Non-simultaneous match-stick geometry

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Abstract

The purpose of this paper is to prove that every finite set of points that can be constructed in the Euclidean plane by using a compass and a ruler can also be constructed by using unitary match-sticks in a non-simultaneous way and following to a certain set of postulates. To prove this, we will deduce the Euclidean axioms for our defined set of axioms.

1 Introduction

In 1939 T.R. Dawson proved (see \cite{1}) that every point in the plane that can be constructed with a compass and a ruler can be constructed by laying match-sticks according to some rules. One of his most important postulates is, that two sticks may be laid simultaneously. Some similarly interesting constructions are done in \cite{2}.

Our aim in the following is to define a set of postulates that are equivalent to the Euclidean postulate systems. These postulates will be similar to Dawson’s, except that no two match-sticks can be used at the same time. Thus we call our constructions "non-simultaneous”.

We will prove that

\begin{itemize}
  \item we can extend every bounded line segment
  \item we can draw a line between any two given points
  \item we can construct right angles
  \item there exists a unique parallel line to a given line and through a given point
  \item we can describe circles geometrically. Since the constructed geometry is a point-geometry, it is sufficient to prove that we can construct the geometrical intersection between 2 given circles, respectively one circle and one line.
\end{itemize}

The first point to prove is necessary since we only work on bounded match-sticks. This point will in fact occupy the major part of this work. Since the constructed geometry is included in the Euclidean space and the definition of lines coincide, the fourth statement can be reduced to the constructibility of the only parallel line.

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2 The postulates

The following are the postulates we will be working on for the rest of this paper.

i. One may lay a match-stick through any two given points at distance equal or less than the unit, where one or even two of the points may be chosen as extremity of the stick segment. This procedure also permits us to determine whether the distance of two given points is strictly less, equal or strictly greater than the unit.

ii. One may choose an arbitrary point on a given stick segment, and it can be chosen such that it is none of the extremities.

iii. From a given point, one may use a match-stick as a unitary ruler. This ruler does not leave a trace, but can be used to determine a point with unit distance to the given point and relatively to a given set of match-sticks. No two match-sticks can be used as a compass at the same time.

Notice that postulates i. and iii. are directly related to the non-simultaneousness of our construction constraints. Postulate ii. is a simple rule that seems necessary, but natural.

3 Basic constructions

We start by making some elementary constructions. These construction schemes will serve as building bricks for the more sophisticated constructions that follow. Most of these constructions directly use the proposed postulates. The first result will solve the problem that our postulates only construct bounded objects. We have to prove the following proposition:

**Proposition 1** Every bounded line can be extended to an arbitrary length.

**Proof**  Let \( A, B \) be the extremities of a given stick. Using postulate ii. we may choose a point \( P \) in the interior of the stick. Using postulate i. we lay down a stick having \( P \) as an extremity and passing through \( B \), and its other extremity shall be denoted by \( Q \). A third stick having \( B \) as one extremity and passing through \( Q \), is a translation of the first and has exactly one point in common with it. By reusing this algorithm, we can enlarge any given segment. Therefore, we may in the following draw lines in the classical way. \( \square \)

Remark that this allows us to determine the intersection point of two given (i.e. determined by respective line segments) lines in a finite time. It does however not enable us to discover whether two lines are parallel. We will give another way for this later on.

Now we construct perpendicular lines. Notice that the nature of constraints in non-simultaneousness renders unaplicable most common constructions, based for example on the perpendicularity of diagonals in a rhombus.

**Proposition 2** For any line \( d \) and any \( A \in d \), we can construct the line perpendicular to \( d \) at \( A \).
Figure 1: Construction of a perpendicular line

**Proof** Choose a unit segment (1) having $A$ as one of its extremities and not lying on $d$. Let us call $Q$ the second extremity of (1). (2) is laid such that its second extremity $R$ lies on $d$, but is different from $A$. (3) is the extension of (2) as shown on the picture. As we have $1 = QA = QR = QS$ and $Q = \text{mid}(R, S)$, Thales’ theorem implies that $\angle SAR = 90^\circ$. An easy computation shows that this construction only works if the angle formed by the initial line and the first match-stick is less than $30^\circ$. Note that the maximal distance between $A$ and $S$ is less than the hypotenuse, which has length of two. Therefore, by doing this construction 3 times with different starting segments $(1)_1 \neq (1)_2 \neq (1)_3$, the pigeonhole principle yields that at least two of the points $S_1, S_2, S_3$ are on the same side of $d$. The above geometrical argument shows that they both have to lie on the perpendicular line. As their maximal distance from $A$ is 2, either two of the constructed points are closer than one unit from each other, or one of them has a distance from $A$ which is less than one unit. In both cases, Postulate i. and Proposition 1. allow us to construct the perpendicular line to $A$. □

The next step will enable us to construct coordinate grids. We will call coordinate grid the covering of the plane with unit squares that are all parallel to each other. Since we work on a finite number of steps, the aim will be to define a grid that contains each point after a finite number of steps.

