DEEP OCEAN INFLUENCE ON UPPER OCEAN
BAROCLINIC INSTABILITY SATURATION*

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Abstract. In this paper we extend earlier results regarding the effects of the lower layer of the ocean (below the thermocline) on the baroclinic instability within the upper layer (above the thermocline). We confront quasigeostrophic baroclinic instability properties of a 2.5-layer model with those of a 3-layer model with a very thick deep layer, which has been shown to predict spectral instability for basic state parameters for which the 2.5-layer model predicts nonlinear stability. We compute and compare maximum normal-mode perturbation growth rates, as well as rigorous upper bounds on the nonlinear growth of perturbations to unstable basic states, paying particular attention to the region of basic state parameters where the stability properties of the 2.5- and 3-layer model differ substantially. We found that normal-mode perturbation growth rates in the 3-layer model tend to maximize in this region. We also found that the size of state space available for eddy-amplitude growth tends to minimize in this same region. Moreover, we found that for a large spread of parameter values in this region the latter size reduces to only a small fraction of the total enstrophy of the system, thereby allowing us to make assessments of the significance of the instabilities.

Key words: layer model, reduced-gravity, stability, instability saturation

1. Introduction

Observations indicate that most of the world oceans variability is confined in a thin layer limited from below by the permanent thermocline. There,

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the density is approximately uniform in the vertical but has important horizontal gradients. The latter imply the existence of a considerable reservoir of potential energy within this layer, stored in the isopycnals tilt and available to feeding baroclinic instability processes (Gill et al., 1974). Haine and Marshall (1998) have argued that these processes are of utmost importance for the dynamics of the upper ocean. These authors pointed out that baroclinic instability waves can be efficient transport agents capable of stopping convective processes, thereby exerting a large influence in the thermodynamic state of the ocean.

Because the total depth of the ocean is much larger than that of the upper thermocline layer, the reduced-gravity setting has been commonly adopted to studying the upper ocean baroclinic instability (Fukamachi et al., 1995; Ripa, 1995; Young and Chen, 1995; Beron-Vera and Ripa, 1997; Ripa, 1999b; Olascoaga and Ripa, 1999; Ripa, 1999c; Ripa, 1999a; Ripa, 2000a; Ripa, 2000b; Ripa, 2001). In this setting the active fluid layer is considered as floating on top of a quiescent, infinitely deep layer. Olascoaga (2001) showed, however, that a thick—but finite—abyssal active layer can substantially alter the stability properties of the upper ocean for certain baroclinic zonal flows, such as the Atlantic North Equatorial Current (ANEC) (Beron-Vera and Olascoaga, 2003).

Olascoaga (2001) considered spectral (i.e. linear, normal-mode), formal (or Arnold), and nonlinear (or Lyapunov) stability (Holm et al., 1985; cf. also McIntyre and Shepherd, 1987) in a 3-layer quasigeostrophic (QG) model. Primary attention was given to the limit of a very thick bottom layer. The stability results were compared with those from a reduced-gravity 2-layer (or 2.5-layer) model (Olascoaga and Ripa, 1999), and assessments were made of the influence of the deep ocean on upper ocean baroclinic instability.

To make further assessments, in this paper we turn our attention to baroclinic instability saturation. Employing Shepherd’s (1988) method, we establish and confront rigorous bounds on nonlinear instability saturation in 2.5- and 3-layer models. This method, which builds on the existence of a nonlinear stability theorem, has been previously used to compute saturation bounds in 2.5- (Olascoaga and Ripa, 1999) and 3-layer (Paret and Vanneste, 1996) models. In addition to considering more general model configurations than in these earlier works, we focus on the size of state space available for the growth of eddies in the region of parameter space where the models present discrepancies in their stability properties. Also, unlike Paret and Vanneste (1996), who computed numerical energy-norm bounds based on both Arnold’s first and second theorems, we derive analytical expressions for enstrophy-norm bounds based on Arnold’s first theorem. Maximum normal-mode perturbation growth rates in the 3-layer model are also calculated and contrasted with those in the 2.5-layer model.
The reminder of the paper has the following organization. Section 2 presents the 3-layer model, from which the 2.5-layer model follows as a limiting case. Normal-mode perturbation growth rates are computed in §3, along with an exposition of the main results of formal and nonlinear stability analyses. Nonlinear saturation bounds are then derived in §4. We want to remark that the number of basic state parameters that define a 3-layer flow is too large to be explored in full detail. To facilitate the analysis we reduce in some cases this number by fixing certain parameters to values that can be taken as “realistic,” because of being found appropriate for a region of the ocean mainly dominated by the ANEC, which is a good example of a major zonal current. Section 5 presents a discussion and the conclusions. Appendices A and B are reserved for mathematical details relating to the computation of the saturation bounds in the 3- and 2.5-layer models, respectively.

