CONTINUITY OF LYAPUNOV EXPONENTS
IN THE $C^0$ TOPOLOGY

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Abstract. We prove that the Bochi-Mañé theorem is false, in general, for linear cocycles over non-invertible maps: there are $C^0$-open subsets of linear cocycles that are not uniformly hyperbolic and yet have Lyapunov exponents bounded from zero.

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1. Introduction

Bochi [4, 5] proved that every continuous $SL(2)$-cocycle over an aperiodic invertible system can be approximated in the $C^0$ topology by cocycles whose Lyapunov exponents vanish, unless it is uniformly hyperbolic. The (harder) version of the statement for derivative cocycles of area-preserving diffeomorphisms on surfaces had been claimed by Mañé [18] almost two decades before. Bochi [4, 5] also completed the proof of this harder claim, based on an outline by Mañé. These results were then extended to arbitrary dimension by Bochi, Viana [7] and Bochi [6].

In this paper, we prove that the Bochi-Mañé theorem does not hold, in general, for cocycles over non-invertible systems: surprisingly, in the non-invertible setting there exist $C^0$-open sets of $SL(2)$-cocycles whose exponents are bounded away from zero. Indeed, we provide two different constructions of such open sets.

The first one (Theorem A) applies to Hölder continuous cocycles satisfying a bunching condition. The second one (Theorem B) has no bunching hypothesis but requires the cocycle to be $C^{1+\epsilon}$ and to be hyperbolic at some periodic point. In either case, we assume some form of irreducibility. A suitable extension of the Invariance Principle (Bonatti, Gomez-Mont, Viana [9], Avila, Viana [2]) that we prove here gives that these cocycles have non-zero Lyapunov exponents. We also

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prove that they are continuity points for the Lyapunov exponents, relative to the $C^0$ topology, and thus the claim follows.

Continuity of Lyapunov exponents, especially with respective to finer topologies, has been the object of considerable recent interest. See Viana [22, Chapter 10], Duarte, Klein [13] and references therein. It was conjectured by Viana [22] that Lyapunov exponents are always continuous among Hölder continuous fiber-bunched $\text{SL}(2)$-cocycles, and this has just been proved by Backes, Brown, Butler [3]. In fact, they prove a stronger conjecture also from Viana [22]: Lyapunov exponents vary continuously on any family of $\text{SL}(2)$-cocycles with continuous invariant holonomies. Improving on a construction of Bocker, Viana in [22, Chapter 9], Butler [11] also shows that the fiber-bunched condition is sharp for continuity in some cases.

These and many other related results require the cocycles to some fair amount of regularity, starting from Hölder continuity. In view also of the Bochi-Mañé theorem, continuity in the $C^0$ topology (outside the uniformly hyperbolic realm) as we exhibit here, comes as a bit of a surprise. It seems that the explanation may lie on the fact that existence of an invariant stable holonomy comes for free in the non-invertible case.

2. STATEMENT OF RESULTS

Let $f : M \to M$ be a continuous uniformly expanding map on a compact metric space. By this we mean that there are $\rho > 0$ and $\sigma > 1$ such that, for any $x \in M$,

(i) $d(f(x), f(y)) \geq \sigma d(x, y)$ for every $y \in B(x, \rho)$ and

(ii) $f(B(x, \rho))$ contains the closure of $B(f(x), \rho)$.

Take $f$ to be topologically mixing and let $\mu$ be the equilibrium state of some Hölder continuous potential (see [23, Chapter 11]). Then $\mu$ is $f$-invariant and ergodic, and the support is the whole $M$.

Let $f : \hat{M} \to \hat{M}$ be the natural extension of $f$, that is, the shift map

$$(\ldots, x_{-n}, \ldots, x_{-1}, x_0) \mapsto (\ldots, x_{-n}, \ldots, x_{-1}, x_0, f(x_0))$$

in the space $\hat{M}$ of sequences $(x_n)_n$, such that $f(x_{-n}) = x_{-n+1}$ for every $n \geq 1$. Then $\hat{f}$ is a hyperbolic homeomorphism (see [21, Definition 1.3 and Section 6]). For any $\hat{x} = (x_{-n})_n$ in $\hat{M}$,

- the local stable set $W_{\text{loc}}^s(\hat{x})$ is the fiber $\pi^{-1}(\hat{x})$ of the canonical projection $\pi(\hat{x}) = x_0$, and
- the local unstable set $W_{\text{loc}}^u(\hat{x})$ consists of the points $\hat{y} = (y_{-n})_n$ such that $d(x_{-n}, y_{-n}) < \rho$ for every $n \geq 0$.

Let $\hat{\mu}$ be the lift of $\mu$ to $\hat{M}$, that is, the unique $\hat{f}$-invariant measure that projects down to $\mu$ under $\pi$. Then $\hat{\mu}$ is ergodic and supported on the whole $\hat{M}$, and it has local product structure (see [10, Section 2.2]).

The projective cocycle defined by a continuous map $A : M \to \text{SL}(2)$ over the transformation $f$ is the map $F_A : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$, $F_A(x, v) = (f(x), A(x)v)$. Denote $A^n(x) = A(f^{n-1}(x)) \cdots A(x)$ for every $n \geq 1$. By [13, 15], there exists $\lambda(A) \geq 0$, called Lyapunov exponent, such that

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)\| = \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)^{-1}\| = \lambda(A) \quad \text{for } \mu\text{-almost every } x \in M.
\end{equation}

We say that $A$ is u-bunched if there exists $\theta \in (0, 1]$ such that $A$ is $\theta$-Hölder continuous and

\begin{equation}
\|A(x)\| \|A(x)^{-1}\| \sigma^{-\theta} < 1 \quad \text{for every } x \in M.
\end{equation}

See [9, Definitions 1.11 and 2.2] and [11, Definition 2.2 and Remark 2.3].
Remark 2.1. The function \( d_\theta(x, y) = d(x, y)^\theta \) is also a distance in \( M \), and it satisfies (i) above with \( \sigma \) replaced with \( \sigma^\theta \). Moreover, \( A \) is 1-Hölder continuous with respect to \( d_\theta \) if it is \( \theta \)-Hölder continuous with respect to \( d \). Thus, up to replacing the distance in \( M \), it is no restriction to suppose that \( \theta = 1 \), and we do so.

Let \( \hat{A} : \hat{M} \to SL(2) \) be defined by \( \hat{A} = A \circ \pi \). Assuming that \( A \) is \( u \)-bunched, the cocycle \( \hat{F}_A \) defined by \( \hat{A} \) over \( \hat{f} \) admits invariant \( u \)-holonomies (see [9], Section 1.4) and [1, Section 3]), namely,

\[
h_{\hat{x}, \hat{y}}^n = \lim_{n \to \pm \infty} \hat{A}^n(\hat{f}^{-n}(\hat{y})) \hat{A}^n(\hat{f}^{-n}(\hat{x}))^{-1}
\]

for any \( \hat{y} \in W^u_{\text{loc}}(\hat{x}) \).

As \( \hat{A} \) is constant on local stable sets, \( \hat{F}_A \) also admits trivial invariant \( s \)-holonomies:

\[
h_{\hat{x}, \hat{y}}^n = \text{id} \text{ for any } \hat{y} \in W^s_{\text{loc}}(\hat{x}).
\]

Remark 2.2. It is not difficult to find a distance on \( \hat{M} \) relative to which \( \hat{A} \) is \( s \)-bunched, in addition to being \( u \)-bunched.

A probability measure \( \hat{m} \) on \( \hat{M} \times PR^2 \) is said to be \( u \)-invariant if it admits a disintegration \( \{ \hat{m}_\hat{x} : \hat{x} \in \hat{M} \} \) along the fibers \( \{ \hat{x} \} \times PR^2 \) such that

\[
(3) \quad (h_{\hat{x}, \hat{y}}^n, \hat{m}_\hat{x} = \hat{m}_\hat{y} \text{ for any } \hat{y} \in W^u_{\text{loc}}(\hat{x}).
\]

Similarly, we say that \( \hat{m} \) is \( s \)-invariant if it admits a disintegration \( \{ \hat{m}_\hat{x} : \hat{x} \in \hat{M} \} \) along the fibers \( \{ \hat{x} \} \times PR^2 \) such that

\[
(4) \quad \hat{m}_\hat{x} = \hat{m}_\hat{y} \text{ for any } \hat{y} \in W^s_{\text{loc}}(\hat{x}).
\]

A \( u \)-invariant (respectively \( s \)-invariant) probability measure \( \hat{m} \) is called a \( u \)-state (respectively, an \( s \)-state) if it is also invariant under \( \hat{F}_A \). We call \( \hat{m} \) an \( su \)-state (see [2], Section 4) if it is both a \( u \)-state and an \( s \)-state.

Theorem A. If \( A \) is \( u \)-bunched and has no \( su \)-states then \( \lambda(A) > 0 \) and \( A \) is a continuity point for the function \( B \mapsto \lambda(B) \) in the space of continuous maps \( B : M \to SL(2) \) equipped with the \( C^0 \) topology. In particular, the Lyapunov exponent \( \lambda(\cdot) \) is bounded away from zero on a \( C^0 \)-neighborhood of \( A \).

