1 Introduction and main results

The interplay between the topology of a closed manifold and the combinatorics of the critical points
of a real valued function of class $C^2$ defined on the manifold is a well known fact of Morse Theory.
It is natural to expect a similar relationship for foliated manifolds. This became evident for the first
time with the following result of G. Reeb ([10]), a consequence of his Stability Theorem ([1],[9],[11]):

\textbf{Theorem 1.} Let $M$ be a closed oriented and connected manifold of dimension $m \geq 2$. Assume
that $M$ admits a $C^1$ transversely oriented codimension one foliation $\mathcal{F}$ with a non empty set of
singularities all of them centers. Then the singular set of $\mathcal{F}$ consists of two points and $M$ is
homeomorphic to the $m$-sphere.

Later on Eells and Kuiper classified the closed manifolds admitting a $C^3$ function with exactly
three non-degenerated singular points ([3]):

\textbf{Theorem 2 ([3]).} Let $M$ be a connected closed manifold (not necessarily orientable) of dimension
$m$. Suppose $M$ admits a Morse function $f: M \to \mathbb{R}$ of class $C^3$ with exactly three singular points.
Then:

(i) $m \in \{2, 4, 8, 16\}$

(ii) $M$ is topologically a compactification of $\mathbb{R}^m$ by an $\frac{m}{2}$-sphere

(iii) If $m = 2$ then $M$ is diffeomorphic to $\mathbb{R}P(2)$. For $m \geq 4$ $M$ is simply-connected and has the
integral cohomology structure of the complex projective plane ($m = 4$), of the quaternionic
projective plane ($m = 8$) or of the Cayley projective plane ($m = 16$).

We will call these manifolds \textit{Eells – Kuiper} manifolds.

In both situations we have a closed manifold endowed with a foliation with Morse singularities
where the number of centers is greater than the number of saddles. In [2] we proved that, in the
case that the manifold is orientable of dimension three, this implies it is homeomorphic to the
3-sphere. The aim of this paper is to consider the n-dimensional case. We proceed to define the
main notions we use. A codimension one foliation with isolated singularities on a compact manifold
$M$ is a pair $\mathcal{F} = (\mathcal{F}_0, \text{sing } \mathcal{F})$ where $\text{sing } \mathcal{F} \subset M$ is a discrete subset and $\mathcal{F}_0$ is a regular foliation
of codimension one on the open manifold $M \setminus \text{sing } \mathcal{F}$. We say that $\mathcal{F}$ is of class $C^k$ if $\mathcal{F}_0$ is of class
$C^k$, $\text{sing } \mathcal{F}$ is called the singular set of $\mathcal{F}$ and the leaves of $\mathcal{F}$ are the leaves of $\mathcal{F}_0$ on $M \setminus \text{sing } \mathcal{F}$.
A point $p \in \text{sing } \mathcal{F}$ is a Morse type singularity if there is a function $f_p: U_p \subset M \to \mathbb{R}$ of class
$C^2$ in a neighborhood of $p$ such that $\text{sing } \mathcal{F} \cap U_p = \{p\}$, $f_p$ has a non-degenerate critical point
at $p$ and the levels of $f_p$ are contained in leaves of $\mathcal{F}$. By the classical Morse Lemma there are
local coordinates $(y_1, \ldots, y_m)$ in a neighborhood $U_p$ of $p$ such that $y_j(p) = 0$, $\forall j \in \{1, \ldots, m\}$
and $f(y_1, \ldots, y_m) = f(p) - (y_1^2 + \cdots + y_{r(p)}^2) + y_{r(p)+1}^2 + \cdots + y_m^2$. The number $r(p)$ is called the Morse
index of $p$. The singularity $p$ is a center if $r(p) \in \{0, m\}$ and it is a saddle otherwise. The leaves of
\( \mathcal{F} \) in a neighborhood of a center are diffeomorphic to the \((m-1)\)-sphere. Given a saddle singular point \( p \in \text{sing} \ \mathcal{F} \) we have leaves of \( \mathcal{F}|_{U_p} \), that accumulate on \( p \), they are contained in the cone \( \tau_p \): 
\[ y_1^2 + \cdots + y_{r(p)}^2 = y_{r(p)+1}^2 + \cdots + y_m^2 \neq 0 \]
and there are two possibilities: either \( r(p) = 1 \) or \( m-1 \) and then \( \tau_p \) is the union of two leaves of \( \mathcal{F}|_{U_p} \), or \( r(p) \neq 1 \) and \( m-1 \) and \( \tau_p \) is a leaf of \( \mathcal{F}|_{U_p} \).

