Linear simultaneous measurements of position and momentum with minimum error-trade-off in each minimum uncertainty state

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So-called quantum limits and their achievement are important themes in physics. Heisenberg’s uncertainty relations are the most famous of them but are not universally valid and violated in general. In recent years, the reformulation of uncertainty relations is actively studied, and several universally valid uncertainty relations are derived. On the other hand, several measuring models, in particular, spin-1/2 measurements, are constructed and quantitatively examined. However, there are not so many studies on simultaneous measurements of position and momentum despite their importance. Here we show that an error-trade-off relation (ETR), called the Branciard-Ozawa ETR, for simultaneous measurements of position and momentum gives the achievable bound in minimum uncertainty states. We construct linear simultaneous measurements of position and momentum that achieve the bound of the Branciard-Ozawa ETR in each minimum uncertainty state. To check their performance, we then calculate probability distributions and families of posterior states, sets of states after the measurements, when using them. The results of the paper show the possibility of developing the theory of simultaneous measurements of incompatible observables. In the future, it will be widely applied to quantum information processing.

I. INTRODUCTION

In quantum physics, uncertainty relations and construction of measurement models are important themes since Heisenberg [1] and von Neumann [2]. In the last forty years, quantum measurement theory has developed. There has been a great deal of study of quantum measurement focused on applications to quantum information technology nowadays. Above all, the theory of uncertainty relations [3–19], the central topic of the paper, has advanced dramatically in the last two decades. Experimental tests of uncertainty relations [20–29] also have been performed due to the rapid improvement of experimental techniques in recent years. In the paper, we present linear simultaneous measurements of position and momentum with minimum error-trade-off in each minimum uncertainty state. The construction of measurements of observables with minimum uncertainty in some class of states is significant but there are few examples. In fact, such measurements are given for spin [22,23] and position [30]. Therefore, we believe that the results of the paper are an important contribution.

Here we consider a one-dimensional nonrelativistic single-particle system \( S \) whose position \( Q_1 \) and momentum \( P_1 \) are defined as self-adjoint operators on \( \mathcal{H}_S = L^2(\mathbb{R}) \) and satisfy the canonical commutation relation \([Q_1, P_1] = i\hbar \). A unit vector \( \psi \) in \( \mathcal{H}_S \) is called a minimum uncertainty state if it satisfies \( \sigma(Q_1 \| \psi) \sigma(P_1 \| \psi) = \hbar / 2 \). Throughout the paper, we suppose that the state \( \psi \) of \( S \) is a minimum uncertainty state with \( \langle Q_1 \| \psi \rangle = q_1 \), \( \langle P_1 \| \psi \rangle = p_1 \) and \( \sigma(Q_1 \| \psi) = \sigma_1 \), i.e.,

\[
\psi(x) = \sqrt{\frac{1}{2\pi\sigma_1^2}} e^{-\frac{(x-q_1)^2}{2\sigma_1^2}} e^{i\frac{p_1}{\hbar}x} \tag{1}
\]

in the coordinate representation. Minimum uncertainty states appear in Heisenberg’s original paper [1] and are also called Gaussian wave packets.

In order to define linear simultaneous measurements of \( Q_1 \) and \( P_1 \), we prepare a probe system \( P \) whose positions \( Q_2, Q_3 \) and momenta \( P_2, P_3 \) are described by self-adjoint operators on \( \mathcal{H}_P = L^2(\mathbb{R}^2) \) and satisfy \([Q_j, P_k] = 0 \) for \( j, k = 2, 3 \), and whose states are described by density operators on \( \mathcal{H}_P \). \( P \) is supposed to be a one-dimensional nonrelativistic two-particle system or a two-dimensional nonrelativistic single-particle system. \( Q_2 \) and \( P_3 \) are used as the meters to measure \( Q_1 \) and \( P_1 \), respectively. In considering linear simultaneous measurements of position and momentum from now on, we ignore the intrinsic dynamics of \( S \) and \( P \). Here we adopt the following interaction Hamiltonian, the measurement interaction between \( S \) and \( P \):

\[
H_{int} = K[\alpha_1 Q_1 P_2 + \beta_1 P_1 Q_2 + \gamma_1 (Q_1 P_1 - Q_2 P_2) + \alpha_2 Q_2 P_3 + \beta_2 P_2 Q_3 + \gamma_2 (Q_2 P_2 - Q_3 P_3) + \alpha_3 Q_3 P_1 + \beta_3 P_3 Q_1 + \gamma_3 (Q_3 P_3 - Q_1 P_1)], \tag{2}
\]

where \( K \) is a positive real number, the coupling constant, and \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \) and \( \gamma_3 \) are real numbers. This interaction is a natural extension of linear measurements given by Ozawa [31] to simultaneous measurements. His model is exactly solvable and contains both the error-free linear position measurement [32] and von Neumann’s model [2]. In particular, the former contributed to the resolution of the dispute on the sensitivity limit to the gravitational wave detector (see also [33–37]).
We treat an error-trade-off relation (ETR) based on
the noise-operator based q-rms error ε(A) for each obser-
vable A. This error is considered standard and is de-
defined later. For every simultaneous measurement of Q1
and P1, the errors ε(Q1) of Q1 and ε(P1) of P1 in ψ then
satisfy
\[
ε(Q1)^2σ(P1)^2 + σ(Q1)^2ε(P1)^2 \geq \hbar^2/4,
\]
which is a special case of the Branciard-Ozawa ETR. We
say that a simultaneous measurement of Q1 and P1 has
the minimum error-trade-off in ψ if it achieves the lower
bound of Eq. (3) in ψ, that is to say, it satisfies
\[
ε(Q1)^2σ(P1)^2 + σ(Q1)^2ε(P1)^2 = \hbar^2/4
\]
in ψ. As suggested by the existence of the error-free
linear position measurements, Heisenberg’s ETR, one of
his uncertainty relations,
\[
ε(Q1)ε(P1) ≥ \hbar/2
\]
is violated in general. Its violation always occurs when
we use linear simultaneous measurements of Q1 and P1
with the minimum error-trade-off in each minimum un-
certainty state. A famous example of simultaneous mea-
surement of position and momentum is the Arthurs-Kelly
model (see Methods). Since their model is moti-
vated by von Neumann’s model and satisfies Heisenberg’s
ETR, it has been considered plausible. On the other
hand, our discussion is based on the general description
of measuring processes in modern quantum measurement
theory. The general theory of quantum measurement
tells us that a broader class of simultaneous measure-
ment models besides the Arthurs-Kelly model is physi-
cally valid. We expect that our models introduced in the
paper become the new, good example.

