Higher spin fields with reversed spin-statistics relation

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Abstract

A construction of massive free fields with arbitrary spin and reversed spin-statistics relation is presented. In the construction of these fields normal (physical) fields with the same spin are taken as starting point, thus a correspondence between fields obeying normal and reversed spin-statistics relation is established. In particular, the polarization states, (anti)commutators, Feynman propagators and Green functions of the fields with reversed spin-statistics relation can be expressed in terms of those of the normal fields to which they correspond. The cases of the scalar and Dirac fields are discussed in additional detail.
1 Introduction

It is well-known that in quantum field theory, in addition to clearly physical fields, other fields that are not in every respect physical also have important role. For example, the Faddeev–Popov ghost fields \([1, 5, 8]\) appearing in quantized gauge field theories are of this kind. The main non-physical features of these fields are that although they are scalar fields, they are fermionic (anticommuting), and the Fock space generated by their modes has indefinite scalar product.

In this paper an elementary construction of massive free fields with arbitrary spin and reversed spin-statistics relation is presented. In relation to the latter property, the Fock space generated by the modes of these fields also has indefinite scalar product. Regarding other properties, specifically locality and the non-negativity of the energy spectrum, the fields with reversed statistical properties are not different from normal fields.

The construction is given in a formalism that was introduced in \([9]\), in which elementary (irreducible) higher spin fields \(\psi(\alpha)\) are obtained in the form

\[
\psi(\alpha) = D[\partial]^{\alpha \beta} \Psi_{\beta},
\]

where \(\Psi\) is an auxiliary multi-component Klein–Gordon field, which satisfies canonical (anti)-commutation relations, and \(D[\partial]\) is a differential operator that projects out the appropriate degrees of freedom from \(\Psi\). This formalism provides canonical field variables and thus it allows the derivation of canonical Hamiltonian equations of motion, if the interaction Hamiltonian operator as a local expression of fields is known. In this respect the formalism of \([9]\) extends that of \([2, 4]\). The calculation of Feynman propagators, (anti)commutators and Green functions of higher spin fields is also simple if they are given in the form (1.1). A further feature of the formalism of \([9]\) is that it exhibits the reality properties of the fields and of various tensors in a clear and covariant manner.

We focus, as in \([9]\), on fields transforming according to the representations \((n/2, m/2) \oplus (m/2, n/2)\) and \((n/2, n/2)\), \(n, m \in \mathbb{Z}, n, m \geq 0\), of \(SL(2, \mathbb{C})\) (the covering group of the Lorentz group), since these are the irreducible real representations of this group. Nevertheless, other types of fields, like the Rarita–Schwinger \([6]\) or the Wigner–Bargmann \([7]\) fields, could be treated in a similar way.

The fields with reversed spin-statistics properties are constructed using data that specifies normal fields. This shows, in particular, that to any normal field a corresponding field with the same spin but reversed spin-statistics relation can be associated.

Our aim is mainly to get an overview of the fields that may in principle appear in a quantum field theoretical model, and specifically to extend the survey of fields given in \([2, 3, 4]\) by considering fields with reversed spin-statistics relation. We stress that we do not make any restrictive assumption about the role of these fields; in particular we do not assume that they are Faddeev–Popov–De Witt type ghost fields.

Higher spin fields have attracted much interest for several decades, and their investigation continues also in the present days. For an incomplete collection of recent results we refer the reader to \([12, 10]\). The constant interest in higher spin fields can be explained partially by the fact that it is not easy to construct consistent models for their interactions. We restrict ourselves to the construction of free fields and thus deal with a much simpler problem, nevertheless we hope
that our elementary discussion can still be of interest. As far as we know, a similar discussion of higher spin fields with reversed spin-statistics relation cannot be found in the literature.

The paper is organized as follows. In Section 2 we recall in a shortened form the framework that we presented in [9] for describing normal fields. In Section 3 we present our construction of higher spin fields with reversed spin-statistics relation. In Sections 4 and 5 we discuss the special cases of the scalar and Dirac fields, respectively. A summary is given in Section 6.

The signature of the Minkowski metric $g_{\mu\nu}$ is taken to be $(-+++).$ Upper indices are used for Minkowski (space-time) vectors and lower indices for Minkowski covectors. Spinor indices and similar indices that label vector components in the representation spaces of $SL(2, \mathbb{C})$ are often written explicitly, but they are suppressed at other places in the text, depending on which notation appears clearer. Such indices are not raised or lowered. For vector spaces and representations the star $^\star$ is used to denote the dual vector space and representation. The dagger $^\dagger$ is used to denote the adjoint of operators in Hilbert spaces, and an analogous operation in the context of Lagrange formalism. The term Poincaré group refers in this paper to the simply connected covering group of the usual connected Poincaré group generated by the translations and Lorentz transformations of the Minkowski space-time.

2 Formalism for higher spin fields

2.1 Multi-component Klein–Gordon fields

The first step of the construction of higher spin fields described in [9] is to introduce multi-component (vector valued) bosonic and fermionic (i.e. commuting and anticommuting) Klein–Gordon fields $\Psi_\alpha,$ which we will also refer to as auxiliary fields. These fields are to some extent analogous to the electromagnetic vector potential in the Gupta–Bleuler formalism.

The Hamiltonian operator for the multi-component Klein–Gordon fields is

$$H = \int d^3x : \left[ \Pi_\alpha \Pi_\beta \epsilon^{\alpha\beta} + (\partial_\alpha \Psi_\beta^\dagger)(\partial_\beta \Psi_\alpha) \epsilon^{\alpha\beta} + \mu^2 \Psi_\alpha^\dagger \Psi_\alpha \epsilon^{\alpha\beta} \right] :,$$

where $\mu$ is the mass parameter and $\Pi_\alpha$ is the momentum field corresponding to $\Psi_\alpha.$ The components $\Psi_\alpha$ of $\Psi$ are indexed by $\alpha = 1, \ldots, N.$ In the fermionic case $N$ is even and $\epsilon^{\alpha\beta}$ is a nondegenerate antisymmetric and purely imaginary matrix:

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha},$$

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha^\dagger}. \quad (2.2)$$

In the bosonic case $\epsilon^{\alpha\beta}$ is a nondegenerate symmetric and real matrix:

$$\epsilon^{\alpha\beta} = \epsilon^{\beta\alpha},$$

$$\epsilon^{\alpha\beta} = \epsilon^{\alpha\beta^\dagger}. \quad (2.4)$$

We denote the inverse of $\epsilon^{\alpha\beta}$ by $\epsilon_{\alpha\beta};$ thus:

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma. \quad (2.6)$$
\( \epsilon^{\alpha\beta} \) and \( \epsilon_{\alpha\beta} \) are not used to raise or lower indices. \( \Psi_\alpha \) and \( \Pi_\alpha \) satisfy, by definition, the canonical equal time (anti)commutation relations

\[
[\Psi_\alpha(x,t), \Pi_\beta(x',t)]_{\pm} = i \epsilon_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{x}') \
[\Psi_\alpha(x,t), \Pi_\beta(x',t)]_{\pm} = i \epsilon_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{x}') ;
\]

(2.7)

(2.8)

all other (anti)commutators of \( \Psi_\alpha, \Psi_\alpha^\dagger, \Pi_\alpha, \Pi_\alpha^\dagger \) are 0. Here and in the following the notation \([], \[],_{\pm} \) is used to indicate that either a commutator (corresponding to the sign \(-\)) or an anticommutator (corresponding to the sign \(+\)) is meant; commutators apply to the case of bosons and anticommutators to the case of fermions.

\( \Psi_\alpha \) has the mode expansion

\[
\Psi_\alpha(x,t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2}\pi)^3} \omega(k)} [e^{i\mathbf{k}\cdot\mathbf{x}} e^{i\omega(k)t} a_\alpha^\dagger(k) + e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega(k)t} b_\alpha(k)] ,
\]

(2.9)

where \( a_\alpha^\dagger(k) \), \( b_\alpha^\dagger(k) \) and \( a_\alpha(k) \), \( b_\alpha(k) \) are creation and annihilation operators, respectively, \( k \) denotes 3-dimensional momentum, and \( \omega(k) = \sqrt{\mu^2 + k^2} \). The non-zero (anti)commutators of these creation and annihilation operators are

\[
[a_\alpha(k), a_\beta^\dagger(k')]_{\pm} = \epsilon_{\alpha\beta} \delta^3(k - k') \omega(k) 
\]

(2.10)

\[
[b_\alpha(k), b_\beta^\dagger(k')]_{\pm} = \epsilon_{\alpha\beta} \delta^3(k - k') \omega(k) .
\]

(2.11)

The Hamiltonian operator can be expressed in terms of these operators as

\[
H = \int \frac{d^3k}{\omega(k)} \omega(k) [a_\alpha^\dagger(k) a_\beta(k) \epsilon^{\alpha\beta} + b_\alpha^\dagger(k) b_\beta(k) \epsilon^{\alpha\beta}] .
\]

(2.12)

The Heisenberg equations of motion for \( \Psi_\alpha \) and \( \Pi_\alpha \) are

\[
[iH, \Psi_\alpha] = \partial_t \Psi_\alpha = \Pi_\alpha 
\]

(2.13)

\[
[iH, \Pi_\alpha] = \partial_t \Pi_\alpha = \partial_x \partial_x \Psi_\alpha - \mu^2 \Psi_\alpha,
\]

(2.14)

which imply that \( \Psi_\alpha \) satisfy the Klein–Gordon equation

\[
\partial_t \partial_t \Psi_\alpha - \partial_x \partial_x \Psi_\alpha + \mu^2 \Psi_\alpha = 0 .
\]

(2.15)

The Lagrangian for \( \Psi_\alpha \) is

\[
L = \int d^3x \mathcal{L} = \int d^3x \left[ (\partial_t \Psi_\alpha^\dagger(\partial_t \Psi_\beta) \epsilon^{\alpha\beta} - (\partial_x \Psi_\alpha^\dagger(\partial_x \Psi_\beta) \epsilon^{\alpha\beta} - \mu^2 \Psi_\alpha^\dagger \Psi_\beta \epsilon^{\alpha\beta} \right] ,
\]

(2.16)

where \( \Psi_\alpha \) is a complex valued field in the bosonic case and a complex Grassmann-algebra valued field in the fermionic case. In the bosonic case the dagger \( \dagger \) denotes simple complex conjugation, but to fermionic quantities the rule \( (\varphi \chi) = \chi^\dagger \varphi^\dagger \) applies instead of \( (\varphi \chi) = \varphi^\dagger \chi^\dagger \). The Euler–Lagrange equation corresponding to (2.16) is the Klein–Gordon equation above.
The canonical momentum for $\Psi_\alpha$ is
\[ \tilde{\Pi}^\alpha = \frac{\partial L}{\partial (\partial_t \Psi_\alpha)} = \epsilon^{\alpha\beta} \partial_t \Psi_\beta^\dagger, \] (2.17)
where $L$ denotes the Lagrangian density. The canonical equal time (anti)commutation relations are then
\[ [\Psi_\alpha(x,t), \tilde{\Pi}^\beta(x',t)] = i \delta_\alpha^\beta \delta^3(x - x') \] (2.18)
in the bosonic case, and
\[ \{\Psi_\alpha(x,t), \tilde{\Pi}^\beta(x',t)\} = -i \delta_\alpha^\beta \delta^3(x - x') \] (2.19)
in the fermionic case.

The relation between $\tilde{\Pi}^\alpha$ and $\Pi_\alpha$ is
\[ \Pi_\alpha = \epsilon_\alpha^\beta \tilde{\Pi}^\beta \] (2.20)
in the bosonic case and
\[ \Pi_\alpha = -\epsilon_\alpha^\beta \tilde{\Pi}^\beta \] (2.21)
in the fermionic case.

### 2.1.1 Fock space

The Fock space is spanned by the vacuum state $|0\rangle$, which is annihilated by the operators $a_\alpha(k)$, $b_\alpha(k)$, and by the multi-particle states
\[ |k_{a1}, p_{a1}; k_{a2}, p_{a2}; \ldots; k_{aj}, p_{aj}; k_{b1}, p_{b1}; k_{b2}, p_{b2}; \ldots; k_{bl}, p_{bl}\rangle = \left( \prod_{i=1}^j p_{a_i}^\alpha a_\alpha^\dagger(k_{ai}) \right) \left( \prod_{i=1}^l p_{b_i}^\alpha b_\alpha^\dagger(k_{bi}) \right) |0\rangle, \] (2.22)
where $k_{a1}, k_{a2}, \ldots, k_{aj}, k_{b1}, k_{b2}, \ldots, k_{bl}$ are the momenta of the particles and $p_{a1}, p_{a2}, \ldots, p_{aj}, p_{b1}, p_{b2}, \ldots, p_{bl}$ are their polarization vectors. The polarization vectors have $N$ components that can take arbitrary complex values.

