Fundamental solution of degenerated Fokker - Planck equation

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ABSTRACT

Fundamental solution of degenerated Fokker - Planck equation is built by means of the Fourier transform method. The result is checked by direct calculation.

Keywords
Fokker-Planck equation, fundamental solution, Fourier transform, exact solution

The object of our considerations is a special case of Fokker - Planck equation.

The Fokker - Planck equation describes evolution of 3D continuum of non-interacting particles imbedded in a dense medium without outer forces. The interaction between particles and medium causes combined diffusion in physical space and velocities space. The only force, which acts on particles, is damping force proportional to velocity. Fundamental solution of Fokker - Planck equation is built in our work [1].

In this paper we consider the special case of the Fokker - Planck equation with zero damping force. We call it degenerated Fokker - Planck equation. For this special case the equation reads:

\[
\frac{\partial n}{\partial t} + v_x \frac{\partial n}{\partial x} + v_y \frac{\partial n}{\partial y} + v_z \frac{\partial n}{\partial z} = k \left( \frac{\partial^2 n}{\partial v_x^2} + \frac{\partial^2 n}{\partial v_y^2} + \frac{\partial^2 n}{\partial v_z^2} \right)
\]  

(1)

where

\( n = n(t, x, y, z, v_x, v_y, v_z) \) - density;
\( t \) - time variable;
\( x, y, z \) - space coordinates;
\( v_x, v_y, v_z \) - velocities;
\( k \) - coefficient of diffusion.

We shall search solution of equation (1) for unlimited space

\[-\infty < x < +\infty, -\infty < y < +\infty, -\infty < z < +\infty, \]

\[-\infty < v_x < +\infty, -\infty < v_y < +\infty, -\infty < v_z < +\infty. \]

(2)
Let us denote by $n_0$ initial density

$$n_0(x, y, z, v_x, v_y, v_z) = n(0, x, y, z, v_x, v_y, v_z).$$  \hfill (3)

We shall use the Fourier transform method. Let us denote by $N$ Fourier transform of density

$$N = N(t, p_x, p_y, p_z, q_x, q_y, q_z) =$$

$$\frac{1}{(2\pi)^6} \int \int \int \int \int \int \exp(-i(xp_x + yp_y + zp_z + v_x q_x + v_y q_y + v_z q_z)) ndxdydzdv_xdv_ydv_z.$$  \hfill (4)

and by $N_0$ - Fourier transform of initial density:

$$N_0(p_x, p_y, p_z, q_x, q_y, q_z) = N(0, p_x, p_y, p_z, q_x, q_y, q_z)$$  \hfill (5)

where $p_x, p_y, p_z$ - space coordinates momentum variables;

$q_x, q_y, q_z$ - velocities momentum variables.

Multiplying (1) by $\exp(-i(xp_x + yp_y + zp_z + v_x q_x + v_y q_y + v_z q_z))$ and integrating over whole space and velocities, we obtain

$$\frac{\partial N}{\partial t} + p_x \frac{\partial N}{\partial q_x} + p_y \frac{\partial N}{\partial q_y} + p_z \frac{\partial N}{\partial q_z} = -k \left( q_x^2 + q_y^2 + q_z^2 \right) N.$$  \hfill (6)

The equation (6) is a linear differential equation of first order, so it can be solved by the method of characteristics. Relations on the characteristics are

$$dt = \frac{dp_x}{0} = \frac{dp_y}{0} = \frac{dp_z}{0} = \frac{-dq_x}{p_x} = \frac{-dq_y}{p_y} = \frac{-dq_z}{p_z} = -\frac{dN/N}{k (q_x^2 + q_y^2 + q_z^2)}. \hfill (7)$$

First three equations (7) have three integrals:

$$p_x = \text{const};$$  \hfill (8)

$$p_y = \text{const};$$

$$p_z = \text{const}.$$  \hfill (9)

If we combine (8) with next three equations (7), we get three further integrals

$$q_x + p_x t = \text{const};$$

$$q_y + p_y t = \text{const};$$

$$q_z + p_z t = \text{const}.$$  \hfill (10)

We solve (9) for $q_x, q_y, q_z$ and obtain

$$q_x = q_{x0} - p_x t;$$

$$q_y = q_{y0} - p_y t;$$

$$q_z = q_{z0} - p_z t.$$  \hfill (10)