In Sec. II measuring process and the noise-operator
based q-rms error are defined. Linear simultaneous mea-
surement of position and momentum is then defined. In
Sec. III we first present a theorem that gives a necessary
and sufficient condition for a linear simultaneous mea-
surement of position and momentum to satisfy Eq. (4)
in ψ. Next, we give four families of linear simultaneous
measurements of position and momentum which satisfy
Eq. (4) in ψ. We then investigate probability distribu-
tions and states after the measurement when using such
families of linear simultaneous measurements of position
and momentum. In Sec. IV the results of the paper are
examined. In Sec. V we prove the theorem and show a
systematic construction of linear simultaneous measure-
ments of position and momentum which satisfy Eq. (4)
in ψ.

Conventions. Let H be a Hilbert space. For every
self-adjoint operator X on H, E^X denotes its spectral
measure. Let n be a natural number, and X_1, · · · , X_n
mutually commuting self-adjoint operators on H, and φ
a unit vector in H. The expectation value and standard
deviation of an observable X in a vector state φ are de-
noted by
\[
\langle X \rangle = \langle X \rangle_φ = \langle φ | X | φ \rangle = \langle φ | X | φ \rangle,
\]
\[
\sigma(X) = \sigma(X|φ) = \langle φ | (X - \langle X \rangle)^2 | φ \rangle^{1/2},
\]
respectively. Then the (joint) probability measure
\[
μ_{X_1, · · · , X_n}^X (I_1 × · · · × I_n) = \langle φ | E^{X_1} (I_1) · · · E^{X_n} (I_n) | φ \rangle
\]
for all intervals (more generally, all Borel sets) I_1, · · · , I_n
of \( \mathbb{R} \). \( p_{X_1, · · · , X_n} (x_1, · · · , x_n) \) denotes the probability den-
sity function of \( μ_{X_1, · · · , X_n}^X \) with respect to the Lebesgue
measure on \( \mathbb{R}^n \) if it exists. For every linear operator X
and Y on H and K, linear operators X ⊗ Y, X ⊗ 1 and
1 ⊗ Y on H ⊗ K are abbreviated as XY, X, and Y,
respectively.

II. PRELIMINARIES

A. Measuring process

First, we shall define a measuring process for S, which
is a quantum mechanical modeling of the probe part \( \mathbf{P}_0 \)
of a measuring apparatus \( \mathbf{A}_0 \). Let n be a natural num-
ber. Here a \((n + 3)\)-tuple \( \mathbb{M}_0 = (K, \zeta, M_1, · · · , M_n, U) \)
called a n-meter measuring process for S (or for H_S)
if it satisfies the following conditions: (1) K is a Hilbert
space. (2) \( \zeta \) is a unit vector of K, the vector state of \( \mathbf{P}_0 \),
(3) \( M_1, · · · , M_n \) are mutually commuting self-adjoint
operators on K as meters, mutually compatible observables
of \( \mathbf{P}_0 \). (4) U is a unitary operator on H_S ⊗ K, the measur-
ing interaction which turns on at time 0 and turns off at
time \( τ \) between S and \( \mathbf{P}_0 \). We then adopt the following
notation for every linear operator Z on H_S ⊗ K:
\[
Z(0) = Z, \quad Z(τ) = U^\dagger Z U.
\]
A 2-meter measuring process \((K, \zeta, M_1, M_2, U)\) for S is
called a simultaneous measurement of position \( Q_1 \) and
momentum \( P_1 \) or a simultaneous \((Q_1, P_1)\)-measurement
if \( M_1 \) and \( M_2 \) are used to measure \( Q_1 \) and \( P_1 \), re-
spectively.

Let n be a natural number. Let \( X_1, · · · , X_n \) be
observables of S, φ a vector state of S, and \( \mathbb{M}_0 = (K, \zeta, M_1, · · · , M_n, U) \)
a n-meter measuring process for S. We consider that \( X_1, · · · , X_n \) are measured in terms of \( M_0 = (K, \zeta, M_1, · · · , M_n, U) \), and that \( X_1, · · · , X_n \) are compared with \( M_1, · · · , M_n \), respectively. The noise-
operator based q-rms error \( ε(X_j) = ε(X_j, \mathbb{M}_0, φ) \) of \( X_j \)
is then defined by
\[
ε(X_j) = ε(X_j, \mathbb{M}_0, φ) = \langle N_j^2 \rangle^1/2_φ φ\zeta
\]
for all \( j = 1, · · · , n \), where \( N_j \) is the noise operator de-
noted by
\[
N_j = N(X_j, \mathbb{M}_0) = M_j(τ) - X_j(0)
\]
for all \( j = 1, \ldots, n \). The error defined here is applicable to the case where \( X_j(0) \) and \( M_j(\tau) \) does not commute, and is considered standard.

For every simultaneous \((Q_1, P_1)\)-measurement \( M_0 = (\mathcal{K}, \zeta, M_1, M_2, U) \), Eq. (3) holds in \( \psi \) for

\[
\varepsilon(Q_1) = \varepsilon(Q_1, M_0, \psi) = \langle (M_1(\tau) - Q_1(0))^2 \rangle_{\psi, \zeta}^{1/2} \tag{12}
\]

\[
\varepsilon(P_1) = \varepsilon(P_1, M_0, \psi) = \langle (M_2(\tau) - P_1(0))^2 \rangle_{\psi, \zeta}^{1/2}. \tag{13}
\]

### B. Linear simultaneous measurement of position and momentum

A 2-meter measuring process \( M = (\mathcal{H}, \xi, Q_2, P_3, U(\tau)) \) for \( S \) is called a linear simultaneous measurement of position \( Q_1 \) and momentum \( P_1 \) or a linear simultaneous \((Q_1, P_1)\)-measurement if \( Q_2 \) and \( P_3 \) are used to measure \( Q_1 \) and \( P_1 \), respectively, where \( \xi \) is a unit vector of \( \mathcal{H} = L^2(\mathbb{R}^2) \) satisfying \( ||Q_2^n P_3^m P_3^n \xi|| < +\infty \) for all non-negative integers \( m_2, n_2, n_3, \tau(> 0) \) is the time the measurement finishes and \( U(t) \) is defined by \( U(t) = e^{-it\mathcal{H}_{\text{int}}/\hbar} \) for all \( t \in \mathbb{R} \). Since we ignore the intrinsic dynamics of \( S \) and \( P \), \( K \) contributes only to the time scale of the measurement time. For simplicity, we assume \( K = 1 \) in the paper. For every observable \( Z \) of \( S + P \) at time 0 and \( t \in \mathbb{R} \), the same observable \( Z(t) \) at time \( t \) is given by

\[
Z(t) = U(t)\dagger ZU(t) \tag{14}
\]

for all \( t \in \mathbb{R} \). This is consistent with the notation before, Eq. (9). By solving Heisenberg’s equations of motion, we have

\[
\begin{pmatrix}
Q_1(t) \\
Q_2(t) \\
Q_3(t)
\end{pmatrix} = e^{iR} \begin{pmatrix}
Q_1(0) \\
Q_2(0) \\
Q_3(0)
\end{pmatrix},
\]

