Local scaling asymptotics
in phase space and time
in Berezin-Toeplitz quantization

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Abstract

This paper deals with the local semiclassical asymptotics of a quantum evolution operator in the Berezin-Toeplitz scheme, when both time and phase space variables are subject to appropriate scalings in the neighborhood of the graph of the underlying classical dynamics. Global consequences are then drawn regarding the scaling asymptotics of the trace of the quantum evolution as a function of time.

1 Introduction

1.1 Preamble

This paper is concerned with the local asymptotics of the quantization $\Phi^\hbar_\tau$ ($\tau \in \mathbb{R}$) of a varying Hamiltonian symplectomorphism of a symplectic manifold, $\phi_\tau : M \to M$, in the Berezin-Toeplitz scheme [B], [BG], [Sch], [Z1]. Here the asymptotics are taken in the semiclassical regime $\hbar \to 0^+$, and ‘varying’ means that, rather than considering one fixed symplectomorphisms $\phi_\tau$ at time say $\tau = \tau_0 \in \mathbb{R}$, we are concerned with the behaviour of $\Phi^\hbar_\tau$ when $\tau - \tau_0 \to 0$ at different rates with respect to $\hbar \to 0^+$. Thus we shall look at the asymptotics of the distributional kernels of $\Phi^\hbar_\tau$ when both ‘time’ and ‘phase space’ variables are suitably rescaled in terms of $\hbar$.

These distributional kernels are the Berezin-Toeplitz analogues of quantum evolution operators, and are therefore a fundamental and natural object of study. Broadly stated, our purpose is to investigate how their local scaling

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asymptotics and geometric concentration relate to the underlying classical dynamics.

This is close in spirit to the study of near-diagonal scaling asymptotics of equivariant Bergman-Szegö kernels and their ensuing analogues for Toeplitz operators (see [BSZ], [SZ], [MM]), and their generalization to (fixed) quantized Hamiltonian symplectomorphisms [P3]; however, rather than restricting our analysis to ‘phase space’ scaling asymptotics, here we shall also consider scaling asymptotics in the ‘time’ variable. The approach we shall take is to look at the (local) asymptotics of $\Phi_{\tau_0 + \sqrt{\hbar}}^h$ for $\tau$ suitably bounded in terms of $\hbar$.

One motivation for including ‘time variable’ scaling asymptotics is related to trace computations. Namely, the semiclassical asymptotics of $\text{trace}(\Phi_{\tau}^h)$ are controlled by the geometry of the fixed locus $M_\tau \subseteq M$ of $\phi_\tau$, and certain Poincaré type data along it. Therefore, they have discontinuities (both in order of growth and leading order term, say) in the presence of jumping behavior of $M_\tau$ as a function of $\tau$. It is therefore of interest to examine the nature of these discontinuities in time in terms of the underlying symplectic dynamics. Under mild assumptions, they can be related fairly explicitly to the critical data of $f$ on $M_{\tau_0}$.

1.2 The symplectic context and its quantization

Let us emphasize from the outset that there is a broader scope for the approach in this article than the one we shall present here, as by introducing the appropriate microlocal framework from [BG] and [SZ] one might cover compact and integral almost Kähler manifolds. For the sake of simplicity, we shall however confine ourselves to the complex projective setting.

Let us describe our geometric set-up. Our archetype of a ‘classical phase space’ will be a connected complex Kähler manifold $(M, J, \omega)$, of complex dimension $d$. So $J$ is a complex structure on $M$, and $\omega$ is a compatible symplectic structure (thus, of type $(1, 1)$), such that $g(\cdot, \cdot) =: \omega(\cdot, J\cdot)$ is a Riemannian metric, and $h = g - i\omega$ is an Hermitian metric.

In the Kähler setting, the symplectic and Riemannian volume forms, $S_M$ and $R_M$, coincide, and we shall generally denote them by $\mu_M =: \omega^d/d!$. We shall also denote by $dV_M = |\mu_M|$ the Riemannian density (and the corresponding measure) on $(M, g)$. When dealing with general symplectic submanifolds $F \subseteq M$, we shall need to distinguish between $S_F$ and $R_F$ (§4.1).

Any classical observable $f \in C^\infty(M; \mathbb{R})$ determines a Hamiltonian vector field $v_f$, and the corresponding flow of Hamiltonian symplectomorphisms

$$
\phi_f^M : \tau \in \mathbb{R} \mapsto (\phi_f^M : M \to M).
$$
In most cases, $\phi^M$ is not family of biholomorphisms of $(M,J)$; this happens if and only if $v_f$ is the real component of a holomorphic vector field on $(M,J)$ \cite{KN}. In this case, we shall say that $f$ is compatible (with the Kähler structure).

In this framework, the subclass of quantizable phase spaces, according to the Berezin-Toeplitz scheme, is given by those $(M,J,\omega)$ that are actually Hodge manifolds \cite{GH}; we may then assume without any essential loss that $\omega$ be cohomologically integral. Therefore, there exists a positive holomorphic Hermitian line bundle $(A,h)$ on $M$, such that its unique compatible connection $\nabla_A$ has curvature $\Theta = -2i\omega$ \cite{GH}. Often $(A,h)$ is called a quantizing line bundle of $(M,J,\omega)$ \cite{CGR, W, GS, Sch}.

For $k \geq 1$, let $H^0(M,A^{\otimes k})$ be the finite dimensional vector space of holomorphic sections of $A^{\otimes k}$. Then $h$ and $dV_M$ endow $H^0(M,A^{\otimes k})$ with a natural Hermitian structure $H_k$ \cite{GH}. Namely, if $\sigma, \tau \in H^0(M,A^{\otimes k})$ set

$$H_k(\sigma, \tau) =: \int_M h_m(\sigma(m), \tau(m)) \, dV_M(m),$$

where we also denote by $h$ the Hermitian structure induced on $A^{\otimes k}$ by $h$. In Berezin-Toeplitz quantization, the Hilbert space $\mathcal{H}_k =: (H^0(M,A^{\otimes k}), H_k)$ is viewed as a quantization of $(M,J,\omega)$ at Planck’s constant $\hbar = 1/k$. Obviously, $\mathcal{H}_k$ is a closed subspace of the Hilbert space of all $L^2$-summable sections, $L^2(M,A^{\otimes k})$.

The quantization of $f \in C^\infty(M; \mathbb{R})$ should be the assignment of self-adjoint operators $\mathcal{T}(f)_k : \mathcal{H}_k \to \mathcal{H}_k$ for every $k \geq 1$. The Berezin-Toeplitz approach is to set $\mathcal{T}(f)_k =: P_k \circ \mathcal{M}_k \circ P_k$, where $P_k : L^2(M,A^{\otimes k}) \to H^0(M,A^{\otimes k})$ is the $L^2$-orthogonal projector, and $\mathcal{M}_f$ denotes multiplication by $f$. This choice leads expected semiclassical properties and induces a star product on the algebra of observables \cite{Gi}. One thinks of $\mathcal{T}(f) =: \bigoplus_{k \geq 0} \mathcal{T}(f)_k$ as acting on the $L^2$ direct sum $L(A) =: \bigoplus_{k \geq 0} L^2(M,A^{\otimes k})$.

The quantization of $\phi^M_t : M \to M$, on the other hand, should be a family of unitary operators $\Phi_{\tau,k} : \mathcal{H}_k \to \mathcal{H}_k$. For compatible Hamiltonians, the choice of $\Phi_{\tau,k}$ is straightforward (for this reason, compatible classical observables on Hodge manifolds are alternatively dubbed quantizable \cite{CGR}). The general situation is more subtle, as we now recall.

The phase flow $\phi^M_t$ lifts in a natural manner to flows $\phi^{A^{\otimes k}}_\tau : A^{\otimes k} \to A^{\otimes k}$ for every $k$; each $\phi^{A^{\otimes k}}_\tau$ is a fiberwise linear, metric and connection preserving diffeomorphism. These geometric data determine unitary operators

$$\mathcal{V}_{\tau,k} : \sigma \in L^2(M,A^{\otimes k}) \mapsto \phi^{A^{\otimes k}}_\tau \circ \sigma \circ \phi^M_t \in L^2(M,A^{\otimes k}).$$

When $f$ is compatible, both $\phi^{A^{\otimes k}}_\tau$ and $\phi^M_t$ are biholomorphisms for every $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$; therefore $\mathcal{V}_{\tau,k}$ preserves $\mathcal{H}_k \subseteq L^2(M,A^{\otimes k})$ and induces by
restriction a unitary operator \( \tilde{\Phi}_{\tau,k} : \mathcal{H}_k \to \mathcal{H}_k \). It is convenient to regard \( \tilde{\Phi}_{\tau,k} \) as defined on all of \( L^2(M, A^\otimes k) \), and vanishing on the \( L^2 \)-orthocomplement of \( \mathcal{H}_k \); with this understanding, \( \tilde{\Phi}_{\tau,k} = \mathcal{P}_k \circ \mathcal{V}_{\tau,k} \circ \mathcal{P}_k \).

In general, however, \( \mathcal{V}_{\tau,k}(\mathcal{H}_k) \not\subseteq \mathcal{H}_k \); to obtain an endomorphism of \( \mathcal{H}_k \), we may then consider the composition \( \mathcal{P}_k \circ \mathcal{V}_{\tau,k} \circ \mathcal{P}_k \) as before, but the latter need not unitary. Nonetheless, by the theory of [Z1], one does obtain a 1-parameter family of unitary operators (at least for \( k \gg 0 \)) by setting

\[
\tilde{\Phi}_{\tau,k} =: \mathcal{T}_{\tau,k} \circ \mathcal{P}_k \circ \mathcal{V}_{\tau,k} \circ \mathcal{P}_k, \tag{2}
\]

where \( \mathcal{T}_{\tau} \) is a canonical 1-parameter family of \( S^1 \)-invariant zeroth order Toeplitz operators, and \( \mathcal{T}_{\tau,k} \) is its restriction to \( \mathcal{H}_k \). In the linear case, this principle goes back to Daubechies [Dau], who introduced Toeplitz multipliers to construct unitary projective representations of the linear symplectic group.

An example of invariant zeroth order Toeplitz operator on \( X \) is the quantum observable \( \mathcal{T}(f) = \bigoplus_{k \geq 0} \mathcal{T}(f)_k \) introduced above; up to a smoothing term, any invariant zeroth order Toeplitz operator has the same form, with \( f \in C^\infty(M) \) replaced by an asymptotic expansion \( f(k) \sim f_0 + k^{-1} f_1 + \ldots \) [G].

Our focus here is on the local asymptotics of \( \tilde{\Phi}_{\tau_k} \) with \( \tau_k = \tau_0 + \tau/\sqrt{k} \), where \( \tau \) is suitably bounded in terms of \( k \). We shall however need to adopt a slightly different view of this picture, as we next explain.

### 1.3 The unit circle bundle and the Szegö kernel

We follow in this work the general approach to Berezin-Toeplitz quantization from [BG], [Z1], [Z2], [KS], [BSZ], [SZ]. Thus the natural framework for the present analysis is the unit circle bundle \( X \subseteq A^\vee \) in the dual line bundle to \( A \), with its structure \( S^1 \)-action \( \tau : S^1 \times X \to X \) given by scalar multiplication. This is a principal \( S^1 \)-bundle over \( M \), with projection \( \pi : X \to M \). Then \( \nabla_A \) corresponds to a connection 1-form \( \alpha \) on \( X \), and \( d\alpha = 2 \pi^*(\omega) \). The pair \( (X, \alpha) \) is a contact manifold, and \( \mu_X =: (1/2\pi) \alpha \wedge \pi^*(d\nu_M) \) is a volume form on \( X \), with induced measure \( d\nu_X = |\mu_X| \). We shall write \( L^2(X) =: L^2(X, d\nu_X) \).

There is on \( X \) a natural Riemannian structure \( \tilde{g} \). This is defined by declaring the horizontal and vertical tangent bundles of \( X \), \( \mathcal{H}(\alpha) =: \ker(\alpha) \) and \( \mathcal{V}(\pi) =: \ker(d\pi) \subseteq TX \), to be orthogonal, \( \pi \) to be a Riemannian submersion, and the generator \( \partial/\partial \theta \) of the structure \( S^1 \)-action to have unit norm\(^2\).

\(^1\)f is real valued if and only if \( \mathcal{T}(f) \) is self-adjoint.

\(^2\)\( \theta \) will always denote an ‘angular’ coordinate which is translated by the \( S^1 \)-action.
One advantage of dealing with $X$ is that sections of powers of $A$ naturally and unitarily correspond to functions on $X$, of the appropriate $S^1$-equivariance. Namely, for any $k \in \mathbb{Z}$ consider the $k$-th isotype

$$C^\infty(X)_k = \{ s \in C^\infty(X; \mathbb{C}) : s(r_g(x)) = g^k s(x) \ \forall \ x \in X, \ g \in S^1 \},$$

and similarly for $L^2(X)_k \subseteq L^2(X)$. Then

$$(C^\infty(X)_k, dV_X) \cong (C^\infty(M, A^{\otimes k}), H_k)$$
as Hilbert spaces in a natural manner [Z2], [SZ].

