FANO THREEFOLDS WITH SECTIONS IN $\Omega^1_V(1)$

PRISKA JAHNKE AND IVO RADLOFF

Introduction

Let $V$ be a Fano manifold of Picard number one, and let $O_V(1)$ be an ample generator of Pic($V$). Usually $H^0(V, \Omega^1_V(1)) = H^0(V, \Omega^1_V \otimes O_V(1)) = 0$. The existence of a form $0 \neq \theta \in H^0(V, \Omega^1_V(1))$ is therefore a special condition. Two particular cases are well known: firstly, if $\dim V = 2r + 1$ is odd and $\theta \in H^0(V, \Omega^1_V(1))$ induces a bundle sequence

\[ 0 \to F \to T_V \xrightarrow{\theta} O_V(1) \to 0 \]

with maximal non–integrable kernel $F$, then $V$ is a so called Fano contact manifold, and it is conjectured that $V$ is homogeneous in this case. Secondly, if $0 \neq \theta \in H^0(V, \Omega^1_V(1))$ and $d\theta \wedge \theta \in H^0(V, \Lambda^3 \Omega^1_V \otimes O_V(2))$ is the zero section, then the kernel of $\theta$ induces a foliation on $V$, which is again a quite special situation.

In general, a section $\theta \in H^0(V, \Omega^1_V(1))$ will neither induce a bundle sequence like (0.1), nor will $d\theta \wedge \theta \in H^0(V, \Lambda^3 \Omega^1_V \otimes O_V(2))$, the section deciding on integrability, be either free of zeroes or completely zero. In general, both $\theta$ and $d\theta \wedge \theta$ will have nontrivial vanishing loci, and the interesting question is in how far these reflect the geometry of $V$. We note that $\theta$ cannot vanish on a divisor, since $V$ has no holomorphic 1–forms by assumption.

Using Iskovskikh’s classification of Fano threefolds, the coarse picture is as follows:

**Theorem.** Let $V$ be a Fano threefold of Picard number one and index $r$, and denote by $O_V(1)$ an ample generator of Pic($V$). If we have on $V$ a holomorphic section $0 \neq \theta \in H^0(V, \Omega^1_V(1))$, then $V$ is in one of the following families

1.) $V_{22}$. If $V$ is general in the family, then $H^0(V, \Omega^1_V(1)) = \mathbb{C}^3$ and $d\theta \wedge \theta \in H^0(V, O_V(1))$ vanishes.
2.) $V_{18}$. Here $H^0(V, \Omega^1_V(1)) = \mathbb{C}$ and $d\theta \wedge \theta \in H^0(V, O_V(1))$ is non–vanishing for any member of the family.

For particular members of the family, where a more detailed description of the Fano manifold in question is available, we can say far more. In 1.), the special member $V_{22}$, the Mukai–Umemura threefold, is almost homogeneous. Here $H^0(V_{22}, \Omega^1_{V_{22}}(1)) = \mathbb{C}^3$, and $d\theta \wedge \theta \in H^0(V_{22}, O_{V_{22}}(1))$ either cuts out precisely the divisor of lines on $V_{22}$, or vanishes completely, defining an almost homogeneous foliation. In contrast to this special case, $d\theta \wedge \theta$ always vanishes on a general $V_{22}$.

By Mukai’s classification, a $V_{18}$ is a complete intersection of two hyperplanes in a 5 dimensional homogeneous contact manifold $M$. Here the space $H^0(V, \Omega^1_V(1)) = \mathbb{C}$.

Date: 14th November 2018.

The authors were supported by the Deutsche Forschungsgemeinschaft.
is simply generated by the pull back of the contact form on $M$ to $V$. On $M$, using the contact sequence, vector fields and hyperplane sections may be identified. If we think in this way of $V$ being the complete intersection of the hyperplanes corresponding to $X_1, X_2 \in H^0(M, T_M)$, then $H^0(V, \Omega^1_V(1))$ is generated by the restriction of $X_1 \wedge X_2$, and the nonvanishing section that decides on integrability corresponds to $[X_1, X_2]$.

1. Existence of sections in $\Omega^1_V(1)$

We will use both Iskovskikh’s and Mukai’s classification to determine all Fano threefolds $V$ with Picard number one which admit a holomorphic section in $\Omega^1_V(1)$. For the convenience of the reader we have added the classification from [I] and [M] in the appendix.

Some notations: denote the index of $V$ by $r$, i.e. $-K_V = rH$, where $\mathcal{O}_V(H)$ is the fundamental divisor on $V$. By Kobayashi and Ochiai’s criterion, $1 \leq r \leq 4$ and $r = 3, 4$ if and only if $V \simeq Q_3, \mathbb{P}^3$, respectively. It remains hence to classify the cases $r = 1$ and $r = 2$. Let $d = H^3$ be the degree of $V$. A Fano threefold of degree $d$ and index 1 we call $V_d$, by $V_{2,d}$ we denote a Fano threefold of index 2 and degree $d$.

Iskovskikh uses the method of double projection from a line for his classification. The existence of lines was proved by Shokurov in [Sh]. Key of Iskovskikh’s method is [I], Theorem 3.3, where he proves the generatedness of the anticanonical divisor. Then $|-K_V|$ determines a morphism

$$\varphi_{-K_V} : V \longrightarrow \mathbb{P}^{g+1},$$

where $g = \frac{1}{2}(-K_V)^3 + 1$ is called the genus of $V$. Moreover, $\varphi_{-K_V}$ is either an embedding, or a $2:1$–cover of some smooth variety. By [I], Theorem 7.2., the latter case is very special. The genus is bounded. Iskovskikh shows $2 \leq g \leq 12$, $g \neq 11$ for $r = 1$ and $g = 5, 9, 13, 17, 21$ for $r = 2$. Except for the cases $r = 1$ and $g = 7, 9, 10, 12$ he obtains the description of each Fano threefold as a complete intersection in a (weighted) projective space as listed in the table in the appendix.

Mukai later developed the vector bundle method to classify Fano threefolds. This method leads in particular to a more detailed description in the case of anticanonical embedded Fano threefolds. Our remaining cases $r = 1$ and $g = 7, 9, 10, 12$ are of this type. We have added Mukai’s realisation in the table in the appendix for these 4 cases.

The reason why we restrict to $\Omega^1_V(1)$ and do not consider higher twists as well, is simply the following. The Euler sequence on projective space $\mathbb{P}^n$ says

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus (n+1)} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$ 

Applying the functor $\wedge^{n-1}$, using $\wedge^{n-1} T_{\mathbb{P}^n}(-1) \simeq \Omega^1_{\mathbb{P}^n}(2)$, we get a surjection from a sum of $\mathcal{O}_{\mathbb{P}^n}$’s to $\Omega^1_{\mathbb{P}^n}(2)$. In this way we see that $\Omega^1_{\mathbb{P}^n}(2)$ is spanned. If now, for example, $V$ is Fano as above and if $\mathcal{O}_V(1)$ is very ample, then the induced embedding $V \hookrightarrow \mathbb{P}^n$ defines a map

$$\Omega^1_{\mathbb{P}^n}(2) \longrightarrow \Omega^1_V(2) \longrightarrow 0,$$

which shows that $\Omega^1_V(2)$ is spanned as well. By Iskovskikh’s classification, $\mathcal{O}_V(1)$ is very ample, except for the cases no. 3,4,8 and 10.
1.1. Proposition. Let $V$ be an index $r$ Fano threefold of Picard number one and genus $g$. Denote by $\mathcal{O}_V(1) \in \text{Pic}(V)$ an ample generator. The existence of a holomorphic section of $\Omega^1_M(1)$ implies $r = 1$ and $g = 10$ or $12$.

We start by proving some general lemmas on the cohomology of twisted 1–forms, which will later cover all threefolds from the classification.

1.2. Lemma. Let $M$ be a projective manifold of dimension $n \geq 4$. Let $\mathcal{O}_M(1)$ be an ample divisor on $M$ and $V \in \{\mathcal{O}_M(d)\}$ be a smooth hypersurface. Define $\mathcal{O}_V(1) = \mathcal{O}_M(1)|_V$. Assume $H^0(M, \Omega^1_M(1)) = 0$. If one of the following conditions holds

1.) $d \geq 2$,
2.) $d = 1$ and $b_2(M) = 1$,
then $H^0(V, \Omega^1_V(1)) = 0$.

