Algebraic Connectivity Under Site Percolation in Finite Weighted Graphs

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Abstract

We study the behavior of algebraic connectivity in a weighted graph that is subject to site percolation, random deletion of the vertices. Using a refined concentration inequality for random matrices we show in our main theorem that the (augmented) Laplacian of the percolated graph concentrates around its expectation. This concentration bound then provides a lower bound on the algebraic connectivity of the percolated graph. As a special case for \((n, d, \lambda)\)-graphs (i.e., \(d\)-regular graphs on \(n\) vertices with non-trivial eigenvalues less than \(\lambda\) in magnitude) our result shows that, with high probability, the graph remains connected under a homogeneous site percolation with survival probability \(p \geq 1 - C_1 n^{-C_2/d}\) with \(C_1\) and \(C_2\) depending only on \(\lambda/d\).

Index Terms

site percolation, algebraic connectivity, matrix concentration inequalities

1 Introduction

Consider a connected weighted graph \(G = (V = [n], E)\) with (non-negative) edge weights \(\{w_{i,j}\}_{1 \leq i, j \leq n}\) and no self-loop (i.e., \(w_{i,i} = 0\) for all \(1 \leq i \leq n\)) and suppose that each vertex \(i\) of \(G\) is deleted independently with probability \(1 - p_i\). These types of random graph models can describe certain phenomena in random media and are studied under percolation theory [6] in mathematics and statistical physics. The process of vertex deletion, as described above, is usually referred to as site percolation whereas bond percolation refers to the process of random deletion (or addition) of the edges of a graph. In this paper we establish a lower bound on algebraic connectivity of the surviving subgraph in the described site percolation model. The algebraic connectivity of a graph \(G\) is \(a = \lambda_2(L)\), the second smallest eigenvalue of the graph Laplacian

\[
L \overset{\text{def}}{=} \sum_{1 \leq i < j \leq n} w_{i,j}(e_i - e_j)(e_i - e_j)^T,
\]

where \(e_i\)'s are canonical basis vectors. Algebraic connectivity and its analog for normalized Laplacians are important because they provide a bound on isoperimetric constants of graphs through Cheeger’s inequality [see e.g., 13] and they are critical in approximation of the mixing rate of continuous-time Markov chains [12; 15, Ch. 20].

Properties such as connectivity, spectral gap, and emergence of a giant component (i.e., a connected component with \(\Omega(n)\) vertices) have received more attention and are better understood for bond percolation models compared to site percolation models. Perhaps, the main reason is that in bond
percolation edges are removed independently whereas in site percolation edge deletions are dependent since they share a common vertex which lead to more intricate behavior.

In this paper we focus on algebraic connectivity of the surviving subgraph of a weighted graph under (inhomogeneous) site percolation. Using a delicate matrix concentration inequality (Proposition 6), in our main result (Theorem 1) we show that the “augmented” Laplacian concentrates around its expectation. This result allows us to find a non-trivial lower bound on the algebraic connectivity in a straightforward way. For concreteness, we also apply the general result of the Theorem 1 to obtain a threshold for connectivity in the special case of \((n, d, \lambda)\)-graphs under uniform site percolation. In particular, Corollary 3 below shows that if the vertices of an \((n, d, \lambda)\)-graph are removed independently with probability \(1 - p\) then, with high probability, the surviving graph is connected if \(p \geq 1 - n^{-O(\frac{1}{d})}\) with the hidden constants depending only on \(\frac{1}{d}\).

1.1 Related work

In [3] the bond percolation model with a uniform edge survival probability \(p\) is studied. With \(d_i\) denoting the degree of vertex \(i\), it is shown in [3] that asymptotically almost surely a giant component survives (or not) if \(p > (1 + \epsilon) \sum d_i/d_i^2\) (or \(p < (1 - \epsilon) \sum d_i/d_i^2\)). Furthermore, the spectral gap under bond percolation is studied in [2] and [14], where the latter established a sharper bound by means of concentration inequalities for random matrices.

A more relevant problem to our work is the problem of network (un)reliability [4] where the goal is to estimate the probability that a percolated graph remains connected. Under the bond percolation model, [8] proposes a method to approximate the network reliability through a fully polynomial-time approximation scheme. Approximation algorithms for the same problem with better computational complexity were proposed later in [7] and [9].