2. The Layer Models

Let \( \mathbf{x} \) denote the horizontal position with Cartesian coordinates \( x \) (eastward) and \( y \) (northward), let \( t \) be the time, and let \( D \) be an infinite (or periodic) zonal channel domain on the \( \beta \) plane with coasts at \( y = \pm \frac{1}{2}W \). The unforced, inviscid evolution equations for QG motions in a 3-layer model, with rigid surface and flat bottom, are given by (cf. e.g. Ripa, 1992)

\[
\partial_t q_i = \hat{\mathbf{z}} \cdot \nabla q_i \times \nabla \psi_i, \quad \dot{\gamma}^\pm_i = 0,
\]

where \( \psi_i, \) being a nonlocal function of \( \mathbf{q} := (q_i)^T \) and \( \mathbf{\gamma} := (\gamma^\pm_i)^T, \) is uniquely determined by

\[
\nabla^2 \psi_i - \sum_j R_{ij} \psi_j = q_i - f
\]

on \( D, \) where

\[
R := \frac{1}{(1 + r_1) R^2} \begin{bmatrix}
1 & -1 & 0 \\
-r_1 & (1 + s)r_1 & -sr_1 \\
0 & -sr_1 & sr_1 r_2 + r_1
\end{bmatrix},
\]

and

\[
\int \mathrm{d}x \partial_y \psi_i = \gamma_i^\pm, \quad \partial_x \psi_i = 0
\]

at \( y = \pm \frac{1}{2}W. \) Here, \( q_i(\mathbf{x}, t), \psi_i(\mathbf{x}, t), \) and \( \gamma_i^\pm = \text{const.} \) denote the QG potential vorticity, streamfunction, and Kelvin circulation along the boundaries of the channel, respectively, in the top \( (i = 1), \) middle \( (i = 2) \) and bottom \( (i = 3) \) layers. The Coriolis parameter is represented as \( f = f_0 + \beta y, \) the
Nabla operator $\nabla = (\partial_x, \partial_y)$, and $\hat{z}$ denotes the vertical unit vector. The quantities

$$R^2 := \frac{g_1 \bar{H}_1 \bar{H}_2}{f_0^2 \bar{H}}, \quad s := \frac{g_1}{g_2}, \quad r_1 := \frac{\bar{H}_1}{\bar{H}_2}, \quad r_2 := \frac{\bar{H}}{\bar{H}_3},$$

(2.2)

where $g_i$ is the buoyancy jump at the interface of the $i$-th and $(i + 1)$-th layers, and $\bar{H} := \bar{H}_1 + \bar{H}_2$ with $\bar{H}_i$ the $i$-th layer reference thickness.

The 2.5-layer model follows from (2.1) in the limit of infinitely thick ($r_2 \to 0$) and quiescent ($\psi_3 \to 0$) lower layer. In the latter case, $(1 + r_1)(r_1/s)^{1/2} R$ and $R$ are equal to the first (equivalent barotropic) and second (baroclinic) deformation radius, respectively, in the limit of weak internal stratification ($s \to 0$).

The evolution of system (2.1) is constrained by the conservation of energy, zonal momentum, and an infinite number of vorticity-related Casimirs, which are given by

$$\mathcal{E} := -\frac{1}{2} \langle \psi_i q_i \rangle, \quad \mathcal{M} := \langle y q_i \rangle, \quad \mathcal{C} := \langle C_i(q_i) \rangle$$

(2.3)

(modulo Kelvin circulations along the boundaries), where $C_i(\cdot)$ is an arbitrary function and $\langle \cdot \rangle := \sum_i \bar{H}_i \int_D d^2x (\cdot)$.

