Realizations of Non-commutative Rational Functions Around a Matrix Centre, II: The Lost-Abbey Conditions

Motke Porat and Victor Vinnikov

Abstract. In a previous paper the authors generalized classical results on minimal realizations of non-commutative (nc) rational functions, using nc Fornasini–Marchesini realizations which are centred at an arbitrary matrix point. In particular, it was proved that the domain of regularity of a nc rational function is contained in the invertibility set of the corresponding pencil of any minimal realization of the function. In this paper we prove an equality between the domain of a nc rational function and the domain of any of its minimal realizations. As for evaluations over stably finite algebras, we show that the domain of the realization w.r.t any such algebra coincides with the so called matrix domain of the function w.r.t the algebra. As a corollary we show that the domain of regularity and the stable extended domain coincide. In contrary to both the classical case and the scalar case—where every matrix coefficients which satisfy the controllability and observability conditions can appear in a minimal realization of a nc rational function—the matrix coefficients in our case have to satisfy certain equations, called linearized lost-abbey conditions, which are related to Taylor–Taylor expansions in nc function theory.

Contents

Introduction 2
1. Preliminaries 7
  1.1. NC Rational Expressions and Functions 8
  1.2. Realization Theory Around a Matrix Centre 9
  1.3. NC Functions 12
2. The Linearized Lost-Abbey Conditions 14
3. NC Difference-Differential Calculus 22

The research of both authors was partially supported by the US–Israel Binational Science Foundation (BSF) Grant No. 2010432, Deutsche Forschungsgemeinschaft (DFG) Grant No. SCHW 1723/1-1, and Israel Science Foundation (ISF) Grant No. 2123/17.
Introduction

Non-commutative (nc, for short) rational functions are a skew field of fractions—more precisely, the universal skew field of fractions—of the ring of nc polynomials, i.e., polynomials in noncommuting indeterminates (the free associative algebra). Essentially, they are obtained by starting with nc polynomials and applying successive arithmetic operations; a considerable amount of technical details is necessary here since in contrast to the commutative case there is no canonical coprime fraction representation for a nc rational function. NC rational functions originated from several sources: the general theory of free rings and of skew fields (see [20–25,45,54–56] for comprehensive expositions, and [57,66] for good surveys); the theory of rings with rational identities (see [6], also [16] and [67, Chapter 8]); and rational former power series in the theory of formal languages and finite automata (see [29–31,52,73,74] and [18] for a good survey).

Much like in the case of rational functions of a single variable [12,51] (and unlike the case of several commuting variables [32,46]), nc rational functions that are regular at 0 admit a good state space realization theory, see in particular Theorem 1 below. This was first established in the context of finite automata and recognizable power series, and more recently reformulated, with additional details, in the context of transfer functions of multidimensional systems with evolution along the free monoid (see [4,7–11]). State space realizations of nc rational functions have figured prominently in work on robust control of linear systems subjected to structured possibly time-varying uncertainty (see [13,14,58]). Another important application of nc rational functions appears in the area of Linear Matrix Inequalities (LMI s, see, e.g., [60,61,68]). Most optimization problems of system theory and control are dimensionless in the sense that the natural variables are matrices, and the problem involves nc rational expressions in these matrix variables which have therefore the same form independent of matrix sizes (see [5,19,35,36,41]). State space realizations are exactly what is needed to convert (numerically unmanageable) rational matrix inequalities into (highly manageable) linear matrix inequalities (see [39,42,43]).

Coming from a different direction, the method of state space realizations, also known as the linearization trick, found important recent applications in free probability, see [15,40,75,76]. Here it is crucial to evaluate nc rational expressions on a general algebra—which is stably finite in many
important cases—rather than on matrices of all sizes. Stably finite algebras appeared in this context in the work of Cohn [25] and they play an important and not surprising role in our analysis.

Here is a full characterization of nc rational functions which are regular at 0 and their (matrix) domains of regularity, in terms of their minimal realizations (for the proofs, see [7, 8, 29–31, 49, 50, 52]).

**Theorem 1.** If \( R \) is a nc rational function of \( x_1, \ldots, x_d \) and \( R \) is regular at 0, then \( R \) admits a unique (up to unique similarity) minimal nc Fornasini–Marchesini realization

\[
R(x_1, \ldots, x_d) = D + C \left( I_L - \sum_{k=1}^d A_k x_k \right)^{-1} \sum_{k=1}^d B_k x_k,
\]

where \( A_1, \ldots, A_d \in \mathbb{K}^{L \times L}, B_1, \ldots, B_d \in \mathbb{K}^{L \times 1}, C \in \mathbb{K}^{1 \times L}, D = R(0) \in \mathbb{K} \) and \( L \in \mathbb{N} \). Moreover, for all \( m \in \mathbb{N} : X = (X_1, \ldots, X_d) \in (\mathbb{K}^{m \times m})^d \) is in the domain of regularity of \( R \) if and only if \( \det (I_{Lm} - X_1 \otimes A_1 - \cdots - X_d \otimes A_d) \neq 0 \); in that case

\[
R(X) = I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^d X_k \otimes A_k \right)^{-1} \sum_{k=1}^d X_k \otimes B_k.
\]

Here a realization is called minimal if the state space dimension \( L \) is as small as possible; this is equivalent to the realization being observable, i.e.,

\[
\bigcap_{0 \leq k} \bigcap_{1 \leq i_1, \ldots, i_k \leq d} \ker(CA_{i_1} \cdots A_{i_k}) = \{0\},
\]

and controllable, i.e.,

\[
\bigvee_{0 \leq k} \bigvee_{1 \leq i_1, \ldots, i_k, j \leq d} A_{i_1} \cdots A_{i_k} B_j = \mathbb{K}^L.
\]

Theorem 1 is strongly related to expansions of nc rational functions which are regular at 0 into formal nc power series around 0; that is why it is not applicable for all nc rational functions. For example, the nc rational expression \( R(x_1, x_2) = (x_1 x_2 - x_2 x_1)^{-1} \) is not defined at 0, nor at any pair \((y_1, y_2) \in \mathbb{K}^2\), therefore one can not consider realizations of \( R \) which are centred at 0 as in Theorem 1, nor at any scalar point (a tuple of scalars). A realization theory for such expressions (and hence functions) is required in particular for all of the applications mentioned above. Such a theory is presented in our first paper [64] and continues here, using the ideas of the general theory of nc functions. Other types of realizations of nc rational functions that are not necessarily regular at 0 have been considered (see [26, 27], [83], and also the recent papers [69–72]).

The theory of nc functions has its roots in the works by Taylor [77, 78] on noncommutative spectral theory. It was further developed by Voiculescu [79–81] and Kalyuzhnyi–Verbovetskyi–Vinnikov [47], including a detailed discussion on nc difference-differential calculus. The main underlying idea is that a function of \( d \) non-commuting variables is a function of \( d \)-tuples of square matrices of all sizes that respects direct sums and simultaneous similarities.
See also the work of Helton–Klep–McCullough [37,38], of Popescu [62,63], of Muhly–Solel [59], and of Agler–McCarthy [1–3].

A crucial fact [47, Chapters 4–7] is that nc functions admit power series expansions, called Taylor–Taylor series in honor of Brook Taylor and of Joseph L. Taylor, around an arbitrary matrix point in their domain. However, a main difference between the scalar and the non-scalar centre cases, is that the coefficients in the Taylor–Taylor series are not arbitrary and must satisfy some compatibility conditions, called the lost-abbey conditions (see [47, equations (4.14)–(4.17)], or equations (1.13)–(1.16) below). This motivates us to generalize realizations as in Theorem 1 above, to the case where the centre is a $d$-tuple of matrices rather than $0$ or a $d$-tuple of scalars. We refer also to [53] for a recent application of the lost-abbey conditions for the study of germs of nc functions.

This is the second in a series of papers with the goal of generalizing the theory of Fornasini–Marchesini realizations centred at $0$ (or any other scalar point), to the case of Fornasini–Marchesini realizations centred at an arbitrary matrix point in the domain of regularity of a nc rational function. In the first paper of the series ([64]), we proved the following main result:

**Theorem 2.** ([64], Corollary 2.18 and Theorem 3.3) If $\mathcal{R}$ is a nc rational function of $x_1,\ldots,x_d$ over $\mathbb{K}$, then for every $Y = (Y_1,\ldots,Y_d) \in \text{dom}_s(\mathcal{R})$ there exists a unique (up to unique similarity) minimal (observable and controllable) nc Fornasini–Marchesini realization

$$
\mathcal{R}(X) = D + C \left( I_L - \sum_{k=1}^d A_k (X_k - Y_k) \right)^{-1} \sum_{k=1}^d B_k (X_k - Y_k)
$$

centred at $Y$, such that

$$
\text{dom}_{sm}(\mathcal{R}) \subseteq \Omega_{sm}(\mathcal{R}) := \left\{ X \in (\mathbb{K}^{sm \times sm})^d : \right.$$

$$
\det \left( I_{Lm} - \sum_{k=1}^d (X_k - I_m \otimes Y_k) A_k \right) \neq 0 \bigg\}$$

and

$$
\mathcal{R}(X) = \mathcal{R}(X) = I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^d (X_k - I_m \otimes Y_k) A_k \right)^{-1}
$$

$$
\sum_{k=1}^d (X_k - I_m \otimes Y_k) B_k
$$

for every $X = (X_1,\ldots,X_d) \in \text{dom}_{sm}(\mathcal{R})$ and $m \in \mathbb{N}$.

Here $C, D$ are matrices of appropriate sizes, while $A_1,\ldots,A_d, B_1,\ldots,B_d$ are linear mappings from $\mathbb{K}^{s \times s}$ to matrices of appropriate sizes, see Sect. 1.2 below for the exact description.

This is a partial generalization of Theorem 1, as it only shows the inclusion of the domain of a nc rational function in the domain of a minimal nc Fornasini–Marchesini realization that the function admits. The difficulty
is that in contrast to Theorem 1, a minimal nc Fornasini–Marchesini realization of a nc rational expression centred at a matrix point is no longer a nc rational expression by itself. An equality between the two domains does hold (Theorem 4.7), but the proof requires more tools from the theory of nc functions, including their nc generalized power series expansions and difference-differential calculus, which are developed in this paper.

Notice that we made a change of notations from [64], where what is denoted here by $\Omega_{sm}(\mathcal{R})$, was denoted by $DOM_{sm}(\mathcal{R})$.

Outline and Main Results: In Sect. 2 we provide a linearization of the lost-abbey ($LA$) conditions, in the case where the nc generalized power series admits a minimal nc Fornasini–Marchesini realization (Lemma 2.2). Then we prove that for any nc Fornasini–Marchesini realization $\mathcal{R}$ such that its coefficients satisfy these linearized lost-abbey ($L-LA$) conditions, the evaluation of $\mathcal{R}$ is a nc function on the invertibility set $\Omega(\mathcal{R})$ of the corresponding pencil, which is an upper admissible nc set (Theorem 2.6).

Then, in Sect. 3, we use the fact that $\mathcal{R}$ is a nc function on the upper admissible nc set $\Omega(\mathcal{R})$, to apply the nc difference-differential calculus and to show that $\Omega(\mathcal{R})$ is similarity invariant (Theorem 3.5), under the assumptions that $\mathcal{R}$ is controllable and observable, and its coefficients satisfy the $L-LA$ conditions. Some formulas such as (3.2), that appear in the proof of Lemma 3.3 might be of separate interest for applications in free analysis.

Finally, in Sect. 4 we show that if a realization is controllable and observable, and its coefficients satisfy the $L-LA$ conditions, then the realization is actually the restriction of a nc rational function (Theorem 4.4). In the proof of Theorem 4.4 we make an extensive use of the results from Sects. 2 and 3, as well as of [82, Lemma 3.9] (cf. Lemma 4.3).

As a corollary of Theorem 4.4, we prove one of the main results in the paper, that is that the domain of a nc rational function coincides with—and not only contains as in Theorem 2—the domain of any of its minimal realizations, centred at an arbitrary matrix point, which allows us to evaluate the nc rational function on all of its domain using the evaluation of the realization. The corresponding result when evaluating over stably finite algebras is given too, while one has to modify our definition of the $A$–domain of the function and consider its $A$–matrix domain.

**Theorem 3.** (Theorems 4.7, 4.14) If $\mathcal{R}$ is a nc rational function of $x_1, \ldots, x_d$ over $K$, then for every $\mathbf{Y} = (Y_1, \ldots, Y_d) \in dom_s(\mathcal{R})$, there exists a unique minimal nc Fornasini–Marchesini realization $\mathcal{R}$ centred at $\mathbf{Y}$, such that $dom_{sm}(\mathcal{R}) = \Omega_{sm}(\mathcal{R})$ for every $m \in \mathbb{N}$ and

$$\mathcal{R}(\mathbf{X}) = I_m \otimes D + (I_m \otimes C) \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \right)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k$$

for every $\mathbf{X} \in dom_{sm}(\mathcal{R})$. Moreover, for every $n \in \mathbb{N}$,

$$dom_{n}(\mathcal{R}) = \{ \mathbf{X} \in (\mathbb{K}^{n \times n})^d : I_s \otimes \mathbf{X} \in \Omega_{sn}(\mathcal{R}) \}$$

and $I_s \otimes \mathcal{R}(\mathbf{X}) = \mathcal{R}(I_s \otimes \mathbf{X})$ for every $\mathbf{X} \in dom_{n}(\mathcal{R})$. Furthermore,

$$dom_{\mathcal{A}}^{Mat}(\mathcal{R}) = \{ \mathbf{a} \in \mathcal{A}^d : I_s \otimes \mathbf{a} \in \Omega_{\mathcal{A}}(\mathcal{R}) \}$$
and $I_s \otimes \mathcal{R}(a) = \mathcal{R}^A(I_s \otimes a)$ for every $a \in \text{dom}^{\text{Mat}}_A(\mathcal{R})$, whenever $\mathcal{A}$ is a stably finite $\mathbb{K}$–unital algebra.

Here $\Omega_\mathcal{A}(\mathcal{R})$ is the invertibility set of the corresponding matrix pencil over the algebra $\mathcal{A}$ (see (1.8) for the definition), $\text{dom}^{\text{Mat}}_A(\mathcal{R})$ denotes a variation of the domain of the rational function $\mathcal{R}$ over the algebra $\mathcal{A}$ called the matrix domain (see (4.17) for the precise definition), and an algebra is called stably finite if every square left invertible matrix over the algebra is also right invertible (see Definition 1.2). We also present in Corollary 4.8 a version of this result using realizations which are independent of a matrix centre, in the spirit of Cohn, Reutenauer and Fliess.

We finish this section by giving a full characterization of all the minimal nc Fornasini–Marchesini realizations of a nc rational function, which are centred at $Y \in \text{dom}(\mathcal{R})$, using the $\mathcal{L} – \mathcal{L}A$ conditions:

**Theorem 4.** (Theorem 4.9) Let $\mathcal{R}$ be a nc Fornasini–Marchesini realization centred at $Y \in (\mathbb{K}^{s \times s})^d$ that is described by $(L; D, C, \mathcal{A}, \mathcal{B})$, and suppose $\mathcal{R}$ is both controllable and observable. The following are equivalent:

1. There exists a nc rational function $\mathcal{R} \in \mathbb{K}(\langle x \rangle)$ regular at $Y$, such that $\text{dom}_{\text{sm}}(\mathcal{R}) = \Omega(\mathcal{R})$ and $\mathcal{R}(X) = \mathcal{R}(X)$, for every $X \in \text{dom}_{\text{sm}}(\mathcal{R})$ and $m \in \mathbb{N}$.

2. The coefficients of $\mathcal{R}$ satisfy the $\mathcal{L} – \mathcal{L}A$ conditions (cf. equations (2.3)–(2.8)).

There is a different notion of domain for nc rational functions, the so called extended domain, which is based on evaluations on generic matrices. It was discovered in [47] and a priori it contains the usual domain of regularity of the function. The extended domain by itself is problematic as it is not closed w.r.t direct sums, but this can be fixed by introducing the stable extended domain (see [82]).

In Sect. 5 we use the crucial fact that if the coefficients of a nc Fornasini–Marchesini realization, that is controllable and observable, satisfy the $\mathcal{L} – \mathcal{L}A$ conditions, then the realization defines a nc function on an upper admissible nc set and we can apply to this function the difference-differential calculus. The main result—which is a generalization of [82, Theorem 3.10], in which the function is assumed to be regular at some scalar point—then is that the stable extended domain of any nc rational function coincides with its usual domain of regularity:

**Theorem 5.** (Corollary 5.5) For every nc rational function $\mathcal{R} \in \mathbb{K}(\langle x \rangle)$, we have

$\text{edom}^{\text{st}}(\mathcal{R}) = \text{dom}(\mathcal{R})$.

Moreover, for every $n \in \mathbb{N}$ we have

$\text{edom}^{\text{st}}_n(\mathcal{R}) = \text{dom}_n(\mathcal{R}) = \{ X \in (\mathbb{K}^{n \times n})^d : I_s \otimes X \in \Omega_{\text{sm}}(\mathcal{R}) \}$,

whenever $\mathcal{R}$ is a minimal realization of $\mathcal{R}$, centred at a point from $\text{dom}_{\text{sm}}(\mathcal{R})$. 
In the next paper [65], we will use the theory of realizations with a matrix center developed in [64] and in the present paper, to characterize explicitly the ring of rational nc generalized power series with matrix center \( Y \). More precisely, we will establish a generalization of the Fliess-Kronecker theorem in which the characterization is given in terms of two conditions: the \( \mathcal{L}A \) conditions and a finiteness condition on the rank of an infinite Hankel matrix.

As part of the tools we will construct a functional model and use it to provide a different one step proof (based on Hankel realizations, see [51] for the classical case and [7] for the scalar nc case) for the existence of a (minimal) realization formula for nc rational functions, without using synthesis. Furthermore, we present an explicit construction of the free skew field \( \mathbb{K}(\langle x \rangle) \), with a self-contained proof that it is the universal skew field of fractions of the ring of nc polynomials.

1. Preliminaries

**Notations:** \( d \) will stand for the number of noncommuting variables, which will be usually denoted by \( x_1, \ldots, x_d \), we often abbreviate non-commuting by nc. For a positive integer \( d \), we denote by \( G_d \) the free monoid generated by \( d \) generators \( g_1, \ldots, g_d \), we say that a word \( \omega = g_{i_1} \cdots g_{i_l} \in G_d \) is of length \( |\omega| = l \) if \( l \geq 1 \) and \( \omega = \emptyset \) is of length 0. For a field \( \mathbb{K} \) and \( n \in \mathbb{N} \), let \( \mathbb{K}^{n \times n} \) be the vector space of \( n \times n \) matrices over \( \mathbb{K} \), let \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) be the standard basis of \( \mathbb{K}^n \) and let \( \mathcal{E}_n = \{E_{ij} = \varepsilon_i \varepsilon_j^T : 1 \leq i, j \leq n\} \) be the standard basis of \( \mathbb{K}^{n \times n} \). The tensor (Kronecker) product of two matrices \( P \in \mathbb{K}^{n_1 \times n_2} \) and \( Q \in \mathbb{K}^{n_3 \times n_4} \) is the \( n_1 n_3 \times n_2 n_4 \) block matrix \( P \otimes Q = [p_{ij} Q]_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \).

The range of a matrix \( P \), that is the span of all of its columns, is denoted by \( \text{Im}(P) \). For any two square matrices \( X \) and \( Y \) of the same size, their commutator is defined by \( [X, Y] = XY - YX \).

We denote operators on matrices by bold letters such as \( \mathbf{A}, \mathbf{B} \), and the action of \( \mathbf{A} \) on \( X \) by \( \mathbf{A}(X) \). If \( \mathbf{A} \) is defined on \( s \times s \) matrices we extend \( \mathbf{A} \) to act on \( sm \times sm \) matrices for any \( m \in \mathbb{N} \), by viewing an \( sm \times sm \) matrix \( X \) as an \( m \times m \) matrix with \( s \times s \) blocks and by evaluating \( \mathbf{A} \) on the \( s \times s \) blocks (cf. equation (1.6)); in that case we denote the evaluation by \( (X) \mathbf{A} \). If \( C \) is a constant matrix and \( \mathbf{A} \) is an operator, then \( C \cdot \mathbf{A} \) and \( \mathbf{A} \cdot C \) are two operators, defined by \( (C \cdot \mathbf{A})(X) := C \mathbf{A}(X) \) and \( (\mathbf{A} \cdot C)(X) := \mathbf{A}(X)C \). For every \( n_1, n_2 \in \mathbb{N} \), we define the permutation matrix

\[
E(n_1, n_2) = [E_{ij}^T]_{1 \leq i \leq n_1, 1 \leq j \leq n_2} \in \mathbb{K}^{n_1 n_2 \times n_1 n_2}
\]

and use these matrices to change the order of factors in the Kronecker product of two matrices by the following rule

\[
P \otimes Q = E(n_1, n_3)(Q \otimes P)E(n_2, n_4)^T,
\]

for all \( n_1, n_2, n_3, n_4 \in \mathbb{N} \), \( Q \in \mathbb{K}^{n_1 \times n_2} \), and \( P \in \mathbb{K}^{n_3 \times n_4} \); for more details see [44, pp. 259–261].
then we use the notation
\[ P \odot_s Q := \left[ \sum_{k=1}^{m} P_{ik} \otimes Q_{kj} \right]_{1 \leq i,j \leq m} \]
for the so-called faux product of \( P \) and \( Q \), viewed as \( m \times m \) matrix over the tensor algebra of \( \mathbb{K}^{s \times s} \), see [28] for its origins in operator spaces. If \( X = (X_1, \ldots, X_d) \in (\mathbb{K}^{sm \times sm})^d \) and \( \omega = g_{i_1} \ldots g_{i_t} \in G_d \), then
\[ X^{\odot_s \omega} := X_{i_1} \odot_s \ldots \odot_s X_{i_t}. \] (1.2)
Recall that \( X \) is called jointly nilpotent, if there exists \( \kappa \in \mathbb{N} \) such that \( X^{\odot_s \omega} = 0 \) for every \( \omega \in G_d \) satisfying \( |\omega| \geq \kappa \). For every matrix \( Z \) over \( \mathbb{K} \) and a unital \( \mathbb{K} \)-algebra \( \mathcal{A} \), we use the notation \( Z_A := Z \otimes 1_A \), where \( 1_A \) is the unit element in \( \mathcal{A} \). Notice that if \( X^{i,j} = (X_1^{i,j}, \ldots, X_d^{i,j}) \in (\mathbb{K}^{n_1 \times n_2})^d \) for \( 1 \leq i, j \leq 2 \), we use the notation
\[ \left( \begin{array}{c}
X_{1,1}^{1,2} \\
X_{2,1}^{1,2} \\
X_{2,2}^{1,2} \\
X_{2,2}^{2,2}
\end{array} \right) := \left( \begin{array}{c}
X_{1,1}^{1,1} X_{1,2}^{1,2} \\
X_{2,1}^{1,1} X_{1,2}^{2,2} \\
X_{2,2}^{1,1} X_{1,2}^{2,2} \\
X_{2,2}^{2,2}
\end{array} \right) \in (\mathbb{K}^{(n_1+n_2) \times (n_1+n_2)})^d. \]
We use \( \mathfrak{R}, \mathcal{R}, R, \) and \( \mathfrak{r} \) for nc rational function, nc Fornasini–Marchesini realization, nc rational expression, and matrix valued nc rational function, respectively. Likewise, we use \( a \) to denote elements in an algebra \( \mathcal{A} \) and \( \mathfrak{A} \) to denote matrices over \( \mathcal{A} \). All over the paper, we use underline to denote vectors or \( d \)-tuples.

### 1.1. NC Rational Expressions and Functions

Let \( \mathbb{K} \) be a field, \( d \geq 1 \) an integer, and \( x_1, \ldots, x_d \) noncommuting variables. We denote by \( \mathbb{K}<x_1, \ldots, x_d> \) the \( \mathbb{K} \)-algebra of nc polynomials in the \( d \) nc variables \( x_1, \ldots, x_d \) over \( \mathbb{K} \). We obtain nc rational expressions by applying successive arithmetic operations (addition, multiplication and taking inverse) on \( \mathbb{K}<x_1, \ldots, x_d> \).

For a nc rational expression \( R \) and \( n \in \mathbb{N} \), the set \( \text{dom}_n(R) \) consists of all \( d \)-tuples of \( n \times n \) matrices over \( \mathbb{K} \) for which all the inverses in \( R \) exist. The domain of regularity of \( R \) is then defined by
\[ \text{dom}(R) := \prod_{n=1}^{\infty} \text{dom}_n(R). \]
For example, \( R(x_1, x_2) = (x_1 x_2 - x_2 x_1)^{-1} \) is a nc rational expression in \( x_1, x_2 \), with \( \text{dom}_1(R) = \emptyset \) and \( \text{dom}_n(R) = \{(X_1, X_2) \in (\mathbb{K}^{n \times n})^2 : \det(X_1 X_2 - X_2 X_1) \neq 0\} \neq \emptyset \) for any \( n > 1 \).

A nc rational expression \( R \) is called non-degenerate if \( \text{dom}(R) \neq \emptyset \) (for instance the nc rational expression \( (x_1 - x_1)^{-1} \) is degenerate). Let \( R_1 \) and \( R_2 \) be nc rational expressions in \( x_1, \ldots, x_d \) over \( \mathbb{K} \). We say that \( R_1 \) and \( R_2 \) are \( (\mathbb{K}^d)_{nc} \)-evaluation equivalent, if \( R_1(X) = R_2(X) \) for every \( X \in \text{dom}(R_1) \cap \text{dom}(R_2) \).

For example the nc rational expressions \( R_1(x_1, x_2) = x_2 (x_1 x_2)^{-1} x_1 + x_1 x_2, R_2(x_1, x_2) = x_2^{-1} (x_2 + x_2 x_1 x_2) \) and \( R_3(x_1, x_2) = 1 + x_1 x_2 \), are \( (\mathbb{K}^3)_{nc} \)-evaluation equivalent.
**Definition 1.1. (NC Rational Functions)** A **nc rational function** of \(x_1, \ldots, x_d\) over \(K\) is an equivalence class of non-degenerate nc rational expressions, w.r.t the \((\mathbb{K}^d)_{nc}\)—evaluation equivalence relation. For every nc rational function \(R\) of \(x_1, \ldots, x_d\), define its **domain of regularity**

\[
dom(R) := \bigcup_{R \in \mathcal{R}} \dom(R). \tag{1.3}
\]

For every \(X \in \dom(R)\), we say that \(R\) is **regular** at \(X\) and its evaluation is given by

\[
R(X) = R(X)
\]

for every nc rational expression \(R \in \mathcal{R}\) such that \(X \in \dom(R)\).

The \(K\)—algebra of all nc rational functions of \(x_1, \ldots, x_d\) over \(K\) is denoted by \(\mathbb{K}\langle x_1, \ldots, x_d \rangle\) and it is a skew field, called the free skew field. Moreover, \(\mathbb{K}\langle x_1, \ldots, x_d \rangle\) is the universal skew field of fractions of \(\mathbb{K}\langle x_1, \ldots, x_d \rangle\). See [6,16,21,22,67] for the original proofs and [25] for a more modern reference, while a proof of the equivalence with the evaluations over matrices is presented in [49,50].

