Solution of the Conformable Angular Equation of the Schrodinger Equation

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Abstract

In this work, the conformable Schrodinger equation in spherical coordinates is separated into two parts; radial and angular part, the angular part of the Schrodinger equation is solved. The normalized Spherical harmonics function is obtained as a solution of the angular part.

Keywords: conformable derivative, spherical harmonics, Schrodinger equation, conformable partial derivative.

1 Introduction

In quantum mechanics, the Schrodinger equation represents a key result to obtain the wave function, and it is the quantum counterpart of Newton’s second law in classical mechanics, and to solve it with three-dimensional spherical coordinates, the method of separating the variables was used. It resulted in two equations, the first is a radial equation and the second is an angular equation so that the solution to the radial equation depends on knowing the potential and the solution to the angular equation is using the special functions, specifically the associated Legendre equation [1], where the associated Legendre equation is a generalization of the Legendre differential equation and the solutions $P_m^l(x)$ to this equation are called the associated Legendre polynomials [2].

The fractional derivative which is a derivative of arbitrary order is as old as calculus. L’Hospital asked the Leibniz about the possibility that the order of the derivative to be $\frac{1}{2}$ in 1695. Since then, many researchers tried to put a definition of fractional derivative. Leibniz was the first to offer the idea of a symbolic approach, employing the symbol \( \frac{d^n}{dx^n} = D^n y \) for the nth derivative, where n is a non-negative integer [3]. In addition, one of the well-known fractional derivatives is the R.L. fractional derivative [4], and
the second one is the Caputo derivative [5] In physics, mathematics, and engineering sciences, the fractional derivative has played an essential role [6–15].

In 2014, Khalil et.al [16], was introduced a new definition of derivative of \( \alpha \) order called the conformable derivative, where \( 0 < \alpha \leq 1 \). This definition is a natural extension of the usual derivative and satisfies the standard properties of the traditional derivative i.e the derivative of the product and the derivative of the quotient of two functions and satisfies the chain rule. The conformable calculus has many applications in several fields, for example in physics, it was used in quantum mechanics to study The effect of fractional calculus on the formation of quantum-mechanical operators [17], and an extension of the approximate methods used in quantum mechanics was made [18–20], and the of conformable harmonic oscillator is quantized using the annihilation and creation operators [21], besides, the effect of deformation of special relativity studied by conformable derivative [22], and the conformable Laguerre and associated Laguerre differential equations using conformable Laplace transform are solved [23].

In this work, the conformable Schrodinger equation is separated into two parts radial which depends on the knowing the potential and angular part which we solved and we obtained the conformable spherical harmonic. Besides, as an explanation we plotted \( |Y_{1\alpha}^2| \) in two and three dimensions.

## 2 Conformable derivative

We start by presenting some definitions related to our work.

**Definition 2.1.** The conformable derivative of \( f \) with order \( 0 < \alpha \leq 1 \) is defined by [16]

\[
T_\alpha (f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},
\]

where \( f \in [0, \infty) \to \mathbb{R} \).

**Definition 2.2.** The conformable partial derivative of \( f \) with order \( 0 < \alpha \leq 1 \) is defined by [24]

\[
\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, \ldots, x_m) = \lim_{\epsilon \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + \epsilon x_i^{1-\alpha}, \ldots, x_m) - f(x_1, \ldots, x_m)}{\epsilon}
\]

## 3 Conformable spherical harmonics

In terms the conformable derivative, we consider the Schrodinger equation as [25]

\[
\hat{p}_\alpha^2 \psi_\alpha(x, t) = (E^\alpha - V_\alpha(\hat{x}_\alpha)) \psi_\alpha(x, t).
\]

and \( h_\alpha = \frac{\hbar}{(2\pi)^{\frac{\alpha}{2}}} \). The coordinate and the momentum operators are defined as

\[
\hat{x}_\alpha = x, \quad \hat{p}_\alpha = -i h_\alpha \nabla^\alpha.
\]
To read more about the conformable quantum mechanics see ref [17, 25]. In terms the conformable derivative, the Schrödinger equation in spherical coordinates can be written as

\[
\nabla^2 \alpha - \frac{2m^\alpha}{\hbar^2} (V_\alpha (r^\alpha) - E^\alpha) \psi_\alpha (r^\alpha, \theta^\alpha, \varphi^\alpha) = 0.
\]

