QUANTUM HAMILTONIAN REDUCTION FOR POLAR REPRESENTATIONS

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Dedicated to the memory of our friend and coauthor Tom Nevins

Abstract. Let $G$ be a reductive complex algebraic group with Lie algebra $\mathfrak{g}$ and suppose that $V$ is a polar $G$-representation. We prove the existence of a radial parts map $\text{rad} : \mathcal{D}(V)^G \to A_\kappa$ from the $G$-invariant differential operators on $V$ to the spherical subalgebra $A_\kappa$ of a rational Cherednik algebra. Under mild hypotheses $\text{rad}$ is shown to be surjective.

If $V$ is a symmetric space, then $\text{rad}$ is surjective, and we determine exactly when $A_\kappa$ is a simple ring. When $A_\kappa$ is simple, we also show that $\ker(\text{rad}) = (\mathcal{D}(V)^G)^G$, where $\tau : \mathfrak{g} \to \mathcal{D}(V)$ is the differential of the $G$-action.

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1. Introduction

Throughout the paper except stated to the contrary, the base field will be $\mathbb{C}$. We fix a connected reductive complex algebraic group $G$ with Lie algebra $\mathfrak{g} = \text{Lie}(G)$.

The idea of using radial parts to understand invariant differential operators goes back (at least) to Harish-Chandra [HC1, HC2], where he defined a radial parts homomorphism $HC : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with Weyl group $W$. This was the fundamental algebraic step in proving his theorems on the regularity of invariant eigendistributions, and hence the regularity of the characters of representations of real forms of $G$. In [Wa] and [LS1] the map $HC$ was actually shown to be surjective, enabling the authors to provide much easier
proofs of those key results of Harish-Chandra. Further applications of these ideas to Springer theory and Weyl group representations appear in [HK, LS3, Wa].

Harish-Chandra’s radial parts map (and resulting regularity results) was then generalised in [Se, LS4, GL] to symmetric space representations $V$ of $G$ satisfying Sekiguchi’s “niceness” condition. The definitions of symmetric spaces and of Sekiguchi’s niceness condition can be found in Section 8. Again for nice symmetric spaces it was shown in [LS4] that, for a certain abelian subalgebra $\mathfrak{a} \subset V$ with associated little Weyl group $W$, there is a radial parts map $\text{rad}: \mathcal{D}(V)^G \to \mathcal{D}(\mathfrak{a}/W)$, although the precise nature of the image was unknown.

Much more recently, the notion of radial parts maps has been very fruitful in the theory of Cherednik algebras or, more precisely, of their spherical subalgebras. Here one can use the radial parts map to relate (via the functor of Hamiltonian reduction) equivariant $\mathcal{D}$-modules on certain $G$-representations to representations of the (spherical) Cherednik algebra. This interplay is useful, both for understanding the equivariant $\mathcal{D}$-modules on $V$ and the representation theory of Cherednik algebras; see for example [BG, EG, GG, GGS, Lo1, MN, Ob].

Given the applications one can derive from the existence of a radial parts map, it is important to understand how generally this result will hold. This is the goal of the article – we describe what we expect is the most general setting in which the theory can be developed. As explained below, developing the theory in this generality allows us to prove new results even in the previously “well-understood” case of symmetric spaces.

**Polar representations.** To motivate the general setting, let us consider the classical example of rotation invariant differential operators on $\mathbb{R}^3$. Here the radial parts decomposition allows one to understand easily the action of the Laplacian on invariant functions. This is possible because restriction of functions to a line through the origin is an isomorphism between rotation invariant functions on $\mathbb{R}^3$ and functions on the line invariant under the involution $v \mapsto -v$. This latter isomorphism holds precisely because the line is orthogonal to every $SO(3)$-orbit. As a consequence, in order to understand the action of the Laplacian on rotation invariant functions, we are reduced to studying differential operators on the line that are invariant under $\mathbb{Z}_2$.

In general, we begin with a slice $\mathfrak{h} \subset V$ intersecting transversely the (closed) $G$-orbits in $V$. Then the key to the existence of a radial parts map is that restriction to $\mathfrak{h}$ should provide a version of the Chevalley isomorphism $\mathbb{C}[V]^G \supseteq \mathbb{C}[\mathfrak{h}]^W$, where $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ is a finite group. The representations where one can find a slice with this property are precisely the polar representations $V$ of [DK]. Thus, it is natural to ask whether a radial parts map holds for $\mathcal{D}(V)^G$ when $V$ is a polar $G$-representation. In this paper we show that this indeed the case. Indeed, we prove that there exists such a radial parts map $\text{rad}$ for any polar representation $V$ (see Theorem 1.2) and, under mild conditions, $\text{rad}$ is surjective (see Theorem 1.3). In this generality the image of $\text{rad}$ will now be some spherical subalgebra $A_\mathfrak{h}(W)$ of a Cherednik algebra, realised as a subalgebra of $\mathcal{D}(\mathfrak{h}_{reg})^W$ via the Dunkl embedding.

In order to state these results precisely, we need some definitions. Following Dadok and Kac [DK], a representation $V$ of $G$ is called polar if there exists a semisimple element $v \in V$ such that

$$\mathfrak{h} := \{ x \in V \mid g \cdot x \subset g \cdot v \}$$

satisfies $\dim \mathfrak{h} = \dim V//G$. One of the fundamental properties of polar representations is that the finite group $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ acts as a complex reflection group on $\mathfrak{h}$ and, by restriction, defines a Chevalley isomorphism $\gamma: V//G \cong \hbar/W$; see [DK].
Theorem 2.9]. The group $W$ is called the Weyl group associated to this data. Thus, polar representations are a natural situation—perhaps the most general one—where one can hope to have a radial parts map.

It is immediate that invariant differential operators act on invariant functions, and so there is a natural map $\phi : D(V)^G \to D(V/G) \cong D(\mathfrak{h}/W)$, but its image is almost never in $D(\mathfrak{h})^W$. Instead, Harish-Chandra’s morphism HC differs from $\phi$ by conjugation with $\delta^{1/2}$ for an appropriate element $\delta$ (the discriminant).

As was already apparent in previous applications to the study of Cherednik algebras, it is important to be able to twist the radial parts map and our version of the radial parts map will be defined in this context. This is constructed in Section 4, but in outline the definition is as follows. For any complex reflection group $(W, \mathfrak{h})$, there exists a discriminant $h \in \mathbb{C}[\mathfrak{b}]^W$ such that the regular locus $\mathfrak{h}_{reg} = (h \neq 0) \subset \mathfrak{h}$ is precisely the locus where $W$ acts freely. We then define the discriminant $\delta = g^{-1}(h)$ on $V$ with regular locus $V_{reg} = (\delta \neq 0) \subset V$. Let $\tau : \mathfrak{g} \to \text{Der}(\mathbb{C}[V])$ be the differential of the action of $G$ on $V$, where $\text{Der}(\mathbb{C}[V])$ denotes the space of $\mathbb{C}$-linear derivations. If $\chi$ is a character of $\mathfrak{g}$ we set

$$\mathfrak{g}_\chi = \{\tau(x) - \chi(x) : x \in \mathfrak{g}\}.$$

One would expect to define a radial parts map such that the localization to the regular locus $V_{reg}$ is a surjection

$$(1.1) \quad (D(V_{reg})/D(V_{reg})\mathfrak{g}_\chi)^G \twoheadrightarrow D(\mathfrak{h}_{reg})^W \cong D(\mathfrak{h}_{reg}/W).$$

It is soon apparent that any twisted radial parts map satisfying (1.1) is governed by the factorisation $\delta = \delta_1^{m_1} \cdots \delta_k^{m_k}$ of the discriminant into irreducible, pairwise distinct polynomials. Then, for complex numbers $(\varsigma_1, \ldots, \varsigma_k) \in \mathbb{C}^k$, we define $\text{rad}_\varsigma : D(V)^G \to D(\mathfrak{h}_{reg})^W$ by

$$\text{rad}_\varsigma(D)(z) = (\delta^{-\varsigma}D(g^{-1}(z)\delta^\varsigma))|_{\mathfrak{h}_{reg}},$$

where $\delta^\varsigma := \delta_1^{\varsigma_1} \cdots \delta_k^{\varsigma_k}$ and $D \mapsto \delta^{-\varsigma}D\delta^\varsigma$ is the automorphism of $D(V_{reg})^G$ given by conjugation by $\delta^{-\varsigma}$. Twisting by $\delta^\varsigma$ can also be thought of as identifying $D(V_{reg})$ with the ring of differential operators on the free $\mathbb{C}[V_{reg}]$-module $\mathbb{C}[V_{reg}]\delta^\varsigma$, as is described in more detail in Section 4. Since $G$ is connected, each $\delta_i$ is a $G$-semi-invariant of weight $\theta_i$ say. This means that the character $\chi = \chi(\varsigma)$ associate to $\varsigma$ in (1.1) must be $\sum_{i=1}^k \varsigma_i d\theta_i$.

With this notation to hand, we can state the first main result of this paper.

**Theorem 1.2.** (Theorem 5.21 and Lemma 5.4) Assume that $V$ is a polar representation for $G$. Then there exists a parameter $\kappa = \kappa(\varsigma)$ for the spherical subalgebra $A_\kappa(W)$ of the rational Cherednik algebra $H_\kappa(W)$ such that $\text{Im}(\text{rad}_\varsigma) \subseteq A_\kappa(W)$.

The localization of $\text{rad}_\varsigma$ to $V_{reg}$ induces the surjection (1.1).

Fundamental to the proof of this theorem is the fact that the parameter $\kappa$ can be computed by taking rank one slices in $V$. Following [Le3], the rank one case can be understood explicitly, and is described in Section 4. Using those slices, the proof of the theorem is then completed in Section 5. In explicit examples, the method of reduction to rank one slices provides a powerful, and quick, way to compute the parameter $\kappa$. This completely avoids the arduous calculations one normally needs to perform to compute the image of $\text{rad}_\varsigma$.

We remark that, if the representation $V$ is stable, then (1.1) is actually an isomorphism; see Corollary 6.9. This will be important in [BNS].

Naturally, our second main results describes situations in which the radial parts map $\text{rad}_\varsigma : D(V)^G \to A_\kappa(W)$ is surjective.
Theorem 1.3. (Theorem 7.1 and Corollary 4.21.) Assume that $V$ is a polar representation for $G$. Then the map $\text{rad}_\kappa : \mathcal{D}(V)^G \to A_\kappa(W)$ is surjective in each of the following cases:

1. when $A_\kappa(W)$ is a simple algebra;
2. when $V$ is a symmetric space;
3. when the associated complex reflection group $W$ is a Weyl group with no summands of type $E$ or $F$;
4. when the rank of $(W, h)$ is at most one.

We do not know of any polar representations where $\text{rad}_\kappa$ is not surjective.

If $\zeta$ and $\zeta'$ are two choices of twist then the images of $\text{rad}_\zeta$ and $\text{rad}_{\zeta'}$ are not, in general, equal as subalgebras of $\mathcal{D}(\mathfrak{b}_{\text{reg}})^W$. However, as shown in Corollary 6.7, they are isomorphic if $\chi(\zeta) = \chi(\zeta')$ since the kernel of $\text{rad}_\zeta$ equals that of $\text{rad}_{\zeta'}$ in this case.

Symmetric spaces. In the companion paper [BNS] we will apply the map $\text{rad}_\kappa$ to study the representation theory of admissible $\mathcal{D}$-modules on $V$ and of modules over $A_\kappa(W)$. The theory developed there only works when $A_\kappa(W)$ is a simple ring, and so it is is important to understand when that is the case. In Sections 8 and 9 we study the case where $V$ is a symmetric space, at least for the most important case when the character $\chi$ is zero. The precise result is given in Theorem 8.23 and depends upon a case-by-case analysis. But to summarise, if $V$ is an irreducible symmetric space and $\zeta = 0$, then $A_\kappa$ is simple if and only if either $V$ is nice or the parameter $\kappa$ takes only integer values. The symmetric spaces satisfying these properties are itemised in the tables in Appendix B.

It is easy to see that the kernel of $\text{rad}_\kappa$ contains $(\mathcal{D}(V)^G)^G$. Much harder than surjectivity is to decide when the kernel of $\text{rad}_\kappa$ equals $(\mathcal{D}(V)^G)^G$. This is known for the adjoint representation of a reductive Lie algebra $G$ on $\mathfrak{g} = \text{Lie}(G)$ [LS2] and, more generally, for nice symmetric spaces [LS4, Theorem A]. In Theorem 9.10 we generalise this to prove:

Theorem 1.4. Let $V$ be a symmetric space and take $\zeta = 0$. If the algebra $A_\kappa = \text{Im}(\text{rad}_0)$ is simple, then $\ker(\text{rad}_0) = (\mathcal{D}(V)^G)^G$.

One significance of this theorem is that it has the following vanishing result as an easy consequence. In the special case when $V = \mathfrak{g}$ is the adjoint representation, this is a classic result of Harish-Chandra [HC1, Theorem 3].

Corollary 1.5. (Corollary 9.14.) Assume that the hypotheses of Theorem 1.4 are satisfied and pick a real form $(G_0, \mathfrak{p}_0)$. Let $T$ be a locally invariant eigendistribution on an open subset $U \subset \mathfrak{p}_0$ with support contained in $U \setminus U'$, where $U'$ is the set of regular semisimple elements in $U$. Then $T = 0$.

Applications. We stated earlier in the introduction that our motivation for constructing the radial parts map in this generality was so that we can peruse the many resulting applications. The applications of this work are indeed numerous, to the extent that we have left them to the companion paper [BNS]. The main application explored there is to use the radial parts map to the understanding of the Harish-Chandra $\mathcal{D}$-module $\mathcal{G}$ (whose distributional solutions are precisely the invariant eigendistributions) on $V$. For instance, under the assumption that $V$ is stable and $A_\kappa$ is simple, we show that:

1. the Harish-Chandra $\mathcal{D}$-module $\mathcal{G}$ is the minimal extension of its restriction to $V_{\text{reg}}$; and
2. $\mathcal{G}$ is semisimple if and only if the associated Hecke algebra $\mathcal{H}_{\mathfrak{q}(\zeta)}(W)$ is semisimple.
Moreover, we study the dependence of $\mathcal{G}$ on the character $\chi$ when $\mathcal{G}$ is semisimple. It is shown that under their natural labelling, the simple summands of $\mathcal{G}$ are permuted by the KZ-twist when we vary the parameter $\zeta$.

**Acknowledgements:** The third author was partially supported by NSF grants DMS-1502125 and DMS-1802094 and a Simons Foundation Fellowship. The fourth author is partially supported by a Leverhulme Emeritus Fellowship.

Part of this material is based upon work supported by the National Science Foundation under Grant DMS-1440140, while the third author was in residence at the Mathematical Sciences Research Institute in Berkeley, California during the Spring 2020 semester.

Finally Gwyn Bellamy, Thierry Levasseur and Toby Stafford would like to acknowledge the great debt they hold for Tom Nevins who tragically passed away while this paper was in preparation. He was a dear friend and a powerful mathematician who taught us so much about so many parts of mathematics.

2. **Spherical rational Cherednik algebras**

In this section we define and prove various basic facts, many of which are well-known, about rational Cherednik algebras and their spherical subalgebras. Throughout the paper, $\otimes$ will implicitly mean $\otimes_{\mathbb{C}}$.

First, if $V$ is an affine algebraic variety, we denote by $\mathbb{C}[V]$ the algebra of regular functions on $V$. The symmetric algebra of a vector space $U$ is denoted $\text{Sym} U$. Then, following [GGOR] or [EG], the Cherednik algebra is defined as follows.

**Definition 2.1.** Let $W$ be a complex reflection group with reflection representation $\mathfrak{h}$ and reflections $S \subset W$. To each $s \in S$ is associated a reflecting hyperplane $H = \ker(1 - s) \subset \mathfrak{h}$. Let $A$ denote the set of all reflecting hyperplanes and for each $H \in A$, let $W_H \subset W$ denote the point-wise stabiliser of $H$. This is a cyclic group of order $\ell_H$ for which we fix a generator $s_H$.

The elements
\[ e_{H,i} = \frac{1}{\ell_H} \sum_{s \in W_H} \det(s) s \]
form an idempotent basis of $\mathbb{C} W_H$ with $s e_{H,i} = \det(s) s^{-i} e_{H,i}$. Choose $\alpha_H \in \mathfrak{h}^*$ such that $H = \ker(\alpha_H)$ and $\alpha_H^\vee \in \mathfrak{h}$ is an eigenvector of $s_H$ with eigenvalue not equal to one. Then $(\alpha_H, \alpha_H) \neq 0$. For $H \in A$ and $0 \leq i \leq \ell_H - 1$, choose $\kappa_{wH,i} \in \mathbb{C}$ subject to $\kappa_{wH,i} = \kappa_{H,i}$ for all $w \in W$, $H$ and $i$. Finally, let $T(\mathfrak{h}^* \oplus \mathfrak{h})$ denote the free algebra on the vector space $\mathfrak{h}^* \oplus \mathfrak{h}$ with skew group algebra $T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W$.

Following [GGOR, (3.1)], the rational Cherednik algebra (with parameter $t = 1$) is defined to be the quotient algebra $H_\kappa = H_\kappa(W, \mathfrak{h})$ of $T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W$ by the relations $[x, x'] = [y, y'] = 0$ and
\begin{equation}
[y, x] = \langle y, x \rangle + \sum_{H \in A} \ell_H \frac{\langle \alpha_H^\vee, x \rangle \langle y, \alpha_H \rangle}{\langle \alpha_H, \alpha_H \rangle} \sum_{j=0}^{\ell_H-1} (\kappa_{H,j+1} - \kappa_{H,j}) e_{H,j},
\end{equation}
for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. This algebra was first introduced in [EG] and, as a vector space, $H_\kappa \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \text{Sym} \mathfrak{h}$, by [EG, Theorem 1.3].

**Remark 2.3.** We impose neither the condition $\sum_{j=0}^{\ell_H-1} \kappa_{H,j} = 0$ nor $\kappa_{H,0} = 0$. This will be important later when it comes to identifying the image of the radial parts map.

We always identify $\mathbb{C}[\mathfrak{h}^*]$ with $\text{Sym} \mathfrak{h}$ using the natural pairing, and regard $\mathfrak{h}^*$ as the natural space of generators for $\mathbb{C}[\mathfrak{h}]$. However, it will be more convenient to use $\text{Sym} \mathfrak{h}$ in place of $\mathbb{C}[\mathfrak{h}^*]$ and, similarly, $\text{Sym} V$ in place of $\mathbb{C}[V^*]$. Let $e =$
formally, the spherical subalgebra $A_\kappa(W)$ is the algebra $A_\kappa(W) = eH_\kappa(W)e$. For brevity, $A_\kappa(W)$ will often be written $A_\kappa$. Note that
\[
\mathbb{C}[\mathfrak{h}]W \cong e\mathbb{C}[\mathfrak{h}]e \subset A_\kappa \quad \text{and} \quad (\text{Sym } \mathfrak{h})W \cong e(\text{Sym } \mathfrak{h})e \subset A_\kappa.
\]
We note for later the following result.

**Lemma 2.4.** The algebra $A_\kappa$ is a free $(\mathbb{C}[\mathfrak{h}]W, (\text{Sym } \mathfrak{h})W)$-bimodule of finite rank equal to $|W|$.

**Proof.** Set $E = (\text{Sym } \mathfrak{h})W$. Since $(W, \mathfrak{h})$ is a complex reflection group, $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ is a free module of rank $|W|^2$ over the subring $\mathbb{C}[\mathfrak{h}]W \otimes \mathbb{C} E$. As $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$ is a graded direct summand of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ as a $(\mathbb{C}[\mathfrak{h}]W, E)$-bimodule, it follows that $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$ is itself a free $(\mathbb{C}[\mathfrak{h}]W, E)$-bimodule of finite rank. Since $A_\kappa$ has associated graded ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$, the ring $A_\kappa$ is also free of finite rank as a $(\mathbb{C}[\mathfrak{h}]W, E)$-bimodule. □

**The Dunkl embedding.** We begin by recalling some standard facts regarding complex reflection groups; for references, see [Br]. Recall that the point-wise stabiliser $W_H$ of $H \in A$ is a cyclic group of order $\ell_H$. The element $\prod_{H \in A} \alpha_H$ acts as the inverse determinant character $\det^{-1}: W \to \mathbb{C}^\times$; see [St, Theorem 2.3]. Moreover, the **discriminant** on $\mathfrak{h}$ is defined to be
\[
h = \prod_{H \in A} \alpha_H^\ell_H \in \mathbb{C}[\mathfrak{h}]W;
\]
this polynomial is $W$-invariant, and is reduced as an element of $\mathbb{C}[\mathfrak{h}]W$.

**Notation 2.6.** Let $\mathfrak{h}_{\text{reg}}$ denote the open subset of $\mathfrak{h}$ where $W$ acts freely. It is a consequence of Steinberg’s Theorem that $\mathfrak{h}_{\text{reg}}$ is precisely the non-vanishing locus of $h$. A vector $x \in \mathfrak{h}_{\text{reg}}$ is called **regular**.

Set
\[
h_H = \alpha_H^{-\ell_H} h.
\]
Since both $h$ and $\alpha_H^\ell_H$ are $W_H$-invariant, $h_H$ is a $W_H$-invariant. Let
\[
A_\kappa(W_H)_{h_H} = A_\kappa(W_H)[h_H^{-1}]
\]
denote the localisation of $A_\kappa(W_H)$ at the Ore set $\{h_H^k\}$.

As in [EG, Proposition 4.5], the **Dunkl embedding** $H_\kappa(W) \hookrightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W$ is given by $x \mapsto x$, $w \mapsto w$ and
\[
y \mapsto T_y := \partial_y + \sum_{H \in A} \frac{\langle y, \alpha_H \rangle}{\alpha_H} \sum_{i=0}^{\ell_H-1} \ell_H^{i} \kappa_{H,i} \chi_{H,i}.
\]
for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$, where $\partial_y$ is $\text{Der } \mathbb{C}[\mathfrak{h}]$ is the vector field defined by $y$. We remark that the Dunkl operator $T_y$ preserves $\mathbb{C}[\mathfrak{h}] \subset \mathbb{C}[\mathfrak{h}_{\text{reg}}]$ only if $\kappa_{H,0} = 0$ for all $H \in A$. We do not impose this condition.

Using the Dunkl embedding, we will always regard $H_\kappa(W)$ and $H_\kappa(W_H)$ as subalgebras of $\mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W$. Since $eH_\kappa(W)e$ is realised as an algebra of $\text{ad}(\mathbb{C}[\mathfrak{h}_{\text{reg}}]W)$-nilpotent operators in $\text{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}_{\text{reg}}]W)$, it follows that $eH_\kappa(W)e$ acts on $\mathbb{C}[\mathfrak{h}_{\text{reg}}]W$ as differential operators. This defines a map
\[
\Phi: A_\kappa(W) = eH_\kappa(W)e \hookrightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W = \mathcal{D}(\mathfrak{h}_{\text{reg}})W \subset \mathcal{D}(\mathfrak{h}_{\text{reg}}).
\]
The map $\Phi_H: A_\kappa(W_H) \hookrightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})$ is defined similarly. Thus, we can and will consider $A_\kappa(W)$, $A_\kappa(W_H)$ and $A_\kappa(W_H)_{h_H}$ as subalgebras of $\mathcal{D}(\mathfrak{h}_{\text{reg}})$.

The following result is standard.
Lemma 2.9. (1) Under the Dunkl embedding, $H_\kappa(W)[h^{-1}] = \mathcal{D}(\mathfrak{b}_{reg}) \rtimes W$ and hence $A_\kappa(W)[h^{-1}] = \mathcal{D}(\mathfrak{b}_{reg})^W$.

(2) The canonical map $\mathcal{D}(\mathfrak{b}_{reg})^W \to \mathcal{D}(\mathfrak{b}_{reg}/W)$, given by restriction of operators, is an isomorphism of filtered algebras.

(3) The centre of $A_\kappa(W)$ reduces to $\mathbb{C}$.

Proof. (1) Since the $\alpha_H$ are units in $H_\kappa(W)[h^{-1}]$, the identity $H_\kappa(W)[h^{-1}] = \mathcal{D}(\mathfrak{b}_{reg}) \rtimes W$ follows from (2.8). Thus

$$A_\kappa[h^{-1}] = e H_\kappa(W)[h^{-1}] e = e \mathcal{D}(\mathfrak{b}_{reg} \rtimes W)e = \mathcal{D}(\mathfrak{b}_{reg})^W.$$

(2) Since $W$ acts freely on $\mathfrak{b}_{reg}$, the quotient map $\mathfrak{b}_{reg} \to \mathfrak{b}_{reg}/W$ is étale. Therefore, [Le1, Corollaire 4] says that, for each $\ell \geq 0$, there is an equivariant isomorphism $\mathcal{D}_\ell(\mathfrak{b}_{reg}) \to C[\mathfrak{b}_{reg}] \otimes_{C[\mathfrak{b}_{reg}]} \mathcal{D}_\ell(\mathfrak{b}_{reg}/W)$. Taking $W$-invariants gives $\mathcal{D}_\ell(\mathfrak{b}_{reg})^W \to \mathcal{D}_\ell(\mathfrak{b}_{reg}/W)$.

(3) By (1), $A_\kappa(W)[h^{-1}] = \mathcal{D}(\mathfrak{b}_{reg})^W$ is a simple ring with centre $\mathbb{C}$. Thus the centre of any spherical algebra $A_\kappa(W)$ also reduces to $\mathbb{C}$. \qed

Corollary 2.10. Let $W = W_1 \times W_2$ be the product of two complex reflection groups and $A_\kappa(W_i), i = 1, 2$, be spherical algebras attached to $W_i$. Then, if $\kappa = \kappa_1 \times \kappa_2$, the spherical algebra $A_\kappa(W) = A_{\kappa_1}(W_1) \otimes A_{\kappa_2}(W_2)$ is simple if and only if $A_{\kappa_1}(W_1)$ and $A_{\kappa_2}(W_2)$ are both simple.

Proof. By Lemma 2.9(3), the centre of a spherical algebra reduces to $\mathbb{C}$, the result is therefore consequence of [MR, Lemma 9.6.9]. \qed

Example 2.11. In the simplest case, $\mathfrak{h} = \mathbb{C}\{y\}$ and $W = \mathbb{Z}/\ell\mathbb{Z} = \langle s \rangle$, where $s(y) = \omega y$, for $\omega$ a fixed primitive $\ell$th root of unity. If $x \in \mathfrak{h}^*$ is such that $\langle y, x \rangle = 1$, the Dunkl embedding (2.8) sends $y$ to

$$T_y = \frac{\partial}{\partial x} + \frac{\ell}{x} \sum_{i=0}^{\ell-1} \kappa_i e_i$$

where the idempotent $e_i \in \mathbb{C}W$ satisfies $se_i = \omega^{-i}e_i$. The defining relation is

$$[y, x] = 1 + \ell \sum_{i=0}^{\ell-1} (\kappa_{i+1} - \kappa_i)e_i,$$

where the subscripts are interpreted modulo $\ell$; thus $\kappa_0 = 0$. In this case, $A_\kappa(W)$ is the algebra generated by $X = ex$, $Y = ey$, $Z = e(xy - \ell \kappa_0)$, together with $\Phi(X) = z = x^\ell$, $\Phi(Z) = \ell z \partial_z = x \partial_x$ and

$$\Phi(Y) = \ell \prod_{i=0}^{\ell-1} \left( z \partial_z + \kappa_i + \frac{i}{\ell} + (\delta_{i,0} - 1) \right) = x^{-\ell} \prod_{i=0}^{\ell-1} \left( x \partial_x + \ell \kappa_i + i + \ell (\delta_{i,0} - 1) \right),$$

where $\delta_{i,0} = 1$ if $i = 0$ and 0 otherwise.

Notation 2.12. Given a differential operator $D \in \mathcal{D}(X)$, for some smooth variety $X$, we let $\text{ord}_X(D)$ denote the order of $D$ as a differential operator, and write $\text{ord} D = \text{ord}_X D$ if it is unambiguous. Set $\mathcal{D}_\ell(X) = \{ d \in \mathcal{D}(X) : \text{ord} d \leq \ell \}$ and

$$\text{gr} \mathcal{D}(X) = \text{gr}_{\text{ord}} \mathcal{D}(X) = \bigoplus_{\ell} \mathcal{D}_\ell(X)/\mathcal{D}_{\ell-1}(X).$$

Given $d \in \mathcal{D}\ell(X) \setminus \mathcal{D}_{\ell-1}(X)$, let $\sigma(d) = [p + \mathcal{D}_{\ell-1}] \in \mathcal{D}_\ell(X)/\mathcal{D}_{\ell-1}(X)$ denote the symbol of $d$. 


Remark 2.13. The algebra $H_\kappa(W)$ has an order filtration given by putting $\mathbb{C}[h] \rtimes W$ in degree zero and $h \subseteq \text{Sym } h$ in degree one. As noted in [EG, p. 281] this agrees, via the Dunkl embedding, with the filtration induced from the order filtration on $\mathcal{D}(h_{reg}) \rtimes W$. Give $\mathcal{D}(h_{reg})^W$ the filtration induced from the order filtration on $\mathcal{D}(h_{reg})$. Then the filtration induced on $A_{\kappa}(W)$ as a subalgebra of $H_{\kappa}(W)$ (again called the order filtration) agrees with the filtration on $A_{\kappa}(W)$ considered as a subalgebra of $\mathcal{D}(h_{reg})^W$ via $\Phi$; see [EG, page 282]. The PBW Theorem for $H_{\kappa}(W)$, as described in [EG, Theorem 1.3], then identifies their associate graded rings:

$$\text{gr } H_{\kappa}(W) = (\mathbb{C}[h] \otimes \text{Sym } h) \rtimes W \quad \text{and} \quad \text{gr } A_{\kappa}(W) = (\mathbb{C}[h] \otimes \text{Sym } h)^W.$$ 

A localisation of $H_{\kappa}$. Fix a reflecting hyperplane $H \in A$ and recall that $W_H$ denotes the pointwise stabiliser of $H$. In order to define the radial parts map in the next section, we will use Luna slices to reduce the problem to the spherical algebra $A_{\kappa}(W_H)$ for suitably defined parameters. For this to work we need to prove that, after localising at a suitable Ore set $\mathfrak{C}$, we have $A_{\kappa}(W) \subseteq A_{\kappa}(W_H) \mathfrak{C}$; see Proposition 2.18 for the precise result. The proof of this result follows the ideas of [BE] and we begin with the appropriate notation.

Let $\mathcal{C}W_H \subseteq R \subseteq S$ be $\mathbb{C}$-algebras. As in [BE, Section 3.2], let $F(S) := \text{Fun}_{W_H}(W,S)$ denote the space of all $W_H$-equivariant maps $W \to S$; these are the maps $f$ with $f(uw) = uf(w)$ for all $u \in W_H, w \in W$. Set $E(S) = \text{End}_F(F(S))$. Fixing a set of left coset representatives of $W_H$ in $W$ shows that $F(S)$ is a free right $S$-module and hence identifies $E(S)$ with the matrix ring $\text{Mat}_{(W/W_H)}(S)$.

We denote by $e$ and $e_D$ the trivial idempotents in $\mathcal{C}W_H$ and $\mathcal{C}W_H$, respectively. Let $f_0 \in F(S)$ be given by $f_0(w) = e_0$ for all $w \in W$. We define $\Psi : \mathcal{C}W \to E(S)$ by $(\Psi(u)(f))(w) = fu(w)$ for $u, w \in W$. It is an embedding, and using $\Phi$ we think of $e$ as an element of $E(CW_H) \subseteq E(S)$; explicitly $e(f) = \frac{1}{|W_H|} \sum_{w \in W} u \cdot f$ for $f \in F(S)$.

It follows that $e(f_0) = f_0$.

Note that $F(\mathcal{C}W_H) \subseteq F(R) \subseteq F(S)$ and $F(S) = F(R) \otimes_R S$. Set

$$E(R,S) = \{D \in E(S) \mid D(F(\mathcal{C}W_H)) \subset F(R)\},$$

and define $\phi : E(R,S) \to E(R)$ by $\phi(D)(f \otimes r)(w) = D(f)(w)r$, for $f \otimes r \in F(\mathcal{C}W_H) \otimes_{W_H} R = F(R)$.

The basic properties of these objects is given by the following result.

Lemma 2.14. Keep the above notation.

1. The morphism $\phi$ is an algebra isomorphism.
2. There is an algebra isomorphism $\varphi : eE(S)e \cong e_0Se_0$, which is defined in the proof. This morphism satisfies $(eD)e(eDe) = f_0\varphi(eDe)$ for $D \in E(S)$.
3. Moreover $\varphi(eE(R,S)e) = e_0Re_0$.

Proof. (1) After fixing a set of left coset representatives $\{g_j : 1 \leq j \leq \ell\}$ of $W_H$ in $W$, a direct computation shows that $\phi$ is indeed an isomorphism.

(2) This is stated in [BE, Lemma 3.1(ii)] under slightly different hypotheses and so we will just outline the argument.

Write $ef$ in place of $e(f)$ for $f \in F(S)$. Expanding $(ef)(x)$ for $f \in F(S)$ and $x \in W$ shows that

$$(ef)(x) = e_0g(x) = f_0(x)\alpha(x) \quad \text{where} \quad \alpha(x) = \frac{|W_H|}{|W|} \sum g(x) \in S.$$ 

Hence $ef = f_0\alpha(x)$. Now $f_0 = f_0e_0$. Consequently, if $D \in E(S)$, then

$$(eDe)(f_0) = e(D(f_0)) = e(D(f_0)) = f_0\alpha(D(f_0)) = f_0e_0\alpha(D(f_0))e_0.$$ 

Hence if we define $\varphi(eDe) = e_0(\alpha(D(f_0)))e_0$, for $D \in E(S)$, then $(eDe)(f_0) = f_0\varphi(eDe)$, as claimed.
The fact that \( \varphi \) is an isomorphism of sets follows from Part (3) for the special case \( R = S \), in which case \( E(R, S) = E(S) \). Moreover, since \( \varphi \) is actually defined as an equality of subsets of \( E(S) \), this also implies that it is an isomorphism of algebras.

(3) Now let \( D \in E(R, S) \). By definition, \( \alpha(D(f_0)) = \frac{|W_\mathfrak{h}|}{|W_\mathfrak{e}|} \sum_i D(f_0)(g_i) \) with \( D(f_0) \in F(R) \) from which it follows that \( D(f_0)(g_i) \in R \) for each \( g_i \). Thus \( \alpha(D(f_0)) \in R \) and therefore \( \varphi(e_{D(e)}) \in e_0R_0 \).

