CONFORMAL CAPACITY OF HEDGEHOGS

DIMITRIOS BETSAKOS, ALEXANDER SOLYNIN AND MATTI VUORINEN

Abstract. We discuss problems concerning the conformal condenser capacity of “hedgehogs”, which are compact sets $E$ in the unit disk $D = \{ z : |z| < 1 \}$ consisting of a central body $E_0$ that is typically a smaller disk $D_r = \{ z : |z| \leq r \}$, $0 < r < 1$, and several spikes $E_k$ that are compact sets lying on radial intervals $I(\alpha_k) = \{ te^{i\alpha_k} : 0 \leq t < 1 \}$. The main questions we are concerned with are the following: (1) How does the conformal capacity $\text{cap}(E)$ of $E = \bigcup_{k=0}^n E_k$ behave when the spikes $E_k$, $k = 1, \ldots, n$, move along the intervals $I(\alpha_k)$ toward the central body if their hyperbolic lengths are preserved during the motion? (2) How does the capacity $\text{cap}(E)$ depend on the distribution of angles between the spikes $E_k$? We prove several results related to these questions and discuss methods of applying symmetrization type transformations to study the capacity of hedgehogs. Several open problems, including problems on the capacity of hedgehogs in the three-dimensional hyperbolic space, will also be suggested.

In memoriam Jukka Sarvas (1944-2021).

Contents

Notation 1
1. Introduction 2
2. Preliminary results on the conformal capacity 5
3. Hedgehogs with geometric restrictions on the number of spikes 14
4. Extremal properties of hedgehogs on evenly distributed radial intervals 27
5. Symmetrization transformations in hyperbolic metric 32
6. Hedgehog problems in $\mathbb{R}^3$ 41
References 42

Notation

- $\mathbb{C}$ - complex plane.
- $D = \{ z \in \mathbb{C} : |z| < 1 \}$ - open unit disk centered at $z = 0$.
- $\lambda_D(z)$ - density of the hyperbolic metric in $D$.
- $\ell_D(\gamma)$ - hyperbolic length of $\gamma$.
- $A_D(E)$ - hyperbolic area of $E$.
- $p_D(z_1, z_2) = |(z_1 - z_2)/(1 - z_1 \overline{z}_2)|$ - pseudo-hyperbolic metric in $D$.

2010 Mathematics Subject Classification. 30C85, 31A15, 51M10.

Key words and phrases. Conformal capacity, hyperbolic metric, hyperbolic transfinite diameter, potential function, hedgehogs, polarization, symmetrization, hyperbolic dispersion.
The main theme discussed in this paper is the dependence of the condenser capacity \( \text{cap}(\mathbb{D}, E) \) on the geometric structure and characteristics of a compact set \( E \subset \mathbb{D} \), where \( \mathbb{D} \) is the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) in the complex plane \( \mathbb{C} \). For brevity, we call \( \text{cap}(\mathbb{D}, E) \) the conformal capacity of \( E \) or the capacity of \( E \) and denote it by \( \text{cap}(E) \).

Due to the conformal invariance of the capacity it is natural to equip \( \mathbb{D} \) with the hyperbolic metric. Indeed, very recently it was shown in [30] that isoperimetric inequalities in hyperbolic metric yield simple upper and lower bounds for the capacity in the case when \( E \) is a finite union of hyperbolic disks. In the subsequent work [29, 31, 32] these ideas were developed further, and it was also pointed out that similar ideas were also applied by F.W. Gehring [19] and R. Kühnau [26] fifty years earlier.

In most cases, we deal with compact sets \( E = \cup_{k=0}^{n} E_k \) consisting of a central body \( E_0 \), which can be absent, and spikes \( E_k \), \( k = 1, \ldots, n \), that are closed intervals or any compact sets lying on \( n \) radial intervals \( I(\alpha_k) \), where \( I(\alpha) = \{ te^{i\alpha} : 0 \leq t < 1 \} \), \( I = I(0) \). This type of compact sets have appeared in several research papers, for instance, in a recent paper [22] by J.-W. M. Van Ittersum, B. Ringeling, and W. Zudilin, where the term “hedgehog” was suggested for this shape of compact sets. Interestingly enough, estimates of the capacity and other characteristics of hedgehogs appeared to be useful in studies on the Mahler measure and Lehmer’s problem. Beside the above mentioned work of three authors, hedgehog structures appeared in the paper [34] by I. Pritsker and, earlier, the same hedgehog structure appeared in [39].

The hyperbolic metric in \( \mathbb{D} \) is defined by the element of length

\[
\lambda_\mathbb{D}(z) \, |dz| = \frac{2|dz|}{1 - |z|^2}.
\]

Then the hyperbolic length \( \ell_\mathbb{D}(E) \) of a compact subset \( E \) of a rectifiable curve can be calculated as

\[
\ell_\mathbb{D}(E) = \int_E \frac{2|dz|}{1 - |z|^2}.
\]

Furthermore, the hyperbolic geodesics are circular arcs in \( \mathbb{D} \) that are orthogonal to the unit circle \( \mathbb{T} = \partial \mathbb{D} \) at their end points. The hyperbolic distance between points \( z_1 \) and \( z_2 \) in \( \mathbb{D} \), that is equal to the hyperbolic length \( \ell_\mathbb{D}([z_1, z_2]) \) of the closed hyperbolic interval \([z_1, z_2] \) joining these points, is given by
(1.1) 
\[ d_\mathbb{D}(z_1, z_2) = \log \frac{1 + p_\mathbb{D}(z_1, z_2)}{1 - p_\mathbb{D}(z_1, z_2)}, \]

where \( p_\mathbb{D}(z_1, z_2) \) is the pseudo-hyperbolic metric defined as

(1.2) 
\[ p_\mathbb{D}(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - z_1 \overline{z}_2} \right|. \]

Everywhere below, \([z_1, z_2]_h\) and \((z_1, z_2)_h\) stand, respectively, for the closed and open hyperbolic intervals with end points \(z_1\) and \(z_2\). Similarly, notations \([z_2, z_2]\) and \((z_1, z_2)\) will be reserved for the closed and open Euclidean intervals with end points \(z_1\) and \(z_2\). If \(z_1\) and \(z_2\) lie on the same diameter of \(\mathbb{D}\) then, of course, \([z_1, z_2]_h = [z_1, z_2]\) and \((z_1, z_2)_h = (z_1, z_2)\).

When \(z_1 = 0\) and \(z_2 = r\), \(0 < r < 1\), the hyperbolic length \(\tau = \tau(r)\) of the interval \([0, r]\) and its Euclidean length \(r = r(\tau)\) are connected via the following formulas, which are often used in calculations:

(1.3) 
\[ \tau = \log \frac{1 + r}{1 - r}, \quad r = \frac{e^\tau - 1}{e^\tau + 1}. \]

The hyperbolic area of a Borel measurable subset \(E\) of \(\mathbb{D}\) is

(1.4) 
\[ A_\mathbb{D}(E) = \int_E \lambda^2_{\mathbb{D}}(z) \, dm, \]

where \(dm\) stands for the 2-dimensional Lebesgue measure. In particular, the hyperbolic area of the disk \(\mathbb{D}_r = \{z : |z| < r\}, 0 < r < 1\), is given by the following formula:

\[ A_\mathbb{D}(\mathbb{D}_r) = \frac{4\pi r^2}{1 - r^2} = 4\pi \sinh^2(\tau/2). \]

For the properties of geometric quantities defined above, we recommend the monograph of A. Beardon [9].

Our main focus in this paper will be on the quantity, which we call the \textit{conformal capacity}, or just \textit{capacity} that is related to the hyperbolic capacity studied in [19] and [46].

**Definition 1.5.** Let \(E\) be a compact set in \(\mathbb{D}\). The conformal capacity \(\text{cap}(E)\) of \(E\) is defined as

(1.6) 
\[ \text{cap}(E) = \inf \int_{\mathbb{D}} |\nabla u|^2 \, dm, \]

where the infimum is taken over all Lipschitz functions \(u\) such that \(u = 0\) on the unit circle \(\mathbb{T} = \partial \mathbb{D}\) and \(u(z) \geq 1\) for \(z \in E\).

In terminology used in electrostatics, the conformal capacity \(\text{cap}(E)\) is usually referred to as the capacity of the condenser \((\mathbb{D}, E)\) with plates \(E\) and \(\mathbb{D}^* = \overline{\mathbb{C}} \setminus \mathbb{D}\) and field \(\mathbb{D} \setminus E\). Therefore, many properties and theorems known for the capacity of a physical condenser can be applied to the conformal capacity as well. In Section 2, we collect some of these properties, which will be needed for the purposes of this work. Several available methods to prove the above mentioned properties also will be discussed in Section 2.
Explicit expressions for the conformal capacity of compact sets are available in a few cases only. Here are three examples, in which the conformal capacity is expressed in terms of both Euclidean and hyperbolic characteristics of the set.

**Example 1.7.** The conformal capacity of the closure of the disk $D_r$, with Euclidean radius $r$, $0 < r < 1$, and hyperbolic radius $\tau > 0$, is given by
\[
\text{cap}(\overline{D}_r) = \frac{2\pi}{\log(1/r)} = \frac{2\pi}{\log((e^\tau + 1)/(e^\tau - 1))}.
\]

**Example 1.8.** The conformal capacity of the interval $[-r, r]$, with hyperbolic length equal to $2\tau = \ell_D([-r, r]) = 2\log((1 + r)/(1 - r))$, is
\[
\text{cap}([-r, r]) = 8\frac{K(r^2)}{K'(r^2)} = 8\frac{K((e^\tau - 1)/2)}{K'(((e^\tau - 1)/2)^2)}.
\]

Here and below, $K(k)$ and $K'(k) = K(\sqrt{1 - k^2})$ stand for the complete elliptic integrals of the first kind.

**Example 1.9.** The conformal capacity of the interval $[0, r]$, with hyperbolic length equal to $\tau = \ell_D([0, r]) = \log((1 + r)/(1 - r))$, is
\[
(1.10) \quad \text{cap}([0, r]) = 4\frac{K(r)}{K'(r)} = 4\frac{K((e^\tau - 1)/(e^\tau + 1))}{K'(((e^\tau - 1)/(e^\tau + 1)))}.
\]

As Examples 1.8 and 1.9 demonstrate, even for compact sets as simple as a hyperbolic interval, the conformal capacity can not be expressed in terms of elementary functions. Thus, estimates in terms of Euclidean characteristics of a set and numerical computations are important when working with this capacity.

This project originated with the following question raised by the third-listed author of this paper. This question arose in the course of recent work [29]-[32] and it was experimentally studied in [24].

**Problem 1.11.** Suppose that $0 < r < s < t < 1$ and $0 < u < 1$ are such that the sets $E = [0, r] \cup [s, t]$ and $E_1 = [0, u]$ have equal hyperbolic lengths. Is it true that conformal capacity of $E$ is greater than conformal capacity of $E_1$?

It appears that this question has many interesting ramifications. Thus, we decided to team up to discuss these questions, answer several of them demonstrating available technique and to point out a few remaining open questions. In the context of Problem 1.11, it is natural to consider compact sets lying on any finite number of radial intervals. This is how geometric shapes resembling animals with spikes, and therefore the term “hedgehog”, appeared in our study. Typically, the central body $E_0$ will be a disk $\overline{D}_r$, $0 < r < 1$, or an emptyset and $E_k$, $1 \leq k \leq m$, will be a collection of closed intervals attached to the central body. In this case, our compact sets $E$ look more like the sea creatures called “stilocidaris affinis”, that is shown in our Figure 1, than like hedgehogs as everyone knows them. But, because the term “hedgehog” was already applied in the context of our study in mathematical literature, we will stick with it in our paper.

Our main results in Sections 3 and 4 deal with several extremal problems for the capacity of compact sets in the unit disk $D$, where hedgehogs possessing certain symmetry properties play
the role of the extremal configuration. Thus, in these sections we mainly work with compact sets in $\mathbb{D}$ having components lying on a finite number of radial intervals. In Section 3, we first demonstrate our methods on simple cases, considered in Lemmas 3.1 and 3.4, when a compact set lies on the radial segment or on the diameter of $\mathbb{D}$. In particular, Lemma 3.1 provides an affirmative answer to the question stated in Problem 1.11. Then, in several theorems presented in Section 3, we extend our proofs to the case of compact sets lying on several radial intervals. In Section 4, we deal with several extremal problems on the conformal capacity for compact sets lying on a finite number of radial intervals evenly distributed over the unit disk.

As is well known, symmetrization type transformations (such as Steiner symmetrization, Schwarz symmetrization, Pólya circular symmetrization, Szegő radial symmetrization, polar-ization and other) provide a standard tool to estimate capacities and many other characteristics of sets. Most of the classical results on symmetrization can be found in the fundamental study by G. Pólya and G. Szegő [33]. More recent approaches to symmetrization were developed by A. Baernstein II [5], V. Dubinin [17], J. Sarvas [36] and also in the papers [37], [13], [40] and [41]. In Section 5, we will discuss hyperbolic counterparts of some of these transformations and how they can be applied in problems about conformal capacity.

Finally, in Section 6, we will mention possible generalizations of our results for conformal capacity in hyperbolic spaces of dimension $n \geq 3$.

2. Preliminary results on the conformal capacity

In this section, we recall properties of the conformal capacity needed for our work. We have already mentioned in the Introduction the connection of the conformal capacity with the condenser capacity. A condenser is a pair $(D, E)$, where $D$ is a domain in the plane and $E$ is a compact subset of $D$. The capacity of the condenser $(D, E)$ is defined by

\begin{equation}
\text{cap}(D, E) = \inf \int_D |\nabla u|^2 \, dm,
\end{equation}
where the infimum is taken over all Lipschitz functions \( u \) such that \( u \leq 0 \) on \( \partial D \) and \( u \geq 1 \) on \( E \). These functions will be called \textit{admissible} for the condenser \((D, E)\). We want to stress here that if \( D = \mathbb{D} \), then the infimum in (2.1) can be taken over all admissible functions as above with an additional requirement that \( u(z) = 0 \) for all \( z \in \mathbb{T} \).

By Theorem 3.8 of Ziemer [50], the capacity of the condenser \((D, E)\) is equal to the modulus \( M(\Gamma) \) of the family \( \Gamma \) of all curves in \( D \setminus E \) joining \( E \) with \( \partial D \). For the definition and the basic properties of the modulus of curve families, we refer to [23, Chapter II] and [21, Chapter 7].

The following invariance property of the conformal capacity, that we often use in our proofs below, is immediate from the well-known invariance property of the capacity of a condenser, see, for instance, [17, Theorem 1.12].

**Proposition 2.2.** The conformal capacity is invariant under the Möbius self maps of \( \mathbb{D} \) and it is invariant under reflections with respect to hyperbolic geodesics. Thus, if \( E \) is a compact subset of \( \mathbb{D} \) and \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) is a Möbius automorphism or a reflection with respect to a hyperbolic geodesic, then \( \text{cap}(\varphi(E)) = \text{cap}(E) \).

Similar to the “sets of measure zero” in measure theory, there are small sets that can be neglected when working with the conformal capacity. These sets are known in the literature as “polar sets” or “sets of zero logarithmic capacity”. For our purposes the second name is more appropriate and many other authors used it in a similar context. We recall that the logarithmic capacity \( \log \text{cap}(E) \) of a set \( E \subset \mathbb{C} \), not necessarily compact, is given by

\[
\log \text{cap}(E) = \sup_{\mu} I(\mu) \quad \text{with} \quad I(\mu) = \iint \log |z - w| \ d\mu(z) d\mu(w),
\]

where the supremum is taken over all Borel probability measures \( \mu \) on \( \mathbb{C} \) whose support is a compact subset of \( E \). For the properties of the logarithmic capacity, we refer to Chapter 5 in T. Ransford book [35] and to the monographs [3], [27]. In Proposition 2.3 below we collect results identifying “sets of zero logarithmic capacity” as sets negligible for the value of the conformal capacity. For the proofs of these results we refer to H. Wallin’s paper [48].

**Proposition 2.3.** The following hold:

1. If \( E \subset \mathbb{D} \) is compact, then \( \text{cap}(E) = 0 \) if and only if \( \log \text{cap}(E) = 0 \).

2. Let \( E_1, E_2 \) be compact sets in \( \mathbb{D} \) such that \( E_1 \subset E_2 \). Then \( \text{cap}(E_1) = \text{cap}(E_2) \) if and only if \( \log \text{cap}(E_2 \setminus E_1) = 0 \).

3. If \( E \subset (-1, 1) \) is such that \( \log \text{cap}(E) = 0 \), then \( \ell_D(E) = 0 \).

We note that the set \( E_2 \setminus E_1 \) in part (2) and the set \( E \) in part (3) of Proposition 2.3 are not necessarily compact. We stress here that the inverse statement for part (3) of this proposition is not true, in general. For example, if \( K \subset [0, 1] \) is the standard Cantor set, then its scaled version \( K_{1/3} = \{ z : 3z \in K \} \) has zero hyperbolic length but positive logarithmic capacity, see, for example, [35, p.143]. A compact set of logarithmic capacity zero is of zero Hausdorff dimension [48].

Next, we will state a proposition about the existence of a function minimizing the Dirichlet integral in equation (2.1), and therefore in equation (1.6) as well. This proposition follows from classical potential theoretic results; see, for instance, [7, Theorem 1], [27, p.97].
Proposition 2.4. Let $E$ be a compact set in $\mathbb{D}$. There is a unique function $u_E$, called the potential function of $E$, that minimizes the integral in (2.1); i.e. such that
\[
\text{cap}(E) = \int_{\mathbb{D}} |\nabla u_E|^2 \, dm.
\]
Moreover, $u_E$ possesses the following properties:

1. $u_E$ is harmonic in $\mathbb{D} \setminus E$ and continuous on $\overline{\mathbb{D}}$, except possibly for a subset of $E$ of zero logarithmic capacity.

2. $u_E(z) = 0$ for $z \in \mathbb{T}$ and $u_E(z) = 1$ for all $z \in E$, except possibly for a subset of $E$ of zero logarithmic capacity.

3. If every point of $\partial E$ is regular for the Dirichlet problem in $\mathbb{D} \setminus E$, then $u_E$ is continuous on $\mathbb{D}$ and $u_E = 1$ on $E$.

Below are two examples of sets exceptional in the sense of parts (1) and (2) of Proposition 2.4.

Example 2.5. Consider a compact set $E_e = \{0\} \cup (\bigcup_{n=1}^{\infty} I_n)$, where $I_n$ is an interval $[e^{-n}, e^{-n} + e^{-n^3}]$ with length $l_n = e^{-n^3}$. Since
\[
\sum_{n=1}^{\infty} \frac{n}{\log(2/l_n)} = \sum_{n=1}^{\infty} \frac{n}{n^3 + \log 2} < \infty,
\]
Wiener’s criterion (see [35, Theorem 5.4]) implies that the point $z = 0 \in E_e$ is irregular for the Dirichlet problem. Therefore, the function $u_{E_e}(z) = 1$ is not a barrier at $z = 0$ (see [35, Definition 4.1.4]). The latter implies that the limit $\lim_{z \to 0} u_{E_e}(z)$ does not exist. Since points $z \in E_e, z \neq 0$, are regular for the Dirichlet problem, $z = 0$ is the only point in $\mathbb{D}$, where $u_{E_e}$ is not continuous.

To obtain a compact set $F_e$ such that $u_{F_e}$ has infinite number of discontinuities, we modify our previous example as follows. For $n \in \mathbb{N}$, let $E^n_e = \{(z+1)/2^n : z \in E_e\}$. Thus, $E^n_e$ is obtained by translating and scaling the set $E_e$. Let $F_e = \{0\} \cup (\bigcup_{n=1}^{\infty} E^n_e)$. Our previous argument can be applied to show that $u_{F_e}$ is not continuous at an infinite set of points $z_n = 2^{-n}, n \in \mathbb{N}$. Thus, $F_e$ has an infinite subset exceptional in the sense of part (1) of Proposition 2.4.

