PANSU PULLBACK AND EXTERIOR DIFFERENTIATION FOR SOBOLEV MAPS ON CARNOT GROUPS

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Abstract. We show that in an $m$-step Carnot group, a probability measure with finite $m^{th}$ moment has a well-defined Buser-Karcher center-of-mass, which is a polynomial in the moments of the measure, with respect to exponential coordinates. Using this, we improve the main technical result of [KMX20] concerning Sobolev mappings between Carnot groups. As a consequence, a number of rigidity and structural results from [KMX20, KMX, KMX21b, KMX21a] hold under weaker assumptions on the Sobolev exponent. We also give applications to quasiregular mappings following [Res89, HH97, Vod07b], extending earlier work in the 2-step case to general Carnot groups.

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1. Introduction

This is part of a series of papers on geometric mapping theory in Carnot groups, in which we establish regularity, rigidity, and partial rigidity results for bilipschitz, quasiconformal, or more generally, Sobolev mappings, between Carnot groups \cite{KMX20, KMX, KMX21b, KMX21a}. In \cite{KMX20} we showed that Reshetnyak’s theorem on pullbacks of differential forms has a partial generalization to mappings between Carnot groups (see also \cite{Dai99, Vod99, Vod07b}). Our aim here is to strengthen the pullback theorem from \cite{KMX20} by relaxing the assumptions on the Sobolev exponent. This yields new applications to quasiregular mappings, in addition to stronger versions of results from \cite{KMX20, KMX, KMX21b, KMX21a}. We expect further applications to geometric mapping theory in Carnot groups, in particular to understanding the threshold between flexibility and rigidity. We refer the interested reader to \cite{KMX20} for more background.

Before stating our results, we briefly recall some facts and notation; see Section 2 for more detail.

Let $G$ be a Carnot group with Lie algebra $\mathfrak{g}$, grading $\mathfrak{g} = \bigoplus_{j \geq 1} V_j$, and dilation group $\{ \delta_r : G \to G \}_{r \in (0, \infty)}$. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism, with inverse $\log : G \to \mathfrak{g}$. Without explicit mention, in what follows all Carnot groups will be equipped with Haar measure and a Carnot-Caratheodory metric denoted generically by $d_{CC}$. If $f : G \supset U \to G'$ is a Sobolev mapping between Carnot groups, where $U$ is open, then $f$ has a well-defined approximate Pansu differential $D^*_P f(x) : G \to G'$ for a.e. $x \in U$, which is a graded group homomorphism (Theorem 2.41); By abuse of notation, we also denote the associated homomorphism of graded Lie algebras by $D^*_P f(x) : \mathfrak{g} \to \mathfrak{g}'$; furthermore, for the sake of brevity we will often shorten “approximate Pansu differential” to “Pansu differential”. If $\omega$ is a differential form defined on the range of $f$, then the Pansu pullback $f^*_P \omega$ is given by $f^*_P \omega(x) = (D^*_P f(x))^* \omega(f(x))$ for a.e. $x \in U$.

Let $G$ be a Carnot group with Lie algebra $\mathfrak{g}$. For every $x \in G$ we let $\log_x : G \to \mathfrak{g}$ be the logarithm map “centered at $x$”, i.e. $\log_x(y) := \log(x^{-1} y)$. We recall \cite{BK81, KMX} that if $\nu$ is a compactly supported probability measure in $G$, then $\nu$ has a well-defined Buser-Karcher center of mass $\text{com}_\nu$, which is characterized as the unique point $x \in G$ such that $\nu$ is “balanced” with respect to logarithmic coordinates.
centered at \( x \): \( \int_G \log_x \, d\nu = 0 \). Our first result is a generalization of this center of mass to the case of measures with noncompact support.

**Theorem 1.1** (Theorem 3.4). Suppose \( G \) is an \( m \)-step Carnot group, and \( \nu \) is a probability measure on \( G \) with finite \( m \)-th moment, i.e., for some \( x \in G \) we have
\[
\int_G d_{CC}^m(x, y) \, d\nu(y) < \infty.
\]
Then for every \( x \in G \) the map \( \log_x \) is integrable w.r.t. \( \nu \), and there is a unique point \( \text{com}_\nu \in G \) such that
\[
\int_G \log(\text{com}_\nu) \, d\nu = 0.
\]
Moreover, \( \log(\text{com}_\nu) \) is a polynomial in the polynomial moments of the pushforward measure \( (\log x_0)_* \nu \), for any \( x_0 \in G \).

**Remark 1.2.** With minor modifications, the same proof works for general simply connected nilpotent groups.

Applying Theorem 1.1 in a standard way, one may define a mollification process for \( L^m_{loc} \)-mappings into an \( m \)-step Carnot group, which yields a family of smooth approximations.

For a Sobolev mapping \( f \) with mollification \( f_\rho \), our main result relates the ordinary pullback \( f_\rho^* \omega \) of a differential form \( \omega \) with the Pansu pullback \( f^*_P \omega \), as defined above. To state the result, we require the notion of the weight \( wt(\alpha) \) of a differential form \( \alpha \); this is defined using the decomposition of \( \Lambda^*g \) with respect to the diagonalizable action of the Carnot scaling, see Subsection 2.2.

**Theorem 1.3** (Approximation theorem). Let \( G, G' \) be Carnot groups, and \( f : U \to G' \) be a map in \( W^{1,p}_{loc}(U, G') \), where \( U \subset G \) is open. Suppose:

- \( \eta \in \Omega^k(G) \), \( \omega \in \Omega^\ell(G') \) are differential forms, where \( k + \ell = N := \dim G \).
- \( \eta \) is left-invariant.
- \( \omega \) is continuous and bounded.
- \( wt(\omega) + wt(\eta) \leq -\nu \), where \( \nu \) is the homogeneous dimension of \( G \).
- \( p \geq -wt(\omega) \).
- \( \frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu} \), where \( G' \) has step \( m \).

Then
\[
f_\rho^* \omega \wedge \eta \to f^*_P \omega \wedge \eta \quad \text{in} \ L^s_{loc}(U) \quad \text{with} \quad s = \frac{p}{-wt(\omega)},
\]
where $f_\rho$ is the mollification of $f$ at scale $\rho$, see Section 3.2. In particular, when $\omega \in \Omega^N(G')$ and $\mathrm{wt}(\omega) \leq -\nu$, then

$$f_\rho^* \omega \to f^* \omega \quad \text{in } L^\frac{\nu}{\nu-1}(U).$$

We refer the reader to Section 4 for more refined statements.

Although the overall outline of the proof of Theorem 1.3 is the same as for [KMX20, Theorem 1.18], the fact that $p \leq \nu$ creates several complications: a Sobolev mapping $f \in W^{1,p}_{\text{loc}}(U,G')$ as in the theorem need not be either (classically) Pansu differentiable almost everywhere or continuous; in particular, the argument cannot be localized in the target.

As immediate consequences of Theorem 1.3, the rigidity and partial rigidity results from [KMX20] hold under weaker assumptions on the exponent. For instance:

1. Let $\{G_i\}_{1 \leq i \leq n}, \{G'_j\}_{1 \leq j \leq n'}$ be collections on Carnot groups where each $G_i, G'_j$ is nonabelian and does not admit a nontrivial decomposition as a product of Carnot groups. Let $G = \prod_i G_i$, $G' = \prod_j G'_j$. Set $K_i := \{k \in \{1, \ldots, n\} : G_k \simeq G_i\}$ and assume that

$$p \geq \max\{\nu_i - 1 : |K_i| \geq 2\}$$

where $\nu_i$ denotes the homogeneous dimension of $G_i$. Assume that $f : G \supset U \to G'$ is a $W^{1,p}_{\text{loc}}$ mapping, $U = \prod_i U_i$ is a product of connected open sets $U_i \subset G_i$, and the (approximate) Pansu differential $D_p f(x)$ is an isomorphism for a.e. $x \in G$. Then $f$ coincides almost everywhere with a product mapping, modulo a permutation of the factors, see Theorem 6.1 below. In [KMX20, Theorem 1.1] the result was proved under the stronger hypothesis $p > \sum_i \nu_i$. If each $G_i$ is either a higher Heisenberg group $\mathbb{H}_{m_i}$ (with $m_i \geq 2$) or any complex Heisenberg group $\mathbb{H}^C_{m_i}$ (with $m_i \geq 1$) then the condition (1.4) can be improved to $p \geq 2$, see Corollary 6.4 below.

2. If $\mathbb{H}^C_m$ is the complexification of the $m^{th}$ Heisenberg group $\mathbb{H}_m$, $U \subset \mathbb{H}^C_m$ is open and connected and $f : U \to \mathbb{H}^C_m$ is a $W^{1,2m+1}_{\text{loc}}$-mapping such that the (approximate) Pansu differential $D_p f(x)$ is an isomorphism for a.e. $x$, then $f$ coincides almost everywhere with a holomorphic or antiholomorphic mapping, cf. [KMX20, Theorem 1.6] for the same result under the stronger condition $p > 4m + 2$. 

Another application of Theorem 1.3 is to quasiregular mappings between Carnot groups, addressing questions originating in [Ric93, HH97]. We recall that a fundamental step in Reshetnyak’s approach to quasiregular mappings in $\mathbb{R}^n$ is showing that the composition of an $n$-harmonic function with a quasiregular mapping is a solution to a quasilinear elliptic PDE; this “morphism property” depends crucially on the fact that pullback commutes with exterior differentiation [Res89]. Using Theorem 1.3, we are able to extend earlier work of [Vod07b] (see also [HH97]), so as to generalize a portion of Reshetnyak’s theory to all Carnot groups. In particular, if a Carnot group $G$ has homogeneous dimension $\nu$ and $f : G \supset U \to G$ is a quasiregular mapping, then (see Section 5 for more details):

- The “morphism” property, which was first shown by Reshetnyak in the $\mathbb{R}^n$ case, holds for locally Lipschitz $\nu$-harmonic functions: if $u : G \to \mathbb{R}$ is a locally Lipschitz $\nu$-harmonic function then the composition $u \circ f$ is $A$-harmonic.
- $f \in W^{1,\nu'}$ for some $\nu' > \nu$.
- $f$ is Hölder continuous, Pansu differentiable almost everywhere, and maps null sets to null sets.

We conclude with some open questions.

**Question 1.5.** What is the exponent threshold for rigidity/flexibility in the results mentioned above?

For instance, suppose $f : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}$ is a $W^{1,p}$-mapping whose Pansu differential is an isomorphism almost everywhere. For which $p$ must $f$ agree with a product mapping almost everywhere? Are there counterexamples when $p = 1$? Optimal Sobolev exponents were obtained for an analogous product rigidity question in the Euclidean setting in [KMSXb, KMSXa].

We recall that a Carnot group $G$ is rigid in the sense of Ottazzi-Warhurst if for any connected open subset $U \subset G$, the family of smooth contact embeddings $U \to G$ is finite dimensional. We conjectured [KMX20, Conjecture 1.10] that quasiconformal homeomorphisms of rigid Carnot groups are smooth. One may ask if there is a rigidity/flexibility threshold for these groups.

**Question 1.6.** Let $f : G \supset U \to G$ be a $W^{1,p}$-mapping, where $U$ is an open subset of an Ottazzi-Warhurst rigid Carnot group, and $D_Pf(x)$ is an isomorphism for a.e. $x \in U$ (recall that $D_Pf(x)$ denotes the
approximate Pansu differential). For which $p$ can we conclude that $f$ is smooth? What if $f$ is a homeomorphism?

Motivated by [IM93, Iwa92] one may ask about minimal regular requirements for quasiregular mappings.

**Question 1.7.** Suppose $f : G \supset U \rightarrow G$ is a weakly quasiregular mapping, i.e. $f \in W^{1,p}$ and for some $C$ we have $|D_h f|^p \leq C \det D_P f$ almost everywhere. For which $p < \nu$ can we conclude that $f \in W^{1,\nu}$?

**Organization of the paper.** We review some background material on Carnot groups and Sobolev mappings in Section 2. Section 3 establishes existence and estimates for the center of mass for measures which satisfy a moment condition, and establishes bounds for the associated mollification procedure. The proof of the main approximation theorem and some applications to the exterior derivative are proven in Section 4. Section 5 gives applications to quasiregular mappings. For the convenience of the reader, we have included proofs of some background results in the appendices. In Appendix A we give a new direct proof of the $L^p$ Pansu differentiability of Sobolev mappings; see Subsection 2.3 and Appendix A for a comparison with the original proof by Vodopyanov. In Appendix B we prove the compact Sobolev embedding, and in Appendix C we discuss Sobolev spaces defined using weak upper gradients, collecting some results from the literature, and comparing with with distributional approach of Reshetnyak and Vodopyanov.

2. **Preliminaries**

2.1. **Carnot groups.** In this subsection we recall some standard facts about Lie groups, in particular nilpotent Lie groups and Carnot groups, and prove a simple estimate for the nonlinear term in the Baker-Campbell-Hausdorff (BCH) formula. This will be useful to define the center of mass for probability measures which do not necessarily have compact support, but only satisfy bounds on certain moments.

Our main interest is in Carnot groups and the reader may focus on this case. Since the construction of the center of mass extends to connected, simply connected nilpotent groups without additional effort, we include a short discussion of nilpotent groups as well. Since the facts

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1Ottazi-Warhurst showed that $C^2$ quasiconformal homeomorphisms of rigid Carnot groups are smooth. Recently Jonas Lelmi improved this result, replacing the $C^2$ regularity assumption with $C^1$ (or even Euclidean bilipschitz); the same result was shown by Alex Austin for the $(2, 3, 5)$ distribution [Lel, Aus].
Let $G$ be a Lie group of dimension $N$. In this paper we will only consider connected, simply connected Lie groups. We give the tangent space $T_e G$ at the identity the structure of a Lie algebra $\mathfrak{g}$ in the usual way: each tangent vector $X \in T_e G$ can be extended to a left-invariant vectorfield $\tilde{X}$ through push-forward by left translation $\ell_a(g) = ag$, i.e $\tilde{X}(a)f = X(f \circ \ell_a)$. For two left-invariant vector fields $\tilde{X}$ and $\tilde{Y}$ one easily sees that the commutator $[\tilde{X}, \tilde{Y}] = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$ is a left-invariant vectorfield. We define the Lie bracket on $T_e G$ by $[X,Y] = [\tilde{X}, \tilde{Y}](e)$. In the following we do not distinguish between $X$ and $\tilde{X}$. Similarly we identify left-invariant differential $k$-forms on $G$ with $\Lambda^k \mathfrak{g}$.

The descending series of the Lie algebra is defined by $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$ where the right hand side denotes the linear space generated by all brackets of the form $[X, Y]$ with $X \in \mathfrak{g}$, $Y \in \mathfrak{g}_i$. We say that $G$ is a nilpotent group of step $m$ if $[\mathfrak{g}]_m \neq \{0\}$ and $[\mathfrak{g}]_{m+1} = \{0\}$. We say that $G$ is a Carnot group of step $m$ if, in addition, $\mathfrak{g}$ is graded, i.e. if we are given a direct sum decomposition $\mathfrak{g} = \bigoplus_{j=1}^m V_j$ (as a vectorspace) with $V_{j+1} = [V_1, V_j]$ for $1 \leq j \leq m - 1$. In the general nilpotent case it will be convenient to introduce subspaces $W_1, \ldots, W_m$ such that $\mathfrak{g}_i = W_i \oplus \mathfrak{g}_{i+1}$. There is no canonical choice of the spaces $W_i$ (except for $W_m$), but for our analysis any choice will do (see also Remark 2.19 below). In the Carnot and general cases, respectively, we have

\begin{equation}
[V_i, V_j] \subset \mathfrak{g}_{i+j} \quad \text{and} \quad [W_i, W_j] \subset \bigoplus_{k=i+j}^m W_k.
\end{equation}

By uniqueness of solutions of ordinary differential equations the integral curve $\gamma_X : \mathbb{R} \to G$ of a left invariant vectorfield $X$ with $\gamma_X(0) = e$ is a subgroup. We define the exponential map $\exp : \mathfrak{g} \to G$ by $\exp X(e) = \gamma_X(1)$. Thus $\exp : \mathfrak{g} \to G$ is smooth. By the Baker-Campbell-Hausdorff (BCH) Theorem for sufficiently small $X$ and $Y$ we have $\exp X \exp Y = \exp(X + Y + P(X,Y))$ where $P(X,Y)$ is a series in iterated Lie brackets of $X$ and $Y$, see, e.g. [CG90] eqn. (2), p. 12 or [Mic08] Thm. 4.29.

For a nilpotent Lie group of step $m$ the Lie brackets of order $m+1$ and higher vanish. Then the expression $P(X,Y)$ is a polynomial, the exponential map is a diffeomorphism and the BCH formula holds for all $X$ and $Y$ [CG90 Thm. 1.21]. We often write $\log = \exp^{-1}$, and denote the induced group action on $\mathfrak{g}$ by

$$X \ast Y := \log(\exp X \exp Y) = X + Y + P(X,Y).$$
One can use $\log : G \to \mathfrak{g}$ as a global chart for $G$ with the group operation given by $*$, but we will usually not do this. We denote by $\pi_i$ the projection $\mathfrak{g} \to V_i$ (or $\mathfrak{g} \to W_i$ for nilpotent groups). It follows from (2.1) that $\pi_i[X,Y]$ depends only on $\pi_1(X), \ldots, \pi_{i-1}(X)$ and $\pi_1(Y), \ldots, \pi_{i-1}(Y)$. Thus the differential of $P$ with respect to the first or second variable is block lower triangular with respect to the decompositions $\mathfrak{g} = \bigoplus_{i=1}^m V_i$ or $\mathfrak{g} = \bigoplus_{i=1}^m W_i$, with zero entries on the block diagonal. It follows that the Lebesgue measure $\mathcal{L}^N$ on $\mathfrak{g}$ is invariant under the left and right group operation $*$. Thus the push-forward measure $\exp_* \mathcal{L}^N$ is the bi-invariant Haar measure on $G$ (up to a factor).

The horizontal bundle $\mathcal{H} \subset TG$ is the span of the left-invariant vectorfields $X$ which satisfy $X(e) \in V_1$ (or $X(e) \in W_1$ in the nilpotent case). We fix a scalar product on $\mathfrak{g}$. This induces a left-invariant metric on $G$ by left-translation. The Carnot-Carathéodory distance on $G$ as the shortest length of horizontal curves, i.e.

$$d_{CC}(x, y) = \inf \{ \int_a^b |\gamma'(t)| \, dt : \gamma : [a, b] \to G \text{ rectifiable}, \gamma'(t) \in \mathcal{H} \}.$$  

Push-forward by left translation preserves the horizontal bundle. Thus the left translation of a horizontal curve is horizontal and the metric $d_{CC}$ is left-invariant. By Chow’s theorem every two points in $G$ can be connected by a horizontal curve of finite length so that $d_{CC}(x, y) < \infty$ for all $x, y \in G$. Moreover $d_{CC}$ induces the usual manifold topology on $G$ [Mon02, Thm 2.1.2 and Thm 2.1.3].

We now focus on Carnot groups. We define a one parameter group of dilations $\delta_r : \mathfrak{g} \to \mathfrak{g}$ by $\delta_r X = r^j X$ for $X \in V_j$ and linear extension. Then $\delta_r[X,Y] = [\delta_r X, \delta_r Y]$ so that $\delta_r$ is a Lie algebra homomorphism. Since $P(X,Y)$ is a sum of iterated Lie brackets it follows that $\delta_r(X * Y) = (\delta_r X) * (\delta_r Y)$. Thus exp $\circ \delta_r \circ \exp^{-1} : G \to G$ is a group homomorphism which we also denote by $\delta_r$. Then $\delta_r(\ell_a x) = \ell_{\delta_r a} \delta_r x$. Since $\delta_r$ as a map on $\mathfrak{g}$ preserves $V_1$ and is scaling by $r$ on $V_1$ it follows that $\delta_r$ maps horizontal curves to horizontal curves and

$$d_{CC}(\delta_r x, \delta_r y) = rd_{CC}(x, y).$$

Since $d_{CC}$ is also left-invariant, the bi-invariant measure of a ball $B(x, r)$ in the $d_{CC}$ metric is given by

$$\mu(B(x, r)) = \mu(B(e, r)) = \mu(\delta_r B(e, 1))$$

$$= \mathcal{L}^N(\delta_r \log B(e, 1)) = r^n \mu(B(e, 1))$$
where
\begin{equation}
\nu := \sum_{j=1}^{m} j \dim g_j \text{ is the homogeneous dimension of } G.
\end{equation}

We define a Euclidean norm \(| \cdot |_e\) on \(V_j\) by restriction of the scalar product on \(g\) to \(V_j\). Recall that \(\pi_j\) denotes the projection from \(g\) to \(V_j\). To reflect the action of \(\delta_r\) on \(g\) we introduce the ‘homogeneous norm’
\begin{equation}
|X| := \left( \sum_{i=1}^{m} |\pi_i X|_{e}^{2m!} / i \right)^{1/2m!}.
\end{equation}

Then
\begin{equation}
|\delta_r X| = r|X|.
\end{equation}

Note that \(|X|\) is comparable to \(\sum_{j=1}^{m} |\pi_j X|_{e}^{1/j}\) and that \(|\cdot|\) does not satisfy the triangle inequality but only the weaker estimate \(|X + Y| \leq C|X| + C|Y|\). It follows from the ball-box theorem, see e.g. \cite[Theorem 2.4.2]{Mon02}, that there exists constant \(C_1\) and \(C_2\) such that
\begin{equation}
C_1 d_{CC}(e, \exp X) \leq |X| \leq C_2 d_{CC}(e, \exp X) \quad \forall X \in g.
\end{equation}

In fact, in Carnot groups the ball-box theorem follows immediately from the seemingly weaker statement that the Riemannian distance \(d\) and \(d_{CC}\) induce the same topology on \(G\). Indeed, together with the fact that \(\exp\) is a homeomorphism from \(\mathbb{R}^n\) to \(G\) equipped with \(d\) this implies that the set \(S := \{X \in g : d_{CC}(e, \exp X) = 1\}\) is compact. Thus \(|X|\) attains its minimum and maximum on \(S\) and the inequality \((2.9)\) follows since all terms scale by \(r\) if we replace \(X\) by \(\delta_r X\).

Note also that
\begin{equation}
C^{-1} \sum_{j=1}^{m} |\pi_j X|_{e}^{2} \leq |X|_{e}^{2} \leq C \sum_{j=1}^{m} |\pi_j X|_{e}^{2} \quad \forall X \in g
\end{equation}
since all norms on a finite-dimensional vector space are equivalent. In fact we can take \(C = 1\) if we choose a scalar product on \(g\) such that the subspaces \(V_j\) are orthogonal.

One of our main goals is to construct a center of mass for probability measures \(\nu\) on \(G\) which is invariant under left-translation and group homomorphisms. Equivalently, we want to construct a center of mass for probability measures on \(g\) which is compatible with the group action \(\ast\). Since we want to allow measures which do not have compact support but only satisfy suitable moment bounds we need good control of the nonlinear term \(P(X, Y)\) in the BCH formula in terms of \(|X|\) and \(|Y|\). To write the estimate we use the following notation for a multiindex
Proposition 2.11. Let $J = j_1, \ldots, j_k$ with $k \geq 1$ and $j_i \in \mathbb{N} \setminus \{0\}$. We set $\#J = k$ and $|J| = \sum_{i=1}^{k} j_i$.

In particular

$$M_j^i(\pi_{i_1}X, \ldots, \pi_{i_j}X) \leq C|X|^{|I|}, \quad L_j^i(\pi_{i_1}Y, \ldots, \pi_{i_k}Y) \leq C|Y|^{|J|}.$$

In particular

$$|P(X, Y)| \leq C(R)(1 + |Y|^{m-1}) \quad \text{for all } X \text{ with } |X| \leq R$$

and the derivatives of $P$ with respect to the first variable satisfy

$$|D_X^k P(X, Y)(\dot{X}, \ldots, \dot{X})| \leq C(R)(1 + |Y|^{m-1})$$

for all $X, \dot{X}$ with $|X| \leq R$ and $|\dot{X}| \leq 1$.

Proof. Since $X = \sum_i \pi_iX$ and $Y = \sum_i \pi_iY$ and since $P(X, Y)$ is a multilinear expression in $X$ and $Y$ it is clear that $P$ can be expanded into sums of products of multilinear terms as in (2.12). To show (2.12) we only have to show that the terms corresponding to $|I| + |J| > m$ vanish. This follows immediately from the fact that $P(X, Y)$ is a sum of iterated Lie brackets and the first inclusion in (2.1)

Since $M_j^i$ is a multilinear form it follows that $M_j^i(\pi_{i_1}X, \ldots, \pi_{i_j}X) \leq C \prod_{k=1}^{j} |\pi_{i_k}X|$. Now by the definition of the homogeneous norm we have $|\pi_{i_k}X| \leq |X|^{|i_k|}$. This implies the estimate for $M_j^i$ and the same argument applies to $L_j^i$. The estimate (2.14) is an immediate consequence of (2.12), (2.13), the fact that only terms with $|J| \leq m - 1$ appear in (2.12) and the estimate $a^k \leq 1 + a^{m-1}$ for $1 \leq k \leq m - 1$.

