Bulk and Shear Viscosity in Hagedorn Fluid

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Assuming that the Hagedorn fluid composed of known particles and resonances with masses \( m < 2 \text{ GeV} \) obeys the first-order theory (Eckart) of relativistic fluid, we discuss the transport properties of QCD confined phase. Based on the relativistic kinetic theory formulated under the relaxation time approximation, expressions for bulk and shear viscosity in thermal medium of hadron resonances are derived. The relaxation time in the Hagedorn dynamical fluid exclusively takes into account the decay and eventually van der Waals processes. We comment on the in-medium thermal effects on bulk and shear viscosity and averaged relaxation time with and without the excluded-volume approach. As an application of these results, we suggest the dynamics of heavy-ion collisions, non-equilibrium thermodynamics and the cosmological models, which require thermo– and hydro–dynamics equations of state.

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I. INTRODUCTION

Since the discovery of new matter at the relativistic heavy-ion collider (RHIC) \(^1\), an immense number of experimental and theoretical works have been invested to explore the properties of non-confined strongly coupled matter, the quark-gluon plasma (QGP). Characterizing QCD matter through transport and collective behavior and the equation of state is the ultimate goal. Also, the confined QCD matter has been subject of various papers \(^2\)–\(^5\).

Shear viscosity \( \eta \) characterizes the elliptic flow and is directly proportional to energy density \( \varepsilon \) and inversely to scattering cross section \( \sigma \) and simultaneously reflects how particles interact and collectively move in many particle medium. Strongly interacting matter, like Hagedorn gas \(^6\) is conjectured to have smaller \( \eta \) than the weakly interacting one. Using perturbative and non-perturbative methods, \( \eta \), (and normalized to entropy \( s \)) has been estimated for non-confined and confined QCD matter, respectively.

At temperature \( T >> T_c \), bulk viscosity \( \xi \) has been studied using perturbative QCD \(^8\) and found to be negligible comparing to \( \eta \) and regarding to the collective evolution of the many body system, as QCD turns to be conformal invariance. Recent lattice QCD simulations \(^7\) show that bulk viscosity \( \xi \) is not negligible near \( T_c \). Its rapid increase at \( T_c \) is apparently associated with a fast growth of the trace anomaly, \( (\varepsilon - 3p)/T^4 \), of the energy-momentum tensor \( T^{\mu \nu} \). Below \( T_c \), various hadron scales likely provide the conformal invariance with bad symmetries (QCD conformal anomalies). It is therefore natural to expect that \( \xi \) at this energy scale is not negligible \(^10\).

II. BULK AND SHEAR VISCOSITY

The relativistic kinetic theory gives the transport equations for classic and colored particles in a non–Abelian external field. In this letter, we discuss the bulk and shear viscosity of the Hagedorn dynamical fluid using the relativistic kinetic theory in the relaxation time approximation \(^2\)–\(^4\) and the explicit implementation of the hadronic mass spectrum and excluded–volume approach.

Disregarding all interactions but decay and repulsion, the thermal change of the relaxation time in the Hagedorn fluid has been studied, which complete a set of thermo \(^17\) and hydrodynamic equations of state needed to characterize the evolution equation in early universe, for instance.

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We model the Hagedorn fluid of QCD confined phases (fermion and boson resonances) as a noninteracting gas of hadron resonances. To do so we sum over all Fermi and Bose resonances. The main motivation of doing this is that it refers to all relevant degrees of freedom of the confined strongly interacting matter. It implicitly includes the interactions that likely result in resonance formation \[18\]. Natural units \(c = \hbar = k = 1\) are applied here. The Hagedorn mass spectrum \(\rho(m)\) implies growth of the hadron mass spectrum with increasing the resonance masses.

\[
\rho(m) = c (m_0^2 + m^2)^{k/4} \exp(m/T_H),
\]

with \(k = -5\), \(c = 0.5\) GeV\(^{3/2}\), \(m_0 = 0.5\) GeV and \(T_H = 0.195\) GeV.

