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To cite this version:
Pascal Ochem, Matthieu Rosenfeld. Avoidability of Palindrome Patterns. The Electronic Journal of Combinatorics, 2021, 28 (1), pp.#1.4. 10.37236/9593. lirmm-03371500

HAL Id: lirmm-03371500
https://hal-lirmm.ccsd.cnrs.fr/lirmm-03371500v1
Submitted on 8 Oct 2021

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Avoidability of palindrome patterns

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Submitted: May 17, 2020; Accepted: Dec 17, 2020; Published: Jan 15, 2021
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Abstract

We characterize the formulas that are avoided by every \( \alpha \)-free word for some \( \alpha > 1 \). We show that the avoidable formulas whose fragments are of the form \( XY \) or \( XYX \) are 4-avoidable. The largest avoidability index of an avoidable palindrome pattern is known to be at least 4 and at most 16. We make progress toward the conjecture that every avoidable palindrome pattern is 4-avoidable.

Mathematics Subject Classifications: 68R15

1 Introduction

A pattern \( p \) is a non-empty finite word over an alphabet \( \Delta = \{ A, B, C, \ldots \} \) of capital letters called variables. An occurrence of \( p \) in a word \( w \) is a non-erasing morphism \( h : \Delta^* \to \Sigma^* \) such that \( h(p) \) is a factor of \( w \) (a morphism is non-erasing if the image of every letter is non-empty). The avoidability index \( \lambda(p) \) of a pattern \( p \) is the size of the smallest alphabet \( \Sigma \) such that there exists an infinite word over \( \Sigma \) containing no occurrence of \( p \). Since there is no risk of confusion, \( \lambda(p) \) will be simply called the index of \( p \).

A variable that appears only once in a pattern is said to be isolated. Following Casaigne [5], we associate a pattern \( p \) with the formula \( f \) obtained by replacing every isolated variable in \( p \) by a dot. The factors between the dots are called fragments.

An occurrence of a formula \( f \) in a word \( w \) is a non-erasing morphism \( h : \Delta^* \to \Sigma^* \) such that the \( h \)-image of every fragment of \( f \) is a factor of \( w \). As for patterns, the index \( \lambda(f) \) of a formula \( f \) is the size of the smallest alphabet allowing the existence of an infinite word containing no occurrence of \( f \). Clearly, if a formula \( f \) is associated with a pattern \( p \), every
word avoiding $f$ also avoids $p$, so $\lambda(p) \leq \lambda(f)$. Recall that an infinite word is recurrent if every finite factor appears infinitely many times and that any infinite factorial language contains a recurrent word [8, Proposition 5.1.13]. If there exists an infinite word over $\Sigma$ avoiding $p$, then there exists an infinite recurrent word over $\Sigma$ avoiding $p$. This recurrent word also avoids $f$, so that $\lambda(p) = \lambda(f)$. Without loss of generality, a formula is such that no variable is isolated and no fragment is a factor of another fragment.

Let us define the types of formulas we consider in this paper. A pattern is doubled if it contains every variable at least twice. Thus it is a formula with only one pattern. A formula $f$ is nice if for every variable $X$ of $f$, there exists a fragment of $f$ that contains $X$ at least twice. Notice that a doubled pattern is a nice pattern. A formula is an $xyx$-formula if every fragment is of the form $XYX$, i.e., the fragment has length 3 and the first and third variable are the same. A formula is hybrid if every fragment has length 2 or is of the form $XYX$. Thus, an $xyx$-formula is a hybrid formula.

In Section 3, we consider the avoidance of nice formulas. In Section 4, we find some formulas $f$ such that every recurrent word avoiding $f$ over $\Sigma_{\lambda(f)}$ is equivalent to a well-known morphic word. In Section 5, we consider the avoidance of $xyx$-formulas and hybrid formulas. In Section 6, we consider the avoidance of patterns that are palindromes.

## 2 Preliminaries

Given a pattern $p$, the Zimin operator constructs the pattern $Z(p) = pXp$ where $X$ is a variable that is not contained in $p$. For every fixed $t$, $Z^t(p)$ denotes the pattern obtained by applying $t$ times the Zimin operator to $p$. Notice that a recurrent word avoids $Z^t(p)$ if and only if it avoids $p$.

We say that a formula $f$ divides a formula $f'$ if every recurrent word avoiding $f$ also avoids $f'$. We denote by $f \preceq f'$ the fact that $f$ divides $f'$. By previous discussion, $p \preceq Z^t(p)$ and $Z^t(p) \preceq p$ for every pattern $p$. The basic case of divisibility is that $f \preceq f'$ if $f'$ contains an occurrence of $f$, that is, if there exists a non-erasing morphism $h$ such that the $h$-image of every fragment of $f$ is a factor of a fragment of $f'$. Another case of divisibility obtained by transitivity: in order to obtain $f \preceq p$, it is sufficient to prove $f \preceq Z^t(p)$, since $Z^t(p) \preceq p$. We use this trick in the proof of Lemma 6 and Theorem 17. Of course, divisibility is related to avoidability: if $f \preceq f'$, then $\lambda(f) \geq \lambda(f')$.

Let $\Sigma_k = \{0, 1, \ldots, k - 1\}$ denote the $k$-letter alphabet. We denote by $\Sigma^n_k$ the $k^n$ words of length $n$ over $\Sigma_k$.

The operation of splitting a formula $f$ on a fragment $\phi$ consists in replacing $\phi$ by two fragments, namely the prefix and the suffix of length $|\phi| - 1$ of $\phi$. A formula $f$ is minimally avoidable if splitting any fragment of $f$ gives an unavoidable formula. The set of every minimally avoidable formula with at most $n$ variables is called the $n$-avoidance basis.