Example 2.3. Take \( f : S^1 \to S^1 \), \( f(x) = kx \mod Z \), for some integer \( k \geq 2 \), and \( m \) to be the Lebesgue measure on \( S^1 \). Let \( A : S^1 \to SL(2) \) be given by \( A(x) = A_0 R_x \), where \( A_0 \in SL(2) \) is a hyperbolic matrix and \( R_x \) is the rotation by angle \( x \). \( A \) is 1-Hölder continuous and, in view of the definition (2), it is \( u \)-bunched provided \( k > \|A_0\| \cdot \|A_0^{-1}\| \).

We claim that \( A \) has no \( su \)-states if \( k \) is large enough. Indeed, suppose that \( \hat{F}_A \) has some \( su \)-state \( \hat{m} \). Then, by [2], Proposition 4.8], \( \hat{m} \) admits a continuous disintegration \( \{ \hat{m}_\hat{x} : \hat{x} \in \hat{M} \} \) which is simultaneously \( s \)-invariant, \( u \)-invariant and \( \hat{F}_A \)-invariant. By \( s \)-invariance, we may write the disintegration as \( \{ \hat{m}_x : x \in M \} \) instead. Continuity and invariance under the dynamics imply that \( (A_0)_0, \hat{m}_0 = \hat{m}_0 \). Since \( A_0 \) is hyperbolic, this means that \( \hat{m}_0 \) is either a Dirac mass or a convex combination of two Dirac masses. Thus, by holonomy invariance, either every \( \hat{m}_x \) is a Dirac mass or every \( \hat{m}_x \) is supported on exactly 2 points.

In the first case, \( \xi(x) = \text{supp} \hat{m}_x \) defines a map \( \xi : S^1 \to PR^2 \) which is continuous and invariant under the cocycle:

\[
\xi(f(x)) = A_0 R_x \xi(x) \text{ for every } x \in S^1.
\]

It follows that the degree \( \deg \xi \) satisfies \( k \deg \xi = \deg \xi + 2 \) (the term 2 comes from the fact that \( S^1 \to PR^2, x \mapsto R_x v \) has degree 2 for any \( v \)). This is impossible when \( k \geq 4 \), and so this first case can be disposed of. In the second case, \( \xi(x) = \text{supp} \hat{m}_x \) defines a continuous invariant section with values in the space \( PR^{2,2} \) of pairs of
distinct points in $\mathbb{P} \mathbb{R}^2$. This can be reduced to the previous case by considering the 2-to-1 covering map $S^1 \to S^1$, $z \mapsto 2z \mod Z$ (notice that $f$ is its own lift under this covering map). Thus, this second case can also be disposed of. This proves our claim that $A$ has no $su$-states.

By Theorem [A] it follows that $\lambda(B) > 0$ for every continuous $B : S^1 \to \text{SL}(2)$ in a $C^0$-neighborhood of $A$. Incidentally, this shows that [S] Corollary 12.34 is not correct: indeed, the “proof” assumes the Bochi-Mañé theorem in the non-invertible case.

Now let $f : M \to M$ be a $C^{1+\epsilon}$ (that is, $C^1$ with H"{o}lder continuous derivative) expanding map on a compact manifold $M$ and $A : M \to \text{SL}(2)$ be a $C^{1+\epsilon}$ function. All the other objects, $\mu, F_A, \pi, M, \hat{f}, \hat{\mu}, \hat{\pi}, \hat{A}$ and $\hat{F}_A$ are as before. An invariant section is a continuous map $\hat{\xi} : M \to \mathbb{P} \mathbb{R}^2$ or $\hat{\xi} : M \to \mathbb{P} \mathbb{R}^{2,2}$ such that $\hat{A}(\hat{x})\hat{\xi}(\hat{x}) = \hat{\xi}(\hat{f}(\hat{x}))$ for every $\hat{x} \in \hat{M}$.

**Theorem B.** If $A$ admits no invariant section then it is continuity point for the function $B \mapsto \lambda(B)$ in the space of continuous maps $B : M \to \text{SL}(2)$ equipped with the $C^0$ topology. Moreover, $\lambda(A) > 0$ if and only if there exists some periodic point $p \in M$ such that $A^{\text{per}(p)}(p)$ is a hyperbolic matrix. In that case, $\lambda(\cdot)$ is bounded from zero for all continuous cocycles on a $C^0$-neighborhood of $A$.

This applies, in particular, in the setting of Young [25] and thus Theorem [B] contains a much stronger version of the main result in there: the Lyapunov exponent is $C^0$-continuous at every $C^2$ cocycle in the isotopy class; moreover, it is non-zero if and only if the cocycle is hyperbolic on some periodic orbit (an open and dense condition).

All the cocycles we consider are of the form $\hat{F}_B(\hat{x},v) = (\hat{f}(\hat{x}), B \circ \pi(\hat{x})v)$ for some continuous $B : M \to \text{SL}(2)$ and so they all have (trivial) $s$-holonomy. Thus the notion of $s$-invariant measure, as defined in [H], makes sense for such cocycles. In Section 3 we study certain properties of such measures. We do not assume the cocycle to be $u$-bunched, and so the conclusions apply for both theorems. In Section 4 we deduce Theorem [A].

The proof of Theorem [B] is similar, but we have to deal with the fact that $u$-holonomies need not exist, since we do not assume $u$-bunching. The first step, in Section 5, is to explain what we mean by a $u$-invariant measure and a $u$-state. Next, we need a suitable version of the Invariance Principle of [G]. This we prove in Section 6, using ideas from [20]. In Section 7 we check that the assumptions ensure that there are no $su$-states. In Section 8 we wrap up the proof.

3. $s$-IN Variant Measures

Let $\mathcal{M}^s$ be the space of probability measures on $\hat{M} \times \mathbb{P} \mathbb{R}^2$ that project down to $\hat{\mu}$ and are $s$-invariant. Let $\mathcal{M}$ be the space of probability measures on $M \times \mathbb{P} \mathbb{R}^2$ that project down to $\mu$. Both spaces are equipped with the weak* topology. Consider the map $\Psi : \mathcal{M} \to \mathcal{M}^s$ defined as follows: given any $m \in \mathcal{M}$ and a disintegration $\{m_x : x \in M\}$ along the fibers $\{x\} \times \mathbb{P} \mathbb{R}^2$, let $\hat{m} = \Psi(m)$ be the measure on $\hat{M} \times \mathbb{P} \mathbb{R}^2$ that projects down to $\hat{\mu}$ and whose conditional probabilities $\hat{m}_{\hat{x}}$ along the fibers $\{\hat{x}\} \times \mathbb{P} \mathbb{R}^2$ are given by

$$\hat{m}_{\hat{x}} = m_{\pi(\hat{x})}.$$  

It is clear from the definition that $\hat{m} \in \mathcal{M}^s$. Moreover, if $\{m'_x : x \in M\}$ is another disintegration of $m$ then, by essential uniqueness of the disintegration, $m_x = m'_x$ for $\mu$-almost every $x$. Recalling that $\hat{\mu}$ is the lift of $\mu$, this implies that $m_{\pi(\hat{x})} = m'_{\pi(\hat{x})}$ for $\hat{\mu}$-almost every $\hat{x}$. Thus $\hat{m}$ does not depend on the choice of the disintegration. This shows that $\Psi$ is well-defined. We are going to prove:
Proposition 3.1. \( \Psi : M \to M^\ast \) is a homeomorphism.

Proof. We use the fact that \( \hat{\mu} \) has local product structure (see Section 2.2)).

For each \( \hat{p} \in \hat{M} \), let \( p = \pi(\hat{p}) \) and consider the neighborhood \( \hat{V}_\rho = \pi^{-1}(B(p, \rho)) \).

We may identify \( \hat{V}_\rho \) to the product

\[
B(p, \rho) \times \pi^{-1}(p) = W^u_{\text{loc}}(\hat{p}) \times W^s_{\text{loc}}(\hat{p})
\]

through a homeomorphism, in such a way that \( \pi \) becomes the projection to the first coordinate. Local product structure gives that the restriction of \( \hat{\mu} \) may be written as

\[
\hat{\mu} | \hat{V}_\rho = \rho \hat{\mu}^u \times \hat{\mu}^s,
\]

where \( \rho : \hat{V}_\rho \to (0, \infty) \) is a continuous function, \( \hat{\mu} = \mu | B(p, \rho) \) and \( \mu^s \) is a probability measure on \( W^s_{\text{loc}}(\hat{p}) \).

