Density Profile of the One-Dimensional Partially Asymmetric Simple Exclusion Process with Open Boundaries

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Abstract
The one-dimensional partially asymmetric simple exclusion process with open boundaries is considered. The stationary state, which is known to be constructed in a matrix product form, is studied by applying the theory of $q$-orthogonal polynomials. Using a formula of the $q$-Hermite polynomials, the average density profile is computed in the thermodynamic limit. The phase diagram for the correlation length, which was conjectured in [1] (J. Phys. A 32 (1999) 7109), is confirmed.

[Keywords: asymmetric simple exclusion, exact solution, density profile, $q$-orthogonal polynomials]

1 Introduction
The one-dimensional asymmetric simple exclusion process (ASEP) [2, 3] is a system of particles which hop preferentially in one direction on a one-dimensional lattice with hard-core exclusion interaction. The ASEP has been studied extensively since it is one of the few models which show rich non-equilibrium behaviors and is exactly solvable [4]. Besides, the ASEP has applications to many interesting problems such as the hopping conductivity, growth processes and the traffic flows [5].

In this article, we consider the stationary state of the ASEP with open boundary conditions. That is, the system is connected to particle reservoirs at boundaries. The case where particles can hop only in one direction, which we refer to as the “totally asymmetric” case in the sequel, was solved in [6, 7]. The current and the density profile were calculated exactly in the thermodynamic limit. The phase diagram for the current and the correlation length were identified. The system exhibits phase transitions depending on the parameters at the boundaries. Recently the obtained phase diagram was discussed from the point of view of the domain wall dynamics [8].
The partially asymmetric case with the open boundary conditions was partially solved in [1]. The current was evaluated in the thermodynamic limit. The phase diagram for the current was identified. It turned out to be the same as the one obtained by mean-field approximation [9] or by employing a plausible assumption [10]. The phase diagram for the correlation length was also obtained by assuming that the correlation length is given by the logarithm of the ratio of the largest and the second largest eigenvalue of a certain matrix which plays a similar role as a transfer matrix does in equilibrium statistical mechanical models. It was shown that the phase diagram has a richer structure than that for the totally asymmetric case. The average density profile was, however, not calculated in [1]. In this sense the obtained phase diagram for the correlation length has remained a conjecture. The purpose of this paper is the confirmation of this phase diagram. By using the explicit formula for the Poisson kernel of the $q$-Hermite polynomials, the average density profile in the thermodynamic limit is calculated for the partially asymmetric case. It turns out that the phase diagram was correctly predicted in [1].

In this article, we only consider the case where hoppings of particles at the boundaries and those at the bulk part of the system are compatible. In other words, when we allow the particle input at the left boundary and the particle output at the right boundary, the hopping rate to the right is assumed to be larger than that to the left. When hoppings at the boundaries and those at bulk is incompatible, the current becomes zero in the thermodynamic limit. The situation seems to be similar to the closed boundary condition where particles can not enter or go out of the system [11]. Of course, when we consider the finite chain, the current remains to be positive. We remark that the asymptotic current for this case was evaluated in [12].

The paper is organized as follows. In the next section, the definition of the model is given in terms of the master equation. The so-called matrix product ansatz, which gives the stationary state in the form of matrix product, is also explained. Some properties of the $q$-Hermite polynomials and the relationship to the matrix product ansatz are explained in section 3. The section 4 is the main section of this article. First, the one-point function is represented in the form of double integrals. Second, the average density profile in the thermodynamic limit are summarized whereas the evaluation of the integrals are relegated to Appendices. The phase diagram for the correlation length is identified. The concluding remarks are given in the last section.

2 Definition of Model and Matrix Product Ansatz

The one-dimensional asymmetric simple exclusion process (ASEP) is defined as follows. During the infinitesimal time interval $dt$, each particle jumps to the right nearest neighboring site with probability $p_R dt$ and to the left nearest neighboring site with probability $p_L dt$. If the chosen site is already occupied, the particle does not move due to the exclusion rule. More than one particle can not be on the same site. Each site can be either empty or occupied. The case where particles can hop only in one direction, i.e., the case where either $p_L = 0$ or $p_R = 0$ is called the “totally asymmetric” case. The $p_R = p_L$ case is
called the “symmetric” case whereas the case where particles hop in both directions with different rates will be referred to as the “partially asymmetric” case. In addition, we allow the particle input at the left end of the chain with rate \( \alpha \) and allow the particle output at the right end of the chain with rate \( \beta \) (Fig. 1). Here the length of the chain is denoted by \( L \). In this article, we restrict our attention to the partially asymmetric case since the totally asymmetric case and the symmetric case was already solved in [6, 7] and in [13] respectively. The restrictions on the parameters are \( 0 < p_L < p_R \) and \( \alpha, \beta > 0 \).

More formally, the process is defined in terms of the master equation. Each configuration of the system is indicated by \( \{\tau_1, \tau_2, \ldots, \tau_L\} \) where \( \tau_j (j = 1, 2, \ldots, L) \) denotes the particle number at site \( j \). Namely \( \tau_j = 0 \) if the site \( j \) is empty whereas \( \tau_j = 1 \) if the site \( j \) is occupied. Let \( P(\tau_1, \tau_2, \ldots, \tau_L; t) \) denote the probability that the system has the configuration \( \{\tau_1, \tau_2, \ldots, \tau_L\} \) at time \( t \). Then the time evolution of the ASEP is described by the following master equation,

\[
\begin{align*}
\frac{d}{dt}P(\tau_1, \tau_2, \ldots, \tau_L; t) &= \alpha (2\tau_1 - 1)P(0, \tau_2, \ldots, \tau_L; t) \\
&+ \sum_{j=1}^{L-1} (\tau_j - \tau_{j+1}) [p_L P(\tau_1, \tau_2, \ldots, 0, 1, \ldots, \tau_L; t) \\
&- p_R P(\tau_1, \tau_2, \ldots, 1, 0, \ldots, \tau_L; t)] \\
&+ \beta (1 - 2\tau_L) P(\tau_1, \tau_2, \ldots, \tau_{L-1}, 1; t).
\end{align*}
\tag{2.1}
\]

For instance, the master equation for \( L = 2 \) case reads

\[
\begin{align*}
\frac{d}{dt} & \begin{bmatrix} P(00; t) \\ P(01; t) \\ P(10; t) \\ P(11; t) \end{bmatrix} = \\
& \begin{bmatrix} \alpha & -\beta & 0 & 0 \\ 0 & \alpha + p_L + \beta & -p_R & 0 \\ -\alpha & -p_L & p_R & -\beta \\ 0 & -\alpha & 0 & \beta \end{bmatrix} \\
& \times \begin{bmatrix} P(00; t) \\ P(01; t) \\ P(10; t) \\ P(11; t) \end{bmatrix}.
\end{align*}
\tag{2.2}
\]

One can confirm himself that the dynamics of the ASEP is correctly encoded in the master equation (2.1).