Conversely, let \( a \in e_0R_0 \). If \( f \in F(S) \), we can define \( D'(f) \in F(R) \) by \( D'(f)(w) = a f(w) \). A simple calculation shows that \( D' \in E(R, S) \) with \( \varphi(eD'e) = a \), as required.

\[ \]

Lemma 2.15. Set \( F = F(\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H) \) and \( E = E(\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H) \). For \( f \in F, u, w \in W \) and \( D \in \mathcal{D}(\mathfrak{h}_{\text{reg}}) \), the rules

\[ \Psi(D(\otimes 1)(f))(w) = w(D)f(w), \quad \Psi(1 \otimes u)(f)(w) = f(wu) \]

define an injective algebra map \( \Psi: \mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W \to E \). Moreover \( \Psi \) induces a commutative diagram

\[ \begin{array}{c}
\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W & \xrightarrow{\Psi} & E \\
\downarrow j & & \downarrow j_0 \ \\
\mathcal{D}(\mathfrak{h}_{\text{reg}})W & \xrightarrow{\Psi_0} & \mathcal{D}(\mathfrak{h}_{\text{reg}})W_H,
\end{array} \]

where \( j(D) = e_0De \) and \( j_0(D) = e_0De_0 \).

Proof. The fact that \( \Psi \) is a well-defined ring homomorphism is a routine computation that is given in Appendix C. By [Mo, Theorem 2.5 and Corollary 2.6], the maps \( j \) and \( j_0 \) are isomorphisms. The same results show that the ring \( e(\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W)e \) is simple and so the nonzero morphism \( \Psi \) is injective. The isomorphism \( \varphi \) is given by Lemma 2.14(2) and then the fact that the diagram commutes is another direct computation, which is again given in the appendix.

For \( y \in \mathfrak{h} \), we temporarily write \( T^W_y \) for the Dunkl embedding of \( y \in H_e(W) \) in \( \mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W \) defined by (2.8). We denote by the same letter \( \kappa \) the restriction of \( \kappa \) to \( W_H \) and set \( A_\kappa(W_H) = e_0H_e(W_H)e_0 \), for the corresponding Cherednik algebra \( H_\kappa(W_H) \). Let \( T^W_{yH} \) denote the Dunkl embedding of \( y \in \mathfrak{h} \subset H_\kappa(W_H) \) in \( \mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H \); thus (2.8) simplifies to give:

\[ T^W_{yH} = \partial_y + \left( \frac{y, \alpha_H}{\alpha_H} \right) \sum_{i=0}^{\ell_H - 1} \ell_H \kappa_{H,i} e_{H,i}. \]

Recall from (2.7) that \( h_H = \alpha_H^{-\ell_H} h \in h_H \subset \mathbb{C}[h]^W_H \subset A_\kappa(W_H) \). This implies that \( h_H \) acts locally ad-nilpotently on \( A_\kappa(W_H) \) and we can therefore form the localisation \( A_\kappa(W_H)_{h_H} = A_\kappa(W_H)[h_H^{-1}] \) at the associated Ore set \( \mathcal{C} = \{ h_H \} \). Write

\[ E(H, \mathcal{D}) := E(H_\kappa(W_H)_{h_H}, \mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H), \]

which is a subalgebra of \( E = E(\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H) \).

Lemma 2.16. \( \Psi(H_\kappa(W_H)) \subset E(H, \mathcal{D}) \) and \( \varphi(eE(H, \mathcal{D})e) = j_0(A_\kappa(W_H)_{h_H}). \)

Proof. This is similar to [BE, Theorem 3.2]. The key to the proof is to show that, for \( y \in \mathfrak{h} \) and \( w \in W \), one has

\[ (2.17) \quad \langle \Psi(T^W_y f)(w) \rangle = \langle T^W_{w(y)} f(w) \rangle + \sum_{H' \in A \atop H' \neq H} \frac{\langle w(y), \alpha_H \rangle}{\alpha_H} \sum_{i=0}^{\ell_H - 1} \ell_{H' \kappa_{H',i}} f(e_{H',i}w), \]
where, with a slight abuse of notation, we write
\[ f(e_{H'}, w) := \frac{1}{\ell_{H'}} \sum_{s \in W_{H'}} \det_y(s)^i f(sw). \]

The remark after [BE, Theorem 3.2] argues that one can deduce their analogue of (2.17) from general results about sheaves of Cherednik algebras, and this allows one to guess the formula in question. Similar arguments apply to (2.17), but since the resulting computations are still intricate, we prefer to give a detailed proof. This is given in Appendix C.

Once (2.17) has been established, the proof of the lemma is easy. Indeed, note that every denominator \( \alpha_{H'} \) appearing on the right hand side of (2.17) is a factor of \( h_H \). Therefore, \( \Psi(T_y^W) \) belongs to the subalgebra \( E(H, \mathcal{D}) \) of \( E(\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H) \).

Since \( H_\alpha(W) \) is generated by \( \mathbb{C}[\mathfrak{h}] W \) and the Dunkl operators \( T_y^W \), it follows that \( \Psi(H_\alpha(W)) \subseteq E(H, \mathcal{D}) \).

Finally, by Lemma 2.14(3),
\[ \varphi(eE(H, \mathcal{D})e) = e_0 H_\alpha(W_H) h_H e_0 = j_0(A_\alpha(W_H) h_H) \]
inside \( e_0(\mathcal{D}(\mathfrak{h}_{\text{reg}}) \times W_H)e_0 \).

We can now give the inclusion of algebras mentioned at the beginning of the subsection.

**Proposition 2.18.** Let \( H \in A \). Under the embedding \( \mathcal{D}(\mathfrak{h}_{\text{reg}})^W \hookrightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})^{W_H} \), the image of \( A_\alpha(W) \) is contained in \( A_\alpha(W_H) h_H \).

**Proof.** First note that, by Lemma 2.16,
\[ (e\Psi)(j(A_\alpha(W))) = (e\Psi)(eH_\alpha(W)e) \subseteq eE(H, \mathcal{D})e = \varphi^{-1}(j_0(A_\alpha(W_H) h_H)). \]

By Lemma 2.15, this pulls back to the desired inclusion \( A_\alpha(W) \subseteq A_\alpha(W_H) h_H \). □

**Notation 2.19.** Let \( \mathfrak{h}^0 = \mathfrak{h} \setminus \bigcup_{H_\alpha \neq H_\beta \in A} H_\alpha \cap H_\beta \); equivalently, \( \mathfrak{h}^0 \) is the set of points of \( \mathfrak{h} \) that lie on at most one hyperplane. The complement to \( \mathfrak{h}^0 \) in \( \mathfrak{h} \) has codimension two.

**Theorem 2.20.** In \( \mathcal{D}(\mathfrak{h}_{\text{reg}})^W \),
\[ A_\alpha(W) = \bigcap_{H \in A} A_\alpha(W_H) h_H. \]

**Proof.** By Proposition 2.18,
\[ A_\alpha(W) \subseteq \bigcap_{H \in A} A_\alpha(W_H) h_H, \]
as subalgebras of \( \mathcal{D}(\mathfrak{h}_{\text{reg}}) \). By Remark 2.13 and Lemma 2.9, the order filtrations on both \( A_\alpha(W) \) and \( A_\alpha(W_H) h_H \) are given by restriction of the order filtration on \( \mathcal{D}(\mathfrak{h}_{\text{reg}}) \). Thus the inclusion of \( A_\alpha(W) \hookrightarrow \bigcap_{H \in A} A_\alpha(W_H) h_H \) is filtered and so it suffices to show that the associated graded map is an equality.

The associated graded map is
\[ \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_W \hookrightarrow \bigcap_{H \in A} \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_{h_H}^W. \]

Let \( f \) be an element in the right hand side. Then \( f \in X := \bigcap_{H \in A} \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_{h_H} \) and so \( f \) is regular on \( \mathfrak{h}^0 \times \mathfrak{h}^* \). Since the complement to \( \mathfrak{h}^0 \times \mathfrak{h}^* \) has codimension two in \( \mathfrak{h} \times \mathfrak{h}^* \), it follows that \( X = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \). Moreover, \( f \in \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_{h_H}^W \) for each \( H \in A \) and \( h_H \) is a \( W_H \)-invariant. Thus, \( f \in X^W_H = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_{h_H}^W \). Since the group \( W \) is generated by all the \( W_H \), it follows that \( f \in \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_W \). □
We note an interesting consequence of Theorem 2.20. First, we recall that to the symplectic quotient singularity \((\mathfrak{h} \times \mathfrak{h}^*)/W\) one can associate Namikawa’s Weyl group \(\Gamma\), which plays an important role in the Poisson deformation theory of the singularity. In this particular case, it is shown in [BST, Lemma 4.1] that \(\Gamma = \prod_{[H] \in A/W} \mathfrak{S}_H\) is a product of symmetric groups acting on the permutation representation with basis \(\{\kappa_H,i : [H] \in A/W, 0 \leq i \leq \ell_H - 1\}\). We think of \(\kappa_H,i\) as a variable and let \(\mathbb{C}[\kappa]\) be the polynomial ring in the variables \(\kappa_H,i\). Then \(\Gamma\) acts on \(\mathbb{C}[\kappa]\). We define a twisted action of \(\Gamma\) on \(\mathbb{C}[\kappa]\) by setting

\[
\sigma \star \kappa_{H,i} = \kappa_{H,\sigma(i)} + \frac{\sigma(i) - i}{\ell} + (\delta_{\sigma(i),0} - \delta_{i,0})
\]

Consider the generic spherical algebra \(A_\kappa \subset \mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa]\); it is a \(\mathbb{C}[\kappa]\)-algebra. Make \(\Gamma\) act on the algebra \(\mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa]\) by twisted action \(\star\) on the coefficients \(\mathbb{C}[\kappa]\) and trivially on \(\mathcal{D}(\mathfrak{h}_{\text{reg}})^W\). The following should be viewed as the analogue of the Harish-Chandra isomorphism (of the centre of the enveloping algebra of a reductive Lie algebra) for spherical Cherednik algebras.

**Corollary 2.21.** The subalgebra \(A_\kappa\) is stable under \(\Gamma\) and the centre of \(A_\kappa^\Gamma\) equals \(\mathbb{C}[\kappa]^{(\Gamma,\star)}\).

**Proof.** The proof of Theorem 2.20 goes through verbatim with the complex parameters \(\kappa\) replaced by the variables \(\kappa\). Fix a hyperplane \(H \in A\) and think of \(A_\kappa(W_H)\) as a subalgebra of \(\mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa]\) as usual. The key point about the twisted action is that it follows from Example 2.11 that

\[
\Phi(Y) = \ell \prod_{i=0}^{\ell-1} \left( z \partial_z + \kappa_{H,i} + \frac{i}{\ell} + (\delta_{i,0} - 1) \right)
\]

and \(\Phi(Z) = \ell z \partial_z\) belong to \(A_\kappa(W_H) \cap (\mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa])^\Gamma\). That is, \(\sigma \star \Phi(Y) = \Phi(Y)\) for all \(\sigma \in \Gamma\). Since \(A_\kappa(W_H)\) is generated as a \(\mathbb{C}[\kappa]\)-algebra by \(\mathbb{C}[\mathfrak{h}]^{W_H}\), \(\Phi(Z)\) and \(\Phi(Y)\), it follows that \(A_\kappa(W_H)\) is generated by \(A_\kappa(W_H) \cap (\mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa])^\Gamma\), and \(\mathbb{C}[\kappa]\) as a \(\mathbb{C}\)-subalgebra of \(\mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa]\). This implies that both \(A_\kappa(W_H)\) and \(A_\kappa(W_H)_{\kappa H}\) are stable under \(\Gamma\). It follows from Theorem 2.20 that \(A_\kappa\) is also stable under \(\Gamma\).

Finally, we note that the centre of \((\mathcal{D}(\mathfrak{h}_{\text{reg}})^W[\kappa])^\Gamma\) equals \(\mathbb{C}[\kappa]^{(\Gamma,\star)}\) and the former is the localization of \(A_\kappa^\Gamma\) with respect to the powers of \(\delta\). It follows that the centre of \(A_\kappa^\Gamma\) equals \(\mathbb{C}[\kappa]^{(\Gamma,\star)}\) too.

Returning to complex valued parameters, we see that:

**Corollary 2.22.** For each \(\sigma \in \Gamma\), \(A_{\sigma \kappa}(W) = A_\kappa(W)\) as subalgebras of \(\mathcal{D}(\mathfrak{h}_{\text{reg}})^W\).

**Proof.** This is similar to the first part of the proof of Corollary 2.21.

Corollary 2.22 is a generalization of [BC, Proposition 5.4], since the group \(G\) of that proposition can be realised as a subgroup of \(\Gamma\), with the same twisted action on parameters.

3. Background on Polar Representations

In this section we give the basic definitions and results for polar representations, for which our main reference is [DK].

We begin with the relevant notation. Fix a reductive group \(G\) with a finite dimensional representation \(V\). Let \(\pi : V \to V/G\) be the categorical quotient; that is, the morphism defined by the inclusion of algebras \(\mathbb{C}[V]^G \to \mathbb{C}[V]\). An element \(v \in V\) is called semisimple if the orbit \(G \cdot v\) is closed. In this case \(G \cdot v\) is affine and the stabiliser \(G_v\) is a reductive subgroup of \(G\). A semisimple element \(v\) is \(V\)-regular if \(\dim G \cdot v\) is maximal amongst orbits of semisimple elements. The set of semisimple
elements, respectively $V$-regular elements in $V$ is denoted by $V_{ss}$, respectively $V_r$. Set
\begin{equation}
(3.1) \quad m = \max \{ \dim G \cdot v : v \in V \} \quad \text{and} \quad s = \max \{ \dim G \cdot x : x \in V_{ss} \}.
\end{equation}
Thus $s \leq m \leq \dim V - \dim V//G$.

Pick a $V$-regular element $v$ and define $h_v = \{ x \in V \mid g \cdot x \subseteq g \cdot v \}$. Then [DK, Lemma 2.1 and Proposition 2.2] say that $h_v$ consists of semisimple elements and the map $\pi : h_v \to V//G$ is finite. In particular, $\dim h_v \leq \dim V//G$. If $\dim h_v = \dim V//G$ for some such $v$, then $V$ is called polar and $h_v$ is a Cartan subspace of $V$. If $G^s$ is the connected component of the identity, the representation $(G, V)$ is polar if and only if $(G^s, V)$ is polar, see [DK, Remark, p. 510].

We will assume for the rest of the paper that the group $G$ is connected.

Assume that $V$ is polar, and fix a Cartan subspace $h$ of $V$. Let
\begin{equation}
(3.2) \quad N_G(h) = \{ g \in G \mid g \cdot h = h \} \quad \text{and} \quad Z_G(h) = \{ g \in G \mid g \cdot x = x, \forall x \in h \}.
\end{equation}
Then $W = N_G(h)/Z_G(h)$ is a finite group by [DK, Lemma 2.7], called the Weyl group of $V$. By [DK, Theorem 2.10], $W$ is a complex reflection group and the map $h \to V//G$ factors to give an isomorphism $h/W \xrightarrow{\sim} V//G$ [DK, Theorem 2.9]. Write $g : \mathbb{C}[V] \to \mathbb{C}[h]$ for restriction of functions; thus $g$ induces a graded isomorphism $\mathbb{C}[V]^G \xrightarrow{\sim} \mathbb{C}[h]^W$, which will again be written $g$.

As in [DK, page 515], the rank of the polar representation $(G, V)$ is defined to be
\begin{equation}
(3.3) \quad \text{rank}(G, V) = \dim h - \dim h^g.
\end{equation}
Recall from Notation 2.6 that $x \in h$ is regular if the stabiliser of $x$ in $W$ is trivial and that the set regular elements in $h$ is denoted $h_{\text{reg}}$.

**Lemma 3.4.** If $v \in V$ is $V$-regular and $h = h_v$, then $G_v \subseteq N_G(h)$. Hence, if $u \in h$ is regular then $h = h_u$ and $g \cdot h = g \cdot u = g \cdot v$ with $G_u = Z_G(h)$.

**Proof.** By definition, $h = h_v = \{ x \in V \mid g \cdot x \subseteq g \cdot v \}$. If $g \in G_v$ then $g \cdot (g \cdot v) = g \cdot v$ which implies that $g \cdot (g \cdot x) = g \cdot (g \cdot x) \subseteq g \cdot v$ for all $x \in h$ and hence $g \cdot x \in h$. Thus $G_v \subseteq N_G(h)$.

If $u \in h$ is regular then [DK, Theorem 2.12] implies that it is $V$-regular. In particular, $\dim G \cdot u = \dim G \cdot u$. Since $g \cdot u \subseteq g \cdot v$, it follows that $g \cdot u = g \cdot v$ and hence $h_u = h$. By the previous paragraph, $G_u \subseteq N_G(h)$. The image of $G_u$ in the quotient $W = N_G(h)/Z_G(h)$ is $W_u$. The latter is trivial by definition of regularity. Thus, $G_u \subseteq Z_G(h)$. Conversely, $Z_G(h) \subseteq G_x$ for all $x \in h$. \qed

**Remark 3.5.** An element $x \in h$ can be regular, or it can be $V$-regular when thought of as an element of $V$. It is conjectured that these concepts coincide; see [DK, p. 521].

**Notation 3.6.** Since $W$ is a complex reflection group, the open set $h_{\text{reg}}/W$ is principal; it is the non-vanishing locus of the discriminant $h \in \mathbb{C}[h]^W$ defined in (2.5). We define the regular locus $V_{reg}$ to be the preimage of $h_{\text{reg}}/W$ under the quotient map $\pi : V \to V//G \cong h/W$. Equivalently, $V_{\text{reg}} = \{ \delta \neq 0 \}$ where
\begin{equation}
(3.7) \quad \delta = g^{-1}(h) \in \mathbb{C}[V]^G
\end{equation}
is defined to be the discriminant of $(G, V)$.

Finally, we recall that there exists a Hermitian inner product $(-, -)$ on $V$, invariant under the action of a maximal compact subgroup of $G$, such that every vector in $h$ has minimal length in its orbit and $h$ is perpendicular to $g \cdot v$ for all $v \in h$. See [DK, Lemma 2.1 and Remark 1.4] for the details.
We can take slices in polar representations. Let $p \in \mathfrak{h}$ and recall that $p$ is semisimple. Define the slice $S_p$ at $p$ to be the orthogonal complement, with respect to $(\cdot,\cdot)$, in $V$ to $g \cdot p$. Then $G_p$ acts on $S_p$, and $(G_p, S_p)$ is again polar, with Cartan subspace $\mathfrak{h}$; see [DK, Lemma 2.1(ii) and Theorem 2.4] for the details.

As in [DK, Corollary 2.5], we can pick a $Z_G(\mathfrak{h})$-invariant complement $U$ to $\mathfrak{h} \oplus g \cdot \mathfrak{h}$ in $V$. Thus the decomposition
\begin{equation}
V = \mathfrak{h} \oplus g \cdot \mathfrak{h} \oplus U
\end{equation}
is $Z_G(\mathfrak{h})$-invariant and it is not difficult to see that this decomposition is $N_G(\mathfrak{h})$-invariant. Furthermore, since Lemma 3.4 implies that $G_u = Z_G(\mathfrak{h})$ for a regular element $u \in \mathfrak{h}$, the proof of [DK, Corollary 2.5] implies that $U//Z_G(\mathfrak{h}) = \{pt\}$.

**Lemma 3.9.** Let $p \in \mathfrak{h}$ with slice representation $(G_p, S_p)$.

1. One has $S_p = \mathfrak{h} \oplus g \cdot \mathfrak{h} \oplus U$.

2. $p$ is $V$-regular $\iff g_p = Z_G(\mathfrak{h}) \iff \text{rank}(G_p, S_p) = 0 \iff S_p = \mathfrak{h} \oplus U$.

**Proof.**

1. The inclusion $(\mathfrak{h} + g_p \cdot \mathfrak{h}) \oplus U \subset S_p$ follows from [DK, Eq. 1, p.511]. Pick $x \in V_\mathfrak{h}$ such that $\mathfrak{h} = h_x$ and hence $g \cdot \mathfrak{h} = g \cdot x$. Then, as in the proof of [DK, Theorem 2.4], $g \cdot x = g \cdot p \oplus g_p \cdot x$ and $g_p \cdot x = g_p \cdot h$. We then get:

\[
\dim \mathfrak{h} + \dim g \cdot p + \dim g_p \cdot h = \dim \mathfrak{h} + \dim g \cdot x = \dim \mathfrak{h} = (\dim V - \dim \mathfrak{h} - \dim U) = \dim V - \dim U.
\]

Since $V = g \cdot p \oplus S_p$ and thus $\dim S_p = \dim V - \dim g \cdot p$, it follows that $S_p = \mathfrak{h} \oplus g_p \cdot h \oplus U$.

2. Define $s$ by (3.1) and note that $s = \dim g - \dim Z_p(\mathfrak{h})$ by Lemma 3.4. We have $Z_p(\mathfrak{h}) \subseteq g_p$ for all $p \in \mathfrak{h}$. From $\dim G \cdot p = \dim g - \dim g_p$, we deduce:

\[
p \in V_s \iff \dim G \cdot p = s \iff \dim g_p = \dim Z_p(\mathfrak{h}) \iff g_p = Z_p(\mathfrak{h}) \subseteq g_p \subseteq Z_p(\mathfrak{h}).
\]

This gives the first equivalence.

By definition, $\text{rank}(G_p, S_p) = 0$ is equivalent to $\mathfrak{h} = h^p$. In other words, $\text{rank}(G_p, S_p) = 0 \iff g_p \subseteq Z_p(\mathfrak{h})$, giving the second equivalence.

Finally, from Part (1), $S_p = \mathfrak{h} \oplus g_p \cdot h \oplus U = \mathfrak{h} \oplus U \iff g_p \cdot h = 0$, which certainly implies that $g_p \subseteq Z_p(\mathfrak{h})$. Conversely, if $p \in V_s$, then $\mathfrak{h} = \mathfrak{h}^p$ and so $g_p \cdot h = 0$ follows from [DK, Lemma 2.1(iii)]. This gives the final equivalence. \hfill \Box

We end the section with some comments on the discriminant $\delta$. Factorise
\begin{equation}
\delta = \delta_1^{m_1} \cdots \delta_k^{m_k}
\end{equation}
in $C[V]$, where the $\delta_i$ are pairwise coprime irreducible homogeneous polynomials. Since we have assumed that $G$ is connected, the $\delta_i$ are semi-invariants, say of weight $\theta_i$, and the highest common factor of the $m_i$ is one. In general, one can have $m_i > 1$; this happens, for example, in [BNS, Example 15.1].

**Remark 3.11.** We assume in this remark that the group $G$ is semisimple. Then the space of linear characters $X^*(G)$ is trivial and any semi-invariant polynomial is $G$-invariant. It follows that the decomposition $\delta = \prod \delta_j^{m_j}$ coincides with the decomposition of $\delta$ into a product of irreducible polynomials in the polynomial algebra $C[V]^G$. Since $\theta : C[V]^G \rightarrow C[h]^W$ is an isomorphism, the factors $\delta_j$ are the $p^{-1}(h_j)$ where $\prod h_j^{m_j}$ is the decomposition of the discriminant $h = \prod_{H \in \mathcal{A}} \alpha_H^{m_H}$ as product of irreducible elements in $C[h]^W$. This decomposition depends on the number of $W$-orbits of hyperplanes in $\mathcal{A}$ (see [Br, Theorem 4.18]).

As an explicit example, suppose that the group $W$ is the Weyl group of an irreducible root system $R$. In this case, there is one orbit of hyperplanes in $\mathcal{A}$ if $R$
is simply laced and two orbits otherwise. Then, $\delta$ is irreducible when $R$ is simply laced. If $R$ is not simply laced, choose a set of positive roots $R_+$ and set

$$h_{\text{sh}} = \prod_{\alpha \in R_+, \alpha \text{ short}} \alpha^2, \quad \text{and} \quad h_{\text{lg}} = \prod_{\alpha \in R_+, \alpha \text{ long}} \alpha^2,$$

and write $\delta_{\text{sh}} = g^{-1}(h_{\text{sh}})$ and $\delta_{\text{lg}} = g^{-1}(h_{\text{lg}})$. Then $\delta = \delta_{\text{sh}}\delta_{\text{lg}} \in \mathbb{C}[V]$ where $\delta_{\text{sh}}$ and $\delta_{\text{lg}}$ are irreducible.

4. The radial parts map: the one-dimensional case

In this section, we define the radial parts map $\text{rad}_{\varsigma}$. We consider in detail the case where $\dim V//G = 1$ and show that the image of $\text{rad}_{\varsigma}$ is always a spherical subalgebra of a rational Cherednik algebra in this case.

The radial parts map. Fix a polar representation $V$ for a connected, reductive group $G$. As in (3.10), $\delta_1, \ldots, \delta_k$ denote the pairwise distinct irreducible factors of $\delta$ in $\mathbb{C}[V]$. The $\delta_i$ are homogeneous semi-invariant functions of weight $\theta_i$ that are invertible on $V_{\text{reg}}$.

For each $i$, choose $\varsigma_i \in \mathbb{C}$ and set

$$\chi = \sum_{i=1}^k \varsigma_i d\theta_i. \tag{4.1}$$

Recall that a weakly equivariant left $\mathcal{D}(V_{\text{reg}})$-module is a left $\mathcal{D}(V_{\text{reg}})$-module $M$ that is also a rational $G$-module such that the action $\mathcal{D}(V_{\text{reg}}) \otimes M \rightarrow M$ is $G$-equivariant. Since each $\delta_i$ is a $G$-semi-invariant, one can define a weakly $G$-equivariant $\mathcal{D}(V_{\text{reg}})$-module, denoted by $\mathbb{C}[V_{\text{reg}}]\delta_i$, as follows. As a $\mathbb{C}[V_{\text{reg}}]$-module it is free of rank one with basis $\delta_i := \delta_1^1 \cdots \delta_k^k$. The group $G$ acts trivially on the generator $\delta_i$ and

$$\partial \ast \delta = \sum_i \varsigma_i \frac{\partial(\delta_i)}{\delta_i} \delta,$$

for each derivation $\partial \in \mathcal{D}(V_{\text{reg}})$. The action of a general differential operator then follows from the fact that $\mathcal{D}(V_{\text{reg}})$ is generated by derivations and $\mathbb{C}[V_{\text{reg}}]$. We remark that, here and elsewhere, we often use $\ast$ to denote the action of a differential operator on functions to distinguish it from multiplication of operators. One can also regard $\mathbb{C}[V_{\text{reg}}]\delta_i$ as the rank one integrable connection defined by the logarithmic one-form $d \log \delta_i$.

Denote by $D \mapsto \delta^{-\varsigma_i} D \delta_i$ the conjugation by $\delta^{-\varsigma_i}$; that is, the automorphism of $\mathcal{D}(V_{\text{reg}})$ defined by:

$$\delta^{-\varsigma_i} f \delta_i = f \quad \text{if} \quad f \in \mathbb{C}[V_{\text{reg}}],$$

$$\delta^{-\varsigma_i} v \delta_i = v + \sum_{i=1}^k \varsigma_i \frac{v(\delta_i)}{\delta_i} \quad \text{if} \quad v \in \text{Der}(\mathbb{C}[V_{\text{reg}}]).$$

Then, $D \in \mathcal{D}(V_{\text{reg}})$ acts on $f \delta_i \in \mathbb{C}[V_{\text{reg}}]\delta_i$ by $f \delta_i \mapsto (\delta^{-\varsigma_i} D \delta_i)(f) \delta_i$. We also adopt the notation $(\delta^{-\varsigma_i} D \delta_i)(f) = \delta^{-\varsigma_i} D(f \delta_i)$. Recall from Notation 3.6 that $h = g(\delta)$ is the discriminant on $\mathfrak{h}$ and set $h_i := \delta_i |_{\mathfrak{h}}$ and $h^i := h_{\mathfrak{i}}^1 \cdots h_{\mathfrak{k}}^k$. The analogous definitions will be given for the rank one $\mathcal{D}(h_{\text{reg}})$-module $\mathbb{C}[h_{\text{reg}}]h^i$ and the automorphism $D \mapsto h^{-\varsigma_i} D h^i$ of $\mathcal{D}(h_{\text{reg}})$ given by conjugation by $h^{-\varsigma_i}$.

Recall that the morphism $g : \mathbb{C}[V] \rightarrow \mathbb{C}[\mathfrak{h}]$, given by restriction $f \mapsto f|_{\mathfrak{h}}$, induces graded isomorphisms $\mathbb{C}[V]^G \sim \mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[V_{\text{reg}}]^G \sim \mathbb{C}[h_{\text{reg}}]^W$, which are also denoted $g$.

Since the module $\mathbb{C}[V_{\text{reg}}]\delta_i$ is weakly $G$-equivariant, the algebra $\mathcal{D}(V)^G$ preserves the subspace $(\mathbb{C}[V_{\text{reg}}]\delta_i)^G = \mathbb{C}[V_{\text{reg}}]^G \delta_i$ and we can identify the latter, via $g$, with
the rank one free \( \mathbb{C}[h_{\text{reg}}/W] \)-module generated by \( h^\varsigma \). Now each \( z \in \mathbb{C}[h_{\text{reg}}/W] \cong \mathbb{C}[V_{\text{reg}}]^G \) acts ad-nilpotently on \( \mathcal{D}(V)^G \). Thus, combined with Lemma 2.9(2), this identification defines a map
\[
\text{rad}_c \colon \mathcal{D}(V)^G \longrightarrow \mathcal{D}(h_{\text{reg}}/W) = \mathcal{D}(h_{\text{reg}})^W,
\]
which is the radial parts map in this context. Explicitly,
\[
(4.3) \quad \text{rad}_c(D)(z) = (\delta^- \circ D(g^{-1}(z)\delta^+))|_{h_{\text{reg}}} \quad \text{for } z \in \mathbb{C}[h_{\text{reg}}]^W \text{ and } D \in \mathcal{D}(V)^G.
\]

**Remark 4.5.** Let \( \chi \) be as in (4.1) and set
\[
\mathfrak{g}_\chi = \{ \tau(x) - \chi(x) | x \in \mathfrak{g} \} \subset \mathcal{D}(V).
\]
Then \( \mathfrak{g}_\chi \) annihilates \( \mathbb{C}[V_{\text{reg}}]^G \delta^\varsigma \) and hence the two-sided ideal \( (\mathcal{D}(V)\mathfrak{g}_\chi)^G \) of \( \mathcal{D}(V)^G \) is contained in the kernel of \( \text{rad}_c \).

**The one-dimensional case.** In this subsection \( K \) will be a reductive group, which need not be connected, and \( S \) will denote a finite-dimensional, faithful \( K \)-module such that \( \dim S/K = 1 \). Much of this subsection parallels [Le3], although that paper is not directly applicable since [Le3] only considers the case where \( \varsigma = 0 \) and \( K \) connected.

**Lemma 4.6.** The pair \((K, S)\) is a polar representation with \( \mathbb{C}[S]^K = \mathbb{C}[u] \) for some homogeneous polynomial \( u \). The rank of \((K, S)\) is at most one.

**Remark 4.7.** This choice of polynomial \( u \) will be fixed throughout the subsection.

**Proof.** Since \( \dim S/K > 0 \), there exists some non-zero semisimple element in \( S \). Therefore, there exists a non-zero \( V \)-regular element \( v \in S \). Set \( \mathfrak{h} = \mathbb{C}v \). Then it follows directly from the definition that \( \mathfrak{h} \) is a Cartan subspace of \( V \) and hence that \((G, V)\) is polar. The rest of the lemma follows from [DK, Theorem 2.9]. \( \square \)

Let \( \ell = \deg u \) and write \( \mathbb{C}[\mathfrak{h}] = \mathbb{C}[x] \). Then \( z := u|_{\mathfrak{h}} \) is a homogeneous polynomial of degree \( \ell \) and so we may assume that \( z = x^\ell \). This implies that the Weyl group of \((K, S)\) is \( W = \mathbb{Z}/\ell \mathbb{Z} \). The following result is presumably well-known, but we do not know of a suitable reference.

**Lemma 4.8.** Let \( L \) be an affine algebraic group acting linearly on a vector space \( U \). Then the ring \( \mathbb{C}[U]^{[L,L]} \) is a unique factorisation domain.

**Proof.** Assume that \( \mathbb{C}[U]^{[L,L]} \) is not a UFD. Then we can find \( u \in \mathbb{C}[U]^{[L,L]} \) having two distinct factorisations; say
\[
u = u_{p_1} \cdots u_{p_s} = v_{r_1} \cdots v_{r_t}.
\]
Moreover, we may assume that \( \sum_i p_i \) is minimal. This means that, up to scalars, \( u_i \neq v_j \) for all \( i, j \). Let \( f_1 \) be an irreducible factor of \( v_\alpha \) in \( \mathbb{C}[U] \). Then \( f_1 \) divides some \( u_i \), say \( f_1 | u_\tau \). By the proof of [Kr, II 3.3, Satz 2], \( f_1 \) is a \( L^0 \)-semi-invariant. The orbit \( \{[f_1], \ldots, [f_m]\} \) of \([f_1]\) under \( L/L^0 \) in \( \mathcal{P}(\mathbb{C}[U]) \) is finite. In other words, if \( l \in L \) then \( l \cdot f_1 = cf_j \) for some \( c \in \mathbb{C}^\times \) and some \( j \). This implies that \( f_1 \cdots f_m \) is an \( L \)-semi-invariant dividing \( v_\alpha \). Since \( v_\alpha \) is irreducible in \( \mathbb{C}[U]^{[L,L]} \), we have \( v_\alpha = \alpha f_1 \cdots f_m \) for some \( \alpha \in \mathbb{C}^\times \). By the same argument \( u_\tau = \beta f_1 \cdots f_m \). But this means we can cancel one copy of \( v_\alpha = u_\tau \) in the factorisation of \( u \), contradicting the minimality of \( \sum_i p_i \). \( \square \)

We say that a polynomial \( f \in \mathbb{C}[S] \) is **equivariantly irreducible** if \( f \) is a \( K \)-semi-invariant and no proper factor of \( f \) is a semi-invariant. Lemma 4.8 says that every \( K \)-semi-invariant admits a unique factorisation into equivariantly irreducible polynomials, up to scalar and permutation of factors. Factorise
\[
u = u_{p_1} \cdots u_{p_s}.
\]
into equivariantly irreducible polynomials.

As in [Le3], fix a unitary inner product \((-,-)\) on \(S\) and let \(\{s_1, \ldots, s_d\}\) be an orthonormal basis of \(S^*\) with dual basis \(\{\partial_1, \ldots, \partial_d\}\) for \(S\). The form \((-,-)\) defines a sesquilinear isomorphism \(\mathbb{C}[S] \xrightarrow{\sim} \text{Sym} \ S\) given explicitly by

\[
\sum_i a_i s^i \mapsto \sum_i p_i \partial^i.
\]

If \(f \in \mathbb{C}[S]\), write \(f_\lambda(D) \in \text{Sym} \ S\) for the image of \(f\). As shown in [Ki, Proposition 2.21], if \(f \in \mathbb{C}[S]\) is a \(\theta\)-semi-invariant then \(f_\lambda(D)\) is a \(\theta\)-semi-invariant (this part of the proof of [Ki, Proposition 2.21] does not require that \(S\) is a prehomogeneous vector space). This implies that \((\text{Sym} \ S)^G = \mathbb{C}[\Delta]\), where \(\Delta := u_\ast(D)\).

