Root Structures of Infinite Gauge Groups and Supersymmetric Field Theories

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Abstract.

We show the relationship between critical dimensions of supersymmetric fundamental theories and dimensions of certain Jordan algebras. In our approach position vectors in spacetime or in superspace are endowed with algebraic properties that are present only in those critical dimensions. A uniform construction of super Poincaré groups in these dimensions will be shown. Some applications of these algebraic methods to hidden symmetries present in the covariant and interacting string Lagrangians and to superparticle will be discussed. Algebraic methods we develop will be shown to generate the root structure of some infinite groups that play the role of gauge groups in a second quantized theory of strings.

1. Introduction

In the following we will put together a few remarks pertaining to some mathematical aspects of physical theories. They will revolve around a theme relating space-time structure to internal symmetries. Chiral symmetry, analyticity properties related to causality and conformal invariance, supersymmetry between elementary excitations belong to those set of ideas. Properties like analyticity and color are generalized in terms of division algebras. In turn these lead to mathematical structures that unify Minkowski space and internal charge space only within certain geometries with critical dimensions that occur in supergravity theories as well as in string and superstring models.

As an introduction to this kind of analysis we shall exploit the relation of generalized integers to some Lie groups. This will be followed by the representations of the Lorentz and Poincaré groups by means of division algebras in higher dimensions. Then we will take a closer look at their extensions to conformal groups and super Poincaré groups. Jordan algebras will be shown to arise naturally. Finally we will give examples of applications to particle dynamics and two-dimensional field theories that can be related to string theories.

The interest in exceptional structures arose in physics from a simple observation that the color group $SU(3)$ which is exact, and most probably not accidental, is connected with the octonion algebra [1].
Indeed if one tries to form three anti commuting Grassmann numbers by using three pairs of octonionic imaginary units, the algebra of these split units admits $SU(3)$ as its automorphism group. One is therefore led to construct quark states with such units and reformulate QCD in terms of these octonion valued fields [2].

A phenomenological manifestation of octonionic structure (based on Cayley numbers) has been found in connection with quark dynamics inside hadrons leading to an effective dynamical supersymmetry that was shown in our earlier papers [3],[4] (for further reviews and applications, see also [5], [6] and [7]). It is based on $SU(m/n)$ supergroups yielding combined classification of mesons and baryons. Also, further applications of split octonions to supersymmetric potential models and skyrmions have recently been discussed in literature [8],[9],[10]. Octonionic structures also appear in constructions of extended Hilbert spaces and associated projective geometries [11], [12], [13]. For a brief history of dynamical supersymmetries we refer to an article by Francesco Iachello [14], and for an extensive paper on octonions and their applications to a paper by John Baez [15].

The fitting of color into larger groups involving also flavor degrees of freedom in grand unified theories can then be done by embedding octonions into larger algebras that would have grand unified theory groups as their automorphism groups. For instance the automorphism group of the algebra of $2 \times 2$ octonionic hermitian matrices is $SO(10)$ while $E_6$ leaves invariant the algebra of $3 \times 3$ hermitian octonionic matrices, the so called exceptional Jordan algebra.

Cancellation of anomalies in 10 dimensional supergravity or superstring theories [16] also requires the restriction of internal symmetries of the Yang-Mills matter to very special group like $E_8 \times E_8$ associated with a remarkable 16-dimensional lattice. All those occurrences suggest that unified theory of gravity and gauge interactions might best be formulated by means of an extension of such algebras that would also incorporate supersymmetry from the start.

Such theories can only be formulated in very special dimensions like $d = 3, 4, 6$ and 10 for supersymmetric theories [17] and $d = 5, 8, 14$ and 26 for their bosonic counterparts. It now seems that $d = 10$ and $d = 26$ associated with the octonionic structures are the only ones that can be quantified consistently although other possibilities might exist [18]. There is another kind of invariance that involves auxiliary variables. In relativistic theories, there are unphysical degrees of freedom, like the $s = 0$ component of a 4-vector field that is necessary for a covariant description of $s = 1$ particle. They contribute to the energy with the wrong sign and cause the occurrence of negative probabilities. They are either eliminated by subsidiary conditions that will in general destroy the covariance or the gauge invariance of the theory or both. Or, alternatively they are kept and further unphysical (ghost) degrees of freedom are added to cancel their contributions. The ghost enlarged action may now behave like the BRST symmetry [19],[20] that transforms ghosts, unphysical particles and physical particles into each other. That generalized action may now exhibit an exact supersymmetry which has nothing to do with the physical spectrum or the physical invariance of the Lagrangian. In this case the symmetry is used for the covariant and gauge invariant quantization of the theory. It is useful and elegant rather than fundamental.

