Higher-Order Correct Multiplier Bootstraps for Count Functionals of Networks

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Abstract
Subgraph counts play a central role in both graph limit theory and network data analysis. In recent years, substantial progress has been made in the area of uncertainty quantification for these functionals; several procedures are now known to be consistent for the problem. In this paper, we propose a new class of multiplier bootstraps for count functionals. We show that a bootstrap procedure with a multiplicative weights exhibits higher-order correctness under appropriate sparsity conditions. Since this bootstrap is computationally expensive, we propose linear and quadratic approximations to the multiplier bootstrap, which correspond to the first and second-order Hayek projections of an approximating U-statistic, respectively. We show that the quadratic bootstrap procedure achieves higher-order correctness under analogous conditions to the multiplicative bootstrap while having much better computational properties. We complement our theoretical results with a simulation study and verify that our procedure offers state-of-the-art performance for several functionals.

1 Introduction
Network data, which represent complex relationships between different entities, have become ubiquitous in many fields. Examples of network data abound; we name a few below to provide the reader with an impression of the prevalence of such data. Facebook networks represent friendships between users. Protein-protein interaction networks are undirected networks consisting of pairs of bait-prey proteins that bind to each other during coaffinity purification experiments arising in mass spectrometry analysis. Finally, brain networks represent association between regions of interest in the brain.

Count functionals play a pivotal role in the analysis of network data. In biological networks, it is believed that certain subgraphs represent functional subunits within the larger system (Chen and Yuan, 2006; Daudin et al., 2008; Kim et al., 2014; Milo et al., 2002). In social networks, the frequency of triangles provides information about the likelihood of mutual friendships (Myers et al., 2014; Newman, 2001; Ugander et al., 2011). At a more theoretical level, count
functionals may be viewed as network analogs of the moments of a random variable; under some conditions, a method of moments approach may be used to estimate the underlying model ([Bickel et al. 2011]). Furthermore, convergence of graph sequences (at least dense sequences) may be stated in terms of the convergence of a collection of subgraph frequencies ([Borgs et al. 2008]).

Given their practical and theoretical importance, quantifying the uncertainty of count functionals is naturally of substantial interest; such estimates can be used to construct confidence intervals for underlying parameters and conduct two-sample tests. In other settings, the bootstrap, introduced by Efron (1979), is a powerful inferential tool that operates on the principle of resampling the data to gauge variability. In the present work, we consider the problem of developing fast and accurate multiplier bootstrap procedures for count functionals of networks. Our procedure is based on the idea that, under the sparse graphon model (see Section 3.1), a count functional may be viewed as a (perturbed) U-statistic that is perturbed by asymptotically negligible Bernoulli noise. Viewing a count statistic as a (perturbed) U-statistic naturally allows the adaptation of bootstrap methods for U-statistics (see, for example, Bose and Chatterjee (2018)) to network data.

Recently, Levin and Levina (2019) proposed a two-step bootstrap procedure for count functionals that involves estimating the latent positions with the adjacency spectral embedding (see, for example, Athreya et al. (2018)) in the first step and in the second step, resampling the corresponding U-statistic with the estimated positions. However, for this estimation, the rank of the underlying network model is required to be known and finite. While it may be possible to relax this assumption, it is also likely that the procedure performs worse with an estimated rank. In any case, this estimation step complicates the theoretical analysis of their procedure.

In contrast, our bootstrap procedure involves directly multiplying each potential subgraph in the network by a multiplier random variable. Applying the multiplier directly mimics the data generating process and we do not have to estimate the latent positions. We show that our multiplier bootstrap is higher-order correct under appropriate sparsity conditions. Since this higher-order correct bootstrap is computationally expensive, we propose linear and quadratic approximations to the multiplier bootstrap, which correspond to the first and second-order Hayek projections of the U-statistic, respectively. Using only the linear term of the Hayek projection was also also considered in Levin and Levina (2019), but we see that this procedure sacrifices higher-order accuracy at the cost of faster computation. In contrast, the quadratic multiplier may be viewed as a computationally efficient approximation to our multiplicative multiplier that retains higher-order correctness under the same conditions. Our observation is also in agreement with previous work on Edgeworth expansions for U-statistics ([Bentkus et al. 1997] Lai and Wang [1993] Maesono [1997]), which suggest that the bootstrap must preserve the first two terms of the Hayek decomposition for higher-order correctness.

To establish higher-order correctness of our bootstrap procedure, we build upon the recent work of Zhang and Xia (2020), who establish an Edgeworth expansion for network moments. Their result serves as a point of reference.
for our Edgeworth expansions. The authors show that the network noise has a “smoothing” effect that allows them to bypass the typical Cramer condition, which is restrictive in the network setting. We are also able to bypass the Cramer condition for the bootstrap, but by using a different approach. We choose a continuous multiplier that matches the first three moments of the data; it is well-known that continuous random variables satisfy Cramer’s condition. To derive our Edgeworth expansion for the bootstrap, we also build upon results from Wang and Jing (2004) for order two U-statistics. It turns out that network noise, particularly when the graph is sparse, causes certain terms related to our Edgeworth expansion to blow up. While the details are technical, we are able to show that a valid Edgeworth expansion is still possible, with the sparsity level directly affecting the convergence rate.

We will now provide a roadmap for the rest of the paper. In Section 2, we discuss related work, focusing on the emerging area of resampling methods for network data. The problem setting and our bootstrap proposal is introduced in Section 3. In Section 4, we present our main results, which establish higher-order correctness for our bootstrap procedures. Finally, in Section 5, we present a simulation study, which shows that our procedure exhibits strong finite-sample performance in a variety of settings.

2 Related Work

The first theoretical result for resampling network data was attained by Bhat-tacharyya and Bickel (2015). Their subsampling proposals involve expressing the variance of a count functional in terms of other count functionals and estimating the non-negligible terms through subsampling. Lunde and Sarkar (2019) show that it is also possible to conduct inference using quantiles of the subsampling distribution as in Politis and Romano (1994). Green and Shalizi (2017) propose a bootstrap based on the empirical graphon; Zhang and Xia (2020) establish conditions under which the empirical graphon bootstrap exhibits higher order correctness. They require Cramer’s condition for the leading term of the Hoeffding projection, which is restrictive for network models. As discussed before, Levin and Levina (2019) study a network bootstrap closely related to our proposal, with one of the main differences being the estimation of latent positions.

Lin et al. (2020) establish the validity of the network jackknife for count functionals. The jackknife is also considered as a variance estimator for an empirical Edgeworth expansion in Zhang and Xia (2020). The empirical Edgeworth expansion proposal, which has been considered in other settings (see, for example, Putter and Van Zwet (1998) and Maesono (1997)), involves studentizing by a variance estimate and plugging in estimated moments into an Edgeworth expansion. For reasons that will be discussed shortly, Zhang and Xia (2020) are able to establish sharper rates of convergence than what we establish for our multiplier bootstraps. While computationally more demanding, work in other settings suggests that the bootstrap may have some favorable properties over empirical Edgeworth expansions (see for example, Hall (1990)).
On the mathematical side, the analysis of our multiplier bootstrap involves Edgeworth expansions for weighted sums. Prior work (c.f. Bai and Zhao (1986) and Liu (1988)) suggests that establishing sharp rates of convergence for the independent but non-identically distributed sequences is more difficult, with the above references establishing a $o(1/\sqrt{n})$ error bound instead of the $O(1/n)$ bound for i.i.d. sequences. Edgeworth expansions for multiplier bootstraps of (degree 2) U-statistics are also considered in Wang and Jing (2004).

3 Problem Setup and Notation

3.1 The Sparse Graphon Model

Let $\{A\}_{n \in \mathbb{N}}$ denote a sequence of $n \times n$ binary adjacency matrices and let $w : [0,1]^2 \to \mathbb{R}$ be a symmetric measurable function such that $\int_0^1 \int_0^1 w(u,v) \, du \, dv = 1$ and $w(u,v) \leq C$ for some $1 \leq C < \infty$. We assume that $A^{(n)}$ is generated by the following model:

$$A_{ij}^{(n)} = A_{ji}^{(n)} \sim \text{Bernoulli}(\rho_n w(X_i, X_j)) \quad (1)$$

where $X_i, X_j \sim \text{Unif}[0,1]$, $\rho_n \to 0$, and $A_{ii}^{(n)} = 0$. While closely related models were considered by Bollobas et al. (2007), Hoff et al. (2002), and Borgs et al. (2019), this particular parameterization was introduced by Bickel and Chen (2009). We will refer to (1) as the sparse graphon model. Sparse graphons are a very rich class of models, subsuming many widely used models, including stochastic block models and their variants (Airoldi et al., 2008; Holland et al., 1983; Karrer and Newman, 2011), and (generalized) random dot product graphs (Rubin-Delanchy et al., 2018; Young and Scheinerman, 2007). More generally, sparse graphons are natural models for graphs that exhibit vertex exchangeability; the functional form is motivated by representation theorems for exchangeable arrays established by Aldous (1981) and Hoover (1979). The parameter $\rho_n = P(A_{ij} = 1)$ determines the sparsity level of the sequence $\{A^{(n)}\}_{n \in \mathbb{N}}$. Many real world graphs are thought to be sparse, with $o(n^2)$ edges; $\rho_n \to 0$ is needed for graphs generated by (1) to exhibit this behavior.

While boundedness of the graphon is a common assumption in the statistics literature (see, for example, the review article by Gao and Ma (2019)), it should be noted that unbounded (integrable) graphons are known to be more expressive. As noted by Borgs et al. (2019), unboundedness allows graphs that exhibit power-law degree distributions, a property that bounded graphons fail to capture. For mathematical expedience, in the present article we focus on the bounded case, but we believe that our analysis may be extended to sufficiently light-tailed unbounded graphons as well.

3.2 Count Functionals

Now we will introduce notation related to our functional of interest. Let $R$ denote the adjacency matrix of a subgraph of interest, with $r$ vertices and $s$
Thus, we may view \( \hat{\theta} \) when there is no ambiguity, is formed by averaging over all \( r \) where we say that \( A \) is a \( r \)-tuple in the graph. Let \( A_{i_1, \ldots, i_r} \) denote the adjacency matrix formed by the node subset \( \{i_1, \ldots, i_r\} \) and for each such \( r \)-tuple, define the following function:

\[
H(A_{i_1, \ldots, i_r}) := \mathbb{1}(A_{i_1, \ldots, i_r} \cong R)
\]

where we say that \( A_{i_1, \ldots, i_r} \cong R \) if there exists a permutation function \( \pi \) such that \( A_{\pi(i_1), \ldots, \pi(i_r)} = R \). Our count functional, which we denote \( \hat{T}_n \), is formed by averaging over all \( r \)-tuples in the graph.

\[
\hat{T}_n := \frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} H(A_{i_1, \ldots, i_r})
\]

Now let \( V(Q) \subset \{1, \ldots, n\} \) denote the vertex set associated with a graph \( Q \) and \( E(Q) \subset V(Q) \times V(Q) \) the corresponding vertex set. Furthermore, let \( E(Q) \) denote the complement of \( E(Q) \) given the universal set \( V(Q) \times V(Q) \). Define the following kernel:

\[
h_n(X_{i_1}, \ldots, X_{i_r}) := \mathbb{E}[H(A_{i_1, \ldots, i_r}) \mid X_{i_1}, \ldots, X_{i_r}]
\]

\[
= \sum_{Q \sim R, V(Q) = \{i_1, \ldots, i_r\}} \prod_{(i,j) \in E(Q)} \rho_n w(X_i, X_j) \prod_{(i,j) \in E(Q)} (1 - \rho_n w(X_i, X_j))
\]

(3)

For readability, we will suppress the \( n \) in \( h_n \) in what follows. Now, define the following (conventional) U-statistic:

\[
T_n := \frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} h(X_{i_1}, \ldots, X_{i_r})
\]

For notational convenience we will refer to \( h(X_{i_1}, \ldots, X_{i_r}) \) by \( h(X_S) \), where \( S \) is the subset \( \{i_1, \ldots, i_r\} \). Denote \( \theta_n := \mathbb{E}[h(X_S)] \). We see that \( \rho_n \theta_n \rightarrow \mu \). This can be thought of as a normalized subgraph density that we want to infer. The normalization by \( \rho_n \) is to ensure that our functional converges to an informative non-zero quantity.

By a central limit theorem for U-statistics ([Hoeffding 1948], it can be shown that \( (T_n - \theta_n)/\sigma_n \) is asymptotically Gaussian. Here \( \sigma_n^2 = r^2 \tau_n^2 / n \), and \( \tau_n^2 = \text{var}(\mathbb{E}[h(X_S)|X_1]) \). Furthermore, [Bickel et al. 2011] show that, for acyclic graphs and \( p \)-cycles, under the condition \( n \rho_n \rightarrow \infty \), \( (T_n - T_n)/\sigma_n = o_P(1) \). Thus, we may view \( (\hat{T}_n - \theta_n)/\sigma_n = (T_n - T_n)/\sigma_n + (T_n - \theta_n)/\sigma_n \) as a U-statistic perturbed by asymptotically negligible noise.

