Low-Rank Discriminative Least Squares Regression for Image Classification

Zhe Chen, Xiao-Jun Wu, He-Feng Yin, and Josef Kittler, Life Fellow, IEEE

Abstract—Latest least squares regression (LSR) methods mainly try to learn slack regression targets to replace strict zero-one labels. However, the difference of intra-class targets can also be highlighted when enlarging the distance between different classes, and roughly persuading relaxed targets may lead to the problem of overfitting. To solve above problems, we propose a low-rank discriminative least squares regression model (LRDLSR) for multi-class image classification. Specifically, LRDILSR class-wisely imposes low-rank constraint on the intra-class regression targets to encourage its compactness and similarity. Moreover, LRDLSR introduces an additional regularization term on the learned targets to avoid the problem of overfitting. These two improvements are helpful to learn a more discriminative projection for regression and thus achieving better classification performance. Experimental results over a range of image databases demonstrate the effectiveness of the proposed LRDLSR method.

Index Terms—least squares regression, low-rank regression targets, overfitting, image classification

I. INTRODUCTION

LEAST squares regression (LSR) is a very popular method in the field of multicategory image classification. LSR aims at learning a projection to transform the original data into corresponding zero-one labels with minimum loss. Over the past decades, many LSR based variants have been developed, such as locally weighted LSR [1], local LSR [2], LASSO regression [3], kernel ridge LSR [4], kernel LSR [5], weighted LSR [6], a least-squares support vector machine (LS-SVM) [7] and partial LSR [8]. Besides, linear regression, sparse representation, collaborative representation, and probabilistic collaborative representation based classification methods (LRC, SRC, CRC and ProCRC) [9][10][11][12] also took advantages of the LSR framework to solve the representation coefficients.

However, there still exists many issues in the above LSR based methods. First, taking the zero-one label matrix as the regression targets is too strict and not suitable for classification, and calculating the least squares loss between the extracted features and the binary targets cannot ideally reflect the classification performance of regression model, especially under multi-class conditions. Such as the Euclidean distance of any two of the inter-class regression targets is a constant, i.e., $\sqrt{2}$, and for each sample, the difference value between the targets of the true and the false class identically equals to 1. These go against the expectation that the transformed inter-class features should be as far as possible. To solve this problem, some representative algorithms, i.e., discriminative LSR (DLSR) [13], retargeted LSR (ReLSR) [14], and group-wise ReLSR [15], were proposed to learn relaxed regression targets to replace original binary targets. Concretely, DLSR utilized the $\varepsilon$-dragging technique to encourage the inter-class regression targets changing in the opposite directions, thus enlarging the distances between different classes. Different to DLSR, ReLSR learned the regression targets from original data rather than directly adopting the zero-one labels of samples, in which the margins between classes are performed to be greater than 1. Lately, Wang et al. [14] proved that DLSR is a special model of ReLSR with the translation values setting to zeros and proposed a new formulation for ReLSR. With the new formulation, GRReLSR introduced a groupwise constraint to guarantee the intra-class samples have similar translation values. Besides, traditional LSR based methods also did not take into account the data correlations during the projection learning procedure, which may lose some useful structural information of data and result in the problem of overfitting. To explore more underlying relationships, Fang et al. [16] constructed a class-compactness-graph to ensure that the projected intra-class features are compact so that the overfitting problem can be mitigated to a degree. Wen et al. [17] proposed a novel framework called inter-class sparsity based DLSR (ICS_DLSR), which introduces an inter-class sparsity constraint on the DLSR model to make the projected features of each class have sparse structure. In fact, both of the RRLRLR and ICS_DLSR algorithms are based on the model of DLSR, that is the $\varepsilon$-dragging technique was adopted.

All the improved algorithms mentioned above are beneficial to improve the classification performance. But only using the $\varepsilon$-dragging technique or margin constraint to relax the label matrix will also amplify the difference among the intra-class regression targets, which would deteriorate the classification performance. In this paper, a novel relaxed targets based regression model named low-rank discriminative least squares regression (LRDLSR) is proposed to learn a more discriminative projection. Based on the model of DLSR, LRDLSR class-wisely imposes low-rank constraint on the relaxed regression targets to ensure the intra-class targets are compact and similar. In this way, the $\varepsilon$-dragging technique will be utilized more perfectly that both the intra-class similarity and the inter-class separability of regression targets can be guaranteed. Moreover, LRDLSR minimizes the energy of dynamic regression targets
to avoid the problem of overfitting.

