Taub-NUT solutions in conformal electrodynamics

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We construct a novel charged Taub-NUT spacetime, providing a first non-trivial example of a self-gravitating solution to the recently proposed ModMax theory [1, 2], the most general (1-parametric) theory of non-linear electrodynamics that is characterized by both the conformal symmetry and the SO(2) duality-rotation invariance. The spacetime features non-trivial magnetic fields, in their presence the non-linearity of the field becomes apparent and the solution is distinguished from that of the Maxwell theory. Thermodynamics of the new solution and the charged black hole limit of vanishing NUT parameter are also briefly discussed.

I. INTRODUCTION

Maxwell’s electromagnetism is a relativistic theory of linear electrodynamics which enjoys many fundamental symmetries, the two distinguished being the conformal invariance and the invariance under the SO(2) duality-rotation transformations. Over the years, various non-linear extensions of this theory that modify the behavior of the field in a strong field regime have been considered. Perhaps the best known example of a theory of non-linear electrodynamics is the Born–Infeld theory [3]. Although invariant under duality transformations [4], the Born–Infeld theory contains a dimensionful parameter, and is not conformally invariant.

Remarkably, Maxwell’s theory is not the only (four-dimensional) theory that enjoys conformal and duality-rotation symmetries. As shown recently [1, 2], there is a 1-parametric generalization of the Maxwell theory with these properties. To stress its conformal invariance, in this paper we call this theory conformal (non-linear) electrodynamics. The corresponding Lagrangian density reads [1, 2]

\[ \mathcal{L} = -\frac{1}{2} \left( S \cosh \gamma - \sqrt{S^2 + P^2} \sinh \gamma \right), \]

where \( \gamma \) is a free dimensionless parameter, and \( S \) and \( P \) are the two invariants of the electromagnetic field,

\[ S = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad P = \frac{1}{2} F_{\mu\nu} (*F)^{\mu\nu}, \]

with the field strength \( F_{\mu\nu} \) given by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), in terms of the vector potential \( A_\mu \). For \( \gamma = 0 \), the theory reduces to the linear Maxwell theory. As discussed in [1] when \( \gamma \neq 0 \) a birefringe phenomenon occurs; apart from the light-like polarization mode there is another mode which is subluminal for \( \gamma > 0 \) and superluminal for \( \gamma < 0 \), hinting on a physical restriction \( \gamma \geq 0 \).

It is the aim of this paper to construct a first non-trivial self-gravitating solution of the conformal electrodynamics. As can be seen from the Lagrangian [1], the non-linear character of the theory only appears for solutions with magnetic fields for which the invariant \( P \) is non-trivial. In what follows, we construct ‘the simplest possible’ such solution — the charged Taub-NUT(-AdS) spacetime. Even in the absence of magnetic charges, this spacetime features charged Misner strings and associated with them magnetic fields. In their presence, the invariant \( P \) becomes non-trivial, the non-linearity of the field becomes apparent, and the solution is distinguished from that of the Maxwell theory.

The Taub–NUT spacetime [5, 6] is a remarkable solution of the Einstein equations that has been a subject of controversy for many years, e.g. [7, 8]. Although predominantly studied in the Euclidean setting (see e.g. recent [9] and references therein), it also provides an example of a cosmological model and a black hole spacetime. In what follows we concentrate on the (Lorentzian) black hole case, maintaining the Misner string singularities on the axes [10, 22]. Such a spacetime is geodesically complete and (despite the presence of regions with closed timelike curves) it is free of causal pathologies for geodesic observers [11, 12, 14]. Moreover, as shown recently, the corresponding black hole thermodynamics can be understood by standard thermodynamic techniques [15, 16] and yields a consistent first law, even in the presence of rotation and charges [17, 18, 21].

