ON UNIFORM LOG K-STABILITY FOR CONSTANT SCALAR CURVATURE KÄHLER CONE METRICS

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Abstract. We prove that the existence of constant scalar curvature Kähler metrics with cone singularities along a divisor implies log $K$-polystability and $G$-uniform log $K$-stability, where $G$ is the automorphism group which preserves the divisor. We also show that a constant scalar curvature Kähler cone metric along an ample divisor of sufficiently large degree always exists. We further show several properties of the path of constant scalar curvature Kähler cone metrics and discuss uniform log $K$-stability of normal varieties.

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1. INTRODUCTION

In Kähler geometry, the existence of canonical metrics is conjectured to be equivalent to algebro-geometric stability. Geodesic stability is used by Chen–Cheng [16] in the recent proof of Donaldson’s geodesic stability conjecture for the existence of constant scalar curvature Kähler (cscK) metric. In the logarithmic (log) setting, the uniqueness of cscK cone (cscKc for short) metric was proven in [73]. Later, in [74], several existence results for cscK cone metrics were proven, extending existence results in [16]. In particular, Theorem 1.8 in [74] shows that the existence of cscK cone metric is equivalent to log geodesic stability, see Section 3.3.2 for the definition.

In this article, we further study the log Yau–Tian–Donaldson (YTD) conjecture for cscK cone metrics in terms of various stability notions, to extend most of the stability results to the log setting as continuation of [73, 74].

$K$-stability was defined by Tian [67] and generalised by Donaldson in [27]. Log $K$-stability was introduced by Donaldson in [29]. The log $K$-stability has a natural connection to the minimal model program in birational geometry, which has been studied deeply in [44, 52] etc.

Let $X$ be a Kähler manifold, and $D$ be a smooth effective divisor on $X$ (we also consider the case $D$ is a simple normal crossing divisor, but for the most of the paper $D$ is smooth). We further set $L$ to be an ample line bundle and denote by $\Omega$ the Kähler class associated with $L$. The pair of $X$ and $D$, together with the polarisation $L$ will be called a polarised pair and denoted as $((X, L); D)$.

**Conjecture 1.1** (Log YTD conjecture). *The polarised pair $((X, L); D)$ admits a cscK cone metric if and only if it is log $K$-polystable.*

The precise definition of log $K$-polystability is given in Section 3.1 and the definition of cscK cone metrics was given in [73, Definition 3.1]. We let $\beta$ be a number in $(0, 1]$. A *Kähler cone metric* $\omega$ of cone angle $2\pi\beta$ along the divisor $D$, is a smooth Kähler metric on the regular part $M := X \setminus D$, and quasi-isometric to the cone flat metric in the coordinate chart centred at a point on $D$. Roughly speaking, a *cscK cone metric* of cone angle $2\pi\beta$ along the divisor $D$ is defined to be a Kähler cone metric, which has constant scalar curvature outside $D$. 
Let $G$ be the identity component of the group of holomorphic automorphisms of $X$ which fix the divisor $D$. The uniqueness of cscK cone metrics is proven in [73], that is the cscK cone metric is unique up to automorphisms in $G$. Combining the uniqueness result with an asymptotic formula of the log $K$-energy [11], we will show in (1) of Theorem 4.1 that

**Theorem 1.2** (Theorem 4.1). Suppose that $((X, L); D)$ admits a cscK cone metric of angle $2\pi \beta$. Then $((X, L); D)$ is log $K$-semistable with angle $2\pi \beta$.

**Remark 1.3.** When $\beta = 1$, Theorem 1.2 was proven in Donaldson [28] by making use of Kodaira embedding and asymptotic expansion of Bergman kernel. Donaldson’s method is extended to twisted cscK metric by Dervan [23].

**Remark 1.4.** Donaldson’s semistability result [28] is extended to orbifolds by Ross–Thomas in [57], where they also ask whether this result could be further extended to cscK metrics with cone singularities. Theorem 1.2 provides an positive answer to their question, in the case when the cone singularities are formed along a divisor, by using a completely different approach.

**Remark 1.5.** The proof of [57] embeds polarised orbifolds into weighted projective space via a weighted version of Kodaira embedding. Similar idea was expected for general cone metrics. However, it is only successful in very limited case, e.g. on a Riemann surface [3, 63] and the projective completion of an ample line bundle over a cscK base [62]. Theorem 1.2 is also related to [63, Conjecture 1.1].

**Remark 1.6.** We believe that Theorem 1.2 also holds for a general compact Kähler manifold, with the Kähler class that is not associated to an ample line bundle, by following the argument [24,25,58,59], but in this paper we decide not to discuss the details which could be technical.

It is shown in [74] that the existence of a cscK cone metric on $(X, L)$ is equivalent to the $d_{1, G}$-coercivity of the log $K$-energy. With the help of this existence result, Theorem 1.2 is strengthened to log $K$-polystability. As a result, we prove one direction of Conjecture 1.1.

**Theorem 1.7** (Theorem 4.1). Suppose that $((X, L); D)$ admits a cscK cone metric of angle $2\pi \beta$. Then $((X, L); D)$ is log $K$-polystable with angle $2\pi \beta$.

The $K$-stability condition is also expected to be replaced by uniform stability.

**Conjecture 1.8** (Log YTD conjecture for uniform stability). The polarised pair $((X, L); D)$ admits a cscK cone metric if and only if it is uniformly log $K$-stable.
This existence result, together with Theorem 3.15, implies uniform log $K$-stability, see Section 3.2.

**Theorem 1.9** (Theorem 4.1). Suppose that $((X, L); D)$ admits a cscK cone metric of angle $2\pi \beta$. Then $((X, L); D)$ is $G$-uniformly log $K$-stable with angle $2\pi \beta$, with $G = \text{Aut}_0((X, L); D)$.

While the above results hold for a general polarised pair $((X, L); D)$, we get more precise results when the cohomology class of $D$ is a multiple of $\mathcal{C}_1(L)$. One result in this direction is the following, which ensures that the cscK cone metrics always exist when the divisor $D$ is ample and of sufficiently large degree.

**Theorem 1.10** (Theorem 5.2). For any $0 < \beta < 1$, there exists $m_0 \in \mathbb{N}$ which depends only on $X$, $L$, and $\beta$, such that $((X, L); D)$ admits a constant scalar curvature Kähler metric with cone singularities of cone angle $2\pi \beta$ along a generic member of the linear system $|mL|$ if $m \geq m_0$.

Further precise quantitative relation between the multiplicity $m$ and the cone angle $\beta$ is given in Theorem 5.25.

When we have $D \in |mL|$ and $m$ is large enough, as above, it turns out that the average value $S^D_1$ of the scalar curvature of Kähler metrics in $C_1(L|D)$ over $D$ satisfies $S^D_1 \ll 0 < \beta$. We have a result on the log $K$-instability in the situation that is somewhat complementary to the above.

**Theorem 1.11** (Theorem 4.18). Suppose that $D \in |L|$ is smooth. Then $((X, L); D)$ is log $K$-unstable with angle $2\pi \beta$ if $\beta$ satisfies

$$\beta < \frac{S^D_1}{n(n-1)}.$$ 

We also prove various sufficient conditions for the uniform log $K$-stability. We believe that comparing the theorem below with Theorem 1.10 (or more precisely (5.1) that is used in its proof) may give an interesting observation for the log YTD conjecture.

**Theorem 1.12** (Theorem 6.19). Suppose that $((X, L); D)$ is a smooth polarised pair such that $D \in |mL|$ for some $m \in \mathbb{N}$, such that it satisfies

$$S_1 \leq mn \quad \text{and} \quad (n+1)\lambda \leq S_1 + m,$$

where $S_1$ is the average scalar curvature of $(X, L)$ and $\lambda \in \mathbb{R}$ is the nef threshold of $L$ (see Definition 5.15).

Then $((X, L); D)$ is uniformly log $K$-stable with cone angle $2\pi \beta$ satisfying the constraint

$$1 - \frac{(n+1)\lambda - S_1}{m} \leq \beta < \beta_u,$$

where $\beta_u$ is a constant defined in terms of the log alpha invariant and $m$ as in Definition 5.24.
We also present some results for the uniform log $K$-stability for singular varieties. Suppose that $X$ is a $\mathbb{Q}$-Gorenstein normal projective variety, $L$ is an ample Cartier divisor on $X$, and $\triangle$ is an effective integral reduced Cartier divisor on $X$. We also write $\underline{S}_{\beta}$ for the constant defined by (6.1), which is the average log scalar curvature when $X$ and $\triangle$ are smooth. We have the following results.

**Theorem 1.13 (Theorem 6.9).** If $(X, (1 - \beta)\triangle)$ is log canonical, we have the following.

- Suppose $\underline{S}_{\beta} < 0$ and that there exists $\eta \geq 0$ such that
  \[
  \begin{aligned}
  (i) & \quad 0 \leq \eta < \frac{n+1}{n} \alpha_{\beta}, \\
  (ii) & \quad \eta L + K_X + (1 - \beta)\triangle \text{ is ample}, \\
  (iii) & \quad -(n-1)(K_X + (1 - \beta)\triangle) - (\underline{S}_{\beta} - \eta)L \text{ is ample},
  \end{aligned}
  \]

  where $\alpha_{\beta}$ is the log alpha invariant in Definition 6.4. Then $(X, L; \triangle)$ is uniformly log $K$-stable with angle $2\pi \beta$.

- Suppose that $\underline{S}_{\beta} < (n+1)\alpha_{\beta}$, and $-\underline{S}_{\beta}L - (n+1)(K_X + (1 - \beta)\triangle)$ is nef. Then $(X, L; \triangle)$ is uniformly log $K$-stable with angle $2\pi \beta$.

The second item above can be seen as a variant of Dervan’s result [22, 23]; see Remark 6.10 for more details. We also prove the following result, which partially generalises the result of Odaka–Sun [52, Theorem 6.1] by relaxing the hypothesis on the polarisation $L$.

**Theorem 1.14 (Theorem 6.18).** Suppose that

- $(X, (1 - \beta)\triangle)$ is log $\mathbb{Q}$-Fano, i.e. $-K_X - (1 - \beta)\triangle$ is ample, and

  - $\underline{S}_{\beta}L - n(-K_X + (1 - \beta)\triangle)$ is nef.

Then $(X, (1 - \beta)\triangle)$ is Kawamata log terminal if $(X, L; \triangle)$ is log $K$-semistable with angle $2\pi \beta$.

**Organisation of the paper.** We review the basic materials on the cscK cone metrics, its variational characterisation, and the automorphism group and the Futaki invariant in Section 2. Section 3 is devoted to the review of various stability notions that are important in this paper. In particular, we recall various log $K$-stabilities in algebraic geometry, as well as the log properness result on the analytic side which was proved in [74] and plays a crucially important role in this paper. In Section 4 we prove one of our main results Theorem 4.1, by applying the important results of Boucksom–Hisamoto–Jonsson [10, 11] and the log properness result [74]. We review the alpha invariant in Section 5, and prove Theorem 5.2 as an application of it, as well as other existence results. We finally discuss the uniform log $K$-stability for a possibly singular normal variety in Section 6.
Notation. We comment on the notational conventions that we use in this paper.

In this paper we shall mainly consider a smooth polarised pair, denoted by \((X, L; D)\), but many topics extend to the case of a more general compact Kähler manifold \(X\) with a smooth effective divisor \(D\), with the fixed Kähler class \(\Omega\); the extension is sometimes straightforward, but there are nontrivial cases in which the extension to general Kähler manifolds is only conjectural (at least to the best of the authors’ knowledge). The convention that we use is as follows: we write \(\Omega\) for the Kähler class, which is equal to \(C_1(L)\) when the manifold \(X\) under consideration is a polarised smooth projective variety, but \(\Omega\) stands for a general Kähler class when it is explicitly stated that \(X\) may not be projective.

We shall use the additive notation for the tensor product of line bundles, unless explicitly stated otherwise. We consistently write \(L_D\) for the line bundle \(\mathcal{O}_X(D)\) on \(X\) defined by the divisor \(D\).

While \(X\) is supposed to be smooth in much of this paper, we also consider a pair \((X, \Delta)\) where \(X\) and \(\Delta\) may also be singular. See Section 6 for more details on the singularities of \((X, \Delta)\).

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2. CscK cone metrics

Let \(X\) be a (possibly non-projective) compact Kähler manifold and \(\Omega\) be a Kähler class, and \(D\) be a smooth effective divisor in \(X\). In this section, we will review recent progress on cscK cone metrics [73, 74]. We also obtain many properties of corresponding energy functionals and the cscK cone path, which extend paralleling results on Kähler–Einstein cone metrics.

2.1. Definition of cscK cone metrics. We recall the definition of cscK cone metrics. We let

\[ C_1(X, D) := C_1(X) - (1 - \beta)C_1(L_D). \]

and choose a smooth form \(\theta \in C_1(X, D)\). We let \(s\) be the defining section of \(D\) and \(h\) is a Hermitian metric on the associated line bundle
L_D. We denote by $\Theta_D$ the curvature form
$$\Theta_D = -i\partial \bar{\partial} \log h.$$ Since $\theta \in C_1(X, D)$, by cohomology condition, there exists a smooth function $h_0$ such that
$$\text{Ric}(\omega_0) = \theta + (1 - \beta)\Theta_D + i\partial \bar{\partial} h_0. \quad (2.1)$$

The reference Kähler cone metric $\omega_\theta \in \Omega$ is obtained by solving the following equation
$$\text{Ric}(\omega_\theta) = \theta + 2\pi(1 - \beta)[D], \quad (2.2)$$
equivalently,
$$\omega^n_\theta = e^{h_0}|s|^{2\beta-2} \omega^n_0, \quad (2.3)$$
and a normalisation condition $\int_X \omega^n_\theta = V$.

**Definition 2.1.** ([73, Definition 3.1]) A cscK cone metric in $\Omega$ is defined to be a solution to the coupled system
$$\omega^n_{\text{cscK}} \omega^n_\theta = e^F, \quad \Delta_{\omega_{\text{cscK}}} F = \text{tr}_{\omega_{\text{cscK}}} \theta - \mathcal{S}_\beta. \quad (2.4)$$

Here, the constant $\mathcal{S}_\beta$ is a topological constant, equal to
$$\mathcal{S}_\beta = n \frac{\int_X C_1(X, D)\Omega^{n-1}}{\int_X \Omega^n} = n \frac{\int_X \theta \wedge \omega^{n-1}_0}{V}, \quad (2.5)$$
with the volume $V = \int_X \omega^n_0$.

Putting the formula of $F$ from the first equation to the second equation, we could see that the scalar curvature of $\omega_{\text{cscK}}$ equals to the constant $\mathcal{S}_\beta$ on the regular part $X \setminus D$.

**Remark 2.2.** The cone metric and the cscK cone metric can be described more explicitly in terms of the moment polytope when $X$ is toric; see e.g. [21, Section 2] for more details.

2.2. CscK cone path.

**Definition 2.3.** ([74, Definition 8.1]) The cscK cone path is defined to be the one-parameter $\beta$ family of cscK cone metrics with cone angle $2\pi\beta$.

Many constructions of cscK cone metrics with small positive cone angle were given in [38].

The existence result (see Theorem 3.15) leads to the following openness of the cscK cone path, when there are no nontrivial holomorphic vector fields which preserve $D$ (i.e. $\text{aut}((X, L); D) = 0$ in the notation of Definition 2.18).
Theorem 2.4. ([74, Theorem 8.2]) Suppose that there are no nontrivial holomorphic vector fields which preserve $D$. The cscK cone path is open when $0 < \beta \leq 1$. Precisely, if there exists a cscK cone metric with cone angle $2\pi \beta_0 \in (0, 2\pi)$, then there is a constant $\delta > 0$ such that for all $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$, there exists a cscK cone metric with cone angle $2\pi \beta$. Especially, when $\beta = 1$, then interval becomes $(1 - \delta, 1]$.

The hypothesis on the automorphism group is indeed necessary; see [35, Theorem 1.7 and Remark 1.8] and also Proposition 2.20.

As a result, we recall the definition of the maximal cone angle of the cscK cone metric.

Definition 2.5.
\[ \beta_{\text{cscKc}}((X, \Omega); D) := \sup_{0 < \beta \leq 1} \{ \exists \text{ a cscK cone metric with cone angle } 2\pi \beta \} . \]

2.3. Energy functionals in the log setting.

2.3.1. Space of Kähler cone potentials.

Definition 2.6. We set $\mathcal{H}(\omega_0)$ to be the space of all Kähler cone potentials $\varphi$ such that $\omega_\varphi := \omega_0 + i\partial\bar{\partial}\varphi$ is a Kähler cone metric of angle $2\pi \beta$.

Similarly, we write $\mathcal{H}(\omega_0)$ for the space of Kähler potentials that are smooth globally over $X$. Let $V = \int_X \omega_0^n$. The $d_1$-distance on $\mathcal{H}(\omega_0)$ is defined to be the norm $V^{-1} \int_X |\psi|^2 \omega_\varphi^n$ over the tangent space $T\mathcal{H}$ of $\mathcal{H}$. The definition is the same for $\mathcal{H}_\beta$. It is important to note that there exists a completion of the above space $\mathcal{H}$, which is denoted by $\mathcal{E}^1$.

Definition 2.7. A $\omega_0$-psh function $\varphi \in PSH(X, \omega_0)$ is an element in the space $\mathcal{E}^1(X, \omega_0)$, if

- $\varphi$ has full Monge–Ampère mass, i.e.
  \[ \lim_{j \to \infty} \int_{\{\varphi > -j\}} (\omega_0 + i\partial\bar{\partial} \max\{\varphi, -j\})^n = V. \]

Then the Monge–Ampère operator is well-defined for such $\varphi$.

- $\varphi \in L^1(\omega_0 + i\partial\bar{\partial}\varphi)$.

The reader is referred to the lecture notes [19] by Darvas or the monograph [33] by Guedj–Zeriahi for more details and many important properties of $\mathcal{E}^1$. A particularly important observation for our paper is that we have $\mathcal{H}_\beta(\omega_0) \subset \mathcal{E}^1$ for all $0 < \beta \leq 1$, according to the Definition 2.7.

Actually, $\mathcal{E}^1$ is also the completion of $\mathcal{H}_\beta(\omega_0)$ under the $d_1$-distance. This could be seen by using Donaldson’s model cone metric $\omega_0 + \delta i\partial\bar{\partial}|s|^{2\beta}_h$, here $\delta$ is a small positive constant, $s$ is a defining section of $D$ and $h$ is an Hermitian metric on $L_D$. Then we see that for any given Kähler potential $\varphi$, $\varphi_j = \varphi + j^{-1}|s|^{2\beta}_h$ is a Kähler cone metric.
for sufficiently large \( j \). The sequence \( \varphi_j \) decreases to \( \varphi \) and converges in \( d_1 \)-distance, which means any smooth Kähler potential is approximated by a sequence of Kähler cone potential and the \( d_1 \)-completion of \( \mathcal{H}_\beta(\omega_0) \) is also \( \mathcal{E}_1 \).

2.3.2. Log entropy.

**Definition 2.8** (Log entropy). The **log entropy** on \( \mathcal{H}_\beta \) is defined to be

\[
E_\beta(\varphi) := \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega_0^n |s|_h^{2\beta-2} e^{\rho_0} \omega_\varphi^n},
\]

where \( \omega_\varphi \) is the Kähler potential. When \( \beta = 1 \), it coincides with the classical entropy functional of Kähler metric \( \omega_\varphi \).

