Mass concentration and uniqueness of ground states for mass subcritical rotational nonlinear Schrödinger equations

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Abstract

This paper considers ground states of mass subcritical rotational nonlinear Schrödinger equation

\[-\Delta u + V(x)u + i\Omega(x^\perp \cdot \nabla u) = \mu u + \rho^{p-1}|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^2,
\]

where \(V(x)\) is an external potential, \(\Omega > 0\) characterizes the rotational velocity of the trap \(V(x)\), \(1 < p < 3\) and \(\rho > 0\) describes the strength of the attractive interactions. It is shown that ground states of the above equation can be described equivalently by minimizers of the \(L^2\)-constrained variational problem. We prove that minimizers exist for any \(\rho \in (0, \infty)\) when \(0 < \Omega < \Omega^\ast\), where \(0 < \Omega^\ast := \Omega^\ast(V) < \infty\) denotes the critical rotational velocity of \(V(x)\). While \(\Omega > \Omega^\ast\), there admits no minimizers for any \(\rho \in (0, \infty)\). For fixed \(0 < \Omega < \Omega^\ast\), by using energy estimates and blow-up analysis, we also analyze the limit behavior of minimizers as \(\rho \to \infty\). Finally, we prove that up to a constant phase, there exists a unique minimizer when \(\rho > 0\) is large enough and \(\Omega \in (0, \Omega^\ast)\) is fixed.

Keywords: Rotational nonlinear Schrödinger equations; Ground states; Mass concentration; Local uniqueness.

1 Introduction

In this paper, we study ground states of the following time-independent nonlinear Schrödinger equation

\[-\Delta u + V(x)u + i\Omega(x^\perp \cdot \nabla u) = \mu u + \rho^{p-1}|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^2, \quad (1.1)
\]

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where $V(x)$ is a trapping potential, $\Omega > 0$ characterizes the rotational velocity of the trap $V(x)$, $x^\perp = (-x_2, x_1)$ with $x = (x_1, x_2) \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ is the chemical potential, $\rho > 0$ describes the strength of the attractive interactions, and $1 < p < 3$. The rotational nonlinear Schrödinger equation (1.1) appears in many aspects of physics, such as nonlinear optics, plasma physics and so on [31,33]. In particular, equation (1.1) with $p = 3$ is also known as Gross-Pitaevskii equation, which models two dimension attractive Bose-Einstein condensates in a trap $V(x)$ rotating at the velocity $\Omega$, see [1,8,11,32] and the references therein.

As illustrated by Theorem A.1 in the Appendix, ground states of (1.1) can be described equivalently by minimizers of the following mass subcritical ($L^2$-subcritical) constraint variational problem

$$I(\rho) := \inf_{\{u \in \mathcal{H}, \|u\|_2^2 = 1\}} E_\rho(u),$$

where the Gross-Pitaevskii (GP) energy functional $E_\rho(u)$ is defined by

$$E_\rho(u) := \int_{\mathbb{R}^2} \left( |\nabla u|^2 + V(x)|u|^2 \right) dx - \frac{2\rho^{p-1}}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} dx - \Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u) dx.$$  (1.3)

Here $(iu, \nabla u) = (u \nabla \bar{u} - \bar{u} \nabla u)/2$, and the space $\mathcal{H}$ is defined as

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^2, \mathbb{C}) : \int_{\mathbb{R}^2} V(x)|u|^2 dx < \infty \right\}$$  (1.4)

with the associated norm $\|u\|_\mathcal{H} = \left\{ \int_{\mathbb{R}^2} \left( |\nabla u|^2 + (1 + V(x))|u|^2 \right) dx \right\}^{\frac{1}{2}}$. To discuss equivalently ground states of (1.1), in this paper, we shall therefore focus on investigating (1.2), instead of (1.1).

For the non-rotational case $\Omega = 0$, it is easy to obtain from the diamagnetic inequality $|\nabla u(x)| \geq |\nabla u|(x)|$ that minimizers of (1.2) are essentially real-valued. Many qualitative properties of the real-valued minimizers were studied in the last decades, see [5,7,21,26,28,29] and the references therein. Indeed, because $I(\rho)$ is a $L^2$–subcritical problem ($1 < p < 3$), one can use the celebrated concentration-compactness lemma [28] to prove that real-valued minimizers of $I(\rho)$ always exist for any $\rho \in (0, \infty)$. Moreover, by applying the energy method developed in [19][20][22], [26] also obtained the mass concentration behavior and the local uniqueness of minimizers for $I(\rho)$ as $\rho \to \infty$ recently.

On the other hand, minimizers of $I(\rho)$ are no longer real-valued in the rotational case $\Omega > 0$. To our knowledge, the existence of complex-valued minimizers for problem (1.2) was obtained earlier in [10], which however focused mainly on the special case where $V(x) = |x|^2$ and $\Omega = 2$. Subsequently, [6] proved the stability of complex-valued minimizers for $I(\rho)$ in $\mathbb{R}^3$. Recently, much attention has been attracted to the problem $I(\rho)$ again due to its significance on rotating BEC theory [2][17][18][25].
According to [25], the model of two dimensional rotating BEC can be described by the $L^2$-critical version of $I(\rho)$, for which the exponent $1 < p < 3$ was replaced by $p = 3$ in the functional $E_\rho(u)$. For this $L^2$-critical problem, [18][25] proved that there exist critical constants $0 < \Omega^* := \Omega^*(V) \leq \infty$ and $\rho^* > 0$, such that for any $\Omega \in (0, \Omega^*)$, minimizers of $I(\rho)$ exist if and only if $\rho < \rho^*$. Moreover, by developing the method of inductive symmetry, [18] also proved the uniqueness and free-vortex of minimizers for $I(\rho)$ as $\rho \nearrow \rho^*$ for some suitable class of radial potential $V(x) = V(|x|)$. More recently, the authors in [17] have generalized the local uniqueness result of [18] to the non-radially symmetric case of $V(x)$.

Compared with those analysis of the real-valued minimizers in [21][26][29], we remark that there are some new problems need to be solved in the rotational case $\Omega > 0$ for $I(\rho)$ (see [17][18][25]). One of the key issue is to obtain a refined estimates of the imaginary part of the complex-valued minimizers, for which some new phase transformations about minimizers and some properties of the linearized operator was fully utilized in [17][18][25]. Moreover, the $L^2$-critical exponent $p = 3$ also plays an important role in their analysis and calculations. Therefore, a natural question to ask is that whether those results in [17][18], which focused on studying $I(\rho)$ with $p = 3$, still holds true for any $1 < p < 3$? The main purpose of this paper is to settle this problem, we shall investigate the existence and nonexistence, the limit behavior and local uniqueness of complex-valued minimizers of (1.2) for any $1 < p < 3$ and $\Omega > 0$.

Throughout the paper, we always assume that the trapping potential $V(x)$ satisfies

$$0 \leq V(x) \in L^\infty_{lo}(\mathbb{R}^2), \quad \lim_{|x| \to \infty} \frac{V(x)}{|x|^2} > 0,$$

and we denote the critical rational speed $\Omega^*$ by

$$\Omega^* := \sup \left\{ \Omega > 0 : V(x) - \frac{\Omega^2}{4}|x|^2 \to \infty \text{ as } |x| \to \infty \right\}. \quad (1.6)$$

To state our main results, we first recall the following Gagliardo-Nirenberg inequality [34]

$$\|u\|_{p+1}^{p+1} \leq \frac{p+1}{2} \left( \frac{2}{p-1} \right)^{\frac{p-1}{2}} \|\nabla u\|_2 \|u\|_2, \quad u \in H^1(\mathbb{R}^2, \mathbb{R}), \ 1 < p < 3, \quad (1.7)$$

where $w$ is the unique (up to translations) positive radial solution of the following nonlinear scalar field equation [12][24][34]

$$- \Delta u + u - u^p = 0 \text{ in } \mathbb{R}^2, \quad u \in H^1(\mathbb{R}^2, \mathbb{R}), \quad (1.8)$$

and the equality in (1.7) is achieved at $u = w$. Note also from [12 Proposition 4.1] that $w = w(|x|) > 0$ satisfies

$$w(x), \ |\nabla w(x)| = O(|x|^{-\frac{1}{2}}e^{-|x|}) \text{ as } |x| \to \infty. \quad (1.9)$$
Moreover, it follows from [5, Lemma 8.1.2] that \( w \) satisfies
\[
\int_{\mathbb{R}^2} |\nabla w|^2 dx = \frac{p-1}{p+1} \int_{\mathbb{R}^2} w^{p+1} dx = \frac{p-1}{2} \int_{\mathbb{R}^2} w^2 dx. \tag{1.10}
\]

Recall also from [27] the following diamagnetic inequality: for \( \mathcal{A} = \Omega x^\perp \),
\[
|\nabla u|^2 - \Omega x^\perp \cdot (iu, \nabla u) + \frac{\Omega^2}{4} |x|^2 |u|^2 = |(\nabla - i\mathcal{A})u|^2 \geq |\nabla |u||^2, \quad u \in H^1(\mathbb{R}^2, \mathbb{C}). \tag{1.11}
\]

Applying the Gagliardo-Nirenberg inequality (1.7) and the diamagnetic inequality (1.11), our first result on the existence and nonexistence of minimizers for \( I(\rho) \) is stated as the following theorem.

**Theorem 1.1.** Suppose \( V(x) \) satisfies (1.5), then we have

1. If \( 0 < \Omega < \Omega^* \), then \( I(\rho) \) admits at least one minimizer for any \( \rho \in (0, \infty) \);
2. If \( \Omega > \Omega^* \), then \( I(\rho) \) admits no minimizer for any \( \rho \in (0, \infty) \).

By variational theory, any minimizer \( u_\rho \) of \( I(\rho) \) satisfies the following Euler-Lagrange equation
\[
-\Delta u_\rho + V(x)u_\rho + i\Omega (x^\perp \cdot \nabla u_\rho) = \mu_\rho u_\rho + \rho^{p-1} |u_\rho|^{p-1} u_\rho \quad \text{in} \quad \mathbb{R}^2, \tag{1.12}
\]
where \( \mu_\rho \in \mathbb{R} \) is a suitable Lagrange multiplier satisfying
\[
\mu_\rho = I(\rho) - \frac{p-1}{p+1} \rho^{p-1} \int_{\mathbb{R}^2} |u_\rho|^{p+1} dx, \quad \rho \in (0, \infty). \tag{1.13}
\]

By making full use of the equation (1.12), we next focus on investigating the limit behavior of minimizers for \( I(\rho) \) as \( \rho \to \infty \), for which we define

**Definition 1.1.** The nonnegative function \( h(x) : \mathbb{R}^2 \to \mathbb{R}^+ \) is homogeneous of degree \( s \in \mathbb{R}^+ \) (about the origin), if there exists some \( s > 0 \) such that
\[
h(tx) = t^s h(x) \quad \text{in} \quad \mathbb{R}^2 \quad \text{for any} \quad t > 0.
\]

This definition implies that if \( h(x) \in C(\mathbb{R}^2) \) is homogeneous of degree \( s > 0 \), then
\[
0 \leq h(x) \leq C |x|^s \quad \text{in} \quad \mathbb{R}^2, \quad \text{where} \quad C := \max_{x \in \partial B_1(0)} h(x).
\]

Furthermore, if \( h(x) \to \infty \) as \( |x| \to \infty \), then the origin is the unique minimum point of \( h(x) \).

Following the above definition, we next assume that \( V_\Omega(x) := V(x) - \frac{\Omega^2}{4} |x|^2 \) satisfies
(V₁). \( V_Ω(x) ≥ 0, \{ x ∈ ℝ^2 : V_Ω(x) = 0 \} = \{0\} \), and there exists a \( κ > 0 \) such that
\[
V_Ω(x) + |∇V_Ω(x)| ≤ Ce^{κ|x|} \quad \text{as} \quad |x| → ∞.
\]

(V₂). There exists a homogeneous function \( h(x) ∈ C^1(ℝ^2) \) of degree \( 1 < s ≤ 2 \), which satisfies \( \lim_{|x| → ∞} h(x) = +∞ \) and \( H(y) := \int_{ℝ^2} h(x + y)w^2(x)dx \) admits a unique global minimum point \( y_0 ∈ ℝ^2 \), such that as \(|x| → 0, V_Ω(x) = h(x) + o(|x|^s), \frac{∂V_Ω(x)}{∂x_j} = \frac{∂h(x)}{∂x_j} + o(|x|^{s-1}) \), where \( j = 1, 2 \). (1.14)

Under the assumptions (1.5), (V₁) and (V₂), we now give the following theorem on the limit behavior of minimizers for \( I(ρ) \) as \( ρ → ∞ \).

**Theorem 1.2.** Suppose \( V(x) \) satisfies (1.3), (V₁) and (V₂), and assume \( 0 < Ω < Ω^* \), where \( Ω^* > 0 \) is defined by (1.6). Denote \( a^* := ||w||_2^2 \), where \( w(x) \) is the unique positive solution of (1.8). Let \( u_ρ \) be a minimizer of \( I(ρ) \), then we have
\[
\lim_{ρ → ∞} ε_ρ u_ρ(ε_ρ(x + y_0))e^{-i(\frac{1}{2}Ωx^2y_0^2 - θ_ρ)} = \frac{w(x)}{√a^*}
\]
strongly in \( H^1(ℝ^2, C) ∩ L^∞(ℝ^2, C) \), where \( y_0 \) is defined in (V₂), \( θ_ρ ∈ [0, 2π) \) is a properly chosen constant, and \( ε_ρ \) is defined as
\[
ε_ρ := \left( \frac{ρ}{√a^*} \right)^{-\frac{p-1}{2-p}}.
\]

**Theorem 1.2** gives a detailed description of the limit behavior of minimizers \( u_ρ \) for (1.2). We shall encounter some new problems in the proof of Theorem 1.2. The first problem is that one cannot use the Gagliardo-Nirenberg inequality directly to establish the optimal energy estimates in \( L^2 \)—subcritical case here. In order to solve this problem, we need to introduce the following new constraint variational problem
\[
\hat{I}(ρ) := \inf_{u ∈ H^1(ℝ^2, ℝ), ||u||_2^2 = 1} \hat{E}_ρ(u),
\]
where \( \hat{E}_ρ(u) \) is defined by
\[
\hat{E}_ρ(u) := \int_{ℝ^2} |∇u|^2 dx - \frac{2ρ^{p-1}}{p+1} \int_{ℝ^2} |u|^{p+1} dx.
\]
We shall establish a refined energy estimate of \( I(ρ) \) in Lemma 3.2 by analyzing the energy estimate of \( \hat{I}(ρ) \) and choosing some suitable test functions. Another problem is that how to deal with the rotational term \( Ω \int_{ℝ^2} x^+ · (iu_ρ, ∇u_ρ)dx \) in (1.3) as \( ρ → ∞ \), for
which we need to make use of the refined energy estimate of \( I(\rho) \) and some properties of the linearized operator (3.50) in Section 3 below.