The rest of this paper is organized as follows. First, related works are briefly introduced in Section II. The proposed LRDLSR model and the corresponding optimization procedure are described in Section III. The experimental results are presented in Section IV and Section V concludes this paper.

II. RELATED WORKS

Let $X = [x_1, x_2, ..., x_n] \in R^{d \times n}$ denotes the $n$ training samples from $c$ classes, where $d$ is the dimensionality of samples. $X_i \in R^{d \times n_i}$ denotes the sample subset of the $i$th class. $H = [h_1, h_2, ..., h_n] \in R^{c \times n}$ denotes the binary label matrix of $X$, where each column of $H$, i.e., $h_i = [0, ..., 0, 1, 0, ..., 0]^T \in R^c$, corresponding to a certain training sample, i.e., $x_i$. If sample $x_i$ belongs to the $p$th class, then the $p$th element of $h_i$ is 1 and all the others are 0.

A. Original LSR

The main idea of LSR is to learn a projection matrix that can well project the original training samples into binary label space, the objective function of LSR can be formulated as

$$\min_Q \|QX - H\|_F^2 + \lambda \|Q\|_F^2,$$

(1)

where $\|\cdot\|_F^2$ is the Frobenius norm of matrix and $\lambda$ is a positive regularization parameter and $Q$ is the projection matrix. The first term in problem (1) is a least squares loss function, while the second term is used to avoid the problem of overfitting. Obviously, (1) has a closed-form solution as

$$Q = HX^T (XX^T + \lambda I)^{-1}$$

(2)

Given a new sample $y$, LSR calculates its label as $k = \arg \max_j (Qy)_j$ where $(Qy)_j$ is the $j$th value of $Qy$.

B. DLSR and ReLSR

As previously said, making the regression features to pursue a strict zero-one labels is inappropriate for classification tasks. Unlike original LSR, DLSR [13] and ReLSR [14] aim to learn relaxed regression targets rather than using the binary labels $H$ as their targets. The main idea of DLSR is to enlarge the distance between the true and the false classes by using an $\varepsilon$-dragging technique, its regression model can be formulated as

$$\min_{Q,T,B} \|QX - (H + B \odot M)\|_F^2 + \lambda \|Q\|_F^2, \ s.t. \ M \geq 0$$

(3)

where $\odot$ denotes the Hadamard-product operator. $M \in R^{c \times n}$ is a non-negative $\varepsilon$-dragging label relaxation matrix. $B \in R^{c \times n}$ is a constant matrix which is defined as

$$B_{ij} = \begin{cases} +1, & \text{if } H_{ij} = 1 \\ -1, & \text{if } H_{ij} = 0 \end{cases}$$

(4)

Compared to original LSR, the regression targets of DLSR are extended to be $H' = H + B \odot M$ and the distances of inter-class targets are greater than $\sqrt{2}$ rather than a constant.

Likewise, ReLSR directly learns relaxed regression targets from original data to ensure samples being correctly classified with large margins. The ReLSR model is defined as

$$\min_{Q,T,B} \|T - QX - be_n\|_F^2 + \lambda \|Q\|_F^2$$

(5)

where $r_j$ indicates the true label of sample $x_j$. $T$ is optimized from $X$ with a large margin constraint which can reveal the class separability, thus ReLSR can perform more flexible than DLSR.

III. FROM DLSR AND RELSR TO LRDLSR

A. Problem Formulation and New Regression Model

Although DLSR and ReLSR can learn soft targets and maintain the closed-form solution for projection, overly seeking the projection with large margins will also result in the overfitting problem. As said before, exploring the data correlations are helpful to learn discriminative data representation. In the view of classification, both of the intra-class similarity and the inter-class incoherence of regression targets should be guaranteed. However, DLSR and ReLSR ignore the former, because their relaxation values are dynamic and random. Such as, the $\varepsilon$-dragging technique in DLSR and the margin constrain method in ReLSR will also promote the intra-class regression targets to be discrete. If the intra-class similarity of learned targets is weakened, it will definitely lose some discriminative information of data. Therefore, based on the model of DLSR, we propose a low-rank discriminative least squares regression model (LRDLSR) as follows

$$\min_{Q,T,M} \frac{1}{2} \|QX - T\|_F^2 + \alpha \|T - (H + B \odot M)\|_F^2 + \beta \sum_{i=1}^{c} \|T_i\|_* + \gamma \|T\|_F^2 + \lambda \|Q\|_F^2, \ s.t. M \geq 0$$

