adjoint in \( L_2^H \) formulas (1.11) define the self-adjoint in \( L_2^H \) operators \( \hat{D}^j H(t), j = 3, 4. \)

In particular, this implies that the operator \( \hat{K}_S^{(N)}(t, h) \) is unitary for \( N = 2 \) (up to \( \mathcal{O}(h^{3/2}) \)):

\[
\langle (\hat{K}_S^{(2)}(t, h) H_{\nu})_L^0 = \langle H_{\nu} | (\hat{K}_S^{(2)}(t, h))^+ \hat{K}_S^{(2)}(t, h) | H_{\nu} \rangle_L^0 = \delta_{\nu, 0} + \mathcal{O}(h^{3/2}) \]

(1.12).

By using (1.10) for any arbitrary operator \( \hat{A}_h(t) = A(\vec{p}, \vec{x}, t) \) with Weyl symbol \( A(\vec{p}, \vec{x}, t) \) possessing a classical analog, we can find its explicit form (up to \( \mathcal{O}(h^{3/2}) \)) in the TC-representation

\[
\hat{A}_h = (\hat{K}_S^{(2)}(t, h))^{-1} \hat{A}_h(t) \hat{K}_S^{(2)}(t, h) + \mathcal{O}(h^{3/2}) = \hat{A}(t) + \hat{D}^2 A(t) - i \sqrt{\hbar} \mathcal{O}(\hat{D}^1 A(t) \hat{\pi}^1 - \hat{\pi}^1 \hat{D}^1 A(t)) = \mathcal{O}(h^{3/2}),
\]

(1.13)

where by \( \hat{D}^j A(t) \), similarly (1.4.8), we denote the operator of the form

\[
\Delta \vec{\beta} = -i\hbar \nabla + Q(t) \Delta \vec{x},
\]

(1.14)

and the function \( A(t) \) is the classical observable \( A(\vec{p}, \vec{x}, t) \), corresponding to the operator \( \hat{A}_h(t) \), calculated in the point \( z(t, z_0) \) on a phase trajectory, \( A(t) = A(\vec{p}(t, z_0), \vec{x}(t, z_0), t) \).

\section{Theorem about semiclassically concentrated states of the Dirac equation}

Let us consider a relativistic particle described by the Dirac equation with anomalous Pauli interaction

\[
\hat{L}_D \psi = 0, \quad \hat{L}_D = -i\hbar \partial_t + \mathcal{H}_D,
\]

(2.1)

where Hamiltonian \( \mathcal{H}_D \) has the form [23]:

\[
\mathcal{H}_D = \hat{\mathcal{H}}_0 + (\hbar \hat{\mathcal{H}}_1, \quad \hat{\mathcal{H}}_0 = c \langle \vec{\alpha}, \hat{\mathcal{P}} \rangle + \rho_3 m_0 c^2 + e \Phi(\vec{x}, t),
\]

\[
\mathcal{H}_1 = \frac{i e_0 (g - 2)}{4 m_0 c} \left[ \rho_3 (\vec{\Sigma}, \hat{\mathcal{H}}(\vec{x}, t)) + \rho_2 (\vec{\Sigma}, \vec{E}(\vec{x}, t)) \right],
\]

\[
\vec{\alpha} = \rho_1 \vec{\Sigma}, \quad \vec{\Sigma} = \left( \begin{array}{cc} \hat{\sigma} & 0 \\
0 & \hat{\sigma} \end{array} \right), \quad e_0 = -e,
\]

\[
\rho_1 = \left( \begin{array}{cc} 0 & I \\
I & 0 \end{array} \right), \quad \rho_2 = \left( \begin{array}{cc} 0 & iI \\
iI & 0 \end{array} \right), \quad \rho_3 = \left( \begin{array}{cc} I & 0 \\
0 & -I \end{array} \right).
\]

(2.2)

Here \( \hat{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices, 0 and I are the zero and unit 2 \( \times \) 2 matrices, \( \vec{\Sigma}, \vec{E}(\vec{x}, t) \) are magnetic and electric fields, \( \vec{\mathcal{H}}(\vec{x}, t) \) is a magnetic field, \( \vec{E}(\vec{x}, t) = -\nabla \Phi(\vec{x}, t) - c^{-1} \partial_t A(\vec{x}, t) \) is the electric field, \( g \) is the gyromagnetic ratio.

Precisely as in the nonrelativistic case, we assume that the electromagnetic potentials \( \Phi(\vec{x}, t) \) and \( \mathcal{H}(\vec{x}, t) \) are smooth functions of \( \vec{x} \in \mathbb{R}^3 \), \( t \in \mathbb{R}^1 \) and as \( |\vec{x}| \to \infty \) increase uniformly in \( t \in \mathbb{R}^1 \) together with their derivatives not faster than any power of \( |\vec{x}| \).

By analogy with the Schrödinger equation, we define the semiclassical concentrated for the Pauli equation:

\textbf{Definition 1} The state \( \psi \) will be called semiclassically concentrated on the phase trajectory \( z(t) = (\vec{p}(t), \vec{x}(t)) \) of the class \( \mathbb{C}(N) = \mathbb{C}(N, z(t), h) \) if

(1)

\[
\hat{L}_D \psi = 0;
\]
for generality, one can set
\[ |\Xi| = \frac{1}{\hbar} |\hat{\Delta}_{\alpha,\beta}|. \]

Proof.

1. We consider the spectral properties of the principal symbol of the Hamiltonian \( \mathcal{H}_D \), i.e., of the matrix \( \mathcal{H}_0(p, \vec{x}, t) \) (2.2), \( p \in \mathbb{R}^3_p, \vec{x} \in \mathbb{R}^3_x \). The equation
\[
\det |(\mathcal{H}_0(p, \vec{x}, t) - \lambda I_{4 \times 4})| = 0
\]
is equal to [28] (see Property 2, Appendix A) two eigenvalues \( \lambda \) of multiplicity 2 for all \( p \in \mathbb{R}^3_p, \vec{x} \in \mathbb{R}^3_x, t \in \mathbb{R}^1 \):
\[
\lambda^{(\pm)}(p, \vec{x}, t) = e^{\Phi(\vec{x}, t)} \pm \varepsilon(p, \vec{x}, t)
\]
\[
\varepsilon(p, \vec{x}, t) = \sqrt{c^2 \vec{\beta}^2 + m_0^2 c^4}, \quad \vec{\beta} = \vec{p} - \varepsilon \hat{\mathcal{A}}(\vec{x}, t)
\]
(2.7)

We use the eigenvectors \( f_j^+(p, \vec{x}, t), j = 1, 2 \), corresponding to \( \lambda^{(\pm)}(p, \vec{x}, t) \), into 4 \( \times \) 2 Pauli matrices \( \Pi_{\pm}(p, \vec{x}, t) \):
\[
\mathcal{H}_0(p, \vec{x}, t) \Pi_{\pm}(p, \vec{x}, t) = \lambda^{(\pm)}(p, \vec{x}, t) \Pi_{\pm}(p, \vec{x}, t),
\]
\[
\Pi_{+}(p, \vec{x}, t) = (2\varepsilon(\varepsilon + m_0 c^2))^{-1/2} \begin{pmatrix} \varepsilon + m_0 c^2 \\ c(\vec{\sigma}, \vec{p}) \end{pmatrix},
\]
\[
\Pi_{-}(p, \vec{x}, t) = (2\varepsilon(\varepsilon + m_0 c^2))^{-1/2} \begin{pmatrix} c(\vec{\sigma}, \vec{p}) \\ -m_0 c^2 - \varepsilon \end{pmatrix}.
\]
(2.8)

2. We expand the function \( \Psi \) with respect to eigenvectors (on the trajectory \( \vec{p}(t), \vec{x}(t) \)) of the principal symbol of the Hamiltonian:
\[
\Psi(\vec{x}, t, h) = N_h(\Pi_{+}(t)J^{(+)}(\vec{x}, t, h) + \Pi_{-}(t)J^{(-)}(\vec{x}, t, h)),
\]
(2.9)

where \( \Pi_{\pm}(t) = \Pi_{\pm}(\vec{p}(t), \vec{x}(t), t) \), \( N_h \) is const. The spinors \( J^{(\pm)} \) are represented as follows
\[
J^{(\pm)}(\vec{x}, t, h) = \begin{pmatrix} Q_{1}^{(\pm)}(\vec{x}, t, h) \exp(i\Xi_1^{(\pm)}(\vec{x}, t, h)) \\ Q_{2}^{(\pm)}(\vec{x}, t, h) \exp(i\Xi_2^{(\pm)}(\vec{x}, t, h)) \end{pmatrix},
\]
(2.10)

where \( \Xi_1^{(\pm)}, Q_{k}^{(\pm)}, k = 1, 2 \), are real functions. The condition (2.3), (2.4) implies that, without loss of generality, one can set
\[
Q_{k}^{(\pm)}(\vec{x}, t, h) = \left( \prod_{l=1}^{3}\alpha_l(h) \right)^{-1/2} \nu_{k}^{(\pm)}(\vec{\xi}, t, h), \quad \xi_j = \Delta x_j/a_j(h),
\]
(2.11)
where \( a_j(h) \) are certain nonnegative functions of \( h \), such that

\[
\lim_{h \to 0} a_j(h) = \lim_{h \to 0} \frac{h}{a_j(h)} = 0. \tag{2.12}
\]

Since there exist momenta \( ^{(j)} \Delta^{cl(0)}_0 \), the following limits are exist:

\[
\lim_{h \to 0} \left( \Xi^{(\pm)}_k(\xi, t, h) - \Xi^{(\pm)}_l(\xi, t, h) \right), \quad \lim_{h \to 0} \left( \Xi^{(\mp)}_k(\xi, t, h) - \Xi^{(\mp)}_l(\xi, t, h) \right),
\]

\( k, l = 1, 2 \). Hence,

\[
\Xi^{(\pm)}_k(\xi, t, h) = \Phi_0(\xi, t, h) + \Phi^{(\pm)}_k(\xi, t, h), \quad k = 1, 2, \tag{2.13}
\]

where \( \Phi^{(\pm)}_k(\xi, t, h) \) regularly depend on \( h \).

Similarly, we get for the wave function in the \( p \)-representation:

\[
\tilde{J}^{(\pm)}(\vec{p}, t, h) = \left( \prod_{i=1}^3 a_i(h) \right)^{-1/2} \left( \rho_{1}^{(\pm)}(\vec{q}, t, h) \exp(-i\Xi^{(\pm)}_{1}(\vec{q}, t, h)) \right), \tag{2.14}
\]

\[
\eta_j = \Delta p_j/b_j(h), \quad \lim_{h \to 0} b_j(h) = \lim_{h \to 0} \frac{h}{b_j(h)} = 0, \quad j = \overline{1, 3},
\]

and

\[
\Xi^{(\pm)}_k(\vec{q}, t, h) = \tilde{\Phi}_0(\vec{q}, t, h) + \tilde{\Phi}^{(\pm)}_k(\vec{q}, t, h), \quad k = 1, 2, \tag{2.15}
\]

where \( \tilde{\Phi}^{(\pm)}_k(\vec{q}, t, h) \) regularly depend on \( h \).

Precisely as in the nonrelativistic case, the conditions

\[
\| \Delta x_j \psi \| = \| \Delta \hat{x}_j \psi \| \sim a_j(h), \quad \| \Delta \hat{p}_j \psi \| = \| \Delta p_j \psi \| \sim b_j(h), \quad j = 1, 2, 3,
\]

implies

\[
0 \leq \lim_{h \to 0} \frac{h}{a_j(h)b_j(h)} < \infty, \quad j = 1, 2, 3, \tag{2.16}
\]

as well as

\[
\Phi_0(\vec{\xi}, t, h) = \Phi_0(t, h) + \left( \langle \vec{p}(t), \Delta \vec{x} \rangle + S(\vec{\xi}, t, h) \right)/h,
\]

\[
\tilde{\Phi}_0(\vec{\eta}, t, h) = \tilde{\Phi}_0(t, h) + \left( \langle \vec{x}(t), \Delta \hat{p} \rangle + \tilde{S}(\vec{\eta}, t, h) \right)/h,
\]

and

\[
S(\vec{\xi}, t, h) \sim \tilde{S}(\vec{\eta}, t, h) \sim C(h),
\]

\[
C(h) = \min \{ a_1(h)b_1(h), a_2(h)b_2(h), a_3(h)b_3(h) \}. \]

3. We find the limit (as \( h \to 0 \)) of the averaged value of the operator \( \tilde{L}_D \) with respect to the state (2.9), (2.10), (2.13), (2.17). By (2.1), (2.2), we get

\[
0 = \left( \lim_{h \to 0} \frac{\text{d}\Phi_0(t, h)}{\text{d}t} - \langle \vec{p}(t), \vec{x}(t) \rangle + \lambda^{(+)}(t) \right) \lim_{h \to 0} \| J^{(+)}(\vec{x}, t, h) \|^2 +
\]

\[
+ \left( \lim_{h \to 0} \frac{\text{d}\Phi_0(t, h)}{\text{d}t} - \langle \vec{p}(t), \vec{x}(t) \rangle + \lambda^{(-)}(t) \right) \lim_{h \to 0} \| J^{(-)}(\vec{x}, t, h) \|^2. \tag{2.18}
\]

The expressions in parentheses in (2.18) cannot vanish simultaneously, thus,

\[
\Phi_0(t, h) = \frac{1}{h} S_0(t, h),
\]

\[
S_0(t, 0) = \int_0^t \left( \langle \vec{p}(t), \vec{x}(t) \rangle - \lambda^{(\pm)}(t) \right) \text{d}t, \tag{2.19}
\]

\[
\lim_{h \to 0} \| J^{(\mp)}(\vec{x}, t, h) \| = 0.
\]

4. For definiteness, we put

\[
S_0(t, 0) = \int_0^t \left( \langle \vec{p}(t), \vec{x}(t) \rangle - \lambda^{(+)}(t) \right) \text{d}t, \quad \lim_{h \to 0} \| J^{(-)}(\vec{x}, t, h) \| = 0. \tag{2.20}
\]
By (2.3), $\lim_{\hbar \to 0} \| J^+(\vec{x}, t, h) \| = 1$. Then, averaging the operator $\hat{L}_D$ with respect to the state (2.19) in the higher orders of $\hbar$, we obtain

$$
0 \equiv \left( \frac{dS(t, h)}{dt} + \lambda^{(+)}(t) - \langle \check{\rho}(t), \check{\rho}(t) \rangle \right)^{(+)} + \sum_{k=1}^{3} \left[ \langle \sigma^{cl(1)}_{\xi_k}(t, h) a_k(h) \left[ \check{p}_k - eC_{\xi_k} - \frac{\hbar^2}{\varepsilon(t)} \langle \vec{P}_D, \left( - \frac{e}{c}\vec{A}_{\xi_k} \right) \rangle \right] + \langle \sigma^{cl(1)}_{\eta_k}(t, h) b_k(h) \left[ \check{x}_k - \frac{\hbar^2}{\varepsilon(t)} \check{P}_k \right] \rangle \right] + \ldots . \tag{2.21}
$$

Here

$$
\langle \sigma^{cl(0)}_{\xi_k}(t, h) \rangle = \| J^+(\vec{x}, t, h) \|, \quad \langle \sigma^{cl(1)}_{\xi_k}(t, h) \rangle = \frac{1}{a_k(h)} \int (J^+(\vec{x}, t, h))^\dagger \Delta x_k J^+(\vec{x}, t, h) d^3x, \tag{2.22}
$$

$$
\langle \sigma^{cl(1)}_{\eta_k}(t, h) \rangle = \frac{1}{b_k(h)} \int (J^+(\vec{x}, t, h))^\dagger \Delta \check{p}_k J^+(\vec{x}, t, h) d^3x. \tag{2.23}
$$

We took into account that, under the conditions (2.20), the order (with respect to $\hbar$) of the functions $\langle \sigma^{cl(1)}_{\xi_k}(t, h) \rangle$ and $\langle \sigma^{cl(1)}_{\eta_k}(t, h) \rangle$ is less than that of the function (2.22). Since the functions $\langle \sigma^{cl(1)}_{\xi_k}(t, h) \rangle$ and $\langle \sigma^{cl(1)}_{\eta_k}(t, h) \rangle$ are independent, Eq. (2.21) yields

$$
\dot{\check{p}} = -\lambda^{(+)}(\vec{p}, \vec{x}, t), \quad \dot{\vec{x}} = \lambda^{(+)}(\vec{p}, \vec{x}, t). \tag{2.24}
$$

If in (2.20) we replace $(+)$ by $(-)$ and $(-)$ by $(+)$, then in (2.23) we get $\lambda^{(+)} \to \lambda^{(-)}$. Thus Theorem 1 is proved.

