The best extending cover-preserving geometric lattices of semimodular lattices∗

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Abstract

In 2010, Gábor Czédli and E. Tamás Schmidt mentioned that the best cover-preserving embedding of a given semimodular lattice is not known yet [A cover-preserving embedding of semimodular lattices into geometric lattices, Advances in Mathematics 225 (2010) 2455-2463]. That is to say: What are the geometric lattices G such that a given finite semimodular lattice L has a cover-preserving embedding into G with the smallest |G|? In this paper, we propose an algorithm to calculate all the best extending cover-preserving geometric lattices G of a given semimodular lattice L and prove that the length and the number of atoms of every best extending cover-preserving geometric lattice G equal the length of L and the number of non-zero join-irreducible elements of L, respectively. Therefore, we comprehend the best cover-preserving embedding of a given semimodular lattice.

AMS classification: 06C10; 06B15

Keywords: Finite atomistic lattice; Semimodular lattice; Geometric lattice; Cover-preserving embedding

∗Supported by the National Natural Science Foundation of China (nos.11901064 and 12071325)
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1 Introduction

Let \( L \) be a lattice. For all \( a, b \in L \), \( a \parallel b \) denotes that \( a \nless b \) and \( a \nleq b \), and \( a \nparallel b \) denotes that \( a \geq b \) or \( a \leq b \). \( a \less b \) means that \( a < b \) and there is no element \( c \in L \) such that \( a < c < b \), and \( a \preceq b \) represents that \( a < b \) or \( a = b \). The set of non-zero join-irreducible elements and the set of atoms of \( L \) will be denoted by \( J(L) \) and \( A(L) \), respectively. The length of \( L \), that is, \( \sup \{ n : L \text{ has an (}n+1\text{-element chain} \} \), will be denoted by \( \ell(L) \). Let \( A \) and \( B \) be two sets. We define \( A - B = \{ x \in A : x \notin B \} \).

We assume that the readers are familiar with the basic notions of lattices such as a partially ordered set (poset), a chain, a lattice, a distributive lattice, a modular lattice, a semimodular lattice etc.. Here, we just recall a necessary concept from the theory of lattices (see, e.g., [3, 9]). We say a lattice \( L \) is (upper) semimodular if \( a \prec b \) implies \( a \lor c \preceq b \lor c \) for all \( a, b, c \in L \). We know from Crawely and Dilworth [3, Theorem 3.7] (see also [9, Theorem 1.7.1]) that for a strongly atomic algebraic lattice \( L \), a semimodularity is equivalent to Birkhoff’s condition:

\[
\text{for all } a, b \in L, \text{ if } a \land b \prec a \text{ and } a \land b \prec b, \text{ then } a \prec a \lor b \text{ and } b \prec a \lor b. 
\]

It is well known that if \( L \) is a semimodular lattice with \( \ell(L) = m \) and \( |J(L)| = n \), then \( n \geq m \) (see [9]).

Classically semimodular lattices arise out of certain closure operators satisfying what is now usually called the Steinitz-Mac Lane exchange property. A semimodularity is one of the most important links between combinatorics and lattice theory (see, e.g., [19]), and the structure of a semimodular lattice plays an important role in lattice theory (see, e.g., [6, 7]). A particular interest is deserved by geometric lattices, originally called matroids, which are semimodular atomistic lattices of finite length.

The Dilworth Embedding Theorem states that each finite lattice \( L \) can be embedded in a finite geometric lattice (see [3]). Further, P. Pudlák, J. Túma [8] proved that each finite lattice \( L \) can be embedded in a finite partition lattice (finite partition lattices are geometric lattices). In 1986, G. Grätzer and E. W. Kiss [5] showed that each finite semimodular lattice \( L \) has a cover-preserving embedding into a finite geometric lattice. Recently, G. Czédli and E. T. Schmidt [4] extended the results in [5], and they proved that each semimodular lattice \( L \) of finite length has a cover-preserving embedding into a geometric lattice \( G \) of the same length and the number of atoms of \( G \) equals the number of non-zero join-irreducible elements of \( L \). That is, they proved the following theorem.

**Theorem 1.1** ([4]) Let \( L \) be a semimodular lattice of finite length. Then there exists a geometric lattice \( G \) such that \( L \) is a cover-preserving sublattice of \( G \), \( |J(L)| = |A(G)| \) and \( \ell(L) = \ell(G) \).
Finally, they mentioned that the best cover-preserving embedding is not known yet. That is to say: What are the geometric lattices $G$ such that a given finite semimodular lattice $L$ has a cover-preserving embedding into $G$ with the smallest $|G|$? In this paper, we shall construct all the best cover-preserving embeddings of a given finite semimodular lattice $L$ into geometric lattices $G$ and prove that the length and the number of atoms of every best extending cover-preserving geometric lattice $G$ equal the length of $L$ and the number of non-zero join-irreducible elements of $L$, respectively.

For the detailed information on semimodular lattices and partially ordered sets the readers are referred to [1, 3, 6, 9]. We use the terminologies and notations of [1, 3].

2 Atomistic partially ordered sets

In this section, we shall introduce the concept of an atomistic partially ordered set and then investigate some of its basic properties.

**Definition 2.1** Let $(P, \leq)$ be a finite partially ordered set and

$$
\ell(P) = \sup \{ n : P \text{ has an } (n + 1)\text{-element chain} \}.
$$

Then we say that $\ell(P)$ is the length of $P$.

If $P$ has the minimum element 0, then let $\ell_P(x)$, or $\ell(x)$ for brevity, denote the length of $[0, x]$ for each element $x \in P$. Thus, $\ell(0) = 0$ and $\ell(1) = \ell(P)$ when 1 is the maximum element of $P$.

Similar to the definitions of atoms of lattices, an element that covers the least element 0 of a partially ordered set $P$ is referred to as an atom of $P$, and denoted by $A(P)$ the set of atoms of $P$, i.e., $A(P) = \{ x \in P : x \succ 0 \}$. In particular, $A_P(y) = A([0, y])$, or $A(y) = A([0, y])$ for brevity, for each $y \in P$.

**Example 2.1** The Hasse diagram of a partially ordered set $P$ is shown as Fig.1.
Fig. 1 The partially ordered set $P$.
In Fig. 1, $A(P) = A(1) = \{a, b, c\}$, $A(x) = A(y) = \{a, b\}$ and $A(0) = \emptyset$.

Definition 2.2 A finite partially ordered set $P$ with the minimum element $0$ is atomistic if and only if the following two conditions are satisfied: for all $x, y \in P$,
(1) $x < y$ implies that $A(x) \subsetneq A(y)$;
(2) $x \parallel y$ yields that $A(x) \subsetneq A(y)$ and $A(y) \subsetneq A(x)$.

Fig. 2 The atomistic partially ordered set $P$.

By Definition 2.2, one can check that Fig. 1 is not atomistic since $x \parallel y$ but $A(x) = A(y)$, and Fig. 2 is atomistic. Clearly, every finite atomistic lattice is an atomistic partially ordered set, but the inverse is not true generally. For example, Fig. 2 is an atomistic partially ordered set, but it is not a finite atomistic lattice since it is not a lattice. However, the following lemma is clear.

Lemma 2.1 If a finite atomistic partially ordered set $P$ is a lattice, then $P$ is an atomistic lattice.
Let $\mathcal{P}(X)$ be the power set of a nonempty set $X$. Then we easily verify the following lemma.

**Lemma 2.2** Let $|X| < \infty$ and $P \subseteq \mathcal{P}(X)$. If $\emptyset \in P$ and $\{x\} \subseteq P$ then $(P, \subseteq)$ is a finite atomistic partially ordered set.

For convenience, in the following, if $P$ is a finite atomistic partially ordered set then we denote $\mathcal{S}_P = \{A(x) : x \in P\}$.

**Lemma 2.3** If $P$ is a finite atomistic partially ordered set, then $(P, \leq) \cong (\mathcal{S}_P, \subseteq)$.

**Proof.** For $x \in P$, define $f : P \to \mathcal{S}_P$ to be a map such that

$$f(x) = A(x) \text{ for any } x \in P.$$  

We will show that the map $f$ is an isomorphism of partially ordered sets.

