Anomalous dimensions at four loops in $\mathcal{N} = 6$ superconformal Chern-Simons theories

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Abstract

In arXiv:0908.2463 we computed the four-loop correction to a function depending on the ’t Hooft coupling(s) that appears in the magnon dispersion relation of the spin chains derived from single trace operators in $\mathcal{N} = 6$ superconformal Chern-Simons theories. In this paper we give detailed descriptions of this calculation and the computation of the four-loop wrapping corrections for a length four operator in the 20 of $SU(4)$, the $R$-symmetry group for these theories. Here, we give all relevant Feynman diagrams and loop integrals explicitly, and also demonstrate the cancellation of double poles in the logarithm of the renormalization constant.

Keywords: Anomalous dimensions; Integrability; Wrapping
1 Introduction and summary of results

The ABJM model is an $\mathcal{N} = 6$ superconformal Chern-Simons (CS) theory coupled to matter \[1\]. The CS theory has gauge group $U(N) \times U(N)$, where the levels of the first and second gauge group are respectively given by $k$ and $-k$. The matter multiplets transform in bifundamental representations of the two groups and in the fundamental or antifundamental representations of $SU(4)$, the $R$-symmetry group. This model also has a parity symmetry, where the spatial inversion is accompanied by an exchange of the two gauge groups, with a corresponding interchange of gauge representations for the matter fields.

The gravity dual for the ABJM model is $M$-theory on $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}_k$ \[1\], where the compact space $\text{S}^7/\mathbb{Z}_k$ is equivalent to $\text{CP}^3$ with a $U(1)$ fibration. In the limit of large $k$ the circle of the $U(1)$ shrinks to vanishing size and the $M$ theory reduces to type IIA string theory on $\text{AdS}_4 \times \text{CP}^3$.

This AdS/CFT correspondence for the ABJM model at large $k$ closely parallels the more familiar correspondence relating $SU(N) \mathcal{N}=4$ Super Yang-Mills (SYM) to type IIB string theory on $\text{AdS}_5 \times \text{S}^5$. In both cases, the gauge theories are superconformal, with underlying supergroups $\text{PSU}(2,2|4)$ and $\text{OSP}(6|4)$ for the $\mathcal{N}=4$ SYM and the $\mathcal{N}=6$ CS theories respectively. In particular, the spectra of the gauge theories should match the spectra of their string duals. On the gauge theory side the spectrum is given by the anomalous dimensions of the gauge invariant composite operators built from the elementary fields of the theory. In the planar limit, where $N \to \infty$ the respective 't Hooft coupling constant $\lambda = g_{\text{YM}}^2 N$ for $\mathcal{N}=4$ SYM theory and $\lambda = \frac{N}{k}$ for the ABJM model is kept fixed. The relevant gauge invariant operators in this limit involve a single trace of the elementary adjoint fields in the case of $\mathcal{N}=4$ SYM theory, or a trace of an alternating product of bifundamental matter fields in the ABJM case.

In both theories these composite operators mix under renormalization, and the anomalous dimensions are given as the eigenvalues of the respective dilatation operator, which is built from the pole part of the corresponding matrix-valued renormalization constant. This mixing of the composite operators can be mapped to a long range integrable Hamiltonian acting on a spin chain. This allows one to find the anomalous dimension eigenvalues by solving appropriate Bethe ansätze \[2–4\]. The spin chains have ground states that correspond to the chiral primary operators in their respective gauge theories. These are operators whose dimensions are protected by supersymmetry and thus have zero anomalous dimension. A convenient choice for a spin chain ground state in $\mathcal{N}=4$ is $\text{tr}(Z^L)$, where $Z$ is one of the complex scalar fields. In the ABJM model a convenient choice is $\text{tr}((Y^1Y_4^\dagger)^L)$, where $Y^1$ is one of the scalars making up the 4 of the $R$-symmetry group and $Y_4^\dagger$ is one of the scalars in the $\bar{4}$ \[5–7\]. This choice of a ground state breaks the supergroups to the subgroups $SU(2|2) \times SU(2|2)$ for $\mathcal{N}=4$ SYM and $SU(2|2)$ for the ABJM model. The other single trace operators are then constructed by introducing magnons that change the fields in the chain. The magnons themselves transform in short representations of the respective unbroken subgroups.

The presence of the $SU(2|2)$ structures imposes severe constraints on the magnon dispersion relations of the respective Bethe ansätze. As was shown in \[8\], these dispersion
relations must have the form

\[ E(p) = \sqrt{Q^2 + 4h^2(\lambda) \sin^2 \frac{p}{2} - Q}, \quad (1.1) \]

where \( p \) is the momentum of the magnon on the spin chain, \( h^2(\lambda) \) is a function depending on the 't Hooft coupling, and \( \lambda \). \( Q \) is the \( R \)-charge of the magnon. For \( \mathcal{N} = 4 \) SYM the charge for a fundamental magnon is \( Q = 1 \), while in ABJM theory the charge is \( Q = \frac{1}{2} \).

The 't Hooft coupling enters the asymptotic Bethe equations only through this same function \( h^2(\lambda) \) \[9,10\]. To the best of our knowledge, integrability makes no prediction for \( h^2(\lambda) \), but there exist some alternative analyses not based on integrability \[11,12\]. At weak coupling, it can be obtained order by order from perturbative calculations in the gauge theory, and the strong coupling expansion follows from the analysis of classical string states and their quantum corrections in the dual string theory. In \( \mathcal{N} = 4 \) SYM, all known results are consistent with \( h^2(\lambda) = \frac{\lambda}{(4\pi)^2} \). However, in the ABJM model it is known at the level of two-loop perturbation theory that \( h^2(\lambda) = \lambda^2 + \mathcal{O}(\lambda^4) \) \[6,13\], while at large coupling we know by taking the BMN limit of type IIA string theory on \( \text{AdS}_4 \times \mathbb{CP}^3 \) \[6,14\] and from one-loop string corrections \[15\] that \( h^2(\lambda) = \frac{1}{2} \ln \frac{\sqrt{3\sqrt{\lambda}} + \mathcal{O}(1)}{\sqrt{2\pi}} \). In fact, since the perturbative expansion is even in \( \lambda \), the function \( h^2(\lambda) \) should have a square root branch cut along the negative real axis.

The calculation we present here also considers the ABJ modification of the ABJM model \[16\], where the gauge group is generalized to \( U(M) \times U(N) \), but with the levels of the gauge group kept at \( k \) and \( -k \). In this case there are now two 't Hooft couplings defined as

\[ \lambda = \frac{M}{k}, \quad \hat{\lambda} = \frac{N}{k}, \quad (1.2) \]

but the superconformal group is still \( OSp(6|4) \). It was shown at the two-loop level in the scalar sector \[17\] and in the full \( OSp(6|4) \) sector \[18\] that \( \lambda^2 \) is replaced by \( \lambda \hat{\lambda} \), but otherwise the dilatation operator is the same, and is therefore still integrable\[\text{1}\]. If in the planar limit the integrability is to persist to higher loop orders for general values of \( \lambda \) and \( \hat{\lambda} \), then the only modification that can occur is in the function \( h^2 \). In general it will be a two parameter function, \( h^2(\lambda, \sigma) \), where we define \( \bar{\lambda} \) and \( \sigma \) as

\[ \bar{\lambda} = \sqrt{\lambda \hat{\lambda}}, \quad \sigma = \frac{\lambda - \hat{\lambda}}{\lambda}. \quad (1.3) \]

Because the numbers of colour for the two gauge groups are different, the ABJ theory is not invariant under parity. Since a parity transformation exchanges the fundamental and antifundamental representations of the gauge groups, it has the effect of reversing the order of the fields inside the trace. In other words, it acts as a parity inversion on the spin chain. This spin-chain parity switches the odd and the even sites, or equivalently exchanges \( \lambda \) with \( \hat{\lambda} \). Clearly, the two loop result is invariant under parity\[\text{2}\], but at higher loops we should allow for different dispersion relations \[11\] for magnons at odd and even spin chain sites, with each having the form

\[ E_{\text{odd}}(p) = \sqrt{Q^2 + 4h^2(\lambda, \sigma) \sin^2 \frac{p}{2} - Q}, \quad E_{\text{even}}(p) = E_{\text{odd}}(p) \big|_{\sigma \to -\sigma}. \quad (1.4) \]

\[\text{1}\]In the ABJM case, hints for a breakdown of integrability beyond the planar limit have explicitly been found in \[19\].

\[\text{2}\]A breakdown of parity invariance beyond the planar limit has been observed in \[20\].
The ABJM case is captured by the special value $\sigma = 0$, where the dispersion relations for odd and even sites automatically coincide.

Up to four loops, the expansion of $h^2(\bar{\lambda}, \sigma)$ is then assumed to have the following form

$$h^2(\bar{\lambda}, \sigma) = \bar{\lambda}^2 + \bar{\lambda}^4 h_4(\sigma).$$

The two loop contribution is independent of $\sigma$ and the dispersion relation is obviously the same for even and odd magnons. At four loops, terms which are odd under $\sigma \to -\sigma$ break parity and cause differences in the two dispersion relations. However, there could also be $\sigma$ dependent terms which preserve the parity.

In this paper, we present in detail our computation of $h^2(\bar{\lambda}, \sigma)$ to four-loop order. We already presented the final result in [21].

In section 2 we show how it can be extracted from the four-loop dilatation operator in the flavour $SU(2) \times SU(2)$ subsector. The dilatation operator itself is reconstructed from an explicit perturbative four-loop renormalization of the respective composite operators as is summarized in section 3. The calculation can be simplified by considering only those graphs which give rise to non-trivial permutations of the flavour degrees of freedom. Nevertheless, we have to find and then evaluate explicitly more than one hundred diagrams plus their reflections where necessary. In section 4 we present the classification of these diagrams, and their explicit evaluations. The dilatation operator and the result for $h^2(\bar{\lambda}, \sigma)$ is then extracted in section 5.

In section 6 we present our computation of the wrapping correction for length four operators in the $SU(2) \times SU(2)$ flavour sector. Our result matches prediction of Gromov, Kazakov and Vieira [23] which they obtained by means of the proposed $Y$-system [24–26] for the ABJM model. Finally, in section 7 we present our conclusions.

Several details about the calculation have been delegated to various appendices. In appendix A we list the underlying component Feynman rules, as well as present useful effective Feynman rules which allow us to resolve the complicated spacetime-tensor structure of the graphs involving gauge bosons or fermions. In appendix B we summarize the flavour permutations which appear in the various Feynman graphs. Some properties of the appearing permutation structures are briefly discussed in appendix C. The transformation properties of the Feynman graphs are discussed in appendix D. We then summarize in appendix E our two-loop computation of the scalar self-energy, which besides the known pole parts also yields the required finite parts. In appendix F an important sign is fixed by the two-loop renormalization of the permutation part of the six-scalar vertex. In appendix G as a consistency check for our four-loop computation, we explicitly demonstrate the cancellation of double poles in the logarithm of the renormalization constant. The $G$-functions and triangle relations for an evaluation of the underlying loop integrals are summarized respectively in appendices H and I. Finally, appendix J contains the lists of the underlying several hundred two- three- and four-loop integrals which we need for the computation and for cross checks.

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3Previous versions of this paper had an incorrect result for $h^2(\bar{\lambda}, \sigma)$ due to several sign errors arising from inconsistent Feynman rules. A reexamination of the result was triggered by inconsistencies with an ongoing computation of $h^2(\bar{\lambda}, \sigma)$ using the $\mathcal{N} = 2$ superspace formalism [22].
2 Extraction of $h^2(\tilde{\lambda}, \sigma)$

To compute the function $h^2(\tilde{\lambda}, \sigma)$ in a field theory calculation, it is only necessary to consider a single scalar magnon either placed on the odd or the even sites of the spin chain. Thus, we may restrict our study to the $SU(2) \times SU(2)$ sector of the full $OSp(6|4)$ group, where the magnons in the first $SU(2)$ live on the odd sites and those in the other $SU(2)$ live on the even sites. The two different types of magnons do not interact with each other until the six-loop level \[10\] so for our purposes we may treat them as noninteracting.

The dispersion relation for magnons at odd sites \[1.4\] yields the following expansion in a power series in $\tilde{\lambda}$

$$E_{\text{odd}}(p) = 2\tilde{\lambda}^2(1 - \cos p) + 2\tilde{\lambda}^4(h_4(\sigma) - 3 + (4 - h_4(\sigma)) \cos p - \cos 2p) + O(\tilde{\lambda}^6) \tag{2.1}$$

where we have set $Q = 1/2$ and we have reexpressed powers of $\sin \frac{p}{2}$ in terms of cosine. The exponentials which are contained in the cosine functions are powers of $e^{ip}$, which is the eigenvalue of the shift operator that moves the magnon over by two sites on the spin chain. Hence, at the four-loop level there is a maximal shifting of four sites, which does not depend on $h_4$. However, for shifts of two sites, there is an $h_4$ dependence.

Without interactions between odd and even site magnons, the energy $E_{\text{odd}}(p)$ is given as the eigenvalue of the dilatation operator, when acting on the (open) momentum / shift operator eigenstate

$$\psi_p = \sum_{k=0}^{L} e^{ipk} (Y^1_4 Y^1_4)^k Y^2_4 Y^1_4 (Y^1_4 Y^1_4)^L k-1$$

(2.2)

of a single magnon at odd sites. The momentum dependence in the expansion \[2.1\] is a result of acting with certain products of permutations on the above state, where a permutation exchanges the fields between two nearest neighbour either odd or even sites. Similar to the $\mathcal{N} = 4$ SYM case, we introduce the permutation structures

$$\{a_1, a_2, \ldots, a_m\} = \sum_{i=1}^{L} P_{2i+a_1} 2i+a_1+2 P_{2i+a_2} 2i+a_2+2 \ldots P_{2i+a_m} 2i+a_m+2 \tag{2.3}$$

where we identify $L + i \simeq i$ when we act on a cyclic state of length $L$. Some details about these structures and the permutation basis at four loops can be found in appendix \[C\] To four loops, the dilatation operator expands as

$$D = L + \tilde{\lambda}^2 D_2 + \tilde{\lambda}^4 D_4(\sigma) + O(\tilde{\lambda}^6) \tag{2.4}$$

At each order, the dilatation operator components decompose into a direct sum as

$$D_k = D_{k,\text{odd}} + D_{k,\text{even}} + D_{k,\text{mixed}} \tag{2.5}$$

where the individual parts act non-trivially respectively on odd and even sites only or mix odd and even sites. The permutation structures \[2.3\] which appear in the three parts therefore either carry only odd or even argument or both types of arguments. The
fact that even and odd site magnons are non-interacting to four-loop order immediately requires the absence of a mixing term $D_{4,\text{mixed}}$ involving the structures $\{1, 2\}$ and $\{2, 3\}$ of the basis (C.2). We make an ansatz for $D_{4,\text{odd}}$ and apply it to the state (2.2). The permutation structures thereby produce the following phase factors

$$
\{\} \rightarrow 1, \quad \{1\} \rightarrow e^{ip} + e^{-ip}, \quad \{1, 3\} \rightarrow e^{2ip} + 2e^{-ip}, \quad \{3, 1\} \rightarrow e^{-2ip} + 2e^{ip}. \tag{2.6}
$$

This yields the eigenvalue as a function of the momentum $p$. Its comparison with the expansion of the single-magnon energy (2.1) then fixes the dilatation operator to

$$
D_{2,\text{even}} = \{\} - \{1\},
$$

$$
D_{2,\text{odd}} = \{\} - \{2\},
$$

$$
D_{4,\text{odd}}(\sigma) = (h_4(\sigma) - 4)\{\} + (6 - h_4(\sigma))\{1\} - \{1, 3\} - \{3, 1\},
$$

$$
D_{4,\text{even}}(\sigma) = (h_4(-\sigma) - 4)\{\} + (6 - h_4(-\sigma))\{2\} - \{2, 4\} - \{4, 2\}. \tag{2.7}
$$

We have added the part acting on even sites with the coefficients transformed by $\sigma \rightarrow -\sigma$ as required by (1.4). The above expression yields zero energy for the ground state, since the sum of the coefficients of the different permutation structures sum up to zero. The above result allows us to find the function $h_4(\sigma)$ by a direct perturbative computation of the dilatation operator. It is thereby sufficient to focus on $D_{4,\text{odd}}$, and to compute only the contributions which involve non-trivial permutations. The coefficient of the identity part follows from the condition, that the ground state energy is zero. Computing the coefficients of the maximum shuffling term and the absence of a part which mixes odd and even sites serve as some checks. These terms have also been computed in [27] and the maximum shuffling terms furthermore to six loops in [28, 29].

3 The dilatation operator from Feynman graphs

The dilatation operator of the theory can be obtained from a perturbative computation of the one-point functions of the composite operators $O_a$. The appearing divergences due to quantum corrections require a renormalization of the composite operators as

$$
O_{a,\text{ren}} = Z_a^b O_{b,\text{bare}}, \quad Z = 1 + \bar{\lambda}^2 Z_2 + \bar{\lambda}^4 Z_4 + \ldots, \tag{3.1}
$$

where the matrix $Z$ cancels the appearing divergences and in general leads to mixing between the different composite operators. It is given by the negative of the sum of the divergences of the underlying Feynman graphs. In the formalism of dimensional reduction the spacetime dimension is reduced to

$$
D = 3 - 2\varepsilon. \tag{3.2}
$$

The divergences then appear as inverse powers of $\varepsilon$. The coupling constant $\bar{\lambda}$ is accompanied by the 't Hooft mass $\mu$ in the combination $\bar{\lambda}\mu^{2\varepsilon}$ to render the mass dimension of the loop integrals constant. The dilatation operator is then extracted as

$$
D = \mu \frac{d}{d\mu} \ln Z(\bar{\lambda}\mu^{2\varepsilon}, \varepsilon). \tag{3.3}
$$
The logarithm thereby guarantees that all higher order poles in $\varepsilon$ cancel out, such that $\ln Z$ only contains simple $\frac{1}{\varepsilon}$ poles. Inserting (3.1), the expansion reads

$$\ln Z = \bar{\lambda}^2 Z_2 + \bar{\lambda}^4 \left( Z_4 - \frac{1}{2} Z_2^2 \right) + O(\bar{\lambda}^6).$$

(3.4)

As a check for our calculation, in appendix G we demonstrate explicitly, that the double poles in the above four-loop contribution cancel out. The dilatation operator can thus also be obtained as

$$D = \lim_{\varepsilon \to 0} \left[ 2\varepsilon \frac{d}{d\bar{\lambda}} \ln Z(\bar{\lambda}, \varepsilon) \right].$$

(3.5)

Effectively, the above definition extracts the coefficient of the $\frac{1}{\varepsilon}$ pole at a given loop order $L$ and multiplies it by a factor $2L$.

4 Four-loop diagrams

In this section we first classify and then compute all logarithmically divergent simply connected four-loop Feynman diagrams with external scalar legs, which contribute to genuine flavour permutations. We neglect all contributions to the identity and trace operator in flavour space. These omissions are indicated by an arrow. We also stress that equalities between four-loop results hold up to irrelevant finite contributions.

We apply dimensional reduction, i.e. the algebra of the spacetime tensors is performed in exactly three dimensions. After that, the integrand of a given Feynman diagram only contains scalar products of momenta, and the integral can be dimensionally regularized in $D = 3 - 2\varepsilon$.

It is known [30] that this procedure is gauge invariant and hence preserves the Slavnov-Taylor identities at least to two loops. Thereby, in pure CS theory the statement holds for the pole part as well as for the finite contributions, while in matter coupled CS theory only the pole part was considered. Instead in dimensional regularization, where the tensor algebra is also continued to $D = 2 - 3\varepsilon$, it is already observed for the divergent parts at two-loops in pure CS theory, that the Slavnov-Taylor identities are not fulfilled. It has not explicitly been checked whether dimensional reduction in CS theories is gauge invariant to four-loop order, but a breakdown appears very unlikely.

