An Auxiliary 'Differential Measure' for $SU(3)$

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Abstract

A 'differential measure' is used to cast our calculus for the group $SU(3)$ into a form similar to Schwinger’s boson operator calculus for the group $SU(2)$. It is then applied to compute (i) the inner product between the basis states and (ii) an algebraic formula for the Clebsch-Gordan coefficients. These were obtained earlier by us using Gaussian integration techniques.

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1. INTRODUCTION

In a previous paper \cite{2} we set up a calculus for dealing with computations in $SU(3)$. One of the main ingredients of this calculus is an 'auxiliary' Gaussian measure using which all computations on $SU(3)$ are reduced to Gaussian integrations. As an application of this calculus we also derived a closed form algebraic expression for the Clebsch-Gordan coefficients involving direct products of arbitrary irreducible representations of $SU(3)$. Apart from solving this important and long standing problem completely this calculus can also be used advantageously in any situation involving computations over irreducible representations of $SU(3)$. Therefore it would be interesting to develop alternative ways of implementing the tools of this calculus.

In this paper we use an 'auxiliary differential measure' in place of the above mentioned Gaussian measure and obtain all our previous results for $SU(3)$. In contrast to our earlier scalar product which involved integrations using a Gaussian measure here the 'differential measure' involves a differential operator which operates on functions of basis states and gives the scalar product as the end product. This present method corresponds to the operator techniques in field theories as opposed to the path integral techniques of our earlier method. Therefore this is not a different inner product but is rather a different technique of evaluating the same inner product which we used in our earlier paper. A similar measure was used by Schwinger \cite{3} in his systematic derivation of the results of the angular momentum algebra. Later Ruegg \cite{7} used a simplified version of it to obtain the basis states and Clebsch-Gordan coefficients of $SU_q(2)$. We have retained the name 'differential measure', first used by Ruegg in the reference cited, to denote the differential operator that we mentioned in the above. We make use of Schwinger’s evaluation of the scalar products fully and, for this purpose, reproduce his derivation in the appendix to this paper.

The plan of the paper is as follows. In section 2 we review our previous results which we obtained using our calculus for $SU(3)$. Then in section 3 we derive the auxiliary normalizations of our basis states for $SU(3)$ using an 'auxiliary differential measure' and show that
it agrees with our previous results using an ‘auxiliary’ Gaussian measure. Next in section 4 we apply the ‘differential measure’ to obtain an algebraic formula for the Clebsch-Gordan coefficients of $SU(3)$ and show that our results in this case also agree with our previous ones. In section 5 we derive the overall normalizations of the Clebsch-Gordan coefficients. This corresponds to computing the norms of the invarints. In our previous work we have left this to be computed by hand. The last section is devoted to a discussion of the results of this paper.

2. OVERVIEW OF OUR PREVIOUS RESULTS

$SU(3)$ is the group of $3 \times 3$ unitary unimodular matrices $U$ with complex coefficients. It is a group of 8 real parameters. The matrix elements satisfy the following conditions

$$U = (u_{ij}),$$
$$U^\dagger U = I, \quad UU^\dagger = I, \quad \text{where } I \text{ is the identity matrix and},$$
$$\det(U) = 1. \quad (2.1)$$

A. Parametrization.

One well known parametrization of $SU(3)$ is due to Murnaghan [1]. In this we write a typical element of $SU(3)$ as :

$$D(\delta_1, \delta_2, \phi_3)U_{23}(\phi_2, \sigma_3)U_{12}(\theta_1, \sigma_2)U_{13}(\phi_1, \sigma_1), \quad (2.2)$$

with the condition $\phi_3 = -(\delta_1 + \delta_2)$. Here $D$ is a diagonal matrix whose elements are $\exp(i\delta_1), \exp(i\delta_2), \exp(i\phi_3)$ and $U_{pq}(\phi, \sigma)$ is a $3 \times 3$ unitary unimodular matrix which for instance in the case $p = 1, q = 2$ has the form

$$\begin{pmatrix}
\cos \phi & -\sin \phi \exp(-i\sigma) & 0 \\
\sin \phi \exp(i\sigma) & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (2.3)$$
The 3 parameters $\phi_1$, $\phi_2$, $\phi_3$ are longitudinal angles whose range is $-\pi \leq \phi_i \leq \pi$; and the remaining 6 parameters are latitude angles whose range is $\frac{1}{2}\pi \leq \sigma_i \leq \frac{1}{2}\pi$.

Now the transformations $U_{23}$ and $U_{13}$ can be changed into transformations of the type $U_{12}$ whose matrix elements are known, by the following device

\[ U_{13}(\phi_1, \sigma_1) = (2,3)U_{12}(\phi_1, \sigma_1)(2,3), \]
\[ U_{23}(\phi_2, \sigma_3) = (1,2)(2,3)U_{12}(\phi_2, \sigma_3)(2,3)(1,2), \tag{2.4} \]

where $(1,2)$ and $(2,3)$ are the transposition matrices

\[
(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2,3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}
\]

In this way the expression for an element of the $SU(3)$ group becomes

\[
D(\delta_1, \delta_2, \phi_3)(1,2)(2,3)U_{12}(\phi_2, \sigma_3)(2,3)(1,2)U_{12}(\theta_1, \sigma_2)(2,3)U_{12}(\phi_1, \sigma_1)(2,3). \tag{2.6}
\]

B. Irreducible representations.

The parametrization, described in the previous section, provides us with a defining irreducible representation 3 of $SU(3)$ acting on a 3 dimensional complex vector space spanned by the triplet $z_1, z_2, z_3$ of complex variables. The hermitian adjoint of the above matrix gives us another defining but inequivalent irreducible representation $3^*$ of $SU(3)$ acting on the triplet $w_1, w_2, w_3$ of complex variables spanning another 3 dimensional complex vector space. Tensors constructed out of these two 3 dimensional representations span an infinite dimensional complex vector space \(3\).

C. The Constraint

If we impose the constraint
\[ z_1 w_1 + z_2 w_2 + z_3 w_3 = 0 , \quad (2.7) \]

on this space we obtain an infinite dimensional complex vector space in which each irreducible representation of \( SU(3) \) occurs and once and only once. Such a space is called a model space for \( SU(3) \). Further if we solve the constraint \( z_1 w_1 + z_2 w_2 + z_3 w_3 = 0 \) and eliminate one of the variables, say \( w_3 \), in terms of the other five variables \( z_1, z_2, z_3, w_1, w_2 \) we can write a generating function to generate all the basis states of all the IRs of \( SU(3) \) each exactly only once. This generating function is computationally a very convenient realization of the basis of the model space of \( SU(3) \). Moreover we can define an auxiliary scalar product on this space by choosing one of the variables, say \( z_3 \), to be a planar rotor \( \exp(i \theta) \). Thus the model space for \( SU(3) \) is now a Hilbert space with this scalar product between the basis states. But it must be admitted that this scalar product is not \( SU(3) \) invariant though the normalizations obtained from it can be easily related to the ones got from a group invariant scalar product.

The above construction was carried out in detail in a previous paper by us \[2\]. For easy accessability we give a self-contained summary of those results here.

