OBSERVABILITY OF WAVE EQUATION WITH VENTCEL DYNAMIC CONDITION

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(Communicated by Vilmos Komornik)

Abstract. The main purpose of this work is to prove a new variant of Mehrenberger’s inequality. Subsequently, we apply it to establish several observability estimates for the wave equation subject to Ventcel dynamic condition.

1. Introduction. In the present paper, we address the exact controllability of the standard wave equation with mixed boundary condition: dynamic Ventcel condition on one part of the boundary and Dirichlet’s on the other part. More precisely, let $\Omega$ be an open rectangular domain in $\mathbb{R}^n$, $n \geq 2$ with boundary $\Gamma$

$$\Omega = \prod_{i=1}^{n} (0, l_i), \quad l_i > 0.$$  

Let $\Gamma_i$, $i = 1, \ldots, n$, $\Gamma_V$ and $S$ denote the following subsets of $\Gamma$:

$$\Gamma_i = \prod_{j=1}^{i-1} (0, l_j) \times \{0\} \times \prod_{j=i+1}^{n} (0, l_j), \quad i = 1, \ldots, n,$$

$$\Gamma_V = \prod_{j=1}^{n-1} (0, l_j) \times \{l_n\},$$

$$S = \Gamma_V \cap \left( \bigcup_{i=1}^{n} \Gamma_i \right).$$

Given $T > 0$, we consider the initial boundary value problem:

$$\begin{cases}
  v'' - \Delta v = 0 & \text{in } \Omega \times (0, T), \\
  v'' - \Delta_T v + \partial_{\nu} v = 0 & \text{on } \Gamma_V \times (0, T), \\
  v = w_i & \text{on } \Gamma_i \times (0, T), \quad i = 1, \ldots, n, \\
  v = w_s & \text{on } S \times (0, T), \\
  v = 0 & \text{on } \left[ \Gamma \setminus \left( \Gamma_V \cup \left( \bigcup_{i=1}^{n} \Gamma_i \right) \cup S \right) \right] \times (0, T), \\
  v(x, 0) = v_0(x), \quad v'(x, 0) = v_1(x), & x \in \Omega, \\
  v(x, 0) = v_2(x), \quad v'(x, 0) = v_3(x), & x \in \Gamma_V; \\
\end{cases}$$

(1)
where $\Delta_T$ designates the tangential laplacien on $\Gamma_V$, $\nabla_T$ the tangential gradient on $\Gamma_V$, $\partial_\nu$ the normal derivative with $\nu$ denoting the unit outward normal field to $\Gamma$.

The second equation is called dynamic Ventcel boundary condition on $\Gamma_V$. Differential problems with Ventcel conditions are quite interesting from the practical point of view, for they can model several physical processes. We mention as an example, the vibrations of an elastic body with a thin layer of high rigidity at the boundary [15], [16], [17] (see also [19], [7], and the references therein for more on the physical meanings of such problems).

The existence and regularity of solutions to such systems have been first investigated by Lemrabet [15], [16] with static Ventcel condition and later, by Lemrabet and Teniou [17] with dynamic Ventcel condition. Thus, we need not worry about an ill-posed problem. The main issue we are discussing here is the exact controllability of (1); though already studied in [19], [7], it seems that, compared to stabilization (see, e.g. [3], [4], [21]), it has received less attention.

In a recent paper, Gal and Tebou [7] have been able to derive, under some geometrical restrictions, a rather complex Carleman estimate for a problem similar to the one under consideration. Subsequently, using two controls, one on Ventcel portion of the boundary and the other on Dirichlet’s, the authors have applied this estimate to establish the controllability of nonconservative waves.

Thanks to the particular structure of our domain, it’s possible to take a much simpler approach; say the one based on Fourier expansions and Ingham type theorems. Since Ingham’s first inequalities [9], many variants have been introduced and used in control theory [20], [12], [13], [14]. In particular, there is the outstanding generalization due to Mehrenberger [20] which extends Ingham’s theorem to cover $n$-dimensional intervals.

Although working with these estimates, no one has retrieved the optimal time condition given by multiplier techniques; this approach remains significantly simpler to implement than the other methods.

To our knowledge, few are the papers investigating the exact controllability of systems such as ours and none are those with Ingham’s theorems as their tools. The existing literature involves especially Dirichlet and Neumann conditions.

The system (1) is said to be exactly controllable if, for any given initial data $v_0$, $v_1$, $v_2$, $v_3$ (in suitable Hilbert spaces), there exist control functions $w_i$, $i = 1, \ldots, n$, $w_S$ such that

\begin{align*}
v(x, T) &= v'(x, T) = 0 \text{ in } \Omega, \\
v(x, T) &= v'(x, T) = 0 \text{ on } \Gamma_V.
\end{align*}

We shall work out this problem using the celebrated Hilbert Uniqueness Method (HUM) of J.-L. Lions [18] which permits us to prove the controllability of a system through the observability of the corresponding homogeneous problem. Spectral analysis of the latter has shown that the situation here is different from the case where Dirichlet condition is imposed on the whole boundary [20]. This brought out the need for the new version of Mehrenberger’s theorem which we give in theorem 5.3. This adaptation is then used to establish the boundary observability cf. theorem 5.2.

After its first application to the observability of the wave equation with Dirichlet condition [20], Mehrenberger’s inequality has been improved by Komornik and Mira ([13], proposition 3.1, p. 139) and applied to establish additional results on the observability of rectangular membranes. Motivated by the joint work of Komornik
with Loreti [12], we represent our variant of Mehrenberger’s estimate in such a way that it can be employed to attain more observability estimates.

We emphasize that in our first result (cf. thm 5.1), while minding the geometric condition of Bardos-Lebeau-Rauch [1], Dirichlet action has sufficed to steer the system to rest within a finite amount of time. This fact, though the settings are not the same, refutes Gal and Tebou final remarks; namely, the two controls (Dirichlet and Ventcel’s) are equally necessary to kill the vibrations. Interestingly, even if we choose the controls to be supported on both Ventcel and Dirichlet boundary, the system is proven not to be exactly controllable only approximately.

For computational simplicity, the results are given in dimension 2, that is when the domain $\Omega$ is a rectangle. The case of higher dimensions can be handled similarly.

This article is organized as follows:

For the reader’s convenience, in section 2, following HUM we prove the well-posedness of the associated homogeneous problem to (1) and we establish an a priori estimate (direct inequality). In section 3, after providing the spectral properties needed, we expand the solution of the homogeneous problem as a nonharmonic series. The existence of weak solution to problem (1) is given in section 4. Section 5 is devoted to the proof of our adaptation of Mehrenberger’s theorem and its application to the boundary observability. Finally, in section 6 we obtain through this adaptation results on internal and combined internal-boundary observability.

In what follows, the notation

$$A \preceq B \preceq c_2 B$$

for some constants $c_1, c_2 > 0$.

2. Statement of problem. Henceforth, we are restricting ourselves to the case of a rectangular membrane $\Omega = (0, l_1) \times (0, l_2), l_1, l_2 > 0$. Let $\Gamma_1, \Gamma_2, \Gamma_V$ and $\Gamma_D$ denote the following boundary portions

$$\Gamma_1 = \{0\} \times (0, l_2), \quad \Gamma_2 = (0, l_1) \times \{0\}, \quad \Gamma_V = (0, l_1) \times \{l_2\} \quad \text{and} \quad \Gamma_D = \Gamma \setminus \Gamma_V.$$

Let’s introduce the homogeneous problem associated with (1)

$$\begin{cases}
  u'' - \Delta u = 0 & \text{in } \Omega \times (0, T),
  u'' - \Delta_T u + \partial_\nu u = 0 & \text{on } \Gamma_V \times (0, T),
  u = 0 & \text{on } \Gamma_D \times (0, T),
  u(x,0) = u_0(x), \quad u'(x,0) = u_2(x), \quad x \in \Omega,
  u(x,0) = u_1(x), \quad u'(x,0) = u_3(x), \quad x \in \Gamma_V.
\end{cases} \quad (3)$$

Throughout, $\nu, \tau$ stand for the unit outward normal vector on $\Gamma$, and the unit tangent vector oriented outside of $\Gamma_V$ at its endpoints.

The well-posedness of problems with Ventcel boundary conditions has been verified by many authors either by variational techniques or by semi-groups theory (see for instance [17], [7], [19]). Nonetheless, for the reader’s convenience, we’re going over the proof of the existence of solutions as well as some regularity properties required for the application of HUM.

2.1. Well-posedness and regularity. Denoting

$$(u, v)_\Omega = \int\limits_\Omega u(x)v(x) \, dx, \quad \|u\|^2_\Omega = \int\limits_\Omega |u(x)|^2 \, dx,$$

$$(u, v)_{\Gamma_V} = \int\limits_{\Gamma_V} uv \, d\Gamma, \quad \|u\|^2_{\Gamma_V} = \int\limits_{\Gamma_V} |u|^2 \, d\Gamma,$$
the scalar product and norm, respectively on \( L^2(\Omega) \) and \( L^2(\Gamma_V) \), we introduce Hilbert spaces

\[
H^1_{\Gamma_D}(\Omega) = \{ u \in H^1(\Omega); u|_{\Gamma_D} = 0 \},
\]

\[
V = \{ (u_1, u_2) \in H^1_{\Gamma_D}(\Omega) \times H^1_0(\Gamma_V); \; u_2 = u_1|_{\Gamma_V} \},
\]

\[
H = L^2(\Omega) \times L^2(\Gamma_V).
\]

Endowed with the norms

\[
\|(u_1, u_2)\|_V^2 = \|\nabla u_1\|_\Omega^2 + \|\nabla_T u_2\|_{\Gamma_V}^2,
\]

\[
\|(u_1, u_2)\|_H^2 = \|u_1\|_\Omega^2 + \|u_2\|_{\Gamma_V}^2,
\]

\( V \) is dense in \( H \) with continuous injection.

We define on \( H \) the operator \( A_0 \) by

\[
A_0\left(\begin{array}{c}
u_1 \\
u_2
\end{array}\right) = \left(\begin{array}{c}
-\Delta u_1 \\
-\Delta_T u_2 + \partial_\nu u_2
\end{array}\right),
\]

whose domain is given by

\[
D(A_0) = \{ (u_1, u_2) \in V; \; A_0(u_1, u_2) \in H \}
\]

\[
= \{ (u_1, u_2) \in V; \; \Delta u_1 \in L^2(\Omega), \; -\Delta_T u_2 + \partial_\nu u_2 \in L^2(\Gamma_V) \}.
\]

\( A_0 \) is sometimes called the Ventcel Laplacian.

**Proposition 1.** Let \( A_0 \) be the operator defined as above. Then,

\[
D(A_0) = \{ (u_1, u_2) \in (H^2(\Omega) \cap H^1_{\Gamma_D}(\Omega)) \times (H^2(\Gamma_V) \cap H^1_0(\Gamma_V)); \; u_2 = u_1|_{\Gamma_V} \}.
\]

**Proof.** Let \((u, u|_{\Gamma_V})\) \( \in D(A_0) \). Then \( u \in H^1(\Omega) \) and \( \Delta u \in L^2(\Omega) \), which implies that

\[
\partial_\nu u \in H^{-\frac{1}{2}}(\Gamma).
\]

However, \( \Delta_T u - \partial_\nu u \in L^2(\Gamma_V) \). Thus, \( \Delta_T u \in H^{-\frac{1}{2}}(\Gamma_V) \) and as a result

\[
u \in H^{-\frac{1}{2}}(\Gamma_V) \cap H^1_0(\Gamma_V).
\]

This together with the fact that \( u \in H^1_{\Gamma_D}(\Omega) \) and \( \Delta u \in L^2(\Omega) \) yield \( u \in H^2(\Gamma_V) \) (Grisvard [8]).

