Momentum map reduction for nonholonomic systems

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Abstract

This paper presents a reduction procedure for nonholonomic systems admitting suitable types of symmetries and conserved quantities. The full procedure contains two steps. The first (simple) step results in a Chaplygin system, described by an almost symplectic structure, carrying additional symmetries. The focus of this paper is on the second step, which consists of a Marsden–Weinstein–type reduction that generalises constructions in (Balseiro and Fernandez 2015 Nonlinearity 28 2873–912, Cortés Monforte 2002 Geometric, Control and Numerical Aspects of non-Holonomic Systems (Springer)). The almost symplectic manifolds obtained in the second step are proven to coincide with the leaves of the reduced nonholonomic brackets defined in (Balseiro and Yapu-Quispe 2021 Ann. Inst. Henri Poincare C 38 23–60). We illustrate our construction with several classical examples.

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1. Introduction

A basic fact in symplectic geometry, widely used in geometric mechanics, is that a symplectic manifold carrying a (free and proper) symplectic action gives rise to a Poisson bracket on the quotient manifold. Moreover, if the action is Hamiltonian then the Marsden–Weinstein reduced spaces \([41]\) of the symplectic manifold, at different values of the momentum map, coincide with unions of symplectic leaves of the quotient Poisson structure. Given an invariant Hamiltonian function, the dynamics in the quotient Poisson manifold restrict to leaves and hence can be studied by means of Marsden–Weinstein reduction. This paper presents some analogs of these results in the context of nonholonomic systems.

The study of nonholonomic systems with symmetries has a vast literature, see e.g. [10, 17, 19]. In our set-up, a nonholonomic system is geometrically described by an *almost* Poisson structure (the lack of integrability being a consequence of the constraints in velocities \([9, 33]\)) along with a Hamiltonian function. In the presence of symmetries, it is shown in \([7, 31]\) that, if the system admits suitable conserved quantities (called *horizontal gauge momenta* \([8, 26]\)), then there is a modification of the almost Poisson bracket that still codifies the nonholonomic dynamics and has the following key property: the corresponding reduced bracket on the quotient manifold, though generally not Poisson, gives rise to a foliation by almost symplectic leaves that are tangent to the reduced nonholonomic vector field. Our goal in this paper is to study, in this context, a Marsden–Weinstein–type reduction that produces these almost symplectic leaves. This procedure extends the ones in \([4, 9, 17, 43]\) in that we allow for more general conserved quantities as well as modifications of the almost symplectic structure (by *dynamical gauge transformations*) prior to reduction.

We now explain the framework and results in this article in more detail. A nonholonomic system is determined by a configuration manifold \(Q\), a Lagrangian \(L\) and a non-integrable distribution \(D\) on \(Q\) describing the permitted velocities. The submanifold \(\mathcal{M} \subset T^*Q\) given by the image of the Legendre transformation of \(D\) has a natural almost Poisson bracket \([16, 38, 46]\), called the *nonholonomic bracket*, and a Hamiltonian function \(H_M\) defined by \(L\). The nonholonomic dynamics on \(\mathcal{M}\) is determined by the ‘hamiltonian’ vector field of \(H_M\) with respect to the nonholonomic bracket, denoted by \(X_{nh}\). If the nonholonomic system has symmetries given by the (free and proper) action of a Lie group \(G\), then the nonholonomic bracket and the dynamics can be reduced to the quotient manifold \(\mathcal{M}/G\).

In our set-up, we assume that \(G\) admits a closed normal subgroup \(G_W\) so that the nonholonomic system is \(G_W\)-Chaplygin \([36]\) (see also \([30]\)). A consequence of this fact is that, setting \(\bar{Q} := Q/G_W\), the nonholonomic vector field descends to a vector field \(\bar{X}_{nh}\) on the cotangent bundle \(T^*\bar{Q}\), which is the hamiltonian vector field of the reduced hamiltonian function \(\bar{H}\) with respect to the nonholonomic bracket, denoted by \(X_{nh}\). If the nonholonomic system has symmetries given by the (free and proper) action of a Lie group \(G\), then the nonholonomic bracket and the dynamics can be reduced to the quotient manifold \(\mathcal{M}/G\).

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Our goal in this paper is to explain a further reduction of the nonholonomic system \((T^*\bar{Q}, \bar{\Omega}, \bar{H})\) making use of the canonical momentum map for the action of the remaining Lie group \(F := G/G_W\) on the cotangent bundle \(T^*\bar{Q}\). Here a first difficulty is that the vector field \(\bar{X}_{nh}\) is not tangent to the momentum level sets. We fix this problem by using conserved quantities of the system to find suitable \(F\)-invariant submanifolds that substitute the momentum level sets in the reduction procedure. A second difficulty is that \(\bar{\Omega}\) is not basic on these \(F\)-invariant submanifolds. This issue is resolved through a suitable modification of \(\bar{\Omega}\) by a special 2-form. We elaborate on these two key points below.
The F-invariant submanifolds carrying the nonholonomic dynamics. A central assumption in this work is the existence of the maximum possible amount of certain types of first integrals—horizontal gauge momenta—defined by the evaluation of the canonical momentum map \( \hat{J} \) on \( T^* \hat{Q} \) on given \( f \)-valued functions \( \eta_i \) on \( \hat{Q} \), for \( i = 1, \ldots, k \), where \( f \) is the Lie algebra of \( F \). We show that, for \( f \)-valued functions \( \mu = \sum c_i \mu_i \), where \( \mu_i \)'s are dual to the \( \eta_i \)'s and \( c_i \in \mathbb{R} \), we obtain \( F \)-invariant submanifolds

\[
\tilde{J}^{-1}(\mu) := \{ \alpha \in T^* \hat{Q} \mid J(\alpha) = \mu(x) \} \subset T^* \hat{Q}
\]

which are \( F \)-invariant and foliate \( T^* \hat{Q} \) in such a way that \( \tilde{X}_{\eta} \) is always tangent to them, see section 3.3, proposition 3.1.

The modification of \( \tilde{\Omega} \). What is behind the fact that the pull-back of \( \tilde{\Omega} \) to \( \tilde{J}^{-1}(\mu) \) does not descend to the quotient \( J^{-1}(\mu) / F \) is that the infinitesimal generator\(^5 \) \( (\eta_i)_{\tilde{\eta}} \tilde{\eta} \) of \( \eta_i \) is not necessarily the ‘hamiltonian’ vector field associated to the horizontal gauge momentum \( \hat{J}_\eta \). Following [7], we define a 2-form \( \mathcal{B} \) on \( T^* \hat{Q} \) that satisfies

\[
i_{(\eta_i)_{\tilde{\eta}}} (\tilde{\Omega} + \mathcal{B}) = d\hat{J}_\mu,
\]

as well as the dynamical condition \( k_{\hat{X}_{\eta}} \mathcal{B} = 0 \). Note that, by this last condition, our nonholonomic system is equivalently described by the triple \( (T^* \hat{Q}, \tilde{\Omega} + \mathcal{B}, H) \). We prove in theorem 3.8 that the pull-back of the 2-form \( \tilde{\Omega} + \mathcal{B} \) to the manifold \( J^{-1}(\mu) \) is basic and hence defines an almost symplectic form \( \alpha_\mu^B \) on \( J^{-1}(\mu) / F \).

Let us stress that an important point in our construction is that, in general, the \( \mu \)'s are suitable \( f \)-valued functions, not just fixed elements of \( f \) as in the usual hamiltonian case. This is essential for the reduction to be compatible with the nonholonomic dynamics. Comparing with previous constructions, we note that in [4, 9] the conserved quantities are assumed to be defined by fixed elements in the Lie algebra, while in [17] the reduction procedure considers \( f \)-valued function but, due to the lack of the 2-form \( \mathcal{B} \), it was not possible to define a reduced 2-form on the quotients \( J^{-1}(\mu) / F \). The 2-form \( \mathcal{B} \) was defined in [7, 31] in the context of hamiltonisation, and its explicit expression permits a better understanding of the resulting ‘Marsden–Weinstein’ reduced spaces even in the specific cases studied in previous works. In particular, inspired by the hamiltonian case [1, 39] and using the shift-trick, we show that the almost symplectic manifolds \( (J^{-1}(\mu) / F, \omega_\mu^B) \) are diffeomorphic to the manifold \( T^* (\hat{Q} / F) \) with its canonical symplectic 2-form modified by a term \( \tilde{B}_\mu \) that only depends on the 2-form \( \mathcal{B} \), see theorem 4.3.

In section 5, we relate the almost symplectic reduced spaces obtained in our construction with an almost Poisson bracket on the \( M / G \) given by the reduction of a modification of the non-holonomic bracket on \( M \) considered in [7, 31]. As shown in these papers, when a nonholonomic system admits the maximum amount of horizontal gauge momenta, the gauge transformation of the nonholonomic bracket on \( M \) by a suitable 2-form \( \mathcal{B} \) generates a new bracket whose reduction by symmetries gives an almost Poisson bracket \( \{ \cdot, \cdot \}_\text{red} \) on \( M / G \) that admits an almost symplectic foliation. We show in theorem 5.1 that its leaves agree with the connected components of the almost symplectic reduced spaces of theorem 3.8. Having a Marsden–Weinstein–type description of the almost symplectic foliation associated to the reduced bracket \( \{ \cdot, \cdot \}_\text{red} \) is useful to study the dynamics restricted to leaves, to find conformal factors for the reduced brackets \( \{ \cdot, \cdot \}_\text{red}^B \), as well as to study Routh reduction, integrability, Hamilton–Jacobi

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\(^5\) The infinitesimal generator of a \( f \)-valued function \( \eta \) is defined at \( \alpha_x \in T^*_x \hat{Q} \) as the infinitesimal generator of \( \eta(x) \in f \).
theory and even numerical methods (e.g. variational integrators), see [12, 13, 20, 25, 28, 37, 40, 42, 47].

Besides the Chaplygin ball (that was also treated in [4]), in section 6 we study many other examples that could not be treated in [4, 9, 17], starting from the simple example of the nonholonomic particle, the snakeboard [6, 11] and the more sophisticated one describing a solid of revolution rolling on plane [3, 19, 31].

2. Nonholonomic systems and first step reduction

In this section we will define the basic concepts around nonholonomic systems with symmetries and, in particular, the vertical symmetry condition which permits the reduction in two steps.

2.1. Nonholonomic systems with symmetries

A nonholonomic system is a mechanical system on a manifold $Q$ with a lagrangian function $L : TQ \to \mathbb{R}$ and (linear) constraints in the velocities that define a (constant rank) nonintegrable distribution $D$ on $Q$. Throughout this paper we assume that the lagrangian $L$ is of mechanical type: $L = \frac{1}{2}k - U$ where $k$ is the kinetic energy metric and $U$ the potential.

Next, we write the nonholonomic equations of motion in the hamiltonian framework following [9]. The Legendre transformation $\kappa : TQ \to T^*Q$ given by $\kappa^g(X)(Y) := \kappa(X,Y)$, with $X, Y \in TQ$, defines the submanifold $\mathcal{M} := \kappa^g(D)$ of $T^*Q$ and $\tau_M := \tau|_\mathcal{M} : \mathcal{M} \to Q$ is a vector subbundle of $\tau : T^*Q \to Q$. The distribution $D$ induces a (nonintegrable) distribution $\mathcal{C}$ on $\mathcal{M}$, with fiber at $m \in \mathcal{M}$, given by

$$C_m := \{v_m \in T_m \mathcal{M} : T\tau_M(v_m) \in D_q, \text{ for } q = \tau_M(m) \in Q\}. \quad (2.2)$$

Let $H : T^*Q \to \mathbb{R}$ be the hamiltonian function associated to the lagrangian $L$ and $\Omega_Q$ the canonical 2-form on $T^*Q$. Considering $\epsilon : \mathcal{M} \to T^*Q$ the natural inclusion, we denote by $\Omega_M := \epsilon^*\Omega_Q$ and $H_M := \epsilon^*H$ the pull backs of $\Omega_Q$ and $H$ to the submanifold $\mathcal{M}$, respectively. Following [9], the nonholonomic dynamics is described by the integral curves of the nonholonomic vector field $X_{\text{sh}}$ defined on $\mathcal{M}$ given by $i_{X_{\text{sh}}}\Omega_M|_C = dH_M|_C$, where $(\cdot)|_C$ is the point-wise restriction to $C$. During this paper, we will use the triple $(\mathcal{M}, \Omega_M|_C, H_M)$ to define a nonholonomic system.

A free and proper action of a Lie group $G$ on $Q$ is a symmetry of the nonholonomic system if its tangent lift leaves the lagrangian $L$ and the distribution $D$ invariant, or equivalently, if the cotangent lift of the action leaves $\mathcal{M}$ and $H$ invariant. In this case the vector field $X_{\text{sh}}$ is $G$-invariant as well: $T\Psi_g(X_{\text{sh}}(m)) = X_{\text{sh}}(\Psi_g(m))$ for all $m \in \mathcal{M}$, $g \in G$ and $\Psi : G \times \mathcal{M} \to \mathcal{M}$ is the induced action on $\mathcal{M}$.