Under the latter unitary isomorphism, holomorphic sections collectively correspond to the Hardy space $H(X) \subseteq L^2(X)$ of $X$. Namely, let $H(X)_k \subseteq C^\infty(X)_k$ be the subspace corresponding to $H^0(M, A^{\otimes k})$ under the isomorphism $(C^\infty(X)_k, dV_X) \cong C^\infty(M, A^{\otimes k})$. Then $H(X) = \bigoplus_{k \geq 0} H(X)_k$ (the Hilbert space direct sum).

The $L^2$-orthogonal projector $\Pi : L^2(X) \to H(X)$ is called the Szegö projector of $X$, and has been described as a FIO in [BS] (see also the discussion in [SZ]). We have $\Pi = \bigoplus_{k \geq 0} \Pi_k$, where $\Pi_k$ (the $k$-th Fourier component of $\Pi$) is the orthogonal projector onto $H(X)_k$. Hence $\Pi_k$ is a smoothing operator, with Schwartz kernel$^3$

$$\Pi_k(x, y) = \sum_{j=1}^{N_k} s_j^{(k)}(x) s_j^{(k)}(y) \quad (x, y \in X), \quad (3)$$

where $N_k = \dim H^0(M, A^{\otimes k})$ and $(s_j^{(k)})_j$ is an arbitrary orthonormal basis of $H(X)_k$.

### 1.4 The contact flow

As mentioned already, $\phi^M$ lifts to a metric and connection preserving fiberwise linear flow $\phi^A$, which therefore restricts to a contact flow $\phi^X$ on $X$. Infinitesimally, this may be described explicitly as follows. Let $\tilde{u}_f^\alpha$ be the horizontal lift of $u_f$ for $\alpha$; then

$$\tilde{u}_f^\alpha =: u_f^\alpha - f \frac{\partial}{\partial \theta} \quad (4)$$
is the contact vector field on $X$ generating $\phi^X$. Clearly $\phi^X : X \to X$ lifts $\phi^M : M \to M$, and pull-back $(\phi^X)^* : L^2(X) \to L^2(X)$ is a unitary operator. By (4), $\tilde{u}_f$ depends on $f$, and not just on $u_f$.

$^3$we shall systematically use the same symbol for an operator and its distributional kernel
Since $\tilde{v}_f$ is $S^1$-invariant, so is $\phi^X_\tau$. Therefore, $(\phi^X_\tau)^*$ preserves every $L^2(X)_k$. The unitary operators

$$V_{\tau,k} = (\phi^X_\tau)^*|_{L^2(X)_k} : L^2(X)_k \to L^2(X)_k$$

(5)
correspond to $\Pi$ under the isomorphism $L^2(X)_k \cong L^2(M, A^\otimes k)$. If $f$ is compatible, then $V_{\tau,k}(H(X)_k) = H(X)_k$ for every $k$ and $\tau$, and the unitary operators $\Phi_{\tau,k} : H(X)_k \to H(X)_k$ induced by restriction correspond to $\tilde{\Phi}_{\tau,k} : \mathcal{H}_k \to \mathcal{H}_k$.

1.5 Toeplitz operators and quantum evolution

We adopt the definition of Toeplitz operators from [BG]. In view of the geometric structure of the wave front of $\Pi$, for any pseudodifferential operator $Q$ on $X$ the composition $\Pi \circ Q \circ \Pi$ is well-defined.

**Definition 1.1.** A Toeplitz operator $T : C^\infty(X) \to C^\infty(X)$ of order $\ell \in \mathbb{Z}$ on $X$ is a composition of the form $T =: \Pi \circ Q \circ \Pi$, where $Q$ is a pseudodifferential operator of classical type on $X$, of order $\ell$. The principal symbol $\tilde{\rho}_T$ of $T$ is the restriction of the principal symbol of $Q$ to the symplectic cone sprayed by the connection form,

$$\Sigma =: \{(x, r \alpha_x) : x \in X, r > 0\} \subseteq T^*X \setminus \{0\};$$

this is well defined, independently of the choice of $Q$ [BG]. Thus $\tilde{\rho}_T(x, r \alpha_x) = r^\ell \rho_T(x, \alpha_x)$.

**Remark 1.1.** Toeplitz operators are clearly filtered by their order, and if $\tilde{\rho}_T = 0$ as an operator of order $\ell$, then $T$ is really of order $\leq \ell - 1$ in obvious sense.

If $\ell \leq 0$, we can alternatively view $T$ as a continuous endomorphism of $H(X)$, or as a continuous endomorphism of $L^2(X)$ vanishing on the $L^2$-orthocomplement $H(X)^\perp$. If in addition $T$ is $S^1$-invariant (which amounts to saying that $Q$ can be chosen $S^1$-invariant), then it restricts for every $k$ to a uniformly bounded operator

$$T_k = \Pi_k \circ Q \circ \Pi_k : H(X)_k \to H(X)_k.$$  
(6)

If $T$ is zeroth order and $S^1$-invariant, then $\tilde{\rho}_T$ is in a natural manner a $C^\infty$ function on $M$, that we shall call the reduced symbol of $T$. 

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The discussion in §1.2 surrounding (2) can then be rephrased by saying that there exists a canonical $C^\infty$ family $T_\tau$ of invariant Toeplitz operators on $X$ such that
\[
\Phi_\tau := T_\tau \circ \pi \circ (\phi_{-\tau})^\ast \circ \pi = T_\tau \circ (\phi_{-\tau})^\ast \circ \pi : H(X) \to H(X)
\] (7)
is unitary, at least on the orthocomplement of a finite dimensional subspace $[Z_1]$.

We are led by the previous considerations to consider operators of the general form
\[
U_\tau := R_\tau \circ \pi \circ (\phi_{-\tau})^\ast \circ \pi = R_\tau \circ (\phi_{-\tau})^\ast \circ \pi : H(X) \to H(X),
\] (8)
where $R_\tau$ is a $C^\infty$ family of $S^1$-invariant zeroth order Toeplitz operators, with reduced symbol $\varrho_\tau := \varrho_{R_\tau}$, and their equivariant restrictions
\[
U_{\tau,k} := R_{\tau,k} \circ \pi : H(X)_k \to H(X)_k.
\] (9)
These are the general models for ‘adjusted’ quantum evolution operators, according to the philosophy from [Z1] (and [Dau] in the linear case). Adopting the notation in (3), the Schwartz kernel $U_{\tau,k} \in C^\infty(X \times X)$ is
\[
U_{\tau,k}(x,y) = \sum_{j=1}^{N_k} U_{\tau}(s_j^{(k)})(x) \cdot s_j^{(k)}(y) \quad (x,y \in X).
\] (10)

Operator kernels such as (10) are thus crucial to Berezin-Toeplitz geometric quantization; this motivates studying their near-graph local scaling asymptotics and their relation to the underlying symplectic dynamics [P3].

This may be seen as a dynamical generalization of the near diagonal scaling asymptotics of the equivariant components of the Szegő kernel [BSZ], [SZ], [MM].

Broadly speaking, the emphasis in [P3] was on fixed time scaling asymptotics of the form $U_{\tau_0,k}(x + v_1/\sqrt{k}, x_{\tau_0} + v_2/\sqrt{k})$, where $\tau_0 \in \mathbb{R}$, $x \in X$, $x_{\tau_0} = \phi_{-\tau_0}^X(x)$, $v_1 \in T_xX$, $v_2 \in T_{x_{\tau_0}}X$. In the first part of this paper, we shall build on the arguments in [P3] to study scaling asymptotics of the form $U_{\tau,k}(x + v_1/\sqrt{k}, x_{\tau_0} + v_2/\sqrt{k})$, where $\tau_k = \tau_0 + \tau/\sqrt{k}$ and $\tau$ satisfies certain bounds; for example, $\tau$ might be fixed and $\neq 0$. These ‘scaled time’ asymptotics are the object of Theorem 2.1.

In the second part, we shall apply these results to the asymptotics of the traces of $U_{\tau,k}$, $\text{trace}(U_{\tau,k})$, under certain assumptions on the structure of the fixed locus of $\phi_{\tau_0}^X$ and of the critical locus of $f$ on it (Theorem 5.1).
Part I
Local scaling asymptotics

We first focus on the local scaling asymptotics of the ‘quantum evolution’ operators $U_{\tau,k}$. We begin by considering general near graph-asymptotics, in the spirit of Theorem 1.4 of [P3] but now including time rescaling. Then we specialize to the scaling asymptotics in the neighborhood of fixed loci. Throughout our discussion, we shall adopt the short-hand

$$m_\tau =: \phi^M_\tau(m), \quad x_\tau =: \phi^X_\tau(x) \quad (x \in X, m \in M, \tau \in \mathbb{R}).$$

2 Near-graph asymptotics

2.1 Preliminaries

2.1.1 Heisenberg local coordinates

Our scaling asymptotics are best expressed in a Heisenberg local coordinate system (HLCS) for $X$ centered at some $x \in X$,

$$\gamma_x : (\theta, v) \in (-\pi, \pi) \times B_{2d}(0, \delta) \mapsto x + (\theta, v) \in X; \quad (11)$$

here $B_{2d}(0, \delta) \subseteq \mathbb{R}^{2d}$ is the open ball centered at the origin and of radius $\delta > 0$. The additive notation is from [SZ], where HLCS’s were first defined and discussed.

In a HLCS the standard $S^1$-action is expressed by a translation in the angular coordinate $\theta$, and $\gamma_x$ comes with a built-it unitary isomorphism $T_x X \cong \mathbb{R} \oplus T_m M \cong \mathbb{R} \oplus \mathbb{C}^d$, where $m = \pi(x)$; the first summand $\mathbb{R} \oplus \{0\}$ corresponds to the vertical tangent space, and the second, $\{0\} \oplus \mathbb{C}^d$, to the horizontal one. We shall also set $x + v =: x + (0, v)$.

Given a system of HLC centered at $x$, if $m = \pi(x)$ then $v \mapsto m + v = \pi(x + v)$ is a system of preferred local coordinates on $M$ centered at $m$ in the sense of [SZ]; thus the unitary structure of $T_m M$ corresponds to the standard one on $\mathbb{C}^d$. With this understanding, in the notation $x + v$ we may assume either $v \in T_m M$ or $v \in \mathbb{R}^{2d} \cong \mathbb{C}^d$.

Heisenberg local coordinates make apparent the universal nature of the near diagonal scaling asymptotics of the equivariant Szegő kernels $\Pi_k$ [SZ], as well as of the near graph scaling asymptotics of $U_{\gamma_0, k}$ [P3]. The phase of the latter is determined by certain Poincaré type data, as we now recall.
2.1.2 Poincaré type data

As in [P3], the Poincaré type data that controls the phase in our asymptotic expansions is encoded in certain quadratic forms on the tangent bundle of $M$. Suppose $x \in X$, and set $x_{\tau_0} := \phi^X_{-\tau_0}(x)$, $m := \pi(x)$, $m_{\tau_0} := \phi^M_{-\tau_0}(m) = \pi(x_{\tau_0})$. Let us choose HLCS’s on $X$ centered at $x$ and $x_{\tau_0}$, respectively. This determines unitary isomorphisms $T_m M \cong \mathbb{C}^d$, $T_m_{\tau_0} M \cong \mathbb{C}^d$, under which the differential $d_m \phi^M_{-\tau_0} : T_m M \to T_m_{\tau_0} M$ is represented by a $2d \times 2d$ symplectic matrix $A$.

**Definition 2.1.** Let $A$ be a symplectic matrix with polar decomposition $A = OP$. Thus $O$ is symplectic and orthogonal (hence unitary) and $P$ is symplectic and symmetric positive definite. Also, let $J_0 := \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$ be the matrix of the standard complex structure on $\mathbb{R}^{2d} \cong \mathbb{C}^d$. Let us define:

$$Q_A := I + P^2, \quad P_A := O Q_A^{-1} O^t, \quad R_A := O \left( I - P^2 \right) Q_A^{-1} J_0 O^t.$$  

Then $Q_A$, $P_A$ and $R_A$ are symmetric matrices. Also, we shall set:

$$\nu_A := \sqrt{\det(Q_A)}. \quad (12)$$

Finally, we shall define a quadratic form $S_A : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{C}$ by setting

$$S_A(u, w) := -L_A(u, w)^t \left[ P_A + \frac{i}{2} R_A \right] L_A(u, w) - i\omega_0(Au, w), \quad (13)$$

where $L_A(u, w) := A u - w$, and $\omega_0(r, s) := -r^t J_0 s$ is the standard symplectic structure on $\mathbb{R}^{2d}$.

These Definitions may be transferred to our geometric setting, as follows.

**Definition 2.2.** Let us think of $(u, w) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ as representing $(u, w) \in T_m M \times T_m_{\tau_0} M$ in the given choice of HLCS’s. Since $(u, w) \mapsto S_A(u, w)$ is invariant under a change of HLCS’s as above, it yields a well-defined quadratic form $S_{\tau_0, m} : T_m M \times T_m_{\tau_0} M \to \mathbb{C}$.

