The Ricci flow in a class of solvmanifolds

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The Ricci flow

\[ \frac{\partial}{\partial t} g(t) = -2 \text{Rc}(g(t)), \quad g(0) = g. \] (1)

\( g \) is a Ricci soliton if:

\[ \text{Rc}(g) = c g + L_X g, \quad c \in \mathbb{R}, \quad X \in \chi(M) \]

\( g \) Ricci soliton \( \iff \) \( g(t) = (-2ct + 1) \phi^* g \) is a solution of the Ricci flow.
The Ricci flow

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Varying Lie brackets
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\[ \mathcal{L}_n = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric and Jacobi} \} . \]
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\( \text{GL}_n(\mathbb{R}) \) acts on \( \mathcal{L}_n \):

\[ g \in \text{GL}_n(\mathbb{R}) \mapsto (Gg \mu, \langle \cdot, \cdot \rangle) \to (Gg \mu, \langle g \cdot, g \cdot \rangle) \text{ isometry}. \]
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vary Lie brackets \(\leftrightarrow\) vary inner products.
Ricci flow on Lie groups: The bracket flow
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$G$, 

Theorem (\[L3\], 2012) There exist time-dependent diffeomorphisms

$\phi(t): G \rightarrow G_{\mu}(t)$ such that $g(t) = \phi(t)^* g(\mu)(t)$, $\forall t \in (a, b)$. 

The bracket flow in a class of solvmanifolds The normalized bracket flow Negative curvature
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\frac{d}{dt} \langle \cdot, \cdot \rangle_t = -2 \text{Rc}(\langle \cdot, \cdot \rangle_t), \quad \langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle,
\]

\( \mu \in \mathbb{L}_n \), the bracket flow starting at \( \mu \) is:

\[
\frac{d}{dt} \mu(t) = \delta \mu(t)(\text{Ric}_\mu(t)), \quad \mu(0) = \mu,
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Theorem ([L3], 2012) There exist time-dependent diffeomorphisms \( \phi(t): G \rightarrow G_\mu(t) \) such that \( g(t) = \phi(t)^* g_\mu(t) \), \( \forall t \in (a, b) \).
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$$\frac{d}{dt} \mu(t) = \delta_{\mu(t)}(\text{Ric}_{\mu(t)}), \quad \mu(0) = \mu, \quad (4)$$
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where $\delta_\mu(A) = \mu(A \cdot, \cdot) + \mu(\cdot, A \cdot) - A\mu(\cdot, \cdot)$, $A \in \text{GL}_n(\mathbb{R})$, $\mu \in \mathcal{V}_n$. 
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Theorem ([L3], 2012)

There exist time-dependent diffeomorphisms $\varphi(t) : G \to G_{\mu}(t)$ such that $g(t) = \varphi(t)^* g_{\mu}(t)$, $\forall t \in (a, b)$. 
The bracket flow in a class of solvmanifolds

A solvmanifold is a simply connected solvable Lie group endowed with a left invariant Riemannian metric, denoted as $(G, \langle \cdot, \cdot \rangle)$. The purpose of studying the Ricci flow is achieved by employing the bracket flow. This approach is particularly useful for solvmanifolds whose Lie algebras contain an abelian ideal of codimension 1.
The bracket flow in a class of solvmanifolds

Solvmanifold:

Simply connected solvable Lie group endowed with a left invariant Riemannian metric. $(S, \langle \cdot, \cdot \rangle)$

Purpose:
To study the Ricci flow.

How?:
Using the bracket flow.

In which solvmanifolds?:
Solvmanifolds whose Lie algebras have an abelian ideal of codimension 1.
The bracket flow in a class of solvmanifolds

