BICOVARIANT DIFFERENTIAL CALCULUS
ON THE QUANTUM D=2 POINCARÉ GROUP

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Abstract

We present a bicovariant differential calculus on the quantum Poincaré group in two dimensions. Gravity theories on quantum groups are discussed.
Quantum groups, or q-deformed function algebras of classical groups [1], have recently appeared to be the underlying algebraic structures of some two-dimensional and three-dimensional quantum field theories. In this paper we will be concerned with the geometry of quantum groups, and more precisely with the differential geometry of the D=2 q-Poincaré group.

Our objective is to construct a q-deformed gravity as a “gauge” theory of the D=4 q-Poincaré group, which reduces for q=1 to ordinary gravity. The D=2 case serves both as a preliminary step and as a differential geometric setting for D=2 q-gravity.

Why should we want to deform ordinary gravity (or supergravity, superstrings etc.)? Continuous deformations of “classical” theories often have physical relevance: consider e.g. the deformation parameters v/c, h or the De Sitter radius. Also, for distances of the order of the Planck length the smoothness of spacetime is really a mathematical assumption, leading to infinities in quantum gravity. A generalization of Riemannian geometry to “something else” is perhaps necessary, and the geometry of quantum groups seems to offer a way, leading to non-commutative spaces with group structure. For a review on non-commutative geometry (not necessarily linked to quantum groups) and its potential uses in physics, see e.g. ref. [2].

The geometrical aspects of quantum groups have been very recently investigated by a number of researchers [3,4,5,6,7,8], from two main viewpoints:

i) as the (noncommutative) geometry of a representation space for the quantum group action [4,5,6];

ii) as the (noncommutative) geometry of the group space itself [3,7,8].

The construction of a bicovariant * differential calculus on quantum groups has been initiated by Woronowicz in [3], and applied to q-SU(2) [3a] and to the q-Lorentz group [8]. A general framework for the general cases of q-SO(N) and q-SU(N) has been proposed in [7].

Here we use most of the tools developed in ref.s [3], and present a bicovariant differential calculus for the two-dimensional q-Poincaré group. An analogous result for the D=4 q-Poincaré group will be presented elsewhere [9].

Following the general formulation of [3c], we consider the q-commutator algebra:

\[ T_i T_j - \hat{R}^{kl}_{ij} T_k T_l = C^k_{ij} T_k \]  (1)

where:

i) \( T_i \) are linear functionals on the algebra \( A \) of “functions on the quantum group”, the q-analogue of the Lie algebra generators for ordinary groups; they belong to the algebra \( A' \), the dual of \( A \). \( A \) and \( A' \) are Hopf algebras with dual Hopf structures: the product in \( A' \), appearing in (1), is defined by

* i.e. covariant under the left and right actions of the group, see later.
\[ T_i T_j(a) \equiv (T_i \otimes T_j) \Delta(a) \] (2)

with \( \Delta = \) coproduct in \( A \). The unit of \( A' \) is the counit of \( A \) and so on. To fix our notations:

- the coproduct, counit and coinverse of \( A \) are denoted respectively \( \Delta, \varepsilon \) and \( \kappa \). The unit of \( A \) is \( I \).
- the coproduct, counit and coinverse of \( A' \) are denoted respectively \( \Delta', \varepsilon' \) and \( \kappa' \). The unit of \( A' \) is \( 1 \).

ii) \( \hat{R}_{ij}^{kl} \) is the braiding matrix satisfying

\[ \hat{R}_{ij}^{kl} \hat{R}_{lm}^{ks} \hat{R}_{qu}^{rs} = \hat{R}_{im}^{jk} \hat{R}_{kl}^{ik} \hat{R}_{up}^{sl} \] (3)

or, in more compact notation

\[ \hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23} \]

iii) \( C_{ij}^k \) are the q-structure constants satisfying the q-analogue of Jacobi identities:

\[ C_{ni}^l C_{jk}^m - \hat{R}_{lm}^{ij} C_{ri}^m C_{jk}^n = 0 \] (4)

A bicovariant differential calculus on the quantum group associated to (1) exists when, besides (3) and (4), two additional conditions are satisfied (see later):

\[ C_{ni}^l C_{jk}^m - \hat{R}_{lm}^{ij} C_{ri}^m C_{jk}^n + \hat{R}_{im}^{jk} \hat{R}_{kl}^{ir} C_{ps}^n C_{rm}^n \] (5)

\[ C_{ef}^j \hat{R}_{ie}^{pq} \hat{R}_{pq}^{bf} = \hat{R}_{da}^{ij} C_{bc}^d \] (6)

The four relations (2), (3), (4) and (5) are henceforth called the bicovariance conditions. Note that the classical limit of \( \hat{R}_{ij}^{kl} \) is \( \delta_i^j \delta_k^l \) so that relations (3), (5) and (6) become trivial in this limit.