On the other hand, if we solve (9) for $q_{x0}, q_{y0}, q_{z0}$, we obtain
\[ q_{x0} = q_x + p_x t; \]
\[ q_{y0} = q_y + p_y t; \]
\[ q_{z0} = q_z + p_z t. \]

To get the last integral of (7), we replace current velocities momentum variables \( q_x, q_y, q_z \) by initial velocities momentum variables \( q_{x0}, q_{y0}, q_{z0} \) in \( q_x^2 + q_y^2 + q_z^2 \):

\[ q_x^2 + q_y^2 + q_z^2 = q_{x0}^2 + q_{y0}^2 + q_{z0}^2 - 2t \left( q_{x0} p_x + q_{y0} p_y + q_{z0} p_z \right) + t^2 \left( p_x^2 + p_y^2 + p_z^2 \right). \]

Integrating (12) in \( t \), we obtain the last integral

\[ \ln(N) + k \left[ t \left( q_{x0}^2 + q_{y0}^2 + q_{z0}^2 \right) - t^2 \left( q_{x0} p_x + q_{y0} p_y + q_{z0} p_z \right) + \frac{1}{3} t^3 \left( p_x^2 + p_y^2 + p_z^2 \right) \right] = \text{const.} \]

We replace in (13) initial values of velocities momentum variables by their current values

\[ \ln(N) + k \left[ t \left( q_x^2 + q_y^2 + q_z^2 \right) + t^2 \left( q_x p_x + q_y p_y + q_z p_z \right) + \frac{1}{3} t^3 \left( p_x^2 + p_y^2 + p_z^2 \right) \right] = \text{const.} \]

To determine the constant term in (14), we write the same expression for initial values and equate both expressions

\[ \ln(N) + k \left[ t \left( q_{x0}^2 + q_{y0}^2 + q_{z0}^2 \right) + t^2 \left( q_{x0} p_x + q_{y0} p_y + q_{z0} p_z \right) + \frac{1}{3} t^3 \left( p_x^2 + p_y^2 + p_z^2 \right) \right] = \ln(N_0). \]

Solve (15) for \( N \)

\[ N = N_0 \left( p_x, p_y, p_z, q_x + p_x t, q_y + p_y t, q_z + p_z t \right) \times \]
\[ \times \exp \left[ -k \left( t \left( q_x^2 + q_y^2 + q_z^2 \right) + t^2 \left( q_x p_x + q_y p_y + q_z p_z \right) + \frac{1}{3} t^3 \left( p_x^2 + p_y^2 + p_z^2 \right) \right) \right] \]

The \( N_0(\ldots) \) in the (16) means, that one have to calculate \( N_0 \) from initial density according to (5) and then replace values of its arguments by expressions (11).

Let us specify initial density value as product of delta functions

\[ \rho_0 = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(v_x - x_0)\delta(v_y - y_0)\delta(v_z - z_0). \]

The Fourier transform of initial density (17) is

\[ N_0 = \frac{1}{(2\pi)^6} \exp \left[ -i \left( x_0 p_x + y_0 p_y + z_0 p_z + v_x q_x + v_y q_y + v_z q_z \right) \right]. \]

\[ -3- \]
Substituting (11) for \( N_0 \) arguments in (18) gives
\[
\hat{N}_0 = \frac{1}{(2\pi)^3} \exp\left\{-i \left[ x_0 p_x + y_0 p_y + z_0 p_z + v_{x0} (q_x + p_x t) + v_{y0} (q_y + p_y t) + v_{z0} (q_z + p_z t) \right] \right\}
\] (19)

or
\[
\hat{N}_0 = \frac{1}{(2\pi)^3} \exp\left\{-i \left[ p_x (x_0 + v_{x0} t) + p_y (y_0 + v_{y0} t) + p_z (z_0 + v_{z0} t) + (q_x v_{x0} + q_y v_{y0} + q_z v_{z0}) \right] \right\}. \tag{20}
\]

It is clear that (20) is the Fourier transform of (21)
\[
\hat{n}_0 = \delta(x - (x_0 + v_{x0} t)) \delta(y - (y_0 + v_{y0} t)) \delta(z - (z_0 + v_{z0} t)) \delta(v_x - v_{x0}) \delta(v_y - v_{y0}) \delta(v_z - v_{z0}). \tag{21}
\]

To get inverse Fourier transform of \( n \) from its Fourier transform (16) we use two known results ([1]):

1. A product of 2 functions \( A(\omega)B(\omega) \) transforms to convolution \( a(x) * b(x) \).

2. The Gaussian exponent of quadratic form \( e^{-\omega^T A \omega} \) with matrix \( A \) transforms to exponent of quadratic form with inverse matrix \( \frac{\pi}{\det(A)} e^{-\frac{1}{4} \omega^T A^{-1} \omega} \).

