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Continuity results for parametric nonlinear singular Dirichlet problems

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Abstract: In this paper we study from a qualitative point of view the nonlinear singular Dirichlet problem depending on a parameter \( \lambda > 0 \) that was considered in [32]. Denoting by \( S_\lambda \) the set of positive solutions of the problem corresponding to the parameter \( \lambda \), we establish the following essential properties of \( S_\lambda \):

(i) there exists a smallest element \( u^*_\lambda \) in \( S_\lambda \), and the mapping \( \lambda \mapsto u^*_\lambda \) is (strictly) increasing and left continuous;

(ii) the set-valued mapping \( \lambda \mapsto S_\lambda \) is sequentially continuous.

Keywords: Parametric singular elliptic equation, \( p \)-Laplacian, smallest solution, sequential continuity, monotonicity

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1 Introduction

Elliptic equations with singular terms represent a class of hot-point problems because they are mathematically significant and appear in applications to chemical catalysts processes, non-Newtonian fluids, and in models for the temperature of electrical conductors (see [3, 9]). An extensive literature is devoted to such problems, especially focusing on their theoretical analysis. For instance, Ghergu-Rădulescu [18] established several existence and nonexistence results for boundary value problems with singular terms and parameters; Gasinski-Papageorgiou [15] studied a nonlinear Dirichlet problem with a singular term, a \((p-1)\)-sublinear term, and a Carathéodory perturbation; Hirano-Sacco-Shioji [21] proved Brezis-Nirenberg type theorems for a singular elliptic problem. Related topics and results can be found in Crandall-Rabinowitz-Tartar [7], Cîrstea-Ghergu-Rădulescu [6], Dupaigne-Ghergu-Rădulescu [10], Gasinski-Papageorgiou [17], Averna-Motreanu-Tornatore [2], Papageorgiou-Winkert [33], Carl [4], Faria-Miyagaki-Motreanu [11], Carl-Costa-Tehrani [5], Liu-Motreanu-Zeng [26] Papageorgiou-Rădulescu-Repovš [30], and the references therein.

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with a \( C^2 \)-boundary \( \partial \Omega \) and let \( \gamma \in (0, 1) \) and \( 1 < p < +\infty \). Recently, Papageorgiou-Vetro-Vetro [32] have considered the following parametric nonlinear singular Dirichlet problem

\[
\begin{align*}
-\Delta_p u(x) &= \lambda u(x)^{\gamma} + f(x, u(x)) \quad \text{in } \Omega \\
u(x) &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where the operator $\Delta_p$ stands for the $p$-Laplace differential operator

$$\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u) \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

The nonlinear function $f$ is assumed to satisfy the following conditions:

$H(f) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$, $f(x, 0) = 0$, $f(x, s) \geq 0$ for all $s \geq 0$, and

(i) for every $\rho > 0$, there exists $a_{\rho} \in L^{\infty}(\Omega)$ such that

$$|f(x, s)| \leq a_{\rho}(x) \quad \text{for a.e. } x \in \Omega \text{ and for all } |s| \leq \rho;$$

(ii) there exists an integer $m \geq 2$ such that

$$\lim_{s \to +\infty} \frac{f(x, s)}{s^{p-1}} = \tilde{\lambda}_m \quad \text{uniformly for a.e. } x \in \Omega,$$

where $\tilde{\lambda}_m$ is the $m$-th eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$, and denoting

$$F(x, t) = \int_0^t f(x, s) \, ds,$$

then

$$pF(x, s) - f(x, s)s \to +\infty \quad \text{as } s \to +\infty, \quad \text{uniformly for a.e. } x \in \Omega;$$

(iii) for some $r > p$, there exists $c_0 > 0$ such that

$$0 \leq \liminf_{s \to 0^+} \frac{f(x, s)}{s^{r-1}} \leq \limsup_{s \to 0^+} \frac{f(x, s)}{s^{r-1}} \leq c_0 \quad \text{uniformly for a.e. } x \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\tilde{\epsilon}_\rho > 0$ such that for a.e. $x \in \Omega$ the function

$$s \mapsto f(x, s) + \tilde{\epsilon}_\rho s^{p-1}$$

is nondecreasing on $[0, \rho]$.

The following bifurcation type result is proved in [32, Theorem 2].

**Theorem 1.** If hypotheses $H(f)$ hold, then there exists a critical parameter value $\lambda^* > 0$ such that

(a) for all $\lambda \in (0, \lambda^*)$ problem (1) has at least two positive solutions $u_0, u_1 \in \text{int}(C^1_0(\bar{\Omega}));$

(b) for $\lambda = \lambda^*$ problem (1) has at least one positive solution $u^* \in \text{int}(C^1_0(\bar{\Omega}));$

(c) for all $\lambda > \lambda^*$ problem (1) has no positive solutions.

