The 1/3-2/3 Conjecture for Ordered Sets whose Cover Graph is a Forest

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Abstract
A balanced pair in an ordered set \( P = (V, \leq) \) is a pair \((x, y)\) of elements of \( V \) such that the proportion of linear extensions of \( P \) that put \( x \) before \( y \) is in the real interval \([1/3, 2/3]\). We define the notion of a good pair and claim any ordered set that has a good pair will satisfy the conjecture and furthermore every ordered set which is not totally ordered and has a forest as its cover graph has a good pair.

Keywords (Partially) ordered set · Linear extension · Balanced pair · Cover graph · Tree · 1/3-2/3 conjecture

1 Introduction

Throughout, \( P = (V, \leq) \) denotes a finite ordered set, that is, a finite set \( V \) and a binary relation \( \leq \) on \( V \) which is reflexive, antisymmetric and transitive. A linear extension of \( P = (V, \leq) \) is a total ordering \( \preceq \) of \( V \) which extends \( \leq \), i.e. such that for every \( x, y \in V \), \( x \preceq y \) whenever \( x \leq y \).

For a pair \((x, y)\) of elements of \( V \) we denote by \( P(x \prec y) \) the proportion of linear extensions of \( P \) that put \( x \) before \( y \). Call a pair \((x, y)\) of elements of \( V \) a balanced pair in \( P = (V, \leq) \) if \( 1/3 \leq P(x \prec y) \leq 2/3 \). The 1/3-2/3 Conjecture states that every finite ordered set which is not totally ordered has a balanced pair. If true, the example (a) depicted in Fig. 1 would show that the result is best possible. The 1/3-2/3 Conjecture first appeared in a paper of Kislitsyn [7]. It was also formulated independently by Fredman in about 1975 and again by Linial [8].

The 1/3-2/3 Conjecture is known to be true for ordered sets with a nontrivial automorphism [6], for ordered sets of width two [8], for semiorders [2], for bipartite ordered sets [12], for 5-thin posets [4], and for 6-thin posets [10]. See also [9] for recent results. For a survey on the subject the reader is directed to [3].
Recently, the author proved that the 1/3-2/3 Conjecture is true for ordered sets having no $N$ in their Hasse diagram [13]. Using similar ideas we prove that the 1/3-2/3 Conjecture is true for ordered sets whose cover graph is a forest.

Let $P = (V, \leq)$ be an ordered set. For $x, y \in V$ we say that $y$ is an upper cover of $x$ or that $x$ is a lower cover of $y$ if $x < y$ and there is no element $z \in V$ such that $x < z < y$. Also, we say that $x$ and $y$ are comparable if $x \leq y$ or $y \leq x$ and we set $x \sim y$; otherwise we say that $x$ and $y$ are incomparable and we set $x \nparallel y$. We denote by $inc(P)$ the set of incomparable pairs of $P$, that is, $inc(P) := \{(x, y) : x \nparallel y\}$. A chain is a totally ordered set. For an element $u \in V$, set $D(u) := \{v \in V : v < u\}$ and $U(u) := \{v \in V : u < v\}$.

The dual of $P$, denoted by $P^*$, is the order defined on $V$ as follows: $x \leq y$ in $P^*$ if and only if $y \leq x$ in $P$.

**Definition 1** Let $P$ be an ordered set. A pair $(a, b)$ of elements of $V$ is **good** if the following two conditions hold simultaneously in $P$ or in its dual.

1. $D(a) \subseteq D(b)$ and $U(b) \setminus U(a)$ is a chain (possibly empty); and
2. $P(a < b) \leq \frac{1}{2}$.

We notice at once that if $(a, b)$ is a good pair, then $a$ and $b$ are necessarily incomparable.

The relation between good pairs and balanced pairs is stated in the following theorem.

**Theorem 2** A finite ordered set that has a good pair has a balanced pair.

We prove Theorem 2 in Section 2.

