The Gaussian Correlation Conjecture Proof

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October 31, 2013

Abstract

This paper concerns the proof of the Gaussian Correlation Conjecture in its most general form. I present the proof using the localisation on the canonical sphere.

For my wife

1 Introduction

The Gaussian Correlation Conjecture is a very captivating problem in the field of convex geometry. The standard Gaussian measure (denoted by $\gamma_n$) of any measurable subset $A \subseteq \mathbb{R}^n$ is defined by

$$\gamma_n(A) = \frac{1}{(2\pi)^{n/2}} \int_A e^{-|x|^2/2} dx.$$ 

A general mean zero Gaussian measure, $\mu_n$, defined on $\mathbb{R}^n$ is a linear image of the standard Gaussian measure. The Gaussian Correlation Conjecture is formulated as follows:

**Conjecture 1.1** For any $n \geq 1$, if $\mu$ is a mean zero Gaussian measure on $\mathbb{R}^n$, then for $K, M$, convex closed subsets of $\mathbb{R}^n$ which are symmetric around the origin, we have

$$\mu_n(K \cap M) \geq \mu_n(K) \mu_n(M).$$

A less general form of the Gaussian Correlation Conjecture first appeared in 1955 in [3] and the general form appeared a few years later in 1972 by S. Das Gupta, M.L. Eaton, I. Olkin, M. Perlman, L.J. Savage and M. Sobel in [2]. The one-dimensional case of the conjecture was proven by both Khatri and Sidak (independently) in 1966–67 in [12], [19] and [20]. The two-dimensional case was proven by Pitt in 1977 in [17]. Hargé proved the conjecture for the ellipsoids in 1999 (see [9]) and this same author continues to work on different cases of this conjecture. Additionally, Schechtman, Schlumprecht and Zinn proved some partial results in 1998 [18]. The subject is quite active and several specialists have worked on variations of the conjecture. Recently, as it was pointed out to

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me by M.Ledoux and S.Szarek, specialists have chosen to believe that this conjecture could have
counter-examples. I apologise if I don’t include the work of more authors- unfortunately the list
is prohibitavely long.

Here I give a proof of conjecture 1.1 using localisation on the Riemannian sphere. More
precisely, using spherical localisation, I bring the n-dimensional Conjecture 1.1 down to a 2-
dimensional correlation problem, which will indeed simplify the geometry and leave us with a
2-dimensional analytic problem. The main result of this paper, before proving Conjecture 1.1 will
be the following :

**Theorem 1** The Gaussian Correlation Conjecture, for every two centrally symmetric convex sets,
is true if for every two strips $S_1$ and $S_2$ in $\mathbb{R}^2$ we have:

$$
\mu_2(\mathbb{R}^2) \mu_2(S_1 \cap S_2) \geq \mu_2(S_1) \mu_2(S_2),
$$

where $\mu_2$ is the measure $C(n)|y|^{n-2}e^{-(x^2+y^2)/2}dx\,dy$ with respect to the $xy$-coordinates in $\mathbb{R}^2$ and
where

$$
C(n) = \left( \int_{-\pi/2}^{\pi/2} \cos(t)^{n-2}dt \right)^{-1}.
$$

Where the definition of the strips is as below:

**Definition 1.1 (Strips)** A set $S \subset \mathbb{R}^2$ is called a strip if $S$ is open, convex, symmetric with
respect to the origin and if a $u \in S^1$ and $h > 0$ exist such that

$$
S = \{x \in \mathbb{R}^2 : |x.u| < h\}.
$$

$h$ is the width of the strip and $u$ is the unit vector of the axis of the strip. The angle $\theta$ of two
strips $S$ and $S'$ is equal to the angle between the respective unit vectors of the axis of the strips.

I strongly believe that the if in Theorem 1 can be replaced by if and only if, but I am not able to
give a complete proof for this. I will cover this issue in more detail in Section 5. The organisation
of this paper is as follows : in Section 3, I recall the theory of convexly-derived measures and the
localisation on the canonical Riemannian sphere. Section 4 deals with a 2-dimensional correlation
type problem. In Section 5, I prove Theorem 1 and finally Section 6 completes the proof of
Conjecture 1.1

2 Acknowledgement

It should be mentioned that I first learned of this conjecture via Olivier Guédon, whom I thank. I
am also grateful to M.Ledoux who pointed out that I ought to look at [4]. This paper took shape
through the fantastic suggestions of the referee which I sincerely appreciate. Finally, I deeply
thank my wife for her genuine and constant support. Without her (and her red pen) this project
would have been impossible.

3 Localisation on the Sphere

In the past few years, localisation methods have been used to prove very interesting geometric
inequalities. In [13] and [11] the authors prove integral formulae using localisation, and apply
their methods to conclude a few isoperimetric type inequalities concerning the convex sets in the Euclidean space. In [5] the authors study a functional analysis version of the localisation used again on the Euclidean space. The localisation on more general spaces was studied in [8], [7], [15], and [16]. It may seem hard to believe that one may prove Conjecture 1.1 using localisation methods, but this is the path we will take! We shall begin with some definitions:

**Definition 3.1 (Convexly-derived measures)** A convexly-derived measure on $\mathbb{S}^n$ (resp. $\mathbb{R}^n$) is a limit of a vaguely converging sequence of probability measures of the form $\mu_i = \frac{\text{vol}(S_i)}{\text{vol}(S)}$, where $S_i$ are open convex sets.

This class of measures was defined first in [8] and used later on in [1], [15], [14]. In Euclidean spaces, a convexly-derived measure is simply a probability measure supported on a convex set which has a $x^k$-concave density function with respect to the Lebesgue measure. To understand convexly-derived measures on the sphere we need some definitions:

**Definition 3.2 (sin-concave functions)** A real function $f$ (defined on an interval of length less than $2\pi$) is called sin-concave, if, when transported by a unit speed paramatrisation of the unit circle, it can be extended to a 1-homogeneous and concave function on a convex cone of $\mathbb{R}^2$.

**Definition 3.3 (sin$^k$-affine functions and measures)** A function $f$ is affinely sin$^k$-concave if $f(x) = A\sin^k(x + x_0)$ for a $A > 0$ and $0 \leq x_0 \leq \pi/2$. A sin$^k$-affine measure by definition is a measure with a sin$^k$-affine density function.

**Definition 3.4 (sin$^k$-concave functions)** A non-negative real function $f$ is called sin$^k$-concave if the function $f^{1/k}$ is sin-concave.

One can easily check the following:

**Lemma 3.1** A real non-negative function defined on an interval of length less than $\pi$ is sin$^k$-concave if for every $0 < \alpha < 1$ and for all $x_1, x_2 \in I$ we have

$$f^{1/k}(\alpha x_1 + (1 - \alpha)x_2) \geq \left(\frac{\sin(\alpha |x_2 - x_1|)}{\sin(|x_2 - x_1|)}\right)f(x_1)^{1/k} + \left(\frac{\sin((1 - \alpha)|x_2 - x_1|)}{\sin(|x_2 - x_1|)}\right)f(x_2)^{1/k}.$$  

Particularly if $\alpha = \frac{1}{2}$ we have

$$f^{1/k}(\frac{x_1 + x_2}{2}) \geq \frac{f^{1/k}(x_1) + f^{1/k}(x_2)}{2\cos(|x_2 - x_1|)}.$$  

One can use Lemma 3.1 as definition of sin$^k$-concave functions.

**Lemma 3.2** Let $S$ be a geodesically convex set of dimension $k$ of the sphere $\mathbb{S}^n$ with $k \leq n$. Let $\mu$ be a convexly-derived measure defined on $S$ (with respect to the normalised Riemannian measure on the sphere). Then $\mu$ is a probability measure having a continuous density $f$ with respect to the canonical Riemannian measure on $\mathbb{S}^k$ restricted to $S$. Furthermore, the function $f$ is sin$^{n-k}$-concave on every geodesic arc contained in $S$.

