AN ORIENTED MATROID VERSION OF THE COLORFUL
CARATHÉODORY THEOREM

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Abstract. We give the following extension of Bárány’s colorful Carathéodory theorem: Let $M$ be an oriented matroid and $N$ a matroid with rank function $\rho$, both defined on the same ground set $V$ and satisfying $rk(M) < rk(N)$. If every $A \subset V$ with $\rho(V - A) < rk(M)$ contains a positive circuit of $M$, then some independent set of $N$ contains a positive circuit of $M$.

1. Introduction

One of the cornerstones of convexity is Carathéodory’s theorem which states that, given a set $P \subset \mathbb{R}^d$ and a point $x$ in the convex hull of $P$, i.e. $x \in \text{conv} P$, there exists a subset $Q \subset P$ such that $|Q| \leq d + 1$ and $x \in \text{conv} Q$. In 1982, Bárány [2] gave the following generalization of Carathéodory’s theorem.

**Theorem 1.1 (Colorful Carathéodory).** Let $P_1, \ldots, P_{d+1}$ be point sets in $\mathbb{R}^d$. If $x \in \bigcap_{i=1}^{d+1} \text{conv} P_i$, then there exists $p_1 \in P_1, \ldots, p_{d+1} \in P_{d+1}$ such that $x \in \text{conv}\{p_1, \ldots, p_{d+1}\}$.

The name originates from thinking of the $P_i$ as distinct color classes. The conclusion tells us the point $x$ is contained in a “colorful simplex”, that is a simplex whose vertices are of all distinct colors. Notice that Bárány’s theorem reduces to Carathéodory’s theorem when the $P_i$ are equal.

Theorem 1.1 has many applications in discrete geometry [12], and gives rise to interesting variations of linear programming [3]. It is easily seen that the hypothesis of Theorem 1.1 is not a necessary condition, and the following weakening of the hypothesis was recently discovered [1, 10].

**Theorem 1.2 (Strong Colorful Carathéodory).** Let $P_1, \ldots, P_{d+1}$ be non-empty point sets in $\mathbb{R}^d$. If $x \in \bigcap_{i=1}^{d+1} \text{conv}(P_i \cup P_j)$, then there exists $p_1 \in P_1, \ldots, p_{d+1} \in P_{d+1}$ such that $x \in \text{conv}\{p_1, \ldots, p_{d+1}\}$.

For applications of the strengthened version, see [1]. The goal of this paper is to give a twofold generalization of Theorem 1.2.

1. Points in $\mathbb{R}^d$ are replaced by an oriented matroid. Every vector configuration in $\mathbb{R}^d$ gives rise to an oriented matroid, but the converse does not hold. In fact, there are far more non-realizable oriented matroids than realizable ones. The question of whether the Colorful Carathéodory theorems extend to oriented matroids is a natural one, and has been asked ever since Bárány first introduced his result. (The rank 3 case was recently considered in [8].)
(2) The color classes are replaced by a matroid. It was noticed by Kalai and Meshulam \cite{11} that the colorful simplices are actually playing the role of bases of the transversal matroid of the family \( \{P_1, \ldots, P_{d+1}\} \). In the closely related setting of \( d \)-Leray complexes, they showed that the transversal matroid can be replaced by an arbitrary matroid. Our main result (Theorem 1.3) can be thought of as dual to the result of Kalai and Meshulam, and our proof is a modification of theirs.

1.1. Matroids and oriented matroids. A matroid is a combinatorial structure designed to capture the notion of linear independence in vector spaces. There are numerous equivalent (or “cryptomorphic”) axiom systems which define a matroid. For an introduction to matroid theory, see e.g. \cite{14}. The independent sets of a matroid naturally give rise to a simplicial complex whose topology will be of importance to us. For more information in this direction we refer the reader to \cite{4}. For a matroid \( N \) let \( \text{rk}(N) \) denote its rank.

An oriented matroid can be thought of as a combinatorial abstraction of a finite vector configuration spanning a vector space over an ordered field. As for ordinary matroids, there are several cryptomorphic axiom systems which define an oriented matroid. Of importance to us are the positive circuits. In the analogy of vector spaces, these correspond to the minimal positive linear dependencies of the configuration. For details, we refer the reader to \cite{7, 15}. For an oriented matroid \( M \) let \( \text{rk}(M) \) denote the rank of its underlying matroid. (In the sequel all oriented matroids are considered to be loopless.)

