On van Hamme's (A.2) and (H.2) supercongruences

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Abstract. In 1997, van Hamme conjectured 13 Ramanujan-type supercongruences labeled (A.2)–(M.2). Using some combinatorial identities discovered by Sigma, we extend (A.2) and (H.2) to supercongruences modulo $p^4$ for primes $p \equiv 3 \pmod{4}$, which appear to be new.

Keywords: Supercongruences; Hypergeometric series; $p$-Adic Gamma functions

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1 Introduction

In 1913, Ramanujan announced the following identity in his famous letter:

$$\sum_{k=0}^{\infty} (-1)^k(4k + 1) \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k 5 = \frac{2}{\Gamma \left(\frac{3}{4}\right)^4},$$

(1.1)

where $(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1)$ for $n \geq 1$. The formula (1.1) was later proved by Hardy (1924) and Watson (1931).

Motivated by some of such formulas, van Hamme [13] conjectured 13 supercongruences in 1997, which relate the partial sums of certain hypergeometric series to the values of the $p$-adic Gamma functions. These 13 conjectural supercongruences labeled (A.2)–(M.2) are $p$-adic analogues of their corresponding infinite series identities. For instance, van Hamme conjectured that the identity (1.1) has the following interesting $p$-adic analogue, which was first proved by McCarthy and Osburn [5] by use of Gaussian hypergeometric series.

**Theorem 1.1** (See [13], (A.2).) For any odd prime $p$, we have

$$\sum_{k=0}^{p-1} (-1)^k(4k + 1) \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k 5 \equiv \begin{cases} -p\Gamma_p \left(\frac{1}{4}\right)^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

where $\Gamma_p(\cdot)$ denotes the $p$-adic Gamma function recalled in the next section.

Later, Swisher [12] showed that the supercongruence (A.2) also holds modulo $p^5$ for primes $p \equiv 1 \pmod{4}$. It is natural to ask whether the case $p \equiv 3 \pmod{4}$ possesses a modulo $p^5$ extension in terms of the $p$-adic Gamma functions. Interestingly, numerical calculation suggests the following result which partially motivates this paper.
Conjecture 1.2 For primes $p \geq 5$ with $p \equiv 3 \pmod{4}$, we have
\[
\sum_{k=0}^{\frac{p-1}{2}} (-1)^k (4k + 1) \left( \frac{1}{2} \right)_k \left( \frac{1}{k!} \right)^5 \equiv -\frac{p^3}{16} \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^5}.
\]
(1.2)

It is very difficult for us to prove (1.2) thoroughly, but we can show that (1.2) holds modulo $p^4$.

Theorem 1.3 Let $p \geq 5$ be a prime. For $p \equiv 3 \pmod{4}$, we have
\[
\sum_{k=0}^{\frac{p-1}{2}} (-1)^k (4k + 1) \left( \frac{1}{2} \right)_k \left( \frac{1}{k!} \right)^5 \equiv -\frac{p^3}{16} \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^4}.
\]
(1.3)

The other motivating example of this paper is the following supercongruence which was both conjectured and confirmed by van Hamme [13].

Theorem 1.4 (See [13, (H.2)].) For any odd prime $p$, we have
\[
\sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{2} \right)_k \left( \frac{1}{k!} \right)^3 \equiv \begin{cases} -\Gamma_p \left( \frac{1}{4} \right)^4 & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \pmod{p^2}.
\]
(1.4)

Recently, Long and Ramakrishna [4, Theorem 3] obtained a modulo $p^3$ extension of (H.2) as follows:
\[
\sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{2} \right)_k \left( \frac{1}{k!} \right)^3 \equiv \begin{cases} -\Gamma_p \left( \frac{1}{4} \right)^4 & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{p^2}{16} \Gamma_p \left( \frac{1}{4} \right)^4 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \pmod{p^3}.
\]
(1.4)

The second aim of this paper is to establish a modulo $p^4$ extension of (1.4) for primes $p \equiv 3 \pmod{4}$.

