Group distance magic Cartesian product of two cycles

Sylwia Cichacz\textsuperscript{1,*}, Paweł Dyrlaga\textsuperscript{1}, Dalibor Froncek\textsuperscript{2}
\textsuperscript{1}AGH University of Science and Technology, Poland
\textsuperscript{2}University of Minnesota Duluth, U.S.A.

May 14, 2019

Abstract

Let \( G = (V, E) \) be a graph and \( \Gamma \) an Abelian group both of order \( n \). A \( \Gamma \)-distance magic labeling of \( G \) is a bijection \( \ell : V \rightarrow \Gamma \) for which there exists \( \mu \in \Gamma \) such that \( \sum_{x \in N(v)} \ell(x) = \mu \) for all \( v \in V \), where \( N(v) \) is the neighborhood of \( v \).

Froncek showed that the Cartesian product \( C_m \square C_n \), \( m, n \geq 3 \) is a \( \mathbb{Z}_{mn} \)-distance magic graph if and only if \( mn \) is even. It is also known that if \( mn \) is even then \( C_m \square C_n \) has \( \mathbb{Z}_\alpha \times \mathcal{A} \)-magic labeling for any \( \alpha \equiv 0 \pmod{\text{lcm}(m, n)} \) and any Abelian group \( \mathcal{A} \) of order \( mn/\alpha \). However, the full characterization of group distance magic Cartesian product of two cycles is still unknown.

In the paper we make progress towards the complete solution this problem by proving some necessary conditions. We further prove that for \( n \) even the graph \( C_n \square C_n \) has a \( \Gamma \)-distance magic labeling for any Abelian group \( \Gamma \) of order \( n^2 \). Moreover we show that if \( m \neq n \), then there does not exist a \( (\mathbb{Z}_2)^{m+n} \)-distance magic labeling of the Cartesian product \( C_{2^m} \square C_{2^n} \). We also give necessary and sufficient condition for \( C_m \square C_n \) with \( \gcd(m, n) = 1 \) to be \( \Gamma \)-distance magic.

\textsuperscript{*}This work was partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of Ministry of Science and Higher Education.
1 Introduction

All graphs \( G = (V, E) \) are finite undirected simple graphs. For standard graph theoretic notation and definitions we refer to Diestel \[4\].

In 1963 Sedláček \[8\] noticed the following connection between a magic square \( M \) of size \( n \times n \) and an edge labeling of the complete bipartite graph \( K_{n,n} \). Namely, assigning the entry in row \( i \) and column \( j \) of the magic square to the edge connecting the \( i \)-th vertex in one partite set to the \( j \)-th vertex in the other set, we obtain the sum of labels of edges incident with each vertex equal to the magic square constant. This type of labeling became known as magic labeling, supermagic labeling or vertex-magic edge labeling. Another concept in graph labeling that was motivated by the construction of magic squares labels vertices instead. A distance magic labeling of a graph \( G \) of order \( n \) is a bijection \( \ell : V \to \{1, 2, \ldots, n\} \) with the property that there exists a positive integer \( \mu \) such that

\[
w(v) = \sum_{u \in N(v)} \ell(u) = \mu
\]

for all \( v \in V \), where \( N(v) \) is the open neighborhood of \( v \) and \( w(v) \) is called the weight of the vertex \( v \). The constant \( \mu \) is called the magic constant of the labeling \( f \). Any graph admitting a distance magic labeling is called a distance magic graph.

We recall one out of four standard graph products (see \[6\]). Let \( G \) and \( H \) be two graphs. The Cartesian product \( G \square H \) is a graph with vertex set \( V(G) \times V(H) \). Two vertices \( (g, h) \) and \( (g', h') \) are adjacent if and only if either \( g = g' \) and \( h \) is adjacent with \( h' \) in \( H \), or \( h = h' \) and \( g \) is adjacent with \( g' \) in \( G \).

Rao at al. proved the following result for Cartesian product of cycles in \[7\].

**Theorem 1.1** (\[7\]). The Cartesian product \( C_m \square C_n \), \( m, n \geq 3 \) is a distance magic graph if and only if \( m, n \equiv 2 \pmod{4} \) and \( m = n \).

Assume \( \Gamma \) is a finite Abelian group of order \( n \) with the operation denoted by \(+\). For convenience we will write \( ka \) to denote \( a + a + \ldots + a \) (where the element \( a \) appears \( k \) times), \( -a \) to denote the inverse of \( a \), and use \( a - b \) instead of \( a + (-b) \). Moreover, the notation \( \sum_{a \in S} a \) will be used as a short form for \( a_1 + a_2 + a_3 + \cdots + a_t \), where \( a_1, a_2, a_3, \ldots, a_t \) are all elements of the
set \( S \). The identity element of \( \Gamma \) will be denoted by \( 0 \). Recall that any group element \( \iota \in \Gamma \) of order 2 (i.e., \( \iota \neq 0 \) and \( 2\iota = 0 \)) is called an *involution*.

The magic labeling (in the classical point of view) with labels being the elements of an Abelian group has been studied for a long time (see papers by Stanley [9, 10]). Therefore, it was a natural step to label the vertices of a graph \( G \) with elements of an Abelian group also in the case of distance magic labeling. This concept was introduced by Froncek in [5].

A \( \Gamma \)-distance magic labeling of a graph \( G = (V, E) \) with \( |V| = n \) is a bijection \( \ell \) from \( V \) to an Abelian group \( \Gamma \) of order \( n \) such that the weight \( w(v) = \sum_{u \in N(v)} \ell(u) \) of every vertex \( v \in V \) is equal to the same element \( \mu \in \Gamma \), called the *magic constant*. A graph \( G \) is called a *group distance magic graph* if there exists a \( \Gamma \)-distance magic labeling for every Abelian group \( \Gamma \) of order \( |V(G)| \).