Example 2.6. Let $E \subset \mathbb{D}$ be a compact set of positive logarithmic capacity, which contains a nonempty subset $E_0$, each point of which is isolated from other points of $E$. Since $u_E$ is harmonic and bounded, every point $z \in E_0$ is removable, which means that $u_E$ can be extended as a function harmonic at $z$. Since $u_E$ is not constant, it follows that $0 < u_E(z) < 1$ for every point $z \in E_0$. Thus, $E_0 \subset E$ is an exceptional set, possibly infinite, as it was mentioned in part (2) of Proposition 2.4.

Next, we recall a subadditivity property of the conformal capacity, which we need in the following form.

Proposition 2.7. Suppose that $E = \bigcup_{k=1}^{n} E_k$ is the union of $n \geq 2$ compact sets $E_k$, $k = 1, \ldots, n$, in $\mathbb{D}$. Then
\[
\text{cap}(E) \leq \sum_{k=1}^{n} \text{cap}(E_k).
\]
Moreover, if each $E_k$ has positive conformal capacity, then (2.8) holds with strict inequality.

**Proof.** Let $u_k$, $k = 1, \ldots, n$, be an admissible function for the condenser $(D, E_k)$ such that $u_k(z) = 0$ for $z \in T$. Then $u(z) = \max\{u_1(z), \ldots, u_n(z)\}$ is an admissible function for the condenser $(D, E)$ such that $|u(z_1) - u(z_2)| \leq \max_{1 \leq k \leq n} |u_k(z_1) - u_k(z_2)|$ for all $z_1, z_2 \in D$. The latter inequality implies that $|\nabla u(z)| \leq \max_{1 \leq k \leq n} |\nabla u_k(z)|$ for all $z \in D$, where the gradients exist. Therefore,

$$\text{cap}(E) \leq \int_D |\nabla u|^2 \, dm \leq \int_D \max_{1 \leq k \leq n} |\nabla u_k|^2 \, dm \leq \sum_{k=1}^n \int_D |\nabla u_k|^2 \, dm. \tag{2.9}$$

Taking the infimum in this equation over all admissible functions $u_k$, $k = 1, \ldots, n$, with the properties mentioned above in this proof, we obtain the required subadditivity property (2.8).

If $\text{cap}(E_k) > 0$, $k = 1, 2, \ldots, n$, then each of the condensers $(D, E_k)$ has the potential function $u_{E_k}$ that is a non-constant harmonic function in $D \setminus E_k$. Therefore, $|\nabla u_{E_k}| > 0$, almost everywhere in $D \setminus E_k$. Since $u_{E_k}$ is the potential function of $(D, E_k)$ it follows that (2.9) holds with $u_k = u_{E_k}$. In this case, the strict inequality $\max_{1 \leq k \leq n} |\nabla u_k(z)| < \sum_{1 \leq k \leq n} |\nabla u_k(z)|$ holds for all points $z$ in an annulus $\{z : \rho < |z| < 1\}$ with $0 < \rho < 1$ such that $\{z : \rho < |z| < 1\} \subset D \setminus \bigcup_{1 \leq k \leq n} E_k$. The latter implies that if $u_k = u_{E_k}$ then the third inequality in (2.9) is strict. Therefore, (2.8) holds with the sign of strict inequality in the case under consideration. □

The proofs of our main theorems in Sections 3 and 4 rely on the polarization technique and on the geometric interpretation of the conformal capacity in terms of the hyperbolic transfinite diameter.

To define the polarization of compact sets with respect to a hyperbolic geodesic $\gamma$, we need the following terminology. To any hyperbolic geodesic $\gamma$, we can give an orientation by marking one of its complementary hyperbolic halfplanes and call it $H_+$; then the other complementary hyperbolic halfplane is given the name $H_-$. Since every hyperbolic geodesic $\gamma$ is an arc of a circle, we can define the classical symmetry transformation (called also inversion or reflection) with respect to $\gamma$. We note that the symmetry transformation with respect to a hyperbolic geodesic $\gamma$ is a hyperbolic isometry on $D$.

The polarization transformation of compact sets in $D$ can be defined as follows.

**Definition 2.10.** Let $\gamma$ be an oriented hyperbolic geodesic in $D$. Let $H_+, H_-$ be the hyperbolic halfplanes determined by $\gamma$. Let $E$ be a compact set in $D$ and let $\mathcal{R}_\gamma(E)$ denote the set symmetric to $E$ with respect to $\gamma$. The polarization $\mathcal{P}_\gamma(E)$ of $E$ with respect to $\gamma$ is defined by

$$\mathcal{P}_\gamma(E) = ((E \cup \mathcal{R}_\gamma(E)) \cap \overline{H_+}) \cup ((E \cap \mathcal{R}_\gamma(E)) \cap H_-). \tag{2.11}$$

Equation (2.11) can be also written in the form

$$\mathcal{P}_\gamma(E) = ((E \cup \mathcal{R}_\gamma(E)) \setminus H_-) \cup ((E \cap \mathcal{R}_\gamma(E)) \setminus H_+).$$

The polarization transformation was introduced by V. Wolontis in 1952, [49]. Wolontis’ work remained unnoticed until 1984, when V. N. Dubinin used this transformation to solve A. A. Gonchar’s problem on the capacity of a condenser with plates on a fixed straight line interval. The name “polarization” was also suggested by Dubinin [16]. The following proposition, describing the change of the conformal capacity under polarization, is an important ingredient of the proofs in the following sections, see [17, Theorem 3.4], [12, Theorem 2.8].
Proposition 2.12. Let $E$ be a compact set in $\mathbb{D}$ and $\mathcal{P}_\gamma(E)$ be the polarization of $E$ with respect to an oriented hyperbolic geodesic $\gamma$. Then

$$\text{cap}(\mathcal{P}_\gamma(E)) \leq \text{cap}(E).$$

Furthermore, equality occurs in (2.13) if and only if $\mathcal{P}_\gamma(E)$ coincides with $E$ up to reflection with respect to $\gamma$ and up to a set of zero logarithmic capacity.

When working with hedgehog structures, the following particular case of Proposition 2.12 is useful.

Corollary 2.14. Under the assumptions of Proposition 2.12, let $E$ be the closure of the union of a finite or infinite number of non-overlapping closed intervals on the diameter $(-1,1)$. Then (2.13) holds with the sign of strict inequality unless $\mathcal{P}_\gamma(E)$ coincides with $E$ up to reflection with respect to $\gamma$.

One more useful characteristic of compact sets in the hyperbolic plane, the hyperbolic transfinite diameter, was introduced by M. Tsuji [46]. It is defined as follows (see [46, p.94] or [17, Section 1.4]).

Definition 2.15. Let $E \subset \mathbb{D}$ be a compact set in $\mathbb{D}$. The hyperbolic transfinite diameter of $E$ is defined as

$$d_h(E) = \lim_{n \to \infty} \max \prod_{1 \leq j < k \leq n} (p_D(z_j, z_k))^{2/[n(n-1)]},$$

where $p_D(z_j, z_k)$ stands for the pseudo-hyperbolic metric defined by (1.2) and the maximum is taken of all $n$-tuples of points $z_1, \ldots, z_n$ in $E$.

The following relation was established in [46].

Proposition 2.17. Let $E$ be a compact set in $\mathbb{D}$. Then

$$\text{cap}(E) = \left[ -\frac{1}{2\pi} \log d_h(E) \right]^{-1}.$$

Let $E$ be a compact set in $\mathbb{D}$. A map $\varphi : E \to \mathbb{D}$ is called a hyperbolic contraction on $E$ if for every $z_1, z_2 \in E$,

$$d_D(\varphi(z_1), \varphi(z_2)) \leq d_D(z_1, z_2),$$

where $d_D(\cdot, \cdot)$ stands for the hyperbolic metric defined by (1.1). Furthermore, $\varphi : E \to \mathbb{D}$ is called a strict hyperbolic contraction on $E$ if there is $k$, $0 < k < 1$, such that for every $z_1, z_2 \in E$,

$$d_D(\varphi(z_1), \varphi(z_2)) \leq k d_D(z_1, z_2).$$

The following contraction principle is immediate from the Definition 2.15 and Proposition 2.17.

Proposition 2.19. Let $E$ be a compact set in $\mathbb{D}$. Let $\varphi : E \to \mathbb{D}$ be a hyperbolic contraction. Then $\text{cap}(\varphi(E)) \leq \text{cap}(E)$.

Moreover, if $\varphi$ is a strict hyperbolic contraction on $E$, then $\text{cap}(\varphi(E)) < \text{cap}(E)$.
Remark 2.20. We note here that the polarization transformation is not contracting, in general. For example, polarizing the set \( E = \{ \pm i/4, \pm (1-i)/4 \} \subset \mathbb{D} \) with respect to the diameter \( I = (-1,1) \) with its standard orientation, we obtain the polarized set \( P_I(E) = \{ \pm i/4, (\pm 1 + i)/4 \} \). Then for every one-to-one map \( \varphi : E \to P_I(E) \), there is a pair of points \( z_1, z_2 \in E \) such that \( d_\mathbb{D}(\varphi(z_2), \varphi(z_1)) > d_\mathbb{D}(z_2, z_1) \); one can easily verify this inequality by considering each one of the possible maps \( \varphi \).

To study the limit behavior of the conformal capacity \( \text{cap}(E) \), when some of the components of \( E \) tend to the boundary of \( \mathbb{D} \), we need a hyperbolic analog of the dispersion property of the Newtonian capacity discussed in [42]. Let \( E_1, \ldots, E_n \) be disjoint nonempty compact sets in \( \mathbb{D} \), not necessarily connected, and let \( E = \bigcup_{k=1}^n E_k \).

**Definition 2.21.** By a hyperbolic dispersion of \( E = \bigcup_{k=1}^n E_k \) we mean a mapping \( \varphi : E \times [0, \infty) \to \mathbb{D} \) satisfying the following properties:

1. For each \( k \), the restriction \( \varphi : E_k \times [0, \infty) \to \mathbb{D} \) is a rigid hyperbolic motion of \( E_k \), which depends continuously on the parameter \( t \in [0, \infty) \), such that \( \varphi(x,0) = x \) for all \( x \in E \).
2. If \( 0 \leq t_1 < t_2 \), then for each \( k \) and \( j, k \neq j \), the hyperbolic distances between the images \( \varphi(E_k, t) \) and \( \varphi(E_j, t) \) satisfy the following inequalities:
   \[
   d_\mathbb{D}(\varphi(E_k, t_1), \varphi(E_j, t_1)) \leq d_\mathbb{D}(\varphi(E_k, t_2), \varphi(E_j, t_2)).
   \]
3. For each \( k \) and \( j, k \neq j \),
   \[
   d_\mathbb{D}(\varphi(E_k, t), \varphi(E_j, t)) \to \infty \quad \text{as} \ t \to \infty.
   \]

Thus, hyperbolic dispersion of \( E \) is a process moving the subsets \( E_1, \ldots, E_n \) farther and farther from each other, resembling the scattering of galaxies of our Universe.

We stress here that not every finite collection of compact sets admits hyperbolic dispersion. For example, the set \( E = E_1 \cup E_2 \), where \( E_1 = \{ 0 \} \) and \( E_2 = \{ z = re^{i\theta} : |\theta| \leq \pi - \varepsilon \} \) with \( 0 < r < 1 \) and sufficiently small \( \varepsilon > 0 \), cannot be hyperbolically dispersed in the sense of Definition 2.21. On the other hand, any union \( E = E_1 \cup E_2 \) of two non-intersecting compact sets, each of which lies on a radial interval, can be hyperbolically dispersed.

The following useful result is a hyperbolic counterpart of Proposition 5 proved in [42].

**Proposition 2.22.** Let \( \varphi : E \times [0, \infty) \to \mathbb{D} \) be a hyperbolic dispersion of a compact set \( E = \bigcup_{k=1}^n E_k \), as above. Then

\[
\text{cap}(\varphi(E, t)) \to \sum_{k=1}^n \text{cap}(E_k), \quad \text{as} \ t \to \infty.
\]

In the proof of Proposition 2.22, we will need the following elementary arithmetic result.

**Lemma 2.24.** Let \( 0 < \alpha_k < 1 \), \( k = 1, \ldots, n \), be such that \( \sum_{k=1}^n \alpha_k = 1 \). Then there are \( k \) sequences of positive integers \( m_{j,k}, j = 1, 2, \ldots \), such that if \( m_j = \sum_{k=1}^n m_{j,k} \), then \( m_j \to \infty \) and \( m_{j,k}/m_j \to \alpha_k \) as \( j \to \infty \).

**Proof.** Consider rational approximations of \( \alpha_k, k = 1, \ldots, n-1 \); i.e. consider \((n-1)\) sequences

\[
\frac{a_{j,k}}{b_{j,k}} \to \alpha_k, \quad k = 1, \ldots, n-1, \quad \text{as} \ j \to \infty,
\]

where \( a_{j,k} \) and \( b_{j,k} \) are positive integers. Let \( \epsilon > 0 \) be given. There exists a positive integer \( M \) such that for all \( k = 1, \ldots, n-1 \),

\[
\frac{a_{j,k}}{b_{j,k}} < \alpha_k + \epsilon \quad \text{for all} \ j > M.
\]
where \( a_{j,k}, b_{j,k} \) are positive integers. Then consider the sequence
\[
m_j = j \prod_{k=1}^{n-1} b_{j,k} \to \infty
\]
and the sequences
\[
m_{j,k} = \frac{a_{j,k} m_j}{b_{j,k}}, \quad k = 1, \ldots, n - 1.
\]

Clearly,
\[
\frac{m_{j,k}}{m_j} = \frac{a_{j,k}}{b_{j,k}} \to \alpha_k \quad \text{as } j \to \infty.
\]

Also, we put
\[
m_{j,n} = m_j - \sum_{k=1}^{n-1} m_{j,k}.
\]

Then
\[
\frac{m_{j,n}}{m_j} = 1 - \sum_{k=1}^{n-1} \frac{m_{j,k}}{m_j} \to 1 - \sum_{k=1}^{n-1} \alpha_k = \alpha_n > 0 \quad \text{as } j \to \infty.
\]
The latter relation shows that \( m_{j,n} > 0 \) for all \( j \) sufficiently large. Therefore, we can remove a finite number of terms from the sequence \( m_j \) and from the sequences \( m_{j,k} \) and then re-enumerate these sequences to obtain sequences with the required properties. \( \square \)

**Proof of Proposition 2.22.** For \( k = 1, 2, \ldots, n \) and \( t \geq 0 \), we set \( E^t = \varphi(E, t), E_k^t = \varphi(E_k, t) \). Since the conformal capacity is invariant under hyperbolic motions, \( \text{cap}(E_k^t) = \text{cap}(E_k) \) for all \( k = 1, \ldots, n \) and all \( t \geq 0 \). This together with the subadditivity property of Proposition 2.19 implies that
\[
(2.25) \quad \text{cap}(E^t) \leq \sum_{k=1}^{n} \text{cap}(E_k^t) = \sum_{k=1}^{n} \text{cap}(E_k).
\]

We assume without loss of generality that \( \text{cap}(E_k) > 0 \) for all \( k = 1, \ldots, n \). Then we set \( \alpha_k = \text{cap}(E_k)/\sum_{k=1}^{n} \text{cap}(E_k) \) and will use the sequences \( m_{j,k}, k = 1, \ldots, n \), and \( m_j \) defined as in the proof of Lemma 2.24 for our choice of \( \alpha_k \).

Let \( z_{j,k}^s, s = 1, \ldots, m_{j,k}, \) be points in \( E_k \) such that
\[
(2.26) \quad \prod_{1 \leq l < s \leq m_{j,k}} p_D(z_{j,k}^l, z_{j,k}^s) = \max \prod_{1 \leq l < s \leq m_{j,k}} p_D(z_{j,k}^l, z_s),
\]
where the maximum is taken over all \( m_{j,k} \)-tuples of points \( z_1, z^2, \ldots, z^{m_{j,k}} \) in \( E_k \). For \( k = 1, \ldots, n, j = 1, 2, \ldots, s = 1, \ldots, m_{j,k}, \) and \( t \geq 0 \), we set \( z_{j,k}^{s,t} = \varphi(z_{j,k}^s, t) \). Since \( \varphi \) is a hyperbolic motion on each \( E_k \), we have
\[
(2.27) \quad p_D(z_{j,k}^{s,t}, z_{j,k}^s) = p_D(z_{j,k}^l, z_{j,k}^s),
\]
for all points \( z_{j,k}^l, z_{j,k}^s \) defined above and all \( t \geq 0 \).
For our choice of points, it follows from (2.27) and equation (2.16) of Definition 2.15 that

\[ d_h(E^t) \geq \limsup_{j \to \infty} \left[ \prod_j \prod_{k=1}^n \Pi_{j,k} \right]^{2/m_j(m_j-1)}, \]

where

\[ \Pi_{j,k} = \prod_{1 \leq l < s \leq m_{j,k}} p_D(z_{j,k}^l, z_{j,k}^s), \]

and

\[ \Pi_j^t = \prod p_D(z_{j,k1}^{l,t}, z_{j,k2}^{s,t}), \]

where the product in (2.30) is taken over all pairs of points \( z_{j,k1}^{l,t}, z_{j,k2}^{s,t} \) such that \( 1 \leq l \leq m_{j,k1}, 1 \leq s \leq m_{j,k2} \) and \( k_1 \neq k_2 \).

Using equations (2.16), (2.26), and (2.29) and taking into account our choice of points \( z_{j,k}^s \), \( 1 \leq s \leq m_{j,k} \) and the limit relation \( \lim_{j \to \infty} m_{j,k}/m_j = \alpha_k \), we conclude that

\[ \lim_{j \to \infty} \left( \Pi_{j,k} \right)^{2/m_j(m_j-1)} = \lim_{j \to \infty} \left( \prod_{1 \leq l < s \leq m_{j,k}} p_D(z_{j,k}^l, z_{j,k}^s) \right)^{2/m_j(m_j-1)}/m_j(m_j-1) \]

\[ = (d_h(E_k))^\alpha_k^2. \]

Our assumption that \( d_D(E_{k1}^t, E_{k2}^t) \to \infty \) when \( k_1 \neq k_2 \) and \( t \to \infty \) and relations (1.1), (1.2), imply that for every \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that if \( k_1 \neq k_2 \) then

\[ p_D(z_{j,k1}^{l,t}, z_{j,k2}^{s,t}) > 1 - \varepsilon \quad \text{for all } t \geq t_\varepsilon. \]

This inequality together with (2.30) imply that

\[ \left( \Pi_j^t \right)^{2/m_j(m_j-1)} \geq 1 - \varepsilon \quad \text{for all } t \geq t_\varepsilon. \]

Combining (2.28), (2.31), and (2.32), we obtain the following:

\[ d_h(E^t) \geq (1 - \varepsilon) \prod_{k=1}^n (d_h(E_k))^\alpha_k^2 \quad \text{for all } t \geq t_\varepsilon. \]

The latter inequality together with (2.18) implies that for all \( t \geq t_\varepsilon \),

\[ \text{cap}(E^t) = \frac{1}{-\frac{1}{2\pi} \log d_h(E^t)} \geq \frac{1}{-\frac{1}{2\pi} \log \left( (1 - \varepsilon) \prod_{k=1}^n (d_h(E_k))^\alpha_k^2 \right)} \]

\[ = \sum_{k=1}^n \frac{\alpha_k^2/\text{cap}(E_k)}{\sum_{k=1}^n \text{cap}(E_k)} \]

\[ = 1 - \frac{1}{2\pi} \log(1 - \varepsilon) \sum_{k=1}^n \text{cap}(E_k). \]
Since $\varepsilon > 0$ can be chosen arbitrarily small, it follows from (2.33) that

$$
\liminf_{t \to \infty} \cap(E^t) \geq \sum_{k=1}^n \cap(E_k).
$$

Finally, equations (2.25) and (2.34) imply (2.23). \hfill \square

**Remark 2.35.** We note here that the conformal capacity of a compact set $E \subset \mathbb{D}$ is not monotone under hyperbolic dispersion, in general.

