CAUSAL VARIATIONAL PRINCIPLES IN THE INFINITE-DIMENSIONAL SETTING: EXISTENCE OF MINIMIZERS

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Abstract. We provide a method for constructing (possibly non-trivial) measures on non-locally compact Polish subspaces of infinite-dimensional separable Banach spaces which, under suitable assumptions, are minimizers of causal variational principles in the non-locally compact setting. Moreover, for non-trivial minimizers the corresponding Euler-Lagrange equations are derived. The method is to exhaust the underlying Banach space by finite-dimensional subspaces and to prove existence of minimizers of the causal variational principle restricted to these finite-dimensional subsets of the Polish space under suitable assumptions on the Lagrangian. This gives rise to a corresponding sequence of minimizers. Restricting the resulting sequence to countably many compact subsets of the Polish space, by considering the resulting diagonal sequence we are able to construct a regular measure on the Borel algebra over the whole topological space. For continuous Lagrangians of bounded range it can be shown that, under suitable assumptions, the obtained measure is a (possibly non-trivial) minimizer under variations of compact support. Under additional assumptions, we prove that the constructed measure is a minimizer under variations of finite volume and solves the corresponding Euler-Lagrange equations. Afterwards, we extend our results to continuous Lagrangians vanishing in entropy. Finally, assuming that the obtained measure is locally finite, topological properties of spacetime are worked out and a connection to dimension theory is established.

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1. Introduction

In the physical theory of causal fermion systems, spacetime and the structures therein are described by a minimizer (for an introduction to the physical background and the mathematical context, we refer the interested reader to §2.1, the textbook [18] and the survey articles [20, 22]). Causal variational principles evolved as a mathematical generalization of the causal action principle [16, 23], and were studied in more detail in [24]. The starting point in [24] is a second-countable, locally compact Hausdorff space $\mathcal{F}$ together with a non-negative function $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+ := [0, \infty)$ (the Lagrangian) which is assumed to be lower semi-continuous, symmetric and positive on the diagonal. The causal variational principle is to minimize the action $\mathcal{S}$ defined as the double integral over the Lagrangian

$$\mathcal{S}(\rho) = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y)$$

under variations of the measure $\rho$ within the class of regular Borel measures on $\mathcal{F}$, keeping the (possibly infinite) total volume $\rho(\mathcal{F})$ fixed (volume constraint). The aim of the present paper is to extend the existence theory for minimizers of such variational principles to the case that $\mathcal{F}$ is non-locally compact and the total volume is infinite. We also work out the corresponding Euler-Lagrange (EL) equations.

In order to put the paper into the mathematical context, in [14] it was proposed to formulate physics by minimizing a new type of variational principle in spacetime. The suggestion in [14, Section 3.5] led to the causal action principle in discrete spacetime, which was first analyzed mathematically in [15]. A more general and systematic enquiry of causal variational principles on measure spaces was carried out in [16]. In this article, the existence of minimizers is proven in the case that the total volume is finite. In [23], the setting is generalized to non-compact manifolds of possibly infinite volume and the corresponding EL equations are analyzed. However, the existence of minimizers is not proved. This is done in [24] in the slightly more general setting of second-countable, locally compact Hausdorff spaces. In this paper, we extend the results of [24] by developing the existence theory in the non-locally compact setting.
The main difficulty in dealing with non-locally compact spaces is that it is no longer possible to restrict attention to compact neighborhoods. Moreover, it turns out that we can no longer assume that the underlying topological space $\mathcal{F}$ is $\sigma$-compact. As a consequence, at first sight it is not clear how to construct global measures on the whole topological space at all. The way out is to introduce a countable collection of suitable compact subsets, which indeed allows us to construct a global measure $\rho$ on $\mathcal{F}$. For simplicity, we first assume that the Lagrangian is of bounded range (see Definition 3.7). In this case, the minimizing property of the measure $\rho$ is proved in two steps: We first show that $\rho$ is a minimizer under variations of compact support. In a second step, we extend this result to variations of finite volume under the assumption that property (iv) in § 2.2 holds, i.e.

$$\sup_{x \in \mathcal{F}} \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) < \infty.$$ 

Afterwards, we generalize our results to Lagrangians which do not have bounded range, but instead have suitable decay properties. To this end, we consider Lagrangians vanishing in entropy (see Definition 6.2). Introducing spacetime as the support of the measure $\rho$, we finally analyze topological properties of spacetime; moreover, a connection to dimension theory is established.

The paper is organized as follows. In Section 2 we give a short physical motivation (§ 2.1) and recall the main definitions and existence results as obtained in [24] (§ 2.2). In Section 3 we begin by working out important topological properties of infinite-dimensional causal fermion systems (§ 3.1); afterwards, we introduce causal variational principles in the non-locally compact (or infinite-dimensional) setting by considering non-locally compact Polish subspaces of (infinite-dimensional) separable Banach spaces (§ 3.2). Exhausting the underlying Banach space by finite-dimensional subspaces and making use of the results in [24], the existence of minimizers is proven for the causal variational principle restricted to finite-dimensional subspaces (§ 3.3). In Section 4 we provide a method for constructing a regular global measure on the Borel algebra over the whole non-locally compact Polish space. More precisely, we first introduce a countable collection of compact subsets of the underlying topological space (§ 4.1). Next, making use of Prohorov’s theorem and applying Cantor’s diagonal argument, we are able to construct a (possibly non-trivial) regular measure on the whole topological space (§ 4.2). Finally, we derive useful properties of the constructed measure (§ 4.3). In Section 5 we prove that, under suitable assumptions, this measure is a minimizer for continuous Lagrangians of bounded range (see Definition 3.7). More precisely, we first introduce an appropriate assumption on the obtained measure (see condition (B) in § 5.1). Next, we prove that the obtained measure is a minimizer under variations of finite-dimensional compact support (§ 5.2) as well as a minimizer under variations of compact support (§ 5.3). Under additional assumptions we show that the constructed measure is a minimizer under variations of finite volume (§ 5.4). Assuming that the measure under consideration is non-zero, we prove that the corresponding Euler-Lagrange (EL) equations are satisfied (§ 5.5). The goal in Section 6 is to weaken the assumption that the Lagrangian is of bounded range. To this end, we introduce Lagrangians vanishing in entropy (see Definition 6.1) which generalize the notion of Lagrangians decaying in entropy (see Definition 2.8). The concept of Lagrangians vanishing in entropy (§ 6.1) can be extended to non-locally compact topological spaces (see Definition 6.2). For such Lagrangians, we repeat the above construction steps,
thus giving rise to a regular measure on the underlying topological space (§6.2). It is shown that, under suitable assumptions, the considered measure is minimizer of the causal action under variations of compact support as well as under variations of finite volume (§6.3). We finally derive the corresponding EL equations (§6.4). Introducing spacetime as the support of the minimizing measure under consideration, in Section 7 we conclude the paper by analyzing topological properties of spacetime and establishing a connection to dimension theory. To this end, we first recall some concepts from dimension theory (§7.1), and afterwards apply them to causal fermion systems (§7.2). In the appendix we summarize useful results which will be referred to frequently: Appendix A is dedicated to the proof that causal fermion systems are Polish (see Theorem A.1); the main result in Appendix B states that the support of locally finite measures on Polish spaces is $\sigma$-compact (see Lemma B.2).

2. Physical Background and Mathematical Preliminaries

2.1. Physical Context and Motivation. The purpose of this subsection is to outline a few concepts of causal fermion systems and to explain how the present paper fits into the general physical context and the ongoing research program. The reader not interested in the physical background may skip this section.

The theory of causal fermion systems is a recent approach to fundamental physics motivated originally in order to resolve shortcomings of relativistic quantum field theory (QFT). Namely, due to ultraviolet divergences, perturbative quantum field theory is well-defined only after regularization, which is usually understood as a set of prescriptions for how to make divergent integrals finite (e.g. by introducing a suitable “cutoff” in momentum space). The regularization is then removed using the renormalization procedure. However, this concept is not convincing from neither the physical nor the mathematical point of view. More precisely, in view of Heisenberg’s uncertainty principle, physicists infer a correspondence between large momenta and small distances. Because of that, the regularization length is often associated to the Planck length $\ell_P \approx 1.6 \cdot 10^{-35}$ m. Accordingly, by introducing an ultraviolet cutoff in momentum space, one disregards distances which are smaller than the Planck length. As a consequence, the microscopic structure of spacetime is completely unknown. Unfortunately, at present there is no consensus on what the correct mathematical model for “Planck scale physics” should be.

The simplest and maybe most natural approach is to assume that on the Planck scale, spacetime is no longer a continuum but becomes in some way “discrete.” This is the starting point in the monograph [14], where the physical system is described by an ensemble of wave functions in a discrete spacetime. Motivated by the Lagrangian formulation of classical field theory, physical equations are formulated by a variational principle in discrete spacetime. In the meantime, this setting was generalized and developed to the theory of causal fermion systems. It is an essential feature of the approach that spacetime does not enter the variational principle a-priori, but instead it emerges when minimizing the action. Thus causal fermion systems allow for the description of both discrete and continuous spacetime structures.

In order to get the connection to the present paper, let us briefly outline the main structures of causal fermion systems. As initially introduced in [19], a causal fermion system consists of a triple $(\mathcal{H}, \mathcal{F}, \rho)$ together with an integer $n \in \mathbb{N}$, where $\mathcal{H}$ denotes a complex Hilbert space, $\mathcal{F} \subset L(\mathcal{H})$ being the set of all self-adjoint operators on $\mathcal{H}$ of finite rank with at most $n$ positive and at most $n$ negative eigenvalues, and $\rho$ being
a positive measure on the Borel \(\sigma\)-algebra \(\mathcal{B}(\mathcal{F})\) (referred to as *universal measure*). Then for any \(x, y \in \mathcal{F}\), the product \(xy\) is an operator of rank at most \(2n\). Denoting its non-trivial eigenvalues (counting algebraic multiplicities) by \(\lambda_{1}^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}\), and introducing the spectral weight \(|.|\) of an operator as the sum of the absolute values of its eigenvalues, the *Lagrangian* can be introduced as a mapping

\[
\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_{0}^{+}, \quad \mathcal{L}(x, y) = |(xy)^{2}| - \frac{1}{2n} |xy|^{2}.
\]

As being of relevance for this article, we point out that the Lagrangian is a continuous function which is symmetric in the sense that

\[
\mathcal{L}(x, y) = \mathcal{L}(y, x) \quad \text{for all } x, y \in \mathcal{F}.
\]

In analogy to classical field theory, one defines the *causal action* by

\[
S(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) \, d\rho(x) \, d\rho(y).
\]

Finally, the corresponding *causal action principle* is introduced by varying the measure \(\rho\) in the class of regular measures on \(\mathcal{B}(\mathcal{F})\) under additional constraints (which assert the existence of non-trivial minimizers). Given a minimizing measure \(\rho\), *space-time* \(M\) is defined as its support,

\[
M := \text{supp} \, \rho.
\]

As being outlined in detail in [18], critical points of the causal action give rise to Euler-Lagrange (EL) equations, which describe the dynamics of the causal fermion system. In a certain limiting case, the so-called *continuum limit*, one can establish a connection to the conventional formulation of physics in a spacetime continuum. In this limiting case, the EL equations give rise to classical field equations like the Maxwell and Einstein equations. Moreover, quantum mechanics is obtained in a limiting case, and close connections to relativistic quantum field theory have been established (for details see [17] and [21]).

In order for the causal action principle to be mathematically sensible, the existence theory is of crucial importance. If the dimension of the Hilbert space \(\mathcal{H}\) is finite, the existence of minimizers was proven in [16, Section 2] (based on existence results in discrete spacetime [15]), giving rise to minimizing measures \(\rho\) on \(\mathcal{F}\) of finite total volume \(\rho(\mathcal{F}) < \infty\). For this reason, it remains to extend these existence results by developing the existence theory in the case that \(\mathcal{H}\) is infinite-dimensional. Then the total volume \(\rho(\mathcal{F})\) is necessarily infinite (for a counter example see [18, Exercise 1.3]). In the resulting *infinite-dimensional setting* (i.e. \(\dim \mathcal{H} = \infty\) and \(\rho(\mathcal{F}) = \infty\)), the task is to deal with minimizers of infinite total volume on non-locally compact spaces. In preparation, the existence theory of minimizers of possibly infinite total volume \(\rho(\mathcal{F})\) on locally compact spaces is developed in [24] in sufficient generality. The remaining second step, which involves the difficulty of dealing with non-locally compact spaces, is precisely the objective of the present paper.

### 2.2. Causal Variational Principles in the \(\sigma\)-Locally Compact Setting

Before introducing causal variational principles on non-locally compact spaces in Section 3 below, we now recall the main results in the less general situation of causal variational principles in the \(\sigma\)-locally compact setting [24] which are based on results concerning causal variational principles in the non-compact setting as studied in [23, Section 2].
The measure \( \rho \) is symmetric, i.e. \( \mathcal{L}(x, y) = \mathcal{L}(y, x) \) for all \( x, y \in \mathcal{F} \).

(ii) \( \mathcal{L} \) is lower semi-continuous, i.e. for all sequences \( x_n \to x \) and \( y_{n'} \to y \),

\[
\mathcal{L}(x, y) \leq \liminf_{n, n' \to \infty} \mathcal{L}(x_n, y_{n'}) .
\]

The causal variational principle is to minimize the action

\[
S(\rho) = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y)
\]  

under variations of the measure \( \rho \), keeping the total volume \( \rho(\mathcal{F}) \) fixed (volume constraint). The papers \cite{23, 24} mainly focus on the case that the total volume \( \rho(\mathcal{F}) \) is infinite. In order to implement the volume constraint and to derive the corresponding Euler-Lagrange equations, in \cite{24} one makes the following additional assumptions:

(iii) The measure \( \rho \) is locally finite (meaning that any \( x \in \mathcal{F} \) has an open neighborhood \( U \subset \mathcal{F} \) with \( \rho(U) < \infty \)).

(iv) The function \( \mathcal{L}(x, \cdot) \) is \( \rho \)-integrable for all \( x \in \mathcal{F} \) and

\[
\sup_{x \in \mathcal{F}} \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) < \infty .
\]

By Fatou’s lemma, the integral in \eqref{2.2} is lower semi-continuous in the variable \( x \).

A measure on the Borel algebra which satisfies (iii) will be referred to as a Borel measure (in the sense of \cite{28}), and the set of Borel measures on \( \mathcal{F} \) shall be denoted by \( \mathcal{B}_\mathcal{F} \). Moreover, the Borel \( \sigma \)-algebra over \( \mathcal{F} \) is denoted by \( \mathcal{B}(\mathcal{F}) \). A Borel measure is said to be regular if it is inner and outer regular (cf. \cite{10} Definition VIII.1.1). An inner regular Borel measure is also called a Radon measure \cite{38}.

In \cite{23, 24} one varies in the following class of measures:

**Definition 2.1.** Given a regular Borel measure \( \rho \) on \( \mathcal{F} \), a regular Borel measure \( \hat{\rho} \) on \( \mathcal{F} \) is said to be a variation of finite volume if

\[
|\hat{\rho} - \rho|(\mathcal{F}) < \infty \quad \text{and} \quad (\hat{\rho} - \rho)(\mathcal{F}) = 0 ,
\]

where the total variation \( |\hat{\rho} - \rho| \) of two possibly infinite measures \( \rho \) and \( \hat{\rho} \) on \( \mathcal{B}(\mathcal{F}) \) is defined in \cite{24, \S2.2} as follows: We say that \( |\hat{\rho} - \rho| < \infty \) if there exists \( B \in \mathcal{B}(\mathcal{F}) \) with \( \rho(B), \hat{\rho}(B) < \infty \) such that \( \rho|_{\mathcal{F}\setminus B} = \hat{\rho}|_{\mathcal{F}\setminus B} \). In this case,

\[
(\hat{\rho} - \rho)(\Omega) := \hat{\rho}(B \cap \Omega) - \rho(B \cap \Omega)
\]

for any Borel set \( \Omega \subset \mathcal{F} \).

Given a regular Borel measure \( \rho \in \mathcal{B}_\mathcal{F} \) and assuming that (i), (ii) and (iv) hold, for every variation of finite volume \( \hat{\rho} \in \mathcal{B}_\mathcal{F} \) the difference of the actions as given by

\[
(S(\hat{\rho}) - S(\rho)) = \int_{\mathcal{F}} d(\hat{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y)
\]

\[
+ \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d(\hat{\rho} - \rho)(y) \mathcal{L}(x, y) + \int_{\mathcal{F}} d(\hat{\rho} - \rho)(x) \int_{\mathcal{F}} d(\hat{\rho} - \rho)(y) \mathcal{L}(x, y)
\]

is well-defined in view of \cite{23} Lemma 2.1. For clarity, we point out that condition (iii) is not required in order for \eqref{2.4} to hold.
Note that the assumptions (iii) and (iv) are dropped in [24]. The causal variational principle in the \(\sigma\)-locally compact setting [24] is then defined as follows.

**Definition 2.2.** Let \(\mathcal{F}\) be a second-countable, locally compact Hausdorff space, and let the Lagrangian \(L : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+^+\) be a symmetric and lower semi-continuous function (see conditions (i) and (ii) above). Moreover, we assume that \(L\) is strictly positive on the diagonal, i.e.

\[ L(x, x) > 0 \quad \text{for all } x \in \mathcal{F}. \]

The causal variational principle on \(\sigma\)-locally compact spaces is to minimize the causal action (2.1) under variations of finite volume (see Definition 2.1).

We point out that (iv) is a sufficient condition for (2.4) to hold. However, since the conditions (iii) and (iv) are not imposed in [24], it is a-priori not clear whether the integrals in (2.4) exist. For this reason, condition (2.4) is included into the definition of a minimizer:

**Definition 2.3.** A regular Borel measure \(\rho\) on \(\mathcal{F}\) is said to be a minimizer of the causal action under variations of finite volume [24] if the difference (2.4) is well-defined and non-negative for all regular Borel measures \(\tilde{\rho}\) on \(\mathcal{F}\) satisfying (2.3),

\[(S(\tilde{\rho}) - S(\rho)) \geq 0.\]

We denote the support of the measure \(\rho\) by \(M\),

\[M := \text{supp} \rho = \mathcal{F} \setminus \bigcup \{ \Omega \subset \mathcal{F} \mid \Omega \text{ is open and } \rho(\Omega) = 0\}\]

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Theorem 2.6 (Euler-Lagrange equations). Let $\mathcal{F}$ be a second-countable, locally compact Hausdorff space, and assume that $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ is continuous and of compact range. Then there exists a regular Borel measure $\rho$ on $\mathcal{F}$ which satisfies the Euler-Lagrange equations

$$\ell|_{\text{supp} \rho} \equiv \inf_{x \in \mathcal{F}} \ell(x) = 0 ,$$

where $\ell \in C(\mathcal{F})$ is defined by

$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) - 1 .$$

Combining [24, Theorem 4.9 and Theorem 4.10], we obtain the following result.

Theorem 2.7. Assume that $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ is continuous and of compact range. Then there is a regular Borel measure $\rho$ on $\mathcal{F}$ which is a minimizer under variations of compact support [24] (see Definition 2.5). Under the additional assumptions that the Lagrangian $\mathcal{L}$ is bounded and condition (iv) is satisfied (see (2.2)), the measure $\rho$ is a minimizer under variations of finite volume [24] (see Definition 2.3).

In [24, Section 5] it was shown that the assumption that the Lagrangian $\mathcal{L}$ is of compact range can be weakened. To this end, we recall that every second-countable, locally compact Hausdorff space can be endowed with a Heine-Borel metric (for details we refer to the explanations in [24, §3.1 and §5.1]). Given an unbounded Heine-Borel metric on the second-countable, locally compact space $\mathcal{F}$, for any $r > 0$ and $x \in \mathcal{F}$ the closed ball $B_r(x)$ is compact, and hence can be covered by finitely many balls of radius $\delta > 0$. The smallest such number is denoted by $E_x(r, \delta)$ and is called entropy. This gives rise to Lagrangians decaying in entropy, being defined as follows (cf. [24, Definition 5.1]).

Definition 2.8. Assume that $\mathcal{F}$ is endowed with an unbounded Heine-Borel metric $d$. The Lagrangian $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ is said to decay in entropy if the following conditions are satisfied:

(a) $c := \inf_{x \in \mathcal{F}} \mathcal{L}(x, x) > 0$.
(b) There is a compact set $K \subset \mathcal{F}$ such that

$$\delta := \inf_{x \in \mathcal{F} \setminus K} \sup_{x \in \mathcal{F} \setminus K} \left\{ s \in \mathbb{R} : \mathcal{L}(x, y) \geq \frac{c}{2} \text{ for all } y \in B_s(x) \right\} > 0 .$$

(c) The Lagrangian has the following decay property: There is a monotonically decreasing, integrable function $f \in L^1(\mathbb{R}^+, \mathbb{R}_0^+)$ such that

$$\mathcal{L}(x, y) \leq \frac{f(d(x, y))}{C_x(d(x, y), \delta)} \text{ for all } x, y \in \mathcal{F} \text{ with } x \neq y ,$$

where

$$C_x(r, \delta) := C \, E_x(r + 2, \delta) \text{ for all } r > 0 ,$$

and the constant $C$ is given by

$$C := 1 + \frac{2}{c} < \infty .$$

We point out that the above definition of Lagrangians decaying in entropy as introduced in [24, Section 5] requires an unbounded Heine-Borel metric. For a more general definition we refer to §6.1 (see Definition 6.1).
For clarity we note that, if \((\mathcal{H}, \mathcal{F}, \rho)\) is a causal fermion system with \(\dim(\mathcal{H}) < \infty\), the space \(L(\mathcal{H})\) of bounded linear operators on \(\mathcal{H}\) is finite-dimensional. Combining the fact that all norms on finite-dimensional vector spaces are equivalent with the Heine-Borel theorem yields that the Fréchet metric induced by the operator norm on the vector space \(L(\mathcal{H})\) is an unbounded Heine-Borel metric on \(\mathcal{F}\).

Let us now recall the main results in [24] under the assumption that the Lagrangian decays in entropy.

