CONVERGENCE RATE OF SYNCHRONIZATION OF SYSTEMS WITH ADDITIVE NOISE

SHAHAD AL-AZZAWI, JICHENG LIU* AND XIANMING LIU

School of Mathematics and Statistics, Huazhong University of Science and Technology
Wuhan, Hubei 430074, China

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Abstract. The synchronization of stochastic differential equations (SDEs) with additive noise is investigated in pathwise sense, moreover convergence rate of synchronization is obtained. The optimality of the convergence rate is illustrated through examples.

1. Introduction. The synchronization of coupled dynamical systems is a well known phenomenon that has been observed in many sciences like biology, physics and also social science areas. It deals with coupled dynamical systems that has a common dynamical feature in an asymptotic sense. A readable descriptive account of its diversity of occurrence can be found in the Strogatz book [14], which contains an extensive list of references. In particular, synchronization provides an explanation for the emergence of spontaneous order in the dynamical behavior of coupled systems, which in isolation may exhibit chaotic dynamics. The synchronization of coupled dissipative systems has been investigated mathematically in the case of autonomous systems by [1, 3, 13], including asymptotically stable equilibria and general attractors, such as chaotic attractors. Analogous results also hold for nonautonomous systems [8], but require a new concept of a nonautonomous attractor.

Recently, the influence of noise on the synchronization in coupled dissipative dynamical systems has been studied, provided that appropriate concepts of random attractors and stochastic stationary solutions are used instead of their deterministic counterparts. The synchronization of stochastic coupled systems driven by Gaussian noise has been investigated in [4, 5]. Liu et al. in [12] showed that this synchronization phenomenon persists under additive Lévy noises, provided that asymptotically stable stochastic stationary solutions are considered. In [3], the author proved that the presence of additive noise could lead to a strengthening of the synchronization, i.e., the level of trajectories rather than attractors, which does not occur in the absence of noise. Moreover, the effects of discretization on the synchronization of dissipative systems with additive noise were investigated by Kloeden et al. [9], where it was seen that the synchronization effect was preserved.
using the drift-implicit Euler-Maruyama scheme with constant step size applied to the coupled system.

Caraballo and Kloeden in [4] considered the following two Itô stochastic differential equations in \( \mathbb{R}^d \)
\[
\begin{align*}
    dX'_t &= f(X'_t)dt + \alpha dW^1_t, \\
    dY'_t &= g(Y'_t)dt + \beta dW^2_t,
\end{align*}
\]
where \( \alpha, \beta \in \mathbb{R}^d \) are constant vectors with no components equal to zero, \( W^1_t, W^2_t \) are independent two-sided scalar Wiener processes and the continuously differential functions \( f, g \) satisfy the one-sided dissipative Lipschitz conditions
\[
\begin{align*}
    \langle x_1 - x_2, f(x_1) - f(x_2) \rangle &\leq -L \|x_1 - x_2\|^2, \\
    \langle y_1 - y_2, g(y_1) - g(y_2) \rangle &\leq -L \|y_1 - y_2\|^2
\end{align*}
\]
for any \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) and some \( L > 0 \), where \( \|x\|^2 = x_1^2 + \cdots + x_d^2 \) denote the Euclidean norm of \( x \), as well as the following integrability condition: There exists \( m_0 > 0 \) such that
\[
\int_{-\infty}^{t} e^{ms}(|f(u(s))|^2 + |g(u(s))|^2)ds < \infty
\]
for any \( m \in (0, m_0] \), and any continuous function \( u : \mathbb{R} \to \mathbb{R}^d \) with sub-exponential growth (cf. [2]), i.e. for \( \epsilon > 0 \) and \( \omega \in \Omega \) there exists a \( t_0(\epsilon, \omega) \geq 0 \) such that \( |t| \geq t_0(\epsilon, \omega) \) it holds that
\[
|u(t)| \leq e^{\epsilon|t|},
\]
which means that the exponential growth rate of \( u(t) \) is zero. Then each of the above stochastic systems has a pathwise asymptotically stable attractor consisting of a stationary random variable. The dissipatively coupled system to be synchronized corresponding to the SDE (1) is
\[
\begin{align*}
    dX'_t = f(X'_t)dt + \nu(Y'_t - X'_t)dt + \alpha dW^1_t, \\
    dY'_t = g(Y'_t)dt + \nu(X'_t - Y'_t)dt + \beta dW^2_t
\end{align*}
\]
with \( \nu > 0 \). They proved that the synchronized system (4) has a unique stochastic stationary solution \( (\bar{X}_t^\nu, \bar{Y}_t^\nu) \), which is pathwise globally asymptotically stable with
\[
(\bar{X}_t^\nu, \bar{Y}_t^\nu) \to (\bar{Z}_t^\infty, \bar{Z}_t^\infty) \text{ as } \nu \to \infty,
\]
pathwise on finite time intervals \([T_1, T_2]\) of \( \mathbb{R} \), where \( \bar{Z}_t^\infty \) is the unique pathwise globally asymptotically stable stationary solution of the averaged SDEs
\[
d\bar{Z}_t = \frac{1}{2} [f(\bar{Z}_t) + g(\bar{Z}_t)]dt + \frac{1}{2} \alpha dW^1_t + \frac{1}{2} \beta dW^2_t.
\]
When \( \alpha = \beta = 0 \), a unique equilibrium of deterministic coupled dissipative dynamical systems, which is globally asymptotically stable, converges to the unique globally asymptotically stable equilibrium of the “averaged” system. This phenomena is known as synchronization.

The main idea of the proof in [4] is based on the theory of random dynamical systems (cf. [2]) and the fact that such SDEs can be transformed to random ODEs, where the following key Lemma [1] is used to prove the boundedness of stochastic stationary solution \( (\bar{X}_t^\nu, \bar{Y}_t^\nu) \) in \( \nu > 0 \), then by Ascoli Theorem to conclude that there exists a convergence subsequence. Finally, the result was obtained from by proving pathwise all possible subsequences have the same limit and the uniqueness of stationary solution, where the limit of stationary solutions is still stationary by
the continuity of random dynamical systems $\phi(t, \omega, x)$ on $x$. The above mentioned lemma (cf. Lemma 2.3 in [4]) is

**Lemma 1.1.** Let $W_t$ be a two-sided $\mathbb{R}^{-}$-valued Wiener processes, then the integrals

$$v \int_{-\infty}^{t} e^{-\nu(t-s)}dW_s(\omega)$$

are pathwise uniformly bounded in $\nu > 0$ on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$.

However, we prove that the result of Lemma 1.1 does not hold, see the details from the below Lemma 4.2. This will deduce that the absorbed closed balls is not a common ball on $\nu$ (see Section 3), and a lack of the uniformly boundedness of stationary solutions of coupled synchronized system in $\nu$. To overcome this difficulty, in this paper we employ the techniques of the exponential martingale inequality and the Borel-Cantelli lemma. Our main innovation is that by using techniques of the exponential martingale inequality and the Borel-Cantelli lemma, we could deal with the pathwise property of the solutions of coupled synchronized systems, whereas usually it is much hard to handle stochastic differential equation with techniques from deterministic equation. Moreover, we analyze the convergence rate of synchronization on a dissipative coupled systems with additive noise, the optimality of the convergence rate is illustrated through examples. On the other hand, since only fewer kind of SDEs can be converted to random ODE, our method is probable to apply for synchronization of general stochastic differential equations, instead of random differential equations.