**A—Domains and Evaluations.** Let \(A\) be a unital \(K\)—algebra. If \(\mathbf{a} = (a_1, \ldots, a_d) \in A^d\) and \(\omega = g_{i_1} \cdots g_{i_L} \in G_d\), then we use the notations \(a^{\omega} := a_{i_1} \cdots a_{i_L}\) and \(\mathbf{1}_A = \mathbf{1}_A\), where \(\mathbf{1}_A\) is the unit element in \(A\). Evaluations and domains of nc rational expressions over \(A\) are defined in a natural way, see [40] or [64, Sect. 1.2] for more details. We are interested in a certain family of algebras, called stably finite algebras, also called weakly finite in [25].

**Definition 1.2.** A unital \(K\)—algebra \(A\) is called **stably finite** if for every \(m \in \mathbb{N}\) and \(\mathfrak{A}, \mathfrak{B} \in A^{m \times m}\), we have \(\mathfrak{A}\mathfrak{B} = I_m \otimes \mathbf{1}_A\) if and only if \(\mathfrak{B}\mathfrak{A} = I_m \otimes \mathbf{1}_A\).

### 1.2. Realization Theory Around a Matrix Centre

We summarize now the main definitions and main results from [64] about realizations of nc rational expressions and nc rational functions around a matrix centre. Notice we denote here by \(\Omega_{sm}(\mathcal{R})\) what was denoted in [64] by \(DOM_{sm}(\mathcal{R})\).

**Definition 1.3.** ([64], Definition 2.1) Let \(s \in \mathbb{N}, L \in \mathbb{N}, Y = (Y_1, \ldots, Y_d) \in (\mathbb{K}^{s \times s})^d, A_1, \ldots, A_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L}\) and \(B_1, \ldots, B_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s}\) be linear mappings, \(C \in \mathbb{K}^{s \times L}\), and \(D \in \mathbb{K}^{s \times s}\). Then

\[
\mathcal{R}(X_1, \ldots, X_d) = D + C \left( I_L - \sum_{k=1}^{d} A_k(X_k - Y_k) \right)^{-1} \sum_{k=1}^{d} B_k(X_k - Y_k) \tag{1.4}
\]

is called a **nc Fornasini–Marchesini realization** centred at \(Y\) and it is defined for every \(X = (X_1, \ldots, X_d) \in \Omega_s(\mathcal{R})\), where

\[
\Omega_s(\mathcal{R}) := \left\{ X \in (\mathbb{K}^{s \times s})^d : \det \left( I_L - \sum_{k=1}^{d} A_k(X_k - Y_k) \right) \neq 0 \right\}.
\]
In that case we say that the realization $\mathcal{R}$ is described by $(L; D, C, A, B)$ and the corresponding generalized linear pencil centred at $\Lambda$ is

$$\Lambda_{\Lambda, Y}(X) := I_L - \sum_{k=1}^{d} A_k(X_k - Y_k).$$

(1.5)

Let $s_1, s_2, s_3, s_4 \in \mathbb{N}$. If $T : \mathbb{K}^{s_1 \times s_2} \to \mathbb{K}^{s_3 \times s_4}$ is a linear mapping and $m \in \mathbb{N}$, then $T$ can be naturally extended to a linear mapping $T : \mathbb{K}^{s_1m \times s_2m} \to \mathbb{K}^{s_3m \times s_4m}$, by the following rule:

$$X = [X_{ij}]_{1 \leq i, j \leq s} \in \mathbb{K}^{s_1m \times s_2m} \implies (X)T = [T(X_{ij})]_{1 \leq i, j \leq s},$$

(1.6)
i.e., $(X)T$ is an $m \times m$ block matrix with entries in $\mathbb{K}^{s_3 \times s_4}$. Therefore, we can extend the pencil $\Lambda_{\Lambda, Y}$ to act on $d$–tuples of $sm \times sm$ matrices, by

$$\Lambda_{\Lambda, Y}(X) = I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k, \forall X = (X_1, \ldots, X_d) \in (\mathbb{K}^{sm \times sm})^d,$$

hence we extend the realization (1.4) to act on $d$–tuples of $sm \times sm$ matrices: for every

$$X = (X_1, \ldots, X_d) \in \Omega_{sm}(\mathcal{R}) := \left\{ X \in (\mathbb{K}^{sm \times sm})^d : \det(\Lambda_{\Lambda, Y}(X)) \neq 0 \right\},$$

we define

$$\mathcal{R}(X) := I_m \otimes D + (I_m \otimes C)\Lambda_{\Lambda, Y}(X)^{-1}\sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k.$$

In addition, if $\mathcal{A}$ is a unital $\mathbb{K}$–algebra, a linear mapping $T : \mathbb{K}^{s_1 \times s_2} \to \mathbb{K}^{s_3 \times s_4}$ can be also naturally extended to a linear mapping $T^A : \mathcal{A}^{s_1 \times s_2} \to \mathcal{A}^{s_3 \times s_4}$, by the following rule:

$$\mathcal{A} = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} E_{ij} \otimes a_{ij} \in \mathcal{A}^{s_1 \times s_2} \implies (\mathcal{A})T^A = \sum_{i=1}^{s_1} \sum_{j=1}^{s_2} T(E_{ij}) \otimes a_{ij} \in \mathcal{A}^{s_3 \times s_4},$$

(1.7)

where $E_{ij} = e_i e_j^T \in \mathbb{K}^{s_1 \times s_2}$ and $a_{ij} \in \mathcal{A}$. Thus we extend the pencil $\Lambda_{\Lambda, Y}$ to act on $d$–tuples of $s \times s$ matrices over $\mathcal{A}$, by

$$\Lambda_{\Lambda, Y}^A(\mathcal{A}) := I_L \otimes 1_{\mathcal{A}} - \sum_{k=1}^{d} (\mathcal{A}_k - Y_k \otimes 1_{\mathcal{A}})\Lambda_k^A, \forall \mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_d) \in (\mathcal{A}^{s \times s})^d.$$

The $\mathcal{A}$–domain of a nc Fornasini–Marchesini realization $\mathcal{R}$ centred at $\Lambda$, as in (1.4), is then defined to be the subset of $(\mathcal{A}^{s \times s})^d$ given by

$$\Omega_{\mathcal{A}}(\mathcal{R}) := \left\{ \mathcal{A} \in (\mathcal{A}^{s \times s})^d : \Lambda_{\Lambda, Y}^A(\mathcal{A}) \text{ is invertible in } \mathcal{A}^{L \times L} \right\}$$

(1.8)

and for every $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_d) \in \Omega_{\mathcal{A}}(\mathcal{R})$, the evaluation of $\mathcal{R}$ at $\mathcal{A}$ is defined by

$$\mathcal{R}^A(\mathcal{A}) := D \otimes 1_{\mathcal{A}} + (C \otimes 1_{\mathcal{A}})\Lambda_{\Lambda, Y}^A(\mathcal{A})^{-1}\sum_{k=1}^{d} \left[ (\mathcal{A}_k)B_k^A - B_k(Y_k) \otimes 1_{\mathcal{A}} \right].$$

(1.9)
Definition 1.4. ([64], Definition 2.3) Let $R$ be a nc rational expression in $x_1, \ldots, x_d$ over $\mathbb{K}$, $Y \in \text{dom}_s(R)$, $\mathcal{R}$ be a nc Fornasini–Marchesini realization centred at $Y$, and $\mathcal{A}$ be a unital $\mathbb{K}$–algebra. We say that:

1. $R$ admits the realization $\mathcal{R}$, or that $\mathcal{R}$ is a realization of $R$, if
   
   $\text{dom}_{sm}(R) \subseteq \Omega_{sm}(\mathcal{R})$ and $R(X) = \mathcal{R}(X)$, $\forall X \in \text{dom}_{sm}(R)$, $m \in \mathbb{N}$.

2. $R$ admits the realization $\mathcal{R}$ w.r.t $\mathcal{A}$, or that $\mathcal{R}$ is a realization of $R$ w.r.t $\mathcal{A}$, if
   
   $\text{dom}_{A}(R) \subseteq \{a \in \mathcal{A}^d : I_s \otimes a \in \Omega_{A}(\mathcal{R})\}$ and
   
   $I_s \otimes R^A(a) = \mathcal{R}^A(I_s \otimes a)$, $\forall a \in \text{dom}_{A}(R)$.

Definition 1.5. ([64], Definition 2.7) Let $A_1, \ldots, A_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L}$ and $B_1, \ldots, B_d : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s}$ be linear mappings, and $C \in \mathbb{K}^{s \times L}$.

1. $(A, B)$ is called controllable, if
   
   $\bigvee \omega \in \mathcal{G}_{d}, X_1, \ldots, X_{|\omega|+1} \in \mathbb{K}^{s \times s}, 1 \leq k \leq d$ \hspace{1cm} \text{Im}(A^\omega(X_1, \ldots, X_{|\omega|})B_k(X_{|\omega|+1})) = \mathbb{K}^L.$

2. $(C, A)$ is called observable, if
   
   $\bigcap \omega \in \mathcal{G}_{d}, X_1, \ldots, X_{|\omega|} \in \mathbb{K}^{s \times s}$ \hspace{1cm} \ker(CA^\omega (X_1, \ldots, X_{|\omega|})) = \{0\}.$

If $\mathcal{R}$ is a nc Fornasini–Marchesini realization of a nc rational expression $R$, that is centred at $Y$, then it is said to be minimal if the dimension $L$ is the smallest integer for which $R$ admits such a realization, i.e., if $\mathcal{R}'$ is a nc Fornasini–Marchesini realization of $R$ centred at $Y$ of dimension $L'$, then $L \leq L'$.

Theorem 1.6. ([64], Theorem 2.16) Let $R$ be a nc rational expression in $x_1, \ldots, x_d$ over $\mathbb{K}$ and $\mathcal{R}$ be a nc Fornasini–Marchesini realization of $R$ centred at $Y \in (\mathbb{K}^{s \times s})^d$. Then $\mathcal{R}$ is minimal if and only if $\mathcal{R}$ is controllable and observable.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two nc Fornasini–Marchesini realizations, described by $(L_1; D^1, C^1, A_1^1, B_1^1)$ and $(L_2; D^2, C^2, A_2^2, B_2^2)$, respectively, both centred at $Y \in (\mathbb{K}^{s \times s})^d$. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ are said to be uniquely similar, if $L_1 = L_2$, $D^1 = D^2$, and there exists a unique invertible matrix $T \in \mathbb{K}^{L_1 \times L_1}$ such that

$C^2 = C^1 T^{-1}$, $B_k^2 = T \cdot B_k^1$, and $A_k^2 = T \cdot A_k^1 \cdot T^{-1}$, $1 \leq k \leq d$.

Corollary 1.7. ([64], Corollary 2.18) Let $R$ be a nc rational expression in $x_1, \ldots, x_d$ over $\mathbb{K}$ and $Y \in \text{dom}_s(R)$. Then $R$ admits a unique (up to unique similarity) minimal nc Fornasini–Marchesini realization centred at $Y$, that is also a realization of $R$ w.r.t any unital stably finite $\mathbb{K}$–algebra.

Theorem 1.8. ([64], Theorem 3.3) Let $\mathcal{R} \in \mathbb{K}[x_1, \ldots, x_d]$. For every two integers $s, n \in \mathbb{N}$, a point $Y \in \text{dom}_s(\mathcal{R})$, a minimal nc Fornasini–Marchesini realization $\mathcal{R}$ centred at $Y$ of $\mathcal{R}$, and a unital stably finite $\mathbb{K}$–algebra $\mathcal{A}$, we have the following properties:
1. \( \text{dom}_n(\mathcal{R}) \subseteq \{Z \in (\mathbb{K}^{n \times n})^d : I_s \otimes Z \in \Omega_{sn}(\mathcal{R})\} \) and \( I_s \otimes \mathcal{R}(Z) = \mathcal{R}(I_s \otimes Z), \forall Z \in \text{dom}_n(\mathcal{R}) \).

2. If \( s \mid n \), then \( \text{dom}_n(\mathcal{R}) \subseteq \Omega_n(\mathcal{R}) \) and \( \mathcal{R}(Z) = \mathcal{R}(Z), \forall Z \in \text{dom}_n(\mathcal{R}) \).

3. \( \text{dom}_A(\mathcal{R}) \subseteq \{a \in A^d : I_s \otimes a \in \Omega_A(\mathcal{R})\} \) and \( I_s \otimes \mathcal{R}^A(a) = \mathcal{R}^A(I_s \otimes a), \forall a \in \text{dom}_A(\mathcal{R}) \).

### 1.3. NC Functions

The definitions and results mentioned in this subsection are taken from the book [47]. We do not quote those in their full generality, as in the book they appear in the framework of a module over a commutative ring, while here we consider the framework of a vector space \( \mathcal{V} \) over a field \( \mathbb{K} \). Moreover, we will mostly consider the vector space \( \mathcal{V} = \mathbb{K}^d \) for an integer \( d \in \mathbb{N} \).

If \( \mathcal{V} \) is a vector space over a field \( \mathbb{K} \), then \( \mathcal{V}_{nc} := \prod_{n=1}^{\infty} \mathbb{K}^{n \times n} \) is called the nc space over \( \mathcal{V} \) and consists of all square matrices over \( \mathcal{V} \). For every \( X \in \mathcal{V}^{n \times n} \) and \( Y \in \mathcal{V}^{m \times m} \), where \( m, n \in \mathbb{N} \), we define their direct sum as

\[
X \oplus Y := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{V}^{(n+m) \times (n+m)}.
\]

For every \( \Omega \subseteq \mathcal{V}_{nc} \) and \( n \in \mathbb{N} \) we use the notation \( \Omega_n := \Omega \cap \mathcal{V}^{n \times n} \).

**Definition 1.9.**

1. A subset \( \Omega \subseteq \mathcal{V}_{nc} \) is called a nc set if it is closed under direct sums, i.e., if \( X \in \Omega_n \) and \( Y \in \Omega_m \), then \( X \oplus Y \in \Omega_{n+m} \), for every two integers \( n, m \in \mathbb{N} \).

2. A nc set \( \Omega \subseteq \mathcal{V}_{nc} \) is called upper (resp., lower) admissible if for every \( n, m \in \mathbb{N} \), \( X \in \Omega_n \), \( Y \in \Omega_m \), and \( Z \in \mathcal{V}^{n \times m} \) (resp., \( Z \in \mathcal{V}^{m \times n} \)), there exists \( 0 \neq c \in \mathbb{K} \) such that

\[
\begin{bmatrix} X & cZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m} \quad \text{(resp.,)} \quad \begin{bmatrix} X & 0 \\ cZ & Y \end{bmatrix} \in \Omega_{n+m}.
\]

3. A subset \( \Omega \subseteq \mathcal{V}_{nc} \) is called similarity invariant, if for every \( n \in \mathbb{N} \), \( X \in \Omega_n \), and invertible \( T \in \mathbb{K}^{n \times n} \), we have \( T \cdot X \cdot T^{-1} \in \Omega_n \).

Notice that if \( X \in \mathcal{V}^{n \times n} \) and \( T \in \mathbb{K}^{n \times n} \), by the products \( T \cdot X \) and \( X \cdot T \) we mean the standard matrix multiplication and we use the action of \( \mathbb{K} \) on \( \mathcal{V} \). In the special and most common case where \( \mathcal{V} = \mathbb{K}^d \), we have the identification

\[
(\mathbb{K}^d)^{nc} = \prod_{n=1}^{\infty} (\mathbb{K}^d)^{n \times n} \cong \prod_{n=1}^{\infty} (\mathbb{K}^{n \times n})^d,
\]

that is the nc space of all \( d \)-tuples of square matrices over \( \mathbb{K} \). Thus, in this particular case, for every \( X = (X_1, \ldots, X_d) \in (\mathbb{K}^{n \times n})^d \) and \( T \in \mathbb{K}^{n \times n} \), the products \( T \cdot X \) and \( X \cdot T \) are given by

\[
T \cdot X := (TX_1, \ldots, TX_d) \text{ and } X \cdot T := (X_1 T, \ldots, X_d T).
\]

For every nc rational expression \( R \) in \( x_1, \ldots, x_d \) over \( \mathbb{K} \), the domain of regularity of \( R \) is an upper admissible, similarity invariant nc set. We will show eventually that the domain of any nc rational function of \( x_1, \ldots, x_d \) over \( \mathbb{K} \) is an upper admissible, similarity invariant nc set as well (cf. Corollary 4.10). If \( \mathbb{K} = \mathbb{C} \), the domains of nc rational expressions or functions are open.
in the uniformly-open topology (cf. [47, Chapter 4.2] for a discussion on the uniformly-open topology and [43, Lemma A.5] for the proof in the case where regularity at 0 is assumed).

Another important example of a similarity invariant, upper admissible nc set is a nilpotent ball: for every $s \in \mathbb{N}$ and $Y = (Y_1, \ldots, Y_d) \in (\mathbb{K}^{s \times s})^d$, we define the nilpotent ball around $Y$ as follows

$$
\text{Nilp}(Y) := \bigcap_{m=1}^{\infty} \text{Nilp}_{sm}(Y) \subseteq (\mathbb{K}^d)^{nc},
$$

(1.10)

where

$$
\text{Nilp}_{sm}(Y) := \{ X \in (\mathbb{K}^{sm \times sm})^d : (X - I_m \otimes Y) \text{ is jointly nilpotent} \}
$$

(1.11)

for every $m \in \mathbb{N}$.

**Definition 1.10.** Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathbb{K}$ and $\Omega \subseteq \mathcal{V}_{nc}$ be a nc set, then $f : \Omega \to \mathcal{W}_{nc}$ is called a nc function, if

1. $f$ is graded, i.e., if $n \in \mathbb{N}$ and $X \in \Omega_n$, then $f(X) \in \mathcal{W}^{n \times n}$;
2. $f$ respects direct sums, i.e., if $X, Y \in \Omega$, then $f(X \oplus Y) = f(X) \oplus f(Y)$; and
3. $f$ respects similarities, i.e., if $n \in \mathbb{N}$, $X \in \Omega_n$, and $T \in \mathbb{K}^{n \times n}$ is invertible such that $T \cdot X \cdot T^{-1} \in \Omega_n$, then $f(T \cdot X \cdot T^{-1}) = T \cdot f(X) \cdot T^{-1}$.

Conditions 2 and 3 in Definition 1.10 are equivalent to a single one: $f$ respects intertwining, namely if $XS = SY$, then $f(X)S = Sf(Y)$, where $X \in \Omega_n$, $Y \in \Omega_m$, and $S \in \mathbb{K}^{n \times m}$ ($n, m \in \mathbb{N}$).

Every nc rational expression $R$ in $x_1, \ldots, x_d$ over $\mathbb{K}$ is a nc function on $\text{dom}(R)$, with $\mathcal{V} = \mathbb{K}^d$ and $\mathcal{W} = \mathbb{K}$. It is also true that for any nc rational function $\mathcal{R}$ of $x_1, \ldots, x_d$ over $\mathbb{K}$, $\mathcal{R} |_{\Omega}$ is a nc function on $\Omega$, for every nc set $\Omega \subseteq \text{dom}(\mathcal{R})$. For instance we can take $\Omega$ to be the nilpotent ball around a point $Y \in \text{dom}(\mathcal{R})$. Since we will show later that $\text{dom}(\mathcal{R})$ is itself an nc set, it follows also that $\mathcal{R}$ is a nc function on $\text{dom}(\mathcal{R})$ (cf. Corollary 4.10).

**Remark 1.11.** If $\Omega \subseteq \mathcal{V}_{nc}$ is a nc set, then by $\tilde{\Omega}$ we denote the smallest nc set that contains $\Omega$ and that is similarity invariant; $\tilde{\Omega}$ is called the similarity invariant envelope of $\Omega$. If $\Omega$ is upper admissible, then so is $\tilde{\Omega}$. Another important result that we will use later on is the following: if $\Omega \subseteq \mathcal{V}_{nc}$ is a nc set and $f : \Omega \to \mathcal{W}_{nc}$ is a nc function, then there exists a unique nc function $\tilde{f} : \tilde{\Omega} \to \mathcal{W}_{nc}$ such that $\tilde{f} |_{\Omega} = f$ (cf. [47, Appendix A]).

It is shown in [47, Theorem 5.8] that every nc function $f$ on an upper admissible nc set containing the point $Y \in (\mathbb{K}^{s \times s})^d$, admits a Taylor–Taylor power series expansion around $Y$, i.e., an expansion of the form

$$
f(X) = \sum_{\omega \in \Omega_d} (X - I_m \otimes Y)^{\otimes \omega} f_\omega,
$$

(1.12)
where for every $\omega \in G_d$, $f_\omega : (K^{s \times s})^{\omega} \to K^{s \times s}$ is a $|\omega|$–linear mapping, or alternatively a linear mapping from $(K^{s \times s})^{|\omega|}$ to $K^{s \times s}$. Notice that

$$(X - I_m \otimes Y)^{\omega} \in \left((K^{s \times s})^{\otimes |\omega|}\right)^{m \times m},$$

hence we can apply $f_\omega$ to every entry of this matrix yielding a matrix in $(K^{s \times s})^{m \times m} \cong K^{sm \times sm}$, which is where the value $f(X)$ lies; this is how we extend $f_\omega$ to a mapping from $(K^{s \times s})^{m \times m} \otimes |\omega|$ into $K^{sm \times sm}$. The equality in (1.12) holds for every $X \in Nilp(Y)$, as this ensures that the series in (1.12) is actually finite.

One important difference with the case of a scalar centre ($s = 1$) is that the coefficients $(f_\omega)_{\omega \in G_d}$ are not arbitrary multilinear mappings, as they have to satisfy certain compatibility conditions w.r.t $Y$, called the lost-abbey ($\mathcal{LA}$) conditions (see [47, equations (4.14)–(4.17)]:

$$Sf_\emptyset - f_\emptyset S = \sum_{k=1}^{d} f_{g_k}([S,Y_k]) \quad (1.13)$$

and for every $\omega = g_{i_1} \ldots g_{i_\ell} \neq \emptyset$ in $G_d$:

$$Sf_\omega(Z_1, \ldots, Z_\ell) - f_\omega(SZ_1, \ldots, Z_\ell) = \sum_{k=1}^{d} f_{g_k\omega}([S,Y_k], Z_1, \ldots, Z_\ell), \quad (1.14)$$

$$f_\omega(Z_1, \ldots, Z_\ell S) - f_\omega(Z_1, \ldots, Z_\ell)S = \sum_{k=1}^{d} f_{\omega g_k}(Z_1, \ldots, Z_\ell, [S,Y_k]) \quad (1.15)$$

and

$$f_\omega(Z_1, \ldots, Z_{j-1}, Z_jS, Z_{j+1}, \ldots, Z_\ell) - f_\omega(Z_1, \ldots, Z_j, SZ_{j+1}, \ldots, Z_\ell)$$

$$= \sum_{k=1}^{d} f_{g_{i_1} \ldots g_{i_j} g_k g_{i_{j+1}} \ldots g_{i_\ell}}(Z_1, \ldots, Z_j, [S,Y_k], Z_{j+1}, \ldots, Z_\ell), \quad (1.16)$$

for every $Z_1, \ldots, Z_\ell \in K^{s \times s}$, $1 \leq j < \ell$, and $S \in K^{s \times s}$. Conversely, in [47, Theorem 5.1.5] it is shown that given a sequence of multilinear mappings $(f_\omega)_{\omega \in G_d}$ which satisfy the $\mathcal{LA}$ conditions, the power series in (1.12) is a nc function on the nilpotent ball around $Y$.

**Remark 1.12.** If a nc function $f$ is locally bounded, then its Taylor–Taylor series around $Y$, as in (1.12), converges (absolutely and uniformly) on a neighborhood of $Y$ in an appropriate topology, see [47, Corollary 7.5].

### 2. The Linearized Lost-Abbey Conditions

One of the purposes of this section is to reformulate the $\mathcal{LA}$ conditions in the case of nc rational functions, in terms of the coefficients of a minimal nc Fornasini–Marchesini realization of the function.

**Definition 2.1.** Let $f$ be a generalized nc power series around $Y$ of the form (1.12) and let $\mathcal{R}$ be a nc Fornasini–Marchesini realization that is centred
at $Y$ and described by $(L; D, C, A, B)$. We say that the series $f$ admits the realization $\mathcal{R}$, if

$$f_\emptyset = D \text{ and } f_\omega(Z_1, \ldots, Z_\ell) = C A_{i_1}(Z_1) \cdots A_{i_{\ell-1}}(Z_{\ell-1}) B_{i_\ell}(Z_\ell),$$

for every $\emptyset \neq \omega = g_1 \cdots g_\ell \in \mathcal{G}_d$ and $Z_1, \ldots, Z_\ell \in \mathbb{K}^{s \times s}$. This is equivalent to saying that $f(X) = \mathcal{R}(X)$ for every $X \in \text{Nilp}(Y)$.

If $\mathcal{R}$ is a nc rational function that is regular at $Y = (Y_1, \ldots, Y_d) \in (\mathbb{K}^{s \times s})^d$, then Corollary 1.7 and Theorem 1.8 guarantee that $\mathcal{R}$ admits a minimal nc Fornasini–Marchesini realization $\mathcal{R}$ that is centred at $Y$ and described by $(L; D, C, A, B)$. Then, a direct computation (cf. [64, Lemma 2.12]) shows that the Taylor–Taylor power series expansion of $\mathcal{R}$ around $Y$ is

$$\mathcal{R}(X) = \sum_{\omega \in \mathcal{G}_d} (X - I_m \otimes Y)^{\odot s} R_\omega, \ X \in (\mathbb{K}^{sm \times sm})^d$$

(2.1)

where the coefficients $R_\omega : (\mathbb{K}^{s \times s})^{|\omega|} \rightarrow \mathbb{K}^{s \times s}$ are the multilinear mappings given by

$$R_\omega(Z_1, \ldots, Z_\ell) = C A_{i_1}(Z_1) \cdots A_{i_{\ell-1}}(Z_{\ell-1}) B_{i_\ell}(Z_\ell)$$

(2.2)

for $\emptyset \neq \omega \in \mathcal{G}_d$, and $R_\emptyset = D$, while they can also be seen as multilinear mappings on $(\mathbb{K}^{sm \times sm})^{\odot |\omega|}$ by

$$(Z_1 \odot \cdots \odot Z_\ell) R_\omega = (I_m \otimes C)(Z_1) A_{i_1} \cdots (Z_{\ell-1}) A_{i_{\ell-1}}(Z_\ell) B_{i_\ell},$$

if $\omega = g_1 \cdots g_\ell \neq \emptyset$ and $R_\emptyset = I_m \otimes D$, for every $m \in \mathbb{N}$ and $Z_1, \ldots, Z_\ell \in (\mathbb{K}^{sm \times sm})^d$. Therefore, it follows immediately that the series in (2.1), which comes from a nc rational function $\mathcal{R}$ (or even a nc rational expression), admits the minimal nc Fornasini–Marchesini realization $\mathcal{R}$.

