UNBOUNDED CONTINUOUS OPERATORS IN BANACH LATTICES

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Abstract. Motivated by the equivalent definition of a continuous operator between Banach spaces in terms of weakly null nets, we introduce two types of continuous operators between Banach lattices using unbounded absolute weak convergence. We characterize reflexive Banach lattices in terms of these spaces of operators. Furthermore, we investigate whether or not the adjoint of these classes of operators has the corresponding property. In addition, we show that these kinds of operators are norm closed but not order closed. Finally, we show that the notions of an M-weakly operator and a uaw-Dunford-Pettis operator have the same meaning; this extends one of the main results of Erkursun-Ozcan et al. (TJM, 2019).

1. MOTIVATION AND INTRODUCTION

Let us first start with some motivation. Suppose $X$ and $Y$ are Banach spaces and $T : X \to Y$ is an operator. It is known that (see [1, Theorem 5.22] for example), $T$ is continuous if and only if it preserves weakly null nets. On the other hand, unbounded convergences have received much attention recently with some deep and inspiring results (see [2, 4, 5, 6]). In particular, unbounded absolute weak convergence as a weak version of unbounded norm convergence and also as an unbounded version of weak convergence has been investigated in [6]. Furthermore, unbounded absolute weak Dunford-Pettis operators (uaw-Dunford-Pettis operators, in brief), as an unbounded version of Dunford-Pettis operators, have been considered recently in [3].

Before to proceed more, let us consider some preliminaries.

Suppose $E$ is a Banach lattice. A net $(x_\alpha)$ in $E$ is said to be unbounded absolute weak convergent (uaw-convergent, for short) to $x \in E$ if for each $u \in E_+$, $|x_\alpha - x| \wedge u \overset{w}{\to} 0$. $(x_\alpha)$ is unbounded norm convergent (un-convergent, in brief) if $\|x_\alpha - x|\wedge u\| \to 0$. Both convergences are topological. For ample information on these concepts, see [2, 4, 6].

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Now, we consider the following observations as unbounded versions of continuous operators.

Suppose $E$ is a Banach lattice and $X$ is a Banach space. A continuous operator $T : E \to X$ is called **unbounded continuous** if for each bounded net $(x_\alpha) \subseteq E$, $x_\alpha \xrightarrow{\text{uaw}} 0$ implies that $T(x_\alpha) \xrightarrow{w} 0$. Moreover, $T$ is said to be **sequentially unbounded continuous** if for each bounded sequence $(x_n) \subseteq E$, $x_n \xrightarrow{\text{uaw}} 0$ implies that $T(x_n) \xrightarrow{w} 0$.

Observe that a continuous operator $T : E \to F$, where $E$ and $F$ are Banach lattices, is said to be **uaw-continuous** if $T$ maps every norm bounded uaw-null net into a uaw-null net. It is **sequentially uaw-continuous** provided that the property happens for sequences. Consider this point that sequentially uaw-continuous operators were introduced in [3] at first as a beside note.

Moreover, recall that $T$ is **uaw-Dunford-Pettis** if for every norm bounded sequence $(x_n) \subseteq E$, $x_n \xrightarrow{\text{uaw}} 0$ implies that $\|T(x_n)\| \to 0$; see [3] for a detailed exposition on this topic.

In this paper, our attempt is to investigate more about these classes of continuous operators. More precisely, we characterize reflexive Banach lattices in terms of these classes of continuous operators. Also, we consider several conditions under which the adjoint of an unbounded continuous (a uaw-continuous) operator is again unbounded continuous (uaw-continuous). Moreover, we investigate closedness properties for these spaces of operators. Among these, we improve one of the main results recently obtained in [3]; in particular, we show that uaw-Dunford-Pettis operators and $M$-weakly compact operators are in fact the same notions, without any extra conditions.

For undefined terminology and concepts, we refer the reader to [1]. All operators in this note, are assumed to be continuous, unless otherwise stated, explicitly.

