Generalized Schmidt decomposition based on injective tensor norm

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We present a generalized Schmidt decomposition for a pure system with any number of two-level subsystems. The basis is symmetric under the permutation of the parties and is derived from the product state defining the injective tensor norm of the state. The largest coefficient quantifies the quantum correlation of the state. Other coefficients have a lot of information such as the unentangled particles as well as the particles whose reduced states are completely mixed. The decomposition clearly distinguishes the states entangled in inequivalent ways and have an information on the applicability to the teleportation and superdense coding when the given quantum state is used as a quantum channel.

Keywords: multipartite entanglement, separability problem, geometric measure, nonlinear eigenproblem

I. INTRODUCTION

The Schmidt decomposition for bipartite systems[1] is a very important tool in quantum information and quantum computing theories. It shows whether two given states are related by a local unitary transformation or not[2], which states are applicable for perfect teleportation[3] and superdense coding[4], and whether it is possible to transform a given bipartite pure state to another pure state by local operations and classical communications[5]. Many substantial results have been obtained with the help of the Schmidt normal form and its generalization to the multipartite states is a task of prime importance[6, 7].

Carteret et al developed a method for such a generalization for pure states of an n-party system, where the dimensions of the individual state spaces are finite but otherwise arbitrary[8]. The idea of this method is the following. First, one finds the product vector that gives maximal overlap with a given state function. Next, one considers product vectors whose constituents are orthogonal to the constituents of the first product vector and finds among them the product vector that gives maximal overlap with the state function. Continuing in this way, one finds a set of orthonormal product states and presents the state function as a linear combination of these product vectors. Since the first product vector is a stationarity point, the resulting canonical form contains a minimal set of state parameters.
Surprisingly, an arbitrary multipartite quantum state has many canonical forms. The reason is that the stationarity equations (SEQ) defining stationarity points are nonlinear equations and in general have many solutions of different types [9]. For example, in the case of generic three qubit states there are six product vectors that are solutions of SEQ [10]. Moreover, some highly entangled states are surrounded by the one-parametric set of equally distant product states that have the same maximal overlap with the state function [11]. All of these product vectors are equally good in a sense that all resulting canonical forms use minimal number of product vectors from a factorizable orthonormal basis. Therefore, it is necessary to make a physically motivated choice of the product vector that yields the best factorizable basis. More precisely, the problem is the following. Each of bases suggests its own set of invariants which are coefficients of the expansion of the state function. And the task is to specify the set of invariants that gives an effective description of the quantum state.

Our approach is based on the injective tensor norm of quantum states [12]. The overlap of a given state with any complete product state attains maximums at several points. These maximums are eigenvalues and corresponding product states are eigenvectors of SEQ. The injective tensor norm takes the value of one of these maximums and thus makes a unique choice of the eigenvector. This choice classifies pure quantum states by types as follows. Consider three qubit pure states. There are six eigenvectors and eigenvalues of SEQ and therefore the injective tensor norm can have six different expressions. Since it is a single valued function, each of expressions should have its own range of definition in which they are deemed applicable. Consequently, each of solutions of SEQ has its own applicable domain depending on state parameters in which they give the relevant eigenvalue. As a result, the space of the state parameters is split into six regions and each region has its own specific eigenvector defining the injective tensor norm. Each of three qubit pure states should belong to the one of six regions of the parameter space except states that lie on joint surfaces separating different regions. This classification provides a geometric picture of entangled regions.

The main objective of this work is to construct the generalized Schmidt decomposition (GSD) for generic three qubit states. We analyze the complete set of SEQ solutions, specify the relevant solution for a given state and calculate GSD coefficients explicitly in terms of the state parameters. In this way we obtain a new set of invariants of three qubit states. Since there are different sets of local invariants [6, 13], one may wonder what is the advantage of the obtained coefficients. For this purpose we study the properties of the constructed GSD.