Theorem 1.5 Let $p \geq 5$ be a prime. For $p \equiv 3 \pmod{4}$, we have
\[
\sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{2} \right)_k \left( \frac{1}{k!} \right)^3 \equiv -\frac{p^2}{4} \Gamma_p \left( \frac{1}{4} \right)^4 \frac{J_{p-3/4}}{J_n}, \pmod{p^4},
\]
(1.5)

where $J_n = \sum_{k=0}^n \left( \frac{2k}{k} \right)^2 / 16^k$.

Remark. We shall show that $J_{p-3/4} \equiv -\frac{1}{2} \pmod{p}$, which implies (1.5) is indeed a modulo $p^4$ extension of the second case of (1.4). By the Chu-Vandermonde identity, we have
\[
\sum_{k=0}^{n} \binom{2n+1}{k}^2 = \frac{1}{2} \sum_{k=0}^{2n+1} \binom{2n+1}{k}^2 = \frac{1}{2} \left( 4n+2 \right),
\]
Letting $n = \frac{p-3}{4}$ with $p \equiv 3 \pmod{4}$ in the above and noting that
\[
\left(\frac{p-1}{2}\right) \equiv \frac{(2k)}{(-4)^k} \pmod{p} \quad \text{and} \quad \left(\frac{p-1}{p-1}\right) \equiv -1 \pmod{p},
\]
we immediately obtain $J_{(p-3)/4} \equiv -\frac{1}{2} \pmod{p}$.

It is known that Gaussian hypergeometric series (see, for example, [3–6, 12]) and the W-Z method (see, for example, [2, 8, 11, 15]) commonly apply to the Ramanujan-type supercongruences. We refer to [8, 12] for more recent developments on van Hamme’s supercongruences. In this paper, we provide a different approach which is based on some combinatorial identities involving harmonic numbers. All of these identities are automatically discovered and proved by the software package Sigma developed by Schneider [9].

The rest of this paper is organized as follows. The preliminary section, Section 2, is devoted to some properties of the $p$-adic Gamma function. We prove Theorems 1.3 and 1.5 in Sections 3 and 4, respectively.

2 Preliminary results

We first recall the definition and some basic properties of the $p$-adic Gamma function. For more details, we refer to [1, Section 11.6]. Let $p$ be an odd prime and $\mathbb{Z}_p$ denote the set of all $p$-adic integers. For $x \in \mathbb{Z}_p$, the $p$-adic Gamma function is defined as
\[
\Gamma_p(x) = \lim_{m \to x} (-1)^m \prod_{0 < k < m \atop (k, p) = 1} k,
\]
where the limit is for $m$ tending to $x$ $p$-adically in $\mathbb{Z}_{\geq 0}$.

Lemma 2.1 (See [1, Section 11.6].) For any odd prime $p$ and $x \in \mathbb{Z}_p$, we have
\[
\begin{align*}
\Gamma_p(1) &= -1, \quad (2.1) \\
\Gamma_p(x)\Gamma_p(1 - x) &= (-1)^{s_p(x)}, \quad (2.2) \\
\frac{\Gamma_p(x + 1)}{\Gamma_p(x)} &= \begin{cases} -x & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1, \end{cases} \quad (2.3)
\end{align*}
\]
where $s_p(x) \in \{1, 2, \ldots, p\}$ with $s_p(x) \equiv x \pmod{p}$ and $| \cdot |_p$ denotes the $p$-adic norm.

Lemma 2.2 (See [4, Lemma 17, (4)].) Let $p$ be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a + 1, \ldots, a + n - 1$ in $p\mathbb{Z}_p$, then
\[
(a)_n = (-1)^n \frac{\Gamma_p(a + n)}{\Gamma_p(a)}. \quad (2.4)
\]

The following lemma is a special case of a theorem due to Long and Ramakrishna [4].

Lemma 2.3 (See [4, Theorem 14].) For any prime $p \geq 5$ and $a, b \in \mathbb{Z}_p$, we have
\[
\Gamma_p(a + bp) \equiv \Gamma_p(a) (1 + G_1(a)bp) \pmod{p^2}, \quad (2.5)
\]
where $G_1(a) = \frac{\Gamma_p'(a)}{\Gamma_p(a)} \in \mathbb{Z}_p$. 

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3 Proof of Theorem 1.3

In order to prove Theorem 1.3 we establish the following two combinatorial identities.