### 3.3 Proposed Bootstrap Procedures

In order to estimate the subgraph density, we will consider the following multiplier bootstrap procedures. In what follows let \( \xi_1, \ldots, \xi_n \) be i.i.d. continuous random
variables with central moments $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 1$. An example of such a random variable is the following product of two independent random variables.

$$X \sim N(1, 1/2) \quad Y \sim N(1, 1/3) \quad Z = XY \quad (4)$$

Let $\xi_{i_1 \ldots i_r}$ denote $\xi_{i_1} \times \ldots \times \xi_{i_r}$ and define the following multiplicative bootstrap:

$$\hat{T}_{n,M}^* = \hat{T}_n + \frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r} \xi_{i_1 \ldots i_r} \cdot \left( H(A_{i_1, \ldots, i_r}^{(n)}) - \hat{T}_n \right) \quad (5)$$

Our multiplicative bootstrap is motivated by Hoeffding’s decomposition (see Supplementary Section A for details). The first two terms of the decomposition for $\hat{U}_n - \theta_n$ are given by:

$$g_1(X_i) = E[h(X_i, X_{i_2} \ldots X_{i_r}) \mid X_i] - \theta_n$$
$$g_2(X_i, X_j) = E[h(X_i, X_j, X_{i_3} \ldots X_{i_r}) \mid X_i, X_j] - g_1(X_i) - g_1(X_j) - \theta_n,$$

leading to the representation:

$$\hat{T}_n - \theta_n = \frac{r}{n} \sum_{i=1}^n g_1(X_i) + \frac{r(r-1)}{n(n-1)} \sum_{i<j} g_2(X_i, X_j) + O_p \left( \frac{\rho_n}{n} \right) \quad (6)$$

Similarly, conditional on the data, it can be shown that we have the following bootstrap analog. Let:

$$\tilde{g}_1(i) = \frac{1}{(r-1)} \sum_{1 \leq i_2 < \ldots < i_r \leq n, i_u \neq i} (H(A_{i, i_2, \ldots, i_r}^{(n)}) - \hat{T}_n) \quad (7)$$
$$\tilde{g}_2(i, j) = \frac{1}{(r-2)} \sum_{1 \leq i_3 < \ldots < i_r \leq n, i_u \neq i, i_j \neq j} (H(A_{i, i_3, \ldots, i_r}^{(n)}) - \hat{T}_n) \quad (8)$$
$$\hat{g}_2(i, j) = \tilde{g}_2(i, j) - \hat{g}_1(i) - \hat{g}_1(j) \quad (9)$$

Before providing the Hoeffding decomposition for our bootstrap statistic, we first give a short discussion on the Hoeffding decomposition.

The Hoeffding decomposition, introduced in Hoeffding (1948), is a key technical tool for studying the asymptotic properties of U-statistics. In essence, a U-statistic may be represented as a sum of U-statistics of different orders. Only the linear (order one) term is non-negligible; the order 2 term is also needed for higher-order correctness. The Hoeffding decomposition, however, generalizes far beyond U-statistics. We will provide a more general treatment of the Hoeffding decomposition below, following Bentkus et al. (1997). This general decomposition will be used to derive a Hoeffding decomposition for the bootstrap statistic.

Let $X_1, \ldots, X_n$ be independent random variables and $\Omega = \{1, \ldots, n\}$. For any $B \subset \Omega$ and a random variable $T = t(X_1, \ldots, X_n)$, let

$$T_B = E[T \mid X_i \in B] \quad (10)$$
It may be verified that:

\[ T_B = \sum_{C \subseteq B} (-1)^{|B|-|C|} T_C \]

We also have the following inverse relationship:

\[ \mathbb{E}[T \mid X_i \in C] = \sum_{B \subseteq C} T_B \]

which implies the Hoeffding decomposition:

\[ T = \sum_{B \subseteq \Omega} T_B \quad (11) \]

In what follows, we will group terms by their cardinality, leading to a decomposition closely resembling the familiar decomposition for U-statistics.

We will now establish the Hoeffding decomposition for our bootstrapped functional below.

**Lemma 1.** We have the following decomposition:

\[
\hat{T}^*_{n,M} - \hat{T}_n = \frac{r}{n} \sum_{i=1}^{n} (\xi_i - 1) \cdot \hat{g}_1(i) + \frac{r(r-1)}{n(n-1)} \sum_{i<j} (\xi_i \cdot \xi_j - \xi_i - \xi_j + 1) \cdot \tilde{g}_2(i,j) + O_P \left( \rho_n^\delta(n, \rho_n, R) \right),
\]

(12)

where \( \delta(n, \rho_n, R) = \begin{cases} 1 & \text{R is acyclic} \\ \frac{n \rho_n}{1} & \text{R is cyclic} \\ \frac{n \rho_n^{3/2}}{R} & \text{R is cyclic.} \end{cases} \)

Although the quadratic term in the above expansion may seem different from Eq 6, some manipulation yields that \( \sum_{i<j} (\xi_i \cdot \xi_j - \xi_i - \xi_j + 1) \cdot \tilde{g}_2(i,j) \) is equivalent to \( \sum_{i<j} \xi_i \cdot \xi_j - (\xi_i - 1) \cdot \hat{g}_1(i) - (\xi_j - 1) \cdot \hat{g}_1(j) \), which is similar to the corresponding term in the Hoeffding decomposition of the U statistic (see Eq 6).

Viewing \( \hat{g}_1(i) \) and \( \tilde{g}_2(i,j) \) as estimates of \( g_1(X_i) \) and \( g_2(X_i, X_j) \), respectively, it is clear that that our weighted bootstrap version encapsulates important information about \( \hat{T}_n - \theta_n \). The above decomposition also suggests that one may approximate the non-negligible terms more directly. Ignoring the remainder term, we arrive at the linear and quadratic bootstrap estimates:

\[
\hat{T}^*_{n,L} = \hat{T}_n + \frac{r}{n} \sum_{i=1}^{n} (\xi_i - 1) \cdot \hat{g}_1(i)
\]

(13)

\[
\hat{T}^*_{n,Q} = \hat{T}^*_{n,L} + \frac{r(r-1)}{n(n-1)} \sum_{i<j} (\xi_i \cdot \xi_j - \xi_i - \xi_j + 1) \cdot \tilde{g}_2(i,j).
\]

(14)
3.4 Linear, Quadratic, and Multiplicative algorithms (MB-L, MB-Q and MB-M)

Algorithm 1 Construction of bootstrap estimate of CDF

**Input:** Network $A$, motif $R$, number of resamples $B$, choice of bootstrap procedure $a \in \{M, Q, L\}$, parameter $t$

Compute $\hat{T}_n$ (Eq 2), $\{\hat{g}_1(i)\}_{i=1}^n$, $\{\hat{g}_2(i,j)\}_{i=1}^n$ (Eq 7) and $\hat{\tau}_n$ (Eq 19)

**for** $j \in \{1, \ldots, B\}$ **do**

| Generate $n$ weights $\xi^{(j)} = \{\xi_i^{(j)}, i = 1, \ldots, n\}_{j=1}^B$ using Eq 4 |
| If $a = M$ $T_n^*(j) \leftarrow \hat{T}_{n,M}^*$ (using Eq 5) |
| Else if $a = Q$ $T_n^*(j) \leftarrow \hat{T}_{n,Q}^*$ (using Eq 14) |
| Else $T_n^*(j) \leftarrow \hat{T}_{n,L}^*$ (using Eq 13) |
| **End** |

Return $\frac{1}{B} \sum_j 1\left( \frac{T_n^*(j) - \hat{T}_n}{\sqrt{n}\hat{\tau}_n} \leq t \right)$

**end**

For a given network, we first compute $\hat{T}_n$ and $\hat{\tau}_n$ (see Eqs 2, 19). For each algorithm, we generate $B$ samples of $n$ weights $\{\xi_i^{(j)}, i = 1, \ldots, n\}_{j=1}^B$ from the Gaussian Product distribution (see beginning of Section 3.3). For each of these, MB-M, MB-Q, and MB-L respectively values $\hat{T}_{n,M}^*$, $\hat{T}_{n,Q}^*$ and $\hat{T}_{n,L}^*$. From the $B$ values one then constructs the CDF of the statistic in question, after shifting and normalizing it appropriately.

Note that even though we divide by $\sqrt{n}\hat{\tau}_n$, our statistic is not studentized, which is why our expansion differs from previous work. This is because, conditioned on the data, for the bootstrap samples, $\hat{\tau}_n$ is constant.

**Algorithm for functions of count functionals** We explicitly provide the algorithm for the multiplicative bootstrap. Other versions are analogous. Consider a function of $m$ count functionals $f(\theta_1^{(1)}, \ldots, \theta_m^{(m)})$. First compute $\hat{Y} = f(\hat{\theta}_1^{(1)}, \ldots, \hat{\theta}_m^{(m)})$ from the whole graph. Estimate the variance using the Jackknife estimate of variance from the whole graph. Call this $\hat{S}_n$. For $i \in \{1, \ldots, B\}$, draw $\{\xi_i^{(i)}, \ldots, \hat{\xi}_n^{(i)}\}_{B=1}$ samples from the Gaussian Product distribution. Now, for each of these samples, compute the bootstrap estimate of the $m$ count functionals $\hat{T}_{n,M}^*(\theta_1^{(1)}), \ldots, \hat{T}_{n,M}^*(\theta_m^{(m)})$. Now compute $Y^* = f(\hat{T}_{n,M}^*(\theta_1^{(1)}), \ldots, \hat{T}_{n,M}^*(\theta_m^{(m)}))$. For $B$ samples, compute the distribution of $(Y^* - \bar{Y})/\hat{S}_n$. Again, $\hat{S}_n$ is a constant conditioned on the data, making this to be different from Studentized statistics. For the truth, we simply computed the CDF of a statistic normalized using the variance across all different runs. The experimental results suggest that indeed our multiplicative and quadratic procedures achieve higher order accuracy.
MB-M is computationally expensive and since it involves computing the expression in Eq 5 for each sample of the weighted bootstrap. For example, the worst case complexity of evaluating all \( \binom{n}{r} \) subsets of nodes is \( n^r \). For \( B \) bootstrap samples, the worst case timing of MB-M will be \( Bn^r \). In comparison, for MB-L and MB-Q, we can precompute the \( \hat{g}_1(i) \) and \( \hat{g}_2(i,j) \) values in \( O(n^r) \) time. After that the time per bootstrap sample is linear for MB-L, and quadratic for MB-Q. Thus worst case computational complexity for a dense network for MB-M, MB-Q and MB-L is \( O(Bn^r) \), \( Bn^2 \) and \( Bn \) respectively, excluding precomputation time (which is \( O(n^r) \) in the worst case).

4 Main Results

Before stating our theorems, we present some mild assumptions. The assumptions below are similar to those in Zhang and Xia (2020). In what follows, recall \( \tau_n^2 = \text{var}(E[h(X_S)|X_1]) \) denote the asymptotic variance of the U-statistic.

\textbf{Assumption 1.} We assume the following:

(a) \( \tau_n/p_n^s \geq c > 0 \), for some constant \( c \).

(b) For acyclic \( R \), \( \rho_n \gg 1/\sqrt{n} \) and for cyclic \( R \), \( \rho_n \gg 1/n^{1/r} \)

The first condition is a standard non-degeneracy assumption for U-statistics. The second is a nontrivial sparsity assumption that Zhang and Xia (2020) also require for higher-order correctness.

Below, we establish an Edgeworth expansion normalized by the true standard deviation, which is more appropriate for our purposes. Since estimating the variance leads to a non-negligible perturbation, the polynomials in our expansion differ from those established by the above authors. All proofs and details are deferred to Supplement Section B and Section C. In what follows, let \( F_n(t) \) denote the CDF of \( \hat{T}_n \) and \( G_n(t) \) denote the Edgeworth expansion of interest, given by:

\[
G_n(t) = \Phi(t) - \phi(t) \left( \frac{t^2 - 1}{6\sqrt{n}\tau_n^3} \right) \left\{ E[g_1^3(X_1)] + 3(r - 1)E[g_1(X_1)g_1(X_2)g_2(X_1, X_2)] \right\}. \tag{15}
\]

We have the following result:

\textbf{Proposition 2.} [Edgeworth Expansion for Count Functionals] Let \( G_n \) be the Edgeworth expansion defined in Eq (15) and let \( R \) be a fixed subgraph. Then, if Assumption 1 is satisfied, and \( \rho_n = O(1/\log n) \), or Cramer’s condition holds, i.e. \( \limsup_{t \to \infty} |E\left[ e^{itg_1(X_1)/\tau_n} \right] | < 1 \) then we have,

\[
\sup_u |F_n(u) - G_n(u)| = O(M(n, \rho_n, R)) \tag{16}
\]

where:

\[
M(n, \rho_n, R) = \begin{cases} 
\frac{1}{np_n^2} & \text{R is cyclic} \\
\frac{1}{n^2\rho_n^2} & \text{R is cyclic} 
\end{cases} \tag{17}
\]
Now, we will state our bootstrap approximation results. We will first show that, conditioned on the network and latent variables, the CDF of $\mathbb{M}_\mathcal{B}$ matches the asymptotic expansion in Eq 15, where the true moments are replaced by their empirical versions. Define

$$
\hat{G}_n(u) = \Phi(u) - \frac{(u^2 - 1)\phi(u)}{6\sqrt{n}\hat{\tau}_n^3} \left[ g_1^2 + 3(r - 1) g_1(i)g_1(j)g_2(i,j) \right],
$$

where we have:

$$
\hat{\tau}_n^2 = \sum_i \frac{\hat{g}_1(i)^2}{n}, \quad \hat{g}_1^3 = \frac{1}{n} \sum_{i=1}^n \hat{g}_1(i)^3,
$$

$$
\hat{g}_1(i)g_1(j)g_2(i,j) = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \hat{g}_2(i,j)\hat{g}_1(i)\hat{g}_1(j).
$$

**Theorem 3.** If Assumption 1 is satisfied, the weights $\xi_1, \ldots, \xi_n$ are generated from a non-lattice distribution (see Feller (1971) page 539) such that $E[\xi_1] = 1$, $E[(\xi_1 - 1)^2] = 1$, $E[(\xi_1 - 1)^3] = 1$, then with probability tending to one, we have,

$$
\sup_u \left| P^* \left( \frac{T_{n,Q} - \hat{T}_n}{\sigma_n} \right) - \hat{G}_n(u) \right| = o(n^{-1/2}) + O \left( \frac{\log n}{n^{2/3}\rho_n} \right),
$$

where $P^*(\cdot)$ denotes the conditional probability of event $(\cdot)$ conditioned on $A$ and $X$.