2.3.3. \( D \)-functional. The Euler–Lagrange functional for the Monge–Amperè operator is

\[
D_{\omega_0}(\varphi) := \frac{1}{V} \frac{1}{n+1} \sum_{j=0}^n \int_M \varphi \omega_0^j \wedge \omega_\varphi^{n-j},
\]

since direct computation shows that its first variation is

\[
\partial_t D_{\omega_0}(\varphi) = \frac{1}{V} \int_M \partial_t \varphi \omega_\varphi^n.
\]

As a result, we also obtain another expression of \( D_{\omega_0} \),

\[
D_{\omega_0}(\varphi) := \frac{1}{V} \int_0^1 \int_M \partial_t \varphi \omega_\varphi^n dt.
\]

2.3.4. \( j \)-functional. Let \( \chi \) to be a closed \((1,1)\)-form.

**Definition 2.9** \((J_\chi \)-functional\). The **log \( J_\chi \)-functional** is defined to be

\[
J_\chi(\varphi) := j_\chi(\varphi) - \chi \cdot D_{\omega_0}(\varphi).
\]

The **log \( j_\chi \)-functional** is defined to be

\[
j_\chi(\varphi) := \frac{1}{V} \int_M \varphi \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j} \wedge \chi.
\]

Here \( \chi \) is the average of \( \chi \) as follows

\[
\bar{\chi} := \frac{\int_X [\chi] \Omega^{n-1}}{\int_X \Omega^n} = \frac{n \int_X \chi \wedge \omega_0^{n-1}}{V}.
\]

It is direct to see that its first variation is

\[
\partial_t J_\chi(\varphi) = \frac{1}{V} \int_M \partial_t \varphi \left( n \chi \wedge \omega_\varphi^{n-1} - \bar{\chi} \omega_\varphi^n \right).
\]

We further define **Aubin’s \( J \)-functional** as

\[
J_{\omega_0}^A(\varphi) := \frac{1}{V} \int_M \varphi \omega_0^n - D_{\omega_0}(\varphi).
\]
2.3.5. Log $K$-energy.

**Definition 2.10.** (Log $K$-energy [73, Equation 3.9]) The **log $K$-energy** is defined on $H_{\beta}$ as

\begin{equation}
\nu_{\beta}(\varphi) := E_{\beta}(\varphi) + J_{-\theta}(\varphi) + \frac{1}{V} \int_{M} (h + h_0)\omega^n_0,
\end{equation}

where $h := -(1 - \beta) \log |s|^2_h$.

When $\beta = 1$, we choose $\theta = Ric(\omega_0)$. Then the log $K$-energy coincides with Mabuchi’s $K$-energy, which will be denoted by,

\begin{equation}
\nu_1(\varphi) = E_1(\varphi) + j_{-Ric(\omega_0)}(\varphi) + S_1 \cdot D_{\omega_0}(\varphi),
\end{equation}

with the entropy term

\[ E_1(\varphi) = \frac{1}{V} \int_{M} \log \frac{\omega^n_\varphi}{\omega^n_0}. \]

We denote $\nu_{\beta}(\omega_0, \omega_\varphi) := \nu_{\beta}(\varphi)$.

**Proposition 2.11.** The log $K$-energy satisfies the co-cycle condition

\[ \nu_{\beta}(\omega_1, \omega_3) = \nu_{\beta}(\omega_1, \omega_2) + \nu_{\beta}(\omega_2, \omega_3). \]

The critical point of the log $K$-energy is the cscK cone metric ([73, Lemma 3.5]).

The following identity between log $K$-energy and $K$-energy was given in [73], we record it here.

**Proposition 2.12.** The relation between the log $K$-energy and the $K$-energy is given by

\[ \nu_{\beta}(\varphi) = \nu_1(\varphi) + (1 - \beta) \frac{1}{V} \int_{M} \log |s|^2_h (\omega^n_\varphi - \omega^n) \]

\[ + (1 - \beta) j_{\Theta_D}(\varphi) - (1 - \beta) C_1(LD)\Omega^{n-1} \Omega^n. \]

**Proof.** Firstly, the entropy term is simplified to be

\[ E_{\beta}(\varphi) = \frac{1}{V} \int_{M} \log \frac{\omega^n_\varphi}{\omega^n_0 |s|^2_h e^{h_0} \omega^n_\varphi} \]

\[ = E_1(\varphi) - \frac{1}{V} \int_{M} \log (|s|^2_{h^0} e^{h_0}) \omega^n_\varphi. \]

Then rearranging these terms, we have

\[ E_{\beta}(\varphi) + \frac{1}{V} \int_{M} (h + h_0)\omega^n_0 \]

\[ = E_1(\varphi) + \frac{1 - \beta}{V} \int_{M} \log |s|^2_h (\omega^n_\varphi - \omega^n_0) - \frac{1}{V} \int_{M} h_0 (\omega^n_\varphi - \omega^n_0). \]

Secondly, we compute

\[ J_{-\theta}(\varphi) := j_{-\theta}(\varphi) + \theta \cdot D_{\omega_0}(\varphi). \]
Since \( \theta \in C_1(X, D) = C_1(X) - (1 - \beta)C_1(L_D) \), we have
\[
\bar{\theta} := \frac{\int_X [\theta] \Omega^{n-1}}{\int_X \Omega^n} = \mathcal{S}_1 - (1 - \beta) \frac{\int_X C_1(L_D) \Omega^{n-1}}{\int_X \Omega^n}.
\]
Then using (2.1), we see
\[
j_\theta(\varphi) := -\frac{1}{V} \int_M \varphi \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j} \wedge \theta
\]
\[
= -\frac{1}{V} \int_M \varphi \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j} [\text{Ric}(\omega_0) - (1 - \beta) \Theta D - i\partial\bar{\partial}h_0].
\]
By using the definition of \( j_{-\text{Ric}(\omega_0)} \) and \( j_{\Theta D} \), it is equal to
\[
j_{-\text{Ric}(\omega_0)}(\varphi) + (1 - \beta) j_{\Theta D}(\varphi) + \frac{1}{V} \int_M \varphi \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j} \wedge i\partial\bar{\partial}h_0.
\]
Further computation shows that
\[
\frac{1}{V} \int_M \varphi \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j} \wedge i\partial\bar{\partial}h_0
\]
\[
= \frac{1}{V} \int_M h_0 i\partial\bar{\partial} \varphi \wedge \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j}
\]
\[
= \frac{1}{V} \int_M h_0 (\omega_\varphi - \omega) \wedge \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j}
\]
\[
= \frac{1}{V} \int_M h_0 \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-j} - \frac{1}{V} \int_M h_0 \sum_{j=0}^{n-1} \omega_0^{j+1} \wedge \omega_\varphi^{n-1-j}
\]
\[
= \frac{1}{V} \int_M h_0 (\omega_\varphi^n - \omega_0^n).
\]
At last, we complete the proof by adding these identities together and making use of the formula of \( K \)-energy (2.11).

In the following, we obtain an equivalent formula of the log \( K \)-energy.

**Proposition 2.13.** On the divisor \( D \), we set the corresponding volume and the normalisation functional \( D \) to be
\[
\text{Vol}(D) = \int_D \Omega^{n-1}, \quad D_{\omega_0, D}(\varphi) = \frac{n}{V} \int_0^1 \int_D \partial_t \varphi \omega_\varphi^{n-1} dt.
\]
Then we have
\[
\nu_\beta(\varphi) = \nu_1(\varphi) + (1 - \beta) \cdot [D_{\omega_0, D}(\varphi) - \frac{\text{Vol}(D)}{V} \cdot D_{\omega_0}(\varphi)].
\]
Proof. We compute that by using the Poincaré–Lelong equation (c.f. [74, Section 2.3.1])
\[ \frac{n}{V} \int_0^1 \int_D \partial_i \varphi \omega_{\varphi}^{n-1} dt = \frac{n}{V} \int_0^1 \int_X [i \partial \bar{\partial} \log |s_h|^2 + \Theta_D] \partial_i \varphi \omega_{\varphi}^{n-1} dt. \]
By (2.8), it is equal to
\[ \frac{n}{V} \int_0^1 \int_X i \partial \bar{\partial} \log |s_h|^2 \partial_i \varphi \omega_{\varphi}^{n-1} dt + j \Theta_D(\varphi). \]
We further compute that
\[ \frac{n}{V} \int_0^1 \int_X i \partial \bar{\partial} \log |s_h|^2 \partial_i \varphi \omega_{\varphi}^{n-1} dt = \frac{n}{V} \int_0^1 \int_X \log |s_h|^2 i \partial \bar{\partial} \partial_i \varphi \omega_{\varphi}^{n-1} dt \]
\[ = \frac{n}{V} \int_0^1 \int_M \log |s_h|^2 (\omega_{\varphi}^{n-1} - \omega^n) dt, \]
which completes the proof.

The linearity of the log K-energy follows directly from this formula.

**Theorem 2.14.** The log K-energy is linear in the cone angle \( \beta \). Precisely, given \( \beta_1 \leq \beta_2 \) and letting \( \beta = (1 - s)\beta_1 + s\beta_2 \), we have
\[ \nu_{\beta} = (1 - s)\nu_{\beta_1} + s\nu_{\beta_2}. \] (2.12)

2.3.6. Uniqueness of cscK cone metrics and lower bound of the log K-energy.

**Theorem 2.15.** ([73, Theorem 1.10]) The cscK cone metric \( \omega_{\text{cscK}} \) with cone angle \( 2\pi \beta \) is unique up to automorphisms.

The lower bound of the log K-energy is obtained from uniqueness Theorem 2.15 and convexity of the log K-energy along the cone geodesic.

**Theorem 2.16.** ([73, Lemma 3.6]) If a Kähler pair \( (X, \Omega, D) \) admits a cscK cone metric \( \omega_{\text{cscK}} \) with cone angle \( 2\pi \beta \), then the log K-energy \( \nu_{\beta} \) archives minimum at \( \omega_{\text{cscK}} \).

2.4. Automorphism group and log Futaki invariant. We start with the following definition of automorphisms on projective variety.

**Definition 2.17.** Let \( (X, L) \) be a polarised smooth projective variety. We write \( \text{Aut}_0(X, L) \) for the identity component of the group of biholomorphic automorphisms on \( X \) whose action lifts to the total space of \( L \), which is known to be a linear algebraic group. We write \( \text{aut}(X, L) \) for its Lie algebra.

When we consider a polarised pair, a more appropriate definition is given as follows.
Definition 2.18. When $D \subset X$ be a smooth effective divisor, we write $\text{Aut}_0((X, L); D)$ for the subgroup of $\text{Aut}_0(X, L)$ which preserves $D$; its Lie algebra $\text{aut}_0((X, L); D)$ consists of holomorphic vector fields on $X$ that are tangential to $D$.

We now recall the definition of the log Futaki invariant, which is defined for a holomorphic vector field and gives an obstruction to the existence of cscK cone metrics. Let $v$ be an element of $H^0(X, T_X)$ which admits a holomorphy potential; recall that $\theta_v$ is said to be a holomorphy potential of $v$ if it satisfies

$$i_v \omega = i \bar{\partial} \theta_v.$$ 

The sign convention for the right hand side varies in the literature, but we fix the one as above. It is well-known that $v \in H^0(X, T_X)$ admits a holomorphy potential if and only if the real part of $v$ is an element of $\text{aut}(X, L)$.

Definition 2.19. The Futaki invariant of $v \in H^0(X, T_X)$ with the holomorphy potential $\theta_v$ is defined by

$$\text{Fut}(v) := - \int_X \theta_v(S(\omega) - S_1) \frac{\omega^n}{n!},$$

where $S_1$ is as defined in (2.5). For a smooth effective divisor $D \subset X$, the log Futaki invariant with cone angle $2\pi \beta$ is defined by

$$\text{Fut}_{D, \beta}(v) := \frac{1}{2\pi} \text{Fut}(v) + (1 - \beta) \left( \int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{\text{Vol}(D)}{V} \int_X \theta_v \frac{\omega^n}{n!} \right).$$

While the Futaki invariant and the log Futaki invariant are both defined with respect to a Kähler form $\omega$ on $X$, it only depends on its cohomology class $[\omega]$, as proved in [29,32].

Comparing with Proposition 2.13, we see that the log Futaki invariant is the gradient of the log $K$-energy.

We finish off this section by providing some auxiliary results on the log Futaki invariant and the automorphism group which we use later.

Proposition 2.20. Suppose that there exists $v \in \text{aut}((X, L); D)$ such that

$$\int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{\text{Vol}(D)}{V} \int_X \theta_v \frac{\omega^n}{n!} \neq 0.$$ 

Then the cone angle $2\pi \beta$ of the cscK cone metric, if exists, is given by

$$\beta = 1 + \text{Fut}(v) \left( \int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{\text{Vol}(D)}{V} \int_X \theta_v \frac{\omega^n}{n!} \right)^{-1}. $$

Proof. From coercivity of the log $K$-energy, we see that it is bounded below by Theorem 2.16. The derivative of the log $K$-energy $\nu_\beta$ is log Futaki invariant $\text{Fut}_{D, \beta}$. So $\text{Fut}_{D, \beta}$ must vanish, by noting $\text{Fut}_{D, \beta}(-v) = \text{Fut}_{D, \beta}(v)$. Therefore, the cone angle $2\pi \beta$ is given by

$$\beta = 1 + \text{Fut}(v).$$
Fut,\( \beta(v) \). Then by (2.13), the cone angle \( 2\pi\beta \) must satisfy the identity (2.15). □

The condition (2.14) is not vacuous, and satisfied e.g. for the case considered in [35, Theorem 1.7]. The above lemma shows that, when there is a nontrivial holomorphic vector field \( v \) that preserves \( D \) the cone angle is uniquely determined by the vanishing of the log Futaki invariant (as long as \( v \) is “generic” in the sense that it satisfies (2.14)).

On the other hand, such cases happen only if we choose the divisor \( D \) to be non-generic, to the extent that \( D \) is preserved by a nontrivial holomorphic vector field at all. Indeed we expect \( \text{Aut}_0((X,L);D) \) to be trivial for a generically chosen divisor \( D \), even when \( \text{Aut}_0((X,L)) \) is not. We partially confirm this expectation by giving a simple sufficient condition for the triviality of \( \text{Aut}_0((X,L);D) \) in the following proposition.

**Proposition 2.21.** Let \( A \) be an ample line bundle on \( X \). Then there exists \( m_1 \in \mathbb{N} \) depending only on \( X \) and \( A \) such that for all \( m \geq m_1 \) we have \( \text{Aut}_0((X,L);D_m) = 0 \) for a generic member \( D_m \) of the linear system \( |mA| \).

The proposition above can be regarded as a partial generalisation of the result by Song–Wang [60, Theorem 2.8] for Fano manifolds (see also Berman [4, §1.6]), and the proof below follows the same strategy as theirs; note on the other hand that the aforementioned papers [4,60] provide a sharper estimate for the \( m_1 \) above for the Fano case.

**Proof.** We first prove that there exists \( m_0 \in \mathbb{N} \) depending only on \( X \) and \( A \) such that a generic member of the linear system \( |mA| \) admits no nontrivial holomorphic vector fields for all \( m \geq m_0 \). Mori’s cone theorem [48] implies that \( K_X + (n+2)A \) is an ample divisor on \( X \) [41, Theorem 1.5.33 and Example 1.5.35]. Pick \( m_0 \geq n+2 \) so that \( |m_0A| \) is basepoint free, noting that how large \( m_0 \) should be depends only on \( X \) and \( A \). Then, for \( m \geq m_0 \), a generic member \( D_m \in |mA| \) is smooth by Bertini. The adjunction formula gives \( K_{D_m} = (K_X + D_m)|D_m = (K_X + mA)|D_m \), which is ample for all \( m \geq m_0 \) since it is a restriction of an ample divisor to \( D_m \). Thus \( D_m \) admits no nontrivial holomorphic vector fields by [39, Chapter III, Theorem 2.4].

We then prove that there exists \( m_1 \in \mathbb{N} \) depending only on \( X \) and \( A \) such that for all \( m \geq m_1 \) and for a generic \( D_m \in |mA| \) there exist no nontrivial holomorphic vector fields on \( X \) that vanish on \( D_m \). It is necessary and sufficient to prove that there exists \( m_1 \in \mathbb{N} \) such that

\[
H^0(X, T_X \otimes \mathcal{O}_X(-D_m)) = H^0(X, T_X \otimes A^{\otimes(-m)}) = 0
\]

for the holomorphic tangent sheaf \( T_X \) of \( X \) and for all \( m \geq m_1 \); this holds indeed since

\[
H^0(X, T_X \otimes A^{\otimes(-m)}) = H^n(X, K_X \otimes T_X \otimes A^{\otimes m})^\vee = 0
\]
for all large enough \( m \) by the Serre duality and the Serre vanishing, where we note how large \( m \) should be depends only on \( X \) and \( A \) [41, Theorem 1.2.6].

Combined with the previous claim and replacing \( m_1 \) by \( \max \{ m_0, m_1 \} \) if necessary, we then find that, for all \( m \geq m_1 \) and a generic \( D_m \in [mA] \), there exist no nontrivial holomorphic vector fields on \( X \) that preserves \( D_m \). Hence we get \( \text{aut}((X, L); D_m) = 0 \), which obviously implies \( \text{Aut}_0((X, L); D_m) = 0 \).

\[ \square \]

3. Review of various log stabilities

We review various stability notions for a polarised pair \( ((X, L); D) \). Sections 3.1 and 3.2 concern variants of the log \( K \)-stability, which is formulated in an algebro-geometric language. Section 3.3, on the other hand, discusses the log properness and the log geodesic stability which is essentially analytic and play an important role in this paper.

3.1. Test configurations and log \( K \)-stability. We recall the basic definitions and facts concerning the log \( K \)-stability and its variants. The reference is [10, 52]. We start by recalling the following.

\textbf{Definition 3.1 (Test configuration).} A test configuration \( (\mathcal{X}, \mathcal{L}) \) for a polarised Kähler manifold \( (X, L) \) is a scheme \( \mathcal{X} \) with a flat projective morphism \( \pi : \mathcal{X} \to \mathbb{C} \) such that

- \( \mathbb{C}^* \) acts on \( \mathcal{X} \) in such a way that \( \pi \) is \( \mathbb{C}^* \)-equivariant, with a linearisation of the \( \mathbb{C}^* \)-action on \( \mathcal{L} \),
- \( \pi^{-1}(1) = (X, L) \).

The fibre \( \pi^{-1}(0) \) over \( 0 \in \mathbb{C} \) is called the central fibre and written also as \( \mathcal{X}_0 \).

A log test configuration \( ((\mathcal{X}, \mathcal{L}); D) \) for a polarised pair \( ((X, L); D) \) is a test configuration \( (\mathcal{X}, \mathcal{L}) \) for \( (X, L) \) together with a subscheme \( D \subset \mathcal{X} \) obtained by complementing the \( \mathbb{C}^* \)-orbit of \( D \) in \( \mathcal{X} \setminus \pi^{-1}(0) \) with the flat limit over \( 0 \in \mathbb{C} \).

We say that a log test configuration \( ((\mathcal{X}, \mathcal{L}); D) \) is normal if \( \mathcal{X} \) is a normal variety. We also say that \( ((\mathcal{X}, \mathcal{L}); D) \) is product if \( \mathcal{X} \) is isomorphic to \( X \times \mathbb{C} \) and \( D \) is isomorphic to \( D \times \mathbb{C} \). Note that a product log test configuration corresponds one-to-one with a holomorphic vector field \( v \) admitting a holomorphism potential which is tangential to the divisor \( D \subset X \) (i.e. the real part of \( v \) is in \( \text{aut}((X, L); D) \)); \( v \) preserving \( D \) is important, since otherwise we have a log test configuration with \( \mathcal{X} \cong X \times \mathbb{C} \) but \( \mathcal{D} \not\cong D \times \mathbb{C} \). Finally, a log test configuration is said to be trivial if \( \mathcal{X} \) (resp. \( \mathcal{D} \)) is \( \mathbb{C}^* \)-equivariantly isomorphic to \( X \times \mathbb{C} \) (resp. \( D \times \mathbb{C} \)), meaning that the \( \mathbb{C}^* \)-action on \( \mathcal{X} \cong X \times \mathbb{C} \) (resp. \( \mathcal{D} \cong D \times \mathbb{C} \)) induces a trivial \( \mathbb{C}^* \)-action on the first factor \( X \) (resp. \( D \)).
We also recall that we can compactify a test configuration to form a family over $\mathbb{P}^1$ (see also [10, section 2.2]).