**Theorem 2** If $\Psi(t)$ is a semiclassically concentrated state of the class $\mathbb{C}S_D(z(t), N)$, then the mean values of the quantum-mechanical spin operator, i.e., of the Bargmann polarization pseudovector $\hat{S}^\mu$ [19], are (in the limit as $h \to 0$) solutions of the classical-relativistic equations of spin motion (the Bargmann–Mishel–Telegdi) [20].

**Proof.** For definiteness, we assume that $\vec{x}(t)$ and $\check{p}(t)$ satisfy the Hamilton system (2.23) and assumption (2.20) holds. Denote

$$
\zeta(t) = \lim_{h \to 0} \int (J^+(\vec{x}, t, h))^\dagger \delta J^+(\vec{x}, t, h) d\vec{x}, \tag{2.25}
$$

$$
a^\mu(t) = \lim_{h \to 0} \langle \Psi(\vec{x}, t, h)|\hat{S}^\mu|\Psi(\vec{x}, t, h) \rangle_D, \tag{2.26}
$$

where

$$
\hat{S}^\mu = (\hat{S}_0, \hat{S}), \quad \hat{S}_0 = \frac{1}{m_0c}(\vec{S}, \check{P}), \quad \hat{S} = \rho_3 \vec{S} + \frac{1}{m_0c} \rho_1 \check{P}. \tag{2.27}
$$

Then, by (2.20), we have

$$
a_0(t) = \gamma \langle \zeta(t), \vec{\beta} \rangle, \quad \vec{a}(t) = \zeta(t) + \frac{\gamma \vec{\beta}}{1 + \gamma^{-1}} \langle \zeta(t), \vec{\beta} \rangle, \quad a_0(t) = \langle \vec{\beta}, \vec{a}(t) \rangle. \tag{2.28}
$$

The time evolution of mean values $a^\mu(t)$ is described by the Heisenberg equation

$$
\frac{da^\mu}{dt} = \lim_{h \to 0} \langle \Psi | \left[ \frac{\partial \hat{S}^\mu}{\partial t} + \frac{i}{\hbar} [\hat{H}_D, \hat{S}^\mu] \right] \rangle |\Psi \rangle. \tag{2.29}
$$

Commuting the operator $\hat{S}^\mu$ with $\hat{H}_D$, we get (see, for example, [29] and Appendix B):

$$
\frac{\partial \hat{S}_0}{\partial t} + \frac{i}{\hbar} [\hat{H}_D, \hat{S}_0] = \frac{e}{m_0c}(\vec{S}, \check{E}) - \frac{e_0(g - 2)}{2m_0c^2} (\rho_2(\vec{S}, \vec{E} \times \vec{P}) + \rho_3(\vec{S}, \vec{H} \times \vec{P})) + O(h), \tag{2.30}
$$

$$
\frac{\partial \vec{S}}{\partial t} + \frac{i}{\hbar} [\hat{H}_D, \vec{S}] = \frac{e}{m_0c}(\rho_1 \check{E} + \vec{H} \times \vec{S}) - \frac{e_0(g - 2)}{2m_0c^2} (\rho_1 \vec{E} + \vec{H} \times \vec{S} - \rho_3(\vec{S}, \vec{E} \check{P}) m_0c + \frac{\check{P}}{m_0c} \rho_3(\vec{S}, \vec{H})) + O(h). \tag{2.31}
$$

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Substituting (2.29) into (2.28) and taking into account (A.8)–(A.14), we obtain (see Appendix B)
\[ \frac{d\tilde{a}}{dt} = \frac{ge}{2m_0c\gamma}(\dot{\tilde{E}}(\dot{\beta}, \dot{a}) + \dot{\tilde{a}} \times \dot{\tilde{H}}) + \frac{(g - 2)e\gamma}{2m_0c} \tilde{B}(\dot{\beta}, \dot{a} \times \dot{\tilde{H}} + \langle \dot{a}, \ddot{\beta} \rangle \langle \ddot{\tilde{E}} \rangle - \langle \dot{a}, \ddot{E} \rangle). \]  
(2.30)

Passing from (2.30) to the equation for \( a^\mu \), and taking into account (2.27), we get
\[ \frac{da^\mu}{d\tau} = \frac{eg}{2m_0c} F^{\mu\nu} a_\nu + \frac{(g - 2)e}{2m_0c^3} \dot{a}^\mu \dot{a}_\nu F^{\nu\alpha} a_\alpha, \]  
(2.31)
where \( F^{\mu\nu} \) is the tensor of electromagnetic field, \( \tau \) is the proper time. Equation (2.31) is the Bargmann–Michel–Telegdi equation. Thus the theorem is proved.

3 Passage to the two-component theory

Each eigenvalue \( \lambda^{(\pm)}(\vec{p}, \vec{x}, t) \) is associated with its own Hamilton system:
\[ \dot{\vec{p}}_\pm = -\lambda^{(\pm)}(\vec{p}_\pm, \vec{x}_\pm, t), \quad \vec{x}_\pm(0, z_0) = \vec{x}_0, \quad \vec{p}_\pm(0, z_0) = \vec{p}_0, \]
\[ \dot{\vec{x}}_\pm = \lambda^{(\pm)}(\vec{p}_\pm, \vec{x}_\pm, t), \quad z_0 = (\vec{p}_0, \vec{x}_0) \in \mathbb{R}^6_{p,x}, \]
(3.1)
and the corresponding variational system
\[ \begin{align*}
\dot{\vec{W}}_\pm &= -\lambda^{(\pm)}(t)\vec{W}_\pm - \lambda^{(\pm)}_{xx}(t)\vec{Z}_\pm, \\
\dot{\vec{Z}}_\pm &= \lambda^{(\pm)}_{pp}(t)\vec{W}_\pm + \lambda^{(\pm)}_{xp}(t)\vec{Z}_\pm.
\end{align*} \]
(3.2)
The initial conditions for system (3.2) are chosen similarly to the scalar case (see I.1.5). Here we give the explicit form of \( 3 \times 3 \)-matrices in (3.2)
\[ \lambda^{(\pm)}_{xx}(\vec{p}, \vec{x}, t) = \| (e(\Phi_{xx}, \vec{x}) - (i\beta^k, \vec{A}_{xx}, \vec{x})) - \frac{e^2}{c^2} (\vec{A}_{xk}, \lambda^{(\pm)}_{pp}, \vec{A}_{xx}) \|, \]
\[ \lambda^{(\pm)}_{xp}(\vec{p}, \vec{x}, t) = (\lambda^{(\pm)}_{xp}(\vec{p}, \vec{x}, t))^t = -\frac{e}{c}\lambda^{(\pm)}_{pp}(\vec{p}, \vec{x}, t)\| \vec{A}_{xk}(\vec{x}, t) \|, \]
\[ \lambda^{(\pm)}_{pp}(\vec{p}, \vec{x}, t) = \pm \frac{e^2}{\epsilon} \| (\delta_{jk} - \beta^k \beta_j^p) \|, \]
calculated at the points \( z^{\pm}(t, z_0) \) of the phase trajectory.

From here on we restrict ourselves to solutions of “positive frequency” (corresponding to the eigenvalue \( \lambda^{(+)}(\vec{p}, \vec{x}, t) \)). Further, in functions \( \vec{x}_+, \ldots \) related to \( \lambda^{(+)} \), we shall omit the index + in all cases where it does not lead to misunderstanding. The solutions corresponding to \( \lambda^{(-)}(\vec{p}, \vec{x}, t) \) (of “negative frequency”) are obtained by the substitution: \( \lambda^{(+)} \rightarrow \lambda^{(-)} \), \( z^+(t, z_0) \rightarrow z^-(t, z_0) \), \( \Pi_+ \rightarrow \Pi_- \), and \( (\vec{Z}_+, \vec{W}_+) \rightarrow (\vec{Z}_-, \vec{W}_-) \).

Let us quantize the classical system (3.1) by the method of complex germ, i.e., with an arbitrary (but fixed) trajectory of a classical particle \( z(t, z_0) \) we associate the complete set of functions of the form:
\[ |\nu, t\rangle = \prod_{k=1}^{3} \frac{1}{\sqrt{\nu_k!}} (a_k(t))^{\nu_k} |0, t\rangle, \]
(3.3)
where
\[ |0, t\rangle = N_0(J(t, z_0))^{-1/2} \exp \left[ \frac{i}{\hbar} S(\vec{x}, t) \right], \]
\[ S(\vec{x}, t) = \int_0^t (\langle \dot{\vec{x}}(t), \dot{\vec{p}}(t) \rangle - \lambda^{(+)}(t) dt + \langle \vec{p}(t), \Delta \vec{x} \rangle + \frac{1}{2} \langle \Delta \vec{x}, Q(t) \Delta \vec{x} \rangle, \]
\[ N_0(h), J(t), \dot{a}^+(t) \] are defined in (I.2.11), (I.2.12), respectively. It is easy to see (item 2 Part I) that the functions (3.3) satisfy the equation of Schrödinger type:
\[ (-i\hbar \partial_t + \lambda) |\nu, t\rangle = 0, \]
\[ \hat{\lambda} = \lambda^{(+)}(t) + \langle \dot{\vec{p}}(t), \Delta \vec{x} \rangle - \langle \dot{\vec{x}}(t), \Delta \vec{p} \rangle + \frac{1}{2} \left[ \langle \Delta \vec{x}, \lambda^{(+)\prime}(t) \Delta \vec{x} \rangle + \langle \Delta \vec{x}, \lambda^{(+)\prime}(t) \Delta \vec{p} \rangle + \langle \Delta \vec{p}, \lambda^{(+)\prime}(t) \Delta \vec{x} \rangle + \langle \Delta \vec{p}, \lambda^{(+)\prime}(t) \Delta \vec{p} \rangle \right] \]
\[ = \lambda^{(+)}(t) + \delta^1 \lambda^{(+)}(t) + \frac{1}{2} \delta^2 \lambda^{(+)}(t). \]
The eigenvectors (2.8) of the matrix $\mathcal{H}_0(\vec{p}, \vec{x}, t)$ form an orthonormalized basis in $\mathbb{C}^4$ (see Property 11 in Appendix A):

\[
P^\pm(\vec{p}, \vec{x}, t) = \mathbb{I}_{2 \times 2},
\]

\[
P^\pm(\vec{p}, \vec{x}, t) P(\vec{p}, \vec{x}, t) = 0,
\]

\[
\sum_{k=\pm} P_k(\vec{p}, \vec{x}, t) P^+_k(\vec{p}, \vec{x}, t) = \mathbb{I}_{4 \times 4},
\]

and the system of scalar functions (3.3) is complete in $\mathcal{P}_h^t(\mathbb{R}^3, \mathbb{C})$:

\[
\langle t, \nu \rangle |\nu, t\rangle = \delta_{\nu, \nu}; \quad \sum_{|\nu| = 0} |\nu, t\rangle \langle t, \nu| = 1,
\]

and

\[
\mathcal{P}_h^t(\mathbb{R}^3, \mathbb{C}) = \left\{ f, f = \exp \left( \frac{it}{\hbar} (S_0(t) + (\vec{p}(t), \Delta \vec{x}))) \phi(\frac{\Delta^2}{\hbar}, t, h), \quad \phi(\vec{z}, t, h) \in S \right\},
\]

\[
S_0(t) = \int_0^t (\langle \vec{p}(t), \dot{\vec{x}}(t) \rangle - \lambda(t)) dt,
\]

where $\phi(\vec{z}, t, h)$ is a smooth function in $t \in [0, T]$ regularly depends on $h$, and $S$ is a Schwartz space with respect $\vec{z} \in \mathbb{R}^3$. The solution of equation (2.1) will be sought in the form

\[
(+)^\Psi(\vec{x}, t, h) = \Psi(\vec{x}, t, h) = (\Pi_+(t), \Pi_-(t)) \left( \begin{array}{c} \mathbb{U}(\vec{x}, t, h) \\ \mathbb{V}(\vec{x}, t, h) \end{array} \right) = (\Pi_+ \mathbb{U} + \Pi_- \mathbb{V}),
\]

where the matrices (2.8) are calculated at the point $z(t, z_0)$ and the unknown two-component spinors $\mathbb{U}(\vec{x}, t, h) \in \mathcal{P}_h^t(\mathbb{R}^3, \mathbb{C})^2$ and $\mathbb{V}(\vec{x}, t, h) \in \mathcal{P}_h^t(\mathbb{R}^3, \mathbb{C})^2$ must be determined.

Remark 1. Note, that if $\Psi \in \mathcal{P}_h^t(\mathbb{R}^3, \mathbb{C})^4$ (3.5) the solution of the Dirac equation than $\Psi$ is semiclassically concentrated states of class $\mathcal{S}_C D(z(t), \infty)$.