**Proposition 3** For any two points $A, B$ in the plane, there exists a construction sequence of a coordinate grid such that $A$ and $B$ are contained in the grid after a finite number of construction steps.
Proof Let $l_0$ be a line containing $A = A_0$ and $l_1$ be its perpendicular line passing through $A_0$. Chose $A_1$ on $l_1$ having unit distance from $A_0$. Then construct the perpendicular $l_2$ to $l_1$ and passing through $A_2$. Chose a point $A_2$ in a way similar to $A_1$ and construct $l_3$ the perpendicular to $l_2$ through $A_2$. $A_3$ is chosen on $l_3$ as having distance 2 from $A_2$ and such that $[A_2, A_3]$ intersects $l_0$ (which is finitely constructible). At each recursive step, we chose $A_i$ on $l_i$ having distance $\lceil \frac{i}{2} \rceil$ from $A_{i-1}$ and such that $[A_{i-1}, A_i]$ intersects $l_{i-3}$. This then covers the plan "spirally". Since the distance between $A$ and $B$ is finite, both will be in this grid sequence after a finite number of steps. \hfill \Box

4 Combined constructions

We now have the means to make more general constructions. We start by constructing the perpendicular bisector and specific parallel lines.

Proposition 4 If $[A, B]$ is a line segment, we can construct its perpendicular bisector.
Proof Using Proposition 2, we construct the perpendicular lines to $AB$ at $A$ and $B$ respectively. Using Postulate iii., we can choose a point $R = R_0$ on the perpendicular to $A$ with distance less than a unit from $AB$. A second point $S = S_0$ is constructed on the other side of $AB$ and at a unit distance from the first point. Now we know that perpendiculars at these points are both parallel to $AB$ and have unit distance. We then construct $T$ and $U$ the intersection points of these perpendiculars with the perpendicular through $B$ respectively. By using the process shown on the picture, we iteratively construct $R_i, T_i, S_i, U_i$ by laying down unit sticks from $R_{i-1}, T_{i-1}, S_{i-1}$ and $U_{i-1}$ respectively. We stop these two processes when $[R_{i-1}, R_i]$ and $[T_{i-1}, T_i]$ intersect, respectively when $[S_{i-1}, S_i]$ and $[U_{i-1}, U_i]$ intersect. This process then yields respective intersection points $P$ and $Q$ at a distance less than one. Therefore, we can draw the line through those two points. A basic geometrical argument proves that $PQ$ is the perpendicular bisector of $[A, B]$. As a corollary, we deduce that its intersection point with $AB$, called $C$, is the midpoint of the segment. □

Since the construction of parallel lines is closely related to the construction of perpendicular lines, the following result follows naturally:

**Proposition 5** Given a line $d$ and a point $A$ with distance from $d$ less than one unit, we can construct the perpendicular and the parallel lines to $d$ passing through $A$.

**Proof** Determine the two points $P$ and $Q$ with unit distance from $A$ with $d$. Using Proposition 4 and the fact that $\triangle APQ$ is isosceles, we can construct the perpendicular bisector of $[PQ]$, which passes through $A$. The perpendicular to this line at $A$ is parallel to $d$, which concludes the proof. □

This result now gives us the announced method of determining whether two lines are parallel. It is enough to consider the parallel through one of the lines that passes through a point of the other line and check if they both are coincident.

## 5 Advanced constructions

We will now verify the validity of the remaining Euclidean postulates. We will prove the uniqueness of the line passing through 2 given points, as well as the pointwise constructibility of circles with given incenter
Most of the previous constructions have dealt with bounded figures. We now combine those constructions with the unit grid to extend our geometry to non-bounded alignment, parallelism and perpendicular lines.

**Proposition 6** If \( A \) and \( B \) are two distinct points, we can construct their common line. If \( l \) and \( p \) are given lines, we can also construct the parallel to \( l \) which passes through \( B \), as well as the perpendicular to \( p \) which passes through \( B \).