**Definition 3.2.** Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for $(X, L)$. The compactification $\bar{\mathcal{X}}$ of $\mathcal{X}$ is defined by gluing together $\mathcal{X}$ and $X \times (\mathbb{P}^1 \setminus \{0\})$ along their respective open subsets $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{C} \setminus \{0\})$, identified by the canonical $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{C} \setminus \{0\})$. The line bundle $\bar{\mathcal{L}}$ over $\bar{\mathcal{X}}$ can be defined as the natural one via the procedure for the compactification above.

The log Donaldson–Futaki invariant is an algebro-geometric invariant associated to a log test configuration, defined as follows. Let $A_k$ be the infinitesimal generator for the $\mathbb{C}^*$ acting on the vector space $H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0})^\vee$, which is the dual of the set of holomorphic sections of $\mathcal{L}^{\otimes k}|_{\mathcal{X}_0}$. We denote the dimension of $H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k})$ by $d_k$ and the trace of $A_k$ by $w_k$.

According to the Riemann–Roch theorem and its equivariant version, $d_k$ and $w_k$ are given by the polynomials

$$d_k = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$
$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),$$

where $a_0, a_1, b_0, b_1, \ldots$ are rational numbers, for sufficiently large $k$. Restricted to $D$, the corresponding dimension $\tilde{d}_k := \dim H^0(D_0, \mathcal{L}^{\otimes k}|_{D_0})$ and the weight $\tilde{w}_k$ of $\mathbb{C}^* \sim H^0(D_0, \mathcal{L}^{\otimes k}|_{D_0})^\vee$ satisfy similar formulae

$$\tilde{d}_k = \tilde{a}_0 k^n + \tilde{a}_1 k^{n-1} + O(k^{n-2}),$$
$$\tilde{w}_k = \tilde{b}_0 k^{n+1} + \tilde{b}_1 k^n + O(k^{n-1}).$$

**Definition 3.3.** The log Donaldson–Futaki invariant is defined by

$$DF(\mathcal{X}, D, \mathcal{L}, \beta) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1 - \beta) \frac{a_0 \tilde{b}_0 - \tilde{a}_0 b_0}{a_0}.$$

**Remark 3.4.** The sign above is different from the one given e.g. in [66, Definition 6.7], but this is because we considered the action on the dual vector space. The sign changes when we consider the action on $H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0})$, rather than its dual, which is more adapted to the differential-geometric formula as in [66, Proposition 7.15].

Similarly to [28], the log Futaki invariant can be regarded as the limit of the “log” Chow weight.

We are now ready to define the log $K$-stability.

**Definition 3.5** (Log $K$-stability). The polarised pair $((X, L); D)$ is said to be log $K$-semistable with angle $2\pi \beta$ if the log Donaldson–Futaki invariant satisfies $DF(\mathcal{X}, D, \mathcal{L}, \beta) \geq 0$ for any normal log test configuration $((\mathcal{X}, L); D)$. 
We say $((X, L); D)$ is log $K$-polystable with angle $2\pi \beta$ if $((X, L); D)$ is $K$-semistable and $DF(X, D, L, \beta) = 0$ if and only if $((X, L); D)$ is a product log test configuration.

Finally, $((X, L); D)$ is said to be log $K$-stable with angle $2\pi \beta$ if $((X, L); D)$ is $K$-semistable and $DF(X, D, L, \beta) = 0$ if and only if $((X, L); D)$ is a trivial log test configuration.

A subtle point in the definition of the log $K$-semistability above is that we decreed $DF(X, D, L, \beta) \geq 0$ for normal log test configurations. It is known that the usual Donaldson–Futaki invariant decreases when we take the normalisation of $X$ [10, Proposition 3.15], and hence it suffices to test the positivity of $DF(X, L)$ for all normal test configurations. On the other hand, for the log case, how the extra term behaves under the normalisation does not seem completely trivial. While it may happen that a similar result holds for the log case, in this paper we just decree that we only check the Donaldson–Futaki invariant for normal test configurations.

Note moreover that the condition $DF(X, D, L, \beta) = 0$ for product log test configurations is equivalent to the vanishing of the log Futaki invariant (2.13), i.e. $\text{Fut}_{D, \beta}(v) = 0$, for any holomorphic vector field $v$ admitting a holomorphic potential and preserving $D$.

We also need the following quantity which serves as the “norm” of test configurations. This was introduced by Boucksom–Hisamoto–Jonsson [10]. We follow the exposition in [31, Definition 2.3].

**Definition 3.6.** Let $(\mathcal{X}, \mathcal{L})$ be a normal test configuration for $(X, L)$ and

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\Theta} & \tilde{X} \\
\Pi \downarrow & & \downarrow \\
X \times \mathbb{P}^1 & \xrightarrow{\text{pr}_1} & X
\end{array}
$$

be the normalisation of the graph of the birational map $X \times \mathbb{P}^1 \to \tilde{X}$. The **non-Archimedean J-functional** of $(\mathcal{X}, \mathcal{L})$ is defined by

$$J^\text{NA}(\mathcal{X}, \mathcal{L}) := \frac{\Pi^* \text{pr}_1^* L^n \cdot \Theta^* \tilde{L}}{\int_X C_1(L)^n} - \frac{\tilde{L}^{n+1}}{(n + 1) \int_X C_1(L)^n},$$

where $\text{pr}_1 : X \times \mathbb{P}^1 \to X$ is the natural projection and the numerator of the first (resp. second) term is the intersection product on $\tilde{Z}$ (resp. $\tilde{X}$).

The non-Archimedean $J$-functional has some nontrivial and interesting properties that are not immediately obvious from the definition. It is always a nonnegative rational number and zero if and only if $(\mathcal{X}, \mathcal{L})$ is (normal and) trivial [10, Theorem 7.9].

It is also well-known [10, Proposition 7.8 and Remark 7.12] that $J^\text{NA}$ is Lipschitz equivalent to the minimum norm introduced by Dervan...
[23, Definition 4.5], which is defined as

\[(3.6) \quad \| (X, L) \|_m := \frac{\tilde{b}_0 a_0 - \tilde{a}_0 b_0}{a_0},\]

where \(a_0, b_0\) are as in (3.1) and (3.2), and \(\tilde{a}_0, \tilde{b}_0\) are as in (3.3) and (3.4) defined for a generic member of the linear system \(|L|\).

We also note that \(J^{\text{NA}}\) agrees with the asymptotic slope of the functional \(J^A_\omega\) (2.9) along a psh ray corresponding to a test configuration; this fact will be used later in Section 4.2, but for the moment we merely refer the reader to [11, Section 3] for the precise statement and more details.

3.2. \(G\)-uniform log \(K\)-stability. We start by recalling the notion of the uniform log \(K\)-stability, which can be defined following Boucksom–Hisamoto–Jonsson [10] and Dervan [23].

**Definition 3.7 (Uniform log \(K\)-stability).** A polarised pair \(((X, L); D)\) is uniformly log \(K\)-stable with angle \(2\pi \beta\) if there exists a positive constant \(\epsilon > 0\) such that

\[
DF(X, D, L, \beta) \geq \epsilon J^{\text{NA}}(X, L)
\]

for any normal log test configuration \(((X, \mathcal{L}); D)\) for \(((X, L); D)\).

Dervan [23, Definition 4.5] used the minimum norm in the definition of the uniform \(K\)-stability, but it is well-known [10, Remark 8.3] that it is equivalent to the formulation above which uses \(J^{\text{NA}}\).

**Remark 3.8.** The above stability condition is closely related to the uniformly twisted \(K\)-stability introduced by Dervan [23, Definition 2.7], in the sense that it is defined as the uniform log \(K\)-stability when the divisor \(D\) is a general member of a fixed linear system [23, Section 2.1]. We shall return to this point when we later discuss Theorem 5.2.

There is a version of the uniform \(K\)-stability when we have a nontrivial automorphism group \(G\); see [36] and [42, Section 3] for more details. We assume in the following definition that \(G := \text{Aut}_0((X, L); D)\) is reductive, but it makes sense when \(G\) is merely a reductive subgroup of \(\text{Aut}_0((X, L); D)\). We write \(T\) for the complex torus which is the identity component of the centre of \(G\). We define the cocharacter lattice \(N_\mathbb{Z}\) of \(T\) as

\[N_\mathbb{Z} := \text{Hom}(\mathbb{C}^*, T),\]

and write \(N_\mathbb{R} := N_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}\). We recall the following definition from [42, Definition 3.1].

**Definition 3.9 (\(G\)-equivariant test configuration).** We say that a log test configuration \(((X, \mathcal{L}); D)\) is \(G\)-equivariant if \(G\) acts on \(X\) in such a way that it commutes with the \(\mathbb{C}^*\)-action of \(((X, \mathcal{L}); D)\), preserves \(D\), admits a linearisation to \(\mathcal{L}\), and agrees with the action \(G \curvearrowright ((X, L); D)\) when restricted to \(\pi^{-1}(1)\).
A $G$-equivariant test configuration can be twisted by a $T$-action, as in [42, Definition 3.2].

**Definition 3.10.** Let $((X, L); D)$ be a $G$-equivariant log test configuration. For $\xi \in \mathbb{N}_Z$, the $\xi$-twist of $((X, L); D)$, written $((X_\xi, L_\xi); D_\xi)$, is a test configuration with the total space $((X, L); D)$ but with the $\mathbb{C}^*$-action generated by $\eta + \xi$, where $\eta$ is the generator of the $\mathbb{C}^*$-action defining $((X, L); D)$.

We also allow $\xi$ to be an element of $\mathbb{N}_\mathbb{R}$; in this case the resulting $\xi$-twist $((X_\xi, L_\xi); D_\xi)$ is no longer a test configuration strictly speaking, since the $\mathbb{C}^*$-action is no longer algebraic, and hence it is called an $\mathbb{R}$-test configuration in [36] or [42]. We note that its compactification $(\bar{X}_\xi, \bar{L}_\xi)$, as in Definition 3.2, makes sense as a complete analytic space which is in general not isomorphic to $(\bar{X}, \bar{L})$; in particular, $J^\text{NA}(X_\xi, L_\xi)$ is well-defined by considering a bimeromorphic map $X \times \mathbb{P}^1 \rightarrow \bar{X}_\xi$ in Definition 3.6, and depends nontrivially on $\xi$. With this understood, we define the following.

**Definition 3.11 (Reduced J-norm).** Let $(X, L)$ be a $T$-equivariant test configuration. The reduced J-norm of $(X, L)$ is defined by

$$J^\text{NA}_T(X, L) := \inf_{\xi \in \mathbb{N}_\mathbb{R}} J^\text{NA}(X_\xi, L_\xi).$$

Note that a $G$-equivariant test configuration is necessarily $T$-equivariant.

We finally arrive at the definition of the $G$-uniform log $K$-stability.

**Definition 3.12 (G-uniform log K-stability).** A polarised pair $((X, L); D)$ is said to be $G$-uniformly log $K$-stable if there exists $\epsilon > 0$ such that for any $G$-equivariant normal log test configuration $((X, L); D)$ we have

$$DF(X, D, L, \beta) \geq \epsilon J^\text{NA}_T(X, L).$$

3.3. **Review of cscK cone metrics and log properness.** We review the results proved in [74], which many results in this paper rely on. Note that the results below hold for a general compact Kähler manifold $X$ with the Kähler class $\Omega$.

3.3.1. **Log properness.** We set $G := \text{Aut}_0((X, \Omega); D)$. The $G$-orbit of a Kähler cone potential $\varphi \in \mathcal{H}_\beta$ is defined to be

$$\mathcal{O}_\varphi = \{ \hat{\varphi} | \omega_{\hat{\varphi}} = \sigma^* \omega_\varphi, \forall \sigma \in G \}.$$

The $d_{1,G}$-distance between two Kähler cone potentials $\varphi_1, \varphi_2 \in \mathcal{H}_\beta$ is defined to be the infimum of the $d_1$-distance between the corresponding orbits $\mathcal{O}_{\varphi_1}$ and $\mathcal{O}_{\varphi_2}$. Since Kähler cone potential could be approximated by smooth Kähler potentials, it is sufficient to consider properness and coercivity in the space $\mathcal{H}$ of smooth Kähler potentials associated to $\Omega$. 
Definition 3.13. The log $K$-energy $\nu_\beta$ is said to be proper, if for any sequence $\{\varphi_i\} \subset \mathcal{H}$,
\[ \lim_{i \to \infty} d_{1,G}(0, \varphi_i) = \infty \iff \lim_{i \to \infty} \nu_\beta(\varphi_i) = \infty. \]

Definition 3.14. The log $K$-energy $\nu_\beta$ is said to be $d_{1,G}$-coercive, if there exists positive constants $A$ and $B$ such that
\[ \nu_\beta(\varphi) \geq A \cdot d_{1,G}(\varphi, 0) - B \]
for all $\varphi \in \mathcal{H}$.

Clearly, coercivity implies properness. It is a quantitative version of properness.

Theorem 3.15. (Log properness theorem [74, Theorem 1.2, Theorem 7.4]) The Kähler pair $((X, \Omega); D)$ admitting a cscK cone metric with cone angle $2\pi \beta$ is equivalent to the properness of the log $K$-energy $\nu_\beta$, which is also equivalent to the $d_{1,G}$-coercivity of $\nu_\beta$.

In particular, when the automorphism group $\text{Aut}_0((X, \Omega); D)$ is trivial, we have

Theorem 3.16. ([74, Theorem 1.2, Theorem 7.2]) Assume that the automorphism group is discrete. The Kähler pair $((X, \Omega); D)$ admitting a cscK cone metric with cone angle $2\pi \beta$ is equivalent to the properness of the log $K$-energy $\nu_\beta$, and also to the $d_1$-coercivity of $\nu_\beta$.

These existence theorems lead to several applications relating to various notions of log $K$-stabilities, as we will see in the rest of this article.

3.3.2. Log geodesic stability.

Definition 3.17. Suppose $\rho(t) : [0, \infty) \to \mathcal{E}_0^1$ is a unit speed $d_1$-geodesic ray. The $\mathcal{S}$-invariant corresponding to the log $K$-energy along the $d_1$-geodesic ray is defined to be the slope,
\[ \mathcal{S}(\rho(t)) = \lim_{t \to \infty} \frac{\nu_\beta(\rho(t))}{t}. \]

A ray $\rho(t) \in \mathcal{E}_0^1$ starting at $\varphi_0$ is called holomorphic, if it is generated by a one-parameter holomorphic action $\sigma(t) \in \text{Aut}_0(X; D)$, i.e. $\omega_{\rho(t)} = \sigma(t)^*\omega_{\varphi_0}$. Two rays $\rho_1(t), \rho_2(t)$ are parallel if they have uniformly bounded $d_1$-distance. A ray is trivial, if it is parallel to a holomorphic ray.

Definition 3.18 (Log geodesic stability). Let $\rho(t) : [0, \infty) \to \mathcal{E}_0^1$ be any unit speed $d_1$-geodesic starting at $\varphi_0 \in \mathcal{E}_0^1$.

- The point $\varphi_0$ is log geodesic semi-stable, if $\mathcal{S}(\rho(t)) \geq 0$.
- The point $\varphi_0$ is log geodesic stable, if it is geodesic semi-stable and the equality holds when $\rho(t)$ is trivial.

A Kähler class $\Omega$ is geodesic stable (resp. geodesic semi-stable), if every $\varphi_0 \in \mathcal{E}_0^1$ is geodesic stable (resp. geodesic semi-stable).
Actually, in the definition above, the condition that $\rho(t)$ is trivial could be replaced by an equivalent condition, i.e. $\rho(t)$ is holomorphic [59]. It is shown in [74] that

**Theorem 3.19.** ([74, Theorem 1.8]) $(M, \Omega)$ admits a cscK cone metric if and only if it is log geodesic stable.

### 4. Results on stability

#### 4.1. Statement of the results

In this section, we will prove that

**Theorem 4.1.** Suppose that $((X, L); D)$ admits a cscK cone metric of angle $2\pi\beta$. Then the following hold.

1. $((X, L); D)$ is log $K$-semistable with angle $2\pi\beta$;
2. $((X, L); D)$ is log $K$-polystable with angle $2\pi\beta$;
3. $((X, L); D)$ is $G$-uniformly log $K$-stable with angle $2\pi\beta$ for $G = \text{Aut}_0((X, L); D)$;
4. $((X, L); D)$ is uniformly log $K$-stable with angle $2\pi\beta$ if the automorphism group $\text{Aut}_0((X, L); D)$ is further assumed to be trivial.

**Remark 4.2.** Noting that uniform log $K$-stability implies log $K$-stability [23], we find that the item 4 above implies the log $K$-stability of $((X, L); D)$ with a cscK cone metric. Note also that the log $K$-polystability implies the equivariant log $K$-polystability.

Since the log $K$-stability implies the twisted $K$-stability [23, Theorem 1.10], we have

**Theorem 4.3.** Suppose that $((X, L); D)$ admits a cscK cone metric of angle $2\pi\beta$. Then $((X, L); 1-\beta L_D)$ is twisted $K$-stable.

**Remark 4.4.** The slope stability introduced in Ross–Thomas [57] was generalised to the log setting, see [43] for the discussion for Kähler–Einstein cone metrics and also Section 4.5. The log $K$-semistability with angle $2\pi\beta$ implies the log slope semistability with angle $2\pi\beta$.

We also prove a certain sufficient condition for the log $K$-instability in terms of the comparison of the cone angle and the average value of the scalar curvature of the divisor; see Section 4.5, particularly Theorem 4.18, for the more precise statement and the details.

#### 4.2. Proof of Theorem 4.1

We first recall a foundational results by Boucksom–Hisamoto–Jonsson [11] that we use in the proof.

**Theorem 4.5.** (Boucksom–Hisamoto–Jonsson [11, Theorem 4.2]) Suppose that $(\phi_t)_{t \geq 0}$ is a smooth subgeodesic ray in $\mathcal{H}(L)$ which admits a non-Archimedean limit $\phi^{\text{NA}} \in \mathcal{H}^{\text{NA}}(L)$. Then

$$\lim_{t \to +\infty} \frac{\mu_\beta(\phi_t)}{t} = \frac{1}{V} \left( K_{(\mathbb{L}, (1-\beta)D)_{/\mathbb{P}^1}}^{\log} \cdot \mathcal{L}^n + \frac{S_3}{n+1} \mathcal{L}^{n+1} \right)$$
for a normal representative \((\mathcal{X}, \mathcal{L})\) of \(\phi^{\text{NA}}\), where \((\bar{\mathcal{X}}, \bar{\mathcal{L}})\) is the compactification of \((\mathcal{X}, \mathcal{L})\) as in Definition 3.2, \(S_{\beta}\) is as defined in (2.5), and

\[
K_{(\bar{\mathcal{X}}, (1-\beta)\mathcal{D})/\mathbb{P}^1}^{\log} := K_{\bar{\mathcal{X}}/\mathbb{P}^1}^{\log} + (1-\beta)\mathcal{D} = K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \mathcal{X}_{0,\text{red}} - \mathcal{X}_0 + (1-\beta)\mathcal{D},
\]

where \(\mathcal{D}\) is defined as in Definition 3.1 in terms of the divisor \(D\subset \mathcal{X}\).

