De Sitter Breaking through Infrared Divergences

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ABSTRACT

Just because the propagator of some field obeys a de Sitter invariant equation does not mean it possesses a de Sitter invariant solution. The classic example is the propagator of a massless, minimally coupled scalar. We show that the same thing happens for massive scalars with $M_S^2 < 0$, and for massive transverse vectors with $M_V^2 \leq -2(D-1)H^2$, where $D$ is the dimension of spacetime and $H$ is the Hubble parameter. Although all masses in these ranges give infrared divergent mode sums, using dimensional regularization (or any other analytic continuation technique) to define the mode sums leads to the incorrect conclusion that de Sitter invariant solutions exist except at discrete values of the masses.

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1 Introduction

Dimensional regularization \[1\] is a wonderful tool for perturbative computations in quantum field theory but, like all tools, it can be misused. One way for this to happen involves the infamous automatic subtraction: dimensional regularization registers only logarithmic divergences; it sets power law divergences to zero. When dimensional regularization is employed to control an ultraviolet divergence the automatic subtraction is no problem because the right way to deal with ultraviolet divergences is by subtracting them with counterterms. In that case the automatic subtraction merely spares one the labor of explicitly working out the counterterms required to subtract off any power law divergences. However, dimensional regularization can also be used to control infrared divergences, and this leads to errors if one fails to recall that the technique automatically sets power law divergences to zero.

The appearance of an infrared divergence in the answer to a quantum field theoretic question means that something about the question is unphysical. The right thing to do in that case is to revise the question so as to make it more physical. The classic example of this is the infrared divergences one encounters when computing exclusive scattering amplitudes in quantum electrodynamics. Because the photon is massless and all real detectors have finite resolution, one can never exclude the possibility that the final state contains an extra, very low energy photon. Including arbitrary numbers of soft photons in the final state eliminates the infrared divergence \[2\]. Another example is when the vacuum decays, as it does for a massless scalar with a cubic interaction. Veneziano showed that even inclusive scattering amplitudes harbor infrared divergences for this system \[3\]. The problem in this case is that the free scalar vacuum centered around φ = 0 decays, so one cannot assume the final state is even stationary, much less centered about φ = 0. Infrared finite results can be obtained by instead releasing the system at a finite time in a prepared state centered around φ = 0 and then following its evolution \[4\].

A peculiar situation arises in curved backgrounds when parameters of the background geometry change the infrared properties of the particle. For example, consider a homogeneous, isotropic and spatially flat background,

\[ ds^2 = -dt^2 + a^2(t)\mathbf{d}x \cdot \mathbf{d}x \, . \]  \hspace{1cm} (1)
The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ are,

$$H \equiv \frac{\dot{a}}{a}, \quad q \equiv -1 - \frac{\dot{H}}{H^2}.$$ (2)

The spatial plane wave mode functions for a massless, minimally coupled scalar are quite complicated for general $q(t)$ [5] but they take a simple form when $q(t)$ is any constant $q_1$,

$$u(t, k) = \sqrt{-\frac{\pi}{4q_1 Ha}} a^{1 - \frac{D}{2}} H^{(1)}(-k/q_1 Ha) \quad \text{where} \quad \nu = \frac{1}{2} - \left(\frac{D - 2}{2q_1}\right).$$ (3)

The naive mode sum for the scalar propagator between $(t, \vec{x})$ and $(t', \vec{x}')$ is,

$$\int \frac{q^{D-1} k}{(2\pi)^{D-1}} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \left\{ \theta(t-t')u(t, k)u^*(t', k) + \theta(t'-t)u(t, k)u^*(t', k) \right\}.$$ (4)

From the small $k$ behavior of the mode functions,

$$-1 \leq q_1 < 0 \implies uu^* = \frac{4^\nu \Gamma^2(\nu)}{\Gamma(-q_1)1 - 2
u} \frac{a^{2\nu} - \nu - 1}{\nu} \times \frac{1}{k^{2\nu}} \left\{ 1 + O(k^2) \right\},$$ (5)

$$0 < q_1 \implies uu^* = \frac{4^{-\nu} \Gamma^2(-\nu)}{\Gamma(-q_1)1 + 2
u} \frac{a^{2\nu} - \nu - 1}{\nu} \times \frac{1}{k^{-2\nu}} \left\{ 1 + O(k^2) \right\},$$ (6)

one can easily recognize that the naive mode sum (4) possesses infrared divergences for all values of $q_1$ in the range [5, 7],

$$-1 \leq q_1 \leq \left(\frac{D - 2}{D}\right) \implies \text{Infrared Divergences}.$$ (7)

However, most of these infrared divergences are of the power law type which are set to zero by dimensional regularization or any other analytic continuation method. One only encounters logarithmic infrared divergences (either in the leading term or one of the $k^{\pm 2\nu + 2N}$ corrections) for the cases [7],

$$-1 \leq q_1 < 0 \implies q_1 = -\left(\frac{D - 2}{D - 2 + 2N}\right) \text{ for Log Divergence},$$ (8)

$$0 < q_1 \implies q_1 = \left(\frac{D - 2}{D + 2N}\right) \text{ for Log Divergence}.$$ (9)

If one were to incorrectly employ dimensional regularization (or any other analytic continuation method) to control the infrared divergence of the naive
mode sum (4) it would seem to give a finite result for the propagator except at the discrete $q_1$ values (8-9) for which there happens to be a logarithmic infrared divergence [8, 9]. In fact the mode sum is infrared divergent for all values of $q_1$ in the range (7). The right way of dealing with these infrared divergences is not to subtract them but rather to correct whatever unphysical assumption about the system produced them. In this case the problem derives from incorrectly assuming that all modes of the initial state are in coherent Bunch-Davies vacuum, even the ones with super-horizon wavelengths which cannot be controlled by a local observer. The system can be made infrared finite either by starting with the super-horizon modes in some less singular vacuum [10], or else by working on a spatially compact manifold which has no initially super-horizon modes [11]. Both of these procedures augment the naive propagator (4) with extra terms which can mediate important effects [12].

The purpose of this paper is to point out that a similar situation exists on de Sitter background ($a(t) = e^{Ht}$, with $H$ constant) if one considers different values of the mass-squared $M^2$. This has important consequences for the construction of de Sitter invariant propagators. We show that the formally de Sitter invariant mode sums are infrared singular for minimally coupled scalars with $M^2_S \leq 0$, and for transverse vectors with $M^2_V \leq -2(\mathcal{D}-1)H^2$. However, one only encounters logarithmic infrared divergences for the special cases,

$$M^2_S = -N(D-1+N)H^2 \quad \text{and} \quad M^2_V = -(N+2)(D-1+N)H^2. \quad (10)$$

Using dimensional regularization (or any other analytic continuation technique) to evaluate the naive mode sums leads to the incorrect conclusion that a de Sitter invariant propagator exists, except for the “problematic” cases (10). The correct result is rather that the naive mode sums diverge for any scalar with $M^2_S \leq 0$ and for any transverse vector with $M^2_V \leq -2(D-1)H^2$. We make the system infrared finite by working on the compact spatial manifold $T^{D-1}$, and we obtain explicit results for the leading infrared corrections to the propagator. These corrections break de Sitter invariance, just as has long been known occur for the massless, minimally coupled scalar [13].

In section 2 we review our conventions for the de Sitter geometry. Section 3 treats minimally coupled scalars, and section 4 is devoted to transverse vectors. Section 5 summarizes and discusses our results.
2 The de Sitter Geometry

We work on the open conformal coordinate submanifold of $D$-dimensional de Sitter space. A spacetime point $x^{\mu}$ can be decomposed into its temporal $(x^{0})$ and spatial $x^{i}$ components which take values in the ranges,

$$-\infty < x^{0} < 0 \quad \text{and} \quad -\infty < x^{i} < +\infty .$$  \hspace{1cm} (11)

In these coordinates the invariant element is,

$$ds^{2} \equiv g_{\mu\nu}dx^{\mu}dx^{\nu} = a^{2}_{x}\eta_{\mu\nu}dx^{\mu}dx^{\nu},$$ \hspace{1cm} (12)

where $\eta_{\mu\nu}$ is the Lorentz metric and $a_{x} = -1/Hx^{0}$ is the scale factor. The parameter $H$ is known as the “Hubble constant”.

