A Simple Duality Proof for
Wasserstein Distributionally Robust Optimization

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We present a short and elementary proof of the duality for Wasserstein distributionally robust optimization, which holds for any arbitrary Kantorovich transport distance, any arbitrary measurable loss function, and any arbitrary nominal probability distribution, as long as certain interchangeability principle holds.

Key words: Wasserstein metric, distributionally robust optimization, duality

1. Introduction

In this paper, we consider the following problem

\[ \mathcal{L}(\rho) := \sup_{\mathcal{P} \in \mathcal{P}(\mathcal{X})} \left\{ \mathbb{E}_{X \sim \mathcal{P}}[f(X)]: \mathcal{K}_c(\widehat{\mathcal{P}}, \mathcal{P}) \leq \rho \right\}, \]

where \( \rho \in \mathbb{R}_+ \), \( \mathcal{P}(\mathcal{X}) \) is set of all probability distributions on a data space \( \mathcal{X} \), \( f: \mathcal{X} \to \mathbb{R} \) is a loss function, \( X \) is a random variable on \( \mathcal{X} \) having a nominal distribution \( \widehat{\mathcal{P}} \), and \( \mathcal{K}_c \) denotes the Kantorovich transport distance, defined as

\[ \mathcal{K}_c(\widehat{\mathcal{P}}, \mathcal{P}) = \inf_{\gamma \in \Gamma(\widehat{\mathcal{P}}, \mathcal{P})} \mathbb{E}_{(\widehat{X}, X) \sim \gamma}[c(\widehat{X}, X)], \]

where \( \Gamma(\widehat{\mathcal{P}}, \mathcal{P}) \) denotes the set of all probability distributions on \( \mathcal{X} \times \mathcal{X} \) with marginals \( \widehat{\mathcal{P}} \) and \( \mathcal{P} \), and \( c: \mathcal{X} \times \mathcal{X} \to [0, \infty) \) is a transport cost function. The functional \( \mathcal{L} \) represents the robust loss hedging against deviations of data within \( \rho \)-neighborhood of the nominal distribution. When \( c = d^\rho \), where \( d \) is a metric on \( \mathcal{X} \) and \( \rho \in [1, \infty) \), then \( (\mathcal{P}) \) is the inner worst-case problem in the distributionally robust optimization with \( \rho \)-Wasserstein metric, which has raised much interest in operations research and machine learning recently; see [9, 4] for tutorials.

A central question is to derive the dual program of \( (\mathcal{P}) \). Existing duality proofs either rely on advanced conic duality for the problem of moments [14] and impose unnecessary conditions for technical reasons [7, 18], or require lengthy analysis [3, 8] to obtain a more general result. In this paper, we aim to provide a novel proof that not only is elementary and short, but also yields results even more general than all existing literature; see Table 1 and the discussion after Theorem 1 below for a detailed comparison.

The rest of this paper is organized as follows. We present our main results in Section 2. The key idea of our proof is to perform the Legendre transform twice on the robust loss as a function of the radius \( \rho \). Leveraging the concavity of the robust loss, the strong duality holds directly from the Legendre transformation, provided that certain interchangeability principle holds, a notion often studied in dynamic programming (see, e.g., [15]). We discuss this notion in Section 3 and show that it is related to certain measurability conditions ensuring the well-definedness of the dual problem as well as the existence of an approximately optimal probability distribution for the primal problem. We conclude the paper in Section 4.
2. Main Result

Throughout this section, we assume the following situation.

**Assumption 1.** Let \( (\mathcal{X}, \mathcal{F}, \mathcal{P}) \) be a probability space. Let \( \rho \in [0, \infty) \), and \( f : \mathcal{X} \to \mathbb{R} \) be a measurable function with \( \mathbb{E}_{\mathcal{P}}[f] > -\infty \). Let \( c : \mathcal{X} \times \mathcal{X} \to [0, \infty] \) be a measurable transport cost function with \( c(\bar{x}, \bar{x}) = 0 \) if and only if \( x = \bar{x} \).

We start with a simple lemma on the properties of \( \mathcal{L}(\rho) \), whose proof is provided in Appendix A.

**Lemma 1.** Assume Assumption 1 holds. Then \( \mathcal{L}(\rho) \) is lower bounded by \( \mathbb{E}_{\mathcal{P}}[f] \), monotonically increasing, and concave in \( \rho \) on \([0, \infty)\).