A good pair is not necessarily a balanced pair (for an example consider the pair $(y, t)$ in example (c) Fig. 1). The following theorem gives instances of good pairs that are balanced pairs. Before stating our next result we first need a definition. Let $P = (V, \leq)$ be an ordered set. A subset $A$ of $V$ is called **autonomous** (or an interval or a module or a clan) in $P$ if for all $v \not\in A$ and for all $a, a' \in A$

\[ v < a \Rightarrow v < a' \] and \[ a < v \Rightarrow a' < v \]. \tag{1}
Theorem 3 Let $P = (V, \leq)$ be an ordered set and let $(x, y) \in \text{inc}(P)$. Suppose that one of the following propositions holds for $P$ or for its dual.

(i) There exists $z \in V$ such that $x < z$, $x \sim y \sim z$ and $\{x, y\}$ is autonomous in $P \setminus \{z\}$ (see example (a) Fig. 1).

(ii) There are $z, t \in V$ such that $x < z$, $y < t$, $y \sim z$, $x \not\sim t$ and $\{x, y\}$ is autonomous for $P \setminus \{z, t\}$ (see example (b) Fig. 1).

(iii) There are $z, t \in V$ such that $t < x < z$, $y$ is incomparable to both $t$ and $z$, and $\{x, y\}$ is autonomous for $P \setminus \{z, t\}$ (see example (c) Fig. 1).

Then $(x, y)$ is balanced in $P$.

We prove Theorem 3 in Section 3.

A semiorder is an order which does not contain the orders depicted in Figure 1 (b) and 1 (c). Brightwell [2] proved that every semiorder has a pair $(x, y)$ satisfying condition (i) of Theorem 3 and that either the pair $(x, y)$ is balanced, or $P(x \prec z \prec y) > \frac{1}{3}$. Theorem 3 shows that the former always occurs. As a result we obtain this.

Corollary 4 A balanced pair in a semiorder can be found in polynomial time.

The next definition describes a particular instance of a good pair.

Definition 5 Let $P$ be an ordered set. A pair $(a, b)$ of elements of $V$ is very good if the following two conditions hold simultaneously in $P$ or in its dual.

(i) $D(a) = D(b)$; and

(ii) $U(a) \setminus U(b)$ and $U(b) \setminus U(a)$ are chains (possibly empty).

For instance, the pairs $(x, y)$ and $(z, y)$ in example (a) Fig. 1 are very good. So are the pairs $(x, y)$ and $(z, t)$ in example (b) Fig. 1. Also, the pairs $(t, y)$ and $(y, z)$ in example (c) Fig. 1 are very good. Observe that every ordered set of width two has a very good pair. We have already mentioned that a semiorder which is not totally ordered has a very good pair. In [13], the author proved that every $N$-free ordered set which is not totally ordered has a very good pair. We now present another instance of a class of ordered sets that have a very good pair.

The cover graph of an ordered set $P = (V, \leq)$ is the graph $\text{Cov}(P) = (V, E)$ such that $\{x, y\} \in E$ if and only if $x$ covers $y$ in $P$.

Theorem 6 Let $P$ be an ordered set not totally ordered whose cover graph is a forest. Then $P$ has a very good pair, and hence has a balanced pair.

Section 4 is devoted to the proof of Theorem 6.

We mention that an algorithm requiring $O(n^2)$ arithmetic operations for computing the number of linear extensions of an ordered set whose cover graph is a tree was given in [1].

2 Proof of Theorem 2

We recall that an incomparable pair $(x, y)$ of elements is critical if $U(y) \subseteq U(x)$ and $D(x) \subseteq D(y)$. The set of critical pairs of $P$ is denoted by $\text{crit}(P)$. 
Lemma 7 Suppose \((x, y)\) is a critical pair in \(P\) and consider any linear extension of \(P\) in which \(y < x\). Then the linear order obtained by swapping the positions of \(y\) and \(x\) is also a linear extension of \(P\). Moreover, \(\mathbb{P}(x < y) \geq \frac{1}{2}\).

Proof Let \(L\) be a linear extension that puts \(y\) before \(x\) and let \(z\) be such that \(y < z < x\) in \(L\). Then \(z\) is incomparable with both \(x\) and \(y\) since \((x, y)\) is a critical pair in \(P\). Therefore, the linear order \(L'\) obtained by swapping \(x\) and \(y\), that is \(L'\) puts \(x\) before \(y\), is a linear extension of \(P\). Then \(\mathbb{P}(x < y) \geq \frac{1}{2}\).

