Forced Burgers Turbulence in 3–Dimensions

Jahanshah Davoudi, A. Reza Rastegar and M. Reza Rahimi Tabar

a Department of Physics, Tarbiat-Moallem University, Tabriz, Iran
b CNRS UMR 6529, Observatoire de la Côte d’Azur, BP 4229, 06304 Nice Cedex 4, France,
c Dept. of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran.
d Institute for Studies in Theoretical Physics and Mathematics Tehran P.O.Box: 19395-5531, Iran.

We investigate non-perturbative results of inviscid forced Burgers equation supplemented to continuity equation in three–dimensions. The exact two–point correlation function of density is calculated in three-dimensions. The two–point correlator \( \langle \rho(x_1)\rho(x_2) \rangle \) behaves as \( |x_1 - x_2|^{-\alpha_3} \) and in the universal region \( \alpha_3 = 7/2 \) while in the non-universal region \( \alpha_3 = 3 \). In the non-universal region we drive a Kramers-Moyal equation governing the evolution of the probability density function (PDF) of longitudinal velocity increments for three dimensional Burgers turbulence. In this region we prove Yakhot’s conjecture [Phys. Rev. E 57, 1737 (1998)] for the equation of PDF for three dimensional Burgers turbulence. We also derive the intermittency exponents for the longitudinal structure functions and show that in the inertial regime one point \( U_{rms} \) enters in the PDF of velocity difference.

PACS numbers, 47.27.Gs and 47.40.Ki

I. - INTRODUCTION

Recently, tremendous activity has started on the non-perturbative understanding of turbulence [1-22]. A statistical theory of turbulence has been put forward by Kolmogorov [23], and further developed by others [24–26]. The approach is to model turbulence using stochastic partial differential equations. The simplest approach to turbulence is the Kolmogorov’s dimensional analysis, which leads to the celebrated \( k^{-5/3} \) law for the energy spectrum. This is obtained by decreeing that the energy spectrum depends neither on the wavenumber where most of the energy resides, nor on the wavenumber of viscous dissipation. Kolmogorov conjectured that the scaling exponents are universal, independent of the statistics of large–scale fluctuation and the mechanism of viscous damping, when the Reynolds number is sufficiently large. In fact the idea of universality is based on the notion of the ”inertial subrange”. By inertial subrange we mean that for very large values of the Reynolds number there is a wide separation between the scale energy input \( L \) and the typical viscous dissipation scale \( \eta \) at which viscous friction become important and the energy is turned into heat.

In turbulent fluid flows, the physical quantities of interest, such as the velocity and density fields, display highly irregular fluctuations both in space and time. To study such processes, using a statistical approach is most natural. The statistics of any fluctuating quantity are described by its probability density function (PDF). One would, therefore, hope to be able to calculate the PDF’s of the turbulent quantities directly from the equations of motion, i.e. for example to calculate the PDF of velocity fluctuations from Navier-Stokes equation. However this task is highly nontrivial and as of today has not been accomplished not even for the relatively simpler problems. The striking feature of developed turbulence is its intermittent spatial and temporal behavior. The structures that arise in a random flow manifest themselves as high peaks at random places and at random times. The intervals between them are characterized by a low intensity and a large size. Rare high peaks are responsible for PDF tails while the regions of low intensity contribute PDF near zero. Analytically the following properties manifest themselves in the nonlinear behavior of the scaling exponents in the velocity difference structure functions (transvers and longitudinal) \( S_n(r) \) i.e. \( \langle (u(x+r) - u(x))^n \rangle \). It is known that \( S_n(r) \) behaves as \( \sim r^{\zeta_n} \), where \( \zeta_n \) is a non-trivial function of \( n \). This behaviour leads to stongly non-gaussian (intermittent) probability distribution functions of velocity difference [27].

On the other hand Burgers equation, which describes the potential flow of the fluid without pressure, provides a wonderful laboratory for testing new ideas and techniques in view of the study of fully developed turbulence in the Navier–Stokes turbulence. These are two cases of non–linear stochastic equations which share in the same structure of the non–linearity. The important differences come from the nature of interaction (i.e. locality and non-locality) and the large scale structure. In the case of incompressible turbulence the interaction is non–local that is the very existence of the pressure causes the far regions of the flow to be coupled with together [11] and leads to effective energy redistribution between components of the velocity field, while in the case of Burgers equation the pressure effects are absent. The non-linear term \( (v \cdot \nabla)v \) in the Navier-Stokes and Burgers equations tends to form
shocks and enhance intermittency while the additional pressure gradient term in N-S equation depresses intermittency. This scenario has been given addressed by V. Yakhot [11], for constructing an approxiante dynamical theory for N-S equation. In multi-dimensional Burgers turbulence the presence of large scale structures (shocks) forming a d-dimensional forth-like pattern is responsible for extreme case of intermittency causing the saturation of the intermittency exponent to \( \alpha_n = 1 \). Large scale structures in "true" turbulence are similarly thought to be the origin of the experimentally observed intermittency, which is however much milder. The deep reason of this difference is thought to be probably related to the "dimension" of the large scale singularities, which is \( d - 1 \) in the d-dimensional Burgers turbulence and only 1 for vortex lines in hydrodynamical turbulence.