The expansion coefficients exhibit the physically significant properties of pure states. The largest coefficient \( g \) is the injective tensor norm of the state. It is a very useful quantity and defines some entanglement measures [9, 14, 15, 16, 17]. Apart from that, we will present a rigorous proof that if a three qubit pure state has a completely mixed one qubit reduced state, then \( g^2 = 1/2 \). This sheds some new light on the
applicability of the quantum state to the teleportation and superdense coding. There is a conjecture that the criterion for perfect quantum teleportation is $g^2 = 1/2$[18]. Now this conjecture can be reformulated as follows: a pure state can be used for perfect teleportation and superdense coding if it has a completely mixed state as a reduced state. The other expansion coefficient, say $h$, has an information on the presence or absence of an unentangled particle in a given quantum state. We will show in the following that $h = 0$ is a separability criterion for pure states of a general multi-qubit system [19].

This paper is organized as follows. In Sec. II we specify GSD for multiqubit systems. In Sec. III we analyze general properties of GSD in the case of three qubit systems. In Sec. IV we construct GSD for three qubit W-type states. In Sec. V we construct GSD for a class of multiqubit W-type states. In Sec. VI we construct GSD for Greenberger-Horne-Zeilinger(GHZ)-type states. In Sec. VII we make concluding remarks.

II. GENERALIZED SCHMIDT DECOMPOSITION

Consider $n$-partite pure systems with the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. The injective tensor norm $g(\psi)$ of a given $n$-partite pure state $|\psi\rangle$ is defined as

$$g(\psi) = \sup |\langle \chi_1 \chi_2 \cdots \chi_n |\psi\rangle|,$$

where the supremum is over all tuples of vectors $|\chi_k\rangle \in \mathcal{H}_k$ with $||\chi_k|| = 1$ [12]. The nearest product state $|q\rangle = |q_1 q_2 \cdots q_n\rangle$ must satisfy stationarity equations [9, 20]

$$\langle q_1 q_2 \cdots \hat{q}_k \cdots q_n |\psi\rangle = g|q_k\rangle, \quad k = 1, 2, \cdots n$$

where the caret means exclusion. These equations can be regarded as nonlinear eigenvalue problem: all eigenvalues $g$ and eigenvectors $|q\rangle$ satisfy it for a given state function $|\psi\rangle$. In general, Eq.(2) has several different solutions. Hereafter we consider only the solution corresponding to the eigenvalue of largest value. This solution, known as dominant eigenvector, defines the injective tensor norm of a given quantum state.

Consider now $n$-qubit system. For each single-qubit state $|q_k\rangle$ there is, up to arbitrary phase, a unique single-qubit state $|p_k\rangle$ orthogonal to it. From these single-qubit states $|q_k\rangle$ and $|p_k\rangle$ one can form a set of $2^n$ $n$-qubit product states which form a basis in the full Hilbert space $\mathcal{H}$. Any vector $|\psi\rangle \in \mathcal{H}$ can be written as a linear combination of vectors in the set. Then from SEQ (2) it follows that all the coefficients of the product states $|q_1 \cdots q_{k-1} p_k q_{k+1} \cdots q_n\rangle(k = 1, 2, \cdots n)$ are zero. Thus any pure state can be written in terms of $2^n - n$ product states. Furthermore, the phases of vectors $|p_k\rangle$ are free and we can choose them so that all the coefficients $t_k$ of vectors $|p_1 \cdots p_{k-1} q_k p_{k+1} \cdots p_n\rangle(k = 1, 2, \cdots n)$ be positive. Still we have
a freedom to make a phase shift $|p_k⟩ \to e^{2i\pi/(n-1)}|p_k⟩$ which remains unchanged $t_k$ and $g$. We use this freedom to vary the phase $\varphi$ of the component $e^{i\varphi}h|p_1p_2\cdots p_n⟩$ ($h \geq 0$ is understood) within the interval $-\pi/(n-1) \leq \varphi \leq \pi/(n-1)$.

Thus the decomposition has $n+1$ real and $2^n-2n-1$ complex parameters. After taking into account the normalization condition, one can show that $2^{n+1}-3n-2$ real numbers parameterize the sets of inequivalent pure states [21].