It is clear that the map $f$ is well-defined. If $x, y \in P$ and $x \neq y$, then $f(x) = A(x) \neq A(y) = f(y)$ by Definition 2.2. Hence, the map $f$ is injective. Moreover, it is clearly that there exists $x \in P$ such that $U = A(x) = f(x)$ for any $U \in \mathcal{S}_P$ from the definition of $\mathcal{S}_P$, i.e., the map $f$ is surjective. Consequently, the map $f$ is a one-to-one map. Below, we only need to prove that both $f$ and its inverse are order-preserving.

Set $x, y \in P$ and $x < y$, and observe that application of the condition (1) of Definition 2.2 yields $f(x) = A(x) \subset A(y) = f(y)$. Thus the map $f$ is order-preserving. Now suppose that $U, V \in \mathcal{S}_P$ and $U \subset V$. Then there exist $x, y \in P$ such that $U = A(x) \subset V = A(y)$. By Definition 2.2, $x < y$. Thus, the inverse of $f$ is order-preserving. Therefore, $(P, \leq) \cong (\mathcal{S}_P, \subseteq)$. \qed

By Lemma 2.3 every finite atomistic partially ordered set can be considered as a set of sets. For instance, Fig.2 and Fig.3 are isomorphic.
Definition 2.3 Let $P$ be a finite atomistic partially ordered set. A map $I_P$ from $P$ to the power sets of $\mathcal{P}(A(P))$ is called an independent function on $P$ if it has the following two properties: for any $x \in P$,
(1) if $\ell(x) = 0$, then $I_P(x) = \{\emptyset\}$;
(2) if $\ell(x) \geq 1$, then

$$I_P(x) = \{S \cup \{a\} : S \in I_P(y), a \in A(x) - A(y), \ell(x) = \ell(y) + 1 \text{ and } x \succ y\}.$$ 

Clearly, $\bigcup I_P(x) = A(x)$ for any $x \in P$. Let $P$ be a finite atomistic lattice. If $x, y \in P$, $x \parallel y$, then $\sigma \not\in A(y)$ for any $\sigma \in I_P(x)$.

From Definition 2.3 and Theorem 6.5 in [3], the following lemma is obviously.

Lemma 2.4 Let $L$ be a finite geometric lattice and $x \in L$. Then the following three conditions are equivalent:
(1) $S \in \mathcal{I}_L(x)$;
(2) $S$ is a maximal independent set of atoms of $[0, x]$;
(3) $S$ is an independent set of atoms of $L$ and $\bigvee S = x$.

The diamond $M_3$ (see Fig.4) is a geometric lattice and $\mathcal{I}_{M_3}(1) = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. One can verify that $a \lor b = a \lor c = b \lor c = 1$, and $\{a, b\}, \{a, c\}$ and $\{b, c\}$ are maximal independent sets of atoms of $M_3$.  

![Figure 3 The atomistic partially ordered set $S_P$.](image)
3 Constructions of geometric lattices

For the rest of this paper, unless otherwise stated, let $L$ be a fixed finite semimodular lattice. Following the convention of, say, Crawley and Dilworth [3] or Birkhoff [1], we assume that $L$ is non-empty. Let $H(L) = L - A(L) \cup \{0\}$. For $x \in H(L)$, let $\Delta(x)$ be a finite set satisfying that $\Delta(x) \cap L = \emptyset$ and $\Delta(x) \cap \Delta(y) = \emptyset$ while $x \neq y$, where $\Delta(x)$ may be empty set. Insert every element in $\Delta(x)$ into $L$. Extend the original order by $0 \prec x' \prec x$ for every $x' \in \Delta(x)$; this way we obtain a finite partially ordered set $(P, \leq)$ with $P = L \cup \bigcup_{x \in H(L)} \Delta(x)$. Notice that if $(P, \leq)$ is a lattice, then we call it an extending lattice of $L$. The constructions of three new finite partially ordered sets $P_1, P_2$ and $P_3$ are depicted in Fig.5; the black-filled elements are the inserted ones.

**Definition 3.1** If $\Delta(x) \neq \emptyset$ for every $x \in J(L) \cap H(L)$, then $(P, \leq)$ is called an extending standard form of $L$ where $P = L \cup \bigcup_{x \in H(L)} \Delta(x)$.

Now, let $\mathcal{E}(L)$ be the set of all the finite extending standard forms of $L$. In Fig.5, one can check that $P_2, P_3 \in \mathcal{E}(L)$ but $P_1 \notin \mathcal{E}(L)$.
In what follows, we write $L \hookrightarrow P$ when $L$ is a cover-preserving sublattice of a lattice $P$, and symbols $L \hookrightarrow P$, $L \hookrightarrow V P$ and $L \hookrightarrow \wedge P$ stand for that $L$ is a sublattice, a $\lor$-subsemilattice and a $\land$-subsemilattice of a lattice $P$, respectively. Then the following lemma is obvious.

**Lemma 3.1** Suppose that $P \in \mathcal{E}(L)$. Then $P$ is an atomistic lattice, $\ell(L) = \ell(P)$ and $L \hookrightarrow P$.

For convenience, if $P \in \mathcal{E}(L)$, then we denote $\Delta_P(L) = P - L$. It is well known that a finite semimodular lattice $L$ can also be expressed as sets of set (see [2]). Therefore, by Lemma 3.1 and Definition 3.1, if $P \in \mathcal{E}(L)$, then there exists a lattice $(T_P^L, \subseteq)$ with $T_P^L \subseteq S_P$ such that

$$L \cong T_P^L \hookrightarrow S_P \cong P$$

(1)

and

the identity map $\text{id}$ is a cover-preserving embedding map from $T_P^L$ to $S_P$. (2)

In fact, $T_P^L \cup A(S_P) = S_P$.

Consider the semimodular lattice $L$ and $L$'s extending standard form $P_2$ represented in Fig.5 again. Then the two lattices $(T_{P_2}^L, \subseteq)$ and $(S_{P_2}, \subseteq)$ in Fig.6 satisfy formula (1).

By formula (1) and the construction of $L$’s extending standard forms, the following lemma is clearly.

**Lemma 3.2** Let $Q \in \mathcal{E}(L)$ with $|A(Q)| > |J(L)|$. Then there exists an element $r \in \Delta_Q(L)$ such that $Q \setminus \{r\} \in \mathcal{E}(L)$. Further, let $P = Q \setminus \{r\}$. Then we have that $L \cong T_P^Q \hookrightarrow S_P$ and $T_P^Q = \{X \setminus \{r\} : X \in T_L^Q\}$ where $T_L^Q$ satisfies formulas (1) and (2).

![Fig.6 Two lattices](image-url)
The following example illustrates Lemma 3.2.

**Example 3.1** Consider the semimodular lattice $L$ and $L$'s extending standard form $P_3$ represented in Fig. 5 again. Then the two lattices $(\mathcal{T}_L^{P_3}, \subseteq)$ and $(\mathcal{S}_{P_3}, \subseteq)$ in Fig. 7 satisfy formulas (1) and (2).

![Fig. 7 Two lattices $(\mathcal{T}_L^{P_3}, \subseteq)$ and $(\mathcal{S}_{P_3}, \subseteq)$.](image)

Obviously, $P_2 = P_3 - \{d'_1\}$. Then, from Fig. 6, we know that $L \cong \mathcal{T}_L^{P_2} \hookrightarrow \mathcal{S}_{P_2}$ and $\mathcal{T}_L^{P_2} = \{X - \{d_1\} : X \in \mathcal{T}_L^{P_3}\}$.

As a conclusion of this section, we shall supply an algorithm to construct a finite geometric lattice $G$ which satisfies that $L \hookrightarrow G$ and $\ell(G) = \ell(L)$.