Finally, we shall analyze the local uniqueness of minimizers for \( I(\rho) \) as \( \rho \to \infty \). Assume that the unique global minimum point \( y_0 \) of \( H(y) = \int_{\mathbb{R}^2} h(x + y) w^2(x) dx \) is non-degenerate in the sense that

\[
\det \left( \int_{\mathbb{R}^2} \frac{\partial h(x + y_0)}{\partial x_j} \frac{\partial w^2(x)}{\partial x_l} dx \right)_{j,l=1,2} \neq 0. \tag{1.19}
\]

then we have the following result concerning the uniqueness of minimizers.

**Theorem 1.3.** Suppose \( V(x) \in C^{1,\alpha}_{loc}(\mathbb{R}^2)(0 < \alpha < 1) \) satisfies (1.2), (V1) and (V2), and \( \Omega \) satisfies \( 0 < \Omega < \Omega^* \), where \( \Omega^* > 0 \) is defined as in (1.6). Moreover, we assume that the unique global minimum point \( y_0 \) of \( H(y) = \int_{\mathbb{R}^2} h(x + y) w^2(x) dx \) is non-degenerate, then up to the constant phase, there exists a unique complex-valued minimizer for \( I(\rho) \) when \( \rho > 0 \) is large enough.

We remark that the local uniqueness, up to a constant phase, of Theorem 1.3 holds in the following sense: there exists a minimizer \( U_\rho \) of \( I(\rho) \) such that any minimizer \( u_\rho \) of \( I(\rho) \) satisfies \( u_\rho \equiv U_\rho e^{i\theta_\rho} \) in \( \mathbb{R}^2 \) for \( \rho > 0 \) large enough, where \( \theta_\rho \in [0,2\pi) \) is a suitable constant phase depending on \( \rho \) and \( u_\rho \).

The main method of proving Theorem 1.3 is inspired by [3, 9, 14, 16, 17] and the references therein, but there still have some essential difficulties occur in our proof. Indeed, note that the Pohozaev identities play a crucial role on the process of the proof of local uniqueness [3,9,14,16]. However, due to the appearance of the rotation term, the Euler-Lagrange equation (1.12) of minimizers \( u_\rho \) is essentially a coupled system of the real and imaginary parts of \( u_\rho \). Therefore, the first difficulty is that one cannot obtain the Pohozaev identities directly for the complex-valued minimizers of \( I(\rho) \). To overcome this difficulty, we shall construct various Pohozaev identities for the real part of the complex-valued minimizers and analyze all terms produced by the rotation. The second difficulty is that those analysis in [17], which focus on the special case \( p = 4 \), can not generalize to the proof of Theorem 1.3 due to the rational of \( 1 < p < 3 \). Therefore, we need to deal with the nonlinear term \( |u_\rho|^{p-1}u_\rho \) more carefully. Besides, we shall use some technical expansions of the nonlinear term in [4] flexibly to obtain a desired estimates in our proof.

This paper is organized as follows. Section 2 is devoted to proving Theorem 1.1 on the existence and nonexistence of minimizers for \( I(\rho) \). In Section 3, we shall prove Theorem 1.2 on the limit behavior of minimizers for \( I(\rho) \) as \( \rho \to \infty \) by employing energy methods and blow-up analysis. By deriving various Pohozaev identities, we shall complete the proof of the local uniqueness of minimizers in Section 4.
2 Existence of minimizers for $I(\rho)$

This section is concerned with the proof of Theorem 1.1 on the existence and nonexistence of minimizers for $I(\rho)$. We first introduce the following compactness lemma.

**Lemma 2.1.** Suppose that $V(x) \in L^\infty_{loc}(\mathbb{R}^2)$ and $\lim_{|x|\to\infty} V(x) = \infty$. If $2 \leq q < \infty$, then the embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^2, \mathbb{C})$ is compact.

The proof of this lemma is similar to those in [35] and the references therein, so we omit it here.

**Proof of Theorem 1.1.** Since the proof of Theorem 1.1 is overall similar to those in [35, Theorem 1.1], we only give the main idea of its proof here.

1. For any $\rho \in (0, \infty)$ and $0 < \Omega < \Omega^*$. Suppose that $u \in \mathcal{H}$ and $\|u\|_2^2 = 1$. Applying the Gagliardo-Nirenberg inequality (1.7) and the diamagnetic inequality (1.11), we deduce that there exist sufficiently large $R > 0$ and $C(\Omega, \rho, R) > 0$ such that for any $\rho \in (0, \infty)$ and $0 < \Omega < \Omega^*$,

$$E_\rho(u) \geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - C(\Omega, \rho, R),$$

(2.1)

which implies that $E_\rho(u)$ is bounded from below. Let $\{u_n\} \in \mathcal{H}$ be a minimizing sequence of $I(\rho)$ satisfying $\|u_n\|_2^2 = 1$ and $\lim_{n \to \infty} E_\rho(u_n) = I(\rho)$. It then follows from (2.1) that the sequence $\{u_n\}$ is bounded uniformly in $\mathcal{H}$. By the compact embedding in Lemma 2.1, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u_0 \in \mathcal{H}$ such that

$$u_{n_k} \rightharpoonup u_0 \text{ weakly in } \mathcal{H}, \quad u_{n_k} \to u_0 \text{ strongly in } L^q(\mathbb{R}^2, \mathbb{C}) (2 \leq q < \infty).$$

Thus we conclude from above convergence and the weak lower semicontinuity that $\|u_0\|_2^2 = 1$ and $I(\rho) = E_\rho(u_0)$, i.e., $u_0$ is a minimizer of $I(\rho)$. This implies that for any $\rho \in (0, \infty)$ and $0 < \Omega < \Omega^*$, there exists at least one minimizer of $I(\rho)$.

2. For any $\rho \in (0, \infty)$ and $\Omega > \Omega^*$. Let $w = w(|x|)$ be the unique positive solution of (1.8). For any $\tau > 0$, choose a trail function

$$w_\tau(x) := \frac{A_{\tau}}{\|w\|_2^2} w(\tau(x - x_\tau)) \varphi(x - x_\tau)e^{i\Omega S(x)},$$

(2.2)

where $A_{\tau} > 0$ is chosen such that $\|w_\tau\|_2^2 = 1$, $x_\tau \in \mathbb{R}^2$ is chosen such that $V_\Omega(x_\tau) \leq -2\frac{\tau^2 \|w\|_p^{p+1}}{\|w\|_2^2}$, $S(x) = \frac{1}{2} x \cdot x_\tau$, and $\varphi(x) \in C_c^\infty(\mathbb{R}^2)$ is a cut-off function satisfying $\varphi(x) = 1$ if $|x| \leq 1$; $\varphi(x) = 0$ if $|x| \geq 2$; $\varphi(x) \in (0, 1)$ if $1 < |x| < 2$.

Using the exponential decay of $w$ in (1.9) and the identity (1.10), direct calculations give that $E_\rho(w_\tau) \leq -\infty$ as $\tau \to \infty$, which implies that $I(\rho)$ is unbounded from below, and so $I(\rho)$ admits no minimizer for any $\rho \in (0, \infty)$ and $\Omega > \Omega^*$. This then completes the proof of Theorem 1.1.  \[ \square \]
3 Mass concentration as $\rho \to \infty$

This section is devoted to proving Theorem 1.2 on the limit behavior of minimizers for $I(\rho)$ as $\rho \to \infty$. We shall first establish the energy estimates of $I(\rho)$ in Lemma 3.2 and then present a detailed analysis on the limit behavior of minimizers for $I(\rho)$ as $\rho \to \infty$. Based on these energy estimates and analysis, we finally complete the proof of Theorem 1.2.

3.1 Energy estimates of $I(\rho)$

We recall the following energy estimates of $\hat{I}(\rho)$ defined in (1.17) as $\rho \to \infty$.

Lemma 3.1 ([25, Lemma A.3]). Let $\hat{u}_\rho$ be a nonnegative minimizer of $\hat{I}(\rho)$ defined in (1.17). Set $a^* := \|w\|^2_2$, where $w$ is the unique positive solution of (1.8). Then we have

$$\hat{I}(\rho) = -\frac{3}{2} \sum_{\rho}^2 \epsilon_\rho^{-2},$$

and there exists a $x_0 \in \mathbb{R}^2$ such that

$$\hat{u}_\rho(x) = \frac{1}{\sqrt{a^*}} \epsilon_\rho^{-1} w(\epsilon_\rho^{-1} x + x_0),$$

where $\epsilon_\rho$ is defined in (1.10).

Based on Lemma 3.1, we have the following energy estimates of $I(\rho)$.

Lemma 3.2. Suppose that $V(x)$ satisfies (1.5), (V1) and assume $0 < \Omega < \Omega^*$, where $\Omega^* > 0$ is defined in (1.6). Let $u_\rho$ be a minimizer of $I(\rho)$ for any $0 < \rho < \infty$. Then we have

$$0 \leq I(\rho) - \hat{I}(\rho) \to 0 \text{ as } \rho \to \infty,$$

and

$$\int_{\mathbb{R}^2} V_\Omega(x)|u_\rho|^2 dx \to 0 \text{ as } \rho \to \infty.$$  (3.4)

Proof. By (1.3), the diamagnetic inequality (1.11), (1.17) and (V1), we have

$$I(\rho) = \int_{\mathbb{R}^2} |(\nabla - iA)u_\rho|^2 dx + \int_{\mathbb{R}^2} V_\Omega(x)|u_\rho|^2 dx - \frac{2\rho^{p-1}}{p+1} \int_{\mathbb{R}^2} |u_\rho|^{p+1} dx$$

$$\geq \int_{\mathbb{R}^2} |\nabla u_\rho|^2 dx - \frac{2\rho^{p-1}}{p+1} \int_{\mathbb{R}^2} |u_\rho|^{p+1} dx$$

$$\geq \hat{I}(\rho) \text{ as } \rho \to \infty.$$  (3.5)

Taking a test function $w_\tau$ of the form (2.2) with $S(x) \equiv 0$, $x_\tau = 0$ and $\tau = (\frac{\rho}{\sqrt{a^*}})^{\frac{p+1}{3-p}}$. Applying the exponential decay of $w$ in (1.9) and the identity (1.10), direct calculations
give
\[ I(\rho) \leq E_\rho(w_\tau) \leq \frac{p-1}{2} \tau^2 - \left( \frac{\rho}{\sqrt{a^*}} \right)^{p-1} \tau^{p-1} + C \tau^{-2} \]
\[ = \hat{I}(\rho) + o(1) \text{ as } \rho \to \infty. \] (3.6)

Then (3.3) follows from (3.5) and (3.6).

Now, we shall prove (3.4). Combining (1.3), the diamagnetic inequality (1.11), (1.17) and (3.3), one then deduce that
\[ 0 \leq \int_{\mathbb{R}^2} V_\Omega(x)|u_\rho|^2dx = I(\rho) - \int_{\mathbb{R}^2} |(\nabla - iA)u_\rho|^2dx + \frac{2\rho^{p-1}}{p+1} \int_{\mathbb{R}^2} |u_\rho|^{p+1}dx \]
\[ \leq I(\rho) - \left( \int_{\mathbb{R}^2} |\nabla |u_\rho||^2dx - \frac{2\rho^{p-1}}{p+1} \int_{\mathbb{R}^2} |u_\rho|^{p+1}dx \right) \] (3.7)
\[ = I(\rho) - \hat{E}_\rho(|u_\rho|) \]
\[ \leq I(\rho) - \hat{I}(\rho) \to 0 \text{ as } \rho \to \infty, \]
which implies (3.4). This completes the proof of Lemma 3.2. \( \square \)

3.2 Blowup analysis

The main purpose of this subsection is to present a detailed analysis on the limit behavior of minimizers for \( I(\rho) \) as \( \rho \to \infty \). Towards this aim, we first give the following lemma on the refined estimates of minimizers \( u_\rho \) and its Lagrange multiplier \( \mu_\rho \) as \( \rho \to \infty \).

**Lemma 3.3.** Suppose that \( V(x) \) satisfies (1.5), (V1) and assume \( 0 < \Omega < \Omega^* \), where \( \Omega^* > 0 \) is defined in (1.6). Let \( u_\rho \) be a minimizer of \( I(\rho) \), then we have

1. Define
\[ w_\rho(x) := \epsilon_\rho u_\rho(\epsilon_\rho x + z_\rho)e^{-i\left( \frac{\rho \Omega}{2} \cdot z_\rho \right) - \theta_\rho}, \] (3.8)
where \( \epsilon_\rho \) is defined by (1.16), \( z_\rho \) is a global maximum point of \( |u_\rho| \) and \( \theta_\rho \in [0, 2\pi) \) is a proper constant. Then there exists a constant \( \eta > 0 \), independent of \( 0 < \rho < \infty \), such that
\[ \int_{B_2(0)} |w_\rho(x)|^2dx \geq \eta > 0 \text{ as } \rho \to \infty. \] (3.9)

2. \( w_\rho \) satisfies
\[ \lim_{\rho \to \infty} w_\rho(x) = \frac{w(x)}{\sqrt{a^*}} \text{ strongly in } H^1(\mathbb{R}^2, \mathbb{C}), \] (3.10)
where \( a^* := \|w\|_2^2 \) and \( w \) is the unique positive solution of (1.8). Furthermore, any global maximal point \( z_\rho \) of \( |u_\rho| \) satisfies \( \lim_{\rho \to \infty} V_\Omega(z_\rho) = 0 \).
3. $\mu_0 \rho^2 \to -1$ as $\rho \to \infty$.