### 2. MAIN RESULT

It is easy to see that every uaw-Dunford-Pettis operator is unbounded continuous but the converse is not true in general. The inclusion map from $c_0$ into $\ell_\infty$ is unbounded continuous but not uaw-Dunford-Pettis; suppose $(x_\alpha)$ is a bounded uaw-null net in $c_0$. By [6, Theorem 7], $x_\alpha \xrightarrow{w} 0$ in $c_0$ so that in $\ell_\infty$. Now, observe that the standard basis $(e_n)$ is uaw-null but certainly not norm null.

**Theorem 1.** Suppose $E$ is a Banach lattice. If every continuous operator $T : \ell_1 \to E$ is unbounded continuous, then $E$ is reflexive.
Proof. Suppose every continuous operator is unbounded continuous. First we show that $E'$ is order continuous; suppose not. So, $E$ contains a lattice copy of $\ell_1$ (with the embedding $\iota$). The identity operator $I$ on $\ell_1$ is continuous but not unbounded continuous. This implies that the composition map $T : \ell_1 \to E$ defined via $T = \iota o I$ is not unbounded continuous which contradicts our assumption.

Now, we prove that $E$ is a $KB$-space. Suppose on a contrary, it is not. So, it contains a lattice copy of $c_0$; with the embedding $\iota_1$. Consider the following diagram.

\[
\begin{array}{ccc}
\ell_1 & \xrightarrow{T} & L_1[0,1] & \xrightarrow{S} & c_0 & \xrightarrow{\iota} & E,
\end{array}
\]

in which $T(\alpha_1, \alpha_2, \ldots) = \sum_{n=1}^{\infty} \alpha_n r_n^+$ where $(r_n)$ is the sequence of Rademacher functions and $S$ is the classical "Fourier coefficients" due to Lozanovsky defined via $T(f) = (\int_0^1 f(t) \sin t dt, \int_0^1 f(t) \sin 2t dt, \ldots)$; for more details see [1, Exercise 10, Page 289].

We claim that this composition is not unbounded continuous and this would complete our proof. Suppose $(e_n)$ is the standard basis in $\ell_1$ which is certainly $uaw$-null. $Te_n = r_n^+$ which is not weakly null. Moreover, $S(r_n^+)$ is not also weakly null in $c_0$; since $\int_0^1 r_n^+(t) \sin t dt \not\to 0$. Now, it is obvious that $S(r_n^+)$ is not weakly null in $E$.

□

Remark 2. The other implication of Theorem [1] is not true in general. Consider operator $S : \ell_1 \to L_2[0,1]$ defined via $S(\alpha_1, \alpha_2, \ldots) = \sum_{n=1}^{\infty} \alpha_n r_n^+$. In which $(r_n)$ denotes the sequence of Rademacher functions; for more details about this operator consider [1] Example 5.17, page 284]. Note that $S$ is not unbounded continuous; the standard basis $(e_n)$ is $uaw$-null in $\ell_1$ but $S(e_n) = r_n^+$ which is not weakly null since $\int_0^1 r_n^+(t)dt = \frac{1}{2}$.

But there is a good news if we apply $uaw$-continuity.

Proposition 3. Suppose $E$ and $F$ are Banach lattices such that $F'$ is order continuous. Then every $uaw$-continuous operator $T : E \to F$ is unbounded continuous.

Proof. Suppose $T$ is $uaw$-continuous. For every norm bounded $uaw$-null net $(x_\alpha)$ in $E$, $T(x_\alpha) \xrightarrow{\text{uaw}} 0$. By [6, Theorem 7], $T(x_\alpha) \xrightarrow{w} 0$, as claimed. □

Corollary 4. Suppose $E$ is a Banach lattice and $F$ is an AM-space. Then it can be seen easily that an operator $T : E \to F$ is sequentially unbounded continuous if and only if it is sequentially $uaw$-continuous.