To give an example of such non-monotonicity, we consider a family of hedgehogs $E(t), t \geq 0,$ with $E(t)$ consisting of a fixed central body $C_{r_0}(\alpha) = \{r_0 e^{i\theta} : |\theta| \leq \alpha\}, 0 < r_0 < 1, 0 < \alpha < \pi,$ and single varying spike $E(t) = [r_1(t)e^{i(\alpha - t)}, r_2(t)e^{i(\alpha - t)}], \alpha,$ which we define as follows.

We put $r_1(0) = r_1, r_2(0) = r_2, 0 < r_1 < r_2 < 1.$ Using polarization with respect to appropriate hyperbolic geodesics $\gamma = (-e^{i\theta}, e^{i\theta}),$ we find that $\cap(C_{r_0}(\alpha) \cup [r_1(t)e^{i(\alpha - t)}, r_2(t)e^{i(\alpha - t)}])$ strictly decreases, when $t$ varies from 0 to $\alpha.$ At the same time $d_\mathbb{D}(C_{r_0}(\alpha), [r_1e^{i(\alpha - t)}, r_2e^{i(\alpha - t)}])$ and $\ell_\mathbb{D}([r_1e^{i(\alpha - t)}, r_2e^{i(\alpha - t)}])$ are constant for $0 \leq t \leq \alpha.$ Using these properties and the well-known convergence result, which is stated in Proposition 2.36 below, we conclude that there is a strictly increasing function $r_1(t)$ such that $r_1(0) = r_1, 0 < r_1 < r_1(\alpha) < 1,$ and a function $r_2(t), r_1(t) < r_2(t) < 1,$ such that $\cap(C_{r_0}(\alpha) \cup [r_1(t)e^{i(\alpha - t)}, r_2(t)e^{i(\alpha - t)}])$ strictly decreases, $d_\mathbb{D}(C_{r_0}(\alpha), [r_1(t)e^{i(\alpha - t)}, r_2(t)e^{i(\alpha - t)}])$ strictly increases, while the hyperbolic length $\ell_\mathbb{D}([r_1(t)e^{i(\alpha - t)}, r_2(t)e^{i(\alpha - t)}])$ remains constant on $0 \leq t \leq \alpha.$

Now, we put $E(t) = C_{r_0}(\alpha) \cup [r_1(t)e^{i(\alpha - t)}, r_2(t)e^{i(\alpha - t)}]$ for $0 \leq t \leq \alpha$ and, for $t \geq \alpha,$ we define $E(t)$ as $C_{r_0}(\alpha) \cup [r_1(t), r_2(t)]$ with $r_1(t) = (t - \alpha + \alpha r_1(\alpha)/t)$ and $r_2(t)$ such that $\ell_\mathbb{D}([r_1(t), r_2(t)]) = \ell_\mathbb{D}([r_1, r_2]).$ The family of hedgehogs $E(t)$ defines a dispersion of compact sets $C_{r_0}(\alpha)$ and $[r_1e^{i\alpha}, r_2e^{i\alpha}]$ such that $\cap(E(t))$ strictly decreases on $0 \leq t \leq \alpha.$ Furthermore, using polarization with respect to appropriate geodesics, as we will demonstrate later in the proof of Lemma 3.4, one can show that $\cap(E(t))$ strictly increases on the interval $t \geq \alpha.$

For our proofs, we need two results on the sequences of compact sets in $\mathbb{D}$ convergent in an appropriate sense.

**Proposition 2.36** (see, [17, Theorem 1.11]). Let $E_k, k = 1, 2, \ldots,$ be a sequence of compact sets in $\mathbb{D},$ such that $E_{k+1} \subset E_k$ for all $k = 1, 2, \ldots,$ and let $E = \cap_{k=1}^\infty E_k.$ Then

$$
\cap(E_k) \to \cap(E) \quad \text{as } k \to \infty.
$$

To state our next proposition, we recall that the Hausdorff distance between two compact sets $K, L$ in the plane is given by

$$
d_H(K, L) = \max\{\text{dist}(x, K), \text{dist}(y, L) : x \in L, y \in K\}.
$$

The following convergence result follows from [4, Theorem 7].

**Proposition 2.37.** For fixed $\delta > 0,$ let $E_k, k = 1, 2, \ldots,$ be a sequence of compact sets on the diameter $(-1, 1),$ each of which consists of a finite number of closed intervals such that the hyperbolic length of each of these intervals is $\geq \delta.$ If the sequence $E_k$ converges in the Hausdorff metric to a compact set $E \subset \mathbb{D},$ then

$$
\cap(E_k) \to \cap(E) \quad \text{as } k \to \infty.
$$
3. Hedgehogs with geometric restrictions on the number of spikes

We start with the following monotonicity result, which, in particular, answers the question raised in Problem 1.11.

**Lemma 3.1.** Suppose that \(-1 < a < 1\) and \(\tau > 0\) are fixed and \(b\) varies in the interval \([a, 1)\).
Let \(c = c(b)\), \(b < c < 1\), be such that \(\ell_D([b, c]) = \tau\). Let \(E_0 \subset (-1, a]\) be a compact set consisting of a finite number of non-degenerate intervals and let \(E(b) = E_0 \cup [b, c(b)]\). Then \(\text{cap}(E(b))\) is a continuous function that strictly increases from \(\text{cap}(E_0 \cup [a, c(a)])\) to \(\text{cap}(E_0) + \text{cap}([a, c(a)])\), when \(b\) runs from \(a\) to \(1\).

**Proof.** The continuity property of \(\text{cap}(E(b))\) follows from Proposition 2.37. To prove the monotonicity of \(\text{cap}(E(b))\), we consider \(b_1, b_2\) such that \(a \leq b_1 < b_2 < 1\) and note that \((c(b_1)) < (c(b_2))\). Let \(\gamma\) be a hyperbolic geodesic that is orthogonal to the hyperbolic interval \([b_1, c(b_2)])_h\) at its midpoint. We give an orientation to \(\gamma\) by marking its complementary hyperbolic halfplane \(H_+\) with \(b_1 \in H_+\), see Figure 2, which illustrates the proof of this lemma. Notice that under our assumptions, \(E_0 \subset H_+\) and, since reflections with respect to hyperbolic geodesics preserve hyperbolic lengths, the hyperbolic interval \(I_1 = [b_1, c(b_1)]_h\) coincides with the reflection of \(I_2 = [b_2, c(b_2)])_h\) with respect to \(\gamma\). Therefore, the polarization \(P_\gamma(E(b_2))\) of \(E(b_2)\) with respect to \(\gamma\) coincides with the set \(E(b_1)\) if \(b_1 \notin E_0\) and with the set \(E(b_1) \cup \{c(b_2)\}\) otherwise, and the set \(P_\gamma(E(b_2)) \setminus E(b_2) = (c(b_1), c(b_2))_h\) is a non-degenerate interval and thus it has positive logarithmic capacity. Furthermore, since \(E_0 \subset H_+\), it follows that the set \(P_\gamma(E(b_2))\) differs from the reflection of \(E(b_2)\) with respect to \(\gamma\) by a set of positive logarithmic capacity. So, applying Proposition 2.12, we conclude that \(\text{cap}(E(b_1)) < \text{cap}(E(b_2))\). Thus we proved that the function \(\text{cap}(E(b))\) is strictly increasing. The assertion about the range of this function follows from the dispersion property of Proposition 2.22 and from the convergence property stated in Proposition 2.37.

**Remark 3.2.** The proof of Lemma 3.1 remains valid if \(E_0\) is any compact set in the hyperbolic halfplane \(H_+\) defined as in the proof above for the hyperbolic geodesic \(\gamma\) passing through the point \(a\) such that the set \(E(b) = E_0 \cup [b, c(b)]_h\) satisfies the assumptions of Proposition 2.37.

**Remark 3.3.** The non-strict monotonicity property in Lemma 3.1 also follows from the contraction principle of Proposition 2.19.

Next, we will use Lemma 3.1 to prove a lower bound for the conformal capacity of compact sets lying on the diameter of \(D\).

**Lemma 3.4.** Let \(\tau > 0\) and let \(r(\tau)\) be defined as in (1.3). If \(E \subset (-1, 1)\) is a compact set such that \(\ell_D(E) = \tau\), then

\[
\text{cap}([0, r(\tau)]) \leq \text{cap}(E).
\]

Equality occurs here if and only if \(E\) coincides with some interval \([a, b] \subset (-1, 1)\) up to a set of zero logarithmic capacity.

**Proof.** (a) Suppose first that \(E\) consists of \(n \geq 2\) intervals \([a_k, b_k]\), \(0 = a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < 1\). Let \(E_1\) be the compact set obtained from \(E\) by replacing the
Figure 2. Hedgehog with one moving interval.

pair of intervals $[a_{n-1}, b_{n-1}]$ and $[a_n, b_n]$ with a single varying interval $[a_{n-1}, b_{n-1}']$ such that $\ell_D([a_{n-1}, b_{n-1}']) = \ell_D([a_{n-1}, b_{n-1}]) + \ell_D([a_n, b_n])$. It follows from the monotonicity property of Lemma 3.1 that $\text{cap}(E_1) < \text{cap}(E)$. Applying this procedure of merging two intervals into a single interval $n-1$ times, we obtain the inequality (3.5) with the sign of strict inequality.

(b) If $E$ is a more general compact set, not the union of a finite number of intervals, we proceed as follows. Since the subset of isolated points of $E$ has zero logarithmic capacity we can remove it without changing the conformal capacity and the hyperbolic length of $E$. Thus, we assume that $E$ does not have isolated points. Also, since both the conformal capacity and hyperbolic length are invariant under conformal automorphisms of $\mathbb{D}$, we may assume that $\min \{ \Re z : z \in E \} = 0$.

We will use the following approximation argument. The set $(-1, 1) \setminus E$ is an open subset of $(-1, 1)$ and therefore, in the case under consideration, it is a countably infinite union of open disjoint intervals $I_k$, $k = 1, 2, \ldots$. We enumerate these intervals such that $I_1 = (-1, 0)$ and $I_2$ has one of its endpoints at 1. Setting $E_n = (-1, 1) \setminus \cup_{k=1}^{n+1} I_k$, $n = 1, 2, \ldots$, we obtain a sequence of compact sets $E_n$, each consisting of a finite number of disjoint closed intervals. Therefore, $\lim_{n \to \infty} \ell_D(E_n) \to \ell_D(E)$ and, by Proposition 2.36, $\lim_{n \to \infty} \text{cap}(E_n) = \text{cap}(E)$.

Let $F_n = [0, a_n]$ be a closed interval on $[0, 1)$ such that $\ell_D(F_n) = \ell_D(E_n)$. Then, by part (a) of this proof,

(3.6) $\text{cap}(F_n) < \text{cap}(E_n)$.

Furthermore, $F_{n+1} \subset F_n$ for all $n$ and $\bigcap_{n=1}^{\infty} F_n = [0, r(\tau)]$. Thus, $\text{cap}(F_n) \to \text{cap}([0, r(\tau)])$, by Proposition 2.36. Therefore, passing to the limit in (3.6), we obtain (3.5).

(c) Here we prove the equality statement. If $E$ coincides with some interval $[a, b]$ up to a set of zero logarithmic capacity then, by Proposition 2.3, $\ell_D([a, b]) = \ell_D(E) = \tau$ and $\text{cap}([a, b]) =$
\( \text{cap} (E) \). This together with the conformal invariance property of the capacity implies that 
\( \text{cap} (E) = \text{cap} ([0, r(\tau)]) \).

Suppose now that \( (3.5) \) holds with the sign of equality. Let \( a \) and \( b \) denote the infimum and supremum of the set of Lebesgue density points of \( E \). We may assume, without loss of generality, that \( a = 0 \).

If \( r(\tau) < b \), then the set \( [0, r(\tau)] \setminus E \) contains an open interval \( (c_1, c_2) \) with \( 0 < c_1 < c_2 < r(\tau) \). Let \( \mathcal{P}_\gamma (E) \) denote the polarization of \( E \) with respect to a hyperbolic geodesic \( \gamma \) that is orthogonal to the hyperbolic interval \([(c_1 + c_2)/2, b]_h \) at its midpoint and oriented such that \( 0 \in H_+ \). Notice, that under our assumptions, the set \( \mathcal{P}_\gamma (E) \) differs from \( E \) by a set of positive one-dimensional Lebesgue measure, and also it differs from the reflection of \( E \) with respect to \( \gamma \) by a set of positive one-dimensional Lebesgue measure, and therefore by a set of positive logarithmic capacity. Hence, by Proposition 2.12,
\begin{equation}
(3.7) \quad \text{cap} (\mathcal{P}_\gamma (E)) < \text{cap} (E).
\end{equation}
Since the polarization with respect to hyperbolic geodesics preserves hyperbolic length, we have \( l_\mathbb{D} (\mathcal{P}_\gamma (E)) = \tau \). Therefore, it follows from the assumption \( \text{cap} (E) = \text{cap} ([0, r(\tau)]) \) and our proof in parts (a) and (b) above, that \( \text{cap} (E) \leq \text{cap} (\mathcal{P}_\gamma (E)) \), which contradicts equation \( (3.7) \). Since the assumption that \( r(\tau) < b \) leads to a contradiction, we must have \( b = r(\tau) \).

In the latter case, \( [0, r(\tau)] \subset E \) and, since \( \text{cap} (E) = \text{cap} ([0, r(\tau)]) \), it follows from Proposition 2.3 that \( E \setminus [0, r(\tau)] \) has zero logarithmic capacity, which completes the proof of the lemma. \( \square \)

Actually, our proof of Lemma 3.4 gives us a more general result, which we state as the following corollary.

**Corollary 3.8.** For \( \tau > 0 \) and \(-1 < a < 1 \), let \( \rho (a, \tau) \in (a, 1) \) be such that \( \ell_\mathbb{D} ([a, \rho (a, \tau)]) = \tau \). Let \( \gamma \) be a hyperbolic geodesic passing through the point \( a \) orthogonally to the diameter \((-1, 1)\) and oriented such that \(-1 \in \partial H_+ \).

If \( E = E_0 \cup E_1 \) is a compact subset in \( \mathbb{D} \) such that \( E_0 \) is a compact subset of \( H_+ \cup \gamma \) and \( E_1 \subset [a, 1] \) is a compact set such that \( \ell_\mathbb{D} (E_1) = \tau \), then
\begin{equation}
(3.9) \quad \text{cap} (E_0 \cup [a, \rho (a, \tau)]) \leq \text{cap} (E).
\end{equation}

If \( E_0 \) has positive logarithmic capacity, then equality occurs in \( (3.9) \) if and only if \( E_1 \) coincides with the interval \([a, \rho (a, \tau)]\) up to a set of zero logarithmic capacity. Otherwise, equality occurs in \( (3.9) \) if and only if \( E_1 \) coincides with some interval \([b, c] \subset [a, 1] \) such that \( \ell_\mathbb{D} ([b, c]) = \tau \) up to a set of zero logarithmic capacity.

**Proof.** Since the considered characteristics of compact sets are invariant under Möbius transformations, we may assume once more that \( a = 0 \). Notice that the polarization transformations used in the proof of Lemma 3.4 do not change the portion \( E_0 \) of the set \( E \), which lies in the halfspace \( H_+ = \{ z \in \mathbb{D} : \text{Re} \, z \leq 0 \} \). Therefore, our arguments used in the proof of Lemma 3.4 prove this corollary as well. \( \square \)

As concerns upper bounds for the conformal capacity of compact sets \( E \subset (-1, 1) \) having fixed hyperbolic length, it is expected that there are no non-trivial upper bounds in this case. Below we present two examples which confirm these expectations.
Example 3.10. It was shown by M. Tsuji [45] that the standard Cantor set \( K \) has positive logarithmic capacity; more precisely, \( \log \operatorname{cap}(K) \geq 1/9 \), see [35, p. 143]. Hence, by Proposition 2.3, the conformal capacity \( \kappa = \operatorname{cap}(K_{1/3}) \) of the scaled Cantor set \( K_{1/3} = \{ z : 3z \in K \} \) is positive.

For \( n \in \mathbb{N} \), let \( K_{1/3}^n = \varphi_n(K_{1/3}) \) denote the image of the scaled Cantor set \( K_{1/3} \) under the Möbius mapping \( \varphi_n(z) = (z + r_n)/(1 + r_n z) \), where \( r_n = 1 - \frac{1}{n!} \). Since the hyperbolic length and conformal capacity are invariant under Möbius automorphisms of \( \mathbb{D} \), we have \( \ell_D(K_{1/3}^n) = \ell_D(K_{1/3}) = 0 \) and \( \operatorname{cap}(K_{1/3}^n) = \operatorname{cap}(K_{1/3}) = \kappa \) for all \( n \in \mathbb{N} \). A simple calculation shows that

\[
p_D(r_{n+1}, r_n) = \frac{n}{n + 2 - \frac{1}{n!}}, \quad n \in \mathbb{N}.
\]

This implies that the sequence of pseudo-hyperbolic distances \( p_D(r_{n+1}, r_n) \) strictly increases and \( p_D(r_{n+1}, r_n) \to 1 \) as \( n \to \infty \). Therefore, the sequence of hyperbolic distances \( d_D(r_{n+1}, r_n) \) also strictly increases and \( d_D(r_{n+1}, r_n) \to \infty \) as \( n \to \infty \). This implies that \( K_{1/3}^k \) and \( K_{1/3}^l \) are disjoint if \( k \neq l \) and

\[
d_D(K_{1/3}^{n+1}, K_{1/3}^n) \to \infty \quad \text{as} \quad n \to \infty.
\]

Given any \( C > 0 \), we fix \( j \in \mathbb{N} \) such that \( j \kappa > C \). Then, for any \( m \in \mathbb{N} \), we consider the compact sets \( K_{1/3}^{m,j} = \bigcup_{s=m}^{m+j-1} K_{1/3}^s \). Using (3.11) and arguing as in the proof of the limit relation (2.23) of Proposition 2.22, we conclude that \( \operatorname{cap}(K_{1/3}^{m,j}) \to j \kappa \) as \( m \to \infty \). The latter shows that for every constant \( C > 0 \) there are compact sets \( E \subset (-1, 1) \) such that \( \ell_D(E) = 0 \) and \( \operatorname{cap}(E) \geq C \).

In our previous example, the hyperbolic diameters of sets \( K_{1/3}^{m,j} \) tend to \( \infty \) as \( m \to \infty \). For compact sets with hyperbolic diameters bounded by some constant, say for compact sets on the interval \( [a, b] \subset (-1, 1) \) such that \( \ell_D(E) < \ell_D([a, b]) \), we have the strict inequality \( \operatorname{cap}(E) < \operatorname{cap}([a, b]) \), which follows from the fact that \( [a, b] \setminus E \) contains a non-empty open interval that is a set of positive logarithmic capacity. In our next example, we show that for every \( \varepsilon > 0 \) and \( a, b \) and \( \tau \) are such that \(-1 < a < b < 1 \), \( 0 < \tau < \ell_D([a, b]) \), there is a compact set \( E \subset [a, b] \) such that \( \ell_D(E) = \tau \) and \( \operatorname{cap}(E) > \operatorname{cap}([a, b]) - \varepsilon \).

Example 3.12. First, we consider a condenser \( (A(\rho^{-1}, \rho), K_n(l)) \) with the domain \( A(\rho^{-1}, \rho) \), where \( A(\rho_1, \rho_2) = \{ z : \rho_1 < |z| < \rho_2 \}, \quad 0 < \rho_1 < \rho_2 \), and a compact set \( K_n(l) = \bigcup_{j=1}^{n} K_{n,j}(l) \), where \( K_{n,j}(l) = \{ e^{i \theta} : |\theta - 2\pi(j - 1)/n| \leq l/n \}, \quad j = 1, \ldots, n \), \( 0 < l < \pi \).