The $G$-symmetry satisfies the dimension assumption if for each $q \in Q$, $T_qQ = D_q + V_q$, where $V_q$ is the tangent space to the $G$-orbit on $Q$ at $q$ (respectively at each $m \in \mathcal{M}$, $T_m\mathcal{M} = C_m + V_m$ where $V_m$ is the tangent to the orbit associated to the $G$-action on $\mathcal{M}$ at $m$). As usual, we denote by $S$ the (constant rank) distribution on $Q$ defined, for each $q \in Q$, by $S_q := D_q \cap V_q$ (respectively $S$ on $\mathcal{M}$ defined by $S_m := C_m \cap V_m$). Let $\mathfrak{g}$ be the Lie algebra associated to $G$ and consider the trivial bundle $Q \times \mathfrak{g} \to Q$ whose sections can be thought as $\mathfrak{g}$-valued functions, that is, if $\xi \in \Gamma(Q \times \mathfrak{g})$, $\xi_q = \xi(q) \in \mathfrak{g}$. Then, $S$ induces the subbundle $\mathfrak{g}_S \to Q$ of $Q \times \mathfrak{g} \to Q$ with fiber $(\mathfrak{g}_S)_q := \{\xi_q \in \mathfrak{g} : (\xi_q)_q(q) \in S_q\}$, where $(\xi_q)_q(q)$ is the infinitesimal generator of the element $\xi_q \in \mathfrak{g}$ at $q$ (see e.g. [11]). For short, we may denote by $\xi_S(q) := (\xi_q)_q(q)$. The bundle $\mathfrak{g}_S \to Q$ has the same rank as $S$, i.e. $k := \text{rank}(S) = \text{rank}(\mathfrak{g}_S)$. 

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The nonholonomic momentum map \cite{11} is the bundle map \( J^h : \mathcal{M} \to \mathfrak{g}_k^\ast \) given by
\[
\langle J^h(m), \xi(q) \rangle := \iota_{\xi_M} \Theta_M(m) \quad \text{with} \quad \xi \in \Gamma(g_k),
\]
where \( q = \tau_M(m), \Theta_M := t^*\Theta_g : \mathcal{M} \to T^*\mathcal{M} \) is the natural inclusion and \( \Theta_g \) is the Liouville 1-form on \( T^*\mathcal{Q} \). If a function of the type \( J_\xi := \langle J^h, \xi \rangle \), where \( \langle J^h, \xi \rangle(m) = \langle J^h(m), \xi(q) \rangle \) is conserved, i.e. \( X_\xi(J_\xi) = 0 \), then the function \( J_\xi \) is called horizontal gauge momentum \cite{8} and the associated element \( \xi \in \Gamma(g_k) \) is a horizontal gauge symmetry.

**Remark 2.1.** The general existence of horizontal gauge momenta is still an open problem and what is usually done is to assume their existence when needed (for further discussion see \([6, 25, 26]\)).

**Lemma 2.2.** The function \( J_\xi = \langle J^h, \xi \rangle \) is G-invariant on \( \mathcal{M} \) if and only if the section \( \xi \) on \( g_k \to Q \) is Ad-invariant: that is for \( q \in Q \) and \( g \in G \), \( \text{Ad}_g(\xi(\Psi_g(q))) = \xi(q) \), where \( \Psi_g : Q \to Q \) is the G-action on \( Q \).

**Proof.** The function \( J_\xi = \langle J^h, \xi \rangle \) is G-invariant if and only if \( J_\xi(m) = J_\xi(\Psi_g(m)) \) which means that, for \( m_q \in \mathcal{M}_q \subset T_q^*\mathcal{Q} \),
\[
\langle m_q, (\xi(q))_0 \rangle = \langle \Psi_g^{-1}(m_q), (\xi(\Psi_g(q)))_0 \rangle = \langle m_q, \text{Ad}_{\Psi_g^{-1}}(\xi(\Psi_g(q)))_0 \rangle.
\]
Therefore, \( \xi(q) = \text{Ad}_{\Psi_g^{-1}}(\xi(\Psi_g(q)))_0 \).

2.2. Chaplygin systems with an extra symmetry

**Definition 2.3.** \cite{2} Let \( (\mathcal{M}, \Omega_M, G, H_M) \) be a nonholonomic system with a G-symmetry satisfying the dimension assumption, then

(a) a distribution \( W \) is a vertical complement of the constraints \( D \) if \( TQ = D \oplus W \) for \( W \subset V \).
(b) A vertical complement \( W \) satisfies the vertical symmetry condition if there exists a closed normal subgroup \( G_w \) of \( G \) so that for each \( q \in Q \), \( W_q = T_q(\text{Orb}_{G_w}(q)) \).

A vertical complement \( W \) always exists, however the vertical symmetry condition implies that the system is \( G_w \)-Chaplygin and hence the \( G_w \)-reduction of the nonholonomic dynamics is described by the vector field \( X_\text{nh} \) on the quotient manifold \( \mathcal{M}/G_w \simeq T^*\tilde{Q} \), for \( \tilde{Q} := Q/G_w \), given by
\[
\iota_{X_\text{nh}} \tilde{\Omega} = d\tilde{H}, \quad \text{with} \quad \tilde{\Omega} := \Omega_{\tilde{Q}} - B_{\mathcal{K}},
\]
where \( \tilde{H} \) is the reduced hamiltonian and \( B_{\mathcal{K}} \) is the 2-form on \( T^*\tilde{Q} \) defined as follows: let \( K_w \) be curvature (a \( \mathfrak{m} \)-valued 2-form on \( Q \)) associated to the principal connection \( A_w : TQ \to \mathfrak{m} \) where \( \mathfrak{m} \) is the Lie algebra associated to \( G_w \) and \( \text{Ker} A_w = D \), that is \( K_w(\cdot, \cdot) = dA_w(\cdot, \cdot) := dA_w(P_D(\cdot), P_D(\cdot)) \), where \( P_D : D \oplus W \to D \) is the projection to the first factor. Following \cite{36}, we define the 2-form \( \langle J, K_w \rangle \) as the natural pairing of the canonical momentum map \( J : \mathcal{M} \to \mathfrak{m}^\ast \) with the \( \mathfrak{m} \)-valued 2-form \( K_w := \tau_{\mathcal{K},w} K_w \) (here \( (\cdot, \cdot) \) denotes the pairing between \( \mathfrak{m}^\ast \) and \( \mathfrak{m} \)).

The 2-form \( \langle J, K_w \rangle \) was proven to be basic with respect to the principal bundle \( \rho_\alpha : \mathcal{M} \to T^*\tilde{Q} \) and therefore \( B_{\mathcal{K}} \) is the 2-form on \( T^*\tilde{Q} \) such that \( \rho_\alpha^* B_{\mathcal{K}} = \langle J, K_w \rangle \).

If the Lie group \( G_w \) is a closed normal subgroup of \( G \), the quotient \( F := G/G_w \) is a Lie group and the (partially) reduced system \( (T^*\tilde{Q}, \tilde{\Omega}, \tilde{H}) \) inherits a F-symmetry (it is straightforward to see that the F-action is free and proper and leaves the system invariant).
Consider the bundles $g_s : Q \rightarrow \tilde{Q}$ and $\tilde{Q} \times \mathfrak{f} \rightarrow \tilde{Q}$, and the projection to the orbits $\rho_{\tilde{Q}} : \tilde{Q} \rightarrow \tilde{Q}$.

(i) There is a one-to-one correspondence between the $Ad$-invariant sections on $g_s \rightarrow Q$ and $\tilde{Q} \times \mathfrak{f} \rightarrow \tilde{Q}$ so that if $\xi \in \Gamma(g_s)$ then there is a unique $\eta \in \Gamma(\tilde{Q} \times \mathfrak{f})$ such that $T\rho_{\tilde{Q}}(\xi_Q(q)) = \eta_Q(\rho_{\tilde{Q}}(q))$.

(ii) The choice of a horizontal $G$-invariant distribution $H_\sigma \subset D$ such that $TQ = H_\sigma \oplus V$, induces a $F$-invariant splitting on $T\tilde{Q} = H_\sigma \oplus \tilde{V}_f$ where $\tilde{V}_f$ is the tangent to the $F$-orbit on $\tilde{Q}$ and $H_\sigma := T\rho_{\tilde{Q}}(H_\sigma)$. In other words, an equivariant principal connection $A : T\tilde{Q} \rightarrow \mathfrak{g}$ induces an equivariant principal connection $\tilde{A} : T\tilde{Q} \rightarrow \mathfrak{f}$ on $\tilde{Q}$ so that $\rho_{\tilde{Q}} \circ \tilde{A} = \tilde{A} \circ T\rho_{\tilde{Q}}$.

Proof. (i) If $\xi \in \Gamma(g_s)$ is $Ad$-invariant, then the vector field $\xi_Q$ is invariant and for each $x = \rho_{\tilde{Q}}(q) \in \tilde{Q}$, $X(x) = T\rho_{\tilde{Q}}(\xi_Q(q)) \in T_x\tilde{Q}$. Using that $\rho = \rho_{\tilde{Q}/F} \circ \rho_{\tilde{Q}} : M \rightarrow M/G$ for $\rho_{\tilde{Q}/F} : \tilde{Q} \rightarrow \tilde{Q}/F$ the orbit projection, we have that $0 = T\rho(\xi_Q(q)) = T\rho_{\tilde{Q}/F}(X(x))$ and hence $X(x) \in (V_f)_x$, where $V_f$ is the tangent to the $F$-orbit on $\tilde{Q}$. Therefore, for each $x \in \tilde{Q}$ there is $\eta(x) \in \mathfrak{f}$ such that $T\rho_{\tilde{Q}}(\xi_Q(q)) = X(x) = \eta_Q(x)$.

Conversely, if $\eta \in \Gamma(\tilde{Q} \times \mathfrak{f})$, then there exists an (unique) invariant vector field $Y$ on $Q$ such that $Y(q) \in D_q$ and $T\rho_{\tilde{Q}}(Y(q)) = \eta_Q(q)$ (Ker$T\rho_{\tilde{Q}} = W$). Since $T\rho(\eta_Q(q)) = T\rho_{\tilde{Q}/F}(T\rho_{\tilde{Q}}(Y(q)) = T\rho_{\tilde{Q}/F}(\eta_Q(x)) = 0$, then $Y(q) \in V_q$ and therefore $Y(q) \in S_q$. Then, for each $q \in Q$, there is an element $\xi(q) \in g_Q$ such that $Y(q) = \xi_Q(q)$. Moreover $\xi \in \Gamma(g_s)$ is $Ad$-invariant.

(ii) Item (i) asserts that rank$(S) = \text{rank}(V_f)$ and moreover, since Ker$T\rho_{\tilde{Q}} = T(\text{Orb}_{\text{G}_Q})$, we have that rank$(H_\sigma) = \text{rank}(\tilde{H}_\sigma)$ and hence we conclude that $T\tilde{Q} = H_\sigma \oplus \tilde{V}_f$ and $\tilde{A} \circ T\rho_{\tilde{Q}} = \rho_{\tilde{Q}} \circ \tilde{A}$.

2.2.1. The conserved quantity assumption. Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}, [\mathcal{H}_{\mathcal{M}}])$ with a $G$-symmetry and recall that $S = D \cap V$. Next we will make a fundamental assumption that will be used the rest of the paper: the nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{M}}, [\mathcal{H}_{\mathcal{M}}])$ admits $k = \text{rank}(S)$ $G$-invariant (functionally independent) horizontal gauge momenta $\{J_1, \ldots, J_k\}$. Since the corresponding horizontal gauge symmetries $\zeta_i \in \Gamma(g_s)$, such that $J_i := [J^\mu, \zeta_i]$, are linearly independent and globally defined, they define a global basis of $(Ad$-invariant) sections of $g_s \rightarrow Q$ denoted by

$$\mathcal{B}_{\text{HGS}} = \{\zeta_1, \ldots, \zeta_k\}.$$  \hfill (2.4)

As a consequence of lemma 2.4, the global basis $\mathcal{B}_{\text{HGS}}$ induces a corresponding basis of global sections

$$\tilde{\mathcal{B}}_{\text{HGS}} = \{\eta_1, \ldots, \eta_k\},$$  \hfill (2.5)

of the bundle $\tilde{Q} \times \mathfrak{f} \rightarrow \tilde{Q}$ where, for each $i = 1, \ldots, k$, $(\eta_i)(x) := g_\mathfrak{f}((\xi_i)(q))$ for $q \in Q$ and $x = \rho_{\tilde{Q}}(q) \in \tilde{Q}$. We will often see the elements $\eta_i \in \mathcal{B}_{\text{HGS}}$ as $\mathfrak{f}$-valued functions on $\tilde{Q}$. Associated to the basis $\tilde{\mathcal{B}}_{\text{HGS}}$ we can define the functions $\{\tilde{J}_1, \ldots, \tilde{J}_k\}$ on $T^*\tilde{Q}$, given by

$$\tilde{J}_i := \Theta_{\tilde{Q}}(\mathfrak{f}) \Theta_{\tilde{Q}}(\tilde{J}_k),$$  \hfill (2.6)

where $\Theta_{\tilde{Q}}$ is the canonical 1-form on $T^*\tilde{Q}$.  