**Solvmanifold:** simply connected solvable Lie group endowed with a left invariant Riemannian metric.
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We fix $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$.
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We fix \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)\). If \((\mathbb{R}^{n+1}, \mu)\) is a Lie algebra with an abelian ideal of codimension 1,
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We fix \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)\). If \((\mathbb{R}^{n+1}, \mu)\) is a Lie algebra with an abelian ideal of codimension 1, then there exists an orthonormal basis \(\{e_0, e_1, \ldots, e_n\}\) such that:
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\[\mu(e_0, e_i) = Ae_i, \quad i = 1, \ldots, n, \quad A \in \mathfrak{gl}_n(\mathbb{R}),\]
\[\mu(e_i, e_j) = 0, \quad \forall i, j \geq 1.\]
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The Ricci operator of \((G_{\mu_A}, g_{\mu_A})\) w. r. t. \(\{e_0, e_1, \ldots, e_n\}\) is:

\[
\text{Ric}_{\mu_A} = \begin{pmatrix}
- \text{tr}(S(A)^2) & 0 \\
0 & \frac{1}{2}[A, A^t] - \text{tr}(A)S(A)
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Then, using \(\frac{d}{dt}\mu(t) = \delta_{\mu(t)}(\text{Ric}_{\mu(t)})\) and proposing \(\mu_A(t)\) as a solution, we obtain that \(\mu(t) = \mu_A(t)\), with \(A(t)\) that satisfies:
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From now on, \((\mathbb{R}^{n+1}, \mu_A)\) or \(\mu_A\), and \((G_{\mu_A}, \langle \cdot, \cdot \rangle)\), or \((G_{\mu_A}, g_{\mu_A})\).

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Then, using \(\frac{d}{dt}\mu(t) = \delta_{\mu(t)}(\text{Ric}_{\mu(t)})\) and proposing \(\mu_A(t)\) as a solution, we obtain that \(\mu(t) = \mu_A(t)\), with \(A(t)\) that satisfies:

\[
\frac{d}{dt}A = -\text{tr}(S(A)^2)A + \frac{1}{2}[A, [A, A^t]] - \frac{1}{2} \text{tr}(A)[A, A^t].
\] (6)
The bracket flow in a class of solvmanifolds

Let $A := \begin{pmatrix} 0 & x_0 \\ y_0 & 0 \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{R})$, with $x_0y_0 < 0$. 

Then, $\mu(t) = \mu(A(t))$ with $A(t) = \begin{pmatrix} 0 & x(t) \\ y(t) & 0 \end{pmatrix}$ and $x(t) = x_0$, $y(t) = y_0$ satisfy:

$x' = x(x + y)(-\frac{3}{2}x + \frac{1}{2}y)$,

$y' = y(x + y)(-\frac{3}{2}y + \frac{1}{2}x)$. 

(7)
The bracket flow in a class of solvmanifolds

Let \( A := \begin{pmatrix} 0 & x_0 \\ y_0 & 0 \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{R}) \), with \( x_0 y_0 < 0 \). Then, \( \mu(t) = \mu_{A(t)} \) with \( A(t) = \begin{pmatrix} 0 & x(t) \\ y(t) & 0 \end{pmatrix} \) and \( x(t) = x, y(t) = y \) satisfy:

\[
\begin{align*}
x' &= x(x + y)(-\frac{3}{2}x + \frac{1}{2}y), \quad x(0) = x_0, \\
y' &= y(x + y)(-\frac{3}{2}y + \frac{1}{2}x), \quad y(0) = y_0.
\end{align*}
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with $A(t) = \begin{pmatrix} 0 & x(t) \\ y(t) & 0 \end{pmatrix}$ and $x(t) = x, y(t) = y$ satisfy:

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$$

(7)

Flujo de corchetas de $\mu_A$ con $A = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$, con $xy < 0$. 
The bracket flow in a class of solvmanifolds

Question:
The bracket flow in a class of solvmanifolds

**Question:** Limits of solutions?
The bracket flow in a class of solvmanifolds

Question: Limits of solutions?

Lemma

Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

- If $\text{tr}(A) = 0$, then $\lim_{t \to \infty} A(t) \parallel A(t) = A_\infty$.
- If $\text{tr}(A) \neq 0$, then $A(t) \to 0$. 

Sketch of proof.

If $\text{tr}(A) = 0$, we consider $F(A) = \parallel [A, A(t)] \parallel^2 \parallel A \parallel^4$ for $A = A(t)$, and the negative gradient flow of $F$, $\bar{A}(t)$. Then if $A$ is not nilpotent, $\lim_{t \to \infty} \bar{A}(t) \parallel \bar{A}(t) = \lim_{t \to \infty} A(t) \parallel A(t)$.