**Theorem 1.** The $k$th qubit is completely unentangled if and only if $h(\psi) = 0$ and $t_i(\psi) = 0$ for $i \neq k$.

**Proof.** Suppose first qubit is completely unentangled and its state vector is $|q_1⟩$. We have $|\psi⟩ = |q_1⟩ \otimes |\psi'⟩$. Let the product state $|q_2q_3\cdots q_n⟩$ be the nearest state of $|\psi'⟩$. Then GSD of $|\psi'⟩$ takes the form

$$|\psi'⟩ = g'|q_2q_3\cdots q_n⟩ + \sum_{i=2}^{n} t_i'|p_2\cdots p_{i-1}q_i p_{i+1} \cdots p_n⟩ + \cdots + e^{i\varphi'h}|p_2p_3\cdots p_n⟩.$$  \hspace{1cm} (3)

Since the nearest state of the state $|\psi⟩$ is, up to a phase, the product state $|q_1q_2\cdots q_n⟩$, then $g(\psi) = g'$, $h(\psi) = 0$, $t_1(\psi) = h'$ and $t_i = 0, i = 2, 3\ldots n$. The inverse is also true. From $h(\psi) = 0$ and $t_i(\psi) = 0$ for $i \neq 1$ it follows that all the terms in GSD which do not contain $|q_1⟩$ vanish and $|\psi⟩ = |q_1⟩ \otimes |\psi'⟩$. Similarly, theorem is true if any other qubit is unentangled.

Consider now $n = 2$ and $n = 3$ cases. For simplicity we will use notations $|0_i⟩$ and $|1_i⟩$ for vectors $|q_i⟩$ and $|p_i⟩$ respectively. Also we will omit sub-indices $i$ whenever it does not create misunderstanding. In the case of two qubit states the expansion reduces to the Schmidt decomposition $|\psi⟩ = g|00⟩ + h|11⟩$ with $g \geq h \geq 0$. Consider three-qubit case.

### III. THREE QUBIT STATES.

Decomposition takes the form

$$|\psi⟩ = g|000⟩ + t_1|011⟩ + t_2|101⟩ + t_3|110⟩ + e^{i\varphi} h|111⟩.$$ \hspace{1cm} (4)

The coefficients should satisfy conditions

$$g \geq \max(t_1,t_2,t_3,h), \quad t_1 \geq 0, \quad t_2 \geq 0, \quad t_3 \geq 0, \quad h \geq 0, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}.$$ \hspace{1cm} (5)

These conditions do not specify GSD uniquely. Eq.(4) is the GSD normal form of the state $|\psi⟩$ if and only if $g$ is the injective norm of the state $|\psi⟩$. There are highly entangled states which can be written in a form of Eq.(4) in six different bases. One basis, where the largest coefficient is injective tensor norm of the state and corresponding product state is the dominant eigenvector of SEQ, gives true GSD. The others, while the largest coefficient is not injective tensor norm and corresponding product states are eigenvectors but are
not dominant eigenvectors, do not. The example with W-type states, which is given in the next section, illustrates this more clearly.

Now we formulate a theorem which shows whether a given three qubit pure state has a completely mixed one qubit reduced state or not. The proof of Theorem 2 uses a particular lower bound on $g$ given in the next section. In this reason we present the proof in Appendix.

**Theorem 2.** One-particle reduced state of $k$th qubit is completely mixed if and only if

$$t_k = 0 \quad \text{and} \quad g^2 = 1/2.$$ 

Consider now several interesting examples.

**IV. W-TYPE STATES**

Our first example that we shall discuss in detail is a family of four-parametric W-type states [22]

$$|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle + d|111\rangle.$$  \hspace{1cm} (6)

Without loss of generality we consider only the case of positive parameters $a, b, c, d$.

