Abelian Repetitions in Sturmian Words

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Outline

1. Introduction

2. Sturmian words and abelian repetitions
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1 Introduction

2 Sturmian words and abelian repetitions
Notation and definitions

Given a word $w = w[1..n]$ of length $n$ over alphabet $\Sigma = \{a_1, \ldots, a_\sigma\}$ of cardinality $\sigma$ we denote by:

- $w[i]$ its $i$-th symbol
- $w[i..j]$ the factor from the $i$-th to the $j$-th symbols
- $|w|_a$ the number of occurrences of symbol $a$ in $w$
- $P_w = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$ its Parikh vector
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- $|w|_a$ the number of occurrences of symbol $a$ in $w$
- $P(w) = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$ its Parikh vector
Remarks on Parikh vectors

Consider $Pw = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$ then

1. $Pw[i] = |w|_{a_i}$
2. $|Pw| = \sum_{i=1}^{\sigma} Pw[i] = |w|$
3. $Pw \subset Q$ iff $Pw[i] \leq Q[i]$ for every $1 \leq i \leq \sigma$ and $|Pw| < |Q|$
Remarks on Parikh vectors

Consider $\mathcal{P}w = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$ then

- $\mathcal{P}w[i] = |w|_{a_i}$
- $|\mathcal{P}w| = \sum_{i=1}^{\sigma} \mathcal{P}w[i] = |w|$
- $\mathcal{P}w \subseteq \mathcal{Q}$ iff $\mathcal{P}w[i] \leq \mathcal{Q}[i]$ for every $1 \leq i \leq \sigma$ and $|\mathcal{P}w| < |\mathcal{Q}|$
Remarks on Parikh vectors

Consider \( P_w = (|w|_{a_1}, \ldots, |w|_{a_\sigma}) \) then

- \( P_w[i] = |w|_{a_i} \)
- \( |P_w| = \sum_{i=1}^{\sigma} P_w[i] = |w| \)
- \( P_w \subset Q \) iff \( P_w[i] \leq Q[i] \) for every \( 1 \leq i \leq \sigma \) and \( |P_w| < |Q| \)
Remarks on Parikh vectors

Consider \( \mathcal{P}_w = (|w|_{a_1}, \ldots, |w|_{a_\sigma}) \) then

- \( \mathcal{P}_w[i] = |w|_{a_i} \)
- \( |\mathcal{P}_w| = \sum_{i=1}^\sigma \mathcal{P}_w[i] = |w| \)
- \( \mathcal{P}_w \subset Q \) iff \( \mathcal{P}_w[i] \leq Q[i] \) for every \( 1 \leq i \leq \sigma \) and \( |\mathcal{P}_w| < |Q| \)

Example

\( \mathcal{P}_{caen} \)
Remarks on Parikh vectors

Consider $\mathcal{P}_w = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$ then

- $\mathcal{P}_w[i] = |w|_{a_i}$
- $|\mathcal{P}_w| = \sum_{i=1}^{\sigma} \mathcal{P}_w[i] = |w|$
- $\mathcal{P}_w \subset \mathcal{Q}$ iff $\mathcal{P}_w[i] \leq \mathcal{Q}[i]$ for every $1 \leq i \leq \sigma$ and $|\mathcal{P}_w| < |\mathcal{Q}|$

**Example**

$\mathcal{P}_{caen} \subset \mathcal{P}_{carmen}$
Remarks on Parikh vectors

Consider $\mathcal{P}_w = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$ then

- $\mathcal{P}_w[i] = |w|_{a_i}$
- $|\mathcal{P}_w| = \sum_{i=1}^{\sigma} \mathcal{P}_w[i] = |w|
- $\mathcal{P}_w \subset Q$ iff $\mathcal{P}_w[i] \leq Q[i]$ for every $1 \leq i \leq \sigma$ and $|\mathcal{P}_w| < |Q|

Example

$\mathcal{P}_{caen} \subset \mathcal{P}_{carmen} \subset \mathcal{P}_{american}$
Abelian periods

[Constantinescu and Ilie, 2006] introduced the notion of Abelian period.

**Definition**

A word $w$ has Abelian period $(h, p)$ iff $w = u_0u_1 \cdots u_{k-1}u_k$ such that:

- $P_{u_0} \subset P_{u_1} = \cdots = P_{u_{k-1}} \supset P_{u_k}$
- $|P_{u_0}| = h$, $|P_{u_1}| = p$

$u_0$ is called the *head* and $u_k$ is called the *tail*.

