Entangled state is the cornerstone of quantum information theory that has many successful applications in quantum information processing, such as the revolutionary one-way quantum computer, quantum cryptography, dense coding, teleportation, communication protocols and computation. Consequently, the ability to generate and control quantum entangled states has become a far-reaching goal in experimental manipulation as well as theoretical investigation in recent years. In fact, a great number of experiments have been devoted to investigating the production of entangled states of photons (including the hyperentangled photon pairs) via the process of spontaneous parametric down-conversion in a nonlinear optical crystal, particularly for use in the tests of Bell inequalities.

A fundamental notion in quantum computation (QC) is universality: a set of quantum logic gates (i.e., unitary matrices) is said to be "universal for QC" if any unitary matrix can be approximated to arbitrary accuracy by a quantum circuit involving only those gates. For example, an arbitrary $U(2)$ matrix can be obtained by combining the Hadamard gate together with the phase gate. When such a $U(2)$ matrix is prepared, an arbitrary state for a single qubit $\psi = \cos \frac{\theta}{2}|0\rangle + \sin \frac{\theta}{2}e^{i\phi}|1\rangle$ can be immediately generated by acting the $U(2)$ matrix on the initial state $|0\rangle$. Since entangled states are important for quantum information processing, it gives rise to a natural question whether an arbitrary pure entangled state of two qubits (such as the maximally entangled state) can be generated via a universal quantum computation protocol and entanglement swapping.

The Yang–Baxter equation, in principle, can be tested in computation and entanglement swapping. We shall study the problem from the viewpoint of quantum optics and Yang–Baxter equation is given by

$$\hat{R}_i(x)\hat{R}_{i+1}(xy)\hat{R}_i(y) = \hat{R}_{i+1}(y)\hat{R}_i(xy)\hat{R}_{i+1}(x).$$

Here the notation $\hat{R}_i(x) \equiv \hat{R}_{i,i+1}(x)$ is used, $\hat{R}_{i,i+1}(x)$ implies $I_1 \otimes I_2 \otimes I_3 \cdots \otimes I_{i-1} \otimes \hat{R}_i(x) \otimes \cdots \otimes I_n$, $I_j$ represents the unit matrix of the $j$-th particle, and $x = e^{i\theta}$ is a parameter related to the degree of entanglement. Let the unitary Yang–Baxter $\hat{R}$-matrix for two qubits be the form

$$\hat{R}_i(x) = F(x)[I_i + G(x)M_i],$$

where $F(x)$ and $G(x)$ are some functions needed to determine later on, $I_i \equiv I_1 \otimes \cdots \otimes I_{i-1}$, and the Hermitian matrices $M_i$’s (i.e., $M_i = M_i^\dagger$) satisfy the Hecke algebraic relations: $(M_iM_{i+1}M_i - M_{i+1}M_iM_i) + g(M_i - M_{i+1}) = 0$, $M_i^2 = \alpha M_i + \beta I_i$, with $\alpha = d - 2$ and $\beta = g = d - 1$. Substituting Eq. (2) into Eq. (1), one has $G(x) + G(y) + \alpha G(x)G(y) = [1 + gG(x)G(y)] G(xy)$. The unitary condition $\hat{R}_i(x) = \hat{R}_i^{-1}(x) = \hat{R}_i(x^{-1})$ yields $G(x) + G(x^{-1}) + \alpha G(x)G(x^{-1}) = 0$, $F(x)F(x^{-1})[1 + \beta G(x)G(x^{-1})] = 0$. In addition, the initial condition $\hat{R}_1(x) = I_1$ leads to $G(x = 1) = 0$, $F(x = 1) = 1$. As a
result, one has
\[ G(x) = -\frac{x - x^{-1}}{(d - 1)x + x^{-1}}, \quad F(x) = \frac{(d - 1)x + x^{-1}}{d}. \]
In this work, the \(d^2 \times d^2\) matrix \(M\) is realized as
\[ M = \sum_{r=1}^{d-1} P_r \otimes P_r = \sum_{i,j=0}^{d-1} \sum_{r=1}^{d-1} |ij\rangle \langle i + r, j + r|, \quad (3) \]
where \(i + r = \text{Mod}[i + r, d]\), and
\[ P_r = \sum_{r=1}^{d-1} |i\rangle \langle i + r|, \quad r = 0, 1, \ldots, d - 1, \quad (4) \]
are the circulation matrices that transform the basis \(\{|r\rangle, |r + 1\rangle, \ldots, |2\rangle, |1\rangle, |0\rangle\}\) of each qudit to the basis \(\{|0\rangle, |1\rangle, |2\rangle, \ldots, |d - 3\rangle, |d - 2\rangle, |d - 1\rangle\}\) of the other qudit. The operator \(P_r\) can be realized through the multiplication of permutation operators \(P_{k,k+1} = (I - |k\rangle \langle k + 1|)(k + 1) + |k\rangle \langle k + 1| + |k + 1\rangle \langle k|)\) of a single qudit, for example, \(P_1 = P_{d-2,d-1}P_{d-3,d-2}\). Moreover, the traceless matrices \(P_r\)'s satisfy the following interesting relations:
\[ P_r = (P_1)^r, \quad P_m P_n = P_n P_m = P_{\text{Mod}[m+n,d]}. \]
Let \(P_0 = \sum_{i=0}^{d-1} |i\rangle \langle i|\) denote the \(d \times d\) unit matrix, we eventually arrive at the unitary Yang–Baxter matrix for two qudits as
\[ \tilde{R}_i(x) = \frac{1}{d} \left\{ \left( (d - 1)x + x^{-1} \right) P_0 \otimes P_0 - \left( x - x^{-1} \right) \sum_{r=1}^{d-1} P_r \otimes P_r \right\}, \quad (5) \]
which has not been reported in the literature. Our main result of connecting unitary Yang–Baxter matrices with entangled states of two qudits is the following Theorem.