Any leaf of \( \mathcal{F}|_{U_p} \) contained in \( \tau_p \) is called a local separatrix of \( \mathcal{F} \) at \( p \), or a cone leaf at \( p \). Any leaf of \( \mathcal{F} \) such that its restriction to \( U_p \) contains a local separatrix of \( \mathcal{F} \) at \( p \) is called a separatrix of \( \mathcal{F} \) at \( p \). A saddle connection for \( \mathcal{F} \) is a leaf which contains local separatrices of two different saddle points. A saddle self-connection for \( \mathcal{F} \) at \( p \) is a leaf which contains two different local separatrices of \( \mathcal{F} \) at \( p \). A foliation \( \mathcal{F} \) with Morse singularities is transversely orientable if there exists a vector field \( X \) on \( M \), possibly with singularities at sing \( \mathcal{F} \), such that \( X \) is transverse to \( \mathcal{F} \) outside sing \( \mathcal{F} \).

**Definition 1.** A Morse foliation \( \mathcal{F} \) on a manifold \( M \) is a transversely oriented codimension one foliation of class \( C^2 \) with singularities such that: (i) each singularity of \( \mathcal{F} \) is of Morse type and (ii) there are no saddle connections.

Basic examples of Morse foliations are given by the levels of Morse functions \( f : M \to \mathbb{R} \) of class \( C^2 \). Therefore any manifold of class \( C^2 \) supports a Morse foliation, i.e., the existence of a Morse foliation imposes no restriction on the topology of the manifold. Nevertheless, there are restrictions which come from the nature of the singularities of a Morse foliation \( \mathcal{F} \) on \( M \). Indeed, our purpose in this paper is to show the following theorem:

**Theorem 3.** Let \( M \) be a compact connected manifold and \( \mathcal{F} \) a Morse foliation on \( M \) such that the number \( k \) of centers and the number \( \ell \) of saddles in sing \( \mathcal{F} \) satisfy \( k \geq \ell + 1 \). Then we have two possibilities:

(i) \( k = \ell + 2 \) and \( M \) is homeomorphic to the \( m \)-sphere.

(ii) \( k = \ell + 1 \) and \( M \) is an Eells-Kuiper manifold.

**Acknowledgement.** We want to thank José Seade for several valuable suggestions and the ICTP for the hospitality during the elaboration of this work.

## 2 Preliminaries

Let us first fix the notation we use. Let \( \mathcal{F} \) be a Morse foliation on a manifold \( M \) of dimension \( m \geq 3 \). Given a center singularity \( p \in \text{sing} \ \mathcal{F} \) the nearby leaves of \( \mathcal{F} \) are compact diffeomorphic to \( S^{m-1} \). Since \( m-1 \geq 2 \) any such leaf \( L \) has trivial holonomy group and therefore by the Local Stability theorem of Reeb ([1], [5]) there is a fundamental system of open neighborhoods \( V \) of \( L \) such that the restriction \( \mathcal{F}|_V \) is equivalent to a product foliation \( \mathcal{G} \) on \( L \times (-1,1) \) whose leaves are of the form \( L \times \{ t \} \), \( t \in (-1,1) \). We introduce therefore the open subset \( \mathcal{C}(\mathcal{F}) \) as the union of centers in sing \( \mathcal{F} \) and leaves of \( \mathcal{F} \) diffeomorphic to \( S^{m-1} \). The basin of a center \( p \in \text{sing} \ \mathcal{F} \) is the connected component \( \mathcal{C}_p(\mathcal{F}) \) of \( \mathcal{C}(\mathcal{F}) \) that contains \( p \). We have the following basic lemma:

**Lemma 1.** Given centers \( p, q \in \text{sing} \ \mathcal{F} \) the sets \( \mathcal{C}_p(\mathcal{F}) \) and \( \mathcal{C}_q(\mathcal{F}) \) are open in \( M \) and \( \mathcal{C}_p(\mathcal{F}) \cap \mathcal{C}_q(\mathcal{F}) \neq \emptyset \) if and only if \( \mathcal{C}_p(\mathcal{F}) = \mathcal{C}_q(\mathcal{F}) \). Moreover we have \( \mathcal{C}_p(\mathcal{F}) = M \) if and only if \( \partial \mathcal{C}_p(\mathcal{F}) = \emptyset \) and in this case \( M \) is homeomorphic to \( S^m \) provided that \( M \) is orientable.

In particular \( M \) is homeomorphic to \( S^m \) or \( \partial \mathcal{C}_p(\mathcal{F}) \) contains some saddle singularity.

In order to fix notations we shall now introduce the notion of holonomy group of an invariant subset of codimension one. We will consider two notions of holonomy. When we refer to the holonomy of a leaf \( L \) of \( \mathcal{F} \) we mean the holonomy group of \( L \) as a leaf of \( \mathcal{F}_0 \) on \( M \setminus \text{sing} \ \mathcal{F} \). On the
other hand, the notion of holonomy can be extended to invariant subsets of the form $S = \tau \cup \{p\}$, $p \in \text{sing} \mathcal{F}$ and $\tau$ is either a cone leaf or a union of two cone leaves. Notice that, in a small neighborhood of $p$, $\tau$ can consist of two components $\tau_1$ and $\tau_2$, and that this can only happen if $r(p) = 1$ or $m - 1$. In this case $S$ locally divides the manifold into three connected components. One of them, say $R_3$, is the union of (regular) leaves which are hyperboloids of one sheet, and the others, say $R_1$ and $R_2$, are the union of one of the connected components of hyperboloids of two sheets (we can think of $R_1$ as the region surrounded by $\tau_1$ and $R_2$ is the region surrounded by $\tau_2$). Let $\gamma : [0,1] \to S$ be a path on $S$ which passes through the singularity $p$ (from $\tau_1$ to $\tau_2$). In this case the holonomy along $\gamma$ can be defined in the usual manner (lifting paths to leaves) on $R_3$, however, there is no canonical extension of this holonomy to the other side in general. Thus we adopt the following notion of holonomy. Fix a neighborhood $U$ of $p \in \text{sing} \mathcal{F}$ where $\mathcal{F}$ is given by a Morse function $f$ with a single singularity at $p$. Let $\gamma : [0,1] \to S$ be a piecewise smooth path (as a map $\gamma : [0,1] \to M$). Let $T_0$ and $T_1$ be local transversals to $\mathcal{F}$ at $\gamma(0)$ and $\gamma(1)$ respectively. The holonomy along $\gamma$ will be the mapping which assigns $t \in T_0$ to $f^{-1}(f(t)) \cap T_1 \in T_1$. This holonomy map is well-defined even if $\gamma$ is not contained in $\{p\} \cup \tau_1$.

Next we study the possible intersections for the boundaries of two basins.

Lemma 2. Suppose $p_1, p_2 \in \text{sing} \mathcal{F}$ are distinct centers such that $\partial \mathcal{C}_{p_1}(\mathcal{F}) \cap \partial \mathcal{C}_{p_2}(\mathcal{F}) \neq \emptyset$. Then we have the following mutually exclusive possibilities:

(i) $\partial \mathcal{C}_{p_1}(\mathcal{F}) = \partial \mathcal{C}_{p_2}(\mathcal{F})$ and so $M = \mathcal{C}_{p_1}(\mathcal{F}) \cup \mathcal{C}_{p_2}(\mathcal{F})$.