Proof. 1. Denote $\bar{w}_\rho(x) := \rho u_\rho(\rho x + z_\rho) e^{-i \frac{\rho^2}{2} x \cdot x} \rho_\rho$ and $w_\rho(x) := \bar{w}_\rho(x) e^{i \theta_\rho}$, where the parameter $\theta_\rho \in [0,2\pi)$ is chosen properly such that

$$
\|w_\rho - \frac{w}{\sqrt{\rho}}\|_{L^2(\mathbb{R}^2)} = \min_{\theta \in [0,2\pi)} \|e^{i \theta} \bar{w}_\rho - \frac{w}{\sqrt{\rho}}\|_{L^2(\mathbb{R}^2)}. \tag{3.11}
$$

Rewrite

$$
w_\rho(x) = R_\rho(x) + iI_\rho(x), \tag{3.12}
$$

where $R_\rho(x)$ and $I_\rho(x)$ denote the real and imaginary parts of $w_\rho(x)$ respectively. By (3.11), we have

$$
\int_{\mathbb{R}^2} w(x) I_\rho(x) dx = 0. \tag{3.13}
$$

From (1.12) and (3.8), we deduce that $w_\rho$ satisfies

$$
- \Delta w_\rho + i \epsilon_2 \Omega (x_\perp \cdot \nabla w_\rho)
+ \frac{\epsilon_4 \Delta^2}{4} |x|^2 + \epsilon_2 V_\Omega(\rho x + z_\rho) - \epsilon_2 \mu_\rho - (a^*)^{p-1} |w_\rho|^{p-1} w_\rho = 0 \text{ in } \mathbb{R}^2. \tag{3.14}
$$

Set $W_\rho = |w_\rho|^2 \geq 0$. We then obtain from (3.14) that

$$
- \frac{1}{2} \Delta W_\rho + |\nabla w_\rho|^2 - \epsilon_2 \Omega (x_\perp \cdot i w_\rho, \nabla w_\rho)
+ \frac{\epsilon_4 \Delta^2}{4} |x|^2 + \epsilon_2 V_\Omega(\rho x + z_\rho) - \epsilon_2 \mu_\rho - (a^*)^{p-1} W_\rho^{\frac{p}{p-1}} W_\rho = 0 \text{ in } \mathbb{R}^2. \tag{3.15}
$$

Using the diamagnetic inequality (1.11), we derive from (3.15) that

$$
- \frac{1}{2} \Delta W_\rho + [-\epsilon_2 \mu_\rho - (a^*)^{p-1} W_\rho^{\frac{p-1}{p-1}}] W_\rho \leq 0 \text{ in } \mathbb{R}^2. \tag{3.16}
$$

By (1.13), (3.1) and (3.3), we obtain that $\epsilon_2 \mu_\rho \leq \frac{3-p}{2}$ as $\rho \to \infty$. Note that $W_\rho^{\frac{p-1}{p-2}}$ is bounded uniformly in $L^\frac{2}{p-2}(\mathbb{R}^2, \mathbb{R})$, where $1 < \frac{2}{p-2} < \infty$. Then applying De Giorgi-Nash-Moser theory [23, Theorem 4.1] to (3.16) yields that

$$
\int_{B_2(0)} W_\rho(x) dx \geq \max_{x \in B_1(0)} W_\rho(x) \text{ as } \rho \to \infty. \tag{3.17}
$$

Since 0 is a global maximal point of $W_\rho(x)$ for any $0 < \rho < \infty$, we have $-\Delta W_\rho(0) \geq 0$ for all $0 < \rho < \infty$. Using the fact that $\epsilon_2 \mu_\rho \leq \frac{3-p}{2}$ as $\rho \to \infty$, we then obtain from (3.16) that there exists a constant $\beta > 0$, which is independent of $0 < \rho < \infty$, such that

$$
W_\rho(0) \geq \beta > 0. \tag{3.18}
$$
It then follows from (3.17) and (3.18) that (3.9) holds.

2. Using the diamagnetic inequality, we derive from (1.11), (3.1), (3.3) and (3.5) that

\[
-\frac{3-p}{2} = \epsilon_p^2 I(\rho) = \epsilon_p^2 I(\rho) + o(1)
\]

\[
= \int_{\mathbb{R}^2} \left( |\nabla w_\rho|^2 - \epsilon_p^2 \Omega x^\perp \cdot (iw_\rho, \nabla w_\rho) + \frac{\Omega^2}{4} \epsilon_p^4 |x|^2 |w_\rho|^2 + \epsilon_p^2 V_\Omega(\epsilon_p x + z_\rho) |w_\rho|^2 \right)
\]

\[
- \frac{2(a^*)^{p-1}}{p+1} |w_\rho|^{p+1} dx + o(1)
\]

\[
\geq \int_{\mathbb{R}^2} |\nabla w_\rho|^2 dx - \frac{2(a^*)^{p-1}}{p+1} \int_{\mathbb{R}^2} |w_\rho|^{p+1} dx
\]

\[
\geq \hat{I}(\sqrt{a^*}) = \frac{3-p}{2} \text{ as } \rho \to \infty,
\]  

(3.19)

which yields that \(|w_\rho|\) is a minimizing sequence of \(\hat{I}(\sqrt{a^*})\). Note from (3.19) that \(|w_\rho|\) is bounded uniformly in \(H^1(\mathbb{R}^2, \mathbb{R})\), therefore we can assume that up to a subsequence if necessary, \(|w_\rho|\) convergence to \(w_0\) weakly in \(H^1(\mathbb{R}^2, \mathbb{R})\) as \(\rho \to \infty\) for some \(0 \leq w_0 \in H^1(\mathbb{R}^2, \mathbb{R})\). From (3.9), we get that \(w_0 \neq 0\) in \(\mathbb{R}^2\). By the weak convergence, we may assume that \(|w_\rho| \to w_0\) a.e. in \(\mathbb{R}^2\) as \(\rho \to \infty\). Using the Brézis-Lieb lemma, we obtain that

\[
\|w_\rho\|_q^q = \|w_0\|_q^q + \|w_\rho - w_0\|_q^q + o(1) \text{ as } \rho \to \infty, \text{ where } 2 \leq q < \infty,
\]

(3.20)

and

\[
\|\nabla w_\rho\|_2^2 = \|\nabla w_0\|_2^2 + \|\nabla (|w_\rho| - w_0)\|_2^2 + o(1) \text{ as } \rho \to \infty.
\]

(3.21)

Next, we prove that \(|w_0|_2^2 = 1\). On the contrary, we assume that \(|w_0|_2^2 = \lambda\) and \(|w_\rho| - w_0|_2^2 = 1 - \lambda\), where \(\lambda \in (0,1)\). Set \(w_\lambda := \frac{w_0}{\sqrt{\lambda}}\) and \(w_{1-\lambda} := \frac{|w_\rho| - w_0}{\sqrt{1-\lambda}}\). From (3.1), (3.19), (3.20) and (3.21), we then derive that as \(\rho \to \infty\),

\[
-\frac{3-p}{2} = \lim_{\rho \to \infty} \left[ \int_{\mathbb{R}^2} |\nabla w_\rho|^2 dx - \frac{2(a^*)^{p-1}}{p+1} \int_{\mathbb{R}^2} |w_\rho|^{p+1} dx \right]
\]

\[
= \int_{\mathbb{R}^2} |\nabla w_0|^2 dx - \frac{2(a^*)^{p-1}}{p+1} \int_{\mathbb{R}^2} |w_0|^{p+1} dx
\]

\[
+ \lim_{\rho \to \infty} \left[ \int_{\mathbb{R}^2} |\nabla (|w_\rho| - w_0)|^2 dx - \frac{2(a^*)^{p-1}}{p+1} \int_{\mathbb{R}^2} |w_\rho| - w_0|^{p+1} dx \right]
\]

\[
> \lambda \left[ \int_{\mathbb{R}^2} |\nabla w_\lambda|^2 dx - \frac{2(a^*)^{p-1}}{p+1} \int_{\mathbb{R}^2} |w_\lambda|^{p+1} dx \right]
\]

\[
+ (1-\lambda) \lim_{\rho \to \infty} \left[ \int_{\mathbb{R}^2} |\nabla w_{1-\lambda}|^2 dx - \frac{2(a^*)^{p-1}}{p+1} \int_{\mathbb{R}^2} |w_{1-\lambda}|^{p+1} dx \right]
\]

\[
\geq \lambda \hat{I}(\sqrt{a^*}) + (1-\lambda) \hat{I}(\sqrt{a^*}) = \hat{I}(\sqrt{a^*}) = -\frac{3-p}{2},
\]

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which is a contradiction, therefore \( \|w_0\|_2^2 = 1 \) holds. Since \( \|w_\rho\|_2^2 = \|w_0\|_2^2 = 1 \), we obtain from (3.20) that
\[
|w_\rho(x)| \to w_0(x) \text{ strongly in } L^2(\mathbb{R}^2, \mathbb{R}) \text{ as } \rho \to \infty. \tag{3.22}
\]
By the uniform boundedness of \( |w_\rho| \) in \( H^1(\mathbb{R}^2, \mathbb{R}) \) and the interpolation inequality, we derive from (3.22) that \( |w_\rho(x)| \to w_0(x) \) strongly in \( L^q(\mathbb{R}^2, \mathbb{R}) \) \( (2 \leq q < \infty) \) as \( \rho \to \infty \). Moreover, using the weak lower semicontinuity, (3.1) and (3.19), we obtain that \( \nabla|w_\rho(x)| \to \nabla w_0(x) \) strongly in \( L^2(\mathbb{R}^2, \mathbb{R}) \) as \( \rho \to \infty \). Therefore, we deduce from above that
\[
|w_\rho(x)| \to w_0(x) \text{ strongly in } H^1(\mathbb{R}^2, \mathbb{R}) \text{ as } \rho \to \infty. \tag{3.23}
\]
Since \( |w_\rho| \) is a minimizing sequence of \( \hat{I}(\sqrt{a^*}) \) and \( \|w_0\|_2^2 = 1 \), we obtain from (3.23) that \( w_0 \) is a minimizer of \( \hat{I}(\sqrt{a^*}) \). Then we get from (3.22) that \( w_0(x) = \frac{w(x + x_0)}{\sqrt{a^*}} \), where \( x_0 \in \mathbb{R}^2 \). On the other hand, since the origin is a global maximum point of \( |w_\rho| \) for any \( \rho \in (0, \infty) \), it must be also a global maximum point of \( w(x + x_0) \) in view of (3.23), which implies that \( x_0 = 0 \). We conclude that, up to a subsequence if necessary,
\[
|w_\rho(x)| \to \frac{w(x)}{\sqrt{a^*}} \text{ strongly in } H^1(\mathbb{R}^2, \mathbb{R}) \text{ as } \rho \to \infty. \tag{3.24}
\]
Furthermore, since the convergence (3.24) is independent of what subsequence \( \{ |w_\rho| \} \) we choose, we conclude that (3.24) holds for the whole sequence.

Now, we shall prove that \( w_\rho \) is bounded uniformly in \( H^1(\mathbb{R}^2, \mathbb{C}) \) as \( \rho \to \infty \). By the definition of \( w_\rho \) in (3.8), we only need to prove that there exists a constant \( C > 0 \), independent of \( \rho \), such that as \( \rho \to \infty \),
\[
\int_{\mathbb{R}^2} |\nabla w_\rho|^2 dx \leq C. \tag{3.25}
\]
In fact, since \( 0 < \Omega < \Omega^* \) is fixed, the definition (1.6) of \( \Omega^* \) implies that
\[
|x|^2 \leq C(\Omega)\left(V(x) - \frac{\Omega^2}{4}|x|^2\right) \text{ for sufficiently large } |x| > 0. \tag{3.26}
\]
Using (3.26), we then deduce that for any given large constant \( M > 0 \),
\[
\frac{\Omega^2}{4} \epsilon^4 \int_{\mathbb{R}^2} |x|^2 |w_\rho|^2 dx = \frac{\Omega^2}{4} \epsilon^2 \int_{|\epsilon x| \leq M} |\epsilon x|^2 |w_\rho|^2 dx + \frac{\Omega^2}{4} \epsilon^2 \int_{|\epsilon x| > M} |\epsilon x|^2 |w_\rho|^2 dx \leq o(1) + C(\Omega) \int_{|\epsilon x| > M} \epsilon^2 V_\Omega(\epsilon x)|w_\rho|^2 dx \tag{3.27}
\]
\[
= o(1) \text{ as } \rho \to \infty.
\]
By Hölder inequality, it then follows from (3.27) that as \( \rho \to \infty \),
\[
\epsilon^2 \Omega^2 \int_{\mathbb{R}^2} x^1 \cdot (iw_\rho, \nabla w_\rho) dx \leq \left( \epsilon^4 \Omega^2 \int_{\mathbb{R}^2} |x|^2 |w_\rho|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |\nabla w_\rho|^2 dx \right)^{\frac{1}{2}} \tag{3.28}
\]
\[
= o(1) \left( \int_{\mathbb{R}^2} |\nabla w_\rho|^2 dx \right)^{\frac{1}{2}}.
\]
Since $|w_\rho|$ is bounded uniformly in $H^1(\mathbb{R}^2, \mathbb{R})$, we derive from (3.28) and (3.19) that as $\rho \to \infty$,

$$\frac{3-p}{2} \geq \int_{\mathbb{R}^2} |\nabla w_\rho|^2 \, dx - o(1) \left( \int_{\mathbb{R}^2} |\nabla w_\rho|^2 \, dx \right)^{\frac{1}{2}} - C,$$

(3.29)

which implies (3.25) holds true.

Based on (3.24) and (3.25), we may assume that up to a subsequence if necessary,

$$w_\rho \rightharpoonup \hat{w}_0 \text{ weakly in } H^1(\mathbb{R}^2, \mathbb{C}) \text{ as } \rho \to \infty,$$

(3.30)

and

$$w_\rho \to \hat{w}_0 \text{ strongly in } L^q_{loc}(\mathbb{R}^2, \mathbb{C})(2 \leq q < \infty) \text{ as } \rho \to \infty,$$

(3.31)

for some $\hat{w}_0 \in H^1(\mathbb{R}^2, \mathbb{C})$ and $\hat{w}_0 \neq 0$. From (3.24) and (3.31), we derive that

$$\int_{\mathbb{R}^2} |\hat{w}_0|^2 \, dx = \lim_{R \to \infty} \lim_{\rho \to \infty} \int_{B_R(0)} |w_\rho|^2 \, dx = \lim_{R \to \infty} \int_{B_R(0)} \left| \frac{w}{\sqrt{a^*}} \right|^2 \, dx = 1,$$

(3.32)

which together with (3.30) implies that

$$w_\rho \to \hat{w}_0 \text{ strongly in } L^2(\mathbb{R}^2, \mathbb{C}) \text{ as } \rho \to \infty.$$

(3.33)

Since $w_\rho$ is bounded uniformly in $H^1(\mathbb{R}^2, \mathbb{C})$ as $\rho \to \infty$, using the interpolation inequality, we derive from (3.33) that

$$w_\rho \to \hat{w}_0 \text{ strongly in } L^q(\mathbb{R}^2, \mathbb{C})(2 \leq q < \infty) \text{ as } \rho \to \infty.$$

(3.34)

By the weak lower semicontinuity, (1.17), (3.1), (3.28) and (3.34), we deduce from (3.19) that

$$\hat{I}(\sqrt{a^*}) = \frac{3-p}{2} \geq \lim_{\rho \to \infty} \int_{\mathbb{R}^2} \left( |\nabla w_\rho|^2 - \frac{2(a^*)^{p-1}}{p+1} |w_\rho|^{p+1} \right) \, dx$$

$$\geq \int_{\mathbb{R}^2} \left( |\nabla \hat{w}_0|^2 - \frac{2(a^*)^{p-1}}{p+1} |\hat{w}_0|^{p+1} \right) \, dx$$

$$\geq \hat{I}(\sqrt{a^*}),$$

(3.35)

which then yields

$$\lim_{\rho \to \infty} \int_{\mathbb{R}^2} |\nabla w_\rho|^2 = \int_{\mathbb{R}^2} |\nabla \hat{w}_0|^2 \, dx.$$

(3.36)

Using (3.24) and (3.36), we obtain that

$$\int_{\mathbb{R}^2} |\nabla \hat{w}_0|^2 = \int_{\mathbb{R}^2} |\nabla \hat{w}_0|^2 \, dx, \text{ i.e., } |\nabla \hat{w}_0| = |\nabla \hat{w}_0| \text{ a.e. in } \mathbb{R}^2.$$
We thus conclude from (3.24), (3.34), (3.36) and (3.37) that
\[
\rho \to \hat{w}_0 = \frac{w}{\sqrt{a^*}} e^{i\sigma} \text{ strongly in } H^1(\mathbb{R}^2, \mathbb{C}) \quad \text{as } \rho \to \infty \quad (3.38)
\]
for some \( \sigma \in \mathbb{R} \). Furthermore, it follows from (3.13) that \( \sigma = 0 \). Since the convergence of (3.38) is independent of the choice of the subsequence, we conclude that (3.38) holds for the whole sequence and hence (3.10) holds. From (3.4) and (3.10), we obtain that \( \lim_{\rho \to \infty} V_{\Omega}(z_\rho) = 0 \).