Theorem 5. Suppose $E$ is a Banach lattice. If every continuous operator $T : L_1[0,1] \to E$ is $uaw$-continuous, then $E$ is reflexive.
Proof. Suppose on a contrary, $E$ is not reflexive. First, assume that $E$ contains a lattice copy of $c_0$. Consider the operator $T : L_1[0,1] \to c_0$ defined via $T(f) = \left( \frac{\int_0^1 f(t)dt}{n} \right)$. We claim that $T$ is not $uaw$-continuous. Let $(f_n)$ be the bounded sequence in $L_1[0,1]$ defined as follows: $n$ on the interval $[0, \frac{1}{n}]$ and zero otherwise. Observe that by [6, Theorem 4], $uaw$-convergence and $un$-convergence in $L_1[0,1]$ agree and by using [2, Corollary 4.2], we conclude that $f_n \overset{uaw}{\longrightarrow} 0$. But $T(f_n) = (x)$ in which $x$ is the constant sequence $(\frac{1}{n})$; not a $uaw$-null sequence. This implies that $T(f_n)$ is not also $uaw$-null in $E$ which is a contradiction.

Now, suppose $E'$ is not order continuous so that $E$ contains a lattice copy of $\ell_1$. Observe that the operator $T : L_1[0,1] \to \ell_1$ defined via $T(f) = \left( \frac{\int_0^1 f(t)dt}{n^2} \right)$ is not $uaw$-continuous. Consider the sequence $(f_n)$ as the first part of argument. It is $uaw$-null. Nevertheless, $S(f_n)$ is the constant sequence $(y)$ defined via $y = (\frac{1}{n^2})$ which is certainly not $uaw$-null. □

Note that the other implication of Theorem 5 may fail, as well. The operator $S : L_1[0,1] \to \ell_2$ defined by $S(f) = \left( \frac{\int_0^1 f(t)dt}{n^2} \right)$ is not $uaw$-continuous.

Now, we are going to investigate whether or not the converse of Theorem 1 is true. First, we have the following lemma.

\textbf{Lemma 6.} If $E'$ is order continuous, then the identity operator $I$ on $E$ is unbounded continuous. Moreover, if $E$ is a KB-space and $I$ is unbounded continuous, then $E$ is reflexive.

\textbf{Proof.} First, assume that $E'$ is order continuous and $(x_\alpha)$ is a bounded $uaw$-null net in $E$. So, by [6, Theorem 7], $x_\alpha \overset{w}{\to} 0$, as wanted. Now, suppose that $(x_\alpha)$ is a norm bounded $uaw$-Cauchy net in $E$. By [6, Theorem 4] and [4, Theorem 6.4], $(x_\alpha)$ is $uaw$-convergent so that weakly convergent by the assumption. Thus, by [6, Theorem 8], $E$ is reflexive. □

Observe that $KB$-space assumption is necessary in Lemma 6 and can not be omitted. The identity operator $I$ on $c_0$ is unbounded continuous but $c_0$ is not reflexive.

Before we proceed with a kind of converse for Theorem 1 we provide some ideal properties.

\textbf{Proposition 7.} Let $S : E \to F$ and $T : F \to G$ be two operators between Banach lattices $E, F, \text{ and } G$.

(i) If $T$ is Dunford-Pettis and $S$ is sequentially unbounded continuous then $TS$ is $uaw$-Dunford-Pettis.
(ii) If $T$ is a uaw-Dunford–Pettis operator and $S$ is sequentially uaw-continuous, then $TS$ is uaw-Dunford-Pettis.

(iii) If $T$ is continuous and $S$ is an unbounded continuous operator, then $TS$ is also unbounded continuous.

(iv) If $T$ is an onto lattice homomorphism and $S$ is uaw-continuous, then $TS$ is also uaw-continuous.

Proof. (i). Suppose $(x_n)$ is a norm bounded uaw-null sequence in $E$. So, $S(x_n) \xrightarrow{w} 0$. Therefore, $\|TS(x_n)\| \rightarrow 0$.