**D. Labels for the basis vectors.**

(i). Gelfand-Zetlein labels

Normalized basis vectors are denoted by, \( |M, N; P, Q, R, S, U, V \rangle \). All labels are non-negative integers. All Irreducible Representations(IRs) are uniquely labeled by \((M, N)\). For a given IR \((M, N)\), labels \((P, Q, R, S, U, V)\) take all non-negative integral values subject to the constraints:

\[ R + U = M , \quad S + V = N , \quad P + Q = R + S . \quad (2.8) \]

The allowed values can be prescribed easily: \( R \) takes all values from 0 to \( M \), and \( S \) from 0 to \( N \). For a given \( R \) and \( S \), \( Q \) takes all values from 0 to \( R + S \).
(ii). Quark model labels.

The relation between the Gelfand-Zetlein labels and the Quark Model labels is as given below.

\[ 2I = P + Q = R + S, \quad 2I_3 = P - Q, \quad Y = \frac{1}{3}(M - N) + V - U = \frac{2}{3}(N - M) - (S - R). \]

(2.9)

E. Explicit realization of the basis states.

(i). Generating function for the basis states of \( SU(3) \)

The generating function for the basis states of the IR’s of \( SU(3) \) can be written as

\[ g(p, q, r, s, u, v) = \exp(r(pz_1 + qz_2) + s(pw_2 - qw_1) + uz_3 + vw_3). \]

(2.10)

The coefficient of the monomial \( p^P q^Q r^R s^S u^U v^V \) in the Taylor expansion of Eq.(2.10) in terms of these monomials gives the basis state of \( SU(3) \) labelled by the quantum numbers \( P, Q, R, S, U, V \).

(ii). Formal generating function for the basis states of \( SU(3) \)

The generating function Eq.(2.10) can be written formally as

\[ g = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV \rangle, \]

(2.11)

where \( |PQRSUV \rangle \) is an unnormalized basis state of \( SU(3) \) labelled by the quantum numbers \( P, Q, R, S, U, V \).

Note that the constraint \( P + Q = R + S \) is automatically satisfied in the formal as well as explicit Taylor expansion of the generating function.

(iii). Generalized generating function for the basis states of \( SU(3) \)

It is useful, while computing the normalizations of the basis states, to write the above generating function in the following form.
\[ G(p, q, r, s, u, v) = \exp(r_p z_1 + r_q z_2 + s_p w_2 + s_q w_1 + uz_3 + vw_3). \] (2.12)

In the above generalized generating function (2.12) the following notation holds.

\[ r_p = rp, \quad r_q = rq, \quad s_p = sp, \quad s_q = sq. \] (2.13)

F. Notation

In the recapitulation below and in the derivations later we will assume that all our variables (except the \( z_i \) and the \( w_i \)) are real (eventhough we have treated them as complex variables at some places) since we are interested only in extracting the various objects, auxiliary normalization constants of basis states, Clebsch-Gordan coefficients etc., occurring as coefficients in the expansion of the scalar products as power series in these variables.

G. 'Auxiliary' scalar product for the basis functions.

The scalar product to be defined in this section is 'auxiliary' in the sense that it does not give us the 'true' normalizations of the basis astes of \( SU(3) \). However it is computationally very convenient for us as all computations with this scalar product get reduced to simple Gaussian integrations and the 'true' normalizations themselves can then be got quite easily.

(i). Scalar product between generating functions of basis states of \( SU(3) \)

We define the scalar product between any two basis states in terms of the scalar product between the corresponding generating functions as follows:

\[ (g', g) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int \frac{d^2 z_1}{\pi^2} \frac{d^2 z_2}{\pi^2} \frac{d^2 w_1}{\pi^2} \frac{d^2 w_2}{\pi^2} \exp(-\bar{z}_1 z_1 - \bar{z}_2 z_2 - \bar{w}_1 w_1 - \bar{w}_2 w_2) \]

\[ \times \exp((r'(p' z_1 + q' z_2) + s'(p' w_2 - q' w_1)) - \frac{v'}{z_3} (z_1 w_1 + z_2 w_2) + u' \bar{z}_3) \]

\[ \times \exp((r(p z_1 + q z_2) + s(p w_2 - q w_1)) - \frac{v}{z_3} (z_1 w_1 + z_2 w_2) + u z_3), \]
\[ = (1 - v'v)^{-2} \left( \sum_{n=0}^{\infty} \frac{(u'u)^n}{(n!)^2} \right) \exp \left[ (1 - v'v)^{-1} (p'p + q'q)(r'r + s's) \right]. \]  

(2.14)

(ii). Choice of the variable \( z_3 \)

To obtain the Eq.(2.14) we have made the choice

\[ z_3 = \exp(i\theta). \]  

(2.15)

The choice, Eq.(2.15), makes our basis states for \( SU(3) \) depend on the variables \( z_1, z_2, w_1, w_2 \) and \( \theta \).

(iii). Scalar product between the generalized generating functions of the basis states of \( SU(3) \)

For the generalized generating function the scalar product becomes

\[ (G', G) = (1 - v'v)^{-2} \exp \left[ (1 - v'v)^{-1} (r_p'r_p + r_q'r_q + s_p's_p + s_q's_q) \right] \]

\[ \times \left[ \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( u' - v' \frac{(r_p's_q + r_q's_p)}{(1 - v'v)} \right)^n \cdot \left( u - v' \frac{(r_p's_q + r_q's_p)}{(1 - v'v)} \right)^n \right]. \]  

(2.16)

If in this we substitute from Eq.(2.13) for \( r_p, \ldots, s_q \) etc we get back the scalar product Eq.(2.14) between the ordinary generating functions Eq.(2.10).

H. Normalizations

Based on the scalar products defined in the previous subsection we arrive at the normalizations described below.

(a). 'Auxiliary' normalizations of unnormalized basis states

The scalar product between two unnormalized basis states, computed using our 'auxiliary scalar product, is given by

\[ M(PQRSUV) \equiv (PQRSUV|PQRSUV), \]

\[ = \frac{(V + P + Q + 1)!}{P!Q!R!S!U!V!(P + Q + 1)}. \]  

(2.17)
(b). Scalar product between the unnormalized and normalized basis states

The scalar product, computed using our 'auxiliary' scalar product, between an unnormalized basis state and a normalized one is denoted by \((PQRSUV\|PQRSUV >\) and is given below

\[
(PQRSUV\|PQRSUV >= N^{-1/2}(PQRSUV) \times M(PQRSUV).
\]

(c). 'True' normalizations of the basis states

We call the ratio of the 'auxiliary' norm of the unnormalized basis state represented by \(|PQRSUV\rangle\) and the scalar product of the normalized Gelfand-Zeitlin state, represented by \(|PQRSUV >\) with an unnormalized basis state, as 'true' normalization. It is given by

\[
N^{1/2}(PQRSUV) = \frac{(PQRSUV\|PQRSUV)}{(PQRSUV\|PQRSUV >} = \left(\frac{(U + P + Q + 1)!V(P + Q + 1)!}{P!Q!R!S!U!V!(P + Q + 1)}\right)^{1/2}.
\]

I. Generating function for the invariants.

All Clebsch-Gordan coefficients can be extracted from the following generating function of the invariants:

\[
I_{\pm}(j_{12}, j_{23}, j_{31}, j_{21}, j_{32}, j_{13}, j_{\pm}) = \exp\left(j_{12} \cdot \widetilde{z}^1 \cdot \widetilde{w}^2 + j_{23} \cdot \widetilde{z}^2 \cdot \widetilde{w}^3 + j_{31} \cdot \widetilde{z}^3 \cdot \widetilde{w}^1 + j_{21} \cdot \widetilde{z}^3 \cdot \widetilde{w}^3 + j_{32} \cdot \widetilde{z}^2 \cdot \widetilde{z}^2 \times \widetilde{z}^3 \right)
\]

where the \(j_{mn}, m, n = 1, 2, 3, m \neq n \) and \(j_+, j_-\) are any eight complex variables.