**Theorem:** All pure two-qudit entangled states \(|\psi\rangle_{2-\text{qudit}}\) can be generated from an initial separable state \(|00\rangle\) via a universal Yang–Baxter matrix \(\tilde{R}(x)\) if one is assisted by local unitary transformations \(U_A \otimes U_B\) and \(V_A \otimes V_B\), namely,
\[ |\psi\rangle_{2-\text{qudit}} = [V_A \otimes V_B] \tilde{R}(x) [U_A \otimes U_B] |00\rangle. \quad (6) \]
Here the local unitary transformation \(V_A \otimes V_B\) is introduced in order to transform a two-qudit state into its Schmidt-form.

**Proof.** We would like to provide analytical proof for the case with \(d = 2\) and numerical proof for the cases with \(d = 3\) and \(4\).

i) For \(d = 2\), in this case \(P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). When one acts \(\tilde{R}(x)\) directly on the separable state \(|00\rangle\), he then generates the following family of states
\[ |\psi\rangle_{YB} = \frac{1}{2} \left[ (x + x^{-1})|00\rangle - (x - x^{-1})|11\rangle \right]. \quad (7) \]

In Ref. [17], the generalized concurrence (or the degree of entanglement [18]) for two qudits is given by
\[ C = \sqrt{\frac{d}{d - 1} \left( 1 - I_1 \right)}, \quad (8) \]
where \(I_1 = \text{Tr}[\rho_A^2] = \text{Tr}[\rho_B^2] = |\kappa_0|^4 + |\kappa_1|^4 + \cdots + |\kappa_{d-1}|^4\), \(\rho_A\) and \(\rho_B\) are the reduced density matrices for the subsystems. For \(d = 2\), one easily has \(C = 2|\kappa_0\kappa_1|\). Obviously, \(|\psi\rangle_{YB}\) has already been in the form of \(|\psi\rangle_{2-\text{qubit}} = \kappa_0|00\rangle + \kappa_1|11\rangle\), with \(\kappa_0 = (x + x^{-1})/2 = \cos \theta\) and \(\kappa_1 = -(x - x^{-1})/2 = -i \sin \theta\). The degree of entanglement for the state \(|\psi\rangle_{YB}\) equals to \(C = 2|\kappa_0\kappa_1| = |\sin(2\theta)|\), which may range from 0 to 1. Thus, for the case of two qubits, all pure states can be generated from \(|00\rangle\) directly via a universal Yang–Baxter matrix \(\tilde{R}(x)\).

ii) For \(d = 3\), in this case
\[ P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]
When matrix \(\tilde{R}(x)\) is acted directly on \(|00\rangle\), it yields the following family of states
\[ |\psi\rangle_{YB} = \frac{1}{3} \left[ (2x + x^{-1})|00\rangle - (x - x^{-1})(|11\rangle + |22\rangle) \right], \quad (9) \]
whose generalized concurrence reads
\[ C = \sqrt{\frac{3}{2} \left[ 1 - \frac{1}{81} |2x + x^{-1}|^4 - \frac{2}{81} |x - x^{-1}|^4 \right]}. \]
When \(|2x + x^{-1}| = |x - x^{-1}|\), namely \(x = e^{i\pi/3}\), the state \(|\psi\rangle_{YB}\) becomes the maximally entangled state (here we would like to call it as the GHZ state) of two qutrits as \(|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle + |22\rangle)\). In general, if one acts the unitary Yang–Baxter matrix \(\tilde{R}(x) = e^{ix/3}\) on the basis \(|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle\), he will generate nine complete and orthogonal maximally entangled states of two qutrits.