The problem of forced and un-forced Burgers turbulence [28-30] has been attacked recently by various methods [11, 13, 18]. The Burgers equation describes a variety of non-linear wave phenomena arising in the theory of wave propagation, acoustics, plasma physics, surface growth, charge density waves, dynamics of the vortex lines in the high \( T_c \) superconductor, dislocations in the disordered solids and formation of large-scale structures in the universe [30, 32]. According to the recent theoretical [11, 13, 18] and numerical works [33], it is known that the PDF for the velocity difference behaves differently in universal and non-universal regions. In the universal region i.e. the interval \( |\Delta u| << U_{rms} \) and \( r << L \), the PDF can be represented in the universal scaling form

\[
P(\Delta u, r) = \frac{1}{r^x} F(\frac{\Delta u}{r^y})
\]

where \( F(x) \) is a normalizable function and the exponent \( z \) is related to the exponent of random force correlation \( \eta \) as \( \eta = \frac{z+1}{3} \). In the region for \( x = \frac{\Delta u}{r^y} >> 1 \) the universal scaling function \( F(x) \) is given by the expression \( F(x) \sim \exp(-\alpha x^\lambda) \), where \( \alpha \) is some constant in one-dimension and it depends on the cosine of angle between the vectors \( \Delta u \) and \( r \) in the higher dimensions. On the other hands the PDF in the interval \( |\Delta u| >> U_{rms} \) behaves as:

\[
P(\Delta u, r) = r G(\frac{\Delta u}{U_{rms}})
\]

which depends on the single-point \( U_{rms} \) and therefore is not a universal function.

In this paper we consider the inviscid forced Burgers equation in 3-dimensions supplemented to continuity equation. We find the exact two-point correlation functions of density in three-dimensions and show that \( \rho(x_1) \rho(x_2) \) behaves as \( |x_1 - x_2|^{-\alpha_3} \), where \( \alpha_3 = 7/2 \) in the universal region i.e. \( |u(x) - u(x')| << U_{rms} \) where \( U_{rms} \) is the rms value of velocity fluctuations. In the non-universal region i.e. \( |u(x) - u(x')| >> U_{rms} \) we derive the exact Kramers-Moyal equation governing on the evolution of the probability density function (PDF) of longitudinal velocity increments for three dimensional Burgers turbulence. In this region we prove the Yakhot conjecture [11] for equation of the PDF in multi-dimensional Burgers turbulence and prove the existence of the A-term and B-term in the equation for the PDF of longitudinal velocity difference. We also derive the intermittency exponents for the longitudinal structure functions and show that how the \( U_{rms} \) enters in the PDF of the velocity difference and breaks the Galilean invariance. Furthermore we refer to the zero viscosity limit of 3D forced Burgers turbulence. We claim that at least when the simplest terms satisfying the basic symmetries of the problem are introduced for closing the dissipation term the requirements of the homogeneity and isotropy would force the coefficient of the proposed terms to vanish. So within this approximation the viscous term in the vanishing viscosity limit do not contribute in the equation for the PDF’s of velocity difference in the inertial range.

The paper is organized as follows: In section 2 we derive the density-density correlators exponent in inviscid Burgers turbulence and the generating function for longitudinal velocity difference in the universal part of the PDF. In section 3 we determine the small scale statistics of longitudinal velocity difference and derive the exact equation for the PDF’s of longitudinal velocity difference in the non-universal part of the PDF and derive the exact value of intermittency exponents in longitudinal structure functions. We show that the intermittency problem in these systems can be investigated non-perturbatively and derive the deformation of PDF’s in length scales. It is shown that deformation of PDF from large to small scales are completely described by Kramers-Moyal equation.