3. Spectral, Formal, and Nonlinear Stability

In this paper we deal with the stability of a basic state, i.e. equilibrium or steady solution of (2.1), of the form

$$\Psi_i = -U_i y,$$

(3.1)

which represents a baroclinic zonal flow. Here, $U_i = \sum_{i=1}^2 g_i \sum_{i=1}^2 H_{i,y}/f_0 = \text{const.}$, where $H_i(y)$ is the thickness of the $i$-th layer in the basic state, whereas $U_3$ is an arbitrary constant (set here to zero with no loss of generality). The following six parameters are enough to characterize the solutions of the 3-layer model stability problem:

$$\kappa := \sqrt{k^2 + l^2 R}, \quad s := \frac{\beta \bar{H}_1 \bar{H}_2}{f_0 H \bar{H}_{1,y}}, \quad b_T := \frac{H_{y,y}}{H_{1,y}}, \quad \frac{sU_2}{U_s}, \quad r_1, \ r_2.$$ 

(3.2)

The first parameter, $\kappa$, is a nondimensional wavenumber of the perturbation, where $k$ and $l$ are the zonal and meridional wavenumbers, respectively. The second parameter, $s$, is a nondimensional measure of the stratification. The third parameter, $b$, is a planetary Charney number, namely the ratio of the planetary $\beta$ effect and the topographic $\beta$ effect due to the geostrophic slope of the upper interface. Here, $U_s := U_1 - U_2$ is the velocity
jump at the interface (i.e. the current vertical “shear”). The fourth parameter, \( b_T \), is a topographic Charney number given by the ratio of the topographic \( \beta \) effects due to the geostrophic slopes of the lower and upper interfaces. Finally, the fifth (resp., sixth), \( r_1 \) (resp., \( r_2 \)), parameter is the aspect ratio of the upper to intermediate (resp., upper-plus-intermediate to lower) reference layer thicknesses. The problem considered by Olascoaga (2001) had \( r_1 = 1 \). In turn, the 2.5-layer problem treated by Olascoaga and Ripa (1999), which can be recovered upon making \( r_2 \to 0 \) and \( \psi_3 \to 0 \), also had \( r_1 = 1 \).

Choosing a Casimir such that \( \delta (E + C - \alpha M) = 0 \) for any constant \( \alpha \), the pseudo energy–momentum,

\[
H_\alpha[\delta q] := (\Delta - \delta)(E + C - \alpha M)\Psi + \frac{1}{2}(C_1QQ\delta q^2),
\]

(3.3)

where \( \Psi := (\psi_i)^T \), is an exact finite-amplitude invariant, quadratic in the perturbation \( \delta q(x,t) \) on the basic state potential vorticity \( Q_i(y) \). Here, the symbols \( \Delta \) and \( \delta \) stand for total and first variations of a functional, respectively, and \( C_i(Q_i) = \int dQ_i(\alpha - U_i)Y(Q_i) \) [\( Y \) is the meridional coordinate of an isoline of \( Q_i \)], where

\[
\begin{align*}
Q_1 &= f_0 + (b + \rho)yU_s/R^2, \\
Q_2 &= f_0 + [b + \rho r_1(b_T - 1)]yU_s/R^2, \\
Q_3 &= f_0 + (b - \rho^2 r_1 r_2 b_T)yU_s/R^2,
\end{align*}
\]

(3.4) \quad (3.5) \quad (3.6)

with \( \rho := (1 + r_1)^{-1} \). Arnold’s (1965; 1966) method for proving formal stability of \( Q_i \) relies upon the sign-definiteness of \( H_\alpha \). For evaluating the latter, it is useful to make the Fourier expansion \( \delta q = \sum_{k,l} \hat{q}(t)e^{ikx} \sin ly \), which implies \( H_\alpha = \frac{1}{2} \sum_{k,l} (\hat{q}^*)^T H_\alpha \hat{q} \) for certain matrix \( H_\alpha(k,s,b,b_T,r_1,r_2) \) (cf. Beron-Vera and Olascoaga, 2003, §2.2.1), so that the sign of \( H_\alpha \) is determined from the inspection of the elements of \( H_\alpha \) (cf. Mu et al., 1994; Paret and Vanneste, 1996; Ripa, 2000a).