The site percolation model for random \(d\)-regular graphs is analyzed in [5]. Specifically, [5] shows that, with high probability, for vertex deletion probability of the form \(n^{-\gamma}\), the surviving subgraph has a giant component of order \(n - o(n)\) that is an expander graph and, if \(\gamma \geq \frac{1}{d-1}\), then it is connected as well. This result was later improved and generalized in [1]. Recall that an \((n, d, \lambda)\)-graph is a \(d\)-regular graph of order \(n\) with the non-trivial eigenvalue less than \(\lambda\) in magnitude. A phase transition for site percolation on such \((n, d, \lambda)\)-graphs is established in [11]. In particular, the mentioned paper shows that if the vertex survival probability is \(p = \frac{1 - \epsilon}{d}\), then with high probability, the connected components of the surviving subgraph have \(O(\log n)\) vertices; whereas if \(p = \frac{1 + \epsilon}{d}\), \(d = o(n)\), and \(\frac{1}{d}\) is relatively small, then with high probability a giant component with \(\Omega\left(\frac{n}{d}\right)\) vertices survives.

Our main result, Theorem 1, relies on a refined concentration inequality stated in Proposition 6 for random Bernoulli matrices and, consequently, is distinct from most of the previous work mentioned above which rely on combinatorial arguments. We also apply our general result to the special case of \((n, d, \lambda)\)-graphs (Corollary 3), and reproduce bounds comparable to those established in [1; 5]. In particular, [1, Proposition 3.5] shows that any \((n, d, \lambda)\)-graph \(G\) with \(d \geq 3\) and \(\lambda \leq 2\sqrt{d-1} + \frac{1}{10}\), that is also “locally sparse” in the sense that

\[
\max_{H \subseteq G, |V(H)| \leq d+29} \frac{|E(H)|}{|V(H)|} \leq 1,
\]

with high probability, remains connected under a homogeneous site percolation with survival probability \(p > 1 - n^{-\frac{1}{2}}\). Similarly, our result in Corollary 3, guarantees that with probability \(\geq 1 - \frac{4}{n}\), any \((n, d, \lambda)\)-graph remains connected under a homogeneous site percolation with survival probability \(p \geq 1 - C_1 n^{-\frac{2dC_2}{d}}\). The constants \(C_1\) and \(C_2\) depend only on \(\frac{1}{d}\); their exact form is provided in the proof of Corollary 3. If we have \(\frac{1}{d} = 1 - \epsilon\) for some \(\epsilon \in (0, 1)\), then \(- \log C_1 = O\left((1 + \frac{\epsilon}{2})^2\right)\) and \(C_2 = O\left(\epsilon^{-4}\right)\) both of which are decreasing in \(\epsilon\). These quantities can be fairly large for small values of \(\epsilon\), which implies...
that our required lower bound on $p$ would be stricter than that of [1]. However, our analysis does not explicitly assume a bound on $\lambda$ or local sparsity as in [1]. The fact that Corollary 3 leads to suboptimal constants compared to [1, Proposition 3.5] is not surprising; Corollary 3 is an application of a very general bound established in Theorem 1 to the case of $(n, d, \lambda)$-graphs.

1.2 Future directions
There are natural extensions to the connectivity problem studied in this paper that we would like to study through the lens of random matrix theory as done here. For example, an immediate question is to find a bound on the size of the giant component of the site-percolated graph. Furthermore, an interesting research direction is to study other properties of the site-percolated random graphs such as their clique number, chromatic number, etc by means of random matrix theory. While the best results might still be obtained through specifically tailored combinatorial arguments, we believe that the analysis based on algebraic methods and random matrix theory would be more robust to model errors.