We substitute the function (3.5) into (2.1) and expand the obtained expressions with respect to the eigenvectors (2.8) taking into account the relations (see Appendix A)

\[
\begin{align*}
\tilde{\Pi}_\pm(t) &= \frac{i}{2} \Pi_\pm(t) \left( \langle \vec{\sigma}, \vec{\beta} \times \vec{\beta} \rangle + \frac{\gamma}{2} \Pi_\mp(t) \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right) \right), \\
\rho_1(\vec{\sigma}, \vec{\beta}) \Pi_{\pm}(t) &= \pm \langle \vec{\beta}, \vec{\beta} \rangle \Pi_{\pm}(t) + \Pi_\mp(t) \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right), \\
\rho_2(\vec{\Sigma}, \vec{E}) \Pi_{\pm}(t) &= -\Pi_{\pm}(t) \langle \vec{\beta}, \vec{\beta} \times \vec{E} \rangle = i \Pi_\mp(t) \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right), \\
\rho_3(\vec{\Sigma}, \hat{H}) \Pi_{\pm}(t) &= \mp \Pi_\pm(t) \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right) + \langle \vec{\beta}, \hat{H} \rangle \Pi_\mp(t).
\end{align*}
\]

Here $c \vec{\beta} = \vec{x}(t, z_0)$, $\gamma^{-1} = \sqrt{1 - \beta^2}$. As the result we get

\[
(-i\hbar \partial_t + \hat{\mathcal{H}}) \Psi = \Pi_+ \left\{ -i \hbar \partial_t + \lambda(+)(t) + c(\vec{\beta}, \Delta \vec{\beta}) + \frac{\hbar}{2} \left( \langle \vec{\sigma}, \vec{\beta} \times \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right) \mathbb{U}(\vec{x}, t) + \frac{\hbar}{2} \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right) \mathbb{V}(\vec{x}, t) \right\} + \Pi_- \left\{ -i \hbar \partial_t + \lambda(-)(t) + c(\vec{\beta}, \Delta \vec{\beta}) + \frac{\hbar}{2} \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right) \mathbb{U}(\vec{x}, t) + \frac{\hbar}{2} \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \frac{\gamma}{1 + \gamma - 1} \right) \mathbb{V}(\vec{x}, t) \right\}
\]
\[- \frac{e(g-2)\hbar}{4m_0c} \left( - \langle \vec{\sigma}, \vec{\beta} \times \vec{E} \rangle - \langle \vec{\sigma}, \vec{H} \rangle + \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{H} \rangle}{1 + \gamma^{-1}} \right) \right] \nu(x, t) + \\
+ \frac{i\hbar \gamma}{2} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{H} \rangle}{1 + \gamma^{-1}} + \gamma^{-1} \langle \vec{\sigma}, \vec{\beta} \rangle \right) + c \left( \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{\Delta} \vec{P} \rangle \right) - \\
- \frac{e(g-2)\hbar}{4m_0c} \left( i \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{E} \rangle}{1 + \gamma^{-1}} - i \gamma^{-1} \langle \vec{\sigma}, \vec{E} \rangle + \langle \vec{\beta}, \vec{H} \rangle \right) U(x, t) \right\},
\]

where $\Delta \Phi = \Phi(\vec{x}, t) - \Phi(\vec{x}(z_0), t)$, $\Delta \vec{P} = \vec{P} - \vec{P}(t)$.

We transform the obtained equation by expanding the expressions in this equation into a Taylor series in operators $\Delta \vec{x}$ and $\Delta \vec{P}$ and taking into account that in the class of functions $\mathcal{F}_h$ we have

\[\Delta \vec{x} = \hat{O}(\sqrt{\hbar}), \quad \Delta \vec{P} = \hat{O}(\sqrt{\hbar}), \quad (-i\hbar \partial_t + \hat{\lambda}) = \hat{O}(\hbar),\]

as $\hbar \to 0$ (see (I.4.4)). Denote $\hat{R}_1 = \Delta \vec{P} - (e/c) \partial^i \vec{A}(t)$,

\[\hat{R}^{(N)}(\vec{A}) = \frac{i\hbar \gamma}{2} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} + \langle \vec{\sigma}, \vec{\beta} \rangle \right) + \frac{e}{k!} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{\beta} \rangle \right) - \\
- \sum_{k=2}^{N+2} \frac{e}{k!} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{\beta} \rangle \right) - \\
- \langle \vec{\sigma}, \vec{x}, t \rangle - \langle \vec{\sigma}, \vec{d}, t \rangle - \langle \vec{\sigma}, \vec{E} \rangle + \langle \vec{\beta}, \vec{H} \rangle) + \hat{O}(\hbar^{(N+3)/2}),\]

\[\hat{\mathcal{U}}^{(N)}(\vec{A}) = 2\varepsilon(t) - 2e \langle \vec{\beta}, \vec{P} \rangle + (-i\hbar \partial_t + \hat{\lambda}) - \frac{1}{2} \langle \vec{P}, \lambda_{pp}^+ \vec{P} \rangle + \\
+ \sum_{k=3}^{N+2} \frac{e}{k!} \left( \langle \vec{\sigma}, \vec{\beta} \rangle + \langle \vec{\beta}, \vec{d} \rangle \right) + \frac{\hbar}{2} \langle \vec{\beta}, \vec{E} \rangle + \frac{\hbar}{2} \langle \vec{\beta}, \vec{H} \rangle + \hat{O}(\hbar^{(N+3)/2}).\]

Then equation (3.6) takes the form

\[(-i\hbar \partial_t + \hat{R}) \Psi^{(N)}(\vec{x}, t) = \Pi_+ \left\{ - i \hbar \partial_t + \hat{\lambda} - \frac{1}{2} \langle \vec{P}, \lambda_{pp}^+ \vec{P} \rangle + \\
+ \frac{\hbar}{2} \langle \vec{\sigma}, \vec{\beta} \rangle \hat{P} \right\} + \\
+ \frac{\hbar}{2} \langle \vec{\sigma}, \vec{\beta} \rangle \hat{P} \right\} + \\
+ \sum_{k=2}^{N+2} \frac{e}{k!} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \right) - \\
- \frac{e(g-2)\hbar}{4m_0c} \sum_{k=3}^{N} \left( \langle \vec{\sigma}, \vec{d} \rangle - \langle \vec{\sigma}, \vec{d}, \vec{E} \rangle - \langle \vec{\sigma}, \vec{d}, \vec{H} \rangle \right) + \hat{O}(\hbar^{(N+3)/2}).\]

Since the eigenvectors of the principal symbol of the Hamiltonian $\hat{H}_D$, which define the matrices $\Pi_+(t)$ and $\Pi_-(t)$, are linearly independent in $\mathbb{C}^4$, then, by (3.5), we get

\[\gamma^{(N)}(\vec{x}, t) = - \{ \hat{\mathcal{M}}^{(N)} \}^{-1} \hat{R}^{(N)} \mathcal{U}^{(N)}(\vec{x}, t),\]

\[(-i\hbar \partial_t + \hat{F}^{(N)}) \mathcal{U}^{(N)}(\vec{x}, t) = 0,\]

\[\hat{F}^{(N)} = \hat{\lambda} - \frac{1}{2} \langle \vec{P}, \lambda_{pp}^+ \vec{P} \rangle + \sum_{k=3}^{N+2} \frac{e}{k!} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \right) + \\
+ \frac{\hbar}{2} \langle \vec{\sigma}, \vec{\beta} \rangle \hat{P} \right\} + \\
+ \sum_{k=3}^{N+2} \frac{e}{k!} \left( \langle \vec{\sigma}, \vec{\beta} \rangle \right) - \\
- \frac{e(g-2)\hbar}{4m_0c} \sum_{k=3}^{N} \left( \langle \vec{\sigma}, \vec{d} \rangle - \langle \vec{\sigma}, \vec{d}, \vec{E} \rangle - \langle \vec{\sigma}, \vec{d}, \vec{H} \rangle \right) + \hat{O}(\hbar^{(N+3)/2}).\]

The operator $(\hat{\mathcal{M}}^{(N)})^{-1}$ inverse to the operator (3.8) will be found from (1.4.10) where we put $\vec{B} = \hat{\mathcal{M}} - 2\varepsilon$, $\hat{A} = 2\varepsilon$. Then

\[\frac{1}{2\varepsilon} \hat{Q}^{(N)} = - \{ \hat{\mathcal{M}}^{(N)} \}^{-1} \hat{R}^{(N)} = \frac{1}{2\varepsilon} \sum_{k=1}^{N+2} \hat{h}^{k/2} \hat{Q}_k + \hat{O}(\hbar^{(N+3)/2}),\]
where

\[ \sqrt{\hbar} \hat{Q}_1 = e \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{P}_1 \rangle - \langle \hat{\sigma}, \hat{P}_1 \rangle \right). \]  

(3.13)

\[ \hbar \hat{Q}_2 = -\frac{e}{2} \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{d}^2 \hat{A} \rangle - \langle \hat{\sigma}, d^2 \hat{A} \rangle \right) + \frac{i \gamma \hbar}{2} \left( \gamma^{-1} \langle \hat{\sigma}, \hat{\beta} \rangle + \langle \hat{\sigma}, \hat{\beta} \rangle \frac{\langle \hat{\beta}, \hat{\beta} \rangle}{1 + \gamma^{-1}} \right) - \frac{e(g - 2) \hbar}{4 \hbar_0 c} \left[ i \langle \hat{\sigma}, \hat{\beta} \rangle \hat{E}(t) \frac{1}{1 + \gamma^{-1}} + i \gamma^{-1} \langle \hat{\sigma}, \hat{E}(t) \rangle - \langle \hat{\beta}, \hat{H}(t) \rangle \right] - \frac{c \sqrt{\hbar}}{\varepsilon} \langle \hat{\beta}, \hat{P}_1 \rangle \hat{Q}_1, \]

and for \( k > 2 \)

\[ \hbar^{k/2} \hat{Q}_k = -\frac{e}{k!} \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, d^k \hat{A} \rangle - \langle \hat{\sigma}, d^k \hat{A} \rangle \right) + \frac{e(g - 2) \hbar}{4 \hbar_0 c} \left( \frac{1}{k - 2} \right) \left( \langle \hat{\beta}, d^k \hat{H} \rangle - i \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, d^k \hat{E} \rangle - \frac{e}{\varepsilon} \langle \hat{\beta}, \hat{P}_1 \rangle \hat{Q}_{k-1} + \left[ \langle \hat{\beta}, d^2 \hat{A} \rangle + (\lambda \hbar \partial_t + \hat{\lambda}) \right] - \frac{1}{2} \langle \hat{\beta}, \hat{P}_1 \rangle \hat{P}_1 + \frac{e(g - 2) \hbar}{4 \hbar_0 c} \left( \langle \hat{\sigma}, \hat{H}(t) \rangle - \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{\hat{H}} \rangle \right) + \frac{e(g - 2) \hbar}{4 \hbar_0 c} \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{d}^n \hat{A} \rangle \right) + \frac{1}{n!} \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{d}^n \hat{H} \rangle \right) \right] h^{k-2/2} \hat{Q}_{k-2} + \sum_{n=3}^{k-1} \left[ \frac{e}{n!} \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{d}^n \hat{A} \rangle \right] h^{(k-n)/2} \hat{Q}_{k-n}. \]

By taking into account (3.8), . . . , (3.11), the operator \( \hat{F}^{(N)} \) can be represented in the form

\[ \hat{F}^{(N)} = \hat{F}_0 + \sqrt{\hbar} \hat{F}_1^{(N)}, \quad \hat{F}_1^{(0)} = 0, \]

where

\[ \hat{F}_0 = \hat{\lambda} + \frac{\hbar}{2} \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{\beta} \rangle \right) - \frac{\hbar c}{2} \left( \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{\beta} \rangle \right) + \langle \hat{\sigma}, \hat{H}(t) \rangle + \frac{e(g - 2) \hbar}{4 \hbar_0 c} \left( \langle \hat{\sigma}, \hat{H}(t) \rangle - \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{\beta} \rangle \right), \]

(3.14)

and the highest terms have the form \( \hat{F}_1^{(N)} = \hat{O}(\hbar) \) as \( \hbar \to 0 \).

Thus we have

\[ \hat{T}^{(N)}(t) = (\Pi_+(t) + \frac{1}{2 \varepsilon} \Pi_-(t)) \hat{Q}^{(N)}(t) \]

(3.15)

and the problem of constructing asymptotic (up to \( O(\hbar^{N+1/2}) \)) positive-frequency solutions \( (+) \Psi(\vec{x}, t, \hbar) \) to the Dirac equation (2.1) is reduced to solving the equation (3.10) with respect to the two-component spinor \( \mathcal{U}^{(N)}(\vec{x}, t) \). As a result we have

\[ (+) \Psi(\vec{x}, t, \hbar) = \Psi(\vec{x}, t, \hbar) = \hat{T}^{(N)}(t) \mathcal{U}^{(N)}(\vec{x}, t). \]

4 Semiclassical trajectory-coherent representation of the Dirac equation with anomalous Pauli interaction

Now let us consider the construction of asymptotic (as \( \hbar \to 0 \)) solutions of equations (3.10) in the two-component theory:

\[ (-i \hbar \partial_t + \hat{F}_0 + \sqrt{\hbar} \hat{F}_1^{(N)}) \mathcal{U}^{(N)}(\vec{x}, t, \hbar) = 0. \]

(4.1)

In the first approximation as \( \hbar \to 0 \) (neglecting the operator \( \sqrt{\hbar} \hat{F}_1 = \hat{O}(\hbar^{3/2}) \) in (3.10)) we have the following equation for the spinor \( \mathcal{U}^{(0)}(\vec{x}, t) \):

\[ (-i \hbar \partial_t + \hat{F}_0) \mathcal{U}^{(0)}(\vec{x}, t, \hbar) = 0 \]

(4.2)

and hence \( \mathcal{U}^{(0)}(\vec{x}, t, \hbar) = \phi(\vec{x}, t, \hbar) u(t) \),

\[ \left\{ \frac{d}{dt} + \frac{ec}{2\varepsilon} \left( (1 + \hat{\gamma}) \langle \hat{\sigma}, \hat{H}(t) \rangle - \langle \frac{1}{1 + \gamma^{-1}} + \hat{\gamma} \rangle \langle \hat{\sigma}, \hat{\beta} \rangle \langle \hat{\beta}, \hat{E}(t) \rangle - \right) \right\} \mathcal{U}^{(0)}(\vec{x}, t, \hbar) = 0. \]
\begin{equation}
\begin{aligned}
\frac{\tilde{\gamma}_1(\tilde{\beta}, \tilde{\bar{H}}(t))}{1 + \gamma^{-1}} e^{i\tilde{\beta}} u = 0, \\
(-i\hbar \partial_t + \lambda) \phi(\vec{x}, t, \hbar) = 0,
\end{aligned}
\tag{4.3}
\end{equation}

where \( \tilde{\gamma} = (g - 2)/2 \). The derivation of (4.3) was based on the following relation [30]:
\[
\tilde{\beta} = \frac{c\epsilon}{\epsilon}(E + \tilde{\beta} \times \tilde{H} - \tilde{\beta}(\tilde{E}, \tilde{\bar{E}})).
\]

Thus the spinor properties of electrons in the semiclassical limit as \( \hbar \to 0 \) are determined by complex solutions of the linear system (4.3). Assume that at the initial time the spinor \( u(t) \) satisfies the following condition [31]:
\[
\langle \delta, \tilde{\phi} \rangle u(0, \zeta) = \zeta u(0, \zeta), \quad \zeta = \pm 1.
\tag{4.4}
\]

By this assumption we fix that at \( t = 0 \) the spin of a particle is directed along an arbitrary unit vector \( \vec{\ell} \in \mathbb{R}^3 \). Then at any instant of time the solutions \( u(t, \zeta) \) of the Cauchy problem (4.3), (4.4) form an orthonormalized basis in \( \mathbb{C}^2: u^+(t, \zeta')u(t, \zeta) = \delta_{\zeta, \zeta'} \), and, hence, the functions
\[
\mathcal{U}_{\nu, \zeta}(\vec{x}, t, \hbar) = u(t, \zeta)|\nu, t\rangle = |\nu, \zeta, t\rangle
\]
form the complete orthonormalized set of solutions of the equation (4.2)
\[
\langle t, \zeta', \nu' |\nu, \zeta, t\rangle = \delta_{\nu, \nu'} \delta_{\zeta, \zeta'}.
\]

The precision of constructed solutions can be estimated similarly to the case of the Schrödinger equation (see Theorem 2 in [1]).