Our paper is organized as follows. In the next section we review the basic properties of the conformal electrodynamics. In Sec.III we construct the charged Taub-NUT(-AdS) spacetime in this theory and compare it briefly with its Maxwell counterpart. Sec.IV is devoted to the study of the thermodynamics of the obtained solution, while Sec.V discusses the charged black hole limit of vanishing NUT charge. We conclude in Sec.VI.
II. CONFORMAL ELECTRODYNAMICS

As shown in [1, 2] the theory [1] is the most general four-dimensional electrodynamics that possesses both the conformal symmetry and the SO(2) duality-rotation invariance. Introducing the following ‘material’ field strength tensor:

\[ E_{\mu\nu} = \frac{\partial L}{\partial F^{\mu\nu}} = 2 \left( L_S F_{\mu\nu} + L_P * F_{\mu\nu} \right), \tag{3} \]

where

\[ L_S = \frac{\partial L}{\partial \Sigma} = \frac{1}{2} \left( \frac{\Sigma}{\sqrt{\Sigma^2 + P^2}} \sinh \gamma - \cosh \gamma \right), \tag{4} \]

\[ L_P = \frac{\partial L}{\partial \mathcal{P}} = \frac{1}{2} \frac{\mathcal{P}}{\sqrt{S^2 + P^2}} \sinh \gamma, \tag{5} \]

the field equations are simply written as

\[ d * E = 0, \quad dF = 0, \tag{6} \]

and are invariant under the SO(2) duality rotations

\[ \left( \begin{array}{c} F'_{\mu\nu} \\ *F'_{\mu\nu} \end{array} \right) = \left( \begin{array}{c} \cos \theta \sin \theta \\ -\sin \theta \cos \theta \end{array} \right) \left( \begin{array}{c} F_{\mu\nu} \\ *F_{\mu\nu} \end{array} \right). \tag{7} \]

Similarly, under a conformal transformation of the metric, \( g \rightarrow \Omega^2 g \), we find

\[ F \rightarrow F, \quad *F \rightarrow *F, \tag{8} \]

and since \( S \rightarrow \Omega^{-1} S, \mathcal{P} \rightarrow \Omega^{-1} \mathcal{P} \), and \( L_S \rightarrow L_S, L_P \rightarrow L_P \), we also have

\[ E \rightarrow E, \quad *E \rightarrow *E, \tag{9} \]

and the field equations [6] are invariant under conformal transformations.

The electromagnetic stress-energy tensor reads

\[ T^{\mu\nu} = -\frac{1}{4\pi} \left( 2F^{\mu\sigma} F^{\nu\sigma} + \mathcal{P} L_P g^{\mu\nu} - L_S g^{\mu\nu} \right), \tag{10} \]

and, using the fact that

\[ L_S S + L_P \mathcal{P} = L, \tag{11} \]

it is obviously traceless, \( T^{\mu\mu} = 0 \). Employing [11], the energy momentum tensor can be further recast as

\[ T^{\mu\nu} = \frac{1}{4\pi} \left( S g^{\mu\nu} - 2F^{\mu\sigma} F^{\nu\sigma} \right) L_S. \tag{12} \]

In what follows we shall calculate the electric and magnetic charges inside a closed spacelike two-surface \( S \). These are simply given by

\[ q_e = \frac{1}{4\pi} \int_S *E, \quad q_m = \frac{1}{4\pi} \int_S F. \tag{13} \]

These definitions respect the duality rotations of Eq. [7], in the sense that magnetic charges rotate to electric charges and vice-versa.

III. CHARGED TAUB-NUT SOLUTION

Let us now couple the conformal electrodynamics to gravity and construct the corresponding (spherical topology) charged Taub-NUT(-AdS) solution. We thus consider the following bulk action for the coupled theory with gravity:

\[ I = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4\pi} \int d^4x \sqrt{-g} L \right). \tag{14} \]

Here, \( \Lambda \) stands for the negative cosmological constant \( \Lambda = -3/l^2 \).