The adjacency graph $AG(f)$ of the formula $f$ is the bipartite graph such that

- for every variable $X$ of $f$, $AG(f)$ contains the two vertices $X_L$ and $X_R$,
- for every (possibly equal) variables $X$ and $Y$, there is an edge between $X_L$ and $Y_R$ if and only if $XY$ is a factor of $f$. 

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We say that a set $S$ of variables of $f$ is free if for all $X, Y \in S$, $X_L$ and $Y_R$ are in distinct connected components of $AG(f)$. A formula $f$ is said to reduce to $f'$ if it is obtained by deleting all the variables of a free set from $f$, discarding any empty word fragment. A formula is reducible if there is a sequence of reductions to the empty formula. Finally, a locked formula is a formula having no free set.

**Theorem 1** ([3]). A formula is unavoidable if and only if it is reducible.

Let us define here the following well-known pure morphic words. To specify a morphism $m : \Sigma_s \to \Sigma_e$, we use the notation $m = m(0)/m(1)/\cdots/m(s-1)$. Assuming a morphism $m : \Sigma_s \to \Sigma_s$ is such that $m(0)$ starts with 0, the fixed point of $m$ is the right infinite word $m^\omega(0)$.

- $b_2$ is the fixed point of 01/10.
- $b_3$ is the fixed point of 012/02/1.
- $b_4$ is the fixed point of 01/03/21/23.
- $b_5$ is the fixed point of 01/23/4/21/0

We also consider the morphic words $v_3 = M_1(b_5)$ and $w_3 = M_2(b_5)$, where $M_1 = 012/1/02/12/\varepsilon$ and $M_2 = 02/1/0/12/\varepsilon$. The languages of each of these words have been studied in the literature. Let us first recall the following characterization of $b_3$, $v_3$, and $w_3$. We say that two infinite words are equivalent if they have the same set of factors.

**Theorem 2** ([1, 16]).

- Every ternary square-free recurrent word avoiding 010 and 212 is equivalent to $b_3$.
- Every ternary square-free recurrent word avoiding 010 and 020 is equivalent to $v_3$.
- Every ternary square-free recurrent word avoiding 121 and 212 is equivalent to $w_3$.

Interestingly, these three words can be characterized in terms of a forbidden distance between consecutive occurrences of one letter.

**Theorem 3.**

- Every ternary square-free recurrent word such that the distance between consecutive occurrences of 1 is not 3 is equivalent to $b_3$.
- Every ternary square-free recurrent word such that the distance between consecutive occurrences of 0 is not 2 is equivalent to $v_3$.
- Every ternary square-free recurrent word such that the distance between consecutive occurrences of 0 is not 4 is equivalent to $w_3$.

**Proof.**
Another characterization for $b_3$ is that every ternary square-free recurrent word avoiding 1021 and 1021 is equivalent to $b_3$ [1]. This rules out the possibility that the distance between two occurrences of 1 is 3.

Since $v_3$ avoids 010 and 020, the distance between two occurrences of 0 is at least 3.

Since $u_3$ avoids 121 and 212, the distance between consecutive occurrences of 0 is at most 3.

The word $b_3$ is also known to avoid large families of formulas.

**Theorem 4 ([2]).** Every locked formula is avoided by $b_4$.

**Theorem 5 ([5, Proposition 1.13]).** If every fragment of an avoidable formula $f$ has length 2, then $b_4$ avoids $f$.

Theorem 5 will be extended to hybrid formulas, see Theorem 21 in Section 5.

Let us give here a result that will be needed in various parts of the paper.

**Lemma 6.** $ABA.ACA.ABCA.ACBA.ABCBA \preceq AA$.

**Proof.** Indeed, $Z^2(AA) = AABAACAABAA$ contains the occurrence $A \rightarrow A$, $B \rightarrow ABA$, $C \rightarrow ACA$ of $ABA.ACA.ABCA.ACBA.ABCBA$. $\square$

Thus, if $w$ is a recurrent word that avoids a formula dividing $ABA.ACA.ABCA.ACBA.ABCBA$, then $w$ is square-free.

Recall that the repetition threshold $RT(n)$ is the smallest real number $\alpha$ such that there exists an infinite $a^+\!$-free word over $\Sigma_n$. The proof of Dejean’s conjecture established that $RT(2) = 2$, $RT(3) = \frac{7}{5}$, $RT(4) = \frac{7}{4}$, and $RT(n) = \frac{n}{n-1}$ for every $n \geq 5$. An infinite $RT(n)^+\!$-free word over $\Sigma_n$ is called a Dejean word.

## 3 Nice formulas

All the nice formulas considered so far in the literature are also 3-avoidable. This includes doubled patterns [12], circular formulas [9], the nice formulas in the 3-avoidance basis [9], and the minimally nice ternary formulas in Table 1 [15].

**Theorem 7 ([9, 15]).** Every nice formula with at most 3 variables is 3-avoidable.

We have a risky conjecture that would generalize both Theorem 7 and the 3-avoidability of doubled patterns.

**Conjecture 8.** Every nice formula is 3-avoidable.

Theorem 19 in Section 5 shows that there exist infinitely many nice formulas with index 3. It means that Conjecture 8 would be best possible and it contrasts with the case of doubled patterns, since we expect that there exist only finitely many doubled patterns with index 3 [12, 13]. In this section, we make progress toward Conjecture 8 by proving that every nice formula is avoidable and we explain how to get an upper bound on the index of a given nice formula.
3.1 The avoidability exponent

Let us consider a useful tool in pattern avoidance that has been defined in [12] and already used implicitly in [11]. The avoidability exponent \( AE(p) \) of a pattern \( p \) is the largest real \( \alpha \) such that every \( \alpha \)-free word avoids \( p \). We extend this definition to formulas. The corresponding notion for the avoidance of patterns in the abelian setting has also been considered [7].