(ii). Suppose $(x_n)$ is a norm bounded uaw-null sequence in $E$. Therefore, $S(x_n) \xrightarrow{uaw} 0$. Thus, $\|TS(x_n)\| \rightarrow 0$.

(iii). Suppose $(x_\alpha)$ is a norm bounded uaw-null net in $E$. By assumption, $S(x_\alpha) \xrightarrow{w} 0$ so that $TS(x_\alpha) \xrightarrow{w} 0$.

(iv). First, observe that for each $u \in G_+$, there is a $v \in F_+$ with $T(v) = u$. Suppose $(x_\alpha)$ is a norm bounded uaw-null net in $E$. By assumption, $S(x_\alpha) \xrightarrow{uaw} 0$ so that $|S(x_\alpha)| \wedge v \xrightarrow{w} 0$. Therefore,

$|TS(x_\alpha)| \wedge u = T(|S(x_\alpha)| \wedge u) = T(|S(x_\alpha)| \wedge v) \xrightarrow{w} 0$.

□

Theorem 8. For a KB-space $E$, the following are equivalent.

(i) $E$ is reflexive.

(ii) Every continuous operator $T : E \rightarrow E$ is unbounded continuous.

Proof. (i) $\rightarrow$ (ii). Note that for each continuous operator $T$ on $E$, $T = TI$ and then use Proposition 7 and Lemma 6.

(ii) $\rightarrow$ (i). By assumption, The identity operator $I$ on $E$ is unbounded continuous and then use Lemma 6. □

Remark 9. Observe that in general, there are no relations between unbounded continuous operators and weakly compact ones. Consider [3, Example 2.21]; the operator $T : \ell_1 \rightarrow L_2[0,1]$ defined by $T(x_n) = (\sum_{n=1}^\infty x_n)\chi_{[0,1]}$ for all $(x_n) \in \ell_2$ where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. It is weakly compact but not unbounded continuous. Indeed, the standard basis $(e_n)$ in $\ell_1$ is uaw-null but $T(e_n)$ is not weakly null since $\int_0^1 \chi_{[0,1]} dt = 1$. Moreover, the identity operator on $\ell_\infty$ is not weakly compact yet it is unbounded continuous using [6, Theorem 7].
Theorem 10. Suppose $E$ is a Banach lattice and $F$ is an order continuous Banach lattice. Then every weakly compact operator $T : E \to F$ has an unbounded continuous adjoint.

Proof. Assume that $(x'_\alpha)$ is a norm bounded net in $F'$ which is uaw-null. By [6, Proposition 5] $x'_\alpha \overset{w^*}{\rightarrow} 0$. By the Gantmacher theorem [1, Theorem 5.23], $T'(x'_\alpha) \overset{w}{\rightarrow} 0$, as desired. □

Remark 11. Weakly compactness of operator $T$ and also order continuity of $F$ are essential in Theorem 10 and can not be dropped. Consider the identity operator $I : c_0 \to c_0$. $I$ is not weakly compact but $F$ is order continuous. Furthermore, $I$ is also unbounded continuous. Its adjoint, $I' : \ell_1 \to \ell_1$ is not unbounded continuous; assume $(e_n)$ is the standard basis of $\ell_1$. It is uaw-null by [6, Lemma 2]. But, certainly, it is not weakly null in $\ell_1$.

Also, consider the operator $T : L_2[0, 1] \to \ell_\infty$ defined via $T(f) = (\int_0^1 f(t)dt, \int_0^1 f(t)dt, \ldots)$. It is weakly compact but $F$ is not order continuous. Consider the operator $T' : (\ell_\infty)' \to L_2[0, 1]$. It is not unbounded continuous. Consider the standard basis $(e_n)$ which is uaw-null in $(\ell_\infty)'$. But $< T'(e_n), 1 > = < e_n, T(1) > = 1$.