**A. Solutions of stationarity equations**

Stationarity equations (2) have six different solutions for each of these states[10]. Four solutions are simple and represent four terms on the right-hand side of Eq.(6)

$$|q_1q_2q_3\rangle = |100\rangle, \quad g = a; \tag{7a}$$

$$|q_1q_2q_3\rangle = |010\rangle, \quad g = b; \tag{7b}$$

$$|q_1q_2q_3\rangle = |001\rangle, \quad g = c; \tag{7c}$$

$$|q_1q_2q_3\rangle = |111\rangle, \quad g = d. \tag{7d}$$

These four solutions owe their existence to the fact that the states Eq.(6) have a hidden symmetry. Namely, there exist local unitary transformations that interchange the positions of any of two coefficients in Eq.(6). Due to this symmetry any entanglement measure is invariant under the permutations of all of the state parameters $a, b, c, d$. 

The fifth nontrivial solution is
\[
|q_1⟩ = \frac{\sqrt{r_a r_d}}{4S\sqrt{ab + cd}} |01⟩, \quad |q_2⟩ = \frac{\sqrt{r_b r_c}}{4\sqrt{ac + bd}} |12⟩, \quad |q_3⟩ = \frac{\sqrt{r_c r_d}}{4S\sqrt{ab + cd}} |03⟩, \quad g = \frac{L}{S},
\]
where
\[
|q_3⟩ = \frac{\sqrt{r_c r_d}}{4S\sqrt{ab + cd}} |03⟩ + \frac{\sqrt{r_a r_b}}{4S\sqrt{ac + bd}} |13⟩.
\]

and \(S\) is the area of the cyclic quadrangle with sides \((a, b, c, d)\). Note that the quadrangle is not unique as the sides can be arranged in different orders. But all these quadrangles have the same area given by Heron’s formula \(S = \sqrt{(s - a)(s - b)(s - c)(s - d)}\), where \(s = (a + b + c + d)/2\) is semiperimeter.

The sixth solution is obtained from the solution (8) by the interchange
\[
d \to -d, \quad r_d \to -r_d.
\]

Due to symmetry one can make one of the following interchanges instead
\[
a \to -a, \quad r_a \to -r_a;
\]
\[
b \to -b, \quad r_b \to -r_b;
\]
\[
c \to -c, \quad r_c \to -r_c
\]
and obtain the same solution. However, it does not mean that any of parameters becomes negative. All the parameters are positive and the interchange (9) shows the existence of a pair of dual solutions.

B. Dominant eigenvector

We have six different product states \(|q_1 q_2 q_3⟩\) and therefore we can form six sets of factorizable bases from vectors \(|q_1⟩, |q_2⟩, |q_3⟩\) and \(|p_1⟩, |p_2⟩, |p_3⟩\). Since all these product states satisfy SEQ, all the expansions of the state function \(|ψ⟩\) in these bases give the canonical form Eq.(4). Therefore there are six factorizable bases where the state function Eq.(6) has a canonical form Eq.(4). The question at issue is whether there are effective expansions or not. If it is indeed the case, then what are the effective expansions which have physically meaningful coefficients?

The injective tensor norm \(g\) has six different expressions. Since it is a single valued function, each expression has its own applicable domain depending on state parameters and these applicable domains are
split up by separating surfaces. Consequently each of solutions (7) and (8) has its own applicable domain depending on state parameters in which they are dominant eigenvectors. Thus four-dimensional sphere given by normalization condition \(a^2 + b^2 + c^2 + d^2 = 1\) is split into six parts and each part has its own dominant eigenvector.

The above classification suggests a natural choice of a factorizable basis. We choose the product state \(|q_1q_2q_3\rangle\) that is dominant eigenvector in a given part of sphere. Next we form product states from constituents \(|q_i\rangle\) and orthogonal to them vectors \(|p_i\rangle\). Thus it is necessary to separate the validity domains and to make clear which of the solutions should be applied for a given state. It is a nontrivial task and for a thorough analysis we refer to Ref.[10]. Here we present some necessary results.