In the following, for each finite atomistic partially ordered set $P$ with $\ell(P) = m$, we define two maps $\varphi_P$ and $\varphi_P$ from $\{1, \ldots, m\}$ to the power set of $\mathcal{P}(A(P))$ and $\mathcal{P}(P)$ as

$\varphi_P(i) = \{\sigma \in \mathcal{I}_P(x) : \ell(x) = i, x \in P\}$

and

$\varphi_P(i) = \{x \in P : \ell(x) = i\},$

respectively. Let $(\mathcal{O}, \subseteq)$ be an atomistic partially ordered set, and

$\overline{X_{\mathcal{O}}} = \bigvee_{\mathcal{O}} \{T \in A(\mathcal{O}) : T \subseteq X\}$

for any set $X \subseteq 1_{\mathcal{O}}$ when $\bigvee_{\mathcal{O}} \{T \in A(\mathcal{O}) : T \subseteq X\}$ exists. Clearly, if $X \in \mathcal{O}$, then $\overline{X_{\mathcal{O}}} = X$.

Suppose that $P \in \mathcal{E}(L)$ and $\ell(L) = m$. Then the following algorithm’s output is a finite geometric lattice whose proof will be given in the next section.
Algorithm 3.1  

**Input:** $Q = \emptyset$, $R = S_P$, $k = 3$, $t = 0$ and $m = \ell(L)$.  

**Output:** $Q$. 

Step 1.  
$Q := R$ and $t := k$. If there exists $X \in \varphi_Q(k)$ which has a proper subset $U$ satisfying the following three conditions:  
(i1) if $V \in \varphi_Q(t - 1)$ and $Y \subseteq U \cap V$ then $\overline{Y} \subseteq U$;  
(ii) if $\sigma \in \phi_Q(k - 1)$ then $\bigcup \sigma \nsubseteq U$; and  
(iii) if $V \in \varphi_Q(k - 1)$, then $U \nsubseteq V$;  
then $Q := X \cup \{U\}$. Otherwise, $k := k + 1$, and if $k \geq m + 1$ then go to Step 5 and if not, go to Step 1. 

Step 2. If $\ell_Q(U) = k - 1$, then $k := 3$, $R := Q$ and go to Step 1. 

Step 3. If $U$ has a proper subset $W$ which satisfies the following three conditions:  
(j1) if $V \in \varphi_Q(t - 1)$ and $Y \subseteq W \cap V$ then $\overline{Y} \subseteq W$;  
(j2) if $\sigma \in \phi_Q(k - 2)$ then $\bigcup \sigma \nsubseteq W$; and  
(j3) if $V \in \varphi_Q(k - 2)$ then $W \nsubseteq V$;  
then $Q := X \cup \{W\}$. Otherwise, go to Step 1. 

Step 4. If $\ell_Q(W) = k - 2$, then $k := 3$, $R := Q$ and go to Step 1. Otherwise, $k := k - 1$ and go to Step 3. 

Step 5. Stop. 

4 All the finite geometric lattices 

In this section, we shall first prove that the output $Q$ in Algorithm 3.1 is an atomistic lattice, and then verify that $L$ is a cover-preserving sublattice of $Q$. Finally, we shall show that all the extending cover-preserving geometric lattices of $L$ with the same length can be constructed by Algorithm 3.1. 

Below this paper, for convenience, if $(P, \leq)$ is a finite atomistic lattice with $n$ atoms, then we denote $A(P) = \{1, \cdots, n\}$, and if $(S, \subseteq)$ is a finite atomistic lattice with $m$ atoms, then we denote $A(S) = \{\{1\}, \cdots, \{m\}\}$, and observe that 

$$U \land_S V = U \cap V$$

for any $U, V \in S$. 

**Lemma 4.1** Every output $Q$ in Algorithm 3.1 is a finite atomistic lattice. 

**Proof.** Note that inasmuch as Algorithm 3.1 and Lemmas 2.2 and 3.1, we know that every output $Q$ is a finite atomistic partially ordered set and it has the minimum element and the maximum element. Then it suffices to show that the output $Q$ is a $\land$-semilattice. 

Note that $S_P$ is an atomistic lattice by Lemma 3.1. Obviously, the $R$ in Step 3 equals to the $R$ in Step 1. Hence, by Algorithm 3.1 we only need to prove that each partially ordered set $R$ from Steps 2 and 4 returning to Step 1 in Algorithm 3.1 is a $\land$-semilattice.
For convenience, we next denote 

$$\mathcal{D}_{H,G} = \{W \in \mathcal{R} : W \subseteq H \cap G\}.$$ 

One can see that $\mathcal{D}_{H,G} = \mathcal{D}_{G,H}$.

The rest of the proof will be completed in three steps.

A. If $\mathcal{R}$ is in Step 2, then $\mathcal{R} = \mathcal{Q} \cup \{M_{t-1}\}$, in which $\mathcal{Q}$ is an atomistic lattice and $M_{t-1}$ is a proper subset of a certain $X$ in $\varphi_\mathcal{Q}(t)$ where $M_{t-1}$ satisfies the conditions (i1), (i2) and (i3). Let $E, F \in \mathcal{R}$. Then there are three cases.

Case 1. If $E, F \in \mathcal{Q}$, then as $\mathcal{Q}$ is an atomistic lattice, we know that $E \cap \mathcal{Q} F = E \cap F$ is the maximum of $\mathcal{D}_{E,F}$. Therefore, $E \cap \mathcal{R} F = E \cap F \in \mathcal{R}$.

Case 2. If $E \in \mathcal{Q}$ and $F = M_{t-1}$, then suppose that $E \parallel M_{t-1}$. Thus $E \cap \mathcal{R} M_{t-1} = E \in \mathcal{R}$ clearly. Now, assume that $E \parallel M_{t-1}$. If $R \in \mathcal{D}_{E,M_{t-1}}$ then $R \nsubseteq E$ and $R \nsubseteq M_{t-1}$. We claim that $X \nsubseteq E$. Otherwise $X \subseteq E$, which means that $M_{t-1} \subseteq X \subseteq E$, contrary to $E \parallel M_{t-1}$. We can distinguish two subcases.

Subcase 1. If $E \nsubseteq X$, then $\ell_\mathcal{Q}(E) \leq t - 1$ since $X \in \varphi_\mathcal{Q}(t)$.

Subcase 2. If $E \parallel X$, then $E \cap \mathcal{Q} X = E \cap X \nsubseteq X$. Thus $\ell_\mathcal{Q}(E \cap X) \leq t - 1$.

From Subcases 1 and 2, we know that there exists an element $E \cap X \in \mathcal{Q}$ such that $\ell_\mathcal{Q}(E \cap X) \leq t - 1$ and $E \cap M_{t-1} \subseteq E \cap X$ since $M_{t-1} \nsubseteq X$. Thus 

$$E \cap M_{t-1} \subseteq M_{t-1} \cap E \cap X,$$

and there exists an element $K \in \varphi_\mathcal{Q}(t - 1)$ such that $E \cap X \subseteq K$, or $E \cap X \in A(\mathcal{Q})$ by Algorithm 3.1. Then by (i1) of Algorithm 3.1, we have that $(E \cap M_{t-1})_\mathcal{Q} \subseteq M_{t-1}$. Note that $(E \cap M_{t-1})_\mathcal{Q} \subseteq E$. Therefore, 

$$\overline{(E \cap M_{t-1})_\mathcal{Q}} = E \cap M_{t-1} \in \mathcal{D}_{E,M_{t-1}}.$$ 

Consequently, $E \cap \mathcal{R} M_{t-1} = E \cap M_{t-1} \in \mathcal{R}$, i.e., $E \cap \mathcal{R} F \in \mathcal{R}$.

Case 3. If $E, F \in \mathcal{R} - \mathcal{Q}$, then clearly $E \cap \mathcal{R} F = E \cap F \in \mathcal{R}$.

In summary, $\mathcal{R}$ is a finite $\land$-semilattice.

B. If $\mathcal{R}$ is in Step 4 and $\mathcal{R} = \mathcal{Q} \cup \{M_{t-1}, M_{t-2}\}$ in which $M_{t-2}$ is a proper subset of $M_{t-1}$ and it satisfies the conditions (j1), (j2) and (j3). By Algorithm 3.1, we know that $M_{t-2} \nsubseteq M_{t-1} \nsubseteq X$. Suppose that $E, F \in \mathcal{R}$. There are four cases as follows.