1.2. Main result.

**Theorem 1.3.** Let \( M \) be an oriented matroid and \( N \) a matroid with rank function \( \rho \), both defined on the same ground set \( V \) and satisfying \( \text{rk}(M) < \text{rk}(N) \). If every \( A \subset V \) with \( \rho(V - A) < \text{rk}(M) \) contains a positive circuit of \( M \), then some independent set of \( N \) contains a positive circuit of \( M \).

Theorem 1.2 is obtained by letting \( M \) be the oriented matroid of the vector configuration \( V = \{ p - x \mid p \in P_1 \cup \cdots \cup P_{d+1} \} \) and \( N \) the transversal matroid of the family \( \{P_1, \ldots, P_{d+1}\} \). Our proof of Theorem 1.3 uses topological methods: The Folkman-Lawrence representation theorem \cite{9} allows us to represent an oriented matroid as an arrangement of open pseudohemispheres with nice intersection properties. We may therefore pass to the nerve complex of the arrangement whose homology can be determined using the Nerve theorem (see \cite{5}). The rest of the proof (more or less) follows the arguments of Kalai and Meshulam \cite{11}.

2. Preliminaries

Here we collect the basic facts needed for the proof of Theorem 1.3.

2.1. Simplicial complexes and homology. First we review some standard notions from simplicial homology. Let \( X \) be a simplicial complex on \( V \). For \( W \subset V \) let

\[
X[W] = \{ T \in X : T \subset W \}
\]

denote the induced subcomplex on \( W \). For a simplex \( S \in X \) let

\[
\text{lk}(S, X) = \{ T \in X : T \cap S = \emptyset, T \cup S \in X \}
\]
denote the link of $S$. Let $\widetilde{H}_j(X)$ denote the $j$-th reduced homology group of $X$ with rational coefficients, $\tilde{\beta}_j(X) = \dim \widetilde{H}_j(X)$ its $j$-th reduced Betti number, and
\[ \eta(X) = \min\{j : \tilde{\beta}_j(X) \neq 0\} + 1 \]

For simplicial complexes $X$ and $Y$ on disjoint vertex sets, the join $X \ast Y$ is the simplicial complex on the union of their vertex sets defined as
\[ X \ast Y = \{ S \cup T : S \in X, T \in Y \} \]

By the Künneth formula
\[ \widetilde{H}_k(X \ast Y) \cong \bigoplus_{i+j=k-1} \widetilde{H}_i(X) \otimes \widetilde{H}_j(Y) \]

we obtain the following.

**Corollary 2.1.** $\eta(X \ast Y) = \eta(X) + \eta(Y)$

Let $X$ be a simplicial complex on $V$ and suppose $V \notin X$. The Alexander dual $X^*$ is the simplicial complex on $V$ defined as
\[ X^* = \{ T \subset V : V - T \notin X \} \]

The homology of $X$ and $X^*$ are related by Alexander duality, which says that if $V \notin X$ then $\widetilde{H}_i(X^*) \cong \widetilde{H}_{|V| - i - 3}(X)$ for all $-1 \leq i \leq |V| - 2$.

**Corollary 2.2.** If $V \notin X$, $S \notin X^*$, and $T = V - S$, then
\[ \widetilde{H}_i(X^*[S]) \cong \widetilde{H}_{|S| - i - 3}(\text{lk}(T, X)) \]

2.2. **The Nerve theorem.** Let $\mathcal{F} = \{ S_v \}_{v \in V}$ be family of sets. The nerve $N_\mathcal{F}$ is the abstract simplicial complex on $V$ whose simplices consists of those $T \subset V$ such that $\bigcap_{v \in T} S_v \neq \emptyset$. We recall the following version of the Nerve theorem. (For a proof see e.g. [5])

**Theorem 2.3.** Let $\mathcal{F} = \{ S_v \}_{v \in V}$ be a family of open contractible subsets of $\mathbb{R}^n$ such that every non-empty intersection $\bigcap_{v \in W} S_v$ is contractible. Then $\mathcal{F}$ and $N_\mathcal{F}$ are homotopy equivalent.