Similarly, with $\nu_A$ as in (12), we shall set

$$\nu(\tau_0, m) := \nu_A. \quad (14)$$

**Remark 2.1.** In the following arguments, a choice of HLCS’s will be always implicit; we shall systematically abuse notation and avoid distinguishing between $S_{\tau_0, m}$, $u$, $w$... and $S_A$, $u$, $w$..., and similarly for $\Omega_{\tau_0, m}$ and $\Omega_A$ to be introduced below.
2.2  The local scaling asymptotics

Let us consider the generalizations of Theorems 1.2 and 1.4 in [P3] to rescaled times. Given \( \tau_0, \tau \in \mathbb{R} \) we shall set \( \tau_k =: \tau_0 + \tau/\sqrt{k} \) for \( k = 1, 2, \ldots \). Furthermore, we shall denote by

\[
\text{dist}_X : X \times X \to \mathbb{R} \quad \text{and} \quad \text{dist}_M : M \times M \to \mathbb{R}
\]

the Riemannian distance functions on \( X \) and \( M \), respectively. Since \( \pi \) is a Riemannian submersion and its fibers are the \( S^1 \)-orbits in \( X \), we have

\[
\text{dist}_X (x, S^1 \cdot x') = \text{dist}_M (\pi (x), \pi (x')) \quad \forall x, x' \in X.
\]

2.2.1  Off-graph rapid decay

To begin with, we have

**Proposition 2.1.** Suppose \( \tau_0 \in \mathbb{R} \) and choose \( C, E, \epsilon > 0 \). Then, uniformly for

\[
\text{dist}_X (y, S^1 \cdot x_{\tau_0}) \geq C k^{-7/18} \quad \text{and} \quad |\tau| \leq E k^{1/9 - \epsilon},
\]

we have \( U_{\tau_k} (x, y) = O (k^{-\infty}) \) as \( k \to +\infty \).

**Proof.** By Theorem 1.2 of [P3], if \( \tau' \) varies in a bounded interval then \( U_{\tau'} (x, y) = O (k^{-\infty}) \) as \( k \to +\infty \), uniformly for \( \text{dist}_X (y, S^1 \cdot x_{\tau'}) > C' k^{-7/18} \). Thus we need only prove that under the hypothesis above \( \text{dist}_X (y, S^1 \cdot x_{\tau_k}) > C'' k^{-7/18} \), for all \( k \gg 0 \) and some constant \( C' > 0 \) (independent of \( k \)).

There exists \( D > 0 \) such that \( \forall m' \in M \) and \( \tau' \in \mathbb{R} \) we have

\[
\text{dist}_M (m', m_{\tau'}) \leq D |\tau'|,
\]

whence in the given hypothesis

\[
\text{dist}_M (m'_{\tau_0}, m'_{\tau_k}) \leq D |\tau|/\sqrt{k} = O \left( k^{1/9 - \epsilon - 1/2} \right) = o (k^{-7/18}) , \quad (15)
\]

uniformly for \( m' \in M \).

Now if \( m = \pi (x), n = \pi (y) \) we have \( \text{dist}_X (y, S^1 \cdot x_{\tau'}) = \text{dist}_M (n, m_{\tau'}). \)

Therefore,

\[
\text{dist}_X (y, S^1 \cdot x_{\tau_k}) = \text{dist}_M (n, m_{\tau_k}) \geq \text{dist}_M (n, m_0) - \text{dist}_M (m_0, m_{\tau_k}) > C/2 \quad k^{-7/18}
\]

if \( k \gg 0 \). This completes the proof.

\[\Box\]
2.2.2 The near-graph asymptotic expansion

Now we shall revisit Theorem 1.4 of [P3], and show how the line of argument in its proof may adapted to include scaling in the time variable, and remove the transversality assumption on the displacement vectors.

**Theorem 2.1.** Suppose $\tau_0 \in \mathbb{R}$ and $E > 0$. Consider $x \in X$ and let $m =: \pi(x)$. Choose HLCS’s on $X$ centered at $x$ and $x_{\tau_0}$, respectively. Then, uniformly in $(u, w, \tau) \in T_m M \times T_{m_0} M \times \mathbb{R}$ with $\|u\|, \|w\|, |\tau| \leq E k^{1/9}$, the following asymptotic expansion holds as $k \to +\infty$:

$$U_{\tau_0 + \tau \sqrt{k}} \left( x + \frac{u}{\sqrt{k}}, x_{\tau_0} + \frac{w}{\sqrt{k}} \right) \sim e^{i \tau \sqrt{k} f(m)} \frac{2^d}{\nu(\tau_0, m)} \left( \frac{k}{\pi} \right)^d e^{i \tau \nu_m(u, v_f(m) + w) + i \tau \omega_m(v_f(m), w)} \left[ \varphi_{\tau_0}(m) + \sum_{j \geq 1} k^{-j/2} b_j(m, \tau, u, w) \right],$$

where each $b_j$ is $C^\infty$ and a polynomial in $(\tau, u, w)$ of joint degree $\deg(b_j) \leq 3j$.

**Proof.** Before delving into the proof, it is in order to recall the general scheme of the arguments in [P3].

For $\tau' \in \mathbb{R}$, let us set $\Pi_{\tau'} = (\phi_{\tau'}^X)^* \circ \Pi$; in terms of Schwartz kernels,

$$\Pi_{\tau'} = (\phi_{\tau'}^X \times id_X)^* (\Pi).$$

Then by definition $U_{\tau'} = R_{\tau'} \circ \Pi_{\tau'}$. Since by hypothesis these operators are $S^1$-invariant, passing to $S^1$-isotypes we have:

$$U_{\tau', k} = R_{\tau', k} \circ \Pi_{\tau', k} = R_{\tau', k} \circ \Pi_{\tau'} = R_{\tau'} \circ \Pi_{\tau', k}. \quad (16)$$

In terms of Schwartz kernels,

$$U_{\tau', k}(x, y) = \int_X R_{\tau'}(x, z) \Pi_{\tau', k}(z, y) dV_X(z), \quad (17)$$

where $\Pi_{\tau', k}(z, y) = \Pi_k(z_{\tau'}, y)$. By Theorem 1.2 of [P3], $U_{\tau', k}(x, y) = O(k^{-\infty})$ as $k \to +\infty$, unless $y$ belongs to a small (and shrinking) $S^1$-invariant neighborhood of $x_{\tau'}$. Also, as argued in the proof of the same Theorem, only a negligible contribution to the asymptotics is lost in (17) if integration in $z$ is restricted to a small (and shrinking) invariant neighborhood of $x$, so that $z_{\tau'}$ varies in a small invariant neighborhood of $x_{\tau'}$.  

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In these neighborhood, perhaps up to smoothing operators irrelevant to the asymptotics, $R_{\tau'}$ and $\Pi_{\tau'}$ may be represented as FI$\text{O}'$s. In fact, by the theory of [?](see also the discussion in [SZ]) in the neighborhood of $S^1 \cdot x$ and $S^1 \cdot x_{\tau_0}$ we can write, respectively,

$$\Pi (x', x'') = \int_{0}^{+\infty} e^{iu\psi(x', x'')} s (u, x', x'') \, du,$$

and

$$R_{\tau'} (y', y'') = \int_{0}^{+\infty} e^{it\psi(y', y'')} a_{\tau'} (t, y', y'') \, dt,$$

where $\psi$ is a complex phase of positive type, while $s$ and $a_{\tau'}$ are semiclassical symbols of degree $d$. More precisely, there are asymptotic expansions

$$s (u, x', x'') \sim \sum_{j \geq 0} u^{d-j} s_j (x', x''),$$

and

$$a_{\tau'} (t, y', y'') \sim \sum_{j \geq 0} t^{d-j} a_{\tau'j} (y', y'').$$

Suppose now that $\tau' \sim \tau_0$, so that $x_{\tau'} \sim x_{\tau_0}$. In HLCS’s centered at $x$ and $x_{\tau_0}$, respectively, we have:

$$s_0 (x, x) = \frac{1}{\pi d} \text{ and } a_{\tau'0} (x_{\tau_0}, x_{\tau_0}) = \frac{1}{\pi d} q_{\tau'} (m).$$

Let us now apply these considerations with $\tau' = \tau_k$, where $\tau_k =: \tau_0 + \tau/\sqrt{k}$. Inserting (18) and (19) in (17), and following the arguments leading to 3.4 in [?], we obtain

$$U_{\tau_k} \left( x + \frac{u}{\sqrt{E}}, x_{\tau_0} + \frac{w}{\sqrt{k}} \right) \sim \frac{k^2}{2\pi} \int_{1/E}^{E} \int_{1/E}^{E} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\mathbb{C}^d} e^{ik\Psi_1 (z)} A_1' \cdot V (\theta, v) \, dt \, dv \, d\vartheta \, d\theta \, dv.$$

Let us briefly clarify (23).

- The factor $k^2$ is due to a change of integration variables $t \mapsto kt$ and $u \mapsto ku$.
- $\vartheta$ is the angular coordinate on the structure group $S^1$ of $X$.
- $z = x + (\theta, v)$ is a HLCS on $X$ centered at $x$. 

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\[ \gamma_k(z) =: \gamma_1 \left( k^{7/18} \|v\| \right), \text{ where } \gamma_1 \in C_0^\infty (\mathbb{R}^{2d}) \text{ is a bump function } \equiv 1 \text{ in a neighborhood of the origin. Thus integration in } v \text{ is thus over a ball of radius } O \left( k^{-7/18} \right). \]

\[ A'_1 = \varrho(\theta, \vartheta) \cdot A_1, \text{ where } \varrho \text{ and } A_1 \text{ are as follows.} \]

1. \( \varrho(\theta, \vartheta) \) is a bump function on \( \mathbb{R}^2 \), supported where \( \| (\theta, \vartheta) \| \leq \epsilon \) and \( \equiv 1 \) in a neighborhood of the origin.

2. With \( s \) and \( a_\tau \) as in (18) and (19), we have

\[ A_1 = a_\tau \left( k t, x + \frac{u}{\sqrt{k}}, z \right) s \left( k u, r_\varrho(z_\tau), x_\tau_0 + \frac{w}{\sqrt{k}} \right). \] (24)

\[ \text{The phase } \Psi'_1 \text{ is given by} \]

\[ \Psi'_1 = t \psi \left( x + \frac{u}{\sqrt{k}}, x + (\theta, v) \right) + u \psi \left( \phi_-(x + (\theta + \vartheta, v)), x_\tau_0 + \frac{w}{\sqrt{k}} \right) - \vartheta. \] (25)

In rescaled HLC, \( z = x + (\theta, v/\sqrt{k}) \), we obtain (see (6.2) of [?]):

\[ U_{\tau_0 k} \left( x + \frac{u}{\sqrt{k}}, x + \frac{w}{\sqrt{k}} \right) \sim \frac{k^{2-d}}{2\pi} \int_{C^d} \left[ \int_{E} \int_{E} \int_{\epsilon} e^{ik\Psi_2} A_2 \cdot V \left( \frac{\theta}{\sqrt{k}}, \frac{v}{\sqrt{k}} \right) dt du d\vartheta d\vartheta \right] dv. \] (26)

Here \( \Psi_2 \) is \( \Psi'_1 \) in rescaled coordinates, and

\[ A_2 =: \gamma_1 \left( k^{-1/9} \|v\| \right) \cdot A''_1, \]

where \( A''_1 \) is \( A'_1 \) in rescaled coordinates.

In view of (I5), the argument in the proof of Lemma 6.1 of [?] still applies, and we get:

**Lemma 2.1.** There exist constants \( C_1, C_2 > 0 \) such that the locus where \( |\theta| > C_1 k^{-7/18} \) or \( |\vartheta| > C_2 k^{-7/18} \) contributes negligibly to the asymptotics of (22).

Thus we can multiply the integrand in (26) by a cut-off of the form \( \gamma'_1 \left( k^{7/18} (\theta, \vartheta) \right) \), with \( \gamma'_1 \equiv 1 \) near the origin, without changing the asymptotics. In rescaled angular coordinates, \( (\theta, \vartheta) \mapsto (\theta/\sqrt{k}, \vartheta/\sqrt{k}) \), this becomes
\[ \gamma'_{1}(k^{-1/9}(\theta, \vartheta)), \] so that integration in \( d\theta d\vartheta \) is now over a ball of radius \( O(k^{1/9}) \) centered at the origin in \( \mathbb{R}^2 \).

We then get (cfr (6.10) of [P3]):

\[ U_{\tau, k}(x + \frac{u}{\sqrt{k}} x_{0} + \frac{w}{\sqrt{k}}) = \int_{C_{\varepsilon}} I_{k}(\tau, u, w, v) \, dv, \quad (27) \]

where

\[ I_{k}(\tau, u, w, v) \sim \frac{k^{1-d}}{2\pi} \int_{1/E}^{E} \int_{-\infty}^{+\infty} e^{i\sqrt{k}\psi_{\tau}} \mathcal{A}_{\tau} \cdot \mathcal{V} (\frac{\theta}{\sqrt{k}}, \frac{v}{\sqrt{k}}) \, dt \, du \, d\theta \, dv. \quad (28) \]

More precisely, notation in (28) is as follows. First, the phase is

\[ \psi_{\tau}(t, \theta, u, \vartheta) = u (\tau f(m) + \theta + \vartheta) - t \theta - \vartheta. \quad (29) \]

The first factor in the amplitude, on the other hand, is

\[ \mathcal{A}_{\tau} = \exp \left( i \tau \omega_{m}(v_{f}(m), Av) - \frac{t}{2} \theta^{2} - \frac{u}{2} (\tau f(m) + \theta + \vartheta)^{2} \right) \cdot \exp \left( t \psi_{2}(u, v) + u \psi_{2}(Av - \tau v_{f}(m), w) \right) \cdot \mathcal{A}'; \quad (30) \]

here \( \mathcal{A}' \) is \( A_{2} \) expressed in rescaled coordinates, times a factor of the form \( e^{i R_{3}} \), where we have set

\[ R_{3} = R_{3}(t, u) \left( \frac{v}{\sqrt{k}}, \frac{u}{\sqrt{k}}, \frac{w}{\sqrt{k}}, \frac{\tau}{\sqrt{k}} \right), \quad (31) \]

for an appropriate function \( R_{3}(\cdot, \cdot, \cdot, \cdot) \) vanishing to third order at the origin, and depending on \( t \) and \( u \).

