Quantum Coherence in Ultrastrong Optomechanics

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Ultrastong light-matter interaction in an optomechanical system can result in nonlinear optical effects such as photon blockade. The system-bath couplings in such systems play an essential role in observing these effects. Here we study the quantum coherence of an optomechanical system with a dressed-state master equation approach. Our master equation includes photon-number-dependent terms that induce dephasing in this system. Cavity dephasing, second-order photon correlation, and two-cavity entanglement are studied with the dressed-state master equation.

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Introduction.— Cavity optomechanics studies quantum effects induced by light-matter interaction between cavity and mechanical modes [1, 2]. Many such effects have been realized in recent experiments, including the preparation of quantum ground state, the observation of strong optomechanical coupling, and the coherent conversion of photon states via a mechanical interface [3–11]. Among recent theoretical works, studies of optomechanical systems in the single-photon strong (or ultrastrong) coupling regime have predicted many interesting nonlinear optical effects such as photon blockade, phonon sidebands and nonlinear optomechanically-induced transparency [12–23]. With the strength of the single-photon optomechanical coupling comparable to the mechanical frequency and the cavity bandwidth, the optomechanical system can demonstrate strong nonlinearity. It is promising to reach this regime in several experimental systems [4–9, 24, 25]. In addition, recent theoretical works have shown that ultrastrong coupling could be achieved by various quantum engineering schemes [26–30].

The cavity and the mechanical modes in an optomechanical system are subject to environmental noise, which causes decoherence and plays a crucial role in studying the nonlinear optical effects. The system-bath couplings can be treated with a master equation approach. Very often, a standard master equation (SME) is used to describe the damping and thermal excitations. For example, the contributions of the mechanical bath can be in the form of $D[\hat{b}|\rho(t)]$ and $D[\hat{b}^\dagger|\rho(t)]$, where $\hat{b}$ is the annihilation operator of the mechanical mode, $D[\hat{b}|\rho(t)] = \frac{1}{2}[\hat{b}\hat{\rho}(t)\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}(t) + \rho(t)\hat{b}^\dagger\hat{b}]$ is the Lindblad superoperator for operator $\hat{b}$, and $\rho(t)$ is the density matrix of the optomechanical system at time $t$. Such treatment is based on the assumption that the optomechanical coupling is much weaker than the mechanical frequency, and thus does not seriously modify the eigenstates of this system. Under this assumption, each system mode is only affected by their corresponding bath modes. However, in the single-photon strong coupling regime, photons in the eigenstates are strongly dressed by phonon excitations of the mechanical mode, and this assumption is not valid anymore [1–4].

Here we study the quantum coherence and dynamics of an optomechanical system in the ultrastrong coupling regime with an appropriate master equation approach. In our method, we decompose the system operators in terms of the eigenstates (dressed states) of the optomechanical system and derive the master equation under this decomposition. This approach was previously used to study strongly-coupled harmonic oscillators with linear coupling [2, 3] and a mechanical resonator coupled to a two-level-system defect [4]. Our master equation contains photon-number-dependent terms, which are, counter-intuitively, induced by the mechanical bath modes due to the light-matter interaction. We show that our master equation generates faster cavity dephasing and entanglement decay when compared with the SME. The second-order photon correlation given by our master equation also demonstrates more classical behavior than that of the SME, predicting photon bunching in some region of photon antibunching predicted by the SME. Our results indicate that the coherence of an optomechanical system could be strongly influenced by the ultrastrong coupling strength, and the SME may not be sufficient for studying this system.

Dressed-state master equation.— We consider an optomechanical system with one cavity mode and one mechanical mode coupling via radiation-pressure interaction. The Hamiltonian of this system is ($\hbar = 1$)

$$\hat{H}_s = \omega_c \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} - g_0 \hat{a}^\dagger \hat{a} \hat{b} + \hat{b}^\dagger,$$ (1)

where $\omega_c$ ($\omega_m$) is the cavity (mechanical) frequency, $g_0$ is the strength of the single-photon optomechanical coupling, $\hat{a}$ ($\hat{b}$) is the annihilation operator of the cavity (mechanical) mode. The eigenstates of this coupled system
can be written as
\[ |n, k^{(n)}\rangle = |n\rangle_c \otimes e^{n\beta_0} |k\rangle_m \] (2)
with cavity photon number \( n \) and phonon number \( k \) for the mechanical mode. Here the mechanical Fock state \( |k\rangle_m \) is shifted with a displacement \( n\beta_0 \) proportional to the cavity photon number \( n \) and \( \beta_0 = g_0/\omega_m \). In other words, the photon Fock state in the eigenstates is dressed by phonon excitations due to the optomechanical coupling. The corresponding eigenenergies of these states are \( \varepsilon_{n,k} = n\omega_c + k\omega_m - n^2g_0^2/\omega_m \). In the ultrastrong coupling regime with \( g_0 \sim \omega_m \), the dressed states are strongly affected by the optomechanical coupling [12–14].