Let us construct the following (in \( \hbar \to 0 \)) approximations for equation (4.1). In contrast to the nonrelativistic case, the operator of “perturbation” \( \sqrt{\mathcal{F}_1} \) is not self-adjoint in \( L_2(\mathbb{R}^3, \mathbb{C}^2) \) and hence the function \( \mathcal{F}_1(\vec{x}, t, \hbar) \) cannot be considered as the wave function of electron. However, to solve equation (4.1) by the methods of the theory of perturbations and then to go over to the semiclassical TC-representation, it is sufficient that only the operator \( \mathcal{F}_0 \) be self-adjoint. By using formula (I.4.10) with operators
\[
\hat{A} = \partial_t + \frac{i}{\hbar} \mathcal{F}_0, \quad \hat{B} = \frac{i}{\sqrt{\hbar}} \mathcal{F}_1
\]
and taking into account that
\[
\hat{A}^{-1}\phi(t) = \sum_{|\nu|=0}^{\infty} \sum_{\zeta = \pm 1} \langle \nu, \zeta, t \rangle \int_0^t d\tau \langle \tau, \zeta, \nu |\phi(\tau)\rangle,
\]
we obtain the solution of equation (4.1) with precision up to \( O(\hbar^{(N+1)/2}) \):
\[
\mathcal{U}_{\nu, \zeta}(\vec{x}, t, \hbar) = \mathcal{F}(\nu, \zeta, t) + O(\hbar^{(N+1)/2}),
\]
\[
\mathcal{F}(\nu) = \sum_{n=0}^{\infty} \left(-\frac{i}{\sqrt{\hbar}}\right)^n (\mathcal{F}_1)^n,
\tag{4.5}
\]
\[
\mathcal{F}_1(\nu)(\vec{x}) = \sum_{|\nu'|=0}^{\infty} \sum_{\zeta' = \pm 1} \langle \nu', \zeta', t \rangle \int_0^t d\tau \langle \tau, \zeta', \nu' |\mathcal{F}_1(\nu')(\vec{x})|\phi(\tau)\rangle.
\]

Now let us construct the semiclassical TC-representation for the Dirac equation (2.1) following the scheme of constructing the TC-representation for the Schrödinger operator. We introduce the Hilbert space of vector-functions
\[
L^2_h(\mathbb{R}^3, \mathbb{C}^2)
\]
with scalar product
\[
\langle \varphi_1, \varphi_2 \rangle_{L^2_h} = \int d^3x \rho_0^\varphi(\vec{x}, t) \psi_1^+(\vec{x}, t, \hbar) \varphi_2(\vec{x}, t, \hbar),
\]
where the density of measure \( \rho_0^\varphi(\vec{x}, t) \) was defined in (1.2).
We define the operator \( \tilde{K}_D^{(N)}(t, \hbar) : L_h^1 \to L_2(\mathbb{R}^3, \mathbb{C}^4) \), which defines the passage to the semiclassical TC-representation, by the formula:
\[
\tilde{K}_D^{(N)}(t, \hbar) \varphi = \tilde{T}^{(N)}(t) \tilde{F}^{(N)}(t) \tilde{K}_D^{(0)}(t, \hbar) \varphi, \quad \varphi \in L_h^1(\mathbb{R}^3, \mathbb{C}^2),
\]
(4.6)
where the operator \( \tilde{T}^{(N)} \) is defined in (3.15), \( \tilde{K}_D^{(0)}(t, \hbar) = \tilde{K}_S^{(0)}(t, \hbar) \), and \( \tilde{K}_S^{(0)}(t, \hbar) \) is defined in (1.4), in which the symbol \( \mathcal{H}(\vec{p}, \vec{x}, t) \) must be replaced by the relativistic Hamiltonian function \( \lambda^{(+)}(\vec{p}, \vec{x}, t) \).

The operator \( \tilde{K}_D^{(N)}(t, \hbar) \) (with precision up to \( O(h^{N+1}/2) \)) unitarily maps the space \( L_h^1(\mathbb{R}^3, \mathbb{C}^2) \) to the space \( L_2(\mathbb{R}^3, \mathbb{C}^2) \), this means that
\[
\langle \tilde{K}_D^{(N)}(t, \hbar) \varphi_1 | \tilde{K}_D^{(N)}(t, \hbar) \varphi_2 \rangle_{L_2} = \langle \varphi_1 | \varphi_2 \rangle_{L_h^1} + O(h^{N+1}/2) .
\]

By direct calculations it is easy to verify that in the semiclassical TC-representation the Dirac equation (2.1) takes the form:
\[
(\tilde{K}_D^{(N)}(t, \hbar))^{+} \left( -i\hbar \partial_t + \tilde{H}_D \right) \tilde{K}_D^{(N)}(t, \hbar) \varphi = \left[ \tilde{\pi}_0 + (\tilde{\sigma}, \tilde{D}_0(t, z_0)) \right] \varphi + \tilde{O}(h^{N+3/2}),
\]
(4.7)
where the operator \( \tilde{\pi}_0 \) is given by the formula (1.7), in which the symbol \( \mathcal{H}(\vec{p}, \vec{x}, t) \) must be replaced by the relativistic Hamiltonian function \( \lambda^{(+)} \) (2.7), and the vector \( \tilde{D}_0(t, z_0) \) is equal to
\[
\tilde{D}_0(t, z_0) = \frac{\mu_0}{\gamma} \left[ \left( 1 + \tilde{\gamma} \gamma \right) \tilde{H}(t) - \left( \frac{1}{1 + \gamma^{-1}} + \tilde{\gamma} \gamma \right) \beta \right] \times \tilde{E}(t) = - \tilde{\gamma} \beta \left( \beta, \tilde{H}(t) \right),
\]
(4.8)
where \( \mu_0 = h e_0/(2 m e) \) is Bohr magneton.

Thus, with precision up to \( \tilde{O}(h^{N+3/2}) \), the Dirac equation in the semiclassical representation takes the form:
\[
(\tilde{\pi}_0 + (\tilde{\sigma}, \tilde{D}_0(t, z_0))) \varphi = 0.
\]
(4.9)
Precisely as in (1.8), equation (4.9) admits the complete orthonormalized set of solutions of the form:
\[
| H_\nu, \zeta \rangle = u(t, \zeta) \prod_{k=1}^{3} \frac{1}{\sqrt{\nu_k!}} (\hat{\lambda}^{+})_{\nu_k} \cdot 1,
\]
(4.10)
where the operators of creation \( \hat{\lambda}^{+} \) are given by formulas (1.9), in which the complex vectors \( \tilde{W}_j(t) \) and \( \tilde{Z}_j(t) \) are solutions of the variational system (3.2).

We return to the initial (Schrödinger) representation of the Dirac equation (2.1) and, taking into account (4.6) and (4.9), obtain the complete orthonormalized set of semiclassical (up to \( \tilde{O}(h^{N+1/2}) \)) trajectory-coherent states of electron:
\[
\Psi_{\nu, \zeta}^{(N)}(\vec{x}, t, \hbar) = \tilde{K}_D^{(N)}(t, \hbar) | H_{\nu, \zeta} \rangle.
\]
(4.11)
The negative-frequency semiclassical TC-states \( -\Psi_{\nu, \zeta}^{(N)} \) are given by the same formula where we change:
\[
\lambda^{(+)} \longrightarrow \lambda^{(-)}, \quad \Pi_+ (t) \longrightarrow \Pi_-(t), \quad z^+(t, z_0) \longrightarrow z^-(t, z_0),
\]
\[
(\tilde{W}_+(t), \tilde{Z}_+(t)) \longrightarrow (\tilde{W}_-(t), \tilde{Z}_-(t)).
\]

Remark 2 The relations (4.6)–(4.11) allow us to obtain the Green function for Dirac equation in semiclassical trajectory-coherent approximation. Analogously (I.6.4) for the positive-frequency part of a kernel of evolution operator for the equation (2.1)
\[
G_D^{(N)}(\vec{x}, \vec{y}, t, s) = \tilde{T}^{(N)}(t) \tilde{F}^{(N)}(t) G_D^{(0)}(\vec{x}, \vec{y}, t, s) (\tilde{F}^{(N)}(s))^{+} (\tilde{T}^{(N)}(s))^{+},
\]
(4.12)
where
\[
G_D^{(0)}(\vec{x}, \vec{y}, t, s) = G^{(0)}(\vec{x}, \vec{y}, t, s) \sum_{\zeta = \pm 1} u(t, \zeta) u^+(s, \zeta),
\]
(4.13)
u(\zeta) is a solution of equation (4.3) with initial condition (4.4), \( G^{(0)}(\vec{x}, \vec{y}, t, s) \) was defined in (I.A1.1) for \( \mathcal{H}_0 = \lambda \). The operators \( \tilde{T}^{(N)} \) and \( \tilde{F}^{(N)} \) were defined in (3.15) and (4.5), respectively. Analogously to the scalar case, the Green function allows to obtain a solution of Cauchy problem for Dirac equation (2.1) only in the class of semiclassical trajectory-coherent states. Nevertheless the Green function (4.12) can be usefull for calculation of concrete physical effects.
5 The relativistic analog of the Pauli equation

The semiclassical description of a quantum particle with its spin properties taken into account allows (with an arbitrary precision in \( \hbar \to 0 \)) to exclude the interference between the positive-frequency and negative-frequency states (the Schrödinger “zitterbewegung” [28]), i.e., in the subspace of positive-frequency (negative-frequency) states it allows to go over to the one-particle two-component theory, whose Hamiltonian is a self-adjoint operator for any order of \( \hbar \to 0 \).

The unitary operator \( \hat{T}^{(N)} \) (mod \( O(\hbar^{(N+3)/2}) \)) of transition to the two-component theory will be sought in the form:

\[
\hat{T}^{(N)} = \hat{F}^{(N)} \hat{B}^{(N)} + O(\hbar^{(N+3)/2}), \tag{5.1}
\]

where the operator \( \hat{T}^{(N)} \) is defined in (3.15) and the operator \( \hat{B}^{(N)} \) is defined by the condition that the operator \( (+) \hat{H}^{(N)} \) is self-adjoint:

\[
(+) \hat{H}^{(N)} = (\hat{B}^{(N)})^+ ( -i\hbar \partial_t + \hat{F}^{(N)} \hat{B}^{(N)} + i\hbar \partial_t + O(\hbar^{(N+3)/2})).	ag{5.2}
\]

In this case the spinor

\[
\hat{U}_{\nu, \zeta}^{(N)} = (\hat{B}^{(N)})^{-1} \hat{U}_{\nu, \zeta}^{(N)}(\vec{x}, t)
\]

can be considered as the wave function of the one-particle problem:

\[
(\hat{T}^{(N)})^+ ( -i\hbar \partial_t + \hat{H}_D) \hat{T}^{(N)} \hat{U}^{(N)} = ( -i\hbar \partial_t + (+) \hat{H}^{(N)} \hat{U}^{(N)} + O(\hbar^{(N+3)/2}). \tag{5.3}
\]

Let us consider the construction of the operator \( \hat{B}^{(N)} \) in the case \( N = 2 \) more precisely. For this purpose, we represent the not self-adjoint part of the operator \( \hat{F}_1^{(2)} \) as follows:

\[
\frac{\sqrt{\hbar}}{2\varepsilon} \hat{Q}_1 ( -i\hbar \partial_t + \hat{F}_0) \frac{\sqrt{\hbar}}{2\varepsilon} \hat{Q}_1 = \frac{\hbar}{2} \left\{ \left( -i\hbar \partial_t + \hat{F}_0 \right) \left( \frac{1}{2\varepsilon} \hat{Q}_1 \right)^2 + \left( \frac{1}{2\varepsilon} \hat{Q}_1 \right)^2 \left( -i\hbar \partial_t + \hat{F}_0 \right) \right\} + \\
+ \frac{\hbar}{2} \left\{ \left[ \left( \frac{1}{2\varepsilon} \hat{Q}_1 \right), \left( -i\hbar \partial_t + \hat{F}_0 \right) \right] \left( \frac{1}{2\varepsilon} \hat{Q}_1 \right) + \frac{1}{2\varepsilon} \hat{Q}_1 \left[ \left( -i\hbar \partial_t + \hat{F}_0 \right), \left( \frac{1}{2\varepsilon} \hat{Q}_1 \right) \right] \right\},
\]

\[
\quad \quad \quad \quad \quad \quad \left[ \hat{A}, \hat{B} \right] = \hat{A}\hat{B} - \hat{B}\hat{A}.
\]