Let us show that \( AE(ABCBA.CBABC) = \frac{4}{3} \). Suppose for contradiction that a \( \frac{4}{3} \)-free word contains an occurrence \( h \) of \( ABCBA.CBABC \). We write \( y = |h(Y)| \) for every variable \( Y \). The factor \( h(ABCBA) \) is a repetition with period \( |h(ABCB)| \). So we have 
\[
a + b + c + b + a < \frac{4}{3}.
\]
This simplifies to \( 2a < 2b + c \). Similarly, \( CBABC \) gives \( 2c < a + 2b \), \( BAB \) gives \( 2b < a \), and \( BCB \) gives \( 2b < c \). Summing up these four inequalities gives 
\[
a + 4b + 2c < 2a + 4b + 2c,
\]
which is a contradiction. On the other hand, the word 
\[
01234201567876834201234
\]
is \( \left(\frac{4}{3}\right) \)-free and contains the occurrence \( A \rightarrow 01, B \rightarrow 2, C \rightarrow 34 \) of \( ABCBA.CBABC \).

As a second example, we obtain that \( AE(ABCDBACBD) = 1 \). When we consider a repetition \( uvu \) in an \( \alpha \)-free word, we derive that 
\[
|uvu| < \alpha
\]
which gives 
\[
|u| + 1 < |v| \text{ with } \alpha = 1 + \frac{1}{|\beta + 1|}.
\]
We look for the smallest \( \beta \) such that this system has no solution. Notice that \( a \) and \( d \) play symmetric roles. Thus, we can set \( a = d \) and simplify the system.
\[
\begin{align*}
\beta a &\leq a + b + c + d \\
\beta b &\leq c + d \\
\beta c &\leq a + b + d \\
\beta d &\leq a + 2b + c
\end{align*}
\]
We look for the smallest \( \beta \) such that this system has no solution. Notice that \( a \) and \( d \) play symmetric roles. Thus, we can set \( a = d \) and simplify the system.
\[
\begin{align*}
\beta a &\leq a + 2b + c \\
\beta b &\leq a + c \\
\beta c &\leq 2a + b
\end{align*}
\]
Then \( \beta \) is the largest eigenvalue of the matrix \( \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \) that corresponds to the latter system. So \( \beta = \frac{3.060647027...}{2} \) is the largest root of the characteristic polynomial \( x^3 - x^2 - 5x - 4 \). Then \( \alpha = 1 + \frac{1}{\beta + 1} = 1.246266172... \).

This matrix approach is a convenient trick to use when possible. It was used in particular for some doubled patterns such that every variable occurs exactly twice [12]. It may fail if the number of inequalities is strictly greater than the number of variables or if the formula contains a repetition \( uvu \) such that \( |u| \geq 2 \). In any case, we can fix a rational value to \( \beta \) and ask a computer algebra system whether the system of inequalities is solvable. Then we can get arbitrarily good approximations of \( \beta \) (and thus \( \alpha \)) by a dichotomy method.

Of course, the avoidability exponent is related to divisibility.
Lemma 9. If \( f \preceq g \), then \( AE(f) \leq AE(g) \).

The avoidability exponent depends on the repetitions induced by \( f \). We have \( AE(f) = 1 \) for formulas such as \( f = AB.BA.AC.CA.BC \) or \( f = AB.BA.AC.BC.CDA.DCD \) that do not have enough repetitions. That is, for every \( \varepsilon > 0 \), there exists a \((1 + \varepsilon)\)-free word that contains an occurrence of \( f \).

Let us investigate formulas with non-trivial avoidability exponent, that is, \( AE(f) > 1 \). To show that a nice formula has a non-trivial avoidability exponent (see Lemma 10), we first introduce a notion of minimality for nice formulas similar to the notion of minimally avoidable for general formulas. A nice formula \( f \) is minimally nice if there exists no nice formula \( g \) such that \( v(g) \leq v(f) \) and \( g \prec f \). Alternatively, splitting a minimally nice formula on any of its fragments leads to a non-nice formula. The following property of every minimally nice formula is easy to derive. If a variable \( V \) appears as a prefix of a fragment \( \phi \), then

- \( V \) is also a suffix of \( \phi \) (since otherwise we can split on \( \phi \) and obtain a nice formula),
- \( \phi \) contains exactly two occurrences of \( V \) (since otherwise we can remove the prefix letter \( V \) from \( \phi \) and obtain a nice formula),
- \( V \) is neither a prefix nor a suffix of any fragment other than \( \phi \) (since otherwise we can remove this prefix/suffix letter \( V \) from the other fragment and obtain a nice formula),
- Every fragment other than \( \phi \) contains at most one occurrence of \( V \) (since otherwise we can remove the prefix letter \( V \) from \( \phi \) and obtain a nice formula).

Lemma 10. If \( f \) is a nice formula with \( v(f) \geq 3 \), then \( AE(f) \geq 1 + \frac{1}{2v(f)-3} \).

Proof. First remark that if a word \( wuv \) is \((1 + \frac{1}{2v(f)-3})\)-free then \( 2|u| + |v| < (|u| + |v|) \left(1 + \frac{1}{2v(f)-3}\right) \) which implies \((2v(f) - 4)|u| < |v|\).