Lemma 3.2. For any \( m \in M \), the measure \( \hat{m} = \Psi(m) \) satisfies

\[
\hat{m} | \hat{V}_\rho \times \mathbb{R}^2 = \rho (m | B(p, \rho)) \times \hat{\mu}^s \text{ for any } \hat{p} \in \hat{M}.
\]

Proof. Given any bounded measurable function \( g : \hat{V}_\rho \times \mathbb{R}^2 \to \mathbb{R} \),

\[
\int_{\hat{V}_\rho \times \mathbb{R}^2} g \, d\hat{m} = \int_{\hat{V}_\rho} \int_{\mathbb{R}^2} g(\hat{x}, v) \, d\hat{m}_{\hat{x}}(v) \, d\hat{\mu}(\hat{x})
\]

\[
= \int_{W^u_{\text{loc}}(\hat{p})} \int_{W^s_{\text{loc}}(\hat{p})} \int_{\mathbb{R}^2} g(x, \xi, v) \, d\hat{m}_{(x, \xi)}(v) \rho(x, \xi) \, d\hat{\mu}^u(x) \, d\hat{\mu}^s(\xi).
\]

Since \( \hat{m}_{(x, \xi)} = m_x \) for every \( x \in M \), by definition, it follows that

\[
\int_{\hat{V}_\rho \times \mathbb{R}^2} g \, d\hat{m} = \int_{W^u_{\text{loc}}(\hat{p})} \int_{W^s_{\text{loc}}(\hat{p})} \int_{\mathbb{R}^2} g(x, \xi, v) \rho(x, \xi) \, dm_x(v) \, d\hat{\mu}^u(x) \, d\hat{\mu}^s(\xi)
\]

\[
= \int_{W^u_{\text{loc}}(\hat{p})} \int_{W^s_{\text{loc}}(\hat{p})} \int_{W^u_{\text{loc}}(\hat{p}) \times \mathbb{R}^2} g(x, \xi, v) \rho(x, \xi) \, dm(v, x) \, d\hat{\mu}^s(\xi).
\]

This proves the claim. \( \square \)

Let us prove that \( \Psi \) is continuous, that is, that given any sequence \( (m_n)_n \) converging to some \( m \in M \), we have

\[
\int_{M \times \mathbb{R}^2} g \, d\Psi(m_n) \to \int_{M \times \mathbb{R}^2} g \, d\Psi(m)
\]

for every continuous function \( g : \hat{M} \times \mathbb{R}^2 \to \mathbb{R} \). It is no restriction to suppose that the support of \( g \) is contained in \( \hat{V}_\rho \) for some \( \hat{p} \in \hat{M} \), for every continuous function is a finite sum of such functions. Then, by Lemma 3.2,

\[
\int_{M \times \mathbb{R}^2} g \, d\Psi(m_n) = \int_{W^u_{\text{loc}}(\hat{p}) \times \mathbb{R}^2} \int_{W^s_{\text{loc}}(\hat{p})} g(x, \xi, v) \rho(x, \xi) \, d\hat{\mu}^s(\xi) \, dm_x(x, v).
\]

Our hypotheses ensure that \( G(x, v) = \int_{W^u_{\text{loc}}(\hat{p})} g(x, \xi, v) \rho(x, \xi) \, d\hat{\mu}^s(\xi) \) defines a continuous function. Hence, the assumption that \( (m_n) \to m \) implies that

\[
\int_{M \times \mathbb{R}^2} g \, d\Psi(m_n) = \int_{W^u_{\text{loc}}(\hat{p}) \times \mathbb{R}^2} G(x, v) \, dm_x(x, v) \to \int_{W^u_{\text{loc}}(\hat{p}) \times \mathbb{R}^2} G(x, v) \, dm(x, v) = \int_{M \times \mathbb{R}^2} g \, d\Psi(m),
\]

as claimed. We are left to proving that \( \Psi \) is a bijection.

Surjectivity is clear: given \( \hat{m} \in M^\ast \) take \( m \) to be the probability measure on \( M \times \mathbb{R}^2 \) that projects down to \( \mu \) and whose conditional probabilities along the vertical fibers \( \{x\} \times \mathbb{R}^2 \) are given by \( m_x = m_{\hat{x}} \) for any \( \hat{x} \in \pi^{-1}(x) \). This is well
defined, by (3), and it is clear from the definition that \( \Psi(m) = \hat{m} \). Injectivity is a consequence of Lemma 3.2. Indeed, if \( \Psi(m_1) = \Psi(m_2) \) then

\[
\int_{X \times V} \int_{W^u_{\loc}(\hat{p})} \rho(x, \xi) \, d\hat{\mu}^*(\xi) \, dm_1(x, v) = \int_{X \times V} \int_{W^u_{\loc}(\hat{p})} \rho(x, \xi) \, d\hat{\mu}^*(\xi) \, dm_2(x, v)
\]

for any \( \hat{p} \in \hat{M} \) and any \( X \times V \subset B(p, \rho) \times \ PR^2 \). This implies that

\[
H m_1 | B(p, \rho) = H m_2 | B(p, \rho), \quad \text{where} \quad H(x) = \int_{W^u_{\loc}(\hat{p})} \rho(x, \xi) \, d\hat{\mu}^*(\xi).
\]

Noting that \( H \) is positive, it follows that the restrictions of \( m_1 \) and \( m_2 \) to \( B(p, \rho) \) coincide, for every \( p \in M \). Thus \( m_1 = m_2 \).

\[\blacksquare\]

**Corollary 3.3.** \( M^+ \) is non-empty, convex and compact.

**Proof.** Convexity is obvious and the other claims follow directly from Proposition 3.1 since \( M \) is non-empty and compact.

Let \( (B_n)_n \) be a sequence of maps converging uniformly to some \( B \) in the space of continuous maps \( M \rightarrow SL(2) \), and \( (\hat{m}_n)_n \) be a sequence of probability measures on \( M \) converging in the weak* topology to some probability measure \( m \).

**Corollary 3.4.** If \( \hat{m}_n \) is an s-state of \( B_n \) for every \( n \) then \( \hat{m} \) is an s-state of \( B \).

**Proof.** It follows from Corollary 3.3 that \( \hat{m} \in \hat{M}^+ \). It is clear that \( \hat{m} \) is \( \hat{F}_B \)-invariant, because \( m_n \) is \( \hat{F}_{B_n} \)-invariant for every \( n \) and \( \hat{F}_{B_n} \) converges uniformly to \( \hat{F}_B \). \[\blacksquare\]

### 4. Proof of Theorem A

If \( \lambda(A) = 0 \) then, by the Invariance Principle ([2] Theorem D, [9] Théorème 1]), every \( \hat{F}_A \)-invariant probability measure \( \hat{m} \) that projects down to \( \hat{\mu} \) is an su-state. Thus, the hypothesis that \( A \) has no su-states implies that \( \lambda(A) > 0 \).

We are left to proving that \( A \) is a continuity point for the Lyapunov exponent. Define (here \( v \) denotes both a direction and any non-zero vector in that direction)

\[
\phi_B : \hat{M} \times PR^2 \rightarrow \mathbb{R}, \quad \phi_B(\hat{x}, v) = \log \frac{\|B(\hat{x})v\|}{\|v\|}.
\]

**Proposition 4.1.** Every \( B : M \rightarrow SL(2) \) in a \( C^0 \)-neighborhood of \( A \) admits some s-state \( \hat{m}_B \) such that

\[
-\lambda(B) = \int_{\hat{M} \times PR^2} \phi_B \, d\hat{m}.
\]

**Proof.** First, suppose that \( \lambda(B) = 0 \). For every \( (\hat{x}, v) \in \hat{M} \times PR^2 \) and \( n \geq 1 \),

\[
\sum_{j=0}^{n-1} \phi_B(\hat{F}_B^j(\hat{x}, v)) = \log \frac{\|\hat{B}^n(\hat{x})v\|}{\|v\|} \in \left[ -\log \|\hat{B}^n(\hat{x})^{-1}\|, \log \|\hat{B}^n(\hat{x})\| \right].
\]

We also have that, or \( \hat{\mu} \)-almost every \( \hat{x} \in \hat{M} \),

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\hat{B}^n(\hat{x})\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\hat{B}^n(\hat{x})^{-1}\| = \lambda(B).
\]

Thus, for any \( \hat{F}_B \)-invariant measure \( \hat{m} \) that projects down to \( \hat{\mu} \),

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_B(\hat{F}_B^j(\hat{x}, v)) = 0 \quad \text{for} \quad \hat{m} \text{-almost every} \ (\hat{x}, v).
\]

By the ergodic theorem, this implies that

\[
\int_{\hat{M} \times PR^2} \phi_B \, d\hat{m} = 0 = \lambda(B)
\]
and so every $\tilde{F}_B$-invariant measure $\tilde{m}$ that projects down to $\tilde{\mu}$ satisfies the conclusion of the lemma.

Now suppose that $\lambda(B) > 0$. By the theorem of Oseledets [19], there exists an $\tilde{F}_B$-invariant splitting $\mathbb{R}^2 = E^u_x \oplus E^s_x$ defined at $\tilde{\mu}$-almost every point $\tilde{x}$ and such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\tilde{B}^n(\tilde{x})v\| = \lambda(B) \text{ for non-zero } v \in E^u_x \text{ and}$$

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\tilde{B}^n(\tilde{x})v\| = -\lambda(B) \text{ for non-zero } v \in E^s_x.$$ 

(6)

Let $\tilde{m}$ be the probability measure on $\tilde{M} \times \mathbb{P}\mathbb{R}^2$ that projects down to $\tilde{\mu}$ and whose conditional probabilities along the fibers $\{\tilde{x}\} \times \mathbb{P}\mathbb{R}^2$ are given by the Dirac masses at $E^u_x$. Then $\tilde{m}$ is an $s$-state: it is clear that it is $\tilde{F}_B$-invariant; to check that it is $s$-invariant, just note that the subspace $E^s_x$ depends only on the forward iterates, and so it is constant on each $\pi^{-1}(x)$. Moreover, by the ergodic theorem and (6),

$$\int_{\tilde{M} \times \mathbb{P}\mathbb{R}^2} \phi_B \, d\tilde{m} = \int_{\tilde{M} \times \mathbb{P}\mathbb{R}^2} \lim_{n \to \pm\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_B \circ \tilde{F}_B \, d\tilde{m}$$

$$= \int_{\tilde{M}} \int_{\mathbb{P}\mathbb{R}^2} \lim_{n \to \pm\infty} \frac{1}{n} \log \|\tilde{B}^n(\tilde{x})v\| \frac{d\tilde{\mu}}{\|v\|} \, d\tilde{\mu} = \int_{\tilde{M}} -\lambda(B) \, d\tilde{\mu}(\tilde{x}) = -\lambda(B).$$