Since the terms $M_j^i$ are multilinear, their derivatives are uniformly bounded for $|X| \leq R$ and thus (2.15) follows in the same way.

The second estimate in (2.16) follows from the first since $P(X, Y)$ is a linear combination of iterated Lie brackets. To show the first
inequality, assume first that \( X \in V_j, Y \in V_k \). Then \([X,Y] \in V_{j+k}\). Thus
\[
||[X,Y]|| = ||[X,Y]||_e^{\frac{1}{e-\gamma}} \leq C||X||_e^{\frac{1}{e-\gamma}}||Y||_e^{\frac{1}{e-\gamma}} \leq C||X||^{\frac{1}{e}}||Y||^{\frac{1}{e}}
\]
and the estimate follows by Young’s inequality. For general \( X, Y \) the estimate follows by bilinearity of the Lie bracket.

Remark 2.17. The estimates (2.13), (2.14) and (2.15) also hold for nilpotent groups if \( \pi_i \) denotes the projection to the spaces \( W_i \) and the homogeneous norm is defined with this definition of \( \pi_i \). Indeed, we can use the second inclusion in (2.1) to see that also in the nilpotent case the sum in (2.12) only contains terms with \( |I| + |J| \leq m \). The rest of the argument is the same. Instead of (2.16) we have the slightly weaker estimates
\[
(2.18) \quad ||[X,Y]|| \leq C(1 + ||X|| + ||Y||) \quad \text{and} \quad |P(X,Y)| \leq C(1 + ||X|| + ||Y||).
\]
Again the second estimate follows from the first. For the first estimate we first consider \( X \in W_j, Y \in W_k \). Then \([X,Y] \in \bigoplus_{i=j+k}^m W_i \) and thus by Young’s inequality and the previous estimate for \( ||[X,Y]||_e^{\frac{1}{e}} \)
\[
||[X,Y]|| \leq \sum_{i=j+k}^m ||\pi_i[X,Y]||_e^{\frac{1}{e}} \leq C(1 + ||[X,Y]||_e^{\frac{1}{e}}) \leq C(1 + ||X|| + ||Y||).
\]
For general \( X, Y \) the estimate follows by bilinearity.

Remark 2.19. Note that in the nilpotent case the homogeneous norm does not just depend on the group and the scalar product on \( \mathfrak{g} \), but also on the choice of the complementing spaces \( W_1, \ldots, W_m \). Different choices lead, however, to norms which are essentially equivalent in the following sense. Let \( \tilde{W}_i \) be different spaces with \( \mathfrak{g}_i = \tilde{W}_i \oplus \mathfrak{g}_{i+1} \), let \( \tilde{\pi}_i \) be the corresponding projections and let \( |.|_\sim \) be the corresponding homogeneous norm. Then there exists a constant \( C \) such that
\[
(2.20) \quad |X|_\sim \leq C(|X|^{\frac{1}{m}} + |X|) \quad \text{and} \quad |X| \leq C(|X|^{\frac{1}{m}} + |X|_\sim).
\]
It suffices to prove the first inequality, the second follows by reversing the roles of \( W_i \) and \( \tilde{W}_i \). We have \( \tilde{\pi}_i|_{\mathfrak{g}_{i+1}} = 0 \). Since \( \pi_{i+1}X, \ldots, \pi_mX \in \mathfrak{g}_{i+1} \) there exist linear maps \( L_i : \bigoplus_{k=1}^k W_k \rightarrow \tilde{W}_i \) such that \( \tilde{\pi}_iX = L_i(\pi_1X, \ldots, \pi_iX) \). Thus
\[
|\tilde{\pi}_iX|_e \leq C \sum_{k=1}^i |\pi_kX|_e \leq C \sum_{k=1}^i |X|^k
\]
and hence
\[ |X|_\sim \leq C \sum_{i=1}^{m} |\tilde{\pi}_i X|_\sim^2 \leq C \sum_{i=1}^{m} |X|_i^2. \]
From this the assertion easily follows by Young’s inequality.

2.2. Differential forms on Carnot groups. Let \( G \) be a Carnot group with graded Lie algebra \( g = \oplus_i V_i \). The grading defines a simultaneous eigenspace decomposition for the dilations \( \{ \delta_r \}_{r \in (0, \infty)} \). Therefore the action of \( \{ \delta_r \}_{r \in (0, \infty)} \) on \( \Lambda^k g \) also has a simultaneous eigenspace decomposition
\[
(2.21) \quad \Lambda^k g = \oplus_w \Lambda^{k,w} g
\]
where \( \delta_r \) acts on \( \Lambda^{k,w} g \) by scalar multiplication by \( r^w \). In particular, for any \( \alpha \in \Lambda^k g \), we have a canonical decomposition
\[
(2.22) \quad \alpha = \sum_w \alpha_w
\]
where \( (\delta_r)_* \alpha_w = r^w \alpha_w \) for every \( w \). Concretely, if \( X_1, \ldots, X_N \) is a graded basis of \( g \), and \( \theta_1, \ldots, \theta_N \) is the dual basis, then the actions of \( \delta_r \) on \( g \) and \( g^* \) are diagonal with respect to these bases, and the action on \( \Lambda^k g \) is diagonal with respect to the basis given by exterior powers of the \( \theta_i \)s.

**Definition 2.23.** An element \( \alpha \in \Lambda^k(g) \) is **homogeneous with weight** \( w \) if \( \alpha \in \Lambda^{k,w} g \); it has **weight \( \leq w \)** if \( \alpha \in \Lambda^{k,\leq w} \) where
\[
(2.24) \quad \Lambda^{k,\leq w} := \oplus_\leq_{w} \Lambda^{k,\leq w}.
\]
If \( U \subset G \) is open, then a \( k \)-form \( \alpha \in \Omega^k(U) \) is **homogeneous of weight** \( w \) or has **weight \( \leq w \)** if \( \omega(x) \in \Lambda^{k,w} \) or \( \omega(x) \in \Lambda^{k,\leq w} \), for every \( x \in U \), respectively. We let \( \Omega^{k,w}(U) \) and \( \Omega^{k,\leq w} \) denote the homogeneous forms of weight \( w \) and the forms of weight \( \leq w \), respectively, so \( \Omega^k(U) = \oplus_{w} \Omega^{k,w}(U) \). Note that \( 0 \in \Lambda^k(g) \) has weight \( w \) for every \( w \in \mathbb{R} \).

**Lemma 2.25.**

1. If \( \alpha_i \in \Omega^{k_i,w_i} \) for \( 1 \leq i \leq 2 \), then \( \alpha_1 \wedge \alpha_2 \in \Omega^{k_1+k_2,w_1+w_2} \).
2. \( \theta_{i_1} \wedge \ldots \wedge \theta_{i_k} \) is homogeneous of weight \( \sum_j \text{weight}(\theta_{i_j}) \). In particular, such wedge products give a basis for the left invariant \( k \)-forms.
3. If \( \beta \in \Omega^{k,w}(G') \) and \( \Phi : G \to G' \) is a graded group homomorphism, then \( \Phi^* \beta = \Phi_{p*}^* \beta \) belongs to \( \Omega^{k,w}(G) \).
Proof. (1) is immediate from the definitions, and (2) follows from (1).

(3) Since $\Phi$ is a graded group homomorphism, the Pansu derivative and the ordinary derivative coincide. Therefore

$$(\delta_r)_{\ast} \Phi^* \beta = (\delta_{r-1})_{\ast} \Phi^* \beta = \Phi^*(\delta_{r-1})_{\ast} \beta = \Phi_{\ast}(\delta_r)_{\ast} \beta = r^{\gamma_p} \Phi_{\ast}^* \beta.$$ 

$\square$

2.3. Sobolev spaces on Carnot groups. In this subsection we discuss $L^p$ and Sobolev spaces for maps between Carnot groups; in addition to the definitions, we cover two key properties needed for the proof of the approximation theorem – the Poincaré-Sobolev inequality and almost everywhere Pansu differentiability (in an $L^p$ sense). In the literature there are different approaches to Sobolev mappings between Carnot groups – some are based on distribution derivatives, and others on (weak) upper gradients (see [Vod96, Vod99, HKST15]). In this subsection we use the distributional definition of Sobolev mappings, and cover the upper gradient version in Appendix C. In fact, the two definitions are equivalent in our setting (see Appendix C), so one could work equally well work with either.

We treat the case where the domain is an open set in a Carnot group; however most statements and proofs apply without modification to equiregular subriemannian manifolds satisfying a suitable Poincaré inequality.

In this subsection we let $G$ denote a Carnot group with graded Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^m V_i$. Let $X_1, \ldots, X_K$ be an orthonormal basis of the first layer $V_1$. As usual we identify the vectors $X_i \in V_1$ with left-invariant vectorfields on $G$. Then $X_1(p), \ldots, X_K(p)$ is a basis of the horizontal subspace at $p$.

Definition 2.26. Let $U \subset G$ be open. We say that $u : U \to \mathbb{R}$ is in the Sobolev space $W^{1,p}(U)$ if $u \in L^p(U)$ and if the distributional derivatives $X_1 u, \ldots, X_K u$ are in $L^p(U)$, i.e., if there exist $g_i \in L^p(U)$ such that

$$\int_U u X_i \varphi \, d\mu = - \int_U g_i \varphi \, d\mu \quad \text{for all } \varphi \in C_0^\infty(U).$$

We say that $u \in W^{1,p}_{loc}(U)$ if $u \in W^{1,p}(V)$ for every open set $V$ whose closure is compact and contained in $U$. 

We write $X_i u$ for the weak derivatives $g_i$ and we define

$$D_h u = (X_1 u, \ldots, X_K u), \quad |D_h u| := \left( \sum_{i=1}^K |X_i f|^2 \right)^{1/2}.$$  

We recall some basic properties of Sobolev functions.

**Proposition 2.28.** Let $U \subset G$ be open. Then the following assertions hold.

(1) $C^\infty(U)$ is dense in $W^{1,p}(U)$;
(2) if $u \in W^{1,p}(U)$ and $\psi : \mathbb{R} \to \mathbb{R}$ is $C^1$ with bounded derivative then $\psi \circ u - \psi(0) \in W^{1,p}(U)$ and the weak derivatives satisfy the chain rule;
(3) if $u \in W^{1,p}(U)$ then $|u| \in W^{1,p}(U)$ and the weak derivatives satisfy $X_i |u| = \pm X_i u$ a.e. in the set $\{ \pm u > 0 \}$ while $X_i |u| = 0$ a.e. in the set $\{ u = 0 \}$;
(4) if $u,v \in W^{1,p}(U)$ then $\min(u,v) \in W^{1,p}(U)$ and $|D_h \min(u,v)| \leq \max(|D_h u|, |D_h v|)$ a.e.;
(5) if $u_k \in W^{1,p}(U)$ for $k \in \mathbb{N}$ and there exist functions $g,h \in L^p(U)$ such that $|D_h u_k| \leq g$ a.e. and $u_k \geq h$ a.e., for all $k \in \mathbb{N}$ then $u := \inf_k u_k \in W^{1,p}(U)$ and $|D_h u| \leq g$ a.e.

**Proof.** For assertion (1) see Friedrichs [Fri44] or Thm. 1.13 and Thm. A.2 in [GN96]. Assertions (2) and (3) follow from (1) in the same way as in the Euclidean case (see, for example [GT01, Sec. 7.4] for the Euclidean setting). Indeed, for (3) one applies (2) with $\psi_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2} - \varepsilon$ and takes the limit $\varepsilon \to 0$. Assertion (1) follows from (3) since $\min(u,v) = \frac{1}{2}(u + v) - \frac{1}{2}|u - v|$. To prove assertion (5) set $w_k = \inf_{j \leq k} u_j$. It follows from (1) that $w_k \in W^{1,p}(U)$ and $|D_h w_k| \leq g$. Moreover $k \mapsto w_k$ is non-increasing. Since $w_k \geq h$ and $h \in L^p(U)$, the monotone convergence theorem implies that $w_k \to u$ in $L^p(U)$. Moreover a subsequence of the weak derivatives $X_i w_k$ converges weakly in $L^p(U)$ to a limit $h_i$ (for $p = 1$ use the Dunford-Pettis theorem). Thus $u$ is weakly differentiable with weak derivatives $h_i$. By weak lower semicontinuity of the norm we deduce that $(\sum_i h_i^2)^{1/2} \leq |g|$. □

We now consider spaces of $L^p$ functions and Sobolev functions with values in a metric space. We will later only consider a Carnot group $G'$ with the Carnot-Carathéodory metric as the target space, but we state the results for general targets to emphasize that they do not use the structure of a Carnot group. The following definition is due to Reshetnyak [Res97] for open subsets of $\mathbb{R}^n$ or a Riemannian manifold
and has been extended by Vodopyanov [Vod99, Proposition 3, p. 64] to the setting of Carnot groups.

**Definition 2.29.** Let \((X', d')\) be a complete separable metric space and let \(U \subset G\) be open.

1. We say that a map \(f : U \to X'\) is in \(L^p(U, X')\) if \(f\) is measurable and if there exist an \(a \in X'\) such that the map \(x \mapsto d'(f(x), a)\) is in \(L^p(U)\).
2. We say that \(f \in L^p(U, X')\) is in the Sobolev space \(W^{1,p}(U; X')\) if for all \(z \in X'\) the functions \(u_z(\cdot) := d'(f(\cdot), z) - d'(a, z)\) are in \(W^{1,p}(U)\) and if there exists a function \(g \in L^p(U)\) such that
   \[
   |D_hu_z| \leq g
   \]
   almost everywhere.

The spaces \(L^p_{\text{loc}}(U; X')\) and \(W^{1,p}_{\text{loc}}(X')\) are defined as usual.

Note that by the triangle inequality the map \(x \mapsto d'(f(x), z)\) is in \(L^p_{\text{loc}}(U)\) for all \(z \in X'\) if it is in \(L^p_{\text{loc}}(U)\) for one \(z \in X'\); if \(\mu(U) < \infty\) then the same assertion holds for \(L^p(U)\). Note however that the assertion fails for \(L^p(U)\) when \(\mu(U) = \infty\).

Definition 2.29 imposes estimates on the weak derivatives of \(f\) composed with the distance functions \(d(z, \cdot)\). These imply similar estimates on the composition with general Lipschitz functions from \(G'\) to a finite-dimensional linear space:

**Proposition 2.31.** Let \((X', d')\) be a complete separable metric space and let \(U \subset G\) be open. Let \(f \in W^{1,p}(U; G')\) and let \(a \in G'\) be such that \(x \mapsto d'(a, f(x))\) is in \(L^p(U)\). Let \(Y\) be a finite-dimensional \(\mathbb{R}\)-vector space and \(v : G' \to Y\) be Lipschitz. Then \(v \circ f - v(a) \in W^{1,p}(U; Y)\).

**Proof.** It suffices to show the assertion for \(Y = \mathbb{R}\). So let \(u : X' \to \mathbb{R}\) be \(L\)-Lipschitz and let \(D \subset X'\) be a countable dense set. Then \(v \circ f\) in \(L^p(U)\) and
   \[
   v(z) = \inf_{z' \in D} v(z') + d''(z', z).
   \]
Thus the assertion follows from the definition of \(W^{1,p}(U; X')\) and Proposition 2.28 (5). \(\square\)

We will use the following version of the Lebesgue point theorem. Here and elsewhere in the paper we use the standard notation \(\bar{\int}\) to denote an average.

**Lemma 2.32.** Let \(f \in L^p_{\text{loc}}(U, X')\). Then for a.e. \(x \in U\) we have
   \[
   \lim_{r \to 0} \bar{\int}_{B(x, r)} [d'(f(y), f(x))]^p d\mu(y) = 0.
   \]
Proof. This follows easily by applying the usual Lebesgue point theorem to the scalar functions $u_z(y) = d'(f(y), z)$ where $z$ runs through a countable dense subset of $X'$. \hfill \Box

To recall the Poincaré-Sobolev inequality for metric-space-valued maps we define the $L^p$-oscillation as follows.

**Definition 2.34.** Let $X'$ be a metric space. Let $A \subset G$ be a measurable set and let $f \in L^p(A, X')$. The $L^p$ oscillation on $A$ is defined by

$$\text{osc}_p(f, A) := \inf_{a \in X'} \left( \int_A d^p(f(x), a) d\mu(x) \right)^{1/p}. \label{eq:oscillation}$$

There is a general strategy for deducing the Poincaré-Sobolev inequality for metric-space-valued maps from the Poincaré-Sobolev inequality for scalar valued functions. It is based on the derivation of a pointwise estimate for a.e. pair of points and a chaining argument, see Theorem 9.1.15 in [HKST15] for an implementation of this approach in the context of the upper gradient definition. Since we are only interested in Carnot groups as targets we use a more pedestrian approach. Recall that a metric space is doubling if every ball of radius $r > 0$ can be covered by a fixed number $M$ of balls of radius $\frac{r}{2}$. Carnot groups are doubling since by compactness $B(e, 1)$ can be covered by $M$ balls of radius $\frac{1}{2}$. By translation and scaling every ball of radius $r$ can be covered by $M$ balls of radius $\frac{r}{2}$.

**Theorem 2.36.** Let $G$ be a Carnot group of homogeneous dimension $\nu$ and let $X'$ be a complete, separable metric space which is doubling. Let $1 \leq p < \nu$ and define $p^*$ by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}. \label{eq:p-star}$$

There exists a constant $C = C(G, p)$ with the following property. Let $B(x, r)$ be a ball in $G$, let $f \in W^{1, p}(B(x, r), X')$ and let $g \in L^p(B(x, r))$ be the function in Definition 2.29. Then

$$\text{osc}_{p^*}(f, B(x, r)) \leq C\|g\|_{L^p(B(x, r))} \label{eq:oscillation-p-star}$$

and

$$\text{osc}_p(f, B(x, r)) \leq C r \|g\|_{L^p(B(x, r))}. \label{eq:oscillation-p}$$

**Proof.** If suffices to prove the first estimate, since the second follows from the first by Hölder’s inequality. For the Poincaré-Sobolev inequality for scalar functions see [GN96, Corollary 1.6.] or [Lu94, Theorem...
By scaling and translation in $G$ it suffices to show the Poincaré inequality for $X'$-valued maps for the set $B = B(e, 1)$ and we may assume that the Haar measure $\mu$ is normalized so that $\mu(B) = 1$.

Let $f \in W^{1,p}(B, X')$ and for $z \in X'$ define $u_z(x) = d'(z, f(x))$. By Definition 2.29 and the Poincaré inequality for scalar functions we see that $u_z \in L^{p^*}(B)$ for all $z \in D$. Thus $L := \text{osc}_{p^*}(f, B) = \inf_z \|u_z\|_{p^*, B} < \infty$. Let $\bar{a} \in X'$ be such that the infimum is achieved. Then

$$\mu\{x \in B : f(x) \in B(\bar{a}, 2L)\} \geq (1 - 2^{-p^*})\mu(B) \geq \frac{1}{2}.$$

Since $X'$ is doubling there exist $M^2$ balls of radius $L/2$ which cover $B(\bar{a}, 2L)$. Thus there exist $z \in X'$ such that

$$\mu(E) \geq \frac{1}{2}M^{-2} \quad \text{where } E = f^{-1}(B(\bar{a}, \frac{L}{2})).$$

By the Sobolev-Poincaré inequality for scalar-valued functions and the triangle inequality we have

\[
\int_B \int_B |u_z(x) - u_z(y)|^{p^*} d\mu(x) d\mu(y) \leq C\|g\|_{p^*, B}^{p^*}.
\]

Let

$$v(x) = \max(u_z(x) - \frac{L}{2}, 0).$$

For $y \in E$ we have $u_z(y) \leq \frac{L}{2}$ and hence $v(x) \leq |u_z(x) - u_z(y)|$. Restricting the outer integral on the left hand side of (2.39) to the set $E$ we get

$$\int_B v^{p^*} d\mu \leq 2CM^2\|g\|_{p^*, B}^{p^*}.$$

Thus $\|v\|_{p^*, B} \leq C(M, p)\|g\|_{p, B}$. By the definition of the oscillation and the definition of $v$ we have

$$\|v\|_{p^*, B} \geq \|u_z\|_{p^*, B} - \frac{L}{2} \geq \frac{L}{2}.$$

Hence $\text{osc}_{p^*}(f, B) = L \leq 2C(M, p)\|g\|_{p, B}$. 

We finally discuss differentiability results. It is well known that locally Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are differentiable a.e. In fact maps in $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ are differentiable a.e. if $p > n$. For $1 \leq p < n$ the map $f$ is differentiable in an $W^{1,p}$ sense. For maps between Carnot groups, Pansu [Pan89] showed that Lipschitz maps (with respect to the Carnot-Carathéodory metrics) on open sets are a.e. differentiable in the following sense, now known as Pansu differentiability. For a.e.
there exist a graded group homomorphism $\Phi : G \to G'$ such
that the rescaled maps
\begin{equation}
\tag{2.40}
f_{x,r} := \delta_{r^{-1}} \circ \ell_{f(x)^{-1}} \circ f \circ \ell_x \circ \delta_r.
\end{equation}
converge locally uniformly to as $r \to 0$. The corresponding Pansu
differentiability results for Sobolev maps between Carnot groups ha
ve been obtained by Vodopyanov \cite[Theorems 1 and 2, Corollaries
1 and 2]{Vod03}, see also \cite{Vod07a} for extensions to maps between
Carnot manifolds. We will use the following result.

**Theorem 2.41** ($L^{p^*}$ Pansu differentiability a.e., \cite[Corollary 2]{Vod03}).
Let $U \subset G$ be open, let $1 \leq p < \nu$ and define $p^*$ by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}$. Let
$f \in W^{1,p}(U; G')$. For $x \in U$ consider the rescaled maps
\begin{equation}
\tag{2.42}
f_{x,r} = \delta_{r^{-1}} \circ \ell_{f(x)^{-1}} \circ f \circ \ell_x \circ \delta_r.
\end{equation}

Then, for a.e. $x \in U$, there exists a group homomorphism $\Phi : G \to G'$
such that
\begin{equation}
\tag{2.44}
|D_P f| \text{ a.e., where}
\end{equation}
Here $| \cdot |_{V_1}$ and $| \cdot |_{V'_1}$ denote the norms induced by the scalar product
on the first layer of $g$ and $g'$, respectively. Thus the condition (2.30) in
Definition 2.29 holds with $g = |D_P f|$. The short proof of (2.44) can be
found in Appendix A.

**Remark 2.46.** Let $\pi_{G'} : G' \to G'/[G', G']$ denote the abelianization
map. Since $\pi_{G'}$ is globally Lipschitz, the map $\pi_{G'} \circ f$ is in $W^{1,p}(U, G'/[G', G'])$.
It easily follows from Theorem 2.41 that the weak (or distributional)

\begin{equation}
\tag{2.47}
X(\pi_{G'} \circ f)(x) = D_P f(x)X \text{ for a.e. } x,
\end{equation}
see Remark A.6 below. In (2.47) we have identified the abelian group
$G'/[G', G']$ with the first layer $V'_1$ of $g'$.

Vodopyanov’s proof in \cite{Vod03} combines work from a series of ear-
lier papers \cite{VU96, Vod96, Vod99, Vod00}. His argument is based on
Lipschitz approximation on sets of almost full measure, an extension
of Pansu’s result to Lipschitz maps defined on sets $E \subset G$ which are
not open and a careful estimate of the remainder terms at points in \( E \) of density one.

In Appendix A we give a direct alternative proof of \( L^p \) differentiability which is based on blow-up, the Poincaré-Sobolev inequality, the compact Sobolev embedding (which is an immediate consequence of the Poincaré-Sobolev inequality) and the following observation: if \( F : G \rightarrow G' \) is a Lipschitz map with \( F(e) = e \) and the abelianization \( \pi_{G'} \circ F \) is affine (i.e. has constant weak horizontal derivatives) then \( F \) is a graded group homomorphism.

3. Center of mass and mollification

In this section we first define a center of mass for measures on a Carnot group which satisfy a moment condition. We then use this to define a mollification procedure for maps with finite \( L^m \)-oscillation taking values in an \( m \)-step Carnot group.

We will be using the notation and results from Section 2, in particular the ‘homogeneous norm’ \( | \cdot | \) on \( g \) defined in (2.7), and the Euclidean norm \( | \cdot |_e \).

3.1. Center of mass in Carnot groups. Let \( G \) be a \( m \)-step Carnot group (for an extension to connected, simple connected nilpotent Lie groups see Remark 3.18 and Remark 3.26 below). Let \( \nu \) be a Borel probability measure on \( G \). We say that \( \nu \) has finite \( p \)-th moment if

\[
\int_{G} d_{CC}^p(e, y) \, d\nu(y) < \infty.
\]

In view of (2.9) this is equivalent to

\[
\int_{g} |Y|^p \, d(\log_* \nu)(Y) < \infty.
\]

In this subsection we define, for probability measures with finite \( m \)-th moment, a center of mass which is compatible with left translation and group homomorphisms. For probability measures \( \nu \) with compact support our notion of center of mass agrees with the one by Buser and Karcher [BK81, Example 8.1.8]. Their proof of the existence of the center of mass is different. They use the bi-invariant flat connection \( D \) such that left-invariant vector fields are \( D \)-parallel, and base their proof on some estimates for the convexity radius of \( D \) with respect to some auxiliary left invariant Riemannian metric. Here we argue directly on the Lie algebra and also show that there is an explicit recursive formula for (the logarithm of) the center of mass and that the logarithm of the center of mass is a polynomial in certain polynomial moments of \( \log_* \nu \).
The extension from compactly supported measures to measures with finite \( m \)-th moment will be crucial in the next subsection where we use the center of mass to define a group compatible mollification for (Sobolev) functions which may be unbounded.