To be more precise, in this paper we consider the following coupled equations with additive noise in $\mathbb{R}^d$

$$\begin{align*}
\begin{cases}
    dX_t = f(X_t)dt + \alpha dW_t, \\
    dY_t = g(Y_t)dt + \beta dW_t,
\end{cases}
\end{align*}$$

(5)

where $\alpha, \beta \in \mathbb{R}^{d \times n}$ are constant matrices, $W_t$ is a two-sided $\mathbb{R}^n$-valued Wiener processes (cf. [2]), the functions $f, g : \mathbb{R}^d \to \mathbb{R}^d$ are continuously differentiable, satisfies the one-sided dissipative Lipschitz conditions [2] and the integrability condition [3], which ensure the forwards existence and uniqueness of solutions. The coupled system to be synchronized corresponding to the SDE (5) is following

$$\begin{align*}
\begin{cases}
    dX^\nu_t = f(X^\nu_t)dt + \nu(Y^\nu_t - X^\nu_t)dt + \alpha dW_t, \\
    dY^\nu_t = g(Y^\nu_t)dt + \nu(X^\nu_t - Y^\nu_t)dt + \beta dW_t
\end{cases}
\end{align*}$$

(6)

with $\nu > 0$. Compared with the model (4), we use the vector valued noise in our models. We will prove that the coupled synchronized system (6) has a unique stochastic stationary solution $(X^\nu_t, Y^\nu_t)$, which is pathwise globally asymptotically stable. Moreover, $(X^\nu_t, Y^\nu_t) \to (Z_t, Z_t)$ as $\nu$ tends to $\infty$, where $Z_t$ is the unique globally asymptotically stable equilibrium of the “averaged” system

$$dZ_t = \frac{1}{2} [f(Z_t) + g(Z_t)]dt + \frac{1}{2} (\alpha + \beta) dW_t.$$  

(7)

Our main idea is to prove the unique pathwise globally asymptotically stable stationary solution of the coupled synchronized system (6), then the stationary solutions of the coupled synchronized system are uniform bounded and uniformly sub-exponentially growing in $\nu$ and convergence to each other, finally we obtain the convergence rate of synchronization of the coupled systems and discuss the the optimality of the convergence rate through examples.
The structure of the paper is as follows. In Section 2, we recall the background of random dynamical systems. In Section 3, we prove that all the coupled systems, the synchronized systems and the averaged system have a unique stochastic stationary solution, which is globally asymptotically stable. In particular, these systems share the same constant $L$ of one-sided dissipative Lipschitz condition, but we cannot get that all these stationary solutions belong to a common compact absorbing ball for any $\nu > 0$, since the coefficient of the Ornstein-Uhlenbeck stationary processes is dependent on $\nu$. In Section 4, the properties of Ornstein-Uhlenbeck processes are discussed, including a sharp modified version of Lemma 1.1 and an improvement of Lemma 4.1 in [6], which will be used for the calculation of convergence rate. In Section 5 and Section 6, the uniformly boundedness and uniform sub-exponential growth in $\nu > 0$ and asymptotic behaviours are obtained for the stationary solutions of the coupled system, which are the key points in our method. In Section 7, synchronization persists for coupled system and a rate of convergence is established, that is, the stationary solutions of coupled synchronized system (6) converge to the unique pathwise globally asymptotically stable stationary solution of the “averaged” system (7) with the exact rate, the optimality of the convergence rate is illustrated through examples.

Throughout this paper, the capital letter $C$ will denote a constant whose values may change from line to line. Generally, it will depend on the Lipschitz constant $L$, the dimension of the vector, the time of $T_1$ and $T_2$, but not on $\nu$.

2. Random dynamical systems. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Following Arnold [2], a random dynamical system (RDS) $(\theta, \phi)$ on $\Omega \times \mathbb{R}^d$ consists of a metric dynamical system $\theta$ on $\Omega$ and a cocycle mapping $\phi: \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$. A family $\hat{A} = \{A(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $A(\omega)$ of $\mathbb{R}^d$ is called $\phi$-invariant if
\[
\phi(t, \omega, A(\omega)) = A(\theta_t \omega)
\]
for all $t \geq 0$ and is called a random attractor if in addition it is pathwise pullback attracting in the sense that
\[
H^*_d(\phi(t, \theta_-t \omega, D(\theta_-t \omega)), A(\omega)) \to 0
\]
as $t \to +\infty$ for all families $\hat{D} = \{D(\omega), \omega \in \Omega\}$ of nonempty measurable bounded subsets $D(\omega)$ of $\mathbb{R}^d$ in the given attracting universe. Here $H^*_d$ is the Hausdorff semi-distance on $\mathbb{R}^d$. The following result [14,8,2] ensures the existence of a random attractor.

**Theorem 2.1.** Let $(\theta, \phi)$ be an RDS on $\Omega \times \mathbb{R}^d$. If there exists a family $\hat{B} = \{B(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $B(\omega)$ of $\mathbb{R}^d$ and a $T_{D, \omega} > 0$ such that
\[
\phi(t, \theta_-t \omega, D(\theta_-t \omega)) \subset B(\omega), \quad \forall t \geq T_{D, \omega}
\]
for all families $\hat{D} = \{D(\omega), \omega \in \Omega\}$ in the given attracting universe, then the RDS $(\theta, \phi)$ has a random attractor $\hat{A} = \{A(\omega), \omega \in \Omega\}$ with the component subsets defined for each $\omega \in \Omega$ by
\[
A(\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \phi(t, \theta_-t \omega, B(\theta_-t \omega)).
\]
If the random attractor consists of singleton sets, i.e. \( A(\omega) = X^*(\omega) \) for some random variable \( X^* \), then \( X^*_t(\omega) := X^*(\theta_t\omega) \) is a stationary stochastic process, if the driving system \( \theta_t \) is a stationary process.

For our purposes here, \( \theta \) represents the driving noise process and \( \phi \) the state space evolution of the system. In our situation, we choose specifically \( \Omega = C_0(\mathbb{R}, \mathbb{R}) \) the space of continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \) with \( \omega(0) = 0 \), \( \mathcal{F} \) the \( \sigma \)-field on \( \Omega \) generated by the topology of uniform convergence on compact subsets, \( P \) two-sided Wiener measure on \( (\Omega, \mathcal{F}) \) and \( W_t \) be a two-sided \( \mathbb{R}^n \)-valued Wiener processes \( t \in \mathbb{R} \) on the canonical probability space \( (\Omega, \mathcal{F}, P) \), where \( W_t(\omega) = \omega(t) \). We always define for all \( t \in \mathbb{R} \),

\[
\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t).
\]

Let \( X(t, x) \) be the solution of a SDE starting at \( X(0, x) = x \in \mathbb{R}^d \), then

\[
\phi(t, \omega, x) = X(t, x), \quad \forall t \geq 0
\]

is a cocycle of homeomorphisms of \( \mathbb{R} \) on the MDS \( \theta \) (cf. [7]). We can extend \( \phi \) to a map \( \phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \) by defining

\[
\phi(-t, \omega, x) = \phi(t, \theta_{-t}\omega, x), \quad \forall t > 0.
\]

Then \( (\theta, \phi) \) forms an RDS on \( \Omega \times \mathbb{R}^d \).

From Theorem 2.1, to show that a system has a unique stochastic stationary solution, which is globally asymptotically stable, we only need to prove

\[
\|x_t\|^2 \leq \|x_{t_0}\|^2 e^{-L(t-t_0)} + C \int_{t_0}^t e^{-L(t-u)}h_u(\omega)du
\]

and

\[
\int_{-\infty}^t e^{-L(t-u)}h_u(\omega)du < \infty,
\]

which means that the system is asymptotically dissipative and has a pullback attractor \( \{A(\omega), \omega \in \Omega \} \).

\[
\lim_{t \to \infty} |x_1(t) - x_2(t)|^2 = 0,
\]

which means all solutions converge pathwise to each other. The fact that all trajectories converge to each other forwards in time, says the sets in this random attractor are singleton sets, i.e. \( A(\omega) = \{a(\omega)\} \). Then the uniqueness stationary solution is \( X_t(\omega) = a(\theta_t\omega) \).