On the space of polarization vectors, which we denote by $\hat{V}$, we introduce the scalar product
\[ \langle p_1, p_2 \rangle = (p_1)^\alpha \epsilon_\alpha^\beta (p_2)^\beta. \] (2.23)
Here the * denotes componentwise complex conjugation. On the dual space, denoted by $V$, we introduce the scalar product
\[ \langle p_1, p_2 \rangle = (p_1)_\alpha^\dagger (p_2)_\beta \epsilon^{\beta\alpha}. \] (2.24)
With these definitions, if $\hat{u}_i, i = 1, \ldots, N,$ is a basis in $\hat{V}$ such that $\langle \hat{u}_i, \hat{u}_j \rangle = s_i \delta_{ij}$, where $s_i$ is either 1 or $-1$, then the basis $u_i$ dual to $\hat{u}_i$ also satisfies $\langle u_i, u_j \rangle = s_i \delta_{ij}$, and vice versa. (The dual basis is defined by the property $\langle \hat{u}_i \rangle^\alpha (u_j)_\alpha = \delta_{ij}$.) The elements of $V$ are called dual polarization vectors.
The (anti)commutation relation of the creation and annihilation operators for particles in general polarization states is

\[
\begin{align*}
\langle (p_1)^\alpha a_\alpha(k_1), (p_2)^\beta a_\beta(k_2) \rangle_{\pm} &= \langle p_1, p_2 \rangle \delta^3(k_1 - k_2) \omega(k_1) \quad (2.25) \\
\langle (p_1)^\alpha b_\alpha(k_1), (p_2)^\beta b_\beta(k_2) \rangle_{\pm} &= \langle p_1, p_2 \rangle \delta^3(k_1 - k_2) \omega(k_1) \quad (2.26)
\end{align*}
\]

This shows that the creation and annihilation operators of particles that have orthogonal polarizations with respect to \( \langle , \rangle \) (anti)commute.

The scalar product on one-particle states is

\[
\langle k_a, p_a | k'_a, p'_a \rangle = \langle p_a, p'_a \rangle \delta^3(k_a - k'_a) \omega(k_a),
\]

where the subscript \( a \) indicates, as above, that the particles are created by the operators \( a^\dagger \). The same formula applies to particles created by the operators \( b^\dagger \). It is straightforward to extend this formula to multi-particle states; see [9].

The eigenvalue of \( H \) on a state of the form \( |2.22\rangle \) is \( \omega(k_{a1}) + \omega(k_{a2}) + \cdots + \omega(k_{aj}) + \omega(k_{b1}) + \omega(k_{b2}) + \cdots + \omega(k_{bk}) \), in particular the spectrum of \( H \) is non-negative.

As can be seen from \( 2.27 \), the definiteness properties of the scalar product on the Hilbert space defined above are determined by the signature of the scalar product \( \langle , \rangle \) given in \( 2.23 \). In the case of fermions, \( \langle , \rangle \) has the signature \( (N/2, N/2) \), whereas in the case of bosons \( \langle , \rangle \) has the same signature as \( \epsilon_{\alpha\beta} \).

**2.1.2 Feynman propagators, Green functions, (anti)commutators**

The Green function for \( \Psi \) is

\[
\langle 0 | \Psi_\alpha(x, t_x) \Psi_\beta^\dagger(y, t_y) | 0 \rangle = \epsilon_{\alpha\beta} G(x - y, t_x - t_y),
\]

and we have

\[
\langle 0 | \Psi_\alpha^\dagger(x, t_x) \Psi_\beta(y, t_y) | 0 \rangle = \langle 0 | \Psi_\alpha(x, t_x) \Psi_\beta^\dagger(y, t_y) | 0 \rangle
\]

\[
\langle 0 | \Psi_\alpha(x, t_x) \Psi_\beta(y, t_y) | 0 \rangle = \langle 0 | \Psi_\alpha^\dagger(x, t_x) \Psi_\beta^\dagger(y, t_y) | 0 \rangle = 0.
\]

The function \( G \), defined by

\[
G(x, t) = \int \frac{d^3k}{2(2\pi)^3 \omega(k)} e^{-ikx} e^{-i\omega(k)t},
\]

is the Green function of the usual Klein–Gordon field.

The Feynman propagator is

\[
\langle 0 | T \Psi_\alpha(x, t_x) \Psi_\beta^\dagger(y, t_y) | 0 \rangle = \epsilon_{\alpha\beta} D_F(x - y, t_x - t_y),
\]

and we have

\[
\langle 0 | T \Psi_\alpha^\dagger(x, t_x) \Psi_\beta(y, t_y) | 0 \rangle = \langle 0 | T \Psi_\alpha(x, t_x) \Psi_\beta^\dagger(y, t_y) | 0 \rangle
\]

\[
\langle 0 | T \Psi_\alpha(x, t_x) \Psi_\beta(y, t_y) | 0 \rangle = \langle 0 | T \Psi_\alpha^\dagger(x, t_x) \Psi_\beta^\dagger(y, t_y) | 0 \rangle = 0.
\]
The function $D_F$, defined as

$$D_F(x,t) = \int \frac{d^3k \, d\omega}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} e^{-ikx} e^{-i\omega t}, \quad (2.35)$$

is the Feynman propagator of the usual Klein–Gordon field, and $T$ denotes time ordering.

The (anti)commutator $[\Psi_\alpha(x,t), \Psi^\dagger_\beta(x,t)]_\pm$ is

$$[\Psi_\alpha(x,t), \Psi^\dagger_\beta(x,t)]_\pm = \epsilon_{\alpha\beta}[G(x - y, t_x - t_y) - G(y - x, t_x - t_y)]. \quad (2.36)$$

We also have

$$[\Psi^\dagger_\alpha(x,t), \Psi_\beta(x,t)]_\pm = [\Psi_\alpha(x,t), \Psi^\dagger_\beta(x,t)]_\pm \quad (2.37)$$

and

$$[\Psi_\alpha(x,t), \Psi_\beta(x,t)]_\pm = [\Psi^\dagger_\alpha(x,t), \Psi^\dagger_\beta(x,t)]_\pm = 0. \quad (2.38)$$

### 2.2 Lorentz transformation properties

For simplicity, in the previous sections the auxiliary multi-component Klein–Gordon fields were introduced without defining Lorentz transformation properties for them. We proceed now by defining Lorentz-covariant auxiliary multi-component Klein–Gordon fields.

The definition of a Lorentz-covariant multi-component Klein–Gordon field $\Psi$ is as follows.

1) Let $D$ be a finite dimensional real representation of the group $SL(2, \mathbb{C})$. In the bosonic (commuting) case, $D$ should be a bosonic (integer spin) representation, whereas in the fermionic (anticommuting) case $D$ should be fermionic (half-integer spin). A representation $D$ is bosonic or fermionic if its decomposition into irreducible representations contains only bosonic or fermionic, respectively, components. An irreducible representation $(n/2, m/2), n, m \in \mathbb{Z}, n, m \geq 0$, is bosonic if $n + m$ is even and fermionic if $n + m$ is odd. $D$ is the representation according to which $\Psi$ will transform.

2) An invariant complex conjugation in $D$ should be chosen; the reality of $D$ ensures, by definition, that this is possible.

3) An arbitrary basis consisting of real vectors in $D$ should also be chosen; the components $\Psi_\alpha$ are considered as vector components of $\Psi$ with respect to this basis. Thus $\Psi$ has $\dim(D)$ components. In this basis the invariant complex conjugation coincides with componentwise complex conjugation.

4) Further, an invariant tensor $\epsilon^{\alpha\beta}$ should be chosen so that it have the properties required of the $\epsilon^{\alpha\beta}$ tensor used in the previous sections (these properties were described at the beginning of Section 2.1). The properties of $D$ ensure that such an $\epsilon^{\alpha\beta}$ tensor exists. This tensor will serve as the $\epsilon^{\alpha\beta}$ tensor that appears in the general definition of multi-component Klein–Gordon fields.

5) The $SL(2, \mathbb{C})$ transformation rule for $\Psi$ takes the usual form

$$U[\Lambda]^{-1} \Psi_\alpha(x) U[\Lambda] = (\Lambda_D)^\beta_\alpha \Psi_\beta(\Lambda_M^{-1} x), \quad (2.39)$$

where $\Lambda$ is an element of $SL(2, \mathbb{C})$, $U[\Lambda]$ is the unitary operator that represents $\Lambda$ in the Hilbert space, $\Lambda_D$ is the matrix that represents $\Lambda$ in $D$, and $\Lambda_M$ is the matrix that represents $\Lambda$ in
Minkowski space-time. Due to the reality of $\Lambda_D$, the same transformation rule applies to $\Psi^\dagger$.

6) The vacuum $|0\rangle$ is invariant under $SL(2, \mathbb{C})$ transformations, i.e. $U[\Lambda]|0\rangle = |0\rangle$.

It follows from (2.39) that $SL(2, \mathbb{C})$ transformations act on the creation operators $a^\dagger_{\alpha}(k)$ as

$$U[\Lambda]^{-1}a^\dagger_{\alpha}(k)U[\Lambda] = (\Lambda_D)^{\beta\alpha}a^\dagger_{\beta}(\Lambda^T_Mk)$$

(2.40)

and $a_{\alpha}(k)$, $b_{\alpha}(k)$ and $b^\dagger_{\alpha}(k)$ also have the same transformation property. In this formula the $T$ in the superscript denotes transposition, and by writing $\Lambda^T_Mk$ it is meant that $\Lambda^T_M$ acts on the dual four-vector $k_{\mu} = (\omega(k), k)$.

As in [9], we shall concentrate mainly on the cases $D = D^{(n,m)}$ and $D = \tilde{D}^{(n)}$, where $n, m \in \mathbb{Z}$ and

$$D^{(n,m)} = (n/2, m/2) \oplus (m/2, n/2), \quad n \geq m , \quad n \neq 0$$

(2.41)

$$\tilde{D}^{(n)} = (n/2, n/2), \quad n \geq 0$$

(2.42)

nevertheless other representations could be treated similarly to these cases. $D^{(n,m)}$ and $\tilde{D}^{(n)}$ are the real irreducible representations of $SL(2, \mathbb{C})$, therefore they are the simplest possible choices for $D$. Any other finite dimensional real representation can be decomposed into a direct sum of these representations. $D^{(1,0)}$ is the Dirac representation and $\tilde{D}^{(1)}$ is the usual vector representation.

In [9] we also made specific choices for the invariant complex conjugation and $\epsilon^{\alpha\beta}$, which we will not describe here in any detail. Nevertheless, we describe these objects explicitly in the case of the Dirac field in Section 5.

The number of the components of $\Psi$ is $\dim(D^{(n,m)}) = 2(n + 1)(m + 1)$ and $\dim(\tilde{D}^{(n)}) = (n + 1)^2$, respectively, in these cases. In the case of $D = D^{(n,m)}$, $\Psi$ is bosonic if $n + m$ is even and fermionic if $n + m$ is odd. In the case of $D = \tilde{D}^{(n)}$, $\Psi$ is bosonic for any value of $n$.

The transformation properties of $\Psi$ under charge conjugation, space and time reflection ($C$, $P$, $T$) transformations take the form

$$C^{-1}\Psi_{\alpha}(x, t)C = \Psi_{\dagger\alpha}^I(x, t)$$

(2.43)

$$P^{-1}\Psi_{\alpha}(x, t)P = \mathcal{P}_{\alpha\beta}\Psi_{\beta}(-x, t)$$

(2.44)

$$T^{-1}\Psi_{\alpha}(x, t)T = \mathcal{T}_{\alpha\beta}\Psi_{\beta}(x, -t)$$

(2.45)

where $\mathcal{P}_{\alpha\beta}$ and $\mathcal{T}_{\alpha\beta}$ are suitable matrices.

### 2.3 Particle spectrum

In order to determine what kind of particles are described by $\Psi$, the space of allowed polarization vectors for any momentum $k$, which is $D^*$, should be decomposed into irreducible representations of the $SU(2)$ little group that leaves the momentum four-vector $(\omega(k), k)$ invariant. Along with $D^*$ it is worth considering $D$ as well. Due to Lorentz symmetry it is sufficient to focus first on $k = 0$, and then obtain the decomposition corresponding to $k \neq 0$ by boost.

In the following we continue with the case of $D = D^{(n,m)}$, and discuss the case of $D = \tilde{D}^{(n)}$ subsequently.
\[ D = D^{(n,m)}; \]

In the representations \( D^{(n,m)} \) it is possible to find basis vectors \( u_{(s)i}, v_{(s)i}; s = (n-m)/2, (n-m)/2 + 1, \ldots, (n+m)/2; i = 1, \ldots, d_s; \] \( d_s = 2s + 1 \) so that the orthogonality relations

\[
\langle u_{(s)i}, u_{(s')j} \rangle = \delta_{ss'}\delta_{ij} \quad \langle v_{(s)i}, v_{(s')j} \rangle = -\delta_{ss'}\delta_{ij} \quad \langle u_{(s)i}, v_{(s')j} \rangle = 0 \quad (2.46)
\]

hold, where \( \langle \cdot, \cdot \rangle \) denotes the scalar product introduced in Section 2.1.1. These basis vectors also have the property that for any fixed \( s, u_{(s)i}, i = 1, \ldots, d_s \) and \( v_{(s)i}, i = 1, \ldots, d_s \) span two subspaces of \( D^{(n,m)} \) that are (irreducible) spin \( s \) representations with respect to the rotation \((SU(2))\) little group that leaves the momentum covector \((\mu, 0)\) invariant.