We define

\[
\log_x = \log \circ \ell_{x^{-1}}
\]

**Theorem 3.4.** Let \( G \) be an \( m \)-step Carnot group and let \( \nu \) be a Borel probability measure on \( G \) with finite \( m \)-th moment. Then \( \log x \) is \( \nu \) integrable and the map \( C_\nu : G \to \mathfrak{g} \) defined by

\[
C_\nu(x) := \int_G \log_x \, d\nu
\]

is a diffeomorphism. Moreover \( C_\nu \circ \exp : \mathfrak{g} \to \mathfrak{g} \) is a polynomial of degree not larger than \( m - 1 \).

For any \( Z \in \mathfrak{g} \), the equation \( C_\nu(\exp X) = Z \) can be solved recursively and \( \log(C_\nu)^{-1}(Z) \) is a polynomial in \( Z \) and certain polynomial moments of \( \log x \nu \). In particular there exist a \( \mathfrak{g} \)-valued polynomial \( Q \) with \( Q(0, \ldots, 0) = 0 \) and \( \mathfrak{g} \)-valued multilinear forms \( L_1, \ldots, L_K : \mathfrak{g} \to \mathfrak{g} \) such that

\[
\log(C_\nu)^{-1}(0) = Q(A_1, \ldots, A_K),
\]

where

\[
A_i = \int_{\mathfrak{g}} L_i(Y, \ldots, Y) \, d(\log_x \nu)(Y).
\]

Moreover

\[
|L_i(Y, \ldots, Y)|_e \leq C_i(1 + |Y|^m) \quad \text{for } 1 \leq i \leq K.
\]

We call

\[
\text{com}_\nu := (C_\nu)^{-1}(0)
\]

the center of mass of \( \nu \).

**Remark 3.10.** The proof shows the condition that \( \nu \) has finite \( m \)-th moment can be slightly weakened. It suffices to assume that

\[
\int_{\mathfrak{g}} (|Y|^{m-1} + |Y|_e) \, d\log_x \nu < \infty.
\]

The reason is that elements of \( V_m \) do not appear in the BCH term \( P(X, Y) \) and that \( P(X, Y) \) is polynomial in \( Y \) of degree not exceeding \( m-1 \), see also (2.14). Recall from (2.7) that the homogeneous norm \( |X| \) is equivalent to \( \sum_{i=1}^{m} |\pi_j X|^\frac{1}{\ell_j} \) where \( \pi_j : \mathfrak{g} \to V_j = V_j \) is the projection.
to the \( j \)-th layer of the algebra. Thus condition (3.11) is equivalent to the condition that \( |Y|^{m-1} + |\pi_m Y|_e \) is \( \log_* \nu \) integrable.

**Proof of Theorem 3.4.** It is easier to work on the algebra \( \mathfrak{g} \) rather than the group \( G \). We thus define

\[
\tilde{C}_\nu(X) = C_\nu(\exp X).
\]

Then, using the BCH formula, we get

\[
\tilde{C}_\nu(X) = \int_G \log(\exp(-X)y) \, d\nu(y) = \int_\mathfrak{g} \log(\exp(-X) \exp Y) \, d\log_* \nu(Y) = -X + \int_\mathfrak{g} (Y + P(-X,Y)) \, d\log_* \nu(Y).
\]

The integrand is a polynomial of degree at most \( m-1 \) in \( X \). By the definition of the homogeneous norm we have \( |Y|_e \leq C(|Y| + |Y|^m) \). Moreover by (2.14) we have \( |P(X,Y)| \leq C(X)(1 + |Y|^{m-1}) \). Since \( \nu \) is a probability measure with finite \( m \)-th moment the integral in (3.12) exists. The bounds (2.15) on the derivatives imply that differentiation with respect to \( X \) and integration commute. Thus \( \tilde{C}_\nu \) is a polynomial of degree at most \( m-1 \).

We now show that for every \( Z \in \mathfrak{g} \), the equation

\[
\tilde{C}_\nu(X) = Z
\]

has a unique solution which is a polynomial in \( Z \), and moreover depends only on certain polynomial moments of \( \log_* \nu \) of degree at most \( m-1 \). Recall that \( \pi_i \) denotes the projection from \( \mathfrak{g} = \bigoplus_{i=1}^m V_i \) to \( V_i \). Define functions \( P^i \) by \( P^i(X,Y) = \pi_i P(X,Y) \) and set \( X^i = \pi_i X, Y^i = \pi_i Y \).

Applying \( \pi_i \) to (3.13) we get a system of \( m \) equations, namely

\[
X^i + \int_\mathfrak{g} Y^i + P^i(-X,Y) \, d\log_* \nu(Y) = Z^i \quad \text{for } i \in \{1, \ldots, m\}.
\]

Since \( P \) consists of iterated commutators one easily sees that \( P_i(X,Y) \) depends on \( X \) only through \( (X^1, \ldots, X^{i-1}) \). Thus the system can be solved recursively starting with

\[
X^1 = -Z^1 + \int_\mathfrak{g} Y^1 \, d\log_* \nu(Y).
\]

Moreover the solution is polynomial in \( Z \) and in particular smooth.
We finally discuss the dependence of the solution on \( \log^* \nu \). Since 
\([V_i, V_j] \subset V_{i+j}\) we see as in Lemma 2.11 that \( P^i(X,Y) \) can be written as

\[
P^i(X,Y) = \sum_{p=1}^{d_i} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} M^p_{i1+\cdots+i_k} (X^{i_1}, \ldots, X^{i_k}) L^p_{i_{k+1} \cdots i_j} (Y^{i_{k+1}}, \ldots, Y^{i_j}) E_p
\]

where \( E_1, \ldots, E_{d_i} \) is a basis of \( V_i \) and where \( M^p_I \) and \( L^p_I \) are multilinear forms.

Let \( \bar{X} = \log \text{com}_\nu \). Then it follows from (3.14) that \( \bar{X} \) can be recursively computed as

\[
\bar{X}^i = \int g (Y^i + P^i(-\bar{X}, Y)) \, d \log_* \nu(Y)
\]
for \( i \in \{1, \ldots, m\} \), starting with \( \bar{X}^1 = \int_{\mathfrak{g}} Y^1 \, d \log_* \nu(Y) \). Hence \( \bar{X} \) is a polynomial expression in

\[
\bar{Y}^i = \int g Y^i \, d \log_* \nu(Y)
\]
and the polynomial moments

\[
A^i_{ik+1, \ldots, ij} := \int_{\mathfrak{g}} L^i_{ik+1 \cdots ij} (Y^{ik+1}, \ldots, Y^{ij}) \, d \log_* \nu(Y).
\]

Since \( s := \sum_{\ell=k+1}^{j} i_\ell \leq i - k \leq m - 1 \) we have by multilinearity

\[
|L^i_{ik+1 \cdots ij} (Y^{ik+1}, \ldots, Y^{ij})|_e \leq C \prod_{\ell=k+1}^{j} |Y^{i_\ell}|_e
\]

\[
\leq C \prod_{\ell=k+1}^{j} |Y^{i_\ell}|_e \leq C |Y|^s \leq C(1 + |Y|^m).
\]

Moreover \( |Y^i|_e \leq |Y|^i \leq 1 + |Y|^m \). \( \square \)

Remark 3.18. The conclusion of Theorem 3.4 continues to hold if we consider a nilpotent group instead of a Carnot group, and use (3.2) rather than (3.1) to define the \( p \)-th moment, where the homogenous norm is defined as in Remark 2.17 using a (noncanonical) decomposition \( \mathfrak{g} = \oplus W_j \) with \( \mathfrak{g}_j = W_j \oplus \mathfrak{g}_{j+1} \) and denoting by \( \pi_j \) the projection from \( \mathfrak{g} \) to \( W_j \).
Indeed, by Remark 2.17 the bounds in Lemma 2.11 also hold in the nilpotent case. Thus \( C_\nu \circ \exp \) is well-defined and a polynomial of degree at most \( m - 1 \). In view of Remark 2.19 the condition that the probability measure \( \log_* \nu \) has finite \( m \)-th moment is independent of the choice of the auxiliary spaces \( W_i \) since different choices lead to homogeneous norms \( | \cdot |_\sim \) and \( | \cdot | \) which satisfy \( |X|_\sim \leq C(1 + |X|^m) \) and \( |X| \leq C(1 + |X|^m) \).

To see that the equation (3.13) can be solved recursively, consider the projections \( \pi_j : g \to W_j \) and \( \tilde{\pi}_j = \sum_{i=1}^j \pi_i \). Since \( [W_j, W_k] \subset \oplus_{i=j+k}^m W_i \), the expression \( \tilde{\pi}_j [X, Y] \) depends only on \( \tilde{\pi}_{j-1}(X) \) and \( \tilde{\pi}_{j-1}(Y) \) and thus \( \tilde{\pi}_j P(X, Y) \) depends only on \( \tilde{\pi}_{j-1}(X) \) and \( \tilde{\pi}_{j-1}(Y) \). Hence (3.13) can be again solved recursively by successively applying the projections \( \tilde{\pi}_1, \ldots, \tilde{\pi}_m = \text{id} \). The projection \( \tilde{\pi}_i P(X, Y) \) can again be expressed as a sum of products of multilinear terms in \( X \) and \( Y \). The only difference is that the condition \( \sum_{i=1}^j \ell_i = i \) is replaced by \( \sum_{i=1}^j \ell_i \leq i \). Nonetheless the bound (3.17) still holds and this implies (3.5).

We note in passing that one can show the recursive solvability of (3.13) without introducing the spaces \( W_i \), by considering the abstract projections \( \bar{\pi}_1, \ldots, \bar{\pi}_m = \text{id} \) given by \( \bar{\pi}_i : g \to g / g_{i+1} \).

**Remark 3.19.** For a step-2 group \( C_\nu \circ \exp \) is an affine function and thus

\[
\log \text{com}_\nu = \int_{\mathfrak{g}} Y \, d \log_* \nu(Y) \quad \text{for step-2 groups.}
\]

We now show that the center of mass defined by \( \text{com}_\nu = C_\nu^{-1}(0) \) commutes with left translations, inversion and group homomorphisms.

**Lemma 3.21.** Let \( G \) and \( G' \) be Carnot groups of step \( m \) and \( m' \), respectively. Let \( \Phi : G \to G' \) be a group homomorphism (with derivative \( D\Phi : \mathfrak{g} \to \mathfrak{g}' \)), let \( I : G \to G \) be the inversion map given by \( I(x) = x^{-1} \), and \( \nu \) be a Borel probability measure on \( G \).

If \( \nu \) has finite \( m \)-th moment then \( (\ell_z)_* \nu \) and \( I_* \nu \) are Borel probability measures on \( G \) with finite \( m \)-th moment and

\[
\text{com}_{(\ell_z)_* \nu} = \ell_z(\text{com}_\nu), \quad \text{com}_{I_* \nu} = I(\text{com}_\nu).
\]

In particular, if \( \nu \) is reflection symmetric, i.e., if \( I_* \nu = \nu \), then

\[
\text{com}_{(\ell_z)_* \nu} = z.
\]
If \( \nu \) has finite \( p \)-th moment for \( p \geq \max(m, m') \), then \( \Phi_* \nu \) is a Borel probability measure on \( G' \) with finite \( p \)-th moment, and

\[
(3.25) \quad \text{com}_{\Phi_* \nu} = \Phi(\text{com}_\nu).
\]

**Remark 3.26.** The assertion and the proof immediately extend to nilpotent groups. To bound the \( m \)-th moment of the measures \( I_* \nu \) and \((\ell_x)_* \nu\) one uses \((2.18)\) instead of \((2.16)\).

**Proof.** Assume that \( \nu \) has finite \( p \)-th moment. We have \(|\log \circ I(y)| = | - \log y| = | \log y|\). Thus \( I_* \nu \) has finite \( p \)-th moment. Similarly \( \log \circ \ell_x(y) = \log x + \log y + P(\log x, \log y) \). By \((2.16)\) we have \(|P(\log x, \log y)| \leq C(\log x + | \log y|)\). Thus \((\ell_x)_* \nu\) has finite \( m \)-th moment.

To control the moment of \( \Phi_* \nu \) we first note that \( \Phi \) preserves the one-parameter subgroups and thus \( \log_{G'} \circ \Phi \circ \exp_G = D\Phi \). It follows that \( (\log_{G'})_* \Phi_* \nu = (D\Phi)_*(\log_G)_* \nu \). It thus suffices to show that

\[
(3.27) \quad |D\Phi(X)| \leq C(\|X\|^{1 \over m} + |X|).
\]

To show this, observe that in a Carnot algebra the elements \( g_j \) of the descending series are given by \( g_j = \oplus_{i=j}^m V_i \). Since \( D\Phi \) is a Lie algebra homomorphism we have \( D\Phi(g_j) \subset g'_j \). Note also that \( D\Phi \) is linear and hence bounded with respect to the Euclidean norms. Thus we have for \( X \in V_j \)

\[
|D\Phi(X)| \leq C \sum_{i=j}^m |\pi_i D\Phi(X)| \leq C \sum_{i=j}^m |X| \leq C \sum_{i=j}^m |X|^{1 \over i}.
\]

By Young’s inequality we have \( |X|^{1 \over i} \leq C(\|X\|^{1 \over m} + |X|) \) whenever \( i \geq j \). By linearity we get \((3.27)\) for all \( X \).

To prove \((3.22)\) \((3.25)\) it suffices to verify the corresponding transformation rules for \( C_\nu \). We have

\[
\log[(\Phi(x))^{-1}\Phi(y)] = \log[\Phi(x^{-1}y)] = D\Phi(\log(x^{-1}y))
\]

and thus

\[
C_{\Phi_* \nu}(\Phi(x)) = D\Phi(C_\nu(x)).
\]

Setting \( x = \text{com}_\nu \) we get \( C_{\Phi_* \nu}(\Phi(\text{com}_\nu)) = 0 \) and thus \((3.25)\). Similarly the relation

\[
\log[I(x^{-1}I(y)] = \log[I(x^{-1}y)] = - \log(x^{-1}y)
\]

gives \((3.23)\) while the identity

\[
(\ell_z x)^{-1}(\ell_z y) = x^{-1}z^{-1}zy = x^{-1}y
\]

gives \((3.22)\).

Now assume that \( \nu \) is reflection symmetric. Then \((3.23)\) implies that \( \text{com}_\nu = e \). In combination with \((3.22)\) we obtain \((3.24)\). \( \square \)
3.2. Mollifying maps between Carnot groups. In this subsection we define a mollification procedure for $L^p_{\text{loc}}$-mappings into a Carnot group. Traditional mollification of mappings into a linear target is based on averaging; since Carnot groups are not linear spaces, we replace averaging with the center of mass from Subsection 3.1.

Let $\sigma_1$ be a smooth probability measure on a Carnot group $G$ with $\text{spt}(\sigma_1) \subset B(e, 1)$. Thus $\sigma_1 = \alpha \mu$ where $\mu$ is the biinvariant measure on $G$ and $\alpha \in C_c^\infty(B(e, 1))$. We also assume that $\sigma_1$ is symmetric under inversion: $I_* \sigma_1 = \sigma_1$, where $I(x) = x^{-1}$. For $x \in G$, $\rho \in (0, \infty)$, let $\sigma_{\rho}, \sigma_x, \text{ and } \sigma_{x, \rho}$ be the pushforwards of $\sigma_1$ under the the corresponding Carnot scaling and left translation:

\[
\sigma_{\rho} = (\delta_{\rho})_* \sigma_1, \quad \sigma_x = (\ell_x)_* \sigma_1, \quad \sigma_{x, \rho} = (\ell_x \circ \delta_{\rho})_* \sigma_1 = (\ell_x)_* (\sigma_{\rho}).
\]

Let $G$ be any Carnot group and let $G'$ be an $m$-step Carnot group. Recall that $f : G \to G'$ is in $L^m_{\text{loc}}(G, G')$ if $f$ is measurable and

\[
y \mapsto d_{CC,G'}(f(y), e) \text{ belongs to } L^m_{\text{loc}}(G).
\]

In particular every continuous map belongs to $L^m_{\text{loc}}(G, G')$. We will see shortly that $f \in L^m_{\text{loc}}(G, G')$ implies that the push-forward measure $f_\ast \sigma_z$ has finite $m$-th moment. We may then define a mollified map $f_1 : G \to G'$ by

\[
f_1(x) = \text{com}(f_\ast(\sigma_x)),
\]

and maps $f_\rho : G \to G'$ by

\[
f_\rho = \delta_\rho \circ (\delta_{\rho^{-1}} \circ f \circ \delta_\rho)_1 \circ \delta_{\rho^{-1}}.
\]

Recall that for $p \in [1, \infty)$ the $L^p$ oscillation on a set $A$ is defined by

\[
\text{osc}_p(f, A) := \inf_{a \in G'} \left( \int_A d_{CC,G'}^p(f(y), a) \mu(dy) \right)^{1/p}
\]

where $\mu$ is the biinvariant measure on $G$.

**Lemma 3.33.** Let $G$ be a Carnot group, let $G'$ be an $m$-step Carnot group, let $p \in [m, \infty)$ and $f \in L^p_{\text{loc}}(G, G')$. As above let $\sigma_1 = \alpha \mu$ with $\alpha \in C_c^\infty(B(e, 1))$ and $I_* \sigma_1 = \sigma_1$. Define $\sigma_z$ by (3.28) and $f_1$ and $f_\rho$ by (3.30) and (3.31). Then:

1. For all $z \in G$ the measures $f_\ast \sigma_z$ have finite $p$-th moment.
2. For all $\rho \in (0, \infty)$,

\[
\delta_{\rho^{-1}} \circ f_\rho \circ \delta_\rho = (\delta_{\rho^{-1}} \circ f \circ \delta_\rho)_1.
\]
3. For all $a \in G$ and $b \in G'$

\[
(\ell_b \circ f \circ \ell_a)_1 = \ell_b \circ f_1 \circ \ell_a.
\]
(4) For all $\rho > 0$
\[ (\delta_\rho \circ f)_1 = \delta_\rho \circ f_1. \]

(5) Assume that for some $x_0 \in G$ and some $a \in G'$
\[ \int_{B(x_0,1)} d_{CC,G'}^m(f(x),a) \, d\mu \leq R^m. \]
Then
\[ d_{CC,G'}(f_1(x_0),a) \leq CR \quad \text{where } C = C(G,G',\sigma_1). \]

(6) We have
\[ f_\rho \to f \quad \text{a.e. and in } L^p_{loc}(G). \]
If $f$ is continuous, then $f_\rho \to f$ locally uniformly.

(7) If $\text{osc}_m(f,B(x_0,1)) \leq R$ then the (Riemannian) norms of the derivatives of $f_1$ are controlled at $x_0$, i.e.
\[ \|D^i(f_1(x_0))\| < C = C(i,R,G,G',\sigma_1) \quad \text{and} \quad \|D^i((\delta_R^{-1} \circ f_1)(x_0))\| \leq C(i,G,G',\sigma_1). \]

In assertion (7) the 'Riemannian' derivatives are computed with respect to the charts $\varphi = \log_G \circ \ell_{x_0^{-1}} : G \to \mathfrak{g}$ and $\psi = \log_G' \circ \ell_{f_1(x_0)^{-1}} : G' \to \mathfrak{g}'$, i.e. we estimate the derivatives of the map
\[ \psi \circ f_1 \circ \varphi^{-1} = \log \circ \ell_{f_1(x_0)^{-1}} \circ f_1 \circ \ell_{x_0} \circ \exp : \mathfrak{g} \to \mathfrak{g}' \]
at $0$.

Remark 3.34. One can consider more general domains. First, if $U \subset G$ is open, then it follows from the proof that the results extend to maps in $L^m_{loc}(U;G)$, whenever the expressions make sense. In particular we need that $\text{spt} \, \sigma_z \subset U$. Taking $G = \mathbb{R}^N$ we in particular obtain a smoothing operation for maps $U \subset \mathbb{R}^N \to G'$. If $U$ is an open subset in a metric measure space $X$ we can abstractly define the mollification $f_1(z)$ using a general family of compactly supported Borel probability measures $\sigma_z$. In this case there is no notion of left translation to define $\sigma_z$, but the measures $\sigma_z$ should be in a suitable sense concentrated near $z$. Then (1) still holds and it is easy to prove counterpart of (5). Moreover the proof of (6) shows that $z \mapsto f_1(z)$ is locally Lipschitz (with bounds on the local Lipschitz constant in terms of the $L^m$-oscillation of $f$), provided that the measures are such that for each function $h \in L^1_{loc}(X)$ the map $z \mapsto \int_X h \sigma_z$ is Lipschitz.
Proof of Lemma 3.33. We will sometimes denote the Carnot-Caratheodory distance $d_{CC,G'}$ generically by $d$ for brevity.

(1). We have

$$\int_{G'} d_{CC,G'}^p(y',e) \, df\ast \sigma_z(y') = \int_G d_{CC,G'}^p(f(y),e) \, d\sigma_z(y).$$

Now $\text{spt} \, \sigma_z = z \, \text{spt} \, \sigma$ is compact. Thus the right hand side is finite since by assumption $y \mapsto d_{CC,G'}^p(f(y),e)$ is integrable over compact sets. By (2.9) this implies that $f\ast \sigma_z$ has finite $p$-th moment.

(2). This is immediate from the definition.

(3). We have

$$(\ell a)\ast \sigma_z = (\ell a)_*(\ell z) \ast \sigma = (f\ast \sigma z) = (f_1 \circ \ell a)\ast (az) = (f_1 \circ \ell a)(z).$$

The identity $(\ell b \circ f)_1 = \ell b \circ f_1$ follows from (3.22).

(4). This follows from (3.25).

(5). Since mollification commutes with pre- and postcomposition by left translation we may assume that $a = e$ and $x_0 = e$. Since mollification commutes with dilation we may also assume $R = 1$. By (3.6) and (3.7)

$$\log f_1 = \log \text{com} f\ast \sigma$$

is a polynomial of the polynomial moments

$$A_i = \int_{G'} L_i(Y,\ldots,Y) \, d\log f\ast \sigma_1(Y).$$

Here and in the following we just write $\log$ instead of $\log_{G'}$ for ease of notation. It just suffices to prove bounds on the $A_i$ (which only depend on $G$, $G'$ and $\sigma$). Now $\sigma = a\mu$ where $\mu$ is the biinvariant measure on $G$ and $a \in C^\infty_c(B(e,1))$ and thus

$$A_i = \int_G L_i(\log f(z),\ldots,\log f(z)) \, \alpha(z) \, \mu(dz).$$

Now by (3.8) we have $|L_i(Y,\ldots,Y)| \leq C(1 + |Y|^m)$. By (2.9) we have $|\log f(z)| \leq Cd_{CC,G'}^m(f(z),e)$.

Hence

$$|A_i| \leq C \int_{B(e,1)} (1 + d_{CC,G'}^m(f(z),e)) \|\alpha\|_\infty \mu(dz) \leq C,$$

as desired.

(6). Taking $a = f(x_0)$ in assertion (5) and unwinding definitions we see that

$$d^m(f_\rho(x_0), f(x_0)) \leq C \int_{B(x_0,\rho)} d^m(f(x), f(x_0)) \, d\mu(x).$$
Thus at every Lebesgue point $x_0$ we have $f_\rho(x_0) \to f(x_0)$, and by Lemma 2.32 we therefore have $f_\rho \to f$ almost everywhere. If $f$ is continuous, then it is uniformly continuous on compact sets and thus $f_\rho \to f$ locally uniformly. To get convergence in $L^p_{\text{loc}}(G)$ for $f \in L^p_{\text{loc}}(G)$ with $p \geq m$, it suffices to show that the restriction of $y \mapsto d^p_{\text{CC},G'}(f_\rho(y), f(y))$ to any ball $B(e, R)$ is equi-integrable for $0 < \rho \leq 1$, see Proposition 3.38 below. To that end, we first observe that (3.35) gives, by Jensen’s inequality and the triangle inequality:

$$d^p(f_\rho(y), f(y)) = (d^m(f_\rho(y), f(y)))^{\frac{p}{m}}$$

$$\leq \left( C \int_{B(y,\rho)} d^m(f(x), f(y)) \, d\mu(x) \right)^{\frac{p}{m}}$$

$$\leq C_p \int_{B(y,\rho)} d^p(f(x), f(y)) \, d\mu(x)$$

$$\leq C_p \int_{B(y,\rho)} (d^p(f(x), f(e)) + d^p(f(y), e)) \, d\mu(x)$$

$$= C_p \left( \int_{B(y,\rho)} h(x) \, d\mu(x) + h(y) \right)$$

(3.36)

where $h(x) := d^p_{\text{CC},G'}(f(x), e)$ and $C_p$ denotes a generic constant depending on $p$. The right hand side of (3.36) can be written as $C_p(h * \varphi_\rho + h)(y)$ with $(f * g)(y) := \int_G f(x)g(x^{-1}y) \, \mu(dx)$ and $\varphi_\rho(z) = \rho^{-\nu}1_{B(0,1)}(\delta_{\rho^{-1}}z)$. Since $h \in L^1_{\text{loc}}(G)$, the mollifications $h * \varphi_\rho$ converge to $h$ in $L^1_{\text{loc}}(G)$ as $\rho \to 0$. Thus the right hand side of (3.36) is equi-integrable on each ball $B(e, R)$ for $0 < \rho \leq 1$.

