DUAL EQUIVALENCE GRAPHS I: A COMBINATORIAL PROOF OF LL T AND MACDONALD POSITIVITY

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Abstract. We make a systematic study of a new combinatorial construction called a dual equivalence graph. We axiomatize these graphs and prove that their generating functions are symmetric and Schur positive. By constructing a graph on ribbon tableaux which we transform into a dual equivalence graph, we give a combinatorial proof of the symmetry and Schur positivity of the ribbon tableaux generating functions introduced by Lascoux, Leclerc and Thibon. Using Haglund’s formula for the transformed Macdonald polynomials, this also gives a combinatorial formula for the Schur expansion of Macdonald polynomials.

1. Introduction

The immediate purpose of this paper is to establish a combinatorial formula for the Schur coefficients of LL T polynomials which, as a corollary, yields a combinatorial formula for the Schur coefficients of Macdonald polynomials. Our real purpose, however, is not only to obtain these results, but also to introduce a new combinatorial construction, called a dual equivalence graph, by which one can establish the symmetry and Schur positivity of functions expressed in terms of monomials.

The transformed Macdonald polynomials, $\tilde{H}_\mu(x; q, t)$, a transformation of the polynomials introduced by Macdonald [Mac88] in 1988, are defined to be the unique symmetric functions satisfying certain triangularity and orthogonality conditions. The existence of functions satisfying these conditions is a theorem, from which it follows that the $\tilde{H}_\mu(x; q, t)$ form a basis for symmetric functions in two additional parameters. The Kostka-Macdonald coefficients, denoted $\tilde{K}_{\lambda, \mu}(q, t)$, give the change of basis from Macdonald polynomials to Schur functions, namely,

$$\tilde{H}_\mu(x; q, t) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}(q, t) s_\lambda(x).$$

A priori, $\tilde{K}_{\lambda, \mu}(q, t)$ is a rational function in $q$ and $t$ with rational coefficients, i.e. $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{Q}(q, t)$.

The Macdonald Positivity Theorem [Hai01], first conjectured by Macdonald in 1988 [Mac88], states that $\tilde{K}_{\lambda, \mu}(q, t)$ is in fact a polynomial in $q$ and $t$ with nonnegative integer coefficients, i.e. $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$. Garsia and Haiman [GH93] conjectured that the transformed Macdonald polynomials $\tilde{H}_\mu(x; q, t)$ could be realized as the bi-graded characters of certain modules for the diagonal action of the symmetric group $S_n$ on two sets of variables. Once resolved, this conjecture gives a representation theoretic interpretation of Kostka-Macdonald coefficients as the graded multiplicity of an irreducible representation in the Garsia-Haiman module, and hence $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$. Following an idea outlined by Procesi, Haiman [Hai01] proved this conjecture by analyzing the algebraic geometry of the isospectral Hilbert scheme of $n$ points in the plane, consequently establishing Macdonald Positivity. This proof, however, is purely geometric and does not offer a combinatorial interpretation for $\tilde{K}_{\lambda, \mu}(q, t)$.

The LLT polynomial $\tilde{G}_\mu^{(k)}(x; q)$, originally defined by Lascoux, Leclerc and Thibon [LLT97] in 1997, is the $q$-generating function of $k$-ribbon tableaux of shape $\mu$ weighted by a statistic called cospin. By the Stanton-White correspondence [SW85], $k$-ribbon tableaux are in bijection with certain $k$-tuples of tableaux, from which it follows that LLT polynomials are $q$-analogs of products of Schur functions. More recently, an alternative definition of $\tilde{G}_\mu^{(k)}(x; q)$ as the $q$-generating function of $k$-tuples of semi-standard tableaux of shapes $\mu = (\mu_0, \ldots, \mu_{k-1})$ weighted by a statistic called $k$-inversions is given in [HHL+05b].

2000 Mathematics Subject Classification. Primary 05E10; Secondary 05A30, 33D52.

Key words and phrases. LL T polynomials, Macdonald polynomials, dual equivalence graphs, quasisymmetric functions.

Work supported in part by an NSF Graduate Research Fellowship, NSF Grant DMS-0301072, and NSF Postdoctoral Fellowship DMS-0703567.
Using Fock space representations of quantum affine Lie algebras constructed by Kashiwara, Miwa and Stern [KMS95], Lascoux, Leclerc and Thibon [LLT97] proved that \( \tilde{G}^{(k)}(x;q) \) is a symmetric function. Thus we may define the Schur coefficients, \( \tilde{K}^{(k)}(q) \), by

\[
\tilde{G}^{(k)}(x;q) = \sum_{\lambda} \tilde{K}^{(k)}(q)s_{\lambda}(x).
\]

Using Kazhdan-Lusztig theory, Leclerc and Thibon [LT00] proved that \( \tilde{K}^{(k)}(q) \in \mathbb{N}[q] \) for straight shapes \( \mu \). This has recently been extended by Grojnowski and Haiman [GH07] to skew shapes. Again, the proof of positivity is by a geometric argument, and as such offers no combinatorial description for \( \tilde{K}^{(k)}(q) \).

In 2004, Haglund [Hag04] conjectured a combinatorial formula for the monomial expansion of \( \tilde{H}_{\mu}(x;q,t) \). This formula was proven by Haglund, Haiman and Loehr [HHL05a] using an elegant combinatorial argument, but this does not prove that \( \tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t] \) since monomial symmetric functions are not Schur positive. Combining Theorem 2.3, Proposition 3.4 and equation (23) from [HHL05a], Haglund’s formula expresses \( \tilde{H}_{\mu}(x;q,t) \) as a positive sum of LLT polynomials \( \tilde{G}^{(k)}_{\nu}(x;q) \) for certain skew shapes \( \nu \) depending upon \( \mu \). Therefore the LLT positivity result of Grojnowski and Haiman [GH07] provides another proof of Macdonald positivity, though this proof is still non-combinatorial. One of the main purposes of this paper is to give a combinatorial proof of LLT positivity for arbitrary shapes, thereby completing the combinatorial proof of Macdonald positivity from Haglund’s formula.

Combinatorial formulas for \( \tilde{K}^{(k)}_{\lambda,\mu}(q) \) and \( \tilde{K}_{\lambda,\mu}(q,t) \) have been found for certain special cases. In 1995, Carré and Leclerc [CL95] gave a combinatorial interpretation of \( \tilde{K}^{(2)}_{\lambda,\mu}(q) \) in their study of 2-ribbon tableaux, though a complete proof of their result wasn’t found until 2005 by van Leeuwen [vL05] using the theory of crystal graphs. Also in 1995, Fishel [Fis95] gave the first combinatorial interpretation for \( \tilde{K}_{\lambda,\mu}(q,t) \) when \( \mu \) is a partition with 2 columns using rigged configurations. Other techniques have also led to formulas for the 2 column Macdonald polynomials [Zab99, LM03, Hag04], but in all cases, finding extensions for these formulas has proven elusive.

Following a suggestion of Haiman, we consider the dual equivalence relation on standard Young tableaux defined in [Hai92]. From this relation, Haiman suggested defining an edge-colored graph on standard tableaux and investigating how this graph may be related to the crystal graph on semi-standard tableaux. The result of this idea is a new combinatorial method for establishing the Schur positivity of a function expressed in terms of monomials. In this paper, this method is applied to LLT polynomials to obtain a combinatorial formula for \( \tilde{K}^{(k)}_{\lambda,\mu}(q) \), and so, too, for \( \tilde{K}_{\lambda,\mu}(q,t) \) by [HHL05a].

This paper is organized as follows. In Section 2, we review symmetric functions and the associated tableaux combinatorics. The theory of dual equivalence graphs is developed in Section 3, beginning in Section 3.1 by reviewing dual equivalence and constructing the graphs suggested by Haiman. In Section 3.2, we define a general dual equivalence graph and state the structure theorem which says that every dual equivalence graph is isomorphic to one of the graphs from Section 3.1. On the symmetric function level, this shows that the generating function of a dual equivalence graph is symmetric and Schur positive and gives a combinatorial interpretation for the Schur coefficients. The proof of the theorem is left to Section 3.3.

The remainder of this paper contains the first application of this theory, beginning in Section 4 with the construction a graph on \( k \)-tuples of tableaux. We present a reformulation of LLT polynomials in Section 4.1, and use it to describe the vertices and signatures of the graph. The edges are constructed in Section 4.2 using a natural analog of dual equivalence. While these graph are not, in general, dual equivalence graphs, we show in Section 5 that they can be transformed into dual equivalence graphs in a natural way that preserves the generating function. In particular, connected components of these graphs are Schur positive.

Acknowledgements

The author is grateful to Mark Haiman for inspiring and helping to develop many of the ideas contained in this paper and in its precursor [Ass07]. The author also thanks A. Garsia, J. Haglund and G. Musiker for helpful conversations about the constructions introduced here and for helping to implement the algorithms described in Section 5 in Maple.
2. Preliminaries

2.1. Partitions and tableaux. We represent an integer partition \( \lambda \) by the decreasing sequence of its (nonzero) parts

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0.
\]

We denote the size of \( \lambda \) by \( |\lambda| = \sum \lambda_i \) and the length of \( \lambda \) by \( l(\lambda) = \max\{i : \lambda_i > 0\} \). If \( |\lambda| = n \), we say that \( \lambda \) is a partition of \( n \). Let \( \geq \) denote the dominance partial ordering on partitions of \( n \), defined by

\[
\lambda \geq \mu \iff \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \forall i.
\]

A composition \( \pi \) is a finite sequence of non-negative integers \( \pi = (\pi_1, \pi_2, \ldots, \pi_m), \pi_i \geq 0 \).

The Young diagram of a partition \( \lambda \) is the set of points \((i, j)\) in the \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) lattice quadrant such that \( 1 \leq i \leq \lambda_j \). We draw the diagram so that each point \((i, j)\) is represented by the unit cell southwest of the point; see Figure 1. Abusing notation, we will write \( \lambda \) for both the partition and its diagram.

![Figure 1](image1)

**Figure 1.** The Young diagram for \((5, 4, 4, 1)\) and the skew diagram for \((5, 4, 4, 1)/(3, 2, 2)\).

For partitions \( \lambda, \mu \), we write \( \mu \subset \lambda \) whenever the diagram of \( \mu \) is contained within the diagram of \( \lambda \); equivalently \( \mu_i \leq \lambda_i \) for all \( i \). In this case, we define the skew diagram \( \lambda/\mu \) to be the set theoretic difference \( \lambda - \mu \), e.g. see Figure 1. For our purposes, we depart from the norm by not identifying skew shapes that are translates of one another. A connected skew diagram is one where exactly one cell has no cell immediately north or west of it, and exactly one cell has no cell immediately south or east of it. A ribbon, also called a rim hook, is a connected skew diagram containing no \( 2 \times 2 \) block.

A filling of a (skew) diagram \( \lambda \) is a map \( S: \lambda \rightarrow \mathbb{Z}_+ \). A semi-standard Young tableau is a filling which is weakly increasing along each row and strictly increasing along each column. A semi-standard Young tableau is standard if it is a bijection from \( \lambda \) to \([n]\), where \([n] = \{1, 2, \ldots, n\}\). For \( \lambda \) a diagram of size \( n \), define

\[
\text{SSYT}(\lambda) = \{ \text{semi-standard tableaux } T: \lambda \rightarrow \mathbb{Z}_+ \},
\]

\[
\text{SYT}(\lambda) = \{ \text{standard tableaux } T: \lambda \rightarrow [n] \}.
\]

For \( T \in \text{SSYT}(\lambda) \), we say that \( T \) has shape \( \lambda \). If \( T \) contains entries \( 1^{\pi_1}, 2^{\pi_2}, \ldots \) for some composition \( \pi \), then we say \( T \) has weight \( \pi \).

![Figure 2](image2)

**Figure 2.** The standard Young tableaux of shape \((3, 2)\) with their content reading words.

The content of a cell of a diagram indexes the diagonal on which it occurs, i.e. \( c(x) = i - j \) when the cell \( x \) lies in position \((i, j)\) in \( \mathbb{Z}_+ \times \mathbb{Z}_+ \). The content reading word of a semi-standard tableaux is obtained by reading the entries in increasing order of content, going southwest to northeast along each diagonal (on which the content is constant). For examples, see Figure 2.

2.2. Symmetric functions. We have the familiar integral bases for \( \Lambda \), the ring of symmetric functions, from [Mac95]: the monomial symmetric functions \( m_\lambda \), the elementary symmetric functions \( e_\lambda \), the complete homogeneous symmetric functions \( h_\lambda \) and, most importantly, the Schur functions, \( s_\lambda \), which may be defined in several ways. For the purposes of this paper, we take the tableau approach:

\[
s_\lambda(x) = \sum_{T \in \text{SSYT}(\lambda)} x^T,
\]
where \( x^T \) is the monomial \( x_1^{\pi_1} x_2^{\pi_2} \cdots \) when \( T \) has weight \( \pi \). This formula also defines the skew Schur functions, \( s_{\lambda/\mu} \), by taking the sum over semi-standard tableaux of shape \( \lambda/\mu \).

The Kostka numbers, \( K_{\lambda,\mu} \), give the change of basis from the complete homogeneous symmetric functions to the Schur functions and, dually, the change of basis from Schur functions to monomial symmetric functions, i.e.,

\[
h_{\mu} = \sum_{\lambda} K_{\lambda,\mu} s_{\lambda}; \quad s_{\lambda} = \sum_{\mu} K_{\lambda,\mu} m_{\mu}.\]

In particular, \( K_{\lambda,\mu} \) is the number of semi-standard Young tableaux of shape \( \lambda \) and weight \( \mu \). For example, \( K_{(3,2),(1^5)} = 5 \) corresponding to the five standard Young tableaux of shape \((3,2)\) in Figure 2. Since the Schur functions are the characters of the irreducible representations of \( \text{GL}_n \), the Kostka numbers also give weight multiplicities for \( \text{GL}_n \) modules. Throughout this paper, we are interested in certain one- and two-parameter generalizations of the Kostka numbers.

As we shall see in Section 3, it will often be useful to express a function in terms of Gessel’s fundamental quasi-symmetric functions \([\text{Ges}84]\) rather than monomials. For \( \sigma \in \{\pm 1\}^{n-1} \), the fundamental quasi-symmetric function \( Q_{\sigma}(x) \) is defined by

\[
Q_{\sigma}(x) = \sum_{i_1 \leq \cdots \leq i_n \text{ s.t. } \pi_1 + \cdots + \pi_i = +1} x_{i_1} \cdots x_{i_n},
\]

We have indexed quasi-symmetric functions by sequences of \( +1 \)'s and \( -1 \)'s, though by setting \( D(\sigma) = \{i | \sigma_i = -1\}, \) we may change the indexing to subsets of \([n-1]\). Similarly, letting \( \pi(\sigma) \) be the composition defined by setting \( \pi_1 + \cdots + \pi_i \) to be the position of the \( i \)th \(-1\), where we regard \( \sigma_n = -1 \) as the final \(-1\), we may change the indexing to compositions of \( n \).

To connect quasi-symmetric functions with Schur functions, for \( T \) a standard tableau on \([n]\) with content reading word \( w_T \), define the descent signature \( \sigma(T) \in \{\pm 1\}^{n-1} \) by

\[
\sigma(T)_i = \begin{cases} +1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w_T, \\ -1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w_T. \end{cases}
\]

For example, the descent signatures for the tableaux in Figure 2 are \( + - + +, - - + +, + - + - + - + - + + - + - + + - + - + + + - + - + - , \) from left to right. Note that if we replace the content reading word with either the row or column reading word, the resulting sequence in (2.4) remains unchanged.

**Proposition 2.1** ([\text{Ges}84]). The Schur function \( s_T \) is expressed in terms of quasi-symmetric functions by

\[
s_T(x) = \sum_{\sigma(T) \in \text{SSYT}(\lambda)} Q_{\sigma(T)}(x).
\]

Comparing (2.2) with (2.5), using quasi-symmetric functions instead of monomials allows us to work with standard tableaux rather than semi-standard tableaux. One nice advantage of this formula is that unlike (2.2), the right hand side of (2.5) is finite. Continuing with the example in Figure 2, we have

\[
s_{(3,2)}(x) = Q_{+ + +}(x) + Q_{- + +}(x) + Q_{- - +}(x) + Q_{+ - +}(x) + Q_{+ + -}(x).
\]

### 2.3. LLT polynomials

Lascoux, Leclerc and Thibon [\text{LLT97}] originally defined \( \widetilde{G}^{(k)}_{\mu}(x; q) \) to be the \( q \)-generating function of \( k \)-ribbon tableaux of shape \( \mu \) weighted by cospin. Below we give an alternative definition of \( \widetilde{G}^{(k)}_{\mu}(x; q) \) as the \( q \)-generating function of \( k \)-tuples of semi-standard tableaux of shapes \( \mu = (\mu^{(0)}, \ldots, \mu^{(k-1)}) \) weighted by \( k \)-inversions first presented in [\text{HHL}^{+05b}]. For a detailed account of the equivalence of these definitions (actually \( q^a \widetilde{G}^{(k)}_{\mu}(x; q) = \widetilde{G}^{(k)}_{\mu}(x; q) \) for a constant \( a \geq 0 \) depending on \( \mu \)), see [\text{HHL}^{+05b}, \text{Ass07}].

