Reducibility of the Cohen–Wales representation of the Artin group of type $D_n$

Claire Levaillant
clairelevaillant@yahoo.fr
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Abstract. Using knot theory, we construct a linear representation of the CGW algebra of type $D_n$. This representation has degree $n^2 - n$, the number of positive roots of a root system of type $D_n$. We show that the representation is generically irreducible, but that when the parameters of the algebra are related in a certain way, it becomes reducible. As a representation of the Artin group of type $D_n$, this representation is equivalent to the faithful linear representation of Cohen-Wales. We give a reducibility criterion for this representation as well as a conjecture on the semisimplicity of the CGW algebra of type $D_n$. Our proof is computer-assisted using Mathematica.

1 Introduction

1.1 Definitions and history

In 2002, Cohen and Wales showed the linearity of all the Artin groups of finite type [4]. The same result was shown independently by Digne in [8]. Linearity of a group means that there exists a faithful linear representation of this group.

In other words, the group can be identified with a subgroup of $GL_k(F)$ for some integer $k$ and some field $F$. If $M = (m_{ij})_{1 \leq i,j \leq m}$ is a Coxeter matrix of size $m$, the Artin group of type $M$ is by definition the group with generators $s_1, s_2, \ldots, s_m$ and relations

\[
s_i s_j s_i \ldots = s_j s_i s_j \ldots
\]

The Artin group of type $A_{n-1}$ is the braid group $B_n$ on $n$ strands. In the past, several authors had tried to use the $(n-1)$-dimensional Burau representation to show the linearity of $B_n$. However, if this representation is faithful for $n = 3$, it was shown to be unfaithful for $n \geq 5$ (see [27], [25], [1]). It is still unknown

\[\text{\ldots}\]

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whether the Burau representation of $B_4$ is faithful or not. Currently, the only known faithful linear representation of the braid group $B_n$ for $n \geq 4$ is the $\frac{n(n-1)}{2}$-dimensional Lawrence-Krammer representation. This representation originated in the work of Ruth Lawrence in [19] and was later recovered by Daan Krammer in [18]. Hence it is called the Lawrence-Krammer representation. The Lawrence-Krammer space is a vector space spanned by vectors indexed by the $\frac{n(n-1)}{2}$ positive roots of a root system of type $A_{n-1}$. The beautiful result of linearity of the braid group using the Lawrence-Krammer representation is due to Bigelow [2] and independently Krammer [18]. Soon after, Cohen and Wales wanted to show that all the Artin groups of finite types are linear groups. As part of their work in [4], they generalized the Lawrence-Krammer representation to types $D$ and $E$. They then generalized Krammer’s algebraic arguments to show the faithfulness of these newly found representations. We will call these representations the Cohen-Wales representations of respective types $D$ and $E$. Cohen, Gijsbers and Wales shortly after in [5] build even more inequivalent representations of the Artin groups of types $D$ and $E$. Except for the Cohen-Wales representation, it is still unknown whether these representations are faithful or not. These parameter-based representations that include the Cohen-Wales representation all factor through the Cohen-Gijsbers-Wales algebra (abbreviated CGW algebra), an algebra that contains the Artin group. Working with generic parameters, Cohen, Gijsbers and Wales show that these are all the irreducible representations of a certain quotient of ideals of the CGW algebra. In particular their work shows that the Cohen-Wales representation is irreducible for generic parameters. The goal of the present paper is to show that the Cohen-Wales representation of the Artin group of type $D_n$ based

\begin{align*}
\text{1.2 Notations and main results} & \\
\text{Main Theorem.} & \text{Let } n \text{ be an integer with } n \geq 4. \text{ Let } t \text{ and } r \text{ be two non-zero complex numbers. Assume that } \begin{cases} r^{2k} \neq 1 & \text{for every integer } k \text{ with } 1 \leq k \leq n \\ r^{2k} \neq -1 & \text{for every integer } k \text{ with } 1 \leq k \leq n - 1 \end{cases} \\
\text{Then the faithful Cohen-Wales representation of the Artin group of type } D_n \text{ based}
on the parameters \( t \) and \( r \) is irreducible except when
\[
t \in \{ r^{4n-4}, r^{2n-4}, -r^{2n-2}, 1, r^{4}, -1 \},
\]
when it is reducible.

The restrictions on the parameter \( r \) are natural ones and we will explain them later on. They have a crucial role to play in the paper.

We now introduce a few notations that relate to root systems of type \( D_n \). In what follows, \( n \) is an integer with \( n \geq 4 \). The vector space of the Cohen–Wales representation of type \( D_n \) is spanned by vectors indexed by the \( n^2 - n \) positive roots of a root system of type \( D_n \). We will number the nodes of the Dynkin diagram of type \( D_n \) as follows. We shall write \( i \sim j \) if nodes \( i \) and \( j \) are adjacent on the diagram. We denote by \( r_1, \ldots, r_n \) the simple reflections.

If \( \alpha_1, \ldots, \alpha_n \) denote the simple roots, then the positive roots are

- The \( n \) simple roots \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \).
- The \( \binom{n-1}{2} \) positive roots \( \alpha_i + \ldots + \alpha_j \) with \( 2 \leq i < j \leq n \), of a root system of type \( A_{n-1} \) on nodes \( 2, \ldots, n \).
- The \( n-2 \) roots \( \alpha_1 + \alpha_3 + \cdots + \alpha_i \) with \( i \geq 3 \).
- The \( \binom{n-2}{2} \) roots \( \alpha_1 + \alpha_2 + \cdots + \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \) with \( 3 \leq i \leq n-1 \) and \( i+1 \leq j \leq n \).

The spanning vectors of the Cohen-Wales space of type \( D_n \) will be denoted in the same order by \( \hat{w}_{1,2}, \hat{w}_{1,3}, \hat{w}_{2,3}, \ldots, \hat{w}_{n-1,n} \) for the simple roots and by \( \hat{w}_{i-1,j}, \hat{w}_{i,j}, \hat{w}_{i,j} \) for the other positive roots. The significance of the indices will be explained in detail later. For now, notice that a spanning vector carries a hat if and only if \( 1 \) is in the support of the positive root, that is the coefficient of \( \alpha_1 \) in the positive root is nonzero.

Let \( g_1, g_2, g_3, \ldots, g_n \) be the generators of the Artin group \( A(D_n) \) of type \( D_n \). As read on the Dynkin diagram, the defining relations are as follows.

\[
(A) \begin{cases}
\text{The } g_i \text{'s with } 2 \leq i \leq n \text{ satisfy the braid relations} \\
g_1 g_3 g_1 = g_3 g_1 g_3 \\
g_1 g_k = g_k g_1 \text{ when } k \neq 3
\end{cases}
\]

The CGW algebra \( CGW(D_n) \) of type \( D_n \) is an algebra with two parameters \( l \) and \( m \) that contains \( A(D_n) \), and contains other elements \( e_1, e_2, e_3, \ldots, e_n \). These elements are related to the generators \( g_i \)’s by
\[
m e_i = l (g_i^2 + mg_i - 1)
\]
The other defining relations of the algebra are as follows.

\[
(DL) \left\{ \begin{array}{ll}
  g_i e_i &= \frac{1}{r} e_i & \text{for all } i \\
  e_i g_j e_i &= l e_i & \text{when } i \sim j
\end{array} \right.
\]

Some selected immediate consequences of these definitions are the following (see \[5\]).

\[
\begin{align*}
  e_i g_i &= \frac{1}{r} e_i & \text{for all } i \\
  g_i - g_i^{-1} &= m (e_i - 1) & \text{for all } i \\
  e_i e_j e_i &= e_i e_i & \text{when } i \sim j \\
  g_j g_i g_j &= e_j g_i^{-1} = e_j g_i + m (e_j - e_i) & \text{when } i \sim j \\
  g_j e_i e_j &= g_i^{-1} e_j = g_i e_j + m (e_j - e_i) & \text{when } i \sim j \\
  e_i^2 &= \delta e_i & \text{with } \delta = 1 - \frac{r}{m}
\end{align*}
\]

Informations that relate to rank or cellularity can be found in \[7\]. The CGW algebra of type $D_n$ is a generalization of the BMW algebra to type $D_n$. The BMW algebra is named after Birman, Murakami and Wenzl. It was introduced by Birman and Wenzl in \[3\] and independently by Murakami in \[29\]. It has the same defining relations as above except they must be read on a Dynkin diagram of type $A_{n-1}$. The BMW algebra is in connection with a polynomial link invariant, namely the Kauffman polynomial. An important feature of the Kauffman polynomial is that it can distinguish oriented links that the other polynomials can't distinguish. Birman and Wenzl wanted to build an algebra equipped with a trace so that the Kauffman polynomial \[12\] is after appropriate renormalization that trace, in the same way the 2-variable generalization of the Jones polynomial \[16\], namely the HOMFLY polynomial \[10\] was after renormalization the trace on the Hecke algebra of the symmetric group. If we define the Hecke algebra $\mathcal{H}(D_n)$ as the algebra with generators $g_1, \ldots, g_n$ that satisfy the Artin relations (A) above and the relations $g_i^2 + m g_i = 1$ for all $i$, we note that a quotient of the CGW algebra is the Hecke algebra. This quotient of $CGW(D_n)$ will play a critical role when studying the reducibility of the representation of $CGW(D_n)$ that we build. We next give the expression of this representation. Its construction will be explained in detail in §2.2. We introduce a new indeterminate $r$ that is related to $m$ by the relation $m = \frac{1}{r} - r$. When later specialized to non-zero complex numbers, $r$ and $-\frac{1}{r}$ will hence be the two complex roots of the polynomial $X^2 + m X - 1$. We choose $\mathbb{Q}(l, r)$ as our base field. As $r^2 + m r = 1$, the field $\mathbb{Q}(l, r)$ is then a left $\mathcal{H}(D_n)$-module for the action given by $g_i, 1 = r$. We will denote by $V_n$ the Cohen-Wales space of type $D_n$. The space $V_n$ is the vector space spanned over $\mathbb{Q}(l, r)$ by the vectors $w_{ij}$'s and $\tilde{w}_{ij}$'s with $1 \leq i < j \leq n$. It will be convenient to introduce the following notation: by $\tilde{w}_{st}$ for some integers $s$ and $t$ with $1 \leq s < t \leq n$, we mean that $\tilde{w}_{st}$ is either $w_{st}$ or $\tilde{w}_{st}$. For more clarity in the writing, we also sometimes add a coma between the two indices $s$ and $t$. 

4
Theorem 1. The following map \( \nu^{(n)} \)

\[
CGW(D_n) \rightarrow \text{End}_{Q(t,r)}(V_n)
\]

\[
g_i \rightarrow \nu_i
\]
defines a representation of the CGW algebra of type \( D_n \) in the Cohen-Wales space of type \( D_n \). The actions are given as follows. First, the action by \( g_1 \) is special and needs to be formulated apart.

\[
\forall 3 \leq i < j \leq n, \quad \nu_1(w_{ij}) = \begin{cases} 
   m r^{j-5} (w_{1,i} - w_{1,j}) + m r^{j-4} (w_{2,i} - w_{2,j}) + m r^{j-3} (w_{1,j} - w_{1,j}) + m r^{j-2} (w_{2,j} - w_{2,j}) + m(2r^{j+1} + r^{j+2} - r^{j+1} - r^{j+2}) (w_{12} - w_{12}) + r w_{ij} & \text{if } j \geq 3 \\
   w_{2j} & \text{if } j = 3 \\
   w_{12} & \text{if } i = 1 \text{ or } j = 1
\end{cases}
\]

(1)

Second, the action by \( g_2, g_3, \ldots, g_n \) is determined by the following expressions.

\[
\forall t \geq i + 2, \quad \nu_{i+1}(w_{s,t}) = w_{i+1,t} \quad (8)
\]

\[

\nu_{j+1}(w_{s,j}) = \frac{w_{s,j+1}}{w_{s,j+1}} \quad (9)
\]

\[

\nu_{i+1}(w_{i,t}) = \frac{w_{i,t+1}}{w_{i,t+1}} \quad (10)
\]

\[

\nu_{i+1}(w_{i,t}) = \frac{w_{i,t+1}}{w_{i,t+1}} + \frac{m t^{i-j+2} w_{i-1,j} - m w_{i,t} w_{i,t}}{t^{i-j+2}} \quad (11)
\]

\[

\forall s \leq j - 2, \quad \nu_{j}(w_{s,j}) = w_{s,j+1} + \frac{m (t^{j-s-1} w_{j-1,j} - m w_{j,j})}{t^{j-s-1}} \quad (13)
\]

\[

\nu_{i}(w_{i,t}) = w_{i-1,t} + \frac{m t^{i-1} w_{i-1,i} - m w_{i,t}}{t^{i-1}} \quad (14)
\]

In all the other cases, \( \nu_{i}(w_{s,t}) = r w_{s,t} \) (15)

As a representation of the Artin group, up to the change of parameters \( l = t^{3} t^{-1} \) and up to some rescaling of the generators, this representation is equivalent to the Cohen-Wales representation with parameters \( t \) and \( r \) that was built and used in [4] to show the linearity of the Artin group of type \( D_n \).
In what follows, $\mathcal{H}_{F,r^2}(n)$ denotes the Iwahori Hecke algebra of the symmetric group $\text{Sym}(n)$ with parameter $r^2$ over the field $\mathbb{Q}(l, r)$ as defined in [26]. The following theorem gives a reducibility criterion for the above representation under some assumption of semisimplicity for the Hecke algebras $\mathcal{H}_{F,r^2}(n)$ and $\mathcal{H}(D_n)$.

**Theorem 2.** Let $n$ be an integer with $n \geq 4$. Let $l$, $m$ and $r$ be three non-zero complex numbers with $m = \frac{1}{r} - r$.

(i) Assume that the Hecke algebras $\mathcal{H}(D_n)$ and $\mathcal{H}_{F,r^2}(n)$ are semisimple. So assume that $r^{2k} \neq 1$ for every integer $k$ with $1 \leq k \leq n$ and $r^{2k} \neq -1$ for every integer $k$ with $1 \leq k \leq n-1$. Then, $\nu^{(n)}$ is irreducible except when

$$l \in \left\{ \frac{1}{r^{4n-7}}e, \frac{1}{r^{2n-2}}, -\frac{1}{r^{2n-5}}, r^{3}, \frac{1}{r}, -r^{3} \right\},$$

when it is reducible.

(ii) For these values of the parameters and the values for which $r$ has been replaced by $-\frac{1}{r}$, the CGW algebra $\text{CGW}(D_n)$ of type $D_n$ of [53] with parameters $l$ and $m$ over the field $\mathbb{Q}(l, m)$ is not semisimple.

Note Theorem 1 together with point (i) of Theorem 2 imply the Main Theorem. On the way, we further show the following theorems on the dimensions.

**Key assumption:** until the end of the paper, we assume that the Hecke algebras $\mathcal{H}(D_n)$ and $\mathcal{H}_{F,r^2}(n)$ are semisimple.

**Theorem 3. (Existence of a one-dimensional invariant subspace)** Let $n$ be an integer with $n \geq 4$. In $V_n$ there exists a one-dimensional invariant subspace if and only if $l = \frac{1}{r^{4n-7}}$. If so, it is unique and it is spanned over $\mathbb{Q}(l, r)$ by the vector

$$u = \sum_{1 \leq i < j \leq n} r^{i+j} (\hat{w}_{ij} + r^{2n-4} w_{ij})$$

**Theorem 4. (Existence of an irreducible $(n - 1)$-dimensional invariant subspace)** Let $n$ be an integer with $n \geq 5$. In $V_n$ there exists an irreducible $(n - 1)$-dimensional invariant subspace if and only if $l = \frac{1}{r^{2n-2}}$. If so it is unique and it is spanned over $\mathbb{Q}(l, r)$ by the vectors $v_i$'s, $1 \leq i \leq n - 1$ with

$$v_i = (r^{2n-6} - \frac{1}{r^2}) w_{i,i+1} + \sum_{j=i+2}^{n} r^{j-i-4} \left\{ (w_{i+1,j} - r w_{i,j}) + r^2 (\hat{w}_{i+1,j} - r \hat{w}_{i,j}) \right\}$$

$$+ \sum_{s=1}^{i-1} r^{s-i} \left\{ r^{2n-6} (w_{s,i+1} - r w_{s,i}) + (\hat{w}_{s,i+1} - r \hat{w}_{s,i}) \right\}$$

**Theorem 5. (Existence of an irreducible $n$-dimensional invariant subspace)**

(i) Let $n$ be an integer with $n \geq 4$ and $n \neq 5$. If there exists an irreducible $n$-dimensional invariant subspace inside $V_n$, then $l = -\frac{1}{r^{2n-2}}$.

(ii) (Case $n = 5$) If there exists an irreducible 5-dimensional invariant subspace inside $V_5$, then $l \in \left\{ -\frac{1}{r^3}, r^3 \right\}$. If so, it is unique.
Theorem 6. (Existence of an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace). Let $n$ be an integer with $n \geq 6$. If there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace inside $V_n$, then $l = r^3$.

Theorem 7. (Existence of an irreducible $\frac{n(n-1)}{2}$-dimensional invariant subspace). Let $n$ be an integer with $n \geq 5$. If $l = \frac{1}{r}$, there exists a unique irreducible $\frac{n(n-1)}{2}$-dimensional invariant subspace inside $V_n$. Moreover, it is spanned over $\mathbb{Q}(l, r)$ by the vectors $t_{ij} = w_{ij} - \hat{w}_{ij}, 1 \leq i < j \leq n$.

The theorems above have been stated in increasing order for the dimensions when $n \geq 5$. While Theorems 3 and 4 provide necessary and sufficient conditions for the existence, Theorems 5, 6 and Theorem 7 only provide a necessary condition and a sufficient condition respectively. We show along the proof of Theorem 2 point (i) that when the representation is reducible, the action on a proper invariant subspace is a $\mathcal{H}(D_n)$-action. When $\mathcal{H}(D_n)$ is semisimple, which we assume in this paper, the irreducible representations of $\mathcal{H}(D_n)$ are indexed by unordered double partitions $(\lambda, \mu)$ of $n$ as in Theorem 1.5 of [14]. Their degrees is given by the number of standard Young tableaux of shape $(\lambda, \mu)$. By standard, we mean that the integers ranging from 1 to $n$ must be filled in the tableau with the numbers increasing along the rows and down the columns. (0) denotes the empty partition. The classes of irreducible $\mathcal{H}(D_n)$-modules are called Specht modules. In [24], we give the complete classification of the invariant subspaces of the representation in terms of Specht modules.

We end this introduction by presenting a conjecture that relates to point (ii) of Theorem 2. It gives a semisimplicity criterion for the CGW algebra of type $D_n$ in the same spirit as existing criteria for type $A$. Let’s briefly recall these criteria in type $A$. In [35], Hans Wenzl was the first to discuss the semisimplicity of the Birman–Murakami–Wenzl algebra. He considers the BMW algebra with nonzero complex parameters $l$ and $m$. He shows the following result.

Theorem. [Wenzl], 1988 The BMW algebra with nonzero complex parameters $l$ and $m$ is always semisimple except possibly if $r$ is a root of unity of if $l$ is some power of $r$, where $r$ is a complex root of the polynomial $X^2 + mX - 1$.

Some of these powers are identified twenty years later in the Ph.D. thesis of [20] and Theorem 2, point (ii) of the present paper can be viewed as a generalization of Theorem 2 of [22]. We recall below this result in type $A$.

