Classification of double flag varieties of complexity 0 and 1

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Abstract. We obtain a classification of double flag varieties of complexity 0 and 1 and consider an application to the problem of decomposing tensor products of irreducible representations of semisimple Lie groups.

Keywords: semisimple Lie groups, double flag varieties, complexity, linear representations.

§ 1. Introduction

Let $G$ be a semisimple complex algebraic group and let $B$ be a Borel subgroup of $G$. Suppose that $G$ acts on an irreducible complex algebraic variety $X$. This action induces an action of $B$ on $X$.

Definition 1. The complexity of the action of $G$ on $X$ is the codimension $c(X) = c_G(X)$ of a generic orbit $Bx \subseteq X$ of the action of the Borel subgroup.

Remark 1. By Rosenlicht’s theorem, $c(X) = \text{trdeg} \mathbb{C}(X)^B / \mathbb{C}$.

A subgroup $P \subset G$ is said to be parabolic if it contains a Borel subgroup. Double flag varieties are varieties of the form $X = G/P \times G/Q$, where $P$ and $Q$ are parabolic subgroups. We consider the diagonal action of $G$ on $G/P \times G/Q$. This paper is devoted to the classification of double flag varieties of complexity at most 1. Littelmann [1] classified double flag varieties of complexity 0 for maximal parabolic subgroups. Stembridge [2] classified all double flag varieties of complexity 0. Panyushev [3] found the complexities of double flag varieties for all maximal parabolic subgroups. We shall obtain these known results by a uniform method, which is simpler and more conceptual, and complete the classification in the case of complexity 1.

The problem of classifying double flag varieties of complexity 0 and 1 has an application to decomposing tensor products of irreducible representations of $G$ into irreducible summands. Any irreducible $G$-module can be realized as the space of global sections of some line bundle $\mathcal{L}$ over $G/P$ (where $P$ is a parabolic subgroup). The tensor product $H^0(G/P, \mathcal{L}) \otimes H^0(G/Q, \mathcal{M})$ of the space of sections may be regarded as the space of sections of a line bundle over the direct product of $G/P$ and $G/Q$, that is, $H^0(G/P \times G/Q, \mathcal{L} \boxtimes \mathcal{M})$. Here $\mathcal{L} \boxtimes \mathcal{M} \to G/P \times G/Q$ is the line bundle whose fibres are the tensor products of the fibres of $\mathcal{L}$ and $\mathcal{M}$ at the corresponding points: $(\mathcal{L} \boxtimes \mathcal{M})_{(x,y)} = \mathcal{L}_x \otimes \mathcal{M}_y$ for all $x \in G/P$, $y \in G/Q$. If the complexity of $X = G/P \times G/Q$ is equal to 0 or 1, then there is an effective method...
for decomposing the space of sections $H^0(X, \mathcal{N})$ of a line bundle $\mathcal{N} \to X$ into irreducible submodules.

Suppose that the semisimple group $G$ is decomposed into an almost-direct product $G = G_1 \cdots G_s$ of simple subgroups. Then the parabolic subgroups $P, Q \subseteq G$ decompose into almost-direct products of parabolic subgroups $P_i, Q_i \subseteq G_i$. We have

$$c_G(G/P \times G/Q) = c_{G_1}(G_1/P_1 \times G_1/Q_1) + \cdots + c_{G_s}(G_s/P_s \times G_s/Q_s).$$

This reduces the problem of computing the complexity of double flag varieties for semisimple groups to the same problem for simple groups.

Suppose that $G$ is a classical matrix group and $B$ is the subgroup of upper triangular matrices (in the orthogonal and symplectic cases we assume that the bilinear form preserved by $G$ has an anti-diagonal matrix). Then the parabolic subgroups containing $B$ have a block-triangular structure and are determined by the sizes of the diagonal blocks. The group $\text{SO}_n$ with even $n$ is an exception: not all parabolic subgroups are of this form. The other parabolic subgroups take the prescribed form after conjugation by the transposition of the two middle basis vectors. We mark such parabolic subgroups with primes.

In the case of exceptional groups, the parabolic subgroups $P \supseteq B$ are determined by the subsets $\Pi \setminus I$, where $\Pi$ is the system of simple roots of $G$ corresponding to the choice of a maximal torus $T \subseteq B$, and $I \subseteq \Pi$ is the system of simple roots of a standard Levi subgroup in $P$. Simple roots are numbered as in [4].

We shall prove the following classification theorems.

**Theorem 1.** Let $G$ be a classical matrix group $(\text{SL}_n, \text{SO}_n, \text{Sp}_n)$. Then all double flag varieties of complexity 0 and 1 correspond to the pairs of parabolic subgroups given in Tables 1–3 (up to a permutation of the subgroups in a pair in all cases, simultaneous transposition with respect to the secondary diagonal for $\text{SL}_n$, and a diagram automorphism of $G$ for $\text{SO}_{2n}$).

| Number of blocks in $P$ and $Q$ | Complexity 0 | Complexity 1 |
|-------------------------------|-------------|-------------|
|                               | $P$         | $Q$         | $P$         | $Q$         |
| 2, 2                          | $(p_1, p_2)$| $(q_1, q_2)$|             |             |
| 2, 3                          | $(p_1, p_2)$| $(1, q_2, q_3)$| $(3, p_2)$, $p_2 \geq 3$| $(q_1, q_2, q_3)$, $q_1, q_2, q_3 \geq 2$|
|                               | $(p_1, p_2)$| $(q_1, 1, q_3)$| $(p_1, p_2)$, $p_1, p_2 \geq 3$| $(2, 2, q_3)$, $q_3 \geq 2$|
|                               | $(2, p_2)$| $(q_1, q_2, q_3)$| $(p_1, p_2)$, $p_1, p_2 \geq 3$| $(2, q_2, 2)$, $q_2 \geq 2$|
| 2, 4                          | $(2, p_2)$|             | $(q_1, q_2, q_3, q_4)$|             |
|                               | $(p_1, p_2)$, $p_1, p_2 \geq 2$| | $(1, 1, q_4)$| |
|                               | $(p_1, p_2)$, $p_1, p_2 \geq 2$| | $(1, 1, q_3, 1)$| |
| 2, $s$                        | $(1, p_2)$| $(q_1, q_2, \ldots, q_s)$|             |             |
| 3, 3                          | $(1, 1, p_3)$|             | $(q_1, q_2, q_3)$|             |
|                               | $(1, p_2, 1)$| | $(q_1, q_2, q_3)$| |
Table 2. Pairs of parabolic subgroups corresponding to the double flag varieties of complexity 0 and 1 for $SO_n$

| Number of blocks in $P$ and $Q$ | Complexity 0 | Complexity 1 |
|-------------------------------|--------------|--------------|
|                               | $P$ | $Q$       | $P$ | $Q$       |
| 2, 2                          | $(p, p)$ | $(p, p)$ | $(p, p)$ | $(p, p)$ |
| 2, 3                          | $(p, p)$ | $(q_1, q_2, q_1), q_1 \leq 3$ | $(6, 6)$ | $(4, 4, 4)$ |
| 2, 4                          | $(p, p)$ | $(1, q, q, 1)$ | $(4, 4')$ | $(2, 2, 2)'$ |
|                               | $(4, 4')$ | $(2, 2, 2)'$ | $(5, 5)$ | $(2, 3, 3)$ |
| 2, 5                          | $(p, p)$ | $(1, 1, q, 1, 1)$ | $(4, 4')$ | $(1, 2, 2, 2, 1)'$ |
| 2, 6                          | $(p, p)$ | $(1, 1, q, 1, 1)$ | $(4, 4')$ | $(1, 1, 2, 2, 2, 1, 1)'$ |
| 3, 3                          | $(1, p, 1)$ | $(q_1, q_2, q_1)$ | $(2, 2, 2)$ | $(2, 2, 2)$ |
|                               | $(p, 1, p)$ | $(2, p, 2), p > 1$ | $(5, 5)$ | $(q, 1, q)$ |
| 3, 4                          | $(1, p, 1)$ | $(q_1, q_2, q_1)$ | $(2, 2, 2)$ | $(1, 2, 2, 1)$ |
| 3, 5                          | $(1, p, 1)$ | $(q_1, q_2, q_3, q_2, q_1)$ | $(1, 1, 1, 1, 1)$ | |
| 4, 4                          | $(1, 2, 2, 1)$ | $(1, 2, 2, 1)$ | $(1, 2, 2, 1)'$ | |

