The vacuum shell in the Schwarzschild–de Sitter world

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Abstract
We construct the classification scheme for all possible evolution scenarios and find the corresponding global geometries for dynamics of a thin spherical vacuum shell in the Schwarzschild–de Sitter metric. This configuration is suitable for the modeling of vacuum bubbles arising during cosmological phase transitions in the early universe. The distinctive final types of evolution from the local point of view of a rather distant observer are either the unlimited expansion of the shell or its contraction with a formation of black hole (with a central singularity) or wormhole (with a baby universe in interior).

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1. Introduction
One of the promising mechanisms of primordial black holes and wormholes formation is a collapse of vacuum bubbles during phase transitions in the early universe [1–5]. The supermassive black holes in the centers of galaxies may be also originated by this mechanism. We will describe the possible types of dynamical evolution of vacuum bubbles in the Schwarzschild–de Sitter metric by using the thin shell approximation for the boundary between the true and false vacuum. The formalism of thin shells in general relativity was first developed by W Israel [6]. Later this formalism was elaborated and adjusted for the case of cosmological vacuum phase transitions [7].

The boundary of vacuum bubble divides the Schwarzschild–de Sitter spacetime into internal and external regions. In the following, these regions are designated by indexes in and out respectively. Our aim is a full classification of possible evolution scenarios of vacuum bubbles versus of parameters of the Schwarzschild–de Sitter spacetime and initial conditions for a thin shell. It appears that a useful classification quantity for this problem is a mass parameter \( m \) which will be defined below. Depending on these mass parameters, the bubble is either expanding to infinity or contracting with a final formation of black hole or wormhole.
This work is a generalization of the earlier analysis in [13–16], where only some particular scenarios for this problem were investigated.

In previous works, [7–9, 13, 14, 17, 18] were considered only the particular cases of solutions corresponding to the zero value of our inner mass parameter $m_{in} = 0$. We describe a more general case when $m_{in} \neq 0$, and then the new types of solutions appear. The bubbles are originated in the phase transition in the early universe and may contain in principle smaller bubbles inside (see e.g. [8, 9]). To model this in a formal way we include an interior mass parameter $m_{in}$. This parameter may be considered as a seed black hole. An analogous problem was analyzed in $2 + 1$ dimensions in [19]. In general, solutions with $m_{in} \neq 0$ are quite similar to the ones considered in [7–9, 13, 14, 17, 18]. Standard classification methods (e.g. [13]) are straightforwardly extended to the case where an interior mass parameter and energy density of the outer region are added.

In section 2, the equation of motion for a thin shell in the Schwarzschild–de Sitter metric is analyzed. Based on this equation, in section 3 we develop a classification scheme for possible evolution scenarios and also construct the Carter–Penrose diagrams for corresponding global geometries. The concluding remarks are shortly summarized in section 4.

2. The equation of motion

The Schwarzschild–de Sitter metric can be written in the form

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{8\pi}{3}\varepsilon r^2\right)dt^2 - \left(1 - \frac{2M}{r} - \frac{8\pi}{3}\varepsilon r^2\right)^{-1}dr^2 - r^2d\Omega^2,$$

where $M$ is the Schwarzschild mass, $\varepsilon$ is a vacuum energy density and $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$. This metric has the following specific properties. The positive roots of equation

$$1 - \frac{2M}{r} - \frac{8\pi}{3}\varepsilon r^2 = 0$$

(2)

define the radii of event horizons in this metric. The number of positive roots (and so the number of event horizons) depends on the ratio of $M$ and $1/\sqrt{72\pi\varepsilon}$. There are no event horizons, if $M > m_2 = 1/\sqrt{72\pi\varepsilon}$. There is only one event horizon, $r_{h1} = 1/\sqrt{8\pi\varepsilon}$, if $M = m_2$. In the case $M < m_2$ there are two distinctive event horizons:

$$r_{h2} = 2 \sqrt{\frac{p}{3}} \cos \left[ \frac{\pi}{3} + \frac{1}{3} \arctan \sqrt{\frac{4p^3}{27q^2} - 1} \right];$$

(3)

$$r_{h3} = 2 \sqrt{\frac{p}{3}} \cos \left[ \frac{\pi}{3} - \frac{1}{3} \arctan \sqrt{\frac{4p^3}{27q^2} - 1} \right],$$

(4)

where $p = 3/(8\pi\varepsilon)$ and $q = 3M/(4\pi\varepsilon)$. It can be shown that $r_{h2} < r_{h1} < r_{h3} < \sqrt{3}r_{h1}$. See in figure 1 the corresponding Carter–Penrose diagrams for global geometry of the Schwarzschild–de Sitter spacetime [14, 20].