In our case the matrix is (see (16))
\[
A = k \begin{bmatrix} t^3/3 & t^2/2 \\ t^2/2 & t \end{bmatrix} \tag{22}
\]

The determinant is equal to
\[
\det(A) = \frac{1}{12} k^2 t^4. \tag{23}
\]

Let us denote by \( D \) expression
\[
D = \frac{\det(A)}{k^2} = \frac{t^4}{12}. \tag{24}
\]

The inverse matrix is
\[
A^{-1} = \frac{1}{kD} \begin{bmatrix} t & -t^2/2 \\ -t^2/2 & t^3/3 \end{bmatrix} \tag{25}
\]

Combining (21) with (25) we obtain expression for the fundamental solution:
\[
G = \left( \frac{\pi}{kN_D} \right)^3 \hat{n}_0 * \exp\left\{-\frac{1}{4kD} \left[ 4t^2 \left( x^2 + y^2 + z^2 \right) - t^2 (xv_x + yv_y + zv_z) + \frac{t^2}{3} (v_x^2 + v_y^2 + v_z^2) \right] \right\}. \tag{26}
\]

where * means convolution of two functions.

The convolution of arbitrary function with product of delta functions simplifies to substitution delta function arguments for this function arguments. Finally, we obtain
\[ G = \left( \frac{\pi}{k\sqrt{D}} \right)^3 \exp \left[ \frac{-1}{4kD} \left( \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \right) - t^2 \left( \hat{\lambda}_x + \hat{\lambda}_y + \hat{\lambda}_z \right) + \frac{t^3}{3} \left( \hat{\lambda}_x^2 + \hat{\lambda}_y^2 + \hat{\lambda}_z^2 \right) \right]; \quad (27) \]

where

\[ \hat{x} = x - (x_0 + tv_{x0}); \quad \hat{y} = y - (y_0 + tv_{y0}); \quad \hat{z} = z - (z_0 + tv_{z0}); \]

\[ \hat{\lambda}_x = v_x - v_{x0}; \quad \hat{\lambda}_y = v_y - v_{y0}; \quad \hat{\lambda}_z = v_z - v_{z0}. \]

This is the fundamental solution of the degenerated Fokker - Planck equation.

Let us check validity of solution (27-28). Direct differentiation of (27-28) and substitution to (1) leads to cumbersome calculations. Therefore we use "semi-reverse" method. (27) has Gaussian form, so we search Gaussian solutions of (1):

\[ n = \exp \left\{ A(t) \left( x^2 + y^2 + z^2 \right) + 2B(t) \left( xv_x + yv_y + zv_z \right) + C(t) \left( v_x^2 + v_y^2 + v_z^2 \right) + 3Q(t) \right\}. \quad (29) \]

To get rid of exponents we write

\[ l = \ln(n) = A(t) \left( x^2 + y^2 + z^2 \right) + 2B(t) \left( xv_x + yv_y + zv_z \right) + C(t) \left( v_x^2 + v_y^2 + v_z^2 \right) + 3Q(t). \quad (30) \]

\[ l \] must satisfy equation

\[ \frac{\partial l}{\partial t} + v_x \frac{\partial l}{\partial x} + v_y \frac{\partial l}{\partial y} + v_z \frac{\partial l}{\partial z} = k \left( \frac{\partial^2 l}{\partial v_x^2} + \frac{\partial^2 l}{\partial v_y^2} + \frac{\partial^2 l}{\partial v_z^2} \right) + k \left( \frac{\partial l}{\partial v_x} \right)^2 + \left( \frac{\partial l}{\partial v_y} \right)^2 + \left( \frac{\partial l}{\partial v_z} \right)^2 \]

instead of equation (1) for \( n \).