We impose the following interchangeability assumption on \( \mathcal{P} \) and \( \mathcal{X} \) that allows us to exchange expectation and supremum. Conditions ensuring it will be discussed in Section 3.

**Interchangeability Principle (IP) For any \((\mathcal{F} \otimes \mathcal{F})\)-measurable function \( \phi : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \cup \{-\infty\}, \) the function \( \bar{x} \mapsto \sup_{x \in \mathcal{X}} \phi(\bar{x}, x) \) is \( \mathcal{P} \)-measurable\(^1\), and it holds that

\[
\mathbb{E}_{\mathcal{P}} \left[ \sup_{x \in \mathcal{X}} \phi(\bar{x}, x) \right] = \sup_{T \in \mathcal{T}} \mathbb{E}_{\mathcal{P}} \left[ \phi(\bar{x}, T(\bar{x})) \right],
\]

whenever the left-hand side is well-defined\(^2\), and \( \mathcal{T} \) contains all \((\mathcal{F}, \mathcal{F})\)-measurable maps from \( \mathcal{X} \) to \( \mathcal{X} \) such that the expectation \( \mathbb{E}_{\mathcal{P}}[\phi(\bar{x}, T(\bar{x}))] \) is well-defined.

A direct consequence of this assumption is the following lemma, whose proof is given in Appendix A.

**Lemma 2.** Under the setting of (IP), Then it holds that

\[
\mathbb{E}_{\mathcal{P}} \left[ \sup_{x \in \mathcal{X}} \phi(\bar{x}, x) \right] = \sup_{\gamma \in \Gamma_{\mathcal{P}}} \mathbb{E}_{\mathcal{P}} \left[ \phi(\bar{x}, X) \right],
\]

where \( \Gamma_{\mathcal{P}} \) be the set of probability distributions on \( \mathcal{X} \times \mathcal{X} \) whose first marginal is \( \mathcal{P} \), such that \( \mathbb{E}_{\gamma}[\phi(\bar{x}, X)] \) is well-defined.

With Lemmas 1 and 2, we derive the dual of (P) by applying Legendre transform to \( \mathcal{L}(\rho) \) twice.

**Theorem 1.** Assume Assumption 1 and (IP) hold. Then for any \( \rho > 0 \), \( \mathcal{L}(\rho) \) defined in (P) equals

\[
\mathcal{L}(\rho) = \inf_{\lambda > 0} \left\{ \lambda \rho + \mathbb{E}_{\mathcal{P}} \left[ \sup_{x \in \mathcal{X}} \left\{ f(x) - \lambda c(\bar{x}, x) \right\} \right] \right\},
\]

(D)
Proof of Theorem 1. Let \( \mathcal{P} \) denote the set of probability measures on \((\mathcal{X}, \mathcal{F})\) satisfying \( K_c(\widehat{P}, \mathcal{P}) < \infty \) and \( \mathbb{E}_\mathcal{P}[f] > -\infty \). Taking Legendre transform on \( L(\cdot) \) gives that for any \( \lambda > 0 \),

\[
L^*(\lambda) := \sup_{\rho \geq 0} \{ L(\rho) - \lambda \rho \} = \sup_{\rho \geq 0} \sup_{\mathcal{P} \in \mathcal{P}(\mathcal{X})} \left\{ \mathbb{E}_{X \sim \mathcal{P}}[f(X)] - \lambda \rho : K_c(\widehat{P}, \mathcal{P}) \leq \rho \right\} \\
= \sup_{\mathcal{P} \in \mathcal{P}} \left\{ \mathbb{E}_{X \sim \mathcal{P}}[f(X)] - \lambda \inf_{\gamma \in \Gamma(\widehat{P}, \mathcal{P})} \mathbb{E}_{(\tilde{X}, X) \sim \gamma}[^c(\tilde{X}, X)] \right\} \\
= \sup_{\mathcal{P} \in \mathcal{P}, \gamma \in \Gamma(\widehat{P}, \mathcal{P})} \left\{ \mathbb{E}_{X \sim \mathcal{P}}[f(X)] - \lambda \mathbb{E}_{(\tilde{X}, X) \sim \gamma}[^c(\tilde{X}, X)] \right\} \\
= \sup_{\gamma \in \Gamma(\tilde{P})} \left\{ \mathbb{E}_{(\tilde{X}, X) \sim \gamma}[^f(X) - \lambda c(\tilde{X}, X)] \right\},
\]