An example where (3) is satisfied, but (4), (5) and (6) are not, is given by the q-SU(2) algebra:

\[ q^{-1} T_0 T_+ - q T_+ T_0 = T_+ \]
\[ q T_0 T_- - q^{-1} T_- T_0 = -T_- \]
\[ q^{\frac{1}{2}} T_+ T_- - q^{-\frac{1}{2}} T_- T_+ = T_0 \] (7)

obtained from the Drinfeld-Jimbo q-SU(2) [1]:

\[ [H_0, H_\pm] = \pm H_\pm, \quad [H_+, H_-] = \frac{1}{2} \frac{q^{H_0} - q^{-H_0}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \] (8)
via the mapping [10]

\[ T_0 = \frac{1 - q^{-2H_0}}{q - q^{-1}}, \quad T_{\pm} = q^{-\frac{H_0}{2}} \sqrt{\frac{2}{q^\pm + q^{-\pm}}} H_{\pm} \]  

Indeed the differential calculus corresponding to (7) is only left covariant (and is essentially equivalent to the left covariant differential calculus on the q-SU(2) of ref. [3a]).

Let us recall the basic facts about bicovariant differential calculus on quantum groups. It is defined by [3c]:

1) a linear map \( d : A \to \Gamma \), satisfying the Leibniz rule \( d(ab) = (da)b + a(db) \), \( \forall a, b \in A \). \( \Gamma \) is a bimodule over \( a \), which essentially means that it can be multiplied from the left and from the right by elements of \( A \), and is the q-analogue of the space of 1-forms on a Lie group. Every element \( \rho \) of \( \Gamma \) is assumed to be expressible as \( \rho = \sum_k a_k db_k \).

2) the left action of the quantum group on \( \Gamma \):

\[ \Delta_L : \Gamma \to A \otimes \Gamma, \quad \Delta_L(\sum_k a_k db_k) = \sum_k \Delta(a_k)(id \otimes d)\Delta(b_k) \]  

generalizes the ordinary pull-back on forms. If we call \( L_x \) the left multiplication by an element \( X \) of a group \( G \):

\[ L_x \equiv xy \quad \forall x, y \in G \]

then the pull-back \( L_x^* \) on 1-forms \( \omega \) is defined by:

\[ (L_x^* \omega)(y) = \omega(xy)|_y \]

and we can consider the map \( L^* \):

\[ L^* : T^* G \to Fun(G) \times T^* G \]

\[ L^* \omega(x, y) \equiv (L_x^* \omega)(y) \]

The left action \( \Delta_L \) reduces to \( L^* \) in the classical limit. The map \( \Delta_L \) is assumed to satisfy properties that q-generalize the well-known properties of \( L^* \) (see [11] for a detailed discussion).

Left-invariant 1-forms \( \omega \) classically satisfy \( L_x^* \omega = \omega \) or equivalently:

\[ L^* \omega = I \otimes \omega \]  

where \( I \) is the identity in \( A \), i.e. the function on \( G \) that sends all elements of \( G \) into the identity of \( \mathbb{C} \). Indeed

\[ L^* \omega(x, y) = (I \otimes \omega)(x, y) = I(x)\omega(y) = \omega(y) \]
In complete analogy, left-invariant 1-forms $\omega$ on quantum groups are defined to satisfy

$$\Delta_L(\omega) = I \otimes \omega$$ (12)

3) the right action of the quantum group on $\Gamma$:

$$\Delta_R : \Gamma \to \Gamma \otimes A, \quad \Delta_R(\sum_k a_k db_k) = \sum_k \Delta(a_k)(d \otimes id)\Delta(b_k)$$ (13)

generalizes the ordinary pull-back on forms induced by right multiplication $R_x y = yx$.

The discussion in 2) can be repeated for $\Delta_R$.