The vectors \( u_{(s)i}(k), v_{(s)i}(k) \) are defined now as

\[
u_{(s)i}(k) = \Lambda_D(k)u_{(s)i}, \quad i = 1, \ldots, d_s \tag{2.47}
v_{(s)i}(k) = \Lambda_D(k)v_{(s)i}, \quad i = 1, \ldots, d_s \tag{2.48}
\]

where \( \Lambda_D(k) \) represents in \( D \) the unique \( SL(2, \mathbb{C}) \) element \( \Lambda(k) \) determined by the properties that \( \Lambda(k) \) is a continuous function of \( k, \Lambda(0) = I, \) and the Lorentz transformation corresponding to \( \Lambda(k) \) is the Lorentz boost that takes the dual four-vector \((\mu, 0)\) to \((\omega(k), k)\).

For any fixed \( s, u_{(s)i}(k), i = 1, \ldots, d_s, \) and \( v_{(s)i}(k), i = 1, \ldots, d_s, \) span two subspaces of \( D^{(n,m)} \) that are (irreducible) spin \( s \) representations with respect to the rotation \((SU(2))\) little group that leaves the momentum covector \((\omega(k), k)\) invariant.

The vectors dual to \( u_{(s)i}(k), v_{(s)i}(k) \) are denoted by \( \hat{u}_{(s)i}(k), \hat{v}_{(s)i}(k) \). These vectors also satisfy the orthogonality relations

\[
\langle \hat{u}_{(s)i}(k), \hat{u}_{(s')j}(k) \rangle = \delta_{ss'}\delta_{ij} \quad \langle \hat{v}_{(s)i}(k), \hat{v}_{(s')j}(k) \rangle = -\delta_{ss'}\delta_{ij} \quad \langle \hat{u}_{(s)i}(k), \hat{v}_{(s')j}(k) \rangle = 0 \quad (2.49)
\]

where \( \langle \cdot, \cdot \rangle \) denotes again the scalar product introduced in Section 2.1.1 thus they form a complete set of orthogonal polarization vectors. For any fixed \( s, \hat{u}_{(s)i}(k), i = 1, \ldots, d_s, \) and \( \hat{v}_{(s)i}(k), i = 1, \ldots, d_s, \) span two subspaces of \( D^{(n,m)*} \) that are irreducible spin \( s \) representations with respect to the \((SU(2))\) little group that leaves the momentum covector \((\omega(k), k)\) invariant. This follows from the duality between \( u_{(s)i}(k), v_{(s)i}(k) \) and \( \hat{u}_{(s)i}(k), \hat{v}_{(s)i}(k) \), and is in accordance with the fact that for any \( k \) the decomposition of both \( D^{(n,m)*} \) and \( D^{(n,m)} \) into irreducible representations of the corresponding \((SU(2))\) little group is

\[
\left( \frac{n + m}{2} \right) \oplus \left( \frac{n + m}{2} - 1 \right) \oplus \left( \frac{n + m}{2} - 2 \right) \oplus \cdots \oplus \left( \frac{n - m}{2} \right) \oplus \left( \frac{n - m}{2} \right). \quad (2.50)
\]

The following completeness relations can be written down for \( u_{(s)i}(k), v_{(s)i}(k), \hat{u}_{(s)i}(k), \hat{v}_{(s)i}(k) \):

\[
\delta_{\alpha}^{\beta} = \sum_{s} \sum_{i=1}^{d_s} u_{(s)ia}(k) \hat{u}_{(s)i}(k) + \sum_{s} \sum_{i=1}^{d_s} v_{(s)ia}(k) \hat{v}_{(s)i}(k) \tag{2.51}
\]

\[
\delta_{\alpha}^{\beta} = \sum_{s} \sum_{i=1}^{d_s} u_{(s)ia}(k)^* \hat{u}_{(s)i}(k)^* + \sum_{s} \sum_{i=1}^{d_s} v_{(s)ia}(k)^* \hat{v}_{(s)i}(k)^*. \tag{2.52}
\]
\[ (2.52) \] is obtained from \((2.51)\) by complex conjugation.

The orthogonality relations \((2.46)\) imply that
\[
\hat{u}_i^\beta(k) = e^{\beta\alpha}u_{(s)i\alpha}(k)^*, \quad \hat{v}_i^\alpha(k) = -e^{\beta\alpha}v_{(s)i\alpha}(k)^*,
\]
thus \((2.51)\) can also be written in the form
\[
\delta_{\alpha}^\beta = \sum_s \sum_{i=1}^{d_s} u_{(s)i\alpha}(k)\bar{u}_{(s)i}(k) - \sum_s \sum_{i=1}^{d_s} v_{(s)i\alpha}(k)\bar{v}_{(s)i}(k),
\]
where the notation
\[
\bar{u}^\alpha = e^{\alpha\beta}u_\beta^*,
\]
is used. In the case of the Dirac field, \(\bar{u}\) is the Dirac-conjugate of \(u\) (see Section 5), therefore \((2.51)\) has a form that is more familiar from textbooks than \((2.54)\). However, \((2.51)\) is more general than \((2.54)\), since it does not make use of the orthogonality properties of \(u_{(s)i\alpha}(k)\) and \(v_{(s)i\alpha}(k)\).

The creation operators that create particles with polarizations \(\hat{u}_{(s)i}(k)\) and \(\hat{v}_{(s)i}(k)\) are
\[
\begin{align*}
c_{(s)i}^\dagger(k) &= \hat{u}_{(s)i}(k)^\alpha a_{(s)i}^\alpha(k), \quad i = 1, \ldots, d_s \\
d_{(s)i}^\dagger(k) &= \hat{u}_{(s)i}(k)^\alpha b_{(s)i}^\alpha(k), \quad i = 1, \ldots, d_s \\
f_{(s)i}^\dagger(k) &= \hat{v}_{(s)i}(k)^\alpha a_{(s)i}^\alpha(k), \quad i = 1, \ldots, d_s \\
h_{(s)i}^\dagger(k) &= \hat{v}_{(s)i}(k)^\alpha b_{(s)i}^\alpha(k), \quad i = 1, \ldots, d_s.
\end{align*}
\]
\(c_{(s)i}^\dagger(0)\) is related to \(c_{(s)i}^\dagger(0)\) by the formula
\[
c_{(s)i}^\dagger(0) = U[\Lambda(k)]c_{(s)i}^\dagger(0)U[\Lambda(k)]^{-1},
\]
where \(\Lambda(k)\) is the boost defined as above. The same formula applies also to \(d_{(s)i}^\dagger(k)\), \(f_{(s)i}^\dagger(k)\) and \(h_{(s)i}^\dagger(k)\).

It follows from \((2.25)\), \((2.26)\) and \((2.49)\) that the operators \(c_i(k), d_i(k), f_i(k), h_i(k)\) satisfy the (anti)commutation relations
\[
\begin{align*}
[c_{(s)i}(k), c_{(s')j}^\dagger(k')]_{\pm} &= [d_{(s)i}(k), d_{(s')j}^\dagger(k')]_{\pm} = \delta_{ss'}\delta_{ij}\delta^3(k-k')\omega(k) \quad (2.61) \\
[f_{(s)i}(k), f_{(s')j}^\dagger(k')]_{\pm} &= [h_{(s)i}(k), h_{(s')j}^\dagger(k')]_{\pm} = -\delta_{ss'}\delta_{ij}\delta^3(k-k')\omega(k), \quad (2.62)
\end{align*}
\]
all other (anti)commutators of them are zero. This shows that the one-particle states created by \(c_{(s)i}^\dagger(k)\) and \(d_{(s)i}^\dagger(k)\) have positive scalar product with themselves, whereas the one-particle states created by \(f_{(s)i}^\dagger(k)\) and \(h_{(s)i}^\dagger(k)\) have negative scalar product with themselves.
By making use of relations (2.51) and (2.52), $\Psi$ can be expressed in terms of $c_{(s)i}$, $d_{(s)i}$, $f_{(s)i}$, $h_{(s)i}$ as

$$
\Psi_\alpha(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2}\pi)^3}} e^{ikx} e^{i\omega(k)t} \sum_{s} \sum_{i=1}^{d_s} \{ u_{(s)ia}(k) c_{(s)i}^\dagger(k) + v_{(s)ia}(k) f_{(s)i}^\dagger(k) 
\quad \quad + e^{-ikx} e^{-i\omega(k)t} \sum_{s} \sum_{i=1}^{d_s} \{ u_{(s)ia}(k)^* d_{(s)i}(k) + v_{(s)ia}(k)^* h_{(s)i}(k) \} \} .
$$

(2.63)

The values of $s$ over which the summation is done here are $(n-m)/2, (n-m)/2+1, \ldots, (n+m)/2$. This formula, along with the properties of the polarization vectors discussed above, shows that $\Psi$ describes 4 kinds of particles of mass $\mu$ for any value of $s$. Two of them are physical and two are non-physical (in the sense that the one-particle states have negative scalar product with themselves).

The Hamiltonian operator can be expressed as

$$
H = \int \frac{d^3k}{\omega(k)} \sum_{s} \sum_{i=1}^{d_s} \left[ c_{(s)i}^\dagger(k) c_{(s)i}(k) + d_{(s)i}^\dagger(k) d_{(s)i}(k)
\quad \quad - f_{(s)i}^\dagger(k) f_{(s)i}(k) - h_{(s)i}^\dagger(k) h_{(s)i}(k) \right].
$$

(2.64)

$D = \tilde{D}^{(n)}$;

The case of $D = \tilde{D}^{(n)}$ is very similar to the case of $D = D^{(n,m)}$; the main difference is that the basis vectors $u_{(s)i}$ are defined now only for $s \in S_+ = \{n, n-2, n-4, \ldots\}$, whereas $v_{(s)i}$ are defined only for $s \in S_- = \{n-1, n-3, n-5, \ldots\}$. ($S_+$ and $S_-$ can be interchanged by changing the sign of $e^{\alpha\beta}$.) The decomposition of $\tilde{D}^{(n)}$ and $\tilde{D}^{(n)\dagger}$ into irreducible representations of the $SU(2)$ little group corresponding to any momentum $k$ is

$$(n) \oplus (n-1) \oplus (n-2) \oplus \cdots \oplus (1) \oplus (0) .
$$

(2.65)

The completeness relations have to be modified in the following obvious way:

$$
\delta_{\alpha\beta} = \sum_{s \in S_+} \sum_{i=1}^{d_s} u_{(s)ia}(k) \hat{u}_{(s)i}(k) + \sum_{s \in S_-} \sum_{i=1}^{d_s} v_{(s)ia}(k) \hat{v}_{(s)i}(k)
$$

(2.66)

$$
\delta_{\alpha\beta} = \sum_{s \in S_+} \sum_{i=1}^{d_s} u_{(s)ia}(k)^* \hat{u}_{(s)i}(k)^* + \sum_{s \in S_-} \sum_{i=1}^{d_s} v_{(s)ia}(k)^* \hat{v}_{(s)i}(k)^* .
$$

(2.67)

The mode expansion of $\Psi$ takes the form

$$
\Psi = \psi + \psi^- ,
$$

(2.68)

where

$$
\psi_\alpha(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2}\pi)^3}} e^{ikx} e^{i\omega(k)t} \sum_{s \in S_+} \sum_{i=1}^{d_s} u_{(s)ia}(k) c_{(s)i}^\dagger(k)
\quad \quad + e^{-ikx} e^{-i\omega(k)t} \sum_{s \in S_+} \sum_{i=1}^{d_s} u_{(s)ia}(k)^* d_{(s)i}(k) \}
$$

(2.69)
and

\[ (\psi^\dagger) \alpha(x, t) = \int \frac{d^3k}{\sqrt{2(2\pi)^3}} e^{ikx} e^{i\omega(k)t} \sum_{s \in S_+} \sum_{i=1}^{d_s} v(s)_{i\alpha}(k) f^i_{(s)j}(k) \]

\[ + e^{-ikx} e^{-i\omega(k)t} \sum_{s \in S_-} \sum_{i=1}^{d_s} v(s)_{i\alpha}(k)^* h_{(s)j}(k) \, . \]  

(2.70)

In the present case \( \Psi \) describes two kinds of physical particles of mass \( \mu \) for any value of \( s \in S_+ \) and two kinds of non-physical particles of mass \( \mu \) for any value of \( s \in S_- \) (again, the non-physical nature is understood to mean that the one-particle states have negative scalar product with themselves). In particular, \( \psi \) and \( \psi^\dagger \) create the physical particles and \( \psi^- \) and \( \psi^-^\dagger \) the non-physical ones.

The Hamiltonian operator can be written as

\[ H = \int \frac{d^3k}{\omega(k)} \omega(k) \sum_{s \in S_+} \sum_{i=1}^{d_s} \left[ c^\dagger_{(s)j}(k)c_{(s)j}(k) + d^\dagger_{(s)j}(k)d_{(s)j}(k) \right] \]

\[ - \int \frac{d^3k}{\omega(k)} \omega(k) \sum_{s \in S_-} \sum_{i=1}^{d_s} \left[ f^\dagger_{(s)j}(k)f_{(s)j}(k) - h^\dagger_{(s)j}(k)h_{(s)j}(k) \right] \, . \]  

(2.71)

In the following, polarization vectors \( \hat{u}_{(s)j}(k) \), \( \hat{v}_{(s)j}(k) \) having the properties described in this section will be called standard polarization vectors. It should also be noted that these standard polarization vectors are usually chosen so that they have the additional property of being eigenstates of the \( SU(2) \) little group generator \( \Lambda_{D^*}(k)M_{3D^*} \Lambda_{D^*}(k)^{-1} \). More details about the standard polarization vectors can be found in [9].