This completes the proof. \qed

**Lemma 4.2.** If $A$ has no $su$-states then it has exactly one $s$-state.

**Proof.** Existence is contained in Proposition 4.1. To prove uniqueness, we argue as follows. Let $\tilde{m}$ be any $s$-state. As observed before, the hypothesis implies that $\lambda(A) > 0$. Let $\mathbb{R}^2 = E^u_x \oplus E^s_x$ be the Oseledets invariant splitting, defined at $\mu$-almost every point $\tilde{x}$. Let $\tilde{m}^u$ and $\tilde{m}^s$ be the probability measures on $\tilde{M} \times \mathbb{P}\mathbb{R}^2$ that project down to $\mu$ and whose conditional probabilities along the fibers $\{\tilde{x}\} \times \mathbb{P}\mathbb{R}^2$ are the Dirac masses at $E^u_x$ and $E^s_x$, respectively. Then $\tilde{m}^u$ is a $u$-state, $\tilde{m}^s$ is an $s$-state and every $\tilde{F}_A$-invariant probability measure is a convex combination of $\tilde{m}^u$ and $\tilde{m}^s$ (compare [2] Lemma 6.1). Then, keeping in mind that $\tilde{\mu}$ is ergodic, there is $\alpha \in [0, 1]$ such that $\tilde{m} = \alpha \tilde{m}^u + (1 - \alpha)\tilde{m}^s$. If $\alpha > 0$, we may write

$$\tilde{m}^u = \frac{1}{\alpha} \tilde{m} + (1 - \frac{1}{\alpha})\tilde{m}^s$$

and, as $\tilde{m}$ and $\tilde{m}^s$ are s-states, it follows that $\tilde{m}^u$ is an $s$-state. Since $\tilde{m}^u$ is also a $u$-state, this contradicts the hypothesis. Thus $\alpha = 0$, that is, $m = m^s$. \qed

Theorem A is an easy consequence. Indeed, we already know that $\lambda(A) > 0$. Consider any sequence $A_k : M \to \text{SL}(2)$, $k \in \mathbb{N}$ converging to $A$ in the $C^0$ topology. By Proposition 4.1, for each $k$ one can find some $s$-state $\tilde{m}_k$ for $A_k$ such that

$$-\lambda(A_k) = \int_{\tilde{M} \times \mathbb{P}\mathbb{R}^2} \phi_{A_k} \, d\tilde{m}_k.$$ 

Up to restricting to a subsequence, we may suppose that $(\tilde{m}_k)_k$ converges to some probability measure $\tilde{m}$ in the weak* topology. By Corollary 3.3, $\tilde{m}$ is an $s$-state for $A$. Moreover, since $\phi_{A_k}$ converges uniformly to $\phi_A$,

$$\lim_k -\lambda(A_k) = \int_{\tilde{M} \times \mathbb{P}\mathbb{R}^2} \phi_A \, d\tilde{m}.$$ 

(7)
Remark 4.3. The converse to Lemma 4.2 is true when $\lambda(\hat{A}) > 0$.

5. u-states without u-bunching

Next we prove Theorem B. Initially, suppose that $0 \leq \lambda(\hat{A}) < \log \sigma$. Then the cocycle is “nonuniformly u-bunched,” in a sense that was exploited before, in [21 Sections 2.1 and 2.2]. Using those methods, one gets that (compare [21 Proposition 2.5])

$$h^u_{\hat{x},\hat{y}} = \lim_{n} \hat{A}^{n}(\hat{f}^{-n}(\hat{y})) \hat{A}^{-n}(\hat{f}^{-n}(\hat{x}))^{-1}$$

exists for $\hat{\mu}$-almost every $\hat{x}$ and any $\hat{y} \in W^u_{\text{loc}}(\hat{x})$. Then we define a probability measure $\hat{m}$ on $\hat{M} \times \mathbb{PR}^2$ to be u-invariant if it admits a disintegration $\{\hat{m}_x : \hat{x} \in \hat{M}\}$ along the fibers $\{\hat{x}\} \times \mathbb{PR}^2$ such that

$$h^u_{\hat{x},\hat{y}} \hat{m}_x = \hat{m}_y$$

for $\hat{\mu}$-almost every $\hat{x}$ and any $\hat{y} \in W^u_{\text{loc}}(\hat{x})$.

As before, a u-state is an $\hat{F}_A$-invariant probability measure which is u-invariant.

When $\lambda(\hat{A}) \geq \log \sigma$ we have to restrict ourselves to the subclass of $\hat{F}_A$-invariant probability measures whose center Lyapunov exponent is strictly less than $\sigma$. More precisely, we consider only $\hat{F}_A$-invariant probability measures $\hat{m}$ such that

$$\lim_{n} \frac{1}{n} \log \|D \hat{A}^{n}(\hat{x})v\| \leq c < \log \sigma$$

for $\hat{\mu}$-almost every $(\hat{x}, v) \in \hat{M} \times \mathbb{PR}^2$, where $D \hat{A}(\hat{x})v$ denotes the derivative of the projective map $\hat{A}(\hat{x}) : \mathbb{PR}^2 \to \mathbb{PR}^2$.

Remark 5.1. The following elementary bound will be useful:

$$\|\hat{A}(\hat{x})\|^{-1}\|\hat{A}(\hat{x})\|^{-1} \leq \frac{\|D \hat{A}(\hat{x})v\|}{\|v\|} \leq \|\hat{A}(\hat{x})\|\|\hat{A}(\hat{x})\|^{-1}$$

for every $\hat{x}$.

In the next result we use the fact that the natural extension of $f$ admits a $C^{1+\epsilon}$ realization: there exist a $C^{1+\epsilon}$ embedding $g : U \to U$ defined on some open subset $U$ of an Euclidean space, and a topological embedding $\hat{\iota} : \hat{M} \to U$ with $g(\hat{\iota}(\hat{M})) = \hat{\iota}(\hat{M})$ and $g \circ \hat{\iota} = \hat{\iota} \circ \hat{f}$. A proof is given in Appendix A. Identifying $\hat{M}$ with $\hat{\iota}(\hat{M})$ we may view $\hat{f}$ as a restriction of $g$, and so we may apply Pesin theory to it.

Proposition 5.2. If $\hat{m}$ satisfies (3) then for $(\hat{x}, v)$ in a full $\hat{m}$-measure subset $\Lambda$ of $\hat{M} \times \mathbb{PR}^2$ there exists a $C^1$ function $\psi_{\hat{x},v} : W^u_{\text{loc}}(\hat{x}) \to \mathbb{PR}^2$ depending measurably on $(\hat{x}, v)$ such that $\psi_{\hat{x},v}(\hat{x}) = v$ and the graphs $W^u_{\text{loc}}(\hat{x}, v) = \{(\hat{y}, \psi_{\hat{x},v}(\hat{y})) : \hat{y} \in W^u_{\text{loc}}(\hat{x})\}$ satisfy

(a) $\hat{F}^{-1}(W^u_{\text{loc}}(\hat{x}, v)) \subseteq W^u_{\text{loc}}(\hat{F}^{-1}(\hat{x}, v))$ for every $(\hat{x}, v) \in \Lambda$;
(b) $d(\hat{F}^{-n}(\hat{x}, v), \hat{F}^{-n}(\hat{y}, w)) \to 0$ exponentially fast for any $(\hat{y}, w) \in W^u_{\text{loc}}(\hat{x}, v)$.

Proof. The assumption ensures that there exists $\hat{m}$-almost everywhere an Oseledets strong-unstable subspace $E^u_{\hat{x},v} \subset T_\hat{x}U \times \mathbb{R}^2$ that is a graph over the unstable direction $E^u_{\hat{x}} \subset T_\hat{x}U$ of $g$. Then, by Pesin theory, there exists $\hat{m}$-almost everywhere a $C^1$ embedded disk $\hat{W}^u(\hat{x}, v)$ tangent to $E^u_{\hat{x},v}$, and such that

$$\hat{F}^{-n}(\hat{y}, w) \in \overline{W^u_{\text{loc}}(\hat{F}^{-n}(\hat{x}, v))} \quad \text{and} \quad d(\hat{F}^{-n}(\hat{x}, v), \hat{F}^{-n}(\hat{y}, w)) \leq \sigma^{-n}$$

for every $n \geq 0$ and $(\hat{y}, w) \in \overline{W^u_{\text{loc}}(\hat{x}, v)}$. This also implies that $\overline{W^u_{\text{loc}}(\hat{x}, v)}$ is a $C^1$ graph over a neighborhood of $\hat{x}$ inside $W^u(\hat{x})$. While the radius $r(\hat{x})$ of this
neighborhood need not be bounded from zero, in principle, Pesin theory also gives that it decreases sub-exponentially along orbits:

\[
\lim_{n \to \infty} \frac{1}{n} \log \rho(\hat{f}^{-n}(\hat{x})) = 0.
\]

On the other hand, the size of \( \hat{f}^{-n}(W^u_{\text{loc}}(\hat{x})) \) decreases exponentially fast (faster than \( \sigma^{-n} \)). Thus, the projection of \( W^u(\hat{F}^{-n}(\hat{x}, v)) \) contains \( \hat{f}^{-n}(W^u_{\text{loc}}(\hat{x})) \) for any large \( n \). Then \( \hat{F}^n(W^u(\hat{F}^{-n}(\hat{x}, v))) \) is a C^1 graph whose projection contains \( W^u_{\text{loc}}(\hat{x}) \).