Extending prior notation, define

\[
\text{SSYT}_k(\lambda) = \{\text{semi-standard } k \text{-tuples of tableaux of shapes } (\lambda^{(0)}, \ldots, \lambda^{(k-1)})\}, \quad \text{SYT}_k(\lambda) = \{\text{standard } k \text{-tuples of tableaux of shapes } (\lambda^{(0)}, \ldots, \lambda^{(k-1)})\}.
\]

As with tableaux, if \( T = (T^{(0)}, \ldots, T^{(k-1)}) \in \text{SSYT}_k(\lambda) \) has entries \( 1^{\pi_1}, 2^{\pi_2}, \ldots \), then we say that \( T \) has shape \( \lambda \) and weight \( \pi \). Note that a standard \( k \)-tuple of tableaux has weight \( (1^n) \), e.g. see Figure 3, and this...
is not the same as a $k$-tuple of standard tableaux, which has weight $(1^{m_1}, 2^{m_2}, \ldots)$ where $m_i$ is the number of shapes of size at least $i$.

For a $k$-tuple of (skew) shapes $(\lambda^{(0)}, \ldots, \lambda^{(k-1)})$, define the shifted content of a cell $x$ by

$$\tilde{c}(x) = k \cdot c(x) + i$$

when $x$ is a cell of $\lambda^{(i)}$, where $c(x)$ is the usual content of $x$ regarded as a cell of $\lambda^{(i)}$. For $T \in \text{SSYT}_k$, let $T(x)$ denote the entry of the cell $x$ in $T$. Define the set of $k$-inversions of $T$ by

$$\text{Inv}_k(T) = \{(x, y) \mid k > \tilde{c}(y) - \tilde{c}(x) > 0 \text{ and } T(x) > T(y)\}.$$

Then the $k$-inversion number of $T$ is given by

$$\text{inv}_k(T) = |\text{Inv}_k(T)|.$$

For example, suppose $T$ is the 4-tuple of tableaux in Figure 3. Since $T$ is standard, let us abuse notation by representing a cell of $T$ by the entry it contains. Then the set of 4-inversions is

$$\text{Inv}_4(T) = \left\{(9, 7), (9, 8), (7, 3), (8, 3), (8, 2), (3, 2), (3, 1), (2, 1), (11, 1), (11, 5), (6, 4), (12, 4), (12, 10)\right\},$$

and so $\text{inv}_4(T) = 13$.

By [HHL+05b], the LLT polynomial $\widetilde{G}^{(k)}_{\mu}(x; q)$ is given by

$$\widetilde{G}^{(k)}_{\mu}(x; q) = \sum_{T \in \text{SYT}_k(\mu)} q^{\text{inv}_k(T)} x^T,$$

where $x^T$ is the monomial $x_{i_1}^{\pi_1}x_{i_2}^{\pi_2} \cdots$ when $T$ has weight $\pi$.

Notice that when $q = 1$, (2.9) reduces to a product of Schur functions:

$$\sum_{T \in \text{SSYT}_k(\lambda)} x^T = \prod_{i=0}^{k-1} \sum_{T^{(i)} \in \text{SSYT}(\lambda^{(i)})} x^{T^{(i)}} = \prod_{i=0}^{k-1} s_{\lambda^{(i)}}(x).$$

Define the content reading word of a $k$-tuple of tableaux to be the word obtained by reading entries in increasing order of shifted content and reading diagonals southwest to northeast. For the example in Figure 3, the content reading word is $(9, 7, 8, 3, 2, 11, 1, 5, 6, 12, 4, 10)$.

For $T$ a standard $k$-tuple of tableaux, define $\sigma(T)$ analogously to (2.4) using the content reading word. Expressed in terms of quasi-symmetric functions, (2.9) becomes

$$\widetilde{G}^{(k)}_{\mu}(x; q) = \sum_{T \in \text{SYT}_k(\mu)} q^{\text{inv}_k(T)} Q_{\sigma(T)}(x).$$

One of the main goals of this paper is to understand the Schur coefficients of $\widetilde{G}^{(k)}_{\mu}(x; q)$ defined by

$$\widetilde{G}^{(k)}_{\mu}(x; q) = \sum_{\lambda} \tilde{K}^{(k)}_{\lambda, \mu}(q)s_{\lambda}(x).$$

In particular, we will show that $\tilde{K}^{(k)}_{\lambda, \mu}(q)$ is a polynomial in $q$ with nonnegative integer coefficients.
2.4. Macdonald polynomials. The transformed Macdonald polynomials \( \tilde{H}_\mu(x; q, t) \) were originally defined by Macdonald [Mac88] to be the unique symmetric functions satisfying certain orthogonality and triangularity conditions. Haglund’s monomial expansion for Macdonald polynomials [Hag04, HHL05a] gives an alternative combinatorial definition of \( \tilde{H}_\mu(x; q, t) \) as the \( q, t \)-generating functions for fillings of the diagram of \( \mu \), e.g. see Figure 4. Since the proof of the equivalence of these two definitions is purely combinatorial [HHL05a], we will use the latter characterization.

For a cell \( x \) in the diagram of \( \lambda \), define the **arm of** \( x \) to be the set of cells east of \( x \), and the **leg of** \( x \) to be the set of cells north of \( x \). Denote the sizes of the arm and leg of \( x \) by \( a(x) \) and \( l(x) \), respectively. For example, letting \( x \) denote the cell with entry 3 in the filling in Figure 4, the arm of \( x \) consists of the cells with entries 4 and 10 and the leg of \( x \) consists of the cell with entry 14, and so we have \( a(x) = 2 \) and \( l(x) = 1 \).

\[
\begin{array}{cccccc}
5 & 1 & 1 & 4 & 9 & 2 \\
6 & 3 & 4 & 10 & \\
8 & 1 & 13 & 7 & 12 \\
\end{array}
\]

**Figure 4.** A standard filling of shape \((5, 4, 4, 1)\).

Let \( S \) be a filling of a partition \( \lambda \). A **descent** of \( S \) is a cell \( c \) of \( \lambda \), not in the first row, such that the entry in \( c \) is greater than the entry in the cell immediately south of \( c \). Denote by \( \text{Des}(S) \) the set of all descents of \( S \), i.e.

\[
\text{Des}(S) = \{(i,j) \in \lambda \mid j > 1 \text{ and } S(i,j) > S(i,j-1)\}.
\]

Define the **major index** of \( S \), denoted \( \text{maj}(S) \), by

\[
\text{maj}(S) \overset{\text{def}}{=} |\text{Des}(S)| + \sum_{c \in \text{Des}(S)} l(c).
\]

Note that for \( \mu = (1^n) \), this gives the usual major index for the reading word of the filling.

For example, let \( S \) be the filling in Figure 4. As before, let us abuse notation by representing a cell of \( S \) by the entry which it contains. Then the descents of \( S \) are given by

\[
\text{Des}(S) = \{11, 14, 9, 3, 10\},
\]

and so the major index of \( S \) is \( \text{maj}(S) = 5 + (1 + 0 + 0 + 1 + 1) = 8 \).

An ordered pair of cells \((c,d)\) is called **attacking** if \( c \) and \( d \) lie in the same row with \( c \) to the west of \( d \), or if \( c \) is in the row immediately north of \( d \) and \( c \) lies strictly east of \( d \). An **inversion pair** of \( S \) is an attacking pair \((c,d)\) such that the entry in \( c \) is greater than the entry in \( d \). Denote by \( \text{Inv}(S) \) the set of inversion pairs of \( S \), i.e.

\[
\text{Inv}(S) = \left\{ (i,j), (g,h) \in \lambda \mid \begin{array}{l}
j = h \text{ and } i < g \text{ or } j = h + 1 \text{ and } g < i,
\text{ and } S(i,j) > S(g,h)
\end{array} \right\}.
\]

Define the **inversion number** of \( S \), denoted \( \text{inv}(S) \), by

\[
\text{inv}(S) \overset{\text{def}}{=} |\text{Inv}(S)| - \sum_{c \in \text{Des}(S)} a(c).
\]

Note that for \( \mu = (n) \), this gives the usual inversion number for the reading word of the filling.

For our running example, the inversion pairs of \( S \) are given by

\[
\text{Inv}(S) = \left\{ (11,9), (14,2), (9,6), (6,4), (10,1), (13,7), (11,2), (14,6), (9,3), (4,1), (8,1), (13,12) \right\},
\]

and so the inversion number of \( S \) is \( \text{inv}(S) = 17 - (3 + 2 + 1 + 2 + 0) = 9 \).
Remark 2.2. If \( c \in \text{Des}(S) \), say with \( d \) the cell of \( S \) immediately south of \( c \), then for every cell \( e \) of the arm of \( c \), the entry in \( e \) is either bigger than the entry in \( d \) or smaller than the entry in \( c \) (or both). In the former case, \((e, d)\) will form an inversion pair, and in the latter case, \((c, e)\) will form an inversion pair. Thus every triple of cells \((c, e, d)\) with \( d \) immediately south of \( c \) and \( e \) in the arm of \( c \) contributes at least one inversion to \( \text{inv}(S) \), and so \( \text{inv}(S) \) is a non-negative integer.

By [HHL05a], the transformed Macdonald polynomial \( \tilde{H}_\mu(x; q, t) \) is given by

\[
(2.16) \quad \tilde{H}_\mu(x; q, t) = \sum_{S : \mu \to [n]} q^{\text{inv}(S)} t^{\text{maj}(S)} x^S = \sum_{S : \mu \to [n]} q^{\text{inv}(S)} t^{\text{maj}(S)} Q_{\sigma(S)},
\]

where \( \sigma(S) \) is defined analogously to (2.4) using the row reading word of a standard filling \( S \). For example, the row reading word for the standard filling in Figure 4 is \((5, 11, 14, 9, 2, 6, 3, 4, 10, 8, 1, 13, 7, 12) \). Again, our main objective is to understand the Schur coefficients defined by

\[
(2.17) \quad \tilde{H}_\mu(x; q, t) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}(q, t) s_\lambda(x).
\]

In this paper, we will give a combinatorial proof that \( \tilde{K}_{\lambda, \mu}(q, t) \) is a polynomial in \( q \) and \( t \) with nonnegative integer coefficients. This proof follows as a corollary to our proof for \( \tilde{K}^{(k)}_{\lambda, \mu}(q) \) as we now explain.

The expression in (2.16) is related to LLT polynomials as follows. Let \( D \) be a possible descent set for \( \mu \), i.e. \( D \) is a collection of \( \mu/(\mu_1) \). For \( i = 1, \ldots, \mu_1 \), let \( \mu_D^{(i−1)} \) be the ribbon obtained from the \( i \)-th column of \( \mu \) by putting the cell \((i, j)\) immediately south of \((i, j+1)\) if \((i, j+1) \in D \) and immediately east of \((i, j+1)\) otherwise. Translate each \( \mu_D^{(i)} \) so that the southeastern most cell has shifted content \( n+i \) for some (any) fixed integer \( n \). Then each filling \( S \) of shape \( \mu \) with \( \text{Des}(S) = D \) may be regarded as a semi-standard \( \mu_1 \)-tuple of tableaux of shape \( \mu_D \), denoted \( S \). For example, the filling \( S \) of shape \((5, 4, 4, 1) \) in Figure 4 corresponds to the 5-tuple of ribbons of shapes \((3, 3, 3, 2)/(3, 3, 1), (1, 1, 1), (2, 2, 1)/(2), (2, 2, 2)/(2, 1), (1) \); see Figure 5.

![Figure 5](image)

Figure 5. A standard filling of shape \((5, 4, 4, 1)\) transformed into a 5-tuple of ribbons of shapes \((3, 3, 3, 2)/(3, 3, 1), (1, 1, 1), (2, 2, 1)/(2), (2, 2, 2)/(2, 1), (1)\).

For this correspondence, it is crucial that we do not identify skew shapes that are translates of one another. For example, the row reading word of the filling in Figure 4 is precisely the content reading word of 5-tuple in Figure 5, but this is not the case if the first tableau is instead considered to have shape \((3, 2)/(1)\). Furthermore, the inversion pairs of \( S \) as defined in (2.14) correspond precisely with the \( \mu_1 \)-inversions of \( S \) as defined in (2.7). Since the major index statistic depends only on the descent set, for a given descent set \( D \) we may define \( \text{maj}(D) \) by \( \text{maj}(D) = \text{maj}(S) \) for any filling \( S \) with \( \text{Des}(S) = D \). Similarly, define \( a(D) = \sum_{c \in D} a(c) \). Then we may rewrite (2.16) in terms of LLT polynomials as

\[
(2.18) \quad \tilde{H}_\mu(x; q, t) = \sum_{D \subseteq \mu/(\mu_1)} q^{-a(D)} t^{\text{maj}(D)} \tilde{G}_{\mu_D}^{(\mu_1)}(x; q).
\]

Note that each term of \( \tilde{G}_{\mu_D}^{(\mu_1)}(x; q) \) contains a factor of \( q^a \) for some \( a \geq a(D) \) (in fact, this is the same constant mentioned in Section 2.3). In terms of Schur expansions, (2.18) may also be expressed as

\[
(2.19) \quad \tilde{K}_{\lambda, \mu}(q, t) = \sum_{D \subseteq \mu/(\mu_1)} q^{-a(D)} t^{\text{maj}(D)} \tilde{K}_{\lambda, \mu_D}^{(\mu_1)}(q).
\]

By the previous remark, proving \( \tilde{K}_{\lambda, \mu_D}^{(\mu_1)}(q) \in \mathbb{N}[q] \) consequently proves \( \tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t] \).
3. Dual equivalence graphs

3.1. The standard dual equivalence graph. Dual equivalence was first defined by Haiman [Hai92] as a relation on tableaux which is “dual” to jeu de taquin equivalence under the Schensted correspondence. We use this relation to construct a graph with vertices given by standard Young tableaux and connected components indexed by partitions. Using quasi-symmetric functions, we define the generating function on the vertices of these graphs, thereby providing the connection with symmetric functions.

We begin by recalling the definition of dual equivalence on permutations regarded as words on $[n]$, which we extend to standard Young tableaux via the content reading word.

**Definition 3.1 ([Hai92]).** An elementary dual equivalence on three consecutive letters $i-1, i, i+1$ of a permutation is given by switching the outer two letters whenever the middle letter is not $i$:

$$
\ldots i \ldots i \mp 1 \ldots i \mp 1 \ldots i \pm 1 \ldots i \ldots
$$

Two permutations are dual equivalent if they differ by some sequence of elementary dual equivalences. Two standard tableaux of the same shape are dual equivalent if their content reading words are.

Construct an edge-colored graph on standard tableaux of partition shape from the dual equivalence relation in the following way. Whenever two standard tableaux $T, U$ have content reading words that differ by an elementary dual equivalence for $i-1, i, i+1$, connect $T$ and $U$ with an edge colored by $i$. Recall the descent signature of a standard tableau $T$ defined in (2.4):

$$
\sigma(T)_i = \begin{cases} 
+1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w_T \\
-1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w_T
\end{cases}
$$

We also associate to each tableau $T$ the signature $\sigma(T)$. Several examples are given in Figure 6, and several more in Appendix A.

![Figure 6](image-url)

**Figure 6.** The standard dual equivalence graphs $G_5, G_{4,1}, G_{3,2}$ and $G_{3,1,1}$.

The connected components of the graph so constructed are the dual equivalence classes of standard tableaux. Let $G_{\lambda}$ denote the subgraph on tableaux of shape $\lambda$. The following proposition tells us that the $G_{\lambda}$ exactly give the connected components of the graph.
**Proposition 3.2** ([Hai92]). Two standard tableaux on partition shapes are dual equivalent if and only if they have the same shape.

Define the generating function associated to $G_\lambda$ by

$$
\sum_{v \in V(G_\lambda)} Q_{\sigma(v)}(x) = s_\lambda(x).
$$

By Proposition 2.1, this is Gessel’s quasi-symmetric function expansion for a Schur function. In particular, the generating function of any vertex-signed graph whose connected components are isomorphic to graphs $G_\lambda$ is automatically Schur positive. This observation is the main idea behind the method for establishing the symmetry and Schur positivity of a function expressed in terms of fundamental quasi-symmetric functions that follows. We will realize the given function as the generating function for a vertex-signed, edge-colored graph such that connected components of the graph are isomorphic to one of the graphs $G_\lambda$. Therefore the connected components of the larger graph will correspond precisely to terms in the Schur expansion of the given function.

**3.2. Axiomatization of dual equivalence.** The purpose of this section is to characterize the $G_\lambda$ in terms of the edges and signatures so that we can readily identify those graphs that are isomorphic to some $G_\lambda$.

**Definition 3.3.** A signed, colored graph of type $(n, N)$ consists of the following data:

- a finite vertex set $V$;
- a signature function $\sigma : V \to \{\pm 1\}^{N-1}$;
- for each $1 < i < n$, a collection $E_i$ of pairs of distinct vertices of $V$.