Case i. If $E, F \in \mathcal{Q}$, then similar to the proof of Case 1, we have that $E \cap \mathcal{R} F = E \cap F \in \mathcal{R}$.

Case ii. If $E \in \mathcal{Q}$ and $F = M_{t-1}$, then similar to the proof of Case 2, we know that 

$$E \cap \mathcal{R} M_{t-1} = E \cap M_{t-1} \in \mathcal{R}.$$ 

Case iii. If $E \in \mathcal{Q}$ and $F = M_{t-2}$, then suppose that $E \parallel M_{t-2}$. Thus 

$$E \cap \mathcal{R} M_{t-2} = E \cap M_{t-2} \in \mathcal{R}.$$
Now, assume that $E \parallel M_{t-2}$. Obviously, $R \subset E$ and $R \subset M_{t-2}$ for any $R \in \mathcal{D}_{E,M_{t-2}}$, and $E \cap M_{t-2} \subset E$. There are two subcases as follows.

Subcase (i). If $E \parallel M_{t-1}$, then $M_{t-1} \subset E$ or $E \subset M_{t-1}$ since $E \neq M_{t-1}$. We claim that $E \subset M_{t-1}$. Otherwise, $M_{t-2} \subset M_{t-1} \subset E$, contrary to the fact that $E \parallel M_{t-2}$. Thus $E \subset X$, and it follows from $X \in \varphi_Q(t)$ that

$$\ell_Q(E) \leq t - 1.$$  

Subcase (ii). If $E \parallel M_{t-1}$, then similar to the proof of Subcase 2, we know that there exists an element $E \cap X \in Q$ such that

$$M_{t-2} \cap E \subseteq E \cap X$$
and
$$\ell_Q(E \cap X) \leq t - 1$$
since $M_{t-2} \cap E \subseteq M_{t-1} \cap E$.

Subcases (i) and (ii) mean that there exists an element $E \cap X \in Q$ such that

$$\ell_Q(E \cap X) \leq t - 1 \text{ and } E \cap M_{t-2} \subseteq E \cap X.$$ 

Hence

$$E \cap M_{t-2} \subseteq M_{t-2} \cap E \cap X,$$
and there exists an element $K \in \varphi_Q(t-1)$ such that $E \cap X \subseteq K$, or $E \cap X \in A(Q)$ by Algorithm 3.1. Then by (j1) of Algorithm 3.1 we have that $(E \cap M_{t-2})_Q \subseteq M_{t-2}$. Note that $(E \cap M_{t-2})_Q \subseteq E$. Therefore,

$$(E \cap M_{t-2})_Q = E \cap M_{t-2} \in \mathcal{D}_{E,M_{t-2}}.$$ 

Consequently, $E \land_R M_{t-2} = E \cap M_{t-2} \in R$.

Case iv. If $E, F \in R - Q$, then, clearly, $E \land_R F = E \cap F \in R$.

In summary, $R$ is a finite $\land$-semilattice.

C. Analogously, if $R$ is in Step 4 and $R = Q \cup \{M_{t-1}, M_{t-2}, \ldots, M_{t-r}\}$ for $r \in \{3, \ldots, t - 2\}$ where $M_{t-r}$ is a proper subset of $M_{t-(r-1)}$ and it satisfies the conditions (j1), (j2) and (j3), then we can prove that $R$ is a finite $\land$-semilattice.

To sum up, the output $Q$ in Algorithm 3.1 is a finite atomistic lattice. This completes the proof. \qed

Algorithm 3.1, Definition 2.3 and Lemma 4.1 imply the following lemma.

**Lemma 4.2** Let $P \in \mathcal{E}(L)$, and $Q$ be the output of Algorithm 3.1. Then the following three statements hold.

1. $\ell_{S_P}(X) = \ell_Q(X)$ for any $X \in S_P$.
2. If $\sigma \in \mathcal{I}_{S_P}(X)$, then $\sigma \in \mathcal{I}_Q(X)$ for any $X \in S_P$.
3. If $X, Y \in \varphi_Q(k)$ and $X \neq Y$, then $\bigcup \sigma \not\subseteq Y$ for any $\sigma \in \mathcal{I}_Q(X)$. 

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**Lemma 4.3** For every output $Q$ in Algorithm 3.1, $L \leq Q$.

**Proof.** Note that $\ell(L) = \ell(P)$ since $P \in \mathcal{E}(L)$, and by Lemma 4.2, $\ell(Q) = \ell(S_P)$. Thus we have that

$$\ell(Q) = \ell(L) \text{ since } P \cong S_P.$$  

By formula (1) and Algorithm 3.1, there exists a lattice $T^P_L$ such that $T^P_L \subseteq S_P \subseteq Q$ and $L \cong T^P_L \leq Q$. Therefore, we only need to prove that $T^P_L \leq Q$.

First, by Lemma 3.1, $S_P$ is a finite atomistic lattice. Then $E \wedge T^P_L F = E \cap F$ for any $E, F \in T^P_L$ by (2). On the other hand, from Lemma 4.1, we know that $Q$ is a finite atomistic lattice, which follows that $H \wedge Q G = H \cap G$ for any $H, G \in Q$. Consequently, $T^P_L \leq Q$ since $T^P_L \subseteq Q$.

Next, we shall prove that $T^P_L \leq \vee Q$.

Let $M, N \in T^P_L$. Set $M \vee T^P_L N = Z$ and $M \vee Q N = T$. Then $M \vee S_P N = Z$ by (2). Suppose that $T \neq Z$, then $T \in Q - S_P$. Obviously, $T \subseteq Z$ since $T^P_L \subseteq Q$. We claim that $M \parallel N$. Otherwise, $T = Z = M \cup N$, a contradiction. As $T^P_L$ is a finite semimodular lattice, we know that $T^P_L$ contains a sublattice lattice as presented in Fig. 8 (the required coverings $\prec$ and $\subseteq$ in the lattice $T^P_L$ are indicated by one line and double lines in Fig. 8, respectively). Furthermore, by formula (1), Fig. 8 is also a sublattice of $S_P$ and $\ell_{T^P_L}(R) = \ell_{S_P}(R)$ for any $R \in T^P_L$. Therefore, by Lemma 4.2

$$\ell_Q(R) = \ell_{S_P}(R) = \ell_{T^P_L}(R)$$  

(3)

for every $R \in T^P_L$.

Now, set $\ell_{T^P_L}(N) = t$. As $T^P_L$ is a finite semimodular lattice and Fig. 8 is a sublattice of $T^P_L$, we obviously have that $\ell_{T^P_L}(N_k) = t + k$ and $\ell_{T^P_L}(Z) = t + k + 1$, which together with formula (3) imply

$$\ell_Q(N_k) = \ell_{S_P}(N_k) = t + k \text{ and } \ell_Q(Z) = \ell_{S_P}(Z) = t + k + 1.$$  

(4)
Let \( \eta \in I_{SP}(N) \). Then by Definition 2.3 and formula (4), there exists a subset \( \rho \) of \( A_{SP}(M) \) such that \( \eta \cup \rho = \pi \in I_{SP}(N_k) \) since Fig.8 is also a sublattice of \( S_P \). Hence \( \pi \in I_{Q}(N_k) \) by Lemma 4.2.

Using formula (4), \( T \subset Z \) and \( T \in Q - S_P \), clearly, there is a \( T_0 \in Q - S_P \) such that \( M \cup N \subseteq T \subseteq T_0 \subsetneq Z \) and \( \ell_Q(T_0) = t + k \), which follow by Lemma 4.2 that \( \bigcup \sigma \not\subseteq T_0 \) for every \( \sigma \in I_Q(N_k) \) since \( \ell_Q(N_k) = t + k \) and \( T_0 \neq N_k \). However, \( \pi = \eta \cup \rho \subseteq A_{SP}(M) \cup A_{SP}(N) \), and then \( \bigcup \pi \subseteq M \cup N \subseteq T_0 \), contrary to \( \pi \in I_Q(N_k) \).

In summary, \( T_P L \rightarrow \lor Q \). Therefore, \( L \rightarrow \prec Q \). This completes the proof.