When time \( t \) goes to infinity, the system is expected to reach the stationary state. The probability distribution in the stationary state will be denoted as \( P(\tau_1, \tau_2, \ldots, \tau_L) \). For instance, except for the normalization, the stationary state for \( L = 2 \) case is the eigenvector of the \( 4 \times 4 \) matrix in the right hand side of (2.2) with the eigenvalue zero. Explicitly, it reads

\[
\begin{align*}
& \begin{bmatrix} P(00) \\ P(01) \\ P(10) \\ P(11) \end{bmatrix} = \text{Const.} \begin{bmatrix} \frac{1}{\alpha} \\ \frac{1}{\alpha \beta} \\ \frac{1}{\alpha \beta} \\ \frac{1}{\beta^2} \end{bmatrix} \\
& \times \frac{1}{p_R} \left( \frac{p_L}{\alpha \beta} + \frac{1}{\alpha} + \frac{1}{\beta} \right).
\end{align*}
\tag{2.3}
\]
In [6], it was shown that the probability distribution of the ASEP in the stationary state for general \(L\) can be written in the form of the matrix product as

\[
P(\tau_1, \tau_2, \ldots, \tau_L) = \frac{1}{Z_L} \langle W | \prod_{j=1}^{L} (\tau_j D + (1 - \tau_j) E) | V \rangle,
\] (2.4)

where \(D\) and \(E\) are square matrices and \(\langle W |\) and \(| V \rangle\) are vectors satisfying following relations,

\[
\begin{align*}
    p_R D E - p_L E D &= \zeta (D + E), \quad (2.5a) \\
    \alpha \langle W | E &= \zeta \langle W |, \quad \beta D | V \rangle = \zeta | V \rangle. \quad (2.5b)
\end{align*}
\]

Here \(\zeta\) is an arbitrary number. If one defines the matrix \(C\) by

\[
    C = D + E,
\] (2.6)

the normalization \(Z_L\) is given by

\[
    Z_L = \langle W | C^L | V \rangle. \quad (2.7)
\]

Here, for the case of \(L = 2\), we check that the state (2.4) indeed gives the stationary state of the process by using the algebraic relations (2.5a) and (2.5b). For \(L = 2\), (2.4) reads

\[
\begin{bmatrix}
P(00) \\
P(01) \\
P(10) \\
P(11)
\end{bmatrix} = \frac{1}{Z_2} \begin{bmatrix}
    \langle W | E^2 | V \rangle \\
    \langle W | E D | V \rangle \\
    \langle W | D E | V \rangle \\
    \langle W | D^2 | V \rangle
\end{bmatrix}.
\] (2.8)

Three components, \(P(00), P(01), P(11)\) can be calculated by simply using (2.5b). On the other hand, one computes \(P(10)\) first changing the order of matrices \(D, E\) by (2.5a) and then using (2.5b). Hence we get

\[
\begin{bmatrix}
P(00) \\
P(01) \\
P(10) \\
P(11)
\end{bmatrix} = \frac{1}{Z_2} \begin{bmatrix}
    \zeta^2 \\
    \frac{p_R}{\alpha^2} \left( \frac{p_R}{\alpha^2} + \frac{1}{\alpha} + \frac{1}{\beta} \right)
\end{bmatrix}.
\] (2.9)

One can compare this expression with (2.3) to see that this expression indeed gives the stationary state for \(L = 2\) case. One also sees that the arbitrary parameter \(\zeta\) appears in the same way for all components, \(P(00), P(01), P(10), P(11)\). Changing the parameter \(\zeta\) only changes the normalization \(Z_2\). This is true for general \(L\) as well. In this article, the proof that the state (2.4) gives the stationary state for general \(L\) is not given. See [6].
We express several physical quantities in the form of matrix products. The one-point function \( \langle n_j \rangle_L \) is defined as the probability that the site \( j \) is occupied. In other words, \( \langle n_j \rangle_L \) is the average density at site \( j \). The two-point function \( \langle n_j n_k \rangle_L \) is defined as the probability that the sites \( j \) and the site \( k \) are both occupied. Higher correlation functions are defined similarly. In the matrix language, they are computed by

\[
\langle n_j \rangle_L = \frac{\langle W | C^{j-1} D C^{L-j} | V \rangle}{Z_L},
\]

\[
\langle n_j n_k \rangle_L = \frac{\langle W | C^{j-1} D C^{k-j-1} D C^{L-k} | V \rangle}{Z_L},
\]

and so on. The current through the bond between site \( j \) and site \( j + 1 \) is defined by

\[
J_L^{(j)} = p_R \langle n_j (1 - n_{j+1}) \rangle - p_L \langle n_j (1 - n_{j-1}) \rangle.
\]

In the steady state, the current is independent of \( j \) and hence is denoted by \( J_L \). It is given by

\[
J_L = \zeta \frac{\langle W | C^{L-1} | V \rangle}{\langle W | C^L | V \rangle} = \zeta \frac{Z_{L-1}}{Z_L}.
\]

Once one finds a representation of these algebraic relations, by using the above formula, one can in principle calculate the physical quantities such as the particle current \( J_L \), the one-point function \( \langle n_j \rangle_L \), the two-point function \( \langle n_j n_k \rangle_L \) and the higher correlation functions.

We note that the process has an obvious particle-hole symmetry. When we look at holes instead of particles, they tend to hop to the left with rate \( p_R \) and to the right with rate \( p_L \) with hard-core exclusion. In addition, they are injected at right end with rate \( \beta \) and they are removed at the left end with rate \( \alpha \). In other words, the process is invariant under the changes,

\[
\text{particle} \leftrightarrow \text{hole}, \quad \alpha \leftrightarrow \beta
\]

site number \( j \) ↔ site number \( L - j + 1 \).

Due to this symmetry, it is sufficient to obtain the density for the right half of the system. The density for the left half of the system is obtained by using the above symmetry as

\[
\langle n_j \rangle_L(\alpha, \beta) = 1 - \langle n_{L-j+1} \rangle_L(\beta, \alpha),
\]

where the dependence of \( \langle n_j \rangle_L \) on the parameters \( \alpha \) and \( \beta \) are explicitly indicated.

Before closing the section, we present some simulation results (Fig. 2 and Fig. 3). Figure 2 shows the space-time diagrams for several choices of parameters. It is clear that the properties of the system crucially depend on the values of the boundary parameters. The differences become more transparent when we consider the particle current or the the average density profile of the stationary state. In principle, the stationary state is achieved only in the infinite time limit. However, we see from Fig. 2 that the system practically goes into a stationary state after some transient time. Hence if we average the density over a long time after the transient time, it would be regarded as the average density profile of the stationary state practically. The results are shown in Fig. 3. When \( \alpha \) is small and \( \beta \)
is large, the bulk density is low. It decays sharply near the right boundary. This is called
the low density phase. Conversely, when \( \alpha \) is large and \( \beta \) is small, the bulk density is high.
It decays sharply near the left boundary. This is called the high density phase. When
\( \alpha = \beta \) is small, the low density region and the high density region coexist (coexistence
line). Finally, when both \( \alpha \) and \( \beta \) are large enough, the density takes the value 1/2 at bulk
and decays slowly near both the boundaries. This is called the maximal current phase.
Our main tasks in the following are to obtain the average density profiles in Fig. 4 exactly.

3 Representation of Algebra and \( q \)-Hermite Polynomials

First we introduce some notations for later convenience. We introduce the \( q \)-number,
\[
\{n\} = 1 - q^n, \tag{3.1}
\]
and the \( q \)-shifted factorial,
\[
(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}), \tag{3.2a}
\]
\[
(a; q)_0 = 1. \tag{3.2b}
\]
We also define
\[
(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j), \tag{3.3}
\]
for \( |q| < 1 \). Since products of \( q \)-shifted factorials appear so often, we use the notations,
\[
(a_1, a_2, \cdots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty, \tag{3.4}
\]
\[
(a_1, a_2, \cdots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n. \tag{3.5}
\]

Next a representation of the algebraic relation (2.5a) and (2.5b) is given. If we define
\[
D = 1 + d, \quad E = 1 + e, \tag{3.6}
\]
we see that these relations become
\[
de - qed = 1 - q, \tag{3.7a}
\]
\[
\langle W | e = a \langle W |, \quad d | V \rangle = b | V \rangle, \tag{3.7b}
\]
where we put
\[
q = p_L/p_R, \tag{3.8}
\]
\[
a = \frac{1 - \tilde{\alpha}}{\tilde{\alpha}}, \quad b = \frac{1 - \tilde{\beta}}{\tilde{\beta}}, \tag{3.9}
\]
with $\tilde{\alpha} = \alpha / (p_R - p_L), \tilde{\beta} = \beta / (p_R - p_L)$. Since $0 < p_R < p_L$, we have $0 < q < 1$.

