A CONSTRUCTION OF OPEN DESCENDANT POTENTIALS IN ALL GENERA

ALEXANDER ALEXANDROV, ALEXEY BASALAEV, AND ALEXANDR BURYAK

Abstract. We present a construction of an open analogue of total descendant and total ancestor potentials via an “open version” of Givental’s action. Our construction gives a genus expansion for an arbitrary solution to the open WDVV equations satisfying a semisimplicity condition and admitting a unit. We show that the open total descendant potentials we define satisfy the open topological recursion relations in genus 0 and 1, the open string and open dilaton equations. We finish the paper with a computation of the simplest nontrivial open correlator in genus 1 using our construction.

1. Introduction

1.1. The WDVV equations. The WDVV equations, also called the associativity equations, is a system of nonlinear PDEs for one function depending on a finite number of variables. Let $N \geq 1$ and $\eta = (\eta_{\alpha\beta})$ be an $N \times N$ symmetric nondegenerate matrix with complex coefficients. The WDVV equations is the following system of PDEs for an analytic function $F(t_1, \ldots, t_N)$ defined on some open subset $M \subset \mathbb{C}^N$:

$$
\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu
u} \frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F}{\partial t^\delta \partial t^\nu \partial t^\gamma} \eta^{\mu \nu} \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta}, \quad 1 \leq \alpha, \beta, \gamma, \delta \leq N,
$$

where $(\eta^{\alpha\beta}) := \eta^{-1}$ and we use the convention of sum over repeated Greek indices. Equations (1.1) are equivalent to the fact that the tensor $c^{\gamma}_{\beta \gamma} := \eta^{\mu \nu} \frac{\partial^3 F}{\partial t^\mu \partial t^\nu \partial t^\gamma}$ defines the structure of an associative algebra in the tangent bundle $TM$. We will consider the case when this algebra structure has a unit given by a vector field $e = A^\alpha \frac{\partial}{\partial t^\alpha}$, $A^1, \ldots, A^N \in \mathbb{C}$. This condition can be equivalently written as

$$
A^\mu \frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}.
$$

In this case we will say that the solution $F$ to the WDVV equations admits the unit $A^\alpha \frac{\partial}{\partial t^\alpha}$. Such a function $F$ defines the structure of a Dubrovin–Frobenius manifold on $M$ and is also called the Dubrovin–Frobenius manifold potential. This structure appears in different areas of mathematics, including curve counting theories in algebraic geometry (Gromov–Witten theory, Fan–Jarvis–Ruan–Witten theory) and singularity theory. A systematic study of Dubrovin–Frobenius manifolds was first done by Dubrovin [Dub96, Dub99]. The matrix $\eta$ is often called the metric.

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We will say that a solution $F$ to the WDVV equations is \textit{semisimple} at a point $p \in M$, if the algebra structure on $T_p M$ does not have nilpotents. Such solutions to the WDVV equations play a special role in Givental’s theory. Note that the function $F$ comes in equations (1.1) and (1.2) together with the third derivatives. So we will consider solutions to the WDVV equations up to adding a quadratic polynomial in $t^1, \ldots, t^N$.

We will often call the WDVV equations the \textit{closed} WDVV equations in order to distinguish them from the open WDVV equations that will appear below. Correspondingly we will add the superscript “c” to the notation for solutions to (1.1).

1.2. The Givental theory. In our paper, we will mostly consider the case when $M$ is a formal neighbourhood of some point $\bar{t}_{\text{orig}} = (t^1_{\text{orig}}, \ldots, t^N_{\text{orig}}) \in \mathbb{C}^N$ meaning that we will consider solutions to the closed WDVV equations of the form $F^c \in \mathbb{C}[[t^1 - t^1_{\text{orig}}, \ldots, t^N - t^N_{\text{orig}}]]$. Let us introduce the notation

$$R^c_{\text{orig}} := \mathbb{C}[[t^1 - t^1_{\text{orig}}, \ldots, t^N - t^N_{\text{orig}}]].$$

In [Giv04, Giv01a] (see also [Lee05]) Givental introduced a group $G^c_{N,+}$ acting on the space of all solutions $F^c \in \mathbb{C}[[t^1, \ldots, t^N]]$ to the closed WDVV equations admitting a unit. The action does not change the algebra structure on $T_0 \mathcal{M}_N$, and the action is transitive on the subspace of solutions defining a fixed semisimple algebra structure on $T_0 \mathcal{M}_N$. Therefore, any solution to the closed WDVV equations, admitting a unit and that is semisimple at 0, can be obtained from the solution $F^c = \sum_{i=1}^{N} a_i t^i \frac{\partial}{\partial t^i}$, with $\eta_{ij} = \delta_{ij}$ and $e = \sum_{i=1}^{N} a_i t^i \frac{\partial}{\partial t^i}, a_1, \ldots, a_N \in \mathbb{C}^*$, making a linear change of variables $t^i \mapsto M^a_{\beta} t^\beta, M = (M^a_{\beta}) \in \text{GL}(\mathbb{C}^N)$, and acting by an element of the group $G^c_{N,+}$.

Let $t_0, t_1, \ldots$ and $\varepsilon$ be formal variables and consider the Kontsevich–Witten potential [Kon92, Wit91]

$$\mathcal{F}^{KW}(t_0, t_1, \ldots, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}^{KW}_g(t_0, t_1, \ldots),$$

where

$$\mathcal{F}^{KW}_g(t_0, t_1, \ldots) := \sum_{n \geq 1, 2g-2+n>0} \left( \int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \prod_{i=1}^{n} t^{d_i}_{i} \right) \in \mathbb{C}[[t_0, t_1, \ldots]].$$

Here $\overline{M}_{g,n}$ is the moduli space of stable algebraic curves of genus $g$ with $n$ marked points, and $\psi_i \in H^2(\overline{M}_{g,n}, \mathbb{Q})$, $1 \leq i \leq n$, is the first Chern class of the $i$-th tautological line bundle over $\overline{M}_{g,n}$ formed by the cotangent lines at the $i$-th marked point on stable curves from $\overline{M}_{g,n}$. Note that $\mathcal{F}^{KW}_0|_{t_2=0} = \frac{t_0^3}{6}$.

Givental postulated the function $\sum_{i=1}^{N} \mathcal{F}^{KW}(a_i t^i, a_i \varepsilon)$ to be a \textit{closed total ancestor potential} associated to the solution $F^c = \sum_{i=1}^{N} a_i t^i \frac{\partial}{\partial t^i}$ to the closed WDVV equations. He constructed a $G^c_{N,+}$-action on a certain space of formal series of the form

$$(1.3) \quad \mathcal{F}^c(t^\ast_0, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}^c_g(t^\ast_0), \quad \mathcal{F}^c_g \in \mathbb{C}[[t^\ast_0]],$$

where $t_0^\ast = t^0, t_1^\ast, t_2^\ast, \ldots, 1 \leq \alpha \leq N$, and $\varepsilon$ are formal variables. Acting by a linear change of variables and then by an element of $G^c_{N,+}$ on the function $\sum_{i=1}^{N} \mathcal{F}^{KW}(a_i t^i, a_i \varepsilon)$, Givental [Giv01a] constructed a closed total ancestor potential $\mathcal{F}^{c,\text{anc}}(t^\ast_0, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}^{c,\text{anc}}_g(t^\ast_0)$ associated to an arbitrary solution $F^c \in \mathbb{C}[[t^1, \ldots, t^N]]$ to the closed WDVV equations, admitting a unit and that is semisimple at 0 $\in \mathbb{C}^N$. The closed total ancestor potential $\mathcal{F}^{c,\text{anc}}$ satisfies $\mathcal{F}^{c,\text{anc}}_0|_{t^\ast_0=0} = F^c$.

Note that there is actually an infinite dimensional space of closed total ancestor potentials associated to a given solution to the closed WDVV equations.
Givental also introduced another group $G_{N,-}$ acting on solutions $F^c$ to the closed WDVV equations (admitting a unit) of the form $F^c \in \mathcal{R}_N^{\text{orig}}, \mathcal{T}_{\text{orig}}^* \in \mathbb{C}^N$, endowed with a certain additional structure called a \textit{calibration}. The $G_{N,-}$-action essentially acts only on the calibration. Using this action, starting from the closed total ancestor potentials Givental introduced the space of \textit{closed total descendant potentials} $\mathcal{F}^\text{c,desc}$ of the form

$$\mathcal{F}^\text{c,desc} = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}_g^\text{c,desc}, \quad \mathcal{F}_g^\text{c,desc} \in \mathcal{R}_N^{\text{orig}}[[t^*_1]], \quad \mathcal{T}_{\text{orig}}^* \in \mathbb{C}^N.$$  

One can see this construction as an axiomatization of the reconstruction formula for the total descendant potential starting from the total ancestor potential in the Gromov–Witten theory of some target variety $X$.

Remarkably, by a result of Teleman [Tel12], if a semisimple solution $F^c$ to the closed WDVV equations is equal to the generating series of primary genus 0 Gromov–Witten invariants of some closed total descendant potentials $\mathcal{F}$, one of Givental’s closed total descendant potentials associated to a semisimple action, starting from the closed total ancestor potentials $Givental$ introduced the space of \textit{closed total descendant potentials} $\mathcal{F}^\text{c,desc}$ of the form

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admitting a unit. This action is an extension of the Givental $G^*_{N,+}$-action. The action does not change the algebra structure on $T_0 \mathbb{C}^{N+1}$, and we will prove that the action is transitive on the subspace of solutions defining a fixed semisimple algebra structure on $T_0 \mathbb{C}^{N+1}$.

Additionally to $t_0, t_1, \ldots$ and $\varepsilon$, consider formal variables $s_0 = s, s_1, \ldots$ and consider the Pandharipande–Solomon–Tessler [PST14, Tes15] generating series of intersection numbers on the moduli spaces of Riemann surfaces with boundary $\bar{\mathcal{M}}_{g,k,i}$:

$$\mathcal{F}^\text{PST}(t_s, s, \varepsilon) = \sum_{g \geq 0} \varepsilon^{-1} \mathcal{F}^\text{PST}_g(t_s, s), \quad \mathcal{F}^\text{PST}_g(t_s, s) \in \mathbb{C}[[t_s, s]].$$

**Remark 1.1.** To be precise, Pandharipande, Solomon, and Tessler introduced the function $\mathcal{F}^\text{PST}_{s \geq 1} = 0$, from which the full function $\mathcal{F}^\text{PST}$, introduced in [Bur15, Section 1.5], can be reconstructed using the relation

$$\frac{\partial \exp (\mathcal{F}^\text{PST})}{\partial s_n} = \frac{\varepsilon^n}{(n+1)!} \frac{\partial^{n+1} \exp (\mathcal{F}^\text{PST})}{\partial s^{n+1}}.$$

Introducing the dependence on the variables $s_{\geq 1}$ is very natural from the point of view of integrable systems, matrix models [Ale15], and Virasoro constraints [Bur16, Section 1.4].

Note that the functions

$$F^c(t^1) = \frac{F^\text{KW}(t^1)}{(t^1)^3} \big|_{t^1 = 0} = \frac{(t^1)^3}{6}, \quad F^\circ(t^1, t^2) = \frac{F^\text{PST}(t^1, t^2)}{(t^1)^3} \big|_{t^1 = 0} = t^1 t^2 + \frac{(t^2)^3}{6},$$

satisfy the system of closed and open WDVV equations with $\eta = 1$ and admit the unit $\varepsilon = \frac{2}{\partial t^1}$. More generally, the functions

$$F^c(t^1, \ldots, t^N) = \sum_{i=1}^N a_i \frac{(t^i)^3}{6}, \quad F^\circ(t^1, \ldots, t^{N+1}) = a_1 t^1 t^{N+1} + \frac{(t^{N+1})^3}{6},$$

satisfy the system of closed and open WDVV equations with $\eta_{ij} = \delta_{ij}$ and admit the unit $\sum_{i=1}^N a_i^{-1} \frac{\partial}{\partial t^i}, a_i \in \mathbb{C}^*$.

We will introduce a $G^\circ_{N+1,+}$-action on a certain space of pairs $(\mathcal{F}^c, \mathcal{F}^\circ)$, where $\mathcal{F}^c$ has the form (1.3), and $\mathcal{F}^\circ = \mathcal{F}^\circ(t^1, \ldots, t^{N+1}, \varepsilon)$ has the form

$$\mathcal{F}^\circ(t^1, \ldots, t^{N+1}, \varepsilon) = \sum_{g \geq 0} \varepsilon^{-1} \mathcal{F}^\circ_g(t^1, \ldots, t^{N+1}), \quad \mathcal{F}^\circ_g(t^1, \ldots, t^{N+1}) \in \mathbb{C}[[t^1, \ldots, t^{N+1}]].$$

Applying to the pair $\left( \sum_{i=1}^N \mathcal{F}^\text{KW}(a_i t^i, a_i \varepsilon), \mathcal{F}^\text{PST}(a_1 t^1, t^{N+1}, \varepsilon) \right)$ a certain shift, a linear change of variables, and the $G^\circ_{N+1,+}$-action, we obtain a space of pairs $(\mathcal{F}^c_{\text{anc}}, \mathcal{F}^\circ_{\text{anc}})$, where $\mathcal{F}^c_{\text{anc}}$ is a closed total ancestor potential, and $\mathcal{F}^\circ_{\text{anc}}$ is a new function, which we call an open total ancestor potential.

Similarly to the Givental construction, we will introduce a group $G^\circ_{N+1,-}$, and acting by it on pairs $(\mathcal{F}^c_{\text{anc}}, \mathcal{F}^\circ_{\text{anc}})$ we obtain a space of pairs $(\mathcal{F}_{\text{desc}}, \mathcal{F}^\circ_{\text{desc}})$, where $\mathcal{F}_{\text{desc}}$ is a closed total descendant potential and $\mathcal{F}^\circ_{\text{desc}}$ is a new function of the form

$$\mathcal{F}^\circ_{\text{desc}} = \sum_{g \geq 0} \varepsilon^{-1} \mathcal{F}^\circ_{\text{desc} g}, \quad \mathcal{F}^\circ_{\text{desc} g} \in \mathcal{R}_{\text{orig}}[[t^1]], \quad \mathcal{F}_{\text{desc}} \in \mathcal{R}_{\text{orig}}[[t^1]],$$

which we call an open total descendant potential. Note that then $\mathcal{F}^\circ_{\text{desc} g} \in \mathcal{R}_{\text{orig}}[[t^1]]$, where $\pi: \mathbb{C}^{N+1} \rightarrow \mathbb{C}^N$ is the projection to the first $N$ coordinates. Here is our first result.

**Theorem 1.** 1. For any pair $(\mathcal{F}^c_{\text{desc}}, \mathcal{F}^\circ_{\text{desc}})$ of total descendant potentials, the pair $(\mathcal{F}^c, \mathcal{F}^\circ)$, where $F^c := \mathcal{F}^c_{\text{desc}} \big|_{t^1 = 0}$ and $F^\circ := \mathcal{F}^\circ_{\text{desc}} \big|_{t^1 = 0}$, satisfies the system of closed and open WDVV equations, admits a unit, and is semisimple at $\mathcal{F}_{\text{desc}} \in \mathbb{C}^{N+1}$. 
2. For any pair \((F^c, F^o)\) of solutions to the closed and open WDVV equations having the form 
\(F^c \in \mathcal{R}_{N}^{\pi(\text{orig})}, \quad F^o \in \mathcal{R}_{N+1}^{\text{orig}}, \quad \text{orig} \in \mathbb{C}^{N+1},\) admitting a unit, and that is semisimple at 
\(\text{orig} \in \mathbb{C}^{N+1},\) there exists a pair \((F^{c, \text{desc}}, F^{o, \text{desc}})\) of total descendant potentials such that 
\(F^c = F^{c, \text{desc}}|_{t^c_{\geq 1} = 0}\) and \(F^o = F^{o, \text{desc}}|_{t^o_{\geq 1} = 0}.\)

We believe that the space of our open total descendant potentials contains the (appropriately defined) open Gromov–Witten potentials of smooth projective varieties and open FJRW potentials, satisfying the semisimplicity condition, which are rigorously constructed in genera higher than zero only in a limited class of cases. Nevertheless, for such geometrically defined potentials \(F^o = \sum_{g \geq 0} (g-1) F^o_g\), which should come with a closed total descendant potential \(F^c = \sum_{g \geq 0} \varepsilon^{2g-2} F^c_g\), it is believed that \(F^o = F^o_0|_{t^o_{\geq 1} = 0}\) should satisfy the open WDVV equations and admit a unit \(\epsilon = A^a \frac{\partial}{\partial t^a}\), and that \(F^o\) should satisfy the following PDEs:

- the open topological recursion relations in genus 0 (open TRR-0 relations)
  \[
  \frac{\partial^2 F^o_0}{\partial t^a_{a+1} \partial t^b_0} = \frac{\partial^2 F^c_0}{\partial t^a_0 \partial t^b_0} \frac{\partial^2 F^o_0}{\partial t^a_0 \partial t^b_0} + \frac{\partial F^c_0}{\partial t^a_0} \frac{\partial F^o_0}{\partial t^b_0} \frac{\partial^2 F^o_0}{\partial t^a_0 \partial t^b_0} + \frac{\partial^2 F^o_0}{\partial t^a_0 \partial t^b_0}, \quad 1 \leq \alpha, \beta \leq N + 1, \quad a, b \geq 0;
  \]

- the open topological recursion relations in genus 1 (open TRR-1 relations)
  \[
  \frac{\partial F^o_1}{\partial t^a_{a+1}} = \frac{\partial^2 F^c_0}{\partial t^a_0 \partial t^b_0} \frac{\partial F^o_1}{\partial t^a_0} + \frac{\partial F^c_0}{\partial t^a_0} \frac{\partial F^o_1}{\partial t^b_0} + \frac{1}{2} \frac{\partial^2 F^c_0}{\partial t^a_0 \partial t^b_0}, \quad 1 \leq \alpha \leq N + 1, \quad a \geq 0;
  \]

- the open string equation
  \[
  A^\alpha \frac{\partial F^o_0}{\partial t^a_0} = \sum_{d \geq 0} t^a_{d+1} \frac{\partial F^o_0}{\partial t^a_0} + \varepsilon^{-1} t^a_{N+1},
  \]

- the open dilaton equation
  \[
  A^\alpha \frac{\partial F^o_0}{\partial t^a_0} = \sum_{d \geq 0} t^a_{d} \frac{\partial F^o_0}{\partial t^a_0} + \varepsilon \frac{\partial F^o_0}{\partial \varepsilon} + \frac{1}{2}.
  \]

**Theorem 2.** For any pair of total descendant potentials \((F^{c, \text{desc}}, F^{o, \text{desc}})\) equations (1.6), (1.7), (1.8), and (1.9) are satisfied.

Finally, in Section 5, we discuss in more details the action of the group \(G^C_{N+1, +}\) on the space of pairs of closed and open total ancestor potentials and compute explicitly the simplest nontrivial correlator in an arbitrary open ancestor potential in genus 1.

### 1.5. Notation and conventions.

- As we already mentioned before, throughout the text we use the Einstein summation convention for repeated upper and lower Greek indices. Also, tensors of different ranks will often simultaneously appear in some formulas. In this case, we assume that a summation is performed in the range of indices where all the terms in the sum are well defined.