Suppose that \( f \) contradicts the lemma. Then there exists a \( (1 + \frac{1}{2v(f)-3})\)-free word \( w \) containing an occurrence \( h \) of \( f \). Let \( X \) be a variable of \( f \) such that \( |h(X)| \geq |h(Y)| \) for every variable \( Y \). Since \( f \) is nice, \( f \) contains a factor of the form \( XPX \) where \( P \) is a sequence of variables that does not contain \( X \). Remark that \( v(P) \leq v(f) - 1 \).

For any variable \( Z \), let \( |P|_Z \) be the number of occurrences of \( Z \) in \( P \). Let \( Y \) be the variable that maximizes \( |h(Y)| \times |P|_Y \), that is, \( |h(W)| \times |P|_W \leq |h(Y)| \times |P|_Y \) for every variable \( W \) in \( P \). We have

\[
|h(P)| = \sum_{W \in Var(P)} |h(W)| \times |P|_W \leq (v(f) - 1)|h(Y)| \times |P|_Y \leq (v(f) - 1)|h(X)| \times |P|_Y.
\]

If \( |P|_Y = 1 \), then \( |h(P)| \leq (v(f) - 1)|h(X)| \) and the exponent of \( |h(X)| \) is at least

\[
\frac{(v(f) + 1)|h(X)|}{v(f)|h(X)|} = 1 + \frac{1}{v(f)},
\]

which is a contradiction.
If $|P|_Y \geq 2$, then the number of letters of $h(P)$ that do not belong to an occurrence of $h(Y)$ is at most

$$\sum_{W \in \text{Var}(P) \setminus \{Y\}} |h(W)| \times |P|_W \leq (v(f) - 2)|h(Y)| \times |P|_Y.$$ 

Thus there exist two occurrences of $h(Y)$ in $h(P)$ that are separated by at most

$$\frac{(v(f) - 2)|h(Y)| \times |P|_Y}{|P|_Y - 1}$$

letters. Since $h(P)$ is $\left(1 + \frac{1}{2v(f) - 3}\right)$-free, we obtain

$$(2v(f) - 4)|h(Y)| < \frac{(v(f) - 2)|h(Y)| \times |P|_Y}{|P|_Y - 1}.$$ 

This can be simplified to

$$(2v(f) - 4)(|P|_Y - 1) < (v(f) - 2) \times |P|_Y$$

and finally

$$|P|_Y < \frac{2v(f) - 4}{v(f) - 2} = 2,$$

which is a contradiction. \(\square\)

The circular formulas studied in [9] show that $AE(f)$ can be as low as $1 + (v(f))^{-1}$. Moreover, our example $AE(ABCDBACBD) = 1.246266172\ldots$ shows that lower avoidability exponents exist among nice formulas with at least 4 variables.

We will describe below a method to construct infinite words avoiding a formula. This method can be applied if and only if the formula $f$ satisfies $AE(f) > 1$. So we are interested in characterizing the formulas $f$ such that $AE(f) > 1$. By Theorems 9 and 10, if $f$ is a formula such that there exists a nice formula $g$ satisfying $g \preceq f$, then $AE(f) > 1$. Now we prove that the converse also holds, which gives the following characterization.

**Theorem 11.** A formula $f$ satisfies $AE(f) > 1$ if and only if there exists a nice formula $g$ such that $g \preceq f$.

**Proof.** What remains to prove is that for every formula $f$ that is not divisible by a nice formula and for every $\varepsilon > 0$, there exists an infinite $(1 + \varepsilon)$-free word $w$ containing an occurrence of $f$, such that the size of the alphabet of $w$ only depends on $f$ and $\varepsilon$.

First, we consider the equivalent pattern $p$ obtained from $f$ by replacing every dot by a distinct variable that does not appear in $f$. We will actually construct an occurrence of $p$. Then we construct a family $f_i$ of pseudo-formulas as follows. We start with $f_0 = p$. To obtain $f_{i+1}$ from $f_i$, we choose a variable that appears at most once in every fragment of $f_i$. This variable is given the alias name $V_i$ and every occurrence of $V_i$ is replaced by a dot. We say that $f_i$ is a pseudo-formula since we do not try to normalize $f_i$, that is, $f_i$ can contain consecutive dots and $f_i$ can contain fragments that are factors of other fragments. However, we still have a notion of fragment for a pseudo-formula. Since $f$ is not divisible by a nice formula, this process ends with the pseudo-formula $f_{v(p)}$ with no variable and
The goal of this process is to obtain the ordering $V_0, V_1, \ldots, V_{v(p)−1}$ on the variables of $p$.

The image of every $V_i$ is a finite factor $w_i$ of a Dejean word over an alphabet of $\lfloor \varepsilon^{-1} \rfloor + 2$ letters, so that $w_i$ is $(1 + \varepsilon)$-free. The alphabets are disjoint: if $i \neq j$, then $w_i$ and $w_j$ have no common letter. Finally, we define the length of $w_i$ as follows: $|w_{v(p)−1}| = 1$ and $|w_i| = \lfloor \varepsilon^{-1} \rfloor \times |p| \times |w_{i+1}|$ for every $i$ such that $0 \leq i \leq v(p) − 2$. Let us show by contradiction that the constructed occurrence $h$ of $p$ is $(1 + \varepsilon)$-free. Consider a repetition $xyx$ of exponent at least $1 + \varepsilon$ that is maximal, that is, which cannot be extended to a repetition with the same period and larger exponent. Since every $w_i$ is $(1 + \varepsilon)$-free and since two matching letters must come from distinct occurrences of the same variable, then $x = h(x')$ and $y = h(y')$ where $x'$ and $y'$ are factors of $p$. Our ordering of the variables of $p$ implies that $y'$ contains a variable $V_i$ such that $i < j$ for every variable $V_j$ in $x'$. Thus, $|y| \geq |w_i| = \lfloor \varepsilon^{-1} \rfloor \times |p| \times |w_{i+1}| \geq \lfloor \varepsilon^{-1} \rfloor \times |x|$, which contradicts the fact that the exponent of $xyx$ is at least $1 + \varepsilon$.