The basic equation of motion for a thin vacuum shell in the Schwarzschild–de Sitter spacetime, resulting from the matching of the inner and outer metrics on the shell, can be written in the following form [7, 15, 21–26]:

$$4\pi \sigma \rho = \sigma_{in} \sqrt{\rho^2 + 1 - \frac{8\pi}{3}\varepsilon_{in}\rho^2 - \frac{2m_{in}}{\rho}} - \sigma_{out} \sqrt{\rho^2 + 1 - \frac{8\pi}{3}\varepsilon_{out}\rho^2 - \frac{2m_{out}}{\rho}}.$$

(5)
In this equation $\rho = \rho(\tau)$ is a shell radius, $\tau$ is a proper time measured by an observer at rest with respect to this shell, $\dot{\rho}$ is the derivative with respect to the proper time, $\sigma$ is the surface energy density on the shell (in the discussed vacuum case $\sigma = \text{const}$), $m_{\text{in}}$ and $m_{\text{out}}$ is the Schwarzschild mass of the inner and outer region respectively, $\varepsilon_{\text{in}}$ is the energy density of the inner region, $\varepsilon_{\text{out}}$ is the energy density of the outer region and symbols $\sigma_{\text{in,out}} = \pm 1$. These symbols equal to 1, if the radius of a two-dimensional sphere is growing in the direction of an outgoing normal, and equal to $-1$ in the opposite case.

For a further analysis, we rewrite the equation of motion (5) in the ‘energy conservation’ form

$$\left(\frac{1}{2}\right)\dot{\rho}^2 + U(\rho) = 0,$$

with an effective potential

$$U(\rho) = \frac{1}{2} \frac{(m_{\text{out}} - m_{\text{in}})^2}{32\pi^2 \sigma^2 \rho^4} \left(\frac{m_{\text{out}} + m_{\text{in}}}{2\rho} - \frac{(m_{\text{out}} - m_{\text{in}})\varepsilon_{\text{out}} - \varepsilon_{\text{in}}}{12\pi \sigma^2 \rho}ight) - \frac{(\varepsilon_{\text{in}} + \varepsilon_{\text{out}} + 6\pi \sigma^2)^2 - 4\varepsilon_{\text{in}}\varepsilon_{\text{out}}}{18\sigma^2} \rho^2,$$

which is shown in figure 2. Values of $\sigma_{\text{in}}$ and $\sigma_{\text{out}}$ in the equation of motion (5) in the ‘energy conservation’ form are defined by relations

$$\sigma_{\text{in}} = \text{sign} \left[ m_{\text{out}} - m_{\text{in}} + \frac{4\pi}{3} (\varepsilon_{\text{out}} - \varepsilon_{\text{in}}) \rho^3 + 8\pi^2 \sigma^2 \rho^3 \right];$$

$$\sigma_{\text{out}} = \text{sign} \left[ m_{\text{out}} - m_{\text{in}} + \frac{4\pi}{3} (\varepsilon_{\text{out}} - \varepsilon_{\text{in}}) \rho^3 - 8\pi^2 \sigma^2 \rho^3 \right].$$

We will consider the general case when a surface energy density of the shell $\sigma$ may be positive and negative. It must be noted that for the positive value of $\sigma$ there is the exceptional case,
when $\sigma_{in} = -1$ and $\sigma_{out} = 1$. In this case, the equation of motion (5) has no solution. It would be also seen in the following Carter–Penrose diagrams.

It is useful to define the following two quantities, one with a dimension of mass, $m = m_{out} + m_{in}$, and one dimensionless, $\mu = (m_{out} - m_{in})/(m_{out} + m_{in})$. With these definitions it is seen from (7) that $\sigma_{in}$ changes its sign at $\rho(\tau) = \rho_1$, where

$$\rho_1^3 = \frac{3\mu m}{4\pi (\varepsilon_{in} - \varepsilon_{out} - 6\pi \sigma^2)},$$

provided that (i) $\mu > 0$ and $\varepsilon_{in} > \varepsilon_{out} + 6\pi \sigma^2$ or (ii) $\mu < 0$ and $\varepsilon_{in} < \varepsilon_{out} + 6\pi \sigma^2$. Respectively, $\sigma_{out}$ changes its sign when $\rho(\tau) = \rho_2$, where

$$\rho_2^3 = \frac{3\mu m}{4\pi (\varepsilon_{in} - \varepsilon_{out} + 6\pi \sigma^2)},$$

provided, that (iii) $\mu > 0$ and $\varepsilon_{in} > \varepsilon_{out} - 6\pi \sigma^2$ or (iv) $\mu < 0$ and $\varepsilon_{in} < \varepsilon_{out} - 6\pi \sigma^2$.

