LINEARITY PROBLEM FOR NON-ABELIAN TENSOR PRODUCTS

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Abstract. In this paper we give an example of a linear group such that its tensor square is not linear. Also, we formulate some sufficient conditions for the linearity of non-abelian tensor products $G \otimes H$ and tensor squares $G \otimes G$. Using these results we prove that tensor squares of some groups with one relation and some knot groups are linear. We prove that the Peiffer square of a finitely generated linear group is linear. At the end we construct faithful linear representations for the non-abelian tensor square of a free group and free nilpotent group.

1. Introduction

R. Brown and J.-L. Loday [8, 9] introduced the non-abelian tensor product $G \otimes H$ for a pair of groups $G$ and $H$ following works of Miller [5], and Lue [7]. They showed that the third homotopy group of the suspension of an Eilenberg-MacLane space $K(G,1)$ satisfies

$$\pi_3 SK(G,1) \cong J_2(G),$$

where $J_2(G)$ is the kernel of the derived map $\kappa : G \otimes G \to G'$, $g \otimes h \mapsto [g, h] = g^{-1}h^{-1}gh$. Hence there exists the short exact sequence

$$0 \to \pi_3 SK(G,1) \to G \otimes G \to G' \to 1.$$

Also, the non-abelian tensor product is used to describe the third relative homotopy group of a triad as a non-abelian tensor product of the second homotopy groups of appropriate subspaces. More specifically, let a $CW$-complex $X$ be the union $X = A \cup B$ of two path-connected $CW$-subspaces $A$ and $B$ whose intersection $C = A \cap B$ is path-connected. If the canonical homomorphisms $\pi_1(C) \to \pi_1(A)$, $\pi_1(C) \to \pi_1(B)$ are surjective, then, according to [8],

$$\pi_3(X, A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C),$$

where the groups $\pi_2(A, C)$ and $\pi_2(B, C)$ act on one another via $\pi_1(C)$.

The investigation of the non-abelian tensor product from a group theoretical point of view started with a paper by Brown, Johnson, and Robertson [15]. They compute the non-abelian tensor square of all non-abelian groups of order up to 30 using Tietze transformations.

One of the topics of research on the non-abelian tensor products has been to determine which group properties are preserved by non-abelian tensor products. By using
homological arguments, Ellis [10] showed that if \( G \) and \( H \) are finite groups, then \( G \otimes H \) is also finite. Visscher [12] proved that if \( G, H \) are solvable (nilpotent), then \( G \otimes H \) is solvable (nilpotent) and gives a bound on the nilpotency class of \( G \otimes H \). In [17] it was proved that the tensor product of groups of nilpotency class at most \( n \) is a group of nilpotency class at most \( n \), thereby improving the bound given by Visscher. For other results in this direction see the survey of Nakaoka [13].

In this paper we study the linearity problem for non-abelian tensor products. Let \( n \) be a positive integer and let \( P \) be a field. A group \( G \) is said to be linear of degree \( n \) over \( P \) if it is isomorphic with a subgroup of \( GL_n(P) \), the group of all \( n \times n \) non-singular matrices over \( P \) or, equivalently, if it is isomorphic with a group of invertible linear transformations of a vector space of dimension \( n \) over \( P \) (see [19]). We study the following

**Question 1.** Let \( G \) and \( H \) be linear groups. Are the groups \( G \otimes H, G \otimes G \) linear?

We show that in general the answer is negative. More accurately, we prove that the tensor square \( SL_n(\mathbb{Q}) \otimes SL_n(\mathbb{Q}) \) of the special linear group \( SL_n(\mathbb{Q}) \) over the field of rational numbers is not linear for \( n \geq 3 \). On the other side we formulate some sufficient conditions under which the groups \( G \otimes H, G \otimes G \) are linear. Using these conditions, we prove that the non-abelian tensor squares of some groups with one defining relation and groups of fibered knots are linear. If \( G \) is a finitely generated free group or finitely generated free nilpotent group, then we construct concrete faithful linear representations for \( G \otimes G \).

The non-abelian tensor square \( G \otimes G \) is connected to other group constructions: exterior tensor square \( G \wedge G \) and Peiffert square \( G \ast G \). We prove that if \( G \) is finitely generated, then \( G \ast G \) is linear.

We note that the following problems are still open.

**Problem 1.** 1) Let \( G \) be a finitely generated linear group. Is the group \( G \otimes G \) linear? 2) Let \( G \) be a linear group. Is the group \( G \ast G \) linear?