In what follows, we denote

$$\mathcal{L} := \{ \lambda > 0 : \text{problem (1) admits a (positive) solution} \} = (0, \lambda^*],$$

$$S_\lambda = \{ u \in W^{1,p}_0(\Omega) : u \text{ is a (positive) solution of problem (1)} \}.$$

for $\lambda \in \mathcal{L}$. In this respect, Theorem 1 asserts that the above hypotheses, in conjunction with the nonlinear regularity theory (see Liebermann [24, 25]) and the nonlinear strong maximum principle (see Pucci-Serrin [34]), ensure that there holds

$$S_\lambda \subset \text{int}(C^1_0(\bar{\Omega})).$$

Also, we introduce the set-valued mapping $\Lambda : (0, \lambda^*) \to C^1_0(\bar{\Omega})$ by

$$\Lambda(\lambda) = S_\lambda \quad \text{for all } \lambda \in (0, \lambda^*].$$

The following open questions need to be answered:
1. Is there a smallest positive solution to problem (1) for each \( \lambda \in (0, \lambda^* \])?

2. If for each \( \lambda \in (0, \lambda^* \] problem (1) has a smallest positive solution \( u_\lambda^* \), then the function \( \Gamma : (0, \lambda^* \] \rightarrow C_0^1(\overline{\Omega}) \) with \( \Gamma(\lambda) = u_\lambda^* \) is it monotone?

3. If for each \( \lambda \in (0, \lambda^* \] problem (1) has a smallest positive solution \( u_\lambda^* \), then is the function \( \Gamma \) continuous?

4. Is the solution mapping \( \Lambda \) upper semicontinuous?

5. Is the solution mapping \( \Lambda \) lower semicontinuous?

In this paper we answer in the affirmative the above open questions.

**Theorem 2.** Assume that hypotheses \( H(f) \) hold. Then there hold:

(i) the set-valued mapping \( \Lambda : \mathcal{L} \rightarrow 2^{C_0^1(\overline{\Omega})} \) is sequentially continuous;

(ii) for each \( \lambda \in \mathcal{L} \), problem (1) has a smallest positive solution \( u_\lambda^* \in \text{int}(C_0^1(\overline{\Omega}))_+ \), and the map \( \Gamma \) from \( \mathcal{L} \) to \( C_0^1(\overline{\Omega}) \) given by \( \Gamma(\lambda) = u_\lambda^* \) is

(a) (strictly) increasing, that is, if \( 0 < \mu < \lambda \leq \lambda^* \), then

\[ u_\lambda^* - u_\mu^* \in \text{int}(C_0^1(\overline{\Omega}))_+ \]

(b) left continuous.

The rest of the paper is organized as follows. In Section 2 we set forth the preliminary material needed in the sequel. In Section 3 we prove our main results formulated as Theorem 2.

### 2 Preliminaries

In this section we gather the preliminary material that will be used to prove the main result in the paper. For more details we refer to [8, 13, 16, 19, 22, 28, 29, 35].

Let \( 1 < p < \infty \) and \( p' \) be its Hölder conjugate defined by \( \frac{1}{p} + \frac{1}{p'} = 1 \). In what follows, the Lebesgue space \( L^p(\Omega) \) is endowed with the standard norm

\[ \| u \|_p = \left( \int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}} \text{ for all } u \in L^p(\Omega). \]

The Sobolev space \( W_0^{1,p}(\Omega) \) is equipped with the usual norm

\[ \| u \| = \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} \text{ for all } u \in W_0^{1,p}(\Omega). \]

In addition, we shall use the Banach space

\[ C_0^1(\overline{\Omega}) = \{ u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \}. \]

Its cone of nonnegative functions

\[ C_0^1(\overline{\Omega})_+ = \{ u \in C_0^1(\overline{\Omega}) : u \geq 0 \text{ in } \Omega \} \]

has a nonempty interior given by

\[ \text{int}(C_0^1(\overline{\Omega})_+) = \{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega \text{ with } \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0 \}. \]
Proposition 6. The propositions below provide criteria of upper and lower semicontinuity.

Hereafter by \(\langle \cdot, \cdot \rangle\) we denote the duality brackets for \((W^{1,p}(\Omega)^*, W^{1,p}(\Omega))\). Also, we define the nonlinear operator \(A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*\) by

\[
\langle A(u), v \rangle = \int_{\Omega} |\nabla u(x)|^{p-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^n} \, dx \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{2}
\]

The following statement is a special case of more general results (see Gasiński-Papageorgiou [14], Motreanu-Motreanu-Papageorgiou [29]).

**Proposition 3.** The map \(A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*\) introduced in (2) is continuous, bounded (that is, it maps bounded sets to bounded sets), monotone (hence maximal monotone) and of type \((S_+), i.e., if \(u_n \rightharpoonup u \) in \(W^{1,p}(\Omega)\) and

\[
\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,
\]

then \(u_n \to u \) in \(W^{1,p}(\Omega)\).