We now prove Theorem 2.

Proof We prove the theorem by contradiction. Let \(P = (V, \leq)\) be an ordered set having a good pair \((a, b)\). We assume that \(P\) has no balanced pair and we argue to a contradiction.

Then \(U(b) \setminus U(a) \neq \emptyset\) because otherwise \((a, b)\) is a critical pair and hence \(\mathbb{P}(a < b) \geq \frac{1}{2}\) (Lemma 7). Since \((a, b)\) is a good pair \(\mathbb{P}(a < b) \leq \frac{1}{2}\) and hence \(\mathbb{P}(a < b) = \frac{1}{2}\) and therefore \((a, b)\) is balanced which is impossible by assumption.

Say \([U(b) \setminus U(a)] \cup \{b\}\) is the chain \(b = b_1 < \cdots < b_n\). Then

\[
\mathbb{P}(a < b_1) < \frac{1}{3}.
\]

Define now the following quantities

\[
q_1 = \mathbb{P}(a < b_1),
\]

\[
q_j = \mathbb{P}(b_{j-1} < a < b_j)(2 \leq j \leq n),
\]

\[
q_{n+1} = \mathbb{P}(b_n < a).
\]

Lemma.[13] The real numbers \(q_j (1 \leq j \leq n + 1)\) satisfy:

(i) \(0 \leq q_{n+1} \leq \cdots \leq q_1 < \frac{1}{3}\),

(ii) \(\sum_{j=1}^{n+1} q_j = 1\).

Proof Since \(q_1, \cdots, q_{n+1}\) is a probability distribution, all we have to show is that \(q_{n+1} \leq \cdots \leq q_1\). To show this we exhibit a one-to-one mapping from the event whose probability is \(q_{j+1}\) into the event with probability \(q_j\) \((1 \leq j \leq n)\). Notice that in a linear extension for which \(b_j < a < b_{j+1}\) every element \(z\) between \(b_j\) and \(a\) is incomparable to both \(b_j\) and \(a\). Indeed, such an element \(z\) cannot be comparable to \(b_j\) because otherwise \(b_j < z\) in \(P\) but the only element above \(b_j\) is \(b_{j+1}\) which is above \(a\) in the linear extension. Now \(z\) cannot be comparable to \(a\) as well because otherwise \(z < a\) in \(P\) and hence \(z < b = b_1 < b_j\) (by assumption we have that \(D(a) \subseteq D(b)\)). The mapping from those linear extensions in which \(b_j < a < b_{j+1}\) to those in which \(b_{j-1} < a < b_j\) is obtained by swapping the positions of \(a\) and \(b_j\). This mapping clearly is well-defined and one-to-one. This completes the proof of the lemma.

Theorem 2 can be proved now: let \(r\) be defined by

\[
\sum_{j=1}^{r-1} q_j < \frac{1}{2} < \sum_{j=1}^{r} q_j
\]
Since $\sum_{j=1}^{r-1}q_j = \mathbb{P}(a < b_{r-1}) \leq \frac{1}{3}$, it follows that $\sum_{j=1}^{r-1}q_j < \frac{1}{3}$. Similarly, $\sum_{j=1}^{r}q_j = \mathbb{P}(a < b_r)$ must be $> \frac{2}{3}$. Therefore, $q_r > \frac{1}{3}$, but this contradicts $\frac{1}{3} > q_1 \geq q_r$. \hfill \Box

