Onsager algebra and cluster XY-models in a transverse magnetic field

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Abstract

The correlation functions of certain $n$-cluster XY models are explicitly expressed in terms of those of the standard Ising chain in transverse field.

1 Introduction

We start with the Hamiltonian, eq. (1) in [1],

$$
H = -J \sum_{j=1}^{N} \sigma_j^x \left( \prod_{k=j+1}^{j+n} \sigma_k^z \right) \sigma_{j+n+1}^x - H \sum_{j=1}^{N} \sigma_j^z, \quad N \equiv (n+1)N_1,
$$

(1)

with periodic boundary conditions, $\sigma_{j+N}^\alpha \equiv \sigma_j^\alpha$, for $\alpha = x, y, z$. In this note we are interested in the factorization of certain correlations functions in the bulk thermodynamic limit $N \to \infty$. The calculation is easiest, if we use periodic boundary conditions and chain length $N \equiv 0 \mod n + 1$.

Hamiltonian (1) is a special case of the generalized XY-model discussed by Suzuki in the early 1970s [2, 3]. The zero-field XY model with isotropic interactions has already been introduced by Nambu in 1950 [4]. A more detailed study with anisotropic interaction was done by Lieb, Schultz and Mattis [5], while Katsura [6] studied the thermodynamic properties in a magnetic field. The special case of (1) with $n = 0$ is called the Ising chain in transverse field and was treated in more detail by Pfeuty [7].

The case of (1) with $n = 1$ has been studied by many authors and can be shown to be equivalent to the zero-field XY model of [4, 5], using a duality transform. This is already to be expected from the Onsager algebra (60) and (61) in [8]: The zero-field Hamiltonian of [4, 5] is a linear combination of $A_1$ and $A_{-1}$, whereas (1) with $n = 1$ is a linear combination of $A_2$ and $A_0$. Expressing

\footnote{If $N \not\equiv 0 \mod n + 1$, we can modify the boundary conditions far away from the operators of interest, such that the factorization still works. For the case of finite chains with open boundary conditions the factorization can be seen to be exact for finite $N$ also.}
the \( A_k \) in terms of Kaufman’s Gamma operators \([9]\), the map is then just a shift of all \( \Gamma_{2j-1} \rightarrow \Gamma_{2j+1} \), keeping the \( \Gamma_{2j} \) fixed, or equivalently \( P_j \rightarrow P_{j+1}, \quad Q_j \rightarrow Q_j \) at the beginning of section 3 of \([9]\).

Looking at Nambu’s figure 1 \([4]\) one may already get the idea that his XY Hamiltonian \((8)\) splits into two commuting Hamiltonians. Indeed, in section 6 of \([10]\) it is worked out in detail how the more general alternating zero-field XY Hamiltonian splits into two transverse-field Ising chain Hamiltonians that commute, implying the factorization of correlation functions.\(^2\) This means, more generally for the case \( n > 0 \), that we can also calculate the correlation functions studied in \([1]\) in terms of the known results for the transverse-field Ising chain given in \([19]\) and references cited there.

We will show this next in section 2. In Section 3 we shall discuss the more general situation using Onsager’s algebra \([8]\). We close with a conclusion in section 4.

2 Fermionization and factorization

Following Kaufman’s spinor analysis \([9]\), we introduce Clifford algebra operators through the Jordan–Wigner transformation \([20]\),

\[
\Gamma_{2j-1} = \left( \prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^x = P_j, \quad \Gamma_{2j} = \left( \prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^y = Q_j, \quad \sigma_j^z = -i\Gamma_{2j-1}\Gamma_{2j},
\]

satisfying

\[
\Gamma_k\Gamma_l + \Gamma_l\Gamma_k = 2\delta_{kl}1. \quad (3)
\]

This is, in fact, eq. (15) with eq. (6) in \([9]\), omitting the asterisks there. Eq. (2) does not appear explicitly as such in \([20]\); Kaufman took it from eq. (9) in \([21]\) instead.