4.1 Classification of Feynman diagrams

We have to find all simply-connected subdiagrams which, when attached to the composite operator, contribute to the renormalization at four-loops. Such diagrams at $L$ loops with $V_i$ elementary vertices with $i$ legs fulfill

$$2L = \sum_i (i - 2)V_i.$$  

(4.1)

At $L = 4$ loops, we obtain the following combinations of vertices

| type | multiplicity |
|------|--------------|
| $V_6$ | 2 1 1 1 0 0 0 0 0 |
| $V_4$ | 0 2 1 0 4 3 2 1 0 |
| $V_3$ | 0 0 2 4 0 2 4 6 8 |

(4.2)
We only want to compute the diagrams with scalar external lines. They should also contribute to genuine flavour permutations. In this case the only relevant classes \((V_6, V_4, V_3)\) are given by \((2, 0, 0), (1, 2, 0), (1, 1, 2), (1, 0, 4), (0, 4, 0), (0, 3, 2)\). The members of each class in terms of the combinations of the vertices \((A.5)\) of the theory are given in table 1. All other combinations either only contribute to the identity or subtraces in flavour space, or they contain tadpoles.

The diagrams which can be built up from the combinations in table 1 will be ordered according to their flavour structure, i.e. we will classify them with respect to their number of six-scalar vertices and of quartic vertices involving fermions.

### 4.2 Diagrams involving two six-scalar vertices

In the following we evaluate all Feynman diagrams with two six-scalar vertices, which contribute to the coefficient of genuine flavour permutations in the dilatation operator. They build up the class \((2, 0, 0)\) in table 1.

Using the flavour permutation structure \((B.2)\) of the six-scalar vertex, the diagrams

| \((V_6, V_4, V_2)\) | \(V_{(Y^+)^3}\) | \(V_{Y^+\psi^+Y^+\psi^+}\) | \(V_{Y^+Y^+\psi^+\psi^+}\) | \(V_{AY^+AY^+}\) | \(V_{AAAY^+Y}\) | \(V_{A^+\psi^+}\) | \(V_{A^+Y^+Y}\) | \(V_{A^+}\) |
|--------------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| \((2, 0, 0)\)            | \begin{align*} a \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} b \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} |
| \((1, 2, 0)\)            | \begin{align*} a \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} b \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} |
| \((1, 1, 2)\)            | \begin{align*} a \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} b \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} |
| \((1, 0, 4)\)            | \begin{align*} a \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 4 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} b \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 3 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 1 \end{align*} |
| \((0, 4, 0)\)            | \begin{align*} a \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} b \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 4 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} |
| \((0, 3, 2)\)            | \begin{align*} a \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 3 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} b \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 3 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 1 \end{align*} & \begin{align*} 0 \end{align*} |
|                          | \begin{align*} c \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 3 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 0 \end{align*} & \begin{align*} 2 \end{align*} & \begin{align*} 0 \end{align*} |

Table 1: Combination of vertices which lead to planar four-loop diagrams involving permutations. The gauge-ghost vertices are not presented, since it is clear that they contribute whenever an gauge boson loop with cubic gauge vertices is present.
which lead to non-trivial flavour permutations are given by

\[ S_2 = \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 I_4(\{1, 3\} - \{1\}) = \frac{(\hat{\lambda} \lambda)^2}{16} \left( - \frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon} \right) (\{1, 3\} - \{1\}) , \]

\[ S_4 = \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 I_4 \frac{1}{2}(\{1, 2\} + \{2, 3\} - \{1\} - \{2\}) = \frac{(\hat{\lambda} \lambda)^2}{16} \left( - \frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \right) (\{1, 2\} + \{2, 3\} - \{1\} - \{2\}) , \]

\[ S_5 = \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 I_4 \frac{1}{4}(\{1\} + \{2\}) = \frac{\lambda^3 \hat{\lambda} \pi^2}{16 \varepsilon} (\{1\} + \{2\}) \]  

\[ S_6 = \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 (-K(I_2)^2)(\{1\}) = \frac{(\hat{\lambda} \lambda)^2}{16} \frac{1}{\varepsilon^2} \{1\} , \]

\[ S_7 = \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 I_4(\{1\}) = \frac{(\hat{\lambda} \lambda)^2}{16} \left( \frac{1}{2\varepsilon^2} - \frac{2}{\varepsilon} \right) \{1\} . \]

The explicit expressions for the appearing integrals can be found in (J.38). As explained at the beginning of this section, we have neglected all contributions to the trace and identity operator. These truncations are indicated by the arrows. We have also not evaluated the remaining scalar diagrams

\[ S_1 = \]  ,  \[ S_3 = \]  ,  \[ S_8 = \]  ,  \[ S_9 = \]  ,  \[ S_{10} = \]

which only contribute to the trace or identity operator.

Along with the set of diagrams in (4.3), we also need the reflected diagrams of \( S_2, S_4, S_7 \), which are easily obtained by using the reflection operator \( \mathbf{R} \) in (D.3). The diagrams \( S_1 \) and \( S_5 \) contain permutations between odd as well as even sites. To compute the dilatation operator for odd sites and to check the vanishing of the terms which mix odd and even sites, we therefore have to consider the respective odd and mixed contributions from the diagrams \( S(S_4) \) and \( S(S_5) \), which are shifted with the shift operator \( \mathbf{S} \) in (D.2) by one site. We thereby keep only those permutation structures, which have an odd entry as their first entry, which is indicated by the suffix \( \{1, \ldots\} \). The sum of the
relevant terms then reads
\[ S = S_2 + R(S_2) + [S_4 + R(S_4) + S(R(S_4)) + S_5 + S(S_5)] \{1, \ldots \} \]
\[ + S_6 + S_7 + R(S_7) \]
\[ \rightarrow \frac{\lambda \hat{\lambda}}{16} \left[ \lambda \hat{\lambda} \left( \left( -\frac{1}{2 \varepsilon^2} + \frac{2}{\varepsilon} \right) \{1, 3\} + \{3, 1\} + 2\{1, 2\} \right) + \left( \frac{4}{\varepsilon^2} - \frac{1}{\varepsilon} \left( 12 - \frac{1}{4 \pi^2} \right) \right) \{1\} \right] \]
\[ + (\lambda - \hat{\lambda})^2 \frac{1}{8 \pi^2} \{1\} \]  
(4.5)

4.3 Diagrams involving a single six-scalar vertex

In the following we evaluate all Feynman diagrams with a single six-scalar vertex and additional interactions, which promote the diagrams to four loops. Their flavour permutation structure is the same as for the simple two-loop diagram which contains the six-scalar vertex, if the additional interactions inside the diagram form flavour-neutral interactions, i.e. they do not alter the flavour flow. These diagrams build up the classes (1, 2, 0), (1, 1, 2) and (1, 0, 4) in table 1.

4.3.1 Diagrams involving flavour-neutral next-to-nearest neighbour interactions

Flavour-neutral next-to-nearest neighbour interactions can only be realized with gauge boson exchange and at least two cubic gauge scalar vertices at the first and last leg. The relevant diagrams involve the following substructure

\[ = + + + + + . \]  
(4.6)

They have to contain a six-scalar vertex which has the flavour permutation structure given in (B.2), and hence contains a simple permutation. The diagram involving the first of the above structures then spans the entire subclass (1, 1, 2) in table 1. The second and third one belong to (1, 0, 4).

Using the effective Feynman rules (A.7), the only non-vanishing contributions after a convenient choice of the external momenta are given by

\[ S_{v_1} = \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 (I_{4q} + 2I_{4g}) \{1\} = \frac{(\lambda \hat{\lambda})^2 1}{16} \left( 2 - \frac{\pi^2}{6} \right) \{1\} . \]  
(4.7)

The corresponding four-loop integrals are explicitly given in (J.50).

4.3.2 Diagrams involving flavour-neutral nearest-neighbour interactions

An interaction between three scalar field lines

\[ = + + + + + + + + + + + . \]  
(4.8)
appears in diagrams of the type
\[ \text{Diagram} \]
(4.9)

These diagrams belong to the subclasses \((1, 1, 2)_c\) and \((1, 0, 4)_c\) in table II. They are all finite, and in particular can be made vanish, if the external momenta are chosen conveniently, carefully avoiding the appearance of IR divergences.

The flavour-neutral nearest-neighbour interactions involve the following diagrams
\[ \text{Diagrams} \]
(4.10)

In the first term, which contains a fermion loop, the flavour flows from the upper to the respective lower external line as in all the other diagrams. Several of the above subdiagrams vanish identically, as follows from the effective Feynman rules (A.9).

The above subdiagrams appear in four-loop diagrams which also contains a six-scalar vertex, and hence according to (B.2) contributes to a simple permutation. They belong to all subclasses of \((1, 2, 0)_c\), \((1, 1, 2)_c\), \((1, 0, 4)_c\) in table II.

Evaluating all the diagrams in which the six-scalar vertex is connected by the structures (4.10) to one of its nearest neighbours, we obtain with the effective Feynman rules (A.9)
\[ S_{n1} = \text{Diagram} \rightarrow \frac{(4\pi)^4}{k^4} M^3 N \left( -2I_{42bb22} - \frac{1}{2} I_{42bb4ab} + I_{42cb7adbd} \right) \{1\} \]
\[ = \frac{\lambda^3 \hat{\lambda}}{16} \left( -\frac{1}{8\varepsilon^2} - \frac{3}{8\varepsilon^2} \pi^2 \right) \{1\}, \quad (4.11) \]
\[ S_{n2} = \text{Diagram} \rightarrow \frac{(4\pi)^4}{k^4} M^3 N \frac{1}{2} I_{42bb1b1} \{1\} = \frac{\lambda^3 \hat{\lambda}}{16} \left( -\frac{1}{8\varepsilon^2} + \frac{1}{2\varepsilon} \right) \{1\}. \]

In \( S_{n1} \), the contribution from the fermion loop has to be considered, while in \( S_{n2} \) the corresponding integral does not have an overall divergence. Only those diagrams in (4.10) with gauge bosons contribute, which do not have a cubic scalar gauge vertex at one of the lines which is an external line. This can be seen if we set the external momentum at the corresponding line to zero. It is an IR safe choice and makes the corresponding contributions vanish. The reflected copies are found by applying the reflection operator \( R \) in (D.3) that preserves the action of the permutations on respectively odd and even sites. For all inserted substructures (4.10) which lead to logarithmically divergent diagrams, one finds that \( R \) acts by simply exchanging \( \lambda \leftrightarrow \hat{\lambda} \). The sum of these contributions is
then given by
\[
S_{nn} = S_{n1} + R(S_{n1}) + S_{n2} + R(S_{n2})
\]
\[
\rightarrow \frac{\lambda \hat{\lambda}}{16} (2\lambda \hat{\lambda} + (\lambda - \hat{\lambda})^2)\left(-\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{1}{2} - \frac{3}{8\pi^2}\right)\right)\{1\} .
\] (4.12)

The flavour-neutral interactions (4.10) can also involve two external lines of the six-scalar vertex. In this case, the relevant contributions read
\[
S_{v2} = \rightarrow -\frac{1}{2} \frac{(4\pi)^4}{k^4} M^3 N K(I_{22A}) K(I_2)\{1\} = -\frac{\lambda^3 \hat{\lambda}}{16} \frac{1}{4\varepsilon^2}\{1\} .
\] (4.13)

Only the diagram with two quartic scalar gauge vertices in (4.10) contributes in this case. The overall pole part of the four-loop integral is given as the negative of the pole parts of two two-loop integrals, which are given in (J.2) and (J.3).

The flavour-neutral interactions (4.10) can also involve one external and one internal line of the six-scalar vertex. The relevant contributions read
\[
S_{v3} = \rightarrow \frac{(4\pi)^4}{k^4} M N^3 \left(-\frac{1}{2} I_{42bb4ab} + I_{42cb7abd} - 2I_{42bb2}\right)\{1\}
\]
\[
= \frac{\lambda \hat{\lambda}^3}{16} \left(-\frac{1}{8\varepsilon^2} - \frac{3\pi^2}{8\varepsilon}\right)\{1\} .
\] (4.14)

In this case, from (4.10) only the first diagram with a fermion loop and the ones with a quartic gauge-scalar vertex at the external line contribute. The relevant integrals are explicitly given in (J.40), (J.43), (J.39).

Finally, the diagrams with the flavour-neutral interaction (4.10) involving two internal lines of the six-scalar vertex are evaluated as
\[
S_{v4} = \rightarrow \frac{(4\pi)^4}{k^4} K(G(2 - 3\lambda, 1)(M^2 N^2(I_{3gv} + 2I_{3gn})
\]
\[
+ M^2 N(I_{3gb} - K(I_{22A})I_2 + 2I_{3gt}
\]
\[
+ I_{3gs} + 2I_{3ge} + I_{3fb}))\{1\}
\]
\[
= \frac{\lambda \hat{\lambda}}{16} \left[\lambda \hat{\lambda} \frac{1}{\varepsilon} \left(16 - \frac{4}{3}\pi^2\right) + \lambda^2 \left(-\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon}\left(1 - \frac{7}{12}\pi^2\right)\right)\right]\{1\} .
\] (4.15)

This is the most complicated contribution in this class, since it involves the maximum number of different contributions, that individually require intricate evaluations. The above result is expressed in terms of three-loop integrals, which involve the nearest-neighbour interaction (4.10) and the two additional loops built by the four attached propagators. The integrals are explicitly given in (J.37).

We still have to include the reflections of the above contributions, given by acting with R in (D.3). Again, this corresponds to simply exchanging \(\lambda \leftrightarrow \hat{\lambda}\). The sum of the above contributions is given by
\[
S_v = S_{v1} + S_{v2} + R(S_{v2}) + S_{v3} + R(S_{v3}) + S_{v4} + R(S_{v4})
\]
\[
\rightarrow \frac{\lambda \hat{\lambda}}{16} \left[\lambda \hat{\lambda} \left(-\frac{5}{4\varepsilon^2} + \frac{1}{\varepsilon}\left(20 - \frac{41}{12}\pi^2\right)\right) + (\lambda - \hat{\lambda})^2 \left(-\frac{5}{8\varepsilon^2} + \frac{1}{\varepsilon}\left(1 - \frac{23}{24}\pi^2\right)\right)\right]\{1\} .
\] (4.16)
4.3.3 Diagrams involving the scalar two-loop self-energy

The two-loop self-energy correction of the scalar field also appears as a sum of subdiagrams. It reads

\[
\Sigma_Y = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6} \\
\text{Diagram 7} \\
\text{Diagram 8} \\
\text{Diagram 9} \\
\text{Diagram 10} \\
\text{Diagram 11} \\
\text{Diagram 12} \\
\end{array}
\]

(4.17)

To calculate the divergences of the corresponding four-loop diagrams, we need the divergent and finite parts of the two-loop self-energy. The individual contributions are evaluated in appendix E. Apart from the Wick rotation (the result has to be multiplied by a factor \( i^2 = -1 \)), the amputated scalar self-energy contribution becomes

\[
\Sigma_Y = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6} \\
\text{Diagram 7} \\
\text{Diagram 8} \\
\text{Diagram 9} \\
\text{Diagram 10} \\
\text{Diagram 11} \\
\text{Diagram 12} \\
\end{array}
\]

(4.18)

where the last propagator factor on the r.h.s. captures the momentum dependence. Its weight label indicates the exponent of \( \frac{1}{p^2} \), where \( p \) is the external momentum. The pole part of the above result coincides with the result in [17].

The following diagrams contain the two-loop self-energy and contribute to the renormalization of the composite operators (factor \((-i)^4i\) from the four scalar propagators and the six-scalar vertex are included)

\[
S_{s1} = \begin{array}{c}
\text{Diagram 13} \\
\text{Diagram 14} \\
\text{Diagram 15} \\
\end{array} \rightarrow -i \frac{(4\pi)^2}{k^2} MN K(\Sigma_Y) K(I_2)\{1\}
\]

(4.19)
They belong to all subclasses of $(1, 2, 0), (1, 1, 2), (1, 0, 4)$ in table 1. The combination of the above diagrams, which enters the renormalization of the composite operator, reads

\[
S_s = \frac{3}{2} S_{s1} + 3 S_{s2}
\]

\[
\rightarrow \frac{\lambda \hat{\lambda}}{16} \left[ \lambda \hat{\lambda} \left( \frac{9}{2} \varepsilon^2 + \frac{1}{\varepsilon} \left( -33 + \frac{9}{4} \pi^2 \right) \right) + (\lambda - \hat{\lambda})^2 \left( \frac{3}{4} \varepsilon^2 + \frac{1}{\varepsilon} \left( -\frac{3}{2} + \frac{3}{8} \pi^2 \right) \right) \right] \{1\} .
\]

\[
\text{(4.20)}
\]

The sum of all contributions which involve a six-scalar vertex and a flavour-neutral four-loop completion is given by

\[
S_n = S_{nn} + S_v + S_s
\]

\[
\rightarrow \frac{\lambda \hat{\lambda}}{16} \left[ \lambda \hat{\lambda} \left( \frac{11}{4} \varepsilon^2 + \frac{1}{\varepsilon} \left( -12 - \frac{23}{12} \pi^2 \right) \right) + (\lambda - \hat{\lambda})^2 \left( -\frac{1}{8} \varepsilon^2 + \frac{1}{\varepsilon} \left( -\frac{23}{24} \pi^2 \right) \right) \right] \{1\} .
\]

\[
\text{(4.21)}
\]

4.4 Diagrams involving four scalar-fermion vertices

In the following we evaluate all diagrams of the class $(0, 4, 0)$ of table 1. The fermions have to form a single loop, which in this case is a square, to contribute to flavour permutations. The only four-loop diagrams of this class in which four neighboured scalar lines interact are given by

\[
F_{s1} = \quad \rightarrow -\frac{1}{8} \frac{(4\pi)^4}{k^4} M^2 N^2 I_{42bb\text{le}} 8(\{1, 2\} + \{2, 3\} - \{1\} - \{2\})
\]

\[
= -\frac{(\lambda \hat{\lambda})^2}{16} \frac{2}{\varepsilon} (\{1, 2\} + \{2, 3\} - \{1\} - \{2\}) ,
\]

\[
\text{(4.22)}
\]

\[
F_{s2} = \quad \rightarrow -\frac{1}{8} \frac{(4\pi)^4}{k^4} M^3 N I_{42bb\text{b1de}} (\text{id. and tr.})
\]

\[
= \frac{\lambda^3 \hat{\lambda}}{16} \frac{\pi^2}{32\varepsilon} (\text{id. and tr.}) ,
\]

where the r.h.s. follow immediately, using the effective Feynman rule for the fermion square \((A.16)\). The resulting integrals are explicitly given in \((J.42)\) and \((J.40)\), and the flavour permutation parts are taken from \((B.3)\). The second of the above diagrams only contributes to the identity and trace part of the dilatation operator and hence is not considered here.

