The Relationship between Flux Coordinates and Equilibrium-based Frames of Reference in Fusion Theory

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2Deceased. The original idea of the JK reference frame and the development of the new annihilation operator is due strictly to Dr. J.M. Greene.

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The properties of two local reference frames based on the magnetic field and the current density are investigated for magnetized plasmas in toroidal geometry with symmetric angle. The magnetic field-based local frame of reference has been well-studied for example by Dewar and colleagues [Phys. Fluids 27, 1723 (1984)] An analogous frame based on the current density vector is possible because it is also divergence free and perpendicular to the gradient of the poloidal flux. The concept of straightness of a vector is introduced and used to elucidate the Boozer and Hamada coordinate systems. The relationship of these local frames to the more well-known Frenet frame of reference, which specifies a curve in terms of curvature and torsion, is given. As an example of the usefulness of the these formal relationships, we briefly review the ideal MHD theory and their use. We also present a new annihilation operator, useful for eliminating shorter time scales than the time scale of interest, for deriving the inner layer equations of Glasser, Greene, and Johnson [Phys. Fluids 18, 875 (1975)]. Compared to the original derivation that is based on the local frame of reference in terms of the magnetic field, the new annihilation operator that is based on the local frame of reference in terms of the current density simplifies the derivation.

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I. INTRODUCTION

High density, magnetized plasmas, such as tokamak or stellarator plasmas, are characterized by a large number of temporal and spatial scales. Laboratory magnetized plasmas have had great success in creating plasmas with confinement times much longer than the shortest time scales. Understanding plasma behavior on these longer times requires analytic, and computational, techniques for stepping over the shortest time scales to study the time scales of interest. A key issue for magnetized plasmas is resolving the anisotropy of the magnetic field. For example, examination of the Braginskii transport equations reveals different transport time scales for the directions parallel to the magnetic field, perpendicular to the magnetic flux surfaces, and the binormal direction that is perpendicular to both directions [2]. This anisotropy arises consistently in fluid and kinetic theories.

A common method for analytically handling this anisotropy is to use a local frame of reference. By frame of reference, we mean the development of a local coordinate system at a point in space. For example, the three directions above enable the development of a set of basis vectors for a local coordinate system. This local frame of reference has long been used in fluid theory [3, 4] in combination with the use of a flux coordinates [3] appropriate for toroidal systems. Dewar and colleagues [5], hereafter referred to as DMS, greatly improved the understanding of this local reference frame by formalizing many of the relationships between flux coordinates and the local frame that are commonly used in analytic derivation. This frame of reference is also implicit in much of the work of Stix [6] with the “Stix frame” [7] being a common tool in RF theory. That frame, like the more well-known Frenet frame for arbitrary parameterized curves, is expressed in terms of dimensionless vectors. As we shall see, for most of the theory calculations, using dimensional local basis vectors is preferred.

Flux coordinate systems make use of two properties of the equilibrium magnetic field. The first is that
the magnetic field is divergence-free. The second is the equilibrium relationship:

\[ \mathbf{J} \times \mathbf{B} = \nabla P. \]  

(1)

which is the zeroth-order force balance in a magnetized plasma \[8\]. Dotting both sides with \( \mathbf{B} \) shows that \( \mathbf{B} \cdot \nabla P = 0 \). This combined with symmetry allows the definition of a flux function such that \( \mathbf{B} \cdot \nabla \psi = 0 \). Quasineutrality enables the charge continuity equation to be written as \( \nabla \cdot \mathbf{J} = 0 \). The equilibrium relationship also shows that \( \mathbf{J} \cdot \nabla P = 0 \) or that \( \mathbf{J} \cdot \nabla \psi = 0 \) as well. Thus, it is obvious that another local frame of reference, completely analogous to the magnetic field-based frame is possible using the current density vector. To our knowledge, this has never been explored before. In this paper, we formalize the relationships between the two reference frames and flux coordinate systems similar in spirit to the work of the DMS paper. We also relate this frame of reference to the Frenet frame from differential geometry of curves, which expresses a curve in terms of curvature and torsion. Fusion theory has favored magnetic shear over torsion, and these relationships are reviewed.

As an example of the usefulness of formalizing the relationships between flux coordinates and local frames of reference, their usefulness in ideal MHD and resistive MHD instability theory is reviewed. Resistive instabilities, which have a time scale that is a hybrid of the Alfvén and resistive diffusion time scales, is much slower than the Alfvén time scales. Analytically, this requires going to second order in the ordering, which involves significant algebra, especially in studying instabilities in toroidal geometry with complicated flux surface shapes. The rest of the paper reviews the derivation of the inner layer equations in toroidal geometry. The first discussion of how to derive these equations is contained in the appendix of Johnson and Greene \[9\] in 1967, but they were not fully derived and published until Glasser, Greene, and Johnson (GGJ) in 1967. Unfortunately, that paper only focused on the analysis of these equations and only presented them. Part of the point of this paper is that the derivation of these equations contains pedagogical value, and a new method of deriving these equations is presented.

The rest of the paper proceeds as follows. First, we introduce the concept of “equilibrium-based frame of references” and introduce our new frame. Next, following the work of DMS, we provide the mapping of the equilibrium-based frames of references to the flux coordinate systems. We then briefly discuss the relationship of equilibrium reference frames to the Frenet frame and common geometric quantities. With the mathematical discussion complete, we then discuss the ideal MHD formulas, and derivation of the inner layer equations in toroidal geometry. Key to the derivation is the use of annihilation operators \[1, 4\]. For the derivation of the inner layer equations in toroidal geometry, we find that a new annihilation operator, based on the new equilibrium-based coordinate system, is more useful in toroidal geometry. Finally, we conclude by discussing the implications of the derivation, and the relationship to computational approaches.
II. EQUILIBRIUM-BASED FRAMES OF REFERENCE

Frames of references are local coordinate systems that vary throughout space. An example would be the rotating frame of reference describing the forces felt by a person on a merry-go-round commonly used to explain the difference between centripetal and centrifugal forces. Another well-known example from mathematical physics is the Frenet frame of reference which constructs a local set of basis vectors along a curve in terms of the tangent of the curve, the curvature vector which is normal to the tangent, and the binormal vector created from the cross-product of both the tangent and curvature. This creates a local orthonormal set of basis vectors at a given point, but it is not a general coordinate system because it cannot be used to label an arbitrary point in space.

As discussed in the introduction, the most common frame of reference has the magnetic field, the gradient of a flux function, and the binormal direction as the local basis vectors. We term this frame the BC frame and define it as:

\[
\nabla \psi, \quad B, \quad C
\]

where \( \psi \) is the poloidal magnetic flux divided by \( 2\pi \), and \( C \) is constructed to make the coordinate system orthogonal:

\[
C \equiv \frac{\nabla \psi \times B}{|\nabla \psi|^2}
\]

Unlike the Stix frame which normalizes the basis vectors so that they are unit-less (and orders the parallel direction last for convenience) \[7\], the normalizations chosen here seem to be the most useful as will be shown.

In this paper, we define a new equilibrium-based frame of reference, the JK frame

\[
\nabla \psi, \quad J, \quad K
\]

where \( K \) is likewise defined as

\[
K \equiv \frac{\nabla \psi \times J}{P' |\nabla \psi|^2}.
\]

The \( P' \) denotes the derivative of the pressure with \( \psi \). Our use of this normalization is to make \( K \cdot B = 1 \) which is useful. Although using the other vector in the equilibrium relation, \( J \), as the basis of an equilibrium-based frame seems obvious, to our knowledge it has never been studied. In this paper, we show how for long wavelength instabilities at least, it is useful to explicitly treat it as such.

Using vector identities, it is easily seen that the other vector components within each system may be expressed by

\[
B = C \times \nabla \psi, \quad \nabla \psi = \frac{B \times C}{C^2}
\]

\[
J = P'K \times \nabla \psi, \quad \nabla \psi = \frac{J \times K}{P'K^2}
\]

To express an arbitrary vector in terms of these local basis vectors, we borrow notation from the curvilinear flux coordinate systems for simplicity. If one considers these two vector systems as “contravariant-like” basis vectors, \( e^i, i \in \{1, 2, 3\} \), then the concomitant “covariant-like” basis vectors, \( e_i \) are given by \( e_1 = \epsilon^{123} J e^2 \times e^3 \) where \( J = (e^1 \cdot e^2 \times e^3)^{-1} \) is the Jacobian of the system, and \( \epsilon^{123} \) is the Levi-Cevita symbol. The Jacobian is used as a normalization so that \( e^i \cdot e_j = \delta^i_j \) where \( \delta^i_j \) is the Kronecker delta. Any vector is represented in terms of these basis vectors by the following representation \( A = \sum_i A^i e_i = \sum_j A_j e^j, \ i \in \{1, 2, 3\} \). The \( A^i \)'s are the contravariant components and the \( A_j \)'s are the covariant components. By the above relations, the components may clearly be seen to be given by \( A^i = A \cdot e^i \) and \( A_j = A \cdot e_j \).