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2. Preliminaries

In this paper we shall use the following notations. For elements \( x, y \) in a group \( G \), the conjugation of \( x \) by \( y \) is \( x^y = y^{-1}xy \); and the commutator of \( x \) and \( y \) is \([x,y] = x^{-1}y^{-1}xy\). We write \( G' \) for the derived subgroup of \( G \), \( G^{ab} \) for the abelianized group \( G/G' \).

Recall the definition of the non-abelian tensor product \( G \otimes H \) of groups \( G \) and \( H \) (see [8, 9]). This tensor product is defined for any pair of groups \( G \) and \( H \) where each one acts on the other (on the right)

\[
G \times H \rightarrow G, \ (g,h) \mapsto g^h; \ H \times G \rightarrow H, \ (h,g) \mapsto h^g
\]
and on itself by conjugation, in such a way that for all \( g, g_1 \in G \) and \( h, h_1 \in H \),

\[
g^{(g_1 h_1)} = \left( \left( g g_1^{-1} \right)^h \right)^{g_1} \quad \text{and} \quad h^{(g h_1)} = \left( \left( h h_1^{-1} \right)^g \right)^{h_1}.
\]

In this situation we say that \( G \) and \( H \) act \textit{compatibly} on each other. The \textit{non-abelian tensor product} \( G \otimes H \) is the group generated by all symbols \( g \otimes h \), \( g \in G \), \( h \in H \), subject to the relations

\[
    gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})
\]

for all \( g, g_1 \in G \), \( h, h_1 \in H \).

In particular, as the conjugation action of a group \( G \) on itself is compatible, then the tensor square \( G \otimes G \) of a group \( G \) may always be defined. Also, the tensor product \( G \otimes H \) is defined if \( G \) and \( H \) are two normal subgroups of some group \( M \) and actions are conjugations in \( M \).

Recall the main diagram for the non-abelian tensor square (see [8, 9]). Let \( G \) be a group. One of the main tools for studying of the non-abelian tensor square \( G \otimes G \) is the following diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & \downarrow & \downarrow & \\
H_3(G) & \rightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & J_2(G) & \rightarrow & H_2(G) & \rightarrow & 0 \\
\| & \| & \downarrow & \downarrow & \\
H_3(G) & \rightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & G \otimes G & \rightarrow & G \wedge G & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
G' & = & G' & 1 & 1
\end{array}
\]

with exact rows and columns. Here

1) \( H_2(G), \ H_3(G) \) are the second and the third homology groups of \( G \) with the coefficients in the trivial \( ZG \)-module \( Z \). The second homology group \( H_2(G) \) for the group \( G = F/R \), where \( F \) is a free group, can be find by the Hopf formula:

\[
H_2(G) \cong (F' \cap R)/[F, R].
\]

2) \( G \wedge G \) is the exterior product of \( G \) onto itself. For the group \( G = F/R \) it can be presented in the form (see [2])

\[
G \wedge G \cong F'/[F, R].
\]

In particular, if \( G \) is a free group, then \( G \wedge G \cong G' \).

3) \( J_2(G) = \pi_3SK(G, 1) \) is the kernel of the derived map \( \kappa : G \otimes G \rightarrow G' \), which on the generators of \( G \otimes G \) is defined by the rule

\[
g_1 \otimes g_2 \mapsto [g_1, g_2].
\]

The group \( J_2(G) \) lies in the center \( Z(G \otimes G) \) and its elements are invariant under the action of \( G \) onto \( G \otimes G \), which is defined by the formula

\[
(g_1 \otimes g_2)^{g} = g_1^g \otimes g_2^g.
\]
In particular, if \( g_2 = g_1 \), then
\[
(g_1 \otimes g_1)^g = g_1 \otimes g_1
\]
for any \( g, g_1 \in G \).

4) \( \Gamma(G^{ab}) \) is Whitehead’s quadratic functor. The group \( \Gamma(G^{ab}) \) is generated by elements \( \gamma(gG') \) and \( \psi \) is defined by the formula
\[
\gamma(gG') \mapsto g \otimes g.
\]
The image \( \psi \Gamma(G^{ab}) \) is not equal in the general case to the group \( J_2(G) \) since
\[
J_2(G) / \psi \Gamma(G^{ab}) \cong H_2(G);
\]
For the functor \( \Gamma : A \mapsto A \Gamma(G^{ab}) \), where \( A \) is an abelian group it is known that
\begin{itemize}
  
  a) \( \Gamma(A \times B) \cong \Gamma(A) \times \Gamma(B) \times (A \otimes Z B) \), where \( A \otimes Z B \) is the abelian tensor product of abelian groups;
  
  b) \( \Gamma(Z_n) \cong \begin{cases} 
  Z_n & n \equiv 1 \pmod{2}, \\
  Z_{2n} & n \equiv 0 \pmod{2}.
  \end{cases} \)
  
  c) \( \Gamma(Z) \cong Z \).
\end{itemize}
In particular, \( \Gamma(Z^n) \cong Z^{n(n+1)/2} \).