II. - PDF’S OF 3D BURGERS TURBULENCE IN THE UNIVERSAL REGION

Our starting point is the 3D–Burgers equation supplemented to continuity equation:

\[
u \tau + (u \cdot \nabla) u = \nu \nabla^2 u + f(x, t)
\]

\[
\rho_t + \partial_{x \alpha} (\rho u_{\alpha}) = 0
\]

for the Eulerian velocity \( u(x, t) \) and viscosity \( \nu \) and density \( \rho \), in 3-dimensions. The force \( f(x, t) \) is the external stirring force, which injects energy into the system on a length scale \( L \). More specifically one can take, for instance a Gaussian distributed random force, which is identified by its two moments:

\[
<f_{\mu}(x, t)> = 0
\]

\[
<f_{\mu}(x, t) f_{\nu}(x', t')> = k(0) \delta(t - t') k_{\mu \nu}(x - x')
\]
where $\mu, \nu = x, y, z$. The correlation function $k_{\mu\nu}(r)$ is normalized to unity at the origin and decays rapidly enough where $r$ becomes larger or equal to integral scale $L$. The quantity $k(0)$ measures the energy injected into the turbulent fluid per unit time and unit volume. $f(x, t)$ provide also the energy flux in the $k-th$ shell as $\Pi_k = \Pi(r = k^{-1}) \simeq \int_{k}^{\infty} |f(k)|^2$, where $r$ belongs to the inertial range.

Eqs.(3) and (4) exhibit special type of nonlinear interactions. It is hidden in the nonlinear term $(\mathbf{u} \cdot \nabla) \mathbf{u}$. The advective term couples any given scale of motion to all the large scales and the large scale contain most of the energy of the flows. This means that the large–scale fluctuation of turbulence production in the energy–containing range couple to the small–scale dynamics of turbulence flow. In other words, the details of the large–scale turbulence production mechanism are important, leading to the non–universality of probability distribution function (PDF) of velocity difference. However in the case that $|u(x) - u(x')| << U_{rms}$ it is believed that the PDF for the velocity difference is not depended to $U_{rms}$ and therefore the one-point $U_{rms}$ does not appear in the velocity difference PDF. This region is known as the Galilean invariant (GI) region.

Now let us consider the inviscid 3D forced Burgers turbulence that is we intend to neglect from the viscosity contribution in eq.(3). The problem is to understand the statistical properties of velocity and density fields which are the solutions of eq.(3) and (4). We consider the following two–point generating function:

$$F_2(\lambda_1, \lambda_2, x_1, x_2) = < \rho(x_1) \rho(x_2) \exp(\lambda_1 \mathbf{u}(x_1) + \lambda_2 \mathbf{u}(x_2)) >$$

where the symbol $< \cdots >$ means an average over various realizations of the random force.

To derive an equation for $F_2$, we write the eq.(3) and (4) in two points $x_1$ and $x_2$ for $u_1, u_2, u_3$ and $\rho(x)$ and multiply the equations in $\rho(x_1)$, $\lambda_1 \rho(x_1) \rho(x_2)$, $\cdots, \lambda_1 \rho(x_1) \rho(x_2)$ and $\rho(x_1), \lambda_2 \rho(x_1) \rho(x_2), \cdots$, and $\lambda_2 \rho(x_1) \rho(x_2)$, respectively. We add the equations and multiply the result by $\exp(\lambda_1 \mathbf{u}(x_1) + \lambda_2 \mathbf{u}(x_2))$ and make average with respect to external random force, so we find:

$$\partial_t F_2 + \sum_{i=1,2} \lambda_i \partial_{\lambda_i} F_2 = 0$$

where we have used the Novikov’s theorem. The above equation is first driven by Polyakov [4] for the problem of one dimensional Burgers equation in the inviscid limit. In that work the effect of the viscous term is found in the limit of $\nu \rightarrow 0$ and $r \ll L$ by appeal to the self-consistent conjecture of operator product expansion. It is found that there are only two terms generated by the viscous term which are consistent with the symmetries of the problem. These two anomaly terms will modify the master equation governing over the generating function in a way such that a positive anomaly terms will modify the master equation governing over the generating function in a way such that a positive

$$k_{\mu\nu}(x_1 - x_3) = k(0) \left[ 1 - \frac{|x_1 - x_3|^2}{2L^2} \delta_{\mu\nu} - \frac{(x_1 - x_3)_{\mu}(x_1 - x_3)_{\nu}}{L^2} \right]$$

with $k(0), L = 1$. We change the variables as: $x_k = x_1 \pm x_2$, $\lambda_\pm = \lambda_1 + \lambda_2$ and and consider the spherical coordinates, so that $x- : (r, \theta, \varphi)$ and $\lambda_- : (\mu, \theta', \varphi')$. Hence we find that the $F_2$ satisfies the following closed equation for homogeneous and isotropic Burgers turbulence:

$$s \partial_r \partial_{\mu} - \frac{s(1 - s^2)}{r \mu} \partial^2_r + \frac{1 + s^2}{r \mu} \partial_s + \frac{1 - s^2}{\mu} \partial_{\mu} \partial_s + \frac{1 - s^2}{r} \partial_{\mu} \partial_s - r^2 \mu^2 (1 + 2s^2) F_2 = 0$$

where $s = \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')$.