Using this notation for the basis vectors above, the inverse Jacobians of the two systems are

\[
B^2 = C^2 |\nabla \psi|^2 = \nabla \psi \cdot B \times C
\]

\[
\frac{J^2}{P'} = K^2 |\nabla \psi|^2 = \nabla \psi \cdot J \times K
\]
so that the covariant-like basis vectors are:

\[
\begin{align*}
\nabla \psi & : B \frac{B}{B^2}, \quad C \frac{C}{C^2} \\
\nabla \psi & : J \frac{J}{J^2}, \quad K \frac{K}{K^2}
\end{align*}
\] (10)

The use of \(| \nabla \psi |^2\) in the definition of the binormal vectors \(C\) and \(K\) is to make these covariant-like basis vectors have this consistent form. An arbitrary vector \(f\) then can be written as

\[
f = f^\psi \frac{\nabla \psi}{| \nabla \psi |^2} + f^B \frac{B}{B^2} + f^C \frac{C}{C^2}
\]

\[
= f^\psi \frac{\nabla \psi}{| \nabla \psi |^2} + f^J \frac{J}{J^2} + f^K \frac{K}{K^2}.
\] (12)

Both the Stix frame and the Frenet frame use basis unit vectors and decompose vectors into components that have the same units. Using these “contravariant-like” components is perhaps un-intuitive, but they have long been used in stability analyses [4]. As we show in Section VI this is because they identify the correct components that equilibrate quickly due to compressional Alfvèn waves.

Using force balance \(J \times B = \nabla P\), and defining a variable \(\sigma\) for the parallel current, the relationships between the \(BC\) and \(JK\) frames are:

\[
\begin{align*}
J &= \sigma B - P' \frac{C}{C^2}, \\
B &= \sigma B^2 \frac{J}{J^2} + \frac{K}{K^2}
\end{align*}
\] (13)

\[
\begin{align*}
K &= \frac{\sigma}{P'} C + \frac{B}{B^2} \\
C &= \frac{\sigma C^2}{P'} \frac{K}{K^2} - P' \frac{J}{J^2}
\end{align*}
\] (14)

The metrics of this transformation are

\[
\begin{align*}
J \cdot B &= \sigma B^2, \quad J \cdot C = -P' \\
K \cdot B &= 1, \quad K \cdot C = \frac{\sigma C^2}{P'}.
\end{align*}
\] (15)

The angle between the coordinate systems then is related to \(\sigma/P'\), which is a flux function in cylindrical geometry but not in toroidal geometry. To restore symmetry to the coordinate systems, one could also use \(P'\) in the definition of \(C\), or equivalently, use \(\nabla P\) as the radial vector, which has a certain elegance since all three vectors of the equilibrium relation would then be used as the basis vectors. The choice of normalizations used here seem to be the easiest to use in practice however. The contravariant vector components are related by

\[
\begin{align*}
\text{f}^C &= \frac{P'}{J^2} (\sigma B^2 f^K - P' f^J), \\
\text{f}^B &= \frac{1}{K^2} (\sigma C^2 f^J + f^K) \\
\text{f}^K &= \frac{1}{P'B^2} (\sigma B^2 f^C + P' f^B), \\
\text{f}^J &= \frac{1}{C^2} (\sigma C^2 f^B - P' f^C).
\end{align*}
\] (16)

The metrics presented here appear in the Mercier criterion; thus, one sees even here that stability can be viewed as a complicated function of the angle between these two frames.

We summarize the expressions in this section in Table I. While these frames of reference are useful, it is necessary to make use of geometry-based coordinate systems to study the physics in the appropriate magnetic configuration. The rest of this paper focuses on the doubly-periodic systems relevant to tokamaks and symmetric stellarators.
III. FLUX COORDINATES AND THE RELATIONSHIP TO EQUILIBRIUM-BASED FRAMES

While the local frames of reference are useful for getting at the physics of the problem, they do not take advantage of the periodicity of the toroidal systems that we wish to study. Following the work of DMS, the relationship between the equilibrium-based frames of reference and the geometry-based coordinate systems is explicitly examined. We start by briefly reviewing flux coordinates [8], labeled in our notations as the $\psi, \Theta, \zeta$ system. In this paper, only systems with at least one degree of symmetry are considered with $\zeta$ being the ignorable coordinate. This includes helically symmetric systems.

We assume nested flux surfaces and will consider various averages related to those surfaces. A volume average of a quantity $Q$ at a surface $\psi$ is denoted as

$$\langle Q \rangle_V = \frac{1}{V(\psi)} \int_Q dV = \frac{1}{V(\psi)} \int_0^{2\pi} \int_0^{\psi} Q J d\psi' d\Theta d\zeta$$

where $V(\psi)$ is the volume enclosed by a flux surface, $V = \int \int J d\psi' d\Theta d\zeta$. The flux-surface average of a quantity $Q$ at a surface $\psi$ is denoted as

$$\langle Q \rangle = \frac{1}{V'(\psi)} \int_0^{2\pi} \int_0^{2\pi} Q J d\Theta d\zeta$$

where $V'(\psi)$ is the derivative of $V$ with respect to $\psi$, $V' = \int \int J d\Theta d\zeta$. Also, we define the theta average of a quantity $Q$ as

$$\langle Q \rangle_{\Theta} = \frac{1}{2\pi} \int_0^{2\pi} Q d\Theta d\zeta.$$ 

For axisymmetric quantities, this average is closely related to the flux-surface average; i.e., if $Q$ is independent of $\zeta$, then $\langle JQ \rangle_{\Theta} = V'/(4\pi^2)\langle Q \rangle$.

A. Properties of divergence-free fields perpendicular to flux surfaces

Flux coordinates are usually derived using the properties of the divergence-free nature of the magnetic field along with the definition of flux surface as a surface perpendicular to the magnetic field. Our equilibrium-based local frames of reference include two other divergence-free vectors, $J$ and $|\nabla \psi|^2 vC$, so it is useful to first understand general properties of vector-free fields perpendicular to flux surfaces. The divergence-free property of $|\nabla \psi|^2 vC$ can be seen using $J \cdot \nabla \psi = 0$:

$$\nabla \psi \cdot J = \nabla \psi \cdot \nabla \times B = \nabla \cdot B \times \nabla \psi = \nabla \cdot |\nabla \psi|^2 vC = 0.$$  

No similar relation can be found for $K$. This makes this vector fundamentally different from the other basis vectors.

Consider the general properties of a vector $V$ with $\nabla \cdot V = 0$ and $V \cdot \nabla \psi = 0$. These properties allow $V$ to be written in a Clebsch representation:

$$V = \nabla \beta_V \times \nabla \psi.$$  

The most general form for $\beta_V$ in toroidal geometry is [10]:

$$\beta_V = T_V(\psi) \zeta - Z_V(\psi) \Theta + \tilde{\beta}_V(\psi, \Theta, \zeta)$$

where $\tilde{\beta}_V$ is a periodic function of $\Theta$ and $\zeta$. This allows $V$ to be written as

$$V = T_V \nabla \zeta \times \nabla \psi + Z_V \nabla \Theta \times \nabla \psi + \nabla \tilde{\beta}_V \times \nabla \psi$$

$$= J^{-1} \left[ \left( T_V + \frac{\partial \tilde{\beta}_V}{\partial \zeta} \right) e_\Theta - \left( Z_V - \frac{\partial \tilde{\beta}_V}{\partial \Theta} \right) e_\zeta \right]$$
which is the covariant form for $V$.

Using the expressions for the contravariant components of $V$ in Eq. (24), the poloidal flux of $V$ may be easily calculated:

$$\Gamma_{V}^{\Theta}(\psi) \equiv \frac{1}{(2\pi)^2} \int \oint \hat{V} \cdot \nabla \Theta \mathcal{J} d\xi d\psi$$

$$= \frac{1}{2\pi} \int T_{V} d\psi + \frac{1}{(2\pi)^2} \int \oint \frac{\partial \hat{\beta}_{V}}{\partial \zeta} d\xi d\psi.$$ 

The last term is zero due to periodicity. Following a similar procedure for the toroidal flux, $\Gamma_{V}^{\zeta}(\psi) \equiv 1/(2\pi) \int \oint \hat{V} \cdot \nabla \zeta \mathcal{J} d\Theta d\psi$, one obtains the relationships

$$T_{V} = 2\pi \Gamma_{V}^{\Theta} / \Gamma_{V}^{\Theta}'; \quad Z_{V} = 2\pi \Gamma_{V}^{\zeta} / \Gamma_{V}^{\zeta'}.$$ (24)

which relates the contravariant components to the fluxes.