Table 3. Pairs of parabolic subgroups corresponding to the double flag varieties of complexity 0 and 1 for $Sp_n$

| Number of blocks in $P$ and $Q$ | Complexity 0 | Complexity 1 |
|-------------------------------|--------------|--------------|
|                               | $P$ | $Q$       | $P$ | $Q$       |
| 2, 2                          | $(p, p)$ | $(p, p)$ | $(p, p)$ | $(p, p)$ |
| 2, 3                          | $(p, p)$ | $(1, q, 1)$ | $(p, p)$ | $(2, q, 2)$ |
| 2, 4                          | $(2, 2)$ | $(1, 1, 1, 1)$ | $(2, 2)$ | $(1, 1, 1, 1)$ |
| 3, 3                          | $(1, p, 1)$ | $(q_1, q_2, q_1)$ | $(1, p, 1)$ | $(q_1, q_2, q_2, q_1)$ |
| 3, 4                          | $(1, p, 1)$ | $(q_1, q_2, q_3, q_2, q_1)$ | $(1, p, 1)$ | $(q_1, q_2, q_3, q_2, q_1)$ |

**Theorem 2.**
1) There are no double flag varieties of complexity 0 and 1 for the groups $G_2, F_4$ and $E_8$.

2) The varieties of complexity 0 for $E_6$ correspond to the following pairs of parabolic subgroups:

$\{\{\alpha_1\}, \{\alpha_1\}\}, \{\{\alpha_1\}, \{\alpha_2\}\}, \{\{\alpha_1\}, \{\alpha_4\}\}, \{\{\alpha_1\}, \{\alpha_5\}\}, \{\{\alpha_1\}, \{\alpha_6\}\}, \{\{\alpha_2\}, \{\alpha_5\}\}, \{\{\alpha_4\}, \{\alpha_5\}\}, \{\{\alpha_5\}, \{\alpha_6\}\}, \{\{\alpha_1\}, \{\alpha_1, \alpha_5\}\}, \{\{\alpha_5\}, \{\alpha_1, \alpha_5\}\}$. 
The varieties of complexity 1 for $E_6$ correspond to the following pairs of parabolic subgroups:

\[
\{\alpha_1\}, \{\alpha_1, \alpha_2\}, \{\alpha_1\}, \{\alpha_1, \alpha_6\}, \{\alpha_1\}, \{\alpha_4, \alpha_5\}, \{\alpha_1\}, \{\alpha_5, \alpha_6\},
\]

\[
\{\alpha_5\}, \{\alpha_1, \alpha_2\}, \{\alpha_5\}, \{\alpha_1, \alpha_6\}, \{\alpha_5\}, \{\alpha_4, \alpha_5\}, \{\alpha_5\}, \{\alpha_5, \alpha_6\}.
\]

3) The varieties of complexity 0 for $E_7$ correspond to the following pairs of parabolic subgroups:

\[
\{\alpha_1\}, \{\alpha_1\}, \{\alpha_6\}, \{\alpha_1\}, \{\alpha_7\}.
\]

The varieties of complexity 1 for $E_7$ correspond to the following pair of parabolic subgroups:

\[
\{\alpha_1\}, \{\alpha_2\}.
\]

The paper is organized as follows. In §2 we discuss a method of decomposing the space $H^0(X, \mathcal{N})$ of sections of a line bundle $\mathcal{N} \to X$ into irreducible submodules provided that the complexity of $X$ is equal to 0 or 1. We also consider some examples of the decomposition of tensor products of irreducible representations using this method. In §3 we state some general theorems on the complexity of double flag varieties. The classification of double flag varieties of complexity 0 and 1 is obtained in §§4 and 5 for the classical and exceptional groups respectively.

§2. Decomposition of spaces of sections

Suppose that $G$ acts on a normal variety $X$. We consider prime $B$-stable divisors on $X$. For every prime divisor $D$ we have a homomorphism $\text{ord}_D : \mathbb{C}(X)^x \to \mathbb{Z}$, $f \mapsto \text{ord}_D(f)$.

Every line bundle over $X$ can be $G$-linearized [5], and every Cartier divisor $\delta$ is linearly equivalent to a $B$-stable divisor. This can be proved by choosing a $B$-semi-invariant rational section of the corresponding line bundle $\mathcal{O}(\delta)$ [6].

2.1. Case of complexity 0. In this case we have $\mathbb{C}(X)^B = \mathbb{C}$. Hence every $B$-semi-invariant function is uniquely determined up to a scalar multiple by its weight. Since the value $\text{ord}_D(f)$ is not changed under multiplication of $f$ by a constant, the set of $B$-stable prime divisors can be mapped (in general, not injectively) to the group $\text{Hom}(\Lambda, \mathbb{Z})$, where $\Lambda = \Lambda(X)$ is the lattice of eigenweights of $B$-semi-invariant rational functions on $X$. We denote the image of a divisor $D$ under this map by $v_D$. Then $\text{ord}_D f_\lambda = \langle v_D, \lambda \rangle$, where $f_\lambda$ is a function of weight $\lambda$.

There are finitely many prime $B$-stable divisors since they lie in the complement of the open $B$-orbit.

We denote an irreducible $G$-module of highest weight $\lambda$ by $V_\lambda$.

The space of global sections of the line bundle $\mathcal{O}(D)$ can be identified with the space of rational functions $f \in \mathbb{C}^x(X)$ such that the divisor $\text{div} f + D$ is effective. The section corresponding to $f \equiv 1$ is said to be canonical.

We now state the main theorem on the decomposition of spaces of sections.

**Theorem 3** ([7], Proposition 3.3). Let $X$ be a variety of complexity 0, and let $\delta = \sum_i m_i D_i$ be a Cartier divisor, where the $D_i$ are all distinct prime $B$-stable
divisors on $X$. Then

$$H^0(X, \mathcal{O}(\delta)) \simeq \bigoplus_{\lambda \in \mathcal{P}(\delta) \cap \Lambda} V_{\lambda + \pi(\delta)},$$

where $\mathcal{P}(\delta) = \{ \lambda \in \Lambda \otimes \mathbb{Q} \mid \langle v_i, \lambda \rangle \geq -m_i \forall i \}$ is a polytope in $\Lambda \otimes \mathbb{Q}$, $v_i$ is the vector corresponding to $D_i$, and $\pi(\delta)$ is the weight of the canonical section $s_\delta$.

**Proof.** One of the equivalent definitions of a spherical variety $X$ (a variety is said to be spherical if its complexity is equal to zero) requires that the action of $G$ on $H^0(X, \mathcal{L})$ is multiplicity-free for every $G$-line bundle $\mathcal{L} \to X$. Therefore it suffices to describe the set of highest weights. Every $B$-semi-invariant section can be represented as $s = f_\lambda s_\delta$. The condition that the divisor $\text{div } s = \text{div } f_\lambda + \delta$ is effective is equivalent to $\lambda \in \mathcal{P}(\delta) \cap \Lambda$. □

2.2. Case of complexity 1. For varieties of complexity 1, the structure of $B$-stable divisors is rather more complicated. Assume for simplicity that $X$ is a rational variety. Then $\mathbb{C}(X)^B \simeq \mathbb{C}(\mathbb{P}^1)$ by Lüroth’s theorem. Therefore $B$-semi-invariant functions are determined by their weights uniquely up to multiplication by a function in $\mathbb{C}(X)^B \simeq \mathbb{C}(\mathbb{P}^1)$. In other words, every $B$-semi-invariant function can be represented as $f_\lambda q$, where $f_\lambda$ is a fixed function of weight $\lambda$ and $q \in \mathbb{C}(\mathbb{P}^1)$. There is a rational map $X \dashrightarrow \mathbb{P}^1$ whose general fibres are the closures of general $B$-orbits. Therefore one can describe all the prime $B$-stable divisors as follows. Except for finitely many of them, they form a family parametrized by the projective line except for finitely many points.

As in the case of complexity 0, we can associate a vector $v_D$ in $\text{Hom}(\Lambda, \mathbb{Z})$ with every prime $B$-divisor $D$. This is done by restricting $\text{ord}_D$ to $\{ f_\lambda \mid \lambda \in \Lambda \}$, where $f_\lambda$ is chosen in such a way that the map $\lambda \mapsto f_\lambda$ is a group homomorphism. Restricting $\text{ord}_D$ to $\mathbb{C}(X)^B \simeq \mathbb{C}(\mathbb{P}^1)$, we get a valuation on $\mathbb{C}(\mathbb{P}^1)$ with centre $z_D \in \mathbb{P}^1$ and order $h_D \in \mathbb{Z}_+$ of a local coordinate at $z_D$. (If $h_D = 0$, then we can take any point of $\mathbb{P}^1$ for $z_D$.) Then $\text{ord}_D f = \langle v_D, \lambda \rangle + h_D \text{ord}_{z_D} q$. Thus we have a triple $(v_D, z_D, h_D)$ for every divisor $D$. Removing a finite set of points from the projective line, we may assume that for every remaining point $z$ there is a unique prime $B$-stable divisor $D$ with $z_D = z$ and, furthermore, $v_D = 0$, $h_D = 1$.