Now, \( u \in H^2(\Omega) \) implies \( \partial_\nu u \in H^{\frac{1}{2}}(\Gamma_V) \) and so \( \Delta_T u \in L^2(\Gamma_V) \). Since \( u \in H^1_0(\Gamma_V) \), it results

\[
u \in H^2(\Gamma_V).
\]

\( \square \)

**Remark 1.** We can readily verify that \( A_0 \) is self-adjoint and positive on \( H \). Further, applying Lax-Milgram theorem shows that

\[
\forall (f, g) \in H, \; \exists! (u, v) \in D(A_0) \; \text{such that} \; A_0(u, v) = (f, g).
\]

Thus, \( 0 \in \rho(A_0) \) and the inverse operator \( (A_0)^{-1} \) is bounded and compact due to Sobolev injections

\[
H^1(\Omega) \hookrightarrow L^2(\Omega), \; H^1(\Gamma_V) \hookrightarrow L^2(\Gamma_V).
\]

This implies that Hilbert space \( H \) possesses an orthonormal basis formed by eigenvectors of \( A_0 \) ([23], prop. 3.2.12.).
Given the space 

\[ X := V \times H, \]

equipped with the norm

\[ \| (U, \tilde{U}) \|_X^2 = \| \nabla u_1 \|_\Omega^2 + \| \nabla_T u_2 \|_{V'}^2 + \| \tilde{u}_1 \|_\Omega^2 + \| \tilde{u}_2 \|_{V'}^2, \quad U = (u_1, u_2), \quad \tilde{U} = (\tilde{u}_1, \tilde{u}_2). \]

The energy at time \( t \) of the solution \( u \) to problem (3) is defined by

\[ E(t) := \frac{1}{2} \left( \| \nabla u \|_\Omega^2 + \| \nabla_T u \|_{V'}^2 + \| u' \|_\Omega^2 + \| u' \|_{V'}^2 \right). \]

The existence and uniqueness of solution to (3) is provided by

**Theorem 2.1.** Let

\[ (u_0, u_1) \in V, \quad (u_2, u_3) \in H. \]

Then, problem (3) has a unique solution \( u(x, t) \) such that

\[ u \in C(0, T; H^1_0(\Omega)) \cap C^1(0, T; L^2(\Omega)), \]

\[ u|_{\Gamma_V} \in C(0, T; H^1_0(\Gamma_V)) \cap C^1(0, T; L^2(\Gamma_V)). \]  

(6)

Furthermore, if \((u_0, u_0|_{\Gamma_V}), (u_2, u_3)\) \( \in D(A_0) \times V \), then the solution is strong and we have

\[ u \in C(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T; H^1_0(\Omega)) \cap C^2(0, T; L^2(\Omega)), \]

\[ u|_{\Gamma_V} \in C(0, T; H^2(\Gamma_V) \cap H^1_0(\Gamma_V)) \cap C^1(0, T; H^1_0(\Gamma_V)) \cap C^2(0, T; L^2(\Gamma_V)). \]

In both cases, the energy is conserved:

\[ E(t) = E(0), \forall t \in [0, T]. \]  

(7)

**Proof.** Putting \( U := (u, u|_{\Gamma_V}, u', (u|_{\Gamma_V})'), U_0 := (u_0, u_1, u_2, u_3) \). System (3) can be written in the form of an abstract Cauchy problem in \( X \)

\[ U' = AU, \quad U(0) = U_0 \]  

(8)

where the operator \( A \) is defined on \( X \) by

\[ D(A) = D(A_0) \times V; \]

\[ A = \begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ -A_0 & 0_{2 \times 2} \end{pmatrix}. \]  

(9)

For problem (8) to have solutions, we need to show that \( A \) is the infinitesimal generator of a continuous semigroup. However, there is an even better alternative: Stone’s theorem (see e.g. [22], thm. 1.10.8 or [23], thm. 3.6.8.) that characterizes generators of unitary groups to be skew-adjoint.

Simple integrations by parts over \( \Omega \) show that \( A \) is skew-symmetric on \( X \) i.e.

\[ < A\Phi, \Psi >_X = - < \Phi, A\Psi >_X, \quad \forall \Phi, \Psi \in D(A). \]

Let’s show that \( R(A) = X \). Given \( (F, G) \in X \), is there \( (\Phi, \Psi) \in D(A) \) such that

\[ \begin{pmatrix} F \\ G \end{pmatrix} = A \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \Leftrightarrow \begin{cases} F = \Phi \in V, \\ G = -A_0\Phi \in H? \end{cases} \]

Since \( A_0 \) is positive, it is invertible ([23], prop. 3.3.2). Then, there exists \( \Phi \in D(A_0) \) such that \( G = -A_0\Phi \) and \( A \) is onto. Applying prop. 3.7.3. of [23], it results that \( A \) is skew-adjoint and \( 0 \in \rho(A) \). Moreover, by Stone’s theorem \( A \) generates a group of isometries on \( X \). Thus, if \( U_0 \in X \), problem (8) admits a unique solution \( \tilde{U} \in C(\mathbb{R}; X) \) ([11], thm. 2.1) which implies (6).
On the other hand, if $U_0 \in D(A)$, the solution satisfies

$$U \in C(\mathbb{R}; D(A)) \cap C^1(\mathbb{R}; X),$$

$$U' = AU$$ in $X$.

According to proposition 1 we have $H^2$- regularity in $\Omega$ and on $\Gamma_V$. Thus, the solution is a strong one.

The conservation of energy follows from the fact that the group generated by $A$ is isometric.

## 2.2. Direct inequality.

The next result is often referred to as hidden regularity. It plays a key role in the application of HUM. Actually, once proven, we have half of the uniqueness theorem we are seeking.

**Theorem 2.2.** Let $T > 0$. For every $((u_0, u_0|_{\Gamma_V}), (u_2, u_3)) \in X$, there is a constant $c = c(\Omega, T) > 0$ such that the solution of (3) satisfies the inequality

$$\int_0^T \int_{\Gamma_1 \cup \Gamma_2} |\partial_t u(x, t)|^2 d\Gamma dt + \int_0^T |\partial_{\tau} u(0, t_2, t)|^2 dt \leq c \left( \|\nabla u_0\|_{\Omega}^2 + \|u_2\|_{\Omega}^2 + \|\nabla T u_0\|_{\Gamma_V}^2 + \|u_3\|_{\Gamma_V}^2 \right).$$

**Proof.** Since $D(A)$ is dense in $X$, it suffices to show estimate (10) for regular solutions with initial conditions belonging to $D(A)$.

Given $((u_0, u_0|_{\Gamma_V}), (u_2, u_3)) \in D(A)$. Let $q = (q_1, q_2)$ be a vector field of class $C^2(\Omega)$. Multiplying equations

$$u'' - \Delta u = 0,$$

$$u'' - \Delta_T u + \partial_\nu u = 0$$

resp. by $q_k \frac{\partial u}{\partial x_k}$, $q_k \frac{\partial u}{\partial x_k} |_{\Gamma_V}$ and integrating by parts over $\Omega \times (0, T)$ and $\Gamma_V \times (0, T)$, in the same manner as done by Lions [18], lemma 3.7, p. 40, we obtain (convention of summation over repeated indices is used)

$$\frac{1}{2} \int_0^T \int_{\Gamma_1 \cup \Gamma_2} q_k \nu_k |\partial_\nu u|^2 d\Gamma dt = \left[ (u'(t), q_k \frac{\partial u}{\partial x_k}(t))_{\Omega} \right]_0^T + \frac{1}{2} \int_0^T \int_{\Omega} \frac{\partial q_k}{\partial x_k} |u'|^2 dx dt$$

$$- \frac{1}{2} \int_0^T \int_{\Gamma_V} q_k \nu_k |u'|^2 d\Gamma dt + \int_0^T \int_{\Omega} \frac{\partial q_k}{\partial x_k} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} dx dt - \frac{1}{2} \int_0^T \int_{\Omega} \frac{\partial q_k}{\partial x_k} |\nabla u|^2 dx dt$$

$$+ \frac{1}{2} \int_0^T \int_{\Gamma_V} q_k \nu_k |\nabla_T u|^2 d\Gamma dt - \int_0^T \int_{\Gamma_V} \partial_\nu u q_k \frac{\partial u}{\partial x_k} d\Gamma dt,$$

$$- \int_0^T \int_{\Gamma_V} \partial_\nu u q_k \frac{\partial u}{\partial x_k} d\Gamma dt = \left[ (u'(t), q_k \frac{\partial u}{\partial x_k}(t))_{\Gamma_V} \right]_0^T + \frac{1}{2} \int_0^T \int_{\Gamma_V} \frac{\partial q_k}{\partial x_k} |u'|^2 d\Gamma dt$$

$$+ \int_0^T \int_{\Gamma_V} \frac{\partial q_k}{\partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} d\Gamma dt - \frac{1}{2} \int_0^T \int_{\Gamma_V} \frac{\partial q_k}{\partial x_k} |\nabla_T u|^2 d\Gamma dt - \frac{1}{2} \int_0^T \int_{\partial \Gamma_V} q_k \nu_k |u|^2 d\sigma dt.$$
Combining these two equalities leads to the identity
\[ \frac{1}{2} \int_0^T \int_{\Gamma_D} q_k \nu_k |\partial_x u|^2 d\Gamma dt \right. \right. + \left. \left. \frac{1}{2} \int_0^T \int_{\Gamma_D} q_k \tau |\partial_x u|^2 d\sigma dt \right. \right. = \left. \left. \left[ (u'(t), q_k \frac{\partial u}{\partial x_k}(t)) \right]_0^T \right. \right. \\
+ \left[ (u'(t), q_k \frac{\partial u}{\partial x_k}(t)) \right]_0^T + \int_0^T \int_{\Omega} \frac{\partial q_k}{\partial x_k} (|u'|^2 - |\nabla u|^2) dx dt \\
+ \frac{1}{2} \int_0^T \int_{\Gamma} \frac{\partial q_k}{\partial x_k} (|u'|^2 - |\nabla u|^2) d\Gamma dt + \int_0^T \int_{\Omega} \frac{\partial q_k}{\partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} dx dt \\
+ \frac{1}{2} \int_0^T \int_{\Gamma} \frac{\partial q_k}{\partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} d\Gamma dt - \frac{1}{2} \int_0^T \int_{\Gamma} q_k \nu_k (|u'|^2 - |\nabla u|^2) d\Gamma dt. \]

Now, taking \( q_k = x_k e_k, k = 1, 2 \) where \( e_k \) is the \( k \)th vector in the canonical basis of \( \mathbb{R}^2 \), the left-hand side of (11) becomes
\[ \frac{1}{2} \int_0^T \int_{\Gamma_D} |\partial_x u|^2 d\Gamma dt + \frac{1}{2} \int_0^T \int_{\partial \Gamma} |\partial_x u|^2 d\sigma dt. \]

On the other hand, we have
\[ \int_0^T \int_{\Omega} \frac{\partial q_k}{\partial x} \frac{\partial u}{\partial x} dx dt = \int_0^T \int_{\Omega} |\nabla u|^2 dx dt, \]
\[ \int_0^T \int_{\Gamma} \frac{\partial q_k}{\partial x} \frac{\partial u}{\partial x} dx dt = \int_0^T \int_{\Gamma} |\nabla u|^2 d\Gamma dt, \]
\[ \left[ (u'(t), q_k \frac{\partial u}{\partial x_k}(t)) \right]_0^T \leq c(\Omega)(||u'(T)||^2 + ||\nabla u(T)||^2 + ||u'(0)||^2 + ||\nabla u(0)||^2). \]

Proceeding similarly with the remaining terms of the right-hand side of (11) and taking account of (7), we get the desired estimate. \( \square \)

3. **Spectral analysis of the homogeneous problem (3).** In this section, we shall express the solution \( u(x, t) \) as a Fourier series. To do this we are going to show, here below, the infinitesimal generator \( A \) (cf. eq. (9)) to be diagonalizable; namely, its resolvent is nonempty and there exists a Riesz base in \( X \) consisting of eigenvectors of \( A \). So first we determine the eigenvalues and eigenvectors which, upon separation of variables, turn out to be solutions to some Sturm-liouville problems; then we apply theorem 3.1 [11] and obtain an explicit formula of \( u \).