In order to incorporate the twist \(\zeta\) into our picture, we need to introduce a multivariable version of the \(b\)-function for \(S\). We can consider a version of the module \(\mathbb{C}[S_{\text{reg}}]u^\alpha\) defined over a polynomial ring \(\mathbb{C}[\alpha] = \mathbb{C}[\alpha_1, \ldots, \alpha_r]\) where

\[
\partial * u^\alpha = \sum_{i=1}^r \alpha_i \frac{\partial(u_i)}{u_i} u^\alpha \quad \text{for every derivation } \partial \in \text{Der} \mathbb{C}[S].
\]

**Proposition 4.10.** There exists a nonzero polynomial \(b(\alpha) = b(\alpha_1, \ldots, \alpha_r) \in \mathbb{C}[\alpha]\) of total degree \(\ell\) such that

\[
(4.11) \quad \Delta u^{\alpha+1} = b(\alpha) u^\alpha.
\]

**Proof.** We define \(u^\alpha\) and each of the \(\alpha_i\) to be \(K\)-invariant of degree zero. Then the \(\mathcal{D}(S)[\alpha]\)-module \(\mathbb{C}[S_{\text{reg}}][\alpha] u^\alpha\) is \(\mathbb{Z}\)-graded and weakly \(K\)-equivariant since each \(u_i\) is a \(K\)-semi-invariant. Since \(\left(\mathbb{C}[S_{\text{reg}}][\alpha][u^\alpha]\right)^K = \mathbb{C}[\alpha][u^{\pm 1}] u^\alpha\), the \(K\)-invariant, degree zero part \(\left(\mathbb{C}[S_{\text{reg}}][\alpha][u^\alpha]\right)_0^K\) of \(\mathbb{C}[S_{\text{reg}}][\alpha] u^\alpha\) equals \(\mathbb{C}[\alpha] u^\alpha\). But \(\Delta u^{\alpha+1}\) is \(K\)-invariant of degree zero. Therefore, there exists \(b(\alpha) \in \mathbb{C}[\alpha]\) such that (4.11) holds.

Repeating the argument given in the paragraph preceding [Ki, Proposition 2.22] shows that \(b\) is a polynomial of total degree at most \(\ell = \deg u\). Moreover, if we specialise this polynomial to \(\alpha_1 = \cdots = \alpha_r = \alpha\), we must recover the polynomial given in [Ki, Proposition 2.22(1)]; again, this part of the proof of [Ki, Proposition 2.22] does not use the fact that \(S\) is a prehomogeneous space. This shows that the total degree of \(b\) is exactly \(\ell = \deg u\).

**Remark 4.12.** In general it appears difficult to compute the polynomial \(b(\alpha)\) but, based on [Gy1] we expect that

\[
b(\alpha) = \prod_{i=0}^{\ell-1} (a_{1,i} + \cdots + a_{r,i} \alpha_i + c_i),
\]

where \(a_{i,j} \in \mathbb{N}\) and \(c_j \in \mathbb{Q}_{>0}\).

**Notation 4.13.** Choose \(\zeta_i \in \mathbb{C}\) for \(1 \leq i \leq r\) and set \(\alpha_i = t - 1 + \zeta_i\), for an indeterminate \(t\). We factorise

\[
b(\alpha)|_{\alpha_i} = t - 1 + \zeta_i = c \prod_{i=0}^{\ell-1} (t + \lambda_i)
\]

for some \(\lambda_i \in \mathbb{C}\) and \(c \in \mathbb{C}^\times\). Write \(d := \sum_{i=1}^r \zeta_i \deg u_i\).

Finally, recall that the *Euler element* is \(e u_S = \sum_{i=1}^n s_i \partial_s_i \in \mathcal{D}(S)\) which is independent of the choice of basis \(\{s_i\}\) of \(S^*\).

**Corollary 4.14.** Using the definitions from Notation 4.13, we have

\[
\text{rad}_z(u) = z, \quad \text{rad}_z(e u_S) = \ell z \partial_z + d, \quad \text{and} \quad \text{rad}_z(\Delta) = \sum_{i=0}^{\ell-1} (z \partial_z + \lambda_i).
\]
Proof. The equality $\text{rad}_\iota(u) = z$ is immediate since $\text{rad}_\iota$ is simply restriction on functions. An element of $\mathbb{C}[S_{\text{reg}}]^K u^\iota$ is a linear combination of $u^m u^\iota$ for various $m \in \mathbb{Z}$. Since $\ell = \deg u$,

$$e_{\mathcal{S}}(u^m u^\iota) = (\ell m + \mathbf{d})u^m u^\iota$$

and hence $\text{rad}_\iota(e_{\mathcal{S}}) = \ell z \partial_z + \mathbf{d}$. If we set $e_{\iota} := e_{\mathcal{S}} - \mathbf{d}$, then $e_{\iota}(u^m u^\iota) = \ell m u^m u^\iota$ and hence $\text{rad}_\iota(e_{\iota}) = \ell z \partial_z$.

Finally, when restricting $\omega_i$ to $m = 1 + \varsigma_i$ for each $i$, Proposition 4.10 shows that

$$\Delta u^m u^\iota = b(\alpha) u^{m-1} u^\iota = \frac{1}{u} \left( \frac{\ell}{e} \prod_{i=0}^{\ell - 1} (m + \lambda_i) \right) (u^m u^\iota)$$

$$= \frac{1}{u} \left( \frac{\ell}{e} \prod_{i=0}^{\ell - 1} (\ell^{-1} eu_i + \lambda_i) \right) (u^m u^\iota).$$

Since $\ell^{-1} \text{rad}_\iota eu_i = z \partial_z$ this implies that

$$\text{rad}_\iota(\Delta)(z^m) = \frac{1}{z} \left( \frac{\ell}{e} \prod_{i=0}^{\ell - 1} (z \partial_z + \lambda_i) \right) (z^m).$$

Hence $\text{rad}_\iota(\Delta) = \frac{\ell}{e} \prod_{i=0}^{\ell - 1} (z \partial_z + \lambda_i)$, as required. \hfill \Box

As shown in Example 2.11, the image of $A_\kappa(W)$ under the Dunkl embedding $\Phi$ is the algebra generated by $z, \partial_z$ and

$$\Phi(Y) = \frac{\ell}{e} \prod_{i=0}^{\ell - 1} \left( z \partial_z + \frac{k_i}{\ell} + (\delta_{i,0} - 1) \right).$$

Therefore, Corollary 4.14 implies that

Corollary 4.15. Set $\kappa = (\kappa_0, \ldots, \kappa_{\ell - 1})$ where

$$\kappa_i + \frac{i}{\ell} + (\delta_{i,0} - 1) = \lambda_i. \quad \text{(4.16)}$$

Then $A_\kappa(W)$ is the subalgebra of $\mathcal{D}(\mathfrak{b}_{\text{reg}})^W$ generated by

$$z = \text{rad}_\iota(u), \quad \text{rad}_\iota(e_{\mathcal{S}}), \quad \text{and} \quad \nabla = \text{rad}_\iota(\Delta).$$

Remark 4.17. The precise definition of the idempotents $e_{H,i}$ is not consistent over the literature and the definition of the $\kappa_i$ changes accordingly. For example, in [Le3] the idempotents $e_i$ are related to the $e_{H,i}$ by $e_{H,i} = e_{H,i}$ for $0 \leq i \leq \ell - 1$ (with the convention that $e_r = e_0$).

Suppose that $\varsigma = 0$. The parameters $\kappa_i$ from (4.16) then differ from the parameters $\kappa_i$ given in [Le3, (2.5)]: for an appropriate numbering of the roots of the $b$-function, we have $\kappa_0 = k_0 = 0$ and $\kappa_i = k_{i-1}$ for $1 \leq i \leq \ell - 1$.

Lemma 4.18. Let $R$ be a subalgebra of $\mathcal{D}(\mathfrak{b}_{\text{reg}})^W$, containing $A_\kappa = A_\kappa(W)$, and such that the adjoint action of $\mathbb{C}[\nabla]$ on $R$ is locally nilpotent. Then:

1. $A_\kappa[z^{-1}] = R[z^{-1}] \subseteq \mathcal{D}(\mathfrak{b}_{\text{reg}})^W$ and
2. $A_\kappa[\nabla^{-1}] = R[\nabla^{-1}]$.

Proof. We use the generators of $A_\kappa$ defined by Corollary 4.15. Since $e \in A_\kappa \subseteq R$, both $A_\kappa$ and $R$ are $\mathbb{Z}$-graded subalgebras of $\mathcal{D}(\mathfrak{b}_{\text{reg}})^W$.

1. The action of $z$ on $\mathcal{D}(\mathfrak{b}_{\text{reg}})^W$ is locally ad-nilpotent and so this is also the case for $R$. Hence the localisation $R[z^{-1}]$ exists with $A_\kappa[z^{-1}] \subseteq R[z^{-1}] \subseteq \mathcal{D}(\mathfrak{b}_{\text{reg}})^W$. The result now follows from the fact that, by Lemma 2.9(1), $A_\kappa[z^{-1}] = \mathcal{D}(\mathfrak{b}_{\text{reg}})^W$.

2. Write $M$ for the polynomial representation $\mathbb{C}[z^{\pm 1}]$: it is a faithful graded $R$-module. If $p(t) = e \prod_{i=0}^{\ell - 1} (t + \lambda_i)$, then Corollary 4.14 shows that $\nabla(z^m) = \frac{\ell}{e} \prod_{i=0}^{\ell - 1} (z \partial_z + \lambda_i) (z^m)$.
Proposition 4.20. Assume that \( \dim S/K = 1 \) and keep the notation developed above. Let \( R \) be a subalgebra of \( \mathcal{D}(h_{\text{reg}})^W \), containing \( A_\kappa(W) \), such that the adjoint action of \( \mathbb{C}[\nabla] \) on \( R \) is locally nilpotent. Then \( R = A_\kappa(W) \).

Proof. Lemma 4.18 shows that, for any \( D \in R \), there exists \( k \gg 0 \) such that \( z^kD \) and \( \nabla^kD \) belong to \( A_\kappa \). As we now explain, the result will now follow by using the argument from [Le3, Lemma 3.8 and Theorem 3.9]. First, by comparing Example 2.11 with [Le3, Proposition 2.8], \( A_\kappa \) identifies with the ring \( U \) of [Le3], where our \( \nabla \) corresponds to the element \( \delta \) of [Le3], up to a scalar. Thus, the proof of [Le3, Lemma 3.8] can be used to prove that, for any \( r \in R \), one has
GKdim((A_κ + rA_κ)/A_κ) ≤ GKdim A_κ - 2 = 0. The proof of [Le3, Theorem 3.9] can now be used unchanged to prove that \( R = A_κ \).

**Corollary 4.21.** Assume that dim \( S//K = 1 \) and keep the notation developed above. If \( κ \) is defined by Equation (4.16), then \( \text{Im}(\text{rad}_κ) = A_κ(\mathcal{W}) \).

**Proof.** As \( \text{Sym} S \) acts locally ad-nilpotently on \( \mathcal{D}(S) \), certainly \( \mathbb{C}[\Delta] \subseteq (\text{Sym} S)^N \) acts locally ad-nilpotently on \( \mathcal{D}(S)^N \). Thus \( \mathbb{C}[\nabla] = \text{rad}_κ(\mathbb{C}[\Delta]) \) acts locally ad-nilpotently on \( R = \text{rad}_κ(\mathcal{D}(S)^N) \subseteq \mathcal{D}(\mathfrak{h}_{\text{reg}})^\mathcal{W} \). Since \( A_κ(\mathcal{W}) \subseteq R \) by Corollary 4.15, the result follows from Proposition 4.20.

5. The radial parts map: existence

As in the rest of the article, \( V \) will denote a polar representation of a connected reductive group \( G \). Recall that we defined the radial parts map \( \text{rad}_κ : \mathcal{D}(V)^G \rightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})^\mathcal{W} \) in (4.3) and (4.4). The goal of this section is to prove the following theorem, which will be proved as Theorem 5.21.

**Theorem 5.1.** Let \( V \) be a polar representation for the connected, reductive group \( G \). There exists a parameter \( κ = κ(ς) \) such that the image of \( \text{rad}_κ \) is contained in the spherical algebra \( A_κ(\mathcal{W}) \).

The basic idea of the proof follows that in [Le2, Section 4] for the case \( V = \mathfrak{g} \), which is in turn based on the proof in [Sc2, Section 3]. Here is an outline. Recall that \( \mathcal{A} \) denotes the set of reflecting hyperplanes in \( \mathfrak{h} \). For a rational Cherednik algebra \( H_κ(W) \), the parameter \( κ \) is uniquely defined by its values on the rank one parabolic subgroups of \( W \). Such a subgroup is the stabiliser of a generic point \( p \in H \), for some \( H \in \mathcal{A} \). One can take a slice \( S_p \) to the \( G \)-orbit through \( p \). Then \( (G_p, S_p) \) is a rank (at most) one polar representation with Weyl group \( W_H \). In the rank one case one can explicitly compute the image of the radial parts map, as is done in Corollary 4.15, to show that it does indeed lie in the appropriate spherical algebra. This tells us the value of the corresponding parameter \( κ \). By considering all hyperplanes \( H \) one can then explicitly compute \( κ \) in terms of \( ς \) and show that \( \text{Im}(\text{rad}_κ) \subseteq A_κ \).

We note that in all our examples, \( κ \) depends affine linearly on \( ς \), although we do not have a general proof of this fact. See Remark 4.12 for a possible explanation.

Before beginning on the proof of Theorem 5.1, we describe what the radial parts map \( \text{rad}_κ \) does to the regular locus \( V_{\text{reg}} \). For this, it is convenient to rewrite \( \text{rad}_κ \) as a composition of functions. First we have the map \( γ_κ : \mathcal{D}(V_{\text{reg}}) \rightarrow \mathcal{D}(V_{\text{reg}}) \) given by conjugation with the formal element \( δ^-ς \); thus \( γ_κ(D) = δ^-ς D δ^ς \) for \( D \in \mathcal{D}(V_{\text{reg}}) \). Equivalently, using (4.2), \( γ_κ \) is the morphism that is the identity on \( \mathbb{C}[\nabla_{\text{reg}}] \) and maps a derivation \( \partial \in \mathcal{D}(V_{\text{reg}}) \) to \( \partial + \sum _i \frac{\partial (ς)}{ς} \). Thus \( γ_κ \) does indeed map \( \mathcal{D}(V_{\text{reg}}) \) to itself and, as \( δ^-ς \in \mathbb{C}[\nabla_{\text{reg}}] \), is a \( G \)-semi-invariant. \( γ_κ \) also induces a filtered automorphism of \( \mathcal{D}(V_{\text{reg}})^G \). Now let \( η : \mathcal{D}(V_{\text{reg}})^G \rightarrow \mathcal{D}(V_{\text{reg}}//G) \) denote the restriction of operators and write \( \tilde{η} \) for the identification \( \mathcal{D}(V_{\text{reg}}//G) \cong \mathcal{D}(\mathfrak{h}_{\text{reg}}//\mathcal{W}) \) induced from \( φ : \mathbb{C}[V//G] \rightarrow \mathbb{C}[\mathfrak{h}//\mathcal{W}] \). Then (4.2) and (4.3) ensure that

\[
\text{rad}_κ = \tilde{η} ◦ η ◦ γ_κ.
\]

A useful consequence of this discussion is that we can also compose \( \text{rad}_κ \) as

\[
\text{rad}_κ \circ \text{rad}_0 \circ γ_κ : \mathcal{D}(V_{\text{reg}})^G \rightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})^\mathcal{W},
\]

Recall from Remark 2.13 that \( A_κ(\mathcal{W}) \) is given the order filtration induced from that on \( \mathcal{D}(\mathfrak{h}_{\text{reg}}) \). We similarly use the order filtration on \( \mathcal{D}(V_{\text{reg}})^G \) induced from that on \( \mathcal{D}(V_{\text{reg}}) \). Finally, given filtered rings \( A = \bigcup _{n≥0} A_n \) and \( B = \bigcup _{n≥0} B_n \), then a homomorphism \( α : A \rightarrow B \) is called filtered surjective if α is a filtered morphism such that \( B_n = α(A_n) \) for all n. For related concepts see Notation 7.5.
Lemma 5.4. The localization of \( \text{rad}_c \) to \( \mathcal{D}(V_{\text{reg}})^G \) gives a filtered surjective map 
\[ \text{rad}_c : \mathcal{D}(V_{\text{reg}})^G \rightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})^W. \]

Proof. Since \( \text{rad}_c = \text{rad}_0 \circ \gamma_c \), where \( \gamma_c \) is a filtration preserving automorphism of \( \mathcal{D}(V_{\text{reg}})^G \), it suffices to prove the lemma when \( \gamma_c = 0 \). Now \( \text{rad}_0 = \tilde{\gamma} \circ \eta \) by (5.2) and since the restriction of differential operators is a filtered morphism, \( \text{rad}_0 \) is clearly a filtered map.

It remains to prove that \( \text{rad}_0 \) is filtered surjective. Let \( \bar{v} \in V_{\text{reg}}//G \) with semisimple lift \( v \in V_{\text{reg}} \). By [DK, Theorem 2.3], every Cartan subspace of \( V \) is conjugate to \( \mathfrak{h} \) and so we may assume that \( v \in \mathfrak{h} \). Then [Sc1, Theorem 4.9] says (using the notation from [Sc1, (3.24)]) that \( \text{rad}_0 \) is filtered surjective in an affine open neighbourhood of \( \bar{v} \) in \( V//G \) if the radial parts map for a slice at \( v \) is filtered surjective.

A slice of the \( G \)-action at \( v \) is given by \( (Z,S) \) where \( S = \mathfrak{h} \oplus U \) as in Lemma 3.9(2), while \( Z = Z_G(\mathfrak{h}) \) is the centraliser of \( \mathfrak{h} \) acting trivially on \( \mathfrak{h} \) and \( U//Z = \{ pt \} \).

Identifying \( \mathcal{D}(pt) = \mathbb{C} \), the radial parts map for the slice is 
\[ \mathcal{D}(\mathfrak{h}) \otimes \mathcal{D}(U)^Z \rightarrow \mathcal{D}(\mathfrak{h}) \otimes \mathbb{C} = \mathcal{D}(\mathfrak{h}), \]
which is vacuously filtered surjective. \( \square \)

Slice representations in \( V \). In this subsection we examine how to use slice representations to reduce our problem to the one-dimensional case and, ultimately, to prove Theorem 5.1.

Notation 5.5. Let \( \mathcal{A} \) denote the set of reflecting hyperplanes in \( \mathfrak{h} \) and, following Notation 2.19, set \( \mathfrak{h}^0 = \mathfrak{h} \setminus \bigcup_{H \in \mathcal{A}} H_{\mathfrak{h}} \cap H_{\mathfrak{h}} \). Fix \( H \in \mathcal{A} \) and \( p \in H^0 = H \cap \mathfrak{h}^0 \); thus \( p \) lies on just the one hyperplane in \( \mathcal{A} \). The stabiliser of \( p \) in \( W \) equals the stabiliser \( W_H = \{ w \in W \mid w \cdot x = x, \forall x \in H \} \). Let \( K = G_p \) be the stabiliser of \( p \) in \( G \) and note that, as \( p \) is semisimple, \( K \) is reductive. As described before (3.8), we may chose a slice \( S = S_p \). Thus \( S \) is \( K \)-stable complement to \( \mathfrak{g} \cdot p \) in \( V \) with \( S \supseteq \mathfrak{h} \); see Lemma 3.9. This notation will be fixed throughout the subsection.

In general, \( K \) need not be connected; see [BNS, Example 15.3] for an explicit example. Even so, we have:

Lemma 5.6. (1) \((K,S)\) is polar with Cartan subalgebra \(\mathfrak{h}\).

(2) The Weyl group of \((K,S)\) equals \(W_H\).

(3) The restriction map \(\tilde{\gamma}_p\): \(\mathbb{C}[S]^K \rightarrow \mathbb{C}[\mathfrak{h}]^W_H\) is an isomorphism.

(4) \(S^K = \mathfrak{h}^W_H = H\) and \(K = Z_G(H)\).

(5) \((K,S)\) has rank at most one.

Proof. (1) This is [DK, Theorem 2.4].

(2) By definition, the Weyl group of \((K,S)\) equals \(N_K(\mathfrak{h})/Z_K(\mathfrak{h})\). This embeds in \(W\), say with image \(W'\). Let \(w \in W_H\) and choose a lift \(g\) of \(w\) in \(N_G(\mathfrak{h})\). Then \(g\) fixes every point in \(H\) and hence \(g \in K\). Therefore, \(g \in K \cap N_G(\mathfrak{h}) = N_K(\mathfrak{h})\) and hence \(w \in W'\). Conversely, if \(w \in W'\) then \(w\) fixes a generic point of the hyperplane \(H\), so \(w\) fixes every point in \(H\) and thus \(w \in W_H\).

(3) See [DK, Theorem 2.9].

(4) The complex reflection group \(W_H\) has rank one, so \(\dim \mathfrak{h}^W_H = \dim \mathfrak{h} - 1\). This implies that the space of elements of degree 1 in \(\mathbb{C}[\mathfrak{h}]^W_H\) satisfies \(\dim \mathbb{C}[\mathfrak{h}]_1^W_H = \dim \mathfrak{h} - 1\). But, by Part (3), the restriction map \(\mathbb{C}[S]^K \rightarrow \mathbb{C}[\mathfrak{h}]^W_H\) is an isomorphism and so \((S^K)^* = (\mathfrak{h})^*\). It follows that \(S^K = \mathfrak{h}^W_H = H\). Therefore \(K \subseteq Z_G(H)\).

Conversely, since \(p \in H\) we have \(Z_G(H) \subseteq G_p = K\) and so \(K = Z_G(H)\).

(5) Finally, if \(\mathfrak{k} = \text{Lie}(K)\) then \(\text{rank}(S,K) = \dim \mathfrak{h} - \dim \mathfrak{h}_K\) by (3.3). Since \(\mathfrak{h} \cap S^K \subseteq \mathfrak{h}_K\), it follows from Part (4) that \(\dim \mathfrak{h}_K \geq \dim \mathfrak{h} - 1\), as desired. \(\square\)
Remark 5.7. In theory, it is possible for \((K,S)\) to have rank zero and this happens precisely if the point \(p\) is \(V\)–regular but not regular, see Lemma 3.9. (As noted in Remark 3.5, it is conjectured that this cannot happen.) In this case the image of \(K\) in \(GL(S)\) is a finite group; indeed, \(S = \mathfrak{h}\) and the image of \(K\) in \(GL(\mathfrak{h})\) equals \(W_H\).

As before, let \(\mathfrak{k} = \text{Lie}(K)\) and set \(\dim V = n\) and \(d = \dim S\), for \(S = S_p\). Given \(X \in \mathfrak{g}\) and \(u \in V\), write \(\tau(X)_u = X \cdot u\). Pick a basis \(s_1, \ldots, s_d\) for \(S\), extended to a basis \(\{s_1, \ldots, s_n\}\) of \(V\). Under our identification \(V \subset \text{Sym} V \subset \mathcal{D}(V)\), these \(s_i\) become derivations on \(V\), which we write as \(\partial_{s_i}\) to avoid confusion. Let \(X_1, \ldots, X_{n-d}\) be a basis for the \(K\)-stable complement \(L\) to \(\mathfrak{k}\) in \(\mathfrak{g}\). The fact that \(V = \mathfrak{g} \oplus p\) implies that \(s_1, \ldots, s_d, \tau(X_1), \ldots, \tau(X_{n-d})\) is also a basis of \(V\). Then, as the \(\partial_{s_i}\) form a \(\mathbb{C}[V]\)-basis of \(\text{Der}(\mathbb{C}[V])\), we can write
\[
\begin{align*}
\partial_{s_1} \wedge \cdots \wedge \partial_{s_d} \wedge \tau(X_1) \wedge \cdots \wedge \tau(X_{n-d}) &= t \cdot \partial_{s_1} \wedge \cdots \wedge \partial_{s_n},
\end{align*}
\]
for some \(t \in \mathbb{C}[V]\). Finally, set \(U = \{t \neq 0\} \subset V\).

Lemma 5.9. (1) The function \(t\) is non-zero and \(K\)-invariant.
(2) \(U\) is a \(K\)-stable affine open subset of \(V\).
(3) \(U = \{u \in V : V = S \oplus \text{Span}(\tau(X_1)_u, \ldots, \tau(X_{n-d})_u)\}\).

Proof. (1) The left hand side of Equation 5.8 evaluated at \(p\) is non-zero since \(s_1, \ldots, s_d, \tau(X_1), \ldots, \tau(X_{n-d})\) is a basis of \(V\). Therefore \(t(p) \neq 0\) and hence \(t \neq 0\). If \(g \in K\), the fact that \(\tau\) is \(G\)-equivariant implies that
\[
\begin{align*}
g \cdot \partial_{s_1} \wedge \cdots \wedge \partial_{s_d} &= \text{det}(g) \partial_{s_1} \wedge \cdots \wedge \partial_{s_d} \\
g \cdot \tau(X_1) \wedge \cdots \wedge \tau(X_{n-d}) &= \text{det}(g) \tau(X_1) \wedge \cdots \wedge \tau(X_{n-d}) \quad \text{and} \\
g \cdot \partial_{s_1} \wedge \cdots \wedge \partial_{s_n} &= \text{det}(g) \partial_{s_1} \wedge \cdots \wedge \partial_{s_n}.
\end{align*}
\]
Since \(S \oplus L \cong V\) as \(K\)-modules, \(\text{det}(g) \text{det}(p) = \text{det}(g)\). Therefore, comparing the three displayed equations to (5.8), \(g(t) = t\).

Part (2) is immediate from (1) and Part (3) follows from the definition of \(t\).

Lemma 5.10. Let \(I\) denote the kernel of the map \(\mathbb{C}[V] \rightarrow \mathbb{C}[S]\) of restriction to \(S\). Then \(\mathbb{C}[V] = \mathbb{C}[S] \oplus I\) as \(K\)-equivariant \(\mathcal{D}(S)\)-modules.

Proof. Using the \(K\)-module decomposition \(V = S \oplus g \cdot p\), we regard \(\mathcal{D}(S)\) as a subalgebra of \(\mathcal{D}(V)\). Let \(x_1, \ldots, x_{n-d} \in V^*\) be coordinates whose set of common zeros is the subspace \(S\). Then \(I = \sum_{j=1}^{n-d} \mathbb{C}[V]x_j\) and \(\partial(x_j) = 0\) for every derivation \(\partial \in \mathcal{D}(S)\). Therefore, the \(\mathbb{C}[S]\)-module decomposition \(\mathbb{C}[V] = \mathbb{C}[S] \oplus I\) extends to a \(\mathcal{D}(V)\)-module decomposition. The \(K\)-equivariance is immediate.

By Lemma 5.6(3), the restriction map \(\varrho_p : S \rightarrow \mathfrak{h}\) induces an isomorphism \(\varrho_p : \mathbb{C}[S]^K \rightarrow \mathbb{C}[\mathfrak{h}]^W_H\). As such, the discriminant \(h \in \mathbb{C}[\mathfrak{h}]^W\) can be considered as a \(K\)-invariant function on \(S\) (denoted \(\delta_S\)) whose non-vanishing locus \(\mathcal{S}_\text{reg}\) equals \(S \cap \mathcal{V}\). The map \(\varrho_p\) localises to \(\varrho_p : \mathbb{C}[S_{\text{reg}}]^K \rightarrow \mathbb{C}[\mathfrak{h}_{\text{reg}}]^W_H\). Recall from (3.10) that the irreducible factors of \(\delta\) are \(\delta_1, \ldots, \delta_k\). Each \(\delta_i\) restricts to a non-zero \(K\)-semi-invariant \(\delta_{S,i}\) on \(S\) and so the radial parts map \(\text{rad}_{\mathfrak{h}}\) from (4.3) induces a radial parts map
\[
\text{rad}_{\mathfrak{h}} : \mathcal{D}(S)^K \rightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})^{W_H}, \quad \text{rad}_{\mathfrak{h}}(D)(z) = (\delta_{S}^{-1}D(\varrho_p^{-1}(z)\delta_S))|_{\mathfrak{h}_{\text{reg}}},
\]
for \(\delta_S = \delta_{S,1} \cdots \delta_{S,k}\).

Let \(S\) denote the set of all functions in \(\mathbb{C}[V]\) not vanishing on \(\mathfrak{h}_{\text{reg}}^\perp = U \cap \mathfrak{h}_{\text{reg}}\) and \(\mathcal{C}\) the set of all functions in \(\mathbb{C}[\mathfrak{h}]\) not vanishing on \(\mathfrak{h}_{\text{reg}}^\perp\). The following technical Lemma 5.13 is key to the main result. Before stating it, recall that the function \(t\) is defined in (5.8) and that \(\delta_{S,i}\) is defined in Remark 4.5. Also, as the next remark shows, we are free to use \(\delta_{S,i}\) in place of \(\delta_{S,i}\) when defining \(\text{rad}_{\mathfrak{h}}\).
Remark 5.12. Choose any $W$-stable affine open subset $C$ of $\mathfrak{h}_{\text{reg}}$ and set $U = \pi^{-1}(C/W) \subset V_{\text{reg}}$. As $V$ is polar, restriction defines an isomorphism $\varphi: C[U]^{G} \rightarrow \mathbb{C}[C]^{W}$. One can then define

$$\text{rad}_{\ell}(D)(z) = (\delta^{-\ell}D(g^{-1}(z)\delta^{\ell}))|_{C} \quad \text{for } z \in \mathbb{C}[C]^{W} \text{ and } D \in \mathcal{D}(V)^{G}.$$  

Under the inclusion $\mathcal{D}(\mathfrak{h}_{\text{reg}})^{W} \subseteq \mathcal{D}(C)^{W}$ the image of this map $\text{rad}_{\ell}$ is the same as that of (4.4), and so we can relabel $\text{rad}_{\ell} = \text{rad}_{\ell}$ without ambiguity. In other words, it does not matter which “test functions” are used to define the radial parts map.

Recall from Lemma 5.9(2) that $U = (t \neq 0)$ is $K$-stable and that $t$ vanishes nowhere on $\mathfrak{h}_{\text{reg}}^{\perp}$. Thus, the morphism $\text{rad}_{\ell,p}$ restricts to give a map $\mathcal{D}(U)^{K} \rightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}})^{W}$, which we also write as $\text{rad}_{\ell,p}$. It is this second version of $\text{rad}_{\ell,p}$ that is used in the next lemma.

Lemma 5.13. Let $D \in \mathcal{D}(V)^{G}$. There exists $\ell > 0$ and $P \in \mathcal{D}(S)^{K}$ such that

$$\text{rad}_{\ell}(D) = \text{rad}_{\ell,p}((t|S)^{-\ell}P)$$

as elements of $\mathcal{D}(\mathfrak{h})_{\mathfrak{c}}$.

Proof. As in Lemma 5.9 we identify $\mathbb{C}[S] \subseteq \mathbb{C}[V] \subseteq \mathbb{C}[U]$. We note that $\delta, \delta_{S}$ and $t$ belong to $\mathcal{S}$. Recall from Notation 2.12 that $\mathcal{D}_{\mathcal{S}}(X)$ denotes the order filtration on $\mathcal{D}(X)$. It follows from Lemma 5.9 that, locally at any point $u \in U$, we have

$$\text{Der} \mathbb{C}[U] = (\mathbb{C}[U] \otimes \mathbb{C}[S] \text{ Der} \mathbb{C}[S]) + \mathbb{C}[U]t\mathfrak{g}.$$

Therefore, by [BD, Corollary A1], (5.14) holds globally on $U$. We have $\mathbb{C}[U] = \mathbb{C}[U] \otimes \mathbb{C}[S] \mathbb{C}[S] \subseteq \mathbb{C}[U] \otimes \mathbb{C}[S] \mathcal{D}_{1}(S)$, and $\mathbb{C}[U]h_{\mathfrak{c}} + \mathbb{C}[U] = \mathbb{C}[U]t\mathfrak{g} + \mathbb{C}[U]$ inside $\mathcal{D}(U)$ by the definition (4.5) of $\mathfrak{g}_{\mathfrak{c}}$. We therefore deduce that

$$\mathcal{D}_{1}(U) = (\mathbb{C}[U] \otimes \mathbb{C}[S] \mathcal{D}_{1}(S)) + \mathbb{C}[U]h_{\mathfrak{c}}.$$

This implies that $\mathcal{D}(U) = (\mathbb{C}[U] \otimes \mathbb{C}[S] \mathcal{D}(S)) + \mathcal{D}(U)h_{\mathfrak{c}}$ and hence that

$$\mathcal{D}(U)^{K} = (\mathbb{C}[U] \otimes \mathbb{C}[S] \mathcal{D}(S))^{K} + (\mathcal{D}(U)h_{\mathfrak{c}})^{K}.$$

Since $U$ is the principal open set ($t \neq 0$), there exists $P_{0} \in (\mathbb{C}[V] \otimes \mathbb{C}[S] \mathcal{D}(S))^{K}$ and $R \in (\mathcal{D}(V)g_{\mathfrak{c}})^{K}$ such that $t^{\ell}D = P_{0} + R$ for some $\ell \geq 0$. Moreover, by Lemma 5.10 we may further decompose $P_{0} = P + Q$, for $P \in \mathcal{D}(S)^{K}$ and $Q \in (I \otimes \mathbb{C}[S] \mathcal{D}(S))^{K}$.

