On the classification of quaternary optimal Hermitian LCD codes

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Abstract
We propose a method for classifying quaternary Hermitian LCD codes having large minimum weights. For example, we classify quaternary optimal Hermitian LCD codes of dimension 3.

Keywords Quaternary Hermitian code · Linear complementary dual code · Griesmer bound

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1 Introduction

Linear complementary dual (LCD for short) codes intersect with their dual codes trivially. LCD codes were introduced by Massey [13] and gave an optimum linear coding solution for the two user binary adder channels. Recently, much work has been done concerning LCD codes for both theoretical and practical reasons (see, e.g. [5–7, 11] and the references given therein). For example, if there is a quaternary Hermitian LCD \([n, k, d]\) code, then there is a maximal entanglement entanglement-assisted quantum error-correcting \([n, k, d; n - k]\) code (see, e.g. [11]). From this point of view, quaternary Hermitian LCD codes play an important role in the studying maximal entanglement entanglement-assisted quantum error-correcting codes. In addition, Carlet, Mesnager, Tang, Qi, and Pellikkaan [6] showed that any code over \(\mathbb{F}_q\) is equivalent to some Euclidean LCD code for \(q \geq 4\) and any code over \(\mathbb{F}_{q^2}\) is equivalent to some Hermitian LCD code for \(q \geq 3\), where \(\mathbb{F}_q\) denotes the finite field of order \(q\) and \(q\) is a prime power. This is also a motivation of our studying quaternary Hermitian LCD codes.

It is a fundamental problem to determine the largest minimum weight \(d_q(n,k)\) among all quaternary Hermitian LCD \([n, k]\) codes and classify quaternary optimal
Hermitian LCD \([n, k, d_4(n, k)]\) codes for a given pair \((n, k)\). It was shown that \(d_4(n, 2) = \lfloor \frac{4n}{5} \rfloor\) if \(n \equiv 1, 2, 3 \pmod{5}\) and \(d_4(n, 2) = \lfloor \frac{2n}{5} \rfloor - 1\) otherwise for \(n \geq 3\) \([10]\) and \([11]\).

Recently, it has been shown that \(d_4(n, 3) = \lfloor \frac{16n}{21} \rfloor\) if \(n \equiv 5, 9, 13, 17, 18 \pmod{21}\) and \(d_4(n, 3) = \lfloor \frac{16n}{21} \rfloor - 1\) otherwise for \(n \geq 6\) \([2]\) and \([11]\).

More recently, Ishizuka \([8]\) has classified quaternary optimal Hermitian LCD codes of dimension 2.

The authors and Saito \([2]\) gave some conditions on the nonexistence of certain quaternary Hermitian LCD codes having large minimum weights \((\text{2, Theorem 9})\). The aim of this note is to propose a method for classifying quaternary Hermitian LCD codes having large minimum weights \((\text{2, Theorem 9})\). For example, we classify quaternary optimal Hermitian LCD codes of dimension 2.

In this section, we give some definitions, notations and basic results used in this note.

A quaternary \([n, k]\) code \(C\) is a \(k\)-dimensional vector subspace of \(\mathbb{F}_4^n\). The parameters \(n\) and \(k\) are called the length and dimension of \(C\), respectively. A generator matrix of a quaternary \([n, k]\) code \(C\) is a \(k \times n\) matrix such that the rows of the matrix generate \(C\). The weight of a vector \(x \in \mathbb{F}_4^n\) is the number of non-zero components of \(x\). A vector of \(C\) is called a codeword of \(C\). The minimum non-zero weight of all codewords in \(C\) is called the minimum weight of \(C\). A quaternary \([n, k, d]\) code is a quaternary \([n, k]\) code with minimum weight \(d\). Two quaternary \([n, k]\) codes \(C\) and \(C'\) are equivalent, denoted \(C \cong C'\), if there is an \(n \times n\) monomial matrix \(P\) over \(\mathbb{F}_4\) with \(C' = \{ xP \mid x \in C \}\). For any quaternary \([n, k, d]\) code, the Griesmer bound is given by \(n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{4^i} \right\rfloor\).

Throughout this note, we use the following notation:

\[
g_4(n, k) = \max \left\{ d \in \mathbb{Z}_{\geq 0} \left| n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{4^i} \right\rfloor \right. \right\},
\]

where \(\mathbb{Z}_{\geq 0}\) denotes the set of nonnegative integers.

The Hermitian dual code \(C^\perp\) of a quaternary \([n, k]\) code \(C\) is defined as:

\[
C^\perp = \{ x \in \mathbb{F}_4^n \mid \langle x, y \rangle_H = 0 \text{ for all } y \in C \},
\]

where \(\langle x, y \rangle_H = \sum_{i=1}^{n} x_i y_i^*\) for \(x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_4^n\). A quaternary \([n, k]\) code \(C\) is called Hermitian linear complementary dual (Hermitian LCD for short) if \(C \cap C^\perp = \{ \mathbf{0}_n \}\). Note that quaternary Hermitian LCD codes are also called zero radical codes (see, e.g. \([11]\)). Let \(d_4(n, k)\) denote the largest minimum weight among all quaternary Hermitian LCD \([n, k]\) codes. A quaternary Hermitian LCD \([n, k, d_4(n, k)]\) code is called
optimal. In this note, we say that a quaternary Hermitian LCD \([n, k, d]([n, k] - 1)\) code is near-optimal. The minimum weight of the Hermitian dual code \(C^\perp\) of \(C\) is called the (Hermitian) dual distance of \(C\) and it is denoted by \(d^\perp\).

The following characterization gives a criterion for quaternary Hermitian LCD codes and is analogous to [13, Proposition 1].

**Proposition 2.1** ([7, Proposition 3.5]) Let \(C\) be a quaternary code and let \(G\) be a generator matrix of \(C\). Then \(C\) is a Hermitian LCD code if and only if \(GG^\perp\) is nonsingular.

Throughout this note, we use the above characterization without mentioning this.

A quaternary code \(C\) is called Hermitian self-orthogonal if \(C \subset C^\perp\). A quaternary code \(C\) is called even if the weights of all codewords of \(C\) are even. A quaternary code \(C\) is Hermitian self-orthogonal if and only if \(C\) is even [12, Theorem 1]. In addition, a quaternary code \(C\) is Hermitian self-orthogonal if and only if \(GG^\perp = O\) for a generator matrix \(G\) of \(C\).