**Theorem 2.9.** Let \(\mathcal{F}\) be a second-countable, locally compact Hausdorff space, and assume that \(L : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+^+\) is continuous and decays in entropy. Then there exists a regular Borel measure \(\rho\) on \(\mathcal{F}\) which satisfies the Euler-Lagrange equations

\[
\ell|_{\text{supp}\rho} \equiv \inf_{x \in \mathcal{F}} \ell(x) = 0,
\]

where \(\ell \in C(\mathcal{F})\) is defined by (2.7).

The following theorem ensures the existence of minimizing Borel measures.

**Theorem 2.10.** Assume that \(L : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+^+\) is continuous and decays in entropy. Then there is a regular Borel measure \(\rho\) on \(\mathcal{F}\) which is a minimizer under variations of compact support [24]. Under the additional assumptions that the Lagrangian \(L\) is bounded and condition (iv) is satisfied (see (2.2)), the measure \(\rho\) is a minimizer under variations of finite volume [24].

The goal of this paper is to extend the above results to the infinite-dimensional setting.

### 3. Causal Variational Principles in the Non-locally Compact Setting

#### 3.1. Motivation: Infinite-Dimensional Causal Fermion Systems

As explained in Section 1, causal variational principles evolved as a mathematical generalization of the causal action principle in order to develop the existence theory for causal fermion systems. In order to point out the connection to causal variational principles in the non-locally compact setting, let us briefly recall the basic structures of causal fermion systems (for details cf. [18, §1.1.1]). By definition, causal fermion systems are characterized by a separable complex Hilbert space \(\mathcal{H}\), an integer \(s \in \mathbb{N}\) (the so-called *spin dimension*) and a measure \(\rho\) defined on the Borel \(\sigma\)-algebra \(\mathcal{B}(\mathcal{F})\), where \(\mathcal{F} \subset L(\mathcal{H})\) consists of all self-adjoint operators on \(\mathcal{H}\) which have at most \(s\) positive and at most \(s\) negative eigenvalues. This gives rise to a triple \((\mathcal{H}, \mathcal{F}, \rho)\). The set \(\mathcal{F}\) can be endowed with the topology induced by the operator norm on \(L(\mathcal{H})\), thus becoming a topological space. More precisely, denoting the Fréchet metric induced by the operator norm on \(L(\mathcal{H})\) by \(d\), the space \((\mathcal{F}, d)\) is a separable complete metric space (Theorem A.1).

Whenever \(\dim(\mathcal{H}) < \infty\), the topological space \(\mathcal{F} \subset L(\mathcal{H})\) is locally compact. On the contrary, whenever \(\mathcal{F}\) is an infinite-dimensional Hilbert space, the corresponding set \(\mathcal{F} \subset L(\mathcal{H})\) is non-locally compact (see Lemma 3.3 below). In preparation, let us first state the following results.

**Proposition 3.1.** Any locally compact Banach space \(X\) is finite-dimensional.

**Proof.** Let \(x_1 \in X\) with \(\|x_1\| = 1\). Given \(x_1, \ldots, x_r \in X\) linearly independent unit vectors (i.e. \(\|x_i\| = 1\) for all \(i = 1, \ldots, r\)), the space \(G_r = \text{span}\{x_1, \ldots, x_r\}\) is an \(r\)-dimensional subspace of \(X\). Since \(G_r\) is finite-dimensional, it is closed. If \(G_r \subset E\), there exists a unit vector \(x_{r+1} \in X\) with \(\|x_{r+1} - x_i\| \geq 1/2\) for all \(i = 1, \ldots, r\).
If we assume that $X$ is infinite-dimensional, this holds for every $r \in \mathbb{N}$, thus ending up with an infinite sequence $(x_r)_{r \in \mathbb{N}}$ of unit vectors satisfying $\|x_p - x_q\| \geq 1/2$ for each $p \neq q$. In particular, the sequence $(x_r)_{r \in \mathbb{N}}$ admits no convergent subsequence in contradiction to the assumption that $E$ is locally compact.

\textbf{Corollary 3.2.}\ Any infinite-dimensional Banach space $X$ is non-locally compact. The same holds true for open subsets of $X$.

\textit{Proof.}\ This is an immediate consequence of Proposition 3.1 \hfill \Box

\textbf{Lemma 3.3.}\ Let $\mathcal{H}$ be an infinite-dimensional, separable complex Hilbert space, and let $\mathcal{F}^{\text{reg}} \subset L(\mathcal{H})$ be the set of self-adjoint operators which have exactly $s$ positive and exactly $s$ negative eigenvalues for some $s \in \mathbb{N}$. Then $\mathcal{F}^{\text{reg}}$ is non-locally compact.

\textit{Proof.}\ Since $\mathcal{F}^{\text{reg}}$ is a Banach manifold (for details see \cite{25}), it can be covered by an atlas $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ for some index set $A$ (cf. \cite{43} Chapter 73]). In particular, every point $x \in U^{\text{reg}}_\alpha$ is contained in some open set $U_\alpha$, whose image $V_\alpha := \phi_\alpha(U_\alpha)$ is open in some infinite-dimensional Banach space $X_\alpha$. Due to Corollary 3.2, the set $V_\alpha$ is non-locally compact. As the mapping $\phi_\alpha$ is a homeomorphism, we deduce that $U_\alpha \subset \mathcal{F}^{\text{reg}}$ is non-locally compact for any $\alpha \in A$, which proves the claim. \hfill \Box

Considering an infinite-dimensional, separable complex Hilbert space $\mathcal{H}$, then the set $\mathcal{F} \subset L(\mathcal{H})$ as introduced in \cite{18} is non-locally compact and Polish (see Lemma 3.3 and Theorem A.1). Our goal in the following is to prove the existence of a regular (possibly non-locally finite) measure $\rho$ on the Borel algebra $\mathcal{B}(\mathcal{F})$ such that $\rho$ is a minimizer of the corresponding causal action principle, giving rise to a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$. Instead of immediately delving into the corresponding causal action principle (see \cite{18} §1.1.1), we deal with causal variational principles on $\mathcal{H}$, which can be viewed as generalizations of the causal action principle (as introduced in \cite{16}, \cite{23} and considered in more detail in \cite{24}). Corresponding results concerning the causal action principle are then obtained as a special case. With this in mind, it suffices to prove the existence of minimizers of the causal variational principle (3.2) under the constraints (2.3) in the non-locally compact setting as introduced in Definition 3.1.

In order to motivate the basic definitions in §3.2 below, we note that $\mathcal{F} \subset \mathcal{K}(\mathcal{H})$, where by $\mathcal{K}(\mathcal{H}) \subset L(\mathcal{H})$ we denote the set of compact operators on $\mathcal{H}$. Since $\mathcal{H}$ is a separable, infinite-dimensional complex Hilbert space, let us point out that $\mathcal{K}(\mathcal{H})$ is a Banach space (see e.g. \cite{40} Satz II.3.2)] and separable in view of \cite{31} §12.E]. Moreover, making use of Proposition 3.1 Corollary 3.2 and Lemma 3.3, we conclude that $\mathcal{K}(\mathcal{H}) \cap \mathcal{F}$ is infinite-dimensional. This allows us to approximate $\mathcal{K}(\mathcal{H})$ by finite-dimensional subspaces. More precisely, we may apply \cite[Lemma 7.1]{2} to deduce that there is a sequence of finite-dimensional subspaces $(L_n)_{n \in \mathbb{N}}$ in $\mathcal{K}(\mathcal{H})$ with $L_n \subset L_{n+1}$ for all $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} L_n$ is dense in $\mathcal{K}(\mathcal{H})$. From the fact that subspaces of locally compact spaces are again locally compact we conclude that $\mathcal{F}^{(n)} := \mathcal{F} \cap L_n$ is locally compact for every $n \in \mathbb{N}$. Denoting by $d$ the Fréchet metric induced by the operator norm on $L(\mathcal{H})$, the space $(\mathcal{F}, d)$ is Polish (cf. Theorem A.1), i.e. a separable metric space. As a consequence, the subsets $\mathcal{F}^{(n)}$ are separable for every $n \in \mathbb{N}$ due to \cite[Lemma 2.16]{2} or \cite[Corollary 3.5]{1}. Together with the fact that separable metric spaces are second-countable this yields that the set $\mathcal{F}^{(n)}$ is a second-countable, locally compact Hausdorff space for every $n \in \mathbb{N}$. Moreover, from Lemma 3.3 we conclude that $\mathcal{F} \subset L(\mathcal{H})$ is non-locally compact.
In order to treat the corresponding causal variational principle in sufficient generality, it seems reasonable to vary in the class of regular, not necessarily locally finite measures on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{F}) \) (as intended in the textbook [18, §1.1.1]). As mentioned in [18], the causal action principle is ill-posed if the total volume \( \rho(\mathcal{F}) \) is finite and the Hilbert space \( \mathcal{H} \) is infinite-dimensional. However, the causal action principle does make mathematical sense in the so-called infinite-dimensional setting where \( \mathcal{H} \) is infinite-dimensional and the total volume is infinite, i.e. \( \rho(\mathcal{F}) = \infty \). These considerations motivate causal variational principles in the infinite-dimensional (or non-locally compact) setting as defined in the next subsection.

3.2. Basic Definitions. Let us first state the causal variational principle in the non-locally compact setting and discuss its difficulties afterwards.

**Definition 3.4.** Assume that \( X \) is a separable, infinite-dimensional Banach space, and let \( \mathcal{F} \subset X \) be a non-locally compact Polish space. Moreover, assume that the Lagrangian \( \mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_+^+ \) is a symmetric and lower semi-continuous function (see conditions (i) and (ii) in (2.2) which is strictly positive on the diagonal, i.e.

\[
\mathcal{L}(x, x) > 0 \quad \text{for all } x \in \mathcal{F}.
\]

The causal variational principle in the non-locally compact setting\(^1\) is to minimize

\[
\minimize \quad S(\rho) := \int_{\mathcal{F}} \int_{\mathcal{F}} d\rho(x) \, d\rho(y) \, \mathcal{L}(x, y)
\]

under variations of finite volume (see Definition 3.5 below) in the class of all regular measures on \( \mathcal{B}(\mathcal{F}) \) (in the sense of [28], cf. [24]) with \( \rho(\mathcal{F}) = \infty \).

The condition (3.1) is needed in order to avoid trivial minimizers supported at \( x \in \mathcal{F} \) with \( \mathcal{L}(x, x) = 0 \) (see [28, Section 1.2]). Furthermore, condition (3.1) is a plausible assumption in view of [18, Exercise 1.2]. Namely, given a minimizing measure \( \rho \) of the causal action principle (3.2), there exists a real constant \( c \) such that \( \text{tr}(x) = c \) for all \( x \in \text{supp} \rho \) according to [18, Proposition 1.4.1]. Under the reasonable assumption that \( c \neq 0 \) (cf. [18, §1.4.1]), we may conclude that \( \mathcal{L}(x, x) > 0 \) for all \( x \in \mathcal{F} \) in view of [18, Exercise 1.2]. This motivates as well as justifies the assumption that the Lagrangian is strictly positive on the diagonal.

Dropping the assumption that the measures under consideration are locally finite, we slightly adapt the definition of a minimizer of the causal action as follows.

**Definition 3.5.** A regular measure \( \rho \) on \( \mathcal{B}(\mathcal{F}) \) is said to be a minimizer of the causal action under variations of finite volume if the difference (2.4) is well-defined and non-negative for all regular measures \( \tilde{\rho} \) on \( \mathcal{B}(\mathcal{F}) \) satisfying (2.3),

\[
(S(\tilde{\rho}) - S(\rho)) \geq 0.
\]

Given a measure \( \rho \) on the Borel algebra \( \mathcal{B}(\mathcal{F}) \) and assuming that \( \tilde{\rho} \) is a variation of finite volume, in view of Definition 3.5, there exists \( B \in \mathcal{B}(\mathcal{F}) \) such that \( \rho|_{\mathcal{F}\setminus B} = \tilde{\rho}|_{\mathcal{F}\setminus B} \). In particular, the measures \( \rho|_B \) and \( \tilde{\rho}|_B \) are finite. Henceforth, whenever \( \rho \) is locally finite, then the same holds true for the measure \( \tilde{\rho}|_{\mathcal{F}\setminus B} \). From the fact that \( \tilde{\rho}|_B \) is a finite measure we conclude that \( \tilde{\rho}|_B \) is locally finite. Consequently, the measure \( \tilde{\rho} \) is locally finite if \( \rho \) is so. For this reason, Definition 3.5 can be viewed as a generalization of Definition 2.3 (cf. [24, Definition 2.1]). The same holds for Definition 3.6 below.

\(^1\)For clarity we point out that "causal variational principles in the non-locally compact setting" and "causal variational principles in the infinite-dimensional setting" are used synonymously.
Definition 3.6. A regular measure $\rho$ on $\mathcal{B}(\mathcal{F})$ is said to be a minimizer under variations of compact support of the causal action if for any regular measure $\tilde{\rho}$ on $\mathcal{B}(\mathcal{F})$ which satisfies (2.3) such that the signed measure $\tilde{\rho} - \rho$ is compactly supported, the inequality
\[
(S(\tilde{\rho}) - S(\rho)) \geq 0
\]
holds.

Let us now point out some difficulties regarding causal variational principles on non-locally compact spaces. First of all, let us recall that a topological space is called hemicompact if there is a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of $X$ such that any compact set $K \subset X$ is contained in $K_n$ for some $n \in \mathbb{N}$ (see [41, 17I]). Since $\mathcal{F}$ is first-countable and non-locally compact, by virtue of [11, Exercise 3.4.E] we conclude that $\mathcal{F}$ cannot be hemicompact.

Next, by contrast to the $\sigma$-locally compact setting as worked out in [24], it is in general not even possible to assume that $\mathcal{F}$ is $\sigma$-compact, as the following argument shows: Every Polish space (as well as every locally compact Hausdorff space) is Baire according to [31, Theorem (8.4)]\(^2\). In view of [41, 25B], a $\sigma$-compact topological space $X$ is Baire if and only if the set of points at which $X$ is locally compact is dense in $X$. Given an infinite-dimensional Hilbert space $\mathcal{H}$, and defining $\mathcal{F} \subset \mathcal{K}(\mathcal{H})$ in analogy to [18] (see §3.1), then $\mathcal{F}$ is a Polish space (see Appendix A). Consequently, the assumption that $\mathcal{F}$ is $\sigma$-compact implies that there exists $x \in \mathcal{F}$ being contained in a compact neighborhood $K \subset \mathcal{F}$ with $K^\circ \neq \emptyset$. From this we conclude that the intersection $K^{\text{reg}} := K \cap \mathcal{F}^{\text{reg}}$ is a compact set with non-empty interior, where the Banach manifold $\mathcal{F}^{\text{reg}} \subset \mathcal{F}$ is defined in Lemma 3.3 (for details see [25]). Given an atlas $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ of $\mathcal{F}^{\text{reg}}$ (cf. [43]) and making use of the fact that each $\phi_\alpha$ is a homeomorphism mapping to some infinite-dimensional Banach space $X_\alpha$, we deduce that the image of $K^{\text{reg}}$ is a compact subset with non-empty interior in contradiction to [32, Exercise 14.3]. For this reason, it is not possible to assume that the space $\mathcal{F}$ is $\sigma$-compact (by contrast to the setting in [24]).

Next, it is no longer possible to assume that the Lagrangian $L : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ is simultaneously lower semi-continuous and of compact range (see Definition 2.4) as introduced in [24, Definition 3.3]. Namely, due to lower semi-continuity of the Lagrangian, the latter assumption already implies that $\mathcal{F}$ is locally compact.

Finally, it is not possible to assume that the Lagrangian decays in entropy in the sense of [24, Definition 5.1] (see Definition 2.8); indeed, this assumption requires a Heine-Borel metric on $\mathcal{F}$, which clearly does not exist in non-locally compact spaces (otherwise each $x \in \mathcal{F}$ is contained in a corresponding ball with compact closure).

In view of these difficulties in the non-locally compact setting, let us begin by generalizing the assumption that $L$ is of compact range in the following way.

Definition 3.7. Let $(\mathcal{F}, d)$ be a metric space. The Lagrangian $L : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ is said to be of bounded range if every bounded set $B \subset \mathcal{F}$ is contained in a bounded neighborhood $B' \subset \mathcal{F}$ such that
\[
L(x, y) = 0 \quad \text{for all } x \in B \text{ and } y \notin B'.
\]

On proper metric spaces (that is, on spaces with the Heine-Borel property), this definition clearly implies that $L$ is of compact range (see Definition 2.4) as defined \(^2\)For clarity, we recall that a topological space $X$ is said to be Baire if the intersection of each countable family of dense open sets in $X$ is dense (see e.g. [41, Definition 25.1]).
in [24]. For this reason, Definition 3.7 provides a good starting point for dealing with causal variational principles on non-locally compact spaces. As we shall see below, the assumption that the Lagrangian is of bounded range can be weakened (see §6.1).

3.3. Finite-Dimensional Approximation. In the infinite-dimensional setting (see Definition 3.4), the space \( X \) is assumed to be a separable, infinite-dimensional Banach space. Hence we may apply [2 Lemma 7.1] to deduce that there exists a sequence of finite-dimensional subspaces \((X_n)_{n \in \mathbb{N}}\) in \( X \) with \( X_n \subset X_{n+1} \) for all \( n \in \mathbb{N} \) such that \( \bigcup_{n \in \mathbb{N}} X_n \) is dense in \( X \). This allows us to introduce the topological spaces

\[
\mathcal{F}(n) := \mathcal{F} \cap X_n \quad \text{for every } n \in \mathbb{N}.
\]

Since finite-dimensional topological vector spaces are locally compact (see e.g. [33, §15.7]), we conclude that each \( X_n \subset X \) is locally compact for all \( n \in \mathbb{N} \). For ease in notation, we shall denote the restriction of the Lagrangian to \( \mathcal{F}(n) \times \mathcal{F}(n) \) by \( \mathcal{L}^{(n)} \).

Thus for every \( n \in \mathbb{N} \), we are given a second-countable, locally compact Hausdorff space \( \mathcal{F}(n) \subset \mathcal{F} \) together with a symmetric, lower semi-continuous Lagrangian

\[
\mathcal{L}^{(n)} : \mathcal{F}(n) \times \mathcal{F}(n) \to \mathbb{R}^+,
\]

which is strictly positive on the diagonal. Henceforth for every \( n \in \mathbb{N} \) we are exactly in the \( \sigma \)-locally compact setting as worked out in [24].

In the following, we additionally assume that \( \mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^+ \) is continuous and of bounded range (see Definition 3.7). We again consider the above exhaustion \((X_n)_{n \in \mathbb{N}}\) of \( X \) by finite-dimensional subsets with \( X_n \subset X_{n+1} \) for all \( n \in \mathbb{N} \). Let us point out that each \((X_n, \| \cdot \|)\) is a finite-dimensional normed vector space, and all norms on \( X_n \) are equivalent. Due to the Heine-Borel theorem [2 Bemerkungen 2.6 and Satz 2.9], each closed ball \( B_r(x) \subset X_n \) is compact for all \( r > 0 \) and \( x \in X_n \). As a consequence, each bounded set \( A \subset \mathcal{F}(n) \) is contained in some compact ball \( B := B_r(x) \subset \mathcal{F}(n) \)

Definition 3.7 yields the existence of a compact set \( B' \subset \mathcal{F}(n) \) such that \( \mathcal{L}^{(n)}(x,y) = 0 \) for all \( x \in B \) and \( y \notin B' \). These considerations show that, whenever \( \mathcal{L} \) is continuous and of bounded range, for every \( n \in \mathbb{N} \) the restricted Lagrangian \( \mathcal{L}^{(n)} \) is continuous and of compact range (see [24 Definition 3.3] or Definition 2.4). As a consequence, by virtue of Theorem 2.6 (also see [24 Theorem 4.2]), for each \( n \in \mathbb{N} \) there exists a regular Borel measure \( \rho_n \) on \( \mathcal{F}(n) \) such that the following EL equations hold,

\[
\ell_n|_{\text{supp} \rho_n} \equiv \inf_{x \in \mathcal{F}(n)} \ell_n(x) = 0, \quad (3.3)
\]

where \( \ell_n \in C(\mathcal{F}) = C(\mathcal{F}, \mathbb{R}) \) is defined by

\[
\ell_n(x) := \int_{\mathcal{F}(n)} \mathcal{L}^{(n)}(x,y) \ d\rho_n(y) - 1. \quad (3.4)
\]

According to Theorem 2.7 (cf. [24 Theorem 4.10]), each Borel measure \( \rho_n \in \mathcal{B}_{\mathcal{F}(n)} \) is a minimizer of the corresponding causal variational principle

\[
\text{minimize} \quad S^{(n)} := \int_{\mathcal{F}(n)} \int_{\mathcal{F}(n)} \mathcal{L}^{(n)} \ d\rho(x) \ d\rho(y)
\]

under variations of compact support [24] in the class of regular Borel measures on \( \mathcal{F}(n) \) with respect to the constraints (2.3).

We extend the measures \( \rho_n \) by zero on the whole topological space \( \mathcal{F} \),

\[
\rho^{[n]}(A) := \rho_n(A \cap \mathcal{F}(n)) \quad \text{for all } A \in \mathcal{B}(\mathcal{F}). \quad (3.5)
\]
Thus
\[ \ell^{[n]}|_{\text{supp } \rho^{[n]}} \equiv \inf_{x \in \mathcal{F}^{(n)}} \ell^{[n]}(x) = 0 , \]
(3.6)
where the function \( \ell^{[n]} \in C(\mathcal{F}) \) is defined by
\[ \ell^{[n]}(x) := \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho^{[n]}(y) - 1 . \]
(3.7)
This gives rise to a sequence of regular Borel measures \( (\rho^{[n]})_{n \in \mathbb{N}} \) on \( \mathcal{F} \). In particular, whenever condition (iv) is satisfied for \( \rho^{[n]} \) (see (2.2)), that is
\[ \sup_{x \in \mathcal{F}} \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho^{[n]} < \infty \quad \text{for all } n \in \mathbb{N} , \]
(3.8)
each measure \( \rho^{[n]} \) is a minimizer on \( \mathcal{F}^{(n)} \) under variations of finite volume [21] (see Definition [2.1] and Definition [2.3]). In virtue of Theorem [2.10], the same holds true if the Lagrangian \( \mathcal{L}^{(n)} \) decays in entropy for any \( n \in \mathbb{N} \), provided that condition [3.8] is satisfied.