Now we consider a behavior of the effective potential (6). The zeros of this potential, $U(\rho) = 0$, define the bounce points of solution when $\dot{\rho} = 0$. The maximum of potential (6) is at the point $\rho = \rho_{max}$, where

$$\rho_{max}^3 = m_{y_{max}} = \frac{9\sigma^2}{4m} \left\{ 1 + \mu \frac{\varepsilon_{out} - \varepsilon_{in}}{6\pi \sigma^2} \right\} \left[ 1 + \mu \frac{\varepsilon_{out} - \varepsilon_{in}}{6\pi \sigma^2} \right]^2 + \frac{2\mu^2}{9\pi^2 \sigma^4} \left\{ (\varepsilon_{in} + \varepsilon_{out} + 6\pi \sigma^2)^2 - 4\varepsilon_{in}\varepsilon_{out} \right\} / \left\{ (\varepsilon_{in} + \varepsilon_{out} + 6\pi \sigma^2)^2 - 4\varepsilon_{in}\varepsilon_{out} \right\}. $$

This maximum corresponds to the zero of potential, $U(\rho_{max}) = 0$, at $m = m_0$, where

$$m_0 = \sqrt{y_{max}} / \left\{ \frac{9\sigma^2}{4m} \left\{ (\varepsilon_{in} + \varepsilon_{out} + 6\pi \sigma^2)^2 - 4\varepsilon_{in}\varepsilon_{out} \right\} + \frac{\varepsilon_{out} - \varepsilon_{in}}{6\pi \sigma^2} + \frac{\mu^2}{16\pi^2 \sigma^2 y_{max}^2} \right\}^{1/2}. $$

Figure 2. The graphs of potential function $U(\rho)$ from (6) at $m < m_0$ (left) and $m > m_0$ (right).
The potential $U(\rho_{\text{max}}) > 0$ at $m < m_0$ and vice versa. A second derivative of the potential (6) is

$$\frac{d^2U}{d\rho^2} = -\frac{m}{\rho^3} - \frac{4\pi(\varepsilon_{\text{out}} + \varepsilon_{\text{in}}) + 12\pi^2\sigma^2}{3}$$

$$- \frac{9\mu^2m^2}{16\pi^2\sigma^2\rho^6} - \left(\frac{\varepsilon_{\text{out}} - \varepsilon_{\text{in}}}{3\sigma} + \frac{\mu m}{4\pi\sigma\rho^3}\right)^2 < 0. \quad (13)$$

The second derivative of the potential is negative everywhere, and so there is no stable equilibrium point for the equation of motion (5).

At the next step let us define the values of parameter $m$ when radii $\rho_1$ and $\rho_2$ (where $\sigma_{\text{in}}$ and $\sigma_{\text{out}}$ change the sign) coincide with the bounce points of the equation of motion (5). The corresponding solution of equation $U(\rho_1) = 0$ is $m = m_1$, where

$$m_1 = \frac{3\mu}{\sqrt{4\pi(\varepsilon_{\text{in}} - \varepsilon_{\text{out}} - 6\pi\sigma^2)(1 + \mu \varepsilon_{\text{in}} + 6\pi\sigma^2)^3}}. \quad (14)$$

Respectively, the corresponding solution of equation $U(\rho_2) = 0$ is $m = m_3$, where

$$m_3 = \frac{3\mu}{\sqrt{4\pi(\varepsilon_{\text{in}} - \varepsilon_{\text{out}} + 6\pi\sigma^2)(1 + \mu \varepsilon_{\text{in}} + 6\pi\sigma^2)^3}}. \quad (15)$$

By using (12), (14) and (15) it can be shown that both $m_1 < m_0$ and $m_3 < m_0$.

There are degenerate cases when the inner and outer regions with respect to the shell have only one event horizon. The inner Schwarzschild–de Sitter metric has only one event horizon when $m = m_{21}$, where

$$m_{21} = \frac{1}{(1 - \mu)\sqrt{18\pi\varepsilon_{\text{in}}}}. \quad (16)$$

Respectively, the outer Schwarzschild–de Sitter metric has only one event horizon when $m = m_{22}$, where

$$m_{22} = \frac{1}{(1 + \mu)\sqrt{18\pi\varepsilon_{\text{out}}}}. \quad (17)$$

It can be verified that both $m_{21} > m_0$ and $m_{22} > m_0$.