We denote such a graph by $G = (V, \sigma, E_2 \cup \cdots \cup E_{n-1})$ or simply $(V, \sigma, E)$.

It is often useful to focus attention on a restricted set of edges of a signed, colored graph. To be precise, for $m \leq n$ and $M \leq N$, the $(m, M)$-restriction of a signed, colored graph $G$ of type $(n, N)$ consists of the vertex set $V$, signature function $\sigma : V \to \{\pm 1\}^{M-1}$ obtained by truncating $\sigma$ at $M-1$, and edge set $E_2 \cup \cdots \cup E_{m-1}$.

**Definition 3.4.** A signed, colored graph $G = (V, \sigma, E)$ of type $(n, N)$ is a dual equivalence graph of type $(n, N)$ if $n \leq N$ and the following hold:

1. For $w \in V$ and $1 < i < n$, $\sigma(w)_i = -\sigma(w)_i$ if and only if there exists $x \in V$ such that $\{w, x\} \in E_i$.

2. For $w, x \in E_i$, $\sigma(w)_j = -\sigma(x)_j$ for $j = i-1$, $i$, and $\sigma(w)_h = \sigma(x)_h$ for $h < i-2$ and $h > i+1$.

3. For $w, x \in E_i$, $\sigma(w)_i = -\sigma(x)_i$, then $\sigma(w)_{i-2} = -\sigma(w)_{i+1}$, and if $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$, then $\sigma(w)_i = -\sigma(w)_i$.

4. Every connected component of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ appears in Figure 8.$$\footnote{If $n = 4$, this should be interpreted as: Every connected component of $G$ appears in Figure 7.}$$

5. If $\{w, x\} \in E_i$ and $\{x, y\} \in E_j$ for $|i - j| \geq 3$, then $\{w, v\} \in E_j$ and $\{v, y\} \in E_i$ for some $v \in V$.

6. Any two vertices of a connected component of $(V, \sigma, E_2 \cup \cdots \cup E_i)$ may be connected by a path crossing at most one $E_i$ edge.

Note that if $n > 4$, then the allowed structure for connected components of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ dictates that every connected component of $(V, \sigma, E_{i-1} \cup E_i)$ appear in Figure 7.

**Figure 7.** Allowed 2-color connected components of a dual equivalence graph.

Every connected component of a dual equivalence graph of type $(n, N)$ is again a dual equivalence graph of type $(n, N)$. Also, for $m \leq n, M \leq N$, the $(m, M)$-restriction of a dual equivalence graph of type $(n, N)$ is a dual equivalence graph of type $(m, M)$.

Notice that the graph for $G_V$ may be obtained from $G_\lambda$ by conjugating each standard tableau and multiplying the signatures coordinate-wise by $-1$. Therefore the structure of $G_{(2,1,1,1)}, G_{(2,2,1)}$ and $G_{(1,1,1,1,1)}$ is
also indicated by Figure 6. Comparing this with Figure 8, axiom 4 stipulates that the restricted connected components of a dual equivalence graph are exactly the graphs for $G_\lambda$ when $\lambda$ is a partition of 5.

**Proposition 3.5.** For $\lambda$ a partition of $n$, $G_\lambda$ is a dual equivalence graph of type $(n, n)$.

**Proof.** For $T \in \text{SYT}(\lambda)$, $\sigma(T)_{i-1} = -\sigma(T)_i$ if and only if $i$ does not lie between $i-1$ and $i+1$ in the content reading word of $T$. In this case, there exists $U \in \text{SYT}(\lambda)$ such that $T$ and $U$ differ by an elementary dual equivalence for $i-1, i, i+1$. Therefore $U$ is obtained from $T$ by swapping $i$ with $i-1$ or $i+1$, whichever lies further away, with the result that $\sigma(T)_j = -\sigma(U)_j$ for $j = i-1, i$ and also $\sigma(T)_h = \sigma(U)_h$ for $h < i-2$ and $i+1 < h$. This verifies axioms 1 and 2.

For axiom 3, if $\sigma(T)_{i-2} = -\sigma(U)_{i-2}$ as well, then $i$ and $i-1$ must have interchanged positions with $i-2$ lying between, so that $T$ and $U$ also differ by an elementary dual equivalence for $i-2, i-1, i$. The analogous argument holds for $i+1$. Also from this, we obtain an explicit description of double edges, and so axiom 4 becomes a straightforward check. If $|i-j| \geq 3$, then $\{i-1, i, i+1\} \cap \{j-1, j, j+1\} = \emptyset$, so the elementary dual equivalences for $i-1, i, i+1$ and for $j-1, j, j+1$ commute, thereby demonstrating axiom 5.

Finally, for $T, U \in \text{SYT}(\lambda)$, $|\lambda| = i+1$, we must show that there exists a path connecting from $T$ to $U$ crossing at most one $E_i$ edge. Let $C_T$ (resp. $C_U$) denote the connected component of the $(i, i)$-restriction of $G_\lambda$ containing $T$ (resp. $U$). Let $\mu$ (resp. $\nu$) be the shape of $T$ (resp. $U$) with the cell containing $i+1$ removed. Then $C_T \cong G_\mu$, and $C_U \cong G_\nu$. If $\mu = \nu$, then, by Proposition 3.2, $C_T = C_U$ and axiom 6 holds. Assume, then, that $\mu \neq \nu$. Since $\mu, \nu \subset \lambda$ and $|\mu| = |\nu| = |\lambda| - 1$, both cells $\lambda/\mu$ and $\lambda/\nu$ must be northeastern corners of $\lambda$. Therefore there exists $T' \in \text{SYT}(\lambda)$ with $i$ in position $\lambda/\nu$, $i+1$ in position $\lambda/\mu$, and $i-1$ between $i$ and $i+1$ in the content reading word of $T'$. Let $U'$ be the result of swapping $i$ and $i+1$ in $T'$, in particular, $\{T', U'\} \in E_i$. By Proposition 3.2, $T'$ is in $C_T$ and $U'$ is in $C_U$, hence there exists a path from $T$ to $T'$ and a path from $U'$ to $U$ each crossing only edges $E_h$, $h < i$. This establishes axiom 6.

**Remark 3.6.** For partitions $\lambda \subset \rho$, with $|\lambda| = n$ and $|\rho| = N$, choose a tableau $A$ of shape $\rho/\lambda$ with entries $n+1, \ldots, N$. Define the set of standard Young tableaux of shape $\lambda$ augmented by $A$, denoted ASYT$(\lambda, A)$, to be those $T \in \text{SYT}(\rho)$ such that $T$ restricted to $\rho/\lambda$ is $A$. Let $G_{\lambda,A}$ be the signed, colored graph of type $(n, N)$ constructed on ASYT$(\lambda, A)$ with $i$-edges given by elementary dual equivalences for $i-1, i, i+1$ with $i < n$. Then $G_{\lambda,A}$ is a dual equivalence graph of type $(n, N)$, and the $(n, n)$-restriction of $G_{\lambda,A}$ is $G_\lambda$.

Proposition 3.5 is the first step towards justifying Definition 3.4, and also allows us to refer to $G_{\lambda}$ as the standard dual equivalence graph corresponding to $\lambda$. In order to show the converse, we first need the notion of a morphism between two signed, colored graphs.

**Definition 3.7.** A morphism between two signed, colored graphs of type $(n, N)$, say $G = (V, \sigma, E)$ and $\mathcal{H} = (W, \tau, F)$, is a map $\phi : V \to W$ such that for every $u, v \in V$

- for every $1 \leq i < N$, we have $\sigma(v)_i = \tau(\phi(v))_i$, and
- for every $1 < i < n$, if $\{u, v\} \in E_i$, then $\{\phi(u), \phi(v)\} \in F_i$.

A morphism is an isomorphism if it is a bijection on vertex sets.
Remark 3.8. Note that if \( \phi \) is a morphism from a signed, colored graph \( G \) of type \((n,N)\) satisfying axiom 1 to an augmented standard dual equivalence graph \( G_{\lambda,A} \), then \( \phi \) is surjective. Indeed, suppose \( T = \phi(v) \) for some \( T \in \text{ASYT}(\lambda,A) \) and some vertex \( v \) of \( G \). Then for every \( 1 < i < n \), if \( \{T,U\} \in E_i \), then since \( \sigma(v) = \sigma(T) \), by axiom 1 there exists a unique vertex \( w \) of \( G \) such that \( \{v,w\} \in E_i \) in \( G \). Since \( \phi \) is a morphism, we must have \( \{T,\phi(w)\} \in E_i \) in \( G_{\lambda,A} \). Thus by the uniqueness condition of axiom 1, \( \phi(w) = U \), and so \( U \) also lies in the image of \( \phi \). Therefore the \( i \)-neighbor of any vertex in the image of \( \phi \) also lies in the image since \( \phi \) preserves \( i \)-edges. Since \( G_{\lambda,A} \) is connected, \( \phi \) must be surjective.

Next we show that the standard dual equivalence graphs are non-redundant in the sense that they are mutually non-isomorphic and have no nontrivial automorphisms. Both results stem from the observation that \( G_\lambda \) contains a unique vertex such that the composition formed by the lengths of the runs of \( +1 \)'s in the signature gives a maximal partition.

**Proposition 3.9.** If \( \phi : G_\lambda \rightarrow G_\mu \) is an isomorphism, then \( \lambda = \mu \) and \( \phi = \text{id} \).

**Proof.** Let \( T_\lambda \) be the tableau obtained by filling the numbers 1 through \( n \) into the rows of \( \lambda \) from left to right, bottom to top, in which case \( \sigma(T_\lambda) = +\lambda_1^{-1}, \ldots, +\lambda_{\ell-1}^{-1}, \ldots, \). For any standard tableau \( T \) such that \( \sigma(T) = \sigma(T_\lambda) \), the numbers 1 through \( \lambda_1 \), and also \( \lambda_1 + 1 \) through \( \lambda_1 + \lambda_2 \), and so on, must form horizontal strips. In particular, if \( \sigma(T) = \sigma(T_\lambda) \) for some \( T \) of shape \( \mu \), then \( \lambda \leq \mu \) with equality if and only if \( T = T_\lambda \).

Suppose \( \phi : G_\lambda \rightarrow G_\mu \) is an isomorphism. Let \( T_\lambda \) be as above for \( \lambda \), and let \( T_\mu \) be the corresponding tableau for \( \mu \). Then since \( \sigma(\phi(T_\lambda)) = \sigma(T_\lambda) \), \( \lambda \leq \mu \). Conversely, since \( \sigma(\phi^{-1}(T_\mu)) = \sigma(T_\mu) \), \( \mu \leq \lambda \). Therefore \( \lambda = \mu \). Furthermore, \( \phi(T_\lambda) = T_\lambda \). For \( T \in \text{SYT}(\lambda) \) such that \( \{T_\lambda, T\} \in E_i \), we have \( \{T_\lambda, \phi(T)\} \in E_i \), so \( \phi(T) = T \) by dual equivalence axiom 1. Extending this, every tableau connected to a fixed point by some sequence of edges is also a fixed point for \( \phi \), hence \( \phi = \text{id} \) on each \( G_\lambda \) by Proposition 3.2.

The final justification of our axiomatization is Theorem 3.10 below, which is the converse of Proposition 3.5. The uniqueness statement follows from Proposition 3.9.

**Theorem 3.10.** Every connected component of a dual equivalence graph of type \((n,n)\) is isomorphic to \( G_\lambda \) for a unique partition \( \lambda \) of \( n \).

The proof of Theorem 3.10 is postponed until Section 3.3. We conclude this section by interpreting Theorem 3.10 in terms of symmetric functions.

**Corollary 3.11.** Let \( G \) be a dual equivalence graph of type \((n,n)\) such that every vertex is assigned some additional statistic \( \alpha \). Let \( C(\lambda) \) denote the set of connected components of \( G \) that are isomorphic to \( G_\lambda \). If \( \alpha \) is constant on connected components of \( G \), then

\[
\sum_{v \in \lambda(\lambda)} q^{\alpha(v)} Q_{\sigma(v)}(X) = \sum_{\lambda} \sum_{\zeta \in C(\lambda)} q^{\alpha(\zeta)} s_\lambda(X).
\]

In particular, the generating function for \( G \) so defined is symmetric and Schur positive.

We can, of course, include multivariate statistics in (3.2), but as our immediate purpose is to apply this theory to LYT polynomials, a single parameter will suffice.

Equation 3.2 appears to be a difficult formula to work with since, in general, it is difficult to determine when two signed, colored graphs are isomorphic. However, in the case of dual equivalence graphs, the problem is greatly simplified. For each vertex \( v \) of a dual equivalence graph, let \( \pi(v) \) be the composition formed by the lengths of the runs of the \( +1 \)'s in \( \sigma(v) \). As shown in Proposition 3.9, each \( G_\lambda \) contains a unique vertex \( T_\lambda \) with the property that \( \pi(T_\lambda) \) forms a partition and, if \( \pi(T) \) also forms a partition for some \( T \in \text{SYT}(\lambda) \), then \( \pi(T) \leq \pi(T_\lambda) \) in dominance order. Therefore if we know which vertices occur on a given connected component of a dual equivalence graph, determining the \( G_\lambda \) to which the component is isomorphic is simply a matter of comparing \( \pi(v) \) for each vertex of the component.

### 3.3. The structure of dual equivalence graphs.

In order to avoid cumbersome notation, as we investigate the connection between an arbitrary dual equivalence graph and the standard one, we will often abuse notation by simultaneously referring to \( \sigma \) and \( E \) as the signature function and edge set for both graphs.

**Definition 3.12.** Let \( G = (V, \sigma, E) \) be a signed, colored graph of type \((n,N)\) satisfying axiom 1. For \( 1 < i < N \), we say that a vertex \( w \in V \) admits an \( i \)-neighbor if \( \sigma(w)_{i-1} = -\sigma(w)_i \).
For $1 < i < n$, if $\sigma(w)_{i-1} = -\sigma(w)_i$ for some $w \in V$, then axiom 1 implies the existence of $x \in V$ such that $\{w, x\} \in E_i$. That is, if $w$ admits an $i$-neighbor for some $1 < i < n$, then $w$ has an $i$-neighbor in $G$. For $n \leq i < N$, though $i$-edges do not exist in $G$, if $G$ were the restriction of a graph of type $(i+1, N)$ also satisfying axiom 1, then the condition $\sigma(w)_{i-1} = -\sigma(w)_i$ would imply the existence of a vertex $x$ such that $\{w, x\} \in E_i$ in the type $(i+1, N)$ graph. When convenient, $E_i$ may be regarded as an involution on vertices admitting an $i$-neighbor, i.e. if $w$ admits an $i$-neighbor, then $E_i(w) = x$ where $\{w, x\} \in E_i$.

Recall the notion of augmenting a partition $\lambda$ by a skew tableau $A$ and the resulting dual equivalence graph $G_{\lambda, A}$ from Remark 3.6.

**Proposition 3.13.** Let $G = (V, \sigma, E)$ be a connected graph of type $(n, N)$ satisfying dual equivalence graph axioms 1, 2, 3, 4 and 5, and let $\phi$ be a morphism from the $(n, n)$-restriction of $G$ to $G_{\lambda}$ for some partition $\lambda$ of $n$. Then there exists a semi-standard tableau $A$ of shape $\rho/\lambda$, $|\rho| = N$, with entries $n+1, \ldots, N$ such that $\phi$ gives a morphism from $G$ to $G_{\lambda, A}$. Moreover, the position of the cell containing $n+1$ is unique.

**Proof.** By axiom 2 and the fact that $G$ is connected, $\sigma_n$ is constant on $G$ for $h \geq n+1$. Therefore once a suitable cell for $n+1$ has been chosen, the cells for $n+2$, $\ldots$, $N$ may be chosen in any way that gives the correct signature. One solution is to place $j$ north of the first column if $\sigma_{j-1} = -1$ or east of the first row if $\sigma_{j-1} = +1$ for $j = n+2, \ldots, N$.

First consider the case when $\sigma_n$ is constant on $G$. Then we require $\sigma_n$ to be constant with the same value on $G_{\lambda, A}$ as well. Since $n$ may occupy any northeastern corner of $\lambda$, any vertex of $G_{\lambda}$ by Proposition 3.2, the only way for $\sigma_n$ to remain constant is for $n+1$ to be placed north of the first column or east of the first row of $\lambda$. The former ensures that $\sigma_n \equiv -1$, and the latter ensures $\sigma_n \equiv +1$. Therefore there is a unique position for $n+1$ in $A$ that gives the correct signature.

Now assume that $\sigma_n$ is not constant on all of $G$. We will identify the unique inner corner of $\lambda$ to house the cell containing $n+1$. By dual equivalence axiom 2, $\sigma_n$ is constant on connected components of the $(n-1, N)$-restriction of $G$. By Proposition 3.2, a connected component of the $(n-1, n-1)$-restriction of $G_{\lambda}$ consists of all standard Young tableaux where $n$ lies in a particular northeastern cell of $\lambda$. Therefore, for each connected component of the $(n-1, N)$-restriction of $G$, we may identify its image under $\phi$ with $G_{\mu}$ for some partition $\mu \subset \lambda$.