### 3 Background materials

Let \(C\) be a quaternary Hermitian LCD \([n, k, d]\) code. Define an \([n + 1, k, d]\) code \(\hat{C}\) as \(\hat{C} = \{(x, 0) \mid x \in C\}\). The following lemma was given for binary LCD codes and ternary LCD codes [1, Proposition 3]. The argument can be applied to quaternary Hermitian LCD codes trivially.

**Lemma 3.1** (Ishizuka [8]) Let \(C_{n,k,d}\) denote all equivalence classes of quaternary Hermitian LCD \([n, k, d]\) codes. Let \(D_{n,k,d}\) denote all equivalence classes of quaternary LCD \([n, k, d]\) codes with dual distances \(d^\perp \geq 2\). Let \(\hat{C}_{n-1,k,d}\) denote all equivalence classes containing \(\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_t\), where \(\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_t\) denote representatives of \(C_{n-1,k,d}\) and \(t = |C_{n-1,k,d}|\). Then \(C_{n,k,d} = D_{n,k,d} \cup C_{n-1,k,d}\).

For the classification of quaternary Hermitian LCD \([n, k, d]\) codes, by the above lemma, it is sufficient to consider the classification of quaternary Hermitian LCD \([n, k, d]\) codes with dual distances \(d^\perp \geq 2\).

According to [11], we define the \(k \times (\frac{4^k-1}{3})\) \(\mathbb{F}_4\)-matrices \(S_k\) by inductive constructions as follows:

\[
S_1 = 1,
S_k = \begin{pmatrix}
S_{k-1} & 0^T_{\frac{4^{k-1}-1}{3}} & S_{k-1} & S_{k-1} & S_{k-1}
\end{pmatrix}
\begin{pmatrix}
\frac{4^{k-1}-1}{3} & 1 & \frac{4^{k-1}-1}{3} & \frac{4^{k-1}-1}{3} & \frac{4^{k-1}-1}{3}
\end{pmatrix}
\]

The matrix \(S_k\) is a generator matrix of the quaternary simplex \([\frac{4^k-1}{3}, k, 4^k-1]\) code \(C_k\). It is known that \(C_k\) is a constant weight code. More precisely, \(C_k\) contains codewords of weights 0 and \(4^k-1\) only. Thus, for \(k \geq 2\), \(C_k\) is even. By [12, Theorem 1], \(C_k\) is Hermitian self-orthogonal for \(k \geq 2\).

Let \(h_{k,i}\) be the \(i\)-th column of the \(k \times (\frac{4^k-1}{3})\) \(\mathbb{F}_4\)-matrix \(S_k\). For a vector \(m = (m_1, m_2, \ldots, m_{\frac{4^k-1}{3}}) \in \mathbb{Z}_{\geq 0}\), we define a \(k \times \sum_{i=1}^{\frac{4^k-1}{3}} m_i\) \(\mathbb{F}_4\)-matrix \(G_k(m)\), which consists of \(m_i\) columns \(h_{k,i}\) for each \(i\) as follows:
\[ G_k(m) = \left( h_{k,1} \cdots h_{k,1} h_{k,2} \cdots h_{k,2} h_{k,3} \cdots h_{k,3} h_{k,4} \cdots h_{k,4} \right). \]

Here \( m_i = 0 \) means that \( h_{k,i} \) does not appear as a column of \( G_k(m) \). Throughout this note, we denote by \( C_k(m) \) the quaternary code with generator matrix \( G_k(m) \).

**Remark 3.2** By considering all vectors \( m \in \mathbb{Z}_{\geq 0}^{d^4} \) with \( n = \sum_{i=1}^{d^4} m_i \), it is possible to find representatives of all equivalence classes of quaternary \([n, k]\) codes with dual distances \( d^2 \geq 2 \) as \( C_k(m) \).

The following lemma was given for \( k = 2, 3, 4 \) in [11]. The argument can be applied to arbitrary \( k \) trivially.

**Lemma 3.3** Suppose that \( k \geq 2 \) and \( s \) is a positive integer. Let \( m = (m_1, m_2, \ldots, m_{d^4}) \) be a vector of \( \mathbb{Z}_{\geq 0}^{d^4} \) with \( n = \sum_{i=1}^{d^4} m_i \). If \( C_k(m) \) is a quaternary Hermitian LCD \([n, k, d]\) code with a generator matrix \( G_k(m) \), then the quaternary code \( C \) with generator matrix

\[ \left( S_k^s \right) G_k(m) \]

is a quaternary Hermitian LCD \([n + \frac{4k-1}{3}s, k, d + 4k-1s]\) code.

The following lemma was given for \( k \geq 3 \) [2, Lemma 7]. The argument can be applied to \( k = 2 \) trivially.

**Lemma 3.4** Suppose that \( k \geq 2 \). Let \( m = (m_1, m_2, \ldots, m_{d^4}) \) be a vector of \( \mathbb{Z}_{\geq 0}^{d^4} \) with \( n = \sum_{i=1}^{d^4} m_i \). If the quaternary Hermitian LCD \([n, k]\) code \( C_k(m) \) has minimum weight at least \( d \), then

\[ 4d - 3n \leq m_i \leq n - \frac{4k-1 - 1}{3 \cdot 4k-2} d, \]

for each \( i \in \{1, 2, \ldots, \frac{d^4-1}{3}\} \).

The following lemma was given for binary LCD codes and ternary LCD codes [3, Lemmas 4.3 and 4.4]. By following the same line as in the proof of [3, Lemmas 4.3 and 4.4], we have the following lemma trivially.

**Lemma 3.5** Suppose that \( \ell \geq 1 \) and \( k \geq 2 \). Let \( C \) and \( C' \) be quaternary Hermitian LCD \([n, k]\) codes with dual distances \( d(C^\ell) \geq 2 \) and \( d(C'^\ell) \geq 2 \). Suppose that there are quaternary Hermitian LCD \([n, k]\) codes \( D \) and \( D' \) satisfying the following conditions:

(i) \( C \cong D \) and \( C' \cong D' \),

(ii) \( D \) and \( D' \) have generator matrices

\[ G = \left( S_k^\ell \right) G_0 \]

and

\[ G' = \left( S_k^\ell \right) G'_0, \]

where \( G_0 \) and \( G'_0 \) are generator matrices of some quaternary Hermitian LCD \([n - \frac{(d^4-1)\ell}{3}, k]\) codes \( C_0 \) and \( C'_0 \), respectively.
Then $C \cong C'$ if and only if $C_0 \cong C'_0$.