J. Multiplicity labels for the Clebsch-Gordan series.

For given three IRs, \((M^1, N^1), (M^2, N^2), (M^3, N^3)\), construct all the solutions of
\[ N(1, 2) + N(1, 3) + L \varepsilon(L) = M^1, \]
\[ N(2, 3) + N(2, 1) + L \varepsilon(L) = M^2, \]
\[ N(3, 1) + N(3, 2) + L \varepsilon(L) = M^3, \]
\[ N(2, 1) + N(3, 1) + |L| \varepsilon(-L) = N^1, \]
\[ N(3, 2) + N(1, 2) + |L| \varepsilon(-L) = N^2, \]
\[ N(1, 3) + N(2, 3) + |L| \varepsilon(-L) = N^3, \]
\[ 3L = \sum_{n=1}^{3} (M^n - N^n), \quad (2.21) \]

where \( N(a, b), a = b \) are non-negative integers. They provide unambiguous labels for the Clebsch-Gordan series as follows. For given two IRs \((M^a, N^a)\) and \((M^b, N^b)\), construct all \((M^{a''}, N^{a''})\) for which \( N(a, b), a \neq b \) have non-negative integer solutions. Then the reversed pair \((N^{a''}, M^{a''})\) gives all IRs in the Clebsch-Gordan series. Multiplicity of solutions for one \((M^{a''}, N^{a''})\) provides the multiplicity of repeating IRs. Therefore \( N(a, b) \) unambiguously provide the multiplicity labels.

**K. Clebsch-Gordan coefficients.**

(i). \( 3 - SU(3) \) Symbol.

\[ 3 - G \] symbols are related [2] to the Clebsch-Gordan coefficients and have more explicit symmetry than the latter. The \( 3 - SU(3) \) symbol is represented by,

\[
\begin{bmatrix}
N^2 & M^1 & N^3 & M^2 & N^1 & M^3 & N^2 \\
V^1, R^1 & V^3, S^3 & V^2, R^2 & V^1, S^1 & V^3, R^3 & V^2, S^2 \\
N(1, 2) & N(1, 3) & N(2, 3) & N(2, 1) & N(3, 1) & N(3, 2) & L
\end{bmatrix}
\quad (2.22)
\]

Here the top row specifies the multiplicity labels. The second and third rows specify the usual complete set of labels for the basis states of the three IRs.

(ii). **Generating function for the** \( 3 - SU(3) \) **symbol for** \( L > 0 \).
Extract the coefficient of the monomial

\[
\frac{N(1,2)}{J_{12}} \frac{N(1,3)}{J_{13}} \frac{N(3,1)}{J_{31}} \frac{N(2,3)}{J_{23}} \frac{N(3,2)}{J_{32}} \frac{J_L}{J_L} \prod_{\alpha=1}^{3} P^\alpha q^\alpha \tilde{R}^\alpha \tilde{u} U^\alpha \tilde{v} V^\alpha ,
\]

in the expansion of

\[
\|B\|^2 \exp \left[ \|B\| \left( (j_2 r^2 u^3 - r^1 j_{12} s^2 + r^2 j_{21} s^1 + r^1 j_{12} s^2 (u^2 j_{23} v^3 + u^1 j_{13} v^3) \\
- r^2 j_{21} s^1 (u^2 j_{23} v^3 + u^2 j_{23} v^3) + r^1 j_{13} v^3 u^3 j_{32} s^2 - r^2 j_{23} v^3 u^3 j_{31} s^1) \times (p^1 q^2 - p^2 q^1) \\
+ (\text{cyclic}) \right) \right] ,
\]

where

\[
\|B^{-1}\| = \text{det}(1 - \tilde{J} \tilde{V}) ,
\]

\[
\tilde{V} = \begin{pmatrix}
\tilde{v}^1 & 0 & 0 \\
0 & \tilde{v}^2 & 0 \\
0 & 0 & \tilde{v}^3
\end{pmatrix} ,
\]

\[
J = \begin{pmatrix}
j_{31} + j_{21} & -j_{12} & -j_{13} \\
-j_{21} & j_{12} + j_{32} & -j_{23} \\
-j_{31} & -j_{32} & j_{23} + j_{13}
\end{pmatrix} .
\]

and multiply it by the factor

\[
\prod_{\alpha=1}^{3} \left[ P^\alpha q^\alpha R^\alpha S^\alpha U^\alpha V^\alpha (U^\alpha + 2 I^\alpha)(2 I^\alpha + 1) \right]^{1/2}.
\]

This gives the 3 – SU(3) symbol up to an overall normalization depending only on IRs involved.

(iii). **Formula for 3 – SU(3) symbol for** \( L > 0 \).

We have obtained an explicit analogue of the Bargamann's formula for the 3 – \( j \) symbol of SU(2). This formula for \( L > 0 \) is given below
\[
\begin{bmatrix}
N(1, 2) & N(2, 3) & N(3, 1) & L & N(1, 3) & N(3, 2) & N(2, 1) \\
M^1N^1 & M^2N^2 & M^3N^3 \\
I^1I_3^1Y^1 & I^2I_3^2Y^2 & I^3I_3^2Y^3 \\
\end{bmatrix}
\]

(2.28)

\[
= n(N(1, 2), N(2, 3), N(3, 1), N(2, 1), N(3, 2), N(1, 3), L)
\]

\[
\times \prod_{\alpha=1}^{3} \left[ \frac{P^\alpha!Q^\alpha!R^\alpha!S^\alpha!U^\alpha!V^\alpha!(U^\alpha! + 2I^\alpha!1)(2I^\alpha! + 1)}{(V^\alpha! + 2I^\alpha! + 1)} \right]^{1/2}
\]

\[
\times \sum_{e,f,g,k,l,m} \frac{(1 + \sum e() + f() + g() + k() + l() + m() + n()!)}{(1 + \sum(k() + l() + m() + n()! \prod l()!...n())!}
\]

\[
\times (-1)^{\sum \epsilon(k() + n()!) + \sum A m() + \sum f() + g()) ,
\]

(2.29)

where \( e(), f(), g(), l(), k(), m(), n() \) are the powers of the various monomials occurring in the above expansion. The notation is explained in the tables given below. The symbol

\[
n(N(1, 2), N(2, 3), N(3, 1), N(2, 1), N(3, 2), N(1, 3), L) ,
\]

stands for the overall normalization of the Clebsch-Gordan coefficients and therefore it does not depend on any particular IR. It is the norm of the invariants [4].

