TILING WITH MONOCHROMATIC BIPARTITE GRAPHS OF BOUNDED MAXIMUM DEGREE

ANTÓNIO GIRÃO AND OLIVER JANZER

Abstract. We prove that for any \( r \in \mathbb{N} \), there exists a constant \( C_r \) such that the following is true. Let \( \mathcal{F} = \{F_1, F_2, \ldots\} \) be an infinite sequence of bipartite graphs such that \( |V(F_i)| = i \) and \( \Delta(F_i) \leq \Delta \) hold for all \( i \). Then in any \( r \)-edge coloured complete graph \( K_n \), there is a collection of at most \( \exp(C_r \Delta) \) monochromatic subgraphs, each of which is isomorphic to an element of \( \mathcal{F} \), whose vertex sets partition \( V(K_n) \). This proves a conjecture of Corsten and Mendonça in a strong form and generalizes results on the multicolour Ramsey numbers of bounded-degree bipartite graphs.

1. Introduction

The problem of partitioning the vertex set of edge-coloured complete graphs into a small number of certain monochromatic pieces has a very rich history; see [18] for a recent survey. An early example of a problem of this kind is Lehel’s conjecture [3]. The conjecture states that any 2-edge coloured complete graph \( K_n \) contains a red and a blue cycle whose vertex sets partition \( V(K_n) \). Here, the empty graph, singletons and edges are regarded as cycles. This conjecture was proved by Luczak, Rödl and Szemerédi [21] for large \( n \). Later, Allen [1] significantly improved the bound on \( n \). Finally, Bessy and Thomassé [4] proved Lehel’s conjecture for all \( n \).

Regarding more colours, Erdős, Gyárfás and Pyber [13] proved that the vertex set of any \( r \)-edge coloured complete graph can be partitioned to \( O(r^2 \log r) \) monochromatic cycles and conjectured that in fact \( r \) monochromatic cycles should be enough. However, this was disproved by Pokrovskiy [22]. The current best known upper bound is \( O(r \log r) \), due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [19].

Grinshpun and Sárközy [17] considered the more general problem where we want to partition the vertex set of the complete graph to graphs chosen from a fixed bounded-degree sequence. Let \( \mathcal{F} = \{F_1, F_2, \ldots\} \) be a sequence of graphs. A monochromatic \( \mathcal{F} \)-tiling of size \( s \) of an edge coloured complete graph \( K_n \) is a collection of \( s \) monochromatic subgraphs, each isomorphic to an element of \( \mathcal{F} \), whose vertex sets partition \( V(K_n) \). The \( r \)-colour tiling number of \( \mathcal{F} \), denoted by \( \tau_r(\mathcal{F}) \), is the minimal \( s \) such that every \( r \)-edge coloured complete graph has a monochromatic \( \mathcal{F} \)-tiling of size at most \( s \) (if no such \( s \) exists, then we let \( \tau_r(\mathcal{F}) = \infty \)). Call \( \mathcal{F} = \{F_1, F_2, \ldots\} \) a \( \Delta \)-bounded graph sequence if \( v(F_i) = i \) and \( \Delta(F_i) \leq \Delta \) for all \( i \). (Here and below, \( v(G) \)
denotes the number of vertices in \( G \) and \( \Delta(G) \) denotes the maximum degree of \( G \). Also, call \( \mathcal{F} = \{F_1, F_2, \ldots\} \) a bipartite \( \Delta \)-bounded graph sequence if in addition each \( F_i \) is bipartite.

Grinshpun and Sárközy gave an upper bound for the 2-colour tiling number of \( \Delta \)-bounded graph sequences.

**Theorem 1.1** (Grinshpun–Sárközy [17]). There exists an absolute constant \( C \) such that for any \( \Delta \geq 2 \) and any \( \Delta \)-bounded graph sequence \( \mathcal{F} \), \( \tau_2(\mathcal{F}) \leq \exp(C\Delta \log \Delta) \).

They also gave a better bound for the 2-colour tiling number of bipartite \( \Delta \)-bounded sequences.

**Theorem 1.2** (Grinshpun–Sárközy [17]). There exists an absolute constant \( C \) such that for any bipartite \( \Delta \)-bounded graph sequence \( \mathcal{F} \), \( \tau_2(\mathcal{F}) \leq \exp(C\Delta) \).

Moreover, they showed that Theorem 1.2 is tight.

**Theorem 1.3** (Grinshpun–Sárközy [17]). There is an absolute constant \( c > 0 \) such that for any \( \Delta \) there exists a bipartite \( \Delta \)-bounded graph sequence \( \mathcal{F} \) with \( \tau_2(\mathcal{F}) \geq \exp(c\Delta) \).

Regarding more colours, they made the following conjecture.

**Conjecture 1.4** (Grinshpun–Sárközy [17]). For every positive integer \( r \) there exists a constant \( C_r \) such that for any \( \Delta \geq 2 \) and any \( \Delta \)-bounded graph sequence \( \mathcal{F} \), \( \tau_r(\mathcal{F}) \leq \exp(\Delta^{C_r}) \).

Recently, Corsten and Mendonça established the finiteness of \( \tau_r(\mathcal{F}) \) for \( \Delta \)-bounded graph sequences and proved a triple exponential upper bound in \( \Delta \).

**Theorem 1.5** (Corsten–Mendonça [11]). There exists an absolute constant \( C \) such that for every \( \Delta \)-bounded graph sequence \( \mathcal{F} \), \( \tau_r(\mathcal{F}) \leq \exp\left(\exp\left(r^{C_r\Delta^3}\right)\right) \).

They point out that their proof gives a double exponential upper bound for bipartite \( \Delta \)-bounded graph sequences. Moreover, they write that it would be very interesting to prove Conjecture 1.4 for bipartite \( \Delta \)-bounded graph sequences. In this paper, we prove such a result with a stronger bound.

**Theorem 1.6.** For every positive integer \( r \) there exists a constant \( C_r \) such that for any \( \Delta \) and any bipartite \( \Delta \)-bounded graph sequence \( \mathcal{F} \), \( \tau_r(\mathcal{F}) \leq \exp(C_r\Delta) \).

By Theorem 1.3, this result is tight up to the value of \( C_r \).

### 1.1. Connection to the Ramsey numbers of bounded-degree graphs

The results just mentioned are closely related to the study of the Ramsey numbers of bounded-degree graphs. The research on these Ramsey numbers was initiated by Burr and Erdős [5]. They conjectured that for any \( \Delta \) there is a constant \( c(\Delta) \) such that the Ramsey number of every graph \( H \) with \( n \) vertices and maximum degree at most \( \Delta \) satisfies \( R(H) \leq c(\Delta)n \). The conjecture was proved by Chvátal, Rödl, Szemerédi and Trotter [6] using Szemerédi’s regularity lemma. There has been plenty of research on improving the value of \( c(\Delta) \) since. First, Eaton [12] showed that \( c(\Delta) \leq 2^{2^{C\Delta}} \) for some absolute constant \( C \). The bound was further improved by Graham, Rödl and Ruciński [15] who proved that \( c(\Delta) \leq 2^{C\Delta \log^2 \Delta} \). Finally, Conlon, Fox and Sudakov [8] showed that \( c(\Delta) \leq 2^{C\Delta \log \Delta} \), which is the current best bound, although it is conjectured that \( c(\Delta) \leq 2^{C\Delta} \).
When $H$ is bipartite, better bounds are known. Improving on earlier work by Graham, Rödl and Ruciński [16], Conlon [7], and Fox and Sudakov [14], Conlon, Fox and Sudakov [10] showed that when $H$ is bipartite, we can take $c(\Delta) \leq 2^{\Delta+6}$. On the other hand, Graham, Rödl and Ruciński [15, 16] proved that for every $\Delta$ and sufficiently large $n$ there are bipartite graphs $H$ with $n$ vertices and maximum degree $\Delta$ for which $R(H) \geq 2^\Delta n$.

The best known upper bound for the $r$-colour Ramsey number of an $n$-vertex graph $H$ with maximum degree $\Delta$ is $R_r(H) \leq \exp(C_r \Delta^2)n$ (see [9]). Fox and Sudakov [14] showed that if $H$ is also bipartite, then $R_r(H) \leq \exp(C_r \Delta)n$.

Note that our Theorem 1.6 can be viewed as a generalization of the last bound $R_r(H) \leq \exp(C_r \Delta)n$. Indeed, let $H$ be an $n$-vertex bipartite graph with maximum degree at most $\Delta$. Clearly, we can define a bipartite $\Delta$-bounded graph sequence $F = \{F_1, F_2, \ldots\}$ where $F_n = H$ and $F_m$ is a subgraph of $F_n$ for every $m > n$. Now let $r \in \mathbb{N}$ and let $C_r$ be the constant provided by Theorem 1.6. We claim that if $N \geq \exp(C_r \Delta)n$, then any $r$-edge colouring of $K_N$ contains a monochromatic copy of $H$. To see this, note that by Theorem 1.6, any such colouring has a monochromatic $F$-tiling of size at most $\exp(C_r \Delta)$. In particular, it contains a monochromatic copy of $F_m$ for some $m \geq N/\exp(C_r \Delta) \geq n$. Any such $F_m$ contains $H$ as a subgraph, so the colouring contains a monochromatic copy of $H$.

Similarly, Theorems 1.1, 1.2 and 1.5 generalize the corresponding upper bounds on Ramsey numbers. On the other hand, it is clear that the Ramsey bounds do not directly imply tiling results.

1.2. Sketch of the proof of Theorem 1.6. Before we turn to the proof, we provide a brief outline. For each main step of the argument, we shall also give a reference to the corresponding subsection in the proof. As is common in tiling problems, we use the absorption method.

Let us give a rough sketch how the absorption is performed. Let $F = \{F_1, F_2, \ldots\}$ be a bipartite $\Delta$-bounded graph sequence and assume that our $r$-edge coloured complete graph $G$ contains a monochromatic red subgraph $F$, isomorphic to an element of $F$. Let $(X, Y)$ be a bipartition of $F$ and let $Z$ be a set of vertices in $G$, disjoint from $X \cup Y$. Suppose that $Z$ is not much larger than $Y$ and that for each $z \in Z$ there are many $y \in Y$ such that all edges between $z$ and the set $N_F(y)$ are red. The key observation is that if the graph $F$ has a certain structure, then it can absorb $Z$; that is, $G$ has a small collection of pairwise vertex-disjoint monochromatic subgraphs, each isomorphic to an element of $F$, whose union is precisely $X \cup Y \cup Z$. Suppose that for any $u, y \in Y$, there are many disjoint choices for $(v, w) \in Y^2$ such that the edges between $v$ and $N_F(u)$, the edges between $w$ and $N_F(v)$ and the edges between $y$ and $N_F(w)$ are all red. By repeatedly using the upper bound for the multicolour Ramsey number of bounded-degree bipartite graphs (e.g. from [14]), we can cover almost all of $Z$ by a small number of monochromatic subgraphs, each isomorphic to an element of $F$. Since $Z$ is not much larger than $Y$, the remaining uncovered subset $Z' \subset Z$ eventually becomes much smaller than $Y$. Let $Z' = \{z_1, \ldots, z_k\}$. Since $Z'$ is much smaller than $Y$ and for every $z \in Z$ there are many $y \in Y$ such that all edges between $z$ and $N_F(y)$ are red, we can choose distinct $y_1, y_2, \ldots, y_k \in Y$ such that for each $i \in [k]$, all edges between $z_i$ and $N_F(y_i)$ are red. Since $k = |Z'|$ is much smaller than $|Y|$, by the upper bound on Ramsey numbers there exists a monochromatic copy of $F_k$ in $G[Y]$.