Most of the various propagators between points $x^{\mu}$ and $z^{\mu}$ can be expressed in terms of the de Sitter length function $y(x; z)$,

$$y(x; z) \equiv a_{x}a_{z}H^{2}\left[\|\vec{x} - \vec{z}\|^{2} - (|x^{0} - z^{0}| - i\epsilon)^{2}\right].$$ \hspace{1cm} (13)

Except for the factor of $i\epsilon$ (whose purpose is to enforce Feynman boundary conditions) the function $y(x; z)$ is closely related to the invariant length $\ell(x; z)$ from $x^{\mu}$ to $z^{\mu}$,

$$y(x; z) = 4\sin^{2}\left(\frac{1}{2}H\ell(x; z)\right).$$ \hspace{1cm} (14)

Because $y(x; z)$ is a de Sitter invariant, so too are covariant derivatives of it. With the metrics $g_{\mu\nu}(x)$ and $g_{\mu\nu}(z)$, the first three derivatives of $y(x; z)$ furnish a convenient basis of de Sitter invariant bi-tensors [14],

$$\frac{\partial y(x; z)}{\partial x^{\mu}} = H a_{x} \left(y\delta_{\mu}^{0}+2a_{z}H\Delta x_{\mu}\right),$$ \hspace{1cm} (15)

$$\frac{\partial y(x; z)}{\partial z^{\nu}} = H a_{z} \left(y\delta_{\nu}^{0}-2a_{x}H\Delta x_{\nu}\right),$$ \hspace{1cm} (16)

$$\frac{\partial^{2} y(x; z)}{\partial x^{\mu}\partial z^{\nu}} = H^{2} a_{x}a_{z} \left(y\delta_{\mu}^{0}\delta_{\nu}^{0}+2a_{z}H\Delta x_{\mu}\delta_{\nu}^{0}-2a_{x}\delta_{\mu}^{0}H\Delta x_{\nu}-2\eta_{\mu\nu}\right).$$ \hspace{1cm} (17)

Here and subsequently $\Delta x_{\mu} \equiv \eta_{\mu\nu}(x-z)^{\nu}$. Acting more covariant derivatives just gives back the basis tensors, for example [14],

$$\frac{D^2 y(x; z)}{Dx^{\mu}Dx^{\nu}} = H^{2}(2-y)g_{\mu\nu}(x) \quad , \quad \frac{D^2 y(x; z)}{Dz^{\mu}Dz^{\nu}} = H^{2}(2-y)g_{\mu\nu}(z).$$ \hspace{1cm} (18)
Similarly, the contraction of any pair of the basis tensors produces more basis tensors \[14\],

\[
g^{\mu\nu}(x) \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} = H^2(4y - y^2) = g^{\mu\nu}(z) \frac{\partial y}{\partial z^\mu} \frac{\partial y}{\partial z^\nu}, \tag{19}
\]

\[
g^{\mu\nu}(x) \frac{\partial^2 y}{\partial x^\mu \partial x^\nu} = H^2(2 - y) \frac{\partial y}{\partial z^\sigma}, \tag{20}
\]

\[
g^{\rho\sigma}(z) \frac{\partial y}{\partial z^\rho} \frac{\partial^2 y}{\partial x^\mu \partial z^\sigma} = H^2(2 - y) \frac{\partial y}{\partial x^\mu}, \tag{21}
\]

\[
g^{\mu\nu}(x) \frac{\partial^2 y}{\partial x^\mu \partial z^\rho} \frac{\partial^2 y}{\partial x^\nu \partial z^\sigma} = 4H^4 g^{\rho\sigma}(z) - H^2 \frac{\partial y}{\partial z^\rho} \frac{\partial y}{\partial z^\sigma}, \tag{22}
\]

\[
g^{\rho\sigma}(z) \frac{\partial^2 y}{\partial x^\mu \partial z^\rho} \frac{\partial^2 y}{\partial x^\nu \partial z^\sigma} = 4H^4 g^{\mu\nu}(x) - H^2 \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu}. \tag{23}
\]

## 3 Scalars

The purpose of this section is to demonstrate that the de Sitter invariant propagator equation has no de Sitter invariant solution for a scalar with \(M_s^2 \leq 0\), and then to construct the leading de Sitter breaking correction terms. We begin by giving the propagator equation and the plane wave mode functions for Bunch-Davies vacuum. The associated mode sum can be evaluated formally to give a de Sitter invariant hypergeometric function which diverges for isolated values of the scalar mass. By studying the infrared behavior of the mode sum we show that these isolated values are those for which one of the infrared divergences, which are actually present for all \(M_s^2 \leq 0\), happens to become logarithmic. Except at the isolated values of \(M_s^2\), all the infrared divergences are of the power law type which dimensional regularization (or any analytic continuation method) incorrectly sets to zero. We fix the problem by working on a compact spatial manifold which has no initially super-horizon modes, and we derive the leading infrared corrections.

The propagator of a minimally coupled scalar with mass \(M_S\) obeys the equation,

\[
\sqrt{-g(x)} \left[ \square_x - M_s^2 \right] i\Delta(x; z) = i\delta^D(x-z). \tag{24}
\]

The plane wave mode function corresponding to Bunch-Davies vacuum is,

\[
u(x^0, k) \equiv \sqrt{\frac{\pi}{4H}} a_x^{-\frac{D-1}{2}} H^{(1)}_\nu(-kx^0) \quad \text{where} \quad \nu = \sqrt{\left(\frac{D-1}{2}\right)^2 - \frac{M_s^2}{H^2}}. \tag{25}
\]
The Fourier mode sum for the propagator on infinite space is [7],

$$i\Delta_{\text{dS}}(x; z) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k} \cdot (\vec{x} - \vec{z})} \left\{ \theta(x^0 - z^0)u(x^0, k)u^*(z^0, k) + \theta(z^0 - x^0)u(x^0, k)u(z^0, k) \right\}.$$  \hspace{1cm} (26)

When this sum exists the result is de Sitter invariant [15],

$$i\Delta_{\text{dS}}(x; z) = H^{D-2} \left\{ \frac{\Gamma(D-1)}{(4\pi)^{D-2}} \int_0^\infty dk k^{D-2} \frac{J_{D-3}(kD\Delta x)}{(kD\Delta x)^{D-3}} \theta(x^0 - z^0)H_\nu^{(1)}(-kx^0)H_\nu^{(1)}(-kz^0)^* + \theta(z^0 - x^0) \right\}.$$  \hspace{1cm} (27)

The gamma function $\Gamma\left(\frac{D-1}{2} - \nu + n\right)$ on the final line of (29) diverges for,

$$\nu = \frac{(D-1)}{2} + N \iff M_2^2 = -N(D-1+N)H^2.$$  \hspace{1cm} (30)

Its origin can be understood by performing the angular integration in the naive mode sum (26) and then changing to the dimensionless variable $\tau \equiv k/H\sqrt{a_xa_z}$,

$$i\Delta_{\text{dS}}(x; z) = \frac{(a_xa_z)^{-(D-1)/2}}{2\pi^{D-3/2}H} \int_0^\infty dk k^{D-2} \frac{1}{(kD\Delta x)^{D-3/2}}J_{D-3}(kD\Delta x) \left\{ \theta(x^0 - z^0)H_\nu^{(1)}(-kx^0)H_\nu^{(1)}(-kz^0)^* + \theta(z^0 - x^0) \right\}.$$  \hspace{1cm} (31)
\[
\int_0^{\infty} d\tau \, \tau^{D-2} \left( \frac{1}{2} \sqrt{a_x a_z} H \Delta x \tau \right)^{-\frac{D-3}{2}} J_{\frac{D-3}{2}} \left( \sqrt{a_x a_z} H \Delta x \tau \right) \\
\times \left\{ \theta(x^0 - z^0) H^{(1)}_{\nu} \left( \sqrt{\frac{a_z}{a_x}} \tau \right) H^{(1)}_{\nu} \left( \sqrt{\frac{a_x}{a_z}} \tau \right)^* + \theta(z^0 - x^0) \text{(conjugate)} \right\}.
\]