Suppose $E$ is a Banach lattice. The class of all unbounded continuous operators on $E$ is denoted by $B_{uc}(E)$, the class of all uaw-continuous operators on $E$ will get the terminology $B_{uaw}(E)$. In this step, we consider some closedness properties for these classes of continuous operators.

Proposition 12. Suppose $E$ is a Banach lattice. Then $B_{uc}(E)$ is closed as a subspace of the space of all continuous operators on $E$.

Proof. Suppose $(T_\alpha)$ is a net of unbounded continuous operators which is convergent to the operator $T$. We need to show that $T$ is unbounded continuous. For any $\varepsilon > 0$, there is an $\alpha_0$ such that $\|T_{\alpha_0} - T\| < \frac{\varepsilon}{2}$. So, for each $x$ with $\|x\| \leq 1$, $\|T_{\alpha_0}(x) - T(x)\| < \frac{\varepsilon}{2}$. Assume that $(x_\beta)$ is a norm bounded uaw-null net in $E$. This means that $\|T_{\alpha_0}(x_\beta) - T(x_\beta)\| < \frac{\varepsilon}{2}$. Note that $T_{\alpha_0}(x_\beta) \overset{w}{\rightarrow} 0$ so that for a fixed $f \in E_1^*$ and sufficiently large $\beta$, $f(T_{\alpha_0}(x_\beta)) < \frac{\varepsilon}{2}$ so that $f(T(x_\beta)) < \varepsilon$. □

Proposition 13. Suppose $E$ is a Banach lattice. Then $B_{uaw}(E)$ is closed as a subspace of the space of all continuous operators on $E$.

Proof. Suppose $(T_\alpha)$ is a net of $uaw$-continuous operators which is convergent to the operator $T$. We need to show that $T$ is $uaw$-continuous. For any $\varepsilon > 0$, there is an $\alpha_0$...
such that $\|T_{ao} - T\| < \frac{\varepsilon}{2}$. So, for each $x$ with $\|x\| \leq 1$, $\|T_{ao}(x) - T(x)\| < \frac{\varepsilon}{2}$. Assume that $(x_\beta)$ is a norm bounded uaw-null net in $E$. This means that $\|T_{ao}(x_\beta) - T(x_\beta)\| < \frac{\varepsilon}{2}$. Note that $T_{ao}(x_\beta) \xrightarrow{\text{uaw}} 0$. Fix $f \in E^*_+ \text{ and } u \in E_+$. Observe that for $a, b, c \geq 0$ in an Archimedean vector lattice, we have $|a \wedge c - b \wedge c| \leq |a - b| \wedge c$. Therefore,

$$f(|T_{ao}(x_\beta)| \wedge u) - f(|T(x_\beta)| \wedge u) \leq \|T_{ao}(|x_\beta| \wedge u) - (|T(x_\beta)| \wedge u)\| \leq \|T_{ao}(|x_\beta|) - |T(x_\beta)|\| \leq \|T_{ao}(x_\beta) - T(x_\beta)\| \leq \frac{\varepsilon}{2}.$$

On the other hand, for sufficiently large $\beta$, $f(T_{ao}(x_\beta) \wedge u) \to 0$. This, in turn, results in $f(T(x_\beta) \wedge u) \to 0$. \hfill \Box

**Remark 14.** Neither $B_{uc}(E)$ nor $B_{uaw}(E)$ are order closed, in general. Consider the operator $S$ in Remark 2. Put $S_n = P_n S$, in which $P_n$ is the canonical projection on $L_2[0,1]$. Observe that each $S_n$ is finite rank so that in this case, uaw-convergence, weak convergence, and norm one agree. Therefore, we conclude that each $S_n$ is both uaw-continuous and unbounded continuous. Moreover $S_n \uparrow S$ and we have seen in Remark 2 that $S$ is neither unbounded continuous nor uaw-continuous.