Let $z \in \mathbb{C}[\mathfrak{h}_{\text{reg}}]^{W}$ with $g^{-1}(z) \in \mathbb{C}[V_{\text{reg}}]^{G}$. Then (4.4) combined with Remark 5.12 says that

$$\text{rad}_{\ell}(D)(z) = (\delta^{-\ell}D(g^{-1}(z)\delta^{\ell}))|_{\mathfrak{h}_{\text{reg}}^{\perp}} = (\delta^{-\ell}t^{-\ell}(P + Q + R)(g^{-1}(z)\delta^{\ell}))|_{\mathfrak{h}_{\text{reg}}^{\perp}}.$$  

Now $Q(g^{-1}(z)\delta^{\ell}) \in I\delta^{\ell}$. Since $I|_{\mathfrak{h}} = 0$, it follows that $\delta^{-\ell}t^{-\ell}Q(g^{-1}(z)\delta^{\ell})|_{\mathfrak{h}_{\text{reg}}} = 0$. On the other hand, $R(g^{-1}(z)\delta^{\ell})|_{\mathfrak{h}_{\text{reg}}} = 0$ since $g_{\mathfrak{c}}(g^{-1}(z)\delta^{\ell})|_{\mathfrak{h}_{\text{reg}}} = 0$. Hence

$$\text{rad}_{\ell}(D)(z) = (\delta^{-\ell}t^{-\ell}P(g^{-1}(z)\delta^{\ell}))|_{\mathfrak{h}_{\text{reg}}}.$$

Let $I = \sum \mathbb{C}[V]x_{i}$, as in the proof of Lemma 5.10. Then, by that lemma, we can write $g^{-1}(z) = g_{p}^{-1}(z) + \sum_{i=1}^{n-m} x_{i}z_{i}$ for some $z_{i} \in \mathbb{C}[V_{\text{reg}}]$. Since $P \in \mathcal{D}(S)$ acts trivially on the $x_{i}$ and $x_{i}(\mathfrak{h}_{\text{reg}}^{\perp}) = 0$ for each $i$,

$$\text{rad}_{\ell}(D)(z) = (\delta^{-\ell}t^{-\ell}P(g_{p}^{-1}(z)\delta^{\ell}))|_{\mathfrak{h}_{\text{reg}}} + \sum_{i=1}^{n-m} x_{i}\delta^{-\ell}t^{-\ell}P(z_{i}\delta^{\ell})|_{\mathfrak{h}_{\text{reg}}}$$

$$= (\delta^{-\ell}t^{-\ell}P(g_{p}^{-1}(z)\delta^{\ell}))|_{\mathfrak{h}_{\text{reg}}} = (t^{-\ell})|_{\mathfrak{h}_{\text{reg}}} (\delta^{-\ell}P(g_{p}^{-1}(z)\delta^{\ell}))|_{\mathfrak{h}_{\text{reg}}}.$$  

Recall that $P \in \mathcal{D}(S)^{K}$, but $\delta, t \in \mathbb{C}[V]$. 

Sublemma 5.16. \((δ−t^−P(\delta^−Z))|_{h_{reg}} = (δ−t^−P(\delta^−Z))|_{h_{reg}}\)

Proof. Since \((t^−f)|_{h_{reg}} = ((t|_{S})^−P)|_{h_{reg}}\), by (5.15) it remains to show that

\[
(δ−P(\delta^−Z))|_{h_{reg}} = (δ−P(\delta^−Z))|_{h_{reg}},
\]

for \(P \in \mathcal{D}(S)^K\). Since \(I_S|_{h_{reg}} = 0\), we must show that

\[
δ−Pδ ≡ δ−Pδ \mod I_S ⊗ 1_{S} \mathcal{D}(S),
\]

for any \(P \in \mathcal{D}(S)\). Formal conjugation by both \(δ^−\) and \(δ^−\) are algebra automorphisms of \(\mathcal{D}(V)\). Therefore, it suffices to prove this identity when \(P = \partial ∈ \text{Der}(\mathbb{C}[S])\). Taking each factor \(δ_i\) of \(δ\) in turn, this reduces to the statement that \(\partial(δ_i)δ_{i}^{-1} ≡ \partial(δ_{S,i})δ_{S,i}^{-1} \mod I_S\). Since \(δ_i = δ_{S,i} + f_i\) for some \(f_i ∈ I\), this follows from the fact that, by Lemma 5.10, \(\partial(f_i) = 0\).

We return to the proof of the lemma. The sublemma implies that \(\text{rad}_c(D)(Z) = \text{rad}_c(D)(Z)\), as required.

For the rest of the subsection we assume that \((K, S)\) has rank equal to one. Let \(S_H\) denote the \(K\)-stable complement to \(S^K\) in \(S\). By Lemma 5.6(4), \(S^K = h^W = H\). If we set \(h_H = h ∩ S_H\), then \((K, S_H)\) is a rank one polar representation with one-dimensional Cartan subalgebra \(h_H\). In particular, \(dim S_H/K = 1\).

We can write \(C[S_H]^K = C[u]\) for some polynomial \(u\), but we need to be precise about the choice of \(u\). Set \(α := α_H ∈ h^∗\). Since \(dim S_H ∩ h = 1\), clearly \(C[S_H ∩ h] = C[α]\) and hence \(α^f ∈ C[S_H ∩ h]^W = C[h^H]^W\), where \(T_H = |W_H|\). From the Chevalley isomorphism \(ϕ : C[S_H]^K ∼ C[h^H]^W\) we can, and will, take \(u\) to be the preimage of \(α^f\) in \(C[S_H]^K\). As in (4.9), fix a factorisation

\[
u = u_1^{p_1} · · · u_r^{p_r}
\]

into equivariantly irreducible polynomials in \(C[S_H]\). Recall from (2.7) that we associate to the hyperplane \(H\) the factor \(h_H\) of the discriminant \(h ∈ C[h]^W\).

Lemma 5.17. Assume that \((K, S)\) has rank one and, as in (3.10), let the \(δ_i\) be the irreducible factors of \(δ\) in \(C[V]\).

1. \(δ|_{S} = gu\), where \(g ∈ C[S]^K\) is a \(K\)-invariant function with \(g|_{h} = h_H\).

2. There exist unique \(\nu_j \geq 0\) such that \(δ|_{S} = g_j \prod_{j=1}^{r} u_j^{\nu_j}\), where \(g_j\) is a \(K\)-invariant function not divisible by \(u\) and \(g_j|_{h} = h_H\).

Proof. (1) Restriction is an isomorphism \(ϕ : C[S]^K ∼ C[h]^W\) of graded polynomial rings. By (2.5) and (2.7), \(h = α^f h_H \in C[h]^W\). Since \(ϕ(u) = α^f\) and \(ϕ(δ|_{S}) = δ|_{h} = α^f h_H\), we deduce that \(δ|_{S} = gu\), where \(g = ϕ^{-1}(h_H)\).

(2) Recall from Lemma 4.8 that \(B := C[S]^K\) is a UFD. We begin by noting that the highest common factor of any \(u_j\) and \(g\) (in \(B\)) is one. Indeed, \(u_j|_{h} = h_H\) divides \(u_j = α^f\), and if \(f\) is a factor of \(g\) then \(f|_{h} = h_H\). But the highest common factor of \(α\) and \(h_H\) in \(C[h]\) is one, as is evident from (2.5) and (2.7).

Let \(g_i\) denote the highest common factor of \(\delta_i|_{S}\) and \(g\) in \(B\) and \(\prod_{j=1}^{r} u_j^{\nu_j}\), the highest common factor of \(\delta_i|_{S}\) and \(u\), again in \(B\). Since \(δ|_{S} = gu\), the previous paragraph implies that \(δ|_{S} = g_i \prod_{j=1}^{r} u_j^{\nu_j}\). The uniqueness of the \(\nu_j\) follows from the fact that \(B\) is a UFD.

Finally, we must show that \(g_i\) is \(K\)-invariant. Recall that \(S = S_H ⊕ H\), with \(S^K = H\), as \(K\)-modules. If \(g_i|_{H} = 0\) then necessarily \(α\) divides \(g_i|_{h}\). But the latter divides \(h_H\) and we already saw that the highest common factor of \(α\) and \(h_H\) is one. Thus, \(g_i|_{H} ≠ 0\). The restriction map \(C[S] → C[H]\) is \(K\)-equivariant and \(g_i\) is a \(K\)-semi-invariant. Since \(K\) acts trivially on \(H\), \(g_i|_{H} ≠ 0\) implies that \(g_i\) is \(K\)-invariant.

\[\square\]
Recall that we are assuming that we are in the rank one case and we now wish to reduce to the one-dimensional situation. Define

\[ \psi = (\psi_1, \ldots, \psi_r) \quad \text{by} \quad \psi_j = \sum_{i=1}^{k} n_{i,j} \kappa_i \]

where the \( n_{i,j} \) are constructed by Lemma 5.17. Then we can define a radial parts map \( \text{rad}_\psi : \mathcal{D}(S)^K \to \mathcal{D}(\mathcal{H}_{\text{reg}})^W_H \) by

\[ \text{rad}_\psi(D)(z) = (u^{-\psi}(D)(q_p^{-1}(z)u^\psi))|_{\mathcal{H}_{\text{reg}}} \quad \text{for} \quad z \in \mathbb{C}[\mathcal{H}_{\text{reg}}]^W_H. \]

If \( A \) is a noetherian algebra and \( a \in A \) is such that \( \{a^n\}_{n \in \mathbb{N}} \) forms an Ore set, then we write \( A_a \) for the localization of \( A \) at this Ore set.

**Lemma 5.19.** Assume that \((K, S)\) has rank one. Define \( g \) and \( \psi \) by Lemma 5.17 and (5.18), respectively and let \( \sigma : \mathcal{D}(S)^K \to \mathcal{D}(S)^K \) be the automorphism given by conjugation with \( g^{-1} \), analogous to the definition of conjugation by \( \delta^{-1} \).

Then there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}(S)^K & \overset{\sigma}{\longrightarrow} & \mathcal{D}(S)^K \\
\text{rad}_{\cdot, p} \downarrow & & \downarrow \text{rad}_{\cdot, q} \\
\mathcal{D}(\mathcal{H}_{\text{reg}})^W_H & & & \mathcal{D}(\mathcal{H}_{\text{reg}})^W_H.
\end{array}
\]

**Proof.** Formally, \( \delta_S = g^* u^\psi \), where \( g^* = g_1^* \cdots g_k^* \) and \( u^\psi = u_1^{\psi_1} \cdots u_r^{\psi_r} \). By (5.11),

\[
\text{rad}_{\cdot, p}(D)(z) = (\delta_S^{-1} D(q_p^{-1}(z)\delta_S))|_{\mathcal{H}_{\text{reg}}}
= (u^{-\psi}(g^{-1} Dg^*)(q_p^{-1}(z)u^\psi))|_{\mathcal{H}_{\text{reg}}}
= (u^{-\psi}(D)(q_p^{-1}(z)u^\psi))|_{\mathcal{H}_{\text{reg}}},
\]

as required. \( \square \)

**General polar representations.** We now return to a general polar representation \((G, V)\). Before proving Theorem 5.21, which is the main result of this section, we begin with a remark on slice representations. This will ensure that the \( \kappa_i \) to be defined in (5.22) are indeed well defined. We recall that \( V_s \) denotes the set of \( V \)-regular elements as defined in Section 3.

**Remark 5.20.** Keep the notation from Notation 5.5 and note that \( H^0 = H \cap h^0 \) is a non empty open subset of \( H \).

1. Suppose that \( H \cap V_s = \emptyset \). Let \( p \in H^0 \); since \( p \) is not \( V \)-regular, Lemmata 3.9 and 5.6 imply that \( \text{rank}(G_p, S_p) = 1 \). Moreover, from the same lemmata we deduce that \( G_p = Z_G(H) \) and \( S_p = h \oplus g_p \cdot h \oplus U = h \oplus Z_G(H) \cdot h \oplus U \) are independent of \( p \in H^0 \). Thus \( \mathbb{C}[S_p] = \mathbb{C}[H] \otimes \mathbb{C}[u] \) and \( \mathbb{C}[h]^W_H = \mathbb{C}[H] \otimes \mathbb{C}[u_h] \) with \( u \) independent of \( p \). Hence, by definition of the \( b \)-function associated to \( \Delta = u_\kappa(D) \) in (4.10), the roots \( \lambda_i \) introduced in Notation 4.13 do not depend on \( p \in H^0 \).

2. Suppose instead that \( H \cap V_s \neq \emptyset \) (as noted in Remark 3.5, it is conjectured that this case never occurs). Then, \( H \cap V_s \) is open in \( H \) since the elements of \( h \) are semisimple and so there exists \( p_0 \in H^0 \cap V_s \). Then, Lemmata 3.9(2) and 5.6(4) applied to \( K = G_{p_0} \) imply that \( g_{p_0} = Z_G(H) = Z_G(h) \).

Now let \( p \in H^0 \) be arbitrary. Then \( g_p = \text{Lie}(G_p) = Z_G(H) \) by Lemma 5.6(4). Since we proved that \( Z_G(H) = Z_G(h) \) it follows that \( g_p = Z_G(h) \) and hence \( p \in H \cap V_s \) by Lemma 3.9(2), again. Therefore \( \text{rank}(G_p, S_p) = 0 \) for all \( p \in H^0 \). In this case \( H \) will contribute nothing to \( \kappa \) (see (5.22)), which is certainly independent of \( p \in H^0 \).

We now come to the main result of this section, announced in Theorem 5.1.
Theorem 5.21. Let $\zeta \in \mathbb{C}^k$. Define the parameter $\kappa = \kappa(\zeta)$ by
\begin{equation}
\kappa_{H,i} = \begin{cases}
\lambda_i - \frac{i}{\ell_H} + (1 - \delta_{i,0}) & \text{for } i = 0, \ldots, \ell_H - 1, \text{ if } H \cap V_\kappa = \emptyset, \\
0 & \text{for all } i, \text{ if } H \cap V_\kappa \neq \emptyset,
\end{cases}
\end{equation}
where the $\lambda_i$ are associated, via Notation 4.13, to the slice representation $(G_p, S_p)$ for $p \in H^\circ$. Then:
$$\text{rad}_c(\mathcal{D}(V)^G) \subseteq A_\kappa(W).$$

Proof. We begin by considering the case when $\text{rank}(G, V) = 0$. Since $G$ is connected, this implies that $V = \mathfrak{h} \oplus U$, where $G$ acts trivially on $\mathfrak{h}$ and $U$ acts on $V$. Note that $\mathcal{D}(U)^G$ acts on $\mathbb{C}[U]^G = \mathbb{C}$ and hence admits a quotient $\mathcal{D}(U)^G \to \mathcal{D}(\mathbb{C}) = \mathbb{C}$. Here the discriminant is equal to one, so there is no twist and $\text{rad}_c(\mathcal{D}(V)^G) \to \mathcal{D}(\mathfrak{h})$ is simply the morphism $\mathcal{D}(\mathfrak{h}) \otimes (U)^G \to \mathcal{D}(\mathfrak{h})$ which is the identity on the first factor and the map $\mathcal{D}(U)^G \to \mathbb{C}$ on the second. Since $A_\kappa(W) = \mathcal{D}(\mathfrak{h})$ for $W = \{1\}$, the theorem holds in this case.

Therefore, we may assume that $\text{rank}(G, V) \geq 1$ and hence that $A \neq \emptyset$. Pick a hyperplane $H \in \mathcal{A}$, an element $p \in H^\circ = H \cap \mathfrak{h}^\circ$ and a corresponding slice representation $(G_p, S_p)$. By Lemma 5.6, $(G_p, S_p)$ is a polar representation with Weyl group $W_H$ and rank at most one.

If $H \cap V_\kappa = \emptyset$, then $p \in V_\kappa$ by Remark 5.20(2). Moreover, by Remarks 5.7 and 5.20 (2), $G_p = W_H$, $S_p = \mathfrak{h}$ and $\text{rank}(G_p, S_p) = 0$. In this case we set $\kappa_{H,i} = 0$ for all $i$. Then $A_\kappa(W_H) = \mathcal{D}(\mathfrak{h})^{W_H} = (S_p)^{G_p}$, so again the theorem holds on this slice.

We are left with the case when $H \cap V_\kappa = \emptyset$ and hence $\text{rank}(G_p, S_p) = 1$ by Remark 5.20 (1). Set $(K, S) := (G_p, S_p)$ with $S = S_H \oplus \mathfrak{h}^{W_H}$. Recall that if we set $\mathfrak{h}_H = \mathfrak{h} \cap S_H$, then $(K, S_H)$ is a rank one polar representation with one-dimensional Cartan subalgebra $\mathfrak{h}_H$, hence $(\mathfrak{h}_H)^{\text{reg}} = \mathfrak{h}_H \sim \{0\}$. As in Lemma 5.17, we have $\delta|_S = g\psi$ where $g|_\mathfrak{h}_H = \mathfrak{h}_H$ and factorisations $\delta|_S = g_k \prod \mu_i^{n_i}$. Define $\psi$ by (5.18). Now choose the parameters $\kappa_{H,i}$ as in the first case of (5.22), set $\kappa_H = \{\kappa_{H,i}\}$ and define a spherical algebra $A_{\kappa_H}(W_H, \mathfrak{h}_H) \subseteq \mathcal{D}(\mathfrak{h}_H)^{W_H}$ in terms of these parameters. Then, Corollary 4.21 shows that $\text{rad}_c(\mathcal{D}(S)^K) = A_{\kappa_H}(W_H, \mathfrak{h}_H)$. Recall that, by definition, $S_H \otimes S^K = S$ and hence, by Lemma 5.6, $\mathfrak{h} = \mathfrak{h}_H \oplus \mathfrak{h}^{W_H}$. Thus we can extend $\text{rad}_c$ to a morphism $\mathcal{D}(S)^K \to (\mathfrak{h}_H)^{W_H}$.

Now $A_{\kappa_H}(W_H, \mathfrak{h}_H) \otimes \mathcal{D}(\mathfrak{h})^{W_H}$ is the spherical algebra constructed inside $\mathcal{D}(\mathfrak{h}_H)^{W_H}$ using the parameters $\kappa_H$, and we call this algebra $A_{\kappa_H}(W_H, \mathfrak{h})$. We deduce that the image of $\mathcal{D}(S)^K$ under $\text{rad}_c$ equals $A_{\kappa_H}(W_H, \mathfrak{h}) \subseteq (\mathfrak{h}_H)^{W_H}$. Therefore, Lemma 5.19 implies that
$$\text{rad}_c(\mathcal{D}(S)^K) \subseteq \text{rad}_c(\mathcal{D}(S)^{K}) = \text{rad}_c(\mathcal{D}(S)^K) \subseteq A_{\kappa_H}(W_H, \mathfrak{h})^{W_H}.$$
Proof. Let $R = \mathbb{C}[\mathfrak{h}]^W$. By Lemma 2.4, the algebra $A_\kappa(W_H)$ is a free $R$-submodule of $\mathcal{D}(\mathfrak{h})$. Since 

$$A_\kappa(W_H)_{h_p} = A_\kappa(W_H) \otimes_R R_{h_p}, \quad \text{and} \quad A_\kappa(W_H)_{h_H} = A_\kappa(W_H) \otimes_R R_{h_H},$$

it therefore suffices to show that 

\begin{equation}
R_h \cap \bigcap_{p \in \mathcal{H}^\circ} R_{h_p} = R_{h_H}.
\end{equation}

Now $R_{h_H}$ lies inside the left hand side of (5.24) since $h_H$ divides both $h$ and $h_p$. Conversely, take $f$ belonging to the left hand side, considered as a $W_H$-invariant rational function on $\mathfrak{h}$. Then $f$ is regular on $h_{\text{reg}}$ and at every point of $\mathcal{H}^\circ$. The set $h_{\text{reg}} \cup H^\circ$ is open in $h$ and its complement in $h_{\text{reg}} \cup H$ has codimension two. Therefore, since $R$ is an integrally closed domain, $f$ is regular on the whole of $h_{\text{reg}} \cup H$. The complement of $h_{\text{reg}} \cup H$ in $h$ is contained inside the zero set of $h_H$ (the latter being the union of all hyperplanes except $H$). Thus, the left hand side of (5.24) is contained inside $R_{h_H}$, as required.

We return to the proof of the theorem. For each $p \in \mathcal{H}^\circ$, we have shown that $\text{rad}_\kappa(\mathcal{D}(V)^G) \subseteq A_\kappa(W_H)_{h_p}$. Moreover, $\text{rad}_\kappa(\mathcal{D}(V)^G) \subseteq \mathcal{D}(h_{\text{reg}})^W$ and, by Lemma 2.9(1), $\mathcal{D}(h_{\text{reg}})^W \subseteq \mathcal{D}(h_{\text{reg}})_{h_H} = A_\kappa(W_H)_{h_H}$. Thus, by Sublemma 5.23, 

$$\text{rad}_\kappa(\mathcal{D}(V)^G) \subseteq A_\kappa(W_H)_{h_H} \cap \bigcap_{p \in \mathcal{H}^\circ} A_\kappa(W_H)_{h_p} = A_\kappa(W_H)_{h_H}.$$ 

Therefore, by Theorem 2.20,

$$\text{rad}_\kappa(\mathcal{D}(V)^G) \subseteq \bigcap_{h \in \mathcal{A}} A_\kappa(W_H)_{h_H} \subseteq A_\kappa(W),$$

as required.

6. Conjugation of radical parts

A natural question raised by the last section is whether the images $\text{Im}(\text{rad}_\kappa)$ are isomorphic as $\varsigma$ varies. In this short section, we show that, for fixed $\chi$, this is indeed the case; see Corollary 6.7. As part of the proof, we also show that the radial parts map $\text{rad}_\kappa$ does induce the surjection (1.1) upon restriction to the regular locus; see Proposition 6.4. We also briefly discuss stable representations, since these will be needed later in the paper.

The generic slice. Let $\mathfrak{g}$ be a Cartan subspace of a polar presentation $(G,V)$ and adopt the notation of Section 3. Set $Z = Z_G(\mathfrak{h})$, $N = N_G(\mathfrak{h})$ and fix an $N$-stable complement $U$ to $\mathfrak{h} \oplus g \cdot \mathfrak{h}$ in $V$ as in (3.8). As shown after that equation, $U/Z = \{p\}$ and so the closed $Z$-orbits in $h_{\text{reg}} \times U$ are precisely the singleton sets $\{(x,0)\}$ for $x \in h_{\text{reg}}$. Recall that a subset $X \subseteq V$ is said to be $G$-saturated if $X = \pi^{-1}(\pi(X))$, where $\pi : V \to V/G$ is the categorical quotient. In particular, $V_{\text{reg}}$ is $G$-saturated.

Proposition 6.1. The map 

$$G \times N_G(\mathfrak{h}) (h_{\text{reg}} \times U) \longrightarrow V_{\text{reg}},$$

given $[g,(x,u)] \mapsto g \cdot (x + u)$, is an isomorphism.

Proof. We have a morphism $\sigma' : G \times N (\mathfrak{h} \times U) \to V$ given by $[g,(x,u)] \mapsto g \cdot (x + u)$ and we write $\sigma$ for the restriction of $\sigma'$ to $G \times N (h_{\text{reg}} \times U)$.

We first show that the image $V' = \text{Im}(\sigma)$ is a $G$-saturated open subset of $V$. Since $V'$ is also the image of $G \times Z (h_{\text{reg}} \times U)$ under the obvious map (which we also call $\sigma$), we work with the latter. For each $x \in h_{\text{reg}}$, the map $\sigma$ is étale at $[1,(x,0)]$. Moreover, since $Z \cdot (x,0)$ is closed in $h_{\text{reg}} \times U$, the orbit $G \cdot [1,(x,0)]$ is closed in
\( G \times Z (\mathfrak{h}_{\text{reg}} \times U) \). Since \( Z = G_x \) by Lemma 3.4, \( \sigma \) maps \( G \cdot [1, (x, 0)] \) bijectively onto \( G \cdot x \subset V \). Therefore, by [Lu, Lemme fondamental, II.2, p.94], there exists a \( G \)-saturated affine open neighbourhood of \( G \cdot [1, (x, 0)] \) whose image in \( V \) is a \( G \)-saturated affine open neighbourhood of \( G \cdot x \).

Now let \( v \in \pi^{-1}(\pi(V')) \) and let \( G \cdot x \) be the unique closed orbit in \( G \cdot v \); thus \( \pi(v) = \pi(x) \). By the last paragraph, there exists an affine open neighbourhood \( \Omega \) of \( G \cdot [1, (x, 0)] \) such that \( \sigma(\Omega) \) is a saturated neighbourhood of \( G \cdot x \). Set \( x' = [1, (x, 0)] \).

Then \( \pi(\sigma(x')) = \pi(x) = \pi(v) \) whence \( v \in \pi^{-1}(\pi(\sigma(x'))) \). Since \( \pi^{-1}(\pi(\sigma(x'))) \subseteq \pi^{-1}(\pi(\sigma(\Omega))) = \pi(\Omega) \), we get that \( v \in \sigma(\Omega) \subseteq Z \). Therefore \( V' = \pi^{-1}(\pi(V')) \) and \( V' \) is indeed \( G \)-saturated.

As noted in [Lu, Remark (2), page 98], even though the map \( G \times Z (\mathfrak{h}_{\text{reg}} \times U) \to V \) need not be bijective onto its image, it follows from [Lu, Lemme fondamental, II.2, p.94] that for each \( x \in \mathfrak{h}_{\text{reg}} \) the induced map \( G \times Z \{\{x\} \times U\} \to \pi^{-1}(\pi(x)) \) is a bijection. In particular, if \( x + u = g \cdot (x + u') \) in \( V \) for some \( u, u' \in U \) then \( g \in G_x = Z \).

Now we return to \( G \times Z \{\mathfrak{h} \oplus U\} \), and again set \( V' = \text{Im}(\sigma) \). Then [Lu, Lemme, p.87] implies that \( V' = \pi^{-1}(A) \) for some open subset \( A \) of \( \mathfrak{h} / \mathfrak{w} \). Necessarily, \( \mathfrak{h}_{\text{reg}} / \mathfrak{w} \subseteq A \), so \( V_{\text{reg}} \subseteq V' \). If this inclusion is proper then the fact that \( V' \) is \( G \)-saturated means that there exists a \( G \) orbit \( \mathcal{O}' \) in \( V' \setminus V_{\text{reg}} \) that is closed in \( V \). Then \( \sigma^{-1}(\mathcal{O}') \) is a (finite) union of closed orbits in \( G \times Z (\mathfrak{h}_{\text{reg}} \times U) \). Let \( \mathcal{O} \) be one of these closed orbits. Then \( \mathcal{O} = G \times Z \mathfrak{O}_0 \) for some closed \( N \)-orbit \( \mathfrak{O}_0 \subseteq \mathfrak{h}_{\text{reg}} \times U \).

Since \( U / Z = \{pt\} \), the closed \( Z \)-orbits in \( \mathfrak{h}_{\text{reg}} \times U \) are precisely the sets \( \{(x, 0)\} \) for \( x \in \mathfrak{h}_{\text{reg}} \). Thus the closed \( N \)-orbits in \( \mathfrak{h}_{\text{reg}} \times U \) are just the ones of the form \( N \cdot (x, 0) \) for \( x \in \mathfrak{h}_{\text{reg}} \). Hence \( \mathcal{O} \subseteq V_{\text{reg}} \) and so \( \sigma \) surjects onto \( V_{\text{reg}} \).

Since \( V_{\text{reg}} \) is smooth, in order to prove that \( \sigma \) is an isomorphism, it suffices to show that it is injective. Let \( [g, (x, u)] \in G \times Z (\mathfrak{h}_{\text{reg}} \times U) \) and suppose that \( g \cdot (x + u) = g_1 \cdot (x_1 + u_1) \) with \( [g_1, (x_1, u_1)] \in G \times Z (\mathfrak{h}_{\text{reg}} \times U) \). Without loss of generality, \( g = 1 \). The unique closed orbit in \( G \cdot (x + u) \) is \( G \cdot x \). Therefore, the identity \( x + u = g_1 \cdot (x_1 + u_1) \) implies that \( G \cdot x = G \cdot x_1 \). It follows that \( W \cdot x = G \cdot x \cap \mathfrak{h} = G \cdot x_1 \cap \mathfrak{h} = W \cdot x_1 \) and hence \( x_1 = k \cdot x \) for some \( k \in N \). Thus

\[
x + u = (g_1k) \cdot x + g_1 \cdot u_1 = (g_1k) \cdot (x + u_2),
\]

where \( u_2 = k^{-1} \cdot u_1 \). By Lemma 3.4 this forces \( z := g_1k \in G_x = Z \). It follows that \( x + u = x + z + u_2 \) and hence \( u = z \cdot u_2 = (zk^{-1}) \cdot u_1 \). Putting all this together gives

\[
[g_1, (x_1, u_1)] = [zk^{-1}, (x_1, u_1)] = [1, (zk^{-1}, x_1, zk^{-1} \cdot u_1)]] = [1, (z \cdot x, u)] = [1, (x, u)],
\]

as required.

\[\square\]

**Lemma 6.2.** Assume that \( \delta_i \) is a \( \theta \)-semi-invariant irreducible factor of \( \delta \) for some character \( \theta \in X^*(G) \). Then:

1. \( \theta(Z) = 1 \);
2. under the identification \( V_{\text{reg}} \cong G \times Z (\mathfrak{h}_{\text{reg}} \times U) \), we have \( \delta_i|_{V_{\text{reg}}} = F_\theta \otimes h_i \otimes 1 \) for some \( W \)-semi-invariant \( h_i \in \mathbb{C}[\mathfrak{h}_{\text{reg}}] \).

**Proof.** (1) By definition, \( \delta \) is nowhere vanishing on \( V_{\text{reg}} \) and so \( \delta_i \) does not vanish on \( V_{\text{reg}} \). Fix \( x \in \mathfrak{h}_{\text{reg}} \). Since \( \mathfrak{h}_{\text{reg}} \subset V_{\text{reg}} \), we have \( \delta_i(x) \neq 0 \). Then, for all \( z \in Z \),

\[
\delta_i(x) = \delta_i(z \cdot x) = \theta(z)\delta_i(x).
\]

Thus \( (1 - \theta(z))\delta_i(x) = 0 \). Since \( \delta_i(x) \neq 0 \) we deduce that \( \theta(z) = 1 \).

(2) Consider the left regular action of \( G \) on itself. As \( \theta \) is a linear character of \( G \), the algebraic Peter-Weyl Theorem [GW, Theorem 4.2.7] implies that, up to scalar, there is a unique \( \theta \)-semi-invariant function \( F_\theta \in \mathcal{O}(G) \). If the group \( N \) acts by (inverse) multiplication on \( G \) from the right, then \( F_\theta \) is a \( \theta^{-1} |_N \)-semi-invariant.
Then each $\theta$-semi-invariant function $\alpha$ on $V_{\text{reg}} = G \times_N (h_{\text{reg}} \times U)$ is of the form $F_\theta \otimes f_\alpha$ for some $N$-semi-invariant function $f_\alpha \in \mathbb{C}[h_{\text{reg}} \times U]$, with character $\theta|_N$. Note that, for $\alpha = \delta_{\iota}$, we have $f_\alpha = \delta_{\iota}|_{h_{\text{reg}} \times U}$. By (1), the character $\theta|_N$ factors through $W$ and so $\delta_{\iota}|_{h_{\text{reg}} \times U}$ is $Z$-invariant. Hence, it is the pullback to $h_{\text{reg}} \times U$ of an invertible $W$-semi-invariant $h_i \in \mathbb{C}[h_{\text{reg}}]$.

**Localisation of radial maps.** In this subsection, we consider the localization of $\text{rad}_c$ to $V_{\text{reg}}$. We write $\mathfrak{z} = \text{Lie}(Z) = \text{Lie}(N)$ and let $\chi$ be the character associated to $\zeta$, as in (4.1). Recall from Proposition 6.1 that $V_{\text{reg}} \cong G \times_N (h_{\text{reg}} \times U)$.

By Lemma 6.2(1), $\chi|_{\mathfrak{z}} = 0$. Therefore, by [BG, Lemma 9.1.2] that there is an identification

$$(6.3) \quad \psi_\chi : (\mathcal{D}(h_{\text{reg}} \times U)/\mathcal{D}(h_{\text{reg}} \times U)\tau_\mathfrak{z}(\mathfrak{z}))^N \rightarrow (\mathcal{D}(V_{\text{reg}})/\mathcal{D}(V_{\text{reg}})\mathfrak{g}_\chi)^G,$$

where $\psi_\chi$ sends $D \in \mathcal{D}(h_{\text{reg}} \times U)^N$ to the image of $1 \otimes D \in \mathcal{D}(G \times h_{\text{reg}} \times U)^G \times N$ in the right hand side. The algebra $(\mathcal{D}(h_{\text{reg}} \times U)/\mathcal{D}(h_{\text{reg}} \times U)\tau_\mathfrak{z}(\mathfrak{z}))^N$ acts on $\mathbb{C}[h_{\text{reg}} \times U]^N \cong \mathbb{C}[h_{\text{reg}}]^W$. This action defines a *split surjective* algebra morphism

$$\nu : (\mathcal{D}(h_{\text{reg}} \times U)/\mathcal{D}(h_{\text{reg}} \times U)\tau_\mathfrak{z}(\mathfrak{z}))^N \rightarrow \mathcal{D}(h_{\text{reg}})^W,$$

with right inverse $\iota$ induced from the natural injection of $\mathcal{D}(h_{\text{reg}})$ into $\mathcal{D}(h_{\text{reg}} \times U)$. This hold simply because the action of $D \in \mathcal{D}(h_{\text{reg}} \times U)^Z$ on $\mathbb{C}[h_{\text{reg}} \times U]^Z \cong \mathbb{C}[h_{\text{reg}}]$ factors through $\mathcal{D}(h_{\text{reg}})$.

We now come to the main result of this section, which also proves the final statement of Theorem 1.2.

**Proposition 6.4.** The localised map

$$\text{rad}_c : (\mathcal{D}(V_{\text{reg}})/\mathcal{D}(V_{\text{reg}})\mathfrak{g}_\chi)^G \rightarrow \mathcal{D}(h_{\text{reg}})^W$$

is a split surjection with $\text{rad}_c \circ \psi_\chi \circ \iota$ equal to conjugation by $h^{-c}$ (in the sense made to be made precise in Sublemma 6.5).

**Proof.** Since $V_{\text{reg}} = G \times_N (h_{\text{reg}} \times U)$ by Proposition 6.1 and $\mathfrak{z} = \text{Lie}(N)$ acts trivially on $h_{\text{reg}}$, one has

$$(\mathcal{D}(V_{\text{reg}})/\mathcal{D}(V_{\text{reg}})\mathfrak{g}_\chi)^G = \left( \frac{\mathcal{D}(G \times h_{\text{reg}} \times U)}{\mathcal{D}(G \times h_{\text{reg}} \times U)\mathfrak{g}_\chi + \mathcal{D}(G \times h_{\text{reg}} \times U)\tau_0(\mathfrak{z})} \right)^{G \times N},$$

where $\tau_0 : \mathfrak{z} \rightarrow \mathcal{D}(G \times U)$ is the derivative of the diagonal $Z$ action on $G \times U$ and the action on $G$ is by right multiplication. The isomorphism (6.3) says that every class in $(\mathcal{D}(V_{\text{reg}})/\mathcal{D}(V_{\text{reg}})\mathfrak{g}_\chi)^G$ can be represented as $1 \otimes D$ for a unique element $D \in (\mathcal{D}(h_{\text{reg}} \times U)/\mathcal{D}(h_{\text{reg}} \times U)\tau_\mathfrak{z}(\mathfrak{z}))^N$. Such an operator will only act on the second factor of a function of the form $F \otimes f$, where $f \in \mathbb{C}[h_{\text{reg}} \times U]$. For clarity, write $\beta : \mathbb{C}[h_{\text{reg}}]^W \rightarrow \mathbb{C}[h_{\text{reg}} \times U]^Z$. For $f \in \mathbb{C}[h_{\text{reg}}]$, we have $\beta(f)(x,u) = f(x)$ and hence $\beta(f) = f \otimes 1$. If $f \in \mathbb{C}[h_{\text{reg}}]$ is a $W$-semi-invariant, $D$ is as above and $r \in \mathbb{C}$, then

$$\nu(\beta(f)^{-r}D\beta(f)^r) = f^{-r}\nu(D)f^r.$$

In other words, $\nu((f \otimes 1)^{-r}D(f \otimes 1)^r) = f^{-r}\nu(D)f^r$.