\[
\begin{pmatrix}
P_1(t) \\
P_2(t) \\
P_3(t)
\end{pmatrix} = e^{-iRT} \begin{pmatrix}
P_1(0) \\
P_2(0) \\
P_3(0)
\end{pmatrix} \tag{16}
\]

for all \( t \in \mathbb{R} \), where

\[
R = \begin{pmatrix}
\gamma_1 - \gamma_3 & \beta_1 & \alpha_3 \\
\alpha_1 & \gamma_2 - \gamma_1 & \beta_2 \\
\beta_3 & \alpha_2 & \gamma_3 - \gamma_2
\end{pmatrix} \tag{17}
\]

and \( R^T \) denotes the transpose of \( R \). We see that \( e^{iR}, e^{-iRT} \in \text{SL}(3, \mathbb{R}) \) for all \( t \in \mathbb{R} \). \( e^{iR} \) and \( e^{-iRT} \) are denoted by \( A = (a_{ij}) \) and \( B = (b_{ij}) \), respectively. When we use a linear simultaneous \((Q_1, P_1)\)-measurement, the noise-operator based q-rms errors \( \varepsilon(Q_1) \) and \( \varepsilon(P_1) \) have the following representations:

\[
\varepsilon(Q_1)^2 = \varepsilon(Q_1, \mathcal{M}, \psi)^2 = (a_{21} - 1)^2 \sigma(Q_1 \| \psi)^2 + \sigma(a_{22} Q_2 + a_{23} Q_3 \| \xi)^2 + ((a_{21} - 1)(Q_1) \psi + a_{22} (Q_2) \xi + a_{23} (Q_3) \xi)^2, \tag{18}
\]

\[
\varepsilon(P_1)^2 = \varepsilon(P_1, \mathcal{M}, \psi)^2 = (b_{31} - 1)^2 \sigma(P_1 \| \psi)^2 + \sigma(b_{32} P_2 + b_{33} P_3 \| \xi)^2 + ((b_{31} - 1)(Q_1) \psi + b_{32} (P_2) \xi + b_{33} (P_3) \xi)^2. \tag{19}
\]

### III. RESULTS

#### A. Characterization theorem

The following theorem is the first result of the paper:

**Theorem.** A linear simultaneous \((Q_1, P_1)\)-measurement \( \mathcal{M} = (\mathcal{H}, \xi, Q_2, P_3, U(\tau)) \) satisfies Eq. (4) in \( \psi \) if and only if it satisfies the following three conditions:

(i) \( a_{21} - 1)(Q_1) \psi + a_{22} (Q_2) \xi + a_{23} (Q_3) \xi = 0 \) and \( b_{31} - 1)(P_1) \psi + b_{32} (P_2) \xi + b_{33} (P_3) \xi = 0 \).

(ii) \( \sigma(a_{22} Q_2 + a_{23} Q_3 \| \xi) = |a_{21} b_{31} |^2 \sigma(Q_1 \| \psi) \) and \( \sigma(b_{32} P_2 + b_{33} P_3 \| \xi) = |a_{21} b_{31} |^2 \sigma(P_1 \| \psi) \).

(iii) \( a_{21} > 0, b_{31} > 0 \) and \( a_{21} + b_{31} = 1 \).

Furthermore, for every \( \nu \in (0, 1) \), there exists a linear simultaneous \((Q_1, P_1)\)-measurement such that

\[
\varepsilon(Q_1)^2 = (1 - \nu) \sigma(Q_1)^2 \quad \text{and} \quad \varepsilon(P_1)^2 = \nu \sigma(P_1)^2 \tag{20}
\]

in \( \psi \).

By the above theorem, any linear simultaneous \((Q_1, P_1)\)-measurement with the minimum error-trade-off in \( \psi \) satisfies

\[
\varepsilon(Q_1) < \sigma(Q_1), \quad \varepsilon(P_1) < \sigma(P_1), \tag{21}
\]

and

\[
\varepsilon(Q_1)\varepsilon(P_1) = \frac{\hbar}{2} \sqrt{\frac{1}{4} - \left( \nu - \frac{1}{2} \right)^2} \leq \frac{\hbar}{2} \leq \frac{\hbar}{2}. \tag{22}
\]

Thus, the range of possible values of the error pairs \((\varepsilon(Q_1), \varepsilon(P_1))\) in the state \( \psi \) is as shown in FIG. [3].
FIG. 1. When the state of S is ψ, possible values of the pair (ε(Q1), ε(P1)) of the errors are indicated by the area with a grid of dotted magenta lines and with magenta boundary except for two points (σ(Q1), 0) and (0, σ(P1)). By the theorem, ε(Q1)^2σ(P1)^2 + ε(P1)^2σ(Q1)^2 = ℏ^2/4 (ε(Q1), ε(P1) > 0), a part of its boundary, is achieved by linear simultaneous (Q1, P1)-measurements, and gives the unbreakable limitation for the pair (ε(Q1), ε(P1)). The cyan line is Heisenberg’s bound, ε(Q1)ε(P1) = ℏ/2. On the other hand, the dashed green line indicates ε(Q1)ε(P1) = ℏ/4.

B. Concrete models

The above theorem does not directly tell us how to construct simultaneous (Q1, P1)-measurements in the minimum error-trade-off in ψ. Notably, in contrast to exactly solvable linear measurements [31], e^{iR} has no more explicit formula. Therefore, we abandon analyzing e^{iR} as it is. We remind the reader that K = 1 is assumed.

We shall give a novel, exactly solvable subclass of linear simultaneous (Q1, P1)-measurements. The following two constraints for R are imposed:

(C1) α_2 = β_2 = γ_1 = γ_3 = 0.

(C2) α_1 β_1 = α_3 β_3.

Under these constraints, R is denoted by S, that is,

\[ S = \begin{pmatrix} 0 & \beta_1 & \alpha_3 \\ \alpha_1 & \gamma_2 & 0 \\ \beta_3 & 0 & -\gamma_2 \end{pmatrix}. \] (23)

Let ν ∈ (0, 1) and κ ∈ R \{0\}. We define a state ξ_{ν,k} of P, which satisfies the following conditions: (1) \( \sigma(Q_2)σ(P_2) = \sigma(Q_3)σ(P_3) = \hbar/2 \) and \( ⟨Q_2Q_3⟩ = ⟨Q_2⟩⟨Q_3⟩ \), (2) \( \sigma(Q_2) = \sqrt{\nu(1-\nu)/(2\nu)} σ_1 \) and \( \sigma(Q_3) = \sqrt{\nu(1-\nu)/(2\nu)} σ_1 \), (3) \( ⟨Q_2⟩ = \frac{1-\nu}{\nu} q_1, \) \( ⟨Q_3⟩ = 0, \) \( ⟨P_2⟩ = 0 \) and \( ⟨P_3⟩ = \frac{\nu}{\sqrt{\nu}} p_1, \) i.e.,

\[ ξ_{ν,k}(x_{2,3}) = \frac{1}{\sqrt{(2πσ^2)}} e^{-\frac{1}{2}(x_{2,3}-μ)^2} G^{-\frac{1}{2}(x_{2,3}-μ)^2}(μ;x_{2,3}) \] (24)

for all \( x_{2,3} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^2 \) in the coordinate representation, where \( u = \begin{pmatrix} \frac{1-\nu}{\nu} q_1 \\ 0 \end{pmatrix} \) and \( G = σ^2 \begin{pmatrix} \frac{1}{2\nu} & 0 \\ 0 & \frac{1}{2\nu} \end{pmatrix} \).