3.1. **Sturm-Liouville problems.** We begin by computing the eigenvectors of the Ventcel Laplacian \( A_0 \) cf. eq. (4), (5). Afterward, we deduce those of the operator \( A \). We know that \( \alpha \in \mathbb{R}^+ \) (\( A_0 \) is self-adjoint and positive) is an eigenvalue of \( A_0 \) if there exists a nonzero vector \( U = (u, u|_{\Gamma_V}) \in D(A_0) \) such that \( A_0 U = \alpha U \). This amounts to
\[ -\Delta u = \alpha u \text{ in } \Omega, \]
\[ -\Delta u + \partial_n u = \alpha u \text{ on } \Gamma_V, \]
\[ u = 0 \text{ on } \Gamma_D. \]
Now, setting \( u(x_1, x_2) = \varphi(x_1)\psi(x_2) \),

the equations above become

\[
-\frac{\varphi''(x_1)}{\psi(x_1)} - \frac{\psi''(x_2)}{\psi(x_2)} = \alpha, \quad (x_1, x_2) \in \Omega,
\]

\[-\varphi''(x_1)\psi(l_2) + \varphi(x_1)\psi'(l_2) = \alpha\varphi(x_1)\psi(l_2), \quad x_1 \in \Gamma_V,
\]

\[\varphi(x_1)\psi(x_2) = 0, \quad (x_1, x_2) \in \Gamma_D.\]

Thus, the eigenvalue problem is reduced to a pair of second-order differential equations

\[
\left\{ \begin{array}{l}
-\varphi''(x_1) = \delta^2 \varphi(x_1), \quad x_1 \in (0, l_1) \\
\varphi(0) = \varphi(l_1) = 0
\end{array} \right. \quad (12)
\]

\[
\left\{ \begin{array}{l}
-\psi''(x_2) = \mu^2 \psi(x_2), \quad x_2 \in (0, l_2) \\
\psi(0) = 0, \quad \psi'(l_2) = \mu^2 \psi(l_2)
\end{array} \right. \quad (13)
\]

The first ODE is a regular Sturm-Liouville equation whose solutions are the classical trigonometric functions

\[\varphi_{k_1}(x_1) = \sqrt{\frac{2}{l_1}} \sin \left( \frac{k_1 \pi}{l_1} x_1 \right), \quad k_1 \geq 1 \quad (14)\]

associated with the eigenvalues

\[\delta_{k_1}^2 = \left( \frac{k_1 \pi}{l_1} \right)^2, \quad k_1 \geq 1. \quad (15)\]

It is well-known that \( \{ \varphi_{k_1}, \quad k_1 \geq 1 \} \) forms an orthonormal basis of \( L^2(0, l_1) \), that is to say a complete orthonormal system in \( L^2(0, l_1) \).

As for the second ODE, notice that its sole apparent difference from (12) is the presence of the eigenvalue parameter in the boundary condition. This type of Sturm-Liouville problems arises when separation of variables is applied to differential equations with dynamic boundary conditions. Many papers have treated such problems, we mention for instance [2], [5], [6], [24].

Let \( H_1 = L^2(0, l_2) \times \mathbb{C} \). Equipped with the inner product

\[
((F, G)) := \int_0^{l_2} F_1(z)\overline{G_1(z)}dz + F_2\overline{G_2} \quad \text{for} \quad F = \left( \begin{array}{c} F_1(z) \\ F_2 \end{array} \right), \quad G = \left( \begin{array}{c} G_1(z) \\ G_2 \end{array} \right) \in H_1,
\]

\( H_1 \) is a Hilbert space.

Spectral properties of problem (13) are listed in the proposition below:

**Proposition 2.** Let \( \{ \mu_{k_2}^2 \}_{k_2=0}^{\infty}, \{ \psi_{k_2} \}_{k_2=0}^{\infty} \) denote, respectively, the eigenvalues and eigenfunctions of (13). We have the following assertions:

1. The eigenvalues \( \{ \mu_{k_2}^2 \}_{k_2=0}^{\infty} \) are positive roots of the transcendental equation

\[\mu_{k_2} \sin(\mu_{k_2} l_2) = \cos(\mu_{k_2} l_2), \quad k_2 \in \mathbb{N}. \quad (16)\]

The corresponding eigenfunctions are given by

\[\psi_{k_2}(x_2) = \sin(\mu_{k_2} x_2), \quad x_2 \in (0, l_2), \quad k_2 \in \mathbb{N}. \quad (17)\]
2. The normalized vectors

\[ \Psi_{k_2}(x_2) = \rho_{k_2} \left( \frac{\sin(\mu_{k_2}x_2)}{\sin(\mu_{k_2}l_2)} \right), \quad k_2 \in \mathbb{N}, \]

where \( \rho_{k_2} := \frac{1}{\|\Psi_{k_2}\|_{H_1}} = \sqrt{\frac{2(1 + \mu_{k_2}^2)}{l_2\mu_{k_2}^2 + l_2 + 1}} \), form an orthonormal basis in \( H_1 \).

3. The cosine functions having the same form as \( \psi_{k_2} \) are orthogonal in \( L^2(0, l_2) \), i.e.

\[ \int_0^{l_2} \cos(\mu_{k_2}x_2) \cos(\mu_{k_2}'x_2) dx_2 = 0, \quad k_2 \neq k_2'. \]

Moreover, we have

\[ \int_0^{l_2} |\cos(\mu_{k_2}x_2)|^2 dx_2 = \frac{l_2\mu_{k_2}^2 + l_2 + 1}{2(1 + \mu_{k_2}^2)}. \quad (18) \]

4. As \( k_2 \to \infty \), we have the asymptotic formulae

\[ \mu_{k_2} = \frac{k_2\pi}{l_2} + O\left(\frac{1}{k_2^2}\right), \quad (19) \]

\[ \rho_{k_2}\psi_{k_2}(x_2) = \sqrt{\frac{2}{l_2}} \sin\left(\frac{k_2\pi}{l_2}x_2\right) + O\left(\frac{1}{k_2}\right). \quad (20) \]

Proof. By Green’s formula, we see that

\[ \int_0^{l_2} |\psi'(x_2)|^2 dx_2 = \mu^2 \left( \int_0^{l_2} |\psi(x_2)|^2 dx_2 + |\psi(l_2)|^2 \right), \]

which confirms that the eigenvalues of (13) are nonnegative. One can easily show that for \( \mu = 0 \), the only possible solution is \( \psi = 0 \). Then \( \mu = 0 \) isn’t an eigenvalue.

Fix \( \mu \neq 0 \). We then have

\[ \psi(x_2) = a \sin(\mu x_2) + b \cos(\mu x_2). \]

Using the condition at \( x_2 = 0 \), it follows \( \psi(x_2) = a \sin(\mu x_2) \). By the second condition, we infer that \( \mu_{k_2}, \ k_2 \geq 0 \) are the positive roots of equation (16) and the corresponding functions are given by (17). Hence we have our first assertion.

Since the second and third properties are related, we work them out at the same time. Let \( k_2, k_2' \in \mathbb{N}, \ k_2 \neq k_2' \), we have

\[ \int_0^{l_2} \sin(\mu_{k_2}x_2) \sin(\mu_{k_2}'x_2) dx_2 = -\sin(\mu_{k_2}l_2) \sin(\mu_{k_2}'l_2) \]

\[ + \frac{1}{\mu_{k_2}} \int_0^{l_2} \cos(\mu_{k_2}x_2) \cos(\mu_{k_2}'x_2) dx_2. \]

Hence for \( \{\Psi_{k_2}\} \) to be orthogonal, it suffices to show

\[ \int_0^{l_2} \cos(\mu_{k_2}x_2) \cos(\mu_{k_2}'x_2) dx_2 = 0. \quad (21) \]
Standard trigonometric relations yield

\[
\int_0^{l_2} \cos(\mu_k x_2) \cos(\mu_k' x_2) d x_2 = \frac{1}{2(\mu_k^2 - \mu_k'^2)} \left( (\mu_k - \mu_k') \sin(\mu_k + \mu_k') l_2 + (\mu_k + \mu_k') \sin(\mu_k - \mu_k') l_2 \right) = \frac{1}{2(\mu_k^2 - \mu_k'^2)} \left( 2\mu_k \sin(\mu_k' l_2) \cos(\mu_k' l_2) - 2\mu_k \sin(\mu_k l_2) \cos(\mu_k l_2) \right).
\]

Using equation (16), (21) follows. Thus, we have proved both sequences \( \{ \Psi_{k_2} \} \), \( \{ \cos \mu_k x_2 \} \) to be orthogonal, resp. in \( H_1 \) and \( L^2(0, l_2) \).

On the other hand,

\[
\| \Psi_{k_2} \|_{H_1}^2 = \int_0^{l_2} |\sin(\mu_k x_2)|^2 d x_2 + |\sin(\mu_k' l_2)|^2 = \frac{l_2}{2} - \frac{1}{4\mu_k} \sin(2\mu_k l_2) + |\sin(\mu_k' l_2)|^2 = \frac{l_2}{2} - \frac{1}{2\mu_k} \cos(\mu_k l_2) \sin(\mu_k' l_2) + |\sin(\mu_k l_2)|^2.
\]

Since \( \cos^2(\mu_k l_2) + \sin^2(\mu_k l_2) = \mu_k^2 \), we get \( \sin^2(\mu_k l_2) = \frac{1}{1 + \mu_k^2} \).

Therefore,

\[
\| \Psi_{k_2} \|_{H_1}^2 = \frac{l_2}{2} + \frac{1}{2(1 + \mu_k^2)}.
\]

Analogous computations lead to (18).

The completeness of \( \{ \Psi_{k_2} \} \) follows from theorem 1 [5].