(6)

where $\alpha, \beta, \gamma$ and $\lambda$ are the regularization parameters and $\|\cdot\|_*$ denotes the nuclear norm (the sum of singular values). $Q$ and $T$ denote the projection matrix and the slack target matrix, respectively. The second term $\|T - (H + B \odot M)\|_F^2$ is used to learn relaxed regression targets with large inter-class margins, the third term $\sum_{i=1}^{c} \|T_i\|_*$ is used to learn similar intra-class regression targets, and the fourth term $\|T\|_F^2$ is used to avoid the overfitting problem of $T$.

With our formulation, we can observe that the difference between our LRDLSR and DLSR is that in LRDLSR we encourage the relaxed regression targets of each class to be low-rank so that the compactness and similarity of the regression targets from each class can be enhanced. By combining the $\varepsilon$-dragging technique, both of the intra-class similarity and inter-class separability of regression targets will be preserved, thus producing discriminative projection.

B. Optimization of LRDLSR

To directly solve optimization problem (6) is impossible because three variables $Q, T$ and $M$ are correlative. Therefore, the alternating direction multipliers method (ADMM) [18]
is exploited to optimize LRDLSR. In order to make (6) separable, we first introduce an auxiliary variable $P$ as follows:

$$
\min_{T,P,Q,M} \frac{1}{2} \|QX - T\|_F^2 + \frac{\alpha}{2} \|T - (H + B \odot M)\|_F^2 + \beta \sum_{i=1}^c \|P_i\|_* + \frac{\gamma}{2} \|T\|_F^2 + \frac{\lambda}{2} \|Q\|_F^2, \text{ s.t. } T = P, M \geq 0
$$

Then we can obtain the augmented Lagrangian function of (7):

$$
L(T, P, Q, M, Y) = \frac{1}{2} \|QX - T\|_F^2 + \frac{\alpha}{2} \|T - (H + B \odot M)\|_F^2 + \beta \sum_{i=1}^c \|P_i\|_* + \frac{\gamma}{2} \|T\|_F^2 + \frac{\lambda}{2} \|Q\|_F^2 + \frac{\mu}{2} \|T - P + \frac{Y}{\mu}\|_F^2
$$

where $Y$ is the Lagrangian multiplier, $\mu > 0$ is the penalty parameter. Next we update variables one by one.

**Update $T$:** By fixing variables $P, Q, M, T$ can be obtained by minimizing the following problem

$$
L(T) = \frac{1}{2} \|QX - T\|_F^2 + \frac{\alpha}{2} \|T - (H + B \odot M)\|_F^2 + \frac{\gamma}{2} \|T\|_F^2 + \frac{\mu}{2} \|T - P + \frac{Y}{\mu}\|_F^2
$$

Obviously, $T$ has a closed-form solution as

$$
T = (1 + \alpha + \gamma + \mu)^{-1} [QX + \alpha(H + B \odot M) + \mu P - Y]
$$

**Update $P$:** Given $T, Q$ and $M$, $P$ can be updated by

$$
L(P) = \beta \sum_{i=1}^c \|P_i\|_* + \frac{\mu}{2} \|T - P + \frac{Y}{\mu}\|_F^2
$$

We can use the singular value thresholding algorithm [19] to class-wisely optimize (11). The optimal solution of $P$ is

$$
P = I_\delta \left( T + \frac{Y}{\mu} \right)
$$

where $I_\delta(\cdot)$ is the singular value shrinkage operator.