We seek the Taub-NUT solution characterized by a single metric function \( f = f(r) \),

\[ ds^2 = -f(dt^2 + 2n \cos \theta d\phi^2) + \frac{dr^2}{f} + (r^2 + n^2)(d\Omega^2 + \sin^2 \theta d\phi^2), \tag{15} \]

and the following gauge potential:

\[ A = a(dt + 2n \cos \theta d\phi), \tag{16} \]

where the gauge function \( a = a(r) \). In the above, we have denoted the NUT parameter by \( n \), and chosen a symmetric distribution for Misner strings, which are now located on both the north-pole and south-pole axes.

With this ansatz we find the following expressions for \( F \) and \( *F \):

\[ F = -a' dt \wedge dr + 2na \cos \theta d\theta \wedge d\phi - 2na \sin \theta d\theta \wedge d\phi, \tag{17} \]

\[ *F = \frac{2na}{n^2 + r^2} dt \wedge dr - \frac{4n^2a}{n^2 + r^2} \cos \theta dr \wedge d\phi - (n^2 + r^2)a' \sin \theta d\theta \wedge d\phi. \tag{18} \]

These are independent of the metric function \( f \) on behalf of the fact that the determinant of the metric \( [15] \) simply reads \( \sqrt{-g} = r^2 + n^2 \). The two electromagnetic invariants take a rather elegant shape

\[ S = -a'^2 + \frac{4n^2a^2}{(n^2 + r^2)^2}, \quad \mathcal{P} = \frac{4naa'}{n^2 + r^2}, \tag{19} \]

and so does the material tensor

\[ E = -a' dt \wedge dr + 2ne^\gamma \cos \theta a'dr \wedge d\phi - 2nae^{-\gamma} \sin \theta d\theta \wedge d\phi. \tag{20} \]

The only non-zero component of the field equation [9] then yields

\[ \epsilon^\gamma \left( (n^2 + r^2) a'' + 2ra' \right) - \frac{4e^{-\gamma}n^2a}{n^2 + r^2} = 0, \tag{21} \]

which when integrated gives

\[ a = c_1 \sin \left( 2e^{-\gamma} \arctan \frac{r}{n} \right) + c_2 \cos \left( 2e^{-\gamma} \arctan \frac{r}{n} \right), \tag{22} \]
where the integration constants $c_1$ and $c_2$ are related to the electric and magnetic charges of the black hole, as described below.

Having obtained the gauge potential, let us now turn towards the Einstein equations with the electromagnetic tensor given by (22). They yield the following equation for the metric function $f$:

$$
\frac{3}{l^2} + \frac{f(n^2 - r^2) + (n^2 + r^2)(rf' - 1)}{(n^2 + r^2)^2} - e^{-\gamma}a^2 = 0 ,
$$

(23)

giving the following solution:

$$
f = \frac{4e^{-\gamma n^2(c_1^2 + c_2^2) - 2mr + r^2 - n^2}}{r^2 + n^2} - \frac{3n^4 - 6n^2r^2 - r^4}{l^2(r^2 + n^2)} ,
$$

(24)

where the integration constant $m$ represents the asymptotic mass of the solution (as can be seen by a large $r$ expansion).

The final step is to determine the constants $c_1$ and $c_2$ in terms of the asymptotic electric and magnetic charges. To this purpose we employ formulæ (13), choosing $S$ to be a sphere of radius $r$. Similar to the Maxwell case, the enclosed charges are radius dependent:

$$
q_e = e^\gamma (n^2 + r^2) a', \quad q_m = 2na ,
$$

(25)

reflecting the fact that the Misner strings carry electromagnetic charges. Evaluating these asymptotically, it is usual to define the asymptotic value of the electric and magnetic charges as the electric and magnetic parameters $e$ and $g$,

$$
\lim_{r \to \infty} q_e = e , \quad \lim_{r \to \infty} q_m = -2ng .
$$

(26)