For every ν ∈ (0, 1), we present four linear simultaneous (Q1, P1)-measurements (\( H_P, \xi_{ν,k}, Q_2, P_3, U(τ) \)) satisfying Eq. (20) herein, denoted by \( Y_{ν}, Y_{ν}^2, Y_{ν}^0 \) and \( Z_{ν} \), respectively. Each model is specified by the triplet of \( τ, S \) and \( κ \) in the following table, Table I.

| τ | S | κ | E |
|---|---|---|---|
| \( \frac{π}{2} \) | \( \begin{pmatrix} 0 & -\frac{2}{\nu} & -\frac{2}{\nu} \\ \frac{2}{\nu} & 1 & 0 \\ 0 & -1 \end{pmatrix} \) | 2 | 1 |
| \( y_{ν} \) | \( \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & 0 \\ 0 & -2 \end{pmatrix} \) | 4 | 0 |
| \( y_{ν}^0 \) | \( \begin{pmatrix} 0 & -\nu & 0 \\ 0 & 0 & 0 \\ \nu & 0 & 0 \end{pmatrix} \) | 1 | 0 |
| \( Z_{ν} \) | \( \log 2 \) | 2 | -1 |

Here \( E \) is a real number defined by

\[ α_1 β_1 = α_3 β_3 = -\frac{3}{2} + \frac{E}{2}, \] (25)

and is used to explicitly solve \( e^{iS} \) for all \( t \in \mathbb{R} \) (see Sec. VA).

C. Probability distributions and families of posterior states

Our next interest is to give probability distributions and families of posterior states when using concrete models \( Y_{ν}, y_{ν}^2, y_{ν}^0 \) and \( Z_{ν} \). First, we show probability distributions related to \( ε(Q_1) \) and \( η(P_1) \), and check the validity of \( X_{ν}, Y_{ν}^2, Y_{ν}^0 \) and \( Z_{ν} \). For every ν ∈ (0, 1), whether we use \( X_{ν}, Y_{ν}^2, Y_{ν}^0 \) or \( Z_{ν} \), we get the following probability density functions:

\[ p_{\psi \otimes ξ_{ν,k}}(z,w) = p_{\nu σ^2}(z - q_1)p_{(1-ν) σ^2}(w - p_1), \] (26)

\[ p_{\psi \otimes ξ_{ν,k}}(x,z) = p_{1-ν σ^2}(x - z)p_{ν σ^2}(z - q_1), \] (27)

\[ p_{\psi \otimes ξ_{ν,k}}(y,w) = p_{ν σ^2}(y - w)p_{(1-ν) σ^2}(w - p_1), \] (28)

where \( \tilde{σ}_1 = \hbar/(2σ_1) \) and \( p_{σ^2}(x) \) denotes the probability density function of the Gaussian probability measure.
with mean 0 and variance $\sigma^2$ (equivalently, standard deviation $\sigma$), i.e.,

$$p_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma^2}e^{-\frac{1}{2\sigma^2}x^2}. \quad (29)$$

We see that all of Eqs. (26), (27) and (28) depend on $\psi$ and $0 < \nu < 1$. Of the three equations, only Eq. (26) can be directly confirmed by any of $X_\nu$, $Y_\nu$, $Z_\nu$ or $\mathbb{Z}_\nu$. The rest two equations, Eqs. (27) and (28), are essential for understanding the performance of $X_\nu$, $Y_\nu$, $Z_\nu$ or $\mathbb{Z}_\nu$. From Eq. (27), the probability density function of the conditional probability measure of $Q_1(0)$ in $\psi \otimes \xi_{\nu,k}$ under the condition that the value $z$ of $Q_2(\tau)$ is given is determined as

$$p_{Q_2(\tau)=z,\psi,\xi_{\nu,k}}(x) = p(1-\nu)\sigma^2(x - z). \quad (30)$$

Since $\varepsilon(Q_1)^2 = (1 - \nu)\sigma^2$, Eq. (30) means that, when the value $z$ of $Q_2(\tau)$ is output, $Q_1(0)$ obeys the Gaussian probability measure with mean $z$ and standard deviation $\varepsilon(Q_1)$. The same argument can be made for Eq. (28) and $\varepsilon(P_1)^2 = \nu\sigma^2$. The noise-operator based q-rms errors $\varepsilon(Q_1)$ and $\varepsilon(P_1)$ are then equal to Gauss' errors $\varepsilon_G(\mu) = \varepsilon_G(\mu_{\psi,\xi_{\nu,k}}) = \varepsilon_G(\mu_{\psi,\xi_{\nu,k}^{\ast}})$, respectively, i.e.,

$$\varepsilon(Q_1) = \varepsilon_G(\mu_{\psi,\xi_{\nu,k}}^{\ast}) = \varepsilon_G(\mu_{\psi,\xi_{\nu,k}}), \quad \varepsilon(P_1) = \varepsilon_G(\mu_{\psi,\xi_{\nu,k}}). \quad (31)$$

Here Gauss' error $\varepsilon_G(\mu)$ for a probability distribution $\mu$ on $\mathbb{R}^2$ is defined by

$$\varepsilon_G(\mu) = \left(\int_{\mathbb{R}^2} (x - y)^2 \, d\mu(x, y)\right)^{\frac{1}{2}}. \quad (32)$$

Following Laplace's pioneering work, Gauss defined his error in 1821. His error is now redefined as above and widely used in the setting of measure-theoretical probability theory.

Next, we consider a family of posterior states, which is the set of the states after the measurement for each output value of the meter (see [40] for the general theory). It is difficult to find families of posterior states for general linear simultaneous ($Q_1$, $P_1$)-measurements with the minimum error-trade-off in $\psi$. Here we shall give them for $\{Y_{\nu}\}_{\nu \in (0,1)}$ and $\{Z_{\nu}\}_{\nu \in (0,1)}$. For every $\nu \in (0,1)$, the family $\{\psi_{y}\}_{y \in \mathbb{R}^2}$ of posterior states for $Y_{\nu}$, $\psi$ is the set of the minimum uncertainty state $\psi_{y}$ with $\langle Q_1 \rangle_{\psi_y} = \frac{y_1(1-\nu)\nu}{\nu - \nu^2}$, $\langle P_1 \rangle_{\psi_y} = \frac{y_2 - \nu p_1}{1 - \nu}$ and $\sigma(Q_1 \| \psi_y) = \sqrt{\frac{1 - \nu}{\nu}}\sigma_1$ for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, i.e.,

$$\psi_{y}(x) = e^{-\frac{\nu}{4(1 - \nu)^2} \left( (x - \frac{y_1(1 - \nu)\nu}{\nu - \nu^2})^2 + i \frac{y_2 - \nu p_1}{1 - \nu} x \right)} \sqrt{\frac{2\pi(1 - \nu)\sigma_1}{\nu}}$$

(33)

for all $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ in the coordinate representation.