In these and subsequent expressions we define \( \Delta x \equiv \| \vec{x} - \vec{z} \| \). That the divergence at (30) is infrared can be seen from the small argument expansion of the Bessel function and from its relation to the Hankel function,

\[
J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n(x)^{\nu+2n}}{n! \Gamma(\nu+n+1)},
\]

and the small \( \tau \) behavior of the integrand (32) derives from three factors, the first of which is \( \tau^{D-2} \). The second factor from the Bessel function takes the form,

\[
\left( \frac{1}{2} \sqrt{a_x a_z} H \Delta x \tau \right)^{-\frac{D-3}{2}} J_{\frac{D-3}{2}} \left( \sqrt{a_x a_z} H \Delta x \tau \right) = \frac{1}{\Gamma(\frac{D-1}{2})} \sum_{n=0}^{\infty} C_1(n) \tau^{2n}.
\]

And the final factor from the Hankel functions is,

\[
H^{(1)}_{\nu} \left( \sqrt{\frac{a_z}{a_x}} \tau \right) H^{(1)}_{\nu} \left( \sqrt{\frac{a_x}{a_z}} \tau \right)^* = \frac{2\Gamma(\nu)\Gamma(2\nu)}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})^2} \sum_{n=0}^{\infty} C_2(n) \tau^{2n}.
\]

One does not need the coefficients \( C_1(n) \) and \( C_2(n) \) to see that the small \( \tau \) expansion of the integrand takes the form,

\[
\tau^{D-2} \times \frac{1}{\Gamma(\frac{D-1}{2})} \sum_{k=0}^{\infty} C_1(k) \tau^{2k} \times \frac{\Gamma^2(\nu)2^{2\nu}}{\pi^{\nu}\Gamma(\nu+\frac{1}{2})^2} \sum_{\ell=0}^{\infty} C_2(\ell) \tau^{2\ell}
\]

\[
= \frac{2\Gamma(\nu)\Gamma(2\nu)}{\pi^{\frac{1}{2}}\Gamma(\frac{D-1}{2})\Gamma(\nu+\frac{1}{2})} \tau^{D-2-2\nu} \sum_{n=0}^{\infty} C_3(n) \tau^{2n}.
\]

Hence the naive mode sum (26) is infrared divergent for

\[
D - 2 - 2\nu \leq -1 \quad \iff \quad M_S^2 \leq 0.
\]

However, there will only be a logarithmic infrared divergence, either from the leading term in (37) or from one of the series corrections at \( n = N \), if one has,

\[
D - 2 - 2\nu + 2N = -1 \quad \iff \quad M_S^2 = -N(D-1+N)\frac{H^2}{2}.
\]
This is precisely the condition (30) for the formal, de Sitter invariant mode sum (29) to diverge.

As emphasized in the Introduction, the appearance of an infrared divergence signals that something is unphysical about the quantity being computed. The correct response to an infrared divergence is not to subtract it off, either explicitly or implicitly with the automatic subtraction of some analytic regularization technique. One must instead understand the physical problem which caused the divergence and then to fix the problem.

The divergence (38) occurs because of the way the Bunch-Davies mode functions (25) depend upon \(k\) for small \(k\). The unphysical thing about having Bunch-Davies vacuum for arbitrarily small \(k\) is that no experimentalist can causally enforce it (or any other condition) for super-horizon modes. This has led to two fixes:

1. One can continue to work on the spatial manifold \(R^{D-1}\) but assume the initial state is released with its super-horizon modes in some less singular condition \([10]\); or

2. One can work on the compact spatial manifold \(T^{D-1}\) with its coordinate radius chosen such that the initial state has no super-horizon modes \([11]\).

We will adopt the latter fix. Of course this makes the mode sum discrete but the integral approximation should be excellent, and gives a simple expression for the propagator which differs from (26) only by having an infrared cutoff at \(k = H\).

From the preceding discussion we see that the infrared corrected propagator \(i\Delta(x; z)\) is just (32) with the lower limit cutoff at \(\tau = 1/\sqrt{a_x a_z}\),

\[
i\Delta(x; z) = \frac{H^{D-2}}{2^D \pi^{D-2} D_x D_z} \int_{\sqrt{a_x a_z}}^\infty \frac{d\tau \tau^{D-2} J_{D-3}(\sqrt{a_x a_z} H \Delta x \tau)}{(\frac{1}{2} \sqrt{a_x a_z} H \Delta x \tau)^{D_x D_z}} \times \left\{ \theta(x^0 - z^0) H_\nu^{(1)} \left( \sqrt{a_x a_z} \tau \right) H_{\nu}^{(1)} \left( \sqrt{a_x a_z} \tau \right)^* + \theta(z^0 - x^0) \text{ (conjugate)} \right\}.
\]

\(^1\)Making the integral approximation does not alter the renormalization of various \(M_s^2 = 0\) scalar models at one loop order \([16, 17, 14, 18]\), or even at two loops \([19, 20, 21, 22]\). Because the physical graviton polarizations have the same mode functions as scalars with \(M_s^2 = 0\) \([23]\), one can also test the integral approximation with the graviton propagator. There is no disruption of powerful consistency checks such as the Ward identity at tree order and one loop \([24]\), or the nature of allowed one loop counterterms \([25, 26, 27]\).
Of course we can express the truncated integral as the full one minus an integral over just the infrared,

\[ \int_0^\infty d\tau = \int_0^\infty d\tau - \int_0^{\sqrt{a_xa_z}} d\tau \quad \Leftrightarrow \quad i\Delta(x; z) \equiv i\Delta^\text{dS}(x; z) + \Delta^\text{IR}(x; z). \] \quad (41)

In this case it does not matter if dimensional regularization is used to evaluate both \( i\Delta^\text{dS}(x; z) \) and \( \Delta^\text{IR}(x; z) \) because the errors we make at the lower limits will cancel.

A further simplification is that \( \Delta^\text{IR}(x; z) \) only needs to include the infrared singular terms which grow as \( a_xa_z \) increases. These terms come entirely from the \( J_{-\nu} \) parts of the Hankel function and they are entirely real,

\[ \Delta^\text{IR}(x; z) = -\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{2\Gamma(\nu)\Gamma(2\nu)}{\Gamma(\nu+\frac{1}{2})} \int_0^{\sqrt{a_xa_z}} d\tau \tau^{D-2} J_{\frac{D-2}{2}}(\sqrt{a_xa_z} H \Delta x \tau) \frac{\Gamma^2(1-\nu)}{2^{2\nu}} J_{-\nu}\left(\sqrt{\frac{a_z}{a_x}} \tau\right) J_{-\nu}\left(\sqrt{\frac{a_x}{a_z}} \tau\right). \] \quad (42)

Before giving the general result for \( \Delta^\text{IR}(x; z) \) it is instructive to work out the first two terms in the small \( \tau \) expansion of the integrand,

\[ \tau^{D-2} J_{\frac{D-2}{2}}(\sqrt{a_xa_z} H \Delta x \tau) \frac{\Gamma^2(1-\nu)}{2^{2\nu}} J_{-\nu}\left(\sqrt{\frac{a_z}{a_x}} \tau\right) J_{-\nu}\left(\sqrt{\frac{a_x}{a_z}} \tau\right) = \frac{\tau^{D-2-2\nu}}{\Gamma\left(\frac{D-1}{2}\right)} \left\{ 1 - \frac{a_xa_zH^2\Delta x^2\tau^2}{2(D-1)} + O(\tau^4) \right\} \left\{ 1 + \frac{(\frac{a_x}{a_z} + \frac{a_z}{a_x})\tau^2}{4(\nu-1)} + O(\tau^4) \right\}. \] \quad (43)

Now use the definition (13) of the de Sitter length function to infer,

\[ y = a_xa_zH^2\left[ \Delta x^2 - \left(\frac{1}{Ha_x} - \frac{1}{Ha_z}\right)^2 \right] \implies a_xa_zH^2\Delta x^2 = (y-2) + \left(\frac{a_x}{a_z} + \frac{a_z}{a_x}\right). \] \quad (44)

Hence we have,

\[ \Delta^\text{IR}(x; z) = -\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{2\Gamma(\nu)\Gamma(2\nu)}{\Gamma\left(\frac{D+1}{2}\right)\Gamma(\nu+\frac{1}{2})} \int_0^{\sqrt{a_xa_z}} d\tau \tau^{D-2-2\nu} \times \left\{ 1 - \left[ \frac{y-2}{2(D-1)} + \frac{\nu-(\frac{D+1}{2})}{2(D-1)(\nu-1)} \left(\frac{a_x}{a_z} + \frac{a_z}{a_x}\right) \right]\tau^2 + O(\tau^4) \right\}. \] \quad (45)
\[
H^{D-2} \frac{\Gamma(\nu)\Gamma(2\nu)}{(4\pi)^{D/2} \Gamma(D-1/2)\Gamma(\nu+1/2)} \left\{ \frac{(a_x a_z)^{\nu-(D-1)/2}}{\nu-(D-1)/2} \right. \\
- \left[ \frac{y-2}{2(D-1)} + \frac{\nu-(D+1)/2}{2(D-1)(\nu-1)} \left( \frac{a_x + a_z}{a_x} \right) (a_x a_z)^{\nu-(D+1)/2} \right] + \ldots \right\}.
\]

(46)

One can see that the first and second terms of (46) respectively cancel the \( N = 0 \) and \( N = 1 \) divergences in the naive mode sum (29).