The above Lemma, proved in [15], completely characterises the class of convexly-derived measures on the sphere. Note the similarity between the Euclidean case and the spherical one.
3.1 Some Useful Properties of $\sin^k$-Concave Measures/Functions

Here I reveal a few properties of $\sin^k$-concave measures and functions which will become useful later on in this paper, and more specifically in the proof of Theorem 2. I shall show the results in the following Lemma, and invite the reader to consult [15] for a complete proof. From now on, let $\nu$ be a general $\sin^k$-concave measure and let $f$ be its density function. Suppose $f$ is defined on a closed interval of $\mathbb{R}$. Then

**Lemma 3.3**

- $f$ admits only one maximum point and does not have any local minima.
- If $f$ is $\sin$-concave and defined on an interval containing 0, then $g(t) = f(|t|)$ is also $\sin$-concave.
- Let $0 < \varepsilon < \pi/2$. Let $\tau > \varepsilon$. $f$ is defined on $[0, \tau]$ and attains its maximum at 0. Let $h(t) = c \cos(t)^k$ where $c$ is chosen such that $f(\varepsilon) = h(\varepsilon)$. Then
  \[
  \begin{cases}
  f(x) \geq h(x) & \text{for } x \in [0, \varepsilon], \\
  f(x) \leq h(x) & \text{for } x \in [\varepsilon, \tau].
  \end{cases}
  \]
  In particular, $\tau \leq \pi/2$.
- Let $\tau > 0$ and $f$ be a nonzero non-negative $\sin^k$-concave function on $[0, \tau]$ which attains its maximum at 0. Then $\tau \leq \pi/2$ and for all $\alpha \geq 0$ and $\varepsilon \leq \pi/2$ we have
  \[
  \int_0^{\min\{\varepsilon, \tau\}} f(t) dt \geq \int_0^{\pi/2} \cos(t)^k dt.
  \]

3.2 A Fundamental Spherical Localisation Lemma

Here I prove the main result of this section. It is a spherical localisation Lemma enabling us to simplify the $n$-dimensional Gaussian Correlation Conjecture to a 2-dimensional correlation problem that still retains its pleasing properties, as we shall see later.

**Lemma 3.4** Let $f_1$, $f_2$, $f_3$, and $f_4$ be four continuous non-negative functions on $\mathbb{S}^n$. Suppose that for every $u \in \mathbb{S}^n$ we have $f_1(u).f_2(u) \leq f_3(u).f_4(u)$. Let $\mu$ denote the normalised Riemannian measure of $\mathbb{S}^n$. If for every probability measure $\nu$ having a $\sin^{n-1}$-affine density function and supported on a geodesic segment $I$ we have
  \[
  \left( \int_I f_1(t) d\nu(t) \right) \left( \int_I f_2(t) d\nu(t) \right) \leq \left( \int_I f_3(t) d\nu(t) \right) \left( \int_I f_4(t) d\nu(t) \right),
  \]
  where the geodesic segment $I$ is parametrised by its arc length $t$ and $f(t)d\nu(t) = f(t)g(t)dt$,
  \[
  g(t) \text{ is the density of the measure } \nu,
  \]
  then we will have:
  \[
  \left( \int_{\mathbb{S}^n} f_1(u) d\mu(u) \right) \left( \int_{\mathbb{S}^n} f_2(u) d\mu(u) \right) \leq \left( \int_{\mathbb{S}^n} f_3(u) d\mu(u) \right) \left( \int_{\mathbb{S}^n} f_4(u) d\mu(u) \right).
  \]
This Lemma is a spherical version of the Localisation Lemma and its corollary, both proved in [13] in the Euclidean case. I first prove the following:

**Lemma 3.5** Let $G_i$ for $i = 1, 2$ be two continuous functions on $\mathbb{S}^n$ such that

$$\int_{\mathbb{S}^n} G_i(u) d\mu(u) > 0,$$

then a $\sin^{n-1}$-affine probability measure $\nu$ supported on a geodesic segment $I$ exists such that

$$\int_I G_i(t) d\nu(t) > 0.$$

**Proof of Lemma 3.5**

We construct a decreasing sequence of convex subsets of $\mathbb{S}^n$ by the following procedure:

- Define the first step cutting map $F_1 : \mathbb{S}^n \rightarrow \mathbb{R}^2$ by

  $$F_1(x) = \left( \int_{x^\vee} G_1(u) d\mu(u), \int_{x^\vee} G_2(u) d\mu(u) \right)$$

  where $x^\vee$ denotes the (oriented) open hemi-sphere pointed by the vector $x$. Apply Borsuk-Ulam Theorem to $F_1$. Hence there exists a $x_1^\vee$ such that

  $$\int_{x_1^\vee} G_1(u) d\mu(u) = \int_{-x_1^\vee} G_1(u) d\mu(u)$$

  $$\int_{x_1^\vee} G_2(u) d\mu(u) = \int_{-x_1^\vee} G_2(u) d\mu(u).$$

  Choose the hemi-sphere, denoted by $x_1^\vee$. Set $S_1 = x_1^\vee \cap \mathbb{S}^n$.

- Define the $i$-th step cutting map by

  $$F_i(x) = \left( \int_{S_i-1 \cap x^\vee} G_1(u) d\mu(u), \int_{S_i-1 \cap x^\vee} G_2(u) d\mu(u) \right).$$

  Applying Borsuk-Ulam Theorem to $F_i$ we get two new hemi-spheres and we choose the one, denoted by $x_i^\vee$. Set $S_i = x_i^\vee \cap S_{i-1}$.

This procedure defines a decreasing sequence of convex subsets $S_i = x_i^\vee \cap S_{i-1}$ for every $i \in \mathbb{N}$. Set:

$$S_\pi = \bigcap_{i=1}^{\infty} (S_i) = \bigcap_{i=1}^{\infty} \text{clos}(S_i),$$

where $\text{clos}(A)$ determines the topological closure of the subset $A$. Furthermore, a convexly derived probability measure $\nu_\pi$ is defined on $S_\pi$. Since $\lim_{i \to \infty} S_i = S_\pi$ (this limit is with respect to Hausdorff topology) the definition of the convexly-derived measures can be applied to define the positive probability measure supported on $S_\pi$ by

$$\nu_\pi = \lim_{i \to \infty} \frac{\mu|S_i}{\mu(S_i)},$$
Hence by the definition of $\nu_\pi$

$$\int_{S_\pi} G_j(x)d\nu_\pi(x) = \lim_{i \to \infty} \int_{S_i} \frac{G_j(x)d\mu(x)}{\mu(S_i)}$$

for $j = 1, 2$, and where the limit is taken with respect to the vague topology defined on the space of convexly-derived measures (see [13]). I recall the following:

**Lemma 3.6** (See [10]). Let $\mu_i$ be a sequence of positive Radon measures on a locally compact space $X$ which vaguely converges to a positive Radon measure $\mu$. Then for every relatively compact subset $A \subset X$ such that $\mu(\partial A) = 0$,

$$\lim_{i \to \infty} \mu_i(A) = \mu(A).$$

By the definition of the cutting maps $F_i(x)$, for every $i \in \mathbb{N}$, $j = 1, 2$ we have

$$\int_{S_i} G_j(u)d\mu(u) > 0.$$  

By applying Lemma 3.6, we conclude that the convexly derived probability measure defined on $S_\pi$ satisfies the assumption of the Lemma 3.5. The dimension of $S_\pi$ is $< n$. Indeed, if it is not the case, then $\text{dim}S_\pi = n$. Since there is a convexly-derived measure with positive density defined on $S_\pi$, and by the construction of the sequence $\{S_i\}$ for every open set $U$ we have

$$\nu_\pi(S_\pi \cap U) = \lim_{i \to \infty} \frac{\mu(S_i \cap U)}{\mu(S_i)} = \lim_{i \to \infty} \frac{\mu(S_\pi \cap U)}{2\mu(S_i)}.$$  

By supposition on the dimension of $S_\pi$, the right hand equality is equal to zero and this is a contradiction with the positive measure $\nu_\pi$ charging mass on $S_\pi \cap U$.

Thus $\text{dim}S_\pi < n$. If $\text{dim}S_\pi = 1$ then Lemma 3.5 is proved. Note that $\text{dim}S_\pi$ can not be equal to zero, since the cutting map in each step cuts the set $S_i$. If $\text{dim}S_\pi = k > 2$ we define a new procedure by replacing $S$ with $S_\pi \cap S$, replacing the normalized Riemannian measure by the measure $\nu_\pi$, and replacing the sphere $S^n$ by the sphere $S^k$ containing $S_\pi$. For this new procedure, in every step we define new cutting maps. Since $k > 2$, by using Borsuk-Ulam Theorem, we get hyperspheres $(\mathbb{S}^{k-1})$ halving the desired (convexly derived) measures. The new procedure defines a new sequence of convex subsets and, by the same arguments given before, a convexly derived measure defined on the intersection of this new sequence satisfying the assumption of the Lemma 3.5. By the same argument as above, the dimension of the intersection of the decreasing sequence of convex sets is $< k$. If the dimension of the intersection of this new sequence is equal to 1, we are done. If not, we repeat the above procedure until arriving to a 1-dimensional set. This proves that a probability measure $\nu$ with a (non-negative) $\sin^{n-1}$-concave density function $f$, supported on a geodesic segment $I$ exists such that:

$$\int_I f(t)G_i(t)dt \geq 0.$$  

(1)