2.3. **The independence complex of a matroid.** Let $\mathcal{N}$ be a matroid on $V$. The independence complex of $\mathcal{N}$ is the simplicial complex $Y_\mathcal{N}$ on $V$ whose simplices are the independent sets of $\mathcal{N}$. In other words,
\[ Y_\mathcal{N} = \{ S \subset V : S \text{ independent in } \mathcal{N} \} \]

The following is a well-known fact. (See e.g. [11].)

**Lemma 2.4.** Let $\mathcal{N}$ be a matroid on ground set $V$ with rank function $\rho$ and $Y = Y_\mathcal{N}$ its independence complex. Then $\eta(Y[S]) \geq \rho(S)$ for every non-empty $S \subset V$.

2.4. **Topological representation of oriented matroids.** Let $\mathcal{M}$ be an oriented matroid of rank $d$ on the ground set $V$. (Oriented matroids are assumed to be loopless.) The Folkman-Lawrence representation theorem [23] states that $\mathcal{M}$ can be represented as an arrangement $\{ S_v \}_{v \in V}$ of oriented pseudospheres in $\mathbb{S}^{d-1}$. Such an arrangement decomposes $\mathbb{S}^{d-1}$ into a regular cell complex whose combinatorial structure encodes the oriented matroid. (See [27] for precise definitions and proofs.) Equivalently, $\mathcal{M}$ can be represented by the collection of open pseudohemispheres $\{ h_v \}_{v \in V}$ in $\mathbb{S}^{d-1}$, which have the $S_v$ as boundaries and lie on the “positive” sides. The crucial fact for us is the following.
Corollary 2.5. Let $\mathcal{M}$ be an oriented matroid on ground set $V$ and $\{h_v\}_{v \in V}$ a topological representation by pseudohemispheres. The intersection $h_W = \bigcap_{w \in W} h_w$ is empty or contractible for every $W \subseteq V$. Moreover, $h_W$ is empty if and only if $W$ contains a positive circuit of $\mathcal{M}$.

2.5. The support complex of an oriented matroid. Let $\mathcal{M}$ be an oriented matroid on $V$. The support complex of $\mathcal{M}$ is the simplicial complex $X_\mathcal{M}$ on $V$ whose simplices are the subsets of $V$ which do not contain positive circuits of $\mathcal{M}$. That is,

$$X_\mathcal{M} = \{ S \subset V : S \text{ contains no positive circuit of } \mathcal{M} \}$$

Proposition 2.6. Let $\mathcal{M}$ be an oriented matroid of rank $r$ and $X = X_\mathcal{M}$ its support complex. The following hold.

1. $\tilde{H}_j(X) = 0$ for all $j \geq r$.
2. $\tilde{H}_j(\text{lk}(S, X)) = 0$ for all $j \geq r - 1$ and non-empty $S \in X$.

Proof. Consider a topological representation of $\mathcal{M}$ by an arrangement of pseudohemispheres $A = \{h_v\}_{v \in V}$ in $S^{r-1}$. Corollary 2.5 implies that $X_\mathcal{M}$ is the nerve of $A$ and that every non-empty intersection $\bigcap_{v \in S} h_v$ is contractible. So by the Nerve theorem, $X$ is homotopic to $\bigcup_{v \in S} h_v \subset S^{r-1}$, hence $\tilde{H}_j(X) = 0$ for all $j \geq r$.

For the second part, let $\emptyset \neq S \in X$ and let $h_S = \bigcap_{v \in S} h_v$. The simplices of $\text{lk}(S, X)$ correspond to subsets $T \subset V \setminus S$ such that the intersection $\bigcap_{v \in T} h_v \cap h_S$ is non-empty. It follows that $\text{lk}(S, X)$ is the nerve of the family $\{h_v \cap h_S\}_{v \in V \setminus S}$, and the Nerve theorem implies that $\text{lk}(S, X)$ is homotopic to $\bigcup_{v \in V \setminus S} (h_v \cap h_S) \subset h_S$. Since $h_S$ is homeomorphic to $\mathbb{R}^{r-1}$ it follows that $\tilde{H}_j(\text{lk}(S, X)) = 0$ for all $j \geq r - 1$.