Combining Theorem 3 with the Hoeffding decomposition in Eq 12, we obtain our next theorem. The argument involves using a smoothing argument using the error term in Eq 12.

**Theorem 4.** If Assumption 1 is satisfied, the weights $\xi_1, \ldots, \xi_n$ are generated from a non-lattice distribution with such that $E[\xi_1] = 1$, $E[(\xi_1 - 1)^2] = 1$, $E[(\xi_1 - 1)^3] = 1$, then with probability tending to one, we have,

$$
\sup_u \left| P^* \left( \frac{T_{n,M} - \hat{T}_n}{\sigma_n} \right) - \hat{G}_n(u) \right| = o(n^{-1/2}) + O \left( \frac{\log n}{n^{2/3}\rho_n} \right),
$$

where $P^*(\cdot)$ denotes the conditional probability of event $(\cdot)$ conditioned on $A$ and $X$.

**Remark 1.** The above theorems shows that $\mathbb{M}_\mathcal{B}$-Q and $\mathbb{M}_\mathcal{B}$-M are both higher-order correct. Note that, for acyclic subgraphs, our rate is not as sharp as the rate in Proposition 2. We believe this can be tightened, and leave this for future work.

We also want to point out that while the normal approximation of count statistic yields $O(1/\sqrt{n\rho_n})$ error (see Bickel et al. (2011)), as long as $\rho_n \gg (\log n)^2/n^{1/3}$, our error term is of smaller order than $O(1/\sqrt{n\rho_n})$. 

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The proofs of Theorems 3 and 4 build upon results from Wang and Jing (2004), which establish higher order correctness of the weighted bootstrap for order-2 U-statistics. However, certain terms that appear as constants in Wang and Jing (2004) blow up when perturbed by sparse network noise. To deal with this issue, we control various terms that are unique to the network setting and use different arguments to control the overall error rate.

**Empirical Edgeworth Expansions** Since $\mathbb{E}\left[g_2(X_1, X_2)g_1(X_1)g_1(X_2)\right] = \mathbb{E}\left[\mathbb{E}[h(X_1) - \theta_n|X_1, X_2]g_1(X_1)g_1(X_2)\right]$, we also have:

**Lemma 5.** Under the assumptions in Assumption 7, we have:

$$\sup_u |\hat{G}_n(u) - G(u)| = O_p(M(n, \rho_n, R))$$

Since the proof is similar to Zhang and Xia (2020), it is deferred to the Supplement Section C.

This leads to a natural question: why not use the empirical edgeworth expansion itself to estimate the sampling distribution of the statistic $\hat{T}_n$. We show that indeed, this achieves smallest error in our experimental section. We call this method the empirical edgeworth expansion (EW).

## 5 Simulation Study

We consider two graphons in our simulation study. The first graphon we consider is a Stochastic Blockmodel (SBM), introduced by Holland et al. (1983). The SBM is a popular model for generating networks with community structure. The SBM may be parameterized by a $K \times K$ probability matrix $B$ and a membership probability vector $\pi$ that takes values in the probability simplex in $\mathbb{R}^K$. Let $Y_1, \ldots, Y_n \in \{1, \ldots, K\}$ be random variables indicating the community membership of the corresponding node, with probability given by the entries of $\pi$. Under this model, we have that $P(A_{ij}^{(n)} = 1 \mid Y_i = u, Y_j = v) = \rho_n B_{uv}$. In our simulations, we consider a two block SBM (SBM-G) with $B_{ij} = 0.6$ for $i = 1, j = 1$ and 0.2 for the rest. $\pi = (0.65, 0.35)$
Figure 1: Y axis has absolute error \( E[F_n(u) - F^*_n(u)] \) of CDFs for all methods, where \( F^*_n(t) \) corresponds to the appropriate resampling distribution. X axis is \( u \in [-3:3] \). The two rows are for triangles (first row) and V-stars (three nodes connected via a V, and not forming a triangle) (second row). The left column is for SBM whereas the right column is for the smooth graphon (SM-G).

The second model we use is a smooth graphon model from Zhang et al. (2017) (SM-G) with \( w(u,v) = (u^2 + v^2)/3 \times \cos(1/(u^2 + v^2)) + 0.15 \). This graphon is continuous and high rank in contrast to the first graphon, which is piece-wise constant and low rank.

To study higher order correctness, we first need an estimate of the true CDF. To this end, conduct Monte Carlo simulations with \( N \) samples generated from each model. Note that, since our goal is to show that the error is better than the Normal approximation, we need \( N \gg (n^{\rho_n})^2 \), which ensures that the error from the Monte Carlo samples is \( o(1/n^{\rho_n}) \). To ease the computational burden, we perform simulations on small networks with \( n = 160 \) nodes. We generate \( N = 10^6 \) Monte Carlo simulations as in Zhang and Xia (2020).

**Competing methods** We compare our algorithms, namely MB-M and MB-Q, with the network resampling procedures discussed in Section 2. In particular, we consider subsampling with subsample size \( b_n = 0.5n \) (SS), the empirical graphon with resample size \( n \) (EG), the latent space bootstrap (LS), and the empirical Edgeworth expansion (EW). For the latent space bootstrap, we treat the latent dimension as known for SBM-G and estimate the latent dimension for SM-G using...
Universal Singular Value Thresholding (USVT) procedure of Chatterjee (2015). We provide a brief description of each algorithm below.

**Empirical Graphon (EG).** We draw $B$ size $n$ resamples $S^*_i$ with replacement from $1, \ldots, n$. We compute the count functional $\hat{T}^*_n,i$ on $A(n)(S^*_i, S^*_i)$. We also compute $\hat{T}_n$ and $\hat{\sigma}_n^2$ on the whole graph. Now for triangles and V stars we compute the CDF of $\{(T^*_n,i - \hat{T}_n)/\hat{\sigma}_n\}^B_{i=1}$. For functions of count functions, we compute the function for each resampled graph, center using the function computed on the whole network, and normalize by the square root of the Jackknife estimate of variance (see Lin et al. (2020)).

**Subsampling (SS).** We draw $B$ size $b$ subsamples $S^*_i$ without replacement from $1, \ldots, n$. We compute the count functional $\hat{T}^*_b,i$ on $A(n)(S^*_i, S^*_i)$. We also compute $\hat{T}_n$ and $\hat{\sigma}_b^2 = n/b \hat{\sigma}_n^2$. Now for triangles and V stars we compute the CDF of $\{(\hat{T}^*_b,i - \hat{T}_n)/\hat{\sigma}_b\}^B_{i=1}$. For functions of count functions, we compute the function for each subsampled graph, center using the function computed on the whole network, and normalize by the square root of the Jackknife estimate of variance (scaled by $\sqrt{n/b}$).

**Latent Space (LS).** We first estimate the latent variables $\hat{X} := \{\hat{X}_1, \ldots, \hat{X}_n\}$ from the given network. For SBM-G, we use the true number of blocks, whereas for smooth graphon SM-G, we use the USVT algorithm to estimate the number of latent variables. We compute the count functional $T_n(\hat{X})$ for $i = 1 \ldots n$, and then compute $T_n(X) = T_n(\hat{X}_1, \ldots, \hat{X}_n)$. Now we simply use the additive variant of bootstrap $T_n(X) + \frac{1}{B} \sum_i (g(\hat{X}_i) - T_n(X))$ (see Bose and Chatterjee (2018); Levin and Levina (2019)). For triangles and V stars, we normalize by the square root of $r^2/n \sum_i g(\hat{X}_i)^2$. For functions of count functionals, we center using the function computed on $\hat{X}$ and normalize by the square root of the Jackknife estimate of variance computed on the full $\hat{X}$.

We compare the performance of the resampling methods for V-stars, triangles and a variant of the transitivity coefficient defined in Example 3 of Bhattacharyya and Bickel (2015), which is essentially an appropriately defined ratio between triangle and V-star.

**Results** In Figure 1, we plot the point-wise (absolute) difference of the bootstrap CDFs from the sampling CDF. We average over 30 independent runs to attain less noisy estimates of the expected difference. We see that the empirical Edgeworth expansion (EW) has strong performance for both V-stars and triangles, outperforming all other methods. Our proposed bootstrap procedures also exhibit strong performance on these functionals, but the advantage of using the bootstrap is most clear for transitivity (Figure 3 (left panel)). For smooth functions of counts such as transitivity, analytical expressions for the Edgeworth expansion are much more difficult to attain; the bootstrap is more user-friendly since the general procedure remains the same.
Figure 2: We present timing results for (A) triangle and (B) two star frequencies for the SBM-G model against sample size $n$.

The empirical graphon (EG) also performs well, which is not surprising given the fact that Zhang and Xia (2020) show higher-order correctness for a closely related method. However, as previously mentioned, higher-order correctness to this point has only been established under a Cramer condition. In contrast, the latent space bootstrap (LS) and the linear bootstrap do not appear to be higher order correct, as suggested by our theory. These two bootstrap procedures track each other closely, suggesting that estimating latent positions did not substantially affect performance for the graphons under consideration. Subsampling had the worst performance; however, it is plausible that subsampling will perform better with a more principled choice of subsample size.

Figure 3: We present $\mathbb{E}|\hat{F}_n(u) - F^*_n(u)|$ for the transitivity coefficient of the SBM-G (A) and SM-G (B) models.

Finally, in Figure 2 we show logarithm of running time for triangle count against growing $n$ for SBM-G and SM-G models respectively. We see that MB-Q offers strong computational performance, outperforming fast methods such as the EG and SS. The additive methods, i.e. LS and MB-L are faster, but not higher order accurate. EW is the fastest, but it cannot be readily adapted for smooth functions of count statistics. Finally, we see that while EG has comparable
performance to MB-Q, it requires recomputation of the count statistic for every 
bootstrap iteration, making it about 50 times slower than MB-Q for $n = 500$
for triangle counting. MB-M is the slowest here, because many computational 
shortcuts can be used for binary matrices which EG benefits from. However, for 
MB-M, the weights make such shortcuts not as simple to apply.

Additional experiments, including timing for triangle and V star frequencies 
for the SM-G model are deferred to Supplement Section D. The experiments are 
run on the Lonestar super computer (1252 Cray XC40 compute nodes, each with 
two 12-core Intel Xeon processing cores for a total of 30,048 compute cores) at 
the Texas Advance Computing Center.

6 Conclusion

In this paper we have proposed a new multiplicative bootstrap (MB-M) method, 
and its faster variants, which use additive (MB-L) and quadratic (MB-Q) weights. 
We have established that MB-M and MB-Q are both higher order correct. We have 
shown that the empirical Edgeworth expansion (EW) is perhaps the fastest and 
most accurate way of estimating the CDF of the count statistic. However, deriving 
an EW for more general functions of count functionals like transitivity requires 
intricate analysis, whereas resampling or subsampling methods automatically 
adapt to the underlying distribution. While we do not present theory for this, our 
simulations suggest that indeed, bootstrap-based methods can be higher-order 
correct for smooth functions. Among competing methods, a close contender is 
the empirical graphon (EG), which is orders of magnitude slower than our fast 
multiplier bootstrap MB-Q.

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**Appendix**

In this supplement we provide proofs of Lemmas and main theorems in the paper.

**A Proof of Lemma**

*Proof.* In what follows, we will consider a projection of $T_{n,M}^*$ with respect to the random variables $\xi_1, \ldots, \xi_n$, conditional on $A$ and $X$. 
Recall that $\xi_i$ follows the Gaussian Product distribution. First, we may express $T_{n,M}^*$ as:

$$T_{n,M}^* = \frac{1}{(n)} \sum_{1 \leq i_1 < i_2 < \ldots < i_r} (\xi_{i_1} \cdot \xi_{i_2} \cdot \ldots \cdot \xi_{i_r} - 1) \cdot \left( H(A_{i_1, \ldots, i_r}^{(n)}) - \hat{T}_n \right)$$

where $\xi_{i_1} \cdot \xi_{i_2} \cdot \ldots \cdot \xi_{i_r}$ denotes the product $\xi_{i_1} \times \ldots \times \xi_{i_r}$. It turns out that applying the Hoeffding decomposition directly to $T_{n,M}^*$ leads to tedious combinatorial calculations; following [Bentkus et al. (1997)], let $\Omega_r$ denote an $r$-tuple of $\{1, \ldots, n\}$. For each summand, we will consider a Hoeffding representation with respect to $\Omega_r$. Note that using the Hoeffding projection (also see [Bentkus et al. (1997)] section 2.8),

$$\prod_{1 \leq i \leq r} \xi_i - 1 = \sum_{k=1}^r \sum_{1 \leq i_1 < \ldots < i_k \leq r} h_k(\xi_{i_1}, \ldots, \xi_{i_k})$$

where for $\Omega_k = \{1, \ldots, k\}$,

$$h_k(\xi_1, \ldots, \xi_k) = \sum_{B \in \Omega_k} (-1)^{k-|B|}E \left[ \prod_{1 \leq i \leq r} \xi_i - 1 \right]$$

Thus the first two terms are given by:

$$h_1(\xi_1) := (\xi_1 - 1)$$

$$h_2(\xi_1, \xi_2) := (\xi_1\xi_2 - 1) - (\xi_1 - 1) - (\xi_2 - 1) = (\xi_1 - 1)(\xi_2 - 1)$$

