TRACE FORMULAS FOR A CONFORMABLE FRACTIONAL DIFFUSION OPERATOR

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Abstract. In this paper, the regularized trace formulas for a diffusion operator which include conformable fractional derivatives of order $\alpha$ ($0 < \alpha \leq 1$) is obtained.

1. Introduction

The fractional derivative has an important place in applied mathematics. Since 1695, the various types of fractional derivatives, usually given through an integral form, have been introduced by many authors (see [1]-[4]).

In 2014, Khalil et al. gave the definition of conformable fractional derivative (see [5]). Shortly after, Abdeljawad and Atangana et al. shown the elementary properties of this derivative (see [6], [7]). The derivative arises in various fields such as quantum mechanics, dynamical systems, time scale problems, diffusions, conservation of mass, etc. (see [10]-[13]).

The trace of a matrix with finite dimensional is the sum of the elements on the main diagonal and is finite. However, the trace of ordinary differential operators with infinite dimensional, which is the sum of all eigenvalues, is not finite. Therefore, the concept of regularize trace for this type operators, which is finite, is mentioned. The regularized trace formulas have great importance, especially, in the solution of the inverse problem according to two spectra.

For about seventy years, the regularized trace formulas for the different types of differential operators have been investigated. Firstly, in 1953, Gel’fand and Levitan obtained the regularized trace formula for the Sturm-Liouville operator with Neumann conditions (see [14]). The study led to the birth of a great and very important theory. After this study, the theory has been continued for the various operators by many researchers (see [15]-[46] and references therein). In 2010, Yang obtained the regularized trace formulas for the diffusion operator i.e., a quadratic pencil of the Schrödinger operator (see [47]). In 2019, Mortazaasl and Jodayree Akbarfam calculated the regularized trace formula for a conformable fractional Sturm-Liouville problem (see [48]).

In the present paper, we consider a diffusion operator which include conformable fractional derivatives of order $\alpha$ ($0 < \alpha \leq 1$) instead of the ordinary derivatives in a traditional diffusion operator. Using the contour integration method, we obtained the regularized trace formulas for this diffusion operator.

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2. Preliminaries

In this section, Firstly, we recall known some concepts of the conformable fractional calculus. Then, we introduce a conformable fractional diffusion operator with the separated boundary conditions on \([0, \pi]\).

**Definition 1.** Let \( f : [0, \infty) \to \mathbb{R} \) be a given function. Then, the conformable fractional derivative of \( f \) of order \( \alpha \) with respect to \( x \) is defined by

\[
D_x^\alpha f(x) = \lim_{h \to 0} \frac{f(x + h^\alpha) - f(x)}{h}, \quad 0 < \alpha \leq 1, \quad x > 0
\]

and

\[
D_x^\alpha f(0) = \lim_{x \to 0^+} D_x^\alpha f(x).
\]

If above limit exist and finite at any point \( x \), \( f \) is \( \alpha \)-differentiable at \( x \) and

\[
D_x^\alpha f(x) = x^{1-\alpha} f'(x).
\]

**Definition 2.** Let \( f : [a, \infty) \to \mathbb{R} \) be a given function. The conformable fractional Integral of \( f \) of order \( \alpha \) is defined by

\[
I_\alpha f(x) = \int_0^x f(t) d_t^\alpha = \int_0^x t^{\alpha-1} f(t) dt, \quad \text{for all } x > 0.
\]

**Lemma 1.** Let \( f : [a, \infty) \to \mathbb{R} \) be any continuous function. Then,

\[
D_x^\alpha I_\alpha f(x) = f(x), \quad \text{for all } x > 0.
\]

**Lemma 2.** Let \( f : (a, b) \to \mathbb{R} \) be any differentiable function. Then,

\[
I_\alpha D_x^\alpha f(x) = f(x) - f(a), \quad \text{for all } x > 0.
\]

**(\alpha\text{-integration by parts):**** Let \( f, g : [a, b] \to \mathbb{R} \) be two conformable fractional differentiable functions. Then,

\[
\int_a^b f(x) D_x^\alpha g(x) d_x^\alpha = \int_a^b f(x) g(x) d_x^\alpha - \int_a^b g(x) D_x^\alpha f(x) d_x^\alpha.
\]

**Lemma 3.** The space \( L_\alpha^2(0, a) \) associated with the inner product for \( f, g \in L_\alpha^2(0, a) \)

\[
\langle f, g \rangle := \int_0^a f(x) g(x) d_x^\alpha x
\]

is a Hilbert space.

**Definition 4.** The Sobolev space \( W_\alpha^2(0, a) \) consists of all functions on \([0, a]\), such that \( f(x) \) is absolutely continuous and \( D_x^\alpha f(x) \in L_\alpha^2(0, a) \).