4 Characterizations of quaternary Hermitian LCD codes

In the rest of this note, we use the following notation:

$$r_{n,k,d} = 4^{k-1}n - \frac{4^k - 1}{3}d,$$  \hspace{1cm} (3)

for a given set of parameters $n, k, d$.

By following the same line as in the proof of Theorem 9 in [2], we have the following theorem.

**Theorem 4.1** Suppose that $4d - 3n \geq 1$ and $4r_{n,k,d} \geq k \geq 2$, where $r_{n,k,d}$ is the integer defined in (3). Then there is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d^\perp \geq 2$ and equivalence classes of quaternary Hermitian LCD $[4r_{n,k,d}, k, 3r_{n,k,d}]$ codes with dual distances $d^\perp \geq 2$.

**Proof** Let $C$ be a quaternary $[n, k, d]$ code with dual distance $d^\perp \geq 2$. Since $d^\perp \geq 2$, by Remark 3.2, there is a vector $m = (m_1, m_2, \ldots, m_{\frac{d^\perp - 1}{3}}) \in \mathbb{Z}_{\geq 0}^{\frac{d^\perp - 1}{3}}$ such that $C \cong C_k(m)$ and $n = \sum_{i=1}^{\frac{d^\perp - 1}{3}} m_i$. Since the minimum weight of $C_k(m)$ is $d$, we have

$$4d - 3n \leq m_i,$$

by Lemma 3.4. Thus, the generator matrix $G_k(m)$ in (1) of $C_k(m)$ consists of at least $4d - 3n$ columns $h_{k,i}$ for each $i \in \{1, 2, \ldots, \frac{d^\perp - 1}{3}\}$. Note that $4d - 3n \geq 1$ from the assumption. Hence, we obtain a matrix $G$ of the following form:

$$G = \left( S_k^{(4d-3n)} G_0 \right),$$  \hspace{1cm} (4)

by permuting columns of $G_k(m)$. Here $G_0$ is a $k \times (n - \frac{(4^k - 1)(4d - 3n)}{3})$ matrix, noting that

$$n - \frac{(4^k - 1)(4d - 3n)}{3} = 4 \left( 4^{k-1}n - \frac{4^k - 1}{3}d \right) = 4r_{n,k,d}.$$

Since $S_k^{(4d-3n)} S_k^{(4d-3n)^T} = O$, we have $GG^T = G_0 G_0^T$. Since $C$ is Hermitian LCD, we have

$$4r_{n,k,d} \geq \text{rank}(G_0) \geq \text{rank}(G_0 G_0^T) = \text{rank}(GG^T) = k.$$  \hspace{1cm} (5)

Let $C_0$ be the quaternary code with generator matrix $G_0$. It follows from (5) that $C_0$ is a quaternary Hermitian LCD $[4r_{n,k,d}, k]$ code. From the assumption $k \geq 2$, the quaternary code with generator matrix $S_k^{(4d-3n)}$ is a Hermitian self-orthogonal $[n', k, d']$ code, where

$$n' = \frac{(4^k - 1)(4d - 3n)}{3} \text{ and } d' = (4d - 3n)4^{k-1}.$$  

By Lemma 3.3, we have
\[ d = d_0 + d' \text{ and } d_0 = 3 \left( 4^{k-1} n - \frac{4^k - 1}{3} d \right) = 3r_{n,k,d}. \]

Hence, if there is a quaternary Hermitian LCD \([n, k, d]\) code \(C\) with dual distance \(d^\perp \geq 2\), then there is a quaternary Hermitian LCD \([n, k, d]\) code \(C'\) such that \(C \cong C'\) and \(C'\) has generator matrix of form (4). In addition, \(G_0\) is a generator matrix of some quaternary Hermitian LCD \([4r_{n,k,d}, k, 3r_{n,k,d}]\) code.

Now let \(C\) and \(C'\) be quaternary Hermitian LCD \([n, k, d]\) codes with dual distances \(d(C^\perp) \geq 2\) and \(d(C'^\perp) \geq 2\). By the above argument, there are quaternary Hermitian LCD \([n, k, d]\) codes \(D\) and \(D'\) satisfying the following conditions:

(i) \(C \cong D\) and \(C' \cong D'\),

(ii) \(D\) and \(D'\) have generator matrices

\[
G = \left( \begin{array}{c} S_k^{(4d-3n)} \end{array} \right) G_0 \quad \text{and} \quad G' = \left( \begin{array}{c} S_k^{(4d-3n)} \end{array} G'_0 \right),
\]

where \(G_0\) and \(G'_0\) are generator matrices of some quaternary Hermitian LCD \([4r_{n,k,d}, k, 3r_{n,k,d}]\) codes \(C_0\) and \(C'_0\), respectively.

It follows from Lemma 3.5 that \(C \cong C'\) if and only if \(C_0 \cong C'_0\). This completes the proof.

The above theorem says that for a given set of parameters \(n, k, d\) the classification of quaternary Hermitian LCD \([n, k, d]\) codes is obtained from that of quaternary Hermitian LCD \([4r_{n,k,d}, k, 3r_{n,k,d}]\) codes, where \(4r_{n,k,d} \leq n\).

At the next step, by following the same line as in the proof of [3, Theorem 4.7], we modify Theorem 4.1 to the form which is used easily by adding some assumption (6) on minimum weights for our study in Section 5 (Theorem 4.3).

Let

\[
n = \frac{4^k - 1}{3} s + t,
\]

where \(s \in \mathbb{Z}_{\geq 0}\) and \(t \in \{0, 1, \ldots, 4^t - 1\}\). In addition, assume the following:

the minimum weight \(d\) is written as

\[
d(s, t) = 4^{k-1} s + a(t),
\]

where \(a(t)\) is a constant depending on only \(t\).

The condition \(4d - 3n \geq 1\) in Theorem 4.1 is equivalent to that \(s \geq s'_{(\frac{4^t - 1}{3}, s+t),k, d(s,t)}} \), where

\[
s'_{(\frac{4^t - 1}{3}, s+t),k, d(s,t)}} = \frac{4r_{(\frac{4^t - 1}{3}, s+t),k, d(s,t)}}{\frac{4^t - 1}{3}} + t.
\]

From (3), we have

\[
r_{(\frac{4^t - 1}{3}, s+t),k, d(s,t)}} = 4^{k-1} \left( d(s, t) - \frac{4^k - 1}{3} s \right) - \frac{4^k - 1}{3} t - 4^t - 1 a(t).
\]

From (7) and (8), we have the following:
Lemma 4.2 Both \( r\left(\frac{d^*-1}{3}s+t\right),k,d(s,t) \) and \( s'\left(\frac{d^*-1}{3}s+t\right),k,d(s,t) \) depend on only \( k, t \) and do not depend on \( s \).