$P_w$ will denote the set of Abelian periods of $w$.  

Abelian periods

\[ w = a b a a b b b a a a b a b a b b a b b a a a \]
Abelian periods

\[ w = \text{a b a a b b b b a a a b a b a b a b b a b b a a} \]

\[ P_{w} = \{(0, 6)\} \]
Abelian periods

$$w = \text{a \ b \ a \ a \ b \ b \ b \ b \ a \ a \ a \ b \ a \ b \ a \ b \ b \ a | b \ b \ a \ a}$$

$$P_w = \{(0,6), (0,10)\}$$
Abelian periods

\[ w = \text{a b a a b b b a a a b} | \text{a b a b a b b a b b a a} \]

\[ P_w = \{(0, 6), (0, 10), (0, 12)\} \]
Abelian periods

\[ w = \begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
\text{a} & \text{b} & \text{a} & \text{a} & \text{b} & \text{b} & \text{b} & \text{b} & \text{b} & \text{a} & \text{a} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{b} & \text{a} & \text{b} & \text{a} & \text{a} \\
\end{array} \]

\[ P_w = \{(0,6), (0,10), (0,12), (1,9)\} \]
Abelian periods

\[ w = \begin{array}{cccccccccccccccccccccc}
\text{a} & \text{b} & \text{a} & \text{a} & \text{b} & \text{b} & \text{b} & \text{b} & \text{a} & \text{a} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{b} & \text{a} & \text{b} & \text{a} & \text{a} & \text{b} \\
\end{array} \]

\[ P_w = \{(0,6), (0,10), (0,12), (1,9), (1,11)\} \]
Abelian periods

\[ w = \text{a b a a b b b a a a b a b a b a b b a a a} \]

\[ P_w = \{(0, 6), (0, 10), (0, 12), (1, 9), (1, 11), (2, 8)\} \]
Abelian periods

\[ w = \begin{array}{ccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
\text{a} & \text{b} & \text{a} & \text{a} & \text{b} & \text{b} & \text{b} & \text{b} & \text{a} & \text{a} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{a} & \text{b} & \text{b} & \text{a} & \text{b} & \text{a} & \text{a}
\end{array} \]

\[ P_w = \{ (0, 6), (0, 10), (0, 12), \\
(1, 9), (1, 11), \\
(2, 8), \\
(3, 9) \} \]
Abelian periods

$w = \begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
a & b & a & a & b & b & b & b & a & a & a & b & a & b & a & b \\
\end{array}$

$P_w = \{(0, 6), (0, 10), (0, 12),$
\[(1, 9), (1, 11),$
\[(2, 8),$
\[(3, 9),$
\[(4, 7)\}
Abelian periods

\[ w = \text{a b a a b b b a a a b a b a b a a b b a b b a a} \]

\[ P_w = \{(0, 6), (0, 10), (0, 12), \\
(1, 9), (1, 11), \\
(2, 8), \\
(3, 9), \\
(4, 7), \\
(5, 7)\} \]
Abelian periods

\[ w = \{a, b, a, a, b, b, b, a, a, a, b, a, b, a, b, a, b, a, b, b, b, a, a\} \]

\[ P_w = \{(0, 6), (0, 10), (0, 12), \]
\[ (1, 9), (1, 11), \]
\[ (2, 8), \]
\[ (3, 9), \]
\[ (4, 7), \]
\[ (5, 7), (5, 9), \ldots\} \]
Abelian periods

\[ w = \overline{a b a a b b b b a a a b a b a b a b a b a b a a} \]

\[ P_w = \{(0, 6), (0, 10), (0, 12), \quad \text{Abelian powers (weak Ap)} \]
\[ (1, 9), (1, 11), \]
\[ (2, 8), \]
\[ (3, 9), \]
\[ (4, 7), \]
\[ (5, 7), (5, 9), \ldots \} \]
Abelian periods

Remark

$a^n$ has $n^2$ Abelian periods.
Motivations

Bioinformatics
- finding CpG islands
- finding clusters of genes
- proteomics: mass spectrometry
- analysis of gene expression time series

Other fields
- approximate pattern matching
- games (letters)
Sturmian words

Definition 1

Infinite words over a binary alphabet that have exactly $n + 1$ distinct factors of length $n$ for each $n \geq 0$
# Fibonacci words

## Fibonacci numbers

\[ F_0 = 0, \quad F_1 = 1, \quad F_j = F_{j-1} + F_{j-2} \quad \text{for} \quad j \geq 2 \]

\( (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots) \)