If in Eq. (5.1) we choose the operator \( \hat{B}^{(2)} \) as

\[
\hat{B}^{(2)} = 1 - \frac{\hbar}{2} \left( \frac{1}{2\varepsilon} \hat{Q}_1 \right)^2, \tag{5.4}
\]

then it can be easily verified that the conditions imposed on the operator \( \hat{T}^{(N)} \) are satisfied. In this case the spinor \( \hat{U}_{\nu, \zeta}^{(2)} \) takes the form:

\[
\hat{U}_{\nu, \zeta}^{(2)}(\vec{x}, t, \hbar) = (\hat{B}^{(2)})^{-1} \hat{U}_{\nu, \zeta}^{(2)}(\vec{x}, t, \hbar) = [1 - i\sqrt{\hbar} \hat{F}_1 - i\hbar \hat{F}_2 - \hbar \hat{F}_1] \hat{U}_{\nu, \zeta}^{(2)}(\vec{x}, t, \hbar) = \hat{F}^{(2)}(\nu, \zeta, t), \tag{5.5}
\]

where

\[
\begin{align*}
\sqrt{\hbar} \hat{F}_1 \varphi(t) & = \frac{1}{\hbar} \sum_{|\nu'| = 0, \zeta' = \pm 1}^{\infty} \int_0^t d\tau \langle \tau, \zeta', \nu' | \hat{F}_1^{(2)} | \varphi(\tau) \rangle, \\
\hbar \hat{F}_2 \varphi(t) & = \frac{1}{\hbar} \sum_{|\nu'| = 0, \zeta' = \pm 1}^{\infty} \int_0^t d\tau \langle \tau, \zeta', \nu' | \hat{F}_1^{(2)} - \hat{F}_1^{(2)} - \hat{F}_2^{(2)} - \hbar \hat{F}_2 \rangle \varphi(\tau) \rangle.
\end{align*}
\]

Thus the functions (5.5) form the complete orthonormalized set of solutions of equation (5.3):

\[
\langle \hat{U}_{\nu, \zeta}^{(2)}(\vec{x}, t, \hbar) | \hat{U}_{\nu, \zeta}^{(2)}(\vec{x}, t, \hbar) \rangle = \delta_{\nu, \nu'} \delta_{\zeta, \zeta'} + O(\hbar^{3/2}),
\]

which can be considered as the relativistic (up to \( O(\hbar^{5/2}) \)) generalization of the Pauli equation. In the case \( N > 2 \) the operator \( \hat{B}^{(N)}(\hat{H}^{(N)}, \hat{T}^{(N)} \) respectively) can be obtained in the same way as the operator (5.4).
Let us explicitly calculate the operator $\tilde{K}^{(2)}_D(t, h)$ which defines the passage to the semiclassical trajectory-coherent representation. By (4.6), (5.4), we get:

$$\tilde{K}^{(2)}_D(t, h) = \left\{ \Pi_+ + \frac{1}{2} \Pi_-(t) \left( \sqrt{\hbar} \hat{Q}_1 + \hbar \hat{Q}_2 + \hbar^{3/2} \hat{Q}_3 + \hbar^2 \hat{Q}_4 \right) \right\} \times \left( 1 - \frac{h}{2} \left( \hat{Q}_1 \right)^2 \right) \left( 1 - i \sqrt{\hbar} \hat{F}_1 - i h \hat{F}_2 - h \hat{F}_1 \right) \hat{K}^{(0)}_D(t, h),$$

(5.6)

where $\hat{K}^{(0)}_D(t, h)$ is defined in (4.6). Since the operator $\hat{T}^{(N)}$ is unitary, the operator (5.6) is also unitary, i.e.:

$$\langle \left( \hat{K}^{(2)}_D(t, h) | H_{\nu'} \zeta' \right) | \left( \hat{K}^{(2)}_D(t, h) | H_{\nu} \zeta \right) \rangle _{L_2} = \langle \zeta' | H_{\nu'} | \zeta \rangle _{L_2} + O(h^{3/2}) = \delta_{\nu,\nu'} \delta_{\zeta,\zeta'} + O(h^{3/2}).$$

Let $\hat{A}_t : L_2(\mathbb{R}^3, \mathbb{C}^4) \rightarrow L_2(\mathbb{R}^3, \mathbb{C}^4)$ be a unitary operator. Then the corresponding operator in the two-component theory can be always represented in the form:

$$\hat{A}_+ = (\hat{T}^{(2)})^+ \hat{A}_t \hat{T}^{(2)} + \hat{O}(h^{3/2}) = \hat{a} + \langle \hat{a}, \hat{A} \rangle + \hat{O}(h^{3/2}),$$

where $\hat{a}$ and $\hat{A}$ are self-adjoint (in $L_2$) operators with symbols $a(\vec{x}, \vec{p}, t)$ and $\hat{A}(\vec{x}, \vec{p}, t)$ respectively. We find the explicit form of the operator $\hat{A}_t(h)$ in the semiclassical TC-representation:

$$\hat{A}' = (\hat{K}^{(2)}_D(t, h))^+ \hat{A}_t \hat{K}^{(2)}_D(t, h) = (\hat{K}^{(0)}_D(t, h))^{(+)} (\hat{K}^{(0)}_D(t, h))^{(+)} \hat{A}_+ \hat{K}^{(2)}_D(t, h) =$$

$$\left( 1 + \hat{D}_1 + \frac{1}{2} \hat{D}_2^2 \right) (a(t) + \langle \hat{a}, \hat{A}(t) \rangle ) - \sqrt{\hbar} \left[ \hat{D}_1 (a(t) + \langle \hat{a}, \hat{A}(t) \rangle ) \hat{\pi}_1 - \hat{\pi}_1 \hat{D}_1 (a(t) + \langle \hat{a}, \hat{A}(t) \rangle ) \hat{\pi}_1 - i \langle \hat{\pi}_2 - \hat{\pi}_1 \hat{\pi}_2, \pi^2 \rangle \langle \hat{\pi}_1 \rangle \right] - \hat{O}(h^{3/2}),$$

(5.7)

where

$$\hat{\pi}_j = (\hat{K}^{(0)}_D(t, h))^{-1} \hat{F}_j \hat{K}^{(0)}_D(t, h), \quad j = 1, 2,$$

and $\hat{D}_j \hat{A}_t$ are defined (1.14).

In particular, in the semiclassical TC-representation, the operators of momentum $\hat{p} = -i \hbar \nabla$, coordinates $\vec{x}$, and spin $^2 \hat{S} = \hbar \hat{\sigma} / 2$ have the form:

$$\hat{X}(t, h) = (\hat{K}^{(0)}_D(t, h))^{-1} \hat{X} \hat{K}^{(0)}_D(t, h) = \vec{x}(t, z_0) + \Delta \vec{x} - i \sqrt{\hbar} \left( \Delta \hat{\pi}_1 - \hat{\pi}_1 \Delta \vec{p} \right) + \hat{O}(h^{3/2}),$$

(5.8)

$$\hat{P}(t, h) = (\hat{K}^{(0)}_D(t, h))^{-1} \hat{P} \hat{K}^{(0)}_D(t, h) = \hat{p}(t, z_0) + \Delta \hat{p} - i \sqrt{\hbar} \left( \Delta \hat{p} \hat{\pi}_1 - \hat{\pi}_1 \Delta \hat{p} \right) + \hat{O}(h^{3/2}),$$

(5.9)

$$\hat{S}(t, h) = (\hat{K}^{(0)}_D(t, h))^{-1} \hat{S} \hat{K}^{(0)}_D(t, h) = \frac{h}{2} \hat{\sigma} \hat{K}^{(0)}_D(t, h) = \frac{h}{2} \hat{\sigma} + \hat{O}(h^{3/2})$$

(5.10)

Let us write the explicit expression for the Hamiltonian of the two-component theory $\hat{H}^{(N)}$ (5.2), in which the operators of the order $\hat{O}(h^{3/2})$ are taken into account, and the expressions for quantum averages of the principal observables in the theory, namely, for operators of coordinates, momenta and spin are calculated for one-particle semiclassical TC-states of electron $| H_{\nu} \zeta \rangle$ (4.10):

$$\hat{H}^{(1)} = (1 + \hat{\delta} + \frac{1}{2!} \hat{\delta}^2 + \frac{1}{3!} \hat{\delta}^3 \lambda (\hat{t}) + \frac{2 \hat{c} \hbar}{e \varepsilon} \left[ \left( \hat{\gamma} \beta + \hat{\gamma} \beta \hat{E} \right) + \frac{1}{1 + \gamma} \right] \hat{\beta} \times \hat{E}(t) + \frac{\hat{y}}{1 + \gamma} \hat{E}(t) \langle \hat{\beta}, \hat{E}(t) \rangle - \frac{\hat{y} \lambda}{1 + \gamma} \hat{E}(t) \langle \hat{\beta}, \hat{E}(t) \rangle - \frac{1}{1 + \gamma} \hat{\beta} \hat{E}(t) \langle \hat{\beta}, \hat{E}(t) \rangle - \frac{1}{1 + \gamma} \hat{\beta} \hat{E}(t) \langle \hat{\beta}, \hat{E}(t) \rangle$$

(5.11)

2In the initial (Schrödinger) representation, the spin operator $\hat{S} = (\hbar/2) \hat{\sigma}$ corresponds (up to $O(h^{3/2})$) to the three-dimensional unit vector of spin $(\hbar/2) \langle \rho_3 \Sigma + \rho_3 (c/e) \beta - c \rho_3 \hat{P} \Sigma \hat{P} \rangle / [ (c/e) + m_0 c^2 ]$ (see [34, 58]).
where \( \sigma = (t, D \frac{D}{t}) + \langle \sigma, D_0(t) \rangle + \langle \sigma, D(t) \Delta x + D_\nu(t) \Delta \tilde{p} \rangle \).

(5.11)

\[
D_\nu(t) \Delta \tilde{p} = - \frac{c}{\varepsilon} \frac{[\beta (\tilde{\beta}, \tilde{E}(t)) + \gamma^{-1} \tilde{E}(t)]}{1 + \gamma^{-1}} - \Delta \tilde{p} -
\]

\[
- \frac{c}{\varepsilon} \frac{[\beta (\tilde{\beta}, \tilde{H}(t)) + \gamma^{-1} \tilde{H}(t)] (\beta, \Delta \tilde{p})}{1 + \gamma^{-1}} -
\]

\[
- \frac{c}{\varepsilon} \frac{\gamma (\beta, \tilde{H}(t)) [\beta (\Delta \tilde{p}, \tilde{H}(t))]}{1 + \gamma^{-1}} - \Delta \tilde{p} -
\]

\[
- \frac{c}{\varepsilon} \frac{\gamma (\beta, \tilde{H}(t)) [\beta (\Delta \tilde{p}, \tilde{H}(t))]}{1 + \gamma^{-1}} - \Delta \tilde{p};
\]

\[
D_{t}(t) \Delta \tilde{p} = - \frac{c}{\varepsilon} \frac{[\beta (\tilde{\beta}, \tilde{E}(t)) + \gamma^{-1} \tilde{E}(t)]}{1 + \gamma^{-1}} - \Delta \tilde{p} -
\]

\[
- \frac{c}{\varepsilon} \frac{[\beta (\tilde{\beta}, \tilde{H}(t)) + \gamma^{-1} \tilde{H}(t)] (\beta, \Delta \tilde{p})}{1 + \gamma^{-1}} -
\]

\[
- \frac{c}{\varepsilon} \frac{\gamma (\beta, \tilde{H}(t)) [\beta (\Delta \tilde{p}, \tilde{H}(t))]}{1 + \gamma^{-1}} - \Delta \tilde{p} -
\]

\[
- \frac{c}{\varepsilon} \frac{\gamma (\beta, \tilde{H}(t)) [\beta (\Delta \tilde{p}, \tilde{H}(t))]}{1 + \gamma^{-1}} - \Delta \tilde{p};
\]

\[
\tilde{X}(t, \zeta, \zeta') = \langle \zeta', H_{\nu}(\tilde{X}(t, \hbar)|H_{\nu}, \zeta) =
\]

\[
=x(t, z_0) \delta_{\zeta', \zeta'} - i \sqrt{n}(\zeta', H_{\nu}|(\Delta \tilde{x} \tilde{p} - \tilde{p} \Delta \tilde{x})|H_{\nu}, \zeta) + \tilde{O}(\hbar^2);
\]

(5.12)

\[
\tilde{P}(t, \zeta, \zeta') = \langle \zeta', H_{\nu}|\tilde{P}(t, \hbar)|H_{\nu}, \zeta =
\]

\[
= \tilde{P}(t, z_0) \delta_{\zeta', \zeta'} - i \sqrt{n}(\zeta', H_{\nu}|(\Delta \tilde{p} \tilde{p} - \tilde{p} \Delta \tilde{p})|H_{\nu}, \zeta) + \tilde{O}(\hbar^2);
\]

(5.13)

\[
\tilde{S}(t, \zeta, \zeta') = \langle \zeta', H_{\nu}|\tilde{S}(t, \hbar)|H_{\nu}, \zeta =
\]

\[
= \frac{\hbar}{2} \tilde{P}(t, \zeta, \zeta').
\]

(5.14)

Here \( \Delta \tilde{x} = - i \hbar \nabla + \frac{Q(t)}{2} \Delta t \), \( \tilde{\eta}(t, \zeta, \zeta') = U^+(t, \zeta') \tilde{U}(t, \zeta) \) is the solution of the Bargmann–Michel–Telegdi equation \( \tilde{D}_0(t, z_0) \) in the rest system

\[
\tilde{\eta} = \frac{2}{\hbar} \tilde{\eta} \times \tilde{D}_0(t, z_0),
\]

where the vector \( \tilde{D}_0(t, z_0) \) is defined in (4.8).

By using the function \( H_{\nu}, \zeta' \) (4.10), it is easy to calculate (with the same precision \( O(\hbar^2), \hbar \to 0 \)) the correlation matrix \( \tilde{D}_0 \), which describes quantum fluctuations of dynamical variables \( \tilde{x}_j(t, \hbar), \tilde{p}_j(t, \hbar) \) with respect to their average values (5.12), (5.14), \( \sigma_{x, x, j}, \sigma_{p, p, j} \), and their correlation \( \sigma_{x, p, j}, i, j = 1, 2, 3 \). Here

\[
\sigma_{AB} = \frac{1}{2} \langle (\hat{A} \hat{B} + \hat{B} \hat{A}) \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle.
\]

We have

\[
\sigma_{xx} = \frac{\hbar}{4} \left[ C(t) D_{\nu}^{-1} C^+(t) + C^+(t) D_{\nu}^{-1} C(t) \right],
\]

\[
\sigma_{pp} = \frac{\hbar}{4} \left[ B(t) D_{\nu}^{-1} B^+(t) + B^+(t) D_{\nu}^{-1} B(t) \right],
\]

(5.15)

\[
\sigma_{px} = \frac{\hbar}{4} \left[ B(t) D_{\nu}^{-1} C^+(t) + B^+(t) D_{\nu}^{-1} C(t) \right],
\]

where \( C(t) \) and \( B(t) \) are defined in (I.1.3), and \( D_{\nu}^{-1} \) denotes the matrix

\[
D_{\nu}^{-1} = \left\| \frac{2\nu_k + 1}{\Im b_k} \delta_{jk} \right\|.
\]
Differentiating the relations (5.12)–(5.15) with respect to $t$ and writing the right-hand sides of obtained relations in terms of variables $\dot{X}$, $\dot{P}$, $\dot{\eta}$, $\Delta_2$, we get (up to $O(\hbar^{3/2})$)

\[
\begin{align*}
\dot{z} &= J\partial_{z} \lambda^{(+)}(z, t) + \frac{1}{2}(\partial_{z}, \Delta_{2}\partial_{z})J\partial_{z} \lambda^{(+)}(z, t) + JD_{z}(z, t)\dot{\eta}, \\
\Delta_{2} &= J(\lambda^{(+)}\tau_{2}^{z} \Delta_{2} - \Delta_{2}(\lambda^{(+)}\tau_{2}^{z} J, \quad \Delta_{2} = \Delta_{2},
\end{align*}
\]

(5.16)

\[
\dot{\eta} = \frac{2}{\hbar} \dot{\eta} \times \vec{D}_{0}(t, z_{0}),
\]

where

\[
\begin{align*}
z &= (\vec{P}(t, \hbar), \vec{X}(t, \hbar)), \\
\Delta_{2} &= \begin{pmatrix} \sigma_{pp} & \sigma_{px} \\ \sigma_{xp} & \sigma_{xx} \end{pmatrix}, \\
D_{z}\dot{\eta} &= (D_{p}\dot{\eta}, D_{x}\dot{\eta}), \\
\beta = \frac{1}{c} \lambda_{p}(z, t),
\end{align*}
\]

vector $\vec{D}_{0}$ is defined in (4.8), and $3 \times 3$ matrices $D_{p}$, $D_{x}$ - in (5.11), $\vec{P}(t, \hbar)$, $\vec{X}(t, \hbar)$ in (5.12), (5.13).

