EXISTENCE OF COUPLED KÄHLER-EINSTEIN METRICS USING THE CONTINUITY METHOD

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ABSTRACT. In this paper we prove the existence of coupled Kähler-Einstein metrics on complex manifolds whose canonical bundle is ample. These metrics were introduced and their existence in the said case was proven by Hultgren and Nyström using calculus of variations. We prove the result using the method of continuity. In the process of proving estimates, akin to the usual Kähler-Einstein metrics, we reduce existence in the Fano case to a $C^0$ estimate.

1. Introduction

Let $(X, \omega_0)$ be a compact Kähler manifold which is either Fano ($c_1(X) > 0$) or anti-Fano ($c_1(X) < 0$). Consider the following equations (the “coupled Kähler-Einstein equations”) on $X$, originally introduced in [8].

\begin{equation}
\operatorname{Ric}(\omega_1) = \operatorname{Ric}(\omega_2) = \ldots = \pm \sum_i \omega_i,
\end{equation}

where $\omega_i$ are Kähler metrics to be solved for in given Kähler classes $[\theta_i]$ satisfying $\pm \sum_i [\theta_i] = c_1(X)$. These equations seem vaguely reminiscent of the bimetric theories of gravity (see [7] and the references therein).

It can easily be shown that (1.1) is equivalent to the following system of Monge-Ampère PDE if $\omega_0$ satisfies $\operatorname{Ric}(\omega_0) = \pm \sum_i \theta_i$. (This can be arranged using Yau’s solution of the Calabi conjecture [12].)

\begin{equation}
(\theta_i + \sqrt{-1} \partial \bar{\partial} \phi_i)^n = C_i e^{-\sum_i \phi_i} \alpha_0^n
\end{equation}

for smooth functions $\phi_i$ satisfying $\sup \phi_2 = \sup \phi_3 = \ldots = \sup \phi_n = 0$ where $C_i = \int_{\omega_0^n} \theta_i$. In [8] the following existence result was proven for anti-Fano $X$.

Theorem 1.1 (Hultgren-Nyström). Let $(X, \omega_0)$ be a compact Kähler manifold which is anti-Fano. Let $[\theta]$ be Kähler classes such that $\sum_i [\theta_i] = -c_1(X)$. Then there exist unique Kähler metrics $\omega_i \in [\theta_i]$ such that

\begin{equation}
\operatorname{Ric}(\omega_1) = \operatorname{Ric}(\omega_2) = \ldots = -\sum_i \omega_i
\end{equation}
Hultegren and Nyström proved theorem 1.1 using calculus of variations. In this paper we prove this theorem using the method of continuity. To do this we establish the following \textit{a priori} estimates.

**Theorem 1.2.** Let $(X, \omega_0)$ be a compact Kähler manifold that is either Fano or anti-Fano such that $\omega_0$ satisfies $\text{Ric}(\omega_0) = \pm \sum_i \theta_i$ where $\theta_i$ are Kähler forms such that $\pm \sum_i [\theta_i] = c_1(X)$. Let $\phi_i$ be a smooth solution of the following system of equations.

\begin{equation}
(\theta_i + \sqrt{-1} \partial \bar{\partial} \phi_i)^n = C_i e^{\mp \sum_i t_i \phi_i} \omega_0^n
\end{equation}

where $C_i = \int \frac{\theta_i^n}{\omega_0^n}$ and $0 \leq t_i \leq 1$.

1. If $X$ is anti-Fano then $\|\phi_i\|_{C^{2,\alpha}} \leq C$ where $C$ is bounded uniformly.
2. If $X$ is Fano then $\|\phi_i\|_{C^{2,\alpha}} \leq C$ where $C$ depends on $\|\phi_1\|_{C^0}$.

Note that at $t_i = 0 \ \forall \ i$, the functions $\phi_i = 0$ solve the equations. By theorem 1.2 the set of $t_i$ for which there exists a solution is closed for anti-Fano manifolds. Theorem 1.1 follows from the following openness result.

**Theorem 1.3.** The set of $0 \leq t < 1$ for which there exists a unique smooth solution to the following system is open.

\begin{equation}
\text{Ric}(\theta_1 \phi_1) = \text{Ric}(\theta_2 \phi_2) = \ldots = \pm \left( \sum t \theta_i \phi_i + \sum (1-t) \theta_i \right)
\end{equation}

Notice that theorems 1.2 and 1.3 reduce the problem for Fano manifolds to the $C^0$ estimate just as in the usual Kähler-Einstein case. In [8] an obstruction to solving the equation akin to K-stability was discovered for Fano manifolds. It is interesting to see if the corresponding $C^0$ estimate can be proven along this continuity path using techniques of [2, 3, 4, 6].