### 3. Stochastic stationary solutions of SDEs

In this section, we will prove the uncoupled equations [5], the averaged SDE [7] and the coupled synchronized system [6] for any \( \nu > 0 \) have unique stochastic stationary solutions respectively, which are globally asymptotically stable. The method is taken from [4], but with a bit more general additive noise, where the dependence noise is allowed.

Let \( \alpha \in \mathbb{R}^{n\times d} \) are constant matrixes, \( W_t \) is a two-sided \( \mathbb{R}^d \)-valued Wiener processes and the Ornstein-Uhlenbeck process in \( \mathbb{R}^d \)

\[
dO_t = -O_t dt + dW_t,
\]

which has the explicit solution

\[
O_t = O_0 e^{-(t-t_0)} + e^{-t} \int_{t_0}^t e^s dW_s, \quad t > t_0.
\]
For fixed $t$, let $t_0$ tends to $-\infty$, the pullback limit gives the Ornstein-Uhlenbeck stochastic stationary process
\[
\overline{O}_t = \int_{-\infty}^{t} e^{-(t-s)} dW_s, \quad t \in \mathbb{R},
\]
which is the stationary solution of equation (8) and satisfies $\overline{O}_t(\omega) = O(\theta_t \omega)$, where $O(\omega) = \int_{-\infty}^{0} e^s dW_s(\omega)$.

It is easy to check that $\alpha \overline{O}_t$ satisfies
\[
dO_t = -O_t dt + \alpha dW_t,
\]
where $\alpha \in \mathbb{R}^{d \times n}$ are constant matrixes, $W_t$ is a two-sided $R^n$-valued Wiener processes. We consider the uncoupled equations in $\mathbb{R}^d$
\[
\begin{align*}
    dX_t &= f(X_t) dt + \alpha dW_t, \\
    dY_t &= g(Y_t) dt + \beta dW_t,
\end{align*}
\]
where $\alpha, \beta \in \mathbb{R}^{d \times n}$ are constant matrixes, $W_t$ is a two-sided $R^n$-valued Wiener processes, continuously differential function $f, g$ satisfies the one-sided dissipative Lipschitz conditions (2) with the constant $L$ and integrability condition (3) with the constant $m_0$. Without loss of generality, we can assume that $L \leq m_0$. Its solution paths are generally not differentiable. Thus we rewrite (9) as
\[
dX_t = [f(X_t) + \alpha \overline{O}_t] dt + \alpha d\overline{O}_t,
\]
Using the transformation
\[
x(t, w) = X_t(w) - \alpha \overline{O}_t(w),
\]
then we transform (9) to the pathwise random ordinary differential equation
\[
\frac{dx}{dt} = F(x, \alpha \overline{O}_t(w)) := f(x(w) + \alpha \overline{O}_t(w)) + \alpha \overline{O}_t(w), \quad (10)
\]
The Vector-field function
\[
\tilde{g}(x, z) = f(x + z) + z,
\]
in the RODE (10) satisfies a one-sided Lipschitz condition in its first variable uniformly in the second variable with the same constant as the original drift coefficient $f$, since we have
\[
\langle x_1 - x_2, \tilde{g}(x_1, z), \tilde{g}(x_2, z) \rangle = \langle x_1 - x_2, f(x_1 + z) + z - f(x_2 + z) + z \rangle = \langle x_1 + z - (x_2 + z), f(x_1 + z) - f(x_2 + z) \rangle \leq -L|x_1 - x_2|^2.
\]
We obtain that any of two solutions of the RODEs (10) satisfy pathwise the differential inequality
\[
\frac{d}{dt}|x_1(t) - x_2(t)|^2 = 2\langle (x_1(t) - x_2(t)), \frac{d}{dt}(x_1(t) - x_2(t)) \rangle \leq -2L|x_1(t) - x_2(t)|^2 \quad (11)
\]
and hence we have
\[
|x_1(t) - x_2(t)|^2 \leq e^{-2Lt}|x_1(0) - x_2(0)|^2
\]
Thus it follows that
\[
\lim_{t \to \infty} |x_1(t) - x_2(t)|^2 = 0,
\]
which means all solutions converge pathwise to each other.

In order to see what they converge to, we first observe that the RODE \([10]\) generates a random dynamical system with \(\phi(t,\omega,x_0) := x(t,\omega)\), the solution of the RODE \([10]\) with deterministic initial value \(x_0\) at time \(t = 0\). Then we need to show that the RODEs \([10]\) is asymptotically dissipative and has a pullback attractor. Omitting \(\omega\) for brevity, we have pathwise

\[
\frac{d}{dt}|x|^2 = 2\langle x, F(x, a\bar{O}_t) \rangle \\
= 2\langle x - 0, F(x, a\bar{O}_t) - F(0, a\bar{O}_t) \rangle + 2\langle x, F(0, \bar{O}_t) \rangle \\
\leq -2L|x|^2 + L|x|^2 + \frac{1}{L}|f(a\bar{O}_t) + a\bar{O}_t)|^2 \\
= -L|x|^2 + \frac{1}{L}|f(a\bar{O}_t) + a\bar{O}_t)|^2.
\]

Integration yields

\[
|x(t)|^2 \leq |x(t_0)|^2 e^{-L(t-t_0)} + \frac{1}{L} \int_{t_0}^{t} e^{-L(t-u)}|f(a\bar{O}_u) + a\bar{O}_u|^2 du.
\]

Now we can use pathwise pullback techniques (i.e. with \(t_0 \to -\infty\)) to show that the closed ball \(B_0(\omega)\) in \(\mathbb{R}^{2d}\) centered at the origin with random radius \(R_0(\omega)\), where is defined by

\[
R_0^2(\omega) := 1 + \frac{1}{L} \int_{-\infty}^{t} e^{-L(t-u)}|f(a\bar{O}_u) + a\bar{O}_u|^2 du.
\]

is a pullback absorbing set for \(t > T_{B,\omega}\) for appropriate families \(\hat{D} = \{D(\omega), \omega \in \Omega\}\) of bounded subsets of \(\mathbb{R}^{d}\) of initial conditions. Theorem \(2.1\) of RDS then gives us a random attractor \(\{A(\omega), \omega \in \Omega\}\) in \(B(\omega)\) for each \(\omega\). Note that all trajectories converge to each other forwards in time. Then the sets in this random attractor are singleton sets, i.e. \(A(\omega) = \{a(\omega)\}\). When transforming back to the SDEs, we have the pathwise singleton set attractor \(a(\theta(\omega)) + \bar{O}(\theta(\omega))\), which is a unique stationary solution the first SDE of \([9]\) and globally asymptotically stable, since the Ornstein-Uhlenbeck process is stationary.