Proof. Write $\mathcal{O}_V(k) = \mathcal{O}_M(k) \otimes \mathcal{O}_V$. The claim follows from standard vanishing theorems applied to the dualized tangent sequence

\[ 0 \rightarrow N^*_V/M = \mathcal{O}_V(-d) \rightarrow \Omega^1_M|_V \rightarrow \Omega^1_V \rightarrow 0, \]

and the ideal sequence of $V$ in $M$, tensorized with $\Omega^1_M(1)$:

\[ 0 \rightarrow \Omega^1_M(1 - d) \rightarrow \Omega^1_M(1) \rightarrow \Omega^1_M(1)|_V \rightarrow 0. \]

1.) Assume $d \geq 2$. Kodaira’s vanishing theorem yields $H^i(V, \mathcal{O}_V(1 - d)) = 0$ for $i = 0, 1$, so $H^0(V, \Omega^1_V(1)) \simeq H^0(V, \Omega^1_M(1)|_V)$ in (1.3). By Serre duality, $H^1(M, \Omega^1_M(1 - d)) \simeq H^n(M, \Omega^{n-1}_M(d - 1))^*$. The latter vanishes by the Kodaira–Akizuki–Nakano vanishing theorem, since $d \geq 2$, $n \geq 3$ and since $\mathcal{O}_M(1)$ is ample. Hence $H^0(M, \Omega^1_M(1)) \rightarrow H^0(V, \Omega^1_M(1)|_V)$ in (1.3) is surjective. Since $H^0(M, \Omega^1_M(1)) = 0$ by assumption, we infer $H^0(V, \Omega^1_M(1)|_V) = 0$, and therefore $H^0(V, \Omega^1_V(1)) = 0$.

2.) Assume $d = b_2(M) = 1$. Then $H^0(M, \Omega^1_M) = 0$ in (1.4), since by assumption $H^0(M, \Omega^1_M(1)) = 0$. By Lefschetz, $h^0(M, \Omega^1_M) = h^1(M, \mathcal{O}_M) = 0$ implies $h^1(V, \mathcal{O}_V) = 0$. From (1.3), twisted by $\mathcal{O}_V(1)$, we infer

\[ h^0(V, \Omega^1_V(1)) = h^0(V, \Omega^1_M(1)|_V) - 1 \leq h^1(M, \Omega^1_M) - 1. \]

But $h^{1,1}(M) = 1$, since $b_2(M) = 1$, yielding $h^0(V, \Omega^1_V(1)) = 0$. \qed

The next lemma requires some basic knowledge on weighted projective spaces $\mathbb{P}(Q) = \mathbb{P}(q_0, \ldots, q_n)$, the Proj of $\mathbb{C}[x_0, \ldots, x_n]$, giving $x_i$ weight $q_i$. For details, in particular concerning the definition of the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(d)$ or $\Lambda^1_{\mathbb{P}(Q)}$, we refer the reader to [D]. Recall that $\mathbb{P}(Q)$ is called well–formed, if the $q_i$’s are pairwise relatively prime, and the greatest common divisor of $q_0, \ldots, q_i, \ldots, q_n$ is 1 for all $i$.

1.5. Lemma. Let $\mathbb{P}(Q) = \mathbb{P}(q_0, \ldots, q_n)$ be a well–formed weighted projective space for some $n \geq 4$. Let $V \in \{\mathcal{O}_{\mathbb{P}(Q)}(d)\}$ be a smooth hypersurface contained in $\mathbb{P}(Q)\text{reg}$, where $d$ is divisible by all the $q_i$’s. Define $\mathcal{O}_V(1) = \mathcal{O}_{\mathbb{P}(Q)}(1)|_V$. Then $H^0(V, \Omega^1_V(1)) = 0$.

Proof. We first conclude $H^0(\mathbb{P}(Q), \Omega^1_{\mathbb{P}(Q)}(1)) = 0$. This follows from the exact Euler sequence on $\mathbb{P}(Q)$, reading for weighted projective spaces ([D], § 2)

\[ (1.6) \quad 0 \rightarrow \Omega^1_{\mathbb{P}(Q)}(1) \rightarrow \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}(Q)}(1 - q_i) \stackrel{\rho}{\rightarrow} \mathcal{O}_{\mathbb{P}(Q)}(1) \rightarrow 0. \]
By [D], 2.3.4. Corollary, we have $H^j(\mathbb{P}(Q), \Omega^l_{\mathbb{P}(Q)}(k)) \neq 0$ only when $j = 0$ and $k > \min_{i_1 < \cdots < i_n}(q_i + \cdots + q_{g_i})$. Hence $H^0(\mathbb{P}(Q), \Omega^1_{\mathbb{P}(Q)}(1)) = 0$.

The sheaf $\mathcal{O}_V(1)$ is free and ample on $V$. We may assume $d \geq 2$, since $d = 1$ implies $q_i = 1$ for all $i$, so $\mathbb{P}(Q) = \mathbb{P}^n$, in which case the proof is analogous to the case 1.) of Lemma 2. For $d \geq 2$, since $V$ is supposed to be contained in the smooth locus of $\mathbb{P}(Q)$, the proof is analogous to 2.) of Lemma 2.

**Proof of Lemma 2**. We prove the claim using the classification, for the notation see the table in the appendix. Since $\dim V = 3$, we have $1 \leq r \leq 4$. By Kobayashi and Ochiai’s criterion, if $r = 4$, then $V \simeq \mathbb{P}^3$, and if $r = 3$, then $V \simeq Q_3$, the quadric hypersurface in $\mathbb{P}^4$. By Bott’s formula, $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(1)) = 0$. In the case of the quadric, Lemma 2 applies, showing $H^0(Q_3, \Omega^1_{Q_3}(1)) = 0$. It remains to consider the cases $r = 1, 2$.

In the case $r = 2$ we have the following 5 possibilities: i) $V \in |\mathcal{O}_{\mathbb{P}(Q)}(6)|$, where $\mathbb{P}(Q) = \mathbb{P}(1,1,1,2,3)$ is a weighted projective space; ii) $V \in |\mathcal{O}_{\mathbb{P}(Q)}(4)|$, $\mathbb{P}(Q) = \mathbb{P}(1,1,1,1,2)$; iii) $V$ is a cubic in $\mathbb{P}^4$; iv) $V \subset \mathbb{P}^6$ is a complete intersection of two quadrics; v) $V$ is the complete intersection of the 6-dimensional Grassmannian $Gr(2,5)$ and 3 hyperplanes in $\mathbb{P}^9$. In the first two cases i) and ii), Lemma 3.5 applies. For the other cases iii) to v) the same is proved by the same is proved by Proposition 3.4. Hence, if $r = 2$, then $H^0(V, \Omega^1_V(1)) = 0$.

In the remaining case $r = 1$ we have $2 \leq g \leq 12$, $g \neq 11$ for the genus $g$ of $V$, and we want to prove $g = 10$ or $g = 12$.

If $g = 2$, then $V \in |\mathcal{O}_{\mathbb{P}(Q)}(6)|$, $\mathbb{P}(Q) = \mathbb{P}(1,1,1,1,3)$, and Lemma 3.5 applies showing $H^0(V, \Omega^1_V(1)) = 0$.

If $g = 3$, then $V$ is either a quartic in $\mathbb{P}^4$, or the following intersection: let $V' \in |\mathcal{O}_{\mathbb{P}(Q)}(8)|$, where $\mathbb{P}(Q) = \mathbb{P}(1,1,1,1,4)$, be a general, hence smooth hypersurface. Then $V \in |\mathcal{O}_{V'}(2)|$ is a quartic, where $\mathcal{O}_{V'}(1) = \mathcal{O}_{\mathbb{P}(Q)}(1)|_{V'}$ by definition. In the first case Lemma 2 applies; for the second case apply Lemma 2.1; then, for $V' \subset V$, to prove the vanishing $H^0(V, \Omega^1_V(1)) = 0$.

If $g = 4, 5$, then $V$ is a complete intersection in some projective space, and if $g = 6, 8$, then $V$ is a complete intersection in some Grassmannian. Both cases are clear by Lemma 2 and Snow’s result on Grassmannians cited above.

If $g = 7, 9$, then $V$ is a linear section in the Hermitian symmetric space $M = G/P$ of type DIII for $g = 7$ and CI for $g = 9$ by a result of Mukai (see [M], § 2 or [IP], § 5.2). For the space DIII, $G = SO(10, \mathbb{C})$ and $G = Sp(6, \mathbb{C})$ in the case CI. The subgroup $P$ of $G$ is maximal parabolic. The cohomology of twisted holomorphic forms on manifolds of these types have been studied by Snow in [Sn2], which gives $H^0(M, \Omega^1_M(1)) = 0$ (see 3.3. Proposition and 2.3. Proposition).