If $\text{tr}(A) \neq 0$, it is easy to see that $A(t) \to 0$ using the spectra of $A$ and $A(t)$. 
The bracket flow in a class of solvmanifolds

Question: Limits of solutions?

Lemma

Let \( A \in \text{gl}_n(\mathbb{R}) \) and consider the bracket flow \( \mu_{A(t)} \) starting at \( \mu_A \). Then:

- If \( \text{tr}(A) = 0 \), then \( \lim_{t \to \infty} \frac{A(t)}{\|A(t)\|} = A_1^\infty \)

Sketch of proof.

If \( \text{tr}(A) = 0 \), we consider

\[
F(A) = \| [A, A(t)] \|_2 \|A\|_4
\]

for \( A = A(t) \), and the negative gradient flow of \( F \), \( \bar{A}(t) \).

Then if \( A \) is not nilpotent \( \lim_{t \to \infty} \bar{A}(t) \parallel \bar{A}(t)\parallel = \lim_{t \to \infty} A(t) \parallel A(t)\parallel \).

If \( \text{tr}(A) \neq 0 \), it is easy to see that \( A(t) \to 0 \) using the spectra of \( A \) and \( A(t) \).
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Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_A(t)$ starting at $\mu_A$. Then:

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Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

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Sketch of proof.

If $\text{tr}(A) = 0$, we consider $F(A) = \| [A, A(t)] \|^2 \|A\|^4$ for $A = A(t)$, and the negative gradient flow of $F$, $\bar{A}(t)$.

Then if $A$ is not nilpotent $\lim_{t \to \infty} \|\bar{A}(t)\| = \lim_{t \to \infty} \|A(t)\|$.

If $\text{tr}(A) \neq 0$, it is easy to see that $A(t) \to 0$ using the spectra of $A$ and $A(t)$. 

The normalized bracket flow

Negative curvature
The bracket flow in a class of solvmanifolds

Question: Limits of solutions?

Lemma

Let $A \in \mathfrak{g}l_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

- If $\text{tr}(A) = 0$, then $\lim_{t \to \infty} \frac{A(t)}{\|A(t)\|} = A_\infty^1 (\leadsto A(t) \to A_\infty)$,
- If $\text{tr}(A) \neq 0$, then $A(t) \to 0$.

Sketch of proof.
The bracket flow in a class of solvmanifolds

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Lemma

Let \( A \in \mathfrak{gl}_n(\mathbb{R}) \) and consider the bracket flow \( \mu_A(t) \) starting at \( \mu_A \). Then:

- If \( \text{tr}(A) = 0 \), then \( \lim_{t \to \infty} \frac{A(t)}{\|A(t)\|} = A^1_\infty (\sim A(t) \to A_\infty) \),
- If \( \text{tr}(A) \neq 0 \), then \( A(t) \to 0 \).

Sketch of proof.

- If \( \text{tr}(A) = 0 \),
The bracket flow in a class of solvmanifolds

Question: Limits of solutions?

Lemma

Let $A \in gl_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

- If $\text{tr}(A) = 0$, then $\lim_{t \to \infty} \frac{A(t)}{\|A(t)\|} = A_1^\infty (\sim \to A(t) \to A_\infty)$,
- If $\text{tr}(A) \neq 0$, then $A(t) \to 0$.

Sketch of proof.

- If $\text{tr}(A) = 0$, we consider $F(A) = \frac{\|[A,A^t]\|^2}{\|A\|^4}$ for $A = A(t)$, and the negative gradient flow of $F$, $\bar{A}(t)$. 

The bracket flow in a class of solvmanifolds

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Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

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- If $\text{tr}(A) = 0$, we consider $F(A) = \frac{\|[A,A^t]\|^2}{\|A\|^4}$ for $A = A(t)$, and the negative gradient flow of $F$, $\bar{A}(t)$. Then if $A$ is not nilpotent $\lim_{t \to \infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|} = \lim_{t \to \infty} \frac{A(t)}{\|A(t)\|}$. 
The bracket flow in a class of solvmanifolds

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Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

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- If $\text{tr}(A) \neq 0$, then $A(t) \to 0$.