From (7), we have

\[
4r\left(\frac{d^*-1}{3}s+t\right),k,d(s,t) = \frac{4^k-1}{3} \left( s'\left(\frac{d^*-1}{3}s+t\right),k,(4^k-1)s+a(t) \right) - 1 \right) + t.
\]

From (8) and (9), we have

\[
3r\left(\frac{d^*-1}{3}s+t\right),k,d(s,t) = 3\left( 4^{k-1}t - \frac{4^k-1}{3} \alpha(t) \right)
= \frac{3}{4} \left( 4^k - \frac{1}{3} \left( s'\left(\frac{d^*-1}{3}s+t\right),k,(4^k-1)s+a(t) \right) - 1 \right) + t
= 4^{k-1} \left( s'\left(\frac{d^*-1}{3}s+t\right),k,(4^k-1)s+a(t) \right) - 1 \right) + 3t
= 4^{k-1} \left( s'\left(\frac{d^*-1}{3}s+t\right),k,(4^k-1)s+a(t) \right) - 1 \right) + \alpha(t).
\]

By Lemma 4.2, (9) and (10), we have the following:

Theorem 4.3 Let \( n = \frac{4^k-1}{3}s + t \geq k \), where \( s \in \mathbb{Z}_{\geq 0} \) and \( t \in \{0, 1, \ldots, 4^k-1\} \). Assume that \( d \) is written as \( d(s,t) = 4^k-1s + \alpha(t) \), where \( \alpha(t) \) is a constant depending on \( t \). Let \( r \) denote the integer \( r\left(\frac{d^*-1}{3}s+t\right),k,d(s,t) \) defined in (3). Let \( s' \) denote the integer \( s'\left(\frac{d^*-1}{3}s+t\right),k,d(s,t) \) defined in (7). Suppose that \( 4r \geq k \geq 2 \). Then there is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD codes with dual distances \( d^* \geq 2 \) and parameters

\[
[4r, k, 3r] = \left[ \frac{4^k-1}{3} \left( s' - 1 \right) + t, k, 4^{k-1} \left( s' - 1 \right) + \alpha(t) \right]
\]

and equivalence classes of quaternary Hermitian LCD code with dual distances \( d^* \geq 2 \) and parameters

\[
\left[ \frac{4^k-1}{3} s + t, k, 4^{k-1} s + \alpha(t) \right],
\]

for every integer \( s \geq s' \).

The above theorem says that for a given set of parameters \( k, t, \alpha(t) \) the classification of quaternary Hermitian LCD \( n = \frac{4^k-1}{3}s + t \) codes yields that of quaternary Hermitian LCD \( \frac{4^k-1}{3}s + t \) codes for every integer \( s \geq s' \). We remark that the assumption (6) on the minimum weight is automatically satisfied for our study in Section 5.

We end this section by giving a consequence of Theorem 4.3.
Corollary 4.4 Let \( n = \frac{d^2 - 1}{3} s + t \geq k \), where \( s \in \mathbb{Z}_{\geq 0} \) and \( t \in \{ 0, 1, \ldots, \frac{d^2 - 1}{3} - 1 \} \). Let \( d(s, t) = 4^{k-1}s + \alpha(t) \), where \( \alpha(t) \) is a constant depending on \( t \). Let \( r \) denote the integer \( \frac{d^2 - 1}{3} s + t + \alpha(t) - s^{-1} \geq \mathbb{Z}_{\geq 0} \) defined in (3). Let \( s' \) denote the integer \( \frac{d^2 - 1}{3} s + t + \alpha(t) - 2 \leq \mathbb{Z}_{\geq 0} \) defined in (7). Suppose that \( 4r \geq k \geq 2 \). If there is no quaternary Hermitian LCD code with dual distance \( d^1 \geq 2 \) and parameters

\[
[4r, k, 3r] = \left[ \frac{4^{k-1}}{3} (s' - 1) + t, k, 4^{k-1}(s' - 1) + \alpha(t) \right],
\]

then there is no quaternary Hermitian LCD code with dual distance \( d^1 \geq 2 \) and parameters

\[
\left[ \frac{4^{k-1}}{3} s + t, k, 4^{k-1}s + \alpha(t) \right],
\]

for every integer \( s \).

Proof For \( s \geq s' \), the assertion follows directly from Theorem 4.3.

Suppose that there is a quaternary Hermitian LCD \( [\frac{d^2 - 1}{3} s'' + t, k, 4^{k-1}s'' + \alpha(t)] \) code with dual distance \( d^1 \geq 2 \) for some \( s'' < s' - 1 \). Then, by Lemma 3.3, there is a quaternary Hermitian LCD \( [\frac{d^2 - 1}{3} (s' - 1) + t, k, 4^{k-1}(s' - 1) + \alpha(t)] \) code with dual distance \( d^1 \geq 2 \). This is a contradiction.

The above corollary is an improvement of [2, Theorem 9 (ii)] by adding some assumption (6) on minimum weights.

5 Quaternary optimal Hermitian LCD codes

In this section, by Theorem 4.3, we classify quaternary optimal Hermitian LCD codes of dimension 2 and quaternary optimal Hermitian LCD codes of dimension 3.

5.1 Classification method

Here we suppose that \( k \in \{ 2, 3 \} \). As described in Remark 3.2, it is possible to find representatives of all equivalence classes of quaternary Hermitian LCD \( [n, k] \) codes with dual distances \( d^1 \geq 2 \) as \( C_k(m) \), by considering all vectors \( m = (m_1, m_2, \ldots, m_{\frac{d^2 - 1}{3}}) \in \mathbb{Z}_{\geq 0}^{\frac{d^2 - 1}{3}} \) satisfying \( n = \sum_{i=1}^{\frac{d^2 - 1}{3}} m_i \) and the condition (2). In addition, any quaternary \( [n, k, d] \) code is equivalent to some code with generator matrix of form \( (I_k A) \), where \( A \) is a \( k \times (n - k) \) matrix and the weight of the first row of \( A \) is exactly \( d - 1 \). Hence, we may assume without loss of generality that

\[
m_1 \geq 1, m_2 \geq 1 \text{ and } \sum_{i \in \mathbb{S}_2} m_i = d \text{ if } k = 2,
\]