**Sublemma 6.5.** If $D$ is defined as above, then $\text{rad}_c(1 \otimes D) = h^{-c} \circ \nu(D) \circ h^c$.

**Proof.** Proposition 6.1 induces an isomorphism $\alpha : \mathbb{C}[h_{\text{reg}}]^W \cong \mathbb{C}[G \times_N (h_{\text{reg}} \times U)]^G$ and we identify the target with $\mathbb{C}[G \times h_{\text{reg}} \times U]^{G \times N}$, where $N$ acts diagonally. If $z \in \mathbb{C}[h_{\text{reg}}]^W$ then $\alpha(z)(g,x,u) = z(x)$; that is, $\alpha(z) = 1 \otimes z \otimes 1 \in \mathbb{C}[G \times h_{\text{reg}} \times U]^{G \times N}$. Now we compute

$$\text{rad}_c(1 \otimes D)(z) = (\delta^{-c}(1 \otimes D)(1 \otimes z \otimes 1)\delta^c)|_{h_{\text{reg}}}.$$
Recall from Section 4 that the irreducible factors of $\delta$ are $\{\delta_1, \ldots, \delta_k\}$, where $\delta_j$ is a semi-invariant of weight $\theta_j$. Since $\delta_j$ is $Z$-invariant, Lemma 6.2(2) implies that $\delta|_{V_{reg}} = F_{\theta_1} \otimes h_1 \otimes 1$, where $h_1 = \delta|_{h_{reg}}$. Therefore,

$$\delta^\chi = \delta_1^\chi \cdot \delta_k^\chi = F_{\theta_1}^\chi \cdot F_{\theta_k}^\chi \otimes h_1^\chi \cdot h_k^\chi \otimes 1$$

and so

$$(1 \otimes D) ((1 \otimes z \otimes 1)\delta^\chi) = F_{\theta_1}^\chi \cdot F_{\theta_k}^\chi \otimes (D(zh_1^\chi \cdot h_k^\chi \otimes 1))$$

$$= F_{\theta_1}^\chi \cdot F_{\theta_k}^\chi \otimes (D(zh^\chi \otimes 1))$$

$$= (1 \otimes (h^{-\chi} \nu(D)h^\chi))(z) \otimes 1) \cdot F_{\theta_1}^\chi \cdot F_{\theta_k}^\chi \otimes h_1^\chi \cdot h_k^\chi \otimes 1$$

$$= (1 \otimes (h^{-\chi} \nu(D)h^\chi))(z) \otimes 1) \cdot \delta^\chi.$$

Hence $(\delta^{-\chi}(1 \otimes D))(1 \otimes z \otimes 1)\delta^\chi)_{h_{reg}} = 1 \otimes (h^{-\chi} \nu(D)h^\chi)(z) \otimes 1$, as required.

We return to the proof of the proposition. By the sublemma, if $D \in \mathcal{D}(h_{reg})^W$ then

$$(6.6) \quad \text{rad}_\chi(1 \otimes \iota(D)) = h^{-\chi} \circ \nu(D) \circ h^\chi = h^{-\chi} \circ D \circ h^\chi$$

or, equivalently, $\text{rad}_\chi \circ \psi_\chi \circ \iota$ equals conjugation by $h^{-\chi}$. Since this is an automorphism of $\mathcal{D}(h_{reg})^W$, it follows that $\text{rad}_\chi$ is a split surjection.

We continue with the notation of Proposition 6.4. Let $\varsigma, \varsigma' \in \mathbb{C}^k$ with associated characters $\chi, \chi'$ as in (4.1).

**Corollary 6.7.** Assume that $\chi = \chi'$ and set $\xi = \varsigma' - \varsigma$.

1. $\text{rad}_\chi(D) = h^{-\chi}(\text{rad}_\chi(D))h^\xi$ for all $D \in \mathcal{D}(V)^G$.
2. Therefore, $\text{Im}(\text{rad}_\chi) = h^{-\chi}(\text{Im}(\text{rad}_\chi))h^\xi \cong \text{Im}(\text{rad}_{\chi'})$.
3. Moreover, $\ker(\text{rad}_\chi) = \ker(\text{rad}_{\chi'})$.

**Proof.** Parts (2) and (3) both follow from Part (1).

In order to prove Part (1), let $D \in \mathcal{D}(V)^G$; as seen in Proposition 6.4, the class of $D$ in $(\mathcal{D}(V_{reg})/\mathcal{D}(V_{reg})_G)^G$ can be represented by $1 \otimes D_0$ where $D_0 = \iota(D) \in \mathcal{D}(h_{reg} \times U)^N/\mathcal{D}(h_{reg} \times U)^N \nu_y(\mathfrak{g})^N$. Note that the choice of $1 \otimes D_0$ depends on $\chi$ but not $\varsigma$. Using Sublemma 6.5 twice we get:

$$\text{rad}_\varsigma(1 \otimes D_0) = (h^{-\varsigma'} h^\varsigma h^{-\varsigma'} \nu(D) h^\varsigma(h^{-\varsigma'} h^\varsigma) = h^{-\xi} \text{rad}_\chi(1 \otimes D_0) h^\xi.$$  

Thus, using (6.6), $\text{rad}_\varsigma(D) = h^{-\xi}(\text{rad}_\chi(D))h^\xi$.

One significance of Corollary 6.7 is the following result.

**Corollary 6.8.** If $V$ is a polar representation of a semisimple group $G$, then the algebras $\text{Im}(\text{rad}_\varsigma)$ are all isomorphic to $\text{Im}(\text{rad}_0)$, and they all have the same kernel.

**Proof.** If $G$ is semisimple, then $X^*(G) = \{1\}$ and hence the character $\chi$ associated to any $\varsigma \in \mathbb{C}^k$ is 0. Now apply Corollary 6.7.

With regard to Corollary 6.7, it is important to note that for general $\chi' \neq \chi$ one can have $\text{Im}(\text{rad}_\varsigma) \neq \theta^{-1}(\text{Im}(\text{rad}_\varsigma))\theta$. This is because the image of $1 \otimes D$ in the proof of Proposition 6.4, depends upon $\chi$. For example, as [BNS, Corollary 14.15] shows for a particular representation, one can easily obtain both simple and nonsimple rings as $\text{Im}(\text{rad}_\varsigma)$ by varying $\varsigma$. 

Stable Representations. We end the section by briefly discussing stable representations.

Recall from (3.1) that \( m \) is the maximum dimension over all orbits in \( V \) and \( s \) the maximum dimension over all semisimple orbits. As there, the set of semisimple elements in \( V \) is denoted by \( V_{ss} \).

Define the set of stable elements in \( V \) to be \( V_{st} = \{ v \in V_{ss} : \dim G \cdot v = m \} \). It is known (and not difficult to see) that \( V_{st} \) is an open subset of \( V_{ss} \). The representation \((G,V)\) is said to be stable if there is a closed orbit whose dimension is maximal amongst all orbits. Equivalently, the representation is stable if \( V_{st} \neq \emptyset \) and one then has \( m = s \) and \( V_s = V_{st} \). Observe also that \( \pi^{-1}(\pi(x)) = G \cdot x \) if \( x \in V_{st} \) since \( \pi^{-1}(\pi(x)) \) contains a unique closed orbit.

By [DK, Corollary 2.5], \( V \) is stable if only if \( V = h \oplus g \cdot h \). Equivalently, \( V \) is polar if and only if the space \( U \) defined at the beginning of the section is zero. Thus for polar representations we obtain the following simpler versions of Propositions 6.1 and 6.4.

Corollary 6.9. Assume that \((G,V)\) is a stable polar representation. Then there is an isomorphism
\[
G \times_N (h_{reg}) \xrightarrow{\sim} V_{reg},
\]
given by \( [g,(x,u)] \mapsto g \cdot (x + u) \).

Corollary 6.10. If \( V \) is a stable representation then the localised map
\[
\text{rad}_{\chi} : (\mathcal{D}(V_{reg})/\mathcal{D}(V_{reg})B_{\chi})^G \longrightarrow \mathcal{D}(h_{reg})^W
\]
is an isomorphism.

Proof. When \( V \) is stable, \( U = 0 \) and so (6.3) simplifies to give an isomorphism
\[
\psi_{\chi} : (\mathcal{D}(h_{reg})^W \xrightarrow{\sim} (\mathcal{D}(V_{reg})/\mathcal{D}(V_{reg})B_{\chi})^G.
\]
Thus the result follows from Proposition 6.4.

7. The radial parts map: surjectivity

In Theorem 5.1 we proved the existence of radial parts maps \( \text{rad}_{\chi} \). In this section, we prove that \( \text{rad}_{\chi} \) is surjective in many cases of interest. Specifically, by combining Theorems 7.10 and 7.19 with Corollary 7.11, we obtain the following result.

Theorem 7.1. Let \( V \) be a polar representation for the connected, reductive group \( G \), with a radial parts map \( \text{rad}_{\chi} : \mathcal{D}(V)^G \to A_{\chi}(W) \), for some spherical algebra \( A_{\chi}(W) \). Then \( \text{rad}_{\chi} \) is surjective in each of the following cases:

1. when \( A_{\chi}(W) \) is a simple algebra;
2. when \( V \) is a symmetric space, in the sense of Section 8;
3. when the associated complex reflection group \( W \) is a Weyl group with no summands of type \( \mathbb{E} \) or \( \mathbb{F} \).

The conclusions of the theorem also hold when \((V,G)\) has rank one and when \( V \) is a visible stable locally free representation; see Corollaries 4.21 and 7.20, respectively. This of course raises the question of whether \( \text{rad}_{\chi} \) is surjective for every polar representation. The authors know of no counterexample to this question and do not even have an opinion as to whether such an example should exist.

The surjectivity of \( \text{rad}_{\chi} \) when \( A_{\chi}(W) \) is simple. Much of the proof of this case involves the relationship between the Euler gradation and the order filtration of \( \mathcal{D}(V)^G \) and so we begin with the relevant notation.
Notation 7.2. As in Notation 4.13, let $euv_1 \in \mathcal{D}(V)$ denote the Euler operator on $V$, so that $[euv_1, x_i] = x_i$ for coordinate vectors $x_i \in V^* \subset \mathbb{C}[V]$, and let $euv_8 \in \mathcal{D}(\mathfrak{h})$ be the analogous operator on $\mathfrak{h}$. Conjugation by these Euler operators induces $\mathbb{Z}$-graded structures on the rings of differential operators, and hence on $A_{\kappa} = A_{\kappa}(W)$. See, for example, [Be, Section 2.4, p.186]. Explicitly,\[
\mathcal{D}(V) = \bigoplus_{k \in \mathbb{Z}} \mathcal{D}(V)_k \quad \text{where} \quad \mathcal{D}(V)^G_k = \{ D \in \mathcal{D}(V)^G : [euv_1, D] = -kD \}.
\]

In order to distinguish the Euler degree from other degree functions, we will write $\deg_{\text{eu}} d = k$ if $d \in \mathcal{D}(V)_k$. The analogous definition applies to $\mathcal{D}(\mathfrak{h})$ and related rings, and we still use the notation $\deg_{\text{eu}}$ for the corresponding degree function. In particular, as the discriminant $h$ is homogeneous, $\mathcal{D}(\mathfrak{h}_{\text{reg}})^W_k = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}(\mathfrak{h}_{\text{reg}})^W_k$ where $\mathcal{D}(\mathfrak{h}_{\text{reg}})^W_k = \{ D \in \mathcal{D}(\mathfrak{h}_{\text{reg}})^W : [euv_8, D] = -kD \}$. When restricted to $A_{\kappa}$, the Euler gradation can also be defined by setting $\deg_{\text{eu}} y = 1$, $\deg_{\text{eu}} x = -1$ and $\deg_{\text{eu}} w = 0$ for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$.

Recall the definitions of the order filtration and $\mathcal{D}_l(X)$ from Notation 2.12. The algebra $A_{\kappa}$ inherits the order filtration from $\mathcal{D}(\mathfrak{h}_{\text{reg}})^W_k$, which we will write as $A_{\kappa} = \bigcup \text{ord}_{\kappa} a_{\kappa}$. As explained in Remark 2.13, this is the same filtration coming from the realisation $A_{\kappa} = eH_{\kappa}(W)c$ of $A_{\kappa}$ as a subalgebra of $H_{\kappa}(W)$, where $H_{\kappa}(W)$ is filtered by placing $\mathbb{C}[\mathfrak{h}]$ and $W$ in degree zero and $\mathfrak{h} \setminus \{0\}$ in degree one.

The following lemma is part of the folklore but we include a proof as we do not know an appropriate reference.

Lemma 7.3. The morphism $\text{rad}_\kappa : \mathcal{D}(V)^G \to A_{\kappa}$ is a filtered homomorphism under the two order filtrations.

Proof. Recall from (5.3) that we can factorize $\text{rad}_\kappa = \text{rad}_\| \circ \gamma_\kappa$, where $\gamma_\kappa$ is conjugation by $\delta^{-\kappa}$. By construction, the morphisms $\gamma_\kappa$ and $\varrho$ are filtered isomorphisms while restricting a differential operator to a subalgebra certainly does not increase the order of an operator. Thus, $\text{rad}_\kappa$ is also a filtered morphism. □

Using these observations it is easy to determine the image $\text{rad}_\kappa(euv_1)$. Recall that $\varrho : \mathbb{C}[V] \to \mathbb{C}[\mathfrak{h}]$ denotes the restriction of functions from $V$ to $\mathfrak{h}$, with the induced Chevalley isomorphism again written $\varrho : \mathbb{C}[V]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$.

Lemma 7.4. (1) There exists $d \in \mathbb{C}$ such that $\text{rad}_\kappa(euv_1) = euv_8 + d$.

(2) The morphism $\text{rad}_\kappa : \mathcal{D}(V)^G \to \mathcal{D}(\mathfrak{h}_{\text{reg}})^W$ preserves the Euler grading.

Proof. (1) This is similar to the proof of Corollary 4.14. Recall from (4.2) that $\delta^\kappa = \delta_1^\kappa \cdots \delta_k^\kappa$. Hence $euv_1 \cdot \delta^\kappa = \left(\sum_i \zeta_i d_i\right) \delta^\kappa = d \delta^\kappa$, where $d_i = \deg_{\text{eu}} d_i$ and $d = \sum \zeta_i d_i$. Next, let $a \in \mathbb{C}[\mathfrak{h}_{\text{reg}}]^W$ be homogeneous, say of degree $p$. Since $\varrho^{-1}(a)$ is still homogeneous of degree $p$, it follows that $euv_1 \left(\varrho^{-1}(a) \delta^\kappa\right) = (p + d) \varrho^{-1}(a) \delta^\kappa$. Therefore, by the definition of $\text{rad}_\kappa$ in (4.4),

$$\text{rad}_\kappa(euv_1)(a) = (\delta^\kappa(euv_1(\varrho^{-1}(a) \delta^\kappa))) |_{\mathfrak{h}_{\text{reg}}} = (d + p)a.$$  

Thus $\text{rad}_\kappa(euv_1) = euv_8 + d$.

(2) Since the two Euler gradings are defined by conjugation by the respective Euler operators, this follows from Part (1). □

We need to be precise about the filtrations and associated graded objects used in this paper and so we will use the following conventions.

Notation 7.5. Let $\mathfrak{A} = \bigcup_{n \geq 0} \Lambda_n$ and $\mathfrak{B} = \bigcup_{n \geq 0} \Gamma_n$ be filtered $\mathbb{C}$-algebras with a filtered morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$; thus $\varphi(\Lambda_n) \subseteq \Gamma_n$ for all $n$. Write the associated graded ring $\text{gr}_\varphi \mathfrak{A} = \bigoplus \mathfrak{A}_n$, where $\mathfrak{A}_n = \Lambda_n/\Lambda_{n-1}$ and define the principal symbol
maps $\sigma_n : \Lambda_n \rightarrow \Lambda_n' \subseteq \text{gr}_{\Lambda} A$ by $\sigma_n(x) = [x + \Lambda_{n-1}]$ for $x \in \Lambda_n$. In this paper any given ring $A$ will only have one filtration on it and so we can write $\text{gr}_{\Lambda} A = \text{gr} A$ without ambiguity. The associated graded map $\text{gr}_{\phi} : \text{gr} A \rightarrow \text{gr} B$ is defined by $\text{gr}_{\phi}(\sigma_n(x)) = \sigma_n(\phi(x)) \in \Gamma_n$ for any $x \in \Lambda_n$. The restriction of $\text{gr}_{\phi}$ to the $\Lambda_n$ will still be denoted by $\text{gr}_{\phi}$. The commutativity $\sigma_n \circ \phi = \text{gr}_{\phi} \circ \sigma_n$ will be used without comment.

Finally, the morphism $\phi$ is called filtered surjective, respectively filtered isomorphism if each map $\phi : \Lambda_n \rightarrow \Gamma_n$ is surjective, respectively an isomorphism. Equivalently, the associated graded map $\text{gr}_{\phi}$ is surjective, respectively, an isomorphism.

**Lemma 7.6.** Assume that $0 \neq f \in A_n(W)$ is homogeneous, with Euler degree $\deg_{eu} f = k$, and ord $f = \ell$ for some $k \geq \ell \geq 0$. Then $k = \ell$ and $f \in (\text{Sym}\ h)^W_n$.

**Proof.** It is easiest to work in $H_n(W)$, with its natural extension of the order filtration given by Remark 2.13. Let $y_1, \ldots, y_n$ be a basis of $h$. We can decompose $f$ uniquely as

$$f = \sum_{\alpha,w,i} f_{\alpha,w,i}(x)y^\alpha w,$$

where $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, while $w \in W$, $i \in \mathbb{Z}$ and $|\alpha| \leq \ell$. Moreover, $f_{\alpha,w,i}(x) \in C[h]$ is chosen to be homogeneous of degree $i$ (and hence Euler degree $-i$).

Clearly $\text{ord}(y^\alpha) = \deg_{eu} y^\alpha = |\alpha|$ while $\deg_{eu}(f_{\alpha,w,i}(x)) < 0$ for $i \neq 0$. Thus, as $f$ is homogeneous with $\deg_{eu} f = k$ the only terms $f_{\alpha,w,i}(x)y^\alpha w$ appearing in $f$ will have $|\alpha| = i = k$. Since $\text{ord}(f) = \ell = k$, the only terms that can appear in $f$ will therefore have $|\alpha| = k$ and $i = 0$. Since $f \neq 0$, this forces $\ell = k$ and then, as $f_{\alpha,w,0}(x) \in C$, we obtain $f \in (\text{Sym}\ h)_k \otimes CW$. Finally since $f \in eH_n(W)e \subseteq H_n(W)$, this means $f \in (\text{Sym}\ h)^W_k e = (\text{Sym}\ h)^W_k$. \hfill $\square$

**Notation 7.7.** Let $v \in h_{\text{reg}}$. Let $U$ be the orthogonal complement, with respect to $(-,-)$, to $h \oplus g \cdot v$ in $V$. Then $V = h \oplus g \cdot v \oplus U$ is a decomposition of $V$ as an $N_C(h)$-module. Recall that the summand $g \cdot v = g \cdot h$ is independent of the choice of $v \in h_{\text{reg}}$ by Lemma 3.4. The projection $V \rightarrow h$ along $g \cdot v \oplus U$ defines an algebra surjection $g^* : \text{Sym}\ V \rightarrow \text{Sym}\ h$. By [BLLT, Proposition 3.2], $g^*$ restricts to a graded algebra isomorphism $g^* : (\text{Sym}\ V)^G \rightarrow (\text{Sym}\ h)^W$.

**Theorem 7.8.** The radial parts map $\text{rad}_c : \mathcal{D}(V)^G \rightarrow A_n(W)$ restricts to the graded isomorphisms $g : \mathbb{C}[V]^G \rightarrow \mathbb{C}[h]^W$ and $\xi : (\text{Sym}\ V)^G \rightarrow (\text{Sym}\ h)^W$.

**Remark 7.9.** We do not claim that $\xi = g^*$ in Theorem 7.8, though we have no reason to doubt this equality.

**Proof.** It is clear from its definition that $\text{rad}_c$ is just the restriction isomorphism $g : \mathbb{C}[V]^G \rightarrow \mathbb{C}[h]^W$ when applied to $\mathbb{C}[V]^G$, so is certainly a graded isomorphism.

We now turn to $(\text{Sym}\ V)^G$. Here, we first prove the injectivity of $\xi$ for $\text{rad}_c$; that is, when $\xi = 0$. This step follows from [Sc1, Corollary 5.10], but we prefer to give an alternate proof. All the earlier results of this paper apply to the case $\xi = 0$; thus $\text{Im}(\text{rad}_c) \subseteq A_{\xi(0)}(W)$ for some $\xi(0)$.

Let $0 \neq D \in (\text{Sym}\ V)^G_n$. As noted in equations (5.2) and (5.3), $\text{rad}_c = \tilde{g} \circ \eta$ where $\tilde{g}$ is the identification $\mathcal{D}(V_{\text{reg}}/G) = \mathcal{D}(h_{\text{reg}}/W)$ and $\eta : \mathcal{D}(V_{\text{reg}}/G) \rightarrow \mathcal{D}(V_{\text{reg}}/G)$ denotes restriction of functions. Therefore, in order to prove that $\text{rad}_c(D) \neq 0$, it suffices to prove that $\eta(D) \neq 0$, or, equivalently, to find $0 \neq f \in \mathbb{C}[V]^G$ with $D \ast f \neq 0$. Now $\mathbb{C}[V]$ is an equivariant left $\mathcal{D}(V)$-module and so the morphism $\text{Sym}\ V \otimes \mathbb{C}[V] \rightarrow \mathbb{C}[V]$ given by $(E, f) \mapsto E \ast f$ is $G$-equivariant. There certainly exists $f \in \mathbb{C}[V]$ such that $D \ast f \in \mathbb{C} \setminus \{0\}$ and, by taking an isotypic component of $\mathbb{C}[V]$, we may further assume that $f$ lies in an irreducible $G$-representation. As
\( \mathbb{C} \) and \( D \) lie in the trivial \( G \)-isotypic component, necessarily \( f \) does too. In other words we have found \( f \in \mathbb{C}[V]^G \) such that \( D \ast f \neq 0 \). Thus \( \eta(D) \neq 0 \) and hence \( \text{rad}_0(D) \neq 0 \).

By Lemma 7.4, \( \text{rad}_0(D) \) is homogeneous of graded degree \( \text{gr}_{\text{even}}(\text{rad}_0(D)) = n \) and, by Lemma 7.3, \( \text{ord}_b(\text{rad}_0(D)) \leq n \). Thus, by Lemma 7.6, \( \text{rad}_0(D) \in (\text{Sym} \mathfrak{h})^W_n \), and hence \( \text{ord}_b(D) = n \). Since \( g \) is a filtered isomorphism, this implies that the order \( \text{ord}_f(G)(\eta(D)) \) of \( \eta(D) \) in \( \mathcal{D}(V//G) \) also equals \( n \).

We now return to the case of general \( \zeta \) and again consider \( 0 \neq D \in (\text{Sym} \mathfrak{V})^G_1 \). Thus \( n = \text{ord}_f(G)(\eta(D)) \) by the last paragraph. We can find elements \( f_1 \in \mathbb{C}[V]^G \) such that the \( n \)-fold commutator \( z = [\cdots [\eta(D), f_1], \cdots, f_n] \neq 0 \). (When \( n = 0 \), interpret this as saying that \( z = \eta(D) \in \mathbb{C}[V]^G \).) In either case, by the definition of order, \( z \in \mathbb{C}[V]^G \).

But \( n \) is also equal to the order \( \text{ord}_V D \) back in \( \mathcal{D}(V) \). Thus, by the definition of order, \( z' = [\cdots [D, f_1], \cdots, f_n] \in \mathbb{C}[V]^G \), where the commutation now takes place in \( \mathcal{D}(V) \). Moreover, \( z' \neq 0 \) as \( \eta(z') = z \). (In fact \( z = z' \), but this is not important.)

Now apply the ring homomorphism \( \text{rad}_z \). By definition, \( \text{rad}_z \) restricts to give the isomorphism \( \psi : \mathbb{C}[V]^G \rightarrow \mathbb{C}[\mathfrak{h}]^W \) when applied to \( \mathbb{C}[V]^G \). Thus

\[
[\cdots [\text{rad}_z(D), \text{rad}_z(f_1)], \cdots, \text{rad}_z(f_n)] = \text{rad}_z(z) \neq 0.
\]

Therefore, \( \text{rad}_z(D) \neq 0 \) and so \( \text{rad}_z \) does indeed restrict to give an injection \( (\text{Sym} \mathfrak{V})^G_n \hookrightarrow A_\kappa(W) \). Finally, as in the case \( \zeta = 0 \), \( \text{rad}_z(D) \) is homogeneous of graded degree \( \text{gr}_{\text{even}}(\text{rad}_z(D)) = n \) by Lemma 7.4, while \( \text{ord}_b(\text{rad}_z(D)) \leq n \) by Lemma 7.3. Thus, by Lemma 7.6 \( \text{rad}_z(D) \in (\text{Sym} \mathfrak{h})^W_n \), with \( \text{ord}_b(D) = n \).

We have therefore proved that the restriction of \( \text{rad}_z \) to \( (\text{Sym} \mathfrak{V})^G_n \) gives an injection \( \xi : (\text{Sym} \mathfrak{V})^G_n \hookrightarrow (\text{Sym} \mathfrak{h})^W_n \) for each \( n \). It remains to prove that \( \xi \) is bijective. However, as observed in Notation 7.7, each graded piece \( (\text{Sym} \mathfrak{V})^G_n \) is finite dimensional, with \( (\text{Sym} \mathfrak{V})^G_n \cong (\text{Sym} \mathfrak{h})^W_n \). Thus in order to prove that \( \xi \) is bijective, it suffices to prove that it is injective. Since this was the conclusion of the last paragraph, \( \xi \) is indeed a graded isomorphism.

Theorem 7.8 immediately implies one of the main aims of this section, by proving Theorem 7.1(1).

**Theorem 7.10.** If \( A_\kappa(W) \) is a simple ring, then the radial parts map

\[
\text{rad}_z : (\mathcal{D}(V)/\mathcal{D}(V)\kappa)^G \rightarrow A_\kappa(W)
\]

induced from \( \text{rad}_z \) is surjective.

**Proof.** Let \( R = \text{Im}(\text{rad}_z) \). Thus, by Theorem 5.21, \( R \) is a subalgebra of \( A_\kappa = A_\kappa(W) \). On the other hand, by Theorem 7.8, \( R \) contains \( \mathbb{C}[\mathfrak{h}]^W \) and \( (\text{Sym} \mathfrak{h})^W \). However, by [BEG, Theorem 4.6] \( A_\kappa = \mathbb{C}[\mathfrak{h}]^W, (\text{Sym} \mathfrak{h})^W \). Hence \( R = A_\kappa \).

There are numerous cases of spherical algebras where it is known that \( A_\kappa \) is generated by \( \mathbb{C}[\mathfrak{h}]^W \) and \( (\text{Sym} \mathfrak{h})^W \) and so, for these examples, Theorem 7.8 implies that \( \text{rad}_z \) is also surjective. A particular case is given by the following result, which is Part (3) of Theorem 7.1.

**Corollary 7.11.** Suppose that \( W \) is a Weyl group with no factors of type \( E \) or \( F \). Then \( A_\kappa(W) \) is generated by \( \mathbb{C}[\mathfrak{h}]^W \) and \( (\text{Sym} \mathfrak{h})^W \) and so any corresponding radial parts map \( \text{rad}_z \) is surjective.

**Proof.** The first assertion follows from [Wa], as noted in the proof of [EG, Proposition 4.9]. Now apply Theorem 7.8.
The surjectivity of \( \text{rad}_c \) for symmetric spaces. Much of the proof for this case involves filtered and graded techniques, so let \( \mathfrak{A} = \bigcup \Lambda_n(\mathfrak{A}) \) and \( \mathfrak{B} = \bigcup \Gamma_n(\mathfrak{B}) \) be filtered algebras with a filtered map \( \phi : \mathfrak{A} \to \mathfrak{B} \), and keep the conventions from Notation 7.5. We further assume that \( \mathfrak{A}_0 = \mathfrak{B}_0 = B \) is a commutative domain, with a multiplicatively closed subset \( C \subseteq B \) which acts ad-nilpotently on both \( \mathfrak{A} \) and \( \mathfrak{B} \). Then \( C \) acts as an Ore set in each of \( \mathfrak{A} \), \( \mathfrak{B} \), \( \text{gr} \mathfrak{A} \) and \( \text{gr} \mathfrak{B} \). We can then filter the localisation \( \mathfrak{A}_c \) by \( \mathfrak{A}_c = \bigcup \Lambda_n, e \), for \( \Lambda_n, e = B e \otimes_B \Lambda_n \), with the analogous definitions apply in the other three cases. One needs to be wise to the fact that it is quite possible for there to exist \( x \in \Lambda_n \setminus \Lambda_{n-1} \) for which \( x \in \Lambda_{n-1} \); this happens precisely when there exists \( e \in \mathfrak{C} \) such that \( cx \in \Lambda_{n-1} \).

The next result is standard, but we give the proof since it will be important later.

**Lemma 7.12.** Keep the above notation. Then

1. The canonical map \( \alpha : \text{gr}(\mathfrak{A}_c) \to (\text{gr} \mathfrak{A})_c \) is an isomorphism of graded rings.
2. Let \( \varphi_c : \mathfrak{A}_c \to \mathfrak{B}_c \) be the induced map. Then there is a commutative diagram of graded rings

\[
\begin{array}{ccc}
\text{gr}(\mathfrak{A}_c) & \xrightarrow{\text{gr}(\varphi_c)} & \text{gr}(\mathfrak{B}_c) \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
(\text{gr} \mathfrak{A})_c & \xrightarrow{(\text{gr} \varphi)_c} & (\text{gr} \mathfrak{B})_c
\end{array}
\]

**Proof.** (1) Since \( B e \) is flat over \( B \), for each \( n \) we have an exact sequence

\[
0 \to B e \otimes_B \Lambda_{n-1} \to B e \otimes_B \Lambda_n \to B e \otimes_B \Lambda_n \to 0.
\]

By definition, \( \Lambda_{n,e} := \Lambda_{n,e}/\Lambda_{n-1,e} = B e \otimes_B \Lambda_n/B e \otimes_B \Lambda_{n-1} \). Therefore, for each \( n \), we have well-defined isomorphisms of \( B e \)-modules

\[
\alpha_n : \Lambda_{n,e} = \Lambda_{n,e}/\Lambda_{n-1,e} \xrightarrow{\sim} B e \otimes \Lambda_n.
\]

Finally, the graded morphism \( \alpha : \text{gr}(\mathfrak{A}_c) \to (\text{gr} \mathfrak{A})_c \) is given by \( \alpha = \bigoplus_n \alpha_n \), and so \( \alpha \) is an isomorphism of \( B \)-modules and also of graded rings.

(2) If \( b \otimes x \in B e \otimes_B \Lambda_n \), then \( \varphi_c(b \otimes x) \in B e \otimes_B \Gamma_n \). Thus \( \varphi_c \) is a filtered morphism and so induces the graded morphism \( \text{gr}(\varphi_c) : \text{gr}(\mathfrak{A}_c) \to \text{gr}(\mathfrak{B}_c) \). The rest of the proof is a simple diagram chase that is left to the reader. \( \square \)

We now return to the rings \( \mathfrak{A} = \mathfrak{D}(V)^G \) and \( \mathfrak{B} = M_n(W) \) with the morphism \( \text{rad}_c \). We always take \( C = \{ \delta^n \}, \subseteq B = \mathfrak{C}[V]^G \cong \mathfrak{C}[h]^W \) and will usually write \( \mathfrak{A}_c = \mathfrak{A}_{\text{reg}} \), etc. Set \( J = \ker(\text{rad}_c) \). Since \( \mathfrak{D}(\mathfrak{h}_{\text{reg}}) \) is a domain, the kernel of \( \text{rad}_c : \mathfrak{D}(V_{\text{reg}})^G \to \mathfrak{D}(\mathfrak{h}_{\text{reg}})^W \) is equal to \( J \otimes_{\mathfrak{C}[V]^G} \mathfrak{C}[V_{\text{reg}}]^G \) and so we can write this as \( J_{\text{reg}} \) without ambiguity.

We will always use the order filtration \( \text{ord} = \text{ord}_{\mathfrak{D}(V)} \) on \( \mathfrak{D}(V) \) with the induced filtration on both the subring \( \mathfrak{D}(V)^G \) and its factor ring \( \mathfrak{D}(V)^G/J \), with the same notation. Similarly, we use the order filtration, again written \( \text{ord} \), on \( \mathfrak{D}(\mathfrak{h}_{\text{reg}}) \) and its subalgebra \( M_n(W) \). Finally, we use the filtration \( \text{ord} \) induced from \( \mathfrak{D}(V)^G/J \) on the localisation \( (\mathfrak{D}(V)^G/J)_e = \mathfrak{D}(V_{\text{reg}})^G/J_{\text{reg}} \). We remark that there is a second filtration on \( \mathfrak{D}(V_{\text{reg}})^G/J_{\text{reg}} \) induced from the order filtration on \( \mathfrak{D}(V_{\text{reg}}) \) but, as the order of an element \( d \in \mathfrak{D}(V) \) does not change upon passage to \( \mathfrak{D}(V_{\text{reg}}) \) it follows easily that this equals our chosen filtration.

As observed in Notation 7.5, we have only one choice of filtration on any given ring \( \mathfrak{A} \) and so we can write the associated graded ring as \( \text{gr} \mathfrak{A} \) without ambiguity. Thus, in the above notation we obtain the induced map

\[
\text{gr}(\mathfrak{D}(V)^G/J) \to \text{gr}(\text{Im}(\text{rad}_c)) \subseteq \text{gr} M_n(W) = (\mathfrak{C}[h] \otimes \text{Sym} \mathfrak{h})^W.
\]
By Lemma 7.12 this commutes with localisation at C. Lemma 7.12(2) implies the equality $gr(J_{\text{reg}}) = (gr J)_{\text{reg}}$ and so we can write this as $gr J_{\text{reg}}$ without ambiguity.

**Notation 7.14.** Set $C = (\mathbb{C}[V] \otimes \text{Sym } V)^G$, graded by the $(\mathbb{C}[V] \otimes \text{Sym } V)^G$ and filtered by $\Lambda_{n,c} = \sum_{j \leq n} (\mathbb{C}[V] \otimes (\text{Sym } V))^G$. We use the analogous notation for $C_{\text{reg}}$ and write $\Gamma_n = \bigcup \Gamma_{n,Q}$ for the analogous filtration on $Q = (\mathbb{C}[h] \otimes \text{Sym } h)^W$.