4) the q-analogue of the fact that left and right actions commute ($L^*_x R^*_y = R^*_y L^*_x$):

$$(id \otimes \Delta_R)\Delta_L = (\Delta_L \otimes id)\Delta_R$$ (14)

We now summarize some important consequences of 1), 2), 3) and 4), derived in [3c].

As in the classical case, the whole of $\Gamma$ is generated by $\{\omega^i\}$, a basis in the vector space $\Gamma_{inv}$ of all left-invariant elements of $\Gamma$ (or by $\{\eta^i\}$, a basis for the right-invariant elements of $\Gamma$, for which $\Delta_R(\eta^i) = \eta^i \otimes I$) Any $\rho \in \Gamma$ is expressible (uniquely) as $\rho = a^i \omega^i$ or also as $\rho = \omega^i b^i$. Therefore we must have:

$$\omega^i b = F^i_j(b) \omega^j, \quad F^i_j : A \to A \text{ linear map}$$

or equivalently

$$\omega^i b = (f^i_j \ast b) \omega^j \equiv (id \otimes f^i_j)\Delta(b)\omega^j$$ (15)

where $f^i_j \in A'$ is the functional on $A$ defined by $f^i_j(a) = \varepsilon(F^i_j(a))$, $\varepsilon$ being the counit of $A$. The same reasoning for right-invariant 1-forms $\eta^i$ yields the equation:

$$b\eta^i = \eta^j(b \ast (f^i_j \circ \kappa)) \equiv \eta^j((f^i_j \circ \kappa) \otimes id)\Delta(b), \quad \kappa = \text{coinverse in } A$$

For consistency, the functionals $f^i_j$ must satisfy the relations:

$$f^i_j(ab) = \sum_k f^i_k(a)f^k_j(b)$$ (16a)

$$f^i_j(I) = \delta^i_j$$ (16b)

$$(f^k_j \circ \kappa)f^i_j = \delta^i_k 1; \quad f^k_j f^j_i \circ \kappa)f^j_k = \delta^i_k 1;$$ (16c)

so that their coproduct, counit and coinverse are given by:

$$\Delta'(f^i_j) = f^i_k \otimes f^k_j$$ (17a)
\[ \varepsilon'(f^i_j) = \delta^i_j \]  

(17b)

\[ \kappa'(f^i_j) = f^i_j \circ \kappa \]  

(17c)

The quantum group element in the adjoint representation can be introduced as follows. It is easy to show that \( \Delta_R(\omega^i) \) belongs to \( \Gamma_{\text{inv}} \otimes A \), and therefore:

\[ \Delta_R(\omega^i) = \omega^j \otimes M_j^i, \quad M_j^i \in A; \]  

(18)

in the classical case, \( M_j^i \) is the adjoint representation of \( G \). This justifies our calling \( M_j^i \) the adjoint representation of the quantum group. The coproduct, counit and coinverse of \( M_j^i \) can be deduced to be [3c]:

\[ \Delta(M_j^i) = M_j^l \otimes M_l^i, \quad \varepsilon(M_j^i) = \delta^i_j \]  

(19)

Moreover one can prove the relation:

\[ M_j^i(a \ast f^i_k) = (f^j_i \ast a)M_k^i \]  

(20)

The last ingredient we need is the exterior product. It is defined by an automorphism \( \hat{R} \) in \( \Gamma \otimes \Gamma \) that generalizes the ordinary permutation operator:

\[ \hat{R}(\omega^i \otimes \eta^j) = \eta^i \otimes \omega^j \]

i) in general \( \hat{R}^2 \neq 1 \), since \( \hat{R}(\eta^i \otimes \omega^j) \) is not necessarily equal to \( \omega^i \otimes \eta^j \). By linearity \( \hat{R} \) can be extended to the whole of \( \Gamma \otimes \Gamma \).

ii) \( \hat{R} \) is invertible and commutes with the left and right actions of \( G \). \( \hat{R}(\omega^i \otimes \omega^j) \) is left-invariant, so that

\[ \hat{R}(\omega^i \otimes \omega^j) = \hat{R}^{ij}_{kl}\omega^k \otimes \omega^l \]  

(21)

iii) we have

\[ \hat{R}^{ij}_{kl} = f^i_k(M^j_l); \]  