This model provides a quite satisfactory description of particle production in heavy-ion collisions \[17, 21, 22\]. The repulsive interactions likely soften the \(T–dependence of the thermodynamic quantities. The excluded–volume approach \[19\] is used to implement the effects of repulsive interactions (van der Waals) by assuming the energy normalized by \(4B\) equals the excluded–volume and the intensive quantity \(T_{pt}\) in the point–type particle approach (and the other thermodynamic quantities \[20\]) have to be corrected as follows.

\[
T = \frac{T_{pt}}{1 - \frac{p_{pt}(T_{pt})}{4B}}, \quad \left( p(T) = \frac{p_{pt}(T_{pt})}{1 - \frac{p_{pt}(T_{pt})}{4B}} \right)
\]

where \(B^{1/4} = 0.34\) GeV stands for the MIT bag constant.

In spherical polar coordinates, the energy–momentum tensor of a single particle with \(p\)– and \(T\)–independent mass \(m\) is defined as

\[
T_{\mu\nu}^1 = \frac{g}{2\pi^2} \rho(m) \int p^2 dp \frac{p_{\mu}p_{\nu}}{\varepsilon} \, n(p, T),
\]

where \(p^\mu = (\varepsilon, \vec{p})\) is momentum four-vector and \(g\) is degeneracy factor of the hadron resonances. The single particle energy is given by the dispersion relation \(\varepsilon = (p^2 + m^2)^{1/2}\).

With the above assumptions on Hagedorn viscous fluid, the overall energy–momentum tensor can be calculated as a sum over energy-momentum tensors \(T_{\mu\nu}^1\) of all hadrons resonances \(^1\),

\[
T_{\mu\nu} = \sum_i T_{\mu\nu}^1
\]

In momentum phase space and assuming that the system is in a state with vanishing chemical potential and near equilibrium, the distribution function \(n(p, T)\) reads

\[
n(p, T) = \frac{1}{\exp \left( \frac{\varepsilon - \vec{p} \cdot \vec{u}}{\varepsilon} \right) \pm 1},
\]

where \(\pm\) stands for fermion and boson statistics, respectively. The local flow velocity \(\vec{u}\) is compatible with the Eckart fluid \[22\], implying that \(T_{\mu\nu} u_\mu u_\nu = \varepsilon\). It is obvious that \(n(p, T)\) satisfies the kinetic theory \[3\] and second law of thermodynamics. The solution of kinetic equation is obtainable by deviating the distribution function from its local equilibrium.

The deviation of energy-momentum tensor from its local equilibrium is corresponding to the difference between the distribution function near and at equilibrium, \(\delta n = n - n_0\). The latter can be determined by relaxation time approximation with vanishing external and self-consistent forces \[3, 24\]

\[
\delta n(p, T) = -\tau \frac{p^\mu}{\varepsilon \vec{p} \cdot \vec{u}} \partial_\mu n_0(p, T)
\]

Then the difference between near and equilibrium energy–momentum tensor reads

\[
\delta T_{\mu\nu}^1 = \frac{g}{2\pi^2} \rho(m) \int_0^{\infty} p^2 dp \frac{p^\mu p^\nu}{\varepsilon^2} \tau \, p^\alpha \partial_\alpha n_0(p, T).
\]

\(^1\) It reflects the algebraic properties, here the addition, of the energy-momentum tensor
Using the symmetric projection tensor $h_{\alpha\beta}$, the components of the derivative $\partial_\alpha$ can be split to parallel and orthogonal to $u^\mu$. $h_{\alpha\beta}$ generates a 3-matic and projects each point into the instantaneous rest space of the fluid.

