Finite dimensional quantum group covariant differential calculus on a complex matrix algebra

R. Coquereaux\(^1\), A. O. García\(^2\), R. Trinchero\(^2\)

\(^1\) Centre de Physique Théorique - CNRS - Luminy, Case 907
F-13288 Marseille Cedex 9 - France

\(^2\) Instituto Balseiro and Centro Atómico Bariloche
CC 439 - 8400 - Bariloche - Río Negro - Argentina

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Abstract

Using the fact that the algebra \(M_3(\mathbb{C})\) of \(3 \times 3\) complex matrices can be taken as a reduced quantum plane, we build a differential calculus \(\Omega(S)\) on the quantum space \(S\) defined by the algebra \(C^\infty(M) \otimes M_3(\mathbb{C})\), where \(M\) is a space-time manifold. This calculus is covariant under the action and coaction of finite dimensional dual quantum groups. We study the star structures on these quantum groups and the compatible one in \(M_3(\mathbb{C})\). This leads to an invariant scalar product on the later space. We analyse the differential algebra \(\Omega(M_3(\mathbb{C}))\) in terms of quantum group representations, and consider in particular the space of 1-forms on \(S\) since its elements can be considered as generalized gauge fields.

Keywords: non commutative geometry, quantum groups, differential calculus, gauge theories.

Anonymous ftp or gopher: cpt.univ-mrs.fr

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\(^1\) Email: coque@cpt.univ-mrs.fr
\(^2\) Email: ariel@cab.cnea.edu.ar
\(^3\) Email: trincher@cab.cnea.edu.ar
1 Introduction

The formulation of physical theories in the framework of noncommutative geometry opens new possibilities and has produced very interesting results. Physical models over a space $S$ described by the tensor product of the commutative algebra of functions over a space-time manifold and the noncommutative space whose “algebra of functions” is given by $\mathcal{M} = M_N(\mathbb{C})$, the algebra of $N \times N$ complex matrices, have been studied by several people \cite{1, 2}, using techniques of noncommutative differential geometry. Such constructions always involve some $\mathbb{Z}$-graded differential algebra, generalizing the usual differential forms. Here, we shall use a differential calculus that has covariance properties with respect to a finite dimensional quantum group. Indeed, the algebra of $N \times N$ complex matrices is the same as the one of a reduced quantum plane with $q^N = 1$ \cite{3}. Hence we naturally have on $\mathcal{M}$ the action of a quantum group $\mathcal{H}$ \cite{4, 5, 6} and the coaction of its dual quantum group $\mathcal{F}$. We construct a differential calculus covariant under the action and coaction of the above mentioned quantum groups (it is a quotient of the Wess-Zumino complex \cite{7}) and study its properties and representation theory (see also \cite{8}). We consider the star operations in $\mathcal{H}$, $\mathcal{F}$ and the covariant one in $\mathcal{M}$. Working with star representations, the above mentioned star structures lead to an invariant scalar product in $\mathcal{M}$.

Although our ultimate interest is to build a gauge theory on $S$ with some invariance property with respect to a quantum group action, the construction of a Lagrangian will not be considered in the present work. In any case it is interesting to remark that a general one form on $S$ involves a vector field $a_\mu$ and two scalar fields $\phi^x$ and $\phi^y$, all of them valued in $\mathcal{M} (= M_N(\mathbb{C}))$.

2 The space of complex matrices as a reduced quantum plane

It has been known for a long time \cite{3} that the algebra of $N \times N$ matrices can be generated by two elements $x$ and $y$ with the relations

\begin{equation}
xy = qyx \tag{1}
\end{equation}

\begin{equation}
x^N = y^N = 1, \tag{2}
\end{equation}

where $q$ denotes a $N$-th root of unity ($q \neq 1$) and $1$ is the unit matrix. From now on we consider the case $N = 3$. For this case we have the following explicit representation of $x$ and $y$:

\[
x = \begin{pmatrix} 1 & 0 & 0 \\
0 & q^{-1} & 0 \\
0 & 0 & q^{-2} \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \end{pmatrix}.
\]

They generate the algebra of $3 \times 3$ complex matrices. The algebra generated by abstract elements $x, y$ with the relations \cite{8} is called the quantum plane $\mathbb{C}_q$, and adding relations \cite{2} leads to the reduced quantum plane $\mathcal{M} \equiv M_3(\mathbb{C})$. This last algebra has the following basis as a vector space of dimension nine: \{${x^r y^s} : \ r, s = 0, 1, 2$\}.

