Sofic equivalence relations

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Abstract

We introduce the notion of sofic measurable equivalence relations. Using them we prove that Connes’ Embedding Conjecture as well as the Measurable Determinant Conjecture of Lück, Sauer and Wegner hold for treeable equivalence relations.

1 Introduction

1.1 Sofic groups and sofic relations

First let us recall the definition of sofic groups. The group $\Gamma$ is sofic if for any real number $0 < \epsilon < 1$ and any finite subset $F \subseteq \Gamma$ there exists a natural number $n$ and a function $\psi_n : \Gamma \rightarrow S_n$ from $\Gamma$ into the group of permutations on $n$ elements with the following properties:

(a) $\#_{\text{fix}}(\phi(e)\phi(f)\phi(ef)^{-1}) \geq (1 - \epsilon)n$ for any two elements $e, f \in F$.
(b) $\phi(1) = 1$.
(c) $\#_{\text{fix}}\phi(e) \leq \epsilon n$ for any $1 \neq e \in F$,

where $\#_{\text{fix}}\pi$ denotes the number of fixed points of the permutation $\pi \in S_n$. The notion of soficity was introduced by Gromov [7] and Weiss [13] as a common generalization of amenability and residual finiteness. Direct products, subgroups, free products, inverse and direct limits of sofic groups are sofic as well. If $N \triangleleft \Gamma$, $N$ is sofic and $\Gamma/N$ is amenable, then $\Gamma$ is also sofic. Residually amenable groups are sofic, however there exist finitely generated non-residually amenable sofic groups as well [3]. It is conjectured that there are non-sofic groups, but no example is known yet (see also the survey of Pestov [11]).

In our paper we introduce the notion of a sofic measurable equivalence relation (SER). First let us briefly recall some basic definitions from [8]. A countable Borel-equivalence relation is a Borel-subspace $E \subset X \times X$, where $E$ is an
Definition 1.1. By an \( r \)-neighborhood we mean an \( r \)-edge-colored oriented multi-graph. That is the out-edges need to have different colors from the set \( \gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}, \ldots, \gamma_r, \gamma_r^{-1} \) and if \( xy \) is colored by \( \gamma_i \) then \( yx \) is colored by \( \gamma_i^{-1} \).

Also, we have a chosen vertex which is called the root such that any vertex is connected to the root via a path of length at most \( r \). It is obvious that up to colored, rooted isomorphisms there are only finitely many different \( r \)-neighborhoods. The set of these will be denoted by \( U^r \).

Given the group action \( \theta \) and a point \( x \in X \) we define its \( r \)-neighborhood \( B_r(x) \) to be the subgraph of \( \mathcal{G} \) spanned by \( \theta(W_r, x) \). Its root is \( x \) and it inherits the edge-coloring from \( \mathcal{G} \).

Definition 1.2. By a \( r \)-labeled \( r \)-neighborhood we mean a \( r \)-neighborhood whose vertices are labeled with words taken from \( \{0, 1\}^r \). Again the isomorphism types of such objects form a finite set which we denote by \( U^{r,r} \).

Given the group action \( \theta \) and a point \( x \in X \) we define its \( r \)-labeled \( r \)-neighborhood \( B^*_r(x) \) to be the \( r \)-neighborhood of \( x \) with labeling defined in the following way: any vertex \( y \in B^*_r(x) \) corresponds to a point \( y' \in X \). The label of \( y \) shall be the unique word \( w \in \{0, 1\}^r \) for which \( y' \in A_w \subset X \).

For a fixed action \( \theta \) and a fix \( \alpha \in U^{r,r} \) it is easy to see that the set \( T(\theta, \alpha) = \{ x \in X : B^*_r(x) \equiv \alpha \} \) forms a Borel subset of \( X \). Hence we can take its measure \( \rho_\alpha(\theta) = \mu(T(\theta, \alpha)) \) which is clearly a number between 0 and 1.
We can repeat everything for any action \( \theta \) of \( F_\infty \) on a finite set \( Y \) whose elements are labeled with elements from \( \{0, 1\}^N \). Then \( p_\alpha(\theta) \) is defined as \( \frac{|T(\theta, \alpha)|}{|Y|} \). We call such vertex labelled sets \( X \)-sets.

**Definition 1.3.** We say that the Borel action \( \theta \) is sofic if there is a sequence of actions \( \theta_n \) of \( F_\infty \) on finite \( X \)-sets \( Y_n \) such that for any \( r \geq 1 \) and \( \alpha \in U^{r, r} \)

\[
\lim_{n \to \infty} p_\alpha(\theta_n) = p_\alpha(\theta).
\]

Note this definition is strongly related to the various notions of graph convergence (see e.g. [3]).

**Remark 1.1.** An action \( \theta \) is sofic if and only if \( \theta|_{\gamma_1, \ldots, \gamma_r} \), its restriction to the first \( r \) generators is sofic. The if part follows from choosing a suitable diagonal sequence from the sequences \( \theta_n^r \) that prove the soficity of each \( \theta^r \). For the only-if part one takes the sofic sequence \( \theta_n \) and restricts it to the first \( r \) generators, thereby obtaining a sequence \( \theta_n^r \) that is obviously sofic for \( \theta^r \).

We call a countable measured Borel-equivalence relation sofic equivalence relation (SER) if it is defined by a sofic action of \( F_\infty \). Obviously, since any countable group is a quotient of \( F_\infty \), Borel-equivalence relations can always be defined by \( F_\infty \)-actions. In Section 2 we shall see that if \( E \) is given by actions \( \theta \) resp. \( \theta' \) and \( \theta \) is sofic, then \( \theta' \) is sofic as well (Theorem 1). That is soficity is not only a property of groups actions, but the property of measurable equivalence relations.

### 1.2 Results

We shall prove that Connes’ Embedding Conjecture holds for the von Neumann algebra of a sofic equivalence relation (Theorem 2). Also, any sofic relation satisfies the Measure-Theoretic Determinant Conjecture of Lück, Sauer and Wegner (Theorem 3). We also show that treeable equivalence relations are always sofic (Theorem 4). Hence we prove that the two conjectures above hold for free actions of free groups.

### 2 Orbit equivalence

**Theorem 1.** If \( \theta_1 \) is a sofic action and \( \theta_2 \) is measured orbit equivalent to \( \theta_1 \) then \( \theta_2 \) is also sofic.

**Proof.** By Remark 1.1 it is enough to prove the statement in the special case when \( \theta_2 \) is obtained from \( \theta_1 \) by adding a generator of the free group whose action does not change the orbit structure of the relation. Indeed, from this statement the general case follows easily: to see that the restriction \( \theta_2^r \) is sofic add the first \( r \) generators of \( \theta_2 \) to \( \theta_1 \), then restrict to the set of \( r \) new generators.

Let \( \gamma_1, \ldots, \gamma_d, \ldots \) generate \( \theta_1 \) and let \( \gamma \) denote the new generator in \( \theta_2 \). Since \( \gamma \) does not change the orbit structure we can find for any point \( x \in X \) words
\( w_x, w'_x \in \langle \gamma_1, \ldots, \gamma_d, \ldots \rangle \) such that \( \theta_2(\gamma, x) = \theta_1(w_x, x) \) and \( \theta_2(\gamma^{-1}, x) = \theta_1(w'_x, x) \). In fact we can do this in a Borel way by taking the shortest and lexicographically smallest \( w_x, w'_x \) of all possible choices.