Notice that the identity map \( i_d \) is a cover-preserving embedding map from \( T_P L \) to \( Q \) by the proof of Lemma 4.3.

Below, denote
\[
\mathcal{S} = \{ Q : Q \text{ is an output of Algorithm 3.1} \}
\]
and
\[
\overline{\mathcal{S}} = \{ Q \in \mathcal{S} : Q \text{ satisfies the condition (M)} \}
\]
in which the condition (M) is as follows.

(M): If \( X \in \varphi_Q(k) \), then \( \ell_Q((X \cup R)_Q) = k + 1 \) for any \( R \in A(Q) - A(X) \).

**Lemma 4.4** Every \( Q \in \overline{\mathcal{S}} \) is a finite geometric lattice.

**Proof.** By Lemma 4.1, we know that \( Q \) is a finite atomistic lattice. Then \([\emptyset, M]\) for any \( M \in Q \) is a geometric lattice when \( \ell_Q(M) \leq 2 \). Now, suppose that \([\emptyset, M]\) is a geometric lattice for every \( M \in Q \) with \( \ell_Q(M) \leq k \). By induction, we shall prove that \([\emptyset, M]\) is a geometric lattice for every \( M \in Q \) with \( \ell_Q(M) = k + 1 \).

Assume that \([\emptyset, M]\) is not a semimodular lattice. Then there exist two elements \( G, H \in [\emptyset, M] \) such that \( G \succ G \cap H, H \succ G \cap H \) but \( G \lor H \not\succ G \) or \( G \lor H \not\succ H \), say, \( G \lor H \not\succ G \). Note that \( G \lor H \in [\emptyset, M] \). We claim that \( G \lor H = M \). Otherwise, \( G \lor H \subsetneq M \). Hence, \( \ell_Q(G \lor H) \leq k \). Therefore, \([\emptyset, G \lor H]\) is a geometric lattice. This follows that \( G \lor H \succ G \) since \( G, H, G \cap H, G \lor H \in [\emptyset, G \lor H] \), a contradiction. Consequently, \( G \lor H = M \), and which yields that \([\emptyset, M]\) contains a sublattice as presented in Fig.9 (the required coverings \( \prec \) and \( \subseteq \) in the lattice \([\emptyset, M]\) are indicated by one line and double lines in Fig.9, respectively).
Let $R \in A(H) - A(G)$. We claim that $R \notin G_{m-1}$. Otherwise, $G_{m-1} \cap H \supseteq G \cap H$. Then $H \supseteq G_{m-1} \cap H \supseteq G \cap H$ together with $H \succ G \cap H$ yields that $G_{m-1} \supseteq H$. Thus $G \lor H \subseteq G_{m-1}$, a contradiction to the fact $G \lor H = M$. Therefore, $R \notin G_{m-1}$, then $R \in A(Q) - A(G_{m-1})$. Note that $\ell_Q(G_{m-1}) \leq k - 1$ by the structure of Fig.9 and $\ell_Q(M) = k + 1$. Then by (9), we know that $\ell_Q((G_{m-1} \cup R)_Q) \leq k$.  

On the other hand, $H \supseteq \overline{(G \cap H) \cup R}_Q \supseteq G \cap H$, it follows from $H \succ G \cap H$ that $H = \overline{(G \cap H) \cup R}_Q$. Thus $(G_{m-1} \cup H)_Q = (G_{m-1} \cup R)_Q$, and then $\ell_Q((G_{m-1} \cup H)_Q) \leq k$ by formula (6). However, $(G_{m-1} \cup H)_Q \supseteq G \lor H = M$ and $\ell_Q(M) = k + 1$, a contradiction. Therefore, $[\emptyset, M]$ is a semimodular lattice.

Consequently, $[\emptyset, M]$ is a finite geometric lattice as $Q$ is a finite atomistic lattice, and the proof of the lemma is complete. \qed

Notice that from Lemmas 4.1, 4.3 and 4.4, we know that every output $Q$ in Algorithm 3.1 with condition (9) is a geometric lattice and $L$ is a cover-preserving sublattice of $Q$.

The following example will illustrate that every output $Q$ in Algorithm 3.1 with condition (9) is a geometric lattice and $L$ is a cover-preserving sublattice of $Q$.

**Example 4.1** Consider the lattices $L$ and $P \in \mathcal{E}(L)$ represented in Fig.10, respectively.
Fig.10 Four lattices $L, P, T^p_L$ and $S_P$.

Obviously, $T^p_L$ and $S_P$ satisfy formula (1), respectively.

**Input**: $Q = \emptyset$, $R = S_P$, $k = 3$, $t = 0$ and $m = 3$.

**Output**: $Q$.

**Step 1**. $Q := R$, $t := 3$, $U_1 = \{1, 2, 4\}$ is a proper subset of $\{1, 2, 3, 4, 5\}$ satisfying (i1), (i2) and (i3), and $Q := Q \cup \{U_1\}$.

**Step 2**. $\ell_Q(U_1) = 2$, $k := 3$ and $R := Q$ (the lattice $R$ as represented in Fig.11).

**Step 3**. $Q := R$, $t := 3$, $U_2 = \{1, 5\}$ is a proper subset of $\{1, 2, 3, 4, 5\}$ satisfying (i1), (i2) and (i3), and $Q := Q \cup \{U_2\}$.

**Step 4**. $\ell_Q(U_2) = 2$, $k := 3$ and $R := Q$ (the lattice $R$ as represented in Fig.12).

**Step 5**. $Q := R$, $t := 3$, $U_3 = \{2, 5\}$ is a proper subset of $\{1, 2, 3, 4, 5\}$ satisfying (i1), (i2) and (i3), and $Q := Q \cup \{U_3\}$.

**Step 6**. $\ell_Q(U_3) = 2$, $k := 3$ and $R := Q$ (the lattice $R$ as represented in Fig.13).

**Step 7**. $Q := R$, $t := 3$, $U_4 = \{4, 5\}$ is a proper subset of $\{1, 2, 3, 4, 5\}$ satisfying (i1), (i2) and (i3), and $Q := Q \cup \{U_4\}$.

**Step 8**. $\ell_Q(U_4) = 2$, $k := 3$ and $R := Q$ (the lattice $R$ as represented in Fig.14).

**Step 9**. $Q := R$, $t := 3$ and $\{1, 2, 3, 4, 5\}$ has no proper subset satisfying (i1), (i2) and (i3), $k = 4 \geq 4$.

**Step 10**. Stop.
Fig. 11 The lattice $\mathcal{R}$.

Fig. 12 The lattice $\mathcal{R}$.
Therefore, the output $Q$ in Algorithm 3.1 is the $\mathcal{R}$ as represented in Fig.14. One can check that $Q \in \mathcal{S}$, $Q$ is a finite geometric lattice and $L \hookrightarrow \mathcal{Q}$.

**Definition 4.1** Let $L_1$ and $L_2$ be two finite atomistic lattices with $L_1 \subseteq L_2$. If $L_1$ satisfies: for any $p \in L_1$,
(e1) $A_{L_1}(p) = A_{L_2}(p)$;
(e2) $\ell_{L_1}(p) = \ell_{L_2}(p)$; and
(e3) $[0, p]_{L_1} = [0, p]_{L_2}$ when $\ell_{L_1}(p) \leq k$,
then we say that $L_1$ is a $k$ order normal subset lattice of $L_2$.

**Lemma 4.5** Let $(Q, \subseteq)$ be a finite geometric lattice with $L \hookrightarrow Q$ and $\ell(L) = \ell(Q)$. Then $Q \in \mathfrak{S}$.

**Proof.** Let $\ell(Q) = m$. As $Q$ is a finite geometric lattice, $Q$ satisfies condition (M). Thus we only need to prove that $Q$ is an output of Algorithm 3.1. Since $L \hookrightarrow Q$, there exists a lattice $T \subseteq Q$ such that $L \sim T \hookrightarrow Q$. Hence there exists a lattice $P \in \mathcal{E}(L)$ such that $S_P = T \cup A(Q)$. Thus there exists a lattice $T_P \subseteq S_P$ such that $T_P \hookrightarrow S_P$ by formula (1).