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**Proposition 2.5.** Recalling that $\rho_{\alpha_{\nu}} : \mathcal{M} \to T^*\tilde{Q}$ is the orbit projection, we have that

(i) the functions $\tilde{J}_i$ on $T^*\tilde{Q}$ are functionally independent and $\rho_{\alpha_{\nu}}^* \tilde{J}_i = J_i$,

(ii) the functions $\tilde{J}_i$ are conserved by the partially reduced dynamics $\tilde{\mathcal{X}}_{\nu_{\alpha}}$.

**Proof.** (i) Let $\xi_i$ be the $Ad$-invariant horizontal gauge symmetry in (2.4) and $\eta_i$ be the corresponding $\mathfrak{f}$-valued functions on $\tilde{Q}$ defined in (2.5). Using (2.6) and that $(\rho_{\alpha_{\nu}}^* \Theta_Q - \Theta_M)|_{\mathcal{C}} = 0$ we obtain that

$$\rho_{\alpha_{\nu}}^* (\tilde{J}_i) = \rho_{\alpha_{\nu}}^* (i_{\eta_i} r_Q \Theta_Q) = i_{(\xi_i)_{\mathcal{M}}} \rho_{\alpha_{\nu}}^* \Theta_Q = i_{(\xi_i)_{\mathcal{M}}} \Theta_M = J_i.$$ 

(ii) It is a consequence of item (i). \hfill $\square$

**Definition 2.6.** The functions $\tilde{J}_i = i_{(\eta_i)_{\mathcal{M}}} \Theta_Q$ for $\eta_i \in \hat{\mathcal{B}}_{HGS}$, are called the partially reduced horizontal gauge momenta and the corresponding $\mathfrak{f}$-valued functions $\eta_i$ are the partially reduced horizontal gauge symmetries.

We conclude that, if the nonholonomic system $(\mathcal{M}, \Omega_M, \mathcal{H}_M)$ admits $k$ (functionally independent) $G$-invariant horizontal gauge momenta, then the partially reduced system $(T^*\tilde{Q}, \Omega_{\tilde{Q}}, \tilde{H})$ inherits $k = \dim(\mathfrak{f})$ partially reduced horizontal gauge momenta.

The vertical space $\tilde{V}_r$ is generated by $\hat{\mathcal{B}}_{HGS}$ and denoting by $\tilde{\mathcal{Y}}^i$ the 1-forms on $\tilde{Q}$ so that $\tilde{\mathcal{Y}}^i ((\eta_i)_{\tilde{Q}}) = \delta_j$ and $\tilde{\mathcal{Y}}^i|_{\mathcal{K}_0} = 0$, the connection $\tilde{\Lambda} : T\tilde{Q} \to \mathfrak{f}$ can be written as

$$\tilde{\Lambda} = \tilde{\mathcal{Y}}^i \otimes \eta_i.$$ 

(2.7)

### 3. Momentum map reduction

In this section, we work with the system $(T^*\tilde{Q}, \tilde{\Omega}, \tilde{H})$ defined in (2.3) and the corresponding symmetry group $F$. We assume the existence of $k = \dim(\mathfrak{f})$ partially reduced horizontal gauge momenta $\{\tilde{J}_1, \ldots, \tilde{J}_k\}$. With these ingredients we will define an almost symplectic foliation—where the reduced nonholonomic dynamics lives—through a Marsden–Weinstein-type reduction.

#### 3.1. The canonical momentum bundle map

Let us consider the canonical momentum map $\tilde{J} : T^*\tilde{Q} \to \mathfrak{f}^*$ as a bundle map, defined by

$$\tilde{J}(\alpha_x), \eta(x)) = i_{\eta_{\tilde{Q}}} \Theta_g(\alpha_x),$$

(3.8)

for $\alpha_x \in T_{Q} \tilde{Q}$ and $\eta(x) \in \mathfrak{f}$ and hence the function $\tilde{J}_\eta \in C^\infty(T^*\tilde{Q})$ is given by

$$\tilde{J}_\eta(\alpha_x) = (\tilde{J}, \eta)(\alpha_x) := (\tilde{J}(\alpha_x), \eta(x)).$$

(3.9)

The partially reduced horizontal gauge momenta $\{\tilde{J}_1, \ldots, \tilde{J}_k\}$ given in definition 2.6 associated to the basis $\mathfrak{g}_H = \{\eta_1, \ldots, \eta_k\}$, are described by the canonical momentum bundle map $\tilde{J}_i := \tilde{J}_{\eta_i} = (\tilde{J}, \eta_i)$, for $i = 1, \ldots, k$.

We denote by

$${\mathfrak{g}}^*_{\text{HGS}} = \{\alpha^1, \ldots, \alpha^k\},$$

(3.10)
the dual basis of $f^*$-valued functions on $\tilde{Q}$ associated to $\tilde{\mathcal{B}}_{\text{HGS}}$, that is, for each $i = 1, \ldots, k$ the $\mu^i$ are $f^*$-valued functions on $\tilde{Q}$ (or sections of the bundle $\tilde{Q} \times f^* \to \tilde{Q}$), such that, at each $x \in \tilde{Q}$, $(\mu^i)(x) \in f^*$ and $\langle \mu^i(x), \eta(x) \rangle = \delta_{ij}$ for $\langle \cdot, \cdot \rangle$ the natural pairing between $f$ and $f^*$.

Let us consider a $f^*$-valued function on $\tilde{Q}$ given by $\mu = c_i \mu^i$ for $c_i$ constants in $\mathbb{R}$ and the level set

$$\tilde{J}^{-1}(\mu) := \{ \alpha_i \in T^*\tilde{Q} : \tilde{J}(\alpha_i) = \mu(x) \}.$$

**Proposition 3.1.** Let $\mu = c_i \mu^i$ be a $f^*$-valued function for $c_i$ constants in $\mathbb{R}$ and $\mu^i \in \tilde{\mathcal{B}}_{\text{HGS}}$. Then the inverse image $\tilde{J}^{-1}(\mu) \subset T^*\tilde{Q}$ coincides with the common level sets of the (partially reduced) horizontal gauge momenta $\tilde{J}_1, \ldots, \tilde{J}_k$ at $c_1, \ldots, c_k$ respectively, i.e., $\tilde{J}^{-1}(\mu) = \bigcap_i \tilde{J}_i^{-1}(c_i)$, and hence $\tilde{J}^{-1}(\mu)$ is a $F$-invariant submanifold of $T^*\tilde{Q}$. Moreover, the collection (of connected components) of the manifolds $\tilde{J}^{-1}(\mu)$ for $\mu \in \text{span}_\mathbb{R}\{\mu^i\}$ defines a foliation of the manifold $T^*\tilde{Q}$.

**Proof.** If $\alpha_i \in T^*\tilde{Q}$ such that $\tilde{J}(\alpha_i) = \mu(x)$, then $\tilde{J}_i(\alpha_i) = \langle \tilde{J}(\alpha_i), \eta_i(x) \rangle = c_i$ for all $i = 1, \ldots, k$ and hence $\tilde{J}^{-1}(\mu) = \bigcap_i \tilde{J}_i^{-1}(c_i)$. Let us consider the $F$-invariant submersion $J := (\tilde{J}_1, \ldots, \tilde{J}_k) : T^*\tilde{Q} \to \mathbb{R}^k$. Since $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, $J^{-1}(c) = \tilde{J}^{-1}(\mu)$ then, for each $\mu \in \text{span}_\mathbb{R}\{\mu^i\}$, $\tilde{J}^{-1}(\mu)$ is a $F$-invariant manifold and the collection of connected components of $\tilde{J}^{-1}(\mu)$ defines a foliation of $T^*\tilde{Q}$. \hfill $\Box$

As a consequence of the previous proposition, we conclude that the (partially reduced) nonholonomic vector field $\tilde{X}_\mu = T\tilde{J}^{-1}(\mu)\tilde{J}_i$ is tangent to the manifolds $\tilde{J}^{-1}(\mu)$ for $\mu = c_i \mu^i$.

However, it is important to note that the bundle map $J$ does not behave as a momentum map on $(T^*\tilde{Q}, \tilde{\Omega})$, not even in the coordinates given by the horizontal gauge symmetries, in the sense that the vector fields $(\eta_i)_\tau$ might not be hamiltonian vector fields associated to the functions $\tilde{J}_i$ (i.e., $(\eta_i)_\tau \cdot \tilde{\Omega}$ can be different from $d\tilde{J}_i$). This observation has a fundamental reflect when we want to study a ‘Marsden–Weinstein reduction’: the pull back of $\tilde{\Omega}$ to the manifold $\tilde{J}^{-1}(\mu)$ is not basic with respect to the bundle $\tilde{J}^{-1}(\mu) \to \tilde{J}^{-1}(\mu)/F$. To solve this problem, we will consider a gauge transformation by a 2-form $B$ (as in [5, 7, 29]) so that we have the desired relation between $(\eta_i)_\tau$, $\tilde{\Omega}$ and the functions $\tilde{J}_i$.

### 3.2. The suitable dynamical gauge transformation $B$

Consider the partially reduced nonholonomic system $(T^*\tilde{Q}, \tilde{\Omega}, \tilde{H})$ and a 2-form $\tilde{B}$ on $T^*\tilde{Q}$. Following [45] (see also [24]) a gauge transformation by a 2-form $\tilde{B}$ of the 2-form $\tilde{\Omega}$ is just considering the 2-form $\Omega + \tilde{B}$.

**Definition 3.2.** [5] A 2-form $\tilde{B}$ on $T^*\tilde{Q}$ induces a dynamical gauge transformation of $\tilde{\Omega}$ if $\tilde{B}$ is semi-basic with respect to the bundle $\tau_\tilde{Q} : T^*\tilde{Q} \to \tilde{Q}$ and $\tilde{X}_\mu \cdot \tilde{B} = 0$.

Definition 3.2 guarantees that $\tilde{\Omega} + \tilde{B}$ is nondegenerate and that the (partially reduced) dynamics $\tilde{X}_\mu$ on $T^*\tilde{Q}$ is also defined by $\tilde{X}_\mu (\tilde{\Omega} + \tilde{B}) = d\tilde{H}$.

**Remark 3.3.** The original definition of gauge transformation appeared in [45] and the dynamical gauge transformation by a 2-form in [5]. In section 5 we will relate definition 3.2 with the nonholonomic bracket.

The goal of considering this extra term given by a 2-form $\tilde{B}$ is that there is a special choice of a 2-form $\tilde{B}$ such that the behaviour of the almost symplectic manifolds $(T^*\tilde{Q}, \tilde{\Omega})$ and $(T^*\tilde{Q}, \tilde{\Omega} + \tilde{B})$ are different: that is, $(\eta_i)_\tau \cdot \tilde{\Omega}$ becomes the hamiltonian vector field of $\tilde{J}_i$ with
respect to $\tilde{\Omega} + \tilde{B}$. The explicit expression of $\tilde{B}$ comes from a 2-form $B$ on $\mathcal{M}$ presented in [7]. More precisely, let $(\mathcal{M}, \Omega_{\mathcal{M}}|_{\mathcal{H}_{\mathcal{M}}})$ be a nonholonomic system with a $G$-symmetry satisfying the dimension assumption. Let $W$ be a vertical complement of the constraints (definition 2.3) and choose a horizontal space $H_\omega \subset D$ so that

$$TQ = H_\omega \oplus V = H_\omega \oplus S \oplus W,$$

(3.11)

(observe that $D = H_\omega \oplus S$). We assume that the nonholonomic system admits $k = \text{rank}(S)$ $G$-invariant (functionally independent) horizontal gauge momenta $\{J_1, \ldots, J_k\}$. The corresponding horizontal gauge symmetries $\{\zeta_1, \ldots, \zeta_k\}$ define the vector fields $\{Y_1, \ldots, Y_k\}$ on $Q$ given by $Y_i := (\zeta_i)_Q$ and the globally defined 1-forms $Y^i$ on $Q$ so that $Y^i|_{H_\omega} = Y^i|_W = 0$ and $Y^i(Y_j) = \delta_{ij}$.

We define the 2-form $B_1$ on $\mathcal{M}$ as

$$B_1 := (J, K_w) + J_i d^2 Y^i$$

(3.12)

where, for $i = 1, \ldots, k$, $Y^i = \tau^*_\mathcal{M} Y^i$ and $d^2 Y^i = \tau^*_\mathcal{M} d^2 Y^i$ with $d^2 Y^i(\cdot, \cdot) = dY^i(P_D(\cdot), P_D(\cdot))$.