Let us fix an \( \varepsilon > 0 \). For this \( \varepsilon \) we can find an integer \( L \) such that \( \mu(X_0) < \varepsilon/2 \) where \( X_0 = \{ x \in X : |w_x| > L \text{ or } |w'_x| > L \text{ or either } w_x \text{ or } w'_x \text{ contains a generator } \gamma_i \text{ where } i > L \} \). Let us look at \( X \setminus X_0 \). It is partitioned into a finite number of Borel subsets \( H_i : 1 \leq i \leq K \) on which \( w_x \) and \( w'_x \) are constant functions of \( x \). We shall define a sequence of Borel subsets \( X_i \subset X \) in a recursive way. We start with \( X_0 \). Then we take \( H_1 \) and approximate it by a finite union of standard closed-open subsets of \( X \) denoted by \( H_i^1 \) so that \( \mu(H_1 \triangle H_i^1) \leq \varepsilon/4 \). (The \( \triangle \) denotes symmetric difference.) Now let \( X_1 = X_0 \cup (H_1 \triangle H_i^1) \) and \( H_i^2 = H_1 \cap H_i^1 \). Next we take \( H_2 \setminus X_1 \), and approximate it by a \( H_i^2 \) which is again a finite union of standard closed-open subsets of \( X \) so that \( \mu(H_2 \setminus X_1 \triangle H_i^2) \leq \varepsilon/8 \), and set \( X_2 = X_1 \cup (H_2 \setminus X_1 \triangle H_i^2) \) and \( H_i'' = (H_2 \setminus X_1) \cap H_i^2 \). We continue this process for all \( H_i \)'s. At each step \( H_i \setminus X_{i-1} \) is completely disjoint from each \( H_i^j : j < i \) so we can always choose \( H_i^j \) to be disjoint from all \( H_i^j : j < i \). So at the end we have a partition \( X = X_K \cup H_i'' \cdots \cup H_1'' \) such that \( \mu(X_K) \leq \varepsilon, H_i'' \subset H_i \cap H_i^1 \). During the whole process we considered some large, but finite number of standard closed-open sets. Each such set is defined by fixing the first few digits of \( x \). Let \( M \geq L \) denote an integer such that none of the used closed-open sets require fixing more than \( M \) digits of \( x \). Now if \( x \in X \setminus X_K \) then the first \( M \) digits of \( x \) determine which \( H_i'' \) it is in, and hence which \( H_i'' \) and which \( H_i \) it is in. This in turn determines \( w_x \) and \( w'_x \).

So in fact we have a Borel splitting \( X = X_K \cup X' \) such that \( \mu(X_K) < \varepsilon \) and for any point \( x \in X' \) the words \( w_x, w'_x \) are determined by the first \( M \) digits of \( x \).

We have the sofic sequence \( G_n \) for \( \theta_1 \). From it we shall construct a sequence \( G_n' \). As a first attempt for each vertex \( g \in G_n \) we read the first \( M \) digits of its label. Then find the corresponding words \( w_x, w'_x \) we defined above, and trace these words in \( G_n \) starting from \( g \). If they end at \( h \) and \( h' \) respectively then we connect \( g \) to \( h \) by an oriented edge labeled \( \gamma \) and to \( h' \) by an oriented edge labeled \( \gamma^{-1} \). At this point the graph \( G_n' \) might not be the graph of a group action: the \( \gamma \) edge going from \( g \) to \( h \) might not be matched by a \( \gamma^{-1} \) edge going from \( h \) to \( g \). Let us temporarily call such \( g \) vertices “bad”. Let us denote by \( \varrho_\varepsilon(n) \) the ratio of bad vertices in \( G_n' \). By the construction of \( G_n' \) the badness of a vertex \( g \) is determined by its \((M, M)\) neighborhood in \( G_n \). Let us call a neighborhood \( \alpha \in U^{M,M}(\theta_1) \) “bad” if its root is a bad vertex. Hence

\[
\varrho_\varepsilon(n) = \sum_{\alpha \text{ is bad}} p_\alpha(G_n).
\]

Then if \( x \in X \) has neighborhood \( \alpha \) then either \( x \) or \( \theta_2(\gamma, x) \) has to lie in \( X_1 \). Hence

\[
\sum_{\alpha \text{ is bad}} p_\alpha(\theta_1) \leq 2\varepsilon.
\]
This means that \( \limsup_{n \to \infty} \varrho_\varepsilon(n) \leq 2\varepsilon \). Let us complete the construction of \( G_n^\varepsilon \) by keeping the \( \gamma \) action for the good vertices, and defining it arbitrarily for the bad vertices to make it a proper action. This can always be done: let us denote the set of good vertices by \( H \). Then \( \gamma(H) \) is the set of \( \gamma \)-neighbors of the elements of \( H \). Obviously \( |H| = |\gamma(H)| \), and hence \( |G_n \setminus H| = |G_n \setminus \gamma(H)| \). So there is a bijection between these last two sets. This bijection shall be the action of \( \gamma \) and its inverse the action of \( \gamma^{-1} \) on \( G_n \setminus H \) and \( G_n \setminus \gamma(H) \) respectively.

Let us fix \( r \) and a neighborhood \( \alpha \in U^{r,r}(\theta_2) \). Let us suppose for a moment that there are no “bad” vertices at all. Then since each \( \gamma \) edge is at most an \( M \)-long path of non-\( \gamma \) edges, the \( r \)-neighborhood of the \( \theta_2 \) action of any vertex is contained in, and determined by the \( r \cdot M \)-neighborhood of the same vertex for the \( \theta_1 \) action. Thus we get a function \( \pi : U^{r,M,r \cdot M}(\theta_1) \to U^{r,r}(\theta_2) \).

Let \( B \pi^{-1}(\alpha) \). Let \( H \subset G_n \) denote those vertices \( x \in G_n \) whose \( r \)-neighborhood \( B_r(x,G_n^\varepsilon) \) contain a “bad” vertex. Then obviously \( x \notin H \) then \( x \in T(G_n^\varepsilon,\alpha) \) if and only if \( x \in \cup_{\beta \in B} T(G_n,\beta) \). In other words \( T(G_n^\varepsilon,\alpha) \triangle (\cup_{\beta \in B} T(G_n,\beta)) \subset H \). On the other hand if \( x \in H \) since \( B_r(x,G_n^\varepsilon) \) contains the “bad” vertex then also \( x \in B_r(y,G_n^\varepsilon) \). Hence \( H \) is covered by the \( r \)-neighborhoods of the “bad” vertices so \( |p_\alpha(G_n^\varepsilon) - \sum_{\beta \in B} p_\beta(G_n)| \leq \varrho_\varepsilon(n) \cdot r^r \).