where \( \Gamma(\tilde{P}) \) denotes the set of probability distributions on \( \mathcal{X} \times \mathcal{X} \) whose first marginal is \( \tilde{P} \) and the expectation of \( f - \lambda c \) is well-defined. Observe from Lemma 1 that \( L(\rho) \) is bounded from below, increasing and concave in \( \rho \geq 0 \). This implies either \( L(\rho) < +\infty \) for all \( \rho > 0 \) or \( L(\rho) = +\infty \) for all \( \rho > 0 \). In the former case, applying Legendre transform (see, e.g., [12, Theorem 12.2]) on the concave function \( L(\cdot) \) yields that for any \( \rho > 0 \),

\[
L(\rho) = \inf_{\lambda > 0} \{ \lambda \rho + L^*(\lambda) \} = \inf_{\lambda > 0} \left\{ \lambda \rho + \sup_{\gamma \in \Gamma(\tilde{P})} \left\{ \mathbb{E}_{(\tilde{X}, X) \sim \gamma}[^f(X) - \lambda c(\tilde{X}, X)] \right\} \right\}.
\]

In the latter case, by definition \( L^*(\lambda) = +\infty \) for all \( \lambda > 0 \), and the above is also true. Finally, by Lemma 2 we have

\[
\sup_{\gamma \in \Gamma(\tilde{P})} \left\{ \mathbb{E}_{(\tilde{X}, X) \sim \gamma}[^f(X) - \lambda c(\tilde{X}, X)] \right\} = \mathbb{E}_{\tilde{X} \sim \tilde{P}} \left[ \sup_{x \in \mathcal{X}} \left\{ f(x) - \lambda c(\tilde{X}, x) \right\} \right],
\]

which completes the proof. \( \square \)

Remark 1. From the proof of Theorem 1 we see that if we impose Assumption 1 only but without \( (\text{IP}) \), then we can prove that

\[
L(\rho) = \inf_{\lambda > 0} \left\{ \lambda \rho + \sup_{\gamma \in \Gamma(\tilde{P})} \left\{ \mathbb{E}_{(\tilde{X}, X) \sim \gamma}[^f(X) - \lambda c(\tilde{X}, X)] \right\} \right\}.
\]

As will be discussed in the next section, all existing results in the literature assume conditions stronger than \( (\text{IP}) \), which enables the expression \( (D) \).

Remark 2. In general, \( (D) \) does not hold at \( \rho = 0 \). Indeed, the right-hand side of \( (D) \) is continuous in \( \rho \in [0, \infty) \), but \( L(\rho) \) may be not right-continuous at 0. For instance, if \( \mathcal{X} = \mathbb{R} \) and \( c(\tilde{x}, x) = |\tilde{x} - x| \), \( f(x) = 1_{\{x \neq 0\}} \) and \( \tilde{P} = \delta_0 \), then \( L(\rho) = 1 \) for any \( \rho > 0 \) and \( L(0) = 0 \). A sufficient condition ensuring the right-continuity of \( L(\rho) \) at 0 is the following: there exists a continuous non-decreasing concave function \( \varphi : [0, \infty] \rightarrow [0, \infty] \) with \( \varphi(0) = 0 \) such that \( f(x) - f(\tilde{x}) \leq \varphi \circ c(\tilde{x}, x) \) for all \( x \in \mathcal{X} \) and \( \tilde{P} \)-a.e.
\( \tilde{x} \in \mathcal{X} \). Indeed, under this condition, for any \( \mathbb{P} \in \mathcal{P} \) with \( K_c(\tilde{\mathbb{P}}, \mathbb{P}) \leq \rho \) and \( \epsilon > 0 \), there exists a \( \gamma \in \Gamma_{\tilde{\mathbb{P}}} \) such that \( E_\gamma[c(\tilde{X}, X)] \leq \rho + \epsilon \), hence \( E_\gamma[f(X) - f(\tilde{X})] = E_\gamma[\varphi \circ c] \leq \varphi(E_\gamma[c]) = \varphi(\rho + \epsilon) \) by Jensen’s inequality, therefore \( L(\rho) \leq \mathbb{E}_\tilde{\mathbb{P}}[f(\tilde{X})] + \varphi(\rho + \epsilon) = L(0) + \varphi(\rho + \epsilon) \), which converges to zero as \( \rho, \epsilon \to 0 \).

When \( c = d^p \), where \( d \) is a metric on \( \mathcal{X} \), this condition implies that \( f \) is upper semicontinuous and satisfies the growth condition imposed in [8].