4. Construction of a Global Measure

In the following, let \( X \) be an infinite-dimensional, separable complex Banach space, and let \( \mathcal{F} \subset X \) be a non-locally compact Polish space endowed with a corresponding metric \( d \) such that \( (\mathcal{F}, d) \) is a separable, complete metric space. By \( \mathcal{O}(\mathcal{F}) \) and \( \mathcal{P}(\mathcal{F}) \) we denote the collection of open subsets of \( \mathcal{F} \) and the power set of \( \mathcal{F} \), respectively. Moreover, the collection of all compact subsets of \( \mathcal{F} \) is represented by \( \mathcal{H}(\mathcal{F}) \).

The goal of this section is to construct a global measure \( \rho \) based on the sequence of regular Borel measures \( (\rho^{[n]})_{n \in \mathbb{N}} \) as obtained in [3.3]. To this end, in a first step we construct a countable set \( \mathcal{D} \subset \mathcal{H}(\mathcal{F}) \) consisting of compact subsets of \( \mathcal{F} \) (§4.1). In a second step, we make use of the set \( \mathcal{D} \) in order to obtain a measure \( \rho \) on \( \mathcal{F} \) by a suitable construction process. In [§4.3] we finally prove that, restricted to suitable relatively compact subsets of \( \mathcal{F} \), the measure \( \rho \) is the weak limit of a subsequence of \( (\rho^{[n]})_{n \in \mathbb{N}} \).

4.1. Construction of a Countable Collection of Compact Sets. To begin with, separability of \( \mathcal{F} \) yields the existence of a countable dense subset \( E := \{ x_j : j \in \mathbb{N} \} \) such that, for every \( n \in \mathbb{N} \) the set \( E^{(n)} := E \cap \mathcal{F}^{(n)} \) is dense in \( \mathcal{F}^{(n)} \).\(^3\) We denote its elements by \( x_j^{(n)} \in E^{(n)} \) with \( j, n \in \mathbb{N} \). Moreover, since \( \mathcal{F}^{(n)} \) is locally compact, for all \( j, k, n \in \mathbb{N} \) there is a compact neighborhood \( V_j^{(n)} \subset E^{(n)} \) of \( x_j^{(n)} \subset E^{(n)} \) such that \( V_j^{(n)} \subset B_{1/k}(x_j^{(n)}) \) and each \( V_j^{(n)} \) being the closure of its interior (in the topology of \( \mathcal{F}^{(n)} \), where the interior of a set \( V \) shall be denoted by \( \text{int } V \)).\(^4\) This gives rise to the set
\[ V^{(1)} := \left\{ V_{j,k}^{(n)} : j, k, n \in \mathbb{N} \right\} . \]
Denoting the union of \( V^{(1)} \) and the empty set \( \emptyset \) by \( \mathcal{D}^{(1)} \), and making use of the fact that a countable union of countable sets is countable (see e.g. [21] Section 2]), we

\(^3\)Since \( \mathcal{F} \) is separable, there exists a countable set \( E^{(0)} \subset \mathcal{F} \) being dense in \( \mathcal{F} \). Similarly, for each \( i \in \mathbb{N} \) there are countable sets \( E^{(i)} \) which are dense in \( \mathcal{F}^{(i)} \). As a consequence, the set \( E := \bigcup_{i=0}^{\infty} \) has the desired properties.

\(^4\)For simplicity, one may consider \( V_{j,k}^{(n)} = B_{1/(2k)}(x_j^{(n)}) \cap \mathcal{F}^{(n)} \) for all \( j, k, n \in \mathbb{N} \).
conclude that \( \tilde{D}^{(1)} \) is countable. Therefore, applying Cantor’s diagonal argument and proceeding iteratively, we conclude that

\[
\tilde{D}^{(i)} := \left\{ D \cup \tilde{D} : D, \tilde{D} \in \tilde{D}^{(i-1)} \right\}
\]

is countable for every \( i \in \mathbb{N} \) with \( i \geq 2 \). As a consequence, the set

\[
\mathcal{D} := \bigcup_{i=1}^{\infty} \tilde{D}^{(i)}
\]

is countable; we denote its members by \((D_m)_{m \in \mathbb{N}}\). In particular, each \( D \in \mathcal{D} \) is a compact subset of \( \mathcal{F} \). Moreover, for every \( n \in \mathbb{N} \) we introduce

\[
\mathcal{D}^{(n)} := \left\{ D \in \mathcal{D} : D \subset \mathcal{F}^{(n)} \right\}
\]

(4.2)

4.2. Construction of a Regular Global Measure. In order to construct a global measure on \( \mathcal{F} \), we proceed similarly to [24] by selecting suitable subsequences of the sequence \((\rho^n)_{n \in \mathbb{N}}\) restricted to compact subsets \( D \in \mathcal{D} \). This allows us to construct a regular measure \( \rho \) on the whole space \( \mathcal{F} \) (see Theorem 4.3 below). In Section 5 we will show that, under suitable assumptions, the measure \( \rho \) is indeed a minimizer of the causal variational principle. In analogy to [24, Lemma 4.1], let us first state the following result.

**Lemma 4.1.** Assume that the Lagrangian \( \mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+ \) is lower semi-continuous and strictly positive on the diagonal (3.1). Furthermore, let \((\rho^n)_{n \in \mathbb{N}}\) be a sequence of measures \( \rho^n : B(\mathcal{F}) \to [0, \infty] \) such that, for every \( x \in \text{supp} \rho^n \),

\[
\int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho^n(y) = 1 \quad \text{for all } n \in \mathbb{N}.
\]

Then for every compact subset \( K \subset \mathcal{F} \) there is a constant \( C_K > 0 \) such that

\[
\rho^n(K) \leq C_K \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** This statement is proven exactly as [24, Lemma 4.1]. \( \square \)

Next, we apply Lemma 4.1 to the compact sets \( D \in \mathcal{D} \). More precisely, restricting the sequence \((\rho^n)_{n \in \mathbb{N}}\) as obtained in (3.5) (cf. 3.3) to the compact set \( D_1 \in \mathcal{D} \), the resulting sequence \((\rho^n|_{D_1})_{n \in \mathbb{N}}\) is bounded (due to Lemma 4.1 as well as uniformly tight (for the definition see [7, Definition 8.6.1])). Since compact subsets of Polish spaces are again Polish, Prohorov’s theorem (see for instance [7, Theorem 8.6.2] or [10, Satz VIII.4.23]) implies that a subsequence of \((\rho^n|_{D_1})_{n \in \mathbb{N}}\) converges weakly on \( D_1 \). Denoting the corresponding subsequence by \((\rho^{[1,n_k]}_{D_1})_{k \in \mathbb{N}}\) and considering its restriction to \( D_2 \in \mathcal{D}, \) the same arguments as before yield the existence of a weakly convergent subsequence \((\rho^{[2,n_k]}_{D_2})_{k \in \mathbb{N}}\) on \( D_2 \). Proceeding iteratively, we denote the resulting diagonal sequence by

\[
\rho^{(k)} := \rho^{[k,n_k]} \quad \text{for all } k \in \mathbb{N}.
\]

(4.3)

Thus by construction, for every \( m \in \mathbb{N} \) the sequence \((\rho^{(k)}|_{D_m})_{k \in \mathbb{N}}\) converges weakly to some measure \( \rho_{D_m} : B(D_m) \to [0, \infty) \),

\[
\rho^{(k)}|_{D_m} \rightharpoonup \rho_{D_m}.
\]

(4.4)
In particular,
\[ \lim_{k \to \infty} \rho^{(k)}(D_m) = \rho_{K_m}(D_m) \quad \text{for all } m \in \mathbb{N}. \]

We point out that each measure \( \rho^{(k)} \) is a minimizer on \( \mathcal{F}^{(n_k)} \). For this reason, we restrict attention to the finite-dimensional exhaustion \( (\mathcal{F}^{(k)})_{k \in \mathbb{N}} \), where for notational simplicity by \( \mathcal{F}^{(k)} \) we denote the sets \( \mathcal{F}^{(n_k)} \) for all \( k \in \mathbb{N} \). Note that the sequence constructed in (4.3) above in general does not converge weakly on arbitrary compact subsets, but only restricted to compact sets \( D \in \mathcal{D} \) (cf. (4.4)). In [24], this problem was resolved by deriving vague convergence of the sequence \( (\rho^{(n)})_{n \in \mathbb{N}} \) to some global measure \( \rho \). In order to obtain a similar situation, let us state the following result.

**Proposition 4.2.** The set function \( \varphi : \mathcal{D} \to [0, \infty) \) defined by
\[ \varphi(D) := \lim_{k \to \infty} \rho^{(k)}(D) < \infty \quad \text{for any } D \in \mathcal{D} \quad (4.5) \]
has the following properties:
1. \( \varphi(D_1) \leq \varphi(D_2) \) for all \( D_1, D_2 \in \mathcal{D} \) with \( D_1 \subset D_2 \),
2. \( \varphi(D_1 \cup D_2) \leq \varphi(D_1) + \varphi(D_2) \) for all \( D_1, D_2 \in \mathcal{D} \), and
3. \( \varphi(D_1 \cup D_2) = \varphi(D_1) + \varphi(D_2) \) for all \( D_1, D_2 \in \mathcal{D} \) with \( D_1 \cap D_2 = \emptyset \).

**Proof.** Given \( D_1, D_2 \in \mathcal{D} \) with \( D_1 \subset D_2 \), property (1) follows from
\[ \varphi(D_1) = \lim_{k \to \infty} \int_{\mathcal{F}} d\rho^{(k)}|_{D_1} \leq \lim_{k \to \infty} \int_{\mathcal{F}} d\rho^{(k)}|_{D_2} = \varphi(D_2). \]
Next, for all \( D_1, D_2 \in \mathcal{D} \), construction of \( \mathcal{D} \) yields \( D_1 \cup D_2 \in \mathcal{D} \). Thus property (2) is a consequence of
\[ \varphi(D_1 \cup D_2) = \lim_{k \to \infty} \int_{\mathcal{F}} d\rho^{(k)}|_{D_1 \cup D_2} \leq \lim_{k \to \infty} \int_{\mathcal{F}} d\rho^{(k)}|_{D_1} + \lim_{k \to \infty} \int_{\mathcal{F}} d\rho^{(k)}|_{D_2} = \varphi(D_1) + \varphi(D_2). \]
Similarly, for all \( D_1, D_2 \in \mathcal{D} \) with \( D_1 \cap D_2 = \emptyset \) we obtain
\[ \varphi(D_1 \cup D_2) = \lim_{k \to \infty} \rho^{(k)}(D_1 \cup D_2) = \lim_{k \to \infty} \rho^{(k)}(D_1) + \lim_{k \to \infty} \rho^{(k)}(D_2) = \varphi(D_1) + \varphi(D_2), \]
which proves property (3).

In order to construct a global measure \( \rho \) on \( \mathcal{F} \), we proceed in analogy to the proof of [10, Satz VIII.4.22]. We point out that, since the underlying topological space \( \mathcal{F} \) is non-locally compact, we cannot employ the Riesz representation theorem as in [24], and Riesz representation theorems on more general Hausdorff spaces as presented in [32, Section 16] do not seem applicable at this stage. Nevertheless, we obtain the following result.

**Theorem 4.3.** Introducing the set function \( \varphi : \mathcal{D} \to [0, \infty) \) by (4.5) and defining the set functions \( \mu : \mathcal{O}(\mathcal{F}) \to [0, \infty] \) and \( \eta : \mathcal{P}(\mathcal{F}) \to [0, +\infty] \) by
\[ \mu(U) := \sup \{ \varphi(D) : D \subset U, D \in \mathcal{D} \} \quad \text{for all } U \subset \mathcal{F} \text{ open}, \]
\[ \eta(A) := \inf \{ \mu(U) : A \subset U, U \subset \mathcal{F} \text{ open} \} \quad \text{for any } A \in \mathcal{P}(\mathcal{F}), \quad (4.6) \]
then the restriction
\[ \rho := \eta|_{\mathcal{B}(\mathcal{F})} \quad (4.7) \]
defines a (possibly non-trivial) measure on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{F}) \). In particular,
\[
\rho(D) = \varphi(D) = \lim_{k \to \infty} \rho^{(k)}(D) \quad \text{for any } D \in \mathcal{D}.
\] (4.8)

Proof. Let us first point out that, by construction, the set function \( \varphi : \mathcal{D} \to [0, \infty) \) defined by (4.5) has the following properties:

\begin{enumerate}[(1)]
  \item \( \varphi(D_1) \leq \varphi(D_2) \) for all \( D_1, D_2 \in \mathcal{D} \) with \( D_1 \subset D_2 \),
  \item \( \varphi(D_1 \cup D_2) \leq \varphi(D_1) + \varphi(D_2) \) for all \( D_1, D_2 \in \mathcal{D} \), and
  \item \( \varphi(D_1 \cup D_2) = \varphi(D_1) + \varphi(D_2) \) for all \( D_1, D_2 \in \mathcal{D} \) with \( D_1 \cap D_2 = \emptyset \).
\end{enumerate}

Indeed, properties (1)–(3) are a consequence of Proposition 4.2. Moreover, \( \varphi(\emptyset) = 0 \) (since \( \emptyset \in \mathcal{D} \)).

Next, similarly to the proof of [10] Satz VIII.4.22, our goal is to show that \( \eta \) is an outer measure,\(^5\) and that every Borel set \( B \in \mathcal{B}(\mathcal{F}) \) is \( \eta \)-measurable. This shall be done in the following by proving that
\[
\eta \text{ is an outer measure, and each closed set } A \subset \mathcal{F} \text{ is } \eta \text{-measurable}.
\] (4.9)

Denoting the \( \sigma \)-algebra of \( \eta \)-measurable sets by \( \mathcal{A}_\eta \), the statement (4.9) implies that the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{F}) \) is contained in \( \mathcal{A}_\eta \), i.e. \( \mathcal{B}(\mathcal{F}) \subset \mathcal{A}_\eta \). Therefore, in view of Carathéodory’s theorem (see e.g. [10] Satz II.4.4), the restriction
\[
\rho := \eta|_{\mathcal{B}(\mathcal{F})}
\]
defines a measure on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{F}) \). Thus it suffices to prove (4.9), which shall be done in several steps in the remainder of the proof.

(a) Let \( A \subset U \subset \mathcal{F} \) with \( A \) closed and \( U \) open. Whenever \( A \subset D \) for some \( D \in \mathcal{D} \), then there exists \( E \in \mathcal{D} \) with \( A \subset E \subset U \).

Proof: Since each \( D \in \mathcal{D} \) is compact, the closed set \( A \subset D \) is compact as well. Moreover, \( D \subset \mathcal{F}^{(n)} \) for sufficiently large \( n \in \mathbb{N} \). Since \( \mathcal{F}^{(n)} \) is locally compact, for every \( x \in A \) there exists \( V_x \in \mathcal{D} \) such that \( x \in V_x^c \subset V_x \subset U \). Since \( A \) is compact, the set \( E := \bigcup_{j=1}^{N} V_{x_j} \in \mathcal{D} \) for some integer \( N = N(A) \) has the desired property.

(b) Whenever \( U, V \subset \mathcal{F} \) open, \( \mu(U \cup V) \leq \mu(U) + \mu(V) \).

Proof: Without loss of generality, let \( U \neq \mathcal{F} \neq V \) and \( \mu(U), \mu(V) < \infty \) (otherwise the inequality is true). For this reason, let \( U, V \subset \mathcal{F} \) be open sets with \( U^c \neq \emptyset \neq V^c \) and \( D \subset U \cup V \) for \( D \in \mathcal{D} \). We then consider the closed sets
\[
A := \{ x \in D : d(x, U^c) \geq d(x, V^c) \} \subset D ,
B := \{ x \in D : d(x, U^c) \leq d(x, V^c) \} \subset D .
\]

Obviously, \( A \subset U \) and \( B \subset V \). Assuming conversely that \( x \in A \setminus U \), we conclude that \( x \in V \), and therefore \( d(x, U^c) = 0 < d(x, V^c) \) because \( V^c \) is closed, giving rise to the contradiction that \( x \notin A \). Similarly, we conclude that \( B \subset V \). Since \( A \subset D \), by
}\(^5\)Given a set \( X \), a set function \( \eta : \mathcal{P}(X) \to \mathbb{R} := [-\infty, +\infty] \) is said to be an outer measure if it has the following properties (see e.g. [10] Definition II.4.1):

(i) \( \eta(\emptyset) = 0 \).
(ii) For all \( A \subset B \subset X \) holds \( \eta(A) \leq \eta(B) \) (monotonicity).
(iii) For every sequence \( (A_n)_{n \in \mathbb{N}} \) of subsets of \( X \) holds \( \eta(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \eta(A_n) \) (\( \sigma \)-subadditivity).
virtue of (a) there exists $E \in \mathcal{D}$ with $A \subset E \subset U$. Similarly, there exists $F \in \mathcal{D}$ such that $B \subset F \subset V$, and $D = A \cup B \subset E \cup F$. Hence (1) and (2) yield

$$
\varphi(D) \leq \varphi(E \cup F) \leq \varphi(E) + \varphi(F) \leq \mu(U) + \mu(V).
$$

Taking the supremum over all $D \in \mathcal{D}$ with $D \subset U \cup V$ gives (b).

(c) For all $n \in \mathbb{N}$ and $U_n \subset \mathcal{F}$ open, $\mu\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu(U_n)$.

Proof: Let $D \in \mathcal{D}$ with $D \subset \bigcup_{n=1}^{\infty} U_n$. Then by compactness of $D$ there exists $p \in \mathbb{N}$ such that $D \subset \bigcup_{n=1}^{p} U_n$. Applying (b) inductively, we conclude that

$$
\varphi(D) \leq \mu\left(\bigcup_{n=1}^{p} U_n\right) \leq \sum_{n=1}^{p} \mu(U_n) \leq \sum_{n=1}^{\infty} \mu(U_n).
$$

Since $D \in \mathcal{D}$ with $D \subset \bigcup_{n=1}^{\infty} U_n$ is arbitrary, we obtain (c).

(d) \(\eta\) is an outer measure.

Proof: As seen before, $\varphi(\emptyset) = 0$, and monotonicity of $\eta$ is a consequence of (1)–(3) and (4.6). In order to prove $\sigma$-subadditivity, let $\varepsilon > 0$ and $M_n \subset \mathcal{F}$ with $\eta(M_n) < \infty$ for all $n \in \mathbb{N}$. In view of (4.6), for every $n \in \mathbb{N}$ there exists an open set $U_n \supset M_n$ with $\mu(U_n) \leq \eta(M_n) + 2^{-n} \varepsilon$. Making use of (4.6) and applying (c) yields

$$
\eta\left(\bigcup_{n=1}^{\infty} M_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu(U_n) \leq \sum_{n=1}^{\infty} \eta(M_n) + \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, we obtain (d).

(e) Each closed set $A \subset \mathcal{F}$ is $\eta$-measurable.

Proof: By definition of measurability (cf. [10] Definition II.4.2), we need to show that, for all $Q \subset \mathcal{F}$,

$$
\eta(Q) \geq \eta(Q \cap A) + \eta(Q \cap A^c) \quad \text{(4.10)}.
$$

Without loss of generality we may assume that $\eta(Q) < \infty$. We first prove (4.10) in the case that $Q = U \subset \mathcal{F}$ is open. To this end, let $\varepsilon > 0$ arbitrary. Given $A \subset \mathcal{F}$ closed, the set $U \cap A^c$ is open and $\mu(U \cap A^c) = \eta(U \cap A^c) < \infty$. In view of (4.6), there exists $D \in \mathcal{D}$ with $\varphi(D) \geq \mu(U \cap A^c) - \varepsilon$. Next, since $U \cap D^c$ is open, we may choose $E \in \mathcal{D}$ with $E \subset U \cap D^c$ and $\varphi(E) \geq \mu(U \cap D^c) - \varepsilon$. Since $D, E$ are disjoint and $D \cup E \subset U$, from (1), (3), (4.6) and the fact that $U \cap D^c \supset U \cap A$ we conclude that

$$
\mu(U) \geq \varphi(D \cup E) = \varphi(D) + \varphi(E) \\
\geq \mu(U \cap A^c) + \mu(U \cap D^c) - 2\varepsilon \\
\geq \eta(U \cap A) + \mu(U \cap D^c) - 2\varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, we obtain (4.10) for $Q = U$ open.

Given arbitrary $Q \subset \mathcal{F}$ with $\eta(Q) < \infty$, for $\varepsilon > 0$ arbitrary we choose $U \supset Q$ open with $\eta(Q) \geq \eta(U) - \varepsilon$ according to (4.6). Then the latter inequality yields

$$
\eta(Q) \geq \eta(U) - \varepsilon \geq \eta(U \cap A) + \eta(U \cap A^c) - \varepsilon \\
\geq \eta(Q \cap A) + \eta(Q \cap A^c) - \varepsilon,
$$

proving (4.10).
As a consequence, the set function $\eta$ is an outer measure according to (d), and each closed set $A \subset \mathcal{F}$ is $\eta$-measurable in view of (e). This yields (4.9), which completes the proof. \hfill \Box

The next result shows that the measure $\rho$ given by (4.7) is regular [10].

**Lemma 4.4.** Let $\rho: \mathcal{B}(\mathcal{F}) \to [0, \infty]$ be the measure defined by (4.7). Then every open subset of $\mathcal{F}$ is inner regular. Moreover, the measure $\rho$ is regular.