Finally, for the asymptotic behavior of \( (\mu_{k_2})_{k_2} \), \( (\mu_{k_2} \psi_{k_2})_{k_2} \), one may check out section 4 of [5].

From the discussion above and remark 1, we draw the following consequence.

**Corollary 1.** Let \( \{ \omega_k^2 \} \), \( \{ U_k \} \), \( k = (k_1, k_2) \in \mathbb{N}^* \times \mathbb{N} \) be the eigenvalues and corresponding eigenvectors of \( A_0 \). Then, we have

\[
\omega_k^2 = \left( \frac{k_1 \pi}{l_1} \right)^2 + \mu_k^2,
\]

\[
U_k(x_1, x_2) = \left( \sin \left( \frac{k_1 \pi}{l_1} x_1 \right) \sin(\mu_k x_2) \right), \quad k \in \mathbb{N}^* \times \mathbb{N}.
\]

Moreover, the normalized vectors \( \{ \zeta_k U_k \} \), with \( \zeta_k^{-2} = \frac{l_1}{2} \frac{l_2 \mu_k^2 + l_2 + 1}{2(1 + \mu_k^2)} \), constitute an orthonormal basis of \( H \).

One significant advantage of \( A_0 \) being diagonalizable is that it allows us to better grasp the structure of Hilbert spaces \( V, H, X \) and their duals. In fact, they can be characterized using the eigenvalues and eigenvectors of \( A_0 \).
Remark 2. Let $Z$ be the linear hull of the basis vectors $U_k$. For fixed $s \in \mathbb{R}$, we define $D^s$ to be the completion of $Z$ with respect to the Euclidian norm

$$\left\| \sum_{k \in \mathbb{N}^* \times \mathbb{N}} a_k U_k \right\|_s^2 = \sum_{k \in \mathbb{N}^* \times \mathbb{N}} \omega_k^s |a_k|^2.$$  

Then, identifying $H$ with its dual, we have (see Komornik [10])

$$D^{-1} = V', \ D^0 = H, \ D^1 = V \text{ and } D^2 = D(A_0).$$

Also, $V \subset H \subset V'$ with dense continuous inclusions.

Remark 3. The dual space $V'$ could equally be viewed as the space of restrictions of elements of $(H^1_{D_0}(\Omega))^' \times H^{-1}(\Gamma_V)$ to Hilbert space $V$ (see [18], remark 5.1., p. 376).

3.2. Fourier series representation. In order to write the solution of (3) as a Fourier series, we need to figure out a Riesz basis for Hilbert space $X$. Since $A$ is defined in terms of Ventcel Laplacian $A_0$, one can easily check that its eigenvalues are

$$\lambda_k = i\omega_k, \ k \in \mathbb{Z}^* \times \mathbb{Z} \tag{22}$$

where $\omega_k = \sqrt{(k_1 \pi \over l_1)^2 + \mu_{k_2}^2}$, $\omega_{-k} = -\omega_k$, and the associated eigenvectors are given by

$$E_{\pm k}(x_1, x_2) = \varrho_k(U_k(x_1, x_2), \lambda_k U_k(x_1, x_2)), \ k \in \mathbb{N}^* \times \mathbb{N}$$

with $\varrho_k^{-2} = \omega_k^2 l_1 \left( \frac{l_2 \mu_{k_2}^2 + l_2 + 1}{2(1 + \mu_{k_2}^2)} \right).$

On the other hand, we have $D(A) \subset X$ densely with continuous compact embedding. Thus, $A$ is skew-adjoint with compact resolvent and the set $\{E_k\}_{k \in \mathbb{Z}^* \times \mathbb{Z}}$ forms an orthonormal basis for $X$. Moreover, $X$ is characterized using spaces $D^s$ as follows

$$X = D^1 \times D^0.$$

The next result provides an explicit formula of $u(x, t)$ solution to (3):

Theorem 3.1. Let

$$u_0 \in H^1_{D_0}(\Omega), \ u_0|_{\Gamma_V} \in H^1_0(\Gamma_V), \ u_2 \in L^2(\Omega) \text{ and } u_3 \in L^2(\Gamma_V).$$

Then the solution $u(x, t)$ of (3) is given by Fourier series

$$u(x_1, x_2, t) = \sum_{k_1, k_2 = 0}^{\infty} \sum_{l_1 = 1}^{\infty} \left( a_{k_l} e^{i\omega_{k_l} t} + a_{-k_l} e^{-i\omega_{k_l} t} \right) \sin \left( \frac{k_1 \pi}{l_1} x_1 \right) \sin(\mu_{k_2} x_2), \tag{23}$$

with suitable coefficients $a_k, k \in \mathbb{Z}^* \times \mathbb{Z}$ depending on the initial data such that

$$\sum_{k_1, k_2 = 0}^{\infty} \sum_{l_1 = 1}^{\infty} \omega_k^2 \left( |a_k|^2 + |a_{-k}|^2 \right) < \infty.$$ 

Proof. Writing the initial data $U_0 = ((u_0, u_0|_{\Gamma_V}), (u_2, u_3)) \in X$ in the base $\{E_k\}$

$$U_0 = \sum_{k \in \mathbb{Z}^* \times \mathbb{Z}} a_k E_k,$$
we obtain, according to theorem 3.1 [11], that the solution of (8) is
\[ U(t) = \sum_{k \in \mathbb{Z}^s \times \mathbb{Z}} \alpha_k E_k e^{i\omega_k t}. \]

We conclude the proof by taking \( a_k := \varphi_k \alpha_k \) in the first component of \( U \) and by noting that \( \varphi_k^2 \cong \omega_k^2 \).

4. Weak solution of system (1). Let \( s \in [0, T] \). Assume that \( u \) and \( v \) are solutions of systems (3) and (1). Then we have
\[ \int_0^s \int_{\Omega} (u'' - \Delta u) vdxdt = 0. \]

If we integrate by parts, formally, in time and in space, we shall arrive at the relation
\[ \int_{\Omega} u'(x, s)v(x, s) - u(x, s)u'(x, s)d\Omega + \int_{\Gamma_V} u'(x, s)v(x, s) - u(x, s)u'(x, s)d\Gamma + \int_0^s \int_{\Gamma_1} \partial_v u w_1(x, t) d\Gamma + \int_0^s \int_{\Gamma_2} \partial_v u w_2(x, t) d\Gamma dt. \]

Thus we are led to considering the existence of a weak solution \( v \) in the sense of the identity above. In fact, we have

**Theorem 4.1.** For any initial and boundary data
\( (v_0, v_2) \in L^2(\Omega) \times L^2(\Gamma_V), \quad (v_1, v_3) \in V', \)
\( (w_1, w_2, w_S) \in L^2(0, T; L^2(\Gamma_1) \times L^2(\Gamma_2) \times \mathbb{R}). \)

There exists a unique weak solution of (1) in the sense of identity (24) such that
\( (v, v|_{\Gamma_V}) \in C(0, T; L^2(\Omega) \times L^2(\Gamma_V)) \cap C^1(0, T; V'). \)

**Proof.** Setting
\[ U(s) = (u(x, s), u|_{\Gamma_V}(x, s), u'(x, s), (u|_{\Gamma_V})'(x, s)), \quad U_0 = U(0) \in X \]
\[ V(s) = (-v'(x, s), -(v|_{\Gamma_V})'(x, s), v(x, s), v|_{\Gamma_V}(x, s)), \quad V_0 = V(0) \in X', \]
we have expression (24) equivalent to
\[ < V(s), U(s) >_{X', X} = < V_0, U_0 >_{X', X} + \int_0^s \int_{\Gamma_1} \partial_v u w_1(x, t) d\Gamma dt + \int_0^s \int_{\Gamma_2} \partial_v u w_2(x, t) d\Gamma dt. \]

Taking into account the direct inequality (10), it follows that, for each \( s \in [0, T] \), the right-hand side of this formula defines a bounded linear form of \( U_0 \in X \).

On the other hand, the operator \( A \) generates a unitary group \( T(s) \) on \( X \). Hence, the right-hand side of (25) is also linear and bounded as a function of \( U(s) \in X \). Denoting this form by \( V(s) \) and noting that it’s continuous with respect to \( s \), we conclude our proof. \( \square \)
Remark 4. From identity (24), we get to define the observability operator $B$ by

$$B : D(B) \subset X \to G,$$

$$D(B) = D(A), \quad G := L^2(\Gamma_1) \times L^2(\Gamma_2) \times \mathbb{R},$$

$$B((y_1, y_2), (\tilde{y}_1, \tilde{y}_2)) := (\partial_x y_1, \partial_a y_1, \partial_x y_2).$$

Hence, estimate (10) means that $B$ is an admissible observability operator.

5. Main result. Our main result concerning the exact controllability of system (1) reads

**Theorem 5.1.** Let

$$(v_0, v_2) \in L^2(\Omega) \times L^2(\Gamma_V), \quad (v_1, v_3) \in V'.$$

Then, there exist $T_0 > 0$ and control functions $(w_1, w_2, w_3)$ of minimal norm in $L^2(0, T; L^2(\Gamma_1) \times L^2(\Gamma_2) \times \mathbb{R})$ such that for $T > T_0$ the solution $(v, v|_{\Gamma_V})$ to (1) reaches its equilibrium state (2) in time $T$.

Following HUM, the task of proving this theorem amounts to making sure that the observability inequality below holds for solutions of the homogeneous problem (3):

**Theorem 5.2.** Let $T_0 = 2(\sqrt{2} + 1) \sqrt{r_1^2 + 4r_2^2}$ and let $(\langle u_0, u_0|_{\Gamma_V} \rangle, (u_2, u_3)) \in X$. Then, for $T > T_0$ there exists a constant $c > 0$ such that

$$E(0) \leq c \int_0^T \int_{\Gamma_1 \cup \Gamma_2} |\partial_x u(x, t)|^2 d\Gamma dt + \int_0^T |\partial_x u(0, t_2, t)|^2 dt. \quad (26)$$

There are many approaches to take when dealing with observability estimates (Multiplier method, Carleman estimates, Microlocal analysis). Here we are adopting the one based on nonharmonic Fourier series. Precisely, we are recasting a generalization of Ingham theorem due to Mehrenberger [20] with the view of employing it to establish (26).

5.1. A variant of Mehrenberger's Ingham theorem. Looking closely, one can observe that, in Mehrenberger's inequality all the variables have the same sequence of weights $(p_k)_{k \in \mathbb{N}^*}$. This seems to fit well with its application to the observability of the wave equation with Dirichlet boundary condition. Here, though, the difference in form of the eigenvalues $\left(\frac{k_1 \pi}{l_1}\right)_{k_1 \in \mathbb{N}^*}$ and $(\mu_{k_2})_{k_2 \in \mathbb{N}}$ leads us to allow each variable its own sequence of weights. The following adaptation shall be justified subsequently.