The splitting (3.11) induces a splitting on $T\mathcal{M}$ so that $T\mathcal{M} = H_\omega \oplus V = H_\omega \oplus S \oplus W$, where $S$ is defined in section 2.1, $(H_\omega)_m = \{v_m \in T_m \mathcal{M} : T\tau_{\mathcal{M}}(v_m) \in (H_\omega)_q\}$ and $(V_m = \{v_m \in V_m : T\tau_{\mathcal{M}}(v_m) \in W_m\}$ at each $m \in \mathcal{M}$, $q = \tau_{\mathcal{M}}(m)$. Let us denote by $A : T\mathcal{M} \to \mathfrak{g}$ the principal connection with corresponding horizontal space $H_\omega$ and denote by $P_V : H_\omega \oplus V \to V$ the projection to the second factor. Finally we also define the 2-form $\mathcal{B}$ on $\mathcal{M}$ as

$$\mathcal{B} := (J, K_V) - \frac{1}{2} (\kappa_\mathfrak{g} \wedge A P_V(X_m)) [K_w + d^2 Y^i \otimes \zeta_i]|_{H_\omega},$$

(3.13)

where $K_V$ is the curvature of $A$, and $\kappa_\mathfrak{g}$ is the $\mathfrak{g}$-valued 1-form on $\mathcal{M}$ given, at each $X \in T\mathcal{M}$, by $\kappa_\mathfrak{g}(X, \eta) = \kappa(T\tau_{\mathcal{M}}(X), \eta_\mathcal{M})$, for $\eta \in \mathfrak{g}$; for more details see [7, section 3.3].

**Proposition 3.4.** [7] The 2-forms $B_1$ and $\mathcal{B}$ defined in (3.12) and (3.13), respectively, are basic with respect to the bundle $\tau_{\mathcal{M}} : \mathcal{M} \to Q$ and $G$-invariant. Moreover, the 2-form $B := B_1 + \mathcal{B}$ satisfies the dynamical condition $\iota_{\mathcal{B}} B = 0$.

We assume now that the vertical complement $W$ verifies the vertical symmetry condition and, following section 2, we consider the system $(T^*Q, \tilde{\Omega}, \tilde{H})$ with the partially reduced horizontal gauge momenta $\{J_1, \ldots, J_k\}$. Next, we will see that $B_1$ and $\mathcal{B}$ descend to well defined 2-forms on $T^*\tilde{Q}$ and, in particular, $B_1$ has an explicit expression on $T^*\tilde{Q}$.

Following lemma 2.4(ii) the splitting (3.11) induces the connection $\tilde{\Lambda} = \tilde{\Omega} \otimes \eta$ on $\tilde{Q}$ as in (2.7). Define the 1-forms $\tilde{\Omega}_i$ on $T^*\tilde{Q}$ such that $\tau_{\tilde{Q}} \tilde{\Omega}_i = \tilde{\Omega}_i$ for $\tau_{\tilde{Q}} : T^*\tilde{Q} \to \tilde{Q}$ for each $i = 1, \ldots, k$ (equivalently $\rho_{\Omega_i} \tilde{\Omega}_i = \tilde{\Omega}_i$). From lemma 2.4 and (2.7) these forms satisfy that $\tilde{\Omega}_i((\eta_i)_{\tau_{\tilde{Q}}}) = \delta_{ij}$.

**Proposition 3.5.** Consider the nonholonomic system $(T^*\tilde{Q}, \tilde{\Omega}, \tilde{H})$.

(i) the 2-forms $B_1$ and $\mathcal{B}$ on $\mathcal{M}$ are basic with respect to the bundle $\rho_{\Omega} : \mathcal{M} \to T^*\tilde{Q}$, i.e. there exist $\tilde{B}_1$ and $\tilde{\mathcal{B}}$ on $T^*\tilde{Q}$ such that $\rho_{\Omega}^* B_1 = \tilde{B}_1$ and $\rho_{\Omega}^* \mathcal{B} = \tilde{\mathcal{B}}$. In particular,

$$\tilde{B}_1 = B_{(\tilde{\Omega})} + J_i d\tilde{\Omega}_i,$$

(3.14)

and $\tilde{\mathcal{B}}$ is basic with respect to the principal bundle $\rho_F : T^*\tilde{Q} \to T^*\tilde{Q}/F$, that is, there is a 2-form $\tilde{\mathcal{B}}$ on $T^*\tilde{Q}/F$ such that $\rho_{\tilde{F}}^* \tilde{\mathcal{B}} = \tilde{\mathcal{B}}$.

(ii) The 2-form $\tilde{\mathcal{B}} := \tilde{B}_1 + \tilde{\mathcal{B}}$ defines a dynamical gauge transformation on $(T^*\tilde{Q}, \tilde{\Omega}, \tilde{H})$. 

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Proof. (i) By construction, the 2-forms $B_1$ and $B$ verify that $i_X B_1 = i_X B = 0$ for all $X \in \Gamma(W)$ (where $W$ is the tangent to the $G$-orbit on $M$). Since they are $G$-invariant (see proposition 3.4), we conclude that they are basic with respect to the bundle $\rho_{\Omega_\text{B}} : M \to T^* \tilde{Q}$.

Therefore, from the expression of $B_1$ in (3.12) and by proposition 2.5 (i), it is straightforward to obtain that $B_1 = B_{(\Omega_\text{B})} + J_i d\tilde{Y}^i$, where $\tilde{Y}^i$ are 1-forms on $T^* \tilde{Q}$ such that $\rho_{\Omega_\text{B}}^* \tilde{Y}^i = Y^i$. Finally, by (3.13) we see that $B$ is semi-basic with respect to the bundle $\rho : M \to M/G$ and $G$-invariant and hence it is basic.

(ii) Since $B$ is semi-basic with respect to the bundle $\tau_M : M \to Q$ then $\tilde{B}$ is semi-basic with respect to $\tau_{\tilde{Q}} : T^* \tilde{Q} \to \tilde{Q}$. The dynamical condition $i_{\tilde{X}_\text{B}}^* B = 0$ is a direct consequence of proposition 3.4.

As a consequence of proposition 3.5 and (2.3), the (partially) reduced nonholonomic vector field $\tilde{X}_\text{B}$ on $T^* \tilde{Q}$ is also determined by $i_{\tilde{X}_\text{B}}^* \Omega_\text{B} = d\tilde{H}$ where $\Omega_\text{B} := \tilde{Q} + B$ and then the triple $(T^* \tilde{Q}, \Omega_\text{B}, \tilde{H})$ describes our (partially reduced) nonholonomic system as well. Note that, by proposition 3.4, the 2-form $\tilde{B}$ is $F$-invariant and hence $F$ is a symmetry of the nonholonomic system $(T^* \tilde{Q}, \Omega_\text{B}, \tilde{H})$.

The following Proposition puts in evidence the need of considering the nonholonomic system described by $(T^* \tilde{Q}, \Omega_\text{B}, \tilde{H})$ instead of $(T^* \tilde{Q}, \tilde{H})$ showing that the canonical momentum map $\tilde{J} : T^* \tilde{Q} \to \tilde{J}^*$ is the map that behaves as a momentum on $(T^* \tilde{Q}, \Omega_\text{B})$ when it is evaluated on the $\tilde{J}$-valued functions on $\tilde{Q}$ of $\mathcal{B}_{\text{HGS}}$ given in (2.5).

Proposition 3.6. Consider the partially reduced nonholonomic system $(T^* \tilde{Q}, \Omega_\text{B}, \tilde{H})$. The (partially reduced) horizontal gauge symmetries $\eta_i \in \mathcal{B}_{\text{HGS}}$ and the momentum map $\tilde{J} : T^* \tilde{Q} \to \tilde{J}^*$ satisfy the relation

$$i_{(\eta_i)_{\tau_{\tilde{Q}}}}^* \Omega_{\text{B}} = d\tilde{H}_i,$$

where $\tilde{H}_i = \tilde{J}_i (\tilde{J}, \eta_i)$ are the partially reduced horizontal gauge momenta.

Proof. Recall that $\Omega_\text{B} = \tilde{Q} + B = \Omega_{\tilde{Q}} - B_{(\Omega_\text{B})} + \tilde{B}_1 + \tilde{B}$. Using (3.14) and the fact that $B$ is basic with respect to $\rho_{\tilde{Q}} : T^* \tilde{Q} \to T^* \tilde{Q}/F$, we have that $i_{(\eta_i)_{\tau_{\tilde{Q}}}}^* \Omega_{\text{B}} = i_{(\eta_i)_{\tau_{\tilde{Q}}}}^* (\Omega_{\tilde{Q}} + \tilde{J}_i d\tilde{Y}^i)$.

Consider now $G$-invariant vector fields $\tilde{X}_1, \ldots, \tilde{X}_n$ so that $\{\tilde{X}_1, \ldots, \tilde{X}_n, (\eta_1)_{\tilde{Q}}, \ldots, (\eta_n)_{\tilde{Q}}\}$ is a basis of vector fields on $\tilde{Q}$ and consider its dual basis of 1-forms on $\tilde{Q}$ given by $\{\tilde{X}^1, \ldots, \tilde{X}^n, \tilde{Y}^1, \ldots, \tilde{Y}^n\}$. Since $\tilde{Y}^i = \tau_{\tilde{Q}}^* Y^i$ for $\tau_{\tilde{Q}} : T^* \tilde{Q} \to \tilde{Q}$ the canonical projection, we have that $\Theta_{\tilde{Q}} = p_\mu \tilde{X}^\mu + p_i \tilde{Y}^i$ where $\tilde{X}^\mu = \tau_{\tilde{Q}}^* X^\mu$ and $\tilde{J}_i = p_i$ (see e.g. [7]), and hence we obtain that $i_{(\eta_i)_{\tau_{\tilde{Q}}}}^* \tilde{Q}_\text{B} = i_{(\eta_i)_{\tau_{\tilde{Q}}}}^* (-d\theta_{\tilde{Q}} + \tilde{J}_i d\tilde{Y}^i) = d\tilde{H}_i$.

Remark 3.7. The particular case where the partially reduced symmetries are given by elements of the Lie algebra $\mathfrak{f}$ and the canonical momentum map behaves as a standard momentum map for $(T^* \tilde{Q}, \Omega_\text{B}, \tilde{H})$ was studied in [4, 32] but we remark these are restrictive conditions, not satisfied in most of the cases as, for instance, the nonholonomic particle, the snakeboard and the solids of revolutions which are examples treated in section 6.

3.3. Almost symplectic reduction

In this section we state one of the main results of the paper: we will perform a reduction of $(T^* \tilde{Q}, \Omega_\text{B})$ using the canonical momentum bundle map $\tilde{J} : T^* \tilde{Q} \to \tilde{J}^*$ following the procedure of a Marsden–Weinstein–type reduction [41] but having into account that the $\tilde{J}$-valued functions $\mu$ considered are, in general, non-constant functions on $\tilde{Q}$.
Theorem 3.8. Let \((\mathcal{M}, \Omega_\mathcal{M}|_{\mathcal{C}}, H_\mathcal{M})\) be a nonholonomic system with a \(G\)-symmetry satisfying the dimension assumption. Suppose that the system admits \(\{J_1, \ldots, J_k\}\), for \(k = \text{rank}(\mathcal{S})\), \(G\)-invariant horizontal gauge momenta and that the vertical complement \(W\) can be chosen so that it satisfies the vertical symmetry condition. Then, for the 2-form \(B\) in proposition 3.5, holds

(i) the partially reduced nonholonomic system \((T^{*\tilde{Q}}\tilde{\Omega}_B, \tilde{H})\) is \(F\)-invariant and it has \(k\) (partially reduced) horizontal gauge momenta \(\{J_1, \ldots, J_k\}\).

(ii) [Almost symplectic reduction] For each \(\mu\)-valued function of the form \(\mu = c_i\mu_i\) for \(c_i \in \mathbb{R}\) and \(\mu_i \in \mathfrak{b}^*_{\text{HGS}}\) \((3.10)\), the manifold \(\tilde{J}^{-1}(\mu)/\tilde{F}\) admits an almost symplectic form \(\omega^\mu\) such that

\[\iota^*_\mu \tilde{\Omega}_B = \rho^*_\mu \omega^\mu,\]

where \(\iota^*_\mu : \tilde{J}^{-1}(\mu) \to T^*\tilde{Q}\) is the inclusion and \(\rho^*_\mu : \tilde{J}^{-1}(\mu) \to \tilde{J}^{-1}(\mu)/\tilde{F}\) is the orbit projection.