The only remaining cases are $g = 10$ and $g = 12$ and we are done.

## 2. Fano threefolds of type $V_{22}$

Throughout this section, by $V$ we denote a Fano threefold with Picard number one of genus 12, i.e. of type $V_{22}$. Then we have natural isomorphisms

\begin{equation}
\Lambda^2 T_V \simeq \Omega^1_V(1) \quad \text{and} \quad \Lambda^3 T_V \simeq \mathcal{O}_V(1)
\end{equation}

and we will sometimes identify these bundles. A general member of the family has a finite automorphism group, hence no vector fields. By [P2], there are three
special types with non-trivial automorphism group: two isolated members $V_22^n$ and $V_22^m$ with one and two dimensional automorphism group, respectively, and the Mukai–Umemura threefold $V_22$ with automorphism group $S_{12}(\mathbb{C})$ moving in a one dimensional family. We first show that there are indeed sections in $\Omega_V^1(1)$.

2.2. Lemma. For $V$ as above of type $V_22$, we have $h^0(V, \Omega_V^1(1)) \geq 3$.

Now let $\theta \in H^0(V, \Omega_V^1(1))$ be a non-zero section. We may consider $\theta$ as a map $\theta: T_V \to \mathcal{O}_V(1)$. Then $\text{im}(\theta) = \mathcal{O}_V(1) \otimes \mathcal{I}_\theta$, where $\mathcal{Z}(\theta) \subset V$ is the zero locus of $\theta$. Defining $\mathcal{F}_\theta = \ker(\theta)$ we get an exact sequence

\[ 0 \to \mathcal{F}_\theta \to T_V \to \mathcal{O}_V(1) \otimes \mathcal{I}_\theta \to 0. \]

Since $\mathcal{O}_V(1) \otimes \mathcal{I}_\theta$ and $\mathcal{F}_\theta$ are torsion free, $\mathcal{F}_\theta$ is even reflexive (see [OSS], 1.1.16 Lemma). The generic rank of $\mathcal{F}_\theta$ is 2. Since $\theta$ cannot vanish on a divisor, $\text{codim}(\mathcal{Z}(\theta), V) \geq 2$. Hence $c_1(\mathcal{F}_\theta) = 0$.

Proof of 2.2 Lemma. A general member $S \in |\mathcal{O}_V(1)|$ is a smooth K3 surface by [SH]. Define $\mathcal{O}_S(1) = \mathcal{O}_V(1)|_S$. The Kodaira–Akizuki–Nakano vanishing theorem implies $h^2(S, \Omega_S^1(1)) = 0$. We will show $H^1(S, \Omega_S^1(1))$ is non-empty: assume to the contrary $h^1(S, \Omega_S^1(1)) = 0$. Then $h^1(S, T_S \otimes N_{S/V}^*) = 0$ by Serre duality, meaning the tangent sequence of $S$ in $V$ splits. This implies

\[ T_V|_S \cong T_S \oplus \mathcal{O}_S(1). \]

From the ideal sequence we compute $h^0(S, \mathcal{O}_S(1)) = h^0(V, \mathcal{O}_V(1)) - 1 = 13$. On the other hand, $h^1(V, T_V(-1)) = h^2(V, \Omega_V^1) = 0$ (see [IP], § 12.2) implies

\[ h^0(S, T_V|_S) = h^0(V, T_V) - h^0(V, T_V(-1)) \leq 3, \]

since $V$ admits at most 3 vector fields, a contradiction. Hence $h^1(S, \Omega_S^1(1)) \geq 1$. By Riemann–Roch on $S$, $\chi(S, \Omega_S^1(1)) = \mathcal{O}_V(1)^3 - 20 = 2$. We obtain

\[ h^0(S, \Omega_S^1(1)) = \chi(S, \Omega_S^1(1)) + h^1(S, \Omega_S^1(1)) \geq 3. \]

From the twisted tangent sequence of $S$ in $V$

\[ 0 \to \mathcal{O}_S \to \Omega_V^1(1)|_S \to \Omega_S^1(1) \to 0, \]

we obtain $h^0(S, \Omega_V^1(1)|_S) = h^0(S, \Omega_S^1(1)) + 1 \geq 4$; the sequence

\[ 0 \to \Omega_V^1 \to \Omega_V^1(1) \to \Omega_V^1(1)|_S \to 0 \]

then gives $h^0(V, \Omega_V^1(1)) \geq h^0(V, \Omega_V^1(1)|_S) - 1 \geq 3$, since $H^1(V, \Omega_V^1) \cong \mathbb{C}$. \hfill \Box

2.4. The Mukai–Umemura threefold $V_22$. A very special member of the $V_22$ family is the almost homogeneous Mukai–Umemura threefold $V_22$. The construction is as follows (see [MU] for details). Let $M_{12} = \mathbb{C}[t_0, t_1]_{12}$ be the $\mathbb{C}$–vector space of homogeneous polynomials in the two variables $t_0, t_1$ of degree 12. View $M_{12} \cong \mathbb{C}^{13}$ as the affine part of $\mathbb{P}(M_{12} \oplus \mathbb{C}) \cong \mathbb{P}^{13}$ and identify $\mathbb{P}(M_{12})$ with the hyperplane at infinity. The natural action of $S_{12}(\mathbb{C})$ on $\mathbb{C}[t_0, t_1]$ induces an action on $\mathbb{P}(M_{12} \oplus \mathbb{C})$. Define

\[ x := t_0t_1(t_0^{10} - 11t_0^5t_1 - t_1^{10}) \in M_{12}. \]

Following Mukai and Umemura, define

\[ V_22^x = S_{12}(\mathbb{C}) \cdot |x + 1|. \]
It is not difficult to see that $V_{22}^s$ is indeed a smooth Fano threefold of genus 12. The action of $\text{Sl}_2(\mathbb{C})$ on $V_{22}^s$ has the 3-dimensional open orbit $O_3 = \text{Sl}_2(\mathbb{C}) \cdot [x + 1]$ and the orbits

$$O_2 = \text{Sl}_2(\mathbb{C}) \cdot [t_0 t_1^1], \quad O_1 = \text{Sl}_2(\mathbb{C}) \cdot [t_1^2]$$

of dimensions 2 and 1, respectively. We have $O_1, O_2 \subset \mathbb{P}(M_{12})$, the hyperplane at infinity. In fact $V_{22}^s = O_1 \cup O_2 \cup O_3$ and $V_{22}^s \cap \mathbb{P}(M_{12}) = O_1 \cup O_2$, i.e. $O_1 \cup O_2 \in |O_{V_{22}^s}(1)|$. The orbit $O_2$ is neither open nor closed, $O_1 \simeq \mathbb{P}^1$ and $O_2 = O_1 \cup O_2$. The hyperplane $O_1 \cup O_2 \in |O_{V_{22}^s}(1)|$ is the hyperplane cut out by lines (cf. [MU], Lemma 6.1.); it is singular along $O_1 \simeq \mathbb{P}^1$, the normalization being $\mathbb{P}^1 \times \mathbb{P}^1$. This can be seen as follows. Taking a general matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}_2(\mathbb{C})$$

to compute $O_2$, we find

$$O_2 = \{((at_0 + bt_1)(ct_0 + dt_1)^{11}) | \ ad - bc = 1 \} \subset \mathbb{P}(M_{12}).$$

The map $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}(M_{12})$ defined by $[a : b] \times [c : d] \to [(at_0 + bt_1)(ct_0 + dt_1)^{11}]$, i.e. by a subsystem of $|O_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 11)|$, is then a normalization map of $\mathcal{O}_2$. Here $\nu$ is equivariant with respect to the action on $\mathcal{O}_2$ and the transposed diagonal action on $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $\nu^\gamma = \nu^\nu$ for any $\gamma \in \text{Sl}_2(\mathbb{C})$. The nonnormal locus of $\mathcal{O}_2 = O_1 \cup O_2$ is $\nu(\Delta) = O_1$, where $\Delta$ denotes the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. We see from this description that $O_1 \cup O_2$ is indeed cut out by lines.