The fact that $V \cdot \nabla \psi = V^{\psi} = 0$ allows the concept of straightness to be defined. A vector $V$ with zero $\psi$ contravariant component is said to be straight if for each flux surface defined by $\psi$, the ratio of the remaining contravariant components is constant; i.e., $V$ is straight if $V^{\zeta}/V^{\Theta} = f(\psi)$. Looking at the contravariant components for $V$ in Eq. (24), it is apparent that this is true, if and only if $\hat{\beta}_{V} = 0$. In this case,

$$\frac{V^{\zeta}}{V^{\Theta}} = \frac{Z_{V}}{T_{V}} = \frac{\Gamma_{V}^{\zeta} / \Gamma_{V}^{\Theta}}{\Gamma_{V}^{\zeta'} / \Gamma_{V}^{\Theta'}} = f(\psi).$$ (25)

Calculations involving a straight vector are obviously greatly simplified because of the simplification of the contravariant components to flux functions.

To convert between $B$ and $C$, and $J$ and $K$, the following identity for the concomitant vector, $\nabla \psi \times V / |\nabla \psi|^2$, is useful:

$$\frac{\nabla \psi \times V}{|\nabla \psi|^2} = \nabla \psi \times (\nabla \beta_{V} \times \nabla \psi)$$

$$= \nabla \beta_{V} - \frac{\nabla \beta_{V} \cdot \nabla \psi}{|\nabla \psi|^2} \nabla \psi$$

$$= \nabla_{t} \beta_{V}$$

where in the last line a new operator is introduced for convenience. The $\nabla_{t}$ operator with respect to $\nabla \psi$ is analogous to the more familiar $\nabla_{\perp}$ operator with respect to $B$. The $t$ notation is used because $\nabla_{t} f$ is tangential to the constant $\psi$ surfaces. Using this notation and the equation for $\beta_{V}$, Eq. (23), the concomitant vector may be written as

$$\frac{\nabla \psi \times V}{|\nabla \psi|^2} = \left(T_{V} + \frac{\partial \hat{\beta}_{V}}{\partial \zeta}\right) \nabla_{t} \zeta - \left(Z_{V} - \frac{\partial \hat{\beta}_{V}}{\partial \Theta}\right) \nabla_{t} \Theta.$$ (26)

Since $V$ has a convenient contravariant form, the concomitant vector will have a convenient covariant form, especially if $V$ is straight.

After the development of this formalism, the calculations for the specific cases of $V \in \{B, J, |\nabla \psi|^2 C\}$ are greatly simplified. In particular, our goal is to find expressions for $\{T, Z, \hat{\beta}\}_{\{B,C,J\}}$.

**B. Flux coordinates**

In this paper, only flux coordinate systems where $B$ is straight ($\hat{\beta}_{B} = 0$) are considered. Because the radial coordinate has been chosen to be $\psi \equiv 2\pi \Gamma_{B}^{\Theta}$, the components of $B$ have the nice form

$$T_{B} = 1; \quad Z_{B} = q(\psi)$$ (27)
where
\[ q(\psi) \equiv \frac{B^\zeta}{B^\Theta} = \frac{\Gamma^\zeta_{\zeta}}{\Gamma^\Theta_{\Theta}}. \] (28)

is the safety factor. This simple form for \( B \) allows \( C \) to be written in the contravariant form
\[ C = \nabla_\zeta - q\nabla_\Theta \] (29)
using Eq. (26).

Considering the covariant form of \( |\nabla \psi|^2 C \) from Eq. (24) and using Eq. (26) to find the contravariant form of \( B \) (Eq. (30)) yields
\[ B = \left( T_C + \frac{\partial \delta_C}{\partial \zeta} \right) \nabla_\zeta + \left( Z_C - \frac{\partial \delta_C}{\partial \Theta} \right) \nabla_\Theta. \] (30)

At this point, the ignorability of \( \zeta \) is used to simplify the \( \zeta \) covariant component of \( B \). We define
\[ F(\psi) \equiv -T_C = -J C^\Theta = -2\pi \Gamma_C^{\Theta'} \] (31)
In axisymmetric systems, \( F \) is known as the toroidal flux function \( F = R B_{toroidal} \). To find a useful expression for \( Z_C \), the covariant form of \( B \) (Eq. (30)) is dotted with the contravariant form of \( B \) to give
\[ B^2 = J^{-1} \left( Z_C - \frac{\partial \delta_C}{\partial \Theta} \right) + J^{-1} qF. \] (32)

For convenience, we define a new variable, \( g \),
\[ g \equiv Z_C - \frac{\partial \delta_C}{\partial \Theta} = JB^2 - qF. \] (33)
Using the \( \Theta \) average allows us to annihilate the second term and solve:
\[ Z_C \equiv \langle g \rangle_{\Theta} = G(\psi) = \langle JB^2 \rangle_{\Theta} - qF = \frac{V'}{4\pi^2} \langle B^2 \rangle - qF; \]
\[ \frac{\partial \delta_C}{\partial \Theta} = \langle JB^2 \rangle_{\Theta} - JB^2 = \frac{V'}{4\pi^2} \langle B^2 \rangle - JB^2. \]

Evidently, to make \( C \) straight, \( JB^2 \) needs to be a flux function. A coordinate system with this property is known as the Boozer flux coordinate system [10, 12]. Using these definitions, the covariant form of \( B \) can be written as \( B = F \nabla_\zeta + g \nabla_\Theta \).

Taking the curl of \( B \) to give the current density gives
\[ J = -F' \nabla \zeta \times \nabla \psi - \left[ \frac{\partial g}{\partial \psi} - \frac{\partial}{\partial \Theta} \left( F \frac{g^\psi_{\zeta}}{g^\psi_{\psi}} + g \frac{\psi_{\Theta}}{g^\psi_{\psi}} \right) \right] \nabla \Theta \times \nabla \psi. \] (34)
The expression for \( T_J \) is simply \( T_J = -F' \); thus, the function \( F \) introduced above is proportional to the poloidal current enclosed by a flux surface as seen from Eq. [24] \( F = -2\pi \Gamma'_\zeta \). Similarly, one can show \( G(\psi) = \langle g \rangle_{\Theta} = 2\pi \Gamma'_\Theta \) is proportional to the toroidal current enclosed by a flux surface.

The above expression for \( J \) was derived using Ampere’s law. The expression for \( Z_J \) can be simplified by deriving another covariant form of \( J \) using Eq. [13] and the covariant forms for \( B \) and \( C \) already derived. Using the covariant forms yields
\[ J = \left( \sigma + \frac{P' F}{B^2} \right) \nabla \zeta \times \nabla \psi - \left( q\sigma - \frac{P' g}{B^2} \right) \nabla \Theta \times \nabla \psi. \] (35)
Equating the \( \Theta \) contravariant components yields an expression for the parallel current:
\[ \sigma = -F' - \frac{P' F}{B^2}. \] (36)
This relation allows us to rewrite the $\zeta$ contravariant component of $J$ in Eq. (35) as

$$h \equiv Z_J - \frac{\partial \hat{\beta}_J}{\partial \Theta} = q\sigma - \frac{P'g}{B^2} = -(P'J + qF')$$

(37)

where $h$ is defined analogously to $g$. Using the $\Theta$ average again allows us to annihilate the second term and solve for each term:

$$Z_J = \langle h \rangle_\Theta = G'(\psi) = -\left(\frac{V'}{4\pi^2}P' + qF'\right) = 2\pi I^\zeta_J;$$

$$\frac{\partial \hat{\beta}_J}{\partial \Theta} = P'(J - \langle J \rangle_\Theta) = P'\left(J - \frac{V'}{4\pi^2}\right).$$

Evidently, to make $J$ straight, the Jacobian needs to be a flux function. A coordinate system with this property is known as the Hamada flux coordinate system.

Equating the $\zeta$ contravariant components yields

$$\frac{\partial g}{\partial \psi} + \frac{\partial}{\partial \Theta} \left(Fg^\psi + g^\psi g\psi\right) = h$$

(38)

which is a generalized Grad-Shafranov equation.