Theorem. [Levaillant-Wales], 2008 Let $n$ be an integer with $n \geq 3$. Let $m, l$ and $r$ be three nonzero complex numbers with $m = \frac{1}{r} - r$.

1) Suppose $n \geq 4$. If $r^{2k} = 1$ for some $k \in \{2, \ldots, n\}$ or if $l$ belongs to the set of values $r, -r^3, \frac{1}{r^3}, \frac{1}{r}, -\frac{1}{r}, -r^2, -r^{n-3}, r^{n-3}, -r^{n-3}, \frac{1}{r}, -\frac{1}{r},$ the BMW algebra $BMW_n$ of type $A_{n-1}$ with parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$ is not semisimple.

2) If $r^4 = 1$ or $r^6 = 1$ or if $l \in \{-r^3, \frac{1}{r^3}, 1, -1\}$, the algebra $BMW_3$ with parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$ is not semisimple.
Simultaneously and independently, using cellularity techniques that were first introduced by John Graham and Gus Lehrer in [11], Hebing Rui and Mei Si find a complete criterion of semisimplicity for the BMW algebra. Their work is based on the groundbreaking work of John Enyang [9], where he constructs a cellular basis for the BMW algebra. Let’s recall here Theorem B of [32].

**Theorem.** [Rui-Si], 2009 Let \( n \) be an integer with \( n \geq 3 \). Let \( B_n \) be the Birman-Murakami-Wenzl with parameters \( l \) and \( r \).

a) Suppose \( l \notin \{-r, \frac{1}{r} \} \). Then \( B_n \) is semisimple if and only if \( o(r) > 2n \) and

\[
l \notin \bigcup_{k=3}^{n} \{ r^{3-2k}, \pm r^{3-k}, -r^{2k-3}, \pm r^{k-3} \}
\]

b) Assume \( l \in \{-r, \frac{1}{r} \} \). Then,

\( B_n \) is not semisimple if \( n \) is either even or odd with \( n \geq 7 \).

\( B_3 \) is semisimple if and only if \( o(r) > 6 \) and \( r^4 \neq -1 \)

\( B_5 \) is semisimple if and only if \( o(r) > 10 \) and \( r^6 \neq -1 \) and \( r^8 \neq -1 \).

In their Theorem, case b) is different and is the case when \( c_3 = 0 \).

Based on Theorem 2 of this paper, and in the spirit of the Theorems [Wenzl] and [Rui-Si] stated above, we give the following conjecture.

**Conjecture.** Let \( l, m \) and \( r \) be three nonzero complex numbers with \( m = \frac{1}{r} - r \). The CGW algebra with parameters \( l \) and \( m \) over the field \( \mathbb{Q}(l, r) \) is semisimple except possibly if \( r \) is a root of unity or if

\[
l \in \bigcup_{k=4}^{n} \{ r^{2k-5}, -r^{5-2k}, -r^{4k-7}, -r^{7-4k}, r^{7-2k}, -r^{2k-7} \}
\]

The values of Theorem 2 point (ii) that don’t depend on \( n \), i.e., \( r^3, -\frac{1}{r}, -r \) and \( \frac{1}{r}, -r^3 \) are obtained with \( k = 4 \) and \( k = 5 \) respectively. As for the values that depend on the integer \( n \), they are obtained with \( k = n \).

The paper is organized as follows. In section 2 we introduce the diagrammatic version of the CGW algebra of type \( D_n \). These diagrams were introduced by the authors in [6]. They show that the CGW algebra of type \( D_n \) is isomorphic to a tangle algebra of type \( D_n \). We next use this tangle algebra to construct the representation of \( CGW(D_n) \) announced in Theorem 1 of the introduction. In §3, we show that when the representation is reducible, the action on a proper invariant subspace is a \( \mathcal{H}(D_n) \)-action. We then investigate the existence of irreducible \( \mathcal{H}(D_n) \)-modules of small dimensions inside the Cohen-Wales space. To finish the proof of reducibility, we use induction on the integer \( n \). In §4, we finish proving the theorems of the introduction.
2 Construction

2.1 The tangle algebra of type $D_n$ of Cohen-Gijsbers-Wales

In [6], Arjeh M. Cohen, Dié A.H. Gijsbers and David B. Wales build a diagram algebra that they show to be isomorphic to the CGW algebra of type $D_n$. Here is how the elements $g_k$’s and $e_k$’s are represented in their tangle algebra.

![Diagram of elements $g_1$ and $e_1$]

The vertical bar on the left hand side is rigid and is called the pole. Among the $g_i$’s and the $e_i$’s, only the elements $g_1$ and $e_1$ have strands twisting around the pole. While the $g_i$’s only have vertical strands, the $e_i$’s contain two horizontal strands. The following relations hold for twists around the pole.

![Diagram of relations]

The vertical bar on the left hand side is rigid and is called the pole.
The first relation says the pole has order two. The second relation says it is indifferent whether the double twist is in the order over-under or in the order under-over. The tangles are defined up to regular isotopy, that is Reidemeister moves II and III are permitted. For the definition of these moves, see [28], page 4. There are seven more defining relations in the tangle algebra. Three of them are independent of the presence of a first node and are the exact same relations as in the algebra of Morton and Traczyk [28], the diagrammatic version of the Birman-Murakami-Wenzl algebra. First, there is the Kauffman skein relation. It is the way you transform an under-crossing into an over-crossing and conversely. It is the diagrammatic version of the algebraic equality $g_i - g_i^{-1} = m(e_i - 1)$.

Next, the defining algebraic relations $(DL)$ are called "delooping relations". Indeed, the way you get rid of a loop on the diagrams is by multiplying by a factor $l$ or $l^{-1}$, as follows.

Finally, to finish with the non pole-related relations, here is how the relation $e_i^2 = \delta e_i$ is conveyed in the diagrams. Each closed loop not intersecting a tangle $T$ can be removed from the tangle by multiplying by a factor $\delta$.

There are now four other relations that involve the pole. They are the diagrammatic interpretations of the algebraic relations $g_1 g_2 = g_2 g_1$, $e_1 g_2 = g_2 e_1$, $e_2 g_1 = g_1 e_2$ and $e_2 e_1 = e_1 e_2$. We call the first of these relations the commuting relation, as in [6]. This relation will be extensively used in the present paper. The other relations are referred to by the authors in [6] as the first pole-related self-intersection relation, the second pole-related self-intersection relation and the first closed pole loop relation respectively. For these, we refer the reader to the diagrams (v), (vi) and (vii) of [6].
2.2 Construction of the representation

To each vector $w_{s,t}$ of the Cohen-Wales space, we associate a tangle. This tangle has two horizontal lines, one at the top joining nodes $s$ and $t$ and one at the bottom joining nodes $n-1$ and $n$. The top horizontal line over-crosses all the vertical strands that it intersects. Moreover, if the vector wears a hat, the top horizontal line twists around the pole, while when the vector does not carry a hat, there is no twist around the pole. If there is one twist around the pole, there should be another twist around the pole. The first possible vertical strand twists around the pole with the twist taking place below the twist of the horizontal strand.

In algebraic terms, to the root $\alpha_1$, one associates the CGW algebra element $e_1 e_{3,n}$. We then build the other positive roots inductively by acting with the simple reflections, except for the positive roots of type $w_{1,j}$. For instance,

$$\alpha_1 + \alpha_2 + 2 \alpha_3 + \cdots + 2 \alpha_i + \alpha_{i+1} + \cdots + \alpha_j = r_1 \cdots r_2 r_j \cdots r_3(\alpha_1)$$

and the associated algebra element is

$$g_{i_1,2} g_{j,3} e_1 e_{3,n}$$

By $g_{i,j}$ or $e_{i,j}$, we understand $g_i \cdots g_j$ or $e_i \cdots e_j$. The reader can check that the corresponding tangle has all the above characteristics. For the positive roots of type $w_{1,j}$, we must also use some inverses of the $g_k$'s. For instance, when $j \geq 3$, the associated CGW algebra element is $g_2^{-1} g_1 g_j e_1 e_{3,n}$. An action by a generator $g_k$ on these tangles can shift one of the horizontal strand’s extremities and/or introduce crossings between the vertical strands. Let $H_n$ be the Hecke algebra of type $A_1 \times D_{n-2}$ with generators $z$ and $g_1, \ldots, g_{n-2}$. Denote by $C_n$ the CGW algebra of type $D_n$.

Claim 1. $M_n = C_n e_n / \langle C_n e_i e_j C_n \cap C_n e_n \rangle_{i \neq j}$ is a right $H_n$-module for the action:

$$\forall 1 \leq k \leq n-2, \forall x \in M_n, x : g_k = x g_k$$
$$\forall x \in M_n, x : z = \frac{1}{g_x} x e_{n,3} e_1 g_2 e_1 e_{3,n}$$
The $g_k$’s act to the right of elements in $M_n$ by simply multiplying them to the right in $M_n$.

$z$ acts to the right of elements in $M_n$ by multiplying them to the right by $\xi$ in $M_n$, where

$$\xi = \frac{1}{\delta^2} e_{n,3} e_1 g_2 e_1 e_{3,n}$$

**Proof of the Claim.** If $x \in M_n$, since for every integer $k$ with $1 \leq k \leq n - 2$, the generator $g_k$ commutes to $e_{n}$, we see that $x . g_k$ is again in $M_n$. Next, we have

$$x . (g_k^2 + mg_k) = x \left(1 + \frac{m}{\delta} e_k\right)$$

But since $e_n$ and $e_k$ commute and $e_k e_n = 0$ in $C_n e_n/\langle C_n e_i e_j C_n \cap C_n e_n \rangle_{i \neq j}$, we see that $g_k^2 + mg_k$ acts like the identity on $x$. Further, $x . z$ belongs to $M_n$. It remains to show that $z^2 + mz$ acts like the identity on $x$. We have

$$x . (z^2 + mz) = x \left(\frac{1}{\delta^4} e_{n,3} e_1 g_2 e_1 e_{3,n} e_{n,3} e_1 g_2 e_1 e_{3,n} + \frac{m}{\delta^4} e_{n,3} e_1 g_2 e_1 e_{3,n}\right)$$

(17)

$$= x \left(\frac{1}{\delta^2} e_{n,3} e_1 (g_2^2 + mg_2) e_1 e_{3,n}\right)$$

(18)

$$= x \left(\frac{1}{\delta^2} e_{n,3} e_1 (1 + \frac{m}{\delta} e_2) e_1 e_{3,n}\right)$$

(19)

$$= x \left(\frac{1}{\delta} e_n\right)$$

(20)

Equality (17) comes from the definition of the action. Equality (18) can be obtained by first using the relation $e_n^2 = \delta e_n$, then applying the relation $e_i e_j e_i = e_i$ for adjacent nodes $i$ and $j$ multiple times, finally by using the fact that $e_1$ and $g_2$ commute and applying $e_1^2 = \delta e_1$. To get (20), observe that $e_1 e_2 = 0$ in $M_n$, then apply the same machinery as before. Now $\frac{1}{\delta} e_n$ acts to the right like the identity on any word ending in $e_n$. This settles the claim.

**Let’s now provide our ground field $F = \mathbb{Q}(l,r)$ with a structure of left $H_n$-module.** We will consider the one-dimensional action given by $g_k . 1 = r$ for every integer $k$ with $1 \leq k \leq n - 2$ and by $z . 1 = r$. Then,

$$C_n e_n/\langle C_n e_i e_j C_n \cap C_n e_n \rangle_{i \neq j} \otimes \mathbb{H}_n \mathbb{Q}(l,r) \in C_n \text{ Mod}$$

Our representation is built inside this CGW algebra left module. We show that the elementary tensors $w_{n-1} \otimes \mathbb{H}_n 1$ are invariant under the action by the generators $g_k$’s. To do so, it will be useful to understand the important role played by the special element $\xi$. Here is how this element $\xi$ is represented in the tangle algebra:
The strand which has the shape of an eight can freely slide along the pole and it can be viewed as a coefficient. It is called $\Xi^+$ in [6] and it has many interesting properties. One of them is that it commutes with another twist around the pole, as shown on Fig. 9 of [6]. Another property of $\Xi^+$ is that it satisfies a Kauffman skein type relation (see equation (2.1) of Lemma 2.11 in [6]). We have $\xi = \frac{1}{\delta^2} \Xi^+ e_n$. Note this simplified expression for $\xi$ allows us to recover the fact from earlier that

$$\xi^2 + m \xi = \frac{1}{\delta} e_n$$

Indeed,

$$\xi^2 + m \xi = \frac{1}{\delta^3} (\Xi^+)^2 e_n + \frac{m}{\delta^2} \Xi^+ e_n$$

$$= \frac{1}{\delta^3} (\delta^2 e_n - m \delta \Xi^+ e_n + \frac{m}{\delta^2} \Xi^+ e_n) + \frac{m}{\delta^2} \Xi^+ e_n$$

$$= \frac{1}{\delta} e_n$$

Equality (21) holds by definition. Equality (22) comes from an application of equality (2.2) of Lemma 2.11 of [6]. Cohen, Gijsbers and Wales define a $(0,0)$-tangle $\Theta$ that consists of two separate loops each of which twists around the pole. The elements $\Xi^+$ and $\Theta$ are interesting in that there is an analogy between $g_i$ (any $i$) and $\Xi^+$ on one hand and $e_i$ (any $i$) and $\Theta$ on the other hand, as is visible on equalities (2.1) and (2.4) of the same lemma. To get equality (23), notice that

$$\Theta e_n = e_{n,3} e_1 e_2 e_{3,n}$$

This equality is illustrated on the following figure.

Now the right member of (24) is zero in $M_n$, which after simplification yields (23). Let’s now mention the key property that we will use extensively to build the representation. A loop around the pole can be suppressed at the cost of a factor $\delta^{-1} \Xi^+$, as shown on Figure 1.
The trick is to add a closed loop by dividing by a factor $\delta$ and to twist it twice around the pole, using the double twist relation. Next, apply the first pole-related self-intersection relation ($v$) of [6] to get the member to the right. In terms of our representation, here is how this tangle property is nicely used. Replace $e_n$ by $\frac{1}{\delta} e_n^2$. Then the CGW algebra element representing the new tangle where the loop around the pole has been removed is obtained from the old CGW algebra element representing the original tangle containing a loop around the pole by multiplying it to the right by $\frac{1}{\delta^2} \Xi^- e_n$. So suppressing the loop around the pole is equivalent to acting to the right by $z$. But recall we are working inside $M_n \otimes_{\mathcal{H}_n} F$, so acting to the right by $z$ on an element of $M_n$ is like acting to the left by $z$ on 1. So, in our representation, a loop around the pole is replaced by a multiplication by $r$. We next show that, if in the loop the crossing has the opposite sign, the loop can be removed at the cost of a factor $\frac{1}{\delta}$. First by the same tangle trick, such a loop can be removed at the cost of a factor $\frac{1}{\delta} \Xi^-$ where $\Xi^-$ is the following tangle.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The (0,0)-tangle $\Xi^-$}
\end{figure}

This tangle was introduced in [6], and as mentioned by the authors, this tangle $\Xi^-$ is not the inverse of $\Xi^+$. We show that multiplying a tangle by $\delta^{-1} \Xi^-$ is in the $C_n$-module $M_n \otimes_{\mathcal{H}_n} F$ a division by $r$. It suffices to show that the product $(\frac{1}{\delta^2} \Xi^- e_n)(\frac{1}{\delta^2} \Xi^+ e_n)$ acts to the right of $e_n$ like the identity. We have

\[ \left( \frac{1}{\delta^2} \Xi^- e_n \right) \left( \frac{1}{\delta^2} \Xi^+ e_n \right) = \frac{1}{\delta^3} \Xi^- \Xi^+ e_n, \]
We show $\Xi^- \Xi^+ = \delta^2$. This is a simple application of the first pole-related self-intersection relation, as shown on the figure below.

The “eight” at the bottom becomes a closed loop, hence a factor $\delta$. The “eight” at the top now has two self-intersections. These intersections are so that an application of Reidemeister’s move II is possible. Thus, you get another closed loop and the announced result. So, the product we considered is $\frac{1}{2} e_n$ and it indeed acts like the identity to the right on $e_n$. We conclude that a loop around the pole with a crossing of opposite sign as the one in Figure 1 can be removed at the cost of a division by $r$.

Using these preliminary remarks, it is now straightforward to see that the action of the generators $g_k$’s leaves the basis consisting of the elementary tensors $w_{\gamma, \ell} \otimes H_n$ invariant. The fact that we can multiply at the bottom by $g_1$ (resp $g_1^{-1}$) at the cost of a division by $r$ (resp a multiplication by $r$) allows us to easily change the extremities of the vertical strand that twists around the pole when these are not well positioned. This is for instance shown on the following example.

When computing the action by $g_3$ on the basis vector $w_{1,j}$, we use Reidemeister’s move III to move the crossing under the horizontal strand, then multiply the tangle at the bottom by $g_1^{-1}$ and simultaneously compensate this addition.
by a factor $r$. To finish, a simple use of the double twist relation, followed by Reidemeister’s move II allows node number 2 at the top to join node number 1 at the bottom with a vertical strand twisting around the pole below the horizontal strands. The final result is $g_3 \cdot \hat{w}_{i,j} = r \hat{w}_{i,j}$. We finish this construction section by describing one of the actions, namely the action by $g_1$ on $\hat{w}_{i,j}$ when $i \geq 3$. This is the most complicated action for our representation. Computing this action involves using the commuting relation a first time, then the double twist relation once, Reidemeister’s move III twice, then acting by $g_2^{-1}$ at the bottom at the cost of a multiplication by $r$ to get this tangle.

The work is not yet over. Indeed, in our basis, the horizontal strand always twists above the vertical twist. An important feature of the commuting relation is that it allows one to pull the bottom twist up and to draw the upper twist down, hence changing the order in which the horizontal strand and the vertical strand twist. In the process, if the horizontal strand that twists around the pole was over-crossing (resp under-crossing) the vertical strand that twists around the pole, it now under-crosses (resp over-crosses) it. So, using the commuting relation a second time, we get

We are now in a situation where the first two vertical strands over-cross the top horizontal strand. We need to transform the four crossings that are involved. These are pointed out on the diagram by circles. We do so by using the Kauffman skein relation. When all the under-crossings have been transformed
into over-crossings, we get the term $\hat{w}_{ij}$ of (1). The rest of the computations must be done step by step with patience. They lead to the result in (1).

We last show another role played by the element $\xi$. When acting by $g_1$ on $w_{12}$, one creates a pole-related self-intersection as in Fig. 13 of [6]. As shown on the same figure, this pole-related self-intersection can be replaced by $\frac{1}{3} \Xi$. From there, replacing $e_n$ by $\frac{e_{3}^2}{3}$, we get a multiplication to the right of $e_n$ by $\xi$. Hence the action by $g_1$ on $w_{12}$ is a multiplication by $r$.

3 Reducibility

3.1 Action on a proper invariant subspace

The following proposition is an easy but crucial statement about the Cohen-Wales representation. It precises the action on a proper invariant subspace when the representation is reducible.

**Proposition 1.** Let $\mathcal{U}$ be a proper invariant subspace of $V_n$. Then,

$$\forall 1 \leq i \leq n, \nu^{(n)}(e_i)(\mathcal{U}) = 0$$

Thus, the action on a proper invariant subspace is a Hecke algebra action.