We can view \( I_{k}(\tau, u, w, v) \) as an oscillatory integral in \( \sqrt{k} \) with real phase \( \psi_{\tau} \). As discussed in [P3], \( \psi_{\tau}(t, \theta, u, \vartheta) \) has a unique stationary point \( P_{\tau} = (1, 0, 1, -\tau f(m)) \), which is non-degenerate and satisfies \( \psi_{\tau}(P_{\tau}) = \tau f(m) \). The Hessian matrix at the critical point has determinant 1 and signature 0.

The Hessian matrix of \( \psi_{\tau} \) and its inverse at the critical point are

\[ H_{P_{\tau}}(\psi_{\tau}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad H_{P_{\tau}}(\psi_{\tau})^{-1} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \quad (32) \]

In particular, \( H_{P_{\tau}}(\psi_{\tau}) \) has zero signature.
Using an integration by parts in $dt \, du$ one can see that, perhaps after disregarding a negligible contribution to the asymptotics, the integrand in (28) may also be assumed to be compactly supported in $(\theta, \vartheta)$ near the critical point \([P3]\).

Applying Taylor expansion, and taking into account the factor $e^{ikR_\vartheta}$, in view of (20), (21), and (24) (with $\vartheta$ rescaled to $\vartheta/\sqrt{k}$) we have for the amplitude in (27) an asymptotic expansion of the form

$$
\frac{k^{1-d}}{2\pi} \cdot A_r \cdot \mathcal{V} \left( \frac{\theta}{\sqrt{k}}, \frac{v}{\sqrt{k}} \right) = e^{\Theta(v, u, w, \tau, t, \theta, \vartheta)} \cdot A' \cdot \mathcal{V} \left( \frac{\theta}{\sqrt{k}}, \frac{v}{\sqrt{k}} \right) \sim k^{d+1} e^{\Theta(v, u, w, \tau, t, \theta, \vartheta)} \cdot \sum_{a \geq 0} \frac{k^{-a/2}}{a!} Q_{a}^{(t,u)} (v, u, w, \tau, \vartheta, \theta),
$$

where

$$
\Theta(v, u, w, \tau, t, \theta, \vartheta) = -\frac{t}{2} \theta^2 - \frac{u}{2} \left( \tau f(m) + \theta + \vartheta \right)^2 + i\tau \omega_m(vf(m), Av) + t \psi_2(u, v) + u \psi_2(Av - \tau vf(m), w),
$$

and $Q_a = Q_{a}^{(t,u)} (v, u, w, \tau, \vartheta, \theta)$ is a polynomial of degree $\leq 3a$.

Let us temporarily use the short-hand

$$
\int =: \int_{1/E}^{E} \int_{1/E}^{E} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty}.
$$

Applying the stationary phase Lemma for real quadratic phase functions \([Dui]\), we obtain from each summand in the second line of (33) an asymptotic expansion of the form

$$
k^{1+d-a/2} \int e^{i\sqrt{k} \Psi} \cdot e^{\Theta(v, u, w, \tau, t, \theta, \vartheta)} \cdot Q_{a}^{(t,u)} (v, u, w, \tau, \vartheta, \theta) \, dt \, du \, d\vartheta \, d\theta
\sim \frac{k^{d-a/2}}{\pi^d} e^{i\sqrt{k} \tau f(m) + B_r(m, u, v)} \cdot \sum_{b \geq 0} \frac{1}{b!} k^{-b/2} \mathfrak{R}^b \left( e^{\Theta} Q_a \right) (v, u, w, \tau, 1, 0, 1, -\tau f(m)),
$$

where $\mathfrak{R}$ is the second order differential operator defined by the inverse Hessian matrix in (32):

$$
\mathfrak{R} = \frac{i}{2} \left( \frac{\partial_t}{\partial_{\vartheta}} \quad \frac{\partial_t}{\partial_{\theta}} \quad \frac{\partial_u}{\partial_{\vartheta}} \quad \frac{\partial_u}{\partial_{\theta}} \right) \left( \begin{array}{cccc} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array} \right) \left( \begin{array}{c} \partial_t \\ \partial_{\vartheta} \\ \partial_u \\ \partial_{\theta} \end{array} \right).
$$
In particular, in $\mathcal{R}$ there are no terms of the form $\partial_\alpha^2$ or $\partial_\alpha^2$; it then follows from the fact that $Q_a$ has degree $\leq 3a$ and the expression (34) for $\Theta$ that

$$\mathcal{R}^b\left(e^\Theta Q_a\right)(v, u, w, \tau, 1, 0, 1, -\tau f(m)) = e^{B_r(m,u,w,v)} S_{a,b}(v, u, w, \tau), \quad (37)$$

where

$$B_r(m, u, w, v) = \Theta(v, u, w, \tau, 1, 0, 1, -\tau f(m))$$

$$= i \tau \omega_m(v_f(m), Av) + \psi_2(u, v) + \psi_2(\tau - \tau v_f(m), w)$$

$$= i \tau \omega_m(v_f(m), w) + \psi_2(u, v) + \psi_2(\tau v_f(m) + w),$$

while $S_{a,b}$ is a polynomial of degree $\leq 3a + 2b \leq 3(a + b)$ (and depending smoothly on $m$). Given this, we obtain for (28) an asymptotic expansion of the form

$$I_k(\tau, u, w, v) \sim \frac{k^d}{\pi^{d/2}} e^{i\sqrt{2}\tau f(m) + B_r(m,u,w,v)} \sum_{j \geq 0} k^{-j/2} I_j(m, \tau, u, w, v), \quad (39)$$

where each $I_j$ is a polynomial in $(\tau, u, w, v)$ of joint degree $\leq 3j$ and, recalling (22), the leading order term is $I_0(m, \tau, u, w, v) = g_{\alpha_0}(m)$. Let us set $v = v' + u$. Then

$$B_r(m, u, w, v) = B_r(m, u, w, v' + u)$$

$$= i \left[ \tau \omega_m(v_f(m), Av) - \omega_m(u, v) - \omega_m(Av' + Au, \tau v_f(m) + w) \right]$$

$$- \frac{1}{2} \|v'\|^2 - \frac{1}{2} \|Av' + Au - w - \tau v_f(m)\|^2.$$

Hence, $\mathcal{R}(B_r(m, u, w, v)) \leq -1/2 \|v'\|^2$. Furthermore, $I_j(\tau, u, w, v' + u)$ splits as a linear combination of terms of the form $\tau^\alpha u^\beta w^\gamma v^\delta$, where $\alpha + |\beta| + |\gamma| + \delta \leq 3j$. After integration in $dv'$ each such term therefore is bounded by $C_j |\tau|^\alpha \|u\|^\beta \|w\|^\gamma \leq C_j k^{3/3}$, where we have used that in our range $|\tau|$, $\|u\|$, $\|w\| \leq E k^{1/3}$. Therefore,

$$\left| k^{-j/2} \int_{C_d} e^{B_r(m,u,w,v)} I_j(\tau, m, u, w, v) \, dv \right| \leq C_j k^{-j/2} k^{3/3} = C_j k^{-j/6}. \quad (41)$$

Furthermore, in view of (33), a similar bound holds for the $J$-th step remainder. Since integration in $dv$ in (27) is actually over a ball centered at
the origin and of radius \( O(k^{1/9}) \) in \( \mathbb{C}^d \), the expansion (39) may be integrated term by term.

Going back to (27), we conclude that there is an asymptotic expansion of the form

\[
U_{\tau, k} \left( x + \frac{u}{\sqrt{k}}, x_{\tau_0} + \frac{w}{\sqrt{k}} \right) \sim \frac{k^d}{\pi^{2d}} e^{i\sqrt{k} f(m)} \cdot \sum_{j \geq 0} k^{-j/2} \int_{\mathbb{C}^d} e^{B_r(m, u, w, v)} I_j(m, \tau, u, w, v) \, dv,
\]

where under the present assumptions the \( j \)-th summand is uniformly bounded by \( D_j k^{-j/2 + j/3} = D_j k^{-(1/6)}j \).

Using (38) and the arguments from (3.15) to Lemma 3.3 of [P3], we obtain for the leading order term:

\[
e^{i\sqrt{k} f(m)} \frac{k^d}{\pi^{2d}} g_{\tau_0}(m) \int_{\mathbb{C}^d} e^{B_r(m, u, w, v)} \, dv \tag{43}
\]

As to the lower order terms, we have for any \( j \geq 1 \) that the integral in (41) is a linear combination of similar integrals, with \( I_j \) replaced by a monomial of the form \( \tau^\alpha u^\beta w^\gamma v^\delta \). Furthermore, we see from (38) that

\[
B_r(m, u, w, v) = i\tau \omega_m(v_f(m), w) - i\omega_0(v, H) - \frac{1}{2} F - \frac{1}{2} v^t Q_A v + v^t \cdot G, \tag{44}
\]

where

\[
F = F(\tau v_f(m), u, w) = : \| \tau v_f(m) + w \|^2 + \| u \|^2 \quad G = G(\tau v_f(m), u, w) = : u + A(\tau v_f(m) + w) \quad H = H(\tau v_f(m), u, w) = : A^{-1} w - u + \tau A^{-1} v_f(m) \]

Let us perform the change of variables \( v = s + r \), with \( r = Q_A^{-1} G \). We obtain from (44) that

\[
B_r(m, u, w, v) = B_r(m, u, w, s + r) \tag{45}
\]

\[
= i\tau \omega_m(v_f(m), w) - \frac{1}{2} F - i\omega_0(r, H) - \frac{1}{2} r^t Q_A r
\]

\[
- i\omega_0(s, H) - \frac{1}{2} s^t Q_A s.
\]

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the proof of Theorem 2.1.

combination of monomials of the form

\[ P \]

auxiliary results.

locus. To this end, we first need to introduce some more terminology and

We now aim to specialize the previous result to asymptotics near the fixed

Hence, if \( F(\cdot) \) denotes the Fourier transform operator, we have

\[
\int_{\mathbb{R}^{2d}} e^{B_r(m,u,w,v)} \, dv = \int_{\mathbb{R}^{2d}} e^{B_r(m,u,w,s+r)} \, ds
\]

\[
= e^{i \tau \omega_m(v_f(m),w)} - \frac{i}{2} F - i \omega_0(r,H) - \frac{1}{2} r^t Q_A r \int_{\mathbb{R}^{2d}} e^{-i \omega_0(s,H) - \frac{1}{2} s^t Q_A s} \, ds
\]

\[
= e^{i \tau \omega_m(v_f(m),w)} - \frac{i}{2} F - i \omega_0(r,H) - \frac{1}{2} r^t Q_A r \cdot (2\pi)^d \, \mathcal{F} \left( e^{\frac{i}{2} s^t Q_A s} \right) \bigg|_{\xi = -J_0 H}
\]

\[
= (2\pi)^d \operatorname{det}(Q_A)^{-1} e^{i \tau \omega_m(v_f(m),w)} - \frac{i}{2} F - i \omega_0(r,H) - \frac{1}{2} r^t Q_A r - \frac{1}{2} (J_0 H)^t Q_A^{-1} (J_0 H).
\]  

Now any monomial \( \tau^\alpha u^{\beta} w^\gamma v^\delta \) with \( \alpha + |\beta| + |\gamma| + |\delta| \leq 3j \) is a linear combination of monomials of the form \( \tau^{\alpha'} u^{\beta'} w^{\gamma'} s^\delta' \) with \( \alpha' + |\beta'| + |\gamma'| + |\delta'| \leq 3j \). On the other hand, for any such \((\alpha', \beta', \gamma', \delta')\) we have

\[
\int_{\mathbb{R}^{2d}} \tau^{\alpha'} u^{\beta'} w^{\gamma'} s^\delta' e^{B_r(m,u,w,s+r)} \, ds \]

\[
= \tau^{\alpha'} u^{\beta'} w^{\gamma'} e^{i \tau \omega_m(v_f(m),w)} - \frac{i}{2} F - i \omega_0(r,H) - \frac{1}{2} r^t Q_A r \int_{\mathbb{R}^{2d}} s^\delta' e^{-i \omega_0(s,H) - \frac{1}{2} s^t Q_A s} \, ds
\]

\[
= \tau^{\alpha'} u^{\beta'} w^{\gamma'} e^{i \tau \omega_m(v_f(m),w)} - \frac{i}{2} F - i \omega_0(r,H) - \frac{1}{2} r^t Q_A r \cdot P_{\delta'} \left( e^{-\frac{i}{2} \xi^t Q_A^{-1} \xi} \right) \bigg|_{\xi = -J_0 H},
\]  

where \( P_{\delta'} \) is a suitable differential polynomial, in the collective variables \( \xi \), of degree \(|\delta'|\). Thus \((47)\) can be decomposed as linear combination of terms of the form

\[
\tau^{\alpha'} u^{\beta'} w^{\gamma'} H^{\delta'} e^{i \tau \omega_m(v_f(m),w)} - \frac{i}{2} F - i \omega_0(r,H) - \frac{1}{2} r^t Q_A r - \frac{1}{2} H^t Q_A^{-1} H
\]

\[
\propto \tau^{\alpha'} u^{\beta'} w^{\gamma'} H^{\delta'} e^{i \tau \omega_m(v_f(m),w)} + S_A(u, \tau v_f(m) + w)
\]

with \( \alpha' + |\beta'| + |\gamma'| + |\delta'| \leq 3j \). Since \( H \) is linear in \( (\tau, u, w) \), this completes the proof of Theorem 2.1.

\[ \square \]

3 Local scaling asymptotics near fixed loci

We now aim to specialize the previous result to asymptotics near the fixed locus. To this end, we first need to introduce some more terminology and auxiliary results.
3.1 Preliminaries

3.1.1 Clean and very clean periods

Let us now make our discussion more precise. To begin with, we clarify the class of periods to which our results will apply.