$$\partial_\mu = Du^\mu + \nabla_\mu,$$

where $D = u^\alpha \partial_\alpha = (\partial_t, 0)$ gives the temporal derivative and $\nabla_\mu = \partial_\mu - u_\mu D = (0, \partial_i)$ gives the spatial derivative. Such an splitting has to guarantee the conservation of equilibrium energy–momentum tensor: $\partial_\mu T^{\mu\nu} = 0$ and fulfill the laws of thermodynamics at equilibrium [3, 25]. In ref [2], the non-equilibrium $n(p, T)$ has been decomposed using the relaxation time approach $n = n_0 + \tau N_1 + \cdots$. Alternatively, as $n(p, T)$ embeds the 1st-rank tensor $u$, $\delta T_1^{\mu\nu}$ can be decomposed into $u$ [3] in order to deduce its spatial components.

$$\partial_k u^i = \frac{1}{2} \left( \partial_k u^i + \partial_i u^k - \frac{2}{3} \delta_{ki} \partial_i u^i \right) + \frac{1}{3} \delta_{ki} \partial_i u^i \equiv \frac{1}{2} W_{kl} + \frac{1}{3} \delta_{ki} \partial_i u^i. \quad (9)$$

Applying the equation of hydrodynamics, then the deviation from equilibrium

$$\delta T_1^{\mu\nu} = \frac{g}{2\pi^2} \rho(m) \tau \int_0^\infty \frac{p^\mu p^\nu}{T} n_0(1 + n_0) \left[ \bar{p} \cdot \bar{\nabla} + \frac{3}{2} \left( \frac{\nabla_p}{p} - \frac{\nabla T}{T} \right) \right] p^2 dp, \quad (10)$$

can be re-written as

$$\delta T_1^{ij} = \frac{g}{2\pi^2} \frac{\tau}{T} \int_0^\infty p^i p^j n_0(1 + n_0) \left[ \left( \frac{c_s^2 \varepsilon}{3} - \frac{\bar{p}^2}{p^2} \right) \partial_i u^j - \frac{\bar{p}^2}{p^2} W_{kl} \right] p^2 dp, \quad (11)$$

where $c_s^2 = \partial \rho/\partial \varepsilon$ is the speed of sound in this viscous fluid. The bulk $\xi$ and shear $\eta$ viscosity can be deduced from Eq. [11] by comparing it with

$$\delta T_1^{ij} = -\xi \delta_{ij} \partial_k u^k - \eta W_{ij}. \quad (12)$$

To find shear viscosity $\eta$, we put $i \neq j$ in Eqs. [9] and [12]. To find bulk viscosity $\xi$, we substitute $i$ with $j$ and $T_0^{\mu\nu}$ with $3P$. The subscript 0 (as in the distribution function $n$) refers to the equilibrium state. Although we keep the gradients of velocity, we put $\bar{u} = 0$ in the final expressions. The intensive quantities $\eta$ and $\xi$ of Hagedorn fluid [2] in the comoving frame read

$$\xi(T) = \frac{g}{2\pi^2} \frac{\tau}{T} \sum_i \rho(m_i) \int_0^\infty n_0(1 + n_0) \left( \frac{c_s^2 \varepsilon}{3} - \frac{\bar{p}^2}{p^2} \right) \frac{2}{p^2} dp \quad (13)$$

$$\eta(T) = \frac{g}{30\pi^2} \frac{\tau}{T} \sum_i \rho(m_i) \int_0^\infty n_0(1 + n_0) \frac{\bar{p}^4}{\varepsilon^4} \frac{2}{p^2} dp. \quad (14)$$

The $T$–dependence of dimensionless ratios $\xi T^4/\tau$ and $\eta T^4/\tau$ is depicted in Fig. [1] With increasing $T$, bulk and shear viscosity increase, significantly. We note that $\xi$ seems to be about one order of magnitude larger than $\eta$. Left panel of Fig. [2] illustrates such a comparison. At low $T$, $\eta$ starts with larger values than $\xi$’s. But with increasing $T$, $\xi$ gets larger [27]

The ratio $\xi/\eta$ has been related to the speed of sound $c_s^2$ in gas of massless pions. Apparently there are essential differences between this system and the one of Hagedorn fluid. According to [24], the ratio of $\xi/\eta$ in $N = 2^*$ plasma is conjectured to remain finite across the second–order phase transition. This behavior seems to be illustrated in Fig. [1] In the Hagedorn fluid, the system is assumed to be drifted away from equilibrium and it should relax after a characteristic time $\tau$. Should we implement a phase transition in the Hagedorn fluid, then $\tau \propto \xi^z$, where $z$ is the critical exponents, likely diverges near $T_c$.