Let the vertex set of this copy be $\{u_1, u_2, \ldots, u_k\}$. Using the assumed property of $Y$, we can choose distinct vertices $v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k$ such that for each $i \in [k]$, the edges between $v_i$ and $N_F(u_i)$, the edges
between $w_i$ and $N_F(v_i)$ and the edges between $y_i$ and $N_F(w_i)$ are all red. Therefore replacing in $F$, for each $i \in [k]$, $u_i$ by $v_i$, $v_i$ by $w_i$, $w_i$ by $y_i$ and $y_i$ by $z_i$, we get a monochromatic red subgraph of $G$ isomorphic to $F$ with vertex set $X \cup (Y \setminus \{u_1, \ldots, u_k\}) \cup \{z_1, \ldots, z_k\}$ (see Figure 1 for an illustration of how the graph $F$ is modified). We had already found a monochromatic graph with vertex set $\{u_1, \ldots, u_k\}$ and a small collection of monochromatic subgraphs partitioning $Z \setminus \{z_1, \ldots, z_k\}$, so we have partitioned $X \cup Y \cup Z$ with a small number of monochromatic subgraphs, each isomorphic to an element of $\mathcal{F}$.

More generally, we can use a similar argument if there exist large pairwise disjoint subsets $Y_1, \ldots, Y_t \subseteq Y$ which together almost cover $Y$ and which have the property that for each $j \in [t]$ and every $u, y \in Y_j$ there are many disjoint choices for $(v, w) \in Y_j^2$ such that the edges between $v$ and $N_F(u)$, the edges between $w$ and $N_F(v)$ and the edges between $y$ and $N_F(w)$ are all red.

The precise description of the absorption is given in Subsection 2.2. Subsections 2.3 and 2.4 are devoted to constructing the absorbers. More specifically, the construction of the subsets $Y_1, \ldots, Y_t$ with the above key property will rely on the results in Subsection 2.3, while the construction of the subgraph $F$ will be presented in Subsection 2.4.

For the latter, we build upon the technique of [14] and [7] that was used for finding a large red subgraph isomorphic to some $F \in \mathcal{F}$ in a host graph $G$ with many red edges. The idea was to use dependent random choice to find a large subset $U \subseteq V(G)$ with the property that all but a tiny proportion of the sets of size $\Delta$ in $U$ have many common red neighbours. Using the condition that $F$ is bipartite and has maximum degree at most $\Delta$, this suffices for finding a red copy of $F$ in $G$ with parts $X \subseteq U$ and $Y$. For our results, we will need to tweak their argument slightly as we need to find a copy of $F$ which also satisfies some additional
properties, such that for every vertex \( w \in V(G) \) which has many red neighbours in \( U \), there should be many vertices \( y \in Y \) such that all edges between \( w \) and \( N_F(y) \) are red. Fortunately, there are many choices to find a red copy of \( F \) with \( X \subset U \), so we will be able to argue that a random embedding will possess the required properties. The precise statement is given in Lemma 2.16.

In Subsection 2.5, we complete the proof of our main result, Theorem 1.6.

Notation and remarks. We will use the standard asymptotic notations \( O \) and \( \Omega \). For positive functions \( f \) and \( g \), \( f = O(g) \) (respectively, \( f = \Omega(g) \)) means that there exists a positive absolute constant \( C \) such that \( f \leq Cg \) (respectively, \( f \geq Cg \)). Also, \( f = O_r(g) \) and \( f = \Omega_r(g) \) mean that the same inequalities hold for some positive \( C \) which only depends on \( r \).

Very occasionally, we treat real numbers as integers when doing so makes no significant difference in the argument.

For the whole Section 2, we fix positive integers \( r \) and \( \Delta \) and a \( \Delta \)-bounded bipartite graph sequence \( F = \{ F_1, F_2, \ldots \} \) (and we do not define them in each lemma).

2. The proof of Theorem 1.6

2.1. Preliminaries. In this subsection we present a few preliminary lemmas which will be used in our proofs. The first one is a density version of the upper bound on the multicolour Ramsey numbers of bipartite bounded-degree graphs.

Proposition 2.1 (Fox–Sudakov [14]). Let \( F \) be a bipartite graph with \( k \) vertices and maximum degree \( \Delta \geq 1 \). If \( \varepsilon > 0 \) and \( G \) is a graph with \( n \geq 32\Delta \varepsilon^{-\Delta} k \) vertices and at least \( \varepsilon \binom{n}{2} \) edges, then \( F \) is a subgraph of \( G \).

Corollary 2.2. If \( n \geq 32\Delta r^\Delta k \), then any \( r \)-edge colouring of \( K_n \) contains a monochromatic copy of \( F_k \).

Proof. There is a colour which appears on at least \( \frac{1}{r} \binom{n}{2} \) edges, and we can apply Proposition 2.1 with \( \varepsilon = 1/r \) for the graph with these edges. \( \square \)

By repeatedly finding large monochromatic subgraphs, we can cover almost all vertices with a small number of monochromatic subgraphs from \( F \).

Corollary 2.3. For any \( 0 < t \leq n \) and any \( r \)-edge colouring of \( K_n \), there exists a collection of at most \( 64\Delta r^\Delta (\log(n/t) + 2) \) pairwise vertex-disjoint monochromatic subgraphs, each of which is isomorphic to an element of \( F \), whose union covers more than \( n - t \) vertices.

In particular, any \( r \)-edge colouring of \( K_n \) has a monochromatic \( F \)-tiling of size at most \( 64\Delta r^\Delta (\log n + 2) \).

Proof. We find suitable subgraphs by the following algorithm. Let us assume that we have already covered all but at most \( s \) vertices of \( K_n \). If \( s \leq 64\Delta r^\Delta \), then cover these vertices by singletons (note that the unique 1-vertex graph belongs to \( F \)); this gives a cover of all vertices. Else, there exists a positive integer \( \ell \) such that \( \frac{s}{2^{\Delta+\Delta}} \leq \ell \leq \frac{2s}{32\Delta r^\Delta} \). Since there are at least \( s \geq 32\Delta r^\Delta \ell \) vertices not yet covered, by Corollary 2.2 there exists a monochromatic copy of \( F_\ell \) whose vertices were not yet covered. Add this to the collection of covering subgraphs.
Eventually, we will have covered all but fewer than \( t \) vertices, and then we stop. We claim that we have used at most \( 64\Delta r^\Delta (\log(n/t) + 2) \) subgraphs. To prove this, it suffices to show that before we terminate the process or get to the stage where \( s \leq 64\Delta r^\Delta \), we used at most \( 64\Delta r^\Delta (\log(n/t) + 1) \) subgraphs. Suppose that we used \( k \) subgraphs to get to this stage. Then, since each time we covered at least \( \frac{1}{64\Delta r^\Delta} \) proportion of the yet uncovered vertices, it follows that

\[
\frac{t - 1}{64\Delta r^\Delta} \leq \left(1 - \frac{1}{64\Delta r^\Delta}\right)^k - 1. \tag{1}
\]

Using the inequality \( 1 - x \leq e^{-x} \), we obtain that \( k \leq 64\Delta r^\Delta \log(n/t) + 1 \), completing the proof of the first assertion. The second assertion follows from the first one by taking \( t = 1 \). \( \square \)

The next lemma is a version of dependent random choice.

**Lemma 2.4.** Let \( k, t \in \mathbb{N} \) and \( 0 < \varepsilon, \delta, \gamma < 1 \) such that \( \delta \varepsilon^k t \geq 2\gamma t \). Let \( G = (A, B) \) be a bipartite graph with \( e(G) \geq \varepsilon |A||B| \). Then there is a set \( S \subset A \) of size at least \( \frac{1}{2}\varepsilon^k |A| \) such that all but at most \( \delta |S|^k \) sets of \( k \) vertices in \( S \) have at least \( \gamma |B| \) common neighbours.

**Proof.** Choose \( t \) vertices from \( B \) at random with replacement and call the set of these vertices \( T \). Let \( S = N(T) \), the common neighbourhood of the vertices in \( T \).

Now

\[
E[|S|] = \sum_{v \in A} (d(v)/|B|)^t \geq |A|(d/|B|)^t \geq |A|\varepsilon^t,
\]

where \( d \) denotes the average degree of the vertices in \( A \). Hence, \( E[|S|] \geq E[|S|^k] \geq \varepsilon^k |A|^k \).

Write \( X \) for the number of sets of \( k \) vertices in \( S \) which have fewer than \( \gamma |B| \) common neighbours. Note that \( E[X] \leq \binom{|A|}{k} \gamma^t \leq |A|^k \gamma^t \). Hence,

\[
E[\delta |S|^k - X] \geq \delta \varepsilon^k |A|^k - \gamma^t |A|^k.
\]

In particular, there exists an outcome with \( \delta |S|^k - X \geq \delta \varepsilon^k |A|^k - \gamma^t |A|^k \). This means on the one hand that \( X \leq \delta |S|^k \), so at most \( \delta |S|^k \) sets of \( k \) vertices in \( S \) have fewer than \( \gamma |B| \) common neighbours. On the other hand, \( \delta |S|^k \geq \frac{1}{2} \delta \varepsilon^k |A|^k \), so \( |S| \geq \frac{2\varepsilon^t}{\gamma^t} |A| \geq \frac{1}{\gamma} \varepsilon^t |A| \). \( \square \)

The next two results are specializations of the previous one and are suited specifically to our needs.

**Lemma 2.5.** There is a constant \( C = C(r) \) such that the following is true. Let \( k \in \mathbb{N} \) and \( 0 < \delta < 1/2 \). Let \( G = (A, B) \) be a bipartite graph with \( e(G) \geq \frac{1}{r} |A||B| \). Then there is a set \( S \subset A \) of size at least \( \delta^C |A| \) such that all but at most \( \delta |S|^k \) sets of \( k \) vertices in \( S \) have at least \( \frac{(1/r)^k}{2} |B| \) common neighbours.

**Proof.** Let \( \varepsilon = \frac{1}{r} \gamma = \frac{(1/r)^k}{2} \) and choose a positive integer \( t \) such that \( 2t^{-2} \leq 1/\delta < 2t^{-1} \). Then it is easy to see that \( \delta \varepsilon^k t \geq 2\gamma t \). Hence, by Lemma 2.4, there exists a set \( S \subset A \) of size at least \( \frac{1}{2}\varepsilon^t |A| \) such that all but at most \( \delta |S|^k \) sets of \( k \) vertices in \( S \) have at least \( \frac{(1/r)^k}{2} |B| \) common neighbours. Since \( \frac{1}{2}\varepsilon^t \geq \delta^{O(1)} \), the proof is complete. \( \square \)

**Lemma 2.6.** Let \( 0 < \varepsilon, \delta < 1 \) such that \( \delta \geq 2\varepsilon^4 \). Let \( G = (A, B) \) be a bipartite graph with \( e(G) \geq \varepsilon |A||B| \). Then there is a set \( S \subset A \) of size at least \( \frac{1}{2}\varepsilon^4 |A| \) such that all but at most \( \delta |S|^2 \) pairs of vertices in \( S \) have at least \( \varepsilon^3 |B| \) common neighbours.
Proof. Let $k = 2$, $t = 4$ and $\gamma = \varepsilon^3$. Then it is easy to see that $\delta \varepsilon^{kt} \geq 2\gamma^t$. Hence, the result follows directly from Lemma 2.4.

Finally, we will use the following version of the Chernoff bound (see, e.g., Theorem A.1.13 in [2]).