In homogeneous and isotropic Burgers turbulence, with stirring correlation as $k(r) \sim 1 - r^n$ (where in the our case i.e. eq.(8) we have $n = 2$), we consider the universal scale–invariant solution of eq.(9) in the following form:

$$F_2(\mu, r) = g(r) F(\mu^\delta)$$

Substituting the following form for the generating function fixes the exponent $\delta$ as $\delta = \frac{3 - n}{n - 1}$ (In our case using eq.(8) we find $\delta = 1$). Invoking to the scaling invariance of the inviscid Burgers equation and continuity equation, we assume the existence of the density-density correlators with the scaling form of introduced in (10). $\alpha_3$ is the exponent of two point correlation and the two point correlation of the density can be found by the generating function in the limit of $\mu \rightarrow 0$. Therefore it is necessary to find such a solution for $F(\mu^\delta)$ which tends to a constant in the limit of $\mu \rightarrow 0$. Proceeding further we focus our attention to the longitudinal velocity components,i.e. $s = 1$, and hence accept the scaling ansatz $F(\mu^\delta) = F(\mu^s)$. The proposed form of the arguments will guarantee that $S_n(r, s) \sim s^n S_n(r)$ when $n < 1$. Rewriting eq.(9) in terms of the variable $z = \mu s$, the following equation is obtained in the limit of $s \rightarrow 1$:

$$z \partial_z^2 F(z) + (3 - \alpha_3) \partial_z F(z) - 3z^2 = 0$$

It is interesting that the above equation was first derived by Polyakov [4] for the problem of one dimensional Burgers equation in the inviscid limit. In that work the effect of the viscous term is found in the limit of $\nu \rightarrow 0$ and $r \ll L$ by appeal to the self-consistent conjecture of operator product expansion. It is found that there are only two terms generated by the viscous term which are consistent with the symmetries of the problem. These two anomaly terms will modify the master equation governing over the generating function in a way such that a positive
satisfies the following equation: we propose the scale invariant solution for the density-density scaling exponent to that the requirement of the positivity on the PDF will fix the value of anomaly coefficient \(^3\). Boldyrev \(^18\) shows that one could find a family of solutions for different values of the b-anomaly coefficient if one relaxed the homogeneity condition for the universal part of the PDF. The value of this coefficient is related to the algebraic decay of the left tail of PDF in the universal regime. Determination of that decay exponent has been a controversial subject for which other methods have been developed and among them recent rigorous methods should be mentioned within which it is fixed to \(7/2\) \(^16\). The interesting point is that our calculations in three dimensions show that when density fluctuations are taken into consideration the coefficient of \(s\) is \(2 - \alpha_2\). Using the positivity and normalizability condition of PDF we find \(F(\mu rs) = \exp(z^2)\) and show that \(\alpha_2 = 5/2\) and \(\gamma = 3/2\). The exact values of the density-density exponents i.e. \(\alpha_2\) and \(\alpha_3\) are the main results of this part of our work which is derived self consistently.

Since the velocity difference PDF is the Laplace transformed of \(F\), one can readily deduce the right tail of the PDF as \(\frac{1}{\Delta u} \exp(- (\frac{\Delta u}{\mu})^2)\) (for \(s = 1\)) in two and three dimensions and in the limit \(\Delta u \to +\infty\). This tail has been confirmed by several other approaches \(^3\) \(^6\). Left tail of the PDF is sensitive to the scaling exponent of the density-density correlator and is given by \(\frac{1}{\Delta u^{\alpha_3 - 1}}\) when \(\Delta u \to -\infty\). At this stage we cannot derive the PDF of density fluctuation. The reason is that we have only two-point correlation functions of the density field while all of moments of density field are needed for this purpose. In the one dimensional decaying Burgers turbulence it has been claimed that the density PDF would have some power law tail \(^12\) but this problem is open for the forced case.