3 Proof of Theorem 3

Let $P = (V, \leq)$ be an ordered set. Denote by $Ext(P)$ the set of all extensions of $P$ (or refinements of the order defined on $P$), that is, all orders $\leq$ on $V$ in which $x \leq y$ whenever $x \leq y$ in $P$. Then $Ext(P)$ is itself ordered: for $Q, R \in Ext(P)$, $Q \subset R$ if $R$ itself is an extension of $Q$. For every pair $(a, b) \in V \times V$, the transitive closure of $P \cup \{(a, b)\}$, denoted by $P \cup (a, b)$, is $P \cup \{(x, y) : x \leq a \text{ and } b \leq y \text{ in } P\}$. As it is well-known, if $(b, a) \notin P$ then this is an order. It is shown in [5] that if $Q$ and $R$ are elements of $Ext(P)$ then $R$ covers $Q$ in $Ext(P)$ if and only if $R$ is obtained from $Q$ by adding the comparability $a < b$ corresponding to a critical pair $(a, b)$ of $Q$. In this case $R = Q \cup \{(a, b)\} = Q \cup \{(a, b)\}$. It turns out that the maximal elements of $Ext(P)$ are the linear extensions of $P$ [11].

In order to prove Theorem 3 we will need the following general result.

**Theorem 8** Let $P$ be an ordered set and let $x, y, z$ be three distinct elements such that $x < z$ and $y$ is incomparable to both $x$ and $z$. Suppose that $(y, z) \in crit(P)$ and let $Q = P \cup \{(y, z)\}$. Then:

$$
\mathbb{P}_Q(x < y) < \mathbb{P}_P(x < y) \leq \frac{2\mathbb{P}_Q(x < y)}{1 + \mathbb{P}_Q(x < y)}. \quad (2)
$$

We should mention here that $\frac{2\mathbb{P}_Q(x < y)}{1 + \mathbb{P}_Q(x < y)} \leq 1$ for every $x, y$ and that $\frac{2\mathbb{P}_Q(x < y)}{1 + \mathbb{P}_Q(x < y)} \leq \frac{2}{3}$ if and only if $\mathbb{P}_Q(x < y) \leq \frac{1}{2}$. The second inequality of (2) above is tight as demonstrated by the example (a) depicted in Fig. 1. Moreover, if $(y, z) \notin crit(P)$, then there exist $y' \leq y$ and $z \leq z'$ such that $(y', z') \in crit(P)$. Obviously, $y'$ is incomparable to $x$ and $z'$.

**Proof** (Of Theorem 8) Denote by $\mathcal{L}(P)$ the set of linear extensions of $P$ and let $a_1 = |\{L \in \mathcal{L}(P) : x <_L y <_L z\}|$, $a_2 = |\{L \in \mathcal{L}(P) : y <_L x\}|$ and $b = |\{L \in \mathcal{L}(P) : z <_L y\}|$. Then

$$
\mathbb{P}_P(x < y) = \frac{b + a_1}{b + a_1 + a_2} \quad \text{and} \quad \mathbb{P}_Q(x < y) = \frac{a_1}{a_1 + a_2}.
$$

Proving the first inequality of Theorem 8 amounts to proving

$$
\frac{a_1}{a_1 + a_2} < \frac{b + a_1}{b + a_1 + a_2},
$$

which is true. Proving the second inequality amounts to proving that $b \leq a_1$ since

$$
\frac{\mathbb{P}_Q(x < y)}{1 + \mathbb{P}_Q(x < y)} = \frac{a_1}{1 + \frac{a_1}{a_1 + a_2}} = \frac{a_1}{2a_1 + a_2},
$$

and

$$
\frac{b + a_1}{b + a_1 + a_2} \leq \frac{2a_1}{2a_1 + a_2} \iff b \leq a_1.
$$
This last inequality is a consequence of Lemma 7. Indeed, there exists an injection from the set of linear extensions in which \( z < y \) and \( x < y \), obtained by swapping the positions of \( y \) and \( z \) in the linear extension. It follows that \( b \leq a_1 \).

We now proceed to the proof of Theorem 3.

**Proof** We consider the three cases separately.

(i) Let \( z \in V \) be such that \( x < z, x \sim y \sim z \) and \( \{x, y\} \) is autonomous in \( P \setminus \{z\} \). Firstly, \( z \) is an upper cover of \( x \). To prove this let \( t \) be such that \( x < t < z \). Then \( t > y \) since \( \{x, y\} \) is autonomous for \( P \setminus \{z\} \). But then \( y < z \), contradicting our assumption.