For the following analysis of the dynamical evolution of the shell, it is important to know the values of potential (6) at the event horizons of both the inner and outer metrics, $\rho_{\text{in}} = (\rho_{\text{in1}}, \rho_{\text{in2}}, \rho_{\text{in3}})$ and $\rho_{\text{out}} = (\rho_{\text{out1}}, \rho_{\text{out2}}, \rho_{\text{out3}})$, respectively. By using equation (2) for the inner event horizon radius, $r = \rho_{\text{in}}$, and equation (6) for potential, after some algebraic manipulation we obtain

$$U(\rho_{\text{in}}) = -\left[\frac{2\left[\varepsilon_{\text{in}} + \rho_{\text{in}}\rho_{\text{b}_{\text{in}}} - m_{\text{in}}\left(1 + \frac{\rho_{\text{in}}}{6\pi\sigma^2}\right)\right]}{8\rho_{\text{in}}(\rho_{\text{in}} - 2m_{\text{in}})}\right]^2 \leq 0. \quad (18)$$

Analogously, for the outer event horizon $r = \rho_{\text{out}}$ we obtain

$$U(\rho_{\text{out}}) = -\left[\frac{2\left[\varepsilon_{\text{out}} + \rho_{\text{out}}\rho_{\text{b}_{\text{out}}} - m_{\text{out}}\left(1 + \frac{\rho_{\text{out}}}{6\pi\sigma^2}\right)\right]}{8\rho_{\text{out}}(\rho_{\text{out}} - 2m_{\text{out}})}\right]^2 \leq 0. \quad (19)$$

The equality in (18) and (19) is achieved only at $m = m_1$ (inner metric), where $\rho_{\text{in}} = \rho_1$, and, respectively, at $m = m_3$ (outer metric), where $\rho_{\text{out}} = \rho_2$. In other words, the points of event horizons of both the inner and outer metrics, $\rho_{\text{in}}$ and $\rho_{\text{out}}$, cannot be below the potential in figure 2. The gravitational radius $\rho_{\text{c}2}$ for both ‘in’ and ‘out’ metrics is changed in the range
Now we have all necessary ingredients for the investigation of possible motions of the thin vacuum shell in the Schwarzschild–de Sitter metric.

3. Dynamical evolution of a vacuum shell

3.1. The case of $\mu > 0$ and $\varepsilon_{\text{out}} > \varepsilon_{\text{in}} + 6\pi \sigma^2$

First of all we consider a simple case, when $\mu > 0$ and $\varepsilon_{\text{out}} > \varepsilon_{\text{in}} + 6\pi \sigma^2$ (another similar case when $\mu < 0$ and $\varepsilon_{\text{in}} > \varepsilon_{\text{out}} + 6\pi \sigma^2$ may be analyzed in a similar way). In this case $\sigma_{\text{in}} = \sigma_{\text{out}} = 1$, as it follows from (7) and (8). Therefore, there are no radii $\rho_1$ and $\rho_2$. It can be shown that in this case $m_{21} > m_{22}$, $\rho_{h_{\text{in}1}} > \rho_{h_{\text{out}1}}$ and $\rho_{h_{\text{in}3}} > \rho_{h_{\text{out}3}}$.

The rolling parameter of our classification scheme is a mass parameter $m$ and we start from the large value of this parameter.

If $m > m_{21}$, then an event horizon is absent and initial expansion (or contraction) of the shell is unbounded (there is no bounce point). The corresponding potential $U(\rho)$ and the Carter–Penrose diagram is shown in figures 3(a) and (b) respectively. Here and further below the Carter–Penrose diagrams are shown only for an initially expanding envelope. The corresponding diagrams for contracting envelope are easily reproduced from the expanding...
ones by symmetry reflection with respect to the median horizontal line (except for some special cases).

At $m = m_{21}$, the first event horizon appears in the inner metric, $\rho_{h in1}$. See figures 3(c) and (d) for the corresponding potential and diagram.

If $m_{22} < m < m_{21}$, then there are two event horizons, $\rho_{h in2}$ and $\rho_{h in3}$. See figure 4(a) for the Carter–Penrose diagram, while the potential has a similar form as in the previous two cases (figures 4(a) and (c)).

At $m = m_{22}$, the first event horizon appears in the outer metric, $\rho_{h out1} < \rho_{h in3}$. We prove that $\rho_{h in2} < \rho_{h out1}$. Indeed, this inequality may be written in the form $(1 + \mu)/(1 - \mu) > 2\epsilon_{out}/(3\epsilon_{out} - \epsilon_{in})$. For $\mu > 0$, if we will prove the inequality $1 > 2\epsilon_{out}/(3\epsilon_{out} - \epsilon_{in})$, then we prove our statement. The last inequality is evident by remembering that $\epsilon_{out} > \epsilon_{in} + 6\pi\sigma^2$. So for $m = m_{22}$ the event horizon $\rho_{h out1}$ is located between $\rho_{h in2}$ and $\rho_{h in3}$. See figures 4(b) and (c) for the corresponding potential and the Carter–Penrose diagram of an expanding shell.