Let $C$ and $D$ be two distinct connected components of the $(n-1, N)$-restriction of $G$ such that there exist vertices $v$ of $C$ and $u$ of $D$ with $\{v, u\} \in E_{n-1}$. Let $\lambda/\mu$ be the position of $n$ in $\phi(u)$, and let $\lambda/\nu$ be the position of $n$ in $\phi(v)$. Since $\{v, u\} \in E_{n-1}$, $\phi(v)$ must have $n-1$ in position $\lambda/\mu$ and $n-2$ lying between $n-1$ and $n$ in the content reading word. Since $\phi$ preserves $E_{n-1}$ edges, $\phi(u)$ must be the result of an elementary dual equivalence on $\phi(w)$ for $n-2, n-1, n$, which will necessarily interchange $n-1$ and $n$. Since $\phi$ preserves signatures, $\lambda/\nu$ lies north of the position of $\lambda/\mu$ if and only if $\sigma(v)_{n-2, n-1} = +$ and $\sigma(u)_{n-2, n-1} = -$. By axiom 3, if $\lambda/\nu$ lies north of the position of $\lambda/\mu$ and $\sigma(v)_n = -1$, then $\sigma(u)_n = -1$, and if $\lambda/\nu$ lies northwest of the position of $\lambda/\mu$ and $\sigma(u)_n = +1$, then $\sigma(v)_n = +1$.

Therefore there exists a unique row such that $\sigma(C)_n = -1$ whenever the $\phi(C)$ has $n$ south of this row and $\sigma(C)_n = +1$ whenever the $\phi(C)$ has $n$ north of this row. In this case, the cell containing $n+1$ must be placed at the eastern end of this pivotal row, and doing so extends $\phi$ to an isomorphism between $(n, n+1)$ graphs.

Once Theorem 3.10 has been proved, Proposition 3.13 may be used to obtain the following generalization of Theorem 3.10 for dual equivalence graphs of type $(n, N)$: Every connected component of a dual equivalence graph of type $(n, N)$ is isomorphic to $G_{\lambda, A}$ for a unique partition $\lambda$ and some skew tableau $A$ of shape $\rho/\lambda$, $|\rho| = N$, with entries $n+1, \ldots, N$. 

\[ \text{Figure 9. Identifying the unique position for } n+1 \text{ from } \phi(v) \text{ and } \phi(u). \]
Lemma 3.14. For a signed, colored graph of type $\left( n, N \right)$ satisfying dual equivalence graph axioms 1, 2 and 3, the connected component of $(V, \sigma, E_{n-2} \cup E_{n-1})$ of a vertex admitting an $n$-neighbor falls into one of the 3 cases (5 graphs) depicted in Figure 10.

![Possible connected components](image)

Figure 10. Possible connected components of $(V, \sigma, E_{n-2} \cup E_{n-1})$ with dotted lines indicating when both endpoints of an $E_{i-1}$ edge admit $i$-neighbors.

Remark 3.15. If $\{w, x\} \in E_{i-1}$ and both $w$ and $x$ admit an $i$-neighbor, then if $G$ is the $(i, N)$-restriction of a graph satisfying axiom 4, then $\{w, x\} \in E_n$ in the type $(i+1, N)$ graph. This motivates the dashed lines in case B of Figure 10. For the remaining vertices admitting an $n$-neighbor, for each of $A$, $B$, and $C$, there is exactly one vertex also admitting an $n-1$-neighbor, and exactly one that does not. For $C$, two identical graphs are shown in Figure 10-C in order to parallel Figure 8.

Proof. Let $w \in V$ admit an $n$-neighbor. If $w$ admits neither an $n-2$-neighbor nor an $n-1$-neighbor, then $w$ is the right-hand side of $A$.

If $w$ admits an $n-1$-neighbor, say $\{w, v\} \in E_{n-1}$, but does not admit an $n-2$-neighbor, then by axiom 2, $v$ must admit an $n-2$-neighbor, say $\{v, u\} \in E_{n-2}$. If $v$ admits an $n$-neighbor then $u$ must also admit an $n$-neighbor by axiom 2, and $w, v, u$ form the left-hand side of $B$. If $v$ does not admit an $n$-neighbor, then by axiom 2 neither does $u$, so $u, v, w$ form the left-hand side of $A$.

If $w$ admits an $n-2$-neighbor, say $\{w, v\} \in E_{n-2}$, but does not admit an $n-1$-neighbor, then $v$ must admit an $n-1$-neighbor by axiom 2, say $\{v, u\} \in E_{n-1}$. Again, by axiom 2, since $w$ admits an $n$-neighbor, so too must $v$. If $u$ admits an $n$-neighbor as well, then $u, v, w$ form the left-hand side of $B$. If $u$ does not admit an $n$-neighbor, then $u, v, w$ are as in $C$.

Finally, consider the case when $w$ admits both an $n-1$-neighbor and an $n-2$-neighbor, say $\{w, v\} \in E_{n-1}$ and $\{w, u\} \in E_{n-2}$. If $u = v$, then $v$ does not admit an $n$-neighbor, and $w, v$ are as in the right-hand side of $B$. Otherwise, $u$ must also admit an $n$-neighbor by axiom 2 and may not admit an $n-1$-neighbor by axiom 3. Similarly, $v$ may not admit an $n-2$-neighbor by axiom 3. If $v$ admits an $n$-neighbor, then $v, w, u$ are as in the left-hand side of $B$, and if $v$ does not admit an $n$-neighbor, then $v, w, u$ are as in $C$. □

Lemma 3.14 justifies defining the $n$-type of a vertex admitting an $n$-neighbor as $A$, $B$ or $C$ according to the characterization of Figure 10. If $v$ admits an $n$-neighbor and is not one of the two leftmost vertices in the left hand side of $B$ (i.e. $v$ is not an endpoint of an $n-1$-edge both of whose endpoints admit an $n$-neighbor), then the connected component of $(V, \sigma, E_{i-1} \cup E_i)$ of $v$ is determined by the $n$-type of $v$ and whether or not $v$ admits an $n-1$-edge.

Remark 3.16. Let $G$ be a signed, colored of type $(n, N)$ satisfying axioms 1, 2 and 3 such that connected components of $(V, \sigma, E_{i-1} \cup E_i)$ all appear in Figure 7. Let $w$ and $x$ be vertices such that $\{w, x\} \in E_i$ and $\{w, x\} \notin E_{i-1}$ for some $i < n$. We may define the $i$-type of $w$ and $x$ by considering the $(i, N)$-restriction of $G$. Note that the condition $\{w, x\} \notin E_{i-1}$ rules out the possibility that $w$ and $x$ are the two leftmost vertices of $B$ in Figure 10. By dual equivalence axioms 1, 2 and 3 and the preceding remarks, $\sigma(w)_h$ and the $i$-type of $w$ are completely determined by $\sigma(x)_h$ and the $i$-type of $x$ for $h \leq i$.

Finally we have all of the ingredients necessary to prove the main result of this section, from which Theorem 3.10 follows.
Theorem 3.17. Let $\mathcal{G}$ be a connected signed, colored graph of type $(n+1, n+1)$ satisfying axioms 1 through 5 such that each connected component of the $(n, n)$-restriction of $\mathcal{G}$ is isomorphic to a standard dual equivalence graph. Then there exists a morphism $\phi$ from $\mathcal{G}$ to $\mathcal{G}_\lambda$ for some unique partition $\lambda$ of $n+1$. Moreover, the fiber over each vertex of $\mathcal{G}_\lambda$ has the same cardinality.

Proof: When $n+1 = 2$, the result follows immediately from Proposition 3.13 since, in this case, $\mathcal{G}$ has no edges. Therefore we proceed by induction, assuming the result for graphs of type $(n, n)$.

By induction, for every connected component $C$ of the $(n, n+1)$-restriction of $\mathcal{G}$, we have an isomorphism from the $(n, n)$-restriction of $C$ to $\mathcal{G}_\mu$ for a unique partition $\mu$ of $n$. By Proposition 3.13, this extends to an isomorphism from $C$ to $\mathcal{G}_\mu, A$ for a unique augmenting tableau $A$, say with shape $\lambda/\mu$. We will show that for any $C$, the shape of $\mu$ augmented with $A$ is the same, and so we may glue these morphisms together to obtain a morphism from the $(n, n+1)$-restriction of $\mathcal{G}$ to the $(n, n+1)$-restriction of $\mathcal{G}_\lambda$.

\[
\begin{array}{c}
\mathcal{C} \\
\phi \\
\mathcal{G}_\mu, A \\
\diamond \\
E_n \\
\downarrow \\
E_{n+1}
\end{array} \quad \begin{array}{c}
\mathcal{D} \\
\psi \\
\mathcal{G}_\nu, B \\
\downarrow \\
D
\end{array}
\]

Figure 11. An illustration of the gluing process.

Suppose $\{w, x\} \in E_n$, and let $\mathcal{C}$ (resp. $\mathcal{D}$) denote the connected component of the $(n, n+1)$-restriction of $\mathcal{G}$ containing $w$ (resp. $x$). Let $\phi$ (resp. $\psi$) be the isomorphism from $\mathcal{C}$ (resp. $\mathcal{D}$) to $\mathcal{G}_\mu, A$ (resp. $\mathcal{G}_\nu, B$), and set $T = \phi(w)$. There are two cases to consider. First suppose $n+1$ lies between $n$ and $n-1$ in the reading word of $T$. By Proposition 3.2, there exists a tableau $T'$ on the $(n-2, n+1)$-restriction of $\mathcal{G}_\mu, A$ such that $n-2$ lies between $n$ and $n-1$. Thus an elementary dual equivalence for $n-2$, $n-1$, $n$ on $T'$ interchanges $n$ and $n-1$, resulting in the tableau $U' = E_{n+1}(T')$. Since $n+1$ also lies between $n$ and $n-1$ in $T'$, we also have $U' = E_n(T')$ in $\mathcal{G}_\lambda$, where $\lambda$ is the shape of $\mu$ augmented by $A$. By axioms 2 and 5, the path taking $T'$ back to $T$ must then take $U'$ to some $U$. Since $\phi$ and $\psi$ are isomorphisms, these paths lift to $\mathcal{C}$ and $\mathcal{D}$. Since $\mathcal{G}$ satisfying axiom 5, each edge in the paths commutes with $E_n$ as indicated in Figure 12, and so we must have $U = \psi(x)$. Therefore $\mathcal{C} = \mathcal{D}$ and, by Proposition 3.9, $\psi = \phi$.

\[
\begin{array}{c}
\leq n-3 \\
n \\
\leq n-3 \\
\leq n-3 \\
U \\
\leq n-3 \\
\end{array} \quad \begin{array}{c}
\leq n-3 \\
\ldots \\
n \\
n \\
\leq n-3 \\
\leq n-3 \\
U' \\
\leq n-3 \\
\end{array}
\]

Figure 12. Illustration of the path from $T$ to $U$ when $n+1$ lies between $n$ and $n-1$.

In the alternative case, $n-1$ must lie between $n$ and $n+1$ in $T$. Let $C_{n-2} \subset E_{n-2}$ be those $n-2$-edges whose endpoints both admit $n$-edges and both have $n$-type $C$, and let $C'$ (resp. $\mathcal{D}'$) be the connected component of $(V, \sigma, E_2 \cup \cdots \cup E_{n-3} \cup C_{n-2})$ containing $w$ (resp. $x$). Note that in $\phi(C')$, the positions of $n$ and $n+1$ are fixed, and though the position of $n-1$ may vary, it will necessarily always lie between $n$ and $n+1$ in the content reading word by the assumption that all $E_{n-2}$ edges connect vertices with $n$-type $C$. Therefore the $(n-2, n-1)$-restriction of $\phi(C')$ is isomorphic to the disjoint union of $\mathcal{G}_\rho, R$ where $\rho$ is a fixed partition of $n-2$ and $R$ ranges over all augmenting tableaux consisting of a single corner cell containing $n-1$ located between the fixed positions of $n$ and $n+1$. In particular, the isomorphism class of the $(n-2, n-1)$-restriction of $\phi(C')$ determines the shape of $\lambda$ along with the positions of two cells occupied by $n$ and $n+1$. Which entry occupies which cell is determined by $\sigma_1$, which is constant on $\phi(C')$ by axioms 2 and 3.

Axioms 4 and 5 together show that $E_n$ gives an isomorphism between $C'$ and $\mathcal{D}'$. In particular, the connected component of the $(n-2, n)$-restriction of $\mathcal{D}$ containing $x$ does not contain an endpoint of a double $E_{n-1}, E_n$ edge, and so $U = \psi(x)$ must also have $n-1$ lying between $n$ and $n+1$. Therefore the above
argument applies to \( \psi(\mathcal{D}') \) as well. The isomorphism between \( \mathcal{C}' \) and \( \mathcal{D}' \) gives an isomorphism between \( \phi(\mathcal{C}') \) and \( \psi(\mathcal{D}') \), and so \( \nu \) augmented by \( \mathcal{B} \) also has shape \( \lambda \), and \( U \) differs from \( T \) by interchanging \( n \) and \( n+1 \), i.e. \( U = E_n(T) \). We now have a well-defined morphism from the \((n, n+1)\)-restriction of \( \mathcal{G} \) to the \((n, n+1)\)-restriction of \( \mathcal{G}_\lambda \) that respects \( n \)-edges. As such, this map lifts to a morphism from \( \mathcal{G} \) to \( \mathcal{G}_\lambda \).

By Remark 3.8, the morphism of Theorem 3.17 is necessarily surjective, though in general it need not be injective. The smallest example where injectivity fails was first observed by Gregg Musiker in a graph of type \((6, 6)\) with generating function \( 2s_{(3,2,1)}(X) \); see Figure 36 in Appendix B.

Proof of Theorem 3.10. Let \( \mathcal{G} \) be a dual equivalence graph of type \((n+1, n+1)\). We aim to show that \( \mathcal{G} \) is isomorphic to \( \mathcal{G}_\lambda \) for a unique partition \( \lambda \) of \( n+1 \). We proceed by induction on \( n+1 \), noting that the result is trivial for \( n+1 = 2 \). Therefore we assume the result for dual equivalence graphs of type \((n, n)\). By Theorem 3.17, there exists a morphism from \( \mathcal{G} \) to \( \mathcal{G}_\lambda \) for a unique partition \( \lambda \) of \( n+1 \). For every connected component \( \mathcal{C} \) of the \((n, n+1)\)-restriction of \( \mathcal{G} \), the \((n, n)\)-restriction of \( \mathcal{C} \) is isomorphic to the standard dual equivalence graph of shape \( \mu \) by the inductive hypothesis, and by Proposition 3.9 we must have \( \mu \subset \lambda \).

Now we invoke axiom 6 to see that the connected components of the \((n, n+1)\)-restriction of \( \mathcal{G} \) are pairwise non-isomorphic. Indeed, suppose \( \mathcal{C} \) and \( \mathcal{D} \) are distinct restricted components with \( \mathcal{C} \cong \mathcal{D} \). By axiom 6, there exist vertices \( w \) in \( \mathcal{C} \) and \( v \) in \( \mathcal{D} \) such that \( \{w, v\} \in E_n \) in \( \mathcal{G} \). Let \( \phi \) denote the isomorphism from the \((n, n)\)-restriction of \( \mathcal{C} \) to \( \mathcal{G}_\mu \). The fact that \( \mathcal{C} \) and \( \mathcal{D} \) are isomorphic ensures that \( n+1 \) lies between \( n \) and \( n-1 \) in the reading word of \( \phi(w) \). From the proof of Theorem 3.17, it follows that \( \mathcal{C} = \mathcal{D} \). Thus the morphism from \( \mathcal{G} \) to \( \mathcal{G}_\lambda \) is injective on the \((n, n+1)\)-restrictions, and so it is injective on all of \( \mathcal{G} \). Surjectivity follows from Remark 3.8, thus the given morphism is an isomorphism.

Remark 3.18. Axiom 6 for general dual equivalence graphs was only needed in order to show that the restricted connected components considered in Theorem 3.10 are pairwise non-isomorphic. Therefore we could replace axiom 6 in Definition 3.4 with this pairwise non-isomorphic condition, which \( \mathcal{G}_\lambda \) clearly satisfies, and Theorem 3.10 would still hold. However, the path condition is, in practice, easier to verify.

4. A GRAPH FOR LLT POLYNOMIALS

4.1. Words with content. In this section we describe a modified characterization of LLT polynomials as the generating function of \( k \)-ribbon words. As Proposition 4.2 shows, these are precisely the content reading words of semi-standard \( k \)-tuples of tableaux.