To obtain the infinite word $w$, we can insert our occurrence of $p$ into a bi-infinite $(1 + \varepsilon)$-free word over an alphabet of $\lfloor \varepsilon^{-1} \rfloor + 2$ new letters. So $w$ is an infinite $(1 + \varepsilon)$-free word over an alphabet of $v(p) (\lfloor \varepsilon^{-1} \rfloor + 2) + 1$ letters which contains an occurrence of $f$.

By Lemma 10, every nice formula is avoidable since it is avoided by a Dejean word over a sufficiently large alphabet. Thus, if a formula is nice and minimally avoidable, then it is minimally nice. This is the case for every formula in the 3-avoidance basis, except $AB.AC.BA.CA.CB$. However, a minimally nice formula is not necessarily minimally avoidable. Indeed, we have shown [15] that the set of minimally nice ternary formulas consists of the nice formulas in the 3-avoidance basis, together with the minimally nice formulas in Table 1 that can be split to $AB.AC.BAC.A.C.B$.

- $ABA.BCB.CAC$
- $ABCA.BCAB.CBAC$ and its reverse
- $ABCA.BAB.CAC$
- $ABCA.BAB.CBC$ and its reverse
- $ABCA.BAB.CBAC$ and its reverse
- $ABCBA.CABC$ and its reverse
- $ABCBA.CAC$

Table 1: The minimally nice ternary formulas that are not minimally avoidable.
3.2 Avoiding a nice formula

Recall that a nice formula $f$ is such that $AE(f) > 1$. We consider the smallest integer $s$ such that $RT(s) < AE(f)$. Thus, every Dejean word over $\Sigma_s$ avoids $f$, which already gives $\lambda(f) \leq s$. Recall that a morphism is $q$-uniform if the image of every letter has length $q$. Also, a uniform morphism $h : \Sigma_s^* \rightarrow \Sigma_e^*$ is synchronizing if for any $a, b, c \in \Sigma_s$ and $v, w \in \Sigma_e^*$, if $h(ab) = vh(c)w$, then either $v = \varepsilon$ and $a = c$ or $w = \varepsilon$ and $b = c$. For increasing values of $q$, we look for a $q$-uniform morphism $h : \Sigma_s^* \rightarrow \Sigma_e^*$ such that $h(w)$ avoids $f$ for every $RT(s)^+\text{-free}$ word $w \in \Sigma_e^*$, where $\ell$ is given by Lemma 12 below. Recall that a word is $(\beta^+, n)$-free if it contains no repetition with exponent strictly greater than $\beta$ and period at least $n$.

Lemma 12. \cite{11} Let $\alpha, \beta \in \mathbb{Q}$, $1 < \alpha < \beta < 2$ and $n \in \mathbb{N}^*$. Let $h : \Sigma_s^* \rightarrow \Sigma_e^*$ be a synchronizing $q$-uniform morphism (with $q \geq 1$). If $h(w)$ is $(\beta^+, n)$-free for every $\alpha^+$-free word $w$ such that $|w| < \max \left(\frac{2\beta}{\beta - \alpha}, \frac{2(q-1)(\beta^2 - 1)}{q(\beta - 1)}\right)$, then $h(w)$ is $(\beta^+, n)$-free for every (finite or infinite) $\alpha^+$-free word $w$.

Given such a candidate morphism $h$, we use Lemma 12 to show that for every $RT(s)^+\text{-free}$ word $w \in \Sigma_s^*$, the image $h(w)$ is $(\beta^+, n)$-free. The pair $(\beta, n)$ is chosen such that $RT(s) < \beta < AE(f)$ and $n$ is the smallest possible for the corresponding $\beta$. If $\beta < AE(f)$, then every occurrence $h$ of $f$ in a $(\beta^+, t)$-free word is such that the length of the $h$-image of every variable of $f$ is upper bounded by a function of $n$ and $f$ only. Thus, the $h$-image of every fragment of $f$ has bounded length and we can check that $f$ is avoided by inspecting a finite set of factors of words of the form $h(w)$.

3.3 The number of fragments of a minimally avoidable formula

Interestingly, the notion of (minimally) nice formula is helpful in proving the following.

Theorem 13. The only minimally avoidable formula with exactly one fragment is $AA$.

Proof. A formula with one fragment is a doubled pattern. Since it is minimally avoidable, it is a minimally nice formula. By the properties of minimally nice formulas discussed above, the unique fragment of the formula is either $AA$ or is of the form $ApA$ such that $p$ does not contain the variable $A$. Thus, $p$ is a doubled pattern such that $p < ApA$, which contradicts that $ApA$ is minimally avoidable. □

By contrast, the family of two-birds formulas, which consists of $ABA.BAB$, $ABCBA.CBABC$, $ABCD.CB.ADCBABC$, and so on, shows that there exist infinitely many minimally avoidable formulas with exactly two fragments. Every two-birds formula is nice. Let us check that every two-birds formula $AB \cdots X \cdots BAX \cdots A \cdots X$ is minimally avoidable. Since the two fragments play symmetric roles, it is sufficient to split on the first fragment. We obtain the formula $AB \cdots X \cdots B.B \cdots X \cdots BAX \cdots A \cdots X$ which divides the pattern $B \cdots X \cdots BAB \cdots X \cdots B = Z(B \cdots X \cdots B)$. This pattern is equivalent to $B \cdots X \cdots B$, which is unavoidable. Thus, every two-birds formula is indeed minimally avoidable.
Concerning the index of two-birds formulas, we have seen that $\lambda(ABA.BAB) = 3$ and $\lambda(ABCBA.CBABC) = 2$ [9]. Computer experiments suggest that larger two-birds formulas are easier to avoid.