2 Problem Setup
For $1 \leq i \leq n$, let $\delta_i \sim \text{Bernoulli}(p_i)$ be the independent random variables that indicate whether or not the corresponding vertex survives. In order to operate on a Laplacian with fixed dimensions we interpret site percolation as removing every edge connected to the affected vertices. The Laplacian of the remaining graph $G_\delta$ is then given by

$$L_\delta = \sum_{1 \leq i < j \leq n} \delta_i \delta_j w_{i,j} (e_i - e_j)(e_i - e_j)^T,$$

which also includes the vertices affected by the site percolation as isolated vertices. We need to take into account the effect of these “ghost vertices” to find a non-trivial bound for the desired algebraic connectivity which we denote by $a_\delta$. To this end, for a coefficient $\alpha \geq 0$, we introduce the augmented Laplacian given by

$$\overline{L}_\delta = L_\delta + \sum_{1 \leq i \leq n} \alpha (1 - \delta_i) e_i e_i^T.$$  \hspace{1cm} (1)

The Laplacian $L_\delta$ and the diagonal matrix $\sum_{1 \leq i \leq n} \alpha (1 - \delta_i) e_i e_i^T$ in (1) are supported on the vertices that survived and the ghost vertices, respectively. Because these two vertex sets are disjoint, the eigenvalues and eigenvectors of the corresponding terms on the right-hand side of (1) constitute the eigendecomposition of $\overline{L}_\delta$. We either have $a_\delta > \alpha$ or $a_\delta \leq \alpha$. If the latter holds, then $a_\delta$ would coincide with the second smallest eigenvalue of $\overline{L}_\delta$ and by Weyl’s eigenvalues inequality we obtain

$$a_\delta = \lambda_2 (L_\delta) \geq \lambda_2 (\mathbb{E}L_\delta) - \left\| L_\delta - \mathbb{E}L_\delta \right\|.$$  \hspace{1cm} (2)

An immediate implication is that

$$a_\delta \geq \min \left\{ \lambda_2 (\mathbb{E}L_\delta) - \left\| L_\delta - \mathbb{E}L_\delta \right\|, \alpha \right\},$$

holds for all $\alpha \geq 0$. Hence, we can obtain a non-trivial lower bound for $a_\delta$ by studying the tail behavior of $\left\| L_\delta - \mathbb{E}L_\delta \right\|$ which also depends on $\alpha$. The lower bound given by (2) can also be optimized with respect to $\alpha$. 
3 Main Result

Our main theorem below provides an upper bound for \( \| \mathbf{L}_\delta - \mathbb{E}\mathbf{L}_\delta \| \). To state the theorem it is necessary to introduce some notation. For each \( 1 \leq i \leq n \), let

\[
K_i = \frac{1}{2} \sqrt{\frac{1 - 2p_i}{\log \frac{1 - p_i}{p_i}}}
\]

(3)
denote the sub-Gaussian parameter of \( \delta_i - p_i \) as used in the Kearns-Saul inequality (Lemma 4 below). Compared to the bounds on the moment generating function used in the Hoeffding and the Bernstein inequalities, the parameter (3) yields tighter bounds, particularly, if \( p_i \) is close to 0 or 1. This property is crucial in our analysis as non-trivial events occur if the vertex survival probabilities (i.e., \( p_i \)'s) are relatively close to 1. We use \( \mathbf{p} = [p_1 \ p_2 \ \cdots \ p_n]^T \) to denote the vector of survival probabilities and \( \mathbf{a}_i \) to denote the \( i \)th column of the adjacency matrix \( \mathbf{A} \). The diagonal matrix whose diagonal entries are given by a vector \( \mathbf{u} \) is denoted by \( \mathbf{D}_\mathbf{u} \). The binary operation \( \circ \) denotes the entrywise (or Hadamard) product.

**Theorem 1.** Let \( \delta_i \sim \text{Bernoulli}(p_i) \) be independent random variables for \( 1 \leq i \leq n \). Furthermore, with \( K_i \) given by (3) define

\[
\sigma^2 = \left\| \sum_i K_i^2 (1 - 2p_i)^2 (\mathbf{a}_i \mathbf{a}_i^T) \right\|, \quad \bar{K} = \max_i \left( \sum_j w_{i,j}^2 K_j^2 \right)^{\frac{1}{2}}.
\]

Then for any \( \varepsilon \in (0, 1) \), with probability \( \geq 1 - \varepsilon \) we have

\[
\| \mathbf{L}_\delta - \mathbb{E}\mathbf{L}_\delta \| \leq 2\bar{K} \sqrt{\log \frac{4n}{\varepsilon}} + \max_i \left| \alpha - \sum_j p_j w_{i,j} \right|
\]

\[
+ \left\| \mathbf{D}_\mathbf{p}(1 - \mathbf{p}) \mathbf{A} \mathbf{D}_\mathbf{p}(1 - \mathbf{p}) \mathbf{D}_\mathbf{p}(1 - \mathbf{p}) \right\| + 2 \left\| \mathbf{D}_\mathbf{p} \mathbf{A} \mathbf{D}_\mathbf{p}(1 - \mathbf{p}) \mathbf{D}_\mathbf{p}(1 - \mathbf{p}) \right\| + \frac{9}{2} \left| \sigma \left( \log \frac{4n}{\varepsilon} \right)^{\frac{1}{2}} \right|.
\]

To evaluate the bound produced using (2) and Theorem 1, we apply the result to two special problems with \((n, d, \lambda)\)-graphs. First we recall the definition of these graphs.