## Fibonacci words

\[ f_1 = b, \quad f_2 = a, \quad f_j = f_{j-1} \cdot f_{j-2} \quad \text{for} \quad j \geq 3 \]

\( (b, a, ab, aba, ababa, abaababa, abaababaabaab, \ldots) \)

Fibonacci words are Sturmian words
Outline

1. Introduction

2. Sturmian words and abelian repetitions
Our starting point

G. Fici, T. L., A. Lefebvre and É. Prieur-Gaston
Computing Abelian periods in words
In J. Holub and J. Žďárek editors, *Proceedings of the Prague Stringology Conference 2011 (PSC 2011)*, Prague, Tcheque Republic, Pages 184–196, 2011

G. Fici, T. L., A. Lefebvre, É. Prieur-Gaston and W. F. Smyth
Quasi-Linear Time Computation of the Abelian Periods of a Word
In J. Holub and J. Žďárek editors, *Proceedings of the Prague Stringology Conference 2012 (PSC 2012)*, Prague, Tcheque Republic, Pages 103–110, 2012

G. Fici, T. L., A. Lefebvre and É. Prieur-Gaston
Algorithms for Computing Abelian Periods of Words.
*Discrete Applied Mathematics* **163**(Part 3) (2014) 287-297
Our starting point

G. Fici, T. Lecroq, A. Lefebvre and É. Prieur-Gaston
Online Computation of Abelian Runs
In: A. Horia Dediu, E. Formenti, C. Martín-Vide and B. Truthe editors, *Proceedings of the 9th International Conference on Language and Automata Theory and Applications (LATA 2015)*, Nice, France, LNCS 8977, Springer, 391–401

G. Fici, T. Lecroq, A. Lefebvre, É. Prieur-Gaston and William F. Smyth
A Note on Easy and Efficient Computation of Full Abelian Periods of a Word
*Discrete Applied Mathematics*, to appear

G. Fici, T. Kociumaka, T. Lecroq, A. Lefebvre and É. Prieur-Gaston
Fast Computation of Abelian Runs
*Theoretical Computer Science*, to appear
### Our starting point 2

| $i$ | $F_i$ | ap |
|-----|-------|----|
| 0   | 0     | 0  |
| 1   | 1     | 1  |
| 2   | 1     | 1  |
| 3   | 2     | 1  |
| 4   | 3     | 2  |
| 5   | 5     | 2  |
| 6   | 8     | 2  |
| 7   | 13    | 3  |
| 8   | 21    | 5  |
| 9   | 34    | 5  |
| 10  | 55    | 5  |
| 11  | 89    | 8  |
| 12  | 144   | 13 |
| 13  | 233   | 13 |
| 14  | 377   | 13 |
| 15  | 610   | 21 |

| $i$ | $F_i$ | ap |
|-----|-------|----|
| 16  | 987   | 34 |
| 17  | 1597  | 34 |
| 18  | 2584  | 34 |
| 19  | 4181  | 55 |
| 20  | 6765  | 89 |
| 21  | 10946 | 89 |
| 22  | 17711 | 89 |
| 23  | 28657 | 144|
| 24  | 46368 | 233|
| 25  | 75025 | 233|
| 26  | 121393| 233|
| 27  | 196418| 377|
| 28  | 317811| 610|
| 29  | 514229| 610|
| 30  | 832040| 610|
| 31  | 1346269| 987|

| $i$ | $F_i$ | ap |
|-----|-------|----|
| 32  | 2178309 | 1597 |
| 33  | 3524578 | 1597 |
| 34  | 5702887 | 1597 |
| 35  | 9227465 | 2584 |
| 36  | 14930352| 4181 |
| 37  | 24157817| 4181 |
| 38  | 39088169| 4181 |
| 39  | 63245986| 6765 |
| 40  | 102334155| 10946 |
| 41  | 165580141| 10946 |
| 42  | 267914296| 10946 |
| 43  | 433494437| 17711 |
| 44  | 701408733| 28657 |
| 45  | 1134903170| 28657 |
| 46  | 1836311903 | 28657 |
| 47  | 2971215073 | 46368 |
Sturmian words

Definition 2

Let \( \alpha \) and \( \rho \), \( \alpha \in (0, 1) \) irrational. The fractional part of \( r \) is defined by \( \{r\} = r - \lfloor r \rfloor \). Therefore, for \( \alpha \in (0, 1) \), one has that \( \{-\alpha\} = 1 - \alpha \).