For the sake of clarity we recall the following notion regarding order.

**Definition 4.** Let \((P, \leq)\) be a partially ordered set. A subset \(E \subset P\) is called downward directed if for each pair \(u, v \in E\) there exists \(w \in E\) such that \(w \leq u\) and \(w \leq v\).

For any \(u, v \in W^{1,p}_0(\Omega)\) with \(u(x) \leq v(x)\) for a.e. \(x \in \Omega\), we set the ordered interval

\[
[u, v] := \{ w \in W^{1,p}_0(\Omega) : u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega \}.
\]

For \(s \in \mathbb{R}\), we denote \(s^+ = \max\{\pm s, 0\}\). It is clear that if \(u \in W^{1,p}_0(\Omega)\) then it holds

\[
u^+ \in W^{1,p}_0(\Omega), \quad u = u^+-u^-, \quad |u| = u^++u^-.
\]

We recall a few things regarding upper and lower semicontinuous set-valued mappings.

**Definition 5.** Let \(X\) and \(Y\) be topological spaces. A set-valued mapping \(F : X \to 2^Y\) is called

(i) upper semicontinuous (u.s.c., for short) at \(x \in X\) if for every open set \(O \subset Y\) with \(F(x) \subset O\) there exists a neighborhood \(N(x)\) of \(x\) such that

\[
F(N(x)) := \bigcup_{y \in N(x)} F(y) \subset O;
\]

if this holds for every \(x \in X, F\) is called upper semicontinuous;

(ii) lower semicontinuous (l.s.c., for short) at \(x \in X\) if for every open set \(O \subset Y\) with \(F(x) \cap O \neq \emptyset\) there exists a neighborhood \(N(x)\) of \(x\) such that

\[
F(y) \cap O \neq \emptyset \text{ for all } y \in N(x);
\]

if this holds for every \(x \in X, F\) is called lower semicontinuous;

(iii) continuous at \(x \in X\) if \(F\) is both upper semicontinuous and lower semicontinuous at \(x \in X\); if this holds for every \(x \in X, F\) is called continuous.

The propositions below provide criteria of upper and lower semicontinuity.

**Proposition 6.** The following properties are equivalent:

(i) \(F : X \to 2^Y\) is u.s.c.;
(ii) for every closed subset $C \subset Y$, the set

$$F^{-}(C) := \{ x \in X \mid F(x) \cap C \neq \emptyset \}$$

is closed in $X$.

**Proposition 7.** The following properties are equivalent:

(a) $F: X \to 2^Y$ is l.s.c.;

(b) if $u \in X$, $(u_{\lambda})_{\lambda \in I} \subset X$ is a net such that $u_{\lambda} \to u$, and $u^* \in F(u)$, then for each $\lambda \in I$ there is $u^*_{\lambda} \in F(u_{\lambda})$ with $u^*_{\lambda} \to u^*$ in $Y$.

### 3 Proof of the main result

In this section we prove Theorem 2. We start with the fact that, for each $\lambda \in \Lambda$, problem (1) has a smallest solution. To this end, we will use the similar technique employed in [12, Lemma 4.1] to show that the solution set $S_\lambda$ is downward directed (see Definition 4).

**Lemma 8.** For each $\lambda \in \Lambda = (0, \lambda^*)$, the solution set $S_\lambda$ of problem (1) is downward directed, i.e., if $u_1, u_2 \in S_\lambda$, then there exists $u \in S_\lambda$ such that

$$u \leq u_1 \quad \text{and} \quad u \leq u_2.$$  

**Proof.** Fix $\lambda \in (0, \lambda^*)$ and $u_1, u_2 \in S_\lambda$. Corresponding to any $\varepsilon > 0$ we introduce the truncation $\eta_{\varepsilon}: \mathbb{R} \to \mathbb{R}$ as follows

$$\eta_{\varepsilon}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{t}{\varepsilon} & \text{if } 0 < t < \varepsilon \\ 1 & \text{otherwise,} \end{cases}$$

which is Lipschitz continuous. It results from Marcus-Mizel [27] that

$$\eta_{\varepsilon}(u_2 - u_1) \in W^{1,p}_0(\Omega)$$

and

$$\nabla(\eta_{\varepsilon}(u_2 - u_1)) = \eta_{\varepsilon}'(u_2 - u_1)\nabla(u_2 - u_1).$$

Then for any function $v \in C^\infty_0(\Omega)$ with $v(x) \geq 0$ for a.e. $x \in \Omega$, we have

$$\eta_{\varepsilon}(u_2 - u_1)v \in W^{1,p}_0(\Omega)$$

and

$$\nabla(\eta_{\varepsilon}(u_2 - u_1)v) = v\nabla(\eta_{\varepsilon}(u_2 - u_1)) + \eta_{\varepsilon}(u_2 - u_1)\nabla v.$$  