If $m_{22} > m > \max(m_0, m^*)$, where $m^*$ is defined from equation $\rho_{h in2} = \rho_{h out2}$, there are two event horizons, $\rho_{h out2}$ and $\rho_{h out3}$, in the outer metric. These event horizons are located between the event horizons of the inner metric $\rho_{h in2}$ and $\rho_{h in3}$. See figure 4(d) for a corresponding diagram.

Figure 4. An effective potential (6) and the corresponding Carter–Penrose diagrams for the case $\mu > 0, \epsilon_{out} > \epsilon_{in} + 6\pi\sigma^2$ and values of the rolling mass parameter $m$ in the range $m_{22} < m < m_{21}$ (graphs (a)), $m = m_{22}$ (graphs (b) and (c)) and $m_{22} > m > \max(m_0, m^*)$ (graph (d)).
Figure 5. An effective potential (6) and the corresponding Carter–Penrose diagrams for the case $\mu > 0$, $\varepsilon_{\text{out}} > \varepsilon_{\text{in}} + 6\pi \sigma^2$ and values of the rolling mass parameter $m$ in the range $m < m_0$ (graphs (a)–(c)), $m^* < m < m_0$ (graphs (b) and (d)) and $m < m^* < m_0$ (graphs (b) and (c)).

If $m^* > m_0$ and $m^* > m > m_0$, then the arrangement of event horizons is $\rho_{\text{out}2} < \rho_{\text{in}2} < \rho_{\text{out}3} < \rho_{\text{in}3}$. The Carter–Penrose diagram for an expanding shell is shown in figure 5(d).

If $m < m_0$, the region appears, where potential $U(\rho)$ is positive (see figure 5(a)). Now solutions for $\rho(\tau)$ have the bounce points. The arrangement of event horizons is $\rho_{\text{out}2} < \rho_{\text{in}2} < \rho_{\text{out}3} < \rho_{\text{in}3}$. The Carter–Penrose diagrams for an expanding and contracting shell are shown in figures 5(b) and (c) respectively.

If $m_0 > m^*$, then in the case $m^* < m < m_0$ the possible Carter–Penrose diagrams for an expanding and contracting shell are shown in figures 5(b) and (d).

In the last case $m < m^*$ the arrangement of event horizons is $\rho_{\text{out}2} < \rho_{\text{in}2} < \rho_{\text{out}3} < \rho_{\text{in}3}$. See in figures 5(b) and (c), the Carter–Penrose diagrams for an expanding and contracting shell respectively. In particular, these diagrams illustrate the formation of a black hole or wormhole after a final contraction of the shell.

3.2. The case of $\mu > 0$ and $\varepsilon_{\text{in}} > \varepsilon_{\text{out}} + 6\pi \sigma^2$

From this point we start classification of a more tangled case, when $\mu > 0$ and $\varepsilon_{\text{in}} > \varepsilon_{\text{out}} + 6\pi \sigma^2$ (the case when $\mu < 0$ and $\varepsilon_{\text{out}} > \varepsilon_{\text{in}} + 6\pi \sigma^2$ may be analyzed in a similar way). Now both
Figure 6. An effective potential (6) and the corresponding Carter–Penrose diagrams for the case $\mu > 0$ and $\epsilon_{in} > \epsilon_{out} + 6\pi \sigma^2$ and values of the rolling mass parameter $m$ in the range $m = m_{22}$ (graphs (a)), $m_{22} > m > m^*$ (graphs (b) and (c)) and $m^* > m > m_0$ (graph (d)).

$\sigma_{in}$ and $\sigma_{out}$ can change the signs at radii $\rho_1$ and $\rho_2$. In the considered case the following inequalities are valid: $\rho_1 > \rho_2$ and $\rho_{h_{out1}} > \rho_{h_{in1}}$.