**Theorem 5.3.** Let $\{\lambda_k, k = (k_1, k_2) \in \mathbb{N}^* \times \mathbb{N}\}$ be a sequence of real numbers. Given $(p_k)_{k_1 \in \mathbb{N}^*}, (q_k)_{k_2 \in \mathbb{N}} \subset \mathbb{C}$, we assume the following gap conditions: there exist $\gamma_1, \gamma_2 > 0$ such that for $k_1, k_1' \in \mathbb{N}^*, k_2, k_2' \in \mathbb{N}$, we have

$$|\lambda_k k_2 - \lambda_{k'} k_2'| \geq \gamma_1 |k_1 - k_1'|,$$

$$|\lambda_k k_2 + \lambda_{k'} k_2'| \geq \gamma_1 |k_1 + k_1'| \quad (27)$$

whenever $|q_{k_2}| \leq \max (|p_{k_1}|, |p_{k_1'}|)$, and

$$|\lambda_k k_2 - \lambda_{k'} k_2'| \geq \gamma_2 |k_2 - k_2'|,$$

$$|\lambda_k k_2 + \lambda_{k'} k_2'| \geq \gamma_2 |k_2 + k_2'| \quad (28)$$

whenever $|p_{k_1}| \leq \max (|q_{k_2}|, |q_{k_2'}|)$. 

Then, for \( T > 2\pi \sqrt{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}} \) there is a constant \( c = c(\gamma_1, \gamma_2, T) > 0 \) such that the following inequality holds for all square summable sequences \((\alpha_k)_{k \in \mathbb{N}^*} \):

\[
\begin{align*}
\int_0^T \sum_{k_2=0}^\infty \sum_{k_1=1}^\infty p_{k_1} (\alpha_k e^{i\lambda_k t} + \alpha_{-k} e^{-i\lambda_k t})^2 \, dt \\
+ \int_0^T \sum_{k_2=0}^\infty \sum_{k_1=1}^\infty q_{k_2} (\alpha_k e^{i\lambda_k t} + \alpha_{-k} e^{-i\lambda_k t})^2 \, dt \\
\geq c \sum_{k \in \mathbb{N}^* \times \mathbb{N}} \left( |p_{k_1}|^2 + |q_{k_2}|^2 \right) \left( |\alpha_k|^2 + |\alpha_{-k}|^2 \right).
\end{align*}
\]

(29)

For the proof, we shall need the following

**Lemma 5.4.** Let us consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
\begin{align*}
f(t) &= \begin{cases} 
\cos\left(\frac{\pi}{T} t\right) & \text{if } |t| \leq \frac{T}{T}, \\
0 & \text{if } |t| > \frac{T}{2}.
\end{cases}
\end{align*}
\]

Then its Fourier transform \( \hat{f} \) is given by

\[
\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{ixt} \, dt = -\frac{2T\pi \cos(xT/2)}{x^2T^2 - \pi^2}, \quad x \neq -\frac{\pi}{T} + \frac{\pi}{T}.
\]

Furthermore, we have for \( \gamma > \frac{2\pi}{T}, \ l \in \mathbb{N}^* \) and \( |x| \geq \gamma T \)

\[
|\hat{f}(x)| \leq \frac{2T}{\pi} \left( \frac{2\pi}{\gamma T} \right)^2 \frac{1}{4l^2 - 1}.
\]

**Proof.** One may turn to [20] or [13] for the proof. \( \square \)

Now, we are ready to take up proving estimate (29) based on the techniques already used by Mehrenberger [20].

**Proof of theorem 11.** To make use of the lemma above, we will effect the integrations over \([-\frac{T}{2}, \frac{T}{2}]\). A classical translation \( s = t + \frac{T}{2} \) enables to recover (29).

For brevity, we set \( k^\prime = (k_1', k_2') \) and

\[
A_{k^2} := \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{k_1=1}^\infty p_{k_1} (\alpha_k e^{i\lambda_k t} + \alpha_{-k} e^{-i\lambda_k t}) \right|^2 \, dt.
\]

Since \( 0 \leq f(t) \leq 1_{[-\frac{T}{2}, \frac{T}{2}]} \), we have

\[
A_{k^2} \geq \int_{\mathbb{R}} |f(t)| \sum_{k_1 \in \mathbb{N}^*, \ |q_{k_2}| \leq |p_{k_1}|} p_{k_1} (\alpha_k e^{i\lambda_k t} + \alpha_{-k} e^{-i\lambda_k t}) + \\
+ \sum_{k_1 \in \mathbb{N}^*, \ |q_{k_2}| > |p_{k_1}|} p_{k_2} (\alpha_k e^{i\lambda_k t} + \alpha_{-k} e^{-i\lambda_k t})^2 \, dt \\
= \sum_{(k_1, k_1') \in \Lambda_1} + \sum_{(k_1, k_1') \in \Lambda_2} + \sum_{(k_1, k_1') \in \Lambda_3} B_k + \\
\]

\[
\sum_{k \in \mathbb{N}^* \times \mathbb{N}} \left( |p_{k_1}|^2 + |q_{k_2}|^2 \right) \left( |\alpha_k|^2 + |\alpha_{-k}|^2 \right).
\]

(29)
where

\[ A_1 := \{ k_1, k_1' \in \mathbb{N}, |q_{k_2}| \leq |p_{k_1}|, |q_{k_2}^*| \leq |p_{k_1}'| \}, \]

\[ A_2 := \{ k_1, k_1' \in \mathbb{N}, |q_{k_2}| \leq |p_{k_1}|, |q_{k_2}^*| > |p_{k_1}'| \}, \]

\[ A_3 := \{ k_1, k_1' \in \mathbb{N}, |q_{k_2}| > |p_{k_1}|, |q_{k_2}^*| \leq |p_{k_1}'| \}, \]

and because \( \tilde{f}(-x) = \tilde{f}(x) \), we have

\[ B_k = p_{k_1}p_{k_1}'((\alpha_k \alpha_{k'} + \alpha_{-k} \alpha_{-k'}) \tilde{f}(\lambda_k - \lambda_{k'}) + (\alpha_k \alpha_{-k'} + \alpha_{-k} \alpha_{k'}) \tilde{f}(\lambda_k + \lambda_{k'})). \]

Seeing as \( f(t) \geq 0 \), we can discard the last term of (30). Now, taking account of the gap assumptions (27), we have for \( T > \frac{2\pi}{\gamma_1} \) (cf. lemma 5.4)

\[
\begin{align*}
|\tilde{f}(\lambda_k - \lambda_{k'})| &\leq \frac{2T}{\pi} \left( \frac{2\pi}{\gamma_1 T} \right)^2 \frac{1}{4(k_1 - k_1')^2 - 1}, \quad \forall \lambda_k \neq \lambda_{k'}, \\
|\tilde{f}(\lambda_k + \lambda_{k'})| &\leq \frac{2T}{\pi} \left( \frac{2\pi}{\gamma_1 T} \right)^2 \frac{1}{4(k_1 + k_1')^2 - 1}, \quad \forall k, k' \in \mathbb{N}^* \times \mathbb{N}.
\end{align*}
\]

On the other hand, using Young inequality, we find

\[
\begin{align*}
p_{k_1}p_{k_1}'((\alpha_k \alpha_{k'} + \alpha_{-k} \alpha_{-k'}) &\leq \frac{1}{2} \left( |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) + |p_{k_1}'|^2 (|\alpha_k'|^2 + |\alpha_{-k'}|^2) \right) \\
p_{k_1}p_{k_1}'((\alpha_k \alpha_{-k'} + \alpha_{-k} \alpha_{k'}) &\leq \frac{1}{2} \left( |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) + |p_{k_1}'|^2 (|\alpha_k'|^2 + |\alpha_{-k'}|^2) \right)
\end{align*}
\]

These estimates allow us to minorize the first three terms of (30) as follows

\[
\begin{align*}
\sum_{(k_1, k_1') \in A_1} B_k &\geq \frac{2T}{\pi} \left( \sum_{k_1 \in \mathbb{N}^*, |q_{k_2}| \leq |p_{k_1}|} |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) - \sum_{(k_1, k_1') \in A_1, k_1 \neq k_1'} C_k \right) - \sum_{(k_1, k_1') \in A_1} D_k, \\
\sum_{(k_1, k_1') \in A_2} B_k &\geq -\frac{2T}{\pi} \sum_{(k_1, k_1') \in A_2} (C_k + D_k), \\
\sum_{(k_1, k_1') \in A_3} B_k &\geq -\frac{2T}{\pi} \sum_{(k_1, k_1') \in A_3} (C_k + D_k)
\end{align*}
\]

where

\[
\begin{align*}
C_k &= \left( \frac{2\pi}{\gamma_1 T} \right)^2 \left( \frac{1}{2} \left( \frac{|p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2)}{2} + \frac{|p_{k_1}'|^2 (|\alpha_k'|^2 + |\alpha_{-k'}|^2)}{2} \right) \right) \\
D_k &= \left( \frac{2\pi}{\gamma_1 T} \right)^2 \left( \frac{1}{2} \left( \frac{|p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2)}{2} + \frac{|p_{k_1}'|^2 (|\alpha_k'|^2 + |\alpha_{-k'}|^2)}{2} \right) \right)
\end{align*}
\]
Recalling that \( \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2} \), we add up these inequalities to obtain for \( T > \frac{2\pi}{\gamma_1} \)

\[
\frac{\pi}{2T} A_{k_2} \geq \sum_{k_1 \in \mathbb{N}^*, |q_{k_1}| \leq |p_{k_1}|} |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) - \sum_{(k_1, k'_1) \in \mathbb{N}^*, k_1 \neq k'_1} C_{k_1} - \sum_{(k_1, k'_1) \in \mathbb{N}^*} D_{k_1} 
\geq \sum_{k_1 \in \mathbb{N}^*, |q_{k_1}| \leq |p_{k_1}|} |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) - \sum_{k_1 = 1}^{\infty} \left( \frac{2\pi}{\gamma_1 T} \right)^2 |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2).
\]

Hence, we get

\[
\frac{\pi}{2T} \sum_{k_2 = 0}^{\infty} A_{k_2} \geq \sum_{k_2 = 0}^{\infty} \sum_{k_1 \in \mathbb{N}^*} \sum_{|q_{k_1}| \leq |p_{k_1}|} |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) - \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 1}^{\infty} \left( \frac{2\pi}{\gamma_1 T} \right)^2 |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2)
= \sum_{k_2 = 0}^{\infty} \sum_{k_1 = 1}^{\infty} \left( \sum_{|q_{k_1}| \leq |p_{k_1}|} \right) - \left( \frac{2\pi}{\gamma_1 T} \right)^2 \times
\times |p_{k_1}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2).
\]

Denoting this time, \( k' = (k_1, k'_1) \) and

\[
\bar{A}_{k_1} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_{k_2 = 0}^{\infty} q_{k_2} (\alpha_k e^{i\lambda_k t} + \alpha_{-k} e^{-i\lambda_k t}) \right|^2 dt,
\]

it is evident that by performing the same computations on \( \bar{A}_{k_1} \), we get for \( T > \frac{2\pi}{\gamma_2} \)

\[
\frac{\pi}{2T} \bar{A}_{k_1} \geq \sum_{k_2 \in \mathbb{N}, \ |q_{k_2}| \leq |q_{k_1}|} |q_{k_2}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) - \sum_{k_2 = 0}^{\infty} \left( \frac{2\pi}{\gamma_2 T} \right)^2 |q_{k_2}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2).
\]

Summing over \( k_1 \) yields

\[
\frac{\pi}{2T} \sum_{k_1 = 1}^{\infty} \bar{A}_{k_1} \geq \sum_{k_1 = 1}^{\infty} \sum_{k_2 = 0}^{\infty} \left( \sum_{|q_{k_2}| \leq |q_{k_1}|} \right) - \left( \frac{2\pi}{\gamma_2 T} \right)^2 \times
\times |q_{k_2}|^2 (|\alpha_k|^2 + |\alpha_{-k}|^2).
\]