**Conjecture 14.** Every two-birds formula with at least 3 variables is 2-avoidable.

## 4 Characterization of some famous morphic words

Our next result gives characterizations of $w_3$, up to renaming, that use just one formula. Then we give similar characterizations of $b_3$ and $b_2$. Let $\sigma = 1/2/0$ be the morphism that cyclically permutes $\Sigma_3$.

**Theorem 15.** Let $f_h = ABA.BCB.ACA$, $f_e = ABA.ABCBA.ACA.ACB.BCA$, and let $f$ be such that $f_h \preceq f \preceq f_e$. Every ternary recurrent word avoiding $f$ is equivalent to $w_3$, $\sigma(w_3)$, or $\sigma^2(w_3)$.

**Proof.** Using Cassaigne’s algorithm [4], we have checked that $w_3$ avoids $f_h$. By divisibility, $w_3$ avoids $f$.

Let $w$ be a ternary recurrent word avoiding $f$. By Lemma 6, $w$ is square-free.

Let $v = 210201202101201021$. A computer check shows that no infinite ternary word avoids $f_e$, squares, $v$, $\sigma(v)$, and $\sigma^2(v)$. So, without loss of generality, $w$ contains $v$. If $w$ contains 121, then $w$ contains the occurrence $A \to 1$, $B \to 2$, $C \to 0$ of $f_e$. Similarly, if $w$ contains 212, then $w$ contains the occurrence $A \to 2$, $B \to 1$, $C \to 0$ of $f_e$. Thus, $w$ avoids squares, 121, and 212. By Theorem 2, $w$ is equivalent to $w_3$.

By symmetry, every ternary recurrent word avoiding $f$ is equivalent to $w_3$, $\sigma(w_3)$, or $\sigma^2(w_3)$.

**Theorem 16.** Let $f$ be such that

- $ABCA.ABA.ACA \preceq f \preceq ABCA.ABA.ACA.ACB.CBA$,
- $ABCA.ABA.BCB.AC \preceq f \preceq ABCA.ABA.ABCBA.ACB$, or
- $ABCA.ABA.BCB.CBA \preceq f \preceq ABCA.ABA.ABCBA.ACB$.

Every ternary recurrent word avoiding $f$ is equivalent to $b_3$, $\sigma(b_3)$, or $\sigma^2(b_3)$.

**Proof.** Using Cassaigne’s algorithm [4], we have checked that $b_3$ avoids $ABCA.ABA.ACA$, $ABCA.ABA.BCB.AC$, and $ABCA.ABA.BCB.CBA$. By divisibility, $b_3$ avoids $f$. Let $w$ be a ternary recurrent word avoiding $f$. By Lemma 6, $w$ is square-free.

Let $v = 20210120210120120$. A computer check shows that no infinite ternary word avoids $ABCA.ABA.ACA.ACB.CBA$ (resp. $ABCA.ABA.ABCBA.ACB$), squares, $v$, $\sigma(v)$, and $\sigma^2(v)$.

So, without loss of generality, $w$ contains $v$. If $w$ contains 010, then $w$ contains the occurrence $A \to 0$, $B \to 1$, $C \to 2$ of $ABCA.ABA.ABCBA.ACB$. Similarly, if $w$ contains 212, then $w$ contains the occurrence $A \to 2$, $B \to 1$, $C \to 0$ of
ABA.ACA.ABCA.ACBA.ABCBA. Thus, \( w \) avoids squares, 010, and 212. By Theorem 2, \( w \) is equivalent to \( b_3 \).

By symmetry, every ternary recurrent word avoiding \( f \) is equivalent to \( b_3 \), \( \sigma(b_3) \), or \( \sigma^2(b_3) \).

Notice that Theorem 16 is a complement to [15, Theorem 2] in which we gave a disjoint set of formulas with the same property. The difference between Theorem 16 and [15, Theorem 2] is that a different occurrence of \( f \) shows that \( f \) divides \( Z^n(AA) \).

Theorem 17. Let \( f_h = AABCAA.BCB \), \( f_e = AABCAAB.AABCAB.AABCB \), and let \( f \) be such that \( f_h \leq f \leq f_e \). Every binary recurrent word avoiding \( f \) is equivalent to \( b_2 \).

**Proof.** Using Cassaigne’s algorithm [4], we have checked that \( b_2 \) avoids \( f_h \). First, \( f_e \leq AAA \) because \( Z(AAA) = AAABAAA \) contains the occurrence \( A \rightarrow A, B \rightarrow A, C \rightarrow B \) of \( f_e \). Second, \( f_e \leq ABABA \) because \( Z(ABABA) = ABABACABABA \) contains the occurrence \( A \rightarrow AB, B \rightarrow A, C \rightarrow C \) of \( f_e \).

Thus, every recurrent word avoiding \( f_e \) also avoids \( AAA \) and \( ABABA \), which means that it is overlap-free. Finally, it is well-known that every binary recurrent word that is overlap-free is equivalent to \( b_2 \).

\[ \square \]

5 **xyx-formulas**

Recall that every fragment of an \( xyx \)-formula is of the form \( XYX \). We associate to an \( xyx \)-formula \( F \) the directed graph \( \overrightarrow{G} \) such that every variable corresponds to a vertex and \( \overrightarrow{G} \) contains the arc \( \overrightarrow{XY} \) if and only if \( F \) contains the fragment \( XYX \). We will also denote by \( G \) the underlying simple graph of \( \overrightarrow{G} \).