It is useful to define the additional five dimensionless parameters: $\tilde{\mu}$, $\mu_{out}$, $\mu_{in}$, $\mu_{1_{out}}$ and $\mu_{1_{in}}$. A first parameter is a solution $\mu = \tilde{\mu}$ of the equation $m_{21} = m_{22}$, where

$$\tilde{\mu} = \frac{\sqrt{\epsilon_{in}} - \sqrt{\epsilon_{out}}}{\sqrt{\epsilon_{in}} + \sqrt{\epsilon_{out}}}.$$  \hfill (20)

($m_{22} < m_{21}$ at $\mu > \tilde{\mu}$ and vice versa). A second parameter is a solution $\mu = \mu_{out}$ of the equation $\rho_2 = \rho_{h_{out1}}$ at $m = m_{22}$, where

$$\mu_{out} = \frac{\epsilon_{in} - \epsilon_{out} + 6\pi \sigma^2}{5\epsilon_{out} - \epsilon_{in} - 6\pi \sigma^2}.$$  \hfill (21)

This parameter exists only if $3\epsilon_{out} > \epsilon_{in} + 6\pi \sigma^2$. Under this condition $\rho_2 > \rho_{h_{out1}}$ at $\mu > \mu_{out}$ and vice versa. In the opposite case, when $3\epsilon_{out} < \epsilon_{in} + 6\pi \sigma^2$, it is always $\rho_2 < \rho_{h_{out1}}$. A third parameter is a solution $\mu = \mu_{in}$ of the equation $\rho_1 = \rho_{h_{in1}}$ at $m = m_{21}$, where

$$\mu_{in} = \frac{\epsilon_{in} - \epsilon_{out} - 6\pi \sigma^2}{5\epsilon_{out} - \epsilon_{in} - 6\pi \sigma^2}.$$  \hfill (22)

($\rho_1 > \rho_{h_{in1}}$ at $\mu > \mu_{in}$ and vice versa). A fourth parameter is a solution $\mu = \mu_{1_{out}}$ of the equation $\rho_{h_{out1}} = \rho_{h_{in3}}$ at $m = m_{22}$, where

$$\mu_{1_{out}} = \frac{\epsilon_{in} - \epsilon_{out}}{5\epsilon_{out} - \epsilon_{in}}.$$  \hfill (23)


This parameter exists only if $\varepsilon_{in} < 3\varepsilon_{out}$. Under this condition $\rho_{h_{out1}} < \rho_{h_{in1}}$ at $\mu > \mu_{1_{out}}$ and vice versa. In the opposite case, when $\varepsilon_{in} > 3\varepsilon_{out}$, it is always $\rho_{h_{out1}} > \rho_{h_{in1}}$. Finally, the fifth parameter is a solution $\mu = \mu_{1_{in}}$ of the equation $\rho_{h_{in1}} = \rho_{h_{out2}}$ at $m = m_{21}$, where

$$\mu_{1_{in}} = \frac{\varepsilon_{in} - \varepsilon_{out}}{5\varepsilon_{in} - \varepsilon_{out}}$$  \hspace{1cm} (24)

($\rho_{h_{in1}} < \rho_{h_{out2}}$ at $\mu < \mu_{1_{in}}$ and vice versa). It is easy to verify that $\rho_{h_{out1}} > \rho_{h_{in2}}$ at $m = m_{22}$ and $\rho_{h_{in1}} < \rho_{h_{out3}}$ at $m = m_{21}$. A mutual arrangement of these five parameters is $\mu_{out} > \mu_{1_{out}} > \mu > \mu_{1_{in}} > \mu_{in}$.

As a first step in classification of the situation, when $\mu > 0$ and $\varepsilon_{in} > \varepsilon_{out} + 6\pi\sigma^{2}$, we consider the case when $\mu > \tilde{\mu}$ (the case $\mu < \tilde{\mu}$ is quite a similar). Again we begin from the large value of the rolling parameter $m$.

If $m > m_{21}$, the event horizons are absent and there are only two radii $\rho_{1}$ and $\rho_{2}$, where the signs of $\sigma_{in, out}$ are changed. The potential $U(\rho)$ is similar to that shown in figure 3(a) (where $\rho_{1}$ and $\rho_{2}$ are not shown). A diagram for an expanding shell is analogous to that shown in figure 3(b).

The next case is $m = m_{21}$. Now a first event horizon $\rho_{h_{in1}}$ appears, which is at left to $\rho_{1}$ (i.e. $\rho_{h_{in1}} < \rho_{1}$). Now potential is similar to that in figure 3(c). The crucial point is that now a shell intersects the radius $\rho_{h_{in1}}$, when $\sigma_{in} = 1$. The Carter–Penrose diagram is similar to that shown in figure 3(d).

If $m_{21} > m > m_{22}$, there are two event horizons $\rho_{h_{in2}}$ and $\rho_{h_{in3}}$, but qualitatively this case is similar to the preceding one. The Carter–Penrose diagram for an expanding shell is shown in figure 4(a).

The case $m = m_{22}$ is divided into subcases. We begin from the subcase $\varepsilon_{in} > 3\varepsilon_{out}$, when inequalities $\rho_{h_{out1}} > \rho_{h_{in3}}$ and $\rho_{h_{out1}} > \rho_{2}$ are fulfilled. The resulting diagram for an expanding shell is shown in figure 6(a). The other subcase will be considered later.