(ii) $\partial \mathcal{C}_{p_1}(\mathcal{F}) \neq \partial \mathcal{C}_{p_2}(\mathcal{F})$ and there is a saddle point $q \in \partial \mathcal{C}_{p_1}(\mathcal{F}) \cap \partial \mathcal{C}_{p_2}(\mathcal{F})$ with Morse index $1$ or $m - 1$, without self connection.

Proof. Since $\partial \mathcal{C}_{p_1}(\mathcal{F}) \cap \partial \mathcal{C}_{p_2}(\mathcal{F}) \neq \emptyset$, by Lemma 1 there is a saddle singular point $q \in \partial \mathcal{C}_{p_1}(\mathcal{F}) \cap \partial \mathcal{C}_{p_2}(\mathcal{F})$. If the Morse index of $q$ is different from $1$ and $m - 1$ then in suitable local coordinates $(y_1, \ldots, y_m)$ we have $q = (0, \ldots, 0)$ and the local separatrix $\tau_q$ through $q$ is given by $y_1^2 + \cdots + y_r^2 = y_{r+1}^2 + \cdots + y_m^2 \neq 0$ where $r \notin \{1, m - 1\}$. In particular $\tau_q$ is connected.

Thus, if $C$ is the separatrix of $\mathcal{F}$ at $q$ we have $\partial \mathcal{C}_{p_1}(\mathcal{F}) \cap \partial \mathcal{C}_{p_2}(\mathcal{F}) = \overline{C} = C \cup \{q\}$ and we are in case (i).

In case the index of $q$ is $1$ or $m - 1$, and $C$ is a self-connection at $q$, then $\partial \mathcal{C}_{p_1}(\mathcal{F}) = \partial \mathcal{C}_{p_2}(\mathcal{F}) = \overline{C}$ and we are again in case (i). The remaining case is index of $q$ is $1$ or $m - 1$ and $q$ has no self-saddle connection. Consider local coordinates $(y_1, \ldots, y_m)$ with $q = (0, \ldots, 0)$ and $\mathcal{F}$ given by the levels of the function $-y_1^2 + y_2^2 + \cdots + y_m^2$. The level zero of this function bounds the regions $R_1$: $y_1 < 0$, $y_1^2 > y_2^2 + \cdots + y_m^2$, $R_2$: $y_1 > 0$, $y_1^2 > y_2^2 + \cdots + y_m^2$ and $R_3$: $y_1^2 < y_2^2 + \cdots + y_m^2$. Then $\mathcal{C}_{p_i}(\mathcal{F}) \cap R_i = \emptyset$, $i=1,2$, because otherwise we would have a saddle self-connection at $q$. On the other hand $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_1 \neq \emptyset$ implies $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_2 = \emptyset$ by the same reason. Therefore $\partial \mathcal{C}_{p_1}(\mathcal{F}) \neq \partial \mathcal{C}_{p_2}(\mathcal{F})$. This proves (ii).

Proposition 1. Let $\mathcal{F}$ be a Morse foliation on a closed connected manifold $M$ of dimension $m \geq 3$. Assume that $k = 2$ and $\ell = 1$, i.e., $\mathcal{F}$ has exactly two centers and one saddle singularity. Then $M$ is homeomorphic to an Eells-Kuiper manifold.

