Isoparametric hypersurfaces in Randers space forms

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Abstract: In this paper, we discuss anisotropic submanifolds and isoparametric hypersurfaces in a Randers space form \((N, F)\) with the navigation datum \((h, W)\). We find that \((N, F, d\mu_{BH})\) and \((N, h)\) have the same isoparametric hypersurfaces although, in general, their isoparametric functions are different. This implies that the classification of isoparametric hypersurfaces in a Randers space form is the same as that in Riemannian case. Lastly, we give some examples of isoparametric functions in Randers space forms.

Key words: isoparametric hypersurfaces, Randers space form, principal curvature, anisotropic submanifolds.

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1 Introduction

In Riemannian geometry, the study of isoparametric hypersurfaces has a long history. Since 1938, E. Cartan had begun to study the isoparametric hypersurfaces in real space forms with constant sectional curvature \(c\) systematically. The classification of isoparametric hypersurfaces in space forms is a classical geometric problem with a history of almost one hundred years. Isoparametric hypersurfaces in Euclidean and hyperbolic spaces were classified in 1930’s (\([1, 3]\)). For the classification of isoparametric hypersurfaces in a unit sphere, which is the most difficult case, there are many important results (as like \([4, 5]\), etc.) and it was recently completely solved in \([6]\).

In Finsler geometry, the concept of isoparametric hypersurfaces has been introduced in \([7]\). Let \((N, F, d\mu)\) be an \(n\)-dimensional Finsler manifold with volume form \(d\mu\). A function \(f\) on \((N, F, d\mu)\) is said to be isoparametric if there are \(\tilde{a}(t)\) and \(\tilde{b}(t)\) such that

\[
\left\{
\begin{align*}
F(\nabla f) &= \tilde{a}(f), \\
\Delta f &= \tilde{b}(f),
\end{align*}
\right.
\]

(1.1)

where \(\nabla f\) and \(\Delta f\) denote the nonlinear gradient and Laplacian of \(f\) with respect to \(d\mu\), respectively (see Section 2.1 and 2.3 for details).

Studing and classifying isoparametric hypersurfaces in Finsler space forms are interesting problems naturally generalized from Riemannian geometry. In \([7]\), the authors studied

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isoparametric hypersurfaces in Finsler space forms, and obtained the Cartan type formula and some classifications on the number of distinct principal curvatures or their multiplicities. For some very special Finsler space forms, such as Minkowski space (with zero flag curvature) and Funk space (a special Rander space with negative constant flag curvature), the isoparametric hypersurfaces have been completely classified (7-9). Xu in [11] studied a special class of isoparametric hypersurfaces in a Randers sphere (with positive constant flag curvature).

Randers manifolds play a fundamental role in Finsler geometry. Those with constant flag curvature were classified in [10], using Zermelo’s navigation method. A forward (resp. backward) complete and simply connected Randers manifold with constant flag curvature $c$ is called a forward (resp. backward) Randers space form, which is denoted by $(N(c), F)$. In fact, the most known examples of Finsler space forms with non-zero flag curvature are Randers space forms. So it is natural to consider the isoparametric hypersurfaces in Randers space forms. Unlike the Riemannian case, there are infinitely many Randers space forms, which are not isometric or even are not homothetic to each other. The classification problems of isoparametric hypersurfaces are far from being fully resolved.

In this paper, we will give the complete classifications of isoparametric hypersurfaces in a forward (or backward) Randers space form $(N^n(c), F)$. By using navigation process, we find the following

**Theorem 1.1.** Let $(N, F)$ be a forward (or backward) Randers space form with the navigation datum $(h, W)$. Then $(N, F, d\mu_{BH})$ and $(N, h)$ have the same isoparametric hypersurfaces, and the number of distinct principal curvatures and the multiplicities of each principal curvature are also the same. So the isoparametric hypersurfaces in $(N, F, d\mu_{BH})$ can be completely classified (see Table 1 for the accurate classifications).

The contents of this paper are organized as follows. In section 2, some fundamental concepts and formulas are given for later use. In section 3, we consider the principal curvatures of submanifolds with respect to $F$ and $h$, respectively, and derive the classification of isoparametric hypersurface in Randers space forms. In section 4, we consider the relation between isoparametric functions with respect to $F$ and $h$ and give some examples of isoparametric functions in special Randers space forms.

2 Preliminaries

2.1 Finsler-Laplacian

Let $(N, F)$ be an $n$-dimensional oriented smooth Finsler manifold and $TN$ be the tangent bundle over $N$ with local coordinates $(x, y)$, where $x = (x^1, \cdots, x^n)$ and $y = (y^1, \cdots, y^n)$. 