We determine $I$ to have minimal length. If $f$ is $\sin^{n-1}$-affine on $I$ then we are done. We suppose this is not the case. We choose a subinterval $J \subset I$, maximal in length, such that a $\sin^{n-1}$-concave
function $f$ satisfying (1) exist such that $f$ additionally is $\sin^{n-1}$-affine on the subinterval $J$. The existence of such $J$ and $f$ follows from a standard compactness argument. We can assume that the length of $I$ is $< \pi/2$. Consider the Euclidean cone over $I$. Let $a, b \in I$ be the end points of $I$ and take the Euclidean segment $[a, b]$ in $\mathbb{R}^2$ (basically the straight line joining $a$ to $b$). By definition of $\sin^{n-1}$-concave functions, the function $f$ is the restriction of a one-homogeneous $x^{n-1}$-concave function $F$ on the circle (a $x^{n-1}$-concave function $F$ is a function such that $F^{1/(n-1)}$ is concave). Transporting the entire problem on $\mathbb{R}^n_{+1}$, we started with two homogeneous functions $\bar{G}_i$ on $\mathbb{R}^n_{+1}$ such that \( \int_{\mathbb{R}^n_{+1}} \bar{G}_i \mathbb{d}x > 0 \) and we proved that there exists a 2-dimensional cone over a segment $[a, b]$, a one-homogeneous $x^{n-1}$-concave function $F$ on $[a, b]$ and a subinterval $[\alpha, \beta] \subset [a, b]$ such that $F^{1/(n-1)}$ is linear on $[\alpha, \beta]$ (this due to the fact that by definition, the restriction of a one-homogeneous $x^{n-1}$-affine function on a 2-dimensional Euclidean cone defines a $\sin^{n-1}$-affine function on the circle) and such that \( \int_{[a, b]} \bar{G}_i(t) F(t) \mathbb{d}t \geq 0. \)

We mimick the same arguments given in [13] (pages 21 – 23) (with the only difference that every construction there drops one dimension). This drop of dimension is due to the fact that every construction has to preserve the homogeneity, or in other words, one dimension has to be preserved for the 2-dimensional cone defined on $[a, b]$. Hence the proof of Lemma 3.5 follows.

\[ \square \]

**Proof of Lemma 3.4**

We prove this Lemma by contradiction, if we have :

\[
\left( \int_{\mathbb{S}^n} f_1(u) d\mu(u) \right) \left( \int_{\mathbb{S}^n} f_2(u) d\mu(u) \right) > \left( \int_{\mathbb{S}^n} f_3(u) d\mu(u) \right) \left( \int_{\mathbb{S}^n} f_4(u) d\mu(u) \right).
\]

We can assume that for $i = 1, 2, 3, 4$, $\int_{\mathbb{S}^n} f_i(u) d\mu(u) > 0$. Fix a constant $C$ such that

\[
\frac{\int_{\mathbb{S}^n} f_1(u) d\mu(u)}{\int_{\mathbb{S}^n} f_3(u) d\mu(u)} > C > \frac{\int_{\mathbb{S}^n} f_4(u) d\mu(u)}{\int_{\mathbb{S}^n} f_2(u) d\mu(u)} > 0.
\]

Then we have

\[
\int_{\mathbb{S}^n} (f_1(u) - Cf_3(u)) d\mu(u) > 0
\]

and on the other hand

\[
\int_{\mathbb{S}^n} (Cf_2(u) - f_4(u)) d\mu(u) > 0.
\]
Applying Lemma 3.5, we find a geodesic segment $I \subset S^n$ and a $\sin^{n-1}$-affine probability measure defined on $I$ such that

$$\int_I (f_1(t) - Cf_3(t))d\nu(t) > 0$$

and

$$\int_I (Cf_2(t) - f_4(t))d\nu(t) > 0.$$

From the above inequalities, we find that for $j = 1, 2$ we have

$$\int_I f_j(t)d\nu(t) > 0,$$

and then:

$$\left(\int_I f_1(t)d\nu(t)\right)\left(\int_I f_2(t)d\nu(t)\right) > C\left(\int_I f_3(t)d\nu(t)\right)\left(\int_I f_2(t)d\nu(t)\right) > \left(\int_I f_1(t)d\nu(t)\right)\left(\int_I f_2(t)d\nu(t)\right),$$

which is a contradiction to the assumption of Lemma 3.4. This ends the proof of our localisation Lemma 3.4.

\[\square\]

### 4 A 2-dimensional Correlation Problem

In this section, I present a correlation problem in $\mathbb{R}^2$. The ideas of this section are to be compared with the ideas from [4]. (This was pointed out to me by M. Ledoux). I shall use almost the same notations which are used in [4] for the rest of this section.

**Definition 4.1 (Strips)** A set $S \subset \mathbb{R}^2$ is called a strip if $S$ is open, convex, symmetric with respect to the origin and if a $u \in S^1$ and $h > 0$ exist such that

$$S = \{x \in \mathbb{R}^2 : |x.u| < h\}.$$  

$h$ is the width of the strip and $u$ is the unit vector of the axis of the strip. The angle $\theta$ of two strips $S$ and $S'$ is equal to the angle between the respective unit vectors of the axis of the strips.

**Definition 4.2 (Angular-Length function)** Let $E$ be an open set containing the origin. The function $\theta_E : (0, \infty) \to [0, \pi/2]$ is defined as

$$\theta_E(r) = \frac{1}{4} \frac{H^1(E \cap \partial B_r)}{r},$$

where $H^1$ stands for the 1-dimensional Hausdorff measure.

It is clear that for every $r > 0$, $\theta_E(r) \leq \pi/2$.  

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**Definition 4.3 (Width decreasing sets)** A set $E \subset \mathbb{R}^2$ is said to be width-decreasing if $E$ is open, contains the origin and is symmetric with respect to it, and for every $r > 0$ if $\theta_E(r) < \pi/2$ then $\theta_E \leq \theta_S$ on $(r, \infty)$ where $S$ is any strip for which $\theta_E = \theta_S(r)$.

Denote $\mathbb{S}^1_+$ by a half unit circle. A $\sin^{n-1}$-affine measure has its support on a half circle $\mathbb{S}^1_+$. We extend this measure on the whole $\mathbb{S}^1$ in such a way that for every $u \in \mathbb{S}^1$, $g(u) = g(-u)$ where $g$ is the density function of this measure. In the next theorem, we are supposing that the $\sin^{n-2}$-concave measures and functions are extended to $\mathbb{S}^1$.

**Theorem 2** Let $\mu_2$ be the measure defined as $r^{n-1}e^{-r^2/2}g(\theta)dr \wedge d\theta$ in polar coordinates of $\mathbb{R}^2$ where $g(\theta)d\theta$ is a $\sin^{n-2}$-affine measure. Let $\nu_1 = g(\theta)d\theta$ be the (extended) $\sin^{n-2}$-affine measure on $\mathbb{S}^1$. Let $K_1$ and $K_2$ be two centrally symmetric convex bodies in $\mathbb{R}^2$. Then two strips $S_1$ and $S_2$ in $\mathbb{R}^2$ exist such that

\[
\frac{\mu_2(K_1 \cap K_2)}{\mu_2(K_1) \mu_2(K_2)} \geq \frac{\mu_2(S_1 \cap S_2)}{\mu_2(S_1) \mu_2(S_2)},
\]

**Proof of Theorem 2**

According to Lemma 3.3, the function $g(t)$ has a unique maximum point. Let the $xy$-coordinate be such that the point $(0, 1)$ coincides with the maximum point of $g(t)$. We assume $\mathbb{S}^1$ is parametrised canonically by $\theta$. Therefore $\theta = 0$ corresponds to $(1, 0)$ and $\theta = \pi/2$ corresponds to the maximum point of the function $g$. For every $r > 0$ let $I_{K_i}(r) = \text{pr}(\partial B(0, r) \cap K_i)$ for $i = 1, 2$ and where $\text{pr}$ is the radial projection of $\partial B(0, r)$ to $\mathbb{S}^1$. Let $\mathcal{M}_0$ be the set of all open sets containing the origin and symmetric with respect to it. Let $\theta \in \mathbb{S}^1$. We define a map $s_\theta : \mathcal{M}_0 \to \mathcal{M}_0$ which we call the $\theta$-double cap symmetrisation map and is defined as follows: let $E \in \mathcal{M}_0$, define

\[
s_\theta(E) = \bigcup_{r > 0}\{re^{i\phi} : |\phi - \theta| \leq \varepsilon_E(r)\} \cup \{re^{i\phi} : |\phi - (\theta + \pi)| \leq \varepsilon_E(r)\},
\]

where $re^{i\phi} = (r \cos(\phi), r \sin(\phi)) \in \mathbb{R}^2$ and $\varepsilon_E(r)$ is such that

\[
\int_{I_{E}(r)} g(t) dt = 2 \int_{\theta-\varepsilon_E(r)}^{\theta+\varepsilon_E(r)} g(t) dt.
\]

Unfortunately, the image of the set of convex sets (or width-decreasing sets) by $s_\theta$ is not necessarily the set of convex sets (or width-decreasing sets). However one has the following:

**Lemma 4.1** Let $E$ be a symmetric convex set containing the origin. Let $\theta$ be such that for every $r > 0$ we have

\[
\varepsilon_E(r) \leq \theta_E(r),
\]

then $s_\theta(E)$ is a width-decreasing set.