**Definition 5.** Let \( y(x) \) and \( z(x) \) be \( \alpha \)-differentiable functions on \([0, \pi]\). The fractional Wronskian of \( y(x) \) and \( z(x) \) is defined as

\[
W_\alpha[y(x), z(x)] := \begin{vmatrix} y(x) & z(x) \\ D_x^\alpha y(x) & D_x^\alpha z(x) \end{vmatrix} = y(x) D_x^\alpha z(x) - z(x) D_x^\alpha y(x).
\]

More detail knowledge about the conformable fractional calculus can be seen in [5-9].
Now, let us consider the boundary value problem \( L_\alpha(p(x), q(x), h, H) = L_\alpha \), called as the conformable fractional diffusion operator (CFDO), of the form

\[
\begin{align*}
(1) & \quad \ell_\alpha y := -D_x^\alpha D_x^\alpha y + [2\lambda p(x) + q(x)]y = \lambda^2 y, \quad 0 < x < \pi \\
(2) & \quad U(y) := D_x^\alpha y(0) - hy(0) = 0 \\
(3) & \quad V(y) := D_x^\alpha y(\pi) + Hy(\pi) = 0
\end{align*}
\]

where \( \lambda \) is the spectral parameter, \( h, H \in \mathbb{R} \), \( p(x), D_x^\alpha p(x), q(x) \in W_\alpha^2(0, \pi) \) are real valued functions, \( 0 < \alpha \leq 1 \) and \( p(x) \neq \text{const.} \).

Let the functions \( \varphi(x, \lambda) \) and \( \psi(x, \lambda) \) be the solutions of the equation (1) satisfying the initial conditions

\[
\begin{align*}
(4) & \quad \varphi(0, \lambda) = 1, \quad D_x^\alpha \varphi(0, \lambda) = h, \\
(5) & \quad \psi(\pi, \lambda) = 1, \quad D_x^\alpha \psi(\pi, \lambda) = -H
\end{align*}
\]

respectively. It is clear that \( U(\varphi) = 0 \), \( V(\psi) = 0 \).

Denote

\[
\Delta(\lambda) = W_\alpha[\psi(x, \lambda), \varphi(x, \lambda)].
\]

It is proven in [49] that \( W_\alpha \) does not depend on \( x \) and putting \( x = 0 \) and \( x = \pi \) in (6) it can be written as

\[
\Delta(\lambda) = V(\varphi) = -U(\psi).
\]

**Definition 6.** The function \( \Delta(\lambda) \) is called as the characteristic function of the problem \( L_\alpha \).

**Lemma 4.** For \( |\lambda| \to \infty \) and each fixed \( \alpha \) the following asymptotic formulas hold:

\[
\begin{align*}
(8) & \quad \varphi(x, \lambda) = \cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) + O \left( \frac{1}{|\lambda|^\frac{\alpha}{2}} \exp \left( \frac{|\lambda|^\frac{\alpha}{2}}{\alpha} x^\alpha \right) \right), \\
(9) & \quad D_x^\alpha \varphi(x, \lambda) = - (\lambda - p(x)) \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) + O \left( \exp \left( \frac{|\lambda|^\frac{\alpha}{2}}{\alpha} x^\alpha \right) \right)
\end{align*}
\]

where \( \lambda = \sigma + i\tau \) and

\[
Q(x) := \int_0^x p(t) d_\alpha t.
\]

**Proof.** Firstly, we rewritten equation (1) as

\[
D_x^\alpha D_x^\alpha y + \frac{D_x^\alpha p(x)}{\lambda - p(x)} D_x^\alpha y + (\lambda - p(x))^2 y = (q(x) + p^2(x)) y + \frac{D_x^\alpha p(x)}{\lambda - p(x)} D_x^\alpha y.
\]

It is easily shown that the system of functions \( \{ \cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right), \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) \} \) is a fundamental system for the following differential equation

\[
D_x^\alpha D_x^\alpha y + \frac{D_x^\alpha p(x)}{\lambda - p(x)} D_x^\alpha y + (\lambda - p(x))^2 y = 0.
\]

Thus, the solution of equation (1) satisfying the initial conditions (4) provides the following integral equations

\[
\varphi(x, \lambda) = \cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) + \frac{h}{\lambda - p(0)} \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)
\]

\[
+ \int_0^x \sin \left[ \frac{\lambda}{\alpha} \left( x^\alpha - t^\alpha \right) - Q(x) + Q(t) \right] \frac{(q(t) + p^2(t)) \varphi(t, \lambda) + \frac{D_x^\alpha p(t)}{\lambda - p(t)} D_x^\alpha \varphi(t, \lambda)}{\lambda - p(t)} dt.
\]
and by $\alpha$–differentiating (13) with respect to $x$

$$D_x^\alpha \varphi(x, \lambda) = - (\lambda - p(x)) \left\{ \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) - \frac{h}{\lambda - p(0)} \cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) \right\} \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)$$

(14) $$+ \int_0^x \frac{\cos \left( \frac{\lambda}{\alpha} (x^\alpha - t^\alpha) - Q(t) \right)}{\lambda - p(t)} \left[ (q(t) + p^2(t)) \varphi(t, \lambda) + \frac{D_x^\alpha p(t)}{\lambda - p(t)} D_t^\alpha \varphi(t, \lambda) \right] d_\alpha t \right\}.$$