In highly entangled region parameters \((a, b, c, d)\) form a cyclic quadrilateral, injective tensor norm is expressed in terms of the the circumradius of the quadrangle and corresponding solution is Eq.(8). In slightly entangled region injective tensor norm is the largest coefficient and corresponding solutions are Eq.(7). Also there are states in between for which both formulae are valid. These states, called second type shared quantum states, separate slightly and highly entangled states and can be ascribed to both types. Another specific states, called first type shared quantum states, are those for which injective tensor norm is a constant and is defined by \(g^2 = 1/2\). These states allow perfect quantum teleportation and superdense coding scenario.

C. Highly entangled W-type states

Highly entangled region is defined by inequalities

\[
r_a \geq 0, \quad r_b \geq 0, \quad r_c \geq 0, \quad r_d \geq 0.
\]

The constituents of the dominant eigenvector are given by Eq.(8) and the task is to present the state function as a linear combination of product vectors

\[
|\psi\rangle = g|q_1q_2q_3\rangle + t_1|q_1p_2p_3\rangle + t_2|p_1q_2p_3\rangle + t_3|p_1p_2q_3\rangle + e^{i\varphi} h|p_1p_2p_3\rangle,
\]

where \(\langle p_i|q_i\rangle = 0, \quad i = 1, 2, 3\).

The calculation of the coefficients requires advanced mathematical technique. One has to factorize polynomials of degree ten. We would like to suggest a simple way. First one convinces oneself that each factor below is a root for the corresponding polynomial and next finds the proportionality coefficient in some particular case. The derivation of \(h\) is the most complicated out of all coefficients and one can use the
hint: if \( a = b + c + d \), then \( r_b = r_c = r_d = -r_a \). The resulting answer is

\[
g = \frac{L}{2S}, \quad t_1 = \frac{Lr_1}{4S(ad + bc)} , \quad t_2 = \frac{Lr_2}{4S(bd + ac)}, \quad t_3 = \frac{Lr_3}{4S(cd + ab)}, \quad \varphi = \frac{\pi}{2}, \quad h = \frac{\sqrt{r_ar_br_cr_d}}{4LS}, \quad (12)
\]

where \( r_1, r_2, r_3 \) are the lengths of the Bloch vectors of first, second and third qubits

\[
r_1 = |b^2 + c^2 - a^2 - d^2|, \quad r_2 = |a^2 + c^2 - b^2 - d^2|, \quad r_3 = |a^2 + b^2 - c^2 - d^2|. \quad (13)
\]

In fact, this set gives a fruitful description of the state. The invariant \( g \) is expressed in terms of the circum-radius of the cyclic quadrangle \( a, b, c, d \) and gives geometric and Groverian entanglement measures of the state. First type shared states are defined by \( r_1r_2r_3 = 0 \) which is to say that the reduced state of one of particles is completely mixed. It is quite obvious that one of coefficients \( t_i \) must vanish for these states. On the other hand if \( r_k = 0 \), then \( g^2 = 1/2 \) and the corresponding state allows teleportation (and dense coding) scenario. For perfect teleportation the receiver should choose \( k \)th particle at initial stage in order to perform the task. Thus the coefficients \( t_i \) contain an information on the applicability to the teleportation and precisely indicate which particle the receiver should choose. Second type shared states lie on the separating surface and this surface is defined by \( r_ar_br_cr_d = 0 \), i.e \( h = 0 \). We conclude that \( h > 0 \) for highly entangled states and \( h = 0 \) for second type shared states.

To complete the analysis let us consider the remaining slightly entangled cases.

D. Slightly entangled W-type states

At least one of quantities \( r_a, r_b, r_c \) and \( r_d \) should be negative in this case. Note that two of them can not be negative simultaneously. Indeed,

\[
br_a + ar_b = 2(ac + bd)(bc + ad) \geq 0
\]

and therefore \( r_a \) and \( r_b \) can not be negative together. Similarly, any pair of quantities \( r_a, r_b, r_c, r_d \) can not be negative simultaneously. Thus one of them should be negative and others should be positive and there are four possibilities. Consider these cases in turn.

a. Solution (7a). The first case is

\[
r_a \leq 0.
\]

The dominant eigenvector is given by (7a). In order to obtain GSD one has to simply relabel the basis of the first qubit \( |0_1\rangle \leftrightarrow |1_1\rangle \). Then the final GSD coefficients are

\[
g = a, \quad h = 0, \quad t_1 = d, \quad t_2 = c, \quad t_3 = b.
\]

(16)
b. Solution (7b). The second case is

\[ r_b \leq 0. \]  \hspace{1cm} (17)

The dominant eigenvector is given by (7b). Now one has to relabel the basis of the second qubit \(|0_2\rangle \leftrightarrow |1_2\rangle\).