For varieties of complexity 1 we have the following theorem on the decomposition of spaces of sections.

**Theorem 4** [9]. Let $X$ be a rational variety of complexity 1, and let $\delta = \sum m_i D_i$ be a Cartier divisor, where the sum is taken over all distinct $B$-stable prime divisors on $X$ (we assume that only finitely many of the $m_i$ are non-zero). Then

$$H^0(X, \mathcal{O}(\delta)) \simeq \bigoplus_{\lambda \in \mathcal{P}(\delta) \cap \Lambda} m(\delta, \lambda) V_{\lambda + \pi(\delta)},$$

Here $\pi(\delta)$ is the weight of the canonical section and $\mathcal{P}(\delta)$ is given by $\mathcal{P}(\delta) = \{ \lambda \in \Lambda \otimes \mathbb{Q} \mid \langle v_i, \lambda \rangle \geq -m_i \forall i \text{ with } h_i = 0 \}$, where $(v_i, z_i, h_i)$ is the triple corresponding to $D_i$. The multiplicity $m(\delta, \lambda)$ of the module $V_{\lambda + \pi(\delta)}$ in the decomposition above is equal to

$$m(\delta, \lambda) = \max \left( 1 + \sum_{z \in \mathbb{P}^1} m_z, 0 \right),$$

where $m_z = \min_{z_i = z, h_i \neq 0} \left[ \frac{\langle v_i, \lambda \rangle + m_i}{h_i} \right]$ for all $z \in \mathbb{P}^1$. 
2.3. Examples. We give some examples to illustrate the methods used. A systematic study of decompositions by these methods will be pursued elsewhere.

Example 1. Suppose that $G = \text{Sp}_n$ and $n = 2l$. Consider the double flag variety $X = G/P \times G/Q$ corresponding to the pair $(1, 2l-2, 1)$, $(l, l)$ of parabolic subgroups. This is a variety of complexity 0. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$, $\varepsilon_i$ the weights of $e_i$ with respect to the diagonal maximal torus $T$, and $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ the fundamental weights. Write $\ell$ and $S$ for lines and $l$-dimensional subspaces corresponding to the points of $G/P$ and $G/Q$. Denote the Plücker coordinates on $G/P$ and $G/Q$ by $x_i$ and $y_{i_1, \ldots, i_l}$ respectively. Let $E_k$ be the $B$-stable subspace $\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$.

Here is a list of $B$-stable prime divisors $D_i$ (determined by geometric conditions on $\ell$ and $S$), their equations $F_i$ in the Plücker coordinates and the degrees and weights of the $F_i$:

\begin{align*}
D_1 & \cup E_{n-1} \\
D_2 & S \cap E_l \neq 0 \\
D_3 & (S + \ell) \cap E_{l-1} \neq 0 \\
D_4 & (S + \ell) \cap E_l \neq 0
\end{align*}

\begin{align*}
x_n & \quad (1, 0) \quad \omega_1 \\
y_{l+1, \ldots, n} & \quad (0, 1) \quad \omega_l \\
\sum_{j \geq 1} (-1)^j x_{i_j} y_{i_1, \ldots, i_j} & \quad (1, 1) \quad \omega_{l-1} \\
\sum_{i \leq l} (x_i y_{i+1, \ldots, n} + \sum_{j > l} (-1)^{j+1} x_j y_{i, l+1, \ldots, j, \ldots, n}) x_{n+1-l} & \quad (2, 1) \quad \omega_l
\end{align*}

The points lying in the complement of these divisors belong to the open orbit. Indeed, take a point outside $D_2$ and consider a matrix whose columns form a basis of $S$. We can assume that the lower $l \times l$ submatrix is the identity matrix and make all other entries equal to zero by the action of $B$. If the point also lies outside $D_1$, then the bottom entry of the column generating $\ell$ is non-zero. Then we can make the entries at the positions $l+1, \ldots, n-1$ in this column equal to zero. If the point also lies outside $D_3$, then the $l$th entry of the column is non-zero and we can make the entries at the positions $2, \ldots, l-1$ equal to zero. If the point also lies outside $D_4$, then the first entry is non-zero. Acting by $B$, we can make all the three non-zero entries equal to 1. Hence any such point has a unique canonical form.

Up to a scalar multiple, the $B$-semi-invariant functions are ratios of products of $F_i$ such that the degrees of the numerator and denominator coincide for every group of Plücker coordinates. Hence we can find the lattice $\Lambda(X)$: it is generated by the weights $\varepsilon_1 - \varepsilon_l$ and $\varepsilon_1 + \varepsilon_l$. We can take $f_{\varepsilon_1-\varepsilon_l} = F_1 F_3$ and $f_{\varepsilon_1+\varepsilon_l} = F_1 F_2$, for the basis weight functions. Our divisors correspond to the vectors $v_{D_1} = (1, 1)$, $v_{D_2} = (0, 1)$, $v_{D_3} = (1, -1)$, $v_{D_4} = (-1, 0)$ in the basis dual to the weights of the weight functions listed above.

Every divisor is equivalent to a linear combination of the pre-images of Schubert divisors: $\delta = pD_1 + qD_2$. The space of sections of the line bundle $\mathcal{O}(\delta)$ is the tensor product of the spaces of sections of $\mathcal{O}(p\pi_1(D_1))$ and $\mathcal{O}(q\pi_2(D_2))$, where $\pi_1, \pi_2$ are the projections of $X$ to $G/P$, $G/Q$ and $\pi_1(D_1), \pi_2(D_2)$ are Schubert divisors. Since the spaces of sections of $\mathcal{O}(p\pi_1(D_1))$ and $\mathcal{O}(q\pi_2(D_2))$ are isomorphic to $V_{p\omega_1}$ and $V_{q\omega_l}$ respectively, we see that to decompose the product $V_{p\omega_1} \otimes V_{q\omega_l}$ it suffices to calculate $H^0(G/P \times G/Q, \mathcal{O}(pD_1 + qD_2))$. The weight polytope is given by

$$
\mathcal{P}(\delta) = \{ \lambda = -a\varepsilon_1 - b\varepsilon_l \mid 0 \leq b \leq a \leq p, a + b \leq 2q \}.
$$
Using Theorem 3, we get a decomposition
\[ V_{\omega_1} \otimes V_{\omega_l} = \bigoplus_{0 \leq b \leq a \leq p} V_{(p+q-a)\varepsilon_1+q\varepsilon_2+ \cdots +q\varepsilon_{l-1}+(q-b)\varepsilon_l}. \]
This formula can be obtained from the results of Littelmann ([1], Proposition 1.3, Table 1), who found such decompositions for all spherical double flag varieties corresponding to maximal parabolic subgroups. One can also deduce this decomposition from the symplectic Pieri rule (see, for example, [8], Proposition 1.4).

**Example 2.** Suppose that \( G = \text{SL}_n \). Consider the double flag variety corresponding to the pair \((3, p_2), (q_1, q_2, q_3)\) of parabolic subgroups, where \( q_1, q_2, q_3 \geq 3 \). This is a variety of complexity 1. We use notation similar to that in Example 1. Assume that \( \omega_0 = \omega_n = 0 \) and write \( \lambda^* \) for the highest weight of the module dual to \( V_\lambda \). The map \( \lambda \mapsto \lambda^* \) extends by linearity to all weights. Then \( \varepsilon_i^* = -\varepsilon_{n+1-i} \).