Lemma 3.1 For any odd integer \( n \), we have

\[
\sum_{k=0}^{n}(-1)^{k}(4k+1)\frac{\left(\frac{1}{2}\right)_{k}^{3}(-n)_{k}(n+1)_{k}}{(1)_{k}^{3}(n+\frac{3}{2})_{k}(-n+\frac{1}{2})_{k}} = 0. \tag{3.1}
\]

Proof. We begin with the following identity [6 (2.1)]:

\[
_{6}F_{5}\left[a, \frac{a}{2}+1, b, c, d, e; \frac{a}{2}, a-b+1, a-c+1, a-d+1, a-e+1; -1\right] = \frac{\Gamma(a-d+1)\Gamma(a-e+1)}{\Gamma(a+1)\Gamma(a-d-e+1)}_{3}F_{2}\left[a-b-c+1, d, e; a-b+1, a-c+1+1\right].
\]

Letting \( a = d = e = \frac{1}{2}, b = -n \) and \( c = n+1 \) in the above gives

\[
\sum_{k=0}^{n}(-1)^{k}(4k+1)\frac{\left(\frac{1}{2}\right)_{k}^{3}(-n)_{k}(n+1)_{k}}{(1)_{k}^{3}(n+\frac{3}{2})_{k}(-n+\frac{1}{2})_{k}} = _{3}F_{2}\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; n+\frac{3}{2}, -n+\frac{1}{2}; 1\right]/\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right). \tag{3.2}
\]

Recall the following Whipple’s identity [10 page 54]:

\[
_{3}F_{2}\left[a, 1-a, c; e, 1+2c-e; 1\right] = \frac{2^{1-2c}\pi\Gamma(c)\Gamma(1+2c-e)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{3}{2}e-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+c-\frac{1}{2}e+\frac{1}{2}a\right)\Gamma\left(1-c-\frac{1}{2}e-\frac{1}{2}a\right)}. \tag{3.3}
\]

Assume that \( n \) is odd. Letting \( a = c = \frac{1}{2} \) and \( e = n + \frac{3}{2} \) in (3.3) and noting that both \( 1 - \frac{1}{2}(n+1) \) and \( \frac{1}{2}(1-n) \) are non-positive integers, we conclude that

\[
_{3}F_{2}\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; n+\frac{3}{2}, -n+\frac{1}{2}; 1\right] = 0. \tag{3.4}
\]

Combining (3.2) and (3.4), we complete the proof of (3.1). \( \square \)

Lemma 3.2 For any odd integer \( n \), we have

\[
\sum_{k=0}^{n}(-1)^{k}(4k+1)\frac{\left(\frac{1}{2}\right)_{k}^{3}(-n)_{k}(n+1)_{k}}{(1)_{k}^{3}(n+\frac{3}{2})_{k}(-n+\frac{1}{2})_{k}} \sum_{j=1}^{k}\left(\frac{1}{(2j)^{2}} - \frac{1}{(2j-1)^{2}}\right)
= 4^{n-2}(2n+1)/n^{2}\left(\frac{n-1}{n-2}\right)^{2}. \tag{3.5}
\]
Proof. Actually, we can automatically discover and prove (3.5) by use of the software package \texttt{Sigma} developed by Schneider (see [7, Section 5] and [9, Section 3.1] for a similar approach to finding and proving identities of this type).

After loading \texttt{Sigma} into Mathematica, we insert:

\[
\ln[1] := \text{mySum} = \sum_{k=0}^{2n+1} (-1)^k (4k + 1) \left( \frac{3}{2} \right)_k \left( -2n + 1 \right)_k \left( 2n + 2 \right)_k \sum_{j=1}^{k} \left( \frac{1}{(2j)^2} - \frac{1}{(2j - 1)^2} \right)
\]

We compute the recurrence for the above sum:

\[
\ln[2] := \text{rec} = \text{GenerateRecurrence}[\text{mySum}, n][[1]]
\]