GSD coefficients are

\[ g = b, \quad h = 0, \quad t_1 = c, \quad t_2 = d, \quad t_3 = a. \]  \hspace{1cm} (18)

c. Solution (7c). The third case is

\[ r_c \leq 0. \]  \hspace{1cm} (19)

The dominant eigenvector is given by (7c) and one has to relabel the basis of the third qubit \(|0_3\rangle \leftrightarrow |1_3\rangle\).

GSD coefficients are

\[ g = c, \quad h = 0, \quad t_1 = b, \quad t_2 = a, \quad t_3 = d. \]  \hspace{1cm} (20)

d. Solution (7d). The fourth case is

\[ r_d \leq 0. \]  \hspace{1cm} (21)

The dominant eigenvector is given by (7d) and one has to relabel the bases of all qubits \(|0_1\rangle \leftrightarrow |1_1\rangle, \ |0_2\rangle \leftrightarrow |1_2\rangle, \ |0_3\rangle \leftrightarrow |1_3\rangle\). Resulting GSD coefficients are

\[ g = d, \quad h = 0, \quad t_1 = a, \quad t_2 = b, \quad t_3 = c. \]  \hspace{1cm} (22)

All of four cases can be summarized as follows. If the state Eq.(6) is slightly entangled, then

i) its last coefficient \(h = 0\) vanishes and \(g\) takes the value of the largest coefficient

\[ h \equiv 0, \quad g = \max(a, b, c, d) \]  \hspace{1cm} (23)

ii) GSD takes the form

\[ |\psi\rangle = g|000\rangle + t_1|011\rangle + t_2|101\rangle + t_3|110\rangle, \]  \hspace{1cm} (24)

iii) All the inequalities (15), (17), (19) and (21) become the same lower bound on \(g\)

\[ g^2 \geq t_1^2 + t_2^2 + t_3^2 + 2t_1t_2t_3 g. \]  \hspace{1cm} (25)

The obvious conclusion is that \(h \neq 0\) only for the highly entangled states and identically vanishes for the slightly entangled states.
Note that four slightly entangled cases (15), (17), (19) and (21) together with the highly entangled case (10) cover the four-dimensional sphere given by the normalization condition. Thus the sixth solution does not create its own factorizable basis. However, it does not mean at all that the solution is unphysical. The state Eq.(6) is a four parametric state whiles generic three qubit states should have six parameters. Thus the state Eq.(6) is not generic and the applicable domain of the sixth’s solution has been shrunk to the separating surface \( r_1 r_2 r_3 = 0 \) in this case. It is easy to check that the sixth solution results the same GSD coefficients Eq.(12) for first type shared states. But it will create its own GSD for generic states.

Let’s now examine multiqubit W-states.

V. MULTIQUBIT W-TYPE STATES

In this section we consider one-parametric n-qubit W-states

\[
|\psi\rangle = a \left( |100 \cdots 0\rangle + |010 \cdots 0\rangle + \cdots + |00 \cdots 10\rangle \right) + b |00 \cdots 01\rangle.
\]

(26)

Slightly entangled region is given by \( r_n = (n - 1)a^2 - b^2 < 0 \) [23]. In this region the last product state \( |0 \cdots 01\rangle \) is the nearest separable state and

\[
g = b, \quad h = 0.
\]

(27)