**III. PDF’S OF 3D BURGERS TURBULENCE IN NON–UNIVERSAL REGION**

In this section we consider the 3-dimensional Burgers turbulence in the non–universal region i.e. \(|u(x) - u(x')| >> U_{\text{rms}}\), and derive the PDF of the longitudinal velocity difference and therefore the exponent of velocity structure functions. As mentioned in the introduction recent works \(^3\) \(^6\) \(^11\) \(^17\) \(^18\) indicate that in the non-universal region the PDF of velocity difference depends on the one-point \(U_{\text{rms}}\) and therefore is not universal which is meant to be sensitive on the details of large scale forcing. This problem is known as the break down of Galilean invariance in the non-universal region. The force free Burgers equation is invariant under space–time translation, parity and scaling transformation. Also it is invariant under Galilean transformation, \(x \to x + V t\) and \(v \to v + V\), where \(V\) is the constant velocity of the moving frame. Both boundary conditions and forcing can violate some or all of symmetries of force free Burgers equation. However it is, usually assumed that in the high Reynolds number flow all symmetries of the Burgers equation are restored in the limit \(r \to 0\) and \(r >> \eta\), where \(\eta\) is the dissipation scale where the viscous effects become important. This means that in this limit the root–mean square velocity fluctuations \(U_{\text{rms}} = \sqrt{\langle v^2 \rangle}\) is not invariant under the constant shift \(V\), cannot enter the relations describing moments of velocity difference. Therefore the effective equations for the inertial–range velocity correlation functions must have the symmetries of the original Burgers equations. For many years this assumption was the basis of turbulence theories. But based on the recent understanding of turbulence, some
of the constraints on the allowed turbulence theories can be relaxed 

According to recent Yakhot modeling of Burgers and N–S turbulence this symmetry breaks in a hard way i.e. the $U_{rms}$ is entered explicitly in the equation of PDF for velocity difference. In the following we aim to show how this symmetry breaks in the sense that the one-point $U_{rms}$ enters in the argument of the PDF in non-universal region. Also since we are interested in the scaling of the longitudinal structure function $S_q = \langle (u(x + r) - u(x))^q \rangle \gg \langle u^q \rangle$, where $u(x)$ is the $x$-component of the three-dimensional velocity field and $r$ is the displacement in the direction of the $x$-axis and the probability density $P(u, r)$ for homogeneous and isotropic turbulence. In the non–universal region using the eq.(11) one can observe that the PDF for the velocity difference in the inviscid Burgers turbulence in two and three dimensions satisfy the following closed equation:

$$\frac{\alpha d s}{r} \partial_u u P - s \partial_r u \partial_r P - \frac{s(1 - s^2)}{r} \partial_r^2 P + \frac{d - 2 + s^2}{r} \partial_r P - \frac{\alpha d}{r} (1 - s^2) \partial_s P + (1 - s^2) \partial_s \partial_r P$$

$$- \frac{(1 - s^2)}{r} \partial_s u \partial_r P + r^2 (2 + 2s^2) \partial_s^2 P = 0 \quad (13)$$

where $s$ equal to $\cos(\theta - \varphi)$ in two-dimensions and $s = \cos \gamma = \cos \theta \cos \psi' + \sin \theta \sin \psi' \cos(\varphi - \varphi')$ in three-dimensions. For determining the reason for neglecting the forcing term is that the forcing contribution is in the order of $r^3/L^2$ so in the inertial range we can safely drop the corresponding term. However we need to consider the effect of forcing by matching the PDF in the inertial range with the PDF in the integral scale. Since we are interested in equation of the longitudinal PDF we have to consider the above equation in the limit of $s \to 1$. This limiting is not trivial and needs to be considered more carefully. The contributions of different terms of the PDF equation in the limiting when $s \to 1$ is determined by the corresponding equation of the structure function.

Assuming that the all of the moments of velocity difference exist, the structure functions $S_n$ for given angle $\gamma$ ( or $s = \cos(\gamma)$) satisfies the following closed equation:

$$[sn + (1 - s^2)] \partial_r \partial_s S_n - ns \alpha d S_n$$

$$-s(1 - s^2) \partial_s S_n + (d - 2 + s^2) \partial_r S_n$$

$$- \alpha d(1 - s^2) \partial_s S_n + n(1 - s^2) \partial_s^2 S_n$$

$$+ r^3 n(n - 1) (n - 2) (1 + 2s^2) S_{n-3} = 0 \quad (14)$$

The forcing contribution to the above equation is the last term i.e. $r^3 n(n - 1) (n - 2) (1 + 2s^2) S_{n-3}$ and this term has not any contribution in the exponent of structure function. However the amplitude of the structure functions do depend on the details of forcing. We mean that the scaling ( multifractal ) exponent of the structure function is not related to the forcing term and is related to the structure of non-linearity and the transverse contributions in the Burgers equation. For solving the eq.(20) one can separate the angular and scale dependent parts of $S_n(r, s)$. The calculations give rise to the result that for $s \to 1$ the structure functions have the following form