Because $Q$ is geometric, the following four statements hold.

C1. For every $M \in Q$, $[\emptyset, M]_Q$ is a geometric lattice.

C2. If $M, N \in Q$, then $(M \cap N)_Q = M \cap N$.

C3. If $M, N \in \varphi_Q(k)$ and $M \neq N$, then $M \notin N$ and $N \notin M$.

C4. If $\sigma \in \phi_Q(k)$, $M \in \varphi_Q(k)$ and $\bigvee_Q \sigma \neq M$, then $\sigma \notin \mathcal{J}_Q(M)$ and $\bigcup \sigma \notin M$.

The rest of the proof will be completed in three steps.

(I). Let $R_1 = S_P$. Then by Definition 4.1, $R_1$ is a 2 order normal subset lattice of $Q$. Suppose that $X \in R_1$ and $\ell_{R_1}(X) = 3$. Let $U \in Q - R_1$, $U \subset X$ and $\ell_Q(U) = 2$. If $V \in \varphi_{R_1}(2)$, then $(U \cap V)_{R_1} = (U \cap V)_Q$ by (e3). It follows from C2 that

$$\frac{(U \cap V)_{R_1}}{R_1} \subseteq U.$$ (7)

Obviously, by C4 and (e2) and (e3) in Definition 4.1,

$$\bigcup \sigma \notin U$$ (8)

for any $\sigma \in \phi_{R_1}(2)$. Moreover, by (e2) in Definition 4.1 and C3, we have

$$U \notin V$$ (9)

for every $V \in \varphi_{R_1}(2)$ since $U \neq V$. Thus, by formulas (7), (8) and (9), we know that $U$ satisfies (i1), (i2) and (i3) in Algorithm 3.1 Therefore, $R_1 \cup \{U\}$ is an atomistic lattice by the proof of Lemma 4.1. Clearly, $R_1 \cup \{U\}$ is a 2 order normal subset lattice of $Q$.

Suppose that $E \in R_1 \cup \{U\}$ and $\ell_{R_1 \cup \{U\}}(E) = 3$. Let $U_1 \in Q - (R_1 \cup \{U\})$, $U_1 \subset E$ and $\ell_Q(U_1) = 2$. Similar to the proof of the preceding paragraph, we can prove that $R_1 \cup \{U\} \cup \{U_1\}$ is an atomistic lattice which is a 2 order normal subset lattice of $Q$ since $R_1 \cup \{U\}$ is a 2 order normal subset lattice of $Q$.
Repeating the process as above, we can obtain an atomistic lattice
\[ \mathcal{R}_2 = \mathcal{R}_1 \cup \bigcup \{ [\emptyset, M]_\mathcal{Q} : \ell_{\mathcal{R}_1}(M) = 3 \}. \]

Obviously,
\[ \mathcal{R}_2 = \mathcal{R}_1 \cup \bigcup \{ [\emptyset, M]_\mathcal{Q} : \ell_{\mathcal{Q}}(M) = 3, M \in \mathcal{R}_1 \} \] (10)
by (e2) in Definition 4.1. Therefore, \( \mathcal{R}_2 \) is a 3 order normal subset lattice of \( \mathcal{Q} \), and for any \( F \in \mathcal{R}_2 \) with \( \ell_{\mathcal{R}_2}(F) \leq 3 \), \( F \) has no proper subset \( N \) satisfying (i1), (i2) and (i3) in Algorithm 3.1.

(II). Suppose that \( X \in \mathcal{R}_2 \) and \( \ell_{\mathcal{R}_2}(X) = 4 \). Let \( U \in \mathcal{Q} - \mathcal{R}_2 \), \( U \subsetneq X \) and \( \ell_{\mathcal{Q}}(U) = 3 \). There are two cases as below.

Case 1. If there exists \( E \in \varphi_{\mathcal{R}_2}(2) \) such that \( E \subsetneq U \). Similar to the proof of formulas (7), (8) and (9), we can verify that \( U \) satisfies (i1), (i2) and (i3) in Algorithm 3.1. Therefore, \( \mathcal{R}_2 \cup \{ U \} \) is an atomistic lattice by the proof of Lemma 4.1. Clearly, \( \mathcal{R}_2 \cup \{ U \} \) is a 2 order normal subset lattice of \( \mathcal{Q} \). Thus, similar to the proof of (10), we can obtain an atomistic lattice \( \mathcal{R}_2 \cup [\emptyset, U]_\mathcal{Q} \) which is a 3 order normal subset lattice of \( \mathcal{Q} \).

Case 2. If there is no element \( E \in \varphi_{\mathcal{R}_2}(2) \) such that \( E \subsetneq U \), then there exists \( U_1 \in \mathcal{Q} - \mathcal{R}_2 \) such that \( U_1 \subsetneq U \) and \( \ell_{\mathcal{Q}}(U_1) = 2 \) since \( \mathcal{Q} \) is geometric. Thus we have that the following three results.

(a1) By (e2), (e3), C1 and C2, \( \overline{U_1 \cap V}_{\mathcal{R}_2} \subseteq U_1 \) for any \( V \in \varphi_{\mathcal{R}_2}(3) \);
(a2) By (e2) and C4, \( \bigcup \sigma \notin U_1 \) for any \( \sigma \in \phi_{\mathcal{R}_2}(2) \);
(a3) By (e2) and C3, \( U_1 \notin V \) for any \( V \in \varphi_{\mathcal{R}_2}(2) \).
Therefore, \( U_1 \) satisfies (j1), (j2) and (j3) in Algorithm 3.1. This follows that \( \mathcal{R}_2 \cup \{ U, U_1 \} \) is an atomistic lattice which is a 2 order normal subset lattice of \( \mathcal{S} \). Analogous to the proof of Case 1, we can obtain an atomistic lattice \( \mathcal{R}_2 \cup [\emptyset, U]_\mathcal{Q} \) which is a 3 order normal subset lattice of \( \mathcal{Q} \).

From Cases 1 and 2, we always obtain an atomistic lattice \( \mathcal{R}_2 \cup [\emptyset, U]_\mathcal{Q} \) which is a 3 order normal subset lattice of \( \mathcal{Q} \).

Continuing as above, we can obtain an atomistic lattice
\[ \mathcal{R}_3 = \mathcal{R}_2 \cup \bigcup \{ [\emptyset, M]_\mathcal{Q} : \ell_{\mathcal{R}_2}(M) = 4 \}. \]

Obviously,
\[ \mathcal{R}_3 = \mathcal{R}_2 \cup \bigcup \{ [\emptyset, M]_\mathcal{Q} : \ell_{\mathcal{Q}}(M) = 4, M \in \mathcal{R}_2 \} \] by (e2) in Definition 4.1. Therefore, \( \mathcal{R}_3 \) is a 4 order normal subset lattice of \( \mathcal{Q} \), and for any \( G \in \mathcal{R}_3 \) with \( \ell_{\mathcal{R}_3}(G) \leq 4 \) there is no element \( H \subsetneq G \) such that \( H \) satisfies (i1), (i2) and (i3) in Algorithm 3.1.

(III). Repeating the preceding proof, we finally obtain an atomistic lattice
\[ \mathcal{R}_{m-1} = \mathcal{R}_{m-2} \cup \bigcup \{ [\emptyset, M]_\mathcal{Q} : \ell_{\mathcal{Q}}(M) = m, M \in \mathcal{R}_{m-2} \}, \]
and for any $W \in \mathcal{R}_{m-1}$ with $\ell_{\mathcal{R}_{m-1}}(W) \leq m$ there is no element $Z \subseteq W$ such that $Z$ satisfies (i1), (i2) and (i3) in Algorithm 3.1. Consequently, by $\ell_Q(M) = m$, we know that $[\emptyset, M]_Q = Q = \mathcal{R}_{m-1}$, and $Q$ is an output of Algorithm 3.1, completing the proof. 

Notable that Lemmas 4.1, 4.3, 4.4 and 4.5 deduce that we can construct all the finite extending cover-preserving geometric lattices of $L$ with the same length by Algorithm 3.1. However, applying the method suggested by G. Czédli and E. T. Schmidt in [4] to the $L$ as depicted in Fig.10, one can only obtain the finite extending cover-preserving geometric lattice as is shown by Fig.15.