To find the general form of \( \Delta^{IR}(x; z) \) we first note (from its expression as an integral over \( k \)) that it is annihilated by \( -M^2 \). We next note from (46) that \( \Delta^{IR}(x; z) \) consists of a series of terms, each one of which has the form \( (a_x a_z)^{\nu-(D-1)/2} \) times a series involving powers of \( (y - 2) \) and \( (a_x + a_z)/a_x \). The contributions at fixed \( N \) must be separately annihilated by \( -M^2 \). The coefficient of the highest power of \( (y - 2) \) at fixed \( N \) derives entirely from the \( N \)th order term in the expansion of the Bessel function \( J_{D-3/2} \). These two facts imply,

\[
\Delta^{IR}(x; z) = H^{D-2} \frac{\Gamma(\nu)\Gamma(2\nu)}{(4\pi)^{D/2} \Gamma(D-1/2)\Gamma(\nu+1/2)} \times \sum_{N=0}^{\infty} \frac{(a_x a_z)^{\nu-(D-1)/2-N}}{\nu-(D+1)/2-N} \sum_{n=0}^{N} \left( \frac{a_x + a_z}{a_x} \right)^{n} \sum_{m=0}^{\lfloor \frac{N-n}{2} \rfloor} C_{Nnm} (y-2)^{N-n-2m},
\]

(47)

where the coefficients \( C_{Nnm} \) are,

\[
C_{Nnm} = \frac{(-1/4)^N}{m!n!(N-n-2m)!} \times \frac{\Gamma(D-1/2+N+n-\nu)}{\Gamma(D-1/2+N-\nu)} \times \frac{\Gamma(D-1)}{\Gamma(D-1/2+N-2m)} \times \frac{\Gamma(1-\nu)}{\Gamma(1-\nu+n+2m)} \times \frac{\Gamma(1-\nu)}{\Gamma(1-\nu+m)}. \]

(48)

Of course there is no point in extending the sum over \( N \) to values \( N > \nu - (D-1)/2 \) for which the exponent of \( a_x a_z \) becomes negative. Those terms rapidly approach zero, and they can be dropped without affecting the propagator equation because they are separately annihilated by \( -M^2 \).

We conclude this section by discussing three special cases which occur with such frequency as to merit special notation. These are

\[
M^2_S = (D-2)H^2 \implies \nu = \frac{D-3}{2} \implies i\Delta_B(x; z) = B(y),
\]

(49)
\[ M_S^2 = 0 \implies \nu = \frac{D-1}{2} \implies i\Delta_A(x; z) = A(y) + \delta A(a_x, a_z, y), \quad (50) \]
\[ M_S^2 = -DH^2 \implies \nu = \frac{D+1}{2} \implies i\Delta_W(x; z) = W(y) + \delta W(a_x, a_z, y). \quad (51) \]

Although the \( B \)-type propagator is de Sitter invariant, its \( A \)-type and \( W \)-type cousins have de Sitter breaking parts,

\[ \begin{align*}
\delta A &= k \ln(a_x a_z), \\
\delta W &= k \left\{ (D-1)^2 a_x a_z - \left( \frac{D-1}{2} \right) \ln(a_x a_z) (y^2 - 1) - \left( \frac{a_x}{a_z} + \frac{a_z}{a_x} \right) \right\}. \quad (53) 
\end{align*} \]

The constant \( k \) is,

\[ k \equiv \frac{H^{D-2}}{(4\pi)^D} \frac{\Gamma(D-1)}{\Gamma\left(\frac{D}{2}\right)}. \quad (54) \]

The main, de Sitter invariant parts of each propagator consist of a few, potentially ultraviolet divergent terms (at \( y = 0 \)), plus an infinite series,

\[ 
\begin{align*}
B(y) &= \frac{H^{D-2}}{(4\pi)^D} \left\{ \Gamma\left(\frac{D}{2} - 1\right) \left( \frac{4}{y} \right)^{\frac{D}{2} - 1} \\
&\quad + \sum_{n=0}^\infty \left[ \frac{\Gamma(n+\frac{D}{2})}{(n+1)!} \left( \frac{y}{4} \right)^{n-\frac{D}{2} + 2} \frac{\Gamma(n+D-2)}{\Gamma(n+\frac{D}{2})} \left( \frac{y}{4} \right)^n \right] \right\}, \\
A(y) &= \frac{H^{D-2}}{(4\pi)^D} \left\{ \Gamma\left(\frac{D}{2} - 1\right) \left( \frac{4}{y} \right)^{\frac{D}{2} - 1} + \frac{\Gamma\left(\frac{D}{2} + 1\right)}{\frac{D}{2} - 2} \left( \frac{4}{y} \right)^{\frac{D}{2} - 2} + A_1 \\
&\quad - \sum_{n=1}^\infty \left[ \frac{\Gamma(n+\frac{D}{2} + 1)}{(n-\frac{D}{2} + 2)(n+1)!} \left( \frac{y}{4} \right)^{n-\frac{D}{2} + 2} - \frac{\Gamma(n-D-1)}{n\Gamma(n+\frac{D}{2})} \left( \frac{y}{4} \right)^n \right] \right\}, \\
W(y) &= \frac{H^{D-2}}{(4\pi)^D} \left\{ \Gamma\left(\frac{D}{2} - 1\right) \left( \frac{4}{y} \right)^{\frac{D}{2} - 1} + \frac{\Gamma\left(\frac{D}{2} + 2\right)}{(\frac{D}{2} - 2)(\frac{D}{2} - 1)} \left( \frac{4}{y} \right)^{\frac{D}{2} - 2} \\
&\quad + \frac{\Gamma\left(\frac{D}{2} + 3\right)}{2(\frac{D}{2} - 3)(\frac{D}{2} - 2)} \left( \frac{4}{y} \right)^{\frac{D}{2} - 3} + W_1 + W_2 \left( \frac{y-2}{4} \right) \\
&\quad + \sum_{n=2}^\infty \left[ \frac{\Gamma(n+\frac{D}{2} + 2)}{(n-D+2)(n-D+1)(n+1)!} - \frac{\Gamma(n+D)}{n(n-1)\Gamma(n+\frac{D}{2})} \left( \frac{y}{4} \right)^n \right] \right\}. \quad (56) 
\end{align*} \]

And the \( D \)-dependent constants \( A_1, W_1 \) and \( W_2 \) are,

\[ A_1 = \frac{\Gamma(D-1)}{\Gamma\left(\frac{D}{2}\right)} \left\{ -\psi\left(1 - \frac{D}{2}\right) + \psi\left(\frac{D-1}{2}\right) + \psi(D-1) + \psi(1) \right\}, \quad (58) \]
\[ W_1 = \frac{\Gamma(D+1)}{\Gamma\left(\frac{D}{2}+1\right)} \left\{ \frac{D+1}{2D} \right\}, \quad (59) \]

\[ W_2 = \frac{\Gamma(D+1)}{\Gamma\left(\frac{D}{2}+1\right)} \left\{ \psi\left(-\frac{D}{2}\right) - \psi\left(\frac{D+1}{2}\right) - \psi(D+1) - \psi(1) \right\}. \quad (60) \]

The infinite series terms of \( B(y), A(y) \) and \( W(y) \) makes expressions (55-57) seem intimidating. However, note that each pair of terms in the infinite sums cancels for \( D = 4 \), so they only need to be retained when multiplying a potentially divergent quantity. Further, because \( y^n \) vanishes more and more strongly at coincidence as \( n \) increases, only a handful of the smallest \( n \) terms ever need to be included. This makes loop computations manageable. For a massless, minimally coupled scalar with a quartic self-interaction, two loop results have been obtained for the expectation value of the stress tensor [19], for the scalar self-mass-squared [20] and for the quantum-corrected mode functions [28]. In Yukawa theory it has been used to compute the expectation value of the coincident vertex function at two loop order [29], and it has been used for a variety of two loop computations in scalar quantum electrodynamics [21, 22].