Given a word \( w \) and a non-decreasing sequence of integers \( c \) of the same length, define the \( k \)-descent set of the pair \((w, c)\), denoted \( \text{Des}_k(w, c) \), by

\[
\text{Des}_k(w, c) = \{(i, j) \mid w_i > w_j \text{ and } c_j - c_i = k\}.
\]

Definition 4.1. A \( k \)-ribbon word is a pair \((w, c)\) consisting of a word \( w \) and a non-decreasing sequence of integers \( c \) of the same length such that if \( c_i = c_{i+1} \), then there exist integers \( h, j \) such that \((h, i), (i, j), (h, i+1) \notin \text{Des}_k(w, c) \). In other words, \( c_h = c_i - k \) and \( w_i < w_h \leq w_{i+1} \) while \( c_j = c_i + k \) and \( w_i \leq w_j < w_{i+1} \).

Proposition 4.2. The pair \((w, c)\) is a \( k \)-ribbon word if and only if there exists a \( k \)-tuple of (skew) semi-standard tableaux such that \( w \) is the content reading word of the \( k \)-tuple and \( c \) gives the corresponding contents.

Proof. Suppose first that \( w \) is the content reading word of some \( k \)-tuple of semi-standard tableaux with corresponding contents given by \( c \). If \( c_i = c_{i+1} \), then in the \( k \)-tuple there must exist entries \( w_i \) and \( w_j \) as shown in Figure 13. The semi-standard condition ensures that \( w_i < w_h \leq w_{i+1} \) and \( w_i \leq w_j < w_{i+1} \). Therefore the conditions of Definition 4.1 are met.

![Figure 13](image-url)

**Figure 13.** Situation when \( c_i = c_{i+1} \) for a \( k \)-tuple of semi-standard Young tableaux.
Now suppose that \((w, c)\) is a \(k\)-ribbon word. For each \(j\), arrange all \(w_i\) such that \(c_i = j\) into cells along a southwest to northeast diagonal in increasing order. Align the southwest corner of the diagonal for \(j - k\) immediately north (resp. west) of the southwest corner of the diagonal for \(j\) whenever the smallest letter with content \(j - k\) is greater than (resp. less than or equal to) the smallest letter with content \(j\).

We must show that the result is a \(k\)-tuple of (skew) shapes whose entries satisfy the semi-standard condition. Consider two adjacent diagonals \(j - k\) and \(j\). By construction, the southwestern most cells of the diagonals form a partition shape and satisfy the semi-standard condition. By induction, assume that the entries in diagonal \(j - k\) through \(w_h\) and the entries in diagonal \(j\) through \(w_i\) form a semi-standard tableau of skew shape, with \(w_h\) immediately west or immediately north of \(w_i\).

Suppose that \(c_{i+1} = c_i\), noting that the case when \(c_{i+1} = c_h\) may be solved similarly. If \(w_h > w_i\), then we must show that \(w_h \leq w_{i+1}\). By Definition 4.1, there exists an integer \(l\) such that \((l, i+1) \not\in \Des_k(w, c)\), and therefore \(w_l \leq w_i\). Since \(c_i = j - k\), we have \(w_h \leq w_l \leq w_{i+1}\). If \(w_h \leq w_i\), then we must show that \(c_{i+1} = j - k\) and \(w_l < w_{i+1} \leq w_{i+1}\). By Definition 4.1, there exists an integer \(l\) such that \((l, i) \in \Des_k(w, c)\) and \((l, i+1) \not\in \Des_k(w, c)\). Therefore \(c_i = j - k\) and \(w_h \leq w_i \leq w_{i+1}\). The non-decreasing condition on \(c\) implies that \(c_{i+1} = j - k\), and so there exists an integer \(m\) such that \((h, m) \in \Des_k(w, c)\) and \((h, m) \not\in \Des_k(w, c)\), i.e. \(w_h \leq w_i \leq w_{i+1}\) with \(c_m = j\). The only way to satisfy these two conditions is to have \(m = i\) and \(l = i+1\).

For \(\mathbf{T}\) and \(\mathbf{U}\) two \(k\)-tuples of semi-standard tableaux, let \((w_\mathbf{T}, c_\mathbf{T})\) and \((w_\mathbf{U}, c_\mathbf{U})\) denote the corresponding \(k\)-ribbon words. Then \(\mathbf{T}\) and \(\mathbf{U}\) have the same shape if and only if \(\Des_k(w_\mathbf{T}) = \Des_k(w_\mathbf{U})\) and \(c_\mathbf{T} = c_\mathbf{U}\). In particular, if we let \(\text{WRib}_k(c, D)\) denote the set of \(k\)-ribbon words with content vector \(c\) and \(k\)-descent set \(D\), then we have established a bijective correspondence

\[
WRib_k(c, D) \sim \text{SSYT}_k(\mu).
\]

Define the set of \(k\)-inversions and the \(k\)-inversion number of a pair \((w, c)\) by

\[
\text{Inv}_k(w, c) = \{(i, j) \mid w_i > w_j \text{ and } k > c_j - c_i > 0\},
\]

\[
\text{inv}_k(w, c) = |\text{Inv}_k(w, c)|.
\]

Recalling (2.7), we have

\[
\text{Inv}_k(w_\mathbf{T}, c_\mathbf{T}) = \text{Inv}_k(\mathbf{T}).
\]

Therefore we may express LLT polynomials in terms of \(k\)-ribbon words as follows.

**Corollary 4.3.** Let \(\mu\) be a (skew) shape, and let \(c, D\) be the content vector and \(k\)-descent set corresponding to \(\mu\) by (4.2). Then

\[
\tilde{G}_\mu^{(k)}(x; q) = \sum_{(w, c) \in \text{WRib}_k(c, D)} q^{\text{inv}_k(w, c)} x^w = \sum_{(w, c) \in \text{WRib}_k(c, D)} q^{\text{inv}_k(w, c)} Q_{\sigma(w)}(x),
\]

where \(x^w\) is the monomial \(x_1^{i_1} x_2^{i_2} \cdots \) when \(w\) has weight \(\pi\), and \(\sigma(w)\) is defined as in (2.4).

### 4.2. Dual equivalence for tuples of tableaux.

Let \(V_{\cdot, D}^{(k)}\) denote the set of permutations \(w\) such that \((w, c)\) is a standard \(k\)-ribbon word with \(\text{Des}_k(w, c) = D\).

Define the distance between two letters \(i\) and \(j\) of \(w \in V_{\cdot, D}^{(k)}\) by

\[
dist(w_i, w_j) = |c_i - c_j|,
\]

with the obvious extension \(\text{dist}(a_1, \ldots, a_l) = \max_{i,j} \{\text{dist}(a_i, a_j)\}\). Note that if \((w, c)\) is a standard \(k\)-ribbon word, then none of \(i - 1, i, i+1\) may occur with the same content.

Similar to Definition 3.1, define involution \(d_i\) and \(\tilde{d}_i\) on permutations in which \(i\) does not lie between \(i-1\) and \(i+1\) by

\[
d_i(\cdots i \cdots i \pm 1 \cdots i \mp 1 \cdots) = \cdots i \mp 1 \cdots i \pm 1 \cdots i \cdots,
\]

\[
\tilde{d}_i(\cdots i \cdots i \pm 1 \cdots i \pm 1 \cdots) = \cdots i \pm 1 \cdots i \mp 1 \cdots i \cdots,
\]
where all other entries remain fixed. Note that the former involution is precisely Haiman’s dual equivalence on permutations. For fixed $k$, combine these two maps into an involution $D_i^{(k)}$ by
\begin{equation}
D_i^{(k)}(w) = \begin{cases} 
\frac{d_i(w)}{d_i(w)} & \text{if } \text{dist}(i-1, i, i+1) > k \\
\frac{d_i(w)}{d_i(w)} & \text{if } \text{dist}(i-1, i, i+1) \leq k 
\end{cases}.
\end{equation}

**Proposition 4.4.** For $w$ a permutation, $c$ a content vector and $k \geq 0$ an integer, we have
\begin{align}
\text{Des}_k(w, c) &= \text{Des}_k(D_i^{(k)}(w), c), \\
\text{inv}_k(w, c) &= \text{inv}_{k}(D_i^{(k)}(w), c).
\end{align}

In particular, $D_i^{(k)}$ is a well-defined involution on $V_{c,D}^{(k)}$ that preserves the number of $k$-inversions.

**Proof.** If $i$ lies between $i-1$ and $i+1$ in $w$, then the assertion is trivial. Assume then that $i$ does not lie between $i-1$ and $i+1$ in $w$. If dist$(i-1, i, i+1) > k$ in $w$, then $\text{Des}_k(w, c) = \text{Des}_k(d_i(w), c)$ and $\text{Inv}_k(w, c) = \text{Inv}_k(d_i(w), c)$. Similarly, if dist$(i-1, i, i+1) \leq k$ in $w$, then $\text{Des}_k(w, c) = \text{Des}_k(d_i(w), c)$ and $\text{inv}_k(w, c) = \text{inv}_{k}(d_i(w), c)$ (though $\text{Inv}_k(w, c) \neq \text{Inv}_k(d_i(w), c)$). The result now follows.

For each content vector $c$ of length $n$, and $k$-descent set $D$, we construct a signed, colored graph $G_{c,D}^{(k)}$ of type $(n, n)$ on the vertex set $V_{c,D}^{(k)}$ as follows. Define the signature function $\sigma : V_{c,D}^{(k)} \to \{\pm 1\}^{n-1}$ by
\begin{equation}
\sigma(w)_i = \begin{cases} 
+1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w \\
-1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w
\end{cases}.
\end{equation}

By (4.10), $D_i^{(k)}$ is an involution on vertices of $V_{c,D}^{(k)}$ admitting an $i$-neighbor. Therefore for $1 < i < n$, we may define the $i$-colored edges $E_i^{(k)}$ to be the set of pairs $\{v, D_i^{(k)}(v)\}$ for each $v$ admitting an $i$-neighbor. Finally, we define
\begin{equation}
G_{c,D}^{(k)} = \left(V_{c,D}^{(k)}, \sigma, E_i^{(k)}\right).
\end{equation}

An example of $G_{c,D}^{(k)}$ is given in Figure 14, and additional examples may be found in Appendix B.

```
\begin{figure}[h]
\centering
\begin{tikzpicture}[scale=1]
\node[circle, draw] (v1) at (0,0) {1};
\node[circle, draw] (v2) at (1,0) {2};
\node[circle, draw] (v3) at (2,0) {3};
\node[circle, draw] (v4) at (0,1) {4};
\node[circle, draw] (v5) at (1,1) {3};
\node[circle, draw] (v6) at (2,1) {2};
\draw[->] (v1) -- (v2);
\draw[->] (v2) -- (v3);
\draw[->] (v3) -- (v4);
\draw[->] (v4) -- (v5);
\draw[->] (v5) -- (v6);
\end{tikzpicture}
\caption{The graph $G_{(-1,0,1,2),((1,1))}^{(2)}$ on domino tableaux of shape $(2,1,1)$.}
\end{figure}
```

By Corollary 4.3 and (2.11), the generating function for $G_{c,D}^{(k)}$ weighted by inv$_k(-, c)$ is given by
\begin{equation}
\sum_{v \in V_{c,D}^{(k)}} q^{\text{inv}_k(v, c)} Q_{\sigma(v)}(x) = \widetilde{G}_{c,D}^{(k)}(x; q).
\end{equation}

In particular, a formula for the Schur coefficients of the generating function for $G_{c,D}^{(k)}$ gives a formula for the Schur coefficients of the LLT polynomial $\widetilde{G}_{c,D}^{(k)}(x; q)$. For example, since the graph in Figure 14 is a dual equivalence graph, we have
\begin{equation}
\widetilde{G}_{(2),((1,1))}^{(2)}(x; q) = q s_{3,1}(x) + q^2 s_{2,1,1}(x).
\end{equation}

In general, $G_{c,D}^{(k)}$ will not satisfy dual equivalence graph axioms 4 or 6; see Appendix B for instances of this. However, these graphs do satisfy the other axioms as well as the following weakened version of axiom 4.

**Definition 4.5.** A signed, colored graph $G = (V, \sigma, E)$ is *locally Schur positive* if for every connected component $C$ of $(V, \sigma, E_{i-1} \cup E_i)$ and every connected component $D$ of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$, the restricted degree 4 and degree 5 generating functions
\begin{equation}
\sum_{v \in C} Q_{\sigma_{i-2,i-1,i}(v)}(x) \quad \text{and} \quad \sum_{v \in D} Q_{\sigma_{i-3,i-2,i-1,i}(v)}(x)
\end{equation}
are symmetric and Schur positive.
Comparing Figures 7 and 8 with the standard dual equivalence graphs of sizes 4 and 5 (see Figure 6), dual equivalence graph axiom 4 implies that $G_\lambda$ is locally Schur positive.

**Proposition 4.6.** For each content vector $c$ and $k$-descent set $D$, the graph $G_{c,D}^{(k)}$ satisfies dual equivalence graph axioms 1, 2, 3 and 5, is locally Schur positive, and the $k$-inversion number is constant on connected components.

**Proof.** Axiom 1 follows from the construction of $E^{(k)}$ using (4.9), and axiom 2 can easily be seen from equations (4.7) and (4.8). For axiom 3, suppose $\{w, x\} \in E_{i-1}^{(k)}$ and $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$. If $x = d_i(w)$, then both $i-2$ and $i+1$ must lie between $i-1$ and $i$. In particular, $\sigma(w)_{i-2} = -\sigma(w)_{i+1}$, $\sigma(w)_{i-2} = -\sigma(x)_{i+1}$. If $x = d_i(w)$, then $i-2$ must lie between the position of $i-1$ in $w$ and the position of $i-1$ in $x$. In particular, $i-2$ must lie between $i-1$ and $i$ in both $w$ and $x$, and so again $\sigma(w)_{i-2} = -\sigma(w)_{i+1}$. The result for $\{w, x\} \in E_i^{(k)}$ with $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$ is completely analogous. Axiom 5 follows from the fact that if $w$ admits both an $i$-neighbor and a $j$-neighbor for some $|i-j| \geq 3$, then $D_i^{(k)} D_i^{(k)}(w) = D_i^{(k)} D_i^{(k)}(w)$.

To establish local Schur positivity, a tedious but straightforward diagram chase shows that there are exactly 25 possible non-isomorphic connected components of $(V, \sigma, E_{i-1} \cup E_{i+1} \cup E_i)$ in $G_{c,D}^{(k)}$. Of these, 7 correspond to the standard dual equivalence graphs of type $(5,5)$ and the remaining 18 are locally Schur positive. Alternately, note that it suffices to consider $G_{c,D}^{(k)}$ for all content vectors $c$ of length 5 with $c_1 = 1$ and $c_5 \leq 25$ and all $k \leq 5$ and all possible $k$-descent sets. This gives finitely many cases to computer verify. Finally, the $k$-inversion number is constant on connected components of $G_{c,D}^{(k)}$ by Proposition 4.4.

As foreshadowed by Definition 4.5, the generating function of a connected component of the signed, colored graph for LLT polynomials is not, in general, a single Schur function, though it is always Schur positive. In Section 5, we describe an algorithm by which the edges of every connected component of $G_{c,D}^{(k)}$ can be transformed so that the resulting graph is indeed a dual equivalence graph. We do this inductively by constructing a sequence of signed, colored graphs

$$G_{c,D}^{(k)} = G_2, \ldots, G_n = G_{c,D}^{(k)}$$

on the same vertex set with the same signature function with the following properties. For each $i = 2, \ldots, n-1$, the graph $G_i$ satisfies dual equivalence graph axioms 1, 2, 3 and 5 as well as axiom 4’, and the $(i+1,N)$-restriction of $G_i$ satisfies axioms 4 and 6 (and so is a dual equivalence graph). Furthermore, vertices paired by $E_i$ in $G_i$ have the property that they lie on the same connected component of $(V, \sigma, E_2 \cup \cdots \cup E_i)$ in $G_{i-1}$. This construction therefore proves the following.

**Theorem 4.7.** For $\mu$ a $k$-tuple of (skew) shapes, let $c, D$ be the corresponding pair by (4.2), and let $G_{c,D}^{(k)}$ be the signed, colored graph constructed above. Then for every connected component $C$ of $G_{c,D}^{(k)}$, the sum

$$\sum_{v \in V(C)} Q_{\sigma(v)}(X)$$

is symmetric and Schur positive.

**Corollary 4.8.** Let $G_{c,D}^{(k)}$ be the dual equivalence graph resulting from the transformation of the graph $G_{c,D}^{(k)}$. Then for $\lambda$ a partition, we have

$$K_{\lambda, \mu}^{(k)}(q) = \sum_{C \in G_{c,D}^{(k)}} q^{\text{inv}_C(C)},$$

where the sum is taken over all connected components $C$ of $G_{c,D}^{(k)}$ that are isomorphic to $G_{\lambda}$. In particular, $K_{\lambda, \mu}^{(k)}(q) \in \mathbb{N}[q]$, and, by (2.19), $K_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$.

The proof of Theorem 4.7 is content of Section 5. Before delving into the proof, we consider two extremal cases of $G_{c,D}^{(k)}$, where the connected components have particularly nice Schur expansions that can be proved by more elementary means.