- When it does not lead to a confusion, we use the symbol * to indicate any value, in the appropriate range, of a sub- or superscript.

- For a Lie group \(G\) acting on a smooth manifold \(M\), we denote by \(g.x \in M\) the result of the action of \(g \in G\) on a point \(x \in M\). If \(g\) is the Lie algebra corresponding to \(G\) and \(\text{ad}_g \in \mathfrak{g}\), then we denote by \(h.x \in T_x M\) the result of the infinitesimal action of \(h\) on \(x\).
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2. A Givental-type theory for solutions to the open WDVV equations

The goal of this section is to introduce the space of open descendant potentials $F_0^c$ that will be the genus 0 part of the open total descendant potentials that we will define in the next section. We will also introduce groups $G^{\infty}_{N+1,\pm}$ together with their actions on appropriate subspaces of the space of open descendant potentials. This will be done as a reduction of the Givental-type theory for flat F-manifolds developed in [ABLR20].

2.1. Flat F-manifolds. Let $L \geq 1$. The oriented WDVV equations are the following PDEs for $L$ analytic functions $F^1, \ldots, F^L$ on an open subset $M \subset \mathbb{C}^L$:

$$
\frac{\partial^2 F^\alpha}{\partial t^{j_1} \partial t^{j_2}} \frac{\partial^2 F^\mu}{\partial t^{j_3} \partial t^{j_4}} = \frac{\partial^2 F^\alpha}{\partial t^{j_3} \partial t^{j_4}} \frac{\partial^2 F^\mu}{\partial t^{j_1} \partial t^{j_2}}, \quad 1 \leq \alpha, \beta, \gamma, \delta \leq \mathbb{L}.
$$

(2.1)

The functions $F^\alpha$ will be considered up to adding a linear polynomial in $t^1, \ldots, t^L$. Equations (2.1) are equivalent to the fact that the tensor $e_{\beta \gamma} := \frac{\partial F^0}{\partial t^\beta \partial t^\gamma}$ defines the structure of an associative algebra in the tangent bundle $TM$. Suppose that this algebra structure has a unit $e$ of the form $e = A^\alpha \frac{\partial}{\partial t^\alpha}$, $A^\alpha \in \mathbb{C}$. Then the $L$-tuple of functions $\mathcal{F} = (F^1, \ldots, F^L)$ defines the structure of a flat F-manifold on $M$ and is called the vector potential of this flat F-manifold.

Note that if $L = N$ and $F^\alpha = \eta_{\alpha \beta} \frac{\partial F^0}{\partial t^\beta}$, where $\eta = (\eta_{\alpha \beta})$ is an $N \times N$ symmetric nondegenerate matrix and $F^c = F^c(t^1, \ldots, t^N)$ is an analytic function, then equations (2.1) are equivalent to the closed WDVV equations for the function $F^c$.

By an observation of Paolo Rossi, if $L = N + 1$ and $F^c = F^c(t^1, \ldots, t^N)$, $F^\alpha = F^\alpha(t^1, \ldots, t^{N+1})$ are two analytic functions, then equations (2.1) for $F^\alpha := \eta_{\alpha \beta} \frac{\partial F^0}{\partial t^\beta}$, $1 \leq \alpha \leq N$, $F^{N+1} := F^\alpha$ are equivalent to the system of closed and open WDVV equations for the pair $(F^c, F^\alpha)$.

2.2. Descendant vector potentials of flat F-manifolds. Let us fix $L \geq 1$, a nonzero vector $\mathcal{A} = (A^1, \ldots, A^L) \in \mathbb{C}^L$, and a point $t_{\text{orig}} = (t^1_{\text{orig}}, \ldots, t^L_{\text{orig}}) \in \mathbb{C}^L$.

In [ABLR20, Section 2.1] the authors introduced the notion of a sequence of descendant vector potentials of a flat F-manifold. An equivalent description is given by [ABLR20, Proposition 2.2]: a sequence of $L$-tuples of functions $\mathcal{F}_0^\alpha = (F_0^{1,a}, \ldots, F_0^{L,a})$, $F_0^{\alpha,a} \in \mathcal{R}_{L}^{\text{orig}}([t^a_{\text{orig}}])$, $a \geq 0$, is a sequence of descendant vector potentials of a flat F-manifold if and only if the following equations are satisfied:

$$
\sum_{b \geq 0} q_{b+1}^{\beta} \frac{\partial F_0^{\alpha,a}}{\partial q_{b}^{\beta}} = - F_0^{\alpha,a-1}, \quad 1 \leq \alpha \leq L, \quad a \in \mathbb{Z},
$$

(2.2)

$$
\sum_{b \geq 0} q_{b}^{\beta} \frac{\partial F_0^{\alpha,a}}{\partial q_{b}^{\beta}} = F_0^{\alpha,a}, \quad 1 \leq \alpha \leq L, \quad a \in \mathbb{Z},
$$

(2.3)

$$
\frac{\partial^2 F_0^{\alpha,0}}{\partial q_{b+1}^{\beta} \partial \bar{q}_{c}^{\bar{\beta}}} = \frac{\partial F_0^{\alpha,0}}{\partial \bar{q}_{c}^{\bar{\beta}}} \frac{\partial F_0^{\alpha,0}}{\partial q_{b}^{\beta}}, \quad 1 \leq \alpha, \beta, \gamma, \leq L, \quad b, c \geq 0,
$$

(2.4)

$$
\frac{\partial F_0^{\alpha,a+1}}{\partial q_{b}^{\beta}} + \frac{\partial F_0^{\alpha,a}}{\partial q_{b+1}^{\beta}} \frac{\partial F_0^{\mu,0}}{\partial q_{b}^{\beta}} = \frac{\partial F_0^{\alpha,a}}{\partial q_{b}^{\beta}} \frac{\partial F_0^{\mu,0}}{\partial \bar{q}_{c}^{\bar{\beta}}}, \quad 1 \leq \alpha, \beta \leq L, \quad a, b \geq 0,
$$

(2.5)
where we use the notation
\[ F_0^{α,a} := (-1)^{α+1}q_{α,a-1}, \quad \text{if } a < 0; \quad q_β := t_β^α - Aβδ_{b,1}. \]
The vector potential of the corresponding flat F-manifold is given by \( \overline{F} = (F^1, \ldots, F^L) \) where \( F^α := F_0^{α,0} \bigg|_{t^α_2 = 0} \), with the unit \( Aαβ\frac{∂}{∂t_β}. \)

A collection of descendant vector potentials \( \overline{F}_0^{α} \) is called ancestor, if \( \overline{F}_{\text{orig}} = 0 \) and \( \frac{∂F_0^{α,a}}{∂t_β} \bigg|_{t^α_2 = 0} = 0 \) for all \( a,b \geq 0 \). For a given solution \( \overline{F} = (F^1, \ldots, F^L) \), \( F^α \in \mathbb{C}[[t^α]] \), to the oriented WDVV equations admitting a unit, there exists a unique sequence of ancestor vector potentials \( \overline{F}_0^{α} \) such that \( F^α = F_0^{α,0} \bigg|_{t^α_2 = 0} \).

**Lemma 2.1.** Consider an arbitrary collection of ancestor vector potentials \( \overline{F}_0^{α} \). Then the coefficient of \( t^α_1 \ldots t^α_n \) in \( \overline{F}_0^{α,d} \) is zero if \( d + \sum d_i \geq n - 1 \).

**Proof.** The statement is obviously true for \( d = d_1 = \ldots = d_n = 0 \). Then the statement is proved by the induction on \( d + \sum d_i \) using equations (2.4) and (2.5). \( \square \)

**Remark 2.2.** It is easy to see that equations (2.3) and (2.5) imply that the whole collection of descendant vector potentials can be uniquely reconstructed from the \( L \)-tuple \( (\overline{F}_0^{1,0}, \ldots, \overline{F}_0^{L,0}) \).

### 2.3. Group actions on descendant vector potentials.

#### 2.3.1. Linear changes of variables.** By [ABLR20, Section 2.1], there is a \( \text{GL}(\mathbb{C}^L) \)-action on the space of descendant vector potentials of flat F-manifolds of dimension \( L \) given by \( \overline{F}_0^{α} \mapsto M \overline{F}_0^{α} \),

\[ M, \overline{F}_0^{α,a} := M_μ^{α} \overline{F}_0^{α,a} \bigg|_{t^α_0 \mapsto \overline{F}_0^{α,a},} \]

and the unit \( \overline{A} = (A^1, \ldots, A^L) \) changes by the formula \( \overline{A} \mapsto M\overline{A} \).

#### 2.3.2. Loop group actions. Consider the following groups:

\[ G_{L,+} := \left\{ R(z) = \text{Id} + \sum_{i \geq 1} R_i z^i \in \text{End}(\mathbb{C}^L)[[z]] \right\}, \]
\[ G_{L,-} := \left\{ S(z) = \text{Id} + \sum_{i \geq 1} S_i z^{-i} \in \text{End}(\mathbb{C}^L)[[z^{-1}]] \right\}, \]

and denote by \( \mathfrak{g}_{L,+} \) and \( \mathfrak{g}_{L,-} \) the corresponding Lie algebras.

In [ABLR20] the authors constructed an action of the group \( G_{L,+} \) on the space of ancestor vector potentials. Infinitesimally, it is given by the formula (see [ABLR20, Proposition 2.9])

\[ r(z).\overline{F}_0^{α,a} = \frac{d}{dt} \left( e^{θ(r)} \overline{F}_0^{α,a} \right) \bigg|_{θ=0} \]
\[ = \sum_{i \geq 1} (-1)^i (r_i)^α_μ \overline{F}_0^{α,a+i}_μ \sum_{i \geq 1, j \geq 0} (-1)^{i-j-1} (r_i)^ν_j \frac{∂\overline{F}_0^{α,a}}{∂q_j^ν} \overline{F}_0^{α,j-1} \]
\[ = \sum_{i \geq 1} (-1)^i (r_i)^α_μ \overline{F}_0^{α,a+i}_μ \sum_{j, k \geq 0} (-1)^k (r_{j+k+1})^ν_j \frac{∂\overline{F}_0^{α,a}}{∂q_j^ν} \overline{F}_0^{α,k} - \sum_{i \geq 1, ℓ \geq 0} (r_i)^μ_ν \frac{∂\overline{F}_0^{α,a}}{∂q_ℓ^ν} q_ℓ^{α}, \]

where \( r(z) \in \mathfrak{g}_{L,+} \) and \( a \in \mathbb{Z}_{\geq 0} \).
In [ABLR20] the authors also constructed an action of the group $G_{L,-}$ on the space of descendant vector potentials $F_0^a, F_0^{a} \in \mathcal{R}_L^{\text{orig}}[[t^*_{\geq 1}]]$. If $S(z) \in G_{L,-}$, then $S(z) \cdot F_0^{a} \in \mathcal{R}_L^{\text{orig}-S_1T_z}[[t^*_{\geq 1}]]$ and (see [ABLR20, Proposition 2.10])

$$S(z) \cdot F_0^{a} = e^{-\widetilde{s}(z)} \left( F_0^{a} + \sum_{i=1}^{a} (-1)^i (S_i)_\mu F_0^{\mu,a-i} + (-1)^{a+1} \sum_{i \geq a+1} (S_i)_\mu q_i^{\mu,a-1} \right),$$

where $s(z) := \log S(z)$ and $\widetilde{s}(z) := \sum_{i \geq 1} (s_i)_\mu q_i^{\mu,a-1}$. The corresponding formula for the infinitesimal action is

$$s(z) \cdot F_0^{a} = - \sum_{i,j \geq 1} (s_i)_\mu q_i^{\mu} \frac{\partial F_0^{a}}{\partial q_j} + \sum_{i=1}^{a} (-1)^i (s_i)_\mu F_0^{\mu,a-i} + (-1)^{a+1} \sum_{i \geq a+1} (s_i)_\mu q_i^{\mu,a-1}.$$

2.3.3. Shifts and the $G_{L,-}$-action. The $G_{L,-}$-action can be used to describe how the ancestor vector potentials change when we shift the variables, $t^a \mapsto t^a + \theta^a$, in a vector potential $F = (F_1, \ldots, F_L)$.

Let $F = (F_1, \ldots, F_L), F^\alpha \in \mathbb{C}[[t^1, \ldots, t^L]]$, be a vector potential of a flat F-manifold and let $F_0^a$ be the corresponding ancestor vector potentials. Define $L \times L$ matrices $\Omega_j(t^*) = (\Omega_{j,\beta}^\alpha(t^*))$, $j \geq 0$, by

$$\Omega_j^\alpha(\theta^*) := \frac{\partial F_0^{\alpha,j}}{\partial \theta^j} \bigg|_{\theta^1 = 0} \in \mathbb{C}[[t^*]].$$

Consider a family of vector potentials $\mathcal{F}_\theta^a$, depending on formal parameters $\theta^1, \ldots, \theta^L$, $\theta = (\theta^1, \ldots, \theta^L)$, defined by

$$F_\theta^a := F^a \big|_{\theta^1=\hat{\theta}^1+\theta^1} \in \mathbb{C}[[\hat{\theta}^1]][[t^*]].$$

Denote by $\mathcal{F}_\theta^a$ the corresponding family of ancestor vector potentials. By [ABLR20, Lemma 2.12] we have

$$\mathcal{F}_{\theta^1}^a = \left( \text{Id} + \sum_{j \geq 1} (-1)^j \Omega_{j-1}(\theta^*) z^{-j} \right)^{-1} F_0^a.$$

2.3.4. The semisimplicity condition. The above group actions preserve the semisimplicity condition of the corresponding flat F-manifold at $\mathcal{R}_L^{\text{orig}}$. Indeed, $\text{GL}(\mathbb{C}^L)$ acts on $T_{\text{orig}} \mathbb{C}^L$ by changing the basis, while the groups $G_{L,\pm}$ do not change the algebra structure on $T_{\text{orig}} \mathbb{C}^L$. So the semisimplicity condition is preserved.

2.3.5. Transitivity in the semisimple case. In [ABLR20] the authors proved that the group $G_{L,+}$ acts transitively on the space of ancestor vector potentials defining a fixed semisimple algebra structure on $T_0 \mathbb{C}^L$. Their approach is constructive, and we recall it, because we will need it for the proof of Theorem 1.

We consider a solution $F = (F_1, \ldots, F_L), F^\alpha \in \mathbb{C}[[t^*]]$, to the oriented WDVV equations, admitting a unit $A^\alpha \frac{\partial}{\partial t^\alpha}$ and that is semisimple at $0 \in \mathbb{C}^L$. Canonical coordinates $u^1, \ldots, u^L$ are formal coordinates on $\mathbb{C}^L$ around $0$, $u^i = u^i(t^*) \in \mathbb{C}[[t^*]]$, such that the vector fields $\frac{\partial}{\partial u^i}$ are the idempotents for the algebra structure in $T \mathbb{C}^L$ around $0$ defined by $F$. Canonical coordinates are defined uniquely up to permutations and shifts $u^i \mapsto u^j + a_i$, $a_i \in \mathbb{C}$.

Choosing canonical coordinates, consider the matrix $\Psi = \Psi(t^*)$ defined by $\Psi := \left( \frac{\partial u^i}{\partial \theta^a} \right)$. Consider the diagonal matrix $U := \text{diag}(u^1, \ldots, u^L)$. There exists a unique diagonal matrix $D$
whose entries are one-forms and a unique matrix $\tilde{\Gamma} = \tilde{\Gamma}(t^*)$ with vanishing diagonal part such that

$$\tilde{D} + \tilde{[\tilde{\Gamma}, dU]} := d\tilde{\Psi} \cdot \tilde{\Psi}^{-1}.$$  

Define a diagonal matrix $H = H(t^*) = \text{diag}(H_1, \ldots, H_L)$, $H_i(0) \neq 0$, by $dH \cdot H^{-1} := -\tilde{D}$. The functions $H_i \in \mathbb{C}[[t^*]]$ are defined uniquely up to the rescaling $H_i \mapsto \lambda_i H_i$ for $\lambda_i \in \mathbb{C}^*$. Define matrices $\Gamma = \Gamma(t^*) = (\gamma^i_j)$ and $\Psi = \Psi(t^*)$ by $\Gamma := H \tilde{\Gamma} H^{-1}$ and $\Psi := H \tilde{\Psi}$. Note that the functions $\gamma^i_j$ can be expressed in terms of the functions $H_k$ as

$$\gamma^i_j = H^{-1}_j \frac{\partial H_i}{\partial u^j}, \quad i \neq j.$$  

There exists a sequence of matrices $R_i = R_i(t^*)$, $i \geq 0$, $R_0 = \text{Id}$, satisfying the relations

$$(2.9) \quad dR_{k-1} + R_{k-1} [\Gamma, dU] = [R_k, dU], \quad k \geq 1.$$  

In more details, the system of equations $(2.9)$ is equivalent to the following system of recursive relations:

$$(2.10) \quad (R_{m+1})^i_j = (R_m)^i_j \gamma^i_j - \frac{\partial (R_m)^i_j}{\partial u^i}, \quad i \neq j, \quad m \geq 0,$$

$$(2.11) \quad d(R_{m+1})^i_i = -\sum_{j \neq i} (R_{m+1})^i_j (du^i - du^j), \quad m \geq 0,$$

which, given matrices $R_0, \ldots, R_m$, determine the matrix $R_{m+1}$ uniquely up to integration constants in the diagonal entries that can be arbitrary.

For $1 \leq \alpha \leq L$, denote by $F_{\text{quad}}^\alpha$ the quadratic part of the formal power series $F^\alpha$. The functions $F_{\text{quad}}^\alpha$ are uniquely (up to linear polynomials in $t^*$) determined by the structure constants of the algebra structure on $T_0 \mathbb{C}^L$, they satisfy the oriented WDVV equations and admit the same unit $A^\alpha \frac{\partial}{\partial q^\alpha}$. By [ABLR20, proof of Theorem 2.13], the sequence of ancestor vector potentials corresponding to the vector potential $\tilde{F}$ can be obtained from the sequence of ancestor vector potentials corresponding to the vector potential $F_{\text{quad}} := (F_{\text{quad}}^1, \ldots, F_{\text{quad}}^M)$ by the action of the element $R(z) \in G_{L,+}$ given by

$$R(z) = \Psi^{-1}(0) R^{-1}(-z, 0) \Psi(0),$$

where $R(z, t^*) := \text{Id} + \sum_{i \geq 1} R_i(t^*) z^i$.