The cavity and the mechanical modes couple to environmental degrees of freedom which induce damping and thermal excitations in the optomechanical system. The system-bath couplings can be written as \( \hat{H}_I^t = \hat{H}_{cb}^t + \hat{H}_{mb}^t \) in the interaction picture with [1]
\[
\hat{H}_I^t = \hat{a}^\dagger(t)\hat{\Gamma}_c(t) + \hat{\Gamma}_c^\dagger(t)\hat{a}(t); \\
\hat{H}_{mb}^t = [\hat{b}(t) + \hat{b}^\dagger(t)][\hat{\Gamma}_m(t) + \hat{\Gamma}_m^\dagger(t)].
\] (3)
\[
(4)
The system operator \( \hat{a}(t) = e^{iH_s^t\hat{a}e^{-iH_s^t}} \) can be decomposed in terms of the eigenstates as
\[
\hat{a}(t) = \sum_{n,k,j} e^{-i\Delta_{k,j}^{(n)}t} A_{j,k}^{(n)} |n - 1, j^{(n-1)}\rangle \langle n, k^{(n)}|,
\] (5)
where the Franck-Condon factors \( A_{j,k}^{(n)} = \sqrt{n} (j^{(n-1)}|k^{(n)}\rangle \) are finite for \( j \neq k \), indicating that \( \hat{a}(t) \) contains many phonon sidebands, and \( \Delta_{k,j}^{(n)} = (\varepsilon_{n,k} - \varepsilon_{n-1,j}) \). The operator \( \hat{b}(t) = e^{iH_s^t\hat{b}e^{-iH_s^t}} \) can be simplified as
\[
\hat{b}(t) = e^{-i\omega_m^t}(\hat{b} - \beta_0\hat{N}_c) + \beta_0\hat{N}_c
\] (6)
with \( \hat{N}_c = \hat{a}^\dagger\hat{a} \) being the photon number operator. The operator \( \hat{\Gamma}_c(t) \) is the cavity (mechanical) bath operator with \( \hat{\Gamma}_c(t) = \sum_j g_{cj} e^{-i\omega_j^t\hat{c}_j} \) in terms of the annihilation operator \( \hat{c}_j \) (frequency \( \omega_j \)) and coupling constant \( g_{cj} \) of the bath modes. With \( \omega_c \gg \omega_m \), the cavity bath spectral density \( J_c(\omega) = \sum_j |g_{cj}|^2 \delta(\omega - \omega_j) \) can be assumed to be flat over the whole range of relevant phonon sidebands with \( J_m(\omega) = \kappa^2/2\pi \) and \( \kappa \) being the cavity damping rate. We also assume that the mechanical bath spectral density \( J_m(\omega) = \sum_j |g_{mj}|^2 \delta(\omega - \omega_m^j) \) is of Ohmic form with \( J_m(\omega) = (\gamma_m\omega/2\pi\omega_m) \) and \( \gamma_m \) being the mechanical damping rate. At high temperature, this spectral density corresponds to a white noise on the mechanical mode [5].

Under the Born-Markov and the rotating wave approximations (RWA), we then derive the full master equation of this system using the dressed-state operator decomposition given in Eqs. (5) and (6). The master equation in the Schrödinger picture has the form
\[
\frac{d\rho(t)}{dt} = -i[\hat{H}_s, \rho(t)] + \gamma_m(n_{th} + 1)\hat{D}[\hat{b} - \beta_0\hat{N}_c]\rho(t) + \kappa\hat{D}\hat{a}\rho(t) + \gamma_m n_{th} \hat{D}|\hat{b}\rangle\langle\hat{b}| \rho(t) + 4\gamma_m(k_B T/\omega_m)\beta_0^2 \hat{D}[\hat{N}_c]\rho(t),
\] (7)
where \( n_{th} \) is the thermal phonon occupation number at temperature \( T \) and \( \hat{D}[\hat{a}]\rho(t) \) is the Lindblad superoperator. Below we call this master equation the dressed-state master equation (DSME). The last term in this master equation is due to the low-frequency part of the mechanical noise [5] and could induce dephasing between different photon number states. Detailed derivation of the DSME can be found in the Supplementary Materials [36]. In the limit of weak single-photon optomechanical coupling with \( \beta_0 \ll 1 \), the \( \beta_0 \)-dependent terms in the DSME can be neglected. The DSME then becomes
\[
\frac{d\rho(t)}{dt} = -i[\hat{H}_s, \rho(t)] + \gamma_m(n_{th} + 1)\hat{D}[\hat{b} - \beta_0\hat{N}_c]\rho(t) + \kappa\hat{D}\hat{a}\rho(t) + \gamma_m n_{th} \hat{D}|\hat{b}\rangle\langle\hat{b}| \rho(t),
\] (8)
which has the familiar form of the SME often seen in the literature.