### 4.3. Special cases.

Since $\text{dist}(i-1, i, i+1) \geq 2$ for every $w \in V_{c,D}^{(k)}$, $D_i^{(1)}$ is just the standard elementary dual equivalence on $i-1, i, i+1$. Therefore $G_{c,D}^{(1)}$ is isomorphic to the standard dual equivalence graph $G_\lambda$ for a unique partition $\lambda$.

When $k \geq 3$, $E_i^{(k)}$ will not give the edges of a dual equivalence graph. For instance, if $w$ has the pattern 2431 with $\text{dist}(1, 2, 3) \leq k$, then $D_2^{(k)}(w)$ contains the pattern 3412. By axiom 4, a dual equivalence graph...
must have \( \{ w, D_2^{(k)}(w) \} \in E_2^{(k)} \cap E_3^{(k)} \). However, \( D_2^{(k)}(w) \neq D_3^{(k)}(w) \), so this is not the case for \( G_c^{(k)} \). Therefore for \( k \geq 3 \), Theorem 4.7 is the best we can hope for. When \( k = 2 \), however, this problematic case does not arise, and we have the following result.

**Theorem 4.9.** The graph \( G_c^{(2)} \) on 2-ribbon words with content vector \( c \) and 2-descent set \( D \) is a dual equivalence graph of type \((n,n)\) for which the 2-inversion number is constant on connected components.

**Proof.** By Proposition 4.6, it suffices to show that dual equivalence axiom 4 holds. Since \( k = 2 \), if \( x = d_i(w) \), then \( \sigma(w)_j = \sigma(x)_j \) for all \( j \neq i-1, i \). In particular, if \( \{ w, x \} \in E_i \) and \( \sigma(w)_{i-2} = -\sigma(x)_{i-2} \), then \( d_i(w) = x = d_{i-1}(w) \). This establishes axiom 4 when \( n \neq 4 \).

To prove that connected components of \((V_c^{(2)}, \sigma, E_c^{(2)} \cup E_c^{(2)} \cup E_c^{(2)})\) have the correct form, note that it suffices to show that if \( x = D_i^{(2)}(w) = D_i^{(2)}(w) \) and \( x \) admits an \( i \)-neighbor, then letting \( y = D_i^{(2)}(D_i^{(2)}(x)) \), we have \( D_i^{(2)}(y) = D_i^{(2)}(y) \). In this case, \( x \) must have \( i-2 \) and \( i+1 \) lying between \( i \) and \( i-1 \) which have contents more than 2 apart. Then in \( D_i^{(2)}(x) \), \( i-3, i-1 \) and \( i+1 \) will all lie between \( i \) and \( i-2 \) which must also have contents more than 2 apart. In \( y = D_i^{(2)}(D_i^{(2)}(x)) \), \( i-3 \) and \( i \) will both lie between \( i-2 \) and \( i-1 \) which must have contents more than 2 apart. Therefore \( D_i^{(2)}(y) = d_{i-2}(y) = d_{i-1}(y) = D_i^{(2)}(y) \). □

Since Theorem 4.9 does not use the transformations of Section 5, we obtain a simple proof of positivity of LLT polynomials when \( k = 2 \), and also of Macdonald polynomials indexed by a partition with at most 2 columns.

Next consider the case when \( k \geq c_n - c_1 \) and so \( D_i^{(k)} = d_i^{(k)} \) for all \( i \). In this case, there will be no double edges in \( G_c^{(k)} \). For the standard dual equivalence graphs, it is easy to see that \( G_\lambda \) has no double edges if and only if \( \lambda \) is a hook, i.e. \( \lambda = (m, 1^{n-m}) \) for some \( m \). Therefore the generating function for a dual equivalence graph with no double edges is a sum of Schur functions indexed by hooks. The analog of this fact for \( G_c^{(k)} \) is that the generating function will be a sum of skew Schur functions indexed by ribbons.

Let \( \nu \) be a ribbon of size \( n \). Label the cells of \( \nu \) from 1 to \( n \) in increasing order of content. Define the descent set of \( \nu \), denoted \( \text{Des}(\nu) \), to be the set of indices \( i \) such that the cell labelled \( i+1 \) lies south of the cell labelled \( i \). Define the major index of a ribbon by

\[
\text{maj}(\nu) = \sum_{i \in \text{Des}(\nu)} i.
\]

Notice that if \( R \) is a filling of a column, and we reshape \( R \) into a semi-standard ribbon as described in Section 2.4, say of shape \( \nu \), then (4.17) agrees with (2.13) in the sense that \( \text{maj}(\nu) = \text{maj}(R) \).

Any connected component of \( G_c^{(k)} \) such that \( D_i^{(k)} = d_i^{(k)} \) on the entire component not only has constant \( k \)-inversion number, but the relative ordering of the first and last letters of each vertex is constant as well. That is, for \( C \) a connected component of \( G_c^{(k)} \), \( w_1 > w_n \) for some \( w \in V(C) \) if and only if \( w_1 > w_n \) for all \( w \in V(C) \). In the affirmative case, say that \((1, n)\) is an inversion in \( C \).

**Theorem 4.10.** Let \( G_c^{(k)} \) be the signed, colored graph of type \((n,n)\) on \( k \)-ribbon words with contents \( c \) and \( k \)-descent set \( D \). Let \( C \) be a connected component of \( G_c^{(k)} \) such that \( D_i^{(k)}(v) = d_i^{(k)}(v) \) for all \( v \in V(C) \). Then

\[
\sum_{v \in V(C)} Q_{\sigma(v)}(x) = \sum_{v \in \text{Rib}(C)} s_v,
\]

where \( \text{Rib}(C) \) is the set of ribbons of length \( n \) with major index equal to \( \text{inv}_k(C) \) such that \( n-1 \) is a descent if and only if \((1, n)\) is an inversion in \( C \).

**Proof.** From the hypotheses on \( C \), we may assume that \( k = n, c = (1, \ldots, n) \) and \( D = \emptyset \). Therefore \( V_c^{(k)} \) is just the set of permutations of \([n]\) thought of as words. In this case, \( k \)-inversions are just the usual inversions for a permutation. By earlier remarks, for \( w, v \in V(C) \), \( \text{inv}(w) = \text{inv}(v) \) and \((1, n) \in \text{Inv}(w) \) if and only if \((1, n) \in \text{Inv}(v) \). In fact, it is an easy exercise to show that this necessary condition for two vertices to coexist in \( V(C) \) is also sufficient. That is to say, \( V(C) \) is the set of words \( w \) with \( \text{inv}(w) = \text{inv}(C) \) and \((1, n) \in \text{Inv}(w) \) if and only if \((1, n) \) is an inversion of \( C \).

Recall Foata’s bijection on words [Foaa8]. For \( w \) a word and \( x \) a letter, \( \phi \) is built recursively using an inner function \( \gamma_x \) by \( \phi(wx) = \gamma_x(\phi(w))x \). From this structure it follows that the last letter of \( w \) is the same.
as the last letter of $\phi(w)$. Furthermore, $\gamma_x$ is defined so that the last letter of $w$ is greater than $x$ if and only if the first letter of $\gamma_x(w)$ is greater than $x$. It is shown in [FS78] that $\phi$ preserves the descent set of the inverse permutation, i.e. $\sigma(w) = \sigma(\phi(w))$. Finally, the bijection satisfies

$$\text{maj}(w) = \text{inv}(\phi(w)).$$

Summarizing these properties, $\phi$ is a $\sigma$-preserving bijection between the following sets:

$$\{w \mid \text{inv}(w) = j \text{ and } (1, n) \in \text{Inv}(w)\} \sim \{w \mid \text{maj}(w) = j \text{ and } n-1 \in \text{Des}(w)\},$$

$$\{w \mid \text{inv}(w) = j \text{ and } (1, n) \notin \text{Inv}(w)\} \sim \{w \mid \text{maj}(w) = j \text{ and } n-1 \notin \text{Des}(w)\}.$$

A standard filling of a ribbon $\nu$ is just a permutation $w$ such that $\text{Des}(w) = \text{Des}(\nu)$. Therefore by (2.5), the Schur function $s_\nu$ may be expressed as

$$(4.19) \quad s_\nu(x) = \sum_{\text{Des}(w) = \text{Des}(\nu)} Q_{\sigma(w)}(x).$$

Applying $\phi$ to this formula yields (4.18).

\[\square\]

5. Transformation into a dual equivalence graph

5.1. **Resolving axiom 4.** In this section, we present an algorithm to alter edges of a graph until dual equivalence graph axiom 4 is satisfied. Axiom 4 can be thought of as restricting the lengths of 2-color strings in the following way. Figure 7 forces the number of edges of a nontrivial connected component of $E_{i-1} \cup E_i$ to be two, either with three distinct vertices or forming a cycle with two vertices. Similarly, Figure 8 forces the number of edges of a nontrivial connected component of $E_{i-2} \cup E_i$ to be one (in the case of $i$-type A) or four, where there are either five distinct vertices (in the case of $i$-type B) or four vertices forming a cycle (in the case of $i$-type C). In order to resolve dual equivalence graph axiom 4 for $G_{c,D}^{(k)}$, it will be useful track the following two conditions, both of which hold for $G_{c,D}^{(k)}$.

**Definition 5.1.** A signed, colored graph $G$ satisfies axiom 4' if the following two conditions hold:

(ax4'a) every nontrivial connected component of $E_{i-1} \cup E_i$ has either two or four edges, and

(ax4'b) every nontrivial connected component of $E_{i-2} \cup E_i$ has either one or at least four edges.

The key ingredients to resolving dual equivalence graph axiom 4 from axiom 4' are two transformations, the first of which alters the $i$-edges of connected components of $E_{i-1} \cup E_i$ which do not conform to Figure 7, and the second alters the $i$-edges of connected components of $E_{i-2} \cup E_{i-1} \cup E_i$ which do not conform to Figure 8. For the first transformation, note that axiom 4'a is equivalent to the condition that every connected component of $E_{i-1} \cup E_i$ either appears in Figure 7 (equivalently satisfies axiom 4) or appears in Figure 15. Assuming dual equivalence graph axioms 1, 2, 3, the left hand component of Figure 15 has degree 4 generating function $s_{3,1} + s_{2,2}$ or $s_{2,1,1} + s_{2,2}$, depending on whether the leftmost vertex has signature $- - +$ or $- - +$, and the right hand component has degree 4 generating function $2s_{2,2}$. In particular, these components are locally Schur positive.

**Figure 15.** The involution $\varphi_i$ used to redefine $E_i$ so that connected components of $(V, \sigma, E_{i-1} \cup E_i)$ conform to Figure 7.

The goal is to re-define $i$-edges of $G$ so that the number of instances of connected components appearing in Figure 15 decreases. From Figure 15 and dual equivalence graph axioms 1, 2 and 3, it follows that a connected component of $(V, \sigma, E_{i-1} \cup E_i)$ occurs in Figure 7 if and only if it does not contain a vertex $w$.
admitting an $i − 1$-neighbor such that $\sigma(w)_i = -\sigma(E_{i-1}(w))_i$ but $E_{i-1}(w) \neq E_i(w)$. Define $W_i(\mathcal{G})$ to be the set of all such vertices bearing witness to the failure of Figure 7, i.e.

$$(5.1) \quad W_i(\mathcal{G}) = \{ w \in V \mid \{ w, v \} \in E_{i-1} \setminus E_i \text{ and } \sigma(w)_i = -\sigma(v)_i \text{ for some } v \in V \}.$$ 

From the figure, it is easy to believe (and is proven below) that axioms 1, 2 and 3 will remain true if the vertical $E_i$ edges are removed and replaced with the dashed edges labelled by $\varphi_i$. However, making this change will not necessarily preserve axiom 5. For that, we introduce the notion of the $i$-package of a vertex.

**Definition 5.2.** For $w$ a vertex of a signed, colored graph of type $(n, N)$ satisfying axioms 1, 2, 3 and 5, the $i$-package of $w$ is the connected component of $(V, \sigma, E_2 \cup \cdots \cup E_{i-3} \cup E_{i+3} \cup \cdots \cup E_{n-1})$ containing $w$.

By axiom 2, both $\sigma_{i-1}$ and $\sigma_i$ are constant on $i$-packages. Therefore $w$ admits an $i$-neighbor, in the sense of Definition 3.12, if and only if every vertex of the $i$-package of $w$ admits an $i$-neighbor. Furthermore, by axiom 5, knowing $E_i(w)$ determines $E_i$ on the entire $i$-package of $w$. For this reason, each time we modify the $i$-edge of a vertex, we must also alter the $i$-edges of each vertex on the same $i$-package. The following result ensures that this is well-defined for the proposed alteration.

**Lemma 5.3.** Let $\mathcal{G} = (V, \sigma, E)$ be a signed, colored graph of type $(n, N)$ satisfying dual equivalence graph axioms 1, 2, 3 and 5, and suppose that the $(i − 1, N)$-restriction of $\mathcal{G}$ is a dual equivalence graph and that $E_{i-2} \cup E_{i-1}$ and $E_{i-1} \cup E_i$ satisfy axiom 4. Then for any $w \in W_i(\mathcal{G})$, there exists an isomorphism between the $(i-2, i-2)$-restrictions of the $i$-packages of $w$ and $E_{i-1}(w)$. Moreover, if the $(i, N)$-restriction of $\mathcal{G}$ satisfies dual equivalence graph axiom 4, then this isomorphism preserves the $i$-type of corresponding vertices.

**Proof.** Recall that $E_{i-1}$ may be regarded as an involution on vertices that admit an $i − 1$-neighbor. Regarded as such, by axioms 1, 2 and 5, $E_{i-1}$ in fact gives an isomorphism between the $(i − 3, i − 3)$-restrictions of the $i$-packages of $w$ and $E_{i-1}(w)$. If $\sigma(w)_{i-3} = -\sigma(E_{i-1}(w))_{i-3}$, then Figure 7 and axiom 3 imply that $E_{i-1}(w) = E_{i-2}(w)$ as well. Thus by axiom 2, we have $\sigma(w)_i = \sigma(E_{i-1}(w))_i$, and so $w \notin W_i(\mathcal{G})$. Therefore $\sigma_{i-3}$ is preserved by $E_{i-1}$ at vertices of $W_i(\mathcal{G})$. By axiom 5, regarded as involutions, $E_{i-1}$ commutes with $E_h$ for all $h \leq i − 4$, and so every vertex of the $i − 1$-package of $w$ lies in $W_i(\mathcal{G})$. Hence we have an isomorphism between the $(i-3, i-2)$-restrictions of the corresponding $i$-packages.

By Proposition 3.13, there exists an isomorphism between the $(i − 3, i − 2)$-restrictions of the $i$-packages of $w$ and $E_{i-1}(w)$ and the augmented dual equivalence graph $\mathcal{G}_{\mu, A}$ for a unique partition $\mu$ of $i − 3$ and a unique augmenting tableau $A$ with a single cell containing $i − 2$. By Theorem 3.17, these isomorphism extend consistently across $E_{i-3}$ edges to give an isomorphism between the $(i−2, i−2)$-restrictions of the connected components containing $w$ and $u$ and $\mathcal{G}_\lambda$ where $\lambda$ is the shape of $\mu$ augmented by $A$. In particular, the $(i−2, i−2)$-restrictions of the $i$-packages of $w$ and $u$ are isomorphic.

In a dual equivalence graph, if $T$ is a vertex admitting both an $i$-neighbor and an $i − 1$-neighbor such that $E_i(T) = E_{i-1}(T)$, then no vertex on the $i$-package of $T$ has $i$-type $C$. Therefore, since the surjective morphism to $\mathcal{G}_\lambda$ necessarily preserves $i$-type, no vertex on the $i$-package of any vertex of $W_i(\mathcal{G})$ can have $i$-type $C$. The remaining $i$-types are distinguished by whether the vertex in question admits an $i − 2$-neighbor, and so since the morphism preserves $\sigma_{i-3}$ and $\sigma_{i-2}$, it must also preserve $i$-type in this case.

For any vertex $w \in W_i(\mathcal{G})$, denote the isomorphism of Lemma 5.3 by $\phi$. We use $\phi$ to define an involution $\varphi_i^w$ on all vertices of $V$ admitting an $i$-neighbor as follows.

$$(5.2) \quad \varphi_i^w(u) = \begin{cases} 
\phi(u) & \text{if } u \text{ lies on the } i\text{-package of } w \text{ or } E_{i-1}(w), \\
E_i\phi E_i(u) & \text{if } E_i(u) \text{ lies on the } i\text{-package of } w \text{ or } E_{i-1}(w), \\
E_i(u) & \text{otherwise.}
\end{cases}$$

Define $E'_i$ to be the set of pairs $\{ v, \varphi_i^w(v) \}$ for each $v$ admitting an $i$-neighbor. Define a signed, colored graph $\varphi_i^w(\mathcal{G})$ of type $(n, N)$ by

$$(5.3) \quad \varphi_i^w(\mathcal{G}) = (V, \sigma, E_2 \cup \cdots \cup E_{i-3} \cup E'_{i} \cup E_{i+1} \cup \cdots \cup E_{n-1}).$$

**Proposition 5.4.** Let $\mathcal{G}$ be a locally Schur positive graph of type $(n, N)$ satisfying dual equivalence graph axioms 1, 2, 3 and 5 and axiom 4', and suppose that the $(i − 1, N)$-restriction of $\mathcal{G}$ is a dual equivalence graph and the $(i, N)$-restriction of $\mathcal{G}$ satisfies dual equivalence graph axiom 4. Then for any $w \in W_i^\mathcal{G}$, the graph $\varphi_i^w(\mathcal{G})$ satisfies axioms 1, 2, 3 and 5, axiom 4', and $W_i(\varphi_i^w(\mathcal{G}))$ is a proper subset of $W_i(\mathcal{G})$. 