\[
m_1 \geq 1, m_2 \geq 1, m_6 \geq 1 \text{ and } \sum_{i \in \mathbb{S}_2} m_i = d \text{ if } k = 3,
\]
where $S_k$ denotes the support of the first row of $S_k$. In this way, we found all quaternary Hermitian LCD $[n, k, d]$ codes which must be checked further for equivalences. For calculations of determinants of $G^T$ for generator matrices $G$, the NTL function determinant [15] was used. To test equivalence of quaternary codes, we used the algorithm given in [9, Section 7.3.3] as follows. For a quaternary $[n, k]$ code $C$, define the digraph $\Gamma(C)$ with vertex set $V$ and arc set $A$, where

$$V = C \cup \{(1, 2, \ldots, n) \times (\mathbb{F}_4 \setminus \{0\})\},$$

$$A = \{(c, (j, c_j)) : c = (c_1, c_2, \ldots, c_n) \in C, j \in \{1, 2, \ldots, n\}\} \cup \{((j, y), (j, oy)): j \in \{1, 2, \ldots, n\}, y \in \mathbb{F}_4 \setminus \{0\}\}.$$

Then, two quaternary $[n, k]$ codes $C$ and $C'$ are equivalent if and only if $\Gamma(C)$ and $\Gamma(C')$ are isomorphic. We used nauty [14] for digraph isomorphism testing.

All computer calculations in this section were done by programs in the language C. In addition, few verification was done by MAGMA [4]. Let $C_{n,k,d}$ denote our equivalence classes of quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d^2 \geq 2$ obtained by the above method. Especially, we verified by MAGMA that $C$ is a quaternary Hermitian LCD $[n, k, d]$ code with dual distance $d^2 \geq 2$ for $C \in C_{n,k,d}$, and $C$ and $C'$ are inequivalent for $C, C' \in C_{n,k,d}$ with $C \neq C'$.

### 5.2 Quaternary optimal Hermitian LCD $[n, 2]$ codes

The largest minimum weights $d_q(n, 2)$ were determined in [2], where the values of $d_q(n, 2)$ are listed in Table 1. Recently, Ishizuka [8] has classified quaternary optimal Hermitian LCD codes of dimension 2. Here we present a different approach to the classification by using Theorem 4.3.

For $n \geq 2$, write $n = 5s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, \ldots, 4\}$. Let $r = r_{5s+t,2,d_q(5s+t,2)}$ and $s' = s'_{5s+t,2,d_q(5s+t,2)}$ be the integers defined in (3) and (7), respectively. For each $5s + t$, we list $d_q(5s + t, 2), s'$ and $r$ in Table 1. Then $d_q(5s + t, 2)$ is written as $4s + \alpha(t)$, where $\alpha(t)$ is a constant depending on only $t$. Since $d_q(5s + t, 2)$ satisfies the assumption (6) in Theorem 4.3, we have the following:

**Proposition 5.1** There is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^2 \geq 2$ and equivalence classes of quaternary Hermitian LCD $[5s + t, 2, d_q(5s + t, 2)]$ code with dual distances $d^2 \geq 2$ for every integer $s \geq s'$, where $r$ and $s'$ are listed in Table 1.

By the method given in Section 5.1, our computer search classified quaternary optimal Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^2 \geq 2$ for $r$ listed in Table 1. The

| $n$     | $d_q(n, 2)$ | $s'$ | $r$ | $n$     | $d_q(n, 2)$ | $s'$ | $r$ |
|---------|-------------|------|-----|---------|-------------|------|-----|
| $5s$    | $4s - 1$    | 5    | 5   | $5s + 3$| $4s + 2$    | 2    | 2   |
| $5s + 1$| $4s$        | 4    | 4   | $5s + 4$| $4s + 2$    | 5    | 6   |
| $5s + 2$| $4s + 1$    | 3    | 3   |         |             |      |     |
numbers $N_q(4r, 2)$ of all inequivalent quaternary optimal Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^\perp \geq 2$ are listed in Table 2. In addition, our computer search classified quaternary optimal Hermitian LCD $[5s + t, 2, d_q(5s + t, 2)]$ codes with dual distances $d^\perp \geq 2$ for $s < s\text{'}$, where $s\text{'}$ is given in Table 1. The numbers $N_q(5s + t, 2)$ of all inequivalent quaternary optimal Hermitian LCD $[5s + t, 2, d_q(5s + t, 2)]$ codes with dual distances $d^\perp \geq 2$ are also listed in Table 2. From Proposition 5.1 and Table 2, we have the following:

**Proposition 5.2**

(i) Suppose that $t \in \{0, 1\}$. Then there are 2 inequivalent quaternary optimal Hermitian LCD $[5s + t, 2]$ codes with dual distances $d^\perp \geq 2$ for every integer $s \geq 2$.

(ii) Suppose that $t \in \{2, 3\}$. Then there is a unique quaternary optimal Hermitian LCD $[5s + t, 2]$ codes with dual distances $d^\perp \geq 2$, up to equivalence, for every integer $s \geq 0$.

(iii) There are 5 inequivalent quaternary optimal Hermitian LCD $[5s + 4, 2]$ codes with dual distances $d^\perp \geq 2$ for every integer $s \geq 3$.

Of course, the above classification coincides with that given in [8].

### 5.3 Quaternary Hermitian LCD $[n, 2, d_q(n, 2) - 1]$ codes

Similar to the previous subsection, here we classify quaternary near-optimal Hermitian LCD $[n, 2, d_q(n, 2) - 1]$ codes. Similar to Proposition 5.1, we have the following:

**Proposition 5.3** There is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^\perp \geq 2$ and equivalence classes of quaternary Hermitian LCD $[5s + t, 2, d_q(5s + t, 2) - 1]$ code with dual distances $d^\perp \geq 2$ for every integer $s \geq s\text{'}$, where $r$ and $s\text{'}$ are listed in Table 3.

| $n$ | $d_q(n, 2) - 1$ | $s\text{'}$ | $r$ | $n$ | $d_q(n, 2) - 1$ | $s\text{'}$ | $r$ |
|-----|----------------|-----------|-----|-----|----------------|-----------|-----|
| $5s$ | $4s - 2$ | 9 | 10 | $5s + 3$ | $4s$ | 10 | 12 |
| $5s + 1$ | $4s - 1$ | 8 | 9 | $5s + 4$ | $4s + 1$ | 9 | 11 |
| $5s + 2$ | $4s$ | 7 | 8 |
Our computer search classified quaternary near-optimal Hermitian LCD \([4r, 2, 3r]\) codes with dual distances \(d^1 \geq 2\) for \(r\) listed in Table 3 and quaternary near-optimal Hermitian LCD \([5s + t, 2, d_4(5s + t, 2) − 1]\) codes with dual distances \(d^1 \geq 2\) for \(s < s'\), where \(s'\) is given in Table 3. In Table 4, we list the numbers \(N'_4(4r, 2)\) of the inequivalent quaternary near-optimal Hermitian LCD \([4r, 2, 3r]\) codes and the numbers \(N'_4(5s + t, 2)\) of the inequivalent quaternary near-optimal Hermitian LCD \([5s + t, 2, d_4(5s + t, 2) − 1]\) codes. From Proposition 5.3 and Table 4, we have the following:

**Proposition 5.4**

(i) There are 15 inequivalent quaternary near-optimal Hermitian LCD \([5s, 2, 4s - 2]\) codes with dual distances \(d^1 \geq 2\) for every integer \(s \geq 7\).