First result on \( \Gamma \)-distance magic labeling of Cartesian product of cycles was proved by [5]:

**Theorem 1.2.** ([5]) The Cartesian product \( C_m \square C_n, \ m, n \geq 3 \) is \( \mathbb{Z}_{mn} \)-distance magic graph if and only if \( mn \) is even.

The result was later improved by Cichacz [2].

**Theorem 1.3.** ([2]) Let \( m \) or \( n \) be even and \( l = \text{lcm}(m, n) \). Then \( C_m \square C_n \) has a \( \mathbb{Z}_{\alpha} \times A \)-magic labeling for any \( \alpha \equiv 0 \pmod{l} \) and any Abelian group \( A \) of order \( mn/\alpha \).

The following related results were also proved in the respective papers.

**Theorem 1.4.** ([5]) The graph \( C_{2^n} \square C_{2^n} \) has a \( (\mathbb{Z}_2)^{2n} \)-distance magic labeling for \( n \geq 2 \) and \( \mu = (0, 0, \ldots, 0) \).

**Theorem 1.5.** ([2]) If \( m, n \) are odd, then \( C_m \square C_n \) is not \( \Gamma \)-distance magic graph for any Abelian group \( \Gamma \) of order \( mn \).

The following general problem is still widely open.

**Problem 1.6.** ([5]) For a given graph \( C_m \square C_n \), determine all Abelian groups \( \Gamma \) such that the graph \( C_m \square C_n \) admits a \( \Gamma \)-distance magic labeling.
Note that if a graph $G$ of order $n$ is distance magic, then it is $\mathbb{Z}_n$-distance magic. Moreover there are infinitely many distance magic graphs that at the same time are group distance magic [1]. Hence Cichacz and Froncek stated the following conjecture.

**Conjecture 1.7** ([3]). If $G$ is a distance magic graph, then $G$ is group distance magic.

In the paper we make some progress towards solution of Problem 1.6 by proving some necessary conditions as well as some new existence results. In particular, we prove that for $n$ even the graph $C_n \square C_n$ has a $\Gamma$-distance magic labeling for any Abelian group $\Gamma$ of order $n^2$. Moreover we show that if $m \neq n$, then there does not exist a $(\mathbb{Z}_2)^{m+n}$-distance magic labeling of the Cartesian product $C_{2m} \square C_{2n}$. We prove a necessary and sufficient condition for $C_m \square C_n$ with $\gcd(m, n) = 1$ to be $\Gamma$-distance magic. Observe that the Cartesian product $C_{2m} \square C_{2n}$ is $\mathbb{Z}_{2m+n}$-distance magic by Theorem 1.3 but is not distance magic by Theorem 1.1. Therefore, this result is the first example that shows that assumptions in Conjecture 1.7 cannot be relaxed, that is, the statement that if a graph $G$ of order $n$ is $\mathbb{Z}_n$-distance magic then it is group distance magic is not true.

## 2 Sufficient conditions

Recall that the **exponent** $\exp(\Gamma)$ of a group $\Gamma$ of order $q$ with elements $a_1, a_2, \ldots, a_q$ is the smallest possible $r$ such that $ra_i = 0$ for any $a_i \in \Gamma$. It is well known that in Abelian groups, $r = \text{lcm}(o_1, o_2, \ldots, o_q)$ where $o_i$ is the order of $a_i$ for $i = 1, 2, \ldots, q$.

It also well known that if $\Gamma$ has an even order, then there is an element $a_i$ of even order, and hence $\exp(\Gamma) = r$ must be even. Because the non-existence of $\Gamma$-labelings of $C_m \square C_n$ for $|\Gamma| = mn$ odd follows from Theorem 1.5, we will from now on only consider the case where $|\Gamma| = mn$ is even. Consequently, we will always have $\exp(\Gamma) = r$ even.

We start with the following general theorem for Cartesian product of graphs:

**Theorem 2.1.** Let $\Gamma_1$ and $\Gamma_2$ be Abelian groups with exponents $r_1$ and $r_2$, respectively. Let $a_1$ and $a_2$ be some positive integers. If an $a_1 r_1$-regular graph $G_1$ is $\Gamma_1$-distance magic and an $a_2 r_2$-regular graph $G_2$ is $\Gamma_2$-distance magic, then the Cartesian product $G_1 \square G_2$ is $\Gamma_1 \times \Gamma_2$-distance magic.
Proof. Let $\ell_i : V(G_i) \to \Gamma_i$ be a $\Gamma_i$-distance magic labeling, and $\mu_i$ the magic constant for the graph $G_i$, $i \in \{1, 2\}$. Define the labeling $\ell : V(G_1 \square G_2) \to \Gamma_1 \times \Gamma_2$ for $G_1 \square G_2$, as:

$$\ell((x, y)) = (\ell_1(x), \ell_2(y)).$$

Obviously, $\ell$ is a bijection and moreover, for any $(u, w) \in V(G_1 \square G_2)$:

$$w(u, w) = \sum_{(x, y) \in N_{G_1 \square G_2}((u, w))} \ell(x, y) = \left( \sum_{x \in N_{G_1}(u)} \ell_1(x) + a_2r_2\ell(u), \sum_{y \in N_{G_2}(w)} \ell_2(y) + a_1r_1\ell(w) \right)$$

$$= (\mu_1, \mu_2) = \mu,$$

which settles the proof.

Theorem 2.1 implies the following observation.

Observation 2.2. Let $d \equiv 0 \mod 4$. A hypercube $Q_d$ is $\Gamma$-distance magic for any Abelian $\Gamma$ of order $2^d$ with $\exp(\Gamma) \leq 4$.