### 2.4. The Givental theory as a reduction of the general case. Let $L = N$.

#### 2.4.1. Closed descendant potentials. We will say that a collection of descendant vector potentials is of **closed type**, if there exists a constant symmetric nondegenerate matrix $\eta = (\eta_{a\beta})$ such that

$$\frac{\partial (\eta_{a\beta} F_{0}^{\mu, a})}{\partial t_{b}^\beta} = \frac{\partial (\eta_{a\beta} F_{0}^{\mu, b})}{\partial t_{a}^\alpha}, \quad 1 \leq \alpha, \beta \leq N, \quad a, b \geq 0.$$  

In this case we define a function $F_{0}^c \in \mathcal{R}_{N}^{\text{toric}}[[t^*_{\geq 1}]]$ by

$$F_{0}^c := \frac{1}{2} \sum_{a \geq 0} q^a_0 \eta_{a\mu} F_{0}^{\mu, a},$$

which satisfies

$$\frac{\partial F_{0}^c}{\partial t_{b}^\beta} = \eta_{b\mu} F_{0}^{\mu, b}, \quad \sum_{a \geq 0} q^a_0 \frac{\partial F_{0}^c}{\partial q^a_0} = 2 F_{0}^c.$$  

We call $F_{0}^c$ a **closed descendant potential**. The function $F_{0}^c|_{t^*_{\geq 1} = 0}$ then satisfies the closed WDVV equations and admits the unit $A^a \frac{\partial}{\partial q^a}$. If $F_{0}^c$ comes from a sequence of ancestor vector potentials, then $F_{0}^c$ is called a **closed ancestor potential**.
The space of closed descendant potentials can be alternatively described as the space of solutions \( \mathcal{F}_0^c \in \mathcal{R}_N^{\text{orig}} [[t_1^*, \ldots, t_n^*]] \) to the following equations:

\[
\sum_{n \geq 0} q_{n+1}^\gamma \frac{\partial F_0^c}{\partial q_n^\alpha} = -\frac{1}{2} \eta_{\alpha \beta} t_0^\alpha t_0^\beta, \tag{2.12}
\]

\[
\sum_{b \geq 0} q_b^\beta \frac{\partial F_0^c}{\partial q_b^\alpha} = 2 F_0^c, \tag{2.13}
\]

\[
\frac{\partial^2 F_0^c}{\partial q_n^\alpha \partial q_b^\beta} = \frac{\partial^2 F_0^c}{\partial q_n^\alpha \partial q_b^\beta}, \tag{2.14}
\]

This matches the Givental approach to the descendant potentials from [Giv04]. Equations (2.12), (2.13), and (2.14) are called, respectively, the closed string, dilaton, and topological recursion relations in genus 0. The subspace of ancestor potentials is specified by the conditions \( \tilde{t}_{\text{orig}} = 0 \) and \( \frac{\partial^2 F_0^c}{\partial q_a^\alpha \partial q_b^\beta} = 0 \) for all \( 1 \leq \alpha, \beta \leq N \) and \( a, b \geq 0 \). From Lemma 2.1 it follows that the coefficient of \( t_{d_1}^a \ldots t_{d_n}^a \) in an arbitrary closed ancestor potential \( F_0^c \) is zero if \( \sum d_i \geq n - 2 \).

**Example 2.3.** A basic example of a closed ancestor potential is the function

\[
F_0^c = F_0^{KW} = \frac{\kappa_3}{6} + \frac{\kappa_4}{6} + \left( \frac{\kappa_5}{6} + \frac{\kappa_6}{24} \right) + O((t_*)^6),
\]

the corresponding metric is \( \eta = 1 \) and the unit is \( e = \frac{\partial}{\partial \tilde{t}} \). There is a simple formula for the coefficients of \( F_0^{KW} \):

\[
\frac{\partial^n F_0^{KW}}{\partial t_{d_1} \ldots \partial t_{d_n}} \bigg|_{t_* = 0} = \begin{cases} \frac{(n-3)!}{\prod d_i!}, & \text{if } n \geq 3 \text{ and } \sum d_i = n - 3, \\ 0, & \text{otherwise.} \end{cases}
\]

2.4.2. **Group actions.** The GL(\( \mathbb{C}^N \))-action on the space of descendant vector potentials preserves the subspace of descendant vector potentials of closed type and acts on the metric \( \eta \) and on the closed descendant potential \( F_0^c \) by

\[
M \cdot \eta_{\alpha \beta} = (M^{-1})^\mu_\alpha \eta_{\mu \nu} (M^{-1})^\nu_\beta, \quad M \cdot F_0^c = F_0^c \bigg|_{\tilde{t}_\eta \rightarrow (M^{-1})^\beta_\eta \tilde{t}_\eta},
\]

for \( M \in \text{GL}(\mathbb{C}^N) \). Note that if \( M = \exp(m) \), for some matrix \( m = (m_{\alpha \beta}) \), then

\[
M \cdot F_0^c = \exp(-\tilde{m}) F_0^c,
\]

where

\[
\tilde{m} := \sum_{d \geq 0} m_{d}^\beta \frac{\partial}{\partial t_{d}^\beta}.
\]

For a fixed matrix \( \eta \) consider the following subgroups of the groups \( G_{N, \pm} \):

\[
G_{N, \pm}^c : = \left\{ M(z) = \text{Id} + \sum_{i \geq 1} M_i z^{\pm i} \in \text{End}(\mathbb{C}^N)[[[z^{\pm 1}]]] \bigg| M^T(-z) \eta M(z) = \eta \right\} \subset G_{N, \pm}.
\]

The corresponding Lie algebras are given by

\[
g_{N, \pm}^c = \left\{ m(z) = \sum_{i \geq 1} m_i z^{\pm i} \in z^{\pm 1} \text{End}(\mathbb{C}^N)[[[z^{\pm 1}]]] \bigg| \eta_{\mu \nu} (m_i)^\nu_{\mu} \eta_{\beta \alpha} = (-1)^{i+1} (m_i)^i \beta \right\}.
\]
Let us introduce the following differential operators:
\[
\tilde{r}(z)^c := - \sum_{i \geq 1, j \geq 0} (r_i)_{\mu}^\alpha q_j^{\beta} \frac{\partial}{\partial t_{i+j}} + \frac{\varepsilon^2}{2} \sum_{i,j \geq 0} (-1)^j (r_{i+j+1})_{\mu}^\alpha \eta_{\mu \beta} \frac{\partial^2}{\partial q_i^\alpha \partial q_j^\beta}, \quad r(z) \in g_{N,+}^c,
\]
\[
\tilde{s}(z)^c := - \sum_{i \geq 1, j \geq 0} (s_i)_{\beta}^\alpha q_{i+j}^{\alpha} \frac{\partial}{\partial q_j} + \frac{\varepsilon^2}{2} \sum_{i,j \geq 0} (-1)^j (s_{i+j+1})_{\mu}^\alpha \eta_{\mu \beta} q_i^{\alpha} q_j^{\beta}, \quad s(z) \in g_{N,-}^c.
\]

The fact that the group \( G_{N,+}^c \) (\( G_{N,-}^c \)) preserves the space of ancestor (descendant) vector potentials of closed type is well known from [Giv04] (see also [Lee05]). However, for completeness, let us present a quick derivation of this result.

**Lemma 2.4.**

1. The group \( G_{N,+}^c \) (\( G_{N,-}^c \)) preserves the space of ancestor (descendant) vector potentials of closed type. We, therefore, denote by \( r(z),\mathcal{F}_0^c \) (\( s(z),\mathcal{F}_0^c \)) the resulting infinitesimal action of \( g_{N,+}^c \) (\( g_{N,-}^c \)) on a closed ancestor (descendant) potential \( \mathcal{F}_0^c \).

2. The corresponding action on the space of closed ancestor (descendant) potentials is given by

\[
\begin{align*}
\tilde{r}(z)^c \exp(\varepsilon^2 \mathcal{F}_0^c) &= \varepsilon^2 r(z) \cdot \mathcal{F}_0^c + O(\varepsilon^6), \\
\tilde{s}(z)^c \exp(\varepsilon^2 \mathcal{F}_0^c) &= \varepsilon^2 s(z) \cdot \mathcal{F}_0^c + O(\varepsilon^6),
\end{align*}
\]

**Proof.** For the group \( G_{N,+}^c \) we compute

\[
\eta^{\alpha \varepsilon} \frac{\partial}{\partial t_0} \text{Coef}_{\varepsilon^{-2}} \left( \frac{r(z)^c \exp(\varepsilon^2 \mathcal{F}_0^c)}{\exp(\varepsilon^2 \mathcal{F}_0^c)} \right) =
\]

\[
= \eta^{\alpha \varepsilon} \frac{\partial}{\partial t_0} \left( - \sum_{i \geq 1, j \geq 0} (r_i)_{\mu}^\alpha q_j^{\beta} \frac{\partial \mathcal{F}_0^c}{\partial t_{i+j}} + \frac{\varepsilon^2}{2} \sum_{i,j \geq 0} (-1)^j (r_{i+j+1})_{\mu}^\alpha \eta_{\mu \beta} \frac{\partial^2 \mathcal{F}_0^c}{\partial q_i^\alpha \partial q_j^\beta} \right) =
\]

\[
= - \sum_{i \geq 1} \eta^{\alpha \nu} (r_i)_{\mu}^\alpha \frac{\partial \mathcal{F}_0^c}{\partial t_0} - \sum_{i \geq 1, j \geq 0} (r_i)_{\mu}^\alpha q_j^{\beta} \frac{\partial \mathcal{F}_0^c}{\partial q_j^{\beta}} + \sum_{i,j \geq 0} (-1)^j (r_{i+j+1})_{\mu}^\alpha \eta_{\mu \beta} \frac{\partial \mathcal{F}_0^c}{\partial q_i^\alpha \partial q_j^\beta} =
\]

\[
= \sum_{i \geq 1} (-1)^i (r_i)_{\mu}^\alpha \mathcal{F}_0^{\mu, i+a} - \sum_{i \geq 1, j \geq 0} (r_i)_{\mu}^\alpha q_j^{\beta} \frac{\partial \mathcal{F}_0^{\alpha, a}}{\partial t_{i+j}} + \sum_{i,j \geq 0} (-1)^j (r_{i+j+1})_{\mu}^\alpha \eta_{\mu \beta} \frac{\partial \mathcal{F}_0^{\alpha, a}}{\partial q_i^\alpha \partial q_j^\beta} = r(z) \cdot \mathcal{F}_0^{\alpha, a},
\]

and also notice that \( \text{Coef}_{\varepsilon^{-2}} \left( \frac{r(z)^c \exp(\varepsilon^2 \mathcal{F}_0^c)}{\exp(\varepsilon^2 \mathcal{F}_0^c)} \right) \) is an eigenvector for the operator \( \sum_{a \geq 0} q_a^\alpha \frac{\partial}{\partial q_a^\beta} \) with eigenvalue 2. The proof for the group \( G_{N,-}^c \) is similar. \( \square \)

**2.4.3. A transitivity statement for the \( G_{N,+}^c \)-action.** Givental proved that the group \( G_{N,+}^c \) acts transitively on the space of closed ancestor potentials defining a fixed semisimple algebra structure on \( T_0 C^N \). Let us see how to deduce this result from the analogous result about the transitivity of the \( G_{N,+}^c \)-action described in Section 2.3.5.

Indeed, we consider a closed ancestor potential \( \mathcal{F}_0^c \), the associated solution \( F^c \) to the closed WDVV equations, the associated sequence of ancestor vector potentials, and follow the constructions from Section 2.3.5. A standard result in the theory of Dubrovin–Frobenius mani-

ifolds [Dub96] says that the symmetric bilinear form \( \eta = \frac{1}{2} \eta_{\alpha \beta} dt^\alpha dt^\beta \) becomes diagonal in the canonical coordinates, \( \eta = \sum_{i=1}^N g_i (du^i)^2 \), and one can choose the functions \( H_i \) such that
In this case the matrices $R_i$ can be chosen [Giv01b] in such a way that they satisfy the additional orthogonality condition

$$R(−z, t^*)^T R(z, t^*) = \text{Id}.$$ 

Then $R(z) = \Psi^−1(0) R^−1(−z, 0) \Psi(0) \in G_{N+,+}$. If we denote by $F_{\text{cub}}^c$ the cubic part of the formal power series $F^c$, then the closed ancestor potential $F_0^c$ is obtained from the closed ancestor potential associated to $F_{\text{cub}}^c$ by the action of the element $R(z) \in G_{N+,+}$.

2.5. Open descendant potentials. Let $L = N + 1$.

2.5.1. Definition. We say that a sequence of descendant vector potentials $\mathcal{F}_0^\alpha = (\mathcal{F}_0^{1,\alpha}, \ldots, \mathcal{F}_0^{N+1,\alpha})$ is of open-closed type, if the functions $\mathcal{F}_0^{1,\alpha}, \ldots, \mathcal{F}_0^{N,\alpha}$ do not depend on $t^{N+1}$ and the sequence of $N$-tuples of functions $(\mathcal{F}_0^{1,\alpha}, \ldots, \mathcal{F}_0^{N,\alpha})$ is a sequence of descendant potentials of closed type, with some matrix $\eta$ and closed descendant potential $F_0^c$. The sequence of descendant vector potentials $(\mathcal{F}_0^{1,\alpha}, \ldots, \mathcal{F}_0^{N,\alpha})$ is called the closed part of the vector potentials $\mathcal{F}_0^\alpha$. We will denote $\mathcal{F}_0^{\alpha,0} := \mathcal{F}_0^{N+1,\alpha}$ and $\mathcal{F}_0^0 := \mathcal{F}_0^0$.

By Remark 2.2, the whole collection of descendant vector potentials $\mathcal{F}_0^\alpha$ of open-closed type is uniquely determined by the pair $(F_0^c, F_0^0)$. The function $F_0^0$ is called the open descendant potential. Note that equation (2.4) written in terms of the functions $\mathcal{F}_0^\alpha$ and $\mathcal{F}_0^0$ looks as follows:

$$\frac{\partial^2 \mathcal{F}_0^c}{\partial t_{a+1} \partial t_b^a} = \frac{\partial^2 \mathcal{F}_0^c}{\partial t_{c+1} \partial t_b^c} + \frac{\partial \mathcal{F}_0^c}{\partial t_{a+1}} \frac{\partial^2 \mathcal{F}_0^c}{\partial t_b^a}, \quad 1 \leq \alpha, \beta \leq N + 1, \quad a, b \geq 0,$$

and one immediately recognizes here the open TRR-0 relations (1.6). The open descendant potential $F_0^0$ is called ancestor if the associated collection of descendant vector potentials is ancestor. From Lemma 2.1 it follows that the coefficient of $t_{d_1} \cdots t_{d_n}$ in an arbitrary open ancestor potential $F_0^0$ is zero if $\sum d_i \geq n - 1$.

Example 2.5. A basic example of a pair of closed and open ancestor potentials is the pair $(F_0^{\text{KW}}(t^1_*), F_0^{\text{PST}}(t_*^1, t_*^2))$, the metric is $\eta = 1$ and the unit is $\frac{\partial}{\partial t_*}$ [PST14] (see also [BB19]). The first few terms of the formal power series $F_0^{\text{PST}} = F_0^{\text{PST}}(t_*, s_*)$ are given by

$$F_0^{\text{PST}} = t_0 s_0 + t_1 s_1 + \frac{s_3^3}{6} + \frac{t_0 t_2^3 s_2}{2} + \frac{t_1 s_3^3}{3} + \frac{t_2^3 t_2 s_0}{2} + t_0 t_2^3 s_0 + t_2^3 t_1^3 s_1 + \frac{t_0^3 s_2}{6} + \ldots,$$

where the dots contain the monomials $\prod_{l=1}^l t_{d_1} \prod_{l=1}^l s_{m_j}$ with $l + k \geq 5$. An explicit formula for the coefficients of $F_0^{\text{PST}}$ was found in [PST14]:

$$\frac{\partial^{l+k} F_0^{\text{PST}}}{\partial t_{d_1} \cdots \partial t_{d_k} \partial s_{i_1} \partial s_{i_2}}|_{t_* = s_* = 0} = \frac{(2 \sum d_i - l + 1)!}{(l(2d_l - 1))!!}, \quad \text{if } 2l + k \geq 3, \quad 2 \sum d_i = 2l + k - 3, \quad \text{and } d_i \geq 1.$$

All the remaining coefficients of $F_0^{\text{PST}}$ can be found using the open string equation and the relation (1.5).

For an arbitrary pair of closed and open descendant potentials $(\mathcal{F}_0^c, F_0^0)$ the functions $F^c := F_0^c|_{t_* = 0}$ and $F^0 := F_0^0|_{t_* = 0}$ satisfy the closed and open WDVV equations and admit the unit $A^c \frac{\partial}{\partial t_*}$. Conversely, for arbitrary solutions $F^c, F^0 \in R^{T_{\text{res}}}_{N+1}$ to the closed and open WDVV equations admitting a unit, there exists a pair of closed and open descendant potentials $(\mathcal{F}_0^c, F_0^0)$ such that $F^c = F_0^c|_{t_* = 0}$ and $F^0 = F_0^0|_{t_* = 0}$. This is proved in [BB19, Lemma 3.2] under additional homogeneity assumptions for $F^c$ and $F^0$, because the authors there also require that $\mathcal{F}_0^c$ and $\mathcal{F}_0^0$ satisfy additional homogeneity conditions. However, the same approach works in the general, nonhomogeneous, case that we consider here.
2.5.2. Group actions. Consider the $\text{GL}(\mathbb{C}^{N+1})$-action on the space of descendant vector potentials. Clearly, the subgroup $\widetilde{\text{GL}}^o(\mathbb{C}^{N+1}) \subset \text{GL}(\mathbb{C}^{N+1})$ given by

$$\widetilde{\text{GL}}^o(\mathbb{C}^{N+1}) := \set{M \in \text{GL}(\mathbb{C}^{N+1}) | M_{N+1} = 0}$$

preserves the space of descendant vector potentials of open-closed type. For an $(N+1) \times (N+1)$ matrix $M$ we denote by $\pi(M)$ the $N \times N$ matrix obtained by deleting the last row and the last column in $M$. For any $M \in \widetilde{\text{GL}}^o(\mathbb{C}^{N+1})$ we have the following formula:

$$M. (\mathcal{F}_0, \mathcal{F}_0) = (\pi(M). \mathcal{F}_0, M. \mathcal{F}_0^o), \quad \text{where} \quad M. \mathcal{F}_0^o = \left( M_{N+1}^o \mathcal{F}_0 + M_{N+1} \eta^{\alpha \mu} \frac{\partial \mathcal{F}_0^o}{\partial \eta^\mu_0} \bigg|_{t^a \rightarrow (M^{-1})^a_{\beta}} \right).$$

For our future construction of open total descendant potentials, it is useful to consider in details the action of a certain subgroup of $\text{GL}(\mathbb{C}^{N+1})$. Define a subgroup $\text{GL}^o(\mathbb{C}^{N+1}) \subset \widetilde{\text{GL}}^o(\mathbb{C}^{N+1})$ by

$$\text{GL}^o(\mathbb{C}^{N+1}) := \set{M \in \text{GL}^o(\mathbb{C}^{N+1}) | M_{N+1} = 1}.$$

Embedding the groups $\text{GL}(\mathbb{C}^N)$ and $\mathbb{C}^N$ into $\text{GL}^o(\mathbb{C}^{N+1})$ by

$$\text{GL}(\mathbb{C}^N) \ni M \mapsto M^\text{ext} := \begin{pmatrix} M & 0 & \vdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 \end{pmatrix} \in \text{GL}^o(\mathbb{C}^{N+1}),$$

$$\mathbb{C}^N \ni \mathbf{v} = (v_1, \ldots, v_N) \mapsto B_{\mathbf{v}} := \begin{pmatrix} \text{Id} & 0 & \vdots & 0 \\ v_1 & \ldots & v_N & 1 \end{pmatrix} \in \text{GL}^o(\mathbb{C}^{N+1}),$$

we have the decomposition

$$\text{GL}^o(\mathbb{C}^{N+1}) = \mathbb{C}^N \rtimes \text{GL}(\mathbb{C}^N).$$

For $M \in \text{GL}(\mathbb{C}^N)$ we have the formula

$$\log \left( \exp(-\hat{m}) \exp(\varepsilon^{-2} \mathcal{F}_0 + \varepsilon^{-1} \mathcal{F}_0) \right) = \varepsilon^{-2} M. \mathcal{F}_0^o + \varepsilon^{-1} M^\text{ext}. \mathcal{F}_0^o + O(\varepsilon^0),$$

where $\exp(m) = M$ and we recall that $\hat{m}$ is defined by (2.15). For $\mathbf{v} \in \mathbb{C}^N$ we have the formula

$$\log \left( \exp(-\hat{\mathbf{v}}) \exp(\varepsilon^{-2} \mathcal{F}_0^o + \varepsilon^{-1} \mathcal{F}_0^o) \right) = \varepsilon^{-2} \mathcal{F}_0^o + \varepsilon^{-1} B_{\hat{\mathbf{v}}}. \mathcal{F}_0^o + O(\varepsilon^0),$$

where

$$\hat{\mathbf{v}} := \sum_{d \geq 0} v_{d} t^d \frac{\partial}{\partial t_{d+1}} - \varepsilon v_{d} \eta^{\alpha \beta} \frac{\partial}{\partial \eta^\beta_0}.$$

Let us now study transformations from the groups $G_{N+1,+}$ ($G_{N+1,-}$) that preserve the space of ancestor (descendant) vector potentials of open-closed type. Consider such a collection of ancestor (descendant) vector potentials. Looking at formula (2.6) (formula (2.7)), one can see that if $(r_1)_{N+1} = 0$ ($(s_1)_{N+1} = 0$) for $1 \leq \alpha \leq N$, then the functions $\mathcal{F}_0^{\alpha d}$ do not contribute to the deformation of the functions $\mathcal{F}_0^{\alpha d}$ with $1 \leq \alpha \leq N$. If we further require that $\pi(r(z)) \in \mathfrak{g}_{N,+}$ ($\pi(s(z)) \in \mathfrak{g}_{N,-}$), then our action restricts to the classical Givental action on the closed part of our descendant vector potentials.