**Proof of Lemma 4.1**

Define (as in [4]):

\[
\varepsilon'_E(r) = \lim_{\delta \to 0^+} \varepsilon_E(r + \delta) - \varepsilon_E(r) \over \delta.
\]
It is shown in [4] that in order for $s_\theta(E)$ to be a width-decreasing set, it is sufficient to show that for every $r > 0$ such that $\varepsilon_E(r) < \pi/2$, we have
\[ \varepsilon'_E(r) \leq -\frac{\tan(\varepsilon_E(r))}{r} \leq 0. \]

Let $r > 0$ be such that $\theta_E(r) < \pi/2$ and let
\[ pr(\partial B(0, r) \cap E) = \bigcup_{i=1}^{N} I_i \cup J_i, \]
where $I_i$ is a subarc of $S^1$ and $J_i = \{-x|x \in I_i\}$.

\[ \int_{I_{E}(r)} g(t)dt = \sum_{i=1}^{N} \int_{I_i} g(t)dt. \]

Let $P = (0, 1) \in \mathbb{R}^2$. Since the function $g(t)$ is supposed to be $\sin^n$-affine, without loss of generality we can suppose that the circle $S^1 - \{P\}$ is the interval $[-\pi, \pi]$ and the function $g(t) = \cos(t)^n$. The maximum point of the function $g(t)$ corresponds to the point $P = (0, 1) \in \mathbb{R}^2$ which also corresponds to $t = 0$ in the interval $[-\pi, \pi]$.

There exist $a, a_i, b_i$ for $i \in \{1, \cdots, N\}$ such that
\[ \int_{a + \varepsilon(r)}^{a - \varepsilon(r)} \cos(t)^n dt = \sum_{i=1}^{N} \int_{a_i}^{b_i} \cos(t)^n dt. \]

For every $r > 0$ such that there exists an interval $I_j = [a_j, b_j]$ such that
\[ |\cos(a_j)^n - \cos(b_j)^n| \geq |\cos(\varepsilon - a)^n - \cos(\varepsilon + a)^n|, \]
we get:
\[ \varepsilon'(r) \leq \theta'_j(r) \leq -\frac{\tan(\theta)}{r} \leq -\frac{\tan(\varepsilon)}{r}. \]

For every $r > 0$ such that such an interval $I_j$ (for every $j \in \{1, \cdots, N\}$) satisfying the above inequality of $2$ does not exist, thanks to the $\sin^n$-concavity of $\cos(x)^n$ (and to $\theta \geq \varepsilon$), we have:
\[ \varepsilon'(r) \leq \sum_{i=1}^{N} \theta'_i(r)|\cos(a_i)^n - \cos(b_i)^n| \leq \sum_{i=1}^{N} N \tan(\theta)|\cos(a_i)^n - \cos(b_i)^n| \leq -\frac{N \tan(\theta)}{r}|\cos(a + \varepsilon)^n - \cos(a - \varepsilon)^n| \leq -\frac{\tan(\varepsilon)}{r}. \]

This ends the proof of the Lemma 4.1.
According to Lemma 4.1 and Lemma 3.3, for the vertical double cap symmetrisation (i.e. $s_{\pi/2}$), we are sure that the image of a convex set under $s_{\pi/2}$ is always a width-decreasing set.

Coming back to the proof, we apply the vertical double cap symmetrisation map, $s_{\pi/2}$ to $K_i$ for $i = 1, 2$. Let $r_i$ be the radius of the biggest disk inscribed in $K_i$. By definition, it is clear that for $r \geq r_i$, we have

$$\theta_i(r) \leq \theta_{S_i}(r),$$

where $S_i$ is a strip of width equal to $r_i$. Remember that the measure which with we work is not rotationally invariant, so the measure of strips of equal width but different axis are all different. Since we adjust the maximum of the density to be on the point $(1, 0)$, the vertical strip of width equal to $r_i$ has the largest measure among other strips of width equal to $r_i$, where the unit vector of the axis of the strip varies from $\theta = \pi$ to $\pi/2$. It is clear by Lemma 3.3 that for $\pi/2 \leq \theta_1 \leq \theta_2 \leq \pi$ we have $s_{\pi/2}(S_{\theta_2}) \subseteq s_{\pi/2}(S_{\theta_1})$.

Define now the following procedure (algorithm):

- First step: If $s_{\pi}(K_i)$ is width decreasing for $i = 1$ or $i = 2$, end this procedure by performing the following symmetrisation: $s_{\pi}(K_i)$ and $s_{\pi/2}(K_j)$ for $i \neq j$. Otherwise, go to the next step.

- Second step: Set $\theta_i$ to be the largest $\pi/2 \leq \theta \leq \pi$ such that $s_{\pi/2}(K_i) \subset s_{\pi/2}(S_{\theta, r_i})$ for $i = 1, 2$. Perform the following symmetrisation: $s_{\theta_i}(K_i)$ and $s_{\pi/2}(K_j)$ for $i \neq j$. If

$$\mu_2(K_1 \cap K_2) \geq \mu_2(s_{\pi/2}(K_1) \cap s_{\theta_2}(K_2)),$$

stop this procedure. Otherwise, go to the next step.

- Third step: Set $\theta_j$ to be the smallest $0 \leq \theta \leq \pi/2$ such that $s_{\pi/2}(K_j) \subset s_{\pi/2}(S_{\theta_j, r_j})$. Perform the following symmetrisation: $s_{\theta_i}(K_i)$ and $s_{\theta_j}(K_j)$ for $i \neq j$. Stop here.

We denote the results after the above procedure by $s_i(K_i)$ for $i = 1, 2$. It is clear, according to Lemma 4.1 that in every step, the symmetric sets defined after the cap symmetrisation operation remain width-decreasing sets. What is important is to compare the measure of the intersection after the symmetrisation procedure, and the measure of the intersection before this procedure. This is provided by the following:

**Lemma 4.2** For every step of the previous procedure, we have:

$$\mu_2(K_1 \cap K_2) \geq \mu_2(s_1(K_1) \cap s_2(K_2))$$

**Proof of Lemma 4.2**

Suppose the procedure ends at the first step. In this case, the lemma is proved by the inclusion-exclusion principle. Indeed, in case for a $r > 0$ we have

$$\nu_1(s_1(I_1(r)) \cap s_2(I_2(r))) \neq 0,$$
reminding that \( \nu_1(I_1(r)) = \nu_1(s_i(I_1(r))) \), we have
\[
\nu_1(s_1(I_1(r)) \cap s_2(I_2(r))) = \nu_1(s_1(I_1(r))) + \nu_1(s_2(I_2(r))) - 1 \\
\leq \nu_1(I_1(r)) + \nu_1(I_2(r)) - \nu_1(I_1(r) \cup I_2(r)) \\
= \nu_1(I_1(r) \cap I_2(r)).
\]
This proves the lemma for the first procedure.

For the second procedure, the lemma is settled by definition. It remains to prove this lemma for the third step.

It is clear that if \( \theta_j = \pi/2 - \theta_i \) then the proof is similar to the first step via the inclusion-exclusion principle. Suppose then \( \theta_i - \theta_j \leq \pi/2 \) and suppose by contradiction that we have:
\[
\mu_2(K_1 \cap K_2) \leq \mu_2(s_1(K_1) \cap s_2(K_2)).
\]
This means that there exists a \( r > 0 \) such that \( \varepsilon_1(r) + \varepsilon_2(r) \leq \pi/2 \) and such that
\[
\nu_1(I_1(r) \cap I_2(r)) \leq \nu_1(s_1(I_1(r)) \cap s_2(I_2(r))).
\]
Therefore two sets \( J_r \) and \( J_r' \) exist such that
\[
J_r \subset s_1(I_1(r)) \cap s_2(I_2(r)),
\]
and
\[
J_r' \subset (s_1(I_1(r)) \cup s_2(I_2(r)))^c,
\]
where \( A(r)^c \) is the complementary of the set \( A(r) \) in \( \partial B(0,r) \) and \( J_r' \subset I_1(r) \) and \( J_r' \not\subset I_2(r) \) and \( \nu_1(J_r) = \nu_1(J_r') \). This fact shows that we can find \( \theta < \theta_1 \) such that the cap symmetrisation \( s_\theta(K_1) \) is a width-decreasing set, which is a contradiction with the definition of \( \theta_1 \).