An analogous situation holds for the second equation of the uncoupled equations \([9]\) and for the averaged SDE \([7]\). Now we deal with the coupled system to be synchronized corresponding to the SDE \([6]\)

\[
\begin{align*}
\frac{dX_t^\nu}{dt} &= f(X_t^\nu) + \nu(Y_t^\nu - X_t^\nu)dt + \alpha dW_t, \\
\frac{dY_t^\nu}{dt} &= g(Y_t^\nu) + \nu(X_t^\nu - Y_t^\nu)dt + \beta dW_t,
\end{align*}
\]

with \(\nu > 0\). We will consider the coupled system \([12]\) as a system like \([9]\) in a higher dimensional space \(\mathbb{R}^{2d}\), and prove that the coupled system has essentially the same structure as \([9]\) in \(\mathbb{R}^{2d}\) for any fixed \(\nu > 0\), that is, coefficients of the coupled system also satisfies a dissipative one-sided Lipschitz condition with the same constant \(L\). If so, by the above discussion, we know that the coupled system \([12]\) have a unique stationary solution and is globally asymptotically stable for any fixed \(\nu > 0\). In fact, set

\[
\xi = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(\xi) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \quad B = \begin{pmatrix} -I_d & I_d \\ I_d & -I_d \end{pmatrix}, \quad D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
\]
with the $d \times d$ identity matrix $I_d$. Then the coupled synchronized system (12) is now

$$d\xi_t = G(\xi_t) + DdW_t$$

with $G(\xi) = F(\xi) + \nu B\xi$. It is easy to check that

$$\xi^T B\xi = -|x - y|_2^2$$

and $G(\xi)$ satisfies a dissipative one-sided Lipschitz condition with the same constant $L$, since for any $\xi = (x, y)^T$, $\eta = (x', y')^T \in \mathbb{R}^{2d}$, we have

$$\langle \xi - \eta, G(\xi) - G(\eta) \rangle_{2d}$$

$$= \langle (x - x', y - y')^T, (f(x) - f(x'), g(y) - g(y'))^T \rangle_{2d} + \nu \langle \xi - \eta \rangle^T B(\xi - \eta)$$

$$= \langle x - x', f(x) - f(x') \rangle_d + \langle y - y', g(y) - g(y') \rangle_d - \nu \langle (x - x') - (y - y') \rangle_2^2$$

$$\leq -L([|x - x'|_2^2 + |(y - y')|_2^2] = -L(|\xi - \eta|_2^2).$$

We also use the notation $(x(t), y(t))$ to denote the unique stationary solution of the coupled system (12), which are globally asymptotically stable for any fixed $\nu > 0$. In this case,

$$\frac{d}{dt} |\xi(t)|^2 = 2\langle \xi, G(\xi, D\overline{\omega}_t) \rangle$$

$$= 2\langle \xi - 0, G(\xi, \overline{\omega}_t) - G(0, D\overline{\omega}_t) \rangle + 2\langle \xi, G(0, D\overline{\omega}_t) \rangle$$

$$\leq -L|\xi|^2 - \nu|x - y|^2 + \frac{2}{L} |F(D\overline{\omega}_t) + D\overline{\omega}_t|^2, + \frac{2}{L} |(\nu B + I)D\overline{\omega}_t|^2,$$

where we still use the notation $\overline{\omega}_t$ by the Ornstein-Uhlenbeck stochastic stationary process in $\mathbb{R}^{2d}$. Integration yields

$$|x(t)|^2 + |y(t)|^2 + \nu \int_{t_0}^t e^{-L(t-u)} |x(u) - y(u)|^2 du$$

$$\leq (|x(t_0)|^2 + |y(t_0)|^2) e^{-L(t-t_0)} + \frac{2}{L} \int_{t_0}^t e^{-L(t-u)} |F(D\overline{\omega}_u) + D\overline{\omega}_u|^2 du$$

$$+ \frac{2}{L} \int_{t_0}^t e^{-L(t-u)} |(\nu B + I)D\overline{\omega}_u|^2 du.$$

The pathwise pullback techniques give

$$|x(t)|^2 + |y(t)|^2 \leq R^2_{\nu,l}(\theta, \omega)$$

for $t > T_{\overline{\omega}}$ for appropriate families $\hat{D} = \{D(\omega), \omega \in \Omega\}$ of bounded subsets of $\mathbb{R}^d$ of initial conditions, where $R_{L,\nu}(\omega)$ is defined by

$$R^2_{L,\nu}(\omega) := 1 + \frac{2}{L} \int_{-\infty}^0 e^{Lu} (|f(\alpha\overline{\omega}_u) + \alpha\overline{\omega}_u|^2 + |g(\beta\overline{\omega}_u) + \beta\overline{\omega}_u|^2) du$$

$$+ \frac{2}{L} \int_{-\infty}^0 e^{Lu} |(\nu B + I)D\overline{\omega}_u|^2 du.$$

We denote closed ball in $\mathbb{R}^{2d}$ centered at the origin with random radius $R_{\nu,L}(\omega)$ by $B_{\nu,L}(\omega)$, which is dependent of parameter $\nu$. Furthermore,

$$R^2_{L,\nu}(\theta, \omega) = 1 + \frac{2}{L} \int_{-\infty}^t e^{-L(t-u)} (|f(\alpha\overline{\omega}_u) + \alpha\overline{\omega}_u|^2 + |g(\beta\overline{\omega}_u) + \beta\overline{\omega}_u|^2) du$$
+ \frac{2}{L} \int_{-\infty}^{t} e^{-L(t-u)}[(\nu B + I)D\bar{O}_u]^2 du.

Noticing that $R_i^2$ is decreasing on $0 < l \leq L$ and the condition \[3\], we have for any $\epsilon > 0$, take $l < \epsilon$

$$\lim_{t \to \pm \infty} e^{-\epsilon |t|} R_{l,\nu}^2(\theta \omega) = 0 \quad (13)$$

for any $\nu$, which means that the stationary solution $(x(t), y(t))$ of the coupled system \[12\] has sub-exponential growth. From now on, $(X_t^\nu, Y_t^\nu)$ denotes the pair stationary solution of the coupled SDE \[12\]. Note that the stationary Ornstein-Uhlenbeck process $\bar{O}_t$ has sub-exponential growth, $(X_t^\nu, Y_t^\nu)$ also has sub-exponential growth.

We emphasize here that the uniformly boundedness of the stationary solutions in $\nu$ is still not established, since the closed ball $B_\nu(\omega)$ depends on $\nu$.

4. The Ornstein-Uhlenbeck processes and Brownian motion. Let the Ornstein-Uhlenbeck process

$$dO^{(\nu,\alpha)}_t = -\nu O^{(\nu,\alpha)}_t dt + \alpha dW_t \quad (14)$$

and the initial value

$$O^{(\nu,\alpha)}_0 = \alpha \int_{-\infty}^{0} e^{\nu s} dW_s,$$

where $\nu > 0$ and $W_t$ is a two-sided $\mathbb{R}$-valued Wiener processes. The solution of equation \[14\] can be expressed as

$$O^{(\nu,\alpha)}_t = \alpha \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s$$

$$= \alpha e^{-\nu t} \int_{-\infty}^{t} e^{\nu s} dW_s$$

$$= e^{-\nu t} \left(O^{(\nu,\alpha)}_0 + \alpha \int_{0}^{t} e^{\nu s} dW_s\right).$$

which is the stationary solution of equation \[14\]. On the other hand, from \[14\] we have

$$O^{(\nu,\alpha)}_t = O^{(\nu,\alpha)}_s - \nu \int_{s}^{t} O^{(\nu,\alpha)}_u du + \alpha (W_t - W_s). \quad (15)$$

It is easy to check that $O^{(\nu,\alpha)}_t$ converges to zero in $L^2(\Omega)$ as $\nu$ tends to infinity for any $t$. In fact,

$$E \left(O^{(\nu,\alpha)}_t\right)^2 = E \left(\alpha \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s\right)^2$$

$$= \alpha^2 \int_{-\infty}^{t} e^{-2\nu(t-s)} ds$$

$$= \alpha^2 \frac{1}{2\nu} \to 0, \text{ as } \nu \to \infty.$$ 

In particular, if $s = 0$, from \[15\] we obtain that

$$\lim_{\nu \to \infty} \nu \int_{0}^{t} O^{(\nu,\alpha)}_s ds = \alpha W_t \text{ in } L^2(\Omega).$$

Moreover, we claim that
Lemma 4.1. For any $\gamma < \frac{1}{2}$, there exists a random variable $G$ such that
\[
O_t^{(\nu,\alpha)} \leq G(\omega)\nu^{-\gamma}, \text{ a.s.}
\]
for any $t \in \mathbb{R}$, where $EG^\gamma(\omega) < \infty$. In turn, $O_t^{(\nu,\alpha)}$ converges to zero almost surely as $\nu$ tends to infinity for any $t \in \mathbb{R}$. Moreover, from (15) we have
\[
\lim_{\nu \to \infty} \nu \int_0^t O_s^{(\nu,\alpha)} ds = \alpha W_t, \text{ a.s.}
\]
for any $t \in \mathbb{R}$.