\textbf{2}
The fundamental form $g$ of $(N, F)$ is given by

$$g = g_{ij}(x, y)dx^i \otimes dx^j, \quad g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}.$$

The projection $\pi : TN \to N$ gives rise to the pull-back bundle $\pi^*TN$ and its dual bundle $\pi^*T^*N$ over $TN \setminus 0$. Recall that on the pull-back bundle $\pi^*TN$ there exists a unique Chern connection $\nabla$ with $\nabla \frac{\partial}{\partial x^i} = \omega^j_i \frac{\partial}{\partial x^j} = \Gamma^j_k dx^k \otimes \frac{\partial}{\partial x^j}$ satisfying (12)

$$dg_{ij} - g_{ik} \omega^k_j - g_{kj} \omega^k_i = 2FC_{ijk} \delta y^k,$$

$$\delta y^i := \frac{1}{F}(dy^i + N^i_j dx^j), \quad N^i_j := \frac{\partial G^i}{\partial y^j} = \Gamma^i_{jk} y^k,$$

where $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ is called the Cartan tensor and

$$G^i = \frac{1}{4} g^{it} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^i} \}$$

are the geodesic coefficients of $(N, F)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field. Then the covariant derivative of $X$ along $v = v^i \frac{\partial}{\partial x^i} \in T_x N$ with respect to a reference vector $w \in T_x N \setminus 0$ for the Chern connection is defined by

$$D^w_v X(x) := \left\{ v^i \frac{\partial X^i}{\partial x^j}(x) + \Gamma^j_{ik}(w)v^j x^k(x) \right\} \frac{\partial}{\partial x^i}. \quad (2.1)$$

Let $\mathcal{L} : TN \to T^* N$ denote the Legendre transformation, satisfying $\mathcal{L}(\lambda y) = \lambda \mathcal{L}(y)$ for all $\lambda > 0$, $y \in TN$, and (13)

$$\mathcal{L}(y) = F(y)[F]_{y^i}(y) dx^i, \quad \forall y \in TN \setminus \{0\}, \quad (2.2)$$

$$\mathcal{L}^{-1}(\xi) = F^*(\xi)[F^*]_{\xi^i}(\xi) \frac{\partial}{\partial x^i}, \quad \forall \xi \in T^* N \setminus \{0\}, \quad (2.3)$$

where $F^*$ is the dual metric of $F$. In general, $\mathcal{L}^{-1}(-\xi) \neq -\mathcal{L}^{-1}(\xi)$. For a smooth function $f : N \to \mathbb{R}$, the gradient vector of $f$ at $x$ is defined as $\nabla f(x) = \mathcal{L}^{-1}(df(x)) \in T_x N$. Set $N_f = \{ x \in N | df(x) \neq 0 \}$ and $\nabla^2 f(x) = D^\nabla f(\nabla f)(x)$ for $x \in N_f$. The Finsler-Laplacian of $f$ with respect to the volume form $d\mu = \sigma(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is defined by

$$\Delta f = \text{div}_\sigma(\nabla f) = \text{tr}_{g_{\nabla f}}(\nabla^2 f) - S(\nabla f), \quad (2.4)$$

where

$$S(x, y) = \frac{\partial G^i}{\partial y^j} - y^i \frac{\partial}{\partial x^i}(\ln \sigma(x)) \quad (2.5)$$

is the $S$-curvature.
2.2 Anisotropic submanifolds

Let \((N, F)\) be an \(n\)-dimensional Finsler manifold and \(\phi : M \to (N, F)\) be an \(m\)-dimensional immersion. Here and from now on, we will use the following convention of index ranges unless otherwise stated:

\[ 1 \leq i, j, \ldots \leq n; \quad 1 \leq a, b, \ldots \leq m < n; \quad m + 1 \leq \alpha, \beta, \ldots \leq n. \]

Let \(N M = \{(x, n) | x \in \phi(M), n \in T_x(N), \mathcal{L}_n(X) = 0, \forall X \in T_xM\}\), which is called the normal bundle of \(\phi\) or \(M\). Note that in general, it is not a vector bundle. We call \(\{(M, g_n) | n \in N M\}\) an anisotropic submanifold of \((N, F)\) to distinguish it from an isometric immersion submanifold \((M, \phi^*F)\). Moreover, we denote the unit normal bundle by \(N^0 M = \{n \in N M | F(n) = 1\}\).

For any \(X \in T_xM\) and \(n \in N^0 M\), we define the shape operator \(A_n : T_xM \to T_xM\) by

\[ A_n(X) = - (D^n_X n)^\top g_n. \quad (2.6) \]

Then it is easy to show that

\[ g_n(A_n(X), Y) = g_n(X, A_n(Y)). \quad (2.7) \]

The eigenvalues of \(A_n\), \(\lambda_1, \lambda_2, \ldots, \lambda_m\), and \(\hat{H}_n = \sum_{a=1}^{m} \lambda_a\) are called the principal curvatures and the anisotropic mean curvature with respect to \(n\), respectively. If \(\lambda_1 = \lambda_2 = \cdots = \lambda_m\) for any \(n \in N M\), then \(M\) is called to be anisotropic-totally umbilic. If \(\hat{H}_n = 0\) for any \(n \in N M\), then \(M\) is called an anisotropic-minimal submanifold of \((N, F)\).