Proof. Note that in the factorization of $\Gamma$ we have only factors $Z_2$ and $Z_4$ since $\exp(\Gamma) \leq 4$. The proof is by induction on $d$. Because $Q_4 \cong C_4 \square C_4$ we obtain by Theorems 1.3 and 1.4 that $Q_4$ is $\Gamma$-distance magic for any Abelian $\Gamma$ of order 16 with $\exp(\Gamma) \leq 4$. Recall that for $d \geq 8$ the hypercube $Q_d$ can be also defined recursively in terms of the Cartesian product of two graphs as $Q_d = Q_{d-4} \square Q_4$. Obviously $Q_d$ is $d$-regular. Therefore we are done by Theorem 2.1.

Let $V(C_m \square C_n) = \{x_{i,j} : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$, where $N(x_{i,j}) = \{x_{i,j-1}, x_{i,j+1}, x_{i+1,j}, x_{i-1,j}\}$ and the operations on the first and second suffix are performed modulo $m$ and $n$, respectively. Without loss of generality we can assume $m \leq n$. By a diagonal $D_j$ of $C_m \square C_n$ we mean a sequence of vertices $(x_{0,j}, x_{1,j+1}, \ldots, x_{m-1,j+m-1}, x_{0,j+m}, x_{1,j+m+1}, \ldots, x_{m-1,j-1})$ of length $l$. It is easy to observe that $l = \text{lcm}(m, n)$, the least common multiple of $m$ and $n$. We denote the $j$-th diagonal by $D_j = (d^1_{j,m}, d^2_{j,m}, \ldots, d^j_{l,m})$ and call $D^0$ the main diagonal.

Now we slightly strengthen Theorem 1.3.
Theorem 2.3. Let \( mn \) be even and \( l = \text{lcm}(m, n) \). Then \( C_m \Box C_n \) has a \( \mathbb{Z}_\alpha \times A \)-magic labeling for any \( \alpha \equiv 0 \pmod{l/2} \) and any Abelian group \( A \) of order \( mn/\alpha \).

Proof. Notice that \( l = 2k \) for some \( k \) and \( \alpha = kh \) for some \( h \). Notice that \( d = \gcd(m, n) \) is the number of diagonals of \( C_m \Box C_n \).

For \( \alpha \equiv 0 \pmod{l} \) the claim follows from Theorem 1.3. Hence, we can only look at the case when \( \alpha \equiv l/2 \pmod{l} \).

Let \( r = mn/\alpha \) and \( \Gamma \cong \mathbb{Z}_\alpha \times A \), thus if \( g \in \Gamma \), then we can write that \( g = (j, a) \) for \( j \in \mathbb{Z}_\alpha \) and \( a \in A \) for \( i = 0, 1, \ldots, r - 1 \). We can assume that \( a_0 \) is the identity in \( A \). Let \( \ell(x) = (l_1(x), l_2(x)) \) where \( l_1(x) \in \mathbb{Z}_\alpha \) and \( l_2(x) \in A \).

There exists a subgroup \( \langle h' \rangle \) of \( \mathbb{Z}_\alpha \) of order \( k = l/2 \), therefore the element \( h = (h', a_0) \) generates a subgroup \( H \) in \( \Gamma \) of order \( k \). Let \( b_0, b_1, \ldots, b_{2d-1} \) be the set of coset representatives for \( \Gamma/H \). Notice that in any cyclic group \( \mathbb{Z}_{2j} \), \( j \geq 1 \) there exists an element \( g \neq 0 \) such that there is no \( a \in \mathbb{Z}_{2j} \) satisfying \( 2a = g \) (for instance take \( g = 1 \in \mathbb{Z}_{2j} \)). Thus by Fundamental Theorem of Abstract Algebra, because \( |\Gamma/H| \) is even, we can assume without loss of generality that \( b_1 \in \Gamma/H \) is such that \( b_1 \neq 2b \) for any \( b \in \Gamma/H \). Moreover we can partition \( \Gamma/H \) in to \( d \) pairs \( (h_i, h'_i) \), where \( h_i + h'_i = b_1 \) and \( h_i \neq h'_i \) for \( i = 0, 1, \ldots, d - 1 \).

Label the vertices of \( D^0 \) as follows:

\[
\ell(d^0_i) = ih + h_0, \quad \ell(d^0_{i+1}) = -ih - h_0 + b_1
\]

for \( i = 0, 1, \ldots, k - 1 \).

The vertices in \( D^1, D^2, D^3, \ldots, D^{d-1} \) will be labeled as

\[
\ell(d^j_g) = l_1(d^{j-1}_g) + h_{j+1} \text{ if } g \equiv 1 \pmod{2}, \\
\ell(d^j_g) = l_1(d^{j-1}_g) - h_{j+1} \text{ if } g \equiv 0 \pmod{2}.
\]

Observe that the labeling \( \ell \) is a bijection because \( h_j \neq h_i + b_1 \) for any \( i \neq j \). Moreover,

\[
\ell(d^j_{2i}) + \ell(d^j_{2i+1}) = b_1 \quad \text{and} \\
\ell(d^j_{2i+1}) + \ell(d^j_{2i+2}) = h + b_1
\]

for any \( i \).
If \( d > 2 \), then the vertex \( x_{i', j'} = d_i^{j'} \) has in \( C_m \square C_n \) neighbors \( d_i^{j-1}, d_{i+1}^{j-1}, d_{i-1}^{j+1} \) and \( d_i^{j+1} \). Therefore \( w(d_i^{j}) = h + 2b_1 \) and the labeling is \( \Gamma \)-distance magic as desired.