The same holds for \( X \) if \( X_0 \) happens to be empty then \( p_\alpha(\theta_2) = \sum_{\beta \in B} p_\beta(\theta_1) \). However \( X_0 \) might not be empty, and in this case \( T(\alpha,\theta_2) \) is not necessarily the same as \( \cup_{\beta \in B} T(\beta,\theta_1) \). But if the \( r \)-neighborhood (by \( \theta_2 \)) of a point \( x \in X \) is disjoint from \( X_K \), then it cannot belong to the symmetric difference of the two sets above. Hence

\[
|p_\alpha(\theta_2) - p_\alpha(G_n^\varepsilon)| \leq \sum_{\beta \in B} |p_\beta(\theta_1) - p_\beta(G_n)| + (\mu(X_K) + \varrho_\varepsilon(n)) \cdot r^r.
\]

So letting \( n \to \infty \) we get that if \( \alpha \in U^{r,r} \) then

\[
\limsup_{n \to \infty} |p_\alpha(G_n^\varepsilon) - p_\alpha(\theta_2)| \leq 3\varepsilon \cdot r^r.
\]

Hence letting \( \varepsilon \to 0 \) we can choose a suitable diagonal sequence \( G_n^\varepsilon \) from the \( G_n^\varepsilon \)'s to get a sofic sequence for \( \theta_2 \).

\[
\square
\]

**Corollary 2.1.** In the definition of soficity we can take actions of \( F_2 \ast F_2 \ast \cdots = F_2^{(\ast \infty)} \) instead of \( F_\infty \).

**Proof.** By Remark 1.1 it is sufficient to show this on the level of finitely generated actions. Let us take an action \( \theta \) of \( F_d \) on \( X \) and consider the underlying simple graphing. It has bounded degree (in fact \( 2d \) is a bound), hence it can be properly Borel edge-colored by at most \( (d^2 + 1) \) colors (see e.g. \[4\], section 5.3). Hence the same equivalence relation can be generated as an action \( \theta' \) of \( F_2^{d'} \) where \( d' = (d^2 + 1) \). Then according to Theorem 1.1 \( \theta \) is sofic if and only if \( \theta' \) is sofic. \( \square \)
3 The von Neumann algebra of a measurable equivalence relation

In this section we briefly recall the notion of the von Neumann algebra of an equivalence relation ([5], [9]). Let $R \subset X \times X$ be a countable Borel-equivalence relation with an invariant measure $\mu$. Then one has a natural $\sigma$-finite measure $\hat{\mu}$ on the space $R$ which is $\mu$ restricted on $X$ ($X \subset R$ is given by the diagonal embedding). The groupoid ring of $R$; $C^{\ast}R$ is defined as follows. Let $L^{\infty}(R, \mathbb{C})$ be the Banach-space of essentially bounded functions on $R$ with respect to $\hat{\mu}$. Then

$C^{\ast}R := \{ K \in L^{\infty}(R, \mathbb{C}) \mid$ there exists $w_{K} > 0$ such that for almost all $x \in X: K(x, y) \neq 0$ or $K(y, x) \neq 0$ only for $w_{K}$ amount of $y$’s.$\}$

The $\ast$-ring structure and a trace is given by:

- $(K + L)(x, y) = K(x, y) + L(x, y)$
- $KL(x, y) = \sum_{z \sim x} K(x, z)L(z, y)$
- $K^{\ast}(x, y) = \overline{K(y, x)}$
- $\text{tr}_{N(R)}(f) = \int_{X} K(x, x) d\mu(x)$

The von Neumann algebra is constructed by the GNS-construction. The inner product $\langle K, L \rangle = tr_{N(R)}(L^{\ast}K)$ defines a pre-Hilbert structure on $C^{\ast}R$ and by $K \mapsto KL$ we obtain a representation of $C^{\ast}R$ on the closure $\mathcal{H}$ of this pre-Hilbert space. The weak closure of $C^{\ast}R$ in the operator algebra $B(\mathcal{H})$ is the von Neumann algebra $N(R)$. The trace $\text{tr}_{N(R)}$ extends to $N(R)$ weakly continuously to a finite trace on $N(R)$.

In Section 6 we shall study the matrix ring $Mat_{d \times d}(N(R))$ as well. Therefore in our paper we use the following version of the groupoid ring of $R$. Let

$C_{d}R := \{ K \in L^{\infty}(R, \mathbb{C}) \mid$ there exists $w_{K} > 0$ such that for almost all $x \in X: K(x, y) \neq 0$ or $K(y, x) \neq 0$ only for $w_{K}$ amount of $y$’s.$\}$

Then $C_{d}R$ is isomorphic to $Mat_{d \times d}(\mathbb{C})$. The normalized trace $\text{tr}_{Mat_{d \times d}(N(R))}(K)$ is defined by

$\text{tr}_{Mat_{d \times d}(N(R))}(K) := \int_{X} \frac{TrK(x, x)}{d} d\mu(x),$

where $Tr$ is the usual trace on $Mat_{d \times d}(\mathbb{C})$. Observe that $Mat_{d \times d}(N(R))$ can be obtained via the GNS-construction directly as a weak closure of $C_{d}R$. 

6
4 Approximation theorems

4.1 The subalgebra of finite type operators

Let $\mathcal{R}$ be a sofic equivalence relation on our standard space $(X, \mu)$ given by a sofic Borel-action $\theta : \mathbb{F}_\infty \curvearrowright X$. Let $\theta_n : \mathbb{F}_\infty \curvearrowright Y_n$ be a sofic approximation as in the Introduction. We define the subalgebra $\mathcal{F}_\theta$ (the subalgebra of finite type operators) the following way. Call an element $K \in \mathbb{C}d\mathcal{R}$ r-fine, $K \in \mathcal{F}_\theta^r$ if for any $\alpha \in U_{r,r}'$, $K(y_1, x_1) = K(y_2, x_2)$ if $x_1, x_2 \in T(\theta, \alpha)$ and $y_1 = wx_1, y_2 = wx_2$ for some $w \in W_r$. The following properties are easy to check:

- $\mathcal{F}_\theta^1 \subset \mathcal{F}_\theta^2 \subset \ldots$
- If $K \in \mathcal{F}_\theta^r, L \in \mathcal{F}_\theta^s$ then $K + L \in \mathcal{F}_\theta^{\max(r,s)}$, $KL \in \mathcal{F}_\theta^{r+s}$, $K^* \in \mathcal{F}_\theta^{2r}$, $Id \in \mathcal{F}_\theta^1$.

That is $\mathcal{F}_\theta = \bigcup_{r=1}^\infty \mathcal{F}_\theta^r$ is a unital $*$-subalgebra of $\mathbb{C}d\mathcal{R}$.

**Proposition 4.1.** $\mathcal{F}_\theta$ is weakly dense in $\mathbb{C}d\mathcal{R}$.

**Proof.** If $K \in \mathbb{C}d\mathcal{R}$ then let $s_K = \sup_{x,y} \|K(x, y)\|$, where $\|\|$ is the usual matrix norm. We say that $\{L_n\}_{n=1}^\infty \subset \mathbb{C}d\mathcal{R}$ converge to $L$ in measure $(L_n \xrightarrow{\mu} L)$. If:

- there exist bounds $w$ and $s$ such that for any $n \geq 1, s_{L_n} \leq s, w_{L_n} \leq w$.
- for any $\varepsilon > 0$, $\lim_{n \to \infty} \mu(A_{\varepsilon}(n)) = 0$, where $A_{\varepsilon}(n) := \{x \in X \mid \|L(y, x) - L_n(y, x)\| > \varepsilon, \text{ for some } y\}$

**Lemma 4.1.** If $L_n \xrightarrow{\mu} L$, then $\{L_n\}_{n=1}^\infty$ weakly converges to $L$.

**Proof.** We need to prove that for any $K \in \mathbb{C}d\mathcal{R}$, $\text{tr}_{\text{Mat}_{d \times d}(\mathcal{R})} K(L_n - L) \to 0$. We use the inequality $\frac{1}{d} |\text{tr}(AB)| \leq ||A|| ||B||$.

