ON THE LINEAR FRACTIONAL SELF-ATTRACTING DIFFUSION*

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Abstract. In this paper, we introduce the linear fractional self-attracting diffusion driven by a fractional Brownian motion with Hurst index $1/2 < H < 1$, which is analogous to the linear self-attracting diffusion. For 1-dimensional process we study its convergence and the corresponding weighted local time. For 2-dimensional process, as a related problem, we show that the renormalized self-intersection local time exists in $L^2$ if $1/2 < H < 3/4$.

1. Introduction

In 1991, Durrett and Rogers [7] studied a system that models the shape of a growing polymer. Under some conditions, they established asymptotic behavior of the solution of stochastic differential equation

$$X_t = B_t + \int_0^t \int_0^s \Phi(X_s - X_u)duds,$$

where $B$ is a $d$-dimensional standard Brownian motion and $\Phi$ Lipschitz continuous. If $\Phi(x) = \Psi(x)x/\|x\|$ and $\Psi(x) \geq 0$, $X_t$ is a continuous analogue of a process introduced by Diaconis and studied by Pemantle [20]. The path dependent stochastic differential equation can be considered as polymer model. In 1995, Cranston and Le Jan [5] extended the model and introduced self-attracting diffusions, where for $d = 1$ two cases of are studied: the linear interaction where $\Phi$ is a linear function and the constant interaction in dimension 1, where $\Phi(x) = \sigma \text{sign}(x)$ for positive $\sigma$, and in both cases the almost sure convergence of $X_t$ is proved. Herrmann-Roynette [9], Herrmann-Scheutzowb [10] generalized these results.

On the other hand, the statistical properties of fractional Brownian motion (fBm) are used to construct a path integral representation of the conformations of some polymers (see, for examples, Chakravarti and Sebastian [3], Cherayil and Biswas [4], Sebastian [21]). Thus, as a natural extension to (1.1) one may consider the path

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dependent stochastic differential equation of the form

\begin{equation}
X_t^H = B_t^H + \int_0^t \int_0^s \Phi(X_s^H - X_u^H) duds,
\end{equation}

where $B_t^H$ is a $d$-dimensional fractional Brownian motion with Hurst index $H \in (0,1)$ and $\Phi$ Lipschitz continuous. Then it is not difficult to show that the above equation admits a unique strong solution. We will call the solution the fractional self-attracting diffusion driven by fBm. In this paper, we consider only a particular case as follows (the linear fractional self-attracting diffusion):

\begin{equation}
X_t^H = B_t^H - a \int_0^t \int_0^s (X_s^H - X_u^H) duds + \nu t
\end{equation}

with $a > 0$, $\nu \in \mathbb{R}^d$ and $\frac{1}{2} < H < 1$. Our aims are to study the convergence and local times of the processes given by (1.3) with $d = 1$. As a related problem, for the two dimensional process we shall show that the renormalized self-intersection local time exists in $L^2$ if $\frac{1}{2} < H < \frac{3}{4}$.

The structure of this paper is as follows. In Section 2 we briefly recall fBm and related the Itô type stochastic integral. In Section 3 we investigate convergence of the linear fractional self-attracting diffusion. We show that the process converges with probability one as $t$ tends to infinity. In Section 4 we define the weighted local time of the process and obtain a Meyer-Tanaka type formula. Finally, in Section 5 for 2-dimensional process we show that its renormalized self-intersection local time exists in $L^2$ if $\frac{1}{2} < H < \frac{3}{4}$.

2. Fractional Brownian motion and the Itô type formula

In this section, we briefly recall the definition and properties of stochastic integral with respect to fBm. Throughout this paper we assume that $\frac{1}{2} < H < 1$ is arbitrary but fixed. Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space such that a fractional Brownian motion with Hurst index $H$ is well defined. For simplicity we let $C$ stand for a positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to $c$.

Recall that a centered continuous Gaussian process $B_t^H = \{B_t^H, t \geq 0\}$ with the covariance function

\[ E[B_t^H B_s^H] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad s, t \geq 0, \]

is called the fBm with Hurst index $H$. Here $E$ denotes the expectation with respect to the probability law of $B_t^H$ on $\Omega$. This process was first introduced by Kolmogorov and studied by Mandelbrot and Van Ness [16], where a stochastic integral representation in terms of a standard Brownian motion was established. The definition of stochastic integrals with respect to the fBm has been investigated by several authors. Here, we refer to Duncan et al [6] and Hu-Øksendal [15] (see also Elliott-Van der Hoek [8], Hu [11], Nualart [17, 18], Øksendel [19]) for the definition and the
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properties of the fractional Itô integral

\[ \int_0^t u_s dB^H_s \]

of an adapted process \( u \). For \( 1/2 < H < 1 \) we define the function \( \phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[ \phi(s, t) = H(2H - 1)|s - t|^{2H - 2}, \quad s, t \geq 0. \]