**Proof.** Let $\mathcal{U}$ be a proper invariant subspace of $V_n$. Fix $i$ with $2 \leq i \leq n$. An action by $e_i$ on any element of the Cohen-Wales space is always proportional to the vector $w_{i-1,i}$. Similarly, an action by $e_1$ on any element of the Cohen-Wales space is always proportional to the vector $w_{12}$. Hence in the first case, if the action by $e_i$ on $\mathcal{U}$ is non-trivial, then the vector $w_{i-1,i}$ belongs to $\mathcal{U}$. And in the second case, if the action by $e_1$ on $\mathcal{U}$ is non-trivial, then the vector $w_{12}$ belongs to $\mathcal{U}$. But by construction (see the beginning of §2.2), the vectors $w_{s,t}$'s are all of the form $y e_3 e_{3,n}$ with $y$ a certain product composed of $g_k$'s and $g_k^{-1}$'s. Then, obviously if one of the $w_{s,t}$ belongs to $\mathcal{U}$, since $\mathcal{U}$ is invariant, $\mathcal{U}$ is then the whole space. This is in contradiction with $\mathcal{U}$ proper. So, the proposition holds.

The goal now is to study which irreps of $\mathcal{H}(D_n)$ can occur in the Cohen-Wales space. First we recall some basic representation theory of the Hecke algebra of type $D_n$ and study further the degrees of the irreps of $\mathcal{H}(D_n)$ that are less than $n^2 - n$, the degree of our representation $\nu^{(n)}$.

3.2 Degrees of the irreps of $\mathcal{H}(D_n)$

In this part, we assume that the Hecke algebra of type $D_n$ is semisimple and we study the degrees of its irreducible representations. We work over the field $F = \mathbb{Q}(l, r)$ which has characteristic zero. Up to some rescaling of the generators, our algebra $\mathcal{H}(D_n)$ is the algebra $\mathcal{H}_{2}(D_n)$ of [14]. If $\mathcal{H}(D_n)$ is semisimple, then by the proof of Theorem 1.5 of [14], we have $f_n(r^2) \neq 0$. Pallikaros defines
\[ f_n(r^2) \text{ in his definition 2.12 of } [30] \text{ as } \]
\[ f_n(r^2) = 2 \prod_{k=1}^{n-1} (1 + r^{2k}) \]

If \( f_n(r^2) \neq 0 \), then \( r^{2k} \neq -1 \) for every integer \( k \) with \( 1 < k < n - 1 \). We will make this assumption in the remainder of this paper. In the past forty years, many authors have studied the representation theory of the Hecke algebra of type \( D_n \) \([14, 30, 12, 31]\) to only cite a few of them. It seems the study finds its origin in the Canadian Ph.D. thesis of P.N. Hoefsmit \([12]\). Our work is based on the existing theories that classify the irreducible \( \mathcal{H}(D_n) \)-modules.

The main result that we use has been copied here from \([31]\). We use however the notations of our own paper.

**Theorem. (Hoefsmit), 1974** The modules \( S^{(\alpha,\beta)} \), where \((\alpha, \beta)\) runs over all unordered pairs of partitions such that \( \alpha \neq \beta \) and \(|\alpha| + |\beta| = n \) and, when \( n \) is even the modules \( S^{(\alpha,\alpha)}^+ \) and \( S^{(\alpha,\alpha)}^- \), where \( \alpha \) runs over all partitions such that \( 2|\alpha| = n \), form a complete set of non-isomorphic irreducible modules for \( \mathcal{H}(D_n) \).

This Theorem says the non-isomorphic irreducible \( \mathcal{H}(D_n) \) are indexed by unordered double partitions of \( n \). They are called Specht modules. To keep the notations lighter, we will sometimes write \( S^{\alpha,\beta} \) instead of \( S^{(\alpha,\beta)} \) and \( S^{\alpha,\alpha}^\pm \) (resp \( S^{(\alpha,\alpha)}^\pm \)) instead of \( S^{(\alpha,\alpha)}^\pm \) (resp \( S^{(\alpha,\alpha)}^- \)), that is we omit the parenthesis around the partitions. Since the double partitions are unordered, if \( (\lambda, \mu) \) is a double partition of \( n \), we can assume without loss of generality that \( |\lambda| \leq |\mu| \) and so when \( n \) is odd \( |\lambda| < \frac{n}{2} \). Moreover, the dimension of \( S^{(\lambda,\mu)} \) is given by the number of standard tableaux of shape \( (\lambda, \mu) \), except in the case when \( n \) is even and \( \lambda = \mu \) (and so \( 2|\lambda| = n \)). To describe the latter case, let’s introduce the notations of \([31]\). When \( \alpha \neq \beta \), the Specht modules \( S^{(\alpha,\beta)} \) are spanned by vectors \( v_L \) indexed by standard tableaux of shape \( (\alpha, \beta) \). When \( \alpha = \beta \), the Specht modules \( S^{(\alpha,\alpha)}^+ \) and \( S^{(\alpha,\alpha)}^- \) are respectively spanned by vectors

\[
\begin{align*}
 v_L^+ &= v_L + v_\sigma L \\
 v_L^- &= v_L - v_\sigma L
\end{align*}
\]

where \( L \) is a standard tableau of shape \( (\alpha, \alpha) \) and where \( \sigma \) is the map sending the standard tableau \( L = (L_\alpha, L_\beta) \) of shape \( (\alpha, \beta) \) to the tableau \( \sigma L = (L_\beta, L_\alpha) \) of shape \( (\beta, \alpha) \). Let’s take an example. Suppose \( n = 4 \). The standard tableaux of shape \((2),(2)\) are:

\[
\begin{align*}
 L_1 &= (\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}) & \sigma L_1 &= (\begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array}) \\
 L_2 &= (\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}) & \sigma L_2 &= (\begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array}) \\
 L_3 &= (\begin{array}{cc} 1 & 4 \\ 2 & 3 \end{array}) & \sigma L_3 &= (\begin{array}{cc} 2 & 3 \\ 1 & 4 \end{array})
\end{align*}
\]

We have \( \dim(S^{(2),(2)}^+) = \dim(S^{(2),(2)}^-) = 3 \) and \( S^{(2),(2)} = S^{(2),(2)}^+ \oplus S^{(2),(2)}^- \).

Recall the Cohen-Wales representation of \( CGW(D_n) \) has degree \( n(n-1) \), the
number of positive roots of a root system of type $D_n$. Our goal in this part is to find all the dimensions of the Specht modules that have dimension less than $n(n-1)$ for a given $n \geq 4$. We prove the following result.

**Theorem 8.** Let $n$ be an integer with $n \geq 4$. Assume that $H(D_n)$ and $H_{F,r^2}(n)$ are semisimple.

(i) Assume that $n \neq \{4, 8\}$. Then, the irreducible representations of $H(D_n)$ have degrees

$$1, n-1, n, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2} \text{ or degrees greater than } \frac{(n-1)(n-2)}{2}$$

(ii) The irreducible representations of $H(D_4)$ have degrees 1, 2, 3, 6 or 8.

(iii) The irreducible representations of $H(D_8)$ have degrees 1, 7, 14, 20, 21 or degrees greater than 21.

(iv) For sufficiently large $n$, an irreducible representation of $H(D_n)$ has degree

$$1, n-1, n, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}, \frac{n(n-1)}{2}, n(n-2)$$

or degree greater than or equal to $n(n-1)$.

**Proof.** Suppose $(\lambda, \mu)$ is a double partition of $n$ with $|\lambda| = k, |\mu| = n - k$ and $n \geq 2k$. We study the possible dimensions, depending on the value of $k$ and then $n$.

* If $k = 0$, $\lambda$ is the empty partition and $\mu$ is a partition of $n$. We want to count the number of standard Young tableaux of shape $\mu$. This number is the same as the dimension of the Specht module $S^\mu$, where $S^\mu$ denotes a class of irreducible $H_{F,r^2}(n)$-module for each partition $\mu$ of $n$. By Corollary 2 of [22], when $H_{F,r^2}(n)$ is semisimple, the irreps of $H_{F,r^2}(n)$ have degree $1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}$ or degrees greater than $\frac{(n-1)(n-2)}{2}$, except when $n = 4$ (resp $n = 8$) when their degrees are 1, 2, 3 (resp 1, 7, 14, 20, 21 or degree greater than 21).

* If $k = 1$, the Ferrers diagram of $\lambda$ is just one box. There are $n$ possible choices to fill it. Once this single box is filled, there are $(n-1)$ integers to fill in a standard tableau of size $(n-1)$. Notice that $\frac{n(n-1)(n-4)}{2} \geq n(n-1)$ as soon as $n \geq 6$. So, using the previous case, when $n \geq 6$ and $n \neq 9$, the only possible degrees are $n$ and $n(n-2)$. The case $n = 9$ is in fact not an exception since $9 \times 14 > 9 \times 8$. When $n = 5$, the possible degrees are 5, 10 and 15. Finally, when $n = 4$, the possible degrees are 4 and 8, so the case $n = 4$ is not exceptional.

* If $k = 2$, suppose first $n \geq 5$. There are two possible partitions for $\lambda$ and $\frac{n(n-1)}{2}$ ways to fill in the two boxes. Once this is achieved, there are $(n-2)$ integers to fill in a standard tableau of size $(n-2)$. Suppose first $n \neq 6$ and $n \neq 10$. When $n \geq 5$, we have $\frac{n(n-1)(n-3)}{2} \geq n(n-1)$, so the only possibility is to have a degree equal to $\frac{n(n-1)(n-3)}{2}$. Consequently also, the case
First, if \( n \) have dimensions of \( S \) and \( \mu \) and their two respective conjugates all have dimension less than \( 35 \). We have \( \dim(S^{(2,1),(2,1)\pm}) = \binom{5}{3} \times 2 \times 2 = 40 > 30 \)

Next, if \(|\lambda| = |\mu|\) but \( \lambda \neq \mu \), there are \( \binom{3}{2} = 3 \) ways to choose two tableaux out of three. First if \( (\lambda, \mu) = (3), (1, 1, 1) \), the dimension of \( S^{(\lambda, \mu)} \) is 20. Second if \( \lambda = (2, 1) \), the dimension is too big: 2 \times \binom{5}{3} = 40 > 30 = 6 \times 5 \). When \( n = 7 \), we have \( 2 \times \binom{7}{3} = 70 > 42 = 7 \times 6 \), so we must have \( \lambda \in \{ (3), (1^3) \} \) and \( \mu \in \{ (4), (1^4) \} \). The dimension of \( S^{(\lambda, \mu)} \) is then 35.

\* If \( k = 4 \) and \( n \geq 8 \), we have

\[
\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{6 \times 4} \geq n(n-1)
\]

Hence, except possibly when \( n = 8 \), there are no irreducible \( \mathcal{H}(D_n) \)-modules of dimension less than \( n(n-1) \). When \( n = 8 \), the Specht modules \( S^{(4),(4)\pm}, S^{(4),(4)\pm} \) and their two respective conjugates all have dimension 35 and there are the only irreducible \( \mathcal{H}(D_8) \)-modules of dimension less than 56.

\* If \( 5 \leq k \leq \frac{n}{2} \), the following inequality holds.

**Lemma 1.** For every integers \( k \) and \( n \) such that \( n \geq 10 \) and \( 5 \leq k \leq \frac{n}{2} \), we have

\[
\frac{1}{n(n-1)} \binom{n}{k} \geq 1
\]

**Proof of the Lemma.** The member to the left of the inequality is

\[
\frac{(n-2) \ldots (n-k+1)}{k \ldots 4 \cdot 6}
\]
In this fraction, there are \((k - 2)\) terms in the nominator and there are \((k - 2)\) terms in the denominator. Moreover, we have 
\(n \geq \frac{n}{2} + 2\), so that 
\(n \geq k + 2\). Further, we have 
\(n - k + 1 \geq 6\) as 
\(n \geq \frac{n}{2} + 5\). \(\square\)

When \(k < \frac{n}{2}\), the smallest possible dimension is \(\binom{n}{k}\), which by the lemma 
is greater than or equal to \(n(n - 1)\). Thus, it remains to inspect the case 
when \(k \geq 5\) and 
\(n = 2k\). In that case, the smallest possible dimension is 
\(\frac{1}{2}\binom{n}{k}\). But when 
\(n \geq 12\), we have 
\(n \geq \frac{n}{2} + 6\) so that 
\(n - \frac{n}{2} + 2 \geq 8\). Then,

\[
\frac{1}{2} \binom{n}{k} = \frac{(n - 2) \ldots (n - \frac{n}{2} + 2)(n - \frac{n}{2} + 1)}{\frac{n}{2} \ldots 5 \cdot 6} \geq 1
\]

and so

\[
\frac{1}{2} \binom{n}{k} \geq n(n - 1)
\]

And when \(n = 10\), this inequality still holds by a direct computation.

We conclude that when \(5 \leq k \leq \frac{n}{2}\), there does not exist any irreducible 
\(\mathcal{H}(D_n)\)-modules that have dimension less than \(n(n - 1)\).

In summary, we have found the following degrees in the generic case:

\[
1, n - 1, n, \frac{n(n - 3)}{2}, \frac{(n - 1)(n - 2)}{2}, \frac{n(n - 1)}{2}, \frac{n(n - 2)}{2}, n(n - 2)
\]

If the study is complete for the non-zero \(k\)'s, it is incomplete when 
\(k = 0\). Indeed, we don’t have a complete list of the degrees of the irreps of 
\(\mathcal{H}_{F,r}(n)\) when these degrees are between 
\(\frac{(n - 1)(n - 2)}{2}\) and \(n(n - 1)\). We only know that 
when \(n\) is large enough, an irreducible 
\(\mathcal{H}_{F,r}(n)\)-module either belongs to 
\(R_n(3)\) or has dimension greater than \(n^3\). This result comes from Theorem 5 of [15], 
applied with \(m = 3\). James’ work deals with the irreducible representations of 
the symmetric group \(Sym(n)\), but can be applied to the irreducible representations 
of \(\mathcal{H}_{F,r}(n)\). Indeed, in characteristic zero when 
\(\mathcal{H}_{F,r}(n)\) is semisimple, the degrees of the irreps of 
\(Sym(n)\) are the same as the degrees of the irreps of 
\(\mathcal{H}_{F,r}(n)\). James denotes by 
\(R_n(m)\) the classes of irreducible Specht modules 
\(S^\mu\) with \(\mu \geq n - m\), where \(\mu = (\mu_1, \mu_2, \ldots)\) is a partition of \(n\), or their conjugates. A Specht module 
\(S^\mu\) belongs to 
\(R_n(3)\) if the first row or the first column of the Ferrers diagram of the partition \(\mu\) contains 
\(n - 3, n - 2, n - 1\) or \(n\) boxes. A straightforward application of the Hook formula (see for instance [34]) shows that

\[
M \in R_n(1) \quad \Rightarrow \quad \dim M \in \{1, n - 1\} \quad (a)
\]

\[
M \in R_n(2) \setminus R_n(1) \quad \Rightarrow \quad \dim M \in \left\{ \frac{n(n - 3)}{2}, \frac{(n - 1)(n - 2)}{2} \right\} \quad (b)
\]

\[
M \in R_n(3) \setminus R_n(2) \quad \Rightarrow \quad \dim M \in \left\{ \frac{n(n - 1)(n - 5)}{6}, \frac{n(n - 2)(n - 4)}{3}, \frac{(n - 1)(n - 2)(n - 3)}{6} \right\} \quad (c)
\]

For (c), we give below the Ferrers diagrams and the hook lengths. The first row of each diagram contains \((n - 3)\) boxes.
The dimensions of the respective Specht modules are obtained by taking $n!$ over the product of the hook lengths. When $n \geq 11$, the three quotients in (c) are greater than or equal to $n(n - 1)$. Then, point (iv) of Theorem 8 holds. This settles the Theorem.

### 3.3 Existence of a one-dimensional invariant subspace

In this part we investigate the existence of a one-dimensional invariant subspace inside $V_n$. We show the only values of $l$ and $r$ for which that happens are those such that $l = \frac{1}{r}$. Assume such a space exists. Let $u$ be a spanning vector and so there exists scalars $\lambda_1, \ldots, \lambda_n$ such that $\nu_i(u) = \lambda_i u$ for each $i$. Further, since $(\nu_i^2 + m \nu_i)(u) = u$, we get $\lambda_1 \in \{ r, -\frac{1}{r}\}$. We show that all the $\lambda_i$'s must in fact be equal to $r$. First, all the $\lambda_i$'s are equal to say $\lambda$. Indeed, applying the braid relation $g_1g_3g_1 = g_3g_1g_3$, we get $\lambda_1 = \lambda_3$ and applying further the same braid relation on nodes 2, $\ldots$, $n$, we get that all the $\lambda_i$'s with $2 \leq i \leq n$ are equal. Let's write a general form for $u$ as follows.

$$u = \mu_{ij} w_{ij} + \sum_{1 \leq i < j \leq n} \mu_{ij} \hat{w}_{ij}$$

**Lemma 2.** Let $i$ be an integer with $2 \leq i \leq n$. If $\nu_i(u) = \lambda u$, then $\mu_{i,j} = \lambda \mu_{i-1,j}$ for every $j > i$ and $\mu_{k,i} = \lambda \mu_{k,i-1}$ for every $k < i - 1$.

**Proof.** Let's for instance prove the first equality. It suffices to look at the term in $w_{i-1,j}$ (resp $\hat{w}_{i-1,j}$) in $\nu_i(u) = \lambda u$. For all $j > i$, an action by $g_i$ creates a term in $w_{i-1,j}$ (resp $\hat{w}_{i-1,j}$) only when it acts on $w_{i,j}$ (resp $\hat{w}_{i,j}$). Hence the result.