Besides the above diagrams, there are the following diagrams in which only three
neighbouring lines interact. They can contain a bubble and are given by

\[ F_{sb1} = \quad \rightarrow -\frac{1}{2} \frac{(4\pi)^4}{k^4} M^2 N^2 (\mathcal{I} - I_{42bb4ad}) 8\{1\} = -\frac{\lambda^2}{16} \frac{1}{\varepsilon^2} \{1\} , \]

\[ F_{sb2} = \quad \rightarrow \frac{1}{8} \frac{(4\pi)^4}{k^4} M^2 N^2 (\mathcal{I} - I_{42bb4ad})(-16\{1\}) = -\frac{\lambda^2}{16} \frac{1}{2\varepsilon^2} \{1\} , \]

\[ F_{sb3} = \quad \rightarrow \frac{1}{2} \frac{(4\pi)^4}{k^4} M^2 N^2 I_{42b\varepsilon} 8\{1\} = -\frac{\lambda^2}{16} \frac{1}{\varepsilon^2} \{1\} , \]

\[ F_{sb4} = \quad \rightarrow -\frac{1}{8} \frac{(4\pi)^4}{k^4} M^2 N^2 I_{2b\varepsilon}(16\{1\}) = -\frac{\lambda^2}{16} \frac{1}{2\varepsilon^2} \{1\} , \]

\[ F_{sb5} = \quad \rightarrow \frac{1}{2} \frac{(4\pi)^4}{k^4} M^2 N^2 I_{2b\varepsilon 3acbe} 8\{1\} = -\frac{\lambda^2}{16} \frac{1}{\varepsilon^2} \{1\} , \]

\[ F_{sb6} = \quad \rightarrow \frac{1}{8} \frac{(4\pi)^4}{k^4} M^2 N^2 I_{2b\varepsilon 3acbe}(-16\{1\}) = -\frac{\lambda^2}{16} \frac{1}{2\varepsilon^2} \{1\} , \]

where the r.h.s. follow immediately, using the effective Feynman rules for the fermion squares with bubble (A.17). The resulting integrals are explicitly given in (J.40), (J.42), (J.41), and the flavour permutation part is taken from (B.3). The diagrams with four scalar-fermion vertices can also contain triangles and read

\[ F_{st1} = \quad \rightarrow -\frac{1}{2} \frac{(4\pi)^4}{k^4} M^2 N^2 K(-I_{42qAdBb} + I_{42qAbBd} - I_{42qABbd}) 4\{1\} \]

\[ = \frac{\lambda^2}{16} \left( \frac{\varepsilon^2}{16} + \frac{1}{\varepsilon} \left( -4 + \frac{2}{3} \pi^2 \right) \right) \{1\} , \]

\[ F_{st2} = \quad \rightarrow \frac{1}{2} \frac{(4\pi)^4}{k^4} M^2 N^2 I_{42bb4de} 4\{1\} = \frac{\lambda^2}{16} \frac{1}{\varepsilon^2} \{1\} , \]

where the r.h.s. follow immediately, using the last of the effective Feynman rules in
The resulting integrals are explicitly given in (J.45) and (J.40), and the flavour permutation part is taken from (B.3).

Summing up the above contributions, considering also the contribution from the diagram \( F_{s1} \) shifted by one site with the shift operator \( S \) in (D.2) and allowing for factors of two that come from reflections of the diagrams \( F_{sb1} \) to \( F_{sb6} \) and \( F_{st2} \), we obtain

\[
F_s = [F_{s1} + S(F_{s1})\{1,...\} + 2(F_{sb1} + F_{sb2} + F_{sb3} + F_{sb4} + F_{sb5} + F_{sb6}) + F_{st1} + 2F_{st2}
\rightarrow \frac{(\lambda\hat{\lambda})^2}{16} \left[ -\frac{4}{\varepsilon}\{1,2\} + \left( -\frac{6}{\varepsilon^2} + \frac{2}{3\varepsilon}\pi^2 \right)\{1\} \right],
\]

where we have only kept those permutation structures, which have an odd entry as their first entry as indicated by the suffix \( \{1,\ldots\} \).

### 4.5 Diagrams involving three scalar-fermion vertices

In the following we evaluate all diagrams of the class \((0,3,2)\) of table 1.

The four-loop diagrams in which three scalar-fermion vertices form a fermion triangle also involve a single gauge boson propagator. According to the flavour permutation structures in (B.3) they contribute to the coefficient of a single permutation. The gauge boson propagator can end on two cubic gauge-fermion vertices. The corresponding diagrams build up the class \((0,3,2)_a\) and are given by

\[
F_{ts1} = \quad \rightarrow 2z\frac{(4\pi)^4}{k^4}(M - N)M^2NI_{422bb3be}c{d}\{1\}
= \frac{1}{16}(\lambda - \hat{\lambda})\lambda^2\hat{\lambda}\left( -\frac{1}{2\varepsilon^2} \right)\{1\},
\]

\[
F_{ts2} = \quad \rightarrow 2z\frac{(4\pi)^4}{k^4}(M - N)M^2NI_{422bb4ab}\{1\}
= \frac{1}{16}(\lambda - \hat{\lambda})\lambda^2\hat{\lambda}\left( -\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon} \right)\{1\},
\]

\[
F_{tv1} = \quad \rightarrow -2z\frac{(4\pi)^4}{k^4}M^3N(-I_{422cb6ade})\{1\}
= \frac{\lambda^3\hat{\lambda}}{16}\left( \frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon}\left( 1 - \frac{\pi^2}{4} \right) \right)\{1\},
\]

\[
F_{tv2} = \quad \rightarrow -2z\frac{(4\pi)^4}{k^4}M^3N(2I_{422bb3abe} - I_{42bb3ad})\{1\} = \frac{\lambda^3\hat{\lambda}}{16}\left( \frac{1}{2\varepsilon^2} \right)\{1\},
\]

where in the first and third line the equalities between the graphs are to be understood for the pole parts only. The r.h.s. of the diagrams involving the fermionic self-energy follow immediately from the effective Feynman rule (A.12). The r.h.s. of the other diagrams are obtained from the effective Feynman rule (A.13). The resulting integrals are explicitly given in (J.41), (J.40), (J.43).
The gauge boson propagator can also end on one cubic gauge fermion vertex and one cubic gauge-scalar vertex. The corresponding diagrams build up the class \((0, 3, 2)_b\) and are given by \(^4\)

\[
F_{tv3} = \rightarrow -\frac{1}{4} \left(\frac{(4\pi)^4}{k^4}\right) M^2 N^2 \left( I_{422qAabd} - I_{422qAba} \right) \left( -8 \{1\} \right)
\]

\[
= z \left( \frac{\lambda \hat{\lambda}}{16} \right)^2 \frac{1}{\varepsilon} \left( 6 - \frac{2}{3\pi^2} \right) \{1\},
\]

\[
F_{tv4} = \rightarrow -\frac{1}{4} \left(\frac{(4\pi)^4}{k^4}\right) M^2 N^2 \left( I_{422qAaBd} - I_{422qAAbd} \right) \left( -8 \{1\} \right)
\]

\[
= z \left( \frac{\lambda \hat{\lambda}}{16} \right)^2 \frac{1}{\varepsilon} \left( 3 - \frac{\pi^2}{4} \right) \{1\},
\]

\[
F_{tv5} = \rightarrow \frac{1}{4} \left(\frac{(4\pi)^4}{k^4}\right) M^2 N^2 \left( -I_{422cb6e6d} + I_{422cb6ecde} \right) \left( -8 \{1\} \right)
\]

\[
= z \left( \frac{\lambda \hat{\lambda}}{16} \right)^2 \frac{1}{\varepsilon} \left( -1 + \frac{\pi^2}{4} \right) \{1\}.
\]

The r.h.s. of the diagrams are obtained from the effective Feynman rules \((A.15)\). The resulting integrals are explicitly given in \((J.41), (J.40), (J.43), (J.45)\) and \((J.46)\).

The remaining diagrams, in which the gauge boson propagator is attached to the diagram via two cubic vertices involving scalars build up the class \((0, 3, 2)_c\) and can be made vanish by a convenient choice of the external momenta.

The above results depend on a sign \(z = \pm 1\) which is not uniquely given in the literature \([7, 31]\). In appendix \([F]\) we fix \(z\) by computing the renormalization of the six-scalar vertex at two loops. It should vanish for the correct sign choice of \(z\) to ensure superconformal invariance. This yields \(z = 1\).

We also have to consider the reflected diagrams. Their contributions are obtained by applying the reflection operator \(R\) in \((D.3)\). We obtain for the sum of the respective diagrams

\[
F_t = 2(F_{ts1} + R(F_{ts1})) + F_{ts2} + R(F_{ts2}) + 2(F_{tv1} + R(F_{tv1}))
\]

\[
+ F_{tv2} + R(F_{tv2}) + 2(F_{tv3} + F_{tv4} + F_{tv5})
\]

\[
\rightarrow z \left[ \frac{\lambda \hat{\lambda}}{16} \left( \frac{3}{\varepsilon^2} + \frac{1}{\varepsilon} \left( 12 - \frac{\pi^2}{3} \right) \right) \right] + (\lambda - \hat{\lambda})^2 \frac{\pi^2}{2\varepsilon} \{1\}.
\]

The presence of the reflected diagrams which do not differ by a sign from the original diagrams, but only by an exchange \(\lambda \leftrightarrow \hat{\lambda}\) thereby guarantees that the result only depends quadratically on the difference of the couplings and hence on the parameter \(\sigma\) defined in \((1.3)\).

\(^4\)Because of inconsistent Feynman rules these three diagrams had a sign error in previous versions of this paper.
5 Result

Fixing $z = 1$ as determined in appendix F, the part of $\mathcal{Z}$ involving non-trivial permutations is given as the negative of the sums of (4.5), (4.21), (4.25), and (4.28).

\[
\bar{\lambda}^4 \mathcal{Z}_4 = -S - S_n - F_s - F_t
\]

\[
\rightarrow \bar{\lambda}^4 \left[ \left( \frac{1}{2\varepsilon^2} - \frac{2}{\varepsilon} \right) \{(1, 3) + \{3, 1\} \} + \frac{1}{\varepsilon^2} \{(1, 2) \} \right. \\
\left. + \left( -\frac{15}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( 12 + \frac{4}{3} \pi^2 \right) + \sigma^2 \left( \frac{1}{8\varepsilon^2} + \frac{1}{3\varepsilon^2} \right) \right) \{1\} \right].
\] (5.1)

By the arrow we indicate that we have neglected all contributions to the identity and trace operator in flavour space, and also that we have only written half of all contributions, namely those with permutation structures with an odd entry as their first entry. In the above result the simple poles of the structure $\{1, 2\}$, which couples magnons at odd and even sites, between the scalar contribution (4.5) and the one with fermion square (4.25) cancel out. Only a double pole part is left. The neglected terms which contain permutations that have an even entry as their first entry can easily be reconstructed by acting with the shift operator $S$ in (4.2) on the above result.

In appendix G we explicitly demonstrate that all the double poles cancel against each other when we add the two-loop contribution and take the logarithm as in (3.4).

According to the prescription (3.5), the dilatation operator for odd sites is the coefficient of the $\frac{1}{\varepsilon}$ pole terms in (5.1) multiplied by 8. We must still add the neglected identity part, which we fix by demanding that the ground state has zero eigenvalue. We then obtain

\[
D_{4,\text{odd}} = -\{(4 + 4\zeta(2) + \sigma^2\zeta(2))\} + \{(6 + 4\zeta(2) + \sigma^2\zeta(2))\} - \{1, 3\} - \{3, 1\},
\] (5.2)

where $\zeta(2) = \frac{\pi^2}{6}$. By comparing this result with (2.7), we immediately see that the coefficients of the maximal shuffling terms match. The four-loop term of the function $h^2(\bar{\lambda}, \sigma)$ in (1.5) is read off as

\[
h_4(\sigma) = -(4 + \sigma^2)\zeta(2).
\] (5.3)

6 Four-loop wrapping interactions

To find the correct anomalous dimensions of a length four state at four loops, we have to consider the wrapping interactions [32, 33]. Recently, Gromov, Kazakov and Vieira have made a prediction for the four-loop wrapping contribution in terms of $h^2(\lambda)$ for scalar operators in the 20 representation of $SU(4)$ [24]. The highest weight of this representation is in the $SU(2) \times SU(2)$ sector. These authors find their result by applying the thermodynamic Bethe ansatz (TBA), first introduced for the original AdS/CFT correspondence in [34, 35], using the predicted asymptotic Bethe equations [10]. In particular, the TBA is formulated in terms of a $Y$-system [24, 26], a series of difference equations, that lend themselves to an efficient order by order solution.
Following the strategy of \([36, 37]\), we first subtract from the asymptotic dilatation operator \(2.7\) the range five interactions (given by \(S_2\) and the reflected diagram). This leaves for \(D_{4,\text{odd}}\) in \((5.2)\)

\[
D_{4,\text{odd}}^{\text{sub}} = (4 + 4\zeta(2) + \sigma^2\zeta(2))(\{1\} - \{}\). \tag{6.1}
\]

We then have to add the following wrapping diagrams which contribute to genuine flavour permutations

\[
\begin{align*}
W_1 = & \quad \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 I_4(-\{1\}) \\
& = \frac{(\lambda\lambda)}{16} \left( -\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon}\right)(-\{1\}),
\end{align*}
\]

\[
\begin{align*}
W_2 = & \quad \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 \frac{1}{2} I_{4\text{bbbd}}2\{1\} \\
& = \frac{(\lambda\lambda)}{16} \frac{\pi^2}{2\varepsilon}\{1\},
\end{align*}
\]

\[
\begin{align*}
W_3 = & \quad \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 \frac{1}{2} I_{42\text{bbd}}4\{1\} \\
& = -\frac{(\lambda\lambda)}{16} \frac{\pi^2}{2\varepsilon}\{1\},
\end{align*}
\]

\[
\begin{align*}
W_4 = & \quad \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 \frac{1}{2} I_{42\text{bc}}8\{1\} \\
& = \frac{(\lambda\lambda)}{16} \frac{1}{\varepsilon}\{1\},
\end{align*}
\]

\[
\begin{align*}
W_5 = & \quad \rightarrow \frac{(4\pi)^4}{k^4} M^2 N^2 2(I_{422q\text{AbBd}} - I_{422q\text{ABbd}})\{1\} \\
& = \frac{(\lambda\lambda)}{16} \frac{2}{\varepsilon}\left(1 - \frac{\pi^2}{12}\right)\{1\}.
\end{align*}
\]

We then sum the above diagrams as

\[
W = W_1 + W_2 + 2W_3 + W_4 + W_5 = \frac{(\lambda\lambda)}{16} \left[ \frac{1}{2\varepsilon^2} \{1\} + \frac{2}{\varepsilon} \left(4 - \frac{\pi^2}{3}\right)\{1\}\right], \tag{6.3}
\]

where we include a factor of two for \(W_3\) since its reflection cannot be mapped to itself by a cyclic rotation. To this sum we add the identity terms needed to give zero wrapping for a chiral primary. Taking the negative of the sum \(W\), extracting the residues of \(\frac{1}{\varepsilon}\) and multiplying by 8 as required by \((3.5)\), we find that the wrapping contribution to the odd site dilatation operator is given by

\[
D_{4,\text{odd}}^w = -(4 - 2\zeta(2))(\{1\} - \{}\). \tag{6.4}
\]

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The 20 in the $SU(2) \times SU(2)$ sector is a completely antisymmetric state, so the eigenvalue of $\{1\} - \{\}$ is $-4$, and we have to double this to take into account also the contribution from the even site dilatation operator $D_{4,\text{even}}^w$. Hence the four-loop wrapping contribution to the anomalous dimension is

$$\gamma_{20}^w = (32 - 16\zeta(2))\bar{\lambda}^4,$$

which agrees with the GKV $Y$-system prediction $[23]$. It does not depend on $\sigma^2$, which is expected from the topology of the diagrams and the fact that the two-loop result does not depend on $\sigma^2$. Adding $D_{4,\text{sub}}^w$ to $D_{4}^w$, we find that the anomalous dimension of the 20 is given by

$$\gamma_{20} = 4 + 8\bar{\lambda}^2 - 8(6 + \sigma^2)\zeta(2)\bar{\lambda}^4,$$

where we have included the lower orders $[5]$. Intriguingly, the rational terms of (6.1) and (6.4) cancel, such that the four-loop contribution only consists of terms with maximum transcendentality.

### 7 Conclusions

Our four-loop calculation of the dilatation operator shows that the unknown function $h^2(\bar{\lambda}, \sigma)$ with the expansion (1.5) has the form

$$h^2(\bar{\lambda}, \sigma) = \bar{\lambda}^2 + \bar{\lambda}^4 h_4(\sigma), \quad h_4(\sigma) = h_4 + \sigma^2 h_{4,\sigma}.$$

It is an even function of $\sigma$, and hence to four loops there are no parity breaking effects which could lead to different dispersion relations for magnons at odd and even sites as in (1.4). The coefficients are explicitly given by

$$h_4 = -4\zeta(2) \approx -6.58, \quad h_{4,\sigma} = -\zeta(2).$$

The negative sign for $h_4$ is sensible, as this will dampen the quadratic behaviour found at small $\bar{\lambda}$ to the linear behaviour at large $\bar{\lambda}$. Interestingly, the rational contributions to $h_4(\sigma)$ cancel, leaving a maximally transcendental result. Inserting the definitions (1.3), it becomes obvious that due to the relative coefficient between $h_4$ and $h_{4,\sigma}$ the four loop term depends on the coupling constants only via a factor $(\lambda + \bar{\lambda})^4$.

The calculation is rather complicated and contains large numbers of contributions. We therefore have performed several cross-checks on the integral tables, and we have reproduced the known results at two loops. Furthermore, by checking the cancellation of double poles of the four-loop contributions in the logarithm of the renormalization constant, we have another check for our final result. Finally, also the four-loop anomalous dimension of the non-protected length four state matches the result obtained from the $Y$-system.

---

5The same result is found using the $\mathcal{N} = 2$ superspace formalism $[22]$. 

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Acknowledgments

We would like to thank O. Bergman, T. Klose and K. Zarembo for very helpful discussions. The research of J. A. M. is supported in part by the Swedish research council. The research of C. S. is supported in part by the European Marie Curie Research and Training Network ENRAGE (MRTN-CT-2004-005616). J. A. M. thanks the CTP at MIT and the Galileo Institute in Florence for kind hospitality during the course of this work. Support for these visits comes from the STINT foundation and INFN. O. O. S. thanks NBI and C. S. thanks Uppsala University for hospitality during the course of this work.

A Action and Feynman rules

In three dimensional spacetime with metric $\eta_{\mu\nu} = \text{diag}(-,+,+)$ The antisymmetric tensor is normalized as $\epsilon^{012} = 1$. The product of $\gamma$-matrices having index positions $(\gamma_\mu)_{\alpha\beta}$ is given by

$$\gamma_\mu \gamma_\nu = \eta_{\mu\nu} + z\epsilon_{\mu\nu\rho}\gamma^\rho , \quad (A.1)$$

where $z = \pm 1$ represents a sign. The Clifford algebra is given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \quad (A.2)$$

The kinetic and interaction parts of the action are given by

$$S_{\text{kin}} = \frac{k}{4\pi} \int d^3x \text{tr} \left[ \frac{1}{\zeta} \left( \delta A \left( \epsilon^{\alpha\beta}\partial_\beta - \frac{1}{\zeta}\partial_\alpha\partial_\gamma \right) A_\gamma - \hat{A}_\alpha \left( \epsilon^{\alpha\beta}\partial_\beta - \frac{1}{\zeta}\partial_\alpha\partial_\gamma \right) \hat{A}_\gamma ight) ight]$$

$$+ \left( \frac{2}{3} i\epsilon^{\alpha\beta\gamma}(A_\alpha A_\beta A_\gamma - \hat{A}_\alpha \hat{A}_\beta \hat{A}_\gamma) - i A_\mu \hat{A}_\mu Y^A Y^A - i \hat{A}_\mu A_\mu Y^A Y^A + 2Y^A \hat{A}_\mu I_{\alpha} \hat{A}_A \hat{A}_A ight)$$

$$+ \psi^B \hat{A} \psi_B + \hat{A}_\mu \psi^B \gamma^\mu \psi_B - i A^\mu \left( \hat{I}_\alpha \hat{A}_\alpha \hat{A}_\alpha \right) - i \hat{A}^\mu \left( \hat{I}_\alpha \partial_\alpha \hat{A}_\alpha \right)$$

$$+ \frac{1}{12} Y^A Y^B Y^C Y^D \left( \delta_{A}^{B} \delta_{C}^{D} \delta_{E}^{F} + \delta_{A}^{F} \delta_{C}^{D} \delta_{E}^{B} - 6 \delta_{A}^{B} \delta_{C}^{E} \delta_{D}^{F} + 4 \delta_{A}^{D} \delta_{C}^{E} \delta_{D}^{B} \right)$$

$$- \frac{i}{2} \left( \frac{1}{2} \epsilon^{ABC} \psi_D - \frac{1}{2} \epsilon^{ABC} \psi_D \right) \left( \delta_{A}^{B} \delta_{C}^{D} - 2 \delta_{B}^{A} \delta_{C}^{D} \right)$$

$$+ \frac{i}{2} \epsilon^{ABCD} Y^A \psi_B Y^C \psi_D - \frac{i}{2} \epsilon^{ABCD} Y^A \psi_B Y^C \psi_D \right] . \quad (A.3)$$
A.1 Component Feynman rules

We obtain the propagators for the gauge, scalar, fermion and ghost fields from $S_{\text{kin}}$ as

\[
\begin{align*}
\begin{array}{c}
\beta \\
\hline
\beta
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\beta \\
\hline
\beta
\end{array} & = \langle A_\alpha(p)A_\beta(-p) \rangle = -\langle \hat{A}_\alpha(p)\hat{A}_\beta(-p) \rangle = -\frac{1}{2} \frac{1}{p^2} \left( \epsilon_{\alpha\gamma\beta} p^\gamma - i \zeta \frac{p_\alpha p_\beta}{p^2} \right), \\
B & = \langle Y^A(p)Y^*_B(-p) \rangle = -i \frac{\delta^A_B}{p^2}, \\
B & = \langle \psi_A(p)\psi^*_B(-p) \rangle = i \frac{\delta^B_A}{p}, \\
\begin{array}{c}
\beta \\
\hline
\beta
\end{array} & = \langle c(p)c^*(-p) \rangle = -\frac{1}{p^2},
\end{align*}
\]  

(A.4)

where diagonality in the gauge group indices and a factor $\frac{4\pi}{k}$ for each propagator have been suppressed, and we will fix the gauge as $\zeta = 0$ in the following.