Ingredients of the theory, exact conformal invariance in the 2-dimensional submanifold, presence of new space-time symmetries that mix states with different spin, exact duality symmetry between exchange and resonance channels, a non-locality of the basic objects (open or closed charged strings) that interact locally, were all suggested as approximate or phenomenological feature of a theory based on the QCD Lagrangian. It turns out that such theories are linked with unique exceptional mathematical structures that enjoy unique algebraic properties, like surprising relations to the octonion algebra.

In short, inspired by the intricate symmetries appearing in the rich world of hadronic physics, we were led to invent a theory that has all these new symmetries as basic ingredients, and hope that such a maximally symmetric theory would be capable of unification of all fundamental forces, including gravity. The constraints on such a theory are so great that if some very restricted solutions exist they have a good chance of being related to some exceptional structures in mathematics that unify various aspects of algebra, geometry, analysis and number theory.
2. Relation of some remarkable groups to division algebras

Hurwitz’ Theorem [21] assures us that there are four division algebras with quadratic norm, namely \( \mathcal{R} \) (reals), \( \mathcal{C} \) (complex numbers), \( \mathcal{H} \) (quaternions) and \( \mathcal{O} \) (octonions).

The first two are commutative and associative. The third is associative but not commutative while the last is neither commutative nor associative. Each algebra has a subring of integers. In the case of complex numbers we have the Gaussian integers of the form \( m + in \) with \( m \) and \( n \) real integers. The theory of quaternionic and octonionic integers with norm and twice the scalar part being integers was developed by Hurwitz, Wedderburn, Dickson and Coxeter.

We can first consider Lie groups that leave the norm and the trace form (twice the scalar part) of the elements of the 4 division algebras invariant. Then we look at their automorphism group and finally we associate generalized integers of unit norm with the root system of Lie groups. Results are in the following table:

| Group      | \( \mathcal{R} \) | \( \mathcal{C} \) | \( \mathcal{H} \) | \( \mathcal{O} \) |
|------------|-------------------|-------------------|-------------------|-------------------|
| Norm       | \( \mathcal{O}(2) \sim U(1) \) | \( \mathcal{O}(4) \sim SU(2) \times SU(2) \) | \( \mathcal{O}(8) \) |
| Trace      | \( \mathcal{O}(3) \sim SU(2) \) | \( \mathcal{O}(7) \) |
| Automorphism | \( SU(2) \) | \( G_2 \) |
| Root System| \( \mathcal{O}(3) \) | \( \mathcal{O}(4) \) | \( \mathcal{O}(8) \) | \( E_8 \) |

Table I - Lie Groups associated with division algebras

The first three lines are well known. The last one requires some explanation. Consider a semisimple Lie group \( G \) and its Cartan subalgebra \( \mathcal{H} = (H_1, \ldots, H_r) \) where \( r \) is the rank of \( G \). In the Cartan basis we consider the non hermitian generators \( E_k \) and their conjugates \( E_{-k} \). \( E_k \) is associated with the root vector \( r^k_m \) such that

\[
[H_m, E_{\pm k}] = \pm r^{(k)}_m E_{\pm k}.
\]

When the rank \( r \) is 1, 2, 4 or 8 we can combine the operators \( H_m \) and its ”eigenvalues” \( r^{(k)}_m \) into elements of a division algebra with imaginary units \( e_i \):

\[
H = H_0 + e_i H_i, \quad r^{(k)} = r^{(k)}_0 + e_i r^{(k)}_i
\]

so that

\[
[H, E_{\pm k}] = \pm r^{(k)} E_{\pm k}.
\]

In the case of \( G = SU(2) \), \( O(4) \), \( O(8) \) and \( E_8 \) all roots have the same length that can be normalized to unity. It turns out that in all four cases the roots are integer elements of the Hurwitz algebras.

In the case of \( SU(2) \) there are two \( E \) generators

\[
E_{\pm 1} = J_1 \pm iJ_2
\]

and one Cartan subalgebra operator \( H = J_3 \) with eigenvalues \( \pm 1 \) which are real integers with unit norm.

\( O(4) \) has roots \( \pm (1, 0) \) and \( (0, \pm 1) \) that combine into the four Gaussian integers \( \pm 1 \) and \( \pm i \) that are the rational points on the circle \( S^1 \).

Similarly the 24 roots of \( O(8) \) correspond to the integer quaternions \( \pm e_1, \pm e_2, \pm e_3, \) \( (8 \text{ such roots}) \) and the remaining ones of the form
(16 such roots). They are the rational points on the unit sphere $S^4$. The principal positive roots can be chosen to be

$$e_1 = e^{\frac{\pi}{2} e_1}, \quad e_2 = e^{\frac{\pi}{2} e_2}, \quad e_3 = e^{\frac{\pi}{2} e_3}$$

and

$$\ell = \exp\left(\frac{\pi}{3} e_1 + e_2 + e_3\right) = \frac{1}{2}(1 + e_1 + e_2 + e_3)$$

corresponding to the four points of the Dynkin diagram

![Dynkin diagram of O(8)](image)

**Fig. 1. Dynkin diagram of O(8)**

The triality of the $O(8)$ group reflects the symmetry of the three quaternionic units $e_1 = -i \sigma_i$, where $\sigma_i$ are the Pauli matrices.