Note that \(K_{(\bar{\mathcal{X}}, (1-\beta)\mathcal{D})/\mathbb{P}^1}^{\log}\) makes sense as a Weil divisor in \(\mathcal{X}\) since it is normal.

We decide not to explain in this paper what a “non-Archimedean limit” or \(H_{\text{NA}}(\mathcal{L})\) is, since it is rather technical. The only fact that we need is that a test configuration naturally defines a smooth subgeodesic ray \((\phi_t)_{t\geq 0}\) in \(H(\mathcal{L})\) for which the non-Archimedean limit \(\phi_{\text{NA}}^{\nu}\) exists and Theorem 4.5 holds. The reader is referred to [10, 11] for more details.

Theorem 4.1 can be obtained essentially as a consequence of the above result of Boucksom–Hisamoto–Jonsson [11], the log properness result (Theorems 3.15 and 3.16), and the log geodesic stability result (Theorem 3.19) of the third author [74].

4.2.1. Proof of log \(K\)-semistability. We start by proving (1) of Theorem 4.1. As we shall see below, for this purpose we only need to assume that \(\nu_\beta\) is bounded below, which is weaker than the existence of conic \(cscK\) metrics.

Proof of (1) in Theorem 4.1, log \(K\)-semistability. By Theorem 3.19 we know that \(\nu_\beta\) is geodesically stable for all finite energy geodesics in \((\mathcal{E}^1(\mathcal{L}), d_1)\), i.e.

\[
\lim_{t\to +\infty} \frac{\nu_\beta(\psi_t)}{t} \geq 0
\]

for any finite energy geodesic \((\psi_t)_{t\geq 0}\); note that the above inequality holds if \(\nu_\beta\) is merely bounded below. Recall that, for any smooth subgeodesic ray \((\phi_t)_{t\geq 0}\) which extends to the total space of a normal test configuration \((\bar{\mathcal{X}}, \bar{\mathcal{L}})\), we can construct a finite energy geodesic \((\psi_t)_{t\geq 0}\) such that

\[
\lim_{t\to +\infty} \frac{\nu_\beta(\phi_t)}{t} = \lim_{t\to +\infty} \frac{\nu_\beta(\psi_t)}{t},
\]

which follows from the result proved by Berman [5], Chen–Tang [17], and Phong–Sturm [53–55], by also noting that the uniform convergence as \(t \to +\infty\) proved in these papers and [11, Lemma 3.9] give (4.1); see in particular [5, Proposition 2.7 and the proof of Proposition 3.6] for more details, and [7, Theorem 6.6] for the generalisation to \(\mathcal{E}^{1,\text{NA}}\).

Suppose that we have a normal test configuration \((\mathcal{X}, \mathcal{L})\). Then we can take a smooth subgeodesic \((\phi_t)_{t\geq 0}\) in \(H(\mathcal{L})\) which admits a non-Archimedean limit \(\phi_{\text{NA}} \in H_{\text{NA}}(\mathcal{L})\) represented by \((\bar{\mathcal{X}}, \bar{\mathcal{L}})\); see [11, Section 3.1] for more details. Note also that the choice of a divisor
$D \subset X$ naturally defines a log test configuration $((\mathcal{X}, \mathcal{L}); \mathcal{D})$ as in Definition 3.1. By (4.1) and Theorem 4.5 we find

$$0 \leq \lim_{t \to +\infty} \frac{\nu_\beta(\phi_t)}{t}$$

$$= \frac{1}{V} \left( K_{(\mathcal{X},(1-\beta)\mathcal{D})/\mathbb{P}^1} \cdot \tilde{\mathcal{L}}^n + \frac{S_\beta}{n+1} \tilde{\mathcal{L}}^{n+1} \right) - \frac{1}{V} (\mathcal{X}_0 - \mathcal{X}_{0,\text{red}}) \cdot \tilde{\mathcal{L}}^n$$

$$\leq \frac{1}{V} \left( K_{(\mathcal{X},(1-\beta)\mathcal{D})/\mathbb{P}^1} \cdot \tilde{\mathcal{L}}^n + \frac{S_\beta}{n+1} \tilde{\mathcal{L}}^{n+1} \right)$$

$$= DF(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta),$$

where the inequality in the third line follows from $\mathcal{X}_0 - \mathcal{X}_{0,\text{red}}$ being an effective divisor supported on the central fibre and $\tilde{\mathcal{L}}$ being $\pi$-semiample on $\mathcal{X}$. The last line follows from [52, Theorem 3.7]; see also [11, Definition 3.17].

4.2.2. Proof of log $K$-polystability. We improve the above result to prove the log $K$-polystability (2) in Theorem 4.1. We follow the argument by Berman–Darvas–Lu [9, Section 4]. We start by generalising the ingredients of the proof of their result to the conic setting.

**Lemma 4.6.** (see [9, Lemma 4.1]) Suppose that we fix a smooth Kähler metric $\omega$ on $X$. Let $u_0 \in E^1 \cap D_\omega^{-1}(0)$ be a cscK cone potential and $\{u_t\}_{t \geq 0} \subset E^1 \cap D_\omega^{-1}(0)$ be a finite energy geodesic ray emanating from $u_0$, where $D_\omega$ is the functional defined in (2.7). If there exists $C > 0$ such that

$$\inf_{g \in G} J^A_{\omega_0}(g \cdot u_t) < C$$

for all $t \geq 0$ for the functional defined in (2.9), there exists a real holomorphic Hamiltonian vector field $v \in \text{isom}(X, \omega_0)$ preserving $D \subset X$ such that $u_t = \exp(tJv) \cdot u_0$, where $J$ is the complex structure on $X$.

**Remark 4.7.** Inspection of the proof below immediately gives that the right hand side of (4.2) only needs to be of order $C + o(t)$, where $o(t)$ stands for the (nonnegative) quantity which goes to zero after dividing by $t$ and taking the limit $t \to +\infty$, i.e. $\lim_{t \to +\infty} o(t)/t = 0$.

Before we start the proof, we first recall the definition of the Hölder space $C^{4,\alpha,\beta}$, where $0 < \alpha < 1$ is a (fixed) Hölder exponent, introduced in [73, Section 4] for any $0 < \beta \leq 1$, (half angle case $0 < \beta < \frac{1}{2}$ in [38, Section 2] and [47, Section 2]), in such a way that generalises the Hölder space $C^{2,\alpha,\beta}$ defined by Donaldson [29]. While we do not give a precise definition of it in this paper, we only point out that the potential functional $u_0$ of a cscK cone metric of angle $2\pi/\beta$ is an element of $C^{4,\alpha,\beta}$, which in particular implies that $u_0$ is a continuous function on $X$ and that there exists a constant $C > 0$ such that

$$C^{-1} |s|^{2\beta-2} \omega^n_0 \leq \omega^n_0 \leq C |s|^{2\beta-2} \omega^n_0 \quad \text{on } X \setminus D,$$
where $\omega_0$ is reference Kähler metric which is smooth on $X$, as in Section 2, and $s \in H^0(X, L_D)$ is the global section that defines $D$ by $D = \{ s = 0 \}$ (its norm $|s|$ is computed with respect to some fixed Hermitian metric on $L_D$). It is also well-known that we have $C^{4,\alpha,\beta} \subset \mathcal{E}^1$ when $0 < \beta < 1$. We only need these properties of $C^{4,\alpha,\beta}$ in what follows, and the reader is referred for more details to [73] and the references above.

**Proof.** Suppose that we take a sequence $\{ g_k \}_{k \in \mathbb{N}} \subset G$ such that $J_\omega(g_k \cdot u_k) < C$. By [73, Theorem 6.1], $\mathfrak{g} = \text{aut}((X, L); D)$ is the complexification of the isometry group $\text{isom}(X, \omega_{u_0})$ of the cscK cone metric $\omega_{u_0}$, where $u_0 \in C^{4,\alpha,\beta}$. Thus by the global Cartan decomposition (see also [20, Proposition 6.2]) we may write $g_k = h_k \exp(-J v_k)$ for some $h_k \in \text{Isom}_0(X, \omega_{u_0})$ and $v_k \in \text{isom}(X, \omega_{u_0})$. We then find

$$k - C < d_1(u_0, \exp(J v_k) \cdot u_0) < k + C$$

exactly as in [9, Proof of Lemma 4.1], which in turn relies on [20, Proposition 5.5, Lemmas 5.9 and 5.10].

We now claim that the re-scaled vector fields $\{ J v_k/k \}_{k \in \mathbb{N}}$ contain a convergent subsequence in $C^\infty$. We first write $\psi_k \in \mathcal{H} \cap D^{-1}_\omega(0)$ for the potential satisfying $\exp(J v_k)^* \omega = \omega + dd^c \psi_k$, which clearly equals the holomorphy potential of $J v_k$. We then find $\exp(J v_k) \cdot u_0 = \psi_k + \exp(J v_k)^* u_0$, as pointed out in [20, Lemma 5.8]. The above inequality and [18, Theorem 3] (see also [20, Proposition 5.4]) implies that there exists $C_1 > 1$ such that

$$C_1^{-1}(k - C) < \int_X |\psi_k + \exp(J v_k)^* u_0 - u_0| \omega_{u_0}^{n} + \int_X |\psi_k + \exp(J v_k)^* u_0 - u_0| (\exp(J v_k)^* \omega_{u_0})^{n} < C_1(k + C).$$

Noting that $\| \exp(J v)^* u_0 - u_0 \|_{C^0} < \max_X u_0 - \min_X u_0$ and that $u_0$ is continuous on $X$ as $u_0 \in C^{4,\alpha,\beta}$, we find that there exists $C_2 > 1$ such that for all large enough $k$ we have

$$C_2^{-1} < \frac{1}{k} \int_X |\psi_k| \omega^n < C_2,$$

since $\psi_k$ vanishes on $D \subset X$ with order at least 1 as it is a holomorphy potential of a holomorphic vector field which preserves $D$, and

$$C_3^{-1}|s|^{2\beta-2} \omega^n \leq \omega_{u_0}^{n} \leq C_3|s|^{2\beta-2} \omega^n$$

for some $C_3 > 1$ (and similarly for $(\exp(J v_k)^* \omega_{u_0})^{n}$), again by $u_0 \in C^{4,\alpha,\beta}$ and (4.3). Note that the subset of $\mathcal{H} \cap D^{-1}_\omega(0)$ consisting of holomorphy potentials for $\mathfrak{g}$ is a finite dimensional vector space isomorphic to $\mathfrak{g}$. Thus the uniform bound as above means that the sequence
\{\psi_k/k\}_{k\in\mathbb{N}} is contained in a compact subset of \(g\) and hence there exists \(v \in \text{isom}(X,\omega_0)\), \(v \neq 0\), such that \(v_k/k\) converges smoothly to \(v\) as \(k \to \infty\). We then argue as in [9, Proof of Lemma 4.1], which relies on [8, Proposition 5.1], to show that \(u_t = \exp(tJv) \cdot u_0 \in C^{4,\alpha,\beta}\).

\[\square\]

**Proof of (2) in Theorem 4.1, log \(K\)-polystability.** With the above ingredients, the proof is a word-by-word repetition of [9, Proof of Theorem 1.6] by Berman–Darvas–Lu, but we provide a sketch proof for the sake of the reader’s convenience.

Suppose that we pick a normal test configuration \((X, L)\) and a smooth subgeodesic ray \((\phi_t)_{t \geq 0}\), emanating from the conic cscK potential, which extends to the total space of \((X, L)\), just as in Section 4.2.1. Recalling that (1) in Theorem 4.1 implies \(DF(X, D, L, \beta) \geq 0\), it remains to show that \(((X, L); D)\) is product if \(DF(X, D, L, \beta) = 0\), in order to prove (2) in Theorem 4.1.

The log properness theorem (Theorem 3.15) implies that we have

\[
0 \leq \lim_{t \to +\infty} \frac{\nu_\beta(\phi_t)}{t} \leq DF(X, D, L, \beta) = 0,
\]

as we saw in Section 4.2.1. Taking a finite energy geodesic ray \((\psi_t)_{t \geq 0}\) which satisfies (4.1), we thus find \(\nu_\beta(\psi_t) = o(t)\) in the notation of Remark 4.7. Combined with Theorem 3.15, we see that there exists a constant \(C\) such that

\[
\inf_{g \in G} J^A_{\omega_0}(g \cdot (\psi_t - D_\omega(\psi_t))) < C + o(t)
\]

holds for all \(t \geq 0\), noting that \(J^A_{\omega_0}\) is invariant under an additive constant. We then invoke Lemma 4.6 (and also Remark 4.7) and conclude that there exists \(v \in \text{isom}(X,\omega_0)\) such that \(\psi_t = \exp(tJv) \cdot \psi_0\).

We observe that [9, Lemma 4.2] applies to the situation under consideration since \(\psi_t = \exp(tJv) \cdot \psi_0 \in C^{4,\alpha,\beta} \subset \mathcal{E}^1\) for all \(t \geq 0\), which implies that \(v\) lifts to the total space of the line bundle \(L\). We further apply [9, Proposition 4.3], which was originally proved by Berman [5, Lemma 3.4], to conclude that \(v\) is the generator of an (algebraic) \(\mathbb{C}^*\)-action and that \(X\) is isomorphic to \(X \times \mathbb{C}\), as required.

\[\square\]

The necessary direction for the YTD Conjecture 1.8 in terms of the uniform stability is obtained by applying [74, Theorem 1.2 (Theorem 7.4)] and [11, Corollary 4.5].

4.2.3. **Proof of \(G\)-uniform log \(K\)-stability.** Recalling the definition of the \(G\)-uniform log \(K\)-stability from Section 3.2, we prove the \(G\)-uniform log \(K\)-stability by following the argument of Hisamoto [36] and C. Li [42].

**Proof of (3) in Theorem 4.1, \(G\)-uniform log \(K\)-stability.** This is a consequence of the \(d_{1,G}\)-coercivity of the log \(K\)-energy (3.7) proved in [74] (see Theorem 3.15). Recall first that the existence of a cscK cone
metric implies that the automorphism group $G := \text{Aut}_0((X, L); D)$ is reductive by [73, Theorem 6.1] (see also [47, Theorem 4.1] for the half angle case $0 < \beta < \frac{1}{2}$). We thus write it as a complexification of its maximal compact subgroup $K$, which we identify with the isometry group of the cscK cone metric, and write $Z$ for the identity component of the centre of $K$. We further define $T$ as the complexification of $Z$ for any $\varphi \in E^1$, where $A > 0$ and $B \in \mathbb{R}$ are fixed real numbers. This in turn implies that we have

$$\nu_\beta(\varphi) \geq A \cdot d_{1,N_G(K)}(\varphi, 0) - B$$

for any $K$-invariant Kähler potential $\varphi \in (E^1)^K$, where $(E^1)^K$ is a totally geodesic subspace of $E^1$ consisting of $K$-invariant potentials, by [42, Proposition B.1] which is due to J. Yu.

Pick a $G$-equivariant log test configuration $((\mathcal{X}, \mathcal{D}); \mathcal{L})$. Then, as we did in Section 4.2.1, we can take a smooth subgeodesic ray $(\phi_t)_{t \geq 0} \subset E^1$ which admits $\phi_{\text{NA}} \in \mathcal{H}^\text{NA}(L)$ as non-Archimedean limit, where $\phi_{\text{NA}}$ is represented by $(\mathcal{X}, \mathcal{L})$ (recall that $\mathcal{D}$ is automatically determined by $(\mathcal{X}, \mathcal{L})$ and $\mathcal{D} \subset X$). By taking the average over $K$ if necessary, we may assume that $(\phi_t)_{t \geq 0} \subset (E^1)^K$ and that its non-Archimedean limit is still represented by $(\mathcal{X}, \mathcal{L})$, since $((\mathcal{X}, \mathcal{D}); \mathcal{L})$ (and hence $(\mathcal{X}, \mathcal{L})$) is assumed to be $G$-equivariant. Then the argument is very similar to what we did in the proof of (1) in Theorem 4.1 (see Section 4.2.1): we divide (3.7) by $t$ and take the limit $t \to +\infty$, to get

$$DF(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) \geq \lim_{t \to +\infty} \frac{\nu_\beta(\phi_t)}{t} \geq A \cdot \lim_{t \to +\infty} \frac{d_{1,T}(\phi_t)}{t}.$$ 

Finally, the result of Hisamoto [36, Theorem B] (see also [42, Theorem 3.14] by C. Li) implies that the limit of the right hand side agrees with $J^{\text{NA}}_T(\mathcal{X}, \mathcal{L})$. We thus conclude, for any $G$-equivariant log test configuration $((\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta)$, that

$$DF(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) \geq A' J^{\text{NA}}_T(\mathcal{X}, \mathcal{L})$$

for some $A' > 0$ as required, by noting that the $d_1$-distance on $E^1$ is Lipschitz equivalent to the functional $J_{\omega_0}^\text{na}$ in (2.9) [20, Proposition 5.5].

4.2.4. Proof of uniform log $K$-stability. Given all the results established so far, it is easy to establish the final item of Theorem 4.1.
Proof of (4) in Theorem 4.1, uniform log K-stability. We simply repeat the proof in section 4.2.3 when \( G = \text{Aut}_0((X, L); D) \) is trivial. Alternatively, we can also use the the coercivity of \( \nu_\beta \) in the proof of (1) Theorem 4.1, given in section 4.2.1. The coercivity of \( \nu_\beta \) implying the uniform log K-stability is also written in [11, Corollary 4.5]. □

Remark 4.8. We believe that Theorem 4.1 holds for a general Kähler class that is not necessarily the first Chern class of an ample line bundle, by adapting the argument of Dervan–Ross [25] and Sjöström Dyrefelt [58, 59], although we decide not to discuss this problem further in this paper as it will involve subtle technical points.

4.3. Openness of stabilities along cscK cone path. Then combined with Theorem 2.4, (4) in Theorem 4.1 directly implies that

Proposition 4.9. If \( \text{Aut}_0((X, L); D) = 0 \) and there exists a cscK cone metric with cone angle \( 2\pi\beta_0 \in (0, 2\pi) \), then there is a constant \( \delta > 0 \) such that \( ((X, L); D) \) is uniformly log K-stable with angle \( 2\pi\beta \) for all \( \beta \in (\beta_0 - \delta, \beta_0 + \delta) \). Especially, when \( \beta = 1 \), then interval becomes \( (1 - \delta, 1] \).

The following conclusion follows directly from the linearity of the log K-energy Theorem 2.14.

Proposition 4.10. Assume that the log K-energy is proper at cone angle \( \beta_0 > 0 \) and bounded below at cone angle \( \beta_1 > \beta_0 \). Then \( ((X, \Omega); D) \) is the log K-energy is proper for any \( \beta \in [\beta_0, \beta_1) \).

Making use of Theorem 3.15 and 3 in Theorem 4.1, we also have

Proposition 4.11. Assume that \( ((X, \Omega); D) \) admits a cscK cone metric with cone angle \( \beta_0 > 0 \) and the log K-energy is bounded below at cone angle \( \beta_1 > \beta_0 \). Then for any \( \beta \in [\beta_0, \beta_1) \)

- \( ((X, \Omega); D) \) is \( d_{1,G} \)-proper;
- \( ((X, \Omega); D) \) admits a cscK cone metric;
- \( ((X, \Omega); D) \) is \( G \)-uniformly log K-stable with angle \( 2\pi\beta \).

4.4. Comparison between maximal cone angles. The results above directly lead to an estimate of the maximal cone angle \( \beta_{\text{cscK}_{c}} \) of cscK cone metrics, see Definition 2.5.