### III. RELAXATION TIME

The relaxation time depends on the relative cross section as

$$\tau(T) = \frac{1}{n_f(T) \langle v(T) \sigma(T) \rangle}, \quad (15)$$

\[ As we assume a vanishing chemical potential, the heat conductivity vanishes as well. \]
where \( v(T) \) and \( n_i(T) \) is the relative velocity of two particles in case of binary collision and the density of each of the two species, respectively. The thermal-averaged transport rate or cross section is \( \langle v(T)\sigma(T) \rangle \). The transport equation of single-particle distribution function in the momentum space, \( n(r, p, t) \),

\[
\frac{\partial n}{\partial t} + \vec{v} \cdot \nabla_x n = -\int \frac{d^3p_2 d^3p_1}{(2\pi)^6} \frac{d^3p_1'}{d^3p_2'} \sigma_{ij} \left[ n n_2 (1-n_1') (1-n_2') - n_1 n_2 (1-n_1)(1-n_2) \right]
\]

(16)

First line in r.h.s. of Eq. (16) gives the Boltzmann collision term. The second line adds the Uehling–Uhlenbeck factors. The third line accounts for the Pauli-blocking of the final states. The total derivative of \( \langle v(T)\sigma(T) \rangle \) remains unchanged as well.

When fitting the decay widths \( \Gamma_i \) of \( i \) hadron resonances, then the decay relaxation times \( \tau_i \) in GeV\(^{-1} \) read \( \tau_i \equiv \Gamma_i^{-1} = (0.151 m_i - 0.058)^{-1} \approx (0.151 m_i - 0.058)^{-1} \). As the resonance mass \( m \) is conjectured to remain constant in thermal and dense medium, this linear fit apparently implies that \( \tau \) remains unchanged as well.

In the Hagedorn fluid, where the inter-particle collisions as in Eq. (17) are minimized, we are left with specific processes to estimate \( \tau \) (decay and repulsion for instance). Formation from free space vacuum and decay to stable resonances; \( P_1 + P_2 \leftrightarrow P_3 \) are examples. The constrains on this process have been discussed in Ref. [22].

In rest of frame of the particle \( P_3 \) boosting from the laboratory frame, the kinetic equation for the time evolution of the number density \( n_3(T) \) reads

\[
\frac{d}{dt} n_3(T) = \frac{d}{dV/dt} (W_{12\rightarrow 3} - W_{3\rightarrow 12})
\]

(18)

The backward (inverse) direction is also valid. Note that \( n(T), \) Eq. (18), and \( n(p, T), \) Eq. (19), are related with each other via \( n(T) = N(T)/V = g/(2\pi^2) \sum_i n_i(m_i) \int p^2 dp n_i(p, T) \) and therefore \( n(p, T) \) is a Lorentz scalar whereas \( n(T) \) not. The thermal decay and production rate \( dW/dV/dt \) have been discussed in Ref. [22]. In Boltzmann limit and assuming that the repulsive interaction does not contribute meaningfully to the overall relaxation time, the decay time in rest frame is given in textbooks.

\[
\tau = \frac{8\pi m_3^2 g_3 I}{p \sum_{spin} |\langle \vec{p}, -\vec{p} | M | m_3 \rangle|^2 \langle \varepsilon_3 \rangle / m_3},
\]

(19)
where \( p \) is the pressure. \( I \) is a step function for particle distinguishability; \( I = 2 \) for indistinguishable and \( I = 1 \) for distinguishable particles. \( M \) is the hadronic reaction matrix.