3. From (1.10), (1.13), (3.8) and (3.34), we deduce that as \( \rho \to \infty \),
\[
\mu_\rho \epsilon_\rho^2 = \epsilon_\rho^2 [I(\rho) - \frac{p-1}{p+1} \rho^{p-1} \int_{\mathbb{R}^2} |u_\rho|^{p+1} dx] \\
= \epsilon_\rho^2 I(\rho) - \frac{p-1}{p+1} (a^*)^{\frac{p-2}{2}} \int_{\mathbb{R}^2} |w_\rho|^{p+1} dx \quad (3.39)
\]
\[
\to -1.
\]
This therefore completes the proof of Lemma 3.3.

**Lemma 3.4.** Under the assumptions of Theorem 1.2 and let \( u_\rho \) be a minimizer of \( I(\rho) \) and \( w_\rho \) be defined in Lemma 3.3. Then we have

1. \( w_\rho \) decays exponentially in the sense that
\[
|w_\rho(x)| \leq C e^{-\frac{4}{3}|x|} \text{ in } \mathbb{R}^2 / B_\rho(0) \text{ as } \rho \to \infty, \quad (3.40)
\]
where \( C > 0 \) is a constant independent of \( \rho \) and \( R > 0 \).

2. The global maximum point \( z_\rho \) of \( |u_\rho| \) is unique as \( \rho \to \infty \), and \( w_\rho(x) \) satisfies
\[
w_\rho(x) \to \frac{w(x)}{\sqrt{a^*}} \text{ uniformly in } L^\infty(\mathbb{R}^2, \mathbb{C}) \text{ as } \rho \to \infty. \quad (3.41)
\]

Since the proof of this lemma is similar to those in [18, Proposition 3.3], we omit it here.

**Proof of Theorem 1.2.** In view of Lemmas 3.3 and 3.4 in order to complete the proof of Theorem 1.2 it remains to prove that
\[
\lim_{\rho \to \infty} \frac{z_\rho}{\epsilon_\rho} = y_0, \quad (3.42)
\]
where \( z_\rho \) is the unique global maximum point of \( |u_\rho|, \epsilon_\rho \) and \( y_0 \) are defined by (1.16) and \( (V_2) \) respectively.
Setting \( \tilde{u}_\rho(x) := \frac{1}{\sqrt{\alpha^*}} \epsilon^{-1}_\rho w(\epsilon^{-1}_\rho x - y_0) \). Recall from \((3.2)\) that \( \tilde{u}_\rho \) is a nonnegative minimizer of \( \hat{I}(\rho) \), it then follows from \((1.9)\), \((V_1)\) that as \( \rho \to \infty \),

\[
I(\rho) - \hat{I}(\rho) \leq E_\rho(\tilde{u}_\rho(x)e^{i\frac{\epsilon}{\alpha^*}x \cdot y_0}) - \hat{E}_\rho(\tilde{u}_\rho(x))
= \frac{\epsilon^2}{a^*} \frac{\Omega^2}{4} \int_{\mathbb{R}^2} |x|^2 w^2(x)dx + \frac{1}{a^*} \int_{\mathbb{R}^2} V_\Omega(\epsilon\rho x + \epsilon\rho y_0)w^2(x)dx
= \begin{cases} \frac{\epsilon^2}{a^*}(1 + o(1)) \int_{\mathbb{R}^2} h(x + y_0)w^2(x)dx & \text{if } 1 < s < 2, \\
\frac{\epsilon^2}{a^*}(1 + o(1)) \int_{\mathbb{R}^2} \frac{\Omega^2}{4} |x|^2 w^2(x) + h(x + y_0)w^2(x)dx & \text{if } s = 2.
\end{cases}
\]

(3.43)

On the other hand, we derive from \((1.2)\), \((1.17)\) and \((3.8)\) that as \( \rho \to \infty \),

\[
I(\rho) - \hat{I}(\rho) \geq E_\rho(u_\rho) - \hat{E}_\rho(|u_\rho|)
\geq \int_{\mathbb{R}^2} V_\Omega(\epsilon\rho x + z_\rho)|w_\rho|^2dx + \frac{\Omega^2}{4} \epsilon^2_\rho \int_{\mathbb{R}^2} |x|^2 |w_\rho|^2dx - \Omega \int_{\mathbb{R}^2} x^\perp \cdot (iw_\rho, \nabla w_\rho)dx.
\]

(3.44)

Next, we claim that

\[
\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iw_\rho, \nabla w_\rho)dx = o(\epsilon^{1+s}_\rho) \text{ as } \rho \to \infty.
\]

(3.45)

Actually, one can derive from \((1.2)\), \((3.1)\), \((3.8)\) and \((3.43)\) that

\[
C \epsilon^{2+s}_\rho \geq \epsilon^2_\rho I(\rho) - \epsilon^2_\rho \hat{I}(\rho)
\geq \int_{\mathbb{R}^2} (|\nabla w_\rho|^2dx - \frac{2(a^*)^{\frac{s-1}{p}}}{p+1} \int_{\mathbb{R}^2} |w_\rho|^{p+1}dx) - \epsilon^2_\rho \Omega \int_{\mathbb{R}^2} x^\perp \cdot (iw_\rho, \nabla w_\rho)dx
+ \frac{3 - p}{2} \text{ as } \rho \to \infty.
\]

(3.46)

Note from \((3.12)\), without loss of generality, we assume that \( \|R_\rho\|^2_{L^2} = \lambda, \|I_\rho\|^2_{L^2} = 1 - \lambda \), where \( \lambda \in (0, 1] \). In addition, it follows from \((3.41)\) that \( R_\rho \to \frac{w}{\sqrt{\alpha^*}} \) and \( I_\rho \to 0 \) uniformly in \( \mathbb{R}^2 \) as \( \rho \to \infty \). Then, we derive from \((1.17)\), \((3.1)\), \((3.2)\) and above that
as $\rho \to \infty$,}

$$
\int_{\mathbb{R}^2} |\nabla w_\rho|^2 dx - \frac{2(a^*)^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}^2} |w_\rho|^{p+1} dx
= \int_{\mathbb{R}^2} \left[ |\nabla R_\rho|^2 + |\nabla I_\rho|^2 \right] dx - \frac{2(a^*)^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}^2} \left[ R_\rho^{p+1} + \frac{p+1}{2} R_\rho^p I_\rho^2 + o(I_\rho^2) \right] dx
= \lambda \left( \int_{\mathbb{R}^2} |\nabla R_\rho|^2 \frac{dx}{\sqrt{\lambda}} - \frac{2(a^*)^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}^2} |\nabla I_\rho|^2 \frac{dx}{\sqrt{\lambda}} \right) + \frac{2(a^*)^{\frac{p+1}{2}}}{p+1} (\lambda - \lambda^{\frac{p+1}{2}}) \int_{\mathbb{R}^2} |\nabla I_\rho|^2 \frac{dx}{\sqrt{\lambda}}
+ \int_{\mathbb{R}^2} |\nabla I_\rho|^2 dx - (a^*)^{\frac{p+1}{2}} \int_{\mathbb{R}^2} R_\rho^{p-1} I_\rho^2 dx + o(\|I_\rho\|_2^2)
\geq \lambda \hat{I}(\sqrt{a^*}) + \frac{2(a^*)^{\frac{p+1}{2}}}{p+1} (1 - \lambda) \int_{\mathbb{R}^2} |\nabla I_\rho|^2 \frac{dx}{\sqrt{\lambda}}
+ \int_{\mathbb{R}^2} |\nabla I_\rho|^2 dx - (a^*)^{\frac{p+1}{2}} \int_{\mathbb{R}^2} R_\rho^{p-1} I_\rho^2 dx + o(\|I_\rho\|_2^2)
= -\frac{3-p}{2} + \int_{\mathbb{R}^2} I_\rho^2 dx + \int_{\mathbb{R}^2} \nabla I_\rho^2 dx - \int_{\mathbb{R}^2} w^{p-1} I_\rho^2 dx + o(\|I_\rho\|_2^2), \quad (3.47)
$$

where $\|R_\sqrt{\lambda}\|_2^2 = 1$ is used in the “$\geq$”. On the other hand, it follows from (3.40) and (3.41) that

$$
\epsilon_\rho^2 \Omega \int_{\mathbb{R}^2} x^\perp \cdot (iw_\rho, \nabla w_\rho) dx = \epsilon_\rho^2 \Omega \int_{\mathbb{R}^2} x^\perp \cdot (R_\rho \nabla I_\rho - I_\rho \nabla R_\rho) dx
= 2\epsilon_\rho^2 \Omega \int_{\mathbb{R}^2} x^\perp \cdot (R_\rho \nabla I_\rho) dx \leq C\epsilon_\rho^2 \|\nabla I_\rho\|_{L^2}. \quad (3.48)
$$

Combining (3.46)–(3.48), we obtain that

$$
C\epsilon_\rho^{2+s} \geq \int_{\mathbb{R}^2} I_\rho^2 dx + \int_{\mathbb{R}^2} |\nabla I_\rho|^2 dx - \int_{\mathbb{R}^2} w^{p-1} I_\rho^2 dx + o(\|I_\rho\|_2^2) - C\epsilon_\rho^2 \|\nabla I_\rho\|_{L^2}
= (\mathcal{L} I_\rho, I_\rho) + o(\|I_\rho\|_2^2) - C\epsilon_\rho^2 \|\nabla I_\rho\|_{L^2}, \quad (3.49)
$$

where the linearized operator $\mathcal{L}$ is defined by

$$
\mathcal{L} := -\Delta - w^{p-1} + 1. \quad (3.50)
$$

From [27] Corollary 11.9 and Theorem 11.8] and (3.13), we deduce that there exists a constant $M > 0$ such that

$$
(\mathcal{L} I_\rho, I_\rho) \geq M\|I_\rho\|_{H^1(\mathbb{R}^2)}^2 \text{ as } \rho \to \infty. \quad (3.51)
$$

Substituting (3.51) into (3.49), we have

$$
C\epsilon_\rho^{2+s} \geq \frac{M}{2} \|I_\rho\|_{H^1(\mathbb{R}^2)}^2 + o(\|I_\rho\|_2^2) - C\epsilon_\rho^2 \|\nabla I_\rho\|_{L^2} \text{ as } \rho \to \infty. \quad (3.52)
$$
Then, we deduce from \((3.52)\) that
\[
\left\| I_\rho \right\|_{H^1(\mathbb{R}^2)} \leq C \epsilon_\rho^{1 + \frac{s}{2}} \quad \text{as} \quad \rho \to \infty.
\] (3.53)

Applying \((3.40)\), \((3.41)\) and \((3.53)\), one can deduce that
\[
\int_{\mathbb{R}^2} x^\perp \cdot (iw_\rho, \nabla w_\rho) \, dx = 2 \int_{\mathbb{R}^2} R_\rho(x^\perp \cdot \nabla I_\rho) \, dx = 2 \int_{\mathbb{R}^2} \frac{w}{\sqrt{a^2}} (x^\perp \cdot \nabla I_\rho) \, dx + o(\epsilon_\rho^{1 + \frac{s}{2}})
\] (3.54)
\[
= 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} I_\rho(x^\perp \cdot \nabla \frac{w}{\sqrt{a^2}}) \, dx + o(\epsilon_\rho^{1 + \frac{s}{2}}) = o(\epsilon_\rho^{1 + \frac{s}{2}}) \quad \text{as} \quad \rho \to \infty,
\]
which yields \((3.45)\). It then follows from \((3.44)\) and \((3.45)\) that as \(\rho \to \infty\),
\[
I(\rho) - \hat{I}(\rho)
\geq \int_{\mathbb{R}^2} V_\Omega(\epsilon_\rho x + z_\rho) |w_\rho|^2 \, dx + \frac{\Omega^2}{4} \epsilon_\rho^2 \int_{\mathbb{R}^2} |x|^2 |w_\rho|^2 \, dx + o(\epsilon_\rho^{1 + \frac{s}{2}})
\]
\[
= \left\{ \begin{array}{ll}
\frac{s^2}{a^2} (1 + o(1)) \int_{\mathbb{R}^2} h(x + \frac{z_\rho}{\epsilon_\rho}) w^2(x) \, dx & \text{if} \quad 1 < s < 2, \\
\frac{s^2}{a^2} (1 + o(1)) \int_{\mathbb{R}^2} \left( \frac{\Omega^2}{4} |x|^2 w^2(x) + h(x + \frac{z_\rho}{\epsilon_\rho}) w^2(x) \right) \, dx & \text{if} \quad s = 2.
\end{array} \right.
\] (3.55)

Since \(V_\Omega(x) \to \infty\) as \(|x| \to \infty\), one can check that \(\left\{ \frac{z_\rho}{\epsilon_\rho} \right\}\) is bounded uniformly in \(\rho\). Combining \((3.43)\) and \((3.55)\), one can verify that, passing to a subsequence if necessary,
\[
\lim_{\rho \to \infty} \frac{z_\rho}{\epsilon_\rho} = y_0.
\] (3.56)

Moreover, since the convergence \((3.56)\) is independent of what subsequence \(\left\{ \frac{z_\rho}{\epsilon_\rho} \right\}\) we choose, hence \((3.56)\) holds for whole sequence, which implies that \((3.42)\) holds true. The proof of Theorem 1.2 is thus completed. 

\[\square\]