In the following, let us compare our proof technique with existing duality results in the literature. Constructive proofs [7, 3, 18, 17] rely on advanced convex duality theory. More specifically, Esfahani and Kuhn [7], Zhao and Guan [18] exploit advanced conic duality [14] for the problem of moments that requires the nominal distribution \( \tilde{\mathbb{P}} \) to be finitely supported and the space \( \mathcal{X} \) to be convex, along with some other assumptions on the transport cost \( c \) and the loss function \( f \); Blanchet and Murthy [3] use an approximation argument that represents the Polish space \( \mathcal{X} \) as an increasing sequence of compact subsets, on which the duality holds for any Borel distribution \( \tilde{\mathbb{P}} \) thanks to Fenchel conjugate on vector spaces [10], under certain semicontinuity assumptions on the transport cost \( c \) and loss function \( f \); using the same infinite dimensional convex duality, Sinha et al. [17, Theorem 5] provide a simplified analysis by assuming the function \( (X, \tilde{X}) \mapsto 4\varphi(X, \tilde{X}) - f(X) \) is a normal integrand [13]. Compared with these non-constructive duality proofs, our (non-constructive) proof uses only Legendre transform, namely, the convex duality for univariate real-valued functions. The constructive proof developed by Gao and Kleywegt [8] provides a result at a similar level of generality as [3] without using convex duality theory, by constructing an approximately worst-case distribution using the first-order optimality condition of the weak dual problem. Although both their proof and ours do not use advanced minimax theorems, our analysis is shorter and more elementary.

### 3. Discussion on Interchangeability Principle

In this section, we discuss conditions ensuring the interchangeability principle (IP) required by Theorem 1. Interested readers may refer to [15] for a discussion on other important situations.

The following proposition suggests that measurability play an important role in ensuring (IP).

**Proposition 1.** Let \( \mathcal{F}_{\tilde{\mathbb{P}}} \) be the completion of \( \mathcal{F} \) under \( \tilde{\mathbb{P}} \) [1, Definition 1.11]. Suppose \((\mathcal{X}, \mathcal{F}, \tilde{\mathbb{P}}) \) satisfies the following two conditions:

**Proj** [Measurable Projection] For any measurable set \( A \in \mathcal{F} \cap \mathcal{F}_{\tilde{\mathbb{P}}} \),

\[
\text{Proj}_\mathcal{X}(A) := \{ \tilde{x} \in \mathcal{X} : (\tilde{x}, x) \in A \text{ for some } x \in \mathcal{X} \} \in \mathcal{F}_{\tilde{\mathbb{P}}},
\]

**Sel** [Measurable Selection] For any set-valued function \( E : \mathcal{X} \to \mathcal{F} \setminus \{ \emptyset \} \) with a measurable graph

\[
\text{Graph}(E) := \{(\tilde{x}, x) \in \mathcal{X} \times \mathcal{X} : x \in E(\tilde{x}) \} \in \mathcal{F} \cap \mathcal{F}_{\tilde{\mathbb{P}}},
\]

there exists a \( (\mathcal{F}_{\tilde{\mathbb{P}}}, \mathcal{F}) \)-measurable map \( T : \mathcal{X} \to \mathcal{X} \) such that \( T(\tilde{x}) \in E(\tilde{x}) \) for all \( \tilde{x} \in \mathcal{X} \).

Then (IP) holds.

As can be seen from the proof in Appendix A, (Proj) ensures the measurability of the function \( \tilde{x} \mapsto \sup_{x \in \mathcal{X}} \phi(\tilde{x}, x) \) in the dual problem (D), and (Sel) ensures the existence of an approximately optimal probability distribution in the primal problem (P).

The two examples below show that existing results rely on assumptions strictly stronger than (IP), therefore our result strictly generalizes existing results in the literature.