In highly entangled region \( r_n > 0 \) and, consequently, \( S_n = (n - 1)^2 a^2 - b^2 > 0 \). The constituent states for the closest separable states are respectively

\[
|q_1\rangle = \cdots = |q_{n-1}\rangle = \frac{a\sqrt{(n-1)(n-2)}|0\rangle + \sqrt{r_n}|1\rangle}{\sqrt{S_n}}, \quad |q_n\rangle = \frac{\sqrt{(n-1)r_n}|0\rangle + b\sqrt{n-2}|1\rangle}{\sqrt{S_n}}.
\]

(28)

Straightforward calculation gives

\[
g = (1 - b^2)^{\gamma+1/2} \left[ \frac{n-2}{S_n} \right]^{\gamma}, \quad t_n = \sqrt{(n-2)r_n} \left[ \frac{r_n}{S_n} \right]^{\gamma}, \quad h = b\sqrt{n-1} \left[ \frac{r_n}{S_n} \right]^{\gamma}, \quad \varphi = \frac{\pi}{n-1},
\]

(29)

where

\[
\gamma = \frac{n-2}{2}.
\]

Expressions (29) have the same meanings as in the three-qubit case. First, \( r_n = 0 \) forces \( g^2 = 1/2 \). Second, \( g^2 > 1/2 \) and \( h = 0 \) means the state is slightly entangled. Third, \( g^2 < 1/2 \) and \( h = 0 \) means \( b = 0 \) and, therefore, the last qubit is unentangled. Fourth, we conjecture that: all the states with \( r_n = 0 \) allow the teleportation scenario and the receiver should choose \( n \)th qubit. In summary, suggested GSD indicates the applicability to the teleportation and distinguishes the unentangled particles as well as completely entangled particles.
VI. GHZ-TYPE STATES.

Consider now the extended GHZ state [24]

\[ |\psi\rangle = a|000\rangle + b|001\rangle + c|110\rangle + d|111\rangle. \tag{30} \]

Stationarity equations have two solutions[11]

\[ |q_1\rangle = |0\rangle, \quad |q_2\rangle = |0\rangle, \quad |q_3\rangle = \frac{a|0\rangle + b|1\rangle}{\sqrt{a^2 + b^2}}, \quad g = \sqrt{a^2 + b^2}; \tag{31a} \]

\[ |q_1\rangle = |1\rangle, \quad |q_2\rangle = |1\rangle, \quad |q_3\rangle = \frac{c|0\rangle + d|1\rangle}{\sqrt{c^2 + d^2}}, \quad g = \sqrt{c^2 + d^2}. \tag{31b} \]

The state function takes the canonical form Eq.(4) in two bases. In both cases nonzero coefficients of the decomposition can be presented as follows

\[ g = \max\left(\sqrt{a^2 + b^2}, \sqrt{c^2 + d^2}\right), \quad t_3 = \frac{ac + bd}{g}, \quad h = \frac{|ad - bc|}{g}. \tag{32} \]

This set of GSD coefficients describes the extended GHZ-type states almost in the same way as bipartite systems. Since \( g^2 \geq 1/2 \), there is no highly entangled region for GHZ-type states. In this sense W-state is more entangled than GHZ-state. When the extended GHZ-state is most entangled, i.e. \( g^2 = 1/2 \), it is applicable for both teleportation and dense coding and the situation is same in the case of bipartite systems. In contrast to W-type case, there is no region where \( h \) is identically zero. Only on condition \( ad = bc \) the canonical coordinate \( h \) vanishes. Thus if \( h \) vanishes, then the state is biseparable and again the same is true for two-qubit systems. The only difference from two-qubit case is that there is an extra term with the coefficient \( t_3 \). It shows that the third particle is unentangled when \( h = 0 \).