The need for de Sitter breaking terms in \( i\Delta_A(x; z) \) has long been recognized [13], and ours reproduces the classic and well known result for the coincidence limit of the propagator [30]. The de Sitter breaking terms also show up in the differential equations obeyed by the de Sitter invariant parts of the various propagators,

\[ (4y-y^2)B'' + D(2-y)B' - (D-2)B = 0, \quad (61) \]

\[ (4y-y^2)A'' + D(2-y)A' = (D-1)k, \quad (62) \]

\[ (4y-y^2)W'' + D(2-y)W' + DW = \frac{1}{2}(D+1)(D-1)k(2-y). \quad (63) \]

Whereas the equation for \( B(y) \) is homogeneous, the equations for \( A(y) \) and \( W(y) \) both possess inhomogeneous terms which are cancelled by \( -M_S^2 \) acting on the de Sitter breaking terms \( \delta A(a_x, a_z, y) \) and \( \delta W(a_x, a_z, y) \). Finally, we give some differential relations between the de Sitter invariant parts which follow from the series expansions [35, 57, 31],

\[ (2-y)A' - k = 2B', \quad (64) \]

\[ (2-y)W'' + \frac{1}{2}(D-1)k = 2A''. \quad (65) \]
4 Vectors

The purpose of this section is to demonstrate that the de Sitter invariant propagator equation possesses no de Sitter invariant solution for a massive vector with $M^2_V \leq -2(D-1)H^2$. We begin by explaining how the full vector propagator (longitudinal plus transverse) can be written as the transverse vector propagator plus a double gradient of the difference of two known scalar propagators. We then derive a formal, de Sitter invariant solution for the transverse vector propagator in terms of scalar propagators. One of these scalar propagators possesses the infrared divergences we found in the previous section. We show how the problem can be corrected, and we derive the leading infrared correction.

The full propagator for a massive vector obeys the equation,

$$\sqrt{-g(x)} \left[ \Box_x - (D-1)H^2 - M^2_V \right] i[\mu \Delta_\nu](x; z) = g_{\mu\nu} i\delta^D(x-z).$$  \hspace{1cm} (66)

Although this object does appear in certain projection operators, there is greater physical interest in its transverse part which obeys,

$$g^{\mu\nu}(x) \frac{D}{Dx^\mu} i[\nu \Delta^T_\rho](x; z) = 0 = g^{\rho\sigma}(z) \frac{D}{Dz^\rho} i[\mu \Delta^T_\sigma](x; z).$$  \hspace{1cm} (67)

An excellent early study of the massive, transverse vector propagator was carried out by Allen and Jacobson [32]. A minor error in their work is that the source term is not transverse. When this is corrected the propagator equation reads [33],

$$\sqrt{-g(x)} \left[ \Box_x - (D-1)H^2 - M^2_V \right] i[\mu \Delta^T_\nu](x; z)$$

$$= g_{\mu\nu} i\delta^D(x-z) + \sqrt{-g(x)} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial z^\nu} i\Delta_A(x; z).$$ \hspace{1cm} (68)

Although $i\Delta_A(x; z)$ contains a de Sitter breaking part, $k \ln(a_xa_z)$, this makes no contribution when the propagator is differentiated on both of its arguments. Therefore, equation (68) is fully de Sitter invariant.

Given the transverse vector propagator one can construct the full vector propagator by adding the double gradient of a longitudinal part,

$$i[\mu \Delta_\nu](x; z) = i[\mu \Delta^T_\nu](x; z) + \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial z^\nu} i\Delta^L_\nu(x; z).$$  \hspace{1cm} (69)
The equation obeyed by $i\Delta L(x; z)$ follows from substituting (69) in (66), commuting some derivatives and using (68) to conclude,

$$\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial z^\nu} \left\{ i\Delta A(x; z) + [\Box_x - M_\mathcal{V}^2] i\Delta L(x; z) \right\} = 0. \quad (70)$$

A solution to (70) is [31],

$$i\Delta L(x; z) = \frac{1}{M^2_\mathcal{V}} \left[ i\Delta A(x; z) - i\Delta M(x; z) \right], \quad (71)$$

where $i\Delta M(x; z)$ is the scalar propagator with mass $M^2_S = M^2_V$.

Because we can always construct the full vector propagator from its transverse part by the procedure just described, we will henceforth concentrate on the transverse vector propagator. The most general de Sitter invariant, symmetric bi-tensor which obeys the transversality condition (67) can be expressed using an arbitrary function $\gamma(y)$ and the basis tensors of section 2 [33],

$$i\left[ \mu^\alpha \right](x; z) = \left[ -\frac{(4y-y^2)\gamma' - (D-1)(2-y)\gamma}{4(D-1)H^2} \right] \frac{\partial^2 y(x; z)}{\partial x^\mu \partial z^\nu} \left[ \frac{(2-y)\gamma' - (D-1)\gamma}{4(D-1)H^2} \right] \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial z^\nu}. \quad (72)$$

Substituting (72) into (68) gives the following differential equation for $\gamma(y)$ away from coincidence (that is, away from $x^\mu = z^\mu$) [33],

$$(4y-y^2)\gamma'' + (D+2)(2-y)\gamma' - \left[ 2(D-1) + \frac{M^2_V}{H^2} \right] \gamma = 2(D-1)B'(y). \quad (73)$$

Recovering the delta function in (68) additionally requires that the most singular term for $y \to 0$ must be [33],

$$\gamma(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \left\{ \left( \frac{D-1}{2} \right) \Gamma \left( \frac{D}{2} - 1 \right) \left( \frac{4}{y} \right)^{D/2 - 1} + O(y^{2-D}) \right\}. \quad (74)$$

At this point it helps to note that the left hand side of relation (73) can be expressed as the derivative of a scalar kinetic operator,

$$\left(4y-y^2\right)\gamma'' + (D+2)(2-y)\gamma' - \left[ 2(D-1) + \frac{M^2_V}{H^2} \right] \gamma$$

$$= \frac{1}{H^2} \frac{\partial}{\partial y} \left[ \Box_x - (D-2)H^2 - M^2_\mathcal{V} \right] I[\gamma]. \quad (75)$$
where \( I[y] \) stands for the indefinite integral of \( \gamma(y) \) with respect to \( y \). Hence \( I[\gamma] \) obeys the scalar propagator equation,

\[
\Box - (D-2)H^2 - M^2_V I[\gamma] = 2(D-1)H^2 B.
\]  

(76)

This is very like the equation we just solved for the longitudinal part of the vector propagator so it should not seem surprising that the unique solution for relations (73-74) is,

\[
\gamma(y) = 2(D-1)H^2 \frac{\partial}{\partial y} \left[ E(y) - B(y) \right],
\]  

(77)

where \( B(y) \) is the scalar propagator with mass \( M^2_S = (D-2)H^2 \) and \( E(y) \) is the scalar propagator with mass \( M^2_S = (D-2)H^2 + M^2_V \). To verify (77), first note the two leading terms in the series expansions of \( B(y) \) and \( E(y) \), from expressions (55) and (29) with \( \nu^2 = \left( \frac{D-3}{2} \right)^2 - \frac{M^2_V}{H^2} \),

\[
B(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \left\{ \Gamma\left( \frac{D}{2} - 1 \right) \left( \frac{1}{y} \right)^{\frac{D}{2} - 1} + \Gamma\left( \frac{D}{2} \right) \left( \frac{1}{y} \right)^{\frac{D}{2} - 2} + \ldots \right\},
\]  

(78)

\[
E(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \left\{ \Gamma\left( \frac{D}{2} - 1 \right) \left( \frac{1}{y} \right)^{\frac{D}{2} - 1} + \left( \nu^2 - \frac{1}{4} \right) \Gamma\left( \frac{D}{2} - 2 \right) \left( \frac{1}{y} \right)^{\frac{D}{2} - 2} + \ldots \right\}.
\]  

(79)

This establishes that (77) has the correct singularity (74) near coincidence. To check the off-coincidence condition (73), note that \( B(y) \) obeys (61) and \( E(y) \) obeys,

\[
(4y-y^2)E'' + D(2-y)E' - \left[ (D-2) + \frac{M^2_V}{H^2} \right] E = 0.
\]  

(80)

Differentiating (61) and (80) with respect to \( y \) gives,

\[
(4y-y^2)B''' + (D+2)(2-y)B'' - \left[ 2(D-1) + \frac{M^2_V}{H^2} \right] B' = - \frac{M^2_V}{H^2} B',
\]  

(81)

\[
(4y-y^2)E''' + (D+2)(2-y)E'' - \left[ 2(D-1) + \frac{M^2_V}{H^2} \right] E' = 0.
\]  

(82)

Subtracting (81) from (82) and multiplying by \( 2(D-1) \frac{H^2}{M^2_V} \) shows that our ansatz (77) indeed obeys equation (73).