Let \( \Gamma = \Gamma(\rho, n, l) \) denote the family of curves in \( A(\rho^{-1}, \rho) \) joining the boundary circles of \( A(\rho^{-1}, \rho) \) with the set \( K_n(l) \) and let \( \Gamma_0 = \Gamma_0(\rho, n, l) \) denote the family of curves in the annular sector \( S_n(\rho) = \{ z : 1 < |z| < \rho, \quad |\arg z| < \pi/n \} \) joining the arc \( \{ z = re^{i \theta} : |\theta| \leq \pi/n \} \) with the set \( K_{n,1}(l) \). It follows from Ziemer’s relation between the capacity of a condenser and the modulus of an appropriate family of curves (see [50, Theorem 3.8]) and from symmetry properties of the modulus of family of curves (see [1, Theorem 4] for an equivalent form of this symmetry property given in terms of the extremal length) that

\[
\operatorname{cap}(A(\rho^{-1}, \rho), K_n(l))) = M(\Gamma) = 2nM(\Gamma_0).
\]
To find \( M(\Gamma_0) \), we consider a function \( \varphi(z) = \varphi_2 \circ \varphi_1(z) \) with
\[
\varphi_1(z) = i(nK(k)/\pi) \log z \quad \text{and} \quad \varphi_2(\zeta) = \text{sn}(\zeta,k),
\]
where the parameter \( k, 0 < k < 1 \), of the elliptic sine function is defined by the equation
\[
\frac{K'(k)}{K(k)} = \frac{2n \log \rho}{\pi}.
\]
It is a well-known property of elliptic integrals [2, Theorem 5.13(1)] that if \( K'(k)/K(k) \to \infty \), then \( k \to 0 \) and the following expansion holds:
\[
\frac{K'(k)}{K(k)} = \frac{2}{\pi} \log \frac{4}{k} + o(1).
\]
From (3.15) and (3.16), we obtain that
\[
\log \frac{1}{k} = n \log \rho - \log 4 + o(1),
\]
where \( o(1) \to 0 \) when \( n \to \infty \).

The function \( w = \varphi(z) \) maps \( S_n(\rho) \) conformally onto the semidisk \( \mathbb{D}^+_R = \{ w \in \mathbb{D}_R : \text{Im } w > 0 \} \) with \( R = 1/\sqrt{k} \). Furthermore, this function maps the arc \( K_{n,1}(l) \) onto an interval \([-c_n(l),c_n(l)]\) with \( 0 < c_n(l) < 1 \). To find \( c_n(l) \), we note that \( K(k) \to \pi/2 \) as \( k \to 0 \) and that \( \text{sn}(\zeta,k) \) converges to \( \sin \zeta \) uniformly on compact subsets of \( \mathbb{C} \) as \( k \to 0 \). Using these relations and equations (3.14), we find that \( c_n(l) = \sin(l/2) + o(1) \) and, therefore, we have the following asymptotic formula for the logarithmic capacity of the interval \([-c_n(l),c_n(l)]\):
\[
\log \text{cap}([-c_n(l),c_n(l)]) = \frac{1}{2} \sin(l/2) + o(1),
\]
where \( o(1) \to 0 \) as \( n \to \infty \).

Let \( \Gamma_1 = \Gamma_1(\rho,n,l) \) denote the family of curves in \( \mathbb{D}_R \) joining the circle \( \mathbb{T}_R \) with the interval \([-c_n(l),c_n(l)]\). Conformal invariance and symmetry properties of the modulus of family of curves imply that
\[
M(\Gamma_0) = \frac{1}{2} M(\Gamma_1).
\]
We have the following limit relation between the modulus of \( \Gamma_1 \) and the logarithmic capacity of \([-c_n(l),c_n(l)]\):
\[
- \frac{1}{2\pi} \log(\log \text{cap}([-c_n(l),c_n(l)])) = (M(\Gamma_1))^{-1} - \frac{1}{2\pi} \log R + o(1),
\]
where \( o(1) \to 0 \) as \( R \to \infty \).

Using relations (3.18) and (3.20) with \( R = 1/\sqrt{k} \) and with \( k \) as in the equation (3.17), we find that
\[
M(\Gamma_1) = \left( -\frac{1}{2\pi} \log \frac{\sin(l/2)}{2} + \frac{1}{4\pi} \left( n \log \rho - \log 4 \right) + o(1) \right)^{-1}.
\]
Finally, combining equations (3.13), (3.19) and (3.21), we obtain the following asymptotic formula for the capacity of the condenser \((A(\rho^{-1}, \rho), K_n(l))\):

\[
\text{cap}(A(\rho^{-1}, \rho), K_n(l)) = \frac{4\pi}{\log \rho} + o(1),
\]

where \(o(1) \to 0\) when \(\rho\) and \(l\) are fixed and \(n \to \infty\).

Let \(s = s(\rho), 0 < s < 1\), be such that

\[
\text{cap}([-s, s]) = \text{cap}(\mathbb{D}_\rho, \mathbb{D}) = \frac{2\pi}{\log \rho}. \tag{3.23}
\]

Then there is a unique function \(\psi(z)\) mapping \(A(1, \rho)\) conformally onto \(\mathbb{D} \setminus [-s, s]\) such that \(\psi(\rho) = 1\). This function can be extended to a function continuous on \(\overline{A(1, \rho)}\) and such that \(\psi(z) = \overline{\psi(z)}\) for all \(z \in \overline{A(1, \rho)}\). Thus, \(\psi(z)\) maps \(K_n(l)\) onto a compact set \(\psi(K_n(l)) \subset [-s, s]\).

The same conformal invariance and symmetry properties, which we used earlier in this example, together with equations (3.22) and (3.23) imply that

\[
\text{cap}(\psi(K_n(l))) = \frac{1}{2} \text{cap}(A(\rho^{-1}, \rho), K_n(l)) = \frac{2\pi}{\log \rho} + o(1).
\]

Notice that there is a constant \(C > 0\) such that \(|\psi'(e^{i\theta})| \leq C\) for all \(\theta \in \mathbb{R}\). This implies that for every \(\tau > 0\) there is \(l_0, 0 < l_0 < 2\pi\), such that \(\ell_{\mathbb{D}}(\psi(K_n(l))) \leq \tau\) for every \(l, 0 < l \leq l_0\), and all \(n \geq 2\).

We fix \(l, 0 < l \leq l_0\), and \(n \geq 2\) and consider a compact set \(E \subset [-s, s]\) such that \(\psi(K_n(l)) \subset E\) and \(\ell_{\mathbb{D}}(E) = \tau\). Then, if \(n\) is large enough, we have

\[
\text{cap}([-s, s]) > \text{cap}(E) \geq \text{cap}(\psi(K_n(l))) = \frac{2\pi}{\log \rho} + o(1).
\]

The latter equation shows that for every \(\tau, 0 < \tau < \ell_{\mathbb{D}}([-s, s])\), and every \(\varepsilon > 0\), there is a compact set \(E \subset [-s, s]\) such that \(\ell_{\mathbb{D}}(E) = \tau\) and \(\text{cap}(E) > \text{cap}([-s, s]) - \varepsilon\). Since the hyperbolic length and conformal capacity are invariant under Möbius automorphisms of \(\mathbb{D}\), compact sets with similar properties exist for every interval \([a, b], -1 < a < b < 1\).

**Remark 3.24.** Our construction of a compact set in Example 3.12 is similar to the construction used in [38] to provide a counterexample for P.M. Tamrazov’s conjecture on the capacity of a condenser with plates of prescribed transfinite diameters. In turn, a counterexample used in [38] is based on the following result, that is example 5) in [28, Ch. II, §4]:

Let \(K^+_n(l) \subset [-1, 1]\) denote the orthogonal projection of the set \(K_n(l)\) introduced earlier onto the real axis. Then

\[
\log \text{cap}(K^+_n(l)) = \frac{1}{2} (\sin(l/2))^{2/n}.
\]

Taking the limit in this equation as \(n \to \infty\), we conclude that for every \(0 < s < 2\) and every \(\varepsilon > 0\), there exists a compact set \(E \subset [-1, 1]\) with Euclidean length \(s\) such that

\[
\log \text{cap}(E) > \log \text{cap}([-1, 1]) - \varepsilon = \frac{1}{2} - \varepsilon.
\]

In particular, this answers a question raised in Problem 2 in [10] by showing that the supremum of the transfinite diameters of compact sets \(E \subset [-1, 1]\) with Euclidean length \(s, 0 < s < 2\), is equal to \(1/2\).
As Examples 3.10 and 3.12 show, there are no upper bounds for the capacity expressed in terms of the hyperbolic length of a compact set $\mathcal{E}$, in general. In a particular case, when $\mathcal{E}$ is connected, a non-trivial upper bound exists and is given in the following lemma.

**Lemma 3.25.** Let $L \subset \mathbb{D}$ be a Jordan arc having the hyperbolic length $\tau > 0$. Then

$$cap(L) \leq cap([0, r(\tau)]) = \frac{K((e^\tau - 1)/(e^\tau + 1))}{K'(((e^\tau - 1)/(e^\tau + 1)))}. \tag{3.26}$$

**Proof.** The proof repeats the well-known proof for the logarithmic capacity, see, for example, [35, Theorem 5.3.2]. We consider a parametrization $T : [0, r(\tau)] \to L$ of $L$ by the hyperbolic arc-length. Then $T$ is contractive in the hyperbolic metric. Therefore, (3.26) follows from Proposition 2.19. \hfill \Box

The polarization technique used in the proof of Lemma 3.1 can be applied in a more general situation as we demonstrate in our next theorem.

**Theorem 3.27.** Consider $1 \leq n \leq 4$ distinct radial intervals $I_k = I(\alpha_k)$, $k = 1, \ldots, n$, of the unit disk $\mathbb{D}$. Suppose that $E_k \subset I_k$, $k = 1, \ldots, n$, is a compact set on $I_k$ such that $\ell_\mathbb{D}(E_k) = l_k$ and that $E_k^* \subset I_k$ is a hyperbolic interval having one end point at $z = 0$ such that $\ell_\mathbb{D}(E_k^*) = l_k$.

If each of the angles formed by the radial intervals $I_k$ and $I_j$, $k \neq j$, is greater than or equal to $\pi/2$, then

$$cap \left( \bigcup_{k=1}^n E_k \right) \geq cap \left( \bigcup_{k=1}^n E_k^* \right). \tag{3.28}$$

Equality occurs in (3.28) if and only if for each $k$, $E_k$ coincides with $E_k^*$ up to a set of zero logarithmic capacity.

**Proof.** The proof is the same for all $n$. Thus, we assume that $n = 4$. Rotating, if necessary, we may assume that $I_1 = [0, 1)$. The diameter $(-i, i)$ is a hyperbolic geodesic, which we orient such that $1/2 \in H_+$. If $n = 4$, then the angles between the neighboring intervals are equal to $\pi/2$ and therefore the set $K = \bigcup_{k=1}^4 E_k$ can be represented as the union $K = E_0 \cup E_1$ with $E_0 = \bigcup_{k=2}^4 E_k \subset \overline{H_+}$. This shows that the sets $E_0$, $E_1$ and $K_1 = E_1^* \cup E_2 \cup E_3 \cup E_4$ satisfy the assumptions of Corollary 3.8. Therefore, by this corollary,

$$cap(K_1) = cap(E_0 \cup E_1^*) \leq cap(E_0 \cup E_1) = cap(K)$$

with the sign of equality if and only if $E_1$ coincides with $E_1^*$ up to a set of zero logarithmic capacity.

The same argument can be applied successively to the sets $K_1$, $K_2 = E_1^* \cup E_2^* \cup E_3 \cup E_4$, $K_3 = E_1^* \cup E_2^* \cup E_3^* \cup E_4$ and $K_4 = E_1^* \cup E_2^* \cup E_3^* \cup E_4^*$ to obtain the inequalities

$$cap \left( \bigcup_{k=1}^4 E_k^* \right) = cap(K_4) \leq cap(K_3) \leq cap(K_2) \leq cap(K_1) \leq cap(K) = cap \left( \bigcup_{k=1}^4 E_k \right).$$

Moreover, equality occurs in any one of these inequalities if and only if the corresponding sets $E_k$ and $E_k^*$ coincide up to a set of zero logarithmic capacity. Thus, the theorem is proved. \hfill \Box

**Remark 3.29.** We want to stress once more that the inequality (3.28) also follows from the contraction principle of Proposition 2.19. Indeed, under the assumptions that all angles between radial intervals are $\geq \pi/2$, there is always a contraction $\varphi : \bigcup_{k=1}^n E_k \to \bigcup_{k=1}^n E_k^*$. 
Figure 3. Hyperbolic capacity of two intervals.

If the angle between some radial intervals $I_k$ and $I_j$ is smaller than $\pi/2$, then both our proofs, with polarization or with the contraction principle, fail even in the simplest case of two intervals $I_1$ and $I_2$ and when each of the sets $E_1 \subset I_1$ and $E_2 \subset I_2$ is a hyperbolic interval. However, the graphs of the results of numerical experiments performed by Dr. Mohamed Nasser, which are presented in Figure 3, suggest that the monotonicity property of the conformal capacity of two intervals remains in place for all angles. Therefore, we suggest the following.

**Problem 3.30.** Given fixed $0 < r < 1$, $0 < s < \infty$, and $0 < \alpha < \pi/2$ and varying $0 \leq t < 1$, let $E(t) = [0, r] \cup \{\tau e^{i\alpha} : t \leq \tau \leq d(s, t)\}$ with $d(s, t)$ such that $\ell_D([t, d(s, t)]) = s$. Prove (or disprove) that $\text{cap}(E(t))$ strictly increases on the interval $0 \leq t < 1$.

As one can see from our proof of Theorem 3.27, the restriction $n \leq 4$ on the number of radial intervals and restriction on angles between them is needed because otherwise polarization with respect to hyperbolic geodesics may destroy the radial structure of compact sets under consideration. Also, if at least one angle between radial intervals is $< \pi/2$, then the contraction principle of Proposition 2.19 can not be applied, in general. Still, under some additional assumptions on the hyperbolic lengths and angles, we have the following more general version.

**Theorem 3.31.** Let $E_k$, $k = 1, \ldots, n$, be compact sets on the radial intervals $I_k = [0, e^{i\beta_k})$, $0 = \beta_1 < \beta_2 < \ldots < \beta_n < \beta_{n+1} = 2\pi$, such that

$$\ell_D(E_k) \geq 2 \log \cot \frac{\alpha}{2}, \quad k = 1, \ldots, n,$$

where $\alpha$ stands for the minimal angle between the intervals $I_k$. Then

$$\text{cap}\left(\bigcup_{k=1}^{n} E_k\right) \geq \text{cap}\left(\bigcup_{k=1}^{n} E^*_k\right),$$

where $E^*_k \subset I_k$ is a hyperbolic interval having one end point at $z = 0$ such that $\ell_D(E^*_k) = \ell_D(E_k), k = 1, \ldots, n$.

Equality occurs here if and only if for each $k$, $E_k$ coincides with $E^*_k$ up to a set of zero logarithmic capacity.
Proof. (a) Let \( \tau_\alpha = \log \cot(\alpha/2), \ r_\alpha = r(\tau_\alpha) = \sec \alpha - \tan \alpha \) and let \( \gamma_\alpha \) be the hyperbolic geodesic orthogonal to the interval \((-1,1)\) at \( z = r_\alpha \) and oriented such that 0 belongs to the halfplane \( H_+ = H_+(\gamma_\alpha) \). An easy calculation shows that \( \gamma_\alpha \) has its endpoints at the points \( e^{\pm \alpha} \). Consider sets \( E^+_\alpha = E_1 \cap H_+, \ E^-_\alpha = E_1 \setminus H_+ \) and \( E_0 = E^+_\alpha \cup (\cup_{k=2}^n E_k) \). Let \( \tilde{E}^-_\alpha \) denote the closed hyperbolic interval with the initial point at \( z = r(\alpha) \) such that \( \tilde{E}^-_\alpha \subset [r_\alpha, 1) \) and \( \ell_\alpha(\tilde{E}^-_\alpha) = \ell_\alpha(E^-_\alpha) \) and let \( \tilde{E}^- = E^+_\alpha \cup \tilde{E}^-_\alpha \). Since the minimal angle between the intervals \( \ell_k \) is \( \alpha \) and \( \gamma_\alpha \) has its endpoints at \( e^{\pm \alpha} \), it follows that \( E_0 \subset H_+ \). Therefore, we can apply Corollary 3.8 to obtain the following:

\[
(3.33) \quad \cap(E_0 \cup \tilde{E}^-_\alpha) = \cap(\tilde{E}_1 \cup (\cup_{k=2}^n E_k)) \leq \cap(E)
\]

with the sign of equality if and only if \( E^-_\alpha \) coincides with \( \tilde{E}^-_\alpha \) up to a set of zero logarithmic capacity.

If \( E^+_\alpha = [0, r_\alpha] \), then \( E^+_\alpha = E^+_\alpha \cup \tilde{E}^-_\alpha \) and the inequality in (3.33) is equivalent to the inequality

\[
(3.34) \quad \cap(E \cup (\cup_{k=2}^n E_k)) \leq \cap(E)
\]

with the sign of equality if and only if \( E^+_\alpha \) coincides with \( E \) up to a set of zero logarithmic capacity.

(b) If \( E^+_\alpha \neq [0, r_\alpha] \), then \( [0, r_\alpha] \setminus E^+_\alpha \) contains an open interval. In this case \( \ell_\alpha(\tilde{E}^-_\alpha) > \ell_\alpha([0, r_\alpha]) \). Let \( b_1, r_\alpha < b_1 < 1 \), denote the end point of the interval \( \tilde{E}^-_\alpha \) and let \( c_1 \) denote the midpoint of the hyperbolic interval \([0, b_1]_h\). Notice that under our assumptions \( r_\alpha < c_1 \). Let \( \gamma_1 \) be the hyperbolic geodesic orthogonal to the diameter \((-1,1)\) at \( z = c_1 \) and oriented such that 0 belongs to \( H_+ (\gamma_1) \). Let \( \tilde{E}_1 = \mathcal{P}_{\gamma_1}(\tilde{E}_1) \) and \( \tilde{E} = \mathcal{P}_{\gamma_1}(\tilde{E}_1 \cup (\cup_{k=2}^n E_k)) \) denote polarizations of the corresponding sets with respect to \( \gamma_1 \). Since \( r_\alpha < c_1 \) and therefore \( \cup_{k=2}^n E_k \subset H_+(\gamma_1) \), we have \( \tilde{E}_1 \supset [0, r_\alpha] \) and \( \tilde{E} = \tilde{E}_1 \cup (\cup_{k=1}^n E_k) \). Applying Proposition 2.12 and using (3.33), we conclude that

\[
(3.35) \quad \cap(\tilde{E}_1 \cup (\cup_{k=2}^n E_k)) < \cap(\tilde{E}_1 \cup (\cup_{k=2}^n E_k)) \leq \cap(E)
\]

with the sign of strict inequality in the first inequality because \( \tilde{E}_1 \) differs from \( \tilde{E} \) and from its reflection with respect to \( \gamma_1 \) by an open interval and therefore by a set of positive logarithmic capacity.

Since \( \ell_\alpha(\tilde{E}_1) = \ell_\alpha(E_1) \) and \( \tilde{E}_1 \supset [0, r_\alpha] \), using (3.35) and applying the same arguments as in part (a) of this proof to the set \( \tilde{E} = \tilde{E}_1 \cup (\cup_{k=2}^n E_k) \), we conclude that in the case \( E^+_\alpha \neq [0, r_\alpha] \) the inequality (3.34) remains true with the same statement on the equality cases.