If $m_{22} > m > m^{*}$, where $m^{*}$ is a solution of the equation $\rho_{h_{out2}} = \rho_{h_{in3}}$. Now instead of one event horizon $\rho_{h_{out2}}$ there are two event horizons, $\rho_{h_{out2}}$ and $\rho_{h_{out3}}$. The corresponding potential and the Carter–Penrose diagram are shown in figure 6(b) and (c) respectively. It must be noted that a moving shell intersects the event horizons $\rho_{h_{out}}$ when $\sigma_{out} = -1$. Therefore, a shell intersects these event horizons in the region $R_{-}$. In contrast, a shell intersects the event horizons $\rho_{h_{in}}$ in the region $R_{+}$, when $\sigma_{in} = 1$.

The only distinguishing feature of the case $m^{*} > m > m_{0}$ from the preceding one is swapping round the event horizons $\rho_{h_{out2}}$ and $\rho_{h_{in3}}$. The arrangement of radii is ($\rho_{h_{in2}}, \rho_{2}) < (\rho_{h_{out2}} < \rho_{h_{in3}} < (\rho_{1}, \rho_{h_{out3}})$. A corresponding diagram is shown in figure 6(d).

If $m_{0} > m > m_{3}$, the potential intersects the axis $U = 0$ and a shell will bounce from the potential. The arrangements of radii is the same as in the previous case. The diagrams for the expanding and contracting shells are shown in figures 7(a) and (b) respectively.

If $m_{3} > m > m_{1}$, the radii $\rho_{h_{out2}}$ and $\rho_{2}$ are swapped round. The potential is shown in figure 7(c) (the event horizons $\rho_{h_{out3}}$ and $\rho_{h_{in2}}$ are not shown). A diagram for an expanding shell is the same as in figure 7(a). A contracting shell intersects the event horizon $\rho_{h_{out2}}$ in the region $R_{+}$ (when $\sigma_{out} = 1$). A diagram for a contracting shell is shown in figure 5(d).

If $m < m_{1}$, the radii $\rho_{h_{in3}}$ and $\rho_{1}$ are swapped round. Two radii $\rho(1)$ and $\rho(2)$ (where signs of $\sigma_{in, out}$ are changed) is now under the potential graph. The diagram for a contracting shell is the same as in the previous case (see figure 5(d)). A corresponding diagram for an expanding shell is shown in figure 7(d).

Now we return to another subcase of the case $m = m_{22}$, when $\varepsilon_{in} + 6\pi\sigma^{2} > 3\varepsilon_{out} > \varepsilon_{in}$, the inequality $\rho_{2} < \rho_{h_{out1}}$ is fulfilled and parameter $\mu_{1_{out}}$ exists. If $\mu > \mu_{1_{out}}$, then $\rho_{h_{out1}} < \rho_{h_{in3}}$. The arrangement of radii is ($\rho_{h_{in2}}, \rho_{2}) < \rho_{h_{out1}} < \rho_{h_{in3}} < \rho_{1}$. A diagram for an expanding shell is shown in figure 8(a).
If \( m_{22} > m > \max(m_0, m^*) \), where \( m^* \) is defined from equation \( \rho_{h\text{in}3} = \rho_{h\text{out}1} \), instead of the one event horizon \( \rho_{h\text{out}1} \) there are two event horizons \( \rho_{h\text{out}2,3} \). The arrangement of radii is \( (\rho_{h\text{in}2}, \rho_2) < \rho_{h\text{out}2} < \rho_{h\text{out}3} < \rho_{h\text{in}3} < \rho_1 \). A corresponding diagram for an expanding shell is shown in figure 8(b).

If \( m^* > m > m_0 \), the arrangement of radii is \( (\rho_{h\text{in}2}, \rho_2) < \rho_{h\text{out}2} < \rho_{h\text{in}3} < (\rho_1, \rho_{h\text{out}3}) \). The only difference with preceding subcase is swapping round the event horizons \( \rho_{h\text{in}3} \) and \( \rho_{h\text{out}3} \). A diagram for an expanding shell shown in figure 6(d).