3 Quantum group coaction on $\mathcal{M}$

Consider the following transformations between “coordinate functions”,

\[
\delta_L \left( \begin{array}{c} x \\ y \end{array} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \delta (\begin{array}{c} x' \\ y' \end{array}) \quad \text{left coaction ,} \tag{3}
\]

and

\[
\delta_R (x \ y) = (x \ y) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\tilde{x} \ \tilde{y}) = \delta (\begin{array}{c} \tilde{x} \\ \tilde{y} \end{array}) \quad \text{right coaction .} \tag{4}
\]

These coactions extend to the whole of $\mathcal{M}$ using the homomorphism property $\delta(fg) = \delta(f)\delta(g)$ ($f, g \in \mathcal{M}$), for both $L$ and $R$ coactions. The elements $a, b, c, d$ should satisfy an algebra such that

\begin{align}
\delta_L(xy - qyx) &= 0, \tag{5} \\
\delta_R(xy - qyx) &= 0. \tag{6}
\end{align}

\footnote{This method of obtaining the product relations was introduced in \cite{9}.}
Hence one obtains

\[
\begin{align*}
qba &= ab & qdb &= bd \\
qca &= ac & qdc &= cd \\
cb &= bc & ad - da &= (q - q^{-1})bc,
\end{align*}
\]

which are the product relations of what is called \( \text{Fun}(GL_q(2)) \). The element \( D \doteq da - q^{-1}bc = ad - qbc \) is a central element (it commutes with all the elements of \( \text{Fun}(GL_q(2)) \)); it is called the \( q \)-determinant and we set it equal to 1, getting \( \text{Fun}(SL_q(2, \mathbb{C})) \).

This algebra is a Hopf algebra with the following structure:

- **Coproduct:**
  \[
  \Delta a = a \otimes a + b \otimes c, \quad \Delta b = a \otimes b + b \otimes d, \quad \Delta c = c \otimes a + d \otimes c, \quad \Delta d = c \otimes b + d \otimes d
  \]
  \[
  (\Delta(AB) = \Delta A \Delta B)
  \]
- **Antipode:**
  \[
  S a = d, \quad S b = -q^{-1}b, \quad S c = -qc, \quad S d = a
  \]
  \[
  (S(uv) = S(v)S(u))
  \]
- **Counit:**
  \[
  \epsilon(a) = 1, \quad \epsilon(b) = 0, \quad \epsilon(c) = 0, \quad \epsilon(d) = 1.
  \]

If we further include the relations

\[
\begin{align*}
x^{a3} &= 1, & x^{a3} &= 1, & y^{a3} &= 1, & y^{a3} &= 1,
\end{align*}
\]

we should impose

\[
\begin{align*}
a^3 &= 1, & d^3 &= 1, & b^3 &= c^3 = 0,
\end{align*}
\]

that define an ideal and coideal. The quotiented Hopf algebra so obtained, \( \mathcal{F} \), has dimension 27.

4 Quantum group action on \( \mathcal{M} \)

The dual of the Hopf algebra \( \mathcal{F} \) can be obtained as a quotient of \( U_q(sl(2)) \), the dual of \( \text{Fun}(SL_q(2, \mathbb{C})) \). In order to fix notation and conventions we recall the defining relations of \( U_q(sl(2)) \) in terms of its generators \( X_+, X_- \) and \( K \).

- **Multiplication:**
  \[
  K X_+ = q^{+2} X_+ K, \quad [X_+, X_-] = \frac{K - K^{-1}}{q - q^{-1}}
  \]
- **Comultiplication:**
  \[
  \Delta X_+ \doteq X_+ \otimes 1 + K \otimes X_+, \quad \Delta X_- \doteq X_- \otimes K^{-1} + 1 \otimes X_-
  \]
  \[
  \Delta K \doteq K \otimes K, \quad \Delta K^{-1} \doteq K^{-1} \otimes K^{-1}
  \]
- **Antipode:**
  \[
  S 1 = 1, \quad S K = K^{-1}, \quad S K^{-1} = K, \quad S X_+ = -K^{-1} X_+, \quad S X_- = -X_- K.
  \]
- **Counit:**
  \[
  \epsilon(1) = 1, \quad \epsilon(K) = 1, \quad \epsilon(K^{-1}) = 1, \quad \epsilon(X_\pm) = 0.
  \]

The dual \( \mathcal{H} \) of \( \mathcal{F} \) is obtained by taking the quotient of \( U_q(sl(2)) \) with the (Hopf) ideal and coideal defined by

\[
X_+^3 = X_-^3 = 0, \quad K^3 = 1.
\]

The duality relations between \( \mathcal{F} \) and \( \mathcal{H} \) are given explicitly by the pairing between generators:

\[
\begin{align*}
< K, a > &= q & < K, b > &= 0 & < K, c > &= 0 & < K, d > &= q^{-1} \\
< X_+, a > &= 0 & < X_+, b > &= 1 & < X_+, c > &= 0 & < X_+, d > &= 0 \\
< X_-, a > &= 0 & < X_-, b > &= 0 & < X_-, c > &= 1 & < X_-, d > &= 0
\end{align*}
\]

This pairing interchanges multiplication and comultiplication via the relations

\[
\begin{align*}
< X_1 X_2, u > &= < X_1 \otimes X_2, \Delta u >, & < \Delta X, u_1 \otimes u_2 > &= < X, u_1 u_2 >.
\end{align*}
\]

Since \( \mathcal{F} \) coacts on \( \mathcal{M} \) there is a natural definition of an action of \( \mathcal{H} \) on \( \mathcal{M} \). Using the pairing and the right coaction \( \delta_R \), the left action of \( \mathcal{H} \) on \( \mathcal{M} \) is defined by

\[
h(z) \equiv h.z \doteq (id \otimes < h, \cdot >) \circ \delta_R z; \quad h \in \mathcal{H}, z \in \mathcal{M}.
\]
This implies, in particular, that

\[ h(1) = \varepsilon(h) 1 \]
\[ h(zw) = \Delta(h)(z \otimes w) \]  