\[
|\text{tr}_{\text{Mat}_{d \times d}(\mathcal{R})} K(L_n - L)| = \left| \int_X \frac{1}{d} \sum_{x \sim z} \text{tr}(K(x, z)(L_n - L)(z, x)) \mu(x) \right| \leq
\]

\[
\leq \int_{A_{\varepsilon}(n)} \frac{1}{d} \sum_{x \sim z} \text{tr}(K(x, z)(L_n - L)(z, x)) \mu(x) + \varepsilon w_K s_K \leq
\]

\[
\leq \mu(A_{\varepsilon}(n)) w_K s_K (s + s_L) + \varepsilon w_K s_K,
\]

where $s$ is the bound on the norms of the operators $\{L_n - L\}_{n=1}^\infty$. \hfill $\square$

Now for $K \in \mathbb{C}d\mathcal{R}$ we construct a sequence in $\mathcal{F}_\theta$ converging to $K$ in measure. First let $K'_n \in \mathbb{C}d\mathcal{R}$ be defined the following way. Let $K'_n(y, x) = K(y, x)$ if there exists $w \in W_n$ such that $y = wx$, otherwise let $K'_n(y, x) = 0$. Clearly, $K_n \xrightarrow{\mu} K$.

Now fix $n \geq 1$. It is easy to see there exist operators $\{K_w\}_{w \in W_n} \subset \mathbb{C}d\mathcal{R}$ such that
Let \( f_w(x) = K_w(wx, x) \). Then we have an approximating function \( f'_w \) such that

- \( \mu(x \in X \mid \|f_w(x) - f'_w(x)\| > \frac{1}{n}) < \frac{1}{n|W_n|} \)
- \( f'_w \) is constant on the sets \( T(\theta, \alpha) \), if \( \alpha \in U^r w^{-r} w \), where \( r_w \) is some integer depending on \( w \).

Now let \( K_n(y, x) = \sum_{w \in W_n} K'_w(y, x) \), where \( K'_w(wx, x) = f'_w(x) \) and \( K'_w(y, x) = 0 \) if \( y \neq wx \). Clearly \( K_n \in \mathcal{F}_\theta \) and \( \mu(x \in X \mid \|K_n(y, x) - K(y, x)\| > \frac{1}{n}) < \frac{1}{n} \). Therefore \( K_n \to K \).

### 4.2 Norm estimates

Let \( A \in \mathbb{C}_d \mathbb{R} \) and denote by \( L_A \) the left-multiplication by \( A \) on the groupoid ring \( \mathbb{C}_d \mathbb{R} \). We give a norm estimate for \( L_A \) in terms of \( w_A \) and \( s_A \).

**Proposition 4.2.** \( \|L_A\| \leq K_d w_A s_A \), where \( K_d \) is a constant depending only the dimension \( d \).

For a matrix \( X \in \text{Mat}_{d \times d}(\mathbb{C}) \), \( \|X\|_{(d)} \) denote the Frobenius norm, that is \( \frac{\text{Tr}(X^*X)}{d} = \|X\|_{(d)}^2 \). We have \( \|X\|_{(d)} \leq k_d \|X\| \) and \( \|X\| \leq k_d \|X\|_{(d)} \) for some constant \( k_d \), where \( \|X\| \) is the usual matrix norm (the \( l^2 \)-norm). Now let \( B \in \mathbb{C}_d \mathbb{R} \). Then \( \|B\|^2 = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{C})}(B^*B) = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{C})}(BB^*) \) that is

\[
\|B\|^2 = \int_X \sum_{x \sim y} \frac{\text{Tr}B(x, y)B^*(y, x)}{d} d\mu(x) = \int_X \sum_{x \sim y} \frac{\text{Tr}B(x, y)B(x, y)}{d} d\mu(x) = \int_X \sum_{x \sim y} \|B(x, y)\|_{(d)}^2 d\mu(x) = \int_X t_x d\mu(x),
\]

where \( t_x = \sum_{x \sim z} \|B(x, z)\|_{(d)}^2 \). On the other hand, \( \|L_AB\|^2 = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{C})}(B^*A^*AB) = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{C})}(A^*ABB^*) \). Hence,

\[
\|L_AB\|^2 = \int_X \sum_{x \sim y} \frac{\text{Tr}A^*A(x, y)B^*B(y, x)}{d} d\mu(x) \leq \int_X \sum_{x \sim y} \|A^*A(x, y)\|_{(d)} \|BB^*(y, x)\|_{(d)} d\mu(x).
\]

Observe that

\[
\|BB^*(y, x)\|_{(d)} = \| \sum_{z \sim x} B(x, z)B(x, z)\|_{(d)} \leq \]
\[
\leq k_d^2 \sum_{x \sim z} B(y, z)B(x, z) \leq k_d^2 \sum_{x \sim z} (\|B(x, z)\|^2 + \|B(y, z)\|^2).
\]
Therefore we have the following inequality:
\[
\|L_AB\|^2 \leq k_d^4 s_{A^*A} \int_X \frac{1}{2} \sum_{x \sim y, A^*(x, y) \neq 0} (\hat{t}_x + \hat{t}_y) \, d\mu(x),
\]
where \( \hat{t}_x = \sum_{x \sim z} \|B(x, z)\|^2 \). Therefore,
\[
\|L_AB\|^2 \leq k_d^4 s_{A^*A} w_{A^*A} \|B\|^2.
\]
Since \( w_{A^*A} \leq w_d^2, s_{A^*A} \leq s_d^2 \) our proposition follows.

The previous proposition can be applied in the case of finite sets as well. Let \( T \) be a finite set and \( K : T \times T \to Mat_{d \times d}(\mathbb{C}) \) be matrix-valued kernel function. These kernels form an algebra analogous to \( \mathbb{C}d \mathcal{R} \). Again we can define \( s_K := \sup_{x, y} \|K(x, y)\| \) and the width \( w_K \) as the supremal number such that for any \( x \in T, K_T(x, y) \neq 0 \) resp. \( K_T(y, x) \neq 0 \) for at most \( w_K \) \( y's \). The normalized trace \( \text{Tr}_s(K) \) is defined as
\[
\text{Tr}_s(K) = \sum_{x \in T} \frac{\text{Tr}K(x, x)}{|T|}.
\]
Again we have the inner product \( \langle K, L \rangle = \text{Tr}_s(L^* K) \) and \( L_A(B) = AB \). The following lemma is the finite version of Proposition 4.2.

**Lemma 4.2.** \( \|L_K\| \leq K_d w_K s_K \).

Finally, we prove a simple lemma about convergence in measure.

**Lemma 4.3.** If \( L_n \overset{\mu}{\to} L \) in \( \mathbb{C}d \mathcal{R} \) then \( \lim_{n \to \infty} \text{tr}_{\text{Mat}_{d \times d}(\mathbb{R})}(L_n^i) = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{R})}(L^i) \).

**Proof.** The fact that \( \lim_{n \to \infty} \text{tr}_{\text{Mat}_{d \times d}(\mathbb{R})}(L_n) = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{R})}(L) \) directly follows from the definition. Since \( (L_n^i - L^i) = (L_{n-1}^i - L^i - 1) + (L_{n-1}^i - L^i - 1) \) a simple induction implies that \( L_n^i \overset{\mu}{\to} L^i \).

