Time for dithering: fast and quantized random embeddings via the restricted isometry property

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Abstract

Recently, many works have focused on the characterization of non-linear dimensionality reduction methods obtained by quantizing linear embeddings, e.g., to reach fast processing time, efficient data compression procedures, novel geometry-preserving embeddings or to estimate the information/bits stored in this reduced data representation. In this work, we prove that many linear maps known to respect the restricted isometry property (RIP), can induce a quantized random embedding with controllable multiplicative and additive distortions with respect to the pairwise distances of the data points being considered. In other words, linear matrices having fast matrix-vector multiplication algorithms (e.g., based on partial Fourier ensembles or on the adjacency matrix of unbalanced expanders), can be readily used in the definition of fast quantized embeddings with small distortions. This implication is made possible by applying right after the linear map an additive and random dither that stabilizes the impact of a uniform scalar quantization applied afterwards.

For different categories of RIP matrices, i.e., for different linear embeddings of a metric space \((K \subset \mathbb{R}^n, \ell_q)\) in \((\mathbb{R}^m, \ell_p)\) with \(p, q \geq 1\), we derive upper bounds on the additive distortion induced by quantization, showing that this one decays either when the embedding dimension \(m\) increases or when the distance of a pair of embedded vectors in \(K\) decreases. Finally, we develop a novel bi-dithered quantization scheme, which allows for a reduced distortion that decreases when the embedding dimension grows, independently of the considered pair of vectors.

1 Introduction

Since the advent of the Johnson and Lindenstraus lemma in 1984 and its numerous extensions \cite{25, 13}, non-adaptive dimensionality reduction techniques for high dimensional data obtained through random constructions are now ubiquitous in numerous fields ranging from data mining and database management \cite{14}, machine learning algorithms \cite{47}, numerical linear algebra \cite{44, 42}, signal processing and compressive sensing (CS) \cite{11, 3, 15}.

In the quest for such efficient techniques, a sub-field of research has grown to study non-linear embeddings obtained by quantizing the output of random linear mappings. This is particularly important for reducing the number of bits required to store the image of those maps by integrating quantization in the whole embedding analysis. For instance, combinations of random linear maps with 1-bit sign operators \cite{6, 24, 36}, universal quantization \cite{5}, dithered uniform quantization \cite{19, 21} or local sketching methods \cite{2} were proved to be either as efficient...
as unquantized linear embeddings, or to allow us better encoding of the signal set geometry \cite{38,9}.

In this context, a difficult question is the design of non-linear maps with fast encoding time (e.g., in log-linear time) and providing embeddings of large sets of \( \mathbb{R}^n \), possibly continuous, such as the set of sparse vectors, the one of low-rank matrices or other low-complexity vector sets. To the best of our knowledge, a few fast non-linear maps already exist in the case of 1-bit quantization (e.g., built on a fast circulant linear operator), but these are currently restricted to the embedding of finite sets \cite{30,48,49}.

A priori, the lack of quantized random maps with fast encoding schemes and provable embedding properties of general set is rather frustrating when we know the large availability of fast linear random embeddings, e.g., as those developed since 2004 in the CS literature for satisfying the celebrated Restricted Isometry Property (RIP), that precisely controls the quality of these embeddings (see e.g., \cite{16,41,39,40} and Sec. 3).

This work aims to fill this gap by showing that for a \textit{dithered} uniform scalar quantization, random linear maps known to satisfy the RIP determine, with high probability, a non-linear embedding simply obtained by quantizing their image. We show that this is possible for all vector sets on which this RIP is known to hold.

In a nutshell, the proofs developed in this work are not technical and are all based on the same architecture: the RIP allows us to focus on the embedding of the image of a low-complexity vector set (obtained through the corresponding linear map) into a quantized domain thanks to a randomly dithered-quantization. This is made possible by softening the discontinuous distances evaluated in the quantized domain according to a mathematical machinery inspired by \cite{36} in the case of 1-bit quantization and extended in \cite{20} for dithered uniform scalar quantization. Actually, this softening allows us to extend the concentration of quantized random maps on a finite covering of the low-complexity vector set \( \mathcal{K} \) to this whole set by a continuity argument.

As already determined in many previous works, a key aspect in our study is the way distances are measured in both the quantized embedding space \( \mathcal{E} \) and in the original vector space \( \mathcal{K} \).

In particular, denoting by the \((\ell_p, \ell_q)\text{-RIP}(\mathcal{K}, \epsilon)\) (with \( p, q \geq 1 \)) the property of a matrix \( \Phi \) to approximate the \( \ell_q \)-norm of all vectors in \( \mathcal{K} \) through the \( \ell_p \)-norm of their projections up to a multiplicative distortion \( \epsilon > 0 \) (see \cite{1} for the precise definition), we will basically show that measuring distances in \( \mathcal{E} \) with the \( \ell_1 \)-norm allows us to build, with high probability, quantized embeddings inherited from the \((\ell_1, \ell_q)\text{-RIP}(\mathcal{K} - \mathcal{K}, \epsilon)\) of their linear map. In this case, both the additive and the multiplicative distortions of this embedding are proportional to \( \epsilon \) provided the embedding dimension \( m \) is large before the dimension of \( \mathcal{K} \) (as measured by the Kolmogorov entropy). Alternatively, if the distance of vectors in \( \mathcal{E} \) is measured with a squared \( \ell_2 \)-norm, as classically done for linear embeddings based on the RIP, we prove that a similar embedding inherited from the \((\ell_2, \ell_2)\text{-RIP}(\mathcal{K} - \mathcal{K}, \epsilon)\) holds with high probability, but the additive distortion now vanishes only for arbitrarily close pairs of embedded vectors. Finally, we show that one can recover a quantized embedding with low additive distortion scaling like \( \epsilon \) and connected to the \((\ell_2, \ell_2)\text{-RIP}(\mathcal{K} - \mathcal{K}, \epsilon)\) of its linear part by designing a \textit{bi-dithered} quantization procedure that implicitly doubles the embedding dimension.

The rest of this paper is structured as follows. We first end this introduction by giving useful mathematical notations. Sec. 2 presents carefully the three main results of this paper and their important preliminary concepts. Sec. 3 discusses the portability of those results and their connection to the existing “market” of RIP matrices, including those with fast/low-complexity vector encoding schemes, before making connections with existing works and providing perspectives and open problems. Finally, Sec. 4, Sec. 5 and Sec. 6 are dedicated to proofs, with few useful lemmata postponed to the Appendices A and B.
Conventions: We find useful to introduce now the conventions and notations used throughout this paper. We will denote vectors and matrices with bold symbols, e.g., $\Phi \in \mathbb{R}^{m \times n}$ or $u \in \mathbb{R}^{n}$, while lowercase light letters are associated to scalar values. The identity function and the identity matrix in $\mathbb{R}^{n}$ read $\text{Id}$ and $I_{n}$, respectively, while $1_{n} := (1, \ldots , 1)^{T} \in \mathbb{R}^{n}$ is the vector of ones. The $i$th component of a vector (or of a vector function) $u$ reads either $u_{i}$ or $(u)_{i}$, while the vector $u_{i}$ may refer to the $i$th element of a set of vectors. The set of indices in $\mathbb{R}^{d}$ is $[d] := \{1, \ldots , d\}$ and for any $\mathcal{S} \subset [d]$ of cardinality $\mathcal{S} = |\mathcal{S}|$, $u_{\mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$ denotes the restriction of $u$ to $\mathcal{S}$, while $B_{\mathcal{S}}$ is the matrix obtained by restricting the columns of $B \in \mathbb{R}^{d \times d}$ to those indexed in $\mathcal{S}$. For any $p \geq 1$, the $\ell_{p}$-norm of $u$ is $\|u\|_{p} = (\sum_{i} |u_{i}|^{p})^{1/p}$ with $\|\cdot\|_{1} = \|\cdot\|_{2}$, and we write also $D_{\ell_{p}}(a, a') := \frac{1}{d} \|a - a'\|_{p}^{p}$ the averaged $p$th power of the $\ell_{p}$-distance. The $(n-1)$-sphere in $\mathbb{R}^{n}$ is $S^{n-1} = \{x \in \mathbb{R}^{n} : \|x\| = 1\}$ while the unit ball is denoted $\mathbb{B}^{n} = \{x \in \mathbb{R}^{n} : \|x\| \leq 1\}$.

We use the simplified notation $\mathcal{P}^{m \times n}$ and $\mathcal{P}^{m}$ to denote an $m \times N$ random matrix or an $m$-length random vector, respectively, whose entries are identically and independently distributed (or iid) as the probability distribution $\mathcal{P}$, e.g., $N^{m \times n}(0, 1)$ (or $\mathcal{U}^{m}(0, 1)$) is the distribution of a matrix (resp. vector) whose entries are iid as the standard normal distribution $N(0, 1)$ (resp. the uniform distribution $\mathcal{U}([0, 1])$). We also use extensively the sub-Gaussian and sub-exponential characterization of random variables (or r.v.) and of random vectors detailed in [15]. The sub-Gaussian and the sub-exponential norms of a random variable $X$ are thus noted $\|X\|_{\psi_{2}}$ and $\|X\|_{\psi_{1}}$, respectively, with the Orlicz norm $\|X\|_{\psi_{\alpha}} := \sup_{p > 1} p^{-1/\alpha} (\mathbb{E}|X|^{p})^{1/p}$ for $\alpha \geq 1$. The random variable $X$ is thus said sub-Gaussian (or sub-exponential) if $\|X\|_{\psi_{2}} < \infty$ (resp. $\|X\|_{\psi_{1}} < \infty$). We also write $\|X\|_{\psi_{\infty}} = \inf\{s > 0 : \mathbb{P}(|X| \leq s) = 1\}$, with the useful relation $\|X\|_{\psi_{\alpha}} \leq \|X\|_{\infty}$. Roughly speaking, we will sometimes write that an event holds with high probability (or w.h.p.) if its probability of failure decays exponentially with some specific dimensions depending on the context (e.g., with the embedding dimension of a map). The common flooring and ceiling operators are denoted $[\cdot]$ and $\lceil \cdot \rceil$, respectively, and the positive thresholding function is defined by $(\lambda)_{+} := \frac{1}{2}(\lambda + |\lambda|)$ for any $\lambda \in \mathbb{R}$. Finally, an important feature of our study is that we do not pay any particular attention to constants in the many bounds developed in this paper. For instance, the symbols $C, C', C'', \ldots , c, c', c'', \ldots > 0$ are positive constants whose values can change from one line to the other. We will, however, limit the use of such constants and we find convenient to use the (non-asymptotic) ordering notations $A \lesssim B$ (or $A \gtrsim B$), if there exist a $c > 0$ such that $A \leq cB$ (resp. $A \geq cB$), for two quantities $A$ and $B$.

## 2 Main results

The Restricted Isometry Property (RIP) is a well known criterion for characterizing the distortion induced by a linear map $\Phi \in \mathbb{R}^{m \times n}$, with $m$ generally smaller than $n$, when this one is used to embed a “low-complexity” set $\mathcal{K} \subset \mathbb{R}^{n}$ of the metric space $(\mathbb{R}^{n}, \ell_{p})$ into $(\mathbb{R}^{m}, \ell_{p})$. In particular, given $0 < \epsilon < 1$, we say that $\Phi$ respects the $(\ell_{p}, \ell_{q})$-Restricted Isometry Property over $\mathcal{K}$, or $(\ell_{p}, \ell_{q})$RIP($\mathcal{K}, \epsilon$) if, for all $x \in \mathcal{K}$,

$$
(1 - \epsilon)\|x\|_{q}^{p} \leq \mu_{\Phi} \|\Phi x\|_{p}^{p} \leq (1 + \epsilon)\|x\|_{q}^{p},
$$

for some $\mu_{\Phi} > 0$ independent of $n$ and $m$ but possibly dependent on $\mathcal{K}$ and $\epsilon$, as in the case $p = q = 1 + O(1)$ and $\mathcal{K} = \Sigma_{s,n} := \{u \in \mathbb{R}^{n} : |\text{supp } u| \leq s\}$ [4]. For simplicity, we consider henceforth that $\mu_{\Phi} = 1$, up to a rescaling of the sensing matrix $\Phi$.

The RIP can be shown to hold for random matrix constructions, e.g., when the entries of $\Phi$ are iid sub-Gaussian [27] or when this matrix is connected to the adjacency matrix of some unbalanced expanders [4]. Many random constructions with fast vector encoding time, i.e., low-complexity matrix-vector multiplication, based on e.g., random Fourier/Basis ensembles [16],
random convolutions [39, 40], spread-spectrum techniques [37] are also well known in the case  
\( p = q = 2 \). Notice that, at least for homogeneous sets \( \mathcal{K} \) (i.e., such that \( \lambda \mathcal{K} \subset \mathcal{K} \) for all \( \lambda \geq 0 \)), the embedding distortion explained by this RIP is multiplicative and represented by the  
factors \((1 \pm \epsilon)\).