(11)

Several properties of this action (with other conventions) have been studied in \([10]\). Explicitly, one gets

|   |   |   |
|---|---|---|
| \(x^2\) | \(q^2x^2\) | 0 |
| \(xy\) | \(xy\) | \(qx^2\) |
| \(y^2\) | \(qy^2\) | \(-q^2xy\) |
| \(x\) | \(qx\) | 0 |
| \(y\) | \(qy^2\) | \(x\) |
| \(x^2y^2\) | \(x^2y^2\) | \(-qy\) |
| \(xy^2\) | \(qxy^2\) | \(-xy^2\) |
| \(1\) | \(1\) | 0 |

(12)

We conclude this section recalling some facts about the representation theory of \(\mathcal{H}\).

The left regular representation of \(\mathcal{H}\) was studied in \([4]\). It is the same \([6]\) as the 27-dimensional non semisimple Hopf algebra \(M_3 \oplus (M_{2|1}(\Lambda^2))_0\) whose elements are given explicitly by

\[
\mathcal{H} = m_3 \oplus \begin{pmatrix}
\alpha_{11} + \beta_{11}\theta_1\theta_2 & \alpha_{12} + \beta_{12}\theta_1\theta_2 & \gamma_{13}\theta_1 + \delta_{13}\theta_2 \\
\alpha_{21} + \beta_{21}\theta_1\theta_2 & \alpha_{22} + \beta_{22}\theta_1\theta_2 & \gamma_{23}\theta_1 + \delta_{23}\theta_2 \\
\gamma_{31}\theta_1 + \delta_{31}\theta_2 & \gamma_{32}\theta_1 + \delta_{32}\theta_2 & \alpha_{33} + \beta_{33}\theta_1\theta_2
\end{pmatrix},
\]  

(13)

where the \(\alpha\)'s, \(\beta\)'s and the coefficients of the 3 \(\times\) 3 matrix \(m_3\) are complex numbers. \(\theta_1\) and \(\theta_2\) denote Grassmann variables.

The representations of \(\mathcal{H}\) can be obtained from the action of the left regular representation matrix of a generic element on its own columns. The first three columns of the 6 \(\times\) 6 matrix \([\mathbf{13}]\) give equivalent representations that correspond to a three dimensional irreducible representation denoted by \(3_1\). The first two columns of the \((M_{2|1}(\Lambda^2))_0\) give equivalent representations of dimension six denoted by \(6_0\), while the last column leads to another six-dimensional representation denoted by \(6_0\). These two 6-dimensional inequivalent representations are indecomposable but not irreducible, their lattice of subrepresentations \([\mathbf{13}]\) is

\[
0 \quad \overset{2}{\longrightarrow} \quad \{3_0^\lambda\} \quad \overset{4_0}{\longrightarrow} \quad 6_o
\]

\[
0 \quad \overset{1}{\longrightarrow} \quad \{3_1^\lambda\} \quad \overset{5_0}{\longrightarrow} \quad 6_o
\]

where (column vectors):

\[
6_o = (\alpha + \beta\theta_1\theta_2, \alpha' + \beta'\theta_1\theta_2, \gamma\theta_1 + \delta\theta_2) \quad 6_o = (\gamma\theta_1 + \delta\theta_2, \gamma'\theta_1 + \delta'\theta_2, \alpha + \beta\theta_1\theta_2)
\]

\[
4_0 = (\beta\theta_2, \beta'\theta_2, \gamma\theta_1 + \delta\theta_2) \quad 5_0 = (\gamma\theta_1 + \delta\theta_2, \gamma'\theta_1 + \delta'\theta_2, \beta\theta_1\theta_2)
\]

\[
3_1^\lambda = (\beta\theta_2, \beta'\theta_2, \gamma\theta_1) \quad 3_0^\lambda = (\gamma\theta_1, \gamma'\theta_1, \beta\theta_1\theta_2)
\]

\[
2 = (\beta\theta_2, \beta'\theta_2, 0) \quad 1 = (0, 0, \beta\theta_1\theta_2)
\]

Here \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\), \(\lambda_1 = \lambda_1\theta_1 + \lambda_2\theta_2\), and \(\lambda_1, \lambda_2 \in \mathbb{C}\). The families of representations \(\{3^\lambda\}\) are actually parametrized by \(\lambda = \frac{\lambda_1}{\lambda_2} \in \mathbb{C}P^1\).

5 Stars on \(\mathcal{F}, \mathcal{H}\) and \(\mathcal{M}\)

For the case \(|q| = 1 (q \neq 1)\) there is only one \(*\)-Hopf structure on \(Fun(SL_q(2, \mathbb{C}))\) \([1]\), up to equivalences\([\mathbf{14}]\). It is given by

\[
a^* = a, \quad b^* = b, \quad c^* = c, \quad d^* = d.
\]  

(14)

In the next name, \(\Lambda^2\) is the Grassmann algebra with two generators and \(0\) denotes the even part of the graded matrix algebra \(M_{2|1}\).