Definition 4.12. For the smooth polarised pair \( ((X, L); D) \), we define the following invariant,

\[
\beta_{\text{K}^*}((X, L); D) := \sup_{\beta} \{(X, L); D \text{ is log K-stable with angle } 2\pi\beta\}.
\]

Clearly, \( ((X, L); D) \) is log K-stable with cone angle \( 2\pi\beta < 2\pi\beta_{\text{K}^*}((X, L); D) \) and log K-unstable with cone angle \( 2\pi\beta > 2\pi\beta_{\text{K}^*}((X, L); D) \)

Definition 4.13. We define the supremum of the cone angle such that \( \beta_{\text{ulK}^*} := \sup_{\beta} \{(X, L); D \text{ is } G\text{-uniformly log K-stable with angle } 2\pi\beta\} \).
Definition 4.14. We could also define a “stronger” version of the maximal cone angle $\beta_{lKs}$ such the log $K$-stability is satisfied on the whole interval $(0, \beta)$, i.e.

$$\beta_{lKs} = \sup_{\gamma} \{((X, L); D) \text{ is log } K\text{-stable with angle } 2\pi \beta \text{ for all } \beta \in (0, \gamma]\}. $$

and define similarly for the maximal cone angle $\beta_{cscK}$ of existence of $cscK$ cone metrics and the maximal cone angle $\beta_{ulKs}$ of the $G$-uniform log $K$-stability.

Theorem 4.1 implies that

**Theorem 4.15.** $\beta_{cscK} \leq \beta_{ulKs} \leq \beta_{lKs}$, $\beta_{cscK} \leq \beta_{ulKs} \leq \beta_{lKs}$.

The positive answer to the following question, which is equivalent to proving Conjectures 1.1 and 1.8, is expected.

**Question 4.16.** $\beta_{cscK} = \beta_{ulKs}$? $\beta_{cscK} = \beta_{ulKs}$?

We could also compare these two types of maximal cone angles.

**Question 4.17.** $\beta_a = \overline{\beta_a}$, $a \in \{cscK, lKs, ulKs\}$?

Note that the answer to Question 4.17 is indeed affirmative for $a \in \{lKs, ulKs\}$ if we assume that $((X, L); D)$ is log $K$-semistable with angle 0. Indeed, we can argue exactly as in Proposition 4.11 by noting

$$DF(X, D, L, \beta) = (1 - s)DF(X, D, L, \beta_1) + sDF(X, D, L, \beta_2),$$

for $\beta = (1 - s)\beta_1 + s\beta_2 \in (\beta_1, \beta_2)$, analogously to Theorem 2.14.

4.5. Average scalar curvature and log $K$-instability. In this section we study how the value of the average scalar curvature on $D$ affects the log $K$-stability. Let $(X, L)$ be an $n$-dimensional polarised manifold as before. Assume that there is a smooth hypersurface $D$ satisfying $D \in |L|$. We write $\hat{L}$ for the ample line bundle on $D$ defined by $\hat{L} := L|_D$. Recall that the average scalar curvature $S_1$ of a Kähler metric on $X$ in the Kähler class $C_1(L)$ can be computed as

$$S_1 := \frac{n \int_X C_1(-K_X)C_1(L)^{n-1}}{\int_X C_1(L)^n},$$

and the average scalar curvature $S^D$ of a one on $D$ in the class $C_1(\hat{L})$ can be computed as

$$S^D := \frac{(n - 1) \int_D C_1(-K_D)C_1(\hat{L})^{n-2}}{\int_D C_1(\hat{L})^{n-1}},$$

where $-K_D$ is the anticanonical line bundle over $D$.

The main result of this section is the following sufficient condition for the log $K$-instability in terms of $S^D$. 
Theorem 4.18. Suppose that $D \in |L|$ is smooth. Then $((X, L); D)$ is log $K$-unstable with angle $2\pi/\beta$ if $\beta$ satisfies

$$\beta < \frac{S^D}{n(n-1)}.$$ 

Remark 4.19. In the theorem above and in the rest of this section, the cone angle $\beta$ may take any real value and may not be constrained in the interval $(0, 1)$, unlike what is discussed previously.

Remark 4.20. The argument in this section works even when $X$ and $D$ are singular, as long as $X$ is $\mathbb{Q}$-Gorenstein (i.e. $K_X$ is $\mathbb{Q}$-Cartier) and there exists $D \in |L|$ such that $D$ is $\mathbb{Q}$-Gorenstein and $\tilde{L}$ is $\mathbb{Q}$-Cartier on $D$. We present here only the smooth case, however, since assuming $\tilde{L}$ to be $\mathbb{Q}$-Cartier on $D$ seems slightly artificial. See Section 6 for more discussions on singular varieties.

The above theorem can be regarded as partially complementing Theorem 5.2, in the sense that we have therein $S^D \ll 0 < \beta$ when the degree of $D$ is very large. We also recall a well-known result of Sun [61].

Theorem 4.21 (Sun [61]). If $C_1(\tilde{L})$ admits a scalar-flat Kähler metric, then $((X, L); D)$ is strictly log $K$-semistable with angle $\beta = 0$.

We note that Theorem 4.21 was stated in [61] when $X$ is a Fano manifold and $D$ is a smooth anticanonical divisor, but the proof carries over word-by-word to the general polarised case, which is also pointed out in [63, page 5527].

Combined with Theorem 5.2 proved later, the above theorems indicate that $((X, L); D)$ is log $K$-stable if $\beta$ is very large compared to $S^D$, strictly log $K$-semistable if $\beta = 0$ (and $D$ has a cscK metric with $S^D = 0$), and log $K$-unstable if $\beta$ is smaller than $S^D/n(n-1)$.

Remark 4.22. As pointed out in [61], Theorem 4.21 can be regarded as an algebro-geometric counterpart of the differential-geometric result of Tian–Yau [68] which shows the existence of a complete Ricci-flat Kähler metric on $X \setminus D$, if we consider the case $L = -K_X$ for a Fano manifold $X$. Note that in the case of Tian–Yau [68], Yau showed that $D$ admits a Ricci-flat Kähler metric [69].

From this observation, the following problem arises: if $D$ has a constant scalar curvature Kähler metric with $S^D = 0$, does $X \setminus D$ admit a complete scalar-flat Kähler metric? This problem is another version of [1] which deals with the case that $0 < 3S^D < n(n-1)$. Note that Autuay [2] established a version of the converse of this problem for Kähler metrics of Poincaré type.

Before we start the proof of Theorem 4.18, we observe the following proposition.
Proposition 4.23. For any polarised Kähler manifold \((X, L)\), we have
\[
S_1 \leq n(n + 1).
\]

Proof. This is a consequence of Mori’s cone theorem [48] that was also used in the proof of Proposition 2.21. Here we use that \(K_X + (n + 1)L\) is a nef divisor on \(X\) [41, Theorem 1.5.33 and Example 1.5.35]. This implies that we have
\[
\int_X C_1(L)^{n-1} C_1(K_X + (n + 1)L) \geq 0,
\]
by noting that there exists \(m > 0\) such that the Poincaré dual of the cohomology class \(mC_1(L)^{n-1}\) contains a smooth complete intersection curve in \(X\), by a repeated application of Bertini’s theorem.\(\Box\)

It seems natural to ask the following question concerning the equality case of Proposition 4.23, which can be seen as a generalisation of Fujita’s result [30] to general polarised varieties, but we decide not to discuss it any further in this paper.

Problem 4.24. Suppose that \((X, L)\) admits a constant scalar curvature Kähler metric and the equality \(S_1 = n(n + 1)\) holds. Is \((X, L)\) necessarily isomorphic to \((\mathbb{P}^n, O(1))\)?

We now give a proof of Theorem 4.18.

Proof of Theorem 4.18. Recall that \(D \in |L|\) and the average values of scalar curvatures \(S\) and \(S^D\) are given by (4.4) and (4.5). We then observe
\[
S^D = (n - 1) \left( \frac{\int_X C_1(-K_X)C_1(L)^{n-1}}{\int_X C_1(L)^n} - 1 \right) = \frac{n - 1}{n} S_1 - (n - 1),
\]
by the adjunction formula \(K_D = (K_X + D)|_D\).

Following [56] and [61], we recall the construction of the test configuration by using the deformation to the normal cone of \(D\). By blowing up \(X \times \mathbb{C}\) along \(D \times \{0\}\), we obtain a family \(\pi : \mathcal{X} \to \mathbb{C}\). The exceptional divisor \(P \subset \mathcal{X}\) corresponds with \(\mathbb{P}(\nu_D \oplus \mathbb{C})\), where \(\nu_D\) is the normal bundle of \(D\) in \(X\). The central fibre \(X_0\) is given by gluing \(P\) to \(X\) along \(D = \mathbb{P}(\nu_D)\). The \(\mathbb{C}\)-action on \(\mathcal{X}\) is obtained by considering the trivial action on \(X\) and the standard \(\mathbb{C}\)-action on \(\mathbb{C}\). Let \(\mathcal{D}\) be the proper transform of \(D \times \{0\}\). \(\mathcal{D}\) is \(\mathbb{C}\)-equivariant and its intersection with the central fibre is the zero section \(\mathbb{P}(\mathbb{C})\). We define an ample line bundle on \(\mathcal{X}\) by \(\mathcal{L}_c := L - cP\) for a rational number \(c \in (0, 1)\) (see [56]). Thus, we obtain \(\mathbb{C}\)-equivariant family \(((\mathcal{X}, \mathcal{D}); \mathcal{L}_c)\) parametrised by \(c \in (0, 1)\). Let \(t\) be the standard holomorphic coordinate on \(\mathbb{C}\). Following [56] (see also [61]), we have the following decomposition
\[
H^0(\mathcal{X}_0, \mathcal{L}_c^k) = H^0(X, L^{(1-c)k}) \oplus \bigoplus_{i=0}^{c-1} t^{k-i} H^0(D, \tilde{L}^{(k-i)}).
\]
Here, we assume that $ck$ is a sufficiently large integer by taking $k \in \mathbb{N}$ to be sufficiently large and divisible. Note that this is the weight decomposition with respect to the $\mathbb{C}^*$-action above.

Recalling that we have the following short exact sequence

$$0 \to H^0(X, L \otimes (i-1)) \to H^0(X, L \otimes i) \to H^0(D, \tilde{L} \otimes i) \to 0,$$

when $i \in \mathbb{N}$ is large enough, we have

$$\dim H^0(X, L \otimes k_{c}) = \dim H^0(X, L \otimes (1-c)k_{c}) + \sum_{i=0}^{ck-1} \dim H^0(D, \tilde{L} \otimes (k-i))$$

for all sufficiently large and divisible $k \in \mathbb{N}$. This equality implies the flatness of the family $((X, D); \mathcal{L}_c)$, and hence $((X, D); \mathcal{L}_c)$ is a test configuration for $((X, L); D)$.

Let $((X, D); \mathcal{L}_c)$ be the test configuration for $((X, L); D)$ defined as above by the deformation to the normal cone of $D$.

By the Riemann–Roch theorem, we have

$$a_0 = \frac{1}{n!} \int_X C_1(L)^n$$

and

$$a_1 = \frac{1}{2(n-1)!} \int_X C_1(-K_X) \cup C_1(L)^{n-1}$$

$$= \frac{1}{2(n-1)!} \left( \frac{S^D}{n-1} + 1 \right) \int_X C_1(L)^n$$

$$= \frac{na_0}{2} \left( \frac{S^D}{n-1} + 1 \right),$$

in the notation of Section 3.1.

Since the weight of the $\mathbb{C}^*$-action is $-1$ on $t$, the decomposition above (4.6) implies that we can compute the total weight as follows:

$$w_k = -\sum_{i=0}^{ck-1} (ck - i) \dim H^0(D, \tilde{L} \otimes (k-i))$$

$$= -\sum_{i=0}^{ck-1} (ck - i) \left( (k-i)^{n-1} na_0 + \frac{S^D}{2(n-1)!} \int_D C_1(\tilde{L})^{n-1} + O(k^{n-3}) \right)$$

$$= -\sum_{i=0}^{ck-1} (ck - i) \left( (k-i)^{n-1} na_0 + na_0 \frac{S^D}{2} (k-i)^{n-2} + O(k^{n-3}) \right)$$

$$= -na_0 \sum_{i=0}^{ck-1} (ck - i) \left( (k-i)^{n-1} + \frac{S^D}{2} (k-i)^{n-2} + O(k^{n-3}) \right).$$
These terms can be expanded in $k$ as
\[
\sum_{i=0}^{ck-1} (ck - i)(k - i)^{n-1} = k^{n+1} \int_0^c (c-x)(1-x)^{n-1} \, dx + \frac{c}{2} k^n + O(k^{n-1}),
\]
and
\[
\int_0^c (c-x)(1-x)^{n-1} \, dx = -\frac{1}{n} \left( \frac{1-(1-c)^n}{n+1} - c \right).
\]
Thus, we can compute $b_0$ and $b_1$ in the notation of Section 3.1 as
\[
b_0 = \left( \frac{1-(1-c)^{n+1}}{n+1} - c \right) a_0
\]
\[
b_1 = \frac{na_0}{2} \left( -c + \frac{S^D}{n-1} \left( \frac{1-(1-c)^n}{n} - c \right) \right).
\]
By following [61], we have
\[
H^0(D_0, L^\otimes c) = H^0(D, L^\otimes c) / t = \frac{t c^k}{H^0(D, \tilde{L}^\otimes k)}.
\]
This implies
\[
\tilde{a}_0 = \frac{1}{(n-1)!} \int_D C_1(\tilde{L})^{n-1} = na_0,
\]
and
\[
\tilde{b}_0 = -c \frac{1}{(n-1)!} \int_D C_1(\tilde{L})^{n-1} = -cna_0.
\]
Substituting the above in the definition of the log Futaki invariant, we get
\[
DF(\mathcal{X}, D, L_c, \beta) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1-\beta) \left( \tilde{b}_0 - \frac{\tilde{a}_0 b_0}{a_0} \right)
\]
\[
= na_0 \left( \frac{1-(1-c)^{n+1}}{n+1} + \frac{S^D}{n-1} \left( \frac{1-(1-c)^n}{n+1} - \left( \frac{1-(1-c)^n}{n} \right) \right) \right)
\]
\[
+ na_0 \left( 1-\beta \left( -c + \frac{1-(1-c)^{n+1}}{n+1} \right) \right)
\]
\[
= na_0 \left( \frac{1-(1-c)^{n+1}}{n+1} \right) \left( \beta + \frac{S^D}{n-1} \left( 1 - \left( \frac{n+1}{n} \right) \left( \frac{1-(1-c)^n}{1-(1-c)^{n+1}} \right) \right) \right).
\]
Noting that for all $c \in (0, 1)$ we have
\[
\frac{S^D}{n-1} \left( 1 - \left( \frac{n+1}{n} \right) \left( \frac{1-(1-c)^n}{1-(1-c)^{n+1}} \right) \right) \in \left( -\frac{S^D}{n(n-1)}, 0 \right),
\]
there exists $c \in (0, 1)$ such that $DF(\mathcal{X}, D, L_c, \beta) < 0$ if $\beta$ is less than $\frac{S^D}{n(n-1)}$. \hfill \Box
5. Results on existence of cscK cone metrics

In this section, we will derive some existence results of cscK cone metrics, the key tool is the criteria of the properness of log $K$-energy, [74, Proposition 4.22]).

5.1. CscK cone metrics along ample divisors of large degree.
We will let the divisor stay in some multiple of the linear system of an ample line bundle.

**Definition 5.1.** We say a polarised pair $((X,L); D)$ is a **standard polarised pair** and denoted by $((X,L); D, m)$, if

- $X$ is a compact Kähler manifold;
- $L$ is an ample line bundle and denote by $\Omega$ the Kähler class associated with $L$;
- $D$ is a smooth effective divisor in the linear system $|mL|$ with $m > 0$.

The main result that we prove in this section is the following.

**Theorem 5.2.** For any $0 < \beta < 1$, there exists $m_0 \in \mathbb{N}$ which depends only on $X$, $L$, and $\beta$, such that the standard polarised pair $((X,L); D, m)$ admits a constant scalar curvature Kähler metric with cone singularities of cone angle $2\pi \beta$ along $D$, as long as $D$ is a generic member of the linear system $|mL|$, if $m \geq m_0$.

Thus, any smooth projective variety admits a cscK cone metric, as long as $D$ is a divisor which is ample, generic, and of sufficiently large degree. This can be regarded as a conic analogue of the results for the twisted cscK metrics [34, Theorem 1], whose algebro-geometric counterpart was proved by Dervan–Ross [26, Theorems 1.2 and 3.12].

In fact, on the algebro-geometric side, it is not surprising that there should be a parallel between twisted cscK metrics and the cscK cone metrics, because Dervan’s definition of the twisted $K$-stability [23, Definition 2.3] can be regarded as the log $K$-stability for a generic member of the linear system; see [23, §3.4] for more details. On the other hand, however, the parallel is not so obvious on the analytic side, since a twisted cscK metric is globally smooth on $X$ whereas a cscK cone metric has a subtle boundary condition near the divisor $D$. The above theorem establishes the expected parallel on the analytic side.

5.2. Log alpha invariant and properness. We first review the key ingredient in the proof of Theorem 5.2.

**Definition 5.3.** ([74, Definition 4.9]) The log alpha invariant in $\mathcal{H}_\beta$ is defined as

$$\alpha_\beta := \sup\{\alpha > 0 | \exists C \text{ such that } \sup_{\varphi \in \mathcal{H}_\beta(\omega_0)} \int_M e^{-\alpha(\varphi - \sup_X \varphi)} \omega_0^n \leq C\}.$$
Here $\mathcal{H}_\beta$ is the space of all $\omega_0$ pluri-subharmonic functions. The reference metric $\omega_0$ is a solution to the equation

$$\omega_0^n = e^{h_0} s^n n^2 - \omega_0^n,$$

in which, $\omega_0$ is a smooth Kähler metric, $h_0$ is a smooth function (see (2.1)), $s$ is the defining section of $D$ and $h$ is a Hermitian metric on $L_D$.

The definition of the log alpha invariant does not depend on the choice of $\omega_0, h_K, h$, and also $\omega_\theta$. It is also important to note that there is an equivalent definition (Definition 6.4) in terms of algebraic geometry, and that it generalises to the case when $X$ has some mild singularities.

The key ingredient in the proof is the following proposition proved in [74, Proposition 4.22], which involves the log $\alpha$-invariant.

**Proposition 5.4.** ([74, Proposition 4.22]) Let $C_1(L) = [\omega_0]$ be a Kähler class. Assume that there is a constant $\eta$ satisfies that

$$\begin{align*}
&\text{(i)} \quad 0 \leq \eta < \frac{n+1}{n} \alpha_\beta, \\
&\text{(ii)} \quad C_1(X, D) < \eta C_1(L), \\
&\text{(iii)} \quad (S_\beta - \eta) C_1(L) < (n - 1) C_1(X, D).
\end{align*}$$

Then the log K-energy is $J$-coercive in $\Omega$, i.e. there exists a constant $C$ such that

$$\nu_\beta(\varphi) \geq \left( \frac{n+1}{n} \alpha_\beta - \eta \right) J_0^A(\varphi) - C, \quad \forall \varphi \in \mathcal{H}_\beta(\omega_0).$$

Here $J_0^A$ is Aubin’s $J$-functional defined in (2.9).

**5.3. Proof of Theorem 5.2.** We prove that the conditions (5.1) are satisfied when $D \in |mL|$ for large enough $m$.

**Lemma 5.5.** The first Chern class for the pair $(X, D)$ is

$$C_1(X, D) = C_1(X) - m(1 - \beta) C_1(L),$$

and the average scalar curvature for a Kähler cone metric becomes

$$S_\beta = S_1 - mn(1 - \beta).$$

Here $S_1 = \frac{\int_X C_1(X)^n}{\int_X n}$ is the average scalar curvature for any smooth Kähler metric in $\Omega$, which is a topological constant.