If \( m_0 > m > m_3 \), the potential intersects the axis \( U = 0 \) and the bounce point appears. The arrangement radii is a similar to the previous case, but a shell now can bounce from the potential. The diagrams for an expanding and contracting shells are shown in figures 7(a) and (b) respectively. At \( m_0 > m^* \) and at \( m_0 > \max(m_3, m^*) \) there will be the following arrangement of radii : \( (\rho_{h\text{in}2}, \rho_2) < \rho_{h\text{out}2} < \rho_{h\text{out}3} < \rho_{h\text{in}3} < \rho_1 \). The diagrams for the expanding and contracting shells are shown in figures 8(c) and (b) respectively. The next two subcases depend on the relation \( m_3 \leq m^* \) and are described in a similar way. The remaining cases for \( m < (m_3, m^*) \) are similar to those for \( \varepsilon_{\text{in}} > 3\varepsilon_{\text{out}} \).

If \( \mu_{1\text{out}} > \mu > \bar{\mu} \) in the case \( \varepsilon_{\text{in}} + 6\pi \sigma^2 > 3\varepsilon_{\text{out}} > \varepsilon_{\text{in}} \), then \( \rho_{h\text{out}1} > \rho_{h\text{in}3} \). As a result this case is reduced to the case, when \( \varepsilon_{\text{in}} > 3\varepsilon_{\text{out}} \).

If \( 3\varepsilon_{\text{out}} - 6\pi \sigma^2 > \varepsilon_{\text{in}} \) for \( m = m_{22} \), then the inequality \( \rho_{h\text{out}2} > \rho_{h\text{in}2} \) is fulfilled and besides the parameter \( \mu_{1\text{out}} \) there exists also \( \mu_{\text{out}} \). At \( \mu > \mu_{\text{out}} \), the inequalities \( \rho_{h\text{out}1} < \rho_{h\text{in}3} \), and \( \rho_2 > \rho_{h\text{out}2} \) are valid. As a result the arrangement of radii is \( \rho_{h\text{in}2} < \rho_{h\text{out}1} < (\rho_2, \rho_{h\text{in}3}) < \rho_1 \). A corresponding diagram for an expanding shell is shown in figure 4(c).

If \( m_{22} > m > m_0 \), instead of the one event horizon \( \rho_{h\text{out}1} \) there are two event horizons \( \rho_{h\text{out}2} \) and \( \rho_{h\text{out}3} \) and the arrangement of radii is \( \rho_{h\text{in}2} < \rho_{h\text{out}2} < \rho_{h\text{out}3} < (\rho_2, \rho_{h\text{in}3}) < \rho_1 \). A corresponding diagram for an expanding shell is shown in figure 4(d).
Figure 8. An effective potential (6) and the corresponding Carter–Penrose diagrams for the case \( \mu > \mu_0, \epsilon_{\text{out}} > \epsilon_{\text{in}} + 6\pi \sigma^2 \) and values of the rolling mass parameter \( m \) in the range \( \mu > \mu_{1\text{out}} \) and \( \mu > \mu_{1\text{out}} \) (graphs (a)), \( m_{22} > m > \max(m_0, m^*) \) (graphs (b)) and \( m_0 > m > m_3 \) (graphs (c)).

If \( m_0 > m > m_3 \), the potential intersects the axis \( U = 0 \) and a shell can bounce from the potential. The corresponding diagrams for an expanding and contracting shells are shown in figures 5(b) and (d) respectively.

If \( m_3 > m > m^* \), where \( m^* \) is a solution of the equation \( \rho_{\text{out}3} = \rho_{\text{in}3} \), the event horizon \( \rho_{\text{out}3} \) changes the place with \( \rho_2 \). The coincidence of these radii occurs at the radius, where \( U(\rho) = 0 \). The radius \( \rho_2 \) now is under the potential curve. A shell will intersect \( \rho_{\text{out}3} \) with \( \sigma_{\text{out}} = -1 \). A corresponding diagram for an expanding shell is shown in figure 8(c).

The case \( m^* > m \) is similar to that for \( \epsilon_{\text{in}} > 3\epsilon_{\text{out}} \). The case \( \mu_{\text{out}} > \mu > \mu_{1\text{out}} \) is similar to that for \( \epsilon_{\text{in}} + 6\pi \sigma^2 > 3\epsilon_{\text{out}} > \epsilon_{\text{in}} \). Finally, the case \( \mu_{1\text{out}} > \mu > \bar{\mu} \) is similar to that for \( \epsilon_{\text{in}} > 3\epsilon_{\text{out}} \). Analogously is the considered case when \( \bar{\mu} > \mu > 0 \), and the case, when only one of the two radii exists, \( \rho_1 \) or \( \rho_2 \). This case completes the classification.

4. Conclusion

We classified all possible evolution scenarios of a thin vacuum shell in the Schwarzschild–de Sitter metric and constructed the Carter–Penrose diagrams for corresponding global geometries. These geometries illustrate the possibilities for final formation of black holes and wormholes or eventual expansion of bubbles.
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