3. Linearity problem

In this section we will use a result of Malcev [19] (see also [20, Chapter 2]) on the linearity of abelian groups. To formulate it recall some definitions. If \( G \) is any group \( \tau(G) \) is the subgroup of \( G \) generated by all the periodic normal subgroups of \( G \); that is \( \tau(G) \) is the maximum periodic normal subgroup of \( G \) has finite rank at most \( n \) if every finite subset of \( G \) is contained in an \( n \)-generator subgroup of \( G \). If \( G \) is abelian and periodic then \( G \) has finite rank at most \( n \) if and only if for each prime \( p \) the Sylow \( p \)-subgroup of \( G \) is a direct product of at most \( n \) cyclic and Prüfer \( p \)-groups (a Prüfer \( p \)-group is a \( C_{p^\infty} \)-group). If \( \pi \) is any set of primes and \( G \) is a group with a unique maximal \( \pi \)-subgroup we denote this maximal \( \pi \)-subgroup by \( G_{\pi} \).

Malcev proved:
\begin{itemize}
  
  i) An abelian group \( A \) has a faithful representation of degree \( n \geq 1 \) over some field of characteristic zero if and only if \( \tau(G) \) has rank at most \( n \).
  
  ii) An abelian group \( A \) has a faithful representation of degree \( n \geq 1 \) over some field of characteristic \( p > 0 \) if and only if \( \tau(G)p' \) (here \( p' \) denotes all primes except \( p \)) has finite rank \( r \) and \( \tau(A)_{p'} \) has finite exponent \( p^e \) satisfying
\[
p^{e-1} + \max\{1, r\} < n + 1.
\]
\end{itemize}

We are ready to prove the following

**Proposition 3.1.** There is a linear group \( G \) such that \( G \otimes G = G \wedge G \) is not linear.

**Proof.** For a perfect group \( G = G' \) it follows from the main diagram (see section [2]) that \( G \otimes G = G \wedge G \) and the sequence
\[
0 \to H_2(G, Z) \to G \otimes G \to G \to 0
\]
is exact.
For $n \geq 3$ the group $SL_n(\mathbb{Q})$ is perfect and its second homology group coincides with the $K_2$-group of the field $\mathbb{Q}$, 

$$H_2(SL_n(\mathbb{Q}), \mathbb{Z}) = K_2(\mathbb{Q}),$$

see [6, Corollary 11.2].

Next,

$$K_2(\mathbb{Q}) = \{\pm 1\} \times \prod_{p \text{ odd prime}} (\mathbb{Z}/p)^\times$$

by [6, Theorem 11.6], so that $K_2(\mathbb{Q})$ contains an abelian 2-group of infinite rank and unbounded exponent. Using Maltsev’s criterion we conclude that such a group can not be linear. Therefore the group

$$SL_n(\mathbb{Q}) \otimes SL_n(\mathbb{Q}) = SL_n(\mathbb{Q}) \wedge SL_n(\mathbb{Q})$$

is not linear as well. 

To study the linearity problem for the non-abelian tensor product we can use a presentation of a tensor product as a central extension (see, for example, [17].) The derivative subgroup of $G$ by $H$ is defined to be the following subgroup

$$D_H(G) = \langle g^{-1}g^h \mid g \in G, h \in H \rangle.$$

The map $\kappa : G \otimes H \rightarrow D_H(G)$ defined by $\kappa(g \otimes h) = g^{-1}g^h$ is a homomorphism, its kernel is the central subgroup of $G \otimes H$ and $G$ acts on $G \otimes H$ by the rule $(g \otimes h)x = g^x \otimes h^x$, $x \in G$. There exists the short exact sequence

$$1 \rightarrow A \rightarrow G \otimes H \rightarrow D_H(G) \rightarrow 1.$$ 

In this case $A$ can be viewed as a $\mathbb{Z}[D_H(G)]$-module via conjugation in $G \otimes H$, i. e. under the action induced by setting

$$a \cdot g = x^{-1}ax, \; a \in A, x \in G \otimes H, \kappa(x) = g.$$

We can formulate some sufficient conditions for the linearity of $G \otimes H$. It is well known that the tensor product $G \otimes H$ with trivial actions is isomorphic to the abelian tensor product $G_{ab} \otimes \mathbb{Z}H_{ab}$. Hence, in this case the question on the linearity of $G \otimes H$ is equivalent to the question on the linearity of the abelian tensor product and the answer follows from the Malcev theorem.