Of course using our ansatz (77) in (72) only makes sense if the two scalar propagators \( B(y) \) and \( E(y) \) exist! Recall that they correspond to masses
$M_2^2 = (D - 2)H^2$ and $M_2^2 = (D - 2)H^2 + M_V^2$, respectively. For $D > 2$ the $B$-type propagator is safe but the $E$-type propagator can have problems if $M_V^2$ is sufficiently negative. Now consider its formal series expansion (29) with $\nu^2 = \left(\frac{D - 3}{2}\right)^2 - \frac{M_2^2}{H^2}$, and in particular, the problematic gamma function, $\Gamma(\frac{D - 1}{2} - \nu + n)$. Because it is really the derivative $E^3(y)$ which appears in our ansatz (77), the problem at $n = 0$ drops out, but there are divergences at $n = N + 1$ for all non-negative integers $N$. This corresponds to the following vector masses,

$$\nu = \left(\frac{D - 1}{2}\right) + N + 1 \quad \iff \quad M_V^2 = -(N + 2)(N + D - 1)H^2. \quad (83)$$

As emphasized in the previous section, condition (83) only gives the logarithmic infrared divergences. For $M_V^2 \leq -2(D - 1)H^2$ there will be power law infrared divergences which analytic continuation techniques incorrectly subtract off. We turn now to fixing this problem.

Because the infrared divergences we have found for $M_V^2 \leq -2(D - 1)H^2$ arise from assuming a de Sitter invariant solution to the propagator equation (68), they can only be corrected by abandoning that assumption. However, there is no need to discard the formal de Sitter invariant solution $i[\mu \Delta_{\nu}^dS](x; z)$. Just like the scalar of the preceding section, we need only to add to it a de Sitter breaking, infrared correction,

$$i\left[\mu \Delta_{\nu}^T\right](x; z) \equiv i\left[\mu \Delta_{\nu}^{dS}\right](x; z) + \left[\mu \Delta_{\nu}^{IR}\right](x; z). \quad (84)$$

Of course the infrared correction must be symmetric and it must obey the transversality condition (67). In analogy with the scalar, we also want it to be annihilated by the kinetic operator $\Box - (D - 1)H^2 - M_V^2$.

Abandoning de Sitter invariance in $[\mu \Delta_{\nu}^{IR}](x; z)$ affects the tensor structure as well as the scalar coefficient functions. If we preserve spatial homogeneity and isotropy (as we did for the scalars of the previous section) then the only extra basis tensors we require are the derivatives of $u(x; z) \equiv \ln(a_x a_z)$ with respect to $x^\mu$ and $z^\nu$,

$$\frac{\partial u}{\partial x^\mu} = H a_x \delta^0_\mu, \quad \frac{\partial u}{\partial z^\nu} = H a_z \delta^0_\nu. \quad (85)$$

The most general homogeneous and isotropic tensor with the correct symmetries is,

$$\left[\mu \Delta_{\nu}^{IR}\right](x; z) = F_1(a_x, a_z, y) \frac{\partial^2 y}{\partial x^\nu \partial z^\nu} + F_2(a_x, a_z, y) \frac{\partial y}{\partial x^\nu} \frac{\partial y}{\partial z^\nu}$$
\[ F_3(a_x, a_z, y) \frac{\partial u}{\partial x} \frac{\partial y}{\partial z} + F_3(a_z, a_x, y) \frac{\partial u}{\partial x} \frac{\partial y}{\partial z} + F_4(a_x, a_z, y) \frac{\partial u}{\partial x} \frac{\partial u}{\partial z}. \] (86)

The coefficient functions \( F_1, F_2 \) and \( F_4 \) must be symmetric under interchange of \( a_x \) and \( a_z \). \( F_3 \) does not have this symmetry; when the argument lists are suppressed (to save space) we will indicate the interchange of \( a_x \) and \( a_z \) by a superscript \( T \),

\[ F_T^3(a_x, a_z, y) \equiv F_3(a_z, a_x, y). \] (87)

We will also use a prime to denote differentiation with respect to \( y \),

\[ F'_i(a_x, a_z, y) \equiv \frac{\partial}{\partial y} F_i(a_x, a_z, y). \] (88)

The right way to think about the coefficient functions \( F_i(a_x, a_z, y) \) is that enforcing transversality determines \( F_3 \) and \( F_4 \) in terms of \( F_1 \) and \( F_2 \). Then \( F_1 \) and \( F_2 \) are fixed (up to normalization of each independent term) by demanding that \( [\Box - (D-1)H^2 - M^2] \) annihilates \( [\mu \Delta_{IR}^\mu](x; z) \).

Covariant derivatives of the new tensor involve some extra identities in addition to those of section 2,

\[ \frac{D^2 u}{Dx^\mu Dx^\nu} = -H^2 g_{\mu\nu}(x) - \frac{\partial u}{\partial x^\mu} \frac{\partial u}{\partial x^\nu}, \quad \frac{D^2 u}{Dz^\mu Dz^\nu} = -H^2 g_{\mu\nu}(z) - \frac{\partial u}{\partial z^\mu} \frac{\partial u}{\partial z^\nu}. \] (89)

There are also some new contraction identities,

\[ g^{\mu\nu}(x) \frac{\partial u}{\partial x^\mu} \frac{\partial u}{\partial x^\nu} = -H^2, \] (90)

\[ g^{\mu\nu}(x) \frac{\partial u}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} = -H^2 [y - 2 + 2ax/a_x], \] (91)

\[ g^{\mu\nu}(x) \frac{\partial u}{\partial x^\mu} \frac{\partial^2 y}{\partial x^\nu \partial z^\rho} = -H^2 \left[ \frac{\partial y}{\partial z^\rho} + 2ax \frac{\partial u}{\partial z^\rho} \right]. \] (92)

The covariant divergence \( g^{\mu\nu}(x) D_\mu \Delta_{IR}^\nu(x; z) \) produces one term proportional to \( \partial y/\partial z^\rho \) and another proportional to \( \partial u/\partial z^\rho \). The term proportional to \( \partial y/\partial z^\rho \) gives an equation for the coefficient function \( F_3(a_x, a_z, y) \) in terms of \( F_1(a_x, a_z, y) \) and \( F_2(a_x, a_z, y) \),

\[ -(2-y)F'_3 + DF_3 + a_x \frac{\partial F_3}{\partial a_x} + 2a_z \frac{a_x}{a_x} F'_3 = (2-y)F'_1 - DF_1 - a_x \frac{\partial F_1}{\partial a_x} \]

\[ + (4y-y^2)F'_2 + (D+1)(2-y)F_2 + (2-y)a_x \frac{\partial F_2}{\partial a_x} - 2a_z \frac{\partial F_2}{\partial a_x}. \] (93)
Because the equation involves only first derivatives with respect to $y$ and $a_x$, the general solution can be given using the Method of Characteristics. Once $F_3(a_x, a_z, y)$ is known, the transversality relation proportional to $\partial u / \partial z^\mu$ gives an equation for the remaining coefficient function $F_4(a_x, a_z, y)$,

$$-(2-y)F_4' + (D-1)F_4 + a_x \frac{\partial F_4}{\partial a_x} + 2a_z \frac{\partial F_4}{\partial a_z} F_4' = -2a_z \frac{\partial F_1}{\partial a_z} - 2a_z F_3$$

$$+(4y-y^2)F_3'' + D(2-y)F_3' + (2-y)a_x \frac{\partial F_3}{\partial a_x} - 2a_z \frac{\partial F_3}{\partial a_z}. \quad (94)$$

This equation can also be solved by the Method of Characteristics.