**Definition 3.1.** Let $\tau_0 \in \mathbb{R}$.

- We shall let $M_{\tau_0} =: \text{Fix} (\phi^M) \subseteq M$ be the fixed locus of $\phi^M : M \to M$.
- We shall say that $\tau_0$ is a *clean period* of the phase flow $\phi^M$ if
  1. $M_{\tau_0}$ is a submanifold of $M$;
  2. at each $m \in M_{\tau_0}$, we have $T_m M_{\tau_0} = \ker (d_m \phi^M - \text{id}_{T_m M_{\tau_0}})$.
- We shall say that $\tau_0$ is a *very clean period*, or a *symplectically clean period*, of $\phi^M$ if
  1. $\tau_0$ is a clean period of $\phi^M$;
  2. $M_{\tau_0} \subseteq M$ is a symplectic submanifold.

**Remark 3.1.** If $\tau_0 \in \mathbb{R}$ is a clean period of $\phi^M$, then it is very clean if and only if, in addition,

$$\ker (\text{id}_{T_m M} - d_m \phi^M_{\tau_0}) \cap \text{im} (\text{id}_{T_m M} - d_m \phi^M_{\tau_0}) = \{0\}$$

(48)

for every $m \in M_{\tau_0}$. In fact, the latter space on the left hand side of (48) is the symplectic normal space of $T_m M_{\tau_0}$ (§4 of [DG]).

Our local scaling asymptotics will apply to all very clean periods.

**Definition 3.2.** Suppose that $P \subseteq M$ is a submanifold, and $T_p P \subseteq T_p M$ is its tangent subspace at a point $p \in P$. We shall denote by $N^g_P = (T_p P)^{\perp_g} \subseteq T_p M$ the Riemannian orthocomplement of $T_p P$ with respect to the Riemannian metric $g$, and by $N^\omega_P = (T_p P)^{\perp_\omega} \subseteq T_p M$ the symplectic normal space with respect to the symplectic structure $\omega$.

Suppose $\tau_0$ is a very clean period of $\phi^M$ and $F \subseteq M_{\tau_0}$ is a connected component of the fixed locus. If $m \in F$, then by the above

$$N^\omega_m F = \text{im} (\text{id}_{T_m M} - d_m \phi^M_{\tau_0})$$

(49)

**Remark 3.2.** Unless $P$ is a complex submanifold, $N^g_P \neq N^\omega_P$ in general. Except when $f$ is compatible, $M_{\tau_0}$ needn’t be a complex submanifold.
3.1.2 Poincaré data along clean fixed loci

Let us specialize the considerations of 2.1.2 to the case of a fixed point of $m \in M_{\tau_0}$. If $m = \pi(x)$ and we fix a HLCS centered at $x$, $d_m : T_m \tau_0 \rightarrow T_m M$ corresponds to a symplectic matrix $A$.

**Definition 3.3.** Restricting the quadratic form (13) to the diagonal subspace in $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$, we obtain a quadratic form $Q_A : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ given by

$$Q_A(u) = -S_A(u, u).$$

If $m = m_{\tau_0}$, with the previous choices $Q_A$ clearly corresponds to the well-defined quadratic form on $T_{m_{\tau_0}}M$

$$\Omega_{\tau_0,m}(v) = -S_{\tau_0,m}(v, v)$$

(Definition 2.2).

**Remark 3.3.** Since $P_A$ is clearly positive definite, $\Re(\Omega_{\tau_0,m}(v)) \geq 0$ for every $v \in T_{m_{\tau_0}}M$. In addition, if $\tau_0$ is very clean then by Remark 3.1, Definition 3.2, (13) and (49) there exists $C = C_{\tau_0} > 0$ such that for any connected component $F$ of $M_{\tau_0}$ we have

$$\Re(\Omega_{\tau_0,m}(v)) \geq C \|v\|^2 \text{ if } m \in F \text{ and } v \in N^\omega_m F;$$

(51)

here $\| \cdot \| = \| \cdot \|_m$ is the pointwise norm for $g$ at $m \in M$.

**Definition 3.4.** If $F$ is a connected component of $M_{\tau_0}$, and $m \in F$, then by Remark 3.3 the restriction $Q_{\tau_0,m}^{\text{nor}}$ of $Q_{\tau_0,m}$ to $N^\omega_m F$ has positive definite real part. We shall denote by $\det(Q_{\tau_0,m}^{\text{nor}})$ its determinant with respect to any orthonormal basis of $N^\omega_m F$ for the restricted Euclidean structure of $T_m M$.

The square root $\det(Q_{\tau_0,m}^{\text{nor}})^{1/2}$ is defined in a standard manner [?].

**Remark 3.4.** More generally, in the same situation we can consider the quadratic expression $S_{\tau_0,m}(u, \tau v_f(m) + u)$ for $(u, \tau) \in N^\omega_m F \times \mathbb{R}$. Since $A - I_{2d}$ restricts to an isomorphism of $N^\omega_m F$ and $v_f(m) \in T_m F$, in this case in place of (51) we have

$$\Re\left(S_{\tau_0,m}(u, \tau v_f(m) + u)\right)$$

$$= -L_A(u, \tau v_f(m) + u)^t P_A L_A(u, \tau v_f(m) + u)$$

$$= -[(A - I_{2d})u - \tau v_f(m)]^t P_A [(A - I_{2d})u - \tau v_f(m)]$$

$$\leq -C \left(\|u\|^2 + \tau^2 \|v_f(m)\|^2\right)$$

(52)

for some $C > 0$ depending only on $\tau_0$. 

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If \( m \in M_{\tau_0} \), then \( m = m_{\tau_0} \); therefore, if \( \pi(x) = m \) then \( \pi(x_{\tau_0}) = m_{\tau_0} = \pi(x) \), whence \( x_{\tau_0} \in S^1 \cdot x \) (the \( S^1 \)-orbit through \( x \)). Thus there exists a unique \( h(x) \in S^1 \) such that \( x_{\tau_0} = r_{h(x)}(x) \). If \( h(x) = e^{i\vartheta_0} \), we have \( x_{\tau_0} = x + (\vartheta_0, 0) \). Thus a HLCS centered at \( x \) obviously induces a HLCS centered at \( x_{\tau_0} \), given by \( x_{\tau_0} + (\theta, v) =: x + (\vartheta_0 + \theta, v) \).

If \( F_1, \ldots, F_{\ell_{\tau_0}} \) are the connected components of \( M_{\tau_0} \), then \( h(x) \) is constant over each \( F_a \). Let \( h_a \in S^1 \) be the constant value of \( h(x) \) for \( \pi(x) \in F_a \). Then for any \( x \in F_a \), \( k \in \mathbb{N} \) and \( \tau \in \mathbb{R} \) we have

\[
U_{\tau,k}(x + v, x + v) = U_{\tau,k}(x + v, h_a^{-1} \cdot (x_{\tau_0} + v)) = h_a^k \cdot U_{\tau,k}(x + v, x_{\tau_0} + v). \tag{53}
\]

### 3.2 Diagonal rapid decrease away from clean fixed loci

For \( \tau \in \mathbb{R} \), let us set \( X_\tau =: \pi^{-1}(M_\tau) \) (in general, \( X_\tau \) is not the fixed locus of \( \phi^X_\tau \), but contains it). Since \( \pi : X \to M \) is a Riemannian submersion, we have

\[
dist_X(y, X_\tau) = dist_M(\pi(y), M_\tau) \tag{54}
\]

for any \( y \in X \).

**Proposition 3.1.** Suppose that \( \tau_0 \) is a clean period of \( \phi^M_\tau \), and choose \( C, E, \epsilon > 0 \). Uniformly in \( y \in X \) and \( \tau \in \mathbb{R} \) satisfying

\[
dist_X(y, X_{\tau_0}) > C k^{-7/18} \quad \text{and} \quad |\tau| < E k^{1/9 - \epsilon}, \tag{55}
\]

we have

\[
U_{\tau_0 + \tau/\sqrt{k}}(y, y) = O(k^{-\infty}).
\]

**Proof.** Let us set as before \( \tau_k = \tau_0 + \tau/\sqrt{k} \). Adapting the argument of Proposition 2.1, we are now reduced to proving that, under the previous assumptions, for some \( D > 0 \) we have

\[
dist_X(y, S^1 \cdot y_{\tau_0}) \geq D k^{-7/18}. \tag{56}
\]

To this end, write \( n = \pi(y) \) and remark that \( \forall \tau' \in \mathbb{R} \)

\[
dist_X(y, X_{\tau'}) = dist_M(n, M_{\tau'}), \quad dist_X(y, S^1 \cdot y_{\tau'}) = dist_M(n, n_{\tau'}); \tag{57}
\]

on the other hand, using first that \( \tau_0 \) is a clean period and then \( \tau \leq E k^{1/9 - \epsilon} \), we have

\[
dist_M(n, n_{\tau_0}) \geq C_1 \cdot dist_M(n, M_{\tau_0}) \geq (C C_1) k^{-7/18}. \tag{58}
\]
Putting together (57), (58) and (15), we get if \( \text{dist}_X (y, S^1 \cdot y_{\tau_k}) \geq C k^{-7/18} \):

\[
\text{dist}_X (y, S^1 \cdot y_{\tau_k}) = \text{dist}_M (n, n_{\tau_k}) \\
\geq \text{dist}_M (n, n_{\tau_0}) - \text{dist}_M (n_{\tau_k}, n_{\tau_0}) \geq \frac{1}{2} CC_1 k^{-7/18}
\]

if \( k \gg 0 \). This establishes (56), hence proves the Proposition. \( \square \)

3.3 The asymptotic expansion near a fixed locus

If we specialize Theorem 2.1 to the asymptotics near a very clean fixed locus, in view of (53) we immediately obtain:

**Corollary 3.1.** Suppose that \( \tau_0 \) is a very clean period of \( \phi^M \) and \( E > 0 \). Let \( F \subseteq M_{\tau_0} \) be a connected component of the fixed locus. Then, uniformly in \( x \in \pi^{-1}(F) \subseteq X_{\tau_0} \), in the choice of a HLCS centered at \( x \), and in \( (u, \tau) \in N_{\omega}(x) \times \mathbb{R} \) with \( \|u\|, |\tau| \leq E k^{1/9} \), the following asymptotic expansion holds as \( k \to +\infty \):

\[
U_{\tau_0,k} \left( x + \frac{u}{\sqrt{k}}, x + \frac{u}{\sqrt{k}} \right) \sim \left( \frac{k}{\pi} \right)^d \frac{\varphi_{\tau_0}(m)}{\nu(\tau_0, m)} e^{i\tau \sqrt{f(m)}}
\]

\[
: h_k^d e^{i\omega_m(v_f(m), u)} : + S_{\tau_0,m}(u, \tau v_f(m) + u) \left[ 1 + \sum_{j \geq 1} k^{-j/2} b_j(m, \tau, u) \right],
\]

where \( m = \pi(x) \), and \( b_j \) is \( C^\infty \) and a polynomial in \( (\tau, u) \) of joint degree \( \deg(b_j) \leq 3j \).

We can further specialize this at a fixed time \( \tau_0 \), that is with \( \tau = 0 \), and obtain the following, which is also a consequence of Theorem 1.4 of [P3]:

**Corollary 3.2.** Under the assumptions of Corollary 3.1 as \( k \to +\infty \) the following asymptotic expansion holds uniformly for \( m \in F_a \subseteq M_{\tau_0} \), \( x \in \pi^{-1}(x) \) and \( \|v\| \leq C k^{1/9} \) with \( v \in N_m \omega F_a \):

\[
U_{\tau_0,k} \left( x + \frac{v}{\sqrt{k}}, x + \frac{v}{\sqrt{k}} \right) \sim \left( \frac{k}{\pi} \right)^d \frac{\varphi_{\tau_0}(m)}{\nu(\tau_0, m)} e^{i\tau_0 \sqrt{f(m)}}
\]

\[
: h_k^d e^{-\Omega_A(v)} : + S_{\tau_0,m}(u, \tau v_f(m) + u) \left[ 1 + \sum_{j \geq 1} k^{-j/2} a_j(m, \tau_0, v) \right],
\]

where \( a_j \) is a polynomial in \( v \) of parity \( j \), and degree \( \leq 3j \).