**Proof.** The definition of $L_D$ implies $C_1(L_D) = mC_1(L)$, which in turn implies that we have

$$C_1(X, D) = C_1(X) - (1 - \beta) C_1(L_D)$$

which implies (5.3) and (5.4), by using

$$S_\beta = n \frac{\int_X C_1(X, D) C_1(L)^{n-1}}{\int_X C_1(L)^n}$$
Lemma 5.6. There exists \( m_2(X, L, \beta) \in \mathbb{N} \) depending only on \( X \), the polarisation \( L \), and \( 0 < \beta < 1 \), such that for all \( m \geq m_2(X, L, \beta) \) the conditions (5.1) are satisfied with \( \eta = 0 \), and \( C_1(X, D) = C_1(X) - (1 - \beta)C_1(L_D) \) with \( L_D = L \otimes^m \).

Proof. By Lemma 5.5, \( S_\beta \cdot C_1(L) - (n - 1)C_1(X, D) \) equals

\[
[S_1 - mn(1 - \beta)] \Omega - (n - 1)C_1(X).
\] (5.5)

Recalling that \( C_1(L) \) is a Kähler class, the equation (5.3) implies that there exists \( m_2(X, L, \beta) \in \mathbb{N} \) such that the second condition of (5.1) is satisfied for all \( m \geq m_2(X, L, \beta) \) and for \( \eta = 0 \).

Likewise, (5.5) implies that the third condition of (5.1) is satisfied for all \( m \geq m_2(X, L, \beta) \) and for \( \eta = 0 \).

It thus suffices to check that the first condition of (5.1) holds for any \( 0 < \beta < 1 \) and any \( m \geq m_2(X, L, \beta) \). By Berman’s formula [4, Proposition 6.2], we have

\[
\alpha_\beta := \alpha(L, (1 - \beta)D) \\
= m\alpha(L_D, (1 - \beta)D) \\
\geq m \min\{\beta, \alpha(L_D), \alpha(L_D|D)\} \\
= \min\{m\beta, \alpha(L), \alpha(L_D|D)\},
\] (5.6)

where \( \alpha(L_D) \) (resp. \( \alpha(L_D|D) \)) is the \( \alpha \)-invariant of the polarisation \( L_D = L \otimes^m \) on \( X \) (resp. \( L_D|D \) on \( D \)). It is a foundational result of Tian [67, Proposition 2.1] (see also Hörmander [37, Theorem 4.4.5]) that we have \( \alpha(L) > 0 \) and \( \alpha(L_D|D) > 0 \), and hence we get \( \alpha_\beta > 0 \) as required. \( \square \)

Given what we have established so far, it is now straightforward to derive the main result of this section.

Proof of Theorem 5.2. We recall from Proposition 2.21 that there exists \( m_1 \in \mathbb{N} \), depending only on \( X \) and \( L \), such that \( \text{Aut}_0((X, L); D) = 0 \) for a generic member \( D \in |mL| \) if \( m \geq m_1 \). We take \( m_2 \in \mathbb{N} \), which depends on \( X \), \( L \), and \( \beta \), to be as in Lemma 5.6. We set \( m_0 := \max\{m_1, m_2\} \), and get Theorem 5.2 as an immediate consequence of Theorem 3.16 and Proposition 5.4, (see Theorem 5.7). \( \square \)

5.4. Uniformly log \( K \)-stable manifolds. We collect several other applications of Proposition 5.4 to find cscK cone metrics and log stable manifolds.

Proposition 5.4 implies properness of log \( K \)-energy in the Kähler class \( \Omega \). Then, we have a cscK cone metric in \( \Omega \) from the existence result Theorem 3.16. These arguments have been seen in the proof.
of Theorem 5.2. We collect this step here, before we write down its further differential-geometric application.

**Theorem 5.7.** Let $C^1(L) = [\omega_0]$ be a Kähler class. Assume that the Condition (5.1) holds.

Then the automorphism group $\text{Aut}_0((X,L);D)$ is trivial. Moreover, there exists a unique cscK cone metric in $\Omega$ and $((X,L);D)$ is also uniformly log $K$-stable.

**Proof.** It suffices to show that if the log $K$-energy is $J$-proper in the sense of (5.2), i.e. $\nu_\beta(\varphi) \geq \left(\frac{n-1}{n} \alpha_\beta - \eta \right) J^{A}_{\omega_0}(\varphi) - C$, then the automorphism group is trivial. Assume that the automorphism group is non-trivial, then there exists a non-zero holomorphic vector field $X$ and it generates a one-parameter group of holomorphic transformation $\sigma_t$.

Given a Kähler cone metric $\omega$, the automorphism $\sigma_t$ induces a cone geodesic $\sigma_t^* \omega$ (see [73, Section 2] for the definition for cone geodesic). But this cone geodesic generated from the Kähler cone metric $\omega$ may not have enough regularity, or appropriate asymptotic rate along the divisor to proceed the convexity argument below. The key observation here is that, instead of a general Kähler cone metric $\omega$, we need to alternately make use of the reference Kähler cone metric $\omega_\theta$, which solves (2.2), to generate the cone geodesic we need. This reference metric has nice growth along $D$, due to the sharp asymptotic analysis for complex Monge–Ampère equation, established in [70]. Following the proof in [73, Section 6], we see that the value of the log $K$-energy remains the same along the cone geodesic $\sigma_t^* \omega_\theta$. Meanwhile, as shown in [73, Lemma 3.12], the $J$-functional is also strictly convex along the cone geodesic, if $X$ is non-trivial, that leads to the contradiction of the $J$-coercivity of the log $K$-energy. Hence, we complete the proof. □

In the next Section 6, these differential-geometric results will be further extended to the algebro-geometric setting for a pair of normal variety $(X,\Delta)$. Moreover, we will also explore the Condition (5.1) in a quantitative way in terms of the cone angle $\beta$.

**Remark 5.8.** This is a particular case of Proposition 4.37 in [74], where the cohomology class $\Omega$ is allowed to be big and nef, not necessary Kähler.

**Remark 5.9.** In order to consider the case when the automorphism group $G = \text{Aut}_0((X,L);D)$ is non-trivial, it is sufficient to consider the $G$-invariant version $J_G$ of the $J$-functional (2.9) i.e. $J_G(\cdot) = \inf_{\sigma \in G} J(\sigma \cdot)$, and its corresponding gradient flow, to obtain the same $J_G$-properness (5.2) for $G$-invariant log $K$-energy.
Recall the following definitions:

\[ C_1(X, D) = C_1(X) - (1 - \beta)C_1(L_D), \quad S_\beta = n \frac{\int_X C_1(X, D)C_1(L)^{n-1}}{\int_X C_1(L)^n}. \]

A related important quantity is the slope \( \mu \) (see [22, Theorem 1.3]), which is defined as

\[ \mu = \mu_\beta((X, L); D) := \frac{S_\beta}{n}. \]

We then obtain the following several existence theorems in terms of the sign of \( C_1(X, D) \).

**Corollary 5.10.** Assume \( C_1(X, D) = 0 \) and \( C_1(L) \) is a Kähler class. There exists a unique log Calabi–Yau metric in \( C_1(L) \) and \((X, L); D)\) is also uniformly log \( K \)-stable.

**Corollary 5.11.** Assume \( C_1(X, D) \leq 0 \). For any Kähler class \( C_1(L) \), if there exists a constant \( \eta \) satisfying \( 0 \leq \eta < \frac{n-1}{n} \alpha_\beta \) such that

\[ (S_\beta - \eta)C_1(L) < (n-1)C_1(X, D), \]

Then \((X, L); D)\) admits a cscK cone metric and is also uniformly log \( K \)-stable.

**Corollary 5.12.** Assume \( C_1(X, D) \geq 0 \). If there exists a constant \( \eta \) satisfying \( 0 \leq \eta < \frac{n-1}{n} \alpha_\beta \) such that

\[ C_1(X, D) < \eta C_1(L) \]

Then \((X, \Omega); D)\) admits a cscK cone metric and is also uniformly log \( K \)-stable.

**Remark 5.13.** In the special situation for \( D \) staying in some multiple of \(-K_X \) i.e. the Kähler–Einstein cone metric, similar results are shown in [52].

**Remark 5.14.** The parallel results for twisted cscK metric are given in [23].

5.5. **Entropy threshold and \( J \)-threshold.** We further discuss an application of the condition (5.1).

**Definition 5.15.** We set constants \( \Lambda, \lambda \in \mathbb{R} \) such that

\[ \lambda = \sup_C \{ C \cdot C_1(L) \leq C_1(X) \}, \quad \Lambda = \inf_C \{ C_1(X) \leq C \cdot C_1(L) \}. \]

We observe that \( \lambda \) is exactly the nef threshold of \( L \).

Recall the log \( K \)-energy (2.10) is defined to be

\[ \nu_\beta(\varphi) = E_\beta(\varphi) + J_{-\theta}(\varphi) + \frac{1}{V} \int_M (h + h_0)\omega_0^\varphi, \]

(5.9)
where $\mathfrak{h} := -(1 - \beta) \log |s|^2$. Following the same argument as Page 2809 in [71], we could add and subtract the term $\eta J_\omega$ for some constant $\eta$ in the log $K$-energy

$$
\nu_\beta(\varphi) = E_\beta(\varphi) - \eta J_\omega(\varphi) + J_{-\theta}(\varphi) + \eta J_\omega(\varphi) + \frac{1}{V} \int_M (\mathfrak{h} + h_0) \omega^0_n.
$$

Firstly, the existence of the global minimiser of the functional

$$
J_{-\theta, \eta}(\varphi) := J_{-\theta}(\varphi) + \eta J_\omega(\varphi)
$$

is then given in [71]. The goal there is to obtain the coercivity of the $K$-energy, by using the lower bound of the twisted functional $J_{-\theta, \eta}$.

These results were generalised to the conical setting in [74]. We collect related results as follows.

**Proposition 5.16.** ([74, Theorem 4.20, Proposition 4.22]) Suppose that the constant $\eta$ satisfies the following conditions,

$$
\begin{align*}
(i) & \quad C_1(X, D) < \eta C_1(L), \\
(ii) & \quad \sum_\beta C_1(L) - (n - 1)C_1(X, D) < \eta C_1(L).
\end{align*}
$$

Then the functional $J_{-\theta, \eta}(\varphi)$ has lower bound for any $\varphi \in \mathcal{H}_\beta$.

Secondly, from (2.6) and (2.3), we have

$$
E_\beta(\varphi) = \frac{1}{V} \int_M \log \frac{\omega^n_\varphi}{\omega^n_\theta} \omega^n_\varphi.
$$

It lower bound is obtained directly by the well-known inequality

$$
E_\beta(\varphi) \geq \frac{n + 1}{n} \alpha_\beta J_\omega(\varphi) - C, \quad \forall \omega_\varphi \in [\omega_\theta].
$$

This inequality is obtained from the Jensen inequality, the definition of the log alpha invariant, see Lemma 5 in [71].

Considering the optimal constant $\eta$ brings us to the following definition of threshold of both the entropy $E_\beta$ and the $J$-functional.

**Definition 5.17.** We set the entropy threshold

$$
e := \sup_\eta \{ \exists C \text{ such that } E_\beta \geq \eta \cdot J_{\omega_\theta} - C, \forall \omega_\varphi \in [\omega_\theta] \}
$$

and the $J$-threshold to be

$$
\mathcal{J} = \inf_\eta \{ \exists C \text{ such that } J_{-\theta} + \eta \cdot J_{\omega_\theta} \geq -C, \forall \omega_\varphi \in [\omega_\theta] \}.
$$

Here, we have

$$
J_{\omega_\theta} = I_{\omega_\theta}^A - J_{\omega_\theta}^A.
$$

**Remark 5.18.** We could use $\omega_\theta$ instead of $\omega_\theta$ is the definition above. Note that $J_{\omega_\theta}$ and $J_{\omega_\theta}$ only differ by a constant depending on $||\varphi_\theta||_\infty$. The relation between these different $J$ functionals is $J_{\omega_\theta} = I_{\omega_\theta}^A - J_{\omega_\theta}^A$. 
Remark 5.19. Note that both $E$ and $J$ only depend on the Kähler cone metric $\omega_\phi$, not $\phi$. Precisely, letting $C$ be any constant, we have
\[ E(\phi + C) = E(\phi), \quad J_\chi(\phi + C) = J_\chi(\phi). \]
Here $\chi$ is a closed form. In contrast,
\[ D_\omega(\phi + C) = D_\omega(\phi) + C. \]
Moreover, it is direct to see from (5.11) that

**Lemma 5.20.** The lower bound of $e$ is $e \geq \frac{n+1}{n} \alpha_\beta$.

Also,

**Lemma 5.21.** If $e > J$, then the log $K$-energy is $J$-coercive.

In conclusion, we have a corollary of Proposition 5.16.

**Proposition 5.22.** Assume that
\[ e > \max\{\Lambda, S_\beta - (n-1)\lambda\}, \]
then the log $K$-energy is $J$-coercive, moreover, $C_1(L)$ admits a cscK cone metric and is uniformly log $K$-stable.

**Proof.** Since $C_1(X, D) \leq \Lambda C_1(L)$ and
\[ S_\omega C_1(L) - (n-1)C_1(X, D) \leq [S_\beta - (n-1)\lambda]C_1(L), \]
the sufficient conditions of $\eta$ to get (5.10) is to set
\[ \eta > \Lambda, \quad \eta > [S_\beta - (n-1)\lambda]. \]
Consequently, the threshold $J$, which is defined to be the infimum of such $\eta$, has upper bound
\[ J \leq \max\{\Lambda, S_\beta - (n-1)\lambda\}. \]
Then we have obtained the conclusion from Lemma 5.21 and Theorem 5.7.

The following result is a log adaption of [6, Proposition 4.11], which suggests that (5.12) has an alternative formula.

**Proposition 5.23.**
\[ e = \sup_\eta \{ \exists C \text{ s.t. } \| e^{-\eta} \|_{L^n(\omega_\phi)} \leq C e^{-D_\omega(\phi)}, \quad \forall \omega_\phi \in [\omega_\theta] \}. \]

Here, $\| \cdot \|_{L^n(\omega_\theta)} = (V^{-1} \int_M | \cdot |^n \omega_\theta^n)^{\frac{1}{n}}$.

**Proof.** From (5.14), we have
\[ \int_M e^{-\eta(\phi - D_\omega(\phi)) - \log \frac{\omega_{\phi}}{\omega_\theta}} \omega_\phi^n \leq C. \]
Then Jensen’s inequality gives $E_\beta(\phi) + C \geq \frac{1}{V} \int_M [-\phi + D_\omega(\phi)] \omega_\phi^n = \eta J_\omega(\phi)$, that is (5.12).

For the other direction from (5.12) to (5.14), we need to show
\[ V^{-1} \int_M e^{-\eta \phi} \omega_\theta^n \leq C e^{-\eta D_\omega(\phi)}. \]
We first find an auxiliary function $u$ as suggested in [6, Lemma 2.11]. Let us set $\mathcal{f} = V^{-1} \int$ and consider the following functional

\[(5.16) \quad A(u) := E_\beta(u) + \eta \int_M \varphi \omega_u^n = \int_M [\log \frac{\omega_u^n}{\omega_{\varphi}^n} + \eta \varphi] \omega_u^n.\]

Its first variation is

\[\delta A(u, \delta u) := \int_M [\log \frac{\omega_u^n}{\omega_{\varphi}^n} + \eta \varphi] \Delta_u(\delta u) \omega_u^n\]

and the critical point equation is solvable

\[(5.17) \quad \log \frac{\omega_u^n}{\omega_{\varphi}^n} + \eta \varphi = C_0 \quad \text{with} \quad V = \int_X e^{C_0 - \eta \varphi} \omega_{\varphi}^n.\]

Moreover, the critical point $\omega_\varphi$ is a Kähler cone metric and has geometric polyhomogeneity, c.f. [46, 70]. That enables us to see that the critical point is a minimiser, i.e.

\[\delta^2 A(v, \delta u) := \int_M |\Delta_v(\delta u)|^2 \omega_v^n \geq 0.\]

Then we make use of the equations (5.17) and (5.16) of the auxiliary function $\phi$ to see,

\[L.H.S. := \int_M e^{-\eta \varphi} \omega_{\varphi}^n (5.17) = e^{-C_0} (5.16) = e^{-A(v)}.\]

We next set the other functional

\[B(u) := J_{\omega_\varphi}(u) + \int_M \varphi \omega_u^n\]

and apply the assumption $E_\beta(v) \geq \eta \cdot J_{\omega_\varphi}(v) - C$ in (5.12), to get

\[L.H.S. \leq C e^{-\eta B(v)}.\]

Recall that $D_{\omega_\varphi}(\varphi) = J_{\omega_\varphi}(\varphi) + \int_M \varphi \omega_\varphi^n$. It remains to check that

\[B(v) \geq B(\varphi) = D_{\omega_\varphi}(\varphi).\]

At last, we check this inequality by direct computation

\[\delta B(u, \delta u) = n \int_M \delta u (\omega_\varphi - \omega_u) \wedge \omega_u^{n-1}.\]

So the critical point is $\omega_u = \omega_\varphi$. Moreover, it is a minimiser, since

\[\delta^2 B(\varphi, \delta u) = \int_M |\partial(\delta u)|^2 \omega_\varphi^n \geq 0.\]

Therefore, we have

\[L.H.S. \leq C e^{-\eta B(\varphi)}\]

and (5.15) is proved. \qed
5.6. **Small angle solution for cscK cone path.** In this section, we choose $\beta$ sufficiently small to prove existence of cscK cone metric with the small angle $\beta$. This is actually an update of Theorem 5.2 in a quantitative way.

**Definition 5.24.** Let $((X, L); D, m)$ be a standard polarised pair. We define the critical cone angle to be the invariant

$$
\beta_u((X, L); D, m) = \min \left\{ 1, \frac{n + 1}{n} \min \{ \alpha(L), m\alpha(L_D) \} \right\} + 1 - \frac{S_1}{mn}.
$$

**Theorem 5.25.** Let $((X, L); D, m)$ be a standard polarised pair. Suppose one of the following conditions holds,

1. Fix a sufficiently large $m$ such that

$$
S_1 < mn + (n - 1)\lambda, \quad \Lambda < m;
$$

2. $m \in \mathbb{N}$ is an integer which satisfies

$$
S_1 \leq mn, \quad \Lambda \leq \frac{S_1}{n} \leq \lambda + m(1 - \beta_u).
$$

Then $((X, L); D, m)$ admits a cscK cone metric of cone angle $2\pi\beta$ along $D$ for any $\beta$ in the following range for each of two cases above:

1. in case 1, $\beta$ can take any value in the range

$$
0 < \beta \leq \min \left\{ 1, 1 - \frac{\Lambda}{m}, 1 - \frac{S_1 - (n - 1)\lambda}{mn} \right\};
$$

2. in case 2, the range is

$$
0 < \beta \leq \beta_u.
$$

We comment on the condition (5.18), before we give the proof.

**Remark 5.26.** From the second inequality in (5.19), we see that $\beta_u$ must be less than or equal to 1. That is the reason why $\beta_u$ is defined in the way as in Definition 5.24. We also see that when $X$ is Fano and $L = -K_X$, $\lambda = \Lambda = 1$. The assumptions in (5.19) are satisfied automatically. In general, when the difference $\Lambda - \lambda > 0$ is large, the multiplicity $m$ must be very large and we also need $\beta_u < 1$

**Proof.** It is sufficient to verify the Condition (5.1). Then, the automorphism group is trivial by Proposition 2.21, and also Theorem 5.7. Then the conclusion follows from Theorem 5.7.