(22)

thus the quantities \( f^i_k \), \( M^j_l \) characterizing the bimodule \( \Gamma \) are dual in the sense of eq. (22) and determine the exterior product:

\[ \rho \wedge \rho' \equiv \rho \otimes \rho' - \hat{R}(\rho \otimes \rho') \]

\[ \omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \hat{R}^{ij}_{kl}\omega^k \otimes \omega^l \]  

(23)
Having the exterior product we can define the exterior differential
\[ d : \Gamma \to \Gamma \wedge \Gamma \]
\[
d(\sum_k a_k db_k) = da_k \wedge db_k \quad (\Rightarrow d^2 = 0)
\]
which can easily be extended to \( \Gamma^\wedge n \) (\( d : \Gamma^\wedge n \to \Gamma^\wedge(n+1) \)) and can be shown to commute with the left and right actions of \( G \), as in the classical case [3c].

Finally, the hypotheses 1), 2), 3), 4) imply the existence of the \( T_i \) functionals satisfying eq. (1), and of the corresponding Cartan-Maurer equations for the left-invariant forms \( \omega^i \) [3c]. In eq. (1) the \( \hat{R} \) matrix is the same that defines the wedge product in (23). One can prove that the \( C \) and \( \hat{R} \) tensors do indeed satisfy the four conditions (3), (4), (5), (6). The action of \( T_i \) on \( M^k_j \) is found to be:

\[
T_i(M^k_l) = C^l_{ki}
\]  

(25)

In the sequel we also need the relations [3c,12,11]

\[
T_i(ab) = \Delta'(T_i)(a \otimes b)
\]  

(26)

\[
da = (T_i * a)\omega^i
\]  

(27)

See for ex. [11] for a detailed discussion on how to obtain (5) and (6), and eq. (25); all these relations are already implicitly contained in [3c] (see also [12]).

Using (22) and (25), we see that the conditions (3), (5) and (6) can be cast into operator form as:

\[
\hat{R}^{ab}_{\, \, nm} f^a_c f^m_d = f^a_n f^b_m \hat{R}^{nm}_{\, \, cd}
\]  

(28a)

\[
f^a_c T_b + C^a_{\, \, nm} f^m_c f^m_b = f^a_n C^{n}_{\, \, cb} + T_n f^a_m \hat{R}^{nm}_{\, \, cb}
\]  

(28b)

\[
T_k f^n_l = \hat{R}^{ij}_{\, \, kl} f^n_i T_j
\]  

(28c)

whereas condition (4) follows from the definition of the \( q \)-commutations (1) applied to \( M^j_i \). In fact, the bicovariance conditions define a quasi triangular quantum Lie algebra, whose associated quantum group admits a bicovariant differential calculus *(see e.g.[12]). By “associated quantum group” we mean the Hopf algebra with Hopf structures dual to

* Strictly speaking, (3), (5) and (6) do not necessarily imply that eqs. (28) hold on the whole algebra \( A \), since the algebra generated by the \( M^j_i \) is in general a subalgebra of \( A \).
those of the quantum Lie algebra. At this juncture we note that the coproduct, counit
and coinverse of the q-algebra (1) can be consistently defined as:

\[ \Delta'(T_i) = T_j \otimes f^j_i + 1 \otimes T_i \]  \hspace{1cm} (29a)

\[ \varepsilon'(T_i) = 0 \]  \hspace{1cm} (29b)

\[ \kappa'(T_i) = -T_j \kappa'(f^j_i) \]  \hspace{1cm} (29c)

Viceversa, suppose that we find some tensors \( \hat{R}^{ij}_{kl} \) and \( C^i_{jk} \) satisfying the bico-
variance conditions (3), (4), (5), (6). Then we have a quasi triangular quantum Lie
algebra, and we can construct a bicovariant calculus on the quantum group dual to the
q-commutator algebra (1). We illustrate the method in the example of D=2 q-Poincaré.
Consider the 3-parameter deformed algebra:

\[ T_0 T_+ - T_+ T_0 = r T_+ \]

\[ T_0 T_- - T_- T_0 = -s T_- \]

\[ q^{\frac{1}{2}} T_+ T_- - q^{-\frac{1}{2}} T_- T_+ = 0 \]  \hspace{1cm} (30)

which reduces to the D=2 Poincaré algebra for \( r \to 1, s \to 1 \) and \( q \to 1 \). The corre-
sp\( \text{ponding } \hat{R} \text{ and } C \text{ tensors are} \)