6. A NEW \( u^- \)IN-VARIANCE PRINCIPLE

Here we prove the following form of the Invariance Principle, where the main novelty is that no \( u^- \)-bunching is assumed:

**Theorem 6.1.** Every \( \hat{F}_A^- \)-invariant probability measure \( \hat{m} \) satisfying

\[
\lim_{n \to \infty} \frac{1}{n} \log \| D\hat{A}^u(\hat{x})v \| \leq 0
\]

for \( \hat{m} \)-almost every \( (\hat{x}, v) \in \hat{M} \times \mathbb{R}^2 \), is a \( u^- \)-state.

We are going to extend to our setting an approach introduced by Tahzibi, Yang [20] for bunched cocycles. This is based on the notion of partial entropy, which may be defined as follows (see [16, 24] for more information).

Let \( \mathcal{R} \) be a Markov partition of \( \hat{f} \) with diameter small enough that \( \mathcal{R}(\hat{x}) \subset \hat{V}_\delta \) for every \( \hat{x} \in \hat{M} \), where \( \mathcal{R}(\hat{x}) \) denotes the element of \( \mathcal{R} \) that contains \( \hat{x} \). (Actually, elements of \( \mathcal{R} \) may intersect along their boundaries but, since the boundaries are nowhere dense and have zero \( \hat{m} \)-measure, we may ignore the trajectories through them.) Let \( \xi^u(\hat{x}) \subset \hat{M} \) be the connected component of \( \mathcal{R}(\hat{x}) \cap W^u_{\text{loc}}(\hat{x}) \) that contains \( \hat{x} \). For \( v \in \mathbb{R}^2 \) such that \( (\hat{x}, v) \in \Lambda \), let \( \Xi^u(\hat{x}, v) \) be the connected component of \( (\mathcal{R}(x) \times \mathbb{R}^2) \cap W^u_{\text{loc}}(\hat{x}, v) \) that contains \( (\hat{x}, v) \).

The family \( \xi^u \) is an adapted partition for \( (\hat{f}, \hat{\mu}) \): its elements are pairwise disjoint and, for \( \hat{\mu} \)-almost every \( \hat{x} \),

- \( \hat{f}^{-1}(\xi^u(\hat{x})) \subset \xi^u(\hat{f}^{-1}(\hat{x})) \)
- \( \xi^u(\hat{x}) \) contains a neighborhood of \( \hat{x} \) inside \( W^u(\hat{x}) \).

Analogously, \( \Xi^u \) is an adapted partition for \( (\hat{F}, \hat{m}) \). The corresponding partial entropies are defined by

\[
(\hat{h}^u_{\hat{f}}, W^u) = H_{\hat{\mu}}(\hat{f}^{-1} \xi^u | \xi^u) \quad \text{and} \quad (\hat{h}^u_{\hat{F}_A}, W^{u}) = H_{\hat{m}}(\hat{F}^{-1}_{\Lambda} \Xi^u | \Xi^u).
\]

### 6.1. \( \epsilon \)-invariant measures

Let \( \{\hat{\mu}^u_x : \hat{x} \in \hat{M}\} \) and \( \{\hat{m}^u_{\hat{x},v} : (\hat{x}, v) \in \hat{M} \times \mathbb{R}^2\} \) be disintegrations of, respectively, \( \hat{\mu} \) relative to \( \xi^u \) and \( \hat{m} \) relative to \( \Xi^u \). Let \( p : \hat{M} \times \mathbb{R}^2 \to \hat{M} \) be the canonical projection. We call \( \hat{m} \) \( \epsilon \)-invariant if

\[
(h^u_{\hat{f},v,w}, \hat{m}^u_{\hat{x},v}) = \hat{m}^u_{\hat{x},w} \quad \text{for} \hat{m} \text{-almost every } (\hat{x}, v) \text{ and } (\hat{x}, w),
\]
where \( h^c_{x,v,w} : \Xi^u(\hat{x}, v) \to \Xi^u(\hat{x}, w) \) is the bijection defined by \( p \circ h^c_{x,v,w} = p \).

Equivalently, the measure \( \hat{m} \) is \( c \)-invariant if
\[
    p_\nu(\hat{m}^u_{x,v}) = \hat{\mu}^u_x \quad \text{for } \hat{m} \text{-almost every } (\hat{x}, v).
\]

**Proposition 6.2.** The measure \( \hat{m} \) is \( u \)-invariant if and only if it is \( c \)-invariant.

**Proof.** Let us start with a model: let \( \nu \) be a probability measure on a product \( X \times Y \) of two separable metric spaces, and let \( \{ \nu^1_y : y \in Y \} \) and \( \{ \nu^2_x : x \in X \} \) be disintegrations of \( \nu \) relative to the partition into horizontals \( X \times \{ y \} \) and the partition into verticals \( \{ x \} \times Y \), respectively. We call \( \nu \) \( v \)-invariant (respectively, \( h \)-invariant) if the disintegrations may be chosen such that \( \nu^1_y \) is independent of \( y \) (respectively, \( \nu^2_x \) is independent of \( x \)).

**Lemma 6.3.** \( \nu \) is \( v \)-invariant if and only if it is \( h \)-invariant.

**Proof.** Suppose that \( \nu \) is \( v \)-invariant and let \( \nu^1 \) be such that \( \nu^1_y = \nu^1 \) for every \( y \). Let \( \nu^2 \) be the quotient of \( \nu \) relative to the horizontal partition or, equivalently, the projection of \( \nu \) to the second coordinate. Then, by the definition of disintegration,
\[
    \nu = \nu^1 \times \nu^2.
\]
This implies that \( \nu^1 \) is the projection of \( \nu \) to the first coordinate and \( \nu^2 = \nu^2 \) defines a disintegration of \( \nu \) relative to the vertical partition. In particular, \( \nu \) is \( h \)-invariant. The proof that \( h \)-invariance implies \( v \)-invariance is identical.

To deduce the proposition we only have to reduce its setting to that of Lemma 6.3. Consider the partitions \( \Xi^c \) and \( \Xi^{cu} \) of \( M \times \mathbb{R}^2 \) defined by
\[
    \Xi^c(\hat{x}, v) = p^{-1}(\hat{x}) \quad \text{and} \quad \Xi^{cu}(\hat{x}, v) = p^{-1}(\xi^u(\hat{x}))
\]
Let \( \{ \hat{m}^c_{x,v} : (\hat{x}, v) \in M \times \mathbb{R}^2 \} \) and \( \{ \hat{m}^{cu}_{x,v} : (\hat{x}, v) \in M \times \mathbb{R}^2 \} \) be disintegrations of \( \hat{m} \) relative to \( \Xi^c \) and \( \Xi^{cu} \), respectively. Both \( \Xi^c \) and \( \Xi^{cu} \) refine \( \Xi^u \). So, by essential uniqueness of the disintegration,
\[
\begin{align*}
    (i) & \quad \{ \hat{m}^u_{y,w} : (\hat{y}, w) \in \Xi^u(\hat{x}, v) \} \quad \text{is a disintegration of } \hat{m}^{cu}_{\hat{x},v} \text{ with respect to the partition } \Xi^u \mid \Xi^{cu}(\hat{x}, v)\text{ and} \\
    (ii) & \quad \{ \hat{m}^u_{y,w} : (\hat{y}, w) \in \Xi^{cu}(\hat{x}, v) \} \quad \text{is a disintegration of } \hat{m}^{cu}_{\hat{x},v} \text{ with respect to the partition } \Xi^{cu} \mid \Xi^{cu}(\hat{x}, v),
\end{align*}
\]
for \( \hat{m} \)-almost every \( (\hat{x}, v) \). This will be used a few times in the following.

Now consider the map
\[
    \Psi_{\hat{x},v} : \Xi^{cu}(\hat{x}, v) \to \xi^u(\hat{x}) \times \mathbb{R}^2, \quad \Phi_{\hat{x}}(\hat{y}, w) = (\hat{y}, w)
\]
where \( z \) is such that \( (\hat{x}, z) \) is the point where \( \Xi^u(\hat{y}, w) \) intersects \( \Xi^c(\hat{x}, v) \). Since \( \Lambda \) has full \( \hat{m} \)-measure, \( \Psi_{\hat{x},v} \) is defined \( \hat{m}^{cu}_{\hat{x},v} \)-almost everywhere for \( \hat{m} \)-almost every \( (\hat{x}, v) \). Clearly, it is an invertible measurable map sending atoms of \( \Xi^u \mid \Xi^{cu}(\hat{x}, v) \) to horizontals \( \xi^u(\hat{x}) \times \{ z \} \) and atoms of \( \Xi^c \mid \Xi^{cu}(\hat{x}, v) \) to verticals \( \{ \hat{y} \} \times \mathbb{R}^2 \).