Secondly \( (y, z) \in \text{crit}(P) \). To prove this let \( u < y \). Then \( u < x \) since \( \{x, y\} \) is autonomous for \( P \setminus \{z\} \). By transitivity we get \( u < z \). Now let \( z < v \). Again by transitivity we have \( x < v \). Hence, \( y < v \) since \( \{x, y\} \) is autonomous for \( P \setminus \{z\} \).

Consider \( Q := P \setminus \{(y, z)\} \) and notice that \((x, y)\) and \((y, x)\) are critical in \( Q \). It follows that \( P_{Q}(x < y) = \frac{1}{2} \). From Theorem 8 we deduce that \((x, y)\) is balanced in \( P \).

(ii) Let \( z, t \in V \) be such that \( x < z, y < t, y \sim z, x \sim t \) and \( \{x, y\} \) is autonomous for \( P \setminus \{z, t\} \). Similar arguments as in (i) yield that \( z \) is an upper cover of \( x, t \) is an upper cover of \( y \) and \( [(y, z), (x, t)] \subseteq \text{crit}(P) \). Consider \( Q := P \setminus \{(y, z)\} \) and observe that \((y, x)\) is critical in \( Q \). Then \( P_{Q}(x < y) < \frac{1}{2} \) (Lemma 7). Moreover, \( \{x, y\} \) is autonomous for \( Q \setminus \{t\} \) which implies that \((x, y)\) is balanced in \( Q \). Hence \( P_{Q}(x < y) \geq \frac{1}{2} \) (this is because \( Q \) satisfies condition \((i)\) of Theorem 3). Apply Theorem 8.

(iii) Let \( z, t \in V \) be such that \( t < x < z, y \) is incomparable to both \( t \) and \( z \) and \( \{x, y\} \) is autonomous for \( P \setminus \{z, t\} \). Similar arguments as in (i) yield that \( z \) is an upper cover of \( x \), \( t \) is a lower cover of \( x \), \( [(t, y), (y, z)] \subseteq \text{crit}(P) \). Consider \( Q := P \setminus \{(y, z)\} \) and observe that \((y, x)\) is critical in \( Q \). Therefore \( P_{Q}(x < y) < \frac{1}{2} \). Moreover, \( \{x, y\} \) is autonomous for \( Q \setminus \{t\} \) which implies that \((x, y)\) is balanced in \( Q \). Hence \( P_{Q}(x < y) \geq \frac{1}{2} \). Apply Theorem 8.

\[ \square \]

### 4 Proof of Theorem 6

Before getting to the proof of Theorem 6 we will need few definitions and preliminary results.

A **fence** (of length \( n \)) is any order isomorphic to the order defined on \( \{f_0, \ldots, f_n\}, n \geq 0 \), where the elements with even subscript are minimal, the elements with odd subscript are maximal (or vice versa), and elements \( f_i \) and \( f_j \) are comparable if and only if \( i = j \) or \( |i - j| = 1 \).

A **crown** (of length \( n \)) is any order isomorphic to the order defined on \( \{c_1, \ldots, c_{2n}\}, n \geq 2 \), where the elements with even subscript are minimal, the elements with odd subscript are maximal and elements \( c_i \) and \( c_j \) are comparable if and only if \( i = j \) or \( |i - j| = 1 \) or \( i = 1 \) and \( j = 2n \).

A **diamond** is any order isomorphic to the order defined on \( \{d_1, d_2, d_3, d_4\} \) where \( d_1 < d_2 < d_4 \) and \( d_1 < d_3 < d_4 \) are the only cover relations among these elements.
The ordered set $P = (V, \leq)$ is crown-free, if either $P$ has no subset isomorphic to a crown of length $n \geq 2$ or $P$ has a subset $\{c_1, c_2, c_3, c_4\}$ isomorphic to a crown of length 2 and there is an element $z \in V$ such that $c_2 < z < c_1$ and $c_4 < z < c_3$. We also say that $P$ is diamond-free if there is no subset isomorphic to a diamond.