We denote the subspaces corresponding to the points of \( G/P \) and \( G/Q \) by \( R_i \) and \( S_j \), where the subscript stands for the dimension of the subspace. Here is a list of \( B \)-stable prime divisors \( D_i \) (determined by geometric conditions on \( R_i \) and \( S_j \)) along with the degrees of their equations \( F_i \) in the Plücker coordinates and the weights of the \( F_i \).

| \( D \) | \( R_i \cap E_{n-3} \neq 0 \) | \( (1, 0, 0) \)  | \( \omega_1^* \) |
|---|---|---|---|
| \( D_1 \) | \( S_{q_1} \cap E_{n-q_1} \neq 0 \) | \( (0, 1, 0) \)  | \( \omega_{q_1}^* \) |
| \( D_2 \) | \( S_{q_1+q_2} \cap E_{n-(q_1+q_2)} \neq 0 \) | \( (0, 0, 1) \)  | \( \omega_{q_1+q_2}^* \) |
| \( D_{3,5,6} \) | \(< R_{3} \cap E_{n-3+k} + S_{q_1} \cap E_{n-3+k} > \cap E_{n-q_1-k} \neq 0, k = 1, 2, 3 \) | \( (1, 1, 0) \)  | \( \omega_{3-k}^* + \omega_{q_1}^* \) |
| \( D_{7,9} \) | \(< R_{3} \cap E_{n-3+k} + S_{q_1+q_2} \cap E_{n-3+k} > \cap E_{n-(q_1+q_2)-k} \neq 0, k = 1, 2, 3 \) | \( (1, 0, 1) \)  | \( \omega_{3-k}^* + \omega_{(q_1+q_2)+k}^* \) |
| \( D_{10} \) | \(< R_{3} \cap E_{n-1} + S_{q_1} \cap E_{n-1} > \cap E_{n-q_1-1} + S_{q_1+q_2} \cap E_{n-(q_1+q_2)-1} \neq 0 \) | \( (1, 1, 1) \)  | \( \omega_1^* + \omega_{q_1+1}^* + \omega_{(q_1+q_2)+1}^* \) |
| \( D_{11} \) | \(< R_{3} + S_{q_1} > \cap E_{n-q_1-2} + S_{q_1+q_2} \cap E_{n-(q_1+q_2)+1} \neq 0 \) | \( (1, 1, 1) \)  | \( \omega_{q_1+2}^* + \omega_{(q_1+q_2)+1}^* \) |
| \( D_{12} \) | \(< R_{3} + S_{q_1} > \cap E_{n-q_1-1} + S_{q_1+q_2} \cap E_{n-(q_1+q_2)-1} \neq 0 \) | \( (1, 1, 1) \)  | \( \omega_{q_1+1}^* + \omega_{(q_1+q_2)+2}^* \) |
| \( D_{13} \) | \(< R_{3} \cap E_{n-2} + (R_{3} + S_{q_1}) > \cap E_{n-q_1-2} + S_{q_1+q_2} \cap E_{n-(q_1+q_2)+2} \neq 0 \) | \( (2, 1, 1) \)  | \( \omega_2^* + \omega_{q_1+2}^* + \omega_{(q_1+q_2)+2}^* \) |
| \( D(z) \) | \( F_4 F_8 F_{11} - z F_5 F_7 F_{12} \) | \( (3, 2, 2) \)  | \( \omega_1^* + \omega_2^* + \omega_{q_1+1}^* + \omega_{q_1+2}^* + \omega_{q_1+2}^* \) |

Regard polynomials in \( F_i \), \( i = 1, \ldots, 13 \), as polynomials in the Plücker coordinates. The subspace of polynomials of weight \( \omega_1^* + \omega_2^* + \omega_{q_1+1}^* + \omega_{q_1+2}^* + \omega_{q_1+2}^* + \omega_{q_1+q_2}^* + \omega_{q_1+2}^* \) and multidegree \((3,2,2)\) with respect to the groups of Plücker coordinates has dimension 2 and is linearly spanned by the polynomials \( F_4 F_8 F_{11}, F_5 F_7 F_{12}, F_{10} F_{13} \). Weight subspaces of lower degrees have dimension 1. These three polynomials are linearly dependent. Multiplying \( F_i \) by constants, we may assume that the equation of linear dependence is
\[ F_4 F_8 F_{11} - F_5 F_7 F_{12} + F_{10} F_{13} = 0. \]
The polynomials in this weight subspace may be regarded as linear forms on $\mathbb{P}^1$, where $F_4 F_8 F_{11}$ and $F_5 F_7 F_{12}$ are taken as homogeneous coordinates.

The valuation corresponding to the divisor $D(z)$ has order $h = 1$ and centre $z$. The vector $v_{D(z)}$ corresponding to this divisor is equal to zero. The valuations corresponding to the divisors $D_4$, $D_8$, $D_{11}$ (resp. $D_5$, $D_7$, $D_{12}$) have order $h = 1$ and centre 0 (resp. $\infty$). The valuations corresponding to $D_{10}$ and $D_{13}$ have order $h = 1$ and centre 1. For other divisors $D_i$, the corresponding valuations have order $h = 0$.

The $B$-semi-invariant functions are constructed in the same way as in Example 1, but only up to multiplication by a function in $\mathbb{C}(X)^B$. All $B$-invariant functions are ratios of homogeneous polynomials of equal degrees in the coordinates $F_4 F_8 F_{11}$ and $F_5 F_7 F_{12}$ of the projective line. Hence the field $\mathbb{C}(X)^B$ is generated by the function $F_4 F_8 F_{11} / F_5 F_7 F_{12}$.

The lattice $\Lambda(X)$ is generated by the weights $\varepsilon_i - \varepsilon_j$, where $i$ and $j$ belong to distinct triples $(1, 2, 3), (q_1 + 1, q_1 + 2, q_1 + 3)$ and $(q_1 + q_2 + 1, q_1 + q_2 + 2, q_1 + q_2 + 3)$.

We take the following functions for the basis weight functions:

\[
\begin{align*}
F_4 F_{11} & = f_{\varepsilon_5 - \varepsilon_1}, & F_3 F_7 F_{11} & = f_{\varepsilon_3 - \varepsilon_1}, & F_1 F_5 & = f_{\varepsilon_{q_1+1} - \varepsilon_1}, & F_1 & = f_{\varepsilon_{q_1+2} - \varepsilon_1}, \\
F_5 & = f_{\varepsilon_{q_1+3} - \varepsilon_1}, & F_1 F_3 F_5 & = f_{\varepsilon_{q_1+q_2+1} - \varepsilon_1}, & F_1 F_2 & = f_{\varepsilon_{q_1+q_2+2} - \varepsilon_1}, & F_3 & = f_{\varepsilon_{q_1+q_2+3} - \varepsilon_1}.
\end{align*}
\]

Let $a_i$ be the coordinates of $\lambda \in \Lambda \otimes \mathbb{Q}$ in the basis of weights of these $B$-semi-invariant functions. Take $\delta = m_1 D_1 + m_2 D_2 + m_3 D_3$. Then we obtain the following inequalities for the coordinates that determine the polytope $\mathcal{P}(\delta)$:

\[
a_2 \geq -m_1, \quad a_3 \leq m_2, \quad a_6 \leq m_3, \quad a_5 \geq 0, \quad a_8 \geq 0,
\]

and the following decomposition:

\[
V_{m_1 \omega_3} \otimes V_{m_2 \omega_{q_1} + m_3 \omega_{q_1+q_2}} = \bigoplus \mathcal{m}(\bar{a}) V_{\lambda(\bar{a}, \bar{m})},
\]

where

\[
m(\bar{a}) = \max(0, 1 + \min(-a_1 - a_2 - a_5 - a_6, -a_2, a_3 + a_7) \\
+ \min(a_1, -a_3 - a_8, a_1 + a_2 + a_4 + a_6) + \min(-a_1 - a_4 - a_7, 0)),
\]

\[
\lambda(\bar{a}, \bar{m}) = m_1 \omega_3 + m_2 \omega_{q_1} + m_3 \omega_{q_1+q_2} - (a_1 + \cdots + a_8) \varepsilon_1 + a_1 \varepsilon_2 + a_2 \varepsilon_3 \\
+ a_3 \varepsilon_{q_1+1} + a_4 \varepsilon_{q_1+2} + a_5 \varepsilon_{q_1+3} + a_6 \varepsilon_{q_1+q_2+1} + a_7 \varepsilon_{q_1+q_2+2} + a_8 \varepsilon_{q_1+q_2+3}
\]

and the sum is taken over all $a_i$ that satisfy the inequalities above.

This decomposition can also be obtained from the Littlewood–Richardson rule.

\section{Some theorems on the complexity of double flag varieties}

We now state some theorems. The main tool in our computation of the complexity of double flag varieties is the following theorem of Panyushev.

\textbf{Theorem 5 [3].} Suppose that $P$ and $Q$ are decomposed into a semidirect product of the standard Levi radical and the unipotent radical: $P = L \ltimes P_u$, $Q = M \ltimes Q_u$. Then the complexity of the action of $G$ on $G/P \times G/Q$ is equal to the complexity of the action of $L \cap M$ on $p_u \cap q_u$, where $p_u$ and $q_u$ are the Lie algebras of $P_u$ and $Q_u$. 
Lemma 1. The complexity does not change when $P$ and $Q$ are interchanged.

Lemma 2. Let $P' \subseteq P$, $Q' \subseteq Q$ be parabolic subgroups. Then the complexity of the double flag variety for the pair $(P', Q')$ is not less than that for the pair $(P, Q)$.

Proof. There is a $G$-equivariant surjective morphism

$$G/P' \times G/Q' \to G/P \times G/Q.$$ 

Hence the codimension of a general $B$-orbit on $G/P \times G/Q$ is not greater than the corresponding codimension on $G/P' \times G/Q'$. □

Lemma 3. We have

$$c \geq \frac{1}{2} (\dim G - \dim L - \dim M - \dim T),$$

where $c$ is the complexity of the corresponding action.