Then the proof of Lemma 4.2 follows.

\[\square\]

At this point, we have succeeded to prove the following inequality:
\[
\frac{\mu_2(K_1 \cap K_2)}{\mu_2(K_1)\mu_2(K_2)} \geq \frac{\mu_2(s_1(K_1) \cap s_2(K_2))}{\mu_2(s_1(K_1))\mu_2(s_2(K_2))}.
\]
Set
\[
r_0 = \inf \{ r > 0 : \theta_1(r) + \theta_2(r) \leq \pi/2 \}
\]
and without loss of generality, we could consider that \( \theta_1(r_0) + \theta_2(r_0) \leq \pi/2 \), \( 0 < \theta_1(r_0) < \pi/2 \) and \( 0 < \theta_2(r_0) < \pi/2 \).

Let \( S_1 \) be a cap such that the unit vector of its axis is given by \( \theta_1 \) and such that \( \theta S_1(r_0) = \theta_1(r_0) \). It is clear that
\[
s_1(K_1)/B(0,r_0) \subset S_1.
\]
\[
S_1 \cap B(0,r_0) \subset s_1(K_1) \cap B(0,r_0).
\]
Define $S_2$ to be a strip such that the unit vector of its axis be given by $\theta_2$ and such that $\theta_{S_2}(r_0) = \theta_2(r_0)$. Similarly we have

\begin{align*}
S_2(K_2)/B(0,r_0) &\subset S_2 \\
S_2 \cap B(0,r_0) &\subset s_2(K_2) \cap B(0,r_0).
\end{align*}

Observe that if we set

\begin{align*}
E &= (s_1(K_1) \cap B(0,r_0)) \cup (S_1/B(0,r_0)) \\
F &= (s_2(K_2) \cap B(0,r_0)) \cup (S_2/B(0,r_0)).
\end{align*}

Then we have $E \cap F = s_1(K_1) \cap s_2(K_2)$ and clearly we obtain the following inequality

\[
\frac{\mu_2(s_1(K_1) \cap s_2(K_2))}{\mu_2(s_1(K_1))\mu_2(s_2(K_2))} \geq \frac{\mu_2(E \cap F)}{\mu_2(E)\mu_2(F)}.
\]

And then

\[
\frac{\mu_2(E \cap F)}{\mu_2(E)\mu_2(F)} \geq \frac{\mu_2(E \cap F) - \mu_2(E/S_1)}{(\mu_2(E) - \mu_2(E/S_1))(\mu_2(F))} = \frac{\mu_2(S_1 \cap S_2)}{\mu_2(S_1)\mu_2(F)} \geq \frac{\mu_2(F \cap S_1) - \mu_2(F/S_2)}{\mu_2(S_1)(\mu_2(F) - \mu_2(F/S_2))} = \frac{\mu_2(S_1 \cap S_2)}{\mu_2(S_1)\mu_2(S_2)}.
\]

This ends the proof of Theorem 2.

\[\square\]

5 From the $n$-Dimensional Conjecture to a 2-Dimensional Problem

The aim of this section is to bring the $n$-dimensional Gaussian Correlation Conjecture to a 2-dimensional problem. More precisely, our goal is to prove the following:

**Theorem 3** The Gaussian Correlation Conjecture, for every two centrally symmetric convex sets, is true if for every two symmetric strips $S_1$ and $S_2$ in $\mathbb{R}^2$ we have:

\[
\mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2) \geq \mu_2(S_1)\mu_2(S_2),
\]

where $\mu_2$ is the measure $C(n)|y|^{n-2}e^{-(x^2+y^2)/2}dx\,dy$ with respect to the $xy$-coordinates in $\mathbb{R}^2$ and where

\[
C(n) = \left(\int_{-\pi/2}^{\pi/2} \cos(t)^{n-2}dt\right)^{-1}.
\]
Proof of Theorem 1

Let $K_1$ and $K_2$ be two centrally symmetric convex sets in $\mathbb{R}^n$.

For $i = 1, 2$ define the functions $f_i : u \in S^{n-1} \to \mathbb{R}$ by
\[ f_i(u) = \int_0^{x_i(u)} e^{-t^2/2} t^{n-1} dt \]

where $x_i(u)$ is the length of the segment issuing from the origin in the direction $u$ where the other ends touches the boundary of $K_i$. Define the function $f_3 : u \in S^{n-1} \to \mathbb{R}$ by
\[ f_3(u) = \int_0^{\min\{x_1(u), x_2(u)\}} e^{-t^2/2} t^{n-1} dt. \]

and finally, the constant function $f_4 : u \in S^{n-1} \to \mathbb{R}$ by
\[ f_4(u) = \int_0^{\infty} e^{-t^2/2} t^{n-1} dt. \]

It is clear that for every $u \in S^{n-1}$ we have
\[ f_4(u) f_3(u) \geq f_1(u) f_2(u). \]

Applying Theorem 2 for every $\sin^{n-2}$-affine probability measure $\nu$ supported on a geodesic segment $I = S_1^+ \subset S^n$, we have
\[ \frac{\left( \int f_4 d\nu \right) \left( \int f_3 d\nu \right)}{\left( \int f_1 d\nu \right) \left( \int f_2 d\nu \right)} \geq \frac{\mu_2(\mathbb{R}^2_+) \mu_2(S_1^+ \cap S_2^+)}{\mu_2(S_1^+) \mu_2(S_2^+)}, \]

where $S_1$ and $S_2$ are two appropriate strips. $\mathbb{R}^2_+$ is the half-plane oriented by the unit vector $(0, 1)$ and for any open set $K$ containing the origin, we define $K_+ = \mathbb{R}^2_+ \cap K$. We suppose that the point $(0, 1)$ corresponds to the maximum point of the density function of the measure $\nu$.

Through the assumptions of Theorem 1 and by the fact that every $\sin^n$-affine measure on $S^1$ is a rotation of the measure $C(n) \cos(t)^{n-1} dt$, we deduce that if for every two strips $S_1$ and $S_2$ we have:
\[ \frac{\mu_2(\mathbb{R}^2_+) \mu_2(S_1^+ \cap S_2^+)}{\mu_2(S_1^+) \mu_2(S_2^+)} \geq 1, \]

where $\mu_2$ is the measure $C(n)|y|^{n-2}e^{-(x^2+y^2)/2}dx dy$ with respect to the $xy$-coordinates in $\mathbb{R}^2$ and where
\[ C(n) = \left( \int_{\pi/2}^{\pi/2} \cos(t)^{n-2} dt \right)^{-1}, \]

then for every $\sin^{n-2}$-affine probability measure $\nu$ supported on a geodesic segment $S_1^+ \subset S^n$, we have
\[ \frac{\left( \int f_4 d\nu \right) \left( \int f_3 d\nu \right)}{\left( \int f_1 d\nu \right) \left( \int f_2 d\nu \right)} \geq 1. \]
Hence, applying the spherical localisation Lemma 3.4 to $f_i$, $i = 1, 2, 3, 4$ as defined above, we deduce that

$$\frac{\gamma_n(K_1 \cap K_2)}{\gamma_n(K_1)\gamma_n(K_2)} = \frac{\int f_4d\mu}{\int f_1d\mu}\frac{\int f_3d\mu}{\int f_2d\mu} \geq 1,$$

where $d\mu$ is the normalised Riemannian measure of $S^{n-1}$. Therefore, the result of the Gaussian Correlation Conjecture follows. This completes the proof of Theorem 1.

5.1 The “If and Only If” Case

Suppose the Gaussian Correlation Conjecture is true for every two centrally-symmetric convex bodies. Let $K_1, K_2 \subset \mathbb{R}^n$, two centrally-symmetric convex bodies are given. Suppose a (measurable) partition of the sphere $S^{n-1}$ by geodesic segments is also given in the following way: Take two points $x, -x \in S^{n-1}$ and join these two points by all geodesic segments. This clearly defines a (measurable) partition of $S^n - \{x, -x\}$ into geodesic segments. Therefore, a probability measure (canonically induced from the normalised Riemannian measure of the sphere) is defined on each segment of this partition. The normalised Riemannian volume of $S^{n-1}$ can be disintegrated on each segment of this partition. The result gives a family of $\sin^{n-2}$-affine probability measures indexed by the segments of the partition. Since the Gaussian Correlation Conjecture is assumed to be true, by repeating the arguments in the proof of Lemma 3.4 we can conclude that a segment $I$-element of the partition, a $\sin^{n-2}$-probability measure $\nu$ defined on $I$ exists such that

$$\frac{\int f_4d\nu}{\int f_1d\nu}\frac{\int f_3d\nu}{\int f_2d\nu} \geq 1,$$

where $f_i$, $i = 1, 2, 3, 4$ are defined previously. Take the 2-dimensional plane containing the origin and containing the segment $I$. Denote this plane by $P$. Then for $i = 1, 2$, $P_i = P \cap K_i$ are centrally symmetric convex sets. According to Theorem 2, two strips $S_1$ and $S_2$ in $\mathbb{R}^2$ exist such that

$$\frac{\mu_2(\mathbb{R}^2)\mu_2(P_1 \cap P_2)}{\mu_2(P_1)\mu_2(P_2)} \geq \frac{\mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2)}{\mu_2(S_1)\mu_2(S_2)}.$$

In order to give a proof for the “If and Only If” case, one has to somehow show that the result of Theorem 2 is sharp and that:

$$\frac{\mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2)}{\mu_2(S_1)\mu_2(S_2)} \geq 1.$$

I am unable to prove this at this time, however I believe it is true.