Recall that the Malliavin \( \phi \)-derivative of the function \( U : \Omega \to \mathbb{R} \) defined in [6] as follows:

\[ D^\phi U = \int_0^\infty \phi(r, s)D_r U dr, \]

where \( D_r U \) is the fractional Malliavin derivative at \( r \). Define the space \( L^{1,2}_{\phi} \) to be the set of measurable processes \( u \) such that \( D^\phi_t u_s \) exists for a.a. \( s, t \geq 0 \) and

\[ \|u\|_{L^{1,2}_{\phi}} := \mathbb{E} \left[ \int_0^\infty \int_0^\infty D^\phi_s u_t D^\phi_t u_s dsdt + \int_0^\infty \int_0^\infty u_s u_t \phi(s, t) dsdt \right] < \infty. \]

Thus, the integral \( \int_0^\infty u_s dB^H_s \) can be well defined as an element of \( L^2(\mu) \) if \( u \) satisfies (2.1). For the integral process \( \eta_t = \int_0^t u_s dB^H_s \), we have (see, for examples, Duncan et al. [6], Hu [11])

\[ D^\phi_s \eta_t = \int_0^t D^\phi_s u_r dB^H_r + \int_0^t u_r \phi(s, r) dr. \]

In particular, if \( u \) is deterministic, then \( D^\phi_s \eta_t = \int_0^t u_r \phi(s, r) dr. \)

**Theorem 2.1** (Duncan et al. [6], Hu [11]). Let \( F \in C^2(\mathbb{R}) \) having polynomial growth and let the process \( X \) be given as follows:

\[ dX_t = v_t dt + u_t dB^H_t, \quad X_0 = x \in \mathbb{R}, \]

where \( u \in L^{1,2}_{\phi} \) and measurable process \( v \) satisfies \( \int_0^t |v_s| ds < \infty \) a.s. Then we have, for all \( t \geq 0 \)

\[ F(X_t) = F(x) + \int_0^t \frac{\partial}{\partial x} F(s, X_s) dX_s + \int_0^t \frac{\partial^2}{\partial x^2} F(s, X_s) u_s D^\phi_s X_s ds. \]

3. **Convergence**

In this section, we consider convergence of the solution of the equation (1.3), the so-call linear fractional self-attracting diffusion. The method used here is essentially due to M. Cranston and Y. Le Jan [5].

**Proposition 3.1.** The solution to the equation (1.3) can be expressed as

(3.1) \[ X^H_t = X^H_0 + \int_0^t h(t, s) dB^H_s + \nu \int_0^t h(t, s) ds, \]

where

(3.2) \[ h(t, s) = \begin{cases} 1 - ae^a \frac{1}{s^a} \int_s^t e^{-\frac{1}{2} a u} du, & t \geq s, \\ 0, & t < s \end{cases} \]

for \( s, t \geq 0 \).
This proposition can also be obtained by the same method as Cranston and Le Jan [5]. It follows from the Ito type formula that

\[
F(X^H_t) = F(0) + \int_0^t F'(X^H_s) dX^H_s + \int_0^t F''(X^H_s) D^\phi_s X^H_s ds
\]

\[
= F(0) + \int_0^t F'(X^H_s) dX^H_s
\]

\[
+ 2H(2H - 1) \int_0^t F''(X^H_s) ds \int_0^s h(s,m)(s-m)^{2H-2} dm
\]

for \(F \in C^2(\mathbb{R})\) having polynomial growth. On the other hand, an elementary calculation yields

\[
h(s) = \lim_{t \to \infty} h(t,s) = 1 - ase^{\frac{a}{2}s^2} \int_s^\infty e^{-\frac{a}{2}u^2} du,
\]

which is continuous on \([0, \infty)\).

**Theorem 3.2.** The solution \(X^H_t\) to (1.3) converges in \(L^2(\mu)\) to the following element as \(t \to \infty\):

\[
X^H_\infty \equiv \int_0^\infty h(s) dB^H_s + \nu \int_0^\infty h(s) ds.
\]

**Proof.** Clearly, we have

\[
|h(t, s_1) - h(s_2)| \leq \frac{1}{t^2}s_1s_2e^{\frac{a}{2}(s_1^2+s_2^2)}e^{-at^2}
\]

for \(s_1, s_2 \leq t\) and \(\left| \int_0^t [h(t,s) - h(s)] ds \right| \leq \frac{1}{at} \to 0 \ (t \to \infty)\). From (2.1) it follows that

\[
E \left| \int_0^t [h(t,s) - h(s)] dB^H_s \right|^2 \leq \frac{2H}{at^{2H-2}} \to 0
\]

as \(t \to \infty\), which proves

\[
E \left| X^H_t - X^H_\infty \right|^2 \leq 2E \left| \int_0^t [h(t,s) - h(s)] dB^H_s \right|^2 + 2 \left| \int_0^t [h(t,s) - h(s)] ds \right|^2 \to 0.
\]

This completes the proof. \(\square\)