Proof. We shall first prove that the nonsingular foliation $\mathcal{F}_0 = \mathcal{F}\big|_{M_0}$ on $M_0 = M \setminus \text{sing} \mathcal{F}$ is a proper stable foliation. There are several equivalent conditions that define a stable foliation (5). We shall prove that given any leaf $L_0$ of $\mathcal{F}_0$ there is a fundamental system of open neighborhoods of $L_0$ in $M_0$ saturated by $\mathcal{F}_0$. By hypothesis we have $M = \overline{\mathcal{C}_{p_1}(\mathcal{F}) \cup \mathcal{C}_{p_2}(\mathcal{F})} = \mathcal{C}_{p_1}(\mathcal{F}) \cup \mathcal{C}_{p_2}(\mathcal{F}) \cup C \cup \{q\}$ where $\partial \mathcal{C}_{p_1}(\mathcal{F}) \cap \mathcal{C}_{p_2}(\mathcal{F}) = \overline{C}$. Thus if $r(q) = 1$ or $m - 1$ then $C$ is a self connection. We proceed to show that this cannot occur. Suppose that $\mathcal{C}_{p_1}(\mathcal{F}) \cap R_3 \neq \emptyset$, then $\mathcal{C}_{p_2}(\mathcal{F})$ will have nonempty intersection
with \( R_1 \) and with \( R_2 \). Taking a small closed ball \( B_q(t) \) of radius \( t > 0 \) centered at \( q \) then for leaves \( L_i \) of \( F \), \( L_i \subset C_{p_i}(F) \) for \( i \in \{1, 2\} \), close enough to \( C \), we have that the intersection \( L_1 \cap (M \backslash B_q(t)) \) is a union of two disjoint \((m - 1)\)-discs. Moreover \( L_2 \cap (M \backslash B_q(t)) \) is the complement of two disjoint \((m - 1)\)-discs in an \((m - 1)\)-sphere. Since both manifolds \( L_1 \cap (M \backslash B_q(t)) \) and \( L_2 \cap (M \backslash B_q(t)) \) are homeomorphic to \( C \backslash (C \cap B_q(t)) \) then we obtain a contradiction. Thus, \( r \neq 1, m - 1 \) and \( C \cap B_q(t) \) is connected for \( t \) small. Given a leaf \( L_0 \) of \( F \) we have two possibilities, either \( L_0 \subset C_{p_i}(F) \) for some \( i \in \{1, 2\} \) and \( L_0 \) is homeomorphic to \( S^{m-1} \), or \( L_0 = C \). In case \( L_0 \) is in \( C_{p_i}(F) \) then, by the Reeb stability theorem, \( L_0 \) has a fundamental system of saturated neighborhoods consisting of compact leaves. This shows that the leaves of \( F_0 \) in \( M_0 \setminus C \) are stable. It suffices to show that \( C \) is also a stable leaf.

By Lemma 3, the holonomy group of \( C \cup \{q\} \) is a finite group conjugated to a subgroup of \( \text{Diff}(\mathbb{R}, 0) \). Indeed, \( C \backslash B_q(t) \) is a disc and therefore simply-connected; on the other hand in \( B_q(t) \) the foliation has a first integral as \( f = -(y_1^2 + \cdots + y_r^2) + y_{r+1}^2 + \cdots + y_m^2 \), so that the holonomy group of \( F \cap B_q(t) \) is finite. Since \( C \cup \{q\} \) has a finite holonomy group, which is a subgroup of \( \text{Diff}(\mathbb{R}, 0) \), the holonomy group of \( C \cup \{q\} \) is either trivial or, in case \( F \) is not transversely orientable, it has order 2. By the classical argument on stability of leaves we conclude that finite holonomy implies that the leaf \( C \) is stable and \( F_0 \) is stable in \( M_0 \).

Since \( F_0 \) is stable in \( M_0 \), the leaf space \( M_0 / F_0 =: \mathcal{X}_{F_0} \) is Hausdorff and therefore it is a 1-manifold. The choice of a differentiable submersion \( \mathcal{X}_{F_0} \to \mathbb{R} \) gives then a differentiable first integral \( F_0 : M_0 \to \mathbb{R} \) for \( F_0 \). Clearly \( F_0 \) can be modified in order to admit a differentiable (radial) extension to the center singularities \( p_1, p_2 \in \text{sing} F \). It remains to show that \( F_0 \) can be modified in order to admit a differentiable extension to \( q \). This is a consequence of the fact that by the triviality of the holonomy group of \( C \cup \{q\} \) we can extend the local first integral \( f = -\sum_{j=1}^r y_j^2 + \sum_{k=r+1}^m y_k^2 \) from a ball \( B_q(t) \) to a neighborhood \( T \) of \( C \cup \{q\} \) in \( M \) in such a way that \( \partial T \) is a union of leaves of \( F \), each leaf diffeomorphic to an \((m - 1)\)-sphere and contained in some basin \( C_{p_j}(F) \).