This work addresses the question of combining such RIP matrix constructions with a quantization procedure in order to reach efficient non-linear embedding procedures, i.e., with a controllable level of distortion between the original distance of any pair of vectors in \( \mathbb{R}^n \) and some suitably defined distance of their quantized projections. This happens to be important for efficiently storing, transmitting or even processing the vectors undergoing this non-linear embedding procedure in order to reach efficient non-linear embedding procedures, consistent with the deterministic relation \(|Q(\lambda) - \lambda| \leq \delta/2\) induced by the very definition of \( Q \), a straightforward application of this quantization on the \((\ell_p, \ell_q)\)-RIP(\(K - K, \epsilon\)) quickly leads to

\[
(1 - \epsilon)^{1/p}\|x - x'\|_q - \delta \leq \frac{1}{\sqrt{m}}\|Q(\Phi) - \Phi\|_q \leq (1 + \epsilon)^{1/p}\|x - x'\|_q + \delta, \tag{2}
\]

for all \( x, x' \in K \). However, this deterministic derivation has a severe drawback: the new additive distortion \( \pm \delta \) brought on both sides of (2) is constant, irrespectively of the embedding dimension \( m \). This seems rather counterintuitive since, for instance, for vectors \( x \) and \( x' \) that are consistent, i.e., \( Q(\Phi)x = Q(\Phi)x' \), it seems harder for them to stay far apart when \( m \) increases, i.e., the set of constraints imposed by this consistency is increasingly hard to satisfy, while (2) just provides \( \|x - x'\|_q \leq (1 - \epsilon)^{-1/p} \delta \approx \delta \).

The objective of this work is rather to show that, up to a randomization of the quantization process through a common dithering procedure [18, 46], the resulting quantized embeddings indeed both a multiplicative and an additive distortion of the distances, just like in (2), but this additive distortion, which is not present in the RIP of linear embeddings, can be shown to decay when \( m \) increases or when the pairwise vector distance decays. More specifically, our aim is to analyze the embedding properties of the quantized map \( A : \mathbb{R}^n \to \mathbb{Z}_\delta^m \) such that, for \( x \in \mathbb{R}^n \),

\[
A(x) = A(x, \Phi, \xi) := Q(\Phi + \xi), \tag{3}
\]

where \( \xi \) is the uniform dither whose entries are iid as \( \mathcal{U}([0, \delta]) \), i.e., \( \xi \sim \mathcal{U}([0, \delta]) \). Notice that such a dithered and quantized random map is not new and some of its properties have already been studied, e.g., in the context of locality-sensitive hashing of finite vector sets (e.g., for Gaussian or \( \alpha \)-stable random matrix distributions) [2, 14] or for universal encoding strategies when \( Q \) is replaced by a non-regular quantizer or a periodic function [5, 8, 9] (see Sec. 3.2).

In order to make this analysis more formal, the embedding space \( \mathcal{E} = \mathbb{Z}_\delta^m \) is endowed with a pre-metric \( D_\mathcal{E} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_+ \) that is assumed homogeneous of degree \( p_\epsilon \), i.e., for all \( \lambda \in \mathbb{R} \),

\[
D_\mathcal{E}(\lambda a, \lambda a') := |\lambda|^{p_\epsilon}D_\mathcal{E}(a, a') \quad \text{for all } a, a' \in \mathbb{R}^m.
\]

For instance, we will consider \( D_{\mathcal{E}} = D_{\ell_p} \) with

\[
D_{\ell_p}(a, a') := \frac{1}{m}\|a - a'\|_p^p \quad \text{for all } a, a' \in \mathbb{R}^m \text{ and } p = p_\epsilon \in \{1, 2\}.
\]

The function \( D_{\ell_p} \) is a pre-metric but not a metric since it does not respect the triangle inequality.

Within this context, we are thus interested in the following characterization of the map \( A \).

**Definition 1 (Quantized RIP).** For \( q \geq 1 \), \( 0 < \epsilon < 1 \) and some additive distortion \( \rho : (\epsilon, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \to \rho(\epsilon, s) \in \mathbb{R}_+ \) such that

\[
\rho(0, 0) = 0 \quad \text{and} \quad \lim_{s \to +\infty} s^{-p_\epsilon} \rho(\epsilon, s) = 0, \tag{4}
\]
a map $A : \mathbb{R}^n \to \mathcal{E}$ respects the $(\mathcal{D}_\mathcal{E}, \ell_q)$-quantized restricted isometry property, or $(\mathcal{D}_\mathcal{E}, \ell_q)$-Q RIP($\mathcal{K}, \epsilon, \rho$), if for all $x, x' \in \mathcal{K}$

$$(1 - \epsilon)\|x - x'\|_q^p - \rho(\epsilon, \|x - x'\|_q) \leq \mathcal{D}_\mathcal{E}(A(x), A(x')) \leq (1 + \epsilon)\|x - x'\|_q^p + \rho(\epsilon, \|x - x'\|_q). \quad (5)$$

Notice that this definition is voluntarily general in order to simplify the presentation of our results. In particular, this paper will consider only the cases where $p_e = 1$ or 2. The pre-metric $\mathcal{D}_\mathcal{E}$ will be proportional to the $\ell_1$-distance, the squared $\ell_2$-distance in $\mathbb{R}^m$ or to a pre-metric derived from the $\ell_1$-distance as will be clearer below. We can directly remark that the targeted Q RIP, through the requirement imposed on $\rho(\epsilon, s)$, i.e., its disappearance when $s = 0$ or $\epsilon = 0$ or its decay relatively to $s^p$ for large values of $s$, is stronger than what is provided by the straightforward derivation \cite{2}. By raising this last relation to the $p_e$th power, as in \cite{5}, the resulting additive distortion $\rho$ contains an independent term in $s^p$ that does not vanish if $\epsilon$ or the vector distance $\|x - x'\|_q$ tends to zero.

In this clarified context, the gist of this paper is thus to show that we can leverage the $(\ell_p, \ell_q)$-RIP($\mathcal{K}, \epsilon$) of certain matrix constructions, if this exists for some $p, q \geq 1$, to prove that, with high probability provided $m$ is large compared to the complexity of $\mathcal{K}$, the random map $A$ in \cite{3} respects a $(\mathcal{D}_\mathcal{E}, \ell_q)$-Q RIP($\mathcal{K}, \epsilon, \rho$), for suitable $\mathcal{D}_\mathcal{E}$ and $\rho = \rho(\epsilon)$. The dither $\xi$ introduced in the definition $A$ plays a crucial role in this analysis. Not only it allows to assimilate the impact of quantization to a mere additive uniform noise, as generally realized in quantization theory \cite{13}, but assuming that $\Phi$ satisfies the $(\ell_p, \ell_q)$-RIP($\mathcal{K}, \epsilon$), this dither also enables us to reach embedding distortions $(\epsilon, \rho)$ that decay when the embedding dimension $m$ increases or when the pairwise distance of vectors in $\mathcal{K}$ decreases.

As an important ingredient, the main requirement on $m$ on which our results are built depends on the dimension of the considered low-complexity space $\mathcal{K} \subset \mathbb{R}^n$, as characterized by its Kolmogorov entropy. In the proofs, this allows us to bound the cardinality of their covering according to their radius. In particular, given $q \geq 1$ and assuming $\mathcal{K}$ is bounded, an $(\ell_q, \eta)$-net $\mathcal{K}_{\ell_q, \eta} \subset \mathcal{K}$ of radius $\eta > 0$ in the $\ell_q$-metric is a finite set of points that covers $\mathcal{K}$, i.e.,

$$\forall x \in \mathcal{K}, \quad \min\{|\|x - u\|_p : u \in \mathcal{K}_{\ell_q, \eta}\} \leq \eta.$$  

Assuming that $\mathcal{K}_{\ell_q, \eta}$ is optimal, i.e., there is no smaller set covering $\mathcal{K}$, the Kolmogorov entropy of $\mathcal{K}$ is then defined as

$$\mathcal{H}_q(\mathcal{K}, \eta) := \log |\mathcal{K}_{\ell_q, \eta}|.$$  

Interestingly, this entropy can be estimated for many low-complexity spaces:

- **Union of low-dimensional subspaces (ULS) ($q \geq 1$):** Let us consider the ULS model where $\mathcal{K} = \bigcup_{i=1}^{T} S_i$ is a union of $T$ $s$-dimensional subspaces $S_i$ in $\mathbb{B}_{\ell_q}^n$. This model includes, for instance, a single subspace of $\mathbb{R}^n$, the set $D(S_{s,d}) \cap \mathbb{B}_{\ell_q}^n$ of sparse signals in some dictionary $D \in \mathbb{R}^{n \times d}$ with $d \geq n$ (with orthonormal bases and frames as special cases) where $\log T \leq s \log(\frac{n}{s})$ (from the Stirling approximation $\binom{n}{s} \leq (en/s)^s$). Under the ULS model we have $\mathcal{H}_q(\mathcal{K}, \eta) \leq \log(T) + s \log(3/\eta) \leq s \log(3T/\eta)$, e.g., $\mathcal{H}_q(\Sigma_{s,n,q}, \eta) \lesssim s \log(n/sn)$.

This can be easily established by observing that an $(\ell_q, \eta)$-net of each such $s$-dimensional subspace in $\mathbb{B}_{\ell_q}^n$ has no more than $(1 + \frac{2}{\eta})^s$ points (see, e.g., \cite{34} Lemma 4.10). Therefore, a (non-optimal) covering of their union can be obtained from the union of the nets, whose cardinality is smaller than $T(1 + \frac{2}{\eta})^s$, which provides the bound.

Consequently, if $\mathcal{K}$ is a bounded ULS, we can always assume

$$\mathcal{H}_q(\mathcal{K}, \eta) \lesssim C_{\mathcal{K}} \log c_{\mathcal{K}}/\eta, \quad (6)$$
for some \( C_K, c_K > 0 \) only depending on \( K \) (e.g., \( C_K = s \) and \( c_K \leq n/s \) for \( K = \Sigma_{s,n,q}^* \)). Henceforth, we say that a set is \textit{structured} if (6) is respected.

Notice that all \( K \) that respect a ULS model are necessary \textit{homogeneous} and also \textit{symmetric}, i.e., \( K = -K \).

- **Other structured sets** (\( q = 2 \)): In the case \( q = 2 \), the analysis above extends to other structured sets \( K \), as explained, e.g., in [31]. For instance, for the set of rank-\( r \) matrices in \( \mathbb{R}^{\sqrt{n} \times \sqrt{n}} \simeq \mathbb{R}^n \) (see e.g., [10] Lemma 3.1) or even more advanced models using group-sparsity, we still have
  \[
  \mathcal{H}_2(K, \eta) \lesssim C_K \log c_K / \eta,
  \]
  for some \( C_K, c_K > 0 \) only depending on \( K \).

- **General bounded sets** (\( q = 2 \)): Finally, still in the case \( q = 2 \), for any bounded set \( K \subset \mathbb{R}^n \), the Sudakov minoration provides
  \[
  \mathcal{H}_2(K, \eta) \leq \eta^{-2} w^2(K),
  \]
  where
  \[
  w(K) := \mathbb{E}_g \sup\{|\langle g, u \rangle| : u \in K\}, \quad \text{with } g \sim \mathcal{N}^n(0, 1),
  \]
  is the Gaussian mean width of \( K \). This last quantity is known for many sets on which directly estimating \( \mathcal{H}_2 \) is not easy [12, 39, 35, 20]. This includes the convex set of compressible vectors \( C_s := \{ u \in \mathbb{R}^n : \|u\|_1 \leq s\} \) with \( w^2(C_s) \lesssim s \log(n/s) \) and the set of rank-\( r \) matrices \( R_r := \{ U \in \mathbb{R}^{\sqrt{n} \times \sqrt{n}} \simeq \mathbb{R}^n : \text{rank} \, U \leq r \} \) with \( w^2(R_r) \lesssim r \sqrt{n} \). Notice that for ULS and structured sets, the bound (7) is still valid (when \( w(K) \) can be bounded) but it provides a much looser bound for small values of \( \eta \).

Our first main result analyzes the following RIP inheritance.

**Proposition 1** (QRIP inherited from the \((\ell_1, \ell_q)\)-RIP). \textit{Assume that } \( \Phi \in \mathbb{R}^{m \times n} \) \textit{respects the } \((\ell_1, \ell_q)\)-\textit{RIP}(\( K - K, \epsilon \)) \textit{for some } \( 0 < \epsilon < 1 \) \textit{and } \( q \geq 1 \). \textit{If}

\[
  m \gtrsim \epsilon^{-2} \mathcal{H}_q(K, \delta \epsilon^2),
  \]

\textit{then, for some } \( C, c > 0 \), \textit{the quantized random embedding } \( A \) \textit{satisfies the } \((D_{\xi}, \ell_q)\)-\textit{QRIP}(\( K - K, \epsilon, \rho \)) \textit{with } \( \rho \lesssim \delta \epsilon \), \( D_{\xi} = D_{\ell_1} \) (i.e., \( p_q = 1 \)) \textit{and with probability exceeding } \( 1 - C \exp(-c m \epsilon^2) \). \textit{In other words, under these conditions and with the same probability,}

\[
  (1 - \epsilon) \| x - x' \|_q - c \delta \epsilon \leq \frac{1}{m} \| A(x) - A(x') \|_1 \leq (1 + \epsilon) \| x - x' \|_q + c \delta \epsilon,
  \]

\textit{for all } \( x, x' \in K \) \textit{and some } \( c > 0 \).

The full proof of this proposition is postponed to Sec. 4. However, its sketch is intuitively simple. Using measure concentration, the dithering enables us to show that, with probability exceeding \( 1 - C \exp(-c' m \epsilon^2) \), the (pseudo) distance \( D(a + \xi, a' + \xi) := D_{\ell_1}(Q(a + \xi), Q(a' + \xi)) \) concentrates around \( D_{\ell_1}(a, a') \) with only an additive distortion in \( \delta \epsilon \) for a fixed pair of vectors \( a, a' \in \mathbb{R}^m \). The rest of the proof consists in extending this result to vectors taken in the full set \( \Phi K \subset \mathbb{R}^m \) and then to use the \((\ell_1, \ell_q)\)-RIP(\( K - K, \epsilon \)) to connect \( D_{\ell_1}(a = \Phi x, a' = \Phi x') \) to \( \| x - x' \|_q \), up to a multiplicative distortion \( (1 \pm \epsilon) \).