The sequence \( \{n\alpha + \rho\}, n > 0 \), defines an infinite word \( s_{\alpha, \rho} = a_1(\alpha, \rho)a_2(\alpha, \rho) \cdots \) by the rule

\[
a_n(\alpha, \rho) = \begin{cases} 
  b & \text{if } \{n\alpha + \rho\} \in [0, \{-\alpha\}), \\
  a & \text{if } \{n\alpha + \rho\} \in [\{-\alpha\}, 1). 
\end{cases}
\]

For \( \alpha = \phi - 1 \) and \( \rho = 0 \), \( \phi = (1 + \sqrt{5})/2 \), \( f = abaababaabaabab \cdots \)
The Sturmian bijection 1

**Proposition**

For any $n, i$, with $n > 0$, if $\{- (i + 1) \alpha \} < \{- i \alpha \}$ then

$$a_{n+i} = a \iff \{n \alpha + \rho\} \in [\{- (i + 1) \alpha \}, \{- i \alpha \})$$

whereas if $\{- i \alpha \} < \{- (i + 1) \alpha \}$ then

$$a_{n+i} = a \iff \{n \alpha + \rho\} \in [0, \{- i \alpha \}) \cup [\{- (i + 1) \alpha \}, 1).$$

When $\alpha = \phi - 1 \approx 0.618$ (thus $\{- \alpha \} \approx 0.382$) for $i = 1$. If $\{n \alpha + \rho\} \in [0, \{- \alpha \}) \cup [\{- 2 \alpha \}, 1)$, then $a_{n+1} = a$; otherwise $a_{n+1} = b$. 
The subintervals of the Sturmian bijection obtained for $\alpha = \phi - 1$ and $m = 6$. Below each interval there is the factor of $s_\alpha$ of length 6 associated with that interval. For $\rho = 0$ and $n = 1$, the prefix of length 6 of the Fibonacci word is associated with $[c_4(\alpha, 6), c_5(\alpha, 6))$, which is the interval containing $\alpha$. 

The Sturmian bijection 2

\[
\begin{array}{cccccccc}
  c_0(\alpha, 6) & & c_1(\alpha, 6) & & c_2(\alpha, 6) & c_3(\alpha, 6) & c_4(\alpha, 6) & \alpha \\
  0 & & 0.145... & & 0.291...0.381... & 0.527... & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
  c_5(\alpha, 6) & & c_6(\alpha, 6) & c_7(\alpha, 6) & \alpha & & & \\
  0.763... & & 0.909... & & 1 & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
  (a_n) & & (a_n+1) & & (a_n+2) & (a_n+3) & (a_n+4) & (a_n+5) \\
  b & & a & & b & a & a & b \\
  a & & a & & a & b & b & a \\
  b & & a & & a & b & a & b \\
  a & & b & & b & a & a & b \\
  a & & a & & a & a & a & b \\
\end{array}
\]
The Sturmian bijection and abelian repetitions

All factors of length $m$ to the right of $\{-m\alpha\}$ have the same Parikh vector.
All factors of length $m$ to the left of $\{-m\alpha\}$ have the same Parikh vector.
The two Parikh vectors are different.
Main Idea

All the points in the sequence
\( \{n\alpha\}, \{(n + m)\alpha\}, \{(n + 2m)\alpha\}, \ldots, \{(n + km)\alpha\} \) are one after the other in the unitary thorus with step \( |\{-m\alpha\}| \), i.e. the distance between \( \{(n + im)\alpha\} \) and \( \{(n + (i + 1)m)\alpha\} \) is \( |\{-m\alpha\}| \) in the unitary thorus.

HENCE, if \( |\{-m\alpha\}| \) is small and \( \{n\alpha\} \) is close to zero, there is a big number \( k \) such that all previous points are all to the left of \( \{-m\alpha\} \), in the unitary interval.

In turn, by the Sturmian bijection, the factors of length \( m \) starting at letters \( a_n, a_{n+m}, a_{n+2m}, \ldots, a_{n+km} \) have the same Parikh vector. We have an abelian power of exponent \( k \) (and conversely).
Main result

Theorem

Let $m$ be a positive integer such that $\{m\alpha\} < 0.5$ (resp. $\{m\alpha\} > 0.5$). Then: In $s_\alpha$ there is an abelian power of period $m$ and exponent $k \geq 2$ if and only if $\{m\alpha\} < \frac{1}{k}$ (resp. $\{-m\alpha\} < \frac{1}{k}$).
Idea of the Proof

⇒
There is an abelian power of period \(m\) and exponent \(k \geq 2\) at position \(n\) in \(s_\alpha\).
Then the \(k\) points \(\{(n + im)\alpha\}, 0 \leq i \leq k - 1\), are naturally ordered and lie all either in \([0, \{-m\alpha\})\) or in \([\{-m\alpha\}, 1)\).