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2. \textit{A priori estimates on solutions to equation 1.2}

As is often the case in fully nonlinear PDE, we prove lower order estimates and improve upon them. In what follows unless clarity demands otherwise, we denote arbitrary uniform (in the time parameters in the method of continuity) constants by $C$.

We first prove a $C^0$ estimate in the anti-Fano case.
Lemma 2.1. If \( c_1(X) < 0 \) then any smooth solution \( \phi_i \) satisfying \( \sup \phi_2 = \sup \phi_3 = \ldots = 0 \) of the system

\[
(\theta_i + \sqrt{-1} \partial \bar{\partial} \phi_i)^n = C_i e^{t_i \phi_i} \omega_0^n,
\]

where \( C_i = \frac{e^{\theta_i}}{\int e^{\theta_i} \omega_0^n} \) and \( 0 \leq t_i \leq 1 \) satisfies \( \| \phi_i \|_{C^0} \leq C \).

Proof. If \( |\phi_1|_{C^0} \leq C \) then by the assumption that \( \phi_i \leq 0 \forall i \geq 2 \), and either the Alexandrov-Bakelmann-Pucci (ABP) maximum principle \([1]\) or \( L^p \) stability for \( p > 1 \) \([9]\) we can see that \( \| \phi_i \|_{C^0} \leq C \) for all \( 1 \leq i \leq n \). In addition, the maximum principle shows that \( \phi_1 \geq -C \). So we just need to prove that \( \phi_1 \leq C \).

Choosing a positive Green’s function \( G \) for the Laplacian of \( \omega_0 \), we see using the representation formula (page 49 in \([10]\) for instance) that

\[
u(x) - C \leq \int e^{t_i \phi_i} \omega_0^n V,
\]

for every \( u \) satisfying \( \sqrt{-1} \partial \bar{\partial} u \geq -C \omega_0 \), where \( V \) is the volume of \( \omega_0 \). Taking \( u = \sum t_i \phi_i \) and using Jensen’s inequality we get

\[
\sum t_i \phi_i(x) - C \leq \ln \left( \int e^{t_i \phi_i} \omega_0^n \right) \Rightarrow \sum t_i \phi_i(x) \leq C.
\]

Therefore, \( \| \sum t_i \phi_i \|_{L^p} \leq C_p \) for all \( p > 1 \). Thus by the the ABP estimate as before we see that \( -C \leq \phi_1 \leq C \). \( \square \)

We proceed to prove a bound on the Laplacian in both, the Fano, and the anti-Fano cases.

Lemma 2.2. Any smooth solution \( \phi_i \) of the system 1.2 satisfies \( \| \Delta \phi_i \| \leq C \).

Proof. Let \( u_i = e^{-\lambda \phi_i} (n + \Delta_0 \phi_i) \). We shall assume that \( \| \phi_i \|_{C^0} \leq C \) in what follows. Just as in Yau’s proof \([12]\) we write the following inequality (inequality 2.3 from \([5]\) for instance) for solutions of \( \omega_0^n = (\omega + \sqrt{-1} \partial \bar{\partial} v)^n = e^{F-\lambda v} \omega_0^n \)

\[
\Delta_{\omega_0}(\exp(-C_1 v)(n + \Delta v)) \geq \exp(-C_1 v) [\Delta F - C_2 - C_1(n + \Delta v)] + \exp(-C_3 v - \frac{F}{n - 1}) C_4(n + \Delta v)^{n/(n-1)}.
\]

Replacing \( C_1 \) by \( \lambda \), \( F \) by \( F + a \sum \phi_j \omega \) by \( \theta_i \), and \( v \) by \( \phi_i \) in the above inequality we get (after a couple of easy estimates) the following. Note that
$a = 1$ or $a = -1$ depending on whether the manifold is anti-Fano or Fano respectively.

$$\Delta_{\theta_i} u_i \geq -C + \bar{C} u_i^{n/(n-1)} + a \sum_{i \neq j} t_j e^{-\lambda \phi_j} \Delta_{\theta_i} \phi_j$$

(2.5)

$$= -C + \bar{C} u_i^{n/(n-1)} + a \sum_{i \neq j} e^{-\lambda \phi_j} t_j \frac{\theta_j^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \phi_j}{\theta_i^n}.$$ 

At this point we analyse the two cases $a = \pm 1$ separately.

(1) $a = 1$. In this case we may continue inequality 2.5 further as follows.

$$\Delta_{\theta_i} u_i \geq -C + \bar{C} u_i^{n/(n-1)} - \sum_{i \neq j} t_j e^{-\lambda \phi_j} \frac{\theta_j^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \phi_j}{\theta_i^n}.$$ 

Therefore by the maximum principle $u_i \leq C \forall 1 \leq i \leq n$.