Further we will assume that the action of $G$ on $H$ or the action of $H$ on $G$ is non-trivial. We have the following short exact sequence

$$0 \rightarrow A \rightarrow G \otimes H \rightarrow D_H(G) \rightarrow 1.$$ 

(1)

Note that $A$ is the kernel of the natural map $G \otimes H \rightarrow D_H(G)$, $g \otimes h \rightarrow g^{-1}g^h$, $g \in G, h \in H$, and is a central subgroup of $G \otimes H$.

**Proposition 3.2.** Let the following conditions hold

1) $A, D_H(G)$ are linear groups;
2) $H^2(D_H(G), A) = 0$, in particular, this condition holds if $A$ is divisible or $D_H(G)$ is a free group.

Then $G \otimes H = A \times D_H(G)$ is the direct product and is a linear group.
Proof. It is well known that if \( H^2(D_H(G), A) = 0 \), then the sequence
\[
0 \rightarrow A \rightarrow G \otimes H \rightarrow D_H(G) \rightarrow 1,
\]
splits. In particular, this condition holds if \( A \) is divisible or \( D_H(G) \) is a free group.
Since \( A \) is a central subgroup, then \( G \otimes H \cong A \times D_H(G) \) and is a linear group as a direct product of linear groups.
\( \square \)

The main problem in the use of this theorem is the description of the central subgroup \( A \). For the tensor square we can use another approach.

Let us formulate some sufficient conditions under which \( G \otimes G \) is a direct product of the commutator subgroup \( G' \) and the Whitehead group \( \Gamma(G^{ab}) \).

Theorem 3.3. Let \( H_2(G) = H_3(G) = H_2(G') = 0 \) and one of the following conditions hold
1) \( H^2(G', \Gamma(G^{ab})) = 0 \);
2) \( \Gamma(G^{ab}) \) is a divisible group;
3) \( G'/G'' \) is a free abelian group.

Then
\[
G \otimes G \cong \Gamma(G^{ab}) \times G'.
\]
If moreover \( G \) is finitely generated and \( G' \) is linear, then \( G \otimes G \) is linear.

Proof. Since, \( H_2(G) = H_3(G) = 0 \), then the main diagram has the form
\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow & & & & \\
0 & \rightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & J_2(G) & \rightarrow & 0 & \rightarrow & 0 \\
\| & \downarrow & & & & & \downarrow & & \\
0 & \rightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & G \otimes G & \rightarrow & G \wedge G & \rightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & G' & = & G' & & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 1 & & 1 & & & \\
\end{array}
\]
From this diagram \( J_2(G) = \Gamma(G^{ab}) \) and \( G \wedge G = G' \). Hence we have the short exact sequence
\[
0 \rightarrow \Gamma(G^{ab}) \rightarrow G \otimes G \rightarrow G' \rightarrow 1.
\]
If \( H^2(G', \Gamma(G^{ab})) = 0 \), then this sequence is splittable:
\[
G \otimes G = \Gamma(G^{ab}) \times G'.
\]
As we know if \( \Gamma(G^{ab}) \) is divisible or \( G' \) is free, then \( H^2(G', \Gamma(G^{ab})) = 0 \). Let us show that if \( G'/G'' \) does not have torsion, then \( H^2(G', \Gamma(G^{ab})) = 0 \). Indeed, by the universal coefficient theorem there is the following short exact sequence
\[
0 \rightarrow Ext_\mathbb{Z}(H_1(G'), \Gamma(G^{ab})) \rightarrow H^2(G', \Gamma(G^{ab})) \rightarrow Hom_\mathbb{Z}(H_2(G'), \Gamma(G^{ab})) \rightarrow 0.
\]
Since \( H_2(G') = 0 \) we have the short exact sequence
\[
0 \rightarrow Ext_\mathbb{Z}(H_1(G'), \Gamma(G^{ab})) \rightarrow H^2(G', \Gamma(G^{ab})) \rightarrow 0.
\]
Hence $H^2(G', \Gamma(G^{ab})) = 0$ if and only if $\text{Ext}_\mathbb{Z}(H_1(G'), \Gamma(G^{ab})) = 0$. It is known that if $H_1(G')$ is free abelian, then $\text{Ext}_\mathbb{Z}(H_1(G'), \Gamma(G^{ab})) = 0$.