**Definition 2 ((n, d, λ)-graphs).** An \((n, d, \lambda)\)-graph is a \( d \)-regular graph with \( n \) vertices whose adjacency matrix has no non-trivial eigenvalue with magnitude greater than \( \lambda \).

Below, we assume that the vertex deletion probabilities are identical, i.e., \( p_1 = p_2 = \ldots = p_n = p \). This assumption also implies that \( K_1 = K_2 = \ldots = K_n = K = \frac{1}{2} \sqrt{\frac{1 - 2p}{\log \frac{1}{2} p}} \). Also we assume all the edge weights \( w_{i,j} \) are \( \{0, 1\} \)-valued and effectively indicate existence of an edge in \( G \). We need to quantify or bound \( \lambda_2 \left( \mathbb{E}\mathbf{L}_\delta \right) \) as well as the parameters \( \sigma \) and \( \bar{K} \). Using Theorem 1, the following corollary basically shows that \( p = 1 - \left( \frac{4n}{\varepsilon} \right)^{-O(\frac{1}{d})} \), could suffice for any \((n, d, \lambda)\)-graph affected by the prescribed site percolation to remain connected with probability \( \geq 1 - \varepsilon \).

**Corollary 3.** Let \( G \) be an arbitrary \((n, d, \lambda)\)-graph. There are positive constants \( C_1 \) and \( C_2 \) depending only on \( \frac{\lambda}{d} \) such that under the site percolation model with vertex survival probability of

\[
p \geq 1 - C_1 \left( \frac{4n}{\varepsilon} \right)^{-C_2/d}
\]

the surviving subgraph of \( G \) is connected with probability \( \geq 1 - \varepsilon \).
Proof: With \( A \) and \( L \) denoting the adjacency and Laplacian matrices of the \((n, d, \lambda)\)-graph \( G \), the expected value of the augmented Laplacian under the considered site percolation would be

\[
\mathbb{E} \bar{\mathcal{L}}_\delta = p^2 L + \alpha (1 - p) I = \left( p^2 d + \alpha (1 - p) \right) I - p^2 A.
\]

Let \( \alpha = pd \). It follows from the definition of the graph and the equation above that

\[
\lambda_2 \left( \mathbb{E} \bar{\mathcal{L}}_\delta \right) \geq p^2 (d - \lambda) + p (1 - p) d = pd - p^2 \lambda.
\]

Furthermore, the parameters \( \sigma^2 \) and \( \mathcal{K} \) can be expressed as

\[
\sigma^2 = K^2 (1 - 2p)^2 \left\| \sum_i (\alpha_i \alpha_i^T) \right\| \quad \text{and} \quad \mathcal{K} = K \sqrt{d}.
\]

Finally, we have

\[
\left\| D_{p,\delta}^{1/2} A D_{p,\delta}^{1/2} \right\| = p (1 - p) \| A \| = p (1 - p) d, \quad \left\| D_{p,\delta}^{1/2} A D_{p,\delta}^{1/2} \right\| = p^{3/2} (1 - p)^{1/2} d,
\]

and

\[
\max_i \left| \alpha - \sum_j p_j w_{i,j} \right| = 0.
\]

We can now invoke Theorem 1 and apply the above bounds to obtain

\[
\left\| \bar{\mathcal{L}}_\delta - \mathbb{E} \bar{\mathcal{L}}_\delta \right\| \leq 2K \sqrt{d \log \frac{4n}{\varepsilon} + p (1 - p) d + 2p^2 (1 - p)^{1/2} d + \frac{9}{2} \sqrt{K} |1 - 2p| d^{3} \left( \log \frac{4n}{\varepsilon} \right)^{1/2}}.
\]