**Lemma 9** Let $P = (V, \leq)$ be an ordered set which is crown-free and diamond-free. If $P$ contains a fence of length $n$, then $P$ contains a fence of length $n$ whose minimal elements are minimal in $P$ and whose maximal elements are maximal in $P$.

**Proof** Let $F := \{f_0, \ldots, f_n\}$, $n \geq 0$, be a fence of length $n$ and let $f_i$ be a minimal element of $F$. If $f_i$ is not minimal in $P$, then let $f < f_i$ be a minimal element in $P$. Since $P$ is crown-free and diamond-free, $f$ is incomparable to all elements of $F \setminus \{f_i\}$ except the upper cover(s) of $f_i$ in $F$. Hence $(F \setminus \{f_i\}) \cup \{f\}$ is a fence of length $n$.

**Lemma 10** Let $P = (V, \leq)$ be an ordered set which is crown-free and diamond-free, $x \in V$, and let $F := \{x = f_0, f_1, \ldots, f_n\}$, $n \geq 2$, be a fence of maximum length among those fences starting at $x$ and assume that $f_n$ is minimal in $F$. Then

(i) $U(f_{n-2}) \cap U(f_n)$ has a unique minimal element and this minimal element is less or equal to $f_{n-1}$.

(ii) If $m_{n-2,n}$ is the unique minimal element of $U(f_{n-2}) \cap U(f_n)$, then every element $f$ such that $f_n \leq f < m_{n-2,n}$ has a unique upper cover and this upper cover is comparable to $m_{n-2,n}$. In particular, every element larger or equal to $f$ is comparable to $m_{n-2,n}$.

**Proof** (i) Suppose that $U(f_{n-2}) \cap U(f_n)$ has two distinct minimal elements $y_1$ and $y_2$. Then $\{f_{n-2}, f_n, y_1, y_2\}$ would be a crown in $P$. Say $m_{n-2,n}$ is the unique minimal element of $U(f_{n-2}) \cap U(f_n)$. Then $m_{n-2,n} \leq f_{n-1}$ because otherwise $m_{n-2,n} \sim f_{n-1}$ and hence $\{f_{n-2}, f_n, m_{n-2,n}, f_{n-1}\}$ would be a crown in $P$.

(ii) Let $f$ be such that $f_n \leq f < m_{n-2,n}$ and $t$ be an upper cover of $f$. We assume that $t \sim m_{n-2,n}$ and we will argue to a contradiction. We will prove that $F' := F \cup \{t\}$ is a fence. Then $F'$ is a fence that starts at $x$ and is of length larger than that of $F$ and this is a contradiction. We start by proving that $t$ is incomparable to both $f_{n-2}$ and $f_{n-1}$. Indeed, if not, then $\{f_{n-2}, f, t, m_{n-2,n}\}$ would be a crown in $P$ or $\{f, t, m_{n-2,n}, f_{n-1}\}$ would be a diamond in $P$ which is not possible. Now suppose there exists $0 \leq l \leq n - 3$ such that $t \sim f_l$. Then $f_l < t$ (indeed by assumption $f_n < t$ and $f_n$ is incomparable to all elements of $\{x = f_0, f_1, \ldots, f_{n-2}\}$ hence $t \not\sim f_j$). Choose $0 \leq l \leq n - 3$ maximal such that $f_l < t$. If $f_l$ is minimal in $F$, then the set $\{f_l, \ldots, f_n, t\}$ is a crown in $P$. Else if $f_l$ is maximal in $F$, then the set $\{f_l, \ldots, f_n, t\}$ is a crown in $P$. This is a contradiction. Hence we have proved that $t$ is comparable to $m_{n-2,n}$, that is $t \leq m_{n-2,n}$ (this is because $f < m_{n-2,n}$ and $t$ is an upper cover of $f$). From our assumption that $P$ is diamond-free we deduce that $\{u : f \leq u \leq m_{n-2,n}\}$ is a chain. It follows then that the set of upper covers of $f$ is a chain and therefore $f$ has a unique upper cover. Finally we prove that if $t' \geq f$, then $t' \sim m_{n-2,n}$. If $m_{n-2,n} \leq t'$, there is nothing to prove. Next we suppose that $m_{n-2,n} \not\sim t'$. Let $f'$ be the largest element verifying $f_n \leq f' < m_{n-2,n}$ and $f' < t'$. It follows from our previous discussion that $f'$ has a unique upper cover and that this upper cover is comparable to $m_{n-2,n}$. Hence, $t' < m_{n-2,n}$ and we are done. This completes the proof of the lemma.\[\square\]
The following corollary gives a characterization of ordered sets whose cover graph is a forest.