The system (5.16) is a closed system of ordinary differential equations for quantum averages in Dirac theory. The problem of correspondence between the obtained equations and the well-known equations for a particle with spin (e.g., Frenkel etc. [36, 37]) requires a more detail consideration. The initial conditions for system (5.16) is choosen as follows:

\[
\begin{align*}
\left| z \right|_{t=0} &= z_{0}, \\
\left| \Delta_{2} \right|_{t=0} &= \Delta_{2},
\end{align*}
\]

(5.17)

where $\vec{k} = (0, 0, 1)$, $\vec{\ell}$ and $\Delta_{2}$ are defined in (4.4) and (I.5.23), respectively. Let us write the system (5.16) in the form

\[
\begin{align*}
\dot{z} &= J\partial_{z} \lambda^{(+)}(z, t) + \frac{1}{2}(\partial_{z}, A\Delta_{2}^{0}A^{+}\partial_{z})J\partial_{z} \lambda^{(+)}(z, t) + JD_{z}(z, t)\dot{\eta} \dot{\eta}, \\
\dot{\eta} &= J(\lambda^{(+)}\tau_{2}^{z} \Delta_{2} - \Delta_{2}(\lambda^{(+)}\tau_{2}^{z} J, \quad \Delta_{2} = \Delta_{2},
\end{align*}
\]

(5.18)

\[
\begin{align*}
\left[ \frac{d}{dt} + \frac{1}{\hbar} \langle \sigma, \vec{D}_{0} \rangle \right] u = 0.
\end{align*}
\]

The initial conditions for spinor $\mathcal{U}$ are defined in (4.4) and

\[
A_{t=0} = \begin{pmatrix} B_{0} & B_{0}^{*} \\ C_{0} & C_{0}^{*} \end{pmatrix}
\]

where $3 \times 3$ matrices $B_{0}$ and $C_{0}$ satisfy the conditions (I.1.4) and (I.1.5).

6 Conclusions

Let us briefly review our consideration. To our opinion, from the physical point of view, the “passage to the classics” in quantum mechanics must inevitably cause the introduction of the notion of classical trajectory, which primarily (on the postulational level) is alien to quantum mechanics and must be taken from the outside. It is essential that one can construct the complete set of approximate solutions of the Dirac equation with the property: as $\hbar \to 0$, the quantum-mechanical average values of coordinates and momenta are general solutions of classical Hamiltonian equations. In literature [38] it was repeated over and over that this possibility is not obvious. Usually (see, for example, [24], p. 69–70) the authors restrict themselves to a verbal formulation of conditions imposed on the relativistic wave function of semiclassical type, having implicitly in mind that these conditions can be met without any essential difficulties. However, as a rule, such states were not presented. Our consideration shows that in order to construct such states explicitly, one can use the method of complex germ [39].

Not only the principal possibility of obtaining approximate (in $\hbar \to 0$) solutions of the Dirac equations (up to $O(\hbar^{N/2})$) for any $N$ is shown, but also a constructive method for obtaining the corresponding higher approximations is given. It is essential that the expansion into an asymptotic series with respect to $\hbar$ contains half-integer powers of $\hbar$, i.e., there are series in $\sqrt{\hbar}$ (in contrast to the standard semiclassical expansion in $\hbar$, $\hbar \to 0$, given in all manuals of quantum mechanics).
The possibility to construct such states (called trajectory-coherent states) explicitly leads to nontrivial conclusions. For example, one can obtain, in the most transparent and natural way, the “classical equations of motion” for average values of quantities, which cannot be exactly well-defined in the classical sense (e.g., spin). It seems natural that, for the classical vector of spin, we obtain the Bargman–Michel–Telegdi equation. However, in the case of arbitrary (and not only homogeneous) electromagnetic fields, one can explicitly show, which is nontrivial, that in this equations the fields must be taken on classical trajectories. Up to now (see, for example, [34]), this fact was justified only by verbal arguments.

In conclusion, we note that, by using the Maslov complex canonical operator, one can go over to the approximate Hamiltonian of quantum theory. The possibility to construct such states (called trajectory-coherent states) explicitly leads to nontrivial conclusions. For example, one can obtain, in the most transparent and natural way, the “classical equations of motion” for average values of quantities, which cannot be exactly well-defined in the classical sense (e.g., spin). It seems natural that, for the classical vector of spin, we obtain the Bargman–Michel–Telegdi equation. However, in the case of arbitrary (and not only homogeneous) electromagnetic fields, one can explicitly show, which is nontrivial, that in this equations the fields must be taken on classical trajectories. Up to now (see, for example, [34]), this fact was justified only by verbal arguments.

In conclusion, we note that, by using the Maslov complex canonical operator, one can go over to the approximate trajectory-coherent representation, in which the classical trajectory is considered already for the approximate Hamiltonian of quantum theory.

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Appendix A. Properties of matrices $\Pi_\pm(t)$ and $\lambda_{pp}^{(+)}(t)$

**Property 1** The following relations hold:

\[ [\lambda_{pp}^{(+)}(t)]^{1/2} = \frac{c}{\sqrt{\varepsilon(t)}} \left( \frac{\beta_i \beta_j}{1 + \gamma^{-1} - \delta_{ij}} \right), \quad (A.1) \]
\[ [\lambda_{pp}^{(+)}(t)]^{-1/2} = -\frac{\sqrt{\varepsilon(t)}}{c} \left( \frac{\beta_i \beta_j}{1 + \gamma^{-1}} \right), \quad (A.2) \]
\[ [\lambda_{pp}^{(+)}(t)]^{-1} = \frac{\varepsilon(t)}{c^2} \| \delta_{jk} + \gamma \beta_j \beta_k \|_{3x3}, \quad (A.3) \]

where

\[ \bar{\beta} = \frac{1}{c} \bar{\varepsilon}(t, z_0), \quad \gamma^{-1} = (1 - \beta^2)^{1/2}, \quad \lambda_{pp}^{(+)} = \frac{\varepsilon^2}{\varepsilon} \| \delta_{ij} - \beta_i \beta_j \|. \]

**Proof.** We can verify directly that

i)
\[
\sum_{i=1}^{3} \frac{c}{\sqrt{\varepsilon(t)}} \left( \frac{\beta_i \beta_j}{1 + \gamma^{-1} - \delta_{ij}} \right) \left( \frac{\beta_i \beta_k}{1 + \gamma^{-1} - \delta_{ik}} \right) = \frac{c^2}{\varepsilon(t)} \left( \frac{\beta_j \beta_k}{1 + \gamma^{-1}} \right) \left( \frac{\beta^2}{1 + \gamma^{-1}} - 2 + \delta_{jk} \right) = \frac{c^2}{\varepsilon(t)} (\delta_{jk} - \beta_j \beta_k) = \lambda_{pp}^{(+)}(t), \quad \beta^2 = 1 - \gamma^{-2}.
\]

ii)
\[
\sum_{i=1}^{3} \frac{c}{\sqrt{\varepsilon(t)}} \left( \frac{\beta_i \beta_i}{1 + \gamma^{-1} - \delta_{ij}} \right) \left( -\frac{\sqrt{\varepsilon(t)}}{c} \right) \left( \frac{\beta_i \beta_k}{1 + \gamma^{-1}} - \delta_{ik} + \gamma \beta_i \beta_k \right) = -\left[ \frac{\beta_j \beta_k}{1 + \gamma^{-1}} \left( \frac{\gamma \beta^2}{1 + \gamma^{-1}} - \gamma + 1 \right) - \delta_{jk} \right] = \delta_{jk}.
\]

iii)
\[
\sum_{i=1}^{3} \frac{c^2}{\varepsilon(t)} (\delta_{ij} - \beta_i \beta_j) \frac{\varepsilon(t)}{c^2} (\delta_{ik} + \gamma^2 \beta_i \beta_k) = \delta_{jk} + \beta_j \beta_k (\gamma^2 - 1 - \gamma^2 \beta^2) = \delta_{jk},
\]

as was to be proved.
Property 2 The following relation holds:

\[ \mathcal{H}_0(t) \Pi_{\pm}(t) = \lambda^{(\pm)}(t) \Pi_{\pm}(t), \]  

(A.4)

where \( \mathcal{H}_0(t) \), \( \lambda^{(\pm)}(t) \), and \( \Pi_{\pm}(t) \) are defined in (2.2), (2.7), and (2.8), respectively.

Proof. Since \( \tilde{\beta} = c\tilde{p}/\varepsilon \), we get

\[
\mathcal{H}_0(t) \Pi_{\pm}(t) = \frac{1}{\varepsilon \sqrt{2 + 2\gamma^{-1}}} \begin{pmatrix} e\Phi + m_0c^2 & c(\vec{\sigma}, \vec{P}) \\ c(\vec{\sigma}, \vec{P}) & e\Phi - m_0c^2 \end{pmatrix} \begin{pmatrix} \varepsilon + m_0c^2 \\ c(\vec{\sigma}, \vec{P}) \end{pmatrix} = \frac{1}{\varepsilon \sqrt{2 + 2\gamma^{-1}}} \begin{pmatrix} (e\Phi + m_0c^2)(\varepsilon + m_0c^2) + c^2(\vec{\sigma}, \vec{P})^2 \\ c(\vec{\sigma}, \vec{P})(\varepsilon + m_0c^2 + e\Phi - m_0c^2) \end{pmatrix} = (\varepsilon + e\Phi) \Pi_{\pm}(t).
\]

Here we took into account that \( c^2(\vec{\sigma}, \vec{P})^2 = c^2 \tilde{p}^2 = \varepsilon^2 - m_0^2c^4 \). For the lower index, relation (A.4) can be proved similarly.

Property 3 The following relation holds:

\[
\rho_1(\vec{\Sigma}, \vec{P}) \Pi_{\pm}(t) = \langle \vec{\alpha}, \vec{P} \rangle \Pi_{\pm}(t) = \pm \langle \vec{\beta}, \vec{P} \rangle \Pi_{\pm}(t) + \Pi_{\mp}(t) \left( \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{P} \rangle \right) = \pm \langle \vec{\beta}, \vec{P} \rangle \Pi_{\pm}(t) + \frac{\sqrt{\varepsilon(t)} \Pi_{\mp}(t)}{c} \langle \vec{\sigma}, (\lambda^{(+)}_p(t))^{1/2} \vec{P} \rangle.
\]  

(A.5)

Proof. Actually,

\[
\langle \vec{\alpha}, \vec{P} \rangle \Pi_{\pm}(t) = \frac{1}{\sqrt{2 + 2\gamma^{-1}}} \begin{pmatrix} \langle \vec{\sigma}, \vec{P} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle \\ \langle \vec{\sigma}, \vec{P} \rangle (1 + \gamma^{-1}) \end{pmatrix}.
\]

\[
\langle \vec{\sigma}, \vec{P} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle = \langle \vec{\beta}, \vec{P} \rangle + i(\vec{\sigma}, \vec{P} \times \vec{\beta}) = [1 + (1 + \gamma^{-1}) - (1 + \gamma^{-1})] \langle \vec{\beta}, \vec{P} \rangle - i(\vec{\sigma}, \vec{P} \times \vec{\beta}) = (1 + \gamma^{-1}) \langle \vec{\beta}, \vec{P} \rangle + [(1 - \gamma^{-1}) \langle \vec{\beta}, \vec{P} \rangle - \langle \vec{\sigma}, \vec{\beta} \rangle \langle \vec{\sigma}, \vec{P} \rangle] = (1 - \gamma^{-1}) = \frac{\vec{\beta}^2}{1 + \gamma^{-1}} = \frac{1}{1 + \gamma^{-1}} \langle \vec{\sigma}, \vec{\beta} \rangle^2 = (1 + \gamma^{-1}) [\langle \vec{\beta}, \vec{P} \rangle + \langle \vec{\sigma}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{P} \rangle].
\]

Thus, we obtain the first relation in (A.5) for upper indices. Similarly

\[
\langle \vec{\alpha}, \vec{P} \rangle \Pi_{\mp}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \begin{pmatrix} -\langle \vec{\sigma}, \vec{P} \rangle (1 + \gamma^{-1}) \\ \langle \vec{\sigma}, \vec{P} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle \end{pmatrix};
\]

\[
-\langle \vec{\sigma}, \vec{P} \rangle (1 + \gamma^{-1}) = \langle \vec{\beta}, \vec{P} \rangle \langle \vec{\beta}, \vec{\sigma} \rangle + \langle \vec{\beta}, \vec{P} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle - (1 + \gamma^{-1}) \langle \vec{\sigma}, \vec{P} \rangle = \langle \vec{\sigma}, \vec{\beta} \rangle [-\langle \vec{\beta}, \vec{P} \rangle] + (1 + \gamma^{-1}) \left[ \frac{\langle \vec{\sigma}, \vec{\beta} \rangle \langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{P} \rangle \right] = \langle \vec{\sigma}, \vec{\beta} \rangle [-\langle \vec{\beta}, \vec{P} \rangle] + \langle \vec{\sigma}, \vec{\beta} \rangle \left[ \frac{\langle \vec{\beta}, \vec{P} \rangle}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{P} \rangle \right],
\]

as was to be proved. The second relation in (A.5) immediately follows from (A.1).