Compared with SME, the extra terms in the DSME originate from the interaction between the cavity and the mechanical modes. This interaction results in the expression in Eq. (6). From Eq. (6) together with Eq. (S2), we see that the mechanical resonator-bath coupling generates two physical processes: (i) the exchange of phonons between the system and bath modes in the shifted basis, which gives rise to the \( \hat{D}[\hat{b} - \beta_0\hat{N}_c] \) and \( \hat{D}[\hat{b}^\dagger - \beta_0\hat{N}_c] \) terms.
in Eq. (7); (ii) the shift of the mechanical displacement that depends on the photon number, which yields the last term in Eq. (7). With \( \beta_0 \sim 1 \) in the ultrastrong coupling regime, the extra terms can have a strong impact on the coherence and dynamics of the optomechanical system.

**Cavity dephasing.**—The dynamics of an optomechanical system governed by the DSME could be quite different from the dynamics governed by the SME. We first study the dephasing of cavity states. Consider the optomechanical system in an initial state \(|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle_c + |3\rangle_c) |0\rangle_m\), with both the cavity and the mechanical modes in a pure state. We numerically simulate the time evolution of the density matrix of this system using the package in Ref. [37]. We then calculate the off-diagonal matrix element \( \rho_{03}(t) \equiv |c(0)\langle T_m |\rho(t)|3\rangle_c| \) of the density matrix \( \rho(t) \), where \(|0\rangle_c, |3\rangle_c\) are photon number states and \( T_m \) is a trace operation over the mechanical mode. This matrix element directly reflects the coherence of the cavity mode. In Fig. 2, \( \rho_{03}(t) \) from the DSME as well as from the SME is plotted. At \( n_{th} = 0 \) \( (T = 0) \), the DSME result predicts stronger cavity coherence than that of the SME, with \( \rho_{03}(t) \) decreasing at a slower rate with the DSME. However, at \( n_{th} = 20 \), opposite behavior can be observed with \( \rho_{03}(t) \) decreasing at a faster rate with the DSME than that with the SME. These results indicate that the dephasing of the cavity is strongly affected by the \( \beta_0 \)-dependent terms in Eq. (7) even at moderate thermal occupation number, and the SME is not sufficient to correctly describe the time evolution of this system.

To explain the above results, we write the master equations in the interaction picture, which can be found in the Supplementary Materials [36]. In the interaction picture, the bath-induced terms in the DSME are exactly the same as that in Eq. (7), only with \( \rho(t) \) replaced by the density matrix \( \rho^I(t) \) in the interaction picture. Whereas in the SME, with all other terms staying the same as that in the DSME, the \( D[N_c] \) term has a different coefficient: \( \gamma_m (2n_{th} + 1) \beta_0^2 \). Hence at \( n_{th} = 0 \) \( (T = 0) \), the SME has one more term than the DSME: \( \gamma_m \beta_0^2 |D[N_c]| \rho^I(t) \), which explains the stronger coherence predicted by the DSME. At \( n_{th} = 20 \) \( (finite T) \), the coefficient of the \( D[N_c] \) term in the DSME becomes larger than that in the SME, which gives faster dephasing with the DSME.

The time evolution of the photon number average, in contrast, is not affected by the \( \beta_0 \)-dependent terms in the master equation. It can be shown that with DSME, \( \langle N_c(t) \rangle = \exp (-\kappa t) \langle N_c(0) \rangle \), which is the usual photon exponential decay at the decay rate \( \kappa \).

**Second-order photon correlation.**—Photon correlation can be strongly affected by the radiation-pressure interaction in an optomechanical system with ultrastrong coupling [13, 16, 17]. The second-order photon correlation at equal times defined as \( g^{(2)}(0) = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle_{ss} / \langle \hat{a}^\dagger \hat{a} \rangle_{ss}^2 \) is a widely used quantity to identify quantum features of a photon state such as antibunching. Here we study the behavior of \( g^{(2)}(0) \) of an optomechanical system governed by the DSME and the SME. The system is under a weak driving on the cavity mode. With the driving, the Hamiltonian \( \hat{H}_s \) in Eq. (7) needs to be replaced by \( \hat{H}'_s = \hat{H}_s + E_0 (\hat{a}^\dagger e^{i\omega dt} + \hat{a} e^{-i\omega dt}) \), where \( E_0 (\omega_d) \) is the amplitude (frequency) of the driving field. In our numerical calculation [37], we choose the detuning of the driving field \( \Delta_0 \equiv \omega_c - \omega_d \) to be at the single-photon resonance with \( \Delta_0 = g_0^2 / \omega_m \), i.e., the driving field can resonantly excite the transition between the ground state and the state \( |1, 0^{(1)}\rangle \) [12, 13]. We derive the photon correlation by solving the steady state of the master equations.

The photon correlation \( g^{(2)}(0) \) is plotted in Fig. 3 as a function of the dimensionless constant \( \beta_0 = g_0 / \omega_m \).
Similar to that in previous works [13, 17], $g^{(2)}(0)$ demonstrates oscillating behavior with peak positions at $\beta_0 = \sqrt{k/2}$ for integer number $k$. These peaks correspond to two-photon resonances at given phonon sidebands. At $n_{th} = 0$, the result with the DSME gives smaller $g^{(2)}(0)$ values and indicates more quantumness in the photon state than that with the SME. On the other hand, at $n_{th} = 10$, $g^{(2)}(0)$ from the DSME is always larger than that from the SME, indicating less antibunching and weaker photon blockade. In particular, in the vicinity of $\beta_0 = 1.7$ and several other values, the SME gives $g^{(2)}(0) < 1$; while the DSME gives an opposite result of $g^{(2)}(0) > 1$, which shows that photon blockade does not occur. These numerical results can be explained by our previous analysis of the master equations in the interaction picture and also agree with the results for cavity dephasing. Our results imply that in the ultrastrong regime, the second-order photon correlation depends sensitively on the coupling $\beta_0$ and could be strongly affected by the $D[N_c]$ term in the DSME.