The vertices are obtained by taking the functional derivatives of $i$ times the action

\[6\text{A previous version of this paper had inconsistent Feynman rules. The propagators in (A.4) and an overall sign in the scalar-gauge three-point vertices in (A.5) have been corrected. As a consequence this alters the overall signs in (A.6), (A.11), (A.14) and (A.15).}\]
w.r.t. the corresponding fields. We obtain for the cubic vertices

\[ \gamma = V_{A^3} = -V_{\tilde{A}^3} = -2\epsilon^{\alpha\beta\gamma} \text{tr} \left( T^a \left[ T^b, T^c \right] \right), \]

\[ \mu = V_{AYY^\dagger} = i \text{tr} \left( T^a B^b B_c \right) \left( p_Y - p_{Y^\dagger} \right) \delta_B^A, \]

\[ \mu = V_{\tilde{A}Y^\dagger Y} = i \text{tr} \left( T^a B_b B^c \right) \left( p_{Y^\dagger} - p_Y \right) \delta_B^A, \]

\[ \mu = V_{\psi^\dagger A \psi} = -i \text{tr} \left( B_b T^a B^c \right) \gamma^\mu \delta_C^B, \]

\[ \mu = V_{\tilde{A} \psi \psi^\dagger} = i \text{tr} \left( T^a B_b B^c \right) \gamma^\mu \delta_C^B, \]

\[ \mu = V_{A^c \nu} = V_{\tilde{A} \omega} = \frac{1}{2} \text{tr} \left( T^a \left[ T^b, T^c \right] \right), \]

\[ \mu = V_{AYY^\dagger} = 2i \text{tr} \left( T^a B^b B_c T^d \delta_B^A \delta_B^C \delta^D_{\mu\nu} \right), \]

\[ Y_{Y^\dagger Y^\dagger} = V_{\tilde{A} Y^\dagger Y} = -i \text{tr} \left( T^a B^b B_c T^d \delta_B^A \delta_B^C \delta^D_{\mu\nu} \right), \]

\[ Y_{Y^\dagger \psi^\dagger Y^\dagger} = -V_{Y^\dagger Y^\dagger \psi^\dagger} = -\frac{1}{2} \text{tr} \left( B_a B^b B_c B_d \delta_B^A \delta_B^C \delta_B^D - 2\delta_B^A \delta_B^C \delta_B^D \right), \]

\[ Y_{Y^\dagger \psi^\dagger Y^\dagger} = \epsilon^{ABCD} \left( \text{tr} \left( B_a B_b B_c B_d \right) - \text{tr} \left( B_a B_d B_b B_b \right) \right), \]

where the momenta carry as suffix the respective field line label at which they enter the vertex. We have suppressed a factor \( \frac{k}{4\pi} \) for each vertex. The dependence on the coupling constant can be easily restored at the end of the calculation.
A.2 Effective Feynman rules

The gauge propagator in (A.4) and the cubic gauge vertex in (A.5) contain $\epsilon$ tensors which appear in the numerators and are contracted with each other or with loop and external momenta. The same happens in presence of fermion fields. Dimensional reduction requires, that the $\epsilon$-tensors are reduced to scalar products in strictly $D = 3$ dimensions, before the integral is dimensionally regularized by switching to $D = 3 - 2\varepsilon$ dimensions. It turns out to be advantageous to introduce effective Feynman rules, in which the corresponding tensors have already been reduced, such that all gauge bosons and fermions appear as scalar propagators with momenta in their numerators.

In the following all substructures are given with amputated external legs. All prefactors (apart from powers $\frac{4\pi}{k}$) of the propagators (A.4) and vertices (A.5), symmetry factors (like $-1$ for a fermion loop) and factors for internal flavour loops are included in the prefactors. The corresponding factors for the propagators which have to be attached at the external legs are not included when the subdiagram is replaced by its scalar representative.

Since the free vector indices of the gauge boson propagators are perpendicular to the momentum, we can define the following effective vertices (extracting the factors of $i$)

\[ Y \mu \rightarrow -2i Y \mu , \]

\[ Y^{\dagger} \mu \rightarrow -2i Y^{\dagger} \mu , \]

\[ Y \rightarrow -2i Y , \]

\[ Y^{\dagger} \rightarrow -2i Y^{\dagger} . \]

Reexpressing the appearing products of two $\epsilon$-tensors in the numerators of the loop integrals in terms of the metric, the flavour-neutral interactions via gauge bosons between three scalar field lines are simplified as

\[ \gamma = 2i \left( - \frac{1}{k^2} + \frac{1}{k} \right) , \]

\[ \gamma = 4i \left( - \frac{1}{k^2} + \frac{1}{k} \right) . \]

In the above relations factors $-i$ for external propagators have not been included in the prefactor. The internal scalar propagators on the r.h.s. have no non-trivial prefactors.

For flavour-neutral interactions between two scalar field lines involving the gauge bosons we need the cubic gauge vertex with propagators. It is reexpressed in terms of the metric as follows

\[ \gamma \rightarrow \epsilon^{\delta \mu \nu} \epsilon_{\alpha \rho \sigma} \epsilon_{\beta \gamma \kappa} = \eta_{\alpha \sigma} \eta_{\beta \gamma} \eta_{\rho \kappa} - \eta_{\alpha \rho} \eta_{\beta \gamma} \eta_{\sigma \kappa} - \eta_{\alpha \sigma} \eta_{\beta \kappa} \eta_{\rho \gamma} + \eta_{\alpha \gamma} \eta_{\beta \kappa} \eta_{\rho \sigma} - \eta_{\alpha \beta} \eta_{\gamma \kappa} \eta_{\rho \sigma} + \eta_{\alpha \kappa} \eta_{\gamma \sigma} \eta_{\beta \rho} + \eta_{\alpha \beta} \eta_{\gamma \rho} \eta_{\sigma \kappa} - \eta_{\alpha \gamma} \eta_{\beta \rho} \eta_{\sigma \kappa} . \]
The effective Feynman rules then become

\[
\begin{align*}
\begin{align*}
\text{Diagram 1}&: & = & \frac{1}{2} \\
\text{Diagram 2}&: & = & \frac{1}{2} \left( \left( \begin{array}{c} \text{Diagram 3} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 4} \end{array} \right) \right) = 0 , \\
\text{Diagram 5}&: & = & - \left( \begin{array}{c} \text{Diagram 6} \end{array} \right) , \\
\text{Diagram 7}&: & = & - \left( \begin{array}{c} \text{Diagram 8} \end{array} \right) = 0 , \\
\text{Diagram 9}&: & = & 2 \left( \left( \begin{array}{c} \text{Diagram 10} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 11} \end{array} \right) \right) = 0 , \\
\text{Diagram 12}&: & = & 4 \left( \left( \begin{array}{c} \text{Diagram 13} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 14} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 15} \end{array} \right) + \left( \begin{array}{c} \text{Diagram 16} \end{array} \right) + \left( \begin{array}{c} \text{Diagram 17} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 18} \end{array} \right) \right) , \\
\text{Diagram 19}&: & = & 2 \left( \left( \begin{array}{c} \text{Diagram 20} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 21} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 22} \end{array} \right) + \left( \begin{array}{c} \text{Diagram 23} \end{array} \right) + \left( \begin{array}{c} \text{Diagram 24} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 25} \end{array} \right) \right) , \\
\text{Diagram 26}&: & = & \left( \begin{array}{c} \text{Diagram 27} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 28} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 29} \end{array} \right) + \left( \begin{array}{c} \text{Diagram 30} \end{array} \right) + \left( \begin{array}{c} \text{Diagram 31} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 32} \end{array} \right) , \\
\text{Diagram 33}&: & = & 4 \left( \left( \begin{array}{c} \text{Diagram 34} \end{array} \right) - \left( \begin{array}{c} \text{Diagram 35} \end{array} \right) \right) ,
\end{align*}
\end{align*}
\]

(A.9)

where we have omitted simple factors of either \( M \) or \( N \) from the colour running in the loop. In the last diagram we have summed up the scalar, fermion, gauge boson and ghost contribution to the gauge boson self-energy.

For the fermion loop we find (considering also the factors 4 for the flavour contraction, \(-1\) for the fermion loop and another \(-1\) for changing the direction of one momentum factor in the numerator)

\[
\left( \begin{array}{c} \text{Diagram 36} \end{array} \right) = -2 \left( \begin{array}{c} \text{Diagram 37} \end{array} \right) .
\]

(A.10)

The fermion fields get finite self-energy corrections at one loop. We can summarize the one-loop fermionic self-energy as

\[
\Sigma_\psi = \psi -$\bullet$ $\psi$ $\psi = -$\bullet$ $\psi$ $\psi + -$\bullet$ $\psi$ $\psi = -iz(M - N)$ $\psi$ $\psi$ $\psi$ $\psi ,
\]

(A.11)

where we have also included the non-trivial colour factor. The above substructure has to be considered in certain diagrams with a fermion loop. The relevant effective Feynman rule for the substructure is

\[
\left( \begin{array}{c} \text{Diagram 38} \end{array} \right) = z \frac{i}{4} (M - N)M$\bullet$ $\psi$ $\psi$ $\psi$ $\psi$ $\psi .
\]

(A.12)
The constant $z = \pm 1$ thereby parameterizes a sign discrepancy in the literature. For a
gauge interaction between two sites of the triangle we obtain with $z^2 = 1$ the following rule

$$
- \frac{i}{4} \left( \begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\right) = z = \pm 1 \left( \begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\right) . 
$$

(A.13)

The above rule simplifies, if the external momentum at one of the three scalar-fermion
interactions can be set to zero. Two of the first three contributions on the r.h.s. then
cancel against each other, and the last term vanishes. This is the case in all required
diagrams at four-loops.

The subdiagram

$$
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\end{array}
= \frac{i}{8} \int \frac{d^D k}{(2\pi)^2} \frac{d^D l}{(2\pi)^2} \text{tr}((l - k)\gamma^\mu(p_1 - k)(p_4 - k)(p_3 - k))\epsilon_{\nu\rho\mu\nu}(p_1 - l)^\rho(l - p_2)^\nu
$$

(A.14)

is only needed in two special cases, in which either $p_1 = p_4$ or $p_3 = p_4$. The effective
Feynman rules respectively read

$$
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\end{array}
= - \frac{i}{4} \left( \begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\right) ,
$$

(A.15)

$$
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\end{array}
= - \frac{i}{4} \left( \begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\right) .
$$

(A.16)

The substructures involving only scalar-fermion vertices lead to the following effective
Feynman rule for a fermion square

$$
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\end{array}
= - \frac{1}{8} \left( \begin{array}{c}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\end{array}
\right) .
$$

(A.16)
We then immediately obtain the rules for the remaining relevant subdiagrams

\[
\begin{align*}
Y^\dagger & \quad Y^\dagger = -\frac{i}{2} \left( \begin{array}{c}
Y \quad Y \quad Y \quad Y \\
\end{array} \right), \\
Y^\dagger & \quad Y^\dagger = \frac{i}{8} \left( \begin{array}{c}
Y \quad Y \quad Y \quad Y \\
\end{array} \right), \quad (A.17) \\
Y^\dagger & \quad Y^\dagger = \frac{i}{2} \left( \begin{array}{c}
Y \quad Y \quad Y \quad Y \\
\end{array} \right).
\end{align*}
\]

As in the case of a fermion triangle, also in this case the rules simplify if the external momentum entering one of the vertices which participates only in a single loop of the full four-loop diagram is set to zero.

**B Flavour structures**

Besides the momentum-loop and colour factors of the diagrams, we also have to work out their flavour structures. Therefore, in the following we focus on the flavour non-neutral vertices and their combinations and compute the flavour flow through the different structures. For our computation we only need the terms which involve non-trivial flavour permutations without any subtraces.

The flavour structure of the six-scalar vertex explicitly reads

\[
\frac{1}{12} (F_{ACE}^{BDE} + F_{CEA}^{DFB} + F_{EAC}^{FBD}) = \frac{1}{4} \left( \delta_A^B \delta_C^D \delta_E^F + \delta_A^F \delta_C^B \delta_E^D - 2 \delta_A^B \delta_C^F \delta_E^D - 2 \delta_A^F \delta_C^D \delta_E^B - 2 \delta_A^B \delta_C^B \delta_E^F + 4 \delta_A^D \delta_C^F \delta_E^B \right).
\]

It can be visualized as

\[
\begin{array}{c}
\bigstar \quad \left( \quad + \quad \right) \quad -2 \quad \left( \quad -2 \quad \right) \quad -2 \quad \left( \quad +4 \quad \right) \\
\end{array}
\]

The last term indicates the non-trivial permutation of the flavour at two next-to-nearest neighboured sites. It appears with prefactor one. The other terms can also lead to non-trivial permutations without subtraces when two six-scalar vertices are combined as in (4.3).

We have to work out the flavour structures of the following contractions of vertices
involving fermions

\[ \begin{align*}
\hat{\mathcal{C}} &= -4 \left( +4 \right) + 4 \left( +4 \right) + 4 \left( +4 \right) + 4 \left( +4 \right) + 4 \left( +4 \right) + 4 \left( +4 \right) \\
&\quad + 4 \left( +4 \right) + 4 \left( +4 \right) \\
&\quad - 8 \left( +4 \right) - 8 \left( +4 \right) - 8 \left( +4 \right) - 8 \left( +4 \right) + 16 \left( +4 \right), \\
\hat{\mathcal{B}} &= 2 \left( -4 \right) \left( -4 \right) \left( -8 \right) \left( +8 \right), \\
\hat{\mathcal{A}} &= -4 \left( +8 \right) \left( +8 \right) \left( +20 \right) \left( -16 \right), \\
\hat{\mathcal{D}} &= -2 \left( -4 \right) \left( -4 \right) \left( +4 \right) \left( +4 \right), \\
\hat{\mathcal{E}} &= -2 \left( +4 \right) \left( +4 \right) \left( +4 \right) \left( +4 \right) - 8 \left( +4 \right).
\end{align*} \]

(B.3)

In case of the first equality, the structures in the last line on the r.h.s. can either lead to a permutation of field flavours or to flavour contractions. This depends on how the scalar lines are connected to the composite operator in the full diagram. In the other equalities, only the respective last term on the r.h.s. is a non-trivial permutation.

C Permutation structures

Since the operators in the \( SU(2) \times SU(2) \) subsector are free of subtraces, the corresponding dilatation operator is given by an expansion in terms of the permutation structures which are defined in (2.3).

The definition is similar as in the \( \mathcal{N} = 4 \) SYM case, \[3\], but each permutation in the product permutes neighbouring fields at either odd or even sites. Also, the permutation structures are shifted by two sides when inserted along the chain and not by one site as in the \( \mathcal{N} = 4 \) SYM case. Permutations at even and odd sites commute with each other, and only do not commute if they have one position in common, i.e. if the integers in the argument lists of the permutation structures differ by two. The rules for manipulation which have to be modified compared to the ones in the \( \mathcal{N} = 4 \) SYM case are

\[ \begin{align*}
\{ \ldots, a, b, \ldots \} &= \{ \ldots, b, a, \ldots \}, \quad |a - b| \neq 2, \\
\{ a, \ldots, b \} &= \{ a + 2n, \ldots, b + 2n \}.
\end{align*} \]

(C.1)

We can shift all odd integers to the left and all even integers in the argument lists to
the right. The basis of permutation structures at four loops hence reads
\[ \{ \}, \{ 1 \}, \{ 2 \}, \{ 1, 2 \}, \{ 2, 3 \}, \{ 1, 3 \}, \{ 3, 1 \}, \{ 2, 4 \}, \{ 4, 2 \}, \] (C.2)
where the first three elements also appear at two loops. The dilatation operator decomposes into three parts, acting on even, on odd and on mixed sides, respectively. These parts are defined by containing permutation structures with respectively only odd, only even and both odd and even arguments.

D Transformations of the Feynman diagrams

Let \( c(\lambda, \hat{\lambda})\{a_1, \ldots, a_m\} \) be the result associated with a certain Feynman graph. We can easily construct the contribution from the analogous graph shifted by one step along the chain of elementary fields from the corresponding operator. This exchanges all fields as
\[
A \leftrightarrow \hat{A}, \quad Y \leftrightarrow Y^\dagger, \quad \psi \leftrightarrow \psi^\dagger, \quad c \leftrightarrow \hat{c}, \quad c^* \leftrightarrow \hat{c}^*,
\] (D.1)
and it exchanges the colour loops of both gauge groups, i.e. it exchanges \( \lambda \leftrightarrow \hat{\lambda} \). We also have to consider several sign changes due to the exchange of certain vertices and of the propagators of the two gauge fields. By \( P_x \) and \( V_x \) we denote the multiplicities with which the corresponding propagator or vertex of type \( x \) appears in the graph. In \( x \) we thereby do not distinguish between the pairs of fields related as in (D.1), since the multiplicities have to consider all propagators or vertices that are related by one or several of the transformations in (D.1) and by reorderings of the fields. A shift by one side, denoted by the operator \( S \), then transforms the contribution from a Feynman graph as
\[
S(c(\lambda, \hat{\lambda})\{a_1, \ldots, a_m\}) = (-1)^{P_A + V_{A^2} + V_{A^2} + V_{Y^2} + V_{\psi^2}} c(\hat{\lambda}, \lambda)\{-a_1, \ldots, -a_m\},
\] (D.2)
Furthermore, the reflection of a given non reflection-symmetric Feynman graph contributes to the perturbation series. A reflection exchanges \( A \leftrightarrow \hat{A} \) and it changes also the sign of all loop momenta. The type of the other field lines is kept fixed. The order of \( Y \) and \( Y^\dagger \) at the composite operator and at the external lines does not change under reflection, if the number of interacting neigboured fields is odd. If this number is even, a reflection of the diagram exchanges the fields with their conjugates. If the composite operator is kept fixed in the latter case, a reflection of the subdiagram not involving the composite operator then requires a shift by one site, before it can be reattached to the composite operator. The so defined reflection, denoted by the operator \( R \), transforms a Feynman graph as
\[
R(c(\lambda, \hat{\lambda})\{a_1, \ldots, a_m\}) = (-1)^{P_A + V_{AY^2} + V_{AY^2} + V_{Y^2} + V_{\psi^2}} c(\hat{\lambda}, \lambda)\{-a_1, \ldots, -a_m\},
\] (D.3)
and it preserves the type of the flavour permutation, i.e. its action on even or odd sites.

The shift (D.2) by one site realizes a parity transformation. An individual Feynman diagram has definite parity whenever \( c(\lambda, \hat{\lambda}) = \pm c(\hat{\lambda}, \lambda) \) with an eigenvalue given by this sign and the additional sign determined from its propagator and vertex content. There are several diagrams which individually do not have definite parity. In some cases
where the diagram is not reflection symmetric, the linear combination together with the reflected diagram has definite parity. In cases of diagrams which involve more than three legs in the interaction, the diagram contains permutation structures which contribute to the dilatation operator at odd and at even sites (and also to the mixed component). Then, also the shifted diagram has to be considered, and their sum has again definite parity.