As a corollary, if one of the $\mu_{ij}$'s (resp $\mu_{ij}$'s) is zero, then all the $\mu_{ij}$'s (resp $\mu_{ij}$'s) are zero. We show that it is impossible to have all the $\mu_{ij}$'s equal to zero. Indeed, if so, then all the $\mu_{ij}$'s are non-zero. Acting with $\nu_1$ on $w_{1,j}$ creates a term in $\hat{w}_{2,j}$. Moreover, as shown by the equations (1) - (7), this is the only way to create a term in $\hat{w}_{2,j}$ when acting with $\nu_1$. This yields a contradiction. Thus, all the $\mu_{ij}$'s are non-zero. From there, it is easy to see that $\lambda$ must be equal to $r$, not $-\frac{1}{r}$. Indeed, look at an action of $g_1$ on $\hat{w}_{34}$ and notice this is the only way to get a term in $\hat{w}_{34}$ when acting with $\nu_1$. Since the term $\hat{w}_{34}$ is multiplied by $r$, we see that $\lambda$ must be equal to $r$. The goal next is to find the relationship between the hat coefficients and the non-hat coefficients. For that, we look at the coefficient of $w_{1,j}$ for $j \geq 3$ in $\nu_1(u) = r u$. We get

$$r \mu_{1,j} = \mu_{2,j} - m \mu_{1,j} - m \sum_{i=3}^{j-1} s^{i-3} \mu_{i,j} - m \sum_{k=j+1}^{n} s^{k-5} \mu_{j,k}$$
By simplifying this expression and using the relations between the coefficients, we derive
\[
\frac{1}{r} \mu_{1,j} = r \hat{\mu}_{1,j} - m \sum_{i=3}^{j-1} r^{i-3} \hat{\mu}_{1,j} - m \sum_{k=j+1}^{n} r^{k-5} r^{k-1} \hat{\mu}_{1,j}
\]
After evaluating the two sums of powers of \( r \) and simplifying, we obtain
\[
\mu_{1,j} = r^{2n-4} \hat{\mu}_{1,j}
\]
(25)
Let’s now look at the coefficient of \( \hat{w}_{12} \) in \( \nu_1(u) = ru \). We have
\[
\left(\frac{1}{1-r}\right) \mu_{12} + m \sum_{j=3}^{n} r^{j-3} \mu_{2,j} + m \sum_{j=3}^{n} r^{j-4} \mu_{1,j} + m^2 \left(\frac{1}{r^{17}} + \frac{1}{r^9}\right) \sum_{j=1}^{n} r^{2i} \sum_{j=i+1}^{n} r^{2j} \mu_{12} = 0
\]
where we used the relation \( \hat{\mu}_{ij} = r^{i-1} r^{j-2} \hat{\mu}_{12} \). Also, we have
\[
\mu_{2,j} = r^{2n-3} \mu_{1,j} = r^{2n-5+j} \hat{\mu}_{12}
\]
\[
\mu_{1,j} = r^{2n-4} \mu_{1,j} = r^{2n-6+j} \hat{\mu}_{12},
\]
where the equalities to the left hold by (25). After evaluating the sums, simplifying and dividing by \( \mu_{12} \) which is known to be a non-zero scalar, all the terms in (26) simplify nicely. It yields
\[
l = \frac{1}{r^{4n-7}}
\]
Conversely, suppose \( l = \frac{1}{r^{4n-7}} \) and let
\[
u(u) = \sum_{1 \leq i < j \leq n} r^{i+j} \hat{w}_{ij} + r^{2n-4} \sum_{1 \leq i < j \leq n} r^{i+j} w_{ij}
\]
It is a tedious but straightforward verification that the \( g_k \)'s act on \( u \) by multiplying it by \( r \). Theorem 3 is thus proven.

3.4 Existence of an irreducible \((n-1)\)-dimensional invariant subspace

The goal of this part is to prove Theorem 4 announced in the introduction. We still assume that \( \mathcal{H}(D_n) \) and \( \mathcal{H}_{F,\mathbb{R}}(n) \) are semisimple. By §3.2, except when \( n = 4 \) and when \( n = 6 \), there are exactly two inequivalent irreducible representations of \( \mathcal{H}(D_n) \) of degrees \((n-1)\). In [29], page 53, we provide matrix representations \( (M_i)_{1 \leq i \leq n-1} \) for \( S^{(n-1,1)} \) (resp \( (N_i)_{1 \leq i \leq n-1} \) for \( S^{(2,1^{n-2})} \)) when we work with \( \mathcal{H}_{F,\mathbb{R}}(n) \). To get a matrix representation \( (H_i)_{1 \leq i \leq n} \) for \( S^{(0),(n-1,1)} \) (resp \( (K_i)_{1 \leq i \leq n} \) for \( S^{(0),(2,1^{n-2})} \)), it suffices to take \( H_{i+1} = M_i \) for all \( i \) with
\[ 1 \leq i \leq n - 1 \text{ and } H_1 = H_2 \text{ (resp } K_{i+1} = N_i \text{ for all } i \text{ with } 1 \leq i \leq n - 1 \text{ and } K_1 = K_2). \] We show that it is impossible to have a basis \( v_1, \ldots, v_{n-1} \) of vectors of the Cohen-Wales space such that

\[ \nabla \]

- (a) \( \nu_1(v_1) = r \cdot v_1 \)
- (b) \( \nu_1(v_2) = -r \cdot v_1 - \frac{1}{r} \cdot v_2 \)
- (c) \( \nu_1(v_t) = -\frac{1}{r} \cdot v_t \quad \forall t \geq 3 \)

\[ i \geq 2 \]

- (d) \( \nu_i(v_1) = -r \cdot v_{i-1} - \frac{1}{r} \cdot v_i \)
- (e) \( \nu_i(v_{i-1}) = r \cdot v_{i-1} \)
- (f) \( \nu_i(v_{i-2}) = -\frac{1}{r} \cdot (v_{i-2} + v_{i-1}) \)
- (g) \( \nu_i(v_i) = -\frac{1}{r} \cdot v_i \quad \forall t \notin \{i, i-1, i-2\} \)

In other words, the Specht module \( S^{(0),(2,n-2)} \) cannot occur in the Cohen-Wales space. First, we claim that for \( n \geq 8 \), the result is obvious. Indeed, we have by equation (g) of (\nabla\)

\[ \forall t \geq 4, \nu_t(v_1) = -\frac{1}{r} \cdot v_1 \quad (27) \]

Then, for every \( t \geq 4 \), all the terms in \( v_1 \) must have indices starting or ending in \( t-1 \) or \( t \). This is not possible as soon as \( n \geq 8 \). Thus, it remains to deal with the cases \( n \in \{4, 5, 6, 7\} \). When \( n = 7 \), the contradiction comes almost immediately as by (27), we must have \( v_1 = \lambda_{46} w_{46} + \lambda_{45} w_{45} \). But then \( \nu_7(v_1) \neq -\frac{1}{r} v_1 \).

When \( n = 6 \), we have by the same arguments as above

\[ v_1 = \lambda_{35} w_{35} + \lambda_{45} w_{45} + \lambda_{46} w_{46} \]

By \( \nu_4(v_1) = -\frac{1}{r} v_1 \), we can reduce further the expression to the first three terms:

\[ v_1 = \lambda_{35} w_{35} + \lambda_{45} w_{45} \]

Then acting with \( \nu_6 \) closes the case. When \( n = 5 \), by (27) with \( t \in \{4, 5\} \), we have

\[ v_1 = \lambda_{45} w_{45} + \lambda_{34} w_{34} + \lambda_{35} w_{35} \quad (28) \]

Apply Lemma 2 with \( \lambda = -\frac{1}{r} \) and \( i = 5 \) to get \( \lambda_{35} = 0 \) and \( \lambda_{34} = -\frac{1}{r} \lambda_{34} \).

Apply again Lemma 2 with \( \lambda = -\frac{1}{r} \) and \( i = 4 \) to further get \( \lambda_{45} = -\frac{1}{r} \lambda_{35} \).

Because the three coefficients that are involved are thus related, they are all non-zero. By the linearity in the \( \nu_i \)'s in the relations (\nabla\), we can set without loss of generality \( \lambda_{34} = 1 \). Then,

\[ v_1 = w_{34} - \frac{1}{r} w_{35} + \frac{1}{r^2} w_{45} \quad (29) \]

We can now conclude. By \( v_2 = -r \cdot \nu_3(v_1) - v_1 \), there is no term in \( w_{15} \) in \( v_2 \). But there is a non-zero term in \( w_{15} \) in \( \nu_2(v_2) \), namely \( w_{15} \). This contradicts \( \nu_2 v_2 = -r \cdot v_1 - \frac{1}{r} v_2 \) and finishes the case \( n = 5 \).

Let’s deal with the case \( n = 4 \).

First, by (27) with \( t = 4 \), there are no terms in \( \hat{w}_{34} \) and \( \hat{w}_{12} \) in \( v_1 \). Hence, a general form for \( v_1 \) is

\[ v_1 = \lambda_{34} w_{34} + \lambda_{13} \hat{w}_{13} + \lambda_{14} \hat{w}_{14} + \lambda_{23} \hat{w}_{23} + \lambda_{24} \hat{w}_{24} \quad (30) \]
We next apply Lemma 2 with the relations \( \nu_4(v_1) = -\frac{1}{r} v_1 \) and \( \nu_2(v_1) = r v_1 \) to get the set of relations
\[
\begin{align*}
\lambda_{23} & = r \lambda_{13} \\
\lambda_{14} & = -\frac{1}{r} \lambda_{13} \\
\lambda_{24} & = -\lambda_{13}
\end{align*}
\]
Consequently also, at least one of \( \lambda_{13} \) or \( \lambda_{13} \) must be non-zero. Otherwise, \( v_1 \) would be a multiple of \( w_{34} \). Then \( v_2 = -r \lambda(w_{24} + m w_{23} + r w_{34}) \) for some non-zero scalar \( \lambda \). This expression is not compatible with \( \nu_3(v_2) = r v_2 \) since the term in \( w_{23} \) in \( \nu_3(v_2) \) is \( -m r^2 \lambda - \frac{m \lambda}{r} \lambda \) and \( \frac{m \lambda}{r} \lambda \neq 0 \). Further, we see on the defining relations for the representation that (5) is the only way to get a term in \( w_{24} \) when acting with \( v_1 \) on \( v_1 \). This fact together with (a) implies that \( \lambda_{14} = r \lambda_{24} \). Thus, both \( \lambda_{13} \) and \( \lambda_{13} \) are non-zero. Without loss of generality, set \( \lambda_{13} = 1 \). Then, by
\[
v_2 = -r \nu_3(v_1) - v_1,
\]
we see that the coefficient of \( w_{12} \) in \( v_2 \) is \( -r \). Thus, the coefficient of \( w_{12} \) in \( -r v_1 - \frac{1}{r} v_2 \) is 1, while the coefficient of \( w_{12} \) in \( \nu_2(v_2) \) is \( -r^2 \). Since \( r^2 \neq -1 \), this contradicts equality (d) with \( i = 2 \). So, we are done with all the cases and conclude that the Specht module \( S^{(0),(2,1^{n-2})} \) cannot occur in the Cohen-Wales space. We now show that the conjugate Specht module \( S^{(0),(n-1,1)} \) can occur in the Cohen-Wales space for the values \( l = \frac{1}{\sqrt{r^2-1}} \). If in the Cohen-Wales space there exists an irreducible invariant subspace isomorphic to \( S^{(0),(n-1,1)} \), there must exist a basis \( v_1, \ldots, v_{n-1} \) such that the \( v_i \)'s satisfy the relations
\[
\begin{align*}
(a') \ & \nu_1(v_1) = -\frac{1}{r} v_1 \\
(b') \ & \nu_1(v_2) = \frac{1}{r} v_1 + v_2 \\
(c') \ & \nu_1(v_1) = r v_t \quad \forall t \geq 3 \quad i \geq 2 \\
(d') \ & \nu_i(v_1) = \frac{1}{r} v_{i-1} + r v_i \\
(e') \ & \nu_i(v_{i-1}) = -\frac{1}{r} v_{i-1} \\
(f') \ & \nu_i(v_{i-2}) = r (v_{i-2} + v_{i-1}) \\
(g') \ & \nu_i(v_t) = r v_t \quad \forall t \notin \{i, i-1, i-2\}
\end{align*}
\]
The relations (\( \Delta \)) are the conjugate relations of (\( \nabla \)) where \( r \) has been replaced by \( -\frac{1}{r} \). We show that these relations force \( l = \frac{1}{\sqrt{r^2-1}} \). Let’s first use (\( c' \)) with \( i = 2 \) to see that in \( v_1 \), there are no terms in \( w_{s,t} \) with \( s \geq 3 \) and there is no term in \( w_{12} \). So,
\[
v_1 = \sum_{i=2}^{n} \mu_{1t} w_{1t} + \sum_{t=3}^{n} \mu_{1t} w_{1t} + \sum_{t=3}^{n} \mu_{2t} w_{2t} + \sum_{t=3}^{n} \mu_{2t} w_{2t} \quad (31)
\]
To get the explicit expression for \( v_1 \), it suffices now to juggle with equations (2) – (5). Look at the term in \( w_{12} \) in \( \nu_1(v_1) = -\frac{1}{r} v_1 \) and get
\[
r \mu_{12} + m \sum_{j=3}^{n} r^{j-3} \mu_{2j} - m \sum_{j=3}^{n} r^{j-4} \mu_{1j} = -\frac{1}{r} \mu_{12}
\]
Using $\mu_{1j} = -\frac{1}{r} \hat{\mu}_{2j}$, now derive

$$\mu_{12} = -\frac{m}{r} \sum_{t=3}^{n} r^{t-3} \hat{\mu}_{2t}$$  \hspace{1cm} (32)$$

For $t \geq 4$, we have $\nu_t(v_1) = r v_1$. By Lemma 2, it follows that $\hat{\mu}_{2t} = r \hat{\mu}_{2,t-1}$ for every $t \geq 4$. Using these relations in (32) yields

$$\mu_{12} = \frac{\mu_{23}}{r^2} (r^{2n-4} - 1)$$  \hspace{1cm} (33)$$

Look at the coefficient of $w_{13}$ in $\nu_1(v_1) = -\frac{1}{r} v_1$ and get using defining relations (4) and (5)

$$\mu_{23} = m \mu_{13} - \frac{1}{r} \hat{\mu}_{13}$$

Replace $\mu_{13} = -\frac{1}{r} \hat{\mu}_{23}$ and $\hat{\mu}_{13} = -r \hat{\mu}_{23}$ (by (e') with $i = 2$ and Lemma 2) to get

$$\mu_{23} = \frac{1}{r^2} \hat{\mu}_{23}$$  \hspace{1cm} (34)$$

All the coefficients in $v_1$ are now determined by $\hat{\mu}_{23}$. Setting $\hat{\mu}_{23} = 1$, we get the expression of $v_1$ given in Theorem 4:

$$v_1 = (r^{2n-6} - \frac{1}{r^2}) w_{12} + \sum_{j=3}^{n} r^{j-5} \left( (w_{2j} - r w_{1j}) + r^2 (w_{2j} - r w_{1j}) \right)$$  \hspace{1cm} (35)$$

Once $v_1$ is known, all the $v_i$'s for $2 \leq i \leq n - 1$ are recursively determined by the formula $(f')$. At this point, it is tempting to look at the coefficient of $w_{12}$ in $\nu_2(v_1) = -\frac{1}{r} v_1$. However, the terms in $\frac{1}{r} v_1$ simplify and this relation appears to be a tautology. Thus, we cannot bypass involving the basis vector $v_2$. In fact, we show that the relations involving $v_1$ and $v_2$ are enough to force a relationship between the parameters $l$ and $r$. It suffices to look at the term in $w_{12}$ in $(d')$ with $i = 2$. We derive

$$l = \frac{1}{r^{2n-7}}$$  \hspace{1cm} (36)$$

Using inductively $(f')$ with (35) and replacing $l$ by its value, we obtain the formulas of Theorem 4.

To finish proving the necessary condition of Theorem 4, we still need to deal with the special case $n = 6$. When $n = 6$ there are two more inequivalent irreducible representations of $\mathcal{H}(D_6)$ of degrees 5. The corresponding double partitions are $((0), (3, 3))$ and $((0), (2, 2, 2))$. We show that it is impossible to have an irreducible 5-dimensional invariant subspace that is isomorphic to $S^{(0), (3, 3)}$ or to $S^{(0), (2, 2, 2)}$. First we show the following lemma.
Lemma 3. Let \( n \) be an integer with \( n \geq 5 \).

(i) If in \( V_n \) there exists an irreducible invariant subspace that is isomorphic to the Specht module \( S^{(0),(n-2,2)} \), then \( l = \nu^3 \). When \( n = 5 \), it is unique and it is spanned over \( \mathbb{Q}(l, r) \) by the vectors

\[
v_4 = r (w_{14} + r^2 \overline{w}_{14}) - (w_{24} + r^2 \overline{w}_{24}) + r(w_{23} + r^2 \overline{w}_{23}) - r^2 (w_{13} + r^2 \overline{w}_{13})
\]  

\[
v_5 = g_5 \cdot v_4
\]

\[
v_1 = g_1 \cdot v_4 - r \cdot v_4
\]

\[
v_2 = g_3 \cdot v_5 - r \cdot v_5
\]

\[
v_3 = g_4 \cdot v_2 - r \cdot v_2
\]

(ii) In \( V_n \), there does not exist any irreducible invariant subspace that is isomorphic to the Specht module \( S^{(0),(2,2,1^{n-4})} \).

Proof of the Lemma. In [20], we found matrix representations \( (P_i)_{1 \leq i \leq 4} \) and \( (Q_i)_{1 \leq i \leq 4} \) of degree 5 for respectively \( S^{(3,2)} \) and \( S^{(2,2,1)} \). This is Fact 1 page 77 of [20]. These are matrix representations of \( \mathcal{H}_{r,s}(5) \). To get matrix representations of \( \mathcal{H}(D_5) \), take \( S_1 = S_2 = P_1 \) and \( S_i = P_{i-1} \) (resp \( T_1 = T_2 = Q_1 \) and \( T_i = Q_{i-1} \)) for each \( i \in \{3, 4, 5\} \).

Let’s prove (i). We will need to use the branching rule. Branching rules for Hecke algebras of type \( D_n \) are stated in [13]. Precisely, we use the results of Theorems 2.5 and 2.6 and Corollary 2.8. Jun Hu studies the decompositions into irreducible modules of the socle of the restriction of each irreducible \( \mathcal{H}(D_n) \)-representation to \( \mathcal{H}(D_{n-1}) \), that is for every irreducible \( \mathcal{H}(D_n) \)-module \( D_i \) he describes \( \text{socle}(D_{\downarrow\mathcal{H}(D_{n-1})}) \). When we assume that \( \mathcal{H}(D_n) \) is semisimple, the socle of a \( \mathcal{H}(D_n) \)-module is the module itself. Suppose \( W \) is an irreducible invariant subspace of \( V_n \) that is isomorphic to \( S^{(0),(n-2,2)} \). Then, by the branching rule, the restriction of \( W \) to \( \mathcal{H}(D_5) \) is isomorphic to a direct sum of Specht modules with one of the summands being \( S^{(0),(3,2)} \). The latter Specht module is obtained by \( (n-5) \) successive removals of one box on the first row of the Ferrers diagram of the partition \( (n-2, 2) \). Then, there exists in \( W \) a family of five linearly independent vectors \( (v_i)_{1 \leq i \leq 5} \) such that the action by the \( g_i \)’s with \( 1 \leq k \leq 5 \) on these vectors is given by the matrices \( S_i \)’s that were introduced above. We work with these matrices to derive the results of (i). We read on the matrices \( S_1, S_2 \) and \( S_4 \) that

\[
g_i(v_4) = -\frac{1}{r} v_4 \quad \forall i \in \{1, 2, 4\}
\]  

These relations simplify greatly the shape of \( v_4 \) and we immediately have

\[
v_4 = \overline{w}_{13} w_{13} + \overline{w}_{14} w_{14} + \overline{w}_{23} w_{23} + \overline{w}_{24} w_{24}
\]  

By the same relations (42) and using Lemma 2, the hat-coefficients are all related, and so are the non-hat coefficients. Moreover, by looking at the coefficient of \( \overline{w}_{24} \) in (42) with \( i = 1 \), the non-hat coefficients are related to the hat.
Look at the term in \( \hat{r}_r \) must be related in a certain way. Combining the uniqueness part when \( l \leq m \leq n \).