(5)

where \( \nabla^2 \alpha \) in spherical coordinates is given by

\[
\nabla^2 \alpha = \frac{1}{r^2 \alpha} D^\alpha_r [r^2 \alpha D^\alpha_r] + \frac{1}{r^{2\alpha} \sin (\theta^\alpha)} D^\alpha_\theta [\sin (\theta^\alpha) D^\alpha_\theta] + \frac{1}{r^{2\alpha} \sin^2 (\theta^\alpha)} D^\alpha_\varphi.
\]

(6)

After substituting in eq.(5), we get

\[
\frac{1}{R_\alpha} D^\alpha_r [r^2 \alpha D^\alpha_r R_\alpha] + \frac{1}{Y_\alpha \sin (\theta^\alpha)} D^\alpha_\theta [\sin (\theta^\alpha) D^\alpha_\theta Y_\alpha] + \frac{1}{Y_\alpha \sin^2 (\theta^\alpha)} D^\alpha_\varphi Y_\alpha
\]

- \( \frac{2m^\alpha r^{2\alpha}}{\hbar^2} (V_\alpha (r^\alpha) - E^\alpha) = 0. \)

(7)

The first part of this equation that depends on \( r^\alpha \) and equal to a constant is given as

\[
\frac{1}{R_\alpha} D^\alpha_r [r^2 \alpha D^\alpha_r R_\alpha] - \frac{2m^\alpha r^{2\alpha}}{\hbar^2} (V_\alpha (r^\alpha) - E^\alpha) = \alpha^2 \ell (\ell + 1).
\]

(8)

This equation is called conformable radial equation and the solution of this equation depends on the potential \( V_\alpha (r^\alpha) \).

The second part of equation (7) reads as

\[
\frac{1}{Y_\alpha \sin (\theta^\alpha)} D^\alpha_\theta [\sin (\theta^\alpha) D^\alpha_\theta Y_\alpha] + \frac{1}{Y_\alpha \sin^2 (\theta^\alpha)} D^\alpha_\varphi Y_\alpha = -\alpha^2 \ell (\ell + 1).
\]

(9)

Using separation of variable \( Y_\alpha (\theta^\alpha, \varphi^\alpha) = \Theta_\alpha (\theta^\alpha) \Phi_\alpha (\varphi^\alpha) \) to solve this equation, we get

\[
\frac{1}{\Theta_\alpha \sin (\theta^\alpha)} D^\alpha_\theta [\sin (\theta^\alpha) D^\alpha_\theta \Theta_\alpha] + \frac{1}{\Phi_\alpha \sin^2 (\theta^\alpha)} D^\alpha_\varphi \Phi_\alpha = -\alpha^2 \ell (\ell + 1),
\]

(10)

after multiplied this equation by \( \sin^2 (\theta^\alpha) \), we get

\[
\frac{\sin (\theta^\alpha)}{\Theta_\alpha} D^\alpha_\theta [\sin (\theta^\alpha) D^\alpha_\theta \Theta_\alpha] + \alpha^2 \ell (\ell + 1) \sin^2 (\theta^\alpha) + \frac{1}{\Phi_\alpha} D^\alpha_\varphi \Phi_\alpha = 0.
\]

(11)

The part of this equation that depends on \( \varphi^\alpha \) and equal to a constant is given as

\[
\frac{1}{\Phi_\alpha} D^\alpha_\varphi \Phi_\alpha = -\alpha^2 m^2,
\]

(12)

thus, the solution of this equation is given by

\[
\Phi_\alpha (\varphi^\alpha) = A e^{im \varphi^\alpha} + Be^{-im \varphi^\alpha}.
\]