\[
P^\alpha = \sum (l(\alpha - -) + k(\alpha - - -) + m(\alpha - - - -) + n(\alpha - - - - -));
\]

\[
Q^\alpha = \sum (l(-\alpha -) + k(-\alpha - -) + m(-\alpha - - -) + n(-\alpha - - - -));
\]

\[
R^\alpha = \sum (l(\alpha - -) + l(-\alpha -) + k(- \alpha - -) + m(- \alpha - - -) + n(- \alpha - - - -));
\]

\[
S^\alpha = \sum (k(- - \alpha -) + m(- - \alpha - -) + n(- - \alpha - - -));
\]

\[
U^\alpha = \sum (l(- \alpha) + m(- - - \alpha -) + n(- - - - \alpha -) + e(- \alpha -) + f(\alpha - -) + f(- - \alpha -) + 2g(\alpha \beta \gamma));
\]

\[
V^\alpha = \sum (m( - - - - \alpha -) + n( - - - - \alpha -) + e(\alpha) + f(- \alpha -) + f(- - \alpha) + g(- \alpha -) + g(- \alpha));
\]
\[ L^\alpha = \sum (- - -); \]
\[ N(\alpha, \beta) = \sum (k(- - \alpha \beta) + m(- - \alpha \beta - - ) + m(- - - - \alpha \beta) + n(- - \alpha \beta - - ) + n(- - - \alpha \beta) + e(\alpha \beta) + f(\alpha \beta - - ) + f(- - \alpha \beta) + g(\alpha \beta - ) + g(\alpha - \beta)); \quad (2.31) \]

Here \( \sum \) stands for summation over all allowed arguments in the blank spaces.

| \( \alpha, \beta, \gamma \cdots = 1, 2 \text{ or } 3 \) | Description |
| --- | --- |
| \( l(\alpha \beta \gamma) \) | \( (\alpha \beta \gamma) \) is a permutation of \((123)\) |
| \( k(\alpha \beta \gamma \delta) \) | \( (\alpha \beta) \) is same or transpose of \((\gamma \delta)\) |
| \( m(\alpha \beta \gamma \delta \epsilon \phi) \) | : (is a permutation of \((123)\); \( (\alpha \beta) \) is same or transpose of \(\gamma \delta\) ); \( \phi \) is either \(\gamma\) or \(\delta\). |
| \( n(\alpha \beta \gamma \delta \epsilon) \) | \( (\gamma \delta \epsilon) \) is a permutation of \((123)\); |
| \( e(\alpha\beta) : \alpha \neq \beta \) | \( \beta = \gamma \) and \( (\alpha \beta \delta) \) is a permutation of \((123)\); |
| \( f(\alpha\beta\gamma\delta) \) | \( (\alpha \beta \gamma \delta ) \) only even permutation of \((123)\); |
| \( g(\alpha\beta\gamma) \) | \( (\alpha \beta \gamma) \) only even permutation of \((123)\); |

| Monomial | \( j + \bar{p}^1 q^2 r^1 \bar{r}^2 u^3 \) | \( \bar{p}^2 q^1 \bar{r}^1 \bar{r}^2 u^3 \) | \( \bar{p}^3 q^2 \bar{r}^1 j_{12} s^2 \) | \( \bar{p}^2 q^1 \bar{r}^1 j_{12} s^2 \) |
| --- | --- | --- | --- | --- |
| Order used in label | \( \bar{p}^1 q^2 \bar{u}^3 \) | \( \bar{p}^2 q^1 \bar{u}^3 \) | \( \bar{p}^1 q^2 r^1 s^2 \) | \( \bar{p}^2 q^1 r^1 s^2 \) |
| Power | \( l(123) \) | \( l(213) \) | \( k(1212) \) | \( k(2112) \) |

\[ \bar{p}^1 q^2 r^2 j_{12} s^1 \] \[ \bar{p}^1 q^2 r^1 j_{12} s^2 \] \[ \bar{p}^1 q^2 r^1 j_{13} s^2 u^1 j_{13} \bar{v}^3 \] \[ \bar{p}^1 q^2 r^1 j_{13} s^2 u^1 j_{12} \bar{v}^3 \] \[ \bar{p}^1 q^2 r^1 j_{13} s^2 u^1 j_{12} \bar{v}^3 j_{13} s^2 \] \[ \bar{p}^1 q^2 r^2 j_{21} s^1 \] \[ \bar{p}^1 q^2 r^1 j_{12} s^2 \] \[ \bar{p}^1 q^2 r^1 j_{12} s^2 u^1 j_{23} \bar{v}^3 \] \[ \bar{p}^1 q^2 r^1 j_{13} s^2 u^1 j_{23} \bar{v}^3 \] \[ \bar{p}^1 q^2 r^1 j_{13} s^2 u^1 j_{23} \bar{v}^3 j_{12} s^2 \]
3. An Auxiliary 'Differential Measure'

For computing the inner product between two arbitrary basis states of \( SU(3) \) we have to find out the inner product between two generating functions, one in primed and another in unprimed variables, of these generating functions. This we propose to do in the following.

Let us first note that

\[
g(p,q,r,s,u,v;z_1,z_2,z_3,w_1,w_2) = \exp \left\{ r(pz_1 + qz_2) + s(pw_2 + qw_1) + uz_3 - \frac{v}{z_3} (z_1 w_1 + z_2 w_2) \right\}
\]

\[
= \exp \left\{ p(r's') \left( \begin{array}{c} z_1 \\ w_2 \end{array} \right) + q(r's') \left( \begin{array}{c} z_2 \\ -w_1 \end{array} \right) - \frac{v}{z_3} \left( \begin{array}{c} z_1 \\ w_2 \end{array} \right) \right\}, \tag{3.1}
\]

where

\[
\begin{pmatrix} x & y \\ y' & x' \end{pmatrix} = (xy' - yx'). \tag{3.2}
\]

Define

\[
T^{(2)} = (g(p',q',r',s',u',v';z_1,z_2,z_3,w_1,w_2)\psi_0, \ g(p,q,r,s,u,v;z_1,z_2,z_3,w_1,w_2)\psi_0)
\]

\[
= \left( \exp \left\{ p'(r's') \left( \begin{array}{c} z_1 \\ w_2 \end{array} \right) + q'(r's') \left( \begin{array}{c} z_2 \\ -w_1 \end{array} \right) - \frac{v'}{z_3} \left( \begin{array}{c} z_1 \\ w_2 \end{array} \right) \right\} \psi_0, \right.
\]

\[
\left. \exp \left\{ p(r's') \left( \begin{array}{c} z_1 \\ w_2 \end{array} \right) + q(r's') \left( \begin{array}{c} z_2 \\ -w_1 \end{array} \right) - \frac{v}{z_3} \left( \begin{array}{c} z_1 \\ w_2 \end{array} \right) \right\} \psi_0 \right), \tag{3.3}
\]

where

\[
\frac{\partial}{\partial z_i} \psi_0 = \frac{\partial}{\partial w_i} \psi_0 = 0, \quad i, j = 1, 2, 3. \tag{3.4}
\]
To begin computing the above scalar product we have to note that $z_i$, $w_i$ and $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial w_i}$ are adjoints of each other [4], for each $i = 1, 2, 3$, under the scalar product. We also note that adjoints of sums and products are sum of adjoints and product of adjoints, respectively.

But we will not be required to prove it here since our scalar product for the basis functions under this assumption reproduces our previous results.