It suffices to prove the estimate for the derivatives of $\delta_{R^{-1}} \circ f_1$ at $x_0$. Then the other estimate follows from the chain rule since $\delta_R$ is smooth. In view of assertion (4) we may in addition assume $R = 1$. In view of assertion (3) we may assume without loss of generality that $x_0 = e$ and $f_1(x_0) = e$. Thus we have to estimate the derivatives of the map

$$g = \log_{G'} \circ f_1 \circ \exp_G.$$  

We begin with the following observation. Let $h \in L^1_{\text{loc}}(G)$ and define

$$\bar{h}(x) := \int_G h \sigma_x.$$  

Then $\bar{h}$ is smooth and the derivatives are uniformly controlled. In particular

$$\sup_{1 \leq j \leq k} |D^k(\bar{h} \circ \exp)(0)| \leq C(G, k, \|\alpha\|_{C_k}, \|g\|_{L^1(B(0,1))}).$$

(3.37)
Indeed, using the definition of $\sigma_x$ and the fact that $\log_* \mu = L^N$ we get

$$
(\bar{h} \circ \exp)(X) = \int_G h(y)\alpha((\exp X)^{-1} y) \, d\mu(y) \\
= \int_{\log B(e,1)} (h \circ \exp)(Y) (\alpha \circ \exp)((-X) \ast Y) \, dL^N(Y).
$$

Here $(-X) \ast Y = -X + Y + P(-X, Y)$ is the induced group operation on the Lie algebra. Since $\alpha$ has compact support in $B(e,1)$ and the group operation $\ast$ is continuous, it follows that $Y \mapsto (\alpha \circ \exp)((-X) \ast Y)$ is supported in a fixed compact subset of $\log B(e,1)$ for all sufficiently small $X$. Hence differentiation and integration commute and the assertion follows since $\|h \circ \exp\|_{L^1(\log B(e,1))} = \|h\|_{L^1(B(e,1))}$.

By (3.6) and (3.7) the quantity $\log f_1(x) = \log \text{com}_{f_\ast(\sigma_x)}$ is a polynomial in the polynomial moments

$$
A_i(x) = \int_G L_i(Y, \ldots, Y) \, d\log_* f_\ast(\sigma_x)(Y) = \int_G L_i(\log f, \ldots, \log f) \, d\sigma_x,
$$

and by (3.8) we have

$$
|L_i(Y, \ldots, Y)| \leq C(1 + |Y|^m).
$$

By (2.9) we have $|Y| \leq C d_{CC,G'}(\exp Y)$. Thus the function $h$ defined by $h(y) := L_i(\log f(y), \ldots, \log f(y))$ is in $L^1_{loc}(G)$. Hence by (3.37), the map $x \mapsto A_i(x)$ is smooth with uniform bounds in terms of $G$, $\alpha$ and $\|d_{CC,G'}^m(f(\cdot), e)\|_{L^1}$. Since $\log f_1$ is a polynomial (depending on $G'$) in the $A_i$ it is also smooth and the derivatives are controlled in terms of the same quantities and $G'$. It only remains to show that $\|d_{CC,G'}^m(f(\cdot), e)\|_{L^1}$ is controlled by a constant, taking into account the normalisations $\bar{R} = 1$ and $f(e) = e$.

By assumption there exists an $a \in G'$ such that

$$
\int_{B(e,1)} d_{CC,G'}^m(f(x), a) \, \mu(dx) \leq 1.
$$

Thus assertion (5) yields

$$
d_{CC,G'}(f_1(e), a) \leq C(G, G', \sigma_1).
$$

Since $f_1(e) = e$ it follows from the triangle inequality that

$$
\int_{B(e,1)} d_{CC,G'}^m(f(x), e) \, \mu(dx) \leq (1 + C(G, G', \sigma))^m.
$$

This concludes the proof of assertion (7). \qed
For the proof of assertion we used the following standard extension of the dominated convergence theorem. Let \((A, \mathcal{A}, \mu)\) be a measure space with \(\mu(A) < \infty\). We say that a family of integrable functions \(f_\alpha : E \to \mathbb{R}\) is equi-integrable if for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that \(\mu(E) < \delta\) implies \(\int_E |f_\alpha| d\mu < \varepsilon\) for all \(k\). Clearly every finite family of integrable functions is equi-integrable and thus every \(L^1\)-convergent sequence of functions is equi-integrable.

**Proposition 3.38.** Let \((A, \mathcal{A}, \mu)\) be a measure space with \(\mu(A) < \infty\).
Assume that \(f_k \to f\) a.e. in \(A\) and that for some \(s \in [1, \infty)\) the family \(\{|f_k|^s\}\) is equi-integrable. Then \(f_k \to f\) in \(L^s(A)\).

**Proof.** An equi-integrable sequence is in particular bounded in \(L^1(A)\). Thus by Fatou’s lemma \(f \in L^s(A)\). Since \(|f_k - f|^s \leq 2^s(|f_k|^s + |f|^s)\) and since the right hand side is integrable it suffices to consider the case \(s = 1, f = 0, f_k \geq 0\). Pick \(\rho > 0\). Let \(F_{k,\rho} : \{x \in A : f_k > \rho\}\). Then \(\mu(F_{k,\rho}) \to 0\) as \(k \to \infty\), so by equi-integrability of \(\{f_k\}\) we have

\[
\limsup_{k \to \infty} \int_A f_k \leq \rho \mu(A) + \limsup_{k \to \infty} \int_{F_{k,\rho}} f_k \leq \rho \mu(A).
\]

Since \(\rho > 0\) was arbitrary, we get \(\|f_k\|_{L^1(A)} \to 0\). \(\square\)

**Lemma 3.39.** Let \(G\) be a Carnot and let \(G'\) be an \(m\)-step Carnot group.

(1) If \(f : G \to G'\) is a group homomorphism, then \(f_1 = f\).

(2) If \(\{f_k : G \to G'\}\) is a sequence of continuous maps, and \(f_k \to f_\infty\) in \(L^m_{loc}(G, G')\), i.e. \(d_{CC,G'}(f_k, f_\infty) \to 0\) in \(L^m_{loc}(G)\), then the sequence of mollified maps \(\{(f_k)_1\}\) converges in \(C^1_{loc}\) (with respect to the Riemannian structure) to \((f_\infty)_1\), for all \(j\).

**Proof.** (1). This follows directly from (3.25) and (3.24).

(2). The main point is to show that

\[
(3.40) \quad (f_k)_1(x_0) \to (f_\infty)_1(x_0) \quad \forall x_0 \in G.
\]

Then \(C^1_{loc}\) convergence will follow from the uniform bounds in Lemma 3.33.

Since mollification commutes with pre- and postcomposition by left-translation we may assume that \(x_0 = e\) and \((f_\infty)(e) = e\). To prove the pointwise convergence \((f_k)_1(e) \to e\) we use the following fact. Suppose that \(\varphi : G' \to \mathbb{R}\) is continuous and \(\varphi(y) \leq Cd^m_{CC,G'}(y, e)\).
Then
\begin{equation}
\varphi \circ f_k \to \varphi \circ f_\infty \quad \text{in } L^1_{\text{loc}}(G).
\end{equation}
This follows easily from Proposition 3.38 by first passing to an a.e. converging sequence and then using uniqueness of the limit.

Now recall that \( \log(f_k)_1(e) \) is a polynomial \( P \) in the polynomial moments
\[
A^k_i := \int_{g'} L_i(Y, \ldots, Y)(\log f_k)_* \sigma_1(Y) = \int_G Q_i(\log f_k, \ldots, \log f_k) \sigma_1.
\]
It thus suffices to show that \( \lim_{k \to \infty} A^k_i = A^\infty_i \). In view of (3.8), this follows from (3.41) applied to the function \( \varphi(y) = L_i(\log y, \ldots, \log y) \) since \( \sigma_1 = \alpha \mu \) and \( \alpha \) is bounded and compactly supported. \( \square \)

### 4. Pansu pullback and mollification

We now consider the behavior of pulling back using a mollified map between Carnot groups \( G \) and \( G' \). For an open set \( U \subset G \) define
\[
U_\rho := \{ x \in U : \text{dist}(x, G \setminus U) > \rho \}.
\]
Assertion (2) of the following lemma provides a key connection between convergence of the mollified pullback \( f^*_\rho \alpha \) and Pansu differentiability.

**Lemma 4.1.** Let \( U \subset G \) be open and let \( f \in L^m_{\text{loc}}(U, G') \). Suppose that \( \alpha \in \Omega^{k, w_\alpha}(G') \) and \( \gamma \in \Omega^{N-k, w_\gamma}(G) \) are left-invariant forms. In particular, if \( k = N \) then \( \gamma \) is a constant zero-form, i.e. a constant function and \( w_\gamma = 0 \). Then

1. For every \( x \in U_\rho \),
\[
(f^*_\rho \alpha \wedge \gamma)(x) = \rho^{-(\nu + w_\alpha + w_\gamma)}(h^*_1 \alpha \wedge \gamma)(\delta_{\rho^{-1}}(x)),
\]
where \( h = \delta_{\rho^{-1}} \circ f \circ \delta_\rho \).

2. For every \( x \in U_\rho \),
\[
(f^*_\rho \alpha \wedge \gamma)(x) = \rho^{-(\nu + w_\alpha + w_\gamma)}((\delta_{\rho^{-1}} \circ f \circ \delta_\rho)^*_1 \alpha \wedge \gamma)(e),
\]
where \( f_x = \ell_{f(x)^{-1}} \circ f \circ \ell_x \).

3. If \( x \in U_\rho \) and \( \text{osc}_m(f, B(x, \rho)) \leq C \rho^{1 + \frac{\nu}{m}} \), then
\[
\|(f^*_\rho \alpha) \wedge \gamma)(x)\| \leq C' \rho^{-w_\alpha} \rho^{-(\nu + w_\alpha + w_\gamma)} \|\alpha\| \|\gamma\|.
\]

**Proof.** The proof of the first two assertions is exactly the same as the proof of the corresponding assertions in Lemma 6.4 in [KMX20]. We include the short calculation for the convenience of the reader.

(1). Note that \( \{ z : B(z, 1) \subset \delta_{\rho^{-1}}U \} = \delta_{\rho^{-1}}U_\rho \) and thus
\[
h : \delta_{\rho^{-1}}U \to G', \quad h_1 : \delta_{\rho^{-1}}U_\delta \to G'.
\]
For \( x \in U_\rho \) we have
\[
(f^*_\rho (\alpha) \wedge \gamma)(x) = ((\delta_\rho \circ h_1 \circ \delta_\rho^{-1})^* \alpha \wedge \gamma)(x)
\]
\[
= (\delta_\rho^{-1} h_1^* \delta_\rho^* \alpha \wedge \delta_\rho^{-1} \delta_\rho^* \gamma)(x)
\]
\[
= \rho^{-(w_\alpha + w_\gamma)} ((\delta_\rho^{-1} h_1^* \alpha \wedge \delta_\rho^{-1} \gamma)(x)
\]
\[
= \rho^{-(w_\alpha + w_\gamma)} ((\delta_\rho^{-1} (h_1^* \alpha \wedge \gamma))(x)
\]
\[
= \rho^{-(w_\alpha + w_\gamma)} (h_1^* \alpha \wedge \gamma)(\delta_\rho^{-1} x).
\]

In the last step we used that \( h_1^* \alpha \wedge \gamma \) is a multiple of the volume form, which has weight \(-\nu\).

(2). With \( h \) as in (1) we get
\[
h = \delta_\rho^{-1} \circ f \circ \delta_\rho
\]
\[
= (\delta_\rho^{-1} \circ \ell(f(x)) \circ \delta_\rho) \circ \delta_\rho^{-1} \circ \ell(f(x))^{-1} \circ f \circ \ell(x) \circ \delta_{\rho} \circ (\delta_{\rho}^{-1} \circ \ell(x)^{-1} \circ \delta_{\rho})
\]
\[
= \ell_{\delta_{\rho}^{-1} f(x)} \circ \delta_{\rho}^{-1} \circ f \circ \delta_{\rho} \circ \ell_{\delta_{\rho}^{-1} x^{-1}}
\]
and so
\[
h_1 = \ell_{\delta_{\rho}^{-1} f(x)} \circ (\delta_{\rho}^{-1} \circ f \circ \delta_{\rho})_1 \circ \ell_{\delta_{\rho}^{-1} x^{-1}}.
\]
Since \( \alpha \) and \( \gamma \) are left invariant we have for \( x \in U_\rho \)
\[
(h_1^* \alpha \wedge \gamma)(\delta_{\rho}^{-1}(x))
\]
\[
= \ell_{\delta_{\rho}^{-1} x^{-1}}^* [((\delta_{\rho}^{-1} \circ f \circ \delta_{\rho})_1^* \alpha \wedge \gamma](\delta_{\rho}^{-1}(x))
\]
\[
= [((\delta_{\rho}^{-1} \circ f \circ \delta_{\rho})_1^* \alpha \wedge \gamma](e).
\]
Combining (1) with (4.2) gives (2).

(3). Note that our assumptions imply that \( \text{osc}_m(h, B(\delta_{\rho}^{-1} x, 1)) \leq C \).
Thus Lemma 3.33 (3) implies that
\[
\|D(\delta_{\rho}^{-1} \circ h_1)(\delta_{\rho}^{-1} x)\| \leq C'.
\]
Using assertion (1) we get
\[
\|(f^*_\rho \alpha \wedge \gamma)(x)\|
\]
\[
= \|\rho^{- (\nu + w_\alpha + w_\gamma)} ((\delta_{\rho} \circ (\delta_{\rho}^{-1} \circ h_1))^* \alpha \wedge \gamma)(\delta_{\rho}^{-1}(x))\|
\]
\[
= C^{-w_\alpha} \|\rho^{- (\nu + w_\alpha + w_\gamma)} ((\delta_{\rho}^{-1} \circ h_1)^* \alpha \wedge \gamma)(\delta_{\rho}^{-1}(x))\|
\]
\[
\leq C' C^{-w_\alpha} \rho^{- (\nu + w_\alpha + w_\gamma)} \|\alpha\| \|\gamma\|.
\]

\[\square\]

**Theorem 4.3** (Approximation theorem). *Let \( G \) be a Carnot group of topological dimension \( N \) and homogeneous dimension \( \nu \) and let \( G' \) be an \( m \)-step Carnot group. Let \( U \subset G \) and \( U' \subset G' \) be open. Suppose*
that \( \omega \in \Omega^{k,w}(U') \) has continuous and bounded coefficients and \( \gamma \in \Omega^{N-k,w}(G) \) is a left-invariant form such that

\[(4.4) \quad w_\omega + w_\gamma \leq -\nu. \]

Assume that \( p \geq -w_\omega \) and \( \frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu} \). Let \( f : U \to U' \) be a map in \( W^{1,p}_{\text{loc}}(U,G') \). Let \( \bar{\omega} \) denote the extension of \( \omega \) to \( G' \setminus U' \) by zero. Then

\[(4.5) \quad f^*\bar{\omega} \land \gamma \to f^*_P\omega \land \gamma \quad \text{in } L^s_{\text{loc}}(U) \quad \text{with } s = \frac{p}{-w_\omega}. \]

Equivalently, we have convergence of weight \( w \) components

\[(f^*_P\omega)_w \to (f^*_P\omega)_w \]

for \( w \geq w_\omega \), see Remark 4.9 below.

In particular we have

\[(4.6) \quad f^*_P\omega \to f^*_P\omega \quad \text{in } L^p_{\text{loc}}(U) \quad \text{if } \omega \in \Omega^N(U'). \]

Remark 4.7. The mollifications \( f_\rho \) may take values outside \( U' \). This is why we need to extend \( \omega \) outside \( U' \) to define the pull-back by \( f_\rho \). The proof shows that convergence in \( (4.5) \) does not depend on which extension we choose. More precisely, if \( \widetilde{\omega} \) is any extension of \( \omega \) which is everywhere defined, bounded, measurable and satisfies \( \widetilde{\omega}(x) \in \Lambda^{k,w}(G) \) at each point, then

\[(4.8) \quad f^*_\rho\widetilde{\omega} \land \gamma \to f^*_P\omega \land \gamma \quad \text{in } L^s_{\text{loc}}(U) \quad \text{with } s = \frac{p}{-w_\omega}. \]

Remark 4.9. The convergence in \( (4.5) \) in connection with the condition \( (4.4) \) is equivalent to convergence of weight \( w \) components

\[(f^*_P\omega)_w \to (f^*_P\omega)_w \]

for \( w \geq w_\omega \). To see this, note that for \( \omega \) fixed and \( w \geq w_\omega \), we may choose a basis \( \{\gamma_i\} \) of the space of left-invariant forms \( \gamma \in \Omega^{N-k,-\nu-w}(G) \), and this is dual via the wedge product to a basis \( \{\alpha_{w,i}\} \) for the left invariant forms in \( \Omega^{k,w}(G) \). Thus \( (4.5) \) applied to each \( \gamma_i \) yields convergence

\[(f^*_\rho\omega)_{w,i} \to (f^*_P\omega)_{w,i} \]

in \( L^s_{\text{loc}} \) where the notation \( (\beta)_{w,i} \) for a form \( \beta \) is defined by \( \beta_w = \sum_i (\beta)_{w,i} \alpha_{w,i} \). In particular, if \( G = G' \) and the weight of \( \omega \) is minimal among nonzero forms of degree \( k \), then all components converge and thus \( f^*_\rho\omega \to f^*_P\omega \) in \( L^s_{\text{loc}}(U) \).
Remark 4.10. We now comment on the assumptions on the exponent $p$. The obvious estimate for the pullback is
\[ |f^* P_\omega|(|x) \leq C |D_p f(x)|^{-w_\omega}|(f(x))_.\]
Therefore, in general, one would expect $p \geq -w_\omega$ to be the optimal lower bound on the Sobolev exponent. However if $w_\omega + w_\gamma < -\nu$ then some improvement is possible, see Corollary 4.16 below. In the abelian case we have $w_\omega = -k$ and it is known that the condition $p \geq k$ is necessary to have $L^1_{\text{loc}}$ convergence of $f^*_\rho \omega$. Typical counterexamples are given by suitable 0-homogeneous functions. For example, if $G = G' = \mathbb{R}^N$ and $\omega = dy_1 \wedge \ldots \wedge dy_N$ one can take $f = \frac{x}{|x|}$. Then $f^* \omega = 0$, $f \in W^{1,p}(U; \mathbb{R}^N)$ for all $p < N$, but it is easily seen, e.g. by a degree argument, that $f^*_\rho \omega$ weak* converges to the Dirac mass $\mu(B(0, 1))\delta_0$ as $\rho \to 0$, where we identify top degree forms with measures. We do not know the optimal exponent $p$ for which the conclusion $f^*_\rho \omega \wedge \gamma \to f^*_\rho \omega \wedge \gamma$ in $L^1_{\text{loc}}(U)$ holds.

Proof of Theorem 4.3. We will prove the result using the dominated convergence theorem. In brief, this is implemented as follows. Pointwise convergence almost everywhere follows from the formula in Lemma 4.1 (2), Pansu differentiability a.e. (in an $L^m$ sense) and the fact the mollification improves $L^m$-convergence to $C^1$-convergence. The majorant is obtained from the estimate in Lemma 4.1 (3) and the Sobolev-Poincaré inequality which provides a uniform estimate of the $L^m$ oscillation in terms of the maximal function of the $p$-th power of the (horizontal) derivative.

We begin with some preparations. Since we only want to prove convergence in $L^1_{\text{loc}}$ we may assume that $f \in W^{1,p}(U; G')$. By linearity it suffices to verify the theorem for forms $\omega = a\alpha$ where $\alpha$ is a left-invariant form with $w_\alpha + w_\gamma \leq -\nu$ and $a$ is a continuous and bounded function. We denote by $\overline{a}$ the extension of $a$ by zero to $G' \setminus U'$. Set $w_\alpha = \text{wt}(\alpha)$ and $w_\gamma = \text{wt}(\gamma)$. Fix a compact set $K \subset U$. We next show pointwise convergence a.e. in $K$. Recall that $U_\rho := \{x \in U : \text{dist}(x, G \setminus U) > \rho\}$. For $\rho > 0$ small enough we have $K \subset U_\rho$. By Lemma 4.1 (2) we have for $x \in K$

\[ (f^*_\rho (\overline{a} \wedge \gamma))(x) = (\overline{a} \circ f_\rho)(x) (f^*_\rho \alpha \wedge \gamma)(x) = (\overline{a} \circ f_\rho)(x) \rho^{-(\nu + w_\alpha + w_\gamma)} ((\delta_\rho^{-1} \circ f_\rho \circ \delta_\rho)^{1}_\gamma \alpha \wedge \gamma)(e). \]
By Theorem 2.41 and the condition $\frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu}$ we have for a.e. $x \in K$ the convergence $\delta_{\rho^{-1}} \circ f_x \circ \delta_{\rho} \overset{L^1}{\longrightarrow} D_p f$. (Recall that we are using the notation $D_p f(x)$ to denote a graded Lie algebra homomorphism $g \to g'$ and a homomorphism of Carnot groups $G \to G'$, depending on the context.) By Lemma 3.39 we get $D(\delta_{\rho^{-1}} \circ f_x \circ \delta_{\rho})_1(e) \to D_p f(x)$ as $\rho \to 0$. Moreover by Lemma 3.33 (6) we have $f_\rho(x) \to f(x)$ almost everywhere.

Let $N \subset K$ be a null set such that for all $x \in K \setminus N$ we have $f_\rho(x) \to f(x)$ and $D(\delta_{\rho^{-1}} \circ f_x \circ \delta_{\rho})_1(e) \to D_p f(x)$. Since $U'$ is open, for each $x \in K \setminus N$ there exist a $\rho_0(x) > 0$ such that $f_\rho(x) \in U'$ for all $\rho < \rho_0(x)$. Since $\overline{a}$ is continuous in $U'$ (and agrees there with $a$) it follows that $\overline{a} \circ f_\rho(x) \to a \circ f(x)$ for all $x \in K \setminus N$. Note that this convergence is independent of how we extend $a$ outside $U'$.

Now if $w_\alpha + w_\gamma = -\nu$, then

$$
(f_\rho * \overline{\omega} \wedge \gamma)(x) \to (a \circ f)(x) ((D_p f(x))^* \alpha)(x) \wedge \gamma
$$

so we have pointwise convergence in this case. If $w_\alpha + w_\gamma < -\nu$, then $(f_\rho * \overline{\omega} \wedge \gamma)(x) \to 0$ as $\rho \to 0$, while

$$(f_\rho * \overline{\omega} \wedge \gamma)(x) = (a \circ f)(x) ((D_p f(x))^* \alpha)(x) \wedge \gamma.$$ 

Now by Lemma 2.23 (3) we deduce that $((D_p f(x))^* \alpha)(x) \wedge \gamma$ is a form of weight strictly less than $-\nu$ and hence zero. Thus if $w_\alpha + w_\gamma < -\nu$ we have $(f_\rho * \overline{\omega} \wedge \gamma)(x) = 0$. Hence we have shown that $(f_\rho * \overline{\omega} \wedge \gamma)(x) \to (f_\rho * \overline{\omega} \wedge \gamma)(x)$ for a.e. $x \in K$.

By Proposition 3.38 it remains only to show that $|f_\rho * \overline{\omega} \wedge \eta|^s$ is equi-integrable for $s = \frac{q}{q-1}$. If $m \geq 2$ define $q > 1$ by $\frac{1}{q} = \frac{1}{m} + \frac{1}{\nu}$ (if $m = 1$, i.e., if $G$ is abelian, take $q = 1$; then (4.14) below follows directly from the Poincaré inequality). Set $\psi = |D_h f|^q$. Then $\psi \in L^\infty(U)$. By the Sobolev-Poincaré inequality (2.31) we have for $x \in K$

$$
\rho^{-\frac{m}{q}} \text{osc}_m(f, B(x, \rho)) \leq C \rho \left( \rho^{-\nu} \int_{B(x, \rho)} \psi \right)^{\frac{1}{q}} = C \rho \psi^\frac{1}{q}_\rho(x)
$$

where

$$
\psi_\rho := \psi \ast \rho^{-\nu} 1_{B(0, \rho)}
$$

and

$$(f \ast g)(x) := \int_G f(xy^{-1})g(y) \mu(dy) = \int_G f(y)g(y^{-1}x) \mu(dy).$$
Since $\psi \in L^{p\over q}_{\text{loc}}(G)$ we have
\begin{equation}
\psi_{\rho} \to \psi \quad \text{in} \quad L^{p\over q}(K)
\end{equation}
as $\rho \to 0$. Moreover
\begin{equation}
\text{osc}_m(f, B(x, \rho)) \leq C \rho^{1 + w_\omega} \psi_\rho^\frac{1}{p}(x).
\end{equation}
Now let $s = \frac{p}{-w_\omega}$. Then by Lemma 4.1 (3)
\begin{equation}
|f^{*}_{\rho} \omega \wedge \eta|^s(x) \leq \|\bar{a}\|^s \|(f^{*}_{\rho} \alpha)(x) \wedge \eta|^s \\
\leq C \psi_\rho^{-s(w_\omega + w_\gamma)} \rho^{-s(w_\omega + w_\gamma)} \|\alpha\|^s \|\gamma\|^s \|\bar{a}\|^s \\
\leq C \rho^{-s(w_\omega + w_\gamma)} \psi_\rho^s.
\end{equation}
In view of (4.13) the family $\psi_\rho^s$ is equi-integrable, so (4.15) gives the desired equi-integrability of $|f^{*}_{\rho} \omega \wedge \eta|^s$. Note also that the argument used only the fact that the extension $\bar{a}$ is bounded. \hfill \Box
The argument above shows easily that we have better convergence results if $wt(\omega) + wt(\gamma) < -\nu$. We summarize these as follows.