$$S_n(r, s) \to s^n S_n(r), \quad (15)$$

where $S_n(r) = \langle (u(x + r) - u(x))^n \rangle \gg \langle u^n \rangle$. Factorizing the angle and scale dependences is known also for the N–S turbulence too [11,24]. The proposed form for structure functions dictates that in the limit when $s \to 1$ the probability distribution of velocity increments satisfies the scaling form, $P(r, u, s) \to \frac{1}{S} P(r, u/s)$ in a sufficient way. It is clear that when $s \to 1$, $P(r, u) = P(-r, -u)$ and it satisfies the following equation in $d=2$ and $3$-dimensions.

$$- \frac{\partial}{\partial u} u - B \frac{\partial}{\partial r} P + A \frac{\partial}{\partial u} u P + 3r^2 \partial_r^3 P = 0 \quad (16)$$

where $P$ is the longitudinal velocity difference PDF, and $B$ approaches to zero as $O(1 - s^2)$ and giving address to $\zeta_3 = 1$ would give $A=1$. The above equation is correct for three and two dimensions and it does not depend on the dimensionality. This form of equation for the PDF has been conjectured recently by Yakhot for the multi-dimensional Burgers turbulence [11]. It is noted that the forcing contribution in the above equation is $3r^2 \partial_r^3 P$ and it is irrelevant in the small scale $r \to 0$. We will deal with the forcing contribution as a boundary condition of PDF in the large scales and this term is responsible for the breakdown of the Galilean invariance . Also in the non-universal region we find the density-density exponent to be $\alpha = d$ in the two and three-dimensions. Now it is easy to see that the eq.(8) can be written as a scale–ordered exponential:

$$P(u, r) = T(e_r \int_0^r dr'L_{K M}(u, r') P(u, r_0))$$

where $L_{K M}$ can be obtained by computing the inverse operator [34]. Using the properties of scale–ordered exponentials the conditional probability density will satisfy the Chapman-Kolmogorov equation. Equivalently we derive that the probability density and as a result the conditional probability density of longitudinal velocity increments satisfies a Kramers-Moyal evolution equation:

$$- \frac{\partial P}{\partial r} = \sum_{n=1}^{\infty} (-1)^n \frac{\partial^n}{\partial u^n} (D^{(n)}(r, u) P) \quad (17)$$

Where $D^{(n)}(r, u) = \frac{\sigma_n}{r} u^n$ [34]. We find that the coefficients $\sigma_n$ depend on $A$ and $B$ through the relation
\[ \sigma_n = (-1)^n \frac{A}{(n+1)(n+2)(n+3) \cdots B+m}. \] The same equation (i.e., eq.(23)) obviously governs over the conditional PDF too but with another boundary condition, i.e., \[ P(u, r | u', r) = \delta(u - u'). \] For a simple case we proceed to find velocity difference PDF \[ P(u, r), \] just by using the same line of reasoning and the well known Bayesian rule. So

\[ P(u, r) = \int P(u, r | u', r) P(u', r) du' \quad (18) \]

where \[ P(u', r) \] is assumed a Gaussian in the integral scale. Since

\[ P(U, r | u', r) = T(e^{r_+} \int e^{r_L} \delta(u - u')) \quad (19) \]

The proposed KM operator has an important property, i.e.,

\[ L_{KM}^+ u^m = \zeta_m u^m \quad (20) \]

where \( \zeta_m \) is the scaling exponent of the longitudinal velocity difference \( S_m \). Plugging the scale ordered form of the conditional PDF in the eq.(24), we will get:

\[ P(u, r) = \left( \frac{r}{L} \right)^{L_{KM}^+} P(u, L) \quad (21) \]

Expanding the assumed gaussian form of \( P(u, L) \) in terms of \( u \) and using (26) ends
with,

\[ P(u, r) = \sum_{m=0}^{\infty} \exp\left[ \ln\left( \frac{r}{L} \right) \zeta_m \right] \left( \frac{u}{U_{rms}} \right)^{2m} (-1)^m \frac{m!}{n!} \]

\[ = \frac{r^{\zeta_m}}{L} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \frac{u}{U_{rms}} \right)^{2m} \]

\[ = \frac{r^{\zeta_m}}{L} \exp\left( -\left( \frac{u}{U_{rms}} \right)^2 \right) \]