Sketch of proof.

- If $\text{tr}(A) = 0$, we consider $F(A) = \frac{\|[A,A^t]\|^2}{\|A\|^4}$ for $A = A(t)$, and the negative gradient flow of $F$, $\tilde{A}(t)$. Then if $A$ is not nilpotent $\lim_{t \to \infty} \frac{\tilde{A}(t)}{\|\tilde{A}(t)\|} = \lim_{t \to \infty} \frac{A(t)}{\|A(t)\|}$. 
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The bracket flow in a class of solvmanifolds

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The bracket flow in a class of solvmanifolds

**Lemma**

Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_A(t)$ starting at $\mu_A$. Then:

\[ A(t) = a(t) \phi_t A \phi_t^{-1}, \]

$a(t)$ is a real valued function, and $\phi_t \in \text{GL}_n(\mathbb{R})$.

As $t \to \infty$, $\text{tr}(S(A(t)))^2 \to 0$. Moreover, $\text{tr}(S(A(t)))^2$ is strictly decreasing if $A$ is not skew-symmetric.
Lemma

Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

- $A(t) = a(t)\varphi_t A \varphi_t^{-1}$, $a(t)$ is a real valued function, and $\varphi_t \in \text{GL}_n(\mathbb{R})$. 

$\Rightarrow$ Spec($A^\infty$) = Spec($A^\infty$).

$A(t)$ is defined $\forall t \in [0, \infty)$. 

$\text{tr}(\mathcal{S}(A(t)))$ is strictly decreasing if $A$ is not skew-symmetric. Moreover, $\text{tr}(\mathcal{S}(A(t))) \to 0$ as $t \to \infty$.
Lemma

Let \( A \in \mathfrak{gl}_n(\mathbb{R}) \) and consider the bracket flow \( \mu_{A(t)} \) starting at \( \mu_A \). Then:

\[ A(t) = a(t)\phi_t A \phi_t^{-1}, \text{ } a(t) \text{ is a real valued function, and } \phi_t \in GL_n(\mathbb{R}). (\sim \text{ Spec}(A_\infty) = a_\infty \text{ Spec}(A).) \]
The bracket flow in a class of solvmanifolds

Lemma

Let $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the bracket flow $\mu_{A(t)}$ starting at $\mu_A$. Then:

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Lemma

Let \( A \in \mathfrak{gl}_n(\mathbb{R}) \) and consider the bracket flow \( \mu_A(t) \) starting at \( \mu_A \). Then:

- \( A(t) = a(t)\varphi_t A \varphi_t^{-1} \), \( a(t) \) is a real valued function, and \( \varphi_t \in GL_n(\mathbb{R}). \) (\( \sim \) \( \text{Spec}(A_\infty) = a_\infty \text{Spec}(A).)\)
- \( A(t) \) is defined \( \forall t \in [0, \infty) \).
- \( \text{tr}(S(A(t))^2) \) is strictly decreasing if \( A \) is not skew-symmetric. Moreover, \( \text{tr}(S(A(t))^2) \rightarrow 0 \) as \( t \rightarrow \infty \).
The bracket flow in a class of solvmanifolds

Corollary

There exists a sequence \( (G_{\mu A(t_k)}, g_{\mu A(t_k)}) \) which converges in the pointed (Cheeger - Gromov) sense to a manifold locally isometric to \( (G_{\mu A_{\infty}}, g_{\mu A_{\infty}}) \), which is flat.
The bracket flow in a class of solvmanifolds

Corollary

There exists a sequence \((G_{\mu_A(t_k)}, g_{\mu_A(t_k)})\) which converges in the pointed (Cheeger - Gromov) sense to a manifold locally isometric to \((G_{\mu_A\infty}, g_{\mu_A\infty})\), which is flat.