6 Gaussian Correlation Conjecture Proof

Theorem 1 gives a necessary (and presumably sufficient) condition for the Gaussian Correlation Conjecture to hold. Indeed, to prove the Gaussian Correlation Conjecture, it remains to demonstrate:
Lemma 6.1 For every two symmetric strips $S_1$ and $S_2$ in $\mathbb{R}^2$ we have:

$$\mu_2(\mathbb{R}^2) \mu_2(S_1 \cap S_2) \geq \mu_2(S_1) \mu_2(S_2).$$

Where $\mu_2 = C(n)|x|^{n-2}e^{-(x^2+y^2)/2}dx\,dy$ and where

$$C(n) = \left(\int_{-\pi/2}^{\pi/2} \cos(t)^{n-2}dt\right)^{-1}.$$

Proof of Lemma 6.1

This Lemma perhaps perfectly sums up the difficulty of the Gaussian Correlation Conjecture. For me, the Gaussian Correlation Conjecture is sort of a Gaussian generalisation of the Bishop-Gromov inequality in Riemannian Geometry. As I Come from a geometry background, this was the primary reason for my interest to this problem. I got brushed off when I tried to explain this to a few mathematicians but c’est la vie! Quite honestly, the proof of the Bishop-Gromov inequality boils down to the following Lemma (and we shall need it later on here):

**Lemma 6.2 (Gromov)** Let $f, g$ be two positive functions defined on $[0, \infty)$. If $f/g$ is non-increasing, then the function:

$$\frac{\int_0^\infty f(t)dt}{\int_0^\infty g(t)dt}$$

is non-increasing on $[0, \infty)$.

Let us begin the proof of Lemma 6.1. Throughout the proof, we shall see the resemblance of this Lemma and the Bishop-Gromov inequality and we shall also see why the Gaussian Correlation Conjecture is complicated in some cases (and so sharp).

At first, assume $S_1$ and $S_2$ be orthogonal symmetric strips and their axis coincide with the $xy$-axis of the Cartesian plane. In this case, there exist $a, b \in \mathbb{R}_+$ such that:

$$\mu_2(S_1) = C(n) \int_{-\infty}^{+\infty} \int_{-a}^{+a} |x|^{n-2}e^{-(x^2+y^2)/2}dx\,dy$$

$$= C(n) \left(\int_{-\infty}^{+\infty} e^{-y^2/2}dy\right) \left(\int_{-a}^{+a} |x|^{n-2}e^{-x^2/2}dx\right).$$

And

$$\mu_2(S_2) = C(n) \int_{-\infty}^{+\infty} \int_{-b}^{+b} |x|^{n-2}e^{-(x^2+y^2)/2}dy\,dx$$

$$= C(n) \left(\int_{-b}^{+b} e^{-y^2/2}dy\right) \left(\int_{-\infty}^{+\infty} |x|^{n-2}e^{-x^2/2}dx\right).$$

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Therefore:

$$\mu_2(S_1)\mu_2(S_2) = C(n)^2 \left( \int_{-\infty}^{+\infty} e^{-y^2/2} dy \right) \left( \int_{-\infty}^{+\infty} |x|^{n-2} e^{-x^2/2} dx \right)$$

$$\left( \int_{-b}^{+b} e^{-y^2/2} dy \right) \left( \int_{-\infty}^{+\infty} |x|^{n-2} e^{-x^2/2} dx \right)$$

$$= C(n)^2 \left( \int_{-\infty}^{+\infty} e^{-y^2/2} dy \right) \left( \int_{-\infty}^{+\infty} |x|^{n-2} e^{-x^2/2} dx \right)$$

$$\left( \int_{-a}^{+a} |x|^{n-2} e^{-x^2/2} dx \right) \left( \int_{-b}^{+b} e^{-y^2/2} dy \right)$$

$$= \left( C(n) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x|^{n-2} e^{-(x^2+y^2)/2} \, dx \, dy \right)$$

$$\left( \int_{-b}^{+b} \int_{-a}^{+a} |x|^{n-2} e^{-(x^2+y^2)/2} \, dx \, dy \right)$$

$$= \mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2).$$

Hence, for the very special case of orthogonal symmetric strips parallel to coordinate axis, we have equality in Lemma 6.1. Now let’s consider a slightly harder case: Assume one of the symmetric strip, $S_2$ has $y$-axis as its axis and the other strip is arbitrary. We like to verify that the correlation inequality of our Lemma is verified for this case.

There exist $a, b, \theta$ such that

$$\frac{1}{2}\mu_2(S_2) = C(n) \int_{0}^{+\infty} \int_{-a}^{+a} |x|^{n-2} e^{-(x^2+y^2)/2} \, dx \, dy,$$

and

$$\frac{1}{2}\mu_2(S_1) = C(n) \int_{0}^{+\infty} \int_{-b+\tan(\theta)x}^{+b+\tan(\theta)x} |x|^{n-2} e^{-(x^2+y^2)/2} \, dy \, dx.$$

Define the function:

$$k(x) = \frac{|x|^{n-2} e^{-x^2/2} \int_{-b+\tan(\theta)x}^{+b+\tan(\theta)x} e^{-y^2/2} \, dy}{|x|^{n-2} e^{-x^2/2} \int_{-\infty}^{+\infty} e^{-y^2/2} \, dy}$$

This function is clearly decreasing on $[0, +\infty)$, due to the fact that $e^{-y^2}$ is a decreasing function. Therefore, we can apply Lemma 6.2 and conclude that the function:

$$\int_{0}^{X} \left( |x|^{n-2} e^{-x^2/2} \int_{-b+\tan(\theta)x}^{+b+\tan(\theta)x} e^{-y^2/2} \, dy \right) \, dx$$

$$\int_{0}^{X} \left( |x|^{n-2} e^{-x^2/2} \int_{-\infty}^{+\infty} e^{-y^2/2} \, dy \right) \, dx$$
is a non-increasing function. The consequence of this fact is that:

\[
\frac{\mu_2(S_1 \cap S_2)}{\mu_2(S_1)} = \frac{\int_0^\infty \left( |x|^{n-2}e^{-x^2/2} \int_{-b+\tan(\theta)x}^{b+\tan(\theta)x} e^{-y^2/2} dy \right) dx}{\int_0^\infty \left( |x|^{n-2}e^{-x^2/2} \int_{-b+\tan(\theta)x}^{b+\tan(\theta)x} e^{-y^2/2} dy \right) dx} \geq \frac{\mu_2(S_1)}{\mu_2(S_2)}.
\]

Unfortunately, it is impossible to prove the correlation inequality for all the different cases the same way as we did above. If it was the case, I believe the Gaussian Correlation Conjecture would be solved long before this paper. To see the difficulty, let us try to solve a case where one of the symmetric strip has axis and the other is an arbitrary strip. In this case, in order to apply Lemma 6.2 same way we did previously, we would have to define a function:

\[
k(y) = \frac{e^{-y^2/2} \int_{-\infty}^{b+cot(\theta)y} |x|^{n-2}e^{-x^2/2} dx}{e^{-y^2/2} \int_{-\infty}^{+\infty} |x|^{n-2}e^{-x^2/2} dx}.
\]

However, the function \(k(y)\) is not necessarily non-increasing. The behaviour of this function depends on the number \(b\) and if \(b < b_{max}\), where \(b_{max}\) is the point where the function \(|x|^{n-2}.e^{-x^2/2}\) attains its maximum, then \(k(y)\) is certainly not non-increasing. Therefore, as we can see, we need to work harder in order to prove Lemma 6.1.

In order to give an overview for the rest of the paper, I list (quickly) the plan of the proof from here:

- We begin by proving Lemma 6.1 for the special case where the two symmetric strips \(S_1\) and \(S_2\) have orthogonal axis.
- In order to cover all the symmetric strips, we rotate the \(xy\)-axis by \(\theta\), and for each \(\theta\) we prove the correlation inequality for the case when one of the strips has its axis parallel to the \(y(\theta)\)-axis.
- Since we have already checked the validity of Lemma 6.1 for the case \(\theta = 0\), and one strip has its axis parallel to the \(y\)-axis, combining the previous two cases, the proof of Lemma 6.1 follows.