For every $\nu \in (0,1)$, the family $\{\psi_{y}\}_{y \in \mathbb{R}^2}$ of posterior states for $Z_{\nu}$ is the same as that for $\{Y_{\nu}\}_{\nu \in (0,1)}$.

IV. DISCUSSION

A. The Arthurs-Kelly model

Here we shall mention the differences between this paper and the paper of Arthurs and Kelly, an important previous study, on the treatment of simultaneous measurements of position and momentum. They use the 2-meter measuring process $M_{AK} = (\mathcal{H}_P, \xi, Q_2, Q_3, U_{AK}(K^{-1}))$ for $\mathcal{S}$, where $U_{AK}(t) = e^{-it\mathcal{H}_{AK}/\hbar}$ is a one-parameter group on $\mathcal{H}_S \otimes \mathcal{H}_P = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^2)$ with $H_{AK} = K(Q_1P_2 + P_1P_3)$, and use $Q_2$ and $Q_3$ to measure $Q_1$ and $P_1$, respectively. Their interaction Hamiltonian is obtained from that of the linear simultaneous $(Q_1, P_1)$-measurement with $\alpha_1 = -\alpha_3 = 1$ and $\alpha_2 = \beta_1 = \beta_2 = \beta_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$ by replacing $Q_3$ and $P_3$ by $-P_3$ and $Q_3$, respectively. Then we have

$$U_{AK}(K^{-1})Q_2U_{AK}(K^{-1}) = Q_1 + Q_2 + \frac{1}{2}P_3, \quad (34)$$

$$U_{AK}(K^{-1})Q_3U_{AK}(K^{-1}) = P_1 - \frac{1}{2}P_2 + Q_3, \quad (35)$$

so that the q-rms errors $\varepsilon(Q_1) = \varepsilon(Q_1, M_{AK}, \psi)$ and $\varepsilon(P_1) = \varepsilon(P_1, M_{AK}, \psi)$ satisfy Heisenberg's ETR, Eq. (5). This result shows that the Arthurs-Kelly model is not what we desire.

On the other hand, the measuring interaction of Ozawa's exactly solvable linear measurements is given by

$$H_O = K[\alpha Q_1P_2 + \beta P_1Q_2 + \gamma(Q_1P_1 - Q_2P_2)], \quad (36)$$

where $K$ is a positive real number, the coupling constant, and $\alpha$, $\beta$ and $\gamma$ are real numbers. In [51], Ozawa systematically analyzed his exactly solvable measuring models using this interaction, and calculated the noise-operator based q-rms error and the disturbance-operator based q-rms disturbance. His investigation motivated the author just as von Neumann's work inspired Arthurs and Kelly.

B. The Branciard-Ozawa ETR and the noise-operator based q-rms error

The reformulation of uncertainty relations is a currently developing project. As part of this research project, this study has the significance of connecting the recent knowledge about uncertainty relations with the construction of measurement models. After Ozawa's inequality

$$\varepsilon(X)\varepsilon(Y) + \varepsilon(X)\sigma(Y) + \sigma(X)\varepsilon(Y) \geq C_{XY} \quad (37)$$

was proved, the study of uncertainty relations became active, where $C_{XY} = |\text{Tr}(\rho [X,Y])|/2$ and $\rho$ is a density operator on $L^2(\mathbb{R})$ describing the state of $\mathcal{S}$. Note, however, that the noise-operator based q-rms error $\varepsilon(X)$
the standard deviation $\sigma(Y)$ are defined for $\rho$. The tightest ETR, which is now known, is the Branciard-Ozawa ETR

$$\varepsilon(X)^2\sigma(Y)^2 + \sigma(X)^2\varepsilon(Y)^2 + 2\varepsilon(X)\varepsilon(Y)\sqrt{\sigma(X)^2\sigma(Y)^2 - D_{XY}^2} \geq D_{XY}^2,$$ (38)

where $D_{XY} = \text{Tr}[^2\sqrt{\rho(X,Y)}]/2$ satisfies $D_{XY} \geq C_{XY}$ (see [12]). This inequality is first proved for pure (vector) states by Branciard [10], and is extended to mixed states by Ozawa [12]. Eq. (3) is the case where $X = Q_1, Y = P_1$ and the state of $S$ is $\psi$.

There is a claim that the use of the noise-operator based q-rms error is questionable because it sometimes vanishes for inaccurate measurements of observables (see [8] for example). In contrast to such a claim, it is shown in [12] that the q-rms error satisfies satisfactory conditions except for the completeness. A q-rms error is said to be complete if it never vanishes for inaccurate measurements of observables in each state [12]. The noise-operator based q-rms error is regarded as a straightforward generalization of Gauss’ error to quantum measurement. Instead of sticking to the noise-operator based q-rms error only, its improved versions that satisfy the completeness are also proposed in [12]. In statistics and information theory, various quantitative measures are defined for different purposes. In that sense, it is valid that we use the noise-operator based q-rms error as a standard, and that we use its improved versions as alternatives when its use is problematic.

V. METHODS

As in standard textbooks of quantum mechanics, $Q_j$ and $P_k$ satisfy

$$\langle Q_j f(x_1, x_2, x_3) = x_j f(x_1, x_2, x_3),$$

$$\langle P_k g(x_1, x_2, x_3) = \hbar \frac{\partial}{\partial x_k} g(x_1, x_2, x_3),$$

respectively, in the coordinate representation for every $j, k = 1, 2, 3$, and for appropriate functions $f$ and $g$ on $\mathbb{R}^3$. We do not explicitly use the above representation in the paper.