**Update $Q$:** Analogously, $Q$ can be solved by minimizing

$$
L(Q) = \frac{1}{2} \|QX - T\|_F^2 + \frac{\lambda}{2} \|Q\|_F^2
$$

We set the derivative of $L(Q)$ with respect to $Q$ to zero, and obtain the following closed-form solution

$$
Q = TX^T(XTX^T + \lambda I)^{-1}
$$

**Update $M$:** After optimizing $T, P$ and $Q$, we can update relaxation matrix $M$ by

$$
\min_M \|T - (H + B \odot M)\|_F^2, \text{ s.t. } M \geq 0
$$

Let $R = T - H$, according to [13], the optimal solution of $M$ can be calculated by

$$
M = max(B \odot R, 0)
$$

The optimization procedures of LRDLSR are concluded in Algorithm 1. Next, we analyze the computational complexity of Algorithm 1. Similar to [17], we do not calculate the complexity of matrix addition, subtraction and multiplication operations. The main time-consuming steps of Algorithm 1 are:

1. Singular value decomposition in Eq. (11).
2. Matrix inverse in (14).

The complexity of singular value decomposition in Eq. (11) is $O(n^3)$. The complexity of pre-computing $X^T(XTX^T + \lambda I)^{-1}$ in Eq. (14) is $O(d^3)$. Thus the final time complexity for Algorithm 1 is about $O(d^3 + \tau n^3)$, where $\tau$ is the number of iterations.

**Algorithm 1. Optimizing LRDLSR by ADMM**

**Input:** Normalized training samples $X$ and its label matrix $H$; Parameters $\alpha, \beta, \gamma, \lambda$.

**Initialization:** $T = P = H, Q = 0, M = I^{c \times n}$, $Y = 0^{c \times n}$, $\mu_{max} = 10^8$, $tol = 10^{-6}$, $\mu = 10^{-5}$, $\rho = 1.1$.

While not converged do:

1. Update $T$ by using Eq. (10).
2. Update $P$ by using Eq. (11).
3. Update $Q$ by using Eq. (14).
4. Update $M$ by using Eq. (16).
5. Update Lagrange multipliers $Y$ as

$$
Y = Y + \mu(T - P).
$$

6. Update penalty parameter $\mu$ as

$$
\mu = \min(\mu_{max}, p\mu).
$$

7. Check convergence:

$$
if \|T - P\|_\infty \leq tol.
$$

End While

**Output:** $Q, T$ and $M$.

**IV. Experiments**

In order to verify the effectiveness of the proposed LRDLSR model, we compare it with three latest LSR model based classification methods, including DLSR [13], ReLSR [14], GReLSR [15], and three representation based classification methods, including LRC [9], CRC [11], ProCRC [12], on five real image datasets. For LRDLSR, DLSR, ReLSR, and GReLSR, we use the NN classifier. To make fair comparisons, we directly utilize the released codes of compared methods to implement experiments and seek the best parameters for them as much as possible. For our LRDLSR, the optimal values of $\alpha$ and $\beta$ are selected from the candidate set $\{0.0001, 0.001, 0.01, 0.1, 1\}$, and the optimal values of $\gamma$ and $\lambda$ are selected from the candidate set $\{0.001, 0.01, 0.1\}$, by using the cross-validation method. All the experiments are repeated ten times with the random splits of training and test samples. And then the average results and the standard deviations are reported. In experiments, the used image datasets can be divided into two types:

1. Face dataset: the AR [20], the CMU PIE [21], the Extended Yale B [22], and the Labeled Faces in the Wild (LFW) [23] datasets;
2. Object dataset: the COIL-20 [24] dataset.

**A. Experiments for Object Classification**

In this section, we validate the performance of our LRDLSR model on the COIL-20 object dataset which has 1440 images
TABLE I
RECOGNITION ACCURACIES (MEAN± STD%) OF DIFFERENT METHODS ON THE COIL-20 OBJECT DATABASE.

| Train No. | LRC    | CRC    | ProCRC | DLSR   | ReLSR  | GReLSR | LRDLSR(ours) |
|-----------|--------|--------|--------|--------|--------|--------|--------------|
| 10        | 92.30±1.15 | 89.09±1.48 | 90.61±0.95 | 93.27±1.43 | 93.65±1.94 | 90.98±1.62 | 95.12±1.22   |
| 15        | 94.89±1.33 | 92.58±1.27 | 94.53±0.85 | 96.25±0.75 | 96.75±0.72 | 93.60±0.83 | 97.78±0.86   |
| 20        | 97.49±0.51 | 94.15±1.15 | 96.17±0.82 | 97.52±0.67 | 98.17±0.67 | 95.65±0.82 | 98.51±0.85   |
| 25        | 98.32±0.60 | 94.99±1.24 | 97.53±0.68 | 98.67±0.53 | 98.90±0.85 | 96.30±0.84 | 99.24±0.59   |