### 2.4 Elementary higher spin fields

In this section the last step in the construction of higher spin fields is described. We continue to focus on the representations \( D = D^{(n,m)} \) and \( \bar{D} = \bar{D}^{(n)} \), treating these cases together.

As was seen in Section 2.3 the fields \( \Psi \) can create states that have negative scalar product with themselves. Moreover, \( \Psi \) can generally create states with several different spins, except in the cases \( D = D^{(n,0)} \). An elementary complex higher spin field \( \psi_{(s)} \) that creates states with a specific spin \( s \) and having positive scalar product with themselves can be obtained from \( \Psi \) by applying to it a suitable real Poincaré-invariant differential operator \( \mathcal{D}_{(s)}[\partial] \), which projects out the desired modes:

\[ \psi_{(s)} = \mathcal{D}_{(s)}[\partial] \Psi \, . \]  

(2.72)

These \( \mathcal{D}_{(s)}[\partial] \) operators are orthogonal projections in the space of the solutions of the Klein–Gordon equation (2.15), i.e. \( \mathcal{D}_{(s)}[\partial] \mathcal{D}_{(s')}[\partial] \Psi = \mathcal{D}_{(s)}[\partial] \Psi \) and \( \mathcal{D}_{(s)}[\partial] \mathcal{D}_{(s')}[\partial] \Psi = 0, \) \( s \neq s' \), hold. Thus \( \psi_{(s)} \) satisfies in addition to the Klein–Gordon equation

\[ (\partial^2_t - \partial^2_x + \mu^2)\psi_{(s)} = 0 \]  

(2.73)
Equations (2.78) and (2.79) follow by complex conjugation from (2.76) and (2.77) and from the differential equations

$$D_{(s)[\partial]}[\partial]D_{(s)[\partial]} = \psi_{(s)}$$

(2.74)

$$D_{(s')[\partial]}[\partial]D_{(s)[\partial]} = 0, \quad s' \neq s.$$  

(2.75)

$D_{(s)[\partial]}$ has the following action on the standard polarization modes of $\Psi$:

$$D_{(s)}[ik]^{\beta\alpha}u(s')_{i\alpha}(k) = \delta_{ss'}u_{(s)i\alpha}(k)$$

(2.76)

$$D_{(s)}[ik]^{\beta\alpha}v(s')_{i\beta}(k) = 0$$

(2.77)

$$D_{(s)}[-ik]^{\beta\alpha}u(s')_{i\alpha}(k)^* = \delta_{ss'}u_{(s)i\alpha}(k)^*$$

(2.78)

$$D_{(s)}[-ik]^{\beta\alpha}v(s')_{i\beta}(k)^* = 0.$$  

(2.79)

Equations (2.78) and (2.79) follow by complex conjugation from (2.76) and (2.77) and from the reality of $D_{(s)[\partial]}$. The notation $D_{(s)}[ik]$ means that $\partial_\mu$ in the expression of $D_{(s)[\partial]}$ is replaced by the momentum covector $(i\omega(k), ik)$. $D_{(s)[\partial]}$ is a polynomial of $\partial_\mu$, with coefficients that are real $SL(2, \mathbb{C})$-invariant tensors. The property that $D_{(s)[\partial]}$ is an invariant differential operator implies that the Poincaré transformation properties of $\psi_{(s)}$ are the same as those of $\Psi$.

A real invariant linear differential operator $D^-[\partial]$ which projects out the non-physical modes from $\Psi$ (i.e. that part of $\Psi$ which creates one-particle states that have negative scalar product with themselves) can also be introduced. On the standard polarization modes of $\Psi$ it has the following action (for all values of $s$):

$$D^-[ik]^{\beta\alpha}u(s')_{i\alpha}(k) = 0$$

(2.80)

$$D^-[ik]^{\beta\alpha}v(s')_{i\beta}(k) = v_{(s)i\beta}(k)$$

(2.81)

$$D^-[-ik]^{\beta\alpha}u(s')_{i\alpha}(k)^* = 0$$

(2.82)

$$D^-[-ik]^{\beta\alpha}v(s')_{i\beta}(k)^* = v_{(s)i\beta}(k)^*.$$  

(2.83)

Equations (2.82) and (2.83) follow by complex conjugation from (2.80) and (2.81) and from the reality of $D^-[\partial]$.

It follows from (2.76), (2.77) and (2.81), (2.81) that $D_{(s)}[ik]^{\alpha\beta}$ and $D^-[-ik]^{\alpha\beta}$ can be expressed in terms of standard polarization vectors as

$$\sum_{i=1}^{d_s} u_{(s)i\alpha}(k)\tilde{u}_{(s)i\alpha}(k) = D_{(s)}[ik]^{\alpha\beta}$$

(2.84)

$$\sum_{s} \sum_{i=1}^{d_s} v_{(s)i\alpha}(k)\tilde{v}_{(s)i\alpha}(k) = D^-[ik]^{\alpha\beta}. $$

(2.85)

Complex conjugation of these formulas gives

$$\sum_{i=1}^{d_s} u_{(s)i\alpha}(k)^*\tilde{u}_{(s)i\alpha}(k)^* = D_{(s)}[-ik]^{\alpha\beta}$$

(2.86)

$$\sum_{s} \sum_{i=1}^{d_s} v_{(s)i\alpha}(k)^*\tilde{v}_{(s)i\alpha}(k)^* = D^-[-ik]^{\alpha\beta}. $$

(2.87)
Formulas of this type are also known as spin sums. They could be rewritten in a more familiar but less general form in the same way as (2.54).

\[ \mathcal{D}_{(s)}[\partial] \] and \( \mathcal{D}^-[\partial] \) form a complete set of projectors on the space of the solutions of the Klein–Gordon equation:

\[ \sum_{s} \mathcal{D}_{(s)}[\partial]_{\alpha}^{\beta} \Psi_{\beta} + \mathcal{D}^-[\partial]_{\alpha}^{\beta} \Psi_{\beta} = \Psi_{\alpha}. \]  

(2.88)

\( \psi_{(s)} \) also satisfies the differential equation

\[ \mathcal{D}^-[\partial] \psi_{(s)} = 0. \]  

(2.89)

The properties (2.70)–(2.71) of \( \mathcal{D}_{(s)}[\partial] \) imply that \( \psi_{(s)} \) has the mode expansion

\[ \psi_{(s)}(x, t) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega(k)}} \sum_{\mu} \left[ e^{ikx} e^{-i\omega(k)t} u_{(s)\alpha}^{\mu}(k)c_{(s)j}^{\dagger}(k) 
+ e^{-ikx} e^{i\omega(k)t} u_{(s)\alpha}^{\mu}(k)d_{(s)j}(k) \right]. \]  

(2.90)

This shows that \( \psi_{(s)} \) describes a pair of particles of mass \( \mu \) and spin \( s \).

The (anti)commutation relations of \( \psi_{(s)} \) are

\[ [\psi_{(s)\alpha}(x, t_x), \psi_{(s)\beta}^{\dagger}(y, t_y)]_\pm = [\psi_{(s)\alpha}(x, t_x), \psi_{(s)\beta}(y, t_y)]_\pm = [\mathcal{D}_{(s)}[\partial(t_x, x)]_{\alpha}^{\beta}\Psi_{\beta}(x, t_x), \Psi_{\beta}^{\dagger}(y, t_y)]_\pm = \mathcal{D}_{(s)}[\partial(t_x, x)]_{\alpha}^{\delta} \epsilon_{\delta\beta} \delta \epsilon_\beta G(x - y, t_x - t_y) - G(y - x, t_x - t_y)]_\pm. \]  

(2.91)

Here the subscript in \( \partial(t_x, x) \) indicates the variables with respect to which the differentiation should be done.

The Green function of \( \psi_{(s)} \) is

\[ \langle 0|\psi_{(s)\alpha}(x, t_x)\psi_{(s)\beta}^{\dagger}(y, t_y)|0 \rangle = \langle 0|\psi_{(s)\alpha}(x, t_x)\psi_{(s)\beta}(y, t_y)|0 \rangle = \langle 0|\psi_{(s)\alpha}(x, t_x)\Psi_{\beta}^{\dagger}(y, t_y)|0 \rangle = \langle 0|\mathcal{D}_{(s)}[\partial(t_x, x)]_{\alpha}^{\delta} \epsilon_{\delta\beta} G(x - y, t_x - t_y) \rangle. \]  

(2.92)

The covariant part of the Feynman propagator

\[ \langle 0|T\psi_{(s)\alpha}(x, t_x)\psi_{(s)\beta}^{\dagger}(y, t_y)|0 \rangle = \langle 0|T\psi_{(s)\alpha}(x, t_x)\psi_{(s)\beta}(y, t_y)|0 \rangle = \langle 0|T\psi_{(s)\alpha}(x, t_x)\Psi_{\beta}^{\dagger}(y, t_y)|0 \rangle = \langle 0|T\mathcal{D}_{(s)}[\partial(t_x, x)]_{\alpha}^{\delta} \psi_{\beta}(x, t_x) \Psi_{\beta}^{\dagger}(y, t_y)|0 \rangle \]  

(2.93)

of \( \psi_{(s)} \) is

\[ \mathcal{D}_{(s)}[\partial(t_x, x)]_{\alpha}^{\delta} \epsilon_{\delta\beta} DF(x - y, t_x - t_y), \]  

(2.94)
where $D_F$ denotes the Feynman propagator of the scalar Klein-Gordon field. (2.94) can also be written as

$$D_F(x - y, t_x - t_y) \propto \int \frac{d^3k \, dk_0 \, i \mathcal{D}(\alpha)[i(k_0, k)]_\alpha^\delta \epsilon_{\delta\beta} e^{-ik(x-y)} e^{-ik_0(t_x-t_y)}}{(2\pi)^4 \, k_0^2 - k^2 - \mu^2 + i\epsilon} \, .$$

The notation $\mathcal{D}(\alpha)[i(k_0, k)]$ indicates that $\partial_\mu$ is replaced in the expression for $\mathcal{D}(\alpha)[\partial]$ by the momentum covector $-i(k_0, k)$.

Generally it is an algebraic problem to find the $\mathcal{D}(\alpha)[\partial]$ and $\mathcal{D}^-(\partial)$ operators. Without going into details, we note that for small values of $m$ or $n$ it is not difficult to find these operators by direct calculations, and relevant general results for any spin can be found in [2, 3, 4] (see also [9]). Explicit expressions for relevant projection operators for the Rarita–Schwinger fields are also available in the literature, e.g. in [10, 11, 13].

We note that real fields can be obtained simply by taking the real or imaginary part of the complex fields: for a complex field $\Psi$, $\Phi$ also available in the literature, e.g. in [10, 11, 13]. Explicit expressions for relevant projection operators for the Rarita–Schwinger fields are also available in the literature, e.g. in [10, 11, 13].

We note that real fields can be obtained by projection on the complex fields: for a complex field $\Psi$, $\Phi_1 = \frac{1}{\sqrt{2}}(\Psi + \Psi^\dagger)$ and $\Phi_2 = \frac{1}{\sqrt{2}}(\Psi - \Psi^\dagger)$ are the corresponding real fields. The real momentum fields $\Pi_1$ and $\Pi_2$ can be obtained in the same way. The real and imaginary parts $\phi_{(s)1}$ and $\phi_{(s)2}$ of a higher spin field $\psi_{(s)}$ can also be defined in the same way. Since the operators $\mathcal{D}_{(s)}[\partial]$ are real, $\phi_{(s)1}$ and $\phi_{(s)2}$ are related to $\Phi_1$ and $\Phi_2$ in the same way as $\psi_{(s)}$ is related to $\Psi$, i.e. $\phi_{(s)1} = \mathcal{D}_{(s)}[\partial] \Phi_1$, $\phi_{(s)2} = \mathcal{D}_{(s)}[\partial] \Phi_2$. These real fields describe half as many particles as the complex fields do.

Fields transforming according to the representations $(n/2, m/2)$, $n \neq m$ (i.e. chiral fields) can also be obtained from the fields that transform according to $D^{(n,m)}$ by projection on the $(n/2, m/2)$ or $(m/2, n/2)$ component. More details on real and chiral fields can be found in [9].

The $C$, $P$, $T$ transformation properties of $\psi_{(s)}$ are determined by (2.43), (2.44), (2.45) and (2.72). Generally $P$, $T$ and $\mathcal{D}_{(s)}[\partial]$ should be chosen so that the $C$, $P$, $T$ transformation properties of $\psi_{(s)}$ be the same as those of $\Psi$.

Regarding the construction of interaction Hamiltonians from higher spin fields, we refer the reader to [4, 2, 13], and to [9], especially Section 5.4.

3 Fields of arbitrary spin with reversed spin-statistics relation

We turn now to the description of the construction of fields with reversed spin-statistics relation (abbreviated hereafter as RSS fields) in the framework described in Section 2. The RSS fields will be constructed from normal fields; more precisely, data that specify RSS fields will be constructed from data used to specify the normal fields. We do not introduce extra notation to distinguish RSS fields from normal fields.