### 4.3 Sofic approximation

For \( K \in \mathcal{F}_\theta^d \) and \( n \geq 1 \) let \( K_n : Y_n \times Y_n \to \mathbb{C} \) be defined the following way. Let \( K_n(q, p) := K(y, x) \) if \( p \in T(\theta_n, \alpha), x \in T(\theta, \alpha) \) and \( wp = q, wx = y \) for some \( w \in W_T \). We call \( \{K_n\}_{n=1}^\infty \) the sofic approximation of \( K \).

**Proposition 4.3.** Let \( K \in \mathcal{F}_\theta^+, L \in \text{FFT}^+ \) then
1. \( \|K_n + L_n - (K + L)n\|_\alpha(n) \to 0 \) , where \( \|A\|_\alpha(n) = \text{Tr}_s(A^*A) \).
2. \( \|K_n L_n - (KL)n\|_\alpha(n) \to 0 \).
3. \[ \|K_n^* - (K^*)_n\|_{\alpha(a)} \to 0. \]

4. \[ \text{Id}_n = \text{Id} \]

5. There exists \( C_K > 0 \) such that \( \|K_n\| \leq C_K \), where \( \|A\| \) denotes the usual norm.

6. \[ \lim_{n \to \infty} \frac{\text{Tr}_s(K_n^*)}{|Y_n|} = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{C})} K^i. \]

**Proof.** We call a sequence \( L_n : Y_n \times Y_n \to \text{Mat}_{d \times d}(\mathbb{C}) \) negligible if

- \( \{s_{L_n}\}_{n=1}^\infty \) and \( \{w_{L_n}\}_{n=1}^\infty \) are bounded above.
- \( \lim_{n \to \infty} \frac{|Q_n|}{|Y_n|} = 1 \), where \( Q_n = \{ x \in Y_n \mid L_n(x,y) = 0, L_n(y,x) = 0 \text{ for any } y \in Y_n \} \).

It is easy to see that if \( \{L_n\}_{n=1}^\infty \) is negligible then

\[ \lim_{n \to \infty} \text{Tr}_s(L_n) = 0 \quad \text{and} \quad \text{Tr}_s(L_n^* L_n) = 0. \]

Observe that \( \{K_n + L_n - (K + L)_n\}_{n=1}^\infty \), \( \{K_n L_n - (KL)_n\}_{n=1}^\infty \) and \( \{K_n^* - (K^*)_n\}_{n=1}^\infty \) are all negligible sequences. Hence (1.), (2.) and (3.) hold. The fourth statement is trivial and the fifth one immediately follows from Lemma 4.2.

Since \( \text{Tr}_s(K_n^* - (K_n)^i) \to 0 \) in order to prove (6.) one only needs to show that

\[ \lim_{n \to \infty} \text{Tr}_s(K_n) = \text{tr}_{\text{Mat}_{d \times d}(\mathbb{R})}(K). \]

The right hand side is equal to

\[ \sum_{\alpha \in U^{rr}} \mu(T(\theta,\alpha))c(K,\alpha), \]

where \( c(K,\alpha) = \text{Tr}K(x,x) \) if \( x \in T(\theta,\alpha) \) and \( K \in F_{\theta}^r \). On the other hand the left hand side of the equation is equal to

\[ \sum_{\alpha \in U^{rr}} \frac{T(\theta_n,\alpha)}{|Y_n|}c(K,\alpha). \]

Thus by the sofic property (6.) follows. \( \square \)

5. **Connes’ Embedding Conjecture**

In this section we prove Connes’ Embedding Conjecture for the von Neumann algebras of sofic equivalence relations. First let us very briefly recall the conjecture based on the survey of Pestov [11] (see also [10]). Let \( R \) be the hyperfinite factor. Let \( \mathcal{G} \) be a non-principal ultrafilter on the natural numbers and \( \lim_{\mathcal{G}} \)
be the corresponding ultralimit. Consider the algebra $B_R \subset \prod_{n=1}^{\infty} R$, where \( \{a_i\}_{n=1}^{\infty} \in B_R \) iff \( \sum_{i \geq 1} ||a_i|| < \infty \). Let $J \subset B_R$ be the ideal of those elements \( \{a_i\}_{n=1}^{\infty} \) such that \( \lim_{G} Tr_{\mathcal{R}}(a_i^* a_i) = 0 \), where $Tr_{\mathcal{R}}$ is the unique finite trace on $R$. Then $R^\omega := B_R/J$ is the tracial ultrapower of $R$, a von Neumann algebra factor with trace 

\[ Tr_\mathcal{G} \{[a_i]\}_{i=1}^{\infty} = \lim_{\mathcal{G}} Tr_R(a_i) . \]

**Conjecture 5.1 (Connes’ Embedding Conjecture).** Every separable factor of type II$_1$ embeds into $R^\omega$.

We confirm the conjecture in the case of von Neumann algebras of sofic equivalence relations.

**Theorem 2.** Let $\mathcal{R}$ be a sofic equivalence relation. Then $\mathcal{N}(\mathcal{R})$ embeds into $R^\omega$.

**Proof.** By the result of [12] it is enough to prove that the weakly dense $*$-algebra $\mathcal{F}_\theta$ has a trace preserving $*$-homomorphism into $R^\omega$. Therefore it is enough to construct (see [10]) unital maps $\psi_n : \mathcal{F}_\theta \to Mat_{i_n \times i_n}(\mathbb{C})$ for some sequence of integers \( \{i_n\}_{n=1}^{\infty} \) such that for each $K, L \in \mathcal{F}_\theta$ the following conditions are satisfied.

- \( \lim_{\mathcal{G}} \|\psi_n(K) + \psi_n(L) - \psi_n(K + L)\|_{(i_n)} = 0 \).
- \( \lim_{\mathcal{G}} \|\psi_n(K)\psi_n(L) - \psi_n(KL)\|_{(i_n)} = 0 \).
- \( \lim_{\mathcal{G}} \|\psi_n(K^*) - (\psi_n(K))^*\|_{(i_n)} = 0 \).
- \( \|\psi_n(K)\|_{(i_n)} \) is a bounded sequence.