Given the inequalities (2), (4), and the assumption that \( \alpha = pd \), we are naturally interested in values of \( p \) for which the right-hand side of the inequality above is strictly smaller than \( pd - p^2 \lambda \). Specifically, we would like to find \( p \) for which we have

\[
pd - p^2 \lambda > 2K \sqrt{d \log \frac{4n}{\varepsilon} + p (1 - p) d + 2p^2 (1 - p)^{1/2} d + \frac{9}{2} \sqrt{K} |1 - 2p| d^{3} \left( \log \frac{4n}{\varepsilon} \right)^{1/2}},
\]

or equivalently

\[
(1 - \frac{\lambda}{d}) p > 2 \left( \frac{\log \frac{4n}{\varepsilon}}{d} \cdot K^2 \right)^{1/2} + 2 \sqrt{p(1 - p)} + \frac{9}{2p} \sqrt{|1 - 2p|} \left( \log \frac{4n}{\varepsilon} \right)^{1/2} \left( \frac{\log \frac{4n}{\varepsilon}}{d} \cdot K^2 \right)^{1/2}.
\]

(5)

For \( p \geq \frac{1}{2} \) we have \( K \leq \frac{1}{2 \sqrt{\log \frac{1}{1 - p}}} \). Therefore, if we parameterize \( p \) by \( \beta \geq 1 \) as \( p = 1 - e^{-\beta^2} \) we have \( K^2 \leq \frac{\beta^{-4}}{4} \). Furthermore, we can write \( p \geq \frac{1}{1 + \beta^{-2}} \geq 1 - \beta^{-2} \), \( 2 \sqrt{p(1 - p)} \leq 2e^{-\beta^4/2} \leq 2 \beta^{-2} \), and \( \frac{1}{2p} \sqrt{|1 - 2p|} \leq 1 \). Therefore, to guarantee (5) it suffices to have

\[
(1 - \frac{\lambda}{d})(1 - \beta^{-2}) > 2 \left( \frac{\beta^{-4} \log \frac{4n}{\varepsilon}}{4d} \right)^{1/2} + 2 \beta^{-2} + 9 \left( \frac{\beta^{-4} \log \frac{4n}{\varepsilon}}{4d} \right)^{1/2} \left( \frac{\log \frac{4n}{\varepsilon}}{d} \right)^{1/2} \beta^{-1},
\]

which is equivalent to

\[
1 - \frac{\lambda}{d} > \left( \frac{\log \frac{4n}{\varepsilon}}{d} \right)^{1/2} + 3 - \frac{\lambda}{d} \beta^{-2} + \frac{9}{\sqrt{2}} \left( \frac{\log \frac{4n}{\varepsilon}}{d} \right)^{1/2} \beta^{-1}.
\]
The inequality above holds for
\[
\beta^2 > \max \left\{ 2 \left( 1 - \frac{\lambda}{d} \right)^{-1} \left( \frac{\log \frac{4n}{\epsilon}}{d} \right)^{\frac{1}{2}} + 3 - \frac{\lambda}{d} \right) , 81 \left( 1 - \frac{\lambda}{d} \right)^{-2} \left( \frac{\log \frac{4n}{\epsilon}}{d} \right)^{\frac{1}{2}} \right\}
\]
which, for the sake of simpler expressions, can be further relaxed to
\[
\beta^4 \geq 81^2 \left( 1 - \frac{\lambda}{d} \right)^{-4} \frac{\log \frac{4n}{\epsilon}}{d} + 8 \left( 1 - \frac{\lambda}{d} \right)^{-2} \left( 3 - \frac{\lambda}{d} \right)^2 .
\]
The desired results follows immediately by setting \( C_1 = \exp \left( -8 \left( 1 - \frac{\lambda}{d} \right)^{-2} \left( 3 - \frac{\lambda}{d} \right)^2 \right) \) and \( C_2 = 81^2 \left( 1 - \frac{\lambda}{d} \right)^{-4} \).

4 Proof of Theorem 1

In this section we prove our main result. The lemmas and other technical tools we use are summarized below in Appendix A.