**Theorem 3.3.** The solution \(X^H_t\) to (1.3) converges to \(X^H_\infty\) almost surely as \(t \to \infty\).

**Proof.** Without loss of generality, we may assume \(\nu = 0\). Note that by Proposition \ref{prop1}

\[
X^H_t - X^H_\infty = \int_0^t [h(t,s) - h(s)] dB^H_s - \int_0^\infty h(s) dB^H_s
\]

\[
\equiv Y^H_t - \int_t^\infty h(s) dB^H_s, \quad t \geq 0.
\]

Thus, it is enough to show that \(Y^H_t\) converges to 0 almost surely as \(t \to \infty\).
For integer numbers $n, k$, $0 \leq k < n$ we set $Z_{n,k}^H = Y_{n+\frac{k}{n}}^H$. Then $Z_{n,k}^H$ is Gaussian, and we have

$$E \left[ (Z_{n,k}^H)^2 \right] = E \left[ \int_0^{n+\frac{k}{n}} [h(t,s) - h(s)] dB_s^H \right]^2 \leq \frac{2H}{an^{2-2H}}$$

by (3.4), and for any $\varepsilon > 0$

$$P(\left| Z_{n,k}^H \right| > \varepsilon) \leq 2 e^{-\frac{\varepsilon^2 an^{2-2H}}{2H}}.$$  

Furthermore, for $s \in (0,1)$ we set $R^n_{s,k} = Y_{n+\frac{k}{n}} - Y_{n+\frac{k+1}{n}}$. Then $R^n_{s,k}$, $0 \leq s \leq 1$ is a Gaussian process and

$$E \left[ (R^n_{s,k} - R^n_{s,k'})^2 \right] \leq \frac{(s-s')^{2H}}{n^{2H}} (1 + a) = \frac{1 + a}{n^{2H}} E \left[ (B_s^H - B_{s'}^H)^2 \right]$$

by (3.4). For any $\varepsilon > 0$,

$$P \left( \sup_{0 \leq s \leq 1} |R^n_{s,k}| > \varepsilon \right) \leq P \left( \frac{\sqrt{1 + a}}{n^{2H}} \sup_{0 \leq s \leq 1} |B_s^H| > \varepsilon \right) \quad \text{(by Slepian’s lemma)}$$

$$\leq \frac{(1 + a) E \left[ \sup_{0 \leq s \leq 1} |B_s^H|^2 \right]}{\varepsilon^2 n^{2H}}$$

$$\leq C_H (1 + a) \frac{1}{\varepsilon^2 n^{2H}}.$$  

Thus, the convergence with probability one follows from the Borel-Cantelli Lemma and

$$\left\{ \sup_{n+\frac{k}{n} < t < n+\frac{k+1}{n}} |Y_t| > \varepsilon \right\} \subseteq \{ |Z_{n,k}^H| > \varepsilon/2 \} \cup \left\{ \sup_{0 \leq s \leq 1} |R^n_{s,k}| > \varepsilon/2 \right\}$$

for all $k, n \geq 0$. This completes the proof of the theorem. \qed

4. LOCAL TIME AND MEYER-TANAKA TYPE FORMULA

In this section, we consider the linear fractional self-attracting diffusion $X^H = \{X_t^H, 0 \leq t \leq T \}$ with $\nu = 0$. It follows that the process is a centered Gaussian process. We study the usual local time and weighted local time of the process and obtain the Meyer-Tanaka type formula of the weighted local time.

For $T \geq t \geq s \geq 0$ we put

$$\sigma^2_t = E \left[ (X_t^H)^2 \right], \quad \sigma^2_{t,s} = E \left[ (X_t^H - X_s^H)^2 \right].$$

Then

$$\sigma^2_t = \int_0^t \int_0^t h(t,u)h(t,v)\phi(u,v) du dv, \quad 0 \leq t \leq T$$

and

\begin{equation}
\sigma^2_{t,s} = \int_0^t \int_0^s [h(t,u) - h(s,u)] [h(t,v) - h(s,v)] \phi(u,v). \tag{4.1}
\end{equation}

Noting that for all $t \geq s \geq 0$,

$$\int_0^t \int_0^s \phi(u,v) du dv = t^{2H}, \quad e^{-\frac{\varepsilon^2}{2} (t^2-s^2)} \leq h(t,s) \leq 1,$
we get

\[ e^{-\frac{a}{2}t^2}t^{2H} \leq \sigma_t^2 = \int_0^t \int_0^t h(t, u)h(t, v)\phi(u, v)du \, dv \leq t^{2H} . \]

**Lemma 4.1.** For all \( t \geq s \geq 0 \) we have

\[ c_{a,H,T}(t-s)^{2H} \leq \sigma_{t,s}^2 \leq C_{a,H,T}(t-s)^{2H} , \]

where \( C_{a,H,T}, c_{a,H,T} > 0 \) are two constants depending on \( a, H, T \).