(c) Now, when (3.34) is proved in all cases, we can apply the iterative procedure as in the proof of Theorem 3.27 to conclude that the inequality (3.32) holds with the sign of equality if and only if, for each \( k, E_k \) coincides with \( E_k^\alpha \) up to a set of zero logarithmic capacity. \( \square \)

For compact sets \( E_1 \) and \( E_2 \), lying on two orthogonal diameters of \( \mathbb{D} \), we have the following result.

**Theorem 3.36.** Let \( E_1 \subset (-1,1), \ E_2 \subset (-i,i) \) be compact sets and let \( r_k, 0 < r_k < 1, \) and \( l_k > 0 \) be such that

\[
\ell_\mathbb{D}(E_k) = \ell_\mathbb{D}([-r_k, r_k]) = 2l_k, \quad k = 1, 2.
\]
Then

(3.37) \[ \text{cap}(E_1 \cup E_2) \geq \text{cap}([-r_1, r_1] \cup [-ir_2, ir_2]) \geq \text{cap}([-r_0, r_0] \cup [-ir_0, ir_0]), \]

where \(0 < r_0 < 1\) is such that \(\ell_D([-r_0, r_0]) = l_1 + l_2\).

Equality occurs in the first inequality if and only if \(E_1\) coincides with \([-r_1, r_1]\) and \(E_2\) coincides with \([-ir_2, ir_2]\) up to a set of zero logarithmic capacity. Equality occurs in the second inequality if and only if \(l_1 = l_2\).

**Proof.** Let \(0 \leq r_k^+ < 1, k = 1, 2\), be such that the following holds:

\[
\ell_D([0, r_1^+]) = \ell_D(E_1 \cap [0, 1]), \quad \ell_D([0, -r_1^-]) = \ell_D(E_1 \cap [0, -1]), \\
\ell_D([0, r_2^+]) = \ell_D(E_2 \cap [0, i]), \quad \ell_D([0, -r_2^-]) = \ell_D(E_2 \cap [0, -i]).
\]

Then, by Theorem 3.27,

(3.38) \[ \text{cap}(E_1 \cup E_2) \geq \text{cap}([0, r_1^+] \cup [0, -r_1^-] \cup [0, i r_2^+] \cup [0, -i r_2^-]) \]

with the sign of equality if and only if the sets in the left and right sides of this inequality coincide up to a set of zero logarithmic capacity.

Suppose that \(r_1^+ \neq r_1^-\), say \(r_1^+ > r_1^-\). Then \(r_1^+ > r_1 > r_1^-\). Let \(P_\gamma\) denote polarization with respect to the geodesic \(\gamma\) that is orthogonal to the hyperbolic interval \([-r_1, r_1^-]\) at its midpoint \(c\), \(0 < c < r_1\). We assume here that \(\gamma\) is oriented such that \(0 \in H^+_1\). Under our assumptions, \(P_\gamma([-r_1, r_1^+] \cup [-ir_2^-, ir_2^+]) = [-r_1, r_1] \cup [-ir_2^-, ir_2^+]\). Since the set \([-r_1, r_1^+] \setminus [-r_1, r_1]\) has positive logarithmic capacity, it follows from Proposition 2.12 that

(3.39) \[ \text{cap}([-r_1, r_1^+] \cup [-ir_2^-, ir_2^+]) > \text{cap}([-r_1, r_1] \cup [-ir_2^-, ir_2^+]). \]

The same polarization argument can be applied to show that, if \(r_2 \neq r_2^+\), then

(3.40) \[ \text{cap}([-r_1, r_1] \cup [-ir_2^-, ir_2^+]) > \text{cap}([-r_1, r_1] \cup [-ir_2, ir_2]). \]

Combining inequalities (3.38)–(3.40), we obtain the first inequality in (3.37) with the sign of equality if and only if \(E_1\) coincides with \([-r_1, r_1]\) and \(E_2\) coincides with \([-ir_2, ir_2]\) up to a set of zero logarithmic capacity.

To prove the second inequality in (3.37), we use the conformal mapping

\[
g(z) = \sqrt{(z^2 + r_2^2)/(1 + r_2^2 z^2)}
\]

from the doubly connected domain \(\mathbb{D} \setminus ([-r_1, r_1] \cup [-ir_2, ir_2])\) onto \(\mathbb{D} \setminus \{r\}\) with

\[
r = \sqrt{(r_1^2 + r_2^2)/(1 + r_1^2 r_2^2)}. 
\]

We note that conformal mappings preserve capacity of condensers and that the function \(\text{cap}(\{r\}) = \text{cap}([-r_1, r_1] \cup [-ir_2, ir_2])\) is strictly increasing on the interval \(0 \leq r < 1\). Furthermore, it follows from formulas (1.3) that the sum \(l_1 + l_2\) of the hyperbolic lengths defined in the theorem is constant if and only if the following product is constant:

(3.41) \[ \frac{1 + r_1}{1 - r_1} \cdot \frac{1 + r_2}{1 - r_2} = C, \]

where \(C\) is constant.
Our goal now is to minimize the function \( F = F(r_1, r_2) \), defined by
\[
F = \frac{r_1^2 + r_2^2}{1 + r_1^2 r_2^2},
\]
under the constraint (3.41).

Introducing new variables \( u = (1 + r_1)/(1 - r_1) \), \( v = (1 + r_2)/(1 - r_2) \) and \( w = u^2 + v^2 \), we can express \( F \) in terms of these variables as follows:
\[
F = \frac{(u^2 + v^2) + (1 - 4C + C^2)}{(u^2 + v^2) + (1 + 4C + C^2)} = \frac{w + (1 - 4C + C^2)}{w + (1 + 4C + C^2)},
\]
which we have to minimize \( F \) under the constraint \( uw = C \). Differentiating, we find that
\[
\frac{d}{dw} F = \frac{4C}{(w + (1 + 4C + C^2))^2} > 0.
\]

Therefore, \( F \) takes its minimal value when \( w = u^2 + y^2 \) is as small as possible. By the classical arithmetic-geometric mean inequality \( u^2 + v^2 \geq 2w = 2C \), unless \( u = v \). Therefore, the minimal value of \( F \) under the constraint \( uw = C \) occurs when \( u = v \). The latter implies that \( \text{cap}([-r_1, r_1] \cup [-ir_2, ir_2]) \geq \text{cap}([-r_0, r_0] \cup [-ir_0, ir_0]) \) with the sign of equality if and only if \( r_1 = r_2 \). This proves the second inequality in (3.37).

\[\square\]

**Remark 3.42.** The inequality obtained in Theorem 3.36 is stronger, in general, than the inequality obtained by the classical Steiner symmetrization, which will be discussed in Section 5.

To give an example, we consider two sets: \( E_1 = [0, 1/2] \cup [0, i/2] \) and \( E_2 = [0, 1/2] \cup [0, i/4] \). Two Steiner symmetrizations, chosen appropriately, transform these sets to the sets \( E_1^* = [-1/4, 1/4] \cup [-i/4, i/4] \) and \( E_2^* = [-1/4, 1/4] \cup [-i/8, i/8] \), respectively. The first inequality in (3.37) compares the conformal capacities of \( E_1 \) and \( E_2 \) with the conformal capacities of the sets
\[
E_1^* = [-2 - \sqrt{3}, 2 - \sqrt{3}] \cup [-i(2 - \sqrt{3}), i(2 - \sqrt{3})],
\]
\[
E_2^* = [-2 - \sqrt{3}, 2 - \sqrt{3}] \cup [-i(4 - \sqrt{15}), i(4 - \sqrt{15})].
\]
Numerical computation gives the following approximation and bounds for \( \text{cap}(E_k) \), \( k = 1, 2 \),
\[
\text{cap}(E_1^*) \approx 3.62589 < \text{cap}(E_2^*) \approx 3.77702 < \text{cap}(E_1) \approx 4.28254,
\]
\[
\text{cap}(E_2^*) \approx 3.19333 < \text{cap}(E_2^*) \approx 3.29244 < \text{cap}(E_2) \approx 3.60548.
\]

In our previous lemmas and theorems of this section, the extremal configurations were hedge-hogs with spikes issuing from the central point. Similar results for compact sets with spikes emanating from a certain compact central body \( E_0 \) sitting in the disk \( \overline{D}_r \), \( 0 < r < 1 \), as it is shown in Figure 4, also may be useful in applications.

**Theorem 3.43.** Let \( E_0 \) be a compact set in the disk \( \overline{D}_r \), \( 0 < r < 1 \). Let \( E_k, k = 1, \ldots, n \), be compact sets on the radial intervals \( I_k = [r e^{i\beta_k}, e^{i\beta_k}], 0 = \beta_1 < \beta_2 < \ldots < \beta_n < \beta_{n+1} = 2\pi \), such that
\[
\ell_D(E_k) \geq 2 \log \left( \cot \frac{\alpha}{2 \frac{1 - r}{1 + r}} \right), \quad k = 1, \ldots, n,
\]
where \( \alpha \) stands for the minimal angle between the intervals \( I_k \). Then
\[
\text{cap}(\bigcup_{k=0}^n E_k) \geq \text{cap}(\bigcup_{k=0}^n E_k^*),
\]
where $E_0^* = E_0$ and $E_k^* \subset I_k$ is a hyperbolic interval having one end point at $z = re^{i\beta_k}$ such that $\ell_D(E_k^*) = \ell_D(E_k)$, $k = 1, \ldots, n$.

Equality occurs in (3.44) if and only if for each $k = 1, \ldots, n$, $E_k$ coincides with $E_k^*$ up to a set of zero logarithmic capacity.

**Proof.** The proof of this theorem is essentially the same as the proof of Theorem 3.31. The only new thing we need is the following observation. Let $\rho(r) = \rho(\theta) \geq 0$ be an upper semicontinuous function on $E_0^*$. For $0 < r < 1$, let $E(r)$ be a compact set in $D$ such that the intersection $E(r) \cap [0, e^{i\theta}]$ is empty if $e^{i\theta} \notin E_0$ and it is an interval $[re^{i\theta}, se^{i\theta}]$ with $s, r \leq s < 1$, such that $\ell_D([r, s]) = \rho(\theta)$, if $e^{i\theta} \in E_0$.

Most of the results presented above in this section can be proved by two methods, either using the polarization technique or the contraction principle. Now, we give an example of a result, when polarization does not work but the contraction principle is easily applicable.

**Theorem 3.45.** Let $E_0$ be a compact subset of $\mathbb{T} = \partial \mathbb{D}$ and let $\rho(\theta) \geq 0$ be an upper semicontinuous function on $E_0$. For $0 < r < 1$, let $E(r)$ be a compact set in $D$ such that the intersection $E(r) \cap [0, e^{i\theta}]$ is empty if $e^{i\theta} \notin E_0$ and it is an interval $[re^{i\theta}, se^{i\theta}]$ with $r \leq s < 1$, such that $\ell_D([r, s]) = \rho(\theta)$, if $e^{i\theta} \in E_0$. 

**Figure 4.** Hedgehogs with central body $E_0$ and spikes on five intervals.
Then the conformal capacity $\text{cap}(E(r))$ is an increasing function on the interval $0 \leq r < 1$.

**Proof.** Let $r_0$, $0 < r_0 < 1$, be fixed and let $\tilde{A}(r_0) = \{z : r_0 \leq |z| < 1\}$. Consider the function $\varphi_r : \tilde{A}(r_0) \to \mathbb{D}$ defined as follows: if $z = se^{i\theta} \in \tilde{A}(r_0)$, then $\arg \varphi_r(z) = \theta$ and $d_{\mathbb{D}}(re^{i\theta}, \varphi_r(z)) = d_{\mathbb{D}}(r_0e^{i\theta}, z)$. We claim that, if $0 \leq r \leq r_0$, then $\varphi_r$ is a hyperbolic contraction on $\tilde{A}(r_0)$.

To prove this claim, we fix two points $z_1, z_2 \in \tilde{A}(r_0)$ and consider the hyperbolic distance $d_{\mathbb{D}}(\varphi_r(z_1), \varphi_r(z_2))$. Our claim will be proved if we show that this hyperbolic distance or, equivalently, the pseudo-hyperbolic distance

$$
(3.46) 
$$

$$
p_{\mathbb{D}}(\varphi_r(z_1), \varphi_r(z_2)) = \left| \frac{\varphi_r(z_1) - \varphi_r(z_2)}{1 - \varphi_r(z_1)\varphi_r(z_2)} \right|
$$

is an increasing function of $r$ on the interval $0 \leq r \leq r_0$.

Rotating, if necessary, we may assume that

$$
z_1 = a_0e^{i\alpha}, \quad z_2 = b_0e^{-i\alpha}, \quad \text{where} \quad r_0 \leq a_0, b_0 < 1, \quad 0 \leq \alpha \leq \pi/2.
$$

Then

$$
\varphi_r(z_1) = ae^{i\alpha}, \quad \varphi_r(z_2) = be^{-i\alpha},
$$

where $a = a(r)$, $b = b(r)$ are functions of $r$.

Consider the function $F(r) = (p_{\mathbb{D}}(ae^{i\alpha}, be^{-i\alpha}))^2$. After some algebra, we find that

$$
(3.47) 
$$

$$
F(r) = \frac{a^2(r) - 2ar(b(r)\cos \alpha + b^2(r))}{(1 - a^2(b(r))^2)}.
$$

Since for fixed $s$ and $\theta$, $r_0 \leq s < 1$, $\theta \in \mathbb{R}$, the hyperbolic distance between the points $re^{i\theta}$ and $\varphi_r(se^{i\theta})$ is the same for all $r$ in the interval $0 \leq r \leq r_0$, it follows that

$$
(3.48) 
$$

$$
\frac{da}{dr} = \frac{1 - a^2}{1 - r^2} \quad \text{and} \quad \frac{db}{dr} = \frac{1 - b^2}{1 - r^2}.
$$

Differentiating (3.47) and using formulas (3.48), we find:

$$
(3.49) 
$$

$$
\frac{dF}{dr} = \frac{2}{(1 - r^2)(1 - ab)^3} \Phi(a, b, \alpha),
$$

where $\Phi = \Phi(a, b, \alpha)$ is the following function:

$$
\Phi = (a(1 - a^2) - a(1 - b^2) + b(1 - a^2))\cos \alpha + b(1 - b^2)(1 - ab) + (a^2 - 2ab\cos \alpha + b^2)(a(1 - b^2) + b(1 - a^2)).
$$

It is clear that $\Phi(a, b, \alpha)$ is an increasing function of $\alpha$ on the interval $0 \leq \alpha \leq \pi/2$. After simple calculation, we find that $\Phi(a, b, 0) = 0$ and therefore

$$
\Phi(a, b, \alpha) \geq \Phi(a, b, 0) = 0.
$$

This, together with (3.49), implies that $\frac{dF(r)}{dr} \geq 0$ and therefore the pseudo-hyperbolic distance in equation (3.46) decreases, when $r$ decreases from $r_0$ to 0. Therefore, our claim that $\varphi_r$ is a hyperbolic contraction is proved. By Proposition 2.19, the latter implies that $\text{cap}(E(r))$ decreases when $r$ decreases from $r_0$ to 0, which proves the theorem. \(\square\)
4. Extremal properties of hedgehogs on evenly distributed radial intervals

In this section we consider problems with extremal configurations lying on \( n \geq 2 \) radial intervals \( I_k^* = \{ z = te^{2\pi i (k-1)/n} : 0 \leq t \leq 1 \} \), \( k = 1, \ldots, n \). Since the intervals \( I_k \) are evenly distributed in \( \mathbb{D} \) it is expected that extremal configurations possess rotational symmetry by angle \( 2\pi/n \).

First, we prove a theorem that generalizes the second inequality of Theorem 3.36 for sets lying on \( 2n \geq 4 \) diameters.

**Theorem 4.1.** Let \( 0 < r_k < 1, \ k = 1, 2 \) and let \( r, 0 < r < 1 \), be such that

\[
\tau = \ell_{\mathbb{D}}([0, r]) = \frac{1}{2}(\ell_{\mathbb{D}}([0, r_1]) + \ell_{\mathbb{D}}([0, r_2]))
\]

and, for \( n \geq 2 \), let

\[
E_n(r_1, r_2) = \left( \bigcup_{k=0}^{n-1} e^{\pi i k/n} [-r_1, r_1] \right) \cup \left( \bigcup_{k=0}^{n-1} e^{\pi i (2k+1)/2n} [-r_2, r_2] \right).
\]

Then

\[
cap(E_n(r_1, r_2)) = 8n \frac{K(\kappa)}{K'(\kappa)}, \quad \text{where} \quad \kappa = \frac{r_1^{2n} + r_2^{2n}}{1 + r_1^{2n}r_2^{2n}}.
\]

Furthermore, the following inequality holds:

\[
cap(E_n(r_1, r_2)) \geq \cap(\bigcup_{k=0}^{2n-1} e^{\pi i k/2n} [-r, r]).
\]

Equality occurs in (4.4) if and only if \( r_1 = r_2 \).

**Proof.** To establish (4.3), we use the function \( \phi = \phi_2 \circ \phi_1 \), where \( \phi_1(z) = z^{2n} \), \( \phi_2(z) = (z + r_2^{2n})/(1 + r_2^{2n}z) \), which maps the sector \( S = \{ z \in \mathbb{D} : |\arg z| < \pi/2n \} \) conformally onto \( \mathbb{D} \) slit along the interval \([-1, 0]\). Using symmetry properties of \( E_n(r_1, r_2) \), we find that

\[
cap(E_n(r_1, r_2)) = 2n \cap(\{ 0, (r_1^{2n} + r_2^{2n})/(1 + r_1^{2n}r_2^{2n}) \})
\]

This together with (1.10) gives (4.3).

One way to prove the monotonicity property of \( \cap(E_n(r_1, r_2)) \) is to differentiate the function in (4.3) and check if its derivative is positive. Here, we demonstrate a different approach, which may be useful when an explicit expression for the derivative is not known.

The proof presented below is similar to the proof of the second inequality in (3.37) in Theorem 3.36. We use the function

\[
g(z) = \frac{z^{2n} + r_2^{2n}}{1 + r_2^{2n}z^{2n}}
\]

to map the domain \( \{ z \in \mathbb{D} \setminus E_n(r_1, r_2) : |\arg z| < \pi/2n \} \) conformally onto the unit disk slit along the interval \((-1, F_n] \), where \( F_n = F_n(r_1, r_2) \) is defined as

\[
F_n = \frac{r_1^{2n} + r_2^{2n}}{1 + r_1^{2n}r_2^{2n}}.
\]

Since \( E_n(r_1, r_2) \) possesses \( 2n \)-fold rotational symmetry about 0, it follows from the symmetry principle for the module of family of curves, that

\[
cap(E_n(r_1, r_2)) = 2n \cap(\{ 0, F_n(r_1, r_2) \}).
\]
Therefore, to minimize \( \text{cap}(E_n(r_1, r_2)) \) under the constraint (4.2), we can minimize \( F_n \) under the same constraint.