In what follows, we will also denote $A_{i_1}^{(n)}, \ldots, A_{i_r}^{(n)}$ by $A_{S}^{(n)}$, where $S = \{i_1, \ldots, i_r\}$. Let

$$\hat{H}_2(i,j) = \frac{1}{(n-2)} \sum_{S, i, j \in S} H(A_S). \quad (A.20)$$

Thus $T_{n,M}^*$ can be written as follows:

$$T_{n,M}^* = \frac{1}{(n)} \sum_{1 \leq i_1 < i_2 < \ldots < i_r} (\xi_{i_1} \cdot \xi_{i_2} \cdot \ldots \cdot \xi_{i_r} - 1) \cdot \left( H(A_{i_1, \ldots, i_r}^{(n)}) - \hat{T}_n \right)$$

$$= \frac{1}{(n)} \sum_{1 \leq i_1 < i_2 < \ldots < i_r} \sum_{k=1}^r \sum_{1 \leq i_1 < \ldots < i_k \leq i_r} h_k(\xi_{i_1}, \ldots, \xi_{i_k}) \cdot \left( H(A_{i_1, \ldots, i_r}^{(n)}) - \hat{T}_n \right)$$

$$= \frac{1}{(n)} \sum_{k=1}^r \sum_{1 \leq i_1 < \ldots < i_k \leq n} h_k(\xi_{i_1}, \ldots, \xi_{i_k}) \cdot \sum_{S} \left( H(A_{S}^{(n)}) - \hat{T}_n \right) 1(i_1, \ldots, i_k \in S)$$

$$= \frac{1}{(n)} \sum_{k=1}^r \frac{\binom{n-k}{r-k}}{\binom{n}{r-k}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} h_k(\xi_{i_1}, \ldots, \xi_{i_k}) \sum_{S} \left( H(A_{S}^{(n)}) - \hat{T}_n \right) 1(i_1, \ldots, i_k \in S)$$

$$= \frac{r}{n} \sum_{i} (\xi_i - 1) \hat{g}_1(i) + \frac{r(r-1)}{n(n-1)} \sum_{1 \leq i < j \leq n} (\xi_i - 1)(\xi_j - 1) \frac{\left( \hat{H}_2(i,j) - \hat{T}_n \right)}{\hat{g}_2(i,j)} + R_n \quad (A.21)$$
Now, it remains to show that the remainder of \((T_n^{*} - \tilde{T}_n)/\hat{\sigma}_n\) as \(O(\delta(n, \rho_n, R))\), where:
\[
\delta(n, \rho_n, R) = \begin{cases} \frac{1}{n\rho_n^2} & \text{R is acyclic} \\ \frac{1}{n\rho_n} & \text{R is cyclic.} \end{cases}
\]

The residual \(R_n\) is a sum of higher order Hoeffding projections, which are all uncorrelated. Therefore, we see that the variance of the \(u^{th}\) order term is
\[
\frac{\sigma^2(u)}{\sum 1 \leq i_1 < i_2 \ldots \leq i_u} \tilde{g}_u(i_1, i_2, \ldots, i_u)^2.
\]

We will now obtain expressions for \(3 \leq u \leq r\).

Consider any term \(\tilde{g}_u(1, \ldots, u)\). We will now bound \(\mathbb{E}[\tilde{g}_u(1, \ldots, u)^2]\).
\[
\mathbb{E}[\tilde{g}_u(1, \ldots, u)^2] \leq 2(\text{var}(\tilde{g}_u(1, \ldots, u)) + \text{var}(\hat{T}_n))/\text{O}(\rho_n^2/n).
\]

The bound on the second term follows from \cite{Bickel et al. 2011} and will be smaller than that of the first term. Let \(S_{r,u}\) denote all subsets of size \(r - u\), not containing \(1, \ldots, u\). For any subset \(S \in S_{r,u}\), also define, \(S_u = S \cup \{1, \ldots, u\}\). For the first part, we have:
\[
\text{var}(\tilde{g}_u(1, \ldots, u)) = \sum_{S,T \in S_{r,u}} \text{cov}(H(A_{S_u}), H(A_{T_u}))/(n-u)^2.
\]

Note that the dominating term here will indeed be the one where \(|S \cap T| = 0\). Number of such terms is \(\binom{n}{2r-u}\). Also the covariance of those terms will be \(\rho_n^{2s-E(A_{1,\ldots,u})}\), where \(E(A_{1,\ldots,u})\) denotes the intersection of the edgeset of \(A_{1,\ldots,u}\) and the subgraph we are counting. This number can be at most \(u-1\) for acyclic \(R\) and \(u\) for cyclic \(R\). For \(|S \cap T| = k\), the number of terms is \(\binom{n}{2r-2u-k}\) and the exponent on \(\rho_n\) is at most \(2s-(u+k-1)\). Thus, for an acyclic subgraph, we have,
\[
\text{var}(\tilde{g}_u(1, \ldots, u)) \leq \frac{\sum_{k=0}^r \binom{n}{2r-2u-k} \rho_n^{2s-(u+k-1)}}{(n-u)^2} \\
\leq \sum_{k=0}^r n^{-k} \rho_n^{2s-(u+k-1)} = \rho_n^{2s-(u-1)} \left(1 + \sum_{k>0} \frac{1}{(n\rho_n)^k}\right).
\]

The cyclic one is worse by a factor of \(\rho_n\). Thus the contribution of the \(u^{th}\) element of the Hoeffding decomposition is
\[
n \cdot \frac{\tilde{g}_u(i_1, \ldots, i_u)^2}{\binom{n}{u}^2 / r_n} = \begin{cases} O P \left(\frac{1}{(n\rho_n)^{u-1}}\right) & \text{R acyclic} \\ OP \left(\frac{\rho_n^{-1}}{(n\rho_n)^{u-1}}\right) & \text{R cyclic} \end{cases}
\]

This shows that the third term contributes the most to \(R_n\) in Eq \([A.21]\). By Markov’s inequality, and the definition of \(OP(.)\) notation, it is easy to see that \(R_n = OP(\delta(n, \rho_n, R))\).
B  Proof of Proposition 2

In what follows, we derive an Edgeworth expansion for a standardized count functional. Our argument here is closely related to Zhang and Xia (2020), but we present it for completeness.

Proof. Consider the following decomposition:

\[ \tilde{T}_n := \frac{\hat{T}_n - \mu_n}{\sigma_n} = \frac{T_n - \mu_n}{\sigma_n} + \frac{\hat{T}_n - T_n}{\sigma_n} = T_{n,1} + T_{n,2} + OP\left(\frac{1}{n}\right) + R_n, \quad (A.22) \]

where

\[ T_{n,1} = \frac{1}{\sqrt{n} \tau_n} \sum_{i=1}^{n} g_1(X_i), \quad T_{n,2} = \frac{r - 1}{\sqrt{n(n-1)\tau_n}} \sum_{i<j} g_2(X_i, X_j), \quad R_n = \frac{\hat{T}_n - T_n}{\sigma_n}. \]

We will begin by bounding \( R_n \). Theorem 3.1(b) of Zhang and Xia (2020) establishes a central limit theorem for \( (\tilde{T}_n - T_n)/\sigma_n \); similar to the theory for U-statistics, the behavior is largely determined by a linear term.

Let:

\[ R_{n,1} = \text{Linear part of } \frac{\hat{T}_n - T_n}{\sigma_n}. \]

where the linear part has the form:

\[ R_{n,1} = \frac{1}{\binom{n}{2}} \sum_{i<j} c_{ij} \left( A_{ij} - E[A_{ij} | X_i, X_j] \right) \quad (A.23) \]

for \( c_{ij} = c_{ij}(X_i, X_j, \rho_n) \approx \rho_n^{-1} n^{-1/2} \) defined in Section 7 of the above reference. Theorem 3.1(b) of the above authors establishes that:

\[ R_n - R_{n,1} = OP(M(n, \rho_n, R)), \quad (A.24) \]

Under the assumed sparsity conditions, given \( X \), the distribution of \( R_{n,1} \) permits the following (uniform) approximation by a Gaussian-distributed variable \( Z_n \):

\[ \sup_u \left| \int_{R_{n,1}} F_{R_{n,1}}(u) - F_{Z_n} \right| = OP\left(\frac{1}{\sqrt{\rho_n n}}\right), \quad (A.25) \]

where \( Z_n \sim N(0, \frac{\sigma_w^2}{n \rho_n}) \) and \( \sigma_w^2 \) is defined as the variance of Eq B. Note that \( \sigma_w \approx 1 \) when \( n \to \infty \).

Now to prove our theorem, we will show the three equations below.

\[ \sup_u \left| F_{T_n}(u) - F_{T_{n,1}+T_{n,2}+R_n} \right| = O(\epsilon(\rho_n, n)), \quad (A.26) \]

\[ \sup_u \left| F_{T_{n,1}+T_{n,2}+R_n}(u) - F_{T_{n,1}+T_{n,2}+Z_n} \right| = O\left(\frac{1}{\sqrt{\rho_n n}}\right), \quad (A.27) \]
We first prove Eq A.26 and Eq A.27 by assuming Eq A.28 holds.

We proceed by stating the following lemma, which is closely related to the proof of Lemma 8.2 of [Zhang and Xia (2020)]. A proof is omitted, as it closely follows the proof of (8.32), pg 42 of the above reference.

**Lemma B.1.** Suppose that Eq A.28 holds. Then, \( F_{T_{n,1} + T_{n,2} + Z_n} \) is Lipschitz.

Now continuing with our proof, since \( R_n - Z_n = R_n - R_{n,1} + R_{n,1} - Z_n = O_P(M(n, \rho_n R)) \) and \( F_{T_{n,1} + T_{n,2} + Z_n} \) is Lipschitz, applying Lemma 8.2 of Zhang and Xia (2020) yields

\[
\sup_u \left| F_{T_n(u)} - F_{T_{n,1} + T_{n,2} + Z_n(u)} \right| = O_P(M(n, \rho_n, R)),
\]

which proves Eq A.27.

Using Eq A.22, Eq A.24, and Eq A.25, we have

\[
\hat{T}_n = T_{n,1} + T_{n,2} + Z_n + O_P\left(\frac{1}{n \sqrt{\rho_n}}\right) + O_P\left(\frac{1}{n}\right) + O_P(M(n, \rho_n, R)).
\]

Applying the same lemma to \( \hat{T}_n \) and \( T_{n,1} + T_{n,2} + Z_n \) gives us

\[
\sup_u \left| F_{\hat{T}_n(u)} - F_{T_{n,1} + T_{n,2} + Z_n(u)} \right| = O_P(M(n, \rho_n, R)).
\]

Combining Eq A.27 and Eq A.30, we have Eq A.26.

\[
\sup_u \left| F_{\hat{T}_n(u)} - F_{T_{n,1} + T_{n,2} + R_n} \right| 
\leq \sup_u \left| F_{\hat{T}_n(u)} - F_{T_{n,1} + T_{n,2} + Z_n(u)} \right| + \sup_u \left| F_{T_{n,1} + T_{n,2} + R_n(u)} - F_{T_{n,1} + T_{n,2} + Z_n(u)} \right| 
= O(M(n, \rho_n, R)).
\]

We have proved Eq A.26 and Eq A.27 assuming Eq A.28 holds. Now we prove Eq A.28.

We prove Eq A.28 using Esseen’s smoothing lemma from Section XVI.3 in Feller (1971),

\[
\sup_u \left| F_{T_{n,1} + T_{n,2} + Z_n(u)} - G_n(x) \right| 
\leq c_1 \int_{-\gamma}^{\gamma} \frac{1}{t} \left| \psi_{F_{T_{n,1} + T_{n,2} + Z_n}}(u) - \psi_{G_n}(t) \right| dt + c_2 \sup_u \frac{G_n'(u)}{\gamma},
\]

(A.31)
where $\psi$ is the characteristic function. $\gamma$ is set to $n$. Breaking the integral into $|t| \in (0, n), (n', n^{1/2})$ and $(n^{1/2}, n)$ and using the same arguments of lemma 8.3 part (b) and part (c) from Zhang and Xia (2020), we have

\[
\int_{C_1n^{1/2}}^{n} \frac{E[e^{it(T_{n,1}+T_{n,2}+Z_n)}]}{t} dt = O_P(n^{-1}), \quad (A.32)
\]

\[
\int_{n^{1/2}}^{C_1n^{1/2}} \frac{E[e^{it(T_{n,1}+T_{n,2}+Z_n)}]}{t} dt = O_P(n^{-1}), \quad (A.33)
\]

for small enough constant $C_1$, and $\rho_n \leq c_p (\log n)^{-1}$ for small enough $c_p$.