In this article, we take the following representation for the matrices $d, e$ and the vectors $\langle W |, | V \rangle$,

$$d = \begin{bmatrix}
0 & \{1\}^{1/2} & 0 & 0 & \cdots \\
0 & 0 & \{2\}^{1/2} & 0 & \cdots \\
0 & 0 & 0 & \{3\}^{1/2} & \cdots \\
\vdots & & & & \ddots
\end{bmatrix}, \quad e = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
\{1\}^{1/2} & 0 & 0 & 0 & \cdots \\
0 & \{2\}^{1/2} & 0 & 0 & \cdots \\
0 & 0 & \{3\}^{1/2} & 0 & \cdots \\
\vdots & & & & \ddots
\end{bmatrix},$$

$$(3.10a)$$

$$\langle W | = \kappa_c \langle a | = \kappa \left(1, \frac{a}{\sqrt{(q; q)_1}}, \frac{a^2}{\sqrt{(q; q)_2}}, \ldots \right), \quad | V \rangle = \kappa | b \rangle_c = \kappa \left(\frac{1}{b}, \frac{\sqrt{(q; q)_1}}{b^2}, \frac{\sqrt{(q; q)_2}}{b^2}, \ldots \right).$$

$$(3.10b)$$

The constant $\kappa$ is taken as $\kappa^2 = (ab; q)_\infty$ so that $\langle W | V \rangle = 1$. It should be noticed that there exists another useful representation of the algebraic relations $(2.5a)$ and $(2.5b)$. It was first given in [6] and was used to obtain the phase diagram of the correlation length in [1].

The advantage of the representation $(3.10a)$ and $(3.10b)$ is that the commutation relation of the matrices $d$ and $e$ turns out to be a simple diagonal matrix. We have

$$de - ed = (1 - q) \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & q & 0 & 0 & \cdots \\
0 & 0 & q^2 & 0 & \cdots \\
0 & 0 & 0 & q^3 & \cdots \\
\vdots & & & & \ddots
\end{bmatrix}. \quad (3.11)$$

This fact will play an important role for the calculation of the average density profile in the next section.

Next we list some properties of the continuous $q$-Hermite polynomials. The proofs can be found for instance in [14, 15]. The continuous $q$-Hermite polynomials $\{H_n(x|q)| n = 0, 1, 2, \ldots \}$ are defined by the three term recurrence relation,

$$H_{n+1}(x; q) + (1 - q^n)H_{n-1}(x; q) = 2xH_n(x; q),$$

$$(3.12)$$

with the initial condition,

$$H_{-1}(x; q) = 0, \quad H_0(x; q) = 1. \quad (3.13)$$

They are explicitly given by the formula,

$$H_n(\cos \theta |q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} e^{i(n-2k)\theta}. \quad (3.14)$$
The orthogonality relation reads
\[
\int_0^\pi H_n(\cos \theta|q)H_m(\cos \theta|q)(e^{2i \theta}, e^{-2i \theta}; q)_\infty d\theta = 2\pi \frac{(q;q)_n}{(q;q)_\infty} \delta_{mn}.
\] (3.15)

The generating function is also known and is given by
\[
\sum_{n=0}^\infty \frac{H_n(\cos \theta|q)}{(q;q)_n} \lambda^n = \frac{1}{(\lambda e^{i \theta}, \lambda e^{-i \theta}; q)_\infty}
\] (3.16)

for \(|\lambda| < 1\). To calculate the average density profile, we also need the so-called Poisson kernel,
\[
\sum_{n=0}^\infty \frac{H_n(\cos \theta|q)H_n(\cos \phi|q)r^n}{(q;q)_n} = \frac{(r^2; q)_\infty}{(re^{i(\theta+\phi)}, re^{-i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{-i(\theta-\phi)}; q)_\infty}.
\] (3.17)

This formula is called \(q\)-Mehler formula in the mathematics literature.

Here we notice that, if we introduce
\[
p_n(x) = \frac{H_n(x|q)}{\sqrt{(q;q)_n}},
\] (3.18)

the three-term recurrence relation is rewritten into the form,
\[
\begin{bmatrix}
0 & \{1\}_{\frac{1}{2}} & 0 & 0 & \cdots \\
\{1\}_{\frac{1}{2}} & 0 & \{2\}_{\frac{1}{2}} & 0 & \cdots \\
0 & \{2\}_{\frac{1}{2}} & 0 & \{3\}_{\frac{1}{2}} & \cdots \\
& \ddots & \ddots & \ddots & \ddots \\
& & & & \ddots
\end{bmatrix}
\begin{bmatrix}
p_0(x) \\
p_1(x) \\
p_2(x) \\
\vdots \\
\vdots
\end{bmatrix}
= 2x
\begin{bmatrix}
p_0(x) \\
p_1(x) \\
p_2(x) \\
\vdots \\
\vdots
\end{bmatrix}.
\] (3.19)

In other words, \(|p(x)\rangle = \{p_0(x), p_1(x), \ldots\}\) is formally an eigenvector of the matrix \(d + e\) with eigenvalue \(2x\). This is the basic relationship between the representation of the algebra (2.5a),(2.5b) and the theory of \(q\)-orthogonal polynomials. Finally, the completeness of the continuous \(q\)-Hermite polynomials reads
\[
1 = \frac{(q; q)_\infty}{2\pi} \int_0^\pi d\theta (e^{i \theta}, e^{-i \theta}; q)_\infty |p(\cos \theta)\rangle \langle p(\cos \theta)|.
\] (3.20)

4 Calculation of Density Profile

In this section, the average density profile is calculated by using the formula (3.17) of the continuous \(q\)-Hermite polynomials. We first recall the asymptotic behaviors of the normalization \(Z_L\) and the current in the thermodynamic limit \(J = \lim_{L \to \infty} J_L\). The asymptotic expressions for \(Z_L\) were given in (4.1) and are summarized as follows:

- For phase \(A\) (low-density phase; \(a > 1\) and \(a > b\); \(\tilde{\alpha} < \frac{1}{2}\) and \(\tilde{\alpha} < \tilde{\beta}\))
  \[
  Z_L \approx \frac{(a^{-2}; q)_\infty}{(b/a; q)_\infty} [(1 + a)(1 + a^{-1})]_L^L,
  \] (4.1)
• For phase B (high-density phase; \( b > 1 \) and \( a < b \); \( \tilde{\beta} < \frac{1}{2} \) and \( \tilde{\alpha} > \beta \))

\[
Z_L \simeq \frac{(b^{-2}; q)_{\infty}}{(a/b; q)_{\infty}}[(1 + b)(1 + b^{-1})]^L,
\]

(4.2)

• For phase C (maximal current phase; \( 0 < a, b < 1 \); \( \tilde{\beta} > \frac{1}{2} \) and \( \tilde{\alpha} > \frac{1}{2} \))

\[
Z_L \simeq \frac{(ab; q)_{\infty}(q; q^3)_{\infty}4^{L+1}L^2}{\sqrt{\pi}(a, b; q^2_{\infty})L^2}.
\]

(4.3)

• On the coexistence line (\( a = b > 1 \); \( \tilde{\alpha} = \tilde{\beta} < \frac{1}{2} \))

\[
Z_L \simeq \frac{(a - a^{-1})(a^{-2}; q)_{\infty}L}{(q; q)_{\infty}}[(1 + a)(1 + a^{-1})]^{L-1}.
\]

(4.4)

Using (2.12), the current in the thermodynamic limit is readily computed as

• For phase A (\( \tilde{\alpha} < \frac{1}{2} \) and \( \tilde{\beta} > \tilde{\alpha} \))

\[
J = (p_R - p_L)\tilde{\alpha}(1 - \tilde{\alpha}),
\]

(4.5)

• For phase B (\( \tilde{\beta} < \frac{1}{2} \) and \( \tilde{\alpha} > \tilde{\beta} \))

\[
J = (p_R - p_L)\tilde{\beta}(1 - \tilde{\beta}),
\]

(4.6)

• For phase C (\( \tilde{\alpha} > \frac{1}{2} \) and \( \tilde{\beta} > \frac{1}{2} \))

\[
J = \frac{p_R - p_L}{4}.
\]

(4.7)

The phase diagram for the current is depicted in Fig. 4.