The equivariance of $\nu$ implies the following: we have a map

$$H^0(\Delta, T_{\Delta}) \hookrightarrow H^0(\mathbb{P}^1 \times \mathbb{P}^1, T_{\mathbb{P}^1 \times \mathbb{P}^1})$$

defined as follows. For $X \in H^0(\Delta, T_{\Delta})$ define $i(X)(p, q) = (X(p), X(q)) \in p_1^*T_{\mathbb{P}^1} \oplus p_2^*T_{\mathbb{P}^1} = T_{\mathbb{P}^1 \times \mathbb{P}^1}$, where $p_i$ denote the projections. Then for any $Y \in H^0(V_{22}^s, T_{V_{22}^s})$ we have

$$\nu^*Y \in \text{im}(H^0(\Delta, T_{\Delta}) \hookrightarrow H^0(\mathbb{P}^1 \times \mathbb{P}^1, T_{\mathbb{P}^1 \times \mathbb{P}^1}) \hookrightarrow H^0(\mathbb{P}^1 \times \mathbb{P}^1, \varphi^*T_{V_{22}^s})).$$

2.5. Proposition.

1.) Let $X, Y \in H^0(V_{22}^s, T_{V_{22}^s}) \simeq \text{sl}_2(\mathbb{C})$ be linearly independent vector fields and define $\theta_{X, Y} = X \wedge Y \in H^0(V_{22}^s, \Lambda^2 T_{V_{22}^s})$. Then

1.1.) $\mathcal{Z}(\theta_{X, Y})_{\text{red}} = O_1 \cup (\text{rational curve}) \subset O_2$,

1.2.) $\mathcal{Z}(d\theta_{X, Y} \wedge \theta_{X, Y}) = V$ or $O_1 \cup O_2$, depending on whether $X$ and $Y$ generate a subalgebra of $\text{sl}_2(\mathbb{C})$ or not.

1.3.) $\mathcal{F}_{\theta_{X, Y}} \simeq \mathcal{O}_{V_{22}^s}$, i.e. we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{V_{22}^s} \longrightarrow T_{V_{22}^s} \longrightarrow \mathcal{O}_{V_{22}^s}(1) \otimes \mathcal{I}_{\mathcal{Z}(\theta_{X, Y})} \longrightarrow 0.$$

2.) $H^0(V_{22}^s, \Lambda^2 T_{V_{22}^s}) \simeq \Lambda^2 H^0(V_{22}^s, T_{V_{22}^s}) \simeq \mathbb{C}^3$, meaning that any section in $H^0(V_{22}^s, \Omega^1_{V_{22}^s}(1))$ is as in 1.)

2.6. Remark. In 1.2.), if $X, Y \in H^0(V_{22}^s, T_{V_{22}^s}) \simeq \text{sl}_2(\mathbb{C})$ define a subalgebra of $\text{sl}_2(\mathbb{C})$, then $d\theta_{X, Y} \wedge \theta_{X, Y} \equiv 0$, and we have a foliation. The leaves are the orbits of the corresponding subgroup of $\text{Sl}_2(\mathbb{C})$. In general, however, we will have $d\theta_{X, Y} \wedge \theta_{X, Y} \neq 0$, and $\mathcal{Z}(d\theta_{X, Y} \wedge \theta_{X, Y}) = O_1 \cup O_2$. 


Proof of Proposition. We write $V$ instead of $V_{22}$ for simplicity.

1.1.) Let $X, Y \in H^0(V, T_V)$ be two linearly independent vector fields. Note $H^0(V, T_V) = \mathbb{C}^3$. Using (2.1), we may think of $X \wedge Y$ as a section of $\Omega^1_V(1)$. The zero set of this section is $Z = \{ p \in V \mid (X \wedge Y)(p) = 0 \}$. We know $\dim_C Z \leq 1$. Since $T_V|_{O_3}$ is generated by three sections, $O_3 \cap Z = \emptyset$. Hence, set theoretically, $Z \subset O_1 \cup O_2$. From above:

$$\nu^*X, \nu^*Y \in \text{im}(H^0(\Delta, T_\Delta) \rightarrow H^0(\mathbb{P}^1 \times \mathbb{P}^1, \nu^*T_V)).$$

It is then clear from this description that $\Delta$ is part of the zero locus of $\nu^*(X \wedge Y)$. It is moreover clear that

$$\nu^*(X \wedge Y) \in \text{im}(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \wedge^2 T_{\mathbb{P}^1 \times \mathbb{P}^1}) \rightarrow H^0(\mathbb{P}^1 \times \mathbb{P}^1, \nu^*\wedge^2 T_V)).$$

From $\wedge^2 T_{\mathbb{P}^1 \times \mathbb{P}^1} = p_1^*O_{\mathbb{P}^1}(2) \otimes p_2^*O_{\mathbb{P}^1}(2)$ we infer the vanishing locus of $\nu^*(X \wedge Y)$ is either $2\Delta$ or $\Delta + \Delta'$, where $\Delta' \in |O_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$. In the first case, set theoretically, $Z = \nu(2\Delta)$, in the latter case $Z = \nu(\Delta) \cup \nu(\text{rational curve of degree 12})$. In any case, $Z_{\text{red}} = O_1 \cup (\text{rational curve})$. This proves 1.1.)

1.3.) and 2.) Define $W = \{ X \wedge Y \mid X, Y \in H^0(V, T_V) \} \subset H^0(V, \wedge^2 T_V)$. Three generating vector fields in $H^0(V, T_V)$ are pairwise independent on $O_3$, implying $W \cong \wedge^2 H^0(V, T_V)$, a three dimensional vector space. We want to show $W = H^0(V, \Omega^1_V(1))$. Using the notation from (2.3) we prove the equivalences

$$\theta \in W \setminus \{ 0 \} \iff h^0(V, F_\theta) \geq 2 \iff F_\theta \cong O^0_V.$$  

The equivalences imply 1.3.)

We first prove $h^0(V, F_\theta) \geq 2$ implies $F_\theta \cong O^0_V$. From $F_\theta \rightarrow T_V$ we infer $h^0(V, F_\theta) \leq 3$. Three vector fields generate $T_V$ on $O_3$. Then they cannot be all contained in $H^0(V, F_\theta)$, since $F_\theta$ is generically of rank two. Hence $h^0(V, F_\theta) = 2$. Let $X_0, Y_0 \in H^0(V, T_V)$ be generators of $H^0(V, F_\theta)$, i.e. $\theta(X_0) = \theta(Y_0) = 0$. Define $Z_0 = \{ p \in V \mid (X_0 \wedge Y_0)(p) = 0 \}$. Then $\text{codim}(Z_0, V) = 2$, since $X_0 \wedge Y_0$ vanishes on a curve by 1.). This gives a map $O^0_V \rightarrow F_\theta$, which is surjective away from $Z_0$. This shows $O^0_V \cong F_\theta$, since $F_\theta$ is reflexive and $c_1(F_\theta) = 0$.

Now assume $F_\theta \cong O^0_V$. We prove that then $\theta \in W \setminus \{ 0 \}$. Indeed, using the notation from above, we may assume $H^0(V, F_\theta)$ is generated by two vector fields $X_0, Y_0$. By construction, the map $i : F_\theta \cong O^0_V \rightarrow T_V$ is then defined by $(f, g) \mapsto fX_0 + gY_0$. Consider on the other hand $\theta_0 : T_V \rightarrow O_V(1) \otimes I_{Z_0}$ defined by $X_0 \wedge Y_0$. Denote the kernel by $F_0$. Then $F_0 \cong O^0_V$ as above, and the inclusion $F_0 \rightarrow O^0_V \hookrightarrow T_V$ is the same map as $i$. Therefore the cokernel maps must coincide, meaning $\theta = \lambda X_0 \wedge Y_0$ for some $\lambda \in \mathbb{C}^*$ (and $Z(\theta) = Z_0$).

Finally assume $0 \neq \theta \in W$. Then $F_\theta \cong O^0_V$ as above, hence $h^0(V, F_\theta) = 2$.

To finally prove $W = H^0(V, \Omega^1_V(1))$, consider some $\theta_0 \in W \setminus \{ 0 \}$. Then $F_{\theta_0} \cong O^0_V$, as we have seen. Since $\Omega^1_V$ is rigid, for $\theta_1$ chosen from some (analytically) open neighborhood $U(\theta_0) \subset H^0(V, \Omega^1_V(1))$ of $\theta_0$, we also have $F_{\theta_1} \cong O^0_V$. The above equivalences show $U(\theta_0) \subset W$, implying $W = H^0(V, \Omega^1_V(1))$. Point 2.) is proved.