**Remark 15.** Suppose $E$ and $F$ are Banach lattices and $T, S : E \to F$ are continuous operators such that $0 \leq S \leq T$. It can be easily verified that if $T$ is either uaw-continuous or unbounded continuous, then so is $S$.

**Theorem 16.** Suppose $E$ is a Grothendieck space and $F$ is an order continuous Banach lattice. Moreover, assume that $T : E \to F$ is a sequentially unbounded continuous operator. Then $T' : F' \to E'$ is also sequentially unbounded continuous.

**Proof.** Suppose $(x_n')$ is a norm bounded uaw-null sequence in $F'$. By [6, Proposition 5], $x_n' \xrightarrow{w^*} 0$. So, $T'(x_n') \xrightarrow{w^*} 0$ in $E'$. By the Grothendieck property, we have $T'(x_n') \xrightarrow{w^*} 0$. \hfill \Box

Observe that the Grothendieck property of $E$ is essential in Theorem 16 and can not be removed. Consider again the identity operator $I$ on $c_0$. We have seen in Lemma 6 that $I$ is unbounded continuous but its adjoint is not. Note that $c_0$ does not have the Grothendieck property.

**Theorem 17.** Suppose $E$ is a Banach lattice whose dual is order continuous and atomic and $F$ is an order continuous Banach lattice. Then every positive operator $T : E \to F$ has a uaw-continuous adjoint.
Proof. Consider positive operator $T' : F' \to E'$. Suppose $(x'_\alpha)$ is a positive bounded net in $F'$ such that $x'_\alpha \xrightarrow{uaw} 0$. By [6, Proposition 5], $x'_\alpha \xrightarrow{w^*} 0$ so that $T'x'_\alpha \xrightarrow{w^*} 0$. By [4, Proposition 8.5] and [6, Theorem 7], we see that $T'x'_\alpha \xrightarrow{uaw} 0$, as claimed. \hfill \square

Furthermore, when the operator $T$ is not positive, we may consider the following.

**Proposition 18.** Suppose $E$ is an order continuous Banach lattice whose dual is also order continuous and atomic and $F$ is an order continuous Banach lattice. Then every continuous operator $T : E \to F$ has a uaw-continuous adjoint.

Proof. Consider operator $T' : F' \to E'$. Suppose $(x'_\alpha)$ is a positive bounded net in $F'$ such that $x'_\alpha \xrightarrow{uaw} 0$. By [6, Proposition 5], $x'_\alpha \xrightarrow{w^*} 0$ so that $T'x'_\alpha \xrightarrow{w^*} 0$. By [4, Theorem 8.4] and [6, Theorem 7], we see that $T'x'_\alpha \xrightarrow{uaw} 0$, as claimed. \hfill \square

In the following, we prove one of the main results of [3] under less hypotheses; more precisely, we show that uaw-Dunford-Pettis operators and $M$-weakly compact ones are in fact the same.

Recall that an operator $T : E \to X$, where $E$ is a Banach lattice and $X$ is a Banach space, is called $M$-weakly compact if for every norm bounded disjoint sequence $(x_n)$ in $E$, $\|T(x_n)\| \to 0$. Moreover, $T$ is said to be $o$-weakly compact if $T[0, x]$ is a weakly relatively compact set for every $x \in E_+$.

**Theorem 19.** Suppose $E$ is a Banach lattice and $X$ is a Banach space. If $T : E \to X$ is an $M$-weakly compact operator then it is uaw-Dunford-Pettis.

Proof. Suppose $(x_n)$ is a positive norm bounded uaw-null sequence in $E$. This means that $x_n \land u \xrightarrow{w^*} 0$ for any positive $u \in E$. Since $T$ is also $o$-weakly compact (by [1, Theorem 5.57] due to Dodds), using [1, Exercise 3, Page 336], convinces us that $\|T(x_n \land u)\| \to 0$. Observe that by [1, Theorem 5.60] due to Meyer-Nieberg, the proof would be complete. \hfill \square

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