Now we only need to prove similar arguments of their part (a) and (d) to our $G_n(u)$ defined in [15]. We want to show our $G_n(u)$ satisfies that

\[
\int_{n^{1/2}}^{n} \left| \frac{\psi_{G_n}(t)}{t} \right| dt = O_P \left( \frac{1}{n} \right), \quad (A.34)
\]

\[
\int_{0}^{n^{1/2}} \left| \frac{E[e^{it(T_{n,1}+T_{n,2}+Z_n)}] - \psi_{G_n}(t)}{t} \right| dt = O_P \left( \frac{1}{n\rho_n} \right) \quad (A.35)
\]

To prove $G_n(u)$ satisfies these conditions, we first state the form of the characteristic function of $G_n(u)$ below:

**Proposition B.2.** We have:

\[
\psi_{G(n)}(t) := \int e^{itu}dG_n(u) = e^{-\frac{|t|^2}{2}} \left( 1 - \frac{it^3}{6\sqrt{n}} \right) \{E[g_1^3(X_1)] + 3(r-1)E[g_1(X_1)g_1(X_2)g_2(X_1,X_2)] \}.
\]

Now we first verify $\psi_{G_n}$ satisfies Eq [A.34]. Using then fact that $t^k e^{-\frac{|t|^2}{2}} \leq C_k$ for some constant $C_k$ depending on $k$ when $t > 1$ for $k = -1, 0, 1, 2, 3, ...,$, we have

\[
\int_{n^{1/2}}^{n} \left| \frac{\psi_{G_n}(t)}{t} \right| dt \leq (C_1 + C_2) \int_{n^{1/2}}^{\infty} e^{-t^2/3} dt = O_P \left( \frac{1}{n} \right).
\]

The rest of the proof verifies $\psi_{G_n}$ satisfies Eq [A.35]. Similar to Zhang and Xia (2020), we write $E[e^{it(T_{n,1}+T_{n,2}+Z_n)}]$ into

\[
E[e^{it(T_{n,1}+T_{n,2}+Z_n)}] = E \left[ e^{it(T_{n,1}+T_{n,2})} \right] \left\{ 1 - \frac{\sigma^2_w t^2}{\rho_n n} + O_P \left( \frac{\sigma^4_w t^4}{\rho_n^2 n^2} \right) \right\} \quad (A.36)
\]

\[
\int_0^{n^{1/2}} \frac{\sigma^4_w t^4}{\rho_n^2} t^{-1} dt = O_P(n^{-1}) \quad \text{when } \rho_n = \Omega(\frac{1}{n^{1/2}}), \text{ and } \epsilon = \delta/2. \quad \text{We write}
\]

\[
E \left[ e^{it(T_{n,1}+T_{n,2})} \right] = E \left[ e^{iT_{n,1}} \right] \left\{ 1 + it T_{n,2} + O_P(t^2 T_{n,2}^2) \right\} \quad (A.37)
\]

Since $T_{n,2}$ is $O_P(n^{-1/2})$, the last term $O_P \left( T_{n,2}^2 t^2 \right)$ is $O_P(n^{-1})$, thus ignorable. From Petrov (2012) Section VI Lemma 4 and Zhang and Xia (2020).
define \( \phi_n(t) = E \left[ e^{it \frac{g(X)}{\sqrt{n} \rho_n}} \right] \), denote \( \phi_n^k(t) \) as the characteristic function of \( n \) independent sums of \( g_1(X_i) \), \( P_0, \ldots, P_k \) are fixed polynomials of \( t \), then

\[
\phi_n(t) = e^{-\frac{t^2}{2}} \left( 1 - \frac{E[g_1(X_1)^3]t^3}{6 \sqrt{n} \tau_n^3} \right) + O_P(n^{-1}e^{-\frac{t^2}{2}} P_0(t))
\]

\[
\phi_n^{-k}(t) = \phi_n^k(t) + O_P(n^{-1}e^{-\frac{t^2}{2}} P_k(t)).
\]

Thus, we have the first term in the RHS above is

\[
E[e^{it\hat{T}_{n,1}}] = \phi_n^0(t) = e^{-\frac{t^2}{2}} \left( 1 - \frac{E[g_1(X_1)^3]t^3}{6 \sqrt{n} \tau_n^3} \right) + O_P(n^{-1}e^{-\frac{t^2}{2}} Poly(t)).
\]

\[
(A.38)
\]

\[\text{Zhang and Xia (2020)}\] gives the second term as

\[
E[e^{it\hat{T}_{n,2}}] = e^{-\frac{t^2}{2}} \times \frac{-it^3(r - 1)}{2 \sqrt{n} \tau_n^3} E[g_1(X_1)g_1(X_2)g_2(X_1, X_2)]
\]

\[
+ O_P(n^{-1}e^{-\frac{t^2}{2}} |t|Poly(t)),
\]

\[
(A.39)
\]

Putting back Eq \[A.38\] and Eq \[A.39\] back to Eq \[A.37\] we have

\[
E \left[ e^{it\hat{T}_{n,1} + \hat{T}_{n,2}} \right] = e^{-\frac{t^2}{2}} \left( 1 - it^3 \frac{1}{6 \sqrt{n} \tau_n^3} [E[g_1^3(X_1)]
\]

\[
+ 3(r - 1)E[g_1(X_1)g_1(X_2)g_2(X_1, X_2)]]
\]

\[
+ O_P(n^{-1}e^{-\frac{t^2}{2}} |t|Poly(t)).
\]

Finally, the integral of the remainder term in Eq \[A.36\] follows same argument in \[Zhang and Xia (2020)\],

\[
\int_0^\tau \left| E \left[ e^{it\hat{T}_{n,1} + \hat{T}_{n,2}} \frac{\sigma_n^2 t^2}{\rho_n n} \right] \frac{1}{t} \right| = O(\rho_n^{-1}n^{-1}).
\]

\[
(A.41)
\]

Thus Eq \[A.35\] follows from Eqs \[A.36\] \[A.40\] \[A.41\] and Lemma \[B.2\]. We have proved that our \( G_n \) satisfies Eq \[A.34\] and Eq \[A.35\].

Now combining Eqs \[A.32\] \[A.33\] \[A.34\] and Eq \[A.35\] into Essens’s smoothing lemma (see Eq \[A.31\]) proves Eq \[A.28\] under the condition \( \rho_n = O_P((\log n)^{-1}) \).

Then combining Eqs \[A.28\] \[A.27\] \[A.26\] yields Proposition \[2\].

\[\square\]

C \hspace{1cm} Edgeworth Expansion for Weighted Bootstrap - Proofs of Theorems \[3\] and \[4\]

Using Eq \[10\], we express our quadratic bootstrap statistic as:

\[
\hat{T}_{n, Q} = \sum_{i} (\xi_i - 1) \hat{g}_1(i) + (r - 1) \sum_{1 \leq i < j \leq n} (\xi_i \xi_j - \xi_i - \xi_j + 1) \hat{g}_2(i, j)
\]

\[
(A.42)
\]
We will first prove Theorem 3. However in order to prove it we state a slightly different version of Theorem 3.1 in Wang and Jing (2004). The main difference is that one condition in the original lemma is not fulfilled in our case. In particular, Bernoulli noise with $\rho_n \to 0$ blows up some terms that are needed to bound the error associated with the Edgeworth expansion. However, a thorough examination reveals that the argument carries through with some modifications.

Let

\[ K_{2,n} = \frac{1}{n^{3/2}B_n^2} \sum_{1 \leq i < j \leq n} b_{nj} d_{nij} \mathbb{E}[Y_1 Y_2 \psi(Y_1, Y_2)] \tag{A.43} \]

\[ L_{1,n}(x) = \sum_{j=1}^{n} \left( \mathbb{E}\Phi(x - b_{nj} Y_j / B_n) - \Phi(x) \right) - \frac{1}{2} \Phi''(x) \tag{A.44} \]

\[ L_{2,n}(x) = -K_{2,n} \Phi''(x) \tag{A.45} \]

\[ E_{2n}(x) = \Phi(x) + L_{1,n}(x) + L_{2,n}(x), \tag{A.46} \]

**Lemma C.1.** Consider the following expression.

\[ V_n = \frac{1}{B_n} \sum_{j} b_{nj} Y_j + \frac{1}{n^{3/2}} \sum_{i<j} d_{nij} \psi(Y_i, Y_j), \tag{A.47} \]

where $B_n^2 = \sum_j b_{nj}^2$. Let $\beta := \mathbb{E}|Y_1|^3$ and $\lambda = \mathbb{E}[\psi^2(Y_1, Y_2)]$, and let $\mathbb{E}[Y_1] = 0$, $\mathbb{E}[Y_1^2] = 1$ and $\kappa(X_1) > 0$. Furthermore, let $\mathbb{E}[\psi(Y_1, Y_2)|Y_t] = 0$ for all $1 \leq t \leq n$. For some constants $\ell_1, \ell_2, \ell_3$ the sequence $b_{n,i}$ satisfies

\[ \frac{1}{n} \sum_{i=1}^{n} b_{n,i}^2 \geq \ell_1 > 0, \quad \frac{1}{n} \sum_{i=1}^{n} |b_{n,i}|^3 \leq \ell_2 \leq \infty, \tag{A.48} \]

Furthermore, define $\alpha_{n,i} := \frac{1}{n} \sum_{j \neq i} d_{nij}^2$ and for sufficiently large $k$, define:

\[ l_{4,n} = \frac{1}{n} \sum_{i=1}^{n} \alpha_i, \quad s_{n}^2 = \frac{1}{n} \sum_{i} \alpha_i^2 - (l_{4,n})^2, \quad l_{5,n} = l_{4,n} + ks_n \tag{A.49} \]

If $\beta, \kappa(Y_1)$ and $\lambda$ are bounded, then,

\[ \sup_x |P(V_n \leq x) - E_{2n}(x)| = O \left( \frac{l_{5,n} \log n}{n^{2/3}} \right), \]

Intuitively, arguments for establishing rates of convergence for the Edgeworth expansions require comparing the characteristic function of the random variable of interest with the Fourier transform of the Edgeworth expansion. To this end, the respective integrals are broken up into several pieces. The bounds required in (A.48) are used to estimate the error of the Edgeworth expansion in some of these steps, but appear as constants and are suppressed in the Big-O notation.
On the other hand, as previously mentioned, it turns out that certain terms that appear as constants in Wang and Jing (2004) blow up when perturbed by sparse network noise and appear in the rate. In particular, the term $l_{5,n}$ arises from needing to bound $\frac{1}{m} \sum_{i=1}^{m} \alpha_i$ for all $m \leq M$ for some $M$ large enough.

Since the data is fixed, we may view $\alpha_1, \ldots, \alpha_n$ as constants. We therefore have the liberty of choosing a “good set” in which $\alpha_i$ are well-behaved. Without loss of generality, we may label these elements $\{\alpha_1, \ldots, \alpha_M\}$; the corresponding multiplier random variables are still independent. Even when there is no randomness, it turns out that a large proportion of $\{\alpha_1, \ldots, \alpha_n\}$ must be within $k$ sample standard deviations of the sample mean $l_{4,n}$ for $k$ large enough. This observation, which we believe is novel in the bootstrap setting, allows us to establish a tight bound for $\frac{1}{m} \sum_{i=1}^{m} \alpha_i$ for all $m \leq M$. We state this lemma below.

**Lemma C.2.** Let $x_1, \ldots, x_n$ be constants in $\mathbb{R}$ and let $\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $s_n^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$. Define the set:

$$\Gamma_k = \{ x_i \geq \bar{x}_n + ks_n \}$$  \hspace{1cm} (A.50)

Then,

$$|\Gamma_k| \leq \frac{n}{k^2}$$

**Proof.** Observe that:

$$s_n^2 \geq \frac{1}{n} \sum_{i \in \Gamma_k} (x_i - \bar{x}_n)^2 \geq \frac{1}{n} \sum_{i \in \Gamma_k} k^2 s_n^2 \implies |\Gamma_k| \leq \frac{n}{k^2}$$

\[\square\]

**Remark C.1.** Our lemma is closely related to concentration of sums sampled without replacement from a finite population. In fact, it implies the without-replacement Chebyshev inequality; see, for example, Corollary 1.2 of Serfling (1974).

We will show that $\hat{T}_{n,Q}^*$ can be written as Eq A.47 with carefully chosen $\{b_{ni}\}$ and $\{d_{nij}\}$’s. We now present some accompanying Lemmas to show that Eq A.48 is satisfied with probability tending to 1. Proofs of Lemmas C.1, C.4, and C.5 are provided in following subsections.

**Lemma C.3.** Under the sparsity assumptions in Assumption 4, we show that, for large enough $C$,

$$P \left( \frac{1}{n^2} \sum_i \sum_{j \neq i} \tilde{g}_{ij}^2 \geq C p_n^{2s-1} \right) \to 1$$

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Lemma C.4. **Under the sparsity conditions in Assumption [1]**, we show that, for some arbitrary \( \epsilon > 0 \),

\[
P \left( \frac{\sum_i |\hat{g}_1(i)/\rho_{ni}|^3}{n} \leq \epsilon \right) \to 1
\]

\[
P \left( \frac{\sum_i |\hat{g}_1(i)/\rho_{ni}|^2}{n} \geq \epsilon' \right) \to 1,
\]

for positive constants \( c,c' \) not depending on \( n \).

Lemma C.5. Let \( \xi_1 \) be generated from the Gaussian product distribution. We have \( E[|\xi_1 - 1|^3] < \infty \).

Now we are ready to provide the proof.

**Proof of Theorem 3**. It is easy to see from Eq A.42 that \( \hat{T}^{*}_{n,Q} \) can be expressed as:

\[
\hat{T}^{*}_{n,Q} = \sum_i b_{n,i} Y_i + \frac{1}{\sqrt{n(n-1)}} \sum_{1 \leq i < j \leq n} \psi(Y_i,Y_j) d_{n,ij},
\]

where we have:

\[
Y_i = \xi_i - 1,
\]

\[
b_{n,i} = \frac{\hat{g}_1(i)}{\rho_{ni}},
\]

\[
B^2 = \sum_{i=1}^n b_{n,i}^2,
\]

\[
d_{n,ij} = \frac{r - 1}{\hat{\tau}_n} \hat{g}(i,j) \times \frac{n}{n-1},
\]

\[
\psi(\xi_i,\xi_j) = \xi_i \xi_j - \xi_i - \xi_j + 1.
\]

Note that since \( \hat{\tau}^2 = \sum_i \hat{g}_1(i)^2/n \). Thus we use \( B^2 = n \hat{\tau}^2 \). Thus, \( B^2 = \sum_i b_{n,i}^2 \). Furthermore, Lemma C.5 shows that our \( \xi_i - 1 \) random variables have finite \( E[|\xi_i - 1|^3] \).

Lemma C.4 shows that the conditions in Eq A.48 are satisfied on a high probability set of \( A, X \).

Using Lemma C.4, we see that the first two conditions in Eq A.48 are satisfied with probability tending to one under Assumption 1. Since \( B^2/n = \sum_i b_{n,i}^2/n \) converges to a positive constant (see Lin et al. [2020]; Zhang and Xia [2020]), the first condition holds. Now, we need to bound \( \ell_{k,n} \) and \( s_n \) as defined Eq A.49. First, let \( \alpha_{n,i} := \sum_{j \neq i} \hat{g}_{ij}/n \hat{\tau}_n \). Let \( \beta_{n,i} := \sum_{j \neq i} \hat{g}_{ij}/n \) and \( \bar{\beta}_n = \sum_i \beta_{n,i}/n \).