Equivalently, following Jordan and Wigner \([20]\), Nambu \([4]\) and Lieb, Schultz and Mattis \([5]\), we could have used the fermion creation and annihilation operators,

\[
c_j = \frac{1}{2}(\Gamma_{2j-1} - i\Gamma_{2j}), \quad c_j^\dagger = \frac{1}{2}(\Gamma_{2j-1} + i\Gamma_{2j}),
\]

but that is less convenient for our purpose. We note that in \([4]\) the \( \Gamma_k \) have been written as \( x_k \) and that in \([5]\) the \( P_j \) and \( Q_j \) have been called \( A_j \) and \(-iB_j \). In \([10]\) we used \( \gamma_j = \Gamma_j/\sqrt{2} \). More recently these operators are also called Majorana fermions in reference to \([22]\). However, these operators appeared already as eq. (I) on p. 650 in \([20]\), identifying \( P_j = \alpha_j \) and \( Q_j = \alpha_{N+j} \).

\(^2\)We have used similar factorizations in several other papers, see for example section 7 of \([11]\), eq. (3.11) of \([12]\), below eq. (7) in \([13]\), eq. (58) of \([14]\), and eq. (22) of \([15]\). Similar to figure 1 in \([4]\) for case \( n = 0 \), the factorization for case \( n = 1 \) is also implicitly present in figure 3 of \([16]\), which used \( a_k \) for what Kaufman called \( \Gamma_k \). These factorizations closely parallel related factorizations in 2D classical spin models, see e.g. section 10.3 of \([17]\). This is not surprising, as such relationships between \( d \)-dimensional quantum systems and \( (d+1) \)-dimensional classical systems was particularly advertised by Suzuki \([18]\).
As a result, the Hamiltonian becomes
\[ H = \sum_{j=1}^{N} \left[ iJ \Gamma_{2j} \Gamma_{2j+2n+1} + iH \Gamma_{2j-1} \Gamma_{2j} \right], \] (5)
up to a boundary term that we can ignore in the thermodynamic limit for the quantities we discuss in this paper. This is so, as long as we stay with operators in the “even sector” with \( \Gamma_j \) operators clearly grouped in pairs. It fails if the “odd sector” becomes important, see e.g. [10, 23]. This complication does not show up in the open boundary case, but then one has to deal with boundary effects.

Similar to eq. (6.8) in [10], we can next relabel the operators according to
\[ \Gamma_{2k+1}^{(p)} = \Gamma_{2p+2k(n+1)+1}, \quad \Gamma_{2k+2}^{(p)} = \Gamma_{2p+2k(n+1)+2}, \] (6)
for \( p = 0, \ldots, n, \ k = 0, \ldots, N_1 - 1 \), and satisfying
\[ \Gamma_{k}^{(p)} \Gamma_{l}^{(q)} + \Gamma_{l}^{(q)} \Gamma_{k}^{(p)} = 2 \delta_{pq} \delta_{kl} \mathbf{1}. \] (7)

For given \( \Gamma_j \), we can find the \( p \) and \( k \) in (6) using
\[ p = \left\lfloor \frac{(n+1) \{ j - \frac{1}{2} \}}{2(n+1)} \right\rfloor, \quad k = \left\lfloor \frac{j - \frac{1}{2} \lfloor j \rfloor}{2(n+1)} \right\rfloor, \] (8)
where \( \lfloor x \rfloor \) stands for the floor or integer part of \( x \) and \( \{ x \} \) is the fractional part of \( x \). The extra +1 or +2 in (6) corresponds to \( j \) being odd or even. We find
\[ H = \sum_{p=0}^{n} H^{(p)}, \quad H^{(p)} = \sum_{k=1}^{N_1} \left[ iJ \Gamma_{2k}^{(p)} \Gamma_{2k+1}^{(p)} + iH \Gamma_{2k-1}^{(p)} \Gamma_{2k}^{(p)} \right]. \] (9)

We can now define
\[ \sigma_{j}^{x(p)} = -i \Gamma_{2j-1}^{(p)} \Gamma_{2j}^{(p)}, \] (10)
\[ \sigma_{j}^{z(p)} = \prod_{k=1}^{j-1} \sigma_{k}^{z(p)} \Gamma_{2j-1}^{(p)}, \quad \sigma_{j}^{y(p)} = \prod_{k=1}^{j-1} \sigma_{k}^{y(p)} \Gamma_{2j}^{(p)}, \] (11)
so that
\[ H^{(p)} = -J \sum_{j=1}^{N_1} \sigma_{j}^{x(p)} \sigma_{j+1}^{x(p)} - H \sum_{j=1}^{N_1} \sigma_{j}^{z(p)}, \quad p = 0, \ldots, n. \] (12)