As mentioned in the discussion above, the coefficients $(R_\omega)_{\omega \in \mathcal{G}_d}$ in (2.2) must satisfy the $\mathcal{L}A$ conditions and hence we get the $\mathcal{L}A$ conditions in terms of $A_1, \ldots, A_d, B_1, \ldots, B_d, C$, and $D$, in the case where $\mathcal{R}$ admits a minimal realization centred at $Y$, as formulated next:

**Lemma 2.2.** Let $f$ be a generalized nc power series around $Y = (Y_1, \ldots, Y_d) \in (\mathbb{K}^{s \times s})^d$ of the form (1.12). Suppose that $f$ admits a nc Fornasini–Marchesini realization $\mathcal{R}$ that is centred at $Y$ and described by $(L; D, C, A, B)$, and that $\mathcal{R}$ is controllable and observable. Then the series $f$ is a nc function on $\text{Nilp}(Y)$ if and only if the following equations hold

$$SD - DS = C \sum_{k=1}^{d} B_k([S, Y_k]),$$

(2.3)

$$SCB_{i_1}(Z_1) - CB_{i_1}(SZ_1) = C \left( \sum_{k=1}^{d} A_k([S, Y_k]) \right) B_{i_1}(Z_1),$$

(2.4)

$$SCA_{i_1}(Z_1) - CA_{i_1}(SZ_1) = C \left( \sum_{k=1}^{d} A_k([S, Y_k]) \right) A_{i_1}(Z_1),$$

(2.5)

$$B_{i_1}(Z_1 S) - B_{i_1}(Z_1) S = A_{i_1}(Z_1) \sum_{k=1}^{d} B_k([S, Y_k]),$$

(2.6)
\[ A_{i_1}(Z_1S)B_{i_2}(Z_2) - A_{i_1}(Z_1)B_{i_2}(SZ_2) = A_{i_1}(Z_1) \left( \sum_{k=1}^{d} A_k([S,Y_k]) \right) B_{i_2}(Z_2), \]

and

\[ A_{i_1}(Z_1S)A_{i_2}(Z_2) - A_{i_1}(Z_1)A_{i_2}(SZ_2) = A_{i_1}(Z_1) \left( \sum_{k=1}^{d} A_k([S,Y_k]) \right) A_{i_2}(Z_2), \]

for every \( S, Z_1, Z_2 \in \mathbb{K}^{\times \times s} \) and \( 1 \leq i_1, i_2 \leq d \).

We call Eqs. (2.3)–(2.8) the **linearized lost-abbey** \((\mathcal{L} - \mathcal{L}A)\) **conditions**.

**Proof.** From the assumption that \( f \) admits the realization \( \mathcal{R} \), we know that

\[ f_{\emptyset} = D \quad \text{and} \quad f_\omega(Z_1, \ldots, Z_\ell) = CA_{i_1}(Z_1) \cdots A_{i_{\ell-1}}(Z_{\ell-1})B_{i_\ell}(Z_\ell) \quad (2.9) \]

for every \( \emptyset \neq \omega = g_{i_1} \ldots g_{i_\ell} \in G_d \) and \( Z_1, \ldots, Z_\ell \in \mathbb{K}^{\times \times s} \). Therefore, the series \( f \) is a nc function in \( Nilp(Y) \) if and only if the coefficients \( (f_\omega)_{\omega \in G_d} \) in (2.9) satisfy equations (1.13)–(1.16). It is only left to show that this is equivalent to the fact that equations (2.3)–(2.8) hold.

This part of the proof is mainly technical, so we show it only for one of the equations, while stressing where the minimality of \( \mathcal{R} \) is coming into play. While skipping the computations, we mention briefly how to obtain all the other equations:

- (2.3) is obtained from (1.13);
- (2.4) is obtained from (1.14), by taking \( \ell = |\omega| = 1 \);
- (2.5) is obtained from (1.14), by taking \( \ell = |\omega| > 1 \) and using the controllability of \((A, B)\);
- (2.6) is obtained from (1.15), by taking any \( \ell \geq 1 \) and using the observability of \((C, A)\);
- (2.7) is obtained from (1.16), by taking \( j = \ell - 1 \) and using the observability of \((C, A)\);
- (2.8) is obtained from (1.16), by taking \( j < \ell - 1 \) and using both the observability of \((C, A)\) and the controllability of \((A, B)\).

We now show the last equivalence in this list: as

\[
\begin{align*}
  f_\omega(Z_1, \ldots, Z_{j-1}, Z_j S, Z_{j+1}, \ldots, Z_\ell) &= \sum_{k=1}^{d} A_k([S,Y_k]) \left( \sum_{k=1}^{d} A_k([S,Y_k]) \right) B_{i_\ell}(Z_\ell), \\
  f_\omega(Z_1, \ldots, Z_{j}, S Z_{j+1}, \ldots, Z_\ell) &= \sum_{k=1}^{d} A_k([S,Y_k]) \left( \sum_{k=1}^{d} A_k([S,Y_k]) \right) B_{i_\ell}(Z_\ell), \\
\end{align*}
\]

and

\[
\begin{align*}
  f_{g_{i_1} \ldots g_{i_j} g_{i_{j+1}} \ldots g_{i_\ell}}(Z_1, \ldots, Z_j, [S,Y_k], Z_{j+1}, \ldots, Z_\ell) &= \sum_{k=1}^{d} A_k([S,Y_k]) \left( \sum_{k=1}^{d} A_k([S,Y_k]) \right) B_{i_\ell}(Z_\ell), \\
\end{align*}
\]

for every \( 1 < j < \ell - 1 \), equation (1.16) holds if and only if
\[ CA_{i_1}(Z_1) \cdots A_{i_{j-1}}(Z_{j-1})A_{i_j}(Z_j)S \cdot A_{i_{j+1}}(Z_{j+1}) \cdots A_{i_{\ell-1}}(Z_{\ell-1})B_{i_\ell}(Z_{\ell}) \]

\[ -CA_{i_1}(Z_1) \cdots A_{i_j}(Z_j)A_{i_{j+1}}(SZ_{j+1}) \cdots A_{i_{\ell-1}}(Z_{\ell-1})B_{i_\ell}(Z_{\ell}) \]

\[ = \sum_{k=1}^d CA_{i_1}(Z_1) \cdots A_{i_j}(Z_j)A_k([S,Y_k])A_{i_{j+1}}(Z_{j+1}) \cdots A_{i_{\ell-1}}(Z_{\ell-1})B_{i_\ell}(Z_{\ell}), \]

which is equivalent to

\[ CA_{i_1}(Z_1) \cdots A_{i_{j-1}}(Z_{j-1}) \]

\[ \left( A_{i_j}(Z_j)S \cdot A_{i_{j+1}}(Z_{j+1}) - A_{i_j}(Z_j)A_{i_{j+1}}(SZ_{j+1}) \right) \]

\[ - \sum_{k=1}^d A_{i_j}(Z_j)A_k([S,Y_k])A_{i_{j+1}}(Z_{j+1}) \]

\[ A_{i_{j+2}}(Z_{j+2}) \cdots A_{i_{\ell-1}}(Z_{\ell-1})B_{i_\ell}(Z_{\ell}) = 0. \]

Due to the controllability of \((A, B)\) and the observability of \((C, A)\), the last equation is equivalent to the vanishing of the middle piece, i.e., to

\[ A_{i_j}(Z_j)S \cdot A_{i_{j+1}}(Z_{j+1}) - A_{i_j}(Z_j)A_{i_{j+1}}(SZ_{j+1}) \]

\[ = A_{i_j}(Z_j) \left( \sum_{k=1}^d A_k([S,Y_k]) \right) A_{i_{j+1}}(Z_{j+1}), \]

which is exactly equation (2.8).

\[ \square \]

**Remark 2.3.** If \(\mathcal{H}\) is an infinite dimensional Hilbert space, \(A_k : \mathbb{K}^{s \times s} \rightarrow \mathcal{L}(\mathcal{H})\) and \(B_k : \mathbb{K}^{s \times s} \rightarrow \mathcal{L}(\mathbb{K}^s, \mathcal{H})\) for every \(1 \leq k \leq d\), \(C \in \mathcal{L}(\mathcal{H}, \mathbb{K}^s)\) and \(D \in \mathbb{K}^{s \times s}\), such that conditions (2.3)–(2.8) hold, then the coefficients \((f_\omega)_{\omega \in \mathcal{G}_d}\) which are given in (2.9) satisfy the \(L\mathcal{A}\) conditions, hence the power series in (1.12) defines a nc function on \(\text{Nilp}(\mathcal{Y})\) and it admits the realization described by \((D, C, A, B)\). This follows from using the same arguments as in the proof of Lemma 2.2, where (formally) \(\mathcal{H}\) replaces the space \(\mathbb{K}^L\).

**Remark 2.4.** Notice that even if we remove the assumptions on the realization being controllable and observable, we get that satisfying the \(L - L\mathcal{A}\) conditions implies that the series is a nc function, however these assumptions on the realization are required for the other direction.

**Remark 2.5.** Recall that for every \(n, m \in \mathbb{N}\) and \(1 \leq k \leq d\), the linear mappings \(A_k : \mathbb{K}^{s \times s} \rightarrow \mathbb{K}^{L \times L}\) and \(B_k : \mathbb{K}^{s \times s} \rightarrow \mathbb{K}^{L \times s}\) are naturally extended to mappings \(A_k : \mathbb{K}^{sn \times sm} \rightarrow \mathbb{K}^{Ln \times Lm}\) and \(B_k : \mathbb{K}^{sn \times sm} \rightarrow \mathbb{K}^{Ln \times sm}\), just by acting on the block matrices of size \(s \times s\). It takes some straightforward computations (which are omitted here) to show that if the \(L - L\mathcal{A}\) conditions hold, then they can be extended to \(sn \times sm\) matrices, i.e., we have

\[ S(I_m \otimes D) - (I_n \otimes D)S = (I_n \otimes C) \sum_{k=1}^d (S(I_m \otimes Y_k) - (I_n \otimes Y_k)S) B_k, \]

\[ S(I_m \otimes C)(Z_i)B_{i_1} - (I_n \otimes C)(SZ_1)B_{i_1} \]

\[ \text{for every } 1 \leq k \leq d. \]
\begin{equation}
(I_n \otimes C) \left( \sum_{k=1}^{d} (S(I_m \otimes Y_k) - (I_n \otimes Y_k) S) A_k \right) (Z_1) B_{i_1}, \tag{2.11}
\end{equation}

\begin{equation}
S(I_m \otimes C)(Z_1) A_{i_1} - (I_n \otimes C)(SZ_1) A_{i_1}
= (I_n \otimes C) \left( \sum_{k=1}^{d} (S(I_m \otimes Y_k) - (I_n \otimes Y_k) S) A_k \right) (Z_1) A_{i_1}, \tag{2.12}
\end{equation}

\begin{equation}
(Z_2S)B_{i_2} - (Z_2)B_{i_2} S = (Z_2)A_{i_2} \sum_{k=1}^{d} (S(I_m \otimes Y_k) - (I_n \otimes Y_k) S) B_k,
\tag{2.13}
\end{equation}

\begin{equation}
(Z_2S)A_{i_2}(Z_1)B_{i_1} - (Z_2)A_{i_2}(SZ_1)B_{i_1}
= (Z_2)A_{i_2} \left( \sum_{k=1}^{d} (S(I_m \otimes Y_k) - (I_n \otimes Y_k) S) A_k \right) (Z_1)B_{i_1}, \tag{2.14}
\end{equation}

\begin{equation}
(Z_2S)A_{i_2}(Z_1)A_{i_1} - (Z_2)A_{i_2}(SZ_1)A_{i_1}
= (Z_2)A_{i_2} \left( \sum_{k=1}^{d} (S(I_m \otimes Y_k) - (I_n \otimes Y_k) S) A_k \right) (Z_1)A_{i_1}, \tag{2.15}
\end{equation}

for every $S \in \mathbb{K}^{s n \times s m}$, $Z_1 \in \mathbb{K}^{s m \times s m}$, $Z_2 \in \mathbb{K}^{s n \times s n}$, and $1 \leq i_1, i_2 \leq d$.

As a corollary of Lemma 2.2 and Remark 2.4, if equations (2.3)–(2.8) hold, then the coefficients in the power series expansion of the realization around $\bar{Y}$ must satisfy the $\mathcal{L}A$ conditions, hence the nc Fornasini–Marchesini realization $\mathcal{R}$ defines a nc function on $\text{Nilp}(\bar{Y})$.

Moreover, we now show that the realization is a nc function on a larger set, that is $\Omega(\mathcal{R})$, the invertibility set of the realization. This will be the first step of our approach in which we start from an arbitrary nc Fornasini–Marchesini realization, that is controllable and observable, for which its coefficients satisfy the $\mathcal{L} - \mathcal{L}A$ conditions (cf. equations (2.3)–(2.8)) and eventually find a nc rational function which this realization admits.

**Theorem 2.6.** Let $\mathcal{R}$ be a nc Fornasini–Marchesini realization centred at $\bar{Y}$ that is described by $(L; D, C, A, B)$. If the coefficients of $\mathcal{R}$ satisfy the $\mathcal{L} - \mathcal{L}A$ conditions, then $\Omega(\mathcal{R})$ is an upper admissible nc subset of $(\mathbb{K}^d)^{nc}$ and the function $\mathcal{R} : \Omega(\mathcal{R}) \to \mathbb{K}^{nc}$ that is given by

$$
\mathcal{R}(X) = I_m \otimes D + (I_m \otimes C) \Lambda_{A, Y}(X)^{-1} \sum_{k=1}^{d} (X_k - I_m \otimes Y_k) B_k, \tag{2.16}
$$

\forall X \in \Omega_{sm}(\mathcal{R})$, $m \in \mathbb{N}$, is a nc function (that is defined only in levels which are multiples of $s$, i.e., if $s \mid n$ then the domain of $\mathcal{R}$ at the level of $d$–tuples of matrices in $\mathbb{K}^{n \times n}$ is empty), with $\bar{Y} \in \Omega_s(\mathcal{R})$.

**Proof.** $\Omega(\mathcal{R})$ is an upper admissible nc subset of $(\mathbb{K}^d)^{nc}$: If $m, \tilde{m} \in \mathbb{N}$, $X \in \Omega_{sm}$
(\mathcal{R}), \bar{X} \in \Omega_{s\bar{m}}(\mathcal{R}), \text{ and } Z \in \left(\mathbb{K}^{sm \times s\bar{m}}\right)^d, \text{ then}

$$\Lambda_{A,Y}(\begin{bmatrix} X & Z \\ 0 & \bar{X} \end{bmatrix}) = \begin{bmatrix} \Lambda_{A,Y}(X) - \sum_{k=1}^{d} (Z_k)A_k \\ \Lambda_{A,Y}(\bar{X}) \end{bmatrix}$$

is invertible, i.e., \[ \begin{bmatrix} X & Z \\ 0 & \bar{X} \end{bmatrix} \in \Omega_{s(m+\bar{m})}(\mathcal{R}). \]

\(\mathcal{R}\) is a nc function on \(\Omega(\mathcal{R})\): \(\mathcal{R}\) is clearly graded, so it is left to show the crucial part which is that \(\mathcal{R}\) respects intertwining. Let \(m, \bar{m} \in \mathbb{N}, X \in \Omega_{sm}(\mathcal{R}), \bar{X} \in \Omega_{s\bar{m}}(\mathcal{R}), \text{ and } T \in \mathbb{K}^{sm \times s\bar{m}}\) such that \(X \cdot T = T \cdot \bar{X}\), i.e., that \(X_kT = T\bar{X}_k\) for all \(1 \leq k \leq d\). For simplifications let us define the matrices

\[
M_B := \sum_{k=1}^{d} \left( T(I_m \otimes Y_k) - (I_m \otimes Y_k)T \right) B_k,
\]

\[
M_A := \sum_{k=1}^{d} \left( T(I_m \otimes Y_k) - (I_m \otimes Y_k)T \right) A_k
\]

and recall that as equations (2.3)–(2.8) hold, we know that equations (2.10)–(2.15) hold as well (cf. Remark 2.5). From (2.13) we get

\[
\sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_kT = \sum_{k=1}^{d} \left( (X_k - I_m \otimes Y_k)T \right) B_k
\]

\[- (X_k - I_m \otimes Y_k)A_kM_B \]

\[= \sum_{k=1}^{d} \left( T\bar{X}_k - T(I_m \otimes Y_k) + T(I_m \otimes Y_k) - (I_m \otimes Y_k)T \right) B_k \]

\[- \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_kM_B \]

\[= \sum_{k=1}^{d} \left( T(\bar{X}_k - I_m \otimes Y_k) \right) B_k + \Lambda_{A,Y}(X)M_B \]

and hence, using (2.10)

\[\mathcal{R}(X)T = (I_m \otimes D)T + (I_m \otimes C)M_B + (I_m \otimes C)\Lambda_{A,Y}(X)^{-1} \times \sum_{k=1}^{d} \left( T(\bar{X}_k - I_m \otimes Y_k) \right) B_k \]

\[= T(I_m \otimes D) + (I_m \otimes C)\Lambda_{A,Y}(X)^{-1} \sum_{k=1}^{d} \left( T(\bar{X}_k - I_m \otimes Y_k) \right) B_k. \]

Next, from (2.14) it follows that
\[
\begin{align*}
&\left(\sum_{k=1}^{d} ((X_k - I_m \otimes Y_k)T) A_k\right) \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k \\
&\quad - \left(\sum_{k=1}^{d} (X_k - I_m \otimes Y_k) A_k\right) M_A \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k \\
&\quad = \left(\sum_{k=1}^{d} (X_k - I_m \otimes Y_k) A_k\right) \sum_{k=1}^{d} T(\tilde{X}_k - I_m \otimes Y_k) B_k \\
&\quad = -\Lambda_{A,Y}(X) \sum_{k=1}^{d} T(\tilde{X}_k - I_m \otimes Y_k) B_k + \sum_{k=1}^{d} \left(T(\tilde{X}_k - I_m \otimes Y_k)\right) B_k,
\end{align*}
\]

So
\[
\sum_{k=1}^{d} (T(\tilde{X}_k - I_m \otimes Y_k)) B_k
\]
\[
\quad = \left(\sum_{k=1}^{d} ((X_k - I_m \otimes Y_k)T) A_k\right) \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k \\
\quad - M_A \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k + \Lambda_{A,Y}(X) M_A \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k \\
\quad + \Lambda_{A,Y}(X) \sum_{k=1}^{d} (T(\tilde{X}_k - I_m \otimes Y_k)) B_k \\
\quad = \Lambda_{A,Y}(X) \sum_{k=1}^{d} (T(\tilde{X}_k - I_m \otimes Y_k)) B_k
\]

and hence, using (2.11),
\[
(I_m \otimes C) \Lambda_{A,Y}(X)^{-1} \sum_{k=1}^{d} (T(\tilde{X}_k - I_m \otimes Y_k)) B_k
\]
\[
= (I_m \otimes C) \left[\sum_{k=1}^{d} (T(\tilde{X}_k - I_m \otimes Y_k)) B_k + M_A \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k\right] \\
\quad + (I_m \otimes C) \Lambda_{A,Y}(X)^{-1} \left(\sum_{k=1}^{d} ((X_k - I_m \otimes Y_k)T) A_k - M_A\right) \\
\quad \times \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k \\
\quad = T(I_m \otimes C) \sum_{k=1}^{d} (\tilde{X}_k - I_m \otimes Y_k) B_k
\]
\[(I_m \otimes C)\Lambda_{A,Y}(X)^{-1} \left( \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k \right) \]

\[\times \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) B_k.\]

From (2.15) we know that

\[\sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k\]

\[= \left( \sum_{k=1}^{d} (X_k - I_m \otimes Y_k) A_k \right) \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k\]

\[+ \Lambda_{A,Y}(X) \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k\]

\[= \left( \sum_{k=1}^{d} ((X_k - I_m \otimes Y_k)T) A_k \right) \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k\]

\[\left. \left. - \left( \sum_{k=1}^{d} (X_k - I_m \otimes Y_k) A_k \right) \right) M_A \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k\right\}\]

\[+ \Lambda_{A,Y}(X) \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k\]

\[= \left( \sum_{k=1}^{d} ((X_k - I_m \otimes Y_k)T) A_k \right) \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k\]

\[+ \Lambda_{A,Y}(X) M_A \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k\]

\[- M_A \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k + \Lambda_{A,Y}(X) \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k\]

\[= \left( \sum_{k=1}^{d} ((X_k - I_m \otimes Y_k)T) A_k - M_A \right) \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k\]

\[+ \Lambda_{A,Y}(X) \left[ M_A \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k + \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k \right]\]

therefore, by (2.12),

\[(I_m \otimes C)\Lambda_{A,Y}(X)^{-1} \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k\]

\[= (I_m \otimes C)\Lambda_{A,Y}(X)^{-1} \left( \sum_{k=1}^{d} (T(\tilde{X}_k - \widetilde{I_m} \otimes Y_k)) A_k \right) \sum_{k=1}^{d} (\tilde{X}_k - \widetilde{I_m} \otimes Y_k) A_k\]
\[ + (I_m \otimes C) \left[ M_A \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) A_k + \sum_{k=1}^{d} \left( T(\tilde{X}_k - I_\tilde{m} \otimes Y_k) \right) A_k \right] \]

\[ = (I_m \otimes C) \Lambda_{A,Y}(X)^{-1} \left( \sum_{k=1}^{d} \left( T(\tilde{X}_k - I_\tilde{m} \otimes Y_k) \right) A_k \right) \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) A_k \]

\[ + T(I_\tilde{m} \otimes C) \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) A_k. \]

Then,

\[ (I_m \otimes C) \Lambda_{A,Y}(X)^{-1} \left( \sum_{k=1}^{d} \left( T(\tilde{X}_k - I_\tilde{m} \otimes Y_k) \right) A_k \right) \Lambda_{A,Y}(\tilde{X}), \]

which implies that

\[ (I_m \otimes C) \Lambda_{A,Y}(X)^{-1} \sum_{k=1}^{d} \left( T(\tilde{X}_k - I_\tilde{m} \otimes Y_k) \right) A_k \]

\[ = T(I_\tilde{m} \otimes C) \left( \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) A_k \right) \Lambda_{A,Y}(\tilde{X})^{-1}. \]

Therefore,

\[ T(I_\tilde{m} \otimes D) + T(I_\tilde{m} \otimes C) \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) B_k \]

\[ + T(I_\tilde{m} \otimes C) \left( \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) A_k \right) \Lambda_{A,Y}(\tilde{X})^{-1} \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) B_k \]

\[ = T(I_\tilde{m} \otimes D) + T(I_\tilde{m} \otimes C) \Lambda_{A,Y}(\tilde{X})^{-1} \sum_{k=1}^{d} (\tilde{X}_k - I_\tilde{m} \otimes Y_k) B_k, \]

i.e., \( R(X) T = TR(\tilde{X}) \).

**Remark 2.7.** The idea of that part of the proof, where we showed that \( R \) respects intertwining, is very similar to the proof of [47, Lemma 5.12], with the only difference that instead of a realization (centred at \( Y \)) they had a power series (around \( Y \)).

### 3. NC Difference-Differential Calculus

In this section we prove that if the coefficients of a nc Fornasini–Marchesini realization \( R \), that is controllable and observable, satisfy the \( \mathcal{L} - \mathcal{L}A \) conditions, then the set \( \Omega(R) \) is similarity invariant (cf. Theorem 3.5). It turns out to be an important ingredient in obtaining one of our main results (cf. Theorem 4.7).
In the proof of Theorem 3.5 below we use some techniques from the general theory of nc functions and their difference-differential calculus; we hereby recall some facts.

Let \( A_1, \ldots, A_d, B_1, \ldots, B_d, C, \) and \( D \) satisfy the \( \mathcal{L} - \mathcal{LA} \) conditions. For every \( 1 \leq j \leq d \), the \( j \)-th right partial nc difference-differential operator \( \Delta_j \) is defined using the nc difference-differential operator \( \Delta := \Delta_R \), by

\[
\Delta_j f(X^1, X^2)(Z) = \Delta f(X^1, X^2)(Z^{(j)}),
\]

for every nc function \( f, n_1, n_2 \in \mathbb{N}, X^1 \in (\mathbb{K}^{n_1 \times n_1})^d \) and \( X^2 \in (\mathbb{K}^{n_2 \times n_2})^d \) in the domain of \( f \), and \( Z \in \mathbb{K}^{n_1 \times n_2} \), where \( Z^{(j)} := (0, \ldots, 0, Z, 0, \ldots, 0) \in (\mathbb{K}^{n_1 \times n_2})^d \) is the \( d \)-tuple which consists of all zero matrices except for the matrix \( Z \) at the \( j \)-th position (see [47, Sects. 2.2–2.6] for the precise definitions and properties of \( \Delta \) and \( \Delta_j \)). To continue, we need the following technical lemma.

**Lemma 3.1.** Let \( \mathcal{R} \) be a nc Fornasini–Marchesini realization centred at \( Y \in (\mathbb{K}^{s \times s})^d \) that is described by \( (L; D, C, A, B) \) and suppose its coefficients satisfy the \( \mathcal{L} - \mathcal{LA} \) conditions. Let \( \mathcal{U} = (\mathbb{K}^{s \times s})^d \) and define

\[
F_1 : \mathcal{U}_{nc} \to (\mathbb{K}^{s \times s})_{nc}, \text{ by } F_1(X) = I_m \otimes D, \forall X \in (\mathbb{K}^{s m \times s m})^d \\
F_2 : \mathcal{U}_{nc} \to (\mathbb{K}^{s \times L})_{nc}, \text{ by } F_2(X) = I_m \otimes C, \forall X \in (\mathbb{K}^{s m \times s m})^d \\
F_3 : \Omega(\mathcal{R}) \to (\mathbb{K}^{L \times L})_{nc}, \text{ by } F_3(X) = \Lambda_{A, Y}(X)^{-1}, \forall X \in \Omega_{sm}(\mathcal{R})
\]

and

\[
F_4 : \mathcal{U}_{nc} \to (\mathbb{K}^{L \times s})_{nc}, \text{ by } F_4(X) = \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k, \forall X \in (\mathbb{K}^{s m \times s m})^d.
\]

Then \( F_1, F_2, \) and \( F_4 \) are nc functions on \( \mathcal{U}_{nc} \), and \( F_3 \) is a nc function on \( \Omega(\mathcal{R}) \subseteq \mathcal{U}_{nc} \), where the nc space is considered as a module over \( \mathbb{K} \) and not over \( \mathbb{K}^{s \times s} \). Moreover, for every \( 1 \leq j \leq d \), we have \( \Delta_j F_1 = 0, \Delta_j F_2 = 0, \Delta_j F_3 = 0, \Delta_j F_4 = 0 \) for every \( m_1, m_2 \in \mathbb{N}, X^1 \in \Omega_{sm_1}(\mathcal{R}), X^2 \in \Omega_{sm_2}(\mathcal{R}), \tilde{X}^1 \in \mathcal{U}^{m_1 \times m_1}, \tilde{X}^2 \in \mathcal{U}^{m_2 \times m_2}, \) and \( Z \in \mathbb{K}^{sm_1 \times sm_2} \).

It is easily seen that \( \mathcal{R} = F_1 + F_2 F_3 F_4 \), so the calculations from Lemma 3.1 will be used later (cf. Lemma 3.4) to calculate the (higher order) nc derivatives of \( \mathcal{R} \). Notice that while \( \mathcal{R} \) is a nc function on a nc subset of \((\mathbb{K}^d)_{nc}\) to \( \mathcal{U}_{nc} \) and is only defined on levels of the form \( n = sm \), the functions \( F_1, \ldots, F_4 \) are nc functions on nc subsets of \((\mathbb{K}^{s \times s})^d)_{nc}\).

**Proof.** In Theorem 2.6 we proved that \( \Omega(\mathcal{R}) \) is an upper admissible nc subset of \((\mathcal{U}^d)_{nc}\), while it is easily seen that \( F_1, F_2, F_3, \) and \( F_4 \) are graded and respect direct sums, it is only left to show that they respect similarities.