Now we turn to consider the average density profile. As you can see from the figures in Fig. 4, the average density is almost constant at bulk part except on the coexistence line. Hence we are interested in the average bulk density and how the density decays near the boundaries. When the average density decays like \( e^{-r/\xi} \) with \( r \) distance from the boundary, we refer to \( \xi \) as the correlation length in this paper. As for the average density near the boundaries, it is sufficient to compute the density near the right boundary due to the symmetry (2.13). The relation (2.14) enables us to know the average density profile near the left boundary. On the other hand, the coexistence line should be treated separately. Since it is easier to calculate the density difference than the density itself, we rewrite the density at site \( j \) as

\[
\langle n_j \rangle_L = \sum_{k=j}^{L-1} (\langle n_k \rangle_L - \langle n_{k+1} \rangle_L) + \langle n_L \rangle_L.
\]

(4.8)
At the right boundary, we have

\[ \langle n_L \rangle_L = \frac{1}{Z_L} \langle W|C^{L-1}DC^*V \rangle \]

\[ = \frac{1}{Z_L} \frac{\beta}{Z_L} \rightarrow \frac{J}{\beta} \quad (L \to \infty) \]

(4.9)

Using (4.3)-(4.7), the density at the right boundary is easily calculated.

Next we notice that

\[ \langle n_k \rangle_L - \langle n_{k+1} \rangle_L = \frac{1}{Z_L} \langle (W|C^{k-1}DC^{L-k}|V) - \langle W|C^kDC^{L-k-1}|V \rangle \rangle \]

\[ = \frac{1}{Z_L} \langle W|C^{k-1}(DC - CD)C^{L-k-1}|V \rangle \]

\[ = \frac{1}{Z_L} \langle W|C^{k-1}(DE - ED)C^{L-k-1}|V \rangle \]

\[ = \frac{1}{Z_L} \langle W|C^{k-1}(de - ed)C^{L-k-1}|V \rangle . \]

(4.10)

Now one can represent \( \langle W|C^{k-1}(de - ed)C^{L-k-1}|V \rangle \) in the form of double integrals. The calculation proceeds as follows. First we notice that

\[ \langle W|C^{k-1}(de - ed)C^{L-k-1}|V \rangle \]

\[ = \kappa^2(q; q)_\infty^2 \int_0^\pi \frac{d\theta}{2\pi} (e^{2i\theta} e^{-2i\theta}; q)_\infty \int_0^\pi \frac{d\phi}{2\pi} (e^{2i\phi} e^{-2i\phi}; q)_\infty \]

\[ \times c \langle a|C^{k-1}p(cos \theta) \rangle \langle p(cos \theta)|\langle de - ed\rangle p(cos \varphi) \rangle \langle p(cos \varphi)|b \rangle \]

\[ = (ab; q)_\infty^2 \int_0^\pi \frac{d\theta}{2\pi} \int_0^\pi \frac{d\phi}{2\pi} (e^{2i\theta} e^{-2i\theta} e^{2i\phi} e^{-2i\phi}; q)_\infty [2(1 + \cos \theta)]^{k-1}[2(1 + \cos \varphi)]^{L-k-1} \]

\[ \times c \langle a|p(cos \theta) \rangle \langle p(cos \theta)|\langle de - ed\rangle p(cos \varphi) \rangle \langle p(cos \varphi)|b \rangle . \]

(4.11)

Here the formula (3.14) gives

\[ c \langle a|p(cos \theta) \rangle = \frac{1}{(ae^{i\theta}, a^{-i\theta}; q)_\infty}, \quad \langle p(cos \varphi)|b \rangle = \frac{1}{(be^{i\theta}, b^{-i\theta}; q)_\infty}. \]

(4.12)

The remaining term \( \langle p(cos \theta)|\langle de - ed\rangle p(cos \varphi) \rangle \) can also be computed by using the fact (3.11) and the formula (3.17). We see that

\[ \langle p(cos \theta)|\langle de - ed\rangle p(cos \varphi) \rangle = (1 - q) \sum_{n=0}^\infty p_n(cos \theta) p_n(cos \varphi) q^n \]

\[ = (1 - q) \sum_{n=0}^\infty \frac{H_n(cos \theta|q) H_n(cos \varphi|q) q^n}{(q; q)_n} \]

\[ = (q; q)^\infty \frac{\langle qe^{i(\theta + \varphi)}, qe^{-i(\theta + \varphi)}, qe^{i(\theta - \varphi)}, qe^{-i(\theta - \varphi)}; q \rangle}{(qe^{i(\theta + \varphi)}, qe^{-i(\theta + \varphi)}, qe^{i(\theta - \varphi)}, qe^{-i(\theta - \varphi)}; q)^\infty} \]

(4.13)
Hence we have

\[
I_{1} = \frac{1}{4}(ab; q)_{\infty}(q; q)^{3} \int_{0}^{\pi} \frac{dz_{1}}{2\pi iz_{1}} \int_{0}^{\pi} \frac{dz_{2}}{2\pi iz_{2}} \frac{(z_{1}^{2}, z_{1}^{-2}, z_{2}^{2}, z_{2}^{-2}; q)_{\infty}}{(a_{1}, a_{1}^{-1}, qz_{1}z_{2}, qz_{1}^{-1}z_{2}^{-1}, qz_{1}z_{2}^{-1}, qz_{1}^{-1}z_{2}, bq_{1}^{-1}; q)_{\infty}(z_{2} + z_{2}^{-1} - z_{1} - z_{1}^{-1})},
\]

and

\[
I_{2} = \frac{1}{4}(ab; q)_{\infty}(q; q)^{3} \int_{0}^{\pi} \frac{dz_{1}}{2\pi iz_{1}} \int_{0}^{\pi} \frac{dz_{2}}{2\pi iz_{2}} \frac{(z_{1}^{2}, z_{1}^{-2}, z_{2}^{2}, z_{2}^{-2}; q)_{\infty}}{(a_{1}, a_{1}^{-1}, qz_{1}z_{2}, qz_{1}^{-1}z_{2}^{-1}, qz_{1}z_{2}^{-1}, qz_{1}^{-1}z_{2}, bq_{1}^{-1}; q)_{\infty}(z_{2} + z_{2}^{-1} - z_{1} - z_{1}^{-1})},
\]

When \(0 < a, b < 1\), the contours of \(z_{1}\) and \(z_{2}\) are both unit circles. For other values of the parameters, the contours are deformed as poles in the integrands move in and out of the unit circles.

The evaluation of the integrals \(I_{1}\) and \(I_{2}\) in the thermodynamic limit for each phase is relegated to Appendices. Basically, the integral \(I_{1}\) gives the density at bulk region whereas...
the integral $I_2$ gives the density near the right boundary. The results are summerized in the following. We thus obtain the phase diagram shown in Fig. 5. Notice that this phase diagram was correctly predicted in [1]. Therein the correlation length for each phase was assumed to be given by the logarithm of the ratio of the largest and the second largest eigenvalue of the matrix $C$.

- **Phase C** ($\tilde{\alpha} > 1/2$ and $\tilde{\beta} > 1/2$; $0 < a, b < 1$)

In this phase, the average density at bulk is $1/2$. The average density decays near the right boundary as

$$\langle n_j \rangle_L = \frac{1}{2} - \frac{1}{2\sqrt{\pi l^2}}. \quad (4.18)$$

Here and in the following, we set $l = L - j + 1$. The density decays algebraically and hence the correlation length is infinite. This average density profile is exactly the same as that for the totally asymmetric case. The density decay at the left boundary can be obtained from the symmetry relation (2.14).

