A new Generating Function of Exponential Polynomials and its Applications to the Related Polynomials and Numbers

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Abstract

In this paper we use computational method based on operational point of view to prove a new generating function of exponential polynomials. We give its applications involving geometric polynomials, Bernoulli and Euler numbers.

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1 Introduction, Definitions and Notations

The exponential (one variable Bell polynomials) polynomials, defined by

\[ \phi_n(x) := \sum_{k=0}^{n} \binom{n}{k} x^k, \]

where \( \binom{n}{k} \) is the second kind Stirling numbers, are first studied by Ramanujan in his unpublished notebooks. For example, Ramanujan obtained the exponential generating function of \( \phi_n(x) \) as,

\[ \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} = e^{x(e^t-1)}, \]

and proved the relation

\[ \phi_{n+1}(x) = x \left( \phi_n(x) + \phi'_n(x) \right). \]
See [5, Part 1, Chapter 3] for details of Ramanujan’s work. Later these polynomials are studied by Bell [3] and Touchard [15, 19]. Besides Grunert obtained (14)

$$(xD)^n e^x = \phi_n(x) e^x$$

for the evaluation of the series

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{k!},$$

where $(xD)f(x) = xf^\prime(x)$. Moreover setting $x = 1$ in (11) these polynomials are reduced to Bell numbers $\phi_n$, defined by (4, 10, 11, 17)

$$\phi_n := \phi_n(1) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}.$$

So investigating the properties of these polynomials are useful in combinatorics.

Geometric polynomials (also known as Fubini polynomials) are defined as follows (6):

$$F_n(x) := \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k! x^k.$$  

By setting $x = 1$ in (5) one can obtain geometric numbers (or preferential arrangement numbers, or Fubini numbers) $F_n$ (12, 15, 13) as

$$F_n := F_n(1) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!.$$  

Geometric and exponential polynomials are connected by the relation (6)

$$F_n(x) = \int_{0}^{\infty} \phi_n(x, \lambda) e^{-\lambda} d\lambda.$$  

Using (7) one can easily get the exponential generating function of the geometric polynomials, is given by (8)

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}.$$  

Geometric polynomials also have close relationship with Apostol-Bernoulli numbers $\beta_n(\lambda)$ and Euler numbers $E_n$, as (9)

$$F_n \left( \frac{-1}{2} \right) = E_n,$$  

$$\beta_n(\lambda) = \frac{n}{\lambda-1} F_n \left( \frac{\lambda}{1-\lambda} \right), \quad \lambda \neq 1.$$
Euler and Apostol-Bernoulli numbers are defined by means of generating function (2, 10, 11)
\[
\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi
\]
and
\[
\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!}, \quad \lambda \in \mathbb{C}, \quad |t + \log \lambda| < 2\pi,
\]
respectively. Here, for \( \lambda = 1 \) in (12), \( \beta_n(1) = B_n \), where \( B_n \) is the Bernoulli numbers. Thereby studying properties of geometric polynomials is vital for gaining the known and unknown properties of Bernoulli and Euler numbers which have an important position in the number theory.

The starting point of this study is the formula (3), which can be written as
\[
\phi_{n+1}(x) = \hat{M}\phi_n(x),
\]
where \( \hat{M} := (x + xD) \). The operator \( \hat{M} \) can be recognized as rising operator acting on the polynomials \( \phi_n(x) \). So, the \( \phi_n(x) \) can be explicitly constructed the action of \( \hat{M}^n \) on \( \phi_0(x) = 1 \):
\[
\phi_n(x) = \hat{M}^n\{1\}.
\]
From this motivation we obtain a operational formula for an arbitrary \( f \) as
\[
\hat{M}^n f(x) = \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k}(x) (xD)^k f(x)
\]
to give a direct proof of Boyadzhiev’s identity (8)
\[
\phi_{n+m}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{m}{j} j^{n-k} x^j \phi_k(x)
\]
and to obtain a new generating function of exponential polynomials. Besides we deal with geometric polynomials because of close relationship with exponential polynomials. Finally as a special case of \( F_n(x) \), we get new generating functions of Bernoulli and Euler numbers. Using these generating functions give us recurrence relations for Bernoulli and Euler numbers as
\[
B_{n+m} = (n - 1)! (n + m) \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{k+1} k!}{(n + k + 1)!} B_{n+k+1}(k),
\]
\[
E_{n+m} = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k k!}{2^k} E_{n+k+1}(k)
\]
respectively, where \( m \in \mathbb{N} \cup \{0\} \).

Now we state our results.
2 An Operational Formula and Its Results for Exponential Polynomials

In this section for an arbitrary function \( f \) we obtain an operational formula involving exponential polynomials and examine the known and unknown properties of these polynomials. First we give the following proposition.

**Proposition 1** The following operational formula holds for an arbitrary \( f \):

\[
\hat{M}^n f(x) = \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k}(x) (xD)^k f(x),
\]

where \( \hat{M} := (x + xD) \).