Two-cavity entanglement. — Consider an optomechanical system made of two cavity modes coupling to a common mechanical resonator with the total radiation-pressure interaction $H_{int} = -\sum_i g_i a_i^\dagger \dot{a}_i (\hat{b} + \hat{b}^\dagger)$, where $g_i$’s are the coupling constants and $a_i$’s are the annihilation operators for the cavity modes with $i = 1, 2$. Here we study the entanglement between the two cavity modes.

The DSME for this system can be derived as

$$\frac{d\rho(t)}{dt} = -i[H_s, \rho(t)] + \gamma_m(n_{th} + 1)D[\hat{b} - \hat{N}_c]\rho(t) + \sum_i \kappa_i D[\hat{a}_i]\rho(t) + \gamma_m n_{th} D[\hat{b}^\dagger - \hat{N}_c]\rho(t) + 4\gamma_m (k_B T/\omega_m) D[\hat{N}_c]\rho(t), \quad (9)$$

where $H_s$ is the total Hamiltonian with the interaction $H_{int}$ given above, $\kappa_i$ is the damping rate of each cavity mode, and $\hat{N}_c = \hat{N}_{c1} + \hat{N}_{c2}$ with $\beta_i = g_i/\omega_m$ and $\hat{N}_{ci} = \hat{a}_i^\dagger \hat{a}_i$. The difference between Eq. (9) and Eq. (7) is that the $\hat{N}_c$ terms in the master equation contain contributions from both cavities. Details of the derivation are presented in the Supplementary Materials [36].

We study the time dependence of the entanglement between the two cavity modes using the master equations. The system starts with an initial state $|\varphi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{c1}|1\rangle_{c2} + |1\rangle_{c1}|0\rangle_{c2})|0\rangle_{m}$ with the cavities in a fully entangled state. We characterize the entanglement with the logarithmic negativity [38, 39]: $E_N(t) = \log_2 \| (T_A[\rho(t)])^T_m \|$, where the superscript $T_A$ denotes the partial transpose of the reduced density matrix $T_m[\rho(t)]$ and $\| \cdot \|$ denotes the trace norm of the matrix $\hat{o}$. The logarithmic negativity $E_N(t)$ is plotted in Fig. 4. For equal coupling strength $\beta_{1,2} = 1.5$, the results from the DSME and from the SME are exactly the same and without oscillations in the amplitudes. This is because $\hat{N}_c = \hat{N}_{c1} + \hat{N}_{c2}$ at equal coupling, proportion to the total photon number in the cavities; and our initial state is in a superposition of two states $|0\rangle_{c1}|1\rangle_{c2}$ and $|1\rangle_{c1}|0\rangle_{c2}$, which have equal total photon number. Hence, the $\hat{N}_c$-dependent terms in the master equations generate equal phase fluctuations on these two states, and induce no extra dephasing in this special case. However, when the couplings are different, e.g., for $\beta_1 = 1.5$ and $\beta_2 = 0.5$, the DSME and the SME give different results.

At $n_{th} = 20$, $E_N(t)$ derived from the DSME decays faster than that from the SME, similar to the behavior of cavity dephasing shown in Fig. 2, due to the larger $D[\hat{N}_c]$ terms in the DSME. This indicates that the mechanical noise is transferred to the cavity modes via the optomechanical coupling and degrades the entanglement. Note that although the time envelopes in Fig. 2 and Fig. 4 all show exponential decay, their time scales and detailed behaviors are quite different. The similarity in the time envelopes is due to the forms of the dissipative terms in the master equations, which induce this generic behavior in both cavity dephasing and entanglement.

Conclusions. — To summarize, we study the quantum coherence in an optomechanical system in the single-photon strong coupling regime with a dressed-state master equation approach. Compared with the standard approach, our master equation takes the modification of the eigenstates due to the optomechanical coupling between the cavity and the mechanical modes into account, and predicts different behavior in cavity dephasing, second-order photon correlation and cavity entanglement. Our results show that the ultrastrong coupling can play a significant role in the open system dynamics of an optomechanical system. This work could be useful for future studies of nonlinear optical effects in optomechanical systems.

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Supplementary materials for “Quantum Coherence in Ultrastrong Optomechanics”

This document consists of detailed derivation of the dressed-state master equations (DSME), i.e., Eqs. (7) and (9) of the main paper.