(iii) [The reduced dynamics] The reduced nonholonomic vector field \(X_{\text{red}}\) on \(\mathcal{M}/G\) is tangent to the manifold \(\tilde{J}^{-1}(\mu)/\tilde{F}\) for \(\mu = c_i\mu_i\), with \(c_i \in \mathbb{R}\) and \(\mu_i \in \mathfrak{b}^*_{\text{HGS}}\), and its restriction to this leaf is a Hamiltonian vector field for the 2-form \(\omega^\mu\) and the Hamiltonian function \(H_\mu := (\iota^*_\mu)^* H_{\text{red}}, \) for \(\mu_i \in \tilde{J}^{-1}(\mu)/\tilde{F} \to T^*\tilde{Q}/\tilde{F}\) the inclusion.

Proof. (i) Since the nonholonomic system \((\mathcal{M}, \Omega_\mathcal{M}|_{\mathcal{C}}, H_\mathcal{M})\) is \(G\)-invariant and \(\mathfrak{b}\) is \(F\)-invariant, then the partially reduced system \((T^{*\tilde{Q}}\tilde{\Omega}_B, \tilde{H})\) is invariant by the \(F\)-action. Moreover, the canonical momentum map \(\tilde{J} : T^*\tilde{Q} \to \mathfrak{b}^*\) and the basis \(\mathfrak{b}^*_{\text{HGS}}\) define the functions \(\tilde{J}\), which are partially reduced horizontal gauge momenta, see lemma 2.4, proposition 2.5 and definition 2.6.

(ii) Following proposition 3.1, for each \(\mu = c_i\mu_i\) (for \(c_i \in \mathbb{R}\) and \(\mu_i \in \mathfrak{b}^*_{\text{HGS}}\), \(\tilde{J}^{-1}(\mu)\) is a \(F\)-invariant manifold and, since the \(F\)-action is free and proper, the quotient space \(\tilde{J}^{-1}(\mu)/\tilde{F}\) is a well defined manifold. Let us denote by \(\tilde{\Omega}_B^\mu\) the pull back of \(\tilde{\Omega}_B\) to \(\tilde{J}^{-1}(\mu)\), i.e. \(\tilde{\Omega}_B^\mu := \iota^*_\mu \tilde{\Omega}_B\). Next, we will show that \(\tilde{\Omega}_B^\mu\) is basic with respect to the bundle \(\tilde{J}^{-1}(\mu) \to \tilde{J}^{-1}(\mu)/\tilde{F}\). That is, as a consequence of proposition 3.6, we will prove that, for each \(\alpha \in \tilde{J}^{-1}(\mu) \subset T^*\tilde{Q}\),

\[\text{Ker}(\tilde{\Omega}_B^\mu(\alpha)) = T_\alpha \text{(Orb}_F(\alpha)),\]

(3.15)

where \(\text{Orb}_F(\alpha)\) is the orbit of the \(F\)-action at \(\alpha\). First, we claim that for all \(\alpha \in \tilde{J}^{-1}(\mu)\) and \(\mu = c_i\mu_i\), \(T_\alpha \tilde{J}^{-1}(\mu) = (T_\alpha \text{Orb}_F(\alpha))\tilde{\Omega}_B\). In fact, the flow \(\phi^X_t\) of a vector field \(X\) on \(\tilde{J}^{-1}(\mu)\), satisfies that \(\tilde{J}(\phi^X_t(\alpha)) = \mu\) for all \(t\). Then, for \(\eta_i \in \mathfrak{b}^*_{\text{HGS}}\) and using proposition 3.6, we have that

\[\tilde{\Omega}_B(X, (\eta_i)_T) = -\frac{d}{dt}\tilde{J}(\phi^X_t(\alpha), \eta_i) \bigg|_{t=0} = -\frac{d}{dt}\tilde{J}(\phi^X_t(\alpha), \eta_i) \bigg|_{t=0} = 0.\]

Since \((\eta_i)_T\) for \(i = 1, \ldots, k\) form a basis of \(T_\alpha \text{Orb}_F(\alpha)\), then we obtain that \(X(\alpha) \in (T_\alpha \text{Orb}_F(\alpha))\tilde{\Omega}_B\). Finally, \(T_\alpha \tilde{J}^{-1}(\mu) = (T_\alpha \text{Orb}_F(\alpha))\tilde{\Omega}_B\) since both spaces have the same dimension.

Now we prove the identity (3.15). Let \(\alpha \in \text{Ker}(\tilde{\Omega}_B^\mu(\alpha))\), then \(\tilde{\Omega}_B^\mu(X(\alpha), Y(\alpha)) = 0\) for all \(Y(\alpha) \in T_\alpha \tilde{J}^{-1}(\mu) = (T_\alpha \text{Orb}_F(\alpha))\tilde{\Omega}_B\). Then \(X(\alpha) \in [(T_\alpha \text{Orb}_F(\alpha))\tilde{\Omega}_B]\tilde{\Omega}_B = T_\alpha \text{Orb}_F(\alpha)\) (since
the 2-form $\tilde{\Omega}_\mu$ is nondegenerate). Conversely, for $\eta_i \in \mathfrak{\hat{B}}_{\mathrm{HGS}}$, $\tilde{\Omega}_\mu^\#(\eta_i, \langle \rho^{\mu}_0(\alpha), X(\alpha) \rangle) = 0$ for all $X(\alpha) \in T_\alpha J^{-1}(\mu)$ and then $\eta_i \in \ker(\tilde{\Omega}_\mu^\#(\cdot))$ for all $i = 1, \ldots, k$.

Therefore, by (3.15), the 2-form $\tilde{\Omega}_\mu^\#$ descends to an almost symplectic 2-form $\omega^\#_\mu$ on $J^{-1}(\mu)/F$ such that $\rho^*_\mu \omega^\#_\mu = \tilde{\Omega}_\mu^\#$.

(iii) If $J_i$ are conserved quantities for the partially reduced dynamics $\tilde{X}_{\alpha i}$ on $T^* \mathcal{Q}$, then the flow $\tilde{\phi}_t$ of the vector field $X_{\alpha i}$ satisfies that, for $\alpha \in J^{-1}(\mu)$, $\tilde{\phi}_t^\#(\alpha) \in J^{-1}(\mu)$ for all $t$. Therefore, by the $G$-invariance of the dynamics, we conclude that $\phi^\#_{\alpha i}(\rho^\mu_{\alpha i}(\alpha)) \in J^{-1}(\mu)/F$ where $\phi^\#_{\alpha i}$ is the flow of the reduced nonholonomic dynamics $X_{\alpha i}$ and $\rho_{\alpha i} : J^{-1}(\mu) \to J^{-1}(\mu)/F$ is the corresponding orbit projection. Denoting by $X^\#_{\alpha i}$ the restriction of $X_{\alpha i}$ to the leaf $J^{-1}(\mu)/F$, we can see that $\kappa_{\alpha i} \omega^\#_\mu = dH_{\mu}$ as a direct consequence of item (ii) and proposition 3.5 (ii).

Remark 3.9. The fact that we consider $G$-invariant horizontal gauge momenta $J_1, \ldots, J_k$ and not only $G_\alpha$-invariant, permits us to reduced the manifold $J^{-1}(\mu)$ by the action of the Lie group $F$ (without taking into account any ‘isotropy group’). Under this assumption, we may denote by $J_i$ the functions on $T^* \mathcal{Q}/F$ such that $\rho^\#_\mu J_i = J_i$ for $\rho^\mu_\mu : T^* \mathcal{Q} \to T^* \mathcal{Q}/F$ the corresponding orbit projections (or equivalently $\rho^\mu J_i = J_i$). Therefore, $J^{-1}(\mu)/F$ coincides with the common level sets of the reduced horizontal gauge momenta $J_i$, i.e. $J^{-1}(\mu)/F \simeq \cap_i J^{-1}(c_i)$.

4. The identification of $(J^{-1}(\mu)/F, \omega^\#_\mu)$ with the canonical symplectic manifold

In the hamiltonian framework, when working on a canonical symplectic manifold $(T^* \mathcal{Q}, \Omega_\mathcal{Q})$ (and when $G = G_\alpha$) we have the identification of the Marsden–Weinstein reduced symplectic manifolds with the cotangent manifold $T^* (\mathcal{Q}/G)$ and its canonical symplectic form plus a magnetic term that depends on a chosen connection, see e.g. [1, 39]. In this section we show an analogous identification but carrying on the information of the nonholonomic character of the system. That is, we take into account that the nonholonomic dynamics takes place on the almost symplectic manifold $(T^* \mathcal{Q}, \Omega_\mathcal{Q})$ and that the 2-form $\tilde{\mathcal{B}}$ also depends on a chosen connection. Hence we obtain an identification of the ‘Marsden–Weinstein’ reduced spaces $(J^{-1}(\mu)/F, \omega^\#_\mu)$ with the cotangent manifold $T^* (\mathcal{Q}/F)$ and its canonical symplectic form modified by a ‘magnetic’ term, i.e. a 2-form that, in this case, does not come from a 2-form on $\mathcal{Q}/F$ (as in the hamiltonian case) and it depends only on the 2-form $\tilde{\mathcal{B}}$. This extra term carries the nonholonomic character of the reduced system since, contrary to hamiltonian systems, its differential can be different from zero. Moreover, in examples 6.1, 6.2 and 6.4 we will see that $\mathcal{B} = 0$ and then the manifolds $(J^{-1}(\mu)/F, \omega^\#_\mu)$ are diffeomorphic to the canonical symplectic manifold $(T^* (\mathcal{Q}/F), \Omega_\mathcal{Q}/F)$ (showing a genuine hamiltonisation).

Note that, we consider, as usual, a nonholonomic system $\mathcal{M}, \Omega, \mathcal{C} \subset H\mathcal{M}$ with a $G$-symmetry admitting $k = \mathrm{rank}(\mathcal{S})$ horizontal gauge momenta and with the vertical symmetry condition. Then the partially reduced nonholonomic system $(T^* \mathcal{Q}, \tilde{\mathcal{B}}, \mathcal{H})$ admits a symmetry given by the action of $F$ with $k$ partially reduced horizontal gauge momenta (recall that the dimension of the Lie algebra $\mathfrak{f}$ is also $k$).

4.1. Identification at the zero-level

Following [39], we consider the zero level set of the momentum bundle map $\tilde{J} : T^* \mathcal{Q} \to \mathfrak{f}^*$ and the map $\tilde{\varphi}_0 : J^{-1}(0) \to T^* \mathcal{Q}$, with $\mathcal{Q} := \mathcal{Q}/F$, given by

$$\langle \tilde{\varphi}_0(\alpha_\mu), T_{\alpha_\mu} \mathcal{Q}(v_\mu) \rangle = (\alpha_\mu, v_\mu),$$

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for $\alpha \in \tilde{J}^{-1}(0) \subset T^{*}\tilde{Q}$, $\psi \in T_{\psi}\tilde{Q}$ and $\rho_{\tilde{Q}} : \tilde{Q} \to \mathcal{Q}$ the orbit projection. Since the map $\tilde{\varphi}_{0}$ is $F$-invariant, it is shown also in [39] that there is a well defined diffeomorphism

$$\varphi_{0} : \tilde{J}^{-1}(0)/F \to T^{*}\mathcal{Q},$$

so that $\varphi_{0} \circ \rho_{0} = \tilde{\varphi}_{0}$ for $\rho_{0} : \tilde{J}^{-1}(0) \to \tilde{J}^{-1}(0)/F$ the canonical projection. Next, we show that this map is, in fact, the diffeomorphism that links the 2-form $\omega_{0}^{B}$ on $\tilde{J}^{-1}(0)/F$ from theorem 3.8 (at $\mu = 0$) with the canonical 2-form $\Omega_{\mathcal{Q}}$ on $T^{*}\mathcal{Q}$.

Recall, from propositions 3.4 and 3.5, that the 2-form $\mathcal{B}$ can be written as $\mathcal{B} = \tilde{B}_{1} + \tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}$ is the 2-form on $T^{*}\tilde{Q}/F$ such that $\rho_{\tilde{Q}}^{*}\tilde{\mathcal{B}} = \mathcal{B}$, for $\rho_{\tilde{Q}} : T^{*}\tilde{Q} \to T^{*}\tilde{Q}/F$.

**Proposition 4.1.** The diffeomorphism $\varphi_{0} : \tilde{J}^{-1}(0)/F \to T^{*}\mathcal{Q}$ satisfies that

$$\varphi_{0}^{*}\Omega_{\mathcal{Q}} = \omega_{0}^{B} - \mathcal{B}_{0},$$

where $\Omega_{\mathcal{Q}}$ is the canonical 2-form on $T^{*}\mathcal{Q}$ and $\mathcal{B}_{0} := (i_{0}^{\omega})^{*}\mathcal{B}$, for $i_{0}^{\omega} : \tilde{J}^{-1}(0)/F \to T^{*}\tilde{Q}/F$ the natural inclusion. In particular, if $\dim(\mathcal{Q}) = 1$, then $\varphi_{0}^{*}\Omega_{\mathcal{Q}} = \omega_{0}^{B}$.