**Proof.** Let us first prove that every open subset of $\mathcal{F}$ is inner regular in view of (4.6). Namely, considering arbitrary $U \in \mathcal{O}(\mathcal{F})$ and $K \in \mathcal{R}(U)$, according to (4.6) and (4.8) we obtain

$$\rho(K) \leq \rho(U) = \sup \{ \varphi(D) : D \in \mathcal{D}, D \subseteq U \} = \sup \{ \rho(D) : D \in \mathcal{D}, D \subseteq U \} \leq \sup \{ \rho(K) : K \subseteq U \text{ compact} \}.$$ 

Taking the supremum on the left hand side yields

$$\rho(U) = \sup \{ \rho(K) : K \subseteq U \text{ compact} \} \quad \text{for all } U \in \mathcal{O}(\mathcal{F}).$$

From this we conclude that every open set $U \in \mathcal{O}(\mathcal{F})$ is inner regular (in the sense of [10] Definition VIII.1.1).

In view of (4.6), we are given $\rho(U) = \mu(U)$ for any $U \in \mathcal{O}(\mathcal{F})$, and the above considerations show that every open set is inner regular. From this we conclude that the measure $\rho$ is regular in the sense of [10] Definition VIII.1.1. \hfill \Box

As a matter of fact, in general it seems possible the regular measure $\rho$ obtained in Theorem 4.3 to be zero. Nevertheless, the following remark gives a sufficient condition for the measure $\rho$ defined by (4.7) to be non-zero.

**Remark 4.5.** Let $(\mathcal{F}^{(n)})_{n \in \mathbb{N}}$ be a finite-dimensional approximation of $\mathcal{F}$ (see §3.3). By construction of $\rho^{(n)}$ we are given $\text{supp} \, \rho^{(n)} \subseteq \mathcal{F}^{(n)}$ for every $n \in \mathbb{N}$. Assuming that the Lagrangian is bounded and of bounded range, for every $x \in \mathcal{F}$ and $\delta > 0$ there exists $B_x \subseteq \mathcal{F}$ bounded and closed such that $\mathcal{L} (\tilde{x}, y) = 0$ for all $\tilde{x} \in B_\delta(x)$ and all $y \notin B_x$. Furthermore, in view of boundedness of the Lagrangian we introduce the upper bound $\mathcal{C} < \infty$ by

$$\mathcal{C} := \sup_{x,y \in \mathcal{F}} \mathcal{L}(x, y) > 0.$$ 

Thus for any $n \in \mathbb{N}$ we deduce that $\mathcal{L}^{(n)}(\tilde{x}, y) = 0$ for all $\tilde{x} \in B_\delta^{(n)}(x) := B_\delta(x) \cap \mathcal{F}^{(n)}$ and all $y \notin B_x^{(n)} := B_x \cap \mathcal{F}^{(n)}$. Hence the EL equations (3.6) and (3.7) imply that

$$1 \leq \int_{\mathcal{F}^{(n)}} \mathcal{L}^{(n)}(\tilde{x}, y) \, d\rho^{(n)} = \int_{B_x^{(n)}} \mathcal{L}^{(n)}(\tilde{x}, y) \, d\rho^{(n)} \leq \sup_{y \in B_x^{(n)}} \mathcal{L}^{(n)}(\tilde{x}, y) \, \rho^{(n)}(B_x^{(n)})$$

for every $n \in \mathbb{N}$. Thus positivity (3.1) yields

$$\rho^{(n)}(B_x^{(n)}) \geq \mathcal{C}^{-1} > 0 \quad \text{for all } n \in \mathbb{N}.$$ 

For each $n \in \mathbb{N}$ and arbitrary $\varepsilon > 0$, by regularity of $\rho^{(n)}$ there exists $D_n \in \mathcal{D}$ such that $\rho^{(n)}(D_n) > \mathcal{C}^{-1} - \varepsilon$. Moreover, $\hat{D}_N := \bigcup_{n=1}^N D_n \in \mathcal{D}$ for every $N \in \mathbb{N}$. Whenever there exists $N \in \mathbb{N}$ such that $\rho^{(n)}(\hat{D}_N) \geq c$ for almost all $n \in \mathbb{N}$ and some $c > 0$, then the measure $\rho$ defined by (4.7) is non-zero. If this holds true for an infinite number of disjoints sets $(\hat{D}_N)_{n \in \mathbb{N}}$, the measure $\rho$ possibly has infinite total volume.
Next, in agreement with [32, Theorem 16.7] and the remark thereafter, it is not clear whether \( \rho \) as given by (4.7) is a locally finite measure. Nevertheless, the following results provide sufficient conditions for \( \rho \) as obtained in (4.7) to be locally finite.

**Lemma 4.6.** Let \( \rho : \mathcal{B}(\mathcal{F}) \to [0, \infty] \) be defined by (4.7). Assuming that \( \rho(K) < \infty \) for all \( K \in \mathcal{R}(\mathcal{F}) \), then the measure \( \rho \) is locally finite and thus a Borel measure in the sense of [28]. In this case, \( \rho \) is regular and moderate.

**Proof.** We point out that the space \( \mathcal{F} \) is first-countable. Thus under the assumption that \( \rho(K) < \infty \) for all \( K \in \mathcal{R}(\mathcal{F}) \), the statement that \( \rho \) is locally finite is a consequence of Lemma 4.4 and [10, Folgerungen VIII.1.2 (d)]. The last statement follows from Ulam’s theorem [10, Theorem VIII.1.16].

**Remark 4.7.** We point out that, if \( \mathcal{F} \) in Definition 3.4 were locally compact, then the measure \( \rho \) constructed in the proof of Theorem 4.3 would be locally finite, i.e. a Borel measure in the sense of [28]. Namely, whenever \( x \in \mathcal{F} \), there exists a compact neighborhood \( V \) of \( x \). Hence Lemma 4.7 implies that \( \rho^{(n)}(V) \leq C_V \) for all \( n \in \mathbb{N} \) and some \( C_V > 0 \). Choosing \( U_x \subset V \) open with \( x \in U \), we conclude that

\[
\rho(U_x) = \sup \{ \varphi(D) : D \subset U_x, D \in \mathcal{D} \} \leq \sup \left\{ \lim_{n \to \infty} \rho^{(n)}(D) : D \subset V, D \in \mathcal{D} \right\} \leq C_V
\]

as desired.

In the remainder of this subsection, we discuss the properties (iii) and (iv) in (2.2). Neither condition (iii) nor condition (iv) do hold in general, but the following result establishes a connection between condition (iv) in (2.2) and locally finite measures.

**Lemma 4.8.** Assume that the Lagrangian \( \mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^{+} \) is lower semi-continuous, symmetric and strictly positive on the diagonal (3.1), and let \( \rho \) be a measure on \( \mathcal{B}(\mathcal{F}) \). Under the assumption that condition (iv) in (2.2) holds (see (2.2)), i.e.

\[
\sup_{x \in \mathcal{F}} \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) < \infty,
\]

the measure \( \rho \) is locally finite (i.e. condition (iii) in (2.2) is satisfied).

**Proof.** Assume conversely that there exists \( x \in \mathcal{F} \) such that \( \rho(U) = \infty \) for any open neighborhood \( U \) of \( x \). Then \( \mathcal{L}(x, x) > 0 \) due to strict positivity on the diagonal (3.1). Moreover, by lower semi-continuity of the Lagrangian there exists an open neighborhood \( U_x \) of \( x \) such that \( \mathcal{L}(x, y) > \mathcal{L}(x, x)/2 > 0 \) for all \( y \in U_x \). Consequently,

\[
\int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) \geq \int_{U_x} \mathcal{L}(x, y) \, d\rho(y) > \mathcal{L}(x, x)/2 \rho(U_x) = \infty
\]

in contradiction to condition (iv) in (2.2).

4.3. **Convergence on Relatively Compact Subsets.** In Section 5 below our goal is to show that, under suitable assumptions, the measure \( \rho \) as given by (4.7) is a minimizer under variations of finite volume (see Definition 3.5). To this end, we provide some useful tools which shall be worked out in the remainder of this section. For clarity, we point out that for every \( D \in \mathcal{D} \) there exists some \( n' \in \mathbb{N} \) such that \( D'^{\circ} \neq \emptyset \) in the relative topology of \( \mathcal{F}^{(n)} \) for all \( n \leq n' \) and \( D^{\circ} = \emptyset \) in the relative topology of \( \mathcal{F}^{(n')} \) for all \( n > n' \). Considering the restriction \( \rho|_{D^{\circ}} \) shall always be understood in the sense of a restriction to the interior of \( D \) in the relative topology of \( \mathcal{F}^{(n')} \). In order to
derive weak convergence on suitable relatively compact sets (see Lemma 4.11 below), we require some properties of the measures $\rho_D$ with $D \in \mathcal{D}$ as obtained in (4.4).

Given $D, E \in \mathcal{D}$ with $D \subset E$ and denoting the interior of $D$ by $D^\circ$, we claim that

$$\int_D f \, d\rho_D = \int_E f \, d\rho_E \quad \text{for all } f \in C_c(D^\circ).$$

(4.11)

Namely, in view of $C_c(D^\circ) \subset C_b(D) \cap C_b(E)$, weak convergence (4.4) yields

$$\int_D f \, d\rho_D = \lim_{k \to \infty} \int_D f \, d\rho^{(k)}|_D \overset{(*)}{=} \lim_{k \to \infty} \int_E f \, d\rho^{(k)}|_E = \int_E f \, d\rho_E$$

for all $f \in C_c(D^\circ)$, where in $(*)$ we made use of the fact that $\text{supp } f \subset D^\circ \subset E$.

This allows us to prove the following result.

**Proposition 4.9.** Whenever $D \in \mathcal{D}$ and $\Omega \subset D^\circ$ open, then

$$\sup \left\{ \rho_D|_{D^\circ} (\hat{D}) : \hat{D} \subset \Omega, \hat{D} \in \mathcal{D} \right\} = \sup \left\{ \rho_D|_{D^\circ} (\hat{D}) : \hat{D} \subset \Omega, \hat{D} \in \mathcal{D} \right\}.$$  

(4.12)

**Proof.** In order to prove (4.12), we need to show that for all $E, F \in \mathcal{D}$ with $E, F \subset \Omega$ there exist sets $\tilde{E}, \tilde{F} \in \mathcal{D}$ with $\tilde{E}, \tilde{F} \subset \Omega$ such that

$$\rho_E|_{D^\circ}(E) \leq \rho_D|_{D^\circ}(\tilde{E}) \quad \text{and} \quad \rho_D|_{D^\circ}(F) \leq \rho_{\tilde{F}}|_{D^\circ}(\tilde{F}).$$

Whenever $E \subset \Omega$ compact, there exists $\eta \in C_c(\Omega; [0,1])$ with $\eta|_E \equiv 1$ and $\tilde{E} \in \mathcal{D}$ with $\text{supp } \eta \subset \tilde{E}^\circ \subset \Omega$, and by weak convergence (4.4) we are given

$$\rho_E|_{D^\circ}(E) = \int_E d\rho_E = \lim_{n \to \infty} \int_{\mathcal{D}} d\rho^{(n)}|_E \leq \lim_{n \to \infty} \int_{\mathcal{D}} \eta \, d\rho^{(n)}|_D = \int_{\mathcal{D}} \eta \, d\rho_D \leq \rho_D|_{D^\circ}(\tilde{E}).$$

On the other hand, whenever $F \in \mathcal{D}$ with $F \subset \Omega$, there exists $\eta \in C_c(\Omega; [0,1])$ with $\eta|_F \equiv 1$ as well as $\tilde{F} \in \mathcal{D}$ with $\text{supp } \eta \subset \tilde{F}^\circ \subset \Omega$. Thus by (4.11) we obtain

$$\rho_D|_{D^\circ}(F) = \int_F d\rho_D \leq \int_D \eta \, d\rho_D = \int_F \eta \, d\rho_{\tilde{F}} \leq \rho_{\tilde{F}}|_{D^\circ}(\tilde{F}),$$

which completes the proof.

**Lemma 4.10.** For every $D \in \mathcal{D}$, the measures $\rho|_{D^\circ}$ and $\rho_D|_{D^\circ}$ coincide. Moreover,

$$\rho^{(n)}|_{D^\circ} \rightharpoonup \rho|_{D^\circ} \quad \text{vaguely.}$$

(4.13)

**Proof.** According to (4.4) and (4.5), the measure $\rho_D|_{D^\circ} : \mathcal{B}(D^\circ) \to [0, \infty]$ is finite for any $D \in \mathcal{D}$. As a consequence, it is locally finite and thus Borel in the sense of [28]. Moreover, since open subsets of Polish spaces are Polish (see [1 §26]), it is regular in view of Ulam’s theorem [10 Satz VIII.1.16] (alternatively, regularity follows by [7 Corollary 7.1.9] and the fact that each metrizable space is perfectly normal [1 Corollary 3.21]). Similar arguments yield regularity of $\rho|_{D^\circ}$ for any $D \in \mathcal{D}$. Thus for any $D \in \mathcal{D}$, we may approximate arbitrary Borel sets $A \in \mathcal{B}(D^\circ)$ by compact sets from inside,

$$\rho_D|_{D^\circ}(A) = \sup \{ \rho_D|_{D^\circ}(K) : K \subset A \text{ compact} \},$$

$$\rho|_{D^\circ}(A) = \sup \{ \rho|_{D^\circ}(K) : K \subset A \text{ compact} \}.$$
Whenever $D \in \mathcal{D}$ and $\Omega \subset D$ open, for each $K \subset \Omega$ compact there exists $\tilde{D} \in \mathcal{D}$ such that $K \subset \tilde{D} \subset \Omega$ by construction of $\mathcal{D}$. From this we conclude that

$$
\rho_D|_{D^\circ}(\Omega) = \sup \{ \rho_D|_{D^\circ}(K) : K \subset \Omega \text{ compact} \} = \sup \{ \rho_D|_{D^\circ}(\tilde{D}) : \tilde{D} \subset \Omega, \tilde{D} \in \mathcal{D} \},
$$

$$
\rho|_{D^\circ}(\Omega) = \sup \{ \rho|_{D^\circ}(K) : K \subset \Omega \text{ compact} \} = \sup \{ \rho|_{D^\circ}(\tilde{D}) : \tilde{D} \subset \Omega, \tilde{D} \in \mathcal{D} \}.
$$

Moreover, for any $\tilde{D} \in \mathcal{D}$, by (4.4) and (4.8) we obtain

$$
\varphi(\tilde{D}) = \lim_{k \to \infty} \rho^{(k)}(\tilde{D}) = \lim_{k \to \infty} \rho^{(k)}|_{\tilde{D}}(\tilde{D}) = \rho_{\tilde{D}}(\tilde{D}).
$$

Given $D \in \mathcal{D}$ and $\Omega \subset D^\circ$ open, we conclude that $\rho|_{D^\circ}$ as well as $\rho_D|_{D^\circ}$ are regular finite Borel measures on $\mathcal{B}(D^\circ)$, implying that

$$
\rho|_{D^\circ}(\Omega) = \sup \{ \varphi(\tilde{D}) : \tilde{D} \subset \Omega, \tilde{D} \in \mathcal{D} \} = \sup \{ \rho_{\tilde{D}}(\tilde{D}) : \tilde{D} \subset \Omega, \tilde{D} \in \mathcal{D} \}
$$

$$
= \sup \{ \rho_D|_{D^\circ}(\tilde{D}) : \tilde{D} \subset \Omega, \tilde{D} \in \mathcal{D} \} = \sup \{ \rho_D|_{D^\circ}(\tilde{D}) : \tilde{D} \subset \Omega, \tilde{D} \in \mathcal{D} \}
$$

$$
= \rho_D|_{D^\circ}(\Omega).
$$

As a consequence, the measures $\rho|_{D^\circ}$ and $\rho_D|_{D^\circ}$ coincide on all open sets $\Omega \subset D^\circ$. Making use of [7 Lemma 7.1.2], we conclude that $\rho|_{D^\circ}$ and $\rho_D|_{D^\circ}$ already coincide on all Borel sets, i.e.

$$
\rho|_{D^\circ} = \rho_D|_{D^\circ} \quad \text{for all } D \in \mathcal{D}.
$$

Given $f \in C_c(D^\circ)$, we thus obtain

$$
\lim_{n \to \infty} \int_{\mathcal{F}} f \, d\rho^{(n)}|_{D^\circ} = \lim_{n \to \infty} \int_{\mathcal{F}} f \, d\rho^{(n)}|_D = \int_{\mathcal{F}} f \, d\rho_D = \int_{\mathcal{F}} f \, d\rho_D|_{D^\circ}.
$$

Since $f \in C_c(D^\circ)$ was arbitrary, we obtain vague convergence

$$
\rho^{(n)}|_{D^\circ} \overset{v}{\to} \rho|_{D^\circ}.
$$

This completes the proof.

Having proved vague convergence on open subsets of $D \in \mathcal{D}$, the following result even yields weak convergence on suitable relatively compact subsets $V \subset \mathcal{F}$ (so-called continuity sets, cf. [7 Section 8.2]).

**Lemma 4.11.** For every $D \in \mathcal{D}$ there exists $E \in \mathcal{D}$ with $D \subset E^\circ$ as well as a relatively compact, open subset $V \subset E^\circ$ with $D \subset V$ such that

$$
\rho^{(n)}|_V \rightharpoonup \rho|_V \quad \text{weakly}.
$$

Similarly, whenever $D \in \mathcal{D}^{(n)}$ and $U \supset D$ open, there exists a relatively compact, open subset $V \subset \mathcal{F}^{(n)}$ with $D \subset V \subset U$ such that (4.16) holds.

**Proof.** Given $D \in \mathcal{D}$, by construction of $\mathcal{D}$ we know that $D$ is compact and thus contained in the interior of some $E \in \mathcal{E}$ with $\rho(E) < \infty$ in view of (4.5) and (4.8). Therefore, $\rho_E : \mathcal{B}(E) \to [0, \infty)$ is a nonnegative finite Borel measure, where $\mathcal{B}(E)$ denotes the Borel $\sigma$-algebra on the topological space $E$. Since $E$ is metrizable, it is completely regular, implying that the class $\Gamma_E$ of all Borel sets in $E$ with boundaries of $\rho_E$-measure zero contains a base (consisting of open sets) of the topology of $E$ (for details see [7 Proposition 8.2.8]). Since $E$ is compact, the set $D \subset E$ can be covered by finitely many relatively compact, open sets $V_1, \ldots, V_N \subset E^\circ$ in $\Gamma_E$. By
construction, the closure of the set \( V := \bigcup_{i=1}^{N} V_i \subset E^0 \) is compact. Considering the restriction \( \rho^{(n)}|_{V} \), for any \( f \in C_b(V) \) we obtain
\[
\lim_{n \to \infty} \int_{V} f \, d\rho^{(n)}|_{V} = \lim_{n \to \infty} \int_{\Gamma} f \, d\rho^{(n)}|_{E^0} = \int_{\Gamma} f \, d\rho|_{E^0} = \int_{V} f \, d\rho|_{V} ,
\]
proving vague convergence
\[
\rho^{(n)}|_{V} \rightharpoonup \rho|_{V} . \tag{4.17}
\]
Since \( \Gamma_{\rho|_{E}} \) is a subalgebra in \( \mathcal{B}(E) \) (see [7, Proposition 8.2.8]), the set \( V \) is also contained in \( \Gamma_{\rho|_{E}} \), implying that \( \rho(\partial V) = \rho|_{E}(\partial V) = 0 \). Note that the measures \( (\rho^{(n)}|_{E^0}) \) as well as \( \rho|_{E^0} \) are regular Borel measures and thus Radon [38]. Therefore, making use of vague convergence (4.17) and applying [31, Theorem 30.2], for the relatively compact, open set \( V \subset E^0 \) and the compact set \( \hat{V} \subset E^0 \) we obtain
\[
\rho(V) = \rho(\hat{V}) \geq \limsup_{n \to \infty} \rho^{(n)}(\hat{V}) \geq \limsup_{n \to \infty} \rho^{(n)}(V) \geq \liminf_{n \to \infty} \rho^{(n)}(V) \geq \rho(V) ,
\]
proving that
\[
\rho(V) = \lim_{n \to \infty} \rho^{(n)}(V) . \tag{4.18}
\]
Let us point out that, for each \( n \in \mathbb{N} \), the measure \( \rho^{(n)}|_{V}/\rho^{(n)}(V) \) is normalized in the sense that \( \rho^{(n)}|_{V}(V)/\rho^{(n)}(V) = 1 \) (cf. [24, §3.2]). Furthermore, applying vague convergence (4.17) as well as (4.18), for any \( f \in C_b(V) \) we are given
\[
\lim_{n \to \infty} \int_{V} f \, d\rho^{(n)}|_{V}/\rho^{(n)}(V) = \int_{V} f \, d\rho|_{V}/\rho(V) .
\]
As a consequence, the sequence of normalized measures \( (\rho^{(n)}|_{V}/\rho^{(n)}(V))_{n \in \mathbb{N}} \) converges vaguely to the normalized measure \( \rho|_{V}/\rho(V) \), and from [31, Corollary 30.9] we deduce that \( (\rho^{(n)}|_{V}/\rho^{(n)}(V))_{n \in \mathbb{N}} \) converges weakly to the normalized measure \( \rho|_{V}/\rho(V) \). Thus in view of
\[
\lim_{n \to \infty} \int_{V} f \, d\rho^{(n)}|_{V} = \lim_{n \to \infty} \rho^{(n)}(V) \int_{V} f \, d\rho^{(n)}|_{V}/\rho^{(n)}(V) = \rho(V) \int_{V} f \, d\rho^{(n)}|_{V}/\rho(V) = \int_{V} f \, d\rho^{(n)}|_{V}
\]
for any \( f \in C_b(V) \), we finally obtain weak convergence \( \rho^{(n)}|_{V} \rightharpoonup \rho|_{V} \). \( \square \)

By contrast to [24], it is not reasonable to consider vague convergence \( \rho^{(n)} \rightharpoonup \rho \) in view of [32, Exercise 14.4].

5. Minimizers for Lagrangians of Bounded Range

5.1. Preliminaries. This section is devoted to the proof that, under suitable assumptions, the measure \( \rho \) as defined in (4.17) is a minimizer of the causal variational principle (4.2) under variations of finite volume (see Definition 3.5). This is accomplished in §5.4. To this end, we proceed in several steps. In this subsection, we introduce an additional assumption on the measure \( \rho \) obtained in Theorem 4.3 (see §5.1). Afterwards we prove that \( \rho \) is a minimizer on suitable compact subsets (see §5.2 and §5.3). Assuming that \( \rho \neq 0 \) is locally finite, we finally show that \( \rho \) satisfies corresponding Euler-Lagrange (EL) equations (see §5.5), which have the same structure as the
EL equations obtained in [24]. Throughout this section, we shall assume that the Lagrangian is of bounded range (see Definition 3.7).