Since $\Gamma(G^{ab})$ is a central subgroup, this product is the direct product:

$$G \otimes G = \Gamma(G^{ab}) \times G'.$$

If $G$ is a finitely generated, then $G^{ab}$ is finitely generated abelian group and $\Gamma(G^{ab})$ is also a finitely generated abelian group. Then $G \otimes G$ is linear as a direct product of two linear groups. □

As consequence we get the following result

**Corollary 3.4.** [2] Let $F_n$ be a free group of rank $n$. Then

$$F_n \otimes F_n \cong \mathbb{Z}^{n(n+1)/2} \times (F_n)'$$

4. **Groups with one defining relation and knot groups**

Let $G$ be a group with one defining relation:

$$G = \langle X \parallel r = 1 \rangle,$$

where $r \notin F'$, $F = \langle X \rangle$. Then $H_k(G) = 0$, $k \geq 2$ (see [3, p. 49]). Hence, there exists the following short exact sequence:

$$0 \rightarrow \Gamma(G^{ab}) \rightarrow G \otimes G \rightarrow G' \rightarrow 1.$$ 

If $G^{ab}$ does not have torsion, then $G^{ab}$ is a free abelian group and $\Gamma(G^{ab})$ is a free abelian group. Then, if $H^2(G') = 0$, then $H^2(G', \Gamma(G^{ab})) = 0$, which follows from the decomposition

$$H^k(S, A \oplus B) = H^k(S, A) \oplus H^k(S, B)$$

for every group $S$ and all $S$-modules $A$ and $B$.

From Theorem 3.3 follows

**Proposition 4.1.** Let $G$ be a group with one defining relation:

$$G = \langle X \parallel r = 1 \rangle,$$

where $r \notin F'$, $F = \langle X \rangle$ such that $H^2(G') = 0$. If one from the following conditions hold:

1) $G^{ab}$ does not have torsion;
2) $G'/G''$ is a free abelian group.

Then $G \otimes G = \Gamma(G^{ab}) \times G'$. If moreover $G$ is finitely generated and $G$ is linear, then $G \otimes G$ is linear.

It is well known that if $K$ is a tame knot in 3-sphere $S^3$ and $G_K = \pi_1(S^3 \setminus K)$ its group, then $H_n(G_K) = 0$ for $n > 1$ (see, for example [18, p. 5]). Recall that a knot $K$ is called fibered if there is a 1-parameter family $F_t$ of Seifert surfaces for $K$, where the parameter $t$ runs through the points of the unit circle $S^1$, such that if $s$ is not equal to $t$ then the intersection of $F_s$ and $F_t$ is exactly $K$. The commutator subgroup $G'_K$ for the fibered knot $K$ is a free group of finite rank [14] and $G_K$ is linear [1].
Proposition 4.2. Let \( K \) be a tame fibered knot in 3-sphere \( S^3 \), then \( G_K \otimes G_K = G'_K \times \mathbb{Z} \) and has a faithful linear representation into \( GL_2(\mathbb{Z}[t,t^{-1}]) \).

Proof. It is well known that \( G_{ab}^K = \mathbb{Z} \) and then \( \Gamma(G_{ab}^K) = \mathbb{Z} \). From Theorem 3.3 it follows that \( G_K \otimes G_K = \mathbb{Z} \times G'_K \).

To construct a linear representation, use the fact that \( G'_K \) is a free group of finite rank and by Sanov’s theorem [11, Chapter 5] it has a faithful linear representation into \( SL_2(\mathbb{Z}) \leq GL_2(\mathbb{Z}[t,t^{-1}]) \). Define a linear representation of \( G_{ab}^K = \mathbb{Z} = \langle \gamma \rangle \) into \( GL_2(\mathbb{Z}[t,t^{-1}]) \) by the rule

\[
\gamma \mapsto \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.
\]

Since the image of \( \gamma \) is a scalar matrix, i.e. lies in the center of \( GL_2(\mathbb{Z}[t,t^{-1}]) \), we constructed a faithful linear representation of \( G_K \otimes G_K \). \( \square \)

Example 4.3. 1) The braid group \( B_3 \) on 3 strings has presentation

\[
B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle,
\]

and is the group of trefoil knot. The commutator subgroup \( B'_3 \) is a free group of rank 2. Hence the tensor square \( B_3 \otimes B_3 = \mathbb{Z} \times F_2 \) has a faithful linear representation into \( GL_2(\mathbb{Z}[t,t^{-1}]) \).