The main ingredients in the definition of a normal higher spin field are the representation $D$, the bilinear form $\epsilon^{\alpha\beta}$, the invariant complex conjugation, the real basis vectors in $D$, the standard polarization vectors, and the projection operator $\mathcal{D}_{(s)}[\partial]$. Assuming that these objects are given, a corresponding new $SL(2, \mathbb{C})$ representation, bilinear form, complex conjugation, real basis vectors, standard polarization vectors and projection operator, which will characterize the RSS field that corresponds to the original normal field, can be defined, as described below. The RSS fields obtained in this way will fit entirely in the framework of Section 2 with the only
difference that bosonic RSS fields will transform according to half-integer spin representations of $SL(2, \mathbb{C})$ and fermionic RSS fields according to integer spin representations.

For the representation $D_{RSS}$ according to which the RSS auxiliary field $\Psi$ transforms we take the direct sum of two copies $D_A$ and $D_B$ of the representation $D$:

$$D_{RSS} = D_A \oplus D_B,$$

where the subscripts $A$ and $B$ are introduced to identify the two components of the direct sum.

We use the notation

$$U = \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$

for vectors in $D_{RSS}$, i.e. we denote by $U_A$ and $U_B$ the projection of a vector $U$ on the components $D_A$ and $D_B$, respectively, of $D_{RSS}$. For dual vectors (which are in $D^*_{RSS}$, and which have, according to the definitions in Section 2.1.1, the role of polarization vectors) we use a similar notation

$$\hat{U} = (\hat{U}^A, \hat{U}^B).$$

We define the invariant complex conjugation in $D_{RSS}$ in a straightforward way via the complex conjugation in the two components $D_A$ and $D_B$:

$$U^* = \begin{pmatrix} U^*_A \\ U^*_B \end{pmatrix}.$$ (3.4)

For dual vectors we have then

$$\hat{U}^* = (\hat{U}^{*A}, \hat{U}^{*B}).$$ (3.5)

With the above choice of the complex conjugation in $D_{RSS}$, if $e_i$ are real basis vectors in $D$, then the vectors $\begin{pmatrix} e_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_i \end{pmatrix}$ are real basis vectors in $D_{RSS}$. This applies if, in particular, \{e_i\} is the fixed real basis in $D$ mentioned in Section 2.2, thus the fixed real basis in $D_{RSS}$ can be chosen to be \{\begin{pmatrix} e_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_i \end{pmatrix}\}.

For the vector components of vectors in $D_{RSS}$ we use the notation $U_{\alpha}$, where $\alpha = 1, 2, \ldots, 2\dim(D)$. This means that we do not introduce a new type of index for vectors in $D_{RSS}$, nevertheless this should not cause confusion.

For the invariant bilinear form on $D_{RSS}$ we take the tensor $\bar{\epsilon}$ defined as

$$\bar{\epsilon}(U, V) \equiv \bar{\epsilon}^{\alpha\beta} U_\alpha V_\beta = i[\epsilon(U_A, V_B) - \epsilon(U_B, V_A)] \equiv i[\epsilon^{\alpha\beta} U_{A\alpha} V_{B\beta} - \epsilon^{\alpha\beta} U_{B\alpha} V_{A\beta}],$$

where $\epsilon$ denotes the invariant bilinear form on $D$. $\bar{\epsilon}$ can be written in block matrix form as

$$\bar{\epsilon} = \begin{pmatrix} 0 & i \epsilon \\ -i \epsilon & 0 \end{pmatrix}.$$ (3.7)

If $\epsilon$ is real and symmetric, then $\bar{\epsilon}$ is purely imaginary and antisymmetric, and vice versa. This implies that the normal auxiliary field, to which $D$ and $\epsilon$ belong, and the corresponding RSS auxiliary field have opposite commutation properties.
The spectrum of particle states that can be created by $\Psi$ and $\Psi^\dagger$ as in Section 2.3.

### 3.1 Polarization vectors and particle spectrum

The inverse of $\tilde{\epsilon}$ is given by

$$\tilde{\epsilon}^{-1}(\tilde{U}, \tilde{V}) \equiv \tilde{\epsilon}_{\alpha\beta} \tilde{U}^\alpha \tilde{V}^\beta = i[\epsilon^{-1}(\tilde{U}^A, \tilde{V}^B) - \epsilon^{-1}(\tilde{U}^B, \tilde{V}^A)] \equiv i[\epsilon_{\alpha\beta} \tilde{U}^A \tilde{V}^B - \epsilon_{\alpha\beta} \tilde{U}^B \tilde{V}^A] \quad (3.8)$$

$\tilde{\epsilon}^{-1}$ can be written in block matrix form as

$$\tilde{\epsilon}^{-1} = \begin{pmatrix} 0 & i\epsilon^{-1} \\ -i\epsilon^{-1} & 0 \end{pmatrix} \quad (3.9)$$

The (anti)commutation relations

$$[\Psi_\alpha(x, t), \Pi^\dagger_\beta(x', t)]_{\pm} = i\epsilon_{\alpha\beta} \delta^3(x - x') \quad (3.10)$$

$$[\Psi_\beta(x, t), \Pi^\dagger_\alpha(x', t)]_{\pm} = i\epsilon_{\alpha\beta} \delta^3(x - x') \quad (3.11)$$

of the RSS auxiliary field $\Psi$ and of the corresponding momentum field $\Pi$ can be written in terms of the components $\Psi_A$, $\Psi_B$, $\Pi_A$, $\Pi_B$ as

$$[\Psi_{A\alpha}(x, t), \Pi^\dagger_{B\beta}(x', t)]_{\pm} = -\epsilon_{\alpha\beta} \delta^3(x - x') \quad (3.12)$$

$$[\Psi_{B\alpha}(x, t), \Pi^\dagger_{A\beta}(x', t)]_{\pm} = \epsilon_{\alpha\beta} \delta^3(x - x') \quad (3.13)$$

$$[\Psi^\dagger_{A\alpha}(x, t), \Pi_{B\beta}(x', t)]_{\pm} = -\epsilon_{\alpha\beta} \delta^3(x - x') \quad (3.14)$$

$$[\Psi^\dagger_{B\alpha}(x, t), \Pi_{A\beta}(x', t)]_{\pm} = \epsilon_{\alpha\beta} \delta^3(x - x') \quad (3.15)$$

The Feynman propagator for $\Psi$ is given by the formulas (3.12)-(3.15) with the replacement $\epsilon \to \tilde{\epsilon}$. In terms of the component fields $\Psi_A$ and $\Psi_B$ we have

$$\langle 0 | T \Psi_{A\alpha}(x, t_x) \Psi^\dagger_{B\beta}(y, t_y) | 0 \rangle = i\epsilon_{\alpha\beta} D_F(x - y, t_x - t_y) \quad (3.16)$$

$$\langle 0 | T \Psi_{B\alpha}(x, t_x) \Psi^\dagger_{A\beta}(y, t_y) | 0 \rangle = -i\epsilon_{\alpha\beta} D_F(x - y, t_x - t_y) \quad (3.17)$$

$$\langle 0 | T \Psi^\dagger_{A\alpha}(x, t_x) \Psi^\dagger_{B\beta}(y, t_y) | 0 \rangle = 0 \quad (3.18)$$

$$\langle 0 | T \Psi^\dagger_{B\alpha}(x, t_x) \Psi^\dagger_{A\beta}(y, t_y) | 0 \rangle = 0 \quad (3.19)$$

### 3.1 Polarization vectors and particle spectrum

The spectrum of particle states that can be created by $\Psi$ and $\Psi^\dagger$ can be found in the same way as in Section 2.3.

We define for all values of $i$ the polarization vectors $\hat{\Psi}_{(s)i}^u(k)$, $\hat{\Psi}_{(s)i}^u(k)$, $\hat{\Psi}_{(s)i}^v(k)$, $\hat{\Psi}_{(s)i}^v(k)$ in terms of $\check{u}_{(s)i}(k)$, $\check{v}_{(s)i}(k)$ in the following way:

$$\hat{\Psi}_{(s)i}^u(k) = \frac{1}{\sqrt{2}} (\check{u}_{(s)i}(k), -i\check{v}_{(s)i}(k)) \quad (3.20)$$

$$\hat{\Psi}_{(s)i}^v(k) = \frac{1}{\sqrt{2}} (\check{v}_{(s)i}(k), i\check{u}_{(s)i}(k)) \quad (3.21)$$

$$\hat{\Psi}_{(s)i}^{v*}(k) = \frac{1}{\sqrt{2}} (\check{v}_{(s)i}(k), -i\check{u}_{(s)i}(k)) \quad (3.22)$$

$$\hat{\Psi}_{(s)i}^{u*}(k) = \frac{1}{\sqrt{2}} (\check{u}_{(s)i}(k), i\check{v}_{(s)i}(k)) \quad (3.23)$$
The complex conjugate of this relation is the form
\[ \langle \hat{U}^u_{(s)}(k), \hat{U}^u_{(s')}\rangle(k) \rangle = \delta_{ss'}\delta_{ij} , \quad \langle \hat{V}^u_{(s)}(k), \hat{V}^u_{(s')}\rangle(k) \rangle = -\delta_{ss'}\delta_{ij} . \] (3.24)

The corresponding dual polarization vectors are
\[ U^u_{(s)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} u_{(s)}(k) \\ iu_{(s)}(k) \end{pmatrix} , \quad V^u_{(s)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} u_{(s)}(k) \\ -iu_{(s)}(k) \end{pmatrix} , \] (3.26)
\[ U^v_{(s)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} v_{(s)}(k) \\ iv_{(s)}(k) \end{pmatrix} , \quad V^v_{(s)}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} v_{(s)}(k) \\ -iv_{(s)}(k) \end{pmatrix} . \] (3.27)

In the case when \( D = D^{(n,m)} \), the completeness relation for these polarization vectors takes the form
\[ \sum_s \sum_{i=1}^{d_s} [U^u_{(s)\alpha}(k)\hat{U}^u_{(s)\beta}(k) + V^u_{(s)\alpha}(k)\hat{V}^u_{(s)\beta}(k) + U^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta}(k) + V^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta}(k)] = \delta_{\alpha\beta} . \] (3.28)

The complex conjugate of this relation is
\[ \sum_s \sum_{i=1}^{d_s} [U^u_{(s)\alpha}(k)\hat{U}^u_{(s)\beta} + V^u_{(s)\alpha}(k)\hat{V}^u_{(s)\beta} + U^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta} + V^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta}] = \delta_{\alpha\beta} . \] (3.29)

In the case when \( D = \hat{D}^{(n)} \), the completeness relations for the polarization vectors take the form
\[ \sum_s \sum_{i=1}^{d_s} [U^u_{(s)\alpha}(k)\hat{U}^u_{(s)\beta} + V^u_{(s)\alpha}(k)\hat{V}^u_{(s)\beta}] + \sum_s \sum_{i=1}^{d_s} [U^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta} + V^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta}] = \delta_{\alpha\beta} , \] (3.30)
\[ \sum_s \sum_{i=1}^{d_s} [U^u_{(s)\alpha}(k)\hat{U}^u_{(s)\beta} + V^u_{(s)\alpha}(k)\hat{V}^u_{(s)\beta}] + \sum_s \sum_{i=1}^{d_s} [U^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta} + V^v_{(s)\alpha}(k)\hat{V}^v_{(s)\beta}] = \delta_{\alpha\beta} . \] (3.31)
The creation operators corresponding to the above polarization vectors can be defined as
\[
c^U_{(s)i}(k) = \hat{U}^u_{(s)i}(k)\alpha^\dagger(k),
\]
\[
c^V_{(s)i}(k) = V^u_{(s)i}(k)\alpha^\dagger(k),
\]
\[
d^U_{(s)i}(k) = \hat{U}^v_{(s)i}(k)\alpha^\dagger(k),
\]
\[
d^V_{(s)i}(k) = V^v_{(s)i}(k)\alpha^\dagger(k),
\]
\[
f^U_{(s)i}(k) = \hat{U}^u_{(s)i}(k)\alpha(k),
\]
\[
f^V_{(s)i}(k) = V^u_{(s)i}(k)\alpha(k),
\]
\[
h^U_{(s)i}(k) = \hat{U}^v_{(s)i}(k)\alpha(k),
\]
\[
h^V_{(s)i}(k) = V^v_{(s)i}(k)\alpha(k),
\]
\[i = 1, \ldots, d_s.
\]

These operators satisfy the (anti)commutation relations
\[
[c^U_{(s)i}(k), c^U_{(s')j}(k')^\dagger] = [d^U_{(s)i}(k), d^U_{(s')j}(k')^\dagger] = \delta_{ss'}\delta_{ii}\delta^3(k-k')\omega(k)
\]
\[
[c^V_{(s)i}(k), c^V_{(s')j}(k')^\dagger] = [d^V_{(s)i}(k), d^V_{(s')j}(k')^\dagger] = -\delta_{ss'}\delta_{ii}\delta^3(k-k')\omega(k)
\]
\[
[f^U_{(s)i}(k), f^U_{(s')j}(k')^\dagger] = [h^U_{(s)i}(k), h^U_{(s')j}(k')^\dagger] = -\delta_{ss'}\delta_{ii}\delta^3(k-k')\omega(k)
\]
\[
[f^V_{(s)i}(k), f^V_{(s')j}(k')^\dagger] = [h^V_{(s)i}(k), h^V_{(s')j}(k')^\dagger] = \delta_{ss'}\delta_{ii}\delta^3(k-k')\omega(k);
\]

all other (anti)commutators of them are zero. This shows that the one-particle states created by \(c^U_{(s)i}(k)\) and \(d^U_{(s')j}(k)\) have positive scalar product with themselves, the one-particle states created by \(c^V_{(s)i}(k)\) and \(d^V_{(s')j}(k)\) have negative scalar product with themselves, whereas \(f^U_{(s)i}(k)\) and \(h^U_{(s)i}(k)\) create one-particle states that have negative scalar product with themselves, and \(f^V_{(s)i}(k)\) and \(h^V_{(s)i}(k)\) create one-particle states that have positive scalar product with themselves.