We see that the subgroup $\tilde{G}_{N+1,+}^o \subset G_{N+1,+}$ defined by

$$\tilde{G}_{N+1,+}^o := \set{R(z) = \text{Id} + \sum_{t \geq 1} R_t z^t \in \text{End}(\mathbb{C}^{N+1})[[z]] | \pi(R(z)) \in \mathfrak{g}_{N,+}^o, (R_t)_{N+1} = 0}$$
preserves the space of ancestor vector potentials of open-closed type, and the subgroup $G_{N+1,-}^0 \subset G_{N+1,-}$ defined by

$$G_{N+1,-}^0 := \left\{ S(z) = \text{Id} + \sum_{i \geq 1} S_i z^{-i} \in \operatorname{End}(C^{N+1})[[z^{-1}]] \mid \pi(S(z)) \in G_{N,-}^0, (S_i)_{i \leq N+1} = 0 \right\}$$

preserves the space of descendant vector potentials of open-closed type.

Let us now discuss the possibility to express the action of the group $\tilde{G}_{N+1,+}^0$ on pairs of ancestor (descendant) potentials $(F_0^c, F_0^o)$ in a form similar to (2.16) and (2.17).

Let us first discuss the group $\tilde{G}_{N+1,+}^0$, whose Lie algebra we denote by $\tilde{g}_{N+1,+}^0$. Given $r(z) \in \tilde{g}_{N+1,+}^0$, its action on $F_0^c$ is given by equation (2.16), while the action of $r(z)$ on the function $F_0^o$ is given by

$$r(z).F_0^o = \sum_{i \geq 1} (-1)^i (r_i)_{\mu}^{N+1} \frac{\partial F_0^o}{\partial q_{i+1}^{\mu}} + \sum_{j,k \geq 0} (-1)^k (r_{j+k+1})^{\nu} \frac{\partial F_0^o}{\partial q_j^{\nu}} \frac{\partial F_0^o}{\partial q_k^{\nu}} - \sum_{i \geq 1} (r_i)_{\mu}^{\nu} \frac{\partial F_0^o}{\partial q_i^{\nu}} q_i^{\mu}.

We see that the functions $F_0^{o,i}$ with $i \geq 1$ appear in general on the right-hand side of this formula. However, note that if we require that $(r_i)_{N+1}^{j+1} = 0$, then the functions $F_0^{o,i}$ with $i \geq 1$ do not appear on the right-hand side of (2.20), and we get

$$r(z).F_0^o = \sum_{i \geq 1} (-1)^i (r_i)_{\mu}^{N+1} \frac{\partial F_0^o}{\partial q_{i+1}^{\mu}} + \sum_{j,k \geq 0} (-1)^k (r_{j+k+1})^{\nu} \frac{\partial F_0^o}{\partial q_j^{\nu}} \frac{\partial F_0^o}{\partial q_k^{\nu}} - \sum_{i \geq 1} (r_i)_{\mu}^{\nu} \frac{\partial F_0^o}{\partial q_i^{\nu}} q_i^{\mu},

where we use the following nonstandard (!) notation:

$$(r_i)^{\alpha \beta} := \begin{cases} \text{if } \beta \leq N, \\ \text{if } \beta = N+1 \text{ and } \alpha \leq N, \\ 0, \text{ if } \alpha = \beta = N+1. \end{cases}$$

We therefore introduce the following subgroup of $\tilde{G}_{N+1,+}^0$:

$$G_{N+1,+}^0 := \left\{ R(z) = \text{Id} + \sum_{i \geq 1} R_i z^{-i} \in \operatorname{End}(C^{N+1})[[z]] \mid \pi(R(z)) \in G_{N,+}^0, (R_i)_{i \leq N+1} = 0 \right\} \subset G_{N+1,+}^0,$$

and denote by $g_{N+1,+}^0$ the corresponding Lie algebra. Remarkably, if for any $r(z) \in g_{N+1,+}^0$ we introduce the differential operator

$$\tilde{r}(z)^{o} := - \sum_{i \geq 1} (r_i)_{\nu}^{\alpha} \frac{\partial}{\partial q_{i+1}^{\alpha}} + \varepsilon \sum_{i \geq 1} (-1)^i (r_i)_{\mu}^{N+1, \nu} \frac{\partial}{\partial q_i^{\nu}} + \varepsilon^2 \sum_{i,j \geq 0} (-1)^j (r_{i+j+1})^{\alpha \beta} \frac{\partial^2}{\partial q_i^{\alpha} \partial q_j^{\beta}},$$

then formulas (2.16) and (2.21) are combined in the following way:

$$\frac{\tilde{r}(z)^{o}}{\exp(\varepsilon^{-2} F_0^c + \varepsilon^{-1} F_0^o)} = \exp(-2 \pi(r(z)).F_0^c + \varepsilon^{-1} r(z).F_0^o + O(\varepsilon^0)).$$

Consider now the group $G_{N+1,-}^0$, whose Lie algebra we denote by $g_{N+1,-}^0$. Given $s(z) \in g_{N+1,-}^0$, its action on $F_0^c$ is given by equation (2.17), while the action of $s(z)$ on the function $F_0^o$ is given by

$$s(z).F_0^o = - \sum_{i \geq 1} (s_i)_{\beta}^{\alpha} \frac{\partial F_0^o}{\partial q_{i+1}^{\alpha}} - \sum_{i \geq 1} (s_i)_{\mu}^{N+1} q_{i-1}^{\mu}. \varepsilon^{-2} \sum_{i,j \geq 0} (-1)^j (s_{i+j+1})^{\alpha \beta} q_i^{\alpha} q_j^{\beta},$$

Again, if for any $s(z) \in g_{N+1,-}^0$ we introduce the differential operator

$$\tilde{s}(z)^{o} := - \sum_{i \geq 1} (s_i)_{\beta}^{\alpha} \frac{\partial}{\partial q_{i+1}^{\alpha}} - \varepsilon^{-1} \sum_{i \geq 1} (s_i)_{\mu}^{N+1, \beta} q_{i-1}^{\mu} + \varepsilon^2 \sum_{i,j \geq 0} (-1)^j (s_{i+j+1})^{\alpha \beta} q_i^{\alpha} q_j^{\beta}.$$
then formulas (2.17) and (2.23) are combined as follows:
\[
\frac{s(z)^{\circ} \exp(\varepsilon^{-2}F_0^c + \varepsilon^{-1}F_0^o)}{\exp(\varepsilon^{-2}F_0^c + \varepsilon^{-1}F_0^o)} = \varepsilon^{-2}\pi(s(z))\cdot F_0^c + \varepsilon^{-1}s(z)\cdot F_0^o + O(\varepsilon^0).
\]

In the following proposition, we summarize what we have just obtained.

**Proposition 2.6.** The group \(G_{N+1,+}^N\) preserves the space of ancestor (descendant) vector potentials of open-closed type. Moreover, the corresponding action on the space of pairs of closed and open ancestor (descendant) potentials \((F_0^c, F_0^o)\) is given by equations (2.22) and (2.24).

Let us present the following useful technical result about the structure of the operators \(\exp(s(z)^{\circ})\) for \(s(z) \in g_{N+1,-}\).

**Lemma 2.7.** For any \(s(z) \in g_{N+1,-}\) we have
\[
\exp\left(s(z)^{\circ}\right) = \exp\left(\varepsilon^{-1}\sum_{i \geq 1} ((S^{-1})_i)_a^{N+1} q_i^{\alpha} \frac{\varepsilon^{-2}}{2} \sum_{a,b \geq 0} q_a^\beta q_b^\gamma \text{Coef}_{z_1^{-a}z_2^{-b}} \left(S(z_1)^{\mu}_{\alpha} \eta_{\mu\nu} S(z_2)^{\nu}_{\beta} - \eta_{\alpha\beta}\right) \right) \times \exp\left(- \sum_{i \geq 1, j \geq 0} (s_i)^{\alpha}_b q_{i+j}^\beta \frac{\partial}{\partial q_j^\alpha}\right),
\]
where \(S(z) := \exp(s(z))\).

**Proof.** We apply part (a) of Lemma A.1 with
\[
X := - \sum_{i \geq 1, j \geq 0} (s_i)^{\alpha}_b q_{i+j}^\beta \frac{\partial}{\partial q_j^\alpha}
\]
and \(Y := Y_1 + Y_2\) where
\[
Y_1 := -\varepsilon^{-1} \sum_{i \geq 1} (s_i)^{N+1}_\mu q_i^{\mu}, \quad Y_2 := \frac{\varepsilon^{-2}}{2} \sum_{i,j \geq 0} (-1)^{j+1} (s_{i+j+1})^{\alpha}_\eta q_i^\alpha q_j^\beta.
\]

Then \(s(z)^{\circ} = X + Y\) and moreover it is well known that
\[
\sum_{n \geq 0} \frac{1}{(n+1)!} \text{ad}_X^n Y_2 = \frac{\varepsilon^{-2}}{2} \sum_{a,b \geq 0} q_a^\alpha q_b^\gamma \text{Coef}_{z_1^{-a}z_2^{-b}} \left(S(z_1)^{\mu}_{\alpha} \eta_{\mu\nu} S(z_2)^{\nu}_{\beta} - \eta_{\alpha\beta}\right).
\]
The fact that
\[
\sum_{n \geq 0} \frac{1}{(n+1)!} \text{ad}_X^n Y_1 = \varepsilon^{-1} \sum_{i \geq 1} ((S^{-1})_i)^{N+1}_a q_i^{\alpha}
\]
is proved by a simple direct computation. □
2.5.3. A transitivity statement for the $G^o_{N+1,+}$-action.

**Proposition 2.8.** The group $G^o_{N+1,+}$ acts transitively on the space of pairs of closed and open ancestor potentials $(\mathcal{F}_0^o, \mathcal{F}_0^c)$ defining a fixed semisimple algebra structure on $T_0 \mathbb{C}^{N+1}$.

**Proof.** We consider an arbitrary pair of ancestor potentials $(\mathcal{F}_0^o, \mathcal{F}_0^c)$ defining a semisimple algebra structure on $T_0 \mathbb{C}^{N+1}$. Let $F^c$ and $F^o$ be the associated solutions to the closed and open WDVV equations. Denote by $\overline{F} = (F^1, \ldots, F^{N+1})$ the vector potential of the associated flat $F$-manifold. Note that the semisimplicity condition at $0 \in \mathbb{C}^{N+1}$ implies that $\left. \frac{\partial^2 F^o}{\partial t^1 \partial t^j} \right|_{t^1 = 0} \neq 0$.

Let $u^1(t^1, \ldots, t^N), \ldots, u^N(t^1, \ldots, t^N)$ be canonical coordinates around $0 \in \mathbb{C}^N$ for the Dubrovin–Frobenius manifold given by the potential $F^c$. Making an appropriate shift we can assume that $u^i(0) = 0$.

**Lemma 2.9.** The functions $u^1, \ldots, u^N$ together with the function $u^{N+1}(t^1, \ldots, t^{N+1}) := \frac{\partial F^o}{\partial t^1}$ are canonical coordinates around $0 \in \mathbb{C}^{N+1}$ for the flat $F$-manifold given by the pair $(F^c, F^o)$.

**Proof.** Canonical coordinates of our flat $F$-manifold can be found as $N + 1$ distinct solutions of the system of PDEs

$$\frac{\partial u}{\partial t^\alpha} \frac{\partial u}{\partial t^\beta} = c^\gamma_{\alpha \beta} \frac{\partial u}{\partial t^\gamma}, \quad 1 \leq \alpha, \beta \leq N + 1,$$

satisfying $u(0) = 0$, where $c^\alpha_{\beta \gamma} := \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$. The functions $u^1, \ldots, u^N$ obviously satisfy this system, so it remains to check that

$$\frac{\partial^2 F^o}{\partial t^1 \partial t^i} \frac{\partial^2 F^o}{\partial t^j \partial t^k} = c^\gamma_{\alpha \beta} \frac{\partial^2 F^o}{\partial t^\gamma \partial t^\alpha \partial t^\beta},$$

which immediately follows from (1.4). \hfill $\Box$

Let us choose the canonical coordinates given by this lemma and follow the construction from Section 2.3.5. Clearly, we have $\Psi^{\leq N}_{N+1} = \Psi^{\leq N}_{N+1} = \Gamma^{\leq N}_{N+1} = 0$. Using this, the system of equations (2.10) immediately implies by induction that $(R_{m})^{\leq N}_{N+1} = 0$. Note that the recursive relations for the entries $(R_{m+1})^i_j$ with $1 \leq i, j \leq N$ involve only the functions $\gamma^i_l$ and $(R_{m})^i_k$ with $1 \leq k, l \leq N$. So we choose the functions $H_1, \ldots, H_N$ such that $\eta = \frac{1}{2}H_{0 \alpha \beta} dt^\alpha dt^\beta = \sum_{i=1}^{N} H_i^2 dt^i$, and then choose the integration constants for $(R_{m})^i_k$ with $1 \leq i \leq N$ is such a way that $\pi(R(R(z, t)^i) R' R(z, t)^i) = \text{Id}$. Note that the system of equations (2.11) gives $d(R_{m})^{N+1}_{N+1} = 0$, and thus we can choose $(R_{m})^{N+1}_{N+1} = 0$ for $m \geq 1$. It is now obvious that we have $\mathcal{R}(z) = \Psi^{-1}(0)R^{-1}(-z, 0)\Psi(0) \in G^o_{N+1,+}$. If we denote by $F^c_{\text{cub}}$ the cubic part of $F^c$ and by $F^o_{\text{quad}}$ the quadratic part of $F^o$, then the element $\mathcal{R}(z) \in G^o_{N+1,+}$ transforms the pair of ancestor potentials corresponding to $(F^c_{\text{cub}}, F^o_{\text{quad}})$ to the pair $(F^c_0, F^o_0)$, which proves the proposition. \hfill $\Box$

2.5.4. Commutation relations for the operators $\widehat{r}(z)$ and $\widehat{s}(z)$. For $r(z), \overline{r}(z) \in \mathfrak{g}^{o}_{N+1,+}$ let

$$G_p(r(z), \overline{r}(z)) := \sum_{i,j \geq 1 \atop i+j=p} \left( (-1)^{i+1} - (-1)^{j+1} \right) (r_1)^{N+1}_{i} \eta^{\mu \nu}(\overline{r}_j)^{N+1}_{\nu}, \quad p \geq 2.$$  

Clearly, $G_p(r(z), \overline{r}(z)) = 0$ if $p$ is even.

**Lemma 2.10.** We have the following commutation relations:

1. $\left[ \widehat{r}(z), \widehat{r}(z) \right] = [r(z), \overline{r}(z)] + \sum_{p \geq 3} G_p(r(z), \overline{r}(z)) \left( \varepsilon \frac{\partial}{\partial q^p} + \frac{\varepsilon^2}{2} \sum_{i,j \geq 0 \atop i+j=p-1} (-1)^{i+1} \frac{\partial^2}{\partial q_i \partial q_j} \right),$

for $r(z), \overline{r}(z) \in \mathfrak{g}^{o}_{N+1,+};$

2. $\left[ \widehat{s}(z), \widehat{s}(z) \right] = [s(z), \overline{s}(z)]$, for $s(z), \overline{s}(z) \in \mathfrak{g}^{o}_{N+1,-}$. 