![Figure 4: Construction of the line between two points](image)

**Proof** We start by constructing a line \( l \) passing through \( A \) and let \( p \) be the perpendicular to \( l \) passing through \( A \). If \( B \) lies on any of those lines, the proof is finished. Otherwise, we construct, using Proposition 3, a coordinate unit grid square having \((A,l,p)\) as determining lines and being sufficiently large to contain \( B \). Let \( l_i, l_{i+1} \) and \( p_j, p_{j+1} \) be the lines defining the unit square that contains \( B \). As the distance of \( B \) from any four of those lines is less than 1 unit, we may use Proposition 5 and draw a line \( d \) perpendicular with \( p_j \) (and hence to \( p \)) passing through \( B \). Similarly, we can obtain the line perpendicular with \( d \) through \( B \). As \( d \) is also perpendicular to \( p \), let \( C \) be its intersection with \( l \). Construct the midpoints \( A_1 \) of \([AC]\) and \( B_1 \) of \([BC]\). Using the intercept theorem, we know that the segment drawn between those two points should be parallel to \([AB]\) and have half of its length. Reusing this algorithm, we can go on by constructing the midpoints of the new segments and so on. After a finite number of steps, the distance between the two actual midpoints is less than a unit. Therefore, we can draw the line between those two points. Using again the previous construction, we can now construct the parallel to this line and through \( B \), which passes also through \( A \). \( \square \)

We will now move on to the circular intersections. Note that there are two different definitions of a circle. On one hand, a circle can be given as the set composed of its center and one of its points. On the other hand, it can also be given as the set composed of the center and a segment defining its radius and not necessarily having the center as one of its extremities. The construction of a parallelogram enables us to translate the given segment to a suitable position. Therefore, whenever in the following, we are referring to a circle, we consider that we know its center and one of its points. We first start by intersecting a line and a circle.

**Proposition 7** Let \( AB \) be a line and \( \Gamma \) be a circle with centre \( O \) and \( S \in \Gamma \). Then we can construct the intersection of \((AB)\) and \( \Gamma \), respectively determine whether this intersection is empty.
Figure 5: Intersection of a circle and a line

**Proof** If $d(O, S) = 1$, the (possibly empty) intersection is obtained by the simple use of postulate iii. Otherwise, choose $S'$ on $(OS)$ such that $OS' = 1$. Let $d$ be the parallel line to $AO$ that goes through $S'$. Let $S''$ be the intersection point of $(AS)$ and $d$. Let $O'$ be the intersection point of $AO$ with the parallel to $OS$ that passes through $S''$. There are 0, 1 or 2 points on $AB$ with unit distance from $O'$, determining the number of intersection points between $AB$ and $\Gamma$. If such a point $R'$ at unit distance exists, then the intersection point of $AB$ and the parallel to $O'R'$ is both on $AB$ and on $\Gamma$, which gives the announced construction. □

Now that we can intersect lines and circles, we will intersect two given circles. The idea will be to transform this intersection problem into an intersection problem of the previous type.

**Proposition 8** Let $\Gamma_1$ and $\Gamma_2$ be two circles with centres $O_1$ and $O_2$ and radii $r_1$ and $r_2$. We can determine whether they are intersective. If they are, we can find their intersection points.
Proof Let $X$ and $Y$ be two points on $\Gamma_2$ such that $(O_1O_2)$ and $(XY)$ are not perpendicular. This is a mild assumption since we can easily construct the antipodal and the respective $90^\circ$-rotations of a given point on the circle, producing at least one non-perpendicular line. Using proposition 7, we can construct the intersection point of the circle with centre $X$ and radius $r_1$ and the perpendicular bisector of segment $[XY]$. Call this point $O_3$. Now, consider the circle $\Gamma_3$ with centre $O_3$ and radius $r_1$. The radical axis $a_1$ of $\Gamma_2$ and $\Gamma_3$ is $(XY)$, as $\Gamma_2 \cap \Gamma_3 = \{X, Y\}$. The radical axis $a_2$ of $\Gamma_1$ and $\Gamma_3$ is the perpendicular bisector of the segment $[O_1O_3]$. This can be proved by symmetry, as $\Gamma_1$ and $\Gamma_3$ have same radius. By using the radical axis theorem, there exists a power point $P$ which is the intersection of $a_1$ and $a_2$ (which is non-empty by construction). Therefore, the perpendicular $a_3$ line to $(O_1O_2)$ which passes through $P$ has to be the radical axis of $\Gamma_1$ and $\Gamma_2$. The intersection of this line with $\Gamma_1$ is equal to the (possibly empty) intersection of the two circles.

6 Conclusion

We have proved the equivalence between the Euclidean postulate system and our new bounded system. One main advantage of this method is that it only uses bounded constructions but still obtains the same results. Remark that it is easy to generalize our construction to the case where the unit segment and the ruler have length ratios that are not 1, but any rational number. If however this ratio is an irrational number, it might be interesting to construct a geometry satisfying the postulates, but different from the Euclidean geometry.
References

[1] T. R. Dawson (1939) "Match-stick" geometry. The Mathematical Gazette 23, 161-168.

[2] D. Wells (1991) The Penguin Dictionary of Curious and Interesting Geometry. Penguin Books, London.

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