**Proof.** Firstly it is good to notice that \( \hat{M} \) is not a commuting operator. Then it can be easily proved by induction with the help of relation (3).

Setting \( f(x) = \phi_m(x) \) in (13) and using the definition of exponential polynomials given in (1) we get the Boyadzhiev’s identity (8) in the following corollary.

**Corollary 2** For all \( n, m = 0, 1, 2, \ldots \)

\[
\phi_{n+m}(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} j^{n-k} x^{j} \phi_k(x).
\]

One of the natural question is to find the generating function of \( \phi_{n+m}(x) \), but first we need the following lemma.

**Lemma 3** Let

\[
g(x) = \sum_{k=0}^{\infty} c_k x^k
\]

be an analytical function. Then we have

\[
e^{t \hat{M}} g(x) = e^{x(e^t-1)} g(xe^t).
\]

**Proof.** Acting \( g(x) \) to the \( e^{t \hat{M}} \) operator we get

\[
e^{t \hat{M}} g(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^n}{n!} \hat{M}^n c_k x^k.
\]

When \( f(x) = x^k \), (13) and (15) show that

\[
e^{t \hat{M}} g(x) = e^{x(e^t-1)} g(xe^t).
\]
For the usefulness of this operational formula let set $g(x) = \phi_m(x)$ in (14). Then we obtain the following theorem which gives a new generating function of exponential polynomials.

**Theorem 4** For $m \in \mathbb{N} \cup \{0\}$, the exponential polynomials have the following generating function

$$
\sum_{n=0}^{\infty} \phi_{n+m}(x) \frac{t^n}{n!} = e^{x(e^t-1)} \phi_m(xe^t), \quad t \in \mathbb{C}.
$$

(16)

For $m = 0 \phi_0(x) = 1$, then (16) reduce to (2). Through this identity one can easily obtain all the exponential generating function of exponential polynomials for any indices. Finally, setting $x = 1$ and $t = ki\pi$ $(k \in \mathbb{Z})$ in (16) we get the value of infinite summation of bell numbers

$$
\sum_{n=0}^{\infty} \phi_{n+m}(ki\pi) \frac{n^n}{n!} = \begin{cases} 
\phi_m; & k \text{ even} \\
\phi_m(e^{-1}); & k \text{ odd}
\end{cases}
$$

Let deal with the ordinary generating function of exponential polynomials:

$$
h(x,t) = \sum_{n=0}^{\infty} \phi_n(x) t^n.
$$

This is obtained by Boyadzhiev in [6] by using polyexponential function [7]. Now we will prove it by different mean using the operational formula (15) in the following proposition.

**Proposition 5** The following ordinary generating function of $\phi_n(x)$ holds:

$$
\sum_{n=0}^{\infty} (-1)^n \phi_n(x) t^{n+1} = (-x)^{\frac{1}{t}} \exp(-x) \gamma(-x, \frac{1}{t}), \quad t \in \mathbb{R} \setminus \{0\}.
$$

(17)

where $\gamma(x,s)$ is the upper incomplete gamma function.

**Proof.** Using operational formulas (13) and (14) we have

$$
\sum_{n=0}^{\infty} (-t)^n \phi_n(x) = \frac{1}{1 + t(x + xD)}
$$

$$
= \int_0^{\infty} e^{-s} e^{x(e^{-st}-1)} ds
$$

To obtain the ordinary generating function of exponential polynomials, we should solve the above integration. Changing variable $e^{-st} = z$ we get

$$
\int_0^{\infty} e^{-s} e^{x(e^{-st}-1)} ds = \frac{\exp(-x)}{t} \int_0^1 z^{t-1} e^{xz} dz.
$$

(18)

If we change variable $xz = -y$ in the above integration the desired equation (17) is obtained. ■
Corollary 6 Setting $t = 1$ in (7) we have
\[
\sum_{n=0}^{\infty} (-1)^n \phi_n (x) = \frac{e^x - 1}{xe^x}.
\] (19)

Remark 7 Setting $x = 1$ in (19) the following infinite summation of bell numbers holds:
\[
\sum_{n=0}^{\infty} (-1)^n \phi_n = e - \frac{1}{e}.
\]

Furthermore, multiplying both sides of (19) with the generating function of Bernoulli numbers $B_k$
\[
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k,
\]
and using (11) we have
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \frac{B_k}{k!} x^{m+k}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\sum_{m=0}^{n} (-1)^m \frac{n+m}{m} \frac{B_{k-m}}{(k-m)!}) x^k
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} \left( \sum_{n=m}^{\infty} (-1)^n \frac{n}{m} \frac{B_{k-m}}{(k-m)!} \right) x^k \right).
\]