Up to a reordering of the terms, this is formula (37) in the statement of Lemma 3. Once \( v_4 \) is known, the vectors \( v_1, v_2, v_3 \) and \( v_5 \) are then uniquely determined by the formulas (38)–(41) which follow after a glance at the matrices \( S_i \)'s. The uniqueness part when \( n = 5 \) is then established. Further, we show that \( l \) and \( r \) must be related in a certain way. Combining (39) and the relation \( \nu_1(v_1) = r v_1 + v_4 \), we get

\[
\nu_1 \nu_3(v_4) - r \nu_1(v_4) = r \nu_3(v_4) + (1 - r^2) v_4 \quad (44)
\]

Look at the term in \( \hat{w}_{12} \) in this expression. First, we compute

\[
\nu_3(v_4) = r^2(w_{14} + r^2 \hat{w}_{14}) - (w_{34} + r^2 \hat{w}_{34}) + \frac{r^5}{l} w_{23} + r^4 \hat{w}_{23} + m r^2(w_{13} + r^2 \hat{w}_{13}) - r^2(w_{12} + r^2 \hat{w}_{12}) \quad (45)
\]

Then we use defining relations (1), (4), (5), (6) of the representation to get

\[
r^2 m - r^2 m^2 \left( \frac{1}{r} + r \right) + \frac{r^5}{l} m + m r^2 \frac{m}{r} - r^4 \left( m - m r + m r - r^2 \frac{m}{r} \right) = -r^5
\]

This expression simplifies to yield \( l = r^3 \). This finishes the proof of point (i).

Let’s prove (ii). Suppose \( \mathcal{W} \) is an irreducible invariant subspace of \( V_n \) that is isomorphic to \( S^{(0)}(2,2,1^{n-4}) \). Then, by the branching rule, \( S^{(0)}(2,2,1) \) is a component of \( \mathcal{W} \downarrow_{H(D_5)} \). Hence there exists linearly independent vectors \( v_1, \ldots, v_5 \) so that the actions by \( g_1, \ldots, g_5 \) on these vectors is given by the matrices \( T_1, \ldots, T_5 \) that were introduced at the beginning of the proof. We show the relations force \( v_1 = 0 \), hence a contradiction and the result. Denote the coefficients of \( v_1 \) by \( \lambda_{i,j} \). From

\[
\begin{align*}
g_2 \cdot v_1 &= -\frac{1}{r} v_1 + v_4 \\
g_3 \cdot v_4 &= v_1 - \frac{1}{r} v_4
\end{align*}
\]

derive

\[
g_3 g_2 \cdot v_1 + \frac{1}{r} g_3 \cdot v_1 + \frac{1}{r} g_2 \cdot v_1 = (1 - \frac{r}{r^2}) v_1 \quad (46)
\]

Notice

\[
r^2 + 1 + \frac{1}{r^2} = 0 \iff r = \frac{\pm 1 \pm i \sqrt{3}}{2}
\]

This is impossible when \( (r^2)^3 \neq 1 \). So (46) implies

\[
\overline{\lambda_{i,j}} = 0 \quad \text{for all } i \geq 4 \text{ and all } j \geq 5
\]

Now, use the same trick a second time with

\[
\begin{align*}
g_4 \cdot v_1 &= -\frac{1}{r} v_1 + v_4 \\
g_3 \cdot v_4 &= v_1 - \frac{1}{r} v_4
\end{align*}
\]
and derive
\[ \lambda_{1,j} = 0 \quad \text{for all } j \geq 5 \]
We will keep making \( v_1 \) lighter. For now,
\[ v_1 = \lambda_{12} w_{12} + r \lambda_{12} w_{13} + \lambda_{14} w_{14} + \sum_{j=3}^{n} \lambda_{2,j} w_{2,j} + r \sum_{j=4}^{n} \lambda_{2,j} w_{3,j} \]
where we also used the relation \( g_3. v_1 = rv_1 \) together with Lemma 2. Notice on the matrices that
\[ g_2. v_1 = g_4. v_1 \] (47)
It follows that
\[ \lambda_{2,j} = 0 \quad \text{for all } j \geq 5 \]
Now \( v_1 \) reduces to
\[ v_1 = \lambda_{12} w_{12} + r \lambda_{12} w_{13} + \lambda_{14} w_{14} + \lambda_{23} w_{23} + \lambda_{24} w_{24} + r \lambda_{24} w_{34} \]
By looking at the term in \( w_{13} \) in (47), we further get
\[ \lambda_{23} = \lambda_{14} \] (48)
And by looking at the term in \( w_{14} \) in (47), we get
\[ \lambda_{24} = r \lambda_{12} - m \lambda_{14} \] (49)
We show that \( \lambda_{12} = 0 \). Below, we write the relations that we use and the relations that they imply on the coefficients. We write \( \lambda_{s,t}^{(1)} \) for the coefficient of \( w_{s,t} \) in \( v_1 \). So \( \lambda_{s,t}^{(1)} \) is simply \( \lambda_{s,t} \). We have
\[
\begin{align*}
(\mathcal{R}) & \quad v_2 = g_5. v_1 \quad \implies \quad \lambda_{12}^{(2)} = r \lambda_{12}^{(1)} \\
& \quad v_3 = g_4. v_2 + \frac{1}{r} v_2 \quad \implies \quad \lambda_{12}^{(3)} = (1 + r^2) \lambda_{12}^{(1)} \\
& \quad v_4 = g_4. v_1 + \frac{1}{r} v_1 \quad \implies \quad \lambda_{12}^{(4)} = \left( r + \frac{1}{r} \right) \lambda_{12}^{(1)} \\
& \quad v_5 = g_5. v_4 \quad \implies \quad \lambda_{12}^{(5)} = \left( 1 + r^2 \right) \lambda_{12}^{(1)} 
\end{align*}
\]
Next, we read on the third column of the matrix \( T_5 \) that
\[ g_5. v_3 = \frac{1}{r} v_1 + v_2 - \frac{1}{r} v_3 - \frac{1}{r^2} v_4 - \frac{1}{r} v_5 \] (50)
We look at the coefficient of \( w_{12} \) in (50) and we use (\( \mathcal{R} \)). We obtain
\[ 2(r + \frac{1}{r}) + r^3 + \frac{1}{r} = 0 \quad \text{or} \quad \lambda_{12}^{(1)} = 0 \] (51)

29
Let’s solve the equation in \( r \). With \( X = \frac{1}{r} + r \), the equation is equivalent to

\[
X^3 - X = 0
\]

From before, we know that \( X = 1 \) or \( X = -1 \) are impossible, as \((r^2)^3\) would then be 1. Also, \( X = 0 \) is impossible since it leads to \( r^2 = -1 \), which is excluded. Then, we are forced to have \( \lambda_{12} = 0 \) where we forgot the index (1) to conform to the notations of the beginning. Now, plug back (48) and (49) into the expression for \( v_1 \) and get the newer and simpler expression

\[
v_1 = \lambda_{23}(w_{14} + w_{23}) - m \lambda_{23}(w_{24} + r^{-1} w_{34})
\]  

(52)

This is enough to conclude. Indeed, by looking at the terms in \( \hat{w}_{24} \) and \( \hat{w}_{13} \) in the relation \( g_1 \cdot v_1 = g_4 \cdot v_1 \), we get the respective equations

\[
\begin{aligned}
\lambda_{23} - (1 + m^2 + m^2 r^2) \lambda_{23} &= 0 \\
\lambda_{23} - (1 + m^2) \lambda_{23} &= 0
\end{aligned}
\]

Then, all the coefficients in \( v_1 \) are zero, a contradiction. Therefore, there does not exist any irreducible invariant subspace in \( V_n \) that is isomorphic to the Specht module \( S^{(0),(2,2,1^{n-4})} \) and point (ii) of the lemma is proven. Let’s go back to the proof of Theorem 4. Suppose \( W \) is an irreducible 5-dimensional subspace of \( V_6 \) that is isomorphic to \( S^{(0),(2,2,2)} \). Then, applying the branching rule yields

\[
W \downarrow \mathcal{H}(D_6) \cong S^{(0),(3,2)}
\]

Then, \( W \) is spanned over \( F \) by vectors \( v_1, \ldots, v_5 \) given in equations (37) – (41) of point (i) of Lemma 3. A quick inspection at these vectors shows that node number 6 does not appear in them. However, when acting with \( v_6 \) on \( v_5 \), one creates terms that end in node number 6. This is in contradiction with the fact that \( W \) is spanned by the \( v_i \)'s, \( 1 \leq i \leq 5 \). We conclude that there does not exist any irreducible invariant subspace of \( V_6 \) that is isomorphic to \( S^{(0),(3,3)} \). Suppose that there exists a 5-dimensional irreducible invariant subspace \( W \) of \( V_6 \) that is isomorphic to \( S^{(0),(2,2,2)} \). Then,

\[
W \downarrow \mathcal{H}(D_6) \cong S^{(0),(2,2,1)}
\]

There exists vectors \( v_1, \ldots, v_5 \) of \( W \) such that the action by \( g_1, \ldots, g_5 \) on these vectors is given by the matrices \( T_i \)'s, \( 1 \leq i \leq 5 \). We get the same contradiction as in the proof of Lemma 3 point (ii). We conclude that there does not exist any irreducible 5-dimensional invariant subspace of \( V_6 \) that is isomorphic to \( S^{(0),(2,2,2)} \), and so out of the four irreducible representations of \( \mathcal{H}(D_6) \) of degree 5, only one of them can occur in the Cohen-Wales space \( V_6 \) and this when \( l = \frac{1}{2} \).

The necessary condition of Theorem 4 is now entirely proven for \( n \geq 5 \). Conversely, suppose \( l \) and \( r \) are related as in (36) and define vectors \( v_i \)'s, \( 1 \leq i \leq n - 1 \) as in Theorem 4. Clearly, these vectors are linearly independent and we
can check that they satisfy all the relations \((\Delta)\). Then, they span an irreducible \((n-1)\)-dimensional invariant subspace inside \(V_n\). This ends the proof of Theorem 4 in the case when \(n \geq 5\). In the case when \(n = 4\), as seen in \(\S\)3.2, there are four more non-isomorphic irreducible \(\mathcal{H}(D_4)\)-modules of dimension 3, namely \(S^{(2,2)}_+\) and \(S^{(2,2)}_-\) and their conjugates.

### 3.5 Existence of an irreducible \(n\)-dimensional invariant subspace

The object of this section is to prove Theorem 5 announced in the introduction. Given \(n \geq 4\), our study in \(\S\)3.2 shows that except when \(n = 5\), there are exactly two distinct classes of irreducible \(\mathcal{H}(D_n)\)-modules of dimension \(n\), namely the Specht module \(S^{(1),(n-1)}\) and its conjugate \(S^{(1),(1^{n-1})}\). When \(n = 5\), there are exactly four distinct classes of irreducible \(\mathcal{H}(D_5)\)-modules of dimension 5, namely the ones above and the Specht modules \(S^{(0),(3,2)}\) and \(S^{(0),(2,2,1)}\). The latter Specht modules have been studied in the previous section. We proved in Lemma 3 of that section that \(S^{(0),(3,2)}\) may occur when \(l = r^3\), while \(S^{(0),(2,2,1)}\) can never occur. We found a matrix representation for \(S^{(1),(3)}\) and the proof of Theorem 5 will rely on it. We give this representation in the following theorem.

**Theorem 9.** The matrices

\[
H_1 = \begin{bmatrix}
r & 0 & 0 & 0 \\
-r^2 + \frac{1}{r^2} & \frac{1}{r^2} & 0 & 0 \\
-r^3 + \frac{1}{r^3} & -r^2 + \frac{1}{r^2} & -\frac{1}{r^2} - \frac{1}{r^3} & 0 \\
1 - r^2 & \frac{1}{r} - r & \frac{1}{r} & r
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
r & 0 & 0 & 0 \\
0 & r & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 1 & -\frac{1}{r}
\end{bmatrix}
\]

\[
H_3 = \begin{bmatrix}
r & 0 & 0 & 0 \\
0 & r & 0 & 0 \\
0 & 1 & -\frac{1}{r} & 1 \\
0 & 0 & 0 & r
\end{bmatrix}, \quad H_4 = \begin{bmatrix}
r - \frac{1}{r} & 1 & -\frac{1}{r} & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 0 & r
\end{bmatrix}
\]

define an irreducible matrix representation of \(\mathcal{H}(D_4)\) of degree 4.

**Proof.** It is easy to visualize that \(H_2, H_3\) and \(H_4\) satisfy the usual braid relations on nodes 2, 3, 4 and that \(H_i^2 + m H_i = I\) for each \(i \in \{2, 3, 4\}\), where \(I\) denotes the identity matrix of size 4. Further, we check, for instance with Maple, that \(H_i^2 + m H_i = I, H_1 H_3 H_1 = H_3 H_1 H_3\) and that \(H_3\) commutes to both \(H_2\) and \(H_4\). Hence these matrices define a representation of \(\mathcal{H}(D_4)\) of degree 4. It remains to show that this representation is irreducible. Suppose there exists a one-dimensional invariant subspace spanned by \(u = (u_1, u_2, u_3, u_4)^T\). We must have \(H_i u = ru u\) for all \(i \in \{1, 2, 3, 4\}\). Next, we used Maple to solve this system of equations and got \(u = 0\). So there does not exist any one-dimensional invariant subspace. If we can show that there does not exist any irreducible 2-dimensional invariant subspace as well, then we are done by
using the semisimplicity of \( \mathcal{H}(D_4) \). Up to equivalence, there is a unique irreducible representation of \( \mathcal{H}(D_4) \) of degree 2 and it is defined by the matrices

\[
J_1 = J_2 = J_4 = \begin{pmatrix} -\frac{1}{r} & 1 \\ 0 & r \end{pmatrix} \quad J_3 = \begin{pmatrix} r & 0 \\ 1 & -\frac{1}{r} \end{pmatrix}
\]

So, there exists two non-zero linearly independent vectors \( v_1 \) and \( v_2 \) of \( \mathbb{C}^4 \) so that

\[
\forall i \in \{1, 2, 4\}, \quad \begin{cases} H_i v_1 = -\frac{1}{r} v_1 & (\ast)_i \\ H_i v_2 = v_1 + r v_2 & \end{cases} \quad H_3 v_1 = r v_1 + v_2 \quad H_3 v_2 = -\frac{1}{r} v_2
\]

Relation \((\ast)_i\) applied with \( i = 2, 4 \) suffices to force \( v_1 = 0 \) by using for instance the first two rows of \( H_2 \), the last two rows of \( H_4 \) and the fact that \( r^2 \neq -1 \). Thus, we get a contradiction. This ends the proof of Theorem 9.

Suppose there exists in \( V_n \) an irreducible \( n \)-dimensional invariant subspace \( W \) that is isomorphic to \( S^{(1,1),(n-1)} \). Applying the branching rule \((n-4)\) times yields

\[
W \downarrow_{\mathcal{H}(D_4)} \simeq (n-4) S^{(0),(4)} \oplus S^{(1),(3)} \quad (\ast)
\]

So there must exist vectors \( v_1, v_2, v_3, v_4 \) in \( W \) so that the left actions by \( g_1, g_2, g_3 \) and \( g_4 \) on these vectors are given by the matrices \( H_i \)'s of Theorem 9. The computations are technical. We sketch them here and leave the details to the reader. First we read on the matrix \( H_3 \) that \( g_3 v_3 = -\frac{1}{r} v_3 \). Hence, a general form for \( v_3 \) is

\[
v_3 = \lambda_{23} w_{23} + \lambda_{12} w_{12} - \frac{1}{r} \lambda_{12} w_{13} + \sum_{j=4}^{n} \lambda_{2j} w_{2j} - \frac{1}{r} \sum_{j=4}^{n} \lambda_{2j} w_{3j}
\]

We now use \( v_4 \) to get more relations between these coefficient. First, we have \( g_4 v_4 = r v_4 \) and so \( \lambda_{24}^{(4)} = r \lambda_{23}^{(4)} \). Second, with \( v_4 = g_2 v_3 - r v_3 \), we obtain

\[
\lambda_{24}^{(4)} = -\frac{1}{r} \lambda_{24} \\
\lambda_{23}^{(4)} = -\frac{1}{r} (\lambda_{12} + \lambda_{23})
\]

and

\[
\lambda_{24}^{(4)} = -\frac{1}{r} \lambda_{24} \\
\lambda_{23}^{(4)} = -\frac{1}{r} \lambda_{12}
\]

So we get \( \lambda_{24} = r(\lambda_{12} + \lambda_{23}) \) and \( \lambda_{24}^{(4)} = r \lambda_{12}^{(4)} \). Further, use \( g_1 v_4 = r v_4 \) and \( v_4 = g_2 v_3 - r v_3 \) to derive

\[
g_1 g_2 v_3 = r g_1 v_3 + r g_2 v_3 - r^2 v_3
\]
Look at the coefficient in \( \hat{w}_{1j} \), \( j \geq 4 \) in this relation and get
\[
\forall j \geq 4, \quad \hat{\lambda}_{2j} = -\lambda_{2j}
\]
In particular, doing \( j = 4 \) and using the relations above, we get
\[
\hat{\lambda}_{12} + \lambda_{12} = -\lambda_{23}
\] (53)
Let’s use the relations above to write
\[
v_3 = \lambda_{12} \hat{w}_{12} - \frac{1}{r} \lambda_{12} \hat{w}_{13} - (\lambda_{12} + \hat{\lambda}_{12}) w_{23} + r \lambda_{12} (\hat{w}_{24} - w_{24}) + \lambda_{12} (w_{34} - \hat{w}_{34})
\]
\[
+ \sum_{j=5}^{n} \lambda_{2j} (w_{2j} - \frac{1}{r} w_{3j}) - \sum_{j=5}^{n} \lambda_{2j} (\hat{w}_{2j} - \frac{1}{r} \hat{w}_{3j})
\] (54)
We will now find more relations between these coefficients and a relation involving \( l \). We read on the third column of \( H_1 \) that
\[
\left( \frac{1}{r^2} + \frac{1}{r^4} \right) g_4 v_3 = -\frac{1}{r} g_4 \left( \frac{1}{r^2} + \frac{1}{r^4} \right) v_2 + \left( \frac{1}{r^2} + \frac{1}{r^4} \right) v_2 + \left( \frac{1}{r} + \frac{1}{r^3} \right) v_3,
\] (55)
where we multiplied both sides by \( \frac{1}{r^2} + \frac{1}{r^4} \) and where we used that \( v_1 = g_4 v_2 \). Further, we read on the third column of \( H_1 \) that
\[
\left( \frac{1}{r^2} + \frac{1}{r^4} \right) v_2 = -\frac{1}{r^2} g_2 v_3 + (r - \frac{1}{r^3}) v_3 - g_1 v_3,
\] (56)
where we used that \( v_1 = g_2 v_3 - r v_3 \). Plugging (56) into (55) and simplifying now yields the following equation in \( v_3 \).
\[
\left( \frac{1}{r^2} + 1 \right) g_4 v_3 = \frac{1}{r^2} g_4 g_2 v_3 + \frac{1}{r} g_4 g_1 v_3 - \frac{1}{r^2} g_2 v_3 - g_1 v_3 + (r + \frac{1}{r}) v_3
\] (57)
By looking at the coefficient in \( w_{24} \) in (57), we obtain the relation
\[
- \hat{\lambda}_{12} + \frac{1}{r^2} \lambda_{12} = m \sum_{j=5}^{n} r^{j-6} \lambda_{2j}
\] (58)
We will use this expression of the sum on the right hand side to derive a relation involving \( \lambda_{12}, \hat{\lambda}_{12} \) and \( l \). It suffices to look at the term in \( w_{23} \) in \( g_3 v_3 = -\frac{1}{r} v_3 \).
We get, where we used (53) and (54),
\[
\lambda_{23} \left( \frac{1}{l} + \frac{m}{lr} \right) = m \sum_{j=5}^{n} \lambda_{2j} r^{j-6} (r - \frac{1}{l}) - m \lambda_{12} (1 - \frac{1}{lr})
\] (59)
Replacing \( \lambda_{23} \) as in (53) and replacing the sum of (59) as in (58), we then obtain
\[
lr \lambda_{12} = -\hat{\lambda}_{12}
\] (60)
Note we can already conclude in the case \( n = 4 \). Indeed, in that case it follows from (58) that \( -\lambda_{12} + \frac{1}{l} \lambda_{12} = 0 \) and so we get \( l = -\frac{1}{r} \) by using (60). To solve the general case, we introduce a few notations.
Claim 2.