Let \( \mathbb{X} \in \mathcal{U}_m \), i.e., \( \mathbb{X} = (X_1, \ldots, X_d) \in (\mathbb{K}^{s m \times s m})^d \) and let \( S \in \mathbb{K}^{m \times m} \) be invertible. We recall the actions of \( S \) (on the left) and of \( S^{-1} \) (on the right) on a tuple \( \mathbb{X} \), these are given by

\[
S \cdot \mathbb{X} \cdot S^{-1} = ((S \otimes I_s)X_1(S^{-1} \otimes I_s), \ldots, (S \otimes I_s)X_1(S^{-1} \otimes I_s)).
\]
Therefore
\[ S \cdot F_1(X) \cdot S^{-1} = (S \otimes I_s)(I_m \otimes D)(S^{-1} \otimes I_s) = I_m \otimes D = F_1(S \cdot X \cdot S^{-1}), \]
while from linearity of \( B_k \) we get
\[
S \cdot F_4(X) \cdot S^{-1} = (S \otimes I_L) \left( \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)B_k \right) (S^{-1} \otimes I_s)
\]
\[
g = \sum_{k=1}^{d} ((S \otimes I_s)(X_k - I_m \otimes Y_k)(S^{-1} \otimes I_s))B_k
\]
\[
= \sum_{k=1}^{d} (S \cdot X_k \cdot S^{-1} - I_m \otimes Y_k)B_k = F_4(S \cdot X \cdot S^{-1})
\]
and similarly for \( F_2 \) and \( F_3 \), with the only difference that for \( F_2 \) we begin with \( X \in \Omega(R) \). Moreover, let \( X^1 \in \Omega_{sm_1}(R), X^2 \in \Omega_{sm_2}(R) \), and \( Z \in U_{m1 \times m2} \), thus
\[
F_3 \left( \begin{bmatrix} X^1 & Z^{(j)} \\ 0 & X^2 \end{bmatrix} \right)
\]
\[
= \left( I_{L(m_1 + m_2)} - \sum_{k=1}^{d} \begin{bmatrix} X_k - I_m \otimes Y_k & 0 \\ 0 & X_k^2 - I_m \otimes Y_k \end{bmatrix} \right)^{-1}A_k
\]
\[
= I_{Lm_1} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k
\]
\[
= \begin{bmatrix} I_{Lm_1} & - (Z)A_j \end{bmatrix}^{-1}
\]
\[
\Delta_j F_3(X^1, X^2)(Z) = F_3(X^1)(Z)A_j F_3(X^2)
\]
and similar calculations hold for \( F_4 \).

\[ \square \]

**Definition 3.2.** Let \( \mathcal{W}, \mathcal{X}_0, \ldots, \mathcal{X}_k, \mathcal{Y}_1, \ldots, \mathcal{Y}_k, \mathcal{Z} \) be vector spaces over \( \mathbb{K} \) with a product operation \( \mathcal{X}_0 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{Y}_k \otimes \mathcal{X}_k \rightarrow \mathcal{Z} \ k \in \mathbb{N} \). For every \( 0 \leq j \leq k \), let \( \Gamma_j \) be an upper admissible nc subset of \((\mathbb{C}^d \otimes \mathcal{W})_{nc} \), let \( f_j : \Gamma_j \rightarrow (\mathcal{X}_j)_{nc} \) be a nc function, and for every \( 1 \leq j \leq k \), let \( h_j : \mathcal{W} \rightarrow \mathcal{Y}_j \) be a linear mapping. If \( m_0, \ldots, m_k \in \mathbb{N} \) and \( X^0 \in (\Gamma_0)^{m_0}, \ldots, X^k \in (\Gamma_k)^{m_k} \), we define
\[
(f_0 \otimes \cdots \otimes f_k)_{h_1, \ldots, h_k} : \mathcal{W}^{m_0 \times m_1} \otimes \cdots \otimes \mathcal{W}^{m_k-1 \times m_k} \rightarrow \mathcal{Z}^{m_0 \times m_k}
\]
by
\[
(f_0 \otimes \cdots \otimes f_k)_{h_1, \ldots, h_k}(X^0, \ldots, X^k)(Z^1, \ldots, Z^k)
\]
\[
:= f_0(X^0)h_1(Z^1) \cdots f_{k-1}(X^{k-1})h_k(Z^k)f_k(X^k)
\]
for every \( Z^1 \in \mathcal{W}^{m_0 \times m_1}, \ldots, Z^k \in \mathcal{W}^{m_k-1 \times m_k} \).

The following useful lemma is a generalization of an idea from [47, pp. 41–54], while over there only the special case—in which essentially \( h_1, \ldots, h_k \) are all equal to the identity mapping—is treated. For the definition of nc functions of order \( k \), that is the class \( T^k \), their derivatives and more properties, see [47, Sect. 3.1].
Lemma 3.3. Let $\mathcal{W}, X_0, \ldots, X_k, Y_1, \ldots, Y_k, Z$ be vector spaces over $\mathbb{K}$, let $k \in \mathbb{N}$, and let $f_0, \ldots, f_k, h_1, \ldots, h_k$ and $\Gamma_0, \ldots, \Gamma_k$ be as in Definition 3.2. Then

$$f_0 \otimes \cdots \otimes f_k)_{h_1, \ldots, h_k} \in T^k(\Gamma_0, \ldots, \Gamma_k; Z_{nc}, W_{nc}, \ldots, W_{nc})$$

and

$$(f_0 \otimes \cdots \otimes f_k)_{h_1, \ldots, h_k} = f_0(X^0)h_1(Z^1) \cdots h_k(Z^{k-1})f_k(X^{k'})(Z)$$

for every $1 \leq j \leq d$, $X^0 \in (\Gamma_0)_{m_0}, \ldots, X^{k-1} \in (\Gamma_{k-1})_{m_{k-1}}, X^{k'} \in (\Gamma_k)_{m_k'}, X^{k''} \in (\Gamma_k)_{m''_k}, Z^1 \in W_{m_0 \times m_1}, \ldots, Z^{k-1} \in W_{m_{k-2} \times m_{k-1}}, Z^{k'} \in W_{m_{k-1} \times m'_k}$, and $Z \in W_{m'_k \times m''_k}$.

Notice that the notation introduced in the lemma can be naturally extended to the case of tensor products of higher order nc functions, so that the equality proven in (3.2) can be written as

$$\Delta_j(f_0 \otimes \cdots \otimes f_k)_{h_1, \ldots, h_k} = (f_0 \otimes \cdots \otimes \Delta_j f_k)_{h_1, \ldots, h_k},$$

similarly to [47, equations (3.38)–(3.39)]. We leave the notational details for the reader.

Proof. Let $g := (f_0 \otimes \cdots \otimes f_k)_{h_1, \ldots, h_k}$. It is not hard to check that $g \in T^k(\Gamma_0, \ldots, \Gamma_k; Z_{nc}, W_{nc}, \ldots, W_{nc})$, so we will only prove (3.2). Let $Z^{k''} \in W_{m_{k-1} \times m'_k}$. As (3.1) holds, one can apply [47, equation (3.19)] to obtain that

$$g(X^0, \ldots, X^{k-1}, \begin{bmatrix} X^{k'} & Z^{(j)} \\ 0 & X^{k''} \end{bmatrix})(Z^1, \ldots, Z^{k-1}, [Z^{k'} Z^{k''}]) = \text{row} [I_{11} I_{12}],$$

where $Z^{(j)} = e_j \otimes Z$, $I_{11} = g(X^0, \ldots, X^{k-1}, X^{k'}) (Z^1, \ldots, Z^{k-1}, Z^{k'})$, and $I_{12} = g(X^0, \ldots, X^{k-1}, X^{k''}) (Z^1, \ldots, Z^{k-1}, Z^{k''}) + \Delta_j g(X^0, \ldots, X^{k-1}, X^{k'}, X^{k''}) (Z^1, \ldots, Z^{k-1}, Z^{k'}, Z)$.

On the other hand, we know that

$$g(X^0, \ldots, X^{k-1}, \begin{bmatrix} X^{k'} & Z^{(j)} \\ 0 & X^{k''} \end{bmatrix})(Z^1, \ldots, Z^{k-1}, [Z^{k'} Z^{k''}])$$

$$= f_0(X^0)h_1(Z^1) \cdots f_{k-1}(X^{k-1})h_k(Z^{k''})f_k(\begin{bmatrix} X^{k'} & Z^{(j)} \\ 0 & X^{k''} \end{bmatrix})$$

$$= f_0(X^0)h_1(Z^1) \cdots f_{k-1}(X^{k-1})$$

$$\begin{bmatrix} h_k(Z^{k'}) & h_k(Z^{k''}) \\ f_k(\begin{bmatrix} X^{k'} & Z^{(j)} \\ 0 & X^{k''} \end{bmatrix}) & f_k(\begin{bmatrix} X^{k''} \end{bmatrix}) \end{bmatrix}$$

$$= \text{row} [J_{11} J_{12}],$$

where it is easily seen that $J_{11} = I_{11}$ and $J_{12} = f_0(X^0)h_1(Z^1) \cdots f_{k-1}(X^{k-1})$.
\[
\left( h_k(Z^{k'}) \Delta_j f_k(X^{k'}, X^{k''}) + h_k(Z^{k''}) f_k(X^{k''}) \right) \\
= g(X^0, \ldots, X^{k-1}, X^{k''})(Z^1, \ldots, Z^{k'}, Z^{k''}) \\
+ f_0(X^0) h_1(Z) \ldots h_{k-1}(Z^{k-1}) f_{k-1}(X^{k-1}) h_k(Z^{k'}) \Delta_j f_k(X^{k'}, X^{k''})(Z).
\]

As \( I_{12} = J_{12} \), we get equation (3.2).

As \( \mathcal{R} : \Omega(\mathcal{R}) \to \mathbb{K}_{nc} \) is a nc function and \( \Omega(\mathcal{R}) \) is an upper admissible nc set, it makes sense to consider the difference-differential operators on \( \mathcal{R} \), these are \( \Delta_j \mathcal{R} \) for \( 1 \leq j \leq d \) and also higher order derivatives \( \Delta^\omega \mathcal{R} = \Delta_{j_k} \Delta_{j_{k-1}} \ldots \Delta_{j_1} \mathcal{R} \), where \( \omega = g_{i_1} \ldots g_{i_t} \in \mathcal{G}_d \). Recall that

\[
\Delta_j(fg)(X^1, X^2)(Z) = f(X^1)\Delta_j g(X^1, X^2)(Z) + \Delta_j f(X^1, X^2)(Z)g(X^2),
\]

whenever \( X^1, X^2 \), and \( Z \) are of appropriate sizes and \( f, g \) are nc functions; see [47, pp. 23] for the proof. Using Lemmas 3.1 and 3.3, we get the following important lemma.

**Lemma 3.4.** Let \( \mathcal{R} \) be a nc Fornasini–Marchesini realization centred at \( Y \in \mathbb{K}^{s \times s} \) that is described by \( \{L; D, C, A, B\} \) and suppose its coefficients satisfy the \( \mathcal{L} - \mathcal{L}A \) conditions. For every \( m \in \mathbb{N}, X \in \Omega_{sm}(\mathcal{R}) \), \( k \in \mathbb{N}, Z^1, \ldots, Z^k \in \mathbb{K}^{sm \times sm} \), and \( 1 \leq j_1, \ldots, j_k \leq d \), we have

\[
\Delta_{j_k} \ldots \Delta_{j_1} \mathcal{R}(X, I_m \otimes Y, \ldots, I_m \otimes Y, I_m \otimes Y)(Z^1, \ldots, Z^k) \\
= (I_m \otimes C)F_3(X)(Z^1)A_{j_k} \cdots (Z^{k-1})A_{j_{k-1}}(Z^k)B_{j_k}
\]

(3.3) and

\[
\Delta_{j_k} \ldots \Delta_{j_1} F_3(I_m \otimes Y, \ldots, I_m \otimes Y, X)(Z^1, \ldots, Z^k) \\
= (Z^1)A_{j_k} \cdots (Z^k)A_{j_k} F_3(X),
\]

(3.4)

where \( F_3 \) is given in Lemma 3.1.

**Proof.** Let \( F_1, F_2, F_3, \) and \( F_4 \) be as defined in Lemma 3.1, so

\[
\mathcal{R}(X) = F_1(X) + F_2(X)F_3(X)F_4(X), \forall X \in \Omega(\mathcal{R}).
\]

For every \( X^1 \in \Omega_{sm_1}(\mathcal{R}), X^2 \in \Omega_{sm_2}(\mathcal{R}), Z^1 \in \mathbb{K}^{sm_1 \times sm_2} \), and \( 1 \leq j_1 \leq d, \)

\[
\Delta_{j_1} \mathcal{R}(X^1, X^2)(Z^1) = \Delta_{j_1} F_1(X^1, X^2)(Z^1) + \Delta_{j_1} F_2 F_3 F_4(X^1, X^2)(Z^1) \\
= F_2(X^1) \Delta_{j_1} F_3 F_4(X^1, X^2)(Z^1) + \Delta_{j_1} F_2(X^1, X^2)(Z^1)F_3(X^2)F_4(X^2) \\
= F_2(X^1) [F_3(X^1) \Delta_{j_1} F_4(X^1, X^2)(Z^1) + \Delta_{j_1} F_3(X^1, X^2)(Z^1)F_4(X^2)] \\
= F_2(X^1) F_3(X^1)(Z^1)B_{j_1} + F_3(X^1)(Z^1)A_{j_1} F_3(X^2)F_4(X^2) \\
= F_2(X^1) F_3(X^1)(Z^1)B_{j_1} + F_2(X^1) F_3(X^1)(Z^1)A_{j_1} F_3(X^2)F_4(X^2)
\]

and in particular, as \( I_m \otimes Y \in \Omega_{sm}(\mathcal{R}) \) and \( F_4(I_m \otimes Y) = 0, \)

\[
\Delta_{j_1} \mathcal{R}(X^1, I_m \otimes Y)(Z^1) = F_2(X^1) F_3(X^1)(Z^1)B_{j_1}.
\]

Using the notations in Lemma 3.3, we have

\[
\Delta_{j_1} \mathcal{R}(X^1, X^2)(Z) = \left[ (f_1 \otimes f_2)B_{j_1} + (f_1 \otimes f_3)A_{j_1} \right] (X^1, X^2)(Z),
\]
i.e., \( \Delta_{j_1} \mathcal{R} = (f_1 \otimes f_2)_{B_{j_1}} + (f_1 \otimes f_3)_{A_{j_1}} \), where \( f_1 = F_2 F_3 \), \( f_2 = I_s \) and \( f_3 = F_3 F_4 \) are nc functions. So

\[
\Delta_{j_2} \Delta_{j_1} \mathcal{R}(X^1, X^2, X^3)(Z^1, Z^2) = \Delta_{j_2}(f_1 \otimes f_2)_{B_{j_1}} + (f_1 \otimes f_3)_{A_{j_1}})
\]

\[
+ f_1(X^1)(Z^1)B_{j_1} \Delta_{j_2} f_3(X^2, X^3)(Z^2) + f_1(X^1)(Z^1)A_{j_1} \Delta_{j_2} f_3(X^2, X^3)(Z^2)
\]

\[
= f_1(X^1)(Z^1)A_{j_1} \left[ F_3(X^2) \Delta_{j_2} F_2(X^2, X^3)(Z^2) + \Delta_{j_2} F_4(X^2, X^3)(Z^2) F_4(X^3) \right]
\]

\[
= f_1(X^1)(Z^1)A_{j_1} \left[ F_3(X^2)(Z^2)B_{j_2} + F_3(X^2)(Z^2)A_{j_2} F_3(X^3)F_4(X^3) \right]
\]

\[
= \left[ (f_1 \otimes F_3 \otimes f_2)_{A_{j_1}}B_{j_2} + (f_1 \otimes F_3 \otimes f_3)_{A_{j_1}}A_{j_2} \right] (X^1, X^2, X^3)(Z^1, Z^2),
\]

i.e.,

\[
\Delta_{j_2} \Delta_{j_1} \mathcal{R} = (f_1 \otimes F_3 \otimes f_2)_{A_{j_1}}B_{j_2} + (f_1 \otimes F_3 \otimes f_3)_{A_{j_1}}A_{j_2}.
\]

We continue the proof by induction (on \( k \)). Suppose that

\[
\Delta_{j_k} \ldots \Delta_{j_1} \mathcal{R}(X^1, \ldots, X^{k+1})(Z^1, \ldots, Z^k)
\]

\[
= (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_2)_{A_{j_1}, \ldots, A_{j_{k-1}}}B_{j_k}
\]

\[
+ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_3)_{A_{j_1}, \ldots, A_{j_{k}}} (X^1, \ldots, X^{k+1})(Z^1, \ldots, Z^k),
\]

i.e.,

\[
\Delta_{j_k} \ldots \Delta_{j_1} \mathcal{R} = (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_2)_{A_{j_1}, \ldots, A_{j_{k-1}}}B_{j_k}
\]

\[
+ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_3)_{A_{j_1}, \ldots, A_{j_{k}}} (X^1, \ldots, X^{k+1})(Z^1, \ldots, Z^k),
\]

and hence—using (3.2) in Lemma 3.3

\[
\Delta_{j_{k+1}} \Delta_{j_k} \ldots \Delta_{j_1} \mathcal{R}(X^1, \ldots, X^{k+2})(Z^1, \ldots, Z^{k+1})
\]

\[
= \Delta_{j_{k+1}} \left[ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_2)_{A_{j_1}, \ldots, A_{j_{k-1}}}B_{j_k} \right.
\]

\[
+ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_3)_{A_{j_1}, \ldots, A_{j_{k}}} \left] (X^1, \ldots, X^{k+2})(Z^1, \ldots, Z^{k+1}) \right.
\]

\[
= f_1(X^1)(Z^1)A_{j_k} \ldots (Z^{k-1})A_{j_{k-1}}F_3(X^k) \left[ (Z^k)B_{j_k} \Delta_{j_{k+1}} f_2(X^{k+1}, X^{k+2})(Z^{k+1}) \right.
\]

\[
+ (Z^k)A_{j_k} \Delta_{j_{k+1}} f_3(X^{k+1}, X^{k+2})(Z^{k+1}) \right]
\]

\[
= f_1(X^1)(Z^1)A_{j_k} \ldots (Z^{k-1})A_{j_{k-1}}F_3(X^k) \left[ (Z^k)B_{j_k} \Delta_{j_{k+1}} f_2(X^{k+1}, X^{k+2})(Z^{k+1}) \right.
\]

\[
+ (Z^k)A_{j_k} \Delta_{j_{k+1}} f_3(X^{k+1}, X^{k+2})(Z^{k+1}) \right]
\]

\[
= \left[ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_2)_{A_{j_1}, \ldots, A_{j_{k}}}B_{j_{k+1}} \right.
\]

\[
+ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_3)_{A_{j_1}, \ldots, A_{j_{k+1}}} \left] (X^1, \ldots, X^{k+2})(Z^1, \ldots, Z^{k+1}), \right.
\]

i.e.,

\[
\Delta_{j_{k+1}} \ldots \Delta_{j_1} \mathcal{R} = (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_2)_{A_{j_1}, \ldots, A_{j_{k}}}B_{j_{k+1}}
\]

\[
+ (f_1 \otimes F_3 \otimes \ldots \otimes F_3 \otimes f_3)_{A_{j_1}, \ldots, A_{j_{k+1}}},
\]

which ends the proof by induction. Since \( F_3(I_m \otimes Y) = I_{L_m} \), we know

\[
\Delta_{j_k} \ldots \Delta_{j_1} \mathcal{R}(X^1, I_m \otimes Y, \ldots, I_m \otimes Y, X^{k+1})(Z^1, \ldots, Z^k)
\]

\[
= f_1(X^1)(Z^1)A_{j_k} \ldots (Z^{k-1})A_{j_{k-1}} (Z^k)B_{j_k}
\]

\[
+ f_1(X^1)(Z^1)A_{j_k} \ldots (Z^k)A_{j_k} f_3(X^{k+1}).
\]
Therefore, as $f_3(I_m \otimes Y) = 0$, we obtain formula (3.3). Next, similarly to the above computations, one easily gets

$$
\Delta_{jk} \cdots \Delta_{j_1} F_3(X^1, \ldots, X^{k+1})(Z^1, \ldots, Z^k)
= F_3(X^1)(Z^1) A_{j_1} F_3(X^2) \cdots F_3(X^k)(Z^k) A_{jk} F_3(X^{k+1})
$$

and hence by taking $X^1 = \cdots = X^k = I_m \otimes Y$, we obtain (3.4). \qed

Finally, we are in a position to take advantage of the nc difference-differential calculus analysis developed in this section and prove that $\Omega(\mathcal{R})$ is similarity invariant, a property that will be quite useful for us later on (cf. the proof of Theorem 4.4).

**Theorem 3.5.** Let $\mathcal{R}$ be a nc Fornasini–Marchesini realization centred at $Y \in \mathbb{K}^{s \times s}$ that is described by $(L; D, C, A, B)$. Suppose $\mathcal{R}$ is controllable and observable, and its coefficients satisfy the $L - LA$ conditions. Then $\Omega(\mathcal{R})$ is similarity invariant, i.e., $X \in \Omega_{sm}(\mathcal{R}) \Rightarrow S \cdot X \cdot S^{-1} \in \Omega_{sm}(\mathcal{R})$ for any invertible $S \in \mathbb{K}^{sm \times sm}$ and $m \in \mathbb{N}$.

For the purpose of the proof, we recall the definitions of the controllability and observability matrices, as they appear in [64, Sect. 2.2]. The infinite block matrix

$$
\mathcal{C}_{A, B} := \text{row} \left[ \mathcal{C}^{(\omega, k)}_{A, B} \right]_{(\omega, k) \in \mathcal{G}_d \times \{1, \ldots, d\}}
$$

is the controllability matrix associated with $(A, B)$, where

$$
\mathcal{C}^{(\omega, k)}_{A, B} \in \mathbb{K}^{L \times s^{2(|\omega|+1)+1}}
$$

is given by $\mathcal{C}^{(\omega, k)}_{A, B} := \text{row} \left[ (A^\omega \cdot B_k)(Z) \right]_{Z \in \mathcal{E}_{|\omega|+1}}$

for each $(\omega, k) \in \mathcal{G}_d \times \{1, \ldots, d\}$.

The infinite block matrix

$$
\mathcal{D}_{C, A} := \text{col} \left[ \mathcal{D}^{(\omega)}_{C, A} \right]_{\omega \in \mathcal{G}_d}
$$

is the observability matrix associated to $(C, A)$, where

$$
\mathcal{D}^{(\omega)}_{C, A} \in \mathbb{K}^{s^{2(|\omega|+1)} \times L}
$$

is given by $\mathcal{D}^{(\omega)}_{C, A} := \text{col} \left[ C \cdot A^\omega (Z) \right]_{Z \in \mathcal{E}_{|\omega|+1}}$

for each $\omega \in \mathcal{G}_d$.

**Proof.** If $m \in \mathbb{N}$ and $X \in \Omega_{sm}(\mathcal{R})$, then $\Lambda_{A, Y}(X)$ is invertible. As $(C, A)$ is observable and $(A, B)$ is controllable, it follows from [64, Proposition 2.10] that there exist $\ell \in \mathbb{N}$ for which the matrix

$$
\mathcal{C}_\ell := \text{row} \left[ \mathcal{C}^{(\omega, k)}_{A, B} \right]_{|\omega| \leq \ell, 1 \leq k \leq d}
$$

is right invertible and the matrix

$$
\mathcal{O}_\ell := \text{col} \left[ \mathcal{D}^{(\omega)}_{C, A} \right]_{|\omega| \leq \ell}
$$

is left invertible. Let us denote a right inverse of $\mathcal{C}_\ell$ by $\mathcal{C}^{(R)}_\ell$ and a left inverse of $\mathcal{O}_\ell$ by $\mathcal{O}^{(L)}_\ell$, thus $\mathcal{C}_\ell \mathcal{C}^{(R)}_\ell = \mathcal{O}^{(L)}_\ell \mathcal{O}_\ell = I_L$. As $X \in \Omega_{sm}(\mathcal{R})$, it follows from Lemma 3.4 that

$$
\Delta_{jk} \cdots \Delta_{j_1} \mathcal{R}(X, I_m \otimes Y, \ldots, I_m \otimes Y, I_m \otimes Y)(Z^1, \ldots, Z^k)
$$
\[
= (I_m \otimes C) \Lambda_{A,Y}(X)^{-1}(Z^1, \ldots, Z^{k-1}) A^{\omega_1}(Z^k) B_{jk}
\]
for every choice of \(k \geq 1, 1 \leq j_1, \ldots, j_k \leq d\), and \(Z^1, \ldots, Z^k \in \mathcal{E}_{sm}\), therefore when considering all words \(\omega \in \mathcal{G}_d\) with \(|\omega| \leq \ell\) and putting them all together (as blocks) in a (row) matrix, we get
\[
\Delta^{(R)}_R(X) = (I_m \otimes C) \Lambda_{A,Y}(X)^{-1} e_\ell,
\]
where
\[
\Delta^{(R)}_R(X) := \text{row} \left[ \Delta^{(w)}_{R}(X) \right]_{|\omega| \leq \ell}
\]
and
\[
\Delta^{(w)}_{R}(X) := \text{row} \left[ \Delta^{\omega} \mathcal{R} \left( X, I_m \otimes Y, \ldots, I_m \otimes Y, X \right) \right]_{Z^1, \ldots, Z^{|\omega|} \in \mathcal{E}_{sm}}.
\]

Multiplying both sides of (3.5) on the right by \(e^{(R)}\), to obtain
\[
\Delta^{(R)}_R(X)e^{(R)} = (I_m \otimes C) \Lambda_{A,Y}(X)^{-1} e^{(R)},
\]
Next, we consider all higher derivatives w.r.t words of length at most \(\ell\) of the nc function \(\Delta^{(R)}_R(\cdot)\); by putting them all together (as blocks) in a (column) matrix, and by using (3.6) and (3.4), we observe that
\[
\Delta^{(R)}_R(X) = \text{col} \left[ \Delta^{(w)}_{R}(X) \right]_{|\omega| \leq \ell}
\]
and
\[
\Delta^{(w)}_{R}(X) = \text{col} \left[ \Delta^{\omega} \Delta^{(R)}_R \left( X, I_m \otimes Y, \ldots, I_m \otimes Y, X \right) \right]_{Z^1, \ldots, Z^{|\omega|} \in \mathcal{E}_{sm}}.
\]

Multiplying both sides of (3.7) on the left by \(\sigma^{(L)}\), to obtain \(\sigma^{(L)} \Delta^{(R)}_R(X)c^{(R)} = \Lambda_{A,Y}(X)^{-1}\), i.e., for every \(X \in \Omega_{sm}(\mathcal{R})\) we showed that
\[
\sigma^{(L)} \Delta^{(R)}_R(X)c^{(R)} \Lambda_{A,Y}(X) = I_{Lm}.
\]
Thus, for every invertible \(S \in \mathbb{K}_{sm \times sm}\) such that \(S \cdot X \cdot S^{-1} \in \Omega_{sm}(\mathcal{R})\), i.e., such that
\[
\det \left( \Lambda_{A,Y}(S \cdot X \cdot S^{-1}) \right) = \det \left( I_{Lm} - \sum_{k=1}^{d} (SX_kS^{-1} - I_m \otimes Y_k)A_k \right) \neq 0,
\]
one can apply (3.8) for \(S \cdot X \cdot S^{-1}\) to obtain that
\[
\sigma^{(L)} \Delta^{(R)}_R(S \cdot X \cdot S^{-1})c^{(R)} \Lambda_{A,Y}(S \cdot X \cdot S^{-1}) = I_{Lm}.
\]
We proved that (3.10) holds whenever (3.9) holds and next we show that (3.10) holds for any invertible \(S \in \mathbb{K}_{sm \times sm}^\text{inv}\) even without assuming (3.9).