**Lemma 18.** Let \( F_1 \) and \( F_2 \) be \( xyx \)-formulas associated to \( \overrightarrow{G}_1 \) and \( \overrightarrow{G}_2 \). If there exists a homomorphism \( \overrightarrow{G}_1 \rightarrow \overrightarrow{G}_2 \), then \( F_1 \leq F_2 \).

**Proof.** Since both digraph homomorphism and formula divisibility are transitive relations, we only need to consider the following two cases. If \( G_1 \) is a subgraph of \( G_2 \), then \( F_1 \) is obtained from \( F_2 \) by removing some fragments. So every occurrence of \( F_2 \) is also an occurrence of \( F_1 \) and thus \( F_1 \leq F_2 \). If \( G_2 \) is obtained from \( G_1 \) by identifying the vertices \( u \) and \( v \), then \( F_2 \) is obtained from \( F_1 \) by identifying the variables \( U \) and \( V \). So every occurrence of \( F_2 \) is also an occurrence of \( F_1 \) and thus \( F_1 \leq F_2 \).

For every \( i \), let \( T_i \) be the \( xyx \)-formula corresponding to the directed circuit \( \overrightarrow{C}_i \) of length \( i \), that is, \( T_1 = AAA, T_2 = ABA.BAB, T_3 = ABA.BCB.CAC, T_4 = ABA.BCB.CDC.DAD \), and so on. More formally, \( T_i \) is the formula with \( i \) variables \( A_0, \ldots, A_{i-1} \) which contains the \( i \) fragments of length three of the form \( A_jA_{j+1}A_j \) such that the indices are taken modulo \( i \). Notice that \( T_i \) is a nice formula.

**Theorem 19.** For every \( i \geq 2 \), \( \lambda(T_i) = 3 \).
Proof. We use Lemma 12 to show that the image of every \((7/4^+)\)-free word over \(\Sigma_4\) by the following 58-uniform morphism is \((3/2, 3)\)-free.

\[
\begin{align*}
0 & \rightarrow \ 001221100220102112002210011220120010221102211201022 \\
1 & \rightarrow \ 001221002201022112200110220100211220102211201022 \\
2 & \rightarrow \ 001122100220102112200110220100211220102211201022 \\
3 & \rightarrow \ 001122100220102112200110220100211220102211201022
\end{align*}
\]

In these words, the factor 010 is the only occurrence \(m\) of \(ABA\) such that \(|m(A)| \geq |m(B)|\). This implies that these ternary words avoid \(T_i\) for every \(i \geq 1\), so that \(\lambda(T_i) \leq 3\).

To show that \(\lambda(T_i) \geq 3\), we consider the \(xyx\)-formula \(H = ABA.BAB.ACA.CBC\) associated to the directed graph \(\overrightarrow{D}_3\) on 3 vertices and 4 arcs that contains a circuit of length 2 and a circuit of length 3. Standard backtracking shows that \(\lambda(H) > 2\), and even the stronger result that \(\lambda(ABA.BAB.ACA.CAC.BCB.CBC) > 2\).

For every \(i \geq 2\), the circuit \(\overrightarrow{C}_i\) admits a homomorphism to \(\overrightarrow{D}_3\). By Lemma 18, this means that \(T_i \preceq H\), which implies that \(\lambda(T_i) \geq \lambda(H) \geq 3\).

\[\square\]

**Theorem 20.** For every \(i \geq 1\), \(b_i\) avoids \(T_i\).

**Proof.** Suppose for contradiction that there exist \(i\) and \(n\) such that \(m^n(0)\) contains an occurrence \(h\) of \(T_i\). Further assume that \(n\) is minimal. Notice that in \(b_i\), every even (resp. odd) letter appears only at even (resp. odd) positions. Thus, for every fragment \(XYX\) of \(T_i\), the period \(|h(XY)|\) of the repetition \(h(XXY)\) must be even. This implies that \(|h(X)|\) and \(|h(Y)|\) have the same parity. By contagion, the lengths of the images of all the variables of \(T_i\) have the same parity. Now we proceed to a case analysis.

- Every \(|h(X)|\) is even.
  - Every \(h(X)\) starts with 0 or 2. By taking the pre-image by \(m\) of every \(h(X)\), we obtain an occurrence of \(T_i\) that is contained in \(m^{n-1}(0)\). This contradicts the minimality of \(n\).
  - Every \(h(X)\) starts with 1 or 3. Notice that in \(b_i\), the letter 1 (resp. 3) is in position 1 (mod 4) (resp. 3 (mod 4)). \(m^n(0)\) contains the occurrence \(h'\) of \(T_i\) such that \(h'(X)\) is obtained from \(h(X)\) by adding to the right the letter 1 or 3 depending on its position modulo 4 and by removing the first letter. Since is also contained in \(m^n(0)\) and every \(h'(X)\) starts with 0 or 2, \(h'\) satisfies the previous subcase.

- Every \(|h(X)|\) is odd. It is not hard to check that every factor \(uvu\) in \(b_i\) with \(|v| = 1\) satisfies \(v \in \{1, 3\}\) and \(u \in \{0, 2\}\). So \(|h(X)|\) \(\geq 3\) for every variable \(X\) of \(T_i\). Let \(X_1, \ldots, X_i\) be the variables of \(T_i\). Up to a shift of indices, we can assume that \(j\) and the first and last letters of \(h(X_j)\) have the same parity. We construct the occurrence \(h'\) of \(T_i\) as follows. If \(j\) is odd, then \(h'(X_j)\) is obtained by removing the first letter of \(h(X_j)\). If \(j\) is even, then \(h'(X_j)\) is obtained by adding to the right the letter 1 or 3 depending on its position modulo 4. Since \(h'\) is also contained in \(m^n(0)\) and every \(|h'(X)|\) is even, \(h'\) satisfies the previous case. \[\square\]
Our next result generalizes Theorems 5 and 20. Recall that every fragment of a hybrid formula has length 2 or is of the form $XYX$.