**Corollary 11** Let $P = (V, \leq)$ be an ordered set. The cover graph of $P$ is a forest if and only if $P$ is crown-free and diamond-free.

**Proof** Clearly, if the cover graph of $P$ is a forest, then $P$ is crown-free and diamond-free. For the converse assume $P$ is crown-free and diamond-free and let $F = \{f_0, ..., f_n\}$, $n \geq 0$, be a fence of maximum length in $P$. It follows from Lemma 9 that we can assume that the minimal elements of $F$ are minimal in $P$ and the maximal elements of $F$ are maximal in $P$. By duality we may assume without loss of generality that $f_n$ is minimal in $P$. We claim that $f_n$ has a unique upper cover. If $n \leq 1$, then $P$ is a disjoint sum of chains and we are done. Else if $n \geq 2$, then our claim follows from (ii) of Lemma 10 with $f = f_n$. Now consider the ordered set $P \setminus \{f_n\}$. From our assumption that $P$ is crown-free and diamond-free it follows that $P \setminus \{f_n\}$ is also crown-free and diamond-free. An induction argument on the number of elements of $P$ shows that the cover graph of $P$ is a forest. □

**Lemma 12** Let $P = (V, \leq)$ be an ordered set which is not a chain and whose cover graph is a tree. Let $f \in V$ be such that $U(f)$ is not a chain and set

$$T_f := \{u \in U(f) : u \text{ has a lower cover } z \text{ such that } z \sim f\}.$$ 

If $T_f = \emptyset$, then $P$ has a very good pair $(a, b)$ such that $a$ and $b$ are maximal elements of $U(f)$.

**Proof** From $P$ is diamond-free and $U(f)$ is not a chain we deduce that $U(f)$ has at least two maximal elements (in $P$) and every element of $U(f)$ has a unique lower cover comparable to $f$. If $T_f = \emptyset$, then the lower covers of every element $u \in U(f)$ are comparable to $f$. Hence every element $u \in U(f)$ has a unique lower cover (this is because $P$ is diamond-free). It follows then that any two distinct maximal elements $a$ and $b$ of $U(f)$ verify $D(a) \setminus D(b)$ and $D(b) \setminus D(a)$ are chains and therefore the pair $(a, b)$ is a very good pair and we are done. □

**Lemma 13** Let $P = (V, \leq)$ be an ordered set which is not a chain and whose cover graph is a tree and $x \in V$. For $n \geq 2$ let $F := \{x = f_0, f_1, ..., f_n\}$ be a fence of maximum length among those fences starting at $x$. Furthermore, assume that $f_n$ is minimal in $F$. If $U(f_n)$ is not a chain, then either $P$ has very good pair in $U(f_n)$ or there exists a fence $F' := (F \setminus \{f_{n-1}, f_n\}) \cup \{f'_{n-1}, f'_n\}$ such that $f_{n-2} < f'_{n-1} > f'_n$, $f'_n$ is minimal in $P$ and $U(f'_n)$ is a chain.

**Proof** We recall that $U(f_{n-2}) \cap U(f_n)$ has a unique minimal element $m_{n-2,n}$ and $m_{n-2,n} \leq f_{n-1}$ ((i) of Lemma 10).