If \( d \leq 2 \), then the vertex \( x_{i', j'} \) has in \( C_m \square C_n \) neighbors \( d_i^j, d_{i+1}^j, d_{i-1}^j \) and \( d_i^j \) for \( 0 \leq j \leq 1, 0 \leq a < b \leq l - 1 \). We know that at least one of \( m, n \) is even, so we can assume that \( m = 2s \). Because \( d_a^i = x_{i', j'-1} \) and \( d_b^i = x_{i', j'+1} \), it is clear that \( a = b + qm \) for same \( 1 \leq q < l/m \). But \( m = 2s \) and \( a = b + 2qs \) and hence \( a \) and \( b \) have the same parity. When \( a \) and \( b \) are even, say \( a = 2c \) and \( b = 2f \), then
\[
\begin{align*}
d_a^i + d_{a+1}^i &= d_{2s}^i + d_{2s+1}^i = b_1 \\
d_{b-1}^i + d_b^i &= d_{2f-1}^i + d_{2f}^i = b_1 + h,
\end{align*}
\]
which implies \( w(x_{i', j'}) = h + 2b_1 \).

Now we present a class of group distance magic cycle products, that is, cycle products that are \( \Gamma \)-distance magic for any Abelian group \( \Gamma \) of an appropriate order.

**Theorem 2.4.** Let \( n \) be even. Then \( C_n \square C_n \) has a \( \Gamma \)-distance magic labeling for any Abelian group \( \Gamma \) of order \( n^2 \).

**Proof.** The Fundamental Theorem of Finite Abelian Groups states that a finite Abelian group \( \Gamma \) of order \( m = n^2 \) can be expressed as the direct product of cyclic subgroups of prime-power order. This implies that
\[
\Gamma \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_k^{\alpha_k}} \quad \text{where} \quad n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}
\]
and \( p_i \) for \( i \in \{1, 2, \ldots, k\} \) are primes, not necessarily distinct. This product is unique up to the order of the direct product. Therefore there exists \( H \leq \Gamma \) such that \(|H| = n\). Let \( b_0, b_1, \ldots, b_{n-1} \) be coset representatives of \( \Gamma/H \). Recall that in any Abelian group of even order the number of involutions is odd, therefore \( \Gamma/H \) has \( 2t - 1 \) involutions \( \iota_1, \iota_2, \ldots, \iota_{2t-1} \) for \( t \geq 1 \).

Let \( b_0 \) be the identity element of \( \Gamma/H \) and \( b_i = \iota_i \) for \( i \in \{1, 2, \ldots, 2t-1\} \), and \( b_{i+1} = -b_i \) for \( i \in \{2t, 2t+2, 2t+4, \ldots, n-2\} \).

Observe that because \(|H|\) is even there exists an involution \( \iota \in H \) (\( \iota \neq 0 \) and \( 2\iota = 0 \)). We will define a bijection \( \varphi : H \to H \) such that \( \varphi(x) \neq x \) for any \( x \in H \). For \( n \equiv 0 \) (mod 4) let \( \varphi(x) = x + \iota \). When \( n \equiv 2 \) (mod 4), then let \( \varphi(x) = -x + \iota \). Notice that for \( n \equiv 2 \) (mod 4) there does not exist \( x \in H \) such that \( 2x = \iota \).
Hence we can partition $H$ into $n/2$ pairs $(h_i, h'_i)$ such that $h'_i = \varphi(h_i)$ and $h_i \neq h'_i$ for $i = 0, 1, \ldots, n/2 - 1$.

Now, for $i = 0, 1, \ldots, n - 1$, label the vertices of $D^0$ as
\[
\ell(d^0_i) = h_i, \quad \ell(d^0_{n/2+i}) = \varphi(h_i)
\]
and the vertices of $D^2$ as
\[
\ell(d^2_i) = -\ell(d^0_{i+1}) + b_2 \quad \text{if } i \equiv 0 \pmod{2} \quad \text{and} \quad \ell(d^2_i) = -\ell(d^0_{i+1}) - b_2 \quad \text{if } i \equiv 1 \pmod{2}.
\]

For $r \in \{1, 3\}$ label the vertices of $D^r$ as
\[
\ell(d^r_i) = \ell(d^{r-1}_i) - b_{r-1} + b_r \quad \text{if } i \equiv 0 \pmod{2} \quad \text{and} \quad \ell(d^r_i) = \ell(d^{r-1}_i) + b_{r-1} - b_r \quad \text{if } i \equiv 1 \pmod{2}.
\]

For $r \in \{4, 5, \ldots, n - 1\}$ the vertices in $D^r$ will be labeled as
\[
\ell(d^r_i) = \ell(d^{r-1}_{i+2}) - b_{r-4} + b_r \quad \text{if } i \equiv 0 \pmod{2} \quad \text{and} \quad \ell(d^r_i) = \ell(d^{r-1}_{i+2}) + b_{r-4} - b_r \quad \text{if } i \equiv 1 \pmod{2}.
\]

Observe that
\[
w(d^r_i) = \ell(d^{r-1}_i) + \ell(d^{r-1}_{i+1}) + \ell(d^{r+1}_{i+1}) + \ell(d^{r+1}_i).
\]

Assume first that $r \not\in \{n-1, 0\}$. When $r \equiv 1, 2 \pmod{4}$, then
\[
w(d^r_i) = \ell(d^{r-1}_i) + \ell(d^{r-1}_{i+1}) - \ell(d^{r-1}_i) - \ell(d^{r-1}_{i+1}) + 2\ell = 0.
\]

If $r \equiv 0, 3 \pmod{4}$, then
\[
w(d^r_i) = -\ell(d^{r-3}_{i+1}) - \ell(d^{r-3}_{i+2}) + 2\ell + \ell(d^{r-3}_i) + \ell(d^{r-3}_{i+2}) = 0.
\]