5 The best geometric lattices

In this section, we shall construct all the best extending cover-preserving geometric lattices of $L$. Denote $\mathfrak{S}_k = \{S \in \mathfrak{S} : |A(S)| = k\}$ for any integer $k > 0$. Then we have the following Lemma.

Lemma 5.1 Let $\mathcal{K} \in \mathfrak{S}_k$ with $k > |J(L)|$. Then there exists an element $\mathcal{H} \in \mathfrak{S}_{k-1}$ such that $|\mathcal{H}| < |\mathcal{K}|$. 

Fig.15 The geometric lattice $G$. 


Proof. Since $K \in \mathfrak{K}$, $L \hookrightarrow K$ by Lemma 4.3. Then there exists a lattice $T \subseteq K$ such that $L \cong T \hookrightarrow K$. Hence there exists a lattice $Q \in \mathfrak{E}(L)$ such that $S_Q = T \cup A(K)$. This follows from formula (\ref{1}) that there exists a lattice $T^Q_L$ such that $T^Q_L \subseteq S_Q$ and $L \cong T^Q_L \hookrightarrow S_Q$. By $k > |J(L)|$, we know that there exists an element $r \in \Delta_Q(L)$ such that $Q - \{r\} \in \mathfrak{E}(L)$. Set $P = Q - \{r\}$ and $R = \{r\}$. Then $L \cong T^P_L \hookrightarrow S_P$ and

$$T^P_L = \{F - R : F \in T^Q_L\} \quad (11)$$

by Lemma 3.2 and there exists a set $\sigma \in \mathfrak{I}_K(X)$ such that $\bigcup \sigma \subseteq X - R$ for any $X \in T^Q_L$. Hence, as $T^Q_L \subseteq S_Q$, there exists a set $\sigma \in \mathfrak{I}_K(X)$ such that

$$\bigcup \sigma \subseteq X - R \quad (12)$$

for any $X \in T^Q_L$ by Lemma 4.2. Note that $K$ is a finite geometric lattice. Then by Lemma 2.4, we have that

$$E_K = (E - R)_K = \bigvee_{\kappa} \sigma \quad (13)$$

whenever $E \in T^Q_L$, $\sigma \in \mathfrak{I}_K(E)$ and $\bigcup \sigma \subseteq E - R$.

Set

$$\mathcal{H} = \{W - R : W \in K, (W - R)_K = W_K\}. \quad (14)$$

Then, by formulas (\ref{11}), (\ref{13}) and (\ref{14}), we know that

$$T^P_L \subseteq \mathcal{H}. \quad (15)$$

Now, we shall show that $L \hookrightarrow \mathcal{H}$ and $\mathcal{H}$ is a geometric lattice. The proof is made in three steps.

A. $\mathcal{H}$ is a finite atomistic lattice.

From (\ref{14}), it is clear that $\mathcal{H}$ is a finite atomistic partially ordered set. Thus, it suffices to show that $\mathcal{H}$ is a lattice.

Suppose $M, N \in \mathcal{H}$. Obviously,

$$R \nsubseteq M \text{ and } R \nsubseteq N. \quad (16)$$

If $M \nparallel N$, then $M \wedge_{\mathcal{H}} N = M \cap N \in \mathcal{H}$. Now, suppose that $M \parallel N$ and denote $D_{M,N} = \{G \in \mathcal{H} : G \subseteq M \cap N\}$. There are three cases.

Case i. If $M, N \in K$, then $M \cap N \in K$. Clearly, $(M \cap N - R)_K = (M \cap N)_K = M \cap N$ by (16). From (\ref{14}), $M \cap N \in \mathcal{H}$. Therefore, $M \wedge_{\mathcal{H}} N = M \cap N$.

Case j. If $M \in K$ and $N \notin K$, then, clearly, $N \cup R \in K$ since $N \in \mathcal{H}$. Thus $M \cap (N \cup R) \in K$. By formula (16), $M \cap (N \cup R) = M \cap N = M \cap N - R$, so that
\[ M \cap (N \cup R) \rangle \subseteq (M \cap N) \rangle = (M \cap N - R) \rangle. \] Hence, \( M \cap N \in \mathcal{H} \) by (14). Therefore, \( M \land H \cap N = M \cap N. \)

Case k. If \( M, N \not\in \mathcal{K} \), then similar to the proof of Case j, we have \( M \cup R, N \cup R \in \mathcal{K} \). Then \( (M \cap N) \cup R \in \mathcal{K} \). There are two subcases.

Subcase 1°. If \( [(M \cap N) \cup R] \rangle \subseteq (M \cap N) \rangle \), then by (14), \( M \cap N \in \mathcal{H} \) which is the maximum element of \( 
\mathcal{D}_{M,N} \). Thus, \( M \land H \cap N = M \cap N. \)

Subcase 2°. If \( [(M \cap N) \cup R] \rangle \not\subseteq (M \cap N) \rangle \), then \( (M \cap N) \rangle \subseteq [(M \cap N) \cup R] \rangle \), and \( (M \cap N) \cup R] \rangle = (M \cap N) \cup R \) since \( (M \cap N) \cup R \in \mathcal{K} \). Thus

\[ M \cap N \subseteq (M \cap N) \rangle \subseteq [(M \cap N) \cup R] \rangle = (M \cap N) \cup R, \]

which means that \( M \cap N = (M \cap N) \rangle \in \mathcal{K} \). Then, by (16), we know that \( M \cap N = (M \cap N) \rangle = [(M \cap N) - R] \rangle \). Therefore, \( M \cap N \in \mathcal{H} \) by (14), it follows that \( M \land H \cap N = M \cap N. \)

Subcases 1° and 2° imply that \( M \land H \cap N = M \cap N \) if \( M, N \not\in \mathcal{K} \).

Therefore, from Cases i, j and k, \( \mathcal{H} \) is a finite atomistic lattice.

B. \( \mathcal{H} \) is a finite geometric lattice.

Inasmuch as we possess A it suffices to show that \( \mathcal{H} \) is a semimodular lattice. Let \( M, N \in \mathcal{H} \) and \( M, N \succ \mathcal{H} \cap M \cap N \). Obviously, \( M \parallel N \). Next, we shall prove that \( M \lor \mathcal{H} \cap M \cap N \). There are three cases as follows.

Case a. If \( M, N \in \mathcal{K} \), then \( M \cap N \in \mathcal{K} \). By formulas (14) and (16), \( [\emptyset, M] \rangle = [\emptyset, M] \rangle \) and \( [\emptyset, N] \rangle = [\emptyset, N] \rangle \). Thus \( M, N \succ \mathcal{K} \cap M \cap N \), which together with \( \mathcal{K} \) is a geometric lattice yields that \( M \lor \mathcal{K} \cap M \cap N \).

We claim that \( (M \lor \mathcal{K} \cap N) \rangle = (M \lor \mathcal{K} \cap N - R) \rangle. \)

Otherwise, \( (M \lor \mathcal{K} \cap N - R) \rangle \not\subseteq (M \lor \mathcal{K} \cap N) \rangle \). It is clear that \( M \not\subseteq M \cup N \subseteq (M \lor \mathcal{K} \cap N - R) \rangle \) by (16). Note that \( (M \lor \mathcal{K} \cap N) \rangle = M \lor \mathcal{K} \cap N \) since \( M \lor \mathcal{K} \cap N \in \mathcal{K} \). Thus

\[ M \not\subseteq (M \lor \mathcal{K} \cap N - R) \rangle \subseteq (M \lor \mathcal{K} \cap N) \rangle = M \lor \mathcal{K} \cap N, \]

contrary to the fact that \( M \lor \mathcal{K} \cap N \succ \mathcal{K} \). Therefore, \( (M \lor \mathcal{K} \cap N) \rangle = (M \lor \mathcal{K} \cap N - R) \rangle \), and then by (14), \( M \lor \mathcal{K} \cap N - R \in \mathcal{H} \). Hence, the condition \( M \lor \mathcal{K} \cap N \succ \mathcal{K} \) deduces that

\[ M \lor \mathcal{K} \cap N - R = M \lor \mathcal{H} \cap M \lor \mathcal{K} \cap N. \]

Case b. If \( M, N \not\in \mathcal{K} \), then \( M \cup N, N \cup R, (M \cap N) \cup R \in \mathcal{K} \) since \( M, N, M \cap N \in \mathcal{H} \).