For each fixed $\alpha$, denote

$$\mu_1(\lambda) := \max_{0 \leq x \leq \pi} \left| \varphi(x, \lambda) \exp \left( - \frac{|\tau_\alpha|}{\alpha} x^\alpha \right) \right| \quad \text{and} \quad \mu_2(\lambda) := \max_{0 \leq x \leq \pi} \left| D_x^\alpha \varphi(x, \lambda) \exp \left( - \frac{|\tau_\alpha|}{\alpha} x^\alpha \right) \right|.$$ 

Since $|\cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)| \leq \exp \left( \frac{|\tau_\alpha|}{\alpha} x^\alpha \right)$ and $|\sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)| \leq \exp \left( \frac{|\tau_\alpha|}{\alpha} x^\alpha \right)$, from (13) and (14) we find

$$\mu_1(\lambda) \leq C \left( 1 + \frac{|h|}{|M|} + \frac{\mu_1(\lambda)}{|M|} + \frac{\mu_2(\lambda)}{|M|} \right), \quad \mu_2(\lambda) \leq C \left( |\lambda| + |h| + \mu_1(\lambda) + \frac{\mu_2(\lambda)}{|M|} \right).$$

Hence, we get

$$\mu_1(\lambda) \leq C, \mu_2(\lambda) \leq C |\lambda|$$

or

$$\varphi(x, \lambda) = O \left( \exp \left( \frac{|\tau_\alpha|}{\alpha} x^\alpha \right) \right), \quad D_x^\alpha \varphi(x, \lambda) = O \left( \lambda \exp \left( \frac{|\tau_\alpha|}{\alpha} x^\alpha \right) \right), \quad |\lambda| \to \infty.$$ 

Substituting these into (13) and (14) we obtain (8) and (9). □

Applying successive approximations method to the equations (13), we can have the more detailed asymptotic of the function $\varphi(x, \lambda)$ as follows

$$\varphi(x, \lambda) = \cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right) + O \left( \frac{|h(\pi x) + \lambda|}{|M|} \right) \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)$$

$$+ \frac{1}{\alpha} \left( \lambda + \frac{1}{2} \int_0^x (q(t) + p(t)) d_\alpha t \right) \sin \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)$$

$$+ \frac{1}{\alpha} \int_0^x (q(t) + p^2(t)) \sin \left( \frac{\lambda}{\alpha} (x^\alpha - 2t^\alpha) - Q(x) + 2Q(t) \right) d_\alpha t$$

$$- \frac{1}{\alpha} \int_0^x D_x^\alpha p(t) \cos \left( \frac{\lambda}{\alpha} (x^\alpha - 2t^\alpha) - Q(x) + 2Q(t) \right) d_\alpha t$$

(15) $$+ \frac{1}{\alpha} \left[ \frac{h(p(x) + p(0))}{2} \right]$$

$$+ \frac{1}{\alpha} \left[ \frac{x^\alpha (x + p(0))^2}{2} + \frac{x^\alpha (x - p(0))^2}{2} \right]$$

$$+ \frac{1}{\alpha} \left[ \frac{x^\alpha (x + p(0))^2}{2} + \frac{x^\alpha (x - p(0))^2}{2} \right] - \frac{1}{\alpha} \int_0^x (q(t) + p(t)) d_\alpha t$$

$$- \frac{1}{\alpha} \left[ \int_0^x (q(t) + p^2(t)) d_\alpha t \right] \cos \left( \frac{\lambda}{\alpha} x^\alpha - Q(x) \right)$$

$$+ O \left( \frac{|\tau_\alpha|}{|\alpha|} \exp \left( \frac{|\tau_\alpha|}{\alpha} x^\alpha \right) \right), \quad |\lambda| \to \infty,$$

uniformly with respect to $x \in [0, \pi]$ for each fixed $\alpha$.

The eigenvalues of $L_\alpha$ coincide with the zeros of its characteristic function $\Delta(\lambda) = V(\varphi) = D_x^\alpha \varphi(\pi, \lambda) + H \varphi(\pi, \lambda)$. Thus, using the formula (15) we can
establish the following asymptotic

\[ \Delta(\lambda) = -\lambda \sin \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) + \frac{(p(\pi) + p(0))}{(\alpha + 1)} \sin \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) + c_1 \cos \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) + \frac{\lambda}{\alpha} \sin \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) + \frac{2\alpha}{\lambda} \cos \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) + c_2 \cos \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) + \frac{2\alpha}{\lambda} \cos \left( \frac{\Delta}{\alpha} \pi^\alpha - c_0 \right) \]

\[ + \frac{1}{2} \int_0^\pi \left( q(t) + p^2(t) \right) \sin \left[ \frac{\Delta}{\alpha} \left( \pi^\alpha - 2t^\alpha \right) - Q(\pi) + 2Q(t) \right] \, \, dt \]

\[ + \frac{1}{2} \int_0^\pi D_t^\alpha p(t) \cos \left[ \frac{\Delta}{\alpha} \left( \pi^\alpha - 2t^\alpha \right) - Q(\pi) + 2Q(t) \right] \, \, dt \]