In figure [I] the regions of the \((b,b_T)\)-space labeled “Stable” correspond to basic states for which there exists \( \alpha \) such that \( H_\alpha \) is positive definite (Arnold’s first theorem). The regions labeled “Possibly Stable” are locus of basic states for which there exists \( \alpha \) such that \( H_\alpha \) is negative definite (Arnold’s second theorem) if the channel in which the flow is contained is sufficiently narrow; cf. Mu (1998) and Mu and Wu (2001) for details on optimality issues relating to Arnold’s second theorem. The results, which are independent of the choice of \( s \), are presented for \( r_1 = 0.5 \), a value estimated for the ANEC. The r.h.s. panel in this figure corresponds to the 3-layer model in the limit \( r_2 \to 0 \); the l.h.s. panel corresponds to the 2.5-layer model. Clearly, as \( r_2 \to 0 \) the 3-layer model stable region does not reduce to that of the 2.5-layer model; it also requires \( \delta \psi_3 \to 0 \) (Olascoaga, 2001).
the regions labeled “ERI” no sign-definite $H_\alpha$ can be found. Consequently, the corresponding states are always unstable either through normal-mode perturbations or explosive resonant interaction (ERI) (Vanneste, 1995). By contrast, in the 2.5-layer instability problem all basic states subject to ERI are spectrally unstable. Finally, **nonlinear stability** can be proven for all formally stable states. Namely, the departure from these basic states can be bounded at all times by a multiple of the initial distance.

For **spectral stability** a perturbation is assumed to be infinitesimal and with the structure of a normal mode, i.e. $\hat{q} = \varepsilon \tilde{q} e^{-ikct} + O(\varepsilon^2)$, where $\varepsilon \to 0$. Nontrivial solutions for $\tilde{q}$, which satisfies $H_c \tilde{q} = 0$, require condition $\det H_c = 0$ to be fulfilled. This implies the eigenvalue $c(\kappa; s, b, b_T, r_1, r_2)$ to satisfy $P(c) = 0$, where $P(\cdot)$ is a cubic characteristic polynomial (cf. Beron-Vera and Olascoaga, 2003, appendix B).

Figure 2 shows the 3-layer model maximum perturbation growth rate, $\max_{\kappa}\{\kappa \, \text{Im} \, c\}$, for $r_1 = 0.5$, and different values of parameters $r_2$ and $s$ in the planetary $b$ vs. topographic $b_T$ Charney numbers space. In general, the maximum perturbation growth rate increases with increasing $r_2$ and decreasing $s$. As $b_T$ increases, the maximum perturbation growth rate tends to achieve the largest values in the region where the 2.5-layer model is nonlinearly stable as a consequence of Arnold’s first theorem, even for (realistically) small values of $r_2$ as depicted in the bottom panels of the figure.

Figure 3 shows instability regions in $(\kappa, b_T)$-space for $b = -0.35$, a value estimated for the ANEC, and the same values of parameters $r_1, r_2,$ and $s$
as in figure 2. The area of the region of possible wavenumbers for destabilizing perturbations in the 3-layer model increases with increasing $r_2$ and decreases with decreasing $s$. Thus the likelihood of these instabilities, which are not present in the 2.5-layer model, appear to be quite limited because they are confined only to small bands of wavenumbers for small $s$ and $r_2$. Yet the perturbation growth rates in these bands are not negligible even for very small values of $r_2$ according to Olascoaga (2001), who explained these instabilities as a result of the resonant interaction between a neutral mode of the 2.5-layer model instability problem and a short Rossby wave in the bottom layer of the 3-layer model. In the next section we will see, however, how the existence of nonlinearly stable states contributes to arrest the eddy-amplitude growth, restricting the significance of these instabilities, at least for certain basic state parameters.