Since $u_1, u_2 \in S_\lambda$, there hold

$$
\int_{\Omega} |\nabla u_i(x)|^{p-2}(\nabla u_i(x), \nabla \varphi(x))_{\mathbb{R}^N} dx = \lambda \int_{\Omega} u_i(x)^{-\gamma} \varphi(x) dx + \int_{\Omega} f(x, u_i(x)) \varphi(x) dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega), \ i = 1, 2.
$$
Inserting $\varphi = \eta_\varepsilon (u_2 - u_1)\nu$ for $i = 1$ and $\varphi = (1 - \eta_\varepsilon (u_2 - u_1))\nu$ for $i = 2$, and summing the resulting inequalities yield

\[
\int_{\Omega} |\nabla u_1(x)|^{p-2} (\nabla u_1(x), \nabla (\eta_\varepsilon (u_2 - u_1)\nu)(x)) d\nu + \int_{\Omega} |\nabla u_2(x)|^{p-2} (\nabla u_2(x), \nabla ((1 - \eta_\varepsilon (u_2 - u_1))\nu)(x)) d\nu
\]

\[
= \int_{\Omega} [\lambda u_1(x)^{-\gamma} + f(x, u_1(x))] (\eta_\varepsilon (u_2 - u_1)\nu)(x) d\nu + \int_{\Omega} [\lambda u_2(x)^{-\gamma} + f(x, u_2(x))] (1 - \eta_\varepsilon (u_2 - u_1))\nu)(x) d\nu.
\]

We note that

\[
\int_{\Omega} |\nabla u_1(x)|^{p-2} (\nabla u_1(x), \nabla (\eta_\varepsilon (u_2 - u_1)\nu)(x)) d\nu
\]

\[
= \frac{1}{\varepsilon} \int_{\Omega} |\nabla u_1(x)|^{p-2} (\nabla u_1(x), \nabla (u_2 - u_1)(x)) \nu(x) d\nu + \int_{\Omega} |\nabla u_1(x)|^{p-2} (\nabla u_1(x), \nabla \eta_\varepsilon (u_2 - u_1)(x)) d\nu
\]

and

\[
\int_{\Omega} |\nabla u_2(x)|^{p-2} (\nabla u_2(x), \nabla ((1 - \eta_\varepsilon (u_2 - u_1))\nu)(x)) d\nu
\]

\[
= -\frac{1}{\varepsilon} \int_{\Omega} |\nabla u_2(x)|^{p-2} (\nabla u_2(x), \nabla (u_2 - u_1)(x)) \nu(x) d\nu + \int_{\Omega} |\nabla u_2(x)|^{p-2} (\nabla u_2(x), \nabla (1 - \eta_\varepsilon (u_2 - u_1))(x)) d\nu.
\]

Altogether, we obtain

\[
\int_{\Omega} |\nabla u_1(x)|^{p-2} (\nabla u_1(x), \nabla \eta_\varepsilon (u_2 - u_1)(x)) d\nu + \int_{\Omega} |\nabla u_2(x)|^{p-2} (\nabla u_2(x), \nabla (1 - \eta_\varepsilon (u_2 - u_1))(x)) d\nu
\]

\[
\geq \int_{\Omega} [\lambda u_1(x)^{-\gamma} + f(x, u_1(x))] (\eta_\varepsilon (u_2 - u_1)\nu)(x) d\nu + \int_{\Omega} [\lambda u_2(x)^{-\gamma} + f(x, u_2(x))] (1 - \eta_\varepsilon (u_2 - u_1))\nu)(x) d\nu.
\]

Now we pass to the limit as $\varepsilon \to 0^+$. Using Lebesgue’s Dominated Convergence Theorem and the fact that

\[
\eta_\varepsilon ((u_2 - u_1)(x)) \to \chi_{\{u_1 = u_2\}}(x) \text{ for a.e. } x \in \Omega \text{ as } \varepsilon \to 0^+,
\]
we find
\[
\int \chi_{\{u_1 < u_2\}} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N} dx + \int \chi_{\{u_1 > u_2\}} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N} dx \\
\geq \int \chi_{\{u_1 < u_2\}} [\lambda u_1(x)^{1/p} + f(x, u_1(x))] v(x) dx + \int \chi_{\{u_1 > u_2\}} [\lambda u_2(x)^{1/p} + f(x, u_2(x))] v(x) dx.
\]
(3)

Here the notation \(\chi_D\) stands for the characteristic function of a set \(D\), that is,
\[
\chi_D(t) = \begin{cases} 
1 & \text{if } t \in D \\
0 & \text{otherwise.}
\end{cases}
\]

The gradient of \(u := \min\{u_1, u_2\} \in W^{1,p}_0(\Omega)\) is equal to
\[
\nabla u(x) = \begin{cases} 
\nabla u_1(x) & \text{for a.e. } x \in \{u_1 < u_2\} \\
\nabla u_2(x) & \text{for a.e. } x \in \{u_1 > u_2\}.
\end{cases}
\]

Consequently, we can express (3) in the form
\[
\int \nabla u(x)^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N} dx \geq \int \nabla u(x)^{1/p} v(x) dx
\]
(4)

for all \(v \in C_0^\infty(\Omega)\) with \(v(x) \geq 0\) for a.e. \(x \in \Omega\). Actually, the density of \(C_0^\infty(\Omega)\), in \(W^{1,p}_0(\Omega)\), ensures that (4) is valid for all \(v \in W^{1,p}_0(\Omega)\).