The Cartan subalgebra is more conveniently represented by the quaternionic operator

$$H = H_0 \frac{1 + e_1 + e_2 + e_3}{2} + e_1 H'_1 + e_2 H'_2 + e_3 H'_3.$$ \hspace{1cm} (8)

All the roots can be obtained from the principal positive roots by Weyl reflections. If $\alpha$ and $\beta$ are roots represented by quaternions $\beta' = -\alpha \bar{\beta} \bar{\alpha}$ is the quaternion form of $\beta$ reflected with respect to $\alpha$. Obviously $\beta'$ has unit norm and its trace form is $\pm 1$ or $0$ like that of $\beta$.

The case of $E_8$ was worked out by Coxeter [22] in connection with 8-dimensional regular solids. There are 240 rational points on the unit sphere $S^7$ represented by integer octonions that correspond to the 240 roots of $E_8$. We first introduce octonionic imaginary units $e_\alpha (\alpha = 1, \ldots, 7)$ with multiplication rule

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} e_\gamma$$ \hspace{1cm} (9)

where the octonion structure constants form an antisymmetrical tensor of third rank with

$$\psi_{123} = \psi_{246} = \psi_{345} = \psi_{367} = \psi_{651} = \psi_{572} = \psi_{714} = 1$$ \hspace{1cm} (10)

and other elements zero. Now define the octonions
\[ l_1 = \frac{1}{2}(e_1 - e_4), \quad l_2 = \frac{1}{2}(e_2 - e_5), \quad l_3 = \frac{1}{2}(e_3 - e_6) \]
\[ l_4 = \frac{1}{2}(e_1 + e_4), \quad l_5 = \frac{1}{2}(e_2 + e_5), \quad l_6 = \frac{1}{2}(e_3 + e_6) \]
\[ l_7 = \frac{1}{2}(1 + e_7), \quad l_8 = l_7 = \frac{1}{2}(1 - e_7). \quad (11) \]

We have
\[ l_1 = e_1 l_7 = l_8 e_1, \quad l_2 = e_2 l_7 = l_8 e_2, \]
\[ l_3 = e_3 l_7 = l_8 e_3, \quad l_4 = e_4 l_8 = l_7 e_1, \]
\[ l_5 = e_5 l_8 = l_7 e_2, \quad l_6 = e_6 l_8 = l_7 e_3. \quad (12) \]

Now consider
\[ \rho_{rs} = \pm l_r \pm l_s \quad (r \neq s) \quad (13) \]
and
\[ \sigma = \frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6 \pm l_7 \pm l_8) \quad (14) \]
with an odd number of minus signs in \( \sigma \).

A special choice of \( \sigma \) gives
\[ h = \frac{1}{2}(l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7 - l_8) \]
\[ = \frac{1}{2}(e_1 + e_2 + e_3 + e_7). \quad (15) \]

Special \( \rho \)'s are \( 1, e_7, e_2, e_6 \), which, together with \( h, e_1 h, e_2 h \) and \( e_7 h \) correspond to one possible set of principal positive roots. These are shown in the figure below:

![Fig. 2. The $E_8$ Dynkin Diagram](image)

It is shown easily that the 112 octonions \( \rho_{rs} \) and the 128 octonions \( \sigma \) are integer octonions with norm 1. It is possible to choose the Cartan subalgebra \( H_a \) \( (a = 1, \ldots, 8) \) such that these 240 octonions are the roots of $E_8$ expressed as eigenvalues of the operator.
$H = H_0 + e_\alpha H_\alpha \quad (\alpha = 1, \ldots, 7)$ \hfill (16)

for the adjoint representation. The 112 roots $\rho_{rs}$ are those of the subgroup $Spin(16)$ while $\sigma$ are the weights associated with the 120 dimensional spinor representation of $Spin(16)$.

Now consider integer octonions of length 2, 3 and 4. Their corresponding numbers 2160, 6720 and 17280 have been calculated by Coxeter. They correspond exactly to the number or weights with maximum length 4, 6 and 8 in the fundamental representations $(00000010)$, $(01000000)$ and $(00000001)$ with respective dimensions 3875, 30380 and 147250, provided that the length of the root is normalized to 2 in the adjoint representation. For example in the 3875 dimensional representation the multiplicity of the weight length 2 is 7 and that of weight length zero associated with generators in the Cartan subalgebra is 35. These numbers can be found in the table by Brenner, Moody and Patera [23]. Hence the weight of that representation consists of 2160 integer octonions of norm 2, $240 \times 7 = 1680$ integer octonions of norm 1 so that we have

$$3875 = 35 \times 1 + 240 \times 7 + 2160.$$ \hfill (17)

Thus the representations of $E_8$ are directly related to the number theory of octonions.