In order to prove that \( \rho \) is a minimizer, we impose the following condition:

(B) For any \( \varepsilon > 0 \) and \( B \subset \mathcal{F} \) bounded, there exists \( N \in \mathbb{N} \) such that

\[
\rho(B \setminus B^{(n)}) < \varepsilon \quad \text{for all } n \geq N ,
\]

where \( B^{(n)} := B \cap \mathcal{F}^{(n)} \).

Lemma 5.1. Assume that the measure \( \rho \) defined by (4.7) satisfies condition (B). Then the measure \( \rho \) is locally finite, and any bounded subset of \( \mathcal{F} \) has finite \( \rho \)-measure.

Proof. Assuming that \( B \subset \mathcal{F} \) is bounded, in view of condition (B) there exists \( N \in \mathbb{N} \) such that \( \rho(B \setminus B^{(n)}) < \varepsilon \) for all \( n \geq N \), where \( B^{(n)} := B \cap \mathcal{F}^{(n)} \) is relatively compact in view of [2 Bemerkungen 2.9]. For this reason, \( B^{(n)} \) can be covered by a finite number of compact sets \( D_1, \ldots, D_L \) with \( D_i \in \mathcal{D}^{(n)} \) for all \( i = 1, \ldots, L \), where \( \mathcal{D}^{(n)} \) is given by (4.2). From (4.5) we obtain \( \rho(B^{(n)}) < \infty \), implying that

\[
\rho(B) = \rho(B \setminus B^{(n)}) + \rho(B^{(n)}) < \infty .
\]

Since each compact set \( K \subset \mathcal{F} \) is bounded, the measure \( \rho \) is locally finite in view of Lemma 4.6. \( \square \)

5.2. Minimizers under Variations of Finite-Dimensional Compact Support.

Before proving our first existence result, we point out that the restricted Lagrangian

\[
\mathcal{L}^{(n)} = \mathcal{L}|_{\mathcal{F}^{(n)} \times \mathcal{F}^{(n)}} : \mathcal{F}^{(n)} \times \mathcal{F}^{(n)} \to \mathbb{R}_0^+
\]

is of compact range (see Definition 2.1 and 3.3) for every \( n \in \mathbb{N} \). Therefore, for all \( j, k, n \in \mathbb{N} \), there exist compact subsets \( (V_{j,k}^{(n)})' \subset \mathcal{I}^{(n)} \) such that \( \mathcal{L}(x, y) = 0 \) for all \( x \in V_{j,k}^{(n)} \) and \( y \in \mathcal{I}^{(n)} \setminus (V_{j,k}^{(n)})' \). By construction of \( \mathcal{D} \) (see §4.1), the set \( (V_{j,k}^{(n)})' \) can be covered by a finite number of sets \( (V_{j',k'}^{(n)})_{i=1,\ldots,L} \in \mathcal{D} \), whose union is also contained in \( \mathcal{D} \). For this reason, we may assume that \( (V_{j,k}^{(n)})' \in \mathcal{D} \) for all \( j, k, n \in \mathbb{N} \).

After these preparations, we are now in the position to state our first existence result.

Proposition 5.2. Assume that the Lagrangian \( \mathcal{L} \in C_b(\mathcal{F} \times \mathcal{F} ; \mathbb{R}_0^+) \) is of bounded range, and that condition (3.8) holds. Moreover, assume that the measure \( \rho \) defined by (4.7) satisfies condition (B) in Section 5. Then \( \rho \) is a minimizer under variations in \( \mathcal{D} \) in the sense that

\[
(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) \geq 0
\]

whenever \( \tilde{\rho} \) satisfying (2.3) is a regular measure on \( \mathcal{B}(\mathcal{F}) \) with \( \text{supp} (\tilde{\rho} - \rho) \in \mathcal{D} \).

Proof. Assume that \( \tilde{\rho} : \mathcal{B}(\mathcal{F}) \to [0, \infty] \) is a measure on the Borel \( \sigma \)-algebra of \( \mathcal{F} \) with \( D := \text{supp} (\tilde{\rho} - \rho) \in \mathcal{D} \) such that (2.3) is satisfied, i.e.

\[
0 < \tilde{\rho}(D) = \rho(D) < \infty .
\]

Since \( D \in \mathcal{D} \) is compact, the difference (2.4) is well-defined in view of [24 §4.3],

\[
\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho) = \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) + \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y)
\]

\[
+ \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) .
\]
Making use of the symmetry of the Lagrangian and applying Fubini’s theorem, we can write this expression as

\[
S(\tilde{\rho}) - S(\rho) = 2 \int_V d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) \\
+ 2 \int_V d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F} \setminus V'} d\rho(y) \mathcal{L}(x, y) + \int_V d(\tilde{\rho} - \rho)(x) \int_V d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y),
\]

where the expression

\[
\int_V d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F} \setminus V'} d\rho(y) \mathcal{L}(x, y) = \int_D d(\tilde{\rho} - \rho)(x) \int_{W \setminus W^{(n')}} d\rho(y) \mathcal{L}(x, y)
\]

can be chosen arbitrarily small for sufficiently large \(n' \in \mathbb{N}\). Thus for \(\varepsilon > 0\) arbitrary, we may arrange that

\[
S(\tilde{\rho}) - S(\rho) \geq 2 \int_V d(\tilde{\rho} - \rho)(x) \int_{V'} d\rho(y) \mathcal{L}(x, y) \\
+ \int_V d(\tilde{\rho} - \rho)(x) \int_V d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) - \varepsilon.
\]

Next, in analogy to the proof of [24, Theorem 4.9], for any \(n \in \mathbb{N}\) we introduce the measures \(\tilde{\rho}_n : \mathcal{B}(\mathcal{F}) \rightarrow [0, \infty]\) by

\[
\tilde{\rho}_n := \begin{cases} 
\frac{c_n \tilde{\rho}}{\rho^{(n)}} & \text{on } V' \\
\frac{c_n \tilde{\rho}}{\rho^{(n)}} & \text{on } \mathcal{F} \setminus V 
\end{cases}
\]

with \(c_n := \frac{\rho^{(n)}(V)}{\tilde{\rho}(V)}\) for all \(n \in \mathbb{N}\).

Since \(\rho\) and \(\tilde{\rho}\) coincide on \(\mathcal{F} \setminus D\), by virtue of [11, Lemma 4.11] (see Lemma 4.11) we obtain

\[
\lim_{n \to \infty} c_n = \lim_{n \to \infty} \frac{\rho^{(n)}(V)}{\tilde{\rho}(V)} = \frac{\rho(V)}{\tilde{\rho}(V)} = \frac{\rho(V \setminus D) + \rho(D)}{\tilde{\rho}(V \setminus D) + \tilde{\rho}(D)} = 1.
\]

Making use of the fact that \(V \subset \mathcal{F}\) is separable (see for instance [11, Corollary 3.5]) and applying [6, Theorem 2.8] in a similar fashion to the proof of [24, Theorem 4.9],
we thus arrive at
\[
S(\tilde{\rho}) - S(\rho) \geq \lim_{n \to \infty} \left[ 2 \int_V d(c_n \tilde{\rho} - \rho^{(n)})(x) \int_{V'} d\rho^{(n)}(y) \mathcal{L}(x, y) + \int_V d(c_n \tilde{\rho} - \rho^{(n)})(x) \int_{\tilde{V}} d\rho^{(n)}(y) \mathcal{L}(x, y) \right] - 2\varepsilon.
\]

In view of the fact that \(\tilde{\rho}_n\) and \(\rho^{(n)}\) coincide on \(\mathcal{F}^{(n)} \setminus V\) for sufficiently large \(n \in \mathbb{N}\) and \(\mathcal{L}(x, y) = 0\) for all \(x \in V\) and \(y \notin V'\), the difference \(S(\tilde{\rho}) - S(\rho)\) can finally be estimated by
\[
S(\tilde{\rho}) - S(\rho) \geq \lim_{n \to \infty} \left[ 2 \int_{\mathcal{F}^{(n)}} d(\tilde{\rho}_n - \rho^{(n)})(x) \int_{\mathcal{F}^{(n)}} d\rho^{(n)}(y) \mathcal{L}(x, y) + \int_{\mathcal{F}^{(n)}} d(\tilde{\rho}_n - \rho^{(n)})(x) \int_{2 \mathcal{F}^{(n)}} d(\tilde{\rho}_n - \rho^{(n)})(y) \mathcal{L}(x, y) \right] - 2\varepsilon.
\]

Since \(\rho^{(n)}\) is a minimizer on \(\mathcal{F}^{(n)}\) for every \(n \in \mathbb{N}\) (see (3.3) and (4.2), we are given
\[
(\mathcal{S}_{\mathcal{F}^{(n)}}(\tilde{\rho}_n) - \mathcal{S}_{\mathcal{F}^{(n)}}(\rho^{(n)})) \geq 0 \quad \text{for all } n \in \mathbb{N}.
\]

Taking the limit \(n \to \infty\) on the left hand side of (5.2), one obtains exactly the above expression in square brackets for \(S(\tilde{\rho}) - S(\rho)\). Since \(\varepsilon > 0\) is arbitrary, this implies that
\[
(S(\tilde{\rho}) - S(\rho)) \geq 0.
\]

Hence \(\rho\) is a minimizer under variations in \(\mathcal{D}\). \(\Box\)

Our next goal is to extend the previous result to minimizers under variations of finite-dimensional compact support, which is defined as follows.

**Definition 5.3.** A regular measure \(\rho\) on \(\mathcal{B}(\mathcal{F})\) is said to be a **minimizer under variations of finite-dimensional compact support** if the inequality
\[
(S(\tilde{\rho}) - S(\rho)) \geq 0
\]
holds for any regular measure \(\tilde{\rho}\) on \(\mathcal{B}(\mathcal{F})\) satisfying (2.3) with the following property: There exists \(n' \in \mathbb{N}\) such that, for all \(n \geq n'\),
\[
\text{supp} \,(\tilde{\rho} - \rho) \subset \mathcal{F}^{(n)} \text{ compact and supp} \,(\tilde{\rho} - \rho) \cap \mathcal{F} \setminus \mathcal{F}^{(n)} = \emptyset.
\]

Based on this definition, we may state the following existence result.

**Proposition 5.4.** Assume that \(\mathcal{L} \in C_b(\mathcal{F} \times \mathcal{F}; \mathbb{R}_0^+)\) is of bounded range, and assume that condition (3.8) holds. Furthermore, assume that the measure \(\rho\) defined by (4.7) satisfies condition (B) in Section 3. Then \(\rho\) is a minimizer under variations of finite-dimensional compact support.

**Proof.** Assuming that \(\tilde{\rho}\) is a variation of finite-dimensional compact support, there exists \(n' \in \mathbb{N}\) such that
\[
\text{supp} \,(\tilde{\rho} - \rho) \subset \mathcal{F}^{(n)}
\]
is compact for all \(n \geq n'\), implying that \(\tilde{\rho}(K_\varepsilon) = \rho(K_\varepsilon) < \infty\) (in virtue of Lemma 5.1). According to Lemma 4.4, the measure \(\rho\) given by (4.7) is regular, and by regularity of \(\rho\) and \(\tilde{\rho}\), for \(\varepsilon > 0\) arbitrary we may choose \(U \supset K_\varepsilon\) open such that
\[
\rho(U \setminus K_\varepsilon) < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\rho}(U \setminus K_\varepsilon) < \tilde{\varepsilon}.
\]
By construction of $\mathcal{D}$, there is some compact set $D \in \mathcal{D}$ such that $K_\sharp \subset D \subset U^{(n')}$, where $U^{(n')} := U \cap \mathcal{F}^{(n')}$. In particular,

$$\rho(D \setminus K_\sharp) < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\rho}(D \setminus K_\sharp) < \tilde{\varepsilon}.$$ 

Applying Lemma 4.11 yields the existence of some relatively compact set $V \subset \mathcal{F}$ with $D \subset V \subset U^{(n')}$ such that (4.10) holds. Moreover,

$$\rho(V \setminus K_\sharp) < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\rho}(V \setminus K_\sharp) < \tilde{\varepsilon}.$$ 

Since $\tilde{\rho} - \rho$ is a signed measure of finite total variation and compact support, the difference (2.4) is well-defined (cf. [24, §4.3]),

$$(S(\tilde{\rho}) - S(\rho)) = 2 \int_{K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x,y) + \int_{K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{K_\sharp} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x,y).$$

By adding and subtracting the terms

$$2 \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x,y) + \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{K_\sharp} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x,y)$$

as well as

$$\int_{V} d(\tilde{\rho} - \rho)(x) \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x,y),$$

one easily verifies that

$$(S(\tilde{\rho}) - S(\rho)) = 2 \int_{V} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x,y) + \int_{V} d(\tilde{\rho} - \rho)(x) \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x,y) + 2 \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x,y) + \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{K_\sharp} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x,y).$$

Since the Lagrangian $\mathcal{L}$ is of bounded range (see Definition 3.7), its restriction $\mathcal{L}^{(n')}$ is of compact range, implying that $\mathcal{L}(x,y) = 0$ for all $x \in V$ and $y \notin V'$ for some relatively compact, open set $V' \subset \mathcal{F}^{(n')}$ such that (4.10) holds. Choosing the open set $U \supset K_\sharp$ suitably, one thus can arrange that

$$\left| \int_{V \setminus K_\sharp} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x,y) \right| \leq \sup_{x,y \in V'} \mathcal{L}(x,y) \rho(V') \left( |\tilde{\rho}(V \setminus K_\sharp)| + |\rho(V \setminus K_\sharp)| \right) < \tilde{\varepsilon}$$

is arbitrarily small. Applying similar arguments to all summands of the above term in square brackets, one obtains the estimate

$$(S(\tilde{\rho}) - S(\rho)) \geq \left\{ 2 \int_{V} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x,y) + \int_{V} d(\tilde{\rho} - \rho)(x) \int_{V} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x,y) \right\} - \varepsilon$$

for any given $\varepsilon > 0$. Proceeding in analogy to the proof of Proposition 5.2 by applying weak convergence (4.10) together with [6, Theorem 2.8] (for details we refer to the
Hence, proceeding in analogy to the proof of Proposition 5.4, we conclude that 
\[ \hat{\rho} \] as well as 
\[ \rho > 0 \] was chosen arbitrarily, we finally arrive at 
\[ (S(\hat{\rho}) - S(\rho)) \geq 0 , \]
which proves the claim. \( \Box \)

5.3. Existence of Minimizers under Variations of Compact Support. Having derived the above preparatory results in §5.2, we are now in the position to deal with minimizers under variations of compact support (see Definition 5.6). For clarity, we point out that, whenever \( \hat{\rho} \) is a variation of compact support of the measure \( \rho \) satisfying (2.4), the condition \( |\hat{\rho} - \rho| < \infty \) yields \( \rho(K) = \hat{\rho}(K) < \infty \), where the compact set \( K \subset F \) is defined by \( K := \text{supp}(\hat{\rho} - \rho) \) (for details see Definition 2.1 and the explanations in §2.2).

**Lemma 5.5.** Assume that \( \mathcal{L} \in C_b(F \times F; \mathbb{R}_+^+) \) is of bounded range, and assume that condition (3.8) holds. Moreover, assume that the measure \( \rho \) defined by (1.7) satisfies condition (B) in Section 5. Then \( \rho \) is a minimizer under variations of compact support.

**Proof.** Let \( \hat{\rho} \) be a variation of compact support. Then the set \( K := \text{supp}(\hat{\rho} - \rho) \subset F \) is compact, and \( \rho(K) = \hat{\rho}(K) < \infty \) according to (2.4). Given \( \epsilon > 0 \) arbitrary, by regularity of \( \rho \), \( \hat{\rho} \) there exists \( U \supset K \) open such that \( \rho(U), \hat{\rho}(U) < \infty \) and 
\[ \rho(U \setminus K) < \epsilon \quad \text{and} \quad \hat{\rho}(U \setminus K) < \epsilon . \]

Moreover, in view of (4.6), there exists \( D \in \mathcal{D} \) such that 
\[ \rho(U \setminus D) < \epsilon \quad \text{and} \quad \hat{\rho}(U \setminus D) < \epsilon . \]

Since \( K \subset F \) is compact, the difference (2.4) is well-defined (cf. [24, §4.3]),
\[ (S(\hat{\rho}) - S(\rho)) = 2 \int_K d(\hat{\rho} - \rho)(x) \int_F d\rho(y) \mathcal{L}(x,y) \]
\[ + \int_K d(\hat{\rho} - \rho)(x) \int_K d(\hat{\rho} - \rho)(y) \mathcal{L}(x,y) . \]

Note that, by definition of \( \mathcal{D} \), each \( D \in \mathcal{D} \) is the finite union of finite-dimensional subsets of \( (F^{(n)})_{n \in \mathbb{N}} \) (cf. §4.1). As a consequence, for each \( D \in \mathcal{D} \) there exists \( n' \in \mathbb{N} \) such that \( D \subset F^{(n')} \) for all \( n \geq n' \). Moreover, by construction of \( \mathcal{D} \) there exists \( E \in \mathcal{D} \) such that \( D \subset E^0 \subset U^{(n)} \), where \( U^{(n)} = U \cap F^{(n)} \), and according to Lemma 4.11 there exists a relatively compact, open set \( V \subset E^0 \) such that \( D \subset V \). In particular,
\[ \rho(U \setminus V) < \epsilon , \quad \hat{\rho}(U \setminus V) < \epsilon . \]

Thus adding and subtracting the terms
\[ 2 \int_{U \setminus K} d(\hat{\rho} - \rho)(x) \int_F d\rho(y) \mathcal{L}(x,y) + \int_{U \setminus K} d(\hat{\rho} - \rho)(x) \int_K d(\hat{\rho} - \rho)(y) \mathcal{L}(x,y) \]
as well as
\[ \int_U d(\hat{\rho} - \rho)(x) \int_{U \setminus K} d(\hat{\rho} - \rho)(y) \mathcal{L}(x,y) , \]
and proceeding in analogy to the proof of Proposition 5.4, we conclude that
\[ (S(\hat{\rho}) - S(\rho)) \geq 0 . \]
Hence \( \rho \) is indeed a minimizer under variations of compact support. \( \Box \)
5.4. Existence of Minimizers under Variations of Finite Volume. We now proceed similarly to [24, §4.4] in order to prove the existence of minimizers under variations of finite volume (see Definition 3.5). For the difference of the actions (2.4) to be well-defined, we require the additional property (iv) in §2.2, i.e.

\[
\sup_{x \in F} \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) < \infty.
\]

Then we can state the following result.

**Theorem 5.6.** Assume that \( \mathcal{L} \in C_b(F \times F; \mathbb{R}_+^+) \) is of bounded range, and assume that condition (3.8) holds. Furthermore, assume that the measure \( \rho \) defined by (4.7) satisfies condition (B) in Section 5. Then \( \rho \) is a minimizer under variations of finite volume.

**Proof.** Assume that \( \tilde{\rho} \) be a variation of finite volume satisfying (2.3). Introducing the set \( B := \text{supp}(\tilde{\rho} - \rho) \), we thus obtain \( \rho(B) = \tilde{\rho}(B) < \infty \). Given \( \varepsilon > 0 \) arbitrary, by regularity of \( \rho \), \( \tilde{\rho} \) there exists \( U \supset B \) open such that

\[
\rho(U \setminus B) < \varepsilon \quad \text{and} \quad \tilde{\rho}(U \setminus B) < \varepsilon.
\]

Making use of the additional assumption that condition (iv) in §2.2 is satisfied, the difference of the actions (2.4) is well-defined. Therefore, proceeding in analogy to the proof of Lemma 5.5 finally gives rise to

\[
(S(\tilde{\rho}) - S(\rho)) \geq 0,
\]

which implies that \( \rho \) is a minimizer under variations of finite volume. \( \square \)

The remainder of this section is devoted to the derivation of the corresponding EL equations for minimizers under variations of finite volume (5.5).

5.5. Derivation of the Euler-Lagrange Equations. The strategy in [24] was to derive the EL equations in order to prove the existence of minimizers under variations of finite volume. However, proceeding similar to [24, §4.2] does not seem promising in the infinite-dimensional setting. For this reason, we rather proceed in the opposite direction by first proving the existence of minimizers and then deriving the EL equations. More precisely, under the assumptions that condition (iv) in §2.2 as well as condition (B) in Section 5 are satisfied, Theorem 5.6 shows that the measure \( \rho \) given by (4.7) is a minimizer under variations of finite volume (see Definition 3.5). Under these assumptions, Lemma 4.8 implies that \( \rho \) is locally finite. This allows us to proceed similarly to the proof of [23, Lemma 2.3], thus giving rise to corresponding EL equations. For convenience, let us state the latter result in greater generality.

**Theorem 5.7 (The Euler-Lagrange equations).** Let \( \mathcal{F} \) be topological Hausdorff space, let \( \rho \) be a Borel measure on \( \mathcal{F} \) (in the sense of [28], i.e. a locally finite measure) and assume that \( \mathcal{L} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_+^+ \) is symmetric and lower semi-continuous. If \( \rho \neq 0 \) is a minimizer of the causal variational principle (3.2), (2.3) under variations of finite volume, then the Euler-Lagrange equations

\[
\ell|_{\text{supp} \rho} \equiv \inf_{x \in \mathcal{F}} \ell(x)
\]

hold, where the mapping \( \ell : \mathcal{F} \rightarrow [0, \infty) \) is defined by

\[
\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x, y) \, d\rho(y) - \mathfrak{s}
\]
for some parameter $s \in \mathbb{R}$.