Suppose the 1st case holds.
The distance between any 2 consecutive such points is \(\{-m\alpha\}\).
Therefore, \((k - 1)\{-m\alpha\}\) must be smaller than the size of the interval \([0, \{-m\alpha\})\) which is equal to \(\{-m\alpha\}) = 1 - \{m\alpha\}\).
Thus \(\{m\alpha\} < 1/k\).
Idea of the Proof

\[
\{m\alpha\} < \frac{1}{k}
\]

By the Kronecker Approximation Theorem, the sequence \(\{n\alpha\}_{n \geq 0}\) is dense in \([0, 1)\).

Therefore, one can find a number \(n\) such that \(\{n\alpha\}\) is (arbitrarily) close to 0 and the points \(\{(n + im)\alpha\}, 0 \leq i \leq k - 1\) lie all in \([0, \{-m\alpha\})\).
Let $s_\alpha$ be a Sturmian word. For any integer $q > 1$, let $k_q$ be the maximal exponent of an abelian repetition of period $q$ in $s_\alpha$. Then

$$\limsup \frac{k_q}{q} \geq \sqrt{5},$$

and the equality holds if $\alpha = \phi - 1$. 

Theorem
Idea of the Proof

Take an approximation \( \frac{n}{m} \) of \( \alpha \) such that \( |\frac{n}{m} - \alpha| < \frac{1}{\sqrt{5}m^2} \).

If \( \frac{n}{m} - \alpha < 0 \) then \( m\alpha - n < \frac{1}{\sqrt{5}} \leq 0.5 \).

Thus \( \{m\alpha\} < \frac{1}{\sqrt{5}m} \)

and thus there is in \( s_\alpha \) an abelian power of period \( m \) and exponent \( \geq \sqrt{5}m \)

Since any irrational \( \alpha \) has an infinity of approximations which satisfy

\( |\frac{n}{m} - \alpha| < \frac{1}{\sqrt{5}m^2} \) the statement holds.
Other results on Fibonacci words

**Theorem**

Let \( j > 1 \). The longest prefix of the Fibonacci infinite word that is an abelian repetition of period \( F_j \) has length \( F_j(F_{j+1} + F_{j-1} + 1) - 2 \) if \( j \) is even or \( F_j(F_{j+1} + F_{j-1}) - 2 \) if \( j \) is odd.

**Corollary**

Let \( j > 1 \) and \( k_j \) be the maximal exponent of a prefix of the Fibonacci word that is an abelian repetition of period \( F_j \). Then

\[
\lim_{j \to \infty} \frac{k_j}{F_j} = \sqrt{5}.
\]
Other results on Fibonacci words 2

**Theorem**

For $j \geq 3$, the (smallest) abelian period of the word $f_j$ is the $n$-th Fibonacci number $F_n$, where $n = \lfloor j/2 \rfloor$ if $j = 0, 1, 2 \mod 4$, or $n = 1 + \lfloor j/2 \rfloor$ if $j = 3 \mod 4$.

The list of Fibonacci numbers is:

2, 2, 2, 3, 5, 5, 5, 8, 13, 13, 13, 21, 34, 34, 34, 55, 89, 89, 89

2 is the abelian period of $aba$, $a ba ab$ and of $a ba ab ab ab a a$.

Not of $aba aba baa baa b$ that has abelian period 3.

Instead 5 is the abelian period of $a baaba baaba ababa ababa$ and of $a baaba baaba ababa ababa abaab abaab aab$.
Open problems

1. Is it possible to find the exact value of $\limsup \frac{k_q}{q}$ for other Sturmian words $s_\alpha$ with slope $\alpha$ different from $\phi - 1$?

2. Is it possible to give the exact value of this superior limit when $\alpha$ is an algebraic number of degree 2?
G. Fici, A. Langiu, T. L., A. Lefebvre, F. Mignosi, and É. Prieur-Gaston
Abelian repetitions in sturmian words
In M.-P. Béal and O. Carton, editors, *Proceedings of the 17th International Conference on Developments in Language Theory (DLT 2013)*, volume 7907 of *Lecture Notes in Computer Science*, pages 227–238, Marne-la-Vallée, France, 2013. Springer-Verlag, Berlin

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