If we consider a octonionic spinor with two octonionic components, then integer octonionic spinors are associated with representations of $E_8 \times E_8$. We shall come back to this point later.

It is also possible to select the roots corresponding to the subgroup $E_6$ of $E_8$ which is of some importance in the compactification of superstrings. To this end we introduce a new root $l_0$ associated with the extended Dynkin diagram

$$l_0 = \frac{1}{2}(l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7 + l_8)$$

$$= \frac{1}{2}(1 + e_1 + e_2 + e_3).$$ \hfill (18)

Now consider the $2 \times 36 = 72$ octonionic integers of the form

$$\tau_{ab} = -\tau_{ba} = \pm(l_a - l_b), \quad (a,b = 0,1,\ldots,8)$$ \hfill (19)

These are the subset of the $E_8$ roots that form the root system of $E_6$. The remaining 168 roots of $E_8$ can be put in the form of a symmetrical tensor

$$\zeta_{abc} = \pm(l_a + l_b + l_c - \frac{1}{3} \sum_{r=0}^{8} l_r)$$ \hfill (20)

with $a, b, c$ all different.

3. Lorentz Vectors as $2 \times 2$ Hermitian matrices

Now we consider $2 \times 2$ hermitian matrices of the form

$$V = \begin{pmatrix} \alpha & \bar{v} \\ v & \beta \end{pmatrix} = V^\dagger$$ \hfill (21)

and

$$\bar{V} = \begin{pmatrix} \beta & -\bar{v} \\ -v & \alpha \end{pmatrix}$$ \hfill (22)

with $\alpha, \beta$ real
\[ \alpha = v_0 + v_{d-1}, \quad \beta = v_0 - v_{d-1} \]  

(23)

and \( v \) belonging to a Hurwitz division algebra such that

\[ N^2(V) = V \bar{V} = DetV = v_0^2 - v_1^2 - \cdots - v_{d-1}^2. \]  

(24)

Then \( V \) represents a vector in \( d = 3, 4, 6 \) and \( 10 \) dimensions which are exactly the dimensions in which classical superstrings can exist [24]. The linear transformations that leave \( N(V) \) invariant are the Lorentz transformations in the same dimensions and for \( \mathcal{R}, \mathcal{C} \) and \( \mathcal{H} \) they are isomorphic to [25],[26]

\[
\begin{align*}
    &d = 3 : \quad O(2, 1) \sim SL(2, \mathcal{R}) \\
    &d = 4 : \quad O(3, 1) \sim SL(2, \mathcal{C}) \\
    &d = 6 : \quad O(5, 1) \sim SL(2, \mathcal{H}) \quad \text{or} \quad SL(2, \mathcal{Q})
\end{align*}
\]

These Lorentz transformations can be represented by

\[ T_L V = V' = L V L^\dagger = V'^\dagger \]  

(26)

where

\[ L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \]  

(27)

\[ \text{Det}L = 1, \quad l_{\alpha \beta} \in \mathcal{R}, \mathcal{C}, \mathcal{H} \]  

(28)

In the octonionic case the form (26) is not the most general linear transformation owing to the nonassociativity of octonions. It must be complemented by their automorphism group \( G_2 \) with 14 parameters which requires associators in its expression. In this case of \( d = 10 \), \( L \) has \( 4 \times 8 - 1 = 31 \) parameters that add up to the 45 parameters of \( O(9, 1) \) with the inclusion of \( G_2 \).

This point is not clear in the literature. With this modification we can represent \( Spin(9, 1) \) by \( SL(2, \mathcal{Q}) \).

In all four cases the lowest spinor representations of the Lorentz groups are \( 2 \times 1 \) matrices

\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]  

(29)

where \( \psi_\alpha (\alpha = 1, 2) \) belong to the 4 division algebras and transforms as

\[ T_L \psi = \psi' = L \psi. \]  

(30)

In the octonionic case this must be supplemented by the \( G_2 \) transformations of the imaginary units. It follows that spinors have 2, 4, 8 and 16 real components respectively in \( d = 3, 4, 6 \) and \( 10 \). Such spinors are real in \( d = 3 \), Weyl or Majorana in \( d = 4 \), Weyl in \( d = 6 \) and both Majorana and Weyl in \( d = 10 \).

We also note that vectors in these dimensions are elements of \( 2 \times 2 \) Jordan algebras with the commutative product defined by

\[ W = U \cdot V = \frac{1}{2} \{ U V \} = \frac{1}{2} (U V + V U) \]  

(31)

However, this product is only invariant under the rotation subgroup of the Lorentz group. The parity conjugate vector

\[ \bar{V} = V^{-1} \text{Det}V \]  

(32)

obeys the transformation law
\[ V' = L^\dagger V L^{-1} \]  

In the associative cases we can construct \( 2 \times 2 \) matrices with the transformation law

\[ F = V \hat{U} \rightarrow LFL^{-1} \]  

under Lorentz transformations and the triple product

\[ X = \frac{1}{2} (V \hat{U} W + W \hat{U} V) \rightarrow LX L^\dagger \]  

is again a vector. It can be rewritten by means of Jordan products as

\[ X = (V \cdot \hat{U}) \cdot W + V \cdot (\hat{U} \cdot W) - (V \cdot W) \cdot \hat{U} \]  

This is nothing but the isotopic Jordan product of \( V \) and \( W \) with respect to \( \hat{U} \) introduced by Jacobson [27]. The equation (36) is also valid in the octonionic case for \( O(9,1) \) vectors. It becomes a quadratic Jordan algebra in the case \( V = W \).