- **Phase A$_1$** ($\tilde{\alpha} < \tilde{\beta} < \tilde{\alpha}/[(1 - \tilde{\alpha})q + \tilde{\alpha}]$ and $\tilde{\beta} < 1/2$; $aq < b < a$ and $b > 1$)

The average density near the right boundary decays exponentially as

$$\langle n_j \rangle_L = \tilde{\alpha} - \frac{(b^{-2}q; q)_\infty}{(a^{-1}b^{-1}q, ab^{-1}q; q)_\infty} \left[ \frac{\tilde{\alpha}(1 - \tilde{\alpha})}{\beta(1 - \beta)} \right]^l (1 - 2\tilde{\beta}), \quad (4.19)$$

with the correlation length

$$\xi^{-1} = \ln \frac{\tilde{\beta}(1 - \tilde{\beta})}{\tilde{\alpha}(1 - \tilde{\alpha})}. \quad (4.20)$$

On the other hand, the density near the left boundary takes the constant value $\tilde{\alpha}$. This fact is obtained by combining the average density profile near the right boundary for the high-density phase below and the symmetry (2.13).

- **Phase A$_2$** ($q/(1 + q) < \tilde{\alpha} < 1/2$ and $\tilde{\beta} > 1/2$; $1 < a < q^{-1}$ and $b < 1$)

In the bulk region, the average density takes the constant value $\tilde{\alpha}$. On the other hand, the density profile near the right boundary decays exponentially as

$$\langle n_j \rangle_L = \tilde{\alpha} - \frac{(a - b)(1 - ab)(aq, a^{-1}bq; q)_\infty(q; q)_\infty^4 [4\tilde{\alpha}(1 - \tilde{\alpha})]^l}{(a - 1)^2(b - 1)^2(aq, a^{-1}q, bq; q)_\infty^4 \sqrt{\pi l^2}}, \quad (4.21)$$

with the correlation length

$$\xi^{-1} = -\ln 4[\tilde{\alpha}(1 - \tilde{\alpha})]. \quad (4.22)$$

But the decay is not purely exponential but with algebraic corrections. At the left boundary, the density takes the constant value $\tilde{\alpha}$. 

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• Phase $A_3$ ($\tilde{\beta} > \tilde{\alpha}/[(1 - \tilde{\alpha})q + \tilde{\alpha}]$ and $\tilde{\alpha} < q/(1 + q)$; $a > q^{-1}$ and $b < aq$)

The average density at bulk is $\tilde{\alpha}$. Near the right boundary, the density decays exponentially as

$$\langle n_j \rangle_L = \tilde{\alpha} - \frac{(1 - ab)(1 - (aq)^{-1})}{(1 - b(aq)^{-1})(1 + aq)} \left[ \frac{(1 + aq)(1 + (aq)^{-1})}{(1 + a)(1 + a^{-1})} \right]^l$$

(4.23)

with the correlation length

$$\xi^{-1} = \ln \frac{q}{[\tilde{\alpha} + (1 - \tilde{\alpha})q]^2}.$$  

(4.24)

This density profile has no correspondense for the totally asymmetric case. At the left boundary, the density takes the constant value $\tilde{\alpha}$.

• Phase $B_1$ ($\tilde{\alpha}q/[(1 - \tilde{\alpha}) + q\tilde{\alpha}] < \tilde{\beta} < \tilde{\beta}$ and $\tilde{\alpha} < 1/2$; $bq < a < b$ and $a > 1$)

The average density at bulk is $1 - \tilde{\beta}$. As can be seen from the calculation in Appendix, The density takes the constant value near the right boundary.

$$\langle n_j \rangle_L = 1 - \tilde{\beta}$$

(4.25)

Since this phase is related to the phase $A_1$ through the symmetry (2.13), the density decays exponentially near the left boundary with the correlation length

$$\xi^{-1} = \ln \frac{\tilde{\alpha}(1 - \tilde{\alpha})}{\tilde{\beta}(1 - \tilde{\beta})}.$$  

(4.26)

• Phase $B_2$ ($q/(1 + q) < \tilde{\beta} < 1/2$ and $\tilde{\alpha} > 1/2$; $a < 1$ and $1 < b < q^{-1}$)

This phase is symmetric to phase $A_2$ through the symmetry (2.13). The average density at bulk and near the right boundary is $1 - \tilde{\beta}$. Near the left boundary, the density decays exponentially with the correlation length

$$\xi^{-1} = -\ln 4[\tilde{\beta}(1 - \tilde{\beta})].$$

(4.27)

• Phase $B_3$ ($a < bq$ and $b > q^{-1}$) ($\tilde{\beta} < \tilde{\alpha}q/[1 - \tilde{\alpha} + \tilde{\alpha}q]$ and $\tilde{\beta} < q/(1 + q)$; $a < bq$ and $b > q^{-1}$)

This phase is symmetric to phase $A_3$ through the symmetry (2.13). The average density at bulk and near the right boundary is $1 - \tilde{\beta}$. The density decays exponentially with the correlation length

$$\xi^{-1} = \ln \frac{q}{[\tilde{\beta} + (1 - \tilde{\beta})q]^2},$$

(4.28)

near the left boundary.
• Coexistence line ($\tilde{\alpha} = \tilde{\beta} < 1/2; a = b > 1$)

The average density shows linear profile at bulk,

$$\langle n_j \rangle_L = \tilde{\alpha} + (1 - 2\tilde{\alpha}) \frac{j}{L}. \quad (4.29)$$

This is essentially the same as the result for the totally asymmetric case.

Before closing the section, we present a simulation result for the correlation length to show the difference between the phase $A_2$ and $A_3$. The simulation was done on $\tilde{\beta}$ line. For the totally asymmetric case, the system is in the $A_2$ phase on this line. The correlation length is given by $\xi_{A_2} = 1/\ln4[\alpha(1 - \alpha)]$. On the other hand, for the partially asymmetric case, the system is in the $A_3$ phase when $\tilde{\alpha} < q/(1 + q)$. The correlation length is given by $\xi_{A_3} = 1/\ln[\frac{1}{\beta + (1 - \beta)q}]^2$. The differences between these two expressions become large especially when $q$ and $\alpha$ are small. Especially, as $\alpha \rightarrow 0$, $\xi_{A_2}$ goes to zero whereas $\xi_{A_3}$ goes to $-1/\ln q$. In Fig. 6 the simulation result for the correlation length is shown for $p_R = 1.0, P_L = 0.9, \beta = 10$ which corresponds to $q = 0.9, \tilde{\beta} = 1$. It is clear that the correlation length approaches the finite value as $\alpha \rightarrow 0$ and is well described by the formula for $\xi_{A_3}$.

5 Concluding Remarks

In this article, we have computed the average density profile of the partially asymmetric simple exclusion process with open boundaries. The calculation has been done for a wide range of parameters satisfying $0 < p_L < p_R$ and $\alpha > 0, \beta > 0$. The phase diagram for the correlation length has been obtained. It has turned out that the phase diagram was correctly predicted in the earlier paper [1]. In [1], the phase diagram was obtained by assuming that the correlation length is given by the logarithm of the ratio of the largest and the second largest eigenvalues of the matrix $C$. The discussions were only for the phases with the exponentially decaying profile. In this article, we have not only confirmed this fact but also obtained the asymptotic expressions of the average density profile for all phases.

There are two key facts which allowed us to calculate the average density profile exactly in the thermodynamic limit. One is that the commutation relation of the matrices $D, E$ becomes a simple diagonal matrix and the other is the formula (3.17) of the $q$-Hermite polynomials.

There seems to be many possible applications and generalizations of the analysis of this article. First, it is possible to generalize the analysis in this paper to the partially asymmetric exclusion process on a ring with a single defect particle [16, 17]. The corresponding totally asymmetric case was already solved in [16]. Second, the case where $p_L > p_R$ is also interesting. Although the current was evaluated in [12], more exact results are desirable. Third, it would be interesting to apply the similar analysis to the multi-species models [18, 19, 20, 21, 22]. Compared to the ASEP, much less is known about these models.
Several investigations are now in progress. The results about these will be reported elsewhere.