In terms of these, the constants $c_1$ and $c_2$ above are determined as follows:

$$
c_1 = -g \cos (2e^{-\gamma} \pi) - \frac{e \cos (2e^{-\gamma} \pi)}{2n} ,
$$

(27)

$$
c_2 = -g \sin (2e^{-\gamma} \pi) + \frac{e \sin (2e^{-\gamma} \pi)}{2n} .
$$

(28)

To summarize, the charged Taub-NUT(-AdS) solution constructed in this section takes the form (15), (16), where the metric and gauge functions are given by

$$
f = \frac{e^{-\gamma}(2e^2 + 4n^2g^2) - 2mr + r^2 - n^2}{r^2 + n^2} - \frac{3n^4 - 6n^2r^2 - r^4}{l^2(r^2 + n^2)} ,
$$

(29)

and

$$
a = -g \cos \left[ e^{-\gamma} \left( \pi - 2 \arctan \frac{\pi}{n} \right) \right] - \frac{e}{2n} \sin \left[ e^{-\gamma} \left( \pi - 2 \arctan \frac{\pi}{n} \right) \right] .
$$

(30)

To compare with the charged Taub-NUT solution in the Maxwell theory, we see that at the level of the metric function $f$, the effect of considering the conformal electrodynamics rather than the Maxwell theory corresponds to rescaling the electric and magnetic parameters by $e^{-\gamma/2}$. To compare the gauge functions, we expand $a$ as a power series in $\gamma$:

$$
a = g \frac{r^2 - n^2}{r^2 + n^2} + \frac{er}{r^2 + n^2} + \gamma \left( \frac{e(r^2 - n^2)}{2n(r^2 + n^2)} - \frac{2gnr^2}{(n^2 + r^2)} \right) \left( \pi - 2 \arctan \frac{r}{n} \right) + O(\gamma^2) .
$$

(31)

The leading term corresponds to the solution in the Maxwell theory, while the next term represents a non-linear effect that arises from the theory (11) to first-order in $\gamma$.

IV. THERMODYNAMICS

In this section we discuss some basic properties of the obtained solution and its thermodynamics. We follow the Lorentzian approach developed in (15–21). Namely, we maintain the Misner strings and assign to them physical properties such as temperature/angular velocity. In this way we are able to formulate a full cohomogeneity first law where all the physical parameters of the solution are varied independently. The results of this section can be straightforwardly compared to those for charged Taub-NUT(-AdS) spacetimes in Maxwell’s theory (17).

A. Basic thermodynamic quantities

To start our discussion we first turn to the black hole horizon. This is located at the largest root $r_+$ of $f(r_+) = 0$. It is a Killing horizon generated by the Killing vector

$$
k = \partial_t .
$$

(32)

In what follows we identify the temperature of the spacetime with the black hole horizon temperature:

$$
T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left( 1 + \frac{3(r^2_+ + n^2)}{l^2} - e^{-2\gamma} \frac{r^2 + 4ng^2}{r^2_+ + n^2} \right) ,
$$

(33)

and the entropy of the spacetime with the area of the black hole horizon:

$$
S = \frac{\text{Area}}{4} = \pi(r^2_+ + n^2) .
$$

(34)

In addition to the black hole horizon, there are two additional Killing horizons in the spacetime that are associated with the Misner strings. They are generated by

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1 In this paper we focus on a configuration with symmetric distribution of Misner strings and do not consider the ‘string strength’ as a thermodynamic quantity. Our analysis can be straightforwardly extended to this more general case by following (18).

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the following Killing vectors:

\[ k_{\pm} = \partial_t \mp \frac{1}{2n} \partial_{\phi} . \]  

(35)

Contrary to the vector \( k \), these Killing vectors are not properly normalized at infinity. In what follows we identify the associated surface gravity \( \kappa_{\pm} \) with the ‘Misner potential’ \( \psi \):

\[ \psi = \kappa_{\pm} = \frac{1}{8\pi n} . \]  

(36)

Alternatively 15, 20, \( \psi \) can be attributed a meaning of the angular velocity of the string, c.f. \( (42) \). We call the conjugate quantity to \( \psi \) the Misner charge and denote it by \( N \).