Once one has determined the coefficient functions $F_3$ and $F_4$ from equations (93-94), $F_1$ and $F_2$ are obtained, up to normalization of each independent piece, by requiring $[\mu \Delta_{\nu}^{\text{IR}}](x; z)$ to solve the homogeneous propagator equation,

$$[\square - (D-1)H^2 - M_V^2] [\mu \Delta_{\nu}^{\text{IR}}](x; z) = 0. \quad (95)$$

Of course this gives a relation for each of the five tensor structures present in $[\mu \Delta_{\nu}^{\text{IR}}](x; z)$, however, only two of these five relations are independent. The one proportional to $\partial^2 y / \partial x^\mu \partial z^\nu$ constrains $F_1(a_x, a_z, y)$,

$$(4y-y^2)F_1'' + D(2-y)F_1' - \left[D + \frac{M_V^2}{H^2}\right] F_1 + 2(2-y)F_2 + 2(2-y)a_x \frac{\partial F_1}{\partial a_x}$$

$$-4a_z \frac{\partial F_1}{\partial a_z} - (D-1)a_x \frac{\partial F_1}{\partial a_x} - a_x \frac{\partial}{\partial a_x} \left[ a_x \frac{\partial F_1}{\partial a_x} \right] - 2F_3 = 0. \quad (96)$$

And $F_2(a_x, a_z, y)$ is constrained by the term proportional to $\partial y / \partial x^\mu \partial y / \partial z^\nu$,

$$(4y-y^2)F_2'' + (D+4)(2-y)F_2' - \left[2D + \frac{M_V^2}{H^2}\right] F_2 - 2F_1' + 2(2-y)a_x \frac{\partial F_2}{\partial a_x}$$

$$-4a_z \frac{\partial F_2}{\partial a_z} - (D+1)a_x \frac{\partial F_2}{\partial a_x} - a_x \frac{\partial}{\partial a_x} \left[ a_x \frac{\partial F_2}{\partial a_x} \right] - 2F_3' = 0. \quad (97)$$

The normalization comes from differentiating (96) with respect to $y$ and then subtracting (97) to obtain an equation for the combination $F_1' - F_2$. By making the definition,

$$F_1'(a_x, a_z, y) - F_2(a_x, a_z, y) \equiv \mathcal{E}'(a_x, a_z, y), \quad (98)$$

we can identify this relation as the homogeneous equation for an $E$-type scalar with mass $M_s^2 = (D-2)H^2 + M_V^2$,

$$\frac{\partial}{\partial y} \left[ \square - (D-2)H^2 - M_V^2 \right] \mathcal{E} = 0. \quad (99)$$
Now write the de Sitter invariant part of the propagator in a form similar to the de Sitter breaking part,

$$i [\mu \Delta^d_s](x; z) \equiv F_1(y) \frac{\partial^2 y}{\partial \nu \partial z^\nu} + F_2(y) \frac{\partial y}{\partial \nu} \frac{\partial y}{\partial z^\nu}.$$  \(100\)

From relations \((12, 17)\) and \((17)\) we infer,

$$F_1 - F_2 = - \left[ \frac{(4y - y^2)\gamma'' + (D + 2)(2y - y)\gamma' - 2(D - 1)\gamma}{4(D - 1)H^2} \right] = - E'(y) \frac{2H^2}{\nu}. \quad \text{\((101)\)}$$

Hence we infer,

$$F_1(a_x, a_z, y) - F_2(a_x, a_z, y) = - \frac{1}{2H^2} \frac{\partial}{\partial y} \Delta^{\text{IR}}(x; z),$$

\(102\)

$$= - \frac{1}{2H^2} \frac{H^{D-2}}{(4\pi)^D} \frac{\Gamma(\nu)\Gamma(2\nu)}{\Gamma(\nu + \frac{1}{2})} \sum_{N=0}^{\infty} \frac{(a_x a_z)^{\nu - (\frac{D+1}{2}) - N}}{\nu - (\frac{D+1}{2}) - N} \times \sum_{n=0}^{N+1} \left( \frac{a_x + a_z}{a_x} \right)^n \sum_{m=0}^{[\frac{N+1-\nu}{2}]} (N + 1 - n - 2m) C_{N+1nm}(y; 2)^{N-n-2m}, \quad \text{\((103)\)}$$

where \(\Delta^{\text{IR}}(x; z)\) was defined in equation \((47)\), the coefficients \(C_{Nnm}\) are given in \((48)\), and the index \(\nu\) obeys \(\nu^2 = \left(\frac{D-3}{2}\right)^2 - \frac{M^2}{H^2}\).

It is instructive to give the first \((N = 0)\) term in the series expansions of the four coefficient functions \(F_i(a_x, a_z, y)\),

\[(F_1)_0 = F \left[ (y - 2) + \frac{(\nu - (\frac{D+1}{2}))(\nu + (\frac{D-1}{2}))}{(\nu + (\frac{D-3}{2}))(\nu - 1)} \right] \left( \frac{a_x}{a_z} + \frac{a_z}{a_x} \right) \frac{(a_x a_z)^{\nu - (\frac{D+1}{2})}}{\nu - (\frac{D+1}{2})}, \quad \text{\((104)\)}\]

\[(F_2)_0 = - F \frac{(a_x a_z)^{\nu - (\frac{D+1}{2})}}{\nu - (\frac{D+1}{2})}, \quad \text{\((105)\)}\]

\[(F_3)_0 = - F \frac{(\nu - (\frac{D+1}{2}))}{(\nu + (\frac{D-3}{2}))} \left[ \frac{(\nu + (\frac{D+1}{2}))}{(\nu - 1)} \right] \left( \frac{a_x}{a_z} + \frac{a_z}{a_x} \right) \frac{(a_x a_z)^{\nu - (\frac{D+1}{2})}}{\nu - (\frac{D+1}{2})}, \quad \text{\((106)\)}\]

\[(F_4)_0 = - F \frac{(\nu - (\frac{D+1}{2}))}{(\nu + (\frac{D-3}{2}))} \left[ \frac{(\nu - (\frac{D+3}{2}))}{(\nu - 1)} \right] \left( \frac{a_x}{a_z} + \frac{a_z}{a_x} \right) (y - 2) \]

\[+ 4 \frac{(\nu - (\frac{D+1}{2}))}{(\nu + (\frac{D-3}{2}))} \frac{(a_x a_z)^{\nu - (\frac{D+1}{2})}}{\nu - (\frac{D+1}{2})}. \quad \text{\((107)\)}\]
Here the index $\nu$ obeys $\nu^2 = (\frac{D-3}{2})^2 - \frac{M^2}{H^2}$ and the constant $F$ is,
\[
F \equiv \frac{1}{16H^2} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\nu)\Gamma(2\nu)}{\Gamma(\frac{D+1}{2})\Gamma(\nu+\frac{1}{2})}.
\] (108)

The general series expansion for $F_1(a_x, a_z, y)$ has the form,
\[
F_1 = F \sum_{N=0}^{\infty} \left( \frac{a_x a_z}{a_z a_x} \right)^{\nu - \frac{D+1}{2} - N} \times \sum_{n=0}^{N+1} \left( \frac{a_x + a_z}{a_x a_z} \right)^n \sum_{m=0}^{N-n} F_{Nnm}^1 (y-2)^{N+1-n-2m}. \tag{109}
\]

As with the de Sitter breaking corrections for the scalar, there is no point in extending the series for $N > \nu - \frac{D+1}{2}$. The other coefficient functions have the same first line as (109) so we give only their subsequent forms,
\[
F_2 \rightarrow \sum_{n=0}^{N} \left( \frac{a_x + a_z}{a_x a_z} \right)^n \sum_{m=0}^{N-n} F_{Nnm}^2 (y-2)^{N-n-2m}, \tag{110}
\]
\[
F_3 \rightarrow \sum_{n=0}^{N} \left( \frac{a_x + a_z}{a_x a_z} \right)^n \times \sum_{m=0}^{N-n} \left[ F_{Nnm}^{3a} \left( \frac{a_x + a_z}{a_x a_z} \right) + F_{Nnm}^{3b} \left( \frac{a_x - a_z}{a_x a_z} \right) \right] (y-2)^{N-n-2m}, \tag{111}
\]
\[
F_4 \rightarrow \sum_{n=0}^{N+1} \left( \frac{a_x + a_z}{a_x a_z} \right)^n \sum_{m=0}^{N+1-n} F_{Nnm}^4 (y-2)^{N+1-n-2m}. \tag{112}
\]