1.2.) To determine $d\theta_{X,Y} \wedge \theta_{X,Y}$, consider the map

$$O_V \cong \wedge^2 F_{X,Y} \rightarrow O_V(1)$$
induced by \( \theta_{X,Y} \circ [ - , - ] = X \wedge Y \wedge [ - , - ] \). We see that the zero set of \( d\theta_{X,Y} \wedge \theta_{X,Y} \)

is the zero set of \( X \wedge Y \wedge [X, Y] \), with \( 2.8 \) viewed as a section of \( O_Y(1) \). If \( [X, Y] \in \langle X, Y \rangle_{\mathbb{C}} \), then \( \mathcal{Z}(d\theta_{X,Y} \wedge \theta_{X,Y}) = V \). Otherwise, choose \( Z \) such that \( H^0(V, T_V) = \langle X, Y, Z \rangle_{\mathbb{C}} \). We have to find the zero set of \( X \wedge Z \). On \( O_3 \), the three sections are independent, so they define a nonzero section of \( O_V(1) \), vanishing on the complement of \( O_3 \). We finally conclude \( \mathcal{Z}(d\theta_{X,Y} \wedge \theta_{X,Y}) = O_1 \cup O_2 \in |O_V(1)| \). □

2.7. Family of Fano threefolds of type \( V_{22} \). By Mukai’s construction (see [M], or [IP], §5.2.), any Fano threefold \( V \) of type \( V_{22} \) can be embedded into the Grassmannian \( \text{Gr}(7,3) \) of 3-dimensional quotient spaces of \( \mathbb{C}^7 \). Let \( Q \) be the universal quotient bundle on the Grassmannian. Then \( V \) is defined as zero locus of 3 sections in \( \wedge^2 Q \). The parameter space of \( V_{22} \) is birationally equivalent to the moduli space of curves of genus 3 by [EPS] or [IP], p.114, hence 6-dimensional and irreducible.

Assume that \( V \) is not the Mukai–Umemura threefold. Then the divisor cut out by lines is a reduced, irreducible divisor from \( |O_V(2)| \) (see [IP], §4.2, [PT] and [IS]), and the splitting type of \( T_V \) on a general line is \( (2, 0, -1) \).

2.8. Proposition. Let \( V \) be general of type \( V_{22} \). Then \( h^0(V, O_V^1(1)) = 3 \) and \( d\theta \wedge \theta \equiv 0 \) for any \( \theta \in H^0(V, O_V^1(1)) \).

Proof. We will apply semicontinuity on the family of Fano threefolds of type \( V_{22} \). Let \( V \) be a general member and \( V^s = V_{22}^s \) the Mukai–Umemura threefold, a special member. Then

\[
h^0(V, O_V^1(1)) \leq h^0(V^s, O_V^1(1)) = 3,
\]

by 2.5 Proposition. On the other hand \( h^0(V, O_V^1(1)) \geq 3 \) by 2.2 Lemma, showing \( h^0(V, O_V^1(1)) = 3 \).

Let \( \theta \in H^0(V, O_V^1(1)) \) be a non–zero section. We want to prove \( d\theta \wedge \theta \equiv 0 \). Since \( H^0(V, O_V^1(1)) \) is threedimensional, \( \theta \) is a deformation of some \( \theta_0 \in H^0(V^s, O_V^1(1)) \).

Define the kernels \( \mathcal{F}_\theta \) and \( \mathcal{F}_{\theta_0} \) as in 2.8. By 2.5 Proposition, \( \mathcal{F}_\theta \simeq O_{V^2}^1 \).

On \( V \) we have the exact sequence

\[
0 \longrightarrow \mathcal{F}_\theta \longrightarrow T_V \longrightarrow O_V(1) \otimes \mathcal{I}_{\mathcal{Z}(\theta)} \longrightarrow 0.
\]

We will show that \( \theta \) vanishes in more than one point on a general line \( l \subset V \). First, we may assume that \( l \) does not meet the codimension 3 locus, where \( \mathcal{F}_\theta \) is free. Therefore \( \mathcal{F}_{\theta|l} \) is a rank two vector bundle of degree 0. Let \( l_s \) be a line in \( V^s \), obtained by deforming \( l \). By semicontinuity, \( h^0(l, \mathcal{F}_\theta(-1)|l) \leq h^0(l_0, \mathcal{F}_{\theta_0}(-1)|l_0) = 0 \). This shows \( \mathcal{F}_{\theta|l} \simeq O_l^1 \).

The splitting type of \( T_V \) on \( l \) is \( T_V|l = O_l(2) \oplus O_l(1) \) (IP, Theorem 4.2.7). The restriction \( \mathcal{I}_{\mathcal{Z}(\theta)} \otimes O_l \) might not be torsion free, but nevertheless, the vanishing order of \( \theta \) on \( l \) is exactly the (negative) degree of the free part, since \( \mathcal{Z}(\theta) \) meets \( l \) only in points. The restriction of 2.8 hence looks like

\[
0 \longrightarrow O_l^2 \overset{\alpha}{\longrightarrow} O_l(2) \oplus O_l \longrightarrow O_l(-1) \longrightarrow 0,
\]

where \( \tau \) is a torsion sheaf, and \( a \) is the order of \( \mathcal{Z}(\theta) \cap l \) we are looking for. Computing \( H^1 \), we find \( a = 2 \). We have proved, that \( \theta \) vanishes in 2 points on \( l \).

Consider now \( d\theta \wedge \theta \). Since \( d\theta \wedge \theta \) obviously vanishes in the zeroes of \( \theta \), it vanishes in two points on a general line \( l \). Since \( d\theta \wedge \theta \in |O_V(1)| \), it follows \( d\theta \wedge \theta|_l \equiv 0 \). This implies, that \( d\theta \wedge \theta \) vanishes on the whole divisor cut out by lines, which is an element in \( |O_V(2)| \), if \( V \neq V_{22}^s \). This shows \( d\theta \wedge \theta \equiv 0 \). □
3. Fano threefolds of type $V_{18}$

Let $M$ be the 5 dimensional contact manifold, homogeneous under the exceptional group $G_2$. Naturally embedded in $\mathbb{P}^{13}$, the contact bundle of $M$ is the fundamental divisor $L = \mathcal{O}_M(1) = \mathcal{O}_{\mathbb{P}^{13}}(1)|_M$. We use

$$0 \to F \to T_M \xrightarrow{\theta_M} L \to 0$$

to describe the contact sequence. The contact form $\theta_M \in H^0(M, \Omega^1_M(1))$ is unique up to multiples.

By Maki’s construction, a Fano threefold $V$ of type $V_{18}$ is a complete intersection of two hyperplanes $H_1, H_2 \in |\mathcal{O}_M(1)|$ in our contact manifold $M$. We do not have vector fields on $V$. With this interpretation of $V$, we first prove

**3.1. Proposition.** For $V$ of type $V_{18}$ we have $H^0(V, \Omega^1_V(1)) = \mathbb{C}$, a generating section being the image of $\theta_M$ under $H^0(M, \Omega^1_M(1)) \to H^0(V, \Omega^1_V(1))$.

**Proof.** Since $-d\theta_M = \theta_M([-,-]) : F \times F \to L$ is non–degenerate, Frobenius theorem implies that if $W$ is a submanifold of $M$ and $T_W \subset F|_W$, then $\dim W < 3$. Then $T_V$ cannot be contained in $F|_V$, and from

$$0 \to F|_V \to T_M|_V \xrightarrow{\theta_M|_V} \mathcal{O}_V(1) \to 0,$$

we see that $\theta_M$ is mapped to a non–vanishing section $\theta$ of $\Omega^1_V(1)$ under the natural map $\Omega^1_M(1) \to \Omega^1_V(1)$. Analogously we see that $\theta_M$ induces a non–vanishing section of $\Omega^1_{H_1}(1)$.

To show $H^0(V, \Omega^1_V(1)) = \mathbb{C}$, we use the dualized tangent sequence of $V$ in $H_1$ and the ideal sequence. The first is

$$0 \to \mathcal{O}_V \to \Omega^1_{H_1}(1)|_V \to \Omega^1_V(1) \to 0,$$

yielding $h^0(V, \Omega^1_V(1)) = h^0(V, \Omega^1_{H_1}(1)|_V) - 1$. By adjunction formula and Lefschetz, $H_1$ is a Fano manifold of Picard number one and $h^1(H_1, \Omega^1_{H_1}) = 1$. The ideal sequence, tensorized with $\Omega^1_{H_1}(1)$ reads

$$0 \to \Omega^1_{H_1} \to \Omega^1_{H_1}(1) \to \Omega^1_{H_1}(1)|_V \to 0,$$

and we get $h^0(V, \Omega^1_{H_1}(1)|_V) \leq h^0(H_1, \Omega^1_{H_1}(1)) + 1$. The two estimations yield $h^0(V, \Omega^1_V(1)) \leq h^0(H_1, \Omega^1_{H_1}(1))$.

Analogously, using the same sequences for $H_1$ in $M$, we find $h^0(H_1, \Omega^1_{H_1}(1)) \leq h^0(M, \Omega^1_M(1)) = 1$, and we conclude $h^0(V, \Omega^1_V(1)) \leq 1$.