Now, noting that for fixed \( k_1 \in \mathbb{N}^*, k_2 \in \mathbb{N} \)

\[
1 \{ |q_{k_2}| \leq |q_{k_1}| \} |p_{k_1}|^2 + 1 \{ |q_{k_1}| \leq |q_{k_2}| \} |q_{k_2}|^2 \geq \max \left( \frac{|p_{k_1}|^2}{\gamma_1}, \frac{|q_{k_2}|^2}{\gamma_2} \right) \geq \frac{|p_{k_1}|^2}{\gamma_1} + \frac{|q_{k_2}|^2}{\gamma_2}.
\]
we deduce
\[
\sum_{k_2=0}^{\infty} A_{k_2} + \sum_{k_1=1}^{\infty} A_{k_1} \geq \left( \frac{1}{\gamma_1^2 + \frac{\pi^2}{T^2}} - \left( \frac{2\pi}{T} \right)^2 \right) \times \\
\times \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \left( \frac{|p_{k_1}|^2}{\gamma_1^2} + \frac{|q_{k_2}|^2}{\gamma_2^2} \right) \left( |\alpha_k|^2 + |\alpha_{-k}|^2 \right)
\]
\[
\geq \min \left( \frac{1}{\gamma_1^2 + \frac{\pi^2}{T^2}} - \left( \frac{2\pi}{T} \right)^2 \right) \times \\
\times \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \left( |p_{k_1}|^2 + |q_{k_2}|^2 \right) \left( |\alpha_k|^2 + |\alpha_{-k}|^2 \right).
\]
Whence, for the desired inequality to hold, we should take \( T > 2\pi \sqrt{\frac{1}{\gamma_1^2 + \frac{\pi^2}{T^2}}} \).

5.2. Proof of the exact observability. The plan is to establish the inverse inequality cf. (26) as an application of theorem 5.3. Hence, we should ensure that the eigenvalues \((\lambda_k)_{k \in \mathbb{N}^* \times \mathbb{N}}\) cf. (22) satisfy the gap assumptions (27), (28); that is, we need to prove the following

Lemma 5.5. Let \((\omega_k)_{k \in \mathbb{N}^* \times \mathbb{N}}\) be the sequence defined in (22). Given \(k_1, k_1' \in \mathbb{N}^*, k_2, k_2' \in \mathbb{N},\) we have:

(i) If \(\mu_{k_2} \leq \max\left( \frac{k_1 \pi}{l_1}, \frac{k_1' \pi}{l_1'} \right),\) then there exists \(\gamma_1 = (\sqrt{2} - 1) \frac{\pi}{l_1} > 0\) such that
\[
|\omega_{k_1,k_2} - \omega_{k_1',k_2}| \geq \gamma_1 |k_1 - k_1'|,
\]
\[
|\omega_{k_1,k_2} + \omega_{k_1',k_2}| \geq \gamma_1 |k_1 + k_1'|.
\]

(ii) If \(\frac{k_1 \pi}{l_1} \leq \max(\mu_{k_2}, \mu_{k_2'})\), there exists \(\gamma_2 = \frac{(\sqrt{2} - 1)}{2} \frac{\pi}{l_2} > 0\) such that
\[
|\omega_{k_1,k_2} - \omega_{k_1,k_2'}| \geq \gamma_2 |k_2 - k_2'|,
\]
\[
|\omega_{k_1,k_2} + \omega_{k_1,k_2'}| \geq \gamma_2 |k_2 + k_2'|.
\]

Proof. (i) Let \(k_1, k_1' \in \mathbb{N}^*, k_2 \in \mathbb{N}.\) Assume that \(\mu_{k_2} \leq \max\left( \frac{k_1 \pi}{l_1}, \frac{k_1' \pi}{l_1'} \right)\) and that \(k_1 < k_1',\) then
\[
|\omega_{k_1,k_2} + \omega_{k_1',k_2}| = \left( \frac{k_1 \pi}{l_1} \right)^2 + \mu_{k_2}^2 + \left( \frac{k_1' \pi}{l_1'} \right)^2 + \mu_{k_2'}^2 \geq \frac{\pi}{l_1} |k_1 + k_1'|,
\]
\[
|\omega_{k_1,k_2} - \omega_{k_1',k_2}| = \left( \frac{\pi}{l_1} \right)^2 |k_1 - k_1'| \frac{|k_1 + k_1'|}{\sqrt{\left( \frac{k_1 \pi}{l_1} \right)^2 + \mu_{k_2}^2} + \sqrt{\left( \frac{k_1' \pi}{l_1'} \right)^2 + \mu_{k_2'}^2}} \geq \left( \frac{\pi}{l_1} \right)^2 \frac{|k_1 + k_1'|}{\sqrt{\left( \frac{k_1 \pi}{l_1} \right)^2 + \left( \frac{k_1' \pi}{l_1'} \right)^2 + 2 \frac{k_1 \pi}{l_1} \frac{k_1' \pi}{l_1'}}} |k_1 - k_1'| \geq \frac{\pi}{l_1} \frac{1 + \frac{k_1}{k_1'}}{\sqrt{2} + \left( \frac{k_1}{k_1'} \right)^2} |k_1 - k_1'|.
\]
Putting $x := \frac{k_1}{k_1'} \in (0, 1)$, we define

$$h(x) := \frac{1 + x}{\sqrt{2 + \sqrt{1 + x^2}}}.$$ 

Simple derivation gives

$$h'(x) = \frac{1}{(\sqrt{2 + \sqrt{1 + x^2}})^3} \left( \frac{1}{\sqrt{2 + \sqrt{1 + x^2}}} - \frac{1 - x}{\sqrt{1 + x^2}} \right) > 0, \quad \forall x \in (0, 1);$$

hence, $h$ is strictly increasing and we have

$$h(x) > h(0) = \sqrt{2} - 1, \forall x \in (0, 1).$$

Thus,

$$|\omega_{k_1, k_2} - \omega_{k_1', k_2'}| \geq \left( \frac{\sqrt{2} - 1}{l} \right) \frac{\pi}{l_1} |k_1 - k_1'|.$$

(ii) Fix $k_1 \in \mathbb{N}^*$, $k_2$, $k_2' \in \mathbb{N}$. Assume that $\frac{k_1 \pi}{l_1} \leq \max(\mu_{k_2}, \mu_{k_2'})$ and $k_2 < k_2'$.

Since $\frac{k_2 \pi}{l_2} \leq \mu_{k_2} \leq \frac{(k_2 + 1) \pi}{l_2}$, we get

$$|\omega_{k_1, k_2} + \omega_{k_1, k_2'}| \geq |\mu_{k_2} + \mu_{k_2'}| \geq \frac{\pi}{l_2} |k_2 + k_2'|;$$

on the other hand, we have

$$|\omega_{k_1, k_2} - \omega_{k_1, k_2'}| = \left| \mu_{k_2} - \mu_{k_2'} \right| \frac{\left| \mu_{k_2} + \mu_{k_2'} \right|}{\sqrt{\left( \frac{k_1 \pi}{l_1} \right)^2 + \mu_{k_2}^2 + \sqrt{\left( \frac{k_1 \pi}{l_1} \right)^2 + \mu_{k_2'}^2}}}
\geq \left| \mu_{k_2} - \mu_{k_2'} \right| \frac{\left| \mu_{k_2} + \mu_{k_2'} \right|}{\sqrt{\mu_{k_2}^2 + \mu_{k_2'}^2 + \sqrt{2 \mu_{k_2'}^2}}}
\geq \frac{\left| \mu_{k_2} - \mu_{k_2'} \right|}{\sqrt{2 + \sqrt{1 + \left( \frac{\mu_{k_2}}{\mu_{k_2'}} \right)^2}}}
\geq \left( \frac{\sqrt{2} - 1}{l} \right) \frac{\pi}{\mu_{k_2} - \mu_{k_2'}}.$$

So it remains to bound $|\mu_{k_2} - \mu_{k_2'}|$ from below by $c|k_2 - k_2'|$, $c = cst > 0$. Since $\frac{k_2 \pi}{l_2} \leq \mu_{k_2} \leq \frac{k_2 \pi}{l_2} + \frac{\pi}{2l_2}$, we find that:

if $k_2' = k_2 + 1$, then

$$\mu_{k_2} - \mu_{k_2} \geq \frac{(k_2 + 1) \pi}{l_2} - \frac{k_2 \pi}{l_2} = \frac{\pi}{2l_2} |k_2' - k_2|.$$

if $k_2' \geq k_2 + 2$, then

$$\mu_{k_2'} - \mu_{k_2} \geq \frac{\pi}{l_2} (k_2' - k_2 - 1) \geq \frac{\pi}{2l_2} |k_2' - k_2|.$$

Consequently, $|\omega_{k_1, k_2} - \omega_{k_1, k_2'}| \geq \frac{\left( \frac{\sqrt{2} - 1}{2} \right) \frac{\pi}{l_2}}{|k_2' - k_2|}$.

- We proceed now with the proof of the observability estimate (26):
We know that the solution of (3) is given by Fourier series
\[ u(x_1, x_2, t) = \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} (a_{k_1} e^{i\omega k_1 t} + a_{-k_1} e^{-i\omega k_1 t}) \sin\left(\frac{k_1 \pi}{l_1} x_1\right) \sin(\mu k_2 x_2). \] (31)

Using this expression and the orthogonality of systems \((\cos \frac{k_1 \pi}{l_1} x_1), (\sin \frac{k_1 \pi}{l_1} x_1)\) in \(L^2(0, l_1)\) and that of \((\cos \mu k_2 x_2), (\sin \mu k_2 x_2)\) in \(L^2(0, l_2)\) and \(L^2(0, l_2) \times \mathbb{C}\) (cf. proposition 2), the initial energy is found to satisfy
\[ \|\nabla u_0\|^2_{\Omega} + \|u_2\|^2_{\Omega} + \|\nabla T u_0\|^2_{TV} + \|u_3\|^2_{TV} \]
\[ \approx \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \left( \left(\frac{k_1 \pi}{l_1}\right)^2 + \mu^2 k_2\right) (|a_{k_1}|^2 + |a_{-k_1}|^2). \]

Likewise, we explicate the right side of (26) in terms of Fourier series (31)
\[
\begin{align*}
& \int_0^T \int_{\Gamma_1} |\partial_x u(x, t)|^2 d\Gamma dt + \int_0^T |\partial_x u(0, l_2, t)|^2 dt + \int_0^T |\partial_x u(x, t)|^2 d\Gamma dt \\
& = \int_0^T \int_0^{l_2} \sum_{k_1, k_2} \frac{k_1 \pi}{l_1} \left( a_{k_1} e^{i\omega k_1 t} + a_{-k_1} e^{-i\omega k_1 t} \right) \sin(\mu k_2 x_2) \\ & \quad + \int_0^T \int_0^{l_2} \sum_{k_1, k_2} \frac{k_1 \pi}{l_1} \left( a_{k_1} e^{i\omega k_1 t} + a_{-k_1} e^{-i\omega k_1 t} \right) \sin(\mu k_2 l_2) \\
& \quad + \int_0^T \int_0^{l_2} \sum_{k_1, k_2} \mu k_2 \left( a_{k_1} e^{i\omega k_1 t} + a_{-k_1} e^{-i\omega k_1 t} \right) \sin\left(\frac{k_1 \pi}{l_1} x_1\right) dx_1 dt.
\end{align*}
\]

Once again, the orthogonality of the sine systems implies
\[
\begin{align*}
& \int_0^T \int_{\Gamma_1} |\partial_x u(x, t)|^2 d\Gamma + |\partial_x u(0, l_2, t)|^2 dt + \int_0^T |\partial_x u(x, t)|^2 d\Gamma dt \\
& \propto \int_0^T \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \mu k_2 \left( a_{k_1} e^{i\omega k_1 t} + a_{-k_1} e^{-i\omega k_1 t} \right)^2 dt + \\
& \quad + \int_0^T \sum_{k_2=0}^{\infty} \sum_{k_1=1}^{\infty} \frac{k_1 \pi}{l_1} \left( a_{k_1} e^{i\omega k_1 t} + a_{-k_1} e^{-i\omega k_1 t} \right)^2 dt.
\end{align*}
\]

Taking \(\lambda_k = \omega_k, p_{k_1} = \frac{k_1 \pi}{l_1}, q_{k_2} = \mu k_2, (k_1, k_2) \in \mathbb{N}^* \times \mathbb{N}\) in the statement of theorem 5.3, the inverse inequality (26) follows immediately from (29).