where \( \zeta_1 = 1 \). We should stress that the same structure is tractable for the other PDF's in the integral scale other than the simple Gaussian form. This result could be derived from eq.(22) by direct calculations even without consulting the KM form of the evolution operator too. This form verifies the proposed form of the PDF in the inner scales where \( \eta << r << L \) and resonates with numerical observation in the one dimensional Burgers turbulence \[ \mathbb{[3]} \], where the non-universal part of the PDF fits with \[ P(\Delta u, r) = r G\left( \frac{\Delta u}{U_{rms}} \right). \] The same results was derived recently with additional assumption by Yakhot \[ \mathbb{[1]} \]. An interesting point with respect to the possible GI breaking mechanisms should be referred. In our calculation we have shown that \( U_{rms} \) and the one point information has revealed itself because of the matching between the inertial range velocity increment PDF and the integral scale velocity increment PDF. Because the the variance of the velocity increments PDF in the integral scales and for the large \( L \) is in the order of the variance of the one point PDF we observe such a breakdown mechanism even with the lack of the forcing contributions in the \( L_{KM} \). This way of breaking which we would like to call as soft GI breaking can be accompanied with the effects of the forcing where \( U_{rms} \) will be entered \( (U_{rms} = \langle k(0) L \rangle^{1/3}) \) in the equation of the PDF itself. So the second way of the GI breaking which may be referred to as a hard GI breaking is linked to the explicit dependence of \( L_{KM} \) to \( k(0) \). These parts of the Kramers Moyal operator causes some scale dependence for the Kramers-Moyal coefficients and the explicit calculation of the velocity increment PDF in terms of the conditional ones is not a trivial task. The intricate part of the manipulations is related to the non-commutativity of the GI breaking parts of the evolution operator with the GI invariant parts so that the scale-ordered exponential can not be simply operated on the integral scale PDF. However it has been shown that these parts would cause non-universal behaviours of the amplitudes in the velocity increment structure functions. That is for \( r \to 0 \),

\[ S_n(r) = A_n r^{\zeta_n} + 3A_{n-3} \frac{n(n-1)(n-2)}{n+B} \frac{r^{3+\zeta_{n-3}}}{3 + \zeta_{n-3} - \zeta_n} \quad (22) \]

where \( \zeta_n = \frac{4 n^2}{n+5} \). This leads to the non-universality of the PDF shapes in the inertial range \[ \mathbb{[1]} \]. In general both of these mechanisms may be relevant for the scaling of the inertial range PDF with \( \Delta u/urms \), but their clear roles would be determined by explicit calculations.

The \( A \) coefficient in eq.(22) is responsible for the scaling of the structure functions while the \( B \) term is an infinitesimal coefficient which its value is responsible for \( n \) independence of the scaling exponents. One should note that according to our calculations we have derived the following terms just by writing the whole equation which is governed over the PDF and then taking all the source terms which are proportional to the derivatives of the PDF with respect to the angle \( s \). So it is the most important result of our calculations which resembles that without consulting to the conjectures for introducing the scaling terms in the PDF equation all the conjectured terms \[ \mathbb{[1]} \] could be driven just by carefully writing the \( s \) dependence of PDF and then taking the limit \( s \to 1 \). So the \( A \) term comes out to be \( a_d - (d-1) = 1 \) and \( B \) term approaches to zero as \( 1 - s^2 \). These are in complete coincidence with the proposed values which was derived in Yakhot theory for the Burgers turbulence \[ \mathbb{[1]} \]. Using the above equations one can show that the \( S_n(r) \) scales with \( r \) as \( r^{\zeta_n} \) where \( \zeta_n = a_d - (d-1) = 1 \).

According to Eq.(16) for the universal part in the three dimensions we have found \( \alpha_3 = 7/2 \) while in the nonuniversal part we have got \( \alpha_3 = 3 \). Comparing the value of the density-density exponent \( \alpha_d \) in universal and non-universal regions shows a small deviation in the two parts. The dependence of the density density exponent to the
specific parts of the velocity increment PDF would be 
revealed if one takes a closer look at the original gener-
atting function. Considering the eq.(10), \( g(r) \) is referring 
to density-density correlation function conditioned on a 
fixed value of the velocity increment. So it’s scaling ex-
ponent should depend on the behaviour of the velocity 
increment PDF too.