Let \(\tilde{u}_\lambda\) be the unique solution of the purely singular elliptic problem
\[
\begin{cases} 
-\Delta_p u(x) = \lambda u(x)^{-\gamma} & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Proposition 5 of Papageorgiou-Smyrlis [31] guarantees that \(\tilde{u}_\lambda \in \text{int}(C^1_0(\overline{\Omega}))\). We claim that
\[
\tilde{u}_\lambda \leq u \text{ for all } u \in S_\lambda.
\]
(5)

For every \(u \in S_\lambda\), there holds
\[
\int \nabla u(x)^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N} dx = \int \lambda u(x)^{-\gamma} v(x) dx
\]
(6)
Lemma 9. If hypotheses \(H\) hold and \(\lambda \in \mathcal{L} = (0, \lambda^*)\), then problem (1) has a smallest (positive) solution \(u_\lambda^* \in S_\lambda\), that is,

\[ u_\lambda^* \leq u \quad \text{for all} \; u \in S_\lambda. \]
**Proof.** Fix \( \lambda \in (0, \lambda^*]. \) Invoking Hu-Papageorgiou [22, Lemma 3.10], we can find a decreasing sequence \( \{u_n\} \subset S_\lambda \) such that

\[
\inf S_\lambda = \inf_n u_n.
\]

On the basis of (5) we note that

\[
\tilde{u}_\lambda \leq u_n \quad \text{for all } n.
\] (9)

Next we verify that the sequence \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega). \) Arguing by contradiction, suppose that a relabeled subsequence of \( \{u_n\} \) satisfies \( \|u_n\| \to \infty. \) Set \( y_n = \frac{u_n}{\|u_n\|}. \) This ensures

\[
y_n \to y \quad \text{weakly in } W^{1,p}_0(\Omega) \quad \text{and} \quad y_n \to y \quad \text{strongly in } L^p(\Omega) \quad \text{with} \quad y \neq 0.
\] (10)

From (6) and \( \{u_n\} \subset S_\lambda \) we have

\[
\langle A(y_n), \nu \rangle = \int_\Omega |\nabla y_n(x)|^{p-2}(\nabla y_n(x), \nabla \nu(x)) dx
\]

\[
= \int_\Omega \left[ \frac{\lambda u_n(x)^\gamma}{\|u_n\|^{p-1}} + \frac{f(x, u_n(x))}{\|u_n\|^{p-1}} \right] \nu(x) dx
\] (11)

for all \( \nu \in W^{1,p}_0(\Omega). \) On the other hand, hypotheses \( H(f)\) (i) and (ii) entail

\[
0 \leq f(x, s) \leq c_1(1 + |s|^{p-1}) \quad \text{for a.e. } x \in \Omega \quad \text{and all} \quad s \geq 0,
\] (12)

with some \( c_1 > 0. \) By (10) and (12) we see that the sequence

\[
\left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right\}
\]

is bounded in \( L^p(\Omega). \)

Due to hypothesis \( H(f)\) (ii) and Aizicovici-Papageorgiou-Staicu [1, Proposition 16], we find that

\[
\left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right\} \to \tilde{\lambda}_m y^{p-1} \quad \text{weakly in } L^p(\Omega).
\]

Then inserting \( \nu = y_n - y \) in (11) and using (9) lead to

\[
\lim_{n \to \infty} \langle A(y_n), y_n - y \rangle = 0.
\]

We can apply Proposition 3 to obtain \( y_n \to y \) in \( W^{1,p}_0(\Omega). \) Letting \( n \to \infty \) in (11) gives

\[
\langle A(y), \nu \rangle = \tilde{\lambda}_m \int_\Omega y^{p-1} \nu dx \quad \text{for all } \nu \in W^{1,p}_0(\Omega),
\]

so \( y \) is a nontrivial nonnegative solution of the eigenvalue problem

\[
\begin{cases}
-\Delta_p y(x) = \tilde{\lambda}_m y(x)^{p-1} & \text{in } \Omega \\
y = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Consequently, \( y \) must be nodal because \( m \geq 2 \) and \( y \neq 0, \) which contradicts that \( y \geq 0 \) in \( \Omega. \) This contradiction proves that the sequence \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega). \)