\(^2\) Two star structures, \(\mathbf{1}\) and \(\mathbf{*}\), over a (Hopf) algebra \(H\) are equivalent if there exists a (Hopf) automorphism \(\alpha\) such that \(\alpha(a^\dagger) = (\alpha(a))^*\), \(\forall a \in H\). The Hopf automorphisms of \(\mathcal{F}\) and \(\mathcal{H}\) are \([\mathbf{13}]\).
We denote this last *-Hopf algebra as \( \text{Fun}(SL_2(\mathbb{R})) \), in correspondence with the classical case. This star is compatible with the relations \( a^+ = d^1 = 1 \), \( b^1 = c^1 = 0 \), and therefore extends to the quotient of \( \text{Fun}(SL_2(\mathbb{R})) \) that we continue to denote by \( \mathcal{F} \).

Using the pairing between \( \mathcal{F} \) and \( \mathcal{H} \) we can obtain the corresponding star in \( \mathcal{H} \). In order to get a *-Hopf structure in the dual \( \mathcal{H} \) of \( \mathcal{F} \) one should have

\[
\langle h^*, a \rangle = \langle h, (Sa)^* \rangle ,
\]

where the bar means complex conjugation. This leads to the following star in \( \mathcal{H} \),

\[
X_+^* = -q^{-1}X_+ , \quad X_-^* = -qX_- , \quad K^* = K .
\]

Next we consider the determination of a star in \( \mathcal{M} \). The basic requirement is that the coaction \( \delta \triangleq \delta_R \) should be a *-homomorphism, i.e.,

\[
(\delta z)^* = \delta(z^*) , \text{ for any } z \in \mathcal{M} , \text{ where } (A \otimes B)^* = A^* \otimes B^* .
\]

Pairing the \( \mathcal{F} \) component of this equation with some \( h \in \mathcal{H} \), and recalling (10) and (15), we see that the dual of condition (17) is

\[
h(z^*) = [(Sh)^* x]^* .
\]

It is not difficult to verify that the only family of stars compatible with (18) is

\[
x^* = \alpha x \ , \quad y^* = \alpha y \ , \quad \text{ with } |\alpha| = 1 ;
\]

moreover, we choose \( \alpha = 1 \).

6 Quantum groups and invariant scalar products

It is not straightforward to define the notion of an invariant scalar product on a representation space \( V \) of a quantum group \( \mathcal{H} \). The basic idea is that taking scalar products should in some way commute with the action of the quantum group. This amounts to say that

\[
h \circ (\cdot, \cdot) = (\cdot, \cdot) \circ \tilde{h}
\]

where \( (\cdot, \cdot) \) denotes the scalar product on \( V \) (antilinear in the first variable and linear in the second).

The action of \( \mathcal{H} \) on \( \mathbb{C} \) is the trivial one, given by \( h(\alpha) = \epsilon(h)\alpha \), so that

\[
(h \circ (\cdot, \cdot))(u \otimes v) = h((u, v)) = \epsilon(h)(u, v) .
\]

Finally, \( \tilde{h} \) is an action of \( \mathcal{H} \) on \( V \otimes V \) to be determined.

The natural action of \( \mathcal{H} \) on \( V \otimes V \) is given by

\[
h.(u \otimes v) = \Delta h(u \otimes v) = (h_1 u) \otimes (h_2 v) ,
\]

however this action can not be the one involved in the r.h.s. of requirement (21). This is so since the l.h.s. of this equation — as given by (22) — has to be linear under the replacement \( h \rightarrow \alpha h \), with \( \alpha \in \mathbb{C} \), but the right hand side would be linear if we attach \( \alpha \) to \( h_2 \) and antilinear if we attach \( \alpha \) to \( h_1 \). If there is a star structure in \( \mathcal{H} \) one could try to solve this problem by defining \( \tilde{h}(u \otimes v) \) as \((h_1 u) \otimes h_2 v\). This would fix the sesquilinearity problem. Unfortunately, this choice for \( \tilde{h} \) would not have a definite homomorphism behaviour for the product in \( \mathcal{H} \) whereas the \( \epsilon \) on the r.h.s. of (22) (and thus on the l.h.s. of (21)), is a homomorphism. The solution to this last problem is to employ the natural antihomomorphism given by the antipode. However there are two options,

\[\mathcal{H} : \quad K \rightarrow K , \quad X_+ \rightarrow \beta X_+ , \quad X_- \rightarrow \beta^{-1} X_- , \quad \text{ where } \beta \in \mathbb{C} ;\]

Using the above notion of equivalence of star structures we get the following family of equivalent stars:

\[
\mathcal{F} : \quad a^{*\ast} = a \ , \quad d^{*\ast} = d \ , \quad b^{*\ast} = ub \ , \quad c^{*\ast} = u^{-1}c , \quad \text{ with } |u| = 1
\]

\[
\mathcal{H} : \quad K^{*\ast} = K \ , \quad X_+^{*\ast} = -q^{-1}uX_+ , \quad X_-^{*\ast} = -qu^{-1}X_- .
\]