\[ + O \left( \frac{1}{|\lambda|^\alpha} \exp \left( \frac{|\lambda|}{\alpha} \pi^\alpha \right) \right), \quad |\lambda| \to \infty, \]

where, for each fixed \( \alpha \)

\[ c_0 = Q(\pi) = \int_0^\pi p(t) \, dt, \]

\[ c_1 = h + H + \frac{1}{2} \int_0^\pi \left( q(t) + p^2(t) \right) \, dt, \]

\[ c_2 = \frac{p(\pi)(p(\pi) - p(0))}{\alpha + 1} - \frac{1}{2} \frac{p^{1+n}(\pi) - p^{1+n}(0)}{\alpha + 1} - \frac{1}{2} \frac{(p(\pi) - p(0))^{1+n}}{\alpha + 1} + hH \]

\[ + \frac{h+h}{\alpha} \int_0^\pi \left( q(t) + p^2(t) \right) \, dt + \frac{1}{2} \left( \int_0^\pi \left( q(t) + p^2(t) \right) \, dt \right)^2, \]

\[ c_3 = \frac{(H-h)(p(\pi) - p(0))}{\alpha + 1} + hH \int_0^\pi \left( q(t) + p^2(t) \right) (2p(t) - p(\pi) - p(0)) \, dt. \]

Take a circle \( \Gamma_N = \{ \lambda \mid |\lambda| = \frac{n}{\pi} (N + \frac{1}{2}) \}, \quad N = 0, 1, 2, \ldots \} in the \( \lambda \)-plane. By the standard method using (16) and Rouche’s theorem (see [50]) and taking \( \Delta(\lambda_n) = 0 \) one can prove that in the circle \( \Gamma_N \), there exist exactly \( |n| \) eigenvalues \( \lambda_n \) and have the form

\[ \lambda_n = \frac{n\alpha}{\pi^\alpha - 1} + \frac{nc_0}{\pi^\alpha} + c_1 + A_n \frac{1}{n!} + O \left( \frac{1}{n^2} \right), \quad |n| \to \infty, \]

where \( n \in \mathbb{Z} \) and for each fixed \( \alpha \)

\[ A_n = \frac{1}{\alpha} \int_0^\pi \left( q(t) + p^2(t) \right) \cos \left( \frac{2nt^\alpha}{\pi^\alpha} + \frac{2c_0 t^\alpha}{\pi^\alpha} - 2Q(t) \right) \, \, dt \]

\[ - \frac{1}{\alpha} \int_0^\pi D_t^\alpha p(t) \sin \left( \frac{2nt^\alpha}{\pi^\alpha} + \frac{2c_0 t^\alpha}{\pi^\alpha} - 2Q(t) \right) \, \, dt. \]

**Corollary 1.** According to (17) for each fixed \( \alpha \) and sufficiently large \( |n| \) the eigenvalues \( \lambda_n \) are real and simple.

### 3. Main Results

In this section, using the contour integration method, we will find formulas which are so-called regularized trace formulas.

**Theorem 1.** Let \( \{\lambda_n\}_{n \geq 0} \) be the sequence of the eigenvalues of the problem \( L_\alpha \). Then, for each fixed \( \alpha \), the following trace formulas are valid:

\[ 2 (\lambda_0 - c_0) + \sum_{n=1}^{\infty} [\lambda_n + \lambda_{-n} - 2c_0 - \frac{1}{\alpha (\pi)^{1+\alpha} B_n^\alpha}] = \frac{p(\pi) + p(0) - 2c_0}{\alpha + 1} \]

\[ - \frac{1}{2} \int_0^\pi \left( 1 - \frac{2c_0}{\alpha} \right) (q(t) + p^2(t)) \sin 2Q(t) \, \, dt \]

\[ + \frac{1}{2} \int_0^\pi \left( 1 - \frac{2c_0}{\alpha} \right) D_t^\alpha p(t) \cos 2Q(t) \, \, dt \]
and

\[ 2 (\lambda_0 - c_0)^2 + \sum_{n=1}^{\infty} \left[ (\lambda_n - c_0)^2 + (\lambda_{-n} - c_0)^2 - 2 \left( \frac{n\alpha}{\pi^{\alpha-1}} \right)^2 - \frac{4\alpha c_1}{\pi^{\alpha}} - \frac{2\alpha}{\pi^{\alpha}} C_n \right] \]

\[ = \frac{2\alpha}{\pi} \left( h + H + \frac{1}{2} \int_0^{\pi} (q(t) + p^2(t)) \, d_\alpha t \right) \]

\[ + \frac{\alpha}{\pi} \int_0^{\pi} (q(t) + p^2(t)) \cos 2Q(t) \, d_\alpha t \]