Assume now that $r = n - 1$ and $n \equiv 0 \pmod{4}$. Then
\[
w(d^{n-1}_i) = \ell(d^{n-2}_i) + \ell(d^{n-2}_{i+1}) + \ell(d^{n-2}_{i+2}) + \ell(d^{n-2}_{i+3}) + \ell(d^{n-2}_{i+4})
\]
\[
= \ell(d^2_{i+2(n/4-1)}) + \ell(d^{n-2}_{i+1+2(n/4-1)}) + \ell(d^{n-2}_{i-1}) + \ell(d^{n-2}_{i})
\]
\[
= \ell(d^0_{i-2+n/2}) + \ell(d^0_{i-1+n/2}) + \ell(d^0_{i-1}) + \ell(d^0_{i})
\]
\[
= -\ell(d^{0}_{i-1+n/2}) - \ell(d^0_{i+1+n/2}) + \ell(d^0_{i-1}) + \ell(d^0_{i})
\]
\[
= -h_{i-1} - h_i + h_{i-1} + h_i = 0.
\]
If \( r = n - 1 \) and \( n \equiv 2 \pmod{4} \), then
\[
\begin{align*}
w(d_i^{n-1}) &= \ell(d_i^{n-2}) + \ell(d_{i+1}^{n-2}) + \ell(d_i^0) + \ell(d_{i-1}^0) \\
&= \ell(d_i^{n-(n/2-1)}) + \ell(d_{i+1+(n/2-1)}^0) + \ell(d_{i-1}^0) + \ell(d_i^0) \\
&= \ell(d_i^{n-(n/2)}) + \ell(d_i^{n/2}) + \ell(d_{i-1}^0) + \ell(d_i^0) \\
&= h_{i-1} + h_i + h_{i-1} + h_i \\
&= 0.
\end{align*}
\]

Similarly we obtain \( w(d_i^0) = 0 \). Hence the labeling is \( \Gamma \)-distance magic as desired.

Theorem 2.4 now immediately implies the following.

**Corollary 2.5.** The graph \( C_{2n} \Box C_{2n} \) has a \( \Gamma \)-distance magic labeling for \( n \geq 2 \) and any Abelian group \( \Gamma \) of order \( 2^{2n} \).

### 3 Necessary conditions

Now we present theorems showing that if we have a group \( \Gamma \cong \mathbb{Z}_{p_1}^{\alpha_1} \times \mathbb{Z}_{p_2}^{\alpha_2} \times \cdots \times \mathbb{Z}_{p_k}^{\alpha_k} \) with elements \((g_1, g_2, \ldots, g_k)\) and the exponent \( r = \exp(\Gamma) \) is rather small compared with the length of the diagonal of \( C_m \Box C_n \), then there is no \( \Gamma \)-distance magic labeling of the cycle product. In other words, the results are showing that if some entries \( g_i \) of \((g_1, g_2, \ldots, g_k) \in \Gamma \) would have to repeat too many times, then such labeling does not exist.

For a positive integer \( m \) define a function
\[
\begin{align*}
f(m) &= \begin{cases} 
  m/4 & \text{if } m \equiv 0 \pmod{4}, \\
  m/2 & \text{if } m \equiv 2 \pmod{4}, \\
  m & \text{if } m \equiv 1 \pmod{2}.
\end{cases}
\end{align*}
\]

**Theorem 3.1.** Let \( \Gamma \) be an Abelian group of an even order \( mn \) with exponent \( r \). If \( 2r \min\{f(m), f(n)\} < \text{lcm}(m, n) \), then there does not exist a \( \Gamma \)-distance magic labeling of the Cartesian product \( C_m \Box C_n \).

**Proof.** For the sake of contradiction, suppose that there exists a \( \Gamma \)-distance magic labeling \( \ell \) of the Cartesian product \( C_m \Box C_n \) with magic constant \( \mu \). By our assumption, we have \( mn \) even. Without loss of generality we can
assume that $m < n$ and $\ell(x_{0,0}) = 0$. Let us consider the weights of $x_{0,1}$ and $x_{m-1,2}$:

$$w(x_{0,1}) = \ell(x_{0,0}) + \ell(x_{1,1}) + \ell(x_{m-1,1}) + \ell(x_{0,2})$$

and

$$w(x_{m-1,2}) = \ell(x_{m-1,1}) + \ell(x_{0,2}) + \ell(x_{m-2,2}) + \ell(x_{m-1,3}).$$

Because we assumed that $\ell$ is a $\Gamma$-distance magic labeling, we have $w(x_{0,1}) = w(x_{m-1,2}) = \mu$, which yields

$$\ell(x_{0,0}) + \ell(x_{1,1}) + \ell(x_{m-1,1}) + \ell(x_{0,2}) = \ell(x_{m-1,1}) + \ell(x_{0,2}) + \ell(x_{m-2,2}) + \ell(x_{m-1,3}).$$

and hence

$$\ell(x_{0,0}) + \ell(x_{1,1}) = \ell(x_{m-2,2}) + \ell(x_{m-1,3}).$$

Similarly, comparing weights of vertices $x_{m-2,3}$ and $x_{m-3,4}$ we obtain

$$\ell(x_{m-2,2}) + \ell(x_{m-1,3}) + \ell(x_{m-3,3}) + \ell(x_{m-2,4}) =$$

$$\ell(x_{m-3,3}) + \ell(x_{m-2,4}) + \ell(x_{m-4,4}) + \ell(x_{m-3,5})$$

and thus

$$\ell(x_{m-2,2}) + \ell(x_{m-1,3}) = \ell(x_{m-4,4}) + \ell(x_{m-3,5}).$$

which implies

$$\ell(x_{0,0}) + \ell(x_{1,1}) = \ell(x_{m-4,4}) + \ell(x_{m-3,5}).$$

Repeating that procedure we conclude that

$$\ell(x_{0,0}) + \ell(x_{1,1}) = \ell(x_{-2\alpha,2\alpha}) + \ell(x_{1-2\alpha,1+2\alpha}) = a_0$$

for some $a_0 \in \Gamma$ and any natural number $\alpha$.