As \(\mathcal{R} : \Omega(\mathcal{R}) \to \mathbb{K}_{nc}\) is a nc function, there exists a unique nc function \(\tilde{\mathcal{R}} : \tilde{\Omega}(\mathcal{R}) \to \mathbb{K}_{nc}\) such that \(\tilde{\mathcal{R}} \mid_{\Omega(\mathcal{R})} = \mathcal{R}\), where \(\tilde{\Omega}(\mathcal{R})\) is the similarity invariant envelope of \(\Omega(\mathcal{R})\), that is the smallest nc set that contains \(\Omega(\mathcal{R})\) and is similarity invariant (cf. [47, Appendix A]). Let \(\varphi : \mathbb{K}_{inv}^{sm \times sm} \to \mathbb{K}_{Lm \times Lm}^{Lm \times Lm}\) be the mapping defined by
\[
\varphi(S) = \sigma^{(L)} \Delta^{(R)}_R(S \cdot X \cdot S^{-1})c^{(R)} \Lambda_{A,Y}(S \cdot X \cdot S^{-1}) - I_{Lm}.
\]
for every $S \in \mathbb{K}^{sm \times sm}$, i.e., for every invertible matrix $S \in \mathbb{K}^{sm \times sm}$, where $X, J, Y \in \Omega_{sm}(R)$ imply that $S \cdot X \cdot S^{-1}, I_m \otimes Y \in \Omega_{sm}(R)$ and hence $\Delta^{(l)}_{\Delta R}(S \cdot X \cdot S^{-1})$ is well defined and is entrywise polynomial in (the entries of) $S$ and in $1/\det(S)$, see the proofs of Propositions A.3 and A.5 in [47]. If $S \in \mathbb{K}^{sm \times sm}$ such that (3.9) holds, then $S \cdot X \cdot S^{-1} \in \Omega_{sm}(R)$ which implies that $\Delta^{(l)}_{\Delta R}(S \cdot X \cdot S^{-1}) = \Delta^{(l)}_{\Delta R}(S \cdot X \cdot S^{-1})$ and thus $\varphi(S) = 0$. As the set

\[
Z = \{S \in \mathbb{K}^{sm \times sm} : \det(S) \neq 0, \det(\Lambda_{A, \varphi}(S \cdot X \cdot S^{-1})) \neq 0\}
\]
is a non-empty (since $I_{sm} \in Z$) Zariski open subset of $\mathbb{K}^{sm \times sm}$ on which $\varphi \big|_Z = 0$, while on the other hand the equation $\varphi(S) = 0$ is a polynomial expression w.r.t (entries of) $S$ and to $1/\det(S)$, we deduce that $\varphi(S) = 0$ holds true for every $S \in \mathbb{K}^{sm \times sm}$. Therefore the matrix $\Lambda_{A, \varphi}(S \cdot X \cdot S^{-1})$ is invertible, which means that $S \cdot X \cdot S^{-1} \in \Omega_{sm}(R)$. 

4. Minimal Realizations and NC Rational Functions: Full Characterization

4.1. Linearized Lost-Abbey Conditions w.r.t Algebras

We begin this section by carrying over some of the computations regarding the $L - LA$ conditions to evaluations over algebras. Let $A$ be a unital $\mathbb{K}$–algebra. As in [64] and in Section 1.2, we extend the linear mappings

\[
A_k : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times L} \quad \text{and} \quad B_k : \mathbb{K}^{s \times s} \to \mathbb{K}^{L \times s}, \quad k = 1, \ldots, d
\]
in a natural way to linear mappings

\[
A_k^A : A^{s \times s} \to A^{L \times L} \quad \text{and} \quad B_k^A : A^{s \times s} \to A^{L \times s}, \quad k = 1, \ldots, d
\]
by the following rule: if $A \in A^{s \times s}$, it can be written as $A = \sum_{p, q = 1}^{s} E_{pq} \otimes a_{pq}$ where $a_{pq} \in A$ and $E_{pq} \in E_s$, thus we define

\[
(A)A_k^A := \sum_{p, q = 1}^{s} A_k(E_{pq}) \otimes a_{pq} \in A^{L \times L} \quad \text{and}
\]

\[
(A)B_k^A := \sum_{p, q = 1}^{s} B_k(E_{pq}) \otimes a_{pq} \in A^{L \times s}
\]
for $1 \leq k \leq d$. It is easy to check that if any of the $L - LA$ conditions (cf. equations (2.3)–(2.8)) holds over $\mathbb{K}^{s \times s}$, then it holds also over $A^{s \times s}$. More precisely, the linearized lost-abbey ($L - LA$) conditions w.r.t $A$ are given by

\[
S_A D_A - D_A S_A = C_A \sum_{k = 1}^{d} ([S, Y_k], A) B_k^A,
\]

(4.1)

\[
(SC)_A (A_1) B_1^A - C_A (S_A A_1) B_1^A = C_A \left( \sum_{k = 1}^{d} ([S, Y_k], A) A_k^A \right) (A_1) B_1^A,
\]

(4.2)

\[
(SC)_A (A_1) A_1^A - C_A (S_A A_1) A_1^A = C_A \left( \sum_{k = 1}^{d} ([S, Y_k], A) A_k^A \right) (A_1) A_1^A,
\]
simplicity, we frequently use the notation 

\((2.8)\) implies equation \((4.6)\), while the proofs for the other equations \((4.3)\), \((4.4)\), \((4.5)\), and \((4.6)\) hold. 

\[
(\mathfrak{A}^i \mathcal{A}) B_{i_1}^A - (\mathfrak{A}^i \mathcal{A}) B_{i_2}^A = (\mathfrak{A}^i \mathcal{A}) A_{i_1}^A \left( \sum_{k=1}^{d} ([S, Y_k], A) B_k^A \right),
\]

\[
(\mathfrak{A}^i \mathcal{A}) A_{i_1}^A (\mathfrak{A}^i \mathcal{A}) B_{i_1}^A - (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A = (\mathfrak{A}^i \mathcal{A}) A_{i_1}^A \left( \sum_{k=1}^{d} ([S, Y_k], A) A_k^A \right) (\mathfrak{A}^i \mathcal{A}) B_{i_2}^A,
\]

and 

\[
(\mathfrak{A}^i \mathcal{A}) A_{i_1}^A (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A - (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A
\]

for every \(S \in \mathbb{K}^{s \times s}, \mathfrak{A}^i, \mathfrak{A}^j \in \mathcal{A}^{s \times s}\), and \(1 \leq i_1, i_2 \leq d\), where for the sake of simplicity, we frequently use the notation 

\[
Z_A := Z \otimes 1_A \in \mathcal{A}^{\alpha \times \beta},
\]

for every \(\alpha, \beta \in \mathbb{N}\) and \(Z \in \mathbb{K}^{\alpha \times \beta}\). In the following lemma, we prove that equation \((2.8)\) implies equation \((4.6)\), while the proofs for the other equations are similar, hence omitted.

**Lemma 4.1.** Let \(\mathcal{A}\) be a unital \(\mathbb{K}\)-algebra. If equation \((2.8)\) holds, then \((4.6)\) holds.

**Proof.** If \(\mathfrak{A}^i, \mathfrak{A}^j \in \mathcal{A}^{s \times s}\), one can write 

\[
\mathfrak{A}^i = \sum_{p, q=1}^{s} E_{pq} \otimes a_{pq}^{(1)} \text{ and } \mathfrak{A}^j = \sum_{k, l=1}^{s} E_{kl} \otimes a_{kl}^{(2)},
\]

where \(E_{pq} \in \mathcal{E}_s\) and \(a_{pq}^{(1)}, a_{pq}^{(2)} \in \mathcal{A}\) for \(1 \leq p, q \leq s\), then 

\[
(\mathfrak{A}^i \mathcal{A}) A_{i_1}^A (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A - (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A (\mathfrak{A}^i \mathcal{A}) A_{i_2}^A
\]

\[
= \left( \sum_{p, q=1}^{s} (E_{pq} S) \otimes a_{pq}^{(1)} \right) A_{i_1}^A \left( \sum_{k, l=1}^{s} E_{kl} \otimes a_{kl}^{(2)} \right) A_{i_2}^A
\]

\[
- \left( \sum_{p, q=1}^{s} E_{pq} \otimes a_{pq}^{(1)} \right) A_{i_1}^A \left( \sum_{k, l=1}^{s} (SE_{kl}) \otimes a_{kl}^{(2)} \right) A_{i_2}^A
\]

\[
= \sum_{p, q=1}^{s} \sum_{k, l=1}^{s} \left[ \left( A_{i_1} (E_{pq} S) \otimes a_{pq}^{(1)} \right) \left( A_{i_2} (E_{kl}) \otimes a_{kl}^{(2)} \right) \right]
\]

\[
- \left( A_{i_1} (E_{pq}) \otimes a_{pq}^{(1)} \right) \left( A_{i_2} (SE_{kl}) \otimes a_{kl}^{(2)} \right)
\]

\[
= \sum_{p, q=1}^{s} \sum_{k, l=1}^{s} \left[ \left( A_{i_1} (E_{pq} S) A_{i_2} (E_{kl}) - A_{i_1} (E_{pq}) A_{i_2} (SE_{kl}) \right) \otimes \left( a_{pq}^{(1)} a_{kl}^{(2)} \right) \right]
\]
Applying (2.8), we get
\[
(\mathfrak{A}_1 S_{\mathcal{A}}) \mathbf{A}_i^A (\mathfrak{A}_2) \mathbf{A}_i^A - (\mathfrak{A}_1) \mathbf{A}_i^A (S_{\mathcal{A}} \mathfrak{A}_2) \mathbf{A}_i^A \\
= \sum_{p,q=1}^{s} \sum_{k,l=1}^{s} \left[ \left( \mathbf{A}_{i_1} (E_{pq}) \left( \sum_{k=1}^{d} \mathbf{A}_k ([S, Y_k]) \right) \mathbf{A}_{i_2} (E_{kl}) \right) \otimes \left( \mathbf{a}_{pq}^{(1)} \mathbf{a}_{kl}^{(2)} \right) \right] \\
= \left( \sum_{p,q=1}^{s} \mathbf{A}_{i_1} (E_{pq} \otimes \mathbf{a}_{pq}^{(1)}) \left( \sum_{k=1}^{d} \mathbf{A}_k ([S, Y_k]) \otimes 1_{\mathcal{A}} \right) \right) \sum_{k,l=1}^{s} \mathbf{A}_{i_2} (E_{kl} \otimes \mathbf{a}_{kl}^{(2)}) \\
= (\mathfrak{A}_1) \mathbf{A}_i^A \left( \sum_{k=1}^{d} ([S, Y_k]) \mathbf{A}_k^A \right) (\mathfrak{A}_2) \mathbf{A}_i^A.
\]
\[\blacksquare\]

For a nc Fornasini–Marchesini realization \(\mathcal{R}\) that is centred at \(Y \in \Omega_s(\mathcal{R})\) and described by \((L; D, C, \mathbf{A}, \mathbf{B})\), we recall the following definitions:

- The \(\mathcal{A}\)-domain of \(\mathcal{R}\) (denoted by \(\text{DOM}^A(\mathcal{R})\) in [64, p.9]), that is
  \[\Omega_{\mathcal{A}}(\mathcal{R}) := \left\{ \mathfrak{A} \in (\mathcal{A}^{s \times s})^d : \Lambda_{\mathcal{A}, Y}^A(\mathfrak{A}) \text{ is invertible in } \mathcal{A}^{L \times L} \right\},\]
  where \(Y_k := Y_k \otimes 1_{\mathcal{A}}\) for every \(1 \leq k \leq d\), and
  \[\Lambda_{\mathcal{A}, Y}^A(\mathfrak{A}) := (I_L)_{\mathcal{A}} - \sum_{k=1}^{d} (\mathfrak{A}_k - Y_k) \mathbf{A}_k^A = (I_L)_{\mathcal{A}} - \sum_{k=1}^{d} \left[ (\mathfrak{A}_k) \mathbf{A}_k^A - (\mathbf{A}_k(Y_k))_{\mathcal{A}} \right].\]

- The \(\mathcal{A}\)-evaluation of \(\mathcal{R}\) at any \(\mathfrak{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_d) \in \Omega_{\mathcal{A}}(\mathcal{R})\), which is
  \[\mathcal{R}^A(\mathfrak{A}) := D_{\mathcal{A}} + C_{\mathcal{A}} \Lambda_{\mathcal{A}, Y}^A(\mathfrak{A})^{-1} \sum_{k=1}^{d} \left[ (\mathfrak{A}_k) \mathbf{B}_k^A - (\mathbf{B}_k(Y_k))_{\mathcal{A}} \right].\]

In the next proposition we show that in the special case where \(\mathfrak{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_d) = (I_s \otimes a_1, \ldots, I_s \otimes a_d) \in \Omega_{\mathcal{A}}(\mathcal{R})\) for some \(a_1, \ldots, a_d \in \mathcal{A}\), the matrix \(\mathcal{R}^A(\mathfrak{A})\) commutes with all the matrices of the form \(S_{\mathcal{A}} = S \otimes 1_{\mathcal{A}}\), where \(S \in \mathbb{K}^{s \times s}\), and hence must be scalar. This result is very important, as we will see in the proof of Theorem 4.4.

**Proposition 4.2.** Let \(\mathcal{R}\) be a nc Fornasini–Marchesini realization centred at \(Y \in \mathbb{K}^{s \times s}\) that is described by \((L; D, C, \mathbf{A}, \mathbf{B})\), suppose its coefficients satisfy the \(\mathcal{L} - \mathcal{L}_{\mathcal{A}}\) conditions, and let \(\mathcal{A}\) be a unital \(\mathbb{K}\)-algebra. If \(\mathfrak{a} = (a_1, \ldots, a_d) \in \mathcal{A}^d\) such that
\[\Lambda_{\mathcal{A}, Y}^A(I_s \otimes \mathfrak{a}) = (I_L)_{\mathcal{A}} - \sum_{k=1}^{d} \left[ \mathbf{A}_k(I_s) \otimes a_k - (\mathbf{A}_k(Y_k))_{\mathcal{A}} \right],\]
is invertible in \(\mathcal{A}^{L \times L}\), then \(\mathcal{R}^A(\mathfrak{A}) S_{\mathcal{A}} = S_{\mathcal{A}} \mathcal{R}^A(\mathfrak{A})\) for every \(S \in \mathbb{K}^{s \times s}\), where \(\mathfrak{A} = (\mathfrak{A}_1, \ldots, \mathfrak{A}_d) = I_s \otimes \mathfrak{a}.\) In particular, there exists an element \(f(\mathfrak{a}) \in \mathcal{A}\) such that \(\mathcal{R}^A(I_s \otimes \mathfrak{a}) = I_s \otimes f(\mathfrak{a})\).

The ideas of the proof are very similar to those in the proof of Theorem 2.6, with the difference of using the \(\mathcal{L} - \mathcal{L}_{\mathcal{A}}\) conditions w.r.t algebras now, instead of the \(\mathcal{L} - \mathcal{L}_{\mathcal{A}}\) conditions. Nevertheless, we present the proof fully detailed.
Proof. Let $S \in \mathbb{K}^{s \times s}$. For simplicity, recall the notations $\mathcal{Y}_i := Y_i \otimes 1_A$ for every $1 \leq i \leq d$ and define the matrices

$$\mathfrak{M}_A = \sum_{k=1}^{d} ([S, Y_k]_A) A_k^A \in A^{L \times L}$$

and

$$\mathfrak{M}_B = \sum_{k=1}^{d} ([S, Y_k]_A) B_k^A \in A^{L \times s}.$$

From equations (4.1) and (4.4), it follows that

$$\mathcal{R}^A(\mathfrak{A}) S_A = (DS)_A + C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1} \left( \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) B_i^A \right) S_A$$

$$= (SD)_A + C_A \mathfrak{M}_B + C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1}$$

$$\times \left[ \sum_{i=1}^{d} ((\mathfrak{A}_i - \mathcal{Y}_i) S_A) B_i^A - \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) A_i^A \mathfrak{M}_B \right]$$

$$= \left( SD - C \sum_{k=1}^{d} B_k([S, Y_k]) \right) A + C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1}$$

$$\times \left[ \sum_{i=1}^{d} ((\mathfrak{A}_i - \mathcal{Y}_i) S_A) B_i^A + \Lambda_A^{\mathcal{A} Y}(\mathfrak{A}) \left( \sum_{k=1}^{d} B_k([S, Y_k]) \right) A \right]$$

$$= \left( SD - C \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) S_A - [S, \mathcal{Y}_i]_A \right) B_i^A$$

$$= (SD)_A + C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1} \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) S_A - [S, \mathcal{Y}_i]_A$$

$$= (SD)_A + C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1} \sum_{i=1}^{d} (\mathfrak{A}_i S_A - (S \mathcal{Y}_i)_A) B_i^A$$

$$= (SD)_A + C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1} \sum_{i=1}^{d} (S_A (\mathfrak{A}_i - \mathcal{Y}_i)) B_i^A,$$

i.e.,

$$\mathcal{R}^A(\mathfrak{A}) S_A - (SD)_A = C_A \Lambda_A^{\mathcal{A} Y}(\mathfrak{A})^{-1} \sum_{i=1}^{d} (S_A (\mathfrak{A}_i - \mathcal{Y}_i)) B_i^A.$$

From (4.5) it follows that

$$((\mathfrak{A}_j - \mathcal{Y}_j) S_A) A_j^A (\mathfrak{A}_i - \mathcal{Y}_i) B_i^A - (\mathfrak{A}_j - \mathcal{Y}_j) A_j^A (S_A (\mathfrak{A}_i - \mathcal{Y}_i)) B_i^A$$

$$= (\mathfrak{A}_j - \mathcal{Y}_j) A_j^A \mathfrak{M}_A (\mathfrak{A}_i - \mathcal{Y}_i) B_i^A,$$

thus

$$\left( \sum_{j=1}^{d} ((\mathfrak{A}_j - \mathcal{Y}_j) S_A) A_j^A \right) \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) B_i^A.$$
\[
\sum_{j=1}^{d} \left( \sum_{i=1}^{d} (A_j - Y_j) A_i^A \right) \sum_{i=1}^{d} (S_A (A_i - Y_i)) B_i^A
\]

\[
= \left( \sum_{j=1}^{d} (A_j - Y_j) A_j^A \right) \Omega_A \sum_{i=1}^{d} (A_i - Y_i) B_i^A
\]

and hence

\[
\Lambda_{A,Y}^A (A) \sum_{i=1}^{d} (S_A (A_i - Y_i)) B_i^A = \sum_{i=1}^{d} \left( \sum_{j=1}^{d} (A_j - Y_j) A_j^A \right) \Omega_A \sum_{i=1}^{d} (A_i - Y_i) B_i^A - \left( \sum_{j=1}^{d} ((A_j - Y_j) S_A) A_j^A \right)
\]

\[
\sum_{i=1}^{d} (A_i - Y_i) B_i^A,
\]

which implies that

\[
\Lambda_{A,Y}^A (A)^{-1} \sum_{i=1}^{d} (S_A (A_i - Y_i)) B_i^A
\]

\[
= \sum_{i=1}^{d} (S_A (A_i - Y_i)) B_i^A - \Lambda_{A,Y}^A (A)^{-1}
\]

\[
\left( \sum_{j=1}^{d} (A_j - Y_j) A_j^A \right) \Omega_A \sum_{i=1}^{d} (A_i - Y_i) B_i^A
\]

\[
+ \Lambda_{A,Y}^A (A)^{-1} \left( \sum_{j=1}^{d} ((A_j - Y_j) S_A) A_j^A \right) \sum_{i=1}^{d} (A_i - Y_i) B_i^A
\]

\[
= \sum_{i=1}^{d} (S_A (A_i - Y_i)) B_i^A + \Lambda_{A,Y}^A (A)^{-1}
\]

\[
\left( \sum_{j=1}^{d} ((A_j - Y_j) S_A) A_j^A \right) \sum_{i=1}^{d} (A_i - Y_i) B_i^A
\]

\[
+ \Omega_A \sum_{i=1}^{d} (A_i - Y_i) B_i^A - \Lambda_{A,Y}^A (A)^{-1} \Omega_A \sum_{i=1}^{d} (A_i - Y_i) B_i^A
\]

\[
= \sum_{i=1}^{d} (S_A (A_i - Y_i)) B_i^A + \Omega_A \sum_{i=1}^{d} (A_i - Y_i) B_i^A
\]

\[
+ \Lambda_{A,Y}^A (A)^{-1} \left( \sum_{j=1}^{d} ((A_j - Y_j) S_A - [S, Y_j] A) A_j^A \right) \sum_{i=1}^{d} (A_i - Y_i) B_i^A.
\]
Therefore, using (4.2), we get
\[
\mathcal{R}^{A}(\mathcal{A}) S_{A} - (SD)_{A} = \sum_{i=1}^{d} C_{A} \left[ (S_{A}(\mathcal{A}_{i} - \mathcal{Y}_{i})) B_{i}^{A} + \mathcal{M}_{A}(\mathcal{A}_{i} - \mathcal{Y}_{i}) B_{i}^{A} \right] \\
+ C_{A} \Lambda^{A}_{A,Y}(\mathcal{A})^{-1} \left( \sum_{j=1}^{d} (\mathcal{A}_{j} S_{A} - (SY)_{A}) A_{j}^{A} \right) \sum_{i=1}^{d} (\mathcal{A}_{i} - \mathcal{Y}_{i}) B_{i}^{A} = \sum_{i=1}^{d} (SC)_{A}(\mathcal{A}_{i} - \mathcal{Y}_{i}) B_{i}^{A} + C_{A} \Lambda^{A}_{A,Y}(\mathcal{A})^{-1} \left( \sum_{j=1}^{d} (S_{A}(\mathcal{A}_{j} - \mathcal{Y}_{j})) A_{j}^{A} \right) \sum_{i=1}^{d} (\mathcal{A}_{i} - \mathcal{Y}_{i}) B_{i}^{A}.
\]

We know, from (4.6), that
\[
\left( \sum_{i=1}^{d} (\mathcal{A}_{i} - \mathcal{Y}_{i}) A_{i}^{A} \right) \sum_{j=1}^{d} (S_{A}(\mathcal{A}_{j} - \mathcal{Y}_{j})) A_{j}^{A} = \left( \sum_{i=1}^{d} ((\mathcal{A}_{i} - \mathcal{Y}_{i}) S_{A}) A_{i}^{A} \right) \sum_{j=1}^{d} (\mathcal{A}_{j} - \mathcal{Y}_{j}) A_{j}^{A} - \left( \sum_{i=1}^{d} (\mathcal{A}_{i} - \mathcal{Y}_{i}) A_{i}^{A} \right) \mathcal{M}_{A} \sum_{j=1}^{d} (\mathcal{A}_{j} - \mathcal{Y}_{j}) A_{j}^{A}
\]

and hence, using (4.3),
\[
C_{A} \Lambda^{A}_{A,Y}(\mathcal{A})^{-1} \sum_{j=1}^{d} (S_{A}(\mathcal{A}_{j} - \mathcal{Y}_{j})) A_{j}^{A} = C_{A} \left( \sum_{j=1}^{d} (S_{A}(\mathcal{A}_{j} - \mathcal{Y}_{j})) A_{j}^{A} \right) + C_{A} \mathcal{M}_{A} \sum_{j=1}^{d} (\mathcal{A}_{j} - \mathcal{Y}_{j}) A_{j}^{A} \\
+ C_{A} \Lambda^{A}_{A,Y}(\mathcal{A})^{-1} \left( \sum_{i=1}^{d} ((\mathcal{A}_{i} - \mathcal{Y}_{i}) S_{A}) A_{i}^{A} \right) \sum_{j=1}^{d} (\mathcal{A}_{j} - \mathcal{Y}_{j}) A_{j}^{A} \\
- C_{A} \Lambda^{A}_{A,Y}(\mathcal{A})^{-1} \mathcal{M}_{A} \sum_{j=1}^{d} (\mathcal{A}_{j} - \mathcal{Y}_{j}) A_{j}^{A} = (SC)_{A} \left( \sum_{i=1}^{d} (\mathcal{A}_{i} - \mathcal{Y}_{i}) A_{i}^{A} \right) + C_{A} \Lambda^{A}_{A,Y}(\mathcal{A})^{-1} \left( \sum_{i=1}^{d} (S_{A}(\mathcal{A}_{i} - \mathcal{Y}_{i})) A_{i}^{A} \right) \\
\sum_{j=1}^{d} (\mathcal{A}_{j} - \mathcal{Y}_{j}) A_{j}^{A}.
\]
which implies
\[ C_A \Lambda^A_{A,Y}(A)^{-1} \left( \sum_{j=1}^{d} (S_A(\mathfrak{A}_j - \mathcal{Y}_j)) A^A_j \right) \Lambda^A_{A,Y}(A) = (SC)_A \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) A^A_i \]
and then
\[ C_A \Lambda^A_{A,Y}(A)^{-1} \sum_{j=1}^{d} (S_A(\mathfrak{A}_j - \mathcal{Y}_j)) A^A_j = (SC)_A \left( \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) A^A_i \right) \Lambda^A_{A,Y}(A)^{-1} \]
\[ = (SC)_A \Lambda^A_{A,Y}(A)^{-1} \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) A^A_i. \]
Finally, we see that
\[ R^A(A)S_A - (SD)_A = \sum_{i=1}^{d} (SC)_A(\mathfrak{A}_i - \mathcal{Y}_i) B^A_i \]
\[ + (SC)_A \Lambda^A_{A,Y}(A)^{-1} \left( \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) A^A_i \right) \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) B^A_i \]
\[ = (SC)_A \Lambda^A_{A,Y}(A)^{-1} \sum_{i=1}^{d} (\mathfrak{A}_i - \mathcal{Y}_i) B^A_i, \]
i.e., \( R^A(A)S_A = S_A R^A(A) \). Then it is easily seen—just by choosing \( S = E_{\alpha\beta} \) for all \( 1 \leq \alpha, \beta \leq s \)—that \( R^A(A) \) is a scalar matrix over \( A \), so for every \( \mathfrak{a} \in A^d \) such that \( I_s \otimes \mathfrak{a} \in \Omega_A(R) \), there exists \( f(\mathfrak{a}) \in A \) satisfying \( R^A(I_s \otimes \mathfrak{a}) = I_s \otimes f(\mathfrak{a}) \).

4.2. Conclusions and Main Results
Towards the final step of viewing a nc Fornasini–Marchesini realization \( R \), that is controllable and observable, as a (restriction of a) nc rational function, we use the following technical yet important lemma, see [82, Lemma 3.9].

**Lemma 4.3.** Let \( \mathfrak{r} = (r_{ij})_{1 \leq i,j \leq n} \) be an \( n \times n \) matrix over \( K \langle \mathfrak{x} \rangle \) and let \( X \in \text{dom}(\mathfrak{r}) := \bigcap_{i,j=1,...,n} \text{dom}(r_{ij}) \). If \( \text{det}(\mathfrak{r}(X)) \neq 0 \), then there exist nc rational expressions \( S_{ij} \), \( i,j = 1,...,n \) such that \( X \in \bigcap_{i,j=1,...,n} \text{dom}(S_{ij}) \) and \( S_{ij} \) represents \( (\mathfrak{r}^{-1})_{ij} \) for every \( 1 \leq i,j \leq n \).