Along a relabeled subsequence, we may assume that

\[
u_n \to u_\lambda^* \quad \text{weakly in } W^{1,p}_0(\Omega) \quad \text{and} \quad \nu_n \to u_\lambda^* \quad \text{in } L^p(\Omega),
\] (13)

for some \( u_\lambda^* \in W^{1,p}_0(\Omega). \) In addition, we may suppose that

\[
u_n(x)^\gamma \to u_\lambda^*(x)^\gamma \quad \text{for a.e. } x \in \Omega.
\] (14)
From $\bar{u}_n \in \text{int}(C_0^1(\overline{\Omega}))$ and (5), through the Lemma in Lazer-Mckenna [23], we obtain

$$0 \leq u_n^{-} \leq \bar{u}_n^{-} \in L^{p'}(\Omega).$$  \hfill (15)

On account of (13)-(15) we have

$$u_n^{-} \to (u_A)^{-} \text{ weakly in } L^{p'}(\Omega)$$  \hfill (16)

(see also Gasiński-Papageorgiou [16, p. 38]).

Proof. (i) It follows from [32, Proposition 5] that there exists a solution $u = u_n \in S_n$ and $v = u_n - u_A^* \in W^{1,p}(\Omega)$ in (6), in the limit as $n \to \infty$ we get

$$\lim_{n \to \infty} \langle Au_n, u_n - u_A^* \rangle = 0.$$  \hfill (17)

The property of $A$ to be of type $(S_\mu)$ (according to Proposition 3) implies

$$u_n \to u_A^* \text{ in } W^{1,p}_0(\Omega).$$

The above convergence and Sobolev embedding theorem enable us to deduce

$$\int_\Omega |\nabla u_A^*(x)|^{p-2} (\nabla u_A^*(x), \nabla v(x))_{R^n} dx = \int_\Omega [\lambda u_A^*(x)^{-} + f(x, u_A^*(x))] v(x) dx$$

for all $v \in W^{1,p}_0(\Omega)$. Consequently, we have

$$u_A^* \in S_n \subset \text{int}(C_0^1(\overline{\Omega})) \text{ and } u_A^* = \inf S_n,$$  \hfill (18)

which completes the proof.

In the next lemma we examine monotonicity and continuity properties of the map $\lambda \mapsto u_A^*$ from $\mathcal{L} = (0, \lambda^*)$ to $C_0^1(\overline{\Omega})$.

Lemma 10. Suppose that hypotheses $H(f)$ hold. Then the map $\Gamma: \mathcal{L} = (0, \lambda^*) \to C_0^1(\overline{\Omega})$ given by $\Gamma(\lambda) = u_A^*$ fulfills:

(i) $\Gamma$ is strictly increasing, in the sense that

$$0 < \mu < \lambda \leq \lambda^* \implies u_A^* - u_\mu^* \in \text{int}(C_0^1(\overline{\Omega}));$$

(ii) $\Gamma$ is left continuous.

Proof. (i) It follows from [32, Proposition 5] that there exists a solution $u_\mu \in S_\mu \subset \text{int}(C_0^1(\overline{\Omega}))$ such that

$$u_A^* - u_\mu^* \in \text{int}(C_0^1(\overline{\Omega})).$$

The desired conclusion is the direct consequence of the inequality $u_\mu^* \leq u_A^*$.

(ii) Let $\{\lambda_n\} \subset (0, \lambda^*)$ and $\lambda \in (0, \lambda^*)$ satisfy $\lambda_n \uparrow \lambda$. Denote for simplicity $u_n = u_{\lambda_n} = \Gamma(\lambda_n) \in S_{\lambda_n} \subset \text{int}(C_0^1(\overline{\Omega}))$. It holds

$$\langle Au_n, v \rangle = \int_\Omega [\lambda_n u_n(x)^{-} + f(x, u_n(x))] v(x) dx$$  \hfill (19)

for all $v \in W^{1,p}_0(\Omega)$. By assertion (i) we know that

$$0 \leq u_1 \leq u_n \leq u_A^*.$$  \hfill (20)

Choosing $v = u_n$ in (19) and proceeding as in the proof of Lemma 9, we verify that the sequence $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$. Given $r > N$, it is true that $(u_A^*)^r \in \text{int}(C_0^1(\overline{\Omega}))$, so there is a constant $c_2 > 0$ such that

$$\bar{u}_1 \leq c_2 (u_A^*)^r = c_2 u_A^*,$$
Lemma 11. Assume that hypotheses H(f) hold. Then the set-valued mapping \( \Lambda : \mathcal{L} \to 2^{C^1_0(\overline{\Omega})} \) is sequentially upper semicontinuous.
The same argument as in the proof of Lemma 10 confirms that, for any closed set $D \subset C^1_0(\bar{\Omega})$, one has that

$$A^-(D) := \{ \lambda \in \mathbb{R} : A(\lambda) \cap D \neq \emptyset \}$$

is closed in $\mathbb{R}$. Let $\{\lambda_n\} \subset A^-(D)$ verify $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. So,