A. Proof of Theorem and the construction of models

To begin with, we shall prove Theorem. When the state of $S$ is $\psi$ and a linear $(Q_1, P_1)$-measurement $M = (\hat{H}_P, \xi, Q_2, P_3, U(\tau))$ is used, we have the following evaluation:

$$\varepsilon(Q_1)^2\sigma(P_1)^2 + \sigma(Q_1)^2\varepsilon(P_1)^2 \geq (a_{21} - 1)^2\sigma(Q_1)^2\sigma(P_1)^2 + \sigma(a_{22}Q_2 + a_{23}Q_3)^2\sigma(P_1)^2$$

$$+ (b_{31} - 1)^2\sigma(Q_1)^2\sigma(P_1)^2 + \sigma(b_{32}P_2 + b_{33}P_3)^2$$

$$= \frac{\hbar^2}{4}((a_{21} - 1)^2 + (b_{31} - 1)^2)$$

$$\sigma(a_{22}Q_2 + a_{23}Q_3)^2\sigma(P_1)^2 + \sigma(Q_1)^2\sigma(b_{32}P_2 + b_{33}P_3)^2$$

$$= \frac{\hbar^2}{4}((a_{21} - 1)^2 + (b_{31} - 1)^2)$$

$$+ 2\sigma(Q_1)^2\sigma(P_1)^2 + 2\sigma(a_{22}Q_2 + a_{23}Q_3)^2\sigma(P_1)^2$$

$$+ \sigma(a_{22}Q_2 + a_{23}Q_3)^2\sigma(P_1)^2$$

$$\sigma(b_{32}P_2 + b_{33}P_3)^2$$

$$\geq \frac{\hbar^2}{4}((a_{21} - 1)^2 + (b_{31} - 1)^2)$$

$$+ \hbar\sigma(a_{22}Q_2 + a_{23}Q_3)\sigma(b_{32}P_2 + b_{33}P_3)$$

$$\geq \hbar^2 l(a_{21}, b_{31}),$$ (41)

where $l(a_{21}, b_{31})$ is the function on $\mathbb{R}^2$ defined by

$$l(a_{21}, b_{31}) = \left[\frac{1}{4}((a_{21} - 1)^2 + (b_{31} - 1)^2) + \frac{1}{2}|a_{21}b_{31}|\right],$$ (42)

and takes the minimal value $1/4$ when $a_{21}, b_{31} \geq 0$ and $a_{21} + b_{31} = 1$. By $[Q_2(\tau), P_3(\tau)] = 0$, we have $a_{21}b_{31} + a_{22}b_{32} + a_{23}b_{33} = 0$. We see that $a_{22}Q_2 + a_{23}Q_3$ and $b_{32}P_2 + b_{33}P_3$ satisfy the following commutation relation

$$\left[a_{22}Q_2 + a_{23}Q_3, b_{32}P_2 + b_{33}P_3\right] = i\hbar(-a_{21}b_{31})I.$$ (43)

Therefore, we obtain

$$\sigma(a_{22}Q_2 + a_{23}Q_3)\sigma(b_{32}P_2 + b_{33}P_3) \geq \frac{\hbar}{2}|a_{21}b_{31}|.$$ (44)

A linear simultaneous $(Q_1, P_1)$-measurement $M = (\hat{H}_P, \xi, Q_2, P_3, U(\tau))$ satisfies Eq. (34) in $\psi$ if and only if it satisfies the conditions (i) and

(i.1) $\sigma(P_1)\sigma(a_{22}Q_2 + a_{23}Q_3) = \sigma(Q_1)\sigma(b_{32}P_2 + b_{33}P_3)$.

(i.2) $\sigma(a_{22}Q_2 + a_{23}Q_3)\sigma(b_{32}P_2 + b_{33}P_3) = \left[\frac{1}{2}|a_{21}b_{31}|\right]$.

(iii) $a_{21}b_{31} \geq 0$, $b_{31}b_{33} \geq 0$ and $a_{21} + b_{31} = 1$.

From the conditions (i.1) and (ii.2), we obtain the condition (ii) of the theorem. If $a_{21}b_{31} = 0$, we get

$$\sigma(a_{22}Q_2 + a_{23}Q_3) = \sigma(b_{32}P_2 + b_{33}P_3) = 0.$$ Since at least one of $a_{22}, a_{23}, b_{32}$ and $b_{33}$ is non-zero, $\sigma(a_{22}Q_2 + a_{23}Q_3) = \sigma(b_{32}P_2 + b_{33}P_3) = 0$ never holds for any unit vector $\xi$ of $L^2(\mathbb{R}^2)$. Therefore, $a_{21}b_{31} \neq 0$ must be satisfied, so that we have the condition (iii) of the theorem. We then have

$$\varepsilon(Q_1)^2 = (a_{21} - 1)^2\sigma_1^2 + |a_{21}b_{31}|\sigma_3^2 = (1 - a_{21})\sigma_1^2,$$ (45)

$$\eta(P_1)^2 = (b_{31} - 1)^2\sigma_1^2 + |a_{21}b_{31}|\sigma_3^2 = a_{21}\sigma_1^2.$$ (46)

To complete the proof, for every $\nu \in (0, 1)$, we find $S$ and $\tau > 0$ such that $a_{21} = \nu$ and $b_{31} = 1 - \nu$. $S$ satisfies
\( S^3 = (-E)S \), so that we have
\[
e^{tS} = \begin{cases} 
I + \frac{\sin(t\sqrt{E})}{\sqrt{E}} S + \frac{1 - \cos(t\sqrt{E})}{E} S^2, & (E > 0) \\
I + tS + \frac{1}{2}t^2 S^2, & (E = 0) \\
I + \frac{\sinh(t\sqrt{-E})}{\sqrt{-E}} S + \frac{\cosh(t\sqrt{-E}) - 1}{-E} S^2, & (E < 0) 
\end{cases}
\]
for all \( t \in \mathbb{R} \). Independent of the sign of \( E \), \( e^{\tau S} = (a_{ij}) \) and \( e^{-\tau S^T} = (b_{ij}) \) satisfy \( a_{22} = b_{33} \) and \( a_{23} = b_{32} \). Since \( [Q_2(\tau), P_3(\tau)] = 0 \), we have \( a_{21} b_{31} + 2 a_{22} a_{33} = 0 \). Then, we use \( \xi_{a_{21}, a_{22}} \) as the state of \( P \), i.e., \( \xi_{\nu, \kappa} \) with \( \nu = a_{21} \) and \( \kappa = a_{22} \). \( \xi_{a_{21}, a_{22}} \) is the product of two Gaussian states \( \xi_2 \) and \( \xi_3 \): It has the form \( \xi_{a_{21}, a_{22}}(x_2, x_3) = \xi_2(x_2) \xi_3(x_3) \) in the coordinate representation, where \( \xi_2 \) and \( \xi_3 \) are given by
\[
\xi_2(x_2) = \sqrt{\frac{|a_{22}|}{(2\pi)|a_{23}| \sigma_1^2}} e^{-\frac{|a_{22}|}{(2\pi)|a_{23}| \sigma_1^2} \left( x_2 - \frac{2a_{22}}{a_{23}} \xi_3 \right)^2},
\]
\[
\xi_3(x_3) = \sqrt{\frac{|a_{23}|}{(2\pi)|a_{22}| \sigma_1^2}} e^{-\frac{|a_{23}|}{(2\pi)|a_{22}| \sigma_1^2} \left( x_3 - \frac{2a_{23}}{a_{22}} \xi_2 \right)^2},
\]
respectively, in the coordinate representation. By Eq \((47)\), the cases \( E > 0 \), \( E = 0 \) and \( E < 0 \) must be handled separately.