(19) \[ + \frac{\alpha}{\pi} \int_0^{\pi} D^\alpha_t p(t) \sin 2Q(t) \, d_\alpha t \]

\[ + (p(\pi) - c_0) (p(\pi) - p(0)) - (p(\pi) - c_0)^{1+\alpha} + (p(0) - c_0)^{1+\alpha} \]

\[ - \frac{(p(\pi) - p(0))^{1+\alpha}}{2(1+\alpha)} + 2hH + (h + H) \int_0^{\pi} (q(t) + p^2(t)) \, d_\alpha t \]

\[ + \frac{1}{4} \left( \int_0^{\pi} (q(t) + p^2(t)) \, d_\alpha t \right)^2 \]

where

\[ B_n = \int_0^{\pi} (q(t) + p^2(t)) \sin \left( \frac{2n\alpha t}{\pi^{\alpha-1}} \right) \sin 2Q(t) \, d_\alpha t \]

\[ - \int_0^{\pi} D^\alpha_t p(t) \sin \left( \frac{2n\alpha t}{\pi^{\alpha-1}} \right) \cos 2Q(t) \, d_\alpha t, \]

\[ C_n = \int_0^{\pi} (q(t) + p^2(t)) \cos \left( \frac{2n\alpha t}{\pi^{\alpha-1}} \right) \cos 2Q(t) \, d_\alpha t \]

\[ - \int_0^{\pi} D^\alpha_t p(t) \cos \left( \frac{2n\alpha t}{\pi^{\alpha-1}} \right) \sin 2Q(t) \, d_\alpha t. \]

**Proof.** Firstly, we consider the case \( c_0 = 0. \)

Denote

(20) \[ \Delta_0(\lambda) = -\lambda \sin \left( \frac{\Delta}{\alpha} \right). \]

It is clear that the zeros of the function \( \Delta_0(\lambda) \) is

\[ \mu_n = \frac{n\alpha}{\pi^{\alpha-1}}, \quad n \in \mathbb{Z}, \]

where only \( \mu_0 = 0 \) is double. We note that for each fixed \( \alpha \) and sufficiently large \( N \), the eigenvalues \( \lambda_n \) which are the zeros of \( \Delta(\lambda) \) are inside \( \Gamma_N \) and the numbers \( \mu_n \) do not lie on the contour \( \Gamma_N \).

Let \( \Delta(\lambda) = \lambda (\lambda - \lambda_n) \) and \( \Delta_0(\lambda) = \lambda \left( \lambda - \frac{n\alpha}{\pi^{\alpha-1}} \right) \), then, from the logarithmic derivatives of the functions \( \Delta(\lambda) \) and \( \Delta_0(\lambda) \), we have

\[ \frac{\lambda \Delta(\lambda)}{\Delta(\lambda)} = \frac{2\lambda - \lambda_n}{\lambda - \lambda_n} \quad \text{and} \quad \frac{\lambda \Delta_0(\lambda)}{\Delta_0(\lambda)} = \frac{2\lambda - \frac{n\alpha}{\pi^{\alpha-1}}}{\lambda - \frac{n\alpha}{\pi^{\alpha-1}}}, \]

respectively, where \( \Delta = \frac{d}{d\lambda}. \)

Thus, from the residue theorem, the following equalities are valid:

(21) \[ \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \Delta(\lambda) \, d\lambda = \sum_{n=-N}^{N} \text{Res} \left( \lambda \frac{\Delta(\lambda)}{\Delta(\lambda)}, \lambda_n \right) = \sum_{n=0}^{N} (\lambda_n + \lambda_{-n}) \]
and similarly

\[
\text{(22)} \quad \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \frac{\Delta_0(\lambda)}{\Delta_0(\lambda)} \, d\lambda = \sum_{n=-N}^{N} \text{Re} \left( \lambda \frac{\Delta_0(\lambda)}{\Delta_0(\lambda)} \frac{n\alpha}{\pi n + \tau} \right) = \sum_{n=0}^{N} \left[ \frac{n\alpha}{\pi n + \tau} + \left( -\frac{n\alpha}{\pi n + \tau} \right) \right].
\]

Subtracting (21) and (22) side by side, we get

\[
\sum_{n=0}^{N} (\lambda_n + \lambda_{-n}) = \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \left( \frac{\Delta(\lambda)}{\Delta(\lambda)} - \frac{\Delta_0(\lambda)}{\Delta_0(\lambda)} \right) \, d\lambda
\]

\[
= \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda d \left( \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \right) = -\frac{1}{2\pi i} \oint_{\Gamma_N} \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \, d\lambda.
\]

On the other hand, it follows from (16) and (20) that

\[
\frac{\Delta(\lambda)}{\Delta_0(\lambda)} = 1 - \frac{p(\tau) + p(0) + 2A(\lambda)}{2\lambda} - \frac{c_1 + B(\lambda)}{\lambda} \cot \left( \frac{\Delta_0(\lambda)}{\alpha} \right)
\]

\[
- \frac{c_2}{\alpha} - \frac{c_3}{\alpha^2} \cot \left( \frac{\Delta_0(\lambda)}{\alpha} \right) + O \left( \frac{1}{\alpha} \right), \quad \text{on } \Gamma_N,
\]

where

\[
A(\lambda) = \frac{1}{2} \int_{\alpha}^{\beta} (q(t) + p^2(t)) \sin \left( \frac{2\lambda t}{\alpha} \right) \, dt + \frac{1}{2} \int_{\alpha}^{\beta} D_t^p p(t) \cos \left( \frac{2\lambda t}{\alpha} \right) \, dt,
\]

\[
B(\lambda) = \frac{1}{2} \int_{\alpha}^{\beta} (q(t) + p^2(t)) \cos \left( \frac{2\lambda t}{\alpha} \right) \, dt - \frac{1}{2} \int_{\alpha}^{\beta} D_t^p p(t) \sin \left( \frac{2\lambda t}{\alpha} \right) \, dt.
\]