Also let \( \gamma_n = \sum_i \hat{g}_{n,i}^2/n - \bar{\beta}_n^2 \). Note that using Lemma C.3, we have, with probability tending to one, \( \ell_{k,n} \leq C \rho_{ni}^{\alpha_{n,i}} \). From Zhang and Xia [2020], we have \( \hat{\tau}_n \) is \( \Theta(\rho^{1/2}) \). Furthermore, let \( G_2(i,j) := H_2(i,j) - h_2(X_i, X_j) \).
We have

\[ \hat{G}_2(i, j)^2 = \hat{G}_2(i, j)^2 - \mathbb{E}[\hat{G}_2(i, j)^2 | X] + \mathbb{E}[\hat{G}_2(i, j)^2 | X] \]

We now will establish the \( O(\rho_n^{2s-1}) \) bound stated above for the second term. Let \( S_{ij}^r \) denote all subsets of size \( r \) not containing \( i, j \).

\[ \mathbb{E}[\hat{G}_2(i, j)^2 | X] = \sum_{S, T \in S_{ij}^r} \mathbb{E}\left[H(A_{ij} \cup S)H(A_{ij} \cup T) | X\right] \]

In the above sum the terms with \( |S \cap T| = 0 \) dominate, and for each of them the conditional expectation is bounded a.s. by \( O(\rho_n^{2s-1}) \) because of the boundedness of the graphon. Now note that:

\[ \hat{g}_{ij}^2 \leq 3 \left( (\hat{H}_2(i, j) - h_2(X_i, X_j))^2 + (h_2(X_i, X_j) - \theta_n)^2 + (\hat{T}_n - \theta_n)^2 \right) \]

\[ \beta_{n, i} \leq \frac{1}{n} \sum_{j \neq i} \delta_{ij} + O(\rho_n^{2s-1}) \]

\[ \gamma_n^2 \leq \frac{1}{n} \sum_i \beta_{n, i}^2 \leq \frac{1}{n} \sum_i \left( \frac{1}{n} \sum_{j \neq i} \delta_{ij} + O(\rho_n^{2s-1}) \right)^2 \]

\[ \leq O(\rho_n^{4s-2}) + \frac{1}{n} \sum_i \left( \frac{1}{n} \sum_{j \neq i} \delta_{ij} \right)^2 \]

Now note that, \( \mathbb{E}[\delta_{ij}] = \mathbb{E}[\mathbb{E}[\delta_{ij} | X]] = 0. \) Thus, for all \( i, \)

\[ \mathbb{E}[A] = \frac{1}{n} \sum_i \mathbb{E}\left( \frac{1}{n} \sum_{j \neq i} \delta_{ij} \right)^2 = \frac{1}{n} \sum_i \text{var} \left( \frac{1}{n} \sum_{j \neq i} \delta_{ij} \right) = O(\rho_n^{4s-3}/n) \]

Thus, we have, for a large enough \( C, \)

\[ P \left( \gamma_n^2 \geq C \rho_n^{4s-2} \right) \leq P \left( A \geq C' \rho_n^{4s-2} \right) \leq O \left( \frac{\mathbb{E}(A)}{\rho_n^{4s-2}} \right) = O \left( \frac{1}{n \rho_n} \right) \]

Therefore, we have with probability tending to one, \( l_{n, 4} + ks_n = O(\rho_n^{-1}). \)

Since the first two conditions in eq A.48 are satisfied, from [Wang and Jing (2004) Theorem 3.1], we have,

\[ \sup_u \left| L_{1n}(u) + \frac{E(\xi_i - 1)^3}{6 B_{n}^3} \sum_{i=1}^{n} b_{n, i}^3 \Phi'''(u) \right| = o(n^{-1/2}), \]
Now we see that, using the definitions of $L_{1n}$, $L_{2n}$ in Eq A.43, plugging in definitions of $b_{ni}$ and $d_{nj}$’s from Eq A.51 and using the fact that $E[Y_i Y_j \psi(Y_i, Y_j)] = E[(\xi_i - 1)^2 (\xi_j - 1)^2] = 1$,

$$\sup_u \left| E_{2n}(u) - \hat{G}_n(u) \right| = o(n^{-1/2}).$$

Therefore, putting all the pieces together we see that, with probability tending to 1,

$$\sup_u \left| P^* \left( \frac{\hat{T}_{n,Q} - \hat{T}_n}{\hat{\sigma}_n} \leq u \right) - \hat{G}_n(u) \right| = o(n^{-1/2}) + O \left( \frac{\log n}{n^{2/3}} \rho_n \right)$$  \hspace{1cm} (A.51)

Now we are ready to finish the proof of Theorem 4.

Proof of Theorem 4. Here we take care of the error term in the Hoeffding projection in Eq 12. For this we will use Lemma 8.2 in Zhang and Xia (2020).

Set $X = \hat{T} \ast_{n,M} - \hat{T}_n \hat{\sigma}_n$, $Y = \hat{T} \ast_{n,Q}$. From Eq 12 we see that $X = Y + R_n$, where $R_n = O_P(\delta(n, \rho_n, R))$. Using Eq A.51, we see that on a high probability set,

$$F_Y(u + a) - F_Y(u) \leq |F_Y(u + a) - \hat{G}_n(u + a)| + |\hat{G}_n(u + a) - \hat{G}_n(u)| + |\hat{G}_n(u) - F_Y(u)|$$

$$\leq Ca + O \left( \frac{\log n}{n^{2/3}} \rho_n \right)$$

Therefore, using Lemma 8.2 in Zhang and Xia (2020), on a high probability set of $X, A$,

$$\sup_u \left| P^* \left( \frac{\hat{T}_{n,M} - \hat{T}_n}{\hat{\sigma}_n} \leq u \right) - \hat{G}_n(u) \right| = o(n^{-1/2}) + O \left( \frac{\log n}{n^{2/3}} \rho_n \right).$$

Proof of Lemma 5. As part of their proof of Theorem 3.2, Zhang and Xia (2020) show that,

$$\max \left( \left| \frac{\sum_j \hat{g}_1(i)^3}{n} - E[g_1(X_1)^3] \right|, \right.$$

$$\left| \frac{\sum_i \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i,j)}{\binom{n}{2}} - E[g_1(X_1)g_1(X_2)g_2(X_1, X_2)] \right| \right)$$

$$= O_P \left( \rho_n^3 \sqrt{n, M(n, \rho_n, R)} \right),$$  \hspace{1cm} (A.52)

and

$$|\hat{\tau}_n^3 - \tau_n^3| = O_P(\rho^3 \sqrt{n})$$  \hspace{1cm} (A.53)

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If we can establish this for our empirical moments, we will get the desired result. Note that our empirical moments involve the first term as well as a slight variation of the second term, which is given below.

\[ g_1(i)g_1(j)g_2(i, j) = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) \]

We will show that this follows from Eq \[\text{A.52}\].

\[
\begin{align*}
\sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) & = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \hat{g}_1(i) - \hat{g}_1(j) \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \sum_{i\neq j} \hat{g}_1(i)\hat{g}_1(j)(\hat{g}_1(i) + \hat{g}_1(j)) \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \sum_{i\neq j} \hat{g}_1(i)\hat{g}_1(j) \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \frac{1}{2} \sum_{i\neq j} \hat{g}_1(i)\hat{g}_1(j) \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \frac{1}{2} \sum_{i\neq j} \hat{g}_1(i)\hat{g}_1(j) \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \frac{1}{2} \sum_{i\neq j} \hat{g}_1(i)^3 \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) - \frac{1}{2} \sum_{i\neq j} \hat{g}_1(i)^3 \\
& = \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) + O_P \left( \frac{\rho^3 n}{n} \right)
\end{align*}
\]

(i) uses the fact that \( \sum_i \hat{g}_1(i) = 0 \). (ii) uses the fact that \( \text{E}[g_1(X_1)^3] = O(\rho^3 n) \) along with Eq \[\text{A.52}\]. Hence we have:

\[
\left| \sum_{i<j} \hat{g}_1(i)\hat{g}_1(j)\hat{g}_2(i, j) \right| - \text{E}[g_1(X_1)g_1(X_2)g_2(X_1, X_2)] = \text{max} \left( O_P \left( \frac{\rho^3 n}{n} \right), O_P \left( \rho^3 \sqrt{nM(n, \rho_n, R)} \right) \right)
\]

This, along with Eqs \[\text{A.52}\] and \[\text{A.53}\] yields the result.

\[\square\]

### C.1 Proof of Lemma \[\text{C.3}\]

**Proof.** Recall the definition of \( \hat{H}_2(i, j) \) from Eq \[\text{A.20}\]

\[
\hat{g}_2(i, j) = \hat{H}_2(i, j) - \hat{T}_n = (\hat{H}_2(i, j) - h_2(X_i, X_j)) + (h_2(X_i, X_j) - \theta_n) - (\hat{T}_n - \theta_n)
\]

\[
\hat{g}_2(i, j)^2 \leq 3 \left( (\hat{H}_2(i, j) - h_2(X_i, X_j))^2 + (h_2(X_i, X_j) - \theta_n)^2 + (\hat{T}_n - \theta_n)^2 \right)
\]

\[\text{A.54}\]
Since \( \text{var}(\hat{T}_n) = O(\rho_n^{2s}/n) \) and the second term is bounded a.s. due to our boundedness assumption. We will just prove that \( \sum_{j \neq i}(\hat{H}_2(i, j) - h_2(X_i, X_j))^2/(n - 1)\rho_n^{2s} \) is bounded with high probability. It is not hard to check (also see Zhang and Xia (2020)) that

\[
\mathbb{E}[(\hat{H}_2(i, j) - h_2(X_i, X_j))^2/\rho_n^{2s}] = O(1/\rho_n)
\]

Therefore,

\[
\sum_{j \neq i} \mathbb{E}\hat{g}_2(i, j)^2/(n\rho_n^{2s}) = O(1/\rho_n)
\]

Furthermore, let \( \hat{G}_2(i, j) := \hat{H}_2(i, j) - h_2(X_i, X_j) \). We have

\[
\hat{G}_2(i, j)^2 = \hat{G}_2(i, j)^2 - \mathbb{E}[\hat{G}_2(i, j)^2 | X] + \mathbb{E}[\hat{G}_2(i, j)^2 | X]_{O(\rho_n^{s-1})}
\]

We now will establish the \( O(\rho_n^{2s-1}) \) bound stated above for the second term. Let \( S_{ij}^r \) denote all subsets of size \( r \) not containing \( i, j \).

\[
\mathbb{E}[\hat{G}_2(i, j)^2 | X] = \sum_{S, T \in S_{ij}^r} \mathbb{E}[H(A_{ij,S})H(A_{ij,T}) | X]
\]

In the above sum the terms with \( |S \cap T| = 0 \) dominate, and for each of them the conditional expectation is bounded a.s. by \( O(\rho_n^{s-1}) \) because of the boundedness of the graphon.

We will analyze \( \sum_i \sum_{j \neq i} \delta_{ij} \). Note that \( \mathbb{E}[\delta_{ij} | X] = 0 \).

\[
\text{var} \left( \frac{1}{n^2} \sum_i \sum_{j \neq i} \delta_{ij} | X \right) = \sum_i \sum_{j \neq i} \text{var}(\delta_{ij} | X) + \sum_{i,k,k \neq i} \sum_{j,l,j \neq l} \text{cov}(\delta_{ik}, \delta_{jl} | X)
\]

(A.55)

\[
\delta_{ij} = \frac{1}{(n-2)^2} \sum_{S, T \in S_{ij}^r} H(A_{ij,S})H(A_{ij,T}) - \mathbb{E}[H(A_{ij,S})H(A_{ij,T}) | X]_{H_{ij}^r(S, T)}
\]

(A.56)

For variance, we have:

\[
\text{var}(\delta_{ij}) = \mathbb{E}\text{var}(\delta_{ij} | X) = \sum_{S_1 \neq T_1, S_2 \neq T_2 \in S_{ij}^r} \mathbb{E}\text{cov}(H_{ij}^r(S_1, T_1), H_{ij}^r(S_2, T_2) | X)
\]

\[
= O \left( \frac{(n-2)^4}{(n-2)^4} \rho_n^{4s-1} \right) = O(\rho_n^{4s-1})
\]

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The dominant term in the above sum is the one with $S_1, S_2, T_1, T_2$ all disjoint. For covariance, for $i \neq j \neq k \neq \ell$ we have:

$$\text{cov}(\delta_{ik}, \delta_{j\ell}) = \mathbb{E}\text{cov}(\delta_{ik}, \delta_{j\ell}|X)$$

$$= \sum_{S_1 \neq T_1 \in S_1^{k}, S_2 \neq T_2 \in S_2^{\ell}} \mathbb{E}\text{cov}(H'_{ij}(S_1, T_1), H'_{ik}(S_2, T_2)|X)$$

$$\xrightarrow{(i)} O \left( \frac{(n^4 - 4(r-2) - 2)\rho_n^{4s-3}}{(n^2 - 4)^4} \right) = O(\rho_n^{4s-3}/n^2).$$

Step (i) is true, because conditioned on $X$, there needs to be at least two nodes $u_1, u_2$ in common between $\{i, k \cup S_1 \cup T_1\}$ and $\{j, \ell \cup S_2 \cup T_2\}$ to have a nonzero covariance. This leads to only $4(r-2) - 2$ choices, which dominates the sum. This along with Eq A.55 gives us:

$$\text{var} \left( \frac{1}{n^2} \sum_{i} \sum_{j \neq i} \delta_{ij} \right) = \text{Evar} \left( \sum_{i} \sum_{j \neq i} \delta_{ij}/n^2 | X \right) = O(\rho_n^{4s-3}/n^2).$$