Recall that by the main diagonal of $C_m \square C_n$ we mean the cyclic sequence of vertices $(x_{0,0}, x_{1,1}, \ldots, x_{m-1,m-1}, x_{0,m}, x_{1,m+1}, \ldots, x_{m-1,n-1})$ of length $l = \text{lcm}(m, n)$. We now consider the following system of equations, going along the main diagonal. Notice that the subscripts need to be read modulo $m$ and $n$, respectively. So, for instance, when $m < n$, the vertex denoted by $x_{m,m}$ is in fact $x_{0,m}$.

Analogously, we get

$$\ell(x_{j,j}) + \ell(x_{j+1,j+1}) = \ell(x_{j-2\alpha,j+2\alpha}) + \ell(x_{j+1-2\alpha,j+1+2\alpha}) = a_j$$
for some $a_j \in \Gamma$ and any natural number $\alpha$.

Note that $x_{-2\alpha,2\alpha}$ belongs to the same diagonal as $x_{0,0}$ for $2\alpha \equiv -2\alpha \pmod{m}$ (then $x_{-2\alpha,2\alpha} = x_{2\alpha,2\alpha}$) or $2\alpha \equiv -2\alpha \pmod{n}$ (then $x_{-2\alpha,2\alpha} = x_{-2\alpha,-2\alpha}$), what happens for both $\alpha = f(m)$ and $\alpha = f(n)$.

This implies that taking $k = \min\{f(m), f(n)\}$ we obtain:

$$
\ell(x_{j,j}) + \ell(x_{j+1,j+1}) = \ell(x_{j+2k,j+2k}) + \ell(x_{j+1+2k,j+1+2k}).
$$

(1)

Note that $x_{2rk,2rk} \neq x_{0,0}$ since $2rk < \text{lcm}(m,n)$. Also, the elements $a_0, a_1, \ldots, a_{2k-1}$ are not necessarily all distinct.

| $\ell(x_{0,0})$ | $\ell(x_{1,1})$ | $\ell(x_{2,2})$ | $a_0$ |
|----------------|----------------|----------------|------|
| $\ell(x_{1,1})$ | $\ell(x_{2,2})$ |               | $a_1$ |
| $\ell(x_{2k-1,2k-1})$ | $\ell(x_{2k,2k})$ |               | $a_{2k-1}$ |
| $\ell(x_{2k,2k})$ | $\ell(x_{2k+1,2k+1})$ |               | $a_0$ |
| $\ell(x_{2k+1,2k+1})$ | $\ell(x_{2k+2,2k+2})$ |               | $a_1$ |
| $\ell(x_{4k-1,4k-1})$ | $\ell(x_{4k,4k})$ |               | $a_{2k-1}$ |
| $\ell(x_{4k-1,4k-1})$ | $\ell(x_{4k,4k})$ |               | $a_{2k-1}$ |
| $\ell(x_{2(r-1)k,2(r-1)k})$ | $\ell(x_{2(r-1)k+1,2(r-1)k+1})$ |               | $a_0$ |
| $\ell(x_{2(r-1)k+1,2(r-1)k+1})$ | $\ell(x_{2(r-1)k+2,2(r-1)k+2})$ |               | $a_1$ |
| $\ell(x_{2r-1,2rk-1})$ | $\ell(x_{2rk,2rk})$ |               | $a_{2k-1}$ |

Multiplying every other equation by $-1$, starting with

$$
-\ell(x_{1,1}) - \ell(x_{2,2}) = -a_1,
$$

and adding all equations, we obtain

$$
\ell(x_{0,0}) - \ell(x_{2rk,2rk}) = r \sum_{i=0}^{2k-1} (-1)^i a_i.
$$

Recall that $\ell(x_{0,0}) = 0$. Because $r = \exp(\Gamma)$, we have $r \sum_{i=0}^{2k-1} (-1)^i a_i = 0$. This implies $-\ell(x_{2rk,2rk}) = 0$, which is a contradiction, because the labeling is injective and we have assumed that $x_{2rk,2rk} \neq x_{0,0}$. □
The following theorem gives a similar result in terms of a more obvious bound, using \(\gcd(m, n)\), the number of diagonals in \(C_m \square C_n\).

**Theorem 3.2.** Let \(\Gamma\) be an Abelian group of an even order \(mn\) with exponent \(r\). If \(2r \gcd(m, n) < \text{lcm}(m, n)\), then there does not exist a \(\Gamma\)-distance magic labeling of the Cartesian product \(C_m \square C_n\).

**Proof.** We again use contradiction and assume that there exists a \(\Gamma\)-distance magic labeling \(\ell\) of \(C_m \square C_n\) with magic constant \(\mu\) and \(mn\) is even, \(m < n\) and \(\ell(x_{0,0}) = 0\).

By the first backward diagonal of \(C_m \square C_n\) we mean the cyclic sequence of vertices \((x_{0,1}, x_{1,0}, x_{2,n-1}, \ldots, x_{m-1,n-m+2}, x_{0,n-m+1}, x_{1,n-m}, \ldots, x_{m-1,2})\) of length \(l = \text{lcm}(m, n)\). Similarly, the sequence \((x_{0,2}, x_{1,1}, x_{2,0}, \ldots, x_{m-1,n-m+3}, x_{0,n-m+2}, x_{1,n-m+1}, \ldots, x_{m-1,3})\) is the second backward diagonal and so on.