Thus far, we have been able to drive system (1) to rest through Dirichlet action on two adjacent sides of the boundary \(\Gamma_D\). It seems interesting to consider including \(\Gamma_V\) in the controlled area and seeing if it is possible to steer the system to equilibrium. This called for the following remarks.
**Remark 5.** Let’s consider one particular solution of the homogeneous problem (3)
\[ u_k(x_1, x_2, t) = a_k e^{i\omega_k t} \sin \left( \frac{k_1\pi}{l_1} x_1 \right) \sin(\mu x_2). \]

Then we have
\[ u_k'(x_1, l_2, t) = ia_k \omega_k e^{i\omega_k t} \sin \left( \frac{k_1\pi}{l_1} x_1 \right) \sin(\mu x_2). \]

Note that this could not be minorized in terms of the initial energy because of the factor \( \sin(\mu x_2) \) that goes to zero as \( k_2 \) tends to infinity. Actually, it’s not possible to exactly control system (1) on Ventcel portion of the boundary, because neither the tangential nor the time derivatives of the solution of (3) are observable on this portion (they both contain \( \sin(\mu x_2) \)).

The same could be said if we impose Ventcel condition on the top and right side of \( \Omega \) and attempt to control the system through Ventcel action only. For this time the solution of the associated homogeneous problem is given by
\[ u(x_1, x_2, t) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \left( a_k e^{i\omega_k t} + a_{-k} e^{-i\omega_k t} \right) \sin(\delta_{k_1} x_1) \sin(\mu x_2), \]
where \( \delta_{k_1} = \cot(\delta_{k_1} l_1), \mu_{k_2} = \cot(\mu_{k_2} l_2) \) and \( \omega_k = \sqrt{\delta_{k_1}^2 + \mu_{k_2}^2}. \)

6. **Further results on the observability of system (3).** In this section, we shall employ once more our variant of Mehrenberger’s theorem to the next two results concerning internal and combined internal-boundary observability.

**Theorem 6.1.** Let \((a, b), (c, d)\) be two non-empty subintervals of respectively \((0, l_1)\) and \((0, l_2)\). Given \(((u_0, u_0|_{\Gamma_V}), (u_2, u_3)) \in X\), we have for \( T > 0 \) sufficiently large, the following observability estimate:
\[ E(0) \leq c \left( \int_0^T \int_0^{l_1} |u'(x_1, x_2, t)|^2 \, dx_1 \, dx_2 \, dt \right)^{\frac{1}{2}} + \int_0^T \int_0^{l_2} |u'(x_1, x_2, t)|^2 \, dx_2 \, dx_1 \, dt \]
\[ + \int_0^T \int_0^{l_2} |u'(x_1, l_2, t)|^2 \, dx_1 \, dt. \]  

**Theorem 6.2.** Let \((a, b)\) be a non-empty subinterval of \((0, l_1)\). Let \(((u_0, u_0|_{\Gamma_V}), (u_2, u_3)) \in X\). Then, for \( T \) sufficiently large, the solution of (3) is proved to satisfy
\[ E(0) \leq c \left( \int_0^T \int_0^{l_1} \left| \frac{\partial u}{\partial x_2}(x_1, 0, t) \right|^2 \, dx_1 \, dt \right)^{\frac{1}{2}} + \int_0^T \int_0^{l_2} |u'(x_1, x_2, t)|^2 \, dx_2 \, dx_1 \, dt \]
\[ + \int_0^T \int_0^{l_2} |u'(x_1, l_2, t)|^2 \, dx_1 \, dt. \]

The estimate of theorem 5.3 cannot directly lead to the inequalities above, that’s why we rewrite it in the following way.
Theorem 6.3. Let \((\omega_l)_{l=-\infty}^{+\infty}\) be a sequence of real numbers. Given \((p_l)_{l=-\infty}^{+\infty} \subset \mathbb{C}\), we assume the following partial gap condition: there exist \(\gamma, \eta > 0\) such that for \(l, l' \in \mathbb{Z}\) we have
\[ |\omega_l - \omega_{l'}| \geq \gamma |l - l'| \text{ whenever } \max(|p_l|, |p_{l'}|) \geq \eta. \]

Then, for \(T > \frac{2\pi}{\gamma}\) we get
\[ \int_0^T \left| \sum_{l=-\infty}^{+\infty} \alpha_l e^{i\omega_l t} \right|^2 dt \geq \frac{2T}{\pi} \left( \sum_{|p_l| \geq \eta} |\alpha_l|^2 - \left( \frac{2\pi}{T\gamma} \right)^2 \sum_{l=-\infty}^{+\infty} |\alpha_l|^2 \right), \quad (34) \]
for all square summable sequences \((\alpha_l)_{l=-\infty}^{+\infty}\) of complex numbers.

Proof. We get this inequality by repeating verbatim the computations carried out by Komornik [13], p. 140-141. However, we call attention to the fact that here, the decomposition of the sum is done according to the sequence \((p_l)_{l=-\infty}^{+\infty}\).

Before moving on to the demonstrations of the observability theorems, we take the chance to state the following

Remark 6. Let’s go back to the situation considered in remark 5 when we have two controls, one on the top side of \(\Gamma\) and the other on the left side. The representation given in the theorem above allows us to see that after all system (3) is approximately controllable. Indeed, if we assume that
\[ \int_0^{l_1} \left| u'(x_1, l_2, t) \right|^2 dx_1 + \int_0^{l_2} \left| \frac{\partial u}{\partial x_1}(0, x_2, t) \right|^2 dx_2 + \left| \frac{\partial u}{\partial x_1}(0, l_2, t) \right|^2 = 0, \quad \forall t \in (0, T). \]

Then, according to inequality (34) we have
\[ \left( \left( \frac{k_1\pi}{l_1} \right)^2 + |\omega_k \sin \mu_k l_2|^2 \right) \left( |\alpha_k|^2 + |\alpha_{-k}|^2 \right) = 0, \forall k \in \mathbb{N}^* \times \mathbb{N}, \]
which implies that the solution of (3) is approximately observable on \(\Gamma_1\) and \(\Gamma_Y\).

The gap assumptions satisfied by \((\omega_k)_{k \in \mathbb{N}^* \times \mathbb{N}}\) (cf. lemma 5.5) together with theorem 6.3 help us draw the following

Corollary 2. If \(T > \frac{2\pi}{\gamma_1} = 2 \left( \sqrt{2} + 1 \right) l_1\), then for every \(k_2 \in \mathbb{N}\) we have
\[ \int_0^T \left| \sum_{k_1=1}^{+\infty} \left( \alpha_{k_1} e^{i\omega_{k_1} t} + \alpha_{-k_1} e^{-i\omega_{k_1} t} \right) \right|^2 dt \]
\[ \geq \left( \frac{2T}{\pi} - \frac{8 \left( \sqrt{2} + 1 \right)^2 l_1^2}{T\pi} \right) \sum_{k_1 \in \mathbb{N}^*, \frac{k_1\pi}{l_1} \geq \mu_{k_2}} (|\alpha_{k_1}|^2 + |\alpha_{-k_1}|^2) \]
\[ - \frac{8 \left( \sqrt{2} + 1 \right)^2 l_1^2}{T\pi} \sum_{k_1 \in \mathbb{N}^*, \frac{k_1\pi}{l_1} < \mu_{k_2}} (|\alpha_{k_1}|^2 + |\alpha_{-k_1}|^2). \]
If \( T > \frac{2\pi}{\gamma^2} = 4 (\sqrt{2} + 1) l_2 \), then for every \( k_1 \in \mathbb{N}^* \) we have

\[
\int_0^T \left| \sum_{k_2=0}^\infty \left( \alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t} \right) \right|^2 \ dt
\geq \left( \frac{2T}{\pi} - \frac{32}{\pi^2} \right) \frac{l_1^2}{T} \sum_{k_2 \in \mathbb{N}, \mu_k \geq \frac{k_1 \pi}{l_1}} (\left| \alpha_k \right|^2 + \left| \alpha_{-k} \right|^2)
- \frac{32}{\pi^2} \left( \sqrt{2} + 1 \right) \frac{l_2^2}{T} \sum_{k_2 \in \mathbb{N}, \mu_k < \frac{k_1 \pi}{l_1}} (\left| \alpha_k \right|^2 + \left| \alpha_{-k} \right|^2).
\]

Setting

\[
m_{a,b} := \inf_{n \in \mathbb{N}^*} \int_a^b \sin^2\left( \frac{n\pi}{l_1} y \right) \ dy
\]
\[
m_{c,d} := \inf_{m \in \mathbb{N}} \int_c^d \sin^2(\mu m y) \ dy.
\]

We get by means of classical trigonometric relations (see [13], p. 136) that

\[
0 < m_{a,b} \leq \int_0^{l_1} \sin^2\left( \frac{n\pi}{l_1} y \right) \ dy = \frac{l_1}{2},
\]
\[
0 < m_{c,d} \leq \int_0^{l_2} \sin^2(\mu m y) \ dy \leq \frac{l_2}{2}.
\]

Now, we have the necessary material to conduct the proofs of theorems 6.1 and 6.2.