It is easy to check that the generalized concurrence \(C\) ranges from 0 to 1 when the parameter \(\theta\) runs from 0 to \(\pi\). However, this fact does not mean that \(|\psi\rangle_{YB}\) is an arbitrary pure state of two qutrits, because \(|\psi\rangle_{2-\text{qudit}}\) has at least two free parameters while \(|\psi\rangle_{YB}\) contains only one. Actually, the entanglement property of a two-qutrit system is completely characterized by two entanglement invariants \(I_1 = \text{Tr}[\rho_A^2] = \text{Tr}[\rho_B^2] = |\kappa_0|^6 + |\kappa_1|^6 + |\kappa_2|^6\) and \(I_2 = \text{Tr}[\rho_A^3] = \text{Tr}[\rho_B^3] = |\kappa_0|^8 + |\kappa_1|^8 + |\kappa_2|^8\), or equivalently,
\[ I'_1 = \frac{3}{2} (1 - I_1), \quad I'_2 = \frac{9}{8} (1 - I_2). \quad (10) \]
where the normalized entanglement invariants \( I'_1, I'_2 \in [0,1] \).

In Fig. 1, we have plots points \((I'_1, I'_2)\) for the two-qutrit state \( |\psi\rangle_{2\text{-qutrit}} = \kappa_0|00\rangle + \kappa_1|11\rangle + \kappa_2|22\rangle \) by randomly taking \( 10^7 \) values of \( \kappa_0, \kappa_1, \) and \( \kappa_2 \), see the red region of figure, whose contour lines form a curved triangle \( \Delta OBG \). One may observe that for a fixed value of \( I'_1 \), there are different values for \( I'_2 \), therefore \( I'_1 \) is not enough for characterizing the entanglement property of two qutrits. In the \( I'_1 - I'_2 \) coordinate, the separable states, such as \( |00\rangle \), locate at the point \( O = (0,0) \). The maximally entangled states (or say the GHZ states), such as \( |\psi\rangle_{GHZ} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \), locate at the point \( G = (1,1) \). And the entangled states, such as \( \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \), locate at the point \( B = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \). The contour line OB corresponds to the states \( |\psi\rangle_{2\text{-qutrit}} = \cos \xi|00\rangle + \sin \xi|11\rangle \), the point \((I'_1, I'_2)\) runs from O to B when \( \xi \) runs from 0 to \( \pi/2 \): The contour lines OG and GB correspond to the states \( |\psi\rangle_{2\text{-qutrit}} = \cos \xi|00\rangle + \sin \xi|11\rangle + \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \). The point \((I'_1, I'_2)\) runs from O to G when \( \xi \) runs from 0 to \( \pi/3 \), and from G to B when \( \xi \) runs from \( \pi/3 \) to \( \pi/2 \).

The state \( |\psi\rangle_{YB} \) is one part of the states \( |\psi\rangle_{2\text{-qutrit}} \), when \( \theta \) runs from 0 to \( \pi/2 \), the point \((I'_1, I'_2)\) runs from O to G, then runs along the line GB towards to point B and finally stops at a point \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\), which corresponds to the states \( |\psi\rangle_{YB} = \frac{1}{\sqrt{3}}(|00\rangle - |21\rangle - |22\rangle) \). Namely, when one acts \( \hat{R}(x) \) directly on the state \( |0\rangle_A \otimes (\cos \varphi|0\rangle_B + \sin \varphi|1\rangle_B) \), he cannot get all pure state of two qutrits. However, numerical computation shows that if one acts \( \hat{R}(x) \) on the state \( |0\rangle_A \otimes (\cos \varphi|0\rangle_B + \sin \varphi|1\rangle_B) \), he can indeed obtain all pure two-qutrit states: by randomly taking \( 10^7 \) values of \( \theta \) and \( \varphi \), one may plot points \((I'_1, I'_2)\) for the states \( |\psi\rangle_{YB} = \hat{R}(x) \left( |0\rangle_A \otimes (\cos \varphi|0\rangle_B + \sin \varphi|1\rangle_B) \right) \), which perfectly recover all the red region of figure.