As in Notation 7.7, let $g \otimes g^* : C \to Q$ be the map given by restriction on the first factor and projection on the second factor. Set $I = \ker(g \otimes g^*)$.

Since $C = \text{gr } D(V)^G$ and $Q = \text{gr } A_n(W)$, there exists a second morphism $\text{gr}_{\text{rad}_c} : C \to Q$ induced from $\text{rad}_c$ and, in theory, this may differ from the map $g \otimes g^*$. However, the maps are closely related as the next lemma describes. Let $Y = \text{Im } \text{gr}_{\text{rad}_c}$, a subalgebra of $\text{gr } A_n(W) = Q$.

**Lemma 7.15.** Assume that $gr J_{\text{reg}} \subseteq I_{\text{reg}}$.

1. $gr J \subseteq I = L$, where $L = \ker(\text{gr}_{\text{rad}_c} : C \to Q)$.
2. $\mathbb{C}[h]^W \otimes (\text{Sym } h)^W \subseteq Y \subseteq Q$ and hence $Q$ is a finitely generated module over the noetherian domain $Y$.
3. Moreover, $Q_{\mathbb{C}} = Y_{\mathbb{C}}$ for $C = \{h^n\}_n$ and so $Q$ and $Y$ have the same field of fractions. In particular, they are equivalent orders.

**Remark 7.16.** Assume that $gr J_{\text{reg}} = I_{\text{reg}}$. By Lemma 7.15(1), $gr_{\text{rad}_c}$ induces a map $C/I \to Q$ with image $Y$. For simplicity, we also denote this map by $gr_{\text{rad}_c}$.

**Proof.** (1) As the ring $Q$ is $\mathbb{C}$-torsionfree, $I = C \cap I_{\text{reg}}$ and hence $gr J \subseteq C \cap gr J_{\text{reg}} = C \cap I_{\text{reg}} = I$. For the same reason, $L = I_{\text{reg}} \cap C$. Moreover, $gr J \subseteq L$ by the construction of $gr_{\text{rad}_c}$ and so

$$L = I_{\text{reg}} \cap C \supseteq gr J_{\text{reg}} \cap C = I_{\text{reg}} \cap C = I,$$

as required.

(2) Let $D \in (\text{Sym } h)^W \subseteq A_n(W)$. By Theorem 7.8, $\text{rad}_c$ restricts to give a filtered isomorphism $\xi : (\text{Sym } V)^G \sim \to (\text{Sym } h)^W$ and so $D \in \text{rad}_c((\text{Sym } V)^G)$. Thus, $D = \sigma(D) \in \text{Im}(\text{gr}_{\text{rad}_c})$. By Theorem 7.8, again, $\text{rad}_c$ restricts to give the filtered isomorphism $g : \mathbb{C}[V]^G \sim \to \mathbb{C}[h]^W$ and so the same argument ensures that $\mathbb{C}[h]^W \subseteq \text{Im}(\text{gr}_{\text{rad}_c})$. Therefore, $(\mathbb{C}[h]^W \otimes (\text{Sym } h)^W) \subseteq Y \subseteq Q$. Since $\mathbb{C}[h] \otimes \text{Sym } h$ is a finitely generated $(\mathbb{C}[h]^W \otimes (\text{Sym } h)^W)$-module, Part (2) follows immediately.

(3) Since $f(\mathbb{A}e) = f(\mathbb{A}e)$ for any ring homomorphism $f : \mathbb{A} \to \mathbb{B}$ as above, it follows that $Y_{\mathbb{C}}$ equals the image of $\text{gr}_{(\text{rad}_c)_{\mathbb{C}}} : C_{\text{reg}}/gr J_{\text{reg}} \to Q_{\mathbb{C}}$. It follows from Lemma 5.4 that $\text{gr}_{(\text{rad}_c)_{\mathbb{C}}}$ is surjective. Thus, $Y_{\mathbb{C}} = Q_{\mathbb{C}}$, and so $Y$ and $Q$ certainly have the same field of fractions. Combined with Part (2) this implies that they are equivalent orders.

Recall that the moment map $\mu : V \times V^* \to \mathfrak{g}^*$ is given by $\mu(x, v)(y) = \langle y, x \rangle$. Let $I(\mu^{-1}(0))$ be the ideal in $\mathbb{C}[V] \otimes \text{Sym } V$ of functions vanishing on $\mu^{-1}(0)$.

**Lemma 7.17.** If $V$ is stable then $\text{gr } J_{\text{reg}} = I_{\text{reg}}$.

**Proof.** By Lemma 5.4, $\text{rad}_c : \mathcal{D}(V_{\text{reg}})^G \to \mathcal{D}(h_{\text{reg}})^W$ is filtered surjective. Since $J_{\text{reg}} = \ker(\text{rad}_c)_{\text{reg}}$, this implies that $(\text{rad}_c)_{\mathbb{C}} : \mathcal{D}(V_{\text{reg}})^G/J_{\text{reg}} \to \mathcal{D}(h_{\text{reg}})^W$ is a filtered isomorphism and so the map $gr(\text{rad}_c)_{\mathbb{C}} : C_{\text{reg}}/gr J_{\text{reg}} \to Q_{\text{reg}}$ is a graded isomorphism and $gr J_{\text{reg}}$ is a prime ideal of height $q = \text{Kdim } C - 2 \dim \mathfrak{h}$.

Clearly $(\mathcal{D}(V_{\text{reg}})_{\mathbb{B}h})^G \subseteq J_{\text{reg}}$. By the definition of $\mathbb{B}h$, the spaces of symbols $\{\sigma_1(y) : y \in \mathbb{B}h\}$ and $\{\sigma_1(x) : x \in \mathfrak{g}\}$ are equal. Thus the set of common zeros of the symbols $\{\sigma_1(x) : x \in \mathfrak{g}\}$ is precisely $\mu^{-1}(0) \subseteq T^* V_{\text{reg}}$. Therefore, $\mathcal{K} := I(\mu^{-1}(0))_{\text{reg}}^G \subseteq \text{gr } J_{\text{reg}}$. 


The (reduced) subscheme of \((V \oplus V^*)//G\) defined by the ideal \(I\) is denoted \(C_0//G\) in [BLLT]. Then [BLLT, Proposition 3.5] says that \(C_0//G\) is a closed subscheme of \(\mu^{-1}(0)//G\). Therefore, \(I(\mu^{-1}(0)/G) \subseteq I\) and, in particular, \(\mathcal{X} \subseteq I_{reg}\).

Using the fact that \(V\) is stable, the proof of [BLLT, Lemma 4.2] shows that \(\mu^{-1}(0)\) is irreducible; thus the radical \(\sqrt{\mathcal{X}}\) is a prime ideal. Moreover, as \(C_{reg} = (\mathbb{C}[V_{reg}] \otimes \text{Sym} V)^G\), [BLLT, Proposition 3.5] implies that \(\text{Kdim} C_{reg}/\mathcal{X} = 2 \dim \mathfrak{h} = \dim Q = \text{Kdim} C/I_{reg}\).

As \(\sqrt{\mathcal{X}} \subseteq I_{reg}\), it follows that \(\sqrt{\mathcal{X}} = I_{reg}\), which is therefore a prime ideal of height \(q\). But \(\mathcal{X} \subseteq \text{gr} I_{reg}\) which also has height \(q\). Thus \(\text{gr} I_{reg} = \sqrt{\mathcal{X}} = I_{reg}\).

We are now ready to prove the main result of this section, which is also Part (3) of Theorem 7.1.

**Theorem 7.18.** Assume that \(V\) is stable and the map \(\varrho \otimes \varrho^*\) is surjective. Then the maps

\[
\text{gr}_{\text{rad}} : (\mathbb{C}[V] \otimes \text{Sym} V)^G \to (\mathbb{C}[\mathfrak{h}] \otimes \text{Sym} \mathfrak{h})^W
\]

and

\[
\text{rad} : \mathcal{D}(V)^G \to A_\chi(W)
\]

are both surjective.

**Proof.** It suffices to prove the surjectivity of \(\text{gr}_{\text{rad}}\).

By the hypothesis of the theorem, the map \(C/I \to Q\) induced from \(\varrho \otimes \varrho^*\) is surjective. It is standard, see [BH, Proposition 6.4.1], that \(Q = (\mathbb{C}[\mathfrak{h}] \otimes \text{Sym} \mathfrak{h})^W\) is a normal domain. Thus \(C/I\) is a normal domain (equivalently, a maximal order). By Lemma 7.17, the hypotheses and hence the conclusions of Lemma 7.15 hold.

By Lemma 7.15(2), \(Y = \text{gr}_{\text{rad}}(C/I) \supseteq \mathbb{C}[\mathfrak{h}]^W \otimes (\text{Sym} \mathfrak{h})^W\). Since \(Q\) is a finite module over \(Y\) one has \(\text{Kdim} Y = \text{Kdim} Q = \text{Kdim} C/I\). Then, as \(C/I\) is a domain, \(\text{Kdim} Y = \text{Kdim} C/I\) implies that the surjection \(\text{gr}_{\text{rad}} : C/I \to Y\) is an isomorphism and so \(Y \cong C/I\) is a maximal order.

On the other hand, \(Y \subseteq Q\) and Lemma 7.15(3) says that \(Y\) is an equivalent order to \(Q\). As \(Y\) is a maximal order this forces \(Y = Q\), whence \(\text{gr}_{\text{rad}}\) is surjective.

The significance of Theorem 7.18 is that it has the following important consequence. The definition of symmetric spaces, together with their basic properties, will be given in Section 8.

**Theorem 7.19.** Suppose that \(V\) is a symmetric space for the connected reductive group \(G\). Then the radial parts map \(\text{rad} : \mathcal{D}(V)^G \to A_\chi(W)\) is surjective.

**Proof.** By Lemma 8.2, \(V\) is indeed a stable polar representation of \(G\). Moreover, by [Te, Theorem], the map \((\mathbb{C}[V] \otimes \text{Sym} V)^G \to \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W\) is surjective. Thus the result follows from Theorem 7.18.

It has recently been shown in [BLLT] that \(\mathbb{C}[\mu^{-1}(0)]^G \to \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W\) is an isomorphism for a large class of polar representations and, as a consequence, we get the following result, which proves Theorem 7.1(4). The relevant definitions are as follows. The representation \(V\) of \(G\) is called **locally free** if the stabiliser subgroup of a general point in \(V\) is finite and is said to be **visible** if the nilcone \(\mathcal{N}(V) = \pi^{-1}(0)\) consists of finitely many \(G\)-orbits. Finally, let \(\text{rad}_\chi : (\mathcal{D}(V)/\mathcal{D}(V)\varrho_\chi)^G \to A_\chi(W)\) denote the map induced from \(\text{rad}_\chi\).

**Corollary 7.20.** (1) If the map \((\varrho \otimes \varrho^*) : \mathbb{C}[\mu^{-1}(0)]^G \to \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W\) is an isomorphism then \(\text{rad}_\chi\) is a filtered isomorphism.

(2) Suppose that \(V\) is a visible, stable polar representation of \(G\) that is also locally free. Then \(\text{rad}_\chi\) is a filtered isomorphism.
Proof. (1) As noted in the proof of Lemma 7.17, \( I(\mu^{-1}(0)) \) is generated by the symbols \( \{\sigma_i(x) : x \in \mathfrak{g}_i\} \) and so \( I(\mu^{-1}(0)) \subseteq \text{gr}_{\text{ord}} J \). Combining Lemma 7.17 with the fact that \( \mathfrak{g} \otimes \mathfrak{g}^* \) is an isomorphism therefore shows that \( I(\mu^{-1}(0)) = \text{gr}_{\text{ord}} J = I \). Thus Theorem 7.18 implies that \( \text{gr}_{\text{rad}} \) is surjective with kernel \( \text{gr}_{\text{ord}} J \). It also follows that \( J = (\mathcal{D}(V)_{\mathfrak{g}_i})^G \) and hence \( \text{rad}_J \) is a filtered isomorphism.

(2) This follows from Part (1) combined with [BLLT, Theorem 1.2].

The locally free hypothesis in Corollary 7.20(2) is obviously very strong—for example, it is not even satisfied in the basic case when \( V = \mathfrak{g} \) for a simple algebraic group \( G \). A number of examples satisfying the hypotheses of the corollary are given in [BLLT], with one of their motivating examples being the representation \( V = S^3 \mathbb{C}^3 \) over \( G = SL(3) \); see [BLLT, Example 5.6] and also [BNS, Example 15.3].

8. Reductive symmetric pairs

In the next two sections we discuss one of the main examples of polar representations, that of symmetric spaces. In this section we concentrate on the image of \( \text{rad}_G \) and we determine, at least for \( \varsigma = 0 \), exactly when \( A_{\varsigma} = \text{Im}(\text{rad}_G) \) is a simple ring. This proves the first half of Theorem 1.4 from the introduction. The proof of that result will be completed in Section 9.

Definition 8.1. We begin with the basic definitions. Let \( \tilde{G} \) be a connected, complex reductive algebraic group with Lie algebra \( \tilde{\mathfrak{g}} \). Fix a non-degenerate, \( \tilde{G} \)-invariant symmetric bilinear form \( \varpi \) on \( \tilde{\mathfrak{g}} \) that reduces to the Killing form on \( [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \). Let \( \vartheta \) be an involutive automorphism of \( \tilde{\mathfrak{g}} \) preserving \( \varpi \) and set \( \mathfrak{g} = \ker(\vartheta - I) \), \( \mathfrak{p} = \ker(\vartheta + I) \).

Then, \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{p} \) and the pair \( (\tilde{\mathfrak{g}}, \vartheta) \) (or \( (\tilde{\mathfrak{g}}, \mathfrak{g}) \)) is called a symmetric pair with symmetric space \( V := \mathfrak{p} \). Let \( G \) be the connected reductive subgroup of \( \tilde{G} \) such that \( \mathfrak{g} = \text{Lie}(G) \). Both \( G \) and its Lie algebra \( \mathfrak{g} \) act on \( \mathfrak{p} \) via the adjoint action: \( x \cdot v = \text{ad}(x)(v) = [x, v] \) for all \( x \in \mathfrak{g} \) and \( v \in \mathfrak{p} \).

Finally, let \( \mathfrak{h} \subseteq \mathfrak{p} \) be a Cartan subspace; thus, \( \mathfrak{h} \) is maximal among the subspaces of \( \mathfrak{p} \) consisting of semisimple elements in \( \tilde{\mathfrak{g}} \) which pairwise commute, see [TY, Corollary 37.5.4]. The Weyl group is then defined to be \( W = N_G(\mathfrak{h})/Z_G(\mathfrak{h}) \).

Any symmetric pair \( (\tilde{\mathfrak{g}}, \vartheta) \) is a direct product \( \tilde{\mathfrak{g}} \cong (\tilde{\mathfrak{g}}_1, \vartheta_1) \times \cdots \times (\tilde{\mathfrak{g}}_m, \vartheta_m) \) of irreducible symmetric pairs, which then decomposes \( (G, \mathfrak{p}) \) into a corresponding direct product of sub-symmetric spaces, although when \( (\tilde{\mathfrak{g}}, \vartheta) \) is irreducible, \( \tilde{\mathfrak{g}} \) may be reducible. For a classification of irreducible symmetric spaces, see the tables in [He, Chapter X] or, in a form more convenient for this paper, the tables in Appendix B. Symmetric spaces \( (G, \mathfrak{p}) \) are a natural generalisation of the adjoint representation of \( G \) on \( \mathfrak{g} \). Specifically, in the diagonal case where \( \tilde{G} = G \times G \) with \( \vartheta(x, y) = (y, x) \), one has \( (\tilde{\mathfrak{g}}, \mathfrak{g}) = (\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}) \) with the natural adjoint action of \( G \) on \( \mathfrak{p} = \mathfrak{g} \).

As we next show, symmetric spaces are examples of polar representations. We recall from the discussion at the beginning of Section 3 that \( \mathfrak{p}_s \) denotes the set of \( \mathfrak{p} \)-regular elements of \( \mathfrak{p} \).

Lemma 8.2. The symmetric spaces \( (G, \mathfrak{p}) \) are stable polar representations. Moreover, in the notation of Section 3, one has \( \mathfrak{p}_{\text{reg}} = \mathfrak{p}_s \).

Proof. The representations \( (G, \mathfrak{p}) \) are particular cases of polar representations, see [DK] and [PV, (8.6)]. In our notation, [KR, Remark 6, p.771] says that

\[ \{v \in \mathfrak{p} : \dim G \cdot v \text{ maximal}\} \cap \mathfrak{p}_{\text{ass}} = \mathfrak{p}_{\text{reg}}, \]

and hence that \( \mathfrak{p}_{\text{reg}} = \mathfrak{p}_s \). As \( \mathfrak{p}_{\text{reg}} \neq \emptyset \), this also implies that \( \mathfrak{p} \) is stable. \( \square \)
Our first aim is to understand exactly when the image $A_{\kappa}$ of the radial parts map is simple. Before giving the details we notice a couple of simplifications to the problem.

**Remark 8.3.** In constructing and understanding radial parts maps, one can always restrict to the case of irreducible symmetric pairs. Since this is true more generally, in order to justify this assertion let $(G_1, V_1)$ be polar representations with associated data $\{h_1, W_1, \varsigma_1, \kappa_1\}$, where $\kappa_i$ is the parameter introduced in (5.22) for $h_i$. Then $V = V_1 \oplus V_2$ is a polar representation for the group $G = G_1 \times G_2$, with associated data $h = h_1 \oplus h_2$ and so forth. Note in particular that, from scalars $\varsigma = (\varsigma_1, \varsigma_2) \in \mathbb{C}^{k_1+k_2}$, the parameter $\kappa$ defined in (5.22) for $(G, V)$ is just $\kappa_1 + \kappa_2$. It follows that $A_{\kappa} = A_{\kappa_1} \otimes \mathbb{C} A_{\kappa_2}$ and so, by Theorem 7.19, the map $\text{rad}_p : \mathbb{D}(V)^G \rightarrow A_{\kappa}$ is the map

$$\text{rad}_{\kappa_1} \otimes \text{rad}_{\kappa_2} : \mathbb{D}(V_1)^{G_1} \otimes \mathbb{D}(V_2)^{G_2} \rightarrow A_{\kappa_1} \otimes \mathbb{C} A_{\kappa_2}.$$  

From Corollary 2.10 we then deduce the following.

**Lemma 8.4.** Suppose that $(G, V) = (G_1, V_1) \oplus (G_2, V_2)$ is a direct sum of polar representations and keep the above notation. Then $A_{\kappa}$ is simple if and only if both $A_{\kappa_1}$ and $A_{\kappa_2}$ are simple. \hfill \Box

**Remark 8.5.** Let $(\mathfrak{g}, \vartheta)$ be a symmetric pair with symmetric space $\mathfrak{p}$ and Cartan subspace $\mathfrak{h} \subseteq \mathfrak{p}$. Recall from (3.3) that $\text{rank} \mathfrak{p} = \dim \mathfrak{h} - \dim \mathfrak{h}^\vartheta$. If $\text{rank} \mathfrak{p} = 0$, we define $(\mathfrak{g}, \vartheta)$ to be a trivial symmetric pair, with trivial symmetric space $(G, \mathfrak{p})$. Write $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{z}$, where $\mathfrak{g}^0 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}$ is the centre of $\mathfrak{g}$. The reason for calling the rank zero case trivial is that then $\mathfrak{h} = \mathfrak{h}^\vartheta \subseteq \mathfrak{z}$, from which it follows that $\mathfrak{p} = \mathfrak{h} \subseteq \mathfrak{z}$. In particular, $G$ acts trivially on $\mathfrak{p}$ and so $\mathbb{D}(\mathfrak{p}) = \mathbb{D}(\mathfrak{p})^G = \mathbb{D}(\mathfrak{h})^\vartheta$. Hence any radial parts map for a trivial symmetric space is the identity.

When $\vartheta = 1$, $\mathfrak{g} = \mathfrak{g}$ and so $\mathfrak{p} = 0$, which is certainly trivial. We do not insist that $\vartheta \neq 1$, since the case $\vartheta = 1$ can appear among sub-symmetric pairs of symmetric pairs with $\vartheta \neq 1$.

Let $(\mathfrak{g}, \vartheta)$ be a symmetric pair and recall that our first goal is to understand when $A_{\kappa} = \text{Im}(\text{rad}_\varsigma)$ is simple. When $\mathfrak{g}$ is semisimple, it suffices to study the question when $\varsigma = 0$.

**Proposition 8.6.** Assume that $(\mathfrak{g}, \vartheta)$ is a symmetric pair such that the centre of $\mathfrak{g}$ acts trivially on $\mathfrak{p}$. Then, $\text{Im}(\text{rad}_\varsigma) \cong \text{Im}(\text{rad}_0)$ and $\ker(\text{rad}_\varsigma) = \ker(\text{rad}_0)$ for all choices of $\varsigma$.

**Proof.** Write $G = G_1 Z$ where $G_1$ is semisimple and $Z$ is the connected component of the centre of $G$. By hypothesis one has $\mathbb{C}[\mathfrak{p}]^G = \mathbb{C}[\mathfrak{p}]^{G_1}$ and $\mathbb{D}(\mathfrak{p})^G = \mathbb{D}(\mathfrak{p})^{G_1}$. Since $G_1$ is semisimple, the result follows from Corollary 6.8. \hfill \Box

If the centre of $\mathfrak{g}$ does not act trivially on $\mathfrak{p}$ then understanding for which $\varsigma$ the algebra $\text{Im}(\text{rad}_\varsigma)$ is simple is a more complicated problem and we do not have a general answer. Indeed, even when $(\mathfrak{g}, \mathfrak{g}) = (\mathfrak{sl}(2), \mathfrak{so}(2))$, all infinite dimensional primitive factors of $U(\mathfrak{sl}(2))$ appear as $\text{Im}(\text{rad}_\varsigma)$ for some choice of $\varsigma$; see [BNS, Corollary 14.15]. For simplicity, we will therefore stick to the case $\varsigma = 0$, which is also the case of most interest in the literature (in the diagonal case this will suffice to solve the problem in general, but there are additional subtleties, for which the reader is referred to the discussion after Corollary 9.16).

Thus, by definition, we are interested in the image of

$$(8.7) \quad \text{rad}_0(D)(z) = q\left(D(q^{-1}(z))\right) \quad \text{for all } D \in \mathbb{D}(\mathfrak{p})^G \text{ and } z \in \mathbb{C}[\mathfrak{h}]^\vartheta.$$  

We will write $\text{rad} = \text{rad}_0$ and $\kappa = \kappa(0)$ throughout the remainder of this section.
In order to describe \( \kappa \) we need some standard notation. Given a symmetric pair \((\mathfrak{g}, \mathfrak{h})\), let \( \Sigma = \Sigma(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^* \) denote the (restricted) root system defined by \( \mathfrak{h} \); see [TY, 38.2.3]. (If \((\mathfrak{g}, \mathfrak{h})\) is trivial then, by definition, \( R = \emptyset = \Sigma \).) Let \( \mathcal{R} = R(p, \mathfrak{h}) = \Sigma_{\text{red}} = \{ \alpha \in \Sigma : \alpha / 2 \notin \Sigma \} \) be the reduced root system associated to \( \Sigma \). Choose a set of positive roots \( \Sigma^+ \subset \Sigma \) and set \( \mathcal{R}^+ = \Sigma^+ \cap \Sigma_{\text{red}} \). Then \( \mathcal{A} = \{ H_\alpha = \ker \alpha : \alpha \in \mathcal{R}^+ \} \) and \( W \) is generated by the real reflections \( s_\alpha = s_{H_\alpha} \); in particular \( \ell_{H_\alpha} = 2 \) for all \( \alpha \).

For \( \alpha \in \Sigma \) set

\[
q^\alpha = \{ v \in q : \text{ad}(a)^2(v) = \alpha(a)^2 v \text{ for all } a \in \mathfrak{h} \},
\]

when \( q = \mathfrak{p} \) or \( q = \mathfrak{g} \), but

\[
\tilde{g}^\alpha = \{ v \in \tilde{g} : \text{ad}(a)(v) = \alpha(a)v \text{ for all } a \in \mathfrak{h} \}.
\]

Then, \( \tilde{g}^{-\alpha} = \vartheta(\tilde{g}^\alpha) \) and \( \tilde{g}^\alpha \oplus \tilde{g}^{-\alpha} = \mathfrak{g}^\alpha \oplus \mathfrak{p}^\alpha \) for all \( \alpha \in \Sigma^+ \). The adjoint action of \( \mathfrak{h} \) yields the decompositions (orthogonal with respect to \( \kappa \)):

\[
\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\mu \in \mathcal{R}} (\mathfrak{g}^{\mu} + \mathfrak{g}^{2\mu}) \quad \text{and} \quad \mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\mu \in \mathcal{R}} (\mathfrak{p}^{\mu} + \mathfrak{p}^{2\mu}).
\]

Note that \( \dim \mathfrak{p}^\alpha = \dim \mathfrak{g}^\alpha = \dim \tilde{g}^\alpha \) and we define the multiplicity of \( \alpha \in \Sigma \) by:

\[
m_\alpha = \dim \tilde{g}^\alpha + \dim \tilde{g}^{-2\alpha}
\]

with the convention that \( \tilde{g}^{-2\alpha} = \{ 0 \} \) if \( 2\alpha \notin \Sigma \). Set

\[
k_\alpha = \frac{1}{2} m_\alpha = \frac{1}{2} (\dim \tilde{g}^\alpha + \dim \tilde{g}^{-2\alpha})
\]

**Proposition 8.10.** For a symmetric pair \((\mathfrak{g}, \mathfrak{h})\), define

\[
(8.11) \quad \kappa = (\kappa_{H_\alpha i} : \alpha \in \Sigma^+_{\text{red}}, i = 0, 1), \quad \text{where } \kappa_{H_\alpha,0} = 0 \text{ and } \kappa_{H_\alpha,1} = k_\alpha.
\]

Then \( \text{Im}(\text{rad}) = A_\kappa(W) \).

**Remark 8.12.** If \((\mathfrak{g}, \mathfrak{h})\) is trivial then \( \Sigma = \emptyset \) and we set \( \kappa = 0 \). Of course, by Remark 8.5, \( \text{Im}(\text{rad}) = A_\kappa(W) \) also holds in this case.

**Proof.** This amounts to translating Theorem 5.21 into the present notation. Since \( p_{\text{reg}} = p_s \) by Lemma 8.2, it follows from Lemma 3.9 that \( \text{rank}(G_p, S_p) \geq 1 \) for \( p \in \mathfrak{h} \ominus \mathfrak{h}_{\text{reg}} \). In particular, if \( H \in \mathcal{A} \), then \( \text{rank}(G_p, S_p) = 1 \) for all \( p \in H^0 = H \cap \mathfrak{h}^0 \) by Remark 5.7.

Fix \( \alpha \in \mathcal{R}^+ \) and pick \( p \in H_\alpha^0 \). Thus, \( \alpha(p) = 0 \) and \( \alpha(p) \neq 0 \) for \( p \in \Sigma \ominus \{ \alpha \} \). One can easily show that

\[
\mathfrak{g}_p = Z_{\mathfrak{g}}(H_\alpha) = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha} \quad \text{and} \quad \mathfrak{p}_p = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \mathfrak{p}^{\alpha} \oplus \mathfrak{p}^{2\alpha},
\]

while \( \mathfrak{g} \cdot p = \bigoplus_{\mu \in \Sigma \ominus \{ \alpha \}} \mathfrak{p}^\mu \). Using Remark 5.20, it follows that the slice \((G_p, S_p)\) is given by \( G_p = Z_{\mathfrak{g}}(S_\alpha) \) and \( S_p = \mathfrak{p}_p = \mathfrak{h} \oplus \mathfrak{p}^{\alpha} \oplus \mathfrak{p}^{2\alpha} \) with Weyl group \( W_p = \{ 1, s_\alpha \} \). Let \( \alpha^\vee \) be such that \( s_\alpha(\alpha^\vee) = -\alpha^\vee \) and write \( S_p = S_\alpha \oplus H_\alpha \) where \( S_\alpha = \mathbb{C} \alpha^\vee \oplus \mathfrak{p}^{\alpha} \oplus \mathfrak{p}^{2\alpha} \). Then

\[
\mathbb{C}[S_p]^{G_p} = \mathbb{C}[H_\alpha] \otimes \mathbb{C}[S_\alpha]^{G_p} \cong \mathbb{C}[H_\alpha]^{W_p} = \mathbb{C}[H_\alpha] \otimes \mathbb{C} \otimes \mathbb{C}[\alpha^\vee]^{W_p} = \mathbb{C}[H_\alpha] \otimes \mathbb{C}[\alpha^\vee].
\]

The representation \((G_p, S_\alpha)\) is polar of rank one with

\[
\dim S_\alpha = \dim(p^{\alpha} \oplus p^{2\alpha}) + 1 = m_\alpha + 1.
\]

The restriction of the form \( \kappa \) to \( S_\alpha \) is \( G_p \)-invariant and non-degenerate and so \( \mathbb{C}[S_\alpha]^{G_p} = \mathbb{C}[u] \cong \mathbb{C}[\alpha^\vee]^{W_p} = \mathbb{C}[\alpha^2] \), where \( u(x) = \kappa(x, x) \) for \( x \in S_\alpha \). Therefore the differential operator \( \Delta = u \mid D \in (\text{Sym} S_\alpha)^{G_p} \) used in Proposition 4.10 is the Laplacian operator on the space \( S_\alpha \); that is, \( \Delta = \sum_{i=1}^{m_\alpha+1} \frac{\partial^2}{\partial x_i^2} \) if \( u = \sum_{i=1}^{m_\alpha+1} x_i^2 \).
in an orthonormal coordinate system. The $b$-function attached to $\Delta$ is easy to compute:

$$b(s) = (s+1)(s + \frac{1}{2} \dim S_\alpha) = (s+1)(s + \frac{1}{2}(m_\alpha + 1)).$$

As in Notation 4.13, we set $\lambda_0 = 0$ and $\lambda_1 = \frac{1}{2}(m_\alpha + 1) - 1 = \frac{1}{2}(m_\alpha - 1)$. Then, by Theorem 5.21, the parameter $\kappa$ is given by

$$\kappa_{H_\alpha,0} = 0, \quad \kappa_{H_\alpha,1} = \lambda_1 - \frac{1}{2} + 1 = \frac{1}{2}m_\alpha = k_\alpha,$$

as claimed. Finally, $\text{Im}(\text{rad}) = A_\kappa(W)$ by Theorem 7.19.

**Remark 8.13.** The parameter $\kappa$ defined in (8.11) appears frequently in the theory of symmetric spaces. For instance, it is known that for this parameter the radial components of the invariant constant differential operators $(\text{Sym} p)^G \subset D(p)^G$ can be computed via the Dunkl operators defined by $\kappa$. To give more details it will be convenient to identify the rational Cherednik algebra $H_\kappa(W)$ with the subalgebra of $D(h_{\text{reg}}) \times W$ generated by $C[h]$, $W$ and the Dunkl operators $T_\kappa(y) = T_y$, $y \in h$, as defined in (2.8) for this choice of $\kappa$. The map $T_\kappa$ extends to a morphism on $\text{Sym} h$, and composed with $\varphi^* : (\text{Sym} p)^G \cong (\text{Sym} h)^W$ gives

$$\text{rad}(D) = T_\kappa(\varphi^*(D)) \in A_\kappa(W)$$

for all $D \in (\text{Sym} p)^G \subset D(p)^G$; see, for example, [DL, (6.2)]. If $D \in D(p)^G$ then $\text{rad}(D) \in \text{co} H_\kappa(W) e \subset D(h_{\text{reg}})^W$ can be expressed as a polynomial in elements of $p^*$ and Dunkl operators $T_\kappa(y) = T_y$ for $y \in h$. The formula (8.14) gives this expression when $D \in (\text{Sym} p)^G \subset D(p)^G$. Therefore Proposition 8.10 can be viewed as an extension of that formula.

Let $(G, p)$ be a symmetric space. Set

$$(8.15) \quad \mathcal{K}(p) = \{d \in D(p) : \forall f \in C[p]^G, d(f) = 0\}, \quad \mathcal{L}(p) = \mathcal{K}(p)/D(p) \tau(g)$$

and note that $\mathcal{K}(p) \cap D(p)^G = \ker(\text{rad}_0)$.

**Special classes of symmetric spaces.** Many important properties of the diagonal case $(g \oplus g, g)$ generalise to Sekiguchi’s nice symmetric spaces, defined as follows.

**Definition 8.16.** If the $k_\alpha$ are defined by (8.9), then the symmetric pair $(\tilde{g}, g)$, or the symmetric space $(g, p)$, is **nice** if

$$k_\alpha \leq 1, \quad \text{for all } \alpha \in R.$$

The diagonal case is obviously nice. The reader is referred to [Se, Section 6] for further details and for a classification of nice pairs to [Se, Lemma 6.2]. For a version of these results using notation closer to that of this paper, see [LS4] and especially [LS4, Theorem 2.5].

The main result from [LS4] now gives:

**Theorem 8.18.** Suppose that $(\tilde{g}, \vartheta)$ is a nice symmetric pair with symmetric space $p$ and define $\kappa$ by (8.11). Then $\mathcal{K}(p) = D(p)^G \tau(g)$ and so $\ker(\text{rad}_0) = (D(p)^G \tau(g))^G$. Moreover, the ring $D(p)^G/\ker(\text{rad}_0) \cong A_\kappa(W)$ is simple.

**Proof.** This forms parts of [LS4, Theorems A and D], combined with Proposition 8.10.

The second class of simple spherical algebras arises from [BEG]. Let $\mathcal{H}_q(W)$ be the Hecke algebra associated to a general parameter $\kappa$ as in [BEG, p. 284]. We also need the following definition.
Definition 8.19. Define the symmetric pair \((\tilde{g}, \vartheta)\) and its corresponding symmetric space \((G, p)\) to be integral if the parameter \(\kappa\) from (8.11) takes only integral values; equivalently, if \(k_\alpha \in \mathbb{Z}\) for all \(\alpha \in R\). Define \((\tilde{g}, \vartheta)\) and \((G, p)\) to be robust if \((\tilde{g}, \vartheta)\) is a direct product of nice symmetric pairs and integral symmetric pairs.

A trivial symmetric space is both nice and integral since in this case \(R = \emptyset\), and so the relevant conditions are vacuously satisfied.

The significance of integrality comes from the following result.

Theorem 8.20. [BEG, Theorem 3.1]

1. If \((\tilde{g}, \vartheta)\) is integral then \(\mathcal{H}_\kappa(W)\) is semisimple.
2. If the Hecke algebra \(\mathcal{H}_\kappa(W)\) is semisimple, then the algebra \(A_\kappa(W)\) is simple.