Let \(\phi : M \to N\) be an embedded hypersurface of \((N, F)\). For any \(x \in M\), there exist exactly two unit normal vectors \(n_\pm\). Let \(n\) be a given normal vector of \(N\). Set \(\hat{g} = \phi^*g_n\). From [7], we have the following Gauss-Weingarten formulas

\[ D^n_X Y = \hat{\nabla}_X Y + \hat{h}(X, Y)n, \quad (2.8) \]

\[ D^n_X n = -A_n X, \quad \forall X, Y \in \Gamma(TM). \quad (2.9) \]

Here

\[ \hat{h}(X, Y) := g_n(n, D^n_X Y) = \hat{g}(A_n X, Y) \]

is called the second fundamental form, and \(\hat{\nabla}\) is a torsion-free linear connection on \(M\) satisfying (9)

\[ (\hat{\nabla}_X \hat{g})(Y, Z) = -2C_n(A_n X, Y, Z), \quad \forall X, Y, Z \in \Gamma(TM), \quad (2.10) \]

where \(C_n\) is the Cartan tensor with \(y = n\).
2.3 Isoparametric functions and isoparametric hypersurfaces

Let \( f \) be a non-constant \( C^1 \) function defined on a Finsler manifold \( (N,F,d\mu) \) and smooth on \( N_f \). Set \( J = f(N_f) \). The function \( f \) is said to be \emph{isoparametric} if there exist a smooth function \( \tilde{a}(t) \) and a continuous function \( \tilde{b}(t) \) defined on \( J \) such that (1.1) holds on \( N_f \). All the regular level surfaces \( M_t = f^{-1}(t) \) form an \emph{isoparametric family}, each of which is called an \emph{isoparametric hypersurface} in \( (N,F,d\mu) \). If \( f \) only satisfies the first equation of (1.1), then it is said to be \emph{transnormal}.

Let \( \phi: M \to N \) be a hypersurface. If for any given \( x \in \phi(M) \), there is a neighborhood \( V \) of \( x \) in \( N \) and an isoparametric function \( f \) defined on \( V \) such that \( \phi(M) \cap V \) is a regular level surface of \( f \), then \( M \) is called a \emph{locally isoparametric hypersurface}.

\textbf{Theorem 2.1.} Let \( M \) be a connected and oriented hypersurface in a connected Finsler manifold with constant flag curvature and constant S-curvature. Then \( M \) is locally isoparametric if and only if its principal curvatures are all constant.

\textit{Proof.} From Theorem 4.1 in [7], we only need to prove the sufficiency. Let \( n \) be a smooth unit normal vector field of \( M \). Because \( n \) is smooth and the normal geodesics locally and smoothly depend on \( n \), we know that for any \( x \in \phi(M) \), there exists a neighborhood \( V \) of \( x \) in \( N \), and there exists a smooth distance function \( f \) defined on \( V \) such that \( M = f^{-1}(0) \). In fact, we can define \( f(p) = d_F(M,p) \) or \( f(p) = -d_F(p,M) \) for \( p \in V \setminus M \) such that \( \nabla f|_M = n \). Then \( F(\nabla f) = 1 \), which shows that \( f \) is a transnormal function. From Lemma 3.5 in [14] (which also holds in a local domain), we know that on every regular level surface of \( f \), the principal curvatures are all constant. By Theorem 4.2 in [7], \( f \) is isoparametric on \( V \). \( \square \)

3 Isoparametric hypersurfaces in a Randers space form

3.1 Anisotropic submanifolds in Randers spaces

Let \( (N,F,d\mu_{BH}) \) be an \( n \)-dimensional Randers space, where \( F = \alpha + \beta = \sqrt{a_{ij}y^iy^j} + b_iy^i \), and let its navigation expression be

\[
F = \frac{\sqrt{\lambda h^2 + w_0^2} - w_0}{\lambda} = \frac{\sqrt{\lambda h_{ij}y^iy^j + (w_iy^i)^2} - w_iy^i}{\lambda},
\]

where \( \lambda = 1 - b^2 \), \( b = \|W\|_h \), \( W = w^i \frac{\partial}{\partial x^i} \), \( w_i = h_{ij}w^j \), and

\[
a_{ij} = \frac{1}{\lambda^2}(\lambda h_{ij} + w_iw_j), \quad b_i = -\frac{w_i}{\lambda}.
\]
Then
\[ g_{ij} = \frac{F}{\alpha}(a_{ij} - \alpha_y \alpha_y) + F_y F_y' \]
\[ = \frac{F}{\lambda^2 \alpha} (\lambda h_{ij} + w_i w_j - \lambda^2 \alpha_y \alpha_y) + F_y F_y'. \]  

(3.2)

Denote the dual metric of \( h \) by \( h^* \). Then the dual metric of \( F \) can be expressed as
\[ F^* = h^* + w^0 = \sqrt{h^{ij} v^i v^j + w^i v^i}, \quad v^i dx^i \in T^*_x N. \]  

(3.3)

Let \( \phi : M \to (N^m, F) \) be an \( m \)-dimensional immersion. Take \( n \in N^0(M) \) and \( \nu = L(n) \).