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Appendix A Evaluation of Integral $I_1$

In this appendix, the integral $I_1$,

$$I_1 = \frac{1}{4} (ab; q)_\infty (q; q)_3 \int \frac{dz_1}{2\pi i z_1} \int \frac{dz_2}{2\pi i z_2} \frac{(z_1^2, z_2^2, z_2^{-2}; q)_\infty [(1 + z_1)(1 + z_1^{-1})]^{L-1}}{(az_1, aq^{-1}z_1, b; q)_\infty(z_2 + z_2^{-1} - z_1 - z_1^{-1})}, \tag{A.1}$$

is evaluated. First, for the case where $a, b < 1$, both of the contours of $z_1$ and $z_2$ are unit circles. We have

$$I_1 = I_1^{(0)} = \int_0^{\pi} \frac{d\theta}{2\pi} \int_0^{\pi} \frac{d\phi}{2\pi} \frac{(e^{2i\theta}, e^{-2i\theta}, e^{2i\phi}, e^{-2i\phi}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, qe^{i(\theta + \phi)}, qe^{-i(\theta + \phi)}, qe^{i(\theta - \phi)}, qe^{-i(\theta - \phi)}, be^{i\phi}, be^{-i\phi}; q)_\infty} \times \frac{2(1 + \cos \theta)^{L-1}}{2 \cos \phi - 2 \cos \theta}. \tag{A.2}$$

Second, consider the case where $a$ becomes larger one but $b$ is still smaller than one. We assume

$$a > aq > aq^2 > \cdots > aq^{n(a)} > 1 > aq^{n(a)+1} > \cdots. \tag{A.3}$$

Then the contour of $z_1$ is still a unit circle but the contour of $z_1$ has to be modified to include all poles at $z_1 = aq^k (k = 0, 1, \ldots, n(a))$ and to exclude all poles at $z_1 = (aq^k)^{-1}$ ($k = 0, 1, \ldots, n(a))$. Separating the contributions from poles at $z_1 = aq^k$ and $z_1 = (aq^k)^{-1}$, the integral $I_1$ can be rewritten as

$$I_1 = I_1^{(0)} - \frac{(ab; q)_\infty (q; q)_2}{2a} \sum_{k=0}^{n(a)} \frac{(-)kq^{k(k-1)/2}(a^2q^{2k}, a^{-2}q^{-2k}; q)_\infty \lambda_k^{(a)} L-1}{(q; q)_k (a^2q^k; q)_\infty} \times \int_{C_0} \frac{dz_2}{2\pi i z_2} \frac{(z_2^2, z_2^{-2}; q)_\infty}{(aq^{k+1}z_2, aq^{k+1}z_2^{-1}, a^{-1}q^{-k}z_2, a^{-1}q^{-k}z_2^{-1}, b; q)_\infty}. \tag{A.4}$$

Here the contour $C_0$ denotes the unit circle. The $\lambda_k^{(c)}$s are defined by

$$\lambda_k^{(c)} = (1 + cq^k)(1 + c^{-1}q^{-k}), \tag{A.5}$$
for $c = a, b$ and $k = 0, 1, 2, \ldots$. The integral can be evaluated explicitly by using the general formula,

$$
\int_{C} \frac{dz}{2\pi i z} (az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_{\infty} = \frac{2(abc; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}}. \quad (A.6)
$$

The contour $C$ is such that it includes all poles of the type $fq^k$ and excludes all poles of the type $f^{-1}q^{-k}$ with $f = a, b, c, d$ and $k = 0, 1, 2, \ldots$. The parameters $a, b, c, d$ in this formula have nothing to do with the $a, b, c, d$ which appear in the rest of this article. This formula plays a crucial role in proving the orthogonality relation of the Askey-Wilson polynomials. The proof can be found in [24]. Now we get

$$
I_1 = I_1^{(0)} + I_1^{(a)},
$$

$$
I_1^{(a)} = -\frac{(ab; q)_{\infty}(q; q)_{\infty}}{a} \sum_{k=0}^{n(a)} \frac{(-)^k q^{k(k-1)/2}(b^2 q^{2k}, b^{-2} q^{-2k}; q)_{\infty}}{(q; q)_k(b^2 q^k; q)_{\infty}} \left[\lambda_k^{(a)}\right]^{L-1}. \quad (A.7)
$$

When $b$ also becomes larger than one, there appear the terms which come from poles at $z_2 = bq^k$ and $z_2 = (bq^k)^{-1}$ in (A.1). When (A.3) and

$$
b > bq > bq^2 > \cdots > bq^{n(b)} > 1 > bq^{n(b)+1} > \cdots
$$

hold, we have

$$
I_1 = I_1^{(0)} + I_1^{(a)} + I_1^{(b,0)} + I_1^{(b,b)} + I_1^{(b,a)}, \quad (A.9)
$$

$$
I_1^{(b,0)} = \frac{(ab; q)_{\infty}(q; q)_{\infty}^2}{2b} \sum_{k=0}^{n(b)} \frac{(-)^k q^{k(k-1)/2}(b^2 q^{2k}, b^{-2} q^{-2k}; q)_{\infty}}{(q; q)_k(b^2 q^k; q)_{\infty}}
$$

$$
\times \int_{C_0} \frac{dz_1}{2\pi iz_1} \frac{\left(z_1^2, z_1^{-2}; q\right)_{\infty}[(1+z_1)(1+z_1^{-1})]^{L-1}}{(az_1, az_1^{-1}, bz_1, bz_1^{-1}, cz_1, cz_1^{-1}, dz_1, dz_1^{-1}; q)_{\infty}}, \quad (A.10)
$$

$$
I_1^{(b,b)} = -\sum_{k=0}^{n(b)} \sum_{m=k+1}^{n(b)} \frac{(-)^k q^{k(k-1)/2+m(m-k)(m-k-1)/2} \left[\lambda_m^{(b)}\right]^{L-1}}{b(q; q)_k(q; q)_{m-k-1}}
$$

$$
\times \frac{(q, ab, b^2 q^{2k}, b^{-2} q^{-2k}, b^2 q^{2m}, b^{-2} q^{-2m}; q)_{\infty}}{(q^{m-k}, b^2 q^k, b^2 q^{m+k+1}, b^{-2} q^{-k-m}, abq^m, ab^{-1} q^{-m}; q)_{\infty}}. \quad (A.11)
$$

$$
I_1^{(b,a)} = \sum_{k=0}^{n(b)} \sum_{m: bq^k > qa^m} \frac{(-)^{k+m} q^{k(k+1)/2+m(m+1)/2} \left[\lambda_m^{(a)}\right]^{L-1}}{b(q; q)_k(q; q)_m}
$$

$$
\times \frac{(q, ab, b^2 q^{2k}, b^{-2} q^{-2k}, a^2 q^{2m}, a^{-2} q^{-2m}; q)_{\infty}}{(b^2 q^k, a^2 q^m, abq^{k+1+m}, a^{-1} bq^{k+1-m}, a^{-1} b^{-1} q^{-k-m}; q)_{\infty}}. \quad (A.12)
$$

Lastly, when $a < 1$ and (A.8) holds, we have

$$
I_1 = I_1^{(0)} + I_1^{(b,0)} + I_1^{(b,b)}. \quad (A.13)
$$

Now we turn to the calculation of the asymptotic expression of the integral $I_1$. 