Identifying \( \Xi^{cu}(\hat{x}, v) \) to \( \xi^u(\hat{x}) \times \mathbb{R}^2 \) through \( \Psi_{\hat{x},v} \), (i) and (ii) above correspond to disintegrations of \( \hat{m}^{cu}_{\hat{x},v} \) relative to the horizontal partition and the vertical partition, respectively. Moreover, \( s \)-invariance and \( u \)-invariance translate to \( v \)-invariance and \( h \)-invariance, respectively. Thus the claim follows from Lemma 6.3.

**6.2. A criterion for \( c \)-invariance.** Note that \( h_\mu(\hat{f}) \leq h_{\hat{m}}(\hat{F}_A) \), because \( (\hat{f}, \hat{\mu}) \) is a factor of \( (\hat{F}_A, \hat{m}) \). For the partial entropies the inequality goes the opposite way:

**Proposition 6.4.** \( h_{\hat{m}}(\hat{F}_A, W^u) \leq h_\mu(\hat{f}, W^u) \) and the equality holds if and only if \( \hat{m} \) is \( c \)-invariant.
Proof. Keep in mind that $\xi^u \prec \hat{f}^{-1}\xi^u$ and $\Xi^u \prec \hat{F}^{-1}_A \Xi^u$. By definition,
\begin{equation}
H_{\hat{\mu}}(\hat{f}, W^u) = H_{\hat{\mu}}(\hat{f}^{-1}\xi^u | \xi^u) = \int H_{\hat{\mu}}^{\nu}(\hat{f}^{-1}\xi^u) \ d\hat{\mu}(\hat{x}) \text{ where } \hat{\nu} = \hat{H}^{-1}(\hat{x}\xi^u) \text{ and similarly for } h_{\hat{\mu}}(\hat{F}_A, W^u) \text{ and } \Xi^u.
\end{equation}

**Lemma 6.5.** For $\hat{m}$-almost every $(\hat{x}, v) \in \hat{M} \times \mathbb{R}^2$,

(a) $H_{\hat{m}_{\hat{x}, v}}(\hat{F}^{-1}_A \Xi^u | \Xi^u) \leq H_{\hat{\mu}}(\hat{f}^{-1}\xi^u)$ and

(b) the equality holds if and only if $\hat{m}_{\hat{x}, v}(\hat{F}_A^{-1} \Xi^u(\hat{y}, w)) = \hat{\mu}_u(\hat{f}^{-1}\xi^u(\hat{y}))$ for $\hat{m}_{\hat{x}, v}$-almost every $(\hat{y}, w) \in \Xi^u(\hat{x}, v)$.

Proof. Since $\hat{\mu} = p_v \hat{m}$ and $\Xi^u(\hat{x}, v) = p^{-1}(\xi^u(\hat{x}))$, essential uniqueness of disintegrations gives that $\hat{\mu}_u = p_v(\hat{m}_{\hat{x}, v})$ for $\hat{m}$-almost every $(\hat{x}, v)$. Thus,
\begin{align*}
H_{\hat{\mu}}(\hat{f}^{-1}\xi^u) &= \int -\log \hat{\mu}_u(\hat{f}^{-1}\xi^u(\hat{y})) \ d\hat{\mu}_u(\hat{y}) \\
&= \int -\log \hat{m}_{\hat{x}, v}(\hat{F}^{-1}_A \Xi^u(\hat{y}, w)) \ d\hat{m}_{\hat{x}, v}(\hat{y}, w) = H_{\hat{m}_{\hat{x}, v}}(\hat{F}^{-1}_A \Xi^u)
\end{align*}
for $\hat{m}$-almost every $(\hat{x}, v)$. Moreover, using the relation $\hat{F}_A^{-1} \Xi^u \vee \Xi^u = \hat{F}_A^{-1} \Xi^u$,
\begin{align*}
H_{\hat{m}_{\hat{x}, v}}(\hat{F}^{-1}_A \Xi^u) &\geq H_{\hat{m}_{\hat{x}, v}}(\hat{F}^{-1}_A \Xi^u | \Xi^u) = H_{\hat{m}_{\hat{x}, v}}(\hat{F}^{-1}_A \Xi^u | \Xi^u).
\end{align*}

This proves claim (a). Moreover, the equality holds if and only if the partitions $\hat{F}_A^{-1} \Xi^u$ and $\Xi^u$ are independent relative to $\hat{m}_{\hat{x}, v}$, that is,
\begin{align*}
\hat{m}_{\hat{x}, v}(\hat{F}_A^{-1} \Xi^u(\hat{y}, w)) &= \hat{m}_{\hat{x}, v}(\hat{F}_A^{-1} \Xi^u(\hat{y}, w)) \text{ for } \hat{m}_{\hat{x}, v} \text{-almost every } (\hat{y}, w).
\end{align*}

By the previous observations, this is equivalent to
\begin{align*}
\hat{m}_{\hat{x}, v}(\hat{F}_A^{-1} \Xi^u(\hat{y}, w)) &= \hat{\mu}_u(\hat{f}^{-1}\xi^u(\hat{y})) \text{ for } \hat{m}_{\hat{x}, v} \text{-almost every } (\hat{y}, w),
\end{align*}
as claimed in (b). \hfill \Box

Similarly to [13], we have $H_{\hat{m}_{\hat{x}, v}}(\hat{F}_A^{-1} \Xi^u | \Xi^u) = \int H_{\hat{m}_{\hat{x}, v}}(\hat{F}_A^{-1} \Xi^u) \ d\hat{m}_{\hat{x}, v}(\hat{y}, v)$. So, integrating the inequality in part (a) of the lemma,
\begin{align*}
H_{\hat{m}}(\hat{F}_A^{-1} \Xi^u | \Xi^u) &= \int H_{\hat{m}_{\hat{x}, v}}(\hat{F}_A^{-1} \Xi^u) \ d\hat{m}(\hat{x}, v) = \int H_{\hat{m}_{\hat{x}, v}}(\hat{F}_A^{-1} \Xi^u | \Xi^u) \ d\hat{m}(\hat{x}, v) \\
&\leq \int H_{\hat{\mu}}(\hat{f}^{-1}\xi^u) \ d\hat{\mu}(\hat{x}) = H_{\hat{\mu}}(\hat{f}^{-1}\xi^u | \xi^u).
\end{align*}

Moreover, the equality holds if and only if $\hat{m}_{\hat{x}, v}(\hat{F}_A^{-1} \Xi^u(\hat{x}, v)) = \hat{\mu}_u(\hat{f}^{-1}\xi^u(\hat{x}))$ for $\hat{m}$-almost every $(\hat{x}, v)$. In other words, the equality holds if and only if $p_v \hat{m}_{\hat{x}, v} = \hat{\mu}_u$ restricted to the $\sigma$-algebra generated by $\hat{F}_A^{-1} \Xi^u$.

Replacing $\hat{F}_A$ by any iterate $\hat{F}_A^n$, and noting that
\begin{align*}
h_{\hat{m}}(\hat{F}^n_A, W^u) &= nh_{\hat{m}}(\hat{F}_A, W^u) \quad \text{and} \quad h_{\hat{\mu}}(\hat{F}^n, W^u) = nh_{\hat{\mu}}(\hat{F}, W^u),
\end{align*}
we get that the equality holds if and only if $p_v \hat{m}_{\hat{x}, v} = \hat{\mu}_u$ restricted to the $\sigma$-algebra generated by $\hat{F}_A^{-n} \Xi^u$. Since $\cup_n \hat{F}_A^{-n} \Xi^u$ generates the Borel $\sigma$-algebra of every $\Xi^u(\hat{x}, v)$, this is the same as saying that $p_v \hat{m}_{\hat{x}, v} = \hat{\mu}_u$ for $\hat{m}$-almost every $(\hat{x}, v)$, that is, that $\hat{m}$ is $\sigma$-invariant. \hfill \Box
6.3. Proof of Theorem 6.1 The hypothesis (11) ensures that the Lyapunov exponents of \( \hat{m} \) along the center fibers \( \{ \hat{x} \} \times \mathbb{P} \mathbb{R}^2 \) are non-positive. Then

\[
h_{\hat{m}}(\hat{F}_A) = h_{\hat{m}}(\hat{F}_A, W^u)
\]

(see [17 Corollary 5.3]). Similarly, \( h_\mu(\hat{f}) = h_\mu(\hat{f}, W^u) \). Moreover, \( h_{\hat{m}}(\hat{F}_A) \geq h_\mu(\hat{f}) \) because \((\hat{f}, \hat{\mu})\) is a factor of \((\hat{F}_A, \hat{m})\). This proves that

\[
h_{\hat{m}}(\hat{F}_A, W^u) \geq h_\mu(\hat{f}, W^u).
\]

By Propositions 6.2 and 6.4 this implies that \( \hat{m} \) is \( u \)-invariant, as claimed.

7. Invariant sections and \( su \)-states

We say that an \( \hat{F}_A \)-invariant probability measure \( \hat{m} \) is an \( su \)-state if it is both an \( s \)-state and a \( u \)-state. Here we prove:

Theorem 7.1. Assume that A admits no invariant section and there exists some periodic point \( p \) of \( f \) such that \( A^{\per(p)}(p) \) is hyperbolic. Then \( A \) has no \( su \)-states.