Proof. It is easy to see that $c \geq \dim(p_u \cap q_u) - \dim(L \cap M \cap B)$. We also have

$$\dim(L \cap M \cap B) = \frac{1}{2} (\dim(L \cap M) + \dim T),$$

$$\dim(p_u \cap q_u) = \frac{1}{2} (\dim G - \dim L - \dim M + \dim(L \cap M)).$$

Substituting these equations in the inequality, we get the desired estimate. □

§ 4. The case of classical matrix groups

In this section $G$ stands for $SL_n$, $SO_n$ or $Sp_n$. We assume that the Borel subgroup $B \subseteq G$ consists of upper-triangular matrices, $SO_n$ preserves the quadratic form with the matrix

$$\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix},$$

and $Sp_n$ preserves the skew-symmetric bilinear form with the matrix

$$\begin{pmatrix} 0 & 1 \\ \vdots & \ddots & \vdots \\ -1 & 0 \end{pmatrix}.$$

We compute the complexity using Theorem 5. Therefore we first describe the Levi subgroups, the Lie algebras of unipotent radicals and their intersections.

The Levi subgroup $L$ (or $M$) consists of block-diagonal matrices. For the groups $SO_n$ and $Sp_n$, the blocks are symmetric with respect to the secondary diagonal, and the matrices at symmetric places are $A$ and $(A^S)^{-1}$ (where $S$ stands for transposition with respect to the secondary diagonal) while the central block (occurring when the number of blocks is odd) is an orthogonal or symplectic matrix respectively.
The Lie algebra $\mathfrak{sl}_n$ consists of matrices with trace 0. In the bases chosen above, the Lie algebra $\mathfrak{so}_n$ consists of matrices that are skew-symmetric with respect to the secondary diagonal, and the Lie algebra $\mathfrak{sp}_n$ consists of matrices with the following property. If we divide the matrix into 4 equal square parts, then the upper right and lower left parts are symmetric with respect to the secondary diagonal, and the other parts are obtained from each other by transposing with respect to the secondary diagonal and changing the sign. The matrices in the Lie algebra of the unipotent radical $\mathfrak{p}_u$ have zeros below the diagonal and in the diagonal blocks.

The groups $\text{SO}_n$ with even $n$ admit parabolic subgroups of another class. These subgroups are said to be special and are obtained from the subgroups described above (without central diagonal block) by conjugation by the transposition of the two middle basis vectors. We will treat special parabolic subgroups separately.

The matrices in $L \cap M$ consist of diagonal square blocks $A_1, \ldots, A_r$ of sizes $k_1, \ldots, k_r$, where $k_i = k_{r+1-i}$ for the groups $\text{SO}_n$ and $\text{Sp}_n$. Note that a middle pair of blocks of size 1 for $\text{SO}_n$ is the same as one middle block of size 2. We now assume that the parabolic subgroups are not special (the case of special subgroups will be treated separately). The matrices in $\mathfrak{p}_u \cap \mathfrak{q}_u$ consist of certain submatrices $X_{ij}$. Each $X_{ij}$ is of size $k_i \times k_j$ and stands at the intersection of the rows passing through $A_i$ and the columns passing through $A_j$. We have $X_{ij} = 0$ for $i \geq j$ and in all cases when $L$ or $M$ contains a matrix with non-zero entries at the place of $X_{ij}$. When speaking of ‘blocks’, we shall often mean only non-zero matrices $X_{ij}$. The intersection $L \cap M \cap B$ of $L \cap M$ with the set of upper-triangular matrices is a Borel subgroup of $L \cap M$. The group $L \cap M \cap B$ acts on $\mathfrak{p}_u \cap \mathfrak{q}_u$ by conjugation and sends $X_{ij}$ to $A_i X_{ij} A_j^{-1}$.

The idea is to consider all possible locations of the blocks $X_{ij}$ and, for each location, compute the complexity for all sizes of blocks. The following lemmas will be used to simplify the case-by-case considerations and reduce the number of cases.

**Lemma 4.** The complexity does not change if we simultaneously transpose $P$ and $Q$ with respect to the secondary diagonal.

**Remark 2.** Lemma 4 simplifies things only for $\text{SL}_n$.

**Lemma 5.** Consider an action obtained from the original action by one of the following operations (or a combination of them):

a) remove some blocks $X_{ij}$ (that is, declare the matrices $X_{ij}$ to be equal to 0),

b) remove some matrices $A_i$ and blocks $X_{ij}$ in the corresponding rows and columns.

Then the complexity for the new action does not exceed that for the original one. In other words, we consider only ‘a part of the action’.

**Proof.** The first operation corresponds to restriction of the action to a $G$-stable subvariety. The complexity of the action on a $G$-stable subvariety does not exceed the complexity of the action on the whole variety [10].

The second operation corresponds to considering a quotient representation. The complexity for it can only be less than or equal to the original complexity. □
Lemma 6. Suppose that four non-zero matrices $X_{pq}$ are located at the vertices of a rectangle (that is, they have indices $ij$, $il$, $kj$ and $kl$) and, in the case of $SO_n$, none of them stand on the secondary diagonal. Then the action of $B \cap L \cap M$ has a rational invariant. We call it an invariant of ‘square’ type.

Remark 3. The additional restriction on the position of the blocks in the $SO_n$ case appears because the matrices in $\mathfrak{so}_n$ have zeros on the secondary diagonal.

Proof of Lemma 6. We denote the lower-right entries of the matrices $A_i, A_k$ by $a_i, a_k$, the upper-left entries of $A_j, A_l$ by $a_j, a_l$, and the lower-left entries of $X_{ij}, X_{il}, X_{kj}, X_{kl}$ by $x_{ij}, x_{il}, x_{kj}, x_{kl}$ respectively. Then $x_{pq} \rightarrow a_p x_{pq} a_q^{-1}$, $p = i, k$, $q = j, l$. We easily see that $x_{ij} x_{kj}^{-1} x_{kl} x_{il}^{-1}$ is an invariant. □

Lemma 7. Suppose that three non-zero matrices $X_{pq}$ are located in a special way at the vertices of a right triangle (that is, they have indices $ij$, $ik$, $jk$) and, in the case of $SO_n$, none of them stand on the secondary diagonal. Then the action of $B \cap L \cap M$ has a rational invariant. We call it an invariant of ‘triangle’ type.

Proof. Let $\bar{x}_{ij}$ be the lower row of $X_{ij}$, $x_{ik}$ the left lower entry of $X_{ik}$, and $\bar{x}_{jk}$ the left column of $X_{jk}$. Then we easily see that $\bar{x}_{ij} x_{jk}^{-1} x_{kl} x_{il}^{-1}$ is an invariant. □

Remark 4. The invariants of ‘square’ and ‘triangle’ types for the groups $SO_n$ and $Sp_n$ do not change if we pass to new blocks obtained from the previous ones under transposition with respect to the secondary diagonal.

Lemma 8. Suppose that three non-zero matrices $X_{ij}$ of height at least 2 are located in one row and, in the case of $SO_n$, none of them stand on the secondary diagonal. Then the complexity is at least 1. If there are 4 such matrices, then the complexity is at least 2.

Proof. We shall prove the first assertion for $SL_n$ (the other cases will follow by passing to the corresponding subgroups of $SL_n$). For a general first matrix, we can use the group action to make the lower-left entry and the entry just above it equal to 1 and 0 respectively. To preserve these entries in the course of further reductions, we shall act on the left only by those matrices whose lower-right $2 \times 2$ submatrix is diagonal. Then we can make the corresponding entries of a general second matrix equal to 1. To preserve these four entries, we shall act on the left only by those matrices whose lower-right $2 \times 2$ submatrix is of the form $\lambda E$. Then the corresponding two entries of the third matrix, being generally non-zero, are multiplied by the same number. We can make one of them equal to 1, and then the second entry cannot be changed without changing the other five entries considered. Hence the general orbits depend on at least one continuous parameter, that is, $c(X) \geq 1$. The proof for 4 matrices is similar. □

We now explain the method for computing the complexity of an action of $L \cap M$ on $p_u \cap q_u$. Since the Lie algebras of $SO_n$ and $Sp_n$ have symmetry in their block structure, it suffices to consider only the blocks on and above (or below) the secondary diagonal. Acting by $B \cap L \cap M$, we shall successively bring the blocks into a canonical form and consider the action of the stabilizer of the canonical form on the remaining blocks. The number of remaining parameters is equal to the complexity. The same method was used in the proof of Lemma 8.
The special parabolic subgroups of \( \text{SO}_n \) with even \( n \) will be marked by primes. We may assume that precisely one of the parabolic subgroups is special while the other has no central block (otherwise we use the automorphism of \( \text{SO}_n \) that interchanges the two middle basis vectors). Then the complexity can be estimated from below by the complexity of another action such that both parabolic subgroups are non-special. To do this, we conjugate the special subgroup by the transposition of the two middle basis vectors and replace the two middle blocks by one (we can assume that the sizes of these blocks do not exceed those of the corresponding blocks for the other group). This enlarges the parabolic subgroup and, therefore, the complexity can only decrease.