Changing the star in \( \mathcal{F} \) and \( \mathcal{H} \) to an equivalent one as in the previous footnote, corresponds to a change in the family of stars in \( \mathcal{M} \) given by

\[
x^{*\ast} = \alpha x , \quad y^{*\ast} = u\alpha y , \quad |\alpha| = 1 .
\]
\( \tilde{h}^1(u \otimes v) = (S h_1)^* u \otimes h_2 v \) \hspace{1cm} (24)

or

\( \tilde{h}^2(u \otimes v) = S(h_1^*) u \otimes h_2 v \) \hspace{1cm} (25)

Replacing these for \( \tilde{h} \) in (21), we get the corresponding conditions of invariance, which are

\[ \epsilon(h)(u, v) = ((S h_1)^* u, h_2 v) \] \hspace{1cm} (26)

and

\[ \epsilon(h)(u, v) = (S(h_1^*) u, h_2 v) \] \hspace{1cm} (27)

We will choose to work with \(*\)-representations, i.e., representations of \( \mathcal{H} \) such that

\[ (h u, v) = (u, h^* v) \] \hspace{1cm} (28)

In this case the requirement (26) is automatically fulfilled since

\[ ((S h_1)^* u, h_2 v) = (u, S h_1 h_2 v) = (u, m[S \otimes id] \Delta h]. v) = \epsilon(h)(u, v) \] \hspace{1cm} (29)

This result contrasts with the later condition (27) which is not a consequence of the star representation condition (28). Condition (27) is actually not fulfilled in our case \( (V = \mathcal{M}) \) unless we choose a vanishing scalar product.

Having decided to work with star representations of \( \mathcal{H} \), the scalar product will then be automatically invariant in the sense of requirement (26), i.e., for \( \tilde{h} \).

\subsection*{6.1 Scalar product for the representation of \( \mathcal{H} \) on \( \mathcal{M} \)}

As \( \mathcal{M} \) acts on itself by left multiplication, we will also require that this representation should also be a star representation, i.e., \((z, z') = (\mathbf{1}, z^* z') = (z^* z, \mathbf{1}) \) for any \( z, z' \in \mathcal{M} \). This last requirement implies that it is enough to know the scalar products of the form \((\mathbf{1}, z), z \in \mathcal{M} \). The condition of star representation for the action of \( \mathcal{H} \) on these particular scalar products is

\[ (h^*, \mathbf{1}, z) = (\mathbf{1}, h z), \quad h \in \mathcal{H} \] \hspace{1cm} (30)

Taking \( z = x^r y^s \) and \( h = K \) one concludes from (12) and (30) that the scalar products of the form \((\mathbf{1}, x^r y^s)\) vanish if \( r \neq s \). Taking \( h = X_+ \) one gets that the only one that does not vanish is

\[ (\mathbf{1}, x^2 y^2) \neq 0. \] \hspace{1cm} (31)

Taking \( h = X_- \) gives no further condition. Recalling that we work in a star representation for the action of \( \mathcal{M} \), we get from (31) eight additional nonvanishing scalar products. We fix their value by choosing

\[ (xy, xy) = 1. \] \hspace{1cm} (32)

\section{Covariant differential calculus on \( \mathcal{M} \)}

We will use the covariant differential calculus of [6]. This calculus is built taking the Manin dual of the quantum plane [12] as the algebra of differentials, the algebra between the differentials and quantum plane variables been obtained requiring quadraticity, covariance and consistency. These relations are

\[ \begin{align*}
xy &= q yx, \\
x dx &= q^2 dx x, \\
x dy &= q dy x + (q^2 - 1) dx y, \\
y dx &= q dx y, \\
y dy &= q^2 dy y, \\
dx^2 &= dy^2 = 0, \\
dx dy + q^2 dy dx &= 0. 
\end{align*} \] \hspace{1cm} (33)

In order to extend this algebra to a differential calculus on the reduced quantum plane \( \mathcal{M} \), we should check that the ideal employed in defining \( \mathcal{M} \) is a differential one. Indeed, it is simple to verify that

\[ d(x^3) = 0 = d(y^3). \] \hspace{1cm} (34)
This differential algebra, $\Omega_{WZ}(M)$, is graded with the decomposition
\[
\Omega_{WZ}(M) = \bigoplus_{n=0}^{2} \Omega_{WZ}^n(M) ,
\]
where
\[
\Omega_{WZ}^0(M) = M , \quad \Omega_{WZ}^1(M) = M \, dx \oplus M \, dy , \quad \Omega_{WZ}^2(M) = M \, dx \, dy .
\]

8 The action of $\mathcal{H}$ on $\Omega_{WZ}(M)$

8.1 The action of $\mathcal{H}$ on $\Omega_{WZ}^0(M) = M$

This action is given in [13]. From that table and the representation theory of $\mathcal{H}$ (see the summary at the end of Section 3), we see that $x^2$, $xy$, $y^2$ span the 3-dimensional representations $3^e \subset 6_e$ (which we will call simply $3_e$). Moreover, $x, y, x^2 y^2$ span one of the 3-dimensional representations $3^o \subset 6_o$ (which we will call simply $3_o$). The above mentioned action is represented schematically in the following figures, where dashed arrows stand for the action of $X_-$ and continuous ones for $X_+$:

Note that $3_o$ contains an irreducible 2-dimensional representation spanned by $x, y$, and that $3_o$ contains an irreducible 1-dimensional representation spanned by $1$.