Using the completeness relations (3.28) and (3.29), \(\Psi\) can be expressed in terms of the above creation operators in the case of \(D = D^{(n,m)}\) as
\[
\Psi_\alpha(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2\pi})^3}} \times
\]
\[
\left[ e^{ikx} e^{i\omega(k)t} \sum_{s=1}^{d_s} \left\{ U^u_{(s)ia}(k)c^U_{(s)i}(k) + V^u_{(s)ia}(k)c^V_{(s)i}(k) \right. \\
+ U^u_{(s)ia}(k)f^U_{(s)i}(k) + V^u_{(s)ia}(k)f^V_{(s)i}(k) \right\} \\
+ e^{-ikx} e^{-i\omega(k)t} \sum_{s=1}^{d_s} \left\{ U^u_{(s)ia}(k)^* d^U_{(s)i}(k) + V^u_{(s)ia}(k)^* d^V_{(s)i}(k) \right. \\
+ U^v_{(s)ia}(k)^* h^U_{(s)i}(k) + V^v_{(s)ia}(k)^* h^V_{(s)i}(k) \right\} \right].
\]

In the case of \(D = \tilde{D}^{(n)}\),
\[
\Psi = \psi + \psi^-,
\]

(3.45)
where
\[\psi^\alpha(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2\pi})^3\omega(k)}} \times \]
\[\left[ e^{ikx} e^{i\omega(k)t} \sum_{s \in S_+} \sum_{i=1}^{d_s} \left\{ U^{(s)\alpha}_i(k)c^{U\dagger}_i(k) + V^{(s)\alpha}_i(k)c^{V\dagger}_i(k) \right\} + e^{-ikx} e^{-i\omega(k)t} \sum_{s \in S_-} \sum_{i=1}^{d_s} \left\{ U^{(s)\alpha}_i(k)\ast d^{U\dagger}_i(k) + V^{(s)\alpha}_i(k)\ast d^{V\dagger}_i(k) \right\} \right] \quad (3.46)\]
and
\[\left(\psi^-\right)^\alpha(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2\pi})^3\omega(k)}} \times \]
\[\left[ e^{ikx} e^{i\omega(k)t} \sum_{s \in S_+} \sum_{i=1}^{d_s} \left\{ U^{(s)\alpha}_i(k)f^{U\dagger}_i(k) + V^{(s)\alpha}_i(k)f^{V\dagger}_i(k) \right\} + e^{-ikx} e^{-i\omega(k)t} \sum_{s \in S_-} \sum_{i=1}^{d_s} \left\{ U^{(s)\alpha}_i(k)\ast h^{U\dagger}_i(k) + V^{(s)\alpha}_i(k)\ast h^{V\dagger}_i(k) \right\} \right]. \quad (3.47)\]

The mode expansion formula \((3.44)\), together with the properties of the polarization vectors described above, shows that in the case of \(D = D^{(n,m)}\) \(\Psi\) describes 8 kinds of particles of mass \(\mu\) for any value of \(s\), of which 4 are physical and 4 are non-physical (in the sense that the one-particle states have positive and negative scalar product with themselves, respectively). Formulas \((3.46)\) and \((3.47)\) show that in the case of \(D = \tilde{D}^{(n)}\) \(\Psi\) describes 4 kinds of particles of mass \(\mu\) for any value of \(s \in S_+\) or \(s \in S_-\) of which 2 are physical and 2 are non-physical. The definition of the polarization vectors \(\tilde{U}^{(s)\alpha}_i(k)\), \(\tilde{V}^{(s)\alpha}_i(k)\), \(\tilde{U}^{(s)\alpha}_i(k)\), \(\tilde{V}^{(s)\alpha}_i(k)\) shows that to each normal (non-RSS) one-particle state there corresponds two RSS one-particle states, of which one is physical and one is non-physical (regarding the positivity properties of the scalar products of these states with themselves). The members of these pairs of RSS particles are distinguished in notation by the letters \(U\) and \(V\).

Since the RSS auxiliary field \(\Psi\) fits in the framework of Section \(2.1\), the energy of all states that can be created by \(\Psi\) and \(\Psi^\dagger\) is positive, i.e. in this respect the RSS fields are not non-physical.

### 3.2 Elementary higher spin fields with reversed spin-statistics relation

In this section we turn to the definition of elementary higher spin RSS fields, which have definite spin. For the definition of these fields we take the projection operators
\[\tilde{D}^{(s)}[\partial] = \begin{pmatrix} D^{(s)}[\partial] & 0 \\ 0 & D^{(s)}[\partial] \end{pmatrix}, \quad (3.48)\]
where \(D^{(s)}[\partial]\) is the differential operator that appears in the definition of the normal elementary spin \(s\) field to which the RSS field will correspond. The elementary RSS field of spin \(s\) is thus
given by
\[ \psi_{(s)} = \tilde{D}_{(s)}[\partial] \Psi . \]  

(3.49)

The block diagonal form of \( \tilde{D}_{(s)}[\partial] \) implies that
\[ \psi_{(s)A} = D_{(s)}[\partial] \Psi_A \]  

(3.50)
\[ \psi_{(s)B} = D_{(s)}[\partial] \Psi_B . \]  

(3.51)

We also define the projection operator \( \tilde{D}^-[\partial] \) in the same manner.

\( \tilde{D}_{(s)}[\partial] \) has the following action on the dual polarization vectors introduced in Section 3.1.

\begin{align*}
\tilde{D}_{(s)}[ik]_a^\beta V^u_{(s)\alpha}(k) &= \delta_{su} U^u_{(s)\alpha}(k) \\
\tilde{D}_{(s)}[ik]_a^\beta V^v_{(s)\alpha}(k) &= \delta_{sv} V^v_{(s)\alpha}(k) \\
\tilde{D}_{(s)}[-ik]_a^\beta U^u_{(s)\alpha}(k) &= \delta_{su} U^u_{(s)\alpha}(k) \\
\tilde{D}_{(s)}[-ik]_a^\beta V^v_{(s)\alpha}(k) &= \delta_{sv} V^v_{(s)\alpha}(k) \\
\end{align*}

Here the second four equations are obtained from the first four equations by complex conjugation.

The action of \( \tilde{D}^-[\partial] \) on the dual polarization vectors is
\begin{align*}
\tilde{D}^-[ik]_a^\beta U^u_{(s)\alpha}(k) &= 0 \\
\tilde{D}^-[ik]_a^\beta V^v_{(s)\alpha}(k) &= 0 \\
\tilde{D}^-[-ik]_a^\beta U^v_{(s)\alpha}(k) &= U^v_{(s)\alpha}(k) \\
\tilde{D}^-[-ik]_a^\beta V^v_{(s)\alpha}(k) &= V^v_{(s)\alpha}(k) \\
\tilde{D}^-[-ik]_a^\beta U^u_{(s)\alpha}(k) &= 0 \\
\tilde{D}^-[-ik]_a^\beta V^v_{(s)\alpha}(k) &= 0 \\
\tilde{D}^-[-ik]_a^\beta U^v_{(s)\alpha}(k) &= U^v_{(s)\alpha}(k) \\
\tilde{D}^-[-ik]_a^\beta V^v_{(s)\alpha}(k) &= V^v_{(s)\alpha}(k) \\
\end{align*}

Again, the second four equations are obtained from the first four equations by complex conjugation.
\[ \hat{D}_{(s)}[ik]_{\alpha}^{\beta} \text{ and } \hat{D}^{-}[ik]_{\alpha}^{\beta} \text{ can be expressed in terms of the polarization vectors as} \]
\[ \sum_{i=1}^{d_\nu} (U^u_{(s)\alpha}(k)\hat{U}^u_{(s)i}(k) + V^u_{(s)\alpha}(k)\hat{V}^u_{(s)i}(k)) = \hat{D}_{(s)}[ik]_{\alpha}^{\beta} \quad (3.68) \]
\[ \sum_{s} \sum_{i=1}^{d_\nu} (U^v_{(s)\alpha}(k)\hat{U}^v_{(s)i}(k) + V^v_{(s)\alpha}(k)\hat{V}^v_{(s)i}(k)) = \hat{D}^{-}[ik]_{\alpha}^{\beta}. \quad (3.69) \]

Complex conjugation of these formulas gives
\[ \sum_{i=1}^{d_\nu} (U^u_{(s)\alpha}(k)^*\hat{U}^u_{(s)i}(k)^{\beta*} + V^u_{(s)\alpha}(k)^*\hat{V}^u_{(s)i}(k)^{\beta*}) = \hat{D}_{(s)}[-ik]_{\alpha}^{\beta} \quad (3.70) \]
\[ \sum_{s} \sum_{i=1}^{d_\nu} (U^v_{(s)\alpha}(k)^*\hat{U}^v_{(s)i}(k)^{\beta*} + V^v_{(s)\alpha}(k)^*\hat{V}^v_{(s)i}(k)^{\beta*}) = \hat{D}^{-}[-ik]_{\alpha}^{\beta}. \quad (3.71) \]

\( \hat{D}_{(s)}[\partial] \) and \( \hat{D}^{-}[\partial] \) form a complete set of projectors on the space of the solutions of the Klein–Gordon equation:
\[ \sum_{s} \hat{D}_{(s)}[\partial]_{\alpha}^{\beta} \Psi_\beta + \hat{D}^{-}[\partial]_{\alpha}^{\beta} \Psi_\beta = \Psi_\alpha. \quad (3.72) \]

\( \psi_{(s)} \) has the mode expansion
\[ \psi_{(s)\alpha}(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2}\pi)^3}} \times \]
\[ [e^{ikx}e^{i\omega(k)t}\sum_{i=1}^{d_\nu} \{U^u_{(s)\alpha}(k)c^{U^\dagger}_{(s)i}(k) + V^u_{(s)\alpha}(k)c^{V^\dagger}_{(s)i}(k)\} + e^{-ikx}e^{-i\omega(k)t}\sum_{i=1}^{d_\nu} \{U^v_{(s)\alpha}(k)^*d^{U^\dagger}_{(s)i}(k) + V^v_{(s)\alpha}(k)^*d^{V^\dagger}_{(s)i}(k)\}] ; \quad (3.73) \]

the other modes that are present in \( \Psi \) are eliminated by \( \hat{D}_{(s)}[\partial] \).

In addition to the Klein–Gordon equation, \( \psi_{(s)} \) also satisfies the differential equations
\[ \hat{D}_{(s)}[\partial]\psi_{(s)} = \psi_{(s)} \quad (3.74) \]
\[ \hat{D}_{(s')}[\partial]\psi_{(s')} = 0, \quad s' \neq s \quad (3.75) \]
\[ \hat{D}^{-}[\partial]\psi_{(s)} = 0. \quad (3.76) \]

The expansion (3.73) shows that \( \psi_{(s)} \) describes 4 kinds of particles of mass \( \mu \) and spin \( s \), of which two are physical and two are non-physical (regarding the positivity properties of the scalar product). In accordance with the statistical properties of \( \Psi \) mentioned in the first part of Section [X], these particles are bosonic if \( s \) is half-integer and fermionic if \( s \) is integer, in contrast with the usual spin-statistics relationship that applies to the normal fields.