Let us finally turn our attention to the region of $(b, b_T)$-space where the two models allow for the possibility of instability. The 2.5-layer model acquires its maximum perturbation growth rates for $b = \frac{1}{2} \rho [(1 - b_T) r_1 - 1] = -\frac{1}{6} (1 + b_T)$ and $b_T < 1 + r_1^{-1} = 3$ as $s \to 0$, which corresponds to an exact cancellation of the planetary and topographic $\beta$ effects, i.e. $Q_1 y + Q_2 y = 0$ (Olascoaga and Ripa, 1999). This result does not hold for the 3-layer model because of the presence of instabilities not present in
the 2.5-layer problem. The latter instabilities are confined to very narrow branches in the \((\kappa, b_T)\)-space and were also explained by Olascoaga (2001) as a result of the resonant interplay of a neutral mode in the 2.5-layer model instability problem and a short Rossby wave in the bottom layer of the 3-layer model. The maximum perturbation growth rates associated with these instabilities are larger than those of the 2.5-layer model (Olascoaga, 2001). In the following section we will see, however, that the fraction of the total enstrophy of the system available for eddy-amplitude growth can be much smaller in the 3-layer model than in the 2.5-layer model for certain unstable basic states.

4. Upper Bounds on Instability Saturation

When a basic state is unstable, a priori upper bounds on the finite-amplitude growth of the perturbation to this state can be obtained using Shepherd’s (1988) method. This method relies upon the existence of a nonlinear stability theorem, and the bounds are given in terms of the “distance” between the unstable basic state, \(Q_i^U\), say, and the nonlinearly stable state, \(Q_i^S\), say, in the infinite-dimensional phase space.

Let \(\delta q_i(x,t)\) be that part of the perturbation representing the “waves” or “eddies,” which result upon subtracting from the perturbation its zonal (i.e.
along-channel) average. Let $S$ denote the space of all possible nonlinearly stable basic states, let $\|a\|^2 := \langle a_i^2 \rangle / Z$, where $Z := \langle (Q_i^U)^2 \rangle$, and assume $q_i \approx Q_i^U$ at $t = 0$ so that $Z$ corresponds to the total enstrophy of the system. According to Shepherd (1988), a rigorous enstrophy-norm upper bound on eddy-amplitude growth, based on Arnold’s first theorem, must have the form

$$\|\delta q'\|^2 \leq \frac{1}{Z} \min_{Q_i^S \in S} \left\{ \frac{\max_{Q_{i,y}^S} Q_{i,y}^S \langle (Q_i^U - Q_i^S)^2 \rangle}{Q_{i,y}^S} \right\}. \quad (4.1)$$

We want to mention that bounds—not treated here—on the zonal-mean perturbation, $\delta \bar{q}_i(y, t) := \delta q_i - \delta q_i^t$, or the total perturbation can also be derived as $\delta \bar{q}_i$, $\delta \bar{q}_i$, and $\delta q_i$ satisfy the Pythagorean relationship $\|\delta q\|^2 = \|\delta \bar{q}\|^2 + \|\delta q'\|^2$ (cf. Ripa, 1999c; Ripa, 2000b).

Figure 4 shows the tightest bound on instability saturation corresponding to the 3-layer model (thick curves, cf. appendix A) as a function of the
planetary Charney number $b$, for aspect ratios $r_1 = 0.5$ and $r_2 = 0.1$, and various values of the topographic Charney number $b_T$. The focus on the small value $r_2 = 0.1$ will allow us to make comparisons with the stability properties of the 2.5-layer model. The latter model’s bound (cf. appendix B) is also plotted in the figure (dashed lines) assuming $r_1 = 0.5$ and the same values of $b_T$. It is important to remark that the 2.5-layer bound does not follow from that of the 3-layer model in the limit $r_2 \to 0$; it also requires $\psi_3 = 0$ (Olascoaga, 2001). Both the 2.5- and 3-layer model bounds are independent of the stratification parameter $s$. The 2.5-layer model bound curves are only present in the upper three panels of the figure because the 2.5-layer model predicts nonlinear stability as a consequence of Arnold’s first theorem for $b_T > 1 + r_1^{-1} = 3$ and any value of $b$ (cf. also figure 2, lower-right panel).