8.2 The action of $\mathcal{H}$ on $\Omega_{WZ}^1(M)$

First we consider the action of $\mathcal{H}$ on the Manin dual $M^!$ of $M$, spanned by $dx$ and $dy$. We note that the right coaction of $\mathcal{F}$ on $M^!$ is the same as in [1], replacing $x, y$ by $dx, dy$ [12]. Indeed, one can alternatively define $\text{Fun}(SL_2(C))$ by requiring only [1] and the invariance of the algebraic relations of the Manin dual $M^!$ under the above mentioned coaction. The left action of $\mathcal{H}$ on $M^!$ is again defined by the relation [14], hence we get:
\[
K \, dx = q \, dx , \quad K \, dy = q^{-1} \, dy ; \quad X_+ \, dx = 0 , \quad X_+ \, dy = dx ; \quad X_- \, dx = dy , \quad X_- \, dy = 0 .
\]

This action corresponds to the irreducible representation 2 of $\mathcal{H}$. Therefore, in order to know the transformation properties of a 1-form we should decompose the tensor products $3^e \otimes 2$, $3^e \otimes 2$ and $3^o \otimes 2$ into direct sums of the indecomposable representations. These tensor products are constructed using the coproduct. The result for these decompositions follows.

- The case $3^o \otimes 2 = 3^e \otimes 3^e$. 

| $K$ | $X_+$ | $X_-$ |
|-----|------|------|
| $x^2 y \, dx$ | $q^2 x^2 y \, dx$ | $-q^2 x^2 y^2 \, dx + x^2 y \, dy$ |
| $xy^2 \, dx$ | $xy^2 \, dx$ | $-x^2 y \, dx$ |
| $1 \, dx$ | $q \, dx$ | $0$ |
| $x^2 y \, dy$ | $x^2 y \, dy$ | $q^2 \, dy + qx^2 y \, dx$ |
| $xy^2 \, dy$ | $q xy^2 \, dy$ | $-x^2 y \, dy + q^2 xy^2 \, dx$ |
| $1 \, dy$ | $q^2 \, dy$ | $0$ |
This indeed gives $3_i \oplus 3_o$, since, up to multiplicative factors,

\[
\begin{align*}
q x^2 y \, dx - dy & \rightarrow 0 \\
-x^2 y \, dy + q^2 x y^2 \, dx & \oplus \\
-q \, dx + q \, x y^2 \, dy & \rightarrow 0
\end{align*}
\]

\[
\begin{align*}
& dy \rightarrow 0 \\
& x^2 y \, dy + q x y^2 \, dx \\
& dx \rightarrow 0
\end{align*}
\]

Notice that $3_o$ contains an irreducible 2-dimensional representation spanned by $\{dx, dy\}$.

• The case $3_e \otimes 2 = 3_i \oplus 3_o$.


\[
\begin{array}{c|cc|c}
\text{K} & \text{X+} & \text{X-} \\
\hline
x \, dx & q^4 x \, dx & 0 & q^4 y \, dx + x \, dy \\
y \, dx & y \, dx & x \, dx & y \, dy \\
x^2 y^2 \, dx & q x^2 y^2 \, dx & -q y \, dx & -x \, dx + x^2 y^2 \, dy \\
x \, dy & x \, dy & q x \, dx & q y \, dy \\
y \, dy & q y \, dy & x \, dy + q^2 y \, dx & 0 \\
x^2 y^2 \, dy & q^2 x^2 y^2 \, dy & -q y \, dy + x^2 y^2 \, dx & -q^2 x \, dy
\end{array}
\]

This gives $3_i \oplus 3_o$, since, up to multiplicative factors,

\[
\begin{align*}
y \, dy & \rightarrow 0 \\
q x \, dy + y \, dx & \oplus \\
x^2 y^2 \, dy - x \, dx & \rightarrow 0
\end{align*}
\]

\[
\begin{align*}
& y \, dy \rightarrow 0 \\
& x^2 y^2 \, dy - x \, dx \\
& x \, dy - q y \, dx \\
& y \, dy \rightarrow 0 \\
x \, dx & \rightarrow 0 \\
y \, dy - q^2 x^2 y^2 \, dx & \rightarrow 0
\end{align*}
\]

Note that $3_o$ contains an irreducible 1-dimensional representation spanned by $x \, dy - q y \, dx$.

• The case $3_i \otimes 2 = 6_e$.


\[
\begin{array}{c|cc|c}
\text{K} & \text{X+} & \text{X-} \\
\hline
x^2 \, dx & x^2 \, dx & 0 & -q x y \, dx + x^2 \, dy \\
x y \, dx & q x y \, dx & q x^2 \, dx & y^2 \, dx + x y \, dy \\
y^2 \, dx & q^2 x y^2 \, dx & -q^2 x y \, dx & y^2 \, dy \\
x^2 \, dy & q x^2 \, dy & q^2 x^2 \, dx & -x y \, dy \\
x y \, dy & q^2 x y \, dy & q^2 x^2 \, dy + x y \, dx & q^2 y^2 \, dy \\
y^2 \, dy & q y^2 \, dy & -q^2 x y \, dy + q y^2 \, dx & 0
\end{array}
\]

This is the six-dimensional indecomposable representation $6_e$ (which is projective, cf. [6]). It contains the four-dimensional indecomposable representation $4_e$, a family of indecomposables of the type $3_e^3$, and one irreducible of dimension two, spanned by $\{-q^2 x^2 \, dy + x y \, dx, -q x y \, dy + y^2 \, dx\}$. 