Proposition

If \(\text{Spec}(A) \not\subseteq i\mathbb{R}\) then \(g_{\mu_A(t)} \rightarrow g_{\mu_A\infty}\) smoothly on \(\mathbb{R}^n\).
The bracket flow in a class of solvmanifolds

Proposition

For every $\mu_A$ with $\text{tr}(A^2) \geq 0$, the Ricci flow $g(t)$ with $g(0) = g_{\mu_A}$ is a Type - III solution
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The bracket flow in a class of solvmanifolds

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The bracket flow in a class of solvmanifolds

Theorem

Let $A \in \mathfrak{gl}_n(\mathbb{R})$. Consider $\mu_A(t)$ the bracket flow starting at $\mu_A$ and $g(t)$ the Ricci flow starting at $g_{\mu_A}$. Then:

(i) $g(t)$ is defined $\forall t \in [0, \infty)$.

(ii) $A(t) \to A_\infty$.

(iii) There exists a sequence $(G_{\mu_A(t_k)}, g_{\mu_A(t_k)})$ which converges in the pointed sense to a manifold locally isometric to $(G_{\mu_A_\infty}, g_{\mu_A_\infty})$, which is flat.

(iv) If $\text{Spec}(A) \not\subset i\mathbb{R}$, then $g_{\mu_A(t)} \to g_{\mu_A_\infty}$ smoothly on $\mathbb{R}^n$.

(v) If $\text{tr}(A^2) \geq 0$, then $g(t)$ is a type - III solution for some constant $C_n$ that only depends on the dimension of $V_n$. 
Lemma

Let $A$ be with $\|A\| = 1$ and consider $\mu_A(t)$ the norm-normalized bracket flow starting at $\mu_A$. Then the following are equivalent:

(i) $\mu_A$ is not an algebraic soliton.

(algebraic soliton: $\text{Ric} \mu_A = cI + D$, $c \in \mathbb{R}, D \in \text{Der}(\mu_A)$)

(ii) $d\|\left[A, A_t\right]\|^2 < 0$.

Theorem

Assume that $A(t_k) \to A_\infty$. Then, $A_\infty$ is an algebraic soliton. Moreover, the following are equivalent:

(i) $\text{Spec}(A) \subseteq i\mathbb{R}$.

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Theorem

Assume that $A(t_k) \to A_\infty$. Then, $A_\infty$ is an algebraic soliton. Moreover, the following are equivalent:

(i) $\text{Spec}(A) \subseteq i\mathbb{R}$.

(ii) $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$ is flat.
Negative curvature

If \((M, g)\) we will say that it has negative curvature, and denote it by \(K < 0\), if all sectional curvatures are strictly negative.

If \((g, \langle \cdot, \cdot \rangle)\) is a Lie algebra with an inner product, we will think about sectional curvatures of \((G, g)\).

In the case of \(\mu_A\), we will denote it by \(K_A\).
Question:
Negative curvature

**Question:** How does the curvature evolve along the Ricci flow?
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If $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a Lie algebra with an inner product, we will think about sectional curvatures of $(G, g)$. In the case of $\mu_A$, we will denote it by $K_A$. 
**Theorem**

Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature.
**Negative curvature**

**Theorem**

Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature. If $\mu_A(t)$ is the bracket flow starting at $\mu_A$, then there exists $S \in \mathbb{N}$ such that $K_{A(t)} < 0, \forall t \geq S$. 
Negative curvature

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Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature. If $\mu_A(t)$ is the bracket flow starting at $\mu_A$, then there exists $S \in \mathbb{N}$ such that $K_{\mu_A(t)} < 0, \forall t \geq S$.

Sketch of proof.

- We consider the norm-normalized bracket flow.
Negative curvature

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Sketch of proof.

- We consider the norm-normalized bracket flow.
- If $A(t_k) \rightarrow A_{\infty}$, then $\text{Spec}(A_{\infty}) = \alpha_{\infty} \text{Spec}(A)$, $\alpha_{\infty} > 0$. 

Theorem

Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature. If $\mu_A(t)$ is the bracket flow starting at $\mu_A$, then there exists $S \in \mathbb{N}$ such that $K_{\mu_A(t)} < 0, \forall t \geq S$.

Sketch of proof.