**Proof.** On the one hand, it was shown in [39] that the diffeomorphism $\varphi_{0} : \tilde{J}^{-1}(0)/F \to T^{*}\mathcal{Q}$ satisfies that $\varphi_{0}^{*}\Omega_{\mathcal{Q}} = \omega_{0}$, where $\omega_{0}$ is the symplectic form on $\tilde{J}^{-1}(0)/F$ such that $\rho_{0}^{*}\omega_{0} = i_{0}^{*}\Omega_{\mathcal{Q}}$, for $i_{0} : \tilde{J}^{-1}(0) \to T^{*}\tilde{Q}$ the natural inclusion. On the other hand, theorem 3.8 at the zero-level, implies that $\rho_{0}^{*}\omega_{0} = \tilde{B}_{1}$. From the expression of $\tilde{B}_{1}$ in (3.14), we have that $i_{0}^{*}(B_{\mathcal{Q}} - \tilde{B}_{1}) = -i_{0}^{*}(\partial_{J}d\tilde{Y}) = 0$. Therefore, we obtain that

$$\rho_{0}^{*}\omega_{0}^{B} = \tilde{B} = i_{0}^{*}\Omega_{\mathcal{Q}} + i_{0}^{*}\tilde{B} = \rho_{0}^{*}\omega_{0} + \rho_{0}^{*}\rho_{\tilde{Q}}^{*}\mathcal{B} = \rho_{0}^{*}\omega_{0} + \rho_{0}^{*}(i_{0}^{\omega})^{*}\mathcal{B},$$

where in the last equality we used that $i_{0}^{\omega} \circ \rho_{0} = \rho_{\tilde{Q}} \circ i_{0}$ for $\rho_{\tilde{Q}} : T^{*}\tilde{Q} \to T^{*}\tilde{Q}/F$ the orbit projection. Then $\omega_{0}^{B} = \omega_{0} + (i_{0}^{\omega})^{*}\mathcal{B}$ which implies that $\omega_{0}^{B} = \varphi_{0}^{*}\Omega_{\mathcal{Q}} + \mathcal{B}_{0}$. \qed

**4.2. Identification at the $\mu$-level and the Shift-trick**

Now, using the *Shift-trick* as in [1, 39], we show that each (connected component of the) almost symplectic manifold $(\tilde{J}^{-1}(\mu)/F, \omega_{\mu}^{B})$ obtained in theorem 3.8, is diffeomorphic to $T^{*}\mathcal{Q}$ with its canonical 2-form $\Omega_{\mathcal{Q}}$ properly modified by a ‘magnetic’ term.

As usual, we denote by $\mathcal{B}_{\text{mag}} = \{\eta_{1}, \ldots, \eta_{k}\}$ a global basis of equivanlant $\mathfrak{f}$-valued functions on $\tilde{Q}$ of (partially reduced) horizontal gauge symmetries and $\mathcal{B}_{\text{mag}} = \{\mu^{1}, \ldots, \mu^{k}\}$ the dual basis of $\mathfrak{f}$-valued functions given in (3.10). Recall that $A$ is the induced connection on $T\tilde{Q}$ (see lemma 2.4) and observe that $\tilde{B}_{1}$ in (3.14) is written with respect to this connection: $\tilde{A} = \eta \otimes \tilde{y}^{J}$ and $\tilde{B}_{1} = B_{\mathcal{Q}} + \tilde{J}, d\tilde{y}$ where $\tilde{\mathcal{Y}} = i_{0}^{*}\tilde{Y}$ for $i_{0} : T^{*}\tilde{Q} \to \tilde{Q}$ is the canonical projection.

Now we define the Shift-map that, on $\mu \in \mathcal{B}_{\text{mag}}$, coincides with the one defined in [1, 39]. More precisely, for $\mathfrak{f}$-valued functions $\mu = c_{i}\mu^{i}$, with $c_{i} \in \mathbb{R}$ and $\mu^{i} \in \mathcal{B}_{\text{mag}}$, we define the diffeomorphism $\text{Shift}_{\mu} : T^{*}\tilde{Q} \to T^{*}\tilde{Q}$ given, at each $\alpha \in T^{*}\tilde{Q}$, by

$$\text{Shift}_{\mu}(\alpha) := \alpha - \alpha_{\mu},$$

where $\alpha_{\mu} = \langle \mu, \tilde{A} \rangle$ for $\langle \cdot, \cdot \rangle$ the natural pairing between the $\mathfrak{f}$-valued function $\mu$ and the $\mathfrak{f}$-valued 1-form $\tilde{A}$, i.e. for $x \in \tilde{Q}$, $\alpha_{\mu}(x) = \langle \mu(x), \tilde{A} \rangle \in T_{x}^{*}\tilde{Q}$. 

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Lemma 4.2. If $\mu = c_i \mu'$ for $c_i \in \mathbb{R}$ and $\mu' \in \mathcal{B}_{\text{HGS}}^*$, then

(i) $\text{Shift}_\mu^* \Omega_Q = \Omega_Q + \tau_Q^\mu c_i dY^i$.

(ii) The restricted map $\text{shift}_\mu := \text{Shift}_\mu |_{J^{-1}(\mu)} : J^{-1}(\mu) \to J^{-1}(0)$ is a well defined equivariant diffeomorphism and hence there is a well defined diffeomorphism $\text{Shift}_\mu : J^{-1}(\mu)/F \to J^{-1}(0)/F$ so that the following diagram commutes

$$
\begin{array}{ccc}
T^*\tilde{Q} & \xrightarrow{\iota_\mu} & J^{-1}(\mu) \xrightarrow{\rho_\mu} J^{-1}(\mu)/F \\
\downarrow \text{Shift}_\mu & & \downarrow \text{shift}_\mu \\
T^*Q & \xrightarrow{\iota_0} & J^{-1}(0) \xrightarrow{\rho_0} J^{-1}(0)/F
\end{array}
$$

Proof. (i) Note that if $\mu = c_i \mu'$, for $\mu' \in \mathcal{B}_{\text{HGS}}^*$, then by (2.7), $\langle \mu, \tilde{A} \rangle = c_i \tilde{Y}^i$ and hence $d(\mu, \tilde{A}) = c_i dY^i$. Moreover, following [39], we can also prove that $\text{Shift}_\mu^\ast \Theta_Q = \Theta_Q - \tau_Q^\mu c_i dY^i$.

(ii) It is straightforward to check that $\text{shift}_\mu$ is a diffeomorphism. To see the equivariance, recall that, for $h \in F$, $\Psi_h : \tilde{Q} \to \tilde{Q}$ denotes the $F$-action on $\tilde{Q}$ and $\mu(x)$ denotes the evaluation of the $f^*$-valued function $\mu$ at $x \in \tilde{Q}$. On the one hand, due to the $F$-invariance of the horizontal space $H_{\mathcal{A}}$, the connection $\tilde{A}$ is $Ad$-equivariant: for $x \in \tilde{Q}$, $\Psi_h : \tilde{Q} \to \tilde{Q}$ denotes the evaluation at $h$ of $\tilde{A}$. Moreover, following [39], we can also prove that $\text{Shift}_\mu^\ast \Theta_Q = \Theta_Q - \tau_Q^\mu c_i dY^i$.

Next, for each $f^*$-valued function $\mu = c_i \mu'$ with $c_i \in \mathbb{R}$ and $\mu' \in \mathcal{B}_{\text{HGS}}^*$ we consider the map

$$
\varphi_\mu := \varphi_0 \circ \text{Shift}_\mu : J^{-1}(\mu)/F \to T^*Q,
$$

which, by construction, is a diffeomorphism.

Theorem 4.3. Consider a nonholonomic system $(\mathcal{M}, \Omega_{\mathcal{A}}|_{\mathcal{C}}, H_{\mathcal{A}})$ with a $G$-symmetry satisfying the dimension assumption such that it admits $k = \text{rank}(S)$ $G$-invariant horizontal gauge momenta. Moreover, we assume that the system verifies the vertical symmetry condition. The reduction of the partially reduced system $(T^*\tilde{Q}, \Omega_{\mathcal{A}}, H)$ given in theorem 3.8, induces, for each $\mu = c_i \mu'$ (with $\mu' \in \mathcal{B}_{\text{HGS}}^*$), the almost symplectic manifold $(J^{-1}(\mu)/F, \omega_\mu^\ast \Phi)$ for which the diffeomorphism $\varphi_\mu : J^{-1}(\mu)/F \to T^*Q$ satisfies that

$$
\varphi_\mu^\ast \Omega_Q = \omega_\mu^\ast \mathcal{B}_{\mathcal{A}},
$$

where $\mathcal{B}_{\mathcal{A}} := (\iota_{\mu}^\ast)^\ast \Phi$ for $\iota_{\mu}^\ast : J^{-1}(\mu)/F \to T^*Q/F$ the natural inclusion. In particular, if $\dim(Q) = 1$, then $\varphi_\mu^\ast \Omega_Q = \omega_\mu^\ast$.
First, consider the following two commutative diagrams

\[ \begin{align*}
\tilde{J}^{-1}(0) & \xrightarrow{i_0} T^* \tilde{Q} \\
\tilde{J}^{-1}(\mu) & \xrightarrow{i_\mu} T^* \tilde{Q}
\end{align*} \]

Since \( \rho_\mu^* \tilde{B} = \tilde{B} \) and, as a consequence of theorem 3.8, we see that \( \varphi^*_\mu \Omega_B = \omega^B_\mu - \tilde{B}_\mu \) if and only if

\[ \rho_\mu^* \circ \varphi^*_\mu \Omega_B = \iota^*_\mu(\Omega_B - B_{[\mu,\nu]} + \tilde{B}) - \tau^*_\mu \tilde{B}. \]  

Next, we will prove (4.20). Using the definition of \( \varphi_\mu \) in (4.18) and by (4.17) we have that

\[ \rho_\mu^* \circ \varphi^*_\mu \Omega_B = \rho_\mu^* \circ \text{Shift}_{\mu} \circ \varphi_0^* \Omega_B = \text{Shift}_{\mu} \circ \rho_0^* \circ \varphi_0^* \Omega_B = \text{Shift}_{\mu} \circ \rho_0^* (\omega_0^B - \tilde{B}_0), \]

where in the last equality we used proposition 4.1. Moreover, since \( \rho_0^* (\omega_0^B - \tilde{B}_0) = \iota^*_0 \Omega_B \) (see the proof of proposition 4.1), and using (4.19) and lemma 4.2, we conclude that

\[ \rho_\mu^* \circ \varphi^*_\mu \Omega_B = \text{Shift}_{\mu} \circ \iota^*_0 \Omega_B = \iota^*_\mu \circ \text{Shift}_{\mu} \Omega_B = \iota^*_\mu (\Omega_B + \tau^*_\mu c_1 d\tilde{Y}^i) = \iota^*_\mu (\Omega_B + \tilde{J}_i d\tilde{Y}^i), \]

where in the last equality we also used proposition 3.1. and the fact that \( \tilde{\tilde{Y}}^i = \tau^*_\mu \tilde{Y}^i \). Finally, recalling the expression of \( B_1 \) in (3.14), we see that \( \Omega_B + \tilde{J}_i d\tilde{Y}^i = \Omega_B - B_{[\mu,\nu]} + \tilde{B} \) and, since \( \tilde{B}_1 = B - \tilde{B} \), we arrive to the desired result (4.20).

**Remark 4.4.** Theorem 4.3 identifies each almost symplectic manifold \( (\tilde{J}^{-1}(\mu))/F, \omega_\mu^B \) with \( (T^* \tilde{Q}, \Omega_B + \tilde{B}_\mu) \) where \( \tilde{B}_\mu := (\varphi_\mu^{-1})^* \tilde{B}_\mu \). Observe that \( \tilde{B}_\mu \) is not a magnetic term in the strict sense since it might be non closed and is not coming from a 2-form defined on \( \tilde{Q} \). Moreover, it has no connection with the magnetic term that appears in hamiltonian systems.

## 5. Relation with the nonholonomic bracket

In this section we will show that the almost symplectic foliation \( (\tilde{J}^{-1}(\mu))/F, \omega_\mu^B \) described in theorem 3.8 is the foliation associated to the (twisted Poisson) bracket \( \{ \cdot, \cdot \}_{\text{Ham}} \) on \( \mathcal{M}/G \) described in [7] (see also [5, 29]).

### 5.1. The nonholonomic bracket and reduction

Consider a nonholonomic system \( (\mathcal{M}, \Omega_\mathcal{M}|_\mathcal{C}, H_\mathcal{M}) \) as in section 2. The nonholonomic bracket [16, 38, 46] (see also [21, 33]) is the almost Poisson bracket given, for each \( f \in C^\infty(\mathcal{M}) \), by

\[ \{ \cdot, f \}_{\text{Ham}} = X_f \quad \text{if and only if} \quad \iota_{X_f} \Omega_\mathcal{M}|_\mathcal{C} = (df)|_\mathcal{C} \]  

and it describes the nonholonomic dynamics since \( X_{\text{Ham}} = \{ \cdot, H_\mathcal{M} \}_{\text{Ham}} \).