**Proof of Theorem 1:** Splitting \( L_\delta - E L_\delta \) into the sum of diagonal and off-diagonal terms as
\[
L_\delta - E L_\delta = \sum_i \left( \delta_i (\sum_j \delta_j w_{i,j}) - p_i (\sum_j p_j w_{i,j}) - \alpha (\delta_i - p_i) \right) e_i e_i^T
+ \sum_{i<j} (\delta_i \delta_j - p_i p_j) (e_i e_j^T + e_j e_i^T)
\]
and applying triangle inequality yields
\[
\left\| L_\delta - E L_\delta \right\| \leq \max_i \left| \delta_i (\sum_j \delta_j w_{i,j}) - p_i (\sum_j p_j w_{i,j}) - \alpha (\delta_i - p_i) \right|
+ \left| \sum_{i<j} (\delta_i \delta_j - p_i p_j) w_{i,j} (e_i e_j^T + e_j e_i^T) \right| .
\]

Our goal is to bound the two terms on the right-hand side of the inequality above. To lighten the notation we use \( \xi_i = \delta_i - p_i \) for \( i = 1, 2, \ldots, n \) and let
\[
S_1 = \max_i \left| \delta_i (\sum_j \delta_j w_{i,j}) - p_i (\sum_j p_j w_{i,j}) - \alpha (\delta_i - p_i) \right|
\]
and
\[
S_2 = \left| \sum_{i<j} (\delta_i \delta_j - p_i p_j) w_{i,j} (e_i e_j^T + e_j e_i^T) \right| .
\]

It is straightforward to verify that
\[
\left| \delta_i (\sum_j \delta_j w_{i,j}) - p_i (\sum_j p_j w_{i,j}) - \alpha (\delta_i - p_i) \right| = \left| \delta_i \sum_j (\delta_j - p_j) w_{i,j} - (\delta_i - p_i) (\alpha - \sum_j p_j w_{i,j}) \right|
\leq \sum_j \xi_j w_{i,j} + \left| \alpha - \sum_j p_j w_{i,j} \right|
\]
using which we obtain
\[
S_1 \leq \max_i \left| \sum_j \xi_j w_{i,j} \right| + |\alpha - \sum_j p_j w_{i,j}|. 
\]

By Chernoff’s inequality and Lemma 4, for each \( i \) we have
\[
\left| \sum_j \xi_j w_{i,j} \right| \leq 2 \left( \sum_j w^2_{i,j} K^2_j \right)^{\frac{1}{2}} \sqrt{\log \frac{4n}{\varepsilon}},
\]
with probability \( \geq 1 - \frac{\varepsilon}{2n} \). Then by union bound we have
\[
S_1 \leq \max_i \left( \sum_j w^2_{i,j} K^2_j \right)^{\frac{1}{2}} \sqrt{\log \frac{8n}{\varepsilon}} + |\alpha - \sum_j p_j w_{i,j}| \leq 2K \sqrt{\log \frac{4n}{\varepsilon}} + \max_i |\alpha - \sum_j p_j w_{i,j}|, \quad (6)
\]
with probability \( \geq 1 - \frac{\varepsilon}{2} \).

Expressing \( S_2 \) in terms of \( \xi_i \)s and applying the triangle inequality reveals that
\[
S_2 = \left\| \sum_{i<j} (\xi_i \xi_j - p_i \xi_j - \xi_i p_j) w_{i,j} (e_i e_j^\top + e_j e_i^\top) \right\|
\]
\[\leq \left\| \sum_{i<j} \xi_i \xi_j w_{i,j} (e_i e_j^\top + e_j e_i^\top) \right\| + \left\| \sum_{i<j} (p_i \xi_j + p_j \xi_i) w_{i,j} (e_i e_j^\top + e_j e_i^\top) \right\|
\]
\[= \left\| D_\xi A D_\xi \right\| + 2 \left\| D_p A D_\xi \right\|, \quad (7)
\]
with \( D_\xi \) and \( D_p \) respectively denoting diagonal matrices with \( \xi_i \)s and \( p_i \)s on their diagonals.