**DSME FOR SINGLE CAVITY SYSTEM**

In this section, we present the details of the derivation of the DSME given in Eq. (7) of the main paper. The coupling between the system modes and the bath modes can be described by the Hamiltonian $\hat{H}^I = \hat{H}^I_{cb} + \hat{H}^I_{mb}$ in the interaction picture, where

$$\hat{H}^I_{cb} = \hat{a}^\dagger(t)\hat{\Gamma}_c(t) + \hat{\Gamma}^+_c(t)\hat{a}(t),$$

$$\hat{H}^I_{mb} = [\hat{b}(t) + \hat{b}^+(t)][\hat{\Gamma}_m(t) + \hat{\Gamma}^+_m(t)]$$

are for the cavity-bath and mechanical mode-bath couplings, respectively. Under the Born-Markov approximation, the master equation for the reduced density matrix $\rho^I(t)$ of the optomechanical system in the interaction picture can be derived as [1–4]

$$\frac{d\rho^I(t)}{dt} = -\int_0^{\infty} ds \text{Tr}_b[\hat{H}_c^I(t), \hat{H}_b^I(t)]\rho^I(t)\rho^I_c(t)\rho^I_m(t),$$

where $\text{Tr}_b$ denotes the trace operation over the bath modes and $\rho_c$ ($\rho_m$) is the density matrix of the cavity (mechanical) bath modes in their thermal state. As the cavity bath and the mechanical bath are independent from each other, the above master equation can be written as

$$\frac{d\rho^I(t)}{dt} = \mathcal{L}^I_c\rho^I(t) + \mathcal{L}^I_m\rho^I(t),$$

where $\mathcal{L}^I_c$ and $\mathcal{L}^I_m$ are superoperators acting on the density matrix of the system. By applying the rotating wave approximation (RWA) to remove fast oscillating terms such as the $e^{\pm 2i\omega_s t}$ terms, the cavity bath contribution becomes

$$\mathcal{L}^I_c\rho^I(t) = \int_0^{\infty} ds \mathcal{R}_-(s)\hat{a}^\dagger(t-s)\rho^I(t)\hat{a}(t)$$

$$-\int_0^{\infty} ds \mathcal{R}_-(s)\hat{a}\rho^I(t-s)\hat{a}\rho^I(t)$$

$$+\int_0^{\infty} ds \mathcal{R}_+(s)\hat{a}\rho^I(t-s)\hat{a}\rho^I(t)$$

$$-\int_0^{\infty} ds \mathcal{R}_+(s)\hat{a}^\dagger(t-s)\rho^I(t)\hat{a}^\dagger(t-s)\rho^I(t)$$

$$+ h.c.$$  

with bath correlation functions defined as

$$\mathcal{R}_-(s) = \text{Tr}_b\left[\hat{\Gamma}_c^+(t)\hat{\Gamma}_c(t-s)\rho_c\right],$$

$$\mathcal{R}_+(s) = \text{Tr}_b\left[\hat{\Gamma}_c^+(t)\hat{\Gamma}_c(t-s)\rho_c\right].$$

For the mechanical bath, we have

$$\mathcal{L}^I_m\rho^I(t) = \int_0^{\infty} ds \mathcal{R}_m(s)\hat{\chi}(t-s)\rho^I(t)\hat{\chi}(t)$$

$$-\int_0^{\infty} ds \mathcal{R}_m(s)\hat{\chi}(t)\rho^I(t-s)\rho^I(t)$$

$$+ h.c.$$  

with the time-dependent operators

$$\hat{\chi}(t) = \hat{b}(t) + \hat{b}^+(t),$$

$$\hat{\chi}_m(t) = \hat{\Gamma}_m(t) + \hat{\Gamma}^+_m(t)$$

and the correlation function for the mechanical bath

$$\mathcal{R}_m(s) = \text{Tr}_b\left[\hat{\chi}_m(t)\hat{\chi}_m(t-s)\rho_m\right].$$

**Cavity bath contribution**

We first derive the time-dependent operator $\hat{a}(t)$. With $\hat{a}(t) = e^{i\hat{H}_c^I t}\hat{a}e^{-i\hat{H}_c^I t}$, we have

$$\hat{a}(t) = \sum_{n,k,j} e^{-i\Delta^{(n)}_{j,k} t}\hat{A}^{(n)}_{j,k}(t)$$

with the operator

$$\hat{A}^{(n)}_{j,k} = \sqrt{n}J^{(n-1)}|j[n]\rangle|k[n]\rangle - 1, j^{(n-1)}|n, k[n]\rangle$$

and the energy separation $\Delta^{(n)}_{j,k} = (\varepsilon_n - \varepsilon_{n-1,j})$, where $\varepsilon_n$’s are eigenenergies given in the main paper. It can be shown that the $\Delta^{(n)}_{j,k}$ is $\omega_c + (k-j)\omega_m + (1-2n)g^2_j/\omega_m$, which includes phonon sidebands $(k-j)\omega_m$.