4 Local uniqueness of minimizers

In this section, we shall focus on the proof of the local uniqueness of minimizers for \(I(\rho)\) as \(\rho \to \infty\). By contradiction, suppose that there exist two different minimizers \(u_{1\rho}\) and \(u_{2\rho}\) of \(I(\rho)\) as \(\rho \to \infty\) in the sense that \(u_{1\rho} \neq u_{2\rho} e^{i\theta}\) for any constant phase \(\theta = \theta(\rho) \in [0, 2\pi]\). Define for \(j = 1, 2\),
\[
\tilde{u}_{j\rho}(x) := \epsilon_\rho u_{j\rho}(\epsilon_\rho(x + y_0)) e^{-i(\frac{z_\rho}{2} x - y_0 - \varphi_{j\rho})} = \tilde{R}_{j\rho}(x) + i \tilde{I}_{j\rho}(x),
\] (4.1)

where \(\epsilon_\rho\) is given in \((1.16)\), \(y_0\) is defined by \((V_2)\), \(\tilde{R}_{j\rho}(x)\) and \(\tilde{I}_{j\rho}(x)\) denote the real and imaginary parts of \(\tilde{u}_{j\rho}(x)\), respectively, and the constant phase \(\varphi_{j\rho} \in [0, 2\pi]\) is chosen properly such that
\[
\int_{\mathbb{R}^2} w(x) \tilde{I}_{j\rho}(x) \, dx = 0, \quad j = 1, 2.
\] (4.2)
From (4.12) and (4.14), we obtain that $\tilde{\eta}_j(x)$ satisfies the following equation

$$
- \Delta \tilde{u}_{j\rho} + i\epsilon^2 \Omega(x^+) \cdot \nabla \tilde{u}_{j\rho} + \left[ \frac{\epsilon^4 \Omega^2 |x|^2}{4} + \epsilon^2 V_{\Omega}(\epsilon_{\rho}(x + y_0)) \right] \tilde{u}_{j\rho} = \epsilon^2 \mu_{j\rho} \tilde{u}_{j\rho} + (a^+)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} \tilde{u}_{j\rho} \text{ in } \mathbb{R}^2, \ j = 1, 2,
$$

(4.3)

where $\mu_{j\rho} \in \mathbb{R}$ satisfies

$$
\mu_{j\rho} = I(\rho) - \frac{(p-1)(a^+)^{\frac{p-1}{2}}}{(p+1)\epsilon^2} \int_{\mathbb{R}^2} |\tilde{u}_{j\rho}|^{p+1} dx.
$$

(4.4)

Furthermore, it follows from Lemmas 3.3 and 3.4 the following Proposition.

**Proposition 4.1.** Under the assumption of Theorem 1.3, let $\tilde{u}_{j\rho}$ be defined by (4.1), then we have

1. The function $\tilde{u}_{j\rho}$ satisfies

$$
\lim_{\rho \to \infty} \tilde{u}_{j\rho}(x) = \frac{w(x)}{\sqrt{a^+}} \text{ strongly in } H^1(\mathbb{R}^2, \mathbb{C}) \cap L^\infty(\mathbb{R}^2, \mathbb{C}),
$$

(4.5)

where $a^+ := \|w\|_2^2$ and $w$ is the unique positive solution of (1.8).

2. $\mu_{j\rho} \epsilon^2 \rightarrow -1$ as $\rho \to \infty$, and $\tilde{u}_{j\rho}$ decays exponentially in the sense that

$$
|\tilde{u}_{j\rho}(x)| \leq Ce^{-\frac{2}{C}|x|}, \quad |\nabla \tilde{u}_{j\rho}(x)| \leq Ce^{-\frac{1}{c}|x|} \text{ uniformly in } \mathbb{R}^2 \text{ as } \rho \to \infty,
$$

(4.6)

where $C > 0$ is a constant independent of $0 < \rho < \infty$.

Since $u_{1\rho} \neq u_{2\rho} e^{i\theta}$ holds for any constant phase $\theta = \theta(\rho) \in [0, 2\pi)$, we define

$$
\tilde{\eta}_\rho(x) := \frac{\tilde{u}_{2\rho}(x) - \tilde{u}_{1\rho}(x)}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)}} = \tilde{\eta}_{1\rho}(x) + i\tilde{\eta}_{2\rho}(x),
$$

(4.7)

where $\tilde{\eta}_{1\rho}(x)$ and $\tilde{\eta}_{2\rho}(x)$ denote the real and imaginary parts of $\tilde{\eta}_\rho(x)$, respectively. From (4.3) and (4.7), we derive that $\tilde{\eta}_\rho$ satisfies

$$
- \Delta \tilde{\eta}_\rho + i\epsilon^2 \Omega(x^+) \cdot \nabla \tilde{\eta}_\rho + \epsilon^2 V_{\Omega}(\epsilon_{\rho}(x + y_0)) \tilde{\eta}_\rho + \frac{\epsilon^4 \Omega^2 |x|^2}{4} \tilde{\eta}_\rho = \epsilon^2 \mu_{j\rho} \tilde{\eta}_\rho + (a^+) \left( \frac{p-1}{2} |\tilde{u}_{2\rho}|^{p-1} \tilde{\eta}_\rho - \frac{p-1}{p+1} \tilde{u}_{2\rho} \int_{\mathbb{R}^2} \tilde{C}_\rho(x) \left( |\tilde{u}_{2\rho}|^{\frac{p+1}{2}} + |\tilde{u}_{1\rho}|^{\frac{p+1}{2}} \right) dx \right)
$$

$$
+ \tilde{D}_\rho(x) \tilde{u}_{1\rho} \quad \text{in } \mathbb{R}^2,
$$

where

$$
\tilde{C}_\rho(x) := (a^+) \left( \frac{p+1}{2} \frac{|\tilde{u}_{2\rho}|^{\frac{p+1}{2}} - |\tilde{u}_{1\rho}|^{\frac{p+1}{2}}}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)}} \right)
$$

$$
= \frac{p+1}{4} (a^+) \left[ \tilde{\eta}_{1\rho} (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) + \tilde{\eta}_{2\rho} (\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \right] \cdot \int_0^1 \left[ t|\tilde{u}_{2\rho}|^2 + (1-t)|\tilde{u}_{1\rho}|^2 \right]^{\frac{p-3}{4}} dt,
$$

(4.8)
and
\[
\tilde{D}_\rho(x) := (a^*)^{\frac{p-1}{2}} \frac{|\bar{u}_{2\rho}|^{p-1} - |\bar{u}_{1\rho}|^{p-1}}{\|\bar{u}_{2\rho} - \bar{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)}} = \frac{p-1}{2} (a^*)^{\frac{p-1}{2}} \left[ \tilde{\eta}_{1\rho}(\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) + \tilde{\eta}_{2\rho}(\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \right] + \int_0^1 \left[ t|\tilde{u}_{2\rho}|^2 + (1-t)|\tilde{u}_{1\rho}|^2 \right]^\frac{p-3}{2} dt. \tag{4.10}
\]

### 4.1 The convergence of \( \tilde{\eta}_\rho \) as \( \rho \to \infty \)

In this subsection, we shall give the convergence of \( \tilde{\eta}_\rho \) in (4.8) as \( \rho \to \infty \). We first address the following \( L^\infty \)-uniform estimates of \( \tilde{\eta}_\rho \) and \( \nabla \tilde{\eta}_\rho \) as \( \rho \to \infty \).

**Lemma 4.2.** Suppose that \( \tilde{\eta}_\rho \) is defined by (4.7), then there exists a constant \( C > 0 \), independent of \( 0 < \rho < \infty \), such that
\[
|\tilde{\eta}_\rho(x)| \leq Ce^{-\frac{1}{4}|x|}, \quad |\nabla \tilde{\eta}_\rho(x)| \leq Ce^{-\frac{1}{4}|x|} \quad \text{uniformly in } \mathbb{R}^2 \text{ as } \rho \to \infty. \tag{4.11}
\]

The proof of Lemma 4.2 is similar to those in [17, Lemma 3.2], so we omit it here.

Based on Lemma 4.2, we now give the following convergence of \( \tilde{\eta}_\rho \) as \( \rho \to \infty \).

**Lemma 4.3.** Suppose \( \tilde{\eta}_\rho \) is defined by (4.7). Then passing to a subsequence if necessary, there exist some constants \( b_0, b_1 \) and \( b_2 \) such that as \( \rho \to \infty \),
\[
\tilde{\eta}_\rho(x) \to b_0 (w + \frac{p-1}{2} x \cdot \nabla w) + \sum_{j=1}^2 b_j \frac{\partial w}{\partial x_j} + M_\rho(x) \quad \text{uniformly in } \mathbb{R}^2, \tag{4.12}
\]
where \( M_\rho(x) \) satisfies that as \( \rho \to \infty \),
\[
|M_\rho(x)| \leq C_M(\epsilon_\rho)e^{-\frac{1}{4}|x|}, \quad |\nabla M_\rho(x)| \leq C_M(\epsilon_\rho)e^{-\frac{1}{4}|x|} \quad \text{uniformly in } \mathbb{R}^2, \tag{4.13}
\]
for some constants \( C_M(\epsilon_\rho) > 0 \) satisfying \( C_M(\epsilon_\rho) = o(1) \) as \( \rho \to \infty \).

**Proof.** Firstly, we claim that passing to a subsequence if necessary, there exist \( b_0, b_1 \) and \( b_2 \) such that as \( \rho \to \infty \),
\[
\tilde{\eta}_\rho(x) \to b_0 (w + \frac{p-1}{2} x \cdot \nabla w) + \sum_{j=1}^2 b_j \frac{\partial w}{\partial x_j} \quad \text{uniformly in } C^1_{loc}(\mathbb{R}^2, \mathbb{C}). \tag{4.14}
\]

In fact, we note from Lemma 4.2 that \( (x^\perp \cdot \nabla \tilde{\eta}_\rho) \) is bounded uniformly and decays exponentially for sufficiently large \( |x| \) as \( \rho \to \infty \). By the definition of \( \tilde{C}_\rho(x) \) and \( \tilde{D}_\rho(x) \), we obtain from Proposition 4.1 that there exists a constant \( C > 0 \) such that
\[
\|\tilde{D}_\rho(x)\|_{L^\infty(\mathbb{R}^2)} \leq C \quad \text{and} \quad \left| \int_{\mathbb{R}^2} \tilde{C}_\rho(x) (|\tilde{u}_{2\rho}|^{\frac{p-1}{2}} + |\tilde{u}_{1\rho}|^{\frac{p-1}{2}}) dx \right| \leq C. \tag{4.15}
\]
Using the standard elliptic regularity, one then obtain from (4.8) and (4.15) that
\( \tilde{\eta}_p \in C^{1,\alpha}_{loc} \) and \( \| \tilde{\eta}_p \|_{C^{1,\alpha}_{loc}} \leq C \) uniformly as \( \rho \to \infty \) for some \( \alpha \in (0,1) \). By (4.8), we obtain from above that passing to a subsequence if necessary,
\[
\tilde{\eta}_p := \tilde{\eta}_{1,\rho} + i\tilde{\eta}_{2,\rho} \to \tilde{\eta}_0 := \tilde{\eta}_1 + i\tilde{\eta}_2 \text{ uniformly in } C^{1}_{loc}(\mathbb{R}^2, \mathbb{C}) \text{ as } \rho \to \infty,
\]
where \( \tilde{\eta}_0 \) solves
\[
-\Delta \tilde{\eta}_0 + \tilde{\eta}_0 - (p-1)w^{p-1}\tilde{\eta}_1 - w^{p-1}\tilde{\eta}_0 = -\left( \frac{p-1}{a^*} \int_{\mathbb{R}^2} w^p \tilde{\eta}_1 dx \right) w \text{ in } \mathbb{R}^2.
\]
This implies that \( (\tilde{\eta}_1, \tilde{\eta}_2) \) satisfies
\[
\begin{align*}
\mathcal{N}\tilde{\eta}_1 &= -\left( \frac{p-1}{a^*} \int_{\mathbb{R}^2} w^p \tilde{\eta}_1 dx \right) w \text{ in } \mathbb{R}^2, \\
\mathcal{L}\tilde{\eta}_2 &= 0 \text{ in } \mathbb{R}^2,
\end{align*}
\]
where \( \mathcal{L} \) is defined in (3.50) and the linearized operator \( \mathcal{N} \) is defined by
\[
\mathcal{N} := -\Delta - pw^{p-1} + 1.
\]
Recall from [24,30] that
\[
\ker \mathcal{N} = \left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2} \right\}.
\]
Furthermore, one can also check that
\[
\mathcal{N} \left( w + \frac{p-1}{2} x \cdot \nabla w \right) = -(p-1)w.
\]
Note from (4.2) and (4.7) that \( \int_{\mathbb{R}^2} w(x)\tilde{\eta}_{2,\rho}(x)dx = 0 \), which together with (4.16) implies that
\[
\int_{\mathbb{R}^2} w(x)\tilde{\eta}_2(x)dx = 0.
\]
From [27] Corollary 11.9 and Theorem 11.8 and (4.18)-(4.21), one then derive that
\[
\tilde{\eta}_1 = b_0 \left( w + \frac{p-1}{2} x \cdot \nabla w \right) + \sum_{j=1}^{2} b_j \frac{\partial w}{\partial x_j} \text{ and } \tilde{\eta}_2 \equiv 0 \text{ in } \mathbb{R}^2,
\]
which implies the claim (4.14) holds true.

On the other hand, using (1.9) and (4.11), we deduce that for any fixed sufficiently large \( R > 0 \), as \( \rho \to \infty \),
\[
\left| \tilde{\eta}_1 - \left[ b_0 \left( w + \frac{p-1}{2} x \cdot \nabla w \right) + \sum_{j=1}^{2} b_j \frac{\partial w}{\partial x_j} \right] \right| \leq Ce^{-\frac{1}{4}R}e^{-\frac{1}{4}|x|} \text{ in } \mathbb{R}^2/B_R(0),
\]
and
\[
\left| \nabla \tilde{\eta}_1 - \nabla \left[ b_0 \left( w + \frac{p-1}{2} x \cdot \nabla w \right) + \sum_{j=1}^{2} b_j \frac{\partial w}{\partial x_j} \right] \right| \leq Ce^{-\frac{1}{4}R}e^{-\frac{1}{4}|x|} \text{ in } \mathbb{R}^2/B_R(0).
\]
Because \( R > 0 \) is arbitrary, combining (4.14), (4.22) and (4.23), one then conclude that (4.13) holds and we are done.
4.2 The refined estimate of $\tilde{\eta}_\rho$ in (4.8)

In this subsection, the refined estimate of $\tilde{\eta}_\rho$ as $\rho \to \infty$ shall be established in Lemma 4.5. In order to prove Lemma 4.5, we shall firstly give the following estimates of the imaginary part $\tilde{I}_{j\rho}$ for $\tilde{u}_{j\rho}$ as $\rho \to \infty$.