From (2.3) and (3.3), we know that
\[ n = F^*_\xi(n) = \frac{h^{ij} \nu_j}{h^*(\nu)} + w^i. \]  

(3.4)

Denote \( \bar{n} = \frac{h^{ij} \nu_j}{h^*(\nu)} \frac{\partial}{\partial x^i} \). Then \( \bar{n} \) is a unit normal vector field of \( M \) with respect to \( h \). Thus
\[ n = \bar{n} + W. \]  

Let \( (u^a) = (u^1, \cdots, u^m) \) be the local coordinates on \( M \) and \( d\phi = \phi_a' du^a \otimes \frac{\partial}{\partial x^i} \). Then
\[ F_{y'}(n) = \alpha_{y'}(n) - \frac{w_i}{\lambda}, \]  

(3.5)

\[ \alpha_{y'}(n) \phi_a' = F_{y'}(n) \phi_a + \frac{w_i \phi_a'}{\lambda} = \frac{w_i \phi_a'}{\lambda}. \]  

(3.6)

It follows from \( F(n) = 1 \) that
\[ \lambda \alpha(n) = \lambda - \lambda \beta(n) = \lambda + \langle n, W \rangle_h = 1 + \langle \bar{n}, W \rangle_h. \]  

(3.7)

Combing (3.2), (3.4), (3.6) and (3.7), yields
\[ \hat{g}_{ab} = g_{ij}(n) \phi_a' \phi_b' \]
\[ = \frac{1}{\lambda^2 \alpha(n)} (\lambda \bar{h}_{ab} + w_i w_j \phi_a' \phi_b' - \lambda^2 \alpha_{y'}(n) \alpha_{y'}(n) \phi_a' \phi_b' ) \]
\[ = \frac{1}{\lambda \alpha(n)} \bar{h}_{ab}, \]

where \( \bar{h}_{ab} = h_{ij} \phi_a' \phi_b' \). Thus we have the following
Lemma 3.1. Let $\phi : M \to (N, F)$ be an anisotropic submanifold in a Randers space $(N, F)$ with the navigation datum $(h, W)$. Then every induced metric $\hat{g}_n = \phi^*g_n$, for any $n \in N^0(M)$, is conformal to $\bar{h} = \phi^*h$ and satisfies

$$\hat{g}_n = \frac{1}{\lambda(n)} \bar{h} = \frac{1}{1 + \langle n, W \rangle h} \bar{h}. \quad (3.8)$$

Denote

$$r_{ij} = \frac{1}{2}(w_{ij} + w_{ji}), \quad s_{ij} = \frac{1}{2}(w_{ij} - w_{ji}),$$

$$r_j = w^i r_{ij}, \quad s_j = w^i s_{ij},$$

$$s^i = h^{ik} s_k, \quad r^i = h^{ik} r_k, \quad s^i_j = h^{ik} s_{kj},$$

$$s^i_0 = s^i_j y^j, \quad s_0 = s^i y^i, \quad r_0 = r^i y^i, \quad r_{00} = r_{ij} y^i y^j,$$

where "|" denotes the covariant differential about $h$. From [15], we know that

$$G^i = \bar{G}^i - F s^i_0 - \frac{1}{2} F^2(r^i + s^i) + \frac{1}{2} (\frac{y^i}{F} - w^i)(2 F r_0 - r_{00} - F^2 r),$$

where $G^i$ and $\bar{G}^i$ are the geodesic coefficients of $F$ and $h$, respectively. Then

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \bar{N}^i_j - F y^i s^i_0 - F s^j s^i_j - FF_{yi}(r^i + s^i)$$

$$+ \frac{1}{2} (\frac{\delta^i_j}{F} - \frac{1}{F^2} y^i F_{y^j})(2 F r_0 - r_{00} - F^2 r)$$

$$+ \frac{1}{2} (\frac{y^i}{F} - w^i)(2 F_{yi} r_0 - 2 F r_j - 2 r_{0j} - 2 FF_{yi} r). \quad (3.9)$$