Thus we have for large enough $C$, we have

$$P \left( \frac{1}{n^2} \sum_{i} \sum_{j \neq i} \hat{g}_{ij} \geq C\rho_n^{2s-1} \right) \leq P \left( \left| \sum_{i} \sum_{j \neq i} \delta_{ij}/n^2 + O(\rho_n^{2s-1}) \right| \geq C\rho_n^{2s-1} \right)$$

$$\leq P \left( \left| \sum_{i} \sum_{j \neq i} \delta_{ij}/n^2 \right| \geq C'\rho_n^{2s-1} \right)$$

$$\leq C''\rho_n^{4s-3}/n^2 = O \left( \frac{1}{n^2\rho_n} \right).$$

C.2 Proof of Lemma C.4

Proof. Let $\Delta_i := |\hat{g}_1(i) - g_1(X_i)|/\rho_n^s$. We have:

$$\sum_{i} |\hat{g}_1(i)/\rho_n^s|^3 \leq \frac{\sum_{i} |g_1(X_i)/\rho_n^s|^3}{n} + \frac{3\sum_{i} |g_1(X_i)/\rho_n^s|^2 \Delta_i}{n} + \frac{\sum_{i} |g_1(X_i)/\rho_n^s|^3}{n}$$

$$= B_1 + B_2 + B_3 + B_4 \quad \text{(A.57)}$$

First note that using the boundedness condition on the graphon, $|g_1(X_i)/\rho_n^s|$ is bounded. Hence $B_4 \leq c$ a.s. Using the proof of Theorem 3.2 in Zhang and
we know that $E(\Delta_i)^2 = O(1/n\rho_n)$. Since $\sum_i \Delta_i \leq \sqrt{n} \sum_j \Delta_j^2$, for the second term we have, for some $C > 0$:

\[ P(B_2 \geq \epsilon) \leq \frac{\sqrt{n}E \sum_i \Delta_i^2}{n\epsilon^2} \leq \frac{C}{\sqrt{n}\rho_n}\epsilon^2 \quad (A.58) \]

Furthermore,

\[ P(B_3 \geq \epsilon) \leq \frac{E \sum_i \Delta_i^2}{n\epsilon^2} \leq \frac{C}{n\rho_n}\epsilon^2. \quad (A.59) \]

By repeated application of Cauchy Schwarz inequality, we have ($\sum_i x_i^3 \leq \sum_i x_i^2 \sum_j x_j^4 \leq (\sum_i x_i^2)^3$), we also have:

\[ P(B_1 \geq \epsilon) \leq \frac{E \sum_i \Delta_i^3}{n\epsilon^2} \leq \frac{\sqrt{(\sum_i E\Delta_i^2)^3}}{n\epsilon^2} \leq \frac{C}{n\rho_n^{3/2}}\epsilon^2 \quad (A.60) \]

Therefore, using the sparsity conditions in Assumption 1, we see that the first equation in the lemma statement is proved.

For the second, we use:

\[ \frac{\sum_i |g_1(i)/\rho_n|^2}{n} \geq \frac{\sum_i |g_1(X_i)/\rho_n|^2}{n} + \frac{\sum_i \Delta_i^2}{n} - 2 \frac{\sum_i |g_1(X_i)/\rho_n|\Delta_i}{n} = C_1 + \alpha B_2 - \beta B_3, \]

where $\alpha, \beta$ are positive constants, and $B_2, B_3$ were defined in Eq A.57. Using Assumption 2 part 1, we see $C_1 > 0$, a.s. Also, now for a small enough constant $\epsilon$, using Eqs (A.58) and (A.59) we see that the second equation in the lemma statement is proven.

C.3 Proof of Lemma C.1

Proof. Define the following quantities.

\[ \gamma_j(t) = \mathbb{E}[\exp(itb_n Y_j/B_n)] \]

\[ \phi_{1,n}(t) = e^{-t^2/2} \left( 1 + \sum_j [\gamma_j(t) - 1] + \frac{t^2}{2} \right) \]

\[ \phi_{2,n}(t) = -t^2 K_{2,n} e^{-t^2/2} \]

\[ S_n = \frac{1}{B_n} \sum_j b_{nj} Y_j \]

\[ \Delta_{n,m} = \frac{1}{m^{3/2}} \sum_{i=1}^{m-1} \sum_{j=i+1}^n d_{nij} \psi(Y_i, Y_j) \]

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As in the original proof,
\[
\int_{-\infty}^{\infty} e^{itx} d[\Phi(x) + L_{1,n}]dx = \phi_{1,n}(t) \quad (A.61)
\]
\[
\int_{-\infty}^{\infty} e^{itx} dL_{2,n}dx = it\phi_{2,n}(t) \quad (A.62)
\]
\[
\int_{-\infty}^{\infty} e^{itx} E_{2n}(x) = \phi_{1,n}(t) + it\phi_{2,n}(t)
\]

Now, for some \(c > 0\) to be chosen later, from Esseen’s smoothing lemma \cite{Petrov2012} and Eq \(A.61\) we have:
\[
\sup_x |P(V_n \leq x) - E_{2n}(x)|
\leq \int_{|t| \leq n^{1-c}} |t|^{-1} |Ee^{itV_n} - \phi_{1,n}(t) - it\phi_{2,n}(t)|dt + Cn^{c-1} \sup_x \left| \frac{dE_{2n}(x)}{dx} \right|
\leq \int_{|t| \leq n^{1-c}} |t|^{-1} |Ee^{itV_n} - \phi_{1,n}(t) - it\phi_{2,n}(t)|dt + \frac{C_1(|K_{2,n}| + \beta)}{n^{1-c}} \quad (A.63)
\]

The last line is true due to the following argument. Note that, for some \(v_j\) in the \(|b_{n_j}Y_j/B_n|\) ball in the neighborhood of \(x\), for \(j \in \{1, \ldots, n\}\),
\[
\frac{dL_{1,n}(x)}{dx} = \sum_{j=1}^{n} (E\phi(x - b_{n_j}Y_j/B_n) - \phi(x)) - \frac{1}{2} \Phi'''(x)
\]
\[
= \sum_{j=1}^{n} E\left(-b_{n_j}Y_j/B_n\phi'(x) + b_{n_j}^2 Y_j^2/2B_n^2\phi''(x) - b_{n_j}^3 Y_j^3/6B_n^3\phi'''(v_j)\right) - \frac{1}{2} \Phi'''(x)
\]
\[
\sup_x \left| \frac{dL_{1,n}(x)}{dx} \right| \leq \sum_{j=1}^{n} C_1 \left( b_{n_j}^2/B_n^2 |\phi''(x)| + C_2 |b_{n_j}/B_n|^3 |Y_j^3||\phi'''(v_j)| \right) + \frac{1}{2} |\Phi'''(x)|
\]
\[
\leq C + E|X_1|^3 \left( \sum_j |b_{n_j}/B_n|^3 \right) + C'
\]
\[
\leq C + \beta/\sqrt{n} \leq C\beta \quad \text{Since } \beta \geq 1
\]

Also note that, for any \(\epsilon > 0\), for \(n\) large enough,
\[
\int_{|t| > n^\epsilon} |\phi_{1,n}(t)/t|dt = O(1/n^{1-\epsilon})
\]
\[
\int_{|t| > n^\epsilon} |\phi_{2,n}(t)/t|dt = O(|K_{2,n}|/n^{1-\epsilon})
\]
Thus the main idea is that \( E[e^{itV_n}] \) behaves like \( E[itS_n] + itE[itS_n\Delta_{n,n}] \).

\[
\int_{|t|\leq n^{1-c}} |t|^{-1} |Ee^{itV_n} - \phi_{1,n}(t) - \phi_{2,n}(t)| dt \leq \sum_{j=1}^{4} I_{j,n}
\]

Going back to Eq \([A.63]\) we break up the first part of the RHS into four parts, and the remainder gets absorbed into \( O(|K_{2,n} + \beta|/n^{1-c}) \) term in Eq \([A.63]\).

\[
|I_{1,n}| = \int_{|t|<n^c} |t|^{-1} \left| Ee^{itV_n} - E[itS_n] - itE[itS_n\Delta_{n,n}] \right| dt
\]

\[
|I_{2,n}| = \int_{|t|<n^c} |t|^{-1} \left| Ee^{itS_n} - \phi_{1,n}(t) \right| dt
\]

\[
|I_{3,n}| = \int_{|t|<n^c} \left| E\Delta_{n,n} e^{itS_n} - \phi_{2,n}(t) \right| dt
\]

\[
|I_{4,n}| = \int_{n^c \leq |t|<n^{1-c}} |t|^{-1} |Ee^{itV_n}| dt
\]

First we will bound some terms which will be used frequently. Since \( ab \leq (a^2 + b^2)/2 \).

\[
|K_{2,n}| \leq \frac{C}{n^{3/2}B_n^2} \sum_{1 \leq i < j \leq n} (b_{ni}^2 b_{nj}^2 + d_{nij}^2)(1 + \lambda)
\]

\[
\leq \frac{C(1 + \lambda)}{n^{3/2}B_n^2} \left( \sum_j b_{nj}^2 \right)^2 + \sum_{i<j} d_{nij}^2
\]

\[
\leq \frac{C(1 + \lambda)}{n^{3/2}B_n^2} (B_n^4 + l_{4,n}n^2)
\]

\[
\leq \frac{C'(1 + \lambda)l_{4,n}}{\sqrt{n}} \quad (A.64)
\]

As for \( \Delta_{n,n} \), we have:

\[
E\Delta_{n,n}^2 = \frac{\lambda}{n^3} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{nij}^2 = \frac{\lambda l_{4,n}}{n} \quad (A.65)
\]

Furthermore we will use:

\[
R(z) := e^{iz} - 1 - iz \quad |R(z)| \leq |z|^\alpha \quad \forall \alpha \in [1, 2] \quad (A.66)
\]

We will first bound \( I_{1,n} \). Using Taylor expansion, for some \( |\eta| \leq 1 \),
\[ |I_{1,n}| \leq \int_{|t|<n} |t|^{-1} t^{2}/2 |E \Delta_{n,n} e^{itS_n} e^{it\Delta_{n,n}}| dt \]
\[ \leq 1/2 \int_{|t|<n} |t| |E \Delta_{n,n}| dt \leq C \frac{(1+\lambda)|t_4,n|}{n^{1-2\epsilon}} \tag{A.67} \]

Next we bound \( I_{2,n} \). Using the argument stated in \cite{Wang and Jing (2004)}, we have:

\[ |I_{2,n}| \leq \frac{C_1}{B^4} \sum_j b_{n,j}^4 + C_2 \left( \frac{1}{B^2} \sum_j |b_{n,j}|^3 |E|X_1|^3 \right)^2 \]
\[ \leq C_1 n^{-2/3} + C_2 \lambda^2 / n \]

Now we do \( I_{3,n} \). Denote \( Z_j = b_{n,j} Y_j / B_n \) and \( \psi_{ij} = d_{nij} \psi(Y_i, Y_j) \). First note that
\[
E[\psi_{ij} e^{it(Z_i+Z_j)}] = -t^2 \ell_{ij} + \theta_{1,ij}(t), \tag{A.68}
\]
we have:
\[
\ell_{ij} = E[\psi_{ij} Z_i Z_j] \leq |b_{n,j} b_{n,j}| d_{nij} / B_n^2 |E|Y_j \psi(Y_i, Y_j)|
\leq \lambda^{1/2} (b_{n,j}^2 + d_{nij}^2) / B_n^2 \tag{A.69}
\]
and using Eq \[A.66\] and the fact that \( E[\psi(Y_i, Y_j)] = 0 \) and \( E[\psi(Y_i, Y_j)|Y_i] = 0 \),
\[
\theta_{1,i,j} = E[\psi_{ij}(it(Z_i R(tZ_i) + Z_j R(tZ_i) + R(tZ_i) R(tZ_j)))]
\leq C |t|^2 \sum \left( |b_{n,j} b_{n,j}^2 + d_{nij} b_{n,j}^2| / B_n^2 \right) \]
\[ \leq C |t|^2 (\lambda\beta)^{1/2} (d_{nij}^2 + b_{n,j}^2 |b_{n,j}|^3 + |b_{n,j}|^3 b_{n,j}^2) n^{-5/4} \tag{A.70}
\]
Using Eq \[A.68\] and setting \( \prod_{k \neq i,j} \gamma_k(t) = e^{-t^2/2} + \theta_{2,i,j} \) we see:
\[
E \Delta_{n,n} e^{itS_n} = n^{-3/2} \sum_{i<j} E[\psi_{ij} e^{itS_n}]
= n^{-3/2} \sum_{i<j} E[\psi_{ij} e^{it(Z_i+Z_j)}] \prod_{k \neq i,j} \gamma_k(t)
= n^{-3/2} \sum_{i<j} (-t^2 \ell_{i,j} + \theta_{1,i,j}(t)) \left( e^{-t^2/2} + \theta_{2,i,j} \right)
= n^{-3/2} \sum_{i<j} (-t^2 \ell_{i,j} e^{-t^2/2} + \theta_{3,i,j}(t))
= -\frac{K_2 t^2 e^{-t^2/2}}{\phi_{2,n}(t)} + n^{-3/2} \sum_{i<j} \theta_{3,i,j},
\]

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where using Lemma A.4 in Wang and Jing (2004), for $|t| < n^{\epsilon} << n^{1/6}$

$$|\theta_{2,i,j}| \leq C \sqrt{n} \left( \beta + \frac{b_{nj}^2 + b_{ni}^2}{\sqrt{n}} \right) (t^2 + t^4)e^{-t^2/8}$$

Furthermore, using Lemma A.4 and $\sum_i |b_{ni}|^3 \leq \ell_2 n$

$$|\theta_{3,i,j}| \leq t^2 |\ell_{i,j} \theta_{2,i,j}| + |\theta_{1,i,j}| \prod_{k \neq i,j} \gamma_k(t)$$

$$\leq t^2 |\ell_{i,j} \theta_{2,i,j}| + 4 |\theta_{1,i,j}| e^{-t^2/8}$$

$$|\ell_{i,j} \theta_{2,i,j}| \leq C \frac{\sqrt{n}}{\ell_{i,j}} |\ell_{i,j}| \left( \beta + \frac{b_{nj}^2 + b_{ni}^2}{\sqrt{n}} \right) (t^2 + t^4)e^{-t^2/8}$$

$$\sum_{i<j} |\ell_{i,j} \theta_{1,i,j}| \leq C \sqrt{\lambda} \left( \beta \frac{B_4^4 + l_{4,n} n^2}{n^{3/2}} + \frac{c}{n^2} \sqrt{\left( \sum_{i<j} d_{ni}^2 \right) \left( \sum_{i<j} b_{ni}^3 b_{nj}^2 \right)} \right) (t^2 + t^4)e^{-t^2/8}$$