We find that the above sum \texttt{SUM}[n] satisfies a recurrence of order 2. Now we solve this recurrence:

\[
\ln[3] := \text{recSol} = \text{SolveRecurrence}[\text{rec}, \text{SUM}[n]]
\]

\[
\text{Out}[3] = \left\{ \left\{ 0, \frac{4(3 + 4n)}{(1 + 2n)^2} \prod_{i=1}^{n} \left( -1 + 2i \right)^2 \right\}, \left\{ 0, \frac{3 + 4n}{(1 + 2n)^2} \left( \prod_{i=1}^{n} \left( -1 + 2i \right)^2 \right) \left( \sum_{i=1}^{n} \frac{4}{i^2} - \sum_{i=1}^{n} \frac{16}{(1 + 2i)^2} \right) \right\}, \{1, 0\} \right\}
\]

Finally, we combine the solutions to represent \texttt{mySum} as follows:

\[
\ln[4] := \text{FindLinearCombination}[\text{recSol}, \text{mySum}, 2]
\]

\[
\text{Out}[4] = \frac{3 + 4n}{4(1 + 2n)^2} \prod_{i=1}^{n} \left( -1 + 2i \right)^2
\]

This implies that both sides of (3.5) are equal. \hfill \Box

Proof of Theorem 1.3. Assume that \( p \equiv 3 \pmod{4} \). Letting \( n = \frac{p-1}{2} \) in (3.1) yields

\[
\sum_{k=0}^{p-1} (-1)^k (4k + 1) \left( \frac{3}{2} \right)_k \left( -2n + 1 \right)_k \left( 2n + 2 \right)_k = 0.
\]

(3.6)

We next determine the following product modulo \( p^4 \):

\[
\frac{\left( -2n + 1 \right)_k \left( 1 + \frac{p}{2} \right)_k}{\left( 2n + 2 \right)_k} = \prod_{j=1}^{k} \frac{(2j - 1)^2 - p^2}{(2j)^2 - p^2}.
\]

From the following two Taylor expansions:

\[
\frac{(2j - 1)^2 - x^2}{(2j)^2 - x^2} = \left( \frac{2j - 1}{2j} \right)^2 - \frac{4j - 1}{(2j)^4} x^2 + \mathcal{O}(x^4),
\]

and

\[
\prod_{j=1}^{k} (a_j + b_j x^2) = \prod_{j=1}^{k} a_j \left( 1 + x^2 \sum_{j=1}^{k} \frac{b_j}{a_j} \right) + \mathcal{O}(x^4),
\]