From [15], $F$ has isotropic $S$-curvature if and only if $W$ satisfies

$$r_{ij} = -2k(x) h_{ij}.$$

Using the above formulas, we get

$$s^i_j = w^i_{ij} + 2k \delta^i_j, \quad r_j = -2k w_j, \quad (3.10)$$

$$r_0 = -2k w_0, \quad r_{00} = -2k h^2, \quad r = -2k b^2. \quad (3.11)$$

Lemma 3.2. Let $\phi : M \to (N, F)$ be an anisotropic submanifold in a Randers space $(N, F)$ with the navigation datum $(h, W)$. If $F$ has isotropic $S$-curvature $S = (n + 1)k(x)F$, then for any $n \in N^0(M)$ and $X \in TM$,

$$D^h_X n = \nabla^h_X n - k(x)d\phi X. \quad (3.12)$$
Proof. Set \(X = X^a \frac{\partial}{\partial u^a}\). By (3.4) \(\sim (3.11)\), we have

\[
D^n_X n = (n^i_x + N^i_j(n)) \phi^a X^a \frac{\partial}{\partial x^i}
\]

\[
= \left( n^i_x + N^i_j(n) - s^i + \frac{1}{2} \delta^i_j (2r_0 - r_{00} - r) + \frac{1}{2} (n^i - w^i)(2r_j - 2r_{j0}) \right) \phi^a X^a \frac{\partial}{\partial x^i}
\]

\[
= \nabla^h_X (\hat{n} + W) - (\nabla^h_X W + 2kd\phi X) + k(-2w_i n^i + h(n)^2 + b^2)d\phi X
\]

\[
+ k(-2w_i \phi^a X^a + 2h_{ij} n^i \phi^a X^a))\hat{n}
\]

\[
= \nabla^h_X \hat{n} - 2kd\phi X + k(-2 \langle n, W \rangle_h + \langle n, n \rangle_h + \|W\|_h^2)d\phi X
\]

\[
+ 2k(-\langle W, d\phi X \rangle_h + \langle n, d\phi X \rangle_h)\hat{n}
\]

\[
= \nabla^h_X \hat{n} - 2kd\phi X + k|\hat{n}|^2d\phi X
\]

\[
= \nabla^h_X \hat{n} - kd\phi X.
\]

Thus, we have the following

**Theorem 3.3.** Let \(M\) be an anisotropic submanifold in a Randers space \((N,F,d\mu_{BH})\) with the navigation datum \((h,W)\). If \(F\) has isotropic \(S\)-curvature \(S = (n+1)k(x)F\), then for any \(n \in \mathcal{N}^0(M)\), the shape operators of \(M\) in Randers space \((N,F)\) and Riemannian space \((N,h)\), \(A_n\) and \(\bar{A}_n\), have the same principal vectors and satisfy

\[
\lambda = \bar{\lambda} + k(x),
\]

where \(\lambda\) and \(\bar{\lambda}\) are the principal curvatures of \(M\) in Randers space \((N,F)\) and Riemannian space \((N,h)\), respectively.

**Proof.** Set \(X = X^a \frac{\partial}{\partial u^a}\) and \(\phi_a = d\phi \frac{\partial}{\partial u^a}\). By (2.6) and (3.12), we know that

\[
A_n X = -[D^n_X n^r]_{g_n} = -g_n(\nabla^h_X \hat{n}, \phi_a)(\hat{g}_n)^{ab} \frac{\partial}{\partial u^b} + kX.
\]
From (3.1) to (3.12) and (2.10), we have

\[ -g_n(\nabla^h_X \hat{n}, \phi_a)(\hat{\gamma}_n)^{ab} \frac{\partial}{\partial u^b} = -\frac{1}{\lambda^2}(\lambda \hat{h}_{ij} + w_i w_j - \lambda^2 \alpha_y(n) \alpha_y(n)) \phi^i_a \hat{n}_c^j X^c(\hat{\gamma}_n)^{ab} \frac{\partial}{\partial u^b} \]

\[ = -\frac{1}{\lambda}(\lambda \hat{h}_{ij} \phi^i_a + w_i \phi^i_a(w_j - \lambda \alpha_y(n)) \hat{n}_c^j X^c \hat{h}^{ab} \frac{\partial}{\partial u^b} \]

\[ = -[\nabla^h_X \hat{n}^\top_h + \langle \phi_a, W \rangle_h F_y(n) \hat{n}_c^j X^c \hat{h}^{ab} \frac{\partial}{\partial u^b} \]

\[ = \hat{A}_n X + (W)_h^\top g_n(n, \nabla^h_X \hat{n}) \]

\[ = \hat{A}_n X + (W)_h^\top g_n(n, D^h_X n) \]

\[ = \hat{A}_n X. \]

Thus \( A_n X = \hat{A}_n X + k(x) X \), and the proof is completed. \( \square \)

**Corollary 3.4.** In a Randers space \((N, F, d\mu_{BH})\) with constant \(S\)-curvature and the navigation datum \((h, W)\), the principal curvatures of an anisotropic submanifold \(M\) are all constant if and only if its principal curvatures in Riemannian space \((N, h)\) are all constant. Especially, when the \(S\)-curvature vanishes, \(M\) is anisotropic minimal if and only if it is minimal in Riemannian space \((N, h)\).