Lemma 2.7. Let $X = \sum_{i=1}^n X_i$, where $X_i = 1$ with probability $p_i$ and $X_i = 0$ with probability $1 - p_i$, and all $X_i$ are independent. Let $\mu = \mathbb{E}[X] = \sum_{i=1}^n p_i$. Then for any $0 < \delta < 1$,

$$\mathbb{P}(X \leq (1 - \delta)\mu) \leq e^{-\mu \delta^2 / 2}.$$  

2.2. The absorption process. We now define our absorbing structure.

Definition 2.8. Let $G$ be an edge-coloured complete graph and let $F$ be a bipartite subgraph with parts $X$ and $Y$. We say that $F$ is $(\eta, \theta)$-good if there exist pairwise disjoint sets $Y_1, \ldots, Y_t \subseteq Y$ such that the following holds.

1. $F$ is monochromatic.
2. $F$ is isomorphic to an element of $\mathcal{F}$.
3. For every $i \in [t]$, $|Y_i| \geq \eta |Y|$.
4. $|Y \setminus \bigcup_{i \in [t]} Y_i| \leq \theta |Y|$.
5. For any $i \in [t]$ and any distinct $y_0, y_3 \in Y_i$, there exist at least $|Y_i|$ pairwise disjoint pairs $(y_1, y_2) \in Y_i^2$ of distinct vertices such that the edges between $y_0$ and $N_F(y_1)$, the edges between $y_1$ and $N_F(y_2)$ and the edges between $y_2$ and $N_F(y_3)$ are all of the same colour as $F$.

The key result of this subsection is that good subgraphs can indeed be used to absorb certain sets.

Lemma 2.9. Let $\eta, \theta, K, C > 0$ satisfy $\eta \geq \exp(-C\Delta)$, $\theta \geq \exp(-\exp(C\Delta))$, $K \leq \exp(\exp(C\Delta))$. Then there exists $C' = C'(C, r)$ such that the following holds.

Let $G$ be an $r$-edge coloured complete graph and let $F = (X, Y)$ be an $(\eta, \theta)$-good subgraph in colour red. Let $Z$ be a subset of $V(G)$ such that $|Z| \leq K |Y|$ and for every $z \in Z$, the number of $y \in Y$ with $N_F(y) \subseteq N_{\text{red}}(z)$ is at least $2\theta |Y|$.

Then $G[X \cup Y \cup Z]$ has a monochromatic $\mathcal{F}$-tiling of size at most $\exp(C'\Delta)$.

To prove this result, we use the following technical lemma.

Lemma 2.10. Let $\eta, \theta, K, C > 0$ satisfy $\eta, \theta \geq \exp(-\exp(C\Delta))$ and $K \leq \exp(\exp(C\Delta))$. Then there exists a constant $C' = C'(C, r)$ such that the following is true.

Let $G$ be an $r$-edge coloured complete graph, let $Y$ and $Z$ be disjoint subsets of $V(G)$ and let $\sim$ be a binary relation defined over $Y \times (Y \cup Z)$ (i.e. a subset of $Y \times (Y \cup Z)$) which satisfies that $y \sim y$ for all $y \in Y$. Assume that for any distinct $x, y \in Y$, there exist at least $\eta |Y|$ pairwise disjoint pairs $(z, w) \in Y^2$ of distinct vertices such that $x \sim z$, $z \sim w$ and $w \sim y$. Assume also that $|Z| \leq K |Y|$ and that for every $z \in Z$, there are at least $\theta |Y|$ vertices $y \in Y$ such that $y \sim z$.

Then there exists an injective map $f : Y \to Y \cup Z$ such that $y \sim f(y)$ for all $y \in Y$ and $G[Y \cup Z \setminus f(Y)]$ has a monochromatic $\mathcal{F}$-tiling of size at most $\exp(C'\Delta)$. 

7
Proof. Let \( t = \min(\frac{n}{100}|Y|, \theta|Y|, \frac{1}{10 \Delta^2} |Y|, |Z|) \). Applying Corollary 2.3, we find that there exists a collection of at most \( 64 \Delta r^\Delta (\log(|Z|/t) + 2) \) pairwise vertex-disjoint monochromatic subgraphs in \( G[Z] \), each isomorphic to an element of \( F \), whose union is \( Z \setminus T \) for some \( T \subset Z \) of size at most \( t \).

Since \( |T| \leq t \leq \theta|Y| \) and for every \( u \in T \) there are at least \( \theta|Y| \) vertices \( y \in Y \) with \( y \sim u \), we can find an injective map \( g : T \to Y \) such that \( g(u) \sim u \) for all \( u \in T \). Also, \( |T| \leq t \leq \frac{1}{10 \Delta^2} |Y| \), so by Corollary 2.2, there exists a monochromatic copy of \( F_{|T|} \) in \( G[Y] \). Let \( R = \{x_1, x_2, \ldots, x_{|T|}\} \) be the vertex set of this copy of \( F_{|T|} \) and let \( g(T) = \{y_1, y_2, \ldots, y_{|T|}\} \subset Y \). After reordering the vertices if necessary, we may assume that there exists \( 0 \leq \ell \leq |T| \) such that \( x_i = y_i \) for all \( \ell < i \leq |T| \) but \( \{x_1, \ldots, x_{\ell}\} \cap \{y_1, \ldots, y_{\ell}\} = \emptyset \). Since \( \ell \leq |T| \leq t \leq \frac{n}{100} |Y| \), we can choose pairwise disjoint vertices \( z_1, w_1, z_2, w_2, \ldots, z_{\ell}, w_{\ell} \in Y \setminus \{x_1, x_2, \ldots, x_{\ell}, y_1, y_2, \ldots, y_{\ell}\} \) such that for every \( 1 \leq i \leq \ell \), \( x_i \sim z_i \), \( z_i \sim w_i \) and \( w_i \sim y_i \).

Now define \( f(v) = v \) for every \( v \in Y \setminus (\bigcup_{1 \leq i \leq \ell} \{x_i, y_i, z_i, w_i\} \cup \bigcup_{|T| < i \leq \ell} \{x_i\}) \), let \( f(x_i) = g^{-1}(y_i) \) for every \( \ell < i \leq |T| \) and let \( f(x_i) = z_i, f(z_i) = w_i, f(w_i) = y_i \) and \( f(y_i) = g^{-1}(y_i) \) for every \( i \leq \ell \). Then \( v \sim f(v) \) holds for all \( v \in Y \). Moreover, \( f \) is a bijection between \( Y \) and \( (Y \setminus R) \cup T \), so \( Y \cup Z \setminus f(Y) = R \cup (Z \setminus T) \).

Hence \( G[Y \cup Z \setminus f(Y)] \) has a monochromatic \( F \)-tiling of size at most \( 64 \Delta r^\Delta (\log(|Z|/t) + 2) + 1 \). It is not hard to see that the choice of \( t \) implies the lemma. \( \Box \)

Proof of Lemma 2.9. First note that we may assume (by replacing \( Z \) with \( Z \setminus (X \cup Y) \) if necessary) that \( Z \) is disjoint from \( X \cup Y \).

Let \( Y_1, \ldots, Y_t \subset Y \) witness the \((\eta, \theta)\)-goodness of \( F \). Fix some \( z \in Z \). By assumption, there are at least \( 2\theta |Y| \) vertices \( y \in Y \) with \( N_F(y) \subset N_{\text{red}}(z) \). By condition (4) of Definition 2.8, we have that the number of \( y \in \bigcup_i Y_i \) such that \( N_F(y) \subset N_{\text{red}}(z) \) is at least \( \theta|Y| \). Thus, there exists \( i \in [t] \) such that the number of \( y \in Y_i \) with \( N_F(y) \subset N_{\text{red}}(z) \) is at least \( \theta|Y_i| \).

Hence, we can obtain a partition \( Z = \bigcup_{i=1}^t Z_i \) such that for every \( z \in Z_i \) the number of \( y \in Y_i \) with \( N_F(y) \subset N_{\text{red}}(z) \) is at least \( \theta|Y_i| \).

For every \( i \in [t] \), we define a binary relation \( \sim_i \) on \( Y_i \times (Y_i \cup Z_i) \) as follows. For \( u \in Y_i \) and \( v \in Y_i \cup Z_i \), we take \( u \sim_i v \) if and only if \( N_F(u) \subset N_{\text{red}}(v) \). Since \( F \) is monochromatic red, we have \( u \sim_i u \) for all \( u \in Y_i \). By condition (5) from Definition 2.8, for any distinct \( x, y \in Y_i \), there exist at least \( \eta |Y_i| \) pairwise disjoint pairs \( (z, w) \in Y_i^2 \) of distinct vertices such that \( x \sim_i z, z \sim_i w, w \sim_i y \). Moreover, for every \( z \in Z_i \), there are at least \( \theta|Y_i| \) vertices \( y \in Y_i \) such that \( y \sim_i z \). Finally, \( |Z_i| \leq |Z| \leq K|Y| \leq K\eta^{-1}|Y_i| \). It follows by Lemma 2.10 that there exists an injection \( f_i : Y_i \to Y_i \cup Z_i \) with \( y \sim_i f_i(y) \) for all \( y \in Y_i \) such that \( G[Y_i \cup Z_i \setminus f_i(Y_i)] \) has a monochromatic \( F \)-tiling of size at most \( \exp(C'' \Delta) \), where \( C'' \) depends only on \( C \) and \( r \).

Since clearly \( t \leq \eta^{-1} \), \( G[\bigcup_{i=1}^t (Y_i \cup Z_i \setminus f_i(Y_i))] \) has a monochromatic \( F \)-tiling of size at most \( \eta^{-1} \exp(C'' \Delta) \). The remaining set of vertices in \( X \cup Y \cup Z \) is precisely \( X \cup (Y \setminus \bigcup_{i=1}^t Y_i) \cup (\bigcup_{i=1}^t f_i(Y_i)) \). We claim that this is the vertex set of a red subgraph isomorphic to \( F \). Indeed, for every \( y \in Y_i \) we have \( y \sim_i f_i(y) \), so the edges in \( G \) between \( f_i(y) \) and \( N_F(y) \) are all red. Hence, we can replace every \( y \in Y_i \) (for every \( i \)) in \( F \) with \( f_i(y) \) and we get another red subgraph isomorphic to \( F \) with vertex set \( X \cup (Y \setminus \bigcup_{i=1}^t Y_i) \cup (\bigcup_{i=1}^t f_i(Y_i)) \). \( \Box \)
2.3. Constructing the absorbers. In view of the previous subsection, we want to find certain good subgraphs. The key condition in Definition 2.8 is property (5). The next lemma will be used to obtain subsets which satisfy this property.

Lemma 2.11. For any $0 < \varepsilon < 1/100$ there exists a positive constant $c = c(\varepsilon) = \Omega(\varepsilon^{27})$ such that the following is true. Let $H = (A,B)$ be a bipartite graph with $|A| \leq |B|$. Suppose that for every $x \in A$, $|N(x)| \geq \varepsilon|B|$. Then there exists a set $S \subset A$ of size at least $\varepsilon|A|$ and a matching $f : S \to B$ in $H$ such that for any distinct $x, y \in S$, there are at least $c|S|$ pairwise disjoint pairs $(z, w) \in S^2$ of distinct vertices such that $f(x) \in N(z)$, $f(z) \in N(w)$ and $f(w) \in N(y)$.