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The (anti)commutator, Green function and the covariant Feynman propagator of the $\psi^{(s)}$ fields is given by the general formulas (2.91), (2.92), (2.95), in which $\tilde{\epsilon}$ and $D_{(s)}[\partial]$ should be used, of course. In terms of the components $\psi^{(s)}_A$ and $\psi^{(s)}_B$ the covariant Feynman propagator is

\[
0\langle T\psi^{(s)}_{A\alpha}(x,t_x)\psi^{(s)}_{B\beta}^\dagger(y,t_y)|0\rangle^C = \]

\[
-\int \frac{d^3k \, dk_0}{(2\pi)^4} \frac{\delta}{\alpha}(k_0,k) \epsilon[\delta,\beta] e^{-ik(x-y)}e^{-ik_0(t_x-t_y)}
\]

\[
= iD_F(x-y,t_x-t_y)^C
\]

\[
\langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{A\beta}^\dagger(y,t_y)|0\rangle^C = \]

\[
\int \frac{d^3k \, dk_0}{(2\pi)^4} \frac{\delta}{\alpha}(k_0,k) \epsilon[\delta,\beta] e^{-ik(x-y)}e^{-ik_0(t_x-t_y)}
\]

\[
= -iD_F(x-y,t_x-t_y)^C
\]

\[
\langle 0\, | T\psi^{(s)}_{A\alpha}(x,t_x)\psi^{(s)}_{A\beta}^\dagger(y,t_y)|0\rangle^C = 0
\]

\[
\langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{B\beta}^\dagger(y,t_y)|0\rangle^C = 0
\]

and we have

\[
\langle 0\, | T\psi^{(s)}_{A\alpha}(x,t_x)\psi^{(s)}_{B\beta}^\dagger(y,t_y)|0\rangle^C = \langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{A\beta}^\dagger(y,t_y)|0\rangle^C
\]

\[
\langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{A\beta}^\dagger(y,t_y)|0\rangle^C = \langle 0\, | T\psi^{(s)}_{A\alpha}(x,t_x)\psi^{(s)}_{B\beta}^\dagger(y,t_y)|0\rangle^C
\]

\[
\langle 0\, | T\psi^{(s)}_{A\alpha}(x,t_x)\psi^{(s)}_{A\beta}^\dagger(y,t_y)|0\rangle^C = \langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{A\beta}^\dagger(y,t_y)|0\rangle^C
\]

\[
\langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{B\beta}^\dagger(y,t_y)|0\rangle^C = \langle 0\, | T\psi^{(s)}_{B\alpha}(x,t_x)\psi^{(s)}_{B\beta}^\dagger(y,t_y)|0\rangle^C
\]

The (anti)commutation relations are

\[
[\psi^{(s)}_{A\alpha}(x,t_x), \psi^{(s)}_{B\beta}^\dagger(y,t_y)]^\pm = iD_{(s)}[\partial_{(t_x,x)}]^\alpha \epsilon_{\delta,\beta}[G(x-y,t_x-t_y) - G(y-x,t_y-t_x)]
\]

\[
[\psi^{(s)}_{B\alpha}(x,t_x), \psi^{(s)}_{B\beta}^\dagger(y,t_y)]^\pm = -iD_{(s)}[\partial_{(t_x,x)}]^\alpha \epsilon_{\delta,\beta}[G(x-y,t_x-t_y) - G(y-x,t_y-t_x)]
\]

\[
[\psi^{(s)}_{A\alpha}(x,t_x), \psi^{(s)}_{A\beta}^\dagger(y,t_y)]^\pm = 0
\]

\[
[\psi^{(s)}_{B\alpha}(x,t_x), \psi^{(s)}_{B\beta}^\dagger(y,t_y)]^\pm = 0
\]

and

\[
[\psi^{(s)}_{A\alpha}(x,t_x), \psi^{(s)}_{B\beta}^\dagger(y,t_y)]^\pm = [\psi^{(s)}_{A\alpha}(x,t_x), \psi^{(s)}_{B\beta}(y,t_y)]^\pm
\]

\[
[\psi^{(s)}_{B\alpha}(x,t_x), \psi^{(s)}_{A\beta}^\dagger(y,t_y)]^\pm = [\psi^{(s)}_{B\alpha}(x,t_x), \psi^{(s)}_{A\beta}(y,t_y)]^\pm
\]

\[
[\psi^{(s)}_{A\alpha}(x,t_x), \psi^{(s)}_{A\beta}^\dagger(y,t_y)]^\pm = [\psi^{(s)}_{A\alpha}(x,t_x), \psi^{(s)}_{A\beta}(y,t_y)]^\pm
\]

\[
[\psi^{(s)}_{B\alpha}(x,t_x), \psi^{(s)}_{B\beta}^\dagger(y,t_y)]^\pm = [\psi^{(s)}_{B\alpha}(x,t_x), \psi^{(s)}_{B\beta}(y,t_y)]^\pm
\]
All Feynman propagators, (anti)commutators and Green functions of two daggered or two non-daggered fields are zero.

Real or chiral RSS fields can be defined in the same way as described at the end of Section 2.4. In the construction of chiral fields the same projection should be applied on the components \( \Psi_A \) and \( \Psi_B \) or \( \psi_A \) and \( \psi_B \). The real elementary RSS fields describe only one pair of particles, consisting of a physical and a non-physical particle (regarding the positivity properties of the scalar product).

The transformation properties of the RSS fields under \( C, P \) and \( T \) are defined so that the components \( \Psi_A \) and \( \Psi_B \), or \( \psi_A \) and \( \psi_B \) do not mix, and the transformation rule for these components is given by the formulas (2.43), (2.44), (2.45).

The fields \( \psi \) are elementary in the sense that they describe particles with definite mass and spin, and their real and imaginary parts \( \phi_{1} \) and \( \phi_{2} \) describe the minimal number, namely two, of such particles. It is not possible for an RSS field to describe a single particle, since that would violate the spin-statistics theorem.

It is also interesting to note that the components \( \phi_{1} \), \( \phi_{1} \) and \( \phi_{2} \), \( \phi_{2} \) are null fields, i.e. they create only states that have zero scalar product with themselves.

4 Scalar fields

In this section we describe the special case of the scalar RSS field, with the aim of showing explicitly how the well-known Faddeev–Popov ghost field fits into the framework described in Section 3.

4.1 Normal scalar field

In the case of the normal (non-RSS) complex scalar field the space of polarization vectors is one-dimensional, and we have

\[
\epsilon = 1 ,
\]

\[
u(k) = 1 , \quad \hat{u}(k) = 1 ,
\]

\[
\epsilon^{\dagger}(k) = a^{\dagger}(k) , \quad d^{\dagger}(k) = b^{\dagger}(k) .
\]

The projection operator is trivial (\( D[\partial] = I \)), thus the auxiliary field is identical to the physical field. The mode expansion of the complex scalar field takes the form

\[
\psi(x,t) = \int \frac{d^{3}k}{\sqrt{2(\sqrt{2\pi})^{3}\omega(k)}} \left[ e^{ikx}e^{i\omega(k)t}c^{\dagger}(k) + e^{-ikx}e^{-i\omega(k)t}d(k) \right] .
\]

The equal-time commutation relations of \( \psi \) are

\[
[\psi^{\dagger}(x,t),\pi(x',t)] = i\delta^{3}(x - x')
\]

\[
[\psi(x,t),\pi^{\dagger}(x',t)] = i\delta^{3}(x - x') ,
\]

where \( \pi = \partial_{t}\psi \). The Feynman propagator of \( \psi \) is

\[
\langle 0|T\psi(x,t_{x})\psi^{\dagger}(y,t_{y})|0 \rangle = \langle 0|T\psi^{\dagger}(x,t_{x})\psi(y,t_{y})|0 \rangle = D_{F}(x - y, t_{x} - t_{y}) ,
\]

where \( D_{F} \) is the function defined in Section 2.1.2.
4.2 Fermionic scalar field

The ϵ tensor for the RSS scalar field is then
\[ \tilde{\epsilon} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \]
(4.8)

and the standard polarization vectors are
\[ \hat{U}^u(k) = \frac{1}{\sqrt{2}} (1, -i) \]
(4.9)
\[ \hat{V}^u(k) = \frac{1}{\sqrt{2}} (1, i). \]
(4.10)

In particular, \( \hat{U}^u(k) \) and \( \hat{V}^u(k) \) are independent of \( k \). The non-zero scalar products of these polarization vectors are
\[ \langle \hat{U}^u(k), \hat{U}^u(k) \rangle = 1, \quad \langle \hat{V}^u(k), \hat{V}^u(k) \rangle = -1. \]
(4.11)

The corresponding dual polarization vectors are
\[ U^u(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad V^u(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \]
(4.12)

The creation operators corresponding to the above polarization vectors are \( c^{U\dagger}(k), c^{V\dagger}(k), d^{U\dagger}(k), d^{V\dagger}(k) \), which have the following non-zero anticommutators:
\[ [c^U(k), c^{U\dagger}(k')]_+ + [d^U(k), d^{U\dagger}(k')]_+ = \delta^3(k-k')\omega(k), \]
(4.13)
\[ [c^V(k), c^{V\dagger}(k')]_+ + [d^V(k), d^{V\dagger}(k')]_+ = -\delta^3(k-k')\omega(k). \]
(4.14)

The projection operator \( \tilde{D}[\partial] \) is again just the unit operator, thus the mode expansion of the RSS scalar field is
\[ \psi(x, t) = \int \frac{d^3k}{\sqrt{2(\sqrt{2} \pi)^3 \omega(k)}} \times \left[ e^{ikx} e^{i\omega(k)t} \left\{ U^u c^{U\dagger}(k) + V^u c^{V\dagger}(k) \right\} + e^{-ikx} e^{-i\omega(k)t} \left\{ U^{us} d^U(k) + V^{us} d^V(k) \right\} \right]. \]
(4.15)

In particular,
\[ \psi_A(x, t) = \int \frac{d^3k}{2(\sqrt{2} \pi)^3 \omega(k)} \times \left[ e^{ikx} e^{i\omega(k)t} \left\{ c^{U\dagger}(k) + c^{V\dagger}(k) \right\} + e^{-ikx} e^{-i\omega(k)t} \left\{ d^U(k) + d^V(k) \right\} \right]. \]
(4.16)
Using (2.95) the propagator we get the usual form of the scalar ghost field Lagrangian (see e.g. [8]):

\[ \pi = c, \bar{c}, \phi, \bar{\phi} \]

where \( \phi, \bar{\phi} \) are real, i.e. \( \phi = c, \bar{c} \) and \( \bar{\phi} = \phi \). By introducing the fields \( \omega_A = \partial_\mu \psi_A \) and \( \omega_B = \partial_\mu \psi_B \). Similarly, the Feynman propagator of \( \psi \) is given by (3.16)-(3.19):

\[ \langle 0|T\psi_A(x, t_x)|\psi_B(y, t_y)\rangle = i\int dF(x - y, t_x - t_y) \]

\[ \langle 0|T\bar{\psi}_A(x, t_x)|\bar{\psi}_B(y, t_y)\rangle = i\int dF(x - y, t_x - t_y) \]

\[ \langle 0|T\bar{\psi}_A(x, t_x)|\psi_B(y, t_y)\rangle = 0 \]

\[ \langle 0|T\psi_B(x, t_x)|\psi_A(y, t_y)\rangle = 0 \]

The Lagrangian (2.16) for the RSS scalar field is

\[ L = \int d^3x \left[ (\partial_\mu \psi_A)(\partial^\mu \psi_B) - (\partial_\mu \bar{\psi}_A)(\partial^\mu \bar{\psi}_B) - \mu^2 \psi_A \bar{\psi}_B + \mu^2 \bar{\psi}_A \psi_B \right] \]

The Lagrangian for a real RSS scalar field \( \phi \) is obtained by omitting the imaginary part of \( \psi \):

\[ L = \int d^3x \left[ (\partial_\mu \phi_A)(\partial^\mu \phi_B) - \mu^2 \phi_A \phi_B \right] \]

\( \phi_A = \frac{1}{\sqrt{2}}(\psi_A + \bar{\psi}_A), \phi_B = \frac{1}{\sqrt{2}}(\psi_B + \bar{\psi}_B) \). The fields \( \phi_A \) and \( \phi_B \) appearing here are obviously real, i.e. \( \phi_A^\dagger = \phi_A, \phi_B^\dagger = \phi_B \). By introducing the fields \( c \) and \( \bar{c} \) as

\[ \phi_A = -i\bar{c}, \ \phi_B = c \]

we get the usual form of the scalar ghost field Lagrangian (see e.g. [8])

\[ L = \int d^3x \left[ (\partial_\mu \bar{c})(\partial^\mu c) - \mu^2 \bar{c}c \right] \]

Using (2.95) the propagator \( \langle 0|Tc(x, t_x)|\bar{c}(y, t_y)\rangle \) is found to be

\[ \langle 0|Tc(x, t_x)|\bar{c}(y, t_y)\rangle = \int d^3k \frac{1}{(2\pi)^4} \frac{i}{k_0^2 - k^2 - \mu^2 + i\epsilon} e^{-ik(x-y)} e^{-ik_0(t_x - t_y)} \]
which, after setting \( \mu = 0 \), is also in agreement with the standard expression (see e.g. [8]). For 
\[ \langle 0 | T \bar{c}(x, t_x) c(y, t_y) | 0 \rangle \]
it is found that 
\[ \langle 0 | T \bar{c}(x, t_x) c(y, t_y) | 0 \rangle = - \langle 0 | T \bar{c}(x, t_x) \bar{c}(y, t_y) | 0 \rangle . \]  
(4.31)

The anticommutation relations of \( c \) and \( \bar{c} \) are 
\[ \{ c(x, t), \partial_t \bar{c}(x', t) \} = i \delta^3(x - x') \]  
(4.32)
\[ \{ \bar{c}(x, t), \partial_t c(x', t) \} = - i \delta^3(x - x') . \]  
(4.33)

The mode expansion of \( c \) and \( \bar{c} \) can be derived from that of \( \psi_A \) and \( \psi_B \) above in a straightforward way.

5 Dirac fields

In this section we discuss the case of the Dirac field in more detail. In particular, we also present the direct Lagrangian formulation (in which the auxiliary field is not used) both for the normal and for the RSS Dirac field.

We use the overbar notation 
\[ \bar{\psi}^\beta = \psi_\alpha^\dagger \epsilon^{\alpha\beta} . \]  
(5.1)

In the following it will be seen that this coincides with the usual Dirac conjugate.