### E Two-loop self-energy of the scalar field

The two-loop self-energy of the scalar fields is given by the following sum of diagrams

$$
\Sigma_Y = \sum_{\text{diagrams}} \text{graph}_1 + \text{graph}_2 + \text{graph}_3 + \text{graph}_4 + \text{graph}_5 + \text{graph}_6 + \text{graph}_7 + \text{graph}_8 + \text{graph}_9 + \text{graph}_{10} + \text{graph}_{11} + \text{graph}_{12} + \text{graph}_{13} + \text{graph}_{14} + \text{graph}_{15} + \text{graph}_{16} + \text{graph}_{17} + \text{graph}_{18} + \text{graph}_{19} + \text{graph}_{20} + \text{graph}_{21} + \text{graph}_{22} + \text{graph}_{23} + \text{graph}_{24} + \text{graph}_{25} + \text{graph}_{26} + \text{graph}_{27} + \text{graph}_{28} + \text{graph}_{29}.
$$

(E.1)

Using the effective Feynman rules (A.9) and (A.10), considering that another internal flavour trace arises in the first diagram giving an additional factor of 4, and that the additional scalar or fermion propagator respectively yield an extra factor of $-i$ or $i$, the individual contributions read

$$
\begin{align*}
\text{graph}_1 &= 8i \left( \frac{(4\pi)^2}{k^2} \right) MN \text{graph}_1 = i \frac{\lambda \hat{\lambda}}{4} \left( \frac{4}{3\epsilon} + \frac{8}{3} (3 - \gamma + \ln 4\pi) \right), \\
\text{graph}_2 &= 12i \left( \frac{(4\pi)^2}{k^2} \right) MN \text{graph}_2 = i \frac{\lambda \hat{\lambda}}{4} \left( \frac{2}{3\epsilon} + 4 (3 - \gamma + \ln 4\pi) \right), \\
\text{graph}_3 &= \frac{1}{2} (-i) \left( \frac{(4\pi)^2}{k^2} \right) M^2 \text{graph}_3 = i \frac{\lambda^2}{4} \left( - \frac{1}{12\epsilon} - \frac{1}{6} (3 - \gamma + \ln 4\pi) \right), \\
\text{graph}_4 &= -2i \left( \frac{(4\pi)^2}{k^2} \right) MN \text{graph}_4 = i \frac{\lambda \hat{\lambda}}{4} \left( \frac{1}{3\epsilon} + \frac{2}{3} (3 - \gamma + \ln 4\pi) \right).
\end{align*}
$$

(E.2)

The relevant two-loop integral is given in terms of $G$-functions (H.3) as $G_2(1, 1)G(-\lambda, 1)$. The contributions which contain quartic as well as cubic vertices all vanish according to the effective Feynman rules with gauge fields (A.9). The same holds for the diagrams in the second line which contain a fermion bubble. The contributions with only cubic
vertices become

\[
\begin{align*}
\text{vertices become} & = 2(-i)\frac{(4\pi)^2}{k^2} MN \left( - \text{vertex} + \text{vertex} - \text{vertex} - \text{vertex} + \text{vertex} \right) \\
& = i\frac{\lambda\hat{\lambda}}{4} \left( \frac{2}{3\varepsilon} - \frac{\pi^2}{2} + \frac{4}{3} \left( 3 - \gamma + \ln 4\pi \right) \right), \\
& = i\frac{\lambda^2}{4} \left( \frac{1}{3\varepsilon} - \frac{\pi^2}{4} + \frac{2}{3} \left( 3 - \gamma + \ln 4\pi \right) \right), \\
& = 4(-i)\frac{(4\pi)^2}{k^2} M^2 \left( - \text{vertex} + \text{vertex} \right) \\
& = i\frac{\lambda^2}{4} \left( - \frac{8}{3\varepsilon} + 8 - \frac{16}{3} \left( 3 - \gamma + \ln 4\pi \right) \right),
\end{align*}
\]

(E.3)

where the integrals have first been expressed as products of two $G$-functions, or have been taken from the tables of integrals in appendix J.1. The expression is then expanded in powers of $\varepsilon$. We thereby have to keep the pole and finite parts.

Summing up the above diagrams and their reflected copies as they appear in (E.1), the amputated scalar self-energy contribution apart from the Wick rotation (the result has to be multiplied by a factor $i^2 = -1$) then becomes

\[
\Sigma_Y = Y \bullet Y^\dagger = i \left[ \frac{\lambda\hat{\lambda}}{4} \left( \frac{3}{2\varepsilon} - \frac{3}{2} \pi^2 + 3 \left( \frac{25}{3} - \gamma + \ln 4\pi \right) \right) \\
+ \frac{(\lambda - \hat{\lambda})^2}{4} \left( \frac{1}{4\varepsilon} - \frac{\pi^2}{4} + \frac{1}{2} \left( 3 - \gamma + \ln 4\pi \right) \right) \right]^{-1+2\varepsilon},
\]

where the last propagator factor on the r.h.s. captures the momentum dependence. Its weight label indicates the exponent of $\frac{1}{p^2}$, where $p$ is the external momentum. The pole part of the above result coincides with the result in [17].

\section{Two-loop renormalization of the six-scalar vertex}

The two-loop renormalization of the six-scalar vertex allows us to fix a sign discrepancy in the literature\footnote{The antisymmetric parts in the product of two $\gamma$ matrices (A.1) in [31] and [7] differ by a sign.} parameterized by $z = \pm 1$ which affects the four-loop diagrams with fermion triangles, c.f. (4.26), (4.27), and (4.28).

We only compute the two-loop renormalization of the component of the scalar six-vertex that permutes the flavour, i.e. which has the flavour structure given by the last
term on the r.h.s. of (B.2), and denote it by \{1\}. The relevant graphs are given by

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array} = -i \frac{(4\pi)^2}{k^2} MN K(I_2)(-\{1\}) = i \frac{\lambda \hat{\lambda}}{4 \varepsilon} \{1\}, \\
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array} = \frac{i}{2} \frac{(4\pi)^2}{k^2} MN K(I_{22B})(8\{1\}) = -i \frac{\lambda \hat{\lambda}}{4 \varepsilon} \{1\}, \\
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array} = \frac{\lambda \hat{\lambda}}{4 \varepsilon} \{1\}, \\
\begin{array}{c}
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\begin{array}{c}
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\end{array} = \frac{\lambda \hat{\lambda}}{4 \varepsilon} \{1\}, \\
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\begin{array}{c}
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\gamma^* \\
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\end{array}
\end{array} = \frac{\lambda \hat{\lambda}}{4 \varepsilon} \{1\}, \\
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array} = 3 \frac{\lambda \hat{\lambda}}{2 \varepsilon} i(z - 1) \{1\}.
\end{align*}
\]

where the relevant two-loop integrals are explicitly given in (J.2), (J.3) and (J.18). The part which contributes to flavour permutations is determined using (B.2) and (B.3).

Neglecting a factor \(i^2 = -1\) for the Wick rotation, the renormalization of the six-scalar vertex is then found as

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma^* \\
\gamma
\end{array}
\end{array} = 3 \frac{\lambda \hat{\lambda}}{2 \varepsilon} i(z - 1) \{1\}.
\end{align*}
\]

Non-renormalization of the coupling and hence superconformal invariance requires \(z = 1\), which corresponds to the sign choice in [31].
Cancellation of double poles

In the expansion of the logarithm (3.4) the four-loop contribution $Z_4$ is combined with the square of the two-loop contribution $Z_4$ at four-loop order. This combination ensures the cancellation of double poles in $Z_4$ which are the remnants from two-loop subdivergences. Here, we explicitly demonstrate the cancellation of these double poles.

The two-loop graphs are given by

\[ \rightarrow \left( \frac{4\pi}{k^2} \right)^2 MN K(I_2) \left( -\frac{1}{2} \{} + \{1\} \right) = \frac{\lambda \hat{\lambda}}{4 \varepsilon} \left( -\frac{1}{2} \{} + \{1\} \right), \]

\[ \rightarrow \left( \frac{4\pi}{k^2} \right)^2 M^2 \frac{1}{2} K(I_{22,4}) \{ \} = \frac{\lambda^2}{4 \varepsilon} \{ \}, \]

\[ = i K(\Sigma_Y) \{ \} = -\left( \frac{\lambda \hat{\lambda}}{4 \varepsilon} \frac{3}{2} + \frac{(\lambda - \hat{\lambda})^2}{4 \varepsilon} \right) \{ \}, \]

where here the arrows indicate that we have neglected contributions to flavour traces, but we kept the contribution to the identity. The integrals are explicitly given in (J.2), (J.3), and the pole part of the scalar self-energy contribution $\Sigma_Y$ can be found in (4.18). We have also included a sign due to the double Wick rotation. The flavour permutation structure of the six-scalar vertex is given in (B.2). In the second diagram, only the second substructure of (4.10) gives a logarithmically divergent contribution to the identity in flavour space.

The above results describe the sum of diagrams, in which the first leg is fixed to an odd side, c.f. the definition of the permutation structures in (2.3). For the final result, we have to include also the sum over even side legs, which is easily obtained employing the shift operator $S$ in (D.2).

The negative of the sum of the diagrams at odd and even sites with a factor $\frac{1}{2}$ for the scalar self-energy contributions then yields the two-loop contribution to the renormalization constant as

\[ \bar{\lambda}^2 Z_2 = -\sum_{i=1}^L \left( \begin{array}{c} 2 \bigotimes \\ \bigoplus \end{array} + \begin{array}{c} 1 \bigotimes \\ \bigoplus \end{array} + \frac{1}{2} \begin{array}{c} 0 \bigotimes \\ \bigoplus \end{array} \right) = \frac{\lambda \hat{\lambda}}{4 \varepsilon} (2\{} - \{1\} - \{2\} \right), \]

where we have indicated the sum over the sites explicitly. The above result is consistent with [17]. It has an obvious decomposition into two parts acting exclusively on even and on odd sites, respectively. The square of the above result can be decomposed as follows

\[ \frac{1}{2} Z_2^2 = Z_{22,dc} + Z_{22,S} + Z_{22,v} + Z_{22,s}. \]
The individual terms are given by

\[
\bar{\lambda}^4 Z_{22,dc} = \sum_{j \geq i + 3}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right) + \sum_{j \geq i + 3}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right) + \frac{1}{2} \sum_{j \leq i - 1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right)
\]

\[
\bar{\lambda}^4 Z_{22,s} = \frac{1}{2} \sum_{i=1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right) + \frac{1}{2} \sum_{i=1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right)
\]

\[
\bar{\lambda}^4 Z_{22,v} = \frac{1}{2} \sum_{i=1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right) + \frac{1}{2} \sum_{i=1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right)
\]

\[
\bar{\lambda}^4 Z_{22,s} = \frac{1}{4} \sum_{i=1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right) + \frac{1}{4} \sum_{i=1}^{L} \left( \begin{array}{c}
\begin{array}{c}
\bullet \vspace{10pt}
\\vspace{10pt}
\end{array}
\end{array}
\right)
\]

\[
\rightarrow \frac{1}{2} \frac{(4\pi)^4}{k^4} (M^2 + N^2) MK(I_2) K(I_{22}) = \frac{\bar{\lambda}^4 (2\lambda \hat{\lambda} + (\lambda - \hat{\lambda})^2)}{16} \frac{1}{2\varepsilon^2} \{1\} ,
\]

where the arrows indicate that we have neglected all contributions to the identity in flavour space, and also that we have only written half of all contributions, namely those with permutation structures with an odd entry as their first entry. It is sufficient to show the cancellation of the double poles for the displayed contributions only. The cancellations work analogously for the terms involving permutations with an even entry as their first entry. The gray line in the middle of the above diagrams reminds us that the diagrams are direct products of the parts above and below that line. The loop integrals are direct products of two loop integrals. The can be read off by contracting the bold gray and black line each to a point, keeping the lines which enter the bold gray line from below as separate external lines of the second integral. We have chosen the above graphical presentation to make an identification with the individual four-loop diagrams more evident.

The contribution $Z_{22,dc}$ cancels the double poles from those diagrams which become disconnected if the composite operator is removed. The corresponding four-loop contributions are themselves squares of two-loop diagrams. They hence only contribute to the double poles and have not been considered in our calculation in the first place. Their
Cancellation in combination with $Z_{22,dc}$ is obvious. The sum of the remaining diagrams is given by

$$Z_{22,s} + Z_{22,v} + Z_{22,s} \rightarrow \frac{1}{10} \frac{1}{\epsilon^2} \left( \frac{1}{2} \{(1,3) + (3,1)\} + \{(1,2) - \frac{15}{4\epsilon^2}(1) + \sigma^2 \frac{1}{8}(1)\} \right).$$

This result precisely cancels the double poles of the respective terms in $Z_4$ in (5.1) in the expansion of the logarithm (3.4).

The above cancellation works for $z = 1$, since according to (F.2) in this case the six-scalar vertex is not renormalized. In a theory where this vertex is renormalized, one has to remember that it is the renormalized six vertex that appears in $Z_2$. There is hence an additional contribution at four loops which comes from the two-loop part of the respective vertex renormalization constant in $Z_2$ itself. If this is taken into account, double pole cancellation should work also in this case.

**H  G-functions**

To compute the required integrals in dimensional reduction in $D$ dimensions, we make use of $G$-functions. The scalar $G$-function is defined as follows

$$G(\alpha, \beta) = \frac{1}{(2\pi)^D} \int \frac{d^Dl}{l^{2\alpha} (l - p)^{2\beta}} \bigg|_{p^2=1} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\alpha + \beta - D/2) \Gamma(D/2 - \alpha) \Gamma(D/2 - \beta)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(D - \alpha - \beta)},$$

where the integral is defined in Euclidean space. To analytically continue to Minkowski space, we have to identify real time as $t = -i\tau$ in terms of Euclidean time, and hence multiply each loop integral by $i$. The non-trivial tensor $G$-functions can be obtained from this by introducing traceless symmetric products of the loop momentum in the numerator. Traceless symmetric tensor indices are embraced by parentheses. We then define

$$\frac{p^{(\mu_1 \ldots \mu_\alpha)}}{p^{2(\alpha + \beta - D/2)}} G_1(\alpha, \beta), \quad G_1(\alpha, \beta) = \frac{1}{(2\pi)^D} \int \frac{d^Dl \{l^{(\mu_1 \ldots \mu_\alpha)}\}}{l^{2\alpha} (l - p)^{2\beta}} \bigg|_{p^2=1}.$$

For a one-loop integral with a single or two contracted momenta in the numerator we define the $G$-functions without parentheses as

$$\begin{align*}
\begin{array}{c}
\alpha \\
\beta
\end{array} & = \frac{1}{p^{2(\alpha + \beta - D/2)}} G(\alpha, \beta), \\
\begin{array}{c}
\alpha \\
\beta
\end{array} & = \frac{p^\mu}{p^{2(\alpha + \beta - D/2)}} G_1(\alpha, \beta), \quad G_1(\alpha, \beta) = \frac{1}{2} \left( G(\alpha, \beta) - G(\alpha, \beta - 1) + G(\alpha - 1, \beta) \right), \\
\begin{array}{c}
\alpha \\
\beta
\end{array} & = \frac{1}{p^{2(\alpha + \beta - 1 - D/2)}} G_2(\alpha, \beta), \quad G_2(\alpha, \beta) = G(\alpha, \beta) - G(\alpha - 1, \beta),
\end{align*}$$

where an arrow on a propagator denotes the appearance of the momentum running along this line in the numerator, and two arrows of identical type denote a contraction of the respective momenta. The function $G_1$ is identical to the function $G_{(1)}$. The function
for a traceless symmetric product of two equal momentum factors in the numerator explicitly reads

\[ G(2)(\alpha, \beta) = \frac{D}{4(D-1)} \left( \left( G(\alpha, \beta) + G(\alpha - 2, \beta) + G(\alpha, \beta - 2) 
+ 2 \left( \frac{D - 2}{D} G(\alpha - 1, \beta) - G(\alpha, \beta - 1) - G(\alpha - 1, \beta - 1) \right) \right) \right). \]  

(H.4)

I Triangle rules

Defining the functions

\[ \Delta(\alpha, \beta) = -\frac{\alpha G(\alpha + 1, \beta) + \beta G(\alpha, \beta + 1)}{\alpha + \beta + 2 - D}, \]
\[ C(\alpha, \beta) = \frac{\alpha}{\alpha + \beta + 2 - D}, \]  

(I.1)

the scalar triangle rule is given by \[38,39\]

\[ \begin{array}{c}
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We then immediately find from (I.2) and (I.4) also the following rule

\[
\begin{align*}
\alpha & \rightarrow \beta \\
\beta & \rightarrow \alpha
\end{align*}
\]

\[
= - \left( \Delta_1(\alpha, \beta) - \frac{1}{2} \Delta_2(\alpha, \beta) \right) \left( \alpha \rightarrow \beta \right)
\]

\[
+ \frac{1}{2(\alpha + \beta + 1 - D)} \Delta_2(\alpha, \beta) \left( \alpha \rightarrow \beta \right)
\]

\[
+ (\tilde{C}(\alpha, \beta) - C(\alpha, \beta)) \left( \alpha \rightarrow \beta \right)
\]

\[
+ C(\alpha, \beta) \left( \alpha \rightarrow \beta \right).
\]

(II.5)

Defining the functions

\[
\Delta_2(\alpha, \beta) = \frac{1}{2(\alpha + \beta + 2 - D)} \left( \alpha \left( G(\alpha + 1, \beta - 1) - G(\alpha + 1, \beta) \right) + \beta \left( G(\alpha - 1, \beta + 1) - G(\alpha, \beta + 1) \right) \right),
\]

(II.6)

we obtain the triangle rule with two contracted momenta in the numerator as

\[
\begin{align*}
\alpha & \rightarrow \beta \\
\beta & \rightarrow \alpha
\end{align*}
\]

\[
= \Delta_2(\alpha, \beta) \left( \alpha \rightarrow \beta \right)
\]

\[
- D(\alpha, \beta) \left( \alpha \rightarrow \beta \right)
\]

\[
+ C(\alpha, \beta) \left( \alpha \rightarrow \beta \right).
\]

(II.7)

### J Tables of integrals

In the following we list all required integrals. The integrals are regularized by dimensional regularization in Euclidean space with dimension

\[
D = 2(\lambda + 1), \quad \lambda = \frac{1}{2} - \varepsilon.
\]

(J.1)

The parameter \( \lambda \) in this appendix always assumes the above value. It has nothing to do with the 't Hooft coupling that appears in the main text and which is also denoted by \( \lambda \). The integrals have a simple dependence on the external momentum \( p_\mu \).
J.1 Two-loop integrals

The simplest two-loop integral is given by

\[ I_2(\alpha) = \frac{1}{p^2(\alpha + 2\lambda - \beta)} = G(1, 1)G(1 - \lambda, \alpha) \, . \]  

(J.2)

It is proportional to \( \frac{1}{p^2(\alpha + 2\lambda - \beta)} \). In the main text we abbreviate \( I_2 = I_2(1) \), which is the simplest logarithmically divergent integral in three dimensions.

The two-loop integrals with a bubble and two contracted momenta in their numerators read

\[
I_{22A} = G_2(1, 1)G(1 - \lambda, 1),
\]

\[
I_{22B} = G_1(1, 1)G(1 - \lambda, 1),
\]

\[
I_{22C} = G_1(1, 1)G_2(2 - \lambda, 1),
\]

\[
I_{22D} = G(1, 1)G_2(2 - \lambda, 1)
\]

(J.3)

They are proportional to \( \frac{1}{p^2(1 - 2\lambda)} \).

We also need the following 2-loop scalar integral which is proportional to \( \frac{1}{p^2(\alpha + 2\lambda - 2\beta)} \)

\[
I_{20\alpha}(\alpha) = G(1, 1)\left(\Delta(\alpha, 1) + C(\alpha, 1)G(2 - \lambda, \alpha + 1) + C(1, \alpha)G(\alpha + 1 - \lambda, 2)\right), \]

(J.4)

where one propagator has weight \( \alpha \), i.e. its squared momentum in the denominator has exponent \( \alpha \).