Proof. First note that by replacing $A$ with a suitable subset of size $\frac{1}{128}\varepsilon^{-13}|A|$, it suffices to prove that the conclusion of the lemma holds for some $c = \Omega(\varepsilon^{14})$ assuming that $|A|/|B| \leq \frac{1}{128}\varepsilon^{-13}$. Moreover, we may assume that $|A| \geq C\varepsilon^{-14}$ for some sufficiently large absolute constant $C$ (else the statement becomes trivial).

Let $\delta$ be chosen so that $\delta + \frac{2\delta}{3} = 1/3$. Then $\delta/8 \geq 2\varepsilon^4$, so by Lemma 2.6 (applied with $\delta/8$ in place of $\delta$), there exists a set $S_0 \subset A$ of size at least $\frac{1}{2}\varepsilon^4|A|$ such that the number of ordered pairs $(x, y) \in S_0^2$ for which $|N(x) \cap N(y)| < \varepsilon^3|B|$ is at most $\frac{1}{2}|S_0|^2$. Let $S_1$ be the collection of those vertices $x \in S_0$ for which there are at most $\frac{1}{2}|S_0|/2$ vertices $y \in S_0$ satisfying $|N(x) \cap N(y)| < \varepsilon^3|B|$. By a double counting argument, $|S_0 \setminus S_1| \leq |S_0|/2$, so $|S_1| \geq |S_0|/2$. Hence, for any $x \in S_1$ there are at most $\delta|S_1|$ vertices $y \in S_1$ for which $|N(x) \cap N(y)| < \varepsilon^3|B|$. Let $B'$ be the set of vertices $u \in B$ for which $|N(u) \cap S_1| \geq \delta|S_1|$. Observe that for any $x \in S_1$, $e(N(x) \setminus B', S_1) \leq |B|\delta|S_1|$, so the number of vertices $y \in S_1$ with $|N(x) \cap N(y) \setminus B'| \geq \frac{\varepsilon^3}{2}|B|$ is at most $\frac{|B|\delta|S_1|}{\varepsilon^3/2|B|} = \frac{2\delta}{3}|S_1|$. It follows that for any $x \in S_1$, the number of vertices $y \in S_1$ with $|N(x) \cap N(y) \cap B'| < \frac{\varepsilon^3}{2}|B|$ is at most $(\delta + \frac{2\delta}{3})|S_1| = |S_1|/3$. This implies that for any $x, y \in S_1$, there are at least $|S_1|/3$ vertices $z \in S_1$ such that $|N(x) \cap N(z) \cap B'| \geq \frac{\varepsilon^3}{2}|B|$ and $|N(y) \cap N(z) \cap B'| \geq \frac{\varepsilon^3}{2}|B|$.

Claim. For each $x \in S_1$, let $g(x)$ be a random neighbour of $x$ in $B'$. Then with probability at least $2/3$, for any $x, y \in S_1$, there are at least $\frac{1}{16}\varepsilon^{10}|S_1|$ pairwise disjoint pairs $(z, w) \in S_1^2$ of distinct vertices such that $g(x) \in N(z)$, $g(z) \in N(w)$ and $g(w) \in N(y)$.

Proof of Claim. Let us estimate the probability that such pairs do not exist for some fixed $x$ and $y$. Condition on the event $g(x) = u$. Since $u \in B'$, we have $|N(u) \cap S_1| \geq \delta|S_1|$. Moreover, for any $z \in N(u) \cap S_1$, there are at least $|S_1|/3$ vertices $w \in S_1$ such that $|N(z) \cap N(w) \cap B'| \geq \frac{\varepsilon^3}{2}|B|$ and $|N(w) \cap N(y) \cap B'| \geq \frac{\varepsilon^3}{2}|B|$. Thus, we can greedily find at least $\delta|S_1|/2$ pairwise disjoint pairs $(z, w) \in S_1^2$ of distinct vertices such that $z \in N(u)$, $|N(z) \cap N(w) \cap B'| \geq \frac{\varepsilon^3}{2}|B|$ and $|N(w) \cap N(y) \cap B'| \geq \frac{\varepsilon^3}{2}|B|$. All but at most two such pairs are disjoint from $\{x, y\}$. For such a pair $(z, w)$, the probability that $g(z) \in N(w)$ and $g(w) \in N(y)$ is at least $(\frac{\varepsilon^3}{2})^2$. Moreover, for disjoint pairs $(z, w)$ these events are independent. Hence, by a standard application of the Chernoff bound (Lemma 2.7), the probability that the number of good pairs $(z, w)$ is less than $\frac{1}{2} \cdot \frac{\delta|S_1|}{2} \cdot (\frac{\varepsilon^3}{2})^2$ is at most $\exp(-\Omega(\delta\varepsilon^6|S_1|)) \leq \exp(-\Omega(\varepsilon^3|S_1|))$. Thus, by the union bound, using that $|S_1| \geq |S_0|/2 \geq \varepsilon^3|A|/4 \geq C'\varepsilon^{-10}$ for some sufficiently large $C'$, we have that with probability at least $2/3$, for any $x, y \in S_1$, there are at least $\frac{1}{2} \cdot \frac{\delta|S_1|}{2} \cdot (\frac{\varepsilon^3}{2})^2 = \frac{1}{16}\varepsilon^{10}|S_1|$ pairwise disjoint pairs $(z, w) \in S_1^2$ of distinct vertices such that $g(x) \in N(z)$, $g(z) \in N(w)$ and $g(w) \in N(y)$. Since $\delta > \varepsilon^4$, the proof of the claim is complete. □
This is close to what we need, but $g$ is not necessarily injective. Let $S$ be a maximal subset of $S_1$ on which $g$ is injective and set $f(x) = g(x)$ for every $x \in S$. Note that $|S_1 \setminus S|$ is at most the number of pairs $(x, y) \in S_1^2$ with $x \neq y$ and $g(x) = g(y)$. For any $x \in S_1$, we have $|N(x) \cap B'| \geq \frac{1}{16}|B'|$, so for any $x \neq y$, the probability that $g(x) = g(y)$ is at most $\frac{2}{|B'|}$. Hence, the expected number of pairs $(x, y) \in S_1^2$ with $x \neq y$ and $g(x) = g(y)$ is at most $|S_1|^2 \cdot \frac{2}{|B'|} \leq \frac{2}{|B'|}|S_1| \leq \frac{1}{64} \epsilon^{10}|S_1|$. By our assumption that $|A|/|B| \leq \frac{1}{128} \epsilon^{13}$. Hence, with probability at least $1/2$, $|S_1 \setminus S|$ is less than $\frac{1}{32} \epsilon^{10}|S_1|$. If this holds and there are at least $\frac{1}{32} \epsilon^{10}|S_1|$ pairwise disjoint pairs $(z, w) \in S_1^2$ of distinct vertices such that $f(x) \in N(z)$, $f(z) \in N(w)$ and $f(w) \in N(y)$, then there are at least $\frac{1}{32} \epsilon^{10}|S_1|$ pairwise disjoint pairs $(z, w) \in S^2$ of distinct vertices such that $f(x) \in N(z)$, $f(z) \in N(w)$ and $f(w) \in N(y)$.

Using the claim and the last paragraph, we find that with probability at least $1/6$, for any $x, y \in S$, there are at least $\frac{1}{32} \epsilon^{10}|S_1| \geq \frac{1}{32} \epsilon^{10}S$ pairwise disjoint pairs $(z, w) \in S^2$ of distinct vertices such that $f(x) \in N(z)$, $f(z) \in N(w)$ and $f(w) \in N(y)$. Since $|S| \geq |S_1|/2 \geq |S_0|/4 \geq \epsilon^4|A|/8$, we may indeed find a suitable $c = \Omega(\epsilon^{14})$.

The next lemma allows us to obtain not just one suitable subset, but an almost-cover by such sets.

**Lemma 2.12.** For any $0 < \epsilon < 1/100$ and $0 < \theta < 1$, there exists a positive constant $\eta = \eta(\epsilon, \theta) = \Omega(\theta \epsilon^{27})$ such that the following is true. Let $H = (A, B)$ be a bipartite graph with $|A| \leq \frac{2}{3}|B|$. Suppose that for every $x \in A$, $|N(x)| \geq \epsilon|B|$. Then there exist pairwise disjoint sets $S_1, S_2, \ldots, S_t \subseteq A$ and an injection $f : A \to B$ such that

1. For every $i \in [t]$, $|S_i| \geq \eta|A|$.
2. For every $x \in A$, $f(x) \in N(x)$.
3. For any $i \in [t]$ and any distinct $x, y \in S_i$, there exist at least $\eta|S_i|$ pairwise disjoint pairs $(z, w) \in S_i^2$ of distinct vertices such that $f(x) \in N(z)$, $f(z) \in N(w)$ and $f(w) \in N(y)$.
4. $|A \setminus \bigcup_{i \in [t]} S_i| \leq \theta|A|$.

**Proof.** Let $c = c(\epsilon/2)$ be the constant provided by Lemma 2.11 and take $\eta = \theta c$. Note that $\eta \leq c$.

We define the sets $S_1, S_2, \ldots$ recursively. Suppose that we have already found pairwise disjoint sets $S_1, S_2, \ldots, S_k$ and matchings $f_i : S_i \to B$ in $H$ such that $f_1(S_1), \ldots, f_k(S_k)$ are pairwise disjoint, $|S_i| \geq \eta|A|$ for every $i \leq k$, and for every $i \leq k$ and distinct $x, y \in S_i$, there exist at least $\eta|S_i|$ pairwise disjoint pairs $(z, w) \in S_i^2$ of distinct vertices such that $f_i(x) \in N(z)$, $f_i(z) \in N(w)$ and $f_i(w) \in N(y)$. Set $A_k = A \setminus \bigcup_{i \leq k} S_i$ and $B_k = B \setminus \bigcup_{i \leq k} f_i(S_i)$. If $|A_k| \leq \theta|A|$, terminate the process. Else, consider the bipartite graph $H_k = (A_k, B_k)$. Observe that $|B \setminus B_k| \leq |A| \leq \frac{2}{3}|B|$. Hence, every $x \in A_k$ has $|N_{H_k}(x)| \geq \frac{2}{3}|B| \geq \frac{2}{3}|B_k|$. Moreover, $|A| \leq |B|$ easily implies that $|A_k| \leq |B_k|$. Therefore, we can apply Lemma 2.11 (with $H_k$ in place of $H$ and $\epsilon/2$ in place of $\epsilon$) to find $S_{k+1} \subseteq A_k$ of size at least $c|A_k|$ and a matching $f_{k+1} : S_{k+1} \to B_k$ in $H_k$ such that for any distinct $x, y \in S_{k+1}$, there are at least $c|S_{k+1}|$ pairwise disjoint pairs $(z, w) \in S_{k+1}^2$ of distinct vertices such that $f_{k+1}(x) \in N_{H_k}(z)$, $f_{k+1}(z) \in N_{H_k}(w)$ and $f_{k+1}(w) \in N_{H_k}(y)$. Note that $|S_{k+1}| \geq c|A_k| \geq c\theta|A| = \eta|A|$.

The process must eventually terminate and then we have pairwise disjoint sets $S_1, S_2, \ldots, S_t \subseteq A$ of size at least $\eta|A|$ each such that $|A \setminus \bigcup_{i} S_i| \leq \theta|A|$. Define $f : A \to B$ as follows. For $x \in S_i$, set $f(x) = f_i(x)$. For every $x \in A \setminus \bigcup_{i} S_i$, define $f(x)$ arbitrarily in a way that $f(x) \in N_{H}(x)$ and $f$ remains injective. This
is possible since for every $x \in A$, $|N_H(x)| \geq \varepsilon|B| \geq |A|$. It is easy to verify that these choices satisfy the conditions of the lemma.