7
8.3 The action of $\mathcal{H}$ on $\Omega^2_{WZ}(\mathcal{M})$

Noting that $\Omega^2_{WZ}(\mathcal{M}) = \mathcal{M} \ d\tilde{x} \ d\tilde{y}$ (and thus it is isomorphic to $\mathcal{M}$), and that $d\tilde{x} \ d\tilde{y}$ is invariant under the action of $\mathcal{H}$ we conclude that $\Omega^2_{WZ}(\mathcal{M})$ decomposes in exactly the same representations as $\mathcal{M}$.

9 Star structure on $\Omega^2_{WZ}(\mathcal{M})$

We now consider the possible stars for $\Omega^2_{WZ}(\mathcal{M})$. Since we have an action of $\mathcal{H}$ on $\Omega^2_{WZ}(\mathcal{M})$, we will look for stars compatible with this action, i.e., the ones satisfying the analog of (38) for this case,

$$h(\omega^*) = ((Sh)\omega)^*, \quad \omega \in \Omega^2_{WZ}(\mathcal{M}).$$

This determines the action of $\mathcal{H}$ on $\Omega^2_{WZ}(\mathcal{M}) \ (\simeq \Omega^2_{WZ}(\mathcal{M}))$. Moreover, we will also require the stars to respect the grading of $\Omega^2_{WZ}(\mathcal{M})$. We already know the star on $\Omega^0_{WZ}(\mathcal{M}) = \mathcal{M}$. For the case of $\Omega^1_{WZ}(\mathcal{M})$ we proceed by writing the following most general expressions for $(d\tilde{x})^*$ and $(d\tilde{y})^*$,

$$
\begin{align*}
  dx^* &= \alpha_{r_1} x^r y^s \tilde{x}^r \tilde{y}^s d\tilde{x} + \alpha_{r_2} x^r y^s \tilde{y}^r d\tilde{y} \\
  dy^* &= \beta_{r_1} x^r y^s \tilde{x}^r \tilde{y}^s d\tilde{x} + \beta_{r_2} x^r y^s \tilde{y}^r d\tilde{y}.
\end{align*}
$$

Replacing this in (37) we obtain equations for the coefficients $\alpha$ and $\beta$. Furthermore, imposing that

$$x^* d\tilde{x}^* = q d\tilde{x}^* x^*,$$

which is the star of one of the relations in (38), it can be seen that the only solution is

$$d\tilde{x}^* = dx, \quad d\tilde{y}^* = dy.$$

We find useful to remark that in spite of being on a noncommutative space this star on $\Omega^2_{WZ}(\mathcal{M})$ shares some properties with the one for real manifolds. It can be easily checked that for all elements $z$ in $\mathcal{M}$, one has $(dz)^* = d(z^*)$. Therefore (with $\phi = z_i d\tilde{x}^i \in \Omega^1_{WZ}$), $d(\phi^*) = -(d\phi)^*$. More generally, $d(\omega^*) = (-1)^p(d\omega)^*$, when $\omega \in \Omega^p_{WZ}$.

Having a star we can write the most general hermitian one-form on $\mathcal{M}$. It is given by a (real) linear combination of the following hermitian forms:

$$
\begin{align*}
  \omega_{3i} &= \alpha_1 (q x^2 y \ d\tilde{x} - \ d\tilde{y}) + \alpha_2 (q^2 y^2 \ d\tilde{x} - x^2 y \ d\tilde{y}) + \alpha_3 (q d\tilde{x} - q x y^2 \ d\tilde{y}) \\
  \omega_{3i}' &= \alpha_1 q^2 y \ d\tilde{y} + \alpha_2 (y \ d\tilde{x} + q x \ d\tilde{y}) + \alpha_3 q y \ d\tilde{x} \\
  \omega_{3c} &= \beta_1 (q \ d\tilde{x} - x^2 y \ d\tilde{y}) + \beta_2 (q^2 y \ d\tilde{x} - q x \ d\tilde{y}) + \beta_3 d\tilde{x} \\
  \omega_{3o} &= \beta_1 q (x \ d\tilde{x} - x^2 y \ d\tilde{y}) + \beta_2 (q^2 y \ d\tilde{x} - q x \ d\tilde{y}) + \beta_3 q (q^2 y^2 \ d\tilde{x} - y \ d\tilde{y}) \\
  \omega_{6c} &= \gamma_1 (x y \ d\tilde{x} - q^2 x^2 \ d\tilde{y}) + \gamma_2 (y^2 \ d\tilde{x} - q x y \ d\tilde{y}) + \gamma_3 q x \ d\tilde{x} + \gamma_4 q y^2 \ d\tilde{y} + \\
  &\quad + \gamma_5 q^2 (x y \ d\tilde{x} + x^2 \ d\tilde{y}) + \gamma_6 q (y x \ d\tilde{y} + y^2 \ d\tilde{y})
\end{align*}
$$
The coefficients \( \alpha_i, \beta_i, \gamma_i \) are arbitrary real numbers, and the subscripts of the \( \omega \)'s refer to the indecomposable representations previously discussed.