IV. - DISCUSSION

The first point which we wish to address is the role 
of viscosity when \( \nu \to 0 \) instead of the inviscid limit. 
It is well known that the viscosity contributions makes 
the generating function equation to become unclosed . 
To our knowledge the only method for dealing with the 
viscosity contribution is the groundbreaking method of 
Polyakov \[3\] in the one dimensional forced Burgers tur-
bulence. In this method the overall contribution of the 
viscosity is assumed to generate some other terms con-
istent with the symmetries of the original equations and 
their presence just renormalises the terms contributed by 
advective derivative in the equation of generating func-
tion. Boldyrev \[18\] generalised the Polyakov’s method 
for the Burgers turbulence supplemented with continuity 
equation in one dimension by invoking to the same strat-
 egy from which behaviour of the PDF tails in one dimen-
sion was given. Keeping the viscous term in the three 
dimensional Burgers equation and with continuity equa-
tion we have examined the same closure for the viscous 
contribution. The only relevant term consistent with the 
symmetries developed as a generalisation of the Boldyrev 
work \[18\] is \( D_2 = a F_2 \) (see \[18\] for more detail). How-
ever one can prove that in this framework the strong con-
straint of homogeneity, i.e. \( < u > = < u(x_1) - u(x_2) > = 0 \) 
will fix the vlue \( a = 0 \). So at least this closure 
has no contribution in PDF equation and the results will 
not be different from the inviscid calculations.

The second problem which we want to discuss about 
is related to the density probability function in general. 
As far as we know there is one simulation being done by 
Gotoh and Kraichnan \[14\] for finding the tail of the den-
sity PDF in one dimension and for the decaying burgers 
turbulence . For the high density regime they have found 
a power law tail for the density PDF. It is generally ac-
cepted that in one dimension the mass accumulates in 
the shock regimes and the shock statistics will determine 
the density profile in the stationary regime. However we 
are not aware of such simulations for the forced problem 
even in one dimension. In higher dimensions because 
the nature of the singularities in the velocity flow are 
more involved \[30\] the simple picture in one dimension 
regarding the mass accumulation in the singularities of 
the velocity profile can not be conducte in a trivial way. 
Apart from the density PDF there is not any simulation 
for investigating any multi-point corralation function of 
the density in higher dimensions. For \( d = 1 \) Boldyrev 
\[18\] has reported about a simulation on which the expo-
nent of the two point correlation function of the density 
has predicted to be \( \sim 2 \). Any attempt for simulating 
the forced Burgers equation with the continuity equation 
would be valuable for clearing out the outcomes of our 
paper.

In this paper we have proved the Yakhot conjecture 
about the PDF equation for the multi-dimensional Burg-
ers turbulence. However he has shown that considering 
\( B \sim 20 \) and using the exact result \( \zeta_A = 1 \) as a bound-
ary condition over final PDF, the incompressible fully 
developed turbulence can be modeled \[11\]. From his the-
ory he finds the multiscaling of fully developed turbu-
ence compatible with the experimental and simulation 
results. In the centepiece of that work it is emphasised 
that the effect of pressure just renormalises the values of 
\( B \) and creates the \( A \) term which behave as source and 
sink. According to our results we have shown that the 
\( A \) term could be generated just by considering the angu-
lar dependences of PDF and even without pressure finite 
scaling exponents are derived and the terms responsible 
for scaling are present in the equation of the stationary 
PDF. It would be illuminating to seek whether, entering 
the effect of pressure just renormalises the values of 
\( A \) and \( B \) coefficients without adding any new terms in 
the PDF equation. We believe that the effect of pres-
sure in the velocity intermittency can be followed if all 
the informations inherent in the correlations of different 
moments of density field are given . Other important 
perspective which we are pursuing is related to the de-
formation of transvers structure functions and a clear 
picture about the transition between the longitudinal to 
transverse PDF. Consequently this would give a thor-
ough understanding about the issue of intermittency in 
the transverse structure functions \[33\].

Our exact results are also applicable to the descrip-
tion of fractal–nature of Interstellar Medium (ISM) if 
we accept that the Burgers equation supplemented with 
continuity equation is a good candidate for modelling it 
\[22\]. In the framework of the Burgers–turbulence the-
ory of interstellar medium we derive the scaling relation 
\( M(r) \sim R_{dH}^{dH+1} \) for the mass on a region of size \( R \), and the 
value of the \( d_H \) can be predicted in this framwork which 
is \( d_H = 3/2 \). The possible relation of ISM and Burgers 
turbulence will be disscused elsewhere \[36\].

We would like to thank Uriel Frisch, Mehran Kardar 
and Victor Yakhot for useful discussions and S. Rouhani 
and D.D. Tsakalaya for important comments. M.R. 
Rahimi Tabar wish to acknowledge the support of the 
French Ministry of Education for his visit to the Obser-
vatoire Cote d’Azur and U. Frisch for his kind hospitality.
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