As promised, let us start by:

**Lemma 6.3** For every two orthogonal symmetric strips \(S_1\) and \(S_2\), we have:

\[
\mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2) \geq \mu_2(S_1)\mu_2(S_2),
\]

where \(\mu_2\) is as it was defined in Lemma 6.1.
Proof of Lemma 6.3

For the case where the strips are parallel to the coordinate axis we have already confirmed the validity of this Lemma. For more general orthogonal strips, we can see their axis as a rotation of the $xy$-axis. Therefore, in order to prove the Lemma, we change the $xy$-coordinates to the new rotated coordinates and write down the integrals in these new coordinates. The function $|x|^n$ changes to $|\cos(\theta)x + \sin(\theta)y|^{n-2}$, where $\theta$ is the angle of rotation. Since any rotation is an isometry, the function $(x^2 + y^2)$ remains invariant in the new coordinates. From here, I will omit writing the constant $C(n)$ in front of the integrals. The results will not be affected since the constants $C(n)$ cancel out from the two sides of equalities (or inequalities). Therefore there exist $a, b$ such that:

$$\mu_2(S_1) = \int_{-\infty}^{+\infty} \int_{-a}^{+a} |\cos(\theta)x + \sin(\theta)y|^{n-2} e^{-(x^2+y^2)/2} dx dy$$

and

$$\mu_2(S_2) = \int_{-\infty}^{+\infty} \int_{-b}^{+b} |\cos(\theta)x + \sin(\theta)y|^{n-2} e^{-(x^2+y^2)/2} dy dx.$$

Suppose first $n - 2$ is even. In this case we have:

$$\mu_2(S_1) = \int_{-\infty}^{+\infty} \int_{-a}^{+a} (\cos(\theta)x + \sin(\theta)y)^{n-2} e^{-(x^2+y^2)/2} dx dy$$

$$= \int_{-\infty}^{+\infty} \int_{-a}^{+a} \left( \sum_{k=0}^{n-2} \binom{n-2}{k} (\cos(\theta)x)^{n-2-k} (\sin(\theta)y)^k e^{-(x^2+y^2)/2} \right) dx dy$$

$$= \sum_{k=0}^{n-2} \binom{n-2}{k} \left( \int_{-\infty}^{+\infty} (\cos(\theta)x)^{n-2-k} e^{-(x^2+y^2)/2} dx \right) \left( \int_{-a}^{+a} (\sin(\theta)y)^k e^{-y^2/2} dy \right).$$

With the same calculation, we have:

$$\mu_2(S_2) = \sum_{j=0}^{n-2} \binom{n-2}{j} \left( \int_{-b}^{+b} (\sin(\theta)y)^j e^{-y^2/2} dy \right) \left( \int_{-\infty}^{+\infty} (\cos(\theta)x)^{n-2-j} e^{-x^2/2} dx \right).$$

Denote:

$$A_i = \int_{-b}^{+b} (\sin(\theta)y)^i e^{-y^2/2} dy$$

$$B_i = \int_{-\infty}^{+\infty} (\sin(\theta)y)^i e^{-y^2/2} dy$$

$$C_i = \int_{-a}^{+a} (\cos(\theta)x)^{n-2-i} e^{-x^2/2} dx$$

$$D_i = \int_{-\infty}^{+\infty} (\cos(\theta)x)^{n-2-i} e^{-x^2/2} dx.$$
Then we have:

\[ \mu_2(S_1)\mu_2(S_2) = \left( \sum_{k=0}^{n-2} \binom{n-2}{k} B_k C_k \right) \left( \sum_{j=0}^{n-2} \binom{n-2}{j} A_j D_j \right) \]

\[ = \sum_{k=0}^{n-2} \sum_{j=0}^{n-2} \binom{n-2}{k} \binom{n-2}{j} A_j D_j B_k C_k \]

\[ = \sum_{k=0}^{n-2} \sum_{j=0}^{n-2} \binom{n-2}{k} \binom{n-2}{j} B_k D_j C_k A_j. \]

For \( j \neq k \), applying directly Lemma 6.2 we get:

\[ A_j D_j B_k C_k + A_k D_k B_j C_j \leq B_k D_k A_j C_j + B_j D_j A_k C_k. \]

Hence:

\[ \mu_2(S_1)\mu_2(S_2) = \sum_{k=0}^{n-2} \sum_{j=0}^{n-2} \binom{n-2}{k} \binom{n-2}{j} B_k D_j A_j C_k \]

\[ \leq \left( \sum_{k=0}^{n-2} \binom{n-2}{k} B_k D_k \right) \left( \sum_{j=0}^{n-2} \binom{n-2}{j} A_j C_j \right) \]

\[ = \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\cos(\theta)x + \sin(\theta)y|^{n-2}e^{-(x^2+y^2)/2} \, dx \, dy \right) \]

\[ \left( \int_{-b}^{+b} \int_{-a}^{+a} |\cos(\theta)x + \sin(\theta)y|^{n-2}e^{-(x^2+y^2)/2} \, dx \, dy \right) \]

\[ = \mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2). \]

When \( n - 2 \) is odd, we have:

\[ \mu_2(S_1) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+a} |\cos(\theta)x + \sin(\theta)y|^{n-2}e^{-(x^2+y^2)/2} \, dx \, dy \]

\[ = \int \int_{R_1 \cap S_1} (\cos(\theta)x + \sin(\theta)y)^{n-2}e^{-(x^2+y^2)/2} \, dx \, dy + \int \int_{R_2 \cap S_1} ((-\cos(\theta)x + \sin(\theta)y))^{n-2}e^{-(x^2+y^2)/2} \, dx \, dy \]

\[ = \int \int_{R_1 \cap S_1} \left( \sum_{k=0}^{n-2} \binom{n-2}{k} (\cos(\theta)x)^{n-2-k} (\sin(\theta)y)^k e^{-(x^2+y^2)/2} \right) \, dx \, dy \]

\[ + \int \int_{R_2 \cap S_1} \left( \sum_{k=0}^{n-2} \binom{n-2}{k} (-1)^k (\cos(\theta)x)^{n-2-k} (\sin(\theta)y)^k e^{-(x^2+y^2)/2} \right) \, dx \, dy \]

\[ = \sum_{l=0}^{n-2} \binom{n-2}{l} \left( \int_{-\infty}^{+\infty} (\sin(\theta)y)^l e^{-y^2/2} \, dy \right) \left( \int_{-a}^{+a} (\cos(\theta)x)^{n-2-l} e^{-x^2/2} \, dx \right), \]

where \( R_1 \) (resp \( R_2 \)) is the region where \( \cos(\theta)x + \sin(\theta)y \geq 0 \) (resp \( < 0 \)) and \( l \) runs over all the even number between 0 to \( n - 2 \). Indeed for the odd powers, the integrals over \( R_1 \cap S_1 \) and \( R_2 \cap S_1 \)
cancels out and only the even powers remain. Thus, with the same calculations as in the previous case, we get:

\[ \mu_2(S_1)\mu_2(S_2) = \sum_{l=0}^{n-2} \sum_{m=0}^{n-2} \binom{n-2}{l} \binom{n-2}{m} B_l D_m A_m C_l \]

\[ \leq \left( \sum_{l=0}^{n-2} \binom{n-2}{l} B_l D_l \right) \left( \sum_{m=0}^{n-2} \binom{n-2}{m} A_m C_m \right) \]

\[ = \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\cos(\theta)x + \sin(\theta)y|^{n-2} e^{-(x^2+y^2)/2} dx dy \right) \]

\[ \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\cos(\theta)x + \sin(\theta)y|^{n-2} e^{-(x^2+y^2)/2} dx dy \right) \]

\[ = \mu_2(\mathbb{R}^2) \mu_2(S_1 \cap S_2). \]

Where \( l \) and \( m \) runs over even numbers between 0 and \( n-2 \).

This ends the proof of Lemma 6.3.

\[ \Box \]

**Question:**
Do we have equality in Lemma 6.3 for every pair of orthogonal symmetric strips? It seems that yes and I am fairly convinced that with some extra work we can manage to prove that the result of Lemma 6.3 is sharp, and the inequality can be replaced by equality.