Remark 3 If in the matrices \( \Pi_{\pm}(t) = \Pi_{\pm}(\vec{p}, \vec{x}, t) \) \( \vec{p} = \vec{p}(t), \vec{x} = \vec{x}(t) \) trajectory \( \vec{x}(t) \), \( \vec{p}(t) \) is not classical, then in the obtained expressions we must put \( \vec{\beta} = c\vec{P}/\varepsilon \).
Property 4 The following relation holds:

\[
\Pi_{\pm}(t) = \frac{i}{2(1 + \gamma^{-1})} \Pi_{\pm}(t) (\dot{\sigma}, \dot{\beta} \times \dot{\beta}) \pm \frac{c}{2 \sqrt{\varepsilon(t)}} \Pi_{\pm}(t) \langle \sigma, (\lambda_{pp}^+(t))^{-1/2} \dot{\beta} \rangle.
\]  

(A.6)

Proof. The latter relation in Eq. (A.6) is a direct consequence of Eq. (A.2). We consider

i) \[
\frac{d}{dt} \Pi_{\pm}(t) \left( \frac{1}{\sqrt{1 + \gamma^{-1}}} \right) = \frac{1}{\sqrt{2}} \left( \frac{d}{dt} \sqrt{1 + \gamma^{-1}} \right) \left( \frac{d}{dt} \sqrt{1 + \gamma^{-1}} (\dot{\sigma}, \dot{\beta}) \right);
\]

\[
\frac{d}{dt} \sqrt{1 + \gamma^{-1}} = \frac{1}{2} \frac{d \gamma^{-1}}{dt} + \frac{1}{2} \frac{1}{\sqrt{1 + \gamma^{-1}}} \frac{d \gamma^{-1}}{dt} \left( \frac{\dot{\beta}^2}{1 + \gamma^{-1}} + \gamma^{-1} \right) = \frac{1}{2 \sqrt{1 + \gamma^{-1}}} \left[ i \langle \dot{\sigma}, \dot{\beta} \dot{\beta} \rangle + \dot{\beta}^2 \frac{d \gamma^{-1}}{dt} + \gamma^{-1} \frac{d \gamma^{-1}}{dt} - i \langle \dot{\sigma}, \dot{\beta} \dot{\beta} \rangle \right] = \frac{1}{\sqrt{1 + \gamma^{-1}}} \left( 1 + \gamma^{-1} \right) \left\{ i \langle \dot{\sigma}, \dot{\beta} \dot{\beta} \rangle \frac{\dot{\beta}^2}{2 (1 + \gamma^{-1})} \right\} - \frac{1}{2} \left( \dot{\sigma}, \dot{\beta} \right) \left[ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{\gamma (\dot{\beta}, \dot{\beta})}{1 + \gamma^{-1}} - \langle \dot{\sigma}, \dot{\beta} \rangle \right];
\]

\[
\frac{d}{dt} \langle \dot{\sigma}, \dot{\beta} \rangle = \frac{1}{2 \sqrt{1 + \gamma^{-1}}} \left\{ 2 \langle \dot{\sigma}, \dot{\beta} \rangle - \frac{1}{1 + \gamma^{-1}} \langle \dot{\sigma}, \dot{\beta} \rangle \frac{d \gamma^{-1}}{dt} \right\} = \frac{1}{2 \sqrt{1 + \gamma^{-1}}} \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{\dot{\beta}^2}{1 + \gamma^{-1}} + (1 + \gamma^{-1}) \right\} + \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \left( \frac{\gamma^{-1}}{1 + \gamma^{-1}} - 1 \right) \right\} = \frac{1}{2 \sqrt{1 + \gamma^{-1}}} \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{\dot{\beta}^2}{1 + \gamma^{-1}} + (1 + \gamma^{-1}) \right\} + \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{\gamma^{-1}}{1 + \gamma^{-1}} - \langle \dot{\sigma}, \dot{\beta} \rangle \frac{d \gamma^{-1}}{dt} \right\} = \frac{1}{\sqrt{1 + \gamma^{-1}}} \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{i \langle \dot{\sigma}, \dot{\beta} \dot{\beta} \rangle}{2 (1 + \gamma^{-1})} + \frac{1}{2} (1 + \gamma^{-1}) \right\} + \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{\gamma (\dot{\beta}, \dot{\beta})}{1 + \gamma^{-1}} + \langle \dot{\sigma}, \dot{\beta} \rangle \right\}.
\]

Similarly,

ii) \[
\frac{d}{dt} \Pi_{-}(t) = \frac{1}{\sqrt{2}} \left( \frac{d}{dt} \sqrt{1 + \gamma^{-1}} \right) \left( \frac{d}{dt} \sqrt{1 + \gamma^{-1}} (\dot{\sigma}, \dot{\beta}) \right);
\]

\[
\frac{d}{dt} \langle \dot{\sigma}, \dot{\beta} \rangle = \frac{1}{2 \sqrt{1 + \gamma^{-1}}} \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{i \langle \dot{\sigma}, \dot{\beta} \dot{\beta} \rangle}{1 + \gamma^{-1}} + (1 + \gamma^{-1}) \right\} + \left\{ \langle \dot{\sigma}, \dot{\beta} \rangle \frac{i \langle \dot{\sigma}, \dot{\beta} \dot{\beta} \rangle}{1 + \gamma^{-1}} + \langle \dot{\sigma}, \dot{\beta} \rangle \right\}.
\]

as was to be proved.
Property 5 The following relation holds:

\[
\rho_3(\vec{\Sigma}, \vec{H})\Pi_{\pm}(t) = \mp \Pi_{\pm}(t) \left[ \langle \vec{\sigma}, \vec{\beta} \rangle \frac{(\vec{\beta}, \vec{H})}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{H} \rangle \right] + \Pi_{\mp}(t) \langle \vec{\beta}, \vec{H} \rangle = \\
= \mp \sqrt{\frac{\varepsilon(t)}{c}} \Pi_{\pm}(t) \langle \vec{\sigma}, (\lambda_{pp}^{(+)}(t))^{-1/2} \vec{H} \rangle + \Pi_{\mp}(t) \langle \vec{\beta}, \vec{H} \rangle.
\] (A.7)

Proof. The latter relation in (A.7) follows directly from Eq. (A.1). Precisely as in Property 3, we get

i) \[
\rho_3(\vec{\Sigma}, \vec{H})\Pi_{+}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \left( \langle \vec{\sigma}, \vec{H} \rangle (1 + \gamma^{-1}) - \langle \vec{\sigma}, \vec{H} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle \right); \\
\langle \vec{\sigma}, \vec{H} \rangle (1 + \gamma^{-1}) = - (1 + \gamma^{-1}) \left[ \langle \vec{\sigma}, \vec{H} \rangle \frac{(\vec{\beta}, \vec{H})}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{H} \rangle \right] + \langle \vec{\sigma}, \vec{\beta} \rangle \langle \vec{\beta}, \vec{H} \rangle; \\
- \langle \vec{\sigma}, \vec{H} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle = - \langle \vec{\sigma}, \vec{\beta} \rangle \left[ \langle \vec{\sigma}, \vec{H} \rangle \frac{(\vec{\beta}, \vec{H})}{1 + \gamma^{-1}} - \langle \vec{\sigma}, \vec{H} \rangle \right] - (1 + \gamma^{-1}) \langle \vec{\beta}, \vec{H} \rangle;
\]

ii) \[
\rho_3(\vec{\Sigma}, \vec{H})\Pi_{-}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \left( \langle \vec{\sigma}, \vec{H} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle \right) (1 + \gamma^{-1}) \langle \vec{\sigma}, \vec{H} \rangle).
\]

Further, the proof coincides with that of Property 3.

Property 6 The following relation holds:

\[
\rho_2(\vec{\Sigma}, \vec{E})\Pi_{\pm}(t) = \Pi_{\pm}(t) (- \langle \vec{\sigma}, \vec{\beta} \times \vec{E} \rangle) + i \Pi_{\mp}(t) \left[ \langle \vec{\sigma}, \vec{\beta} \rangle \frac{(\vec{\beta}, \vec{E})}{1 + \gamma^{-1}} + \gamma^{-1} \langle \vec{\sigma}, \vec{E} \rangle \right] = \\
= - \Pi_{\pm}(t) \langle \vec{\sigma}, \vec{\beta} \times \vec{E} \rangle + i \gamma^{-1} \frac{c}{\sqrt{\varepsilon(t)}} \Pi_{\mp}(t) \langle \vec{\sigma}, (\lambda_{pp}^{(+)}(t))^{-1/2} \vec{E} \rangle.
\] (A.8)

Proof. The latter equality in (A.8) follows from (A.2). We consider the relations

i) \[
\rho_2(\vec{\Sigma}, \vec{E})\Pi_{+}(t) = \frac{i}{\sqrt{2(1 + \gamma^{-1})}} \left( - \langle \vec{\sigma}, \vec{E} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle \right); \\
i \langle \vec{\sigma}, \vec{E} \rangle \langle \vec{\sigma}, \vec{\beta} \rangle = - i \langle \vec{\beta}, \vec{E} \rangle + \langle \vec{\sigma}, \vec{E} \times \vec{\beta} \rangle = \\
= - i (1 + \gamma^{-1}) \langle \vec{\beta}, \vec{E} \rangle + i \gamma^{-1} (\langle \vec{\beta}, \vec{E} \rangle + i \langle \vec{\sigma}, \vec{\beta} \times \vec{E} \rangle) + (1 + \gamma^{-1}) \langle \vec{\sigma}, \vec{E} \times \vec{\beta} \rangle = \\
= (1 + \gamma^{-1}) \left[ - \langle \vec{\sigma}, \vec{E} \times \vec{E} \rangle \right] - i \langle \vec{\sigma}, \vec{\beta} \rangle \left[ \frac{(\vec{\beta}, \vec{E})}{1 + \gamma^{-1}} + \gamma^{-1} \langle \vec{\sigma}, \vec{E} \rangle \right]; \\
i \langle \vec{\sigma}, \vec{E} \rangle (1 + \gamma^{-1}) = i \left\{ \frac{1 + \gamma^{-1}}{\gamma} + \beta^2 \right\} \langle \vec{\sigma}, \vec{E} \rangle = \\
= - i ((\vec{\sigma}, \vec{\beta}) \langle \vec{\sigma}, \vec{E} \rangle - (\vec{\sigma}, \vec{E}) \vec{\beta}^2) + i \langle \vec{\sigma}, \vec{\beta} \rangle \langle \vec{\sigma}, \vec{E} \rangle + \frac{1 + \gamma^{-1}}{\gamma} \langle \vec{\sigma}, \vec{E} \rangle = \\
= \langle \vec{\sigma}, \vec{\beta} \rangle [- \langle \vec{\sigma}, \vec{E} \times \vec{E} \rangle] + i (1 + \gamma^{-1}) \left[ \langle \vec{\sigma}, \vec{\beta} \rangle \frac{(\vec{\sigma}, \vec{E})}{1 + \gamma^{-1}} + \gamma^{-1} \langle \vec{\sigma}, \vec{E} \rangle \right],
\]
as was to be proved.
Property 7 The following relation holds:
\[
(\vec{\Sigma}, \vec{S})\Pi_{\pm}(t) = \Pi_{\pm}(t) \left[ \frac{-e^{-1}}{\sqrt{\sqrt{e^2}}(t)} \left( (\vec{\xi}, \vec{S}) + \frac{\langle \vec{\beta}, \vec{S} \rangle}{1 + \gamma^{-1}} \langle \vec{\xi}, \vec{\xi} \rangle \right) \right] + \Pi_{\pm}(t)[i(\vec{\xi}, \vec{\beta} \times \vec{S})] = \\
= \frac{-e^{-1}}{\sqrt{\sqrt{e^2}}(t)} \Pi_{\pm}(t)(\vec{\xi}, (\lambda_{pp}^{(+)})^{-1/2}) + i\Pi_{\pm}(t)[i(\vec{\xi}, \vec{\beta} \times \vec{S})]. \tag{A.9}
\]

Proof. Since
\[
(\vec{\Sigma}, \vec{S})\Pi_{\pm}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \left( \frac{1 + \gamma^{-1}}{\langle \vec{\xi}, \vec{S} \rangle(1 + \gamma^{-1})} \right) \right) \\
(\vec{\Sigma}, \vec{S})\Pi_{\pm}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \left( \frac{-\langle \vec{\xi}, \vec{S} \rangle \langle \vec{\xi}, \vec{\xi} \rangle}{\langle \vec{\xi}, \vec{S} \rangle(1 + \gamma^{-1})} \right) \\
the further proof coincides with that of Property 6.

Property 8 The following relation holds:
\[
\rho_3\Pi_{\pm}(t) = 1 + \gamma^{-1}\Pi_{\pm}(t) + \Pi_{\pm}(t)(\vec{\xi}, \vec{\xi}). \tag{A.10}
\]

Proof. Since
\[
i) \rho_3\Pi_{\pm}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \left( 1 + \gamma^{-1} \right) \right) \\
(1 + \gamma^{-1}) = \gamma^{-1}(1 + \gamma^{-1}) + \beta^2 = (1 + \gamma^{-1})\gamma^{-1} + (\langle \vec{\xi}, \vec{\xi} \rangle)^2; \\
-\langle \vec{\xi}, \vec{\xi} \rangle = \gamma^{-1}(\langle \vec{\xi}, \vec{\xi} \rangle - (1 + \gamma^{-1})(\langle \vec{\xi}, \vec{\xi} \rangle), \\
as was to be proved.

ii) \rho_3\Pi_{\pm}(t) = \frac{1}{\sqrt{2(1 + \gamma^{-1})}} \left( \frac{\langle \vec{\xi}, \vec{\xi} \rangle}{1 + \gamma^{-1}} \right) \right).

Proof is similar to the preceding one.

Property 9 The following relation holds:
\[
\rho_2\Pi_{\pm}(t) = \mp i\Pi_{\pm}(t). \tag{A.11}
\]

Proof directly follows from the definition of matrices \(\Pi_{\pm}(t)\).

Property 10 The following relation holds:
\[
\rho_1\Pi_{\pm}(t) = \pm \Pi_{\pm}(t)\langle \vec{\xi}, \vec{\xi} \rangle - \gamma^{-1}\Pi_{\pm}(t). \tag{A.12}
\]

Proof. Multiplying the left- and right-hand sides of (A.10) by \(i\rho_2\) and transforming the right-hand side of the obtained expression according to (A.11), we get (A.12).