Now let $\psi_n(K) = K_n$ as in Section 4. Then by Proposition 4.3 all the conditions above are satisfied. \( \square \)

### 6 The Measurable Determinant Conjecture

The goal of this section is to show that the Measurable Determinant Conjecture of Lück, Sauer and Wegner [9] holds for sofic equivalence relations. Let us recall some basic notions from their paper. Let $A \in Mat_{d \times d}(\mathcal{N}(\mathcal{R}))$. Then $AA^* \in Mat_{d \times d}(\mathcal{N}(\mathcal{R}))$ is a positive, self-adjoint element. Let $E(\lambda) = \chi_{[0,\lambda]}(AA^*) \in Mat_{d \times d}(\mathcal{N}(\mathcal{R}))$ be the spectral projection corresponding to the interval $[0,\lambda]$ and $F(\lambda) = tr_{Mat_{d \times d}(\mathcal{N}(\mathcal{R}))} E(\lambda)$ be the associated spectral distribution function. The Fuglede-Kadison determinant is defined as

\[ \det_{Mat_{d \times d}(\mathcal{N}(\mathcal{R}))} (AA^*) = \int_{0^+}^{\infty} \lambda dF(\lambda) . \]

The Measurable Determinant Conjecture states that

\[ \det_{Mat_{d \times d}(\mathcal{N}(\mathcal{R}))} (AA^*) \geq 1 \]
provided that $A \in \text{Mat}_{d \times d}(\mathbb{Z})$, where $\mathbb{Z}dR \subset \mathbb{C}dR$ is defined by

$$\mathbb{Z}dR := \left\{ K \in L^\infty(R, \text{Mat}_{d \times d}(\mathbb{Z})) \mid \text{there exists } w_K > 0 \text{ such that for almost all } x \in X: K(x, y) \neq 0 \text{ or } K(y, x) \neq 0 \text{ only for } w_K \text{ amount of } y \text{'s.} \right\}$$

**Theorem 3.** If $R$ is a sofic equivalence relation, then the measurable determinant conjecture holds.

**Proof.** First let us suppose that $A$ is an operator of finite type. Then $AA^* \in F_0$ and we can consider the sofic approximations $\{A_i\}_{i=1}^\infty$, $\{A_iA_i^*\}_{i=1}^\infty$. Observe that

- $\det(A_iA_i^*) \geq 1$. Indeed $A_iA_i^*$ is a a positive matrix with integer entries (see e.g. the proof of Theorem 3.1 (1) in [9]).
- $\{\|LA_iA_i^*\|\}_{i=1}^\infty$ is uniformly bounded.
- $\lim_{n \to \infty} \text{Tr}_n((A_iA_i^*)^m) = \text{Tr}_{\text{Mat}_{d \times d}N(R)}((AA^*)^m)$.

Then by Lemma 3.2 of [9] $\det_{\text{Mat}_{d \times d}N(R)}(AA^*) \geq 1$ holds.

Now let $A$ be an arbitrary element and $A_nA_n^* \xrightarrow{\text{a.s.}} AA^*$, where $\{A_nA_n^*\}_{n=1}^\infty \subset F_0$.

By the previous observation and Proposition 4.3 the conditions of Lemma 3.2 are satisfied, hence $\det_{\text{Mat}_{d \times d}N(R)}(AA^*) \geq 1$. \hfill $\square$

### 7 Examples of sofic equivalence relations

#### 7.1 The Bernoulli shift

Let $\Gamma$ be a group. We consider the Bernoulli space $\{0, 1\}^\Gamma = \{f : \Gamma \to \{0, 1\}\}$.

The (right) Bernoulli shift $\theta : \{0, 1\}^\Gamma \times \Gamma \to \{0, 1\}^\Gamma$ is defined by $\theta(f, \gamma_1)(\gamma_2) = f(\gamma_1 \cdot \gamma_2)$. $\{0, 1\}^\Gamma$ can be identified with $X = \{0, 1\}^{\mathbb{N}}$ by fixing an enumeration of $\Gamma : \{\gamma_1, \gamma_2, \ldots\}$. Then a $k$-digit label is just a function $\{\gamma_1, \ldots, \gamma_k\} \to \{0, 1\}$.

**Proposition 7.1.** The Bernoulli shift of a sofic group is sofic.

**Proof.** Let $\Gamma$ be a sofic group generated by $s_1, s_2, \cdots \in \Gamma$. Any element $\gamma \in \Gamma$ can of course be expressed as a word in these generators, but this expression is usually not unique. For later use let us fix for each element $\gamma \in \Gamma$ a word $w_\gamma$ that expresses $\gamma$ in terms of the generators. Let us take a sequence of graphs $G_n$ that prove the soficity of $\Gamma$. That is, $G_n$ is a directed graph with each edge being labeled by some $s_i$ such that each vertex has exactly one in-edge and one out-edge labeled with each generator. We can also think of this as a right action of the free group $\mathbb{F}_\infty = \langle s_1, s_2, \cdots \rangle$ on the vertex set of $G_n$. Furthermore the neighborhood statistics of $G_n$ converge to that of $\Gamma$’s Cayley graph on these generators.

We shall label each vertex of $G_n$ with an element of $\{0, 1\}^\Gamma$ so that the labeled neighborhood statistic of $G_n$ will converge to the labeled neighborhood statistic of $\theta$. To do so we first assign to each vertex of each $G_n$ a random bit.
This assignment is simply a random function $\omega : \bigcup_{n=1}^{\infty} G_n \to \{0, 1\}$. Then we take a vertex $g \in G_n$ and assign to it a function $\omega_g : \Gamma \to \{0, 1\}$ by the formula $\omega_g(\gamma) = \omega(g \cdot r)$. Thus now we have an action $\theta_n$ on the $\{0, 1\}^{\Gamma}$-labeled space $G_n$. We claim that $p_\alpha(\theta_n) \rightarrow p_\alpha(\theta)$ for any labeled neighborhood $\alpha$ for a suitable choice of $\omega$ (in fact for almost all $\omega$’s).

In order to prove this, we shall first consider $\{0, 1\}$-labeled neighborhoods, so let us denote by $V^r$ the set of usual $r$-neighborhoods where each vertex is labeled with 0 or 1, up to labeled isomorphism. For an $\alpha \in V^r$ and a $\{0, 1\}$-labeled graph $G$ the notations $T(\alpha, G)$ and $p_\alpha(G)$ extend naturally. In the previous paragraph we described how to obtain a $\{0, 1\}^1$-labeling from an $\{0, 1\}$-labeling for the actions $\theta_n$ on $G_n$. It is clear by that construction that the $U^{r, r}$-neighborhood of a vertex $g$ is determined by the $V^{r+R}$-neighborhood of the same vertex where $R = \max_{i=1, 2, \ldots, r} \lvert \gamma_i \rvert$.

On the other hand there is a natural $\{0, 1\}$-labeling on the points of the Bernoulli-shift: just label each $f : \Gamma \to \{0, 1\}$ by the value of $f$ on the identity element. In this way we can talk about the $V^r$-neighborhoods of points of the Bernoulli-shift, and the $U^{r, r}$-neighborhoods are again determined by the $V^{r+R}$ neighborhoods in the exact same fashion. Hence to finish the proof it is enough to show that $p_\alpha(\theta_n) \rightarrow p_\alpha(\theta)$ for all $\alpha \in V^r$ for almost all $\omega$’s.

First let $\alpha \in V^r$ such that its underlying graph is not isomorphic to the $r$-neighborhood of the identity of $\Gamma$ in the Cayley graph. Since the $G_n$ is a sofic sequence for the Cayley graph, it is immediate that $p_\alpha(\theta_n) \rightarrow 0$. On the other hand the Bernoulli-shift is essentially free, hence almost all orbits are isomorphic to the Cayley graph of $\Gamma$ so $p_\alpha(\theta) = 0$.