To describe its zero locus as well as $d\theta \wedge \theta \in H^0(V, \mathcal{O}_V(1))$, we now briefly recall the group theoretic background of $M$ and its contact structure. We refer to [3] for details.

Instead of considering merely the exceptional group $G_2$, we study an arbitrary simple complex Lie group $G$. Let $\mathfrak{g}$ be its Lie algebra. Note that $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic via the Cartan killing form $(-,-)$ (and because of this we will sometimes write $\mathfrak{g}$ where perhaps $\mathfrak{g}^*$ would be more appropriate in the sequel). There exists exactly one closed orbit $M$ of the adjoint action of $G$ on $\mathbb{P}(\mathfrak{g})$. Let $L = \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)|_M$. 


We briefly sketch the idea of the following well known result: \( M \) carries a contact structure with contact line bundle \( L \) if and only if the dimension of \( M \) is odd.

One direction is trivial. Indeed, if \( M \) carries a contact structure \( \theta_M \) with contact line bundle \( L \), then the pull back of \( \theta_M \) to the total space of \( L \) induces a symplectic structure on \( L \), showing that \( \dim M \) must be odd. To prove that the convers holds in the above situation, we first define this symplectic structure, before showing that it comes from a contact form.

Let \( M^\circ \) be the orbit of \( G \) under the adjoined action of \( G \) on \( \mathfrak{g} \), such that \( \mathbb{P}(M^\circ) = M \). The tangent space \( T_{M^\circ}(Z) \) is canonically isomorphic to \( \mathfrak{g}/\ker \text{ad}(Z) \) for any \( Z \in M^\circ \). On \( M^\circ \) we have a nowhere degenerated symplectic form, locally defined by

\[
\omega_Z : T_{M^\circ}(Z) \times T_{M^\circ}(Z) \longrightarrow \mathbb{C}, \quad (X, Y) \mapsto \langle [X, Y], Z \rangle,
\]

which is nothing but the Kostant–Kirillov symplectic form, usually rather defined via the coadjoined representation. Note that \( \omega_Z \) is well defined at \( Z \) by Jacobi’s formula. The existence of \( \omega_Z \) implies that \( \dim M^\circ \) is even. Now assume \( \dim M \) is odd.

The dimension dropping by one, by going from \( M^\circ \) to \( M \), means \( M^\circ \) is the total space of \( L \) over \( M \). This is the case if and only if \( Z \in M^\circ \) implies \( cZ \in M^\circ \) for any \( Z \in M^\circ \) and \( c \in \mathbb{C}^* \). But \( M^\circ \) is an orbit, so this is the case if and only if \( Z \) and \( cZ \) are conjugated under the adjoined action for any choice of \( c \in \mathbb{C}^* \) and \( Z \in M^\circ \). This holds if and only if for any \( Z \in M^\circ \) there exists an \( H_Z \in \mathfrak{g} \) such that \( [H_Z, Z] = Z \).

The existence of an \( H_Z \in \mathfrak{g} \) for any \( Z \in M^\circ \) such that \( [H_Z, Z] = Z \) implies that for any \( Z \in M^\circ \) we have \( \mathfrak{z}_Z \subseteq Z^\perp \), where

\[
\mathfrak{z}_Z = \{ X \in \mathfrak{g} \mid [X, Z] = \lambda Z \text{ for some } \lambda \in \mathbb{C} \}
\]

and \( Z^\perp = \{ X \in \mathfrak{g} \mid [X, Z] = 0 \} \). Indeed, if \( X \in \mathfrak{z}_Z \) and \( [X, Z] = \lambda Z \), \( \lambda \neq 0 \), then \( [X, Z] = \lambda^{-1}([X, [X, Z]]) = 0 \). If \( X \in \mathfrak{z}_Z \) and \( [X, Z] = 0 \), we pick \( H_Z \) from above satisfying \( [H_Z, Z] = Z \), and we see \( [X, Z] = [X, [H_Z, Z]] = \langle [X, Z], H_Z \rangle = 0 \).

As in the case of \( T_{M^\circ}(Z) \) we have a canonical isomorphism for the tangent space \( T_M([Z]) \) of \( M \) at \( [Z] \in M \), \( Z \in \mathfrak{g} \):

\[
T_M([Z]) \simeq \mathfrak{g}/\mathfrak{z}_Z.
\]

At the point \( [Z] \in M \), the total space of \( O_{\mathbb{P}(\mathfrak{g})}(1) \) is isomorphic to \( \mathfrak{g}/Z^\perp \) (using \( \mathfrak{g} \simeq \mathfrak{g}^* \)). Since \( \mathfrak{z}_Z \subseteq Z^\perp \), we have a well defined surjection \( T_M([Z]) \rightarrow L([Z]) \), which glues, yielding a bundle sequence

\[
0 \longrightarrow F \longrightarrow T_M \theta_M L \longrightarrow 0.
\]

By construction, the pull back of the contact form to the total space of \( L \) is the Kostant–Kirillov form \( \theta_M \), showing that \( \theta_M \) indeed defines a contact structure. Alternatively one may consider the induced map

\[-d\theta_M = \theta_M([-, -]) : F \times F \longrightarrow L, \quad \text{given by} \quad (X, Y) \mapsto [X, Y] \mod Z^\perp.\]

This map is non–degenerate. Indeed, at \( [Z] \in M \) we have \( F = Z^\perp/\mathfrak{z}_Z \). Fix some \( Y \in Z^\perp \) and assume \( [X, Y] \in Z^\perp \) for any \( X \in Z^\perp \). Then \( [\langle Y, Z \rangle, Z] = 0 \), implying that the hyperplane \( \langle -, [Y, Z] \rangle \) is 0 contains the hyperplane \( Z^\perp \). Then \( [Y, Z] = \lambda Z \) for some \( \lambda \in \mathbb{C} \). Then \( Y \in \mathfrak{z}_Z \), showing that the map is indeed non–degenerate.
The construction of the homogeneous contact manifold $M$ shows that the contact sequence induces an isomorphism $H^0(M, T_M) \simeq H^0(M, L)$. Hyperplane sections of $M$ and vector fields may in this way be identified. Assume from now on that $V$ is cut out by the two smooth general hyperplanes $H_1$ and $H_2$, which are in this sense given by the two vector fields

$$X_1, X_2 \in H^0(M, T_M).$$

In our situation $H^0(M, T_M) = \mathfrak{g}$, so we may think of $X_1, X_2$ as elements of $\mathfrak{g}$. In explicit form, $H_i$ is now given by $\{[Z] \in M \mid \langle Z, X_i \rangle = 0\}$ and

$$V = \{[Z] \in M \mid \langle Z, X_i \rangle = 0, \text{ for } i = 1, 2\}.$$ Since $V$ is again a Fano manifold of index 1, we have again canonical isomorphisms

$$\bigwedge^2 T_V \simeq \Omega^1_V(1) \quad \text{and} \quad \bigwedge^3 T_V \simeq \mathcal{O}_V(1)$$

With this description, we can interpret $\theta$ as follows:

**3.4. Proposition.** Let $V$ be Fano of type $V_{18}$, given as above as a complete intersection of hyperplanes $H_1, H_2$ of the homogeneous $G_2$–contact manifold $M$, induced by vector fields $X_1, X_2 \in g_2$ on $M$. Then

$$\theta_{X_1, X_2} = X_1 \wedge X_2 \mid V \in H^0(V; \bigwedge^2 T_V)$$

is non–vanishing and may be thought of as the pull back of the contact structure $\theta_M$. The vanishing locus of $d\theta_{X_1, X_2}$ is the vanishing locus of $[X_1, X_2] \mid V \in H^0(V, \mathcal{O}_V(1)).$

**Proof.** We begin with considering a single smooth general hyperplane section $H_1$ of $M$, cut out by a section corresponding to $X_1 \in H^0(M, T_M) = g_2$, i.e.,

$$H_1 = \{[Z] \in M \mid \langle Z, X_1 \rangle = 0\}$$

as above. The contact form induces a nonzero section $\theta_{H_1} \in H^0(H_1, \Omega^1_{H_1}(1))$. We are interested in finding the points where $\theta_{H_1} : T_{H_1} \rightarrow \mathcal{O}_{H_1}(1)$ drops rank. The tangent space of the hyperplane $H_1$ at a point $[Z] \in H_1$ has the following canonical description

$$(3.5) \quad T_{H_1}([Z]) = [X_1, Z]^\perp / \mathfrak{j}_{[Z]}$$

with $\mathfrak{j}_{[Z]}$ as in (3.3). Note that $\mathfrak{j}_{[Z]} \subset [X_1, Z]^\perp$ for $[Z] \in H_1$. From this description we see: $X_1$ viewed as a vector field on $M$ is contained in $T_{H_1}([Z])$ for every $[Z] \in H_1$, implying

$$X_1 \in H^0(H_1, T_{H_1}).$$

The form $\theta_{H_1}$ drops rank precisely at those points $[Z] \in H_1$, where the contact bundle $F$ and $T_{H_1}$ define the same hyperplane of $T_M$. Hence $\theta_{H_1}$ drops rank precisely at those $[Z] \in H_1$ where $[X_1, Z]^\perp = Z^\perp$, which in turn holds precisely for those $[Z] \in M$ satisfying $[X_1, Z] = \lambda Z$ for some $\lambda \in \mathbb{C}^*$. The latter condition is equivalent to $X_1 \in \mathfrak{j}_{[Z]}$. For the equivalence note again that $\mathfrak{j}_{[Z]} \subset Z^\perp$ and that $H_1$ is smooth.