**Corollary 4.16.** With the notation and assumptions of Theorem 4.3 the following refinements of (4.5) hold.

1. If $\frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu}$, $p \geq -w_\omega$ and $w_\omega + w_\gamma < -\nu$ then
\begin{equation}
\rho^{\nu + w_\omega + w_\gamma} f^{*}_{\rho} \omega \wedge \gamma \to 0 \quad \text{in} \quad L^{s}_{\text{loc}}(U) \quad \text{with} \quad s = \frac{p}{-w_\omega}.
\end{equation}

2. Set $\beta = -\frac{p}{w_\omega}(w_\omega + w_\gamma + \nu)$. If $\beta < -w_\omega$ then
\begin{equation}
f^{*}_{\rho} \omega \wedge \gamma \to 0 \quad \text{in} \quad L^{\frac{p}{-w_\omega - \beta}}_{\text{loc}}(U).
\end{equation}
If $\beta \geq -w_\omega$ then
\begin{equation}
f^{*}_{\rho} \omega \wedge \gamma \to 0 \quad \text{locally uniformly}.
\end{equation}

**Proof.** The first assertion follows directly from the proof of Theorem 4.3. Indeed, (4.11), Pansu differentiability a.e., and the estimates for the mollification imply that $\rho^{\nu + w_\omega + w_\gamma} f^{*}_{\rho} \omega \wedge \gamma \to (a \circ f) f^{*}_{\rho} \alpha \wedge \gamma$ a.e. Moreover $f^{*}_{\rho} \alpha \wedge \gamma = 0$ since forms of weight strictly less than $-\nu$ must vanish. Regarding equi-integrability, (4.15) yields
\begin{equation}
|\rho^{\nu + w_\omega + w_\gamma} f^{*}_{\rho} \omega|^{-w_\omega / (w_\omega - \beta)} \leq C |\psi_\rho|^\frac{p}{s} \to C |\psi|^\frac{p}{s}
\end{equation}in $L^1_{\text{loc}}(U)$. Hence the assertion follows from Proposition 3.38.
To prove the second assertion, we note that Lemma 4.1 (3) yields
\[ |f^*_\rho \omega \wedge \gamma| (x) \leq C \left( \rho^{-(1+\frac{\nu}{m})} \text{osc}_m (f, B(x, \rho)) \right)^{-w_\omega} \rho^e \]
where where $C$ is a constant independent of $x$ and $\rho$, and
\[ e = -(w_\omega + w_\gamma + \nu). \]
By (4.14) and the Poincaré-Sobolev inequality we have
\[ \text{osc}_m (f, B(x, \rho)) \leq C \rho^{1+\frac{\nu}{m} - \frac{\beta}{p}} \| D_h f \|_{L^p(B(x, \rho))}. \]

Recall that $\beta = \frac{ep}{\nu}$. If $\beta < -w_\omega$ then we take (4.21) to the power $-w_\omega - \beta$ and (4.22) to the power $\beta$ to get
\[ |f^*_\rho \omega \wedge \gamma| (x) \leq C \psi_\rho (x) \left( \rho^{w_\omega - \frac{\beta}{q}} \right) \| D_h f \|_{L^p(B(x, \rho))}^{\beta}. \]

Since $\| D_h f \|_{L^p(B(x, \rho))} \to 0$ as $\rho \to 0$, locally uniformly in $x$, and $\psi_\rho$ converges in $L^p_{\text{loc}}(U)$ it follows that $|f^*_\rho \omega \wedge \gamma| \to 0$ in $L^{p-\frac{\beta}{q}}_{\text{loc}}(U)$. If $\beta \geq -w_\omega$ we take (4.22) to the power $-w_\omega$ and get (for $\rho \leq 1$) the estimate $|f^*_\rho \omega \wedge \gamma| (x) \leq C \| D_h f \|_{L^p(B(x, \rho))}^{w_\omega}$. The assertion follows since the right hand side converges locally uniformly to zero.

Next we apply the approximation theorem to show that for certain components the Pansu pullback of differential forms commutes with exterior differentiation. Note that in general the Pansu pullback does not commute with exterior differentiation (see [KMX20]).

**Theorem 4.24.** Let $G$ be a Carnot group of topological dimension $N$ and homogeneous dimension $\nu$, let $G'$ be a $m$-step Carnot group, and $f : G \supset U \to U' \subset G'$ be a $W^{1,p}_{\text{loc}}$-mapping between open subsets. Suppose that $\alpha \in \Omega^{k,w_\alpha}(G')$ has continuous and bounded coefficients such that the weak exterior differential $d\alpha$ also has continuous and bounded coefficients. Let $\beta \in \Omega^{N-k-1,w_\beta}(G)$ be a closed left-invariant form. Assume that
\[ w_\alpha + w_\beta = -\nu + 1. \]
Then the following assertions hold.
(1) If $\alpha$ is weakly closed, $p \geq -w_\alpha$ and $\frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu}$ then $f^*_p(\alpha) \wedge \beta$ is weakly closed, i.e.

$$\int_G f^*_p(\alpha) \wedge \beta \wedge d\varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(U).$$

(2) Assume $\text{wt}(d\alpha) < w_\alpha$, and that $d\alpha = \sum_{s \leq w < w_\alpha} \omega^{(w)}$ is the weight decomposition of $d\alpha$. Assume that $p \geq -s$ and $\frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu}$. Then

$$d(f^*_p\alpha \wedge \beta) = f^*_p(d\alpha) \wedge \beta \quad \text{in the sense of distributions,}$$

i.e.

$$(-1)^N \int_U f^*_p\alpha \wedge \beta \wedge d\varphi = \int_U f^*_p(d\alpha) \wedge \beta \varphi \quad \forall \varphi \in C^\infty_c(U).$$

Remark 4.29. If $G = G'$, $k = N - 1$ and $w_\alpha = -\nu + 1$ we have $d\alpha \in \Omega^N(G')$ and hence $\text{wt}(d\alpha) = -\nu$. Thus for $p \geq \nu$ we can take $\beta \equiv 1$ and we get

$$df^*_p\alpha = f^*_p(d\alpha) \quad \text{if } G = G', \ k = N - 1, \ \text{wt}(\alpha) = -\nu + 1$$

in the sense of distributions. For 2-step groups this was first shown by Vodopyanov, see [Vod07b].

Remark 4.31. If we use Corollary 4.16 then the condition on the exponent in the second assertion can be slightly improved if $s > -w_\alpha + 1$. In that case we can replace the condition $p \geq s$ by

$$p \geq s - \frac{p}{\nu}(s + w_\beta + \nu) = s - \frac{p}{\nu}(s + 1 - w_\alpha),$$

or, equivalently,

$$\frac{s}{p} \leq 1 + \frac{1}{\nu}(s + 1 - w_\alpha).$$

Proof of Theorem 4.24. Since $\beta$ is closed we have $d(\varphi \beta) = d\varphi \wedge \beta$, and hence $\text{wt}(d(\varphi \beta)) \leq \text{wt} \beta - 1$. Using that the (weak) exterior derivative commutes with pullback by smooth functions we get

$$\int_G f^*_p\alpha \wedge \beta \wedge d\varphi = (-1)^N \int_G f^*_p(d\alpha) \wedge \beta \varphi \quad \forall \varphi \in C^\infty_c(U).$$

Hence both assertions follow by applying Theorem 4.3 to both sides of (4.33); on the right hand side the theorem is applied to each component of the weight decomposition of $d\alpha$ separately. Note that the condition $\text{wt}(d\alpha) < w_\alpha$ ensures that Theorem 4.3 can be applied. \qed
5. Quasiregular mappings

In this section we review some results from [Vod07b] which were stated only for 2-step Carnot groups, but which extend immediately to general Carnot groups using the Approximation Theorem 4.3.

In this section we fix a Carnot group $G$ of homogeneous dimension $\nu$, and an open subset $U \subset G$.

**Definition 5.1** ([Vod07b]). A mapping $f : G \supset U \to G$ is **quasiregular** (has bounded distortion) if $f \in W^{1,\nu}_{\text{loc}}$ and there is a constant $C$ such that $|D_h f|^\nu \leq C \det D_P f$ almost everywhere.

We now fix a quasiregular mapping $f : G \supset U \to G$.

Following Reshetnyak [Res89, HH97, Vod07b], we exploit the pull-backs of $\nu$-harmonic functions to control quasiregular mappings.

**Theorem 5.2.** If $u : G \to \mathbb{R}$ is a Lipschitz $\nu$-harmonic function, then the composition $u \circ f$ is $A$-harmonic. See [HH97, Sec. 2], [Vod07b, Subsec. 4.3] for the definition and basic properties of $A$-harmonic functions.

Note that if $f$ takes values in an open subset $U'' \subset G$, then the theorem holds when $u$ is locally Lipschitz, see below.

**Proof.** In the 2-step case, the proof is contained in [Vod07b]. This extends to general Carnot groups using the Approximation Theorem.

We give an outline of the steps, to facilitate reading of [Vod07b]:

- By Remark 4.29 if $\omega$ is a smooth differential form on $G$ with codegree and coweight 1, and both $\omega$ and $d\omega$ are bounded, then

\[
 df_P^* \omega = f_P^* d\omega
\]

distributionally.

- If $\Sigma \subset G$ is a Borel null set, then the (approximate) Pansu differential $D_P f(x)$ satisfies $\det D_P f(x) = 0$ for a.e. $x \in f^{-1}(\Sigma)$ [Vod00]; hence by the bounded distortion assumption in fact $D_P f(x) = 0$ for a.e. $x \in f^{-1}(\Sigma)$. If $\omega$ is a measurable differential form on $G$, and we define $f_P^* \omega(x)$ to be zero whenever $D_P f(x) = 0$, then the Pansu pullback $f_P^* \omega$ is well-defined almost everywhere.

- By an approximation argument [5.3] remains true if $\omega, d\omega \in L^\infty$, see [Vod07b, Corollaries 2.15, 2.18].

- It follows from Proposition 2.31 that the composition $v := u \circ f$ belongs to $W^{1,\nu}_{\text{loc}}(U)$.

- To see that $v$ is $A$-harmonic, it suffices to show that its horizontal differential $d_h v$ satisfies the distributional equation $\delta(Ad_h v) = 0$. 

---

}\text{end of page 39}
This is equivalent to the vanishing of the distributional exterior derivative of \( \star A d_h v \), see [HH97, Section 3, Theorem 3.14]. Since \( \star A d_h v = f_P(\star d_h u) \), this follows from (5.3).

\[ \square \]

The composition \( u \) of the abelianization map \( G \to G/[G,G] \) with a coordinate function is Lipschitz and \( \nu \)-harmonic. Hence by Theorem 5.2 the composition \( u \circ f \) is \( \mathcal{A} \)-harmonic. Following [Res89, BI83, HH97, Vod07b], by applying the Caccioppoli inequality for \( \mathcal{A} \)-harmonic functions and the Poincaré inequality one obtains a number of results, including:

- \( f \in W^{1,\nu'} \) for some \( \nu' > \nu \).
- \( f \) is Hölder continuous, (classically) Pansu differentiable almost everywhere, and maps null sets to null sets.
- A suitable change of variables formula holds for \( f \).

Since \( f \) is continuous, the proof of Theorem 5.2 may be localized in the target:

**Corollary 5.4.** Suppose the image of \( f \) is contained in an open subset \( U' \subset G \), and \( u : U' \to \mathbb{R} \) is a locally Lipschitz \( \nu \)-harmonic function. Then \( u \circ f \) is \( \mathcal{A} \)-harmonic.

If there exists for some \( r > 0 \) a locally Lipschitz \( \nu \)-harmonic function \( u : B(e,r) \setminus \{e\} \to (0,\infty) \) such that \( \lim_{x \to e} u(x) \to \infty \), then the method of Reshetnyak could be applied to show that \( f \) is open and discrete, which would have a number of further consequences, see [Vod07b] (Theorem 4.11 and the ensuing discussion). Unfortunately, the existence of such \( \nu \)-harmonic functions remains an open problem.

6. **Product rigidity**

In this section we show how the results in [KMX20] on product rigidity can be improved by using the improved version of the Pullback Theorem, Theorem 4.24 and a better choice of forms to be pulled back.

**Theorem 6.1** (Product rigidity). Let \( \{G_i\}_{1 \leq i \leq n}, \{G'_j\}_{1 \leq j \leq n'} \) be collections of Carnot groups where each \( G_i, G'_j \) is nonabelian and does not admit a nontrivial decomposition of Carnot groups. Let \( G = \prod_i G_i, G' = \prod_j G'_j \). Set

\[ K_i := \{k \in \{1, \ldots, n\} : G_k \simeq G_i\} \]
and if $|K_i| \geq 2$ for some $i$, assume that

\[(6.2) \quad p \geq \max \{\nu_i - 1 : |K_i| \geq 2\}\]

where $\nu_i$ denotes the homogeneous dimension of $G_i$. Suppose that $f : G \supset U \to G'$ is a $W^{1,p}_{\text{loc}}$-mapping, $U = \prod_i U_i$ is a product of open connected sets $U_i \subset G_i$, and the (approximate) Pansu differential $D_P f(x)$ is an isomorphism for a.e. $x \in U$. Then $f$ is a product of mappings, i.e. $n = n'$ and for some permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ there are mappings $\{f_{\sigma(i)} : U_i \to G'_{\sigma(i)}\}_{1 \leq i \leq n}$ such that

\[(6.3) \quad f(x_1, \ldots, x_n) = (f_1(x_{\sigma^{-1}(1)}), \ldots, f_n(x_{\sigma^{-1}(n)}))\]

for a.e. $(x_1, \ldots, x_n) \in \prod_i U_i$.

For many groups the condition (6.2) on $p$ can be improved.

**Corollary 6.4.** Let $G_i, G'_j, U_i$ be as in Theorem 6.1, assume that $G_j$ is a group of step $m_j$ and set

\[
m = \max \{m_i : |K_i| \geq 2\}, \quad \bar{\nu} = \sum_{i : |K_i| \geq 2} \nu_i.
\]

Assume further for all $i$ with $|K_i| \geq 2$ the Lie algebra $g_i / \oplus_{i=3}^{m_i} V_j$ is not a free Lie algebra. Then the conclusions of Theorem 6.1 hold provided (6.2) is replaced by the weaker conditions

\[(6.5) \quad p \geq 2 \quad \text{and} \quad \frac{1}{p} \leq \frac{1}{m} + \frac{1}{\bar{\nu}}.\]

For example, the conclusion of Theorem 6.1 holds for $p = 2$ if all $G_i$ are isomorphic to a higher Heisenberg group $\mathbb{H}_{d_i}$ with $d_i \geq 2$ or to a complex Heisenberg group $\mathbb{H}_{d_j}^C$ with $d_j \geq 1$. On the other hand, the assumptions are not satisfied if some $G_i$ is a copy of the first Heisenberg group $\mathbb{H}_1$ which is a free Carnot group of step 2.

We use the following result from [Xie13, Prop. 2.5], see also [KMX20].

**Lemma 6.6.** Suppose $g = \oplus_{i \in I} g_i$, $g' = \oplus_{j \in I'} g'_j$ where every $g_i$, and $g'_j$ is nonabelian and does not admit a nontrivial decomposition as a direct sum of graded ideals. Then any graded isomorphism $\phi : g \to g'$ is a product of graded isomorphisms, i.e. there is a bijection $\sigma : I \to I'$ and for every $i \in I$ there exists a graded isomorphism $\phi_i : g_i \to g'_{\sigma(i)}$ such that for all $i \in I$ we have $\pi_{\sigma(i)} \circ \phi = \phi_i \circ \pi_i$.

We also use the following Fubini-type property of Sobolev maps.

**Lemma 6.7.** Let $G_1$, $G_2$, and $G'$ be Carnot groups, let $U_i \subset G_i$ be open sets, let $\iota_i$ be the injections $G_i \to G_1 \times G_2$ and let $\pi_i : G_1 \times G_2 \to G_i$ be the corresponding projections. Let $f$ be the representative of a map
in \( W^{1,p}(U_1 \times U_2; G') \), let \( D_P f \) be a representative of the (approximate) Pansu differential. Then the following assertions hold.

1. For a.e. \( a \in U_1 \) the map \( f_a : U_2 \to G' \) defined by \( f_a(y) = f(a, y) \) is in the Sobolev space \( W^{1,p}(U_2; G') \) and the Pansu differential \( D_P f_a \) satisfies
   \begin{equation}
   D_P f_a = D_P f(a, \cdot) \circ \iota_2 \quad \mu_{G_2}-\text{a.e. in } U_2.
   \end{equation}

2. If, in addition, \( U_2 \) is connected and \( D_P f \circ \iota_2 = 0 \) a.e. then there exists a function \( \bar{f} : U_1 \to G' \) such that \( f = \bar{f} \circ \pi_1 \) a.e.

3. If \( G' = G_1 \times G_2 \), \( U_i \) is connected, and for a.e. \( x \in U \) we have
   \[ \pi_1 \circ D_P f(x) \circ \iota_2 = 0, \quad \text{and} \quad \pi_2 \circ D_P f(x) \circ \iota_1 = 0, \]
   then there exist mappings \( \bar{f}_i : U_i \to G_i \) such that \( f(x_1, x_2) = (\bar{f}_1(x_1), \bar{f}_2(x_2)) \) for a.e. \( (x_1, x_2) \in U_1 \times U_2 \).

Proof. To prove (1), first note that for \( G' = \mathbb{R}^N \) the Pansu differential (viewed as map on the Lie algebra) is given by \( D_P f = D_h f \circ \Pi_1 \) a.e., where \( D_h f \) is the weak horizontal differential and \( \Pi_1 \) is the projection to the horizontal subspace. Thus, for \( G' = \mathbb{R}^N \), assertion (1) follows directly from the definition of the weak horizontal derivative and Fubini’s theorem.

For a general Carnot group \( G' \) there exists a \( p \)-integrable function \( g : U_1 \times U_2 \to \mathbb{R} \) such that \( |D_h d(z, f(\cdot))| \leq g \) a.e. Applying the result for real-valued maps to the maps \( (x, y) \mapsto d(z, f(x, y)) \) for all \( z \) in a countable dense subset \( D \) of \( G' \) we easily conclude that \( |D_h d(z, f_a(\cdot))| \leq g_a \) and hence \( f_a \in W^{1,p}(U_2, G') \) for a.e. \( a \in U_1 \).

Let \( \Pi_{G'} \) denote the abelianization map. Then we can apply the result for \( \mathbb{R}^N \)-valued maps to \( \Pi_{G'} \circ f \) and we get, for a.e \( a \in U_1 \),
\[
\Pi_{G'} \circ D_P f_a = D_P (\Pi_{G'} \circ f_a) = (D_P (\Pi_{G'} \circ f)(a, \cdot)) \circ \iota_2 = \Pi_{G'} \circ (D_P f \circ \iota_2)(a, \cdot) \]
\( \mu_{G_2} \)-a.e. in \( U_2 \). Now if \( \Phi, \Psi : G_2 \to G' \) are graded group homomorphism such that \( \Pi_{G'} \circ \Phi = \Pi_{G'} \circ \Psi \) then \( \Phi = \Psi \). Hence (6.8) holds.

Assertion (2) is an immediate consequence of assertion (1).

Assertion (3) follows by applying assertion (2) to the compositions \( \pi_i \circ f \).

\[ \square \]

Proof of Theorem 6.1. The result was established in [KMX20] under the stronger condition \( p > \nu(\Pi_i G_i) = \sum_i \nu_i \). We first briefly recall the argument in [KMX20] and indicate the strategy to obtain the improved condition (6.2).

First, Lemma 6.6 implies that we may assume without loss of generality that \( n = n' \) and \( g_i = g'_i \) for all \( i \in I \), and so there is a measurable
function $\sigma : U \to \text{Perm}(\{1, \ldots, n\})$ such that $D_P f(x)(g_i) = g_{\sigma(x)(i)}$ for a.e. $x \in U$. Moreover $\sigma(i) = i$ if $|K_i| = 1$. Hence it follows by applying Lemma 6.7(3) repeatedly that we may assume without loss of generality that

$$|K_i| \geq 2 \quad \forall i = 1, \ldots, n.$$  

The main point is to show that there exists a constant permutation $\bar{\sigma}$ such that $\sigma = \bar{\sigma}$ a.e. Then, using again Lemma 6.7, we see $f$ has the desired product structure.

To prove that $\sigma$ is constant a.e. we argue as follows. We choose closed left-invariant forms $\alpha$ such that the pullback $f^* \alpha$ can 'detect' the permutation $\sigma$. Then we use the Pullback Theorem to deduce that for suitable left-invariant forms $\beta$, we have $df^* \alpha \wedge \beta = 0$ in the sense of distributions and conclude that $\sigma$ is constant. In [KMX20] we use for $\alpha$ the volume forms $\omega_i$ of the factors $G_i$. Then Lemma 6.6 implies that $f^*_i \omega_i = \sum_{j \in K_i} a_j \omega_j$ where the $a_j$ are integrable functions. Moreover, for each $x$, exactly one of the functions $a_j$ is different from zero, namely $a_{\sigma^{-1}(i)}$. Let $j' \in \{1, \ldots, n\}$ and $l \neq j'$. For $X \in V_1 \cap g_l$ we apply the Pullback theorem with the closed codegree $N_j' + 1$ test forms $\beta = \omega_i \wedge \bigwedge_{\nu \neq \{j', l\}} \omega_{\nu}$ and easily conclude that $a_{j'}(x_1, \ldots, x_n)$ depends only on $x_{j'}$. Then one easily deduces that $\sigma = \bar{\sigma}$ almost everywhere.

The requirement $p > \nu(\Pi G_i)$ in [KMX20] comes from the hypotheses of the Pullback Theorem [KMX20 Theorem 4.5]. Using the improved Pullback Theorem, Theorem 4.24 we see that this argument works if $p \geq \max_i \nu_i$. The second condition in Theorem 4.24, namely $\frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu}$ is then automatically satisfied since

$$m \leq \max_i \nu_i - 1.$$  

To get the improved condition (6.2) we will apply the pullback theorem not to the volume forms of each factor, but to a codegree one form on each factor, with weight $-\nu_i + 1$.

To set the notation, we assign to a form $\alpha \in \Omega^*(G_i)$ the form $\pi_i^* \alpha \in \Omega^*(G)$ where $\pi_i : G \to G_i$ is the projection map. Note that $\pi_i^* \alpha$ is closed if and only if $\alpha$ is closed. We will usually write $\alpha$ also for the form $\pi_i^* \alpha$ if no confusion can occur. Similar we identify a vector field $X \in TG_i$ with a vector field in $TG$ through the push-forward by the canonical injection $G_i \to G$.

For $i \in \{1, \ldots, n\}$ let $\text{vol}_{G_i}$ denote the volume form in $G_i$ and let $Y \in V_1(G_i) \setminus \{0\}$. By Cartan’s formula and the biinvariance of $\text{vol}_{G_i}$ the $N_i - 1$ form $i_Y \text{vol}_{G_i}$ is closed. Let $\alpha_i = i_Y \text{vol}_{G_i}$. Then the $N_i - 1$ form $\alpha_i$ is left-invariant, closed and has weight $-\nu_i + 1$. 

In view of (6.9) we have $p \geq \nu_i - 1$. For $j \in K_i$ let $X_{j,k}$, $k = 1, \ldots, \dim V_1(G_i)$ be a basis of $V_1(G_j)$. Then $i_{X_{j,k}} \text{vol}_{G_j}$, $k = 1, \ldots, \dim V_1(G_i)$ is a basis of the left-invariant forms on $G_j$ with degree $N_i - 1$ and weight $-\nu_i + 1$. Since pullback by a graded isomorphism preserves degree and weight we have

$$f_p^* \alpha_i = \sum_{j \in K_i} \sum_{k=1}^{\dim V_1(G_i)} a_{j,k} i_{X_{j,k}} \text{vol}_{G_j}$$

with $a_{j,k} \in L^1_{\text{loc}}(U)$. Set

$$a_j := (a_{j,1}, \ldots, a_{j,\dim V_1(G_i)})$$

and

$$E_j = \{x \in U : \sigma^{-1}(x)(i) = j\}.$$ 

Then $U \setminus \bigcup_{j \in K_i} E_j$ is a null set. Since $D_P f(x)$ is a graded automorphism for a.e. $x \in U$, we have, for all $j \in K_i$,

$(6.11) \quad a_j \neq 0$ a.e. in $E_j$, $a_j = 0$ a.e. in $U \setminus E_j$.