\[
\prod_{j=1}^{k} (a_j + b_j x^2) = \prod_{j=1}^{k} a_j \left( 1 + x^2 \sum_{j=1}^{k} \frac{b_j}{a_j} \right) + \mathcal{O}(x^4),
\]
we deduce that for $0 \leq k \leq \frac{p-1}{2}$,
\[
\frac{(\frac{1}{2})_k (\frac{1-p}{2})_k}{(1+\frac{p}{2})_k (1-\frac{p}{2})_k} \equiv \prod_{j=1}^{k} \left( \left( \frac{2j-1}{2j} \right)^2 - \frac{(4j-1)p^2}{(2j)^4} \right) \\
\equiv \left( \frac{(\frac{1}{2})_k}{(1)_k} \right)^2 \left( 1 + p^2 \sum_{j=1}^{k} \left( \frac{1}{(2j)^2} - \frac{1}{(2j-1)^2} \right) \right) \pmod{p^4}. \tag{3.7}
\]
Substituting (3.7) into (3.6) gives
\[
\frac{p-1}{2} \sum_{k=0}^{p-1} (-1)^k (4k + 1) \left( \frac{\frac{1}{2}}{k!} \right)^5 \\
\equiv -p^2 \sum_{k=0}^{p-1} (-1)^k (4k + 1) \left( \frac{\frac{1}{2}}{k!} \right)^5 \sum_{j=1}^{k} \left( \frac{1}{(2j)^2} - \frac{1}{(2j-1)^2} \right) \pmod{p^4}. \tag{3.8}
\]
Furthermore, letting $n = \frac{p-1}{2}$ in (3.5) and noting the fact that
\[(a + bp)_k(a - bp)_k \equiv (a)_k^2 \pmod{p^2}, \tag{3.9}\]
we obtain
\[
\frac{p-1}{2} \sum_{k=0}^{p-1} (-1)^k (4k + 1) \left( \frac{\frac{1}{2}}{k!} \right)^5 \sum_{j=1}^{k} \left( \frac{1}{(2j)^2} - \frac{1}{(2j-1)^2} \right) \\
\equiv p2^{p-5} \left/ \left( \frac{p-1}{2} \right)^2 \left( \frac{p-3}{2} \right)^2 \right. \pmod{p^2}. \tag{3.10}
\]
It follows from (3.8) and (3.10) that
\[
\frac{p-1}{2} \sum_{k=0}^{p-1} (-1)^k (4k + 1) \left( \frac{\frac{1}{2}}{k!} \right)^5 \equiv -p^3 2^{p-5} \left/ \left( \frac{p-1}{2} \right)^2 \left( \frac{p-3}{2} \right)^2 \right. \pmod{p^4}. \tag{3.11}
\]
By the Fermat’s little theorem, we have
\[
2^{p-5} \left/ \left( \frac{p-1}{2} \right)^2 \right. \equiv \frac{1}{4} \pmod{p}. \tag{3.12}
\]
On the other hand, using (2.1), (2.4) and (2.5) we obtain
\[
\left( \frac{p-3}{2} \right)^2 = 2^{p-3} \left( \frac{\frac{1}{2}}{1} \frac{p-3}{p-1} \right)^2 = 2^{p-3} \frac{\Gamma_p \left( \frac{p-1}{4} \right)^2 \Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( \frac{1}{2} \right)^2 \Gamma_p \left( \frac{p+1}{4} \right)^2} \equiv \frac{1}{4} \cdot \frac{\Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( \frac{1}{2} \right)^2 \Gamma_p \left( \frac{1}{4} \right)^2} \pmod{p}. \tag{3.13}
\]
From (3.11)–(3.13), we deduce that
\[
\sum_{k=0}^{p-1} (-1)^k(4k + 1) \left( \frac{\frac{1}{2}}{k!} \right)^5 \equiv -p^3 \frac{\Gamma_p \left( \frac{1}{2} \right) \Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( -\frac{1}{4} \right)^2} \pmod{p^4}.
\]
Since \( p \equiv 3 \pmod{4} \), by (2.2) we have
\[
\Gamma_p \left( \frac{1}{2} \right)^2 = (-1)^{p+1} = 1 \quad \text{and} \quad \Gamma_p \left( -\frac{1}{4} \right)^2 \Gamma_p \left( \frac{5}{4} \right)^2 = 1.
\]
It follows that
\[
\sum_{k=0}^{p-1} (-1)^k(4k + 1) \left( \frac{\frac{1}{2}}{k!} \right)^5 \equiv -p^3 \frac{\Gamma_p \left( \frac{1}{4} \right)^2 \Gamma_p \left( \frac{5}{4} \right)^2}{\Gamma_p \left( -\frac{1}{4} \right)^2} \pmod{p^4}.
\]
Finally, by (2.3) we obtain
\[
\Gamma_p \left( \frac{5}{4} \right) = -\frac{1}{4} \Gamma_p \left( \frac{1}{4} \right).
\]
Substituting the above into (3.14), we complete the proof of (1.3). □

4 Proof of Theorem 1.5

Lemma 4.1 For any odd integer \( n \), we have
\[
\sum_{k=0}^{n} \left( \frac{\frac{1}{2}}{k!} \right) (-n)_k(n+1)_k = 0.
\] (4.1)

Proof. Assume that \( n \) is odd. Letting \( a = -n, e = 1 \) and \( c = \frac{1}{2} \) in (3.3) and noting that \( \frac{1-n}{2} \) is a non-positive integer, we immediately arrive at (4.1). □

Lemma 4.2 For any odd integer \( n \), we have
\[
\sum_{k=0}^{n} \left( \frac{\frac{1}{2}}{k!} \right) (-n)_k(n+1)_k \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = -4^{n-1} \left( \sum_{k=0}^{n-1} \frac{2^{2k}}{16^k} \right)^2 / n^2 \left( \frac{n-1}{2} \right)^2.
\] (4.2)