**Proof.** Proceed in analogy to the proof of [23, Lemma 2.3]. \hfill \Box

For clarity, we point out that Theorem 5.7 requires that $\rho$ is locally finite. Choosing the parameter $s$ suitably, one can arrange that the infimum in (5.3) vanishes:

**Lemma 5.8.** Assume that the measure $\rho$ given by (4.7) is non-zero. Then, under the assumptions of Theorem 5.6, for a suitable choice of $s \geq 0$ in (5.4) the Euler-Lagrange equations (5.3) read

$$\ell|_{\text{supp } \rho} = \inf_{x \in \mathcal{F}} \ell(x) = 0.$$ (5.5)

**Proof.** Assuming that $\rho \neq 0$, we conclude that $\text{supp } \rho \neq \emptyset$. Moreover, under the assumptions of Theorem 5.6 by Lemma 5.8 we know that $\rho$ is locally finite. Next, from Theorem 5.6 and Theorem 5.7 we deduce that $\rho$ satisfies the EL equations (5.3). By assumption, condition (iv) in §2.2 is satisfied, implying that

$$0 \leq \int_{\mathcal{F}} L(x, y) \, d\rho(y) < \infty \quad \text{for every } x \in \mathcal{F}.$$  

This allows us to choose $s \geq 0$ such that (5.5) holds. \hfill \Box

Lemma 5.8 generalizes the results of [23, Section 2] and [24, Section 4] to the infinite-dimensional setting. It remains an open task to prove the existence of a Lagrangian of bounded range such that the measure $\rho$ in (4.7) is non-zero and satisfies condition (iv) in §2.2 as well as condition (B) in Section 5.

6. Minimizers for Lagrangians Vanishing in Entropy

In Section 5 the results from [24, Section 4] were generalized to the non-locally compact setting. This raises the question whether it is possible also to weaken the assumption that the Lagrangian is of bounded range (see Definition 3.7) similarly to Lagrangians decaying in entropy as introduced in [24, Section 5]. It is precisely the objective of this section to analyze this question in detail. To this end, we first generalize the notion of Lagrangians decaying in entropy (§6.1). Afterwards we proceed similarly as in [24, Section 5] and Section 5 to prove that, under suitable assumptions, the measure $\rho$ obtained in Theorem 4.3 is a minimizer under variations of compact support and variations of finite volume (see Definition 3.6 and Definition 3.5).

6.1. Lagrangians Vanishing in Entropy. This subsection is devoted to generalize the notion of Lagrangians decaying in entropy as introduced in [24, Section 5]. More precisely, in order for the constructions in [24, Section 5] to work, the definition of Lagrangians decaying in entropy (see Definition 2.8 and [24, Definition 5.1]) requires an unbounded Heine-Borel metric. On the other hand, any Heine-Borel space (that is, a topological space endowed with a Heine-Borel metric) is $\sigma$-compact and locally compact, see [42]. In particular, every separable Heine-Borel space $X$ is a second-countable, locally compact Hausdorff space, and hemicompact (see [41, Problem 17I]) in view of [11, Exercise 3.8.C]. Accordingly, there is a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of $X$ with $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$ such that any compact set $K \subset X$ is contained in $K_n$ for some $n \in \mathbb{N}$ and $X = \bigcup_{n=1}^{\infty} K_n$ (also see [4, Lemma 29.8]). Moreover, in view of [3, Theorem 31.5], the space $X$ is Polish. These considerations motivate the following procedure.
For any second-countable, locally compact Hausdorff topological space $X$, assume that the Lagrangian $\mathcal{L} : X \times X \to \mathbb{R}_0^+$ is continuous, symmetric and positive on the diagonal $\mathcal{L}(x,x) > 0$. Moreover, assume that the measure $\tilde{\rho}$ on $\mathcal{B}(X)$ is obtained similarly to the constructions in [24, §4.1]. In order for the constructions in [24, Section 5] to apply, one requires that, for any $x \in X$ and arbitrary $\varepsilon > 0$, there exists $K_{x,\varepsilon} \subset X$ compact such that

$$\int_{X \setminus K_{x,\varepsilon}} \mathcal{L}(x, y) \, d\tilde{\rho}(y) < \varepsilon.$$ 

The results in [42] imply that for any separable, locally compact metric space $(X,d)$ there is a Heine-Borel metric $d_{HB}$ on $X$ which generates the same topology. In order to also allow for bounded Heine-Borel metrics in [24, Section 5], it seems preferable to not specify the set $K_{x,\varepsilon}$ in [24, eq. (5.2)] and the calculations thereafter in terms of a possibly bounded Heine-Borel metric $d_{HB}$. For this reason, it seems preferable to formulate the calculations in [24, §5.1] purely in terms of compact subsets rather than in terms of (closed) balls (whose diameter depends on the corresponding Heine-Borel metric). To this end, for any topological space $Y$, we denote the set of all functions $f : Y \to \mathbb{R}$ by $\mathcal{F}(Y)$, and let $\mathcal{F}^+(Y)$ be the subset of non-negative such functions. Given a second-countable, locally compact Hausdorff space $X$, by $\mathcal{F}^+_B(X)$ we denote the subset of non-negative functions vanishing at infinity in the sense that, for any $\varepsilon > 0$, there exists $K \subset X$ compact with $f|_{X \setminus K} < \varepsilon$ (for continuous functions vanishing at infinity we refer to [10, §VIII.2]).

This allows us to generalize the definition of Lagrangians decaying in entropy by restating condition (c) in Definition 2.8 in the following way: Given the compact exhaustion $(K_m)_{m \in \mathbb{N}}$ of $X$ with $K_m \subset K_{m+1}$ for every $m \in \mathbb{N}$ and $X = \bigcup_{m=1}^\infty K_m$, for every $x \in X$ let $N = N(x)$ be the least integer such that $x \in K_m$ for all $m \geq N$. We now introduce the sets $(K_m(x))_{m \in \mathbb{N}}$ by

$$K_m(x) := K_{m+N-1} \quad \text{for all } m \in \mathbb{N}. \quad (6.1)$$

Introducing entropy $E_x(K_m(x), \delta)$ according to [22] as the smallest number of balls of radius $\delta > 0$ covering $K_m(x)$, and replacing (c) in Definition 2.8 by (c’), we define Lagrangians vanishing in entropy as follows.

**Definition 6.1.** Let $(X,d)$ be a second-countable, locally compact metric space. Then the Lagrangian $\mathcal{L} : X \times X \to \mathbb{R}_0^+$ is said to vanish in entropy if the following conditions are satisfied:

(a) $c := \inf_{x \in X} \mathcal{L}(x, x) > 0$.

(b) There is a compact set $K \subset X$ such that

$$\delta := \inf_{x \in X \setminus K} \sup \left\{ s \in \mathbb{R} : \mathcal{L}(x, y) \geq \frac{c}{2} \quad \text{for all } y \in B_x(x) \right\} > 0.$$ 

(c’) The Lagrangian has the following decay property: Given an exhaustion of $X$ by compact subsets $(K_m)_{m \in \mathbb{N}}$, there exists $f : X \times X \to \mathbb{R}_0^+$ with $f(x,\cdot) \in \mathcal{F}^+_B(X)$ for every $x \in X$ such that, for every $x \in X$ and all $m \in \mathbb{N}$,

$$\mathcal{L}(x, y) \leq 2^{-m} \frac{f(x, y)}{C_x(m, \delta)} \quad \text{for all } y \in K_m(x),$$ 

where $(K_m(x))_{m \in \mathbb{N}}$ is defined by (6.1),

$$C_x(m, \delta) := C \cdot E_x(K_{m+2}(x), \delta) \quad \text{for all } x \in \mathcal{F}, m \in \mathbb{N} \text{ and } \delta > 0.$$
(with entropy $E_x(K_m(x), \delta)$ as introduced in [23], and the constant $C$ is given by

$$C := 1 + \frac{2}{c} < \infty.$$  

As mentioned in [24], we may assume that $\delta = 1$ (otherwise we suitably rescale the corresponding metric on $X$). Let us point out that Definition 6.1 by contrast to Definition 2.8 (see [24 Definition 5.1]), does not require a Heine-Borel metric and thus allows for more general applications. Definition 2.8 can be considered as a special case of Definition 6.1.

Under the assumptions (a), (b), (c'), for any $x \in X$ and $\varepsilon > 0$ there exists $K_{x, \varepsilon} \subset X$ compact such that

$$\int_{X \setminus K_{x, \varepsilon}} \mathcal{L}(x, y) \, d\tilde{\rho}(y) < \varepsilon.$$  

To see this, we make use of the fact that $K_n \subset K_{n+1}^c$ for all $n \in \mathbb{N}$. Given $x \in X$ and arbitrary $\varepsilon > 0$, there exists $K_{x, \varepsilon} \subset X$ compact with $f(x, y) < \varepsilon/6$ for all $y \notin K_{x, \varepsilon}$. Since $X$ is hemicompact, there exists $n \in \mathbb{N}$ with $K_{x, \varepsilon} \subset K_n$. We denote the least such integer by $N_0 = N_0(x, \varepsilon)$. Then the compact set (cf. [24 eq. (5.2)])

$$K_{x, \varepsilon} := K_{N_0} \subset X$$  

has the desired property:

$$\int_{X \setminus K_{N_0}} \mathcal{L}(x, y) \, d\tilde{\rho}(y) = \sum_{m=N_0}^{\infty} \int_{K_{m+1} \setminus K_m} \mathcal{L}(x, y) \, d\tilde{\rho}(y) \leq \sum_{m=N_0}^{\infty} \sup_{y \in K_{m+1} \setminus K_m} \mathcal{L}(x, y) \, \tilde{\rho}(K_{m+1} \setminus K_m) \leq \sup_{y \in X \setminus K_{N_0}} f(x, y) \sum_{m=N_0}^{\infty} 2^{-m} < \varepsilon/3.$$  

By definition of $C_\varepsilon(m, \delta)$, we are given

$$\int_{X \setminus K_{x, \varepsilon}} \mathcal{L}(\tilde{x}, y) \, d\tilde{\rho}(y) < \varepsilon/3$$  

for all $\tilde{x}$ in a sufficiently small neighborhood of $x$.

Assuming that the Lagrangian is continuous, we proceed similarly to [24] to prove that the same is true for the measures $\tilde{\rho}^{(n)}$ as given by [24 eq. (4.5)] (where the measures $\tilde{\rho}^{(n)}$ originate in the same manner as in [24 §4.1]). More precisely, for any given $x \in X$ and $\varepsilon > 0$, we introduce the compact sets $A_m(x) \subset X$ by

$$A_m(x) := K_{m+1}(x) \setminus K_m(x)$$  

for all $m \geq N_0 = N_0(x, \varepsilon)$.

Next, regularity of $\tilde{\rho}$ yields the existence of open sets $U_m(x) \supset A_m(x)$ with $U_m(x) \subset K_{m+1}(x) \setminus K_{m-1}(x)$ such that

$$\tilde{\rho}(U_m \setminus (K_{m+1}(x) \setminus K_m(x))) < 2^{-m-1}\varepsilon/3$$  

for all $m \geq N_0$.

In view of [11 Lemma 2.92], for every $m \geq N_0$ there exists $\eta_m \in C_c(U_m(x); [0,1])$ such that $\eta_m|_{A_m(x)} \equiv 1$, implying that $\mathcal{L}(x, \cdot) \eta_m \in C_c(U_m(x))$ for all $m \geq N_0$. Repeating the arguments in [24], we finally arrive at [24 eq. (5.6)], i.e.

$$\int_{X \setminus K_{x, \varepsilon}} \mathcal{L}(\tilde{x}, y) \, d\tilde{\rho}^{(n)}(y) < \varepsilon$$  

and

$$\int_{X \setminus K_{x, \varepsilon}} \mathcal{L}(\tilde{x}, y) \, d\tilde{\rho}(y) < \varepsilon$$  

(6.4)
for all $\bar{x}$ in a small neighborhood of $x$ and sufficiently large $n \in \mathbb{N}$. As a consequence, all results in [24 Section 5] remain valid for Lagrangians decreasing in entropy.

The advantage of Definition 6.1 is that it applies to arbitrary second-countable, locally compact metric spaces. In particular, by contrast to Definition 2.8 it need not be endowed with an unbounded Heine-Borel metric. Furthermore, the concept of Lagrangians vanishing in entropy carries over to possibly non-locally compact metric spaces in the following way.

**Definition 6.2.** Given a metric space $(X, d)$, the Lagrangian $\mathcal{L} : X \times X \to \mathbb{R}_0^+$ is said to vanish in entropy if, for any second-countable, locally compact Hausdorff space $Y \subset X$, its restriction $\mathcal{L}|_{Y \times Y} : Y \times Y \to \mathbb{R}_0^+$ vanishes in entropy with respect to the induced metric $d_Y := d|_{Y \times Y}$ (see Definition 6.1).

6.2. **Preparatory Results.** After these preliminaries we return to causal variational principles in the non-locally compact setting (see Definition 3.4). Accordingly, let $X$ be a separable infinite-dimensional complex Banach space and assume that $\mathcal{F} \subset X$ is a non-locally compact Polish subspace (with respect to the Fréchet metric $d$ induced by the norm on $X$). Then $(\mathcal{F}, d)$ is a separable, complete metric space. In what follows we assume that the Lagrangian $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ vanishes in entropy (see Definition 6.2 and Definition 6.1). Considering the finite-dimensional exhaustion $\mathcal{F}^{(n)} \subset \mathcal{F}$ endowed with the induced metric $d_n := d|_{\mathcal{F}^{(n)} \times \mathcal{F}^{(n)}}$ for all $n \in \mathbb{N}$, the explanations in §3.3 imply that each closed bounded subset of $\mathcal{F}^{(n)}$ (with respect to $d_n$) is compact. Moreover, the restricted Lagrangians $\mathcal{L}^{(n)} : \mathcal{F}^{(n)} \times \mathcal{F}^{(n)} \to \mathbb{R}_0^+$ vanish in entropy (see Definition 6.1). As outlined in §6.1 all results in [24 Section 5] remain valid if we replace “decaying in entropy” by “vanishing in entropy.” In particular, by applying [24 Theorem 5.8] we conclude that for each $n \in \mathbb{N}$, there is some regular Borel measure $\rho^{[n]}$ on $\mathcal{F}^{(n)}$ which is a minimizer of the corresponding action $S^{(n)} := S_{\mathcal{F}^{(n)}}$ under variations of compact support, where

$$S_E(\rho) := \int_E d\rho(x) \int_E d\rho(y) \mathcal{L}(x, y)$$

for any $E \in \mathcal{B}(\mathcal{F})$ (cf. [24 §3.2]). Moreover, in view of [24 Theorem 5.5], for all $n \in \mathbb{N}$ the following Euler-Lagrange equations hold,

$$\ell^{[n]}|_{\text{supp } \rho^{[n]}} \equiv \inf_{x \in \mathcal{F}} \ell^{[n]}(x) = 0,$$

where the mapping $\ell^{[n]} : \mathcal{F} \to \mathbb{R}$ is defined by

$$\ell^{[n]}(x) := \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho^{[n]}(y) - 1.$$ 

We point out that Lemma 4.1 is applicable to the sequence $(\rho^{[n]})_{n \in \mathbb{N}}$, implying that

$$\rho^{[n]}(K) \leq C_K \quad \text{for all } n \in \mathbb{N}.$$ 

For this reason, we may proceed in analogy to Section 4 by introducing a countable set $\mathcal{D} \subset \mathcal{R}(\mathcal{F})$ (see §4.1). Next, in analogy to §4.2 we iteratively restrict the sequence of measures $(\rho^{[n]})_{n \in \mathbb{N}}$ to $D_m \subset \mathcal{F}$ compact with $D_m \subset \mathcal{D}$ for all $m \in \mathbb{N}$ and denote the resulting diagonal sequence by $(\rho^{(k)})_{k \in \mathbb{N}}$ (cf. (4.3)). Defining the corresponding set function $\varphi : \mathcal{D} \to [0, \infty)$ by (4.5) and proceeding in analogy to the proof of Theorem 4.3 we obtain a (possibly trivial) measure $\rho$ on the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{F})$. 

Lemma 4.4 yields that the resulting measure \( \rho : \mathcal{B}(\mathcal{I}) \to [0, +\infty] \) is regular. Moreover, the useful results Lemma 4.10 and Lemma 4.11 still apply.

In analogy to Remark 4.5, the following remark yields a sufficient condition for the measure \( \rho \) obtained in Theorem 4.3 to be non-zero.

**Remark 6.3.** Let \( \mathcal{F}^{(n)} \) be a finite-dimensional approximation of \( \mathcal{F} \) (see §3.3). Assuming that the Lagrangian is bounded and vanishes in entropy (see Definition 6.2), for every \( x^{(n)} \in \mathcal{F}^{(n)} \) and \( 0 < \varepsilon < 1 \) there exists \( K_{x,\varepsilon}^{(n)} \subset \mathcal{F}^{(n)} \) compact such that
\[
\int_{\mathcal{F} \setminus K_{x,\varepsilon}^{(n)}} L(x^{(n)}, y) \, d\rho^{(n)}(y) < \varepsilon.
\]

In view of boundedness of the Lagrangian we introduce the upper bound \( \mathcal{C} < +\infty \) by
\[
\mathcal{C} := \sup_{x, y \in \mathcal{I}} L(x, y) > 0.
\]

Then the EL equations (3.6) and (3.7) yield
\[
1 \leq \int_{\mathcal{I}} L(x^{(n)}, y) \, d\rho^{(n)} = \int_{K_{x,\varepsilon}^{(n)}} L(x^{(n)}, y) \, d\rho^{(n)} + \int_{\mathcal{F} \setminus K_{x,\varepsilon}^{(n)}} L(x^{(n)}, y) \, d\rho^{(n)},
\]

implying that
\[
0 < \frac{1 - \varepsilon}{\mathcal{C}} \leq \rho^{(n)}(K_{x,\varepsilon}^{(n)}) \quad \text{for sufficiently large } n \in \mathbb{N}.
\]

Without loss of generality we may assume that \( K_{x,\varepsilon}^{(n)} \in \mathcal{D} \) for every \( n \in \mathbb{N} \). Moreover, we are given \( K_{x,\varepsilon}(N) := \bigcup_{n=1}^{N} K_{x,\varepsilon}^{(n)} \in \mathcal{D} \) for every \( N \in \mathbb{N} \). Therefore, whenever there exists \( N \in \mathbb{N} \) such that \( \rho^{(n)}(K_{x,\varepsilon}(N)) \geq c \) for almost all \( n \in \mathbb{N} \) and some \( c > 0 \), the measure \( \rho \) as defined by (4.7) is non-zero. If this holds true for an infinite number of disjoints sets \( (K_{x_i,\varepsilon}(N_i))_{i \in \mathbb{N}} \), the measure \( \rho \) possibly has infinite total volume.

The remainder of this section is devoted to the proof that the measure \( \rho \) defined by (4.7) is, under suitable assumptions, a minimizer under variations of compact support as well as under variations of finite volume. For non-trivial minimizers, we shall derive the corresponding EL equations (see §6.4).

### 6.3. Existence of Minimizers

The aim of this subsection is to prove that, under suitable assumptions, the measure \( \rho \) defined by (4.7) is a minimizer of the causal variational principle (3.2), (2.3) under variations of finite volume (see Definition 3.5). To this end, we first show that \( \rho \) is a minimizer of the causal action under variations of compact support (see Definition 3.6).

In order to show that the measure \( \rho \) obtained in Theorem 4.3 is a minimizer under variations of compact support, we need to assume that condition (iv) in §2.2 holds (cf. [24, §5.4]), i.e.
\[
\sup_{x \in \mathcal{I}} \int_{\mathcal{I}} L(x, y) \, d\rho(y) < \infty.
\]

Under the additional assumption that the measure \( \rho \) obtained in Theorem 4.3 also satisfies condition (B) in Section 5, we obtain the following existence result.

**Lemma 6.4.** Assume that the Lagrangian \( L \in C_b(\mathcal{I} \times \mathcal{I}; \mathbb{R}_{>0}) \) vanishes in entropy, and that condition (3.8) holds. Moreover, assume that the measure \( \rho \) obtained in (4.7) satisfies condition (B) in Section 5 and that condition (iv) in §2.2 holds. Then \( \rho \) is a minimizer under variations of compact support.
Proof. Assume that $\tilde{\rho} : \mathcal{B}(\mathcal{F}) \to [0, \infty]$ is a regular Borel measure satisfying (2.3) such that $K := \text{supp}(\tilde{\rho} - \rho)$ is a compact subset of $\mathcal{F}$. Then the difference of actions (2.4) as given by

$$ (S(\tilde{\rho}) - S(\rho)) = 2 \int_K d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) $$

$$ + \int_K d(\tilde{\rho} - \rho)(x) \int_K d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) $$

is well-defined (see the explanations in [24, §4.3]). Assuming that condition (iv) in §2.2 holds, for any $x \in \mathcal{F}$ and $\tilde{\varepsilon} > 0$ there exists an integer $R = R(x, \tilde{\varepsilon})$ such that

$$ \int_{\mathcal{F} \setminus B_R(x)} \mathcal{L}(x, y) \, d\rho(y) < \tilde{\varepsilon}/2. \quad (6.5) $$

By continuity of the Lagrangian, there is an open neighborhood $U_x$ of $x$ such that

$$ \int_{\mathcal{F} \setminus B_R(x)} \mathcal{L}(z, y) \, d\rho(y) < \tilde{\varepsilon} \quad \text{for all} \quad z \in U_x. \quad (6.6) $$

Proceeding in analogy to the proof of [24, Theorem 5.8] by covering the compact set $K \subset \mathcal{F}$ by a finite number of such neighborhoods $U_{x_1}, \ldots, U_{x_L}$ and introducing the bounded set $B_K := \bigcup_{j=1}^L B_R(x_j)$, we conclude that

$$ \int_{\mathcal{F} \setminus B_K} \mathcal{L}(x, y) \, d\rho(y) < \tilde{\varepsilon} \quad \text{for all} \quad x \in K. \quad (6.7) $$

This implies that, by choosing $\tilde{\varepsilon} > 0$ suitably, the last summand in the expression

$$ (S(\tilde{\rho}) - S(\rho)) = \left[ 2 \int_K d(\tilde{\rho} - \rho)(x) \int_{B_K} d\rho(y) \mathcal{L}(x, y) \right] + 2 \int_K d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F} \setminus B_K} d\rho(y) \mathcal{L}(x, y) $$

is arbitrarily small. For this reason, it remains to consider the term in square brackets in more detail. Combining the facts that $\rho$ satisfies condition (B) in Section 5 and that $B_K \subset \mathcal{F}$ is bounded, Lemma 5.1 implies that $\rho(B_K) < \infty$. Proceeding similarly to the proof of Lemma 5.5, we deduce that the term in square bracket is bigger than or equal to zero, up to an arbitrarily small error term. This gives rise to

$$ (S(\tilde{\rho}) - S(\rho)) \geq 0, $$

which proves the claim.