Now let us consider an $\alpha \in V^r$ whose graph looks like the Cayley graph around the identity. We can think that the vertices of $\alpha$ are indexed by those elements of $\Gamma$ that have length at most $r$. Then if $f : \Gamma \to \{0, 1\}$ is a point in the free part of the Bernoulli-shift then $f \in T(\theta, \alpha)$ if and only if $f(\gamma) = \alpha(\gamma)$ for all elements $\lvert \gamma \rvert < r$. (Here we $\alpha(\gamma)$ denotes the label written on the vertex of $\alpha$ corresponding to $\gamma$.) Hence $p_\alpha(\theta) = 1/2^{\lvert \alpha \rvert}$. All we have to prove now is

**Lemma 7.1.** For almost all $\omega$’s $p_\alpha(G_n) \rightarrow 1/2^{\lvert \alpha \rvert}$.

**Proof.** Let us say that a vertex $g \in G_n$ is normal if its $r$-neighborhood is isomorphic as a graph to the $r$-neighborhood of the identity element of the Cayley graph. For any vertex $g \in G_n$ let $X_g$ denote a random variable that is 1 if $g \in T(G_n, \alpha)$ and 0 otherwise. Obviously $P(X_g = 1) = 1/2^{\lvert \alpha \rvert}$ for any normal vertex $g$ and 0 otherwise, and

$$p_\alpha(G_n) = \frac{\sum_{g \in G_n} X_g}{\lvert G_n \rvert}.$$

If all the $X_g$’s were independent, then by the law of large numbers $p_\alpha(G_n)$ would converge to the limit of its expected value with probability 1, and this expected value is simply

$$\lim_{n \to \infty} E(p_\alpha(G_n)) = \lim_{n \to \infty} \sum_{g \in G_n} E(X_g) = \lim_{n \to \infty} \frac{\lvert \{ g \in G_n \text{ normal} \} \rvert}{2^{\lvert \alpha \rvert} \lvert G_n \rvert} = \frac{1}{2^{\lvert \alpha \rvert}}.$$
The $X_g$’s are however not independent, but at least they are independent for $g$’s in different graphs, and also $X_{g_1}, \ldots, X_{g_k}$ are jointly independent if $g_1, \ldots, g_k \in G_n$ are pairwise far from each other, namely $d(g_i, g_j) > r$.

**Lemma 7.2.** There exists a natural number $l > 0$ (depending on $r$) and a partition $\bigcup_{i=1}^{l} B^n_i = G_n$ such that if $x \neq y \in B^n_i$ then the $r$-neighborhoods of $x$ and $y$ are disjoint.

**Proof.** Let $H_n$ be a graph with vertex set $V(G_n)$. Let $(x, y) \in E(H_n)$ if and only if $B_r(x) \cap B_r(y) \neq \emptyset$. Then $\deg(x) \leq r^r$ for any $x \in V(H_n)$. Let $l = r^r + 1$ then $H_n$ is vertex-colorable by the colors $c_1, c_2, \ldots, c_l$. Let $B^n_i$ be the vertices coloured by $c_i$.

Now for a fix $\varepsilon > 0$ let $B^n_{i_1}, \ldots, B^n_{i_q}$ be those elements of the partition for which $|B^n_{i_j}| \geq \varepsilon/l$. Then since $\{X_g : g \in B^n_{i_j}\}$ are jointly independent, by the previous argument we get

$$\lim \frac{B^n_{i_j} \cap T(G_n, \alpha)}{|B^n_{i_j}|} = \lim E \frac{B^n_{i_j} \cap T(G_n, \alpha)}{|B^n_{i_j}|} = \frac{1}{2|\alpha|}$$

almost surely for any choice of $i_j$. An easy calculation now shows that setting $B = \bigcup_{j=1}^{q} B^n_{i_j}$ we have

$$\lim \frac{B \cap T(G_n, \alpha)}{|B|} = \frac{1}{2|\alpha|}$$

for the same set of $\omega$’s. Since $|G_n \setminus B| \leq \varepsilon$, this shows that

$$\frac{1}{2|\alpha|} - \varepsilon \leq \lim \inf p_{\alpha}(G_n) \leq \lim \sup p_{\alpha}(G_n) \leq \frac{1}{2|\alpha|} + \varepsilon$$

almost surely, and finally letting $\varepsilon \to 0$ we get the desired almost sure convergence.

Thus we have $p_{\alpha}(\theta_n) \to p_{\alpha}(\theta)$ almost surely for all $\alpha$’s. Hence there exists an $\omega$ for which $p_{\alpha}(\theta_n) \to p_{\alpha}(\theta)$, hence the Bernoulli shift is sofic.

Note that the fact that for residually amenable groups the Measurable Determinant Conjecture holds for the Bernoulli shift has already been proved in [9].

**7.2 Treeable relations**

Recall [8] that an equivalence relation $E \subset X \times X$ is called treeable if it has an $L$-treeing generated by measure-preserving involutions $S_1, S_2, \ldots$. We prove that all treeable equivalence relations are sofic. The most important examples of such treeable relations are the free actions of free groups.

**Theorem 4.** The action of $\Gamma = < \gamma_1, \gamma_2, \ldots | \gamma_i^2 = 1(i = 1, 2, \ldots) >$ defined by $\theta(\gamma_i, x) = S_i(x)$ is sofic.
Proof. By Remark 1.1 it is again sufficient to work with finitely generated actions. So let us assume $\Gamma$ is generated by $\gamma_1, \ldots, \gamma_d$. Let us fix a large $r$. For any $\alpha, \beta \in U^{r,r}$ and any $1 \leq i \leq d$ let us denote

$$T(\theta, \alpha, i, \beta) = \{x \in T(\theta, \alpha) : S_i(x) \in T(\theta, \beta)\}$$

and it measure (as it is obviously a Borel set)

$$p_{\alpha i \beta}(\theta) = \mu(T(\theta, \alpha, i, \beta)).$$

There numbers together with the $p_{\alpha}(\theta)$’s satisfy certain equations:

$$\sum_{\alpha \in U^{r,r}} p_{\alpha}(\theta) = 1$$

$$\sum_{\beta \in U^{r,r}} p_{\alpha i \beta}(\theta) = p_{\alpha}(\theta) \text{ for any } i$$

$$p_{\alpha i \beta}(\theta) = p_{\beta i \alpha}(\theta) \text{ for any } \alpha, i, \beta.$$

Let us introduce variables $w_{\alpha} : \alpha \in U^{r,r}$ and $w_{\alpha i \beta} : \alpha, \beta \in U^{r,r}, 1 \leq i \leq d$. Then $w_{\alpha} = p_{\alpha}(\theta), w_{\alpha i \beta} = p_{\alpha i \beta}(\theta)$ is a solution to the following set of linear equations:

$$\sum_{\alpha \in U^{r,r}} w_{\alpha} = 1 \quad (1)$$

$$\sum_{\beta \in U^{r,r}} w_{\alpha i \beta} = w_{\alpha} \text{ for any } i \quad (2)$$

$$w_{\alpha i \beta} = w_{\beta i \alpha} \text{ for any } \alpha, i, \beta. \quad (3)$$

Now we use the rational approximation trick of Bowen [2]. Let us fix a small $\varepsilon > 0$. If a set of linear equations with rational coefficients has some solution, then it also has a rational solution in which each variable is at most $\varepsilon$-far from the corresponding value of the initial solution. Further we may also assume that if a variable was 0 in the initial solution then it remains 0 in the new solution. So our set of equations has such a rational solution which will shall simply denote by $w_{\alpha}, w_{\alpha i \beta}$. Since now these numbers are all rational, we may choose a large integer $N$ for which $W_{\alpha i \beta} = N \cdot w_{\alpha i \beta}$ is always an even integer.