### 3.2 Classification of isoparametric hypersurfaces in a Randers space form

**Proof of Theorem 1.1.**

Let \((N, F, d\mu_{BH})\) be a Randers space with the navigation datum \((h, W)\). By Theorem 5.11 in [15], \((N, F)\) has constant flag curvature \(c\) if and only if the Riemannian space \((N, h)\) has constant sectional curvature \(\bar{c}\) and \(W\) is a homothetic vector field with dilation \(k_0\). In this case, \(F\) has constant \(S\)-curvature, that is, \(k(x) = k_0\) in (3.13) and \(c = \bar{c} - k_0^2\). Then Theorem 1.1 follows from Theorem 2.1 and Theorem 3.3. \( \square \)

Up to now, the classifications of the isoparametric hypersurfaces in real space forms have been completely solved ([1], [2], [6], [16] \sim [23]). So according to Theorem 1.1, we can give the complete classifications of isoparametric hypersurfaces in a Randers space form \((N(c), F)\). Note that in general, when \((N(c), F)\) is complete, \((N(c), h)\) is not necessarily complete. In this case, \((N(\bar{c}), h)\) is isometric to an open subset of a real space form \(\bar{N}(\bar{c})\). That is

\[ N(\bar{c}) \cong \{ x \in \bar{N}(\bar{c}) \mid \| W \|_h < 1 \}, \]

where \(\bar{N}(\bar{c}) = \mathbb{R}^n, \mathbb{H}^n(\bar{c})\) or \(\mathbb{S}^n(\frac{1}{\sqrt{\bar{c}}})\). By Proposition 5.4 in [15], \(k_0 = 0\) when \(\bar{c} = c + k_0^2 \neq 0\). Specifically, we have
(1) if \( c = 0 \), then \( \bar{c} = k_0 = 0 \), and \( \overline{N}(\bar{c}) \cong \{ x \in \mathbb{R}^n \mid \|W\|_h < 1 \} \);

(2) if \( c < 0 \), then \( \bar{c} = c + k_0^2 = 0 \) for \( k_0 \neq 0 \), and \( \overline{N}(\bar{c}) \cong \{ x \in \mathbb{H}^n(\bar{c}) \mid \|W\|_h < 1 \} \); or \( \bar{c} = c < 0 \) for \( k_0 = 0 \), and \( \overline{N}(\bar{c}) \cong \{ x \in \mathbb{H}^n(\bar{c}) \mid \|W\|_h < 1 \} \);

(3) if \( c > 0 \), then \( (\overline{N}(c), F) \) is compact and \( \bar{c} = c \), which implies that \( (\overline{N}(\bar{c}), h) \cong S^n(\frac{1}{\sqrt{c}}) \).

Thus the Randers space form \((\overline{N}(c), F)\) is globally isometric to a Randers sphere.

The known classification results are summarized in the following table.

**Table 1: Classification results for isoparametric hypersurfaces in \((\overline{N}(c), F, d\mu_{BH})\)**

| \(K_F = c\) | S-curv. | \(N(c)\) | \(g\) | \(\text{dim } M\) | \(\text{mul.}\) | \(M\) is an open subset of following hypersurfaces |
|-------------|----------|-----------|-----|--------|--------|------------------------------------------------|
| \(c = 0\)  | \(k_0 = 0\) | \(\mathbb{R}^n\) | \(g=1\) | \(n-1\) | \(n-1\) | a hypersphere \(S^{n-1}\) or a hyperplane \(\mathbb{R}^{n-1}\) |
| \(c = -1\) | \(k_0 = 0\) | \(\mathbb{H}^n\) | \(g=1\) | \(n-1\) | \(n-1\) | a sphere \(S^{n-1}\), a hyperbolic \(\mathbb{H}^{n-1}\) or a horosphere \(\mathbb{R}^{n-1}\) |
|             | \(k_0^2 = 1\) | \(\|W\|_h < 1\) | \(g=2\) | \(n-1\) | \(m,n-m-1\) | a cylinder \(S^m \times \mathbb{R}^{n-m-1}\) |
| \(c = 1\)  | \(k_0 = 0\) | \(\mathbb{S}^n\) | \(g=1\) | \(n-1\) | \(n-1\) | a great or small hypersphere |
|             | \(\|W\|_h < 1\) | | \(g=2\) | \(n-1\) | \(m,n-m-1\) | a Clifford torus \(S^m(r) \times S^{n-m-1}(s), r^2 + s^2 = 1\) |
|             | | | \(g=3\) | \(3\) | \(1,1\) | a tube over a standard Veronese embedding of \(\mathbb{F}P\) into \(S^{3m+1}\), where \(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{O}\), for \(m = 1, 2, 4, 8\), respectively. |
|             | | | \(g=4\) | \(2m-2, m \geq 3\) | \(1,m-2\) | OT-FKM type |
|             | | | | \(4m-2, m \geq 2\) | \(2,2m-3\) | |
|             | | | | \(8m-2, m \geq 2\) | \(4,4m-5\) | |
|             | | | | \(8\) | \(2,2\) | homogeneous |
|             | | | | \(18\) | \(4,5\) | homogeneous |
|             | | | | \(14\) | \(3,4\) | OT-FKM type |
|             | | | | \(30\) | \(6,9\) | FKM type or homogeneous |
|             | | | | \(30\) | \(7,8\) | OT-FKM type |
| \(g=6\)    | | | | \(6\) | \(1,1\) | homogeneous |
|             | | | | \(12\) | \(2,2\) | |
3.3 Isoparametric functions of Randers space forms