Now we can ask a question about the nature of the discrete subgroups of the Lorentz groups in these critical dimensions. The case of \( d = 3 \) is well known. Real integer \( 2 \times 2 \) matrices with unit determinant give the modular group \( \Gamma \) and automorphic forms (or theta functions) are representations of the cosets \( SL(2,R)/\Gamma \). In the case of \( SL(2,C) \) the discrete subgroup has \( 2 \times 2 \) matrices of Gaussian integers as elements forming the Klein group. For the quaternonic case \( d = 6 \), the discrete group involves integer quaternions. Those in turn are associated with towers of \( O(8) \) representations which could be related to the infinite roots of an affine Lie algebra associated with \( O(8) \). Similarly we may speculate that unimodular \( 2 \times 2 \) integer octonionic hermitian matrices might be related to the affine group \( E_{10} \).

4. Representation of Poincaré groups by triangular \( 4 \times 4 \) matrices

Let \( X \) be a position vector represented by a hermitian \( 2 \times 2 \) matrix. The Poincaré group is represented by the \( 4 \times 4 \) matrix

\[
\begin{pmatrix}
I & X' \\
0 & I
\end{pmatrix} = \begin{pmatrix}
L & \frac{1}{2}AL^\dagger \\
0 & L^\dagger - 1
\end{pmatrix} \begin{pmatrix}
I & X \\
0 & I
\end{pmatrix} \begin{pmatrix}
L^{-1} & \frac{1}{2}L^{-1}A \\
0 & L^\dagger
\end{pmatrix} = \begin{pmatrix}
I & LX L^\dagger + A \\
0 & I
\end{pmatrix}
\]  

where \( X \) and \( A \) are hermitian and \( I \) is the \( 2 \times 2 \) unit matrix. Then \( dX \) and the momentum \( P \) transform like vectors.

By a Lorentz transformation \( P \) can be diagonalized to take the standard form

\[ \Pi = \begin{pmatrix}
\alpha & 0 \\
0 & \epsilon \alpha
\end{pmatrix}, \]  

where \( \epsilon = 1,0,-1 \) respectively for a time-like, light-like and space-like momentum vector. The subgroup of the Lorentz group that leaves \( \Pi \) invariant is the little group given by the following table:
**Table 2.**

| Massive: $\epsilon = 1$ | $SL(2, \mathbb{R})$ | $SL(2, \mathbb{C})$ | $SL(2, \mathbb{Q})$ | $SL(2, \mathbb{O})$ |
|-------------------------|---------------------|---------------------|---------------------|---------------------|
| $U(1)$                  | $SU(2)$             | $Spin(5) = Sp(2, \mathbb{Q})$ | $Spin(9)$            |
| Light-like: $\epsilon = 0$ | $T_1$               | $E(2) = U(1) \oplus T_2$ | $E(4) = Spin(4) \oplus T_4$ | $E(8) = Spin(8) \oplus T_8$ |
| Tachyon: $\epsilon = -1$ | $SO(1, 1)$         | $SO(2, 1)$         | $SO(4, 1)$         | $SO(8, 1)$         |
| $d = 3$                 | $d = 4$             | $d = 6$             | $d = 10$            |

where $E(d - 2)$ is the euclidean group of motion in $d - 2$ dimensions, $T_{d-2}$ is the translation group and $\oplus$ denotes semi-direct product. The $\epsilon = 1$ groups are the spin groups for massive states. The compact parts of the little groups for $\epsilon = 0$ are the helicity groups for light-like states and the $\epsilon = -1$ groups are little groups for tachyons.