2.4. Finding good subgraphs. Call a hypergraph $\mathcal{G}$ down-closed if for every $e \in E(\mathcal{G})$ and $e' \subseteq e$, we have $e' \in E(\mathcal{G})$. The following lemma is a straightforward modification of Lemma 2.2 from [14], but we provide a proof for completeness. By an embedding of a multihypergraph $\mathcal{H}$ into a simple hypergraph $\mathcal{G}$, we just mean an embedding of $\mathcal{H}'$ into $\mathcal{G}$, where $\mathcal{H}'$ is the simple hypergraph obtained by replacing the multiedges of $\mathcal{H}$ with simple edges.

Lemma 2.13. Let $\mathcal{H}$ be an $m$-vertex multihypergraph with maximum degree at most $\Delta$ in which every hyperedge has size at most $\Delta$. Let $0 < \lambda < 1/(2\Delta)$ and let $\mathcal{G}$ be a down-closed hypergraph with $n \geq 2m$ vertices and more than $(1 - \lambda^\Delta){n \choose \Delta}$ hyperedges of size $\Delta$. Then there are at least $(1 - 2\Delta\lambda)m(n - 1) \cdots (n - m + 1)$ labelled embeddings of $\mathcal{H}$ into $\mathcal{G}$.

Proof. Let us call a set $S \subseteq V(\mathcal{G})$ of size at most $\Delta$ rich if it is contained in more than $(1 - \lambda^\Delta){n - |S| \choose \Delta - |S|}$ hyperedges of size $\Delta$. Moreover, let us call a vertex $v \in V(\mathcal{G}) \setminus S$ rich with respect to $S$ if $S \cup \{v\}$ is rich.

We claim that for any rich set $S$ of size less than $\Delta$, there are at most $\lambda n$ vertices in $V(\mathcal{G}) \setminus S$ which are not rich with respect to $S$. Indeed, if more than $\lambda n$ such vertices exist, then $S$ is contained in at least $\lambda n \cdot \lambda^\Delta{n - |S| \choose \Delta - |S|}/(\Delta - |S|) \geq (1 - 2\Delta\lambda)(n - |S|)^{\Delta - |S|}/(\Delta - |S|)$ sets of size $\Delta$ which are non-edges in $\mathcal{G}$, which contradicts the assumption that $S$ is rich.

Let $V(\mathcal{H}) = \{v_1, \ldots, v_m\}$ and write $V_k = \{v_1, \ldots, v_k\}$ for every $0 \leq k \leq m$. We will prove by induction on $k$ that for any $0 \leq k \leq m$, there are at least $\prod_{0 \leq i \leq k-1}(n - i - \Delta\lambda n)$ injective maps $f : V_k \to V(\mathcal{G})$ such that for every $e \in E(\mathcal{H})$, $f(e \cap V_k)$ is a rich set. For $k = 0$, this follows from the fact that the empty set is rich. Now let $0 \leq k \leq m - 1$ and assume that $f : V_k \to V(\mathcal{G})$ is an injective map such that for every $e \in E(\mathcal{H})$, $f(e \cap V_k)$ is a rich set. It suffices to prove that there are at least $n - k - \Delta\lambda n$ ways to extend $f$ to an injective map $f' : V_{k+1} \to V(\mathcal{G})$ satisfying that for every $e \in E(\mathcal{H})$, $f'(e \cap V_{k+1})$ is a rich set. Note that $\mathcal{H}$ has at most $\Delta$ edges containing $v_{k+1}$; let us call them $e_1, \ldots, e_\ell$. Now if we choose $f'(v_{k+1})$ to be a vertex that does not belong to $f(V_k)$ and which is rich with respect to the set $f(e_i \cap V_k)$ for every $1 \leq i \leq \ell$, then we get a desired extension of $f$. Thus, the number of choices for $f'(v_{k+1})$ which do not give a desired extension is at most $k + \ell\Delta\lambda n \leq k + \Delta\lambda n$. This completes the induction step.

Since $m \leq n/2$, we have $n - i - \Delta\lambda n \geq (1 - 2\Delta\lambda)(n - i)$ for every $0 \leq i \leq m - 1$. Hence, there are at least $\prod_{0 \leq i \leq m-1}(1 - 2\Delta\lambda)(n - i) = (1 - 2\Delta\lambda)m(n - 1) \cdots (n - m + 1)$ injective maps $f : V(\mathcal{H}) \to V(\mathcal{G})$ such that $f(e)$ is rich for every $e \in E(\mathcal{H})$. For any such $f$ and any $e \in E(\mathcal{H})$, $f(e)$ is contained in at least one hyperedge of $\mathcal{G}$, so $f(e) \in E(\mathcal{G})$, since $\mathcal{G}$ is down-closed. Hence, any such $f$ gives a labelled embedding of $\mathcal{H}$ into $\mathcal{G}$. □

Lemma 2.14. Let $\mathcal{H}$ be an $m$-vertex multihypergraph with minimum degree at least one and maximum degree at most $\Delta$ in which every hyperedge has size at most $\Delta$. Let $U$ be a set of size $n \geq 16m$ and let $R \subseteq U$ have size at least $\frac{n}{8\Delta}$. Let $f$ be a uniformly random injective map from $V(\mathcal{H})$ to $U$. Then the probability that there are fewer than $\frac{e(\mathcal{H})}{2\Delta(16\Delta)^2}$ edges of $\mathcal{H}$ whose image under $f$ lie entirely in $R$ is at most $\exp(-\Omega(\frac{m}{4\Delta(16\Delta)^2})).$

Proof. Since every edge of $\mathcal{H}$ has size at most $\Delta$ and the maximum degree of $\mathcal{H}$ is at most $\Delta$, it follows that $\mathcal{H}$ has at least $e(\mathcal{H})/\Delta^2$ pairwise disjoint edges. Choose such edges $e_1, e_2, \ldots, e_k$ with $k \geq e(\mathcal{H})/\Delta^2$. We can
construct f by taking an arbitrary list of the vertices of \( \mathcal{H} \) and mapping them one by one to \( U \), in each step randomly choosing one of the still available vertices. Take an ordering in which the vertices of \( e_1 \) come first, followed by the vertices of \( e_2 \) and so on. After the vertices of \( e_k \), the remaining vertices are mapped in an arbitrary order. Note that \(|V(\mathcal{H})| = m \leq \frac{n \theta}{16r} \leq |R|/2\), so in each step at least \(|R|/2 \geq \frac{n \theta}{16r} \) vertices of \( R \) are still available. Hence, for every \( i \), conditional on any mapping of the vertices of \( e_1, \ldots, e_{i-1} \), the probability that every vertex in \( e_i \) gets mapped to \( R \) is at least \( (\frac{1}{16r})^\Delta \). Thus, the probability that fewer than \( \frac{k}{(32r)^2} \) of the edges \( e_1, e_2, \ldots, e_k \) get mapped to \( R \) is upper bounded by the probability that the binomial random variable \( \text{Bin}(k, (\frac{1}{16r})^\Delta) \) takes value less than \( k/(32r)^\Delta \). Again by a standard application of the Chernoff bound (Lemma 2.7), this probability is \( \exp(-\Omega(k/(16r)^\Delta)) \). Since \( k \geq e(\mathcal{H})/\Delta^2 \) and \( e(\mathcal{H}) \geq m/\Delta \) (the latter holds because of the minimum degree condition), the lemma follows.

**Lemma 2.15.** There exists a constant \( c = c(r) > 0 \) with the following property. Let \( \mathcal{H} \) be an \( m \)-vertex multihypergraph with minimum degree at least one and maximum degree at most \( \Delta \) in which every hyperedge has size at most \( \Delta \). Let \( \mathcal{G} \) be a down-closed hypergraph with \( n \geq 16rm \) vertices and more than \( (1 - c^2)\binom{n}{\Delta} \) hyperedges of size \( \Delta \). Let \( t \leq \exp(c^\Delta m) \) and for every \( 1 \leq i \leq t \), let \( R_i \subseteq V(\mathcal{G}) \) have size at least \( \frac{n}{8r} \).

Then there exists an embedding of \( \mathcal{H} \) into \( \mathcal{G} \) such that for each \( 1 \leq i \leq t \), the number of edges of \( \mathcal{H} \) which are entirely in \( R_i \) is at least \( \frac{e(\mathcal{H})}{\Delta^2 (32r)^2} \).

**Proof.** Let \( c \) be a sufficiently small positive constant, depending only on \( r \). By Lemma 2.13 with \( \lambda = c^\Delta \), there are at least \( (1 - 2\Delta c^\Delta)^m(n-1)\cdots(n-m+1) \) labelled embeddings of \( \mathcal{H} \) into \( \mathcal{G} \). This means that a random injective map \( f : V(\mathcal{H}) \to V(\mathcal{G}) \) defines a valid embedding with probability at least \( (1 - 2\Delta c^\Delta)^m \). Using \( 1 - 2\Delta c^\Delta \geq \exp(-4\Delta c^\Delta) \), we get that \( f \) is an embedding with probability at least \( \exp(-4\Delta c^\Delta m) \).

By Lemma 2.14, for any fixed \( i \), the probability that \( f \) maps fewer than \( \frac{e(\mathcal{H})}{\Delta^2 (32r)^2} \) edges of \( \mathcal{H} \) to subsets of \( R_i \) is at most \( \exp(-\Omega(\frac{m}{\Delta^2 (16r)^2})) \). Taking union bound over all \( 1 \leq i \leq t \), we find that the probability that there exists \( 1 \leq i \leq t \) for which \( f \) maps fewer than \( \frac{e(\mathcal{H})}{\Delta^2 (32r)^2} \) edges of \( \mathcal{H} \) to subsets of \( R_i \) is at most \( \exp(c^\Delta m - \Omega(\frac{m}{\Delta^2 (16r)^2})) \). If \( c \) is small enough, then this is less than \( \exp(-4\Delta c^\Delta m) \), so there exists a choice for \( f \) which defines a valid embedding and which maps at least \( \frac{e(\mathcal{H})}{\Delta^2 (32r)^2} \) edges of \( \mathcal{H} \) to subsets of \( R_i \) for each \( 1 \leq i \leq t \).

Observe that in the proof of Theorem 1.6, we may assume that every \( F_k \in \mathcal{F} \) has at most one isolated vertex (else we may add edges, keep the graph bipartite and keep the maximum degree at most \( \Delta \)). In what follows, let \( \mathcal{F} \) satisfy this condition. For an edge-coloured graph \( G \) and a vertex \( w \in V(G) \), we write \( N_{\text{red}}(w) \) for the set of vertices \( u \in V(G) \) such that \( wu \) is a red edge.

**Lemma 2.16.** Let \( \varepsilon = \Omega_r(1)^\Delta \) satisfy \( \varepsilon < 1/100 \), let \( \theta = \frac{1}{2\Delta^2 (32r)^2} \) and let \( c = c(r) \) be the constant from Lemma 2.15. Let \( G \) be an \( r \)-edge coloured complete graph. Let \( U, V \) and \( W \) be subsets of \( V(G) \) such that \( U \cap V = \emptyset \) and \( |U| \geq 100r^2 \). Assume that for every \( w \in W \), we have \( |N_{\text{red}}(w) \cap U| \geq \frac{|U|}{8r} \). Moreover, assume that more than \( (1 - c^\Delta) \binom{|U|}{\Delta} \) of the \( \Delta \)-sets in \( U \) have at least \( \varepsilon |V| \) common red neighbours in \( V \). Finally, assume that \( |U| \leq \frac{1}{2}|V| \) and that \( |W| \leq \exp(\frac{c^\Delta}{800 r^2} |U|) \).