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The case $a < 1$ and $b < 1$

In this case, $I_1 = I_1^{(0)}$. We evaluate the asymptotic behavior of $I_1^{(0)}$ by employing the steepest decent method. First we change the variable from $\theta, \varphi$ to $u, y$ as

$$1 + \cos \theta = 2e^{-u/L},$$

$$1 + \cos \varphi = 2ye^{-u/L},$$

(A.14)

(A.15)

to obtain

$$I_1 = -\frac{4L^2}{L^2} \int_0^\infty du u^2 e^{-u} \sqrt{\frac{1 - e^{-u/L}}{u/L}} e^{-u/L} \int_0^{\infty/L} dy \frac{y^2 \sqrt{1 - ye^{-u/L}}}{1 - y}$$

$$\times \frac{(e^{2i\theta}, e^{-2i\theta}, e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, qe^{i(\theta+\varphi)}, qe^{-i(\theta+\varphi)}, qe^{i(\theta-\varphi)}, qe^{-i(\theta-\varphi)}, be^{i\varphi}, be^{-i\varphi}; q)_\infty}. \quad (A.16)$$

Here $\theta$ and $\varphi$ are considered as functions in $u$ and $y$ through (A.14) and (A.15) respectively. Now we can take the limit $L \to \infty$ in the integrand. Changing the variable $y$ back to $\varphi$ by

$$1 + \cos \varphi = 2y,$$

(A.17)

we have

$$I_1 \simeq -\frac{\sqrt{\pi} (q; q)_\infty^2 4L^2}{2(a; q)_\infty^2 L^2} \int_0^\pi d\varphi \frac{(e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(e^{i\varphi}, e^{i\varphi}, qe^{-i\varphi}, qe^{i\varphi}, be^{i\varphi}, be^{i\varphi}; q)_\infty}. \quad (A.18)$$

Using the formula (A.6), we get

$$I_1 \simeq -\frac{\pi^2 (1 - b) 4^{L+1}}{a, b; q)_\infty^2 L^2}. \quad (A.19)$$

The case $a < 1$ and $b > 1$

The integral $I_1$ is given by (A.13). The main contributions come from $I_1^{(0)}$ and $I_1^{[b,0]}$. Each contribution behaves as $4^L$. whilst the normalization $Z_L$ behaves as $[[1+b](1+b^{-1})]^L$ for this case. Since $(1+b)(1+b^{-1})$ is larger than 4, $I_1$ is negligible compared to $Z_L$. So we do not compute the explicit expression.

The case $a > 1$

The main contribution comes from the $k = 0$ term in the summation of $I_1^{(a)}$. We have

$$I_1 \simeq -\frac{a(1 - ab)(a^{-2}; q)_\infty [(1 + a)(1 + a^{-1})]^{L-1}}{(a^{-1}b; q)_\infty}. \quad (A.20)$$
Appendix B  Evaluation of Integral $I_2$

In this Appendix, the integral $I_2$, 

$$I_2 = \frac{1}{4} (ab; q)_\infty (q; q)_\infty^3 \int \frac{dz_1}{2\pi i z_1} \int \frac{dz_2}{2\pi i z_2} \frac{(z_1^2, z_1^{-2}, z_2^2, z_2^{-2}; q)_\infty [(1 + z_1)(1 + z_1^{-1})]^{j-1}[1 + z_2)(1 + z_2^{-1})]^{L-j}}{(az_1, a_1^{-1}, q_1 z_2, a z_1^{-1} z_2^{-1}, q_1 z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, q z_2^{-1}, q z_1^{-1} z_2^{-1}, (B.1)$$

is evaluated. First, for the case where $a, b < 1$, both of the contours of $z_1$ and $z_2$ are unit circles. We have

$$I_2 = I_2^{(0)} = \int_0^\pi \frac{d\theta}{2\pi} \int_0^\pi \frac{d\varphi}{2\pi} \frac{(e^{2i\theta}, e^{-2i\theta}, e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, q e^{i(\theta+\varphi)}, q e^{-i(\theta+\varphi)}, q e^{i(\theta-\varphi)}, q e^{-i(\theta-\varphi)}, be^{i\varphi}, be^{-i\varphi}; q)_\infty \times [2(1 + \cos \theta)]^{j-1}[2(1 + \cos \varphi)]^{L-j}}. (B.2)$$

Since we will calculate the average density profile near the right boundary, we set $l = L - j + 1$. Similarly to the case of the integral $I_1$, when $a$ or $b$ or both of them become larger than one, there appear other contributions besides $I_2^{(0)}$. When (A.3) and (A.8) hold,
the result is

\[
I_2 = I_2^{(0)} + I_2^{(a,0)} + I_2^{(a,a)} + I_2^{(a,b)} + I_2^{(b,0)} + I_2^{(b,b)} + I_2^{(b,a)},
\]

\[
I_2^{(a,0)} = -\frac{(ab;q)_\infty (q;q)_\infty^2}{2a} \sum_{k=0}^{n(a)} \frac{(-)^k q^k (k-1/2)^2 (a^2 q^{2k}, a^{-2} q^{-2k}; q)_\infty [\lambda_k^{(a)}]_{L-l}}{(q; q)_k (a^2 q^k; q)_\infty}
\times \int_{\mathcal{C}_0} \frac{dz_2}{2\pi i z_2} \frac{(z_2^2, z_2^{-2}; q)_\infty [(1+z_2)(1+z_2^{-1})]^{l-1}}{(aq^{k+1} z_2, aq^{k+1} z_2^{-1}, a^{-1} q^{-k} z_2, a^{-1} q^{-k} z_2^{-1}, b z_2, b z_2^{-1}; q)_\infty},
\]

\[
I_2^{(a,a)} = \sum_{k=0}^{n(a)} \sum_{m=k+1}^{n(a)} \frac{(-)^k q^k (k+1/2)^2 (m(m+k)q^{-k} m^{-1} q^{-k}; q)_\infty [\lambda_k^{(a)}]_{L-l} [\lambda_m^{(a)}]_{L-l}}{a(q; q)_k (q; q)_m}
\times \frac{(aq^{m-k}, a^2 q^m, a^{-2} q^{-m}, ab q^{-m}, b^{-1} q^{-m}; q)_\infty}{(q^m, a^2 q^m, a^2 q^{m+k+1}, a^{-2} q^{-k-m}, ab q^{-m}, b^{-1} q^{-m}; q)_\infty},
\]

\[
I_2^{(a,b)} = \frac{(ab;q)_\infty (q;q)_\infty^2}{2b} \sum_{k=0}^{n(b)} \frac{(-)^k q^k (k-1/2)^2 (b^2 q^{2k}, b^{-2} q^{-2k}; q)_\infty [\lambda_k^{(b)}]_{L-l}}{(q; q)_k (b^2 q^k; q)_\infty}
\times \int_{\mathcal{C}_0} \frac{dz_1}{2\pi i z_1} \frac{(z_1^2, z_1^{-2}; q)_\infty [(1+z_1)(1+z_1^{-1})]^{l-1}}{(az_1, a z_1^{-1}, b q^{k+1} z_1, b q^{k+1} z_1^{-1}, b^{-1} q^{-k} z_1, b^{-1} q^{-k} z_1^{-1}; q)_\infty},
\]

\[
I_2^{(b,0)} = \sum_{k=0}^{n(b)} \sum_{m=k+1}^{n(b)} \frac{(-)^k q^k (k+1/2)^2 (m(m-k)q^{-1} k^{-1}; q)_\infty [\lambda_k^{(b)}]_{L-l} [\lambda_m^{(b)}]_{L-l}}{b(q; q)_k (q; q)_m}
\times \frac{(q, ab, b^2 q^{2k}, b^{-2} q^{-2k}, b^2 q^{-2k}, b^{-2} q^{2k}; q)_\infty}{(aq^{-k}, a^2 q^{-k}, a^{-2} q^{k}, ab q^{-k}, b^{-1} q^{-k}, a^{-1} b^{-1} q^{-k}; q)_\infty},
\]

\[
I_2^{(b,b)} = \sum_{k=0}^{n(b)} \sum_{m=k+1}^{n(b)} \frac{(-)^k q^k (k+1/2)^2 (m(m-k)q^{1-k} m^{-1}; q)_\infty [\lambda_k^{(b)}]_{L-l} [\lambda_m^{(b)}]_{L-l}}{b(q; q)_k (q; q)_m}
\times \frac{(aq^{m-k}, a^2 q^m, ab q^{m+k+1}, a^{-2} q^{-m-k}, ab q^{-m-k}, a^{-1} b^{-1} q^{-m-k}; q)_\infty}{(b^2 q^k, a^2 q^{-k}, ab q^{k+1} m^{-1}, a^{-1} b q^{-k-1} m^{1}, ab^{-1} q^{-m-k}, a^{-1} b^{-1} q^{-m-k}; q)_\infty}.
\]

Now we consider the asymptotic expression of the integral \(I_2\). We take the limit \(L \to \infty\) at first and then take the limit \(l \to \infty\).