The contravariant form of $J$ can be derived using Eq. (13) and the covariant forms for $B$ and $C$ already derived. Using the contravariant forms yields

$$J = \left(\sigma F - \frac{P'}{C^2}\right) \nabla_\zeta + \left(\sigma g + \frac{qP'}{C^2}\right) \nabla_\Theta.$$  

(39)

Using the contravariant and covariant forms of $J$, expressions for $K$ may be easily derived using Eq. [5]. The results are

$$K = -\frac{1}{P'} \left[\left(\sigma F - \frac{P'}{C^2}\right) \nabla_\zeta \times \nabla \psi \left(\sigma g + \frac{qP'}{C^2}\right) \nabla_\Theta \times \nabla \psi\right]$$

(40)

and

$$K = -\frac{1}{P'} \left[F'\nabla_\zeta \zeta + h\nabla_\zeta \Theta\right]$$

(41)

We summarize the expressions which were derived above for all four of our basis vectors in Table I.

C. Helical angle coordinates

In addition to the $\psi, \Theta, \zeta$ coordinate system, another useful coordinate system is the $\psi, \Theta, \alpha$ coordinate system where $\alpha$ is the Clebsch angle for $B$

$$\alpha \equiv \beta_B = \zeta - q\Theta.$$  

(42)

such that $B = \nabla \alpha \times \nabla \psi$. If $\zeta$ is an ignorable coordinate, then $\alpha$ is an ignorable coordinate also. The primary advantage of this coordinate system is that the tangential gradient of $\alpha$ is equal to $C$ as can easily be seen by Eq. (29):

$$C = \nabla_\zeta \alpha = \nabla \alpha - \frac{g^\psi \alpha}{g^\psi \psi} \nabla \psi.$$  

(43)

Expressions for the basis vectors using this coordinate system can be easily evaluated. We summarize the expressions which were derived above for all four of our basis vectors in Table I.
Examination of the equation for $\nabla_t \alpha$, Eq. (43), in more detail, shows that the last term may be written as

$$\frac{g^{\psi \alpha}}{g^{\psi \psi}} = \frac{1}{|\nabla \psi|^2} (\nabla \zeta \cdot \nabla \psi - \nabla (q \Theta) \cdot \nabla \psi) = - \left( q' \Theta + \frac{gg^{\psi \Theta} - g^{\psi \zeta}}{g^{\psi \psi}} \right).$$

The $R$ term is known as the residual shear and will be discussed in the next section. As evidenced by the $q' \Theta$ term in the previous equation, $\alpha$ is not a periodic coordinate. While this coordinate system is still useful for short wavelength modes, it is often more useful to have a coordinate system which is periodic, yet closely related to $\alpha$. We define the coordinate $u$ as

$$u \equiv \zeta - q_s \Theta = \alpha + (q - q_s) \Theta \quad (44)$$

where $q_s$ is the value of $q$ at a particular surface. For a vector $F \perp \nabla \psi$, $F \cdot \nabla u = F \cdot \nabla \alpha + (q - q_s) F \cdot \nabla \Theta$ which may be used to find the $u$ contravariant components and following a similar derivation as above, the covariant components as well. The summary of these relationships is shown in vectors in Table 1 as well.

D. Special cases of flux coordinate systems

Until this point, $\Theta$ and $\zeta$ are only specified such that the magnetic field lines are straight:

$$\frac{B \cdot \nabla \zeta}{B \cdot \nabla \Theta} = q(\psi). \quad (45)$$

This is one constraint on two quantities. To completely specify a straight-field line coordinate system, another requirement is needed; generally the Jacobian is specified.

Previously we mentioned two specific coordinates, the Boozer and Hamada coordinate systems, and here we describe their Jacobians and properties. Recall that Hamada coordinates specify the constraint that $J$ is straight,

$$\frac{J \cdot \nabla \zeta}{J \cdot \nabla \Theta} = f(\psi), \quad (46)$$

and Boozer coordinates specify that the magnetic binormal vector is straight $^1$,

$$\frac{C \cdot \nabla \zeta}{C \cdot \nabla \Theta} = f(\psi). \quad (47)$$

In addition to these two coordinate systems, we also discuss the “symmetry coordinates”. Practical aspects related to calculating the coordinate systems numerically may be found in DMS.\[5\]

1. Hamada Coordinate System

Hamada coordinates are elegant because it makes the current density field lines straight as well, thus continuing the analogous relationships between $B$ and $J$. From Eq. (37), we see that

$$\frac{J \cdot \nabla \zeta}{J \cdot \nabla \Theta} = \frac{h}{F^r} = q + J \frac{P'}{F^r} \quad (48)$$

To make these straight, we need to make the Jacobian a flux-function. This flux function may be found from the relationship:

$$\langle J^{-1} \rangle = \frac{4\pi^2}{\sqrt{r}}. \quad (49)$$

---

$^1$ A useful mnemonic: Allen Boozer made $C$ straight as simple as ABC. Hamada Is $J$ straight as simple as HIJ.
which implies that the Jacobian for the Hamada coordinate system may be written as

\[ J_H = \frac{V'}{4\pi^2}. \] (50)

\[ J = -F'(\psi)\nabla \zeta \times \nabla \psi - G'(\psi)\nabla \Theta \times \nabla \psi. \] (51)

with \( G'/F' = q + 4\pi^2/(F'V') \).

\[ -qF' - G' = \frac{V'}{4\pi^2} P'. \] (52)

which is the Grad-Shafranov equation in Hamada coordinates (using \( \psi \) as the dependent variable). This relationship is used extensively in Greene and Johnson [3], and in the original derivation of the inner layer equations of GGJ.

2. Boozer Coordinate System

Boozer coordinates have nice properties because \( C \) is straight as well as \( B \). From Eq. (33), we see that

\[ \frac{C \cdot \nabla \zeta}{C \cdot \nabla \Theta} = -\frac{q}{F} = q - \frac{J B^2}{F} \] (53)

To make these straight, we need to have

\[ J_B B^2 = G(\psi) \] (54)

To find this flux function, we have

\[ \left\langle \frac{J_B^{-1}}{B^2} B^2 \right\rangle = \frac{J_B^{-1}}{B^2} \left\langle B^2 \right\rangle = \frac{4\pi^2}{Vr} \] (55)

or

\[ J_B B^2 = G(\psi) = \frac{V'}{4\pi^2}. \] (56)

3. Symmetry Coordinates

In an axisymmetric system, the Boozer and Hamada surfaces of constant \( \zeta \) will not correspond to surfaces of constant \( \phi \) where \( \phi \) is the “symmetry” angle. That is, given a cylindrical \( R, \phi, Z \) coordinate system, one is interested in the case of \( \zeta = -\phi \).

In this coordinate system, \( \nabla \zeta \cdot \nabla \zeta = 1/R^2 \), \( \nabla \zeta \cdot \nabla \psi = 0 \), and \( \nabla \zeta \cdot \nabla \Theta = 0 \). This means that the covariant and contravariant components are parallel:

\[ \nabla \Theta \times \nabla \psi = -J^{-1} e_\zeta = J^{-1} R^2 \nabla \zeta \] (57)

which allows us to write the magnetic field as

\[ B = J^{-1} q R^2 \nabla \zeta + \nabla \zeta \times \nabla \psi. \] (58)

Because of the orthogonality of the \( \zeta \) surfaces and \( \psi \) surfaces in this coordinate system, the tangential derivative is the same such that we can write

\[ B = F \nabla \zeta + g \nabla \Theta. \] (59)

\[ \text{The minus sign is to enable the right handed-ness to remain in the (} R, \phi, Z \text{) and (} \psi, \Theta, \zeta \text{) coordinate systems.} \]
Equating the two expressions for the covariant components allows us to write the familiar form for the symmetric angle \( B \) \[13\].

\[
B = F \nabla \zeta + \nabla \zeta \times \nabla \psi, \quad (60)
\]

with the Jacobian in this system given as

\[
\mathcal{J} = \frac{q R^2}{F}. \quad (61)
\]

This is sometimes referred to as the PEST angle because the first paper describing it was in a paper by Grimm, Greene, and Johnson \[14\] describing the PEST ideal MHD code.
IV. GEOMETRIC QUANTITIES AND FRENET’S LOCAL FRAME OF REFERENCE

Expressions using the quantities of the first section such as $\sigma$ or $P'$, or the metric elements of the flux coordinate system, can be difficult to use for gaining physical intuition. In this section, common geometric quantities are introduced and related to the reference frames of the previous sections.