10 Incorporation of space-time

Let \( \Lambda \) be the algebra of usual differential forms over a space-time manifold \( M \) (the De Rham complex) and \( \Omega_{WS} = \Omega_{WS}(M) \) the differential algebra over the reduced quantum plane introduced in Section 7. Remember that \( \Omega_{WS}^0 = \mathcal{M}, \Omega_{WS}^1 = \mathcal{M} \, dx + \mathcal{M} \, dy \), and that \( \Omega_{WS}^2 = \mathcal{M} \, dx \, dy \). We call \( \Xi \) the graded tensor product of these two differential algebras:

\[
\Xi = \Lambda \otimes \Omega_{WS}
\]

- A generic element of \( \Xi^0 = \Lambda^0 \otimes \Omega_{WS}^0 \) is a 3\( \times \)3 matrix with elements in \( C^\infty(M) \). It can be thought as a scalar field valued in \( M_3(\mathbb{C}) \).
- A generic element of \( \Xi^1 = \Lambda^0 \otimes \Omega_{WS}^1 = \Lambda^1 \otimes \Omega_{WS}^0 \) is given by a triplet \( \omega = \{ a_\mu, \phi^a, \phi^b \} \), where \( a_\mu \) determines a one-form (a vector field) on the manifold \( M \) with values in \( M_3(\mathbb{C}) \) (that we can consider as the Lie algebra of the Lie group \( GL(3, \mathbb{C}) \)), and where \( \phi^a, \phi^b \) are \( M_3(\mathbb{C}) \)-valued scalar fields. Indeed \( \phi^a(\mu) \, dx^\mu + \phi^b(\mu) \, dy^\mu \in \Lambda^0 \otimes \Omega_{WS}^1 \).
- A generic element of \( \Xi^2 = \Lambda^0 \otimes \Omega_{WS}^2 = \Lambda^1 \otimes \Omega_{WS}^1 = \Lambda^2 \otimes \Omega_{WS}^0 \) consists of
  - a matrix-valued 2-form \( F_{\mu\nu} dx^\mu \wedge dx^\nu \) on the manifold \( M \), i.e., an element of \( \Lambda^2 \otimes \Omega_{WS}^0 \)
  - a matrix-valued scalar field on \( M \), i.e., an element of \( \Lambda^0 \otimes \Omega_{WS}^2 \)
  - two matrix-valued vector fields on \( M \), i.e., an element of \( \Lambda^1 \otimes \Omega_{WS}^1 \)

The algebra \( \Xi \) is endowed with a differential (of square zero, of course, and obeying the Leibniz rule) defined by \( d = d \otimes id \pm id \otimes d \). Here \( \pm \) is the (differential) parity of the first factor of the tensor product upon which \( d \) is applied, and the two \( d \)'s appearing on the right hand side are the usual De Rham differential on antisymmetric tensor fields and the differential of the reduced Wess-Zumino complex, respectively.

11 Concluding remarks

Fundamental interactions are usually described by gauge theories (abelian or not) that may have two kinds of invariances: global and local symmetries. In both cases, the geometrical interpretation is clear and is ultimately described in terms of group actions (gauged or not).

On the other hand, quantum groups —either specialized at roots of unity or not— have been used many times, during the last decade, in the physics of integrable models and in conformal theories, but not in space-time physics. There is no good reason (no known reason), however, to discard quantum groups from the toolbox used to construct four-dimensional classical—or quantum—field theories. Several related attempts have been made to replace the four-dimensional space-time itself by a quantum space on which a quantum group would act, or to develop the analog of gauge theories for quantum groups. Our present goal, in contrast, is to develop a new type of field theories, where global symmetries are described by a quantum group (we use the adjective “global” since we are not at all trying to gauge —whatever it means— the action of the quantum group). This program requires a new understanding of several basic notions, including reality structure (hermiticity) and differential calculus; this was discussed here. Other aspects shall be discussed in a forthcoming review paper.

Besides casting some light on representation theory of a specific class of non semi-simple finite dimensional Hopf algebras, the present work provides a first example of a particular type of generalized differential forms, defined on a space-time manifold, and having covariance properties both with respect to the Lorentz group (or any group acting on the chosen space-time \( M \)) and with respect to a finite dimensional quantum group acting on “internal indices”. More precisely, if \( G \) is a Lie group acting on \( M \), and if \( \mathcal{U} \) denotes the enveloping algebra of its Lie algebra, our construction gives an action of the Hopf algebra \( \mathcal{U} \otimes \mathcal{H} \) on the differential algebra \( \Xi \).

As stated in the Introduction, we did not undertake in this work, the construction of any kind of Lagrangian model, but we hope that the present paper will trigger some interesting ideas in that direction.
12 Acknowledgements

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