(13)
In this solution we will adopt the part \(A e^{im\varphi^\alpha}\) because \(\Phi_\alpha\) is a single valued function where \(m\) is integer, so we get
\[
\Phi_\alpha(\varphi^\alpha) = A e^{im\varphi^\alpha}. \tag{14}
\]
The part of eq.\((11)\) that depends on \(\theta^\alpha\) and equal to a constant is given as
\[
\frac{\sin (\theta^\alpha)}{\Theta_\alpha} D^\alpha_\ell \left[\sin (\theta^\alpha) D^\alpha_\ell \Theta_\alpha\right] + \alpha^2 \ell (\ell + 1) \sin^2 (\theta^\alpha) = \alpha^2 m^2. \tag{15}
\]
Multiplying this equation by \(\Theta_\alpha\), we get
\[
\sin (\theta^\alpha) D^\alpha_\ell \left[\sin (\theta^\alpha) D^\alpha_\ell \Theta_\alpha\right] + \alpha^2 \left[\ell (\ell + 1) \sin^2 (\theta^\alpha) - m^2\right] \Theta_\alpha = 0. \tag{16}
\]
Let \(\Theta_\alpha(\theta^\alpha) = X_\alpha(x^\alpha), x^\alpha = \cos (\theta^\alpha) \rightarrow \alpha x^{\alpha-1} dx = -\alpha \theta^{\alpha-1} \sin (\theta^\alpha) \rightarrow D^\alpha_\ell = -\sin (\theta^\alpha) D^\alpha_\ell.\) After substituting in this equation, we get
\[
(1 - x^{2\alpha}) D^\alpha_\ell \left[-(1 - x^{2\alpha}) D^\alpha_\ell X_\alpha\right] + \alpha^2 \left[\ell (\ell + 1) (1 - x^{2\alpha}) - m^2\right] X_\alpha = 0. \tag{17}
\]
After multiplied this equation by \(1/(1 - x^{2\alpha})\), we get
\[
(1 - x^{2\alpha}) D^\alpha_\ell D^\alpha_\ell X_\alpha - 2\alpha x^\alpha D^\alpha_\ell X_\alpha + \alpha^2 \left[\ell (\ell + 1) - \frac{m^2}{(1 - x^{2\alpha})}\right] X_\alpha = 0. \tag{18}
\]
This equation is called conformable associated Legendre differential equation and its solution is given by \([26]\)
\[
X_\alpha = P^m_\ell(\alpha) = \frac{(-1)^m (1 - x^{2\alpha})^\frac{\ell}{2}}{\alpha^{\ell+1}} D^{(\ell+m)\alpha} \left(x^{2\alpha} - 1\right). \tag{19}
\]
So, the solution for eq.\((9)\) is given as
\[
Y^{\alpha}_{\ell m}(\theta^\alpha, \varphi^\alpha) = N^m_{\ell\alpha} e^{im\varphi^\alpha} P^m_\ell(\cos (\theta^\alpha)), \tag{20}
\]
where \(N^m_{\ell\alpha}\) is normalization constant, can be calculated using normalization condition
\[
\int |Y^{\alpha}_{\ell m}|^2 d^\alpha \Omega = |N^m_{\ell\alpha}|^2 \int P^m_\ell(\cos (\theta^\alpha)) P^m_\ell(\cos (\theta^\alpha)) d^\alpha \Omega \tag{21}
\]
where \(d^\alpha \Omega = \sin (\theta^\alpha) d\theta d^\alpha \varphi.\)
Using the orthogonality of conformable associated Legendre functions \([26]\), we get
\[
\int |Y^{\alpha}_{\ell m}|^2 d^\alpha \Omega = |N^m_{\ell\alpha}|^2 \frac{(2\pi)^\alpha \alpha^{2m-1} 2(\ell + m)!}{(2\ell + 1)(\ell - m)!} = 1,
\]
then, the normalization constant is equal \(N^m_{\ell\alpha} = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{\alpha^{2m-2}(\ell + m)!(2\pi)^\alpha}}.\) Thus the orthonormal spherical harmonic
\[
Y^{\alpha}_{\ell m} = \frac{(2\ell + 1)(\ell - m)!}{\alpha^{2m-2}(\ell + m)!(2\pi)^\alpha} e^{im\varphi^\alpha} P^m_\ell(\cos (\theta^\alpha)). \tag{22}
\]
3.1 The relation between $Y_{\ell\alpha}^{m\alpha}$ and $Y_{\ell\alpha}^{-m\alpha}$