(A.71)

To bound (A) we see:

$$(A) \leq \sqrt{(n^2 l_{4,n} l_{\ell_2})(n \ell_2)} \sum_{i<j} b_{ni}^3 b_{nj}^2 \leq n^{3/2} \sqrt{l_{2} l_{4,n}} \sqrt{\left( \sum_i b_{ni}^3 \right) \left( \sum_j b_{nj}^2 \right)}$$

$$\leq C' n^{5/2} \sqrt{l_{4,n} l_{\ell_2}}$$

Plugging this back in Eq[A.71] and assuming WLOG $l_{4,n} \geq 1$,

$$\sum_{i<j} |\ell_{i,j} \theta_{2,i,j}| \leq C' \sqrt{\lambda} \left( \beta \frac{B_4^4 + l_{4,n} n^2}{n^{3/2}} + \frac{1}{n^2 n^{5/2} l_{2} l_{4,n}^{1/2}} \right) (t^2 + t^4)e^{-t^2/8}$$

$$\leq C' l_{4,n} \sqrt{\lambda} n^{1/2} (t^2 + t^4)e^{-t^2/8}$$

(A.72)

Finally, we also have:

$$\sum_{i<j} |\theta_{1,i,j}| \leq C |t|^{2.5} (\lambda \beta)^{1/2} (l_{4,n} n^2 + 2 \ell_2 n B_n^2) n^{-5/4} \leq C |t|^{2.5} (\lambda \beta)^{1/2} l_{4,n} n^{1/4}$$

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Finally we have, since \( t^4 \leq |t| + |t|^6 \), and \( |t|^2.5 \leq |t| + |t|^6 \),

\[
R_{n,4} \leq n^{-3/2} \sum_{i < j} |\theta_{3,i,j}| \leq n^{-3/2} \left( \sum_{i < j} t^2 |\ell_{i,j} \theta_{2,i,j}| + 4 \sum_{i < j} |\theta_{1,i,j}| e^{-t^2/8} e^{-t^2/8} \right) \\
\leq \left( t^2 l_{4,n} \sqrt{\lambda t^2 + t^4} e^{-t^2/8} n^{-1} + |t|^2.5 (\lambda \beta)^{1/2} l_{4,n} n^{-3/4} e^{-t^2/8} \right) \\
\leq l_{4,n} \left( \sqrt{\lambda \beta n^{-1} + (\lambda \beta)^{1/2} n^{-3/4}} \right) (|t| + |t|^6) e^{-t^2/8} \\
\leq C't_{4,n} \left( \beta^2 n^{-1} + (\lambda + \beta)n^{-3/4} \right) (|t| + |t|^6) e^{-t^2/8}
\]

Finally, for \( I_{3,n} \), we have:

\[
|I_{3,n}| \leq \int_{|t| \leq n^s} |t|^{-1} R_{n,4} dt \\
\leq C't_{4,n} \left( \beta^2 n^{-1} + (\lambda + \beta)n^{-3/4} \right) \int_{|t| \leq n^s} (1 + |t|^6) e^{-t^2/8} dt \\
\leq C't_{4,n} \left( \beta^2 n^{-1} + (\lambda + \beta)n^{-3/4} \right)
\]

Now we will bound \( I_{4,n} \).

Define \( \Omega := \{ k : \min(1/2, e_2/e_1^{3/2}) \leq \sqrt{n} d_{n,k} / B_n \leq 2e_2/e_1^{3/2} \} \). Using Lemma A.5 in [Wang and Jing (2004)], we see that \( |\Omega| \geq c_0 n \), for some \( c_0 \in (0,1) \).

Now, let \( \Gamma := \{ i : \alpha_i \geq \bar{\alpha} + ks_n \} \). Applying Lemma C.2 and setting \( k = \sqrt{2/c_0} \), we see that \( |\Gamma^c| \geq n(1 - c_0/2) \). Therefore, \( |\Gamma^c \cap \Omega| \geq nc_0/2 \). Let \( k_0 = [c_0/2] \).

WLOG assume \( b_{n,1} \ldots b_{n,k_0} \in \Omega \cap \Gamma^c \) and \( e_2/e_1^{3/2} \geq 1/2 \). Now for \( m \in [2, k_0 n] \), we have:

\[
S_m = \frac{1}{B_n} \sum_{k=1}^{m} b_{nk} Y_k \\
S_m^{i,j} := \frac{1}{B_n} \sum_{k \neq i,j} b_{nk} X_k
\]

For \( 1, \ldots, m \leq k_0 n \), we have:

\[
\frac{1}{mn} \sum_{i=1}^{m} \sum_{1 \leq j \neq i} d_{nij}^2 = \frac{1}{m} \sum_{i=1}^{m} \alpha_i \leq l_{4,n} + ks_n =: \ell_{5,n}
\]

As for \( \Delta_{n,m} \), we have:

\[
E\Delta_{n,m}^2 = \frac{\lambda}{n^3} \sum_{i=1}^{m-1} \sum_{j=i+1}^{n} d_{nij}^2 \leq \lambda\ell_{5,n} \frac{m}{n^2}
\]

(A.73)
Now we use the decomposition in Bickel et al. [1986] (17)-(22).

\[
\mathbb{E}[e^{itV_n}] = \mathbb{E}[e^{it(V_n - \Delta_{n,m})}e^{it\Delta_{n,m}}] \\
= \mathbb{E}[e^{it(V_n - \Delta_{n,m})}(1 + it\Delta_{n,m})] + R_{n,5} \\
= \mathbb{E}[e^{it(V_n - \Delta_{n,m})}(1 + it\Delta_{n,m})] + Ct^2\lambda_{5,n}m/n^2 \\
= \mathbb{E}[e^{it(V_n - \Delta_{n,m})}] + \frac{it}{n^{3/2}}\sum_{i=1}^{m}\sum_{j=i+1}^{n}\mathbb{E}[e^{it(V_n - \Delta_{n,m})}\psi_{ij}] + Ct^2\lambda_{5,n}\frac{m}{n^2}
\]

where the last line is obtained using Eqs [A.66] and [A.73] as follows:

\[
R_{n,5} \leq |\mathbb{E}[e^{it(V_n - \Delta_{n,m})}t^2\Delta_{n,m}^2]| \leq Ct^2\lambda_{5,n}m/n^2
\]

Note that \(V_n - \Delta_{n,m}\) can be written as \(S_{m-1} + Y_{m,n}\), where \(Y_{m,n}\) does not depend on \(Y_1, \ldots, Y_{m-1}\). So we will write:

\[
\left|\sum_{i=1}^{m-1}\sum_{j=i+1}^{n}(D_{ij})\right| = \left|\sum_{i=1}^{m-1}\sum_{j=i+1}^{n}\mathbb{E}[e^{it(S_{i-1}^{(m-1)} + \psi_{ij} + Y_{m,n})}\psi_{ij}]\right| \\
= \left|\sum_{i=1}^{m-1}\sum_{j=i+1}^{n}\mathbb{E}[e^{itS_{i-1}^{(m-1)}}]\mathbb{E}[e^{it\psi_{ij} + Y_{m,n})}\psi_{ij}]\right| \\
\leq \sup_{i<j}|\mathbb{E}[e^{itS_{i-1}^{(m-1)}}]|\sum_{i=1}^{m-1}\sum_{j=i+1}^{n}|\mathbb{E}[\psi_{ij}]| \\
\leq \sup_{i<j}|\mathbb{E}[e^{itS_{i-1}^{(m-1)}}]|\sum_{i=1}^{m-1}\sum_{j=i+1}^{n}|d_{nij}|\mathbb{E}[|\psi(Y_i, Y_j)|] \\
\leq \sqrt{\lambda}\sup_{i<j}|\mathbb{E}[e^{itS_{i-1}^{(m-1)}}]|\sqrt{mn}\sum_{i=1}^{m-1}\sum_{j=i+1}^{n}d_{nij}^2 \\
\leq \sqrt{\lambda_l\lambda_{5,n}}\sup_{i<j}|\mathbb{E}[e^{itS_{i-1}^{(m-1)}}]|mn
\]

Plugging it back, we have:

\[
|\mathbb{E}[e^{itV_n}]| \leq |\mathbb{E}[e^{itS_{m-1}^{(m-1)}}]| + \frac{|t|}{n^{3/2}}\sqrt{\lambda_{5,n}}\sup_{i<j}|\mathbb{E}[e^{itS_{i-1}^{(m-1)}}]|mn + Ct^2\lambda_{5,n}\frac{m}{n^2}
\]

(A.74)

Now, we have for \(|t| \leq 1/4\sqrt{\lambda_l/\mathbb{E}[Y_1]^3}\)

\[
|\mathbb{E}[e^{itS_{m}^{(m-1)}}]| \leq e^{-c_0m^2/n} \quad \mathbb{E}[e^{itS_{i-1}^{(m-1)}}] \leq e^{-c_0(m-2)t^2/n}
\]

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Taking \( m = \lfloor 6n \log n/c_0 t^2 \rfloor + 1 \) (for a large enough \( \epsilon \), this is still smaller than \( k_0 n \)), from Eq A.74 we have:

\[
\int_{n^\epsilon \leq |t| < 1/4 \sqrt{n/E}|Y_1|^3} |t|^{-1} \left| \mathbb{E} e^{itV_n} \right| dt \\
\leq \int_{n^\epsilon \leq |t| < 1/4 \sqrt{n/E}|Y_1|^3} \left( \frac{e^{-c_0 m t^2/n}}{|t|} + \frac{m \sqrt{\lambda l_{5,n} e^{-c_0(m-2)t^2/n}}}{n} + C|t|\lambda l_{5,n} \frac{m}{n^2} \right) dt \\
\leq C' \lambda l_{5,n} \frac{\log^2 n}{n}
\]

Now we will deal with the range \( 1/4 \sqrt{n/E}|Y_1|^3 \leq |t| \leq n^{1-\epsilon} \). Since \( \kappa(Y_1) > 0 \), and hence for large enough \( n \),

\[
|\gamma_k(t)| \leq 1 - \kappa(Y_1) \\
|\mathbb{E}[e^{itS_n}]| \leq e^{-c_0 \kappa(Y_1)} \\
|\mathbb{E}[e^{itS_{n}^2}]| \leq e^{-(m-2)c_0 \kappa(Y_1)}
\]

Using this in conjunction with Eq A.74 and setting \( m = \lfloor 4 \log n/\kappa(Y_1) \rfloor + 2 \),

\[
\int_{1/4 \sqrt{n/E}|Y_1|^3 \leq |t| \leq n^{1-\epsilon}} |t|^{-1} \left| \mathbb{E} e^{itV_n} \right| dt \\
\int_{1/4 \sqrt{n/E}|Y_1|^3 \leq |t| \leq n^{1-\epsilon}} \left( \frac{e^{-\kappa(Y_1)m}}{|t|} + \frac{m \sqrt{\lambda l_{5,n} e^{-\kappa(Y_1)(m-2)}}}{n} + C|t|\lambda l_{5,n} \frac{m}{n^2} \right) \\
\leq C' \frac{\rho n l_{5,n} \log n}{\kappa(Y_1)} \frac{\log n}{n^{2(1-\epsilon)}} = C' \frac{\lambda l_{5,n} \log n}{\kappa(Y_1)} n^{2(1-\epsilon)}
\]

Thus, using the bounds on \( I_{1,n}, I_{2,n}, I_{3,n} \) and \( I_{4,n} \) along with Eq A.63 Eq A.64 we get:

\[
\sup_x |P(V_n \leq x) - E_{2n}(x)| \\
\leq \sum_{i=1}^{4} I_{n,i} + C' \frac{\beta + (1 + \lambda) l_{4,n}/\sqrt{n}}{n^{1-\epsilon}} \\
\leq C \left( \frac{(1 + \lambda) l_{4,n}}{n^{1-2\epsilon}} + l_{4,n}(\lambda + \beta) n^{-3/4} + \frac{\lambda l_{5,n} \log n}{\kappa(Y_1)} \frac{\log n}{n^{2\epsilon}} \right) + C' \frac{\beta + (1 + \lambda) l_{4,n}/\sqrt{n}}{n^{1-\epsilon}} \\
\leq (l_{4,n} + k n) \frac{\log n}{n^2/3}
\]

The last line assumes \( \beta, \lambda \) and \( \kappa(Y_1) \) are all bounded. \( \square \)
C.4 Proof of Lemma C.5

Proof. Let $X \sim N(1, c_1^2)$ and $Y \sim N(1, c_2^2)$ be two independent random variables. We have $\xi_1 = XY$.

$$E|XY - 1|^3 \leq E|XY|^3 + 1 + 3E|X^2|Y| + 3E||X||Y|^2$$

$$= E(||X||^3E|Y|^3 + 3E[X^2]E|Y|) + 3E|X||E|Y|^2$$

$$< \infty$$

The last step is true because both $E||X||^3$ and $E||Y||^3$ are bounded for bounded $c_1, c_2$.

D Additional experiments

In this section we provide detailed algorithm descriptions that were left out from the main text for space concerns.

We have shown timing results for triangle and twostar frequencies for the SBM model against sample size $n$ in the main paper. Now we show below the computation time of triangles and V-star frequencies in SM-G.

![Graph A.4](image)

Figure A.4: From left to right, we present log of computation time of triangle frequencies in SM-G and V-star frequencies of SM-G against sample size $n$. 

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