The position vector $X$ can be expressed in terms of the $2 \times 2$ components $\Psi$ and $\Sigma$ of a "twistor" in the following way ($\Psi$ is non singular)

\[
\begin{pmatrix}
1 & X \\
0 & I
\end{pmatrix} = \begin{pmatrix}
\Psi & \Sigma \\
0 & \Psi^{-1}
\end{pmatrix} \begin{pmatrix}
I & I \\
0 & I
\end{pmatrix} \begin{pmatrix}
\Psi^{-1} & \Sigma^\dagger \\
0 & \Psi^\dagger
\end{pmatrix}
\]

(39)

where the triangular matrix made of $\Psi$ and $\Sigma$ has the Poincaré transformation law

\[
\begin{pmatrix}
\Psi' & \Sigma' \\
0 & \Psi'^{-1}
\end{pmatrix} = \begin{pmatrix}
L & \frac{1}{2} A L^{-1} \\
0 & L^{-1}
\end{pmatrix} \begin{pmatrix}
\Psi & \Sigma \\
0 & \Psi^\dagger
\end{pmatrix}
\]

(40)

so that

\[
\Psi' = L \Psi, \quad \Sigma' = L \Sigma + \frac{1}{2} A L^{-1} \Psi^\dagger
\]

(41)

We have

\[
X = \Psi \Psi^\dagger + \Psi \Sigma + \Sigma \Psi^\dagger
\]

(42)

giving

\[
X' = \Psi' \Psi'^\dagger + \Psi' \Sigma + \Sigma' \Psi'^\dagger = L X L^\dagger + A
\]

(43)

If we put

\[
S = \begin{pmatrix}
\Psi & \Sigma \\
0 & \Psi^{-1}
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

(44)

\[
\bar{S} = \Gamma S^\dagger \Gamma = \begin{pmatrix}
\Psi^{-1} & \Sigma^\dagger \\
0 & \Psi^\dagger
\end{pmatrix}
\]

(45)

and

\[
\Xi = \bar{\Xi} = \begin{pmatrix}
I & X \\
0 & I
\end{pmatrix}
\]

(46)

\[
H = \bar{H} = \begin{pmatrix}
I & I \\
0 & I
\end{pmatrix}, \quad H^n = \begin{pmatrix}
I & n I \\
0 & I
\end{pmatrix}
\]

(47)

we can write

\[
\Xi = SH \bar{S} = Z \bar{Z} \quad \text{with} \quad Z = SH^2.
\]

(48)
5. The Super Poincaré Group and its imbedding

We now replace the position $X$ by the superspace coordinates $Z(X, \theta, \theta^\dagger)$ where $\theta$ is a spinor. Because in $d = 3, 4, 6, 10$, $X$ can be regarded as a $2 \times 2$ hermitian matrix and $\theta$ as a $2 \times 1$ matrix over the Hurwitz algebras, the super-position vector can be represented by a $3 \times 3$ hermitian matrix

$$J = \begin{pmatrix} X & \theta \\ \theta^\dagger & s \end{pmatrix} = J^\dagger \quad (49)$$

with the addition of a scalar $s$. The superspace variable is now upgraded to the $6 \times 6$ matrix

$$Z = \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \quad (I = 3 \times 3 \text{ unit matrix}) \quad (50)$$

where $J$ is an element of a graded Jordan algebra. In a super-Lorentz transformation we have

$$T_\Lambda Z = Z' = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^\dagger \end{pmatrix} \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda^\dagger \end{pmatrix}$$

and in a generalized translation

$$T_B Z = \begin{pmatrix} I & \frac{1}{2}B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \begin{pmatrix} I & \frac{1}{2}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & J + B \\ 0 & I \end{pmatrix} \quad (52)$$

$\Det Z$ is invariant under this transformation. Since $dJ$ is invariant under translations in superspace, $\Det (dJ)$ is super Poincaré invariant.

$$d = 3 \quad d = 4 \quad d = 6 \quad d = 10$$

$$(\text{dim.}\, J) - 1 : \quad 5 \quad 8 \quad 14 \quad 26$$

$$(3 \times 1 + 2) \quad (3 \times 2 + 2) \quad (3 \times 4 + 2) \quad (3 \times 8 + 2) \quad (53)$$

Superspace vectors will transform like $dJ$. Let

$$W = \begin{pmatrix} V \\ \psi^\dagger_L \\ g \end{pmatrix} \quad (54)$$

Under translations only the argument of $W$ will transform if $W$ depends on space-time variables. Under the Lorentz group

$$\Lambda(L) = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}, \quad \Lambda(L)^\dagger = \begin{pmatrix} L^\dagger & 0 \\ 0 & 1 \end{pmatrix} \quad (55)$$

so that

$$W' = \Lambda(L) W \Lambda(L)^\dagger = \begin{pmatrix} LV L^\dagger & L \psi_L \\ \psi^\dagger_L L^\dagger & g \end{pmatrix} \quad (56)$$

which shows $V, \psi_L$ and $g$ to transform respectively like a vector, a left handed spinor and a scalar. Under left super-transformations
\[ \Lambda(\xi_L) = \exp \begin{pmatrix} I & \xi_L \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & \xi_L \\ 0 & 1 \end{pmatrix} \]  \tag{57} \\

and

\[ \Lambda(\xi_L)\dagger = \begin{pmatrix} I & 0 \\ \xi_L & 1 \end{pmatrix} \]  \tag{58} \\

we have

\[ W'' = \Lambda(\xi_L) W \Lambda(\xi_L)\dagger = \\
\begin{pmatrix} V + \psi_L\xi_L\dagger + \xi_L\psi_L\dagger + g\xi_L\xi_L\dagger & \psi_L + g\xi_L \\
\psi_L + g\xi_L & g \end{pmatrix} \]  \tag{59} \\

If we take \( g = 1 \), this coincides with the super Poincaré transformation.