If \( A \) was linear (i.e., if \( Q = \text{Id} \)), this would be done by covering the space \( K \), here according to the \( \ell_q \)-metric, with a sufficiently dense but finite set of vectors (i.e., a \( \eta \)-net with radius \( \eta > 0 \)). Knowing that the set cardinality is bounded by \( \exp(\mathcal{H}_q(K, \eta)) \), a union bound provides
that the concentration of the r.v. $\mathcal{D}_\varepsilon(A(x), A(x'))$ can hold simultaneously for all vectors of the $\eta$-net if $[3]$ is respected. This can be extended to $\mathcal{K}$ by continuity of the linear map (see e.g., [3] for an application of this method to proving the RIP of sub-Gaussian matrices). However, the argument breaks here as $Q$ and thus $\mathcal{D}_\varepsilon(A(\cdot), A(\cdot))$, is actually discontinuous.

From a method initially developed in [36] for 1-bit quantization (with a sign operator) and extended later in [20] for uniform quantization, this situation is fortunately overcome by introducing a soft (pseudo) distance $\mathcal{D}^t(a, a')$, whose softening degree is controlled by $t \in \mathbb{R}$ with $\mathcal{D}^0 = D$ (see Sec. 3 for the precise definition). It happens that the variations of $\mathcal{D}^t(a, a')$ can be better controlled in a small neighborhood of $a, a' \in \Phi \mathcal{K}$ provided that a little perturbation of $t$, adjusted to the neighborhood size, is allowed. In the case where $\Phi$ respects the $(\ell_1, \ell_2)$-RIP, the neighborhood size can be connected to the radius of an $\eta$-covering of $\mathcal{K}$, and a careful setting of $\eta$ in function of $\varepsilon$ finally enables the extension of the concentration result to the whole set $\mathcal{K}$, hence proving the QRIP above by eventually setting $t = 0$.

As a second result, we are able to show, up to a stronger distortion $\rho(\varepsilon, s) \lesssim \delta s + \delta^2 \varepsilon$ that only vanishes when both $\varepsilon$ and $s$ tend to zero, that the $(\ell_2, \ell_2)$-RIP involves a $(\mathcal{D}_\varepsilon, \ell_2)$-QRIP, with $\mathcal{D}_\varepsilon = \mathcal{D}_{t_2}$.

**Proposition 2** ($(\ell_2, \ell_2)$-QRIP inherited from the $(\ell_2, \ell_2)$-RIP). Assume that $\Phi \in \mathbb{R}^{m \times n}$ respects the $(\ell_2, \ell_2)$-RIP$(\mathcal{K} - \mathcal{K}, \varepsilon)$ for some $0 < \varepsilon < 1$. If

$$m \gtrsim e^{-2H_2(K, \delta \varepsilon^{3/2})},$$

then, for some $C, c > 0$ and with probability exceeding $1 - C \exp(-cm^2)$, the quantized random embedding $A$ satisfies the $(\mathcal{D}_\varepsilon, \ell_2)$-QRIP$(\mathcal{K}, \varepsilon, \rho)$, with $\mathcal{D}_\varepsilon = \mathcal{D}_{t_2}$ (i.e., $p_\varepsilon = 2$) and $\rho(\varepsilon, s) \lesssim \delta s + \delta^2 \varepsilon$. In other words, under these conditions and with the same probability,

$$\frac{1}{m} ||A(x) - A(x')||^2_2 - ||x - x'||^2_2 \lesssim \varepsilon ||x - x'||_2^2 + \delta ||x - x'||_2 + \delta^2 \varepsilon,$$

for all $x, x' \in \mathcal{K}$.

The proof architecture of Prop. 2 developed in Sec. 5 is similar to the one of Prop. 4 except that special care must be given to the quadratic nature of $\mathcal{D}_\varepsilon = \mathcal{D}_{t_2}$ and of the softened version of $\mathcal{D}_{t_2}(A(\cdot), A(\cdot))$.

Notice that the reason of this strongly distorted embedding can be traced back to the expectation of $\mathbb{E}_\varepsilon[Q(a + \varepsilon) - Q(b + \varepsilon)]^2$ for $a, b \in \mathbb{R}$, which determines the expectation of $\frac{1}{m} ||A(x) - A(x')||^2_2$ by separability of $\mathcal{D}_{t_2}$. Indeed, for sufficiently close $a$ and $b$, the random variable $X := \frac{1}{m} Q(a + \varepsilon) - Q(b + \varepsilon) \in \{0, 1\}$ is binary, i.e., $\mathbb{E}X^q = \mathbb{E}X$ for all $q > 0$. Thus $\mathbb{E}_\varepsilon[Q(a + \varepsilon) - Q(b + \varepsilon)]^2 = \delta \mathbb{E}_\varepsilon[Q(a + \varepsilon) - Q(b + \varepsilon)] = \delta |a - b|$, which clearly deviates from the quadratic dependence in $|a - b|$ reached by $\mathbb{E}_\varepsilon[Q(a + \varepsilon) - Q(b + \varepsilon)]^2 \geq |a - b|^2(1 - 2\delta |a - b|^2) \approx |a - b|^2$ when $|a - b| \gg \delta$. In this context, $\mathcal{D}_\varepsilon(A(\cdot), A(\cdot))$ cannot escape a strong deviation to $\mathcal{D}_{t_2}(\cdot, \cdot)$ in asymptotic regime (i.e., large value of $m$) when pairwise signal distances are small compared to $\delta$.

This change of regime between small- and large-distance embeddings also meets similar behaviors described in a version of the Johnson-Lindenstrauss lemma in the quantized embedding of finite sets by [39, Sec. 5] or in more complex non-linear random embeddings as observed in [9] (see Sec. 3.2).

**Bi-dithered quantized maps:** As a last result of this paper, we provide a novel and simple way to extend the dithering procedure of [36] in order to improve the distortion properties of the embedding of Prop. 2. This new scheme relies on a bi-dithered quantized map, which, up to
an implicit doubling of the number of measurements, is able to remove the strong distortion of the previous \((D_{l_2}, \ell_2)\)-quantized embedding, i.e., reducing it from \(\rho \lesssim \delta s + \delta^2 \epsilon\) to \(\rho \lesssim \delta^2 \epsilon\).

The best way to introduce it is to first generalize \(3\) to a matrix map over \(\mathbb{R}^{n \times 2}\), i.e., we now define \(A : X \in \mathbb{R}^{n \times 2} \rightarrow \mathbb{Z}_\delta^{m \times 2}\) such that

\[
A(X) = Q(\Phi X + \Xi),
\]

where \(Q\) is applied entrywise on its matrix argument, \(\Phi \in \mathbb{R}^{m \times n}\) is as before the linear part of the map \(A\) and the dither \(\Xi \sim U^{m \times 2}([0, \delta])\) is now composed of two independent columns.

Defining the operation \((\cdot)^\circ : B \in \mathbb{R}^{m \times d} \rightarrow B^\circ \in \mathbb{R}^m\) such that \((B^\circ)_i = \Pi_{j=1}^d B_{ij}\) together with the pre-metric \(\|B\|_{1,0} = \|B^\circ\|_1\), which is a norm only for \(d = 1\), it is easy to see from the independence of the columns of \(\Xi\) that

\[
\mathbb{E}_\Xi \|A(X) - A(X')\|_{1,0} = \|\Phi(X - X')\|_{1,0}.
\]

In particular, using the compact notation \(\tilde{u} := u 1_2^\circ \in \mathbb{R}^{n \times 2}\) for any vectors \(u \in \mathbb{R}^n\), and defining the quantized map

\[
\tilde{A} : x \in \mathbb{R}^n \mapsto \tilde{A}(x) := A(x) \in \mathbb{Z}_\delta^{m \times 2},
\]

we have

\[
\mathbb{E}_\Xi \|\tilde{A}(x) - \tilde{A}(x')\|_{1,0} = \|\Phi(x - x')\|^2.
\]

Therefore, up to the identification \(\mathbb{Z}_\delta^{m \times 2} \simeq \mathbb{Z}_2^{2m}\), the map \(\tilde{A}\) amounts to doubling the number of measurements \(m\), with the use of two random dithers per action of a single row of \(\Phi\) on \(x\), compared to the single dither of the initial map \(3\). As described below this is, however, highly beneficial for reducing the distortion of the quantized embedding of \((K, \ell_2)\) in \((\mathbb{Z}_\delta^{m \times 2}, D_0)\), with \(D_0(\cdot) := \frac{1}{m}\|\cdot\|_{1,0}\), which is reached with high probability by the map \(\tilde{A}\) once \(\Phi\) has the \((\ell_2, \ell_2)\)-RIP.

**Proposition 3** \((\ell_2, \ell_2)\)-RIP involves bi-dithered quantized embedding. There exists a \(0 < \epsilon_0 < 1\) such that if \(\Phi \in \mathbb{R}^{m \times n}\) respects the \((\ell_2, \ell_2)\)-RIP \((K, \ell_2, \epsilon)\) with \(0 < \epsilon < \epsilon_0\) and

\[
m \gtrsim \epsilon^{-2} H_2(K, \delta \epsilon^2),
\]

then, with probability exceeding \(1 - C \exp(-c m \epsilon^2)\) for some \(C, c > 0\), the quantized random embedding \(\tilde{A}\) satisfies the \((D_\varepsilon, \ell_2)\)-QRIPT \((K, \epsilon, \rho)\) with \(D_\varepsilon = D_0\) (i.e., \(p_\varepsilon = 2\)) and \(\rho(\epsilon, s) \lesssim \delta^2 \epsilon\). In other words, under the same conditions and with the same probability,

\[
(1 - \epsilon)\|x - x'\|^2 \leq \frac{1}{m}\|A(x) - A(x')\|_{1,0} \leq (1 + \epsilon)\|x - x'\|^2 + c \epsilon s^2,
\]

for all \(x, x' \in K\) and some \(c > 0\).

The proof of this proposition, postponed to Sec. 6 actually starts by showing that there is an additive distortion \(\rho \lesssim \delta^2 \epsilon + \delta s \epsilon\) for \(0 < \epsilon < 1\). However, this one is equivalent to \(15\), up to a rescaling of \(\epsilon\). Indeed, if \(s \leq \delta\), then \(\rho \lesssim \delta^2 \epsilon + \delta s \epsilon\) involves \(\rho \lesssim \delta^2 \epsilon\), while if \(s > \delta\), the second term \(\delta s \epsilon \lesssim \epsilon s^2\) of the bound on \(\rho\) can be integrated to the multiplicative distortion scaling like \(\epsilon s^2\) (with \(s = \|x - x'\|\)).

### 3 Portability, Discussion and Perspectives

The main interest of the three propositions introduced in Sec. 2 is to connect the existence of quantized embeddings of low-complexity vector sets to the one of linear embeddings of the same sets, as derived from specific instances of the RIP. We find useful to emphasize in this section the portability of these results, linking them to efficient (e.g., fast) RIP matrix constructions, as well as highlighting both their connection with existing works and a few related open problems.
Table 1: Non-exhaustive list of quantized embeddings inherited from RIP-matrix constructions. The name of the construction and its references, the linear embedding type, the spaces that are embeddable by this linear embedding, the linear encoding complexity, the type of RIP inferred by this linear embedding and the corresponding propositions are provided. In this table, SGMW means spaces with small Gaussian mean width (before $m$), $\Sigma_{s,n}$ stands for $s$-sparse in a basis with low mutual-coherence with respect to the basis used for sensing (see cited references). (*): Only if $K$ is homogeneous.

**3.1 Portability of the results**

We provide below a brief summary of interesting RIP constructions that reveal themselves useful for at least one of our three main propositions. The connection between these constructions, their characteristics and their links with our main results is also summarized in Table 1. Moreover, we stress below important aspects of quantized embeddings, such as the achievability of fast 1-bit quantized embeddings.

**Random sub-Gaussian ensembles (RSGE):** The first constructions of linear maps known to respect the RIP with high probability, which traces back to the randomized maps used to prove the JL Lemma [13][11], are the random sub-Gaussian ensembles. These are associated to a linear map $\Phi \in \mathbb{R}^{m \times n}$ with $\Phi_{ij} \sim \text{iid } X$, with a sub-Gaussian random variable $X$ with unit variance and zero expectation, such that $X \sim N(0, 1)$ or $X \in \{\pm 1\}$ a Bernoulli random variable with $\mathbb{P}(X = 1) = 1/2$.

It was rapidly established in the CS literature that, provided $m \gtrsim \epsilon^{-2} s \log n/s$, such a matrix $\Phi$ respects the $(\ell_2, \ell_2)$-RIP($\Psi_{\Sigma_{s,n}, \epsilon}$) with high probability and for any orthonormal basis (ONB) $\Psi \in \mathbb{R}^{n \times n}$, i.e., RSGE are thus universal in that sense [11]. Hence, RSGE is of course a first class of matrix construction that can be used for inducing dithered or bi-dithered quantized embeddings of $\Psi_{\Sigma_{s,n}}$, as stated by Prop. 2 and Prop. 3, respectively.

However, it is also observed in [28][26] that for any sets $K \subset S^{n-1}$, we have, with probability exceeding $1 - \exp(-c \epsilon^2 m)$,

$$\sup_{u \in K} \|\frac{1}{m} \Phi u\| - 1 \leq \epsilon,$$

provided $m \gtrsim \epsilon^{-2} w^2(K)$, with $w(K)$ the Gaussian mean width defined in Sec. 2.