Considerations of a particle tracing out a curve in three-dimensional Euclidean space lead to studies in the mid-19th century of the geometry of curves. This led to the development of the Frenet frame of reference, which uses as its local basis vectors the tangent of the curve, the curvature vector which is normal to the tangent, and the binormal vector. The fundamental theorem of space curves states that in three-dimensional space every curve with non-zero curvature has its shape completely determined by curvature and torsion. Because of the importance of the $BC$ frame of reference, the relationship of this frame to the curvature and torsion has been considered for some time [5, 15]. That is, while the curvature and torsion of $J$ might also be of interest, we will only consider the curvature and torsion of the magnetic field. Our goal is to express the Frenet quantities of torsion and curvature in terms of our quantities from the $BC$ and $JK$ reference frames.

Hegna [15] related the quantities of torsion to the normalized version of our $BC$ frame (also known as the Stix frame). The relationships are:

\[
\begin{align*}
\hat{b} \cdot \nabla \hat{n} &= -\kappa_n \hat{b} - \tau_n \hat{n} \times \hat{b} \\
\hat{b} \cdot \nabla \hat{b} &= \kappa_n \hat{n} - \kappa_g \hat{n} \times \hat{b} \\
\hat{b} \cdot \nabla \hat{n} \times \hat{b} &= \tau_n \hat{n} + \kappa_g \hat{b}
\end{align*}
\]

The curvature, rather than acting as the normal component in the Frenet frame, is now decomposed into the normal curvature, $\kappa_n$ that is perpendicular to a flux surface, and the geodesic curvature, $\kappa_g$, that is parallel to the magnetic binormal direction. In these formulas, the only component of torsion that matters is the normal torsion, $\tau_n$. Here, we briefly review the relationship of these quantities to our frames, and derive other important geometric relationships.

A. Magnetic Shear

Although a single curve can be described by curvature and torsion, it does not convey how neighboring curves behave. In magnetically confined systems, how a magnetic field line moves relative to a neighboring field line is important for modes that have finite width. The most useful measure of this variation is the magnetic shear, $S$, which is defined as [9, 16]:

\[
S \equiv -\frac{1}{C^2} C \cdot \nabla \times C.
\]

Definition of local shear often differs from this definition by either a minus sign [15] or the factor of $C^2$ [9, 16].

Although the $C$ component is generally the most useful component of $\nabla \times C$, we will consider the other components as well. The normal component can be shown to be zero using the divergence-free nature of $B$:

\[
\nabla \cdot B = \nabla \cdot (C \times \psi) = \nabla \psi \cdot \nabla \times C = 0.
\]

Using the relation between $C$ and $\alpha$ (Eq. 43), the curl of $C$ may be written as

\[
\nabla \times C = \left( |\nabla \psi|^2 C \cdot \nabla \frac{g^{\psi \alpha}}{g^{\psi \psi}} \right) \frac{B}{B^2} - \left( B \cdot \nabla \frac{g^{\psi \alpha}}{g^{\psi \psi}} \right) \frac{C}{C^2}
\]

Using Eq. (44), the local magnetic shear then is given by

\[
C^2 S = -B \cdot \nabla \frac{g^{\psi \alpha}}{g^{\psi \psi}} = J^{-1} \left( q' + \frac{\partial R}{\partial \theta} \right)
\]
Thus \( q' \) is termed the global shear, and \( R \) is termed the integrated residual shear because its derivative gives the variation of the local shear within a flux surface.

Performing a similar calculation for the parallel component of the curl of \( C \) allows us to write the total curl as

\[
\nabla \times C = -C^2 \left[ FS \frac{B}{B^2} + S \frac{C}{C^2} \right] 
\]

\( \text{(66)} \)

### B. Torsion

The normal component of the torsion is given by

\[
\tau_n = \hat{b} \cdot \nabla \frac{C}{|C|} \cdot \frac{\nabla \psi}{|\nabla \psi|} = \frac{1}{B^2} \mathbf{B} \cdot \nabla C \cdot \nabla \psi. 
\]

\( \text{(67)} \)

Using the identity

\[
2 \mathbf{a} \cdot \nabla \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \nabla (\mathbf{b} \cdot \mathbf{c}) - \mathbf{b} \cdot \nabla (\mathbf{c} \cdot \mathbf{a}) + \mathbf{c} \cdot \nabla (\mathbf{a} \cdot \mathbf{b})
\]

the normal torsion can be related to other geometric quantities

\[
2 \mathbf{B} \cdot \nabla C \cdot \nabla \psi = \mathbf{C} \times \nabla \psi \cdot \nabla \times \mathbf{B} - \nabla \psi \times \mathbf{B} \cdot \nabla \times \mathbf{C}
\]

or, relating the normal component of the torsion to the above quantity:

\[
2\tau_n = \sigma + \mathcal{S}. 
\]

\( \text{(68)} \)

This relationship is discussed in both Ref. [17] and [15] and can be viewed as a type of Grad-Shafranov equation.

### C. Curvature

The curvature of the magnetic field line is defined as

\[
\kappa \equiv \left( \hat{b} \cdot \nabla \right) \hat{b}. 
\]

\( \text{(69)} \)

Using the equilibrium force balance \( (\mathbf{J} \times \mathbf{B} = \nabla P) \) and Ampere’s law \( (\mathbf{J} = \nabla \times \mathbf{B}) \), the magnetic curvature can be expressed as

\[
\kappa = \frac{1}{B^2} \nabla_\perp \left( P + \frac{B^2}{2} \right). 
\]

\( \text{(70)} \)

Because the curvature vector is perpendicular to the magnetic field, we can write the curvature vector in the contravariant form: \( \kappa = \kappa_\psi \nabla \psi + \kappa_C \mathbf{C} \). The covariant components are trivially related to the normal and geodesic curvatures by normalization constants; \( \kappa_n = \kappa_\psi |\nabla \psi|, \kappa_\theta = \kappa_C |\mathbf{C}| \). Using Eq. \text{(70)} expressions for the covariant components of the curvature may be easily calculated:

\[
\kappa_\psi = \frac{1}{B^2} \frac{\partial}{\partial \psi} \left( P + \frac{B^2}{2} \right) + \frac{g^{\psi \theta}}{g^{\psi \psi}} \frac{1}{B^2} \frac{\partial}{\partial \theta} \left( \frac{B^2}{2} \right) 
\]

\( \text{(71)} \)

\[
\kappa_C = \frac{1}{C^2 B^2} \mathbf{C} \cdot \nabla \theta \frac{\partial}{\partial \theta} \left( \frac{B^2}{2} \right) = -F \mathcal{J}^{-1} \frac{1}{B^2} \frac{\partial}{\partial \theta} \left( \frac{B^2}{2} \right). 
\]

\( \text{(72)} \)
This will be used in the next section.

An important relation for the geodesic curvature can be derived using $\nabla \cdot J = 0$ and Eqs. (13), 21, 72:

$$\nabla \cdot J = \nabla \cdot \left( \sigma B - P' C \frac{C}{2} \right) = B \cdot \nabla \sigma - \nabla \cdot \frac{P'}{B^2} \mid \nabla \psi \mid^2 C = B \cdot \nabla \sigma + P' 2\kappa_C$$

(73)

or,

$$2\kappa_C = -B \cdot \nabla \frac{\sigma}{P'}.$$  

(74)

Thus we can see that the angle between the $BC$ coordinate system and the $JK$ coordinate system which was shown to be related to $\sigma/P'$ is related to the geodesic curvature.

D. Curl($K$)

Due to its relationship to the local magnetic shear, the curl of $C$ is an important quantity. The curl of $K$ is also an important quantity and here we examine its components. Similar to the curl of $C$, the curl of $K$ can be shown to be perpendicular to $\nabla \psi$ using $\nabla \cdot J = 0$ which allows us to decompose the curl using $J$ and $K$ components, or $B$ and $C$ components. The $K$ component is the definition of the shear of the current density field. Experience shows that this is not a very useful quantity, so we opt for examining the $BC$ components of the curl. Calculating the curl of $K$:

$$\nabla \times K = \nabla \times \frac{B}{B^2} + \nabla \times \frac{\sigma}{P'} C = \sigma \frac{B}{B^2} + \frac{B \times \nabla (P + B)}{B^4} - C \times \nabla \frac{\sigma}{P'} + \frac{\sigma}{P'} \nabla \times C$$

(75)

The parallel component is

$$B \cdot \nabla \times K = \sigma - B^2 \frac{\partial}{\partial \psi} \left( \frac{\sigma}{P'} \right) - \frac{\sigma C^2}{P'} F S$$

(76)

where we have used Eq. (66). The perpendicular component is

$$C \cdot \nabla \times K = -2\kappa_{\psi} + \frac{P'}{B^2} - \frac{\sigma C^2}{P'} S$$

(77)

Using the generalized Grad-Shafranov equation (Eq. (38)) and equations for the normal curvature (Eq. (71)) and shear (Eqs. (65) and (44)), one can derive

$$C \cdot \nabla \times K = J^{-1} \frac{\partial}{\partial \psi} J + J^{-1} \frac{\partial}{\partial \Theta} \left( J \frac{g^{\psi \Theta}}{g^{\psi \psi}} \right) + \frac{F' C^2}{P'} S$$

(78)

The importance of this component is discussed in the derivation of the inner layer equations.