We can also consider the right supertransformations

\[ \Lambda(\eta_R) = \exp \begin{pmatrix} 0 & 0 \\ \eta_R\dagger & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \eta_R & 1 \end{pmatrix} \]  \tag{60} \\

and

\[ \Lambda(\eta_R)\dagger = \begin{pmatrix} I & \eta_R \\ 0 & 1 \end{pmatrix} \]  \tag{61} \\

so that

\[ W''' = \Lambda(\eta_R) W \Lambda(\eta_R)\dagger = \\
\begin{pmatrix} V & \psi_L + V\eta_R \\
\psi_L + \eta_R\dagger V + g\eta_R\psi_L + \eta_R\dagger V\eta_R & \eta_R \end{pmatrix} \]  \tag{62} \\

Under this transformation the vector field is invariant while the scalar field and the spinor field transform.

Thus, the super-Lorentz transformations include the super Poincaré transformations as a sub-supergroup.

We can also consider the transformation of \( W \) under dilatation

\[ W'''' = \Lambda(\sigma) W \Lambda(\sigma)\dagger \]  \tag{63} \\

with

\[ \Lambda(\sigma) = \exp \begin{pmatrix} -\frac{1}{2}I\sigma & 0 \\ 0 & -\sigma \end{pmatrix} = \begin{pmatrix} e^{-\frac{\sigma}{2}} & e^{-\frac{\sigma}{2}} \\ e^{-\frac{\sigma}{2}} & e^{-\sigma} \end{pmatrix} \]  \tag{64} \\

such that the super determinant of \( \Lambda \) is one. Letting \( \lambda = e^\sigma \) we have

\[ W'''' = \begin{pmatrix} \lambda^{-1}V & \lambda^{-\frac{3}{2}}\psi_L \\
\lambda^{-\frac{3}{2}}\psi_L & \lambda^{-2}g \end{pmatrix} \]  \tag{65} \\

These are correct transformation laws for vector and spinor fields. Note that \( g \) transforms like a metric field under dilatations and is invariant under super-Poincaré transformations.
Under the Super-Poincarè group, dilatations and left-supertransformations the super determinant of \( W \) is invariant.

Let us diagonalize \( W \)

\[
W = \begin{pmatrix}
I & g^{-1}\psi_L \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
V & -g^{-1}\psi_L\psi_L^\dagger \\
0 & g \\
\end{pmatrix} \begin{pmatrix}
I & 0 \\
g^{-1}\psi_L^\dagger & 1 \\
\end{pmatrix}
\]

(66)

We have

\[
\text{SDet} W = \text{SDet} \begin{pmatrix}
V & -g^{-1}\psi_L\psi_L^\dagger \\
0 & g \\
\end{pmatrix} = g^{-1}\text{Det}(V - g^{-1}\psi_L\psi_L^\dagger).
\]

(67)

On the other hand, using \( \hat{\psi}_L = -i\sigma_2\psi_L^* \),

\[
\text{Det} V = \bar{V}V
\]

(68)

\[
\text{Det}(V - g^{-1}\psi_L\psi_L^\dagger) = \text{Det} V - \frac{1}{2}g^{-2}\bar{\psi}_L^*\psi_L^\dagger\hat{\psi}_L + g^{-1}\psi_L^\dagger\psi_L
\]

(69)

and

\[
\text{SDet} W = g^{-1}\bar{V}V + g^{-2}\psi_L^\dagger\psi_L - \frac{1}{2}g^{-3}\bar{\psi}_L^*\psi_L^\dagger\hat{\psi}_L
\]

(70)

which is obviously invariant under dilatations. Note that \( g^2\text{SDet} W \) is still super Poincarè invariant and transforms like a scalar density of weight 4 under dilatations. We have

\[
g^2 \text{SDet} W = g\bar{V}V + \psi_L^\dagger\bar{V}\psi_L - \frac{1}{2}g^{-1}\psi_L^\dagger\psi_L^\dagger\hat{\psi}_L
\]

(71)

Let us note that if the components of \( \psi_L \) are not treated like Grassmann variables, in the case one assumes that \( \psi_\alpha, \psi_\alpha^* \) (\( \alpha = 1, 2 \)) all commute, but nevertheless transform like a spinor under Lorentz transformations, then, the invariant under the 3 \( \times \) 3 Lorentz transformation matrix would be

\[
\text{Det} W = g\bar{V}V - \psi_L^\dagger\bar{V}\psi_L
\]

(72)

with no quartic spinor term.