Therefore,

$$\frac{\partial}{\partial p'} T^{(2)} = p(r'r' + ss') T^{(2)} + \frac{v\bar{v}'}{\bar{z}_3 z_3} \frac{\partial}{\partial p'} T^{(2)}, \quad (3.5)$$

and

$$(1 - \frac{v\bar{v}'}{\bar{z}_3 z_3}) \frac{\partial}{\partial p'} T^{(2)} = p(r'r' + ss') T^{(2)}. \quad (3.6)$$

Similarly,

$$(1 - \frac{v\bar{v}}{z_3 \bar{z}_3}) \frac{\partial}{\partial q'} T^{(2)} = q(r'r' + ss') T^{(2)}. \quad (3.7)$$

The solution of these equations is,

$$T^{(2)} = \exp \left( \frac{(pp' + qq')(rr' + ss')}{1 - \frac{v\bar{v}'}{z_3 \bar{z}_3}} \right) \times T^{(2)}_0, \quad (3.8)$$

where

$$T^{(2)}_0 = \left( \exp v \left( \begin{array}{cc} z_1 & w_2 \\ -w_1 \end{array} \right) \psi_0, \quad \exp v' \left( \begin{array}{cc} z_2 & w_1 \\ -w_1 \end{array} \right) \psi_0, \right),$$

$$= (\psi_0, \quad Q\psi_0), \quad (3.9)$$

and

$$Q = \exp \left( \frac{v}{z_3} \left( \begin{array}{cc} \frac{\partial}{\partial z_1} & \frac{\partial}{\partial w_2} \\ \frac{\partial}{\partial w_1} \end{array} \right) \right) \cdot \exp \frac{v'}{z_3} \left( \begin{array}{cc} z_1 & w_2 \\ -w_1 \end{array} \right). \quad (3.10)$$

To evaluate $T^{(2)}_0$ we follow Schwinger’s procedure [3]. For the convenience of the reader the derivation of Schwinger for the general case $T^{(n)}_0$ is given in the appendix. We simply quote the result below (see appendix Eq.A.13),
\[ T^{(2)}_0 = \frac{1}{|1 + \kappa \bar{\lambda}|}, \]  

where the vertical bars indicate a determinant.

For us

\[ \kappa = \begin{pmatrix} 0 & \frac{v'}{z_3} \\ -\frac{v'}{z_3} & 0 \end{pmatrix}, \]

\[ \bar{\lambda} = \begin{pmatrix} 0 & \frac{\bar{v}}{\bar{z}_3} \\ -\frac{\bar{v}}{\bar{z}_3} & 0 \end{pmatrix}. \]  

Therefore

\[ |1 + \kappa \bar{\lambda}| = (1 - \frac{v'\bar{v}}{z_3\bar{z}_3})^2, \]

\[ T^{(2)}_0 = \frac{1}{(1 - \frac{v'\bar{v}}{z_3\bar{z}_3})^2}. \]  

This gives us the scalar product between the generating functions for the basis states as

\[ T^{(2)} = \frac{1}{(1 - \frac{v'\bar{v}}{z_3\bar{z}_3})^2} \exp \left( \frac{(pp' + qq')(rr' + ss')}{1 - \frac{v'\bar{v}}{z_3\bar{z}_3}} \right). \]  

As in our previous work, we observe that \(|z_3|^2\) appears in the denominators in Eq.(3.14). We therefore use the same choice for \(z_3\) namely set it to,

\[ z_3 = \exp(i\theta), \]  

so that \(|z_3|^2 = 1\).

With this choice the scalar product now yeilds

\[ T^{(2)} = \frac{1}{(1 - v'\bar{v})^2} \exp \left( \frac{(pp' + qq')(rr' + ss')}{1 - v'\bar{v}} \right). \]  

We still have to specify the measure over the \(\theta\) variable. We keep this as we had done in our previous paper. That is for us also the scalar product over the angle variable \(\theta\) is an averaging over the angle. So that,

\[ T_{\theta} = \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \exp u'(\exp(-i\theta)) \exp u(\exp(i\theta)) = \sum_{n=0}^{\infty} \frac{(u'u)^n}{n!^2}. \]  

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Therefore taking into account the $T_\theta$ factor we now write the final expression for the scalar product between the generating functions for the basis as

$$T^{(2)} = (g(p', q', r', s', u', v'; z_1, z_2, z_3, w_1, w_2) \psi_0, \ g(p, q, r, s, u, v; z_1, z_2, z_3, w_1, w_2) \psi_0) ,$$

$$= \frac{1}{(1 - v'\bar{v})^2} \exp \left( \frac{(pp' + qq')(rr' + ss')}{1 - v'\bar{v}} \right) \sum_{n=0}^{\infty} \frac{(u'u)^n}{n!^2} .$$

(3.18)

Similarly the scalar product between two generalized generating functions is

$$(G', G) = (1 - v'v)^{-2} \exp \left[ (1 - v'v)^{-1} (r_p'r_p + r_q'r_q + s_p's_p + s_q's_q) \right]$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!^2} \left( u' - v \frac{(r_p's_q + r_q's_p)}{(1 - v'v)} \right)^n \cdot \left( u - v' \frac{(r_p's_q + r_q's_p)}{(1 - v'v)} \right)^n .$$

(3.19)

4. COMPUTATION OF THE CLEBSCH-GORDAN COEFFICIENTS USING THE 'DIFFERENTIAL MEASURE'

There are several methods of computing the Clebsch-Gordan coefficients of $SU(3)$. One of these is the method of vector invariants first used by van der Waerden to compute the Clebsch-Gordan coefficients for $SU(2)$. In this way of computing these coefficients one has to first write down the most general vector invariant of $SU(3)$ in terms of the basis states of three pairs of $\mathbf{3}$ and $\mathbf{3}^*$ of $SU(3)$. This invariant may be expanded in terms of product of basis states of three irreducible representations of $SU(3)$ and the expansion coefficients are the required Clebsch-Gordan coefficients of $SU(3)$. We borrow from our previous paper [2] the expression for the generating function for the most general general vector invariants of $SU(3)$ Eq. (2.1). We recall here that the basis states of $SU(3)$ are also available to us in the form of a generating function (2.10). We extract the Clebsch-Gordan coefficients by making use of 'auxiliary scalar product'. In view of this the computation of the Clebsch-Gordan coefficients of $SU(3)$ reduces to the problem of computing the scalar product,
\[ \int_\pm = (g_1 g_2 g_3, \mathcal{I}_\pm) . \]  

(4.1)

where \( g^a \) is the generating function for the basis states in the variables \( z^a_i, w^a_i, \ a, i = 1, 2, 3 \) and \( \mathcal{I}_+ \) is the generating function for the invariants involving the trilinear invariant in the \( z^a_i \) whereas \( \mathcal{I}_- \) is the generating function for the invariants involving the trilinear invariant in the \( w^a_i \). Throughout the following the upper index denotes one of the three IR’s and the lower index denotes the components of the vectors in the give IR. It is enough for us to compute the scalar product involving \( \mathcal{I}_+ \) the other one can then be deduced from this by a simple argument \[2\].