Using the variables \( u = (1 + r_1)/(1 - r_1) \), \( v = (1 + r_2)/(1 - r_2) \), constrained by the condition \( uv = C \), we express \( F_n \) as follows:

\[
F_n = \frac{(C - 1 + (u - v))^{2n} + (C - 1 + (v - u))^{2n}}{(C + 1 + (u + v))^{2n} + (C + 1 - (u + v))^{2n}}.
\]

To minimize \( F_n(u, v) \) under the constraint \( uv = C \), we introduce Lagrange’s function

\[
L = F_n(u, v) + \lambda uv, \quad \lambda \in \mathbb{R}.
\]

Differentiating this function, we find

\[
(4.5) \quad \frac{\partial L}{\partial u} - \lambda v = \frac{\partial L}{\partial v} - \lambda u = M(u, v, C),
\]

where \( M = M(u, v, C) \) is defined as follows:

\[
M = 2n ((C + 1 + (u + v))^{2n} + (C + 1 - (u + v))^{2n})^{-2} (M_1 N_1 - M_2 N_2),
\]

\[
M_1 = (C + 1 + (u + v))^{2n} + (C + 1 - (u + v))^{2n},
\]

\[
M_2 = (C - 1 + (u - v))^{2n} + (C - 1 + (v - u))^{2n},
\]

\[
N_1 = (C - 1 + (u - v))^{2n-1} - (C - 1 + (v - u))^{2n-1},
\]

\[
N_2 = (C + 1 + (u + v))^{2n-1} - (C + 1 - (u + v))^{2n-1}.
\]

It follows from equation (4.5) that if \((u, v)\) is a critical point of the minimization problem under consideration, then \( u = v = \sqrt{C} \). In this case

\[
F_n(\sqrt{C}, \sqrt{C}) = \frac{2(C - 1)^{2n}}{((\sqrt{C} + 1)^{4n} + (\sqrt{C} - 1)^{4n})}.
\]

Since \( u \geq 1 \), \( v \geq 1 \), \( uv = C \) and there is only one critical point \((u, v) = (\sqrt{C}, \sqrt{C})\) of the minimization problem under consideration and since \( F(u, v) = F(v, u) \), it follows that \( F(u, v) \) achieves its minimal value either at the point \((u, v) = (\sqrt{C}, \sqrt{C})\) or at the point \((u, v) = (1, C)\).

In the latter case, we have

\[
F_n(1, C) = \frac{(C - 1)^{2n}}{(C + 1)^{2n}}.
\]

The inequality \( F_n(\sqrt{C}, \sqrt{C}) < F_n(1, C) \) is equivalent to the following inequality:

\[
((\sqrt{C} + 1)^{4n} + (\sqrt{C} - 1)^{4n}) > 2(C + 1)^{2n}.
\]

Using binomial expansion, this inequality can be written as

\[
\sum_{k=0}^{2n} \binom{4n}{2k} C^{2n-k} > \sum_{k=0}^{2n} \binom{2n}{k} C^{2n-k}.
\]

Since \( \binom{2n}{k} > \binom{n}{k} \) for all \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \), the latter inequality holds true.

Thus, \( F_n(u, v) \) takes its minimal value when \( u = v = \sqrt{C} \) and therefore the inequality (4.4) is proved. \( \square \)
Remark 4.6. The binomial inequalities similar to the one used in the proof of Theorem 4.1 are known to the experts. To prove that \( \binom{2n}{2k} > \binom{n}{k} \), we can argue as follows.

Let \( n \geq 2 \) and \( 0 \leq k \leq n - k \). If \( k = 0 \), then \( \binom{2n}{2k} = \binom{n}{0} = 1 \). Suppose that \( \binom{2n}{2k} \geq \binom{n}{k} \) for \( 0 \leq k \leq n - k - 1 \). Then

\[
\binom{2n}{2(k+1)} = \binom{2n}{2k} \frac{(2n-2k)(2n-2k-1)}{(2k+1)(2k+2)} \geq \binom{n}{k} \frac{2(n-k)(2n-2k-1)}{2(k+1)(2k+1)}
\]

\[
= \binom{n}{k+1} \frac{2n-2k-1}{2k+1} > \binom{n}{k+1}.
\]

Now the required inequality follows by induction.

It was shown in the proof of Theorem 4.1 that there is only one critical point in the minimization problem considered in that theorem. Therefore, the following monotonicity result is also proved.

Corollary 4.7. Under the assumptions of Theorem 4.1, suppose that \( r_1 = s \) and \( r_2 = r_2(s) \) is such that condition (4.2) holds. Then \( \text{cap}(E_n(s, r_2(s))) \) strictly decreases from \( 8n^2 \frac{\mathcal{K}(\kappa_0)}{\mathcal{K}(\kappa_1)} \) to \( 16n^2 \frac{\mathcal{K}(\kappa_1)}{\mathcal{K}(\kappa_1)} \), when \( s \) runs from 0 to \( r \), where

\[
\kappa_0 = \left( \frac{e^{2\tau} - 1}{e^{2\tau} + 1} \right)^2, \quad \kappa_1 = \left( \frac{e^\tau - 1}{e^\tau + 1} \right)^4.
\]

Since the conformal capacity is conformally invariant, the conformal capacity of an interval \( E \) on \((-1, 1)\) remains constant when \( E \) moves along the diameter \((-1, 1)\) so that its hyperbolic length is fixed. For \( n \) intervals situated on \( n \) equally distributed radial intervals the latter property is not true any more but, as our next theorem shows, if all these intervals have equal hyperbolic lengths and move synchronically, the conformal capacity of their union changes monotonically. Actually, the non-strict monotonicity is already established in Theorem 3.45. Thus, our intention here is to prove the strict monotonicity result and relate it with certain properties of relevant transcendental functions.

Theorem 4.8. Let \( E_\tau(r) \) be a closed subinterval of \([0, 1)\) with the initial point at \( 0 < r < 1 \) and hyperbolic length \( \tau > 0 \). For \( n \in \mathbb{N} \), let \( E_\tau^n(r) = \bigcup_{k=1}^n e^{2\tau (k-1)/n} E_\tau(r) \). Then the conformal capacity \( \text{cap}(E_\tau^n(r)) \) is given by

\[
\text{cap}(E_\tau^n(r)) = 4n \frac{\mathcal{K}(\kappa)}{\mathcal{K}'(\kappa)},
\]

where

\[
\kappa = \frac{\rho^n - r^n}{1 - r^n \rho^n}, \quad \rho = \frac{e^\tau (1 + r) - (1 - r)}{e^\tau (1 + r) + (1 - r)}.
\]

Furthermore, \( \text{cap}(E_\tau^n(r)) \) strictly increases from \( 4n \frac{\mathcal{K}(\kappa_0)}{\mathcal{K}(\kappa_1)} \) with \( \kappa_0 = (e^\tau - 1)^n / (e^\tau + 1)^n \) to \( 4n \frac{\mathcal{K}(\kappa_1)}{\mathcal{K}(\kappa_1)} \) with \( \kappa_1 = (e^\tau - 1) / (e^\tau + 1) \), when \( r \) varies from 0 to 1.
Proof. As in the proof of Theorem 4.1, we use the function \( \varphi = \varphi_2 \circ \varphi_1 \), where \( \varphi_1(z) = z^n \), \( \varphi_2(z) = (z - r^n)/(1 - r^n z) \). Then \( \varphi \) maps the sector \( S = \{ z \in \mathbb{D} : |\arg z| < \pi/n \} \) conformally onto \( \mathbb{D} \) slit along the interval \([-1, 0]\). Furthermore, \( \varphi \) maps \( E_\tau(r) \) onto the interval \([0, \rho]\) with \( \rho \) defined as in (4.10). Using symmetry properties of \( E_\tau^n(r) \), we find that

\[
\cap([r^n, \rho^n]) = \cap([0, (\rho^n - r^n)/(1 - r^n \rho^n)]).
\]

This equation together with (1.10) gives (4.9).

As we have mentioned in the proof of Theorem 4.1, we know two approaches to prove the monotonicity property of the conformal capacity in that theorem. The same approaches can be used to prove the monotonicity statement of Theorem 4.8. Here, we demonstrate one more approach, which also may be useful when an explicit expression for the derivative is not known. We first note that \( \cap(E_\tau^n(r)) \) is an analytic function of \( r \). This follows from equation (4.9). Since \( \cap(E_\tau^n(r)) \) is not constant and analytic, it follows that \( \cap(E_\tau^n(r)) \) is not constant on any subinterval of \([0, 1]\). Furthermore, it follows from Theorem 3.45 that \( \cap(E_\tau^n(r)) \) is a non-decreasing function. Since it is non-decreasing and not constant on any interval, it is strictly increasing.

Our next theorem can be considered as a counterpart of the subadditivity property of the conformal capacity discussed in Proposition 2.7.

**Theorem 4.11.** Let \( E_k, k = 1, \ldots, n, \) be compact sets on the interval \( I = [0, 1] \) having positive logarithmic capacities and such that every point of \( E_k \) is regular for the Dirichlet problem in \( \mathbb{D} \setminus E_k, k = 1, 2, \ldots, n. \) Then

\[
\frac{1}{n} \sum_{k=1}^{n} \cap(\bigcup_{j=1}^{n} e^{2\pi i(j-1)/n} E_k) \leq \cap(\bigcup_{k=1}^{n} e^{2\pi i(k-1)/n} E_k) < \sum_{k=1}^{n} \cap(E_k).
\]

Equality occurs in the first inequality if and only if for each \( k \) and \( j \), \( E_k \) coincides with \( E_j \) up to a set of zero logarithmic capacity.

**Proof.** The second inequality is just the subadditivity property of the conformal capacity stated in Proposition 2.7.

To prove the first inequality, we use the method of separation of components of a condenser in the style of Dubinin’s paper [15]. Let \( u \) denote the potential function of the condenser \((\mathbb{D}, \bigcup_{k=1}^{n} e^{2\pi i(k-1)/n} E_k)\). Since every point of \( E_k \) is regular for the Dirichlet problem, it follows that \( u \) is continuous on \( \overline{\mathbb{D}} \). Let \( u_k' \) and \( u_k'' \) denote the functions obtained from \( u \), first by restricting \( u \), respectively, onto the sector \( S_1 = \{ z \in \mathbb{D} : \pi(2k - 3)/n \leq \arg z \leq 2\pi(k - 1)/n \} \) or onto the sector \( S_2 = \{ z \in \mathbb{D} : 2\pi(k - 1)/n \leq \arg z \leq \pi(2k - 1)/n \} \) and then extending this restriction by symmetry on the whole unit disk. Then each of the functions \( u_k' \) and \( u_k'' \) is admissible for the condenser \((\mathbb{D}, \bigcup_{j=1}^{n} e^{2\pi i(j-1)/n} E_k)\). Furthermore, each of these functions possesses \( n \)-fold rotational symmetry about 0 and is symmetric with respect to the real axis. Therefore, the following inequality holds:

\[
\frac{1}{n} \cap(\bigcup_{j=1}^{n} e^{2\pi i(j-1)/n} E_k) \leq \int_{S_1 \cup S_2} |\nabla u_k|^2 \, dm, \quad k = 1, \ldots, n.
\]

Summing up all the inequalities in (4.13), we obtain the first inequality in (4.12).
Furthermore, since every point of the sets $E_j$, $j = 1, \ldots, n$, is regular for the Dirichlet problem, it follows that $u_k'$ or $u_k''$ defined above in the proof is a potential function of $(\mathbb{D}, \bigcup_{j=1}^m e^{2\pi i (j-1)/n} E_k)$ if and only if $E_j = E_k$ for all $j = 1, \ldots, n$. Therefore, if $E_j \neq E_k$ for some $j \neq k$, then we have the strict inequality in (4.13) and in the first inequality in (4.12) as well.

Above we discussed results on the conformal capacity of compact sets lying on a fixed number of radial intervals. In our next theorem, we work with compact sets on $m \geq n \geq 2$ radial intervals that are “densely spread” over $\mathbb{D}$ in the sense that the angle between any two neighboring intervals is $\leq 2\pi/n$.

**Theorem 4.14.** Let $E_0 = \overline{D}_r$, $0 < r < 1$, and let $E_1 \subset [r, 1)$ be a compact set of hyperbolic length $\tau > 0$. Let $0 = \alpha_0 = \alpha_1 < \alpha_2 < \ldots < \alpha_m < \alpha_{m+1} = 2\pi$ be such that $\alpha_{k+1} - \alpha_k \leq 2\pi/n$, $k = 1, \ldots, m$, $m \geq n \geq 2$. Consider compact sets $E = \bigcup_{k=0}^m e^{i\alpha_k} E_1$, $E^* = \bigcup_{k=0}^m e^{i2\pi (k-1)/n} E_1$, and $E_n(r, \tau) = E_0 \cup (\bigcup_{k=1}^m e^{i2\pi (k-1)/n} [r, r_1])$, where $r_1$ is such that $r < r_1 < 1$ and $\ell_\mathbb{D}([r, r_1]) = \tau$. Then

\begin{equation}
\text{cap}(E) \geq \text{cap}(E^*) \geq \text{cap}(E_n(r, \tau)).
\end{equation}

Equality in the first inequality occurs if and only if $m = n$.

**Proof.** The first inequality in (4.15), together with the statement on the equality cases, follows from Theorem 5 in [6]. Then the second inequality follows from the contraction principle stated in Theorem 3.45. \hfill \Box

Polarization and the contraction principle are customarily applied when lower bounds for the conformal capacity are needed. In our next theorem, we present a result with an upper bound for this capacity.

**Theorem 4.16.** For $0 \leq r \leq a < b < 1$, let $E_0 = \overline{D}_r$ and let $E_1 \subset [a, b]$ be a compact set of hyperbolic length $\tau > 0$, $0 < \tau < \ell_\mathbb{D}([a, b])$. Let $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha_{n+1} = 2\pi$. Let

\[ E = E_0 \cup (\bigcup_{k=1}^n e^{i\alpha_k} E_1), \quad E^* = E_0 \cup (\bigcup_{k=1}^n e^{i2\pi (k-1)/n} E_1), \quad E^{a,b} = E_0 \cup (\bigcup_{k=1}^n e^{i2\pi (k-1)/n} [a, b]). \]

Then

\begin{equation}
\text{cap}(E) \leq \text{cap}(E^*) < \text{cap}(E^{a,b}).
\end{equation}

Equality in the first inequality in (4.17) occurs if and only if $\alpha_k = 2\pi (k-1)/n$, $k = 1, \ldots, n$.

**Proof.** The first inequality in (4.17) together with the statement on the cases of equality follows from Dubinin’s dissymmetrization results; see Theorem 4.14 in [17]. Then, since the conformal capacity is an increasing function of a set, the second inequality follows. \hfill \Box

**Remark 4.18.** If $E_1$ in Theorem 4.16 is an interval $[c, d] \subset [a, b]$, then the upper bound in (4.17) can be replaced with $\text{cap}(E_0 \cup (\bigcup_{k=1}^n e^{i2\pi (k-1)/n} [s, b]))$ with $s$ in $(a, b)$ such that $\ell_\mathbb{D}([s, b]) = \tau$. This follows, for instance, from the monotonicity property stated in Theorem 3.45.
5. Symmetrization transformations in hyperbolic metric

In this section, we discuss possible counterparts of classical symmetrization-type transformations applied with respect to the hyperbolic metric. We note that Steiner, Schwarz and circular symmetrizations destroy hedgehog structures, in general, and therefore their applications to problems studied in previous sections of this paper are limited. We define symmetrizations using the following notations, which are convenient for our purposes: \( L_α = \{ z : \text{Im}(e^{-iα}z) = 0 \} \), \( L^a = \{ z : \text{Im} z = a \} \), \( C_r = \{ z : |z| = r \} \), \( R_α = \{ z = te^{iα} : t ≥ 0 \} \). First, we mention results obtained with Steiner symmetrization.

**Definition 5.1.** Let \( E ⊂ \mathbb{C} \) be a compact set. The Steiner symmetrization of \( E \) with respect to the imaginary axis is defined to be the compact set

\[
E^∗ = \{ z = x + iy : E ∩ L^y ≠ \emptyset, |x| ≤ (1/2)\ell(E ∩ L^y) \},
\]

where \( \ell(·) \) stands for the one-dimensional Lebesgue measure.

Furthermore, the Steiner symmetrization of \( E \) with respect to the line \( L_α \) is defined to be the compact set \( E^*_α = e^{i(α−π/2)}(e^{i(π/2−α)}E)^∗ \).

For the properties and results obtained with the Steiner symmetrization, the interested reader may consult [33], [17], [5].

We note that if \( E \) is a compact set in \( \mathbb{D} \), then \( E^*_α ⊂ \mathbb{D} \) for all \( α ∈ \mathbb{R} \) and therefore \( \text{cap}(E^*_α) \) is well defined. As is well known, Steiner symmetrization does not increase the capacity of a condenser and therefore it does not increase the conformal capacity. Also, Steiner symmetrization preserves Euclidean area but, in general, it strictly decreases the hyperbolic area \( A_\mathbb{D}(E) \) of \( E \) that is defined by (1.4). Therefore, it is not an equimeasurable rearrangement with respect to the hyperbolic metric. Thus, to study problems on the hyperbolic plane, a version of Steiner symmetrization, which preserves the hyperbolic area and does not increase the conformal capacity, is needed. To define this symmetrization, we will use the function

\[
w = ϕ_0(z) = \log \frac{1 + z}{1 - z}, \quad ϕ_0(0) = 0,
\]

which maps \( \mathbb{D} \) conformally onto the horizontal strip \( Π = \{ w : |\text{Im } w| < π/2 \} \). We note that \( ϕ \) maps the hyperbolic geodesic \((-1, 1)\) onto the real axis and it maps the curves equidistant from \((-1, 1)\), (which are circular arcs in \( \mathbb{D} \) joining the points 1 and \(-1\)), onto the horizontal lines \( \{ w : \text{Im } w = c \}, 0 < |c| < π/2, \) in \( Π \).

**Definition 5.2.** Let \( E \) be a compact set in \( \mathbb{D} \). The hyperbolic Steiner symmetrization of \( E \) with respect to the hyperbolic geodesic \((-i, i)\), centered at \( z = 0 \), is defined as

\[
E^*_h = ϕ_0^{-1}((ϕ_0(E))^∗),
\]

where \((ϕ_0(E))^∗\) stands for the Steiner symmetrization as in Definition 5.1.

Furthermore, the hyperbolic Steiner symmetrization of \( E \) with respect to a hyperbolic geodesic \( γ \), centered at the point \( a ∈ γ \), is defined as

\[
E_h(γ, a) = ψ^{-1}((ψ(E))^*_h),
\]

where \( ψ(z) \) denotes the Möbius automorphism of \( \mathbb{D} \), which maps \( γ \) onto the hyperbolic geodesic \((-i, i)\) such that \( ψ(a) = 0 \).
The hyperbolic Steiner symmetrization was introduced by A. Dinghas [14]. We note that this transformation symmetrizes sets along the hyperbolic equidistant lines and not along the hyperbolic geodesics. Previously it was used in several research papers, see, for instance, [25], [20]. In our next theorem, we collect properties of the hyperbolic Steiner symmetrization that are relevant to our study.

**Theorem 5.3** (see [25]). Let $E_h(\gamma, a)$ be the image of a compact set $E \subset \mathbb{D}$ under the hyperbolic Steiner symmetrization with respect to a hyperbolic geodesic $\gamma$, centered at $a \in \gamma$. Let $\gamma_\perp \ni a$ be the hyperbolic geodesic orthogonal to $\gamma$. Then the following hold true:

1. If $E$ is a hyperbolic disk, then $E_h(\gamma, a)$ is a hyperbolic disk of the same hyperbolic area as $E$ having its center on $\gamma$.
2. $E_h(\gamma, a)$ is a compact set that is symmetric with respect to $\gamma$.
3. $\ell_\mathbb{D}(E_h(\gamma, a) \cap \gamma_\perp) = \ell_\mathbb{D}(E \cap \gamma_\perp)$.
4. $A_\mathbb{D}(E_h(\gamma, a)) = A_\mathbb{D}(E)$.
5. $\text{cap}(E_h(\gamma, a)) \leq \text{cap}(E)$ with the sign of equality if and only if $E_h(\gamma, a)$ coincides with $E$ up to a set of zero logarithmic capacity and up to a Möbius automorphism of $\mathbb{D}$ that preserves $\gamma_\perp$.

**Proof.** We use the notation that we set in the definitions of Steiner and hyperbolic Steiner symmetrizations. We equip the strip $\Pi$ with the hyperbolic metric $\lambda_\Pi(w)|dw|$ induced by the hyperbolic metric in $\mathbb{D}$ via the conformal mapping $\varphi_0 : \mathbb{D} \to \Pi$. That is, we have

$$\lambda_\Pi(w)|dw| = \lambda_\mathbb{D}(z)|dz|, \quad w = \varphi_0(z), \ z \in \mathbb{D}, \ w \in \Pi.$$

So, trivially, $\varphi_0$ is a hyperbolic isometry from $\mathbb{D}$ to $\Pi$.