Substituting (29) for \( A, B, C, Q \) in (30) and collecting of similar terms leads to following equations

\[ \frac{dA}{dt} = 4k B^2; \quad (32) \]

\[ \frac{dB}{dt} = 4k \ B \ C - A; \quad (33) \]

\[ \frac{dC}{dt} = 4k \ C^2 - 2B; \quad (34) \]

\[ \frac{dQ}{dt} = 2k \ C. \quad (35) \]

It is not easy to solve nonlinear system (32-35), but our task is simpler. We have only to check, that

\[ D(t) = \frac{t^4}{12}; \quad (36) \]

\[ A(t) = \left( \frac{-1}{4kD} \right) t = -\frac{3}{k} t^3; \quad (37) \]
\[ B(t) = \left( \frac{1}{4kD} \right) \frac{1}{2} t^2 = \frac{3}{2k} t^2; \quad (38) \]
\[ C(t) = \left( -\frac{1}{4kD} \right) \frac{1}{3} t^3 = -\frac{1}{k} t^{-1}; \quad (39) \]
\[ Q(t) = -2 \ln(t); \quad (40) \]
satisfies both equations (32-35). This result is obvious.

We proved validity of (27) for the special case
\[ x_0 = 0; y_0 = 0; z_0 = 0; v_{x0} = 0; v_{y0} = 0; v_{z0} = 0. \quad (41) \]

For the common case we prove that differential operators (see [2])
\[ \frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}; \quad (42) \]
and
\[ t \frac{\partial}{\partial x} + \frac{\partial}{\partial v_x}; \quad t \frac{\partial}{\partial y} + \frac{\partial}{\partial v_y}; \quad t \frac{\partial}{\partial z} + \frac{\partial}{\partial v_z}; \quad (43) \]
are symmetries of PDE (1). This statement is trivial for (43) because (1) does not contain \( x \) explicitly. To prove the statement for (43) we build prolongation of differential operator according to Lie prolongation formula (ref. [3])
\[ \delta \left( \frac{\partial u^a}{\partial x^i} \right) = D_i(\delta u^a) - \frac{\partial u^a}{\partial x^i} D_i(\delta x^i); \quad (44) \]
where
\( u^a \) - a set of dependent variables;
\( x^i \) - a set of independent variables;
\( D_i \) - full derivation on \( x^i \) operator;
\( \delta u^a, \delta x^i \) - actions of infinitesimal symmetry operator on variables. We use this non-standard notation instead of usual \( \zeta^a, \zeta^i \) to emphasize their nature as small variations of variables.
\[ \delta \left( \frac{\partial u^a}{\partial x^i} \right) \] - induced action of infinitesimal symmetry operator on derivatives.

For operators (43) Lie formula gives
\[ t \frac{\partial}{\partial x} + \frac{\partial}{\partial v_x} - \frac{\partial n}{\partial x} \frac{\partial}{\partial n}; \quad (45) \]
\[ t \frac{\partial}{\partial y} + \frac{\partial}{\partial v_y} - \frac{\partial n}{\partial y} \frac{\partial}{\partial n}; \quad (46) \]
\[ t \frac{\partial}{\partial z} + \frac{\partial}{\partial v_z} - \frac{\partial n}{\partial z} \frac{\partial}{\partial n}; \quad (47) \]
where \( n_i = \frac{\partial n}{\partial t} \). For second order derivatives we have \( \delta(n_{uu}) = 0, \delta(n_{vv}) = 0, \delta(n_{ww}) = 0 \) (see [3]).
It is easy to calculate, that the action of (45-47) on (1) is identical zero. We proved, that differential operators (42-43) are symmetries of PDE (1).

The action of operators (42-43) on solutions (41) increases variables \(x_0, y_0\ldots\) from zero to arbitrary values. In this way we get from (41) solutions (27-28).

We checked the solution (27-28).

**DISCUSSION**

The main result of these considerations is the closed form expression for fundamental solution of the degenerated Fokker - Planck equation without forces. To obtain this result we could not simply substitute \(\alpha = 0\) to previous formulae. Some calculations are necessary. Their result is expression for fundamental solution. This expression have the same Gaussian form, as the fundamental solution of full equation. Its coefficient one could get by passage to the limit \(\alpha \to 0\), using first terms of exponent Taylor expansion. All coefficients are polynomials on time variable, no exponential saturation effect is present.

Fundamental solution of the full Fokker - Planck equation contains time \(t\) only as product \(\alpha t\) with damping coefficient. Therefore the limit \(\alpha \to 0\) also depicts the limit \(t \to 0\) behavior of the fundamental solution.

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