We now consider particular cases. The pictures show the location of the non-zero blocks \( X_{ij} \) (grey) and matrices \( A_i \) (black). The complexity is denoted by \( c \). We successively consider all possible locations of the blocks \( X_{ij} \) and omit the cases when Lemmas 5–8 yield the estimate \( c \geq 2 \). We also omit the cases that are symmetric to those already considered. The results of our considerations are presented below. We indicate only the cases when the complexity does not exceed 1.

### 4.1. The group \( \text{SL}_n \)

We list the results for \( \text{SL}_n \).

1. \[ \begin{array}{c c c} & & \\
& & \\
\end{array} \] \quad \( k_1, k_r \) are arbitrary, \( c = 0 \).

2. \[ \begin{array}{c c c c}
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \] \quad \begin{align*}
k_1 &\leq 2, & k_{r-1}, k_r &\text{ are arbitrary, } \ c = 0; \\
k_1 &\text{ is arbitrary, } & k_{r-1} &= 1 \text{ or } k_r = 1, \quad c = 0; \\
k_1 &\geq 3, & k_{r-1} &= k_r = 2, \quad c = 1; \\
k_1 &= 3, & k_{r-1}, k_r &\geq 2, \quad c = 1. \end{align*}

To give an example, we consider case 2 in more detail. There is no loss of generality in assuming that \( k_{r-1} \geq k_r \). Acting on the right, we bring two general blocks into the following form. The entries on the secondary diagonal coming from the lower-left corner are equal to 1, and the entries on the right of this diagonal are equal to 0. Using left multiplication, we can also make the entries of the first block above the secondary diagonal coming from the lower-left corner equal to 0. Let us find the stabilizer of such blocks. \( A_1 \) has zeros everywhere outside the diagonal and the upper-left \( \max(k_1 - k_{r-1}, 0) \times \max(k_1 - k_{r-1}, 0) \) submatrix. The same holds for \( A_{r-1} \) and \( A_r \), but the submatrix is now lower right and of size \( \max(k_i - k_1, 0) \times \max(k_i - k_1, 0) \), where \( i = r - 1, r \) respectively, and the diagonal entries are equal to those of \( A_1 \) (in order to preserve the ones in the blocks). Now, in the second block, we can make all the entries of the first column in the rows \( 2, \ldots, \min(k_1, k_{r-1}) \) equal to 1. Then all the diagonal entries in the stabilizer that lie outside the submatrices considered above must be equal to the same number \( \lambda \).

If \( k_1 \leq 2 \) or \( k_r = 1 \), then we can make all the entries of the first column in the second block equal to 1. This leaves no free parameters. Hence all general points lie in the orbit of any point of the form described and, therefore, \( c = 0 \).

If \( k_1 = 3 \) and \( k_{r-1} \geq k_r \geq 2 \), then we can make the entry in the first column and third row (from below) of the second block equal to 1 (if it is not already equal to 1). Then \( A_1 = \lambda E \) and the uppermost two diagonal entries of \( A_r \) are also equal to \( \lambda \). Hence we cannot change the entry in the first column and third row (from below)
of the second block. Thus a general orbit depends on one continuous parameter, that is, \( c = 1 \).

We now suppose that \( k_{r-1} = k_r = 2 \), \( k_1 \geq 4 \). Consider the submatrix of the second block above the second row from the bottom. We can multiply it on the left by any upper-triangular matrix. The action on the right reduces to multiplication of all entries by the same number. We can make the lower \( 2 \times 2 \) submatrix of this submatrix equal to \( \begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix} \) and make all the entries above it equal to 0. The entry * cannot be changed, whence \( c = 1 \).

It remains to consider the case when \( k_1 \geq 4 \), \( k_{r-1} \geq 3 \), \( k_r \geq 2 \). We cannot change the entries in the second column and rows 3 and 4 (from the bottom) of the second block. Hence \( c \geq 2 \).

3. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
\( k_1 = 1 \), \( k_{r-2}, k_{r-1}, k_r \) are arbitrary, \( c = 0 \);
\( k_1 = 2 \), \( k_{r-2}, k_{r-1}, k_r \) are arbitrary, \( c = 1 \);
\( k_1 \geq 3 \), \( k_{r-2} = k_{r-1} = k_r = 1 \), \( c = 1 \).

If we add blocks to the first row, then Lemma 8 yields that their height cannot be greater than 1.

4. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
\( k_1 = 1 \), \( c = 0 \).

5a. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
at least two of \( k_1, k_2, k_3 \) are equal to 1, \( c = 1 \).

5b. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
\( k_1 = 1 \) or \( k_4 = 1 \), \( k_2, k_3 \) are arbitrary, \( c = 0 \);
\( k_1 = k_4 = 2 \), \( k_2, k_3 \) are arbitrary, \( c = 1 \);
\( k_1 = 2 \), \( k_4 \geq 3 \), \( k_2 = 1 \), \( k_3 \) is arbitrary, \( c = 1 \);
\( k_1 \geq 3 \), \( k_4 = 2 \), \( k_2 \) is arbitrary, \( k_3 = 1 \), \( c = 1 \).

6. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
\( k_1 = 1 \), \( k_2 = 1 \) or \( k_4 = 1 \), \( c = 1 \).

7. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
Case 7 occurs only when the number of blocks in the upper row is \( s \leq 3 \). This is either Case 5 or Case 6.

8. \[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} & \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]
\( k_1 = k_2 = 1 \) or \( k_{r-1} = k_r = 1 \), \( c = 1 \).
9a. $c \geq 2$ because there are independent invariants of ‘square’ and ‘triangle’ type.

9b. $k_1 = 1, k_2 = 1$, $k_3$ is arbitrary, $k_4, k_5$ are arbitrary, $c = 1$; $k_1 = 1, k_2$ is arbitrary, $k_3$ is arbitrary, $k_4 = k_5 = 1$, $c = 1$.

10. Case 10 occurs only when the number of blocks in the upper row is $s \leq 4$. If $s = 3$, this is Case 9. If $s = 4$, then $c \geq 2$ because there are independent invariants of ‘square’ and ‘triangle’ type.

If there are at least three blocks in the second row, then there are two independent invariants of ‘square’ type, that is, $c \geq 2$.

11a. $c \geq 2$ because there are two invariants of ‘triangle’ type.

11b. $k_1 = k_5 = 1$, $c = 1$.

12. Case 12 occurs only when the number of blocks in the top row is $s \leq 4$. If $s = 3$, this is Case 11. If $s = 4$, then $c \geq 2$ because there are two invariants of ‘triangle’ type.

13. $c \geq 2$ because there are invariants of ‘square’ and ‘triangle’ types.

14. Case 14 occurs only when the number of blocks in the upper row is $s \leq 4$. If $s = 3$, this is Case 13. If $s = 4$, then $c \geq 2$ because this case reduces to Case 13 by Lemma 5.
Cases 15, 16 occur only for $|s - m| \leq 1$ (where $s$ and $m$ are the numbers of non-zero blocks in the first row and last column respectively). If $s, m \geq 4$ (these are the remaining cases), then the complexity is at least 2 because there are either two invariants of ‘triangle’ type (Case 15) or invariants of ‘triangle’ and ‘square’ types (Case 16).

Combining all cases together, we obtain the classification shown in Table 1. We recall that the classification is made up to transposition with respect to the secondary diagonal and permutation of the parabolic subgroups.

### 4.2. The group $\text{SO}_n$.

Consider the cases when $P$ and $Q$ are non-special.

1. $c = 0$.

2a. $k_1 = 1$, $k_2$ is arbitrary, $c = 0$; $k_1$ is arbitrary, $k_2 = 1$, $c = 0$; $k_1 = 2$, $k_2 = 2$, $c = 1$.

2b. $k_1 \leq 3$, $k_2$ is arbitrary, $c = 0$; $k_1$ is arbitrary, $k_2 = 1$, $c = 0$; $k_1 = 4$, $k_2 = 2$, $c = 1$.

3a. $k_1 = 1$, $k_2$ is arbitrary, $c = 0$; $k_1 = 2$, $k_2 = 1$, $c = 1$.