Proceeding similarly to the proof of Lemma 6.4, we obtain the following result.

**Theorem 6.5.** Assume that the Lagrangian $\mathcal{L} \in \mathcal{C}_b(\mathcal{F} \times \mathcal{F}; \mathbb{R}_0^+)$ vanishes in entropy, and that condition (3.8) holds. Moreover, assume that the measure $\rho$ obtained in (4.7) satisfies condition (B) in Section 5 and that condition (iv) in §2.2 holds. Then $\rho$ is a minimizer under variations of finite volume.

**Proof.** Assume that $\tilde{\rho} : \mathcal{B}(\mathcal{F}) \to [0, \infty]$ is a regular measure satisfying (2.3). By virtue of Lemma 5.1 the measure $\rho$ is locally finite, implying that $\tilde{\rho}$ is also a locally finite measure (see the explanations after Definition 3.3). Introducing $B := \text{supp}(\tilde{\rho} - \rho)$,
we are given \( \rho(B) = \tilde{\rho}(B) < \infty \). Since condition (iv) in §2.2 holds, the difference of actions (2.4) is well-defined and given by

\[
(S(\tilde{\rho}) - S(\rho)) = 2 \int_B d(\tilde{\rho} - \rho)(x) \int_x d\rho(y) \mathcal{L}(x, y) + \int_B d(\tilde{\rho} - \rho)(x) \int_B d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y).
\]

Making use of regularity of \( \rho \) and \( \tilde{\rho} \), for arbitrary \( \tilde{\varepsilon} > 0 \) there is \( U \supset B \) open such that

\[
\rho(U \setminus B) < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\rho}(U \setminus B) < \tilde{\varepsilon}.
\]

Approximating \( U \) from inside by compact sets \( K \) such that

\[
\rho(U \setminus K) < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\rho}(U \setminus K) < \tilde{\varepsilon}
\]

and proceeding in analogy to the proof of [24, Theorem 5.9] and Lemma 6.4, we finally may deduce that

\[
(S(\tilde{\rho}) - S(\rho)) \geq 0,
\]

which proves the claim.

Theorem 6.5 concludes the existence theory in the non-locally compact setting.

6.4. Derivation of the Euler-Lagrange Equations. Under the assumptions of Theorem 6.5, for non-trivial measures \( \rho \neq 0 \) we are able to deduce the corresponding Euler-Lagrange equations. More precisely, in analogy to [23, Lemma 2.3] we obtain the following result.

**Theorem 6.6.** Assume that the Lagrangian \( \mathcal{L} \in \mathcal{C}_b(\mathcal{F} \times \mathcal{F}; \mathbb{R}_0^+) \) vanishes in entropy (see Definition 6.2), and that condition (3.8) holds. Moreover, assume that the regular measure \( \rho \) obtained in (4.7) is non-zero and satisfies condition (B) in Section 5 as well as condition (iv) in §2.2. Then the following Euler-Lagrange equations hold,

\[
\ell|_{\text{supp } \rho} \equiv \inf_{x \in \mathcal{F}} \ell(x) = 0,
\]

where \( \ell \in \mathcal{C}(\mathcal{F}) \) is defined by (5.4) for a suitable parameter \( s \in \mathbb{R}_0^+ \).

**Proof.** Under the assumptions of Theorem 6.6, the measure \( \rho \) constructed in (4.7) is locally finite in view of Lemma 4.8 and a minimizer under variations of finite volume. Assuming that \( \rho \neq 0 \) and arguing similarly to the proof of Lemma 5.8. Theorem 5.7 gives rise to (6.8).

Theorem 6.6 generalizes the results of [24, Section 5] to the infinite-dimensional setting. It remains an open task to prove the existence of Lagrangians vanishing in entropy such that the measure \( \rho \) given by (4.7) is non-zero and satisfies condition (iv) in §2.2 as well as condition (B) in Section 5.

7. Topological Properties of Spacetime

The goal of this section is to derive topological properties of spacetime and to work out a connection to dimension theory. To this end, we let \( \mathcal{F} \) be a non-locally compact Polish space in the non-locally compact setting (see Definition 3.4). Under suitable assumptions on the Lagrangian (see Theorem 5.6 and Theorem 6.5), the measure \( \rho \) obtained in (4.7) is a minimizer of the corresponding variational principle (3.2). In order to obtain dimension-theoretical statements on its support, let us first recall
some basic results from dimension theory (§7.1). Afterwards we specialize the setting by applying the obtained results to causal fermion systems (7.2).

7.1. Dimension-Theoretical Preliminaries. To begin with, let us first point out that there are several notions of “dimension” of a topological space, among them the small inductive dimension \(\text{ind}\), the large inductive dimension \(\text{Ind}\), the covering dimension \(\text{dim}\), the Hausdorff dimension \(\text{dim}_H\) and the metric dimension \(\mu\text{dim}\) (for details we refer to [3], [12], [29] and [35]). For a separable metric space \(X\), the relation

\[ \text{dim} X \leq \text{dim}_H X \tag{7.1} \]

holds in view of [29, Section VII.4]. Moreover, for every separable metrizable space \(X\) we have \(\text{ind} X = \text{Ind} X = \text{dim} X\) (see [12, Theorem 4.1.5]), and \(\mu\text{dim} Y = \text{dim} Y\) for every compact metric space \(Y\) (see e.g. [3]).

For a metric space \(X\), the local dimension \(\text{dim}_\text{loc}: X \rightarrow [0, \infty]\) is given by

\[ \text{dim}_\text{loc}(x) = \inf \{ \text{dim}_H(B_\varepsilon(x)) : \varepsilon > 0 \} \]

(see [9, §2]), where

\[ \text{dim}_H(A) = \inf \{ s \geq 0 : H^s(A) = 0 \} \]

is the Hausdorff dimension of \(A \subset X\), and \(H^s\) is the \(s\)-dimensional Hausdorff measure (the interested reader is referred to [13, Section 2.10], [29, Chapter VII] and [37]). Whenever \(X\) is a separable metric space, then one can show that

\[ \text{dim}_H(X) = \sup_{x \in X} \text{dim}_\text{loc}(x). \]

Moreover, if \(X\) is compact then the supremum is attained (cf. [9, Proposition 2.7]).

**Definition 7.1.** A normal space \(X\) is locally finite-dimensional if for every \(x \in X\) there exists a normal open subspace \(U\) of \(X\) such that \(x \in U\) and \(\text{dim} U < \infty\). See [12, Section 5.5].

In order to apply the above preliminaries to minimizers of the causal variational principle, let us summarize some general topological properties of the support of a locally finite measure \(\mu\) on a Polish space \(F\) in the next statement.

**Lemma 7.2.** Let \(X\) be a Polish space, and assume that \(\mu\) is a locally finite measure on \(B(X)\). Then \(\text{supp} \mu \subset X\) is \(\sigma\)-compact, and there exists a locally finite-dimensional subspace \(F\) being dense in \(\text{supp} \mu\). Whenever \(\text{supp} \mu\) is hemicompact, then \(\text{supp} \mu\) is locally compact and thus locally finite-dimensional. Moreover, in the latter case there exists a (Heine-Borel) metric on \(\text{supp} \mu\) such that each bounded subset in \(\text{supp} \mu\) is finite-dimensional.

**Proof.** According to Lemma [13.2] \(\text{supp} \mu\) is a \(\sigma\)-compact separable metric space, and there is a dense subset \(F \subset \text{supp} \mu\) such that each \(x \in F\) is contained in a compact neighborhood \(N_x\). In view of [9, Proposition 2.7] we conclude that \(\text{dim}_H N_x < \infty\) for every \(x \in \text{supp} \mu\). Thus Definition [7.1] together with (7.1) gives the first statement.

Whenever \(\text{supp} \mu\) is hemicompact, it is locally compact according to Lemma [13.2]. Thus each \(x \in \text{supp} \mu\) is contained in a compact neighborhood \(N_x\). Making use of [9, Proposition 2.7] we deduce that \(\text{dim}_H N_x < \infty\). Since \(x \in \text{supp} \mu\) is arbitrary and the interior of \(N_x\) is open, from Definition [7.1] we obtain that \(\text{supp} \mu\) is locally finite-dimensional. Moreover, due to Lemma [13.2] the space \(\text{supp} \mu\) can be endowed with a Heine-Borel metric. Accordingly, whenever \(B \subset \text{supp} \mu\) is bounded (with respect to
the Heine-Borel metric), its closure is compact. Covering the resulting compact set by a finite number of compact neighborhoods, we conclude that \( \dim_H(B) < \infty \). □

7.2. Application to Causal Fermion Systems. In the remainder of this section, we finally apply the previous results to the case of causal fermion systems. To this end, let \( \mathcal{F} \) be an infinite-dimensional separable complex Hilbert space. For a given spin dimension \( n \in \mathbb{N} \), the set \( \mathcal{F} \subset L(\mathcal{F}) \) (for details see [18, Definition 1.1.1]) is a non-locally compact Polish space (see Theorem A.1 and Lemma 3.3). Assuming that the Lagrangian \( \mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+ \) is symmetric, lower semi-continuous and strictly positive on the diagonal \( \mathcal{F} \times \mathcal{F} \), we are exactly in the non-locally compact setting as introduced in §3.2 (see Definition 3.4). Therefore, by virtue of Theorem 4.3, there exists a regular measure \( \rho : \mathcal{B}(\mathcal{F}) \to [0, \infty] \). Assuming in addition that the Lagrangian is continuous, bounded and of bounded range and that condition (B) in Section 5 is satisfied, the measure \( \rho \) is a minimizer of the causal variational principle (3.2), (2.3) under variations of compact support by virtue of Lemma 5.5. Under the additional assumption that condition (iv) in §2.2 is satisfied, the measure \( \rho \) is a minimizer of the causal variational principle under variations of finite volume due to Theorem 5.6. Under these assumptions, the same is true for Lagrangians vanishing in entropy (see Theorem 6.4 and Theorem 6.5). As a consequence, we are given a causal fermion system \( (\mathcal{H}, \mathcal{F}, \rho) \), and spacetime \( M \) is defined as the support of the universal measure \( \rho \),

\[
M := \text{supp} \rho.
\]

Combining the results of Lemma 7.2 and Lemma B.2, we arrive at the following main results of this section.

**Theorem 7.3.** Assume that \( \mathcal{L} \in C_b(\mathcal{F} \times \mathcal{F}; \mathbb{R}_0^+) \) is of bounded range or vanishes in entropy. Moreover, assume that the measure \( \rho \) defined by (4.7) satisfies condition (B) in Section 5. Then spacetime \( M \) is \( \sigma \)-compact and contains a locally finite-dimensional dense subspace. Under the additional assumption that spacetime \( M \) is hemicompact, it is a locally finite-dimensional, \( \sigma \)-locally compact Polish space.

**Proof.** Assuming that condition (B) in Section 5 is satisfied, the measure \( \rho \) is locally finite in view of Lemma 5.1. Henceforth the statement is a consequence of Lemma B.2 and Lemma 7.2. □

In [5] the question is raised whether the support of minimizing measures always is compact. Theorem 7.3 indicates that the support should in general at least be \( \sigma \)-compact.

Under the assumption that the measure \( \rho \) is locally finite (for sufficient conditions see Lemma B.6, Lemma 1.8 and Lemma 5.1), we obtain the following result.

**Theorem 7.4.** Assume that the measure \( \rho : \mathcal{B}(\mathcal{F}) \to [0, \infty] \) given by (4.7) is locally finite. Then the interior of spacetime \( M = \text{supp} \rho \) is empty (in the topology of \( \mathcal{F} \)).

**Proof.** Assume that \( M^\circ \neq \emptyset \) in the topology of \( \mathcal{F} \). Then \( U^\text{reg} := M^\circ \cap \mathcal{F}^\text{reg} \) is open in the relative topology. Since \( \mathcal{F}^\text{reg} \) is a Banach manifold (see [25]), it can be covered by an atlas \( (U_\alpha, \phi_\alpha)_{\alpha \in A} \) for some index set \( A \) (cf. [43, Chapter 73]). In particular, each \( x \in U^\text{reg} \) is contained in some open set \( U_\alpha \), whose image \( V_\alpha := \phi_\alpha(U_\alpha) \) is open in some infinite-dimensional Banach space \( X_\alpha \). From Lemma B.2 we know that there exists a dense subset \( F \subset \text{supp} \rho \) such that each \( x \in F \) has a compact neighborhood. Given \( x \in F \) and choosing a compact neighborhood \( N_x \subset U_\alpha \) for some \( \alpha \in A \), from
the fact that the mapping $\phi_0$ is a homeomorphism we conclude that $\phi_0(N_x) \subset X_0$ is a compact neighborhood of $\phi_0(x) \in X_0$ which contains a non-empty open subset in contradiction to Exercise 14.3. This gives the claim.

Theorem 7.4 generalizes Theorem 3.16 to the infinite-dimensional setting.

**Appendix A. Topological Properties of Causal Fermion Systems**

The goal of this appendix is to prove the following result:

**Theorem A.1.** Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system. Then $\mathcal{F}$ is a Polish space.\[6\]

Throughout this section we assume that $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension $s \in \mathbb{N}$. More precisely, we consider a (possibly infinite-dimensional) separable complex Hilbert space $\mathcal{H}$ endowed with a scalar product $\langle \cdot | \cdot \rangle_\mathcal{H}$. Denoting the set of all bounded linear operators on $\mathcal{H}$ by $L(\mathcal{H})$, we let $\mathcal{F} \subset L(\mathcal{H})$ be the subset consisting of those operators $A \in L(\mathcal{H})$ which are self-adjoint with respect to the scalar product $\langle \cdot | \cdot \rangle_\mathcal{H}$ on $\mathcal{H}$ and have at most $s$ positive and at most $s$ negative eigenvalues (see [18 §1.1.1]). The proof of Theorem A.1 is split up in two parts: We first point out that $\mathcal{F}$ is separable (§A.1). Afterwards, we prove that $\mathcal{F}$ is completely metrizable (§A.2). The result can be immediately generalized to the case of operators which have at most $p$ positive and at most $q$ negative eigenvalues.

**A.1. Separability.** In order to prove separability of $\mathcal{F}$, we employ the following argument: Given an infinite-dimensional, separable complex Hilbert space $\mathcal{H}$, the set of linear operators on $\mathcal{H}$, denoted by $L(\mathcal{H})$, is a non-separable Banach space (see [31 §3.A and §12.E]). Since each compact linear operator on $\mathcal{H}$ is bounded, from [31 §12.E] we infer that the class of compact operators $\mathcal{K}(\mathcal{H}) \subset L(\mathcal{H})$ is separable (as well as a Banach space according to [40 Satz II.3.2]). Applying the previous results to a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ shows that $\mathcal{K}(\mathcal{H})$ is separable. Since each $A \in \mathcal{F}$ has finite rank (see e.g. [34 Chapter 15]), from [40 Section II.3] we conclude that $A$ is compact, implying that $\mathcal{F} \subset \mathcal{K}(\mathcal{H})$. Since $L(\mathcal{H})$ is metrizable by the Fréchet metric induced by the operator norm on $L(\mathcal{H})$, the set $\mathcal{K}(\mathcal{H})$ is metrizable, and hence $\mathcal{F}$ is separable in view of [1 Corollary 3.5].

**A.2. Completeness.** The aim of this subsection is to show that $\mathcal{F}$ is completely metrizable with respect to the Fréchet metric induced by the operator norm on $L(\mathcal{H})$. To this end, we proceed as follows. Given a sequence of operators $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$, our task is to prove that its limit $A \in \mathcal{K}(\mathcal{H})$ is self-adjoint (with respect to the scalar product $\langle \cdot | \cdot \rangle_\mathcal{H}$ on $\mathcal{H}$) and has at most $n$ positive and at most $n$ negative eigenvalues.

**A.2.1. Self-Adjointness.** We start by proving that $A$ is self-adjoint in the case of a general Hilbert space $H$.

**Lemma A.2.** Let $(H, \langle \cdot | \cdot \rangle_H)$ be a Hilbert space, and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of self-adjoint operators in $L(H)$ converging in norm to some $A \in L(H)$. Then $A$ is self-adjoint.

6More precisely, endowed with the Fréchet metric $d$ induced by the operator norm on $L(\mathcal{H})$, the space $(\mathcal{F}, d)$ is a separable, complete metric space.
Proof. For any \( u, v \in H \), applying the Cauchy-Schwarz inequality and making use of the fact that \( A_n \) is self-adjoint for every \( n \in \mathbb{N} \) yields
\[
\left| \langle u \mid A^* v \rangle_H - \langle u \mid A v \rangle_H \right| = \left| \langle A u \mid v \rangle_H - \langle u \mid A v \rangle_H \right| = \left| \langle A u \mid v \rangle_H - \langle A_n u \mid v \rangle_H + \langle A_n u \mid v \rangle_H - \langle u \mid A v \rangle_H \right| \\
\leq 2 \| A - A_n \|_{L(H)} \| u \| \| v \| \to_{n \to \infty} 0.
\]
This completes the proof. \( \square \)

A.2.2. Operators, Resolvents and Spectra. The remainder of this section is dedicated to the proof that the limit \( A \in L(H) \) of a sequence \( (A_n)_{n \in \mathbb{N}} \) in \( \mathcal{F} \) (with respect to the operator norm) has at most \( s \) positive and at most \( s \) negative eigenvalues. To this end, we will essentially make use of results in [30], which we now briefly recall.

For Banach spaces \( X \) and \( Y \), by \( \mathcal{B}(X,Y) \) and \( \mathcal{C}(X,Y) \) we denote the set of all bounded and closed operators from \( X \) to \( Y \), respectively. Then \( \mathcal{B}(X,Y) \) is a Banach space, and we let \( \mathcal{B}(X) := \mathcal{B}(X,X) \) and \( \mathcal{C}(X) := \mathcal{C}(X,X) \). We denote the domain of an operator \( T \) from \( X \) to \( Y \) by \( D(T) \), and its graph \( G(T) \) is by definition the subset of \( X \times Y \) consisting of all elements of the form \( (u,Tu) \) with \( u \in D(T) \). Note that \( G(T) \) is a closed linear subspace of \( X \times Y \) if and only if \( T \in \mathcal{C}(X,Y) \) (see [30, III-§5.2]).

In what follows, let \( X, Y \) be complex Banach spaces, and let \( H \) be a complex Hilbert space. For \( \zeta \in \mathbb{C} \) and \( T \in \mathcal{C}(X) \), we introduce the operator
\[
T_\zeta := T - \zeta 1.
\]
Then the resolvent set \( \rho(T) \) is defined to consist of all \( \xi \in \mathbb{C} \) for which \( T_\xi \) has an inverse, denoted by
\[
R(\xi) = R(\xi,T) := (T - \xi)^{-1}.
\]
We call \( R(\zeta,T) \) the resolvent of \( T \) (see [30, III-§6] and [36, Definition 8.38]). The spectrum \( \sigma(T) \) of \( T \) is given by the complementary set of the resolvent set in the complex plane, \( \sigma(T) := \mathbb{C} \setminus \rho(T) \). Note that the spectrum of a compact operator \( T \) in a Banach space \( X \) has a simple structure analogous to that of an operator in a finite-dimensional space. Namely, for compact operators, each non-zero eigenvalue is of finite multiplicity:

**Theorem A.3.** Let \( T \in \mathcal{B}(X) \) be compact. Then \( \sigma(T) \) is a countable set with no accumulation point different from zero, and each nonzero \( \lambda \in \sigma(T) \) is an eigenvalue of \( T \) with finite multiplicity.

**Proof.** See [30, Theorem III-6.26]. \( \square \)

Moreover, the spectrum \( \sigma(T) \) of a selfadjoint operator \( T \) in \( H \) is a subset of the real axis.

An isolated point of the spectrum is referred to as isolated eigenvalue [30, III-§6.5]. Concerning compact operators \( T \) on \( X \), we may state the following remark.

**Remark A.4.** Let \( T \) be a compact operator. Then every complex number \( \lambda \neq 0 \) belongs to \( \rho(T) \) or is an isolated eigenvalue with finite multiplicity. See [30, Remark III-6.27].
A.2.3. **Projection and Decomposition.** Let $X,Y$ be Banach spaces, and let $M \subset X$ be a linear subspace (or “manifold” in the terminology of [30]). As usual, an idempotent operator $P \in \mathcal{B}(X)$ ($P^2 = P$) is called a projection, giving rise to the decomposition

$$X = M \oplus N,$$

(A.1)

where $M = PX$ and $N = (1 - P)X$ are closed linear subspaces of $X$ which are referred to as complementary (see [30, III.§3.4]). Then each $x \in X$ can be uniquely expressed in the form $u = u' + u''$ with $u' \in M$ and $u'' \in N$. The vector $u'$ is called the projection of $u$ on $M$ along $N$, and $P$ is called the projection operator (or simply the projection) on $M$ along $N$. Accordingly, the operator $1 - P$ is the projection on $N$ along $M$.

The range of $P$ is $M$ and the null space of $P$ is $N$. For convenience we often write $\dim P$ for $\dim M = \dim \mathcal{R}(P)$, where $\mathcal{R}(P)$ denotes the range of $P$. Since $P u \in M$ for every $u \in X$, we have $P P u = P u$, implying that $P$ is idempotent: $P^2 = P$.

Next, a linear subspace $M$ is said to be invariant under an operator $T \in \mathcal{B}(X)$ if $T M \subset M$. In this case, $T$ induces a linear operator $T_M$ on $M$ to $M$, defined by $T_M u = Tu$ for $u \in M$. The operator $T_M$ is called the part of $T$ in $M$. If there are two invariant linear subspaces $M, N$ for $T$ such that $X = M \oplus N$, the operator $T$ is said to be decomposed (or reduced) by the pair $M, N$.