(i) There exists a unique 4-tuple of scalars \((\eta_1, \eta_2, \eta_3, \eta_4)\) such that the action by \(g_1, g_2, g_3, g_4\) on \(X = g_5 v_3 + \eta_1 v_1 + \eta_2 v_2 + \eta_3 v_3 + \eta_4 v_4\) is a multiplication by \(r\).

(ii) For each integer \(k \geq 6\), there exists a unique 4-tuple of scalars \((\eta^k_1, \eta^k_2, \eta^k_3, \eta^k_4)\) such that the action by \(g_1, g_2, g_3, g_4\) on \(X_k = g_k v_3 + \eta^k_1 v_1 + \eta^k_2 v_2 + \eta^k_3 v_3 + \eta^k_4 v_4\) is a multiplication by \(r\).

**Proof.** Immediate with \((\ast)\)

Before we go further, we will need to have a better knowledge of \(v_4, v_2\) and \(v_1\). We compute \(v_4\) with the relation \(v_4 = g_2 v_3 - r v_4\) and we get

\[
v_4 = \left\{ \left( \frac{1}{r} - r \right) \left( \lambda_{12} + m r \lambda_{12} - m \sum_{j=5}^{n} \lambda_{2j} r^{j-4} \right) - m \left( \lambda_{12} + \lambda_{12} \right) \right\} w_{12}
\]

\[\quad- \lambda_{12} (w_{24} - w_{24}) + r \lambda_{12} (w_{14} - w_{14}) + \frac{1}{r} \lambda_{12} (w_{23} - w_{23}) + \lambda_{12} (w_{13} - w_{13})
\]

\[\quad+ \sum_{j=5}^{n} \lambda_{2j} (w_{1j} - w_{1j}) - \frac{1}{r} \sum_{j=5}^{n} \lambda_{2j} (w_{2j} - w_{2j})
\]

(61)

Next, \(v_2\) is given by formula (56). In what follows, a term carries a star if it is multiplied by a factor \(\frac{1}{r} + \frac{1}{r}\). So we have by (56),

\[
v_2^* = -g_1 v_3 + \left( r - \frac{1}{r} - \frac{1}{r} \right) v_3 - \frac{1}{r} v_4
\]

We will study the coefficients of \(w_{1j}\) and \(w_{2j}\) for \(j \geq 4\) in \(v_2^*\) and show that these coefficients are all zero. Going back to the expression of \(v_3\) in (54), we see that the coefficient of \(w_{13}\) in \(-g_1 v_3\) is \((1 + \frac{m}{r}) \lambda_{2j} = \frac{1}{r} \lambda_{2j}\). This coefficient cancels with the coefficient \(-\frac{1}{r} \lambda_{2j}\) of \(w_{13j}\) in \(-\frac{1}{r} v_4\). Similarly, the coefficient of \(w_{14}\) in \(-g_1 v_3\) is \(-m + r) \lambda_{12} = -\frac{1}{r} \lambda_{12}\), which cancels to \(\frac{1}{r} \lambda_{12}\), the coefficient of \(w_{14}\) in \((r - \frac{1}{r} - \frac{1}{r}) v_3 - \frac{1}{r} v_4\). When \(j \geq 5\), the coefficient of \(w_{2j}\) in \(v_2^*\) is \((\frac{1}{r} - r) \lambda_{2j} + \left( r - \frac{1}{r} - \frac{1}{r} \right) \lambda_{2j} + \frac{1}{r} \lambda_{2j}\), that is zero. And the coefficient of \(w_{24}\) in \(v_2^*\) is \(-mr \lambda_{12} - (r - \frac{1}{r} - \frac{1}{r}) r \lambda_{12} - \frac{1}{r} \lambda_{12}\), that is zero. Since \(v_1 = g_4 v_2\), there are obviously no terms in \(w_{1j}\) or \(w_{2j}\) for \(j \geq 5\) in \(v_1\) either. We are now in a position to use the claim. First, by point (ii) of Claim 2, we have for any \(k \geq 6\)

\[
g_2 X_k = r X_k
\]

(62)

Look at the coefficient of the term in \(w_{2,k-1}\) in (62). Using the discussion above, it comes

\[
r(\lambda_{2k} + \eta^k_1 \lambda_{2,k-1} - \frac{m}{r} \lambda_{2,k-1}) = -m \lambda_{2,k} - m \eta^k_1 \lambda_{2,k-1} + \lambda_{2,k-1} \eta^k_4 + \frac{m}{r} \lambda_{2,k-1} \eta^k_4,
\]
from which we derive

$$\lambda_{2,k} = \lambda_{2,k-1} \left( (r + \frac{1}{r}) \eta_{4}^{k} - \eta_{3}^{k} \right)$$  \hspace{1cm} (63)$$

Look now at the coefficient of the term in $w_{14}$ in the same equation (62). In order to get the coefficient on the left hand side, we must in particular look at the coefficient in $w_{24}$ in $g_{4}v_{2}$, so using the discussion above, we must look at the coefficient in $w_{23}$ in $v_{2}$, as there is no term in $w_{24}$ in $v_{2}$. Up to a division by a factor $\frac{1}{r^{2}} + \frac{1}{r^{4}}$, this coefficient is

$$-(m(\lambda_{12} + \lambda_{12}) - \frac{1}{r} \lambda_{12} + m \lambda_{12} - m \sum_{j=3}^{n} \lambda_{2,j} r^{j-5} - (r - \frac{1}{r^{2}} - \frac{1}{r})(\lambda_{12} + \lambda_{12}) - \frac{1}{r^{2}}(\frac{1}{r} \lambda_{12})$$

After replacing the sum as in (58), all the terms simplify nicely and yield

$$\left( \frac{1}{r} + \frac{1}{r^{3}} \right) \lambda_{12}$$

Further the coefficient of the term in $w_{14}$ in $g_{4}v_{2}$ is given by the coefficient of the term in $w_{13}$ in $v_{2}$. Up to a division by a factor $\frac{1}{r^{2}} + \frac{1}{r^{4}}$, this coefficient is

$$-\frac{m}{r} \lambda_{12} - \frac{m}{r} \lambda_{12} + m \sum_{j=5}^{n} r^{j-6} \lambda_{2,j} + \frac{\lambda_{12}}{r}$$

By using again (58) and simplifying, this coefficient is simply

$$\frac{1}{r} \left( \frac{1}{r} + \frac{1}{r^{3}} \right) \lambda_{12}$$

So, the coefficients of the term in $w_{14}$ in $\eta_{1}^{k} g_{2} v_{1}$ and in $r \eta_{1}^{k} v_{1}$ respectively cancel each other. We thus obtain

$$-r^{2} \lambda_{12} - r \lambda_{12} \eta_{3}^{k} + \lambda_{12} \eta_{4}^{k} = -r^{2} \lambda_{12} \eta_{4}^{k}$$

Equivalently,

$$((1 + r^{2}) \eta_{4}^{k} - r \eta_{3}^{k}) \lambda_{12} = r^{2} \lambda_{12}$$  \hspace{1cm} (64)$$

Assume for now that $\lambda_{12}$ is nonzero. Then we get after dividing also by $r$,

$$\forall k \geq 6, \ (r + \frac{1}{r}) \eta_{4}^{k} - \eta_{3}^{k} = r$$  \hspace{1cm} (65)$$

and so by (63),

$$\forall k \geq 6, \ \lambda_{2,k} = r \lambda_{2,k-1}$$  \hspace{1cm} (66)$$

We derive now from (58)

$$-\lambda_{12} + \frac{1}{r^{2}} \lambda_{12} = m \lambda_{25} \sum_{j=5}^{n} r^{j-6} r^{j-5}$$
And after evaluating the sum, it comes
\[ -\lambda_{12} + \frac{1}{r^2} \lambda_{12} = \left( \frac{1}{r^2} - r^{2n-10} \right) \lambda_{25} \] (67)

To get more relations between the coefficients, we use point (i) of Claim 2. First, we look at the coefficient of the term in \( w_{25} \) in \( g_2 X = r X \). We get after simplifications,
\[ r \lambda_{24} - m r \lambda_{12} = \lambda_{25} \left( \frac{m}{r} - \frac{\eta_3}{r} + (1 + \frac{1}{r^2}) \eta_4 \right) \]

Recall from earlier that \( \lambda_{24} = -\lambda_{24} = -r \lambda_{12} \) (see page 34 of the present paper). Hence, we get
\[ -\lambda_{12} = \lambda_{25} \left( \frac{1}{r^2} - 1 - \frac{\eta_3}{r} + (1 + \frac{1}{r^2}) \eta_4 \right) \] (68)

In particular, since we assumed that \( \lambda_{12} \) is non-zero, it follows that \( \lambda_{25} \) is also non-zero. Further, look at the coefficient of the term in \( w_{16} \) still in \( g_2 X = r X \) and derive after using (66) with \( k = 6 \) and simplifying by \( \lambda_{25} \)
\[ \eta_4 = 1 \] (69)

Furthermore, look at the coefficient of the term in \( w_{26} \) this time in the same equation \( g_2 X = r X \). After simplifying and using (69), we get
\[ \eta_3 = \frac{1}{r} \] (70)

Plugging (69) and (70) in (68) now yields
\[ \lambda_{25} = -r^2 \lambda_{12} \] (71)

Next, by plugging (71) into (67), we obtain
\[ \lambda_{12} = r^{2n-6} \lambda_{12} \] (72)

Combining (60) and (72), we derive immediately
\[ l = -\frac{1}{r^{2n-5}} \]

This is the value announced in Theorem 5. It remains to show that our assumption that \( \lambda_{12} \neq 0 \) indeed holds. If \( \lambda_{12} = 0 \), equation (60) implies that \( \lambda_{12} = 0 \). Then many terms in \( v_3 \) and in \( v_4 \) vanish. Further, by (68), we get \( \lambda_{25} = 0 \) or \( \eta_3 - (r + \frac{1}{r}) \eta_4 = \frac{1}{r} - r \). Suppose the second equality holds. Looking at the coefficient of the term in \( w_{2j}, j \geq 6 \) in \( g_2 X = r X \) yields
\[ \lambda_{2j} \left( 1 + \frac{\eta_3}{r} - (1 + \frac{1}{r^2}) \eta_4 \right) = 0, \]
and so $\frac{1}{r} \lambda_{2j} = 0$. Then, $\lambda_{2j} = 0$ for all $j \geq 6$. Next, look at the coefficient of the term in $w_{16}$ in

$$g_2 X_6 = r X_6$$

(73)

where we used the notations of Claim 2. Since $\lambda_{26} = 0$, we simply get $\lambda_{25} = 0$. But then $v_3$ is zero, and this is a contradiction. So looking back up, we must in fact have $\lambda_{25} = 0$. We show this implies inductively that all the $\lambda_{2j}$'s, $j \geq 6$ are zero. Let $k \geq 6$ and suppose that $\lambda_{2,k-1} = 0$. Let’s show that $\lambda_{2,k} = 0$ then. It suffices to look at the coefficient of $w_{1,k}$ in

$$g_2 X_k = r X_k$$

Get

$$-m \lambda_{2k} + \eta_3 \lambda_{2k} - \frac{1}{r} \eta_4 \lambda_{2k} = r \eta_4 \lambda_{2k},$$

which after simplification rewrites

$$\left(m + \left(1 - \frac{r}{\eta_3 - \eta_4}\right) \eta_4 - \eta_3\right) \lambda_{2k} = 0$$

(74)

As we have seen that $\eta_3 - (r + \frac{1}{r}) \eta_4 \neq m$, this forces $\lambda_{2k} = 0$, as announced. The fact that all the $\lambda_{2k}$, $k \geq 5$ are zero is again a contradiction. So our initial assumption $\lambda_{12} = 0$ is not possible. A consequence also is that without loss of generality, $\lambda_{12}$ can be set to 1. Then $\lambda_{12}$ is uniquely determined by (60). And $\lambda_{25}$ is uniquely determined by (71). In turn, the $\lambda_{2k}$'s, $k \geq 6$ are uniquely determined by (66). Thus, $v_3$ is uniquely determined. And so is $v_4$. Then, $v_2$ is uniquely determined by (56) and $v_1$ is in turn completely determined by $v_1 = g_4 v_2$. And so we have the following intermediate result.

**Result 1.** If in the Cohen-Wales space $V_n$, there exists an irreducible invariant subspace that is isomorphic to $S^{(1)}(1^n-1)$, then $l = -\frac{1}{2}$ and there exists in $V_n$ a unique irreducible $\mathcal{H}(D_4)$-module that is isomorphic to $S^{(1)}(1^n-3)$.

To finish the proof of Theorem 5 stated in the introduction, it remains to show that the Specht module $S^{(1)}(1^n-1)$ cannot occur in the Cohen-Wales space. Suppose it does, and let $\mathcal{W}$ be an irreducible invariant subspace of $V_n$ that is isomorphic to $S^{(1)}(1^n-1)$. Then,

$$\mathcal{W} \downarrow_{\mathcal{H}(D_4)} \simeq (n - 4) S^{(0)}(1^4) \oplus S^{(1)}(1^7)$$

We show that it is impossible to have vectors $y_1, y_2, y_3, y_4$ such that the matrices of the left actions by the $g_k$’s with $k \in \{1, 2, 3, 4\}$ on these vectors is given by the matrices $H_k$’s above, where $r$ has been replaced by $-\frac{1}{r}$. Let’s call these conjugate matrices the $K_k$’s. First, the set of relations

$$
\begin{cases}
  g_2 y_1 &= -\frac{1}{r} y_1 \\
  g_3 y_1 &= -\frac{1}{r} y_1
\end{cases}
$$
forces without loss of generality
\[ y_1 = w_{13} - \frac{1}{r} w_{23} - r w_{12} \] (75)

From there, it is very easy to conclude. Indeed, \( y_2 \) is given by the first column of \( K_4 \), then \( y_3 \) is provided by the second column of \( K_3 \) and finally \( y_4 \) is given by third column of \( K_2 \). Since an action by the \( g_i \)’s with \( i = 2, 3, 4 \) on “non-hat terms” will never create a “hat term” by defining equations (8), (9), (11), (14), (15), (16) of the representation, we see with (75) that the \( y_i \)’s do not contain any hats. However, an action by \( g_1 \) on \( y_1 \) creates a term in \( \hat{w}_{23} \) with coefficient 1. So \( g_1 y_1 \) cannot be a linear combination of \( y_1, y_2, y_3 \) and \( y_4 \). This is a contradiction. Hence we are done with the complete proof of Theorem 5, points (i) and (ii).

### 3.6 Proof of the Main Theorem

#### 3.6.1 Proof of the necessary condition

In this part, we assume that \( \mathcal{H}_{F, r^2}(n) \) and \( \mathcal{H}(D_n) \) are semisimple and we show that if \( \nu^{(n)} \) is reducible, then \( l \in \{ \frac{1}{r^{n-1}}, \frac{1}{r^{n-2}}, -\frac{1}{r^{n-2}}, r^3, -r^3, \frac{1}{r} \} \). We solve the small cases \( n \in \{4, 5, 6, 7, 8, 9\} \) by computer means when \( n \in \{4, 5, 6, 7\} \) and by hand when \( n \in \{8, 9\} \). We explain them later. For now suppose the necessary condition above holds in these cases and fix \( n \geq 10 \). We proceed by induction. We assume that the necessary condition above holds at ranks \( n - 2 \) and \( n - 1 \) and show it then holds at rank \( n \). Suppose \( \nu^{(n)} \) is reducible and let \( \mathcal{W} \) be an irreducible proper invariant subspace of \( V_n \). Suppose \( l \not\in \{ \frac{1}{r^{n-1}}, \frac{1}{r^{n-2}}, -\frac{1}{r^{n-2}} \} \). Then, by Theorems 3, 4, 5 and Theorem 8, point (i), we must have

\[ \dim(\mathcal{W}) \geq \frac{n(n - 3)}{2} \]

When \( n \geq 10 \), we claim that the dimension of \( \mathcal{W} \) is large enough so that the intersection spaces \( \mathcal{W} \cap V_{n-2} \) and \( \mathcal{W} \cap V_{n-1} \) are nonzero. Indeed, if \( \mathcal{W} \oplus V_{n-2} \), then it comes \( \dim(\mathcal{W}) \leq \dim(V_n) - \dim(V_{n-2}) = 4n - 6 \). And if \( \mathcal{W} \oplus V_{n-1} \), then it comes \( \dim(\mathcal{W}) \leq \dim(V_n) - \dim(V_{n-1}) = 2n - 2 \). But

\[ \frac{n(n - 3)}{2} > 4n - 6 \iff n \geq 10 \quad \text{and} \quad \frac{n(n - 3)}{2} > 2n - 2 \iff n \geq 7 \]

So when \( n \geq 10 \), both intersections are nonzero. Moreover, both intersections are proper, because if \( \mathcal{W} \) contains \( V_{n-1} \) or \( V_{n-2} \), then it is quite easy to see on the representation that \( \mathcal{W} \) would be the whole space \( V_n \). From there, we get that both \( \nu^{(n-1)} \) and \( \nu^{(n-2)} \) are reducible, so applying the induction hypothesis yields

\[
\begin{cases}
  l \in \left\{ \frac{1}{r^{n-1}}, \frac{1}{r^{n-2}}, -\frac{1}{r^{n-2}}, r^3, -r^3, \frac{1}{r} \right\} \\
  \&
\end{cases}
\]

\[
\begin{cases}
  l \in \left\{ \frac{1}{r^{n-1}}, \frac{1}{r^{n-2}}, -\frac{1}{r^{n-2}}, r^3, -r^3, \frac{1}{r} \right\} \\
  \&
\end{cases}
\]

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With our restrictions on \( r \), we see that \( l \) must then belong to the set of values \( \{ r^3, -r^3, \frac{1}{r} \} \). This finishes the proof in the general case.

Let’s now deal with the case \( n = 9 \), still assuming the result holds for the smaller values of \( n \). Suppose \( \nu^{(9)} \) is reducible and let \( \mathcal{W} \) be an irreducible invariant subspace of \( V_9 \). Using part §3.2, the degrees less than 72 of the irreps of \( \mathcal{H}(D_8) \) are

\[
1, 8, 9, 27, 28, 36, 42, 48, 56, 63, 70
\]

If \( l \not\in \{ \frac{1}{r^3}, \frac{1}{r^3}, -\frac{1}{r^3} \} \), then by Theorems 3, 4, 5 we must have \( \dim(\mathcal{W}) \geq 27 \). First, if \( \dim(\mathcal{W}) \geq 36 > 2(18 - 3) = 30 \), then \( \mathcal{W} \cap V_7 \neq \{0\} \) and the general technique applies. Hence suppose \( \dim(\mathcal{W}) \in \{27, 28\} \). Define \( \mathcal{W}_8 = \mathcal{W} \cap V_8 \).