- We consider the norm-normalized bracket flow.
- If $A(t_k) \to A_\infty$, then $\text{Spec}(A_\infty) = \alpha_\infty \text{Spec}(A)$, con $\alpha_\infty > 0$.
- As $\mu_A$ admits an inner product with $K < 0$, the $\text{Re}(\text{Spec}(A)) > 0$ or $\text{Re}(\text{Spec}(A)) < 0$ ([H]).
Theorem

Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature. If $\mu_A(t)$ is the bracket flow starting at $\mu_A$, then there exists $S \in \mathbb{N}$ such that $K_{\mu_A(t)} < 0, \forall t \geq S$.

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- Then $\text{Re}(\text{Spec}(A_\infty)) > 0$ or $\text{Re}(\text{Spec}(A_\infty)) < 0$, and $A_\infty$ is normal because it is an algebraic soliton.
Theorem

Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature. If $\mu_A(t)$ is the bracket flow starting at $\mu_A$, then there exists $S \in \mathbb{N}$ such that $K_{A(t)} < 0, \forall t \geq S$.

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- We consider the norm-normalized bracket flow.
- If $A(t_k) \to A_\infty$, then $\text{Spec}(A_\infty) = \alpha_\infty \text{Spec}(A)$, con $\alpha_\infty > 0$.
- As $\mu_A$ admits an inner product with $K < 0$, the $\text{Re}(\text{Spec}(A)) > 0$ or $\text{Re}(\text{Spec}(A)) < 0$ ([H]).
- Then $\text{Re}(\text{Spec}(A_\infty)) > 0$ or $\text{Re}(\text{Spec}(A_\infty)) < 0$, and $A_\infty$ is normal because it is an algebraic soliton. Then $K_{A_\infty} < 0$. 

Finally, it is easy to see that the theorem is true for the norm-normalized bracket flow and then for the bracket flow.
Theorem

Let $\mu_A$ be a solvable Lie algebra that admits an inner product with negative curvature. If $\mu_A(t)$ is the bracket flow starting at $\mu_A$, then there exists $S \in \mathbb{N}$ such that $K_{A(t)} < 0, \forall t \geq S$.

Sketch of proof.

- We consider the norm-normalized bracket flow.
- If $A(t_k) \to A_\infty$, then $\text{Spec}(A_\infty) = \alpha_\infty \text{Spec}(A)$, con $\alpha_\infty > 0$.
- As $\mu_A$ admits an inner product with $K < 0$, the $\text{Re}(\text{Spec}(A)) > 0$ or $\text{Re}(\text{Spec}(A)) < 0$ ([H]).
- Then $\text{Re}(\text{Spec}(A_\infty)) > 0$ or $\text{Re}(\text{Spec}(A_\infty)) < 0$, and $A_\infty$ is normal because it is an algebraic soliton. Then $K_{A_\infty} < 0$.
- Finally, it is easy to see that the theorem is true for the norm-normalized bracket flow and then for the bracket flow.
Negative curvature

Question:

We consider \((\mu, \lambda, \alpha, \langle \cdot, \cdot \rangle)\) defined as follows:

\[
\mu_{\lambda, \alpha}(e_0, e_i) = \alpha \left( \lambda^{-1} - \lambda \right) e_i, \quad \mu_{\lambda, \alpha}(e_1, e_2) = e_3.
\]

\((\mu, \lambda, \alpha, \langle \cdot, \cdot \rangle)\) is an algebraic soliton \(\iff\)

\[
\alpha = \sqrt{3} \sqrt{2(\lambda^2 + (1 - \lambda)^2 + 1)}.
\]

\[
K(e_1, e_3) = \frac{1}{4} - \frac{3}{2} \lambda \lambda^2 + (1 - \lambda)^2 + 1.
\]

\[
K(e_1, e_3) \geq 0 \iff \lambda \leq 2 - \sqrt{3} \text{ or } \lambda \geq 2 + \sqrt{3}.
\]

If \(0 < \lambda \leq 2 - \sqrt{3}\), then \(0 < 1 - \lambda\) and so

\[
\text{Re}(\text{Spec}(\text{ad}(e_0))) > 0.
\]

Then \(\mu_{\lambda, \alpha}\) admits an inner product with negative curvature (\([H]\)).