**Proof.** For all \( t \geq s \geq 0 \) we have

\[ \sigma_{t,s}^2 = \int_0^t \int_0^t [h(t, u) - h(s, u)] [h(t, v) - h(s, v)] \phi(u, v)du \, dv \]

\[ = \int_s^t \int_s^t h(t, u)h(t, v)\phi(u, v)du \, dv + \int_s^t \int_0^s h(t, u) [h(t, v) - h(s, v)] \phi(u, v)du \, dv + \int_0^s \int_s^t [h(t, u) - h(s, u)] h(t, v)\phi(u, v)du \, dv + \int_0^s \int_0^s [h(t, u) - h(s, u)] [h(t, v) - h(s, v)] \phi(u, v)du \, dv \]

\[ = A_{[s,t]^2} + A_{[s,t] \times [0,s]} + A_{[0,s] \times [s,t]} + A_{[0,s]^2} . \]

On the other hand, for all \( t \geq s \geq 0 \) we have

\[ A_{[s,t] \times [0,s]} = A_{[0,s] \times [s,t]} \]

\[ = - \left( \int_s^t e^{-\frac{a}{2}u^2}du \right) \int_0^svue^{\frac{2}{2}u^2}du \int_s^t h(t, v)\phi(u, v)dv , \]

\[ A_{[0,s]^2} = 2 \left( \int_s^t e^{-\frac{a}{2}u^2}du \right) \int_0^s a^2ue^{\frac{2}{2}u^2}du \int_0^u ve^{\frac{2}{2}u^2}\phi(u, v)dv . \]

But, some elementary calculus can show that

\[ C_{a,T}a^2e^{-aT^2}s^{2+2H}(t-s)^2 \leq A_{[0,s]^2} \leq 2a^2s^{2+2H}(t-s)^2 , \]

\[ e^{-\frac{a}{2}(t^2-s^2)}(t-s)^{2H} \leq A_{[s,t]^2} \leq (t-s)^{2H} , \]

and

\[ \lim_{s \uparrow t} \frac{A_{[s,t] \times [0,s]}}{(t-s)^{2H}} = 0 , \quad \lim_{s \downarrow 0} \frac{A_{[s,t] \times [0,s]}}{(t-s)^{2H}} = 0 , \]

which lead to

\[ \lim_{s \uparrow t} \frac{\sigma_{t,s}^2}{(t-s)^{2H}} = 1 , \quad \lim_{s \downarrow 0} \frac{\sigma_{t,s}^2}{(t-s)^{2H}} \leq t^{2H} \]

It follows that there are two constants \( c_{a,H,T}, C_{a,H,T} > 0 \) such that

\[ c_{a,H,T}(t-s)^{2H} \leq \sigma_{t,s}^2 \leq C_{a,H,T}(t-s)^{2H} . \]

This completes the proof. \( \square \)
From the lemma above, we see that
\[
\int_0^t \int_0^t E \left[ (X_u^H - X_v^H)^2 \right]^{-1/2} \, du \, dv < \infty
\]
holds for all \( t \geq 0 \), and furthermore, we can show that the process \( X^H = (X_t^H)_{0 \leq t \leq T} \) is local nondeterminism for every \( 0 < T < \infty \), i.e. for \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T \),

(4.4) \[
\text{Var} \left( \sum_{j=2}^n u_j (X_{t_j}^H - X_{t_{j-1}}^H) \right) \geq \kappa_0 \sum_{j=2}^n u_j^2 \sigma_{t_j, t_{j-1}}^2
\]

with a constant \( \kappa_0 > 0 \). Combining this with Berman [1, 2], we obtain

**Proposition 4.2.** If \( \nu = 0 \), then the solution \( X^H \) of the equation (1.3) has continuous local time \( L_x^t \), \( t \geq 0, x \in \mathbb{R} \) such that

\[
L_x^t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[x-\varepsilon, x+\varepsilon]}(X_s^H) \, ds = \int_0^t \delta(X_s^H - x) \, ds,
\]

where \( \delta(X_s^H - \cdot) \) denotes the delta function of \( X_s^H \).

For \( t \geq 0, x \in \mathbb{R} \) we now set

\[
L_x^t = 2H(2H-1) \int_0^t \delta(X_s^H - x) \, ds \int_0^s h(s, m)(s-m)^{2H-2} \, dm.
\]

Then \( L_x^t \) is well-defined and

\[
L_x^t = \int_0^t \delta(X_s^H - x) D_s^\phi X_s^H \, ds.
\]

The process \( (L_x^t)_{t \geq 0} \) is called the weighted local time of \( X^H \) at \( x \in \mathbb{R} \).