(a) Then there exist \( \eta = \Omega_r(1)^\Delta \) and a \( (\eta, \theta) \)-good subgraph \( F = (X, Y) \) in \( G[U \cup V] \) with at most \( |U|/(16r^2) \) vertices such that for every \( w \in W \), the number of \( y \in Y \) with \( N_F(y) \subseteq N_{\text{red}}(w) \) is at least \( 2\theta |Y| \).
(b) Moreover, if $|W| \leq \exp(\exp(O_r(\Delta)))|U|$, then for any $Z \subset W$, $G[X \cup Y \cup Z]$ has a monochromatic $\mathcal{F}$-tiling of size at most $O_r(1)^{\Delta}$.

Proof. Let $k = \left\lfloor \frac{|U|}{16r^2} \right\rfloor$. Since $F_k$ has at most one isolated vertex, we can choose a bipartition $(X', Y')$ of $F_k$ such that every vertex in $X'$ has degree at least one. Also, since $F_k$ has at most one isolated vertex, $e(F_k) \geq \frac{k-1}{2}$, so that the maximum degree is at most $\Delta$, we have $|X'|, |Y'| \geq \frac{k-1}{2\Delta}$. Define a multihypergraph $\mathcal{H}$ whose vertex set is $X'$ and whose hyperedges are $N_{F_k}(y)$ for every $y \in Y'$ (with repetition). Since $F_k$ has maximum degree at most $\Delta$, it follows that $\mathcal{H}$ has maximum degree at most $\Delta$ and every hyperedge in $\mathcal{H}$ has size at most $\Delta$. Moreover, since every vertex in $X'$ has degree at least one in $F_k$, the minimum degree of $\mathcal{H}$ is at least one. Write $m = |V(\mathcal{H})| = |X'|$.

Define a hypergraph $\mathcal{G}$ whose vertex set is $U$ and whose edges are those subsets of $U$ which have at least $\varepsilon|V|$ common red neighbours in $V$. It is clear that $\mathcal{G}$ is down-closed. Writing $n = |V(\mathcal{G})|$, we have $n = |U| \geq 16r^2k \geq 16rm$. By the assumption on the red common neighbourhood of $\Delta$-sets in $U$, $\mathcal{G}$ has more than $(1 - c\Delta^2)(\binom{n}{\Delta})$ hyperedges of size $\Delta$.

For every $w \in W$, let $R_w = N_{\text{red}}(w) \cap U$. By assumption, $|R_w| \geq \frac{n}{8r}$ holds for all $w \in W$. Since $m = |X'| \geq \frac{k-1}{2\Delta} \geq \frac{|U|}{80r^2\Delta}$, we have $|W| \leq \exp(\frac{c\Delta^2}{80r^2\Delta}|U|) \leq \exp(c\Delta m)$.

By Lemma 2.15, there exists an embedding $g$ of $\mathcal{H}$ into $\mathcal{G}$ such that for every $w \in W$, the number of edges of $\mathcal{H}$ whose images are entirely in $R_w$ is at least $\frac{c(\mathcal{H})}{\Delta^2(32r)^{\Delta}}$.

Let us define a bipartite graph $H$ as follows. The parts of $H$ are defined to be $A = E(\mathcal{H})$ and $B = V$. For every $e \in E(\mathcal{H})$, the neighbourhood of $e$ in $B$ is defined to be the common red neighbourhood inside $V$ of the vertices in $g(e)$ (note that $g(e) \subset U$). For any $e \in E(\mathcal{H}) = A$, $g(e) \in E(\mathcal{G})$, so $g(e)$ has at least $\varepsilon|V|$ common red neighbours in $V$, hence $|N_{\mathcal{H}}(e)| \geq \varepsilon|V| = \varepsilon|B|$. Moreover, $|A| = |E(\mathcal{H})| = |Y'| \leq k \leq |U| \leq \frac{5}{2}|V| = \frac{5}{2}|B|$, so we may apply Lemma 2.12. Let pairwise disjoint sets $S_1, \ldots, S_t \subset A$ and an injection $f : A \to B$ satisfy the four properties in Lemma 2.12 with some $\eta = \Omega(\varepsilon 27)$. Clearly, $\eta = O_r(1)^{\Delta}$.

Let $X = g(X') \subset U$ and let $Y = f(A) \subset V$. Define a bipartite graph $F$ with parts $X$ and $Y$ as follows. For every $y \in Y$, let the neighbourhood of $y$ in $F$ be $g(f^{-1}(y)) \subset X$. We claim that $F$ is $(\eta, \theta)$-good. It is clear by the definition that $F$ is monochromatic red and is isomorphic to $F_k$. For each $i$, let $Y_i = f(S_i)$. Clearly, $|Y_i| = |S_i| \geq \eta|A| = \eta|Y|$. Furthermore, $|Y \setminus \bigcup_i Y_i| = |A \setminus \bigcup_i S_i| \leq \theta|A| = \theta|Y|$. It remains to check property (5) from Definition 2.8. Let $y_b, y_3$ be distinct vertices in $Y_i$ for some $i \in [t]$. Let $e_0 = f^{-1}(y_b)$ and let $e_3 = f^{-1}(y_3)$. By condition (3) from Lemma 2.12, there exist at least $\eta|S_i|$ pairwise disjoint pairs $(e_1, e_2) \in S_i^2$ of distinct vertices such that for each $0 \leq b \leq 2$, $f(e_b) \in N_{\mathcal{H}}(e_{b+1})$. For $j \in \{1, 2\}$, let $y_j = f(e_j)$. It suffices to prove that for every $0 \leq b \leq 2$, the edges between $y_b$ and the elements of $N_{F}(y_{b+1})$ are all red. By definition, $N_F(y_{b+1}) = g(f^{-1}(y_{b+1})) = g(e_{b+1})$. But $f(e_b) \in N_{\mathcal{H}}(e_{b+1})$ means precisely that every edge between $y_b = f(e_b)$ and the elements of $N_{F}(y_{b+1}) = g(e_{b+1})$ is red, completing the proof that $F$ is $(\eta, \theta)$-good.

Now let $w \in W$. The number of $y \in Y$ with $N_{F}(y) \subset R_w$ is equal to the number of edges $e \in E(\mathcal{H})$ such that $g(e) \subset R_w$. We know that this number is at least $\frac{c(\mathcal{H})}{\Delta^2(32r)^{\Delta}} = 2\theta e(\mathcal{H}) = 2\theta|Y|$. Since $R_w = N_{\text{red}}(w) \cap U$, the proof of (a) is complete.
For part (b), we shall use Lemma 2.9. If suffices to verify that $|W| \leq K|Y|$ holds for some $K \leq \exp(\exp(O_r(\Delta)))$. However, $|Y| = |Y'| \geq \frac{k-1}{2\Delta} \geq \frac{|U|}{800r^2\Delta}$, so the required inequality follows from $|W| \leq \exp(\exp(O_r(\Delta)))|U|$. 

The next lemma follows fairly easily by a repeated application of Lemma 2.16 (b). In an edge-coloured graph $G$ whose colours are labelled by the first $r$ positive integers, we write $N_i(w)$ for the set of vertices $u \in V(G)$ such that the edge $wu$ has colour $i$.

**Lemma 2.17.** Let $\varepsilon = \Omega_r(1)^\Delta$ satisfy $\varepsilon < 1/100$ and let $c = c(r)$ be the constant from Lemma 2.15. Let $G$ be an $r$-edge coloured complete graph with the colours being the first $r$ positive integers. Let $k \in [r]$ and let $U, V_1, \ldots, V_k$ and $W$ be subsets of $V(G)$ such that $U \cap V_i = \emptyset$ for each $1 \leq i \leq k$. Assume that for every $w \in W$, there exists some $i \in [k]$ such that $|N_i(w) \cap U| \geq \frac{|U|}{10r}$. Moreover, assume that for each $i \in [k]$, fewer than $4^{-\Delta}c^{2\Delta}(\frac{|U|}{\Delta})$ sets of size $\Delta$ in $U$ have fewer than $\varepsilon|V_i|$ common neighbours in colour $i$ within $V_i$. Finally, assume that $|W| \leq \exp(\exp(O_r(\Delta)))|U|$ and that for each $i \in [k]$, $|U| \leq \frac{\varepsilon}{8}|V_i|$.

Then, there is a set $D \subset V(G)$ of size at most $|U|/(16r)$ such that for any $Z \subset W$, $G[D \cup Z]$ has a monochromatic $F$-tiling of size at most $O_r(1)^\Delta$.

**Proof.** First suppose that $|U| < 200r^2\Delta$ or $|W| > \exp(\frac{\varepsilon^2}{100r^2\Delta}|U|)$. Then, since $|W| \leq \exp(\exp(O_r(\Delta)))|U|$, it follows that $|U| \leq \exp(\exp(O_r(\Delta)))$, and hence also $|W| \leq \exp(\exp(O_r(\Delta)))$. Now note that for any $Z \subset W$, $Z$ has size at most $\exp(O_r(1)^\Delta)$ and therefore Corollary 2.3 implies that $G[Z]$ has a monochromatic $F$-tiling of size at most $O_r(1)^\Delta$. Hence, in this case we can take $D = \emptyset$.

Now assume that $|U| \geq 200r^2\Delta$ and $|W| \leq \exp(\frac{c^{2\Delta}(\frac{|U|}{\Delta})}{1600r^2\Delta}|U|)$.

For each $i \in [k]$, let $W_i$ be the set of vertices $w \in W$ for which $|N_i(w) \cap U| \geq \frac{|U|}{10r}$.

**Claim.** There exist pairwise disjoint sets $D_1, D_2, \ldots, D_k \subset V(G)$ of size at most $|U|/(16r^2)$ each, such that for every $1 \leq i \leq k$ and every $Z \subset W_i$, $G[D_i \cup Z]$ has a monochromatic $F$-tiling of size at most $O_r(1)^\Delta$.

**Proof of Claim.** We construct the sets one by one. Suppose that suitable $D_1, D_2, \ldots, D_{i-1}$ have already been found. Let $\tilde{U} = U \setminus (D_1 \cup \cdots \cup D_{i-1})$ and $\tilde{V}_i = V_i \setminus (D_1 \cup \cdots \cup D_{i-1})$. We shall apply Lemma 2.16 (b) with $\tilde{U}$ in place of $U$, $\tilde{V}_i$ in place of $V$ and $W_i$ in place of $W$. Since $|D_1 \cup \cdots \cup D_{i-1}| \leq r|U|/(16r^2) = |U|/(16r^2)$, it follows that every $w \in W_i$ satisfies that $|N_i(w) \cap \tilde{U}| \geq \frac{|U|}{10r} \geq \frac{|U|}{8r}$. Moreover, since $|\tilde{U}| \geq 2\Delta$ and $|\tilde{U}|/|U| \geq 1/2$, we have $\frac{|\tilde{U}|}{|V_i|} \geq \frac{\frac{|U|}{10r}}{|V_i|} \geq \frac{|U|}{8r}$. So there are fewer than $c\Delta^2\frac{|\tilde{U}|}{|V_i|}$ sets of $\Delta$ vertices in $\tilde{U}$ which have fewer than $\varepsilon|V_i|$ common neighbours in colour $i$ inside $V_i$. Since $|D_1 \cup \cdots \cup D_{i-1}| \leq |U|/(16r)$ and $|U| \leq \frac{\varepsilon}{8}|V_i|$, it follows that $|D_1 \cup \cdots \cup D_{i-1}| \leq \frac{|U|}{8r}$. Hence, there are fewer than $c\Delta^2\frac{|\tilde{U}|}{|V_i|}$ sets of $\Delta$ vertices in $\tilde{U}$ which have fewer than $\frac{\varepsilon}{8}|V_i|$ common neighbours in colour $i$ inside $V_i$. Also, $|\tilde{U}| \leq |U| \leq \frac{\varepsilon}{8}|V_i| \leq \frac{\varepsilon}{8}|V_i|$. Moreover, $|\tilde{U}| \geq |U|/2 \geq 100r^2$, $|W_i| \leq |W| \leq \exp(\frac{\varepsilon^2}{100r^2\Delta}|U|) \leq \exp(\frac{\varepsilon^2}{100r^2\Delta}|\tilde{U}|)$ and $|W_i| \leq \exp(\exp(O_r(\Delta)))|\tilde{U}|$. Thus, we can apply Lemma 2.16 (b) (with $\varepsilon/2$ in place of $\varepsilon$), and we can take $D_i = X \cup Y$ for the sets $X$ and $Y$ provided by that lemma. This completes the proof of the claim. 