(ii) There are 7 inequivalent quaternary near-optimal Hermitian LCD \([5s + 1, 2, 4s - 1]\) codes with dual distances \(d^1 \geq 2\) for every integer \(s \geq 5\).

(iii) There are 8 inequivalent quaternary near-optimal Hermitian LCD \([5s + 2, 2, 4s]\) codes with dual distances \(d^1 \geq 2\) for every integer \(s \geq 5\).

(iv) There are 22 inequivalent quaternary near-optimal Hermitian LCD \([5s + 3, 2, 4s]\) codes with dual distances \(d^1 \geq 2\) for every integer \(s \geq 8\).

(v) There are 12 inequivalent quaternary near-optimal Hermitian LCD \([5s + 4, 2, 4s + 1]\) codes with dual distances \(d^1 \geq 2\) for every integer \(s \geq 6\).

For each \(C\) of the quaternary codes listed in Table 4, there is a vector \((m_1, m_2, \ldots, m_5) \in \mathbb{Z}^5_{\geq 0}\) with \(C_2(m) \cong C\). For 

\[(n, d) = (32, 24), (36, 27), (40, 30), (44, 33), (48, 36),\]

the corresponding vectors \((m_1, m_2, \ldots, m_5)\) are listed in Table 5. For the codes listed in Table 4, let \(V_{5s+t}\) be the set of the corresponding vectors \((m_1, m_2, \ldots, m_5)\) for the

**Table 4 \(N'_4(n, 2)\)**

| \(n\)   | \(N'_4(4r, 2)\) | \(N'_4(5s + t, 2)\) (\(s < s'\)) |
|--------|----------------|---------------------------------|
| 5s     | \(N'_4(40, 2) = 15\) | \(N'_4(5, 2) = 1\) \(N'_4(10, 2) = 5\) \(N'_4(15, 2) = 9\) |
|        |                 | \(N'_4(20, 2) = 11\) \(N'_4(25, 2) = 13\) \(N'_4(30, 2) = 14\) |
|        |                 | \(N'_4(35, 2) = 15\)           |
| 5s + 1 | \(N'_4(36, 2) = 7\) | \(N'_4(6, 2) = 2\) \(N'_4(11, 2) = 3\) \(N'_4(16, 2) = 5\) |
|        |                 | \(N'_4(21, 2) = 6\) \(N'_4(26, 2) = 7\) \(N'_4(31, 2) = 7\) |
| 5s + 2 | \(N'_4(32, 2) = 8\) | \(N'_4(7, 2) = 2\) \(N'_4(12, 2) = 4\) \(N'_4(17, 2) = 6\) |
|        |                 | \(N'_4(22, 2) = 7\) \(N'_4(27, 2) = 8\) |
| 5s + 3 | \(N'_4(48, 2) = 22\) | \(N'_4(8, 2) = 2\) \(N'_4(13, 2) = 7\) \(N'_4(18, 2) = 12\) |
|        |                 | \(N'_4(23, 2) = 16\) \(N'_4(28, 2) = 18\) \(N'_4(33, 2) = 20\) |
|        |                 | \(N'_4(38, 2) = 21\) \(N'_4(43, 2) = 22\) |
| 5s + 4 | \(N'_4(44, 2) = 12\) | \(N'_4(4, 2) = 1\) \(N'_4(9, 2) = 2\) \(N'_4(14, 2) = 7\) |
|        |                 | \(N'_4(19, 2) = 8\) \(N'_4(24, 2) = 10\) \(N'_4(29, 2) = 11\) |
|        |                 | \(N'_4(34, 2) = 12\) \(N'_4(39, 2) = 12\) |
inequivalent quaternary near-optimal Hermitian LCD $[5s + t, 2, d_4(5s + t, 2) - 1]$ codes with dual distances $d_4 \geq 2$. We verified that $V_{5(s-1)+t}$ is obtained as:

$$\{(m_1 - 1, m_2 - 1, \ldots, m_5 - 1) \in \mathbb{Z}_5^5 \mid (m_1, m_2, \ldots, m_5) \in V_{5s+t}\}$$

for $s \leq s'$. The corresponding vectors $(m_1, m_2, \ldots, m_5)$ can be also obtained electronically from http://www.math.is.tohoku.ac.jp/~mharada/qLCD/.

### 5.4 Quaternary optimal Hermitian LCD $[n, 3]$ codes

The largest minimum weights $d_4(n, 3)$ were determined in [2], where the values of $d_4(n, 3)$ are listed in Table 6. In this subsection, we classify quaternary optimal Hermitian LCD codes of dimension 3.

We apply Theorem 4.3 to $k = 3$. For $n \geq 3$, write $n = 21s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, \ldots, 20\}$. Let $r = r_{21s+t,3,d_4(21s+t,3)}$ and $s' = s'_{21s+t,3,d_4(21s+t,3)}$ be the integers defined in (3) and (7), respectively. For each $21s + t$, we list $d_4(21s + t, 3), s'$ and $r$ in Table 6. Then $d_4(21s + t, 3)$ is written as $16s + \alpha(t)$, where $\alpha(t)$ is a constant depending on only $t$. Since $d_4(21s + t, 3)$ satisfies the assumption (6) in Theorem 4.3, we have the following:

**Proposition 5.5** There is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[4r, 3, 3r]$ codes with dual distances $d_4 \geq 2$ and equivalence classes of quaternary Hermitian LCD $[21s + t, 3, d_4(21s + t, 3)]$ code with dual distances $d_4 \geq 2$ for every integer $s \geq s'$.
By the method given in Section 5.1, our computer search classified quaternary optimal Hermitian LCD $[4r, 3, 3r]$ codes with dual distances $d^2 \geq 2$ for $r$ listed in Table 6. The numbers $N_d(4r, 3)$ of all inequivalent quaternary optimal Hermitian LCD $[4r, 3, 3r]$ codes with dual distances $d^2 \geq 2$ are listed in Table 7. In addition, our computer search classified quaternary optimal Hermitian LCD $[21s + t, 3, d_4(21s + t, 3)]$ codes with dual distances $d^2 \geq 2$ for $s < s'$, where $s'$ is given in Table 6. The numbers $N_d(21s + t, 3)$ of all inequivalent quaternary optimal Hermitian LCD $[21s + t, 3, d_4(21s + t, 3)]$ codes with dual distances $d^2 \geq 2$ are also listed in Table 7. From Proposition 5.5 and Table 7, we have the following:

**Theorem 5.6**

(i) Suppose that $t \in \{0, 12\}$. Then there are 7 inequivalent quaternary optimal Hermitian LCD $[21s + t, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 2$ if $t = 0$ and $s \geq 1$ if $t = 12$.

(ii) There are 12808 inequivalent quaternary optimal Hermitian LCD $[21s + 1, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 5$.

(iii) Suppose that $t \in \{2, 11\}$. Then there are 318 inequivalent quaternary optimal Hermitian LCD $[21s + t, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 3$.

(iv) Suppose that $t \in \{3, 15\}$. Then there are 147 inequivalent quaternary optimal Hermitian LCD $[21s + t, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 3$ if $t = 3$ and $s \geq 2$ if $t = 15$.

(v) Suppose that $t \in \{4, 13\}$. Then there are 4 inequivalent quaternary optimal Hermitian LCD $[21s + t, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 1$.

(vi) Suppose that $t \in \{5, 9, 17, 18\}$. Then there is a unique quaternary optimal Hermitian LCD $[21s + t, 3]$ code with dual distance $d^2 \geq 2$, up equivalence, for every integer $s \geq 1$ if $t = 5$ and $s \geq 0$ if $t \in \{9, 17, 18\}$.

(vii) There are 2162 inequivalent quaternary optimal Hermitian LCD $[21s + 6, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 4$.

(viii) Suppose that $t \in \{7, 16\}$. Then there are 44 inequivalent quaternary optimal Hermitian LCD $[21s + t, 3]$ codes with dual distances $d^2 \geq 2$ for every integer $s \geq 2$. 

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| $n$ | $d_4(n, 3)$ | $s'$ | $r$ | $n$ | $d_4(n, 3)$ | $s'$ | $r$ |
|-----|-------------|------|-----|-----|-------------|------|-----|
| 21s | 16s – 1     | 5    | 21  | 21s | 16s + 11    | 6    | 29  |
| 21s + 1 | 16s – 1 | 8    | 37  | 21s + 12 | 16s + 8 | 5    | 24  |
| 21s + 2 | 16s       | 7    | 32  | 21s + 13 | 16s + 9 | 4    | 19  |
| 21s + 3 | 16s + 1   | 6    | 27  | 21s + 14 | 16s + 9 | 7    | 35  |
| 21s + 4 | 16s + 2   | 5    | 22  | 21s + 15 | 16s + 10 | 6    | 30  |
| 21s + 5 | 16s + 3   | 4    | 17  | 21s + 16 | 16s + 11 | 5    | 25  |
| 21s + 6 | 16s + 3   | 7    | 33  | 21s + 17 | 16s + 12 | 4    | 20  |
| 21s + 7 | 16s + 4   | 6    | 28  | 21s + 18 | 16s + 13 | 3    | 15  |
| 21s + 8 | 16s + 5   | 5    | 23  | 21s + 19 | 16s + 13 | 6    | 31  |
| 21s + 9 | 16s + 6   | 4    | 18  | 21s + 20 | 16s + 14 | 5    | 26  |
| 21s + 10 | 16s + 6 | 7    | 34  |
Table 7  \( N_{4}(n, 3) \)

| \( n \) | \( N_{4}(4r, 3) \) | \( N_{4}(21s + t, 3) (s < s') \) |
|---|---|---|
| 21s | \( N_{4}(84, 3) = 7 \) | \( N_{4}(21, 3) = 5 \) | \( N_{4}(42, 3) = 7 \) | \( N_{4}(63, 3) = 7 \) |
| 21s + 1 | \( N_{4}(148, 3) = 12808 \) | \( N_{4}(22, 3) = 1871 \) | \( N_{4}(43, 3) = 9793 \) | \( N_{4}(64, 3) = 12405 \) |
| 21s + 2 | \( N_{4}(128, 3) = 318 \) | \( N_{4}(23, 3) = 135 \) | \( N_{4}(44, 3) = 288 \) | \( N_{4}(65, 3) = 318 \) |
| 21s + 3 | \( N_{4}(108, 3) = 147 \) | \( N_{4}(24, 3) = 73 \) | \( N_{4}(45, 3) = 138 \) | \( N_{4}(66, 3) = 147 \) |
| 21s + 4 | \( N_{4}(88, 3) = 4 \) | \( N_{4}(4, 3) = 0 \) | \( N_{4}(25, 3) = 4 \) | \( N_{4}(46, 3) = 4 \) |
| 21s + 5 | \( N_{4}(68, 3) = 1 \) | \( N_{4}(5, 3) = 0 \) | \( N_{4}(26, 3) = 1 \) | \( N_{4}(47, 3) = 1 \) |
| 21s + 6 | \( N_{4}(132, 3) = 2162 \) | \( N_{4}(6, 3) = 2 \) | \( N_{4}(27, 3) = 937 \) | \( N_{4}(48, 3) = 1948 \) |
| 21s + 7 | \( N_{4}(112, 3) = 44 \) | \( N_{4}(7, 3) = 1 \) | \( N_{4}(28, 3) = 30 \) | \( N_{4}(49, 3) = 44 \) |
| 21s + 8 | \( N_{4}(92, 3) = 23 \) | \( N_{4}(8, 3) = 1 \) | \( N_{4}(29, 3) = 18 \) | \( N_{4}(50, 3) = 23 \) |
| 21s + 9 | \( N_{4}(72, 3) = 1 \) | \( N_{4}(9, 3) = 1 \) | \( N_{4}(30, 3) = 1 \) | \( N_{4}(51, 3) = 1 \) |
| 21s + 10 | \( N_{4}(136, 3) = 947 \) | \( N_{4}(10, 3) = 13 \) | \( N_{4}(31, 3) = 589 \) | \( N_{4}(52, 3) = 889 \) |
| 21s + 11 | \( N_{4}(116, 3) = 318 \) | \( N_{4}(73, 4) = 947 \) | \( N_{4}(94, 3) = 947 \) | \( N_{4}(115, 3) = 947 \) |
| 21s + 12 | \( N_{4}(96, 3) = 7 \) | \( N_{4}(11, 3) = 13 \) | \( N_{4}(32, 3) = 220 \) | \( N_{4}(53, 4) = 309 \) |
| 21s + 13 | \( N_{4}(76, 3) = 4 \) | \( N_{4}(12, 3) = 2 \) | \( N_{4}(33, 3) = 7 \) | \( N_{4}(54, 3) = 7 \) |
| 21s + 14 | \( N_{4}(140, 3) = 5736 \) | \( N_{4}(13, 3) = 2 \) | \( N_{4}(34, 3) = 4 \) | \( N_{4}(55, 3) = 4 \) |
| 21s + 15 | \( N_{4}(120, 3) = 147 \) | \( N_{4}(14, 3) = 156 \) | \( N_{4}(35, 3) = 3562 \) | \( N_{4}(56, 3) = 5398 \) |
| 21s + 16 | \( N_{4}(100, 3) = 44 \) | \( N_{4}(77, 3) = 5709 \) | \( N_{4}(98, 3) = 5736 \) | \( N_{4}(119, 3) = 5736 \) |
| 21s + 17 | \( N_{4}(80, 3) = 1 \) | \( N_{4}(15, 3) = 28 \) | \( N_{4}(36, 3) = 118 \) | \( N_{4}(57, 3) = 147 \) |
| 21s + 18 | \( N_{4}(60, 3) = 1 \) | \( N_{4}(78, 3) = 147 \) | \( N_{4}(99, 3) = 147 \) | \( N_{4}(59, 3) = 1 \) |
| 21s + 19 | \( N_{4}(124, 3) = 947 \) | \( N_{4}(16, 3) = 10 \) | \( N_{4}(37, 3) = 39 \) | \( N_{4}(58, 3) = 44 \) |
| 21s + 20 | \( N_{4}(104, 3) = 23 \) | \( N_{4}(17, 3) = 1 \) | \( N_{4}(38, 3) = 1 \) | \( N_{4}(59, 3) = 1 \) |