Thus \( H \) is decomposed into \( n + 1 \) commuting Ising chains in transverse field, with identical coupling \( J \) and field \( H \) and factorizing \( \exp(\beta H) \), as we can again ignore the boundary effect in the large \( N_1 \) limit. This causes the partition function and the spin correlations to factorize in the thermodynamic limit.\(^3\)

\(^3\)For finite \( N \) the spin correlations for the system (1), with periodic boundary conditions and \( N = N_1(n+1) \), become ratios of sums with four factorized terms, cf. [9, eqs. (35), (39)]. If we had applied open boundary conditions, the correlations would simply factorize, even for \( N \neq 0 \) mod \( n + 1 \).
Let us now consider the equilibrium pair correlation functions
\[ X^{(c)}(k) = \langle \sigma_j^x \sigma_{j+k}^x \rangle, \quad Y^{(c)}(k) = \langle \sigma_j^y \sigma_{j+k}^y \rangle, \quad Z^{(c)}(k) = \langle \sigma_j^z \sigma_{j+k}^z \rangle, \tag{13} \]
for cluster model (1) in the large \( N \) limit.\(^4\) Here \( \langle O \rangle \) stands for either the ground state expectation of \( O \), or the thermal expectation \( \langle O \rangle = \text{Tr} \, O \, e^{-\beta H} / \text{Tr} \, e^{-\beta H} \). Now
\[ \sigma_j^x \sigma_{j+k}^x = -i \Gamma_{2j} \left( \prod_{l=j+1}^{j+k-1} (-i \Gamma_{2l-1} \Gamma_{2l}) \right) \Gamma_{2j+2k-1}, \tag{14} \]
\[ \sigma_j^y \sigma_{j+k}^y = -i \Gamma_{2j-1} \left( \prod_{l=j+1}^{j+k-1} (-i \Gamma_{2l-1} \Gamma_{2l}) \right) \Gamma_{2j+2k}, \tag{15} \]
\[ \sigma_j^z \sigma_{j+k}^z = (-i \Gamma_{2j-1} \Gamma_{2j}) (-i \Gamma_{2j+2k-1} \Gamma_{2j+2k}). \tag{16} \]
Therefore, we immediately conclude that
\[ X^{(c)}(k) = Y^{(c)}(k) = 0, \quad \text{if } k \neq 0 \text{ mod } n + 1, \tag{17} \]
as then \( \Gamma_{2j} \) and \( \Gamma_{2j+2k-1} \) (and similarly \( \Gamma_{2j-1} \) and \( \Gamma_{2j+2k} \)), belong to different \( p \) values, causing odd numbers of \( \Gamma \) to fall into the two corresponding \( H^{(p)} \).

Let us introduce
\[ X(k) = \langle \sigma_j^x \sigma_{j+k}^x \rangle, \quad Y(k) = \langle \sigma_j^y \sigma_{j+k}^y \rangle, \quad Z(k) = \langle \sigma_j^z \sigma_{j+k}^z \rangle, \tag{18} \]
for pair correlations in the Ising chain in transverse field with coupling \( J \) and field \( H \), and
\[ X^*(k) = \langle \sigma_j^x \sigma_{j+k}^x \rangle, \quad Y^*(k) = \langle \sigma_j^y \sigma_{j+k}^y \rangle, \quad Z^*(k) = \langle \sigma_j^z \sigma_{j+k}^z \rangle, \tag{19} \]
for the dual case with coupling \( H \) and field \( J \), and obtained after the duality transform \( \Gamma_{1} \rightarrow \Gamma_{1}^{-1} \), so that \( \sigma_j^x = \sigma_{j+1}^* \sigma_{j+1}^x \) and \( \sigma_j^z \sigma_{j+1}^x = \sigma_{j+1}^* \), see [8, p. 123] and [9, p. 1237].