More difficult to obtain are the two-loop integrals in which the central line has a propagator weight differing from one. To compute the needed graphs, we work with \( p \)-space weights in \( p \)-space, not with \( p \)-space weights in \( x \)-space as in the rest of the paper. The corresponding special 2-loop graph in \( p \)-space looks the same as in \( x \)-space, but two lines which are connected at one point are fused with the \( G \)-functions, while the weights of two lines forming a loop can be directly added to yield the weight of the single line which replaces the loop. First, we can use the point transformations described in [40] to relate

\[
\alpha \quad = \quad 2\lambda + 1 - \alpha - \beta \quad = \quad 2\lambda - \beta \quad = \quad \beta + 1 - \lambda \quad = \quad 3\lambda - \alpha - \beta \quad = \quad \frac{1}{p^2(\alpha + 2\lambda - \beta)} \, . \]

(J.5)

where in the first, second and third equality the conformal transformation of inversion, the adding of an external line to make the vertex unique, and the conformal transformation of inversion, have been respectively used. We can then use the transformation
to evaluate the following integrals

\[
\begin{align*}
\frac{3\lambda - 1}{\lambda} &= G(1, 1)^2 = \frac{1}{(8\pi)^2} \pi^2, \\
\frac{3\lambda - 2}{\lambda} &= I_{20a}(1) = \frac{1}{(8\pi)^2} \left( -\frac{2}{\varepsilon} - 4(1 - \gamma + \ln 4\pi) \right).
\end{align*}
\] (J.6)

The first integral is finite, and thus its finite part is identical to the finite part of one of the required integrals. We obtain

\[
I_{20a} = \frac{1 - \lambda}{1 - \lambda} = \frac{1}{(8\pi)^2} \pi^2
\] (J.7)

up to terms of order \(\mathcal{O}(\varepsilon)\).

The second of the above integrals is correctly reproduced by the \(F\)-function of [11] as

\[
\frac{1}{(4\pi)^3} F(1, 3\lambda - 2, 1),
\]
which is given as a converging double series for the given values. We can then use the \(F\)-function to also compute another required integral with a different \(\varepsilon\)-dependence of the central line. It reads

\[
I_{20b} = \frac{1 - \lambda}{1 - \lambda} = \frac{1}{(4\pi)^3} F(1, -\lambda, 1) = \frac{1}{(8\pi)^2} \left( \frac{2}{\varepsilon} + 4(1 - \gamma + \ln 4\pi) \right).
\] (J.8)

The above results are used to compute the following integrals which are proportional to \(\frac{1}{p^{\varepsilon-2\lambda}}\)

\[
\begin{align*}
I_{22a} &= \frac{1 - \lambda}{1 - \lambda} = \frac{1}{2} \left( 2G(1 - \lambda, 1)G(2 - 2\lambda, 1) - I_{20a} \right) \\
&= \frac{1}{(8\pi)^2} \left( \frac{1}{\varepsilon} + 6 - \frac{\pi^2}{2} - 2\gamma + 2 \ln 4\pi \right), \\
I_{22b} &= \frac{1 - \lambda}{1 - \lambda} = \frac{1}{2} I_{20b} = \frac{1}{(8\pi)^2} \left( \frac{1}{\varepsilon} + 2 - 2\gamma + 2 \ln 4\pi \right), \\
I_{22c} &= \frac{1 - \lambda}{1 - \lambda} = \frac{1}{2} \left( 2G(1 - \lambda, 1)G(2 - 2\lambda, 1) - I_{20b} \right) = \frac{1}{(8\pi)^2} \pi^4, \\
I_{22d} &= \frac{1 - \lambda}{1 - \lambda} = I_{22a} - I_{22b} = \frac{1}{(8\pi)^2} \left( 4 - \frac{\pi^2}{2} \right).
\end{align*}
\] (J.9)
\[ I_{222\beta} = \frac{1}{2}(I_{22\beta} - G_2(2 - \lambda, 1)G(2 - 2\lambda, 1) - G_1(2 - \lambda, 1)G(2 - 2\lambda, 1)) , \]

\[ I_{222\beta} = I_{222\beta} - I_{22\beta} . \]  

We need the following integrals with a contraction of one loop momentum in the numerator with the external momentum which are proportional to \( \frac{1}{p^{2(\alpha + 1 - 2\lambda)}} \)

\[ I_{21\alpha}(\alpha) = \frac{1}{2}(G(\alpha, 1)G(\alpha + 1 - \lambda, 1) + I_{20\alpha}(\alpha) - I_{20\alpha}(\alpha - 1)) , \]

\[ I_{21\beta}(\alpha) = \frac{1}{2}(G(\alpha, 1)G(\alpha + 1 - \lambda, 1) - I_{20\alpha}(\alpha) - I_{20\alpha}(\alpha - 1)) , \]

\[ I_{21\epsilon}(\alpha) = \frac{1}{2}(\Delta_-(\alpha, 1) + \Delta_+(\alpha, 1) + \tilde{C}(\alpha, 1)G_1(2 - \lambda, \alpha + 1) - \tilde{C}(1, \alpha)G_1(\alpha + 1 - \lambda, 2)) , \]

\[ I_{21\mu}(\alpha) = \frac{1}{2}(G(1, 1)G(2 - \lambda, \alpha) + G(1, 1)G(\alpha + 1 - \lambda, 1) + I_{20\alpha}(\alpha)) , \]

\[ I_{21\nu}(\alpha) = \frac{1}{2}(G(1, 1)G(2 - \lambda, \alpha) + G(1, 1)G(\alpha + 1 - \lambda, 1) - I_{20\alpha}(\alpha)) , \]

We need the following integrals with two contracted loop momenta in the numerator
which are proportional to \( \frac{1}{\nu^{(\alpha+1-2\lambda)}} \),

\[
I_{22a}(\alpha) = \frac{1}{2}(G(\alpha, 1)G(\alpha + 1 - \lambda, 1) - I_{20a}(\alpha) + I_{20a}(\alpha - 1)) ,
\]

\[
I_{22b}(\alpha) = \frac{1}{2}(G(\alpha, 1)G(\alpha + 1 - \lambda, 1) - G(1, 1)G(\alpha + 1 - \lambda, 1)) ,
\]

\[
I_{22c}(\alpha) = \frac{1}{2}(-G(\alpha, 1)G(\alpha + 1 - \lambda, 1) + G(1, 1)G(\alpha + 1 - \lambda, 1)) ,
\]

\[
I_{22d}(\alpha) = I_{22a}(\alpha) - I_{22b}(\alpha) ,
\]

\[
I_{22e}(\alpha) = \frac{1}{2}(G(1, 1)G(\alpha, 1) - G(1, 1)G(2 - \lambda, \alpha) + I_{20a}(\alpha - 1)) ,
\]

\[
I_{22f}(\alpha) = I_{22a}(\alpha) - I_{22c}(\alpha) ,
\]

\[
I_{22g}(\alpha) = \frac{1}{2}(-G(1, 1)G(\alpha, 1) + G(1, 1)G(2 - \lambda, \alpha) + I_{20a}(\alpha - 1)) ,
\]

\[
I_{22h}(\alpha) = \frac{1}{2}(-G(1, 1)G(\alpha, 1) + G(\alpha, 1)G(\alpha + 1 - \lambda, 1)
- G(1, 1)G(\alpha + 1 - \lambda, 1)) ,
\]

\[
I_{22i}(\alpha) = \frac{1}{2}(-G(1, 1)G(\alpha, 1) - G(1, 1)G(2 - \lambda, \alpha) + I_{20a}(\alpha - 1)) ,
\]

\[
I_{22j}(\alpha) = \frac{1}{2}(G(1, 1)G(2 - \lambda, \alpha) + G(1, 1)G(\alpha + 1 - \lambda, 1) - I_{20a}(\alpha)) ,
\]

We need the following integrals in which two loop momenta in the numerator are
contracted with the external momentum and which are proportional to \( \frac{1}{\mu^{\alpha-2\lambda}} \)

\[
I_{211\alpha}(\alpha) = \frac{1}{2} (G_1(1, \alpha)G_1(\alpha + 1 - \lambda, 1) + G(1, \alpha)G_1(1, \alpha + 1 - \lambda)
- I_{21\alpha}(\alpha) - I_{21\alpha}(\alpha - 1)) ,
\]

\[
I_{211\beta}(\alpha) = \frac{1}{2} (G_1(1, \alpha)G_1(\alpha + 1 - \lambda, 1) + I_{21\alpha}(\alpha) - I_{21\alpha}(\alpha - 1)) ,
\]

\[
I_{211\gamma}(\alpha) = \frac{1}{2} (-G(1, 1)G_1(2 - \lambda, \alpha) + G(1, 1)G_1(1, \alpha + 1 - \lambda) + I_{21\alpha}(\alpha)) ,
\]

\[
I_{211\delta}(\alpha) = \frac{1}{2} (-G(1, 1)G_1(2 - \lambda, \alpha) + G(1, 1)G_1(1, \alpha + 1 - \lambda) - I_{21\alpha}(\alpha)) ,
\]

\[
I_{211\epsilon}(\alpha) = \frac{1}{2} (G_1(1, \alpha)G_1(\alpha + 1 - \lambda, 1) - I_{21\alpha}(\alpha) - I_{21\alpha}(\alpha - 1)) ,
\]

\[
I_{211\zeta}(\alpha) = \frac{1}{2} (G_1(1, 1)G_1(\alpha, 2 - \lambda) - G(1, 1)G_1(1, \alpha + 1 - \lambda, 1) + I_{21\epsilon}(\alpha)) ,
\]

\[
I_{211\eta}(\alpha) = \frac{1}{2} (G_1(1, 1)G_1(\alpha, 2 - \lambda) - G(1, 1)G_1(1, \alpha + 1 - \lambda, 1) - I_{21\epsilon}(\alpha)) ,
\]

\[
I_{211\theta}(\alpha) = \frac{1}{2} (-G_1(1, 1)G_1(2 - \lambda, \alpha) - G_1(1, 1)G_1(1, \alpha + 1 - \lambda, 1) + I_{21\epsilon}(\alpha)) ,
\]

\[
I_{211\iota}(\alpha) = \frac{1}{2} (-G_1(1, 1)G_1(2 - \lambda, \alpha) - G_1(1, 1)G_1(1, \alpha + 1 - \lambda, 1) - I_{21\epsilon}(\alpha)) ,
\]

\[
I_{211\kappa}(\alpha) = \frac{1}{2} (-G_1(1, 1)G_1(2 - \lambda, \alpha) + G_1(1, 1)G_1(1, \alpha + 1 - \lambda, 1)
+ G(1, 1)G_1(1, \alpha + 1 - \lambda) - I_{21\delta}(\alpha)) ,
\]

\[
(I.13)
\]

We need the integrals in which two loop momenta in the numerator are contracted with each other and one loop momentum is contracted with the external momentum.
The integrals are proportional to $\frac{1}{p^{(\alpha-2\lambda)}}$ and read

\begin{align*}
I_{221ac}(\alpha) &= \frac{1}{2} (G_1(1, \alpha) G_1(\alpha + 1 - \lambda, 1) - I_{21a}(\alpha) + I_{21c}(\alpha - 1)) , \\
I_{221ad}(\alpha) &= \frac{1}{2} (G(\alpha, 1) G_1(1, \alpha + 1 - \lambda) - I_{21d}(\alpha) + I_{21d}(\alpha - 1)) , \\
I_{221ae}(\alpha) &= \frac{1}{2} (-G(\alpha, 1) G_1(\alpha + 1 - \lambda, 1) - I_{21e}(\alpha) + I_{21e}(\alpha - 1)) , \\
I_{221be}(\alpha) &= \frac{1}{2} (-G(1, 1) G_1(\alpha + 1 - \lambda, 1) + G(1, 1) G_1(\alpha + 1 - \lambda, 1)) , \\
I_{221bc}(\alpha) &= \frac{1}{2} (G(1, 1) G_1(1, \alpha) - G_1(1, 1) G(\alpha, 1) \\
&\quad + G_1(1, \alpha) G_1(\alpha + 1 - \lambda, 1) + G_1(1, 1) G_1(\alpha + 1 - \lambda, 1)) , \\
I_{221bd}(\alpha) &= \frac{1}{2} (G_1(1, 1) G_1(1, \alpha) + G(\alpha, 1) G_1(1, \alpha + 1 - \lambda) \\
&\quad - G(1, 1) G_1(1, \alpha + 1 - \lambda) - G_1(1, 1) G_1(\alpha + 1 - \lambda, 1)) , \\
I_{221be}(\alpha) &= \frac{1}{2} (-G(1, 1) G(\alpha, 1) - G(\alpha, 1) G_1(\alpha + 1 - \lambda, 1) \\
&\quad + G_1(1, 1) G_1(\alpha + 1 - \lambda, 1)) , \\
I_{221cc}(\alpha) &= \frac{1}{2} (-G(1, 1) G_1(1, \alpha) + G_1(1, 1) G(\alpha, 1) + G_1(1, \alpha) G_1(\alpha + 1 - \lambda, 1) \\
&\quad - G_1(1, 1) G_1(\alpha + 1 - \lambda, 1)) , \\
I_{221ce}(\alpha) &= \frac{1}{2} (G_1(1, 1) G(\alpha, 1) - G(\alpha, 1) G_1(\alpha + 1 - \lambda, 1) \\
&\quad - G_1(1, 1) G_1(\alpha + 1 - \lambda, 1)) , \\
I_{221dc}(\alpha) &= I_{221ac}(\alpha) - I_{221bc}(\alpha) , \\
I_{221dd}(\alpha) &= I_{221ad}(\alpha) - I_{221bd}(\alpha) ,
\end{align*}

(J.14)
\[ I_{221eb}(\alpha) = \frac{1}{2}(-G(1,1)G_1(\alpha, 1) + G(1,1)G_1(\alpha, 2 - \lambda) - I_{21b}(\alpha - 1)) , \]
\[ I_{221ed}(\alpha) = \frac{1}{2}(G_1(1,1)G(\alpha, 1) - G_1(1,1)G_1(2 - \lambda, \alpha) + I_{21d}(\alpha - 1)) , \]
\[ I_{221ee}(\alpha) = \frac{1}{2}(-G_1(1,1)G(\alpha, 1) + G_1(1,1)G_1(2 - \lambda, \alpha) + G(1,1)G_1(\alpha, 2 - \lambda) + I_{21e}(\alpha - 1)) , \]
\[ I_{221fe}(\alpha) = I_{221ae}(\alpha) - I_{221ee}(\alpha) , \]
\[ I_{221gd}(\alpha) = \frac{1}{2}(-G_1(1,1)G(\alpha, 1) + G_1(1,1)G_1(2 - \lambda, \alpha) + I_{21d}(\alpha - 1)) , \]
\[ I_{221ha}(\alpha) = \frac{1}{2}(-G(1,1)G_1(\alpha, 1) + G_1(1,\alpha)G_1(\alpha + 1 - \lambda, 1) + G(\alpha, 1)G_1(1,\alpha + 1 - \lambda) - G(1,1)G_1(1,\alpha + 1 - \lambda)) , \]

where e.g. we have the relations
\[ I_{221ad}(\alpha) - I_{221ae}(\alpha) = I_{22a}(\alpha) , \]
\[ I_{221bd}(\alpha) - I_{221be}(\alpha) = I_{22b}(\alpha) , \]
\[ I_{221ed}(\alpha) - I_{221ee}(\alpha) = I_{22e}(\alpha) , \]

We need the integrals in which four loop momenta in the numerator are pairwise
contracted with each other. The integrals are proportional to \( \frac{1}{p^{\alpha - 2} \lambda} \) and read

\[
I_{222ae}(\alpha) = \frac{1}{2} (-G(1, 1)G_2(\alpha, 1) + I_{22a}(\alpha - 1) + G(1, 1)G_2(2 - \lambda, \alpha)) ,
\]

\[
I_{222af}(\alpha) = \frac{1}{2} (-G_1(\alpha, 1)G_2(\alpha + 1 - \lambda, 1) - I_{22f}(\alpha) + I_{22f}(\alpha - 1)) ,
\]

\[
I_{222ah}(\alpha) = \frac{1}{2} (G(1, 1)G_2(\alpha, 1) - G_2(\alpha, 1)G(\alpha - \lambda, 1)
- G_1(\alpha, 1)G_2(\alpha + 1 - \lambda, 1) + G(1, 1)G_2(\alpha + 1 - \lambda, 1)) ,
\]

\[
I_{222ai}(\alpha) = \frac{1}{2} (G(1, 1)G_2(\alpha, 1) + I_{22a}(\alpha - 1) + G(1, 1)G_2(2 - \lambda, \alpha)) ,
\]

\[
I_{222aj}(\alpha) = \frac{1}{2} (-G(\alpha, 1)G_2(\alpha + 1 - \lambda, 1) - I_{22j}(\alpha) + I_{22j}(\alpha - 1)) ,
\]

\[
I_{222bf}(\alpha) = \frac{1}{2} (-G_1(1, 1)G_1(\alpha, 1) - G_1(\alpha, 1)G_2(\alpha + 1 - \lambda, 1)
+ G(1, 1)G_2(\alpha + 1 - \lambda, 1) + G_1(1, 1)G(\alpha - \lambda, 1)) ,
\]

\[
I_{222bg}(\alpha) = \frac{1}{2} (G_1(1, 1)G_1(\alpha, 1) + G_1(\alpha, 1)G(\alpha - \lambda, 1)
- G_1(1, 1)G(\alpha - \lambda, 1)) ,
\]

\[
I_{222bh}(\alpha) = \frac{1}{2} (G_1(1, 1)G_1(1, \alpha) + G_1(1, \alpha)G_2(\alpha + 1 - \lambda, 1)
+ G_1(1, 1)G_2(\alpha + 1 - \lambda, 1) ,
\]

\[
I_{222bi}(\alpha) = \frac{1}{2} (G_1(1, 1)G_1(1, \alpha) + I_{22b}(\alpha - 1) - G_1(1, 1)G(1 - \lambda, \alpha)) ,
\]

\[
I_{222bj}(\alpha) = \frac{1}{2} (-G_2(1, 1)G(\alpha, 1) - G(\alpha, 1)G_2(\alpha + 1 - \lambda, 1)
+ G_2(1, 1)G(\alpha - \lambda, 1) + G_1(1, 1)G_2(\alpha + 1 - \lambda, 1)) ,
\]

\( (J.17) \)
\[ I_{222cc}(\alpha) = \frac{1}{2}(-G_1(1, 1)G_1(1, \alpha) + G_1(1, \alpha)G_2(\alpha + 1 - \lambda, 1)
+ G_1(1, 1)G_2(\alpha + 1 - \lambda, 1)), \]
\[ I_{222ce}(\alpha) = \frac{1}{2}(G_1(1, 1)G_1(1, \alpha) + I_{22c}(\alpha - 1) - G_1(1, 1)G(1 - \lambda, \alpha)), \]
\[ I_{222de}(\alpha) = \frac{1}{2}(-G_1(1, 1)G_1(1, \alpha) + I_{22d}(\alpha - 1) + G_1(1, 1)G(1 - \lambda, \alpha)
+ G(1, 1)G_2(2 - \lambda, \alpha)), \]
\[ I_{222df}(\alpha) = I_{222a} - I_{222f}, \]
\[ I_{222dh}(\alpha) = \frac{1}{2}(G_1(1, 1)G_1(1, \alpha) - G_1(1, \alpha)G(\alpha - \lambda, 1)
- G(\alpha, 1)G_2(\alpha + 1 - \lambda, 1) + G_1(1, 1)G_2(\alpha + 1 - \lambda, 1)), \]
\[ I_{222di}(\alpha) = \frac{1}{2}(G_1(1, 1)G_1(1, \alpha) + I_{22d}(\alpha - 1) + G(1, 1)G_2(2 - \lambda, \alpha)
+ G_1(1, 1)G(1 - \lambda, \alpha)), \]
\[ I_{222e}(\alpha) = \frac{1}{2}(-G_1(1, 1)G_1(1, \alpha, 1) + G_1(1, 1)G_2(2 - \lambda, \alpha) + I_{22f}(\alpha - 1)), \]
\[ I_{222eg}(\alpha) = \frac{1}{2}(G(1, 1)G_2(\alpha, 1) - G_1(1, 1)G_1(\alpha, 1) + I_{22e}(\alpha - 1)
- G_1(1, 1)G_2(2 - \lambda, \alpha)), \]
\[ I_{222e}(\alpha) = \frac{1}{2}(-G_2(1, 1)G(\alpha, 1) + I_{22j}(\alpha - 1) + G_2(1, 1)G(1 - \lambda, \alpha)
+ G_1(1, 1)G_2(2 - \lambda, \alpha)), \]
\[ I_{222f}(\alpha) = \frac{1}{2}(G_1(1, 1)G_1(\alpha, 1) + I_{22f}(\alpha - 1) + G_1(1, 1)G_2(2 - \lambda, \alpha)), \]

\[(J.18)\]
J.2 Three-loop integrals

J.2.1 Three-loop integrals with central cubic vertex

The positions for the numerator momenta are indicated as follows

\begin{equation}
\begin{aligned}
& a \quad b \\
& \quad \quad \quad \quad f \\
& \quad \quad \quad \quad d \\
& \quad \quad \quad \quad \quad \quad \quad \quad g \\
& \quad \quad \quad \quad e \\
\end{aligned}
\end{equation}