Thus we have proved that \( F \) is given by a Morse function \( F : M \to \mathbb{R} \) and therefore, by Eells-Kuiper Theorem \( M \) is an Eells-Kuiper manifold.

For any \( r > 0 \) we will write \( B(r) = D(r) \times I(r) \subset \mathbb{R}^{m-1} \times \mathbb{R} \), where \( D(r) \) and \( I(r) \) are closed discs of radius \( r \) centered at zero. The foliation on \( B(r) \) given by the submersion \( (x, t) \mapsto t \) will be denoted by \( \mathcal{H} \). Let \( p \in M \) be a center and let \( q \in \partial C_{p_i}(F) \) be a saddle point of \( F \). We will say that \( p, q \) form a trivial pairing if there are open neighborhoods \( V \supset V' \supset C_{p_i}(F), p, q \in V' \) and a diffeomorphism \( \varphi : V \to B(1) \), onto \( B(1) \), such that \( \varphi(V') = B(1/2) \) and \( F|_{\varphi^{-1}(V')} = \varphi^* \mathcal{H} \).

Lemma 3. Suppose \( p_1, p_2 \in M \) are two different centers such that \( \partial C_{p_1}(F) \cap \partial C_{p_2}(F) \neq \emptyset \) and let \( q \) be the saddle point contained in this intersection. Assume that the index of \( q \) is one and that there is no saddle self-connection at \( q \). Then, either \( p_1, q \) or \( p_2, q \) form a trivial pairing.

Proof. In a neighborhood \( q \in U \) there are local coordinates \((y_1, ..., y_m)\) such that \( q = (0, ..., 0) \) and the leaves of \( F|_U \) are given by the levels of the function \( f(y_1, ..., y_m) = -y_1^2 + (y_2^2 + ... + y_m^2) \) on \( \mathbb{R}^m \). As before the cone \( -y_1^2 + (y_2^2 + ... + y_m^2) = 0 \) divides \( U \) in three regions. The regions \( R_1 \) and \( R_2 \) are defined by \( R_1 : y_1 < 0, y_1^2 > y_2^2 + ... + y_m^2 \), \( R_2 : y_1 > 0, y_1^2 < y_2^2 + ... + y_m^2 \). The cone leaves \( \tau_1 \) and \( \tau_2 \) of \( F|_U \) bound \( R_1 \) and \( R_2 \) respectively and since there is no self-connection at \( q \) there are different leaves of \( F \), \( \ell_1 \) and \( \ell_2 \), such that \( \ell_1 \supset \tau_1 \) and \( \ell_2 \supset \tau_2 \). Since we have two center basins \( C_{p_1}(F) \) and \( C_{p_2}(F) \) and three regions \( R_1, R_2, R_3 \) then some \( C_{p_i}(F) \) will intersect \( R_1 \) or \( R_2 \). Suppose that \( C_{p_1}(F) \cap R_1 \neq \emptyset \). Then \( C_{p_1}(F) \cap R_2 = \emptyset \) and \( C_{p_2}(F) \cap R_3 = \emptyset \) because both \( R_2 \) and \( R_3 \) have \( \ell_2 \) in their boundary and if either \( C_{p_1}(F) \cap R_2 \neq \emptyset \) or \( C_{p_2}(F) \cap R_3 \neq \emptyset \) this would imply a saddle self-connection at \( q \). Thus \( \partial C_{p_1}(F) = T_1 \). We claim that
Lemma 3 we have either