In right panel of Fig. 2 the relaxation time in fm, Eq. (19), is given as function of \( T \) in GeV. We note that increasing the temperature \( T \) leads to reducing the relaxation time \( \tau \). It might mean that the decay processes get faster when \( T \) increases. Near \( T_c \), the effect of excluded–volume approach is considerable.

As an application of these results, we mention the cosmological viscous models \([33]\), which require a complete set of thermo\([17]\) and hydrodynamic equations of state in order to solve the evolution equation in early universe and study the nucleosynthesis.

### IV. RATIO OF SHEAR VISCOSITY OVER ENTROPY DENSITY

Using Anti de Sitter space/Conformal Field Theory (AdS/CFT) methods \([34]\), it has been argued that the ratio \( \eta/s \) seems to have a universal lower bound in any physical system. The bound is \( \eta/s \geq 1/4\pi \). It has been found to this value saturated for a large class of strongly interacting systems with a dual description (string theory in anti-de Sitter space) \([12, 35]\). Also, at temperatures below \( T_c \), i.e. Hagerdorn-type models, this ratio has been analyzed using chiral perturbation theory \([36]\), coupled Boltzmann equations of pions and nucleons in low baryon number densities \([5]\), gas of massless pions \([37]\), relativistic mean field models with scaled hadron masses and couplings \([16]\) and viscous relativistic hydrodynamics \([38]\). In Fig. 3 we draw \( \eta/s \) calculated in the present model as a function of \( T \). It is clear that \( \eta/s \) starts from a much higher values than the AdS/CFT lower bound one. It comes closer to it with increasing \( T \).

Also, it is obvious that \( \eta/s \) would be reduced with increasing the Hagedorn mass spectrum. It is a universal property that the quantities which are depending on thermodynamics are suppressed with increasing the mass. Shear viscosity \( \eta \) apparently follows this behavior, which can be realized primarily due to the enhancement of the massive resonances that leads to a decrease in the mean free path of a particle and a corresponding increase in the average binary collision cross section. The results are compatible with the ones introduced in Ref. \([15]\) which are compared with the UrQMD simulations. Nevertheless, we notice however, that for Hagedorn resonance gas treated within the rate equation approach, an upper limit of \( \eta/s \) is found to be as small as the KSS lower bound near \( T_c \) \([13]\).

The UrQMD model calculations give saturated \( \eta/s \)–values in the region \( 0.125 < T > 0.166 \) GeV \([15]\). Within this interval, \( \eta \) quantitatively equal to the entropy density \( s \). Another difference between UrQMD and the present work is graphically shown in the \( T \)–region, \( 0.06 < T > 0.16 \) GeV, where it appears that the UrQMD model drastically underestimates the values of \( \eta/s \). It is clear that the massive resonances likely contribute dominantly, especially high \( T \). The ratio \( \eta/s \) gradually reduces.

The effects of excluded volume on \( \eta/s \), which as discussed previously takes into consideration – at least – the van der Waals repulsive interactions, are also illustrated graphically in the same figure. The viscosity coefficients are enhanced due to this additional interaction. Finally, we notice that the values of \( \eta/s \) very close to \( T_c \) are close to the upper bound of 0.24 obtained from viscous hydrodynamic calculations \([39]\) of elliptic flow.

To compare with the lattice QCD calculations below \( T_c \) \([40]\), we need to rearrange the configurations the Hagedorn fluid to be compilable with the lattice QCD configurations. Since, it is not trivial to match the two configurations, we
think that a future study should be devoted to this subject. At higher $T$, the universal properties of bulk viscosity near the QCD phase transition are introduced in Ref. [9]. On the other hand, there are several lattice QCD estimations for the lower bound [41]. It is very essential to mention that these lattice QCD calculations have two limitations. First is the use of quench approximation, i.e. without quark pair creation-annihilation effects on vacuum, and second is the use of an ansatz for the spectral function.