A. Evaluation of \( \int_+ \) - The \( \theta \) Variable Part

We now evaluate \( \int_+ \). We have,

\[ \int_+ = (g_1 g_2 g_3, \mathcal{I}_+) . \]  

(4.2)

Here,

\[ g^a = \exp \left( r^a_+ z^a_1 + s^a_+ w^a_1 + s^a_+ w^a_2 + u^a e^{i \theta^a} - v^a e^{-i \theta^a} (z^a_1 w^a_1 + z^a_2 w^a_2) \right) \]  

\[ a=1,2,3 . \]  

(4.3)

In Eq.(4.3) we have to use the meaning of the variables implied by the Eq.(2.13). Also, in \( \mathcal{I}_+ \) we have,

\[ z^a \cdot w^a = z^a_1 w^a_1 + z^a_2 w^a_2 - \exp(i \theta^a - i \theta^b)(z^b_1 w^b_1 + z^b_2 w^b_2) , \]  

(4.4)

and

\[ \vec{z}_1 \cdot \vec{z}_2 \times \vec{z}_3 = e^{i \theta^1} (z^2_1 z^3_2 - z^3_1 z^2_2) + (\text{cyclic}) . \]  

(4.5)

We have no new way of evaluating the \( \theta \) integrations required to be done in the above scalar product. Hence for this part we borrow the results of \[3\].
B. Evaluation of $f_+$ - The $z^a_i$ and $w^a_i$ Part, where $a = 1, 2, 3$, and $i = 1, 2$

Next we use this scalar product to extract the Clebsch-Gordan coefficients of $SU(3)$ from the remaining part of the most general expression for the invariants of $SU(3)$ formed out of the pair of vectors $z^i_j$ and $w^i_j$ where $i, j = 1, 2, 3$. The upper index denotes the IR and the lower one denotes the component in each IR.

The remaining part of the scalar product after doing the $\theta$ integrations is denoted by $T$ and is given below.

$$T = \left( \exp \left\{ \sum_{i,j} V^{ij} (z^1_1 w^1_1 + z^1_2 w^1_2) + p^i (r^i z^i_1 + s^i w^i_2) + q^i (r^i z^i_2 - s^i w^i_1) \right\} \psi_0, \right.$$

$$\left. \exp \left\{ \sum_{m,n} -J_{mn} (z^1_m w^1_n + z^2_m w^2_n) + j z^1_m A_{mn} z^2_n \right\} \psi_0 \right),$$

where

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$  \hfill (4.7)

Next differentiating $T$ with respect to $p_i$ we get

$$\frac{\partial T}{\partial p^i} = \left( \exp(\cdots) \psi_0, \right.$$

$$\left. (r^i \frac{\partial}{\partial z^i_1} + s^i \frac{\partial}{\partial w^i_2}) \exp(\cdots) \psi_0 \right),$$

$$= \left( \exp(\cdots) \psi_0, \right.$$

$$\left. \left\{ \sum_{m,n} -r^i J_{mn} w^m_1 \delta_{mi} - s^i J_{mn} z^m_2 \delta_{mi} + j q^i A_{mn} z^m_2 \delta_{mi} \right\} \exp(\cdots) \psi_0 \right),$$

$$= \left( \left\{ \sum_n -r^i J_{in} \frac{\partial}{\partial w^1_n} - \sum_m s^i J_{ni} \frac{\partial}{\partial z^1_m} + j + \sum_n r^i A_{in} \frac{\partial}{\partial z^2_n} \exp(\cdots) \right\} \psi_0, \exp(\cdots) \psi_0 \right),$$

where the ellipses for the arguments in the exp function stand for the same arguments of the exp functions in the Eq.(4.6) above.

We will first evaluate the effect of the term involving $A$. So let,

$$T' = \sum_n r^i A_{in} \left( \frac{\partial}{\partial z^2_n} \exp(\cdots) \psi_0, \exp(\cdots) \psi_0 \right).$$  \hfill (4.9)
This gives us,

\[
\sum_m \left(1 - V J^T\right)_{mn} T' = \left(\sum_a r^i A_{ia} r^a_q\right) \psi_0, \quad (4.10)
\]

Therefore

\[
\frac{\partial T}{\partial p^i} = \left(\sum_n \frac{-r^i J_{in}}{\partial w^n_1} - \sum_m s^i J_{mi} \frac{\partial}{\partial z^m_2} \exp(\cdots)\right) \psi_0, \quad \exp(\cdots) \psi_0 + j^i T',
\]

\[
= \left(\sum_n \frac{-r^i J_{in}}{\partial w^n_1} - \sum_m s^i J_{mi} \frac{\partial}{\partial z^m_2} \exp(\cdots)\right) \psi_0, \quad \exp(\cdots) \psi_0
\]

\[+ j^i \left(\sum_{m,n} A_{mn} \frac{1}{(1 - V J^T)_{mn} r^n_q}\right) T,\]

\[= j^i \left(\sum_{m,n} A_{mn} \frac{1}{(1 - V J^T)_{mn} r^n_q}\right) T - \left(\sum_a (r^i J_{ia} s^a_q + s^i J_{ai} r^a_q)\right) T
\]

\[+ \sum_a (V J^T)_{ai} \frac{\partial T}{\partial p^a}.
\]

Hence

\[
\frac{\partial T}{\partial p^i} - \sum_{a,b} V_{ab} J^T_{bi} \frac{\partial T}{\partial p^a} = \sum_{a,b} \left(j^i r^a A_{ia} \frac{1}{(1 - V J^T)_{ab} r^n_q} - r^i J_{ia} s^a_q - s^i J_{ai} r^a_q\right) T. \quad (4.12)
\]

Expressed in matrix notation, the solution of this equation is

\[
T = \exp \left[-S_q^T J^T (1 - V J^T)^{-1} R_p - S_p^T J^T (1 - V J^T)^{-1} R_p\right]
\]

\[+ j^i R^T q J^T (1 - V J^T)^{-1} A^T (1 - V J^T)^{-1} R_p \right] T_{0(6)}^0, \quad (4.13)
\]

where

\[
R_p = \begin{pmatrix} r^1_p \\ r^2_p \\ r^3_p \end{pmatrix}, \quad (4.14)
\]

and similar notation follows for \( R_q, S_p, S_q \).

In the above

\[
T_{0(6)}^0 = \left(\exp \sum_{i,j} -V_{ij} (z_1^i w^i_1 + z_2^i w_2^i)\right) \psi_0, \quad \left\{\exp \sum_{m,n} -J'_{mn} (z_1^m w^m_1 + z_2^m w_2^m) + j^i z_1^m A_{mn} z_2^m\right\} \psi_0
\]

\[= \left(\exp \sum_{\mu,\nu=1}^6 V_{\mu\nu} A_{\mu} A_{\nu}\right) \psi_0, \quad \left\{\exp \sum_{\mu,\nu=1}^6 J'_{\mu\nu} A_{\mu} A_{\nu}\right\} \psi_0, \quad (4.15)
\]

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where

\[
A_1 = (z_1^1 z_2^1), \quad A_2 = \begin{pmatrix} w_2^1 \\ -w_1^1 \end{pmatrix}, \quad A_3 = (z_1^2 z_2^2),
\]
\[
A_4 = \begin{pmatrix} w_2^2 \\ -w_1^2 \end{pmatrix}, \quad A_5 = (z_1^3 z_2^3), \quad A_6 = \begin{pmatrix} w_2^3 \\ -w_1^3 \end{pmatrix},
\]

(4.16)

\[
[xy] = x_1 y_2 - x_2 y_1, \text{ with } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } y = (y_1 \ y_2),
\]