We take $U$ small enough so that $\bar{\ell}_1 \cap U = \bar{\pi}_1$. Thus, for any leaf $L \subset C_p\ell(\mathcal{F})$ close enough to $\bar{\ell}_1$, $L \cap U$ is connected and $f|_{L \cap U} = \delta < 0$, a constant. Write $L = L_\delta$. Then, as $\delta \to 0$ $L_\delta \setminus U$ approaches $\bar{\ell}_1 \setminus \ell_1$ and this implies that $\bar{\ell}_1 \setminus \ell_1$ is homeomorphic to $L_\delta \setminus U$, i.e., to an $(m - 1)$–disc. Therefore $\bar{\ell}_1$ is homeomorphic to $S^{m-1}$. Moreover for $\epsilon > 0$ small enough each leaf $f^{-1}(\delta)\cap \mathcal{F}|_{R_1 \cup R_3}$, $-2\epsilon \leq \delta \leq 2\epsilon$, bounds an $(m - 1)$–disc $D_\delta$ close to $\bar{\ell}_1 \setminus \ell_1$, with $D_0 = \bar{\ell}_1 \setminus \ell_1$. The union $T_{2\epsilon} = \bigcup_{-2\epsilon \leq \delta \leq 2\epsilon} \overline{D_\delta}$ is a trivially foliated neighborhood of $\bar{\ell}_1 \setminus \ell_1$. We can extend the function $f$ to $U \cup T_{2\epsilon}$ by writing $f|_{\mathcal{F}|_{T_{2\epsilon}}} = \delta$. We define a saturated neighborhood $V_0$ of $\bar{\ell}_1 \setminus \ell_2$ by $V_0 = f^{-1}([-\epsilon, \epsilon])\cap C_p\ell(\mathcal{F})$ and define $g = f|_{V_0 \setminus C_p\ell(\mathcal{F})}$. Consider now a Riemannian metric defined on $M$ and a normal vector field to $\mathcal{F}$ that in $U$ takes the form $\mathcal{N} = -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + \ldots + y_m \frac{\partial}{\partial y_m}$. For $a > 0$ small enough the submanifold $e = (y_1 = a) \cap \ell_2$ is well defined and diffeomorphic to $S^{m-2}$. Let $\Sigma$ be a cross section to $\mathcal{F}$, over $e$, i.e. $\Sigma \cap \ell_2 = e$, $\Sigma$ contained in $V_0$ and invariant by $\mathcal{N}$. We take $\Sigma$ diffeomorphic to $e \times [-\epsilon, \epsilon]$ by means of a map that takes each $e \times \delta$, $-\epsilon \leq \delta \leq \epsilon$, to the leaf $(f = \delta)\cap \Sigma$ of $\mathcal{F}|_{\Sigma}$.

Consider the region $V \subset V_0$ bounded by $(g = -\epsilon) \cup (g = \epsilon)$ and $\Sigma$ and define a neighborhood $\partial V \subset W \subset V$, as $W = (-\epsilon \leq g \leq -\epsilon/2) \cup (\epsilon/2 \leq g \leq \epsilon) \cup N$ where $N$ is a neighborhood of $\Sigma \subset V$ invariant by $\mathcal{N}$.

The foliation $\mathcal{F}|_W$ is trivial in the sense that on $(-\epsilon \leq g \leq -\epsilon/2) \cup (\epsilon/2 \leq g \leq \epsilon)$ the leaves of $\mathcal{F}$ are levels of $g$, diffeomorphic to $D(1)$ and on $N$ the leaves of $\mathcal{F}$ are levels of $g - \epsilon/2 \leq g \leq \epsilon/2$ diffeomorphic to $D(1) \setminus D(1/2)$. Moreover in $W$ the foliation $\mathcal{F}$ and the trajectories of $\mathcal{N}$ are everywhere transverse. Thus a diffeomorphism $\varphi: W \to B(1) \setminus B(1/2)$ can be easily constructed by sending leaves of $\mathcal{F}|_W$ at the level $(g = \alpha\epsilon)$ onto leaves of $\mathcal{H}|_{B(1) \setminus B(1/2)}$ at the level $(t = \alpha)$ and orbits of $\mathcal{N}|_W$ to orbits of $\partial_\mathcal{F}$. Then $\varphi$ is extended to $V$. □