In particular, this result establishes that $\Phi$ can respect w.h.p. an $(\ell_2, \ell_2)$-RIP($K - K, \epsilon$) if $K \subset \mathbb{R}^n$ is homogeneous, i.e., if $K \subset K$ for all $\lambda \geq 0$. Indeed, for such a set $K$, $u/\|u\| \in K \cap S^{n-1}$ for all $u \in K \setminus \{0\}$. Therefore, if $m \gtrsim \epsilon^{-2} w^2(K^*) \geq \epsilon^{-2} w^2(K \cap S^n)$ with $K^* := K \cap \mathbb{B}^n \supset K \cap S^{n-1}$,
then (15) holds on $u/\|u\| \in K \cap S^{n-1}$ for all $u \in K$, which shows that $\Phi$ respects, w.h.p., the $(\ell_2, \ell_2)$-RIP($K, \epsilon$). Applying the same argument on the homogeneous set $K - K \subset \mathbb{R}^n$, and observing that $(K - K) \cap B^n \subset K^* - K^*$ with $w(K^* - K^*) \leq 2w(K^*)$, this shows that $\Phi$ respects w.h.p. the $(\ell_2, \ell_2)$-RIP($K - K, \epsilon$) if $m \gtrsim \epsilon^{-2}w^2(K^*)$.

The portability of Prop. 2 and Prop. 3 is therefore guaranteed to those sets, which include for instance the ULS models described in Sec. 2 or other structured space as the set of low-rank matrices in $\mathbb{R}^{n \times \sqrt{n}}$ (when identified with $\mathbb{R}^n$).

In particular, w.h.p., the requirements of Prop. 2 and Prop. 3 will be satisfied on a homogeneous set $K$ if we have jointly

$$m \gtrsim \epsilon^{-2}w^2(K^*) \quad \text{and} \quad m \gtrsim (\epsilon \eta(\epsilon, \delta))^{-2}w^2(K^*),$$

while, from (14), for both homogeneous and structured sets, these requirements are more easily verified if

$$m \gtrsim \epsilon^{-2}w^2(K^*) \quad \text{and} \quad m \gtrsim C_K \epsilon^{-2} \log c_K / \eta(\epsilon, \delta),$$

with $\eta(\epsilon, \delta) = \delta \epsilon^{3/2}$ for Prop. 2 and to $\eta(\epsilon, \delta) = \delta^2$ for Prop. 3. As described in Sec. 2 the values $C_K, c_K > 0$ depend on the structure of the set, with e.g., $C_K = s$ and $c_K = n/s$ for $K = \Sigma_{s,n}$ or $C_K = r\sqrt{n}$ and $c_K = O(1)$ for the set of rank-$r\sqrt{n} \times \sqrt{n}$ matrices [10].

A similar result to Prop. 1 in the case of $q = 2$, i.e., for $(\mathcal{D}_{\ell_1}, \ell_2)$-quantized embeddings, was established before for dithered quantized maps specifically built on RSGE matrices [20]. Using the notations of the present work, it was shown that the resulting map $A$ satisfies, w.h.p., the $(\mathcal{D}_{\ell_1}, \ell_2)$-QRIP($K, \epsilon, \delta^2 + \rho_{sg}$) provided $m \gtrsim \delta^{-2}e^{-\delta^2 \epsilon^2}w^2(K)$ and with an additional distortion $\rho_{sg}$ that is non-zero for a sub-Gaussian but non-Gaussian random map $\Phi$ and whose value is small, but not zero, for “not too sparse” vectors in $K$, i.e., with small $\ell_2$ norm ratio.

Interestingly, this work improves this former result for homogeneous sets. Indeed, as an extension to (19), it has been proved in [33] (see also [36]) that for $K \subset S^{n-1}$, we have with probability exceeding $1 - \exp(-c\epsilon^2 m)$

$$\sup_{u \in K} \left| \frac{1}{m} \sqrt{n} \| \Phi u \|_1 - 1 \right| \leq \epsilon,$$

provided $m \gtrsim \epsilon^{-2}w^2(K)$, i.e., as for $(\ell_2, \ell_2)$-linear embeddings. Therefore, applying on (19) the same observations than the one made on (16), this shows that for homogeneous sets $K$, if $m \gtrsim \epsilon^{-2}w^2(K^*)$, $\Phi$ respects the $(\ell_1, \ell_2)$-RIP($K - K, \epsilon$) (with $\mu_\Phi = \sqrt{\epsilon/2}$).

Consequently, w.h.p., all the conditions of Prop. 1 can be satisfied with $\eta = \delta \epsilon^2$ if (17) holds when $K$ is homogeneous, of if (18) holds if $K$ is also structured. In these cases, the non-linear map $\Phi$ respects w.h.p. the $(\mathcal{D}_{\ell_1}, \ell_2)$-QRIP($K, \epsilon, \delta \epsilon$).

Therefore, compared to [20] and under almost similar requirements on $m$, we can construct here a quantized embedding of $(K - K, \ell_2)$ in $(\mathbb{Z}_2^n, \mathcal{D}_{\ell_1})$ based on RSGE with no additional distortion induced by the non-Gaussianity of the RSGE! The additive distortion is merely upper bounded by $c\delta \epsilon$ for some $c > 0$ and can be made arbitrarily close to 0 by adjusting $\epsilon$.

To conclude this paragraph dedicated to RSGE, we can highlight a potential multi-dithered generalization $A$ in Prop. 3. The non-linear map $A$ defined in (15) can indeed be extended over $\mathbb{R}^{n \times d}$ with $d \geq 2$ and $E \sim \mathcal{U}^{n \times d}([0, \delta])$, i.e., associating each row of $\Phi$ with $d$ different dithers. Following a similar proof to the one developed for Prop. 3, the independence of the $d$ columns of $E$ could then lead to a connection between matrices satisfying the $(\ell_d, \ell_q)$-RIP for $d > 2$ and quantized random maps respecting the $(\mathcal{D}_E, \ell_q)$-QRIP with $\mathcal{D}_E = \mathcal{D}_\phi$. This is the case of matrices drawn according to random Gaussian ensembles, which are known to respect $(\ell_d, \ell_q)$-RIP($\Sigma_{s,n}, \epsilon$) provided $m \geq m_0$ with $m_0 = O((s \log n/s)^{d/2})$ [22]. However, as there is no other known matrix construction satisfying this RIP, we prefer to not investigate here this potential generalization.
Structural random matrix constructions (SRMC): As explained in the Introduction, there is a growing literature interested in the development of fast quantized embeddings. Many works focus on the possibility to define fast 1-bit embeddings by replacing $Q$ with a sign operator without considering a pre-quantization dithering \cite{30,18,49}. However, to the best of our knowledge, such fast 1-bit embeddings are currently available for finite sets only, with sometimes strong restrictions between their cardinality, the ambient dimension $n$ and the embedding dimension $m$ (e.g., $m < n^{1/2}$ in \cite{30}).

This paper aims thus at showing that if we accept a different quantization process, i.e., a simple uniform quantization, a suitable pre-quantization dither allows us to leverage the now large market of random matrix constructions known to respect the RIP. This includes, for instance, structured random matrix constructions (SRMC) with fast vector encoding schemes such as random orthonormal basis ensembles \cite{11,16}, random convolutions \cite{39,40}, spread-spectrum \cite{37}, and scrambled block Hadamard ensembles \cite{17}. Moreover, some of those SRMC known to only respect the\footnote{Actually, a multiresolution version of the RIP satisfied by most of those SRMC \cite{32}.} $\ell_2$-RIP($\Sigma_{s,n}, \epsilon$), generally provided $m \gtrsim s \log(n)O(1) \log(s)O(1)$, can be extended to the embedding of general sets with small Gaussian mean width (SGMW) by a simple random sign flipping of the encoded vector \cite{32}, i.e., similarly to the effect of spread-spectrum \cite{37}.

Finally, let us mention that the adjacency matrices of high-quality unbalanced expander graphs have been shown to provide\footnote{1 Actually, a multiresolution version of the RIP satisfied by most of those SRMC \cite{32}.} $(\ell_2, \ell_2)$-RIP($\Sigma_{s,n}, \epsilon$), generally provided $m \gtrsim s \log(n)O(1) \log(s)O(1)$, can be extended to the embedding of general sets with small Gaussian mean width (SGMW) by a simple random sign flipping of the encoded vector \cite{32}, i.e., similarly to the effect of spread-spectrum \cite{37}.

Fast 1-bit quantized embeddings of low-complexity sets: Combined with Prop. 1 and Prop. 3, the SRMC above can thus provide fast quantized embeddings of low-complexity sets of $\mathbb{R}^n$, e.g., with log-linear vector encoding time, that hold w.h.p. provided $m$ respects the respective requirements of these propositions.

Removal of an invertible $E \in K \prime$ the adjacency matrices of high-quality unbalanced expander graphs have been shown to provide\footnote{Actually, a multiresolution version of the RIP satisfied by most of those SRMC \cite{32}.} $(\ell_2, \ell_2)$-RIP($\Sigma_{s,n}, \epsilon$) matrix constructions with $1 \leq p \leq 1 + O(1/\log n)$ and fast vector encoding schemes. Combined with Prop. 1 such matrices determine a fast quantized map satisfying, w.h.p., the $(D_1, 1)$-QRIP($\Sigma_{s,n}, \epsilon, c\delta\epsilon$) for some $c > 0$, provided $m \gtrsim \epsilon^{-2} \log 1/(\delta^2)$.

3.2 Connections with “representation coding signal geometry”

Removing the condition on $\rho$, the QRIP defined in \cite{9} can be seen as a particular instance of the (distorted) embedding concept introduced in \cite{9}. In this work, a non-linear map $F : K \rightarrow \mathcal{E}$ is a $(g, \epsilon, \rho)$-embedding of $(K, D_K)$ in $(\mathcal{E}, D_{\mathcal{E}})$ if for some invertible $g : \mathbb{R} \rightarrow \mathbb{R}$, we have

\begin{equation}
(1 - \epsilon)g(D_{K}(u, u')) - \rho \leq D_{\mathcal{E}}(F(u), F(u')) \leq (1 + \epsilon)g(D_{K}(u, u')) + \rho, \quad (20)
\end{equation}

for all $u, u' \in K$. From this definition, a $(D_{\mathcal{E}}, \ell_q)$-QRIP($K, \epsilon, \rho$) map $A$ is thus a $(\ell_q, \epsilon, \rho)$-embedding of $(K, D_K)$ in $(\mathcal{E}, D_{\mathcal{E}})$ with the additional requirement \footnote{Actually, a multiresolution version of the RIP satisfied by most of those SRMC \cite{32}.} \footnote{Actually, a multiresolution version of the RIP satisfied by most of those SRMC \cite{32}.} on $\rho$.

In \cite{9}, the authors show that, under certain conditions, if $F$ is w.h.p. a $(g, \epsilon, \rho)$-embedding of $(K, D_K)$ in $(\mathcal{E}, D_{\mathcal{E}})$ on an arbitrary pair of vectors $u, u' \in K$, then, for some $c > 1$ depending on $f$ and $g$, $F$ is also w.h.p. a $(g, \epsilon, c\rho)$-embedding of the whole set $K$. The conditions for this to happen are that we can control both the size of any covering of $K$ (i.e., with its Kolmogorov
entropy) and the probability $P_T$ to get $T$ discontinuity points of $F$ in any balls of radius $r$, i.e., in order to show that $P_T$ quickly decays if $r$ decreases and $T$ increases.

From this general result, the authors are then able to study the alteration of the random linear map $\Phi$ (with rows iid as some suitable vector distribution) defined through $x \mapsto A_f(x) = f(\Phi x + \xi)$ for some 1-periodic function $f$, possibly discontinuous, and with $\xi \sim \mathcal{U}^m([0,1])$. In particular, the function $f$, through its Fourier series, and the distribution of the rows of $\Phi$ determine together an explicit “kernelization” of the space $\mathcal{K}$, i.e., it enables a geometric encoding of $\mathcal{K}$ where small pairwise vector distances in $\mathcal{K}$ are better encoded than large ones.

By considering a uniform quantizer $Q$ as a discontinuous map, it could seem that the procedure described in [9], and in particular the Theorem 3.2 inside, could be used to prove our results on specific random matrix constructions (e.g., RSIE). However, even if this is potentially possible, we believe that such an adaptation would not be straightforward.

The standpoint of [9] is indeed fully probabilistic, i.e., in [20] the stochastic behavior of $F$ is characterized from the space $\mathcal{K}$ by the combined action of a random matrix $\Phi$ and of a random dithering $\xi$. In our work, we prefer to assume (an instance of) the RIP of the linear map $\Phi \in \mathbb{R}^{m \times n}$ to first embed $\mathcal{K}$ in $\mathbb{R}^m$, with an appropriate $\ell_p$-distance, and then characterize the embedding of $\mathbb{R}^m$ in the quantized domain $\mathbb{Z}_q^m$ thanks to the dithered-quantization. As explained above, this trick allows us to directly integrate, for our specific case, a much broader class of random matrix constructions, with possibly log-linear vector encoding time.

Another difference between [9] and our work comes from the treatment reserved to the discontinuities of the non-linear map. As explained above, in [9], the authors need to characterize the Lipschitz continuity of $f$ “by part”, i.e., between its discontinuity points. Then, the probability $P_T$ of having $T$ discontinuities in the neighborhood of radius $r$ of any $x \in \mathcal{K}$, i.e., that exactly $T$ discontinuity frontiers will pass through this neighborhood, must be bounded and proved to decay rapidly when $r$ decreases and $T$ increases. This is indeed important to get a very general framework valid for any non-linear map $F$. Therefore, considering a $r$-covering of $\mathcal{K}$ of cardinality $C_r$ with balls of radius $r$ (whose cardinality is controlled by the Kolmogorov entropy), the slicing of all those balls in these $T$ parts related to the discontinuities of $f$ is still a $r$-covering of $\mathcal{K}$. Taking arbitrarily one point per ball slice thus defines another $r$-net of $\mathcal{K}$ with cardinality at most $TC_r$, on which [20] can hold w.h.p. by union bound, which provides an easy extension to the whole set $\mathcal{K}$ by continuity. A side effect of this elegant and general analysis is, however, the necessity to control $P_T$.