We know from Theorem 1.1 that the Randers space form \((N,F,d\mu_{BH})\) with the navigation datum \((h,W)\) and the Riemannian space form \((N,h)\) have the same isoparametric hypersurfaces, but in general, they have different isoparametric families. For example, from Table 1, it is easy to see that every Euclidean sphere is isoparametric in Minkowski-Randers space or Funk space \((N^n,F)\). But in general, its isoparametric family in \((N,F)\) is a family of spheres with variational center \([7,9]\), while its isoparametric family in \((N,h)\) is a family of concentric spheres. Thus the corresponding isoparametric functions are different. On the other hand, the metrics on an isoparametric hypersurface induced from Randers metric \(F\) and Riemannian metric \(h\) are different, and their geometric characteristics are different accordingly. It is still necessary to study the properties of isoparametric functions in Randers space forms.

Theorem 3.5. Let \((N,F,d\mu_{BH})\) be a Randers space with the navigation datum \((h,W)\), where \(W\) is a homothetic vector field, and let \(f\) be an isoparametric function of \((N,h)\). Then \(f\) is an isoparametric function of \((N,F)\) if and only if there is a smooth function \(\varphi\) such that \(df(W) = \varphi(f)\).

Proof. From [9], we know that in a Randers space \((N,F,d\mu)\) with the navigation datum \((h,W)\), \(f\) is an isoparametric function if and only if there exist two functions \(\tilde{a}(t)\) and \(\tilde{b}(t)\) such that \(f\) satisfies

\[
\begin{aligned}
|df|_h + \langle df, W^* \rangle_h &= \tilde{a}(f), \\
\frac{1}{|df|_h} \Delta^h f + \text{div}_W + \frac{1}{|df|_h} \langle d\langle df, W^* \rangle_h, df \rangle_h &= \frac{\tilde{b}(f)}{\tilde{a}(f)}. \\
\end{aligned}
\]  

(3.14)

If \(f\) is an isoparametric function of \((N,h)\), then there exist two functions \(a(t)\) and \(b(t)\) such that \(f\) satisfies

\[
\begin{aligned}
|df|_h &= a(f), \\
\Delta^h_a f &= b(f). \\
\end{aligned}
\]  

(3.15)

Furthermore, if \(f\) is also an isoparametric function of \((N,F)\), then the desired conclusion follows directly from the first equation of (3.14).

Conversely, suppose there is a smooth function \(\varphi : f(M) \to R\) such that \(df(W) = \varphi(f)\). Since \(W\) is a homothetic vector field, we have

\[
\text{div}_W W = h^{ij} w_{ij} = -2nk_0,
\]

\[
\langle d\langle df, W^* \rangle_h, df \rangle_h = \varphi'(f)|df|_h^2.
\]
From the above formulas and (3.15), we obtain that

\[
\begin{align*}
&|df|_h + \langle df, W^*_h \rangle_h = a(f) + \varphi(f), \\
&\frac{1}{|df|_h^2} \Delta^h f + \text{div}_\sigma W + \frac{1}{|df|_h^2} \langle d(df, W^*_h)h, df \rangle_h = \frac{b(f)}{a(f)} - 2nk_0 + \varphi'(f).
\end{align*}
\]

Then by (3.14), \( f \) is an isoparametric function of \((N,F)\) with

\[
\begin{align*}
\tilde{a}(f) &= a(f) + \varphi(f), \\
\tilde{b}(f) &= b(f) - a(f) (2nk_0 - \varphi'(f)).
\end{align*}
\]

Remark 3.6. In [11], the author obtained this result for a special case, \( df(W) = 0 \), in a different way.

By Theorem 3.5, we can find some examples of isoparametric functions in special Randers space forms. From [15], if \( F \) is a Randers metric with the navigation datum \((h,W)\), then \( F \) has constant flag curvature if and only if the Riemannian metric \( h \) has constant sectional curvature \( \bar{c} \) and the vector field \( W \) satisfies

\[
W = \begin{cases}
-2k_0x + xQ + e, & \bar{c} = 0, \\
xQ + e + \bar{c}(e,x)x, & \bar{c} \neq 0,
\end{cases}
\]

(3.16)

where \( Q \) is an antisymmetric matrix, \( e \in \mathbb{R}^n \) is a constant vector and \( k_0 \) is a constant, and if \( \bar{c} \neq 0 \), then \( k_0 = 0 \), that is, \( W \) is a Killing vector field.