Proof. We briefly explain the notation from [BEG]. First, the parameters \(c_\alpha\) from [BEG] are just the negative of our \(k_\alpha\) (compare (8.11) with Remark A.2). The hypothesis \(c \in \mathbb{C}[R]_{reg}^W\) in [BEG, Theorem 3.1] simply means that the corresponding Hecke algebra \(\mathcal{H}_\kappa(W)\) is semisimple. As observed immediately before [BEG, Corollary 2.3], the hypothesis \(c \in \mathbb{Z}[R]^W\) (meaning that each \(c_\alpha \in \mathbb{Z}\)) also implies that \(c \in \mathbb{C}[R]_{reg}^W\), which gives Part (1). Part (2) of the theorem is then [BEG, Theorem 3.1].

Remark 8.21. Retain the notation from the proof of Theorem 8.20. If \((\tilde{g}, \vartheta)\) is not integral, then one of the parameters \(\kappa_\alpha = -c_\alpha\) belongs to \(\frac{1}{2} + \mathbb{Z}\). It then follows from [Gy2, §3.8 and Theorem 3.9] that the associated Hecke algebra \(H_\kappa(W)\) is not semisimple. Thus the converse of Theorem 8.20(1) also holds.

Corollary 8.22. Assume that the symmetric pair \((\tilde{g}, \vartheta)\) is robust. Then the algebra \(A_\kappa = \text{Im}(\text{rad})\) is simple.

Proof. By Proposition 8.10, \(\text{Im}(\text{rad}) = A_\kappa(W)\). Now combine Lemma 8.4 with Theorems 8.18 and 8.20.

In fact Corollary 8.22 exhausts the symmetric pairs for which \(A_\kappa\) is simple, since we will prove the following stronger result.

Theorem 8.23. Let \((\tilde{g}, \vartheta)\) be a symmetric pair. Then the spherical algebra \(A_\kappa\) is simple if and only if \((\tilde{g}, \vartheta)\) is robust.

Proof. The proof of consists of a case-by-case analysis to show that, for the symmetric spaces not covered by Corollary 8.22, the algebra \(A_\kappa\) is not simple. This is deferred to Appendix A.

9. The kernel of the radial parts map for symmetric spaces

As has been remarked in the introduction, given a polar representation \((G, V)\) then \(\ker(\text{rad}_0)\) always contains \(\langle \mathcal{D}(V)\tau(g) \rangle^G\), but checking equality is an important but much more subtle problem. The main aim of this section is to examine this problem for symmetric spaces and thereby to complete the proof Theorem 1.4. In Corollaries 9.12 and 9.14 we give applications of this result to equivariant eigendistributions. The reader is referred to the discussion after Corollary 9.16 for applications to the diagonal case.

Since \(\ker(\text{rad})\) is described by Theorem 8.18 for nice symmetric spaces, it remains to consider the integral symmetric spaces, as we do below. Suppose that \(\tilde{g}\) is semisimple. Then \((\tilde{g}, g)\) is isomorphic to a product of trivial pairs and irreducible symmetric spaces \((\tilde{g}_i, g_i)\) where either \((\tilde{g}_i, g_i) \cong (s \times s, s)\), with \(s\) simple, or \(g_i\) is simple. From the tables in Appendix B, the integral irreducible symmetric spaces \((\tilde{g}, g)\) are the following:
Lemma 9.1. A symmetric pair $(\mathfrak{g}, \vartheta)$ is integral if and only if it is isomorphic to a product of pairs of type (1)–(4) and trivial pairs.

In our finer analysis of integral symmetric pairs it will be convenient to exclude pairs $(s \times s, s)$ with $s$ simple; partly because, by Remarks 9.3(2), a useful technical condition (9.5) usually fails for such a pair. This will not affect the results since the pairs $(s \times s, s)$ are also nice symmetric spaces. We therefore make the following definition.

Definition 9.2. The symmetric pair $(\mathfrak{g}, \vartheta)$ satisfies (†) if it is an integral symmetric pair with no summands of the form $(s \times s, s)$ for $s$ simple. As usual, the corresponding symmetric space $(G, p)$ satisfies (†) if $(\mathfrak{g}, \vartheta)$ satisfies (†).

For standard definitions concerning $\mathcal{D}(p)$-modules, in particular for the support $\text{Supp} M$ and characteristic variety $\text{Ch} M$ of a $\mathcal{D}$-module $M$, we refer the reader to [LS4, Section 3]. We will frequently identify $p$ with its dual $p^*$ through the non-degenerate $G$-invariant bilinear form $\langle \cdot, \cdot \rangle$. As in [LS4, Section 2], the commuting variety of $p$ is the closed subvariety of $T^*p = p \times p^* \cong p \times p$ defined by $C(p) = \{(x, y) \in p \times p : \langle x, y \rangle = 0\}$.

If $b \in p$, then the centraliser of $b$ in a subset $X$ of $\mathfrak{g}$ is denoted by $X_b$. Recall that $x \in p$ is called nilpotent if $x \in [\mathfrak{g}, \mathfrak{g}]$ and $ad_x^p(x)$ is nilpotent; this is equivalent to $f(x) = 0$ for all $f \in C[p]^\mathfrak{g}$. We denote by $N(p)$ the set of nilpotent elements.

By Lemma 8.2, any symmetric space $p$ is stable. Thus, by the discussion before Corollary 6.9, $p_s = p_{st}$ and hence, by (3.8) and in the notation of (3.1),

$$s = m = \max_{v \in p} \dim G \cdot v = \dim p - \dim \mathfrak{h}.$$  

We set $p_m = \{v \in p : \dim G \cdot v = m\}$. It is useful to note that, in the notation of Section 3, $p_{reg} = p_m \cap p_{st}$ (written in our notation, this is the assertion of [KR, Remark 6]).

We say that $x$ is regular nilpotent if $x \in p_m \cap N(p)$ and write $N(p)^{reg} = \{x \in N(p) : x$ is regular nilpotent$\}$. We say that $x \in N(p)$ is distinguished, if the centraliser $p_x$ does not contain non-central semisimple elements and put $N(p)^{dist} = \{x \in N(p) : x$ is distinguished$\}$. By [Pa, Lemma 1.3] one has $N(p)^{reg} \subset N(p)^{dist}$.

Remarks 9.3. (0) If $(\mathfrak{g}, \vartheta)$ is trivial, then $p = \mathfrak{h}$ and so $N(p) = \{0\}$.

(1) If $x \in p_{reg} \cap N(p)$ and write $N(p)^{reg} = N(p)^{reg} = N(p)_{\mathfrak{g}}$ since $p_x = \mathfrak{h}$ for all $x \in p \setminus \{0\} = \mathbb{C}^{n-1} \setminus \{0\}$; see, for example, [SY, Proposition 4].

(2) If $(\mathfrak{g}, \vartheta) = (s \times s, s)$ with $s$ simple, then $p \cong s$ is the adjoint representation of $s$ and it is known that $N(p)^{reg} = N(p)^{dist}$ only in the case where $s = sl(n)$, see [BC].

Suppose that $b \in p$ is semisimple. Then, as in [KR, L6], the decomposition $\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{b}_0$ defines a sub-symmetric pair $(\mathfrak{g}_0, \vartheta_0)$ inside $(\mathfrak{g}, \vartheta)$ with sub-symmetric space $(\mathfrak{g}_0, \vartheta_0)$.

Lemma 9.4. If the symmetric pair $(\mathfrak{g}, \vartheta)$ satisfies (†) then the following condition holds:

$(9.5)$\hspace{1cm} $N(p_s)^{dist} = N(p_s)^{reg}$ for each sub-symmetric pair $(\mathfrak{g}_0, \vartheta_0)$ of $(\mathfrak{g}, \vartheta)$. Furthermore, every sub-symmetric pair of $(\mathfrak{g}, \vartheta)$ satisfies (†).
Proof. It suffices to prove the claims when \((\bar{g}, g)\) is irreducible, with the trivial case being obvious. For type \(\text{Dil}_p\), it follows from Remarks 9.3(1), since a sub-symmetric pair is either trivial (when \(b \neq 0\)) or equal to \((\bar{g}, g)\) (if \(b = 0\)). The assertion is proved in \([Pa, \text{Theorem 3.2}]\) for type \(\text{AII}_n\) and \([Pa, \text{Proof of Proposition 3.4}]\) for type \(\text{ElIV}\).

Set \(Z(p) = N(p) \setminus N(p)^{\text{dist}}\). Observe that, for a symmetric space \((G, p)\) satisfying \((\dagger)\), we have \(Z(p) = N(p) \setminus N(p)^{\text{reg}}\).

**Proposition 9.6.** Let \((\bar{g}, g)\) be a symmetric pair. Suppose that \(M\) is a holonomic \(G\)-equivariant \(\mathcal{D}(p)\)-module such that

\[
\text{Ch} M \subseteq \mathcal{C}(p) \cap (Z(p) \times N(p)),
\]

where \(\text{Ch} M\) is the characteristic variety of \(M\). Then \(M = \{0\}\).

**Proof.** We follow the proof of \([LS4, \text{Theorem 3.8}]\). We may assume that \(0 \neq M\) is simple. Then, as loc. cit., one shows that the support of \(M\) is the closure of a single nilpotent orbit \(G \cdot x\). By hypothesis \(G \cdot x \subseteq Z(p)\), thus \(x\) is not distinguished.

Pick an irreducible component \(X\) of \(\text{Ch} M\) containing a point of the form \((x, \xi)\). Then, it follows from \([LS4, \text{Lemma 2.2(ii)}]\) that \(\dim X < \dim p\). But since \(M\) is holonomic, we must have \(\dim X = \dim p\), giving a contradiction.

Recall from (8.15) the \(\mathcal{D}(p)\)-modules \(\mathcal{K}(p)\) and \(\mathcal{L}(p)\). We will need the following general lemma.

**Lemma 9.7.** Let \((G, p)\) be a symmetric space.

1. Let \(d \in \mathcal{D}(p)\) and suppose that \(dp \in \mathcal{D}(p)\tau(g)\) for some \(p \in (\text{Sym} p)^G\). Then that \(p^nd \in \mathcal{D}(p)\tau(g)\) for some \(n > 0\). In particular, if \(F \subseteq (\text{Sym} p)^G\) is an ideal such that \(dF \subseteq \mathcal{D}(p)\tau(g)\), then \(F^td \subseteq \mathcal{D}(p)\tau(g)\) for some \(t > 0\).

2. The support of \(\mathcal{L}(p) = \mathcal{K}(p) / \mathcal{D}(p)\tau(g)\) in \(p\) is contained in \(p \setminus p_{\text{reg}}\) and hence in \(p \setminus p_{\text{reg}}\). In particular, for any \(\theta \in \mathcal{L}(p)\) there exists \(n > 0\) such that \(\delta^n \theta = 0\).

**Proof.** (1) The proof of \([LS2, \text{Lemma 5.4}]\) proves the existence of the integer \(n\). Let \(F = \sum_{j=1}^{r} (\text{Sym} p)^{G} p_j\) and pick \(n \in \mathbb{N}^*\) such that \(p_j^nd \in \mathcal{D}(p)\tau(g)\) for all \(j\). Then \(F^td \in \mathcal{D}(p)\tau(g)\).

(2) The idea behind the proof is standard (see, for example, \([LS3, \text{Lemma 6.7}]\) or the proof of \([LS4, \text{Lemma 4.5}]\)) so some details will be left to the reader. Since \(p_{\text{reg}} \supseteq p_{\text{reg}}\) by \([KR, \text{Remark 6}]\), it suffices to prove the first assertion of Part (2).

Let \(v \in p_{\text{reg}}\) if \(M\) is a \(\mathbb{C}[p]\)-module, denote by \(M_v\) the localisation of \(M\) at the maximal ideal \(m_v\) defined by \(v\). Pick algebraically independent homogeneous elements \(p_j\) such that \(\mathbb{C}[p]^G = \mathbb{C}[p_1, \ldots, p_t]\). By \([KR, \text{Theorem 13}]\) the morphism \(\varpi : p \to \mathbb{C}[G]\) defined by \(\varpi(x) = (p_1(x), \ldots, p_t(x))\), has rank \(\ell = \dim \mathfrak{h}\) at \(v\). It follows that \(T_v(G \cdot v) = \ker d_v \varpi = \mathbb{C}\tau(g)|_v\) = \([v, \xi] : \xi \in g\). Furthermore, there exist scalars \(\lambda_j\) such that \(\{z_j = p_j - \lambda_j : 1 \leq j \leq \ell\}\) forms a part of a system of parameters of the local ring \(\mathbb{C}[p]_v\). Using \([MR, \text{Corollary 15.1.12}]\) we can find derivations \(\partial_i \in \text{Der} \mathbb{C}[p]_v\) such that \(\partial_i(z_j) = \partial_i(p_j) = \delta_{i,j}\) for \(1 \leq i, j \leq \ell\). Set \(N = \sum_{i=1}^{\ell} \mathbb{C}[p]_v \partial_i\) and denote by \(N_{\mid v} \subseteq T_v p\) the space of tangent vectors defined by the elements \(\partial_i\). From \(d_v \varpi = \mathbb{C}\tau(g)|_v\), one deduces that

\[
\text{Der} \mathbb{C}[p]_v / (\text{Der} \mathbb{C}[p]_v \partial_i, \text{Der} \mathbb{C}[p]_v) = T_v p = N_{\mid v} \oplus \mathbb{C}\tau(g)|_v.
\]

Then, Nakayama’s Lemma implies that \(\text{Der} \mathbb{C}[p]_v = N + \mathbb{C}[p]_v \tau(g)\).

Let \(P \in \mathcal{K}(p)_v := \mathbb{C}[p]_v \otimes_{\mathbb{C}[p]} \mathcal{K}(p)\). Then one can write \(P = P_0 + P_1\) with \(P_0 \in \mathcal{D}(p)_v \tau(g)\) and (with the usual notation) \(P_1 \in \sum_{i \geq 0} \mathbb{C}[p]_v \partial^i\). If \(P_1 \neq 0\), we can find some element \(f \in \mathbb{C}[p]^G = \mathbb{C}[p_1, \ldots, p_t]\) such that \(P_1 * f \neq 0\). Since \(\tau(g) * \mathbb{C}[p]^G = 0\), it follows that \(P * f \neq 0\), contradicting the fact that \(P \in \mathcal{K}(p)_v\). Thus, \(P = P_0 \in \mathcal{D}(p)_v \tau(g)\). This proves that \(\mathcal{L}(p)_v = \{0\}\), as required.

\(\square\)
In order to understand the kernel of rad, and more generally $\mathcal{K}(p)$, we need to reduce to the case of irreducible symmetric spaces and so we need to understand how $\mathcal{K}(p)$ relates to the corresponding objects for summands of $p$. This will form the content of the next couple of results.

We begin with an abstract result in which $\otimes$ will mean $\otimes_C$. For $i = 1, 2$, let $A_i$ be a $C$-algebras with a left ideal $J_i$. Given a left $A_1$-module $M$ and $0 \neq f \in A_1$, then $M$ is called $f$-torsionfree if $fm \neq 0$ for all $0 \neq m \in M$. Set $A = A_1 \otimes A_2$ and identify $A_1 = A_1 \otimes 1 \subseteq A$ and $A_2 = 1 \otimes A_2 \subseteq A$. Write

$$L = AJ_1 + AJ_2 = J_1 \otimes A_2 + A_1 \otimes J_2.$$

**Lemma 9.8.** Keep the notation as above. For $i = 1, 2$, let $f_i \in A_i$ and assume that $A_i/J_i$ is $f_i$-torsionfree. Then:

1. $A/L$ is $f_1$-torsionfree;
2. if $f = f_1f_2 \in A$ then $A/L$ is $f$-torsionfree.

**Proof.** (1) As $C$-vector spaces, choose a complementary summand $J^+_2$ of $J_2$ inside $A_2$ and pick $C$-bases $\{\theta_j\}$ of $J_2$, respectively $\{\phi_j\}$ of $J^+_2$. Suppose that $D \in A$ satisfies $f_1^sD \in L$ for some $s \geq 1$. We can write $D$ uniquely as

$$D = \sum_p \alpha_p \otimes \theta_p + \sum_q \beta_q \otimes \phi_q \quad \text{for some } \alpha_p, \beta_q \in A_1.$$

Now $L = A_1 \otimes J_2 + J_1 \otimes J^+_2$ and so we can write $f^sD = \sum_p \mu_p \theta_1 + \sum_j \nu_j \phi_1$ for some $\mu_1 \in A_1$ and $\nu_j \in J_1$. On the other hand,

$$f^sD = \sum_p (f^s\alpha_p) \otimes \theta_p + \sum_q (f^s\beta_q) \otimes \phi_q,$$

and so, by uniqueness, $f^s\alpha_p = \mu_p$ and $f^s\beta_q = \nu_q$ for all $p, q$. In particular each such $f^s\beta_q = \nu_q \in J_1$ and hence $\beta_q \in J_1$ as $A_1/J_1$ is $f_1$-torsionfree. Therefore,

$$D = \sum_p \alpha_p \otimes \theta_p + \sum_q \beta_q \otimes \phi_q \in A_1 \otimes J_2 + J_1 \otimes A_2 = L;$$

as required.

(2) Suppose that $L \ni f^sD = f^s_1f^s_2D$ for some $D \in A$ and $s \geq 1$. Then part (1) implies that $f^s_2D \in L$. By the analogue of part (1) for $f_2$, it follows that $D \in L$. □

We now return to symmetric spaces. Let $(G, p) = (G_1, p_1) \oplus (G_2, p_2)$ be a direct sum of symmetric spaces, and keep the resulting notation from Remark 8.3. To simplify the notation, identify $\mathcal{D}(p_1)$ with $\mathcal{D}(p_1) \otimes 1$ inside $\mathcal{D}(p) = \mathcal{D}(p_1) \otimes \mathcal{D}(p_2)$ and similarly for $\mathcal{D}(p_2)$. Define $\mathcal{K}(p_1)$ analogously to (8.15); thus

$$\mathcal{K}(p_1) = \{D \in \mathcal{D}(p_1) : D(\mathbb{C}[p_1])^{G_1} = 0\}.$$

**Lemma 9.9.** Let $(G, p) = (G_1, p_1) \oplus (G_2, p_2)$ be a sum of symmetric spaces. Then

$$\mathcal{K}(p) = (\mathcal{K}(p_1) \otimes \mathcal{D}(p_2)) \cup (\mathcal{D}(p_1) \otimes \mathcal{K}(p_2)).$$

**Proof.** Set $A_i = \mathcal{D}(p_i)$; thus $A = A_1 \otimes A_2 \simeq \mathcal{D}(p)$. Writing $J_i = \mathcal{K}(p_i)$ for each $i$, we need to prove that $L := AJ_1 + AJ_2$ equals $\mathcal{K}(p)$. Since $\mathbb{C}[p_1]^{G_1} \otimes \mathbb{C}[p_2]^{G_2} = \mathbb{C}[p]^{G}$, at least $L \subseteq \mathcal{K}(p)$.

If $0 \neq f_i \in \mathbb{C}[p_i]$ then we claim that the left $A_i$-module $A_i/J_i$ is $f_i$-torsionfree. Indeed suppose that $D \in A_i$ satisfies $f_i^sD \in J_i$ for some $s$; thus $f_i^sD(\psi) = 0$ for all $\psi \in \mathbb{C}[p_i]^{G_i}$. Since $\mathbb{C}[p_i]$ is a domain this implies that $D(\psi) = 0$ for all such $\psi$. In other words, $D \in J_i$ and so $A_i/J_i$ is $f_i$-torsionfree, as claimed.

Now let $\delta_i \in \mathbb{C}[p_i]$ be the discriminant of the pair $(G_i, p_i)$ and note that $\delta = \delta_1\delta_2$ is the discriminant of $(G, p)$. By the previous paragraph each $A_i/J_i$ is $\delta_i$-torsionfree. Now let $D \in \mathcal{K}(p)$. Then, by Lemma 9.7(2), there exists $s$ such that
\( \delta^*D \in \mathcal{D}(p)\tau(g) \). Since \( \tau(g) \subset J \), clearly \( \mathcal{D}(p)\tau(g) \subseteq L \) and hence \( \delta^*D \in L \). Thus Lemma 9.8(2) implies that \( D \in L \). Hence \( L = \mathcal{K}(p) \), as required.

We can now prove the main result of this section.

**Theorem 9.10.** Let \((G, p)\) be a robust symmetric space. Then \( \mathcal{K}(p) = \mathcal{D}(p)\tau(g) \), and so \( \mathcal{D}(p)\tau(g)^G = \ker(\text{rad}_0) \).

**Proof.** It suffices to prove that \( \mathcal{L}(p) = 0 \). By Lemma 9.9 it suffices to prove this when \((\tilde{g}, \tilde{\vartheta})\) is irreducible. Since the nice symmetric spaces are covered by Theorem 8.18, it remains to consider the case when \((\tilde{g}, \tilde{\vartheta})\) satisfies (i). We argue by induction on the dimension \( \dim \mathfrak{g} \) of sub-symmetric pairs \((\tilde{g}_0, \tilde{g}_0)\) of \((\tilde{g}, \tilde{g})\).

If \((\tilde{g}_0, \tilde{g}_0)\) is trivial, the assertion is clear, so assume not. Suppose that \( 0 \neq b \in \mathfrak{p} \) is a semisimple element. By Lemma 9.1, the centraliser \( \mathfrak{g}_0 \) cannot equal \( \mathfrak{g} \). Thus, the slice \((\tilde{g}_0, \tilde{p}_0)\) is a proper sub-symmetric pair and so, by induction and Lemma 9.4, the theorem holds for \((\tilde{g}_0, \tilde{p}_0)\). Therefore, \([\text{LS4}, \text{Lemma 4.1}(1)]\) implies that \( \text{Supp} \mathcal{L}(p) \subseteq \mathcal{N}(p) \). Now the characteristic variety \( \mathcal{L}(p) \) is contained in \( \text{CH}(\mathcal{L}(\mathfrak{p})/\mathcal{D}(p)\tau(g)) \), and hence in \( \mathcal{C}(p) \). Thus \( \text{CH} \mathcal{L}(p) \subseteq \mathcal{C}(p) \cap (\mathcal{N}(p) \times p) \). Since \((\tilde{g}, \tilde{g})\) satisfies (i), Lemma 9.7(2) then implies that

\[
\text{Supp} \mathcal{L}(p) \subseteq \mathcal{N}(p) \cap (\mathfrak{p} \times p_m) = \mathcal{N}(p) \cap (\mathcal{N}(p) \times \mathfrak{p}) = \mathcal{Z}(p) \times \mathfrak{p}.
\]

Therefore \( \text{CH} \mathcal{L}(p) \subseteq \mathcal{C}(p) \cap (\mathcal{Z}(p) \times \mathfrak{p}) \).

If we show that

\[
(9.11) \quad \text{CH} \mathcal{L}(p) \subseteq \mathcal{C}(p) \cap (\mathcal{Z}(p) \times \mathfrak{N}(p))
\]

then Proposition 9.6 will force \( \mathcal{L}(p) = 0 \), thereby proving the theorem.

Set \( \mathfrak{m} = (\text{Sym} \mathfrak{p})^G \). Recall that \( x \in \mathcal{N}(p) \) if and only if \( f(x) = 0 \) for all \( f \in \mathbb{C}[\mathfrak{p}]^G \). However, we have identified \( \mathfrak{p} \times \mathfrak{p}^* = \mathfrak{p} \times \mathfrak{p} \) using \( \times \). Therefore, in order to prove (9.11), it remains to prove that any element of \( \mathcal{L}(p) \) is annihilated, on the right, by a power of \( \mathfrak{m} \). To prove this we use the argument of \([\text{LS4}, \text{Corollary 3.9}(2)]\).

Observe that both \( \mathcal{D}(p)/\mathcal{D}(p)\tau(g) \) and \( \mathcal{L}(p) \) are right \( \mathcal{D}(p)^G \)-modules. Denote by \( I \) the annihilator of \( \mathcal{L}(p) \) as a \( \mathcal{D}(p)^G \)-module; it is a graded ideal in the naturally graded ring \( \mathcal{D}(p)^G \), see \([\text{LS4}, \text{page 1725}]\). Since \( \text{CH} \mathcal{L}(p) \subseteq \mathcal{C}(p) \cap (\mathcal{N}(p) \times \mathfrak{p}) \) by \([\text{LS4}, \text{Lemma 2.2}(1)]\), \( \mathcal{L}(p) \) is a holonomic left \( \mathcal{D}(p) \)-module. Thus \( \mathcal{D}(p)\mathcal{L}(p) \) has finite length and so \( \text{End}_{\mathcal{D}(p)}(\mathcal{L}(p)) \) is finite dimensional. From \( \mathcal{D}(p)^G/I \subseteq \text{End}_{\mathcal{D}(p)}(\mathcal{L}(p)) \) we conclude that \( I \cap (\text{Sym} \mathfrak{p})^G \) has finite codimension in \( (\text{Sym} \mathfrak{p})^G \). Since \( I \) is a graded ideal, so is \( I \cap (\text{Sym} \mathfrak{p})^G \) and hence \( I \cap (\text{Sym} \mathfrak{p})^G \supseteq F = \mathfrak{m}^s \) for some \( s \). Hence \( dF \in \mathcal{D}(p)\tau(g) \) for all \( d \in \mathcal{K}(p) \). Lemma 9.7(1) then implies that \( F^t d \in \mathcal{D}(p)\tau(g) \) for some \( t \). In other words, (9.11) holds, which is enough to prove the theorem.

**Applications of Theorem 9.10.** The detailed description of \( \mathcal{K}(p) \) given by Theorem 9.10 has a number of applications, which we now describe.

First, as an immediate consequence of Theorem 9.10 one obtains a generalization of a fundamental result of Harish-Chandra [HC2, Theorem 5] from the diagonal case to robust symmetric pairs. For nice symmetric spaces this was proved in \([\text{LS4}]\).

**Corollary 9.12.** Let \((\tilde{g}, \tilde{g})\) be a robust symmetric pair that is the complexification of a real symmetric pair \((\tilde{g}_0, \tilde{g}_0)\). Write \( G_0 \) for the connected algebraic group satisfying \( \text{Lie}(G_0) = \tilde{g}_0 \). Let \( U \subset \tilde{p}_0 \) be a \( G_0 \)-stable open subset and \( T \) be a \( G_0 \)-invariant distribution on \( U \). Then \( \mathcal{K}(p)^G \cdot T = 0 \).

With a little extra work we are able to generalise a second result of Harish-Chandra [HC1, Theorem 3] to robust symmetric spaces. We begin with a subsidiary result.

**Corollary 9.13.** Let \((G, p)\) be a robust symmetric space. If \( \mathfrak{n} \subset (\text{Sym} \mathfrak{p})^G \) is a maximal ideal and \( f \in \mathbb{C}[\mathfrak{p}]^G \setminus \{0\} \), then \( \mathcal{D}(p) = \mathcal{D}(p)f + \mathcal{D}(p)\tau(g) + \mathcal{D}(p)\mathfrak{n} \).
Proof. By Corollary 8.22, \( \mathcal{D}(p)^G / \ker(\text{rad}_0) \cong A_\kappa \) is simple and so any non-zero \( A_\kappa \)-module \( Z \) has annihilator \( \text{ann}_A Z = 0 \). Thus [Lo2, Theorems 1.1 and 1.3] imply that \( \text{GKdim}_A Z \geq \dim \mathfrak{h} \) for any finitely generated \( A_\kappa \)-module \( Z \).

Set \( N = \mathcal{D}(p)/(\mathcal{D}(p)f + \mathcal{D}(p)\tau(g) + \mathcal{D}(p)n) \). Then Theorem 9.10 implies that \( N^G = A_\kappa/(A_\kappa f + A_\kappa n) \). By the first paragraph of this proof, in order to prove that \( N^G = 0 \) and therefore \( N = 0 \), it suffices to prove that \( \text{GKdim} M \leq \dim \mathfrak{h} \).

We prove this by mimicking the proof of [LS4, Corollary 5.7]. Under the order filtration from Remark 2.13, the associated graded ideal is \( \text{gr} \, n = (\text{Sym} \, p)_G^C \) and so

\[
\text{Ch} \, N \subseteq X := \mathcal{C}(p) \cap (p \times N(p)) \cap ((f = 0) \times p).
\]

By [LS4, Proposition 2.1], \( X//G \) identifies with the subvariety \( (f = 0) \) of \( p//G \). Since \( p//G \) is an irreducible variety of dimension \( \ell = \dim \mathfrak{h} \), this means that \( \dim \, X//G < \ell \) and hence that the Gelfand-Kirillov dimension

\[
\text{GKdim} \, N^G = \dim \text{Ch} \, N//G \leq \dim X//G < \ell,
\]

as required. \( \square \)

The significance of Corollary 9.13 is that, for robust symmetric pairs, it proves a result proved in [LS4, Corollary 5.8] for nice pairs, which in turn generalises [HC1, Theorem 3]. We just state the result; the notation and proof are essentially the same as those used in [LS4]. In the case when \( I = \mathbb{C}[p]^G \), this reduces to Corollary 1.5.

**Corollary 9.14.** Assume that the symmetric space \( (G, p) \) is robust and pick a real form \( (G_0, p_0) \) of \( (G, p) \). Let \( T \) be a \( G \)-invariant distribution on an open subset \( U \subseteq p_0 \). Suppose that:

(1) there exists an ideal \( I \subset \mathbb{C}[p]^G \) of finite codimension such that \( I : T = 0 \);

(2) \( \text{Supp}(T) \subseteq U \setminus U \cap p_{\text{reg}} \).

Then \( T = 0 \). \( \square \)

We end the section with several comments and applications of the earlier results.

First, by combining Proposition 8.6 with Theorems 8.23 and 9.10, we deduce the following.

**Corollary 9.15.** Assume that \( (\tilde{g}, g) \) is a robust symmetric pair such that the centre of \( g \) acts trivially on \( p \). Then, \( \text{Im}(\text{rad}_c) \) is simple and \( \ker(\text{rad}_c) = (\mathcal{D}(g)\tau(g))^G \) for all choices of \( \varsigma \). \( \square \)

In particular, suppose that \( (\tilde{g}, g) = (g \oplus g, g) \) is of diagonal type with \( g \) reductive. Then the centre of \( g \) certainly acts trivially on \( g = p \) and so we conclude:

**Corollary 9.16.** If \( (\tilde{g}, g) = (g \oplus g, g) \) is of diagonal type then \( \text{Im}(\text{rad}_c) \cong \text{Im}(\text{rad}_0) \) is a simple ring for all choices of \( \varsigma \). Also, \( \ker(\text{rad}_c) = \ker(\text{rad}_0) = (\mathcal{D}(g)\tau(g))^G \). \( \square \)

Corollary 9.16 is significant because the radial parts map that appears in the literature for the diagonal case is not \( \text{rad}_0 \). In more detail, the Harish-Chandra homomorphism \( HC : \mathcal{D}(g)^G \to \mathcal{D}(h)/W \) studied in [HC1, LS1, LS2, Sc2, Wa] differs from rad by conjugation with the element \( \delta^\frac{1}{2} \); thus, in our notation, \( HC = \text{rad}_\delta^\frac{1}{2} \).

The details can be found, for example, in [HC1, Theorem 1] or [Wa, Theorem 3.1]. This is important since, as Harish-Chandra noticed in [HC1], it is only after taking this shift that one obtains \( \text{Im}(HC) \subseteq \mathcal{D}(h)^W \). It was then proved in [LS1, LS2] that \( \text{Im}(HC) = \mathcal{D}(h)^W \) and \( X(g) = \mathcal{D}(g)\tau(g) \), thus \( \ker(HC) = (\mathcal{D}(g)\tau(g))^G \). Up to an isomorphism, Corollary 9.16 recovers these result.

We end this section with a remark about a possible generalisation of the previous results. A large class of polar representations is given by Vinberg’s \( \theta \)-representations [Vi]. These are defined as follows. Let \( \tilde{G} \) be as above and \( \tilde{\theta} : \tilde{g} \to \tilde{g} \) be an automorphism of order \( 2 \leq m < \infty \). Then \( \theta \) defines a \( (\mathbb{Z}/m\mathbb{Z}) \)-grading \( \tilde{g} = \mathbb{Z}/m\mathbb{Z} \).
the complexification of an euclidean vector space and $W$ reflections. Let $R$ if Question (2) holds, then so does Question (1), as can be seen by examining the proof of Theorem 4.4. Remark A.2. If Question (2) holds, then so does Question (1), as can be seen by examining the proof of Theorem 7.19.

APPENDIX A. NON-SIMPLECTICITY OF CERTAIN SPHERICAL ALGEBRAS

The main aim of this appendix is to complete the proof of Theorem 8.23, by showing that $A_\kappa$ is not simple for the symmetric spaces not covered by Corollary 8.22.

Definitions. The computations in this appendix become much simpler if we adopt the notation of [BEG, EG], where the rational Cherednik algebra $H_c(W)$ is defined in terms of a multiplicity function $c$ as opposed to $\kappa$ and so we begin with the relevant notation.

We assume that $W$ is the Weyl group associated to a finite root system, hence $\frak h$ is the complexification of an euclidean vector space and $W$ is generated by orthogonal reflections. Let $R \subset \frak h^*$ be the set of roots and make a choice of positive roots $R^+ \subset R$ so that $S = \{s_\alpha : \alpha \in R^+\}$ is the set of reflections in $W$. If $s = s_\alpha$ with $\alpha \in R$, we set $c_\alpha = \alpha$. Let $c : S \rightarrow \mathbb C$ be a $W$-invariant function, setting $c_s = c(s)$ for $s \in S$ and $c_0 = 0$. The set of such maps, called multiplicity functions, is denoted by $\mathcal C(W)$ (it is called $\mathbb C[R]^W$ in [BEG]). The rational Cherednik algebra $H_c(W) = \mathbb C[\mathfrak h^*, \mathfrak h, W]$ associated to $c \in \mathcal C(W)$ is defined in [BEG, Section 1]. For our purposes it suffices to note that the Dunkl operator $T_y(c)$ associated to $y \in \mathfrak h$, as defined in [BEG, page 288] and [CE, §2.1], is

$$T_y(c) = \partial_y - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \frac{1 - s}{\alpha_s} \in \text{End}_\mathbb C \mathbb C[\mathfrak h]. \quad (A.1)$$

Remark A.2. Comparing the element $T_y$ from (2.8) with (A.1) shows that $\kappa_{H,0} = 0$ and $\kappa_{H,1} = -c_0$ for $H = \ker(\alpha)$ and $\alpha \in R^+$.