Assume, by contradiction, that \( \hat{F}_A \) does admit some \( su \)-state \( \hat{m} \). Suppose for a while that \( \hat{m} \) admits a continuous disintegration \( \{ \hat{m}_x : \hat{x} \in \hat{M} \} \) along the vertical fibers \( \{ \hat{x} \} \times \mathbb{P} \mathbb{R}^2 \). The fact that \( \hat{m} \) is \( \hat{F}_A \)-invariant means that \( A(\hat{x}), \hat{m}_x = \hat{m}_{\hat{f}(\hat{x})} \) for \( \hat{m} \)-almost every \( \hat{x} \). Then, by continuity, this must hold for every \( \hat{x} \).

Let \( \hat{p} \) be the fixed point of \( \hat{f} \) in \( \pi^{-1}(p) \) and \( \kappa = \per(p) \) be its period. Then \( A^\kappa(\hat{p}) = A^\per(p) \) is hyperbolic. The fact that \( A^\kappa(\hat{p}), \hat{m}_{\hat{p}} = \hat{m}_\hat{p} \) implies that \( \hat{m}_{\hat{p}} \) is a convex combination of not more than two Dirac masses. Then, by \( su \)-invariance, the same is true about \( \hat{m}_x \) for every \( \hat{x} \). Thus \( \xi(\hat{x}) = \supp \hat{m}_x \) defines an invariant section for \( \hat{F}_A \), which is in contradiction with the hypotheses.

In general, disintegrations are only measurable. In what follows we explain how to bypass that and turn the previous outline into an actual proof of Theorem 7.1.

7.1. Dirac disintegrations. By the definition of \( su \)-state, there are disintegrations \( \{ \hat{m}_x^1 : \hat{x} \in \hat{M} \} \) and \( \{ \hat{m}_x^2 : \hat{x} \in \hat{M} \} \) of \( \hat{m} \) and there exists a full \( \hat{\mu} \)-measure subset \( U_\hat{p} \) of the neighborhood \( V_\hat{p} \approx W^u_{\hat{loc}}(\hat{p}) \times W^s_{\hat{loc}}(\hat{p}) \) such that

(i) \( (h^\per_{\hat{p}})_* \hat{m}_{x_1}^1 = \hat{m}_{\hat{y}_1} \) for every \( \hat{x}_1, \hat{y}_1 \in U_\hat{p} \) with \( \hat{y}_1 \in W^u_{\hat{loc}}(\hat{x}_1) \) (\( u \)-invariance);
(ii) \( \hat{m}_{x_2}^2 = \hat{m}_{\hat{y}_2} \) for every \( \hat{x}_2, \hat{y}_2 \in U_\hat{p} \) with \( \hat{y}_2 \in W^s_{\hat{loc}}(\hat{x}_2) \) (\( s \)-invariance);
(iii) \( \hat{m}_{x_3} = \hat{m}_{\hat{y}_3} \) for every \( \hat{x}_3 \in U_\hat{p} \) (essential uniqueness of disintegrations).

Also, we may choose \( U_\hat{p} \) so that \( \hat{m}_x^1(\Lambda_x) = 1 \) (recall that \( \Lambda_x = \Lambda \cap \{ (\hat{x}) \times \mathbb{P} \mathbb{R}^2 \} \)) for every \( \hat{x} \in U_\hat{p} \).

Since the Pesin unstable manifolds \( W^u(\hat{z}, u) \) vary measurably with the point, we may find compact sets \( \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda \) such that \( \hat{m}(\Lambda_j) \to 1 \) and \( W^u(\hat{z}, u) \) varies continuously on every \( \Lambda_j \). We may choose these compact sets in such a way that \( \hat{F}_A(A_j) \subset A_{j+1} \) for every \( j \geq 1 \). Up to reducing \( U_\hat{p} \) if necessary, \( \hat{m}_x^1(\Lambda_{j, \hat{x}}) \to 1 \) for every \( \hat{x} \in U_\hat{p} \).

Fix any \( \hat{x} \in U_\hat{p} \) such that \( \hat{\mu}_\hat{x}(\xi^{\per}(\hat{x}) \setminus U_\hat{p}) = 0 \). Then define \( \hat{m}_{\hat{x}} = \hat{m}_{x}^1 \) and

(a) \( \hat{m}_{\hat{y}} = (h^\per_{\hat{p}})_* \hat{m}_{x}^1 \) for every \( \hat{y} \in \xi^{\per}(\hat{x}) \);
(b) \( \hat{m}_{\hat{z}} = \hat{m}_{\hat{y}} \) for every \( \hat{z} \in W^s_{\hat{loc}}(\hat{y}) \cap V_\hat{p} \) with \( \hat{y} \in \xi^{\per}(\hat{x}) \).

By (i)-(iii), we have that \( \hat{m}_{\hat{y}} = \hat{m}_{\hat{y}}^1 = \hat{m}_{\hat{y}}^2 \) for every \( \hat{y} \cap \xi^{\per}(\hat{x}) \cap U_\hat{p} \) and \( \hat{m}_{\hat{z}} = \hat{m}_{\hat{z}}^2 \) for every \( \hat{z} \in W^s_{\hat{loc}}(\hat{y}) \cap V_\hat{p} \) with \( \hat{y} \in \xi^{\per}(\hat{x}) \cap U_\hat{p} \). By the choice of \( \hat{x} \) and the fact that \( \hat{\mu} \) has local product structure, the latter corresponds to a full \( \hat{\mu} \)-measure subset of points \( \hat{z} \in V_\hat{p} \). In particular, \( \{ \hat{m}_{x} : \hat{x} \in V_\hat{p} \} \) is a disintegration of \( \hat{m} \) on \( V_\hat{p} \).

Let us collect some immediate consequences of the definition of \( \hat{m}_{\hat{x}} \). For \( \hat{x}, \hat{y}, \hat{z} \) as in (a)-(b) above, denote \( h^\per_{\hat{y}, \hat{z}} = h^\per_{\hat{y}} \circ h^\per_{\hat{z}, \hat{y}} \) with \( h^\per_{\hat{y}, \hat{z}} : \{ \hat{y} \} \times \mathbb{P} \mathbb{R}^2 \to \{ \hat{z} \} \times \mathbb{P} \mathbb{R}^2 \) given by the identity. For \( j \geq 1 \), denote \( \alpha_j = \hat{m}_{\hat{z}}(\Lambda_{j, \hat{x}}) \); keep in mind that \( \alpha_j \to 1 \).
Corollary 7.2. For each $j \geq 1$,
(a) $\hat{A}_{j, \hat{z}} = h_{\hat{z}, \hat{z}}^j(\hat{A}_{j, \hat{z}})$ is compact and varies continuously with $\hat{z} \in V_{\hat{p}}$;
(b) the measure $\hat{m}_{\hat{z}} | \hat{A}_{j, \hat{z}}$ varies continuously with $\hat{z} \in V_{\hat{p}}$ in the weak$^*$ topology;
(c) $\hat{m}_{\hat{z}}(\hat{A}_{j, \hat{z}}) = \alpha_j$ for every $\hat{z} \in V_{\hat{p}}$.

Since the matrix $\hat{A}_{\hat{p}}(\hat{p})$ is hyperbolic, its action on the projective space $\mathbb{P}^2$ is a North pole-South pole map, that is, a Morse-Smale diffeomorphism with one attractor $a$ and one repeller $r$. We are going to prove:

Proposition 7.3. The support of $\hat{m}_{\hat{p}}$ is contained in $\{a, r\}$.

Proof. Since $\{\hat{m}_{\hat{z}} : \hat{z} \in V_{\hat{p}}\}$ is a disintegration and $\hat{m}$ is $F_{\hat{A}}$-invariant,

$$\tag{16} (F_{\hat{A}}^n) \hat{m}_{\hat{z}} = \hat{m}_{f_{\hat{z}}(\hat{z})} \text{ for } \hat{m}\text{-almost every } \hat{z} \in V_{\hat{p}} \cap \hat{f}^{-\infty}(V_{\hat{p}}).$$

The identity may not hold for $\hat{z} = \hat{p}$, but we are going to show that $\hat{m}_{\hat{p}}$ is at least “almost $F_{\hat{A}}$-invariant,” in a suitable sense:

Lemma 7.4. $\hat{m}_{\hat{p}}(\hat{F}^{-l_\infty}(K)) \geq \hat{m}_{\hat{p}}(K)$ for any compact set $K \subset \hat{A}_{j, \hat{p}}$ and every $l \geq 1$ and $j \geq 1$.

Proof. Fix $K$, $l$ and $j$. For any $\tilde{q}$ close to $\hat{p}$, define

$$h_{\tilde{q}} = h_{\hat{z}, \hat{z}}^l \circ h_{\hat{p}, \hat{p}}^l \circ h_{\hat{z}, \hat{z}}^l$$

where $\hat{y}$ and $\hat{z}$ are the points where $W^u_{\hat{p}}(\hat{z})$ intersects $W^s_{\hat{p}}(\hat{y})$ and $W^s_{\hat{p}}(\hat{q})$, respectively. Keep in mind that the two $s$-holonomies are given by the identity.