$$A(\lambda_n) \cap D \neq \emptyset,$$

hence there exists a sequence $\{u_n\} \subset \text{int}(C^1_0(\bar{\Omega}))$ satisfying

$$u_n \in A(\lambda_n) \cap D \text{ for all } n \in \mathbb{N},$$

in particular

$$\int_{\Omega} |\nabla u_n(x)|^{p-2} (\nabla u_n(x), \nabla v(x))_{\mathbb{R}^N} \, dx = \int_{\Omega} [\lambda_n u_n(x)^\gamma + f(x, u_n(x))] v(x) \, dx \quad (20)$$

for all $v \in W^{1,p}_0(\Omega)$. As in the proof of Lemma 9, we can show that the sequence $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$. Therefore we may assume that

$$u_n \rightarrow u \text{ weakly in } W^{1,p}_0(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^p(\Omega). \quad (21)$$

for some $u \in W^{1,p}_0(\Omega)$. Furthermore, the sequences $\{f(\cdot, u_n(\cdot))\}$ and $\{u_n^\gamma\}$ are bounded in $L^p(\Omega)$ as already demonstrated in the proofs of Lemmas 9 and 10. In (20), we choose $v = u_n - u \in W^{1,p}_0(\Omega)$ and then pass to the limit as $n \rightarrow \infty$. By means of (21) we are led to

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0.$$

Since $A$ is of type $(S_+)$, we can conclude

$$u_n \rightarrow u \text{ in } W^{1,p}_0(\Omega). \quad (22)$$

On account of (20), the strong convergence in (22) and Sobolev embedding theorem imply

$$\int_{\Omega} |\nabla u(x)|^{p-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^N} \, dx = \int_{\Omega} [\lambda u(x)^\gamma + f(x, u(x))] v(x) \, dx$$

for all $v \in W^{1,p}_0(\Omega)$. This reads as $u \in S_\lambda = A(\lambda)$.

It remains to check that $u \in D$. Fix $\lambda \in \mathcal{L}$ such that

$$\underline{A} < \lambda_n \leq \lambda^* \text{ for all } n.$$

By Lemma 10 (i) we know that

$$u^*_\underline{A} < u_{\lambda_n}^* \leq u_n \text{ for all } n.$$

The same argument as in the proof of Lemma 10 confirms that, for $r > N$ fixed, the function $x \mapsto \lambda_n u_n(x)^\gamma + f(x, u_n(x))$ is bounded in $L^r(\Omega)$. Let $g_{\lambda_n}(x) = \lambda_n u_n(x)^\gamma + f(x, u_n(x)) \in L^r(\Omega)$ and consider the linear Dirichlet problem

$$\begin{cases} -\Delta v(x) = g_{\lambda_n}(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial \Omega. \end{cases} \quad (23)$$

The standard existence and regularity theory (see, e.g., Gilbarg-Trudinger [19, Theorem 9.15]) ensure that problem (23) has a unique solution

$$v_{\lambda_n} \in W^{2,r}(\Omega) \subset C^{1,q}_0(\bar{\Omega}) \text{ with } \|v_{\lambda_n}\|_{C^{1,q}_0(\bar{\Omega})} \leq c_n,$$
with a constant $c_3 > 0$ and $\alpha = 1 - \frac{N}{p}$. Denote $w_n(x) = \nabla \nu_{\lambda_n}(x)$ for all $x \in \Omega$. It holds $w_n \in C^{0,\alpha}([\Omega])$ thanks to $\nu_{\lambda_n} \in C^{1,\alpha}([\Omega])$. Notice that

$$
\begin{cases}
-\text{div}(\nabla u_n(x)) = w_n(x) \quad &\text{in } \Omega \\
u_n = 0 &\text{on } \partial \Omega.
\end{cases}
$$

The nonlinear regularity up to the boundary in Lieberman [24, 25] reveals that $u_n \in C^{1,\beta}([\Omega])$ for all $n \in \mathbb{N}$ with some $\beta \in (0, 1)$. The compactness of the embedding of $C^{1,\beta}([\Omega])$ in $C^{1}([\Omega])$ and (22) yield the strong convergence

$$
u_n \to u \quad \text{in } C^{1}([\Omega]).
$$

Recalling that $D$ is closed in $C^{1}([\Omega])$ it results that $u \in \Lambda(\lambda) \cap D$, i.e., $\lambda \in \Lambda^{-}(D)$.

**Lemma 12.** Suppose that hypotheses $H(f)$ hold. Then the set-valued mapping $\Lambda : \mathcal{L} \to 2^{C^{1}([\Omega])}$ is sequentially lower semicontinuous.