The notion of the decomposition of $T$ by a pair $M, N$ of complementary subspaces (see [A.1], [30, III-(3.14)]) can be extended in the following way. An operator $T$ is said to be decomposed according to $X = M \oplus N$ if

$$PD(T) \subset D(T), \quad TM \subset M, \quad TN \subset N,$$

(A.2)

where $P$ is the projection on $M$ along $N$. When $T$ is decomposed as above, the parts $T_M, T_N$ of $T$ in $M, N$, respectively, can be defined. Then $T_M$ is an operator in the Banach space $M$ with $D(T_M) = D(T) \cap M$ such that $T_M u = Tu \in M$, and $T_N$ is defined similarly.

A.2.4. **Generalized Convergence.** Let us briefly recall the definition of convergence in the generalized sense:

**Definition A.5.** Let $T, T_n \in \mathcal{B}(X,Y)$ for all $n \in \mathbb{N}$.

(i) The convergence of $(T_n)_{n \in \mathbb{N}}$ to $T$ in the sense of $\|T_n - T\| \to 0$ is called uniform convergence or convergence in norm.

(ii) Given closed operators $T, S \in \mathcal{C}(X,Y)$, their graphs $G(T), G(S)$ are closed linear subspaces of the product space $X \times Y$. For two closed linear subspaces $M, N$ of a Banach space $Z$ we let $\delta(M, N)$ be the smallest number $\delta$ such that

$$\text{dist}(u, N) \leq \delta \|u\| \quad \text{for all } u \in M.$$

We call $\delta(T, S) := \delta(G(T), G(S))$ the gap between $T$ and $S$. If $\delta(T_n, T) \to 0$, we shall also say that the operator $T_n$ converges to $T$ (or $T_n \to T$) in the generalized sense.

See [30, Chapter III, §3.1] and [30, Chapter IV, §2.1].

The following theorem establishes a connection between convergence in the generalized sense and uniform convergence.

**Theorem A.6.** Let $T, T_n \in \mathcal{C}(X,Y)$ for all $n \in \mathbb{N}$. If $T \in \mathcal{B}(X,Y)$, then $T_n \to T$ in the generalized sense iff $T_n \in \mathcal{B}(X,Y)$ for sufficiently large $n$ and $\|T_n - T\| \to 0$.

**Proof.** See [30, Theorem IV-2.23].
A.2.5. Separation of the Spectrum. Sometimes it happens that the spectrum $\sigma(T)$ of a closed operator $T$ contains a bounded part $\sigma'$ separated from the rest $\sigma''$ in such a way that a rectifiable, simple closed curve $\Gamma$ (or, more generally, a finite number of such curves) can be drawn so as to enclose an open set containing $\sigma'$ in its interior and $\sigma''$ in its exterior. Under such a circumstance, we have the following decomposition theorem.

**Theorem A.7.** Let $\sigma(T)$ be separated into two parts $\sigma'$, $\sigma''$ in the way described above. Then we have a decomposition of $T$ according to a decomposition $X = M' \oplus M''$ of the space (in the sense of (A.2), cf. [30, III-§5.6]) in such a way that the spectra of the parts $T_{M'}$, $T_{M''}$ coincide with $\sigma'$, $\sigma''$ respectively and $T_{M'} \in \mathcal{B}(M')$.

**Proof.** See [30, Theorem III-6.17].

The proof of Theorem A.7 makes use of the so-called eigenprojection

$$P[T] = \frac{1}{2\pi i} \int_{\Gamma} R(\zeta, T) \, d\zeta \in \mathcal{B}(X),$$

(3.3)

which is a projection on $M' = P[T]X$ along $M'' = (1 - P[T])X$.

**Remark A.8 (Finite system of eigenvalues).** Suppose that the spectrum $\sigma(T)$ of $T \in \mathcal{C}(X)$ has an isolated point $\lambda$. Obviously $\sigma(T)$ is divided into two separate parts $\sigma'$, $\sigma''$ where $\sigma'$ consists of the single point $\lambda$; any closed curve enclosing $\lambda$ but no other point of $\sigma(T)$ may be chosen as $\Gamma$. Then the spectrum of the operator $T_{M'}$ described in [30, Theorem III-6.17] (see Theorem A.7) consists of the single point $\lambda$.

If $M'$ is finite-dimensional, $\lambda$ is an eigenvalue of $T$. In fact, since $\lambda$ belongs to the spectrum of the finite-dimensional operator $T_{M'}$, it must be an eigenvalue of $T_{M'}$ and hence of $T$. In this case, $\dim M'$ is called the (algebraic) multiplicity of the eigenvalue $\lambda$ of $T$.

For brevity, a finite collection $\lambda_1, \ldots, \lambda_s$ of eigenvalues with finite multiplicities will be called a finite system of eigenvalues.

A.2.6. Continuity of the Spectrum. This paragraph is devoted to proof that the spectrum of a sequence of operators converging in the generalized sense behaves continuously.

**Theorem A.9.** Let $T \in \mathcal{C}(X)$ and let $\sigma(T)$ be separated into two parts $\sigma'(T)$, $\sigma''(T)$ by a closed curve $\Gamma$ as in (A.2, 3) (cf. [30, III-§6.4]). Let $X = M'(T) \oplus M''(T)$ be the associated decomposition of $X$. Then there exists $\delta > 0$, depending on $T$ and $\Gamma$, with the following properties. Any $S \in \mathcal{C}(X)$ with $\delta(S, T) < \delta$ has spectrum $\sigma(S)$ likewise separated by $\Gamma$ into two parts $\sigma'(S)$, $\sigma''(S)$ (via itself running in $\rho(S)$). In the associated decomposition $X = M'(S) \oplus M''(S)$, $M'(S)$ and $M''(S)$ are isomorphic with $M'(T)$ and $M''(T)$, respectively. In particular, $\dim M'(S) = \dim M'(T)$, $\dim M''(S) = \dim M''(T)$ and both $\sigma'(S)$ and $\sigma''(S)$ are nonempty if this is true for $T$. The decomposition $X = M'(S) \oplus M''(S)$ is continuous in $S$ in the sense that the projection $P[S]$ of $X$ onto $M'(S)$ along $M''(S)$ tends to $P[T]$ in norm as $\delta(S, T) \to 0$.

**Proof.** See [30, Theorem IV-3.16].

**Lemma A.10.** Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of compact operators in $\mathcal{K}(X)$, and suppose that $T_n \to T$ in norm for some operator $T \in \mathcal{K}(X)$. Moreover, let $\sigma'(T)$ be a finite system of eigenvalues, separated from the rest $\sigma''(T)$ of $\sigma(T)$ by a closed curve $\Gamma$ in the manner of [30, III-§6.4]. Then $\sigma(T_n)$ is separated by $\Gamma$ into $\sigma'(T_n)$, $\sigma''(T_n)$ such
that each \(\sigma'(T_n)\) also consists of \(m\) eigenvalues of \(T_n\), provided that \(n\) is sufficiently large.

Proof. Since \(T\) is a compact operator, Remark A.4 states that each non-zero eigenvalue of \(T\) is isolated (see [30, Remark III-6.27]). For this reason, each non-zero eigenvalue of \(T\) as well as all positive or all non-zero eigenvalues of \(T\) can be enclosed by a closed curve \(\Gamma\) running in \(\rho(T)\) as described in §A.2.5 (cf. [30, III-§6.4]). We denote the set of eigenvalues lying within \(\Gamma\) by \(\sigma'(T)\), and the set of eigenvalues without \(\Gamma\) by \(\sigma''(T)\); thus \(\Gamma\) encloses a finite system of eigenvalues. By virtue of Theorem A.7, we have decomposition of \(T\) according to \(X = M' \oplus M''\) such that the spectra of the parts \(T_{M'}\), \(T_{M''}\) coincide with \(\sigma'(T)\), \(\sigma''(T)\). Since \(T_n, T \in \mathcal{B}(X)\) for all \(n \in \mathbb{N}\), Theorem A.6 ensures that \(T_n \to T\) in the generalized sense, i.e. \(\delta(T_n, T) \to 0\) in the limit \(n \to \infty\). Hence we can apply Theorem A.9 giving rise to some \(N \in \mathbb{N}\) such that \(\Gamma\) separates the spectra \(\sigma(T_n)\) into two parts \(\sigma'(T_n), \sigma''(T_n)\) for all \(n \geq N\). Considering the corresponding decomposition \(X = M'(T_n) \oplus M''(T_n)\) for all \(n \geq N\), by virtue of Theorem A.9, there exist isomorphisms \(M'(T_n) \simeq M'(T)\) for all \(n \geq N\). In particular, we obtain \(\dim M'(T_n) = \dim M'(T)\) for all \(n \geq N\), and the decomposition \(X = M'(T_n) \oplus M''(T_n)\) is continuous in \(n\) in the sense that the projection \(P[T_n]\) (see (A.3)) of \(X\) onto \(M'(T_n)\) along \(M''(T_n)\) tends to \(P[T]\) in norm as \(\delta(T_n, T) \to 0\) which holds true in view of Theorem A.6. A fortiori, the algebraic multiplicity of eigenvalues of \(T\) within \(\Gamma\) coincides with the algebraic multiplicity of eigenvalues of \(T_n\) within \(\Gamma\) for all \(n \geq N\).

Making use of continuity of the spectrum according to Lemma A.10, we may derive the following result.

Lemma A.11. Let \(H\) be a Hilbert space, and let \((A_n)_{n \in \mathbb{N}}\) be a sequence of self-adjoint compact operators in \(\mathcal{K}(H)\) such that each operator \(A_n\) has at most \(s\) positive and at most \(s\) negative eigenvalues. If \((A_n)_{n \in \mathbb{N}}\) converges in norm to some \(A \in \mathcal{L}(H)\), then \(A\) is also a self-adjoint compact operator which has at most \(s\) positive and at most \(s\) negative eigenvalues.

Proof. Given a sequence \((A_n)_{n \in \mathbb{N}}\) in \(\mathcal{K}(H)\) with \(A_n \to A\) in \(\mathcal{L}(H)\), the fact that \(\mathcal{K}(H)\) is a closed subspace of \(\mathcal{L}(H)\) implies that the limit \(A\) is a compact operator. Thus in view of Theorem A.3 and Remark A.3, each non-zero eigenvalue of \(A\) is isolated and has finite multiplicity, and according to Remark A.3, the non-zero eigenvalues of \(A\) form a finite system of isolated eigenvalues. In particular, there is a closed curve \(\Gamma\) as described in Paragraph A.2.5 (cf. [30, III-§6.4]) which encloses all positive (negative) eigenvalues of \(A\). Now assume that \(A\) has \(m > s\) positive (negative) eigenvalues. Then Lemma A.10 yields the existence of some \(N \in \mathbb{N}\) such that the spectrum \(\sigma(A_n)\) is separated by \(\Gamma\) into two parts \(\sigma'(A_n)\) within \(\Gamma\) and \(\sigma''(A_n)\) without \(\Gamma\) for all \(n \geq N\). As a consequence, \(\sigma'(A_n)\) consists of \(m > s\) positive (negative) eigenvalues for all \(n \geq N\) in contradiction to the fact that \(A_n\) has at most \(s\) positive and at most \(s\) negative eigenvalues for all \(n \in \mathbb{N}\). Hence \(A \in \mathcal{K}(H)\) is a selfadjoint operator which has at most \(s\) positive and at most \(s\) negative eigenvalues. This concludes the proof.

A.2.7. Application to Causal Fermion Systems. After these preparations, we finally are in the position to prove Theorem A.1.

Proof of Theorem A.7. Let \((\mathcal{H}, \mathcal{F}, \rho)\) be a causal fermion system. Separability of \(\mathcal{F}\) follows from §A.1. By virtue of Lemma A.11, we conclude that \(\mathcal{F} \subset \mathcal{L}(\mathcal{H})\) is closed.
Since $L(H)$ is a complete metric space with respect to the Fréchet metric induced by the operator norm, we conclude that $F$ is completely metrizable. Taken together, $F$ is a separable, completely metrizable space, and thus Polish [31, Definition (3.1)]. □

More precisely, the space $(F, d)$ is a complete metric space, where $d$ is the Fréchet metric induced by the operator norm on $L(H)$ (cf. [2, §0.7]).

APPENDIX B. SUPPORT OF LOCALLY FINITE MEASURES ON POLISH SPACES

In this section we derive useful topological properties concerning the support of locally finite measures (or Borel measures in the sense of [28]) on Polish spaces (see Lemma B.2 below). To begin with, let us recall the following preparatory result.

**Proposition B.1.** Let $X$ be Polish and $\mu$ a finite measure on $B(X)$. Then $A \subset X$ is $\mu$-measurable if and only if there exists a $\sigma$-compact set $F \subset A$ with $\mu(A \setminus F) = 0$.

**Proof.** See [31, Theorem (17.11)]. □

Moreover, based on [8, Chapter IX], a Borel measure (in the sense of [28]) on a topological Hausdorff space $X$ is said to be moderated if $X$ is the union of countably many open subsets of finite $\mu$-measure (see [10, Chapter VIII]). (Since open sets are measurable, every moderated measure is $\sigma$-finite.) We point out that, due to Ulam’s theorem [10, Satz VIII.1.16], every Borel measure on a Polish space is regular and moderated. (Due to Meyer’s theorem, the same is true for Borel measures on Souslin spaces, see [10, Satz VIII.1.17].) As a consequence, we may derive useful properties of the support of Borel measures on Polish spaces, as the following lemma shows.

**Lemma B.2.** Let $X$ be a Polish space, and assume that $\mu$ is a Borel measure on $B(X)$. Then $\text{supp} \mu \subset X$ is a $\sigma$-compact topological space. Moreover, there exists a dense subset $F \subset \text{supp} \mu$ such that each $x \in F$ has a compact neighborhood in $\text{supp} \mu$.

Whenever $\text{supp} \mu$ is hemicompact, one can arrange that $\text{supp} \mu$ is a Polish space which has the Heine-Borel property. The corresponding Heine-Borel metric can be chosen locally identical to a complete metric on $X$ (in the relative topology of $\text{supp} \mu$).

**Proof.** We make essentially use of the fact that the measure $\mu$ is moderated in view of Ulam’s theorem. As a consequence, there is a sequence of sets $(U_n)_{n \in \mathbb{N}}$ with $U_n \subset X$ open and $\mu(U_n) < \infty$ for all $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} U_n = X$. From the fact that open subsets of Polish spaces are Polish (in the relative topology, see [4, §26] or [31, Theorem (3.11)]), we conclude that each set $U_n \subset X$ is Polish. Thus each $\mu|_{U_n}$ is a finite Borel measure on a Polish space. Due to [7, Proposition 7.2.9], every (finite) Borel measure on a separable metric space has support, implying that

$$\mu|_{U_n}(U_n \setminus \text{supp} \mu|_{U_n}) = 0.$$ 

In view of Proposition B.1, we conclude that $\text{supp} \mu|_{U_n}$ is contained in a $\sigma$-compact set $F_n \subset U_n$, and thus $\text{supp} \mu|_{U_n}$ is $\sigma$-compact for every $n \in \mathbb{N}$. From this we deduce that $\bigcup_{n \in \mathbb{N}} \text{supp} \mu|_{U_n}$ is $\sigma$-compact. Making use of the fact that subsets of $\sigma$-compact spaces are $\sigma$-compact, we conclude that

$$\text{supp} \mu \subset \bigcup_{n \in \mathbb{N}} \text{supp} \mu|_{U_n}$$

is $\sigma$-compact

(where $\mu$ has support in view of [10, Lemma VIII.2.15]). Since $\text{supp} \mu \subset X$ is closed (see [10, §VIII.2.5]), the support $\text{supp} \mu$ is Polish and thus Baire (cf. [3.2]). From [41]
we conclude that there exists a dense subset $F \subset \text{supp } \mu$ such that each $x \in F$ has a compact neighborhood in $\text{supp } \mu$.

Assuming that $\text{supp } \mu$ is hemicompact (see for instance [41, 17]), then $\text{supp } \mu$ is locally compact in view of [11, Exercise 3.4.E] and thus a complete $\sigma$-locally compact space (in the sense of [39]). In this case, due to [42, Theorem 2'] and the explanations in [24, Section 3], the space $\text{supp } \mu$ is metrizable by a Heine-Borel metric which is (Cauchy) locally identical to a complete metric on $X$; endowed with such a metric, the space $\text{supp } \mu$ has the Heine-Borel property, i.e. each closed bounded subset (with respect to the Heine-Borel metric) is compact. □

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References

1. C.D. Aliprantis and K.C. Border, Infinite Dimensional Analysis: A hitchhiker’s guide, third ed., Springer, Berlin, 2006.
2. H.W. Alt, Lineare Funktionalanalysis. Eine anwendungsorientierte Einführung, Fünfte, überarbeitete Auflage, Springer-Verlag, 2006.
3. A.V. Arkhangel’skii and V.V. Fedorchuk, General Topology I: Basic Concepts and Constructions Dimension Theory, vol. 17, Springer Science & Business Media, 2012.
4. H. Bauer, Measure and Integration Theory, De Gruyter Studies in Mathematics, vol. 26, Walter de Gruyter & Co., Berlin, 2001, Translated from the German by Robert B. Burckel.
5. Y. Bernard and F. Finster, On the structure of minimizers of causal variational principles in the non-compact and equivariant settings, Advances in Calculus of Variations 7 (2014), no. 1, 27–57.
6. P. Billingsley, Convergence of Probability Measures, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc., New York, 1999, A Wiley-Interscience Publication.
7. V.I. Bogachev, Measure Theory, Vol. I, II, Springer-Verlag, Berlin, 2007.
8. N. Bourbaki, Integration. II. Chapters 7–9, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004, Translated from the 1963 and 1969 French originals by Sterling K. Berberian.
9. J. Dever, Local Hausdorff measure, arXiv preprint [arXiv:1610.00075] (2016).
10. J. Elstrodt, Maß- und Integrationstheorie, fourth ed., Springer-Lehrbuch. [Springer Textbook], Springer-Verlag, Berlin, 2005, Grundwissen Mathematik [Basic Knowledge in Mathematics].
11. R. Engelking, General Topology, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author.
12. , Theory of Dimensions Finite and Infinite, Sigma Series in Pure Mathematics, vol. 10, Heldermann Verlag, Lemgo, 1995.
13. H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
14. F. Finster, The Principle of the Fermionic Projector, AMS/IP Studies in Advanced Mathematics, vol. 35, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2006.
15. , A variational principle in discrete space-time: existence of minimizers, Calc. Var. Partial Differential Equations 29 (2007), no. 4, 431–453.
16. , Causal variational principles on measure spaces, J. Reine Angew. Math. 646 (2010), 141–194.
17. , Perturbative quantum field theory in the framework of the fermionic projector, J. Math. Phys. 55 (2014), no. 4, 042301, 53.
18. , The Continuum Limit of Causal Fermion Systems, Fundamental Theories of Physics, vol. 186, Springer, 2016, From Planck scale structures to macroscopic physics.
19. F. Finster, A. Grotz and D. Schiefeneder, Causal fermion systems: a quantum space-time emerging from an action principle, (2012), 157–182.
20. F. Finster and M. Jokel, *Causal fermion systems: An elementary introduction to physical ideas and mathematical concepts*, Progress and Visions in Quantum Theory in View of Gravity (2020), 63–92.

21. F. Finster and N. Kamran, *Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles*, arXiv preprint arXiv:1808.03177 (2018).

22. F. Finster and J. Kleiner, *Causal fermion systems as a candidate for a unified physical theory*, Journal of Physics: Conference Series 626 (2015), 012020.

23. *A Hamiltonian formulation of causal variational principles*, Calc. Var. Partial Differential Equations 56 (2017), no. 3, Paper No. 73, 33.

24. F. Finster and C. Langer, *Causal variational principles in the sigma-locally compact setting: Existence of minimizers*, arXiv preprint arXiv:2002.04412 (2020).

25. F. Finster and M. Lottner, *Banach manifold structure and jet spaces for infinite-dimensional causal fermion systems*, in preparation.

26. F. Finster and D. Schiefeneder, *On the support of minimizers of causal variational principles*, Arch. Ration. Mech. Anal. 210 (2013), no. 2, 321–364.

27. S.A. Gaal, *Point Set Topology*, Pure and Applied Mathematics, Vol. XVI, Academic Press, New York-London, 1964.

28. R.J. Gardner and W.F. Pfeffer, *Borel measures*, Handbook of set-theoretic topology (1984), 961–1043.

29. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J., 1941.

30. T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

31. A.S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

32. H. König, *Measure and Integration: An Advanced Course in Basic Procedures and Applications*, Springer Science & Business Media, 2009.

33. G. Köthe, *Topological Vector Spaces. I*, Translated from the German by D. J. H. Garling, Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag New York Inc., New York, 1969.

34. R. Meise and D. Vogt, *Introduction to Functional Analysis*, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press, Oxford University Press, New York, 1997.

35. J.R. Munkres, *Topology*, second ed., Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

36. M. Renardy and R.C. Rogers, *An Introduction to Partial Differential Equations*, Texts in Applied Mathematics, vol. 13, second ed., Springer-Verlag, New York, 2004.

37. C.A. Rogers, *Hausdorff Measures*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1970 original, With a foreword by K. J. Falconer.

38. L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973, Tata Institute of Fundamental Research Studies in Mathematics, No. 6.

39. L.A. Steen and J.A. Seebach, Jr., *Counterexamples in Topology*, Dover Publications, Inc., Mineola, NY, 1995, Reprint of the second (1978) edition.

40. D. Werner, *Funktionalanalysis*, second ed., Springer-Verlag, Berlin, 2000.

41. S. Willard, *General Topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

42. R. Williamson and L. Janos, *Constructing metrics with the Heine-Borel property*, Proc. Amer. Math. Soc. 100 (1987), no. 3, 567–573.

43. E. Zeidler, *Nonlinear Functional Analysis and its Applications. IV: Applications to mathematical physics*, Springer-Verlag, New York, 1988, Translated from the German and with a preface by Juergen Quandt.