(J.19)

The direction of the numerator momenta (arrows) is such that the label is on the r.h.s. if one follows the momentum. The integral with two loop momenta in the numerator which are contracted with the external momentum reads

\[
\begin{align*}
I_{311cde} &= \Delta_-(1,1)(I_{211b}(2-\lambda) - I_{211h}(2-\lambda)) + \Delta_+(1,1)I_{211i}(2-\lambda) \\
& \quad + \tilde{C}(1,1)(I_{311c}(2) + G(2,1)I_{211i}(2-\lambda) - G_1(1,2)I_{211e}(2-\lambda)) \\
& = \frac{1}{(8\pi)^3}\pi \left( -8 + \frac{2}{3}\pi^2 \right).
\end{align*}
\]

(J.20)

We need one integral in which two loop momenta in the numerator are contracted with each other and one loop momentum is contracted with the external momentum. It reads

\[
\begin{align*}
I_{321cdf} &= \frac{1}{2} \left[ -G(1,1)I_{21c}(1-\lambda) - G(1,1)G_1(2-\lambda,1)G_1(3-2\lambda,1) \right] \\
& = \frac{1}{(8\pi)^3}\pi \left( -3 + \frac{\pi^2}{4} \right).
\end{align*}
\]

(J.21)
The integrals in which four loop momenta in the numerator are pairwise contracted read

\[
I_{322ca\text{de}} = \frac{1}{2} \left[ - G_1(1, 1)^2 G_2(2 - \lambda, 2 - \lambda) - 2 G_1(1, 1) I_{22a}(1 - \lambda) \right]
\]
\[
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( -\frac{1}{\varepsilon} - 1 + 3\gamma - \ln 256\pi^3 \right),
\]

\[
I_{322cad\text{ce}} = \frac{1}{2} \left[ - G(1, 1) I_{22b}(1 - \lambda) + G(1, 1) G_2(2 - \lambda, 1) G(2 - 2\lambda, 1)
\right.
\]
\[
+ \frac{1}{2} G(1, 1)^2 G(2 - 2\lambda, 1)
\]
\[
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( -\frac{1}{\varepsilon} - 4 + \frac{\pi^2}{2} + 3\gamma - \ln 256\pi^3 \right),
\]

\[
I_{322cad\text{de}} = \frac{1}{2} \left[ - G(1, 1) G_2(1, 1) G(2 - \lambda, 1 - \lambda) + G_1(1, 1) I_{22g}(1 - \lambda)
\right.
\]
\[
- G(1, 1) I_{22h}(1 - \lambda)
\]
\[
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( -\frac{1}{\varepsilon} - 8 + \frac{\pi^2}{2} + 3\gamma - \ln 256\pi^3 \right),
\]

(J.22)
\[ I_{322cqdg} = \]
\[-\left( \Delta_1(1,1) - \frac{1}{2} \Delta(1,1) \right) I_{222ai}(2 - \lambda) \]
\[-\frac{1}{2(3 - D)} \Delta(1,1)(I_{222di}(2 - \lambda) - I_{222bi}(2 - \lambda)) \]
\[+ \left( \tilde{C}(1,1) - C(1,1) \right) \left( -G(1 - 2\lambda,2)I_{222dh}(1) + G_1(2 - 2\lambda,2)I_{221dc}(1) \right. \]
\[\quad + G_2(1,2) \left( I_{222ef}(2 - \lambda) - \frac{1}{D} I_{22h}(1 - \lambda) \right) \]
\[\left. - G_1(1,2)I_{222ej}(2 - \lambda) \right) \]
\[+ C(1,1) \left( -G_2(2 - 2\lambda,2)I_{221bd}(1) - G_1(2 - 2\lambda,2)(I_{221bd}(1) - I_{221eb}(1)) \right. \]
\[\quad - G_2(3 - 2\lambda,2) \left( I_{221i}(1) - \frac{1}{D} I_{22i}(1) \right) - \frac{1}{D} G(2 - 2\lambda,2)I_{22i}(1) \]
\[\left. + G_2(1,2) \left( I_{222af}(2 - \lambda) - \frac{1}{D} I_{22c}(1 - \lambda) \right) - G_1(1,2)I_{222aj}(2 - \lambda) \right) \]
\[= \frac{1}{(8\pi)^3}( -6 + \frac{2}{3} \pi^2 ) , \]

\[ I_{322eaged} = \]
\[-\left( \Delta_1(1,1) - \frac{1}{2} \Delta(1,1) \right) I_{222de}(2 - \lambda) \]
\[-\frac{1}{2(3 - D)} \Delta(1,1)(I_{222di}(2 - \lambda) + G_1(1,1)G_1(1,2 - \lambda)) \]
\[+ \left( \tilde{C}(1,1) - C(1,1) \right) \left( G(1 - 2\lambda,2)I_{222bg}(1) + G_1(2 - 2\lambda,2)I_{221be}(1) \right. \]
\[\quad + G_2(1,2) \left( I_{222ef}(2 - \lambda) - \frac{1}{D} I_{22h}(1 - \lambda) \right) \]
\[\left. - G_1(1,2)I_{222ej}(2 - \lambda) \right) \]
\[+ C(1,1) \left( -G_2(2 - 2\lambda,2)I_{221bd}(1) - G_1(2 - 2\lambda,2)(I_{221bd}(1) - I_{221eb}(1)) \right. \]
\[\quad - G_2(3 - 2\lambda,2) \left( I_{211i}(1) - \frac{1}{D} I_{21i}(1) \right) - \frac{1}{D} G(2 - 2\lambda,2)I_{21i}(1) \]
\[\left. + G_2(1,2) \left( I_{222af}(2 - \lambda) - \frac{1}{D} I_{22c}(1 - \lambda) \right) - G_1(1,2)I_{222aj}(2 - \lambda) \right) \]
\[= \frac{1}{(8\pi)^3}( -4 + \frac{\pi^2}{2} ) . \]
\[
I_{322\text{c}b\text{d}c\text{e}} = \frac{1}{2} \left[ G_1(1, 1) I_{22\text{g}}(1 - \lambda) + G(1, 1) I_{22\text{b}}(1 - \lambda) - G(1, 1) G_2(1, 1) G(2 - \lambda, 1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 4 - \frac{\pi^2}{2} - 3\gamma + \ln 256\pi^3 \right),
\]

\[
I_{322\text{c}b\text{e}c\text{d}} = \frac{1}{2} \left[ - G_1(1, 1) I_{22\text{a}}(1 - \lambda) + G(1, 1) I_{22\text{b}}(1 - \lambda) - G_1(1, 1)^2 G_2(2 - \lambda, 2 - \lambda) - G_1(1, 1) I_{22\text{e}}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \pi \left( 5 - \frac{\pi^2}{4} \right),
\]

\[
I_{322\text{c}c\text{f}d\text{e}} = \frac{1}{2} \left[ - G(1, 1) G_1(1, 1) G_2(2 - \lambda, 2 - \lambda) - G_1(1, 1) I_{22\text{f}}(1 - \lambda) - G_1(1, 1) I_{22\text{d}}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \pi \left( 3 - \frac{\pi^2}{4} \right),
\]

\[
I_{322\text{c}d\text{c}d\text{e}} = \frac{1}{2} \left[ G_1(1, 1)^2 G_2(2 - \lambda, 2 - \lambda) + 2 G_1(1, 1) I_{22\text{d}}(1 - \lambda) - 2 G(1, 1) I_{22\text{i}}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 9 - 3\gamma + \ln 256\pi^3 \right),
\]

\[
I_{322\text{c}d\text{c}e\text{f}} = \frac{1}{2} \left[ - G(1, 1) G_1(1, 1) G_2(2 - \lambda, 2 - \lambda) + G(1, 1) I_{22\text{i}}(1 - \lambda) + G(1, 1) I_{22\text{h}}(1 - \lambda) - G_1(1, 1) I_{22\text{j}}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \pi \left( -1 - \frac{\pi^2}{4} \right),
\]

\[
I_{322\text{c}d\text{f}\text{e}g} = \frac{1}{2} G(1, 1) \left[ I_{22\text{h}}(1 - \lambda) - \frac{1}{2} G(1, 1) G(2 - 2\lambda, 1) + G_1(2 - \lambda, 1) G(2 - 2\lambda, 1) \right] \\
= \frac{1}{(8\pi)^3} \pi \left( 2 - \frac{\pi^2}{2} \right),
\]

We can then evaluate certain integrals as linear combinations of the above integrals.
They read

\[ I_{322caecd} = \begin{array}{c}
\begin{array}{c}
\text{diagram 1:} \\
\text{diagram 2:} \\
\text{diagram 3:}
\end{array}
\end{array} - 2 \begin{array}{c}
\text{diagram 4:}
\end{array} = I_{322cdgef} - I_{322cde} - 2I_{322cdef} \]

\[ = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( -\frac{1}{\varepsilon} + 3 - \frac{\pi^2}{3} + 3\gamma - \ln 256\pi^3 \right), \]

\[ I_{322edgef} = \begin{array}{c}
\begin{array}{c}
\text{diagram 5:} \\
\text{diagram 6:} \\
\text{diagram 7:}
\end{array}
\end{array} + 2 \begin{array}{c}
\text{diagram 8:}
\end{array} = I_{322cdedge} - I_{311cede} + 2I_{321cdfe} \]

\[ = \frac{1}{(8\pi)^3} \pi \left( 4 - \frac{2}{3}\pi^2 \right), \]

\[ I_{322cde} = \begin{array}{c}
\begin{array}{c}
\text{diagram 9:} \\
\text{diagram 10:}
\end{array}
\end{array} + \begin{array}{c}
\text{diagram 11:}
\end{array} = -I_{322cde} + I_{322edge} \]

\[ = \frac{1}{(8\pi)^3} 5\pi \left( 1 - \frac{\pi^2}{12} \right). \]

(J.25)

J.2.2 Three-loop integrals with central triangle

The positions for the numerator momenta are indicated as follows

\[ \begin{array}{c}
\begin{array}{c}
\text{diagram 12:}
\end{array}
\end{array} \]

(J.26)

The direction of the numerator momenta (arrows) is such that the label is on the r.h.s. if one follows the momentum.

The integrals in which two loop momenta in the numerator are contracted with each
We also need one integral, in which two loop momenta in the numerator are contracted with each other and one loop momentum is contracted with the external momentum. It reads

\[
I_{32tce} = \begin{align*}
I_{32tce} &= - \left( \begin{array}{c}
\text{ contracted momentum}
\end{array} \right) + 2 \left( \begin{array}{c}
\text{ contracted momentum}
\end{array} \right) = -I_{32taf} - G(1, 1)I_{20a} + 2I_{32tad} \\
&= \frac{1}{(8\pi)^3} (-4\pi).
\end{align*}
\]  

(J.27)

We also need several integrals in which four loop momenta in the numerator are contracted with each other. It reads

\[
I_{32tceg} = \begin{align*}
I_{32tceg} &= \frac{1}{2} \left[ - G_1(1, 1)G_2(2 - \lambda, 1)G(\lambda, 2 - 2\lambda, 1) - G_1(\lambda, 1)I_{22e}(1 - \lambda) \\
&\quad - I_{32tce} \right] \\
&= \frac{1}{(8\pi)^3} 2\pi.
\end{align*}
\]  

(J.28)

We also need several integrals in which four loop momenta in the numerator are contracted with each other.
pairwise contracted. They read

\[ I_{322tadce} = \frac{1}{2} \left[ G_1(1, 1) G_1(2 - \lambda, 1) - G_1(1, 1) G_2(1, 1) G(1, 1 - \lambda) - G_1(1, 1) I_{22e}(1 - \lambda) + G_2(1, 1) I_{20\beta} \right] \]

\[ = \frac{1}{(8\pi)^3} \left( -\frac{\pi}{2} \right), \]

\[ I_{322taddg} = \frac{1}{2} \left[ G(1, 1) G_1(1, 1) G_2(2 - \lambda, 1) + G(1, 1) I_{22d}(1 - \lambda) + G_1(1, 1) I_{22\beta} \right. \]

\[ + G(1, 1) I_{22\delta} \]

\[ = \mathcal{O}(\varepsilon), \]

\[ I_{322taecd} = \frac{1}{2} \left[ G_1(1, 1) G_1(1, 1) G_2(2 - \lambda, 1) - G_1(1, 1) I_{22e}(1 - \lambda) \right. \]

\[ + G(1, 1) I_{22\beta} \left] \right. \]

\[ = \frac{1}{(8\pi)^3} \pi \left( \frac{5}{2} - \frac{\pi^2}{4} \right), \]

\[ I_{322tafdg} = -\Delta_-(1, 1) I_{222\gamma \delta} + \Delta_+(1, 1) I_{222\alpha \gamma} \]

\[ - \tilde{C}(1, 1) \left( - G(2, 1) I_{221dd}(2 - \lambda) - G_1(1, 2) I_{222df}(2 - \lambda) \right. \]

\[ - G_1(1, 2) I_{221ad}(2 - \lambda) \]

\[ - G_{(2)}(1, 2) \left( I_{222af}(2 - \lambda) - \frac{1}{D} I_{222e}(1 - \lambda) \right) \right) \]

\[ = \frac{1}{(8\pi)^3} 5\pi \left( 1 - \frac{\pi^2}{12} \right), \]

(J.29)
\[
I_{322t\text{agce}} = \sum_{i} = \Delta_2(1, 1)I_{22\delta} \\
\quad - D(1, 1)(-G(1, 1)G_1(1, 1)G_1(1, 2 - \lambda) \\
\quad - G(1, 1)G_1(1, 1)G_1(2 - \lambda, 1) \\
\quad + G_1(1, 1)I_{22g}(1 - \lambda) - G(1, 1)I_{22\delta}(1 - \lambda) \\
\quad - G_1(1, 1)I_{22f}(1 - \lambda)) \\
\quad + C(1, 1)\left(G_{(2)}(2, 1)\left(I_{222eg}(2 - \lambda) - \frac{1}{D}I_{22i}(1 - \lambda)\right)\right) \\
\quad + \frac{1}{D}G(1, 1)I_{22i}(1 - \lambda) - G_1(2, 1)I_{222de}(2 - \lambda) \\
\quad + G_{(2)}(2, 1)\left(I_{222ef}(2 - \lambda) - \frac{1}{D}I_{22h}(1 - \lambda)\right) \\
\quad + \frac{1}{D}G(1, 1)I_{22h}(1 - \lambda) - G_1(2, 1)I_{222ef}(2 - \lambda)\right) \\
\quad = O(\varepsilon),
\]

\[
I_{322t\text{agdf}} = \sum_{i} = \Delta_2(1, 1)I_{22\gamma} \\
\quad - 2D(1, 1)(G(1, 1)G_1(1, 1)G_1(1, 2 - \lambda) - G_1(1, 1)I_{22g}(1 - \lambda)) \\
\quad + 2C(1, 1)\left(G_{(2)}(2, 1)\left(I_{222eg}(2 - \lambda) - \frac{1}{D}I_{22i}(1 - \lambda)\right)\right) \\
\quad + \frac{1}{D}G(1, 1)I_{22i}(1 - \lambda) - G_1(2, 1)I_{222eg}(2 - \lambda)\right) \\
\quad = \frac{1}{(8\pi)^3}(-2\pi),
\]

(J.30)
\[ I_{322\text{cdce}} = \frac{1}{2} \left[ -G(1, 1)G(1, 1)G(2 - \lambda, 1) + G(1, 1)G(1, 1)G(2 - \lambda, 1) \\
- G(1, 1)I_{22g}(1 - \lambda) + G(1, 1)I_{22\beta} \right] \\
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 2 - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322\text{cdde}} = \frac{1}{2} \left[ G(1, 1)G(1, 1)G(1 - \lambda, 1) + G(1, 1)I_{22\beta}(1 - \lambda) + G(1, 1)I_{22\beta} \right] \\
= \frac{1}{(8\pi)^3} \pi \left( \frac{1}{\varepsilon} + 5 - \frac{\pi^2}{4} - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322\text{cdgg}} = \frac{1}{2} \left[ G(1, 1)G(1, 1)G(2 - \lambda, 1) - G(1, 1)I_{22\alpha} + G(1, 1)I_{22\beta} \\
+ G(1, 1)I_{22\alpha}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( -\frac{1}{\varepsilon} - 10 + \pi^2 + 3\gamma - \ln 256\pi^3 \right), \]

\[ I_{322\text{ceed}} = \frac{1}{2} \left[ -G(1, 1)G(1, 1)G(2 - \lambda, 1) + G(1, 1)I_{22\alpha}(1 - \lambda) + G(1, 1)I_{22\beta} \right] \\
= \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} - 2 - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322\text{cdfg}} = \frac{1}{2} \left[ G(1, 1)^2G(2 - \lambda, 1) - G(1, 1)I_{22\alpha} + G(1, 1)I_{22\beta}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \pi \left( -\frac{1}{\varepsilon} - 4 + \frac{\pi^2}{2} + 3\gamma - \ln 256\pi^3 \right), \]

\[ I_{322\text{ceed}} = \frac{1}{2} \left[ G(1, 1)G(1, 1)G(1 - \lambda, 1) - G(1, 1)^2G(2 - \lambda, 1) \\
+ G(1, 1)I_{22\alpha} + G(1, 1)I_{22\beta}(1 - \lambda) \right] \\
= \frac{1}{(8\pi)^3} \pi \left( \frac{1}{\varepsilon} + \frac{5}{2} - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322\text{cggd}} = \frac{1}{2} \left[ -G(1, 1)^2G(2 - \lambda, 1) + G(1, 1)I_{22\alpha}(1 - \lambda) - G(1, 1)I_{22\beta} \right] \\
= \frac{1}{(8\pi)^3} \left( -\frac{\pi}{2} \right). \]
\[ I_{322t_{cogdf}} = \frac{1}{2} \left[ G_1(1, 1)^2 G_1(2 - \lambda, 1) - G_1(1, 1) I_{22\beta} - G(1, 1) I_{22\alpha}(1 - \lambda) \right] \]
\[ = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( - \frac{5}{\varepsilon} - 1 + 3\gamma - \ln 256\pi^3 \right), \]

\[ I_{322t_{defg}} = \frac{1}{2} \left[ - G(1, 1) G_2(1, 1) G(2 - \lambda, 1) + G(1, 1) I_{22\alpha} + G(1, 1) I_{20\alpha}(-\lambda) \right. \]
\[ - G_1(1, 1) I_{22\alpha}(1 - \lambda) \]
\[ = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 4 - \frac{\pi^2}{2} - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322t_{dfdg}} = \frac{1}{2} \left[ G_1(1, 1)^2 G_1(2 - \lambda, 1) + G_1(1, 1) I_{22\beta} + G(1, 1) I_{22\delta} \right. \]
\[ + G_1(1, 1) I_{22\alpha}(1 - \lambda) \]
\[ = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 7 - \frac{\pi^2}{2} - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322t_{dffg}} = \frac{1}{2} \left[ G(1, 1) G_2(1, 1) G(2 - \lambda, 1) + G(1, 1) I_{22\alpha} - G(1, 1) I_{20\alpha}(-\lambda) \right. \]
\[ + G_1(1, 1) I_{22\alpha}(1 - \lambda) \]
\[ = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 8 - \frac{\pi^2}{2} - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322t_{dgef}} = \frac{1}{2} \left[ + G_1(1, 1)^2 G_1(2 - \lambda, 1) - G_1(1, 1) I_{22\beta} + G(1, 1) I_{22\alpha} \right. \]
\[ - G_1(1, 1) I_{22\alpha}(1 - \lambda) \]
\[ = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{5}{2} - \frac{\pi^2}{4} \right), \]

\[ (J.32) \]
\[ I_{322tcedg} = \]
\[
= -G_1(1, 1)I_{22\beta} - G(1, 1)I_{22\beta} + I_{322tdfdg} + I_{322taddg} - I_{322tafdg}
\]
\[
= \frac{1}{(8\pi)^3} \pi \left( -\frac{13}{2} + \frac{2\pi^2}{3} \right),
\]
\[ I_{322tcecg} = \]
\[
= -I_{322tcedg} + I_{322tagce} = \frac{1}{(8\pi)^3} \pi \left( \frac{13}{2} - \frac{2\pi^2}{3} \right),
\]
\[ I_{322taecg} = \]
\[
= I_{322tcecg} + I_{322tcgde} = \frac{1}{(8\pi)^3} 2\pi \left( 3 - \frac{\pi^2}{3} \right),
\]
\[ I_{322tfgc} = \]
\[
= I_{322tcedg} - I_{322tcecg} = \frac{1}{(8\pi)^3} \pi \left( \frac{1}{\varepsilon} - 11 - 3\gamma + \frac{4\pi^2}{3} + \ln 256\pi^3 \right),
\]
\[ I_{322tfcg} = \]
\[
= -I_{322tdfdg} + I_{322tafdg} = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( -\frac{1}{\varepsilon} + 3 - \frac{\pi^2}{3} + 3\gamma - \ln 256\pi^3 \right),
\]
\[ (J.33) \]

\[ J.2.3 \] Three-loop integrals with central square

The positions for the numerator momenta are indicated as follows

\[ (J.34) \]

The direction of the numerator momenta (arrows) is such that the label is on the r.h.s. if one follows the momentum.