Fig. 3: The shear viscosity to entropy density ratio $\eta/s$ as a function of temperature $T$ for the hadron resonance gas with Hagedorn mass spectrum. Horizontal dashed line give the AdS/CFT lower bound.
[21] F. Karsch, K. Redlich and A. Tawfik, Eur. Phys. J. C 29, 549 (2003); Phys. Lett. B 571, 67 (2003).
[22] P. Braun-Munzinger, D. Magestro, K. Redlich, and J. Stachel, Phys. Lett. B518, 41 (2001); F. Becattini, J. Cleymans, A. Keranen, E Suhonen and K. Redlich, Phys. Rev. C 64, 024901 (2001).
[23] C. Eckart, Phys. Rev. 58, 919 (1940).
[24] F. Reif, Fundamentals of statistical and Thermal Physics, McGraw-Hill, New York, (1965).
[25] Roy Maartens, “Causal thermodynamics in relativity”, arXiv:astro-ph/9609119.
[26] M. Prakash, M Prakash, R. Venugopalan and G. Welke, Phys. Reps., 227, 321 (1993).
[27] S. Gavin, Nucl. Phys. A 435, 826 (1985); J.-W. Chen and J. Wang, Phys. Rev. C 79, 044913 (2009).
[28] H. Stöcker and W. Greiner, Phys. Rep. 137, 277 (1986).
[29] A. Buchel, Phys. Lett. B 663, 286 (2008); A. Buchel and C. Pagnutti, Nucl. Phys. B 816, 62 (2009).
[30] P. Ring and P. Schuck, The nuclear many-body problem, Springer-Verlag Berlin Heidelberg (1980).
[31] I. Senda, Phys. Lett. B 263, 270 (1991).
[32] I. Kuznetsova and J. Rafelski, [arXiv:1002.0375 [hep-th]]
[33] A. Tawfik, AIP Conf. Proc. 1115, 239 (2009); [arXiv:1002.0296 [gr-qc]]; A. Tawfik, M. Wahba, H. Mansour and T. Harko, [arXiv:1001.2814 [gr-qc]]; Invited talk at 7th International Conference on Modern Problems of Nuclear Physics, Tashkent, Uzbekistan, 22-25 Sep 2009, [arXiv:0911.4105 [gr-qc]]; A. Tawfik, H. Mansour and M. Wahba, Talk given at 12th Marcel Grossmann Meeting on General Relativity (MG 12), Paris, France, 12-18 Jul 2009, [arXiv:0912.0115 [gr-qc]].
[34] P. Kovtun, D. T. Son and A. O. Starinets, JHEP 0310, 064 (2003); Phys. Rev. Lett. 94, 111601 (2005).
[35] G. Policastro, D.T. Son, and A.O. Starinets, Phys. Rev. Lett. 87, 081601 (2001); JHEP 0209, 043 (2002); C.P. Herzog, J. High Energy Phys. 0212, 026 (2002); A. Buchel and J.T. Liu, Phys. Rev. Lett. 93, 090602 (2004).
[36] D. Fernandez-Frailea and A. G. Nicolab, Eur. Phys. J. A 31, 848-850 (2007).
[37] J.-W. Chen and J. Wang, Phys. Rev. C 79, 044913 (2009).
[38] N. Demir and S. A. Bass, Phys. Rev. Lett. 102, 172302 (2009).
[39] P. Romatschke and U. Romatschke, Phys. Rev. Lett. 99, 172301 (2007); H. Song and U. Heinz, Phys. Lett. B 658, 279 (2008).
[40] S. Muroya, talk at international Workshop on "Hadron Physics and Property of High Baryon Density Matter", Xi’an, China, 22-25 Nov. (2006). [hep-ph/0702220]
[41] H.B. Meyer, Phys. Rev. D 76, 101701 (2007); A. Nakamura and S. Sakai, Nuclear Physics A 774, 775-778 (2006).