5.1 Normal Dirac field

In the case of the normal Dirac field, the projection operator is 
\[ D_{(1/2)}[\partial] = \frac{1}{2 \mu} (\mu + i \gamma^\mu \partial_\mu) , \]  
(5.2)
and 
\[ D^{-}[\partial] = \frac{1}{2 \mu} (\mu - i \gamma^\mu \partial_\mu) . \]  
(5.3)

The Dirac field \( \psi \) thus satisfies the differential equation 
\[ 2 \mu (\psi - D_{(1/2)}[\partial] \psi) = (\mu - i \gamma^\mu \partial_\mu) \psi = 0 , \]  
(5.4)
which is the well-known Dirac equation. \( \psi \) satisfies the equation \( D^{-}[\partial] \psi = 0 \) as well, but this is also identical to the Dirac equation. In order to have the usual normalization, we define the Dirac field as 
\[ \psi = \sqrt{2 \mu} D_{(1/2)}[\partial] \Psi \]  (i.e. the factor \( \sqrt{2 \mu} \) is inserted into \ref{2.72}). The anticommutation relations that follow from this definition for \( \psi \) are 
\[ \{ \psi_\alpha(x, t), \psi_\beta(x', t) \} = 0 \]  
(5.5)
\[ \{ \psi_\alpha(x, t), \psi_\beta^\dagger(x', t) \} = (\gamma^0)_\alpha^\rho \epsilon_{\rho\beta} \delta^3(x - x') . \]  
(5.6)
The general formula \((2.95)\) gives the Feynman propagator as
\[
\langle 0| T \psi_\alpha(x,t_x) \psi_\beta^\dagger(y,t_y) |0 \rangle = \int \frac{d^3k}{(2\pi)^4} \frac{k_\mu \gamma_\mu}{k_0^2 - k^2 - \mu^2 + i\epsilon} \delta_{\delta\delta} e^{-ik(x-y)} e^{-ik_0(t_x-t_y)},
\]
where \(k_\mu \equiv (k_0, k)\).

In \([9]\) we took a real basis in \(D^{(1,0)}\) with respect to which the gamma matrices have the form
\[
\gamma^0 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},
\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},
\gamma^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},
\gamma^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},
\]
and \(\epsilon\) has the canonical form
\[
\epsilon = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.
\]
This shows, in particular, that the gamma matrices are purely imaginary with respect to the invariant complex conjugation, and thus the differential operators \(D^{(1/2)}[\partial]\) and \(D^{-}[\partial]\) introduced above are real. It is also worth noting that \((\gamma^\mu)_{\alpha}^{\rho} \epsilon_{\rho\beta}\) and \((\gamma^\mu)_{\rho}^{\alpha} \epsilon_{\beta\rho}\) are symmetric in \(\alpha\) and \(\beta\).

The standard dual polarization vectors at \(k = 0\) in \([9]\) were
\[
u_1 = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -i \end{pmatrix}, \quad \nu_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix}, \quad \nu_1 = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ i \end{pmatrix}, \quad \nu_2 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ i \end{pmatrix}.
\]

The corresponding polarization vectors are
\[
\hat{u}_1 = \frac{1}{2} (-1, i, -1, i), \quad \hat{u}_2 = \frac{1}{2} (1, -i, -1, i), \quad \hat{v}_1 = \frac{1}{2} (1, -i, 1, -i), \quad \hat{v}_2 = \frac{1}{2} (1, i, 1, i).
\]

The Dirac equation can be derived from the Lagrangian
\[
L = \int d^3x \mathcal{L} = -\int d^3x \bar{\psi}(\mu - i\gamma^\mu \partial_\mu)\psi,
\]
where
where $\psi$ is understood to be an anticommuting field. The canonical momentum field corresponding to $\psi_\alpha$ is
\[
\tilde{\pi}^\alpha = \frac{\partial L}{\partial (\partial_\alpha \psi_\alpha)} = -i\psi_\gamma (\gamma^0)_\beta^\alpha \epsilon^{\gamma\beta} = -i(\bar{\psi}\gamma^0)^\alpha .
\] (5.14)

The canonical anticommutation relation
\[
\{\psi_\alpha(x,t) , \bar{\pi}^\beta(x',t)\} = -i\delta_\alpha^\beta \delta^3(x-x')
\] (5.15)
together with (5.14), is equivalent to (5.6). The Hamiltonian that can be derived from (5.13) is
\[
H = \int d^3x \bar{\psi}(\mu - i\gamma^\rho \partial_\rho)\psi .
\] (5.16)

5.2 Bosonic Dirac field

Taking into consideration (5.2) and the general definition (3.48) of projection operators for the RSS fields, we have
\[
\tilde{D}_{(1/2)}[\partial] = \frac{1}{2\mu}(\mu + i\gamma^\mu \partial_\mu) ,
\] (5.18)
where \( \tilde{\gamma}^{\mu} \) is
\[
\tilde{\gamma}^{\mu} = \begin{pmatrix}
\gamma^{\mu} & 0 \\
0 & \gamma^{\mu}
\end{pmatrix}.
\] (5.19)

The RSS Dirac field \( \psi = \sqrt{2\mu} \tilde{D}_{(1/2)}[\partial] \Psi \) satisfies the differential equation (3.74), i.e.
\[
2\mu(\psi - \tilde{D}_{(1/2)}[\partial] \psi) = (\mu - i\tilde{\gamma}^{\mu} \partial_{\mu}) \psi = 0,
\] (5.20)
which is equivalent to the Dirac equation for the two components \( \psi_A, \psi_B \):
\[
(\mu - i\tilde{\gamma}^{\mu} \partial_{\mu}) \psi_A = 0 \quad \text{and} \quad (\mu - i\tilde{\gamma}^{\mu} \partial_{\mu}) \psi_B = 0.
\] (5.21)

Equation (3.76) takes the form (2\mu \tilde{D}_{(1/2)} \psi) = (\mu - i\tilde{\gamma}^{\mu} \partial_{\mu}) \psi = 0, thus is identical to (5.20).

From the commutation relations of the auxiliary field \( \Psi \) and from the relation \( \psi = \sqrt{2\mu} \tilde{D}_{(1/2)}[\partial] \Psi \) it follows that \( \psi \) satisfies the commutation relation
\[
[\psi_{\alpha}(x,t), \psi_{\beta}^\dagger(x',t)] = -(\gamma^{0})_{\beta}^{\rho} \varepsilon_{\rho\alpha} \delta^{3}(x-x'),
\] (5.23)
which implies the commutation relations
\[
[\psi_{A\alpha}(x,t), \psi_{B\beta}^\dagger(x',t)] = i(\gamma^{0})_{\beta}^{\rho} \varepsilon_{\rho\alpha} \delta^{3}(x-x') \quad \text{and} \quad [\psi_{B\alpha}(x,t), \psi_{A\beta}^\dagger(x',t)] = -i(\gamma^{0})_{\beta}^{\rho} \varepsilon_{\rho\alpha} \delta^{3}(x-x'),
\] (5.24) (5.25)
for \( \psi_A \) and \( \psi_B \). It is worth noting that \( (\tilde{\gamma}^{\mu})_{\alpha}^{\rho} \varepsilon_{\rho\beta} \) and \( (\tilde{\gamma}^{\mu})_{\rho}^{\alpha} \varepsilon_{\rho\beta} \) are antisymmetric in \( \alpha \) and \( \beta \).

The equation of motion (5.20) can be derived from the Lagrangian
\[
L = -\int d^{3}x \bar{\psi}(\mu - i\tilde{\gamma}^{\mu} \partial_{\mu}) \psi.
\] (5.26)
In terms of the components \( \psi_A, \psi_B \), this Lagrangian takes the form
\[
L = -\int d^{3}x \left[i\epsilon^{\alpha\beta}\psi_{A\alpha}^\dagger(\mu - i\tilde{\gamma}^{\mu} \partial_{\mu})B_{\beta}^{\gamma} \psi_{B\gamma} - i\epsilon^{\alpha\beta}\psi_{B\alpha}^\dagger(\mu - i\tilde{\gamma}^{\mu} \partial_{\mu})A_{\beta}^{\gamma} \psi_{A\gamma}\right].
\] (5.27)

The canonical momentum corresponding to \( \psi_{\alpha} \) is
\[
\tilde{\pi}_{\alpha} = i\psi_{\rho}^\dagger(\gamma^{0})_{\beta}^{\rho} \varepsilon_{\rho\alpha}.
\] (5.28)
The canonical commutation relation
\[
[\psi_{\alpha}(x,t), \tilde{\pi}_{\beta}(x',t)] = i\delta_{\alpha}^{\beta} \delta^{3}(x-x'),
\] (5.29)
together with (5.28), reproduces (5.23).

The Hamiltonian that can be obtained from (5.26) is
\[
H = \int d^{3}x \bar{\psi}(\mu - i\tilde{\gamma}^{\mu} \partial_{\mu}) \psi.
\] (5.30)
The polarization vectors, the Green function and the propagator can be obtained in a straightforward way from the general formulas \((3.20)-(3.23), (3.26), (3.27), (2.92), (2.95)\) and \((3.77) - (3.80)\). In particular, the Feynman propagator in terms of \(\psi_A\) and \(\psi_B\) is

\[
\langle 0 | T \psi_A(x,t) \psi_B^\dagger(y,t) | 0 \rangle = -\int \frac{d^3k \, dk_0}{(2\pi)^4} \frac{(\mu + k_\mu \gamma^\mu)_{\alpha \delta} \epsilon_{\delta \beta}}{k_0^2 - k^2 - \mu^2 + i\epsilon} e^{-ik(x-y)} e^{-ik_0(t_x-t_y)}
\]

(5.31)

\[
\langle 0 | T \psi_B(x,t) \psi_A^\dagger(y,t) | 0 \rangle = \int \frac{d^3k \, dk_0}{(2\pi)^4} \frac{(\mu + k_\mu \gamma^\mu)_{\alpha \delta} \epsilon_{\delta \beta}}{k_0^2 - k^2 - \mu^2 + i\epsilon} e^{-ik(x-y)} e^{-ik_0(t_x-t_y)}
\]

(5.32)

\[
\langle 0 | T \psi(s)_A(x,t) \psi(s)_B^\dagger(y,t) | 0 \rangle = 0
\]

(5.33)

\[
\langle 0 | T \psi(s)_B(x,t) \psi(s)_A^\dagger(y,t) | 0 \rangle = 0
\]

(5.34)

6 Summary

In this paper a construction of free massive quantum fields with arbitrary spin and reversed spin-statistics relation was presented. In relation to their spin-statistics property, these fields (abbreviated as RSS fields) also have the characteristic property that they create one-particle states that form pairs consisting of one state that has positive scalar product with itself and another one that has negative scalar product with itself.

For the construction a framework introduced in \[9\] for higher spin fields was used. This involves the definition of auxiliary higher spin fields, which are constrained only by the Klein-Gordon equation and which are quantized canonically. An elementary higher spin field is obtained from an auxiliary field by the application of a suitable projection operator, which is a differential operator and is closely related to the propagator of the field.

In the framework of \[9\] a higher spin field is specified if certain data, including the representation of \(SL(2, \mathbb{C})\) according to which the field transforms, an invariant tensor \(\epsilon_{\alpha \beta}\), and the projection operator applied to the auxiliary field, is given. In the presented construction of RSS fields these data are created from similar data specifying normal higher spin fields. The details of the construction show that for any normal field there exists a corresponding RSS field with the same spin.

The \(SL(2, \mathbb{C})\) representation for an RSS field, related to some normal field, is taken to be the direct sum of two copies of the representation according to which the normal field transforms, thus any RSS field \(\psi\) has two parts \(\psi_A\) and \(\psi_B\) corresponding to these copies. The statistical properties of a field depend on whether the tensor \(\epsilon^{\alpha \beta}\) belonging to it is symmetric or antisymmetric, and the doubling of the representation according to which the field transforms makes it possible to define an \(e^{\alpha \beta}\) tensor, in terms of the original \(e^{\alpha \beta}\) tensor that belongs to the normal field, with symmetry property opposite to that of the original \(e^{\alpha \beta}\) tensor. It is this definition by which the reversed spin-statistics property of the RSS fields is achieved. The projection operator for the RSS field is just the original projection operator acting separately on the two parts of the RSS field, hence these two parts satisfy in themselves the same field equations as the original
normal field.

As mentioned above, the RSS one-particle states form pairs; for each one-particle state of the normal field there is a corresponding pair of RSS one-particle states of which one has positive scalar product with itself and one has negative scalar product with itself. The Feynman propagators, the (anti)commutators and the Green functions of the RSS field can also be expressed in terms of those of the normal field in a straightforward way.

As illustration the scalar field and the Dirac field were discussed. In the case of the scalar field the aim was to show how the scalar Faddeev–Popov ghost field fits into the framework presented in this paper. The bosonic Dirac field was discussed because it is the simplest bosonic RSS field and because of the importance of the (fermionic) Dirac field in particle physics. It was shown that the RSS Dirac field admits a first order Lagrangian formulation (without the auxiliary field) similar to that of the normal Dirac field. It was also pointed out that by making use of the $\epsilon^{\alpha\beta}$ tensor the Dirac conjugation can be written in an obviously covariant form, and in this form it can be generalized naturally to any other field described in the framework of [9] and of the present paper.

Regarding further work, it would obviously be interesting to find applications of the RSS fields, different from their known application as ghost fields in the Faddeev–Popov–De Witt and related methods.

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