(1) Let $E$ be a hyperbolic disk in $\mathbb{D}$. Then $\psi(E)$ is a hyperbolic disk in $\mathbb{D}$ and $\varphi_0 \circ \psi(E)$ is a hyperbolic disk in $\Pi$. It is easy to observe that hyperbolic disks in $\Pi$ are horizontally convex (namely, their intersection with any horizontal line is either empty or a single horizontal rectilinear interval). It follows from the definition of Steiner symmetrization that $(\varphi_0 \circ \psi(E))^*$ is obtained by a horizontal rigid motion of $\varphi_0 \circ \psi(E)$ and it is a hyperbolic disk in $\Pi$, symmetric with respect to the imaginary axis. Since both $\psi^{-1}$ and $\varphi_0^{-1}$ are hyperbolic isometries and preserve symmetries, $E_h(\gamma, a)$ is a hyperbolic disk in $\mathbb{D}$, symmetric with respect to $\gamma$. Moreover, a horizontal motion in $\Pi$ preserves the hyperbolic area. Therefore, $A_\mathbb{D}(E_h(\gamma, a)) = A_\mathbb{D}(E)$.

(2) It is well known that Steiner symmetrization transforms compact sets to compact sets. So, if $E \subset \mathbb{D}$ is compact, $E_h(\gamma, a)$ is compact, too. Moreover, $(\varphi_0(E))^*$ is a set in $\Pi$, symmetric with respect to the imaginary axis. Hence, $E_h^*$ is a compact set in $\mathbb{D}$, symmetric with respect to the geodesic $(-i, i)$. It follows that $E_h(\gamma, a)$ is symmetric with respect to $\gamma$.

(3) Since $\psi$ is a hyperbolic isometry on $\mathbb{D}$, the set $\psi(E \cap \gamma_\perp)$ is a compact subset of the diameter $(-1, 1)$ having the same hyperbolic length as $E \cap \gamma_\perp$. Therefore, $\varphi_0 \circ \psi(E \cap \gamma_\perp)$ lies on the real axis and $\ell_\Pi((\varphi_0 \circ \psi(E \cap \gamma_\perp)) = \ell_\mathbb{D}(E \cap \gamma_\perp)$. The hyperbolic length (in $\Pi$) on the real axis is proportional to the Euclidean length $\ell$. Hence $\ell_\Pi((\varphi_0 \circ \psi(E \cap \gamma_\perp)) = \ell_\Pi((\varphi_0 \circ \psi(E \cap \gamma_\perp))^*)$. Thus the equality in (3) follows at once.

(4) This is true because $\varphi_0$, $\psi$, and Steiner symmetrization (with respect to the imaginary axis, in $\Pi$) preserve hyperbolic areas.
The conformal maps $\varphi_0$ and $\psi$ preserve the capacity of condensers. So, to prove the inequality in (5), it suffices to show that for every compact subset $F$ of $\Pi$, we have $\text{cap}(\Pi, F) \geq \text{cap}(\Pi, F^*)$. This is a well-known symmetrization theorem [17, Theorem 4.1]. The equality statement follows from [12, Theorem 1]. □

Remark 5.4. Of course, any transformation bearing the name “symmetrization” must possess property (1) of Theorem 5.3; otherwise it is not a symmetrization. On the other side, it was shown by L. Karp and N. Peyerimhoff in [25] and by F. Guéritaud in [20] that even in the best case scenario the hyperbolic Steiner symmetrization changes hyperbolic triangles into sets that are not convex with respect to hyperbolic metric and therefore these sets are not hyperbolic triangles. In fact, the image $T_h^*$ of the hyperbolic triangle $T$ with vertices $z_1 = -r$, $z_2 = r$, $0 < r < 1$, and $z_3 = -is + \sqrt{1 + s^2}e^{i\beta}$, $s > 0$, $\arctan s < \beta < \pi/2$, is a proper subset of the hyperbolic isosceles triangle $T_0$ with vertices $z_1 = -r$, $z_2 = r$, and $z_3 = (\sqrt{1+s^2} - s)i$.

In relation to Theorem 3.43 we suggest the following problem, where two hyperbolic Steiner symmetrizations provide some qualitative information about extremal configuration but these are not enough to give a complete solution of the problem. This problem is a counterpart of the problem for the logarithmic capacity in the Euclidean plane, which was solved in [8].

Problem 5.5. Find the minimal conformal capacity among all compact sets $E \subset \mathbb{D}$ with prescribed hyperbolic diameter $d > 0$ and prescribed hyperbolic area $0 < A < 4\pi \sinh^2(d/4)$. Describe possible extremal configurations.

The Schwarz symmetrization of $E$ in the plane with respect to a point $a$ replaces $E$ by the disk centered at $a$ of the same area. The hyperbolic analog of this symmetrization is the following.

Definition 5.6. Let $E$ be a compact set in $\mathbb{D}$. Then its hyperbolic Schwarz symmetrization with respect to $a \in \mathbb{D}$ is the hyperbolic disk, we call it $E_a^\#$, centered at $a$ and such that $A_D(E_a^\#) = A_D(E)$.

The hyperbolic Schwarz symmetrization was first suggested by F. Gehring [19] and later used in [18] and [11]. In particular, the following result was proved.

Theorem 5.7 (see [19],[18],[11]). If $E_a^\#$ is the hyperbolic Schwarz symmetrization of a compact set $E \subset \mathbb{D}$, then

$$\text{cap}(E_a^\#) \leq \text{cap}(E)$$

with the sign of equality if and only if $E_a^\#$ coincides with $E$ up to a set of zero logarithmic capacity and up to a Möbius automorphism of $\mathbb{D}$.

Next, we will discuss the hyperbolic circular symmetrization of $E$ with respect to the hyperbolic ray $[a, e^{i\alpha}]_h$, $a \in \mathbb{D}$, $\alpha \in \mathbb{R}$. This is defined as the image of the interval $[0,1)$ under a Möbius automorphism $\varphi$ of $\mathbb{D}$ such that $\varphi(0) = a$, $\varphi(1) = e^{i\alpha}$.

Definition 5.8. Let $E$ be a compact set in $\mathbb{D}$. Then its hyperbolic circular symmetrization with respect to the hyperbolic ray $[0,1)_h$ is a compact set $E_h^\circ \subset \mathbb{D}$ such that: (a) $0 \in E_h^\circ$ if and only if $0 \in E$, (b) for $0 < r < 1$, $E_h^\circ \cap C_r = \emptyset$ if and only if $E \cap C_r = \emptyset$, (c) if, for $0 < r < 1$,
$E \cap C_r \neq \emptyset$, then $E_h^o \cap C_r$ is a closed circular arc on $C_r$ (which may degenerate to a point or may be the whole circle $C_r$) centered at $z = r$ such that $\ell_D(E_h^o \cap C_r) = \ell_D(E \cap C_r)$.

Furthermore, the hyperbolic circular symmetrization of $E$ with respect to a hyperbolic geodesic ray $[a, e^{i\alpha})_h$ is defined as

$$
E_h^o(a, \alpha) = \varphi^{-1}((\varphi(E))^o_h),
$$

where $\varphi(z)$ denotes the Möbius automorphism of $\mathbb{D}$, which maps $[a, e^{i\alpha})_h$ onto the interval $[0, 1]$.

**Remark 5.10.** We immediately admit here that the hyperbolic circular symmetrization with respect to $[0, 1)_h$ is exactly the classical circular symmetrization with respect to the positive real axis. The reason for this is that the hyperbolic density $\lambda_D(z)$ is constant on circles centered at $0$. Thus, this transformation will not provide any new information that is not available via classical symmetrization methods. However, the variant in formula (5.9) can be used to obtain additional information that may be rather interesting.

It follows from Definition 5.8 that the hyperbolic area and conformal capacity $\text{cap}(E_h^o(a, \alpha))$ do not depend on $\alpha$. For the Euclidean area of $E_h^o(a, \alpha)$ we have the following result.

**Lemma 5.11.** Let $E \subset \mathbb{D}$ be a compact set of positive area, and let $A(\alpha)$ denote the Euclidean area of $E_h^o(r, \alpha)$, $0 < r < 1$, $0 \leq \alpha \leq \pi$, considered as a function of $\alpha$. If $E_h^o(r, \alpha)$ does not coincide with a hyperbolic disk centered at $r$ up to measure 0, then $A(\alpha)$ is strictly increasing in $\alpha$ on the interval $0 \leq \alpha \leq \pi$; otherwise $A(\alpha)$ is constant for $0 \leq \alpha \leq \pi$.

**Proof.** Let $E_0$ denote the circular symmetrization of $\varphi(E)$ with respect to $[0, 1)_h$, with $\varphi$ defined as in Definition 5.8. Then $E_h^o(r, \alpha) = \psi(E_0)$, where

$$
\psi(z) = (e^{i\beta}z + r)/(1 + e^{i\beta}rz),
$$

with $\beta = \beta(\alpha)$ chosen such that

$$
e^{i\alpha} = \psi(1) = (e^{i\beta} + r)/(1 + e^{i\beta}r).
$$

The function $\beta(\alpha)$ strictly increases from $0$ to $\pi$, when $\alpha$ increases from $0$ to $\pi$. Therefore, to prove monotonicity of a certain characteristic $F$ of $E_h^o(r, \alpha)$ as a function of $\alpha$, we can consider $\alpha = \alpha(\beta)$ as a function of $\beta$ and treat $F$ as a function of $\beta$. Thus, we will work with the function $A_1(\beta) = A(\alpha(\beta))$.

The Euclidean area $A_1(\beta)$ can be found as follows:

$$
A_1(\beta) = \int_{E_0} |\psi'(z)|^2 \, dm = (1 - r^2)^2 \int_0^1 \left( \int_{\theta(-\rho)}^{\theta(\rho)} \frac{d\theta}{1 + r\rho e^{i(\theta + \beta)}} \right) \rho \, d\rho,
$$

where $0 \leq \theta(r) \leq \pi$ is defined by the condition $\gamma(r) = E_0 \cap C_r = \{re^{i\theta} : |\theta| \leq \theta(r)\}$.

If $0 < \theta(r) < \pi$, then $\gamma(r)$ is a non-degenerate proper arc on $C_r$ that is centered at $z = r$.

Using this symmetry and the monotonicity property of the function

$$
g(t) = 1 + 2r\rho \cos t + r^2\rho^2; \quad 0 \leq t \leq \pi,
$$

Theorem 5.13. For Theorem 5.17 are shown in parts (c) and (d) of this figure. Examples of an admissible set and an extremal set of Theorem 5.13 proved in previous sections. Figure 5 illustrates proofs of these results, which are given in Theorems 5.13 and 5.17 below. Consider a function $\varphi(z) = rz/|z|$. It can be easily shown that $\varphi$ is a hyperbolic contraction from $E$ onto $E_{pr}(r)$. Hence, by Proposition 2.19,

$$\cap(E) \geq \cap(E_{pr}(r)).$$

As well known, the circular symmetrization decreases the conformal capacity, precisely, the following holds:

$$\cap(E_{pr}(r)) \geq \cap(C_r(\alpha))$$

with the sign of equality if and only if $E_{pr}(r)$ coincides with $C_r(\alpha)$ up to a set of zero logarithmic capacity and a rotation about 0. Combining (5.15) and (5.16), we obtain (5.14) together with the statement about cases of equality in it.

Theorem 5.17. For $0 < r < 1$, $0 \leq \alpha \leq \pi$, and $\tau(r) = \log((1 + r)/(1 - r))$, let $E \subset \mathbb{D}$ be a compact set such that $\ell_{\mathbb{D}}(E \cap C_r) = 2\alpha r/(1 - r^2)$ and $\ell_{\mathbb{D}}(E_{pr}^o) = \tau$, where $E_{pr}^o$ stands for the circular projection of $E$ onto the radius $[0, 1]$. Then

$$\cap(E) \geq \cap(C_r(\alpha) \cup [0, r(\tau)]),$$

where $r(\tau) = (e^\tau - 1)/(e^\tau + 1)$, with the sign of equality if and only if $E$ coincides with $C_r(\alpha) \cup [0, r(\tau)]$ up to a set of zero logarithmic capacity and a rotation about 0.

Proof. Performing the circular symmetrization as in the proof of Theorem 5.13, we obtain the following inequality:

$$\cap(E) \geq \cap(C_r(\alpha) \cup E_{pr}^o)$$
with the sign of equality if and only if $E$ coincides with $C_r(\alpha) \cup E^o_{pr}$ up to a set of zero logarithmic capacity and up to a rotation about $0$.

If $E^o_{pr}$ does not coincide with the interval $[0, r(\tau)]$ up to a set of zero logarithmic capacity, then we can perform polarization transformations as in the proof of Lemma 3.4 to transform $E^o_{pr}$ into the interval $[0, r(\tau)]$. Furthermore, our assumption that $\ell_D(E^o_{pr}) = \tau \geq \log((1 + r)/(1 - r))$ guarantees that these polarizations do not change $C_\alpha(r)$. Hence, we obtain the inequality

$$\text{cap}(C_r(\alpha) \cup E^o_{pr}) \geq \text{cap}(C_r(\alpha) \cup [0, r(\tau)])$$

with the sign of equality if and only if $E^o_{pr}$ coincides with $[0, r(\tau)]$ up to a set of zero logarithmic capacity. Combining (5.19) and (5.20), we obtain (5.18) together with the statement about cases of equality in it. \qed
The restriction $\tau \geq \tau(r)$ in the previous theorem is due to limitations of methods used in this paper. In relation with this theorem, we suggest two problems.

**Problem 5.21.** Find $\min_E \text{cap}(E)$ over all compact sets $E \subset \mathbb{D}$ satisfying the assumptions of Theorem 5.17 with $0 < \tau < \tau(r)$.

**Problem 5.22.** Let $E \subset \{z : r_1 \leq |z| \leq r_2\}, \ 0 < r_1 < r_2 < 1$, be a compact set such that its radial projection on the unit circle $\mathbb{T}$ coincides with $\mathbb{T}$ and its circular projection onto the radius $[0, 1)$ coincides with the interval $[r_1, r_2]$, as it is shown in Figure 6, and let $E^* = \overline{B}_{r_1} \cup [r_1, r_2)$. Is it true that $\text{cap}(E) \geq \text{cap}(E^*)$? If this is not true then identify the shape of $E$ minimizing the conformal capacity under the assumptions of this problem.

**Remark 5.23.** Let $\gamma = \gamma(r_1, r_2) \ni r_2$ denote the hyperbolic geodesic tangent to the circle $C_{r_1}$. We assume that $\gamma$ touches $C_{r_1}$ at the point $z_1 = r_1 e^{i\theta_0}, \ 0 < \theta_0 < \pi/2$. Let $\alpha$ denote the arc of $\gamma$ with the endpoints $z_1, r_2$ and let $\overline{\alpha} = \{z : \overline{z} \in \alpha\}$. It can be shown with the help of polarization that if $E^*$ minimizes the conformal capacity in Problem 5.22, then $E^*$ belongs to the compact set bounded by the arcs $\alpha, \overline{\alpha}$ and the circular arc $\{z = r_1 e^{i\theta} : \theta_0 \leq \theta \leq 2\pi - \theta_0\}$.

Now we turn to the hyperbolic version of the radial symmetrization which in the Euclidean setting was introduced by G. Szegö [44]. To define this radial symmetrization, we need the following notation. For $0 < r < 1, \alpha \in \mathbb{R}$, and a compact set $E \subset \mathbb{D}$, define $E(r, \alpha) = E \cap [re^{i\alpha}, e^{i\alpha})$.

**Definition 5.24.** Let $E$ be a compact set in $\mathbb{D}$ such that $\overline{B}_r \subset E, \ 0 < r < 1$. Then the radial symmetrization of $E$ with respect to $0$ is the compact set $E^\text{rad} \subset \mathbb{D}$ with the property: for every $\alpha \in \mathbb{R}, \ E^\text{rad}(r, \alpha)$ is a radial interval such that

$$
\int_{E^\text{rad}(r, \alpha)} \frac{|dz|}{|z|} = \int_{E(r, \alpha)} \frac{|dz|}{|z|}.
$$
Equation (5.25) shows that Szegö’s radial symmetrization is a rearrangement that is equimeasurable with respect to the logarithmic metric. It found important applications to several problems in Complex Analysis and Potential Theory. However, because of the usage of the logarithmic metric, its applications to the hedgehog problems studied in this paper are rather limited. Indeed, certain compact sets on the interval \([0,1]\) having a big hyperbolic length are transformed by this symmetrization to the radial intervals with a very small hyperbolic length.

In the search for better estimates for the studied characteristics of compact sets in \(D\), we turned to the following version.

**Definition 5.26.** The hyperbolic radial symmetrization \(E_{h}^{\text{rad}}\) of a compact set \(E \subset D\) with respect to 0 is defined to be a compact set starlike with respect to 0 and such that
\[
\ell_{D}(E_{h}^{\text{rad}} \cap [0,e^{i\alpha}]) = \ell_{D}(E \cap [0,e^{i\alpha}]) \quad \text{for all } \alpha \in \mathbb{R}.
\]

We recall that the compact sets in Theorems 3.27, 3.31, and 3.43 having the minimal conformal capacity among all sets admissible for these theorems can be obtained via the hyperbolic radial symmetrization defined above. Also, the graphs in Figure 3 of the results of numerical computations suggest that the hyperbolic radial symmetrization of two radial intervals reduces the conformal capacity of these intervals. Therefore, the following conjecture sounds plausible.

**Problem 5.27.** Let \(E\) be a compact set in \(D\). Prove (or disprove) that
\[
\text{cap}(E_{h}^{\text{rad}}) \leq \text{cap}(E).
\]

We conclude this section with the following result, which describes how the hyperbolic area and hyperbolic diameter of a compact set \(E\) behave under the hyperbolic radial symmetrization.

**Lemma 5.28.** The hyperbolic area and the hyperbolic diameter do not increase under the hyperbolic radial symmetrization of \(E\).

**Proof.** First we deal with the hyperbolic area \(A_{h}(E)\) of a compact set \(E\) in \(D\). Since every compact set in \(D\) can be approximated by a finite union of polar rectangles, we may assume that \(E\) is a finite union of sets of the form
\[
R = \{re^{it} : r_{1} \leq r \leq r_{2}, t_{1} \leq t \leq t_{2}\},
\]
where \(0 \leq r_{1} < r_{2} < 1\) and \(0 \leq t_{1} < t_{2} \leq 2\pi\). For such an \(E\), the hyperbolic radial symmetrization \(E_{h}^{\text{rad}}\) is again a union of sets of the same form, and moreover, each of the polar rectangles of \(E_{h}^{\text{rad}}\) is obtained by moving a rectangle of \(E\) radially towards the origin, keeping its hyperbolic height \(\tau\) fixed. More precisely, if \(R\) (as defined above) is a rectangle of \(E\), then the corresponding rectangle of \(E_{h}^{\text{rad}}\) has the form
\[
\hat{R} = \{re^{it} : \hat{r}_{1} \leq r \leq \hat{r}_{2}, t_{1} \leq t \leq t_{2}\},
\]
with \(\hat{r}_{1} \leq r_{1}\) and
\[
\ell_{D}([\hat{r}_{1}, \hat{r}_{2}]) = \ell_{D}([r_{1}, r_{2}]) = \tau.
\]
We express \(r_{2}\) as a function of \(r_{1}\) using (5.29) and find that
\[
r_{2} = r_{2}(r_{1}) = (r_{1} + r(\tau))/(1 + r(\tau)r_{1}), \quad \text{where } r(\tau) \text{ is defined in (1.3)}.
\]
Another elementary calculation gives
\[ A_h(R) = \frac{r_2^2 - r_1^2}{2(1 - r_1^2)(1 - r_2^2)}(t_2 - t_1). \]

We consider the function
\[ g(r_1) = \frac{r_2^2 - r_1^2}{(1 - r_1^2)(1 - r_2^2)}, \quad r_2 = r_2(r_1), \quad 0 < r_1 < 1. \]

By straightforward differentiation of \( g(r_1) \) and taking into account (5.30), we find that \( g(r_1) \) is strictly increasing. This implies that \( A_h(R) \geq A_h(\hat{R}) \). Summing up over the hyperbolic areas of all rectangles, we conclude that \( A_h(E) \geq A_h(E_h) \).