Given a nc Fornasini–Marchesini realization \( R \) centred at \( Y = (Y_1, \ldots, Y_d) \in (K^{s \times s})^d \), define the following two matrices of nc rational functions:
\[ \delta_R(x_1, \ldots, x_d) := I_L - \sum_{k=1}^{d} [A_k(I_s)x_k - A_k(Y_k)] \]
\[ (4.8) \]
in \( K \langle \mathfrak{x} \rangle^{L \times L} \), and
\[ \mathfrak{v}_R(x_1, \ldots, x_d) = D \]
\[ + C \left( I_L - \sum_{k=1}^{d} [A_k(I_s)x_k - A_k(Y_k)] \right)^{-1} \sum_{k=1}^{d} [B_k(I_s)x_k - B_k(Y_k)] \]
in $\mathbb{K}[x]_{s \times s}$. We will show in the proof of Theorem 4.4 below that the matrix $\delta_R$, appearing in the formula of $\nu_R$, is indeed invertible over $\mathbb{K}[x]_{L \times L}$.

**Theorem 4.4.** Let $R$ be a nc Fornasini–Marchesini realization that is centred at $Y \in \mathbb{K}^{s \times s}$ and described by $(L; D, C, A, B)$. Suppose that the coefficients of $R$ satisfy the $L - LA$ conditions. Then there exists $f \in \mathbb{K}[x]_s$ such that

$$\nu_R = I_s \otimes f \quad (4.10)$$

Moreover, $\Omega_{sm}(R) \subseteq \text{dom}_{sm}(f)$ and $\mathcal{R}(X) = f(X)$ for every $m \in \mathbb{N}$ and $X \in \Omega_{sm}(R)$. In particular, $Y \in \text{dom}_s(f)$.

**Remark 4.5.** The equality in (4.10) holds in $\mathbb{K}[x]_{s \times s}$, in the sense that $X \in \text{dom}(\nu_R)$ if and only if $X \in \bigcap_{1 \leq i, j \leq s} \text{dom}(R_{ij})$, where $R_{ii}$ (for $1 \leq i \leq s$) are nc rational expressions which represent $f$, and $R_{ij}$ (for $1 \leq i \neq j \leq s$) are nc rational expressions which represent $0$. In that case, the evaluation is given by $\nu_R(X) = (R_{ij}(X))_{1 \leq i, j \leq s}$.

**Proof.** In Theorem 2.6 we showed that $Y \in \Omega_s(R)$, while $\Omega(R)$ is closed under direct sums and similarities, where $R$ is the nc function given by (2.16), thus $I_s \otimes Y \in \Omega_{s^2}(R)$ and as $Y \otimes I_s = E(s, s)(I_s \otimes Y)E(s, s)^{-1}$ (cf. equation (1.1)), we get that $Y \otimes I_s \in \Omega_{s^2}(R)$, i.e., that the matrix

$$\Lambda_{A, Y}(Y \otimes I_s) = I_{Ls} - \sum_{k=1}^{d} (Y_k \otimes I_s - I_s \otimes Y_k)A_k$$

is invertible, which implies that the matrix

$$\delta_R(Y) = I_L \otimes I_s - \sum_{k=1}^{d} [A_k(I_s) \otimes Y_k - A_k(Y_k) \otimes I_s]$$

is invertible. It is well known (e.g., it follows immediately from Lemma 4.3) that an element from $\mathbb{K}[x]_{L \times L}$ is invertible in $\mathbb{K}[x]_{L \times L}$ if and only if its evaluation at some point is invertible. Therefore $\delta_R$ is invertible in $\mathbb{K}[x]_{L \times L}$, since the matrix $\delta_R(Y)$ is invertible.

By applying Proposition 4.2 with the free field $A = \mathbb{K}[x]$ and $a_1 = x_1, \ldots, a_d = x_d$, the invertibility of $\delta_R$ in $\mathbb{K}[x]_{L \times L}$—which is equivalent for equation (4.7)—implies that there exists $f \in \mathbb{K}[x]$ such that $\mathcal{R}^A(I_s \otimes x) = I_s \otimes f(x)$, i.e., $\nu_R = I_s \otimes f$.

Next, let $X \in \Omega_{sm}(R)$, similarly to the arguments above, we get that $\delta_R(X)$ is invertible, thus it follows from Lemma 4.3 that there exist nc rational expressions $S_{ij}$ (for $1 \leq i, j \leq L$), such that $X \in \bigcap_{1 \leq i, j \leq L} \text{dom}(S_{ij})$ and the $L \times L$ matrix valued nc rational expression $(S_{ij})_{1 \leq i, j \leq L}$ represents $\delta_R^{-1}$. Therefore, the $s \times s$ matrix of nc rational expressions given by

$$\tilde{R}(x) = D + C \left(S_{ij}(x)\right)_{1 \leq i, j \leq L} \left(p_{ij}(x)\right)_{1 \leq i \leq L, 1 \leq j \leq s}$$

is...
\[ = \left( d_{ij} + \sum_{k,\ell=1}^{L} c_{ik} S_{k\ell}(x)p_{\ell j}(x) \right)_{1 \leq i, j \leq s}, \]

where \((p_{ij})_{1 \leq i \leq L, 1 \leq j \leq s} = \sum_{k=1}^{d} \left[ B_k(I_s)x_k - B_k(Y_k) \right]\) is an \(L \times s\) matrix of nc polynomials, represents the \(s \times s\) matrix of nc rational functions \(v_R\). As \(X \in \bigcap_{1 \leq i,j \leq L} dom(S_{ij})\), it follows that

\[ X \in dom_{sm}(\xi_1^T \tilde{R}\xi_1) = dom_{sm} \left( d_{11} + \sum_{k,\ell=1}^{L} c_{ik} S_{k\ell}p_{\ell 1} \right), \]

however \(\xi_1^T \tilde{R}\xi_1\) represents the nc rational function \(\xi_1^T v_R\xi_1 = \xi_1^T (I_s \otimes f)\xi_1 = f\), hence \(X \in dom_{sm}(f)\). Moreover, as

\[ \delta_R(X) = I_{Lsm} - \sum_{k=1}^{d} \left[ A_k(I_s) \otimes X_k - A_k(Y_k) \otimes I_{sm} \right] \]

\[ = E(L, sm)^{-1} \left( I_{Lsm} - \sum_{k=1}^{d} \left[ X_k \otimes A_k(I_s) - I_{sm} \otimes A_k(Y_k) \right] \right) E(L, sm) \]

\[ = E(L, sm)^{-1} \Lambda_{A,Y}(X \otimes I_s)E(L, sm), \]

one can evaluate \(v_R(X)\) in the following way

\[ v_R(X) = D \otimes I_{sm} + (C \otimes I_{sm})\delta_R(X)^{-1} \sum_{k=1}^{d} \left[ B_k(I_s) \otimes X_k - B_k(Y_k) \otimes I_{sm} \right] \]

\[ = D \otimes I_{sm} + (C \otimes I_{sm})E(L, sm)^{-1}\Lambda_{A,Y}(X \otimes I_s)^{-1} E(L, sm) \]

\[ \left( \sum_{k=1}^{d} E(L, sm)^{-1} [X_k \otimes B_k(I_s) - I_n \otimes B_k(Y_k)] E(s, sm) \right) \]

\[ = E(s, sm)^{-1} \left( I_{sm} \otimes D + (I_{sm} \otimes C)\Lambda_{A,Y}(X \otimes I_s)^{-1} \right. \]

\[ \sum_{k=1}^{d} (X_k \otimes I_s - I_{sm} \otimes Y_k)B_k \bigg) \]

\[ E(s, sm) = E(s, sm)^{-1} \mathcal{R}(X \otimes I_s)E(s, sm) \]

\[ = E(s, sm)^{-1} \mathcal{R}(E(s, sm) \cdot (I_s \otimes X) \cdot E(s, sm)^{-1}) E(s, sm) \]

\[ = \mathcal{R}(I_s \otimes X) = I_s \otimes \mathcal{R}(X), \]

while \(v_R = I_s \otimes f\) implies \(v_R(X) = I_s \otimes f(X)\), therefore \(f(X) = \mathcal{R}(X)\).  \(\square\)

Before we prove one of the main results of the paper, we prove two useful properties of nc rational functions and expressions:

**Proposition 4.6.** Let \(\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{K}\langle x \rangle\), let \(R\) be a nc rational expression in \(x_1, \ldots, x_d\) over \(\mathbb{K}\), \(n \in \mathbb{N}\), and \(Z \in (\mathbb{K}^{n \times n})^d\).

1. If \(I_m \otimes Z \in dom_{mn}(R)\) for some \(m \in \mathbb{N}\), then \(Z \in dom_{m}(R)\).
2. If \(\mathcal{R}_1(X) = \mathcal{R}_2(X)\) for every \(X \in dom(\mathcal{R}_1) \cap dom(\mathcal{R}_2)\), then \(\mathcal{R}_1 = \mathcal{R}_2\).

Using the terminology in [47, pp. 95–96], item 1 says that \(dom(R)\) is a radical set.
Proof. 1. By synthesis: we only show the step of going from a nc rational expression to its inverse, as the proof for the other parts (going from two nc rational expressions to their sum and product, as well as the case of nc polynomial) are either similar or trivial. Suppose that our statement is known for a nc rational expressions $R$ and let $Z$ be such that $I_m \otimes Z \in \text{dom}_{mn}(R^{-1})$ for some $m \in \mathbb{N}$. Then $I_m \otimes Z \in \text{dom}_{mn}(R)$ and $R(I_m \otimes Z)$ is invertible, but from the assumption (on $R$) we get that $Z \in \text{dom}_n(R)$. As $R$ respects direct sums we get that $R(I_m \otimes Z) = I_m \otimes R(Z)$ is invertible, hence $R(Z)$ is invertible as well, i.e., $Z \in \text{dom}_n(R^{-1})$ as needed.

2. Let $R_1 \in \mathcal{R}_1$ and $R_2 \in \mathcal{R}_2$ be (non-degenerate) nc rational expressions. For every $X \in \text{dom}(R_1) \cap \text{dom}(R_2)$, we have $X \in \text{dom}(R_1) \cap \text{dom}(R_2)$ and hence $R_1(X) = R_1(X) = R_2(X) = R_2(X)$, i.e., $R_1$ and $R_2$ are $(\mathbb{K}^d)_{nc}$-evaluation equivalent, so they represent the same nc rational function, meaning $\mathcal{R}_1 = \mathcal{R}_2$. □

We now show that Theorems 4.4 and 2.6 imply one of our main results, that is—roughly speaking—we can evaluate a nc rational function and its domain using any of its minimal realizations, centred at any chosen point from its domain:

**Theorem 4.7.** Let $\mathcal{R} \in \mathbb{K}[x_1, \ldots, x_d]$. For every integer $s \in \mathbb{N}$, a point $Y \in \text{dom}_s(\mathcal{R})$, and a minimal nc Fornasini–Marchesini realization $\mathcal{R}$ centred at $Y$ of $\mathcal{R}$, we have

1. $\text{dom}_{sm}(\mathcal{R}) = \Omega_{sm}(\mathcal{R})$ and $\mathcal{R}(X) = \mathcal{R}(X)$, for every $X \in \text{dom}_{sm}(\mathcal{R})$ and $m \in \mathbb{N}$.

2. $\text{dom}_n(\mathcal{R}) = \{X \in (\mathbb{K}^{n \times n})^d : I_s \otimes X \in \Omega_{sn}(\mathcal{R})\}$ and $I_s \otimes \mathcal{R}(X) = \mathcal{R}(I_s \otimes X)$, for every $X \in \text{dom}_n(\mathcal{R})$ and $n \in \mathbb{N}$.

Proof. In Theorem 1.8 we proved that

$$\text{dom}_{sm}(\mathcal{R}) \subseteq \Omega_{sm}(\mathcal{R}) \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(X), \forall X \in \text{dom}_{sm}(\mathcal{R}), \forall m \in \mathbb{N}.$$ 

Since $\mathcal{R}$ admits the realization $\mathcal{R}$, it follows from Lemma 2.2 that the coefficients of $\mathcal{R}$ must satisfy the $\mathcal{L} - \mathcal{L}A$ conditions, thus Theorem 4.4 guarantees the existence of a nc rational function $f \in \mathbb{K}[x_1, \ldots, x_d]$ for which

$$\Omega_{sm}(\mathcal{R}) \subseteq \text{dom}_{sm}(f) \quad \text{and} \quad f(X) = \mathcal{R}(X), \forall X \in\Omega_{sm}(\mathcal{R}), \forall m \in \mathbb{N}.$$ 

Therefore, we have

$$\text{dom}_{sm}(\mathcal{R}) \subseteq \text{dom}_{sm}(f) \quad \text{and} \quad \mathcal{R}(X) = f(X), \forall X \in \text{dom}_{sm}(\mathcal{R}). \quad (4.11)$$

Next, we show that (4.11) is true for $sm$ replaced by any $n \in \mathbb{N}$. If $X \in \text{dom}_n(\mathcal{R})$, it follows from Theorem 1.8 that $I_s \otimes X \in \Omega_{sn}(\mathcal{R})$ and $I_s \otimes \mathcal{R}(X) = \mathcal{R}(I_s \otimes X)$, hence $I_s \otimes X \in \text{dom}_{sn}(f)$ and $f(I_s \otimes X) = \mathcal{R}(I_s \otimes X)$. From the first part of Proposition 4.6 it follows that $X \in \text{dom}_n(f)$ and also that $I_s \otimes f(X) = f(I_s \otimes X) = \mathcal{R}(I_s \otimes X) = I_s \otimes \mathcal{R}(X)$, so $f(X) = \mathcal{R}(X)$, i.e., we showed that

$$\text{dom}_n(\mathcal{R}) \subseteq \text{dom}_n(f) \quad \text{and} \quad \mathcal{R}(X) = f(X), \forall X \in \text{dom}_n(\mathcal{R}), \forall n \in \mathbb{N}.$$
Thus from the second part of Proposition 4.6, we obtain that $\mathcal{R} = f$ and thus

$$
\text{dom}_{sm}(\mathcal{R}) = \text{dom}_{sm}(f) = \Omega_{sm}(\mathcal{R}) \quad \text{and} \quad \mathcal{R}(X) = f(X) \quad \text{for every } X \in \text{dom}_{sm}(\mathcal{R}).
$$

Finally, if $I_s \otimes X \in \Omega_{sn}(\mathcal{R}) = \text{dom}_{sn}(\mathcal{R})$, then from the first part of Proposition 4.6, we have $X \in \text{dom}_n(\mathcal{R})$, hence $I_s \otimes \mathcal{R}(X) = \mathcal{R}(I_s \otimes X) = \mathcal{R}(I_s \otimes X)$. □

A direct consequence of the results above, which yields a representation that is independent of matrix centre for all nc rational functions, in the spirit of Cohn and Reutenauer, is now presented.

**Corollary 4.8.** Let $\mathcal{R} \in \mathbb{K}\langle x_1,\ldots,x_d \rangle$.

1. There exist $d_0 \in \mathbb{K}$, $c \in \mathbb{K}^{1 \times L}$, $b_0,\ldots,b_d \in \mathbb{K}^{L \times 1}$, and $M_0,\ldots,M_d \in \mathbb{K}^{L \times L}$, where $L$ is the McMillan degree of $\mathcal{R}$, such that

$$
\mathcal{R}(x_1,\ldots,x_d) = d_0 + c \left( M_0 - \sum_{k=1}^{d} M_k x_k \right)^{-1} \left( b_0 - \sum_{k=1}^{d} b_k x_k \right) \quad (4.12)
$$

in the free skew field and $X \in \text{dom}_n(\mathcal{R})$ if and only if $\det \left( M_0 \otimes I_n - \sum_{k=1}^{d} M_k \otimes X_k \right) \neq 0$.

2. There exist $\tilde{c} \in \mathbb{K}^{1 \times (L+1)}$, $\tilde{M}_0,\ldots,\tilde{M}_d \in \mathbb{K}^{(L+1) \times (L+1)}$, and $\tilde{b} \in \mathbb{K}^{(L+1) \times 1}$ such that

$$
\mathcal{R}(x_1,\ldots,x_d) = \tilde{c} \left( \tilde{M}_0 - \sum_{k=1}^{d} \tilde{M}_k x_k \right)^{-1} \tilde{b} \quad (4.13)
$$

in the free skew field and $X \in \text{dom}_n(\mathcal{R})$ if and only if $\det \left( \tilde{M}_0 \otimes I_n - \sum_{k=1}^{d} \tilde{M}_k \otimes X_k \right) \neq 0$.

The equality in (4.12) means that for every $n \in \mathbb{N}$ and $X \in \text{dom}_n(\mathcal{R})$, we have

$$
\mathcal{R}(X) = d_0 \otimes I_n + (c \otimes I_n) \left( M_0 \otimes I_n - \sum_{k=1}^{d} M_k \otimes X_k \right)^{-1} \left( b_0 \otimes I_n - \sum_{k=1}^{d} b_k \otimes X_k \right). \quad (4.14)
$$

The representation in (4.13) is well known in the literature; it was used extensively by Fliess and presented precisely in the papers [26,27]. We notice that such a representation does not involve a matrix centre at all, in contrary to our theory, while the domain of the rational function coincides with the invertibility set of the corresponding pencil. The representation in (4.13) is clearly not minimal, but it is also not that far from being minimal as the state space dimension differs only by 1 from the minimal dimension. On the other hand, the representation in (4.12) is minimal and good in terms of the precise domain of the function, with the disadvantage that it has not been studied earlier.
Proof. Let $Y \in \text{dom}_{s}(\mathcal{R})$ be any arbitrary point from the domain of $\mathcal{R}$ and as in Theorem 4.7, let $\mathcal{R}$ be a minimal nc Fornasini–Marchesini realization of $\mathcal{R}$ that is centred at $Y$. From the proof of Theorem 4.7 it follows that for every $X \in \text{dom}_{n}(\mathcal{R})$ we have

$$I_{s} \otimes \mathcal{R}(X) = r_{\mathcal{R}}(X) = D \otimes I_{n} + (C \otimes I_{n}) \left( I_{Ln} - \sum_{k=1}^{d} [A_{k}(I_{s}) \otimes X_{k} - A_{k}(Y_{k}) \otimes I_{n}] \right)^{-1}$$

$$\sum_{k=1}^{d} [B_{k}(I_{s}) \otimes X_{k} - B_{k}(Y_{k}) \otimes I_{n}]$$

and hence

$$\mathcal{R}(X) = (e_{1}^{T} \otimes I_{n})r_{\mathcal{R}}(X)(e_{1} \otimes I_{n})$$

$$= (e_{1}^{T}De_{1}) \otimes I_{n} + \left( [e_{1}^{T}C] \otimes I_{n} \right)$$

$$\times \left( \left[ I_{L} + \sum_{k=1}^{d} A_{k}(Y_{k}) \right] \otimes I_{n} - \sum_{k=1}^{d} A_{k}(I_{s}) \otimes X_{k} \right)^{-1}$$

$$\times \left( \left[ - \sum_{k=1}^{d} B_{k}(Y_{k})e_{1} \right] \otimes I_{n} + \sum_{k=1}^{d} [B_{k}(I_{s})e_{1}] \otimes X_{k} \right),$$

that is exactly the form in (4.14) with $d_{0} = e_{1}^{T}De_{1}$, $\zeta = e_{1}^{T}C$, $M_{0} = I_{L} + \sum_{k=1}^{d} A_{k}(Y_{k})$, $b_{0} = -\sum_{k=1}^{d} B_{k}(Y_{k})e_{1}$, and $M_{j} = A_{j}(I_{s})$ and $b_{j} = -B_{j}(I_{s})e_{1}$ for $j = 1, \ldots, d$.

Finally, similarly to the proof of Theorem 4.4, it follows that

$$\det \left( M_{0} \otimes I_{n} - \sum_{k=1}^{d} M_{k} \otimes X_{k} \right) \neq 0$$

$$\iff \det \left( I_{Ln} - \sum_{k=1}^{d} [A_{k}(I_{s}) \otimes X_{k} - A_{k}(Y_{k}) \otimes I_{n}] \right) \neq 0$$

$$\iff \det (\Lambda_{A,Y}(X \otimes I_{s})) \neq 0 \iff X \otimes I_{s} \in \Omega_{sn}(\mathcal{R})$$

$$\iff I_{s} \otimes X \in \Omega_{sn}(\mathcal{R}) \iff X \in \text{dom}_{n}(\mathcal{R}),$$

where we used the fact that $\Omega(\mathcal{R})$ is invariant under similarity and also the second part of Theorem 4.7. Finally, by letting

$$\tilde{\zeta} = [-\zeta \ d_{0}], \quad \tilde{b} = \begin{bmatrix} 0_{L \times 1} \\ 1 \end{bmatrix}, \quad \tilde{M}_{0} = \begin{bmatrix} M_{0} & b_{0} \\ 0 & 1 \end{bmatrix},$$

and $\tilde{M}_{j} = \begin{bmatrix} M_{j} & b_{j} \\ 0 & 0 \end{bmatrix}$ for every $j = 1, \ldots, d$,

one can transform from the representation in (4.12) to the one in (4.13), without changing the invertibility set of the pencils. □
Here is another immediate consequence of Theorems 4.4, 4.7, and Lemma 2.2, which states that given a nc Fornasini–Marchesini realization that is controllable and observable, it is the realization of a nc rational function if and only if its coefficients satisfy the $\mathcal{L} - \mathcal{L}A$ conditions:

**Theorem 4.9.** Let $\mathcal{R}$ be a nc Fornasini–Marchesini realization centred at $Y \in (K^{s \times s})^d$ that is described by $(L; D, C, A, B)$, and suppose $\mathcal{R}$ is both controllable and observable.

The following are equivalent:

1. There exists $\mathcal{R} \in K\langle x \rangle$ that is regular at $Y$, such that $\text{dom}_{sm}(\mathcal{R}) = \Omega_{sm}(\mathcal{R})$ and $\mathcal{R}(X) = \mathcal{R}(X)$, for every $X \in \text{dom}_{sm}(\mathcal{R})$ and $m \in \mathbb{N}$.
2. The coefficients of $\mathcal{R}$ satisfy the $\mathcal{L} - \mathcal{L}A$ conditions (cf. equations (2.3)–(2.8)).
3. There exists $\mathcal{R} \in K\langle x \rangle$ that is regular at $Y$, such that $\text{dom}_{n}(\mathcal{R}) = \{X \in (K^{n \times n})^d : I_s \otimes X \in \Omega_{sn}(\mathcal{R})\}$ and $I_s \otimes \mathcal{R}(X) = \mathcal{R}(I_s \otimes X)$, for every $X \in \text{dom}_{n}(\mathcal{R})$ and $n \in \mathbb{N}$.

**Proof.**

1 $\implies$ 2: This is an immediate corollary of Lemma 2.2, as $\mathcal{R}$ admits the realization $\mathcal{R}$.

2 $\implies$ 1: Suppose that the coefficients of $\mathcal{R}$ satisfy the $\mathcal{L} - \mathcal{L}A$ conditions. By applying Theorem 4.4, there exists $f \in K\langle x \rangle$ such that

$$\Omega_{sm}(\mathcal{R}) \subseteq \text{dom}_{sm}(f) \text{ and } \mathcal{R}(X) = f(X), \forall X \in \Omega_{sm}(\mathcal{R}), m \in \mathbb{N}.$$  
(4.15)

In particular we know that $Y \in \text{dom}_{s}(f)$, thus from Corollary 1.7 and Theorem 4.7, there exists a minimal nc Fornasini–Marchesini realization $\tilde{\mathcal{R}}$ of $f$ that is centred at $Y$, satisfying

$$\text{dom}_{sm}(f) = \Omega_{sm}(\tilde{\mathcal{R}}) \text{ and } f(X) = \tilde{\mathcal{R}}(X), \forall X \in \text{dom}_{sm}(f), m \in \mathbb{N}.$$  
(4.16)

From (4.15) and (4.16), it follows that

$$\Omega_{sm}(\mathcal{R}) \subseteq \Omega_{sm}(\tilde{\mathcal{R}}) \text{ and } \mathcal{R}(X) = \tilde{\mathcal{R}}(X), \forall X \in \Omega_{sm}(\mathcal{R}), m \in \mathbb{N}.$$  

Since $\text{Nilp}_{sm}(Y) \subseteq \Omega_{sm}(\mathcal{R})$ for every $m \in \mathbb{N}$, we obtain that

$$\mathcal{R}(X) = \tilde{\mathcal{R}}(X), \forall X \in \text{Nilp}_{sm}(Y), m \in \mathbb{N}$$

and as both $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are minimal nc Fornasini–Marchesini realizations centred at $Y$, we can use the arguments which appear in the proof of [64, Theorem 2.13] (which is based on the fact that the two realizations give the same value on the nilpotent ball around $Y$ and the uniqueness of the Taylor–Taylor coefficients of their power series expansions around $Y$), to deduce that $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are similar and hence $\Omega_{sm}(\mathcal{R}) = \Omega_{sm}(\tilde{\mathcal{R}})$ for every $m \in \mathbb{N}$. In conclusion, we get that $\text{dom}_{sm}(f) = \Omega_{sm}(\mathcal{R})$ and $f(X) = \mathcal{R}(X)$ for every $X \in \text{dom}_{sm}(f)$ and $m \in \mathbb{N}$, as needed.

1 $\implies$ 3: It follows from Theorem 4.7, as $\mathcal{R}$ admits the minimal realization $\tilde{\mathcal{R}}$.

3 $\implies$ 1: Let $n = sm$ and let $X \in \text{dom}_{sm}(\mathcal{R})$, then $I_s \otimes X \in \text{dom}_{2sm}(\mathcal{R})$ and $I_s \otimes \mathcal{R}(X) = \mathcal{R}(I_s \otimes X)$. As $(I_s \otimes X_k - I_{sm} \otimes Y_k)A_k = I_s \otimes (X_k - I_m \otimes Y_k)A_k$, ...
we have
\[ \Lambda_{\underline{A}, Y}(I_s \otimes X) = I_s \otimes \left( I_{Lm} - \sum_{k=1}^{d} (X_k - I_m \otimes Y_k)A_k \right) = I_s \otimes \Lambda_{\underline{A}, Y}(X) \]
and thus
\[ I_s \otimes X \in \Omega_{s^{2m}}(\mathcal{R}) \implies \det(\Lambda_{\underline{A}, Y}(I_s \otimes X)) \neq 0 \]
\[ \implies \det(\Lambda_{\underline{A}, Y}(X)) \neq 0 \implies X \in \Omega_{s^{2m}}(\mathcal{R}). \]
As \( \mathcal{R} \) obviously respects direct sums, \( I_s \otimes \mathcal{R}(X) = \mathcal{R}(I_s \otimes X) = I_s \otimes \mathcal{R}(X) \)
and hence \( \mathcal{R}(X) = \mathcal{R}(X) \). We showed that for every \( X \in dom_{s^{2m}}(\mathcal{R}) \), we have \( X \in \Omega_{s^{2m}}(\mathcal{R}) \) and that \( \mathcal{R}(X) = \mathcal{R}(X) \), i.e., we showed that \( \mathcal{R} \) admits the minimal nc Fornasini–Marchesini realization \( \mathcal{R} \) and thus Theorem 4.7 implies the assertion in 1.

\[ \square \]

Using our theory of minimal nc Fornasini–Marchesini realizations of nc rational functions, we now provide a short proof for the fact that the domain of regularity of a nc rational function is an upper admissible, similarity invariant nc set. Notice that proving the upper admissibility is a highly non trivial thing to do, however with our methods the proof becomes easy.