(4.17)

\[
V' = \begin{pmatrix} 0 & v^{12} & 0 & 0 & 0 & 0 \\ -v^{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v^{34} & 0 & 0 \\ 0 & 0 & -v^{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v^{56} & 0 \\ 0 & 0 & 0 & 0 & -v^{56} & 0 \end{pmatrix},
\]

\[
= \begin{pmatrix} 0 & v^1 & 0 & 0 & 0 & 0 \\ -v^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v^2 & 0 & 0 \\ 0 & 0 & -v^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v^3 & 0 \\ 0 & 0 & 0 & 0 & -v^3 & 0 \end{pmatrix},
\]

\[
J' = \begin{pmatrix} 0 & J_{12} & j_+ & J_{14} & j_+ & J_{16} \\ -J_{12} & 0 & J_{32} & 0 & J_{52} & 0 \\ -j_+ & -J_{32} & 0 & J_{34} & j_+ & J_{36} \\ -J_{14} & 0 & -J_{34} & 0 & J_{54} & 0 \\ -j_+ & -J_{52} & -j_+ & -J_{54} & 0 & J_{56} \\ -J_{16} & 0 & -J_{36} & 0 & -J_{56} & 0 \end{pmatrix}
\]
\[
\begin{pmatrix}
0 & j_{31} + j_{21} & j_+ & j_{13} & j_+ & j_{13} \\
-(j_{31} + j_{21}) & 0 & j_{21} & 0 & j_{31} & 0 \\
-j_+ & -j_{21} & 0 & j_{12} + j_{32} & j_+ & j_{23} \\
-j_{12} & 0 & -(j_{12} + j_{32}) & 0 & j_{32} & 0 \\
-j_+ & -j_{31} & -j_+ & -j_{32} & 0 & j_{23} + j_{13} \\
-j_{13} & 0 & -j_{23} & 0 & -(j_{23} + j_{13}) & 0 \\
\end{pmatrix}
\]

(4.18)

We borrow the expression for \( T_0^{(6)} \) from Schwinger (see appendix Eq.(A.13)). For us \( \kappa = J' \), \( \lambda = V' \) and, as before, we will assume that both \( J' \) and \( V' \) are real matrices as this makes no difference to our results.

With this identification we get,

\[
T_0^{(6)} = \frac{1}{|1 + J'V'|},
\]

\[
= \frac{1}{|1 + (V')^T(J')^T|},
\]

\[
= \frac{1}{|1 - (V')(J')^T|}.
\]

(4.19)

By expanding this determinant one can show that

\[
T_0^{(6)} = \frac{1}{|1 - V'(J')^T|},
\]

\[
= \frac{1}{(|1 - VJ^T|)^2}.
\]

(4.20)

where \( V, J \) are as defined in Eq.(2.26).

Putting together our results obtained so far we see that,

\[
\int_+ = \exp \left[ -S_q^T J^T (1 - V J^T)^{-1} R_p - S_p^T J^T (1 - V J^T)^{-1} R_q \\
+ j_+ R_q J^T (1 - V J^T)^{-1} A^T (1 - V J^T)^{-1} R_p \right] T_0^{(6)},
\]

\[
= \frac{1}{(|1 - VJ^T|)^2} \exp \left[ -S_q^T J^T (1 - V J^T)^{-1} R_p - S_p^T J^T (1 - V J^T)^{-1} R_q \\
+ j_+ R_q J^T (1 - V J^T)^{-1} A^T (1 - V J^T)^{-1} R_p \right].
\]

(4.21)
We note that this is the expression that we got for \( f_+ \) in our paper \([2]\). The expression for \( S f_- \), which corresponds to the case \( L < 0 \) can be got from the one for \( f_+ \) easily by a simple argument (see \([2]\)).

C. Overall normalization factor of the Clebsch-Gordan coefficients

In our earlier work \([2]\) we obtained a formula for the Clebsch-Gordan coefficients of \( SU(3) \) up to an overall normalization factor, which we denoted by \( n(N(1, 2), N(1, 3), \cdots)\) Eq.(2.31). This factor depends on the three IRs, for which the Clebsch-Gordan coefficients are being calculated, only and not on the individual basis states. It corresponds to the norm of the most general invariant \([4]\) constructed out of three arbitrary IRs of \( SU(3) \). It is a function of the multiplicity labels \( N(1, 2, \cdots, L) \). We now wish to compute the same with help of the tools developed so far. To do this we proceed as before, namely we compute the norm of the generating function, for the invariants, itself. This will give us a generating function for the normalization factors \( n(N(1, 2) \cdots) \). To do this we first have to evaluate the \( \theta \) part of the scalar product. We recall that on earlier occasions we have not handled this using our 'differential' measure. We therefore propose to use, as in our previous work \([2]\), an averaging over the \( \theta \) variable as the scalar product for that part of the scalar product. Now by mimicking the argument in that paper for doing the \( \theta \) integrations word to word we come to the same conclusion, as in that paper, that is that we need not do these \( \theta \) integrations actually but can actually drop them. Therefore with this understanding we write the resulting scalar product over the remaining complex variables \( z_j^i, w_j^i \), where \( i = 1, 2, 3 \) and \( j = 1, 2 \), as below.

\[
T_0' = \left( \exp \left\{ \sum_{m',n'} -J_{m'n'}^{m'n'} (z_1^{m'} w_1^{n'} + z_2^{m'} w_2^{n'}) + j_+ z_1^{m'} A_{m'n'} z_2^{n'} \right\} \left\{ \psi_0, \exp \left\{ \sum_{m,n} -J_{mn}^{mn} (z_1^m w_1^n + z_2^m w_2^n) + j_+ z_1^m A_{mn} z_2^n \right\} \psi_0 \right\} = \right)
\]
\[ T'_0 = \frac{1}{|1 + J'' J'|} \]  \hspace{1cm} (4.22)

where, following the notation in Eq.\((4.15)\), \(J'\) is a 6 matrix whose elements are functions of \(j_{12}, \cdots, j_{+}, L\) and \(J''\) is a similar \(6 \times 6\) matrix whose elements are functions of \(j'_{12}, \cdots, j'_{+}, L'\). The last result follows from Eq.\((A.13)\).

Formally we can write

\[ T'_0 = \sum_{N' (1,2), N' (1,3), N' (2,1), N' (2,3), \cdots, \sum_{j_{+} = 0}} n(N(1,2), N'(1,2) \cdots \times \frac{1}{|1 + J'' J'|} \cdot n(N(1,2), N'(1,2) \cdots \times J_{12} J_{13} J_{21} J_{31} J_{23} J_{32} J_{+} J_{12} J_{13} J_{21} J_{31} J_{23} J_{32} J_{+} \cdot = 0 \hspace{1cm} (4.23) \]

Therefore to get the normalization factor \(n(N(1,2) \cdots)\) we have to expand the right hand side of Eq.\((4.22)\) in a taylor series expansion in the monomials \((j'_{12} j_{12})^{N(1,2)} (j'_{13} j_{13})^{N(1,3)} \cdots (j'_{+} j_{+})^{L} \) and extract its coefficient.

5. DISCUSSION

Making use of Eq.\((4.21)\), one can derive an explicit closed form algebraic expression for the Clebsch-Gordan coefficients of \(SU(3)\) (Eq.\((2.29)\)). This is done by computing the scalar product between the invariants and basis states in two ways, (i) using the formal expressions and, (ii) using the explicit realizations. By equating these two expressions the formula for the Clebsch-Gordan coefficient is obtained. The derivation is given in detail in [2] to which the interested reader is referred.