2) It is known that the group of the figure eight knot has a presentation

\[
G = \langle x, y \mid xy^{-1}yx^{-1} = x^{-1}yx^{-1}, x \rangle
\]

and is a fibered knot. Hence the tensor square \( G \otimes G = \mathbb{Z} \times G' \) has a faithful linear representation into \( GL_2(\mathbb{Z}[t,t^{-1}]) \).

In the first example we shown that \( B_3 \otimes B_3 = \mathbb{Z} \times F_2 \). On the other side \( B_3 \) contains the pure braid group \( P_3 \), which is normal in \( B_3 \), has index 6 and is the direct product of the center, which is isomorphic to \( \mathbb{Z} \), and a free group of rank 2. Hence, \( B_3 \otimes B_3 \) is isomorphic to \( P_3 \) and we proved

Proposition 4.4. There is a non-trivial non-abelian group \( G \) such that the tensor square \( G \otimes G \) is isomorphic to a proper subgroup of \( G \).

Question 2. 1) Is it true that \( B_n \otimes B_n, n \geq 3 \), is linear?

2) Is it true that for arbitrary tame knot \( K \) the group \( G(K) \otimes G(K) \) is linear?

5. On the linearity of the Peiffer product

Recall the definition of the Peiffer product. Given \( G \) and \( H \) acting compatibly on each other, in [21] the Peiffer product \( G \bowtie H \) was defined their as the quotient of the free product \( G \ast H \) by the normal closure \( K \) of all elements of the form

\[
h^{-1}g^{-1}hg^k \text{ or } g^{-1}h^{-1}gh^g
\]

where \( g \in G \) and \( h \in H \). Whitehead [21] posed a question on the asphericity of subcomplexes of aspherical 2-complexes and reformulated it as part of the wider problem of finding conditions under which the groups \( G \) and \( H \) are embedded in \( G \bowtie H \).
In [4] it was proved that if \( \varphi : G \ast H \to G \rtimes H \), then modulo \( K = \text{Ker}(\varphi) \), \( hg \equiv gh^g \), so that every element of \( G \rtimes H \) can be written as \( \varphi(g)\varphi(h) \) for suitable \( g, h \). Denote \( \varphi(g)\varphi(h) \) as \( \langle g, h \rangle \). The relations
\[
\langle g, h \rangle \langle g_1, h_1 \rangle = \langle gg_1, h^g h_1 \rangle = \langle gg_1^{-1}, hh_1 \rangle
\]
are defining relations for \( G \rtimes H \) on the generators \( \langle g, h \rangle \) and so \( G \rtimes H \) is a homomorphic image of both the semidirect products \( G \ltimes H \) and \( G \rtimes H \). The group \( G \rtimes H \) is obtained from \( G \ltimes H \) (or from \( G \rtimes H \)) by imposing the relations
\[
(g^{-1}g^h, 1) = (1, h^{-g}h).
\]
If \( G \) and \( H \) act on one another trivially, then \( G \rtimes H \) is just the direct product \( G \times H \) and \( K = \text{Ker}(G \ast H) \), where \( G \rtimes H \) is the Cartesian subgroup of \( G \ast H \) (the kernel of the canonical homomorphism \( G \ast H \to G \times H \)).

From [4, Proposition 2.1] follows
\[
G \rtimes G \simeq G^{ab} \times G^{ab}.
\]
Using this isomorphism one can prove

**Proposition 5.1.** Let \( G \) be a linear group and \( G \) is finitely generated or \( G = G' \), then \( G \rtimes G \) is linear.

From this proposition it follows that if \( G = \text{SL}_n(\mathbb{Q}) \), \( n \geq 2 \), then \( G \rtimes G \) is linear. On the other side, we know that \( \text{SL}_n(\mathbb{Q}) \otimes \text{SL}_n(\mathbb{Q}) \) and \( \text{SL}_n(\mathbb{Q}) \wedge \text{SL}_n(\mathbb{Q}) \) are not linear for \( n \geq 3 \).