Thus

\[ M \cup R = (M \cup R) \rangle = \overline{M} \rangle, N \cup R = (N \cup R) \rangle = \overline{N} \rangle \]

(17)

and

\[ (M \cap N) \cup R = [(M \cap N) \cup R] \rangle = (M \cap N) \rangle \]

(18)
by (14).

On the other hand, we claim that \( M \cup R \succ_K (M \cap N) \cup R \). Otherwise, there exists an atom \( I \subseteq M - N \) such that \( M \cup R \supseteq [(M \cap N) \cup R \cup I]_K \supseteq (M \cap N) \cup R \). Then by formula (18), \( [(M \cap N) \cup R \cup I]_K = [(M \cap N) \cup I]_K \), which means that \( [(M \cap N) \cup I]_K - R \in \mathcal{H} \). Thus \( M \supseteq [(M \cap N) \cup I]_K - R \supseteq M \cap N \), contrary to the fact that \( M \succ \mathcal{H} M \cap N \). Therefore,

\[
M \cup R \succ_K (M \cap N) \cup R. \quad (19)
\]

Similarly, we have \( N \cup R \succ_K (M \cap N) \cup R \). Thus \( (M \cup R) \lor_K (N \cup R) \succ_K M \cup R, N \cup R \), and which means that

\[
\overline{(M \cup N \cup R)}_K = (M \cup R) \lor_K (N \cup R) \succ_K M \cup R, N \cup R. \quad (20)
\]

We claim that \( [(M \cup N \cup R)_K]_K = [(M \cup N \cup R)_K - R]_K \). Otherwise,

\[
(M \cup N \cup R)_K \supseteq [(M \cup N \cup R)_K - R]_K \supseteq (M \cup N)_K
\]

since \( [(M \cup N \cup R)_K]_K = (M \cup N \cup R)_K \) and \( (M \cup N \cup R)_K - R \supseteq M \cup N \). However, \( (M \cup N)_K \supseteq M_K = M \cup R \) by (17). Hence \( (M \cup N)_K = (M \cup N \cup R)_K \), a contradiction. Therefore, \( [(M \cup N \cup R)_K]_K = [(M \cup N \cup R)_K - R]_K \). This follows that \( (M \cup N \cup R)_K - R \in \mathcal{H} \) by (14). Further, by formula (20),

\[
(M \cup N \cup R)_K - R = (M \cup N)_K - R = M \lor \mathcal{N} M \succ \mathcal{H} M, N.
\]

Case c. If \( M \notin \mathcal{K} \) and \( N \in \mathcal{K} \), or \( M \in \mathcal{K} \) and \( N \notin \mathcal{K} \), say, \( M \notin \mathcal{K} \) and \( N \in \mathcal{K} \), then \( M \cup R, N, M \cap N \in \mathcal{K} \) since \( M, N, M \cap N \in \mathcal{H} \). Hence,

\[
M \cup R = (M \cup R)_K = M_K \quad (21)
\]

by (14).

Similar to the proof of formula (19), we have that \( (M \cup R) \succ_K M \cap N \). On the other hand, similar to the proof of Case a, we know that \( N \succ_K M \cap N \). Thus \( (M \cup R) \lor_K N \succ_K M \cup R, N \), and which means that

\[
(M \cup N \cup R)_K = (M \cup R) \lor_K N \succ_K M \cup R, N. \quad (22)
\]

Analogous to the proof of Case b, we know that

\[
(M \cup N \cup R)_K - R = (M \cup N)_K - R = M \lor \mathcal{N} M \succ \mathcal{H} M, N
\]

by formulas (14), (21) and (22).

In summary, \( \mathcal{H} \) is a finite geometric lattice.

C. \( \mathcal{T}_L^P \) \( \leftarrow \rightarrow \mathcal{H} \).
Let $M, N \in \mathcal{T}_L^P$. Then there are two elements $E, F \in \mathcal{T}_L^Q \subseteq \mathcal{S}_Q$ such that $M = F - R$ and $N = E - R$ by (11). Thus by $L \cong \mathcal{T}_L^P \cong \mathcal{T}_L^Q$ and (11),

$$M \vee_{\mathcal{T}_L^P} N = (F \vee_{\mathcal{T}_L^Q} E) - R.$$ \hfill (23)

As $F, E \in \mathcal{T}_L^Q$, we have that

$$F \vee_{\mathcal{K}} E = \overline{(M \cup N)_{\mathcal{K}}} = \overline{((M \cup N)_{\mathcal{K}} - R)_{\mathcal{K}}}$$

by formula (12) and (13). Hence, by formulas (14),

$$M \vee_{\mathcal{H}} N = \overline{(M \cup N)_{\mathcal{K}}} - R = F \vee_{\mathcal{K}} E - R.$$ \hfill (24)

Clearly, by formula (15), we know that

$$F \vee_{\mathcal{T}_L^P} E = F \vee_{\mathcal{K}} E,$$

which together with formulas (23) and (24) clearly leads to $M \vee_{\mathcal{H}} N = M \vee_{\mathcal{T}_L^P} N$. Therefore, $\mathcal{T}_L^P \hookrightarrow \vee \mathcal{H}$. On the other hand, by formula (11), we know that $M \wedge_{\mathcal{T}_L^P} N = M \cap N$ for any $M, N \in \mathcal{T}_L^P$. Thus $\mathcal{T}_L^P \hookrightarrow \wedge \mathcal{H}$ by the proof of A. Hence, $\mathcal{T}_L^P \hookrightarrow \mathcal{H}$.

Obviously, $L \cong \mathcal{T}_L^P \cong \mathcal{T}_L^Q$ together with (14) means that $\ell(\mathcal{T}_L^P) = \ell(\mathcal{H})$. Therefore, $\mathcal{T}_L^P \hookrightarrow \mathcal{K}$ since $\mathcal{T}_L^P \hookrightarrow \mathcal{H}$, completing the proof of C.

Finally, from B, C and Lemma 4.5, we know that $\mathcal{H} \in \overline{\mathcal{S}}_{k-1}$ and $|\mathcal{H}| < |\mathcal{K}|$. This completes the proof. \qed

Let $G$ be a finite geometric lattice. It is clear that if $L \hookrightarrow \mathcal{H}$, then there exists a sublattice $[x, y]$ of $G$ with $\ell([x, y]) = \ell(L)$ such that $L \hookrightarrow [x, y]$. Clearly, $[x, y]$ is also a geometric lattice. Therefore, by Lemmas 4.1, 4.3, 4.4, 4.5 and 5.1, we have the following theorem.

**Theorem 5.1** Every best extending cover-preserving geometric lattice of $L$ is the best one in $\overline{\mathcal{S}}_{|J(L)|}$.

**Example 5.1** Consider the lattice $L$ in Example 4.1 again. If $U = \{1, 2, 4, 5\}$ in Step 1 of Algorithm 3.1. Then $\mathcal{Q} = \mathcal{S}_P \cup \{U\}$ is the output lattice of Algorithm 3.1 (the lattice $\mathcal{Q}$ as represented in Fig.16).
Obviously, $Q \in \mathfrak{F}_5$. Further, we know that $Q$ is the unique best extending cover-preserving geometric lattice of $L$ in the sense of isomorphism.

6 Conclusions

In this paper, we proposed an algorithm to calculate all the best extending cover-preserving geometric lattice $G$ of a given semimodular lattice $L$ and proved that $|A(G)| = |J(L)|$ and $\ell(G) = \ell(L)$. It is worth pointing out that every different $U$ (resp. $W$) in Algorithm 3.1 leads to a different output, and the computational complexity of Algorithm 3.1 is likely to grow rapidly as $|J(L)|$ and $\ell(L)$ grow.

Data availability statements

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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