(ix) Suppose that \( t \in \{ 8, 20 \} \). Then there are 23 inequivalent quaternary optimal Hermitian LCD \( [21s + t, 3] \) codes with dual distances \( d^{\perp} \geq 2 \) for every integer \( s \geq 2 \) if \( t = 8 \) and \( s \geq 1 \) if \( t = 20 \).

(x) Suppose that \( t \in \{ 10, 19 \} \). Then there are 947 inequivalent quaternary optimal Hermitian LCD \( [21s + t, 3] \) codes with dual distances \( d^{\perp} \geq 2 \) for every integer \( s \geq 3 \).
There are 5736 inequivalent quaternary optimal Hermitian LCD $[21s + 14, 3]$ codes with dual distances $d^⊥ \geq 2$ for every integer $s \geq 4$.

For each $C$ of the quaternary codes listed in Table 7, there is a vector $m \in \mathbb{Z}_{\geq 0}^{21}$ with $C_3(m) \cong C$. The vectors $m$ can be obtained electronically from http://www.math.is.tohoku.ac.jp/~mharada/qLCD/. For the codes listed in Table 7, let $V_{21s+t}$ be the set of the corresponding vectors $(m_1, m_2, \ldots, m_{21})$ for the inequivalent quaternary optimal Hermitian LCD $[21s + t, 3]$ codes with dual distances $d^⊥ \geq 2$. We verified that $V_{21(s−1)+t}$ is obtained as:

$\{(m_1 - 1, m_2 - 1, \ldots, m_{21} - 1) \in \mathbb{Z}_{\geq 0}^{21} | (m_1, m_2, \ldots, m_{21}) \in V_{21s+t}\}$

for $s \leq s'$.

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References

1. Araya, M., Harada, M.: On the classification of linear complementary dual codes. Discrete Math. 342, 270–278 (2019)
2. Araya, M., Harada, M., Saito, K.: Quaternary Hermitian linear complementary dual codes. IEEE Trans. Inform. Theory 66, 2751–2759 (2020)
3. Araya, M., Harada, M., Saito, K.: Characterization and classification of optimal LCD codes. Des. Codes Cryptogr. 89, 617–640 (2021)
4. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system I: The user language. J. Symbolic Comput. 24, 235–265 (1997)
5. Carlet, C., Guilley, S.: Complementary dual codes for counter-measures to side-channel attacks. Adv. Math. Commun. 10, 131–150 (2016)
6. Carlet, C., Mesnager, S., Tang, C., Qi, Y., Pellikaan, R.: Linear codes over $\mathbb{F}_q$ are equivalent to LCD codes for $q > 3$. IEEE Trans. Inform. Theory 64, 3010–3017 (2018)
7. Güneri, C., Özkaya, B., Solé, P.: Quasi-cyclic complementary dual codes. Finite Fields Appl. 42, 67–80 (2016)
8. Ishizuka, K.: Classification of optimal quaternary Hermitian LCD codes of dimension 2. J. Algebra Comb. Discrete Struct. Appl. 7, 229–236 (2020)
9. Kaski, P., Östergård, P.R.J.: Classification Algorithms for Codes and Designs. Springer, Berlin, (2006)
10. Li, R.: Research on quantum codes and self-orthogonal codes. Xi’an Jiaotong University, Postdoctor work report (2008)
11. Lu, L., Li, R., Guo, L., Fu, Q.: Maximal entanglement entanglement-assisted quantum codes constructed from linear codes. Quantum Inf. Process. 14, 165–182 (2015)
12. MacWilliams, F.J., Odlyzko, A.M., Sloane, N.J.A., Ward, H.N.: Self-dual codes over GF(4). J. Combin. Theory Ser. A 25, 288–318 (1978)
13. Massey, J.L.: Linear codes with complementary duals. Discrete Math. 106(107), 337–342 (1992)
14. McKay, B.D., Piperno, A.: Practical graph isomorphism, II. J. Symbolic Comput. 60, 94–112 (2014)
15. V. Shoup, NTL: A Library for doing Number Theory, Available online at http://www.shoup.net/ntl/

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