Proof. The above identity possesses, of course, the same automated proof as (3.5), and we omit the details. □

Proof of Theorem 1.5 Assume that \( p \equiv 3 \pmod{4} \). Letting \( n = \frac{p-1}{2} \) in (4.1), we obtain
\[
\sum_{k=0}^{p-1} \left( \frac{\frac{1}{2}}{k!} \right) \left( \frac{1+p}{2} \right)_k \left( \frac{1-p}{2} \right)_k = 0.
\] (4.3)
Similarly to the proof of (3.7), we can show that for \(0 \leq k \leq \frac{p-1}{2}\),
\[
\frac{\left(\frac{1+p}{2}\right)_k \left(\frac{1-p}{2}\right)_k}{(1)_k^2} \equiv \left(\frac{1}{2}\right) \left(1 - p^2 \sum_{j=1}^{k} \frac{1}{(2j-1)^2}\right) \quad (\text{mod } p^4). \tag{4.4}
\]
Substituting (4.4) into (4.3) gives
\[
\frac{p-1}{2} \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{1}{2}\right)_k \equiv p^2 \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{1}{2}\right)_k \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \quad (\text{mod } p^4). \tag{4.5}
\]
Furthermore, letting \(n = \frac{p-1}{2}\) in (4.2) and then using (3.9), we arrive at
\[
\frac{p-1}{2} \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{1}{2}\right)_k \equiv -4 \frac{p-3}{2} \sum_{j=1}^{\frac{p-3}{2}} \left(\frac{1}{2}\right) \left(\frac{1+p}{2}\right)^2 \left(\frac{p-3}{2}\right)^2 \quad (\text{mod } p^2). \tag{4.6}
\]
By (2.4), we have
\[
4 \frac{p-3}{2} \equiv \left(\frac{p-1}{2}\right)^2 \left(\frac{1+p}{2}\right)^2 \left(\frac{p-3}{2}\right)^2 = \frac{1}{4} \left(\frac{1}{2}\right)^2 \frac{1}{2} \frac{1}{4} \Gamma_p \Gamma_p \left(\frac{1}{2}\right) \Gamma_p \left(\frac{1-p}{2}\right)^2. \tag{4.7}
\]
Since \(p \equiv 3 \pmod{4}\), by (2.1) and (2.2) we obtain
\[
\Gamma_p \left(\frac{1}{4}\right)^2 = 1,
\Gamma_p \left(\frac{1}{2}\right)^2 = (-1)^{\frac{p-1}{2}} = 1,
\Gamma_p \left(\frac{p+3}{4}\right)^2 \Gamma_p \left(\frac{1-p}{4}\right)^2 = 1.
\]
Substituting the above into (4.7) yields
\[
4 \frac{p-3}{2} \equiv \left(\frac{p-1}{2}\right)^2 \left(\frac{1+p}{2}\right)^2 \left(\frac{p-3}{2}\right)^2 = \frac{1}{4} \Gamma_p \left(\frac{1+p}{4}\right)^2 \Gamma_p \left(\frac{1-p}{4}\right)^2.
\]
By (2.5), we have
\[
\Gamma_p \left(\frac{1+p}{4}\right) \equiv \Gamma_p \left(\frac{1}{4}\right) \left(1 + \frac{p}{4} G_1 \left(\frac{1}{4}\right)\right) \quad (\text{mod } p^2),
\Gamma_p \left(\frac{1-p}{4}\right) \equiv \Gamma_p \left(\frac{1}{4}\right) \left(1 - \frac{p}{4} G_1 \left(\frac{1}{4}\right)\right) \quad (\text{mod } p^2).
\]
It follows that
\[
\Gamma_p \left( \frac{1 + p}{4} \right) \Gamma_p \left( \frac{1 - p}{4} \right) \equiv \Gamma_p \left( \frac{1}{4} \right)^2 \pmod{p^2},
\]
and so
\[
4^{p-3} \left( \frac{p-1}{2} \right)^2 \left( \frac{p-3}{4} \right)^2 \equiv \frac{1}{4} \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^2}. \tag{4.8}
\]
Then the proof of \((1.5)\) follows from \((4.5)\), \((4.6)\) and \((4.8)\). \qed

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