We have

\[
\dim(\mathcal{W}_8) \geq \dim(\mathcal{W}) + \dim(V_8) - \dim(V_9)
\]

So, if \( \dim(\mathcal{W}) = 27 \), we get \( \dim(\mathcal{W}_8) \geq 11 \) and if \( \dim(\mathcal{W}) = 28 \), we get \( \dim(\mathcal{W}_8) \geq 12 \). In any case, if \( \mathcal{W}_8 \) is irreducible, we must have \( \dim(\mathcal{W}_8) \geq 14 \) by Theorem 8, point (iii). Assume first that \( \mathcal{W}_8 \) is reducible. If \( \dim(\mathcal{W}_8) > 14 = 2(8 - 1) \), then \( \nu^{(7)} \) is reducible. Then both \( \nu^{(8)} \) and \( \nu^{(7)} \) are reducible and we conclude like in the general case. So, suppose \( \dim(\mathcal{W}_8) = 14 \). Then notice \( \mathcal{W}_8 \oplus V_7 = V_8 \) by an inspection on the dimensions. In particular, there exists elements \( z_1 \in \mathcal{W}_8 \) and \( z_2 \in V_7 \) such that

\[
w_{78} = z_1 + z_2
\]

It then comes

\[
w_{89} = e_9 \cdot w_{78} = e_9 \cdot z_1
\]

as the tangle resulting from an action by \( e_9 \) on any basis vector of \( V_7 \) contains two horizontal strands at the bottom: one joining nodes 6 and 7, the other one joining nodes 8 and 9. By construction of the representation, such an element is zero. Now, \( z_1 \) belongs to \( \mathcal{W} \). So, \( e_9 \cdot z_1 \) belongs to \( \mathcal{W} \). Then, \( w_{89} \) belongs to \( \mathcal{W} \). This implies that \( \mathcal{W} \) is the whole space \( V_9 \), a contradiction.

Suppose now \( \mathcal{W}_8 \) is reducible. By the semisimplicity assumption for \( \mathcal{H}(D_8) \), the fact that \( \dim(\mathcal{W}_8) \geq 11 \) and the uniqueness part in Theorem 3, we must again have \( \dim(\mathcal{W}_8) \geq 14 \). So this case reduces to the previous case. This ends the case \( n = 9 \).

Let’s now deal with the cases \( n \in \{4, 5, 6\} \). Suppose \( \nu^{(n)} \) is reducible and let \( \mathcal{W} \) be such a proper invariant subspace of \( V_n \). By Proposition 1 of §3.1, we know that

\[
e_k \cdot \mathcal{W} = 0 \quad \forall 1 \leq k \leq n
\]
Definition 1. We define algebra elements that are some conjugates of the $e_k$'s.

- $C_{i,i+1} = e_{i+1}$ for all $1 \leq i \leq n - 1$
- $C_{i,j} = g_{j,i+2} e_{i+1} g_{i+2,j}^*$ for all $1 \leq i < j \leq n$ with $j \geq i + 2$
- $\hat{C}_{12} = e_1$
- $\hat{C}_{1,j} = g_{j,3} e_1 g_{3,j}$ for all $3 \leq j \leq n$
- $\hat{C}_{i,j} = g_{i,2} g_{3,3} e_1 g_{3,j} g_{2,i}$ for all $2 \leq i < j \leq n$

By $g_{s,t}$ (resp. $g_{s,t}^*$), we understand the product of the $g_k$'s (resp. the $g_k^{-1}$s), where $k$ lies on the integer path from $s$ up or down to $t$.

Definition 2. Define

$$S(n) = \sum_{1 \leq i < j \leq n} C_{ij} + \sum_{1 \leq i < j \leq n} \hat{C}_{ij}$$

Definition 3. Define

$$K(n) = \left( \cap_{1 \leq i < j \leq n} \text{Ker} \nu^{(n)}(C_{ij}) \right) \cap \left( \cap_{1 \leq i < j \leq n} \text{Ker} \nu^{(n)}(\hat{C}_{ij}) \right)$$

and let $k(n)$ be the dimension of $K(n)$ as a vector space over $\mathbb{Q}(l,r)$.
After this series of definitions, we are back to the proof of the necessary condition. Equalities (76) and the fact that $W$ is invariant imply that the left action by $S(n)$ on $W$ is trivial. Then, since $W \neq \{0\}$, the determinant of this action must be zero. Using the tangles, we computed the matrix of the left action by $S(n)$ in the basis formed by the $w_{ij}$’s and the $\hat{w}_{ij}$’s. Note each row of this matrix corresponds to the action of one of the $C_{ij}$’s or $\hat{C}_{ij}$’s. In particular, ordering the vectors of the basis in such a way that the $2(n-1)$ last vectors of this basis have an extremity ending in $n$ allows us to only have to compute $4(n-1)^3$ entries and not $n^2(n-1)^2$ after rank 4. For $n = 4$, the matrix is of size 12, when $n = 5$ of size 20 and when $n = 6$ of size 30. We used Mathematica to solve the determinant of this matrix equals zero. We obtained the values of Theorem 2 point (i) in each of the cases $n = 4, 5, 6$. When $n = 7$, the number of entries is unreasonably big to do it by hand. Thus, we wrote a program in Mathematica that computes the sum matrix. All the matrices in that program are defined by blocks and inductively. When running the program, we obtained

$$\det S(7) = \frac{(-1 + lr)^{21}(l - r^3)^{14}(l + r^3)^{35}(-1 + lr^7)^6(l + r^9)^7(-1 + lr^{21})}{l^{42} r^{105} (r^2 - 1)^{42}}$$

So, if $\nu(7)$ is reducible, then

$$l \in \left\{ \frac{1}{r^{21}}, \frac{1}{r^7}, -\frac{1}{r^5}, r^3, -r^3, \frac{1}{r} \right\}$$

We get the values of Theorem 2 point (i) for $n = 7$. This terminates the case $n = 7$.

Let’s finish all the cases by doing $n = 8$. Suppose $\nu(8)$ is reducible and let $W$ be an irreducible invariant subspace of $V_8$. By the discussion of §3.2 of this paper and Appendix B of [20], the degrees of the irreps of $H(D_8)$ that are less than 56 are

$$1, 7, 8, 14, 20, 21, 28, 35, 42, 48$$

Suppose $l \not\in \left\{ \frac{1}{r^{21}}, \frac{1}{r^7}, -\frac{1}{r^5} \right\}$. Let’s show that $l \in \left\{ \frac{1}{r^7}, r^3, -r^3 \right\}$ then. By Theorems 3, 4, 5, we have $\dim(W) \geq 14$. Further, we have $2(2 \times 8 - 3) = 26$, so if $\dim(W) \geq 28$, then the general technique applies.

Hence suppose $\dim(W) = 21$ and so $W$ is isomorphic to $S^{(0),(6,1,1)}$ or its conjugate $S^{(0),(3,1^5)}$. Then,

$$W \downarrow_{H(D_8)} \cong S^{(0),(5,1,1)} \oplus S^{(0),(6,1)} \text{ or } W \downarrow_{H(D_8)} \cong S^{(0),(2,1^5)} \oplus S^{(0),(3,1^4)}$$

Now look at $W \cap V_7$. We have

$$\dim(W \cap V_7) \geq 21 + 42 - 56 = 7$$

So $W \cap V_7$ cannot be isomorphic to $S^{(0),(6,1)}$ or to $S^{(0),(2,1^5)}$. Further, $W \cap V_7$ is not $W$ either, as otherwise $W$ would be the whole space $V_8$. Then $W \cap V_7$
must be isomorphic to $S^{(0),(5,1,1)}$ or to $S^{(0),(3,1^4)}$. In any case, the dimension of $W \cap V_7$ is 15. Moreover, since $W \cap V_7$ is a proper invariant subspace of $V_7$, it is annihilated by all the $C_{ij}$’s with $1 \leq i < j \leq 7$. Hence we have $W \cap V_7 \subseteq K(7)$, and so $k(7) \geq 15$.

**Lemma 4.**

$$k(n) = n^2 - n - \text{rank}(S(n)) \quad \forall n \geq 4$$

**Proof of the Lemma.** Obvious by the remark above that each row of the matrix of the left action by $S(n)$ on the basis vectors $\overline{w_{ij}}$ s, $1 \leq i < j \leq n$ corresponds to the action of one of the $C_{ij}$’s, as the kernel of the sum matrix is then $K(n)$.

We computed the rank of $S(7)$ with Mathematica for the different values of $l$ and $r$ present in $\bullet$. Here is what we got.

When $l = \frac{1}{r^7}$, $k(7) = 1$

When $l = \frac{1}{r}$, $k(7) = 6$

When $l = -\frac{1}{r^7}$, $k(7) = 7$

When $l = r^3$, $k(7) = 14$

When $l = \frac{1}{r}$, $k(7) = 21$

When $l = -r^3$, $k(7) = 35$

So, if $k(7) \geq 15$, this forces $l \in \{\frac{1}{r^7}, -r^3\}$.

- Suppose now dim($W$) = 20. By the discussion of §3.2 and the table of Appendix B of [20], the only classes of irreducible $H(D_8)$-modules of dimension 20 are $S^{(0),(6,2)}$ and $S^{(0),(2^2,1^4)}$. Then, by Lemma 3, we get $W \simeq S^{(0),(6,2)}$ and $l = r^3$.

- Suppose finally dim($W$) = 14. Then $W \simeq S^{(0),(4,4)}$ or $W \simeq S^{(0),(2,2,2,2)}$.

  - If $W \simeq S^{(0),(4,4)}$, then $W \downarrow_{H(D_8)} \simeq S^{(0),(4,3)}$ and so $W \downarrow_{H(D_8)} \simeq S^{(0),(3,3)} \oplus S^{(0),(4,2)}$. Then the Specht module $S^{(0),(3,2)}$ is a constituent of $W \downarrow_{H(D_8)}$. Then, the proof of point (i) of Lemma 3 shows that $l$ must be equal to $r^3$.

  - If $W \simeq S^{(0),(2^4)}$, then $W \downarrow_{H(D_8)} \simeq S^{(0),(2^3,1)}$ and so $W \downarrow_{H(D_8)} \simeq S^{(0),(2^3)} \oplus S^{(0),(2,2,1,1)}$. Then the Specht module $S^{(0),(2,2,1,1)}$ is a constituent of $W \downarrow_{H(D_8)}$. The proof of point (ii) of Lemma 3 shows that this cannot happen.

So, we are done with the proof of the necessary condition.

3.6.2 Proof of the sufficient condition

The reducibility of $\nu(n)$ is already known when $l = \frac{1}{r^7}$ by Theorem 3 and when $l = \frac{1}{r^3}$ by Theorem 4. Thus, it remains to show that the representation
Lemma 5. Let $n$ be an integer with $n \geq 4$. The vector space $K(n)$ is a $CGW(D_n)$-submodule of $V_n$.

**Proof of the Lemma.** We want to show that

$$\forall 1 \leq k \leq n$$

Let $x \in K(n)$. We proceed in four steps. Step 1 and Step 2 are almost identical, Step 3 uses Step 2 and Step 4 uses Step 1 and Step 3.

- **Step 1.** We show that $g_k x \in \cap_{1 \leq i < j \leq n} \text{Ker} \ C_{ij}$ for every integer $k$ with $2 \leq k \leq n$. Fix integers $i$ and $j$ with $1 \leq i < j \leq n$ and $j \geq i + 2$. Let's show that $C_{ij} g_k x = 0$.

  First, if $k \in \{i, i + 1, j, j + 1\}$, acting to the right of $C_{ij}$ with $g_k$ shifts one of the extremities of the bottom horizontal strand (use the Kauffman skein relation when necessary) and the result follows. When $j = i + 1$, use the delooping relation for the "middle case".

  Next, if $i + 2 \leq k \leq j - 1$, use Reidemeister’s move III twice to notice that $C_{ij} g_k = g_k C_{ij}$, and so $C_{ij} g_k x = 0$.

  Finally, when $k \leq i - 1$ or $k \geq j + 2$, the elements $C_{ij}$ and $g_k$ also commute, which gives the result in this case as well.

- **Step 2.** We show that $g_k x \in \cap_{1 \leq i < j \leq n} \text{Ker} \ C_{ij}$ for all $k$ with $2 \leq k \leq n$. This case is identical as Step 1, except Reidemeister’s move III must also be used twice when $k \leq i - 1$ and when $j = i + 1$, we use

  $$\widehat{C_{i,j+1}} g_{i+1} x = \delta^{-1} \Xi + \widehat{C_{i,j+1}} x,$$

  after moving the crossing near the pole thanks to Reidemeister’s move III.

- **Step 3.** We show that $g_k x \in \cap_{1 \leq i < j \leq n} \text{Ker} \ C_{ij}$. Because $g_1$ commutes to $C_{ij}$ when $i \geq 3$, we only need to show that $C_{2,j} g_1 x = 0$ and $C_{1,j} g_1 x = 0$. But,

  $$C_{2,j} g_1 = g_1 g_1^{-1} C_{2,j} g_1 = g_1 \widehat{C_{ij}},$$

  where the second equality follows from an application of the double twist relation, as in Figure 3. The second one is more difficult and requires careful manipulations on the tangles. Proceed as follows. Multiply the top of $C_{ij}$ by $g_1 g_1^{-1}$. Use the commuting relation at the top and at the bottom, as on Figure 4. Then use the double twist relation and get all together $g_1$ times a tangle that is almost $\widehat{C_{2,j}}$, except the top horizontal strand and the bottom horizontal strand both under-cross the vertical strand that they first intersect when sliding along the strands from the left hand side extremities. Now, it suffices to multiply at the top by $g_2$ and at the bottom by $g_2^{-1}$ to get $\widehat{C_{1,j}}$ instead. So, we have

  $$g_1 g_1^{-1} C_{1,j} g_1 x = g_1 (g_2^{-1} \widehat{C_{1,j}} g_2) x$$

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But by Step 2, we have \( \widehat{C}_{1j} g_2 x = 0 \), so we are done.

- Step 4. We show that \( g_1 x \in \cap_{1 \leq i < j \leq n} \text{Ker} \widehat{C}_{ij} \). First, we deal with \( \widehat{C}_{1j} \) and \( \widehat{C}_{2j} \). With the Kauffman skein relation, it suffices to show that \( \widehat{C}_{1j} g_1^{-1} x = 0 \) and \( \widehat{C}_{2j} g_1^{-1} x = 0 \). We have

\[
\begin{align*}
\widehat{C}_{1j} g_1^{-1} x &= g_1^{-1} g_1 \widehat{C}_{1j} g_1^{-1} x = g_1^{-1} \widehat{C}_{2j} x = 0 \\
\widehat{C}_{2j} g_1^{-1} x &= 2g_1^{-1} g_2 \widehat{C}_{2j} g_2^{-1} x = 3g_1^{-1} \widehat{C}_{2j} x = 0
\end{align*}
\]

Equality 1 is obtained by using in the respective order, simultaneously at the top and at the bottom the double twist relation followed by a Reidemeister’s move II, then the double twist relation again.

For equality 2, multiply to the left by \( g_1^{-1} g_1 \), use the commuting relation at the top and at the bottom, followed by the double twist relation and a Reidemeister’s move II. Get \( g_1^{-1} \) times the tangle of Figure 5 below. The latter tangle is \( g_2 C_{1j} g_2^{-1} \), hence equality 2.

Equality 3 is obtained by using the Kauffman skein relation and Step 1.

It remains to show that \( \widehat{C}_{ij} g_1 x = 0 \) when \( 3 \leq i < j \leq n \), or which is equivalent \( \widehat{C}_{ij} g_1^{-1} x = 0 \). We chose to do it algebraically. When \( i \geq 4 \), we have

\[ \widehat{C}_{ij} = g_i.4 \widehat{C}_{3j} g_4.4 \]
We compute \( \overline{C_{3,j}} g_1^{-1} \). We have, where the parenthesis point out where the next transformations take place.

\[
\overline{C_{3,j}} g_1^{-1} = g_3 g_2 g_{3,3} e_1 g_{3,j} g_2^{-1} (g_1^{-1} g_1^{-1} g_1^{-1}) \quad (77)
\]

\[
= g_3 g_2 g_{3,3} e_1 g_{3,j} (g_1^{-1} g_1^{-1}) g_1^{-1} g_1^{-1} g_3 \quad (78)
\]

\[
= g_3 g_2 (g_{3,4} e_3 g_{4,j}) g_1^{-1} g_1^{-1} g_1^{-1} \quad (79)
\]

\[
= g_3 g_2 C_{2,j} (g_2^{-1}) g_3^{-1} g_1^{-1} \quad (80)
\]

\[
= g_3 g_2 C_{2,j} g_3^{-1} g_1^{-1} + mg g_2 C_{2,j} g_3^{-1} g_1^{-1} - mg g_2 C_{2,j} g_3^{-1} g_1^{-1} \quad (81)
\]

Equality (78) is obtained by using the braid relation. To get (79), commute \( g_1^{-1} \) to the right of \( g_3^{-1} \), add a factor \( g_1^{-1} \) in between \( e_1 \) and \( g_3^{-1} \), use the braid relation with nodes 1 and 3 and use the first delooping relation (DL). Now derive from the first equality in (6) of Proposition 2.3 of [5] that

\[
e_1 g_1^{-1} g_1^{-1} = g_1^{-1} g_1^{-1} e_3 \quad (82)
\]

Further cancel the product \( g_3 g_3^{-1} \) to the left of \( e_3 \) and cancel the same product to the right of \( e_3 \) after replacing \( e_3 \) with \( l e_3 g_3 \). Commute \( g_1^{-1} \) to the right hand side and use the braid relation with nodes 1 and 3. Cancel the product \( g_3^{-1} g_3 \) of the extreme left. Get (79).

Equality (81) is obtained by applying the Kauffman skein relation. We study the three terms of this sum separately. Let’s call them \( a, b \) and \( c \). We have

\[
a = g_3 g_2 C_{1,j} g_3^{-1} g_1^{-1} = g_2 g_3^{-1} C_{1,j} g_1^{-1}
\]

Now the fact that \( a \) annihilates \( x \) follows from Step 3. Further, we have

\[
b = mg g_2 g_3^{-1} C_{3,j} g_1^{-1} = mg g_2 g_3^{-1} g_1^{-1} C_{3,j}
\]

and so \( b \) also annihilates \( x \). Finally, we have

\[
c = -mg g_2 C_{2,j} g_3^{-1} g_3 e_2 g_3^{-1} = -mg g_2 C_{2,j} g_3^{-1} C_{13} g_1^{-1}
\]

and again the fact that \( c \) annihilates \( x \) follows from Step 3.

This settles Lemma 5. The goal next is to show that this submodule of \( V_n \) is non-zero when \( l \in \{-\frac{1}{r^3}, r^3, -r^3\} \). The results are summarized in the following Theorem.

**Theorem 10.** (Reducibility of the representation \( \nu(n) \) when \( l \in \{r^3, -r^3, -\frac{1}{r^3}\} \)).

(i) When \( l = r^3 \), the vector \( \mathcal{X} = (w_{24} + r^2 w_{24}) - r(w_{14} + r^3 w_{14}) - r(w_{23} + r^2 w_{23}) + r^2 (w_{13} + r^2 w_{13}) \) belongs to \( K(n) \) for all \( n \geq 4 \).