\[ \hat{R}^{++}_{++} = q^{-1} = f^{-}_-(M^+_+), \quad \hat{R}^{+-}_{++} = q = f^+_- (M^-) \]

\[ \hat{R}^{+0}_{00} = 1 = f^+_+ (M^+_0), \quad \hat{R}^{0+}_{00} = 1 = f^0_0 (M^+_+) \]

\[ \hat{R}^{-0}_{0-} = 1 = f^-_- (M^+_0), \quad \hat{R}^{0-}_{0-} = 1 = f^0_0 (M^-) \]

\[ \hat{R}^{++}_{++} = 1 = f^+_+ (M^+_+), \quad \hat{R}^{+-}_{--} = 1 = f^-_- (M^-), \quad \hat{R}^{00}_{00} = 1 = f^0_0 (M^+_0) \]  \hspace{1cm} (31)

\[ C^0_{-0} = s = T_0 (M^-), \quad C^+_0 = -r = T_0 (M^+_+) \]

\[ C^-_{-0} = -s = T_- (M^-_0), \quad C^+_{+0} = r = T_+ (M^+_0) \]  \hspace{1cm} (32)

all other components vanishing.

It is not difficult to check that the bicovariance conditions (3), (4) (5) and (6)
are indeed fulfilled by the tensors given above. In (29) and (30) we have adjoined the
equalities due to eqs. (22) and (25). The matrix \( M^i_j \) can be seen as the generic quantum
group element in the adjoint representation (cfr. eq. (18)). Let us try to give this matrix
an explicit form. To get a feeling of what it should look like, we consider it in the limit
q=1. In the classical case the Lie algebra generators in the adjoint representation are
\[(T_0)^i_j = -C_{0j}^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\]

\[(T_+)^i_j = -C_{+j}^i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\]

\[(T_-)^i_j = -C_{-j}^i = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\] (33)

The generic group element of D=2 Poincaré in the adjoint representation can then be parametrized with exponential coordinates:

\[M = \exp(x^0 T_0) \exp(x^+ T_+) \exp(x^- T_-) = \begin{pmatrix} 1 & x^+ & -x^- \\ 0 & \exp(-x^0) & 0 \\ 0 & 0 & \exp(x^0) \end{pmatrix}\] (34)

as the reader can verify by using eqs. (33).

In the quantum case, the matrix elements \(M_{ij}\) do not commute any more, but satisfy the exchange relations:

\[M_{ij} M_{rq} \hat{R}_{pk} = \hat{R}_{jq} M_{rp} M_{ki}\] (35)

that are obtained by taking \(a = M_p^q\) in eq. (20), and using (22). To obtain further information on the elements of the quantum \(M_{ij}\), we apply the formula (27) to \(M_{ij}\):

\[dM_{ij} = (T_k \ast M_{ij}) \omega^k \equiv (id \otimes T_k) \Delta(M_{ij}) = M_{ij}^k T_k (M_{ij})\] (36)

and find the differentials:

\[dM_0^0 = dM_+^0 = dM_-^0 = 0\]
\[dM_+^+ = r(-M_+^+ \omega^0 + M_+^0 \omega^+), \quad dM_-^- = s(M_-^- \omega^0 - M_-^0 \omega^-)\]
\[dM_+^- = s(M_+^- \omega^0 - M_+^0 \omega^-), \quad dM_-^+ = r(-M_-^+ \omega^0 + M_-^0 \omega^+)\]
\[dM_0^+ = r(-M_0^+ \omega^0 + \omega^+), \quad dM_0^- = s(-M_0^- \omega^0 + \omega^-)\] (37)

From the first line we see that \(M_0^0, M_+^0\) and \(M_-^0\) must be constants (i.e. proportional to the unit of \(A\)); moreover from eq. (35) we find that \(M_+^0 M_-^0 = q^{-1} M_-^0 M_+^0\), which implies that the constant \(M_+^0, M_-^0\) must vanish. The element \(M_0^0\), which according to (35) commutes with all the other elements, can be chosen equal to the unit \(I\). Finally, the elements \(M_+^-\) and \(M_-^+\) can also consistently be taken as vanishing, and we arrive at the quantum matrix:
\[ M_i^j = \begin{pmatrix} I & \frac{1}{r}x^+ & -\frac{1}{s}x^- \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \] (38)

whose elements satisfy the commutations:

\[
\begin{align*}
x^+x^- &= qx^-x^+ \\
u x^- &= qx^-u \\
v x^+ &= q^{-1}x^+v
\end{align*}
\] (39)

all other being trivial. The “coordinates” \( x^i \) belonging to \( A \) are defined by:

\[ T_i(x^j) = \delta^j_i \] (40)

(such \( x^i \in A \) can always be found, see [3c]).