2.6. Colorful simplices. Let $Z$ be a simplicial complex on $V$ and $\bigcup_{i=1}^m V_i$ a partition of $V$. A colorfull simplex of $Z$ is a simplex $S \in Z$ such that $|S \cap V_i| = 1$ for all $1 \leq i \leq m$. Meshulam [13] gave the following sufficient condition for a simplicial complex on $\bigcup_{i=1}^m V_i$ to contain a colorfull simplex. (For a short proof based on the Nerve theorem, see [11])

Proposition 2.7. If for all $\emptyset \neq I \subset [m]$ we have

$$\eta(Z[\bigcup_{i \in I} V_i]) \geq |I|$$

then $Z$ contains a colorfull simplex.

3. Proof of Theorem 1.3

Let $V = \{v_1, v_2, \ldots, v_m\}$, $r = rk(\mathcal{M})$, $X = X_\mathcal{M}$ the support complex of $\mathcal{M}$, and $Y = Y_\mathcal{N}$ the independence complex of $\mathcal{N}$. Make a disjoint copy $V' = \{v'_1, v'_2, \ldots, v'_m\}$ of $V$ and let $Y'$ be an isomorphic copy of $Y$ on $V'$. Consider the join $Z = X^* \ast Y'$ and let $V_i = \{v_i, v'_i\}$ for $1 \leq i \leq m$.

Notice that a colorfull simplex $S \cup T' \in Z$ implies that $T = V - S$ is independent in $\mathcal{N}$. It also implies $T \not\subset X$ and therefore $T$ contains a positive circuit of $\mathcal{M}$. The strategy is therefore to apply Proposition 2.7 to show that $Z$ contains a colorfull simplex.

For $\emptyset \neq I \subset [m]$ set $S = \{v_i : i \in I\}$ and $S' = \{v'_i : i \in I\}$. By Corollary 2.1 and Lemma 2.4 we have
\[ \eta(Z \cup_{i \in I} V_i) = \eta(X^*[S] \star Y'[S']) = \eta(X^*[S]) + \eta(Y[S]) \geq \eta(X^*[S]) + \rho(S) \]

If \( S \in X^* \) then \( X^*[S] \) is contractible, which implies \( \eta(X^*[S]) = \infty > |I| \). We may therefore assume \( S \notin X^* \) and consequently \( T = V - S \in X \). By hypothesis \( \mathcal{M} \) contains positive circuits, hence \( V \notin X \), so by Corollary 2.2 we have

\[ \tilde{H}_i(X^*[S]) \cong \tilde{H}_{|S|-i-3}(\text{lk}(T,X)) \]

There are two cases to consider.

1. If \( S = V \) then \( X^*[S] = X^* \), \( T = \emptyset \), and \( \text{lk}(T,X) = X \). The first case of Lemma 2.6 implies that \( \tilde{H}_i(X^*) = 0 \) for all \( i \leq |S| - r - 3 \), hence

\[ \eta(X^*) \geq |S| - r - 1 \]

By hypothesis \( \rho(V) = rk(N) > r \), which implies

\[ \eta(Z) \geq \eta(X^*) + \rho(V) \geq (|S| - r - 1) + (r + 1) = |V| \]

2. If \( S \) is a proper subset of \( V \), then the second case of Lemma 2.6 implies that \( \tilde{H}_i(X^*[S]) = 0 \) for all \( i \leq |S| - r - 2 \), hence

\[ \eta(X^*[S]) \geq |S| - r \]

Since \( T = V - S \in X \), \( T \) does not contain a positive circuit of \( \mathcal{M} \), so by hypothesis \( \rho(S) = \rho(V - T) \geq r \). Hence

\[ \eta(Z \cup_{i \in I} V_i) \geq \eta(X^*[S]) + \rho(S) \geq (|S| - r) + r = |I| \]

Proposition 2.4 therefore implies that \( Z \) contains a colorful simplex. \( \square \)

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