The central result of our letter [1] is that (i) the capture rate $\eta$ of Levy walks with Poisson distributed targets goes linearly with the target density $\rho$ for all values of the Levy exponent $\alpha$ in space dimension $d \geq 2$. This contradicts results in [2], and has important consequences: (ii) the optimal gain $\eta_{\max}/\eta$ achieved by varying $\alpha$ is bounded in the limit $\rho \to 0$ so that tuning $\alpha$ yields a marginal gain ; (iii) the optimum is realized for a range of $\alpha$ and is controlled by the model-dependent parameters $a$ (detection radius), $l_c$ (restarting distance) and $s$ (scale parameter) (Fig.1).

First, and most importantly, [3] states that our main result (i) is correct, thereby acknowledging that the determination of $\eta$ in [2] is wrong.

Second, [3] opposes that claim (iii) is not new, because earlier publications reported that optimal Levy strategies can be realized for $\alpha \neq 1$. We did acknowledge such observations in [1], where we in fact show that they result from the linear scaling of $\eta$ with $\rho$ for $d \geq 2$ ; this is novel.

Last, [3] disputes claim (ii). Technically, claim (ii) is correct and by no means compromised by [3]. It states that for fixed values of $s, l_c$, the optimal gain $\eta_{\max}/\eta$ is bounded when $\rho \to 0$. This comes from the linear scaling of $\eta$ with $\rho$ (Eq. 5 in [1], whose validity is acknowledged by [3]), and is independent of any determination of $K_d(\alpha, s, l_c)$. In [1], Eq.(3) is used only to derive the scaling of $\eta$ with $\rho$ ; we make no prediction regarding $K_d(\alpha, s, l_c)$. Attempting to deduce $K_d(\alpha, s, l_c)$ from Eq.(3) is the initiative of [3], not ours ; in fact, we agree that Eq.(3) is unsuitable to study $l_c \to a$, which falls out of the validity regime given in [1]. This is certainly not a problem in [1] as argued by [2], simply because we nowhere aimed at determining $K_d(\alpha, s, l_c)$.

Finally, the only aspect in (ii) that [3] disputes is rhetorical – our qualification of the optimum as marginal. The comment is based only on the analysis of the singular limit $s \to 0$ and $l_c \to a$, which can indeed lead to arbitrary large values of $\eta_{\max}/\eta$ for $\alpha \to 1$. This is actually a mere 1d limit (Fig. 1), as noted in [1] ; it is thus expected, and consistent with our findings, to recover the 1d optimum. This by no means contradicts claim (ii) of boundedness when $\rho \to 0$ for fixed $s, l_c$. Last, we summarize the conditions of optimality (CO) of inverse square Levy walks for $d \geq 2$ :

- upon each capture event, a spherical target reappears infinitely fast at the same position,
- the searcher starts the new search infinitely close to the target boundary $(l_c - a \ll a)$,
- the typical scale of its displacements is infinitely smaller than the target $(s \ll a)$.

If any of these conditions is not met, $\alpha = 1$ is not optimal. Given that $s$ and $l_c$ are system-dependent parameters with arbitrary values, CO are generically not met and our conclusion that inverse square Levy walks are not optimal is justified. Additionally, if $l_c, s$ are allowed to vary, as done in [3], the obvious optimal strategy is $l_c = a$, leading to immediate recapture of the same target ; the limit $l_c \to a^+$ in CO is thus artificial.

To our knowledge, CO have never been stated explicitly, nor verified in any experimental system. Given that CO are a mere 1d limit of the problem, the claim that [3] restores the optimality of $\alpha = 1$ for $d \geq 2$ is unfounded ; given that [3] acknowledges that the scaling of $\eta$ with $\rho$ is wrong in [2], stating that [3] restores the validity of [2] is also unfounded.

[1] N. Levernier, J. Téxtor, O. Bénichou, and R. Voituriez, Physical Review Letters 124, 080601 (2020).
[2] G. M. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, Nature 401, 911 (1999).
[3] S. V. Buldyrev et al., Phys Rev Lett (2020).
[4] N. Levernier, O. Bénichou, T. Guérin, and R. Voituriez, Physical Review E 98, 022125 (2018).

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