**Note 1:** the commutation relation of the quantum plane coordinates \( x^+ \) and \( x^- \) is formally similar to the one of ref.s [4,5]. However here we are dealing with light-cone coordinates.

**Note 2:** it is not obvious how one can parametrize the diagonal elements \( u \) and \( v \) with the coordinate \( x^0 \).

**Note 3:** as in the classical case, \( T_i \) can be represented as \( \frac{\partial}{\partial x^i}|_{x=0} \), the partial derivatives being defined by \( da = (T_i \star a)\omega^i \equiv (\frac{\partial}{\partial x^i}a)\omega^i \), \( \forall a \in A \).

**Note 4:** the product of two matrices \( M(x,u,v) \) and \( M(y,w,z) \) of type (34) yields a matrix \( M \) of the same type:

\[ M = \begin{pmatrix} 1 & \frac{1}{r}(y^+ + x^+w) & -\frac{1}{s}(y^- + x^-z) \\ 0 & uw & 0 \\ 0 & 0 & vz \end{pmatrix} \] (41)

The inverse (in the matrix sense) of (34) gives the explicit form of the coinverse of \( M_i^j \), since \( \kappa(M_i^j)M_j^k = \delta^k_i \) (cfr. eq. (19)):

\[ (M^{-1})_i^j = \kappa(M_i^j) = \begin{pmatrix} 1 & -\frac{1}{r}u^{-1}x^+ & \frac{1}{s}v^{-1}x^- \\ 0 & u^{-1} & 0 \\ 0 & 0 & v^{-1} \end{pmatrix} \] (42)

where we have introduced the new elements \( u^{-1} \) and \( v^{-1} \) satisfying \( u^{-1}u = uu^{-1} = v^{-1}v = vv^{-1} = I \). In the Table we have collected all the information on the quantum group \( M \) generated by \( I, x^+, x^-, u, v, u^{-1}, v^{-1} \). This group reduces to the matrix group (38) in the classical limit. The product of two quantum matrices in (41) again satisfies the exchange relations (35), if we recall that all the elements of \( M(x,u,v) \) commute with
all the elements of $M(y,w,z)$. On the other hand, the inverse satisfies the exchange relations (35) with $q \rightarrow q^{-1}$.

**Note 5**: the non-commutativity of the quantum plane arises as part of the geometry of the whole Poincaré quantum group, rather than of a representation space for the action of the D=2 rotation group as in [4,5]. In fact, here the quantum plane has the coset interpretation $q$-Poincaré / $q$-Lorentz, the $q$-Lorentz group in D=2 being the SO(1,1) generated by $T_0$.

The functionals $f^i_j$ are also easy to find: they are determined by their value on the $M^l_k$ (then their value on any power of $M^l_k$ can be found via the rule (16a)). In the adjoint representation they take the form:

$$(f_+)^i_j \equiv f^+(M^i_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}$$ (43a)

$$(f_-)^i_j \equiv f^-(M^i_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (43b)

$$(f_0)^i_j \equiv f^0(M^i_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (43c)

Note the relation $f_+ f_- = q T_0$.

The Cartan-Maurer equations of our deformed D=2 Poincaré algebra are:

$$d\omega^+ - r \frac{1}{2} \omega^+ \wedge \omega^0 = 0$$

$$d\omega^- - s \frac{1}{2} \omega^- \wedge \omega^0 = 0$$

$$d\omega^0 = 0$$ (44)

They can be deduced from the general formula [3c]:

$$d\omega^k + (T_i T_j)(x^k)\omega^i \wedge \omega^j = 0$$ (45)

Setting

$$C^k_{ij} \equiv (T_i T_j)(x^k)$$ (46)

and recalling that $C^k_{ij}$ can be expressed as

$$C^k_{ij} \equiv (T_i T_j - \hat{R}^{kl}_{ij} T_k T_l)(x^k)$$ (47)
(apply (1) to $x^k$ and use (40)) we find the relation:

$$C^k_{ij} = C^k_{ij} - \hat{R}^{lm}_{ij} C^l_{km}$$  \hspace{1cm} (48)

Thus the structure constants $C$ appearing in the Cartan-Maurer equations (45) are in general different from the structure constants $C$ of the q-commutator algebra (1). In fact, they coincide only in the case $\hat{R}^2 = 1$ ($\hat{R}^{ij}_{kl} \hat{R}^{kl}_{rs} = \delta^i_r \delta^j_s$). For our q-algebra (30) we have indeed $\hat{R}^2 = 1$, so that its Cartan-Maurer equations are as given in (44).