Proof. Set $\epsilon = \frac{1}{\nu}$ and $O_t^{(\epsilon,\alpha)} = O_t^{(\frac{1}{\nu},\alpha)}$, Define
\[
\tilde{O}(t, \epsilon) = \begin{cases} O_t^{(\epsilon,\alpha)}, & \text{if } \epsilon > 0, \\ 0, & \text{if } \epsilon = 0. \end{cases}
\]
To prove this lemma, by the Kolmogorov’s criterion (cf. [11] Theorem 2.1), we need only to prove
\[
E|\tilde{O}(t, \epsilon) - \tilde{O}(t, \epsilon')|^p \leq C|\epsilon - \epsilon'|^{p/2}
\]
for all $p, t \in \mathbb{R}$ and $\epsilon > 0$. It is enough to prove the following inequalities
\[
E|O_t^{(\epsilon,\alpha)}|^p \leq C\epsilon^{p/2}
\]
and
\[
E|O_t^{(\epsilon,\alpha)} - O_t^{(\epsilon',\alpha)}|^p \leq C|\epsilon - \epsilon'|^{p/2}.
\]
For (16), we have
\[
E|O_t^{(\epsilon,\alpha)}|^p = E \left| \alpha \int_{-\infty}^t e^{-\frac{1}{2}(t-s)} dW_s \right|^p
\leq C|\alpha|^p E \left( \int_{-\infty}^t e^{-\frac{2}{2}(t-s)} ds \right)^{p/2}
= C|\alpha|^p \left| \frac{\epsilon}{2} \right|^{p/2} \leq C|\alpha|^p |\epsilon|^{p/2}.
\]
For (17), noticing that
\[
\int_{-\infty}^t \left( e^{-\frac{1}{2}(t-s)} - e^{-\frac{1}{2}(t-s)} \right)^2 ds = \frac{\epsilon}{2} + \frac{\epsilon'}{2} - \frac{2\epsilon\epsilon'}{\epsilon + \epsilon'}
= \frac{(\epsilon - \epsilon')^2}{2(\epsilon + \epsilon')}
\leq \frac{1}{2} |\epsilon - \epsilon'|,
\]
we have
\[
E|O_t^{(\epsilon,\alpha)} - O_t^{(\epsilon',\alpha)}|^p = E \left| \alpha \int_{-\infty}^t e^{-\frac{1}{2}(t-s)} - e^{-\frac{1}{2}(t-s)} dW_s \right|^p
\leq C|\alpha|^p E \left( \int_{-\infty}^t \left( e^{-\frac{1}{2}(t-s)} - e^{-\frac{1}{2}(t-s)} \right)^2 ds \right)^{p/2}
\leq C|\alpha|^p |\epsilon - \epsilon'|^{p/2}.
\]
\[\square\]
Lemma 4.2. Let $W_t$ be a two-sided $\mathbb{R}$-valued Wiener processes and the integrals

$$N_{\nu}^\gamma := \nu^\gamma \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s(\omega)$$

pathwise unbounded in $\nu > 0$ for any $\gamma > \frac{1}{2}$ and $t \in \mathbb{R}$.

Proof. Note that for any $\gamma > \frac{1}{2}$

$$N_{\nu}^\gamma = \nu^\gamma \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s(\omega) \sim N\left(0, \frac{1}{2}\nu^{(2\gamma-1)}\right).$$

Hence

$$Ee^{-\left(N_{\nu}^\gamma\right)^2} = \sqrt{\nu^{1-2\gamma} 1 + \nu^{1-2\gamma}} \leq \nu^{1/2-\gamma},$$

which means that $e^{-\left(N_{\nu}^\gamma\right)^2}$ converges to zero in $L^1(\Omega)$ as $\nu$ tends to $\infty$. It follows that there exists a subsequence $e^{-\left(N_{\nu_n}^\gamma\right)^2}$ of $e^{-\left(N_{\nu}^\gamma\right)^2}$ such that $e^{-\left(N_{\nu_n}^\gamma\right)^2}$ converges to zero almost surely as $n$ tends to infinity, that is, $(N_{\nu_n}^\gamma)^2$ converges to infinity almost surely. Therefore, $N_{\nu}^\gamma$ is unbounded in $\nu > 0$ and the proof is complete. □

Remark 1. By the above two lemmas, we have

$$\nu^\gamma \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s(\omega)$$

are pathwise bounded in $\nu > 0$ for any $\gamma < \frac{1}{2}$ and $t > 0$ and pathwise unbounded in $\nu > 0$ for any $\gamma > \frac{1}{2}$ and $t > 0$. Hence the integrals

$$\nu \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s(\omega)$$

are pathwise unbounded in $\nu > 0$ for any $t > 0$, which implies that the result of Lemma 2.3 in [4] does not hold.

Lemma 4.3. Let $W_t$ be a two-sided $\mathbb{R}$-valued Wiener processes and the integrals

$$\hat{N}_{\nu}^\gamma := \frac{1}{\sqrt{\ln t}} \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s(\omega)$$

and

$$\tilde{N}_{\nu}^\gamma := \frac{1}{\sqrt{\ln(-t)}} \int_{-\infty}^{t} e^{-\nu(t-s)} dW_s(\omega).$$

Then

$$\limsup_{t \to \infty} |N_{\nu}^\gamma| = \limsup_{t \to -\infty} |\tilde{N}_{\nu}^\gamma| = \frac{1}{\sqrt{\nu}}$$

pathwise for any $\nu > 0$.

Proof. Set

$$M_t := \sqrt{2\nu} \int_{-\infty}^{t} e^{\nu s} dW_s(\omega),$$

then

$$\hat{N}_{\nu}^\gamma = \frac{1}{\sqrt{2\nu \ln t}} e^{-\nu t} M_t.$$
When 

SetRemark 2. By the Lemma 4.3, both \( \lim_{t \to \infty} \) motion, we obtain the desired result. The second part is similar, we omit it.

Similarly,

\[ \tilde{N}_t = \frac{1}{2\nu \ln t} e^{-\nu t} B_{2\nu t} = \frac{1}{\sqrt{2\nu \ln t}} \frac{1}{\sqrt{2\nu (\ln \ln t - \ln(2\nu))}} B_t \]

\[ = \frac{1}{\sqrt{2\nu \ln (1 - \frac{\ln(2\nu)}{\ln t})}} \frac{1}{\sqrt{2\tau \ln \ln t}} B_t. \]

When \( t \) tends to \( \infty \), \( \tau \) tends to \( \infty \). By the law of the iterated logarithm for Brownian motion, we obtain the desired result. The second part is similar, we omit it. \( \square \)

Remark 2. By the Lemma 4.3, both \( \lim_{t \to \infty} O_t^{\nu,\alpha}(\omega) \) and \( \lim_{t \to \infty} O_t^{\nu,\alpha}(\omega) \) diverge. But both \( \lim_{t \to \infty} t^{-\gamma} O_t^{\nu,\alpha}(\omega) = 0 \) and \( \lim_{t \to \infty} t^{\nu} O_t^{\nu,\alpha}(\omega) = 0 \) for any \( \gamma > 0 \), which is a sharp improvement of Lemma 4.1 in [6].