Vertical dashed lines in each panel of figure 4 indicate the values of $b$ for marginal stability, denoted by $b_{\pm}$. In the 3-layer model, $b_– = \min\{–\rho, \rho r_1(1 – b_T)\}$ and $b_+ = \max\{\rho r_1(1 – b_T), \rho^2 r_1 r_2 b_T\}$, whereas for the 2.5-layer model, $b_– = –\rho$ and $b_+ = \rho r_1(1 – b_T)$. Note that in the 2.5-layer model, while the $b_–$ marginal stability value remains fixed at $b \approx –0.66667$, the $b_+$ moves toward smaller values as $b_T$ increases, until it collapses with $b_–$ at $b_T = 3$ (not shown in the figure). For $b > b_+)$ and $b < b_–$ the basic flow in both the 2.5- and 3-layer models is nonlinearly stable. For $b_+ < b < b_–$ the basic flow is unstable unless the zonal channel flow is narrow enough for Arnold’s second stability theorem to be fulfilled. The latter is not always true in the 3-layer model case, however, since there is a possibility that a spectrally stable basic state could become unstable through ERI.

Three-layer surface-confined flows are susceptible to suffer more destabilization than 2.5-layer flows. However, the state space available, (determined by the fraction of total potential enstrophy), for eddy-amplitude growth in the 3-layer model tends to be smaller than the space available in the 2.5-layer model, at least for certain basic state parameters. This is evident in the upper three panels of figure 4. There is an overall tendency of the 3-layer model bound to decrease as $b_T$ increases. Moreover, for a large set of parameters this bound reduces to only a small fraction of the total enstrophy of the system. In these cases, the significance of the associated instabilities is relative. On the other hand, there are basic state parameters for which this fraction is not negligible. As an example not shown in the figure, for $b = −0.35$ and $b_T = 2.5$, which are appropriate for a region similar to the ANEC, the fraction of total enstrophy is about 45%, which is not negligible. Of course, when the upper bounds are not small enough, no unambiguous conclusion can be drawn about the significance of an instability.

Figure 4 also shows that the 3-layer model bound can be significantly
small for certain potentially ERI unstable flows (cf. lower two panels in
the figure). This also allows us to make an unambiguous assessment of
the significance of these type of instabilities in the sense that they can be
certainly negligible for some basic state parameters.

Before closing this section, two points deserve additional discussion.
First, the result that the bounds for the 3-layer model with a very thick
depth layer are smaller than the 2.5-layer model bounds in the region of
parameters where the two models share similar instability properties might
seem at odds with the fact that the 3-layer model is less constrained than
the 2.5-layer model, which allows for the development of more unstable
states. However, we believe that this result should not be surprising inas-
much as the space over which the minimization is carried out is larger in
the 3-layer model than in the 2.5-layer model, which offers the possibility of
finding tighter bounds (cf. Olascoaga, 2001). Second, Paret and Vanneste
(1996) were not able to draw a conclusion on the significance of ERI in-
stability as in the present paper. These authors computed energy-norm
saturation bounds, according to both Arnold’s first and second theorems,
using numerical minimization algorithms. These bounds, whose analytical
computation appears to be too difficult, were not found to minimize at
basic state parameters for which ERI instability is possible. The analytical
minimization involved in the derivation of the enstrophy-norm of this paper
has shown that the tightest bounds are obtained using stable basic states
whose parameters have quite spread numerical values. The minimization
thus requires to search for a solution in a considerably large space, making
numerical computations extremely expensive. This might explain the diffi-
culty of Paret and Vanneste (1996) to find tighter bounds for potentially
ERI unstable basic flows.

5. Concluding Remarks

A previous study showed that the quasigeostrophic baroclinic instability
properties of a surface-confined zonal current may differ substantially be-
tween a 2.5-layer model and a 3-layer, if the former is considered to be a
simplified 3-layer model with a very thick deep layer. For certain basic state
parameters, the 2.5-layer model predicts nonlinear stability whereas the 3-
layer model spectral instability. That study thus suggested that the effects
of the deep ocean on the baroclinic instability of the upper thermocline
layer of the ocean may be important for certain currents.

In this paper we have made further assessments of the importance of
the deep ocean on upper baroclinic instability. We have achieved this by
analyzing (i) maximum normal-mode perturbation growth rates and (ii)
rigorous enstrophy-norm upper bounds on the growth of perturbations to
unstable basic states, in both 2.5- and 3-layer models of baroclinic instability. The new results show that instabilities, which the 3-layer model predicts in the region of basic state parameters where the 2.5-layer model predicts nonlinear stability, appear to maximize their growth rates. At the same time, however, the saturation bounds tend to minimize in this same region of basic state parameters, thereby reducing the size of state space available for eddy-amplitude growth. Moreover, for a large subset of parameters in the region, the latter reduces to only a small fraction of the total enstrophy of the system. In these cases we have been able to make unambiguous assessments of the significance of the associated instabilities in the sense that they can be certainly negligible.