- The case \(a, b < 1\)

In this case, \(I_2 = I_2^{(0)}\). The evaluation for this case proceeds analogously to the
evaluation of $I_1^{(0)}$. Changing the variables $\theta, \varphi$ to $u, y$ as in (A.14) and (A.13), we get
\[
I_2 = -\frac{4^{L+2}}{L^2} \int_0^\infty du \frac{u^{\frac{1}{2}}}{u/L} e^{-u/L} e^{-u/L} \int_0^{e^{\pi/L}} dy \frac{y^{l-\frac{1}{2}}}{1-y} \frac{e^{2i\varphi} - e^{-2i\varphi} - 2i\varphi}{e^{2i\varphi} - e^{-2i\varphi}}.
\] (B.10)
We take the limit $L \to \infty$ in this expression and change the variable $y$ back to $\varphi$ to get
\[
I_2 \simeq -\frac{\sqrt{\pi}(q; q)_\infty^2 4^{L+2}}{2(a; q)_\infty^2 L^2} \int_0^\pi d\varphi \left[ \frac{1}{2} \right] (1 + \cos \varphi)^l \frac{(q e^{i\varphi} - q e^{-i\varphi} ; q)_\infty}{(q e^{i\varphi} - q e^{-i\varphi} ; q)_\infty}.
\] (B.11)
We can consider the limit $l \to \infty$ by using the steepest descent method. We have
\[
I_2 \simeq -\frac{2 \cdot 4^{L+1} \pi}{(a, b ; q)_\infty^2 L^2 l^2}.
\] (B.12)
- The case $1 < a < q^{-1}$ and $b < 1$
In this case, the integral $I_2$ is given by $I_2 = I_2^{(0)} + I_2^{(a, 0)}$. The main contribution comes from the $k = 0$ term in $I_2^{(a, 0)}$.
\[
I_2 \simeq -(ab, a^{-2} ; q)_\infty (q; q)_\infty^2 [(1+a)(1+a^{-1})]^{L-l} \\
\times \int_{C_0} \frac{dz_2}{2\pi i z_2} \frac{(z_2^2, z_2^{-2} ; q)_\infty [(1+z_2)(1+z_2^{-1})]^{l-1}}{(a^{-1} z_2, a^{-1} z_2^{-1}, a q z_2, a q z_2^{-1}, b z_2, b z_2^{-1}, q)_\infty}.
\] (B.13)
Taking the limit $l \to \infty$ by using the steepest descent method, we get
\[
I_2 \simeq -\frac{4^l (ab, a^{-2} ; q)_\infty (q; q)_\infty^2 [(1+a)(1+a^{-1})]^{L-l}}{a\sqrt{\pi} l^2 (a^{-1}, qa, b)_\infty^2}.
\] (B.14)
- The case $a > q^{-1}$ and $aq > b$
In general, the integral for this case has the expression (B.3). The main contribution comes from the $k = 0, m = 1$ term in $I_2^{(a, a)}$. We have
\[
I_2 \simeq -(1-ab)(1-a^{-2}q^{-2})(a^{-2} ; q)_\infty [(1+a)(1+a^{-1})]^{L-l}[(1+aq)(1+a^{-1}q^{-1})]^{l-1} \\
\times (a^{-1} b q^{-1}, q)_\infty.
\] (B.15)
- The case $a > b > aq$ and $b > 1$
The main contribution for this case comes from the $k = 0, m = 1$ term in $I_2^{(a, b)}$. We have
\[
I_2 \simeq -(1-ab)(q, a^{-2}, b^{-2} ; q)_\infty [(1+a)(1+a^{-1})]^{L-l}[(1+b)(1+b^{-1})]^{l-1} \\
\times a(a^{-1} b, a^{-1} b^{-1}, ab^{-1} q ; q)_\infty.
\] (B.16)
The case $b > 1$ and $b > a$

In this case, the normalization $Z_L$ behaves as $[(1 + b)(1 + b^{-1})]^L$. All contributions to $I_2$ can be neglected compared to $Z_L$. Hence we do not compute the asymptotic expression explicitly for this case.
Figure Captions

Fig. 1 : One-dimensional partially asymmetric simple exclusion process with open boundaries. Particles have hard-core exclusion interaction and tend to hop to the right (resp. left) nearest neighboring site with rate $p_R$ (resp. $p_L$). There are also particle injection (resp. ejection) at the left (resp. right) edge.

Fig. 2 : Space-time diagram of the ASEP from Monte-Carlo simulations. The horizontal axis represents the site number $j$ whereas the vertical axis represents time. The existence of particle is represented as a black point. The lattice length is taken to be $L = 200$. The bulk hopping rates are taken to be $p_R = 1$, $p_L = 0$. After some transient time, the system practically goes to a steady state. The steady state depends crucially on the values of the boundary parameters.

Fig. 3 : Average density profile of the ASEP from Monte-Carlo simulations. The horizontal axis represents the site number $j$ whereas the vertical axis represents the average density. The lattice length is taken to be $L = 200$. The bulk hopping rates are taken to be $p_R = 1$, $p_L = 0$.

Fig. 4 : The phase diagram of the current. Regions $A$, $B$ and $C$ are called the low-density phase, the high-density phase and the maximal current phase respectively.

Fig. 5 : The phase diagram of the correlation length in the $\tilde{\alpha}-\tilde{\beta}$ plane for the partially asymmetric case. The low-density phase (resp. high-density phase) is divided into three phases, $A_1, A_2$ and $A_3$ (resp. $B_1, B_2$ and $B_3$).

Fig. 6 : The correlation length $\xi$ for the case $p_R = 1, p_L = 0.9, \tilde{\beta} = 1$. The solid line is the theoretical prediction given by $\xi = 1/\ln\frac{q}{|\tilde{\beta}+(1-\tilde{\beta})q|^2}$ whereas the black dots are the simulation data.
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\( \bullet \) : Particle
(A) low density phase
$p_R = 1.0, p_L = 0.0$
$\alpha = 0.2, \beta = 1.0$

(C) maximal current phase
$p_R = 1.0, p_L = 0.0$
$\alpha = 1.0, \beta = 1.0$

coexistence line
$p_R = 1.0, p_L = 0.0$
$\alpha = 0.2, \beta = 0.2$

(B) high density phase
$p_R = 1.0, p_L = 0.0$
$\alpha = 1.0, \beta = 0.2$
\( \langle p_j \rangle \)

(A) low density phase
\[ p_R = 1.0, \ p_L = 0.0 \]
\[ \alpha = 0.2, \ \beta = 0.25 \]

\( j \)

(C) maximal current phase
\[ p_R = 1.0, \ p_L = 0.0 \]
\[ \alpha = 1.0, \ \beta = 1.0 \]

\( \dot{j} \)

coexistence line
\[ p_R = 1.0, \ p_L = 0.0 \]
\[ \alpha = 0.2, \ \beta = 0.2 \]

\( j \)

(B) high density phase
\[ p_R = 1.0, \ p_L = 0.0 \]
\[ \alpha = 0.25, \ \beta = 0.2 \]