Our approach benefits of the specialization $f = Q$. In our case, we can soften the distance used in the quantized embedding domain $\mathcal{E} = \mathbb{Z}_q^m$ in order to prevent any estimation of $P_T$. As described in the different proofs provided in Secs. 4.6, we can then directly analyze the continuity of this new distance within a small neighborhood of the image by $\Phi$ of an $\eta$-net of $\mathcal{K}$ (with $\eta$ depending on $\epsilon$ and $\delta$), whose radius can be estimated from the RIP of $\Phi$.

An interesting open problem would consist in the study of a general softening procedure of the discontinuities induced by any non-linear function $f$ in $\mathcal{D}_\mathcal{E}$ through a map $A_f$. We could then analyze if fast linear encoding schemes are compatible with more advanced non-linear maps, where $Q$ in [3] or in [15] is replaced by other quantization schemes, such as non-uniform quantization [18] [23], vector or binned quantization [29] [33], non-regular quantization [5] [9], or by more general periodic functions as in [9] [8].

4 Proof of Proposition 1

Prop. 1 is formally demonstrated by showing first that we can embed, with high probability, $(J := \Phi(\mathcal{K} - \mathcal{K}), \ell_1)$ in $(Q(J + \xi), \ell_1)$ for $\xi \sim \mathcal{U}^{m}([0,\delta])$, and then using the linear $(\ell_1, \ell_2)$-embedding allowed by $\Phi$ (through the assumed RIP) to relate $(J := \Phi(\mathcal{K} - \mathcal{K}), \ell_1)$ to $(\mathcal{K} - \mathcal{K}, \ell_2)$.
As explained in the proof sketch provided after Prop. [11] the first step of the proof is obtained by softening the (pseudo) distance $D_{\ell_1}(Q(a), Q(a')) = \frac{1}{m} \| Q(a) - Q(a') \|_1$, i.e., the distance from which $D_{\ell_1}(A(x), A(x'))$ is derived by taking $a = \Phi x + \xi$ and $a' = \Phi x' + \xi$ and $A$ defined in (3).

According to a procedure defined in [20] that we rewrite here for completeness, this is done by softening each of the $m$ elements composing the sum $mD_{\ell_1}(Q(a), Q(a')) = \sum_{i=1}^m |Q(a_i) - Q(b_i)|$.

Let us first observe that, for $a, a', \in \mathbb{R}$,

$$d(a, a') := |Q(a) - Q(a')| := \delta \sum_{k \in \mathbb{Z}} \mathbb{I}(a - k\delta, a' - k\delta),$$

with $S = \{(a, a') \in \mathbb{R}^2 : aa' < 0\}$ and $\mathbb{I}(a, a')$ is equal to 1 if $(a, a') \in \mathbb{C}$ and 0 otherwise. In words, $d$ counts the number of thresholds in $\delta \mathbb{Z}$ that can be inserted between $a$ and $a'$, i.e., we can alternatively write $d(a, a') = \delta (|\delta \mathbb{Z}| \cap |a, a'|)$.

Introducing the set $S^t = \{(a, a') \in \mathbb{R}^2 : a < -t, a' > t\} \cup \{(a, a') \in \mathbb{R}^2 : a > t, a' < -t\}$ for $t \in \mathbb{R}$, we can define a soft version of $d$ by

$$d^t(a, a') := \delta \sum_{k \in \mathbb{Z}} \mathbb{I}^t(a - k\delta, a' - k\delta),$$

which is clearly a decreasing function of $t$. In fact,

$$d^{[t]}(a, b) \leq d(a, b) \leq d^{-[t]}(a, b)$$

since $S^{[t]} \subset S \subset S^{-[t]}$. Moreover, we can show that [20],

$$|d^t(a, b) - d^s(a, b)| \leq 4(\delta + |t - s|),$$

$$|d^t(a, b) - |a - b|| \leq 4(\delta + |t|),$$

$$|Ed^t(a + \xi, b + \xi) - |a - b|| \leq 4|t|,$$

where $\xi \sim \mathcal{U}([0, \delta])$ and the last inequality is proved in [20] Appendix C].

As expressed in the next lemma, the distance $d^t$ already displays a certain form of continuity that $d$ has not.

**Lemma 1.** For all $a, a' \in \mathbb{R}$, $t \in \mathbb{R}$ and $|r|, |r'| \leq \epsilon$, we have

$$d^{+\epsilon}(a, a') \leq d^t(a + r, a' + r') \leq d^{-\epsilon}(a, a').$$

**Proof.** Considering the definition of the set $S^t$ in (24), we have

$$S^{+\epsilon}(a, a') \subset S^t(a + r, a' + r') \subset S^{-\epsilon}(a, a').$$

Indeed, for instance $a + r < -t$ involves $a \leq -t + \epsilon$, while $a + r > t$ involves $a \geq t - r$, and similarly for all other conditions involved in the definition of $S^{+\epsilon}$, $S^t$ and $S^{-\epsilon}$. Considering these inclusions the result follows since $l_\mathcal{C} \leq l_{\mathcal{C'}}$ if $\mathcal{C} \subset \mathcal{C'}$. 

From the scalar (pseudo) distance $d^t$, we can thus define for $a, a' \in \mathbb{R}^m$,

$$D^t(a, a') = \frac{1}{m} \sum_i d^t(a_i, a'_i),$$

with $D^0(a, a') = D_{\ell_1}(Q(a), Q(a'))$. This distance inherits from $d^t$ the following continuity property.

**Lemma 2** (Continuity of $D^t$ with respect to $\ell_p$-perturbations). Let $p \geq 1$ and $a, a', r, r' \in \mathbb{R}^m$. We assume that $\max(||r||_p, ||r'||_p) \leq \eta \sqrt{m}$ for some $\eta > 0$. Then for every $t \in \mathbb{R}$ and $P \geq 1$ one has

$$D^{t + \eta \sqrt{P}}(a, a') - 8(P^{-1} \delta + \eta) \leq D^t(a + r, a' + r') \leq D^{t - \eta \sqrt{P}}(a, a') + 8(P^{-1} \delta + \eta).$$

(26)
Proof. The proof is inspired by the proof of Lemma 5.5 in [36] valid for 1-bit (sign) quantization, and an adaptation of [20] Lemma 3 to $\ell_p$, perturbations. By assumption, the set
\[ T := \{ i \in [m] : |r_i| \leq \eta \sqrt{P}, |r'_i| \leq \eta \sqrt{P} \} \]
is such that $|T^c| \leq 2m/P = 2\eta^p m \geq \|r\|_{\ell_p}^p + \|r'\|_{\ell_p}^p \geq (\|r\|_{\ell_p} + \|r'\|_{\ell_p})^p + |T^c|\eta^p P \geq |T^c|\eta^p P$. Therefore, considering Lemma 1, we find with $D := \max(|r_i|, |r'_i|)$,
\[ \mathcal{D}^{t+\eta\sqrt{P}}(a, a') = \frac{1}{m} \sum_{i=1}^m d^{i+\eta\sqrt{P}}(a_i, a'_i) \leq \frac{1}{m} \sum_{i \in T} d^{i+\eta\sqrt{P}}(a_i, a'_i) + \frac{1}{m} \sum_{i \in T^c} d^{i+\eta\sqrt{P}}(a_i, a'_i) \leq \frac{1}{m} \sum_{i \in T} d^i(a_i + r_i, a'_i + r'_i) + \frac{1}{m} \sum_{i \in T^c} d^i(a_i + r_i, a'_i + r'_i) \]
leq \[ \frac{1}{m} \sum_{i \in T^c} \left[ d^{i+\eta\sqrt{P}-\rho_i}(a_i + r_i, a'_i + r'_i) - d^i(a_i + r_i, a'_i + r'_i) \right]. \]
Using (22) to bound the last sum of the last expression and since, by definition of $T$, $\rho_i \geq \eta \sqrt{P}$ for $i \in T^c$, we find
\[ \mathcal{D}^{t+\eta\sqrt{P}}(a, a') \leq \mathcal{D}^t(a + r, a' + r') + \frac{1}{m} \sum_{i \in T^c}(\delta + \rho_i - \eta \sqrt{P}) \]
\[ \leq \mathcal{D}^t(a + r, a' + r') + \frac{8\delta}{P} + \frac{1}{m} \sum_{i \in T^c} \rho_i. \]
However,
\[ \sum_{i \in T^c} \rho_i \leq \|r_{T^c}\|_1 + \|r'_{T^c}\|_1 \leq m^{1-1/p}\|r\|_p + m^{1-1/p}\|r'\|_p \leq 2\eta m, \]
so that
\[ \mathcal{D}^{t+\eta\sqrt{P}}(a, a') \leq \mathcal{D}^t(a + r, a' + r') + \frac{8\delta}{P} + 8\eta, \]
which provides the lower bound of (26). For the upper bound,
\[ \mathcal{D}^{t-\eta\sqrt{P}}(a, a') = \frac{1}{m} \sum_{i=1}^m d^{i-\eta\sqrt{P}}(a_i, a'_i) \]
\[ \geq \frac{1}{m} \sum_{i \in T} d^i(a_i + r_i, a'_i + r'_i) + \frac{1}{m} \sum_{i \in T^c} d^{i-\eta\sqrt{P}+\rho_i}(a_i + r_i, a'_i + r'_i) \]
\[ \geq \mathcal{D}^t(a + r, a' + r') - \frac{1}{m} \sum_{i \in T^c} \left[ d^{i-\eta\sqrt{P}+\rho_i}(a_i + r_i, a'_i + r'_i) - d^i(a_i + r_i, a'_i + r'_i) \right], \]
and the last sum is bounded as above. \hfill \square

Before delving into the proof of Prop. 1 and following its aforementioned proof sketch, we now need to study how the random variable $\mathcal{D}^t(a + \xi, a' + \xi)$, for $\xi \sim U^m([0, \delta])$, concentrates around its expected value, and how this one deviates from $D_{\ell_1}(a, a')$.

**Lemma 3.** Let $\xi \sim U^m([0, \delta])$ and $a, a' \in \mathbb{R}^m$. We have
\[ \mathbb{P}_\xi[|\mathcal{D}^t(a + \xi, a' + \xi) - D_{\ell_1}(a, a')| > 4|t| + \epsilon(\delta + |t|)] \lesssim e^{-\alpha^2|t|^2}. \quad (27) \]
\[ |\mathbb{E}\mathcal{D}^t(a + \xi, a' + \xi) - D_{\ell_1}(a, a')| \lesssim |t|. \quad (28) \]

**Proof.** Most of the elements of this proof are already detailed in [20] but we prefer to rewrite them here briefly under our specific notations. Denoting $Z_i^t := d^i(a_i + \xi_i, b_i + \xi_i) - |a_i - b_i|$, (23) provides $\|Z_i^t\|_{\psi_2} \lesssim \|Z_i^t\|_\infty \lesssim \delta + |t|$, which shows that each r.v. $Z_i^t$ is sub-Gaussian. Therefore, from the approximate rotation invariance of sub-Gaussian variables [40], $Z_i^t := \sum_{i} Z_i^t = m\mathcal{D}^t(a + \xi, b + \xi) - \|a - b\|_1$ is sub-Gaussian with norm
\[ \|Z_i^t\|_1^2 \lesssim m(\delta + |t|)^2. \]
Consequently, there is a $c > 0$ such that $\mathbb{P}(|Z^t - \mathbb{E}Z^t| > \varepsilon) \leq e \exp(-c\varepsilon^2/\|Z^t\|_2^2)$, so that

$$
\mathbb{P}(\|\frac{1}{m}Z^t\| > \frac{1}{m}\mathbb{E}Z^t| + \varepsilon(\delta + |t|)) \leq \mathbb{P}(\|\frac{1}{m}(Z^t - \mathbb{E}Z^t)\| > \varepsilon(\delta + |t|)) \leq \exp(-c\varepsilon^2m),
$$

which provides (27) since from (24)

$$
\|\frac{1}{m}\mathbb{E}Z^t| = |\mathbb{E}D^t(a + \xi, b + \xi) - \frac{1}{m}\|a - b\|_1| \leq 4|t|,
$$
as written in (28). \hfill \square

**Proof of Prop. 1.** Let us define an $(\ell_q, \eta)$-covering $\mathcal{K}_\eta$ of the set $\mathcal{K}$ with log $|\mathcal{K}_\eta| \leq \mathcal{H}_q(\mathcal{K}, \eta)$. We fix $0 < \varepsilon < 1$ and $t \in \mathbb{R}$ and assume that $\Phi$ satisfies the $(\ell_1, \ell_q)$-RIP($\mathcal{K} - \mathcal{K}, \varepsilon$). Therefore, for all $x, x' \in \mathcal{K}$ with their respective closest points in $\mathcal{K}_\eta$ being $x_0$ and $x'_0$, we have

$$
D_{\ell_1}(\Phi x, \Phi x_0) \leq m(1 + \varepsilon)\|x - x_0\|_q \leq 2m\eta \quad \text{and} \quad D_{\ell_1}(\Phi x', \Phi x'_0) \leq 2m\eta. \quad (29)
$$

Moreover, since $\Phi$ is fixed, we observe from a union bound applied on (27) that if $m \geq \varepsilon^{-2}\mathcal{H}_q(\mathcal{K}, \eta)$, then, for $P \geq 1$ to be fixed later, both relations

$$
|D_{-n}^t(\Phi x_0 + \xi, \Phi x'_0 + \xi) - D_{\ell_1}(\Phi x_0, \Phi x'_0)| \leq 4|t| + 4\eta P + \varepsilon(\delta + |t| + \eta P), \quad (30)
$$

$$
|D_{+n}^t(\Phi x_0 + \xi, \Phi x'_0 + \xi) - D_{\ell_1}(\Phi x_0, \Phi x'_0)| \leq 4|t| + 4\eta P + \varepsilon(\delta + |t| + \eta P), \quad (31)
$$

hold jointly for all $x_0, x'_0 \in \mathcal{K}_\eta$ (i.e., on no more than $|\mathcal{K}_\eta|^2 \leq \exp(2\mathcal{H}_q(\mathcal{K}, \eta))$ vectors) with probability exceeding $1 - Ce^{-c\varepsilon^2m}$ for some $C, c > 0$.