The Cartesian Coordinates are the \( xy \)-Coordinates. If we rotate the \( xy \)-Coordinates by \( \theta \) counterclockwise, the new (orthogonal) coordinates are denoted \( x(\theta)y(\theta) \). We are interested in the following: for every \( \theta \), we consider symmetric strips where one has \( y(\theta) \) as its axis and the other one is an arbitrary symmetric strip. This is summarised in the following:

**Lemma 6.4** For every \( \theta \), let \( S_1(\theta) \) be a symmetric strip which has \( y(\theta) \) as its axis. Let \( S_2 \) be an arbitrary symmetric strip. Then for \( S_1(\theta) \) and \( S_2 \) Lemma 6.1 is verified.

**Proof of Lemma 6.4**
After suitably choosing the axis \( y(\theta) \), for simplicity we omit writing \( \theta \) everywhere. We assume it being fixed. There exist \( a, \phi \) such that:

\[ \mu_2(S_2) = \int_{x=-\infty}^{x=+\infty} \int_{y=-a+\tan(\phi)x}^{y=+\infty} |\cos(\theta)x + \sin(\theta)y|^{n-2} e^{-(x^2+y^2)/2} dy dx. \]

And there exists \( b \in \mathbb{R}_+ \) such that:

\[ \mu_2(S_1 \cap S_2) = \int_{x=-b}^{x=+b} \int_{y=-a+\tan(\phi)x}^{y=+b} |\cos(\theta)x + \sin(\theta)y|^{n-2} e^{-(x^2+y^2)/2} dy dx. \]
The trick is to consider \( \mu_2(S_2) \) and \( \mu_2(S_1 \cap S_2) \) as functions of \( \phi \) for \( \phi \in [0, \pi/2] \). Thus, set:

\[
k(\phi) = \frac{\mu_2(S_1 \cap S_2)}{\mu_2(S_2)} = \frac{s_1(\phi)}{s_2(\phi)} = \frac{\int_{x=-b}^{x=b} \int_{y=-a+\tan(\phi)x}^{a+\tan(\phi)x} |\cos(\theta)x + \sin(\theta)y|^n e^{-((a+\tan(\phi)x)^2+y^2)/2} dy \, dx}{\int_{x=-\infty}^{x=+\infty} \int_{y=-a+\tan(\phi)x}^{a+\tan(\phi)x} |\cos(\theta)x + \sin(\theta)y|^n e^{-((a+\tan(\phi)x)^2+y^2)/2} dy \, dx}.
\]

For \( \phi = 0 \), applying Lemma 6.3 we get:

\[
k(0) = \frac{\mu_2(S_1)}{\mu_2(\mathbb{R}^2)}.
\]

For \( \phi = \pi/2 \), the two strips have same axis hence one is included in the other and therefore we have:

\[
k(\pi/2) \geq \frac{\mu_2(S_1)}{\mu_2(\mathbb{R}^2)}.
\]

We calculate the derivative of \( k(\phi) \):

\[
k'(\phi) = \left( \int_{x=-b}^{x=b} m(\phi, x) \, dx \right) s_2(\phi) - \left( \int_{x=-\infty}^{x=+\infty} m(\phi, x) \, dx \right) s_1(\phi),
\]

where

\[
m(\phi, x) = c(\phi) \left( A(\phi, x) - B(\phi, x) \right),
\]

where

\[
A(\phi, x) = |\cos(\theta)x + \sin(\theta)(a + \tan(\phi)x)|^{n-2} e^{-((a+\tan(\phi)x)^2+x^2)/2},
\]

\[
B(\phi, x) = |\cos(\theta)x + \sin(\theta)(-a + \tan(\phi)x)|^{n-2} e^{-((-a+\tan(\phi)x)^2+x^2)/2},
\]

and

\[
c(\phi) = 1 + \tan(\phi)^2.
\]

However, for every \( A \in \mathbb{R}^+ \) we have:

\[
\int_{x=-A}^{x=+A} m(\phi, x) \, dx = \int_{x=-A}^{x=+A} A(\phi, x) \, dx - \int_{x=-A}^{x=+A} B(\phi, x) \, dx = 0.
\]
Therefore, since \( c(\phi) > 0 \), the sign of \( k'(\phi) \) is determined by the sign of

\[
\int_{x=-\infty}^{x=+\infty} m(\phi, x) \, dx,
\]

(which I think is equal to 0 anyway). Hence \( k(\phi) \) is monotone and without loss of generality we can suppose that for every \( \phi \in [0, \pi/2] \), we have:

\[
k(\phi) \geq k(\pi/2).
\]

This means that for every \( \phi \in [0, \pi/2] \), we have:

\[
\frac{\mu_2(S_1 \cap S_2)}{\mu_2(S_2)} \geq \frac{\mu_2(S_1)}{\mu_2(\mathbb{R}^2)}.
\]

This ends the proof of Lemma 6.4.

Lemma 6.3 combined with Lemma 6.4 for every \( a \) and \( \theta \) ends the proof of Lemma 6.1.

Remark: I believe that Lemma 6.1 is (very) sharp in the sense that for every symmetric strips \( S_1 \) and \( S_2 \) with different axis, we have the equality, i.e.:

\[
\mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2) = \mu_2(S_1)\mu_2(S_2).
\]

The Proof of the Gaussian Correlation Conjecture is completed.

7 Epilogue

The story does not end with the Gaussian Correlation Conjecture. The spherical localisation technique developed in Section 3 (and mainly Lemma 3.4) can be applied to other problems:

- My first idea (which is very close to correlation type problems) is the following problem: Consider a probability measure \( \mu \) in \( \mathbb{R}^n \) be given. Assume this probability measure has a density function which is a radial function (depending only on the distance like the Gaussian measure). Then we could ask for which kind of such measures, the following inequality:

\[
\mu(K_1 \cap K_2) \geq \mu(K_1)\mu(K_2),
\]

for every pair of centrally-symmetric convex sets \( K_1 \) and \( K_2 \) holds. The results of this paper enable one to have a good test in order to investigate probability measures \( \mu \) satisfying 3. Indeed for every such probability measure \( \mu \) with density function \( f(r) \), if for every pairs of strips \( S_1 \) and \( S_2 \) in \( \mathbb{R}^2 \), the following:

\[
\mu_2(\mathbb{R}^2)\mu_2(S_1 \cap S_2) \geq \mu_2(S_1)\mu_2(S_2),
\]

where \( \mu_2 = C(n)|x|^{n-2}r^{n-1}.f(r)\,dx\,dy \) and \( C(n) \) is the normalisation constant, is satisfied then 3 is also satisfied. This suggests to study measures \( \mu_2 \) in \( \mathbb{R}^2 \) defined as above for which for every two strips \( S_1 \) and \( S_2 \), 4 is satisfied.
• **Mahler Conjecture.** I believe the spherical localisation techniques of Lemma 3.4 can be used to attempt to prove the Mahler Conjecture. Let me explain: The Mahler Conjecture suggests that for $K$ a centrally-symmetric convex body in $\mathbb{R}^n$, we have:

$$\text{vol}_n(K) \text{vol}_n(\hat{K}) \geq \frac{\text{vol}_n(I^n) \text{vol}_n(\hat{I}^n)}{4^n} = \frac{\Gamma(n+1)}{},$$

where $\hat{K}$ is the polar of the symmetric convex body $K$. Let us see how one could prove the above inequality. Apply the following steps:

- Position *well* your convex set $K$ around the origin $0 \in \mathbb{R}^n$ in such a way that for *almost* all the sections $P \cap K$, where $P$ is a 2-dimensional plane containing the origin we have:

$$(P \cap K) = P \cap \hat{K}.$$  

Apparently this good position is called the *John* position (pointed out by R.Vershynin).

- Study the two-dimensional variational problem, which is to minimise:

$$\mu_2(S) \mu_2(\hat{S}),$$

where $\mu_2 = C(n)|x|^{n-2}dx\,dy$ in $\mathbb{R}^2$ over all two-dimensional symmetric convex sets $S$, where $C(n)$ is the appropriate normalisation constant. In order to answer this two-dimensional variational problem, one could consult [6] (pointed out to me by M.Fradelizi), where an answer to this could perhaps even be found (I haven’t checked this myself). If not, I believe experts (such as the amazing French team of Barthe, Fradelizi, Guédon and Meyer) could easily answer this question.

- Denote the minimum obtained in the previous step by $g(n)$. Applying Lemma 3.4 we directly obtain that for every symmetric convex set $K$ in $\mathbb{R}^n$, we have:

$$\text{vol}_n(K) \text{vol}_n(\hat{K}) \geq m(n) = \text{vol}_{n-1}(S^{n-1})^2 g(n),$$

where $\text{vol}_{n-1}(S^{n-1})$ is the $(n-1)$-dimensional volume of the canonical sphere $S^{n-1}$.

- compare $m(n)$ with $\frac{4^n}{\Gamma(n+1)}$. If we have:

$$m(n) \geq \frac{4^n}{\Gamma(n+1)};$$

then the proof of the Mahler Conjecture follows.

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