Proof. 1. One can immediately see that
\[
\begin{align*}
[\tilde{r}^\alpha, \tilde{r}^\beta] &= [\pi(r), \pi(\tilde{r})]^\varepsilon \\
&+ \sum_{i,j \geq 1} \left( -\frac{[r_i, \tilde{r}_{j}]_{\mu}^{N+1} \theta_k \partial_{q_i+j+k} + \varepsilon (-1)^{i+j} [r_i, \tilde{r}_{j}]_{N+1}^{\nu} \partial_{q_i+j+k} \right) + \varepsilon \sum_{p \geq 3} G_p(r, \tilde{r}) \partial_{q_i+j+k} \\
&= \sum_{i,j \geq 0, k \geq 1} \left( -\frac{[r_i, \tilde{r}_{j}]_{\mu}^{N+1} \eta^{\alpha \beta} \tilde{r}_{j}^{\gamma} + \varepsilon (-1)^{i+j} [r_i, \tilde{r}_{j}]_{N+1}^{\nu} \eta^{\alpha \beta} \tilde{r}_{j}^{\gamma} \right) \\
(2.25) + \varepsilon^2 &\sum_{i,j \geq 1, k \geq 1} \left( -\frac{[r_i, \tilde{r}_{j}]_{\mu}^{N+1} \eta^{\alpha \beta} \tilde{r}_{j}^{\gamma} + \varepsilon (-1)^{i+j} [r_i, \tilde{r}_{j}]_{N+1}^{\nu} \eta^{\alpha \beta} \tilde{r}_{j}^{\gamma} \right) \\
(2.26) + \varepsilon^2 &\sum_{i,j \geq 1, k \geq 1} \left( -\frac{[r_i, \tilde{r}_{j}]_{\mu}^{N+1} \eta^{\alpha \beta} \tilde{r}_{j}^{\gamma} + \varepsilon (-1)^{i+j} [r_i, \tilde{r}_{j}]_{N+1}^{\nu} \eta^{\alpha \beta} \tilde{r}_{j}^{\gamma} \right)
\end{align*}
\]

We see that
\[
\begin{align*}
&\sum_{i,j \geq 0, k \geq 1} A^7_{ijk} \frac{\partial^2}{\partial q_i^{N+1} \partial q_j^{N+1}} + \sum_{i,j \geq 1, k \geq 0} C^7_{ijk} \frac{\partial^2}{\partial q_i^{N+1} \partial q_j^{N+1}} = - \sum_{a,b \geq 1, j+k \geq 0} \frac{(-1)^k (r_a)_{\mu}^{N+1} (r_b)_{\nu}^{N+1} \partial^{\alpha \beta} \eta^{\gamma \nu}}{\partial q_j^{N+1} \partial q_k^{N+1}} \\
&\sum_{i,j \geq 0, k \geq 1} B^7_{ijk} \frac{\partial^2}{\partial q_i^{N+1} \partial q_j^{N+1}} + \sum_{i,j \geq 1, k \geq 0} D^7_{ijk} \frac{\partial^2}{\partial q_i^{N+1} \partial q_j^{N+1}} = \sum_{a,b \geq 1, j+k \geq 0} \frac{(-1)^{k} (r_a)_{\mu}^{N+1} (r_b)_{\nu}^{N+1} \partial^{\alpha \beta} \eta^{\gamma \nu}}{\partial q_j^{N+1} \partial q_k^{N+1}},
\end{align*}
\]

and as a result the sum of the terms in lines (2.25) and (2.26) is equal to
\[
\varepsilon^2 \sum_{a,b \geq 1, j+k \geq 0} \frac{(-1)^k [r_a, \tilde{r}_b]_{\mu}^{N+1} \eta^{\alpha \beta} \partial^{\gamma \nu}}{\partial q_j^{N+1} \partial q_k^{N+1}}.
\]

Analogously, we obtain
\[
\sum_{i \geq 1, j \geq 0} E_{ijk} \frac{\partial^2}{\partial q_i^{N+1} \partial q_k^{N+1}} = \sum_{a,b \geq 1, i,j \geq 0} \frac{(-1)^{a+i} (r_a)_{\mu}^{N+1} (r_b)_{\nu}^{N+1} \partial^{\alpha \beta} \eta^{\gamma \nu}}{\partial q_i^{N+1} \partial q_j^{N+1}} =
\]
\[
= \frac{1}{2} \sum_{i \geq 0} \frac{(-1)^i \sum_{j \geq 1} G_{i+j+1} (r, \tilde{r}) \frac{\partial^2}{\partial q_i^{N+1} \partial q_j^{N+1}}},
\]

which completes the proof of Part 1.

2. The proof is easy and we omit it.

2.5.5. Some explicit formulas in canonical coordinates. Here, in addition to Lemma 2.9, we would like to present explicit formulas for the functions \((R_m)^{N+1}_i\) and \(H_{N+1}\) appearing at the end of the proof of Proposition 2.8.

Lemma 2.11. Using the notations from the last paragraph of the proof of Proposition 2.8, we have the following statements.

1. Up to the multiplication by a nonzero constant, we have \(H_{N+1} = \left( \frac{\partial^2 F^\alpha}{\partial q^2 + \alpha^2} \right)^{-1} \).

2. \((R_m)^{N+1}_j = (-1)^{m-1} \frac{\theta_m^{N+1}}{(q_{N+1})^2} \) for \(m \geq 1\) and \(1 \leq j \leq N\).
Proof. The function $H_{N+1}$ is determined by the equation $H_{N+1}^{-1} \cdot dH_{N+1} = -(\tilde{\Psi} \cdot \tilde{\Psi}^{-1})_{N+1}$. Since the matrix $\tilde{\Psi}$ has the form

$$
\tilde{\Psi} = \begin{pmatrix}
\ast & 0 & \cdots & 0 \\
\frac{\partial F^1}{\partial t^1} & \cdots & \frac{\partial F^1}{\partial t^N} & 0 \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 0 & \frac{\partial F^N}{\partial (t^{N+1})^2}
\end{pmatrix},
$$

Part 1 of the lemma is proved.

For Part 2, first note that equation (2.10) immediately gives that $(R_i)_j^i = \gamma_j^i$ for $i \neq j$. Since we chose $(R_m)_{N+1}^j = 0$ for $m \geq 1$, equation (2.10) gives $(R_{m+1})_{N+1}^j = -\frac{\partial (R_m)}{\partial u^{N+1}}$ for $m \geq 1$, which completes the proof.

### 3. Open total descendant potentials

In this section, after recalling the Givental construction of closed total descendant potentials, we present our construction of open total descendant potentials, discuss group actions on the space of these potentials, and prove Theorem 1.

We fix $N \geq 1$.

#### 3.1. The Givental construction of closed total descendant potentials

Let $\overline{A} \in \mathbb{C}^N$ be a nonzero vector and $\eta = (\eta_{\alpha \beta})$ be an $N \times N$ constant symmetric nondegenerate matrix. By Givental, the space of closed total ancestor potentials of rank $N$,

$$
\mathcal{F}^{c,\text{anc}}(t^1_s, \ldots, t^N_s, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}^{c,\text{anc}}(t^1_s, \ldots, t^N_s), \quad \mathcal{F}^{g,\text{anc}} \in \mathbb{C}[t^1_s],
$$

is defined by

$$
\exp(\mathcal{F}^{c,\text{anc}}) := \exp \left( \left( r(z)^c \right) \psi \prod_{i=1}^N t^{KW}(a_i t^i_s, a_i \varepsilon) \right),
$$

where the parameters $r(z) \in \mathfrak{g}_{N,+}^*, a_1, \ldots, a_N \in \mathbb{C}^*$, and $\psi \in \text{End}(\mathbb{C}^N)$ satisfy the conditions

$$
\exp(\psi)^i_a A^a = a_i^{-1}, \quad \exp(\psi)^i_a \eta^{\alpha \beta} \exp(\psi)^j_\beta = \delta^{ij}.
$$

Then the space of closed total descendant potentials $\mathcal{F}^{c,\text{desc}}(t^*_s, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}^{c,\text{desc}}(t^*_s)$ of rank $N$ is given by

$$
\exp(\mathcal{F}^{c,\text{desc}}) := \exp \left( \left( s(z)^c \right) \mathcal{F}^{c,\text{anc}} \right),
$$

where $s(z) \in \mathfrak{g}_{N,-}^*$ and $\mathcal{F}^{c,\text{anc}}$ is an arbitrary closed total ancestor potential. Note that in general the resulting potentials $\mathcal{F}^{c,\text{desc}}$ are formal power series in shifted variables: $\mathcal{F}^{c,\text{desc}} \in \mathcal{R}_N^{-s_i \overline{A}} \mathbb{C}[t^1_s]$. Let us make some remarks about this construction. First of all, note that since the maps $r(z) \mapsto r(z)^c$ and $s(z) \mapsto s(z)^c$ are Lie algebra homomorphisms from the Lie algebras $\mathfrak{g}_{N,+}$ and $\mathfrak{g}_{N,-}$ respectively, to the Lie algebra of differential operators, the Givental construction gives a $G_{N,+}^\mathbb{C}$-action on the space of closed total ancestor potentials and a $G_{N,-}^\mathbb{C}$-action on the space of closed total descendant potentials.

Then note that the function $\sum_{i=1}^N a_i^{-2} \mathcal{F}^{KW}_0(a_i t^i_s)$ is the closed ancestor potential corresponding to the solution $\sum_{i=1}^N a_i(t^i_s)^3$ of the closed WDVV equations, with the metric $(\delta_{ij})$ and the unit $\sum_{i=1}^N a_i^{-1} \frac{\partial}{\partial t^i_s}$. The relations (3.1) imply that the function $\exp \left( \left( s(z)^c \right) \sum_{i=1}^N a_i^{-2} \mathcal{F}^{KW}_0(a_i t^i_s) \right)$
is a closed ancestor potential with the metric $\eta$ and the unit $A^\alpha \partial_\mu$. Formula (2.16) (respectively, (2.17)) shows that the $G_{N,+}^c$-action (respectively, $G_{N,-}^c$-action) on the space of closed total ancestor (respectively, descendant) potentials agrees with the $G_{N,+}^c$-action (respectively, $G_{N,-}^c$-action) on the space of closed ancestor (respectively, descendant) potentials $F_0^c$ discussed in Section 2.4.2.

Remembering also Section 2.3.4, we see that the functions $F_0^{\text{desc}}$ are closed descendant potentials defining a semisimple algebra structure on $\mathcal{T}_{\text{orig}} \mathbb{C}^N$. Conversely, for any closed descendant potential $F_0^c$ defining a semisimple algebra structure on $\mathcal{T}_{\text{orig}} \mathbb{C}^N$ there exists a closed total descendant potential $F^{\text{desc}}_c$ such that $F^{\text{desc}}_c = F_0^c$.

The following equations are the closed analogs of equations (1.7), (1.8), and (1.9), respectively (the closed analog of equation (1.6) is equation (2.14)):

$$\frac{\partial F^{\text{desc}}_0}{\partial t_{\alpha+1}} = \frac{\partial^2 F^{\text{desc}}_0}{\partial t_\alpha \partial t_0} \eta^{\mu\nu} \frac{\partial F^{\text{desc}}_0}{\partial t_0} + \frac{1}{24} \eta^{\mu\nu} \frac{\partial^3 F^{\text{desc}}_0}{\partial t_\alpha \partial t_0^3 \partial t_0}, \quad 1 \leq \alpha \leq N, \quad a \geq 0,$$

(3.2)

$$A^a \frac{\partial F^{\text{desc}}_c}{\partial t_0^a} = \sum_{d \geq 0} t_{d+1}^a \frac{\partial F^{\text{desc}}_c}{\partial t_d} + \frac{\varepsilon-2}{2} \eta_{\alpha\beta} t_0^\alpha t_0^\beta + \frac{1}{2} \text{tr}(r_1),$$

(3.3)

### 3.2. A construction of open total descendant potentials.

Recall [PST14, Bur15] that the coefficient of $\prod_{i=1}^l t_i \prod_{j=1}^k s_{m_j}$ in the formal power series $F^\text{PST}_g$ is zero unless $2 \sum_{i=1}^l d_i + 2 \sum_{j=1}^k m_j = 3g - 3 + k + 2l$.

This implies that $$F^\text{PST}_g \in \mathbb{C}[s_0][[t_*, s_{\geq 1}]].$$

Therefore, for any $\theta \in \mathbb{C}$ the transformation

$$F^\text{PST}_g \mapsto F^\text{PST}_g(\theta) = F^\text{PST}_g|_{s_i \mapsto s_i + \delta_i, \alpha \theta + \sum_{j \geq 1} (-1)^j \left( \frac{\theta^{2j} s_i + j + 1 - (2j - 1)^2 s_i - 1}{2j - 1} \right) + \delta^2 \frac{\theta}{2j - 1} \delta_1 + \delta^2 \frac{\theta}{2j - 1} \delta_2 + \cdots + \delta^2 \frac{\theta}{2j - 1} \delta_1 \cdot \delta_1} - \frac{\varepsilon-1}{6} \delta^3 \left( \frac{\theta^2}{2j - 1} \delta_1 + \frac{\theta^2}{2j - 1} \delta_2 + \cdots + \frac{\theta^2}{2j - 1} \delta_1 \cdot \delta_1 \right)$$

(3.4)

is well defined, and $F^\text{PST}_g(\theta)$ has the form $F^\text{PST}_g(\theta)_g$ where $F^\text{PST}_g(\theta)_g \in \mathbb{C}[[t_*, s_*]]$. We define $\tau^\text{PST}(\theta) := \exp(F^\text{PST}_g(\theta))$.

We fix an open $\mathbb{C}$-algebra: $A = (A^1, \ldots, A^{N+1}) \in \mathbb{C}^{N+1}$ such that $\pi(A) \neq 0 \in \mathbb{C}^N$. Consider the following data: $r(z) \in g_{N+1}^\text{anc}$, $a_1, \ldots, a_N, \theta \in \mathbb{C}^*$, $b = (b_1, \ldots, b_N) \in \mathbb{C}^N$, and $\psi \in \text{End}(\mathbb{C}^N)$ satisfying

$$\exp(\psi)_a A^a = a_i^{-1}, \quad \exp(\psi)_{\alpha \beta} A^\alpha = \delta_{ij}, \quad b_0 A^a + A^{N+1} = 0.$$}

We then have the closed total ancestor potential $F^{\text{anc}, a}$ corresponding to the parameters $\pi(r(z)) \in g_{N+1}^\text{anc}$, $a_1, \ldots, a_N \in \mathbb{C}^*$, and $\psi \in \text{End}(\mathbb{C}^N)$. We define an open total ancestor potential

$$F_0^{\text{anc}}(t_1^*, \ldots, t_N^*, \varepsilon) = \sum_{g \geq 0} \varepsilon^{g-1} F_0^{\text{anc}}(t_1^*, \ldots, t_N^*), \quad F_0^{\text{anc}} \in \mathbb{C}[[t_*]],$$

by

$$\exp(F_0^{\text{anc}} + F^{\text{anc}}) := \exp(r(z)) \exp(b) \exp(\psi) \left( \tau^{\text{PST}}_{t_*^1} (a_i t_*^1, a_i \varepsilon) \prod_{i=1}^N \tau^{\text{KW}}(a_i t_*^i, \varepsilon) \right).$$
Note that there is no rescaling of $\varepsilon$ in $\tau_{(\theta)}^{\text{PST}}$ in this formula. If in addition we have an element $s(z) \in \mathfrak{g}_{N+1,-}^0$, then there is the corresponding closed total descendant potential $\mathcal{F}_c^{\text{desc}}$ given by

$$\exp(\mathcal{F}_c^{\text{desc}}) = \exp\left(\pi(s(z))\right)\exp(\mathcal{F}_c^{\text{anc}}),$$

and we define an open total descendant potential $\mathcal{F}_o^{\text{desc}} = \sum_{g \geq 0} \varepsilon^{g-1} \mathcal{F}_g^{\text{desc}}$ by

$$\exp(\mathcal{F}_o^{\text{desc}} + \mathcal{F}_c^{\text{desc}}) = \exp\left(s(z)\right)\exp(\mathcal{F}_o^{\text{anc}} + \mathcal{F}_c^{\text{anc}}).$$

Note that $\mathcal{F}_g^{\text{desc}} \in \mathcal{R}_{N+1}^{-1}[[t^\pm_1]].$

### 3.3. First properties

We see that a pair of total descendant potentials $(\mathcal{F}_c^{\text{desc}}, \mathcal{F}_o^{\text{desc}})$, with metric $\eta$ and unit $\overline{A} \in \mathbb{C}^{N+1}$ such that $\pi(\overline{A}) \neq 0$, is associated to the data

$$a_1, \ldots, a_N, \theta \in \mathbb{C}^*, \quad \psi \in \text{End}(\mathbb{C}^N), \quad \overline{b} \in \mathbb{C}^N, \quad r(z) \in \mathfrak{g}_{N+1,+}^0, \quad s(z) \in \mathfrak{g}_{N+1,-}^0,$$

satisfying the conditions

$$\exp(\psi)_a^{\alpha} A^a = a_i^{-1}, \quad \exp(\psi)_a^{\alpha} \eta^{\alpha\beta} \exp(\psi)_b^{\beta} = \delta^{ij}, \quad b_{\alpha} A^a + A^{N+1} = 0.$$

Denote by $\text{Desc}_N$ the set of all pairs of total descendant potentials $(\mathcal{F}_c^{\text{desc}}, \mathcal{F}_o^{\text{desc}})$. Choosing $s(z) = 0$, we obtain the set of total ancestor potentials, which we denote by $\text{Anc}_N \subset \text{Desc}_N$. Further choosing $r(z) = 0$, we obtain a smaller subset of $\text{Anc}_N$, which we denote by $\text{Anc}_N^0$. Choosing also $\overline{b} = 0$, we get a subset of $\text{Anc}_N^0$ denoted by $\text{Anc}_N^0$.

Let us first discuss the initial pair of total ancestor potentials

$$\left(\sum_{i=1}^N \mathcal{F}_c^{\text{KW}}(a_i t_i^1, a_i \varepsilon), \mathcal{F}_o^{\text{PST}}(a_i t_i^1, t_i^{N+1}, \varepsilon)\right) \in \text{Anc}_N^0.$$  

Note that the functions

$$\mathcal{F}_c^c = \sum_{i=1}^N a_i^{-2} \mathcal{F}_c^{\text{KW}}(a_i t_i^1), \quad \mathcal{F}_o^o = \mathcal{F}_o^{\text{PST}}(a_i t_i^1, t_i^{N+1})$$

are the closed and open ancestor potentials corresponding to the solutions

$$F_c = \sum_{i=1}^N a_i \frac{(t_i^1)^3}{6}, \quad F_o = a_i t_i^1 t_i^{N+1} + \frac{(t_i^{N+1})^3}{6}$$

to the closed and open WDVV equations, with the metric $(\delta_{ij})$ and the unit $\sum_{i=1}^N a_i^{-1} \frac{\partial}{\partial t_i}$.

The corresponding algebra structure on $T_0 \mathbb{C}^{N+1}$ is not semisimple. However, note that for any $\theta \neq 0$ the algebra structure on the tangent space at $(0, \ldots, 0, \theta) \in \mathbb{C}^{N+1}$ is semisimple. Consider the matrices $\Omega_k$ defined in Section 2.3.3. It is easy to compute that

$$\Omega_k(0, \ldots, 0, \theta) = \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 \\
\frac{\theta_{2k+1}}{(2k+1)!} & \ldots & \frac{\theta_{2k+2}}{2^{k+1}(k+1)!}
\end{pmatrix}.$$  

Using Lemma 2.7, we see that

$$\tau_{(\theta)}^{\text{PST}}(a_i t_i^1, t_i^{N+1}, \varepsilon) \prod_{i=1}^N \tau^{\text{KW}}(a_i t_i^1, a_i \varepsilon) = \exp\left(s(z)\right) \left(\tau_{(\theta)}^{\text{PST}}(a_i t_i^1, t_i^{N+1}, \varepsilon) \prod_{i=1}^N \tau^{\text{KW}}(a_i t_i^1, a_i \varepsilon)\right).$$

### 3.4. Second properties

We see that a pair of total descendant potentials $(\mathcal{F}_c^{\text{desc}}, \mathcal{F}_o^{\text{desc}})$, with metric $\eta$ and unit $\overline{A} \in \mathbb{C}^{N+1}$ such that $\pi(\overline{A}) \neq 0$, is associated to the data

$$a_1, \ldots, a_N, \theta \in \mathbb{C}^*, \quad \psi \in \text{End}(\mathbb{C}^N), \quad \overline{b} \in \mathbb{C}^N, \quad r(z) \in \mathfrak{g}_{N+1,+}^0, \quad s(z) \in \mathfrak{g}_{N+1,-}.$$
where

\[(3.9) \quad s(z) := \log S(z), \quad S(z) := \left(\text{Id} + \sum_{j \geq 1} (-1)^j \Omega_{j-1} z^{-j}\right)^{-1}.
\]

Formulas (2.24) and (2.8) then imply that $F^o_{(\theta),0}$ is the open ancestor potential corresponding to the solutions

\[
F^c = \sum_{i=1}^N a_i (t^i)^3 6, \quad F^o_{(\theta)} = F^o_{|t^{N+1}+\theta} = a_1 t^1 (t^{N+1} + \theta) + \frac{(t^{N+1} + \theta)^3}{6}
\]
to the closed and open WDVV equations. The corresponding algebra structure on $T_0\mathbb{C}^{N+1}$ is semisimple.

