Symplectic integration of post-Newtonian equations of motion with spin

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Outline

Post-Newtonian equations for spinning binary systems

A symplectic PN integrator

Numerical comparisons
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Motivation

spinning binary black holes as probable source of gravit. waves
templates of waveforms for detection? chaotic behaviour?

need long-term simulations

motion described by Einstein eqs. of general relativity, but
post-Newtonian approximation is excellent at large separation
distance or large mass ratio
Post-Newtonian description of spinning binary systs.

masses $m_a$, positions $X_a$, momenta $P_a$ and spins $S_a$ ($a = 1, 2$)
center-of-mass dynamics: $P \equiv P_1 = -P_2$.

Hamiltonian $H(X, P, S_1, S_2)$ (Damour, Jaranowski, Schäfer)

non-canonical Poisson bracket:

$$\{F, G\} = \left( \frac{\partial F}{\partial X} \cdot \frac{\partial G}{\partial P} - \frac{\partial F}{\partial P} \cdot \frac{\partial G}{\partial X} \right) + \sum_{a=1}^{2} \text{det} \left( \frac{\partial F}{\partial S_a}, S_a, \frac{\partial G}{\partial S_a} \right).$$

canonical spin bracket
PN equations of motion

\[
\begin{align*}
\frac{d\mathbf{X}}{dt} &= \{\mathbf{X}, H\} = \frac{\partial H}{\partial \mathbf{P}}, \\
\frac{d\mathbf{P}}{dt} &= \{\mathbf{P}, H\} + \mathbf{F} = -\frac{\partial H}{\partial \mathbf{X}} + \mathbf{F}, \quad \text{F radiation force} \\
\frac{d\mathbf{S}_a}{dt} &= \{\mathbf{S}_a, H\} = \frac{\partial H}{\partial \mathbf{S}_a} \times \mathbf{S}_a,
\end{align*}
\]

conservative case (F = 0): Poisson system with Casimirs |\mathbf{S}_a|

flow \(\varphi_t: (\mathbf{X}(0), \mathbf{P}(0), \mathbf{S}_a(0)) \mapsto (\mathbf{X}(t), \mathbf{P}(t), \mathbf{S}_a(t))\)
preserves the Casimirs and the bracket as

\[
\{F \circ \varphi_t, G \circ \varphi_t\} = \{F, G\} \circ \varphi_t \quad \forall \text{ smooth } F, G
\]

further conserved quantities: total energy \(H\) and total angular momentum \(\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2\)
Aim: structure-preserving integrator

Numerical flow $\Phi_h$ should preserve Casimirs and bracket:

$$\{ F \circ \Phi_h, G \circ \Phi_h \} = \{ F, G \} \circ \Phi_h \quad \forall \text{ smooth } F, G$$

Then, $\Phi_h$ is the exact flow of a Poisson system with the same bracket and a modified, nearby Hamiltonian (up to terms exp. small in $h$)

Construct a Poisson integrator by a splitting approach!
Need to look at the fine structure of the Hamiltonian
The PN Hamiltonian

\[ H(\mathbf{X}, \mathbf{P}, S_1, S_2) = H_{\text{Orb}}(\mathbf{X}, \mathbf{P}) + H_{\text{Spin}}(\mathbf{X}, \mathbf{P}, S_1, S_2) \]

**orbital Hamiltonian:** Kepler and beyond

\[ H_{\text{Orb}} = H_N + H_{\text{PN}} \]

**spin Hamiltonian:** spin-orbit and spin-spin interactions

\[ H_{\text{Spin}} = H_{\text{SO}} + H_{S_1 S_1} + H_{S_1 S_2} + H_{S_2 S_2} \]
Spin Hamiltonian

\[
H_{SO} = 2 \frac{G}{c^2} \frac{S_{\text{eff}} \cdot L}{R^3}
\]

\[
S_{\text{eff}} = \left( 1 + \frac{3}{4} \frac{m_2}{m_1} \right) S_1 + \left( 1 + \frac{3}{4} \frac{m_1}{m_2} \right) S_2,
\]

\[
H_{S_1 S_2} = \frac{G}{c^2} \frac{1}{R^3} \left[ 3 (S_1 \cdot N) (S_2 \cdot N) - (S_1 \cdot S_2) \right],
\]

\[
H_{S_1 S_1} = \frac{1}{2} \frac{G}{c^2} \frac{1}{R^3} \left[ 3 (S_1 \cdot N) (S_1 \cdot N) - (S_1 \cdot S_1) \right] \frac{m_2}{m_1},
\]

\[
H_{S_2 S_2} = 1 \iff 2.
\]

\[
L = \mathbf{X} \times \mathbf{P}
\]

orbital angular momentum

\[
N
\]

is the unit vector \( \mathbf{X}/R \), where \( R = |\mathbf{X}| \).
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Splitting integrator

\[ \varphi_h^H \approx \varphi_{h/2}^{H_{SS}} \circ \varphi_{h/2}^{H_{SO}} \circ \varphi_{h}^{H_{Orb}} \circ \varphi_{h/2}^{H_{SO}} \circ \varphi_{h/2}^{H_{SS}}. \]