The cavity bath contribution to the DSME can be derived from Eq. (S5). With $\hat{\Gamma}_c^+(t) = \sum_{j} g_{cj} e^{-i\omega_{cj} t}\hat{\epsilon}_{cj}$ (given in the main paper),

$$\mathcal{R}_-(s) = \sum_{j} |g_{cj}|^2 n(\omega_c, T)e^{i\omega_{cj}s},$$

$$\mathcal{R}_+(s) = \sum_{j} |g_{cj}|^2 [n(\omega_c, T) + 1]e^{-i\omega_{cj}s},$$

where $n(\omega_c, T)$ is the average occupation number of the corresponding bath mode. Because $\omega_c \gg \omega_m$, we assume that the cavity bath spectral density defined as $J_c(\omega) = \sum_j |g_{cj}|^2\delta(\omega - \omega_{cj})$ is slow-varying near $\omega = \omega_c$, and can thus be written as $J_c(\omega) \approx \kappa/2\pi$ in the full range of the phonon sidebands. Hence,

$$\int_0^{\infty} ds e^{i\Delta^{(n)}_{j,k} s}\mathcal{R}_+(s) \approx \frac{\kappa}{2} [n(\omega_c, T) + 1] \approx \frac{\kappa}{2},$$

$$\int_0^{\infty} ds e^{-i\Delta^{(n)}_{j,k} s}\mathcal{R}_-(s) \approx \frac{\kappa}{2} n(\omega_c, T) \approx 0,$$
where the thermal photon number at the cavity frequency $n(\omega_c, T) \approx 0$. The cavity bath contribution is hence

\[
\mathcal{L}_c \rho'(t) = \frac{\kappa}{2} \sum_{k,j,n,i,l,r} \left\{ 2\left[ e^{-i\Delta_{s,t}^{(n)} t} \hat{A}_j^{(n)} \right] \rho'(t) \left[ e^{i\Delta_{s,t}^{(n)} t} \hat{A}_j^{(n)} \right]^\dagger - \left[ e^{i\Delta_{s,t}^{(n)} t} \hat{A}_j^{(n)} \right] \rho'(t) - \rho'(t) \left[ e^{i\Delta_{s,t}^{(n)} t} \hat{A}_j^{(n)} \right]^\dagger \right\},
\]

(S14)

which is simply \( \mathcal{L}_c \rho'(t) = \kappa \mathcal{D} \hat{a}(\hat{a}) \rho'(t) \). Here \( \mathcal{D} \hat{a} \rho(t) = \frac{1}{2} [2 \hat{a} \rho(t) \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \rho(t) - \rho(t) \hat{a}^\dagger \hat{a}] \) is the Lindblad superoperator for operator \( \hat{a} \). Under the RWA, the fast oscillating terms in this expression can be omitted from the above equation.

By transforming Eq. (S14) to the Schrödinger picture, the cavity bath contribution becomes

\[
\mathcal{L}_c \rho(t) = \kappa \mathcal{D} \hat{a}(t) \rho(t),
\]

(S15)

where \( \mathcal{L}_c \) is the superoperator and \( \rho(t) = e^{-i H_s t} \rho(t) e^{i H_s t} \) is the density matrix of the system modes in the Schrödinger picture. The time dependence in the operator is cancelled due to the transformation. Equation (S15) has exactly the same form as that of a standard master equation (SME).

**Mechanical bath contribution**

The time-dependent operator \( \hat{b}(t) = e^{i H_s t} \hat{b} e^{-i H_s t} \) can be decomposed in the eigenbasis as

\[
\hat{b}(t) = \sum_{n,j} \left[ \sqrt{j} e^{-i \omega_m t} |n, (j - 1) \rangle \langle n, j \rangle + \beta n |n, j \rangle \langle n, j | \right],
\]

(S16)

which can be simplified to be

\[
\hat{b}(t) = e^{-i \omega_m t} (\hat{b} - \beta_0 \hat{N}_c) + \beta_0 \hat{N}_c
\]

(S17)

with \( \hat{N}_c = \hat{a}_c^\dagger \hat{a}_c \) being the photon number operator.

Using the expression \( \mathcal{G}_m(t) = \sum_j g_{mj} e^{-i \omega_m t} \hat{c}_m \), we derive the correlation function \( \mathcal{R}_m(s) \) in Eq. (S8) as

\[
\mathcal{R}_m(s) = \sum_j |g_{mj}|^2 n(\omega_{mj}, T) e^{i \omega_m s} + \sum_j |g_{mj}|^2 n(\omega_{mj}, T) e^{-i \omega_m s},
\]

(S18)

where \( n(\omega_{mj}, T) \) is the thermal occupation number of bath mode \( \hat{c}_{mj} \). We assume that the spectral density of the mechanical bath \( J_m(\omega) = \sum_j |g_{mj}|^2 \delta(\omega - \omega_{mj}) \) is Ohmic and takes the form of \( J_m(\omega) = 2 \pi \omega_m / 

\[
\rho'(t) = \frac{\gamma_m}{\omega_m} \int_0^\infty d\omega e^{-i \omega t} \left[ \coth \left( \frac{\omega}{2 k_B T} \right) + 1 \right],
\]

(S19)

where we have applied the relation \( n(-\omega, T) = -[n(\omega, T) + 1] \). Similar to the calculation for the cavity bath in Sec. , we find

\[
\int_0^\infty ds e^{i \omega_m s} \mathcal{R}_m(s) = \frac{\gamma_m}{2} (n_{th} + 1),
\]

\[
\int_0^\infty ds e^{-i \omega_m s} \mathcal{R}_m(s) = \frac{\gamma_m}{2} n_{th},
\]

\[
\int_0^\infty ds \mathcal{R}_m(s) = \frac{\gamma_m}{2} \left( \frac{k_B T}{\omega_m} \right),
\]

(S20)

where \( n_{th} \equiv n(\omega_m, T) \) is the thermal phonon number at the mechanical resonance.