5. The uniformly boundedness of stationary solutions to the synchronized system. In this section, we will prove the stationary solution of the coupled system is uniformly bounded and sub-exponentially growing in \( \nu > 0 \). Consider the following synchronized system in \( \mathbb{R}^d \)

\[
\begin{align*}
\dot{X}_t &= f(X_t) + \nu(Y_t^\nu - X_t^\nu)dt + \alpha dW_t, \\
\dot{Y}_t &= g(Y_t) + \nu(X_t^\nu - Y_t^\nu)dt + \beta dW_t,
\end{align*}
\]

where \( \alpha, \beta \in \mathbb{R}^{n \times d} \) are constant matrices, \( W_t \) is a two-sided \( \mathbb{R}^n \)-valued Wiener processes, and \( f \) and \( g \) satisfy one-sided dissipative Lipschitz conditions [2] with \( L \leq m_0 \). If set \( x_1 = x, x_2 = 0 \) in [2], then we have the following dissipative monotone conditions

\[ \langle x, f(x) \rangle \leq \langle x, f(0) \rangle - L|x|^2 \leq \frac{1}{2L}|f(0)|^2 + \frac{L}{2}|x|^2 - L|x|^2 = \frac{1}{2L}|f(0)|^2 - \frac{L}{2}|x|^2. \]

Similarly,

\[ \langle y, g(y) \rangle \leq \frac{1}{2L}|g(0)|^2 - \frac{L}{2}|y|^2. \]

Set

\[ \xi_t = \begin{pmatrix} X_t^\nu \\ Y_t^\nu \end{pmatrix}, \quad F(\xi) = \begin{pmatrix} f(X_t^\nu) \\ g(Y_t^\nu) \end{pmatrix}, \quad B = \begin{pmatrix} -I_d & I_d \\ I_d & -I_d \end{pmatrix}, \quad D = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \]

with the \( d \times d \) identity matrix \( I_d \). Then the coupled synchronized system (18) is now

\[ d\xi_t = G(\xi_t)dt + DdW_t \]

with \( G(\xi) = F(\xi) + \nu B\xi \). It is easy to check that

\[ \langle B\xi, \xi \rangle = \xi^T B\xi = -|X_t^\nu - Y_t^\nu|^2 \]

and \( F(\xi) \) satisfies a dissipative monotone condition with the constant \( L/2 \), i.e.,

\[ \langle F(\xi), \xi \rangle \leq \frac{1}{2L}|f(0)|^2 - \frac{L}{2}|\xi|^2. \]
Lemma 5.1. Under the condition (2), we have that both $X_t^\nu$ and $Y_t^\nu$ are pathwise uniformly bounded in $\nu > 0$ on finite time intervals $[T_1, T_2]$ of $\mathbb{R}$ and uniformly sub-exponentially growing in $\nu > 0$, where $(X_t^\nu, Y_t^\nu)$ is a pair stationary solutions of the coupled SDE (18).

Proof. Let $x \in \mathbb{R}^{2d}$ and for $|x|^2$, by Itô’s formula and the monotone conditions, we have

$$
e^{Lt|x|^2} = e^{L_{t_0}|x|^2} + \int_{t_0}^{t} e^{Ls} [2\langle \xi_s^\nu, X_s^\nu \rangle + |D|] ds + M_t$$

where

$$|D|^2 = \text{trace}(D^T D)$$

is the Hilbert-Schmidt norm of the matrix $D$ by $|D|^2 = |\alpha|^2 + |\beta|^2 = \sum_{i=1}^{d} \sum_{j=1}^{n} (\alpha_{ij}^2 + \beta_{ij}^2)$, and

$$M_t = 2 \int_{t_0}^{t} e^{Ls} \langle \xi_s^\nu, DdW_s \rangle.$$

On the other hand, assign $\delta > 0$ arbitrarily, for every integer $n \geq t_0$, using the exponential martingale inequality (i.e. [10] Theorem 1.7.4), one sees that

$$P \left\{ \sup_{t \in [t_0, n\delta]} \left[ M_t - 2e^{-n\delta L} \int_{t_0}^{t} e^{2Ls} |\xi_s^\nu D|^2 ds \right] > 2e^{n\delta L} \ln n \right\} \leq \frac{1}{n^2}.$$

The Borel-Cantelli lemma then yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n(\omega) \geq t_0 + 1$ such that

$$\sup_{t \in [t_0, n\delta]} \left[ M_t - 2e^{-n\delta L} \int_{t_0}^{t} e^{2Ls} |\xi_s^\nu D|^2 ds \right] \leq 2 \ln n$$

whenever $n \geq n_0$. That is,

$$M_t \leq 2e^{-n\delta L} \int_{t_0}^{t} e^{2Ls} |\xi_s^\nu D|^2 ds + 2e^{n\delta L} \ln n$$

for all $t \in [t_0, n\delta]$, $n \geq n_0$ almost surely. Substituting this into (19) deduces that

$$e^{Lt|x|^2} + 2\nu \int_{t_0}^{t} e^{Ls} |X_s^\nu - Y_s^\nu|^2 ds$$

$$\leq e^{L_{t_0}|x|^2} + \frac{e^{Lt}}{L} \left( \frac{1}{L} |F(0)|^2 + |D|^2 \right)$$

$$+ 2e^{n\delta L} \ln n + 2|D|^2 e^{-n\delta L} \int_{t_0}^{t} e^{Ls} (e^{Ls} |\xi|^2) ds,$$
which then implies, by the Gronwall inequality, that

\[
|\xi_t^\nu|^2 
\leq e^{-Lt} \left( e^{Lt_0} |\xi_0^\nu|^2 + \frac{e^{Lt} - e^{Lt_0}}{L} \left[ \frac{1}{L} |F(0)|^2 + |D|^2 \right] + 2e^{n\delta\ln n} \right) e^{\frac{2(D)^2}{L}}
\]  

(20)

for all \( t \in [t_0, n\delta] \), \( n \geq n_0 \) almost surely. By (20) and (13), letting \( t_0 \) tends to infinity, we have pathwise \( t \in [(n-1)\delta, n\delta] \)

\[
|\xi_t^\nu|^2 \leq \left( \frac{e^{Lt}}{L} \left[ \frac{1}{L} |F(0)|^2 + |D|^2 \right] + 2e^{n\delta\ln n} \right) e^{\frac{2(D)^2}{L}}
\]

\[
= \left( \frac{1}{L} \left[ \frac{1}{L} |F(0)|^2 + |D|^2 \right] + 2e^{(n\delta-t)L\ln n} \right) e^{\frac{2(D)^2}{L}}
\]

\[
\leq \left( \frac{1}{L} \left[ \frac{1}{L} |F(0)|^2 + |D|^2 \right] + 2e^{(n\delta-t)L\ln \frac{t}{\delta} + 1} \right) e^{\frac{2(D)^2}{L}} =: R_{L,\delta}(t)
\]

for any \( \delta > 0 \). Moreover, for any \( \epsilon > 0 \),

\[
\lim_{t \to \pm \infty} e^{-\epsilon t} R_{L,\delta}(t) = 0,
\]

(21)

which implies that \( |X_t^\nu|^2 + |Y_t^\nu|^2 = |\xi_t^\nu|^2 \) are pathwise uniformly bounded in \( \nu > 0 \) on finite time intervals \( [T_1, T_2] \) of \( \mathbb{R} \) and uniformly sub-exponentially growing in \( \nu > 0 \), since \( R_{L,\delta}(t) \) is independent of \( \nu \). The proof is complete.

6. The asymptotic behaviour of the synchronized system. In this section, we will prove that the stationary solution of coupled synchronized system is very close when the parameter \( \nu \) is large enough.