**Lemma 4.4.** Under the assumptions of Theorem 1.3, let $\tilde{I}_{j\rho}$ be defined in (4.1) for $j = 1, 2$. Then as $\rho \to \infty$,

$$|\tilde{I}_{j\rho}(x)| \leq C_{j1}(\epsilon_\rho)e^{-\frac{1}{2}|x|}, \quad |\nabla \tilde{I}_{j\rho}(x)| \leq C_{j2}(\epsilon_\rho)e^{-\frac{1}{3}|x|} \text{ uniformly in } \mathbb{R}^2,$$  \hspace{1cm} (4.24)

where the constants $C_{j1}(\epsilon_\rho), C_{j2}(\epsilon_\rho) > 0$ satisfy

$$C_{j1}(\epsilon_\rho) = o(\epsilon_\rho^2) \quad \text{and} \quad C_{j2}(\epsilon_\rho) = o(\epsilon_\rho^2) \quad \text{as} \quad \rho \to \infty. \hspace{1cm} (4.25)$$

**Proof.** Note from (4.1) and (4.3) that the imaginary part $\tilde{I}_{j\rho}$ of $\tilde{u}_{j\rho}$ satisfies

$$L_\rho \tilde{I}_{j\rho}(x) = -\epsilon_\rho^2 \Omega(x^1 \cdot \nabla \tilde{R}_{j\rho}) \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \tilde{I}_{j\rho}(x)w(x)dx = 0, \hspace{1cm} (4.26)$$

where $L_\rho$ is defined as

$$L_\rho := -\Delta + \frac{\epsilon_\rho^2 \Omega |x|^2}{4} + \epsilon_\rho^2 V_\rho(x + y_\rho) - \epsilon_\rho^2 \mu_{j\rho} - (a^*)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1}. \hspace{1cm} (4.27)$$

Applying (1.9), (1.5) and (4.6), we deduce that as $\rho \to \infty$,

$$|\epsilon_\rho^2 \Omega(x^1 \cdot \nabla \tilde{R}_{j\rho}| \leq C(\epsilon_\rho)e^{-\frac{1}{4}|x|} \quad \text{uniformly in } \mathbb{R}^2, \hspace{1cm} (4.28)$$

where the constant $C(\epsilon_\rho) > 0$ satisfies $C(\epsilon_\rho) = o(\epsilon_\rho^2)$ as $\rho \to \infty$.

Based on (4.28), we next prove (4.24) and (4.25). Multiplying (4.26) by $\tilde{I}_{j\rho}$ and integrating over $\mathbb{R}^2$, using the Hölder inequality, we obtain from (4.28) that

$$\int_{\mathbb{R}^2} (L_\rho \tilde{I}_{j\rho})\tilde{I}_{j\rho}dx = -\epsilon_\rho^2 \Omega \int_{\mathbb{R}^2} (x^1 \cdot \nabla \tilde{R}_{j\rho})\tilde{I}_{j\rho}dx = o(\epsilon_\rho^2)|\tilde{I}_{j\rho}|^2_{L^2(\mathbb{R}^2)} \text{ as } \rho \to \infty. \hspace{1cm} (4.29)$$

From (3.51), (4.2) and (4.5), we derive that as $\rho \to \infty$,

$$\int_{\mathbb{R}^2} (L_\rho \tilde{I}_{j\rho})\tilde{I}_{j\rho}dx \geq \int_{\mathbb{R}^2} \left[ (L \tilde{I}_{j\rho})\tilde{I}_{j\rho} - (1 + \epsilon_\rho^2 \mu_{j\rho})\tilde{I}_{j\rho}^2 - ((a^*)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} - w^{p-1})\tilde{I}_{j\rho}^2 \right] dx \hspace{1cm} (4.30)$$

where $L$ is defined in (3.50) and $M > 0$, which is independent of $0 < \rho < \infty$, is given by (3.51). It then follows from (4.29) and (4.30) that

$$\|\tilde{I}_{j\rho}\|_{H^1(\mathbb{R}^2)} = o(\epsilon_\rho^2) \quad \text{as} \quad \rho \to \infty. \hspace{1cm} (4.31)$$
On the other hand, we deduce from (4.29) that $|\tilde{T}_{j\rho}|^2$ satisfies the following equation
\[
\left[-\frac{1}{2}\Delta + \left(\frac{\epsilon_5^2|\rho|^2}{4} + \epsilon_6^2 \Omega(\rho(x + y_0)) - \epsilon_7^2 \mu_{j\rho} - (a^*)^{\frac{p-1}{2}}|\tilde{u}_{j\rho}|^{p-1}\right)\right]|\tilde{T}_{j\rho}|^2 + |\nabla \tilde{T}_{j\rho}|^2
= -\epsilon_8^2 \Omega(x^\perp \cdot \nabla \tilde{R}_{j\rho})\tilde{T}_{j\rho} \quad \text{in } \mathbb{R}^2,
\]
which yields that
\[
-\frac{1}{2}\Delta |\tilde{T}_{j\rho}|^2 - \epsilon^2 \mu_{j\rho} |\tilde{T}_{j\rho}|^2 - (a^*)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} |\tilde{T}_{j\rho}|^2 \leq -\epsilon^2 \Omega(x^\perp \cdot \nabla \tilde{R}_{j\rho})\tilde{T}_{j\rho} \quad \text{in } \mathbb{R}^2. \tag{4.32}
\]
Since $\mu_{j\rho} \epsilon^2 \to -1$ as $\rho \to \infty$, by De Giorgi-Nash-Moser theory [23] Theorem 4.1], we deduce from (4.32) that for any $\xi \in \mathbb{R}^2$,
\[
\sup_{x \in B_1(\xi)} |\tilde{T}_{j\rho}(x)|^2 \leq C \left(\|\tilde{T}_{j\rho}\|^2_{L^2(B_2(\xi))} + \|\epsilon^2 \Omega(x^\perp \cdot \nabla \tilde{R}_{j\rho})\tilde{T}_{j\rho}\|_{L^\infty(B_2(\xi))}\right). \tag{4.33}
\]
Employing Proposition 4.1, we then derive from (4.28), (4.31) and (4.33) that
\[
\|\tilde{T}_{j\rho}\|_{L^\infty(\mathbb{R}^2)} = o(\epsilon^2) \quad \text{as } \rho \to \infty, \tag{4.34}
\]
and hence
\[
|(a^*)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} |\tilde{T}_{j\rho} - \epsilon^2 \Omega(x^\perp \cdot \nabla \tilde{R}_{j\rho})| \leq C_0(\epsilon_8) e^{-\frac{1}{4}|x|} \quad \text{uniformly in } \mathbb{R}^2, \tag{4.35}
\]
where the constant $C_0(\epsilon_8) > 0$ satisfies $C_0(\epsilon_8) = o(\epsilon^2)$ as $\rho \to \infty$. By the comparison principle, we thus deduce from (4.32), (4.34) and (4.35) that
\[
|\tilde{T}_{j\rho}(x)| \leq C_{j1}(\epsilon_8) e^{-\frac{1}{4}|x|} \quad \text{uniformly in } \mathbb{R}^2 \quad \text{as } \rho \to \infty, \tag{4.36}
\]
where the constant $C_{j1}(\epsilon_8) > 0$ satisfies $C_{j1}(\epsilon_8) = o(\epsilon^2)$ as $\rho \to \infty$. Furthermore, by the exponential decay (4.36), applying gradient estimates (see (3.15) in [13]) to (4.26) then yields that the gradient estimate of (4.24) and (4.25) hold true. The proof of Lemma 4.4 is thus completed. \hfill \Box

Combining Lemma 4.3 and Lemma 4.4, we now establish the refined estimates of $\tilde{\eta}_\rho$ as $\rho \to \infty$.

**Lemma 4.5.** Suppose $\{\tilde{\eta}_\rho\}$ is the sequence obtained in Lemma 4.3. Then the imaginary $\tilde{\eta}_{2\rho}$ of $\tilde{\eta}_\rho$ satisfies that as $\rho \to \infty$,
\[
\tilde{\eta}_{2\rho}(x) = \frac{2^2 \Omega}{2} (-b_1 x_2 + b_2 x_1) w(x) + E_\rho(x) \quad \text{uniformly in } \mathbb{R}^2, \tag{4.37}
\]
where $(x_1, x_2) = x \in \mathbb{R}^2$, $b_1$ and $b_2$ are as in Lemma 4.3 and $E_\rho(x)$ satisfies that as $\rho \to \infty$,
\[
|E_\rho(x)| \leq C_E(\epsilon_8) e^{-\frac{1}{4}|x|}, |\nabla E_\rho(x)| \leq C_E(\epsilon_8) e^{-\frac{1}{16}|x|} \quad \text{uniformly in } \mathbb{R}^2 \tag{4.38}
\]
for some constants $C_E(\epsilon_8) > 0$ satisfying $C_E(\epsilon_8) = o(\epsilon^2)$ as $\rho \to \infty$. 

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Proof. We first obtain from (4.2) and (4.8) that
\[
L_2^2 \tilde{\eta} \tilde{\eta} = -\varrho^2 \Omega (x^+ \cdot \nabla \tilde{\eta}) + G_\varphi (x) \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \tilde{\eta} w dx = 0, \tag{4.39}
\]
where \( L_j \) is defined for \( j = 1, 2, \)
\[
L_j := -\Delta + \frac{\varrho^4 \Omega^2 |x|^2}{4} + \varrho^2 \Omega (\epsilon_\varphi (x + y_0)) - \varrho^2 \mu_j (a^{\pm_1} |\tilde{\eta}|^{p-1}), \tag{4.40}
\]
and \( G_\varphi (x) \) is defined as
\[
G_\varphi (x) := -\frac{p-1}{p+1} \tilde{L}_2^2 \int_{\mathbb{R}^2} \tilde{C}_\varphi (x) \left( |\tilde{u}_2|^{p+1} + |\tilde{u}_1|^{p+1} \right) dx + \tilde{D}_\varphi (x) \tilde{L}_1^2,
\]
where \( \tilde{C}_\varphi \) and \( \tilde{D}_\varphi \) are defined in (4.9) and (4.10). It then follows from Lemma 4.3 and (4.39) that
\[
L_2^2 \tilde{\eta} \tilde{\eta} = -\varrho^2 \Omega \left[ -b_1 \frac{\partial w}{\partial x_2} + b_2 \frac{\partial w}{\partial x_1} + \text{Re}(x^+ \cdot \nabla M) \right] + G_\varphi (x) \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \tilde{\eta} w dx = 0, \tag{4.41}
\]
where \( \text{Re}(\cdot) \) denotes the real part. On the other hand, one can check that \( -\frac{1}{2} x_j w \) is the unique solution of the following equation
\[
L u = \frac{\partial w}{\partial x_j}, \quad \int_{\mathbb{R}^2} u w dx = 0, \quad j = 1, 2, \tag{4.42}
\]
where \( L \) is defined in (3.50). Denote
\[
E_\varphi (x) := \tilde{\eta} \varphi - \varphi^2 \left( -b_1 x_2 + b_2 x_1 \right) w(x).
\]
By (4.42), we then derive from (4.41) that \( E_\varphi (x) \) satisfies
\[
L_2^2 E_\varphi (x) = \frac{\varrho^2 \Omega}{2} \left( \mathcal{L} - L_2 \right) \left[ (-b_1 x_2 + b_2 x_1)w(x) \right] - \varrho^2 \Omega \text{Re}(x^+ \cdot \nabla M) + G_\varphi (x) \text{ in } \mathbb{R}^2, \tag{4.43}
\]
and
\[
\int_{\mathbb{R}^2} E_\varphi (x) w dx = 0. \tag{4.44}
\]
Now, we shall estimate the right hand side of (4.43). From the definition of \( L_2^2 \) in (4.40), we obtain that
\[
\frac{\varrho^2 \Omega}{2} \left| \left( \mathcal{L} - L_2 \right) \left[ (-b_1 x_2 + b_2 x_1)w(x) \right] \right| \leq C(\varphi_\varphi) e^{-\frac{1}{2} |x|} \text{ uniformly in } \mathbb{R}^2 \text{ as } \varrho \to \infty,
\]
where \( C(\varphi_\varphi) > 0 \) satisfies
\[
C(\varphi_\varphi) = o(\varrho^2) \text{ as } \varrho \to \infty. \tag{4.45}
\]
By Lemma 4.3, we obtain that
\[
|\text{Re}(e^{2\Omega x^\perp} \cdot \nabla M_\rho)| \leq C(\epsilon_\rho)e^{-\frac{1}{2}|x|} \text{ uniformly in } \mathbb{R}^2 \text{ as } \rho \to \infty,
\] (4.46)
where \(C(\epsilon_\rho) > 0\) also satisfies (4.45). Furthermore, we derive from Lemma 4.4 and (4.15) that
\[
|G_\rho(x)| \leq C(\epsilon_\rho)e^{-\frac{1}{2}|x|} \text{ uniformly in } \mathbb{R}^2 \text{ as } \rho \to \infty,
\] (4.47)
where \(C(\epsilon_\rho) > 0\) also satisfies (4.45). Applying the above estimates, we then derive from (4.43) that
\[
|L_{2\rho}E_\rho(x)| = \left|\frac{\epsilon^2_{\rho}}{2} (L - L_{2\rho}) [(-b_1 x_2 + b_2 x_1) w(x) - \epsilon^2_{\rho} \text{Re}(x^\perp \cdot \nabla M_\rho) + G_\rho(x)] \right|
\leq C(\epsilon_\rho)e^{-\frac{1}{2}|x|} \text{ uniformly in } \mathbb{R}^2 \text{ as } \rho \to \infty.
\] (4.48)

Similar to the argument of proving Lemma 4.4, one can conclude from (4.43), (4.44) and (4.48) that (4.37) and (4.38) hold true. The proof of Lemma 4.5 is thus completed.

In the following, we shall follow the refined estimates of \(\tilde{\eta}_\rho\) to complete the proof of Theorem 1.3 on the local uniqueness of minimizers for \(I(\rho)\) as \(\rho \to \infty\).