(2) In this case we have the following consequence of inequality 2.5.

$$\Delta_{\theta_i} u_i \geq -C + \bar{C} u_i^{n/(n-1)} - C \sum_{i \neq j} t_j u_j.$$ 

Let max $X_i = M_i$. By the maximum principle and inequality 2.6 we see that for every $i$ we have the following inequality.

$$C \left(1 + \sum t_j M_j \right) \geq M_i^{n/(n-1)}.$$ 

Summing 2.7 over all $i$ and using Young’s inequality $a \leq e a^{n/(n-1)} + C(n, \epsilon)$ we get (upon choosing a small enough $\epsilon$),

$$C \left(1 + \epsilon \sum M_j^{n/(n-1)} + \sum C(n, \epsilon) \right) \geq \sum M_i^{n/(n-1)}$$

$$\Rightarrow M_i \leq C \forall 1 \leq i \leq n.$$ 

(2.8)

Finally, we need a $C^{2,\alpha}$ estimate in order to complete the proof of theorem 1.2. Indeed, theorem 1.1 of [11] implies the desired $C^{2,\alpha}$ estimate provided $\|\phi_i\|_{C^1} \leq C$. The latter inequality is true because of the Laplacian bound and $W^{2,p}$ elliptic regularity. We also note that standard elliptic theory (Schauder estimates) and bootstrapping imply that $\|\phi_i\|_{C^k} \leq C$ for any $k$.

3. Uniqueness in the anti-Fano case and openness along the continuity path

The uniqueness part of theorem 1.1 was proven in [8] but we prove it again for the convenience of the reader.
Proposition 3.1. Let $X$ be an anti-Fano manifold. If a solution to the coupled Kähler-Einstein equations exists, then it is unique.

Proof. If $\phi_i' = \phi_i + u^{(i)}$ is another solution of 1.2 then upon subtraction we get

$$L^a_{\bar{b}} u^{(j)} = e^\sum \phi_i (e^\sum u^{(i)} - 1),$$

where $L^a_{\bar{b}} u^{(j)} = (\frac{(\theta + \sqrt{-1} d\bar{\phi}_j + \sqrt{-1} d\phi_i)^{\alpha}}{\omega^\alpha})^n$. Note that $L^a_{\bar{b}}$ is a positive-definite matrix. Multiplying 3.1 by $u^{(j)}$, integrating-by-parts, and summing over $j$ we see that

$$\sum_j u^{(j)} e^\sum \phi_i (e^\sum u^{(i)} - 1) \leq 0.$$

This means that $\sum u^{(j)} = 0$ and $\partial u^{(j)} = \bar{\partial} u^{(j)} = 0$. Therefore $u^{(j)} = 0 \forall j$. □

Now we proceed to prove openness, i.e., theorem 1.3.

Proof of theorem 1.3: Suppose we know that $\omega_i \in [\theta_i]$ solve the system 1.5 for $t$. Then we need to prove that for $t + \delta$ where $\delta$ is in a small open interval, the system can still be solved. We shall in fact consider $t_i$ to be potentially different for different $i$ until the very end of this proof. This is because for the anti-Fano case, one can prove a slightly more general result than the one stated in theorem 1.3. To this end define the following Banach manifolds.

**Definition.** Let $B^k_1$ be the open subset of $C^{4,\alpha}$ functions $\psi_i$ satisfying

$$\int_X \psi_i \omega^\alpha_i = 0$$

and

$$\omega_i + \sqrt{-1} \partial \bar{\partial} \psi_i > 0.$$

Let $B_2$ be the subspace of $C^{3,\alpha}$ real (1,1)-forms $\eta$ of the form

$$\eta = \sqrt{-1} \partial \bar{\partial} f$$

where $f$ is a $C^{2,\alpha}$ function satisfying

$$\int_X f \omega^0_0 = 0.$$

Notice that we have the map $T : U = \prod_{i=1}^k (B^k_1 \times [0, 1]) \to V = B^k_2$ given by

$$T(\psi_1, t_1, \psi_2, t_2, \ldots) = \left( \text{Ric}(\omega_1\psi_1) + a \left( \sum t_i \omega_i \psi_i + \sum (1 - t_i) \theta_i \right), \ldots \right),$$

where $a = \pm 1$ depending on the sign of $-c_1(X)$. Suppose we take a point $p = (0, t_1, 0, t_2, \ldots)$ such that $T(p) = 0$. The implicit function theorem states that if $DT_p(v_1, 0, v_2, 0, \ldots)$ is an isomorphism from $TU$ to $TV$, then $\psi_i$ can be
locally solved for in terms of $t_j$ and therefore the set of $t_j$ for which $T = 0$ is open. The derivative $DT_p$ is

$$DT_p(v_1, 0, v_2, 0, \ldots) = (-\sqrt{-1}\partial\bar{\partial}\Delta_\omega v_1 + a \sum t_i \sqrt{-1}\partial\bar{\partial}v_i, \ldots).$$