(ii) When \( l = -r^3 \), the vector \( \mathcal{Y} = w_{34} - \frac{1}{r} w_{35} + \frac{1}{r} w_{45} \) belongs to \( K(n) \) for all \( n \geq 5 \). The vector \( \mathcal{Z} = r^3 w_{24} - r^2 w_{34} + w_{23} \) belongs to \( K(4) \).

(iii) When \( l = -\frac{1}{r^3} \), the vector

\[
\mathcal{J}_n = (\overline{w}_{12} + r^{2n-6} w_{12}) - \frac{1}{r} (\overline{w}_{13} + r^{2n-6} w_{13}) - (1 + r^{2n-6}) w_{23}
\]

\[+ r \sum_{j=4}^{n} r^{j-5} \left\{ (w_{3,j} - \overline{w}_{3,j}) - r (w_{2,j} - \overline{w}_{2,j}) \right\}
\]

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belongs to $K(n)$ for all $n \geq 4$.

Note $X$ is up to a sign the vector $v_4$ of Lemma 3 in §3.4. Also, $Y$ is the vector $v_1$ of expression (29) of §3.4. As for $Z$ it was found with Mathematica. Finally, $J_n$ is the vector $v_3$ of expression (54) of §3.5, where $\lambda_{12}$ has been set to the value 1 and where the other coefficients are given by (66), (71) and (72).

Figure 5

To prove Theorem 10, we will make an extensive use of the following proposition that provides the action by the $C_{ij}$ on the basis vectors of the Cohen-Wales space.

**Proposition 2.** The following equalities hold.

\[
\begin{align*}
C_{ij} \cdot w_{i-s,i} &= \frac{1}{r(j-i)+(s-2)} \ w_{ij} & \text{LONHNH}_{j-i,s} \\
C_{ij} \cdot \bar{w}_{i-s,i} &= \frac{1}{r(j-i)+(s-2)} \ \bar{w}_{ij} & \text{LONHH}_{j-i,s} \\
\hat{C}_{ij} \cdot w_{i-s,i} &= \frac{l}{r(j-i)+(s-3)} \ w_{ij} & \text{LOHNH}_{j-i,s,l} \\
\hat{C}_{ij} \cdot \bar{w}_{i-s,i} &= \frac{1}{l[r(j-i)+(s-1)]} \ \bar{w}_{ij} & \text{LOHH}_{j-i,s,l}
\end{align*}
\]
\[
C_{ij}, w_{i,j-s} = \frac{1}{r_{s-1}} \ w_{ij} \quad \text{LINHNH}_{s,i}
\]
\[
C_{ij}, \bar{w}_{i,j-s} = \frac{1}{r_{s-2}} \ w_{ij} \quad \text{LINHH}_s
\]
\[
\tilde{C}_{ij}, w_{i,j-s} = \frac{1}{r_{s-1}} \ \bar{w}_{ij} \quad \text{LIHH}_{s,i}
\]
\[
\tilde{C}_{ij}, \bar{w}_{i,j-s} = \frac{1}{r_{s-2}} \ \bar{w}_{ij} \quad \text{LIHNN}_s
\]
\[
C_{ij}, \bar{w}_{i,j-t} = \frac{m_{r^t-s-2}}{l} (1 - l r)(1 + r^2) \ w_{ij} \quad \text{INHNL}_{i,t-s}
\]
\[
\tilde{C}_{ij}, w_{i,j-t} = 0
\]
\[
\tilde{C}_{ij}, \bar{w}_{i,j-t} = 0
\]
\[
\tilde{C}_{ij}, \bar{w}_{i-s,j-t} = \frac{m_{r^{s-t}-2}}{l} (1 - l r)(1 + r^2) \ \bar{w}_{ij} \quad \text{ELOHH}_{j-i,t+s}
\]
\[
C_{ij}, w_{i,s-j-t} = \frac{m_{r^{s-t}-2}}{l} \left( \frac{1}{l} - \frac{1}{r} \right) \ w_{ij} \quad \text{LCNHN}_{i,s+t}
\]
\[
C_{ij}, \bar{w}_{i,s-j-t} = \frac{m_{r^{s-t}-2}}{l} \left( \frac{1}{l} - \frac{1}{r} \right) \ \bar{w}_{ij} \quad \text{LCNHN}_{i,s+t}
\]
\[
\tilde{C}_{ij}, w_{i,s-j-t} = \frac{m_{r^{s-t}-2}}{l} (r - l) \ \bar{w}_{ij} \quad \text{LCHNN}_{i,t+s}
\]
\[
\tilde{C}_{ij}, \bar{w}_{i,s-j-t} = \frac{m_{r^{s-t}-2}}{l} \left( \frac{1}{l} - \frac{1}{r} \right) \ \bar{w}_{ij} \quad \text{LCHHH}_{i,t+s}
\]

In this proposition, the capital letters stand for the following words.

L: left; I: inside; H: hat; NH: non hat; E: extreme; O: outside; C: crossed.

All these equalities were obtained by using the tangles. For now assume that they hold and let us prove the Theorem. Let’s deal with \(i\). The program of the Appendix provides what we called the sum matrix. Running it for \(n = 4\) and for \(n = 5\), we can check that these matrices both annihilate \(X\) and so \(X\) lies in the intersection \(K(4) \cap K(5)\). For larger \(n\), we proceed by induction. Let \(n \geq 6\) and suppose that \(X \in K(n-1)\). We must study the action by \(C_{k,n}\) on the vectors \(\bar{v}_{13}, \bar{v}_{23}, \bar{v}_{14}, \bar{v}_{14}\) and \(\bar{v}_{24}\). First, when \(5 \leq k \leq n-1\), the action by \(C_{k,n}\) on these vectors is zero. We next deal with the actions by \(\overline{C}_{1,n}, \overline{C}_{2,n}, \overline{C}_{3,n}\) and \(\overline{C}_{4,n}\). We see with the second set of formulas above that \(\overline{C}_{1,n}(-r(w_{14} + r^2 \bar{w}_{14}) + r^2(w_{13} + r^2 \bar{w}_{13})) = 0\) and this independently from the values of \(l\) and \(r\). Moreover, the first equality of the third set above implies that \(\overline{C}_{1,n}(r^2 \bar{w}_{24} - r^3 \bar{w}_{23}) = 0\). And so, \(\overline{C}_{1,n}.X = 0\). Also, by \(\text{INHNN}\) and \(\text{IHNN}\), we have \(\overline{C}_{1,n}.(w_{24} + r^2 \bar{w}_{24} - r(w_{23} + r^2 \bar{w}_{23})) = 0\). Hence, \(\overline{C}_{1,n}.X = 0\). For the action by \(\overline{C}_{2,n}\), notice that

\[
\overline{C}_{2,n} = g_2 \overline{C}_{1,n} g_2^{-1}
\]

and \(g_2.X = -\frac{1}{r}X\) when \(l = r^3\).
Hence $C_{2,n} \cdot X = 0$ by the previous case. By the first set of relations above, the action by $C_{3,n}$ on the linear combination of the vectors ending in node 3 in $X$ is zero. And by the last set of relations above, the action by $C_{3,n}$ on the rest of $X$ is also zero. Hence $C_{3,n} \cdot X = 0$. This also implies that $C_{4,n} \cdot X = 0$ after noticing that

$$C_{4,n} = g_4 C_{3,n} g_4^{-1}$$

and $g_4 \cdot X = -\frac{1}{r} X$.

To finish, we compute $C_{k,n} \cdot \left( r^2 w_{24} - r^3 w_{14} - r^3 w_{23} + r^4 w_{13} \right)$, when $5 \leq k \leq n$ and we use the last relation of the third set of relations above. The coefficient is given by

$$r^2 ELOHH_{n-k,2k-6} - r^3 ELOHH_{n-k,2k-5} - r^3 ELOHH_{n-k,2k-5} + r^4 ELOHH_{n-k,2k-4}$$

and we see that it is indeed zero.

Let’s deal with (ii). First the fact that $Z$ belongs to $K(4)$ can be achieved with Mathematica. Likewise, we check that $\mathcal{Y}$ belongs to $K(5)$. Then, it remains to check that for all $j \geq 6$, the algebra elements $C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}$ and $C_{5,j}$ all annihilate the vector $\mathcal{Y}$. For $C_{1,j}$ and $C_{2,j}$, it follows from IHNH. For $C_{3,j}$ and $C_{4,j}$, we also use the first and the last equations of the second set respectively. As for $C_{5,j}$ and $C_{5,j}$, we use the first and the third relations of the first set respectively. Finally, we have using the tables

$$[C_{4,j} \cdot \mathcal{Y}]_{w_{4,j}} = \frac{1}{r^2} \left( -\frac{1}{p} \frac{1}{r^3-j-6} \right) - \frac{m}{r^3-j-6} \left( -\frac{1}{p^3} - \frac{1}{r} \right) + \frac{1}{r^3-j-5} = 0$$

$$[\overline{C_{4,j}} \cdot \mathcal{Y}]_{\overline{w}_{4,j}} = \frac{1}{r^2} \left( -\frac{1}{p^3-j} \right) - \frac{m}{r^3-j-6} \left( r + r^3 \right) - \frac{r^3}{r^3-j-6} = 0$$

Let’s prove (iii). Look at the action of the $C_{s,t}$’s on $J_n$. First, when $s \geq 4$, we cut the sum term in $J_n$ into three parts: a sum from 4 to $s - 1$, a term with the vectors ending in node $s$ and a sum from $s + 1$ to $n$. For the first part, we see with ELOHH that the action is zero. For the middle part, we see with the LO set that the action is zero. The last part however requires more computations with the tangles. These are left to the reader. It remains to check that the actions by $C_{1,k}, C_{2,k}, C_{3,k}$ on $J_n$ are zero. Notice that

$$C_{3,k} = g_3 C_{2,k} g_3^{-1}$$

$$g_3 \mathcal{J}_n = -\frac{1}{r} \mathcal{J}_n \text{ when } l = -\frac{1}{r^2n-5}$$

Hence it suffices to study the actions by $C_{1,k}$ and $C_{2,k}$. This is left to the reader.

This finishes the proof of Theorem 2 point (i). We now give an example of
how to compute the tangles of the table above. We show below how ELOHH is computed. We want to compute $\hat{C}_{ij} \hat{w}_{i-s,j-t}$.

Use Reidemeister's move of type 3 to move the bottom horizontal strand of the upper tangle close to the pole as on the following picture. Multiply the bottom of the tangle successively by the products $g_{i-2} \ldots g_3$ and $g_{j-2} \ldots g_4$ at the cost of divisions by $r^{i-4}$ and $r^{j-5}$ respectively. We must evaluate
where we omitted the top horizontal strand. Apply the commuting relation in the upper left region of the figure and get

Use the Kauffman skein relation twice, then use a Reidemeister move of type 2 twice, multiply at the bottom by $g_1$ at the cost of a division by $r$ and get a tangle that is zero. It remains to compute the four terms arising from the two uses of the Kauffman skein relation. When transforming the under-crossing into an over-crossing on the upper left hand corner of the picture, one must add two terms. The first term contains a loop around the pole that can be "delooped" at the cost of a factor $\frac{1}{r}$. It is then possible to apply the double twist relation. We hence obtain a vertical strand joining nodes $i - t$ at the top and 2 at the bottom. We also obtain a loop that can be suppressed at the cost of a factor $\frac{1}{r}$. The resulting strand is vertical and joins nodes $i - s$ at the top and 3 at the bottom. Use the sequence of Reidemeister moves $R3$, $R2$, $R2$. Multiply at the bottom by $g_3$ at the cost of a division by $r$. This clarifies the first term. When dealing with the second term, use a Reidemeister move of type 3 and get a loop around the pole. Suppress it at the cost of a factor $r$. Further multiply at the bottom by $g_1$ at the cost of a factor $\frac{1}{r}$ and apply the double twist relation twice, with a Reidemeister move of type 2 in between the two moves. Multiply at the bottom by $g_3 g_2$ at the cost of a division by $r^2$ and use a Reidemeister move of type 2 twice. Use a Reidemeister move of type 3. Multiply at the bottom by $g_3^{-1}$ at the cost of a multiplication by $r$ and use a Reidemeister move of type 2. Multiply at the bottom by $g_1^{-1}$ at the cost of a multiplication by $r$, use the double twist relation and a Reidemeister move of type 2. Multiply at the bottom by $g_3^{-1}$ at the cost of a multiplication by $r$. Up to the coefficient, get the same tangle as the one obtained after processing the first term. Gathering all the moves that we did, we must now compute the expression of Figure 6, where we omitted the parts of the pictures that are not of direct interest.
There is very little work that remains to be done on the second tangle of Figure 6. We must still straighten the vertical strands. To that aim, multiply at the bottom successively by the products \( g_{-1} \), \( g_{-1} \), \( ... \), \( g_{-1} \) and \( g_{-1} \), \( g_{-1} \), \( ... \), \( g_{-1} \), at the cost of multiplications by \( r_{i-3} \) and \( r_{i-2} \) respectively.

Finally, there are two more terms to compute. These arise from the first tangle of Figure 6 when we apply the Kauffman skein relation for the second time. In the first term, there is a factor \( \frac{1}{l} \) arising from a loop. For the rest, after applying the adequate moves, we get the same tangle as the one to the right. As for the second term, there is a bit more work to be done. In what follows, we use the abbreviation DT for double twist. The first step is to do a Reidemeister move of type 3. This then allows us to apply the commuting relation. Multiply at the bottom by \( g_{-2} \) at the cost of a multiplication by \( r \), use \( R_{2} \), multiply at the bottom by \( g_{1} \) at the cost of a division by \( r \), do the sequence of moves DT, \( R_{2} \), DT, multiply at the bottom by \( g_{1} \) and in order to do so, divide by \( r^{2} \), multiply at the bottom by \( g_{1} \) and in order to do so multiply by \( r \), use DT, then \( R_{2} \). After doing all these moves, we get the tangle to the right of the picture. The total is

\[
m r^{2i-t-s-5} \left( \frac{1}{l^{2}r^{2}} - r + \frac{1}{l} - \frac{1}{r} \right)
\]

One should not forget the factor \( \frac{1}{r^{i+j-9}} \) from the beginning. All together, it yields the coefficient of Proposition 2.

We end this section by showing that as a representation of the Artin group, the representation \( \nu^{(n)} \) is equivalent to the Cohen-Wales representation. Then, with the change of parameters announced at the end of Theorem 1, the point (i) of Theorem 2 implies the Main Theorem. In [5], the authors built all the inequivalent irreducible representations of the quotient of ideals \( I = \langle e_{1}C_{n} \rangle \).
$C_n C_i C_j C_n > i \neq j$, where $C_n$ denotes the CGW algebra of type $D_n$. Only two of them have degree the number of positive roots of a root system of type $D_n$, which is also the degree of $\nu^{(n)}$. By construction and by Theorem 2, point (i), the representation $\nu^{(n)}$ is an irreducible representation of $I$. Then, it must be equivalent to the representation of $\mathcal{S}_5$. Our $r$ is the $1/2$ of $\mathcal{S}_5$. Further, as a representation of the Artin group, the representation of $\mathcal{S}_5$ is itself equivalent to the representation of $\mathcal{S}_5$, the one that was used to show the linearity of the Artin group. The change of parameters is given in the introduction of $\mathcal{S}_5$ right before Theorem 1.1.

4 End of the proofs of the Theorems

In this last section, we complete the proofs of Theorem 6 and Theorem 7. These theorems provide of course important informations about the structure of the Cohen-Wales representation of type $D_n$ and are extensively used in [24]. In Theorem 6, we must still show that $S^{(0),(4,3)}$ and its conjugate, both of dimensions 14 cannot occur inside $V_7$. In Theorem 7, we must still show that the submodule of $V_n$ spanned by the $\frac{n(n-1)}{2}$ vectors $t_{ij}$’s is irreducible. The latter point uses the first point. We show the following results.

Proposition 3. The Specht modules $S^{(0),(4,3)}$ and its conjugate $S^{(0),(2^{3},1)}$ don’t occur in the Cohen-Wales space $V_7$.

Proposition 4. Proposition 3 and Lemma 3 imply Theorem 6.

PROOF. By § 3.2, when $n \geq 6$, the only irreducible $\mathcal{H}(D_n)$-modules of degree $\frac{n(n-3)}{2}$ are $S^{(0),(n-2,2)}$ and its conjugate $S^{(0),(2,2,1^{n-4})}$, except when $n = 7$, when there are two more irreducibles, namely $S^{(0),(4,3)}$ and $S^{(0),(2^{3},1)}$. This settles Proposition 4. Let’s prove Proposition 3. Suppose there exists in $V_7$ an irreducible invariant subspace $W$ that is isomorphic to $S^{(0),(4,3)}$. Then, we have

$$W \downarrow_{\mathcal{H}(D_7)} \simeq 2 S^{(0),(3,2)} \oplus S^{(0),(4,1)} \quad (\diamond)$$

The proof of Lemma 3, point (i) shows that if there exists an invariant subspace of $W \downarrow_{\mathcal{H}(D_7)}$ that is isomorphic to $S^{(0),(3,2)}$, then it is unique. Hence it is impossible to have (\diamond).

If now $W$ is isomorphic to $S^{(0),(2^{3},1)}$, then

$$W \downarrow_{\mathcal{H}(D_7)} \simeq 2 S^{(0),(2,2,1)} \oplus S^{(0),(2,1,1,1)} \quad (\ast)$$

But, by the proof of Lemma 3, point (ii), the Specht module $S^{(2,2,1)}$ cannot be a constituent of $W \downarrow_{\mathcal{H}(D_7)}$. Thus, (\ast) cannot happen and Proposition 3 holds. This closes the proof of Theorem 6.

Let’s now show that the $\frac{n(n-1)}{2}$-dimensional invariant subspace of Theorem 7, say $T$, is irreducible. Then it must be unique. Excluding $n = 4$, when
\( l = \frac{1}{2} \), the restrictions on \( r \) prevent the existence of an irreducible \( d \)-dimensional invariant subspace of \( V_n \) with \( d \in \{ 1, n - 1, n, 2(n-3)\} \) by Theorems 3, 4, 5, 6 respectively. So, if \( T \) has an irreducible proper invariant subspace, the dimension of this irreducible proper invariant subspace must be greater than or equal to \( \frac{(n-1)(n-2)}{2} = \frac{n(n-3)}{2} + 1 \). But then it has a summand in \( T \) whose dimension is less than or equal to \((n-1)\), impossible. And so Theorem 7 holds when \( n \geq 5 \).

We conclude this paper by proving point \((ii)\) of Theorem 2. For the values of Theorem 2, point \( (i) \), the representation \( \nu^{(n)} \) of \( CGW(D_n) \) that we built is reducible. Moreover, if \( \nu^{(n)} \) were completely reducible, then by Proposition 1, the action of each \( e_i \) on the Cohen-Wales space \( V_n \) would be trivial. This is impossible. Thus, \( CGW(D_n) \) is not semisimple for these values of \( l \) and \( r \). As \( r \) and \(-\frac{1}{2} \) play identical role, \( CGW(D_n) \) is not semisimple either for the values of Theorem 2 point \( (i) \) where \( r \) has been replaced by \(-\frac{1}{2} \).

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