We next show that

$(6.12) \quad Z a_{j'} = 0$ in distributions for all $j' \in K_i$, $Z \in \oplus_{l \neq j'} g_l$.

To prove (6.12), let $\theta_{j',k'}$ be basis of left-invariant one-forms which vanish on $\bigoplus_{l=2}^n V_1(G_j')$ which is dual to the basis $X_{j',k}$ of $V_1(G_j')$, i.e. $\theta_{j',k'}(X_{j,k}) = \delta_{kk'}$. Note that the forms $\theta_{j',k'}$ are closed. For $l \in \{1, \ldots, n\} \setminus \{j'\}$, and $X \in V_1(G_l)$ consider the closed form

$$\beta = \theta_{j',k'} \wedge i_X \text{vol}_{G_l} \wedge (\Lambda_{\neq j',l} \text{vol}_{G_l}).$$

Then, for a.e. $x \in U$,

$$(D_P f)^*(x) \alpha_i \wedge \beta = \pm a_{j',k'} i_X \text{vol}_{G_l}.$$ 

In view of (6.10) and the assumption $p \geq \nu_i - 1$ (recall that we may assume (6.9)) we get from the Pullback Theorem, Theorem 4.24

$$0 = \int_U f_p^* \alpha \wedge \beta \wedge d\varphi = \pm \int_U a_{j',k'} X \varphi \ \text{vol}_{G_l}$$

for all $\varphi \in C_c^\infty(U)$. Since $V_1 \cap g_l$ generates $g_l$ as a Lie algebra, we see that (6.12) holds.

It follows from (6.12) that $a_j(x) = a_j(x_j)$. Thus (6.11) implies that $\chi_{E_j}(x) = \chi_{E_j}(x_j)$ for all $j \in K_i$. Since $\sum_{j \in K_i} \chi_{E_j} = 1$ a.e. there exists one $j_0$ such that $\chi_{E_{j_0}} = 1$ almost everywhere. Thus $\sigma^{-1}(i) = j_0$ almost everywhere. Summarizing, we have shown that for all $i$ the function $\sigma^{-1}(x)(i)$ is constant almost everywhere. Hence $\sigma$ is constant almost everywhere. $\square$
The proof of Corollary 6.4 is very similar to the proof of Theorem 6.1. The main change is that instead of the closed codegree one forms $i_Y \text{vol}_{G_i}$ of weight $-\nu_i + 1$ we pull back certain closed two-forms of weight $-2$ in $G_i$. To identify suitable two-forms we use the setting in [KMX21b]. For a Carnot algebra $g = \oplus_{i=1}^s V_i$, let $\Lambda^i g$ denote the space of one-forms which vanish on the first layer $V_1$ and let $I^*g \subset \Lambda^*g$ be the differential ideal generated by $\Lambda^i_g$. Thus

$$
(6.13) \quad I^*g = \text{span}\{\alpha \wedge \tau + \beta \wedge d\eta : \alpha, \beta \in \Lambda^*g, \tau, \eta \in \Lambda^1_g\}.
$$

The set of $k$-forms in $I^*g$ is denoted by $I^k g = I^*g \cap \Lambda^k g$. We define $J^*g$ to be the annihilator $\text{Ann}(I^*g)$ of $I^*g$, i.e.,

$$
(6.14) \quad J^*g = \{\alpha \in \Lambda^*g : \alpha \wedge \beta = 0 \quad \forall \beta \in I^*g\}.
$$

We will use the following facts about $I^*g$ and $J^*g$ which easily follow from exterior algebra and the formula for $d\alpha$. For the convenience of the reader we include a proof after the proof of Corollary 6.4.

**Proposition 6.15.** Let $g = \oplus_{i=1}^s V_i$ be a Carnot algebra of dimension $N$, homogeneous dimension $\nu$ and step $s \geq 2$. Let $G$ be the corresponding Carnot group. Then the following assertions hold.

1. For all $0 \leq k \leq N$ we have

$$
J^{N-k}g = \text{Ann}(I^k g) \cap \Lambda^{N-k} g = \{\alpha \in \Lambda^{N-k} g \mid \alpha \wedge \beta = 0, \forall \beta \in I^k g\}.
$$

2. If $I^k g = \Lambda^k g$ then $J^k g = \{0\}$. If $I^k g \neq \Lambda^k g$ then the wedge product induces a nondegenerate pairing

$$
\Lambda^k g/I^k g \times J^{N-k}g \xrightarrow{\wedge} \Lambda^N g \simeq \mathbb{R}.
$$

In particular, $\dim \Lambda^k g/I^k g = \dim J^{N-k}g$ and for each basis $\{\tilde{\alpha}_i\}$ of $\Lambda^k g/I^k g$ there exists a dual basis $\{\tilde{\gamma}_j\}$ of $J^{N-k}g$ such that $\tilde{\alpha}_i \wedge \tilde{\gamma}_j = \delta_{ij} \text{vol}_G$.

3. $J^*g$ is a differential ideal, i.e. $\alpha \in J^k g \implies d\alpha \in J^{k+1}g$.

4. If $g'$ is another Carnot algebra and $\Phi : g \rightarrow g'$ is a graded isomorphism then $\Phi^*(I^k g') = I^k g$ and $\Phi^*$ induces an isomorphism from $\Lambda^k g/I^k g$ to $\Lambda^k g'/I^k g'$.

5. If $\gamma \in J^k g \setminus \{0\}$ then $\gamma$ is homogeneous with coweight equal to its codegree, i.e. $\text{wt}(\alpha) = N - k - \nu$.

6. If $\gamma \in J^k g$ then $d\gamma = 0$.

The main new ingredient in the proof of Corollary 6.4 is the following simple observation. For a Carnot algebra $g = \oplus_{i=1}^s V_i$ with $s \geq 3$ let $\pi_{1,2}$ denote the projection $g \rightarrow V_1 \oplus V_2$. We define $\tilde{g} := g/\oplus_{j=3}^s V_j$ as the algebra $V_1 \oplus V_2$ with bracket $[X,Y]_\sim = \pi_{1,2}[X,Y]$. Then $\tilde{g}$ is a Carnot algebra. If $s = 2$, we set $\tilde{g} = g$. 
Proposition 6.16. If $g/\oplus_{j=3}^s V_j$ is not a free Carnot algebra then $I^2(g) \neq \Lambda^2(g)$. Moreover

$$
\Lambda^2 g/I^2 g \simeq (\Lambda^1_h g \wedge \Lambda^1_v g)/(I^2 g \cap (\Lambda^1_h g \wedge \Lambda^1_v g))
$$

where $\Lambda^1_h g$ denotes the space of horizontal one-forms, i.e. one-forms which vanish on $\oplus_{i=2}^s V_i$.

Proof. Since $\Lambda^1 g = \Lambda^1_h g \oplus \Lambda^1_v g$ we have

$$
\Lambda^2 g = (\Lambda^1_h g \wedge \Lambda^1_v g) \oplus (\Lambda^1_v g \wedge \Lambda^1_h g) \oplus (\Lambda^1_v g \wedge \Lambda^1_v g).
$$

Since the second and third summand on the right hand side are contained in $I^2 g$ we get (6.17).

Now assume that $g/\oplus_{j=3}^s V_j = V_1 \oplus V_2$ is not a free Carnot algebra. Let $\{X_i\}_{1 \leq i \leq \dim V_1}$ be a basis of $V_1$. Then there exist coefficients $\{a_{i,j}\}_{1 \leq i < j \leq \dim V_1}$ which do not all vanish such that

$$
\sum_{1 \leq i < j \leq \dim V_1} a_{i,j} [X_i, X_j] = 0.
$$

Since the $a_{i,j}$ do not all vanish there exists $\gamma \in \Lambda^1_h \wedge \Lambda^1_h$ such that

$$
\gamma(\sum_{i < j} a_{i,j} X_i \wedge X_j) \neq 0.
$$

We claim that $\gamma \notin I^2 g$. Otherwise there exist $\alpha \in \Lambda^1$ and $\tau, \eta \in \Lambda^1_v$ such that

$$
0 \neq (\alpha \wedge \tau + d\eta) \left( \sum_{i < j} a_{i,j} X_i \wedge X_j \right) = -\eta \left( \sum_{i < j} a_{i,j} [X_i, X_j] \right).
$$

This contradiction concludes the proof. \qed

Proof of Corollary 6.4. Again we may assume without loss of generality that $|K_i| \geq 2$ for all $i$, that $n = n'$ and that $g'_i = g_i$. Then $G'$ is a step $\bar{m}$ group.

By Propositions 6.15 and 6.16 there exist horizontal two-forms $\alpha_{j,k} \in \Lambda^1_h \wedge \Lambda^1_h$ such that $\alpha_{j,k} + I^2 g_j$ is a basis of $\Lambda^2 g_j/I^2 g_j$ and there exist dual bases $\gamma_{j,k'}$ of $J^{N_j-2} g_j$ such that

$$
\alpha_{j,k} \wedge \gamma_{j,k'} = \delta_{kk'} \text{vol}_{G_j}.
$$

Note also that forms in $\Lambda^1 h \wedge \Lambda^1 h$ are closed since forms in $\Lambda^1_h$ are closed.
Now we can proceed as in the proof of Theorem \[6.1\] Let \(i \in \{1, \ldots, n\}\). Then, for a.e. \(x \in U\) we have \((D_P f)^*(x) \alpha_{i,1} \in \Lambda^1_h(g_{\sigma-1}(i)) \wedge \Lambda^1_h(g_{\sigma-1}(i))\). Thus
\[
f_P^* \alpha_{i,1} = \sum_{j \in K_i} \sum_{k} (a_{j,k} \alpha_{j,k} + \beta_k)
\]
with \(a_{j,k} \in L^1_{\text{loc}}(U)\) and \(\beta_k \in L^1_{\text{loc}}(U; P^2 g_j)\). Set
\[
a_j := (a_{j,1}, \ldots, a_{j,\dim(\Lambda^2(g_j)/P^2 g_j)})
\]
and
\[
E_j = \{x \in U : (\sigma(x))^{-1}(i) = j\}.
\]
Then \(U \setminus \bigcup_{j \in K_i} E_j\) is a null set. By Proposition \[6.15\] we have, for a.e. \(x \in U\), \((D_P f)^*(x) \alpha_{i,1} \notin P^2 g_{\sigma-1}(i)\). Thus, for all \(j \in K_i\),
\[
(6.19) \quad a_j \neq 0 \quad \text{a.e. in } E_j, \quad a_j = 0 \quad \text{a.e. in } U \setminus E_j.
\]

We next show that
\[
(6.20) \quad Z a_j = 0 \quad \text{in distributions for all } j \in K_i, \ Z \in \oplus_{j' \neq j} g_{j'}.
\]

To prove \[6.20\], let \(j' \in K_i\), \(l \in \{1, \ldots, n\} \setminus \{j'\}\), \(X \in V_1(G_l)\), and consider the closed form
\[
\beta = \gamma_{j',k'} \wedge i_X \operatorname{vol}_{G_l} \wedge \Lambda_{\nu-3} \operatorname{vol}_{G_{j'}}.
\]
Then \(\beta\) is a closed form of degree \(N - 3\) and weight \(\nu + 3\), where \(N\) and \(\nu\) are the topological and homogeneous dimension of \(G\), respectively. Moreover
\[
\alpha_{j,k} \wedge \beta = \pm \delta_{jj'} \delta_{kk'} i_X \operatorname{vol}_{G_l}.
\]
Finally, \(\alpha_{i,1}\) is a closed left-invariant two-form of weight \(-2\). Thus the conditions on \(p\) in Corollary \[6.4\] allow us to apply the Pullback Theorem, Theorem \[4.22\], and we get
\[
0 = \int_U f_P^* \alpha_{i,1} \wedge \beta \wedge d\varphi = \pm \int_U a_{j',k'} X \varphi \operatorname{vol}_{G_l}.
\]
for all \(\varphi \in C^\infty_c(U)\), all \(j' \in K_i\), and all \(k'\). As in the proof Theorem \[6.1\] we conclude that \(a_{j,k}(x_1, \ldots, x_n)\) depends only on \(x_j\) and that \(\sigma\) is constant almost everywhere. \qed

Looking back, we see that the arguments in the proofs of Theorem \[6.1\] and Corollary \[6.4\] are exactly analogous. The only difference is that we use different forms in the ideals \(I^*\) and \(J^*\) as forms to be pulled back and as test forms. Indeed, note that \(I^1 g_i = \Lambda^1 g_i\) and thus \(\Lambda^1 g_i/I^1 g_i \simeq \Lambda^1_h g_i\) and \(J^1 g_i = \{i_X \operatorname{vol}_{G_l} : X \in \Lambda^1_h g_i\}\). Thus in the proof of Theorem \[6.1\] we pull back a form in \(J^1 g_i\) and use test forms of the type \(\Lambda^1 g_{j'}/I^1 g_{j'} \wedge i_X \operatorname{vol}_{G_l} \wedge \Lambda_{\nu-3 \setminus \{j',l\}} \operatorname{vol}_{G_{j'}}\), while for the proof of
Corollary 6.4 we pull back forms in $\Lambda^2 g_i/I^2 g_i$ and use test forms of the type $J^2 g_j \wedge i_X \text{vol}_{g_i} \wedge \Lambda_{j' \neq j} \text{vol}_{g_{j'}}$.

We finally provide a proof of Proposition 6.15 for the convenience of the reader.

Proof of Proposition 6.15. Since $I^k g \subset I^* g$ we have $\text{Ann}(I^k g) \cap \Lambda^{N-k} g \supset \text{Ann}(I^* g) \cap \Lambda^{N-k} g$. To establish the opposite inclusion, choose $\alpha \in \text{Ann}(I^k g) \cap \Lambda^{N-k} g$, and $\beta \in I^* g$ for some $0 \leq j \leq N$. If $k < j \leq N$ we have $\alpha \wedge \beta = 0$ since $\deg \alpha + \deg \beta > N$. If $0 \leq j \leq k$ and $\gamma \in \Lambda^{k-j} g$ we have $\beta \wedge \gamma \in I^k g$ since $I^* g$ is an ideal, so

(6.21) 
$$ (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) = 0 $$

because $\alpha \in \text{Ann}(I^k g)$ by assumption. The pairing $\Lambda^{N-(k-j)} g \times \Lambda^{k-j} g \rightarrow \Lambda^N g$ is nondegenerate and $\gamma \in \Lambda^{k-j} g$ was arbitrary, so (6.21) implies that $\alpha \wedge \beta = 0$. Since $\beta \in \Lambda^j g$ was arbitrary, we have $\alpha \in \text{Ann}(I^j g)$. Since $j$ was arbitrary we get $\alpha \in \cap_{0 \leq j \leq N} \text{Ann}(I^j g) = \text{Ann}(I^* g)$.

(2) Since the wedge product $\Lambda^k g \times \Lambda^{N-k} g \rightarrow \Lambda^N g \simeq \mathbb{R}$ is a nondegenerate pairing, (2) follows from (1) and the fact that for any nondegenerate pairing $E \times E' \rightarrow \mathbb{R}$ of finite dimensional vector spaces and any subspace $W \subset E$, there is a nondegenerate pairing $(E/W) \times W^\perp \rightarrow \mathbb{R}$ induced by $b$, where $W^\perp := \{e' \in E' \mid b(e, e') = 0, \forall e \in W\}$.

(3) This follows from the fact that $I$ is a differential ideal and the graded Leibniz rule.

(4) This follows from the facts that $\Phi^*(\Lambda^1 g') = \Lambda^1 g$ and that $d$ commutes with pullback by $\Phi$.

(5) Let $\{X_{i,j}\}$, $i = 1, \ldots, s$, $j = 1, \ldots, \dim V_i$ be a graded basis of $g$, i.e., $X_{i,j} \in V_i$. Let $\theta_{i,j}$ be the dual basis of one-forms. Then the forms $\theta_{i,j}$ are homogeneous with $\text{wt}(\theta_{i,j}) = -i$. Moreover

$$ \Lambda^1_{h} g = \text{span}\{\theta_{1,j} : 1 \leq j \leq \dim V_1\}, $$

$$ \Lambda^1_{v} g = \text{span}\{\theta_{i,j} : i \geq 2, 1 \leq j \leq \dim V_i\}. $$

Set $\tau = \Lambda_{\geq 2,1 \leq j \leq \dim V_i} \theta_{i,j}$. It is easy to see that for $k \geq N - \dim V_1$ every $\gamma \in J^k g$ is of the form

$$ \gamma = \alpha \wedge \tau, \quad \text{with } \alpha \in \Lambda^{k-(N-\dim V_1)}(\Lambda^1_{h} g), $$

and $J^k g = \{0\}$ if $k < N - \dim V_1$. Thus every non-zero element of $J^k g$ is a homogeneous form with weight $-\nu + (N-k)$.

(6) Let $\gamma \in J^k g$ and assume that $d\gamma \neq 0$. By properties (3) and (5) the form $\gamma$ is homogeneous and

(6.22) 
$$ \text{wt}(d\gamma) = \text{wt}(\gamma) - 1. $$
On the other hand for $\gamma \in \Lambda ^k \mathfrak{g}$ we have

$$d\gamma (X_0, \ldots , X_k) = \sum_{0 \leq i < j \leq k} \gamma ([X_i, X_j], X_0, \ldots , \hat{X}_i, \ldots , \hat{X}_j, \ldots , X_k)$$

where $\hat{X}_j$ denotes that $X_j$ is omitted (see, for example, [Mic08, Lemma 14.14]). If $\gamma$ is homogeneous and $d\gamma \neq 0$ it follows that $d\gamma$ is homogeneous and $\text{wt}(d\gamma) = \text{wt}(\gamma)$. This contradicts (6.22). Hence $d\gamma = 0$. □

7. Complexified Carnot algebras

In [KMX20] it was shown that under suitable conditions nondegenerate Sobolev maps of a complexified Carnot group are automatically holomorphic or antiholomorphic. In this section show these results with improved conditions on the Sobolev exponent.

We first recall the setting in [KMX20] to which we refer for further details. Let $H$ be a Carnot group of topological dimension $N$ and homogeneous dimension $\nu$. Let $\mathfrak{h}$ be the corresponding Carnot algebra. Let $\mathfrak{g}$ denote the complexified Carnot algebra, i.e. $\mathfrak{g} = \mathfrak{h}^C$ equipped with the grading $\mathfrak{g} = \bigoplus_j V_j^C$. The corresponding Carnot group $G$ has topological dimension $2N$ and homogeneous dimension $2\nu$. We now denote by $J$ the almost complex structure on $G$ coming from $\mathfrak{g}$; it follows from the Baker-Campbell-Hausdorff formula that $J$ is integrable, i.e. $(G, J)$ is a complex manifold, and the group operations are holomorphic. Also, complex conjugation $\mathfrak{g} \to \mathfrak{g}$ is induced by a unique graded automorphism $G \to G$, since $G$ is simply-connected.

**Theorem 7.1.** Let $U \subset G$ be a connected open subset, let $p \geq \nu$, let $f \in W^{1,p}_{\text{loc}}(U, G)$ and assume that (approximate) Pansu differential $D Pf (x)$ is either a $J$-linear isomorphism or a $J$-antilinear graded isomorphism for a.e. $x \in U$. Then $f$ is holomorphic or antiholomorphic (with respect to the complex structure $J$).

**Corollary 7.2.** Let $U \subset G$ be open, let $p \geq \nu$, let $f \in W^{1,p}_{\text{loc}}(U, G)$. Suppose that any graded isomorphism $\mathfrak{g} \to \mathfrak{g}$ is either $J$-linear or $J$-antilinear and that $D Pf (x)$ is an isomorphism for a.e. $x \in U$. Then $f$ is holomorphic or antiholomorphic (with respect to the complex structure $J$).

The condition that any graded isomorphism $\mathfrak{g} \to \mathfrak{g}$ is either $J$-linear or $J$-antilinear is in particular satisfied for the complexified Heisenberg algebras $\mathfrak{h}^C_m$, see [RR00, Section 6].

**Proof of Theorem 7.1.** The result is proved in [KMX20] under the stronger condition $p > \text{homogeneous dimension of } G = 2\nu$. The key step in the proof is to show that $D Pf$ cannot switch between a $J$-linear and a
J-antilinear map. To show this we use the Pullback Theorem to prove that the pullback of the top degree holomorphic form cannot oscillate between a holomorphic and an anti-holomorphic form. Since the top degree holomorphic form has weight $-\nu$, the improved version of the Pullback Theorem, Theorem 4.24, gives this result already under the weaker condition $p \geq \nu$. Note that $G$ is a group of step $m$ with $m < \nu$ so the condition $\frac{1}{p} \leq \frac{1}{m} + \frac{1}{\nu}$, where $\nu = 2\nu$ is the homogeneous dimension of $G = H^C$, is automatically satisfied.

Thus, under the assumption $p \geq \nu$, we still may assume that $D_P f$ is $J$-linear a.e. (the case that $D_P f$ is $J$-antilinear a.e. being analogous). Let $\pi_G : G \to G/[G,G]$ denote the abelianization map. By Remark 2.46 the map $\pi_G \circ f$ belongs to $W^{1,p\text{loc}}$ and for each horizontal vectorfield $X$ the weak derivative satisfies $X(\pi_G \circ f)(x) = D_P f(x)X$ for a.e. $x \in U$. Since $D_P f(x)$ is $J$-linear, the horizontal anticonformal derivatives $\bar{Z}((\pi_G \circ f)$ vanish. It follows easily that $\pi_G \circ f$ is holomorphic (see, e.g., [KMX20]). In particular $\pi_G \circ f$ is smooth and hence $|D_P f(x)| := \max\{|D_P f(x)X_\theta| : |X_\theta| \leq 1, X \in V_1\}$ is locally bounded. By (2.44), it follows that $f \in W^{1,\infty}_{\text{loc}}(U;G)$. Thus the assertion follows from the result in [KMX20] for $p > 2\nu$.

Proof of Corollary 7.2. This follows immediately from Theorem 7.1.

Appendix A. $L^{p*}$-Pansu-differentiability

Here we give another proof of the following result by Vodopyanov.

Theorem A.1 ($L^{p*}$ Pansu differentiability a.e., [Vod03], Corollary 2). Let $U \subset G$ be open, let $1 \leq p < \nu$ and define $p^*$ by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}$. Let $f \in W^{1,p}(U;G')$. For $x \in U$ consider the rescaled maps $f_{x,r} = \delta_{r^{-1}} \circ f_{(x)}^{-1} \circ f \circ \ell_x \circ \delta_r$.

Then, for a.e. $x \in U$, there exists a graded group homomorphism $\Phi : G \to G'$ such that

$$f_{x,r} \to \Phi \quad \text{in } L^{p^*}_{\text{loc}}(G;G') \text{ as } r \to 0.$$  

We write $D_P f(x) = \Phi$ for the $L^{p*}$ Pansu derivative and we use the same notation to denote the corresponding graded Lie algebra homomorphism given by $D\Phi(e)$. 


Remark A.3. It follows easily from Theorem A.1 that for all $z \in G'$ the functions $u_z := d'(z, f(\cdot))$ satisfy

$$|D_h u_z| \leq |D_P f| \text{ a.e., where}$$

$$|D_P f(x)| = \max \{|D_P f(x)X|_{V_i} : X \in V_1, |X|_{V_i} \leq 1\}.$$  

Here $| \cdot |_{V_1}$ and $| \cdot |_{V'_1}$ denote the norms induced by the scalar product on the first layer of $g$ and $g'$, respectively. Thus the condition (2.30) in Definition 2.29 holds with $g = |D_P f|$. We provide a proof after the proof of Theorem A.1.