**Theorem 21.** Every avoidable hybrid formula is avoided by $b_4$.

*Proof.* Let $f$ be a hybrid formula. If $f$ contains a locked formula or a formula $T_i$, then $b_4$ avoids $f$ by Theorems 4 and 20. If $f$ contains neither a locked formula nor a formula $T_i$, then we show that $f$ is unavoidable. By induction and by theorem 1 it is sufficient to show that $f$ is reducible to a hybrid formula containing neither a locked formula nor a formula $T_i$. Since $f$ is not locked, $f$ contains a free set of variables and thus $f$ has a free singleton $\{X\}$. If $f$ contains a fragment $XYX$, then $\{Y\}$ is also a free singleton of $f$. Using this argument iteratively, we end up with a free singleton $\{Z\}$ such that $f$ contains no fragment $TZT$, since $f$ contains no formula $T_i$.

So we can assume that $f$ contains a free singleton $\{Z\}$ and no fragment $TZT$. Thus, deleting every occurrence of $Z$ from $f$ gives an hybrid sub-formula containing neither a locked formula nor a formula $T_i$. By induction, $f$ is unavoidable. □

So the index of an avoidable $xyx$-formula is at most 4 and we have seen examples of $xyx$-formulas with index 3 in Theorems 15 and 19. The next results give an $xyx$-formula with index 4 and an $xyy$-formula with index 2 that is not divisible by $AAA$.

**Theorem 22.** $\lambda(ABA.BCB.DCD.DED.AEA) = 4$.

*Proof.* By Theorem 21, $ABA.BCB.DCD.DED.AEA$ is 4-avoidable. Notice that $ABA.BCB.DCD.DED.AEA \prec ABA.BCB.ACA$ via the homomorphism $A \rightarrow A$, $B \rightarrow B$, $C \rightarrow C$, $D \rightarrow B$, $E \rightarrow C$. Moreover, $w_3$ contains the occurrence $A \rightarrow 0$, $B \rightarrow 1$, $C \rightarrow 02$, $D \rightarrow 01$, $E \rightarrow 2$ of $ABA.BCB.DCD.DED.AEA$. By Theorem 15, the formula is not 3-avoidable. □

**Theorem 23.** The fixed point of $001/011$ avoids the $xyx$-formula associated to the directed graph on 4 vertices with all the 12 arcs.

*Proof.* We use again Cassaigne’s algorithm. □

### 6 Palindrome patterns

Mikhailova [10] has considered the index of an avoidable pattern that is a palindrome and proved that it is at most 16. She actually constructed a morphic word over $\Sigma_{16}$ that avoids every avoidable palindrome pattern.

We make a distinction between the largest index $P_w$ of an avoidable palindrome pattern and the smallest alphabet size $P_s$ allowing an infinite word avoiding every avoidable palindrome pattern. We obtained [15] the lower bound

$$\lambda(ABCADACBA) = \lambda(ABCA.ACBA) = 4,$$

so that $4 \leq P_w \leq P_s \leq 16$.

The following result is a slight improvement to $\lambda(ABCA.ACBA) = 4$ that is not related to palindromes.
Theorem 24. $\lambda(ABCA.ACBA.ABCBA) = 4$.

Proof. By Lemma 6, every recurrent word avoiding $ABCA.ACBA.ABCBA$ is square-free. A computer check shows that no infinite ternary square-free word avoids the occurrences $h$ of $ABCA.ACBA.ABCBA$ such that $|h(A)| = 1$, $|h(B)| \leq 2$, and $|h(C)| \leq 3$. □

Let us give necessary conditions on a palindrome pattern $P$ so that $5 \leq \lambda(P) \leq 16$.

1. The length of $P$ is odd and the central variable of $P$ is isolated. Indeed, otherwise $P$ would be a doubled pattern and thus 3-avoidable [12].

2. No variable of $P$ appears both at an even and an odd position. Indeed, if $P$ had a variable that appears both at an even and an odd position, then $P$ would be divisible by a formula in the family $AA$, $ABCA.ACBA$, $ABCDEA.AEDCBA$, $ABCDEFGA.AGFEDCBA$, … Such formulas (with an odd number of variables) are locked and thus are avoided by $b_4$ by Theorem 4. So $P$ would be 4-avoidable.

We have found three patterns/formulas satisfying these conditions (see Theorem 25), but they seem to be 2-avoidable. We use again Cassaigne’s algorithm with simple pure morphic words to ensure that they are 4-avoidable. Let $z_3$ be the fixed point of $01/2/20$.

Theorem 25.

1. $ADBDCDAD.DADCDBDA$ is avoided by $b_4$.

2. $ABCDADC.CDADCBA$ is avoided by $z_3$.

3. $ABACDBAC.CABDCABA$ is avoided by $z_3$ and $b_4$.

7 Discussion

Let us briefly mention the things that we have attempted to do in this paper, without success.

- Find a result similar to Theorems 15 and 16 for $v_3$, the morphic word avoiding squares, $010$, and $020$.

- Improve Theorem 23 by showing that some $xyx$-formula on 4 variables and fewer fragments is 2-avoidable.

- Show that the $xyx$-formula associated to the transitive tournament on 5 vertices is 2-avoidable.
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