Consequently, since for any points $x, x' \in \mathcal{K}$, we can always write $x = x_0 + r$ and $x' = x'_0 + r'$ with $x_0, x'_0 \in \mathcal{K}_\eta$ and $\max(\|r\|_q, \|r'\|_q) \leq \eta$, using Lemma 2 with $p = 1$, (30) and again (31), we have with the same probability and for some $c > 0$,

$$
D^t(\Phi x + \xi, \Phi x' + \xi) \leq D_{-n}^t(\Phi x_0 + \xi, \Phi x'_0 + \xi) + 8(P^{-1}\delta + \eta)
$$

$$
\leq D_{\ell_1}(\Phi x_0, \Phi x'_0) + 4|t| + 4\eta P + \varepsilon(\delta + |t| + \eta P) + 8P^{-1}\delta + \eta
$$

$$
\leq D_{\ell_1}(\Phi x, \Phi x') + 4|t| + 2\eta(2P + 1) + \varepsilon(\delta + |t| + \eta P) + 8P^{-1}\delta + \eta
$$

$$
\leq D_{\ell_1}(\Phi x, \Phi x') + c\left(|t| + \delta\varepsilon\right)
$$

$$
\leq (1 + \varepsilon)\|x - x'\|_q + c(|t| + \delta\varepsilon),
$$

where, for some $c > 0$, we set the free parameters to $P^{-1} = \varepsilon$ and $\eta = \delta^2 < \delta\varepsilon$ giving $\eta P = \delta\varepsilon$.

The lower bound is obtained similarly using (41), and Prop. 1 is finally obtained with $t = 0$. \hfill \square

5 \hspace{1em} Proof of Proposition 2

The proof of this proposition is essentially similar to the one of Prop. 1. We formally show first that we can embed, with high probability, $(\mathcal{J} := \Phi(\mathcal{K} - \mathcal{K}), \ell_2)$ in $(\mathcal{Q}(\mathcal{J} + \xi), \ell_2)$ for $\xi \sim \mathcal{U}^m([0, \delta])$, and then, using the linear $(\ell_2, \ell_2)$-embedding allowed by $\Phi$ (through the RIP), we relate $(\mathcal{J} := \Phi(\mathcal{K} - \mathcal{K}), \ell_2)$ to $(\mathcal{K} - \mathcal{K}, \ell_2)$. However, some subtleties come from the quadratic nature of the $D_2 = D_{\ell_2}$. First, conversely to $D_2 = D_{\ell_2}$, where $\mathbb{E}D_{\ell_1}(A(x), A(x)) = \|\Phi(x - x')\|_1$, in the quadratic case $\mathbb{E}D_{\ell_2}(A(x), A(x)) \neq \|\Phi(x - x')\|^2$, i.e., a distortion is already present between $\mathbb{E}D_{\ell_2}(A(x), A(x))$ and $\|\Phi(x - x')\|^2$. Second, the concentration $D_{\ell_2}(A(x), A(x))$ around its mean is not anymore independent of $\|\Phi(x - x')\|$. And finally, the mandatory softening of the pseudo-distance $D_{\ell_2}(A(x), A(x))$, as imposed to reobtain a certain form of continuity in the neighborhoods of $x$ and $x'$, also depends on $\|\Phi(x - x')\|$. We will see below that these three effects worsen the additive distortion of the $(\ell_2, \ell_2)$-quantized embedding.
We start by considering the general soft (pseudo) distance
\[
D_2^\ell(a, a') := \frac{1}{m} \sum_i (d^i(a_i, a'_i))^2,
\]
for \(a, a' \in \mathbb{R}^m\) and \(d^i\) defined as before. We thus have \(D_2^\ell(a, a') = D_{\ell_2}(Q(a), Q(a'))\).

As for \(D^\ell\) in the previous section, \(D_2^\ell\) can be shown to respect a certain form of continuity under \(\ell_2\)-perturbations, i.e., as those induced by the image by \(\Phi\) of any \(\ell_2\)-perturbation of a point in \(K\) around its image, thanks to the \((\ell_2, \ell_2)\)-RIP.

**Lemma 4** (Continuity of \(D_2^\ell\) with respect to \(\ell_2\)-perturbations). Let \(a, a', r, r' \in \mathbb{R}^m\). We assume that \(\max(||r||, ||r'||) \leq \eta \sqrt{m}\) for some \(\eta > 0\). Then for every \(t \in \mathbb{R}\) and \(P \geq 1\) one has
\[
D_2^\ell(a + r, a' + r') - D_2^{\ell - \eta \sqrt{P}}(a, a') \lesssim \frac{\sqrt{P} \delta}{\sqrt{m}} ||a - a'|| + (\delta + |t|) \eta + \delta (\delta + |t|) \frac{2}{\sqrt{P}},
\]

\[
D_2^{\ell + \eta \sqrt{P}}(a, a') - D_2^\ell(a + r, a' + r') \lesssim \frac{\sqrt{P} \delta}{\sqrt{m}} ||a - a'|| + (\delta + |t|) \eta + \delta (\delta + |t|) \frac{2}{\sqrt{P}}.
\]

**Proof.** The result is achieved by simply taking \(A = a 1^T_2, A' = a' 1^T_2, R = r 1^T_2\) and \(R' = r' 1^T_2\) in the more general Lemma 7 given in Sec. 6.

The question is now to study the concentration properties \(D_2^\ell(a + \xi, a' + \xi)\) for \(\xi \sim \mathcal{U}^m([0, \delta])\). We first observe how \((d^i)^2\) behaves under a scalar dithering of its inputs.

**Lemma 5.** Let \(\xi \sim \mathcal{U}([0, \delta])\) and \(a, a' \in \mathbb{R}\). For \(|a - b| \leq \delta - 2|t|\), we have
\[
\delta(|a - a'| - 4|t|)_+ \leq \mathbb{E} (d^i(a + \xi, a' + \xi))^2 \leq \delta(|a - a'| + 4|t|),
\]
while, whatever the value of \(|a - a'|\),
\[
\mathbb{E} (d^i(a + \xi, a' + \xi))^2 \geq (|a - a'| - 4|t|)_+^2, \quad (35)
\]
\[
\mathbb{E} (d^i(a + \xi, a' + \xi))^2 \leq (|a - a'| + 4(\delta + |t|))^2 \delta(|a - a'| + 4|t|)). \quad (36)
\]

**Proof.** Notice that if \(|a - a'| \leq \delta - 2|t|\) for \(a, a' \in \mathbb{R}\), which implicitly imposed \(|t| \leq \delta/2\), \(X := \delta^{-1} d^i(a + \xi, a' + \xi) \in \{0, 1\}\), so that
\[
\mathbb{E}(d^i(a + \xi, a' + \xi))^2 = \delta \mathbb{E}d^i(a + \xi, a' + \xi) \in [\delta(|a - a'| - 4|t|)_+, \delta(|a - a'| + 4|t|)].
\]

By Jensen inequality and using \((24)\) we find also
\[
\mathbb{E}(d^i(a + \xi, a' + \xi))^2 \geq (\mathbb{E}d^i(a + \xi, a' + \xi))^2 \geq (|a - a'| - 4|t|)_+^2.
\]

For the last relation, we notice from \((23)\) that \(0 \leq X \leq \delta^{-1}(|a - a'| + 4(\delta + |t|))\), so that using again \((24)\) we get
\[
\mathbb{E}(d^i(a + \xi, a' + \xi))^2 \leq (|a - a'| + 4(\delta + |t|)) (|a - a'| + 4|t|).
\]

Denoting \(\overline{D}_2^\ell(a, a') := \mathbb{E} D_2^\ell(a + \xi, a' + \xi)\) with \(\xi \sim \mathcal{U}([0, \delta])\), we can now determine how \(\overline{D}_2^\ell(a, a')\) deviates from \(||a - a'||^2\).

**Corollary 1.** Let \(\xi \sim \mathcal{U}^m([0, \delta])\). Then, for \(a, a' \in \mathbb{R}^m\),
\[
\overline{D}_2^\ell(a, a') - D_{\ell_2}(a, a') \geq -8|t| D_{\ell_2}^{1/2}(a, a') + 16|t|^2, \quad (37)
\]
\[
\overline{D}_2^\ell(a, a') - D_{\ell_2}(a, a') \leq (8|t| + 4\delta) D_{\ell_2}^{1/2}(a, a') + 4|t| (4|t| + \delta), \quad (38)
\]
which provides the loose bound
\[
||\overline{D}_2^\ell(a, a') - D_{\ell_2}(a, a')|| \leq (8|t| + 4\delta) D_{\ell_2}^{1/2}(a, a') + 4|t| (4|t| + \delta). \quad (39)
\]
Proof. For any $a, a' \in \mathbb{R}$, Lemma 5 provides
\[
(a - a'| - 4|t|)^2 \lesssim \mathbb{E}(d'(a + \xi, a' + \xi))^2 \lesssim (a - a'| + 4|t|)^2 + 4\delta(a - a'| + 4|t|).
\]
Therefore,
\[
\sum_i \mathbb{E}(d'(a_i + \xi, a' + \xi))^2 \geq \sum_i (a - a'| - 4|t|)^2 \\
\geq \|a - a'|^2 + 16m|t|^2 - 4|t|\|a - a'|_1 \\
\geq \|a - a'|^2 + 16m|t|^2 - 4|t|\sqrt{m}\|a - a'|_2 \\
= m\left(\frac{1}{\sqrt{m}}\|a - a'|_2 - 4|t|\right)^2,
\]
and
\[
\sum_i \mathbb{E}(d'(a_i + \xi, a' + \xi))^2 \leq m\left(\frac{1}{\sqrt{m}}\|a - a'|_2 + 4|t|\right)^2 + 4\delta\|a - a'|_1 + 4\delta|t|m \\
\leq m\left(\frac{1}{\sqrt{m}}\|a - a'|_2 + 4|t|\right)^2 + 4\delta\sqrt{m}\|a - a'|_2 + 4\delta|t|m.
\]
\qed

The last step before proving Prop. 2 is to study the concentration properties of the r.v. $\mathcal{D}_t(a + \xi, a' + \xi)$.

Lemma 6. Given $\xi \sim \mathcal{U}^m([0, \delta])$ and $a, a' \in \mathbb{R}^m$, there exist two constants $c, c' > 0$ such that
\[
P\left[|\mathcal{D}_t(a + \xi, a' + \xi) - \mathcal{D}_t(a, a')| > c'\delta + |t|\mathcal{D}_{\ell_2}(a, a') + c'\delta^2 \right] \lesssim e^{-c\varepsilon^2 m}. \tag{40}
\]

Proof. Denoting $Z_i := (d'(a_i + \xi, a_i' + \xi) - |a_i - a_i'|), we find from (23)
\[
|Z_i| = (d'(a_i + \xi, a_i' + \xi) + |a_i - a_i'|) |d'(a_i + \xi, a_i' + \xi) - |a_i - a_i'|| \\
\lesssim (\delta + |t|)(|a_i - a_i'| + \delta + |t|),
\]
which provides $\|Z_i\|_{\ell_2} \leq \|Z_i\|_\infty \lesssim (\delta + |t|)(|a_i - a_i'| + \delta + |t|)$ and proves the sub-Gaussianity of $Z_i$. Therefore, $Z_i := \sum_i |Z_i| = m(\mathcal{D}_t(a + \xi, a' + \xi))^2 - \|a - a'|^2_{\ell_2} is sub-Gaussian with norm
\[
\|Z_i\|_{\ell_2} \lesssim (\delta + |t|)^2 \sum_i (|a_i - a_i'| + \delta + |t|)^2 \\
= (\delta + |t|)^2 (\|a - a'|^2 + 2(\delta + |t|)\|a - a'|_1 + (\delta + |t|)^2 m) \\
\lesssim m(\delta + |t|)^2(\frac{1}{\sqrt{m}}\|a - a'|_2 + (\delta + |t|))^2, \\
\lesssim m(\delta + |t|)^2(\mathcal{D}_{\ell_2}(a, a') + (\delta + |t|))^2,
\]
from the approximate rotation invariance of sub-Gaussian variable [45]. Consequently, there is a $c > 0$ such that $P(|Z^t - \mathbb{E}Z^t| > \varepsilon) \lesssim \exp(-c\varepsilon^2/\|Z^t\|_{\ell_2}^2$), which provides
\[
P\left[\frac{1}{m}Z^t \left| \left| \frac{1}{m}\mathbb{E}Z^t \right| + \epsilon(\delta + |t|)(\mathcal{D}_{\ell_2}(a, a') + \delta + |t|)\right\right] \\
\leq P\left[\frac{1}{m}(Z^t - \mathbb{E}Z^t)| > \epsilon(\delta + |t|)(\mathcal{D}_{\ell_2}(a, a') + (\delta + |t|))\right] \lesssim \exp(-c\varepsilon^2 m).
\]