**Proof.** In order to refer to Proposition 7, let $\{\lambda_n\} \subset \mathcal{L}$ satisfy $\lambda_n \to \lambda \neq 0$ as $n \to \infty$ and let $w \in S_{\lambda} \subset \text{int}(C^1([\Omega]), \mathcal{L})$. For each $n \in \mathbb{N}$, we formulate the Dirichlet problem

$$
\begin{cases}
-\Delta_p u(x) = \lambda_n w(x) \quad &\text{in } \Omega \\
u > 0 &\text{in } \Omega \\
u = 0 &\text{on } \partial \Omega.
\end{cases}
$$

(24)

In view of $w \geq \bar{u}_\lambda \in \text{int}(C^1([\Omega]), \mathcal{L})$ (see (5)) and

$$
\begin{cases}
\lambda_n w(x) > f(x, w(x)) &\text{for all } x \in \Omega \\
\lambda_n w(x) < f(x, w(x)) &\text{for all } x \in \Omega
\end{cases},
$$

it is obvious that problem (24) has a unique solution $u_0^0 \in \text{int}(C^1([\Omega]), \mathcal{L})$. Relying on the growth condition for $f$ (see hypotheses $H(f)$ (i) and (ii)), through the same argument as in the proof of Lemma 9 we show that the sequence $\{u^0_n\}$ is bounded in $W^{1,p}_0(\Omega)$. Then Proposition 1.3 of Guedda-Véron [20] implies the uniform boundedness

$$
u_0^0 \in L^{\infty}(\Omega) \quad \text{and } \|\nu_0^0\|_{L^{\infty}(\Omega)} \leq c_5 \quad \text{for all } n \in \mathbb{N},$$

with a constant $c_5 > 0$. As in the proof of Lemma 11, we set $g_{\lambda_n}(x) = \lambda_n w(x) + f(x, w(x))$ and consider the Dirichlet problem (23) to obtain that $\{u^0_0\}$ is contained in $C^{1,\beta}([\Omega])$ for some $\beta \in (0, 1)$. Due to the compactness of the embedding of $C^{1,\beta}([\Omega])$ in $C^1([\Omega])$, we may assume

$$
u_n^0 \to u \quad \text{in } C^1([\Omega]) \quad \text{as } n \to \infty,'
$$

with some $u \in C^1([\Omega])$. Then (24) yields

$$
\begin{cases}
-\Delta_p u(x) = \lambda w(x) \quad &\text{in } \Omega \\
u > 0 &\text{in } \Omega \\
u = 0 &\text{on } \partial \Omega.
\end{cases}
$$

(24)

Thanks to $w \in \Lambda(\lambda)$, a simple comparison justifies $u = w$. Since every convergent subsequence of $\{u_n\}$ converges to the same limit $w$, it is true that

$$
\lim_{n \to \infty} u^0_n = w.
$$
Next, for each $n \in \mathbb{N}$, we consider the Dirichlet problem
\[
\begin{align*}
-\Delta_p u(x) &= \lambda_n u_n^0(x)^{-\gamma} + f(x, u_n^0(x)) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Carrying on the same reasoning, we can show that this problem has a unique solution $u^1_n$ belonging to int$(C^1_0(\Omega))$ and that
\[
\lim_{n \to \infty} u^1_n = w.
\]
Continuing the process, we generate a sequence $\{u^k_n\}_{n,k=1}$ such that
\[
\begin{align*}
-\Delta_p u^k_n(x) &= \lambda_n u^{k-1}_n(x)^{-\gamma} + f(x, u^{k-1}_n(x)) \quad \text{in } \Omega \\
u^k_n &> 0 \quad \text{in } \Omega \\
u^k_n &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
and
\[
\lim_{n \to \infty} u^k_n = w \quad \text{for all } k \in \mathbb{N}. \quad (25)
\]
Fix $n \geq 1$. As before, based on the nonlinear regularity [24, 25], we notice that the sequence $\{u^k_n\}_{k=1}$ is relatively compact in $C^1_0(\overline{\Omega})$, so we may suppose
\[
u^k_n \to \nu_n \text{ in } C^1_0(\overline{\Omega}) \text{ as } k \to \infty,
\]
for some $\nu_n \in C^1_0(\overline{\Omega})$. Then it appears that
\[
\begin{align*}
-\Delta_p u_n(x) &= \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x)) \quad \text{in } \Omega \\
u_n &> 0 \quad \text{in } \Omega \\
u_n &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
which means that $\nu_n \in \Lambda(\lambda_n)$.

The convergence in (25) and the double limit lemma (see, e.g., [13, Proposition A.2.35]) result in
\[
u_n \to w \text{ in } C^1_0(\overline{\Omega}) \text{ as } n \to \infty.
\]
By Proposition 7 we conclude that $\Lambda$ is lower semicontinuous. \hfill \Box

**Proof of Theorem 2.** (i) It suffices to apply Lemmas 11 and 12.

(ii) The stated conclusion is a direct consequence of Lemmas 9 and 10. \hfill \Box

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