We can write
\[
\|D_\xi A D_\xi\|^2 = \|D_\xi A D_\xi^2 A D_\xi\|
\]
\[\leq \|D_\xi A D_{p(1-p)} A D_\xi\| + \|D_\xi A D_{\xi(1-2p)} A D_\xi\|
\]
\[\leq \|D_{\frac{1}{2}p(1-p)} A D_{\xi} A D_{\frac{1}{2}p(1-p)}\| + \|A D_{\xi(1-2p)} A\|
\]
\[\leq \|D_{\frac{1}{2}p(1-p)} A D_{p(1-p)} A D_{\frac{1}{2}p(1-p)}\| + \|D_{\frac{1}{2}p(1-p)} A D_{\xi(1-2p)} A D_{\frac{1}{2}p(1-p)}\| + \|A D_{\xi(1-2p)} A\|. \quad (8)
\]
We used the identity \( \delta_i^2 = \delta_i \) or equivalently \( \xi_i^2 = p_i(1 - p_i) + \xi_i(1 - 2p_i) \) followed by a triangle inequality to obtain the first inequality. To obtain the second inequality we simply rearranged the matrices in the first term and used the fact that \( |\xi_i| \leq 1 \) to bound the second term. Applying the identity \( \delta_i^2 = \delta_i \) again yields the third inequality. With \( a_i \) denoting the \( i \)th column of the adjacency matrix \( A \), we can invoke Proposition 6 to guarantee that with probability \( \geq 1 - \frac{\varepsilon}{2} \) we have
\[
\|A D_{\xi(1-2p)} A\| = \left\| \sum_i \xi_i (1 - 2p_i) a_i a_i^\top \right\|
\]
\[\leq 2 \left\| \sum_i K^2_i (1 - 2p_i)^2 (a_i a_i^\top)^2 \right\|^{\frac{1}{2}} \sqrt{\log \frac{4n}{\varepsilon}}
\]
\[= 2\sigma \sqrt{\log \frac{4n}{\varepsilon}}.
\]
On the same event we also have

\[
\left\| D_{p(1-p)}^{\frac{1}{2}} A D_{\xi(1-2p)} \right\| \leq \left\| D_{p(1-p)}^{\frac{1}{2}} \right\| \left\| A D_{\xi(1-2p)} A \right\| \left\| D_{p(1-p)}^{\frac{1}{2}} \right\|
\]
\[
\leq \frac{1}{4} \left\| A D_{\xi(1-2p)} A \right\|
\]
\[
\leq \frac{\sigma}{2} \sqrt{\log \frac{4n}{\epsilon}},
\]

where we used the fact that \( \left\| D_{p(1-p)}^{\frac{1}{2}} \right\| \leq \frac{1}{4} \) in the second line. These probabilistic upper bounds together with (8) ensure that

\[
\left\| D_{\xi(1-2p)} A D_{\xi(1-2p)} \right\| \leq \left\| D_{p(1-p)}^{\frac{1}{2}} A D_{\xi(1-2p)} \right\|^2 + 5 \frac{\sigma}{2} \sqrt{\log \frac{4n}{\epsilon}} + \left( \frac{1}{2} \right)^{\frac{1}{2}},
\]

with probability \( \geq 1 - \frac{\epsilon}{2} \). Furthermore, using similar arguments as above we have

\[
\left\| D_p A D_{\xi(1-2p)} \right\| \leq \left\| D_p A D_{\xi(1-2p)} \right\|^2 + \left\| D_p \right\|^2 \left\| A D_{\xi(1-2p)} A \right\|
\]
\[
\leq \left\| D_p A D_{\xi(1-2p)} \right\|^2 + 2 \sigma \sqrt{\log \frac{4n}{\epsilon}},
\]

and thus

\[
\left\| D_p A D_{\xi(1-2p)} \right\| \leq \left\| D_p A D_{\xi(1-2p)} \right\|^2 + 2 \sigma \left( \log \frac{4n}{\epsilon} \right)^{\frac{1}{2}}.
\]

(10)

It follows from (7), (9), and (10) that

\[
S_2 \leq \left\| D_{p(1-p)}^{\frac{1}{2}} A D_{\xi(1-2p)} \right\| + 2 \left\| D_p A D_{\xi(1-2p)} \right\| + \frac{9}{2} \sigma \left( \log \frac{4n}{\epsilon} \right)^{\frac{1}{2}},
\]

(11)

with probability \( \geq 1 - \frac{\epsilon}{2} \).

The desired result follows immediately using the derived bounds (6) and (11). □

**Appendix A**

**Auxiliary tools and technical lemmas**

We use the following lemma due to [10] which provides a sharp bound for the sub-Gaussian norm of general Bernoulli random variables.