Property 11 For matrices \(\Pi_{\pm}(t)\), the orthonormality and completeness relations hold:
\[
\Pi_{\pm}^*(t)\Pi_{\pm}(t) = I_{2 \times 2}, \quad \Pi_{\pm}(t)\Pi_{\pm}^*(t) = 0_{2 \times 2}; \\
\Pi_{\pm}(t)\Pi_{\pm}^*(t) + \Pi_{\pm}(t)\Pi_{\pm}^*(t) = I_{4 \times 4}. \tag{A.13}
\]

Proof. By the straightforward verification, we get
\[
\Pi_{\pm}^*(t)\Pi_{\pm}(t) = \frac{1}{2(1 + \gamma^{-1})}[(1 + \gamma^{-1})^2 + (\langle \vec{\xi}, \vec{\xi} \rangle)^2] = \\
= \frac{1}{2(1 + \gamma^{-1})}(1 + 2\gamma^{-1} + \gamma^{-2} + \beta^2)I_{2 \times 2} = I_{2 \times 2}; \\
\Pi_{\pm}(t)\Pi_{\pm}^*(t) = \frac{1}{2(1 + \gamma^{-1})}(\langle \vec{\xi}, \vec{\xi} \rangle(1 + \gamma^{-1}) - (1 + \gamma^{-1})\langle \vec{\xi}, \vec{\xi} \rangle) = 0; \\
\Pi_{\pm}(t)\Pi_{\pm}^*(t) + \Pi_{\pm}(t)\Pi_{\pm}^*(t) = \frac{1}{2(1 + \gamma^{-1})} \left[ \left( \frac{(1 + \gamma^{-1})^2}{\langle \vec{\xi}, \vec{\xi} \rangle(1 + \gamma^{-1})} \right) + \\
+ \left( \frac{-\langle \vec{\xi}, \vec{\xi} \rangle}{\langle \vec{\xi}, \vec{\xi} \rangle(1 + \gamma^{-1})} \right) \right] = I_{4 \times 4},
\]
as was to be proved.
Appendix B. The Heisenberg equations for the polarization operator $\hat{S}_\mu$

Let us write the Heisenberg equation for the polarization operator:

$$\frac{d}{dt} \langle \Psi | \hat{S}_\mu | \Psi \rangle_D = \langle \Psi | \frac{\partial \hat{S}_\mu}{\partial t} | \Psi \rangle_D + i \hbar \langle \Psi | [\hat{H}_D, \hat{S}_\mu] | \Psi \rangle_D,$$

where

$$\hat{S}_\mu = (\hat{S}_0, \hat{S}), \quad \hat{S}_0 = \frac{1}{m_0 c} \langle \vec{\Sigma}, \hat{\vec{P}} \rangle, \quad \hat{S} = \rho_3 \vec{\Sigma} + \frac{1}{m_0 c} \rho_1 \hat{\vec{P}}, \quad (B.2)$$

$$\hat{H}_D = \hat{H}_0 - i \hbar \hat{H}_1, \quad \hat{H}_0 = c \langle \hat{\vec{A}}, \hat{\vec{P}} \rangle + \rho_3 m_0 c^2 + \epsilon \Phi,$$

$$\hat{H}_1 = \frac{ie_0 (g - 2)}{4m_0 c} [\rho_3 \langle \vec{\Sigma}, \vec{H} \rangle + \rho_2 \langle \vec{\Sigma}, \vec{E} \rangle]. \quad (B.3)$$

For this purpose we need following auxiliary propositions:

**Lemma 1** The following relation holds:

$$\frac{\partial \hat{S}_0}{\partial t} + i \frac{\hbar}{\epsilon} [\hat{H}_0, \hat{S}_0] = \frac{e}{m_0 c} \langle \vec{\Sigma}, \vec{E} \rangle. \quad (B.4)$$

**Proof.** Actually,

$$\frac{\partial \hat{S}_0}{\partial t} = - \frac{e}{m_0 c^2} \frac{1}{\epsilon} \frac{\partial A}{\partial t},$$

$$[\hat{H}_0, \hat{S}_0] = \frac{i \hbar}{m_0 c} \langle \vec{\Sigma}, \nabla \Phi \rangle.$$

Since $\vec{E} = -\nabla \Phi - \frac{i}{\epsilon} \frac{\partial \vec{A}}{\partial t}$, the property is proved.

**Lemma 2** The following relation holds:

$$\frac{\partial \hat{S}}{\partial t} + i \frac{\hbar}{\epsilon} [\hat{H}_0, \hat{S}] = \frac{e}{m_0 c} (\rho_1 \vec{E} + \vec{H} \times \vec{\Sigma}). \quad (B.5)$$

**Proof.** Actually,

$$\frac{\partial \hat{S}}{\partial t} = - \frac{e}{m_0 c^2} \frac{1}{\epsilon} \frac{\partial A}{\partial t},$$

$$[\hat{H}_0, \rho_1 \vec{\Sigma}] = c \{ (i \rho_2) [\vec{\Sigma} - \vec{\Sigma} \times \vec{P}] - i \rho_2 (\vec{P} + i \vec{\Sigma} \times \vec{P}) \} =$$

$$= -2ic \rho_2 \vec{P},$$

$$\frac{1}{m_0 c} [\hat{H}_0, \rho_1 \vec{P}] = 2ic \rho_2 \vec{P} + \frac{i \hbar}{m_0 c} \rho_1 \nabla \Phi + \frac{1}{m_0} \langle \vec{\Sigma}, \vec{P} \rangle.$$

Since $[\vec{P}_k, \vec{P}_l] = (i \hbar c) (A_{k,l} - A_{l,k}) = (i \hbar c) \varepsilon_{kli} \vec{H}_j$, where $\varepsilon_{kli}$ is absolutely antisymmetric tensor, the property is proved.

**Lemma 3** The following relation holds:

$$[\hat{H}_1, \hat{S}_0] = \frac{-e_0 (g - 2)}{2m_0 c^2} \left\{ \rho_3 \langle \vec{\Sigma}, \vec{H} \times \vec{P} \rangle + \rho_2 \langle \vec{\Sigma}, \vec{E} \times \vec{P} \rangle + \right.$$}

$$\left. + \frac{\hbar}{2} \rho_3 \text{div} \vec{H} + \rho_2 \text{div} \vec{E} + i \rho_3 \langle \vec{\Sigma}, \text{rot} \vec{H} \rangle + i \rho_3 \langle \vec{\Sigma}, \text{rot} \vec{E} \rangle \right\}. \quad (B.6)$$

**Proof.** Actually,

$$[\hat{H}_1, \hat{S}_0] = \frac{-e_0 (g - 2)}{2m_0 c^2} \frac{1}{m_0 c} \left\{ \rho_3 [\langle \vec{\Sigma}, \vec{H} \rangle \langle \vec{\Sigma}, \vec{P} \rangle] - + \rho_2 \langle \vec{\Sigma}, \vec{E} \rangle \langle \vec{\Sigma}, \vec{P} \rangle \right\},$$

$$[\langle \vec{\Sigma}, \vec{H} \rangle, \langle \vec{\Sigma}, \vec{P} \rangle] = - [\langle \vec{\Sigma}, \vec{P} \rangle] - i \langle \vec{\Sigma}, \vec{H} \times \vec{P} \rangle - i \langle \vec{\Sigma}, \vec{P} \times \vec{H} \rangle =$$

$$= 2i \langle \vec{\Sigma}, \vec{H} \times \vec{P} \rangle + i \hbar \text{div} \vec{H} - \hbar \langle \vec{\Sigma}, \text{rot} \vec{H} \rangle,$$

as was to be proved.
Lemma 4 The following relation holds:

\[ [\hat{\mathcal{H}}_1, \hat{S}] = \frac{e_0(g-2)}{2(m_0 c)^2} \left\{ (m_0 c) (-\hat{\Sigma} \times \hat{H} - \hat{E} \rho_1) + \rho_3(\hat{\Sigma}, \hat{E}) \hat{P} - \rho_2(\hat{\Sigma}, \hat{H}) \hat{P} - \frac{ih}{2} \text{grad}(\rho_3(\hat{\Sigma}, \hat{E}) + \rho_2(\hat{\Sigma}, \hat{H})) \right \}. \] (B.7)

Proof. Actually,

\[ [\hat{\mathcal{H}}_1, \rho_3\hat{\Sigma}] = \frac{ie_0(g-2)}{2m_0 c} \left\{ [(\hat{\Sigma}, \hat{H}), \hat{\Sigma}] + i \rho_1[(\hat{\Sigma}, \hat{E}), \hat{\Sigma}] + \frac{ih}{2} \text{grad}(\rho_3(\hat{\Sigma}, \hat{E}) + \rho_2(\hat{\Sigma}, \hat{H})) \right \}, \]

as was to be proved.

Lemma 5 The following relation holds:

\[ \Pi_+^t(t) \left\{ \frac{\partial \hat{S}_0}{\partial t} + \frac{i}{\hbar} [\hat{\mathcal{H}}_0, \hat{S}_0] \right \} \Pi_+(t) = \frac{e}{m_0 c} \gamma \left\{ \langle \hat{\sigma}, \hat{E} \rangle + \gamma \frac{1}{1 + \gamma^{-1}} \langle \hat{\sigma}, \beta \rangle \right \}. \] (B.8)

Proof follows directly from (A.9).

Lemma 6 The following relation holds:

\[ \Pi_+^t(t) \left\{ \frac{\partial \hat{S}}{\partial t} + \frac{i}{\hbar} [\hat{\mathcal{H}}_0, \hat{S}] \right \} \Pi_+(t) = \frac{e}{m_0 c} \gamma \left\{ \langle \hat{\sigma}, \hat{E} \rangle + \gamma \frac{1}{1 + \gamma^{-1}} \langle \hat{\sigma}, \beta \rangle \right \}. \] (B.9)

Proof follows directly from (A.9) and (A.12).

Lemma 7 The following relation holds:

\[ \Pi_+^t(t) [\hat{\mathcal{H}}_1, \hat{S}_0] \Pi_+(t) = -\frac{e_0(g-2)}{(m_0 c)^2} \left\{ \frac{1}{1 + \gamma^{-1}} \langle \hat{\sigma}, \hat{E} \rangle - \gamma^{-1} \hat{H} \times \hat{P} \right \} - \frac{1}{1 + \gamma^{-1}} \langle \hat{\sigma}, \beta \rangle \hat{H} \times \hat{P} \right \} + O(\hbar). \] (B.10)

Proof follows directly from (A.7) and (A.8).

Lemma 8 The following relation holds:

\[ \Pi_+^t(t) [\hat{\mathcal{H}}_1, \hat{S}] \Pi_+(t) = \frac{e_0(g-2)}{(m_0 c)^2} \left\{ -m_0 c \langle \hat{\sigma}, \beta \rangle \hat{E} - \gamma^{-1} \hat{H} \times \hat{\sigma} - \frac{1}{1 + \gamma^{-1}} \langle \hat{\sigma}, \beta \rangle \hat{H} \times \hat{\beta} \right \} - \left\{ \langle \hat{\sigma}, \beta \rangle \frac{\langle \hat{\beta}, \hat{E} \rangle}{1 + \gamma^{-1}} - \langle \hat{\sigma}, \beta \rangle \hat{E} - \langle \hat{\sigma}, \beta \times \hat{H} \rangle \hat{P} \right \} + O(\hbar). \] (B.11)

Proof follows directly from (B.7), (A.7) and (A.8).

Lemma 9 The following relations hold:

\[ \Pi_+^t(t) \hat{S}_0 \Pi_+(t) = \frac{1}{m_0 c} \left\{ \gamma^{-1} \langle \hat{\sigma}, \hat{P} \rangle + \frac{\langle \hat{\beta}, \hat{P} \rangle}{1 + \gamma^{-1}} \langle \hat{\sigma}, \beta \rangle \right \}, \] (B.12)

\[ \Pi_+^t(t) \hat{S} \Pi_+(t) = \frac{\hat{\sigma} - \langle \hat{\sigma}, \beta \rangle}{1 + \gamma^{-1}} + \frac{1}{m_0 c} \hat{P} \langle \hat{\sigma}, \beta \rangle. \] (B.13)

Proof follows directly from (A.9), (A.7) and (A.12).
Lemma 10  The vector
\[ \vec{a} = \vec{\zeta} + \frac{\gamma \vec{\beta}}{1 + \gamma^{-1}} \langle \vec{\zeta}, \vec{\beta} \rangle \]
satisfies the equation
\[ \dot{\vec{a}} = \frac{ge}{2m_0 c \gamma} \langle \vec{a}, \vec{\beta} \rangle \vec{E} + \vec{H} \times \vec{a} \]  \[ + \frac{e(g - 2)}{2m_0 c \gamma} \gamma \langle \vec{a}, \vec{\beta} \rangle \vec{E} + \langle \vec{a}, \vec{\beta} \rangle \langle \vec{\beta}, \vec{E} \rangle + \langle \vec{\beta}, \vec{a} \times \vec{H} \rangle \].  \hfill (B.14)

Proof. We express the vector \( \vec{\zeta} \) in terms of \( \vec{a} \) and obtain
\[ \vec{\zeta} = \vec{a} - \frac{\vec{\beta}}{1 + \gamma^{-1}} \langle \vec{a}, \vec{\beta} \rangle \]  \hfill (B.15)
(see (A.1) and (A.2)). Averaging (.9) with respect to \( J^{(1)} \) and setting \( \hbar \to 0 \), we substitute (.15) into the relation obtained. Then
\[ \frac{e}{m_0 c} \left\{ \gamma^{-1} \langle \vec{a}, \vec{\beta} \rangle \vec{E} + \gamma^{-1} \vec{H} \times \left( \vec{a} - \frac{\vec{\beta}}{1 + \gamma^{-1}} \langle \vec{a}, \vec{\beta} \rangle \right) \right\} = \frac{1}{1 + \gamma^{-1}} \gamma^{-1} \langle \vec{a}, \vec{\beta} \rangle \vec{H} \times \vec{\beta} \right\} = \frac{e}{m_0 c \gamma} \{ \langle \vec{a}, \vec{\beta} \rangle \vec{E} + \vec{H} \times \vec{a} \}.
\]
Similarly, we transform (.11)
\[ \frac{e(g - 2)}{(m_0 c)^2} \left\{ - m_0 c \langle \vec{a}, \vec{\beta} \rangle \vec{E} + \vec{H} \times \vec{a} \right\} \gamma^{-1} - \gamma m_0 c \vec{\beta} \left\{ \gamma^{-1} \langle \vec{a}, \vec{\beta} \rangle \frac{\langle \vec{\beta}, \vec{E} \rangle}{1 + \gamma^{-1}} + \langle \vec{a}, \vec{E} \rangle + \frac{\langle \vec{\beta}, \vec{E} \rangle}{1 + \gamma^{-1}} \langle \vec{a}, \vec{\beta} \rangle - \langle \vec{a}, \vec{\beta} \times \vec{H} \rangle \right\} = \frac{e(g - 2)}{2 m_0 c} \gamma^{-1} \{ \langle \vec{a}, \vec{\beta} \rangle \vec{E} + \vec{H} \times \vec{a} \} + \gamma \vec{\beta} \langle \vec{a}, \vec{E} \rangle + \langle \vec{a}, \vec{\beta} \rangle \langle \vec{\beta}, \vec{E} \rangle - \langle \vec{a}, \vec{\beta} \times \vec{H} \rangle \}.
\]
Substituting the latter relation into (2.30), we get (.14), as was to be proved.

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