Twist by a character. Let $\chi \in \text{Hom}(W, \mathbb C^*)$ be a linear character of $W$. The idempotent $e_\chi \in CW$ associated to $\chi$ is $e_\chi = \frac{1}{|W|} \sum_{w \in W} \chi^{-1}(w)w$. Let $M$ be a $W$-module; denote by $M[\chi]$ the $\chi$-isotypic component of $M$. Then:

$$M[\chi] := \{ x \in M \mid \forall w \in W, \, wx = \chi(w)x \} = \{ e_\chi x : x \in M \} =: e_\chi M.$$ Observe that, since $\mathbb C W \subset H_c(W)$, the above can be applied to any $H_c(W)$-module.
In particular, if \( \chi = \text{triv} \), the trivial character, one gets the trivial idempotent \( e_{\text{triv}} \), also denoted by \( e_W \), and the set of \( W \)-invariants in \( M \):

\[
e_{\text{triv}} = \frac{1}{|W|} \sum_{w \in W} w, \quad M[\text{triv}] = e_{\text{triv}}M = M^W.
\]

The sign character \( \text{sgn} : W \to \{ \pm 1 \} \) is given by \( \text{sgn}(w) = \det_\mathfrak{h}(w) \).

If \( c \in C(W) \) is a multiplicity function, one defines \( e^c \in C(W) \) by \( e^c_s = \chi(s)c_s \) for all \( s \in S \). When \( \chi = \text{sgn} \) we therefore have \( e^{\text{sgn}} = -c \).

The next lemma is a standard result in the literature; see [BC, Section 5.1].

**Lemma A.3.** Let \( \chi \in \text{Hom}(W, \mathbb{C}^*) \). There exists an algebra isomorphism

\[
f_\chi : H_c(W) \xrightarrow{\sim} H_{c,\chi}(W)
\]

given by \( f_\chi(x) = x \), \( f_\chi(y) = y \), \( f_\chi(w) = \chi(w)w \), for all \( x \in \mathfrak{h}^* \), \( y \in \mathfrak{h} \) and \( w \in W \).

The inverse of \( f_\chi \) is \( f_{\chi^{-1}} : H_{c,\chi}(W) \xrightarrow{\sim} H_c(W) \).

**Remark A.4.** Note that \( f_\chi(e_\chi) = e_{\text{triv}} \) and that in terms of Dunkl operators we have \( f_\chi(T_h(e)) = T_h(e^\chi) \).

Let \( M \) be a \( H_c(W) \)-module. The twist of \( M \) by \( \chi \) is the \( H_c(W) \)-module \( M^{\chi} \)

defined by \( M^{\chi} = M \) as a vector space and \( a \cdot m = f_\chi(a)m \) for all \( a \in H_c(W) \) and \( m \in M \). Using this notation one easily proves the following.

**Lemma A.5.** (1) The functor \( M \mapsto M^{\chi} \) is an equivalence of categories between \( H_c(W)\text{-Mod} \) and \( H_{c,\chi}(W)\text{-Mod} \). Its inverse is given by \( N \mapsto N^{\chi^{-1}} \).

(2) Let \( M \) be an \( H_c(W) \)-module. Then, the vector subspace \( M^W = M[\text{triv}] \) of the \( W \)-module \( M \) is equal to the isotypic component \( M^{\chi}[\text{triv}] \) of the \( W \)-module \( M^{\chi} \).

In particular:

\[
M^W = M^{\text{sgn}[\text{sgn}]} \quad \text{for any } H_{-c}(W)\text{-module } M.
\]

Let \( \chi \in \text{Hom}(W, \mathbb{C}^*) \) and set:

\[
(A.6) \quad A_{c,\chi}(W) = e_\chi H_c(W)e_\chi, \quad A_c(W) = A_{c,\text{triv}}(W) = e_{\text{triv}} H_c(W)e_{\text{triv}}
\]

Each \( A_{c,\chi}(W) \subset H_c(W) \) is an algebra whose identity element is the idempotent \( e_\chi \) and \( A_c(W) \) is the usual spherical subalgebra of \( H_c(W) \). Note that if \( M \) is an \( H_c(W) \)-module, the isotypic component \( M[\chi] = e_\chi M \) is a module over the algebra \( A_{c,\chi}(W) \). In particular, \( M^W = e_{\text{triv}}M = e_W M \) is an \( A_c(W) \)-module and \( M[\text{sgn}] \) is a module over \( A_{c,\text{sgn}}(W) \).

Recall that \( f_\text{sgn}(e_\text{sgn}) = e_{\text{triv}} \) and \( e^{\text{sgn}} = -c \); via the isomorphism \( f_\text{sgn} \) one gets the isomorphism of algebras:

\[
(A.7) \quad f_\text{sgn} : A_{c,\text{sgn}}(W) = e_\text{sgn} H_c(W)e_\text{sgn} \xrightarrow{\sim} A_{-c}(W) = e_{\text{triv}} H_{-c}(W)e_{\text{triv}}
\]

We are interested in the (non) simplicity of the spherical algebra for certain Weyl groups and multiplicity functions. We will need some results from [Lo3]. First, recall that the multiplicity \( c \in C(W) \) is said to be totally aspherical if one has the following equivalence: \( M \) is a module of the category \( \mathcal{O}_c(W) \) which is torsion as an \( S(\mathfrak{h}^*) \)-module if and only if \( M^W = e_{\text{triv}}M = \{0\} \); see [BEG] or [Lo3] for a definition of \( \mathcal{O}_c(W) \).

Let \( W' \subset W \) be a parabolic subgroup; that is, a stabiliser in \( W \) of some \( p \in \mathfrak{h} \). Decompose \( \mathfrak{h} \) as \( \mathfrak{h} = \mathfrak{h}^{W'} \oplus \mathfrak{h}' \) where \( \mathfrak{h}' \cong \mathfrak{h}/\mathfrak{h}^{W'} \) is the unique \( W' \)-complement of the set of fixed points \( \mathfrak{h}^{W'} \). Set \( S' = S \cap W' \) and define a \( W' \)-invariant function \( c' : S' \rightarrow \mathbb{C} \) by \( c' = e_{c'} \in C(W') \). One can then construct a Cherednik algebra \( H_{c'}(W') \) from the datum \( (W', \mathfrak{h}', c') \), cf. [BE].

**Proposition A.8** ([Lo3], Proposition 2.7, Lemmata 2.8 & 2.9). The following are equivalent:
(1) $c \in C(W)$ is totally aspherical;
(2) $\epsilon_W M' = \{0\}$ for any parabolic subgroup $W' \neq \{1\}$ and any finite dimensional $H_c(W')$-module, where $\epsilon_{W'}$ is the trivial idempotent in $CW'$;
(3) the spherical algebra $A_c(W)$ is simple.

This proposition, combined with Lemma A.5 and (A.7), implies the following result.

**Corollary A.9.** If there exists a parabolic subgroup $W' \subset W$ and a finite dimensional $H_c(W')$-module $L$ such that $L[\text{sgn}] \neq \{0\}$, then the spherical algebra $A_{-c}(W)$ is not simple.

**Finite dimensional modules in the $A_1$ case.** We assume in this subsection that $R$ is a root system of type $A_1$. Therefore $\mathfrak{h}^* = \mathbb{C} x$, $W = \{1, s\}$ and $c \in C(W)$ is determined by $c = c_s$. The algebra $H_c(W)$ is isomorphic to the $\mathbb{C}$-algebra generated by $x, s, t$ where

$$T = \frac{\partial}{\partial x} - \frac{c}{x} (1 - s) \in \text{End}_\mathbb{C} \mathbb{C}[x].$$

We now make the hypothesis:

**Hypothesis A.** $c = m + \frac{1}{2}$ with $m \in \mathbb{N}$.

Then $T(x^{2m+1}) = 0$ and the standard module $L_c(\text{triv}) \cong \mathbb{C}[x]$ has a unique irreducible quotient, namely: $L_c(\text{triv}) = \mathbb{C}[x]/(\bigoplus_{j \geq 2m+1} \mathbb{C} x^j) = \bigoplus_{i=0}^{m} \mathbb{C} \tilde{c}_i^j$, where $t$ is the class of $x$. Since $s(t) = -t$ we get:

$$L_c(\text{triv})^W = L_c(\text{triv})[\text{triv}] = \bigoplus_{i=0}^{m} \mathbb{C} \tilde{c}_i^j,$$

$$L_c(\text{triv})[\text{sgn}] = \begin{cases} \{0\} & \text{if } m = 0, \\
\bigoplus_{i=0}^{m-1} \mathbb{C} \tilde{c}_i^{j+1} & \text{if } m \geq 1. \end{cases}$$

Therefore, if we set $M = L_c(\text{triv})^{\text{sgn}}$, then the $H_{-c}(W)$-module $M$ satisfies $M^W = \{0\}$ if $c = \frac{1}{2}$ and $M^W \neq \{0\}$ for $m \geq 1$ by Lemma A.5. Thus, we have:

**Corollary A.10.** Let $R$ be a root system of type $A_1$ and, in the above notation, assume that $m \geq 1$, equivalently if $c = \frac{d}{2}$ for an odd integer $d \geq 3$. Then there exists a nonzero finite dimensional module over $A_{-c}(W)$.

When $(\tilde{g}, \tilde{\vartheta})$ is an irreducible symmetric pair of rank 1, the parameter $k$ (given by $\kappa$) is always of the form $k = m$ or $k = m + \frac{1}{2}$ for some $m \in \mathbb{N}$ (see the tables in Appendix B). When $k = \frac{1}{2}$, it is easy to see that $A_\kappa$ is simple, see [LS4, §6], or observe that $(\tilde{g}, \tilde{\vartheta})$ is nice. We therefore deduce the following result.

**Corollary A.11.** If $(\tilde{g}, \tilde{\vartheta})$ is an irreducible symmetric pair of rank 1, then $A_\kappa(W)$ is not simple if and only if $k = m + \frac{1}{2}$ with $m \geq 1$.

**Finite dimensional modules in the $B_p$ case.** In this subsection we recall some results from [CE]. Let $\mathfrak{S}_p$ be the $p$th symmetric group and set $\mu_2 = \{ \pm 1 \}$. We consider here the Weyl group $W = \mathfrak{S}_p \ltimes \mathfrak{g}_p^\mathbb{C}$ of a root system of type $B_p$ (or $C_p$) with $p \geq 2$. It acts on the $p$-dimensional space $\mathfrak{h} = \mathbb{C}^p$. The set $S \subset W$ consists of the elements $s_i$, $1 \leq i \leq p$, and $\sigma^{(m)}_{i,j}$, $1 \leq i < j \leq p$, defined by:

$$s_i(x_1, \ldots, x_i, \ldots, x_p) = (x_1, \ldots, -x_i, \ldots, x_p),$$

$$\sigma^{(m)}_{i,j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_p) = (x_1, \ldots, (-1)^m x_j, \ldots, (-1)^m x_i, \ldots, x_p),$$

for $m = 0, 1$. A multiplicity function $c \in C(W)$ is given by two complex numbers $\{c_1, c_2\}$:

(A.12) $c_1 = c_{s_i}$ for all $i$ and $c_2 = c_{\sigma^{(m)}_{i,j}}$ for all $i, j, m$. 

Remark A.13. When \( R \) is a root system of type \( B_p \), respectively \( C_p \), \( s_i \) is the reflection associated to a short, respectively long, root and \( \sigma_{i,j}^{(0)} \) is associated to a long, respectively short, root. See [Bo, Planches II & III].

We will work under the following hypothesis, see [CE, §4.1].

**Hypothesis B.** Set \( r = 2(c_1 + (p - 1)c_2) \). We assume that \( r \geq 1 \) is an odd integer and write \( r = 2m - 1 \) for \( m \geq 1 \).

Let \( \bar{u} \) be an indeterminate over \( \mathbb{C} \) and define an \( r \)-dimensional vector space by

\[
U_r = \mathbb{C}[\bar{u}]/(\bar{u}^r) = \oplus_{i=0}^{r-1} \mathbb{C}u^i
\]

where \( u \) is the class of \( \bar{u} \) modulo \( (\bar{u}^r) \). Let \( X_r = U_r^{\otimes p} \), thus \( \dim X_r = r^p \). A basis of the vector space \( X_r \) is given by the \( u^j = u^{j1} \otimes \cdots \otimes u^{jp} \) where \( j = (j_1, \ldots, j_p) \in \{0, \ldots, r - 1\}^p \). The space \( X_r \) has a natural structure of graded \( W \)-module which can be described as follows.

The group \( \mu_2 \) acts on \( U_r \) by \( (\pm 1) \cdot u^j = (\pm 1)^{j_1}u^j \) for \( j = 0, \ldots, r - 1 \), and \((e_1, \ldots, e_p) \in \mu_2^p \) acts on \( a = a_1 \otimes \cdots \otimes a_p \in X_r \) via \( \epsilon \cdot a = \epsilon_1 a_1 \otimes \cdots \otimes \epsilon_p a_p \). Let \( w = (\sigma, \epsilon) \in W \) with \( \sigma \in \mathfrak{S}_p \) and \( \epsilon = (e_1, \ldots, e_p) \in \mu_2^p \) then the action of \( w \) on \( a = a_1 \otimes \cdots \otimes a_p \in X_r \) is: \( w \cdot a = \sigma \cdot (\epsilon \cdot a) = \epsilon^{\sigma(1)} \cdot a_{\sigma(1)} \otimes \cdots \otimes \epsilon^{\sigma(p)} \cdot a_{\sigma(p)} \).

Recall that \( M_\epsilon(\text{triv}) = H_\epsilon(W) \otimes_{\mathfrak{g}^{\text{WS}}} \mathbb{C}_\text{triv} \) is the standard representation of \( H_\epsilon(W) \) with trivial lowest weight; this is the polynomial representation, on which the elements of \( \mathfrak{h} \) act via Dunkl operators.

**Proposition A.14** ([CE, Theorem 4.2]). Under Hypothesis B, there exists a quotient \( Y_\epsilon \) of \( M_\epsilon(\text{triv}) \) which is isomorphic to \( X_r \) as a graded \( W \)-module.

In order to apply Lemma A.5, we want to describe \( X_r[\text{sgn}] \), the isotypic component of type \( \text{sgn} \) of the \( W \)-module \( X_r \).

The reflection \( s_k \) identifies with the element \((1, \ldots, 1, -1, \ldots, 1) \in W \), where \( -1 \) is in position \( k \). Therefore, \( s_k \) acts on \( u^j \) by \( s_k \cdot u^j = (-1)^{j_k}u^j \). Let \( X = \sum_j \lambda_j u^j \in X_r \). Then, \( s_k \cdot x = sgn(s_k)x = -x \) if and only if \( \sum_j (-1)^{j_k} \lambda_j u^j = -\sum_j \lambda_j u^j \).

Equivalently, \((-1)^{j_k+1} \lambda_j = \lambda_j \) for all \( j \); that is \( \lambda_j = 0 \) when \( j_k \) is even. This shows that the elements \( u^j \) such that \( j_k \) is odd for all \( k \) give a basis of the space \( X'_r = \{x \in X_r : s_k \cdot x = -x, k = 1, \ldots, p\} \). Since \( j \in \{0, \ldots, r - 1\} \) is odd if only if \( j = 2i - 1 \) with \( i \in \{1, \ldots, m - 1\} \), we have \( \dim X'_r = (m - 1)^p \).

Note that \( X'_r \) is an \( \mathfrak{S}_p \)-submodule of \( X_r \) and, if one sets \( U'_r = \oplus_{i=1}^{m-1} \mathbb{C}u^{2i-1} \subset U_r \), we have \( X'_r = (U'_r)^{\otimes p} \subset X_r \).

Since \( X_r[\text{sgn}] = \{x \in X'_r : \sigma \cdot x = \text{sgn}(\sigma)x \text{ for all } \sigma \in \mathfrak{S}_p\} \), the \( W \)-module \( X_r[\text{sgn}] \) identifies with the space of antisymmetric tensors in \((U'_r)^{\otimes p}\), that is to say with the \( p \)-th exterior product \( \Lambda^p U'_r \).

We can summarise the previous discussion in the next three corollaries.

**Corollary A.15.** Under Hypothesis B and the notation of Proposition A.14:

1. the isotypic component \( Y_r[\text{sgn}] \) is isomorphic to \( \Lambda^p U'_r \) (with the above \( W \)-module structure);
2. \( Y_r[\text{sgn}] = \{0\} \) if and only if \( m < p + 1 \) and \( \dim Y_r[\text{sgn}] = \binom{m-1}{p} \) if \( m \geq p + 1 \).

**Remark A.16.** In terms of the multiplicity function \( c \), the condition \( m \geq p + 1 \) can be rewritten as \( c_1 + (p - 1)c_2 \geq p + \frac{1}{2} \).

**Corollary A.17.** Assume that \( c_1 + (p - 1)c_2 \in \frac{1}{2} + \mathbb{Z} \) and \( c_1 + (p - 1)c_2 \geq p + \frac{1}{2} \). Then:
Proof. Recall that there exist isomorphisms of algebras:

\[ f_{\text{sgn}} : H_{-c}(W) \cong H_c(W), \quad f_{\text{sgn}} : A_{-c}(W) \cong e_{\text{sgn}} A_c(W) e_{\text{sgn}}. \]

Set \( M = Y_{r,\text{sgn}} \), where \( Y_r \) is an in Proposition A.14. Then, \( M \) is an \( H_{-c}(W) \) module; using Lemma A.5 and Corollary A.15 we get \( M^W = Y_r[\text{sgn}] \neq \{0\} \). This proves (1), from which (2) is an immediate consequence. \( \square \)

When \( p = 2 \) the expression \( r = 2(c_1 + (p - 1)c_2) = 2(c_1 + c_2) \) is symmetric in \( c_1 \) and \( c_2 \), and so we obtain:

**Corollary A.18.** Let \( R \) be a root system of type \( B_2 \) or \( C_2 \). Write \( W = \langle s_\alpha, s_\beta \rangle \) where \( \alpha \) and \( \beta \) have different length. Suppose that \( r = 2(c_{s_\alpha} + c_{s_\beta}) = 2m - 1 \) where \( m \) is an integer \( \geq 3 \). Then there exists a finite dimensional \( H_{-c}(W) \)-module \( M \) such that \( e_{\text{tri}}M = M^W \neq \{0\} \).

**Application.** We give here examples of multiplicity functions such that the spherical algebra \( A_{-c}(W) \) is not simple. Our notation for root systems follows [Bo].

Let \( \{\alpha_1, \ldots, \alpha_\ell\} \) be a basis of \( R \). If \( I \) is a subset of \( \{1, \ldots, \ell\} \), denote by \( R_I \) the root system generated by the \( \alpha_i \), \( i \in I \), and let \( W' = W_I \) be the parabolic subgroup generated by the \( \{s_{\alpha_i} : i \in I\} \). Prior to Proposition A.8, for each multiplicity function \( c \in C(W) \) we defined a multiplicity function \( c' = c|_{G \rtimes W'} \in C(W') \) and hence we obtain a Cherednik algebra \( H_{c'}(W') \).

**Proposition A.19.** Adopt the previous notation. Suppose that one of the following hypothesis holds.

(a) Let \( I = \{j\} \) and suppose that \( c_{s_{\alpha_j}} = m + \frac{1}{2} \) with \( m \in \mathbb{N}^* \).

(b) Let \( I = \{j, j + 1\} \) with \( R_I \) of type \( B_2 \) or \( C_2 \) and suppose, for some integer \( m \geq 3 \), that \( r = 2(c_{s_{\alpha_j}} + c_{s_{\alpha_{j+1}}}) = 2m - 1 \).

Then the spherical algebra \( A_{-c}(W) \) is not simple.

**Proof.** If (a) holds, the result follows from Corollary A.10 and Proposition A.8.

If (b) holds, apply Corollary A.18 and Proposition A.8. \( \square \)

**Examples A.20.** For each of the eleven following cases, we indicate the type of \( R \), the conditions on the integers \( p, q, n \), and the values of the multiplicity \( c \). We write \( c_{sh} \), resp. \( c_{bl} \), for \( c_a \) with \( \alpha \) short, respectively long. We then explain how to use Proposition A.19 to obtain the non-simplicity of \( A_{-c}(W) \).

The choice of the multiplicities (and the numbering of the different cases) will be justified in Remark A.22.

(A.20.1) **A_II p,q:** \( R = B_p, \quad p + q = n + 1, \quad 2 \leq p \leq \frac{3}{2}n < q; \quad c_{sh} = n - 2p + \frac{3}{2} = q - p + \frac{1}{2}, \quad c_{bl} = 1. \)

The non-simplicity of \( A_{-c}(W) \) follows from Proposition A.19(a) applied with \( I = \{\alpha_j\} \) where \( \alpha_j \) is short, since \( c_{s_{\alpha_j}} = q - p + \frac{1}{2} \) with \( q - p \geq 1 \).

(A.20.2) **B_II p,q:** \( R = B_p, \quad p + q = 2n + 1, \quad 2 \leq p < n; \quad c_{sh} = n - p + \frac{3}{2} = \frac{1}{2}(q - p), \quad c_{bl} = 1. \)

Take \( I = \{\alpha_j\} \) where \( \alpha_j \) is short. Observe that \( p \pm q \) is odd and \( p < n \) implies \( q \geq p + 3 \). Then, \( c_{s_{\alpha_j}} = \frac{1}{2}(q - p) = m + \frac{1}{2} \) with \( m = \frac{1}{2}(q - p - 1) \) integer \( \geq 1 \). Therefore, Proposition A.19(a) shows that \( A_{-c}(W) \) is not simple.

(A.20.3) **C_II p,q:** \( R = B_p, \quad p + q = n, \quad 2 \leq p \leq \frac{3}{2}(n - 1) < q; \quad c_{sh} = 2n - 4p + \frac{3}{2} = 2q - 2p + \frac{3}{2}, \quad c_{bl} = 2. \)
The non simplicity of $A_{-c}(W)$ follows from Proposition A.19(a) applied with $I = \{\alpha_j\}$ where $\alpha_j$ is short, since $c_{s_{\alpha_j}} = 2q - 2p + 1 + \frac{1}{2}$ with $2q - 2p + 1 > 1$.

(A.20.4) $\text{Cl}_{p,p} \colon R = C_p, 2 \leq p; c_{\tilde{g}} = \frac{1}{2}, c_{sh} = 2.$

The non simplicity of $A_{-c}(W)$ follows from Proposition A.19(a) applied with $I = \{\alpha_j\}$ where $\alpha_j$ is long, since $c_{s_{\alpha_j}} = \frac{1}{2}$.

(A.20.5) $D_{|p,q} \colon R = B_p, p + q = 2n, 2 \leq p \leq n - 2 < q; c_{sh} = n - p = \frac{1}{2}(q - p), c_{\tilde{g}} = \frac{1}{2}.$ Note that $q - p$ is even and $q \geq p + 4$ (since $2p \leq 2n - 4 = p + q - 4)$.

If $I = \{p - 1, p\}$, the root system $R_I$ is of type $B_2$; one has $c_{s_{\alpha_p}} = \frac{1}{2}(q - p)$ and $c_{s_{\alpha_{p-1}}} = \frac{1}{2}$. Apply Proposition A.19(b): we have $r = q - p + 1 = 2m - 1$ where $m = \frac{1}{2}(q - p) + 1$ is an integer $\geq 3$. Thus $A_{-c}(W)$ is not simple.

(A.20.6) $\text{DII}_{4p} \colon R = C_p, 2 \leq p; c_{\tilde{g}} = \frac{1}{2}, c_{sh} = 2.$

If $I = \{p - 1, p\}$, the root system $R_I$ is of type $C_2$; one has $c_{s_{\alpha_p}} = 2$ and $c_{s_{\alpha_{p-1}}} = \frac{1}{2}$. Apply Proposition A.19(b): we have $r = 5 = 2m - 1$ where $m = 3$. Thus $A_{-c}(W)$ is not simple.

(A.20.7) $\text{DII}_{4p+2} \colon R = B_{2p}, 2 \leq p; c_{sh} = \frac{3}{2}, c_{\tilde{g}} = 2.$

The non simplicity of $A_{-c}(W)$ is obtained by applying Proposition A.19(a) with $I = \{\alpha_p\}$, since $c_{s_{\alpha_p}} = \frac{5}{2}.$

(A.20.8) $\text{EII} \colon R = B_2, p = 2; c_{sh} = \frac{2}{3}, c_{\tilde{g}} = 3.$

The non simplicity of $A_{-c}(W)$ is obtained by applying Proposition A.19(a) with $I = \{\alpha_1\}$, since $c_{s_{\alpha_2}} = \frac{5}{2}.$

(A.20.9) $\text{EV} \colon R = F_4, p = 4; c_{sh} = 2, c_{\tilde{g}} = \frac{1}{2}.$

Take $I = \{2, 3\}$, then $R_I$ is of type $B_2$; we have $c_{s_{\alpha_2}} = \frac{1}{2}$ and $c_{s_{\alpha_3}} = 2$. Therefore, $r = 5 = 2m - 1$ with $m = 3$ and Proposition A.19(b) gives the non simplicity of $A_{-c}(W)$.

(A.20.10) $\text{EII} \colon R = C_3, p = 3; c_{\tilde{g}} = \frac{1}{2}, c_{sh} = 4.$

Take $I = \{2, 3\}$, then $R_I$ is of type $C_2$; we have $c_{s_{\alpha_2}} = 4$ and $c_{s_{\alpha_3}} = \frac{1}{2}$. Therefore, $r = 9 = 2m - 1$ with $m = 5$ and Proposition A.19(b) gives the non simplicity of $A_{-c}(W)$.

(A.20.11) $\text{EI} \colon R = F_4, p = 4; c_{sh} = 4, c_{\tilde{g}} = \frac{1}{2}.$

Take $I = \{2, 3\}$, then $R_I$ is of type $B_2$; we have $c_{s_{\alpha_2}} = \frac{1}{2}$ and $c_{s_{\alpha_3}} = 4$. We get $r = 9 = 2m - 1$ with $m = 5$; thus, by Proposition A.19(b), $A_{-c}(W)$ is not simple.

Radial components. Let $\tilde{g}$ be a finite dimensional semisimple complex Lie algebra. We adopt the notation introduced in Section 8 for symmetric pairs $(\tilde{g}, \vartheta) = (\tilde{g}, \vartheta)$ and the associated stable polar representation $(G, V = p)$.

Recall the multiplicity function $k : R \to \mathbb{Z}^\mathbb{N}$ defined by:

\[
\forall \alpha \in R^+, \quad k_\alpha = \frac{1}{2}(\dim \tilde{g}^\alpha + \dim \tilde{g}^{2\alpha}).
\]

Define the multiplicity $c : S \to \mathbb{C}$ by $c_{s_\alpha} = -k_\alpha$ for all $\alpha \in R$. Then the rational Cherednik algebra $H_k(W)$ associated to the symmetric pair $(\tilde{g}, \vartheta)$ and the multiplicity $k$ is isomorphic to the Cherednik algebra $H_{-c}(W)$ and, for the spherical algebras, we have $A_k(W) \cong A_{-c}(W)$.

Remark A.22. Suppose that $(\tilde{g}, \vartheta)$ is irreducible. From the classification of symmetric spaces, see table in Appendix B, the pairs for which the simplicity of $\text{Im}(\text{rad})$ cannot be determined by Corollary 8.22, or the study of the rank one case, are the eleven cases (A.20.1–A.20.11).

By combining Lemma 8.4, Corollary 8.22 and Appendix B with the non simplicity of the spherical subalgebra $A_k(W) \cong A_{-c}(W)$ in the cases (1) to (11) above, we get the following result, announced in Theorem 8.23.
Theorem A.23. Let \((\tilde{\mathfrak{g}}, \vartheta)\) be an irreducible symmetric space. Then the algebra \(\text{Im}(\text{rad}) \cong A_k(W)\) is simple if and only if \((\tilde{\mathfrak{g}}, \vartheta)\) is robust.

Appendix B. Tables of symmetric pairs

We adopt the notation of [He, Chapter X] for the classification of irreducible symmetric pairs \((\tilde{\mathfrak{g}}, \vartheta) = (\tilde{\mathfrak{g}}, \mathfrak{g})\). The multiplicity \(k : \mathbb{R} \to \frac{1}{2}\mathbb{N}\) is defined in (A.21).

The columns of the next two tables give the following information (with minor exceptions in the diagonal case).

1. The first column gives the type of \((\tilde{\mathfrak{g}}, \vartheta)\), using the notation from [He], and the type of the root system \(\mathbb{R}\).
2. Columns (2)–(4) are self-explanatory.
3. The fifth column gives the value \(k_\lambda\) of the multiplicity \(k\) on the root \(\lambda\). When there are two values for \(k_\lambda\), the top number gives \(k_\lambda\) for the long roots and the bottom number gives it for the short roots.
4. The final column describes, by “Y” or “N”, whether \(A_k(W)\) is simple or not and gives a justification of this statement; namely, “nice” if the pair \((\tilde{\mathfrak{g}}, \vartheta)\) is nice, “BEG” if the simplicity follows from Theorem 8.20, “rank 1” or “(x)” if the non simplicity is consequence of Corollary A.11, respectively is proved in case (A.20.x) from Examples A.20.
| Type (g, θ) | g, g | rank | dim p | k_λ | simple |
|------------|------|------|-------|-----|--------|
| diagonal type | s × s, s | rank s | dim s | 1 | Y nice |
| A_n | sl(n), so(n) | n − 1 | 1/(n − 1) + 2 | 1/2 | Y nice |
| A_{n−1} | sl(2n), sp(n) | n − 1 | (n − 1)(2n + 1) | 2 | Y BEG |
| A_{n−1} | sl(p+q), gl(p) × sl(q) | p | 2pq | 1/n − 2p + 3/2 | N (1) |
| B_p | sp(n), gl(n) × sl(n) | n | 2n^2 | 1 | Y nice |
| C_{n−1} | sp(n+1), gl(n) | n | n(n+1) | 2/n | Y nice |
| D_{2p−1, p+1} | sp(p+q), sp(p) × sp(q) | p | 4pq | 2n − 4p + 3/2 | N (3) |
| C_{n} | sp(q+1), sp(q) × sp(q) | 1 | 4q | 2q − 7/2 | N rank 1 |
| D_{2p−1} | sp(2p), sp(p) × sp(p) | p | 4p^2 | 2 | N (4) |
| E_6 | 3 ≤ p | p−1 | p^2−1 | 1 | Y nice |
| E_7 | 3 ≤ p | p | 2p−1 | 1 | Y BEG |
| E_8 | 3 ≤ p | p | 2p−1 | 1 | Y BEG |
| F_4 | 2 ≤ p | p | 2p(2p−1) | 2 | N (6) |
| G_2 | 2 ≤ p | p | 2p(2p+1) | 2 | N (7) |
Appendix C. Detailed proofs for Section 2

In this appendix we give the details for a couple of proofs from Section 2.

Detailed proofs for Lemma 2.15. We keep the notation from the lemma. We first check that the map $\Psi$ is well-defined.

First, let $u_1, u_2, w \in W$ and $f \in F$. Then
\[
(\Psi(u_1 u_2)f)(w) = f(w(u_1 u_2)) = f((wu_1)u_2) = (\Psi(u_2)f)(wu_1) = (\Psi(\Psi(u_1)u_2)f)(w).
\]

Next, if $u, w \in W$ and $D \in \mathcal{D}(\mathfrak{h}_{\text{reg}})$ then
\[
(\Psi((1 \otimes u)(D \otimes 1)(1 \otimes u^{-1}))f)(w) = \Psi(u \cdot D \otimes 1)f(w) = (w \cdot u \cdot D)f(w) = (wu) \cdot Df(w).
\]

This shows that $\Psi$ is well-defined. The fact that it is then a homomorphism of rings is left to the reader.

Next, we check that the diagram in the statement of that lemma is indeed commutative, for which the reader should recall the definition of $f_0$ and $\varphi$ from Lemma 2.14. In particular, $\{g_i\}$ is a choice of left coset representatives of $W_H$ in...
where, with a slight abuse of notation, we write

\[ \varphi(e\Phi(eDe)e) = e_0a(\Phi(D \odot 1)(f_0)) = e_0 \sum_i \frac{|W_i|}{|W|} \left( \Phi(D \odot 1)(f_0) \right)(g_i)e_0 \]

\[ = e_0 \sum_i \frac{|W_i|}{|W|}(g_i \cdot D)f_0(g_i)e_0 = e_0 \sum_i \frac{|W_i|}{|W|}De_0e_0 = e_0De_0, \]

where in the last line we have used that \( g_i \cdot D = D \) since \( D \in \mathcal{D}(V_{reg}^W) \). Since \( e_0De_0 = f_0(D) \), this proves that the diagram commutes. \( \square \)

**Proof of Equation 2.17.** We remind the reader that we are interested in proving the following formula, where the notation is set up prior to Lemma 2.16. For \( y \in \mathfrak{h} \) and \( w \in W \),

\[
(C.1) \quad (\Psi \left(T^W_y \right) f)(w) = T^W_{\omega(y)}f(w) + \sum_{H' \in A, H' \neq H} \frac{w(y), \alpha_{H'}}{\alpha_{H'}} \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}f(e_{H', i}w),
\]

where, with a slight abuse of notation, we write

\[ f(e_{H', i}w) := \frac{1}{\ell_{H'}} \sum_{s \in W_{H'}} \det_h(s)^i f(sw). \]

**Proof.**

\[
(\Psi \left(T^W_y \right) f)(w) = \Psi (\partial_y)(f)(w) + \sum_{H' \in A} \langle y, \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}f(e_{H', i}w)
\]

\[ = \partial_{w(y)}f(w) + \sum_{H' \in A} \langle y, \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}w(\alpha_{H'}^{-1})(\Psi(e_{H', i})f)(w)
\]

\[ = \partial_{w(y)}f(w) + \sum_{H' \in A} \langle y, \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}w(\alpha_{H'}^{-1})(\frac{1}{\ell_{H'}} \sum_{s \in W_{H'}} \det_h(s)^i f(ws))\]

\[ = \partial_{w(y)}f(w) + \sum_{H' \in A} \langle y, \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}w(\alpha_{H'}^{-1})(\frac{1}{\ell_{H'}} \sum_{s \in W_{H'}} \det_h(\alpha_{H'}^{-1})^i f(sw))\]

\[ = \partial_{w(y)}f(w) + \sum_{H' \in A} \langle w(y), \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}f(e_{H', i}w)
\]

\[ = \partial_{w(y)}f(w) + \sum_{H' \in A} \langle w(y), \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}f(e_{H', i}w)\]

\[ + \sum_{H' \neq H} \langle w(y), \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}f(e_{H', i}w)\]

\[ + \sum_{H' \neq H} \langle w(y), \alpha_{H'} \rangle \sum_{i=0}^{\ell_{H'}-1} \ell_{H' \vee H', i}f(e_{H', i}w)\]
\[ T_{w(y)}^{\alpha H}f(w) + \sum_{H' \neq H} (w(y), \alpha_{H'}) \ell_{H'^{-1}} \sum_{i=0}^{\ell_{H'^{-1}}} \ell_{H'^{-1}H'^{-1}H}f(e_{H'^{-1}H}w) \]

This concludes the proof of Equation 2.17 and hence that of Lemma 2.16. \qed

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