The required integral can be easily computed by promoting it first to a logarithmically divergent four-loop integral by adding a scalar line, and then cancelling one of
its scalar lines (after shifting external momenta thereby taking care not to produce IR divergences). We hence find

\[ I_{322scedf} = \frac{1}{(8\pi)^3} \pi \left( -4 + \frac{2}{3} \pi^2 \right), \tag{J.35} \]

where the equality between the two integrals is only valid for the finite parts, they may differ at \( \mathcal{O}(\varepsilon) \). Two arrows of the same type at a single line indicate a factor of squared momentum along that line in the numerator, and hence cancellation of the propagator (which in the above graph means shrinking it to a point).

The required integrals with six pairwise contracted loop momenta in the numerator are then found as

\[ I_{322scfde} = \frac{1}{4} \left[ G(1, 1)^2 G_2(2 - \lambda, 2 - \lambda) + G(1, 1)^2 G_2(1, 1) - G(1, 1) I_{22\alpha} \right] \]
\[ + \frac{1}{2} \left[ -G(1, 1)^2 G_2(2 - \lambda, 1) + G(1, 1) G_1(1, 1) G_2(2 - \lambda, 1) \right. \]
\[ + \left. G(1, 1) G_2(1, 1) G(1 - \lambda, 1) + G_1(1, 1) I_{22\beta} \right] \]
\[ + G_1(1, 1) I_{22\alpha}(1 - \lambda) \]
\[ = \frac{1}{(8\pi)^3} \pi \left( \frac{1}{\varepsilon} + 1 + \frac{\pi^2}{4} - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{322scfde} = \frac{1}{2} \left[ I_{322tcede} + I_{322tcedy} + \frac{1}{2} I_{322scfde} \right] \]
\[ = \frac{1}{(8\pi)^3} \pi \left( \frac{1}{\varepsilon} - 6 + \pi^2 - 3\gamma + \ln 256\pi^3 \right). \tag{J.36} \]
The above results for three-loop integrals with non-trivial numerators are required to determine the following three-loop integrals which enter (4.15)

\[ I_{3fb} = -2G(1, 1)^2G_2(1, 1), \]

\[ I_{3gb} = \frac{1}{2} G_2(1, 1)I_{20\beta} = \frac{1}{(8\pi)^3} \frac{\pi}{2} \left( \frac{1}{\varepsilon} + 2 - 3\gamma + \ln 256\pi^3 \right), \]

\[ I_{3gt} = -G_2(1, 1)I_{20\beta} + I_{322\text{c}d\text{e}d\text{e}} = \frac{1}{(8\pi)^3} \pi \left( 3 - \frac{\pi^2}{4} \right), \]

\[ I_{3gs} = 4 \left[ G_2(1, 1)I_{20\beta} - 2I_{322\text{c}d\text{e}d\text{e}} + 2I_{322\text{s}c\text{c}d\text{e}d\text{e}} - I_{322\text{s}c\text{f}d\text{c}d\text{e}d\text{e}} \right] = \mathcal{O}(\varepsilon), \]

\[ I_{3gn} = 2 \left[ G_2(1, 1)I_{22h} + G_2(1, 1)I_{22b} - I_{322\text{c}g\text{d}e} - I_{322\text{c}d\text{e}g} + I_{322\text{c}d\text{e}g} \right] = \frac{1}{(8\pi)^3} \pi \left( -4 + \frac{\pi^2}{3} \right), \]

\[ I_{3ge} = G_1(1, 1) \left( I_{222\beta} - I_{22\alpha} - I_{22\delta} + I_{22\beta} \right) + I_{322\text{c}a\text{e}c\text{d}l} - I_{322\text{c}a\text{c}d\text{e}} - I_{322\text{c}e\text{f}\text{d}e} + I_{322\text{c}c\text{d}e\text{f}} = \frac{1}{(8\pi)^3} \pi \left( -2 + \frac{\pi^2}{6} \right), \]

\[ I_{3gv} = 4 \left[ -G(1, 1)I_{222\beta\epsilon} + G(1, 1)I_{22\alpha} \right] = \frac{1}{(8\pi)^3} \pi (24 - 2\pi^2). \]

Thereby, the effective Feynman rules (A.9) and (A.10) have been used.
J.3 Four-loop integrals

In this appendix we list the appearing logarithmically divergent four-loop integrals. In contrast to the three-loop integrals, we only need their pole parts and directly give the expressions in which the appearing subdivergences have been subtracted.

The simplest scalar four-loop integrals are given by

\[ I_4 = \frac{1}{8\pi^4} \left( -\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon} \right), \]

\[ I_{4bbb} = \frac{1}{8\pi^4} \left( -\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \right), \]  

(J.38)

J.3.1 Four-loop integrals with three bubbles and two contracted momenta in the numerator

We also need the following integrals

\[ I_{42bbb1} = \frac{1}{8\pi^4} \left( -\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \right), \]  

(J.39)

\[ I_{42bbb2} = \frac{1}{8\pi^4} \left( -\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \right), \]  

and

\[ I_4 = \frac{1}{8\pi^4} \left( -\frac{1}{2\varepsilon^2} + \frac{2}{\varepsilon} \right), \]  

(J.38)
J.3.2 Four-loop integrals with two bubbles

We need the following integrals with two bubbles and two contracted momenta in the numerators

\[ I_{42bb1de} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = K(G(1,1)^2 G_1(1, 2 - 2\lambda) G_2(3 - 3\lambda, 1))
= \frac{1}{(8\pi)^3} \left(-\frac{\pi^2}{4\varepsilon}\right), \]

\[ I_{42bb3ad} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = -K(G(1,1)G_1(1, 1) G_2(2 - \lambda, 1) G(2 - 2\lambda, 1 - \lambda)
- K(G(1,1)G_2(2 - \lambda, 1) I_2)
= \frac{1}{(8\pi)^3} \left(-\frac{1}{4\varepsilon^2}\right), \]

\[ I_{42bb3ag} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = -K(G(1,1) G_1(1, 1) G_2(2 - \lambda, 2 - 2\lambda) G(2 - 3\lambda, 1))
= \frac{1}{(8\pi)^3} \frac{1}{\varepsilon}, \]

\[ I_{42bb3cd} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = -K(G(1,1) G_2(2 - \lambda, 1) G(2 - 2\lambda, 1) G(2 - 3\lambda, 1)
- K(G(1,1) G_2(2 - \lambda, 1)) I_2)
= \frac{1}{(8\pi)^3} \left(-\frac{1}{2\varepsilon^2}\right), \]

\[ I_{42bb4ab} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = K(G(1,1) G_2(2, 1) G(1 - \lambda, 1) G(2 - 2\lambda, 1 - \lambda))
- K(G_2(1,1) G(1 - \lambda, 1)) K(I_2)
= \frac{1}{(8\pi)^3} \left(-\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon}\right), \]

\[ I_{42bb4ad} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = -K(G(1,1) G_2(2 - \lambda, 1) G(2 - 2\lambda, 1) G(2 - 3\lambda, 1)
- K(G(1,1) G_2(2 - \lambda, 1)) I_2)
= \frac{1}{(8\pi)^3} \left(-\frac{1}{4\varepsilon^2}\right), \]

\[ I_{42bb4de} = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} = K(G(1,1) G_2(2 - \lambda, 1) G(2 - 2\lambda, 1) G(2 - 3\lambda, 1)
- K(G(1,1) G_2(2 - \lambda, 1)) I_2)
= \frac{1}{(8\pi)^3} \frac{1}{2\varepsilon^2}, \]

We also need the following integrals with two bubbles and four pairwise contracted...
momenta in their numerators

\[ I_{422bb3acbe} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)G_1(1,1)G_2(2-\lambda,1)G(2-3\lambda,1)
- K(G_1(1,1)G_2(2-\lambda,1))I_2) + \frac{1}{(8\pi)^4} \left(-\frac{1}{4\varepsilon^2}\right), \]

\[ I_{422bb3aebc} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)G_1(1,1)G_2(2-\lambda,1)G(2-3\lambda,1)
- K(G_1(1,1)G_2(2-\lambda,1))I_2) + \frac{1}{(8\pi)^4} \left(-\frac{1}{4\varepsilon^2}\right), \]

\[ I_{422bb3becd} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)G_1(1,1)G_2(2-\lambda,1)G(2-3\lambda,1)
- K(G_2(1,1)G_2(2-\lambda,1))I_2) + \frac{1}{(8\pi)^4} \left(-\frac{1}{4\varepsilon^2}\right). \]

\[ I_{422bb3acbe} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)G_1(1,1)G_2(2-\lambda,1)G(2-3\lambda,1)
- K(G_1(1,1)G_2(2-\lambda,1))I_2) + \frac{1}{(8\pi)^4} \left(-\frac{1}{4\varepsilon^2}\right). \]

\[ I_{422bb3aebc} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)G_1(1,1)G_2(2-\lambda,1)G(2-3\lambda,1)
- K(G_1(1,1)G_2(2-\lambda,1))I_2) + \frac{1}{(8\pi)^4} \left(-\frac{1}{4\varepsilon^2}\right). \]

J.3.3 Four-loop integrals with one bubble

We need the following integrals with one bubble and two contracted momenta in the numerators

\[ I_{42bad} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)I_{22c}(1-\lambda)G(2-3\lambda,1) = \frac{1}{(8\pi)^4} \left(-\frac{\pi^2}{4\varepsilon}\right), \]

\[ I_{42bce} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = -K(G(1,1)I_{22c}(1-\lambda)G(2-3\lambda,1) = \frac{1}{(8\pi)^4} \frac{2}{\varepsilon}, \]

\[ I_{42bce} = \begin{array}{c}
\raisebox{-0.5cm}{\begin{tikzpicture}
\begin{scope}[xscale=-1,yscale=1]
\draw[thick,->] (0,0) -- (0.5,0);
\draw[thick,->] (0,0) -- (0,0.5);
\draw[thick,->] (0,0) -- (-0.5,0);
\draw[thick,->] (0,0) -- (0,-0.5);
\end{scope}
\end{tikzpicture}}
\end{array} = K(G_1(1,1)I_{22c}(1-\lambda)G(2-3\lambda,1) + K(G_1(1,1)G_2(2-\lambda,1))I_2) = \frac{1}{(8\pi)^4} \left(-\frac{1}{4\varepsilon^2}\right). \]

The required integrals with one bubble and four pairwise contracted momenta in
their numerators read

\[ I_{422cb6adbe} = \begin{array}{c}
\includegraphics{diagram1}
\end{array} = K(G(1,1)I_{222ah}(1)G(2-3\lambda,1) - K(I_{222ah}(1))I_2)
\]
\[ = \frac{1}{(8\pi)^4} \left( \frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( -\frac{1}{2} + \frac{\pi^2}{8} \right) \right), \]

\[ I_{422cb6bcde} = \begin{array}{c}
\includegraphics{diagram2}
\end{array} = -K(G(1,1)I_{222bg}(1)G(2-3\lambda,1) = \frac{1}{(8\pi)^4} \left( -\frac{\pi^2}{16\varepsilon} \right), \]

\[ I_{422cb6becd} = \begin{array}{c}
\includegraphics{diagram3}
\end{array} = -K(G(1,1)I_{222bf}(1)G(2-3\lambda,1) = \frac{1}{(8\pi)^4} \left( -\frac{\pi^2}{16\varepsilon} \right), \]

\[ I_{422cb7adbd} = \begin{array}{c}
\includegraphics{diagram4}
\end{array} = \frac{1}{2} K(G(1,1)G(2-2\lambda,1-\lambda)(-G(1,1)G_2(2-\lambda,1)
+ G_1(1,1)G_1(2-\lambda,1)
+ G(1,1)G_2(1,1))
- K(G_2(1,1)G(1,1) + G_2(1,1)G(1-\lambda,1)
+ G_1(1,1)G_2(2-\lambda,1))I_2)
\]
\[ = \frac{1}{(8\pi)^4} \left( -\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{1}{2} + \frac{\pi^2}{8} \right) \right). \]

\[ (J.43) \]

**J.3.4 Four-loop integrals with central quartic vertex**

The positions for the numerator momenta are indicated as follows

\[ (J.44) \]

The direction of the numerator momenta (arrows) is such that the label is on the r.h.s. if one follows the momentum.

With the external momentum entering the central vertex and exiting the lower ver-
tex, the required integrals become

\[ I_{422qABad} = G(2 - 3\lambda, 1)I_{322tadce} = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( -\frac{1}{4} \right), \]

\[ I_{422qABbd} = K(G(2 - 3\lambda, 1)I_{322tcgdf} - K(G_1(1, 1)G_2(2 - \lambda, 1)))I_2 = \frac{1}{(8\pi)^4} \left( \frac{1}{4\varepsilon^2} + \frac{1}{4\varepsilon} \right), \]

\[ I_{422qACbd} = K(G(2 - 3\lambda, 1)I_{322tacdg}) = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( -3 + \frac{\pi^2}{3} \right), \]

\[ I_{422qAaBd} = K(G(2 - 3\lambda, 1)I_{322taccd}) = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( \frac{5}{4} - \frac{\pi^2}{8} \right), \]

\[ I_{422qAbBd} = K(G(2 - 3\lambda, 1)I_{322tacdc} - K(G_1(1, 1)G_2(2 - \lambda, 1)))I_2 = \frac{1}{(8\pi)^4} \left( \frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{5}{4} - \frac{\pi^2}{12} \right) \right), \]

\[ I_{422qAdBb} = K(G(2 - 3\lambda, 1)I_{322tdefg} + K(G(1, 1)G(1 - \lambda, 1)))I_2 = \frac{1}{(8\pi)^4} \left( \frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon} \left( 1 - \frac{\pi^2}{4} \right) \right), \]

\[ I_{422qAbCd} = K(G(2 - 3\lambda, 1)I_{322tacgd}) = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( -2 + \frac{\pi^2}{4} \right), \]

(J.45)
The above final four-loop integrals have also been checked by using the Gegenbauer polynomial $x$-space technique (GPXT) as formulated in [41] and extended in [37, 42].
The four-loop integrals with central ladder structure and four pairwise contracted momenta in their numerators become

\[ I_{422a} = \frac{1}{(8\pi)^4} \left( \frac{1}{4\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{3}{2} - \frac{\pi^2}{8} \right) \right) , \]

\[ I_{422b} = \frac{1}{(8\pi)^4} \left( \frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{1}{2} - \frac{\pi^2}{8} \right) \right) , \]

\[ I_{422c} = \frac{1}{(8\pi)^4} \left( -\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{1}{2} - \frac{\pi^2}{4} \right) \right) , \]

\[ I_{422d} = \frac{1}{(8\pi)^4} \left( -\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon} \left( 1 - \frac{\pi^2}{4} \right) \right) , \]

\[ I_{422e} = G(2 - 3\lambda, 1) I_{322cd} = \frac{1}{(8\pi)^4} \frac{5}{2\varepsilon} \left( 1 - \frac{\pi^2}{12} \right) , \]

\[ I_{422f} = G(2 - 3\lambda, 1) I_{322be} = \frac{1}{(8\pi)^4} \frac{1}{4} \left( \frac{5}{4} - \frac{\pi^2}{8} \right) , \]

\[ I_{422g} = \frac{1}{(8\pi)^4} \left( -\frac{1}{4\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{5}{4} - \frac{\pi^2}{12} \right) \right) . \]

The different ways to reexpress the four-loop integral in terms of a three-loop integral are based on choosing different (IR safe) external points. Cutting then out the scalar propagator which directly connects these points, we obtain the relations between the finite parts of the integrals with a central cubic vertex and with a central triangle. They
read

\[ I_{322cadde} = I_{322tdffg}, \quad I_{322ccfde} = I_{322tagdf}, \quad I_{322cbdce} = I_{322tafgd}, \quad I_{322cdef} = I_{322tafgd}, \quad I_{322cdef} = I_{322tafgd}, \quad (J.48) \]

and hold only for the leading (finite) parts of the respective integrals.

With convenient choices for the external momentum, thereby avoiding IR divergences, the required integrals with central ladder structure and six pairwise contracted momenta in their numerators read

\[
I_{4222l1} = \frac{1}{2} \left( \frac{1}{2} + 2 \right) = \frac{1}{2} \left[ K(G(1, 1)^2 G(1 - \lambda, 1)^2 - 2 K(G(1, 1)G(1 - \lambda, 1)) I_2 \right]
- 2I_{42bb3ad} + I_{42qabcd} + 2I_{42ta} \right] = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( 1 + \frac{\pi^2}{8} \right),
\]

\[
I_{4222l2} = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2} \left[ I_{42qAbcd} + I_{42bb3ad} + I_{42ttb} + I_{42tce} - I_{42td} \right] = \frac{1}{(8\pi)^4} \frac{1}{2\varepsilon},
\]

\[
I_{4222l3} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2} \left[ - I_{42qAabd} + I_{42qAbbd} + I_{42te} + I_{42tf} - I_{42tag} \right] = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( 1 - \frac{\pi^2}{24} \right),
\]

(J.49)
J.3.6 Four-loop integrals for the next-to-nearest-neighbour interactions between three internal legs

Using the effective Feynman rules (A.7), the four-loop integrals which are required in (4.7) are determined as

\[ I_{4gq} = 2(-I_{422qACbd} + I_{422qAbCd}) = \frac{1}{(8\pi)^4} \frac{1}{\varepsilon} \left( 2 - \frac{\pi^2}{6} \right), \]

\[ I_{4gl} = 4(I_{422qACbd} - I_{422qAbCd} - I_{42221} + 2I_{42222} - I_{42223}) = O(\varepsilon^0). \] (J.50)

It is interesting, that \( I_{4gl} \) is finite, as is the corresponding four-loop integral involving \( I_{3gs} \). This means that integrals which involve boxes with gluon propagators do not contribute to the pole parts to four-loop order. It would be interesting to see if this continues to hold to higher loop orders.

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