The relation between $Y_{\ell\alpha}^{m\alpha}$ and $Y_{\ell\alpha}^{-m\alpha}$ is given by

$$Y_{\ell\alpha}^{-m\alpha} = (-1)^m Y_{\ell\alpha}^{m\alpha} \quad (23)$$

**Proof.** In the first step we need to prove the relation between $P_{\ell\alpha}^{m\alpha}$ and $P_{\ell\alpha}^{-m\alpha}$, let us define $P_{\ell\alpha}^{-m\alpha}$ using eq. (19) as,

$$P_{\ell\alpha}^{-m\alpha} = \frac{(-1)^m (1 - x^{2\alpha})^{-\frac{m}{2}}}{\alpha^\ell 2^\ell \ell!} D^{(\ell-m)\alpha}(x^{2\alpha} - 1)^\ell \quad (24)$$

But, $D^{(\ell+m)\alpha}(x^{2\alpha} - 1)^\ell = D^{(\ell+m)\alpha}(x^{\alpha} - 1)^\ell(x^{\alpha}+1)^\ell$, now let $f = x^{\alpha} - 1, g = x^{\alpha}+1$

$$D^{(\ell+m)\alpha}(f)^\ell = D^{(\ell+m)\alpha}(f)\ell^\ell = D^{(\ell+m)\alpha}(f^{\frac{1}{2}})\alpha^\ell (g^{\frac{1}{2}})^\ell$$

Let $w = f^{\frac{1}{2}}, z = g^{\frac{1}{2}} \rightarrow D^{(\ell+m)\alpha}[(w)\alpha^\ell (z)^\alpha]$. Using Leibniz rule [23], we get

$$D^{(\ell+m)\alpha}[(w)\alpha^\ell (z)^\alpha] = \sum_{k=0}^{\ell} \binom{\ell}{k} D^{(\ell-m-k)\alpha}(w)^\ell D^{k\alpha}(z)^\alpha\ell$$

$$= \sum_{k=0}^{\ell} \binom{\ell}{k} D^{(\ell-m-k)\alpha}(w)^\ell D^{k\alpha}(z)^\alpha\ell$$

where $D^{k\alpha}(z)^\alpha = \frac{\alpha^k}{(k-m)!}(z)^{(k-m)\alpha}$, $D^{(\ell+m-k)\alpha}(w)^\ell = \frac{\alpha^{\ell-m-k}}{(k-m)!}(w)^{(k-m)\alpha}$

$$D^{(\ell+m)\alpha}[(w)\alpha^\ell (z)^\alpha] = \sum_{k=m}^{\ell} \binom{\ell}{k} \frac{\alpha^{k\ell}}{(\ell-k)!}(z)^{(k-m)\alpha} \frac{\alpha^{\ell-m-k}}{(k-m)!}(w)^{(k-m)\alpha}(z)^{(k-m)\alpha}$$

Thus, we have

$$D^{(\ell+m)\alpha}[(w)\alpha^\ell (z)^\alpha] = \sum_{k=m}^{\ell} \binom{\ell}{k} \frac{\alpha^{\ell+m-k}}{(\ell-k)!}(w)^{(k-m)\alpha}(z)^{(k-m)\alpha}$$