Proof. Any axioms not involving vertices lying on the $i$-package of $w$, $E_{i-1}(w)$, $E_i(w)$ or $E_iE_{i-1}(w)$ necessarily hold for $\varphi_i^w(G)$ since they hold for $G$. Axiom 1 follows since all vertices in question admit an $i$-neighbor and all remain part of an (albeit different) $i$-edge. Axiom 5 is immediate from the extension of $\varphi_w$ to $i$-packages. For axioms 2 and 3, we may assume that $v = \varphi_w(u)$ and \{u, v\} \notin E_i. It is clear from the definition of $\varphi_w$ that $\sigma(u)_{i-j} = -\sigma(v)_{i-j}$ for $j = i-1, i$. Since $G$ satisfies axiom 2, $\sigma(u)_h = \sigma(v)_h$ for $h < i-3$ and $h \geq i+1$. To see that $\sigma(u)_{i-3} = \sigma(v)_{i-3}$, note that this holds for $u, v$ on the $i$-package of $w$ and $E_{i-1}(w)$ by Lemma 5.3, and so too for $E_i(w)$ and $E_iE_{i-1}(w)$ since $E_i$ preserves $\sigma_{i-3}$ by axiom 2.

If $\sigma(u)_{i-2} = \sigma(v)_{i-2}$, then we must have $u, v \in W_i(G)$, so \{u, v\} $\in E_{i-1}$, and in particular $\sigma(u)_{i-2} = -\sigma(u)_{i-1}$. Note that for $u, v \notin W_i(G)$ or $E_i(W_i(G))$, we have $\sigma(u)_h = \sigma(E_i(v))_h$ and $\sigma(v)_h = \sigma(E_i(u))_h$ for $h = i, i+1$. Thus if $\sigma(u)_{i+1} = -\sigma(v)_{i+1}$, then $u, v \in E_i(W_i(G))$. In this case $\sigma(E_i(u))_{i+1} = \sigma(E_i(v))_{i+1}$ by axiom 2, so we must have $\sigma(u)_{i+1} = -\sigma(E_i(v))_{i+1}$ or $\sigma(v)_{i+1} = -\sigma(E_i(u))_{i+1}$. Either case implies $\sigma(u)_{i+1} = -\sigma(v)_{i}$ by axiom 3. Therefore we have established dual equivalence graph axioms 1, 2, 3 and 5.

By dual equivalence graph axiom 5, as involutions, both $E_i$ and $E_{i-1}$ commute with $E_h$ for $h < i-3$ and $h \geq i+3$. In particular, the connected component of $E_{i-1} \cup E_i$ containing $u$ satisfies axiom 4$'$a if and only if the connected component of $E_{i-1} \cup E_i$ containing $E_h(u)$ satisfies axiom 4$'$a and $u \in W_i(G)$ if and only if $E_h(u) \in W_i(G)$. Figure 15 shows that axiom 4$'$a is preserved at vertices in $W_i(G)$, so the only potential difficulty in extending this result along $i$-packages is when crossing an $E_{i-3}$ edge that is not a double edge with $E_{i-4}$. By dual equivalence axiom 6, this happens at most once on an $i$-package, so we may assume it occurs at $w$.

![Figure 16](image_url)  
**Figure 16.** The two possibilities when $w \in W_i$ has $i-1$-type B.

First suppose that $w$ has $i-1$-type B. Then by dual equivalence graph axioms 1, 2, 3 and 4, we have one of the two situations depicted in Figure 16. Axiom 4$'$b ensures that if $E_{i-1}(u) = E_{i-2}(u)$, then the connected component of $E_{i-1} \cup E_i$ containing $u$ must have only two edges, else the connected component of $E_{i-2} \cup E_i$ containing $u$ will have exactly two edges. Thus for both graphs in Figure 16, connected components of $E_{i-1} \cup E_i$ in $\varphi_i^w(G)$ have zero, two or four edges. Furthermore, the two connected components of $E_{i-2} \cup E_i$ containing $w$ and $E_{i-3}(w)$ will become one connected component with four edges containing $w$ and $E_{i-1}(w)$ and one other component, so again connected components of $E_{i-2} \cup E_i$ will have the correct number of edges in $\varphi_i^w(G)$. If instead $w$ has $i-1$-type C, then we have one of the situations depicted in Figure 17. In either case, connected components of $E_{i-1} \cup E_i$ will have the correct number of edges after applying $\varphi_i^w$, as well as connected components of $E_{i-2} \cup E_i$ assuming the result at $w$ and $E_{i-1}(w)$. This establishes axiom 4$'$ for the $(i+1, N)$-restriction of $\varphi_i^w(G)$ and shows $W_i(\varphi_i^w(G)) \subset W_i(G)$ as a proper subset.

![Figure 17](image_url)  
**Figure 17.** The two possibilities when $w \in W_i$ has $i-1$-type C.
The only remaining consideration for axiom 4’a is connected components of $E_i \cup E_{i+1}$. If no vertex on the connected component of $E_{i-1} \cup E_i$ containing $w$ lies in $W_{i+1}$, then the result is clear. By dual equivalence graph axioms 1, 2 and 3, the only way for a connected component of $E_{i-1} \cup E_i$ with four edges and a connected component of $E_i \cup E_{i+1}$ with four edges to overlap are the two cases shown in Figure 18. A case analysis for which vertices admit $i-1$-neighbors and which admit $i+1$-neighbors shows that $\varphi_i^x$ preserves the number of edges in connected components of $E_i \cup E_{i+1}$. This result extends down $i$-packages since $E_h$ commutes with $E_{i+3}$ for $h \leq i-3$ by dual equivalence graph axiom 5, and it extends up $i$-packages since those vertices will again lie in $W_i$ by previous remarks. Finally, connected components of $E_i \cup E_{i+2}$ have the desired form for vertices of $W_i$ since $E_{i-1}$ commutes with $E_{i+2}$, and the result follows on $i$-packages for the same reasons as before. Therefore axiom 4’ also holds for $\varphi_i^x(G)$.

Figure 18. Possible components when a vertex lies in $W_i \cap W_{i+1}$.

Next, we characterize components of $E_{i-2} \cup E_{i-1} \cup E_i$ that do not appear in Figure 8. This happens if and only if the component contains a vertex $x$ with $i$-type C such that $E_i(x)$ also has $i$-type C and $E_{i-2}$ and $E_i$ do not commute at $x$, i.e. $E_{i-2}E_i(x) \neq E_iE_{i-2}(x)$. For this characterization to make sense without axiom 4, we must alter the notion of the $i$-type of a vertex. As before, $w$ has $i$-type A if and only if $w$ admits an $i$-neighbor but not an $i-2$-neighbor. For $w$ admitting both an $i-2$-neighbor and an $i$-neighbor, $w$ has $i$-type B if and only if $w$ admits an $i-1$-neighbor and $E_{i-2}(w) = E_{i-1}(w)$ or $w$ does not admit an $i-1$-neighbor but $E_iE_{i-2}(w)$ does, and otherwise $w$ has $i$-type C. This extension essentially changes the dotted $i$-edge for a solid $i$-edge and the solid $i-1$-edge for a dotted $i-1$-edge on the left hand side of $i$-type B in Figure 10.

Figure 19. The involution $\psi_i$ used to redefine $E_i$ so that connected components of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ conform to Figure 8.

With this notion of $i$-type, define $X_i(G)$ to be those vertices of $G$ that bear witness to the failure of Figure 8,

$$X_i(G) = \{ x \in V \mid x \text{ and } E_i(x) \text{ have } i\text{-type C, but } E_{i-2}E_i(x) \neq E_iE_{i-2}(x) \}.$$  

Note that this definition makes sense under the assumption of axiom 4’b. To define the map $\psi_i$ as indicated in Figure 19, we must first show that we have the necessary isomorphism of $i$-packages. This clearly holds for edges $E_j$ with $j \geq i+3$ by dual equivalence graph axiom 5, and the following lemma ensures that it holds for $E_h$ with $h \leq i-3$ as well.

**Lemma 5.5.** Let $G = (V, \sigma, E)$ be a locally Schur positive graph of type $(n, N)$ satisfying dual equivalence graph axioms 1, 2, 3 and 5 and axiom 4’, and suppose that the $(i, N)$-restriction of $G$ is a dual equivalence graph. Then for any $x \in X_i(G)$, there exists an isomorphism between the $i$-packages of $x$ and $E_i(x)$ and also between the $i$-packages of $E_{i-2}(x)$ and $E_{i-2}E_i(x)$. Moreover, these isomorphisms preserve the $i$-type of corresponding vertices.
Proof. By axioms 1, 2 and 5, $E_i$ may be regarded as an isomorphism between the $(i-2, i-2)$-restrictions of the $i$-packages of $x$ and $E_i(x)$. By the proof of the latter claim in Lemma 5.3, no vertex of $W_i(G)$ admitting both an $i$-neighbor and an $i-1$-neighbor may have $i$-type $C$. Therefore if $u$ admits an $i$-neighbor and has $i$-type $C$, then $\sigma_{i-2}(v) = \sigma_{i-2}(E_i(v))$ for every $v$ on the $i$-package of $u$. In particular, $E_i$ preserves $\sigma_{i-2}$ on the $i$-package of any vertex in $X_G$. Exactly as in the proof of Lemma 5.3, the isomorphism between the $(i-2, i-1)$-restrictions of the connected components containing $x$ and $E_i(x)$ extends to a morphism of $(i-1, i-1)$-restrictions, which we may now restrict to get an isomorphism between the $(i-2, i-2)$-restrictions of the $i$-packages of $E_{i-2}(x)$ and $E_{i-2}E_i(x)$.

For $x \in X_i(G)$, denote the isomorphism of Lemma 5.5 by $\phi$. Similar to the construction of $\varphi_i^x$, we use $\phi$ to define an involution $\psi_i^x$ on all vertices of $V$ admitting an $i$-neighbor as follows.

$$
\psi_i^x(u) = \begin{cases} 
\phi(u) & \text{if } u \text{ lies on the }\text{-package of } E_{i-2}(x) \text{ or } E_{i-2}E_i(x), \\
E_i\phi E_i(u) & \text{if } E_i(u) \text{ lies on the }\text{-package of } E_{i-2}(x) \text{ or } E_{i-2}E_i(x), \\
E_i(u) & \text{otherwise}.
\end{cases}
$$

(5.5)

Now define $E'_i$ to be the set of pairs $\{v, \psi_i^x(v)\}$ for each $v$ admitting an $i$-neighbor. Define a signed, colored graph $\psi_i^x(G)$ of type $(n, N)$ by

$$
\psi_i^x(G) = (V, \sigma, E_2 \cup \cdots \cup E_{i-1} \cup E'_i \cup E_{i+1} \cup \cdots \cup E_n).
$$

(5.6)

Proposition 5.6. Let $G$ be a locally Schur positive graph of type $(n, N)$ satisfying dual equivalence graph axioms 1, 2, 3 and 5 and axiom 4' such that the $(i, N)$-restriction of $G$ is a dual equivalence graph. Then for any $x \in X_i(G)$, the graph $\psi_i^x(G)$ satisfies axioms 1, 2, 3 and 5 and the $(i+1, N)$-restriction of $\psi_i^x(G)$ is locally Schur positive and satisfies axiom 4'. Moreover, $W_i(\psi_i^x(G)) = W_i(G)$ and $X_i(\psi_i^x(G))$ is a proper subset of $X_i(G)$.

Proof. The $(i, N)$-restriction of $\psi_i^x(G)$ is the $(i, N)$-restriction of $G$, so it is still satisfies the same conditions. As with the proof of Proposition 5.4, axioms not involving new $i$-colored edges necessarily hold for $\psi_i^x(G)$ since they hold for $G$, and axioms 1 and 5 follow easily from the definition of $\psi_i^x$. Axiom 2 follows from the observation that $E_{i-2}(v) \neq E_{i-3}(v)$ for any $v \in X_G$, since axiom 5 for $E_i$ and $E_{i-3}$ would then lead to a contradiction. For axiom 3, note that $\sigma_{i-2}(v) = \sigma_{i-2}(\psi_i^x(v))$ for all $v$ on the $i$-packages of $x$ and $E_i(x)$. If $\sigma(u)_{i+1} = -\sigma(\psi_i^x(u))_{i+1}$ for some $u$ on the $i$-package of $x$, then axiom 2 ensures $\sigma(E_{i-2}(u))_{i+1} = -\sigma(E_{i-2}(\psi_i^x(u)))_{i+1}$. Therefore by axiom 3, $\sigma(E_{i-2}(u))_{i+1} = -\sigma(E_{i-2}(\psi_i^x(u)))_{i+1}$ and so $\sigma(u)_{i+1} = -\sigma(u)_{i+1}$. Furthermore, in this case axioms 2 and 5 dictate that either $\sigma(E_i(u))_{i+1} = -\sigma(u)_{i+1}$ or $\sigma(E_i(\psi_i^x(u)))_{i+1} = -\sigma(\psi_i^x(u))_{i+1}$.

Therefore if $\sigma(z)_{i+1} = -\sigma(\psi_i^x(z))_{i+1}$ for some $z$ on the $i$-package of $x$, then $\sigma(z)_{i+1} = -\sigma(z)_{i+1}$.

The definition of $X_i(G)$ precludes any of the altered $E_i$ edges from changing $\sigma_{i+2}$, and so these edges are final edges for connected components of $(V, \sigma, E_{i-1} \cup E_i)$, as indicated by Figure 19. A quick inspection of the figure shows that the lengths of connected components of $(V, \sigma, E_{i-1} \cup E_i)$ remain unchanged after applying $\psi_i$, and this result persists on $i$-packages. Therefore axiom 4'a remains in tact for the $(i+1, N)$-restriction of $\psi_i^x(G)$ and $W_i(\psi_i^x(G)) = W_i(G)$.

Finally, since $G$ and $\psi_i^x(G)$ have the same signature function and $E_{i-2}$ and $E_{i-1}$ edges, the set of vertices having $i$-type $C$ is the same for both graphs. By Lemma 5.5, since corresponding vertices on the $i$-packages of $x$, $E_i(x)$, $E_{i-2}(x)$ and $E_{i-2}E_i(x)$ have the same $i$-type, the involution $\psi_i^x$ will ensure that for all vertices on these $i$-packages with $i$-type $C$, $E_{i-2}E_i(u) = E_iE_{i-2}(u)$. Since $x \in X_i(G)$, this ensures that $X_i(\psi_i^x(G)) \neq X_i(G)$. If $u$ lies on the $i$-package of $E_iE_{i-2}(x)$ and $u \in X_i(\psi_i^x(G))$, then $u$ must have $i$-type $C$, and so $E_i(u)$ must have also $i$-type $C$ since it lies on the $i$-package of $E_{i-2}(x)$. In this case, axiom 5 ensures that $E_{i-2}E_i(u) \neq E_iE_{i-2}(u)$, else $x \notin X_i(G)$. Therefore $u \in X_i(G)$, and so $X_i(\psi_i^x(G))$ is a proper subset of $X_i(G)$, and axiom 4'b holds for the $(i+1, N)$-restriction of $\psi_i^x(G)$. Since $\psi_i^x$ separates Schur positive components from $E_{i-2} \cup E_{i-1} \cup E_i$, the $(i+1, N)$-restriction remains locally Schur positive.