Define $D = D_1 \cup \cdots \cup D_k$. Clearly, $|D| \leq k|U|/(16r^2) \leq |U|/(16r)$. Let $Z \subset W$. Define $Z_1 = (Z \setminus D) \cap W_1$, $Z_2 = (Z \setminus (D \cup W_1)) \cap W_2$, \ldots, $Z_k = (Z \setminus (D \cup W_1 \cup \cdots \cup W_{k-1})) \cap W_k$. Then $Z_1, \ldots, Z_k$ partition $Z \setminus D$ and $Z_i \subset W_i$ holds for every $i$. By the claim above, for each $1 \leq i \leq k$, $G[D_i \cup Z_i]$ has a monochromatic $F$-tiling
Let $G$ such that for any $\Delta$ neighbours in colour $k$.

Note that $\delta = \exp(-100c^{-1}Cr\Delta^{2(r-k)+3})$ and $\delta' = \exp(-100c^{-1}Cr\Delta^{2(r-k+1)+3})$.

Let $G$ be an $r$-edge coloured complete graph with the colours labelled by the first $r$ positive integers. Let $A, B, V_1, V_2, \ldots, V_{k-1} \subset V(G)$ and $K = \exp(O_r(\Delta))$ satisfy the following properties.

1. $A$ is disjoint from $B \cup V_1 \cup \cdots \cup V_{k-1}$.
2. There are at least $\frac{1}{2}|A||B|$ edges of colour $k$ between $A$ and $B$.
3. For every $i \in [k-1]$, there are at most $\delta(\frac{|A|}{\Delta})$ sets of size $\Delta$ in $A$ which have fewer than $\varepsilon|V_i|$ common neighbours in colour $i$ inside $V_i$.
4. $|A| \leq |B| \leq K|A|$ and $|A| \leq |V_i|$ for all $i \in [k-1]$.

Then there are pairwise disjoint sets $A' \subset A, B' \subset A \cup B$ and $D$ such that for any $Z \subset (A \cup B) \setminus (A' \cup B')$, $G[D \cup Z]$ has a monochromatic $F$-tiling of size at most $O_r(1)^\Delta$; and either $|A' \cup B'| \leq 2$ or there exist $V'_1, V'_2, \ldots, V'_{k} \subset V(G)$ and $K' = \exp(O_r(\Delta))$ such that the following hold.

i. $A'$ is disjoint from $B' \cup V'_1 \cup \cdots \cup V'_{k}$.
ii. There is some $j \in [r] \setminus [k]$ such that there are at least $\frac{1}{2}|A'|||B'||$ edges of colour $j$ between $A'$ and $B'$.
iii. For every $i \in [k]$, there are at most $\delta(\frac{|A'|}{\Delta})$ sets of size $\Delta$ in $A'$ which have fewer than $\varepsilon'|V'_i|$ common neighbours in colour $i$ inside $V'_i$.
iv. $|A'| \leq |B'| \leq K'|A'|$ and $|A'| \leq |V'_i|$ for all $i \in [k]$.

D is disjoint from $A' \cup B' \cup V'_1 \cup \cdots \cup V'_{k}$.

Proof. Let $\delta'' = \exp(-100c^{-1}Cr\Delta^{2(r-k)+2})$. If $|A| < 4\Delta(\delta'')^{-C}$, then by Corollary 2.3 we may take $A' = B' = D = \emptyset$. So assume that $|A| \geq 4\Delta(\delta'')^{-C}$.

By Lemma 2.5 applied to the graph formed by edges of colour $k$ between $A$ and $B$ (with $\delta''$ playing the role of $\delta$), there is a set $U \subset A$ of size at least $(\delta'')^C|A|$ such that at most $\delta''|U|^\Delta$ sets of $\Delta$ vertices in $U$ have fewer than $\varepsilon|B|$ common neighbours in colour $k$ inside $B$. Since $|U| \geq 2\Delta$, it follows that $|U|^\Delta / (\frac{|U|}{\Delta}) \leq (2\Delta)^\Delta$, so at most $(2\Delta)^\Delta\delta''(\frac{|U|}{\Delta})$ sets of $\Delta$ vertices in $U$ have fewer than $\varepsilon|B|$ common neighbours in colour $k$ inside $B$. By replacing $U$ with a random subset of size $[(\delta'')^C|A|]$, we may assume that $(\delta'')^C|A| \leq |U| \leq 2(\delta'')^C|A|$. Again, by $|U| \geq 2\Delta$, we have $(\frac{|U|}{\Delta}) \geq (\frac{|U|}{|A|})^\Delta \geq 2^{-\Delta}(\delta'')^C\Delta$. Note that $\frac{\delta}{2 - \frac{\frac{\delta}{\delta''}|A|}{|A|}} \leq \delta''$, so for every $1 \leq i \leq k-1$, there are at most $\delta''(\frac{|U|}{\Delta})$ sets of size $\Delta$ in $U$ which have fewer than $\varepsilon|V_i|$ common neighbours in colour $i$ inside $V_i$. Setting $V_k = B$, we get that for every $1 \leq i \leq k$, there are at most $(2\Delta)^\Delta\delta''(\frac{|U|}{\Delta})$ sets of $\Delta$ vertices in $U$ which have fewer than $\varepsilon|V_i|$ common neighbours in colour $i$ inside $V_i$. Let $W$ be the set of vertices $w \in A \cup B$ for which there exists $i \in [k]$ with $|N_i(w) \cap U| \geq \frac{|U|}{4r}$.

Note that $|W| \leq |A| + |B| \leq 2K|A| \leq 2K(\delta'')^{-C}|U| \leq \exp(O_r(\Delta))|U|$. Moreover, for every $i \in [k]$, $|U| \leq 2(\delta'')^C|A| \leq \frac{\delta}{\delta''}|A| \leq \frac{\delta}{\delta''}|V_i|$.

Since $(2\Delta)^\Delta\delta'' < 4^{-\Delta}e^{2\Delta}$, Lemma 2.17 implies that there is a set $D \subset V(G)$ of size at most $|U|/(16r)$ such that for any $Z \subset W$, $G[D \cup Z]$ has a monochromatic $F$-tiling of size at most $O_r(1)^\Delta$. 

2.5. Completing the proof of Theorem 1.6.

Lemma 2.18. Let $C > 1$ be the constant from Lemma 2.5 and let $c$ be the constant from Lemma 2.15. Let $k \in [r]$. Let $\varepsilon = \frac{1}{2\Delta}$, $\varepsilon' = \frac{1}{2\Delta}$, $\delta = \exp(-100c^{-1}Cr\Delta^{2(r-k)+3})$ and $\delta' = \exp(-100c^{-1}Cr\Delta^{2(r-k+1)+3})$.

Let $A$ be an $r$-edge coloured complete graph with the colours labelled by the first $r$ positive integers. Let $A, B, V_1, V_2, \ldots, V_{k-1} \subset V(G)$ and $K = \exp(O_r(\Delta))$ satisfy the following properties.
Let $S = (A \cup B) \setminus (D \cup W)$ and let $U' = U \setminus D$. If $|U'| < |S|/2$, let $A' = U'$; otherwise let $A'$ be a uniformly random subset of $U'$ of size $|S|/2$. In either case, let $B' = S \setminus A'$. It is easy to see that $A'$, $B'$ and $D$ are pairwise disjoint. Also, $S \subset A' \cup B'$, so $(A \cup B) \setminus (A' \cup B') \subset (A \cup B) \setminus S \subset D \cup W$. Let $Z \subset (A \cup B) \setminus (A' \cup B')$. By the above, we have $Z \subset D \cup W$, so $Z \setminus D \subset W$. Hence, $G[D \cup Z] = G[D \cup (Z \setminus D)]$ has a monochromatic $\mathcal{F}$-tiling of size at most $O_r(1)\Delta$. This means that we are done if $|A' \cup B'| \leq 2$, so let us assume that $|A' \cup B'| \geq 3$. Then necessarily $|S| > 1$ and $A' \neq \emptyset$.

For every $1 \leq i \leq k$, let $V_i = V_i \setminus D$. It is clear that $D$ is disjoint from $A' \cup B' \cup V'_1 \cup \cdots \cup V'_k$. Moreover, since $A' \subset A$ and $V'_i \subset V_i$ for all $i \in [k]$, it follows that $A'$ is disjoint from $B' \cup V'_1 \cup \cdots \cup V'_k$.

Note that $|A'| \leq |S|/2 \leq |B'|$. Also, if $|U'| < |S|/2$, then $|A'| = |U'| \geq |U|/2 \geq \frac{(\delta'\delta)^G}{2(\Delta+1)} |A| \geq \frac{(\delta'\delta)^G}{2(\Delta+1)} |A \cup B| \geq \frac{(\delta'\delta)^G}{2(\Delta+1)} |B'|$. On the other hand, if $|U'| \geq |S|/2$, then $|A'| = |S|/2$, whereas $|B'| \leq |S|$. Thus, in both cases, $|B'| \leq K|A'|$ for some $K = \exp(\exp(O_r(\Delta)))$. Finally, for every $1 \leq i \leq k$, $|V'_i| \geq |V_i|/2 \geq |A|/2$ and $|A'| \leq |U| \leq |A|/2$, so $|A'| \leq |V'_i|$. Recall that for every $1 \leq i \leq k$, there are at most $(2\Delta)^3\delta''(\frac{|U'|}{\Delta})$ sets of $\Delta$ vertices in $U'$ which have fewer than $\varepsilon|V_i|$ common neighbours in colour $i$ inside $V_i$. Moreover, $(\frac{|U'|}{\Delta})/(\frac{|U'|}{\Delta}) \leq 4\Delta$ (since $|U'| \geq |U|/2 \geq 2\Delta$) and $|D| \leq |U| \leq \frac{\varepsilon}{2}|V_i|$, so there are at most $4^\Delta (2\Delta)^3\delta''(\frac{|U'|}{\Delta})$ sets of $\Delta$ vertices in $U'$ which have fewer than $\frac{\varepsilon}{2}|V_i|$ common neighbours in colour $i$ inside $V_i$. Since $A'$ is a uniformly random subset of $U'$ of certain prescribed size, it follows that for each $1 \leq i \leq k$, the expected number of sets of $\Delta$ vertices in $A'$ which have fewer than $\frac{\varepsilon}{2}|V_i|$ common neighbours in colour $i$ inside $V_i$ is at most $(8\Delta)^3\delta''(\frac{|A'|}{\Delta})$. Hence, by Markov's inequality, for any $1 \leq i \leq k$, the probability that the number of sets of $\Delta$ vertices in $A'$ which have fewer than $\frac{\varepsilon}{2}|V_i|$ common neighbours in colour $i$ inside $V_i$ is at least $6k \cdot (8\Delta)^3\delta''(\frac{|A'|}{\Delta})$ is at most $\frac{1}{k^2 \Delta}$. Thus, by the union bound, it holds with probability at least $5/6$ that for every $1 \leq i \leq k$, the number of sets of $\Delta$ vertices in $A'$ which have fewer than $\frac{\varepsilon}{2}|V_i|$ common neighbours in colour $i$ inside $V_i$ is at most $6k \cdot (8\Delta)^3\delta''(\frac{|A'|}{\Delta})$. Note that $\delta' \leq 6k \cdot (8\Delta)^3\delta''$. 