For it to be surjective we need to solve

$$(-\sqrt{-1}\partial\bar{\partial}\Delta_\omega v_1 + a \sum t_i \sqrt{-1}\partial\bar{\partial}v_i, \ldots) = (\sqrt{-1}\partial\bar{\partial}f_1, \sqrt{-1}\partial\bar{\partial}f_2, \ldots)$$

$$\Rightarrow L(v_1, v_2, \ldots) = (-\Delta_\omega v_1 + a \sum t_i v_i, \ldots) = (f_1, f_2, \ldots).$$

By the Fredholm alternative we simply need to prove that the kernel of $L$ is trivial. The kernel consists of functions such that

$$\Delta_\omega v_1 = a \sum t_i v_i$$
$$\Delta_\omega v_2 = a \sum t_i v_i$$
$$\vdots$$

(3.5) Note that at $t_i = 0 \forall i$ we see that the kernel is obviously trivial and thus openness holds for small $t_i$. Therefore we may assume without loss of generality that $t_i > 0 \forall i$. We observe that the normalised volume forms $\frac{\omega^n}{\int \omega^n}$ are all equal (to some form $dvol$) because the Ricci curvatures of $\omega_i$ are equal. Multiplying the $j$th equation of (3.5) by $t_i v_i dvol$ and integrating the left-hand side by parts

$$- \int_X t_i \langle \nabla_j v_j, \nabla_i v_i \rangle dvol = a \int_X t_i \sum_k t_k v_k dvol.$$

Taking $i = j$ and summing over all $j$ we get

$$- \int_X \sum_j t_j |\nabla_j v_j|^2 dvol = a \int_X \left( \sum_k t_k v_k \right)^2 dvol.$$

(3.7) There are two cases to consider.

(1) $X$ is anti-Fano, i.e. $a = 1$ : Equation 3.7 implies that $\sum t_i v_i = 0$. Therefore by equations 3.5 all the $v_i$ are constants and in fact equal to 0 (because $\int v_i \theta_i^n = 0$).

(2) $X$ is Fano, i.e., $a = -1$ : A Weitzenböck identity (see page 65 of [10] for instance) that

$$\int_X (\Delta_\omega v_i)^2 dvol \geq \int_X \text{Ric}(\omega_i)(\partial v_i, \bar{\partial}v_i) dvol$$

$$\geq \int_X \sum_j t_j |\nabla_j v_i|^2 dvol.$$

(3.8) Assume without loss of generality that none of the $v_i$ are constant. Indeed, if let’s say $v_1$ is a constant, then $\Delta_1 v_1 = \Delta v_1 = 0$ which by
the maximum principle means that all the \( v_i \) are constant and in fact 0 by normalisation. Note that 3.6 implies that

\[
\int_X \langle \nabla_j v_j, \nabla_i v_i \rangle d\text{vol} = \int_X |\nabla_i v_i|^2 d\text{vol}.
\]

Choose normal coordinates for \( \omega_i \) at a point \( p \). Further, assume that \( \omega_j \) is diagonal at \( p \) with eigenvalues \( \lambda_\mu \). Writing the integrand of the left hand side of 3.9 at \( p \) in the said coordinates we get the following.

\[
\langle \nabla_j v_j, \nabla_i v_i \rangle (p) = \sum_\mu \frac{\partial_\mu v_j \partial_\mu v_i}{\lambda_\mu}
\leq \sqrt{\sum_\mu |\partial_\mu v_j|^2} \frac{1}{\lambda_\mu^2} \sqrt{\sum_\mu |\partial_\mu v_i|^2}
= |\nabla_j v_j||\nabla_i v_i|.
\]

Thus using 3.9, 3.10, and the Cauchy-Schwarz inequality we get

\[
\|\nabla_j v_j\| \|\nabla_i v_i\| \geq \|\nabla_i v_i\|^2
\Rightarrow \|\nabla_j v_j\| \geq \|\nabla_i v_i\|
\]

At this point we put \( t_i = t_j = t \) in 3.8, summing over \( i \) and \( j \), and using 3.5 and 3.7 we get,

\[
\sum_{i,j} \int_X \left( |\nabla_j v_j|^2 + |\nabla_i v_i|^2 - |\nabla_j v_j|^2 - |\nabla_i v_i|^2 \right) d\text{vol} \geq 0
\]

Equation 3.12 in conjunction with 3.11 implied that equality holds in all the inequalities above. Therefore all the \( v_i \) are constants and in fact 0 by normalisation. Note that this may not be true for \( t = 1 \) because equality holding in the inequalities would merely mean that \( \nabla_j v_j \) are holomorphic vector fields proportional to each other.

\[\square\]

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