Recall that a Poisson bracket is a Lie bracket on functions with Leibniz identity and hence admits a symplectic foliation. An almost Poisson bracket lacks the Jacobi identity and thus the characteristic distribution of an almost Poisson bracket—the distribution generated by the hamiltonian vector fields—might not be integrable. In particular the characteristic distribution of the nonholonomic bracket is the nonintegrable distribution \( \mathcal{C} \) defined in (2.2). In between,
there is a class of almost Poisson brackets, called twisted Poisson \[35, 45\] that have an almost symplectic foliation and whose failure of the Jacobi identity is encode by a closed 3-form:
\[
\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \Phi(X_f, X_g, X_h), \quad \text{for } f, g, h \in C^\infty(M),
\]
where \(\Phi\) is a closed 3-form on \(M\). Thus, we observe that \(\{\cdot, \cdot\}_\text{ah}\) is not twisted Poisson.

If the nonholonomic system \((M, \Omega_M|_C, H_M)\) has a \(G\)-symmetry then the nonholonomic bracket induces an almost Poisson bracket \(\{\cdot, \cdot\}_\text{red}\) on the reduced manifold \(M/G\) so that, for \(f, g \in C^\infty(M/G)\),
\[
\{f, g\}_\text{red}(\rho(m)) = \{\rho^*f, \rho^*g\}_\text{ah}(m),
\]
for \(m \in M\) and \(\rho: M \to M/G\) the orbit projection, describing the reduced dynamics: \(X_m = \{\cdot, H_m\}_\text{red}\), for \(H_m \in C^\infty(M/G)\) the reduced hamiltonian. We denote a nonholonomic system by the triple \((M, \{\cdot, \cdot\}_\text{ah}, H_M)\) or by \((M/G, \{\cdot, \cdot\}_\text{red}, H_m)\) to refer to the reduced system.

### 5.2. Gauge transformation of the nonholonomic bracket and reduction

Consider a nonholonomic system \((M, \Omega_M|_C, H_M)\) with a \(G\)-symmetry admitting \(k = \text{rank}(S)\) horizontal gauge momenta \(\{J_1, \ldots, J_k\}\) and let \(B\) the 2-form on \(M\) defined in proposition 3.4.

Due to the nondegeneracy of \((\Omega_M + B)|_C\), there is a well defined almost Poisson bracket \(\{\cdot, \cdot\}_B\) on \(M\) given, for \(f \in C^\infty(M)\), by
\[
\{f, g\}_B = \{\cdot, f\}_B \quad \text{if and only if} \quad \iota_{\xi}(\Omega_M + B)|_C = (df)|_C,
\]
(cf.\[5.21\]) which is obtained by the gauge transformation of \(\{\cdot, \cdot\}_B\) by \(B\) (see \[5, 29, 45\] for more details).

The bracket \(\{\cdot, \cdot\}_B\) describes the dynamics: \(X_m = \{\cdot, H_M\}_B\), because \(\iota_{\xi_m}B = 0\).

Since the 2-form \(B\) is \(G\)-invariant, then \(\{\cdot, \cdot\}_B\) descends to an almost Poisson bracket \(\{\cdot, \cdot\}_\text{red}^B\) on \(M/G\) given, at each \(f, g \in C^\infty(M/G)\), and \(m \in M\), by
\[
\{f, g\}_\text{red}^B(\rho(m)) = \{\rho^*f, \rho^*g\}_B(m).
\]

Therefore the nonholonomic system can be equivalently determined by the triple \((M, \{\cdot, \cdot\}_B, H_M)\) and by \((M/G, \{\cdot, \cdot\}_\text{red}^B, H_m)\) to refer to the reduced system. However, it was proven in \[7\] that \(\{\cdot, \cdot\}_\text{red}^B\) is twisted Poisson in contrast to \(\{\cdot, \cdot\}_\text{ah}\) that might not be. In fact, the reduced conserved quantities are Casimirs for \(\{\cdot, \cdot\}_\text{ah}\), that is \(\{\cdot, J_i\}_\text{ah} = 0\).

As usual, \(\mu = c_i\mu^i\) is a \(f^*\)-valued function on \(\tilde{Q}\) where \(\mu^i \in \mathbb{B}_\text{twist}\) and \(c_i \in \mathbb{R}\) for \(i = 1, \ldots, k\).

**Theorem 5.1.** The leaves of the twisted Poisson bracket \(\{\cdot, \cdot\}_\text{red}^B\) on \(M/G\) are (the connected components of) the almost symplectic manifolds \((\tilde{J}^{-1}(\mu))/F, \omega_\mu^B\) obtained in theorem 3.8, where \(\tilde{J}^{-1}(\mu)/F\) coincides with the common level sets of the reduced horizontal gauge momenta \(J_i\) on \(M/G\), i.e. \(\tilde{J}^{-1}(\mu)/F \cong \cap_i \tilde{J}_i^{-1}(c_i)\).

**Proof.** Let \(f \in C^\infty(T^*\tilde{Q}/F)\). We will show that the vector field \(\tilde{X} := \{\cdot, f\}_\text{red}^B\) defined on \(T^*\tilde{Q}/F\) satisfies that, for \(\alpha \in \tilde{J}^{-1}(\mu)/F, \tilde{X}(\tilde{\tau}) \in T_{\tilde{\tau}}(\tilde{J}^{-1}(\mu)/F)\) and
\[
\iota_{\tilde{X}(\tilde{\tau})}\omega_\mu^B = d(\iota_{\mu^B})^f(f(\tilde{\tau})),
\]
where, as usual, \(\iota_{\mu^B}: \tilde{J}^{-1}(\mu)/F \to M/G\) is the natural inclusion. In fact, first observe that, since \(\tilde{X}\) belongs to the characteristic distribution of \(\{\cdot, \cdot\}_\text{red}^B\) then \(\tilde{X}(\tilde{J}_i) = 0\) and hence \(\tilde{X}(\tilde{\tau}) \in T_{\tilde{\tau}}(\tilde{J}^{-1}(\mu)/F)\), using proposition 3.1. Let \(X := \{\cdot, \rho^f\}_B\) and observe that \(T_{\tilde{\tau}}(\tilde{X}) = \tilde{X}(\tilde{\tau}) \in T_{\tilde{\tau}}(\tilde{J}^{-1}(\mu)/F)\) satisfies that, for \(\alpha \in \tilde{J}^{-1}(\mu), \iota_{\tilde{X}(\tilde{\tau})}\tilde{\Omega}_\mu = d\rho^f(\alpha)\) (recall \(\rho^f: T^*\tilde{Q} \to M/G\). Since \(\tilde{X}(\alpha) \in T_{\alpha}(\tilde{J}^{-1}(\mu))\), then \(\iota_{\tilde{X}(\alpha)}\tilde{\Omega}_\mu = d\rho^f(\alpha)\) which, by theorem 3.8, is equivalent to (5.26).\(\square\)
Remark 5.2. (i) As a consequence of the three main theorems (theorems 3.8, 4.3 and 5.1) we conclude that the (connected components of the) almost symplectic leaves of the reduced bracket \{\cdot,\cdot\}^\mathfrak{g}_{\mathfrak{al}} on \mathcal{M}/G are diffeomorphic to \((T^*\mathcal{O}_T,\Omega_\mathcal{Q}+\tilde{B}_1)\), see remark 4.4.

(ii) Following the notation of [7], we have that the reduction of the manifold \((T^*\mathcal{O}_T,\Omega_\mathcal{Q}+\tilde{B}_1)\) gives the Poisson bracket \{\cdot,\cdot\}^\mathfrak{g}_{\mathfrak{al}} on \mathcal{M}/G for which the symplectic leaves are diffeomorphic to \((T^*\mathcal{O}_T,\Omega_\mathcal{Q})\). Moreover, theorem 5.1 puts in evidence the gauge relation of \{\cdot,\cdot\}^\mathfrak{g}_{\mathfrak{al}} and \{\cdot,\cdot\}^\mathfrak{g}_{\mathfrak{al}} since they have the same foliation and the 2-form on each leaf is given by \(\Omega_\mathcal{Q}\)

6. Examples

In this section we study four different examples. All of them admit a vertical complement of the constraints satisfying the vertical symmetry condition and \(k\) horizontal gauge momenta given by the nonholonomic momentum map evaluated in non-constant sections.

6.1. Nonholonomic particle

Consider the classical motion of a particle in \(\mathbb{R}^3\) subjected to \(\ddot{z} = y\dot{x}\) with \(L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\). In particular it is a \(\mathbb{R}^2\)-Chaplygin system and the partially reduced system is \((\mathbb{T}\mathbb{R}^2,\Omega_\mathcal{Q} - B_{\mathcal{H}K},\tilde{H})\) [9]. The residual action of \(F \simeq \mathbb{R}\) on \(\mathbb{R}^2\) is a symmetry of the system.

6.1.1. Momentum map and reduction. The momentum bundle map \(\tilde{J} : T^*\mathbb{R}^2 \rightarrow \mathfrak{f}^*\) encodes the conserved quantity \(J = \frac{1}{\sqrt{1+y^2}}p_x = \tilde{J}_y(\eta)\) with \(\eta = \frac{1}{\sqrt{1+y^2}} \in \Gamma(\mathbb{R}^2 \times \mathfrak{f})\). Following section 3.2, \(B = 0\) and \(\eta_{\mathfrak{f}^*}\) is the hamiltonian vector field associated to the function \(J_y\) for the 2-form \(\Omega = \Omega_\mathcal{Q} - B_{\mathcal{H}K}\).

If \(\mu \in \Gamma(\mathbb{R}^2 \times \mathfrak{f}^*)\) is the dual element associated to \(\eta\) then for \(\mu_c = c\mu\) with \(c \in \mathbb{R}\), \(\tilde{J}^{-1}(\mu_c) = \{(x,y,p_x,p_y) : p_x = c\sqrt{1+y^2}\} = J^{-1}(c)\), recovering proposition 3.1. Following theorem 3.8, \(\mu_c\) \(\tilde{\Omega} = dy \wedge dp_x\) and, on \(\tilde{J}^{-1}(\mu_c)/F \simeq T^*\mathbb{R},\omega_{\mu_c} = dy \wedge dp_x\). Then, each symplectic leaf associated to \{\cdot,\cdot\}^\mathfrak{g}_{\mathfrak{al}} is identified with the canonical symplectic manifold \((T^*\mathbb{R},\omega_{\mathfrak{al}})\) as theorems 4.3 and 5.1 show.

6.2. Snakeboard

The snakeboard describes the dynamics of a skateboard but allowing the axis of the wheels to rotate by the effect of the human rider creating a torque, so that the board spins about a vertical axis; following [10, 11]. The system is modelled on \(Q = \text{SE}(2) \times S^1 \times S^1\) with coordinates \(q = (\theta, x, y, \psi, \phi)\), \(m\) is the mass of the board, \(r\) the distance from the center of the board to the pivot point of the wheel axes and \(J_0, J_0\) the inertia of the rotor and of the board respectively. The lagrangian is \(L(q,\dot{q}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + r^2\dot{\theta}^2) + \frac{1}{2}J\dot{\psi}^2 + J\psi\dot{\theta} + J_0\phi^2\) and the constraint 1-forms can be written as \(c^z = dx + r\cos\theta\cot\phi\ d\theta\) and \(c^x = dy + r\sin\theta\cot\phi\ d\theta\), with \(\phi \neq 0,\pi\). The action of the Lie group \(G = \text{SE}(2) \times S^1\) on \(Q\), given at each \((\alpha,\beta) \in \mathbb{R}\) \(G\) and \((\theta,x,y,\psi,\phi) \in Q\) by \((\theta + \alpha, x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \psi + \beta, \phi)\), is free and proper and it defines a symmetry for the nonholonomic system. In particular, the system is \(\mathbb{R}^2\)-Chaplygin and the partially reduced nonholonomic system takes place in \(T^*\tilde{Q}\) for \(\tilde{Q} \simeq \text{SO}(2) \times S^1 \times S^1\) with coordinates \((\theta,\psi,\phi)\), hamiltonian \(\tilde{H}\) and the 2-form \(\Omega = \Omega_{\mathcal{Q}} + \tilde{B}_1\).
we have that $34e$ is 14

The residual action of $F \simeq S^1 \times S^1$ on $\mathcal{Q}$ is a symmetry of $(T^* \mathcal{Q}, \bar{\Omega}, \bar{H})$.