**Lemma 6.1.** Under the condition [2], for any \( \gamma < \frac{1}{2} \), we have

\[
|X_t^\nu - Y_t^\nu| \leq \frac{C}{\nu^\gamma},
\]

uniformly in \( t \in [T_1, T_2] \) for any bounded \( T_1 \) and \( T_2 \). In particular, we have

\[
\lim_{\nu \to \infty} |X_t^\nu - Y_t^\nu| = 0
\]

pathwise uniformly in \( t \in [T_1, T_2] \) for any bounded \( T_1 \) and \( T_2 \). Moreover, for any \( \gamma < 1 \), pathwise

\[
\int_{-\infty}^{t} e^{-L(t-s)} |X_{\nu}(s) - Y_{\nu}(s)|^2 ds \leq \frac{C}{\nu^\gamma}
\]

(22)

holds.

**Proof.** Using the transformations

\[
X_t^\nu = x_t^\nu + O_t^{(\mu,\alpha)}, \quad Y_t^\nu = y_t^\nu + O_t^{(\mu,\beta)}
\]

with

\[
O_t^{(\mu,\alpha)} = \alpha \int_{-\infty}^{t} e^{-\mu(t-u)} dW_u, \quad O_t^{(\mu,\beta)} = \beta \int_{-\infty}^{t} e^{-\mu(t-u)} dW_u,
\]

which satisfy the SDEs

\[
dO_t^{(\mu,\alpha)} = -\mu O_t^{(\mu,\alpha)} dt + \alpha dW_t, \quad dO_t^{(\mu,\beta)} = -\mu O_t^{(\mu,\beta)} dt + \beta dW_t.
\]
From (16) we have
\begin{align*}
\frac{dx_\nu^\nu(t)}{dt} &= f(X_\nu^\nu) + \nu(y_\nu^\nu(t) - x_\nu^\nu(t)) + \nu(O_\nu^{(\mu,\beta)} - O_\nu^{(\mu,\alpha)}) + \mu O_\nu^{(\mu,\alpha)}, \\
\frac{dy_\nu^\nu(t)}{dt} &= g(Y_\nu^\nu) + \nu(x_\nu^\nu(t) - y_\nu^\nu(t)) + \nu(O_\nu^{(\mu,\alpha)} - O_t^{(\mu,\beta)}) + \mu O_t^{(\mu,\beta)}.
\end{align*}
(23)

From (23), we have
\[ \frac{d}{dt}(x_\nu^\nu(t) - y_\nu^\nu(t)) = f(X_\nu^\nu) - g(Y_\nu^\nu) - 2\nu(x_\nu^\nu(t) - y_\nu^\nu(t)) + (2\nu - \mu)(O_t^{(\mu,\beta)} - O_t^{(\mu,\alpha)}). \]

Let \( \mu = 2\nu \), and define \( x_\nu(t) = x_{2\nu}^\nu(t) \) and \( y_\nu(t) = y_{2\nu}^\nu(t) \), we have
\[ \frac{d}{dt}(x_\nu(t) - y_\nu(t)) = f(X_\nu^\nu) - g(Y_\nu^\nu) - 2\nu(x_\nu - y_\nu) \]
and
\[ \frac{d}{dt}|x_\nu(t) - y_\nu(t)|^2 = 2(x_\nu(t) - y_\nu(t)) \frac{d}{dt}(x_\nu(t) - y_\nu(t)) \]
\[ = -4\nu|x_\nu(t) - y_\nu(t)|^2 + 2(x_\nu(t) - y_\nu(t), f(X_\nu^\nu) - g(Y_\nu^\nu)) \]
\[ \leq -2\nu|x_\nu(t) - y_\nu(t)|^2 + \frac{2}{\nu}(|f(X_\nu^\nu)|^2 + |g(Y_\nu^\nu)|^2) \]
\[ \leq -2\nu|x_\nu(t) - y_\nu(t)|^2 + \frac{1}{\nu}M_{T_1, T_2, \omega}^\nu. \]
with
\[ M_{T_1, T_2, \omega} = \sup_{t \in [T_1, T_2], \nu > 0} (|f(X_\nu^\nu)|^2 + |g(Y_\nu^\nu)|^2). \]

Hence by Lemma 6.1 and the continuity of \( f \) and \( g \), \( M_{T_1, T_2, \omega}^\nu \) is a finite random variable, and thus by the uniformly boundedness of \( X_\nu^\nu, Y_\nu^\nu, O_t^{(2\nu, \beta)} \) and \( O_t^{(2\nu, \beta)} \), we have
\[ |x_\nu(t) - y_\nu(t)|^2 \leq e^{-2\nu(t - T_1)}|x_\nu(T_1) - y_\nu(T_1)|^2 + \frac{1}{\nu} \int_{T_1}^t e^{-2\nu(t - s)}M_{T_1, T_2, \omega}^\nu ds \leq \frac{C}{\nu^2} \]
pathwise uniformly in \( \nu > 0 \), \( t \in [T_1, T_2] \) for any bounded \( T_1 \) and \( T_2 \). Moreover, since
\[ X_\nu^\nu = x_\nu(t) + O_t^{(2\nu, \alpha)}, \ Y_\nu^\nu = y_\nu(t) + O_t^{(2\nu, \beta)}, \]
by Lemma 4.2 and Lemma 6.1 for any \( \gamma < \frac{1}{2} \) we have
\[ |X_\nu^\nu - Y_\nu^\nu| \leq |x_\nu(t) - y_\nu(t)| + |O_t^{(2\nu, \alpha)}| + |O_t^{(2\nu, \beta)}| \leq \frac{C(T_1, T_2, \omega)}{\nu^\gamma} \]
uniformly in \( t \in [T_1, T_2] \) for any bounded \( T_1 \) and \( T_2 \). Thus \( |X_\nu^\nu - Y_\nu^\nu| \) pathwise converges to zero as \( \nu \) tends to \( \infty \) uniformly in \( t \in [T_1, T_2] \).

On the other hand, we have
\[ \int_{T_1}^t e^{-L(t - s)}|x_\nu(s) - y_\nu(s)|^2 ds \]
\[ \leq e^{-L(t - T_1)}|x_\nu(T_1) - y_\nu(T_1)|^2 + \frac{2}{\nu} \int_{T_1}^t e^{-L(t - s)} (|f(X_\nu^\nu(s))|^2 + |g(Y_\nu^\nu(s))|^2) ds, \]
Thus we observe that $\hat{Z}$ is the attracting stationary solution of the averaged SDE (25) of $Z_t$ is the attracting stationary solution of the averaged SDE

\[dZ_t = \frac{1}{2}(f(Z_t) + g(Z_t))dt + \frac{1}{2}(\alpha + \beta)dW_t.\] (24)

Furthermore, if we also assume $f$ and $g$ satisfy the Lipschitz conditions, i.e.,

\[|f(x) - f(y)| + |g(x) - g(y)| \leq C(|x - y|),\] (25)

then we have for any $\gamma < \frac{1}{2}$

\[|X^\nu_t - Z_t| + |Y^\nu_t - Z_t| \leq \frac{C}{\nu^\gamma}\]

pathwise uniformly on bounded time intervals $[T_1, T_2]$ of $\mathbb{R}$.