Now let $1 \leq i \leq k$ and let $z \in S$. Then $z \notin W$, so $|N_i(z) \cap U'| < \frac{|U'|}{\Delta}$. Moreover, $|U'| \geq \frac{3}{8}|U|$, so $|N_i(z) \cap U'| < \frac{3|U'|}{8\Delta}$. Hence, $\sum_{i \in [k], z \in S} |N_i(z) \cap U'| < \frac{k}{3\Delta}|S||U'|$. Recall that either $A' = U'$ or $A'$ is a uniformly random subset of $U'$ of size $|S|/2$. In either case, $E\left[\sum_{i \in [k], z \in S} |N_i(z) \cap A'|ight] < \frac{k}{3\Delta}|S||A'|$. Thus, the probability that $\sum_{i \in [k], z \in S} |N_i(z) \cap A'| \geq \frac{k}{3\Delta}|S||A'|$ is at most $2/3$. But $|S| \leq 2|B'|$, so then the probability that $\sum_{i \in [k], z \in B'} |N_i(z) \cap A'| \geq \frac{k}{3\Delta}|B'||A'|$ is also at most $2/3$. It follows from this and the previous paragraph that there exists an outcome for the random set $A'$ such that the number of edges between $A'$ and $B'$ with colour in $[k]$ is less than $\frac{\varepsilon}{2}|A'||B'|$ and for each $i \in [k]$, the number of sets of $\Delta$ vertices in $A'$ which have fewer than $\varepsilon|V_i'|$ common neighbours in colour $i$ inside $V_i'$ is at most $\delta'(\frac{|A'|}{\Delta})$. Then there is some $j \in [r] \setminus [k]$ such that there are at least $\frac{1}{2}|A'||B'|$ edges of colour $j$ between $A'$ and $B'$. This completes the proof of the lemma.

The next result follows easily from an iterative application of Lemma 2.18.

**Corollary 2.19.** Let $G$ be an $r$-edge coloured complete graph. Then there exist $0 \leq \ell \leq r$, pairwise disjoint sets $D_1, \ldots, D_\ell \subset V(G)$ and sets $T_{\ell+1} \subset T_\ell \subset \cdots \subset T_1 = V(G)$ such that $|T_{\ell+1}| \leq 2$ and for any $1 \leq i \leq \ell$ and $Z \subset T_i \setminus T_{i+1}$, $G[D_i \cup Z]$ has a monochromatic $\mathcal{F}$-tiling of size $O_r(1)\Delta$. 

Proof. Let \( T_1 = V(G) \). If \(|T_1| \leq 2\), then we can take \( \ell = 0 \). Else, let \( A \) be an arbitrary subset of \( V(G) \) of size \( \lceil|V(G)|/2\rceil \) and let \( B = V(G) \setminus A \). Label the most frequent colour between \( A \) and \( B \) by the number 1. Then the conditions (1)-(4) from Lemma 2.18 are satisfied for \( k = 1 \). We can now repeatedly apply the lemma, after each step removing the subset \( D \) from the graph, until we eventually (after at most \( r \) steps) obtain \( A', B' \) with \(|A' \cup B'| \leq 2\). Let \( \ell \) be the number of iterations of Lemma 2.18 before this happens. For \( 1 \leq i \leq \ell \), let \( D_i \) be the set \( D \) obtained by Lemma 2.18 in the \( i \)th iteration, and let \( T_{i+1} \) be the set \( A' \cup B' \) obtained by Lemma 2.18 in the \( i \)th iteration.

It is easy to deduce our main result from this.

Proof of Theorem 1.6. Let \( G \) be an \( r \)-edge coloured complete graph. Choose \( \ell, D_1, \ldots, D_\ell, T_1, \ldots, T_{\ell+1} \) according to Corollary 2.19. For each \( i \in [\ell] \), let \( Z_i = T_i \setminus (T_{i+1} \cup D_1 \cup \cdots \cup D_{\ell}) \). For each \( i \in [\ell] \), \( G[D_i \cup Z_i] \) has a monochromatic \( \mathcal{F} \)-tiling of size \( O_r(1)^\Delta \). Since the sets \( D_1 \cup Z_1, \ldots, D_{\ell} \cup Z_{\ell}, T_{\ell+1} \setminus (D_1 \cup \cdots \cup D_{\ell}) \) partition \( V(G) \) and \(|T_{\ell+1}| \leq 2\), it follows that \( G \) has a monochromatic \( \mathcal{F} \)-tiling of size \( O_r(1)^\Delta \).

3. Concluding remarks

A graph is called \( d \)-degenerate if each of its subgraphs has a vertex of degree at most \( d \). Proving a conjecture of Burr and Erdős [5], Lee [20] showed that the Ramsey number of an \( n \)-vertex \( d \)-degenerate graph is at most \( c(d)n \). Hence, it is natural to wonder whether it is sufficient to assume that each \( F_k \in \mathcal{F} \) is \( d \)-degenerate (rather than that it has maximum degree \( d \)) to guarantee the boundedness of \( \tau_r(\mathcal{F}) \). However, this is not the case, even when each \( F_k \in \mathcal{F} \) is bipartite. In fact, it is not even enough that each \( F_k \) is bipartite with maximum degree at most \( d \) on one side. Indeed, it was observed by Pokrovskiy (see [11] for a proof) that when \( \mathcal{F} = \{S_1, S_2, S_3, \ldots\} \) is the collection of stars, \( \tau_r(\mathcal{F}) = \infty \) holds for any \( r \geq 2 \).

Hence, it is perhaps slightly surprising that using similar techniques as in this paper, we can prove the following result.

Theorem 3.1. For any \( r \in \mathbb{N} \), there is a \( C_r \) such that the following is true. Let \( \mathcal{F} = \{K_1', K_2', K_3', \ldots\} \), where \( K_i' \) denotes the 1-subdivision of \( K_i \) and let \( G \) be an \( r \)-edge coloured complete graph. Then \( G \) has a monochromatic \( \mathcal{F} \)-tiling of size at most \( C_r \).

As the proof would require several additional pages, we omit it.

We conclude by mentioning a question asked by Corsten and Mendonça.

Problem 3.2 (Corsten–Mendonça [11]). Is there a function \( g : \mathbb{N} \to \mathbb{N} \) with \( \lim_{n \to \infty} g(n) = \infty \) such that the following is true for all positive integers \( r \) and \( d' \)? If \( \mathcal{F} = \{F_1, F_2, \ldots\} \) is a sequence of \( d \)-degenerate graphs with \(|V(F_i)| = i \) and \( \Delta(F_i) \leq g(i) \) for all \( i \), then \( \tau_r(\mathcal{F}) < \infty \).

It would also be interesting to answer this question in the case where all \( F_i \) are bipartite.

References

[1] P. Allen. Covering two-edge-coloured complete graphs with two disjoint monochromatic cycles. Combinatorics, Probability and Computing, 17(4):471–486, 2008.
[2] N. Alon and J. H. Spencer. *The probabilistic method*. John Wiley & Sons, 2004. 7

[3] J. Ayel. *Sur l’existence de deux cycles supplémentaires unicolores, disjoints et de couleurs différentes dans un graphe complet bicolore*. PhD thesis, Université Joseph-Fourier-Grenoble I, 1979. 1

[4] S. Bessy and S. Thomassé. Partitioning a graph into a cycle and an anticycle, a proof of Lehel’s conjecture. *Journal of Combinatorial Theory, Series B*, 100(2):176–180, 2010. 1

[5] S. Burr and P. Erdős. On the magnitude of generalized Ramsey numbers for graphs, infinite and finite sets, vol. I, Colloquia Mathematica Societatis Janos Bolyai, 10, 1975. 2, 17

[6] C. Chvátal, V. Rödl, E. Szemerédi, and W. T. Trotter Jr. The Ramsey number of a graph with bounded maximum degree. *Journal of Combinatorial Theory, Series B*, 34(3):239–243, 1983. 2

[7] D. Conlon. Hypergraph packing and sparse bipartite Ramsey numbers. *Combinatorics, Probability and Computing*, 18(6):913–923, 2009. 3, 4

[8] D. Conlon, J. Fox, and B. Sudakov. On two problems in graph Ramsey theory. *Combinatorica*, 32(5):513–535, 2012. 2

[9] D. Conlon, J. Fox, and B. Sudakov. Recent developments in graph Ramsey theory. *Surveys in combinatorics*, 424(2015):49–118, 2015. 3

[10] D. Conlon, J. Fox, and B. Sudakov. Short proofs of some extremal results II. *Journal of Combinatorial Theory, Series B*, 121:173–196, 2016. 3

[11] J. Corsten and W. Mendonça. Tiling edge-coloured graphs with few monochromatic bounded-degree graphs. *arXiv preprint arXiv:2103.16535*, 2021. 2, 17

[12] N. Eaton. Ramsey numbers for sparse graphs. *Discrete mathematics*, 185(1-3):63–75, 1998. 2

[13] P. Erdős, A. Gyárfás, and L. Pyber. Vertex coverings by monochromatic cycles and trees. *Journal of Combinatorial Theory, Series B*, 51(1):90–95, 1991. 1

[14] J. Fox and B. Sudakov. Density theorems for bipartite graphs and related Ramsey-type results. *Combinatorica*, 29(2):153–196, 2009. 3, 4, 5, 11

[15] R. L. Graham, V. Rödl, and A. Ruciński. On graphs with linear Ramsey numbers. *Journal of Graph Theory*, 35(3):176–192, 2000. 2, 3

[16] R. L. Graham, V. Rödl, and A. Ruciński. On bipartite graphs with linear Ramsey numbers. *Combinatorica*, 21(2):199–209, 2001. 3

[17] A. Grinshpun and G. N. Sárközy. Monochromatic bounded degree subgraph partitions. *Discrete Mathematics*, 339(1):46–53, 2016. 1, 2

[18] A. Gyárfás. Vertex covers by monochromatic pieces—a survey of results and problems. *Discrete Mathematics*, 339(7):1970–1977, 2016. 1

[19] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and E. Szemerédi. An improved bound for the monochromatic cycle partition number. *Journal of Combinatorial Theory, Series B*, 96(6):855–873, 2006. 1

[20] C. Lee. Ramsey numbers of degenerate graphs. *Annals of Mathematics*, pages 791–829, 2017. 17

[21] T. Łuczak, V. Rödl, and E. Szemerédi. Partitioning two-coloured complete graphs into two monochromatic cycles. *Combinatorics, Probability and Computing*, 7(4):423–436, 1998. 1

[22] A. Pokrovskiy. Partitioning edge-coloured complete graphs into monochromatic cycles and paths. *Journal of Combinatorial Theory, Series B*, 106:70–97, 2014. 1