The statement about the parity is part of Theorem 1.4 of [P3].
Remark 3.5. In view of Remarks 3.4 and 3.3, the previous Corollaries describe an exponential decay along transverse directions to the fixed locus.

Part II

Trace scaling asymptotics

We shall now apply the local diagonal scaling asymptotics near fixed loci from §3 to study global scaling asymptotics for the trace of $U_{\tau,k}$ near certain well-behaved periods; now the scaling will be solely with respect to the time variable, as it approaches a fixed period $\tau_0$ at a controlled pace. We shall first dwell however on the asymptotics as $k \to +\infty$ of the trace of $U_{\tau_0,k}$. Prior to that, we need to introduce some preliminaries on the natural volume forms and measures on certain submanifolds of $M$.

4 Trace asymptotics at a fixed time

4.1 Symplectic vs Riemannian volume forms

4.1.1 Vector spaces

Let $(V,J_V,\omega_V)$ be an Hermitian vector space of finite complex dimension $d \geq 1$ (thus $J_V$ is a linear complex structure on $V$, and there is a positive definite Hermitian product $h_V : V \times V \to \mathbb{C}$, for which $\omega_V = -\Im(h_V)$). Then $g_V = : \Re(h_V) : V \times V \to \mathbb{R}$ is an Euclidean product on $V$, viewed as a real vector space.

Let $W \subseteq V$ be any oriented $r$-dimensional real vector subspace and let $g_W : W \times W \to \mathbb{R}$ be the Euclidean product on $W$ induced by restriction of $g$; then there is an induced Euclidean volume form $E_W : \bigwedge^r W \to \mathbb{R}$, characterized by $E_W(w_1,\ldots,w_r) = 1$ on any oriented orthonormal basis of $(W,g_W)$. If $(w_j)$ is any oriented basis of $W$, then $E_W(w_1,\ldots,w_r) = \sqrt{\det(h(w_i,w_k))}$.

Suppose that $r = 2\ell$, and $W$ is a symplectic vector subspace of $(V,\omega_V)$, with restricted symplectic form $\omega_W : W \times W \to \mathbb{R}$. Then there is also a symplectic volume form $S_W = (1/\ell!) \omega_W^{\wedge \ell} : \bigwedge^r W \to \mathbb{R}$, which is characterized by $S_W(w_1,\ldots,w_r) = 1$ if $(w_j)$ is any Darboux basis of $(W,\omega_W)$. We shall implicitly consider any symplectic vector space as oriented by its symplectic volume form (equivalently, by the choice of the orientation class of any Darboux basis).
Thus on a symplectic vector subspace $W$ of $(V,J,V,\omega_V)$ we have two naturally induced volume forms $E_W$ and $S_W$.

In particular, any complex $\ell$-dimensional vector subspace $W \subseteq V$ is also Hermitian for the metric $h_W = g_W - i\omega_W$ given by restriction of $h$. It is therefore also a symplectic vector subspace; if $(e_1, \ldots, e_\ell)$ is an orthonormal basis of $(W,h_W)$ over $\mathbb{C}$, then $(e_1, J(e_1), \ldots, e_\ell, J(e_\ell))$ is a Darboux basis of $(W,\omega_W)$ over $\mathbb{R}$, and an oriented orthonormal basis of $(W,g_W)$. Thus $E_W = S_W$, since they both equal one on the latter basis.

For a general symplectic vector subspace $W \subseteq V$, \[ S_W = \zeta(W) \ E_W \] (59) for a unique $\zeta(W) > 0$. Clearly, $\zeta(W) = \det(h(w_j,w_k))^{-1/2}$ if $(w_j)$ is an arbitrary Darboux basis of $W$.

Similarly, if $W^* \subseteq V$ is the symplectic orthocomplement of $W$, then $S_{W^*} = \zeta(W^*) \ E_{W^*}$.

### 4.1.2 Manifolds

Globally, on a complex $d$-dimensional Kähler manifold $(R,J,R,\omega_R)$ the symplectic volume form $S_R = \omega_R^d/d!$ equals the Riemannian volume form $E_R$ of the oriented Riemannian manifold $(R,g_R)$, where $g_R(\cdot,\cdot) = \omega_R(\cdot,J_R(\cdot))$; in this case, we shall write $\mu_R = S_R = E_R$, and denote the corresponding measure by $dV_R$. Obviously, the same applies to any complex submanifold of $R$.

Consider a general symplectic submanifold $\iota: Q \hookrightarrow R$, of (real) dimension $2\ell$ and codimension $2c = 2(d-\ell)$. For $q \in Q$, let $T_qQ \subseteq T_qR$ be the tangent space of $Q$ at $q$, and $N^\omega_qQ \subseteq T_qR$ the symplectic orthocomplement of $T_qQ$. Thus we have a symplectic direct sum $T_qR = T_qQ \oplus N^\omega_qQ$.

Let $\omega_Qq$ and $\eta_Qq$ be the induced symplectic structures on $T_qQ$ and $N^\omega_qQ$, respectively. This defines symplectic structures $\omega_Q$ and $\eta_Q$ on the vector bundles $TQ$ and $N^\omega Q$ on $Q$; clearly, $\omega_Q = \iota^*(\omega_R)$. Correspondingly, on $TQ$ and $N^\omega Q$ there are fiberwise symplectic volume forms $S_Q = (1/\ell!)[\omega_Q^\wedge \ell]$ and $S_{N^\omega Q} = (1/c!)[\eta_Q^\wedge c]$, respectively.

Omitting symbols of pull-back for notational simplicity, along $Q$ we have
\[
\mu_R = \frac{1}{d!} \omega_R^d = \left(\frac{1}{\ell!} \omega_Q^\wedge \ell\right) \wedge \left(\frac{1}{c!} \eta_Q^\wedge c\right) = S_Q \wedge S_{N^\omega Q}
\]
on $\wedge^{2d} TR \cong [\wedge^{2\ell} TQ] \wedge [\wedge^{2c} N^\omega Q]$.

The oriented vector bundles $TQ$ and $N^\omega Q$ also have metric structures, induced by the Riemannian metric of $M$. Therefore, they have Euclidean
(Riemannian) volume forms $\mathcal{E}_Q$ and $\mathcal{E}_{N\cap Q}$, respectively, defined pointwise as above.

If $F$ is a complex submanifold, then $\mathcal{E}_Q = S_Q$ and $\mathcal{E}_{N\cap Q} = S_{N\cap Q}$. For a general symplectic submanifold, as a global version of (59) we now have

$$S_Q = \zeta_Q \mathcal{E}_Q \quad \text{and} \quad S_{N\cap Q} = \zeta_{N\cap Q} \cdot \mathcal{E}_{N\cap Q},$$

for unique $C^\infty$ functions $\zeta_Q, \zeta_{N\cap Q} : Q \to (0, +\infty)$. Therefore along the symplectic submanifold $Q \subseteq R$

$$\mu_M = S_Q \wedge S_R = \zeta_N \cdot S_Q \wedge \mathcal{E}_{N\cap Q} = (\zeta_Q \zeta_{N\cap Q}) \cdot \mathcal{E}_Q \wedge \mathcal{E}_{N\cap Q}. \quad (61)$$

Suppose $q \in Q$, and let $U \subseteq M$ be a sufficiently small open neighborhood of $q$, so that we can find local coordinate systems $\beta_m(u) = m + u$, centered at $m \in U$ and smoothly varying with $m$. Using Taylor expansion in $u$, in additive notation we obtain for $q' \in Q \cap U$:

$$\mu_M \left( q' + \frac{u}{\sqrt{k}} \right) \sim \zeta_{N\cap Q}(q') \left[ 1 + \sum_{j \geq 1} k^{-j/2} A_j(q', u) \right] \cdot S_{Q,q'} \wedge E_{N\cap Q}(q') \quad (62)$$

$$= \zeta_Q(q') \zeta_{N\cap Q}(q') \cdot \left[ 1 + \sum_{j \geq 1} k^{-j/2} B_j(q', u) \right] \cdot E_{Q,q'} \wedge E_{N\cap Q}. $$

Here $A_j$ and $B_j$ are $C^\infty$ in $q' \in U \cap Q$ and homogeneous polynomials of degree $j$ in $u$.

### 4.2 Trace asymptotic expansion at a fixed time

The fixed-time asymptotic expansion for the trace that we shall discuss presently is really a consequence of the local scaling asymptotics at fixed time in [P3], and in fact it was briefly touched upon in the introduction of that paper for the special case of isolated fixed points; this case was also treated in [Ch] by a different approach. We take up the issue here again in more detail and generality than in [P3], both to put things in perspective towards the trace scaling asymptotics to follow, and for its obvious intrinsic interest. In the holomorphic case, the global benchmark is of course the classical Atiyah-Singer Lefschetz fixed point formula; asymptotic approaches based on local scaling expansions, encompassing compositions with Toeplitz operators and isotypic decompositions under Lie group actions, were given in [MZ] and [P1].
Theorem 4.1. Let $\tau_0$ be a very clean period of $\phi^M$, and let $F_1, \ldots, F_{\ell_{\tau_0}}$ be the connected components of $M_{\tau_0} \subseteq M$. For $a = 1, \ldots, \ell_{\tau_0}$, let $2d_a$ be the real dimension of the symplectic submanifold $F_a$. Then

$$\text{trace}(U_{\tau_0,k}) = \sum_{a=1}^{\ell_{\tau_0}} F_a(\tau_0, k),$$

where each summand $F_a(\tau_0, k)$ is given by an asymptotic expansion of the form

$$F_a(\tau_0, k) \sim h_a^k \left( \frac{k}{\pi} \right)^{d_a} \sum_{j \geq 0} k^{-j} F_{aj}(\tau_0), \quad (63)$$

with leading coefficient

$$F_{0a}(\tau_0) = 2^d \int_{M_{\tau_0}} \frac{\theta_{\tau_0}(p)}{\nu(\tau_0, p)} \zeta_{N_a}(p) \det (\Omega_p^\text{nor})^{-1/2} \, dV_F^s(p),$$

where the latter factor in the integrand is as in Definition 3.4, \( \zeta_{N_a} =: \zeta_{N^a_F} : F_a \rightarrow (0, +\infty) \) is as in (61), and we have denoted by $dV_F^s = |S_F|$ the density (or its measure) associated to the symplectic volume form.

Remark 4.1. In fact, $F_a(\tau_0, k)$ is supported near $F_a$, in the sense that it vanishes (up to negligible contributions) if $R_{\tau_0}$ is smoothing in a neighborhood $U$ of $F_a$ .

Proof. The function $x \mapsto U_{\tau', k}(x, x)$ descends for any $\tau' \in \mathbb{R}$ to a well-defined $C^\infty$ function on $M$, call it $U_{\tau', k}$. Thus $U_{\tau', k}(m) = U_{\tau', k}(x, x)$ if $m = \pi(x)$. Furthermore,

$$\text{trace}(U_{\tau', k}) = \int_X U_{\tau', k}(x, x) \, dV_X(x) = \int_M U_{\tau', k}(m) \, dV_M(m). \quad (64)$$

Now let us set $\tau' = \tau_0$ in (64). By Theorem 2.1, up to a negligible contribution the integrand in (64) asymptotically localizes in a shrinking neighborhood of $M_{\tau_0}$. Therefore, for $\tau' = \tau_0$ we can rewrite (64) asymptotically as

$$\text{trace}(U_{\tau_0,k}) \sim \sum_{a=1}^{\ell_{\tau_0}} \int_{M_a} \beta_a(m) \, U_{\tau_0,k}(m) \, dV_M(m), \quad (65)$$
where $M'_a \subseteq M$ is a tubular neighborhood of $F_a$, and $\beta_a \in C^\infty_0(M'_a)$ is identically $= 1$ in a smaller tubular neighborhood $M''_a \subseteq M'_a$ of $F_a$. We aim to estimate asymptotically the $a$-th summand in (65).

For any $p_0 \in F_a$, we can find an open neighborhood $U \subseteq F_a$ of $p_0$, and a smoothly varying family of Heisenberg local coordinates centered at points in $U' = \pi^{-1}(U)$, which we denote $\Gamma : U' \times (-\pi, \pi) \times B_{2d}(0, \delta) \longrightarrow X$. More precisely, we require that for any $x \in U'$ the restriction

$$\gamma_x =: \Gamma(x, \cdot, \cdot) : (-\pi, \pi) \times B_{2d}(0, \delta) \longrightarrow X$$

be a HLCS centered at $x$, that we denote additively by $\gamma_x(\theta, v) = x + (\theta, v)$. In particular, $r_\theta(x + v) = x + (\theta, v)$. In addition, we shall assume without loss that

$$x + (\theta + \vartheta, v) = (x + (\theta, 0)) + (\vartheta, v).$$

This gives a meaning to the expression $p + v$ when $p \in U$ and $v \in \mathbb{R}^{2d}$ has suitably small norm (namely, $p + v =: \pi(x + v)$ if $p = \pi(x)$).

Let $N^*F_a$ be the symplectic normal bundle of $F_a$ (Definition 3.2), and let $N^*F_a|_U$ be its restriction to $U \subseteq F_a$. Let $\mathcal{N}_U \subseteq N^*F_a|_U$ be a suitably small open neighborhood of the zero section. Then $\varsigma_U : (p, u) \in \mathcal{N}_U \mapsto p + u \in M$ is a diffeomorphism onto its image $\mathcal{N}'_U =: \varsigma_U(\mathcal{N}_U) \subseteq M$, which is then a tubular neighborhood of $U$.