However, from (37) and (38),
\[
\frac{1}{m}\mathbb{E}Z^t = \mathcal{D}_t(a, a') - \mathcal{D}_t(a, a') \lesssim (|t| + \delta) (\mathcal{D}_{\ell_2}(a, a') + |t|).
\]

Therefore, up to a rescaling of $c > 0$ and considering that $\varepsilon < 1$, there is a $c' > 0$ such that
\[
P\left[|\mathcal{D}_t(a + \xi, a' + \xi) - \mathcal{D}_t(a, a')| > c'\delta + |t|\mathcal{D}_{\ell_2}(a, a') + c'\delta^2 \right] \lesssim e^{-c\varepsilon^2 m}. \tag{40}
\]
\qed
Proof of Proposition 2. As in the proof of Prop. 1, let us define an \((\ell_2, \eta)\)-covering \(K_\eta \subset K\) of the set \(K\) with \(\log |K_\eta| \leq \mathcal{H}_2(K, \eta)\) and \(\eta\) fixed later. We fix \(0 < \epsilon < 1\) and \(t \in \mathbb{R}\) and assume that \(\Phi\) satisfies the \((\ell_2, \ell_2)\)-RIP \((K - K, \epsilon)\), i.e., for all \(u, u' \in K\),

\[
D_{\ell_2}^2(\Phi u, \Phi u') \leq (1 + \epsilon)\|u - u'\|^2, \quad D_{\ell_2}^{1/2}(\Phi u, \Phi u') \leq (1 + \epsilon)\|u - u'\|.
\]

Therefore, for all \(x, x' \in K\) that \(\Phi\) satisfies the \((\ell_2, \ell_2\rangle\)-RIP \((K - K, \epsilon)\) with \(\Phi\) fixed, we observe from a union bound applied on \(1\) that if \(m \geq \epsilon^{-2}\mathcal{H}_2(K, \eta)\), then, for \(P \geq 1\) to be fixed later, both relations

\[
|D_{\ell_2}^{2-\eta\sqrt{\gamma'}}(\Phi x_0 + \xi, \Phi x_0' + \xi) - D_{\ell_2}(\Phi x_0, \Phi x_0')| \lesssim (\delta + |t - \eta\sqrt{\gamma'}|)D_{\ell_2}^{1/2}(\Phi x_0, \Phi x_0') + \delta^2\epsilon + |t - \eta\sqrt{\gamma'}|)
\]

\[
\lesssim (\delta + |t| + \eta\sqrt{\gamma'})\|x_0 - x_0'\| + \delta^2\epsilon + (|t| + \eta\sqrt{\gamma'})(\delta + |t|)
\]

and

\[
|D_{\ell_2}^{2+\eta\sqrt{\gamma'}}(\Phi x_0 + \xi, \Phi x_0' + \xi) - D_{\ell_2}(\Phi x_0, \Phi x_0')| \lesssim (\delta + |t| + \eta\sqrt{\gamma'})\|x_0 - x_0'\| + \delta^2\epsilon + (|t| + \eta\sqrt{\gamma'}) (\delta + |t|)
\]

hold jointly for all \(x_0, x_0' \in K_\eta\) with probability exceeding \(1 - c\epsilon^c m\) for some \(c, c' > 0\).

Consequently, for any points \(x = x_0 + r, x' = x_0' + r'\) in \(K\) with \(x_0, x_0' \in K_\eta\) and \(\max(\|r\|, \|r'\|) \leq \eta\), we have \(max(\|\Phi r\|, \|\Phi r'\|) \leq 2\sqrt{m}\eta\) from \(\mathcal{H}_2(K, \eta)\). Lemma 4 and \(\mathcal{H}_2(K, \eta)\) give then

\[
D_{\ell_2}^2(\Phi x + \xi, \Phi x' + \xi) - D_{\ell_2}^{2-\eta\sqrt{\gamma'}}(\Phi x_0 + \xi, \Phi x_0' + \xi)
\]

\[
\lesssim 2^4/\sqrt{m}\|\Phi x_0 - \Phi x_0'\| + (\delta + |t|)\eta + \delta(\delta + |t|)\sqrt{\gamma'}/2,
\]

\[
\lesssim (\eta + \delta)\|x_0 - x_0'\| + (\delta + |t|)\eta + \delta(\delta + |t|)\sqrt{\gamma'}/2,
\]

so that, using \(\mathcal{H}_2(K, \eta)\), the \((\ell_2, \ell_2\rangle\)-RIP \((K - K, \epsilon)\) of \(\Phi\), we find

\[
D_{\ell_2}^2(\Phi x + \xi, \Phi x' + \xi) - D_{\ell_2}(\Phi x_0, \Phi x_0')
\]

\[
\lesssim (\delta + |t| + \eta\sqrt{\gamma'})\|x_0 - x_0'\| + \delta^2\epsilon + (|t| + \eta\sqrt{\gamma'}) (\delta + |t| + \eta\sqrt{\gamma'})
\]

\[
\lesssim (\delta + |t| + \eta\sqrt{\gamma'})\|x_0 - x_0'\| + \delta^2(\epsilon + \sqrt{\gamma'}) + (|t| + \eta\sqrt{\gamma'}) (\delta + |t| + \eta\sqrt{\gamma'}) + \delta|t|^2
\]

\[
\lesssim (\delta + |t|)|x_0 - x_0'| + (\delta + |t|)(\delta + |t|)
\]

where we finally set the free parameters to \(P^{-1} = \epsilon\) and \(\eta = \delta \epsilon^{3/2} < \delta\epsilon\), giving \(\eta\sqrt{\gamma'} = \delta\epsilon\). Finally, from \(1\), \(D_{\ell_2}(\Phi x_0, \Phi x_0') \lesssim (1 + \epsilon)|x_0 - x_0'|^2\) while the covering provides \(\|x_0 - x_0'|^2 \geq \|x - x'|^2 - 2\eta\|x - x'|^2 + 4\eta^2\). Therefore,

\[
D_{\ell_2}(\Phi x + \xi, \Phi x' + \xi) - (1 + \epsilon)|x - x'|^2 \lesssim (\delta + |t|)|x - x'| + (\delta + |t|)(\delta + |t|).
\]

The lower bound is obtained similarly using \(4\). \(\square\)
6 Proof of Proposition 3

Proceeding as for the proofs of Props. 1 and 2 proving Prop. 3 requires the following softened distance: let us define, for \( A := (a_1, a_2), A' := (a'_1, a'_2) \in \mathbb{R}^{m \times 2} \), the pre-metric

\[
D^t(A, A') = \frac{1}{m} \sum_{i=1}^m d'(a_{1i}, a'_{1i}) d'(a_{2i}, a'_{2i}).
\]

We directly observe that \( D^0(A, A') = D_0(\mathcal{Q}(A), \mathcal{Q}(A')) \) so that we will be able to set later \( A = \Phi x 1^2 + \xi \) and \( A' = \Phi x' 1^2 + \xi \) for some \( x, x' \in \mathbb{R}^n \).

As shown in the next lemma (proved in Appendix B), this pre-metric displays some form of continuity with respect to \( \ell_2 \)-perturbations of its arguments.

**Lemma 7.** Given \( A, A', R, R' \in \mathbb{R}^{m \times 2} \), we assume that \( \max(||r_1||, ||r_2||, ||r'_1||, ||r'_2||) \leq \eta \sqrt{m} \) for some \( \eta > 0 \), with \( r_i \) and \( r'_i \) the \( i \)-th column of \( R \) and \( R' \), respectively. Then for every \( t \in \mathbb{R} \) and \( P \geq 1 \) one has

\[
\begin{align*}
D^t(A + R, A' + R') - D^{t-\eta \sqrt{P}}(A, A') &\lesssim L(A - A'), \\
D^{t+\eta \sqrt{P}}(A, A') - D^t(A + R, A' + R') &\lesssim L(A - A'),
\end{align*}
\]

with

\[
L(B) := (\eta + \frac{\delta}{\eta \sqrt{P}}) \frac{1}{\sqrt{m}} ||B||_F + (\delta + |t| + \eta) \eta + \delta (\delta + |t|) \frac{2}{P}.
\]

Moreover, when for some some \( a, a' \in \mathbb{R}^n \), \( A = a 1^2 + \Xi \) and \( A' = a' 1^2 + \Xi \) for \( \Xi \sim \mathcal{U}^{m \times 2}([0, \delta]) \), the pre-metric \( D^t(A, A') \), taken as a random variable of \( \Xi \), has an expect value on \( t = 0 \) that matches \( \frac{1}{\sqrt{m}} ||a - a'||_2^2 \), i.e., an effect that was absent from the quantized embedding defined by (3) as explained in Sec. 2 and Sec. 3. As a consequence, we can show that \( D^t(A, A') \) concentrates around \( \frac{1}{\sqrt{m}} ||a - a'||_2^2 \) with reduced distortion.

**Lemma 8.** Given \( a, a' \in \mathbb{R}^m \), \( A = a 1^2 + \Xi \) and \( A' = a' 1^2 + \Xi \) for \( \Xi \sim \mathcal{U}^{m \times 2}([0, \delta]) \), we have for some \( C, c > 0 \) and probability smaller than \( C \exp(-c \epsilon^2 m) \),

\[
|D^t(A, A') - \frac{1}{m} ||a - a'||_2^2| \geq (\epsilon (\delta + |t|) + |t|) \frac{8}{\sqrt{m}} ||a - a'|| + 16 \epsilon (\delta + |t|)^2 + 16 |t|^2.
\]

**Proof.** Notice that we can rewrite

\[
D^t(A, A') - \frac{1}{m} ||a - a'||_2^2 = \frac{1}{m} \sum_i (X_i^t - |a_i - a'_i|)^2,
\]

with \( X_i^t = d'(a_{1i} + \Xi_{1i}, a'_{1i} + \Xi_{1i}) \) and \( Y_i^t = d'(a_{2i} + \Xi_{2i}, a'_{2i} + \Xi_{2i}) \). Moreover, from (24), \( \max(||EX_i^t - |a_i - a'_i||, ||EY_i^t - |a_i - a'_i||) \leq 4|t| \) while (23) gives \( \max(||X_i^t - |a_i - a'_i||_\infty, ||Y_i^t - |a_i - a'_i||_\infty) \leq 4(\delta + |t|) \). Consequently, we can apply Lemma 7 on the concentration of the product of independent and bounded random variables. The r.v.’s \( X_i \) and \( Y_i \) being both bounded by \( L = 4 (\delta + |t|) \), setting \( s = 4|t| \) in this lemma gives the result.

**Proof of Prop. 3.** As in the previous proofs of Prop. 1 and Prop. 2 let us define an \((\ell_2, \eta)\)-covering \( \mathcal{K}_\eta \) of the set \( \mathcal{K} \). We fix \( 0 < \epsilon < 1 \) and \( t \in \mathbb{R} \) and assume that \( \Phi \) satisfies the \((\ell_2, \epsilon \eta)\)-RIP(\( \mathcal{K} - \mathcal{K} \), \( \epsilon \)). Therefore, for all \( x, x' \in \mathcal{K} \) with their respective closest points in \( \mathcal{K}_\eta \), being \( x_0 \) and \( x'_0 \), we have

\[
||\Phi(x - x_0)||_2 \leq 2 \sqrt{m} \eta \quad \text{and} \quad ||\Phi(x' - x'_0)||_2 \leq 2 \sqrt{m} \eta,
\]

with also \( ||\Phi(u - u')||_2^2 - ||u - u'||_2^2 \leq \epsilon ||u - u'||_2^2 \) for all \( u, u' \in \mathcal{K} \).

\(^2\)That is such that \( D^t(A, A') \geq 0 \) and \( D^t(A, A) = 0 \).
Hereafter, we use again the compact notation \( \tilde{\Phi} = u \mathbf{1}_n^T \) for any vectors \( u \). Moreover, since \( \Phi \) is fixed, we observe from a union bound applied on (48) that if \( m \gtrsim \epsilon^{-2} \mathcal{H}_2 (K, \eta) \), then, for \( P \geq 1 \) to be fixed later, both relations

\[
\begin{align*}
|D^{t-\eta D}(\Phi x_0 + \Xi, \Phi x'_0 + \Xi) - D_{\ell_2}(\Phi x_0, \Phi x'_0)| \\
\lesssim (\epsilon \delta + (1 + \epsilon) |t - \eta \sqrt{P}|)^2 \frac{\delta}{m} |\Phi (x_0 - x'_0)| \\
+ 16 \epsilon \delta^2 + (2 \epsilon \delta + (1 + \epsilon) |t - \eta \sqrt{P}|) \lesssim (\epsilon \delta + |t| + \eta \sqrt{P}) |x_0 - x'_0| + \epsilon \delta^2 + (|t| + \eta \sqrt{P})(\epsilon \delta + |t| + \eta \sqrt{P})
\end{align*}
\]

hold jointly for all \( x_0, x'_0 \) in \( K_N \) with probability exceeding 1 - \( ce^{-c' \epsilon^2 m} \) for some \( c, c' > 0 \).

