SMOOTH VERSUS SYMPLECTIC CIRCLE ACTIONS

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Abstract. We construct a 6-manifold $M$ which admits a smooth circle action and a symplectic form $\omega$ such that if $\omega'$ is another symplectic form on $M$ equivalent to $\omega$, then $(M, \omega')$ does not admit a symplectic circle action.

1. Introduction

In this short note we are concerned with a question whether a manifold admitting a symplectic form and a smooth circle action has to admit a structure of a symplectic $S^1$-manifold.

For a symplectic surface the answer is clearly positive. In case of symplectic 4-manifolds the answer is positive if the action has a fixed point \cite{1} and unknown in general. Little research has been conducted in dimensions 6 and above.

The main result of this paper is the following theorem.

Theorem 1.1. There exists a symplectic 6-manifold $(M, \omega)$ such that the underlying smooth manifold $M$ admits a non-trivial smooth circle action, but for any symplectic form $\omega'$ equivalent to $\omega$ the symplectic manifold $(M, \omega')$ admits no non-trivial symplectic circle actions.

We say that two symplectic forms $\omega$ and $\omega'$ on a manifold $M$ are equivalent if there exists a sequence $(\omega, \omega_1, \ldots, \omega_k, \omega')$ of symplectic forms on $M$ such that each two consecutive forms in this sequence are either deformation equivalent or one of them is a pull-back of the other by a self-diffeomorphism of $M$.

Unfortunately, Theorem 1.1 does not answer the question stated above. It is known that manifolds in general may admit non-equivalent symplectic forms (cf. \cite{6} for dimension 4 and \cite{5} for dimension 6).

The proof of Theorem 1.1 above is divided into two steps. First, in Section 2 below, we prove an inequality for Todd genus of arbitrary Hamiltonian 6-manifolds. Then, in Section 3 we construct a symplectic 6-manifold with a non-trivial smooth circle action, which violates this inequality and such that any symplectic circle action on this manifold has to be Hamiltonian. Since Todd genus of a symplectic manifold is invariant with respect to equivalence of symplectic forms, these two results combined constitute a proof of Theorem 1.1.

2. Todd genus of Hamiltonian 6-manifolds

In this section we will prove the following lemma.

Lemma 2.1. If $(M, \omega)$ is a Hamiltonian 6-manifold, then

$$\text{td}(M, \omega) \geq 1 - \frac{b_1(M)}{2},$$

where $b_k$ denotes the $k$th Betti number of $M$. 

By \( \text{td}(M, \omega) \) we denote the Todd genus of \((M, \omega)\), defined as the Todd genus of an almost complex manifold \((M, J)\), where \(J\) is any almost complex structure on \(M\) compatible with \(\omega\). Since the space of all such almost complex structures on \(M\) is contractible, this is in fact well-defined.

Fix any symplectic manifold \((M, \omega)\). For now we do not assume that \(M\) is of dimension 6. Fix a non-trivial Hamiltonian circle action on \(M\) with Hamiltonian function \(\phi\). It is well known that the critical manifolds of \(\phi\), which correspond to the connected components of the fixed point set of the action, are symplectic submanifolds of \(M\) and that exactly one of these submanifolds, which we will denote by \(M_\phi\), corresponds to the minimum of the Hamiltonian. By studying \(\phi\) as a Morse-Bott function, Li [3] concluded that

\[
\pi_1(M_\phi) \cong \pi_1(M).
\]

At the other hand, recall that the Todd genus is an evaluation of a Hirzebruch’s \(\chi_y\)-genus at \(y = 0\). Combining this with the localization formula for \(\chi_y\)-genus of \(S^1\)-manifolds [2] it is not hard to obtain

\[
\text{td}(M_\phi, \omega_{M_\phi}) = \text{td}(M, \omega).
\]

**Proof of Lemma 2.1**. Let now assume that \(\dim M = 6\). As mentioned earlier, \(M_\phi\) is a connected symplectic submanifold of \(M\), so if the action is not trivial, then \(\dim M_\phi \in \{0, 2, 4\}\). If \(\dim M_\phi = 0\), then \(M_\phi\) is a point and

\[
\text{td}(M, \omega) = \text{td}(\bullet) = 1 \geq 1 - \frac{b_1(M)}{2}.
\]

If \(\dim M_\phi = 2\), then \(M_\phi\) is an oriented surface \(F_g\) of genus \(g\) for some non-negative integer \(g\) and \(\pi_1(M) \cong \pi_1(F_g) \cong \#^g\mathbb{Z}^2\). In particular \(b_1(M) = 2g\). Now, the Todd genus of a surface does not depend on a choice of a symplectic (or almost complex) structure and equals \(\text{td}(F_g) = 1 - g\). Thus

\[
\text{td}(M, \omega) = \text{td}(F_g) = 1 - g = 1 - \frac{b_1(M)}{2}.
\]

Finally, if \(\dim(M_\phi) = 4\) then \(M_\phi\) is a symplectic 4-manifold with \(b_1(M_\phi) = b_1(M)\).