Using the well-known ordinary generating function of second kind Stirling numbers (11)
\[
\sum_{n=m}^{\infty} \frac{n}{m} y^n = \frac{y^m}{(1-y)(1-2y)\ldots(1-my)}
\]
for $y = -1$ and comparing the coefficients of $x^k$ we obtain the following identity for Bernoulli numbers:
\[
\sum_{m=0}^{k} \frac{k}{m} (-1)^m \frac{B_{k-m}}{(m+1)} = (-1)^k.
\]

3 Applications to the Related Polynomial and Numbers

In this section we derive the identities of geometric polynomials from exponential polynomials with the help of integral representation (7). Moreover we obtain new generating functions and recurrence relations for Bernoulli and Euler numbers.
Theorem 8 For all \( m = 0, 1, 2, \ldots \)
\[
\sum_{n=0}^{\infty} F_{n+m}(x) \frac{t^n}{n!} = \frac{1}{1 - x(e^t - 1)} F_m\left(\frac{x e^t}{1 - x(e^t - 1)}\right). \tag{20}
\]

Proof. Equation (16) can be written in the form
\[
\sum_{n=0}^{\infty} \phi_{n+m} (x\lambda) \frac{t^n}{n!} = e^{x\lambda(e^t - 1)} \phi_m (x\lambda e^t). \tag{21}
\]

Multiplying both sides with \( e^{-\lambda} \), integrating for \( \lambda \) from zero to infinity and in view of (7) we have
\[
\sum_{n=0}^{\infty} F_{n+m} (x) \frac{t^n}{n!} = \int_{0}^{\infty} \phi_n (x\lambda e^t) e^{-\lambda (1-x(e^t-1))} d\lambda.
\]

Finally, using (11) gives the desired equation (20). \( \Box \)

Now, we will mention two important consequences of (20). The first one is the special case \( x = -\frac{1}{2} \) and the second one is the case rewriting equation (10) as
\[
F_{n+m} \left( \frac{\lambda}{1-\lambda} \right) = \frac{(\lambda - 1) \beta_{n+m+1} (\lambda)}{n + m + 1},
\]
which are given in the following corollary.

Corollary 9 The following generating functions hold:
\[
\sum_{n=0}^{\infty} E_{n+m} \frac{t^n}{n!} = \frac{2}{e^t + 1} F_m \left( \frac{-e^t}{e^t + 1} \right), \quad m = 0, 1, 2, \ldots, \tag{21}
\]
\[
\sum_{n=0}^{\infty} \frac{\beta_{n+m} (\lambda)}{n + m} \frac{t^n}{n!} = \frac{-1}{\lambda (e^t - 2) + 1} F_{m-1} \left( \frac{-\lambda e^t}{\lambda (e^t - 2) + 1} \right), \quad m = 1, 2, \ldots. \tag{22}
\]

Corollary 10 For \( \lambda = 1 \) we have the following generating function for Bernoulli numbers:
\[
\sum_{n=0}^{\infty} B_{n+m} \frac{t^n}{n + m} \frac{t^n}{n!} = \frac{1}{1 - e^t} F_{m-1} \left( \frac{e^t}{1 - e^t} \right). \tag{23}
\]

Remark 11 For \( k \in \mathbb{Z} \) setting \( t = 2k \pi \) in (21) and \( t = (2k + 1) \pi \) in (23), we have
\[
\sum_{n=0}^{\infty} E_{n+m} \frac{(2k\pi)^n}{n!} = E_m, \quad m = 0, 1, 2, \ldots,
\]
\[
\sum_{n=0}^{\infty} \frac{B_{n+m} ((2k+1)\pi)^n}{n + m} \frac{t^n}{n!} = \frac{1}{2} E_{m-1}, \quad m = 1, 2, \ldots,
\]
respectively.
Also, equations (21) and (23) have one more important application.

**Proposition 12** We have the following recurrence relations:

\[
E_{n+m} = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{2^k} k! E_n^{(k+1)}(k), \quad (24)
\]

\[
B_{n+m} = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{k+1} k! (n-1)! (n+m)}{(n+k)} B_{n+k}^{(k+1)}(k). \quad (25)
\]

**Proof.** From (5) we can write (21) as

\[
\sum_{n=0}^{\infty} E_{n+m} \frac{t^n}{n!} = \sum_{k=0}^{m} \binom{m}{k} \left( \frac{2}{e^t+1} \right)^{k+1} e^{kt}.
\]

Using the generating function of higher order Euler polynomials

\[
\sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{2}{e^t+1} \right)^{\alpha} e^{xt}, \quad \alpha \in \mathbb{C}, \quad |t| < \pi
\]

and comparing the coefficients of \( \frac{t^n}{n!} \) we have (24).

Equation (25) can be proved the same method by using the generating function of higher order Bernoulli polynomials

\[
\sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{t}{e^t-1} \right)^{\alpha} e^{xt}, \quad \alpha \in \mathbb{C}, \quad |t| < 2\pi.
\]

Higher order Euler polynomials and higher order Bernoulli polynomials have many explicit expression and recurrence relations in literature hence one can improve the identities for \( E_{n+m} \) and \( B_{n+m} \).

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