We close this section by giving the transverse vector propagator for the important special case of, $M_S^2 = -2(D-1)H^2$. For that mass the associated scalar has $M_V^2 = -DH^2$, corresponding to the $W$-type propagator considered at the end of section 3. For this reason we give the vector propagator a subscript $W$, and we decompose it into a de Sitter invariant part and a de Sitter breaking part,
\[
i[\mu_\nu]_W(x; z) = [\mu W_\nu](x; z) + [\mu \delta W_\nu](x; z). \tag{113}
\]

The de Sitter invariant part is,
\[
[\mu W_\nu](x; z) = \left\{ \frac{(4y-y^2)(W-B)' + (D-1)(2y)(W-B)'}{4(D-1)H^2} \right\} \frac{\partial^2 y}{\partial x^\mu \partial z^\nu}
\]
\begin{align*}
+ \left[ -\frac{(2-y)(W-B)'' + (D-1)(W-B)'}{4(D-1)H^2} \right] \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial z^\nu}, \quad (114)
\end{align*}

And the de Sitter breaking part is,
\begin{align*}
\left[ \kappa \delta W_\nu \right] &= \frac{k}{4(D-1)H^2} \left\{ \left[ \frac{(D-1)^2}{2} \ln(a_xa_z)(y-2) + D \left( \frac{a_x}{a_z} + \frac{a_z}{a_x} \right) \right] \frac{\partial^2 y}{\partial x^{\mu} \partial z^{\nu}} \\
&\quad - \frac{(D-1)^2}{2} \ln(a_xa_z) \frac{\partial y}{\partial x^{\mu}} \frac{\partial y}{\partial z^{\nu}} - \left[ D \frac{a_x}{a_z} + \frac{a_z}{a_x} \right] \frac{\partial u}{\partial x^{\mu}} \frac{\partial u}{\partial z^{\nu}} \\
&\quad - \left[ \frac{a_x}{a_z} + D \frac{a_z}{a_x} \right] \frac{\partial y}{\partial x^{\mu}} \frac{\partial u}{\partial z^{\nu}} + \left( \frac{a_x}{a_z} + \frac{a_z}{a_x} \right) (y-2) \frac{\partial u}{\partial x^{\mu}} \frac{\partial u}{\partial z^{\nu}} \right\}, \quad (115)
\end{align*}

5 Discussion

We have shown that infrared divergences preclude de Sitter invariant solutions for the propagators of either a minimally coupled scalar with mass $M_S^2 \leq 0$, or for a transverse vector with mass $M_V^2 \leq -2(D-1)H^2$. (If one includes the longitudinal part of the vector propagator then infrared divergences occur for $M_V^2 \leq 0$.) However, in most cases these infrared divergences are of the power law type which is automatically subtracted by any regularization which is based upon analytic continuation. (We stress that these considerations apply as well to the standard technique of continuation from Euclidean de Sitter space.) Only the special values of $M_S^2$ and $M_V^2$ given in equation (10) result in logarithmic infrared divergences which show up in analytic regularization techniques. Thus one might reach the incorrect conclusion that de Sitter invariant propagators exist for all scalar and vector masses, except for a few “singular” cases.

That conclusion is wrong because infrared divergences should not be renormalized the way one treats an ultraviolet divergence. The appearance of an infrared divergence signals that an unphysical assumption has been made, and the right response is to identify the problematic assumption and modify it. In our case the unphysical assumption is that the universe can have been prepared in coherent Bunch-Davies vacuum for arbitrarily long wavelength modes. There is no causal process by which this can be accomplished in the de Sitter geometry. When one assumes either that the initially super-horizon modes are in some less singular state [10], or else that the spatial manifold is compact [11], the resulting propagators become infrared finite, but not de Sitter invariant.
In each case the true propagator can be written as the naive, de Sitter invariant result (defined by dimensional regularization) plus a de Sitter breaking infrared correction which is real and obeys the homogeneous propagator equation. For the scalar propagator we have,

$$i\Delta(x; z) = i\Delta_{dS}(x; z) + \Delta_{IR}(x; z)$$

(116)

where $i\Delta_{dS}(x; z)$ is expression (29) and $\Delta_{IR}(x; z)$ is given by relations (47-48). The analogous expression for the transverse vector is,

$$i\left[\mu \Delta^\nu \right](x; z) = i\left[\mu \Delta^\nu_{dS} \right](x; z) + \left[\mu \Delta^\nu_{IR} \right](x; z)$$

(117)

where the de Sitter invariant part is defined by expressions (72) and (77), and the de Sitter breaking terms are given by equations (86) and (108-112). The full vector propagator, including the longitudinal part, is given by equations (69) and (71).

It might be wondered what physical sense it makes to consider the propagators of particles with tachyonic masses. First, there is the mathematical point that they don’t possess de Sitter invariant propagators, despite what one might conclude by erroneously defining these propagators with some analytic continuation technique. Second, there is the important issue of following the time dependent vacuum decay which must occur when symmetry breaking takes place during a phase of inflation. In this respect the infrared correction terms may be quite important, as they sometimes are for the analogous case of FRW geometries with constant deceleration [12].

Another application for our propagators is the projection operators for higher spin propagators, in which case there are no physical particles with tachyonic masses to worry about. For example, consider the graviton $h_{\mu\nu}$ in exact de Donder gauge,

$$\left(\delta^\rho_\mu D^\sigma - \frac{1}{2} D_\mu g^{\rho\sigma}\right)h_{\rho\sigma} = 0$$

(118)

Just as the source term for the transverse vector propagator equation (68) must be consistent with Lorentz gauge (67), so too the source term of the graviton propagator equation must be consistent with (118). The resulting projection operator turns out to involve the full vector propagator,

$$i\left[\mu \nu \mathcal{P}_{\rho\sigma} \right](x; x') = g_{\mu(\rho}g_{\sigma)\nu}i\delta^D(x-x') - \frac{1}{D-2} g_{\mu\nu}g_{\rho\sigma}i\delta^D(x-x')$$

$$+ \frac{1}{2} \sqrt{-g(x)} \left\{ D_\mu D_\nu i[\nu \Delta_\sigma](x; x') + D_\mu D_\nu i[\mu \Delta_\sigma](x; x') \right\}$$

(119)

22
One can easily check that the de Donder gauge condition,

\[
\left[ \delta_a^\rho D^\sigma - \frac{1}{2} D_a^\rho g^\rho_\sigma \right] i_\mu P_\rho \sigma (x; x') = 0. \tag{120}
\]

requires the vector to have mass \( M^2_V = -2(D-1)H^2 \),

\[
\sqrt{-g(x)} \left[ \Box_x + (D-1)H^2 \right] i_\mu \Delta_\nu (x; x') = g_{\mu \nu} i \delta^D(x-x'). \tag{121}
\]

This is not only tachyonic, it actually corresponds to the first of the special cases \( (83) \) for which the transverse part harbors a logarithmic infrared divergence, so the problem would show up even in an analytic regularization technique.

This is all highly relevant to the debate concerning the de Sitter invariance of free gravitons \( [34] \). It has long been obvious to cosmologists that free gravitons cannot be de Sitter invariant because they share the same mode functions as massless, minimally coupled scalars \( [23] \). Indeed, the tensor contribution to the primordial anisotropies of the cosmic ray microwave background derives from precisely the same infrared singular dependence of these mode functions \( [35] \). On the other hand, some relativists insist that free gravitons must be de Sitter invariant because de Sitter invariant solutions exist for the propagator when a gauge fixing term is added to the action \( [36] \).

In previous work we have shown that adding these gauge fixing terms is not valid, owing to the linearization instability \( [31, 37] \). Imposing an exact gauge condition such as \( (118) \) should still be all right, but we have just seen that it leads to an inevitable breaking of de Sitter invariance through the projection operator. (This same breaking can be seen as well in noninvariant gauges by adding the appropriate compensating gauge transformations \( [38] \).) Antoniadis and Mottola long ago discovered a similar problem in another gauge \( [39] \). Before our current work one might have dismissed these examples as “spurious IR divergences for the Feynman propagator in the sense that the IR divergences are absent for other values of gauge parameters” \( [40] \). They now appear as just those cases for which infrared divergences, that are always present and which always break de Sitter invariance, just happen to go from being of the power law type to logarithmic, and hence become visible to analytic continuation techniques. And the correct procedure is not to ignore them or subtract them but rather to remove the erroneous assumption of de Sitter invariance.
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