5.2. Resolving axiom 6. In this section we give an algorithm for transforming a graph satisfying dual equivalence graph axioms 1 through 5 into a dual equivalence graph by resolving axiom 6. Here we consider a graph $G$ satisfying axioms 1, 2, 3 and 5 such that the $(i, N)$-restriction of $G$ is a dual equivalence graph and the $(i+1, N)$-restriction of $G$ satisfies dual equivalence graph axiom 4. Our goal is to redefine $i$-edges so that the $(i+1, N)$-restriction of $G$ satisfies axiom 6 as well and so is a dual equivalence graph.
First observe that the hypotheses on $G$ exactly satisfy the hypotheses of Theorem 3.17. Therefore for each connected component $H$ of the $(i+1, i+1)$-restriction of $G$, there exists a (surjective) morphism $\phi$ from $H$ to $G_\lambda$ for a unique partition $\lambda$ of $i+1$, and the fiber over each vertex of $G_\lambda$ has the same cardinality. By Proposition 3.5 and Theorem 3.10, $H$ satisfies axiom 6 if and only if $\phi$ is an isomorphism. When $\phi$ is not an isomorphism, we have a situation as in Figure 20. For an example that arises from transforming an LLT graph, see Figures 36 and 37 in Appendix B.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure20.png}
\caption{Two possible two-fold covers of a dual equivalence graph $G_\lambda$ with $\alpha, \beta, \gamma, \delta \subseteq \lambda$ the four partitions obtained from $\lambda$ by removing a single cell. Here vertices represent isomorphism classes and edges indicate when two components have vertices connected by an $E_i$ edge.}
\end{figure}

Similar to the previous transformations, we will define an involution $\theta_i$ on vertices of $H$ admitting an $i$-neighbor as indicated in Figure 21 and use it to redefine $i$-edges until no violations of axiom 6 remain. To do this, we need to characterize when two connected components of the $(i, i)$-restriction of $H$ can be paired without violating dual equivalence axiom 3. Recall that partitions of size $i$ contained in a fixed partition of size $i+1$ are totally ordered by dominance and that $\sigma_{i+1}$ is constant on connected components of the $(i, i)$-restriction of $G$ by dual equivalence axiom 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure21.png}
\caption{The involution $\theta_i$ used to redefine $E_i$ so that axiom 6 holds.}
\end{figure}

**Definition 5.7.** Let $H$ be a connected signed, colored graph of type $(i+1, N)$ satisfying dual equivalence graph axioms 1, 2, 3, 4, 5 such that the $(i, N)$-restriction of $H$ is a dual equivalence graph. Let $A$ and $B$ be two connected components of the $(i, N)$-restriction of $H$, say with $A \cong G_\alpha$ and $B \cong G_\beta$ where $\alpha > \beta$ in dominance order. Then $A$ and $B$ are called $i$-incompatible if $\sigma_{i+1}(A) = -1$ and $\sigma_{i+1}(B) = +1$. Otherwise $A$ and $B$ are called $i$-compatible.

The motivation for Definition 5.7 is that $A$ and $B$ are $i$-compatible if and only if axiom 3 is satisfied for $i$-edges pairing a vertex in $A$ with a vertex in $B$. Similar to the argument in the proof of Proposition 3.13, let $w \in A$ and $v \in D$ and suppose $\{w, v\} \in E_i$. Given the assumptions on $\alpha$ and $\beta$, we have $\sigma_{i-1}(w) = --$ and $\sigma_{i-1}(v) = +-. Therefore axiom 3 fails for this edge if and only if $\sigma_{i-1,w}(w) = ++$ and $\sigma_{i-1,v}(v) = --$, equivalently if and only if $A$ and $B$ are $i$-incompatible.

Fix a connected component $H$ of the $(i+1, i+1)$-restriction of $G$. If $\sigma_{i+1}(B) \equiv +1$ for every connected component of the $(i, i)$-restriction of $H$, then let $\mu$ be the maximum partition of $i$ contained in $\lambda$ and let $C$ be any connected component of the $(i, i)$-restriction of $H$ isomorphic to $G_\mu$. Otherwise, let $\mu \subset \lambda$ be the maximum partition (with respect to dominance order) such that there is a connected component of the $(i, i)$-restriction of $H$, say $C$, such that $C \cong G_\mu$ and $\sigma_{i+1}(C) \equiv -1$. Let $E_i(C)$ be the union of all connected components $B$ of the $(i, i)$-restriction of $H$ such that $B \neq C$ and $\{w, u\} \in E_i$ for some $w \in C$ and some $u \in B$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure22.png}
\caption{The involution $\theta_i$ used to redefine $E_i$ so that axiom 6 holds.}
\end{figure}
For each connected component $B'$ of the $(i, i)$-restriction of $H$, let $\phi_{B'}$ be the (unique) isomorphism from $B'$ to some (unique) $B \subset E_i(C)$. Finally, define the involution $\theta_i^C$ as follows.

\[
\theta_i^C(u) = \begin{cases} 
\phi_{B'}(E_i(u)) & \text{if } u \in E_i(C) \text{ and } E_i(u) \in B', \\
E_i(\phi_{B'}(u)) & \text{if } E_i(u) \in E_i(C) \text{ and } u \in B', \\
E_i(u) & \text{otherwise.}
\end{cases}
\]

(5.7)

Define $E_i'$ to be the set of pairs $\{v, \theta_i^C(v)\}$ for all vertices $v$ admitting an $i$-neighbor. Define a signed, colored graph $\theta_i^C(G)$ by

\[
\theta_i^C(G) = (V, \sigma, E_2 \cup \cdots \cup E_{i-1} \cup E_i' \cup E_{i+1} \cup \cdots \cup E_{n-1}).
\]

(5.8)

Proposition 5.8. Let $G$ be a locally Schur positive graph of type $(n, N)$ satisfying dual equivalence graph axioms 1, 2, 3 and 5 and axiom 4' such that the $(i, N)$-restriction of $G$ is a dual equivalence graph, and the $(i+1, N)$-restriction of $G$ satisfies dual equivalence graph axiom 4. Then for any connected component $H$ of the $(i+1, N)$-restriction of $G$ and for any restricted component $C$ chosen as described, the graph $\theta_i^C(G)$ satisfies dual equivalence graph axioms 1, 2, 3 and 5, the $(i, N)$-restriction of $\theta_i^C(G)$ is a dual equivalence graph and the $(i+1, i-1)$-restriction of $\theta_i^C(G)$ satisfies dual equivalence graph axiom 4.

Proof. Since only $i$-edges have been modified, the $(i, N)$-restriction of $\theta_i^C(G)$ remains a dual equivalence graph. The morphism $\phi$ from Theorem 3.17 and axioms 1 – 3 for $G$ along with the canonical choice for $C$ in $H$ ensures all axioms for $\theta_i^C(G)$ except axiom 3. For this, it suffices to show that the new $i$-edges pair $i$-compatible components. If $\sigma_{i+1} \equiv +1$ on all of $H$, then all restricted components are $i$-compatible. Consider $B, A, B', A'$ as indicated in Figure 21, say with $A \equiv A' \equiv G_\alpha$ and $B \equiv B' \equiv G_\beta$ where $\alpha > \beta$ in dominance order. Suppose that $\sigma_{i+1}(A) = -1$. Then by the choice of $C$, we must have $\mu > \alpha$. Also, $C$ and $B$ must be $i$-compatible since they share an $i$-edge in $G$, and so since $\sigma_{i+1}(C) = -1$ and $\mu > \alpha > \beta$, we must have $\sigma_{i+1}(B) = -1$ as well. Therefore $A$ and $B$ must be $i$-compatible. Now suppose that $\sigma_{i+1}(A') = -1$. Again, the choice of $C$ forces $\mu > \alpha > \beta$. Since $C$ and $A$ share an $i$-edge in $G$, they must be $i$-compatible, forcing $\sigma(A) = -1$. Similarly, $A$ and $B'$ share an $i$-edge in $G$, so they must be $i$-compatible as well forcing $\sigma(B') = -1$. Therefore $A'$ and $B'$ must also be $i$-compatible, and so axiom 3 will hold in $\theta_i^C(G)$. \( \square \)

Finally, we give a constructive proof of the following result, from which Theorem 4.7 follows.

Theorem 5.9. The graph $G^{(k)}_{c, D}$ can be modified to produce a dual equivalence graph with the same vertex set and signatures, and with connected components of the dual equivalence graph subdividing the connected components of the original graph.

Proof. As mentioned in Section 4.2, we will use $\varphi_i, \psi_i$ and $\theta_i$ to construct a sequence of graphs

\[
G^{(k)}_{c, D} = G_2, \ldots, G_{n-1} = \bar{G}^{(k)}_{c, D}
\]

on the same vertex set with the same signature function such that the $G_i$ satisfies dual equivalence graph axioms 1, 2, 3 and 5 and the $(i+1, N)$-restriction of $G_i$ satisfies dual equivalence graph axioms 4 and 6 as well, i.e. it is a dual equivalence graph. From the definitions of these maps, it follows that vertices paired by $E_i$ in $G_i$ have the property that they lie on the same connected component of $(V, \sigma, E_2 \cup \cdots \cup E_i)$ in $G_{i-1}$, so this will prove the result.

For the base case $G_2 = G^{(k)}_{c, D}$, note that dual equivalence graph axioms 4 and 6 are vacuous for a graph of type $(3, N)$. Therefore the result in this case follows from Proposition 4.6. While $i < n$, we construct $G_i$ from $G_{i-1}$ using $\varphi_i, \psi_i$ and $\theta_i$, so by Propositions 5.4, 5.6 and 5.8, dual equivalence graph axioms 1, 2, 3 and 5 will still hold for $G_i$. The crux of the argument is to see that local Schur positivity and axiom 4' can be preserved so that dual equivalence graph axioms 4 and 6 will hold for the $(i+1, N)$-restriction of $G_i$.

Let $G = G_i$. If $w \in W_i(G)$ is such that $\varphi^w_i(G)$ is locally Schur positive, then replace $G$ with $\varphi^w_i(G)$. The only way for this to fail is for the single connected component of $E_{i-2} \cup E_{i-1} \cup E_i$ containing $w$ to become two components in $\varphi^w_i(G)$, where one of the components begins with a double edge for $E_{i-2} \cap E_{i-1}$, ends with the new double edge for $E_{i-1} \cap E_i$ and the number of edges of $E_{i-2} \cup E_i$ is congruent to 2 modulo 4, but still larger than 4; for example, see Figure 22. To resolve these situations, apply $\psi_i$ for the appropriate $x$ on this component until no vertex on the connected component of $E_{i-2} \cup E_{i-1} \cup E_i$ containing $w$ lies in $X_i$.

Therefore we can apply $\varphi_i$ until $W_i(G)$ is empty while maintaining axiom 4' and local Schur positivity for the $(i+1, N)$-restriction of $G$, provided we apply $\psi_i$ as needed. From this point, apply $\psi_i$ until $X_i$ is empty.
The resulting graph, say \( \tilde{G} \), will then have \( W_i(\tilde{G}) = \emptyset = X_i(\tilde{G}) \). That is to say, the \((i+1,N)\)-restriction of \( G \) satisfies dual equivalence graph axiom 4. By Proposition 5.8, we can repeatedly apply \( \theta_i \) to \( G \), while maintaining dual equivalence graph axioms \( 1 - 5 \) for the \((i+1,N)\)-restriction of \( G \), each time reducing the number of violations of axiom 6. Therefore it remains only to show that local Schur positivity and axiom 4’ hold, or can be restored, for \( G \) beyond the \((i+1,N)\)-restriction.

By Proposition 5.4, the only potential problem in applying \( \varphi_i \) is that it could create a violation of local Schur positivity for \( E_{i-2} \cup E_{i-1} \cup E_i \), and this situation has already been resolved. The situation for \( \psi_i \) is somewhat more involved. Axiom 4’a for connected components of \( E_i \cup E_{i+1} \) is problematic precisely in the situation depicted in Figure 23 (possibly with one, but not both, of the lower four-edge components of \( E_i \cup X_{i+1} \) instead a double edge for \( E_i \cap X_{i+1} \)). For this case, we apply \( \varphi_{i+1} \) to the lower four-edge components of \( E_i \cup E_{i+1} \), then apply \( \psi_i^x \) as originally intended, and then apply \( \varphi_{i+1} \) again, as shown in Figure 23. Given the weaker hypotheses for Proposition 5.4, this application of \( \varphi_{i+1} \) is well-defined and will preserve axioms 1, 2, 3 and 5 as well as axiom 4’a for \( E_i \cup E_{i+1} \). Axiom 4’b for \( E_i \cup E_{i+2} \) and local Schur positivity for \( E_i \cup E_{i+1} \cup E_{i+2} \) now follow, essentially from dual equivalence graph axiom 5, and local Schur positivity for \( E_{i+1} \cup E_{i+2} \cup E_{i+3} \) can be stored later with \( \psi_{i+1} \) as discussed in the previous case (though, in fact, by comparing Figures 22 and 23, one can show that this is never necessary).

Finally, we consider what problems may arise when applying \( \theta_i \). Similar to the case with \( \psi_i \), if two \( E_i \) edges being interchanged have all four vertices in \( W_{i+1}(\tilde{G}) \), then it is possible that applying \( \theta_i \) will result in one \( E_i \cup X_{i+1} \) component with three edges and another with five edges, thereby violating axiom 4’a for \( E_i \cup E_{i+1} \). The remedy here is as before, first apply \( \varphi_{i+1} \) to the two \( E_i \) edges and then proceed with \( \theta_i \). The result will by a cycle of four edges for the \( E_i \cup X_{i+1} \) component, and so it will not only satisfy axiom 4’a, but axiom 4’b for \( E_i \cup E_{i+2} \) and local Schur positivity for \( E_i \cup E_{i+1} \cup E_{i+2} \) as well since two potentially distinct components have now been combined. Proposition 5.4 ensures the remaining cases for axiom 4’ and local Schur positivity based on \( \varphi_{i+1} \) (again, analyzing this case and Figure 22 shows that further complications do not arise in this case).
APPENDIX A. STANDARD DUAL EQUIVALENCE GRAPHS

Below we give the dual equivalence graphs of type $(6,6)$. As remarked earlier, the graphs for the conjugate shapes may be obtained by transposing each tableau and multiplying the signature coordinate-wise by $-1$. These examples illustrate axiom 5 and its interplay with dual equivalence graph axiom 4.

**Figure 24.** The standard dual equivalence graph $G_6$.

**Figure 25.** The standard dual equivalence graph $G_{5,1}$.
Figure 26. The standard dual equivalence graph $G_{4,2}$.

Figure 27. The standard dual equivalence graph $G_{3,3}$.

Figure 28. The standard dual equivalence graph $G_{4,1,1}$.

Figure 29. The standard dual equivalence graph $G_{3,2,1}$.

Appendix B. Graphs for tuples of tableaux

In this appendix we give several examples of connected components of the graphs $G^{(k)}_{c,D}$ constructed in Section 4 as well as the transformations to these graphs presented in Section 5. The graph in Figure 30 is a
connected component of the graph on domino tableaux of shape \((3, (2, 1))\). Comparing this graph with the examples above, it is isomorphic to \(G_{(4,2)}\). In particular, this demonstrates Theorem 4.9 which states that the graph on domino tableaux is always a dual equivalence graph.

![Figure 30. A connected component of the graph for domino tableaux of shape ((3), (2, 1)).](image)

The graph in Figure 31 is a connected component of the graph for the Macdonald polynomial \(\tilde{H}_{(4,1)}(X; q, t)\). Note that while the generating function of the graph is \(s_{3,2} + s_{(4,1)}\) which indeed is Schur positive, the graph itself is not a dual equivalence graph.

![Figure 31. A connected component of the graph for standard fillings of shape (4, 1).](image)

The graph in Figure 32, which is also not a dual equivalence graph, arises as a connected component of the graph for the Macdonald polynomial \(\tilde{H}_{(5)}(X; q, t)\). The transformation of this graph into a dual equivalence graph requires only \(\varphi_{3}\) and \(\varphi_{4}\). The result of the transformation is the dual equivalence graph given in Figure 33. For this example, axiom 6 is immediate from axiom 4 given the size of the graph, and it is mere coincidence that \(\psi_{4}\) was not needed to resolve axiom 4.

![Figure 32. A connected component for the graph for the 5-tuple ((1), (1), (1), (1), (1)) with generating function \(s_{3,2} + s_{3,1,1} + s_{2,2,1}\).](image)

The graph in Figure 34 is also not a dual equivalence graph and also arises as a connected component of the graph for the Macdonald polynomial \(\tilde{H}_{(5)}(X; q, t)\). Figure 34 shows the resulting dual equivalence graph after implementing the algorithm of Section 5.1, this time requiring \(\psi_{4}\) as well as \(\varphi_{3}\) and \(\varphi_{4}\). Again, axiom 6 is immediate from axiom 4 given the size of the graph.

The final example in Figure 36, first observed by Gregg Musiker, demonstrates the necessity of axiom 6 in the definition of dual equivalence graphs. This graph arises when transforming the graph for the Macdonald polynomial \(\tilde{H}_{(6)}(X; q, t)\). This graph satisfies dual equivalence axioms 1 through 5, but fails
Figure 33. The transformation of the graph in Figure 32 using $\varphi_3$ and $\varphi_4$.

Figure 34. A connected component of the graph for the 5-tuple $((1), (1), (1), (1), (1))$ with generating function $s_4,1 + s_{3,2} + s_{3,1,1}$.

Figure 35. The transformation of the graph in Figure 34 using $\varphi_3, \varphi_4$ and $\psi_4$.

axiom 6. Comparing with the standard dual equivalence graph $G_{(3,2,1)}$ in Appendix A, this graph is a two-fold cover of $G_{(3,2,1)}$ as expected from its generating function $2s_{(3,2,1)}(X)$. Figure 37 gives the isomorphism classes of the $(5,6)$-restriction of this graph, similar to the presentation in Figures 20 and 21.
Figure 36. The smallest graph satisfying dual equivalence graph axioms 1 – 5 but not 6.

Figure 37. The (5,6)-restriction of Figure 36 highlighting the two-fold cover of $G_{4,2,1}$. 