For case 1, we choose $\eta = 0$ in Condition (5.1), that is we need to show that $\alpha_{\beta}$ is positive, and $C_1(X, D)$ and $\sum_1 C_1(L) - (n - 1)C_1(X, D)$ are both negative.

Making use of the formulas (5.3) and (5.5), together with the choice of $m$ in (5.18), we see that

$$
C_1(X, D) = C_1(X) - m(1 - \beta)C_1(L) \leq [\Lambda - m(1 - \beta)]C_1(L)
$$
and
\[ S_\beta \cdot C_1(L) - (n - 1)C_1(X, D) = [S_1 - mn(1 - \beta)] C_1(L) - (n - 1)C_1(X) \leq [S_1 - mn(1 - \beta) - (n - 1)\lambda] C_1(L). \]

are both negative, as long as \( \beta \) is sufficiently small and \( m \) is sufficiently large. As shown above in the proof of Theorem 5.2, \( \alpha_\beta \) is always positive.

For case 2, we choose
\[ \eta = \mu + \epsilon = \frac{S_\beta}{n} + \epsilon = \frac{S_1}{n} - m(1 - \beta) + \epsilon \]
in Condition (5.1) for some positive constant \( \epsilon \) determined below, where we recall that \( \mu \) was defined in (5.7). Now we will show that
\[ \alpha_\beta > \frac{n}{n + 1} \eta, \quad C_1(X, D) < \eta C_1(L), \]
(5.22)
\[ S_\beta \cdot C_1(L) - (n - 1)C_1(X, D) < \eta C_1(L). \]

Firstly, we use the lower bound (5.6) of the alpha invariant, \( \alpha_\beta \geq \min\{m\beta, \alpha(L), m\alpha(L_D|D)\} \). We set \( \check{\alpha} = \min\{\alpha(L), m\alpha(L_D|D)\} \). The first inequality in (5.22) is equivalent to
\[ \check{\alpha} \geq m\beta, \quad m\beta \geq \frac{n}{n + 1}(\mu + \epsilon) \]
or
\[ \check{\alpha} < m\beta, \quad \check{\alpha} \geq \frac{n}{n + 1}(\mu + \epsilon). \]

In the first situation, once we choose
\[ n\epsilon \leq m\beta, \]
the second inequality automatically holds under the assumption that \( S_1 \leq mn \). So we only need the cone angle satisfies
\[ \beta \leq \frac{\check{\alpha}}{m}. \]

In the second situation, after inserting the formula of \( \mu \) in (5.7), we have
\[ \frac{\check{\alpha}}{m} < \beta \leq \frac{n + 1}{n} \frac{\check{\alpha}}{m} + 1 - \frac{S_1}{mn} - \frac{\epsilon}{m}. \]

Putting these two situations together, we have
\[ 0 < \beta \leq \frac{n + 1}{n} \frac{\check{\alpha}}{m} + 1 - \frac{S_1}{mn} - \frac{\epsilon}{m}. \]

Secondly, we check the last two inequalities in (5.22). It is sufficient to show
\[ \Lambda - m(1 - \beta) < \eta, \quad S_1 - mn(1 - \beta) - (n - 1)\lambda < \eta. \]
Equivalently,
\[ \Lambda < \frac{S_1}{n} + \epsilon, \quad \frac{S_1}{n} < \lambda + m(1 - \beta) + \frac{\epsilon}{n - 1}. \]
According to the second assumption in (5.19), we see that the first inequality holds and \( \frac{S_1}{n} \leq \lambda + m(1 - \beta) \) for any \( \beta \leq \beta_u \). So, the second inequality holds for any \( \epsilon > 0 \). Thus, we complete the proof.

The first application of this theorem is an answer to Question 4.17.

**Corollary 5.27.** Let \(((X, L); D, m)\) be a standard polarised pair. Under the assumption of (5.19), we have
\[ \beta_{\text{cscK}} \geq \beta_u, \quad \beta_{\text{cscK}} = \beta_{\text{cscK}}. \]

**Proof.** The first conclusion is direct and the second one follows from Proposition 4.11.

As the second application, this theorem also leads to a result for Kähler–Einstein cone metric, which was proved in [43, Corollary 2.19].

**Corollary 5.28.** On Fano manifold, if \( L = -K_X \), then there exists a Kähler–Einstein cone metric, when \( 0 < \beta \leq \beta_u(-K_X) \), for any \( m \geq 1 \).

**Proof.** In this case, we have \( \Lambda = \lambda = 1 \) and \( S_1 = n \). So, both conditions (5.18) and (5.19) are satisfied.

---

6. **Uniform log K-stability of singular varieties**

In this section, we will present an algebro-geometric criteria of uniform log K-stabilities on a pair of normal variety \((X, \Delta)\), which are the counterpart of the differential-geometric results including Theorem 5.7 in Section 5.4.

The singular cscK metric on the singular pair \((X, \Delta)\) was introduced in [74], which extend the study of singular Kähler–Einstein metrics motivated from the minimal model programme. The existence of the singular cscK metric was shown in [74, Section 4], and we hope enthusiastic readers will find more resources there.

**6.1. Preliminaries.** We collect some materials concerning the singularities of a projective variety. The reader is referred to [40, Section 2.3] for more details. Let \((X, \Delta)\) be a pair consisting of an irreducible normal complex projective variety \(X\) and an effective \(R\)-Weil divisor \(\Delta\) such that \(K_X + \Delta\) is \(R\)-Cartier; note that \(K_X\) is well-defined as a Weil divisor since \(X\) is normal.

**Definition 6.1.** A log resolution \(\pi : \tilde{X} \to X\) of \((X, \Delta)\) gives
\[ \pi^*(K_X + \Delta) = K_{\tilde{X}} - D, \]
in which \(\Delta = -\pi_*D\) and \(D := \sum a_i E_i\) is an \(R\)-Weil divisor and \(\cup_i E_i\) is a simple normal crossing divisor in \(\tilde{X}\).
Recall that $E$ is said to be a **prime divisor over** $X$ if there exists a normal variety $Y$ and a proper birational morphism $\pi : Y \to X$ such that $E$ is a prime (i.e. reduced irreducible Weil) divisor on $Y$. Given a divisor $E$ over $X$, we define the following important quantity.

**Definition 6.2.** Given a prime divisor $E$ over $X$, we define
\[
a(E, X, \triangle) := \text{ord}_E(K_Y - \pi^*(K_X + \triangle))\]
where $\pi : Y \to X$ is a proper birational morphism and $E \subset Y$, with $Y$ normal. We call $a(E, X, \triangle)$ the **discrepancy** of $(X, \triangle)$ along $E$.

Note that the divisor $E_i$ which appears in Definition 6.1 is a divisor over $X$, and $a_i$ therein is precisely $a(E_i, X, \triangle)$. An important point above is that $a(E, X, \triangle)$ depends only on $(X, \triangle)$ and the divisor $E$ over $X$ (see e.g. [40, Remark 2.23]).

**Definition 6.3.** Several classes of mild singularities are defined as follows.

- The pair $(X, \triangle)$ is log canonical, if $a(E, X, \triangle) \geq -1$ holds for any prime divisor $E$ over $X$.
- The pair $(X, \triangle)$ is Kawamata log terminal (klt), if $a(E, X, \triangle) > -1$ holds for any prime divisor $E$ over $X$.
- When $\triangle = 0$, $X$ is said to be log canonical (resp. log terminal), if $(X, 0)$ is log canonical (resp. Kawamata log terminal).

In what follows we will need the log alpha invariant for a klt pair, defined as follows.

**Definition 6.4.** Fixing $0 < \beta < 1$, let $\triangle$ be an effective $\mathbb{R}$-Weil divisor in $X$ such that $K_X + (1 - \beta)\triangle$ is $\mathbb{R}$-Cartier and $L$ be an ample line bundle on $X$. The **log alpha invariant** of the polarised pair $((X, (1 - \beta)\triangle); L)$ is defined by
\[
\alpha((X, (1 - \beta)\triangle); L) := \inf_{m \in \mathbb{N}} \inf_{D_m \in |mL|} \text{lct} \left( (X, (1 - \beta)\triangle); \frac{1}{m}D_m \right),
\]
which we also abbreviate as $\alpha_\beta$, where lct stands for the log canonical threshold defined as
\[
\text{lct} \left( (X, (1 - \beta)\triangle); \frac{1}{m}D_m \right) := \sup \left\{ c \in \mathbb{R} \mid (X, (1 - \beta)\triangle + \frac{c}{m}D_m) \text{ is log canonical} \right\}
\]
We decree $\text{lct} \left( (X, (1 - \beta)\triangle); \frac{1}{m}D_m \right) = -\infty$ if there does not exist $c \in \mathbb{R}$ such that $(X, (1 - \beta)\triangle + \frac{c}{m}D_m)$ is log canonical.

In what follows, we shall mostly consider the case when $(X, (1 - \beta)\triangle)$ is log canonical. Note that $\alpha((X, (1 - \beta)\triangle); L) \geq 0$ if $(X, (1 - \beta)\triangle)$ is log canonical.
When $X$ and $\triangle$ are both smooth, the log alpha invariant defined above agrees with the analytic definition given in Definition 5.3, as proved in [4, Proposition A.4]; see also [13, Appendix A] and [22, Section 5]. Thus, without loss of generality, $\alpha_\beta$ will always stand for the quantity defined above in what follows. Note also that the average scalar curvature (2.5) makes sense as a ratio of intersection numbers

\[ S_\alpha = n \frac{(-K_X + (1 - \beta)\triangle)L^{n-1}}{L^n} \]

over the normal projective variety $X$.

6.2. Computation of log Donaldson–Futaki invariant. Let $X$ be a $\mathbb{Q}$-Gorenstein (i.e. $K_X$ is $\mathbb{Q}$-Cartier) normal projective variety, $\triangle$ an effective integral $\mathbb{Q}$-Cartier divisor, which implies that $K_X + (1 - \beta)\triangle$ is $\mathbb{R}$-Cartier for any $\beta \in (0, 1)$. We recall some results on the Donaldson–Futaki invariant for the log test configurations of $((X, L); \triangle)$. The log Donaldson–Futaki invariant is computed by blowing up flag ideals, as in Odaka–Sun [52]. We apply their formula to obtain several criterion between log stable manifolds and singularity types of normal varieties, which could be seen as algebro-geometric analogues of the differential-geometric results in Section 5.4. Parallel results for twisted cscK metrics were obtained in [23].

In this section we use the blow-up formalism of test configurations, and the reader is referred to the paper of Odaka–Sun [52] for more details. Recall [52, Definition 3.4] that a coherent ideal $\mathcal{I}$ of $X \times \mathbb{C}$ is called a flag ideal if it is invariant under the natural $\mathbb{C}^*$-action on $X \times \mathbb{C}$. We define $\mathcal{B} := \text{Bl}_\mathcal{I}(X \times \mathbb{P}^1)$ and write $\pi : \mathcal{B} \to X \times \mathbb{P}^1$ for the blowdown map; note that $\mathcal{B}$ agrees with the compactification of $\text{Bl}_\mathcal{I}(X \times \mathbb{C})$ over $\mathbb{P}^1$ as in Definition 3.2. Writing $\text{pr}_1 : X \times \mathbb{P}^1 \to X$ for the natural projection, we also define a semiample line bundle $\mathcal{L}' := \pi^*\text{pr}_1^*L$ on $\mathcal{B}$. Fixing a divisor $\triangle \subset X$, we also have $\mathcal{B}_\triangle := \text{Bl}_{\mathcal{I}_{\triangle \times \mathbb{P}^1}}(\triangle \times \mathbb{P}^1)$. It is well-known [52, Proposition 3.5] that for any log test configuration $((\mathcal{X}, \mathcal{L}); \mathcal{D})$ for a polarised pair $((X, L); \triangle)$ as above, there exists a flag ideal $\mathcal{I}$ such that the log Donaldson–Futaki invariant of $((\mathcal{B}, \mathcal{L}' - E); \mathcal{B}_\triangle)$ agrees with that of $((\mathcal{X}, \mathcal{L}); \mathcal{D})$, where $E$ is the exceptional Cartier divisor of $\pi$. Thus without loss of generality we may only consider the log test configurations of the form described above, and moreover we may assume that $\mathcal{B}$ is Gorenstein in codimension 1 (so that $K_\mathcal{B}$ is a well-defined Weil divisor) [52, Corollary 3.6]. Furthermore, the log Donaldson–Futaki invariant of $((\mathcal{B}, \mathcal{L}' - E); \mathcal{B}_\triangle)$ admits the following formula given in terms of intersection numbers.
Theorem 6.5. ([52, Theorem 3.7]) The log Donaldson–Futaki invariant of the blowup \((\mathcal{B}, \mathcal{L}' - E); \mathcal{B}_\Delta\) is given by the following formula
\[(6.2)\]
\[DF(\mathcal{B}, \mathcal{B}_\Delta, \mathcal{L}' - E, \beta) = (\mathcal{L}' - E)^n \cdot \left[\frac{S_\beta}{n+1} (\mathcal{L}' - E) + \pi^*((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1) + K_e\right].\]

Here, \(K_e := (K_{\mathcal{B}/(X,(1-\beta)\Delta)} \times \mathbb{P}^1)_{\text{exc}}\) denotes the exceptional part of the divisor \(K_{\mathcal{B}/(X,(1-\beta)\Delta)} \times \mathbb{P}^1\), where
\[(6.3)\]
\[K_{\mathcal{B}/(X,(1-\beta)\Delta)} \times \mathbb{P}^1 := K_{\mathcal{B}} - \pi^*((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1),\]
and we also note that \((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1 = \text{pr}_1^*(K_X + (1 - \beta)\Delta)\) is an \(\mathbb{R}\)-Cartier divisor on \(X \times \mathbb{P}^1\).

As mentioned above, the log Donaldson–Futaki invariant \(DF(X, D, \mathcal{L}, \beta)\) for the test configuration \(((X, \mathcal{L}); D)\) is equal to \(DF(\mathcal{B}, \mathcal{B}_\Delta, \mathcal{L}' - E, \beta)\).

6.3. Uniform log \(K\)-stability for a singular pair \((X, \Delta)\). We assume as before that \(X\) is a \(\mathbb{Q}\)-Gorenstein normal projective variety, and \(\Delta\) is an effective integral \(\mathbb{Q}\)-Cartier divisor.

6.3.1. Log Calabi–Yau pair \(C_1(X, \Delta) = 0\).

Definition 6.6. We say that \((X, (1 - \beta)\Delta)\) is a log Calabi–Yau pair if the \(\mathbb{R}\)-Cartier divisor \(K_X + (1 - \beta)\Delta\) is \(\mathbb{R}\)-linearly equivalent to zero.

In particular, \(C_1(X, \Delta) := C_1(-K_X - (1 - \beta)\Delta)\) makes sense and equals zero.

It was shown in [52, Corollary 6.3] that if \((X, (1 - \beta)\Delta)\) is a log Calabi–Yau pair, then \((X, (1 - \beta)\Delta)\) is log canonical if and only if \(((X, L); \Delta)\) is log \(K\)-semistable with angle \(2\pi\beta\).

Compared with Theorem 5.10, we actually further have

Theorem 6.7. Suppose that \((X, (1 - \beta)\Delta)\) is a log Calabi–Yau pair. If \((X, (1 - \beta)\Delta)\) is Kawamata log terminal, then \(((X, L); \Delta)\) is uniformly log \(K\)-stable with angle \(2\pi\beta\).

Proof. Under the assumption that \(K_X + (1 - \beta)\Delta\) is \(\mathbb{R}\)-linearly equivalent to zero, we have \(S_\beta = 0\). Then (6.2) becomes
\[DF(\mathcal{B}, \mathcal{B}_\Delta, \mathcal{L}' - E, \beta) = (\mathcal{L}' - E)^n \cdot K_e.\]
Therefore, (2) and (6) in Lemma 6.16 is applied to obtain that the log canonical (resp. Kawamata log terminal) condition implies log \(K\)-semistability (resp. uniform log \(K\)-stability). \(\square\)
6.3.2. Log canonical pair. Actually, these statements could be extended to the case when $C_1(X, \Delta)$ is not necessary vanishing, generalising Theorem 5.7, Corollary 5.11 and Corollary 5.12 in Section 5.4 to singular varieties.

We recall the definition of a nef cohomology class.

**Definition 6.8.** An $\mathbb{R}$-Cartier divisor $F$ on a normal projective variety $X$ is said to be **nef** if for any irreducible curve $C$ in $X$ we have $C.F \geq 0$, or equivalently

$$\int_C C_1(F) \geq 0.$$  

Observing that the nefness depends only on the numerical equivalence class of the divisor, we may abuse the terminology and say that the cohomology class $C_1(F)$ is nef when $F$ is nef.

**Theorem 6.9.** Suppose that $X$ is a $\mathbb{Q}$-Gorenstein normal projective variety, $\Delta$ is an effective integral reduced Cartier divisor on $X$, and $(X, (1 - \beta)\Delta)$ is log canonical. We have the following two conclusions.

- Suppose $S_\beta < 0$ and that the cohomology Condition (5.1) holds,
  
  $$(i) \quad 0 \leq \eta < \frac{n+1}{n} \alpha_\beta,$$
  
  $$(ii) \quad C_1(X, \Delta) < \eta C_1(L),$$
  
  $$(iii) \quad (S_\beta - \eta)C_1(L) < (n-1)C_1(X, \Delta).$$

  Then $(X, L; \Delta)$ is uniformly log $K$-stable with angle $2\pi\beta$.

- Suppose that the following conditions hold
  
  $$(6.4) \quad S_\beta < (n+1)\alpha_\beta, \quad \text{and} \quad -S_\beta C_1(L) + (n+1)C_1(X, \Delta) \text{ is nef}.$$  

  Then $(X, L; \Delta)$ is uniformly log $K$-stable with angle $2\pi\beta$.

Recall that $(X, (1 - \beta)\Delta)$ being log canonical implies $\alpha_\beta \geq 0$.

**Remark 6.10.** Dervan [22, Theorem 1.3] proved that the condition (6.4) implies that $(X, L; \Delta)$ is log $K$-stable with angle $2\pi\beta$. The second part of the theorem above strengthens this result to the uniform log $K$-stability; this may be well-known to the experts since the twisted version already appeared in [23, Theorem 1.9] and the proof that we present below is based on Dervan’s [22, 23], but the statement as above does not seem to have appeared in the literature to the best of the authors’ knowledge (see [23, Remark 3.26 and Theorem 3.27] for the comparison between the log $K$-stability and the twisted $K$-stability).

A direct corollary is given below.

**Corollary 6.11.** Suppose that

$$C_1(X, \Delta) < 0, \quad \text{and} \quad -S_\beta C_1(L) + nC_1(X, \Delta) \text{ is nef}.$$  

If $(X, (1 - \beta)\Delta)$ is log canonical, then $(X, L; \Delta)$ is uniformly log $K$-stable with angle $2\pi\beta$. 
Proof. When $C_1(X, \Delta) < 0$, the first two conditions in Condition (5.1) hold for any sufficiently small $\eta > 0$. The third condition is satisfied, similar to Remark 6.13. Precisely speaking, we have

$$(\Sigma_\beta - \eta)C_1(L) < nC_1(X, \Delta) < (n - 1)C_1(X, \Delta), \quad \forall \eta > 0$$

and $-\Sigma_\beta C_1(L) + (n - 1)C_1(X, D)$ is nef. \qed

6.3.3. Condition (5.1) and Condition (6.4). Before we prove this theorem, we discuss these two conditions. When $X$ is smooth, it is clear that Condition (6.4) is weaker than Condition (5.1).