The commutation relations of the “coordinates” $x^i$ with the left-invariant one-forms $\omega^k$ are deduced from eq. (15), and wedge products of $\omega^i$ are defined in (23). We list them in the Table that summarizes the bicovariant calculus on $D=2$ q-Poincaré.

We point out that other deformations of the $D = 2$ Poincaré algebra have been found [13,14]. However, only the one given in (28) leads to a bicovariant differential calculus.

We conclude by outlining a procedure to “gauge” the quantum (Poincaré) groups that follows from the geometric approach advocated in [15] (see [15b] for a brief introduction).

The idea is to allow the right-hand side of eq. (45) to be nonvanishing, and then to interpret it as the curvature $R^k$ associated to $\omega^k$. This is called “softening” of the group $G$, and formally the same thing can be carried out for quantum groups. The closure of the $d$ operator, and the quantum Jacobi identities (4) lead to differential conditions on the curvatures, the quantum Bianchi identities:

$$dR^k + C^k_{ij} R^i \wedge \omega^j - C^k_{ij} \omega^i \wedge R^j = 0$$  \hspace{1cm} (49)

A quantum Lie derivative [3,12] can be introduced on the “soft quantum group”, using its representation as the operator:

$$l_y = i_y d + d i_y$$  \hspace{1cm} (50)

that holds also in the quantum case, provided the contraction operator $i_y$ is defined appropriately [11] ($y$ belongs to the dual of the soft $A$, and is the q-analogue of a tangent vector).

This allows the definition of “quantum diffeomorphisms”:

$$\delta_y \omega^k = l_y \omega^k = (i_y d + d i_y) \omega^k = (\nabla y)^k + i_y R^k$$  \hspace{1cm} (51)

where $\nabla$ is the quantum covariant derivative whose definition can be read off the Bianchi identities (49) $\nabla R^k = 0$.

The construction of an action invariant under the transformations (51) proceeds as in the classical case. For example, the lagrangian for $D=4$ q-gravity is formally unchanged:

$$L_q = R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd}$$  \hspace{1cm} (52)
$R^{ab}$ refers now to the q-Lorentz part of the q-Poincaré curvature, $V^c$ is the softening of the left-invariant one-form corresponding to the translations, the wedge product is defined as in (23), and the $\varepsilon_{abcd}$ tensor is the quantum generalization of the alternating tensor for the q-Lorentz group. This of course presupposes that one can find a q-Poincaré group containing q-Lorentz as a subgroup, which is indeed the case [16, 9].

\begin{table}
\caption{The bicovariant calculus on D=2 q-Poincaré}
\end{table}

\[ D=2 \text{ q-Poincaré Hopf algebra} \]

\begin{align*}
T_0 T_+ - T_+ T_0 &= r T_+ \\
T_0 T_- - T_- T_0 &= -s T_- \\
q^{\frac{1}{2}} T_+ T_- - q^{-\frac{1}{2}} T_- T_+ &= 0 \\
\Delta'(T_0) &= T_0 \otimes 1 + 1 \otimes T_0 \\
\Delta'(T_\pm) &= T_\pm \otimes f_\pm + 1 \otimes T_\pm \\
\varepsilon'(T_i) &= 0; \quad \kappa'(T_0) = -T_0, \quad \kappa'(T_\pm) = -T_\pm \kappa'(f_\pm)
\end{align*}

\begin{align*}
(T_0)^i_j &= -C^i_{0j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(T_+)^i_j &= -C^i_{+j} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
(T_-)^i_j &= -C^i_{-j} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
(f_+)^i_j &\equiv f_+^i (M_j^i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} \\
(f_-)^i_j &\equiv f_-^i (M_j^i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}