In this paper, which is devoted mostly to rederive our previous results, we have computed the overall normalization \(n(\cdots)\) (Eq.\((4.23)\)) using our ‘auxiliary differential measure’. We recall that this was left uncomputed in our paper using Gaussian integration techniques. In this task as in other tasks Schwinger’s formula has been of great utility to us.
Now let us note a few facts regarding our 'differential measure' for $SU(3)$.

Schwinger [3] while trying to construct basis states of the total angular momentum operator used a scalar product. It is a remarkable fact that the scalar product in both cases turn out to be almost identical to each other though, in the case of Schwinger it provides the actual normalizations for the basis states whereas in our case it provides only auxiliary normalizations. It is only almost exact because in our, $SU(3)$, case we have the scalar product between basis functions which are dependent on four complex variables and a planar rotor variable. In the case of Schwinger there are only four complex variables and for him the scalar product does not involve any other variable. Therefore we in our, $SU(3)$, case have an additional piece, coming from the planar rotor variable part, added to our scalar product. Compare Eq.(C7) of (3) with Eq.(2.14).

The just mentioned relation between the scalar products of the basis functions of the total angular momentum operator and those of the group $SU(3)$ raises the possibility of the corresponding basis functions being related to each other. In fact we can relate every basis state of $SU(3)$, Eq.(2.10), with every basis state of $SU(2) \times SU(2)$ (Eq.(3.35) of [3]) (or equivalently with that of $SO(1, 3)$ (only finite dimensional IRs in this case) and vice versa in a one-to-one fashion. This kind of duality is known already as a general principle. But our model space plus generating function method of studying group representations has made it possible to realize this duality in a simple and concrete way. A particular use of this duality may be the possibility of interpreting the IR and multiplicity labels of $SU(3)$ in terms of angular momentum quantum numbers. Investigations of this nature will form a part of a future publication.

Ruegg [7] in his work on the Clebsch-Gordan coefficients of $SU_q(2)$ used a 'differential measure' to derive the correct normalizations for the basis states and later the $SU_q(2)$ Clebsch-Gordan coefficients. For him it was a necessity since Quantum groups are not

\footnote{manuscript under preparation}
groups in the same sense as Lie groups and therefore a group invariant measure and a scalar product in terms of this measure do not exist. So he first defined a ‘differential measure’ for $SU(2)$ and then extended it to $SU_q(2)$ by making use of the $q$-derivatives, $q$-factorials, and $q$-binomial theorem etc. It is also clear now that the extension of our results to the group $SU_q(3)$ can proceed on lines similar to $SU_q(2)$. In principle one should be able to mimic the procedure of Ruegg designed for $SU_q(2)$ for the group $SU_q(3)$ also using the realization of $SU_q(3)$ algebra in terms of four pairs of boson operators and a planar rotor operator which we obtained in our earlier work (see appendix B of [4]).

Another interesting fact to note is that when one tries extend our results for $SU(3)$ to groups $SU(n), n > 3$ one has to grapple with two problems. One is the construction of a generating function for the basis states and the other is the construction of a measure to be used in the scalar product between the basis states. The nature of the ‘differential measure’ makes it easily adaptable to generalization though as one can see clearly the algebra that one has to go through to compute the scalar products is a little more tedious in the case of the ‘differential measure’ than in the case of Gaussian integrations.

Finally we note that since the present calculus is in complete analogy with that of Schwinger for $SU(2)$ [3] we can try to apply his techniques of obtaining various objects [8,9] on $SU(3)$.

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Appendix : Schwinger’s derivation of the scalar product $T_0^{(n)}$

Schwinger evaluated the scalar product $T_0^{(n)}$ (see Appendix C of [3]). For convenience we reproduce his derivation here almost verbatim.

For this purpose let,

$$T_0^{(n)} = (\psi_0, Q\psi_0), \quad (A.1)$$

where

$$Q = \exp \left( \frac{1}{2} \sum_{\mu,\nu=1}^{n} (\lambda^*)_{\mu\nu} [A_\mu A_\nu] \right) \cdot \exp \left( \frac{1}{2} \sum_{\mu,\nu=1}^{n} \kappa_{\mu\nu} [A_\mu^\dagger A_\nu^\dagger] \right), \quad (A.2)$$

and the square brackets $[]$ have the same meaning as in Eq.(4.17). The $A_\mu$ are $n$ sets of two-component operators obeying

$$[A_\zeta\mu, A_\zeta'^\dagger\nu] = \delta_{\mu\nu} \delta_{\zeta\zeta'}, \quad (A.3)$$

while $\lambda_{\mu\nu}$ and $\kappa_{\mu\nu}$ form antisymmetrical matrices.$^2$

We note the following properties of $Q$

$$\left( \frac{\partial Q}{\partial \lambda^*_{\mu\nu}} \right) = [A_\mu A_\nu]Q, \quad (A.4)$$

$$[[xA_\mu], Q] = -Q \sum_{\nu} \kappa_{\mu\nu} (xA), \quad (A.5)$$

$$[Q, (xA_\mu^\dagger)] = \sum_{\nu} \lambda^*_{\mu\nu} [xA_\nu]Q, \quad (A.6)$$

in which $X$ is an arbitrary constant spinor and the left hand sides of the last two equations are commutators. The above equations can be combined to give

$$\sum_{\nu} (1 + \kappa \lambda^*)_{\mu\nu} [xA_\nu]Q = Q[xA_\nu]Q - \sum_{\beta} \kappa_{\mu\beta} (xA_\beta^\dagger)Q, \quad (A.7)$$

or

$$[xA_\nu]Q = \sum_{\nu\beta} \left( \frac{1}{1 + \kappa \lambda^*} \right)_{\nu\beta} Q[xA_\beta] - \sum_{\nu\beta} \left( \frac{1}{1 + \kappa \lambda^*} \right)_{\nu\beta} (xA_\beta^\dagger)Q. \quad (A.8)$$

Therefore

$^2$This fact is not used in the proof
\[
[A_{\mu} A_{\nu}] Q = \sum_{\beta} \left( \frac{1}{1 + \kappa \lambda^*} \right)_{\nu \beta} [A_{\mu} Q A_{\beta}] - \sum_{\beta} \left( \frac{1}{1 + \kappa \lambda^*} \right)_{\nu \beta} (A_{\beta}^\dagger A_{\mu}) Q \\
-2 \left( \frac{1}{1 + \kappa \lambda^* \kappa} \right)_{\nu \mu} Q,
\]

(A.9)

from which we obtain, since \((A_{\mu} \psi_0 = 0)\)

\[
\left( \frac{\partial}{\partial \lambda^*_\mu} \right) T^{(n)}_0 = -2 \left( \frac{1}{1 + \kappa \lambda^*} \right)_{\nu \mu} J T^{(n)}_0.
\]

(A.10)

Thus with respect to changes in the matrix \(V'\), we have

\[
\delta \log T^{(n)}_0 = \frac{1}{2} \sum_{\mu \nu} \delta \lambda^*_\mu (\frac{\partial}{\partial \lambda^*_\mu}) \log T^{(n)}_0 = -\text{tr} \left( \frac{1}{1 + \kappa \lambda^* \kappa} \delta \lambda^* \right).
\]

(A.11)

On comparing this with the theorem on differentiation of a determinant [3],

\[
\delta \log |M| = \text{tr} (M^{-1} \delta M),
\]

we obtain the desired general result,

\[
T^{(n)}_0 = \frac{1}{|1 + \kappa \lambda^*|},
\]

(A.13)

where the vertical bars indicate a determinant.
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