Hence, as \((\mu, \lambda, \alpha, \langle \cdot, \cdot \rangle)\) is an algebraic soliton, if \(\mu(t)\) is the bracket flow starting at \(\mu_{\lambda, \alpha}\) then \((G \mu(t), g \mu(t))\) has planes with curvature bigger than or equal to zero.
Negative curvature

Question: Is the same true in the general case?
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\mu_{\lambda,\alpha}(e_0, e_i) = \alpha \begin{pmatrix} \lambda & 1 - \lambda \\ 1 - \lambda & 1 \end{pmatrix} e_i, \quad \mu_{\lambda,\alpha}(e_1, e_2) = e_3.
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Negative curvature

Question: Is the same true in the general case?

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1 & 1 & 1
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K(e_1, e_3) \geq 0 \iff \lambda \leq 2 - \sqrt{3} \quad \text{or} \quad \lambda \geq 2 + \sqrt{3}.
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If \(0 < \lambda \leq 2 - \sqrt{3}\), then \(0 < 1 - \lambda\) and so \(\text{Re}(\text{Spec}(\text{ad}(e_0)))) > 0\).
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If \(0 < \lambda \leq 2 - \sqrt{3}\), then \(0 < 1 - \lambda\) and so \(\text{Re}(\text{Spec}(\text{ad}(e_0))) > 0\). Then \(\mu_{\lambda,\alpha}\) admits an inner product with negative curvature ([H]). Hence, as \((\mu_{\lambda,\alpha}, \langle \cdot, \cdot \rangle)\) is an algebraic soliton, if \(\mu(t)\) is the bracket flow starting at \(\mu_{\lambda,\alpha}\) then \((G_{\mu(t)}, g_{\mu(t)})\) has planes with curvature bigger than or equal to zero.
Question:
Negative curvature

**Question:** What happens with the Ricci flow when we start with a metric whose sectional curvatures are all negative?
Negative curvature

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Question: What happens with the Ricci flow when we start with a metric whose sectional curvatures are all negative? Let $\lambda$ be fixed and we consider $\mu(t)$ starting at $\mu_{\alpha,\lambda}$. 

**Negative curvature**
Negative curvature

**Question**: What happens with the Ricci flow when we start with a metric whose sectional curvatures are all negative? Let $\lambda$ be fixed and we consider $\mu(t)$ starting at $\mu_{\alpha,\lambda}$.

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\mu(t)(e_0, e_i) = \alpha(t) \begin{pmatrix} \lambda & 1 - \lambda \\ 1 & 1 \end{pmatrix} e_i, \quad \mu(t) = h(t)e_3,
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with $\alpha(t) = \frac{1}{\sqrt{2c_\lambda t + \alpha^{-2}}}$ and $h(t) = \frac{1}{\sqrt{3t+1}}$. 
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with $\alpha(t) = \frac{1}{\sqrt{2c_\lambda t+\alpha^{-2}}}$ and $h(t) = \frac{1}{\sqrt{3t+1}}$. For each $t$, we have that

$$K(e_1, e_3) = \frac{h^2}{4} - \lambda \alpha^2 = \frac{1}{4(3t+1)} - \frac{\lambda}{2c_\lambda t+\alpha_0^{-2}}$$
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For sufficiently large $\alpha$, $\mu_{\alpha,\lambda}$ has a negative curvature ([H])
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For sufficiently large $\alpha$, $\mu_{\alpha, \lambda}$ has a negative curvature ([H]) but if $0 < \lambda \leq 2 - \sqrt{3}$ then from some $t_0$, $K(e_1, e_3) \geq 0, \forall t \geq t_0$. 

¡Thank you for your attention!
E. Heintze, On homogeneous manifolds of negative curvature, *Math. Ann.* **211**, (1974), 23-34.

J. Lauret, Ricci soliton solvmanifolds, *J. reine angew. Math.* **650**, (2011), 1 - 21.

J. Lauret, Convergence of homogeneous manifolds, *J. London Math. Soc.*, en prensa (arXiv:1105.2082).

J. Lauret, Ricci flow of homogeneous manifolds, arXiv:1112.5900 v2.