Note that this proposition is not true for arbitrary linear group \( G \) since there are linear groups with nonlinear abelization.

**Example 5.2.** 1) (O. V. Bryukhanov) Let \( G = \bigast_{i=2}^{\infty} \mathbb{Z}_i \) be the free product of cyclic groups. Then \( G \) is linear as the free product of linear groups. On the other side, by Malcev criteria (see Section 3) the abelization \( G^{ab} \) is not linear.

2) (J. O. Button) Take the set of matrices
\[
A_i = \begin{pmatrix} 1 & x^i \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}[x]), \ i \in \mathbb{N}.
\]
Then \( A = \langle A_i \mid i \in \mathbb{N} \rangle \) is a free abelian group of countable rank. Put
\[
B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{Q}).
\]
It is easily to check that these matrices satisfy the relations
\[
BA_iB^{-1} = A_i^3, \ i \in \mathbb{N}.
\]
Hence the group generated by \( A_i \) and \( B \) has the presentation
\[
G_2 = \langle A_i, i \in \mathbb{N}, B \mid [A_i, A_j] = 1, BA_iB^{-1} = A_i^3, i, j \in \mathbb{N} \rangle,
\]
which is a subgroup of \( \text{GL}_2(\mathbb{Q}[x]) \), but its abelization \( G_2^{ab} \approx \bigoplus_{i=1}^{\infty} \mathbb{Z}_2 \oplus \mathbb{Z} \) does not have a faithful linear representations over field of characteristic \( p \neq 2 \).
Analogically, take the set of matrices
\[ C_i = \begin{pmatrix} 1 & y^i \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}[y]), \ i \in \mathbb{N}. \]

Then \( C = \langle C_i \mid i \in \mathbb{N} \rangle \) is a free abelian group of countable rank. Put
\[ D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q}). \]

It is easily to check that these matrices satisfy the relations
\[ DC_i D^{-1} = C_4^i, \ i \in \mathbb{N}. \]

Hence the group generated by \( C_i \) and \( D \) has the presentation
\[ G_3 = \langle C_i, i \in \mathbb{N}, D \mid [C_i, C_j] = 1, DC_i D^{-1} = C_4^i, i, j \in \mathbb{N} \rangle, \]
which is a subgroup of \( GL_2(\mathbb{Q}[y]) \), but its abelization \( G_3^{ab} \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}_3 \oplus \mathbb{Z} \) does not have a faithful linear representations over field of characteristic \( p \neq 3 \).

Let us take \( G = G_2 \oplus G_3 \). Then it is metabelian and has a faithful linear representation in \( GL_4(\mathbb{Q}[x, y]) \), but its abelization \( G^{ab} \cong \bigoplus_{i=1}^{\infty} (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus \mathbb{Z} \oplus \mathbb{Z} \) is not linear.

It is evident that the following short exact sequence holds
\[ 1 \rightarrow 1 \times G' \rightarrow G^{ab} \times G \rightarrow G^{ab} \times G^{ab} \rightarrow 1. \]

Since \( G^{ab} \times G \cong G \rtimes G \) we can add in the main diagram new terms.

**Proposition 5.3.** The following diagram holds

\[
\begin{array}{ccccccccc}
0 & 0 & \downarrow & \downarrow & \\
H_3(G) & \rightarrow & \Gamma(G^{ab}) & \overset{\psi}{\rightarrow} & J_2(G) & \rightarrow & H_2(G) & \rightarrow & 1 \\
\cong & \cong & \downarrow & \downarrow & \\
H_3(G) & \rightarrow & \Gamma(G^{ab}) & \overset{\psi}{\rightarrow} & G \otimes G & \rightarrow & G \wedge G & \rightarrow & 1 \\
& & \downarrow & \downarrow & \\
& & G \rtimes G & = & G \rtimes G & & \\
& & \downarrow & \downarrow & \\
& & G^{ab} \times G^{ab} & = & G^{ab} \times G^{ab} & = & H_1(G) \times H_1(G) \\
& & \downarrow & \downarrow & 1 & 1 & \\
& & 1 & 1 & & & \\
\end{array}
\]

6. **Faithful linear representations**

In the paper [2] it was proved:

1) If \( F_n \) is the free group of rank \( n \), then
\[ F_n \otimes F_n \cong \mathbb{Z}^{n(n+1)/2} \times (F_n)', \]
2) If \( N_{n,c} = F_n/\gamma_c F_n \) is the free nilpotent group of rank \( n > 1 \) and class \( c \geq 1 \), then
\[
N_{n,c} \otimes N_{n,c} \cong \mathbb{Z}^{n(n+1)/2} \times (N_{n,c+1})'.
\]

**Proposition 6.1.** Let \( G \) be a free countable group. Then the exterior square \( G \wedge G \) has a faithful representation into \( SL_2(\mathbb{Z}) \) and the tensor square \( G \otimes G \) has a faithful representation into \( GL_2(\mathbb{C}) \).