**Lemma 4** (Kearns-Saul inequality [10]). For \( p \in [0, 1] \) let \( \delta \) be a Bernoulli\((p)\) random variable. Then for all \( t \in \mathbb{R} \) we have

\[
\mathbb{E} e^{t(\delta - p)} = p e^{t(1-p)} + (1-p) e^{-tp} \leq e^{(K(p))t^2},
\]
where

\[ K(p) \overset{\text{def}}{=} \frac{1}{2} \left\lfloor \frac{1 - 2p}{\log \frac{1 - p}{p}} \right\rfloor. \]

We also use the following master tail bound for sums of independent random matrices due to [16].

**Theorem 5.** [16, Theorem 3.6] Consider a finite sequence \{\(Z_i\)\} of independent, random, self-adjoint matrices. For all \(t \in \mathbb{R}\),

\[ \mathbb{P} \left( \lambda_{\max} \left( \sum_{i=1}^{n} Z_i \right) \geq t \right) \leq \inf_{\theta > 0} e^{-\theta t} \text{tr} \left( \sum_{i=1}^{n} \log \mathbb{E} e^{\theta Z_i} \right). \]

In particular, we combine Lemma 4 and Theorem 5 to obtain a sharper analog to the tail bounds for Rademacher series derived in [16], for general centered Bernoulli random variables. As a consequence of the use of Kearns-Saul inequality (i.e., Lemma 4), the improvement over similar bounds obtained via matrix Hoeffding or matrix Bernstein inequalities can be particularly significant if the Bernoulli random variables have means close to 0 or 1.

**Proposition 6.** For \(i = 1, 2, \ldots, n\) let \(\delta_i \sim \text{Bernoulli}(p_i)\) be independent random variables. Furthermore, let \(X_i\) be deterministic \(N \times N\) self-adjoint matrices. Then with \(K_i = K(p_i)\) defined as in Lemma 4, we have

\[ \mathbb{P} \left( \left\| \sum_{i=1}^{n} (\delta_i - p_i) X_i \right\| \geq t \right) \leq 2N e^{-\frac{t^2}{4\sigma^2}}, \]

where

\[ \sigma^2 = \left\| \sum_{i=1}^{n} K_i^2 X_i^2 \right\|. \]

**Proof:** Let \(Z_i = (\delta_i - p_i) X_i\) for \(i = 1, 2, \ldots, n\). For any real number \(\theta\) we have

\[ \mathbb{E} e^{\theta Z_i} = p_i e^{\theta (1 - p_i) X_i} + (1 - p_i) e^{-\theta p_i X_i}. \]

Since \(\theta X_i\) is self-adjoint, it can be diagonalized. Therefore, by applying Lemma 4 to the eigenvalues of \(\theta X_i\) the above equation implies that

\[ \mathbb{E} e^{\theta Z_i} \leq e^{\theta^2 K_i^2 X_i^2}, \]

where the inequality is with respect to the positive semidefinite cone. Therefore, we have

\[ \text{tr exp} \left( \sum_{i=1}^{n} \log \mathbb{E} e^{\theta Z_i} \right) \leq \text{tr exp} \left( \theta^2 \sum_{i=1}^{n} K_i^2 X_i^2 \right) \leq N \exp \left( \theta^2 \left\| \sum_{i=1}^{n} K_i^2 X_i^2 \right\| \right) = N e^{\theta^2 \sigma^2}. \]

Then it follows from Theorem 5 that

\[ \mathbb{P} \left( \lambda_{\max} \left( \sum_{i=1}^{n} Z_i \right) \geq t \right) \leq \inf_{\theta > 0} N e^{-\theta t + \theta^2 \sigma^2} = N e^{-\frac{t^2}{4\sigma^2}}. \]

Replacing \(X_i\) by \(-X_i\) and repeating the above argument we can similarly show that

\[ \mathbb{P} \left( \lambda_{\min} \left( \sum_{i=1}^{n} Z_i \right) \leq -t \right) \leq N e^{-\frac{t^2}{4\sigma^2}}. \]

The union bound then guarantees that

\[ \mathbb{P} \left( \left\| \sum_{i=1}^{n} Z_i \right\| \geq t \right) \leq 2N e^{-\frac{t^2}{4\sigma^2}}, \]

as desired. \(\square\)
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