Also, $K \subset \hat{A}_{j, \hat{p}}$ ensures that $h_{\hat{q}}$ is continuous restricted to $K$. Define $K_{\tilde{q}} = h_{\tilde{q}}(K)$. Then $K_{\tilde{q}}$ is a compact subset of $F(I_{\hat{p}}(\hat{q})) \times \mathbb{P}^2$ such that $\hat{m}_{\hat{p}}(K_{\tilde{q}}) = \hat{m}_{\hat{p}}(K)$.

When $\tilde{q} \to \hat{p}$, the point $f_{\hat{p}}(\hat{q})$ also goes to $\hat{p}$, and then the same is true for $\hat{y}$ and $\hat{z}$. Thus $K_{\tilde{q}} \to K$ as $\tilde{q} \to \hat{p}$.

Choose $\tilde{q}$ close enough to $\hat{p}$ that $f_{\hat{p}}(\hat{q}) \in V_{\hat{p}}$ for $0 \leq n \leq l$ and such that $(F_{\hat{A}})_{\hat{m}_{\hat{q}}}$ is continuous restricted to $\hat{m}_{\hat{q}}(\tilde{\Lambda}_{\hat{q}, \hat{q}})$. It follows that

$$\hat{m}_{\hat{q}}(\hat{F}^{-l_\infty}(K_{\tilde{q}})) = \hat{m}_{\hat{p}}(K_{\tilde{q}}) = \hat{m}_{\hat{p}}(K).$$

Corollary 7.2 gives that $\hat{m}_{\hat{q}}(\tilde{\Lambda}_{\hat{q}, \hat{q}}) = \alpha_k$ for every $k \geq 1$. Thus

$$\hat{m}_{\hat{q}}(\hat{F}^{-l_\infty}(H_{\hat{q}})) = \hat{m}_{\hat{q}}(\tilde{\Lambda}_{\hat{q}, \hat{q}} \cap \hat{F}^{-l_\infty}(K_{\tilde{q}})) \geq \hat{m}_{\hat{p}}(K) + \alpha_k - 1.$$

By parts (a) and (b) of Corollary 7.2, the compact set $\tilde{\Lambda}_{\hat{q}, \hat{q}}$ and the measure $\hat{m}_{\hat{q}} | \tilde{\Lambda}_{\hat{q}, \hat{q}}$ depend continuously on $\hat{q}$. We know that the same is true for $\hat{F}^{-l_\infty}(K_{\tilde{q}})$. Thus, making $\tilde{q} \to \hat{p}$ in (17), we get that

$$\hat{m}_{\hat{p}}(\hat{F}^{-l_\infty}(K)) \geq \hat{m}_{\hat{p}}(K) + \alpha_k - 1.$$ 

Clearly, the left-hand side is less than or equal to $\hat{m}_{\hat{p}}(\hat{F}^{-l_\infty}(K))$. So, making $k \to \infty$ we get the claim. \qed

We are ready to complete the proof of Proposition 7.3. Suppose that $\hat{m}_{\hat{p}}$ is not supported inside $\{a, r\}$. Then, since the $\hat{A}_{j, \hat{p}}$ are a non-decreasing sequence whose union has full $\hat{m}_{\hat{p}}$-measure, for every large $j \geq 1$ the measure $\hat{m}_{\hat{p}} | \hat{A}_{j, \hat{p}}$ is not supported on $\{a, r\}$. Then we can find a compact set $K \subset \hat{A}_{j, \hat{p}}$ contained in a fundamental domain of $\hat{A}_{\hat{p}}(\hat{p})$ with positive $\hat{m}_{\hat{p}}$-measure. By Lemma 7.4 it follows that $\hat{F}^{-l_\infty}(K) \geq \hat{m}_{\hat{p}}(K) > 0$ for every $l \geq 0$. Since these sets are pairwise disjoint, it follows that $\hat{m}_{\hat{p}}$ is an infinite measure, which is a contradiction. \qed
7.2. Proof of Theorem 7.1. By Proposition 7.3, \( \hat{m}_\hat{f} \) is a convex combination of not more than two Dirac masses. Then, in view of the definition of this disintegration, the same is true about \( \hat{m}_\hat{f} \) for every \( \hat{z} \in \hat{V}_\hat{g} \). Then \( \hat{\xi}(\hat{z}) = \text{supp} \hat{m}_\hat{f} \) defines a continuous map on \( \hat{V}_\hat{g} \) with values on \( \mathbb{PR}^2 \) or \( \mathbb{PR}^{2,2} \) and such that \( \hat{A}(\hat{z})\hat{\xi}(\hat{z}) = \hat{\xi}(\hat{f}(\hat{z})) \) for every \( \hat{z} \in \hat{V}_\hat{g} \cap \hat{f}^{-1}(V_\hat{g}) \).

The same argument shows that for any point \( \hat{y} \in \hat{M} \) there exists a continuous disintegration \( \{\hat{m}_{\hat{y},\hat{z}} : \hat{z} \in \hat{V}_\hat{g}\} \) of the \( su \)-state restricted to \( \hat{V}_\hat{g} \). Since disintegrations are essentially unique and the neighborhoods \( \hat{V}_\hat{g} \) overlap on positive \( \hat{\mu} \)-measure subsets, all these conditional measures \( \hat{m}_{\hat{y},\hat{z}} \) must be supported on the same number, 1 or 2, of points. Thus, the map \( \hat{\xi} \) in the previous paragraph extends to a continuous invariant section on the whole \( \hat{M} \), which contradicts the assumptions of Theorem 8.

8. Proof of Theorem 8

If \( \lambda(A) = 0 \) then, trivially, \( A \) is a continuity point. Now assume that \( \lambda(A) > 0 \). Then (see for instance Kalinin [14, Theorem 1.4]) there exists some periodic point \( p \) of \( f \) such that \( A^{\text{per}}(p) \) is hyperbolic. Thus we may use Theorem 7.1 to conclude that there are no \( su \)-states. Now the proof of continuity of the Lyapunov exponents is entirely analogous to Section 4.

The same arguments also prove the converse: if the cocycle is hyperbolic at some periodic point then, again by Theorem 7.1, there are no \( su \)-states and thus the exponent cannot vanish. The proof of Theorem 8 is complete.

Appendix A. Smooth natural extensions

We show that the natural extension of any \( C^k \) local diffeomorphism \( f : M \to M \) on a compact manifold admits a \( C^k \) realization.

Since \( M \) is compact and \( f \) is locally injective, we may find families of open sets \( \{U_i, V_i : i = 1, \ldots, N\} \) such that: \( \{U_1, \ldots, U_N\} \) covers \( M \); every \( V_i \) contains the closure of \( U_i \); and every \( f \mid V_i \) is injective. Take smooth functions \( h_i : M \to [0, 1] \) such that \( h_i \mid U_i \equiv 1 \) and \( h_i \mid V_i^c \equiv 0 \). Define \( h(x) = (h_1(x), \ldots, h_N(x)) \) for \( x \in M \).

Then \( h : M \to [0, 1]^N \) is such that \( h(x) \neq h(y) \) for any pair \( (x, y) \) with \( x \neq y \) and \( f(x) = f(y) \). Since \( f \) is locally injective, the set of such pairs is a compact subset of \( M^2 \). Hence, there is \( \delta > 0 \) such that \( ||h(x) - h(y)|| \geq \delta \) for any \( (x, y) \) with \( x \neq y \) and \( f(x) = f(y) \).

Let \( \phi : M \to \mathbb{R}^m \) be a Whitney embedding of \( M \) into some Euclidean space, and \( \psi : M \times D \to \mathbb{R}^m \) be a tubular neighborhood: \( D \) denotes the open unit ball in \( \mathbb{R}^{m \times \dim M} \) and \( \psi \) is a smooth embedding with \( \psi(x, 0) = \phi(x) \). Identify \( M \times D \) with its image \( U = \psi(M \times D) \) through \( \psi \). Fix \( \lambda < \delta/4N \) and define

\[
g : M \times D \to M \times D, \quad g(x, v) = (f(x), h(x)/2N + \lambda v).
\]

It is clear that \( g \) is well defined and a \( C^k \) local diffeomorphism, and the image \( g(M \times D) \) is relatively compact in \( M \times D \).

Suppose that \( g(x, v) = g(y, w) \). Then \( f(x) = f(y) \) and \( h(x) - h(y) = 2N\lambda(w - v) \). In particular, \( ||h(x) - h(y)|| \leq 4N \lambda < \delta \). By the definition of \( \delta \), this implies that \( x = y \). Then the previous identities imply that \( v = w \). This proves that \( g \) is injective and, consequently, an embedding.

For each \( \hat{x} = (x_n)_{n=1}^\infty \in \hat{M} \) and \( n \geq 1 \) the set \( g^n((x_{-n})_0 \times D) \) is a disk \( D_n(\hat{x}) \) of radius \( \lambda^n \) inside \( [x_0] \times D \). These disks are nested and each \( D_{n+1}(\hat{x}) \) is relatively compact in \( D_n(\hat{x}) \). Thus, the intersection consists of exactly one point, which we denote as \( \psi(\hat{x}) \). By construction, the map \( \psi : \hat{M} \to M \times D \) defined in this way satisfies \( g \circ \psi = \psi \circ \hat{f} \). Moreover, the image \( \psi(M) \) coincides with \( \cap_n g^n(M \times D) \) and so it satisfies \( g(\psi(M)) = \psi(M) \).
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