**Lemma 4.3 (Hu [14]).** Let \( Y \) be normally distributed with mean zero and variance \( \sigma^2 (\sigma > 0) \). Then the delta function \( \delta(Y - \cdot) \) of \( Y \) exists uniquely and we have

(4.5) \[
\delta(Y - x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(Y-x)} \, d\xi, \quad x \in \mathbb{R}.
\]

**Proposition 4.4.** Assume that \( t \in [0, T] \) is given. Then \( L_x^t \) and \( L_x^t \) are square integrable for all \( x \in \mathbb{R} \) and we have

(4.6) \[
E \left[ (L_x^t)^2 \right] \leq C_{a, H, T} t^{2-2H},
\]

(4.7) \[
E \left[ (L_x^t)^2 \right] \leq C_{a, H, T} t^{2H}.
\]

**Proof.** We have by Lemma 4.3

\[
E \left[ (L_x^t)^2 \right] \leq \frac{1}{(2\pi)^2} \int_0^t \int_0^t \, du \, dv \int_{\mathbb{R}^2} E \left[ e^{i(\xi X_u^H + \eta X_v^H)} \right] \, d\xi \, d\eta
\]

\[
\leq \frac{2}{(2\pi)^2} \int_0^t \, du \int_0^t \, dv \int_{\mathbb{R}^2} e^{-\frac{i}{2} \text{Var}(\xi X_u^H + \eta X_v^H)} \, d\xi \, d\eta.
\]

Noting that

\[
\text{Var}(\xi X_u^H + \eta X_v^H) \geq k \left[ \xi^2 \sigma_{u, v}^2 + (\eta + \xi)^2 \sigma_v^2 \right]
\]
for a positive $k > 0$ by local nondeterminacy of the process $X^H$, we get
\[
E \left[ (Z^x_t)^2 \right] \leq \frac{2}{(2\pi)^2} \int_0^t du \int_0^u dv \int_{\mathbb{R}^2} e^{-\frac{k}{2} (\xi^2 \sigma_{u,v}^2 + (\eta - \xi)^2 \sigma_v^2)} d\xi dv
\]
\[
\leq \frac{1}{k^2} \int_0^t du \int_0^u dv \frac{1}{\sigma_{u,v} \sigma_v}
\]
\[
\leq \frac{1}{k^2 \pi c_{a,T}} \int_0^t du \int_0^u \frac{dv}{(u - v)^H\nu^H} \leq C_{a,H,T} T^{2 - 2H}.
\]
by Lemma 4.1. This obtains (4.6). Similarly, one can show that inequality (4.7) holds.

**Theorem 4.5.** Let $X^H$ be the solution to the equation (1.3) with Hurst index $\frac{1}{2} < H < 1$, $X^H_0 = z$, $\nu = 0$ and let $L$ be the weighted local time of $X^H$. Suppose that $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is a convex function having polynomial growth. Then
\[
(4.8) \quad \Phi(X^H_t) = \Phi(z) + \int_0^t D^- \Phi(X^H_s) dX^H_s + \int_\mathbb{R} \mathcal{L}^t \mu_\Phi(dx),
\]
where $D^- \Phi$ denotes the left derivative of $\Phi$ and the signed measure $\mu_\Phi$ is defined by
\[
\mu_\Phi([a, b]) = D^- \Phi(b) - D^- \Phi(a), \quad a < b, a, b \in \mathbb{R}.
\]

**Proof.** For $\varepsilon > 0$ and $x \in \mathbb{R}$ we set
\[
\Phi_\varepsilon(x) = \int_\mathbb{R} p_\varepsilon(x - y) \Phi(y) dy \quad (\varepsilon > 0),
\]
where $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi \varepsilon}} e^{-\frac{1}{2\varepsilon} x^2}$. Then $\Phi_\varepsilon \in C^2$ and we have $\lim_{\varepsilon \to 0} \Phi_\varepsilon(x) = \Phi(x)$, $\lim_{\varepsilon \to 0} \Phi_\varepsilon'(x) = D^- \Phi(x)$ for all $x \in \mathbb{R}$. It follows that for all $\varepsilon > 0$
\[
\Phi_\varepsilon(X^H_t) = \Phi_\varepsilon(z) + \int_0^t \Phi_\varepsilon'(X^H_s) dX^H_s + 2H(2H - 1) \int_0^t \Phi_\varepsilon''(X^H_s) \overline{h}(s) ds.
\]

On the other hand, it is easy to see that $\Phi_\varepsilon(X^H_t)$ converges to $\Phi(X^H_t)$ almost surely, and $\int_0^t \Phi_\varepsilon'(X^H_s) dX^H_s \to \int_0^t D^- \Phi(X^H_s) X^H_s ds$ a.s., and furthermore, $\int_0^t \Phi_\varepsilon'(X^H_s) dB^H_s \to \int_0^t D^- \Phi(X^H_s) dB^H_s$ in $(S)^\ast$.