**Corollary 4.10.** If \( \mathcal{R} \in \mathbb{K}(\mathbb{Z}) \), then its domain of regularity is an upper admissible, similarity invariant nc subset of \((\mathbb{K}^d)_{nc}\).

**Proof.** Fix a point \( Y \in dom_{s\mathcal{R}}(\mathcal{R}) \) and a minimal nc Fornasini–Marchesini realization \( \mathcal{R} \) of \( \mathcal{R} \), that is centred at \( Y \); by using Corollary 1.7 for instance.

\[ \text{dom}(\mathcal{R}) \text{ is similarity invariant: Let } X \in \text{dom}_n(\mathcal{R}) \text{ and } T \in \mathbb{K}^{n \times n} \text{ be invertible. It follows from Theorem 4.7 that } I_s \otimes X \in \Omega_{s^n}(\mathcal{R}), \text{ while from (the similarity invariance of } \Omega(\mathcal{R}), \text{ proved in) Theorem 3.5 it follows that} \]
\[ I_s \otimes (T^{-1} \cdot X \cdot T) = (I_s \otimes T)^{-1} \cdot (I_s \otimes X) \cdot (I_s \otimes T) \in \Omega_{s^n}(\mathcal{R}). \]

By using Theorem 4.7 again, we get that \( T^{-1} \cdot X \cdot T \in \text{dom}_n(\mathcal{R}) \).

\[ \text{dom}(\mathcal{R}) \text{ is upper admissible: Let } X \in \text{dom}_n(\mathcal{R}), X' \in \text{dom}_m(\mathcal{R}), \text{ and } Z \in (\mathbb{K}^{n \times m})^d. \text{ Theorem 4.7 implies that } I_s \otimes X \in \Omega_{s^n}(\mathcal{R}) \text{ and } I_s \otimes X' \in \Omega_{s_m}(\mathcal{R}), \text{ while from (the similarity invariance of } \Omega(\mathcal{R}), \text{ proved in) Theorem 3.5 we have } X \otimes I_s \in \Omega_{s^n}(\mathcal{R}) \text{ and } X' \otimes I_s \in \Omega_{s_m}(\mathcal{R}). \text{ Next, from (the upper admissibility of } \Omega(\mathcal{R}), \text{ proved in) Theorem 2.6, we get} \]
\[ \left[ \begin{array}{c} X \\ Z \\ X' \end{array} \right] \otimes I_s = \left[ \begin{array}{c} X \otimes I_s \\ Z \otimes I_s \\ X' \otimes I_s \end{array} \right] \in \Omega_{s(n+m)}(\mathcal{R}) \]

and thus the similarity invariance of \( \Omega(\mathcal{R}) \) implies that \( I_s \otimes \left[ \begin{array}{c} X \\ Z \\ X' \end{array} \right] \in \Omega_{s(n+m)}(\mathcal{R}). \) Finally, Proposition 4.6 and Theorem 4.6 imply that \( \left[ \begin{array}{c} X \\ Z \\ X' \end{array} \right] \in \text{dom}_{n+m}(\mathcal{R}). \)

\[ \square \]
4.3. Evaluations Over Algebras

For the sake of completeness, one would like to get a similar result to Theorem 4.7 for evaluations w.r.t stably finite algebras. To do so, we must introduce the following definition of a matrix domain of a nc rational function w.r.t an algebra. For every \( R \in \mathbb{K}\langle x \rangle \), define the matrix domain of the function \( R \) w.r.t an algebra \( A \), also called the matrix \( A \)–domain of \( R \), by

\[
\text{dom}^\text{Mat}_A(R) := \{ a \in A^d : a \in \text{dom}^\text{Mat}_A(R) \text{ for some } 1 \times 1 \text{ matrix valued nc rational expression } R \text{ which represents } R \}.
\] (4.17)

Here \( \text{dom}^\text{Mat}_A(R) \) is defined just as in [64, Definition 1.1], using the idea of synthesis, with the only difference that the synthesis may involve matrix valued nc rational expressions; e.g.

\[
R(x_1, x_2) = \begin{pmatrix} 1 & x_1 \\ x_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

with \( \text{dom}^\text{Mat}_A(R) = \{ (a_1, a_2) \in A^2 : \begin{pmatrix} 1_A & a_1 \\ a_2 & 1_A \end{pmatrix} \text{ is invertible in } A^{2 \times 2} \} \). As every nc rational expression is a \( 1 \times 1 \) matrix valued nc rational expression, we automatically have

\[
\text{dom}_A(R) \subseteq \text{dom}^\text{Mat}_A(R).
\]

For more details on matrix valued nc rational expressions and functions, see [50].

**Definition 4.11.** Let \( A \) be a unital \( \mathbb{K} \)–algebra. We say that \( A \) has property \( \mathcal{I} \) (here \( \mathcal{I} \) stands for inversion), if for every \( n \in \mathbb{N} \) and \( \mathfrak{A} = (a_{ij})_{1 \leq i,j \leq n} \in A^{n \times n} \) that is invertible in \( A^{n \times n} \), there exist nc rational expressions \( R_{ij} \) (for \( 1 \leq i, j \leq n \)) in \( n^2 \) nc variables \( \{ x_{ij} : 1 \leq i, j \leq n \} \), such that

\[
a := (a_{11}, \ldots, a_{nn}) \in \text{dom}_A(R_{ij}) \text{ and } (\mathfrak{A}^{-1})_{i,j} = R_{ij}(a),
\]

for every \( 1 \leq i, j \leq n \).

We know certain families of algebras which satisfy Property \( \mathcal{I} \), those are when \( A \) is either commutative, a skew field or a matrix algebra \( A = \mathbb{K}^{n \times n} \) for some \( n \in \mathbb{N} \). The first case is true due to using determinants and an analog of the Cramer’s rule. The second case can be treated by using quasideterminants (as developed and discussed in [33,34]), or by using Schur complements (as in the proof of [82, Lemma 3.9] with a smart way of choosing a non-zero element of the matrix as a pivot). The third case follows directly from [82, Lemma 3.9].

It is easily seen that once \( A \) satisfies Property \( \mathcal{I} \), we have

\[
\text{dom}_A(R) = \text{dom}^\text{Mat}_A(R)
\] (4.18)

for every nc rational function \( R \). Moreover, we get the precise description of all algebras with Property \( \mathcal{I} \), by using the domain and the matrix domain of matrix valued nc rational functions w.r.t \( A \):

**Proposition 4.12.** Let \( A \) be a unital \( \mathbb{K} \)–algebra. Then \( A \) has Property \( \mathcal{I} \) if and only if \( \text{dom}_A(\mathfrak{r}) = \text{dom}^\text{Mat}_A(\mathfrak{r}) \) for every matrix valued nc rational function \( \mathfrak{r} \).
Here the $\mathcal{A}$–domain and the matrix $\mathcal{A}$–domain, respectively, of a matrix valued nc rational function $\mathbf{r} = (r_{ij})$, are defined as the intersection of all the $\mathcal{A}$–domains of the nc rational functions $r_{ij}$ and the union of all the domains of matrix valued nc rational expressions representing $\mathbf{r}$, respectively.

**Proof.** If $\mathcal{A}$ has Property $\mathcal{I}$, then it follows from the definition of the matrix $\mathcal{A}$–domain that for every matrix valued nc rational function $\mathbf{r}$ we have $\text{dom}_\mathcal{A}(\mathbf{r}) = \text{dom}_{\mathcal{A}}^{\text{Mat}}(\mathbf{r})$.

On the other hand, suppose it is true that $\text{dom}_\mathcal{A}(\mathbf{r}) = \text{dom}_{\mathcal{A}}^{\text{Mat}}(\mathbf{r})$ for every matrix valued nc rational function $\mathbf{r}$ and let $\mathfrak{A} = (a_{ij})_{1 \leq i, j \leq n} \in \mathcal{A}^{n \times n}$ be invertible in $\mathcal{A}^{n \times n}$. Consider the $n \times n$ matrix valued nc rational function

$$
\mathbf{r}(x_{11}, \ldots, x_{nn}) = \left( r_{ij}(x_{11}, \ldots, x_{nn}) \right)_{1 \leq i, j \leq n},
$$

which is well defined as the matrix $(x_{ij})_{1 \leq i, j \leq n}$ is invertible over the free skew field $\mathbb{K}\langle x_{11}, \ldots, x_{nn} \rangle$. As $\mathfrak{A}$ is invertible, we know that $\mathfrak{a} = (a_{11}, \ldots, a_{nn}) \in \text{dom}_{\mathcal{A}}^{\text{Mat}}(\mathbf{r})$, so by the assumption we get $\mathfrak{a} \in \text{dom}_\mathcal{A}(\mathbf{r})$, which implies that $\mathfrak{a} \in \text{dom}_\mathcal{A}(r_{ij})$ and $(\mathfrak{r}^\mathcal{A}(\mathfrak{a}))_{i, j} = r_{ij}^\mathcal{A}(\mathfrak{a})$ for every $1 \leq i, j \leq n$. Thus there exist nc rational expressions $R_{ij}$ such that $\mathfrak{a} \in \text{dom}_\mathcal{A}(R_{ij})$ and $r_{ij}^\mathcal{A}(\mathfrak{a}) = R_{ij}^\mathcal{A}(\mathfrak{a})$ for every $1 \leq i, j \leq n$, hence

$$(\mathfrak{A}^{-1})_{i, j} = (\mathfrak{r}^\mathcal{A}(\mathfrak{a}))_{i, j} = r_{ij}^\mathcal{A}(\mathfrak{a}) = R_{ij}^\mathcal{A}(\mathfrak{a}).$$

It is not true that every stably finite algebra satisfies Property $\mathcal{I}$, as the following example—kindly provided by I. Klep and J. Volčič—shows.

**Example 4.13.** Let $\mathfrak{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, $\mathfrak{Z} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$, and define

$$\mathcal{A} := \mathbb{K}\langle x, z \rangle / J = \left\{ p + J : p \in \mathbb{K}\langle x, z \rangle \right\},$$

where $x = (x_{11}, x_{12}, x_{21}, x_{22})$, $z = (z_{11}, z_{12}, z_{21}, z_{22})$, and $J$ is the two sided ideal in $\mathbb{K}\langle x, z \rangle$ generated by the 8 equations obtained from $\mathfrak{X} \mathfrak{Z} = \mathfrak{Z} \mathfrak{X} = I_2$. For convenience denote the 8 equations by $h_j(x, z) = 0$, for $1 \leq j \leq 8$, hence $J = \langle h_j : 1 \leq j \leq 8 \rangle$.

It is known that $\mathcal{A}$ is a free ideal ring (e.g., see [24, Theorem 5.3.9] and [17, Theorem 6.1]) and hence $\mathcal{A}$ is stably finite, as it can be embedded into a skew field which is trivially stably finite.

The natural mapping $\varphi : \mathbb{K}\langle x \rangle \to \mathcal{A}$ given by $\varphi(p) = p + J$ for every $p \in \mathbb{K}\langle x \rangle$, is an embedding from $\mathbb{K}\langle x \rangle$ into $\mathcal{A}$, i.e., $\text{ker}(\varphi) = \{0_\mathcal{A}\}$: Let $f \in \mathbb{K}\langle x \rangle$ such that $\varphi(f) = 0_\mathcal{A}$, then

$$f(x) = \sum_{i=1}^{m} p_i(x, z) h_{k_i}(x, z) q_i(x, z)$$

for some $m \in \mathbb{N}$, $p_i, q_i \in \mathbb{K}\langle x, z \rangle$ and $k_i \in \{1, \ldots, 8\}$ for any $1 \leq i \leq m$. Therefore for every $2 \times 2$ block matrix $X = [X_{ij}]_{1 \leq i, j \leq 2}$ invertible over $\mathbb{K}$,
we let $X^{-1} = [Z_{ij}]_{1 \leq i, j \leq 2}$, and obtain that

$$f(X) = \sum_{i=1}^{m} p_i(X, Z) h_{ki}(X, Z) q_i(X, Z) = 0,$$

where $X = (X_{11}, X_{12}, X_{21}, X_{22})$ and $Z = (Z_{11}, Z_{12}, Z_{21}, Z_{22})$. As $f$ vanishes on a set of matrices that is Zariski dense, it follows that $f$ vanishes on all matrices and therefore $f = 0_{\mathbb{K}(z)}$.

If $f \in \mathbb{K}(z)$ is non-constant, then $\varphi(f)$ is not invertible in $\mathcal{A}$: Suppose $f \in \mathbb{K}(z)$ is non-constant and that $\varphi(f)$ is invertible in $\mathcal{A}$. Thus there exist $g \in \mathbb{K}(z, \bar{z})$ such that $(g + J)\varphi(f) = 1_{\mathcal{A}}$ which implies that $fg - 1 \in J$. Therefore, similarly to (4.20), we get that whenever the matrix $[X_{ij}]_{1 \leq i, j \leq 2}$ is invertible over $\mathbb{K}$, so is the matrix $f(X)$. However, this is a contradiction to a claim proved in [82, page 79], which is the following: For every non-constant $f \in \mathbb{K}(z)$, there exists a tuple of matrices $X = (X_{11}, X_{12}, X_{21}, X_{22})$ for which the matrix $[X_{ij}]_{1 \leq i, j \leq 2}$ is invertible over $\mathbb{K}$, but $f(X)$ is not invertible over $\mathbb{K}$.

Finally, the algebra $\mathcal{A}$ does not satisfy Property $\mathcal{I}$: First, it is obvious that

$$\mathfrak{A} := \varphi(\mathcal{A}) = (\varphi(x_{ij}))_{1 \leq i, j \leq 2} = (x_{ij} + J)_{1 \leq i, j \leq 2}$$

is invertible in $\mathcal{A}^{2 \times 2}$, with its inverse being equal $(z_{ij} + J)_{1 \leq i, j \leq 2}$. Now, suppose $\mathcal{A}$ satisfies Property $\mathcal{I}$, therefore there exist nc rational expressions $R_{ij}$ such that $\mathfrak{a} \in \text{dom}_A(R_{ij})$ and $R_{ij}^A(\mathfrak{a}) = (\mathfrak{A}^{-1})_{ij}$ for every $1 \leq i, j \leq 2$, where $\mathfrak{a} = (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathcal{A}^4$ is given by $a_{ij} = x_{ij} + J$. If the expression $R_{ij}$ contains at least one inversion, then let us take the innermost nested inverse, say $\tilde{R}^{-1}$ with non-constant nc polynomial $\tilde{R}$, which appears in $R_{ij}$, so we must have $\mathfrak{a} \in \text{dom}_A(\tilde{R}^{-1})$. Thus $\tilde{R}^A(\mathfrak{a})$ is invertible in $\mathcal{A}$, while a simple calculation shows that $\tilde{R}^A(\mathfrak{a}) = \tilde{R}^A(\mathfrak{a}) + J = \tilde{R}(\mathfrak{a}) + J = \varphi(\tilde{R})$, hence $\varphi(\tilde{R})$ is invertible in $\mathcal{A}$ and $\tilde{R} \in \mathbb{K}(z)$ is non-constant, which is a contradiction. Therefore the expressions $R_{ij}$ must be polynomials, so we get that $(\mathfrak{A}^{-1})_{ij} = R_{ij}^A(\mathfrak{a}) = \varphi(R_{ij})$ which implies that

$$1_{\mathcal{A}^{2 \times 2}} = \mathfrak{A}^{-1} = (\varphi(x_{ij}))_{1 \leq i, j \leq 2} \cdot (\varphi(R_{ij}))_{1 \leq i, j \leq 2},$$

but as $\varphi$ is an an embedding, we must have $I_2 = (x_{ij})_{1 \leq i, j \leq 2} \cdot (R_{ij})_{1 \leq i, j \leq 2}$, which is a contradiction, since the matrix $(x_{ij})_{1 \leq i, j \leq 2}$ is not invertible over $\mathbb{K}(z)$.

In the next theorem we realize (up to a tensor product with the identity matrix) the matrix $\mathcal{A}$–domain of a nc rational function as the $\mathcal{A}$–domain of any of its minimal realizations, centred at any point from its domain of regularity.

**Theorem 4.14.** Let $\mathfrak{R} \in \mathbb{K}(\mathfrak{X})$. For every integer $s \in \mathbb{N}$, a point $Y \in \text{dom}_s(\mathfrak{R})$, a minimal nc Fornasini–Marchesini realization $\mathcal{R}$ centred at $Y$ of $\mathfrak{R}$, and a unital stably finite $\mathbb{K}$–algebra $\mathcal{A}$, we have

$$\text{dom}^\text{Mat}_A(\mathfrak{R}) = \{ \mathfrak{a} \in \mathcal{A}^d : I_s \otimes \mathfrak{a} \in \Omega_A(\mathcal{R}) \}$$

(4.21)
and $I_s \otimes R^A(a) = R^A(I_s \otimes a)$ for every $a \in \text{dom}_{A}(R_A)$.

Most of the results in [64] can be modified and proven with the setting of the matrix domain (cf. [50]), instead of the usual domain of regularity. We will skip most of the details, as the proofs are pretty much the same as the proofs in the scalar case, yet give the reader the instructions to what exact changes have to be done.

Proof. A parallel version of [64, Theorem 2.4]—that is the existence of a realization formula for any nc rational expression, that is centred at any point from $\text{dom}(R_A)$, of $R_A$ w.r.t. $A$, in the sense that $a \in \text{dom}_{A}(R_A)$ implies $I_s \otimes a \in \Omega_A(R)$ and $I_s \otimes R^A(a) = \tilde{R}^A(I_s \otimes a)$. Furthermore, it can be shown then that it holds for every minimal nc Fornasini–Marchesini realization of $R_A$ centred at any point in $\text{dom}(R_A)$. This proves the first inclusion of (4.21), that is

$$\text{dom}_{A}(\mathcal{R}) \subseteq \{ a \in A^d : I_s \otimes a \in \Omega_A(R) \}.$$ 

On the other hand, let $a \in A^d$ and suppose that $I_s \otimes a \in \Omega_A(R)$, thus the matrix

$$A_{A,Y}^A(I_s \otimes a) = I_L \otimes 1_A - \sum_{k=1}^{d} [(I_s \otimes a_k)A_k^A - A_k(Y_k) \otimes 1_A]$$

is invertible in $A^{L \times L}$. Therefore $a \in \text{dom}_{A}(\mathcal{R})$, where we recall that $r_R$ is the $s \times s$ matrix of nc rational functions given in (4.9). However, as shown in the proof of Theorem 4.4, there exists $f \in \mathbb{K} \langle \mathcal{X} \rangle$ such that $r_R = I_s \otimes f$, while in the proof of Theorem 4.7 we then showed that $f = R$, therefore $r_R = I_s \otimes R$. Finally, as $e_1^T r_R e_1 = R$, we obtain that $a \in \text{dom}_{A}(R_A)$.

As an immediate consequence of Theorem 4.14 and of (4.18), we get the following:

**Corollary 4.15.** Let $R \in \mathbb{K} \langle \mathcal{X} \rangle$. For every integer $s \in \mathbb{N}$, a point $Y \in \text{dom}_s(R)$, a minimal nc Fornasini–Marchesini realization $R$ centred at $Y$ of $R$, and a unital stably finite $\mathbb{K} -$ algebra $A$ which satisfies Property $I$, we have

$$\text{dom}_{A}(R) = \{ a \in A^d : I_s \otimes a \in \Omega_A(R) \} \quad (4.22)$$

and $I_s \otimes R^A(a) = R^A(I_s \otimes a)$ for every $a \in \text{dom}_{A}(R_A)$.

**Remark 4.16.** Let $n \in \mathbb{N}$ and consider the unital $\mathbb{K} -$algebra $A_n = \mathbb{K}^{n \times n}$. It is easily seen that $\text{dom}_{A_n}(R) = \text{dom}_n(R)$ and $R(a) = R^A_n(a)$, for every nc rational expression $R$ and $a \in \text{dom}_n(R)$. Therefore, Corollary 4.15 actually gives us another way of evaluating nc rational functions anywhere on their domain, that is by using the $A_n -$evaluation of the functions.
5. Stable Extended Domain

In this last section we use results from previous sections to show that the so-called stable extended domain of a nc rational function coincides with the domain of regularity of the function. We begin by recalling the definitions and some properties of the extended and stable extended domains of nc rational expressions and functions, see [47, 82] for more details.

Let $\Xi = (\Xi_1, \ldots, \Xi_d)$ be the $d$–tuple of $n \times n$ generic matrices, i.e., the matrices whose entries are independent commuting variables. Let $R$ be a non-degenerate nc rational expression. For every $n \in \mathbb{N}$ such that $\text{dom}_n(R) \neq \emptyset$, let $R[n] := R(\Xi)$, that is an $n \times n$ matrix whose entries are rational functions in $dn^2$ (commutative) variables. If $R_1$ and $R_2$ are $(\mathbb{K}^d)_{nc}$–evaluation equivalent nc rational expressions, then $R_1[n] = R_2[n]$ for every $n \in \mathbb{N}$ such that $\text{dom}_n(R_1), \text{dom}_n(R_2) \neq \emptyset$. Therefore, if $\mathcal{R} \in \mathbb{K} \langle \vec{\mathcal{X}} \rangle$ and $\text{dom}_n(\mathcal{R}) \neq \emptyset$, we define $\mathcal{R}[n] := R[n]$ for any $R \in \mathcal{R}$ such that $\text{dom}_n(R) \neq \emptyset$. Let the extended domain of $\mathcal{R}$ be

$$
edom(\mathcal{R}) := \bigcap_{n=1}^{\infty} \nedom_n(\mathcal{R}),$$

where $\nedom_n(\mathcal{R})$ is defined as the intersection of the domains of all entries in $\mathcal{R}[n]$, for such $n \in \mathbb{N}$ with $\text{dom}_n(\mathcal{R}) \neq \emptyset$ and $\nedom_n(\mathcal{R}) = \emptyset$ otherwise, thus $\nedom_n(\mathcal{R})$ is a Zariski open set in $(\mathbb{K}^{n \times n})^d$. This is the definition of the extended domain (at the level of $d$–tuples of $n \times n$ matrices) as it appears in [82]; in [47] there is a slightly different definition for the extended domain, but it turns out that the stable extended domains coming from these two different definitions do coincide, see Remark 5.6.

As pointed out in [82], the extended domain of regularity of a nc rational function $\mathcal{R}$ is not closed under direct sums, however this can be fixed by considering the stable extended domain of $\mathcal{R}$, that is

$$\nedom^{st}(\mathcal{R}) := \bigcap_{n=1}^{\infty} \nedom^{st}_n(\mathcal{R}),$$

where

$$\nedom^{st}_n(\mathcal{R}) := \left\{ X \in (\mathbb{K}^{n \times n})^d : I_m \otimes X \in \nedom_{mn}(\mathcal{R}) \text{ for every } m \in \mathbb{N} \right\}.$$ 

Thus, we have the relations

$$\text{dom}(\mathcal{R}) \subseteq \nedom^{st}(\mathcal{R}) \subseteq \nedom(\mathcal{R}), \quad (5.1)$$

while in [82, Theorem 3.10] it was shown that

$$\text{dom}(\mathcal{R}) = \nedom^{st}(\mathcal{R}) \quad (5.2)$$

for every $\mathcal{R} \in \mathbb{K} \langle \vec{\mathcal{X}} \rangle$ with $\text{dom}_1(\mathcal{R}) \neq \emptyset$. The proof of (5.2) is done by considering a descriptor realization and applying the ideas from [47], that is showing that both the domain and the stable extended domain of a nc rational function, that is regular at a scalar point, are equal to the invertibility set of the pencil which appears from such a minimal realization.
The purpose of this section is to generalize the equality in (5.2) for an arbitrary nc rational function $\mathcal{R} \in \mathbb{K}\langle \ell, \mathcal{X} \rangle$, by showing that $\text{edom}^{st}(\mathcal{R})$ coincides with the invertibility set of a pencil which corresponds to a minimal nc Fornasini–Marchesini realization of $\mathcal{R}$, while on the other hand the invertibility set coincides with $\text{dom}(\mathcal{R})$, as we already showed in Theorem 4.7. The generalization takes most of its ideas from [47] and [82], where the case of nc rational functions which are regular at a scalar point is considered, while invoking our results on realizations with a centre of an arbitrary size and the corresponding generalizations stated in Sect. 3.

In the spirit of [82, Lemma 3.4], we first show that the stable extended domain of any nc rational function is an upper (and lower) admissible nc set.

**Lemma 5.1.** Let $\mathcal{R} \in \mathbb{K}\langle \mathcal{X} \rangle$, then $\text{edom}^{st}(\mathcal{R})$ is a lower (and upper) admissible nc set.

**Proof.** Let $X^1 = (X^1_1, \ldots, X^1_d) \in \text{edom}^{st}_{n_1}(\mathcal{R})$ and $X^2 = (X^2_1, \ldots, X^2_d) \in \text{edom}^{st}_{n_2}(\mathcal{R})$. As $\text{edom}^{st}(\mathcal{R})$ is closed under direct sums (see [82, Proposition 3.3] for a proof), we have $X^1 \oplus X^2 \in \text{edom}^{st}_{n_1+n_2}(\mathcal{R})$ and hence $X^1 \oplus X^2 \in \text{edom}_{n_1+n_2}(\mathcal{R})$. Let $\Xi^{11} = (\Xi^{11}_1, \ldots, \Xi^{11}_d)$ be a $d$–tuple of $n_1 \times n_1$ generic matrices, $\Xi^{22} = (\Xi^{22}_1, \ldots, \Xi^{22}_d)$ be a $d$–tuple of $n_2 \times n_2$ generic matrices and $\Xi^{21} = (\Xi^{21}_1, \ldots, \Xi^{21}_d)$ be a $d$–tuple of $n_2 \times n_1$ generic matrices. Due to the simple fact that by inverting, taking products and taking sums of lower triangular block matrices, the outcome diagonal blocks only depend on the initial diagonal blocks, the denominators of the entries in the matrix

$$\mathcal{R}[n_1+n_2] \begin{bmatrix} \Xi^{11} & 0 \\ \Xi^{21} & \Xi^{22} \end{bmatrix} = \mathcal{R}[n_1+n_2] \begin{bmatrix} [\Xi^{11}_1 & 0] & \cdots & [\Xi^{11}_d & 0] \\ [\Xi^{21}_1 & \Xi^{22}_1] & \cdots & [\Xi^{21}_d & \Xi^{22}_d] \end{bmatrix},$$

are independent of (the entries of matrices in) $\Xi^{21}$, therefore $X^1 \oplus X^2 \in \text{edom}_{n_1+n_2}(\mathcal{R})$ implies that

$$\begin{bmatrix} X^1_1 & 0 \\ Z_1 & X^2_1 \\ \vdots & \vdots \\ Z_d & X^2_d \end{bmatrix} \in \text{edom}_{n_1+n_2}(\mathcal{R})$$

for every $Z = (Z_1, \ldots, Z_d) \in (\mathbb{K}^{n_2 \times n_1})^d$.