3b. $k_1 = 1$, $k_2$ is arbitrary, $k_3$ is arbitrary, $c = 0$; $k_1 = 2$, $k_2$ is arbitrary, $k_3 = 1$, $c = 1$.

4a. $k_1 = 1$, $k_2$ is arbitrary, $k_3$ is arbitrary, $c = 1$.

4b. $k_1 = 1$, $k_2$ is arbitrary, $k_3$ is arbitrary, $c = 0$.

If there are at least five blocks in the first row, then Lemma 8 yields that their height cannot be greater than 1.

Suppose that non-zero blocks occur only in the first row and last column and denote the number of blocks in the first row by $m$. Suppose that the height of the blocks in the first row is $k_1 = 1$. Denote the complexity in this case by $c_{m,a}$ (if $r = m + 1$) or $c_{m,b}$ (if $r = m + 2$), where $r$ is the number of diagonal blocks. Then we have the following lemma.
Lemma 9. If \( m \geq 4 \), then \( c_{m,a} = c_{m-1,b} + 1 \) and \( c_{m,b} = c_{m-1,a} \).

Proof. Using multiplication on the right, we can bring a general pair of blocks \( X_{1,i} \) and \( X_{1,r+1-i} \), \( i \neq \frac{r+1}{2} \), into the form \((x,0,\ldots,0)\) and \((t,0,\ldots,0,y)\) if the widths of the blocks are at least 2, and to the form \((x)\) and \((y)\) if the widths of the blocks are equal to 1. Here \( t, x \) are any non-zero numbers and \( y \) is uniquely determined by \( x \) since the product \( xy \) is invariant. We can bring a non-zero general block \( X_{1,\frac{r+1}{2}} \) (for odd \( r \)) into the form \((x,0,\ldots,0,y)\) if the width of this block is at least 2 (where \( x \) and \( y \) are as above). If the width of this block is equal to 1, then we cannot change the entry in this block. All these matrices are multiplied on the left by the same number. Thus the complexity is equal to the number of such pairs plus the number of central blocks (0 or 1) minus 1. This proves the lemma. □

Lemma 9 enables us to compute the complexities in Cases 5, 6 below, and the complexity is greater than 1 for \( m \geq 7 \).

5a. \( k_1 = 1, \quad c = 1 \).

5b. \( k_1 = 1, \quad c = 1 \).

6a. \( k_1 = 1, \quad c = 2 \).

6b. \( k_1 = 1, \quad c = 1 \).

7. \( k_1 = 1, \quad k_2 \) is arbitrary, \( c = 0 \);
  \( k_1 \) is arbitrary, \( k_2 = 1, \quad c = 0 \);
  \( k_1 = 2, \quad k_2 = 2, \quad c = 1 \);
  \( k_1 = 2, \quad k_2 = 3, \quad c = 1 \);
  \( k_1 = 3, \quad k_2 = 2, \quad c = 1 \).
If we add blocks only to the first row (and also add the blocks symmetric to them with respect to the secondary diagonal), then these cases do not occur.

9a. \[ k_1 = 1, \ k_2 = 1, \ k_3 = 1, \ c = 1. \]

9b. \[ k_1 = 1, \ k_2 = 1, \ \text{k}_3 \text{ is arbitrary}, \ c = 0; \]
\[ k_1 = 1, \ k_2 = 2, \ \text{k}_3 = 1, \ c = 1; \]
\[ k_1 = 2, \ k_2 = 1, \ \text{k}_3 = 1, \ c = 1. \]

10a. \[ c \geq 2 \text{ because there are two invariants of ‘triangle’ type}. \]

10b. \[ k_1 = 1, \ k_2 = 1, \ k_3 = 1, \ c = 1. \]

11. \[ c \geq 2 \text{ because there are two invariants of ‘triangle’ type}. \]

If we add blocks only to the first row (and also add the blocks symmetric to them), then these cases do not occur.

12a. \[ c \geq 2 \text{ because there are invariants of ‘square’ and ‘triangle’ types}. \]
12b. \( c \geq 2 \) because there are invariants of ‘square’ and ‘triangle’ types.

If we add one or two blocks to the first row (and also add the blocks symmetric to them), then \( c \geq 2 \) by reduction to Case 12. If we add more than two blocks to the first row (and also add the blocks symmetric to them), then these cases do not occur. If we add a block to the second row (and to the first row, respectively, and also add the blocks symmetric to them), then there are two invariants of ‘square’ type.

13. 

\[
\begin{align*}
  k_1 &= 1, \\  k_2 &= 1, \\  k_3 &= 1, \\  c &= 0; \\
  k_1 &= 1, \\  k_2 &= 1, \\  k_3 &= 2, \\  c &= 1; \\
  k_1 &= 1, \\  k_2 &= 2, \\  k_3 &= 1, \\  c &= 1; \\
  k_1 &= 2, \\  k_2 &= 1, \\  k_3 &= 1, \\  c &= 1.
\end{align*}
\]

14. \( k_1 = 1, \ k_2 = 1, \ k_3 = 1, \ c = 1. \)

15. \( c \geq 2 \) because this case reduces to Case 11 by Lemma 5.

If we add blocks to the first row (and also add the blocks symmetric to them), then these cases do not occur. If there are three blocks in the third row, four in the second and five or six in the first, then \( c \geq 2 \) by Case 12a. If there are six or more blocks in the first row, then these cases do not occur. If there are five or more blocks in the second row, then there are two invariants of ‘square’ type. If there are four or more blocks in the third row, then we also have two invariants of ‘square’ type.

We now consider the case of special subgroups.

The cases will be enumerated in accordance with the numbering of the cases for non-special subgroups obtained by replacing a special subgroup by a non-special one. This replacement is carried out by conjugating by the transposition of the two middle basis vectors and replacing the two middle blocks by one central block. It suffices to consider the cases when the size of the two central blocks obtained by this transformation is not greater than the size of the two central blocks for the other subgroup.
0. \[ c = 0. \]

2b. 
\[ k_1 = 1, \quad k_2 \text{ is arbitrary, } \quad c = 0; \]
\[ k_1 = 2, \quad k_2 = 1, \quad c = 0; \]
\[ k_1 = 2, \quad k_2 = 2, \quad c = 1; \]
\[ k_1 = 3, \quad k_2 = 1, \quad c = 1. \]

3a. 
\[ k_1 = 1, \quad k_2 = 1, \quad c = 1. \]

9b. 
\[ k_1 = 1, \quad k_2 = 1, \quad k_3 = 1, \quad c = 1. \]

10b. To ensure complexity not greater than 1, it is necessary to have \( k_3 = 1 \). Since one central block of size 2 is the same as two middle blocks of sizes 1 and 1, we may assume in Case 10b (for non-special subgroups) that both subgroups have two middle blocks. Hence we need not consider this case.

Combining all these cases, we get the classification shown in Table 2.

4.3. The group \( \text{Sp}_n \). The numbering of cases corresponds to the enumeration of cases for \( \text{SO}_n \).

1. \[ c = 0. \]

2a. 
\[ k_1 = 1, \quad k_2 \text{ is arbitrary, } \quad c = 0. \]

2b. 
\[ k_1 = 1, \quad k_2 \text{ is arbitrary, } \quad c = 0; \]
\[ k_1 = 2, \quad k_2 \text{ is arbitrary, } \quad c = 1. \]

3a. 
\[ k_1 = 1, \quad k_2 \text{ is arbitrary, } \quad c = 1. \]

3b. 
\[ k_1 = 1, \quad k_2 \text{ is arbitrary, } \quad k_3 \text{ is arbitrary, } \quad c = 0. \]

If there are at least four blocks in the first row, then Lemma 8 yields that their height cannot be greater than 1.
Suppose that non-zero blocks occur only in the first row and last column and denote the number of blocks in the first row by \( m \). Suppose that the height of the blocks in the first row is \( k_1 = 1 \). Denote the complexity in this case by \( c_{m,a} \) (if \( r = m + 1 \)) or \( c_{m,b} \) (if \( r = m + 2 \)), where \( r \) is the number of diagonal blocks. Then we have the following lemma.

**Lemma 10.** If \( m \geq 3 \), then \( c_{m,a} = c_{m-1,b} + 1 \) and \( c_{m,b} = c_{m-1,a} \).

**Proof.** This is similar to the proof of Lemma 9. □

Lemma 10 enables us to compute the complexities in Cases 4, 5 below, and the complexity is greater than 1 for \( m \geq 6 \).

- **4a.** \( k_1 = 1, \ c = 1 \).
- **4b.** \( k_1 = 1, \ c = 1 \).
- **5a.** \( c \geq 2 \).
- **5b.** \( k_1 = 1, \ c = 1 \).
- **7.** \( k_1 = 1, \ k_2 = 1, \ c = 1 \).
- **8.** \( c \geq 2 \) because there are invariants of ‘triangle’ and ‘square’ types.