The transformation matrix is now a 3 \( \times \) 3 unimodular matrix that belongs to \( SL(3, \mathbb{R}) \) and \( SL(3, \mathbb{C}) \) respectively in the real and complex cases. In the quaternionic case we have \( SL(3, \mathbb{H}) \) which is isomorphic to \( SU^+(6) \). Finally in the octonionic case \( SL(3, \mathbb{O}) \) does not give all the linear transformations on \( \mathbb{O} \) since we can also act on the octonions in \( J \) by associators that come in the expression of the \( G_2 \) automorphism group of octonions. In all cases the dimension of the group is given by

\[
D = 8d + \dim \text{ (aut)}
\]

(73)

where \( d \) is the dimension of the division algebra. The automorphism groups are non trivial only for quaternions (\( SU(2) \) with dimension 3) and octonions (\( G_2 \) with dimension 14). This gives for \( D \)

\[
D(\mathbb{R}) = 8, \quad D(\mathbb{C}) = 16, \quad D(\mathbb{H}) = 35, \quad D(\mathbb{O}) = 78
\]

(74)
which are the dimensions of the Lie algebras of $SL(3, \mathcal{R})$, $SL(3, \mathbb{C})$, $SU^*(6)$ and $E_{6,-26}$ respectively. The non compact group $E_{6,-26}$ has $O(9, 1)$ as a subgroup as well as the automorphism group $F_4$ of the Jordan algebra of $3 \times 3$ traceless octonionic hermitian matrices. $E_{6,-26}$ means that the group has rank 6 and the number of non compact generators minus the number of compact generators is $-26$.

6. Case of $E_{10}$ and future prospects

If instead of $D = 11$ we start from $D = 10$, then the whole $9 + 1$ dimensional space can be regarded as the Cartan torus of $E_{10}$, the hyperbolic extension of $E_8$. This corresponds to a space-time with some periodic boundary conditions, the period being associated with some quantum theoretical cutoff. The actual group of this $9 + 1$ dimensional space-time would be $O(9, 1)$ regarded as a torus it would lie in the Cartan disk of the invisible $E_{10}$.

If we start from $D = 26 = (25 + 1)$ space-time (dimension of the string) then torus compactification to $D = 10$ would give the root lattices of $E_8 \times E_8$ or $O(32)$ as the internal group of the 16 compactified small dimensions. We can compactify the remaining 10 dimensions to a torus with large spatial periods and large or $(\infty)$ time period and identify it with part of $E_{10}$ i.e. with the Lorentz group $O(9, 1)$.

Hence the groups of the compactification $O(9, 1) \times O(8) \times O(8)$, or $O(9, 1) \times O(16)$ are lifted to $E_{10} \times E_8 \times E_8$ or $E_{10} \times O(32)$. Here $E_{10}$ is the external group of space-time group $O(9, 1)$, and in a similar way we identify with $E_8 \times E_8$ and $O(32)$ the internal groups $O(8) \times O(8)$ and $O(16)$, respectively.

External and internal groups are supported respectively by their $10$ dimensional Lorentzian and $16$ dimensional Euclidean lattices. They are imbedded into a $25 + 1$ Lorentzian lattice $L(25, 1)$. This is the Conway-Sloane lattice. It is regarded as corresponding to a new Lorentzian extension of the hyperbolic group $E_{10}$. Cartan generators can be introduced to give the structure of a new infinite Lie algebra with a rank 26 generalization (called $E_\infty$ by Conway and Sloane). It has sublattices $E_{10} \times E_8 \times E_8$, $E_{10} \times O(32)$, $E_{10} \times O(16) \times O(16)$, i.e. the groups of known heterotic string theories. Hence a new string theory could be associated with $E_\infty$ it would fuse external and internal symmetries together. Note that $E_\infty$ also contains the Leech lattice (not a group!) and the Monster.

At this point it may be useful to summarize some known results (mostly due to Kac) about hyperbolic groups:

Characterized by Dynkin diagrams such that if a point is erased the remaining diagram is either an extended Dynkin diagram (associated with affine algebra $\hat{G}$) or a Dynkin diagram of a Lie group.

They stop at rank 10 corresponding to hyperbolic extension of $E_8$. Among those the strictly hyperbolic groups are those which are not $\hat{G}$, but give Lie group $G$ when one point is erased. They stop at rank 4.

Extensions which correspond to even Lorentzian lattices and are not hyperbolic groups stop at rank 26 (the Conway-Sloane lattice) $[8k + 2, \text{families with } k = 1, 2, 3]$. Again we find the critical dimensions $4, 10, 26$ for hyperbolic and Lorentzian algebras.

Construction of the root lattices of $E_8$, $E_9 = \hat{E}_8$ and $E_{10}$, as well as association of the Conway-Sloane lattice with some of the lattices described above and with discrete Jordan algebras over octonions is deferred to another publication. Remarkable associations between symmetries of superstrings and the lattices generated by discrete Jordan algebras suggest that all known superstring theories are related and originate from a more general theory related to the Conway-Sloane transhyperbolic group.

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