Finally, for every $\ell \geq 1$, it is easily seen that $I_\ell \otimes X^1 \in \text{edom}^{st}_{n_1+\ell}(\mathcal{R})$ and $I_\ell \otimes X^2 \in \text{edom}^{st}_{n_2+\ell}(\mathcal{R})$, these imply by the first part of the proof that

$$I_\ell \otimes X^1 \in \text{edom}_{n_1+n_2+\ell}(\mathcal{R}),$$

while on the other hand

$$I_\ell \otimes \begin{bmatrix} X^1 & 0 \\ Z & X^2 \end{bmatrix} = E(\ell, n_1+n_2)^T \begin{bmatrix} E(n_1, \ell) & 0 \\ 0 & E(n_2, \ell) \end{bmatrix},$$

$$I_\ell \otimes \begin{bmatrix} X^1 & 0 \\ Z \otimes I_\ell & X^2 \end{bmatrix} = E(\ell, n_1+n_2)^T \begin{bmatrix} E(n_1, \ell) & 0 \\ 0 & E(n_2, \ell) \end{bmatrix}^T E(\ell, n_1+n_2).$$
Using the fact that $edom(\mathcal{R})$ is closed under simultaneous conjugation, we conclude that
\[
I_\ell \otimes \left[ \begin{array}{c} X_1^1 & 0 \\ Z & X_2^1 \end{array} \right] = \left( I_\ell \otimes \left[ \begin{array}{c} X_1^1 & 0 \\ Z_1 & X_2^1 \end{array} \right], \ldots, I_\ell \otimes \left[ \begin{array}{c} X_1^d & 0 \\ Z_d & X_2^d \end{array} \right] \right)
\in \text{edom}_{(n_1+n_2)\ell}(\mathcal{R}), \forall \ell \geq 1,
\]
i.e., \[ \left[ \begin{array}{c} X_1^1 & 0 \\ Z & X_2^1 \end{array} \right] \in \text{edom}^{st}_{n_1+n_2}(\mathcal{R}) \] for every $Z \in (\mathbb{K}^{n_2 \times n_1})^d$. Similar arguments for upper block triangular matrices imply that $\text{edom}^{st}(\mathcal{R})$ is also upper admissible. □

Remark 5.2. The proof of Proposition 3.3 in [82] uses results from [40] for descriptor realizations (cf. [82, Lemma 3.2]), however if one wants to be self contained, we can do that by using similar arguments regarding nc Fornasini–Marchesini realizations (e.g., see Corollary 1.7 and Theorem 1.8, as the main use of realizations in the proof is the existence of such a realization with a good domination on the domain of the corresponding expression which represents the function).

As $\text{edom}^{st}(\mathcal{R})$ is an upper admissible nc set, we can now consider the nc difference-differential calculus of the nc function $f$, defined on each level by
\[
f[n] = \mathcal{R}[n] |_{\text{edom}^{st}_n(\mathcal{R})}, \forall n \in \mathbb{N}
\]
on its domain that is $\text{edom}^{st}(\mathcal{R})$.

Lemma 5.3. Let $\mathcal{R} \in \mathbb{K} \langle x \rangle$ and let $f$ be as in (5.3). For every two integers $k_1, k_2 \geq 0$ such that $k_1 + k_2 > 0$, $n \in \mathbb{N}$, $Y^1, \ldots, Y^{k_1+k_2} \in \text{dom}_n(\mathcal{R})$, $Z^1, \ldots, Z^{k_1+k_2} \in \mathbb{K}^{n \times n}$, and $\omega \in G_d$ of length $k_1 + k_2$, we have that
\[
\Delta^\omega f(Y^1, \ldots, Y^{k_1}, X, Y^{k_1+1}, \ldots, Y^{k_1+k_2})(Z^1, \ldots, Z^{k_1+k_2})
\]
is a matrix of rational functions which are all regular on $\text{edom}^{st}_n(\mathcal{R})$.

Proof. Let $W \in \text{edom}^{st}_n(\mathcal{R})$. By setting $Z^j = e_j \otimes Z^j$ (for $1 \leq j \leq k_1 + k_2$) and applying Lemma 5.1, we have
\[
\begin{pmatrix}
Y^1 & Z^1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & Y^{k_1} & Z^{k_1} & 0 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix} \in \text{edom}^{st}_{n(k_1+k_2+1)}(\mathcal{R}),
\]
which means that $f(\Xi) = \mathcal{R}[n(k_1+k_2+1)](\Xi)$ is a matrix of rational functions
\[
dn^2(k_1 + k_2 + 1)^2
commuting variables) which are all regular at $P_W$, where $Ξ$ is a $d$–tuple of generic matrices of size $n(k_1 + k_2 + 1) \times n(k_1 + k_2 + 1)$. In particular, we can fix all of them, except for the ones which correspond to the location of $W$ in the block matrix, to obtain that also

$$f[n(k_1 + k_2 + 1)] \begin{pmatrix} Y^1 & Z^1 & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & Y^{k_1} & Z^{k_1} & 0 & \vdots \\ \vdots & 0 & \ddots & Z^{k_1+1} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & Y^{k_1+k_2} \end{pmatrix}$$

is a matrix of rational functions (in $dn^2$ commuting variables) which are all regular at $W$, where $Ξ'$ is a $d$–tuple of generic matrices of size $n \times n$. However, due to [49, Theorem 3.11], we know that the upper most right block matrix of the matrix in (5.5) is equal to

$$\Delta_ω f(Y^1, \ldots, Y^{k_1}, Ξ', Y^{k_1+1}, \ldots, Y^{k_1+k_2})(Z^1, \ldots, Z^{k_1+k_2}),$$

hence the matrix in (5.4) consists of rational functions, all regular at $W$, as needed. \qed

Now we are ready to prove that the stable extended domain of a nc rational function coincides with its domain of regularity, as well as with the invertibility set of any of its minimal realizations centred at a point in $\text{dom}_s(\mathcal{R})$, first on all levels which are multiples of $s$.

**Theorem 5.4.** Let $\mathcal{R} \in \mathbb{K}<X>$ and let $\mathcal{R}$ be a minimal nc Fornasini–Marchesini realization of $\mathcal{R}$, centred at $Y \in \text{dom}_s(\mathcal{R})$. Then

$$\text{edom}_{sm}^st(\mathcal{R}) = Ω_{sm}(\mathcal{R}) = \text{dom}_{sm}(\mathcal{R}), \forall m \in \mathbb{N}. \quad (5.6)$$

**Proof.** As $\mathcal{R}$ is a minimal nc Fornasini–Marchesini realization of $\mathcal{R}$, it follows from Theorem 4.9 that the coefficients of $\mathcal{R}$ satisfy the $L – LA$ conditions (cf. equations (2.3)–(2.8)), while Theorem 2.6 implies that $\mathcal{R} : Ω(\mathcal{R}) \rightarrow \mathbb{K}_{nc}$ is a nc function. In the proof of Theorem 3.5 (cf. equation (3.8)), we showed that $φ(W) = 0$ for every $m \in \mathbb{N}$ and $W \in Ω_{sm}(\mathcal{R})$, where

$$φ(X) := o^{(L)}Δ_{Δ_{R}}^{(L)}(X)c^{(R)}Λ_YY(X) – I_{Lm}, \quad (5.7)$$

$o^{(L)}$ and $c^{(R)}$ are constant matrices, $Λ_YY$ is the generalized linear pencil centred at $Y$, and $Δ_{Δ_{R}}^{(L)}$ is a block matrix with entries of the form

$$Δ_ω R(I_m \otimes Y^1, \ldots, I_m \otimes Y^l, X, I_m \otimes Y^l, \ldots, I_m \otimes Y^l)(Z^1, \ldots, Z^{k_1+k_2}),$$

where $Z^1, \ldots, Z^{k_1+k_2} \in \mathcal{E}_{sm}$. However, from Theorem 4.7 we know that $\mathcal{R}(X) = \mathcal{R}(X)$ for every $X \in Ω_{sm}(\mathcal{R})$ and also that $Ω_{sm}(\mathcal{R}) = \text{dom}_{sm}(\mathcal{R}) \subseteq \mathbb{R}$.
$edom_{sm}^{st}(\mathcal{R})$, thus $\mathcal{R}(X) = \mathcal{R} | _{edom_{sm}^{st}(\mathcal{R})} (X) = f(X)$ for every $X \in \Omega_{sm}(\mathcal{R})$ and $m \in \mathbb{N}$, therefore

$$\tilde{\varphi}(X) = 0$$

(5.8)

for every $X \in \Omega_{sm}(\mathcal{R})$, where

$$\tilde{\varphi}(X) = o^{(L)}(\Delta^{(\ell)} \epsilon^{(R)} \Lambda_{A \cdot Y}(X)) - I_{Lm} = [\tilde{\varphi}_{ij}(X)]_{1 \leq i, j \leq Lm}.$$

By applying Lemma 5.3 with $n = sm$ and $Y^1 = \ldots = Y^{k_1+k_2} = I_m \otimes Y$, we know that

$$\Delta^\omega f(I_m \otimes Y, \ldots, I_m \otimes Y, X, I_m \otimes Y, \ldots, I_m \otimes Y)(Z^1, \ldots, Z^{k_1+k_2})$$

are all matrices of rational functions which are regular on $edom_{sm}^{st}(\mathcal{R})$, therefore all $\tilde{\varphi}_{ij}$ are rational functions, regular on $edom_{sm}^{st}(\mathcal{R})$, for every $1 \leq i, j \leq Lm$.

As seen in Theorem 4.7, $\Omega_{sm}(\mathcal{R}) = dom_{sm}(\mathcal{R})$, so together with the trivial inclusion $dom_{sm}(\mathcal{R}) \subseteq edom_{sm}^{st}(\mathcal{R})$, we know that $\Omega_{sm}(\mathcal{R})$ is a non-empty Zariski open set and hence Zariski dense in $edom_{sm}^{st}(\mathcal{R})$. Therefore, as all the entries $\tilde{\varphi}_{ij}$ of $\varphi$ are rational functions which are regular on $edom_{sm}^{st}(\mathcal{R})$ and the equality in (5.8) holds in $\Omega_{sm}(\mathcal{R})$—which is a Zariski dense subset of $edom_{sm}^{st}(\mathcal{R})$—it follows that (5.8) holds for every $X \in edom_{sm}^{st}(\mathcal{R})$. To conclude, we showed that if $X \in edom_{sm}^{st}(\mathcal{R})$, then $\Lambda_{A \cdot Y}(X)$ is invertible, i.e., that $edom_{sm}^{st}(\mathcal{R}) \subseteq \Omega_{sm}(\mathcal{R})$.

On the other hand, we know from Theorem 4.7 that

$$\Omega_{sm}(\mathcal{R}) = dom_{sm}(\mathcal{R}) \subseteq edom_{sm}^{st}(\mathcal{R}),$$

hence the equality in (5.6). \qed

Finally, we show that the stable extended domain of a nc rational function coincides with its domain of regularity on all levels, hence coincide. Moreover, at each level they can be identified with the domain of the realization, i.e., the invertibility set $\Omega(\mathcal{R})$, up to some tensoring with the identity matrix.

**Corollary 5.5.** Let $\mathcal{R} \in \mathbb{K}[\mathcal{X}]$ and let $\mathcal{R}$ be a minimal nc Fornasini–Marchesini realization of $\mathcal{R}$, centred at $Y \in dom_s(\mathcal{R})$. Then

$$edom_n^{st}(\mathcal{R}) = dom_n(\mathcal{R}) = \{X \in (\mathbb{K}^{n \times n})^d : I_s \otimes X \in \Omega_{sn}(\mathcal{R})\}$$

(5.9)

for every $n \in \mathbb{N}$. Moreover, we get the equality

$$dom(\mathcal{R}) = edom^{st}(\mathcal{R}).$$

(5.10)

**Proof.** From Theorem 4.7 and the relations in (5.1), we know that

$$\{X \in (\mathbb{K}^{n \times n})^d : I_s \otimes X \in \Omega_{sn}(\mathcal{R})\} = dom_n(\mathcal{R}) \subseteq edom_n^{st}(\mathcal{R}).$$

On the other hand, if $X \in edom_n^{st}(\mathcal{R})$, then as $edom^{st}(\mathcal{R})$ is closed under direct sums $I_s \otimes X \in edom_{sn}^{st}(\mathcal{R})$, while Theorem 5.4 implies $I_s \otimes X \in \Omega_{sn}(\mathcal{R})$, as needed. Finally, it follows that

$$edom^{st}(\mathcal{R}) = \bigsqcap_{n \in \mathbb{N}} edom_n^{st}(\mathcal{R}) = \bigsqcap_{n \in \mathbb{N}} dom_n(\mathcal{R}) = dom(\mathcal{R}).$$

\qed
Remark 5.6. The definition of the extended domain in [47] is slightly different than the one in [82]. By the discussion in [47], the only case in which the extended domains might be different is when $K$ is a finite field, $\text{dom}_n(\mathcal{R}) = \emptyset$ but $\mathcal{R}$ can be evaluated on $d-$tuples of $n \times n$ generic matrices. This issue is resolved when moving to the stable extended domain, by taking amplifications as described in the proof of Corollary 5.5. The point is that if one follows the definition of the extended domain from [47], then the corresponding stable extended domain contains the domain of regularity and is closed under amplifications, so similarly to Corollary 5.5 we know that it coincides with the domain of regularity of the function.

Acknowledgements

The authors would like to thank Dmitry Kalyuzhnyi-Verbovetskyi, Roland Speicher, and Juri Volčič for their helpful comments and discussions. A special gratitude is due to Igor Klep and Juri Volčič for the details of Example 4.13. The idea of developing realization theory around a matrix point, along the lines presented here, was first explored by the second author at a talk at MFO workshop on free probability theory in 2015 [48]; the second author would like to thank MFO and the workshop organizers for their hospitality. Finally, we would like to thank the anonymous referee for helpful comment.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Agler, J., McCarthy, J.E.: Global holomorphic functions in several non-commuting variables. Can. J. Math. 67(2), 241–285 (2015)
[2] Agler, J., McCarthy, J.E.: Pick interpolation for free holomorphic functions. Am. J. Math. 137(6), 1685–1701 (2015)
[3] Agler, J., McCarthy, J.E.: Aspects of non-commutative function theory. Concrete Oper. 3(1), 15–24 (2016)
[4] Alpay, D., Kaliuzhnyi-Verbovetskyi, D.S.: On the intersection of null spaces for matrix substitutions in a non-commutative rational formal power series. C.R. Math. 339(8), 533–538 (2004)
[5] Alpay, D., Kaliuzhnyi-Verbovetskyi, D. S.: Matrix $J$-unitary noncommutative rational formal power series. Operator Theory: Advances and Applications 161: pp. 49–113, Birkhäuser-Verlag, Basel (2005)
[6] Amitsur, S.A.: Rational identities and applications to algebra and geometry. J. Algebra 3(3), 304–359 (1966)
[7] Ball, J.A., Groenewald, G., Malakorn, T.: Structured noncommutative multidimensional linear systems. SIAM J. Control. Optim. 44(4), 1474–1528 (2005)

[8] Ball, J.A., Groenewald, G., Malakorn, T.: Conservative structured noncommutative multidimensional linear systems. Operator Theory: Advances and Applications 161: pp. 179–223, Birkhäuser, Basel (2006)

[9] Ball, J.A., Groenewald, G., Malakorn, T.: Bounded real lemma for structured noncommutative multidimensional linear systems and robust control. Multidimension. Syst. Signal Process. 17(2–3), 119–150 (2006)

[10] Ball, J.A., Kaliuzhnyi-Verbovetskyi, D.S.: Conservative dilations of dissipative multidimensional systems: the commutative and non-commutative settings. Multidimension. Syst. Signal Process. 19(1), 79–122 (2008)

[11] Ball, J.A., Vinnikov, V.: Lax–Phillips scattering and conservative linear systems: a Cuntz-algebra multidimensional setting. Memoirs of the American Mathematical Society 178(837) (2005)

[12] Bart, H., Golberg, I., Kaashoek, M.A.: Minimal factorization of matrix and operator functions. Birkhäuser-Verlag, Basel, Operator Theory, Advances and Applications (1979)

[13] Beck, C.: On formal power series representations for uncertain systems. IEEE Trans. Autom. Control 46(2), 314–319 (2001)

[14] Beck, C.L., Doyle, J., Glover, K.: Model reduction of multidimensional and uncertain systems. IEEE Trans. Autom. Control 41(10), 1466–1477 (1996)

[15] Belinschi, S.T., Mai, T., Speicher, R.: Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem. J. für die reine und angewandte Math. (Crelles Journal) 732, 21–53 (2017)

[16] Bergman, G.M.: Skew fields of noncommutative rational functions, after Amitsur. S’eminaire Schützenberger–Lentin–Nivat, Année 16, Paris (1969/70)

[17] Bergman, G.M.: Coproducts and some universal ring constructions. Trans. Am. Math. Soc. 200, 33–88 (1974)

[18] Berstel, J., Reutenauer, C.: Rational Series and their Languages. Monographs on Theoretical Computer Science 12, Springer, Berlin (1988)

[19] Camino, J.F., Helton, J.W., Skelton, R.E., Ye, J.: Matrix inequalities: a symbolic procedure to determine convexity automatically. Integr. Eqn. Oper. Theory 46(4), 399–454 (2003)

[20] Cohn, P.M.: On the embedding of rings in skew fields. Proc. Lond. Math. Soc. 3(1), 511–530 (1961)

[21] Cohn, P.M.: The embedding of firs in skew fields. Proc. Lond. Math. Soc. 3(2), 193–213 (1971)

[22] Cohn, P.M.: Universal skew fields of fractions. Sympos. Math. 8, 135–148 (1972)

[23] Cohn, P.M.: Free Rings and their Relations. London Mathematical Society Monographs 2. Academic Press, London (1971)

[24] Cohn, P.M.: Skew fields. Theory of General Division Rings. Encyclopedia of Mathematics and its Applications 57. Cambridge University Press, Cambridge (1995)

[25] Cohn, P.M.: Free Ideal Rings and Localization in General Rings. New Mathematical Monographs 3. Cambridge University Press, Cambridge (2006)
[26] Cohn, P.M., Reutenauer, C.: A normal form in free fields. Can. J. Math. 46(3), 517–531 (1994)
[27] Cohn, P.M., Reutenauer, C.: On the construction of the free field. Int. J. Algebra Comput. 9(3–4), 307–323 (1999)
[28] Effros, E.G.: Advances in quantized functional analysis, pp. 906–916. Proceedings of the International Congress of Mathematicians, Berkeley (1986)
[29] Fliess, M.: Sur le plongement de l’algèbre des séries rationnelles non commutatives dans un corps gauche. Comptes Rendus de l’Académie des Sciences Ser. A 271: pp. 926–927 (1970)
[30] Fliess, M.: Matrices de Hankel. J. de Math. Pures et Appl. 53(9), 197–222 (1974)
[31] Fliess, M.: Sur divers produits de séries formelles. Bull. Soc. Math. France 102, 181–191 (1974)
[32] Galkowski, K.: Minimal state-space realization for a class of linear, discrete, nD. SISO systems. Int. J. Control 74(13), 1279–1294 (2001)
[33] Gelfand, I., Gelfand, S., Retakh, V., Wilson, R.L.: Quasideterminants. Adv. Math. 193(1), 56–141 (2005)
[34] Gelfand, I., Retakh, V.: Determinants of matrices over noncommutative rings. Funct. Anal. Appl. 25(2), 91–102 (1991)
[35] Helton, J.W.: “Positive” noncommutative polynomials are sums of squares. Ann. Math. 156(2), 675–694 (2002)
[36] Helton, J.W.: Manipulating Matrix Inequalities Automatically. Mathematical Systems Theory in Biology, Communications, Computation, and Finance, Springer, New York, pp. 237–256 (2003)
[37] Helton, J.W., Klep, I., McCullough, S.: Analytic mappings between noncommutative pencil balls. J. Math. Anal. Appl. 376(2), 407–428 (2011)
[38] Helton, J.W., Klep, I., McCullough, S.: Proper analytic free maps. J. Funct. Anal. 260(5), 1476–1490 (2011)
[39] Helton, J.W., Klep, I., McCullough, S., Volčič, J.: Noncommutative polynomials describing convex sets. Foundations of Computational Mathematics, pp. 1–37 (2020)
[40] Helton, J.W., Mai, T., Speicher, R.: Applications of realizations (aka linearizations) to free probability. J. Funct. Anal. 274(1), 1–79 (2018)
[41] Helton, J.W., McCullough, S.: Every convex free basic semi-algebraic set has an LMI representation. Ann. Math. 176(2), 979–1013 (2012)
[42] Helton, J.W., McCullough, S.: Free convex sets defined by rational expressions have LMI representations. J. Convex Anal. 21(2), 425–448 (2014)
[43] Helton, J.W., McCullough, S., Vinnikov, V.: Noncommutative convexity arises from linear matrix inequalities. J. Funct. Anal. 240(1), 105–191 (2006)
[44] Horn, R.A., Johnson, C.R.: Topics in matrix analysis. Corrected reprint of the 1991 original, Cambridge University Press (1994)
[45] Hughes, I.: Division rings of fractions for group rings. Commun. Pure Appl. Math. 23(2), 181–188 (1970)
[46] Kaczorek, T.: Two Dimensional Linear Systems. Lecture Notes in Control and Information Sciences, vol. 68. Springer, Berlin (1985)
[47] Kaliuzhnyi-Verbovetskyi, D.S., Vinnikov, V.: Foundations of Free Noncommutative Function Theory. Mathematical Surveys and Monographs 199, American Mathematical Society, Providence, RI (2014)

[48] Kaliuzhnyi-Verbovetskyi, D.S., Vinnikov, V.: Realization theory for noncommutative rational functions around a matrix point. Oberwolfach Reports, Vol. 12, Issue 2: pp. 1600–1603 (Report No. 28/2015, https://doi.org/10.4171/OWR/2015/28, Free Probability Theory) (2015)

[49] Kaliuzhnyi-Verbovetskyi, D.S., Vinnikov, V.: Noncommutative rational functions, their difference-differential calculus and realizations. Multidimensional. Syst. Signal Process. 23(1–2), 49–77 (2012)

[50] Kaliuzhnyi-Verbovetskyi, D.S., Vinnikov, V.: Singularities of rational functions and minimal factorizations: the noncommutative and the commutative settings. Linear Algebra Appl. 430(4), 869–889 (2009)

[51] Kalman, R.E., Arbib, M.A., Falb, P.L.: Topics in Mathematical Systems Theory. McGraw Hill, New York (1969)

[52] Kleene, S.C.: Representation of events in nerve nets and finite automata. Automata Studies, Annals of Mathematics Studies 34: pp. 3–41, Princeton University Press, Princeton (1956)

[53] Klep, I., Vinnikov, V., Volčič, J.: Local theory of free noncommutative functions: germs, meromorphic functions and Hermite interpolation. Trans. Am. Math. Soc. 373, 5587–5625 (2020)

[54] Lewin, J.: Fields of fractions for group algebras of free groups. Trans. Am. Math. Soc. 192, 339–346 (1974)

[55] Lichtman, A.I.: On universal fields of fractions for free algebras. J. Algebra 231(2), 652–676 (2000)

[56] Linnell, P.A.: Division rings and group von Neumann algebras. Forum Math. 5(5), 561–576 (1993)

[57] Linnell, P.A.: Noncommutative localization in group rings. Non-commutative localization in algebra and topology, vol. 330 of London Mathematical Society Lecture Note Series, pp. 40–59, Cambridge University Press, Cambridge (2006)

[58] Lu, W.M., Zhou, K., Doyle, J.C.: Stabilization of uncertain linear systems: an LFT approach. IEEE Trans. Autom. Control 41(1), 50–65 (1996)

[59] Muhly, P.S., Solel, B.: Progress in noncommutative function theory. Sci. China Math. 54(11), 2275–2294 (2011)

[60] Nemirovskii, A.: Advances in convex optimization: conic programming. Plenary Lecture, International Congress of Mathematicians 1: pp. 413–444, Madrid, Spain (2006)

[61] Nesterov, Y., Nemirovskii, A.: Interior-Point Polynomial Algorithms in Convex Programming. Studies in Applied Mathematics 13, Philadelphia, PA (1994)

[62] Popescu, G.: Free holomorphic functions on the unit ball of $B(H)^n$. J. Funct. Anal. 241(1), 268–333 (2006)

[63] Popescu, G.: Free holomorphic automorphisms of the unit ball of $B(H)^n$. J. für die reine und angewandte Math. (Crelles Journal) 638, 119–168 (2010)

[64] Porat, M., Vinnikov, V.: Realizations of non-commutative rational functions around a matrix centre, I: Synthesis, minimal realizations and evaluation on stably finite algebras. ArXiv preprint, arXiv:1905.11304 (2019)
[65] Porat, M., Vinnikov, V.: Realizations of Non-commutative Rational Functions Around a Matrix Center. Kronecker-Fliess Theorem and the Free Skew Field. To appear, III, Functional Models (2020)

[66] Reutenauer, C.: Malcev-Neumann series and the free field. Expo. Math. 17(5), 469–478 (1999)

[67] Rowen, L.H.: Polynomial Identities in Ring Theory. Pure and Applied Mathematics 84, Academic Press, New York, London (1980)

[68] Skelton, R.E., Iwasaki, T., Grigoriadis, K.M.: A Unified Algebraic Approach to Linear Control Design. Taylor & Francis (1997)

[69] Schrempf, K.: A standard form in (some) free fields: How to construct minimal linear representations. arXiv preprint, arXiv:1803.10627 (2018)

[70] Schrempf, K.: Free fractions: An invitation to (applied) free fields. arXiv preprint, arXiv:1809.05425 (2018)

[71] Schrempf, K.: Linearizing the word problem in (some) free fields. Int. J. Algebra Comput. 28(7), 1209–1230 (2018)

[72] Schrempf, K.: On the factorization of non-commutative polynomials (in free associative algebras). J. Symb. Comput. 94, 126–148 (2019)

[73] Schützenberger, M.P.: On the definition of a family of automata. Inf. Control 4(2–3), 245–270 (1961)

[74] Schützenberger, M.P.: Certain elementary families of automata. In: Proceedings of the Symposium on Mathematical Theory of Automata (New York, 1962), pp. 139–153, Polytechnic Press of Polytechnic Institute of Brooklyn, Brooklyn, New York (1963)

[75] Speicher, R.: Polynomials in asymptotically free random matrices. Acta Phys. Pol. B 46(9), 1611–1624 (2015)

[76] Speicher, R.: Free Probability Theory. The Oxford Handbook of Random Matrix Theory, pp. 452–470, Oxford University Press, Oxford (2011)

[77] Taylor, J.L.: A general framework for a multi-operator functional calculus. Adv. Math. 9, 183–252 (1972)

[78] Taylor, J.L.: Functions of several noncommuting variables. Bull. Am. Math. Soc. 79(1), 1–34 (1973)

[79] Voiculescu, D.V.: Free analysis questions I: duality transform for the coalgebra of $\partial_X$. Int. Math. Res. Not. 16, 793–822 (2004)

[80] Voiculescu, D.V.: Free analysis questions II: the Grassmannian completion and the series expansion at the origin. J. für die reine und Angew. Math. (Crelles Journal) 645, 155–236 (2010)

[81] Voiculescu, D.V., Dykema, K.J., Nica, A.: Free Random Variables. A Noncommutative Probability Approach to free Products with Applications to Random Matrices, Operator Algebras and Harmonic Analysis on Free Groups. CRM Monograph Series 1, American Mathematical Society, Providence, RI (1992)

[82] Volčič, J.: On domains of noncommutative rational functions. Linear Algebra Appl. 516, 69–81 (2017)

[83] Volčič, J.: Matrix coefficient realization theory of noncommutative rational functions. J. Algebra 499, 397–437 (2018)