Finally, we have as $\varepsilon \to 0$
\[
\int_0^t \Phi_\varepsilon''(X^H_s) \overline{h}(s) ds = \int_0^t ds \overline{h}(s) \int_\mathbb{R} \Phi_\varepsilon''(x) \delta(X^H_s - x) dx
\]
\[
\to \frac{1}{2H(2H - 1)} \int_\mathbb{R} \mathcal{L}^t \mu_\Phi(dx).
\]
This completes the proof. \(\square\)

**Corollary 4.6.** Let $X^H$ be the solution to the equation (1.3) with Hurst index $\frac{1}{2} < H < 1$, $X_0^H = z$, $\nu = 0$ and let $L$ be the weighted local time of $X^H$. Then the Tanaka formula
\[
(4.9) \quad |X^H_t - x| = |X^H_0 - x| + \int_0^t \text{sign}(X^H_s - x) dX^H_s + \mathcal{L}^t
\]
holds for all $x \in \mathbb{R}$. \(\square\)
5. Self-intersection local time on \( \mathbb{R}^2 \)

In this section, we shall use the idea of Hu [12] (see also Hu-Nualart [13]) to study the renormalized self-intersection local time of the linear fractional self-attracting diffusion \( X^H = (X^H,1, X^H,2) \) on \( \mathbb{R}^2 \), where \( X^H,j \) \((j = 1,2)\) is the solution of the equation

\[
X^H,j_t = B^H,j_t - a \int_0^t \int_0^u (X^H,j_u - X^H,j_v) dvdu, \quad 0 \leq t \leq T
\]

with \( a > 0 \) and two independent fractional Brownian motions \( B^H,j_t, j = 1,2 \). Then we have

\[
X^H,j_t = \int_0^t h(t,s) dB^H,j_t, \quad j = 1,2
\]

from Section 3, and for all \( s, t \geq 0 \)

\[
h(t, s) = \begin{cases} 
1 - ase^{\frac{1}{2}as^2} \int_s^t e^{-\frac{1}{2}au^2} du, & t \geq s, \\
0, & t < s.
\end{cases}
\]

The renormalized self-intersection local time \( \beta^H_T \) of the process

\[
X^H_t = (X^H,1_t, X^H,2_t), \quad 0 \leq t \leq T
\]

is formally defined as

\[
\beta^H_T = \int_0^T \int_0^t \delta_0(X^H_t - X^H_s) dsdt - E \left[ \int_0^T \int_0^t \delta_0(X^H_t - X^H_s) dsdt \right],
\]

where \( \delta_0 \) is the delta function. For \( \varepsilon > 0 \) we define

\[
\beta^{H,\varepsilon}_T = \int_0^T \int_0^t p_\varepsilon(X^H_t - X^H_s) dsdt,
\]

where

\[
p_\varepsilon(x) = \frac{1}{2\pi\varepsilon} e^{-\frac{|x|^2}{2\varepsilon}}, \quad x \in \mathbb{R}^2
\]

is the heat kernel. Then main object of this section is to explain and prove Theorem 5.1.

**Theorem 5.1.** The random variable \( \beta^{H,\varepsilon}_T - E \left[ \beta^{H,\varepsilon}_T \right] \) converges in \( L^2 \) as \( \varepsilon \) tends to zero if \( \frac{1}{2} < H < \frac{3}{4} \).

In order to prove the theorem we need some preliminaries. For \( t \geq s \geq 0, t' \geq s' \geq 0 \) we now denote

\[
\sigma^2_{t,s} = E \left( X^H,1_t - X^H,1_s \right)^2, \quad \mu = E(X^H,1_t - X^H,1_s)(X^H,1_{t'} - X^H,1_{s'})
\]

and

\[
d_H(s, t, s', t') = \sigma^2_{s,t'} - \sigma^2_{s', t'} - \mu^2.
\]

Then, by Lemma 4.1 and Hu [12] one can establish the following lemma.
Lemma 5.2. (1) For $0 < s < s' < t < t' < T$, we have

$$d_H(s, t, s', t') \geq \kappa \left[(t-s)^{2H}(t'-t)^{2H} + (t'-s')^{2H}(s-s)^{2H}\right].$$

(2) For $0 < s' < s < t < t' < T$, we have

$$d_H(s, t, s', t') \geq \kappa (t-s)^{2H}(t'-s')^{2H}.$$

(3) For $0 < s < t < s' < t' < T$, we have

$$d_H(s, t, s', t') \geq \kappa (t-s)^{2H}(t'-s')^{2H},$$

where $\kappa > 0$ is an enough small constant.

Lemma 5.3. For $0 \leq x < y \leq T$ we set

$$h^*(y, x, u, v) = [h(y, u)1_{(0, y]}(u) - h(x, u)1_{(0, x)}(u)] \left[ h(y, v)1_{(0, y]}(v) - h(x, v)1_{(0, x)}(v) \right],$$

where $h(\cdot, \cdot)$ is defined in Section $\text{[B]}$. Then we have

$$\int_0^{t'} \int_0^{t'} [h^*(t', s, u, v) - h^*(t', t, u, v)] \phi(u, v) \, du \, dv \leq C_{a, H, T} \left[(t'-s)^{2H} - (t'-t)^{2H}\right]$$

for all $0 \leq s \leq t \leq t' \leq T$.