Remark 6.12. The assumptions (i, ii) in Condition (5.1) together imply the upper bound of the average scalar curvature

$$\Sigma_\beta < n\eta < (n + 1)\alpha_\beta.$$ 

Remark 6.13. We could reword (iii) in Condition (5.1) and it says

$$(6.5) \quad -\Sigma_\beta C_1(L) + (n - 1)C_1(X, D) \text{ is nef.}$$

Proposition 6.14. Suppose that $C_1(X, \Delta)$ is nef and that $(X, (1 - \beta)\Delta)$ is a log canonical pair. Then the cohomology Condition (5.1) implies Condition (6.4).

Proof. The first condition is obtained in Remark 6.12. The second condition follows from Remark 6.13 as well. \qed

Remark 6.15. Given a compact subgroup $P$ of the automorphism group $\text{Aut}_0((X, L); D)$, we could restrict ourselves in the $P$-equivariant setting. It is natural to speculate that, by making use of $P$-equivariant log alpha invariant and $G$-equivariant log test configuration, we can obtain corresponding results for uniform log $K$-stability, similar to Theorem 6.9. We may then hope to compare (6.4) and Proposition 2.20 to find a manifold that does not have cscK cone metric. We decide not to pursue this point any further, however, in this paper.

6.4. Proof of Theorem 6.9. We will need the following lemma.

Lemma 6.16. ([51, Lemma 4.2], [22, Section 5], [23, Lemma 3.10, Remark 3.11 and 3.17])

Suppose that $X$ is a $\mathbb{Q}$-Gorenstein normal projective variety, $\Delta$ is an effective integral reduced Cartier divisor on $X$, and $(X, (1 - \beta)\Delta)$ is log canonical.

(1) Let $R$ be a nef divisor on $X$ and $\mathcal{R} = p^*R$, where $p$ is the composition of the blowdown map $\pi : B \to X \times \mathbb{P}^1$ and the natural projection $X \times \mathbb{P}^1 \to X$. Then $(L' - E)^n \cdot \mathcal{R} \leq 0$.

(2) $(L' - E)^n \cdot K_x \geq 0$.

(3) The exceptional divisor $K_x - \alpha \beta E$ is effective and $(L' - E)^n(K_x - \alpha \beta E) \geq 0$.

(4) $(L' - E)^nE > 0$.

(5) $(L' - E)^n(L' + nE) = (n + 1)\|B\|_m$ and $(L' - E)^nE \geq \frac{n + 1}{n}\|B\|_m$. 


(6) \((L' - E)^n \cdot K_e \geq \alpha \frac{n+1}{n} \|B\|_m\).

Recall that in the above \(K_e\) stands for the exceptional part of the divisor \(K_{\mathcal{B}/(X,(1-\beta)\Delta)\times \mathbb{P}^1}\) defined in (6.3), and that \(\|\cdot\|_m\) stands for the minimum norm in (3.6).

**Proof.** We simply indicate where the proof can be found in the literature and omit the details.

The first and the fourth item is exactly as written in [23, Lemma 3.10]; see also [22, Lemma 3.6]. The second and the third item follows form [22, equations (91)–(94)] and [22, Lemma 3.6]. The fifth and the sixth item is exactly as written in [23, page 4770] (where we also use the third item above). \(\square\)

Now we start to prove Theorem 6.9.

**Proof of Theorem 6.9.** We rewrite the log Donaldson–Futaki invariant (6.2) as

\[
\text{DF}(\mathcal{B}, \mathcal{B}_\Delta, L' - E, \beta) = (L' - E)^n \cdot \left\{ (S_\beta - \eta) L' + (n - 1)[\pi^*((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1)] - \frac{S_\beta}{n + 1}(n L' + E) + (n - 2)\eta L' + K_e + (2 - n)[\eta L' + \pi^*((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1)] \right\}.
\]

According to (1), (2) and (3) in Lemma 6.16, we have

\[
\text{DF}(\mathcal{B}, \mathcal{B}_\Delta, L' - E, \beta) \geq (L' - E)^n \cdot \left\{ -\frac{S_\beta}{n + 1}(n L' + E) + (n - 2)\eta L' + K_e \right\},
\]

by noting \(\eta L' + \pi^*((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1) = p^*(\eta L + K_X + (1 - \beta)\Delta)\). We set \(e = -\frac{S_\beta}{n + 1} S_\beta + (n - 2)\eta\) and \(f = -\frac{S_\beta}{n + 1}\). Then

\[
\text{DF}(\mathcal{B}, \mathcal{B}_\Delta, L' - E, \beta) \geq (L' - E)^n[e L' + f E + K_e],
\]

\[
= (L' - E)^n[e(L' + n E) + (f - en)E + K_e].
\]

At last, (5) in Lemma 6.16 implies that

\[
\text{DF}(\mathcal{B}, \mathcal{B}_\Delta, L' - E, \beta) \geq e(n + 1)\|B\|_m + (f - en)\frac{n+1}{n}\|B\|_m + (L' - E)^n K_e
\]

\[
= -\frac{S_\beta}{n} \|B\|_m + (L' - E)^n K_e.
\]

The first conclusion in the theorem is proved directly, since \(S_\beta < 0\) and \((L' - E)^n K_e \geq 0\) from (2) in Lemma 6.16.

The proof of the second conclusion is an analogue of [23, Theorem 1.9] for the twisted case and the result regarding log \(K\)-stability was proven in [22, Theorem 1.3].
Applying (1) and (3) in Lemma 6.16, with the hypothesis that \(-S_\beta \Omega + (n+1)C_1(X, \Delta)\) is nef, and the exceptional divisor \(K_e - \alpha \beta E \geq 0\), we see that
\[
DF(B, B_\Delta, L' - E, \beta) = (L' - E)^n \cdot \left[ -\frac{S_\beta}{n+1}L' + \pi^*((K_X + (1 - \beta)\Delta) \times \mathbb{P}^1) - \frac{S_\beta}{n+1}E + K_e \right]
\]
\[
= (L' - E)^n \cdot p^* \left( \frac{S_\beta}{n+1}L + K_X + (1 - \beta)\Delta \right) + (L' - E)^n \cdot \left( -\frac{S_\beta}{n+1}E + K_e \right)
\]
\[
\geq (L' - E)^n \cdot \left( \alpha \beta - \frac{S_\beta}{n+1} \right) E \geq \frac{n+1}{n} \left( \alpha \beta - \frac{S_\beta}{n+1} \right) \|B\|_m,
\]
which completes the proof, by (5) in Lemma 6.16. □

Remark 6.17. It is shown in [22, Theorem 5.7] that a log canonical \((X, (1 - \beta)\Delta)\) satisfying (6.4) is log \(K\)-stable with cone angle \(2\pi\beta\). The proof is identical to the one given above, where the last inequality is strict, since (4) is used instead of (5) in Lemma 6.16, that is
\[
DF(B, B_\Delta, L' - E, \beta) \geq (L' - E)^n \cdot \left( \alpha \beta - \frac{S_\beta}{n+1} \right) E > 0.
\]

6.5. From log semi-stability to singularity types of varieties.

Let \(X\) be a \(Q\)-Gorenstein normal projective variety and \(\Delta\) be an effective integral \(Q\)-Cartier divisor, as before. The hypotheses on \(X\) can be much weakened (see [50, Definition 1.1] and [52, Section 6]), but we content ourselves with the ones that we have been working with so far.

Theorem 6.18. Suppose that \(((X, L); \Delta)\) is log \(K\)-semistable with angle \(2\pi\beta\) and that we have
\[
C_1(X, \Delta) > 0, \quad S_\beta C_1(L) - nC_1(X, \Delta) \text{ is nef}.
\]
Then \((X, (1 - \beta)\Delta)\) is Kawamata log terminal.

Note also that the log \(K\)-semistability of \(((X, L); \Delta)\) implies that \((X, (1 - \beta)\Delta)\) is log canonical, by a result of Odaka–Sun who proved a more general version [52, Theorem 6.1]. They also proved that \((X, (1 - \beta)\Delta)\) is Kawamata log terminal if \(L\) is a positive multiple of \(-K_X - (1 - \beta)\Delta\); the result above relaxes this hypothesis on \(L\).

Proof. We essentially repeat the argument in [52, Proof of Theorem 6.1] and [23, Proof of Theorems 3.28 and 3.30], which the reader is referred to for the details, and only provide a brief summary here for the reader’s convenience.

The formula (6.2) gives
\[
DF(B, B_\Delta, L' - E, \beta) = (L' - E)^n \cdot \left[ -\frac{S_\beta}{n(n+1)}(L' + nE) + \frac{S_\beta}{n}L' + \pi^*\left( (K_X + (1 - \beta)\Delta) \times \mathbb{P}^1 \right) + K_e \right].
\]
Then, applying (1) and (5) in Lemma 6.16 with the stated assumptions and noting $rac{S_1}{n}L' + \pi^*((K_X + (1-\beta)\Delta) \times \mathbb{P}^1) = p^*(\frac{S_1}{n}\Omega + K_X + (1-\beta)\Delta)$, we get

$$DF(B, B_{\Delta}, L' - E, \beta) < (L' - E)^n \cdot K_e.$$ 

So, the proof is completed once there exists a flag ideal $I$ with $K_e = 0$ when $(X, (1 - \beta)\Delta)$ is not Kawamata log terminal, which can be done exactly as in [52, Proof of Theorem 6.1].

6.6. **Uniform log $K$-stability for standard polarised pair.** We will apply (6.2) to check uniform log $K$-stability for standard polarised pair $((X, L); D, m)$, which was defined in Definition 5.1. In particular, we assume that $X$ and $D$ are both smooth in the rest of this section, and moreover $D$ is chosen to be sitting in the linear system $[mL]$.

For uniform log $K$-stability, the following Theorem 6.19 extends Theorem 5.25 for existence of csCK cone metrics. We will also present results related to Question 4.16. We have a lower bound of $\beta_{ulKs}$, which was defined in Definition 4.13, i.e. the maximal angle $\beta_{ulKs}$ such that $((X, L); D)$ is uniformly log $K$-stable with angle $2\pi \beta$.

**Theorem 6.19.** Let $((X, L); D, m)$ be a standard polarised pair (Definition 5.1). Suppose that

$$S_1 \leq mn \text{ and } (n + 1)\lambda \leq S_1 + m.$$ 

Then $((X, L); D, m)$ is uniformly $K$-stable with cone angle $2\pi \beta$ satisfying the constraint

$$1 - \frac{(n + 1)\lambda - S_1}{m} \leq \beta < \beta_u.$$ 

Here, $\beta_u$ was defined in Definition 5.24 and we fixed the constants $\lambda$ and $\Lambda$ as in Definition 5.15 so that it satisfies $\lambda \Omega \leq C_1(X) \leq \Lambda \Omega$.

Consequently, we also have

$$\beta_{ulKs} \geq \beta_u.$$ 

**Proof.** $(X, (1 - \beta)D)$ being log canonical necessarily implies $1 - \beta \leq 1$, and hence $\beta \geq 0$. We also need $1 - \beta \geq 0$ since $D$ is effective. From (6.7), we have $\beta \geq 0$ by using $(n + 1)\lambda \leq S_1 + m$, and $\beta \leq 1$ by the definition of $\beta_u$.

Then it is sufficient to check the condition (6.4) in Theorem 6.9. From the following two lemmas, Lemma 6.21 and Lemma 6.24, we have

$$\text{(6.7)} \implies \text{(6.8)} \implies \text{(6.4)}.$$ 

Therefore, Theorem 6.9 implies the required result.

**Remark 6.20.** It is interesting to compare conditions (6.6), (6.7) in Theorem 6.19 with the conditions (5.19) and (5.21) in Theorem 5.25. It may suggest certain chance to find some parameters such that some
standard polarised pair \(((X, L); D, m)\) does not admit s \(cscK\) cone metric, but is uniformly log \(K\)-stable with angle \(2\pi\beta\).

We need to interpret the topological condition (6.4) in Theorem 6.9 into a cone angle constraint.

**Lemma 6.21.** The condition (6.4) is deduced from the following bound of the cone angle \(\beta\),

\[
1 - \frac{(n + 1)\lambda - \frac{S_1}{m}}{m} \leq \beta < 1 - \frac{S_1}{mn} + \frac{n + 1}{mn}\alpha_\beta.
\]

**Proof.** We recall by Lemma 5.5 that \(C_1(X, D) = C_1(X) - m(1 - \beta)C_1(L)\) and

\[
S_\beta = S_1 - mn(1 - \beta), \quad S_1 = n\frac{\int_X C_1(X)C_1(L)^{n-1}}{\int_X C_1(L)^n}.
\]

We then check the condition (6.4). The first one becomes,

\[
S_\beta = S_1 - mn(1 - \beta) < (n + 1)\alpha_\beta,
\]

which means

\[
\beta < 1 - \frac{S_1}{mn} + \frac{n + 1}{mn}\alpha_\beta.
\]

While the second condition says for all \(\eta > 0\),

\[
(n + 1)[C_1(X) - m(1 - \beta)C_1(L)] + \eta C_1(L) > [S_1 - mn(1 - \beta)]C_1(L),
\]

that is

\[
(n + 1)[m(1 - \beta) + S_1 - \eta]C_1(L) < (n + 1)C_1(X).
\]

With the lower bound of \(C_1(X)\), \(C_1(X) \geq \lambda \cdot C_1(L)\), the inequality above is strengthened to

\[
\beta > 1 - \frac{(n + 1)\lambda - \frac{S_1}{m}}{m} + \eta.
\]

Thus the lower bound of \(\beta\) is obtained, as \(\eta \to 0\). \(\square\)

**Remark 6.22.** According to the definition (5.15), we have \(n\Lambda \geq S_1\), so the left-hand side of (6.8)

\[
1 - \frac{(n + 1)\lambda - \frac{S_1}{m}}{m} \leq 1 - \frac{(n + 1)\lambda - n\Lambda}{m}.
\]

**Remark 6.23.** If \(C_1(X) \leq 0\), the constant \(\lambda\) is non-positive and we have

\[
\beta \geq 1 + \frac{S_1}{m}.
\]

In the case when \(C_1(X) > 0\), we multiply (6.9) with \(C_1(L)^{n-1}\) and make use of the definition of \(S_1\) to see

\[
\beta \geq 1 - \frac{S_1}{mn}.
\]
Lemma 6.24. Suppose $\mathcal{S}_1 \leq mn$. Then the condition (6.7) implies the condition (6.8).

Proof. We only need to check the right-hand side of (6.8). Recall that the slope $\mu$ defined in (5.7) satisfies $\mu = \frac{S_1}{n} - m(1 - \beta)$. Then (6.8) becomes
\[
-n\lambda + S_1 \leq \frac{n}{n+1} \mu < \alpha\beta.
\]
(6.10)

The lower bound of the log alpha invariant (5.6) gives
\[
\alpha\beta \geq \min\{m\beta, \alpha(L), m\alpha(L_D|D)\}.
\]
Under the assumption of $m$, i.e., $\mathcal{S}_1 \leq mn$, we have $m\beta > \frac{n}{n+1}\mu$. Therefore, we obtain the conditions in Theorem 6.19, as in the argument in Theorem 5.25.

At last, we link Theorem 5.25 and Theorem 6.19 to known results on Kähler–Einstein cone metrics on Fano manifolds.

Corollary 6.25. Let $((X, L); D, m)$ be a standard polarised pair with $m \geq 1$. Suppose $C_1(X) > 0$ and $L = -K_X$, i.e. $X$ is Fano. If $0 < \beta < \beta_u$ with
\[
\beta_u(-K_X) = \min\{1, 1 - \frac{1}{m} + \frac{n+1}{mn} \min\{\alpha(-K_X), \alpha(L_D|D)\}\},
\]
then
1. $((X, -K_X); D, m)$ is uniformly log $K$-stable with angle $2\pi\beta$. Moreover, the maximal existence angle $\beta_{cscKc}$ satisfies that $\beta_{cscKc} \geq \beta_u$.
2. The log $K$-energy is proper. $((X, -K_X); D, m)$ admits a cscK cone metric, which is a Kähler–Einstein cone metric satisfying
\[
\text{Ric}(\omega) = \mu \cdot \omega + 2\pi(1 - \beta)[D], \quad \mu = 1 - m(1 - \beta).
\]
(6.11)

Proof. Under the assumption, we have $\Lambda = \lambda = 1$ and $\mathcal{S}_1 = n$. Then $\mathcal{S}_\beta = n - mn(1 - \beta) = n\mu$ and the cone angle constrain (6.8) becomes
\[
1 - \frac{1}{m} \leq \beta < 1 - \frac{1}{m} + \frac{n+1}{mn} \alpha\beta.
\]
We get an equivalent inequality
\[
0 \leq \frac{n}{n+1} \mu < \alpha\beta.
\]
By (5.6), we know
\[
\alpha\beta \geq \min\{m\beta, \alpha(-K_X), m\alpha(L_D|D)\}.
\]
Clearly, $m\beta \geq \frac{n}{n+1}\mu$. So Theorem 6.19 implies that if
\[
\min\{\alpha(-K_X), m\alpha(L_D|D)\} > \frac{n}{n+1}\mu,
\]
then \(((X, -K_X); D, m)\) is uniformly log \(K\)-stable. The first conclusion follows from rewriting this condition in \(\beta\).

The properness in second conclusion follows from Proposition 5.4, if we show that the condition 6.4 implies the Condition (5.1). Actually, under the assumption, we have \(C_1(X, D) = \mu C_1(X)\). Then (6.4) becomes

\[
n\mu < (n + 1)\alpha_\beta, \quad -n\mu + (n + 1)\mu \geq 0.
\]

While, the topological Condition (5.1) reads

\[
0 \leq \eta < \frac{n + 1}{n}\alpha_\beta, \quad \mu < \eta, \quad (n\mu - \eta) < (n - 1)\mu.
\]

We could choose \(\eta = \mu + \epsilon\) with some sufficiently small \(\epsilon > 0\) and verify that all these three inequalities are satisfied. Thus, (6.4) gives Condition (5.1).

Furthermore, the existence Theorem 5.7 implies there exists cscK cone metric in \(\Omega\), see also Theorem 5.25. Actually, it is Kähler–Einstein cone metric (6.11) by uniqueness, Theorem 2.15 (see [46] and references therein for more results on Kähler–Einstein cone metric).

\[\square\]

**Corollary 6.26.** Suppose that there exists \(D \in |-K_X|\) which is smooth, and take \(m = 1\) in Corollary 6.25. Then we have

\[
0 \leq \beta < \min\{1, \frac{n + 1}{n}\min\{\alpha(-K_X), \alpha(L_D|D)\}\}.
\]

Consequently, the maximal angle for uniform log \(K\)-stability (Definition 4.13) has lower bound

\[
\beta_{ulKs} \geq \min\{1, \frac{n + 1}{n}\min\{\alpha(-K_X), \alpha(L_D|D)\}\}.
\]

**Remark 6.27.** For Kähler–Einstein cone metric, the uniform log \(K\)-stability is also equivalent to the log \(K\)-stability, as a result of the resolution of the YTD conjecture for the Kähler–Einstein metric. In general, these two stabilities are not expected to be equivalent.

**Remark 6.28.** Twisted Kähler–Einstein path \(\omega \in C^1(X)\) is defined to be

\[
\text{Ric}(\omega) = \beta \cdot \omega + (1 - \beta)\omega_0,
\]

with a smooth Kähler metric \(\omega_0 \in C^1(X)\) and \(0 \leq \beta \leq 1\), the lower bound of the maximal existence time \(\beta\) of the twisted Kähler–Einstein path was introduced in [64] and its lower bound was given in [23, Corollary 3.2].

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