For \( E > 0 \) both \( a_{21} = \nu \) and \( b_{31} = 1 - \nu \) are satisfied if and only if it holds that
\[
\frac{\sin(\tau \sqrt{E})}{\sqrt{E}} + \gamma_2 \frac{1 - \cos(\tau \sqrt{E})}{E} = \frac{\nu}{\alpha_1} = 1 - \frac{\nu}{\alpha_3}.
\]
\[\text{(50)}\]

For example, for every \( 0 < \nu < 1 \), \( \tau > 0 \), \( \gamma_2 \geq 0 \) and \( \tau > 0 \), there uniquely exist \( \alpha_1 > 0 \) and \( \alpha_3 < 0 \) satisfying Eq \((50)\), which completes the proof of the theorem. The family \( \{\Xi_\nu\}_{\nu \in (0,1)} \) of linear simultaneous \((Q_1, P_1)\)-measurements are contained in this case.

For \( E = 0 \) both \( a_{21} = \nu \) and \( b_{31} = 1 - \nu \) are satisfied if and only if it holds that
\[
\tau + \frac{1}{2} \gamma_2 \tau^2 = \frac{\nu}{\alpha_1} = 1 - \frac{\nu}{\alpha_3}.
\]
\[\text{(51)}\]

For every \( 0 < \nu < 1 \), \( \gamma_2 > 0 \) and \( \tau > 0 \), there uniquely exist \( \alpha_1 > 0 \) and \( \alpha_3 < 0 \) satisfying Eq \((52)\).

The family \( \{\Xi_\nu\}_{\nu \in (0,1)} \) of linear simultaneous \((Q_1, P_1)\)-measurements are contained in this case.

\[
\gamma S = A = (a_{ij}) \text{ and } e^{-\tau S^T} = B = (b_{ij}) \text{ in each model are then given as follows:}
\]

| \( e^{\gamma S} \) | \( e^{-\tau S^T} \) |
|----------------|----------------|
| \( \nu = \frac{1}{2} \) | \( \nu = \frac{1}{2} \) |
| \( \nu < \frac{1}{2} \) | \( \nu < \frac{1}{2} \) |
| \( \nu > \frac{1}{2} \) | \( \nu > \frac{1}{2} \) |

\[\text{TABLE II.}\]

For every \( 0 < \nu < 1 \), \( E < 0 \), \( \gamma_2 > 0 \) and \( \tau > 0 \), there uniquely exist \( \alpha_1 > 0 \) and \( \alpha_3 < 0 \) satisfying Eq \((52)\).

\[\text{B. Probability distributions and families of posterior states}\]

The characteristic function \( \lambda \) of the probability measure \( \mu \) on \( \mathbb{R}^d \) is defined as the inverse Fourier transform of \( \mu \):
\[
\lambda(k) = \int_{\mathbb{R}^d} e^{i(x,k)} \, d\mu(x),
\]
\[\text{(53)}\]

where \( \langle \cdot, \cdot \rangle \) is the inner product of \( \mathbb{R}^d \). For any observables \( X_1, X_2 \) and vector state \( \phi \), the characteristic function of \( \mu \phi_{X_1, X_2} \) is denoted by \( \lambda_{\phi} \). The characteristic function of a Gaussian measure
\[
d\mu_{V,m}(x) = \frac{1}{\sqrt{(2\pi)^d \det(V)}} e^{-\frac{1}{2} \langle x-m, V^{-1}(x-m) \rangle} \, dx
\]
\[\text{(54)}\]

has the following form:
\[
\lambda_{V,m}(k) = e^{i(m,k) - \frac{1}{4} \langle k, V k \rangle},
\]
\[\text{(55)}\]

where \( V > 0 \) is a covariance matrix and \( m \in \mathbb{R}^d \) is a mean vector. Conversely, if a characteristic function is given by Eq. \((55)\), then the corresponding probability measure is a Gaussian measure given by Eq. \((54)\). We refer the reader to textbooks of probability theory and statistics.

The characteristic function \( \lambda_{\phi \otimes \Xi_{a_{21}, a_{22}}} \) of \( \mu_{\phi \otimes \Xi_{a_{21}, a_{22}}} \)
is given by
\[
\lambda_{\psi_{\otimes \xi_{a_1}}}(Q_1(0), Q_2(\tau), \psi)(k) = \langle \psi | e^{i(k_1 + a_2 k_2) Q_1(0)} e^{i(k_1 + a_2 k_2) Q_2(0)} e^{i(k_1 + a_2 k_2) \xi_2(0)} | \psi \rangle_{\otimes \xi_{a_1 + a_2}}
\]
\[
= e^{i(k_1 + a_2 k_2) Q_1(0)} e^{i(k_1 + a_2 k_2) Q_2(0)} e^{i(k_1 + a_2 k_2) \xi_2(0)}
\]
\[
= e^{i(k_1 + a_2 k_2) - \frac{i}{2} \sigma_1^2 (k_1 + a_2 k_2)^2}
\]
\[
\times e^{i \frac{1}{2} a_2^2 (a_2 k_2 - \frac{1}{2} \sigma_1^2 (a_2 k_2)^2) / e - \frac{i}{2} a_2^2 \sigma_1^2 (a_2 k_2)^2}
\]
\[
= e^{i q_1 (k_1 + a_2 k_2) - \frac{i}{2} \sigma_1^2 (k_1 + a_2 k_2)^2}
\]
\[
= \lambda_{W, q}(k)
\]
(56)
for all \( k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{R}^2 \), where \( q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \) and \( W = \sigma_1^2 \begin{pmatrix} 1 & a_2 \\ a_2 & 1 \end{pmatrix} \). Here we used \( 0 < a_2 < 1, a_2 (1 - a_2) + 2 a_2 a_2 = 0 \), which is obtained from the condition (iii) of the theorem and \( a_2 b_2 + 2 a_2 a_2 = 0 \), and the relation
\[
\langle \psi | e^{i(a Q_2 + b \psi)} | \psi \rangle = e^{i q_1 a - \frac{i}{2} \sigma_1^2 a^2} e^{i p_1 b - \frac{i}{2} \sigma_1^2 b^2}
\]
(57)
for all \( a, b \in \mathbb{R} \). From \( \det(W) = \sigma_1^2 a_2 (1 - a_2) \) and
\[
W^{-1} = \frac{1}{(1 - a_2) \sigma_1^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{a_2 \sigma_1^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
(58)
we obtain
\[
p_{\psi_{\otimes \xi_{a_1}}}(Q_1(0), Q_2(\tau), \psi)(x, z) = p_{a_2 (1 - a_2) \sigma_1^2} (x - z) p_{a_2 \sigma_1^2} (z - q_1).
\]
(59)
Eq. (59) is obtained from Eq. (58) for \( \mathbb{H}_\nu, \mathbb{H}_\nu \), \( \mathbb{S}_\nu \) and \( \mathbb{Z}_\nu \). Similarly, we have
\[
p_{\psi_{\otimes \xi_{a_1}}}(Q_1(0), Q_2(\tau), \psi)(y, w) = p_{a_2 \sigma_1^2} (y - w) p_{(1 - a_2) \sigma_1^2} (w - p_1),
\]
(60)
\[
p_{\psi_{\otimes \xi_{a_1}}}(Q_1(0), Q_2(\tau), \psi)(z, w) = p_{a_2 \sigma_1^2} (z - q_1) p_{a_2 (1 - a_2) \sigma_1^2} (w - p_1).
\]
(61)
In particular, Eqs. (27) and (28) are derived in the same way.