Set \(k = \gcd(m, n)\). Because the length of each backward diagonal is \(\text{lcm}(m, n)\), there are \(k\) backward diagonals, and the vertices \(x_{0,k+1}, x_{1,k}, x_{2,k-1}, \ldots, x_{0,2k+1}, x_{1,2k}, x_{2,2k-1}, \ldots, x_{k,k+1}, \ldots, x_{0,3k+1}, x_{1,3k}, x_{2,3k-1}, \ldots, x_{2k,2k+1}, \ldots\) belong to the same diagonal as \(x_{0,1}\).

We look at weights of vertices of that sequence, starting at \(x_{0,1}\) and going the opposite direction, that is, \(w(x_{0,1}), w(x_{m-1,2}), w(x_{m-2,3}), \ldots, w(x_{1,0})\).

The weights of \(x_{0,1}\) and \(x_{m-1,2}\) are again

\[
w(x_{0,1}) = \ell(x_{0,0}) + \ell(x_{1,1}) + \ell(x_{m-1,1}) + \ell(x_{0,2}) = \mu
\]

and

\[
w(x_{m-1,2}) = \ell(x_{m-1,1}) + \ell(x_{0,2}) + \ell(x_{m-2,2}) + \ell(x_{m-1,3}) = \mu.
\]

Comparing the above equalities, we get

\[
\ell(x_{0,0}) + \ell(x_{1,1}) = \ell(x_{m-2,2}) + \ell(x_{m-1,3}).
\]

Continuing this way, we obtain

\[
\ell(x_{0,0}) + \ell(x_{1,1}) = \ell(x_{2α_0,2α_0}) + \ell(x_{1-2α_0,1+2α_0}) = c_1
\]

for some \(c_1 \in \Gamma\) and any natural number \(α_0\). In particular, because there are \(k\) diagonals and \(x_{2k,2k+1}\) belongs to the same backward diagonal as \(x_{0,1}\), we must also have

\[
\ell(x_{0,0}) + \ell(x_{1,1}) = \ell(x_{2α_0,2α_0}) + \ell(x_{1-2α_0,1+2α_0}) = c_1
\]
\[ \ell(x_0,0) + \ell(x_1,1) = \ell(x_2k,2k) + \ell(x_{2k+1},2k+1) = c_1, \]
\[ \ell(x_0,0) + \ell(x_1,1) = \ell(x_{4k},4k) + \ell(x_{4k+1},4k+1) = c_1, \]
\[ \vdots \]
\[ \ell(x_0,0) + \ell(x_{1,1}) = \ell(x_{2(r-1)k,2(r-1)k}) + \ell(x_{2(r-1)k+1,2(r-1)k+1}) = c_1. \]

Looking at the first backward diagonal again and starting to compare the weights at \( x_{1,0}, x_{0,1}, x_{m-2}, \ldots \) instead, we get
\[ w(x_{1,0}) = \ell(x_{1,n-1}) + \ell(x_{2,0}) + \ell(x_{0,0}) + \ell(x_{1,1}) = \mu \]
and
\[ w(x_{0,1}) = \ell(x_{0,0}) + \ell(x_{1,1}) + \ell(x_{m-1,1}) + \ell(x_{0,2}) = \mu. \]
Comparing these equalities, we obtain
\[ \ell(x_{1,n-1}) + \ell(x_{2,0}) = \ell(x_{m-1,1}) + \ell(x_{0,2}) = d_1 \]
for some element \( d_1 \).

Comparing the weights of remaining vertices, we again have
\[ \ell(x_{1,n-1}) + \ell(x_{2,0}) = \ell(x_{1-2\beta_1,-1+2\beta_1}) + \ell(x_{2-2\beta_1,2\beta_1}) = d_1 \]
for any \( \beta_1 \in \mathbb{Z} \).

Proceeding in the same fashion, we obtain two such equalities for each diagonal. Ordering them conveniently and renaming the elements \( c_i \) and \( d_i \), we again obtain the same system of equations as in the previous theorem. Namely,
For instance, when $m$ is stronger result; when by the property of group exponent $r$
Solving it the same way as before, we again obtain $x$

Corollary 3.3. When

Theorem 3.2. $C$

a labeling of $C$

Corresponding long cycle $m$

ment above. We show that for a given cycle length $n > sm$

if $n$

Follows directly from Theorem 3.2.

Proof. $\Gamma$

Let

The following non-existence result is in a certain sense ‘dual’ to the statement above. We show that for a given cycle length $m$, we can always find a corresponding long cycle $C$

and an Abelian group $\Gamma$ which does not provide a labeling of $C_m \square C_n$. 

\[ \ell(x_{0,0}) + \ell(x_{1,1}) + \ell(x_{2,2}) + \cdots + \ell(x_{2k-1,2k-1}) + \ell(x_{2k,2k}) + \ell(x_{2k+1,2k+1}) + \cdots + \ell(x_{4k-1,4k-1}) + \ell(x_{4k,4k}) + \cdots + \ell(x_{2r-1,2r-1}) + \ell(x_{2r,2r}) = a_0 \]

by the property of group exponent $r$. But because $\ell(x_{0,0}) = 0$ and $x_{2r,2r} \neq x_{0,0}$, we get a contradiction, which completes the proof. \[ \square \]

Notice that each of the above two theorems is useful in different scenarios. For instance, when $m \equiv 0 \pmod{4}$ and $n = mq$, then Theorem 3.1 gives a stronger result; when $m$ is odd and $n \neq mq$, then it is better to use Theorem 3.2.

To illustrate the strength of the theorems above with a concrete example, we present two special cases separately.

Corollary 3.3. Let $\Gamma \cong (\mathbb{Z}_s)^t$ when $s$ is even or $\Gamma \cong (\mathbb{Z}_s)^t \times \mathbb{Z}_2$ when $s$ is odd, and $|\Gamma| = mn$. Then $C_m \square C_n$ does not have a $\Gamma$-distance magic labeling if $n > sm$.