Similarly to the case of surfaces, the Todd genus of a 4-manifold does not depend on the choice of an almost complex structure and equals \(\text{td}(M_\phi) = \frac{1-b_1(M_\phi)+b_2^+(M_\phi)}{2}\), where \(b_2^+(M_\phi)\) is the dimension of the self-dual part of \(H^2(M_\phi; \mathbb{R})\). But \((M_\phi, \omega|_{M_\phi})\) is symplectic, so \([\omega|_{M_\phi}] \in H^2(M_\phi; \mathbb{R})\) is self-dual and \(b_2^+(M_\phi) \geq 1\). In particular

\[
\text{td}(M, \omega) = \text{td}(M_\phi) \geq 1 - \frac{b_1(M_\phi)}{2} = 1 - \frac{b_1(M)}{2}.
\]

\(\square\)

3. Construction of an example

Let \(K\) denote the \(K3\) surface. It is a symplectic manifold with Euler characteristic \(\chi(K) = 24\), signature \(\sigma(K) = -16\), Todd genus \(\text{td}(K) = 2\) and even intersection form. If we take a one point blow up \(K\mathbb{P}^2\) of \(K\), then we obtain a new symplectic manifold, now with \(\chi(K\mathbb{P}^2) = 25, \sigma(K\mathbb{P}^2) = -17, \text{td}(K\mathbb{P}^2) = 2\) and odd intersection form. By Serre’s classification of indefinite unimodular bilinear symmetric forms we see, that the intersection form of \(K\mathbb{P}^2\) coincides with that
We have Lemma 3.3.

$M := (\mathbb{CP}^2 \times \mathbb{CP}^2) \times F_g \cong (3\mathbb{CP}^2 \times 20\mathbb{CP}^2) \times F_g$.

**Lemma 3.1.** Let $\omega$ be a non-trivial symplectic form on $M$. Every symplectic circle action on $(M, \omega)$ is Hamiltonian.

**Proof.** Since $\chi(M) = 50(1 - g) \neq 0$, every circle action on $M$ has fixed points. Recall, that a symplectic $2n$-manifold $(X, \omega_X)$ is said to have the weak Lefschetz property (WLP) if $\wedge[\omega_X]^{n-1} : H^1(X; \mathbb{R}) \rightarrow H^{2n-1}(X; \mathbb{R})$ is an isomorphism. If a symplectic manifold $(X, \omega_X)$ has WLP, then a symplectic circle action on $(X, \omega_X)$ is Hamiltonian if and only if it has fixed points (see [4] Theorem 5.5). So it remains to see that $(M, \omega)$ has WLP.

Using Künneth’s formula we obtain decompositions

\[
H^1(M; \mathbb{R}) \cong H^0(K_{\mathbb{CP}^2}; \mathbb{R}) \otimes H^1(F_g; \mathbb{R}),
\]

\[
H^2(M; \mathbb{R}) \cong H^0(K_{\mathbb{CP}^2}; \mathbb{R}) \otimes H^2(F_g; \mathbb{R}) \oplus H^2(K_{\mathbb{CP}^2}; \mathbb{R}) \otimes H^0(F_g; \mathbb{R}),
\]

\[
H^5(M; \mathbb{R}) \cong H^1(K_{\mathbb{CP}^2}; \mathbb{R}) \otimes H^4(F_g; \mathbb{R}).
\]

In particular, $[\omega]$ decomposes as $[\omega] = 1 \otimes b + a \otimes 1$ for some $a \in H^2(K_{\mathbb{CP}^2}; \mathbb{R})$ and $b \in H^2(F_g; \mathbb{R})$. Take any class $\gamma = 1 \otimes c \in H^1(M; \mathbb{R})$, where $c \in H^1(F_g; \mathbb{R})$. We have

$\gamma \wedge [\omega]^2 = a^2 \otimes c$.

$\wedge[\omega] : H^1(M; \mathbb{R}) \rightarrow H^5(M; \mathbb{R})$ is an isomorphism precisely when $a^2 \neq 0$. But $0 \neq [\omega]^3 = a^2 \otimes b$. \hfill $\square$

We can easily endow $3\mathbb{CP}^2 \times 20\mathbb{CP}^2$ with a non-trivial smooth circle action by taking appropriate linear actions on each component and using equivariant connected sum. Product action on $M$ is non-trivial. Hence, we have shown the following.

**Lemma 3.2.** $M$ admits a non-trivial smooth circle action.

Finally, let $\omega_{K_{\mathbb{CP}^2}}$ be a symplectic form on $K_{\mathbb{CP}^2}$ and let $\omega_{F_g}$ be a symplectic form on $F_g$. Denote the product form on $M$ by $\omega$. Clearly, it is symplectic. Moreover, $\text{td}(M, \omega) = \text{td}(K_{\mathbb{CP}^2}) \text{td}(F_g) = 2(1 - g)$. This gives us the following.

**Lemma 3.3.** $M$ admits a symplectic form $\omega$ such that $\text{td}(M, \omega) = 2(1 - g)$.

Lemmas 2.1, 3.1, 3.2 and 3.3 complete the proof of Theorem 1.1 as outlined in Section 1.

**References**

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