**Proof of Theorem 1.3:** Argue by contradiction, we assume that, up to a constant phase, there exist two different minimizers \(u_{1\rho}\) and \(u_{2\rho}\) of \(I(\rho)\) as \(\rho \to \infty\), that is, \(u_{1\rho} \neq u_{2\rho} e^{i\theta}\) for any constant phase \(\theta = \theta(\rho) \in [0, 2\pi)\). Recall from (4.3) and (4.4) that \(\tilde{u}_{j\rho}(x) := \tilde{R}_{j\rho}(x) + i\tilde{T}_{j\rho}(x)\) defined in (4.1) satisfies the following equation
\[
-\Delta \tilde{u}_{j\rho} + i\epsilon^2_{\rho} \Omega(x^\perp \cdot \nabla \tilde{u}_{j\rho}) + \left[\frac{\epsilon^4_{\rho} \Omega^2 |x|^2}{4} + \epsilon^2_{\rho} V_{\Omega}(\epsilon_{\rho} (x + y_0))\right] \tilde{u}_{j\rho} = \epsilon^2_{\rho} \mu_{j\rho} \tilde{u}_{j\rho} + (a^*)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} \tilde{u}_{j\rho} \text{ in } \mathbb{R}^2, \quad j = 1, 2,
\] (4.49)
where \(\mu_{j\rho} \in \mathbb{R}\) satisfies
\[
\mu_{j\rho} = I(\rho) - \frac{(p-1)(a^*)^{\frac{p-1}{2}}}{(p+1)\epsilon^2_{\rho}} \int_{\mathbb{R}^2} |\tilde{u}_{j\rho}|^{p+1} dx.
\] (4.50)

Then, the real part \(\tilde{R}_{j\rho}\) of \(\tilde{u}_{j\rho}\) satisfies
\[
-\Delta \tilde{R}_{j\rho} - \epsilon^2_{\rho} \Omega(x^\perp \cdot \nabla \tilde{T}_{j\rho}) + \left[\frac{\epsilon^4_{\rho} \Omega^2 |x|^2}{4} + \epsilon^2_{\rho} V_{\Omega}(\epsilon_{\rho} (x + y_0))\right] \tilde{R}_{j\rho} = \epsilon^2_{\rho} \mu_{j\rho} \tilde{R}_{j\rho} + (a^*)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} \tilde{R}_{j\rho} \text{ in } \mathbb{R}^2, \quad j = 1, 2.
\] (4.51)

To obtain a contradiction, we next prove the constants \(b_i = 0 (i = 0, 1, 2)\) defined in (4.12) by constructing Pohozaev identities of \(\tilde{R}_{j\rho}\), where \(j = 1, 2\). This process is divided into the following three steps:
Step 1. We claim that the following Pohozaev identity holds

\[ b_0 \frac{p-1}{2} \int_{\mathbb{R}^2} \frac{\partial h(x+y_0)}{\partial x_l} (x \cdot \nabla w^2) dx + \sum_{j=1}^{2} b_j \int_{\mathbb{R}^2} \frac{\partial h(x+y_0)}{\partial x_j} \frac{\partial w^2}{\partial x_j} dx = 0, \quad l = 1, 2, \quad (4.52) \]

where \( b_0, b_1, \) and \( b_2 \) are defined in (4.12).

In order to prove the claim (4.52), we first multiply (4.51) by \( \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} \) and integrating over \( \mathbb{R}^2 \), where \( j = 1, 2 \) and \( l = 1, 2 \), one then obtain

\[ -\int_{\mathbb{R}^2} \Delta \tilde{R}_{j\rho} \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} - \int_{\mathbb{R}^2} \epsilon_\rho^2 \Omega(x^\perp \cdot \nabla \tilde{I}_{j\rho}) \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} \]

\[ + \int_{\mathbb{R}^2} \frac{1}{2} \left[ \epsilon_\rho^2 \Omega^2 \frac{|x|^2}{4} + \epsilon_\rho^2 V_\Omega (\epsilon_\rho (x+y_0)) \right] \frac{\partial |\tilde{R}_{j\rho}|^2}{\partial x_l} \]

\[ = \int_{\mathbb{R}^2} \frac{1}{2} \epsilon_\rho^2 \mu_{j\rho} \frac{\partial |\tilde{R}_{j\rho}|^2}{\partial x_l} + \int_{\mathbb{R}^2} \left( a^* \right)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} \frac{\partial |\tilde{R}_{j\rho}|^2}{\partial x_l}. \quad (4.53) \]

By the exponential decay (4.6), we calculate that for \( j = 1, 2 \),

\[ -\int_{\mathbb{R}^2} \Delta \tilde{R}_{j\rho} \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} = - \lim_{R \to \infty} \int_{B_R(0)} \Delta \tilde{R}_{j\rho} \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} \]

\[ = - \lim_{R \to \infty} \int_{\partial B_R(0)} \left( \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} - \frac{1}{2} |\nabla \tilde{R}_{j\rho}|^2 \nu_l \right) dS = 0, \]

where \( \nu = (\nu_1, \nu_2) \) denotes the outward unit of \( \partial B_R(0) \), and

\[ \int_{\mathbb{R}^2} \frac{1}{2} \left[ \epsilon_\rho^2 \Omega^2 \frac{|x|^2}{4} + \epsilon_\rho^2 V_\Omega (\epsilon_\rho (x+y_0)) \right] \frac{\partial |\tilde{R}_{j\rho}|^2}{\partial x_l} = - \int_{\mathbb{R}^2} \frac{1}{2} \left[ \epsilon_\rho^2 \Omega^2 \frac{x^l}{2} + \epsilon_\rho^2 V_\Omega (\epsilon_\rho (x+y_0)) \right] |\tilde{R}_{j\rho}|^2, \]

where \( (x_1, x_2) = x \in \mathbb{R}^2 \). It then follows from (4.53) that

\[ \int_{\mathbb{R}^2} \frac{1}{2} \left[ \epsilon_\rho^2 \Omega^2 \frac{x^l}{2} + \epsilon_\rho^2 V_\Omega (\epsilon_\rho (x+y_0)) \right] |\tilde{R}_{j\rho}|^2 \]

\[ = - \int_{\mathbb{R}^2} \epsilon_\rho^2 \Omega(x^\perp \cdot \nabla \tilde{I}_{j\rho}) \frac{\partial \tilde{R}_{j\rho}}{\partial x_l} - \int_{\mathbb{R}^2} \left( a^* \right)^{\frac{p-1}{2}} |\tilde{u}_{j\rho}|^{p-1} \frac{\partial |\tilde{R}_{j\rho}|^2}{\partial x_l}, \quad j = 1, 2. \quad (4.54) \]
Moreover, using (4.1), (4.7) and (4.54), we have

\[
\int_{\mathbb{R}^2} \frac{\epsilon^2 \Omega^2 x_l}{2} (\tilde{\mathcal{R}}_{2\rho} + \tilde{\mathcal{R}}_{1\rho} \tilde{\eta}_{1\rho})
\]

\[
= - \int_{\mathbb{R}^2} \frac{\epsilon^4 \Omega^2 x_l}{4} (\tilde{\mathcal{R}}_{2\rho} + \tilde{\mathcal{R}}_{1\rho} \tilde{\eta}_{1\rho})
\]

\[
- \int_{\mathbb{R}^2} \left\{ \epsilon^2 \Omega (x^+ \cdot \nabla \tilde{\eta}_{2\rho}) \frac{\partial \tilde{\mathcal{R}}_{2\rho}}{\partial x_l} + \epsilon^2 \Omega (x^+ \cdot \nabla \tilde{\mathcal{I}}_{1\rho}) \frac{\partial \tilde{\eta}_{1\rho}}{\partial x_l} \right\}
\]

\[
+ \int_{\mathbb{R}^2} \frac{(a^*)^{\frac{n-1}{2}}}{2} \frac{|\tilde{u}_{2\rho}|^{p-1}}{\partial x_l} \frac{\partial \tilde{\eta}_{2\rho} (\tilde{\mathcal{I}}_{2\rho} + \tilde{\mathcal{I}}_{1\rho})}{\partial x_l}
\]

\[
+ \int_{\mathbb{R}^2} \frac{(p-1)(a^*)^{\frac{n-1}{2}}}{4} \left[ \tilde{\eta}_{1\rho} (\tilde{\mathcal{R}}_{2\rho} + \tilde{\mathcal{R}}_{1\rho}) + \tilde{\eta}_{2\rho} (\tilde{\mathcal{I}}_{2\rho} + \tilde{\mathcal{I}}_{1\rho}) \right]
\]

\[
\cdot \int_0^1 [t|\tilde{u}_{2\rho}|^2 + (1-t)|\tilde{u}_{1\rho}|^2]^{\frac{n-1}{2}} dt \frac{\partial |\tilde{\mathcal{I}}_{1\rho}|^2}{\partial x_l}.
\]

Next, we shall prove the claim (4.52) by estimating all terms of (4.55). Applying Lemma 4.3 and (4.32), we deduce that as \( \rho \to \infty \),

\[
- \int_{\mathbb{R}^2} \frac{\epsilon^4 \Omega^2 x_l}{4} (\tilde{\mathcal{R}}_{2\rho} + \tilde{\mathcal{R}}_{1\rho} \tilde{\eta}_{1\rho})
\]

\[
= - \frac{\epsilon^4 \Omega^2}{2 \sqrt{a^*}} \int_{\mathbb{R}^2} x_l w \left[ b_0 (w + \frac{p-1}{2} x \cdot \nabla w) + \sum_{j=1}^2 b_j \frac{\partial w}{\partial x_j} \right] + o(\epsilon^4)
\]

\[
(4.56)
\]

\[
= - \frac{\epsilon^4 \Omega^2}{4 \sqrt{a^*}} \int_{\mathbb{R}^2} \frac{\partial w^2}{\partial x_l} + o(\epsilon^4)
\]

\[
= \frac{\epsilon^4 \Omega^2 \sqrt{a^*} b_l}{4} + o(\epsilon^4), \ l = 1, 2.
\]

Similarly, we obtain from Lemma 4.5 that as \( \rho \to \infty \),

\[
- \int_{\mathbb{R}^2} \frac{\epsilon^2 \Omega^2 (x^+ \cdot \nabla \tilde{\eta}_{2\rho})}{2} \frac{\partial \tilde{\mathcal{R}}_{2\rho}}{\partial x_l}
\]

\[
= - \frac{\epsilon^2 \Omega}{2 \sqrt{a^*}} \int_{\mathbb{R}^2} (x^+ \cdot \nabla) \left[ \frac{\epsilon^2 \Omega}{2} (-b_1 x_2 + b_2 x_1) w \right] \frac{\partial \left( \frac{w}{\sqrt{a^*}} \right)}{\partial x_l} + o(\epsilon^4)
\]

\[
(4.57)
\]

\[
= - \frac{\epsilon^2 \Omega^2}{2 \sqrt{a^*}} \int_{\mathbb{R}^2} (-b_2 x_2 w - b_1 x_1 w) \frac{\partial w}{\partial x_l} + o(\epsilon^4)
\]

\[
= - \frac{\epsilon^2 \Omega^2 \sqrt{a^*} b_l}{4} + o(\epsilon^4), \ l = 1, 2.
\]

From Proposition 4.1, Lemmas 4.4 and 4.5, we also derive that as \( \rho \to \infty \),

\[
\int_{\mathbb{R}^2} \frac{\epsilon^2 \Omega (x^+ \cdot \nabla \tilde{\mathcal{I}}_{1\rho})}{2} \frac{\partial \tilde{\eta}_{1\rho}}{\partial x_l} = o(\epsilon^4),
\]

(4.58)
\[
\int_{\mathbb{R}^2} \frac{(a^*)^{p-1}}{2} |\tilde{u}_{2\rho}|^{p-1} \partial \left[ \tilde{\eta}_{2\rho}(\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \right] \frac{\partial}{\partial x_l} = o(\epsilon_{\rho}^4), \quad (4.59)
\]
and
\[
\int_{\mathbb{R}^2} \frac{(p-1)(a^*)^{p-1}}{4} \left[ \tilde{\eta}_{\rho}(\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) + \tilde{\eta}_{2\rho}(\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \right] \\
\cdot \int_0^1 \left[ t|\tilde{u}_{2\rho}|^2 + (1-t)|\tilde{u}_{1\rho}|^2 \right] \frac{t^{p-3}}{2} dt \frac{\partial |\tilde{I}_{1\rho}|}{\partial x_l} = o(\epsilon_{\rho}^4). \quad (4.60)
\]

Employing (4.56)-(4.60), (V1), (V2) and Lemma 4.3 one can deduce from (4.55) that for \( l = 1, 2 \), as \( \rho \to \infty \),
\[
o(\epsilon_{\rho}^4) = \int_{\mathbb{R}^2} \frac{c^2}{\alpha} \frac{\partial V_\Omega(\epsilon_{\rho}(x + y_0))}{\partial x_l} (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) \tilde{\eta}_{\rho} \frac{\partial h(x + y_0)}{\partial x_l} dx \\
= \frac{\epsilon_{\rho}^{2+s}}{2} \int_{\mathbb{R}^2} \frac{\partial h(x + y_0)}{\partial x_l} (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) \tilde{\eta}_{\rho} dx + o(\epsilon_{\rho}^{2+s}) \\
= \frac{\epsilon_{\rho}^{2+s}}{2} \int_{\mathbb{R}^2} \frac{\partial h(x + y_0)}{\partial x_l} \left[ b_0 \left( w + \frac{p-1}{2} x \cdot \nabla w \right) + \sum_{j=1}^2 b_j \frac{\partial w}{\partial x_j} \right] dx + o(\epsilon_{\rho}^{2+s}) \quad (4.61)
\]
which implies that the claim (4.52) holds true.