We close remarking that the important issue of making assessments of the accuracy of the saturation bounds as predictors of equilibrated eddy amplitudes is still largely open. This cannot be addressed without performing direct numerical simulations. The importance of this subject relies upon the potential of the bounds in the architecture of transient-eddy parametrization schemes. The treatment of these issues are reserved for future investigation.

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A. Three-Layer Model Bounds

Upon minimizing the r.h.s. of (4.1) over all stable states, we have been able to find, in addition to the trivial bound $\max \| \delta \mathbf{q}' \|^2 = 1$, various sets of possible bounds. A first set involves 9 possibilities, for which $Q^S_{i,y} = \max \{ Q^S_{i,y} \}$ and is given by

$$\max \| \delta \mathbf{q}' \|^2 = \begin{cases} -Q^U_{i,y} (Q^U_{i,y} + Q^U_{I,y}) \\ -\sum_i Q^U_{i,y} \sum_j Q^U_{j,y} \end{cases}$$

\[ \div \frac{1}{4} \sum_j (Q^U_{j,y})^2, \text{ for } i \neq I = 1, 2, 3. \]  

A second set involves other 9 possibilities, for which $\max \{ Q^S_{i,y} \} = Q^S_{I_1,y} = Q^S_{I_2,y}$ and is given by

$$\max \| \delta \mathbf{q}' \|^2 = \begin{cases} (Q^U_{I_1,y})^2 + (Q^U_{I_2,y})^2 \\ \frac{1}{2} (Q^U_{I_1,y} - Q^U_{I_2,y})^2 \\ \frac{1}{2} (Q^U_{I_1,y} - Q^U_{I_2,y})^2 - 2Q^U_{i,y} \sum_j Q^U_{j,y} \end{cases}$$
\[ \frac{1}{4} \sum_j (Q_{j,y}^U)^2 \quad \text{for} \quad i \neq I_1, I_2, \quad \text{where} \quad \{I_1, I_2\} = \{1, 2\}, \{2, 3\}, \{1, 3\}. \]

Another possibility finally results for \( Q_{S_1,y} = Q_{S_2,y} = Q_{S_3,y} \), and is given

\[
\max \|\delta q'\|^2 = 2 \left[ 1 - \frac{Q_{1,y}^U Q_{2,y}^U + Q_{4,y}^U Q_{3,y}^U + Q_{2,y}^U Q_{3,y}^U}{\sum_j (Q_{j,y}^U)^2} \right]. \tag{A.3}
\]

The tightest bound follows as the least continuous bound of the above 20 possible bounds in the 4-dimensional space of unstable basic state parameters, with coordinates \((b, b_T, r_1, r_2)\) (the bounds are independent of \(s\)).

### B. Two-and-a-Half-Layer Model Bounds

In the 2.5-layer model \((r_2 \to 0 \text{ and } \psi_3 \to 0)\) the least bound in the 3-dimensional space of unstable basic state parameters, with coordinates \((b, b_T, r_1)\), is given by

\[
\max \|\delta q'\|^2 = \begin{cases} 
-4Q_{1,y}^U (Q_{1,y}^U + Q_{2,y}^U) & \text{if} \quad -\rho < b < b_1 \\
\frac{1}{2} (Q_{1,y}^U - Q_{1,y}^U)^2 & \text{if} \quad b_1 \leq b \leq b_2 \\
-4Q_{2,y}^U (Q_{1,y}^U + Q_{2,y}^U) & \text{if} \quad b_2 < b < -r_1 \rho (b_T - 1) 
\end{cases} \tag{B.1}
\]

\[
\div \left[ (Q_{1,y}^U)^2 + (Q_{2,y}^U)^2 \right], \quad \text{where} \quad b_1 := -\frac{1}{4} \rho [r_1(b_T - 1) + 3] \quad \text{and} \quad b_2 := -\frac{1}{4} \rho \times [3r_1(b_T - 1) + 1]. \]

This result extends to arbitrary \(r_1\) that of Olascoaga and Ripa (1999).

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