3 Proof of the Theorem

Now we prove Theorem $\square$. We will proceed by induction on the number $\ell$ of saddle singularities. If $\ell = 0$ then Reeb’s Theorem applies and $M$ is homeomorphic to $S^m$. Assume now that $\ell \geq 1$ and the result has been proven for foliations with at most $\ell - 1$ singularities of saddle type. We have $k \geq \ell + 1$. Thus $k \geq 2$. Suppose that $M$ is not homeomorphic to $S^m$. Then for each center $p \in \text{sing}\mathcal{F}$ there must be a saddle $q(p) \in \partial C_p(\mathcal{F})$. Since $k \geq \ell + 1$ and $k \geq 2$ there are two centers $p_1, p_2$ such that $q(p_1) = q(p_2)$, i.e., there is a saddle $q$ such that $q \in \partial C_{p_1}(\mathcal{F}) \cap \partial C_{p_2}(\mathcal{F})$ and by Lemma 3 we have either $M = C_{p_1}(\mathcal{F}) \cup C_{p_2}(\mathcal{F})$ or $q$ has index 1 or $m - 1$ and is not self-connected.

In case $M = C_{p_1}(\mathcal{F}) \cup C_{p_2}(\mathcal{F})$ then clearly $C_p(\mathcal{F}) \cap \text{sing}\mathcal{F} = \{p_1\}$, $i = 1, 2$.

Thus $\text{sing}\mathcal{F} = \{p_1, p_2, q\}$ and by the Proposition $M$ is an Eells-Kuiper manifold. In case $q$ has index 1 or $m - 1$ and no self-connection then by Lemma 3 we can eliminate one saddle and one center replacing $\mathcal{F}$ by a Morse foliation $\mathcal{F}_1$ on $M$ with a number $k_1$ of centers and $\ell_1$ of saddles given by $k_1 = k - 1$ and $\ell_1 = \ell - 1$. Therefore $k_1 \geq \ell_1 + 1$ and $\ell > \ell_1 \geq 0$. By the induction hypothesis $M$ is homeomorphic to $S^m$ or to an Eells-Kuiper manifold. This proves the theorem. □
References

[1] C. Camacho, A. Lins Neto: Geometric Theory of Foliations, Birkhäuser; 1985.

[2] C. Camacho, B. Scárdua: On codimension one foliations with Morse singularities on three-manifolds, to appear in Topology and its Applications.

[3] J. Eells, N. Kuiper: Manifolds which are like projective planes; Pub. I.H.E.S., t.14 (1962), p. 5-46.

[4] J. Eells, N. Kuiper: Closed manifolds which admit nondegenerate functions with three critical points; Indag. Math. 23 (1961), p. 411-417.

[5] C. Godbillon: Feuilletages. Études géométriques. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991.

[6] A. Haefliger; Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. Comment. Math. Helv. 32 1958 248–329.

[7] A. Haefliger; Variétés feuilletées, An. Scuola Norm. Sup. Pisa, 3, vol. 16, 1962, 367-397.

[8] J. Milnor: Morse Theory, Annals of Mathematics Studies Number 51, Princeton, N.J., 1963.

[9] G. Reeb: Sur certaines propriétés topologiques des variétés feuilletées; Actualités Sci. Ind., Hermann, Paris, 1952.

[10] G. Reeb; Sur les points singuliers d’une forme de Pfaff complétement intégrable ou d’une fonction numérique. C.R.A.S. Paris 222, 1946, p.847-849.

[11] E. Wagneur; Formes de Pfaff à singularités non dégénérées. Annales de l’institut Fourier, 28 no. 3 (1978), p. 165-176

César Camacho                        Bruno Scárdua
IMPA-Estrada D. Castorina, 110       Instituto de Matemática
Jardim Botânico                       Universidade Federal do Rio de Janeiro
Rio de Janeiro - RJ                   Caixa Postal 68530
CEP. 22460-320                        21.945-970 Rio de Janeiro-RJ
BRAZIL                                BRAZIL