Let us now discuss the spaces Anc$_N^0$ and Anc$_N^1$. For $H = \hat{\psi}, \psi \in \text{End}(\mathbb{C}^N)$, the transformation

\[(\exp(F^c_{\text{anc}}), \exp(F^o_{\text{anc}})) \mapsto (\exp(-H) \exp(F^c_{\text{anc}}), \exp(-H) \exp(F^c + F^o_{\text{anc}}))
\]
defines a $GL(\mathbb{C}^N)$-action on Anc$_N^0$, while for $H = \hat{\psi}, \psi \in \mathbb{C}^N$, this formula defines a $\mathbb{C}^N$-action on Anc$_N^1$ (the last claim is justified by noting that $[\hat{v}, \hat{w}] = 0$ for any $v, w \in \mathbb{C}^N$). At the level of functions $F^c_{\text{anc}}$ and $F^o_{\text{anc}}$, the formulas for these actions coincide with the formulas (2.18) and (2.19) for the actions of these groups on the space of pairs $(F^c_{\theta}, F^o_{\theta})$ of closed and open ancestor potentials. We conclude that for any pair $(F^c_{\text{anc}}, F^o_{\text{anc}}) \in \text{Anc}_N^1$ the function $F^o_{\text{anc}}$ is an open ancestor potential, and by (3.7) the corresponding metric is $\eta$ and the unit is $A^o \frac{\partial}{\partial \theta}$.

Let us now discuss the space of all pairs of total ancestor potentials Anc$_N$.

**Proposition 3.1.** For any pair $(F^c_{\text{anc}}, F^o_{\text{anc}}) \in \text{Anc}_N$ we have

\[(3.10) \quad \frac{\partial}{\partial t^k_{N+1}} \exp(F^o_{\text{anc}}) = \frac{\varepsilon^k}{(k+1)!} \frac{\partial^{k+1}}{(\partial t^0_{N+1})^{k+1}} \exp(F^o_{\text{anc}}), \quad k \geq 0.
\]

**Proof.** Denote

\[O_k := \frac{\partial}{\partial t^k_{N+1}} - \frac{\varepsilon^k}{(k+1)!} \frac{\partial^{k+1}}{(\partial t^0_{N+1})^{k+1}}, \quad k \geq 0.
\]

From formula (3.6) and the fact that the operator $O_k$ obviously commutes with the operators $\hat{r}(z)^o, \hat{\psi}$, and $\hat{b}$, it follows that it is sufficient to prove equation (3.10) only for $F^o_{\text{anc}} = F_{\text{PST}}(a_1 t^1_s, t^N_{s+1}, \varepsilon)$.

**Lemma 3.2.** Consider an element $s(z) \in g^N_{N+1,-}$ such that $(s_{\geq 2})_{N+1} = 0$. Then we have

\[O_k \circ \exp(s(z)^o) = \exp(s(z)^o) \circ \sum_{j=0}^k \frac{(-s_1)_{N+1}^j}{j!} O_{k-j}, \quad k \geq 0.
\]

**Proof.** For any operators $X, Y$ and a formal variable $t$ we have the identity

\[(3.11) \quad \exp(tX) \circ Y \circ \exp(-tX) = Y + \sum_{n \geq 1} \frac{t^n}{n!} \text{ad}^n_X Y.
\]

Substituting $X = s(z)^o, Y = O_k$, and $t = -1$ in (3.11) and noting that $[s(z)^o, O_k] = (s_1)_{N+1}^o O_{k-1}$ for $k \geq 0$ (we adopt the convention $O_{-1} := 0$), we obtain the desired result.

Equation (3.10) for $F^o_{\text{anc}} = F_{\text{PST}}(a_1 t^1_s, t^N_{s+1}, \varepsilon)$ follows from this lemma, the property (see Remark 1.1)

\[O_j \exp(F_{\text{PST}}(a_1 t^1_s, t^N_{s+1}, \varepsilon)) = 0,
\]
and equation (3.8), where one should note that the property \((s\geq 2)\sum_{i=1}^{N+1} k_i = 0\) is satisfied for \(s(z)\) given by (3.9), because 
\[
(S_k)_{N+1}^N = \frac{1}{k!} ((S_1)_{N+1}^N)^k. 
\]

\[\square\]

**Lemma 3.3.** For any \(r(z), \tilde{r}(z) \in \mathfrak{g}_N^{o, +}\) and \((\mathcal{F}^{c, \text{anc}}, \mathcal{F}^{o, \text{anc}}) \in \text{Anc}_N\) we have
\[
\left( \left[ r(z), \tilde{r}(z) \right] - [r(z), \tilde{r}(z)] \right) \exp(\mathcal{F}^{c, \text{anc}} + \mathcal{F}^{o, \text{anc}}) = 0.
\]

**Proof.** By Lemma 2.10, it is sufficient to check that the expression
\[
\varepsilon \frac{\partial \exp(\mathcal{F}^{o, \text{anc}})}{\partial t^N_{p+1}} + \varepsilon^2 + \frac{(-1)^{i+1} \partial^2 \exp(\mathcal{F}^{o, \text{anc}})}{\partial t^N_{i+1} \partial t^N_{j+1}}
\]
vanishes for any odd \(p\). Indeed, by Proposition 3.1 this expression is equal to
\[
\left( 1 + \frac{1}{2} \sum_{k+l \geq 1} (-1)^k \binom{p+1}{k} \right) \frac{\varepsilon^{p+1} \partial^{p+1} \exp(\mathcal{F}^{o, \text{anc}})}{(p+1)! (\partial t^N_{0})^{p+1}} = 0,
\]
as required. \(\square\)

The lemma immediately implies that the formula
\[
(\exp(\mathcal{F}^{c, \text{anc}}), \exp(\mathcal{F}^{c, \text{anc}} + \mathcal{F}^{o, \text{anc}})) \mapsto
\mapsto \left( \exp \left( \pi(r(z)) \right) \exp(\mathcal{F}^{c, \text{anc}}), \exp \left( \pi(\tilde{r}(z)) \right) \exp(\mathcal{F}^{c, \text{anc}} + \mathcal{F}^{o, \text{anc}}) \right), \quad r(z) \in \mathfrak{g}_N^{o, +},
\]
defines a \(G_0^{N+1,+}\)-action on the space \(\text{Anc}_N\). At the level of functions \(\mathcal{F}_0^{c, \text{anc}}\) and \(\mathcal{F}_0^{o, \text{anc}}\), the formula for action coincides with formula (2.22) for the \(G_0^{o,N+1,+}\)-action on the space of pairs \((\mathcal{F}_0^c, \mathcal{F}_0^o)\) of closed and open ancestor potentials. We thus see that \(\mathcal{F}_0^{o, \text{anc}}\) is an open ancestor potential.

Regarding the whole space \(\text{Desc}_N\), Lemma 2.10 immediately implies that the formula
\[
(\exp(\mathcal{F}^{c, \text{desc}}), \exp(\mathcal{F}^{c, \text{desc}} + \mathcal{F}^{o, \text{desc}})) \mapsto
\mapsto \left( \exp \left( \pi(s(z)) \right) \exp(\mathcal{F}^{c, \text{desc}}), \exp \left( \pi(\tilde{s}(z)) \right) \exp(\mathcal{F}^{c, \text{desc}} + \mathcal{F}^{o, \text{desc}}) \right), \quad s(z) \in \mathfrak{g}_N^{o, -},
\]
defines a \(G_0^{N+1,-}\)-action on the space \(\text{Desc}_N\). Under this action, the point \(\tau_{\text{orig}}\) changes as follows:
\[
\tau_{\text{orig}} \mapsto \tau_{\text{orig}} - \gamma\bar{A},
\]

At the level of functions \(\mathcal{F}_0^{c, \text{desc}}\) and \(\mathcal{F}_0^{o, \text{desc}}\) the formula for the action coincides with (2.24). Therefore, \(\mathcal{F}_0^{o, \text{desc}}\) is an open descendant potential.

3.4. **Proof of Theorem 1.** In Section 3.3 we proved that for any \((\mathcal{F}^{c, \text{desc}}, \mathcal{F}^{o, \text{desc}}) \in \text{Desc}_N\) the pair \((\mathcal{F}_0^{c, \text{desc}}, \mathcal{F}_0^{o, \text{desc}})\) is a pair of closed an open descendant potentials. This implies Part 1 of the theorem.

Part 2 is a corollary of the following result.

**Proposition 3.4.** For any pair of closed an open descendant potentials \((\mathcal{F}_0^c, \mathcal{F}_0^o)\) defining a semisimple algebra structure on \(\tau_{\text{orig}} C_0^{N+1}\) there exists a pair \((\mathcal{F}_0^{c, \text{desc}}, \mathcal{F}_0^{o, \text{desc}}) \in \text{Desc}_N\) such that \(\mathcal{F}_0^{c, \text{desc}} = \mathcal{F}_0^c\) and \(\mathcal{F}_0^{o, \text{desc}} = \mathcal{F}_0^o\).
Proof. It is easy to see that there exists an \((N + 1) \times (N + 1)\) matrix \(s_1\) such that \(s_1\mathbf{A} = \mathbf{t}_{\text{orig}}\) and \(s_1z^{-1} \in g_{0,N+1,\text{ancest}}^0\). Acting on the pair \((F_0^c, F_0^\omega)\) by the element \(\exp(s_1z^{-1}) \in G_{N+1,-}^0\) we shift the point \(\mathbf{t}_{\text{orig}}\) to 0 in \(C^{N+1}\). Therefore, without loss of generality we can assume that \(\mathbf{t}_{\text{orig}} = 0\).

Define \((N + 1) \times (N + 1)\) matrices \(\Omega_j = (\Omega_j^{\alpha, \beta}(t^*))\), \(j \geq 0\), by \(\Omega_j^{\alpha, \beta} := \frac{\partial F_0^\omega}{\partial t^\alpha}\bigg|_{t^0 = 0}\), where \(F^\omega = (F_{1,0}, \ldots, F_{N+1,0})\) are the descendant vector potentials corresponding to \((F_0^c, F_0^\omega)\). By [ABLR20, Proposition 2.10], the element \((\mathbb{I} + \sum_{j \geq 1}(-1)^jz^{-j}\Omega_{j-1}(0))^{-1} \in G_{N+1,-}^0\) transforms the pair \((F_0^c, F_0^\omega)\) to a pair of ancestor potentials. Therefore, without loss of generality we can further assume that our pair of potentials \((F_0^c, F_0^\omega)\) is ancestor.

Denote by \(F^c\) and \(F^\omega\) the corresponding solutions to the closed and open WDVV equations. Let us consider the canonical coordinates on the flat F-manifold given by \((F^c, F^\omega)\) described by Lemma 2.9. The metric of the Dubrovin–Frobenius manifold becomes diagonal in the canonical coordinates: \(\frac{1}{2}g_{\alpha\beta}dt^\alpha dt^\beta = \sum_{i=1}^N g_i(du^i)^2\), where we consider \(g_i\) as a function of \(t^*\), \(g_i = g_i(t^*)\).

It is easy to check that the pair of total ancestor potentials from the space \(\text{Anc}_N\) corresponding to the parameters \(a_1, \ldots, a_N, \theta \in \mathbb{C}^*, b_1, \ldots, b_N \in \mathbb{C}^N\), and \(\psi \in \text{End}(\mathbb{C}^N)\) defined by

\[
a_i := \sqrt{g_i(0)}^{-1}, \quad \theta := \frac{\partial u^{N+1}}{\partial t^{N+1}}(0), \quad \exp(\psi)_\alpha := a_i^{-1}\frac{\partial u^i}{\partial t^\alpha}(0), \quad b_\alpha := \theta^{-1}\left(\frac{\partial u^{N+1}}{\partial t^\alpha}(0) - \frac{\partial u^1}{\partial t^\alpha}(0)\right),
\]

gives a flat F-manifold having the same algebra structure at the origin as the flat F-manifold given by \((F^c, F^\omega)\). By Proposition 2.8, applying an appropriate element of the group \(G_{N+1,+}^0\) we obtain a pair \((F_{\text{anc}}^c, F_{\text{anc}}^\omega) \in \text{Anc}_N\) such that \(F_{\text{anc}}^c = F_0^c\) and \(F_{\text{anc}}^\omega = F_0^\omega\).

\[\square\]

4. Proof of Theorem 2

4.1. The open TRR-0 relations. In Section 3.3 we proved that \(F_{0,\text{desc}}^\omega\) is an open descendant potential and in Section 2.5.1 we showed that any open descendant potential satisfies the open TRR-0 equations. So we are done with this part.

4.2. The open TRR-1 relations. First of all, the system of equations (1.7) is satisfied for \(N = 1\), \(F_g(t^0_1) = F_g^{\text{KW}}(t^0_1)\), and \(F^\omega = F_g^{\text{PST}}(t^1_1, t^0_1)\) [BCT18, Section 6.2.3]. Then it is easy to see that equation (1.7) holds for an arbitrary \(N \geq 1\) and \(F_g = \sum_{i=1}^N a_i^{-2}F_g^{\text{KW}}(a_it^0_i)\), \(F_g = F_g^{\text{PST}}(a_it^0_i, t^0_i, t^{N+1}_1)\).

Let us now prove that equation (1.7) is preserved by the \(G_{N+1,-}^0\)-action on \(\text{Desc}_N\). It is sufficient to check that the group \(G_{N+1,-}^0\) preserves equation (1.7) infinitesimally, i.e.

\[
\frac{\partial (\delta F_0^c)}{\partial t^a_{\alpha+1}} = \frac{\partial^2 (\delta F_0^c)}{\partial t^a_{\alpha} \partial t^0_\nu} \frac{\partial F^c_0}{\partial t^\nu} + \frac{\partial^2 F_0^c}{\partial t^a_{\alpha} \partial t^0_\nu} \frac{\partial (\delta F_0^c)}{\partial t^\nu} + \frac{\partial (\delta F_0^\omega)}{\partial t^a_\alpha} \frac{\partial F^c_0}{\partial t^0_{\alpha+1}} + \frac{\partial F^\omega}{\partial t^a_\alpha} \frac{\partial (\delta F_0^\omega)}{\partial t^0_{\alpha+1}} + \frac{1}{2} \frac{\partial^2 (\delta F_0^\omega)}{\partial t^a_\alpha \partial t^0_{\alpha+1}},
\]

where \(\delta \bullet := s(z) \bullet\) and \(s(z) \in g_{0,N+1,-}^0\). We have

\[
\delta F_0^c = -s(z)F_0^c + \frac{1}{2} \sum_{i,j \geq 0} (-1)^{j+1}(s_i^{\alpha+1})^\beta_{\eta_{ij}} q_j^\beta, \quad \delta F_0^\omega = -s(z)F_0^\omega - \frac{1}{2} \sum_{i,j \geq 0} (s_i^{\alpha+1})^\beta_{\eta_{ij}} q_j^\beta
\]

where we recall our notation \(s(z) = \sum_{i,j \geq 0} (s_i^{\alpha+1})^\beta_{\eta_{ij}} \frac{\partial}{\partial t^\nu} q_j^\beta\). Using the equation obtained from (1.7) by applying the operator \(-s(z)\) to both sides, we see that equation (4.1) is equivalent to

\[
- \sum_{i,j \geq 0} (s_i^{\alpha+1})^\beta_{\eta_{ij}} \frac{\partial F_0^c}{\partial t^0_\beta} = - \sum_{i,j \geq 0} (s_i^{\alpha+1})^\beta_{\eta_{ij}} \left( \frac{\partial^2 F_0^c}{\partial t^0_\beta \partial t^0_\nu} \frac{\partial F_0^c}{\partial t^\nu} + \frac{\partial F_0^c}{\partial t^0_\beta} \frac{\partial F_0^c}{\partial t^0_{\alpha+1}} + \frac{1}{2} \frac{\partial^2 F_0^c}{\partial t^0_\beta \partial t^0_{\alpha+1}}\right) - (s_{\alpha+1})^\beta_{\eta_{ij}} \frac{\partial F_0^\omega}{\partial t^0_\nu}.
\]
Applying equation (1.7) to the terms with \( j \geq 1 \) on the left-hand side, we immediately see that this equation is true. In particular, we obtain that equation (1.7) is true for the pair \( \left( \sum_{i=1}^{N} F^{KW}(a_i t_i, a_i \varepsilon), F^{\text{PST}}_{(0)}(a_i t_i, a_i t_i^{N+1}) \right) \)

The fact that the \( GL(\mathbb{C}^N) \)-action on \( \text{Anc}^0 \) preserves equation (1.7) is obvious. Therefore, equation (1.7) is true for any pair \( (F^{c, \text{anc}}, F^{o, \text{anc}}) \in \text{Anc}^0 \).

Let us prove that equation (1.7) is preserved by the \( \mathbb{C}^N \)-action on \( \text{Anc}^1 \). We again proceed infinitesimally, i.e. we want to prove that equation (4.1) with \( \delta \bullet := \overline{b} \bullet \) and \( \overline{b} \in \mathbb{C}^N \) is satisfied assuming that equation (1.7) is true. For this we compute

\[
\delta F_0^c = 0, \quad \delta F_0^o = - \sum_{d \geq 0} b_d a \frac{\partial F_0^c}{\partial t_i} + b_\alpha a \eta^{\alpha \beta} \frac{\partial F_0^c}{\partial t_0}, \quad \delta F_1 = - \sum_{d \geq 0} b_d a \frac{\partial F_1^c}{\partial t_i} + b_\alpha a \eta^{\alpha \beta} \frac{\partial F_0^c}{\partial t_0}.
\]

Using the equation obtained from (1.7) by applying the operator \(- \sum_{d \geq 0} b_d a \frac{\partial}{\partial t_i} \) to both sides, we see that equation (4.1) is equivalent to

\[
-b_\alpha \frac{\partial F_0^c}{\partial t_i} + b_\gamma \eta^{\gamma \beta} \frac{\partial^2 F_0^c}{\partial t_0^2} = - \frac{\partial^2 F_0^c}{\partial t_0^2} \eta^{\mu \nu} b_{\mu \nu} \frac{\partial F_0^c}{\partial t_0} + \frac{\partial^2 F_0^c}{\partial t_0^2} \eta^{\mu \nu} b_0 \frac{\partial F_0^c}{\partial t_0^2} + \frac{\partial^2 F_0^c}{\partial t_0^2} \eta^{\gamma \beta} b_\gamma
\]

where we adopt the convention \( b_{N+1} := 0 \). Using the open TRR-0 equation and equation (1.7) we see that this equation is true. Hence, equation (1.7) is true for any pair \( (F^{c, \text{anc}}, F^{o, \text{anc}}) \in \text{Anc}^1 \).

Let us prove that equation (1.7) is preserved by the \( G_{N+1, +}^0 \)-action on \( \text{Anc}_{N} \). We proceed infinitesimally, i.e. we want to prove that equation (4.1) with \( \delta \bullet := r(z) \bullet \) and \( r(z) \in \mathfrak{g}_{N+1, +}^0 \) is satisfied assuming that equation (1.7) is true.