Taking Taylor’s expansion formula for the \(\ln(1-u)\) into account, we get

\[
\text{(25)} \quad \ln \left( \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \right) = -\frac{p(\tau) + p(0) + 2A(\lambda)}{2\lambda} - \frac{c_1 + B(\lambda)}{\lambda} \cot \left( \frac{\Delta_0(\lambda)}{\alpha} \right) + O \left( \frac{1}{\alpha^2} \right), \quad \text{on } \Gamma_N,
\]

Thus, substituting (24) into (23) we find

\[
\sum_{n=0}^{N} (\lambda_n + \lambda_{-n}) = \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{p(\tau) + p(0) + 2A(\lambda)}{2\lambda} \, d\lambda
\]

\[
+ \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{c_1 + B(\lambda)}{\lambda} \cot \left( \frac{\Delta_0(\lambda)}{\alpha} \right) \, d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_N} O \left( \frac{1}{\alpha^2} \right) \, d\lambda.
\]

By the well-known formulas such as the generalized Cauchy integral formula, the residue theorem and \(\cot z = \frac{1}{z} + 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2 - \pi^2 z^2} \) (see [51] for more details), the contour integrals in (26) calculate that

\[
\text{(27)} \quad \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{p(\tau) + p(0) + 2A(\lambda)}{2\lambda} \, d\lambda = \frac{p(\tau) + p(0)}{2} + A(0),
\]

\[
\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{c_1 + B(\lambda)}{\lambda} \cot \left( \frac{\Delta_0(\lambda)}{\alpha} \right) \, d\lambda = \frac{c_1}{\alpha} B(0)
\]

\[
+ \frac{1}{\alpha} \sum_{n=1}^{N} \frac{1}{n} \left( B \left( \frac{n\alpha^2}{\pi n + \tau} \right) - B \left( -\frac{n\alpha^2}{\pi n - \tau} \right) \right),
\]

and for sufficiently large \(N\) and each fixed \(\alpha\)

\[
\text{(29)} \quad \left| \oint_{\Gamma_N} O \left( \frac{1}{\alpha^2} \right) \, d\lambda \right| = O \left( \frac{1}{\alpha^2} \right).
\]
From (26)-(29), we get
\[ 2\lambda_0 + \sum_{n=1}^{N} \left[ \lambda_n + \lambda_{-n} - \frac{1}{n\alpha \pi^{1-n}} \left( B \left( \frac{n\alpha^2}{\pi^{n+1}} \right) - B \left( -\frac{n\alpha^2}{\pi^{n+1}} \right) \right) \right] \]
\[ = \frac{\hat{p}(\pi) + \hat{p}(0)}{2} + A(0) + \frac{\alpha}{\pi} B(0) + o \left( \frac{1}{N} \right). \]

For \( N \to \infty \) in (30),
\[ 2\lambda_0 + \sum_{n=1}^{\infty} \left[ \lambda_n + \lambda_{-n} - \frac{1}{n\alpha \pi^{1-n}} \left( B \left( \frac{n\alpha^2}{\pi^{n+1}} \right) - B \left( -\frac{n\alpha^2}{\pi^{n+1}} \right) \right) \right] \]
\[ = \frac{\hat{p}(\pi) + \hat{p}(0)}{2} + A(0) + \frac{\alpha}{\pi} B(0) \]
is obtained.

Now we consider the case \( c_0 \neq 0 \).

It is obvious that we can rewrite the equation (1) as
\[ -D_{x}^{\alpha} D_{x}^{\alpha} y = \left[ 2 \left( \lambda - c_0 \right) p(x) - c_0 + q(x) + 2 c_0 \bar{p}(x) - c_0^2 \right] y = (\lambda - c_0)^2 y. \]

Denote \( \lambda - c_0 = \widetilde{\lambda}, q(x) + 2 c_0 \bar{p}(x) - c_0^2 = \bar{q}(x) \) and \( p(x) - c_0 = \bar{p}(x) \). Thus,
\[ -D_{x}^{\alpha} D_{x}^{\alpha} y = \left[ 2 \widetilde{\lambda} \bar{p}(x) + \bar{q}(x) \right] y = \lambda^2 y. \]