If now we set \( j = 1 \) and replace \( k \) by \( k(n + 1) \) in (14), we can rewrite
\[ \sigma_1^x \sigma_{1+k(n+1)}^x = -i \Gamma_2 (0) \left( \prod_{l=2}^{k} (-i \Gamma_{2l-1} \Gamma_{2l}) \right) \Gamma_{2k+1}^{(0)} \]
\[ \times \prod_{p=1}^{n} \left[ -i \Gamma_1 (p) \left( \prod_{l=2}^{k} (-i \Gamma_{2l-1} \Gamma_{2l-1}) \right) \Gamma_{2k}^{(p)} \right], \tag{20} \]
and an analogous expression for (15). Similar expressions are found for other values of \( j \), but we really only need the result for \( j = 1 \) as Hamiltonian (1) is translationally invariant. Hence, we find the factorizations
\[ X^{(c)}(k(n+1)) = X(k) X^*(k)^n, \quad Y^{(c)}(k(n+1)) = Y(k) X^*(k)^n. \tag{21} \]

\(^4\)Using the methods in e.g. [10, 11], what follows can also be generalized to time-dependent correlations \( \langle \sigma_j^x(t) \sigma_{j+k}^x(t) \rangle \) with \( A(t) \equiv e^{i H t} A e^{-i H t} \).
From (16) and using (8) we see that
\[ \sigma^x_j \sigma^z_{j+k} = \sigma^z_{k_1} \epsilon_k^{(p_2)}, \tag{22} \]
with \( p_1 = p_2 \) only if \( k \) is a multiple of \( n + 1 \). Therefore, we find
\[ Z(c)(k(n+1)) = Z(k), \quad \text{but } Z(c)(m) = M_z^2, \quad \text{if } m \not\equiv 0 \mod n + 1, \tag{23} \]
where \( M_z = \langle \sigma^z_j \rangle \) is the z-magnetization in the Ising chain (12). Now we have only one or two factors remaining, as the other \( n \) or \( n - 1 \) factors are trivially equal to one.

Next we consider the “well-tailored” cluster operators in eq. (24) of [1],
\[ O_j^{(n)} = \left( \prod_{k=1}^{j-1} \sigma_k^z \right) \left( \prod_{k=0}^{\lfloor n/2 \rfloor} \sigma^y_{j+2k} \sigma^z_{j+2k+1} \right), \quad \text{if } n \text{ is odd,} \]
\[ O_j^{(n)} = \sigma_j^x \left( \prod_{k=1}^{\lfloor n/2 \rfloor} \sigma^y_{j+2k-1} \sigma^z_{j+2k} \right), \quad \text{if } n \text{ is even.} \tag{24} \]

From this we find, using \((-1)^{\lfloor (n+1)/2 \rfloor} = (-1)^{n(n+1)/2}, \)
\[ O_j^{(n)} O_{j+r}^{(n)} = (-1)^{n(n+1)/2} (-i)^{n+1} \prod_{l=0}^{n} \Gamma_{2j+2l} \times \left( \prod_{k=j+r+1}^{j+r-1} (-i \Gamma_{2k-1} \Gamma_{2k}) \right) \prod_{l=0}^{n} \Gamma_{2j+2r+2l-1}. \tag{26} \]

Next we use the relabeling (6) and reorder the \( \Gamma^{(p)} \)'s by increasing \( p \). That costs exactly \( \frac{1}{2} n(n+1) \) minus signs. We can set \( j = 1 \), as the correlation to be gotten cannot depend on \( j \). Thus we get
\[ O_1^{(n)} O_{r+1}^{(n)} = \prod_{p=0}^{n} \prod_{k=1}^{\lfloor (r+n-p)/(n+1) \rfloor} (-i \Gamma^{(p)}_k \Gamma_2^{(p)}), \tag{27} \]
Hence, we find
\[ \langle O_j^{(n)} O_{j+r}^{(n)} \rangle = \prod_{p=0}^{n} X \left( \frac{r + p}{n + 1} \right), \tag{28} \]
with \( X(k) = \langle \sigma^x_j \sigma^x_{j+k} \rangle \) for the Ising chain (12) and \( \lfloor x \rfloor \) the integer part of \( x \).