Set $c_a =: d - d_a$, so that $2c_a$ is the (real) codimension of $F_a$ in $M$. Perhaps after replacing $U$ by a smaller open neighborhood of $p_0$ in $F_a$, we may find an orthogonal local trivialization of $N^*F_a$ on $U$ (with respect to the Riemannian metric $g$); thus we may smoothly and unitarily identify $N^*_pF_a \cong \mathbb{R}^{2c_a}$ for $p \in U$. Accordingly, perhaps after restricting $\mathcal{N}_U$, we shall identify $\mathcal{N}_U \cong U \times B_{2c_a}(0, \delta)$ for some $\delta > 0$. Using this, we may think of $\varsigma_U$ as as diffeomorphism

$$\varsigma_U : U \times B_{2c_a}(0, \delta) \cong \mathcal{N}_U; \text{ with } (p, u) \mapsto m = m(p, u) =: p + u.$$
\( \text{a-th summand in (65) as follows:} \)
\[
\int_{M_a} \beta_a(m) \, U_{\tau_0,k}(m) \, dV_M(m)
\]
\[
= \sum_j \int_{\mathbb{R}^{2\alpha}} \beta_{aj}(m) \, \beta_a(m) \, U_{\tau_0,k}(m) \, dV_M(m)
\]
\[
= \sum_j \int_{U_{aj} \times B_{2\alpha}(0,\delta)} \beta_{aj}(p + u) \, \beta_a(p + u) \, U_{\tau_0,k}(p + u) \, dV^{(j)}_M(p + u);
\]
here in the \( j \)-th summand \( p + u = \zeta_{U_j}(p, u) \), and
\[
dV^{(j)}_M(p + u) =: \zeta^{(j)}_U(p, u).
\]

By Proposition 2.1, only a negligible contribution to the asymptotics is lost if the integrand in the \( j \)-th summand in (66) is multiplied by \( \gamma_k(u) =: \gamma_1 \left( k^{7/18} \| u \| \right), \gamma_1 \in C^\infty_0(\mathbb{R}) \) being a bump function \( \equiv 1 \) in a neighborhood of the origin. The factor \( \beta_a(p + u) \) may then be omitted.

If we now compose each \( \zeta_{U_j} \) with rescaling in \( u \) by \( k^{-1/2} \), (66) may be rewritten as follows:
\[
\int_{M_a} \beta_a(m) \, U_{\tau_0,k}(m) \, dV_M(m)
\]
\[
\sim k^{-c_a} \sum_j \int_{U_{aj} \times \mathbb{R}^{2\alpha}} \gamma_1 \left( k^{-1/9} \| u \| \right) \beta_{aj} \left( p + \frac{u}{\sqrt{k}} \right) \cdot U_{\tau_0,k} \left( p + \frac{u}{\sqrt{k}} \right) \, dV^{(j)}_M \left( p + \frac{u}{\sqrt{k}} \right).
\]

Given (62), we have for each \( j \)
\[
dV^{(j)}_M \left( p + \frac{u}{\sqrt{k}} \right) \sim \zeta^{N_a}(p) \cdot \left[ 1 + \sum_{h \geq 1} k^{-h/2} B_{h}^{(j)}(p, u) \right] \\
\cdot (S_{Fa})_p \land du,
\]
where we have written \( N^a \) for \( N^a_Fa \) and \( du \) for the standard volume density on \( \mathbb{R}^{2\alpha} \), and \( B_{h}^{(j)} : U_j \times \mathbb{R}^{2\alpha} \to \mathbb{R} \) is homogeneous of degree \( h \) in \( u \). Similarly,
\[
\beta_{aj} \left( p + \frac{u}{\sqrt{k}} \right) \sim \sum_{l \geq 0} k^{-l/2} \beta_{ajl} \left( p, u \right),
\]
where \( \beta_{ajl} \left( p, u \right) \) is homogeneous in \( u \) of degree \( l \), and \( \beta_{aj0} \left( p, u \right) = \beta_{aj} \left( p \right) \).
Then, multiplying the asymptotic expansions in (68), (69) and Corollary 3.2, we conclude that (67) may be rewritten
\[
\int_{M'} \beta_a(m) U_{\tau_0,k}(m) \, dV_M(m) \sim h_a^k k^{-c_a} \left( \frac{k}{\pi} \right)^d \int_{F_a} \Lambda_k(p) S_{F_a}(p),
\]
where \( \Lambda_k \) has the form
\[
\Lambda_k(p) = \int_{\mathbb{R}^{2c_a}} \gamma_1 \left( k^{-1/9} u \right) e^{-\Omega_{\tau_0,p}(u)} G_k(p, u) \, du;
\]
and the \( c_l \)'s are polynomials in \( u \), of degree \( \leq 3l \) and parity \( l \); the leading term is
\[
c_0(p, u) = \varrho_{\tau_0}(p).
\]

We now aim to estimate \( \Lambda_k \) asymptotically. Since in (71) integration in \( du \) is over a ball centered at the origin and of radius \( O \left( k^{1/9} \right) \), the asymptotic expansion (82) may be integrated term by term. Given this, in view of (51) we only lose a rapidly decreasing contribution if in (71) the cut-off is omitted and integration is taken over all of \( \mathbb{R}^{2c_a} \). Therefore, taking into account the previous assertion about the parity of the \( c_l \)'s,
\[
\Lambda_k(p) \sim \sum_{l \geq 0} k^{-l} \lambda_l(p),
\]
where
\[
\lambda_l(p) =: \frac{2^d}{\nu(\tau_0, p)} \zeta_{N^a}(p) \int_{\mathbb{R}^{2c_a}} c_{2l}(p, u) e^{-\Omega_{\tau_0,p}(u)} \, du.
\]
In particular, the leading order term is
\[
\lambda_0(p) =: \frac{2^d}{\nu(\tau_0, p)} \zeta_{N^a}(p) \varrho_{\tau_0}(p) \cdot \int_{\mathbb{R}^{2c_a}} e^{-\Omega_{\tau_0,p}(u)} \, du
\]
\[
= \frac{2^d}{\nu(\tau_0, p)} \zeta_{N^a}(p) \varrho_{\tau_0}(p) \pi^{c_a} \det \left( \Sigma_p^{\text{nor}} \right)^{-1/2}.
\]
Inserting this in (70) completes the proof of Theorem 4.1.
\[ \square \]
5 Trace asymptotics at rescaled times

5.1 Morse-Bott periods

We next aim to discuss global scaling asymptotics for traces, the scaling being now with respect to the time variable. The result that we shall discuss presently applies to a more restrictive class of periods than very clean ones, which we now define.

Definition 5.1. We shall say that \( \tau_0 \) is a Morse-Bott (very) clean period of \( \phi^M \) if

1. \( \tau_0 \) is a (very) clean period of \( \phi^M \) (Definition 3.1);
2. the restriction of \( f \) to \( M_{\tau_0} \), \( f_{\tau_0} = f|_{M_{\tau_0}} : M_{\tau_0} \to M_{\tau_0} \), is a Morse-Bott function.

Albeit quite restrictive, these conditions are satisfied in many natural situations.

Example 5.1. If \( f : M \to M \) is a Morse-Bott function, then \( 0 \in \mathbb{R} \) is a Morse-Bott very clean period of \( f \).

Example 5.2. Let \((N, \eta)\) be any compact symplectic manifold. If \( \delta \geq 1 \) is an integer, suppose that \( \mu : T^\delta \times M \to M \) is an Hamiltonian action of the compact \( \delta \)-dimensional torus \( T^\delta \) on \((N, \eta)\), with moment map \( \Phi : N \to t^* \cong \mathbb{R}^\delta \). If \( \xi \in t \) (the Lie algebra of \( T \)), set \( f = \Phi_\xi = \langle \Phi, \xi \rangle : N \to \mathbb{R} \); then the Hamiltonian flow of \( f \) is simply the restriction of \( \mu \) to the 1-parameter subgroup generated by \( \xi \). Then \( f \) is, for any choice of \( \xi \), a Morse-Bott function and its critical submanifolds are all symplectic, with even Morse indexes and coindexes. Furthermore, for any \( g \in T^\delta \) the fixed locus \( N_g \subseteq N \) of \( \mu_g \) is a \( T \)-invariant compact symplectic submanifold, and so the restriction of \( f \) to it is a Morse-Bott function (\([\text{Au}],[\text{MS}],[\text{N}],\S 3.5\)).

Definition 5.2. If \( \tau_0 \in \mathbb{R} \) is a Morse-Bott very clean period of \( \phi^M \), we shall adopt the following notation. As before, \((F_a)\) will be the connected components of the fixed locus \( M_{\tau_0} \), \( a = 1, \ldots, \ell_{\tau_0} \). For each \( a \),

- \( d_a =: \frac{1}{2} \dim(F_a) \in \mathbb{Z} \);
- \( f_a : F_a \to \mathbb{R} \) denotes the restriction of \( f \) to \( F_a \).
- \( C_{ab} \subseteq F_a, (1 \leq b \leq s_a) \), is the connected critical submanifolds of \( f_a \) in \( F_a \).
Furthermore, for each pair \((a, b)\) with \(a = 1, \ldots, \ell_{\tau_0}\) and \(b = 1, \ldots, s_a\):

- \(d_{ab} := \frac{1}{2} \dim(C_{ab}) \in \frac{1}{2} \mathbb{Z}\).
- \(f_{ab} \in \mathbb{R}\) is the constant value of \(f\) on \(C_{ab}\).
- \(H_q(f)\) is the (non-degenerate) transverse Hessian of \(f\) at \(q \in C_{ab}\).
- \(q \in C_{ab} \mapsto \det(H_q(f))\) denotes the determinant of \(H_q(f)\) with respect to any orthonormal basis of \(N^q_q(C_{ab}/F_a)\).
- \(\sigma_{ab} \in \mathbb{Z}\) is the constant signature of \(H_q(f)\) along \(C_{ab}\).

**Theorem 5.1.** Suppose that \(\tau_0 \in \mathbb{R}\) is a Morse-Bott very clean period of \(\phi^M\), and adopt the notation in Definition 5.2. Fix \(C > 0\). Then, uniformly for \(C^k - \frac{1}{9} < |\tau| < C^k + \frac{1}{9}\), we have

\[
\text{trace}(U_{\tau_0 + \frac{\tau}{\sqrt{k}}} = \sum_{a=1}^{\ell_{\tau_0}} \sum_{b=1}^{s_a} L_{ab}(k, \tau),
\]

where for \(k \to +\infty\) each summand \(L_{ab}(k, \tau)\) is given by an asymptotic expansion of the form

\[
L_{ab}(k, \tau) \sim h^k a \cdot \frac{k^{1/2(d_a + d_{ab})}}{\pi^{d_a}} \cdot \left(\frac{2\pi}{\tau}\right)^{d_a - d_{ab}} e^{i\sqrt{\tau} f_{ab} + \frac{1}{2}\pi \sigma_{ab}} \cdot \sum_{l, r \geq 0} k^{-l/2} (\tau \sqrt{k})^{-r} A_{abl}(\tau),
\]

with \(A_{abl}(\tau)\) a polynomial in \(\tau\) of degree \(\leq 3(l + r)\). The leading order coefficient is given by

\[
A_{abl0} = \int_{C_{ab}} \varphi(\tau_0(q), \nu(\tau_0, q) \zeta_{F_a}(q) \zeta_{N_a}(q) \cdot \det(\Omega_{q}^{\text{nor}})^{-1/2} \det H_q(f)^{-1/2} \mathcal{E}_{C_{ab}}(q).
\]

Furthermore, \(L_{ab}\) is localized in the neighborhood of \(C_{ab}\), in the sense that it is negligible as soon as \(R_{\tau'}\) is smoothing near \(C_{ab}\), for \(\tau'\) near \(\tau_0\).

**Remark 5.1.** To see why (78) is indeed an asymptotic expansion for \(k \to +\infty\), let us write

\[
\sum_{l, \tau \geq 0} k^{-l/2} (\tau \sqrt{k})^{-r} A_{abl}(\tau) = \sum_{s \geq 0} \left[ \sum_{l + r = s} k^{-l/2} (\tau \sqrt{k})^{-r} A_{abl}(\tau) \right].
\]
and remark that given that \( Ck^{-\frac{7}{9}} < |\tau| \) by (77) if \( l + r = s \) then

\[
k^{l/2} \left( |\tau| \sqrt{k} \right)^r \geq C^r k^{\frac{1}{2}(l+r)-\frac{1}{9}r} \geq C^r k^{\frac{7}{18}s}.
\]

Therefore, since each \( A_{abr}(\tau) \) has degree \( \leq 3s \) in \( \tau \), and we are also assuming \( |\tau| < Ck^{\frac{7}{9}} \) by (77), we have

\[
\left| \sum_{l+r=s} k^{-l/2} \left( \tau \sqrt{k} \right)^{-r} A_{abr}(\tau) \right| \leq C_s k^{-\frac{7}{18}s} k^{\frac{1}{2}3s} = C_s k^{-\frac{1}{9}s}.
\]

**Proof.** Let us backtrack to the proof of Theorem 4.1, and set \( \tau' = \tau_k =: \tau_0 + \tau/\sqrt{k} \) in (64). Then (65), (66), and (67) continue to hold, with \( \tau_0 \) replaced by \( \tau_k \).