**Proof.** As was proven in [9], for the free group \( G \) there are isomorphisms
\[
G \wedge G \cong G', \quad G \otimes G \cong \Gamma(G^{ab}) \times G'.
\]
Since \( G \) is free, its commutator subgroup \( G' \) is free. Hence, by the Sanov result [11, Chapter 5] there is a faithful representation of \( G' \) into \( SL_2(\mathbb{Z}) \) and the first part of the proposition holds.

Further, \( \Gamma(G^{ab}) \) is a free abelian group. Let \( a_k, k \in I \) be its free generators. Take transcendental elements \( t_k, k \in I \) in the field \( \mathbb{C} \), which are algebraically independent over \( \mathbb{Q} \). Then the matrix group
\[
T = \left\langle \left( \begin{array}{cc} t_k & 0 \\ 0 & t_k \end{array} \right) \mid k \in I \right\rangle
\]
is isomorphic to the group \( \Gamma(G^{ab}) \). If \( \varphi : G' \to GL_2(\mathbb{Z}) \) is an embedding, then
\[
\left\langle \varphi G', T \right\rangle \cong \varphi G' \times T.
\]
Hence, the group \( G \otimes G \) has a faithful representation in the matrix group over the ring \( \mathbb{Z}[t_k^\pm 1, k \in I] \).

If \( G = F_\infty \) is countably generated then it has a faithful representation into \( SL_2(\mathbb{Z}) \). To prove that \( \Gamma(F^{ab}_\infty) \) is linear we use the following property
\[
\Gamma(F^{ab}_\infty) = \Gamma(\lim F^{ab}_n) = \lim(\Gamma F^{ab}_n).
\]

For the finitely generated free groups from this theorem follows

**Corollary 6.2.** The tensor square \( F_n \otimes F_n \) has a faithful representation into \( GL_2(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]) \), where \( m = \frac{n(n+1)}{2} \).

For the free nilpotent groups we can prove

**Proposition 6.3.** There is a faithful representation
\[
N_{n,c} \otimes N_{n,c} \to T_{c+2}(\mathbb{C})
\]
into the group of triangular matrices \( T_{c+2}(\mathbb{C}) \).

**Proof.** We noted that
\[
N_{n,c} \otimes N_{n,c} \cong \mathbb{Z}^{n(n+1)/2} \times (N_{n,c+1})'.
\]
Hence, we have to define faithful linear representations for \( \mathbb{Z}^{n(n+1)/2} = \langle a_1, a_2, \ldots, a_m \rangle \), \( m = n(n+1)/2 \) and for \( (N_{n,c+1})' \), where \( N_{n,c+1} = \langle x_1, x_2, \ldots, x_n \rangle \). Let
\[
\tau_1, \tau_2, \ldots, \tau_m, t_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, c + 1,
\]
be complex numbers which are algebraically independent over $\mathbb{Q}$. Define the following maps

$$a_k \mapsto \tau_k E \in T_{c+2}(\mathbb{C}), \ k = 1, 2, \ldots, m,$$

which defines a faithful representation of $\mathbb{Z}^{n(n+1)/2}$ into $T_{c+2}(\mathbb{C})$, and

$$x_i \mapsto \begin{pmatrix}
1 & t_{i1} & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & t_{i2} & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 1 & t_{ic} \\
0 & 0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix} \in UT_{c+2}(\mathbb{C}) \ i = 1, 2, \ldots, n.$$

As Romanovskii proved [16] the map, defined on $x_i$ is a faithful representation of $N_{n,c+1}$ into $UT_{c+2}(\mathbb{C})$. Hence we have a faithful representation of $\mathbb{Z}^{n(n+1)/2} \times N_{n,c+1}$ into $T_{c+2}(\mathbb{C})$. Since $\mathbb{Z}^{n(n+1)/2} \times (N_{n,c+1})'$ is a subgroup of $\mathbb{Z}^{n(n+1)/2} \times N_{n,c+1}$, we have the needed representation. \qed

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