**Step 2.** The coefficient \( b_0 = 0 \) in (4.12).

Multiplying (4.1) by \((x \cdot \nabla \hat{R}_{j\rho})\) and integrating over \( \mathbb{R}^2 \), where \( j = 1, 2 \), we have
\[
- \int_{\mathbb{R}^2} \Delta \hat{R}_{j\rho}(x \cdot \nabla \hat{R}_{j\rho}) - \int_{\mathbb{R}^2} \epsilon_{\rho}^2 \Omega(x^\perp \cdot \nabla \hat{I}_{j\rho})(x \cdot \nabla \hat{R}_{j\rho}) \\
= \int_{\mathbb{R}^2} \left[ \epsilon_{\rho}^4 \hat{R}_{j\rho} - \frac{\epsilon_{\rho}^4 \Omega^2 |x|^2}{4} - \epsilon_{\rho}^2 V_\Omega(\epsilon_{\rho}(x + y_0)) \right] \hat{R}_{j\rho}(x \cdot \nabla \hat{R}_{j\rho}) \\
+ (a^*)^{p-1} \int_{\mathbb{R}^2} |\tilde{u}_{j\rho}|^{p-1} \hat{R}_{j\rho}(x \cdot \nabla \hat{R}_{j\rho}). \quad (4.62)
\]
By the integration by parts, we derive from (4.6) that for \( j = 1, 2 \),
\[
A_{j\rho} := - \int_{\mathbb{R}^2} \Delta \hat{R}_{j\rho}(x \cdot \nabla \hat{R}_{j\rho}) \\
= - \lim_{R \to \infty} \left[ \int_{\partial B_R(0)} \frac{\partial \hat{R}_{j\rho}}{\partial \nu}(x \cdot \nabla \hat{R}_{j\rho}) dS - \int_{B_R(0)} \nabla \hat{R}_{j\rho} \nabla (x \cdot \nabla \hat{R}_{j\rho}) \right] \\
= - \lim_{R \to \infty} \left[ \int_{\partial B_R(0)} \frac{\partial \hat{R}_{j\rho}}{\partial \nu}(x \cdot \nabla \hat{R}_{j\rho}) dS - \frac{1}{2} \int_{\partial B_R(0)} (x \cdot \nu) |\nabla \hat{R}_{j\rho}|^2 dS \right] \\
= 0,
\]

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Thus, we derive from (4.62) and above that

\[
B_{j\rho} := \int_{\mathbb{R}^2} \left[ \frac{c_\rho^4 \Omega^2}{4} |x|^2 - c_\rho^2 \Omega (\epsilon_{\rho}(x + y_0)) \right] \tilde{R}_{j\rho}(x \cdot \nabla \tilde{R}_{j\rho})
\]

\[
= - \int_{\mathbb{R}^2} \tilde{R}_{j\rho}^2 \left[ \frac{c_\rho^4 \Omega^2}{2} |x|^2 - c_\rho^2 \Omega (\epsilon_{\rho}(x + y_0)) - \frac{c_\rho^2}{2} x \cdot \nabla \Omega (\epsilon_{\rho}(x + y_0)) \right],
\]

and

\[
C_{j\rho} := (a^*)^{\frac{p-1}{2}} \int_{\mathbb{R}^2} |\tilde{u}_{j\rho}|^{p-1} \tilde{R}_{j\rho}(x \cdot \nabla \tilde{R}_{j\rho})
\]

\[
= \frac{(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^2} (x \cdot \nabla |\tilde{u}_{j\rho}|^2) - \frac{(a^*)^{\frac{p-1}{2}}}{2} \int_{\mathbb{R}^2} |\tilde{u}_{j\rho}|^{p-1} (x \cdot \nabla \tilde{\tilde{R}}_{j\rho}^2)
\]

Denote

\[
D_{j\rho} := - \int_{\mathbb{R}^2} \frac{c_\rho^2 \Omega (x^+ \cdot \nabla \tilde{\tilde{R}}_{j\rho}) (x \cdot \nabla \tilde{R}_{j\rho})}{|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}|_{L^\infty(\mathbb{R}^2)}} \]

Thus, we derive from (4.62) and above that

\[
\frac{D_{2\rho} - D_{1\rho}}{|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}|_{L^\infty(\mathbb{R}^2)}} = \frac{(B_{2\rho} - B_{1\rho}) + (C_{2\rho} - C_{1\rho})}{|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}|_{L^\infty(\mathbb{R}^2)}}. \quad (4.63)
\]

We next estimate all terms of (4.63) as follows. Applying Lemmas 4.2, 4.4 and 4.5, we obtain that as \( \rho \to \infty \),

\[
\frac{D_{2\rho} - D_{1\rho}}{|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}|_{L^\infty(\mathbb{R}^2)}}
\]

\[
= - c_\rho^2 \int_{\mathbb{R}^2} \Omega (x^+ \cdot \nabla \tilde{\eta}_{2\rho}) (x \cdot \nabla \tilde{R}_{2\rho}) - \frac{c_\rho^2}{2} \int_{\mathbb{R}^2} \Omega (x^+ \cdot \nabla \tilde{\eta}_{1\rho}) (x \cdot \nabla \tilde{\eta}_{1\rho})
\]

\[
= - c_\rho^2 \int_{\mathbb{R}^2} \Omega (x^+ \cdot \nabla \tilde{\eta}_{2\rho}) \left[ x \cdot \left( \frac{w}{\sqrt{a^*}} \right) + x \cdot \nabla \left( \tilde{R}_{2\rho} - \frac{w}{\sqrt{a^*}} \right) \right] + o(\epsilon_\rho^4)
\]

\[
= c_\rho^2 \frac{\Omega}{\sqrt{a^*}} \int_{\mathbb{R}^2} (x^+ \cdot \nabla (x \cdot \nabla w)) \tilde{\eta}_{2\rho} + o(\epsilon_\rho^4) = o(\epsilon_\rho^4).
\]
Similarly, by Proposition 4.1, Lemmas 4.4 and 4.5, we deduce that as $\rho \to \infty$,

$$
\frac{C_{2\rho} - C_{1\rho}}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)} = - \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^2} \frac{|\tilde{u}_{2\rho}|^{p+1} - |\tilde{u}_{1\rho}|^{p+1}}{2} \int_{\mathbb{R}^2} |\tilde{u}_{2\rho}|^{p-1} \left[ x \cdot \nabla [\tilde{\eta}_{2\rho}(\tilde{F}_{2\rho} + \tilde{I}_{1\rho})] \right] 
$$

$$
- \frac{(p-1)(a^*)^{\frac{p-1}{2}}}{4} \int_{\mathbb{R}^2} \left[ \tilde{\eta}_{1\rho}(\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) + \tilde{\eta}_{2\rho}(\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \right] 
$$

$$
\cdot \int_0^1 \left[ t|\tilde{u}_{2\rho}|^2 + (1-t)|\tilde{u}_{1\rho}|^2 \right]^{\frac{p-3}{2}} dt (x \cdot \nabla \tilde{\eta}_{1\rho}^2) 
$$

$$
= - \frac{2(a^*)^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^2} \frac{|\tilde{u}_{2\rho}|^{p+1} - |\tilde{u}_{1\rho}|^{p+1}}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)}} + o(\epsilon^4), \quad (4.65) \]

As for the term containing $B_{j\rho}$, we obtain from the assumption (V2) that as $\rho \to \infty$,

$$
\frac{B_{2\rho} - B_{1\rho}}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left[ \frac{\epsilon^4 \Omega^2 |x|^2}{2} + \epsilon^2 \Omega_1 (\epsilon \rho(x + y_0)) + \frac{\epsilon^2}{2} (x + y_0) \cdot \nabla_2 \Omega_1 (\epsilon \rho(x + y_0)) \right] 
$$

$$
\cdot (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) \tilde{\eta}_{1\rho} dx - J_{\rho} - K_{\rho} 
$$

$$
= \int_{\mathbb{R}^2} \left[ \frac{\epsilon^4 \Omega^2 |x|^2}{2} + \frac{2 + 8 \epsilon^{2+s} h(x + y_0)}{2} (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) \tilde{\eta}_{1\rho} dx - J_{\rho} - K_{\rho} + o(\epsilon^{2+s}) \right] 
$$

$$
= O(\epsilon^{2+s}) - J_{\rho} - K_{\rho}, 
$$

where the fact that $x \cdot \nabla h(x) = sh(x)$ is used in the last equality. Here the terms $J_{\rho}$ and $K_{\rho}$ are defined as

$$
J_{\rho} := \frac{\epsilon^2}{2} \int_{\mathbb{R}^2} [y_0 \cdot \nabla_2 \Omega_1 (\epsilon \rho(x + y_0))] (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) \tilde{\eta}_{1\rho} dx, 
$$

and

$$
K_{\rho} := 2 \epsilon^2 \int_{\mathbb{R}^2} \frac{\tilde{R}_{2\rho}^2}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)}} \frac{\tilde{R}_{1\rho}^2}{\|\tilde{u}_{2\rho} - \tilde{u}_{1\rho}\|_{L^\infty(\mathbb{R}^2)}} dx. 
$$

Using (3.42), we note from the first identity of (4.61) that as $\rho \to \infty$,

$$
J_{\rho} = o(\epsilon^4). 
$$
Applying Lemmas 4.3, 4.4 and 4.5, we obtain from (4.50) that as $\rho \to \infty$,

\[
K_\rho := \epsilon_\rho^2 \int_{\mathbb{R}^2} \| \tilde{u}_{2\rho} \|^2 \mu_{2\rho} - |\tilde{u}_{1\rho}|^2 \mu_{1\rho} - \epsilon_\rho^2 \int_{\mathbb{R}^2} \| \tilde{u}_{2\rho} - \tilde{u}_{1\rho} \|^2 L_\infty(\mathbb{R}^2)
\]

\[
= \epsilon_\rho^2 \frac{(\mu_{2\rho} - \mu_{1\rho})}{\| \tilde{u}_{2\rho} - \tilde{u}_{1\rho} \|^2 L_\infty(\mathbb{R}^2)} - \epsilon_\rho^2 \int_{\mathbb{R}^2} \tilde{\eta}_\rho (\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \mu_{2\rho} - \left( \frac{p - 1}{p + 1} \right) \| \tilde{u}_{2\rho} - \tilde{u}_{1\rho} \|^2 L_\infty(\mathbb{R}^2)
\]

It then follows from above that as $\rho \to \infty$,

\[
B_{2\rho} = B_{1\rho}
\]

\[
= O(2^{+s}) + \frac{(p - 1)(a^*)^{p-1}}{p + 1} \int_{\mathbb{R}^2} \| \tilde{u}_{2\rho} - \tilde{u}_{1\rho} \|^2 L_\infty(\mathbb{R}^2)
\]

From (4.63)-(4.66), we deduce that as $\rho \to \infty$,

\[
O(\epsilon_\rho^{2+s}) = \frac{(3 - p)(a^*)^{p-1}}{p + 1} \int_{\mathbb{R}^2} \| \tilde{u}_{2\rho} - \tilde{u}_{1\rho} \|^2 L_\infty(\mathbb{R}^2)
\]

\[
= \frac{(3 - p)(a^*)^{p-1}}{4} \int_{\mathbb{R}^2} \left\{ \left( |\tilde{u}_{2\rho}|^{p-1} + |\tilde{u}_{1\rho}|^{p-1} \right) \left[ (\tilde{R}_{2\rho} + \tilde{R}_{1\rho}) \tilde{\eta}_\rho + (\tilde{I}_{2\rho} + \tilde{I}_{1\rho}) \tilde{\eta}_{2\rho} \right] - \int_0^1 \left[ t |\tilde{u}_{2\rho}|^2 + (1 - t) |\tilde{u}_{1\rho}|^2 \right] \frac{2}{4} dt \right\} dx
\]

\[
= \frac{(3 - p)}{\sqrt{a^*}} \int_{\mathbb{R}^2} w^p \tilde{\eta}_1 dx + o(1).
\]

Using Lemma 4.3 it then follows from (4.67) that

\[
0 = \int_{\mathbb{R}^2} w^p \tilde{\eta}_1
\]

\[
= \int_{\mathbb{R}^2} w^p \left[ b_0 (w + \frac{p - 1}{2} x \cdot \nabla w) + \sum_{j=1}^2 b_j \frac{\partial w}{\partial x_j} \right]
\]

\[
= b_0 \int_{\mathbb{R}^2} w^{p+1} + \frac{b_0}{2} \frac{p - 1}{p + 1} \int_{\mathbb{R}^2} x \cdot \nabla w^{p+1} + \sum_{j=1}^2 \frac{b_j}{p + 1} \int_{\mathbb{R}^2} \frac{\partial w^{p+1}}{\partial x_j}
\]

\[
= b_0 \int_{\mathbb{R}^2} w^{p+1} - b_0 \frac{p - 1}{p + 1} \int_{\mathbb{R}^2} w^{p+1}
\]

\[
= b_0 \left[ 1 - \frac{p - 1}{p + 1} \right] \int_{\mathbb{R}^2} w^{p+1},
\]
which implies $b_0 = 0$ due to $1 < p < 3$.

**Step 3.** The constants $b_1 = b_2 = 0$.

By step 2, we derive from (4.52) that

$$
\sum_{j=1}^{2} b_j \int_{\mathbb{R}^2} \frac{\partial h(x+y_0)}{\partial x_j} \frac{\partial u^2}{\partial x_j} dx = 0, \quad l = 1, 2,
$$

which implies from the non-degeneracy assumption of $H(y)$ in (1.19) that $b_1 = b_2 = 0$, and the proof of Step 3 is thus completed.

Since $\|\tilde{\eta}_\rho\|_{L^\infty(\mathbb{R}^2)} = 1$, we can deduce from the exponential decay of Lemma 4.2 that $\tilde{\eta}_\rho \to \tilde{\eta}_0 = \tilde{\eta}_1 + i \tilde{\eta}_2 \not\equiv 0$ uniformly in $C^1(\mathbb{R}^2)$ as $\rho \to \infty$. However, Steps 2 and 3 imply that $\tilde{\eta}_0 \equiv 0$, this is a contraction. Therefore, we complete the proof of Theorem 1.3.

A Appendix

In the appendix, we shall prove the equivalence between ground states of equation (1.1) and minimizers of (1.2). We first introduce the definition of ground states of (1.1). Given any $\rho \in (0, \infty)$ and $0 < \Omega < \Omega^*$, the energy functional of (1.1) is defined by

$$
F_{\mu,\rho}(u) := \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + (V(x) - \mu)|u|^2 \right] dx - \frac{2\rho^{p-1}}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} dx - \Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u) dx,
$$

(A.1)

where $\mu \in \mathbb{R}$ is a parameter and the energy functional $E_\rho(u)$ is given by (1.3). Define

$$
S_{\mu,\rho} := \{ u \in \mathcal{H} \setminus \{0\} : \langle F'_{\mu,\rho}(u), \varphi \rangle = 0 \text{ for all } \varphi \in \mathcal{H} \},
$$

where

$$
\langle F'_{\mu,\rho}(u), \varphi \rangle = 2Re \left\{ \int_{\mathbb{R}^2} [\nabla u \nabla \bar{\varphi} + (V(x) - \mu)u \bar{\varphi}] dx - \rho^{p-1} \int_{\mathbb{R}^2} |u|^{p-1} u \bar{\varphi} dx \right.
$$

$$
+ \left. \int_{\mathbb{R}^2} i \Omega (x^\perp \cdot \nabla u) \bar{\varphi} dx \right\},
$$

and

$$
G_{\mu,\rho} := \{ u \in S_{\mu,\rho} : F_{\mu,\rho}(u) \leq F_{\mu,\rho}(v) \text{, for all } v \in S_{\mu,\rho} \}.
$$

If $u \in G_{\mu,\rho}$, we say that $u$ is a *ground state* of (1.1). Now we give the following theorem on the equivalence between ground states of equation (1.1) and minimizers of (1.2).
Theorem A.1. Suppose $\rho \in (0, \infty)$ and $0 < \Omega < \Omega^*$ are given, then any minimizer of (1.2) is a ground state of (1.1) for some $\mu \in \mathbb{R}$; conversely, any ground state of (1.1) for some $\mu \in \mathbb{R}$ is a minimizer of (1.2).

Since the proof of Theorem A.1 is similar to [15, Proposition A.1], we omit it here.

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