A partial listing of the expressions in this section are summarized in Table I.
Summary of equilibrium-based local frames of reference

\[ \nabla \psi = \frac{B \times C}{|B|^2} \]
\[ B = C \times \nabla \psi \]
\[ J = P' K \times \nabla \psi \]
\[ C \equiv \frac{\nabla \times B}{|B|^2} \]
\[ C^2 = \frac{\nabla \times J}{|J|^2} \]
\[ B = \sigma B' J + P' K \]
\[ J' = \sigma B - P' C \]
\[ C = \frac{\sigma^2 K}{P' K^2} - P' J \]
\[ K = \frac{P' C + B}{P' J} \]
\[ f = f^\psi \frac{\nabla \psi}{|\nabla \psi|^2} + f^B \frac{B}{|B|^2} + f^C \frac{C}{|C|^2} \]
\[ f^B = \frac{1}{\sigma g} (\sigma C^2 f^j + f^K) \]
\[ f^j = \frac{1}{\sigma g} (\sigma C^2 f^B - B' f^C) \]
\[ f^K = \frac{1}{\sigma g} (\sigma B^2 f^j - P' f^j) \]
\[ \text{Summary of helical angle coordinate’s relationship to equilibrium-based frames} \]
\[ B = \nabla \alpha \times \nabla \psi = F \nabla \alpha + d \nabla \psi \]
\[ C = \frac{1}{|\nabla \psi|^2} \left[ F \nabla \alpha \times \nabla \psi + g \nabla \alpha \times \nabla \psi \right] = \nabla \alpha \quad \text{Boozer: } d \equiv J B^2 - q F; \]
\[ J = -F' \nabla \alpha \times \nabla \psi - h \nabla \alpha \times \nabla \psi \]
\[ \text{Hamada: } h = G' \Rightarrow J \text{ is straight} \]
\[ K = \frac{1}{|\nabla \psi|^2} \left[ \left( \sigma F - \frac{\mu}{\sigma g} \right) \nabla \alpha \right] + \left( \sigma g + \frac{\mu}{\sigma g} \right) \nabla \psi \]
\[ \text{Brief summary of key geometric quantities} \]
\[ S \equiv -\frac{\mu}{\sigma} C \cdot \nabla \times C \quad \nabla \times C = -C^2 \left[ F S \frac{B}{P' C^2} + S \frac{C}{P' C^2} \right] \]
\[ \tau_n \equiv \frac{1}{2 P' C} B \cdot \nabla C \cdot \nabla \psi \quad 2 \tau_n = \sigma + S \]
\[ \kappa \equiv \left( \frac{b \cdot \nabla}{\sigma} \right) b \quad \kappa = \frac{\sigma}{P' C} \nabla \left( P + \frac{b^2}{2} \right) \]
\[ C \cdot \nabla \times K = -2 \kappa \psi + \frac{P'}{P' C} - \frac{\sigma C^2}{P' C} S \]

Table I: Summary of the flux and helical-angle coordinate systems with demonstration of the special cases.
V. BRIEF EXAMPLE OF EQUILIBRIUM-BASED COORDINATE CALCULATIONS: IDEAL MHD

Before discussing the derivation of the inner layer equations, we give a brief example of how the new formalism simplifies computations that arise in MHD. The energy principle is given by [13, 18]:

\[
W_F = \frac{\mu_0}{2} \int dV \left( \frac{\vec{Q}}{\mu_0} + (\vec{J} \times \hat{n}) (\xi \cdot \hat{n}) \right)^2 + \Gamma P (\nabla \cdot \xi)^2 - 2 |\vec{B} \cdot \nabla \hat{n} \cdot \vec{J} \times \hat{n}| (\xi \cdot \hat{n})^2 \]

Expressing this in terms of our equilibrium-based coordinate systems shows that the vector \( \vec{K} \), which has not been discussed before, features prominently:

\[
W_F = \frac{\mu_0}{2} \int dV \left( \frac{\vec{Q}}{\mu_0} + \frac{\xi \psi}{P'} \vec{K} \right)^2 + \Gamma P (\nabla \cdot \xi)^2 + \left[ 2P' B \mid \nabla \psi \mid \vec{b} \cdot \nabla \hat{n} \cdot \vec{K} \right] (\xi \psi)^2
\]

The fact that \( \vec{K} \) appears in every both the non-compressive stabilizing term, and the de-stabilizing term indicates that this vector is important in MHD studies.

VI. DERIVATION OF INNER LAYER EQUATIONS

Although fusion plasmas are very nearly ideal, resistivity can still change the topology near surfaces where the magnetic winding number is rational. Just as the flow of wind over an airplane wing, which may be analyzed using boundary layer theory, resistive modes can be analyzed using boundary layer theory when the boundary layers occur near rational surfaces. In this theory, the plasma is analyzed in two regions: an “outer region” where the plasma is ideal, and an “inner layer” where dissipation is important.

In this section, linear equations are found for the inner resistive layer based on the narrow-layer-width approximation. The inner-layer equations in cylindrical geometry were originally derived by Coppi, Greene, and Johnson (CGJ). In toroidal geometry, the derivation is much more difficult because the \( 1/R \) dependence of the magnetic field causes the poloidal harmonics to be coupled. The formalism used in references [1, 9] is used in deriving these equations in toroidal geometry, which are presented in the Glasser-Greene-Johnson (GGJ) paper [1]. Here we show how the knowledge of the coordinate systems allows one to more easily derive these relations.
A. Resistive instability ordering

We will consider a narrow layer width at the $q_s = M/N$ rational surface where we adopt the usual resistive layer ordering

$$x \equiv \psi - \psi_s \sim \gamma \sim \eta^{1/3} \sim \epsilon, \quad \frac{\partial}{\partial \psi} \sim \frac{1}{\epsilon}. \quad (83)$$

Equilibrium quantities will be considered to be approximately constant across the layer.

We begin with the linearized version of the MHD equations. Our notation is such that equilibrium quantities are shown with capital letters, and perturbed quantities are denoted with lower case letters. That is, for an arbitrary quantity $Q$, we write

$$Q = Q_0 + \epsilon q_1 + \epsilon^2 q_2 + ... \quad (84)$$

The linearized MHD equations are then:

$$\gamma^2 \rho \xi = (\nabla \times b) \times B + J \times b - \nabla p \quad (85)$$

$$b = \frac{\eta}{\gamma} \nabla \times (\nabla \times b) + \nabla \times (\xi \times b) \quad (86)$$

$$p = \xi \cdot \nabla P + \gamma h P (\nabla \cdot \xi) \quad (87)$$

where $\gamma_h$ is the ratio of specific heats, $\xi$ is the velocity divided by complex growth rate $\gamma$.

Because we are considering perturbed quantities near the rational surface, the $\psi, \Theta, u$ coordinate system is the most useful because it allows easy identification of resonant perturbations when Fourier expanding the perturbed quantities:

$$f = \sum_{m,n} f_{m,n}(\psi) e^{i(m\Theta - n\xi)} e^{\gamma t}$$

$$= \sum_{m,n} f_{m,n}(\psi) e^{-iN\nu} e^{\gamma t} + \sum_{m,n \neq M,N} f_{m,n}(\psi) e^{-iN\nu} e^{i(m-nq_s)\Theta}$$

$$= \sum_n f_n(\psi, \Theta) e^{\gamma t - iN\nu}.$$

where again $q_s = M/N$ is the rational surface of interest. The last line is the procedure we will adopt for simplicity. If after transforming, the Fourier transform of a perturbed quantity, $f_n$, is independent of $\Theta$, then only the resonant harmonic is present.