Next we turn to hyperbolic diameter, which we denote by \( \text{Diam}_h(E) \). We will use the following basic fact of the hyperbolic geometry.

(a) Given a hyperbolic geodesic \( \gamma \) and a point \( a \not\in \gamma \), there is a unique hyperbolic geodesic \( \gamma_\perp \ni a \) orthogonal to \( \gamma \). Let \( \gamma_\perp \) intersect \( \gamma \) at \( z = b \). Then the hyperbolic distance \( d_\mathbb{H}(a, z) \) strictly increases as \( z \) moves along \( \gamma \) from \( b \) to \( T \).

To prove this result, we may assume that \( \gamma = (−1, 1) \) and \( a = is, \ 0 \leq s < 1 \). Then \( \gamma_\perp = (−i, i) \) and the monotonicity of \( d_\mathbb{H}(i s, r) \) for \( 0 \leq r < 1 \) follows after simple calculations.

Furthermore, if \( \gamma = (−1, 1) \) and \( \Re a > 0, \ \Im a > 0 \), then \( \gamma_\perp \) intersects \( \gamma \) at the point \( b \) such that \( 0 < b < \Re a \).

We will also use the following result that can be checked by standard Calculus technique.

(b) Let \( z_1 = r e^{i\theta}, \ 0 < r < 1, \) and \( z_2 = e^{i\alpha} z_1, \ 0 \leq \alpha \leq \pi \). Then \( p_\mathbb{H}(z_1, z_2) = \frac{2r \sin(\alpha/2)}{\sqrt{1 - 2r^2 \cos \alpha + r^4}} \).

Furthermore, the pseudo-hyperbolic distance \( p_\mathbb{H}(z_1, z_2) \), and therefore the hyperbolic distance \( d_\mathbb{H}(z_1, z_2) \), is a strictly increasing function of \( r, 0 \leq r < 1 \) and a strictly increasing function of \( \alpha, 0 \leq \alpha \leq \pi \).

Now let \( z_1, z_2 \) be two points on \( E_\mathbb{H} \) such that \( d_\mathbb{H}(z_1, z_2) = \text{Diam}_h(E_\mathbb{H}) \). Note that \( [0, z_1] \subset E_\mathbb{H} \) and \( [0, z_2] \subset E_\mathbb{H} \). We may assume that \( z_1 = r_1 \in (0, 1) \), and \( z_2 = 0 \) or \( z_2 = r_2 e^{it}, \ 0 < r_2 \leq r_1, \ 0 \leq t \leq \pi \). If \( z_2 = -r_2, \ 0 \leq r_2 \leq r_2 \), then
\[ \text{Diam}_h(E_\mathbb{H}) = \ell_\mathbb{H}([-r_2, r_1]) = \ell_\mathbb{H}(E \cap (-1, 1)) \leq \text{Diam}_h(E) \]
and the required result is proved.

Next, we assume that \( z_2 = r_2 e^{it} \) with \( 0 < r_2 \leq r_1, \ 0 < t < \pi \). The set \( E \) contains points \( \zeta_1 = s_1 \) and \( \zeta_2 = s_2 e^{it} \) with \( s_1 = \max\{|z| : z \in E \cap [0, 1]\}, \ s_2 = \max\{|z| : z \in E \cap [0, e^{it}]\} \) such that \( s_1 \geq r_1, \ s_2 \geq r_2 \). If \( \pi/2 < t < \pi \), then it follows from the monotonicity property stated in (a) that \( d_\mathbb{H}(\zeta_1, \zeta_2) \geq d_\mathbb{H}(z_1, z_2) \) and therefore \( \text{Diam}_h(E) \geq \text{Diam}_h(E_\mathbb{H}) \) in this case.

Now, we consider the case when \( 0 < r_2 \leq r_1, \ 0 < t < \pi/2 \). In this case, the hyperbolic geodesic \( \gamma \nparallel z_1 \) orthogonal to \( (-e^{it}, e^{it}) \) crosses \( (-e^{it}, e^{it}) \) at the point \( z^* = r^* e^{it} \) with \( 0 < r^* < r_1 \). If \( 0 < r_2 \leq r^* \), then
\[ \text{Diam}_h(E_\mathbb{H}) = d_\mathbb{H}(r_1, r_2 e^{it}) < d_\mathbb{H}(r_1, 0) = \ell_\mathbb{H}(E_\mathbb{H} \cap [0, 1]) = \ell_\mathbb{H}(E \cap [0, 1]) \leq \text{Diam}_h(E), \]
where the first inequality follows from the monotonicity property stated in (a).

Thus, we are left with the case \( r^* < r_2 \leq r_1, \ 0 < t < \pi/2 \). We work with the points \( \zeta_1 = s_1 \) and \( \zeta_2 = s_2 e^{it} \) defined above. If \( r^* < s_2 \leq r_1 \leq s_1 \), then using twice the monotonicity property
stated in (a) we obtain the required result:

\[ \text{Diam}_h(E^\text{rad}_h) = d_\mathbb{D}(z_1, z_2) < d_\mathbb{D}(z_1, \zeta_2) \leq d_\mathbb{D}(\zeta_1, \zeta_2) \leq \text{Diam}_h(E). \]

If \( r_1 < s_k \leq s_j, \ k \neq j \), then

\[ \text{Diam}_h(E^\text{rad}_h) = d_\mathbb{D}(z_1, z_2) \leq d_\mathbb{D}(z_1, r_1 e^{i\theta}) \leq d_\mathbb{D}(s_k, s_k e^{i\theta}) \leq d_\mathbb{D}(\zeta_k, \zeta_j) \leq \text{Diam}_h(E), \]

where the first and the third inequalities follow from the monotonicity property stated in (a) and the second inequality follows from the monotonicity property with respect to \( r \) stated in (b). Now, the inequality \( \text{Diam}_h(E^\text{rad}_h) \leq \text{Diam}_h(E) \) is proved in all cases. \( \square \)

6. HEDGEHOG PROBLEMS IN \( \mathbb{R}^3 \)

In this section we briefly mention how some of our results proved in the previous sections can be generalized for compact sets in the ball \( \mathbb{B} = \{ \bar{x} : |\bar{x}| < 1 \} \) in \( \mathbb{R}^3 \). Here, \( \bar{x} = (x_1, x_2, x_3) \), \( |\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \). The conformal capacity of a compact set \( E \subset \mathbb{B} \) is defined as

\[ \text{cap}(E) = \inf \int_{\mathbb{B}} |\nabla u|^3 \, dV, \]

where \( dV \) stands for the three-dimensional Lebesgue measure and the infimum is taken over all Lipschitz functions \( u \) such that \( u = 0 \) on \( \partial \mathbb{B} \) and \( u = 1 \) on \( E \).

The conformal capacity of \( E \) is just the conformal capacity of the condenser \( (\mathbb{B}, E) \) and therefore it is invariant under Möbius automorphisms of \( \mathbb{B} \), see [19]. It is also known that the conformal capacity does not increase under polarization, see [5]. However, it is an open problem to prove that the conformal capacity is strictly decreasing under polarization unless the result of the polarization and the original compact set coincide up to reflection with respect to the plane of polarization and up to a set of zero logarithmic capacity; see, for instance, [43], where this question was discussed in the context of the Teichmüller’s problem in \( \mathbb{R}^3 \). This is why statements on the equality cases will be missing in all results, which we mention below.

Furthermore, the contraction principle, as it is stated in Proposition 2.19, is not available in the context of the conformal capacity in dimensions \( n \geq 3 \). This issue was discussed, for instance, in Section 5.6.2 in [5].

Below we list possible extensions of our results to \( \mathbb{R}^3 \) and to spaces of higher dimension.

1. The non-strict monotonicity property in Lemma 3.1 for a finite number of intervals on the diameter remains true in any dimension \( n \geq 3 \).
2. The inequality (3.5) of Lemma 3.4 holds in any dimension \( n \geq 3 \).
3. The inequality (3.28) of Theorem 3.27 holds true for a set \( E \) lying on \( m, 1 \leq m \leq 6 \), distinct radial intervals \( I_k \) in \( \mathbb{B} \) under the assumption that the angles between these intervals are greater than or equal to \( \pi/2 \).
4. Theorems 3.31 and 3.43 also can be extended to the case of appropriate compact sets in \( \mathbb{B} \).
5. The result of Theorem 4.11 can be extended to the three-dimensional space. For this, under the assumptions of this theorem, we assume additionally that the unit disk \( \mathbb{D} \) is embedded into \( \mathbb{B} \) as follows: \( \mathbb{D} = \mathbb{B} \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\} \). Then, inequality (4.12) remain true for conformal capacities in \( \mathbb{R}^3 \).
Some of the proofs in this paper make essential use of methods only available in the planar case and therefore the corresponding result for dimensions $n \geq 3$ remain open. Next we list a few of these open problems.

1. Our proofs of Theorems 3.36 and 4.1 require computations related to conformal mapping of doubly connected domains, which is not available in higher dimensions.

2. The result stated in Theorem 3.45 easily follows from the contractions principle. But, as we have mentioned above, the contraction principle is not available to problems on conformal capacity in dimensions $\geq 3$.

3. Theorem 4.8 can be proved by two methods: the first one uses the contraction principle and the second method uses explicit calculations. Both methods are not available in higher dimensions.

4. To prove Theorem 4.14 in the three-dimensional setting, we would need inequality (4.15) under the assumption that the central body $E_0$ is a ball $B_r = \{ x \in \mathbb{R}^3 : |x| \leq r \}$, $0 \leq r < 1$, and $E_1 \subset [r, 1)$ is the same as in Theorem 4.18. The proof of Theorem 4.14 is based on Baernstein’s $*$-function method and the contraction principle. Both techniques are still waiting to be developed for the case of this type of problems in $\mathbb{R}^3$. Thus, the counterpart of Theorem 4.14 remains unproved in dimensions $n \geq 3$.

Acknowledgements. We dedicate this paper to the memory of Jukka Sarvas whose work on symmetrization is a standard reference in the potential-theoretic study of isoperimetric inequalities and symmetrization.

We are indebted to Prof. M. M.S. Nasser and Dr. Harri Hakula for kindly providing several numerical results for this paper. We also thank K. Zarvalis for his help with the figures.

We are grateful to the referee for many valuable suggestions.

References

1. L. V. Ahlfors, Lectures on quasiconformal mappings. Second edition. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence, RI, 2006. viii+162 pp.

2. G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, Conformal invariants, inequalities, and quasiconformal maps. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1997.

3. D. H. Armitage and S. J. Gardiner, Classical potential theory. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2001.

4. V. V. Aseev, Continuity of conformal capacity for condensers with uniformly perfect plates. (Russian) Sibirsk. Mat. Zh. 40 (1999), no. 2, 243-253; translation in Siberian Math. J. 40 (1999), no. 2, 205-213

5. A. Baernstein II, Symmetrization in analysis. With David Drasin and Richard S. Laugesen. With a foreword by Walter Hayman. New Mathematical Monographs, 36. Cambridge University Press, Cambridge, 2019. xviii+473 pp

6. A. Baernstein II and A. Yu. Solynin, Monotonicity and comparison results for conformal invariants. Rev. Mat. Iberoam. 29 (2013), no. 1, 91–113.

7. T. Bagby, The modulus of a plane condenser. J. Math. Mech. 17, 1967, 315–329.

8. R. W. Barnard, K. Pearce, and A. Yu. Solynin, An isoperimetric inequality for logarithmic capacity. Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 2, 419–436.

9. A. F. Beardon, The geometry of discrete groups. Graduate texts in Math., Vol. 91, Springer-Verlag, New York, 1983.
10. D. Betsakos, Elliptic, hyperbolic, and condenser capacity; geometric estimates for elliptic capacity. J. Anal. Math. 96 (2005), 37–55.
11. D. Betsakos, Hyperbolic geometric versions of Schwarz’s lemma. Conform. Geom. Dyn. 17 (2013), 119–132.
12. D. Betsakos and S. Pouliasis, Equality cases for condenser capacity inequalities under symmetrization. Ann. Univ. Mariae Curie-Skłodowska Sect. A 66 (2012), no. 2, 1-24.
13. F. Brock and A. Yu. Solynin, An approach to symmetrization via polarization. Trans. Amer. Math. Soc. 352 (2000), no. 4, 1759–1796.
14. A. Dinghas, Minkowskische Summen und Integrale. Superadditive Mengenfunktionale. Isoperimetrische Ungleichungen, (German) Mémor. Sci. Math., Fasc. 149 Gauthier-Villars, Paris 1961 101 pp.
15. V. N. Dubinin, Change of harmonic measure in symmetrization. (Russian) Mat. Sb. (N.S.) 124(166) (1984), no. 2, 272–279.
16. V. N. Dubinin, Transformation of condensers in space. (Russian) Dokl. Akad. Nauk SSSR 296 (1987), no. 1, 18–20; translation in Soviet Math. Dokl. 36 (1988), no. 2, 217–219.
17. V. N. Dubinin, Condenser Capacities and Symmetrization in Geometric Function Theory, Birkhäuser, 2014.
18. A. Fryntov and J. Rossi, Hyperbolic symmetrization and an inequality of Dyn’kin. Entire functions in modern analysis (Tel-Aviv, 1997), 103–115, Israel Math. Conf. Proc., 15, Bar-Ilan Univ., Ramat Gan, 2001.
19. F. W. Gehring, Inequalities for condensers, conformal capacity, and extremal lengths. Michigan Math. J. 18 (1971), 1-20.
20. F. Guéritaud, A note on Steiner symmetrization of hyperbolic triangles, Elem. Math. 58 (2003), 21–25.
21. P. Hariri, R. Klén, and M. Vuorinen, Conformally Invariant Metrics and Quasiconformal Mappings, Springer Monographs in Mathematics, xix+502 pp, 2020.
22. J.-W. M. Van Ittersum, B. Ringeling, and W. Zudilin, Hedgehogs in Lehmer’s problem, Bull. Aust. Math. Soc. 105 (2022), 236 - 242, doi:10.1017/S0004972721000654.
23. J. A. Jenkins, Univalent functions and conformal mapping. Reihe: Moderne Funktionentheorie Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Heft 18. Springer-Verlag, Berlin-Göttingen-Heidelberg 1958 vi+169 pp.
24. E. M. Kalmoun, M. M.S. Nasser and M. Vuorinen, Numerical computation of preimage domains for a strip with rectilinear slits, arXiv:2204.00726.
25. L. Karp and N. Peyerimhoff, Extremal properties of the principal Dirichlet eigenvalue for regular polygons in the hyperbolic plane. Arch. Math. (Basel) 79 (2002), no. 3, 223–231.
26. R. Kühnau, Geometrie der konformen Abbildung auf der hyperbolischen und der elliptischen Ebene. VEB Deutscher Verlag der Wissenschaften, Berlin 1974.
27. N. S. Landkof, Foundations of modern potential theory. Translated from the Russian by A. P. Doohovskoy. Die Grundzüge der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.
28. N. A. Lebedev, The area principle in the theory of univalent functions. Izdat. "Nauka", Moscow, 1975. 336 pp.
29. M. M.S. Nasser and M. Vuorinen, Computation of conformal invariants.- Appl. Math. Comput., 389 (2021), 125617, arXiv:1908.04533.
30. M. M.S. Nasser and M. Vuorinen, Isoperimetric properties of condenser capacity.- J. Math. Anal. Appl. 2021, 499, 125050, arXiv:2010.09704.
31. M. M.S. Nasser, O. Rainio, and M. Vuorinen, Condenser capacity and hyperbolic diameter.- J. Math. Anal. Appl. 508(2022), 125870, arXiv:2011.06293.
32. M. M.S. Nasser, O. Rainio, and M. Vuorinen, Condenser capacity and hyperbolic perimeter.- Comput. Math. Appl. 105(2022), 54–74. arXiv:2103.10237.
33. G. Pólya and G. Szegő, Isoperimetric Inequalities in Mathematical Physics. Ann. Math. Studies 27, Princeton Univ. Press, 1952. MR0043486 (13:270d).
34. I. E. Pritsker, House of algebraic integers symmetric about the unit circle. J. Number Theory 236 (2022), 388–403.
35. Th. Ransford, Potential theory in the complex plane. London Mathematical Society Student Texts, 28. Cambridge University Press, Cambridge, 1995. x+232 pp.
36. J. Sarvas, Symmetrization of condensers in n-space. Ann. Acad. Sci. Fenn. Ser. A. I. 1972, no. 522, 44 pp.
37. A. Yu. Solynin, Polarization and functional inequalities. (Russian, with Russian summary) Algebra i Analiz 8 (1996), no. 6, 148–185; English transl., St. Petersburg Math. J. 8 (1997), no. 6, 1015–1038.
38. A. Yu. Solynin, Extremal configurations in some problems on capacity and harmonic measure. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 226 (1996), Anal. Teor. Chisel i Teor. Funktsii. 13, 170–195, 239; translation in J. Math. Sci. (New York) 89 (1998), no. 1, 1031–1049.
39. A. Yu. Solynin, Harmonic measure of radial segments and symmetrization. (Russian) Mat. Sb. 189 (1998), no. 11, 121–138; translation in Sb. Math. 189 (1998), no. 11–12, 1701–1718.
40. A. Yu. Solynin, Continuous symmetrization via polarization. Algebra i Analiz 24 (2012), no. 1, 157–222; reprinted in St. Petersburg Math. J. 24 (2013), no. 1, 117–166.
41. A. Yu. Solynin, Exercises on the theme of continuous symmetrization. Comput. Methods Funct. Theory 20 (2020), no. 3–4, 465–509.
42. A. Yu. Solynin, Problems on the loss of heat: herd instinct versus individual feelings. Algebra i Analiz 33 (2021), no. 5, 1–50.
43. A. Yu. Solynin, Canonical embeddings of pair of arcs and related problems. Submitted (2022).
44. G. Szegö, On a certain kind of symmetrization and its applications. Ann. Mat. Pura Appl. (4) 40 (1955), 113–119.
45. M. Tsuji, On the capacity of general Cantor sets. Journal of the Mathematical Society of Japan, Vol. 5, No. 2, July, 1953.
46. M. Tsuji, Potential Theory in Modern Function Theory. Chelsea Publishing Co., New York, 1975.
47. M. Vuorinen, Conformal Geometry and Quasiregular Mappings, Lecture Notes in Mathematics, 1319, Springer-Verlag, 1988.
48. H. Wallin, Metrical characterization of conformal capacity zero. J. Math. Anal. Appl. 58 (1977), no. 2, 298–311.
49. V. Wolontis, Properties of conformal invariants. Amer. J. Math. 74 (1952), 587–606.
50. W. P. Ziemer, Extremal length and p-capacity, Michigan Math. J. 16 (1969), 43–51.

DEPARTMENT OF MATHEMATICS, ARISTOTLE UNIVERSITY OF THESSALONIKI, GR–54124 THESSALONIKI, GREECE

Email address: betsakos@math.auth.gr

DEPARTMENT OF MATHEMATICS AND STATISTICS, TEXAS TECH UNIVERSITY, BOX 41042, LUBBOCK, TEXAS 79409, USA

Email address: alex.solynin@ttu.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, FI-20014 UNIVERSITY OF TURKU, FINLAND

Email address: vuorinen@utu.fi