TABLE II
RECOGNITION ACCURACIES (MEAN± STD%) OF DIFFERENT METHODS ON THE EXTENDED YALE B FACE DATABASE.

| Train No. | LRC    | CRC    | ProCRC | DLSR   | ReLSR  | GReLSR | LRDLSR(ours) |
|-----------|--------|--------|--------|--------|--------|--------|--------------|
| 10        | 82.18±0.92 | 91.85±0.61 | 91.74±0.86 | 87.95±1.10 | 89.68±0.94 | 88.46±1.00 | 91.18±0.65   |
| 15        | 89.43±0.58 | 94.76±0.66 | 95.41±0.76 | 93.37±0.99 | 93.98±0.52 | 93.13±0.82 | 95.07±0.66   |
| 20        | 92.00±0.77 | 96.39±0.56 | 96.74±0.26 | 95.73±0.68 | 96.14±0.54 | 95.25±0.50 | 96.84±0.36   |
| 25        | 93.73±0.79 | 97.69±0.40 | 97.58±0.37 | 97.34±0.55 | 97.75±0.64 | 97.06±0.37 | 98.16±0.46   |

TABLE III
RECOGNITION ACCURACIES (MEAN± STD%) OF DIFFERENT METHODS ON THE AR FACE DATABASE.

| Train No. | LRC    | CRC    | ProCRC | DLSR   | ReLSR  | GReLSR | LRDLSR(ours) |
|-----------|--------|--------|--------|--------|--------|--------|--------------|
| 3         | 28.73±0.99 | 71.42±0.59 | 76.16±1.12 | 73.58±1.63 | 73.53±1.47 | 74.77±1.45 | 78.80±0.76   |
| 4         | 37.21±1.13 | 78.50±0.67 | 83.58±0.82 | 80.47±1.36 | 81.46±0.79 | 82.54±1.24 | 86.20±0.45   |
| 5         | 44.69±1.22 | 83.54±0.67 | 87.33±0.74 | 85.33±0.93 | 86.43±0.94 | 87.35±1.21 | 90.16±0.75   |
| 6         | 52.95±1.54 | 86.79±0.71 | 90.32±0.66 | 88.18±0.78 | 88.98±0.99 | 89.96±0.73 | 92.23±0.80   |

TABLE IV
RECOGNITION ACCURACIES (MEAN± STD%) OF DIFFERENT METHODS ON THE CMU PIE FACE DATABASE.

| Train No. | LRC    | CRC    | ProCRC | DLSR   | ReLSR  | GReLSR | LRDLSR(ours) |
|-----------|--------|--------|--------|--------|--------|--------|--------------|
| 10        | 75.67±1.01 | 86.39±0.60 | 89.00±0.37 | 87.54±0.79 | 88.18±0.79 | 86.88±0.72 | 91.57±0.48   |
| 15        | 85.26±0.63 | 91.14±0.43 | 92.18±0.25 | 92.22±0.54 | 92.29±0.42 | 91.21±0.51 | 94.45±0.51   |
| 20        | 89.84±0.48 | 93.08±0.35 | 93.94±0.18 | 94.12±0.27 | 94.23±0.21 | 93.39±0.27 | 95.83±0.35   |
| 25        | 92.55±0.39 | 94.12±0.30 | 94.58±0.21 | 95.25±0.20 | 95.53±0.16 | 94.32±0.31 | 96.59±0.21   |

TABLE V
RECOGNITION ACCURACIES (MEAN± STD%) OF DIFFERENT METHODS ON THE LFW FACE DATABASE.

| Train No. | LRC    | CRC    | ProCRC | DLSR   | ReLSR  | GReLSR | LRDLSR(ours) |
|-----------|--------|--------|--------|--------|--------|--------|--------------|
| 5         | 29.99±2.21 | 31.67±1.16 | 33.19±0.99 | 30.43±1.38 | 31.43±1.13 | 36.76±1.37 | 37.20±1.66   |
| 6         | 32.37±1.36 | 34.27±1.04 | 35.90±0.93 | 32.35±1.62 | 34.46±1.51 | 39.22±0.92 | 39.99±1.22   |
| 7         | 35.53±1.69 | 35.96±1.40 | 36.87±1.55 | 34.67±2.45 | 37.50±2.61 | 43.02±2.19 | 43.82±1.23   |
| 8         | 36.98±1.82 | 37.92±1.50 | 38.24±1.15 | 36.27±1.65 | 38.72±1.22 | 44.39±1.77 | 44.88±1.58   |

of 20 classes. Each class consists of 72 images that are collected at pose intervals with 5 different degrees. In our experiments, all images are resized to 32×32 pixels beforehand. For each class, we randomly choose 10, 15, 20, 25 samples to train model and treat all the remaining images as test set. The average classification accuracies are reported in Table I. As shown in Table I, we can find that our LRDLSR algorithm can achieve much better classification results than all the remaining comparison methods, which proves the effectiveness of LRDLSR for the object classification tasks.