**Proof.** Set $\tilde{Z}^\nu_t = \frac{1}{2}(X^\nu_t + Y^\nu_t)$ and $\tilde{Z}^\nu_t = \frac{1}{2}(X^\nu_t - Y^\nu_t)$, we have

\[X^\nu_t = \tilde{Z}^\nu_t + \tilde{Z}^\nu_t, \quad Y^\nu_t = \tilde{Z}^\nu_t - \tilde{Z}^\nu_t\]

and we observe that $\tilde{Z}^\nu_t$ satisfies the SDE

\[2d\tilde{Z}^\nu_t = [f(\tilde{Z}^\nu_t + \tilde{Z}^\nu_t) + g(\tilde{Z}^\nu_t - \tilde{Z}^\nu_t)]dt + (\alpha + \beta)dW_t.\]

Note that

\[2dZ_t = [f(Z_t) + g(Z_t)]dt + (\alpha + \beta)dW_t.\]

Thus

\[\frac{2d(\tilde{Z}^\nu_t - Z_t)}{dt} = [f(\tilde{Z}^\nu_t + \tilde{Z}^\nu_t) - f(Z_t)] + [g(\tilde{Z}^\nu_t - \tilde{Z}^\nu_t) - g(Z_t)].\]
and
\[ \frac{d|\tilde{Z}_t^\nu - Z_t|^2}{dt} = 2(\tilde{Z}_t^\nu - Z_t, \frac{d(\tilde{Z}_t^\nu - Z_t)}{dt}) \]
\[ = (\tilde{Z}_t^\nu - Z_t, [f(\tilde{Z}_t^\nu) + \tilde{Z}_t^\nu] - f(Z_t)) + [g(\tilde{Z}_t^\nu - Z_t^\nu) - g(Z_t))] \]
\[ = (\tilde{Z}_t^\nu - Z_t, [f(\tilde{Z}_t^\nu + \tilde{Z}_t^\nu) - f(\tilde{Z}_t^\nu)] + [f(\tilde{Z}_t^\nu + \tilde{Z}_t^\nu) - f(\tilde{Z}_t^\nu))] \]
\[ = 2L|\tilde{Z}_t^\nu - Z_t|^2 + \frac{2}{L}|f(Z_t + \tilde{Z}_t^\nu) - f(Z_t)|^2 \]
\[ \leq -2L|\tilde{Z}_t^\nu - Z_t|^2 + \frac{2}{L}|f(Z_t + \tilde{Z}_t^\nu) - f(Z_t)|^2 \]
\[ = -2L|\tilde{Z}_t^\nu - Z_t|^2 + \frac{2}{L}(|f(Z_t + \tilde{Z}_t^\nu) - f(Z_t)|^2 + |g(Z_t - \tilde{Z}_t^\nu) - g(Z_t))^2). \]

Hence
\[ |\tilde{Z}_t^\nu - Z_t|^2 \leq e^{-L(t-T_1)}|\tilde{Z}_{T_1}^\nu - Z_{T_1}|^2 \]
\[ + \frac{2}{L} \int_{t}^{\infty} e^{-L(t-s)}(|f(Z_s + \tilde{Z}_s^\nu) - f(Z_s)|^2 + |g(Z_s - \tilde{Z}_s^\nu) - g(Z_s))^2)ds. \] 

Recall that \(X_t^\nu, Y_t^\nu\) and \(Z_t\) are uniformly sub-exponentially growing in \(\nu > 0\). Letting \(T_1\) tends to infinity, we get
\[ |\tilde{Z}_t^\nu - Z_t|^2 \leq \frac{2}{L} \int_{\infty}^{\infty} e^{-L(t-s)}(|f(Z_s + \tilde{Z}_s^\nu) - f(Z_s)|^2 + |g(Z_s - \tilde{Z}_s^\nu) - g(Z_s))^2)ds. \] 

Noticing that \(\tilde{Z}_t^\nu\) pathwise converges to zero as \(\nu\) tends to \(\infty\) for each \(t\) and the continuity of \(f, g\), by Lebesgue’s dominated convergence theorem, we have
\[ \lim_{\nu \to \infty} |\tilde{Z}_t^\nu - Z_t|^2 = 0. \]

Therefore
\[ \lim_{\nu \to \infty} |X_t^\nu - Z_t| \leq \lim_{\nu \to \infty} |\tilde{Z}_t^\nu - Z_t| + \lim_{\nu \to \infty} |\tilde{Z}_t^\nu| = 0 \]
and
\[ \lim_{\nu \to \infty} |Y_t^\nu - Z_t| \leq \lim_{\nu \to \infty} |\tilde{Z}_t^\nu - Z_t| + \lim_{\nu \to \infty} |\tilde{Z}_t^\nu| = 0, \]
which complete the proof of the first part.

For the second part, from the condition (2) and (25), inequality (26), letting \(T_1\) tends to infinity, we get
\[ |\tilde{Z}_t^\nu - Z_t|^2 \leq C \int_{-\infty}^{t} e^{-L(t-s)}|\tilde{Z}_s^\nu|^2ds. \]

Hence, by (22) for any \(\gamma < \frac{1}{2}\)
\[ |\tilde{Z}_t^\nu - Z_t| \leq \frac{C}{\nu^\gamma}, \]
in turn, by Lemma 6.1, we finally have
\[ |X_t^\nu - Z_t| + |Y_t^\nu - Z_t| \leq \frac{C}{\nu^\gamma}, \]
which ends the proof.

The optimality of the convergence rate can be illustrated through the following example.

**Example 7.2.** Let us consider the linear equation in $\mathbb{R}^2$

\[
\begin{aligned}
    dX_t^{\nu} &= -X_t^{\nu} dt + \nu (Y_t^{\nu} - X_t^{\nu}) dt + \alpha dW_t, \\
    dY_t^{\nu} &= -Y_t^{\nu} dt + \nu (X_t^{\nu} - Y_t^{\nu}) dt + \beta dW_t,
\end{aligned}
\]

where $\alpha^T, \beta^T \in \mathbb{R}^2$ are constant vectors, $W_t$ is a two-sided two dimensional Wiener processes. The averaged SDE

\[
dZ_t = -Z_t dt + \frac{1}{2} (\alpha + \beta) dW_t.
\]

Then

\[
\begin{aligned}
    d(X_t^{\nu} + Y_t^{\nu}) &= -(X_t^{\nu} + Y_t^{\nu}) dt + (\alpha + \beta) dW_t, \\
    d(X_t^{\nu} - Y_t^{\nu}) &= -(2\nu + 1)(X_t^{\nu} - Y_t^{\nu}) dt + (\alpha - \beta) dW_t.
\end{aligned}
\]

Since

\[
X_t^{\nu} - Y_t^{\nu} = (\alpha - \beta) \int_{-\infty}^{t} e^{-(2\nu+1)(t-s)} dW_s
\]

and

\[
X_t^{\nu} + Y_t^{\nu} = 2Z_t = (\alpha + \beta) \int_{-\infty}^{t} e^{-(t-s)} dW_s,
\]

hence

\[
2X_t^{\nu} = (\alpha + \beta) \int_{-\infty}^{t} e^{-(t-s)} dW_s + (\alpha - \beta) \int_{-\infty}^{t} e^{-(2\nu+1)(t-s)} dW_s,
\]

\[
2Y_t^{\nu} = (\alpha + \beta) \int_{-\infty}^{t} e^{-(t-s)} dW_s - (\alpha - \beta) \int_{-\infty}^{t} e^{-(2\nu+1)(t-s)} dW_s
\]

and

\[
2(X_t^{\nu} - Z_t) = -2(Y_t^{\nu} - Z_t) = X_t^{\nu} - Y_t^{\nu} = (\alpha - \beta) \int_{-\infty}^{t} e^{-(2\nu+1)(t-s)} dW_s.
\]

By Lemma 4.2, we know that both $X_t^{\nu}$ and $Y_t^{\nu}$ is bounded in $\nu > 0$, moreover

\[
\lim_{\nu \to \infty} \nu^\gamma |X_t^{\nu} - Y_t^{\nu}| = 0
\]

for any $\gamma < \frac{1}{2}$ and

\[
\lim_{\nu \to \infty} \nu^\gamma |X_t^{\nu} - Y_t^{\nu}| \text{ is divergent}
\]

for any $\gamma > \frac{1}{2}$, which implies that the convergence rate in Theorem 7.1 is sharp.

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E-mail address: shahad_math2014@yahoo.com
E-mail address: jcliu@hust.edu.cn
E-mail address: xmliu@hust.edu.cn