Remark A.6. It follows from Step 2 of the proof of Theorem A.1 that $f_{x,r}$ converges to $\Phi$ in the stronger sense that the horizontal distributional derivatives of the composition $\pi_{G'} \circ f_{x,r}$ converge in $L^p_{\text{loc}}$ to a constant function, where $\pi_{G'} : G' \to G'/[G',G']$ is the abelianization map; more precisely, for a.e. $x \in U$, and a basis $X_i$ of the space of left-invariant horizontal vectorfields

$$X_i(\pi_{G'} \circ f_{x,r}) \rightarrow g_i(x) \quad \text{ in } L^p_{\text{loc}} \text{ where } g_i = X_i(\pi_{G'} \circ f)$$

is the weak horizontal derivative of $\pi_{G'} \circ f$ in direction $X_i$. It follows from (A.2) that $\pi_{G'} \circ f_{x,r} \to \pi_{G'} \circ \Phi$ in $L^p_{\text{loc}}$ and thus

$$g_i(x) = X_i(\pi_{G'} \circ \Phi) \quad \text{on } G$$

Evaluating the right hand side at $e \in G$ and viewing $D_P f(x)$ as a map from $g$ to $g'$ given by $D\Phi(e)$ we get

$$g_i(x) = D_P f(x)X_i.$$ 

Here we used the fact that $D_P f(x)$ maps the first layer of $g$ to the first layer of $g'$ and that $D\pi_{G'}(e)$ is the identity map on the first layer of $g'$. It follows that for every horizontal vectorfield $X$ on $G$ the weak derivative of $(\pi_{G'} \circ f)$ in direction $X$ satisfies

$$X(\pi_{G'} \circ f)(x) = D_P f(x)X \quad \text{for a.e. } x.$$ 

We give the proof using the distributional definition of Sobolev spaces, see Definitions 2.26 and 2.29. The proof uses only the Poincaré-Sobolev inequality (which easily implies compactness of the Sobolev embedding, see Appendix B) and a characterization of group homomorphism by their abelianization (see Lemma A.16 below). Thus the proof applies verbatim if one instead uses the definition of Sobolev spaces by upper gradients, if one replaces the function $g$ in Definition 2.29 by an upper gradient; see [HKST15, Theorem 9.1.15] for the Poincaré-Sobolev
inequality for Sobolev spaces defined by upper gradients. For the convenience of the reader a short discussion of Sobolev spaces defined by upper gradient is given in Appendix C.

The strategy of the proof is as follows. Denote by \( \pi_{G'} : G' \to G' / [G', G'] \approx V_1 \) the abelianization map. By Proposition 2.31 we have \( \pi_{G'} \circ f \in W^{1,p}(U; G' / [G', G']) \). Let \( x \) be a Lebesgue point of \( f \), the function \( g \) in Definition 2.29 and of the weak derivatives \( g_i = X_i(\pi_{G'} \circ f) \).

1. By the compact Sobolev embedding, Theorem B.1 (2), a subsequence \( f_{x,r} \) converges to a Lipschitz map \( \hat{f} \) in \( L^p_{loc} \) with \( \hat{f}(e) = e \).
2. The whole sequence \( \pi_{G'} \circ f_{x,r} \) converges to a linear map \( \hat{u} \), i.e., the weak horizontal derivatives of \( \hat{u} \) are constant.
3. If \( F : G \to G' \) is Lipschitz, \( F(e) = e \) and \( \pi_{G'} \circ F \) is a linear map, then \( F \) is a group homorphism and \( F \) is uniquely determined by \( \pi_{G'} \circ F \); see Lemma A.16 below.
4. Uniqueness of the limit implies that the full sequence \( f_{x,r} \) converges in \( L^p_{loc} \).

We begin by recalling some properties of the abelianization map.

Proposition A.9. Let \( G \) be a Carnot group, equipped with the Carnot-Caratheodory distance, with graded Lie algebra \( \mathfrak{g} = \oplus_{i=1}^m \mathfrak{g}_i \). Then the abelianization homomorphism \( \pi = \pi_G : G \to G / [G, G] \) has the following properties.

1. The map \( \pi \) is graded, i.e., \( \pi(\delta_r g) = r \pi(g) \), and the restriction of \( d\pi(g) \) to the horizontal subspace of \( T_g G \) is an isomorphism onto \( G / [G, G] \). Thus for every \( Y \in G / [G, G] \) there exists a unique horizontal vectorfield \( Z \) on \( G \) with \( d\pi(g)Z(g) = Y \). Moreover \( Z \) is left-invariant.
2. The map \( \pi \) is 1-Lipschitz if \( G / [G, G] \) is equipped with the induced metric \( d_\pi(\pi(a), \pi(b)) := \min_{g, g' \in [G, G]} d_{CC,G}(ag, bg') \). Moreover, the induced metric comes from a norm: \( d_\pi(a', b') = |a' - b'| \).

Proof. (1) By definition, the commutator subgroup \( [G, G] \) is the closure of the set of finite products of commutators \( [xy] = x^{-1}y^{-1}xy \). In particular \( [G, G] \) is a normal subgroup. If \( G \) is a Carnot group then \( [G, G] \) is a Lie subgroup with \( T_e[G, G] = [\mathfrak{g}, \mathfrak{g}] = \oplus_{j=2}^m \mathfrak{g}_j \). Thus \( \ker d\pi(e) = \oplus_{j=2}^m \mathfrak{g}_j \) and hence the restriction of \( d\pi(e) \) to the horizontal subspace \( \mathfrak{g}_1 \) is an isomorphism onto \( G / [G, G] \) (note that we can identify \( T_g G / [G, G] \) and \( G / [G, G] \) since \( G / [G, G] \) is abelian). In particular
$d\pi(e)$ is graded and hence $\pi$ is graded. Since $\pi$ is a group homomorphism we have $d\pi(g) \circ (\ell_g)_* = d\pi(e)$ and the remaining assertions easily follow from this identity.

(2) The fact that $\varphi$ is 1-Lipschitz follows directly from the definition of the induced metric. Since the Carnot-Caratheodory distance is left-invariant, so is the induced metric $d_\pi$. Thus $d_\pi(a, b) = N(a - b)$ since $G/[G, G]$ is abelian. The fact that $\pi$ is graded implies the $N$ is 1-homogeneous and hence a norm.

To show that the limit map $\hat{f}$ is Lipschitz and to ensure that the normalisation $f_{x,r}(e) = e$ implies that $\hat{f}(e) = e$ we use the following facts, which are a simple consequence of the Poincaré inequality.

**Proposition A.10.** Let $X'$ be a complete, separable metric space. Let $U \subset G$ be open, let $f \in W^{1,p}(U; X')$ and let $g : U \to \mathbb{R}$ be the common bound for the weak derivatives of the maps $x \mapsto d(z, f(x))$ in Definition 2.22.

(1) Suppose that $x$ is $p$-Lebesgue point of $f$ and $g$. Then there exists a constant $C_x < \infty$ such that, for all $0 < s < \text{dist}(x, U)$,

$$\int_{B(x,s)} (d')^p(f(y), f(x)) d\mu \leq C_x s^p \quad \text{(A.11)}$$

(2) If $g$ is bounded by $L$ a.e on $B(x, R)$ then $f$ has a representative which is $CL$-Lipschitz in $B(x, R/5)$.

**Proof.** (1) Consider the local maximal function of $g^p$ at $x$:

$$M_x = M_{x,R} := \sup_{0 < r < R} \int_{B(x,r)} g^p(y) \mu(dy).$$

Since $x$ is a $p$-Lebesgue point of $g$ we have $M_x < \infty$. By the Poincaré inequality, for every $r \in (0, R]$ there exists a point $f_r \in X'$ such that

$$\int_{B(x,r)} (d')^p(f(y), f_r) \mu(dy) \leq C r^p M_x \quad \text{(A.12)}$$

By the triangle inequality we have $d'(f_r, f_{r/2}) \leq d'(f(y), f_r) + d'(f(y), f_{r/2})$. Integrating this estimate over $B(x, r/2)$ and using that $\mu(B(x, r)) = 2^r \mu(B(x, r/2))$ we deduce that

$$d'(f_r, f_{r/2}) \leq C r M_x^{1/p}.$$ 

Thus the sequence $i \mapsto f_{2^{-i}}$ is a Cauchy sequence and hence has a limit $f_0$ and $d'(f_r, f_0) \leq C r M_x^{1/p}$. Now $x$ is a $p$-Lebesgue point of $f$ and it follows from (A.12) that $f_0 = f(x)$. This gives the desired estimate with $C_x = C M_x$. 

(2) Since $g$ is bounded we have $M_{x,s} \leq L^p$ for all $x \in B(x, R/5)$ and all $s \leq \frac{4}{5}R$. Let $x_1, x_2 \in B(x, R/5)$ be $p$-Lebesgue points of $f$ and set $r = d'(x_1, x_2)$. Then $r \leq \frac{2}{5}R$. Application of assertion (1) with $C_x = CM_x$ gives for $i = 1, 2$

$$
\int_{B(x, 2r)} (d')^p(f(y), f(x_i))d\mu \leq CL^pr^p.
$$

Now one can integrate the inequality $d'(f(x_1), f(x_2)) \leq d'(f(y), f(x_1)) + d'(f(y), f(x_2))$ over $B(x_1, r) \subset B(x_1, 2r) \cap B(x_2, 2r)$ and use that the measure of the ball $B(x_1, s)$ is $\mu(B(x_1, s)) = s'$ to deduce that

$$
d'(f(x_1), f(x_2)) \leq CLr
$$

Since the Lebesgue points are dense in $B(x, R/5)$ there exists a unique $CL$-Lipschitz function $\tilde{f}$ on $B(x, R/5)$ which agrees with $f$ at all Lebesgue points. 

**Proof of Theorem A.1.** Since the assertion is local we may assume that $U$ is bounded. Let $g : U \to \mathbb{R}$ be the common bound for the weak derivatives of the maps $x \mapsto d'(z, f(x))$ as in Definition [2.29]. Let $u = \pi_{G'} \circ f$. Since $G'/[G', G']$ is a linear space, $\pi_{G'}$ is $1$-Lipschitz and $U$ is bounded, it follows from Proposition [2.31] that $u \in W^{1, p}(U; G'/[G', G'])$. For $j = 1, \ldots, K$ let $g_j = X_j u$ denotes the weak derivatives. Let $x$ be a Lebesgue point for $f, g$ and $g_1, \ldots, g_K$. Fix a ball $B(e, R)$.

Step 1: There exists a subsequence $r_j \to 0$ such that $f_{x, r_j} \to \hat{f}$ in $L^{p^*}(B(e, R), G')$. Moreover $\hat{f} \in W^{1, p}(B(e, R); G')$ and $\hat{f}$ has a representative which is Lipschitz in $B(e, R/5)$ and satisfies $\hat{f}(e) = e$.

Set

$$
G_{x,r} := g \circ \ell_x \circ \delta_r
$$

and let $z \in G'$. It follows directly from the behaviour of the Carnot-Caratheodory metric on $G'$ under left-translation and dilation that $|D_p d'(z, f_{x, r}(\cdot))| \leq G_{x,r}$ in $B(e, R)$ (as long as $Rr < \text{dist}(x, G \setminus U)$).

Since $x$ is a Lebesgue point of $g$ the sequence $G_{x,r}$ converges to a constant:

$$
G_{x,r} \to g(x) \quad \text{in} \quad L^p(B(e, R)). \tag{A.13}
$$

Applying (A.11) to $f$ and $B(x, \rho r)$ and unwinding definitions we see that for all $0 < \rho \leq R$:

$$
\int_{B(e, \rho)} (d')^p(f_{x, r}(y), e) \, d\mu \leq C_x \rho^p. \tag{A.14}
$$

Taking $\rho = R$ and using in addition (A.13) we can apply the compact Sobolev embedding, Theorem [B.1] [2]. Thus there exists a subsequence
$r_j \to 0$ and a map $\hat{f} \in W^{1,p}(B(e, R); G')$ such that

$$f_{x,r_j} \to \hat{f} \text{ in } L^{p^*}(B(e, R); G') \text{ as } j \to \infty.$$ 

In particular $d'(z, f_{x,r_j}(\cdot)) \to d'(z, \hat{f}(\cdot))$, for all $z \in G'$, and it follows from the $L^p$ convergence of $G_{x,r}$ that $|D_h d'(z, \hat{f}(\cdot))| \leq g(x)$.

Thus by Proposition A.10 (2) the map $\hat{f}$ has a representative which is $C_{g(x)}$-Lipschitz in $B(e, R/5)$. Passing to the limit in (A.14) we see that

$$\int_{B(e, \rho)} (d')^p(\hat{f}(y), \epsilon) \, d\mu \leq C_{x,\rho}.$$

Thus a representative of $\hat{f}$ which is $C_{g(x)}$-Lipschitz in $B(e, R/5)$ satisfies $\hat{f}(e) = e$.

Step 2: The functions $\pi_{G'} \circ f_{x,r}$ converge in $L^{p^*}(B(e, R); G'/[G', G'])$ to a linear map $\hat{u}$, i.e. $\hat{u}(e) = 0$ and the (weak) horizontal derivatives of $\hat{u}$ are constant.

Set $u = \pi_{G'} \circ f$ and define $u_{x,r}$ like $f_{x,r}$, i.e., $u_{x,r} = \delta_r^{-1} \circ \ell_{u(x)}^{-1} \circ u \circ \ell_x \circ \delta_r$. Since the target is abelian this can actually be written in the more conventional form:

$$u_{x,r}(y) = \frac{u(x \delta_r y) - u(x)}{r}.$$

Since $\pi_{G'}$ is a graded group homomorphism it follows that $\pi_{G'} \circ f_{x,r} = u_{x,r}$. Since $\pi_{G'}$ is a Lipschitz map we get

$$u_{x,r_j} \to \hat{u} = \pi_{G'} \circ \hat{f} \text{ in } L^{p^*}(B(e, R); G'/[G', G']) \text{ as } j \to \infty.$$

In particular $\hat{u}$ has a representative which is $C_{g(x)}$-Lipschitz in $B(0, R/5)$ and satisfies $\hat{u}(e) = 0$.

In addition, we have

$$X_j u_{x,r} = g_j \circ \ell_x \circ \delta_r$$

and thus $X_j u_{x,r}$ converges to a constant:

$$X_j u_{x,r} \to g_j(x) \text{ in } L^p(B(e, R)).$$

It follows directly from the definition of weak derivatives that $\hat{u}$ has constant weak horizontal derivatives which are given by $X_j \hat{u} = g_j(x)$. Since $\hat{u}(e) = 0$ it follows (e.g., from the Poincaré inequality) $\hat{u}$ is uniquely determined by $g_j(x)$. Thus the whole sequence $u_{x,r}$ converges to $\hat{u}$.

Step 3: Conclusion.

Apply Steps 1 and 2 on balls $B(e, R)$ with $R = 1, 2, \ldots$. For each $R$
choose a further subsequence in Step 1 and finally choose a diagonal sequence. Thus we find a single sequence $r_k \to 0$ such that
\[ f_{x,r_k} \to \hat{f} \text{ in } L^p_{\text{loc}}(G; G'). \]
Moreover $|D_{\theta} d'(z, \hat{f}(\cdot))|$ is bounded by the constant $g(x)$ for all $z \in G'$.

Thus $\hat{f}$ has a Lipschitz representative and we have already shown that this representative satisfies $\hat{f}(e) = e$. In combination with Step 2 we see that
\[ \pi_{G'} \circ \hat{f} = \hat{u} \]
where $\hat{u}$ has constant horizontal derivatives. Now Lemma A.16 below implies that $\hat{f}$ is a graded group homomorphism and $\hat{f}$ is uniquely determined by $\hat{u}$. Uniqueness implies that the full sequence $f_{x,r}$ converges to $\hat{f}$ in $L^p_{\text{loc}}(G)$.

\[ \square \]

Proof of (A.4). Fix $z \in G'$ and write $u = u_z$. Theorem A.1 and Remark A.6 also apply to the map $u : G \to \mathbb{R}$ and thus for a.e. $x \in U$ there exists a linear map $L_x : G \to \mathbb{R}$ (i.e. $L_x(e) = 0$ and the horizontal derivatives of $L$ are constant) such that the maps $u_{x,r}(y) := r^{-1}(u(x) - u(x))$ satisfy
\[ u_{x,r} \to L_x \text{ in } L^p_{\text{loc}}(G) \quad \text{and} \quad X_i L_x \equiv (X_i u)(x). \]
Here $X_i u$ denotes the weak horizontal derivatives. On the other hand the triangle inequality for $d'$ and the behaviour of $d'$ under left-translation and dilation imply that
\[ |u_{x,r}(y)| = \frac{|d'(z, f(x)(\delta_r y)) - d'(z, f(x))|}{r} \leq d(e, f_{x,r}(y)). \]
Passing to the limit $r_j \to 0$ we see that $|L_x(y)| \leq d(e, D_P f(x)(y))$ for a.e. $x \in U$. Taking $y = \exp tX$ and letting $t \to 0$ we obtain (A.4). \[ \square \]

To show that the map $\hat{f}$ constructed in the proof of Theorem A.1 is a graded group homomorphism, we introduce some notation. Let $X'$ be a finite-dimensional, normed, linear space. We say that a map $u : G \to X'$ is affine if for every left-invariant horizontal vector field $X$ the weak derivative $X u$ is constant. We say that a map $f : G \to G'$ is an $L$-map if $\pi_{G'} \circ f = L$ and $L$ is affine.

Lemma A.16. Let $f$ and $f'$ be Lipschitz $L$-maps.
\[ (1) \text{ If } f \text{ and } f' \text{ agree at one point then } f \equiv f'. \]
\[ (2) \text{ If } f(e) = e \text{ then } f \text{ is a graded group homomorphism.} \]
Proof. We first show that the second assertion is an immediate consequence of the first. Let \( f \) be an \( L \)-map with \( f(e) = e \). Since \( \pi_{G'} \) is a graded group homomorphism the maps \( F_a := \ell_{f(a)}^{-1} \circ f \circ \ell_a \) and \( F_r := \delta_{r^{-1}} \circ f \circ \delta_r \) are also \( L \)-maps and \( F_a(e) = F_r(e) = e \). Thus \( F_a = F_r = f \) and hence \( f \) is a graded group homomorphism.

Note that the closure of the group generated by \( \exp g_1 \) is \( G \). Thus to prove the first assertion it suffices to show the following implication:

\[
(A.17) \quad f(x_0) = f'(x_0) \implies f(x_0 \exp X) = f'(x_0 \exp X) \quad \forall X \in g_1.
\]

Let \( Y = XL \) be the constant horizontal derivative of \( L = \pi \circ f = \pi \circ f' \).

Consider the curves

\[
\gamma(t) = f(x_0 \exp tX), \quad \eta(t) = f'(x_0 \exp tX).
\]

Since \( t \mapsto x_0 \exp tX \) is a horizontal Lipschitz curve in \( G \) and since \( f \) and \( f' \) are Lipschitz, the curves \( \gamma \) and \( \eta \) are rectifiable curves in \( G' \) (where \( G' \) is equipped with the Carnot-Carathéodory metric) and hence differentiable a.e. with horizontal derivative, see [Pan89, Proposition 4.1].

Moreover

\[
X(\pi_{G'} \circ \gamma) = X(\pi_{G'} \circ \eta) = \frac{d}{dt}L(x_0 \exp tX) = Y.
\]

By Proposition A.9 there exists a unique horizontal vectorfield \( Z \) on \( G' \) such that \( d\pi'(g')Z(g') = Y \). Thus both \( \gamma \) and \( \eta \) are integral curves of \( Z \). Moreover \( Z \) is left-invariant and hence smooth. Since \( \gamma(0) = \eta(0) \) it follows that \( \gamma \equiv \eta \). Taking \( t = 1 \) we get (A.17). \( \square \)

**Appendix B. Compact Sobolev embeddings**

Here we give a proof of the compact Sobolev embedding. For scalar-valued maps it is observed in [GN96] that the compactness of the Sobolev embedding is an immediate consequence of the Poincaré-Sobolev inequality. The same reasoning applies to Sobolev maps with values in metric spaces and we provide the details for the convenience of the reader.

**Theorem B.1.** Let \( U \subset G \) be open, let \( X' \) be a metric space. Suppose that every closed ball in \( X' \) is compact. Let \( 1 \leq p < \nu \) and define \( p^* \) by

\[
\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}.
\]


Let \( f_k \in W^{1,p}_{\text{loc}}(B(x,r), X') \) and let \( g_k \in L^p(B(x,r)) \) be a common bound for the weak derivatives of the maps \( x \mapsto d(z, f_k(x)) \) as in Definition 2.29. Assume that, for some \( a \in X' \)

\[
\sup_k \|d'(f_k(\cdot), a)\|_{L^p(B(x,r))} + \|g_k\|_{L^p(B(x,r))} < \infty.
\]

Then:

1. there exists a subsequence \( f_{k_j} \) and a map \( f_\infty \) such that

\[
\begin{align*}
\int_{B(x,r)} |d'(f_{k_j}(\cdot), f_\infty(\cdot))| d\mu &< \infty \quad \text{for all } q < p^*, \\
i.e.,
\int_{B(x,r)} |d'(f_{k_j}(\cdot), f_\infty(\cdot))| d\mu &\to 0 \quad \text{in } L^q(B(x,r)) \quad \text{for all } q < p^*.
\end{align*}
\]

2. If, in addition, the sequence \( |g_k|^p \) is equiintegrable then the convergence also holds in \( L^{p^*}_{\text{loc}}(B(x,r)) \).

Recall that a sequence of \( L^1 \) functions \( h_k : U \to \mathbb{R} \) is equiintegrable if there exists a function \( \omega : [0, \infty) \to [0, \infty) \) with \( \lim_{t \to 0} \omega(t) = 0 \) such that for all measurable sets \( A \subset U \) one has \( \int_A |h_k| d\mu \leq \omega(\mu(A)) \). Note that if \( g_k \to g \) in \( L^p(U) \) then \( |g_k|^p \) is equiintegrable.

For the proof we use the following simple covering result.

**Proposition B.5.** Let \( X \) be a metric space with a doubling measure. Then there exists a constant \( C(X) \) with the following property. Let \( F \) be a family of balls of fixed radius \( s > 0 \) in \( X \). Then there exists a disjointed subfamily \( G \) such that

\[
\bigcup_{B \in F} B \subset \bigcup_{B \in G} 5B,
\]

where \( 5B \) denotes the concentric ball of five times the radius. Moreover each point is contained in at most \( C(X) \) of the balls \( 5B \) with \( B \in G \):

\[
\sum_{B \in G} 1_{5B} \leq C(X).
\]

**Proof.** The existence of a subfamily \( G \) is classical, see e.g. [HKST01], Theorem 1.2. To show (B.7) consider \( x \in X \) and let \( G_x = \{B \in G : x \in 5B\} \). Let \( B(a,s) \) be a ball in \( G_x \). Then \( d(x,a) \leq 5s \) and hence \( B(a,s) \subset B(x,6s) \subset B(a,11s) \). Since \( \mu \) is doubling there exist a \( C(X) > 0 \) such that \( \mu(B(x,6s)) \leq \mu(B(a,11s)) \leq C(X) \mu(B(a,s)) \). Since the balls in \( G_x \) are disjoint it follows that the number of ball in \( G_x \) is bounded by \( C(X) \).

**Proof of Theorem B.1.** We first show (1). The argument is essentially the same as in the scalar-valued case, see for example the proof of Theorem 1.28 in [CN96]. By the Poincaré inequality the functions
d(f_k(\cdot), a) are uniformly bounded in $L^{p*}(B(x, r))$. It thus suffices to show convergence in $L^p(B(x, r); X')$. Actually it suffices to show $L^p$ convergence for all $B(x, r')$ with $r' < r$ since the $L^p$ norm in $B(x, r) \setminus B(x, r')$ is small, uniformly in $r'$, if $r'$ is close to $r$. Indeed,

$$\int_{B(x, r) \setminus B(x, r')} d^p(f_k(y), a) \mu(dy) \leq \left(\mu(B(x, r) \setminus B(x, r'))\right)^{1 - \frac{p}{p'}} \|d'(f_k(\cdot), a)\|_{L^{p*}(B(x, r))}^{p}$$

and $\mu(B(x, r) \setminus B(x, r')) = r^{\nu} - (r')^{\nu}$.

Let $r' < r$. To show convergence of a subsequence in $L^p(B(x, r'); X')$ it suffices to show that for every $\varepsilon > 0$ there exist a compact subset $K$ of $L^p(B(x, r'); X')$ such that $\sup_k \text{dist}(f_k, K) \leq \varepsilon$. We will take $K$ as a set of functions which are piecewise constant on a fixed partition of $B(x, r')$. If the partition is taken sufficiently fine then the Poincaré inequality will guarantee that all members of the sequence are $\varepsilon$-close to $K$.

Set

$$M \doteq \sup_k \|d(f_k(\cdot), a)\|_{L^p(B(x, r))} + \|g_k\|_{L^p(B(x, r))}.$$  

Let $j$ be an integer with $j^{-1} < \frac{1}{10} (r - r')$. By Proposition B.3 there exist disjoint balls $B(x_i, j^{-1})$ such that the balls $B(x_i, 5j^{-1})$ cover $B(x, r')$, $B(x_i, 5j^{-1}) \subset B(x, r)$ and each point is contained in at most $C(G)$ of the balls $B(x_i, 5j^{-1})$. Since the balls $B(x_i, j^{-1})$ are disjoint, the collection of balls is finite. Define a partition of $B(x, r')$ recursively by

$$A_1 = B(x_1, 5j^{-1}) \cap B(x, r'), \quad A_{i+1} = B(x_{i+1}, 5j^{-1}) \cap B(x, r') \setminus \bigcup_{k=1}^{i} A_i.$$  

Let

$$\tilde{K}_j = \{f : B(x, r') \to X' : \text{for all } i \text{ the map } f \text{ is constant on } A_i, \forall i\}.$$  

Now the Poincaré inequality implies that there exist $h_k \in \tilde{K}_j$ such that

$$\int_{A_i} d^p(f_k(y), h_k(y)) \mu(dy) \leq \int_{B(x_i, 5j^{-1})} d^p(f_k(y), h_k(y)) \mu(dy) \leq C j^{-p} \int_{B(x_i, 5j^{-1})} g_k^p \mu(dy).$$

Summing over $i$ we see that

$$\text{dist}_{L^p}(f_k, \tilde{K}_j) \leq C(G)^{1/p} C j^{-1} M.$$  

Now set $K_j = \{h \in \tilde{K}_j : \|d'(h(\cdot), a)\|_{L^p(B(x, r'))} \leq M + 1\}$ and choose $j$ so large that $C(G)^{1/p} C j^{-1} M \leq \varepsilon < 1$. Then $K_j$ is compact in
\[ \text{dist}_{L^p}(f_k, K_j) \leq \text{dist}_{L^p}(f_k, \tilde{K}_j) \leq \varepsilon. \]

This finishes the proof of the first assertion.