**Example 1.** If \((\mathcal{X}, \mathcal{F}_{\tilde{\mathbb{P}}}) \) is a discrete measurable space, that is, the discrete set of finite mass, then (Proj) and (Sel) always hold, because every subset of \( \mathcal{X} \) is measurable and every map \( T : \mathcal{X} \to \mathcal{X} \) is measurable. For instance, when \( \mathcal{X} = \mathbb{R}^d \) equipped with Borel \( \sigma \)-algebra, and \( \tilde{\mathbb{P}} \) is finitely supported, then \( \mathcal{F}_{\tilde{\mathbb{P}}} = 2^{\mathbb{R}^d} \) is the collection of all subsets of \( \mathbb{R}^d \), which is the discrete \( \sigma \)-algebra on \( \mathcal{X} \). Thus our result covers the results in [7, 18].
Example 2. If $\mathcal{X}$ is a Suslin space and $\mathcal{F}$ is its Borel $\sigma$-field, then $\text{(Proj)}$ holds due to [5, Theorem III.23], and $\text{(Sel)}$ holds due to [5, Theorem III.22]. In particular, these hold if $\mathcal{X}$ is a Polish space and $\mathcal{F}$ is its Borel $\sigma$-field, which recover the results in [3, 8]. A Suslin space (also known as analytic set) is the image of a Polish space (complete and separable metric space) under a continuous mapping, which is often considered in dynamic programming [2], multistage stochastic programming [6] and minimax stochastic optimization [11], in which the measurability of the optimal value function is concerned.

4. Concluding Remarks

We develop a new duality proof for Wasserstein distributionally robust optimization, which is based on applying Legendre transform twice to the worst-case loss as a function of Wasserstein radius. Although being non-constructive as compared to [8], our proof is elementary and concise. The proof technique may be applicable to other choices of statistical distance in distributionally robust optimization.

Endnotes

1. Throughout the paper, we say an extended real-valued function on $(\mathcal{X}, \mathcal{F})$ is measurable if it is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ measurable, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra on the extended reals, and we say an extended real-valued function on $\mathcal{X}$ is $\bar{\mathcal{F}}$-measurable if it is measurable with respect to the completion of $\mathcal{F}$ under the measure $\bar{\mathcal{P}}$ [1, Definition 1.11]. This is in the same spirit as the universally measurability considered in [16].

2. Well-definedness here means that we require either the positive part or the negative part of the integrand has a finite integral under the measure $\bar{\mathcal{P}}$. The expectation can take values in $\mathbb{R} \cup \{\pm \infty\}$.

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Appendix A: Proofs of Auxiliary Results

Proof of Lemma 1. The monotonicity of $\mathcal{L}(\rho)$ can be seen from the definition. Moreover, since $\mathcal{K}_c(\bar{\mathcal{P}}, \bar{\mathcal{P}}) = 0$,

$$\mathcal{L}(\rho) \geq \mathcal{L}(0) \geq \mathbb{E}_{X - \bar{\mathcal{P}}}[f(X)] > -\infty.$$  

Therefore for all $\rho \geq 0$, $\mathcal{L}(\rho)$ is bounded from below. To verify the concavity, fix $\rho_0, \rho_1 \geq 0$. For any $t \in [0, 1]$ and $\mathcal{P}_0, \mathcal{P}_1 \in \mathcal{P}(\mathcal{X})$ satisfying $\mathcal{K}_c(\bar{\mathcal{P}}, \mathcal{P}_0) \leq \rho_0$, $\mathcal{K}_c(\bar{\mathcal{P}}, \mathcal{P}_1) \leq \rho_1$, denote $\mathcal{P}_t = (1 - t)\mathcal{P}_0 + t\mathcal{P}_1$, then

$$\mathcal{K}_c(\bar{\mathcal{P}}, \mathcal{P}_t) \leq (1 - t)\mathcal{K}_c(\bar{\mathcal{P}}, \mathcal{P}_0) + t\mathcal{K}_c(\bar{\mathcal{P}}, \mathcal{P}_1) \leq (1 - t)\rho_0 + t\rho_1,$$

hence $\mathcal{P}_t$ is a feasible solution to $\text{(P)}$ and

$$\mathcal{L}((1 - t)\rho_0 + t\rho_1) \geq \mathbb{E}_{X - \mathcal{P}_t}[f(X)] = (1 - t)\mathbb{E}_{X - \mathcal{P}_0}[f(X)] + t\mathbb{E}_{X - \mathcal{P}_1}[f(X)].$$

Taking the supremum over $\mathcal{P}_0$ and $\mathcal{P}_1$, we have

$$\mathcal{L}((1 - t)\rho_0 + t\rho_1) \geq (1 - t)\mathcal{L}(\rho_0) + t\mathcal{L}(\rho_1),$$

which completes the proof. \qed
Proof of Lemma 2. On the one hand, using the tower property of the conditional expectation, we have
\[ \sup_{\gamma \in \Gamma_\beta} \left\{ E_{(\bar{\gamma}, X) - \gamma} [\phi(\tilde{X}, X)] \right\} = \sup_{\gamma \in \Gamma_\beta} \left\{ E_{\bar{\gamma} \sim \beta} \left[ E_{X - \gamma \sim \bar{\gamma}} \left[ \phi(\tilde{X}, X) \mid \tilde{X} \right] \right] \right\} \leq E_{\bar{\gamma} \sim \beta} \left( \sup_{x \in \mathcal{X}} \phi(\tilde{X}, x) \right). \]