If we view $X_1$ as a vector field of $M$, then those points $[Z] \in M$, where $X_1 \in \mathfrak{j}_{[Z]}$, are the zeroes of $X_1$. We have proved

Zero locus of $\theta_{H_1} \in H^0(H_1, \Omega^1_{H_1}(1)) = \text{Zero locus of } X_1 \in H^0(H_1, T_{H_1}).$
The tangent bundle of homogeneous $M$ is globally generated. The vanishing locus of a general section consists of points, their number being equal to the highest Chern class of $M$. We finally conclude (see 3.6 Lemma):

Zero locus of $\theta_{H_1} \in H^0(H_1, \Omega^1_{H_1}(1)) = c_4(T_{H_1}) = c_5(T_M) = 6$ points.

Purely in terms of Chern classes, our result contains (and shows) the following equality of Chern classes, which also follow from the tangent sequence combined with the contact sequence:

$$c_4(T_{H_1}) = c_4(F|_{H_1}) = c_4(F^*(1)|_{H_1}) = c_4(\Omega^1_{H_1}(1)) \text{ and } c_5(T_M) = c_4(T_{H_1}).$$

Concerning $d\theta_{H_1} \wedge \theta_{H_1} \in H^0(H_1, \bigwedge^3 \Omega^1_{H_1} \otimes \mathcal{O}_{H_1}(2))$. Since $\bigwedge^3 \Omega^1_{H_1} \otimes \mathcal{O}_{H_1}(2) \simeq T_{H_1}$, we may view $d\theta_{H_1} \wedge \theta_{H_1}$ as a vector field on $H_1$. Writing down an explicit isomorphism, we find that the vanishing locus of $d\theta_{H_1} \wedge \theta_{H_1}$ coincides with the vanishing locus of $X_1 \in H^0(H_1, T_{H_1})$, which are 6 points.

Now consider $V$ from above, the complete intersection of the hyperplanes $H_1, H_2$ corresponding to $X_1, X_2 \in H^0(M, T_M)$. First, since $X_i$ is in the kernel of the map $T_M|_{H_i} \to \mathcal{O}(1)$ on global sections, it is clear from

$$0 \to \bigwedge^2 T_V \to \bigwedge^2 T_{H_i|V} \to T_V(1) \to 0$$

$$0 \to \bigwedge^2 T_{H_1} \to \bigwedge^2 T_M|_{H_i} \to T_{H_1}(1) \to 0$$

that $X_1 \wedge X_2$ indeed defines a nonzero section of $\bigwedge^2 T_V$. Since $H^0(V, \Omega^1_V(1))$ is one dimensional, we may take this section to be the pull back of $\theta_M$. Alternatively, one may derive a description of $T_V([Z])$ and conclude as above, that the pull back of $\theta_M$ on $V$ drops rank precisely at the vanishing points of $X_1 \wedge X_2$.

We use the following identifications to determine $d\theta \wedge \theta$:

$$\bigwedge^2 \Omega^1_V \otimes \mathcal{O}_V(1) \to \bigwedge^3 \Omega^1_{H_1} \otimes \mathcal{O}_{H_1}(2)|_V \to \bigwedge^3 \Omega^1_V \otimes \mathcal{O}_V(2) \to 0$$

$$0 \to T_V \to T_{H_1|V} \to \mathcal{O}_V(1) \to 0$$

The pull back of $d\theta_M \wedge \theta_M$ to $H_1$, using the identification $\bigwedge^3 \Omega^1_{H_1} \otimes \mathcal{O}_{H_1}(2) \simeq T_{H_1}$, yields $X_1$ as we saw above. The image of $X_1$ under $H^0(H_1, T_{H_1}) \to H^0(V, \mathcal{O}_V(1))$ is the section induced by the vector field $[X_1, X_2]$. This is clear from the identification $\bigwedge^2 \Omega^1_V \otimes \mathcal{O}_V(1) = T_V$ and, for example, the pointwise description of the tangent bundle on $V$, analogous to (3.6).

Since $X_1$ and $X_2$ were chosen general, it is clear, that $[X_1, X_2]$ does not vanish, and is different from $X_1$ and $X_2$ as elements in $H^0(M, T_M) = \mathfrak{g_2}$. It therefore defines a non–zero section in $\mathcal{O}_V(1) = \mathcal{O}_{M(1)}|_V$.

We check for entertainment that the zero locus of $X_1 \wedge X_2$, viewed as a section of $\bigwedge^2 T_V$, is indeed contained in the vanishing locus of $[X_1, X_2]$, viewed as a section of $\mathcal{O}_V(1)$. This must necessarily be the case, since $Z(\theta) \subset Z(d\theta \wedge \theta)$.

If the wedge product $X_1 \wedge X_2$ vanishes at a point $[Z] \in V$, then $X_1$ and $X_2$, evaluated at $[Z]$, are dependent, meaning $\lambda Z = \lambda_1[X_1, Z] + \lambda_2[X_2, Z]$ for some $\lambda, \lambda_1, \lambda_2 \in \mathbb{C}$. We may assume $\lambda_1 \neq 0$. Applying $\langle \cdot, X_2 \rangle$, we find on the left hand side $\lambda \langle Z, X_2 \rangle$. Since $[Z]$ is a point on $V$, this is zero. The right hand side then reads $\lambda_1 \langle [X_1, Z], X_2 \rangle + \lambda_2 \langle [X_2, Z], X_2 \rangle = \lambda_1 \langle [X_1, Z], X_2 \rangle = 0$, using $[X_2, X_2] = 0$. Since $\lambda_1 \neq 0$, we conclude $\langle [X_1, X_2], Z \rangle = 0$, as desired. □
3.6. Lemma. Let $M$ be a quotient of the exceptional simple Lie group $G_2$ by a maximal parabolic subgroup. Then $\dim M = 5$ and $c_5(T_M) = 6$.

Proof. Let $G$ be the exceptional group of type $G_2$ and $B \subset G$ a Borel group. Then there are two (maximal) parabolic subgroups $P_1, P_2$ in $G$ containing $B$. The corresponding homogeneous manifolds are $M_1 = G/P_1$, a 5–dimensional quadric, and $M_2 = G/P_2$, the 5–dimensional contact manifold associated to $G$. The dimension of the homogeneous manifold $M_B = G/B$ is 6. We will show that the highest Chern classes of $M_1$ and $M_2$ coincide. We have the following diagram:

\[
\begin{array}{c}
M_B \\
\pi_2 \\
\downarrow \\
M_2 \\
\pi_1 \quad M_1 \\
\end{array}
\]

Let $\mathcal{L}_1, \mathcal{L}_2$ be the (globally generated) fundamental line bundles on $M_B$. Then $-K_{M_B} = 2\mathcal{L}_1 + 2\mathcal{L}_2$ and $\mathcal{L}_i = \pi_i^* L_i$, where $L_i$ is the fundamental line bundle on $M_i$. We have $-K_{M_1} = 5\mathcal{L}_1$ and $-K_{M_2} = 3\mathcal{L}_2$. All these facts can be found for example in [A]. The fibers $F_i = P_i/B$ of $\pi_i$ are so–called $\alpha$–lines in $M_B$, that are smooth rational curves with the property $\mathcal{L}_i F_j = 1$ for $i \neq j$. This can be easily checked or can be found for example in [Ko]. In particular, the projections $\pi_i$ are $\mathbb{P}^1$–bundles. Consider the relative tangent sequences

\[
0 \rightarrow T_{M_B/M_i} \rightarrow T_{M_B} \rightarrow \pi_i^* T_{M_i} \rightarrow 0
\]

for $i = 1, 2$. The relative tangent bundles $T_{M_B/M_i}$ are line bundles, which we get by computing the determinant of the above sequence: $T_{M_B/M_1} = -3\mathcal{L}_1 + 2\mathcal{L}_2$ and $T_{M_B/M_2} = 2\mathcal{L}_1 - \mathcal{L}_2$. Since Chern polynomials in short exact sequences are multiplicative, we have

\[
c_1(\pi_i^* T_{M_i}).(1 + c_1(T_{M_B/M_i})t) = c_i(\pi_i^* T_{M_B}).(1 + c_1(T_{M_B/M_i})t).
\]