For the equation (32), at case \( \widetilde{c}_0 = \int_{0}^{\pi} \bar{p}(t) d_{x} t = 0 \), according to (31) we obtain
\[ 2\widetilde{\lambda}_0 + \sum_{n=1}^{\infty} \left[ \widetilde{\lambda}_n + \widetilde{\lambda}_{-n} - \frac{B_n}{n\alpha \pi^{1-n}} \right] \]
\[ = \frac{\hat{p}(\pi) + \hat{p}(0)}{2} + \widetilde{A}(0) + \frac{\alpha}{\pi} \widetilde{B}(0), \]
where,
\[ A \left( \widetilde{\lambda} \right) := \widetilde{A} \left( \lambda \right) = A \left( \lambda \right), B \left( \widetilde{\lambda} \right) := \widetilde{B} \left( \lambda \right) = B \left( \lambda \right) \] and \( \widetilde{B} \left( \frac{n\alpha^2}{\pi^{n+1}} \right) - \widetilde{B} \left( -\frac{n\alpha^2}{\pi^{n+1}} \right) = B_n. \)

Substituting the expressions of \( \widetilde{\lambda}_n, \bar{q}(x) \) and \( \bar{p}(x) \) into (33), we arrive at (18).

Similarly, we prove that the formula (19) is true.

We consider the case \( c_0 = 0 \) again. Denote \( \Delta(\lambda) = \lambda^2 (\lambda - \lambda_n) \) and \( \Delta_0(\lambda) = \lambda^2 \left( \lambda - \frac{n\alpha}{\pi^{n+1}} \right) \). Then,
\[ \lambda^2 \frac{\Delta(\lambda)}{\Delta(\lambda_0)} = \frac{3\lambda^2 - 2\lambda \lambda_n}{\lambda - \lambda_n} \quad \text{and} \quad \lambda^2 \frac{\Delta_0(\lambda)}{\Delta_0(\lambda_0)} = \frac{3\lambda^2 - 2\lambda \frac{n\alpha}{\pi^{n+1}}}{\lambda - \frac{n\alpha}{\pi^{n+1}}}. \]

Thus, the following equalities are valid:
\[ \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda^2 \frac{\Delta(\lambda)}{\Delta(\lambda_0)} d\lambda = \sum_{n=-N}^{N} \text{Res} \left( \lambda^2 \frac{\Delta(\lambda)}{\Delta(\lambda_0)}, \lambda_n \right) = \sum_{n=0}^{N} \left( \lambda_n^2 + \lambda_{-n}^2 \right) \]
and
\[ \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda^2 \frac{\Delta_0(\lambda)}{\Delta_0(\lambda_0)} d\lambda = \sum_{n=-N}^{N} \text{Res} \left( \lambda^2 \frac{\Delta_0(\lambda)}{\Delta_0(\lambda_0)}, \frac{n\alpha}{\pi^{n+1}} \right) = \sum_{n=0}^{N} \left[ \left( \frac{n\alpha}{\pi^{n+1}} \right)^2 + \left( -\frac{n\alpha}{\pi^{n+1}} \right)^2 \right]. \]
Subtracting (34) and (35) side by side, we get
\[
\sum_{n=0}^{N} \left[ \lambda_n^2 + \lambda_{-n}^2 - \left( \frac{n\alpha}{\pi^2} \right)^2 - \left( -\frac{n\alpha}{\pi^2} \right)^2 \right] = \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda^2 \left( \frac{\Delta(\lambda)}{\Delta_0(\lambda)} - \frac{\Delta_0(\lambda)}{\Delta(\lambda)} \right) d\lambda
\]
\[\text{(36)}\]
\[
\frac{1}{2\pi i} \oint_{\Gamma_N} \lambda^2 d\left( \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \right) \right) = -\frac{1}{2\pi i} \oint_{\Gamma_N} 2\lambda \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda.
\]
From (24), we obtain
\[
\ln \left( \frac{\Delta(\lambda)}{\Delta_0(\lambda)} \right) = -\frac{p(\pi)+p(0)+2A(\lambda)}{2\lambda} - \frac{c_1+B(\lambda)}{\lambda} \cot \left( \frac{\Delta}{\alpha} \pi^\alpha \right)
\]
\[\text{(37)}\]
\[-\frac{c_1}{\lambda^4} - \frac{\pi}{\lambda^3} \cot \left( \frac{\Delta}{\alpha} \pi^\alpha \right) + O \left( \frac{1}{\lambda^2} \right), \text{ on } \Gamma_N.
\]
Thus, the integral on the right side of (36) is written as
\[
-\frac{1}{2\pi i} \oint_{\Gamma_N} 2\lambda \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda = \frac{1}{2\pi i} \oint_{\Gamma_N} (p(\pi) + p(0) + 2A(\lambda)) d\lambda
\]
\[\text{(38)}\]
\[+\frac{1}{2\pi i} \oint_{\Gamma_N} 2(c_1 + B(\lambda)) \cot \left( \frac{\Delta}{\alpha} \pi^\alpha \right) d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2\pi}{\lambda} d\lambda
\]
\[+\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2\pi}{\lambda} \cot \left( \frac{\Delta}{\alpha} \pi^\alpha \right) d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_N} O \left( \frac{1}{\lambda^2} \right) d\lambda.
\]
For the contour integrals in (38), we have
\[
\frac{1}{2\pi i} \oint_{\Gamma_N} [p(\pi) + p(0) + 2A(\lambda)] d\lambda = 0,
\]
\[
\frac{1}{2\pi i} \oint_{\Gamma_N} 2(c_1 + B(\lambda)) \cot \left( \frac{\Delta}{\alpha} \pi^\alpha \right) d\lambda = \frac{2\alpha c_1}{\pi^\alpha} + \frac{4\alpha c_1}{\pi^\alpha} N
\]
\[\text{(39)}\]
\[+\frac{2\pi}{\pi} B(0) + \frac{2\pi}{\pi} \sum_{n=1}^{N} \left[ B \left( \frac{n\alpha^2}{\pi^2} \right) + B \left( -\frac{n\alpha^2}{\pi^2} \right) \right],
\]
\[\text{(40)}\]
\[\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2\pi}{\lambda} d\lambda = 2c_2,
\]
\[\text{(41)}\]
\[\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2\pi}{\lambda} \cot \left( \frac{\Delta}{\alpha} \pi^\alpha \right) d\lambda = 0.
\]
Substituting the expressions of (29) and (39)-(42) into (38), we arrive
\[
-\frac{1}{2\pi} \oint_{\Gamma_N} 2\lambda \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda = \frac{2\alpha c_1}{\pi^\alpha} + \frac{4\alpha c_1}{\pi^\alpha} N
\]
\[+\frac{2\pi}{\pi} B(0) + \frac{2\pi}{\pi} \sum_{n=1}^{N} \left[ B \left( \frac{n\alpha^2}{\pi^2} \right) + B \left( -\frac{n\alpha^2}{\pi^2} \right) \right] + 2c_2 + O \left( \frac{1}{N} \right)
\]
and from (36)
\[
2\lambda_0^2 + \sum_{n=0}^{N} \left[ \lambda_n^2 + \lambda_{-n}^2 - 2 \left( \frac{n\alpha}{\pi^2} \right)^2 - 4\alpha c_1 - 2\alpha \left( B \left( \frac{n\alpha^2}{\pi^2} \right) + B \left( -\frac{n\alpha^2}{\pi^2} \right) \right) \right]
\]
\[\text{(43)}\]
\[= \frac{2\alpha c_1}{\pi^\alpha} + \frac{2\alpha}{\alpha} B(0) + 2c_2 + O \left( \frac{1}{N} \right).
\]
For $N \to \infty$ in (43)