VII. SUMMARY

We have used a general way of constructing generalized Schmidt decomposition proposed in Ref.[8] and specified a unique decomposition for arbitrary composite systems consisting of two-level subsystems. This decomposition suggests a new set of local invariants. Three qubit pure states have following six invariants

\[ g, \ t_1, \ t_2, \ t_3, \ h, \ \varphi. \]

We have considered general three-qubit and W-type \( n \)-qubit systems whose injective tensor norms were already derived analytically. All the invariants have been calculated explicitly and expressed in terms of state parameters \( a, b, c, d \). It is shown that they provide a profound information on the quantum states. The largest
coefficient $g$ gives two entanglement measures and together with the last coefficient $h$ clearly distinguishes the states entangled in inequivalent ways. Namely, for W-type states there is a region where the function $h(a, b, c, d)$ is identically zero and there is no such region for GHZ-type states. Furthermore, isolated zeros of the function $h(a, b, c, d)$ indicate the appearance of the unentangled particles. The coefficients $t_i$ are related to Bloch vectors and reveal the existence of completely mixed reduced states. Thereby they show whether or not a given state is applicable for perfect teleportation (and dense coding) and precisely indicate which particle the receiver should choose at initial stage in order to perform the task.

We have derived a lower bound on injective tensor norm Eq. (25) for slightly entangled states ($h=0$). A natural question now arises: is it possible to derive such a lower bound for arbitrary three qubit states? This question is closely related to the following problem. For every measure, including geometric, there must be a set of states which are maximally entangled, or at least approaching some lower bound. And the problem is to find these states explicitly. If one derives a strong lower bound on $g$, then one can see when the lower bound is saturated and obtain maximally entangled states.

A lower bound on $g$ for generic three qubit states can be derived by making use of GSD form (4). Indeed, consider a state $|\psi_{\text{rest}}\rangle = t_1|011\rangle + t_2|101\rangle + t_3|110\rangle + e^{i\varphi}h|111\rangle$. Suppose $|pr\rangle$ is its nearest product state and $\mu(t_1, t_2, t_3, h)$ is its injective tensor norm. Obviously

$$g \geq |\langle pr |\psi\rangle| = |g\langle pr |000\rangle + \mu|. $$

This is a generic lower bound and its derivation reduces to the derivation of $|pr\rangle$. Unfortunately, known methods do not allow us to solve SEQ for $|\psi_{\text{rest}}\rangle$ and find $|pr\rangle$. Since the phases $\varphi$ can be absorbed in the definitions of basis vectors, $|\psi_{\text{rest}}\rangle$ is a four parametric state. It is interesting to note that the states of type $|\psi_{\text{rest}}\rangle$ are the only remaining four parametric states whose injective norm has not been obtained analytically so far. There is a good reason to calculate this injective tensor norm and obtain a lower bound on it for three qubit states.

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APPENDIX A: PROOF OF THEOREM 2

Proof. Suppose $t_1 = 0$ and $g^2 = 1/2$. Equation (4) allows to express the Bloch vectors of qubits in terms of GSD coefficients. Denote by $r_1$ the norm of the Bloch vector of the first particle. Straightforward calculation yields

$$r_1^2 = 4h^2 t_1^2 + (g^2 + t_1^2 - t_2^2 - t_3^2 - h^2)^2. \tag{A1}$$

Then $t_1 = 0$ and $g^2 = 1/2$ together with the normalization condition $g^2 + t_1^2 + t_2^2 + t_3^2 + h^2 = 1$ force $r_1 = 0$. Therefore the reduced state of the first particle is completely mixed.

Suppose now that the reduced state of the first particle is a completely mixed state. Then $r_1$ should vanish which is to say that

$$ht_1 = 0, \quad g^2 + t_1^2 - t_2^2 - t_3^2 - h^2 = 0. \tag{A2}$$

Either $t_1 = 0$ or $h = 0$. If $h = 0$, then the state Eq.(4) can be transformed to the state Eq.(6) by a local unitary transformation $|0_i\rangle \leftrightarrow |1_i\rangle$. In this reason both states have the same GSD and Eq.(4) coincides with Eq.(24). Thus the inequality Eq.(25) holds for both states. On the other hand $h = 0$ forces $g^2 = t_2^2 + t_3^2 - t_1^2$ and this is contradicted by inequality Eq.(25). Consequently $t_1 = 0$ and the the normalization condition forces $g^2 = 1/2$. This ends the proof of the theorem.