This shows $\pi_i^* c_5(T_{M_i}).c_1(T_{M_B/M_i}) = \pi_i^* c_5(T_{M_B}).c_1(T_{M_B/M_i})$. Since $c_5(T_{M_i})$ are points, the pull–backs $\pi_i^* c_5(T_{M_i})$ are fibers $F_{i,j}$ of $\pi_i$. Hence

\[
\pi_i^* c_5(T_{M_i}).c_1(T_{M_B/M_i}) = (\sum_{j=1}^{\deg c_5(T_{M_i})} F_{1,j}).(-3\mathcal{L}_1 + 2\mathcal{L}_2) = 2 \deg c_5(T_{M_i}),
\]

since $F_{1,j} \mathcal{L}_1 = 0$ and $F_{1,j} \mathcal{L}_2 = 1$ for all $j$. Analogously, $\pi_i^* c_5(T_{M_B}).c_1(T_{M_B/M_i}) = 2 \deg c_5(T_{M_2})$, implying $c_5(T_{M_1}) = c_5(T_{M_2})$, viewed as natural numbers.

It remains hence to compute $c_5(Q)$, where $Q \subset \mathbb{P}^6$ denotes the 5–dimensional quadric. From the tangent sequence we get

\[
c_i(T_Q).(1 + 2c_1(O_Q(1)))t = c_i(T_{p^6} Q) = (1 + c_1(O_Q(1)))t^7,
\]

where $O_Q(1) = O_{p^6}(1)|_Q$. Successively we obtain

\[
c_5(T_Q) = \sum_{i=0}^{5} \binom{7}{i} 2^5 i c_1(O_Q(1))^5 = 3c_1(O_Q(1))^5.
\]

Now $c_1(O_Q(1))^5 = (c_1(O_{p^6}(1)|_Q))^5 = c_1(O_{p^6}(1))^5$. $Q = c_1(O_{p^6}(1))^5$. $Q = c_1(O_{p^6}(1))^5$. $Q = c_1(O_{p^6}(1))^5$. $Q = c_1(O_{p^6}(2)) = 2$, completing the proof. \[\square\]
**Appendix**

The following classification of Fano threefolds with Picard number one is due to Iskovskikh and Mukai, and can be found in [I] and [M], respectively.

| No. | $r$ | $H^3$ | $g$ | Description                                                                 |
|-----|-----|-------|-----|-----------------------------------------------------------------------------|
| 1   | 4   | 1     | 33  | $\mathbb{P}^3$                                                             |
| 2   | 3   | 2     | 28  | $Q_3 \subset \mathbb{P}^4$ the quadric                                    |
| 3   | 2   | 1     | 5   | Hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$                        |
| 4   | 2   | 2     | 9   | Hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$                        |
| 5   | 2   | 3     | 13  | $V_{2,3} \subset \mathbb{P}^4$ a cubic                                    |
| 6   | 2   | 4     | 17  | $V_{2,4} \subset \mathbb{P}^5$ the intersection of 2 quadrics              |
| 7   | 2   | 5     | 21  | $V_{2,5} \subset \mathbb{P}^6$ the intersection of the Grassmannian $\text{Gr}(2,5) \subset \mathbb{P}^9$ by a $\mathbb{P}^6$ |
| 8   | 1   | 2     | 2   | Hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$                        |
| 9   | 1   | 4     | 3   | $V_4 \subset \mathbb{P}^4$ a quartic                                       |
| 10  | 1   | 4     | 3   | Complete intersection of a quadratic cone and a hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,1,2)$ |
| 11  | 1   | 6     | 4   | $V_6 \subset \mathbb{P}^5$ the intersection of a quadric and a cubic       |
| 12  | 1   | 8     | 5   | $V_8 \subset \mathbb{P}^6$ the intersection of three quadrics              |
| 13  | 1   | 10    | 6   | $V_{10} \subset \mathbb{P}^7$ the intersection of the Grassmannian $\text{Gr}(2,5) \subset \mathbb{P}^9$ by a $\mathbb{P}^7$ |
| 14  | 1   | 12    | 7   | $V_{12} \subset \mathbb{P}^8$ the intersection of the Hermitian symmetric space $M = G/P \subset \mathbb{P}^{15}$ of type DIII by a $\mathbb{P}^8$ |
| 15  | 1   | 14    | 8   | $V_{14} \subset \mathbb{P}^9$ the intersection of the Grassmannian $\text{Gr}(2,6) \subset \mathbb{P}^{14}$ by a $\mathbb{P}^9$ |
| 16  | 1   | 16    | 9   | $V_{16} \subset \mathbb{P}^{10}$ is the intersection of the Hermitian symmetric space $M = G/P \subset \mathbb{P}^{19}$ of type CI by a $\mathbb{P}^{10}$ |
| 17  | 1   | 18    | 10  | $V_{18} \subset \mathbb{P}^{11}$ is the intersection the 5–dimensional rational homogeneous contact manifold $G_2/P \subset \mathbb{P}^{13}$ by a $\mathbb{P}^{11}$ |
| 18  | 1   | 22    | 12  | $V_{22} \subset \mathbb{P}^{13}$ is the zero locus of three sections of the rank 3 vector bundle $\bigwedge^2 Q$, where $Q$ is the universal quotient bundle on $\text{Gr}(7,3)$ |

**References**

[A] D.N. Akhiezer: Lie Group Actions in Complex Analysis. Aspects of Mathematics E27 (1995)

[B] A. Beauville: Fano contact manifolds and nilpotent orbits. Comm. Math. Helv. 73, 566–583 (1998)

[D] I. Dolgachev: Weighted projective varieties. Lect. N. Math. 956, 34–71 (1982)

[EPS] G. Ellingsrud, R. Piene, S.A. Strømme: On the variety of nets of quadrics defining twisted cubics. Lect. N. Math. 1266, 84–96 (1987)

[FH] W. Fulton, J. Harris: Representation Theory. Springer 1991
V.A. Iskovskikh: Fano 3–folds I, II. Math. USSR, Izv. 11, 485–527 (1977); 12, 469–506 (1978)

A. Iliev, C. Schuhmann: Tangent scrolls in prime Fano threefolds. Kodai Math. J. 23, 411–431 (2000)

V.A. Iskovskikh, Yu.G. Prokhorov: Algebraic Geometry V: Fano varieties. Springer 1999

J. Kollár: Rational curves. Springer 1996

L. Manivel, D.M. Snow: A Borel–Weil theorem for holomorphic forms. Comp. Math. 103, 351–365 (1996)

S. Mukai: Fano 3–folds. London Math. L. Notes 179, 255–263 (1992)

S. Mukai, H. Umemura: Minimal rational threefolds. Lect. N. Math. 1016, 490–518 (1983)

C. Okonek, M. Schneider, H. Spindler: Vector bundles on Complex Projective Spaces. Progress in Math. 3, 1980

Yu.G. Prokhorov: On exotic Fano varieties. Moscow Univ. Math. Bull. 45, No.4, 36–38 (1990)

Yu.G. Prokhorov: Automorphism groups of Fano manifolds. Russ. Math. Surv. 45, No.3, 222–223 (1990)

V.V. Shokurov: Smoothness of the general anticanonical divisor on a Fano 3–fold. Math. USSR, Izv. 14, 395–405 (1980)

D.M. Snow: Cohomology of Twisted Holomorphic Forms on Grassmann Manifolds and Quadric Hypersurfaces. Math. Ann. 276, 159–176 (1986)

D.M. Snow: Vanishing Theorems on Compact Hermitian Symmetric Spaces. Math. Z. 198, 1–20 (1988)

A.N. Tyurin: Five lectures on three–dimensional varieties. Russ. Math. Surv. 27, No.2, 1–53 (1972)

Mathematisches Institut, Universität Bayreuth, D–95440 Bayreuth/ Germany
E-mail address: priska.jahnke@uni-bayreuth.de
E-mail address: ivo.radloff@uni-bayreuth.de