Now take a set $Y$ with $N$ elements and partition it into subsets $Y_\alpha : \alpha \in U^{r,r}$ with $|Y_\alpha| = W_\alpha$. This can be done because of (1) above. Then fix an index $i$ and do the following: if for a type $\alpha$ the involution $S_i$ is fixing the root, then define $S_i(y) = y$ for all $y \in Y_\alpha$. Otherwise partition $Y_\alpha$ into subsets $Y_{\alpha i \beta}$ of size $W_{\alpha i \beta}$. This can be done because of (2) above. Finally define $S_i$ to be a random bijection between $Y_{\alpha i \beta}$ and $Y_{\beta i \alpha}$, or a random matching in $Y_{\alpha i \alpha}$ (this is where we need that the size of this set is even). This can be done because of (3). Repeat this procedure for each index. Finally for any $\alpha \in U^{r,r}$ and any $y \in Y_\alpha$ look at the label of the root in $\alpha$. This is a word $w \in \{0, 1\}^k$. Label $y$ with any infinite $w' \in \{0, 1\}^\infty$ which starts with $w$. 

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This way we defined an action $\theta'$ of $\Gamma$ on the finite labeled set $Y$. We claim this will be a good approximation to the action $\theta$. To make this precise let us fix an ordering of all possible neighborhood types $\alpha_1, \alpha_2, \ldots$, and for two actions $\theta, \theta'$ let us introduce their statistical distance $d_s(\theta, \theta') = \sum_{i=1}^{\infty} \frac{|p_{\alpha_i}(\theta) - p_{\alpha_i}(\theta')|}{2^i}$. It is easy to see that $\theta_n$ is a sofic sequence for $\theta$ if and only if $d_s(\theta, \theta_n) \to 0$.

**Lemma 7.3.** Let $\nu_q$ denote the ratio of those points in $Y$ through which there is a $\theta'$ cycle of length at most $q$. Then for any fixed $q$ we have $\nu_q \to 0$ in probability when $N \to \infty$.

**Proof.** By the construction of $Y$ the probability of the existence of any particular $xy$ edge is at most $c/N$ for some universal constant $c$ depending only on the $w_{\alpha i \beta}$ numbers. Hence the probability that a particular cycle of length $l$ exists in $\theta'$ is at most $c^l/N^l$, hence the expected number of length $l$ cycles is at most $c^l/N^l \cdot (N/l)^l < c l^l$ which is a constant. So for fixed $q$ and large $N$ the expected number of points through which there is cycle of length at most $q$ is at most some constant $c_q$. Then for any fixed $\varepsilon$ we have $P(\nu_q > \varepsilon) \leq \frac{c_q}{\varepsilon N^q}$ so clearly $P(\nu_q > \varepsilon) \to 0$ as $N \to \infty$.

Then the ratio of those vertices whose $r$-neighborhood is not a tree is at most $d'\nu_{2r}$, since any such neighborhood contains a cycle of length at most $2r$, and hence the root of this neighborhood is at most $r$ steps from a vertex in the cycle.

For a neighborhood $\alpha \in U^{r,r}$ let us denote by $\alpha|_{q} \in U^{q,q}$ the subgraph of $\alpha$ spanned by the vertices that are at most $q$ steps from the root and keeping only the first $q$ digits of the labels. The following is easily verified by induction on $q$:

**Claim 7.1.** If $q \leq r$ and the girth of $\theta'$ at $y \in Y_\alpha$ is greater than $2q$ then $B_q(y) \cong \alpha|_q$.

Now we can estimate $d_s(\theta, \theta')$. Let us fix $r$ and let $j$ denote the index of the first $\alpha_i$ neighborhood in our listing either whose radius is larger than $r$ or its labels have more than $r$ digits.

Let

$$U = \bigcup_{q \leq r} U^{q,q}, \quad U_c = \{\alpha \in U : \alpha \text{ is not a tree}\}, \quad U_t = U \setminus U_c.$$ 

If $\alpha \in U_c$ then $p_\alpha(\theta) = 0$ since $\theta$ is a treeing, and $p_\alpha(\theta') \leq d'\nu_{2r}$ since at most this many vertices can have cycles in their $r$-neighborhood.

If $\alpha \in U_t \cap U^{r,q}$ then

$$|p_\alpha(\theta) - p_\alpha(\theta')| = \left| \sum_{\beta \in U^{r,r} : \beta|_q \cong \alpha} p_{\beta}(\theta) - p_{\beta}(\theta') \right| \leq \left| \sum_{\beta \in U^{r,r} : \beta|_q \cong \alpha} p_{\beta}(\theta) - w_{\beta} \right| + d'\nu_{2r} \leq \varepsilon |U^{r,r}| + d'\nu_{2r}.$$ 

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The term $d^r \nu_{2r}$ appears again because $p_\beta(\theta')$ is not necessarily equal to $w_\beta$: the difference comes from exactly those vertices in $Y_\beta$ whose $2r$ neighborhood is not a tree. And finally

\[
ds(\theta, \theta') = \sum_{i=1}^{\infty} \frac{|p_{\alpha_i}(\theta) - p_{\alpha_i}(\theta')|}{2^i} \leq \sum_{i: \alpha_i \in U_c} \frac{d^r \nu_{2r}}{2^i} + \sum_{i: \alpha_i \in U_t} \frac{\varepsilon |U^{r,r}| + d^r \nu_{2r}}{2^i} + \sum_{i \geq j} \frac{1}{2^i} \leq \varepsilon |U^{r,r}| + 2d^r \nu_{2r} + 1/2^{j-1} \quad (4)
\]

So in order to construct a finite action with $d_s(\theta, \theta') < \delta$ first we choose $r$ so large that $1/2^{j-1} < \delta/3$ in (4). Then we choose an $\varepsilon < \frac{\delta}{3 |U^{r,r}|}$. Then we find a rational solution to our system of equations \[\text{(1,2,3)}.\] Finally we choose $N$ so large, that with positive probability $\nu_{2r} \leq \frac{4}{6 \pi r}$. We pick an action $\theta'$ satisfying this and hence

\[
ds(\theta, \theta') \leq \varepsilon |U^{r,r}| + 2d^r \nu_{2r} + 1/2^{j-1} < 3 \cdot \delta/3 = \delta.
\]

Hence $\theta$ is indeed a sofic action. \[\square\]

Note that the previous theorem combined with Theorem 1 shows the all treeable groups are sofic. Recall that a group is treeable if it has a free treeable action.

### 7.3 Profinite actions

The simplest case of sofic action is arguably the case of profinite actions. Let $\Gamma$ be a countable residually finite group and $\Gamma \supset N_1 \supset N_2 \ldots$ be finite index normal subgroups such that $\bigcap_{i=1}^{\infty} N_i = \{1\}$. Then $G = \lim_{\leftarrow} \Gamma/N_i$ is the profinite closure with respect to the system $\{N_i\}$, a compact group. Then $\Gamma$ is a dense subgroup of $G$ and so it preserves the Haar-measure $\nu$. It is easy to see that $\Gamma \acts (G, \nu)$ is a sofic action.

### 8 Conclusion

We can conclude that the Connes Embedding Conjecture and the Measurable Determinant Conjecture hold for treeable sofic relations, particularly, for relations induced by free actions of free groups. We end our paper with a question related to Question 10.1 of Aldous and Lyons \[\text{[1]}\] on unimodular networks.

**Question 8.1.** Does there exist a measurable equivalence relation that is not sofic ?

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