Consequently, as any points \( x, x' \in K \) can always be written as \( x = x_0 + r, x' = x'_0 + r' \) in \( K \) with \( x_0, x'_0 \in K_N \) and \( \max(\|r\|, \|r'\|) \leq \eta \), using Lemma [7] and [10], we find

\[
D^t(\Phi x + \Xi, \Phi x' + \Xi) - D^{t-\eta D}(\Phi x_0 + \Xi, \Phi x'_0 + \Xi)
\]

\[
\lesssim (\eta + \frac{\delta}{\sqrt{P}}) \frac{1}{m} |\Phi x_0 - \Phi x'_0| (\delta + |t| + \eta \sqrt{P}) |x - x'|
\]

\[
\lesssim (\eta + \frac{\delta}{\sqrt{P}}) \frac{1}{m} |\Phi x_0 - \Phi x'_0| (\delta + |t| + \eta \sqrt{P}) |x - x'|
\]

\[
\lesssim (\eta + \frac{\delta}{\sqrt{P}}) \frac{1}{m} |\Phi x_0 - \Phi x'_0| (\delta + |t| + \eta \sqrt{P}) |x - x'|
\]

\[
\lesssim (\eta + \frac{\delta}{\sqrt{P}}) \frac{1}{m} |\Phi x_0 - \Phi x'_0| (\delta + |t| + \eta \sqrt{P}) |x - x'|
\]

where the used the \((\ell_2, \ell_2)\)-RIP(K - K, \epsilon) and \( \|x_0 - x'_0\| \leq \|x - x'\| + 2 \eta \) from the covering property of \( K_N \). Using again this last bound, the RIP and (51), we get

\[
D^t(\Phi x + \Xi, \Phi x' + \Xi) - D_{\ell_2}(\Phi x_0, \Phi x'_0)
\]

\[
\lesssim (\epsilon \delta + |t| + \eta \sqrt{P}) |x - x'| + \epsilon \delta^2 + (|t| + \eta \sqrt{P})(\epsilon \delta + |t| + \eta \sqrt{P})
\]

\[
\lesssim (\epsilon \delta + |t|) |x - x'| + \epsilon \delta(|t| + \delta) + |t|^2
\]

where we have set the free parameters to \( P^{-1} = \epsilon^2 \) and \( \eta = \delta \epsilon < \delta \epsilon \) (with \( \eta \sqrt{P} = \delta \epsilon \)).

Finally, since \( D_{\ell_2}(\Phi x_0, \Phi x'_0) \leq (1 + \epsilon) \|x_0 - x'_0\|^2 \) and

\[
\|x_0 - x'_0\|^2 - \|x - x'\|^2 \lesssim \epsilon \delta \|x - x'\|^2
\]

we find on \( t = 0 \) where \( D^t(\Phi x + \Xi, \Phi x' + \Xi) = \frac{1}{m} \|A(x) - A(x')\|_{1,o} \),

\[
\frac{1}{m} \|A(x) - A(x')\|_{1,o} - (1 + \epsilon) \|x - x'\|^2 \lesssim \epsilon \delta \|x - x'\|^2 + \epsilon \delta^2
\]

The upper bound in (15) is then achieved by observing that the distortion \( \epsilon \delta \|x - x'\|^2 \) can be assimilated to either a constant additive distortion in \( \delta^2 \epsilon \) or a slight worsening of the multiplicative distortion according to the comparison of \( \|x - x'\| \) to \( \delta \). Indeed, if \( \|x - x'\| \leq \delta \), then

\[
\frac{1}{m} \|A(x) - A(x')\|_{1,o} - (1 + \epsilon) \|x - x'\|^2 \lesssim \epsilon \delta^2
\]

while if \( \|x - x'\| > \delta \)

\[
\frac{1}{m} \|A(x) - A(x')\|_{1,o} - (1 + \epsilon) \|x - x'\|^2 \lesssim \epsilon \|x - x'\|^2 + \epsilon \delta^2,
\]

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which is equivalent to
\[
\frac{1}{m} \| A(x) - A(x') \|_{l,0} - (1 + \epsilon) \| x - x' \|^2 \lesssim \epsilon \delta^2,
\]
up to a rescaling of \( \epsilon \) inducing the requirement \( 0 < \epsilon < \epsilon_0 \) for some \( 0 < \epsilon_0 < 1 \). The lower bound in (15) is obtained similarly using (53). \( \square \)

A \ Concentration of the product of bounded and independent random variables

Lemma 8 relies on the following useful property.

**Lemma 9.** Let \( X_i \) and \( Y_i \) be independent and bounded random variables for \( 1 \leq i \leq m \) with \( \mathbb{E}X_i = \mathbb{E}Y_i = \mu_i \). Assume that for some \( \alpha \in \mathbb{R}^m, |\mu_i - |a_i|| \leq s \) and \( \max(||X_i - |a_i||\|_{\infty}, ||Y_i - |a_i||\|_{\infty}) \leq L \) for some \( L, s > 0 \). Then, for some \( c > 0, \)

\[
\mathbb{P}[\| \sum_i X_iY_i - |a_i|^2 \| \geq \epsilon L (L + \frac{2}{m} \| a \|) + s(s + \frac{2}{m} \| a \|)] \lesssim \exp(-c\epsilon^2 m).
\]

**Proof.** Notice first that, for \( Z_i := X_iY_i - |a_i|^2 \) for \( i \in [m], \)

\[
\|Z_i\|_{\infty} \leq \|X_i(Y_i - |a_i|)\|_{\infty} + \|a_i(X_i - |a_i|)\|_{\infty} \leq L||X_i||_{\infty} + |a_i|L \leq L^2 + 2|a_i|L.
\]

Therefore, each \( Z_i \) is sub-Gaussian with \( \| Z_i \|_{\psi_2} \leq \| Z_i \|_{\infty} \leq L(L + 2|a_i|) \). Moreover, by approximate rotational invariance [15],

\[
\| \sum_i Z_i \|_{\psi_2}^2 \leq \sum_i \| Z_i \|_{\psi_2}^2 \leq L^2 \sum_i (L^2 + 4|a_i| + 4|a_i|^2) \leq L^2(mL^2 + 4\sqrt{m} \| a \| + 4\| a \|^2) \leq mL^2(L + \frac{2}{m} \| a \|)^2.
\]

Consequently, for \( \epsilon > 0 \) and some \( c > 0, \)

\[
\mathbb{P}(\| \sum_i Z_i - \mathbb{E} Z_i \| \geq \epsilon) \lesssim \exp(-c\epsilon^2 m L^2 (L + \frac{2}{m} \| a \|)^2)^{-1}).
\]

However, \( |\mathbb{E} Z_i - |a_i|^2| \leq |\mu_i - |a_i|| |\mu_i + |a_i|| \leq s(2|a_i| + s), \) so that \( |\sum_i Z_i - \mathbb{E} Z_i| \geq |\sum_i Z_i - |a_i|^2| - ms^2 - 2s \sqrt{m} \| a \| \). Therefore

\[
\mathbb{P}(\| \sum_i Z_i - |a_i|^2 \| \geq \epsilon + ms^2 + 2s \sqrt{m} \| a \|) \lesssim \exp(-c\epsilon^2 m L^2 (L + \frac{2}{m} \| a \|)^2)^{-1})
\]

which gives the result by a simple rescaling \( \epsilon \to m\epsilon L(L + \frac{2}{m} \| a \|). \) \( \square \)

B \ Proof of Lemma 7

By assumption, the sets

\[
T_j := \{ i \in [m] : |r_{ij}| \leq \eta \sqrt{P}, |r'_{ij}| \leq \eta \sqrt{P} \}, \quad i \in \{1, 2\},
\]

are such that \( |T_j| \leq 2m/\rho \) as \( 2\eta^2 m \geq ||r_j||_2^2 + ||r'_{j2}||_2^2 \geq ||(r_j)_T||_T^2 + ||(r'_{j2})_T||_T^2 + ||T^c||_T^2 \geq |T^c|\eta^2 P \). Therefore, considering Lemma 1 we find with \( \rho_{ij} := \max(|r_{ij}|, |r'_{ij}|), \)

\[
D_t^{+\sqrt{P}}(A, A') = \frac{1}{m} \sum_{i=1}^m d^{+\sqrt{P}}(a_{i1}, a'_{i1}) d^{+\sqrt{P}}(a_{i2}, a'_{i2}) \leq \frac{1}{m} \sum_{i \in T} d^{+\sqrt{P}}(a_{i1}, a'_{i1}) d^{+\sqrt{P}}(a_{i2}, a'_{i2}) + \frac{1}{m} \sum_{i \in T^c} d^{+\sqrt{P}}(a_{i1}, a'_{i1}) d^{+\sqrt{P}}(a_{i2}, a'_{i2}) \leq \frac{1}{m} \sum_{i=1}^m d(a_{i1} + r_{i1}, a'_{i1} + r'_{i1}) d(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) + \frac{1}{m} \sum_{i \in T^c} R_i,
\]

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with

\[ R_i := d^{+ \eta \sqrt{T} - \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})d^{+ \eta \sqrt{T} - \rho_2}(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) \\
- d^{+ \eta \sqrt{T} - \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) \\
= d^{+ \eta \sqrt{T} - \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})(d^{+ \eta \sqrt{T} - \rho_2}(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) - d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2})) \\
+ d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2})(d^{+ \eta \sqrt{T} - \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1}) - d'(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})). \]

Using (22) and (23), we have for some \( c > 0 \) and \( i \in T^c \),

\[ d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) \leq |a_{i2} - a'_{i2}| + r_{i2} - r'_{i2} + 4\delta + 4|t| \lesssim |a_{i2} - a'_{i2}| + r_{i2} + \delta + |t|, \]

\[ d^{+ \eta \sqrt{T} - \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1}) \lesssim |a_{i1} - a'_{i1}| + r_{i1} + \delta + |t| - \eta \sqrt{T} \lesssim |a_{i1} - a'_{i1}| + r_{i1} + \delta + |t|, \]

since \( \rho_{ij} \geq \eta \sqrt{T} \), while

\[ |d^{+ \eta \sqrt{T} - \rho_1}(a_{ij} + r_{ij}, a'_{ij} + r'_{ij}) - d'(a_{ij} + r_{ij}, a'_{ij} + r'_{ij})| \leq c\rho_{ij} + c(\delta - \eta \sqrt{T}) \lesssim \rho_{ij} + \delta. \]

Therefore

\[ R_i \lesssim (|a_{i1} - a'_{i1}| + r_{i1} + \delta + |t|)(\rho_{i2} + \delta) + (|a_{i2} - a'_{i2}| + r_{i2} + \delta + |t|)(\rho_{i1} + \delta). \]

In the last bound, the first term of the RHS can be bounded as

\[ \sum_{i \in T^c} (|a_{i1} - a'_{i1}| + r_{i1} + \delta + |t|)(\rho_{i2} + \delta) \]

\[ \lesssim |a_{i1} - a'_{i1}| \|\rho_2\|_{T^c} + \|\rho_1\|_{T^c} \|\rho_2\|_{T^c} + (\delta + |t|) \|\rho_2\|_{T^c} \]

\[ \lesssim (\eta + \frac{\delta}{\sqrt{m}})\sqrt{m} |a_{11} - a'_{11}| + (\delta + |t| + \eta)m \eta + \delta(\delta + |t|) \frac{2m}{\eta}, \]

where we used the crude bounds \( \max(\|\rho_1\|_{T^c}, \|\rho_2\|_{T^c}) \leq \sqrt{m} \max(\|\rho_1\|_2, \|\rho_2\|_2) \leq 2m \eta \) and \( \max(\|\rho_1\|_{T^c} \|\rho_2\|_{T^c} \|\rho_2\|_{T^c} \|\rho_2\|_2 \|\rho_2\|_2) \leq 2\eta \sqrt{m} \). From a similar development on the second term of the bound over \( R_i \) above, we find then

\[ \sum_{i \in T^c} R_i \]

\[ \lesssim (\eta + \frac{\delta}{\sqrt{m}})\sqrt{m} |a_{11} - a'_{11}| + (\delta + |t| + \eta)m \eta + \delta(\delta + |t|) \frac{2m}{\eta}, \]

This provides finally

\[ D^{+ \eta \sqrt{T}}(A, A') - D'(A + R, A' + R') \lesssim (\eta + \frac{\delta}{\sqrt{m}})\sqrt{m} |A - A'|_F + (\delta + |t| + \eta)\eta + \delta(\delta + |t|) \frac{2m}{\eta}. \]

The upper bound is obtained similarly by observing that

\[ D^{+ \eta \sqrt{T}}(A, A') = \frac{1}{m} \sum_{i=1}^{m} d^{+ \eta \sqrt{T}}(a_{i1}, a'_{i1})d^{+ \eta \sqrt{T}}(a_{i2}, a'_{i2}) \]

\[ = \frac{1}{m} \sum_{i \in T} d^{+ \eta \sqrt{T}}(a_{i1}, a'_{i1})d^{+ \eta \sqrt{T}}(a_{i2}, a'_{i2}) + \frac{1}{m} \sum_{i \in T^c} d^{+ \eta \sqrt{T}}(a_{i1}, a'_{i1})d^{+ \eta \sqrt{T}}(a_{i2}, a'_{i2}) \]

\[ \geq \frac{1}{m} \sum_{i=1}^{m} d'(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) - \frac{1}{m} \sum_{i \in T^c} |R'_i|, \]

with

\[ R'_i := d^{+ \eta \sqrt{T} + \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})d^{+ \eta \sqrt{T} + \rho_2}(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) \]

\[ - d'(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) \]

\[ = d^{+ \eta \sqrt{T} + \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})(d^{+ \eta \sqrt{T} + \rho_2}(a_{i2} + r_{i2}, a'_{i2} + r'_{i2}) - d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2})) \]

\[ + d'(a_{i2} + r_{i2}, a'_{i2} + r'_{i2})(d^{+ \eta \sqrt{T} + \rho_1}(a_{i1} + r_{i1}, a'_{i1} + r'_{i1}) - d'(a_{i1} + r_{i1}, a'_{i1} + r'_{i1})). \]

which has the same magnitude upper bound than the one found for \( R_i \) above.
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