However, instead of Corollary 3.2 we now need to use Corollary 3.1. Since in our construction \( u \in N_p F_a \) while \( v_f(p) \in T_p F_a \) because \( F_a \) is \( \phi^M \)-invariant, we have \( \omega_p(v_f(p), u) = 0 \). Therefore, in place of (70) we have

\[
\int_{M_a'} \beta_a(m) U_{\tau_0,k}(m) dV_M(m) \sim k^a k^{-c_a} \left( \frac{k}{\pi} \right)^d \int_{F_a} \Lambda_k(p, \tau) S_{F_a}(p), \tag{80}
\]

where now

\[
\Lambda_k(p, \tau) =: e^{i\tau \sqrt{f(p)}} \int_{\mathbb{R}^{2d}} \gamma_1 (k^{-1/9} u) e^{\mathcal{S}_{\tau_0,p}(u, v_f(p) + u)} G_k(p, u, \tau) du; \tag{81}
\]

here

\[
G_k(p, u, \tau) \sim 2^d \nu(\tau_0, p) \zeta_{N^a}(p) \sum_{l \geq 0} k^{-l/2} c_l(p, u, \tau), \tag{82}
\]

and the \( c_l \)'s are polynomials in \((u, \tau)\), of degree \( \leq 3l \); the leading term is

\[
c_0(p, u) = \theta_{\tau_0}(p). \tag{83}
\]

We have on the exponent the estimate (52) in Remark 3.4.

Therefore, in place of (74) we now have

\[
\Lambda_k(p, \tau) \sim \sum_{l \geq 0} k^{-l/2} \lambda_l(p, \tau), \tag{84}
\]

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where
\[
\lambda_l(p, \tau) =: \frac{2^d}{\nu(\tau_0, p)} \xi_{N^a}(p) e^{i\tau \sqrt{k} f(p)} 
\cdot \int_{\mathbb{R}^{2c}} e^{S_{0,p}(u, \tau v_f(p)+u)} c_l(p, u, \tau) \, du
\]
\[
= e^{i\tau \sqrt{k} f(p)} \frac{2^d}{\nu(\tau_0, p)} \xi_{N^a}(p) K_l(p, \tau),
\]
with
\[
K_l(p, \tau) =: \int_{\mathbb{R}^{2c}} e^{S_{0,p}(u, \tau v_f(p)+u)} c_l(p, u, \tau) \, du.
\] (85)

Using this in (80), we obtain
\[
\int_{M^{'\prime}_a} \beta_a(m) \, dV_M(m) \sim h^k_a k^{-c_a} \left( \frac{k}{\pi} \right)^d \sum_{l \geq 0} k^{-l/2} H_{al}(k),
\]
with
\[
H_{al}(k, \tau) =: \int_{F_a} e^{i\tau \sqrt{k} f(p)} \frac{2^d}{\nu(\tau_0, p)} \xi_{N^a}(p) K_l(p, \tau) S_{F_a}(p).
\] (88)

In the range (77), \( \tau \sqrt{k} \to +\infty \) for \( k \to +\infty \), and therefore \( H_{al}(k) \) may be interpreted as an oscillatory integral in \( \tau \sqrt{k} \) with real phase \( f \). Since we are assuming \( |\tau| < C k^{1/9} \) we also have \( |\tau|^2 < C (\tau \sqrt{k})^\delta \) with say \( \delta = 2/5 < 1/2 \). Call \( G_l \) the amplitude in (88); then we see from (85) that in any local coordinate system on \( F_a \)
\[
|\partial^n G_l| = O \left( |\tau|^{3l+2|a|} \right) = O \left( |\tau|^{3l+\frac{2}{5} |a|} \right).
\] (89)

Therefore, we may apply the first part the stationary phase Lemma [Dui], and conclude that the integral (88) asymptotically localizes in the neighborhood of the critical manifolds \( C_{ab} \). More precisely, for every \( a = 1, \ldots, \ell_{\tau_0} \) and \( b = 1, \ldots, s_a \) let \( F_{ab}' \subseteq F_a \) be an open neighborhood of \( C_{ab} \) in \( F_a \) and let \( \gamma_{ab} : F_a \to [0, +\infty) \) be a bump function, compactly supported in \( F_{ab}' \) and \( \equiv 1 \) on some smaller neighborhood \( F_{ab}'' \subseteq F_{ab}' \). Then as \( k \to +\infty \) we have
\[
H_{al}(k, \tau) = \sum_{b=1}^{s_a} H_{abl}(k, \tau),
\] (90)
where
\[
H_{abl}(k, \tau) =: \int_{F_{ab}''} \gamma_{ab}(p) e^{i\tau \sqrt{k} f(p)} \frac{2^d}{\nu(\tau_0, p)} \xi_{N^a}(p) K_l(p, \tau) S_{F_a}(p).
\] (91)
Arguing as in the proof of Theorem 4.1, we can furthermore find a $C^\infty$ finite partition of unity $\delta_{abj} : F'_{ab} \to [0, +\infty)$ subordinate to an open cover $F'_{abj}$, such that the following holds. For each $j$, let $C_{abj} = C_{ab} \cap F_{abj}$; thus $(C_{abj})$ is an open cover of $C_{ab}$. Let $B_{2(d_{ab} - d_{abj})}(0, \delta) \subseteq \mathbb{R}^2(d_{ab} - d_{abj})$ be the open ball centered at the origin and of radius $\delta$. Then for each $j$ and some $\delta > 0$ there is a diffeomorphism $\Phi_{abj} : C_{abj} \times B_{2(d_{ab} - d_{abj})}(0, \delta) \to F'_{abj}$ of the form $(q, n) \mapsto p(q, n) = q + n =: \varphi^{(j)}_q(n)$, where $\varphi^{(j)}_q$ is a local coordinate system on $F_a$ centered at $q$; furthermore, we shall require that for every $q \in C_{abj}$ the differential of $d_q\zeta_q$ induces a unitary isomorphism $\mathbb{R}^{2d} \cong T_q F_a$, under which $\{0\} \times \mathbb{R}^{2(d_{ab} - d_{abj})} \cong N^d_q(C_{ab} / F_a)$; here by $N^d_q(C_{ab} / F_a)$ we denote the Riemannian normal space to $C_{ab}$ at $q$, as a submanifold of $F_a$.

Let us set $B =: B_{2(d_{ab} - d_{abj})}(0, \delta)$. Recalling that $S_{F_a} = \zeta_{F_a} \cdot \mathcal{E}_{F_a}$, we can then rewrite $H_{abl}(k, \tau)$ in (91) as follows:

$$H_{abl}(k, \tau) = \sum_j \int_{C_{abj}} \int_B e^{i\tau \sqrt{k f(q+n)}} F_{abj}(q + n) Z_a(q + n, \tau) \mathcal{E}_{F_a}(q + n),$$

where

$$F_{abj} =: \gamma_{ab} \cdot \delta_{abj}, \quad Z_{al} =: \zeta_{F_a} \cdot \zeta_{F_a} \frac{2^d}{\nu(\cdot, r_0)} \cdot K_l,$$

and in the $j$-th summand we use $\Phi_{abj}$ to give a meaning to $q + n$. Also,

$$\mathcal{E}_{F_a}(q + n) =: \Phi^*_{abj}(\mathcal{E}_{F_a})(q + n) = V_j(q, n) |dn| \mathcal{E}_{C_{ab}}(q),$$

and by construction $V_j(q, n) = 1 + O(n)$. Therefore, we can rewrite (92) as

$$H_{abl}(k, \tau) = \sum_j \int_{C_{abj}} \left[ \int_B e^{i\tau \sqrt{k f(q+n)}} \psi_{abj}(q + n, \tau) |dn| \right] \mathcal{E}_{C_{ab}}(q),$$

where

$$\psi_{abj} =: F_{abj} Z_{al} V_j$$

By the Morse-Bott assumption, if $q \in C_{abj}$ the function $n \in B \mapsto f(q + n)$ has a non-degenerate critical point at the origin. Given this and (89), we may apply the second part of the stationary phase Lemma and obtain, with the
notation in Definition 5.2, an asymptotic expansion of the form

\[
\int_B e^{ir \sqrt{k} f(q+n)} \Psi_{abl}(q + n) \, dn |
\]

\[\sim e^{ir \sqrt{k} f_{ab}} \left( \frac{2\pi}{\tau \sqrt{k}} \right)^{da-dab} \left| \det (H_q(f)) \right|^{-1/2} e^{i \frac{\pi}{4} \sigma_{ab}} \cdot \sum_{r=0}^{+\infty} \left( \tau \sqrt{k} \right)^{-r} \Psi_{abl}(q, \tau),
\]

where \( \Psi_{abl}= R_r(\Psi_{abl})/r! \) and \( R_r \) is a differential operator of degree \( 2r \) in the \( n \)-variables, defined in terms of the transverse Hessian and the third order remainder of \( f_a \) at \( q \). In particular, \( R_0 \) is the identity.

It is then clear from this and (86) that \( \Psi_{abl} \) has the form

\[
\Psi_{abl}(p, \tau) = \int_{\mathbb{R}^{2n}} e^{S_{\tau, a}^p(u, \tau f(p)+u)} c_{abl}(p, u, \tau) \, du \quad (p \in F_{abl}), \tag{97}
\]

with \( c_{abl}(p, u, \tau) \) a polynomial in \( \tau \) of degree \( \leq 3l + 2r \leq 3(l+r) \).

**Lemma 5.1.** If \( q \in C_{ab} \), then \( d_q f = 0 \); equivalently, \( v_f(q) = 0 \).

**Proof.** By construction, \( d_q f_a = 0 \). By assumption, \( F_a \) is a connected symplectic submanifold of \( M \); let \( v_f^{(a)} \in \mathfrak{X}(F_a) \) be the Hamiltonian vector field of \( f_a \). Thus \( v_f^{(a)}(q) = 0 \).

On the other hand, \( M_{\tau_0} \) is invariant under the flow \( \phi^M \) generated by \( v_f \), and therefore so is every connected component \( F_a \) of \( M_{\tau_0} \). Hence \( v_f \) is tangent to \( F_a \), so \( v_f = v_f^{(a)} \) along \( F_a \). In particular, \( v_f = 0 \) on \( C_{ab} \). \( \square \)

By (97) and Lemma 5.1, for \( q \in C_{ab} \) we have

\[
\Psi_{abl}(q, \tau) = \int_{\mathbb{R}^{2n}} e^{-S_{\tau, a}^p(u)} c_{abl}(q, u, \tau) \, du, \tag{98}
\]

which is a polynomial in \( \tau \) of degree \( \leq 3(l+r) \).

Going back to (96), we obtain

\[
\begin{align*}
H_{abl}(k, \tau) &\sim e^{ir \sqrt{k} f_{ab}} \left( \frac{2\pi}{\tau \sqrt{k}} \right)^{da-dab} e^{i \frac{\pi}{4} \sigma_{ab}} \\
&\quad \cdot \sum_{r=0}^{+\infty} \left( \tau \sqrt{k} \right)^{-r} \int_{C_{ab}} \left| \det (H_q(f)) \right|^{-1/2} \Psi_{abl}(q, \tau) \, E_{C_{ab}}(q),
\end{align*}
\]

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where $\Psi_{ablr} =: \sum_j \Psi_{abljr}$. Using this in (87) and (90), we obtain

$$\int_{M'_a} \beta_a(m) U_{\tau, k}(m) \, d\nu_M(m) \quad (100)$$

$$\sim h^k \sum_{b=1}^{s_a} e^{\frac{i}{\hbar} \sqrt{K_{ab}} + i \frac{\pi}{4} \sigma_{ab}} \cdot \frac{k^{1/2}(d_a+d_b)}{\pi d_a} \left( \frac{2\pi}{\hbar} \right)^{d_a-d_{ab}} \cdot \sum_{l,r \geq 0} k^{-1/2} \left( \frac{\tau}{\sqrt{k}} \right)^{-r} \left[ \frac{1}{\pi c_a} \int_{C_{ab}} \det (H_q(f)) \right]^{-1/2} \sum_{l,r} \frac{1}{\pi c_a} \int_{C_{ab}} \det (H_q(f)) \cdot \Psi_{ablr}(q, \tau) \, C_{ab}(q).$$

We see from (100) that (78) holds with

$$A_{ablr}(\tau) =: \pi^{-c_a} \int_{C_{ab}} \left[ \det (H_q(f)) \right]^{-1/2} \Psi_{ablr}(q, \tau) \, C_{ab}(q). \quad (101)$$

Regarding the leading order term, since $c_0(p, u, \tau) = \vartheta_0(p)$, from (86) we get

$$K_0(q, \tau) =: \vartheta_0(q) \int_{\mathbb{R}^{2c_a}} e^{\xi_{\tau_0} \cdot (u, u)} \, du = \vartheta_0(q) \pi^{c_a} \det (\Sigma_{q^{nor}})^{-1/2}. \quad (102)$$

Pairing this with (95) and (101), we obtain

$$A_{ab00}(\tau) = \int_{C_{ab}} \zeta_{N^a}(q) \zeta_{F_a}(q) \frac{2^d}{\nu(q, \tau_0)} \vartheta_0(q) \left| \det (H_q(f)) \right|^{-1/2} \det (\Sigma_q^{nor})^{-1/2} \, C_{ab}(q). \quad (103)$$

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