Using the above results and applying the RWA to omit the fast oscillating terms, we derive the mechanical bath contribution to the DSME:

\[
\mathcal{L}_m \rho'(t) = \gamma_m (n_{th} + 1) \mathcal{D} \hat{b} - \beta_0 \hat{N}_c \rho'(t) + \gamma_m n_{th} \mathcal{D} \hat{b} - \beta_0 \hat{N}_c \rho'(t) + 4 \gamma_m \left( \frac{k_B T}{\omega_m} \right) \beta_0^2 \mathcal{D} \hat{N}_c \rho'(t),
\]

(S21)

With Eq. (S17), \( e^{-i H_s t} \hat{b} (\hat{b} - \beta_0 \hat{N}_c) e^{i H_s t} = e^{i \omega_m t} (\hat{b} - \beta_0 \hat{N}_c) \).

The mechanical bath contribution in the Schrödinger picture \( \mathcal{L}_m \rho(t) \) has exactly the same form as Eq. (S21) with the replacement \( \rho'(t) \rightarrow \rho(t) \).

**Summary**

Here we summarize the results derived in the previous subsections. In the Schrödinger picture, the DSME has the form

\[
\frac{d \rho(t)}{dt} = -i [\hat{H}_s, \rho(t)] + \mathcal{L}_c \rho(t) + \mathcal{L}_m \rho(t),
\]

(S22)

where \( \mathcal{L}_c \rho(t) \) and \( \mathcal{L}_m \rho(t) \) are given by Eqs. (S15) and (S21), respectively. Written explicitly in terms of the system operators, the master equation is

\[
\frac{d \rho(t)}{dt} = -i [\hat{H}_s, \rho(t)] + \kappa \mathcal{D} \hat{a} \rho(t)
\]

\[
+ \gamma_m (n_{th} + 1) \mathcal{D} \hat{b} - \beta_0 \hat{N}_c \rho(t) + \gamma_m n_{th} \mathcal{D} \hat{b} - \beta_0 \hat{N}_c \rho(t) + 4 \gamma_m \left( \frac{k_B T}{\omega_m} \right) \beta_0^2 \mathcal{D} \hat{N}_c \rho(t).
\]

(S23)
In the interaction picture, the DSME becomes

\[
\frac{d\rho(t)}{dt} = \kappa D[\hat{a}(t)]\rho(t) + \gamma_m(n_{th} + 1)D[\hat{b}]\rho(t) + \gamma_m(n_{th} + 1)D[\hat{b}^\dagger - \beta_0\hat{N}_c]\rho(t) + 4\gamma_m \left( \frac{k_B T}{\omega_m} \right) \beta_0^2 D[\hat{N}_c]\rho(t), \tag{S24}
\]

which contains fast oscillating terms with frequency \(O(\omega_m)\) generated by the phonon sidebands. These terms can be omitted under the RWA.

The SME for this system, often used in the literature, has the form

\[
\frac{d\rho(t)}{dt} = -i[\hat{H}_s, \rho(t)] + \kappa D[\hat{a}]\rho(t) + \gamma_m(n_{th} + 1)D[\hat{b}]\rho(t) + \gamma_m(n_{th} + 1)D[\hat{b}^\dagger - \beta_0\hat{N}_c]\rho(t). \tag{S25}
\]

in the Schrödinger picture. Applying the transformation \(\rho(t) = e^{i\hat{H}_s t} \rho(t) e^{-i\hat{H}_s t}\) and omitting the fast oscillating terms including \(e^{\pm i\omega_m t}\), the SME in the interaction picture becomes

\[
\frac{d\rho(t)}{dt} = \kappa D[\hat{a}(t)]\rho(t) + \gamma_m(n_{th} + 1)D[\hat{b}]\rho(t) + \gamma_m(n_{th} + 1)D[\hat{b}^\dagger - \beta_0\hat{N}_c]\rho(t) + \gamma_m(2n_{th} + 1)\beta_0^2 D[\hat{N}_c]\rho(t). \tag{S26}
\]

We have used Eq. (17) and the RWA in deriving this master equation. The difference between Eq. (24) and Eq. (26) is the last term in the master equation, which corresponds to photon dephasing. This difference is proportional to \(\gamma_m \left[ 4 \left( \frac{k_B T}{\omega_m} \right) - 2n_{th} - 1 \right] \beta_0^2\), and is originated from the mechanical bath modes. Because of the strong coupling between the cavity and the mechanical modes, the mechanical noise is transferred to the cavity mode and induces photon dephasing. At high temperature with \(k_B T \gg \omega_m\), the DSME predicts more serious dephasing than the SME. Whereas at low temperature, the DSME in Eq. (24) predicts slower dephasing than the SME. We want to note that the master equations here are all based on a bath correlation function given by Eq. (S19) which corresponds to a white noise on the mechanical mode at high temperature.