On the other hand, recalling that $T(\mathcal{X}, \mathcal{Y})$ is the set of measurable functions from $\mathcal{X}$ to $\mathcal{Y}$, by (IP),
\[ E_{\bar{\gamma} \sim \beta} \left( \sup_{x \in \mathcal{X}} \phi(\tilde{X}, x) \right) = \sup_{T \in T} E_{\bar{\gamma} \sim \beta} \left[ \phi(\tilde{X}, T(\tilde{X})) \right] \]
\[ = \sup_{T \in T} E_{(\bar{\gamma}, X) - (\text{Id} \times T) \sim \beta} \left[ \phi(\tilde{X}, X) \right] \leq \sup_{\gamma \in \Gamma_\beta} \left\{ E_{(\bar{\gamma}, X) - \gamma} [\phi(\tilde{X}, X)] \right\}, \]
where $(\text{Id} \times T) \sim \beta$ means the push-forward of measure $\beta$ via the map $\text{Id} \times T : \tilde{x} \mapsto (\tilde{x}, T(\tilde{x}))$, which belongs to $\Gamma_\beta$. Thus we prove the desired result. \hfill \Box

Proof of Proposition 1. Define $\Phi(\tilde{x}) = \sup_{x \in \mathcal{X}} \phi(\tilde{x}, x)$. For any $\alpha \in \mathbb{R}$, the super level set of $\Phi$ can be regarded as
\[ \{ \tilde{x} : \Phi(\tilde{x}) > \alpha \} = \{ \tilde{x} : \exists x, \phi(\tilde{x}, x) > \alpha \} = \text{Proj}_{\tilde{x}} \{ (\tilde{x}, x) : \phi(\tilde{x}, x) > \alpha \}. \]

By assumption (Proj), $\text{Proj}_{\tilde{x}}$ maps measurable sets to measurable sets. Therefore $\Phi$ is measurable.

Since for any $T : \mathcal{X} \to \mathcal{Y}$, $\phi(\tilde{x}, T(\tilde{x})) \leq \Phi(\tilde{x})$, it is clear that
\[ E_{\tilde{x} \sim \beta} \left[ \Phi(\tilde{X}) \right] \geq \sup_{T \in T} E_{\tilde{x} \sim \beta} \left[ \phi(\tilde{X}, T(\tilde{X})) \right]. \]

To see the other direction, we may assume $E_{\tilde{x} \sim \beta} [\Phi(\tilde{X})] > -\infty$, otherwise the conclusion holds trivially. Define
\[ \Phi_n(\tilde{x}) = n \wedge \left( \Phi(\tilde{x}) - \frac{1}{n} \right), \quad E_n(\tilde{x}) = \{ x \in \mathcal{X} : \phi(\tilde{x}, x) > \Phi_n(\tilde{x}) \}. \]

For each $\tilde{x} \in \mathcal{X}$, $E_n(\tilde{x})$ is nonempty, and its graph
\[ \text{Graph}(E_n) = \{ (\tilde{x}, x) \in \mathcal{X} \times \mathcal{X} : \phi(\tilde{x}, x) > \Phi_n(\tilde{x}) \} \]
is measurable, so by assumption (Sel) we can find a measurable map $T_n \in T$ such that $T_n(\tilde{x}) \in E_n(\tilde{x})$.

Since $\min \{ \Phi - 1, 0 \} \leq \Phi_n \leq \Phi$ and $\Phi_n \uparrow \Phi$ as $n \to \infty$, by monotone convergence theorem
\[ \lim_{n \to \infty} E_{\tilde{x} \sim \beta} \left[ \Phi_n(\tilde{X}) \right] = \lim_{n \to \infty} E_{\tilde{x} \sim \beta} \left[ \Phi_n(\tilde{X}) \right] = E_{\tilde{x} \sim \beta} \left[ \lim_{n \to \infty} \Phi_n(\tilde{X}) \right] = E_{\tilde{x} \sim \beta} \left[ \Phi(\tilde{X}) \right]. \]

This completes the proof. \hfill \Box
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