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\[(f_0)^i_j \equiv f^0_0(M^i_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\]

\[M^i_j = \begin{pmatrix} I & \frac{1}{r}x^+ & -\frac{1}{s}x^- \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix}\]

\[\kappa(M^i_j) = \begin{pmatrix} 1 & -\frac{1}{r}u^{-1}x^+ & \frac{1}{s}v^{-1}x^- \\ 0 & u^{-1} & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}\]

Exchange relations of the algebra \(\mathcal{M}\), generated by \(I, x^+, x^-, u, v, u^{-1}, v^{-1}\)

\[x^+x^- = qx^-x^+\]
\[ux^- = qx^-u\]
\[vx^+ = q^{-1}x^+v\]
\[v^{-1}x^+ = qx^+v^{-1}\]
\[u^{-1}x^- = q^{-1}x^-u^{-1}\]

Hopf structure on \(\mathcal{M}\)

Coproduct:

\[\Delta(I) = I \otimes I, \quad \Delta(x^+) = I \otimes x^+ + x^+ \otimes u, \quad \Delta(x^-) = I \otimes x^- + x^- \otimes v\]
\[\Delta(u) = u \otimes u, \quad \Delta(v) = v \otimes v\]
\[\Delta(u^{-1}) = u^{-1} \otimes u^{-1}, \quad \Delta(v^{-1}) = v^{-1} \otimes v^{-1}\]

Counit:

\[\varepsilon(I) = 1, \quad \varepsilon(x^+) = \varepsilon(x^-) = 0\]
\[\varepsilon(u) = \varepsilon(v) = \varepsilon(u^{-1}) = \varepsilon(v^{-1}) = 1\]

Coinverse:

\[\kappa(I) = 1, \quad \kappa(x^+) = -u^{-1}x^+, \quad \kappa(x^-) = -v^{-1}x^-\]
\[\kappa(u) = u^{-1}, \quad \kappa(v) = v^{-1}, \quad \kappa(u^{-1}) = u, \quad \kappa(v^{-1}) = v\]
Bicovariant differential calculus on $\mathcal{M}$

Exterior derivative:

$$dx^+ = -x^+ \omega^0 + \omega^+, \quad dx^- = -x^- \omega^0 + \omega^-$$

$$du = -ru \omega^0, \quad dv = sv \omega^0, \quad du^{-1} = ru^{-1} \omega^0, \quad dv^{-1} = -sv^{-1} \omega^0$$

The action of $T_i$ and $f^i_j$:

$$T_\pm(x^\pm) = 1, \quad T_0(u) = -r, \quad T_0(v) = s, \quad T_0(u^{-1}) = r, \quad T_0(v^{-1}) = -s$$

$$f_+(v) = q, \quad f_-(u) = q^{-1}, \quad f_+(u) = f_-(v) = f_0(I) = f_0(u) = f_0(v) = 1$$

$$f_+(v^{-1}) = q^{-1}, \quad f_-(u^{-1}) = q, \quad f_+(u^{-1}) = f_-(v^{-1}) = f_0(u^{-1}) = f_0(v^{-1}) = 1$$

Left and right actions

$$\Delta_R(\omega^i) = \omega^j \otimes M^i_j, \quad \omega^i = \text{left invariant one\textendash}forms$$

$$\Delta_L(\eta^i) = \kappa(M_j^i) \otimes \eta^j, \quad \eta^i = \text{right invariant one\textendash}forms$$

Commutation relations of $M^i_j$ and left-invariant forms

$$\omega^+ x^- = qx^- \omega^+, \quad \omega^- x^+ = q^{-1} x^+ \omega^-$$

$$\omega^+ v = qv \omega^+, \quad \omega^- u = q^{-1} u \omega^-$$

all other commutations are trivial

Wedge products

$$\omega^+ \wedge \omega^- = \omega^+ \otimes \omega^- - q \omega^- \otimes \omega^+$$

$$\omega^- \wedge \omega^+ = \omega^- \otimes \omega^+ - q^{-1} \omega^+ \otimes \omega^-$$

$$\Rightarrow \omega^+ \wedge \omega^- = -q \omega^- \wedge \omega^+$$

all other wedge products as in classical case

Cartan-Maurer equations

$$d\omega^+ - r \frac{1}{2} \omega^+ \wedge \omega^0 = 0$$

$$d\omega^- + s \frac{1}{2} \omega^- \wedge \omega^0 = 0$$

$$d\omega^0 = 0$$
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