Similar to the case of Maxwell electrodynamics, the thermodynamic mass \( M \) is simply given by the parameter \( m \),

\[ M = m , \]  

(37)

and the asymptotic angular momentum of the spacetime vanishes.

Since the Misner strings are charged, the electric and magnetic charges depend on the radius of the sphere and are given by \( (35) \). One can easily check that they are related, \( q_e \leftrightarrow q_m \), by the electromagnetic duality:

\[ e \leftrightarrow -2ng , \quad 2ng \leftrightarrow e . \]  

(38)

In particular, we have

\[ Q = e , \quad Q_m = -2ng \]  

(39)

for the asymptotic charges, and

\[ Q^+(r_+) = q_e(r_+) = e^{\gamma}(r_-^2 + r_+^2) a'(r_+) , \]

\[ Q_m^+(r_+) = q_m(r_+) = 2na(r_+) \]  

(40)

for the horizon charges. The electrostatic potential \( \phi \) can be calculated by evaluating \( -\xi_\mu A^\mu \) in the horizon and subtracting its value at infinity,

\[ \phi = -(\xi_\mu A^\mu |_{r=r_+} - \xi_\mu A^\mu |_{r=\infty}) , \]  

yielding

\[ \phi = -a(r_+) - g \]

\[ = g \left( \cos \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) - 1 \right) \]

\[ + \frac{e}{2n} \sin \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) . \]  

(42)

Upon using the electromagnetic duality \( (35) \) we recover the corresponding magnetic potential

\[ \phi_m = \frac{e}{2n} \left( \cos \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) - 1 \right) \]

\[ - g \sin \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) . \]  

(43)

As usual in the framework of extended black hole thermodynamics 23, we identify the cosmological constant with a dynamical pressure

\[ P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi T^2} , \]  

(44)

allowing it to vary in the first law, and define the thermodynamic volume as the corresponding conjugate quantity:

\[ V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,N,...} . \]  

(45)

Finally, we calculate the Gibbs free energy by evaluation the Euclidean action

\[ I = I + I_{\text{GH}} + I_C \]

\[ = \frac{1}{16\pi} \int_M d^4x \sqrt{g} (R - 2\Lambda) - \frac{1}{4\pi} \int d^4x \sqrt{g} L \]

\[ + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} \left( \kappa - \frac{2}{l} - \frac{1}{2} R(h) \right) \]  

(46)

where apart from the bulk action \( I \), \( (13) \), we have also included the Gibbons–Hawking term \( I_{\text{GH}} \) and the AdS counterterm \( I_C \) designed to cancel possible divergences 24. The Gibbs function is given by \( G = I / \beta \), where \( \beta \) is the inverse temperature and periodicity of the time Euclidean coordinate. This yields

\[ G = \frac{m}{2} + \frac{eg}{2} \left( 1 - \cos \left( 2e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) \right) \]

\[- \frac{e^2 - 4n^2 g^2}{8n} \sin \left( 2e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) \]

\[- \frac{r_+(3n^2 + r_+^2)}{2l^2} . \]  

(47)

In the limit \( \gamma \rightarrow 0 \), this expression straightforwardly reduces to that in the Maxwell theory, calculated in 17.