The individual flows in this formula are further approximated in a structure-preserving way.
Orbital integrator

compose

- half-step of symplectic Euler for $H_{PN}$ (implicit in $X$)
- full step of high-order symplectic integrator for Kepler flow (explicit)
- half-step of adjoint symplectic Euler for $H_{PN}$ (implicit in $P$)

implicit, but only few iterations from excellent starting iterates
Spin-orbit integrator

\[ H_{SO}(X, P, S) = S \cdot \frac{L}{R^3}, \quad L = X \times P, \quad R = |X| \]

split

\[ H_{SO} = H_{SO}^1 + H_{SO}^2 + H_{SO}^3 \quad \text{with} \quad H_{SO}^i = \frac{S^i L^i}{R^3} \]

exact flow of \( H_{SO}^i \) is obtained by rotations of \( X, P, S \)

approximate spin-orbit flow by composing the three exact subflows
consider $H^{\text{rot}}(S) = \Omega \cdot S$ with a constant $\Omega$

$$\dot{S} = \Omega \times S$$

solved by Rodrigues formula:

$$S(t) = \mathcal{R}(\Omega, t) S(0)$$

$$= S(0) + \frac{\sin(t\Omega)}{\Omega} \Omega \times S(0) + \frac{1}{2} \left( \frac{1}{2} \frac{\sin(t\Omega)}{\frac{1}{2}\Omega} \right)^2 \Omega \times \Omega \times S(0)$$
Spin-orbit integrator

eqs. of motion for $H_{SO}^1(X, P, S) = S^1 L^1 / R^3$
(with $L = X \times P$ and $R = |X|$):

$$
\begin{align*}
\dot{X} &= S^1 e_1 \times X / R^3 \\
\dot{P} &= S^1 e_1 \times P / R^3 + 3H_{SO}^1 X / R^3 \\
\dot{S} &= L^1 e_1 \times S / R^3
\end{align*}
$$

$$
\frac{d}{dt} R^2 = 2X \cdot \dot{X} = 0 \quad \rightarrow \quad R = \text{const.}
$$

$$
\dot{S}^1 = \dot{S} \cdot e_1 = 0 \quad \rightarrow \quad S^1 = \text{const.}
$$

$$
\dot{L} = \ldots = S^1 e_1 \times L / R^3 \quad \rightarrow \quad \dot{L}^1 = \dot{L} \cdot e_1 = 0 \quad \rightarrow \quad L^1 = \text{const.}
$$

can solve eqs. of motion for $H_{SO}^1$ by rotations
Spin-spin integrator

Spin-spin Hamiltonian is a linear combination of Hamiltonians

$$S_a \cdot S_b / R^3 \quad \text{and} \quad (S_a \cdot N)(S_b \cdot N) / R^3$$

whose exact flows are again obtained from rotations

approximate spin-spin flow by composing the exact subflows
Splitting integrator

\[ \varphi_h^H \approx \varphi_{h/2}^{H_{SS}} \circ \varphi_{h/2}^{H_{SO}} \circ \varphi_h^{H_{Orb}} \circ \varphi_{h/2}^{H_{SO}} \circ \varphi_{h/2}^{H_{SS}}. \]

The individual flows in this formula are further approximated in a structure-preserving way.

Composition of Poisson maps is a Poisson map.

symmetric splitting of order 2, higher order by composition
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Linear vs. quadratic error growth

| Method | $h$  | $t_{CPU}$ | Method  | $h$  | $t_{CPU}$ |
|--------|------|----------|---------|------|----------|
| SPN4   | 64   | 5.1 s    | RK4     | 8    | 9.1 s    |
| SPN4   | 32   | 10.2 s   | RK4     | 4    | 18.5 s   |
| SPN4   | 16   | 20.8 s   | RK4     | 2    | 37.1 s   |
| Linear growth |     |          | Quadratic growth |     |          |
No energy drift for the symplectic integrator

\[ \frac{E - E_0}{10^{-10}} - 10^{-6} t \]
Including the radiation-reaction force

add non-conservative forces by further splitting

retain favourable properties for small dissipative forces
Energy errors in the dissipative regime

Figure: Energy deviation from different initial separations.
Summary

A structure-preserving (symplectic, or Poisson) PN integrator was constructed by splitting. Luckily, the spin Hamiltonian could be split into integrable sub-Hamiltonians.

Favourable long-time properties compared to standard integrators
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