Proof. For $0 < u, v < T$ we have

$$h(t', u)1_{(0, t']}(u) - h(s, u)1_{(0, s]}(u) = h(t', u)1_{(s, t']}(u) + \left[h(t', u) - h(s, u)\right]1_{(0, s]}(u),$$

and

$$h(t', v)1_{(0, t']}(v) - h(s, v)1_{(0, s]}(v) = h(t', v)1_{(s, t']}(v) + \left[h(t', v) - h(s, v)\right]1_{(0, s]}(v).$$

So,

$$h^*(t', s, u, v) = h(t', u)h(t', v)1_{(s, t']}(u, v) + \left[h(t', u) - h(s, u)\right]1_{(0, s]}(u) h(t', v)1_{(s, t']}(u, v) - h(t', u)1_{(s, t']}(u, v) h(s, v) - h(t', v)1_{(s, t']}(u, v) - h(t', v)1_{(s, t']}(u, v) h(s, u) - h(t', u)1_{(s, t']}(u, v).$$

Similarly, we also have

$$h^*(t', t, u, v) = h(t', u)h(t', v)1_{(t', t']}(u, v) + \left[h(t', u) - h(t, u)\right]1_{(0, t]}(u, v) h(t', v)1_{(t', t']}(u, v) - h(t', u)1_{(0, t]}(u, v) h(t, v) - h(t', v)1_{(t, t']}(u, v) - h(t', v)1_{(0, t]}(u, v) h(t, u) - h(t', u)1_{(t, t']}(u, v).$$
On the other hand, for all $0 < u, v \leq s \leq t \leq t' \leq T$ we set
\[
\Delta(t', t, s, u, v) = [h(s, u) - h(t', u)][h(s, v) - h(t', v)]1_{(0, s]}^2(u, v)
- [h(t, u) - h(t', u)][h(t, v) - h(t', v)]1_{[t, t')}^2(u, v)
= a^2uwe^{2(u^2-v^2)} \left\{ \left( \int_s^{t'} e^{-\frac{u}{2}w^2} dw \right)^2 - \left( \int_t^{t'} e^{-\frac{u}{2}w^2} dw \right)^2 \right\}.
\]

Then for all $s < t < t' \leq T$, we have
\[
\lim_{s \to 0} \frac{1}{(t' - s)^{2H} - (t' - t)^{2H}} \int_0^s \int_0^s \Delta(t', t, s, u, v) \phi(u, v) du dv = 0
\]
and
\[
\lim_{s \to t} \frac{1}{(t' - s)^{2H} - (t' - t)^{2H}} \int_0^s \int_0^s \Delta(t', t, s, u, v) \phi(u, v) du dv = 0,
\]
which implies that there is a constant $C_{a,H,T} > 0$ such that
\[
\int_0^s \int_0^s \Delta(t', t, s, u, v) \phi(u, v) du dv \leq C_{a,H,T} \left[ (t' - s)^{2H} - (t' - t)^{2H} \right].
\]
Combining these with
\[
0 \leq h(t, u) - h(t', u) \leq h(s, u) - h(t', u) \leq 2, \quad 0 \leq u \leq s \leq t,
\]
we get
\[
\int_0^t \int_0^{t'} \left[ h^*(t', s, u, v) - h^*(t', t, u, v) \right] \phi(u, v) du dv
\leq \int_0^t \int_0^{t'} h(t, u)h(t', v) \left( 1_{(s,t')}^2(u, v) - 1_{(t,t')}^2(u, v) \right) \phi(u, v) du dv
+ 4 \int_t^{t'} \int_0^{t} \phi(u, v) du dv
+ \int_0^s \int_0^s \Delta(t', t, s, u, v) \phi(u, v) du dv
\leq C_{a,H,T} \left[ (t' - s)^{2H} - (t' - t)^{2H} \right].
\]
This completes the proof. \qed

The proof similar to Lemma 5.3 by decomposing the function
\[
h^*(t', s, u, v) - h^*(t', t, u, v) + h^*(s', t, u, v) - h^*(s', s, u, v),
\]
one can show that the following lemma holds for all $0 \leq s \leq t \leq s' \leq t' \leq T$.

**Lemma 5.4.** Under the assumptions of Lemma 5.3, for all $0 \leq s \leq t \leq s' \leq t' \leq T$ we have
\[
\int_0^t \int_0^{t'} \left[ h^*(t', s, u, v) - h^*(t', t, u, v) + h^*(s', t, u, v) - h^*(s', s, u, v) \right] \phi(u, v) du dv \leq C_{a,H,T} \left[ (t' - s)^{2H} - (t' - t)^{2H} + (s' - t)^{2H} - (s' - s)^{2H} \right].
\]
Lemma 5.5. Let \( \sigma^2_{t,s} \) and \( \mu \) be as in the proof of Theorem 5.1. Then we have
\[
\int_T \frac{\mu^2 dsdt'ds'}{d_H(s,t,s',t')(\sigma^2_{t,s}\sigma^2_{t',s'})} < \infty
\]
if \( \frac{1}{2} < H < \frac{3}{4} \).