As \( r \to \infty \), the above result goes to \( \langle (\sigma^x_j)^2 \rangle^{2(n+1)} \), the \( 2(n+1) \) power of the Ising order parameter \( m_x = \langle \sigma^x_j \rangle \).
3 Onsager algebra for Ising model

In his solution of the 2-dimensional Ising model, Onsager introduced \[8\]

\[
A_n = \sum_{j=1}^{N} \sigma^x_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^x_{j+n}, \tag{29}
\]

\[
G_n = \frac{1}{2i} \sum_{j=1}^{N} \left[ \sigma^x_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^y_{j+n} + \sigma^y_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^x_{j+n} \right]. \tag{30}
\]

We have to assume periodicity \(\sigma^\alpha_{j+N} = \sigma^\alpha_j, \alpha = x, y, z\). In addition, as \((\sigma^z)^2 = 1\),
we have

\[
\prod_{k=j+1}^{j} \sigma^z_k = 1, \quad \prod_{k=j+1}^{j-N} \sigma^z_k = \prod_{k=j-N+1}^{j} \sigma^z_k, \tag{31}
\]

so that, using this and the Pauli matrix product rules, we find

\[
A_0 = -\sum_{j=1}^{N} \sigma^x_j, \quad A_{-n} = \sum_{j=1}^{N} \sigma^y_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^y_{j+n}, \tag{32}
\]

\[
A_{n\pm N} = -PA_n = -A_nP, \quad P \equiv \prod_{k=1}^{N} \sigma^z_k, \tag{33}
\]

\[
G_0 = 0, \quad G_{-n} = -G_n, \quad G_{n\pm N} = -PG_n = -G_nP, \tag{34}
\]

\[
A_{n\pm 2N} = A_n, \quad G_{n\pm 2N} = G_n. \tag{35}
\]

Onsager [8] derived the following commutation rules:

\[
[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0. \tag{36}
\]

From these we also have

\[
[A_j, [A_j, [A_j, A_k]]] = 16[A_j, A_k], \quad [A_j, [A_j, G_k]] = 16G_k, \tag{37}
\]

compare \[24\].

We can expand the algebra introducing \[25\]

\[
A_n^{\alpha\beta} = \frac{1}{2} \sum_{j=1}^{N} \sigma^\alpha_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^\beta_{j+n}, \quad \alpha, \beta = x, y, \tag{38}
\]

so that Onsager’s \(A_n = 2A_n^{xx}, A_{-n} = 2A_n^{yy}, G_n = iA_n^{xy} \equiv i(A_n^{xy} + A_n^{yx})\) are recovered, while the commuting \[26\] \(A_n^{(xy)} \equiv A_n^{xy} - A_n^{yx}, [A_n^{(xy)}, A_n^{(yx)}] = 0, \) has been added.

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\[5\] The comparison with \[8\] requires the identification \(s_j = \sigma^x_j\) and \(C_j = \sigma^z_j\), while using a rotated representation of the Pauli matrices, i.e. \(\sigma^x_j \leftrightarrow \sigma^z_j, \sigma^y_j \rightarrow -\sigma^y_j\).
If we look at the Jordan–Wigner transform (2), we would expect the \( \Gamma_j \) not to be periodic mod \( 2N \), but periodic mod \( 4N \), i.e.,

\[
\Gamma_{j \pm 2N} = P \Gamma_j, \quad \Gamma_{j \pm 4N} = \Gamma_j, \quad P = \prod_{k=1}^{N} (-i \Gamma_{2k-1} \Gamma_{2k}), \quad (39)
\]

with \( P \) given in (33).\(^6\) To see what this implies, let us evaluate, for \( 1 \leq j, k \leq N \),

\[
\sigma^x_j (\prod_{k=j+1}^{N+1} \sigma^z_k) \sigma^x_N = \sigma^x_j (\prod_{k=1}^{j} \sigma^z_k) (\prod_{k=1}^{N} \sigma^z_k) (\prod_{k=1}^{i-1} \sigma^z_k) \sigma^x_l
\]

\[
= \left( \prod_{k=1}^{j-1} \sigma^z_k \right) \sigma^x_j \sigma^x_j P \left( \prod_{k=1}^{l-1} \sigma^z_k \right) \sigma^x_l
\]

\[
= \Gamma_{2j-1} (-i \Gamma_{2j-1} \Gamma_{2j}) P \Gamma_{2l-1}
\]

\[
= -i \Gamma_{2j} P \Gamma_{2l-1} = -i \Gamma_{2j} \Gamma_{2N+2l-1}. \quad (40)
\]