Proof. Follows directly from Theorem 3.2. \[ \square \]
Observation 3.4. For any positive integer $m$ there exists $n$ for which the Cartesian product $C_m \square C_n$ is not group distance magic. That is, there exists an Abelian group $\Gamma$ such that $C_m \square C_n$ is not $\Gamma$-distance magic.

Proof. Let $n = 2m^3$ and $\Gamma \equiv (\mathbb{Z}_m)^4 \times \mathbb{Z}_2$. Then $\exp(\Gamma) \leq 2m$ and $\gcd(m, n) = m$. Hence we have $2r \gcd(m, n) \leq 4m^2 \times 2m^3$, because $m \geq 3$. But $\text{lcm}(m, n) = n = 2m^3$, and the product $C_m \square C_n$ is not $\Gamma$-distance magic by Theorem 3.2. \hfill \square

The following extreme case is worth mentioning.

Observation 3.5. Let $\gcd(m, n) = 1$. There exists a $\Gamma$-distance magic labeling of the Cartesian product $C_m \square C_n$ if and only if $mn$ is even and either $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_{mn/2}$ or $\Gamma \cong \mathbb{Z}_{mn}$.

Proof. For $\Gamma \cong \mathbb{Z}_{mn}$ and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_{mn/2}$ the labelings exist by Theorem 1.7.

Because $\gcd(m, n) = 1$, we cannot have both $m$ and $n$ even, so without loss of generality $m$ is odd. If $n$ is odd, then $C_m \square C_n$ is not $\Gamma$-distance magic graph for any Abelian group $\Gamma$ of order $mn$ by Theorem 1.5.

It is well known that for an Abelian group $\Gamma$ of order $2k$, $\exp(\Gamma) = k$ if and only if $k$ is even and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_k$.

Therefore, if $\Gamma \not\cong \mathbb{Z}_{mn}$ and $\Gamma \not\cong \mathbb{Z}_2 \times \mathbb{Z}_{mn/2}$, we must have $\exp(\Gamma) < mn/2$. On the other hand, we have $\text{lcm}(m, n) = mn$, because $\gcd(m, n) = 1$. Hence $2r \gcd(m, n) < mn = \text{lcm}(m, n)$ and there does not exist a $\Gamma$-distance magic labeling of $C_m \square C_n$ by Theorem 3.2. \hfill \square

Finally, we complement Theorem 1.4 and Observation 3.6 by the following related result.

Theorem 3.6. There exists a $(\mathbb{Z}_2)^{m+n}$-distance magic labeling of the Cartesian product $C_{2m} \square C_{2n}$ if and only if $2 \leq m = n$.

Proof. When $m = n$, then we are done by Theorem 1.4. Suppose that $m \not= n$. Without loss of generality we can assume that $2 \leq m < n$; then $\min\{f(2^m), f(2^n), \gcd(2^m, 2^n)\} = 2^{m-2}$, whereas $\text{lcm}(2^m, 2^n) = 2^n$. Observe that the exponent of $(\mathbb{Z}_2)^{m+n}$ is $r = 2$. Therefore $2r \min\{f(m), f(n)\} = 2^m < 2^n = \text{lcm}(m, n)$. \hfill \square
4 Conclusion

We made some progress towards the full characterization of Abelian groups $\Gamma$ such that $|\Gamma| = mn$ and there exists a $\Gamma$-distance magic labeling of $C_m \Box C_n$.

We improved a previous bound for existence of such labeling, showing that if $\Gamma$ has a cyclic subgroup of order $\text{lcm}(m, n)/2$, then $C_m \Box C_n$ is $\Gamma$-distance magic, lowering the bound from $\text{lcm}(m, n)$.

On the other hand, we have shown that groups with an exponent $\text{exp}(\Gamma)$ that is relatively small compared with $\text{lcm}(m, n)$ do not admit such labeling.

Since we found necessary conditions for existence of such labeling (Theorem 3.1 and 3.2), which in some cases were also sufficient (Observation 3.5), we post now the following conjecture.

**Conjecture 4.1.** Let $\Gamma$ be an Abelian group of an even order $mn$ with exponent $r$. There exists a $\Gamma$-distance magic labeling of the Cartesian product $C_m \Box C_n$ if and only if $2r \min\{f(m), f(n), \gcd(m, n)\} \geq \text{lcm}(m, n)$.

References

[1] M. Anholcer, S. Cichacz, I. Peterin, A. Tepeh, *Group distance magic labeling of direct product of graphs*, Ars Math. Contemp. 9 (2015), 93–108.

[2] S. Cichacz, *Group distance magic labeling of some cycle-related graphs*, Australas. J. Combin. 57 (2013), 235–243.

[3] S. Cichacz, D. Froncek, *Distance magic circulant graphs*, Discrete Math. 339(1) (2016), 84–94.

[4] R. Diestel, *Graph Theory, Third Edition*, Springer-Verlag, Heidelberg Graduate Texts in Mathematics, Volume 173, New York, 2005

[5] D. Froncek, *Group distance magic labeling of Cartesian product of cycles*, Australas. J. Combin., 55 (2013), 167–174.

[6] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition (John Wiley & Sons, New York, 2000).

[7] S.B. Rao, T. Singh and V. Parameswaran, *Some sigma labelled graphs I*, In Graphs, Combinatorics, Algorithms and Applications, eds. S. Arumugam, B.D. Acharya and S.B. Rao, Narosa Publishing House, New Delhi, (2004), 125–133.
[8] J. Sedláček, Problem 27. In: Theory of Graphs and Its Applications (M. Fiedler, ed.). Praha 1964, 163–164.

[9] R.P. Stanley, Linear homogeneous Diophantine equations and magic labelings of graphs, Duke Math. J. 40 (1973), 607–632.

[10] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, Duke Math. J. 43 (1976), 511–531.