6.1. **Proof of theorem 6.1.** We are going to assess the terms of the right-hand side of inequality (32), using the expansion of \( u(x,t) \) as a Fourier series as well as the estimates of corollary 2. We start up with the first term

\[
\int_0^T \int_0^{l_1} |u'(x_1, x_2, t)|^2 \ dx_1 \ dt
\]
\[
= \int_0^T \int_0^{l_1} \left| \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} i\omega_k \left( a_k e^{i\omega_k t} - a_{-k} e^{-i\omega_k t} \right) \sin\left( \frac{k_1 \pi}{l_1} x_1 \right) \sin(\mu_k x_2) \right|^2 \ dx_1 \ dt
\]
\[
= \frac{l_1}{2} \int_0^T \sum_{k_1} \left| \sum_{k_2} \omega_k \left( a_k e^{i\omega_k t} - a_{-k} e^{-i\omega_k t} \right) \sin(\mu_k x_2) \right|^2 \ dt.
\]

Using the second estimate of corollary 2, we get for \( T > 4 (\sqrt{2} + 1) l_2 \)

\[
\int_0^T \int_0^{l_1} |u'(x_1, x_2, t)|^2 \ dx_1 \ dt
\]
\[
\geq \frac{l_1}{2} \sum_{k_1} \left( \frac{2T}{\pi} - \frac{32 (\sqrt{2} + 1)^2 l_2^2}{T \pi} \right) \sum_{k_2, \mu k_2 \geq \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \sin^2(\mu k_2 x_2)
\]
\[
- \frac{l_1}{2} \sum_{k_1} \frac{32 (\sqrt{2} + 1)^2 l_2^2}{T \pi} \sum_{k_2, \mu k_2 < \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \sin^2(\mu k_2 x_2)
\]
\[
= \frac{l_1}{\pi} \left( T - \frac{16 (\sqrt{2} + 1)^2 l_2^2}{T} \right) \sum_{k_1, k_2, \mu k_2 \geq \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right)
\]
\[
- \frac{l_1}{\pi} \frac{8 (\sqrt{2} + 1)^2 l_2^2}{T} \sum_{k_1, k_2, \mu k_2 < \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right).
\]

Integrating over \((c, d)\), it results
\[
\int_0^T \int_0^d \int_0^1 |u'(x_1, x_2, t)|^2 dx_1 dx_2 dt
\]
\[
\geq \frac{l_1}{\pi} \left( T - \frac{16 (\sqrt{2} + 1)^2 l_2^2}{T} \right) m_{c,d} \sum_{k_1, k_2, \mu k_2 \geq \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right)
\]
\[
- \frac{l_1}{\pi} \frac{8 (\sqrt{2} + 1)^2 l_2^2}{T} \sum_{k_1, k_2, \mu k_2 < \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right).
\]

Thus, if \(T^2 > 16 (\sqrt{2} + 1)^2 l_2^2\), we have
\[
\int_0^T \int_0^d \int_0^1 |u'(x_1, x_2, t)|^2 dx_1 dx_2 dt
\]
\[
\geq \frac{l_1}{\pi} \left( T - \frac{16 (\sqrt{2} + 1)^2 l_2^2}{T} \right) m_{c,d} \sum_{k_1, k_2, \mu k_2 \geq \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right)
\]
\[
- \frac{l_1}{\pi} \frac{8 (\sqrt{2} + 1)^2 l_2^2}{T} \sum_{k_1, k_2, \mu k_2 < \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right).
\]

Similarly, we handle the other two terms on the right side of inequality (32) to find that for \(T > 2 (\sqrt{2} + 1) l_1\), we have
\[
\int_0^T \left( \int_0^{l_2} |u'(x_1, x_2, t)|^2 dx_2 + |u'(x_1, l_2, t)|^2 \right) dt
\]
\[
= \frac{1}{2} \sum_{k_2} \frac{l_2 \mu k_2^2 + l_2 + 1}{2(1 + \mu k_2^2)} \left| \sum_{k_1} \omega_k (a_k e^{i \omega_k t} - a_{-k} e^{-i \omega_k t}) \sin \frac{k_1 \pi}{l_1} x_1 \right|^2 dt
\]
\[
\geq \frac{l_2}{2} \sum_{k_2} \left( \frac{2T}{\pi} - \frac{8 (\sqrt{2} + 1)^2 l_2^2}{T \pi} \right) \sum_{k_1, \mu k_2 \geq \frac{\mu_1}{1+\mu_1}} |\omega_k|^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \sin^2 \frac{k_1 \pi}{l_1} x_1.
Integrating with respect to $x_1$ yields

$$
\int_0^T \left( \int_0^b \left| u'(x_1, x_2, t) \right|^2 dx_2 dx_1 + \int_a^b \left| u'(x_1, l_2, t) \right|^2 dx_1 \right) dt
\geq \frac{l_2}{\pi} \left( T - 4 \left( \sqrt{2} + 1 \right)^2 l_1^2 \right) m_{a,b} \sum_{(k_1, k_2), \frac{k_1 \pi}{l_1} \geq \mu_{k_2}} |\omega_k|^2 (|a_k|^2 + |a_{-k}|^2) + \frac{l_1}{\pi} \left( T - 16 \left( \sqrt{2} + 1 \right)^2 l_1^2 \right) m_{c,d} \sum_{(k_1, k_2), \frac{k_1 \pi}{l_1} < \mu_{k_2}} |\omega_k|^2 (|a_k|^2 + |a_{-k}|^2).
$$

Adding up estimates (35), (36) gives

$$
\int_0^T \int_0^1 \int_0^1 \left| u'(x_1, x_2, t) \right|^2 dx_1 dx_2 dx_3 dt + \int_0^T \int_0^1 \int_0^1 \left| u'(x_1, x_2, t) \right|^2 dx_2 dx_1 dt
\geq \left( \frac{l_2}{\pi} \left( T - 4 \left( \sqrt{2} + 1 \right)^2 l_1^2 \right) m_{a,b} - \frac{l_1}{\pi} \left( T - 16 \left( \sqrt{2} + 1 \right)^2 l_1^2 \right) m_{c,d} \right) \sum_{(k_1, k_2), \frac{k_1 \pi}{l_1} \geq \mu_{k_2}} |\omega_k|^2 (|a_k|^2 + |a_{-k}|^2) + \int_0^T \int_0^1 \left| u'(x_1, l_2, t) \right|^2 dx_1 dt
$$

Hence, for us to have inequality (32), it suffices to take

$$
T^2 > \max \left\{ 4 \left( \sqrt{2} + 1 \right)^2 l_1^2 + \frac{8 (\sqrt{2} + 1)^2 l_1 l_2^2}{m_{a,b}}, 16 \left( \sqrt{2} + 1 \right)^2 l_1^2 + \frac{2 (\sqrt{2} + 1)^2 (l_2 + 1) l_2}{m_{c,d}} \right\}.
$$

6.2. Proof of theorem 6.2. This demonstration goes along the lines of the previous one. We have already evaluated the last two terms of the right-hand side of (33); thus to attain the desired inequality, it suffices to treat similarly the first term.

$$
\int_0^T \int_0^1 \left| \frac{\partial u}{\partial x_2} (x_1, 0, t) \right|^2 dx_1 dt
= \int_0^T \sum_{k_1, k_2} \mu_{k_2} \left( a_k e^{i \omega_k t} + a_{-k} e^{-i \omega_k t} \right) \sin \frac{k_1 \pi}{l_1} x_1 \left| dx_1 dt
= \frac{l_1}{2} \sum_{k_1} \sum_{k_2} \mu_{k_2} \left( a_k e^{i \omega_k t} + a_{-k} e^{-i \omega_k t} \right) dt
$$
\[
\begin{align*}
&\geq \frac{l_1}{2} \sum_{k_1} \left( \frac{2T}{\pi} - 32 \left( \sqrt{2} + 1 \right)^2 l_1^2 \right) \sum_{k_2 \in \mathbb{N}, \, \mu_{k_2} \geq \frac{k_1 \pi}{T}} \mu_{k_2}^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \\
&\quad - \frac{l_1}{2} \sum_{k_1} \frac{32 \left( \sqrt{2} + 1 \right)^2 l_1^2}{T} \sum_{k_2 \in \mathbb{N}, \, \mu_{k_2} < \frac{k_1 \pi}{T}} \mu_{k_2}^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \\
&= \frac{l_1}{T} \left( T - 16 \left( \sqrt{2} + 1 \right)^2 l_2^2 \right) \sum_{(k_1, k_2), \, \mu_{k_2} \geq \frac{k_1 \pi}{T}} \mu_{k_2}^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \\
&\quad - \frac{l_1}{T} \frac{16 \left( \sqrt{2} + 1 \right)^2 l_2^2}{T} \sum_{(k_1, k_2), \, \mu_{k_2} \leq \frac{k_1 \pi}{T}} \mu_{k_2}^2 \left( |a_k|^2 + |a_{-k}|^2 \right).
\end{align*}
\]

However, we have \( \mu_{k_2}^2 > \frac{\omega_k^2}{2} \) if \( \mu_{k_2} > \frac{k_1 \pi}{l_1} \) and \( \mu_{k_2}^2 \leq \frac{\omega_k^2}{2} \) if \( \mu_{k_2} \leq \frac{k_1 \pi}{l_1} \). Hence, we deduce

\[
\int_0^T \int_0^{l_1} \left\| \frac{\partial u}{\partial x_2}(x_1, 0, t) \right\|^2 dx_1 dt 
\geq \frac{l_1}{\pi} \left( T - 16 \left( \sqrt{2} + 1 \right)^2 l_2^2 \right) \sum_{(k_1, k_2), \, \mu_{k_2} \geq \frac{k_1 \pi}{T}} \omega_k^2 \left( |a_k|^2 + |a_{-k}|^2 \right) \\
- \frac{l_1}{\pi} \frac{16 \left( \sqrt{2} + 1 \right)^2 l_2^2}{T} \sum_{(k_1, k_2), \, \mu_{k_2} \leq \frac{k_1 \pi}{T}} \omega_k^2 \left( |a_k|^2 + |a_{-k}|^2 \right).
\]

Summing this with inequality (36) from the previous proof, we obtain

\[
\int_0^T \int_0^{l_1} \left\| \frac{\partial u}{\partial x_2}(x_1, 0, t) \right\|^2 dx_1 dt + \int_0^b \int_a^b \left\| u(x_1, x_2, t) \right\|^2 dx_2 dx_1 dt \\
+ \int_0^b \int_a^b \left\| u'(x_1, l_2, t) \right\|^2 dx_1 dt 
\geq \left[ \frac{l_1}{\pi} \left( T - 16 \left( \sqrt{2} + 1 \right)^2 l_2^2 \right) - \frac{l_2 + 12 \left( \sqrt{2} + 1 \right)^2 l_1^3}{T} \right] \\
\times \sum_{(k_1, k_2), \, \mu_{k_2} \geq \frac{k_1 \pi}{T}} \omega_k^2 \left( |a_k|^2 + |a_{-k}|^2 \right) + \\
+ \left[ \frac{l_2}{\pi} \left( T - 4 \left( \sqrt{2} + 1 \right)^2 l_2^2 \right) m_{a,b} - \frac{l_1}{\pi} \frac{16 \left( \sqrt{2} + 1 \right)^2 l_2^2}{T} \right] \\
\times \sum_{(k_1, k_2), \, \mu_{k_2} \leq \frac{k_1 \pi}{T}} \omega_k^2 \left( |a_k|^2 + |a_{-k}|^2 \right).
\]

Finally, choosing \( T > 0 \) such that

\[
T^2 > \max \left\{ 16 \left( \sqrt{2} + 1 \right)^2 l_2^2 + 2 \left( \sqrt{2} + 1 \right)^2 l_1^2 (l_2 + 1), 4 (\sqrt{2} + 1)^2 l_1^2 + \frac{16 (\sqrt{2} + 1)^2 l_1 l_2}{m_{a,b}} \right\}
\]

we reach the desired estimate.
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Received December 2017; revised August 2018.

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