The helical coordinate also facilitates the representation of the parallel gradient operator:

$$B \cdot \nabla f = J^{-1} \left( \frac{\partial f}{\partial \Theta} + (q - q_s) \frac{\partial f}{\partial u} \right)$$

$$\approx J^{-1} \frac{\partial f}{\partial \Theta} - \epsilon i N \lambda x f$$

where $\lambda = J^{-1} q'$ and $x = \psi - \psi_s$. In the last line, $q - q_s$ was Taylor expanded about the rational surface after Fourier expanding to give the approximation $q - q_s \approx \epsilon q' x$. Other directional derivatives may be easily computed in a similar manner

$$J \cdot \nabla f \approx P' i N f - F' J^{-1} \frac{\partial f}{\partial \Theta} - \epsilon F' i N \lambda x f \quad (88)$$

$$|\nabla \psi|^2 C \cdot \nabla f \approx B'^2 i N f - F' J^{-1} \frac{\partial f}{\partial \Theta} - \epsilon F i N \lambda x f \quad (89)$$

The fact that straight-field line coordinate systems give convenient forms for $B \cdot \nabla$ operators has long been recognized. This shows that $C \cdot \nabla$ and $J \cdot \nabla$ are greatly simplified as well.
As is obvious from the above relations, terms of the form $\mathcal{J}^{-1} \partial Q / \partial \Theta$ will often arise. It will be found useful to eliminate these terms which arise due to non-resonant harmonics to reduce the problem from a two-dimensional problem in $\psi, \Theta$ to a one-dimensional problem in $\psi$. To do so, we introduce the averaging operator:

$$\langle Q \rangle = \frac{2\pi}{V} \int Q \mathcal{J} \, d\Theta,$$

(90)

If $Q$ is symmetric such that there is no $\zeta$ dependence, this average is the same as the flux surface average given in Eq. (19).

For the vector quantities, decomposition using the covariant basis vectors is the most convenient. The perturbed quantities are decomposed and ordered as

$$b = b_\psi^{(2)} \frac{\nabla \psi}{|\nabla \psi|^2} + b_B^{(1)} \frac{B}{B^2} + b_C^{(1)} \frac{C}{C^2} \quad \rightarrow \quad b = b_\psi^{(2)} \frac{\nabla \psi}{|\nabla \psi|^2} + b_J^{(1)} \frac{J}{J^2} + b_K^{(1)} \frac{K}{K^2}$$

(91)

$$\xi = \xi_\psi^{(2)} \frac{\nabla \psi}{|\nabla \psi|^2} + \xi_B^{(1)} \frac{B}{B^2} + \xi_C^{(1)} \frac{C}{C^2} \quad \rightarrow \quad \xi = \xi_\psi^{(2)} \frac{\nabla \psi}{|\nabla \psi|^2} + \xi_J^{(1)} \frac{J}{J^2} + \xi_K^{(1)} \frac{K}{K^2}$$

(92)

$$p = p^{(1)}.$$  

(93)

The $\psi$ components of the vector quantities are lower order to help satisfy the divergence criterion as shown below. The ordering also arises from viewing the vectors as arising from a potential of order $\epsilon^2$, and the non-$\psi$ components are of lower order because of the $d/d\psi$ derivative which arises in those components [11].

Because the perturbed magnetic field is divergence-free ($\nabla \cdot b = 0$), the components can be related to each other:

$$\nabla \cdot b = \mathcal{J}^{-1} \left[ \frac{\partial}{\partial \psi} (J b_\psi^{(2)}) + \frac{\partial}{\partial \Theta} \left( \frac{1}{B^2} (b_B^{(1)} - F b_C^{(1)}) \right) + \frac{\partial}{\partial u} (J b_C^{(1)}) \right] + \mathcal{O}(\epsilon^2)$$

(94)

Taking the average of the last expression, we have

$$\frac{\partial}{\partial \psi} \langle b_\psi^{(2)} \rangle = iN \langle b_C^{(1)} \rangle.$$

(95)

The orderings shown in Eq. (91) are motivated by this balance. Looking at the pressure equation, Eq. (93), we see that the first two terms are of order $\epsilon^2$ which implies that the plasma is incompressible to lowest order, $\nabla \cdot \xi \approx 0$. An equation similar to Eq. (95) for the displacement vector can then be derived.

### B. First-order equations and magnetosonic waves

Our goal is to obtain equations for averaged perturbed quantities $\langle b_\psi \rangle, \langle \xi_\psi \rangle, \langle b_B \rangle$ by taking the appropriate projections of the linearized MHD equations and ordering. The details of this calculation are presented in the appendix. Here we summarize the results of the lowest order equations which gives the components whose resonant harmonics are dominant at the rational surface:

$$\nabla \psi \cdot (\text{Momentum Eq.}) \rightarrow p = -b_B$$

(96)

$$\nabla \psi \cdot (\text{Induction Eq.}) \rightarrow \langle \xi_\psi \rangle = \xi_\psi$$

(97)

$$B \cdot (\text{Momentum Eq.}) \rightarrow \langle b_B \rangle = b_B$$

(98)

$$J \cdot (\text{Momentum Eq.}) \rightarrow \langle b_J \rangle = b_J$$

(99)

$$C \cdot (\text{Induction Eq.}) \rightarrow \langle \xi_C \rangle = \xi_C$$

(100)

$$K \cdot (\text{Induction Eq.}) \rightarrow \langle \xi_K \rangle = \xi_K$$

(101)
The first equation is the equilibration of compressional Alfvén waves to lowest order. The second, third, and fifth equations are the equilibration of sound waves to lowest order. The fourth equation is a statement of \( \nabla \cdot \mathbf{J} = 0 \) to lowest order, and the final equation is a statement of \( \nabla \cdot \mathbf{V} = 0 \) to lowest order. Using Eq. 17 and \( \langle b^J \rangle = b^J \), we can find the variation of \( b^C \) within a surface:

\[
b^C = \frac{C^2}{\langle C^2 \rangle} \langle b^C \rangle + \frac{1}{P^i} b^B \left( \sigma C^2 - \langle \sigma C^2 \rangle \frac{C^2}{\langle C^2 \rangle} \right).
\]

Similarly, using Eq. 17 and \( \langle \xi^K \rangle = \xi^K \), we can find the variation of \( \xi^B \) within a surface:

\[
\xi^B = \frac{B^2}{\langle B^2 \rangle} \langle \xi^B \rangle - \frac{1}{P^i} \xi^C \left( \sigma B^2 - \langle \sigma B^2 \rangle \frac{B^2}{\langle B^2 \rangle} \right).
\]

One can find similar relations for \( b^K \) and \( \xi^J \) using Eqs. (16), 98, and 100, but they are not needed for this derivation.

C. Annihilation operators

We are now ready to look at the next order of the equations. The \( C \) component of the momentum equation will yield the same information as the \( \nabla \psi \) component of the magnetic field to lowest order; i.e., that magnetosonic waves need to be eliminated (Eq. ??). To obtain additional information from this equation, we must go to higher order and subtract the two components from each other. A convenient way to do this is to find an operator that annihilates the lowest order information, apply it to the full equation before ordering, and then take the lowest-order terms. This is the concept of annihilation introduced by Kruskal. CGJ [4] and GGJ [1] used as the annihilation operator

\[
\nabla \cdot \mathbf{B} \times \frac{1}{B^2} \times
\]

This operator was so important to MHD theory that the Princeton theory group dubbed it the Grand Old Operator [19]. In toroidal geometry and considering the effects of compressibility [1], this operator leaves a higher order term, \( \xi^C(3) \). To cancel this term, the GGJ equation is formed by

\[
\langle \nabla \cdot \mathbf{B} \times (Momentum \ Eq.) - \frac{\sigma}{P^i} \frac{\partial}{\partial \psi} \mathbf{B} \cdot (Momentum \ Eq.) \rangle
\]

Comparing the equation for the complete formation of the annihilated equation to the relation of \( K \) to \( B \) and \( C \) (Eq. (13)) motivates a new annihilation operator:

\[
\nabla \cdot \mathbf{K} \times
\]

We study the derivation of the annihilated equation in detail to show its advantages. Applying this operator to the momentum equation, we have

\[
\nabla \cdot \mathbf{K} \times \left[ \gamma^2 \rho \xi = (\nabla \times \mathbf{b}) \times \mathbf{B} + \mathbf{J} \times \mathbf{b} - \nabla p \right].
\]

The first term is

\[
\gamma^2 \rho \nabla \cdot \mathbf{K} \times \xi = \gamma^2 \rho \nabla \cdot \xi \frac{\mathbf{J}}{P^i} \left[ \frac{1}{P^i} \right] \frac{\nabla \psi}{\| \nabla \psi \|^2} - \xi^J \frac{\nabla \psi}{P^i} \left[ \frac{1}{P^i} \right] \frac{\nabla \psi}{\| \nabla \psi \|^2} = -\frac{\gamma^2 \rho}{P^i} \frac{\partial \xi^J}{\partial \psi} + \mathcal{O}(\epsilon^2).
\]