Hence, with the identification (39), we find

\[
A_n = -i \sum_{j=1}^{N} \Gamma_{2j} \Gamma_{2j+2n-1}, \quad (41)
\]

\[
G_n = i \sum_{j=1}^{N} (\Gamma_{2j-1} \Gamma_{2j+2n-1} - \Gamma_{2j} \Gamma_{2j+2n}), \quad (42)
\]

\[
A_n^{(xy)} = -i \sum_{j=1}^{N} (\Gamma_{2j-1} \Gamma_{2j+2n-1} + \Gamma_{2j} \Gamma_{2j+2n}). \quad (43)
\]

Note that not all the terms are quadratic in fermion operators due to the identification (39). We will need to follow Onsager [8] and Kaufman [9] and split the state space into a direct sum of even and odd states, corresponding to eigenvalue of \( P \) being +1 or −1. Each \( \Gamma_j \) changes an even state into an odd state and vice versa. The \( A_n \) and \( G_n \) act on the odd sector as quadratic in fermions with cyclic boundary conditions; on the even sector they acquire anticyclic boundary conditions.\(^7\) In general,

\[
A_n = P_- A_n^{(c)} + P_+ A_n^{(ac)}, \quad (44)
\]

with cyclic and anticyclic versions of \( A_n \) and projection operators \( P_{\pm} \),

\[
P_{\pm} = \frac{1}{2} (1 \pm P), \quad (P_{\pm})^2 = P_{\pm}, \quad P_- + P_+ = 1, \quad P_- P_+ = 0. \quad (45)
\]

\(^6\) This \( P \) is called \( \dagger U \) in eq. (36) of [9], with \( U \) from the text above (14) there, not to be confused with \( U \) defined differently in (32) of [9].

\(^7\) Sometimes the sector with periodic boundary conditions is named after Ramond [27] and the one with antiperiodic boundary conditions after Neveu and Schwarz [28] even though these string theory papers were written more than two decades later.
Because of (35), we only have to consider the following cases: When $0 < n < N$,

$$A_n^{(c)} = -i \sum_{j=1}^{N-n} \Gamma_{2j} \Gamma_{2j+2n-1} - i \sum_{j=N-n+1}^{N} \Gamma_{2j} \Gamma_{2j+2n-2N-1}, \quad (46)$$

$$A_n^{(ac)} = -i \sum_{j=1}^{N-n} \Gamma_{2j} \Gamma_{2j+2n-1} + i \sum_{j=N-n+1}^{N} \Gamma_{2j} \Gamma_{2j+2n-2N-1}. \quad (47)$$

When $-N < n < 0$,

$$A_n^{(c)} = -i \sum_{j=1}^{-n} \Gamma_{2j} \Gamma_{2j+2n+2N-1} - i \sum_{j=-n+1}^{-1} \Gamma_{2j} \Gamma_{2j+2n-1}, \quad (48)$$

$$A_n^{(ac)} = +i \sum_{j=1}^{-n} \Gamma_{2j} \Gamma_{2j+2n+2N-1} - i \sum_{j=-n+1}^{-1} \Gamma_{2j} \Gamma_{2j+2n-1}. \quad (49)$$

Finally,

$$A_0^{(c)} = A_0, \quad A_0^{(ac)} = 0, \quad A_N^{(c)} = A_0, \quad A_N^{(ac)} = -A_0. \quad (50)$$

If the Hamiltonian $\mathcal{H}$ is a linear combination of $A_n$’s with periodic boundary conditions, the partition function $Z = \text{Tr} e^{-\beta \mathcal{H}}$ can be rewritten as

$$Z = \frac{1}{2} \text{Tr} e^{-\beta \mathcal{H}^{(c)}} + \frac{1}{2} \text{Tr} e^{-\beta \mathcal{H}^{(ac)}} - \frac{1}{2} \text{Tr} P e^{-\beta \mathcal{H}^{(c)}} + \frac{1}{2} \text{Tr} P e^{-\beta \mathcal{H}^{(ac)}}, \quad (51)$$

using $P_+ + P_+ = 1$ (45). In the limit $N \to \infty$ one can show that the first two terms are asymptotically equal and infinitely larger than the other two terms [6, 9, 29], so that we can replace $Z$ by $\text{Tr} e^{-\beta \mathcal{H}^{(c)}}$ or by $\text{Tr} e^{-\beta \mathcal{H}^{(ac)}}$. Similarly, in equal-time correlations we can also replace $\mathcal{H}$ by $\mathcal{H}^{(c)}$ or $\mathcal{H}^{(ac)}$, following for example [29].