Lemma 3.7. [16] Let \((S^n, h) \hookrightarrow \mathbb{R}^{n+1} (n \geq 2)\) be the standard Euclidean sphere and \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R} \) be a \( k \)-degree homogeneous function. Then

\[
\begin{align*}
(1) \quad \nabla^h \Phi &= \nabla^E \Phi - k\Phi \xi, \\
(2) \quad |\nabla^h \Phi|^2 &= |\nabla^E \Phi|^2 - k^2\Phi^2, \\
(3) \quad \Delta^h \Phi &= \Delta^E \Phi - k(k + n - 1)\Phi,
\end{align*}
\]

(3.17)

where \( \nabla^E \) and \( \Delta^E \) denote the Euclidean gradient and Laplacian in \( \mathbb{R}^{n+1} \), respectively.

Example 3.8. Let \( \tilde{F} \) be a Randers metric with the navigation datum \((\tilde{h},\tilde{W})\), where \( \tilde{h} = \sqrt{\sum \alpha(y^i)^2}, \tilde{W} = xQ + e, x, y \in \mathbb{R}^{n+1}, Q \) is an antisymmetric matrix, and \( e \in \mathbb{R}^{n+1} \) is a constant vector. Set \( \tilde{M} = \{ x \in \mathbb{R}^{n+1} \mid |xQ + e|^2 < 1 \} \). Then \( \tilde{W} \) is a Killing vector field in \( \mathbb{R}^{n+1} \) and thus \((\tilde{M}, \tilde{F})\) has constant flag curvature \( c = 0 \). But it is not a local Minkowski space.

(1) Take

\[ \Phi : \mathbb{R}^{n+1} \to \mathbb{R} \]
\[ x \mapsto \langle x, e_{n+1} \rangle, \quad e_{n+1} = (0, \ldots, 0, 1), \]

and

\[ Q = \begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix}. \]

Then \( \Phi \) is an isoparametric function of \((\tilde{M}, \tilde{h})\) with \( g = 1 \) and

\[ \langle \nabla^E \Phi, W \rangle = \langle e_{n+1}, xQ + e \rangle = \langle e_{n+1}, e \rangle. \]

From Theorem 1.1 and Theorem 3.5, we know that \( \Phi \) is an isoparametric function of Randers space \((\tilde{M}, \tilde{F})\) with \( g = 1 \).

Let \((S^n, h) \hookrightarrow \mathbb{R}^{n+1} (n \geq 2)\) be the standard Euclidean sphere. Take \( \tilde{W} = xQ \), then \( W = \tilde{W}|_{S^n} \) is a Killing vector field on \( S^n \). Let \( F \) be a Randers metric on \( S^n \) with the navigation datum \((h, W)\) and \( f = \Phi|_{S^n} \). Then \( f \) is an isoparametric function of \((S^n, h)\) with \( g = 1 \) and

\[ \langle \nabla^h f, W \rangle_h = \langle \nabla^E \Phi - 2\Phi x, W \rangle = \langle e_{n+1} - \Phi x, xQ \rangle = 0. \]

From Theorem 1.1 and Theorem 3.5, we know that \( f \) is also an isoparametric function of Randers space form \((S^n, F)\) with \( g = 1 \).

(2) Define

\[ \Phi : \mathbb{R}^m \times \mathbb{R}^{n-m+1} \rightarrow \mathbb{R}, \quad (x_1, x_2) \mapsto |x_1|^2 - |x_2|^2. \]

Then from \([16]\), \( f = \Phi|_{S^n} \) is an isoparametric function of \((S^n, h)\) with \( g = 2 \). Take \( W = xQ \), where \( x \in \mathbb{R}^{n+1} \), and

\[ Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}. \]

Then

\[ \langle \nabla^h f, W \rangle_h = \langle \nabla^E \Phi - 2\Phi x, W \rangle = \langle 2(x_1, -x_2) - 2\Phi x, (x_1, x_2)Q \rangle = 0. \]

By Theorem 1.1 and Theorem 3.5, we conclude that \( f \) is also an isoparametric function of Randers space form \((S^n, F)\) with \( g = 2 \).

More generally, we have

**Theorem 3.9.** Let \((S^n, F, d\mu_{BH})\) be a Randers sphere corresponding to the navigation datum \((h, W)\), where \( W = xQ|_{S^n}, \; x \in \mathbb{R}^{n+1}, \) and \( Q \) is a skew symmetric \((n + 1)\)-matrix. Let \( \Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) be a homogeneous function of degree \( k \). Then \( \Phi|_{S^n} \) is an isoparametric function of \((S^n, F, d\mu_{BH})\) if and only if \( \Phi \) satisfies

\[
\begin{cases}
|\nabla^E \Phi - k\Phi x| + \langle \nabla^E \Phi, xQ \rangle = \varphi(\Phi), \\
\frac{\Delta^E \Phi - k(k + n - 1)\Phi}{|\nabla^E \Phi - k\Phi x|} + \frac{\langle \nabla^E(\nabla^E \Phi, xQ), \nabla^E \Phi \rangle}{|\nabla^E \Phi|^2 - k^2\Phi^2} = \psi(\Phi).
\end{cases}
\]
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