### B. Unconstrained thermodynamics

Being in the grandcanonical ensemble, we have

\[ G = G(T, \psi, \phi, Q_m^+, P) \]

\[ = M - TS - \phi Q - \psi N , \]  

(48)

and the Gibbs free energy satisfies

\[ \delta G = -S\delta T - N\delta \psi - Q\delta \phi + \phi_m \delta Q_m^+ + V \delta P . \]  

(49)

These relations can be used to verify the above expressions for \( S, Q \) and \( \phi_m \), as well as to find the remaining missing thermodynamic quantities \( V \) and \( N \). The thermodynamic volume reads

\[ V = \frac{4}{3} \pi r_+^3 \left( 1 + \frac{3n^2}{r_+} \right) . \]  

(50)
and is identical to the Maxwell case. The Misner charge \( N \) can be calculated either from (50), or as
\[
N = -\frac{\partial G}{\partial \psi} \bigg|_{T, \phi, Q_m, P}.
\] (51)
In either case the full expression for \( N \) is not very illuminating and we omit it in this paper.

By construction the obtained thermodynamic quantities obey the full cohomogeneity first law:
\[
\delta M = T \delta S + \phi \delta Q + \psi \delta N + V \delta P,
\] (52)
as well as the corresponding Smarr relation
\[
M = 2TS + \phi Q + \phi_m Q^+_m + 2\psi N - 2PV.
\] (53)

C. Electric first law

Similar to the Maxwell case (17) let us finally consider the constrained thermodynamics, where one imposes an extra ‘regularity condition’,
\[
a(r+) = 0,
\] (54)
requiring that both the gauge potential \( A \) (16) and the magnetic charge \( Q^+_m \), (44) vanish on the horizon. This allows one to eliminate the magnetic parameter according to
\[
g = -\frac{e}{2n} \tan \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right).
\] (55)
With this the electrostatic potential (22) reads
\[
\phi = -g = \frac{e}{2n} \tan \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right),
\] (56)
and the Gibbs free energy simplifies to
\[
\mathcal{G} = \mathcal{G}(T, \psi, \phi, P)
\]
\[
= \frac{m}{2} - \frac{r_+(3n^2 + r_+^2)}{2f^2}
- \frac{e}{2n} \tan \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right).
\] (57)
In this case, the thermodynamic volume takes the same form (50), while the Misner charge simplifies to
\[
N = -\frac{4\pi n^3}{r_+} \left( 1 + \frac{3(n^2 - r_+^2)}{l^2} \right)
- e^2 \left( e^{-\gamma} \sec^2 \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) \right)
- \frac{r_+}{2n} \tan \left( e^{-\gamma} \left( \pi - 2 \arctan \frac{r_+}{n} \right) \right) \). (58)

With these simplifications we obtain the ‘electric first law’
\[
\delta M = T \delta S + \phi \delta Q + \psi \delta N + V \delta P,
\] (59)
accompanied by the following Smarr relation:
\[
M = 2(TS - VP + \psi N) + \phi Q.
\] (60)

V. LIMIT OF VANISHING NUT CHARGE

Following (17) let us now take the limit of a vanishing NUT charge. That is, we want to take the limit
\[
n \to 0,
\] (61)
while preserving both electric and magnetic charges at infinity:
\[
Q = e = \text{const.}, \quad Q_m = -2\pi g \equiv \hat{g} = \text{const}.
\] (62)
Obviously, in order to keep the latter charge constant as \( n \to 0 \) we have to have \( g \to \infty \). The limit of the metric simply reads
\[
d\bar{s}^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]
\[
f = 1 - \frac{2m}{r} - e^{-\gamma}(e^2 + \hat{g}^2) + \frac{r_+^2}{r^2}.
\] (63)

In order to take the limit of the gauge potential \( A \), one has to first ‘renormalize’ the gauge potential by adding a pure gauge, \( A \to A + gdt \), upon which it is straightforward to take the limit and obtain
\[
A = -e^{-\gamma} \frac{e^{-\gamma}}{r} dt + \hat{g} \cos \theta d\phi.
\] (64)

The solution given by (63) and (64) represents the generalization of the (dyonic) charged AdS black hole to the case of the conformal electrodynamics. The thermodynamics of this solution follows from the thermodynamics of the Taub-NUT(-AdS) case discussed in the previous section. In particular, from (17) we obtain the following Gibbs free energy
\[
\mathcal{G} = \frac{m}{2} - \frac{e^{-\gamma} e^2 - \hat{g}^2}{2r_+} - \frac{r_+^2}{2l^2}.
\] (65)
Since the Misner strings are no longer present, the horizon charges coincide with the asymptotic charges and the thermodynamics significantly simplifies.