$$2\lambda_0^2 + \sum_{n=0}^{\infty} \left[ \lambda_n^2 + \lambda_{-n}^2 - 2 \left( \frac{n\alpha}{\pi^{2+\alpha}} \right)^2 - \frac{4\alpha c_1}{\pi^{2+\alpha}} - \frac{2\alpha}{\pi^{2+\alpha}} \left( B \left( \frac{n\alpha}{\pi^{2+\alpha}} \right) + B \left( -\frac{n\alpha}{\pi^{2+\alpha}} \right) \right) \right]$$

(44) $= \frac{2\alpha c_1}{\pi^{2+\alpha}} + \frac{2\alpha}{\pi^{2+\alpha}} B(0) + 2\tilde{c}_2$

is obtained.

In the case $c_0 \neq 0$, from (44) we can write that

$$2\lambda_0^2 + \sum_{n=0}^{\infty} \left[ \lambda_n^2 + \lambda_{-n}^2 - 2 \left( \frac{n\alpha}{\pi^{2+\alpha}} \right)^2 - \frac{4\alpha c_1}{\pi^{2+\alpha}} - \frac{2\alpha}{\pi^{2+\alpha}} C_n \right]$$

(45) $= \frac{2\alpha c_1}{\pi^{2+\alpha}} + \frac{2\alpha}{\pi^{2+\alpha}} \tilde{B}(0) + 2\tilde{c}_2$.

Substituting the known expressions of $\lambda_n$, $\tilde{q}(x)$ and $\tilde{p}(x)$ into (45), we arrive the formula (19), where

$$\tilde{B} \left( \frac{n\alpha}{\pi^{2+\alpha}} \right) + \tilde{B} \left( -\frac{n\alpha}{\pi^{2+\alpha}} \right) = C_n,$$

$$\tilde{c}_1 = c_1,$$

$$\tilde{c}_2 = \frac{(p(\pi)-c_0)(p(\pi)-p(0))}{2} - \frac{(p(\pi)-c_0)^{1+\alpha} + (p(0)-c_0)^{1+\alpha}}{4(1+\alpha)} - \frac{(p(\pi)-p(0))^{1+\alpha}}{4(1+\alpha)} + hH$$

$$+ \frac{(h+H)^2}{2} \int_0^\pi \left( q(t) + p^2(t) \right) \, d_\alpha t + \frac{1}{3} \left( \int_0^\pi \left( q(t) + p^2(t) \right) \, d_\alpha t \right)^2.$$

\[\square\]

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