However, this method does not work for time-dependent correlations of odd operators like $\sigma_j^x$. If one wants to keep translation invariance, one has two options: Following Cheng and Wu [23, 30], one can asymptotically construct the square of the correlation by doubling the number of spins in the correlation, so that one only has even combinations. This leads to infinite (block-Toeplitz) determinants. Otherwise one gets expressions with both $\mathcal{H}^{(c)}$ and $\mathcal{H}^{(ac)}$ in it [10, 31, 32].

4 Conclusion

We have expressed the pair correlations of the $n$-cluster model, discussed in [1], explicitly in terms of those of the standard Ising chain in transverse field. These latter correlations are known in great detail, especially in the ground state [7, 19]. In particular, one can use the 2-dimensional Ising recurrence relations and asymptotic results of section 2 in [19], together with (60)–(63) there to
obtain very accurate results for $X(n)$ and $X^*(n)$ in the ground state. Then the $Y(n)$ and $Y^*(n)$ can be obtained from (37), (38), (71) and (72) for $t = 0$ in [19].

We also factorized the pair correlation of two cluster operators of the form (24) in [1] as a product of $n+1$ factors of the form $X(n)$, which can also be given to great accuracy using [19].

It should be noted that the time-dependent $xx$, $xy$, $yx$ and $yy$ correlations have factorizations similar to those in (17), (21) and (23), so that results from [19] and references cited there can be used.

Next, we note that we can derive similar factorizations for cluster models with Hamiltonians of the form $H = \lambda A_n + \mu A_m$, with $n$ and $m$ arbitrary. Then there is a factorization into $|m-n|$ Ising chains in transverse field similar to (9).

As an example one may consider the zero-field cluster model with Hamiltonian of the form

$$
H = -J_x A_{n+1} - J_y A_{-n-1},
$$

or

$$
H = -\sum_{j=1}^{N} J_x \sigma_j^x \left( \prod_{k=j+1}^{j+n} \sigma_k^z \right) \sigma_{j+n+1}^x - J_y \sum_{j=1}^{N} \sigma_j^y \left( \prod_{k=j+1}^{j+n} \sigma_k^z \right) \sigma_{j+n+1}^y + B \sigma_j^z,
$$

with $N \equiv (n+1)N_1$. In this case there are $2(n+1)$ factors. We leave that as an exercise for the interested reader.

Factorizations can also occur if there are more terms in the Hamiltonian, which is assumed to be of the form $H = \sum_{k=0}^{r} \lambda_k A_{n+k}$. For example,

$$
H = -\sum_{j=1}^{N} \left[ J_x \sigma_j^x \left( \prod_{k=j+1}^{j+n} \sigma_k^z \right) \sigma_{j+n+1}^x + J_y \sigma_j^y \left( \prod_{k=j+1}^{j+n} \sigma_k^z \right) \sigma_{j+n+1}^y + B \sigma_j^z \right],
$$

or

$$
H = -J_x A_{n+1} - J_y A_{-n-1} + BA_0.
$$

Now there are $n+1$ factors corresponding to different anisotropic XY-chains in transverse field. The correlation functions can all be factored in full detail like is done in section 2. It should be remarked that Minami [33] very recently briefly noted this decoupling of Hamiltonian (53), without discussing the factorization of the correlation functions. In two papers [33, 34] he also presented a general construction of models equivalent to the Ising chain in transverse field, in essence giving representations of the Temperley–Lieb algebra [35],

$$
e_i^2 = \sqrt{2} e_i, \quad e_i e_{i+1} e_i = e_i, \quad [e_i, e_j] = 0 \text{ if } |i-j| > 2,
$$

identifying $\eta_i = 1 - \sqrt{2} e_i$ in [33, 34].

Finally, so far we have taken all interaction constants to be uniform. If we relax that, we no longer have the Onsager algebra (36), but a larger algebra instead. However all factorizations still work the same way.

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