In the same way

$$D^{(\ell-m)\alpha}[(w)\alpha^\ell (z)^\alpha] = \sum_{r=0}^{\ell-m} \binom{\ell-m}{r} D^{(\ell-m-r)\alpha}(w)^\ell D^{r\alpha}(z)^\alpha\ell$$

$$= \sum_{r=0}^{\ell-m} \binom{\ell-m}{r} \frac{\alpha^{\ell-m-r}}{(r+m)!}(w)^{(r+m)\alpha} \frac{\alpha^{r\ell}}{(\ell-r)!}(z)^{(r-m)\alpha}$$

(25)
Thus, we have
\[
D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \sum_{r=0}^{\ell-m} \frac{(\ell-m)!\alpha^{\ell-m}(\ell)!}{r!(\ell-m-r)!(r+m)!(\ell-r)!(\ell+r+m)\alpha(z)^{\ell-r}\alpha} (w)^{(r+m)\alpha} (z)^{(\ell-r)\alpha}
\] (26)

Since the omitted terms in the sum vanish \(D^{k\alpha}(f)^r = 0\) if \(k > r\), and change the summation variable to \(k + r + m\) and substituting in eq. (26), we get
\[
D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \sum_{k=m}^{\ell} \frac{(\ell-m)!\alpha^{\ell-m}(\ell)!}{(k-m)!(\ell-k)!((k+m)\alpha(z)^{\ell-k})\alpha} (w)^{(k-m)\alpha} (z)^{(\ell-k)\alpha}
\] (27)

Multiply eq. (27) by \(\alpha^{2m(\ell+m)!\alpha(z)^{m\alpha}}\), we have
\[
D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \frac{\ell-m)!\alpha^{\ell-m}(\ell)!}{(\ell+m)!}\alpha^{2m} \sum_{k=m}^{\ell} \frac{(\ell+m)!\alpha^{\ell+m}(\ell)!}{(k-m)!(\ell-k)!((k+m)\alpha(z)^{\ell-k})\alpha} (w)^{(k-m)\alpha} (z)^{(\ell-k)\alpha}
\] (28)

From eq. (25), we get
\[
D^{(\ell-m)\alpha}[(w)^{\alpha\ell}(z)^{\alpha\ell}] = \frac{(\ell-m)!\alpha^{\ell-m}(\ell)!}{(\ell+m)!}\alpha^{2m} D^{(\ell+m)\alpha}[(w)^{\alpha\ell}]
\] (29)

After substitutions, we have
\[
D^{(\ell-m)\alpha}[(x^{2\alpha}-1)^{\ell}] = \frac{(\ell-m)!\alpha^{\ell-m}(\ell)!}{(\ell+m)!}\alpha^{2m} D^{(\ell+m)\alpha}[(x^{2\alpha}-1)^{\ell}]
\] (30)

Now substituting in eq. (24), we have
\[
P_{\ell\alpha}^{-m\alpha} = \frac{(-1)^{m(\ell-m)!}(1-x^{2\alpha})^{\frac{\alpha}{2}}}{(\ell+m)\alpha^{2m}} D^{(\ell+m)\alpha}[(x^{2\alpha}-1)^{\ell}]
\] (31)

Using eq. (19), we get
\[
P_{\ell\alpha}^{-m\alpha} = \frac{(-1)^{m(\ell-m)!}}{\alpha^{2m}(\ell+m)!} P_{\ell\alpha}^{m\alpha}.
\] (32)

In the second step We define \(Y_{\ell\alpha}^{-m\alpha}\) using eq. (22)
\[
Y_{\ell\alpha}^{-m\alpha} = \sqrt{\frac{(2\ell+1)(\ell+m)!}{\alpha^{2m-2}(\ell-m)!}} e^{-im\varphi} P_{\ell\alpha}^{-m\alpha} (\cos(\theta^\alpha)).
\] (33)