**Analytical solutions of operator averages**

With the DSME given above, the time evolution of some operators can be solved analytically. For the photon number operator \(\hat{N}_c\),

\[
\frac{d\langle \hat{N}_c \rangle}{dt} = -\kappa \langle \hat{N}_c \rangle, \tag{S27}
\]

which yields the solution \(\langle \hat{N}_c(t) \rangle = e^{-\kappa t} \langle \hat{N}_c(0) \rangle\). This result is the exactly same as the time evolution given by the SME, i.e., the dynamics of the photon number operator is not affected by our approach. This is because \(\hat{N}_c\) commutes with both \(\hat{H}_s\) and the extra dephasing term (the last term) in the DSME.

Similarly, for the annihilation operator of the mechanical mode \(\hat{b}\),

\[
\frac{d\langle \hat{b} \rangle}{dt} = -i\omega_m (\langle \hat{b} \rangle - \beta_0 \langle \hat{N}_c \rangle) - \frac{\gamma_m}{2} (\langle \hat{b} \rangle - \beta_0 \langle \hat{N}_c \rangle), \tag{S28}
\]

which depends on the photon number average \(\langle \hat{N}_c \rangle\). Combining Eq. (27) and Eq. (29), we derive

\[
\langle \hat{b}(t) \rangle = e^{-i\omega_m t - \frac{\omega_m t}{2}} \langle \hat{b}(0) \rangle + \frac{i\gamma_0 + \beta_0 \omega_m}{\omega_m + \gamma_m/2 - \kappa} \left( e^{-\kappa t} - e^{-i\omega_m t - \frac{\omega_m t}{2}} \right) \langle \hat{N}_c(0) \rangle, \tag{S29}
\]

which depends on the initial cavity photon number, but is independent of the thermal temperature of the mechanical bath.

**DSME FOR TWO CA VITY SYSTEM**

In this section, we derive the DSME for two cavity modes coupling to a common mechanical mode. The total Hamiltonian of this system can be written as

\[
\hat{H}_s = \sum_{i=1,2} \omega_c i \hat{a}_i^\dagger \hat{a}_i + \omega_m \hat{b} \hat{b} - \sum_{i=1,2} g_i \hat{a}_i^\dagger \hat{a}_i (\hat{b}^\dagger + \hat{b}), \tag{S30}
\]

where \(\hat{a}_i\) is the annihilation operator for the \(i\)th cavity mode, \(\omega_c\) is its frequency, and \(g_i\) is the coupling constant between cavity \(\hat{a}_i\) and the mechanical mode. The eigenstates of this Hamiltonian are

\[
|n_1, n_2, k(n_1, n_2) = |n_1\rangle_c |n_2\rangle_c e^{(\sum_i n_i \beta_i)(\hat{b}^\dagger - \hat{b})} |k\rangle_m \tag{S31}
\]

with \(\beta_i = g_i / \omega_m\). The corresponding eigenenergies are

\[
\varepsilon_{n_1, n_2, k} = n_1 \omega_c + n_2 \omega_c + k \omega_m - (n_1 \beta_1 + n_2 \beta_2) \omega_m. \tag{S32}
\]

To derive the DSME, we consider the time-dependent operators \(\hat{b}(t)\) and \(\hat{a}(t)\). For the mechanical mode, \(\hat{b}(t) = e^{i\hat{H}_s t} \hat{b} e^{-i\hat{H}_s t} = e^{-i\omega_m t} (\hat{b} - \hat{N}_t) + \hat{N}_t \tag{S33}\) with the effective number operator defined as

\[
\hat{N}_t = \beta_1 \hat{a}_1^\dagger \hat{a}_1 + \beta_2 \hat{a}_2^\dagger \hat{a}_2. \tag{S34}
\]

For the cavity mode, \(\hat{a}(t) = e^{i\hat{H}_s t} \hat{a} e^{-i\hat{H}_s t}\), including many terms due to the phonon sidebands. We apply the same assumptions as that in Sec., i.e., the cavity bath spectral density is smooth in the entire range of the
phonon sidebands and the mechanical bath is Ohmic. The DSME of this system can be derived as

\[
\frac{d\rho(t)}{dt} = -i[H_s, \rho(t)] \sum_{i=1,2} \kappa_i \mathcal{D}[\hat{a}_i]\rho(t) + \gamma_m (n_{th} + 1) \mathcal{D}[\hat{b} - \hat{N}_i]\rho(t) + \gamma_m n_{th} \mathcal{D}[\hat{b} - \hat{N}_i]\rho(t) + 4\gamma_m \left( \frac{k_B T}{\omega_m} \right) \mathcal{D}[\hat{N}_i]\rho(t).
\] (S35)

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