Lemma 5.5 is a consequence of the lemmas above and Lemma 15 in Hu [13] (see also Hu [12, pp.245–247]). In fact, Lemma 5.3 and Lemma 5.4 imply that the estimate
\[
\mu \leq C_{a,H,T} |t - s'|^{2H} - |t' - t|^{2H} + |t' - s|^{2H} - |s - s'|^{2H}
\]
holds for all \((s, s', t, t') \in T\). Thus, Lemma 5.5 follows from Lemma 5.2 and Lemma 15 in Hu [13] (see also Hu [12, pp.245–247]).

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. Clearly, as \( \varepsilon \) tends to zero \( \beta^{H,\varepsilon}_T - E[\beta^{H,\varepsilon}_T] \) converges in \( L^2 \) if and only if
\[
(5.4) \quad \text{Var}(\beta^{H,\varepsilon}_T) = E[\beta^{H,\varepsilon}_T]^2 - \left( E[\beta^{H,\varepsilon}_T] \right)^2
\]
tends to a constant. Now let us show that \( \text{Var}(\beta^{H,\varepsilon}_T) \) converges as \( \varepsilon \) tends to zero. Using the classical equality
\[
p_{\varepsilon}(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i<\xi,x>} e^{-\frac{\varepsilon |\xi|^2}{2}} d\xi,
\]
once can obtain
\[
(5.5) \quad \beta^{H,\varepsilon}_T = \frac{1}{(2\pi)^2} \int_0^T \int_0^t \int_{\mathbb{R}^2} e^{i<\xi,X^{H}_t - X^{H}_s>} e^{-\frac{\varepsilon |\xi|^2}{2}} d\xi ds dt.
\]
Combining this with the facts \( \langle \xi, X^{H}_t - X^{H}_s \rangle \sim N(0, |\xi|^2 \sigma^2_{t,s}) \) and
\[
E\left[e^{i<\xi,X^{H}_t - X^{H}_s>}\right] = e^{-\frac{1}{2}|\xi|^2 \sigma^2_{t,s}},
\]
\[
\int_{\mathbb{R}^2} e^{-\frac{1}{2}|\xi|^2(\varepsilon + \sigma^2_{t,s})} d\xi = \frac{2\pi}{\varepsilon + \sigma^2_{t,s}},
\]
we get
\[
E\left[\beta^{H,\varepsilon}_T\right] = \int_0^T \int_0^t E\left(p_{\varepsilon}(X^{H}_t - X^{H}_s)\right) ds dt
\]
\[
(5.6) \quad = \frac{1}{2\pi} \int_0^T \int_0^t (\varepsilon + \sigma^2_{t,s})^{-1} ds dt.
\]
Denote \( T = \{(s, t, s', t') : 0 < s < t < T, 0 < s' < t' < T\} \), then according to the representation (5.5) we get
\[
E\left[(\beta^{H,\varepsilon}_T)^2\right] = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} E_{\varepsilon,i<\xi,X^{H}_t - X^{H}_s>} + i\eta,X^{H}_{t'} - X^{H}_{s'}> e^{-\frac{\varepsilon |\xi|^2 + |\eta|^2}{2}} d\xi d\eta ds dt' ds'.
\]
Noting that
\[
\langle \xi, X^{H}_t - X^{H}_s \rangle + \langle \eta, X^{H}_{t'} - X^{H}_{s'} \rangle \sim N(0, |\xi|^2 \sigma^2_{t,s} + |\eta|^2 \sigma^2_{t',s'} + 2\mu < \xi, \eta>)
\]
for any \( \xi, \eta \in \mathbb{R}^2 \), we can write
\[
E \left[ (\beta_{T, \varepsilon}^H)^2 \right] = \frac{1}{(2\pi)^2} \int_\mathbb{T} \int_{\mathbb{R}^4} e^{-\frac{1}{2} \left( (\sigma_{t,s}^2 + \varepsilon)(\sigma_{t',s'}^2 + \varepsilon) - \mu^2 \right) - d/2} d\xi d\eta ds d\sigma_{t,s} d\sigma_{t',s'}
\]
for all \( \varepsilon > 0 \). It follows from (5.6) that
\[
E \left[ (\beta_{T, \varepsilon}^H)^2 \right] - \left( E \beta_{T, \varepsilon}^H \right)^2 = \frac{1}{(2\pi)^2} \int_\mathbb{T} \left[ (\sigma_{t,s}^2 + \varepsilon)(\sigma_{t',s'}^2 + \varepsilon) - \mu^2 \right]^{-1} - \frac{1}{\mu^2} ds d\sigma_{t,s} d\sigma_{t',s'}
\]
Thus, the theorem follows from Lemma 5.5. \( \Box \)

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