6.2.1. Momentum map and reduction. Following [6], the momentum bundle map $\tilde{J} : T^* \mathcal{Q} \to \mathfrak{g}^*$ encodes the partially reduced horizontal gauge momenta $J_1 = (\tilde{J}, \eta_1) = E(\phi)(p_\theta - p_\phi)$ and $J_2 = (\tilde{J}, \eta_2) = p_\phi$ on $T^* \mathcal{Q}$, where $E(\phi) = \exp \left( r \int \frac{F(\phi)}{m_r} \sin^2 \phi \right)$ and $\eta_1, \eta_2 \in \Gamma(Q \times \mathfrak{g})$ with $\eta_1 = E(\phi)(e_1 - e_2)$ and $\eta_2 = e_2$. From section 3.2 we have that $B = 0$ and then proposition 3.6 is verified for $\Omega$. Let $\mathfrak{g}_{\text{red}}^* = \{\mu^1, \mu^2\}$ so that $\mu^1 = \frac{1}{E(\phi)}e^1$ and $\mu^2 = e^1 + e^2$. For $c = c_1 \mu^1 + c_2 \mu^2 \in \Gamma(Q \times \mathfrak{g}^*)$ with $c_1, c_2 \in \mathbb{R}$,

Computing $\mathcal{L}_\mu^{\gamma} Q$ and, following theorem 3.8 we conclude that the (almost) symplectic form on $\tilde{T}^{-1}(\mu)/F$ is $\omega_{\mu} = d\phi \wedge dp_\phi$. In agreement with theorems 4.3 and 5.1, we identify the leaves of $(\cdot, \cdot)_{\text{red}}$ with the canonical symplectic manifolds $(T^* S^1, \Omega_0)$ (observe that $\dim(Q) = 1$).

6.3. Chaplygin ball

The celebrated Chaplygin ball ([14, 22, 29, 34], see also [32]) consists of a rolling ball on a plane with inhomogeneous mass distribution and non sliding constraints. For $g \in SO(3)$ representing the orientation and $(x, y) \in \mathbb{R}^2$ the position of the center of mass of the ball, the Lagrangian is $L((g, x, y), (\Omega, \dot{x}, \dot{y})) = \frac{1}{2} \langle \Omega, \dot{\Omega} \rangle + \frac{1}{2} d^2 \gamma^2 + \frac{1}{2} d^2 \gamma_3$, where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the angular velocity in body coordinates, $\mathcal{L}$ the inertia tensor represented as a diagonal matrix and $m$ is the mass of the ball. Following the notation of [7], the constraints 1-forms are $\epsilon^1 = dx - r(\beta, \lambda) d\gamma^1 + e^{2} = dy + r(\alpha, \lambda) d\gamma^2$, with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are the first and second rows of $\lambda$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are the left-invariant Maurer Cartan 1-forms on $SO(3)$ and $\epsilon^2$ is the natural pairing in $\mathbb{R}^3$.

As it is known, the system admits a $G$-symmetry given by the action of the Lie group $G = SE(2)$ on $Q$. In particular, the system is $\mathbb{R}^2$-Chaplygin and therefore the partially reduced system is described on $T^* SO(3)$ with coordinates $(g, M)$ by the 2-form $\Omega = \Omega_0 - B_{\text{BCS}} = -d(M_1 \lambda_1) - m^2 \Omega - \gamma_1 \Omega_1 d\lambda_3$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the third row of $g$. Recall that $M = (M_1, M_2, M_3)$ are the angular momenta so that $M = A \Omega + m^2 \gamma \Omega_0$ for $A = I + m^2 Id$. The remaining Lie group $F = G/G_{\text{red}} \simeq S^1$ leaves invariant the system $(T^* SO(3), \Omega_0, H)$. The horizontal gauge moment $J_e = (\mathfrak{g}^e, \mathfrak{g}^e, \mathfrak{g}^e)$ descends to a partially horizontal gauge moment $J_1$ which is encoded by the canonical momentum map $J : T^* SO(3) \to \mathfrak{g}^e$ since $J_1 = (\tilde{J}, 1)$ for $1 \in \mathbb{R}$, see lemma 2.4. Therefore we can perform a standard (almost) symplectic reduction (as it was done in [4, 32]) on the system $(T^* SO(3), \Omega_0, H)$ for $\Omega_0 = \hat{\Omega} + \hat{\mathcal{B}} = -d(M_1 \lambda_1) + m^2 \gamma_1 \Omega_1 d\lambda_3$, since $I_0$ is the hamiltonian vector field associated to $J_1$.

In this example, our focus is to illustrate theorem 4.3 and enlighten the almost symplectic leaves associated to the twisted Poisson bracket described in [5, 14, 29]. In this case, $\mathcal{B} = K + \hat{\mathcal{B}}$ where $B_1 = (J, K_{\mathcal{W}}) + J_1 d\gamma_1$ and $\mathcal{B} = m^2 \gamma_1 \Omega_1$ for $\Phi = d(\gamma, \lambda)$ and $J_1 \Phi_{\mathcal{W}}$ for $\Phi_{\mathcal{W}} = d(\gamma, \lambda) = \gamma_1 d\gamma_2 \wedge d\gamma_3 + \gamma_2 d\gamma_1 \wedge d\gamma_3 + \gamma_3 d\gamma_1 \wedge d\gamma_2$, see [5, 7] for details. Let us consider the basis $\mathfrak{g}_{\mathcal{W}} = \{\gamma, \lambda, \gamma_1, \gamma_2, \gamma_3\}$ of 1-forms on $\mathcal{Q}$ with $(\tilde{p}, \tilde{p}_1, \tilde{p}_2)$ the associated coordinates on $T^* \mathcal{Q}$. Then, using remark 3.9 $J^{-1}(\gamma)/F = J^{-1}(e) = \langle \gamma, \gamma \rangle = 1, \tilde{p} = 0$, $\omega_{\mu} = \Omega_{\mathcal{W}}$, $\omega_{\mu} = \langle 1 \gamma_1 (A_1 - A_2^2) \rangle (\gamma, \gamma) \Phi_{\mathcal{W}}$, where $\Omega_{\mathcal{W}}$ is the
canonical 2-form on $T^*(S^2)$ and $Y(\gamma) = 1 - m^2 \langle A\gamma, \gamma \rangle$. In this case, we may compute a conformal factor $f_\mu$ for each leaf ($T^*(S^2), \Omega_{3\xi} + \tilde{B}_\mu$) to obtain the conformal factor for the bracket $\{\cdot, \cdot\}^B_{\text{red}}$ (see e.g. [4]).

6.4. Solids of revolution on a plane

Let us consider a convex body of revolution rolling on a plane without sliding. This example is interesting because the horizontal gauge momenta cannot be explicitly written. However, we will see that the reduced dynamics leaves on the symplectic manifolds described in theorem 3.8 that are diffeomorphic to $(T^* S^1, \Omega_x)$ as theorem 4.3 asserts.

We follow [3, 18, 19] and keep the notation and framework of the previous example. We assume that the body is invariant under rotations around $I_3$. The total mass of the body is $m$ and the position of the center of mass is represented by the coordinates $x = (x, y, z) \in \mathbb{R}^3$ while the relative position of the body is given by the matrix $g \in SO(3)$. The lagrangian $L$ is given by

$L((g, x), (\Omega, x)) = \frac{1}{2} \langle \Omega, \Omega \rangle + \frac{1}{2} m(x, \dot{x}) - mg(x, e_3)$, where $\Omega$ is the angular velocity of the body in body coordinates and $g$ is the constant of gravity.

Let $s$ be the vector from $x$ to a fixed point on the surface $S$ of the body that is represented as $s : S^2 \rightarrow S$ by $s(\gamma) = (g(\gamma_1, \gamma), g(\gamma_2, \zeta(\gamma_3))$, where $g = g(\gamma_1)$ and $\zeta = \zeta(\gamma_3)$ are the smooth functions defined in [19, chapter 6.7] that depend on the shape of the body. Hence the configuration manifold is written as $Q = \{(g, x) \in SO(3) \times \mathbb{R}^3 : z = - \langle \gamma, s \rangle \} \simeq SO(3) \times \mathbb{R}^2$.

The nonsliding constraints are given as $g'x = -\Omega \times x$ and thus the constraints 1-forms are

\[
\epsilon^1 = dx - (\alpha, s \times \lambda) \quad \text{and} \quad \epsilon^2 = dy - (\beta, s \times \lambda).
\]

For $(g, (x, y))$ coordinates on $Q$ we define the action of $G = S^1 \times SE(2)$ on $Q$ by

\[
\Psi_{((h_1, (h_2, (a, b))))}(g, (x, y)) = (\tilde{h}_2g\tilde{h}_1^{-1}, h_1h_2(x, y)' + (a, b)'),
\]

where $h_1$ and $h_2 \in SO(2)$ are orthogonal $2 \times 2$ matrices and $\tilde{h}_i = \begin{pmatrix} h_i & 0 \\ 0 & 1 \end{pmatrix} \in SO(3)$. Since this action is not free, from now on, we consider $Q$ given by the coordinates $(g, (x, y))$ with $\gamma_3 \neq \pm 1$ and, with this restriction, the action defines a symmetry of this nonholonomic system as in [6]. In particular, the system is Chaplygin for the Lie subgroup $G_u = \mathbb{R}^2$ and the partially reduced nonholonomic system is defined on $T^* SO(3)$ with $\hat{\Omega} = \Omega_{SO(3)} - B_{g, k}$. The action of $F \simeq S^1 \times S^1$ on $T^* SO(3)$, given at each $(g, M) \in T^* SO(3)$ by $\tilde{\Psi}_{((h_1, (h_2, (a, b))))}(g, M) = (\tilde{h}_2g\tilde{h}_1^{-1}, h_1M)$, defines a symmetry on $(T^* SO(3), \hat{\Omega}, \hat{H})$.

6.4.1. Momentum map and conserved quantities. Following [15, 19] (see also [6]), this example admits two $G$-invariant horizontal gauge momenta $J_1, J_2$ on $M$ that descend to two partially horizontal gauge momenta $\tilde{J}_1, \tilde{J}_2$ on $T^* SO(3)$. The bundle map $J : T^* SO(3) \rightarrow \mathfrak{t}^*$ encodes the conserved functions $J_1, J_2$ since there are $\mathfrak{t}$-valued functions $\eta_i = f_i' e_1 + f_i'' e_2$, for $f_i' \in \mathcal{C}^\infty(\hat{Q})$ and $e_1, e_2$ the canonical elements in $\mathfrak{t} \simeq \mathbb{R}^2$, with $i, j = 1, 2$ so that $J_i = \langle J, \eta_i \rangle$.

Following section 3.2 (see [7, 31]) and proposition 3.6, the partially reduced system admits a dynamical gauge transformation by $\mathcal{B} = B_1 = m\varphi(\gamma, s) \langle \Omega, d\lambda \rangle$, so that the infinitesimal generators associated to $\eta_1, \eta_2$ are hamiltonian vector fields of $\tilde{J}_1, \tilde{J}_2$ respectively with respect to the 2-form $\Omega_{\tilde{B}} = \hat{\Omega} + \mathcal{B}$. Therefore, we can perform an almost symplectic reduction as in theorem 3.8. In particular, let $\mathcal{B}_{T^* SO(3)} = \{X^{\hat{\Omega}} = -\gamma^2 \tilde{Y} \hat{Y} + \lambda^2, \hat{\tilde{Y}} = \gamma^2 \tilde{Y} + \lambda^2 \}$ with associated coordinates $(g, p_0, p_1, p_2)$ on $T^* SO(3)$ (observe that $\mathcal{B}_{T^* SO(3)}$ is dual
to $\mathfrak{b}_{\text{TSO}(3)} = \{X_0 = \gamma_1 X_1^2 - \gamma_2 X_1^1, \bar{Y}_1 = (\mathbf{e}_1)_{\text{SO}(3)}, \bar{Y}_2 = (\mathbf{e}_2)_{\text{SO}(3)}\}$. Let $\{\mu^1, \mu^2\}$ be $f^*$-valued functions dual to $\{\eta_1, \eta_2\}$. If $\mu = c_1 \mu^1 + c_2 \mu^2$ for $c_1, c_2 \in \mathbb{R}$ then $\tilde{J}^{-1}(\mu) = \{(g, p_0, p_1, p_2) : \tilde{f}_p = c_1\}$ and is determined by the coordinates $(g, p_0)$. The quotient by the action of $F$ defines the manifold $\tilde{J}^{-1}(\mu)/F$ that is diffeomorphic to $T^* S^1$ and is given by the coordinates $(\gamma_3, p_0)$.

Following theorem 3.8 and since $c^*_\mu \bar{\Omega}_b = c^*_\mu (\Omega_0 + \tilde{J}_d \tilde{\gamma}_0)$, we obtain that $\omega^b = X^0 \wedge dp_0$ which is the canonical 2-form on $T^* S^1$ for $X^0 = \frac{d\gamma_3}{d\gamma_3}$, recalling that $\gamma_3 \neq \pm 1$.

Therefore, as a consequence of theorems 4.3 and 5.1, we conclude that the reduced almost Poisson manifold $(\mathcal{M}, \{\cdot, \cdot\}_b)$, given in section 5, is in fact a Poisson manifold (as it was shown in [3, 31, 44]) with symplectic leaves symplectomorphic to $(T^* S^1, \Omega^b)$.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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