The second term is

\[
\nabla \cdot \mathbf{K} \times (\nabla \times \mathbf{b}) \times \mathbf{B} = \nabla \cdot (\mathbf{K} \cdot \mathbf{B}(\nabla \times \mathbf{b})) - \nabla \cdot (\mathbf{K} \cdot (\nabla \times \mathbf{b}) \mathbf{B})
\]

\[
= \mathbf{B} \cdot \nabla \left( \nabla \times \mathbf{K} \times \mathbf{b} - \mathbf{b} \times \nabla \times \mathbf{K} \right)
\]

\[
= \mathbf{B} \cdot \nabla \left[ \nabla \cdot \left( \frac{b^J \mathbf{J}}{P^i} \left[ \frac{1}{P^i} \right] \frac{\nabla \psi}{\| \nabla \psi \|^2} \right) - \mathbf{b} \cdot \nabla \times \mathbf{K} \right]
\]

\[
= iN\lambda \mathbf{r} \left( \frac{1}{P^i} \frac{\partial b^J}{\partial \psi} - \mathbf{J}^{-1} \frac{\partial}{\partial \Theta} (...) \right) + \mathcal{O}(\epsilon^2).
\]
The cancellation used in going from the first step to the second required \( K \cdot B = 1 \). This is the elimination of the magnetosonic wave required of our annihilation operator.

The third term is

\[
\nabla \cdot K \times (J \times b) = \nabla \cdot b^K J = J \cdot \nabla b^K = -F^* J^{-1} \frac{\partial b^K}{\partial \Theta} + P' iNb^K + O (\epsilon^2). \tag{109}
\]

The fourth term is

\[
\nabla \cdot K \times \nabla p = \nabla p \cdot \nabla \times (J \times b) = \nabla p \cdot \nabla \times K = C \cdot \nabla \times K iNb^K + O (\epsilon^2). \tag{110}
\]

Putting all of the terms together and taking the average, we have

\[
\frac{\gamma^2 P}{P'} \frac{\partial}{\partial \psi} \langle \xi^J \rangle + iP' \langle b^K \rangle + iN \langle C \cdot \nabla \times K \rangle b^K + iN A x \frac{1}{P'} \frac{\partial b^J}{\partial \psi} = 0. \tag{111}
\]

This is the annihilated momentum equation. All that remains is to convert the above variables to the variables we are solving for \( \langle b^\psi \rangle, \langle \xi^\psi \rangle, \langle b^B \rangle \) using (Eqs. (16)-17) and our previous relationships.

### D. The inner layer equations

Similar to the annihilated momentum equation, the parallel induction equation leaves higher order terms in toroidal geometry. The best annihilation operator for this term is \( K \cdot B \) for this equation. Applying it yields

\[
\gamma b^K = \eta \frac{\partial^2 b^K}{\partial \psi^2} - \nabla \cdot \xi + B \cdot \nabla \xi^K + C \cdot \nabla \times K \xi^\psi \tag{112}
\]

Taking the average and substituting in the expression for \( \nabla \cdot \xi \) from the pressure equation yields

\[
\langle b^K \rangle = \frac{\eta}{\gamma} \left( \frac{\partial^2 b^K}{\partial \psi^2} |\nabla \psi|^2 \right) - iN A x \xi^K + \frac{b^K}{\gamma P} + \frac{\xi^\psi}{\gamma} \left( \langle C \cdot \nabla \times K \rangle + \frac{P'}{\gamma_h P} \right) \tag{113}
\]

Again, we need to convert the variables in the above equation. After a bit of algebra (see Appendix), the normalized inner layer equations are:

\[
\Psi_{XX} - H T_X = Q (\Psi - \bar{x} \Xi), \tag{114}
\]

\[
I_X = H \Psi_{XX} + FT_X, \tag{115}
\]

\[
Q^2 \Xi_{XX} - \bar{x}^2 Q \Xi + ET + Q \bar{x} \psi + \Gamma = 0, \tag{116}
\]

\[
1 \quad \Xi_{XX} - \frac{\bar{x}^2}{Q} \Xi + G \Psi + \bar{x} \frac{Q}{Q} \Psi - \bar{x} \frac{Q}{Q^2} \Xi = 0. \tag{117}
\]
Eqs. \[116\] and \[117\] are the normalized annihilated momentum and induction equations. As can be seen by the complicated factors in \(F\) and \(H\), many of the complications appearing in going from Eqs. \[111\] and \[113\] to these equations are due to converting the \(J\) and \(K\) contravariant components which arise naturally into the normal and parallel components. Examination of Eqs. \[16\] and \[17\] shows easily how many of the factors arise.

In GGJ, it shown that a plasma is ideal unstable if \(D_I > 0\) where \(D_I = E + F + H\) and that it is unstable to resistive interchange modes if \(D_R = E + F + H > 0\). From the expression for \(E\) given above it is not obvious that our form reduces to forms given previously in the literature. We now show that it does.

Using the equation for \(\mathbf{C} \cdot \nabla \times \mathbf{K}\) (Eq. \[78\]) and the above relation, one can write
\[
\mathbf{C} \cdot \nabla \times \mathbf{K} = J^{-1} \frac{\partial}{\partial \psi} J + \frac{F'}{P'} C^2 S + J^{-1} \frac{\partial}{\partial \Theta} \left( J \frac{g^{\psi \Theta}}{g^{\psi \psi}} \right).
\]
Taging the average of this equation and substituting into the equation for \(E\), yields
\[
E = -\frac{\langle C^2 \rangle P'}{A^2} \left[ \frac{V''}{V} + \frac{4\pi^2 F'q'}{V'P'} + \Lambda \langle \sigma B^2 \rangle \right]
\]
which gives the more familiar \(V''\) criterion of GGJ \[11, 20\]. The direct calculation of \(E\) is easier in Hamada coordinates because in calculating the curl of \(\mathbf{K}\), it is easier when \(\mathbf{J}\) is straight and \(\mathbf{K}\) has a nice contravariant form seen in Eq. \[11\].

VII. DISCUSSION AND SUMMARY

Two equilibrium-based local frame of references have been introduced. The \(BC\) reference frame is widely used even if not explicitly stated as such. When normalized, it is closely related to the explicitly named Stix frame (differing only in that the parallel direction is last and the binormal changed accordingly) \[7\], and to the Frenet-like frame discussed by Hegna \[15\]. Decomposition of fields into the \(BC\) local frame was explicitly performed by MHD theory in the 60’s \[4\]. The usefulness of having \(\mathbf{C}\) straight was pointed out by Boozer in the early 80’s \[10\] \[12\].

It was DMS \[5\] that made the transformation between flux coordinates and the local \(BC\) frame explicit. This systematic investigation is useful for the algebraic expressions that consistently arise in analytic derivations. The original motivation of the DMS paper is in discussing numerical implementations. Because different analytic forms of the linearized MHD equations can have different convergence properties, being able to analytically transform easily is important.

An equivalent systematic investigation of the transformation properties of the \(JK\) frame has never been performed, and is new to this paper. The fusion theory community has developed a battery of techniques, both analytic and computational, for dealing with the stiff time scales of plasma oscillations, compressional Alfvèn waves, and (not discussed in this paper) cyclotron motion. One of the main methods for dealing with compressional Alfvèn waves analytically, the Grand Old Operator, actually does not quite work in toroidal geometry. It is seen that \(\nabla \cdot \mathbf{K} \times\) is the appropriate annihilation operator, and that the dominant term of \(D_I\) appears naturally. The other terms in \(D_I\) in this derivation are the metric elements of the \(BC\) and \(JK\) frames that arise when converting the annihilated equation variables to the fundamental variables that we wish to solve for. By having a clean separation from the two frames, the derivation is clearer. The magnetic well term, as embodied in the \(V''\) criterion, is shown to be related to the magnetic binormal direction of the curl of the current density binormal. The important role of \(\mathbf{K}\) perhaps should have been obvious sooner given that the ideal MHD potential energy term, \(W_F\), is nicely expressed in terms of \(\mathbf{K}\) and the manipulations thereof.

The presentation here focused on the calculation of \(W_F\) and the inner layer equations, but the formulas and techniques presented here should be useful for any theoretical analysis in toroidal geometry where one wishes to separate the length scales associated with different directions.

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