B. Experiments for Face Classification

In this section, we evaluate the classification performance of LRDLSR on four real face datasets.

(1) The Extended Yale B Dataset: The Extended Yale B database consists of 2414 face images of 38 individuals. Each individual has about 59-64 images. All images are resized to the size of 32×32 pixels in advance. We randomly select 10, 15, 20, and 25 images of each individual as training samples, and set the remaining images as test samples.

(2) The AR Dataset: We select a subset which consists of 2600 images of 50 women and 50 men and use the projected 540-dimensional features provided in [25]. In each individual, we randomly select 3, 4, 5, and 6 images as training samples and the remaining images are set as test samples.

(3) The CMU PIE Dataset: We select a subset of this dataset where each individual has 170 images that are collected under five different poses (C05, C07, C09, C27 and C29). All images
are resized to 32×32 pixels. We randomly select 10, 15, 20, and 25 images of each individual as training samples, and treat the remaining images as test samples.

(4) The LFW Dataset: Similar to [17], we use a subset of this dataset which consists of 1251 images of 86 individuals to implement experiments. Each individual has about 11-20 images. In our experiments, all images are resized to the size of 32×32. We randomly select 5, 6, 7, and 8 images from each individual as training samples and use the remaining images to test.

The average classification results on these four face datasets are reported in Tables II-V, respectively. It can be observed that our LRDLSR almost outperforms all the comparison algorithms on four face datasets. The main reason is that our LRDLSR can simultaneously guarantee the intra-class compactness and the inter-class irrelevance of slack regression targets so that more discriminative information can be preserved during the projection learning. And it is worth noting that the standard deviations in accuracies of LRDLSR are also very competitive which indicates LRDLSR is not sensitive to the combination of samples. Besides, we can find that the performance gain of LRDLSR is significant when the number of training samples per subject is small, which indicates our model is also suitable to the small-sample-size problem. Fig. 1 shows the t-SNE [26] visualization of the features on the Extended Yale B dataset which are extracted by DLSR, ReLSR and LRDLSR, respectively. We randomly select 5 samples for each individual to validate. It is obvious that the features extracted by LRDLSR model present ideal inter-class separability and intra-class compactness which is
favorable to classification.

C. Parameter Sensitivity Analysis

In this section, we analyze the parameter sensitivity of our LRDLSR model. There are four parameters need to be selected in LRDLSR. For convenience, we set the parameters $\gamma$ and $\gamma$ to 0.01 in advance, then just focus on selecting the optimal values of parameters $\alpha$ and $\beta$ from the candidate set \{0.0001, 0.001, 0.01, 0.1, 1\}. The classification accuracy as a function of different parameter values on three datasets are shown in Fig. 2. It is apparent that the optimal parameters are different on respective datasets, but our LRDLSR model is not very sensitive to the values of $\alpha$ and $\beta$. This also demonstrates that the compact and similar intra-class targets are critical to discriminative projection learning and its classification performance does not completely depend on the selection of parameters.

D. Convergence Study

In this section, we verify the convergence of Algorithm 1 on three face datasets. The convergence results are shown in Fig. 3. We can find that Algorithm 1 converges very well, with the value of objective function of LRDLSR monotonically decreasing with the increasing number of iterations. This demonstrates the effectiveness of the used ADMM optimization method.

V. CONCLUSION

In this paper, we propose a low-rank discriminative least squares regression (LRDLSR) model for multi-class image classification. LRDLSR aims to improve the intra-class similarity of the regression targets learned by the $\varepsilon$-dragging technique. This can ensure that the learned targets are not only relaxed but also discriminative, thus leading to more effective projection. Besides, LRDLSR introduces an extra regularization term to avoid the problem of overfitting by restricting the energy of learned regression targets. Experimental results on the object and face databases verify the effectiveness of the proposed method.

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