If one requires the state \( |\psi\rangle_{YB} \) to be the maximally entangled state (or the GHZ state), he must set \( |(d-1)x + x^{-1}| = |x - x^{-1}| \), namely,

\[
\cos(2\theta) = 1 - \frac{d}{2}.
\]

For \( d = 2, d = 3, \) and \( d = 4, \) one has \( \theta = \pi/4, \pi/3, \) and \( \pi/2 \), respectively. However, the above condition is not valid for \( d \geq 5 \), because one will have \( |\cos(2\theta)| > 1 \).
1 when $d \geq 5$. This fact implies that the maximally entangled two-qudit states can be generated when we act $\hat{R}(x)$ directly on the separable state $|00\rangle$ for $d \leq 4$.

Similarly, numerical results show that all pure two-qudit entangled states can be generated in the following way: $|\psi_{2-\text{qudit}}\rangle = [V_A \otimes V_B] |\hat{R}(x)\rangle |U_A \otimes U_B\rangle |00\rangle = [V_A \otimes V_B] |\hat{R}(x)\rangle |\Phi_A \otimes |\Phi_B\rangle$, where $|\Phi_A\rangle = U_A |0\rangle_A = |0\rangle_A$ and $|\Phi_B\rangle = U_B |0\rangle_B = \cos \varphi_1 |0\rangle_B + \sin \varphi_1 \cos \varphi_2 |1\rangle_B + \cdots + \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{d-2}(-d - 2)_{d-2}$. In particular, for $d = 4$, one has $|\Phi_B\rangle = \cos \varphi_1 |0\rangle_B + \sin \varphi_1 \cos \varphi_2 |1\rangle_B + \sin \varphi_1 \sin \varphi_2 |2\rangle_B$. Numerical proof of the Theorem for $d = 4$ is provided in Fig. 2. In Fig. 2, we have plots points $(I'_1, I'_2, I'_3)$ for the two-qudit state $|\psi_{2-\text{qudit}}\rangle = \kappa_0 |00\rangle + \kappa_1 |11\rangle + \kappa_2 |22\rangle + \cdots + \kappa_3 |33\rangle$ by randomly taking $10^7$ values of $\kappa_j$'s, see the blue region of figure. By randomly taking $10^7$ values of $\vartheta$, $\varphi_1$ and $\varphi_2$, one may also plot points $(I'_1, I'_2, I'_3)$ for the states

$$|\psi\rangle_{Y_B} = \hat{R}(x) |\Phi_A \otimes |\Phi_B\rangle,$$  

(11)

which perfectly recover all the blue region of Fig. 2.

In conclusion, we have shown that all pure entangled states of two qudits can be generated from an initial separable state $|00\rangle$ via a universal Yang–Baxter matrix if one is assisted by local unitary transformations. Eventually, we would like to point out that the spirit of a unitary matrix assisted by local unitary transformations as shown in Eq. (8) or Eq. (11) coincides with the spirit of entangling power, which is a quantitative measure how much entanglement capability a given unitary operator has in the context of quantum information. The concept of entangling power is first introduced in Refs. [14, 20], which is defined as

$$e_p(\mathcal{U}) := E(\mathcal{U} | \Phi_A \otimes |\Phi_B\rangle).$$  

(12)

where $E(|\Psi\rangle) = 1 - \text{Tr}[\rho_A^2]$ the overbar stands for the average over all the product states, and it can be simplified as $e_p(\mathcal{U}) = (d/d + 1)^2 [E(\mathcal{U}) + E(\mathcal{US}) - E(S)]$, with $S = \sum_{i,j=0}^{d-1} |ij\rangle\langle ij|$ is the permutation operator of two qudits. The entangling power has been useful for the study of quantum evolutions and Hamiltonians [19, 20, 21, 22, 23, 24, 25], and been also applied to some quantum chaotic systems [26, 27, 28, 29]. Actually, $E(|\Psi\rangle) = 1 - \text{Tr}[\rho_A^2]$ is the entanglement invariant $I'_1$ up to a normalized constant $d/(d - 1)$. Similarly, based on the entangling invariants of two qudits one may define a series of entangling powers as $e_p(\mathcal{U}) := E_j(\mathcal{U} | \Phi_A \otimes |\Phi_B\rangle)$ with $E_j = I'_j$, which we will investigate subsequently.

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