We have

\[
\delta F_0^c = \sum_{i \geq 1} \frac{(-1)^i}{i} \sum_{i \geq 0} \frac{\partial F_0^c}{\partial t_i^{N+1}} + \sum_{i \geq 1} \frac{(-1)^i}{i} \sum_{i \geq 0} \frac{\partial F_0^c}{\partial t_i} \frac{\partial F_0^c}{\partial t_i}, \quad \delta F_1 = \sum_{i \geq 1} \frac{(-1)^i}{i} \sum_{i \geq 0} \frac{\partial F_0^c}{\partial t_i} \frac{\partial F_0^c}{\partial t_i},
\]

where \( H := - \sum_{i \geq 1} \frac{(-1)^i}{i} \sum_{i \geq 0} \frac{\partial F_0^c}{\partial t_i^{N+1}} \).

Let us introduce the following operators:

\[
P_{\alpha, \alpha} := \frac{\partial}{\partial t_i^{N+1}} - \frac{\partial F_0^c}{\partial t_i^{N+1}} \frac{\partial}{\partial t_0} - \frac{\partial F_0^c}{\partial t_0} \frac{\partial}{\partial t_i^{N+1}}, \quad 1 \leq \alpha \leq N + 1, \quad \alpha \geq 0.
\]
Then we have $P_{\alpha,a}\frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^i} = P_{\alpha,a} \frac{\partial F_0^\gamma}{\partial t_0^i} = 0$, $P_{\alpha,a}\frac{\partial F_0^\gamma}{\partial t_0^i} = -\frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+1}}$ and we have to check that

$$P_{\alpha,a}\delta F_1^\alpha = \frac{\partial^2 (\delta F_0^\gamma)}{\partial t_0^\alpha \partial t_0^0} \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} + \frac{\partial (\delta F_0^\gamma)}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{N+1}} + \frac{1}{2} \frac{\partial^2 (\delta F_0^\gamma)}{\partial t_0^\alpha \partial t_0^{N+1}} \Leftrightarrow$$

$$\Leftrightarrow P_{\alpha,a}(H F_1^\alpha + L_1^0 + Q_1^0) = \frac{\partial^2 (H F_0^\gamma + Q_0^0)}{\partial t_0^\alpha \partial t_0^0} \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} + \frac{\partial (H F_0^\gamma + L_0^0 + Q_0^0)}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{N+1}}$$

$$+ \frac{1}{2} \frac{\partial^2 (H F_0^\gamma + L_0^0 + Q_0^0)}{\partial t_0^\alpha \partial t_0^{N+1}}.$$ 

Noting that $P_{\alpha,a} L_1^0 = 0$ and $\frac{\partial t_0^i}{\partial t_0^{i+1}} = 0$, we collect all the terms with the operator $H$ on the left-hand side and obtain the equivalent equation

$$(4.2) \quad P_{\alpha,a} H F_1^\alpha - \frac{\partial^2 (H F_0^\gamma)}{\partial t_0^\alpha \partial t_0^0} \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} - \frac{\partial (H F_0^\gamma)}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{N+1}} - \frac{1}{2} \frac{\partial^2 (H F_0^\gamma)}{\partial t_0^\alpha \partial t_0^{N+1}} =$$

$$= \frac{\partial^2 Q_0^0}{\partial t_0^0 \partial t_0^\alpha} \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} + \frac{\partial (L_0^0 + Q_0^0)}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{N+1}} + \frac{1}{2} \frac{\partial^2 Q_0^0}{\partial t_0^\alpha \partial t_0^{N+1}} - P_{\alpha,a} Q_1^0.$$ 

Let us compute the left-hand side here.

We compute

$$[P_{\alpha,a}, H] = \sum_{i \geq 1} \left( - (r_{i,a})^\mu_{\mu_{i+a+1}} + (r_{i,a})^\mu_{\mu_i} \frac{\partial}{\partial t_0^\mu} \right) + \left( \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^0} \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} + \left( \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{N+1}} + \frac{1}{2} \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{N+1}} \right) \right).$$

Therefore, the left-hand side of (4.2) is equal to

$$\frac{1}{2} H \frac{\partial^2 F_0^0}{\partial t_0^\alpha \partial t_0^{N+1}} - \sum_{i \geq 1} (r_{i,a})^\mu_{\mu_{i+a+1}} \frac{\partial F_1^\mu}{\partial t_0^{i+a+1}} + \sum_{i \geq 1} \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} + \frac{1}{2} \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+a}} \left( \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} \right) +$$

$$+ \left( \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \right) \frac{\partial F_1^0}{\partial t_0^{N+1}} - \left( \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^0} \eta_{\mu\nu} \frac{\partial F_1^\mu}{\partial t_0^0} \right) + \sum_{i \geq 1} (r_{i,a})^\gamma \frac{\partial F_1^\gamma}{\partial t_0^0} - \left( \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \right) \frac{\partial F_1^0}{\partial t_0^{N+1}} - \sum_{i \geq 1} \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+a}} -$$

$$- \sum_{i \geq 1} (r_{i,a})^\gamma \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+a}} \frac{\partial F_1^0}{\partial t_0^{i+a}} =$$

$$= \sum_{i \geq 1} (r_{i,a})^\gamma \left( \frac{\partial F_1^0}{\partial t_0^\alpha} - \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^\gamma}{\partial t_0^0} - \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+a}} - \frac{1}{2} \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+a}} \right)$$

$$+ \sum_{i \geq 1} (-1)^{i+1} \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} + \sum_{i \geq 1} \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} - \frac{1}{2} \frac{\partial^2 F_0^\gamma}{\partial t_0^\alpha \partial t_0^{i+a}} \frac{\partial F_1^0}{\partial t_0^{i+a}} =$$

$$= \sum_{i \geq 1} (-1)^{i+1} \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} + \sum_{i \geq 1} \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} + \frac{\partial F_0^\gamma}{\partial t_0^\alpha} \frac{\partial F_1^0}{\partial t_0^{i+a}} .$$
Therefore, equation (4.2) is equivalent to

\[
(4.3) \quad \sum_{j \geq 1} (-1)^{j+1} \frac{\partial F_{j}^{\gamma}}{\partial t_{j}} \left( r_{j} \right) \mu_{\gamma} \frac{\partial F_{1}^{\gamma}}{\partial t_{1}} + \sum_{j \geq 1} \frac{\partial^{2} F_{j}^{\gamma}}{\partial t_{j}^{2}} \left( r_{j} \right) \mu_{\gamma} \frac{\partial F_{0}^{\gamma}}{\partial t_{0}} = 0.
\]

We further compute

\[
\frac{\partial Q_{0}^{o}}{\partial t_{a}} \frac{\partial F_{0}^{o}}{\partial t_{0}} = \sum_{j,k \geq 0} (-1)^{k} \left( r_{j+k+1} \right) \mu_{\gamma} \left( \frac{\partial F_{0}^{o}}{\partial t_{a}} \frac{\partial F_{0}^{o}}{\partial t_{j+k+1}^{2}} \right),
\]

and

\[
\frac{1}{2} \frac{\partial^{2} Q_{0}^{o}}{\partial t_{a} \partial t_{0}^{2}} - P_{a,a} Q_{1}^{o} = \sum_{j,k \geq 0} (-1)^{k} \left( r_{j+k+1} \right) \mu_{\gamma} \left( \frac{1}{2} \frac{\partial^{2} F_{0}^{o}}{\partial t_{a}^{2}} \frac{\partial F_{0}^{o}}{\partial t_{j+k+1} \partial t_{0}} \right),
\]

where we underlined the terms that cancel each other when we substitute these expressions on the right-hand side of (4.3). So equation (4.3) is equivalent to

\[
(4.4) \quad \sum_{j \geq 1} \left( r_{j} \right) \mu_{\gamma} \left( (-1)^{j+1} \frac{\partial^{2} F_{j}^{\gamma}}{\partial t_{j}^{2}} \left( r_{j} \right) \mu_{\gamma} \frac{\partial F_{1}^{\gamma}}{\partial t_{1}} + \frac{\partial^{2} F_{j}^{\gamma}}{\partial t_{j}^{2}} \left( r_{j} \right) \mu_{\gamma} \frac{\partial F_{0}^{\gamma}}{\partial t_{0}} \right) = \sum_{j,k \geq 0} (-1)^{k} \left( r_{j+k+1} \right) \mu_{\gamma} \frac{\partial F_{0}^{\gamma}}{\partial t_{j} \partial t_{k}} \frac{\partial F_{0}^{\gamma}}{\partial t_{k}} = 0.
\]

Finally note that the left-hand side of (4.4) is equal exactly to \( \sum_{j,k \geq 0} (-1)^{k} \left( r_{j+k+1} \right) \mu_{\gamma} \frac{\partial F_{0}^{\gamma}}{\partial t_{j} \partial t_{k}} \frac{\partial F_{0}^{\gamma}}{\partial t_{k}} \), and using the open TRR-1 equations we conclude that equation (4.4) is true. Therefore, equation (1.7) is true for any pair of total ancestor potentials.

Any pair of total descendant potentials is obtained from a pair of total ancestor potentials by the action of an element of the group \( G_{N+1, -}^{o} \). Since we have already proved that equation (1.7) is preserved by the \( G_{N+1, -}^{o} \), equation (1.7) is true for any pair of total descendant potentials.

4.3. The open string equation. Consider the following operator:

\[
\mathcal{L}_{-1}^{-A} := \frac{\varepsilon^{-2}}{2} \eta_{a \beta} q_{0}^{\alpha} q_{0}^{\beta} + \varepsilon^{-1} q_{0}^{N+1} + \sum_{d \geq 1} q_{d}^{a} \frac{\partial}{\partial q_{d-1}^{a}}.
\]

Using the closed string equation (3.2), we see that equation (1.8) follows from the equation

\[
(4.5) \quad \left( \mathcal{L}_{-1}^{-A} + \frac{1}{2} \text{tr}(r_{1}) \right) \exp(\mathcal{F}_{\text{desc}}^{c} + \mathcal{F}_{\text{desc}}^{o}) = 0,
\]

which we are going to prove.

Lemma 4.1. We have the following commutation relations.
1. \( \left[ s(z)^o, L_{\eta,\underline{A}}^{\eta,\underline{A}} \right] = 0 \) for any \( s(z) \in g_{N+1,-}^0 \).

2. \( \left[ \hat{\psi}, L_{\eta,\underline{A}}^{\eta,\underline{A}} \right] = A^\alpha \psi^\beta \frac{\partial}{\partial q_0^\alpha} + \frac{\varepsilon^2}{2} \left( \psi_\alpha \eta_\beta + \eta_\alpha \psi^\beta \right) t_0^\alpha t_0^\beta \) for any \( \psi \in \text{End}(\mathbb{C}^N) \).

3. \( \left[ \hat{b}, L_{\eta,\underline{A}}^{\eta,\underline{A}} \right] = b^\alpha A^\alpha \frac{\partial}{\partial q_0^\alpha} \) for any \( \hat{b} \in \mathbb{C}^N \).

4. \( \left[ r(z)^o, L_{\eta,\underline{A}}^{\eta,\underline{A}} \right] = \frac{1}{2} \text{tr}(r_1) \) for any \( r(z) \in g_{N+1,+}^0 \).

**Proof.** First of all, note that \( L_{\eta,\underline{A}}^{\eta,\underline{A}} = -\text{Id} \cdot z^{-1} \), which immediately implies Part 1 of the lemma.

Parts 2 and 3 are simple direct computations.

For Part 4, let us decompose \( L_{\eta,\underline{A}}^{\eta,\underline{A}} = A + B \), where \( A := \frac{\varepsilon^2}{2} \eta_\alpha \beta q_0^\alpha \beta + \varepsilon^{-1} q_0^{N+1} \) and \( B := \sum_{d \geq 1} q_d^\alpha \frac{\partial}{\partial q_{d+1}^\alpha} \). We compute

\[
\left[ r(z)^o, A \right] = \sum_{i \geq 0} (r_{i+1})_\nu^\alpha q_0^\nu \frac{\partial}{\partial q_i^\alpha} + \varepsilon \sum_{i \geq 0} (-1)^i (r_{i+1})_{N+1,\alpha} \frac{\partial}{\partial q_i^\alpha} + \frac{1}{2} \text{tr}(r_1),
\]

\[
- \sum_{i \geq 1, j \geq 0} (r_i)_\nu^\alpha q_j^\nu \frac{\partial}{\partial q_i^\alpha} + \varepsilon \sum_{i \geq 1} (-1)^i (r_i)_{N+1,\nu} \frac{\partial}{\partial q_i^\nu} B \] = - \sum_{k \geq 1} (r_k)_\nu^\alpha q_0^\nu \frac{\partial}{\partial q_k^\nu} + \varepsilon \sum_{i \geq 1} (-1)^i (r_i)_{N+1,\nu} \frac{\partial}{\partial q_i^\nu} B,
\]

\[
\left[ \frac{\varepsilon}{2} \sum_{i \geq 0} (-1)^i (r_{i+1})_{\alpha\beta} \frac{\partial^2}{\partial q_i^\alpha \partial q_i^\beta} B \right] = 0,
\]

which completes the proof of the lemma. \( \Box \)

By [BT17, Theorem 1.3], equation (4.5) is true for \( F^o = F^{\text{PST}}(t_1^i, t_2^{N+1}, \varepsilon) \) and \( F^c = \sum_{i=1}^N F^{\text{KW}}(t_i, \varepsilon) \). Using Part 1 of Lemma 4.1, it is easy to see that equation (4.5) is true for \( F^o = F^{\hat{(O)}}(a_1, t_1^i, t_2^{N+1}, \varepsilon) \) and \( F^c = \sum_{i=1}^N F^{\text{KW}}(a_1, t_i^i, \varepsilon) \) for any \( a_1, \ldots, a_N, \theta \in \mathbb{C}^* \).

Part 2 of Lemma 4.1 implies that for any \( \psi \in \text{End}(\mathbb{C}^N) \) we have

\[
\exp \left( \hat{\psi} \right) \circ L_{\eta,\underline{A}}^{\eta,\underline{A}} = L_{\eta,\underline{A}}^{\eta,\underline{A}} \circ \exp \left( \hat{\psi} \right),
\]

where \( \eta_{\alpha\beta} = \sum_{i=1}^N \exp(\psi)^i_\alpha \exp(\psi)^i_\beta \) and \( \underline{A} = (A^1, \ldots, A^{N+1}) \) is given by \( A^\alpha = \sum_{i=1}^N \exp(-\psi)^i_\alpha a_i^{-1} \) for \( 1 \leq \alpha \leq N \), and \( A^{N+1} = 0 \). This proves equation (4.5) for any pair of total ancestor potentials from the space \( \text{Anc}_N^0 \).

Part 3 of Lemma 4.1 implies that

\[
\exp \left( \hat{b} \right) \circ L_{\eta,\underline{A}}^{\eta,\underline{A}} = L_{\eta,\underline{A}}^{\eta,\underline{A}} \circ \exp \left( \hat{b} \right),
\]

which proves equation (4.5) for any pair of total ancestor potentials from the space \( \text{Anc}_N^1 \).

Part 4 of Lemma 4.1 implies that

\[
\exp \left( r(z)^o \right) \circ L_{\eta,\underline{A}}^{\eta,\underline{A}} = \left( L_{\eta,\underline{A}}^{\eta,\underline{A}} + \frac{1}{2} \text{tr}(r_1) \right) \circ \exp \left( r(z)^o \right), \quad r(z) \in g_{N+1,+}^0,
\]

which proves equation (4.5) for any pair of total ancestor potentials. Finally, Part 1 of Lemma 4.1 shows that equation (4.5) is true for any pair of total descendant potentials.
4.4. The open dilaton equation. Consider the following operator:

\[ p^\alpha := \sum_{k \geq 0} q_k^\alpha \frac{\partial}{\partial q_k} + \varepsilon \frac{\partial}{\partial \varepsilon} + \frac{N}{24} + \frac{1}{2} \]

Using the closed dilaton equation (3.3), we see that equation (1.9) follows from the equation (4.6)

\[ p^\alpha \exp(\mathcal{F}^c,\text{desc}) + \mathcal{F}^o,\text{desc}) = 0, \]

which we are going to prove.

Lemma 4.2. We have the following commutation relations.

1. \[ [s(z)^o, \mathcal{P}^\alpha] = [r(z)^o, \mathcal{P}^\alpha] = 0 \] for any \( s(z) \in \mathfrak{g}^{N+1}_0 \) and \( r(z) \in \mathfrak{g}^{N+1}_0 \).

2. \[ [\hat{\psi}, \mathcal{P}^\alpha] = A^\alpha_{\psi^\beta} \frac{\partial}{\partial q^1} \] for any \( \psi \in \text{End}(\mathbb{C}^N) \).

3. \[ [\hat{b}, \mathcal{P}^\alpha] = b_a A^\alpha_{-\frac{\partial}{\partial q^1}} \] for any \( \hat{b} \in \mathbb{C}^N \).

Proof. Part 1 of the lemma is obvious. Parts 2 and 3 are simple direct computations. \( \square \)

Using this lemma, the proof follows exactly in the same way as for the open string equation starting from the fact that equation (4.6) is true for \( \mathcal{F}^o = \mathcal{F}^{\text{PST}}(t^1_s, t^{N+1}_s, \varepsilon) \) and \( \mathcal{F}^c = \sum_{i=1}^{N_1} \mathcal{F}^{\text{KW}}(t^i_s, \varepsilon) \) [BT17, Theorem 1.3].

5. Further properties of the construction and an example in genus 1

In this section, we present more formulas for the actions of the groups \( G^o_{N+1,+} \) and \( \mathbb{C}^N \) on the spaces \( \text{Anc}_N \) and \( \text{Anc}_c \), respectively (see Proposition 5.1 and Lemma 5.2), and then use them for the computation of the coefficient of \( t_0^{N+1} \) in an arbitrary open ancestor potential in genus 1 (see Proposition 5.3).