Next, for every \( \nu \in (0, 1) \), we find the family of posterior states for \( (\mathbb{H}_\nu, \psi) \). We check the following probability density functions via their characteristic functions:
\[
p_{\psi_{\otimes \xi_{\nu}}}(Q_1(\tau), Q_2(\tau), \psi)(x, z, w) = p_{\nu (1 - \nu) \sigma_1^2} (x - z - (1 - \nu) q_1)
\]
\[
\times p_{\nu \sigma_1^2} (z - q_1) p_{\nu (1 - \nu) \sigma_1^2} (w - p_1),
\]
(62)
\[
p_{\psi_{\otimes \xi_{\nu}}}(Q_1(\tau), Q_2(\tau), \psi)(y, z, w) = p_{\nu \sigma_1^2} (y - w - p_1)
\]
\[
\times p_{\nu \sigma_1^2} (z - q_1) p_{\nu (1 - \nu) \sigma_1^2} (w - p_1).
\]
(63)
For example, the characteristic function \( \lambda_{\psi_{\otimes \xi_{\nu}}}(Q_1(\tau), Q_2(\tau), \psi)(\tau) \) is given by
\[
\lambda_{\psi_{\otimes \xi_{\nu}}}(Q_1(\tau), Q_2(\tau), \psi)(\tau)(k) = \langle \psi | e^{i(k_1 + a_2 k_2) Q_1(0) + i(k_1 + a_2 k_2) Q_2(0) + i(k_1 + a_2 k_2) \xi_2(0)} | \psi \rangle_{\otimes \xi_{a_1 + a_2}}
\]
\[
= e^{i(k_1 + a_2 k_2) Q_1(0)} e^{i(k_1 + a_2 k_2) Q_2(0)} e^{i(k_1 + a_2 k_2) \xi_2(0)}
\]
\[
= e^{i(k_1 + a_2 k_2) - \frac{i}{2} \sigma_1^2 (k_1 + a_2 k_2)^2}
\]
\[
\times e^{i \frac{1}{2} a_2^2 (a_2 k_2 - \frac{1}{2} \sigma_1^2 (a_2 k_2)^2) / e - \frac{i}{2} a_2^2 \sigma_1^2 (a_2 k_2)^2}
\]
\[
= e^{i(k_1 + a_2 k_2) - \frac{i}{2} \sigma_1^2 (k_1 + a_2 k_2)^2}
\]
\[
= \lambda_{W, q}(k)
\]
(64)
for all \( k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{R}^3 \), where \( \tilde{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \) and \( Z = \sigma_1^2 \begin{pmatrix} 2 - \nu & 1 \\ 1 & \nu \end{pmatrix} \). From \( \det Z = (1 - \nu) \sigma_1^2 \) and
\[
Z^{-1} = \frac{1}{(1 - \nu) \sigma_1^2} \begin{pmatrix} 1 & \nu \\ -\nu & 1 \end{pmatrix} + \frac{1}{\nu \sigma_1^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
(65)
we obtain Eq. (62). The relation \( \frac{(1 - \nu) \sigma_1^2}{\nu} \cdot \frac{\nu \sigma_1^2}{1 - \nu} = \frac{\hbar^2}{4} \)
implies that the family of posterior states for \( (\mathbb{H}_\nu, \psi) \) is given by Eq. (63) and is unique up to phase. For every \( \nu \in (0, 1) \), the family of posterior states for \( (\mathbb{H}_\nu, \psi) \) is derived in the same way.

For every rectangular, more generally, Borel set \( J \) in \( \mathbb{R}^2 \), we then obtain the state \( \rho_J \) after the measurement under the condition that output values not contained in \( J \) is excluded, which is given by
\[
\text{Tr}[X \rho_J] = \int \frac{U(\tau) (\psi \otimes \xi)}{U(\tau) (\psi \otimes \xi)} U(\tau) (\psi \otimes \xi) \langle X | \psi \rangle \langle \psi | d\mu_{\nu, \nu}(y)
\]
(66)
wherever \( \langle X | \psi \otimes \xi \rangle (1 \otimes E(J) U(\psi \otimes \xi)) \neq 0 \). Here \( E \) is the spectral measure of \( \mathbb{R}^2 \) on \( L^2(\mathbb{R}^2) \) such that \( E_1 \times J_2 = E Q_1(J_1) P_3(J_2) \) for all Borel sets \( J_1, J_2 \) of \( \mathbb{R} \). For every \( \nu \in (0, 1) \), the family \( \{ \psi_y \}_{y \in \mathbb{R}^2} \) of posterior states for \( (\mathbb{H}_\nu, \psi) \) satisfies
\[
\rho_J = \frac{1}{\mu_{\nu, \nu}(J)} \int | \psi_y \rangle \langle \psi_y | d\mu_{\nu, \nu}(y)
\]
(67)
for all Borel set \( J \) of \( \mathbb{R}^2 \), where \( \nu = \begin{pmatrix} \nu \sigma_1^2 & 0 \\ 0 & 1 - \nu \sigma_1^2 \end{pmatrix} \) and \( r = \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \).
VI. SUMMARY AND PERSPECTIVES

We have given a necessary and sufficient condition for a linear simultaneous \((Q_1, P_1)\)-measurement to satisfy Eq. (4), and constructed four families \{\mathcal{X}_\nu\}(0,1), \{\mathcal{Y}_\nu\}(0,1), \{\mathcal{V}_\nu\}(0,1), and \{\mathcal{Z}_\nu\}(0,1) of linear simultaneous \((Q_1, P_1)\)-measurements. Furthermore, we have probability distributions when using \{\mathcal{X}_\nu\}(0,1), \{\mathcal{Y}_\nu\}(0,1), \{\mathcal{V}_\nu\}(0,1), and \{\mathcal{Z}_\nu\}(0,1), and families of posterior states for \{\mathcal{Y}_\nu\}(0,1) and \{\mathcal{Z}_\nu\}(0,1). We believe that the results of the paper have important implications for future research on simultaneous measurements. There are not so many studies on simultaneous measurements of position and momentum since Heisenberg’s paper in spite of their importance. In fact, this paper shows that there is still room for studying simultaneous measurements of position and momentum. The same is true for simultaneous measurements of different components of the spin. It is desirable to study simultaneous measurements more and more actively, in connection with the recent progress of uncertainty relations. We believe that it will make a significant contribution to the resolution of various problems in the field of quantum information through quantitative analysis. In the future, the theory of simultaneous measurements of incompatible observables will be widely applied to quantum information processing.

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