To prove the second assertion, let \( A_i \subset U \) and \( \tilde{K}_j \) be as before. Let \( \omega : [0, \infty) \to [0, \infty) \) be the function in the definition of equi-integrability. The Sobolev-Poincaré inequality yields

\[
\text{osc}_{p^*}^{p^*}(f_k, A_i) \leq C \|g_k\|_{L^p(B(x_i, 5j^{-1}))}^{p^*} \leq C \omega^{p^* - 1}(\mu(B(x_i, 5j^{-1}))) \|g_k\|_{L^p(B(x_i, 5j^{-1}))}^p
\]

Summation over \( i \) yields

\[
\text{dist}_{L^{p^*}(B(x,r'))}(f_k, \tilde{K}_j) \leq \omega^{p^* - 1}(5^p j^{-p}) C(G) CM^p \quad \forall k \in \mathbb{N}
\]

Thus given \( \varepsilon \in (0, 1] \) there exists a \( j \) such that

\[
\text{dist}_{L^{p^*}(B(x,r'))}(f_k, \tilde{K}_j) \leq \varepsilon \quad \forall k \in \mathbb{N}.
\]

By the Sobolev-Poincaré inequality we have

\[
M' := \sup_k \|d(f_k(\cdot), a)\|_{L^p(B(x,r))} < \infty.
\]

Thus setting

\[
K_j = \{ h \in \tilde{K}_j : \|h\|_{L^{p^*}(B(x,r'))} \leq M' + 1 \}
\]

we see that \( K_j \) is compact and \( \text{dist}_{L^{p^*}}(f_k, K_j) = \text{dist}_{L^{p^*}}(f_k, \tilde{K}_j) \leq \varepsilon. \)

\[\square\]

**Appendix C. Sobolev spaces defined by upper gradients**

In this section we first recall the definition of a (weak) upper gradient, the Poincaré-Sobolev inequality for maps which possess a \( p \)-integrable \( p \)-weak upper gradient, and the stability of \( p \)-weak upper gradients under \( L^p \) convergence.

We then show that if the domain \( U \) is an open subset of a Carnot group, then a map is in \( W^{1,p}(U; X') \) in the sense of Definition 2.29 if and only if it has a representative which has \( p \)-integrable \( p \)-weak upper gradient. While this can be shown by combining standard arguments in the field, we are not aware of a specific reference for this result. The corresponding result for scalar-valued functions defined on open subsets of Euclidean space can be found, for example, in [HKST15, Theorem 7.4.5].
A comprehensive introduction to the Sobolev spaces defined via (weak) upper gradients is given in the book [HKST15] by Heinonen, Koskela, Shanmugalingam and Tyson and we closely follow their exposition.

C.1. **Weak upper gradients and the Poincaré-Sobolev inequality.** Let \( X = (X,d,\mu) \) be a metric measure space, i.e. a separable metric space \((X,d)\) with a nontrivial locally finite Borel regular (outer) measure \( \mu \). A curve \( \gamma : I \to X \) is a continuous map from an interval \( I \subset \mathbb{R} \) to \( X \). We say that \( \gamma \) is compact or open if \( I \) is compact or open. We define the length of a compact curve \( \gamma : [a,b] \to X \) as the supremum of the numbers \( \sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1})) \) where the supremum is taken over all choices \( t_0, \ldots, t_k \) with \( a = t_0 < t_1 < \ldots < t_k = b \) and all \( k \in \mathbb{N} \). For a noncompact curve the length is defined as the supremum of the length of all compact subcurves. A curve if rectifiable if it has finite length and locally rectifiable if all compact subcurves have finite length. For a rectifiable curve \( \gamma \) we denote by \( \gamma_s \) its arc-length parametrization. We say that a rectifiable curve \( \gamma : [a,b] \to X \) is absolutely continuous if for every \( \varepsilon > 0 \) we can find a \( \delta > 0 \) such that \( \sum_{i=1}^{k} d(\gamma(b_i), \gamma(a_i)) < \varepsilon \) whenever \((a_i, b_i) \subset [a,b] \) are non-overlapping intervals with \( \sum_{i=1}^{k} (b_i - a_i) < \delta \). For a rectifiable curve \( \gamma : I \to X \) and a Borel function \( \rho : X \to [0,\infty] \) we define the integral \( \int_{\gamma} \rho ds \) by \( \int_{0}^{\text{length}(\gamma)} \rho \circ \gamma_s(t) \, dt \). If \( \gamma \) is locally rectifiable \( \int_{\gamma} \rho ds \) is defined as the supremum of the integrals over all compact subcurves.

For \( p \geq 1 \) the \( p \)-modulus of a family \( \Gamma \) of curves is defined by

\[
\text{mod}_p(\Gamma) := \inf \left\{ \int_X \rho^p \, d\mu : \rho : X \to [0,\infty] \text{ Borel,} \quad \int_{\gamma} \rho \, ds \geq 1 \text{ for all locally rectifiable } \gamma \in \Gamma \right\}
\]

We call the Borel functions \( \rho \) with \( \int_{\gamma} \rho \, ds \geq 1 \) for all \( \gamma \in \Gamma \) admissible densities. Every family of non-locally rectifiable curves has modulus zero and every family which contains a constant curve has modulus \( \infty \). Moreover the modulus is countably subadditive.

We say that a family of curves is \( p \)-exceptional if it has \( p \)-modulus zero. We say that a property holds for \( p \)-a.e. curve if there exists a curve family \( N \) of zero \( p \)-modulus such that the property holds for all which do not belong to \( N \). A set \( E \) is \( p \)-exceptional if the \( p \)-modulus of the family of all nonconstant (rectifiable) curves which meet \( E \) is zero.
We denote the family of all nonconstant compact rectifiable curves by \( \Gamma_{\text{rec}} \).

We give the definition of a \( p \)-weak upper gradient directly in the setting of metric-space-valued maps.

**Definition C.1** ([HKST15], Section 6.2, p. 152). Let \( U \subset G \) be open, let \( 1 \leq p < \infty \) and let \( (X, d') \) be a metric space. Let \( g : U \to [0, \infty] \) be a Borel function. We say that \( g \) is \( p \)-weak upper gradient of a map \( f : U \to X' \) if for \( p \)-a.e. rectifiable curve \( \gamma : [a, b] \to U \)

\[
d'(f(\gamma(b)), f(\gamma(a))) \leq \int_\gamma g \, ds
\]

If \( g \) is \( p \)-integrable then one easily sees that \( \int_\gamma g \, ds < \infty \) on \( p \)-a.e. compact curve. This yields the following result.

**Proposition C.3** ([HKST15], Proposition 6.3.2). Suppose that the Borel function \( g : U \to [0, \infty] \) is a \( p \)-integrable \( p \)-weak upper gradient of \( f : U \to X' \). Then \( p \)-a.e. every compact rectifiable curve \( \gamma \) in \( U \) has the following property: \( g \) is integrable on \( \gamma \) and the pair \( (f, g) \) satisfies the upper gradient inequality (C.2) on \( \gamma \) and each of its compact subcurves. In particular every map \( f : U \to X' \) that has a \( p \)-integrable \( p \)-weak upper gradient is absolutely continuous on \( p \)-a.e. compact curve in \( X \).

To state the Poincaré-Sobolev inequality we recall that for a measurable set \( A \subset G \) and a map \( f : A \to X' \) we defined the \( L^p \) oscillation by

\[
\text{osc}_p(f, A) := \inf_{a \in X'} \left( \int_A d^p(f(x), a) \mu(dx) \right)^{1/p}.
\]

**Theorem C.5** ([HKST15], Thm. 9.1.15). Let \( U \subset G \) be open, let \( X' \) be a metric space. Let \( 1 \leq p < \nu \) and define \( p^* \) by

\[
\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}.
\]

Let \( f \in W^{1,p}(U, X') \) and let \( g \in L^p(U) \) be a \( p \)-weak upper gradient. Then for every ball \( B(x, r) \subset U \)

\[
\text{osc}_{p^*}(f, (B(x, r))) \leq C\|g\|_{L^p(B(x, r))}
\]

and

\[
\text{osc}_p(f, (B(x, r))) \leq Cr\|g\|_{L^p(B(x, r))}.
\]

**Proof.** First note that \( G \) supports a \( p \)-Poincaré inequality (for scalar-valued functions) in the sense of (1.3) in [HKST15]. A short proof is...
due to Varapoulos [Var87, see also [HK00 Proposition 11.17], [SC95, p. 461] or [GN96, Corollary 1.6]]. Note also that a $p$-weak upper gradient $g$ can be approximated in $L^p$ by upper gradients $g_k$ (see Lemma 6.2.2. in [HKST15]).

Now (C.6) follows from Theorem 9.1.15 in [HKST15] and the isometric embedding of $X'$ into the Banach space $\ell^\infty(X')$ of bounded functions on $X'$ given by $i(x) = d'(\cdot, x) - d'(\cdot, a)$ where $a$ is an arbitrary point in $X'$. Indeed, Theorem 9.1.15 in [HKST15] implies that there exists an element $h$ of $\ell^\infty(X')$ such that

$$\left(\int_{B(x,r)} |(i \circ f)(y) - h|^p \, \mu(dy)\right)^{1/p} \leq C\|g\|_{L^p(B(x,r))}.$$ 

Hence there exists a $\bar{y} \in B(x,r)$ s.t. $|(i \circ f)(\bar{y}) - h| \leq C\|g\|_{L^p(B(x,r))}$. Thus by the triangle inequality

$$\left(\int_{B(x,r)} |(i \circ f)(y) - (i \circ f)(\bar{y})|^p \, \mu(dy)\right)^{1/p} \leq 2C\|g\|_{L^p(B(x,r))}.$$ 

This implies (C.6) since $i$ is an isometric embedding.

Finally, (C.7) follows from (C.6) and H"older’s inequality since $\mu(B(x,r)) = cr^\nu$.

□

A key feature of the $p$-weak upper gradient is that it is stable under $L^p$ convergence in the following sense.

**Lemma C.8.** Let $(X', d')$ be a complete metric space. Let $f_k : U \to X'$ be maps and let $g_k : U \to [0, \infty]$ be Borel functions and assume that $g_k$ is a $p$-weak upper gradient of $f_k$. Assume further that there exist a map $f : U \to X'$ and a Borel function $g : U \to [0, \infty]$ such that $d'(f_k, f) \to 0$ in $L^p(U)$ and $g_k \to g$ in $L^p(U)$. Then there exists a subsequence $f_{k_j}$ with the following property. The set $E$ where $f_{k_j}$ does not converge is a $\mu$-null set, the set of curves $\gamma$ such that $f_{k_j} \circ \gamma$ does not converge pointwise is a $p$-exceptional set and if we define

$$(C.9) \quad \bar{f}(x) = \lim_{j \to \infty} f_{k_j}(x) \quad \text{if } x \in U \setminus E,$$

and extend $\bar{f}$ arbitrarily in $E$ then $g$ is a $p$-weak upper gradient of $\bar{f}$ and $\bar{f} = f$ a.e.

The proof of this result uses two standard arguments. The first is Fuglede’s lemma which can be seen as a counterpart of Fubini’s theorem in metric measure spaces.

**Lemma C.10 (Fuglede’s lemma).** Let $X$ be a metric measure space and suppose that $f_k : X \to \mathbb{R}$ is a sequence of Borel functions which
converges in $L^p(X)$ to a Borel function $f$. Then there exists a subsequence $f_{k_j}$ such that

$$\lim_{j \to \infty} \int_{\gamma} |f_{k_j} - f|^p = 0$$

for $p$-a.e. all rectifiable curves $\gamma$ in $X$.

Proof. This follows directly from the definition of the modulus of a curve family if we choose the subsequence such that

$$\int_{E} |f_{k_j} - f|^p \, dx < 2^{-pj-j},$$

see, for example, [V71], Thm. 28.1 or [HKST15], Chapter 5.2. \qed

The second standard tool is an improvement of a.e. properties to properties which hold away from a $p$-exceptional set once we have a $p$-integrable $p$-weak upper gradient (see [HKST15, Lemma 6.3.5, Corollary 6.3.6] for closely related results and arguments). We state the result for a metric measure space $X$. It applies equally to open subsets $U \subset X$ considered as metric measure spaces with the induced metric and measure.

**Proposition C.11.** Let $X$ be a metric measure space. Then the following assertions hold.

1. If $E \subset X$ is a $\mu$-nullset then $p$-a.e. curve has zero length in $E$, i.e. $L^1(\{t \in [0,\text{length}(\gamma)] : \gamma_s(t) \in E\}) = 0$ where $\gamma_s$ denotes the arclength parametrization.

2. Suppose $f : X \to X'$ has a $p$-integrable $p$-weak upper gradient. Assume that there exists $c \in X'$ such that $f = c \mu$-a.e. (or assume $X' = \mathbb{R}$ and $f \geq a \mu$-a.e.). Then there exists a $p$-exceptional set $E$ such that $f = c$ in $X \setminus E$ (or $f \geq a$ in $X \setminus E$).

3. Suppose that the maps $f_j : X \to X'$ have $p$-integrable $p$-weak upper gradients $g_j$. Assume that $f_j \to f \mu$-a.e. and that exists a Borel function $g$ such that $g_j \to g$ in $L^p(X)$. Then there exists a $p$-exceptional set $E$ such that $f_j$ converges in $X \setminus E$.

Proof. **(1).** Let $E' \supset E$ be a Borel null set. Then the assertion follows from the definition of the $p$-modulus if we consider the admissible function $\rho$ which is $\infty$ on $E'$ and zero elsewhere.

** (2).** Assume $f = c \mu$-a.e. and let $E$ be the set where $f \neq c$. By assertion **(1)** we have $f \circ \gamma_s = c \mathcal{L}^1$ a.e. for $p$-a.e. curve. Since $f$ is absolutely continuous on $p$-a.e. curve, it follows that $p$-a.e. curve does not meet $E$. The same reasoning applies if $X' = \mathbb{R}$ and $f \geq a$ a.e.
(3). Let $E$ be set where the sequence $f_j$ does not converge. It follows from Fuglede’s lemma that, for $p$-a.e. rectifiable curve $\gamma$, we have $g_j \circ \gamma_s \to g \circ \gamma_s$ in $L^1([0, \text{length}(\gamma)])$. Thus the functions $f_j \circ \gamma_s$ are equicontinuous on $p$-a.e. curve and one concludes as for assertion \[2\]. □

C.2. Upper gradients and weak derivatives. In this subsection we provide a proof that a map is in the Sobolev space $W^{1,p}(U; X')$ defined by weak derivatives if and only if it has a representative which possesses a $p$-integrable $p$-weak upper gradient. Specifically we show the following results.

**Proposition C.12.** Let $U \subset G$ be open and let $1 \leq p < \infty$. Let $u : U \to \mathbb{R}$ be in $L^p(U)$. Then the following two assertions are equivalent:

1. $u \in W^{1,p}(U)$;
2. $u$ has a representative $\bar{u}$ which has a $p$-integrable $p$-weak upper gradient $g$.

Moreover, if $u \in W^{1,p}(U)$ then every Borel representative of $|D_hu|$ is a $p$-weak upper gradient. Conversely if $g$ is a $p$-weak upper gradient then $|D_hu| \leq g$ a.e.

**Proposition C.13.** Let $U \subset G$ be open and let $1 \leq p < \infty$. Let $X'$ be a complete separable metric space and let $f : U \to X'$ be a measurable function such that $d(f(\cdot), a) \in L^p(U)$ for some $a \in X'$. Then the following three assertions are equivalent:

1. There exists a representative $\bar{f}$ of $f$ which has a $p$-integrable $p$-weak subgradient $g$;
2. for every Lipschitz function $\varphi : X' \to \mathbb{R}$ the function $\varphi \circ f - \varphi(a)$ is in $W^{1,p}(U)$;
3. $f \in W^{1,p}(U; X')$;

Moreover if the above assertions hold, $\bar{f}$ is as in \[4\], and $\bar{g} \in L^p(U)$ is a Borel representative of the function $g$ in Definition \[2.29\] then $\bar{g}$ is a $p$-weak upper gradient for $\bar{f}$. Conversely if $\bar{g}$ is a $p$-weak upper gradient then we can take $g = \bar{g}$ in Definition \[2.29\].

The same conclusions holds if one replaces $L^p(U; X')$ and $W^{1,p}(U; X')$ by $L^p_{\text{loc}}(U; X')$ and $W^{1,p}_{\text{loc}}(U; X')$, respectively.

Similar equivalences using absolute continuity along a.e. horizontal curve $t \mapsto a \exp(tX_j)$ rather than absolute continuity for $p$-a.e. rectifiable curve have been studied by Vodopyanov \[Vod99\] Proposition 3, p. 674].
Proof of Proposition C.12. To show the implication (1) $\implies$ (2) we first note that smooth functions are dense in $W^{1,p}(U)$. This was proved by Friedrichs [Fri44] (in local coordinates) who observed that for a $C^1$ vectorfield $X$ the commutator $\bar{J}_\varepsilon = [X, J_\varepsilon]$, where $J_\varepsilon$ is the usual (Euclidean) mollification, satisfies $\bar{J}_\varepsilon u \to 0$ in $L^p_{\text{loc}}$ for $u \in L^p$; see also Thm. 1.13 and Thm. A.2 in [GN96]).

Now let $u_k$ be a sequence of smooth functions such that $u_k \to u$ and $X_i u_k \to h_i$ in $L^p(U)$ where $h_i$ are the weak horizontal derivatives of $u$. Then $g_k := |D_h u_k|$ is an upper gradient of $u_k$ and $g_k \to (\sum h_i^2)^{1/2} = |D_h u|$. Let $g$ be a Borel representative of $|D_h u|$. Then it follows from Lemma C.8 that $u$ has a representative $\tilde{u}$ such that $g$ is a $p$-weak upper gradient of $\tilde{u}$.

For the converse implication one uses essentially absolute continuity on a.e. curve $t \mapsto \exp(tX_j)$ and Fubini's theorem. For the convenience of the reader we sketch some details. By a partition of unity it suffices to show that the weak derivatives exists in a small neighbourhood of any point in $U$. Let $X$ be a left-invariant vectorfield. Let $B' \subset \mathbb{R}^{N-1}$ be a (smal) ball around 0 and consider a smooth surface $\Psi : B' \to G$ which is transversal to $X$. Then $\Phi(t, x') = \Psi(x') \exp tX$ defines a smooth diffeomorphism of $(-\delta, \delta) \times B'$ to its image if $\delta > 0$ is small enough. Moreover $\partial_t \Phi = X \circ \Phi$. Since the Haar measure $\mu$ is biinvariant, the pull-back measure $\Phi^* \mu$ is invariant under translation in $x_1$ direction, i.e. $\Phi^* \mu = dx_1 \otimes \mu'$. Consider the curves $\gamma_{x'}(t) = \Phi(t, x') = \Psi(x') \exp tX$ and the family $\Gamma_E = \{\gamma_{x'} : x' \in E\}$. Using Fubini’s theorem one easily checks that $\text{mod}_p(\Gamma_E) = 0$ implies $\mathcal{L}^{N-1}(E) = 0$ (or, equivalently, $\mu'(E) = 0$).

Set $\tilde{u} = u \circ \Phi$. Then $t \mapsto \tilde{u}(t, x')$ is absolutely continuous for $L^{N-1}$-a.e. $x'$ and $|\tilde{u}(b, x') - \tilde{u}(a, x')| \leq \int_a^b \tilde{g}(t, x') \, dt$ where $\tilde{g} = g \circ \Phi$. Set $\tilde{u} = u \circ \Phi$. It is then easy to show that the difference quotients $\Delta^s \tilde{u} := s^{-1}(u(t + s, x') - u(t, x'))$ are controlled by a family of one-dimensional convolutions of $g$ and hence a subsequence $s_j \downarrow 0$ converges weakly in $L^p_{\text{loc}}$ (for $p = 1$ use the Dunford-Pettis theorem) to a function $h \in L^p$ with $|h| \leq g$ a.e. Then $h$ is a weak derivative of $\tilde{u}$, i.e. $\int \tilde{u} \partial_i \varphi \, \Phi^* \mu = -\int h \varphi \, \Phi^* \mu$. Unwinding definitions, we see that $h = h \circ \Phi^{-1}$ is the desired weak derivative $X u$. Moreover $|h| \leq g$ a.e. from which we deduce $|D_h u| \leq g$ by considering a countable dense family of left-invariant unit vector fields.

Proof of Proposition C.13. The assertion essentially follows from Theorem 7.1.20 in [HKST15] upon using the isometric embedding of $(X', d')$ into the Banach space $V = \ell^\infty(X')$ of bounded functions on $X'$. We give a self-contained proof for the convenience of the reader.
We only give the argument for $W^{1,p}(U; X')$. The version for $W^{1,p}_{loc}$ is then deduced easily.

(1) $\implies$ (2). Note that $(\text{Lip } \varphi) g$ is a $p$-weak upper gradient of $\varphi \circ f$ and that $\varphi \circ \bar{f} = \varphi \circ f$ almost everywhere. Thus the implication follows from Proposition C.12

(2) $\implies$ (3). This is clear since the map $y \mapsto d'(y, z)$ is 1-Lipschitz.

(3) $\implies$ (1). Set $u_z(x) = d'(z, f(x))$. Let $D \subset Z$ be a countable dense subset. By the definition of $W^{1,p}(U; X')$ there exists a Borel function $\bar{g}$ such that $|D_h u_z| \leq \bar{g}$ almost everywhere. By Proposition C.12 for each $z \in D$ there exists a representative $\bar{u}_z$ such that $\bar{g}$ is a $p$-weak upper gradient of $\bar{u}_z$. The main point is to show that there exist a $p$-exceptional set $E$ and a map $\bar{f} : U \setminus E \to X'$ such that

\[
(C.14) \quad d'(z, \bar{f}(x)) = \bar{u}_z(x) \quad \forall x \in U \setminus E, \quad \forall z \in D
\]

Then the upper gradient inequality for the function $\bar{u}_z$ implies that

\[
|d'(z, \bar{f}(\gamma_s(t))) - d'(z, \bar{f}(\gamma_s(s)))| \leq \int_s^t \bar{g} \circ \gamma_s \, d\mathcal{L}^1
\]

for all $z \in D$ and $p$-a.e. curve $\gamma$. If $z_k \to z$ in $Z$ then the functions $d'(z_k, \cdot)$ converge uniformly to $d'(z, \cdot)$. Thus the inequality holds for all $z \in Z$. Taking $z = \bar{f}(\gamma_s(t))$ we see that $\bar{g}$ is $p$-weak upper gradient of $\bar{f}$.

To construct $\bar{f}$, note that the definition of $u_z$ and the triangle inequality imply that

\[
\inf_{z \in D} u_z = 0, \quad \forall z, z' \in D \quad d'(z, z') - u_{z'} \leq u_z \leq d'(z, z') + u_{z'}.
\]

Since $\bar{u}_z$ agrees with $u_z$ a.e., it follows from and Proposition C.11 (2) that there exists a $p$-exceptional set $E$ such that

\[
(C.15) \quad \inf_{z \in D} u_z = 0 \quad \text{in } U \setminus E
\]

\[
(C.16) \quad d'(z, \bar{z}) - \bar{u}_z \leq \bar{u}_z \leq d'(z, \bar{z}) + \bar{u}_z \quad \text{in } U \setminus E
\]

for all $z, z' \in D$. We claim that for all $x \in U \setminus E$ there exists a unique $\bar{z} = \bar{z}(x) \in Z$ such that

\[
(C.16) \quad d'(z, \bar{z}) = \bar{u}_z(x) \quad \forall z \in D
\]

Fix $x \in U \setminus E$. By definition of $u_z$ there exist $z_k \in D$ such that $\bar{u}_{z_k}(x) \to 0$. Moreover $d'(z_k, z_l) \leq \bar{u}_{z_k}(x) + \bar{u}_l(x)$. Hence $z_k$ is a Cauchy sequence. Thus $z_k \to \bar{z}$ and taking $z' = z_k$ in (C.15) we get (C.16). If also $d'(z, \bar{z}) = \bar{u}_z(x)$ for all $z \in D$ then we get $d'(z, \bar{z}) = d(z, \bar{z})$ for $z \in D$. Thus $\bar{z} = \bar{z}$. 
We now define $\tilde{f}(x) = \bar{z}(x)$ for $x \in U \setminus E$. Since $\bar{u}_z(x) = u_z(x) = d'(z, f(x))$ for a.e. $x \in U$ it follows from the uniqueness for $\bar{z}$ that $\tilde{f} = f$ a.e. $\square$

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