5.1. On the \( G^o_{N+1,+} \)-action on the space \( \text{Anc}_N \). Given \( R(z) \in G^o_{N+1,+} \), let \( r(z) := \log R(z) \) and

\[
L_1 := - \sum_{i \geq 1, j \geq 0} (r_i)^i_j \frac{\partial}{\partial q^{i+j}}, \quad L_2 := \varepsilon \sum_{1 \leq \alpha \leq N} L^\alpha_{a \mu} \frac{\partial}{\partial q^a}, \quad L_3 := \varepsilon \sum_{a \geq 2} L^a_{1 \mu} \frac{\partial}{\partial q^{N+1}},
\]

\[
E := \frac{\varepsilon^2}{2} \sum_{a,b \geq 0} E^{\alpha, a; b, \beta} \frac{\partial^2}{\partial q^a \partial q^b},
\]

where

\[
L^\alpha_{a \mu} := \text{Coeff}_{z^a} R(-z)^{N+1}_{\mu} \eta^{\alpha \mu}, \quad \varepsilon \leq \alpha \leq N, \quad a \geq 1,
\]

\[
L^a_{1 \mu} := - \frac{1}{2} \sum_{i+j=a} \binom{a}{i} ((R^{-1})_{\mu}^{N+1})_{\mu} \eta^{\mu \nu} ((R^{-1})_{\nu}^{N+1}), \quad a \geq 2,
\]

\[
E^{\alpha, a; b, \beta} := \begin{cases} 
\text{Coeff}_{z^{a+b}(1-R^{-1}(z_1)R(z_2))} \eta^{\mu \beta}, & \text{if } \beta \leq N, \\
E^{N+1, b; a, \alpha}, & \text{if } \alpha \leq N \text{ and } \beta = N + 1, \\
0, & \text{if } \alpha = \beta = N + 1.
\end{cases}
\]

Proposition 5.1. For an arbitrary pair \( (\mathcal{F}^{c, \text{anc}}, \mathcal{F}^{o, \text{anc}}) \) of closed and open total ancestor potentials we have

\[
\exp(\hat{r}(z)^o) \exp(\mathcal{F}^{c, \text{anc}} + \mathcal{F}^{o, \text{anc}}) = \exp(L_1) \exp(L_2) \exp(L_3) \exp(E) \exp(\mathcal{F}^{c, \text{anc}} + mg^{o, \text{anc}}).
\]
Proof. Let us apply part (b) of Lemma A.1 with
\[ X := - \sum_{i \geq 1, j \geq 0} (r_i)_{i,j} \frac{\partial}{\partial q_i^{(j)}} + \varepsilon \sum_{i \geq 1} (-1)^i (r_i)^{N+1,\nu} \frac{\partial}{\partial q_i^{(N+1,\nu)}} \quad \text{and} \quad Y := \frac{\varepsilon^2}{2} \sum_{i \geq 1, j \geq 0} (-1)^i (r_i)_{i,j}^{N+1,\nu} \frac{\partial^2}{\partial q_i^{(N+1,\nu)} \partial q_j^{(N+1,\nu)}}. \]

After a long and tedious computation, using Lemma A.2, we obtain
\[ \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \text{ad}_X^n Y = \frac{\varepsilon^2}{2} \sum_{a,b \geq 0} \tilde{E}_{\alpha,a;\beta,b} \frac{\partial^2}{\partial q_a^{\beta} \partial q_b^{\beta}}, \]
where \( \tilde{E}_{\alpha,a;\beta,b} = E^{\alpha,a;\beta,b} \) if \( \alpha \leq N \) or \( \beta \leq N \), and
\[ \tilde{E}_{\alpha,N+1;N+1} = \text{Coef}_{z^1 z^2} \left( \frac{R(z) - 1}{r(z)} \right)^{N+1} \eta^{\mu \nu} r(z)^{N+1} \frac{\partial}{\partial q_a^{\mu} \partial q_b^{\nu}} \exp(E). \]

We therefore obtain
\[ \exp \left( \tilde{E} \right) = \exp \left( - \sum_{i \geq 1, j \geq 0} (r_i)_{i,j} \frac{\partial}{\partial q_i^{(j)}} + \varepsilon \sum_{i \geq 1} (-1)^i (r_i)^{N+1,\nu} \frac{\partial}{\partial q_i^{(N+1,\nu)}} \right) \exp(E). \]

We then again use part (b) of Lemma A.1 with \( X := - \sum_{i \geq 1, j \geq 0} (r_i)_{i,j} \frac{\partial}{\partial q_i^{(j)}} \) and \( Y := \varepsilon \sum_{i \geq 1} (-1)^i (r_i)^{N+1,\nu} \frac{\partial}{\partial q_i^{(N+1,\nu)}} \) and obtain
\[ \exp \left( \tilde{E}_{\alpha,a;\beta,b} \right) = \exp(L_1) \exp(L_2) \exp \left( - \varepsilon \sum_{a \geq 0} \text{Coef}_{z^a} \left( \frac{R(z) - 1}{r(z)} \right)^{N+1} \eta^{\mu \nu} r(z)^{N+1} \frac{\partial}{\partial q_a^{\mu} \partial q_a^{\nu}} \right). \]

Recall that \( \frac{\partial \exp(F^{\mu \nu})}{\partial q_a^{(N+1)}} = \varepsilon^a (a+1)! \frac{\partial^{a+1} \exp(F^{\mu \nu})}{\partial q_a^{(N+1)+1}} \), which gives \( \frac{\partial^2 \exp(F^{\mu \nu})}{\partial q_a^{(N+1)} \partial q_b^{(N+1)}} = \varepsilon^{-1} \frac{\partial \exp(F^{\mu \nu})}{\partial q_a^{(N+1)} \partial q_b^{(N+1)}} \). Therefore, it remains to check that
\[ - \sum_{a \geq 0} \text{Coef}_{z^a} \left( \frac{R(z) - 1}{r(z)} \right)^{N+1} \eta^{\mu \nu} r(z)^{N+1} \frac{\partial^{a+1}}{\partial q_a^{(N+1)+1}} \exp(F^{\mu \nu}) + \frac{1}{2} \sum_{i+j = a-1} (a+1) \tilde{E}_{\alpha,N+1;N+1,j} = L_3^a, \]
which is a simple direct computation based on the following observation:
\[ \sum_{i+j = a} \binom{a+2}{i+1} \text{Coef}_{z^i z^j} P(z_1, z_2) = \sum_{i+j = a+1} \binom{a+1}{i} \text{Coef}_{z^i z^j} (z_1 + z_2) P(z_1, z_2), \]
where \( P(z_1, z_2) \in \mathbb{C}[z_1, z_2] \). \( \square \)

5.2. On the \( \mathbb{C}^N \)-action on the space \( \text{Anc}_1^N \). Recall that given \( \tilde{b} = (b_1, \ldots, b_N) \in \mathbb{C}^N \) the operator \( \hat{\tilde{b}} \) is defined by
\[ \hat{\tilde{b}} = \sum_{d \geq 0} b_d t_d \frac{\partial}{\partial t_d^{N+1}} - \varepsilon b^\alpha \frac{\partial}{\partial t_0^\alpha}, \]
where we introduced the notation \( b^\alpha := \eta^{\alpha \beta} b_\beta \).

Lemma 5.2. We have
\[ \exp(\hat{\tilde{b}}) = \exp \left( \sum_{d \geq 0} b_d t_d \frac{\partial}{\partial t_d^{N+1}} \right) \exp \left( -\varepsilon b^\alpha \frac{\partial}{\partial t_0^\alpha} - \varepsilon b^\alpha b_\beta \frac{\partial}{\partial t_0^{N+1}} \right). \]
Proof. Easy application of part (b) of Lemma A.1 with \( X := \sum_{n \geq 0} b_n t_n^3 \frac{\partial}{\partial t_n^3} \) and \( Y := -\varepsilon b^n \frac{\partial}{\partial b^n} \). \( \square \)

5.3. An example in genus 1. We consider an arbitrary pair of closed and open total ancestor potentials \( (\mathcal{F}^c, \mathcal{F}^o) \) corresponding to parameters \( a_1, \ldots, a_N, \theta \in \mathbb{C}^* \), \( b = (b_1, \ldots, b_N) \in \mathbb{C}^N \), \( \psi \in \text{End}(\mathbb{C}^N) \), and \( r(z) \in \mathfrak{g}_N^{\mathcal{H}} \)

\[
\exp(\mathcal{F}^c, \mathcal{F}^o) := \exp\left( \pi(r(z)) \right) \exp\left( \hat{\psi} \right) \left( \prod_{i=1}^{N} \left( \tau^\text{KW}(a_i t_i^*, a_i \varepsilon) \right) \right),
\]

\[
\exp(\mathcal{F}^c, \mathcal{F}^o) = \exp\left( \pi(r(z)^o) \right) \exp\left( \hat{\psi} \right) \left( \tau^\text{PST}(a_i t_i^1, t_i^{N+1}, \varepsilon) \prod_{i=1}^{N} \left( \tau^\text{KW}(a_i t_i^*, a_i \varepsilon) \right) \right).
\]

Recall that the \( N \times N \) matrix \( \eta_{\alpha \beta} \), used in the expressions for \( \pi(r(z)^c) \), \( \pi(r(z)^o) \), and \( \hat{b} \), is given by \( \eta_{\alpha \beta} = \sum_{i=1}^{N} (\Psi^{-1})_i^\alpha (\Psi^{-1})_i^\beta \), where \( \Psi = (\Psi^a_i) = \exp(\psi) \). The corresponding solutions of the closed and open WDVV equations admit a unit given by \( A^\alpha = \left\{ \sum_{i=1}^{N} (\Psi^{-1})_i^\alpha a_i^{-1}, \text{ if } 1 \leq \alpha \leq N, \right. -\sum_{\beta=1}^{N} b_\beta A^\beta, \left. \alpha = N+1. \right\}

For an \((N+1) \times (N+1)\) matrix \( M = (M^a_\beta) \), we will denote \( M^a_\beta := M^a_\beta A^\beta \).

**Proposition 5.3.** We have
1. \( F^o_{1, t_1^* = 0} = 0 \),
2. \( \text{Coef}_{t_0^{N+1}} F^o_{1} = \frac{\theta}{2} (r_1^2 (a_1 \Psi_1^1 + \theta b_0) + \frac{\theta^2}{2} (r_1)^{N+1} - a_1 \Psi_1^1 b^\alpha - \theta b_\alpha b^\alpha). \)

**Proof.** Consider the total ancestor potentials \( \tilde{\mathcal{F}}^c, \tilde{\mathcal{F}}^o \) given by

\[
\exp(\tilde{\mathcal{F}}^c, \tilde{\mathcal{F}}^o) := \exp\left( \hat{\psi} \right) \left( \prod_{i=1}^{N} \left( \tau^\text{KW}(a_i t_i^*, a_i \varepsilon) \right) \right),
\]

\[
\exp(\tilde{\mathcal{F}}^c, \tilde{\mathcal{F}}^o) := \exp\left( \hat{\psi} \right) \left( \tau^\text{PST}(a_i t_i^1, t_i^{N+1}, \varepsilon) \prod_{i=1}^{N} \left( \tau^\text{KW}(a_i t_i^*, a_i \varepsilon) \right) \right).
\]

Note that the operator \( \exp(L_1) \) acts as the substitution \( t_1^0 \mapsto t_1^0 + \sum_{i=1}^{d} ((R^{-1})_i^0 (t_1^0)^{d - i} \beta_{d - i, 1} A^\beta) \). We therefore introduce an operator \( L'_1 \) by

\[
L'_1 := -\sum_{d \geq 2} ((R^{-1})_i^0)^{d} \frac{\partial}{\partial t_1^0}.
\]

**Lemma 5.4.** We have

\[
F^o_{1, t_1^* = 0} = \exp(\hat{t}_1^0) \left( \hat{\mathcal{F}}^o_{1} + \sum_{a,b \geq 0} F^a_{a; \beta, b} \frac{\partial}{\partial t_a^b} \right) \frac{\partial}{\partial t_1^0} \frac{\partial}{\partial t_1^0} \left. \right|_{t_1^0 = t_1^* = 0} \]

\[
= \exp(\hat{t}_1^0) \left( \hat{\mathcal{F}}^o_{1} + \sum_{a,b \geq 0} F^a_{a; \beta, b} \frac{\partial}{\partial t_a^b} \right) \frac{\partial}{\partial t_1^0} \frac{\partial}{\partial t_1^0} \left. \right|_{t_1^0 = t_1^* = 0} \]

\[
= \exp(\hat{t}_1^0) \left( \hat{\mathcal{F}}^o_{1} + \sum_{a,b \geq 0} F^a_{a; \beta, b} \frac{\partial}{\partial t_a^b} \right) \frac{\partial}{\partial t_1^0} \frac{\partial}{\partial t_1^0} \left. \right|_{t_1^0 = t_1^* = 0} \]

**Proof.** This follows from Proposition 5.1, the property \( \frac{\partial}{\partial t_1^0} = 0 \), and the fact that (see Section 2.4.1)

\[
\text{the coefficient of } t_1^0 \cdots t_1^0 \text{ in } \hat{\mathcal{F}}^c_{1} \text{ is zero if } \sum d_i \geq n - 2. \]

\( \square \)
Lemma 5.5. For \( k = 0 \) or \( k = 1 \), we have
\[
\text{Coef}_{t_0^{N+1},k} F_{1,\text{anc}}^{0} = \text{Coef}_{t_0^{N+1},k} \left( \exp \left( L_1' \right) \widetilde{F}_{1,\text{anc}}^{0} \right).
\]

Proof. This follows from the previous lemma and the fact that (see Section 2.5.1)
\[
(5.2) \quad \text{the coefficient of } t_{d_1}^{a_1} \ldots t_{d_n}^{a_n} \text{ in } \widetilde{F}_{0,\text{anc}}^{0} \text{ is zero if } \sum d_i \geq n - 1.
\]

Define the open \( G \)-function corresponding to \( G_{1,\text{anc}}^{0} \) by
\[
G^{0}(t^1, \ldots, t^{N+1}) := \left. F_{1,\text{anc}}^{0} \right|_{t_0^1 = 0} \in \mathbb{C}[[t^1, \ldots, t^{N+1}]].
\]
Denote \( v^\alpha := \eta^{\alpha 1} \partial^2 F_{0,\text{anc}}^{0} / \partial \alpha \partial \beta \), \( 1 \leq \alpha \leq N \), and \( \phi := \partial F_{0,\text{anc}}^{0} / \partial t_0^0 \). Note that the closed and open string equations imply that
\[
v^\alpha |_{t_0^0 = 0} = t_0^\alpha, \quad 1 \leq \alpha \leq N, \quad \phi |_{t_0^1 = 0} = t_0^{N+1}.
\]
In [BB21b], the authors proved that
\[
F_{1,\text{anc}}^{0} = G^{0}(v^1, \ldots, v^N, \phi) + \frac{1}{2} \log \frac{\partial^2 F_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1}}.
\]
Denote by \( \widetilde{G}^0 \) the open \( G \)-function corresponding to \( \widetilde{F}_{1,\text{anc}}^{0} \).

Lemma 5.6. We have
1. \( \left. F_{1,\text{anc}}^{0} \right|_{t_0^1 = 0} = \left. \widetilde{G}^0 \right|_{t_0^1 = 0}, \)
2. \( \text{Coef}_{t_0^{N+1},t_0^1} F_{1,\text{anc}}^{0} = \text{Coef}_{t_0^{N+1}} \widetilde{G}^0 + \frac{1}{2} \left. (r_1)_2^0 \frac{\partial^4 \widetilde{F}_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1} \partial t_0^{N+1}} \right|_{t_0^1 = 0} \).

Proof. Combining the previous lemma, the formula
\[
\widetilde{F}_{1,\text{anc}}^{0} = \widetilde{G}^{0}(\widetilde{v}^1, \ldots, \widetilde{v}^N, \widetilde{\phi}) + \frac{1}{2} \log \frac{\partial^2 \widetilde{F}_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1}},
\]
the property (5.1), and the vanishing \( \frac{\partial \widetilde{G}^{0}}{\partial t_0^{N+1}} = 0 \), we obtain
\[
\text{Coef}_{t_0^{N+1},t_0^1} F_{1,\text{anc}}^{0} = \text{Coef}_{t_0^{N+1}} \left[ \left. \widetilde{G}^{0}(0, \ldots, 0, \exp(L_1')\phi) + \frac{1}{2} \log \left( \exp(L_1') \frac{\partial^2 \widetilde{F}_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1}} \right) \right|_{t_0^1 = 0} \right]
\]
for \( k = 0 \) or \( k = 1 \).

For part 1, using the property (5.2), we obtain \( \exp(L_1')\phi |_{t_0^1 = 0} = 0 \) and \( \exp(L_1') \frac{\partial^2 \widetilde{F}_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1}} |_{t_0^1 = 0} = 1 \), which immediately proves part 1.

For part 2, we again use the property (5.2) and obtain \( \frac{\partial}{\partial t_0^{N+1}} \exp(L_1')\phi |_{t_0^1 = 0} = 1 \) and
\[
\frac{\partial}{\partial t_0^{N+1}} \log \left( \exp(L_1') \frac{\partial^2 \widetilde{F}_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1}} \right) |_{t_0^1 = 0} = \left. \exp(L_1') \frac{\partial^4 \widetilde{F}_{0,\text{anc}}^{0}}{\partial t_0^0 \partial t_0^{N+1} \partial t_0^{N+1}} \right|_{t_0^1 = 0},
\]
which completes the proof.
\[ \square \]
Using the open TRR-0 equations, the fact that
\[ \tilde{F}_{0}^{\text{o,anc}}|_{t_{1}^{*} = 0} = \sum_{i=1}^{N} a_{i} \frac{(\Psi_{i} t_{0}^{\gamma})^{3}}{6}, \]
and the open string equation, it is easy to compute that
\[ \frac{\partial^{4} \tilde{F}_{0}^{\text{o,anc}}}{\partial t_{0}^{\gamma} \partial t_{0}^{\gamma + 1} \partial t_{0}^{\alpha}}|_{t^{*} = 0} = \left\{ \begin{array}{ll} \theta(a_{1} \Psi_{1}^{\alpha} + \theta b_{\alpha}), & \text{if } 1 \leq \alpha \leq N, \\ \theta^{2}, & \text{if } \alpha = N + 1. \end{array} \right. \]

Therefore, the following lemma completes the proof of the proposition.

**Lemma 5.7.** We have
\[ \tilde{G}^{\alpha} = \sum_{i=1}^{N} a_{i} \frac{(\Psi_{i} t_{0}^{\gamma})^{2}}{2} \Psi_{i}^{\gamma} t^{\gamma} - a_{1} (\Psi_{1}^{\gamma} t^{\gamma} + b_{\gamma} t^{\gamma}) - b_{\alpha} b_{\gamma} \left( a_{1} \Psi_{1}^{\gamma} t^{\gamma} + \theta (t^{N+1} + b_{\gamma} t^{\gamma}) + \frac{1}{2} (t^{N+1} + b_{\gamma} t^{\gamma})^{2} \right). \]

**Proof.** By (3.4), we have \( F_{1}^{\text{PST}}|_{t_{1}^{*} = s_{2}^{*} = 0} = 0. \) Equation (3.5) implies that \( F_{(\theta)^{1}}^{\text{PST}}|_{t_{1}^{*} = s_{2}^{*} = 0} = 0. \) Then the lemma is proved by a careful application of Lemma 5.2.

\[ \square \]

**Appendix A. Technical Lemmas**

**Lemma A.1.** Let \( X \) and \( Y \) be two operators such that \([Y, \text{ad}_{X}^{n} Y] = 0\) for any \( n \geq 0\). Consider also a formal variable \( z \). Then we have

\[ \text{(a)} \ \exp(z(X + Y)) = \exp \left( \sum_{n \geq 0} \frac{z^{n+1}}{(n+1)!} \text{ad}_{X}^{n} Y \right) \exp(zX), \]

\[ \text{(b)} \ \exp(z(X + Y)) = \exp(zX) \exp \left( \sum_{n \geq 0} \frac{(-1)^{n} z^{n+1}}{(n+1)!} \text{ad}_{X}^{n} Y \right). \]

**Proof.** Both formulas are well-known special cases of the Baker–Campbell–Hausdorff formula. Note that the second formula can be obtained from the first one by changing \( X \leftrightarrow X - Y \) and then \( X \leftrightarrow -X \).

\[ \square \]

**Lemma A.2.** For any two operators \( A, B \) and a formal variable \( z \) we have

\[ \sum_{m,n \geq 0} z^{m+n+1} \frac{A^{m}(A + B)B^{n}}{m!n!(m + n + 1)} = \exp(zA) \exp(zB) - 1, \]

\[ \sum_{m,n \geq 0} z^{m+n+1} \frac{A^{m}(A + B)B^{n}}{m!(n+1)!(m + n + 2)} = \exp(zA) \frac{\exp(zB) - 1}{zB} - \frac{\exp(zA) - 1}{zA}. \]

**Proof.** Elementary exercise.

\[ \square \]
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