If we add blocks in the first row (and also add the blocks symmetric to them), then these cases do not occur. If there are at least three blocks in the second row, then \( c \geq 2 \) because there are two invariants of ‘square’ type.

Combining all of these cases, we get the classification shown in Table 3.
§ 5. The case of exceptional groups

We fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Let $\Delta$ be the system of roots with respect to $T$, $\Pi$ the system of simple roots corresponding to the choice of $B$, and $I \subseteq \Pi$ a subset. Any parabolic subgroup containing $B$ coincides with a standard parabolic subgroup $P_I$, whose Lie algebra can be decomposed into the direct sum of the Lie algebra of $T$ and root subspaces corresponding to the positive roots and the roots that are linear combinations of the elements of $I$ with integer coefficients:

$$p_I = t \oplus \bigoplus_{\{\alpha > 0\} \cup \{\alpha \in ZI\}} g_{\alpha}.$$ 

The Lie algebra $p_I$ can be decomposed into the direct sum of the standard Levi subalgebra $l$ and the Lie algebra of the unipotent radical. The roots in $ZI$ correspond to $l$, and the other roots $\{\alpha > 0\} \cap \{\alpha \notin ZI\}$ correspond to the unipotent radical.

Let $P = P_I = L \ltimes P_u$ and $Q = P_J = M \ltimes Q_u$ be parabolic subgroups. Then

$$l \cap m = t \oplus \bigoplus_{\alpha \in Z(I \cap J)} g_{\alpha}, \quad p_u \cap q_u = \bigoplus_{\alpha > 0, \alpha \notin ZI \cup ZJ} g_{\alpha}.$$ 

We now describe a general method for computing the complexity of a linear representation of a reductive group ([11], § 1.4). Let $G$ be a reductive group, $V$ a linear representation of $G$, and $v_\lambda$ a vector of weight $\lambda$. We recall that $\lambda^*$ stands for the highest weight of the module $V^*_\lambda$. Consider a lowest-weight vector $v_{-\lambda^*}$ in $V$. The space $V$ can be decomposed as $V = \langle v_{-\lambda^*} \rangle \oplus W$, where $W$ is $B$-stable. Consider the open $B$-stable subset $\hat{V} = \mathbb{C}^*v_{-\lambda^*} \oplus W$ of $V$. Let $P$ be the parabolic subgroup preserving the line $\langle v_{\lambda^*} \rangle \subseteq V^*$. We decompose $P$ into the semidirect product of the Levi subgroup and the unipotent radical: $P = L \ltimes P_u$. Let $V'$ be an $L$-stable subspace complementary to $p_u v_{-\lambda^*}$ in $W$, that is, $p_u v_{-\lambda^*} \oplus V' = W$. The subset $\hat{V}$ is isomorphic as a $B$-variety to the direct product $\hat{V} = P_u \times (\mathbb{C}^*v_{-\lambda^*} \oplus V')$, where $P_u$ (resp. $B \cap L$) acts on the first factor by left translation (resp. by conjugation) while the action on the second factor is induced from that of $L$. Hence the codimension of a general orbit for the action $B = (B \cap L) \ltimes P_u : \hat{V}$ is equal to that for the action $B \cap L : \langle v_{-\lambda^*} \rangle \oplus V'$. This reduces the question of the complexity of the action $G : V$ to the question of the complexity of an action of the smaller group $L$ on the smaller space $\langle v - \lambda^* \rangle \oplus V'$. We successively construct groups $L^{(i)}$ and spaces $V^{(i)}$:

$$G = L^{(0)} \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(s)}, \quad V = V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V^{(s)}$$

until all the irreducible $L^{(s)}$-submodules $V^{(s)}$ are one-dimensional. Then the action of $L^{(s)}$ on $V^{(s)}$ is determined by the weights $\mu_1, \ldots, \mu_N$, and its complexity is equal to $\dim V^{(s)} - \text{rk}(\mu_1, \ldots, \mu_N) = N - \text{rk}(\mu_1, \ldots, \mu_N)$.

We now explain how to use this method in our case. The intersection $L \cap M$ of the Levi subgroups and the intersection $p_u \cap q_u$ of the Lie algebras of the unipotent radicals are determined by certain subsets of roots $E_1$ and $F_1$ corresponding to the weight subspaces (with non-zero weights) of the Lie algebras. Let $\mu_1$ be a minimal root in $F_1$. We put

$$E'_1 = \{\alpha \in E_1 \mid \alpha + \mu_1 \in F_1\}, \quad F'_1 = \{\alpha + \mu_1 \mid \alpha \in E'_1\}.$$
We also put

\[ E_2 = E_1 \setminus E'_1, \quad F_2 = F_1 \setminus (F'_1 \cup \{\mu_1\}). \]

Using \( E_2 \) and \( F_2 \), we similarly construct \( \mu_2, E_3 \) and \( F_3 \) and so on until \( F_l \) is empty. This yields a set of weights \( \mu_1, \mu_2, \ldots, \mu_N \). The complexity of the action is equal to \( N - \text{rk} \langle \mu_1, \ldots, \mu_N \rangle \).

For every exceptional group \( G \) we first compute the complexity of the double flag varieties for the maximal parabolic subgroups (they correspond to the subsets consisting of all but one of the simple roots). Then we gradually decrease the parabolic subgroups. Of course, we omit the cases when \( c \geq 2 \) by Lemmas 2, 3.

We now consider particular groups.

5.1. The groups \( G_2, F_4 \). The groups \( G_2, F_4 \) admit no roots with estimate of complexity \( \leq 1 \).

5.2. The group \( E_8 \). We have \( c \geq 2 \) for all pairs of parabolic subgroups except possibly the pair corresponding to the roots \( (\alpha_1, \alpha_1) \). For this pair, the complexity is equal to 2. Hence there are no suitable cases.

5.3. The groups \( E_6, E_7 \). Table 4 contains all pairs of roots for which the estimate of complexity for the corresponding parabolic subgroups does not exceed 1, along with the corresponding values of the complexity.

| \( E_6 \) | \( \Pi \setminus I \) | \( \Pi \setminus J \) | complexity |
|---|---|---|---|
| 1 | 1 | 0 |
| 1 | 2 | 0 |
| 1 | 4 | 0 |
| 1 | 5 | 0 |
| 1 | 6 | 0 |
| 2 | 5 | 0 |
| 4 | 5 | 0 |
| 5 | 5 | 0 |
| 5 | 6 | 0 |
| 6 | 6 | 2 |
| 1 | 1, 2 | 1 |
| 1 | 1, 5 | 0 |
| 1 | 1, 6 | 1 |
| 1 | 4, 5 | 1 |
| 1 | 5, 6 | 1 |
| 5 | 1, 2 | 1 |
| 5 | 1, 5 | 0 |
| 5 | 1, 6 | 1 |
| 5 | 4, 5 | 1 |
| 5 | 5, 6 | 1 |

| \( E_7 \) | \( \Pi \setminus I \) | \( \Pi \setminus J \) | complexity |
|---|---|---|---|
| 1 | 1 | 0 |
| 1 | 2 | 1 |
| 1 | 6 | 0 |
| 1 | 7 | 0 |
| 6 | 6 | 2 |
| 1 | 1, 2 | 2 |
| 1 | 1, 6 | 2 |
| 1 | 1, 6 | 2 |
We consider the case of the group $E_6$ and subsets $\Pi \setminus I = \{\alpha_1\}$, $\Pi \setminus J = \{\alpha_5\}$ in more detail. In the notation of [4], Table 1, we have

$$E_1 = \{\varepsilon_i - \varepsilon_j \mid i, j = 2, \ldots, 5; i < j\} \cup \{\varepsilon_6 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \ldots, 5; i < j\},$$

$$F_1 = \{\varepsilon_1 - \varepsilon_6\} \cup \{\varepsilon_1 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \ldots, 5; i < j\} \cup \{2\varepsilon\}.$$ 

Take $\varepsilon_1 - \varepsilon_6$ for $\mu_1$. Then

$$E'_1 = \{\varepsilon_6 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \ldots, 5; i < j\},$$

$$F'_1 = \{\varepsilon_1 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \ldots, 5; i < j\},$$

whence $E_2 = \{\varepsilon_i - \varepsilon_j \mid i, j = 2, \ldots, 5; i < j\}$ and $F_2 = \{2\varepsilon\}$. Taking $2\varepsilon$ for $\mu_2$, we have $F_3 = \emptyset$. The complexity is given by the formula $2 - rk\langle \mu_1, \mu_2 \rangle = 0$.

The final result is stated in Theorem 2.

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