After substituting eq. (32), we get
\[
Y_{\ell\alpha}^{-m\alpha} = \sqrt{\frac{(2\ell+1)(\ell+m)!}{\alpha^{2m-2}(\ell-m)!}} e^{-im\varphi} \frac{(-1)^{m(\ell-m)!}}{\alpha^{2m}(\ell+m)!} P_{\ell\alpha}^{m\alpha} (\cos(\theta^\alpha))
\]
\[
= (-1)^{m} \sqrt{\frac{(2\ell+1)(\ell-m)!}{\alpha^{2m-2}(\ell+m)!}} e^{-im\varphi} P_{\ell\alpha}^{m\alpha} (\cos(\theta^\alpha))
\]
\[
= (-1)^{m} Y_{\ell\alpha}^{m\alpha},
\] (34)

Some of the low-lying conformable spherical harmonic functions are enumerated in the table below, as derived from the above formula.
Table 1: the first nine conformable spherical harmonics $Y^{m\alpha}_{\ell\alpha}$

| $\ell$ | $m$ | $Y^{m\alpha}_{\ell\alpha}$ |
|-------|-----|---------------------------|
| 0     | 0   | $\sqrt{\frac{\alpha^2}{2(2\pi)^{\frac{\alpha}{2}}}}$ |
| 1     | -1  | $\alpha \sqrt{\frac{3}{4(2\pi)^{\frac{\alpha}{2}}}} e^{-i\phi} \sin (\theta^\alpha)$ |
|       | 0   | $\sqrt{\frac{3\alpha^2}{2(2\pi)^{\frac{\alpha}{2}}}} \cos (\theta^\alpha)$ |
|       | 1   | $-\alpha \sqrt{\frac{3}{4(2\pi)^{\frac{\alpha}{2}}}} e^{i\phi} \sin (\theta^\alpha)$ |
| 2     | -2  | $\sqrt{\frac{15\alpha^2}{16(2\pi)^{\frac{\alpha}{2}}}} \sin^2 (\theta^\alpha) e^{-i2\phi}$ |
|       | -1  | $\alpha \sqrt{\frac{15}{4(2\pi)^{\frac{\alpha}{2}}}} e^{-i\phi} \cos (\theta^\alpha) \sin (\theta^\alpha)$ |
|       | 0   | $\sqrt{\frac{5\alpha^2}{8(2\pi)^{\frac{\alpha}{2}}}} (3 \cos^2 (\theta^\alpha) - 1)$ |
|       | 1   | $-\alpha \sqrt{\frac{15}{4(2\pi)^{\frac{\alpha}{2}}}} e^{i\phi} \cos (\theta^\alpha) \sin (\theta^\alpha)$ |
|       | 2   | $\sqrt{\frac{15\alpha^2}{16(2\pi)^{\frac{\alpha}{2}}}} \sin^2 (\theta^\alpha) e^{i2\phi}$ |

The conformable spherical harmonic density for $Y^{1\alpha}_{2\alpha}$ and for different values of $\alpha$ are plotted in 3D and 2D using Mathematica as follows,
Figure 1: Plot $|Y_{2\alpha}^{1\alpha}|^2$ with different value of $\alpha$ from 0.1 to 0.9 in 3d

Figure 2: $|Y_{2\alpha}^{1\alpha}|^2$ when $\alpha = 1$

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Figure 3: Plot $|Y_{123}^{\alpha}|^2$ with different value of $\alpha$ from 0.1 to 0.9 in polar plot

Figure 4: $|Y_{123}^{\alpha}|^2$ when $\alpha = 1$ in polar plot
4 Conclusions

We have solved the angular part of the conformable Schrödinger equation, and we obtained the conformable spherical harmonic function as solution of this part. We observed that the conformable spherical harmonics goes to spherical harmonic function when $\alpha$ goes to 1. To illustrate our calculation we have drawn the conformable spherical harmonic function for $\ell = 2$ and $m = 1$ in 3D and 2D, with different values of $\alpha$. We observed that in figures 1 the density function gradually converts to the traditional density function given in figures 2. Also the same thing has been seen for density function in polar plot.

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