VI. CONCLUSION

We have obtained an exact axially symmetric black hole solution for Einstein gravity coupled to a maximally symmetric non-linear electromagnetic theory (11). While the black hole metric calculated above is identical to the Einstein–Maxwell case, except for the rescaling of the charges, a salient feature of the solution is that the electric and magnetic fields in this spacetime do not correspond to simple rescalings. As a consequence, it is in principle possible to detect any deviations from linear electrodynamics by placing charged test particles in a curved spacetime. Whether this observed behaviour for the solutions is exclusive to Taub-NUT class or is a general feature for other spacetimes in this theory remains to be explored.
Moreover, we have extended the recent work on the thermodynamics of Lorentzian NUTs to this novel theory of electromagnetism, hence adding to the program of re-habilitating these spacetimes. Future work in these lines could aim towards studying the impact of non-linearity of the electromagnetic field on phase transitions of the Taub-NUT solutions, as well as exploring the thermodynamics of other exact black hole solutions in this theory.

Note added. After the work on the present manuscript was largely complete, it came to our attention that [25] had presented a similar solution to the one introduced here, albeit in a simpler spherically symmetric setup.

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[1] Igor Bandos, Kurt Lechner, Dmitri Sorokin, and Paul K. Townsend, “A non-linear duality-invariant conformal extension of Maxwell’s equations,” (2020), arXiv:2007.09092 [hep-th].
[2] B.P. Kosyakov, “Nonlinear electrodynamics with the maximum allowable symmetries,” Phys. Lett. B 810, 135840 (2020).
[3] Max Born and Leopold Infeld, “Foundations of the new field theory,” Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 144, 425–451 (1934).
[4] GW Gibbons and DA Rasheed, “Electric-magnetic duality rotations in non-linear electrodynamics,” Nuclear Physics B 454, 185–206 (1995).
[5] Abraham H Taub, “Empty space-times admitting a three parameter group of motions,” Annals of Mathematics, 472–490 (1951).
[6] Ezra Newman, L Tamburino, and T Unti, “Empty-space generalization of the schwarzschild metric,” Journal of Mathematical Physics 4, 915–923 (1963).
[7] Charles W Misner, “The flatter regions of newman, unti, and tamburino’s generalized schwarzschild space,” Journal of Mathematical Physics 4, 924–937 (1963).
[8] Charles W Misner, “Taub-nut space as a counterexample to almost nothing,” Relativity theory and astrophysics 1, 160 (1967).
[9] Luca Ciambelli, Cristóbal Corral, José Figueroa, Gastón Giribet, and Rodrigo Olea, “Topological Terms and the Misner String Entropy,” (2020), arXiv:2011.11044 [hep-th].
[10] William B Bonnor, “A new interpretation of the nut metric in general relativity,” in Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 66 (Cambridge University Press, 1969) pp. 145–151.
[11] JG Miller, Martin D Kruškal, and Brenton B Godfrey, “Taub-nut (newman, unti, tamburino) metric and incompatible extensions,” Physical Review D 4, 2945 (1971).
[12] V.S. Manko and E. Ruiz, “Physical interpretation of NUT solution,” Class. Quant. Grav. 22, 3555–3560 (2005).
[13] Gérard Clément, Dmitri Gal’tsov, and Mourad Guenouche, “Rehabilitating space-times with NUTs,” Phys. Lett. B 750, 591–594 (2015).
[14] Gérard Clément, Dmitri Gal’tsov, and Mourad Guenouche, “NUT wormholes,” Phys. Rev. D 93, 024048 (2016).
ter,“ (2020). arXiv:2011.10836 [gr-qc]