**Claim 1:** $\{t : f_n < t \text{ and } t \sim m_{n-2,n}\} = \emptyset$.

**Proof of Claim 1:** Follows from (ii) of Lemma 10.

**Claim 2:** $U(m_{n-2,n})$ is a chain if and only if $U(f_n)$ is a chain.

**Proof of Claim 2:** Obviously, if $U(f_n)$ is a chain, then $U(m_{n-2,n})$ is also a chain. Now suppose that $U(m_{n-2,n})$ is a chain. From Claim 1 we deduce that in order to prove $U(f_n)$ is a chain it is enough to prove that the set $\{x : f_n \leq t \leq m_{n-2,n}\}$ is a chain. This is true since $P$ is diamond-free. This completes the proof of claim 2.
Suppose that $U(f_n)$ is not a chain. It follows from Claim 2 that $U(m_{n-2,n})$ is not a chain. We consider the set $T_{m_{n-2,n}}$. If $T_{m_{n-2,n}} = \emptyset$ we apply Lemma 12 to obtain the required conclusion. Else if $T_{m_{n-2,n}} \neq \emptyset$ then let $y$ be a maximal element of $T_{m_{n-2,n}}$. If $U(y)$ is not a chain then we apply Lemma 12 to $T_y$ to obtain the required conclusion. For the remainder of the proof of the lemma we assume that $U(y)$ is a chain. Let $z$ be a lower cover of $y$ such that $z \sim m_{n-2,n}$. In particular $z \notin \{f_{n-2},f_n\}$.

**Claim 3:** For all $z' \leq z$, $z'$ is incomparable to all elements of $\{m_{n-2,n}\} \cup D(m_{n-2,n})$.

**Proof of Claim 3:** Suppose there exists $u \in \{m_{n-2,n}\} \cup D(m_{n-2,n})$ and $u \sim z'$. If $z' < u$, then it follows from our assumption that $z \sim m_{n-2,n}$ that $z \neq z'$ and hence $\{z', m_{n-2,n}, z, y\}$ is a diamond in $P$. Else if $u < z'$, then it follows from our assumption that $z' \leq z$ and $z \sim m_{n-2,n}$ that $u \neq m_{n-2,n}$ and hence $\{u, m_{n-2,n}, z, y\}$ is a diamond in $P$. In both cases we obtain a contradiction. This completes the proof of Claim 3.

**Claim 4:** For all $z' \leq z$, $F' := \{x = f_0, f_1, ..., f_{n-2}, y, z\}$, $n \geq 2$, is a fence of maximum length among those fences starting at $x$.

**Proof of Claim 4:** From our assumption that $F$ is fence follows that $F \setminus \{f_{n-1}, f_n\}$ is fence. Hence in order to prove Claim 4 all we have to prove is that $y$ is incomparable to all elements of $F' \setminus \{f_{n-2}, y, z\}$ and $z'$ is incomparable to all elements of $F' \setminus \{y, z\}$. From our assumption that $P$ is crown-free and diamond-free follows easily that $y$ is incomparable to all elements of $F' \setminus \{f_{n-2}, y, z\}$. We now prove that $z'$ is incomparable to all elements of $F' \setminus \{y, z\}$. Suppose there exists $0 \leq l \leq n-2$ such that $z' \sim f_l$. Then $l \neq n-2$ (follows from Claim 3) and $z' < f_l$ (this is because $z' < y$ and $y$ is incomparable to all elements of $\{x = f_0, f_1, ..., f_{n-2}\}$ and hence $f_l \neq z'$). Choose $0 \leq l \leq n-3$ maximal such that $z' < f_l$. If $f_l$ is minimal in $F$, then the set $\{z', f_{l+1}, ..., f_{n-2}, y\}$ is a crown in $P$. Else if $f_l$ is maximal in $F$, then the set $\{z', f_l, ..., f_{n-2}, y\}$ is a crown in $P$. This is a contradiction. The proof of Claim 4 is now complete.

**Claim 5:** Let $t$ be such that $m_{n-2,n} \leq t < y$ and let $z' \leq z$. Then $t \sim z'$.

**Proof of Claim 5:** Suppose not. Then $z' < t$ (this is because $z' \leq z$ and $z \sim m_{n-2,n}$) and hence $z' \neq z$ (this is because $z$ is a lower cover of $y$ and $t < y$). It follows then that $\{z', t, z, y\}$ is a diamond in $P$ which is impossible. This completes the proof of Claim 5.

**Claim 6:** For every $z' \leq z$, if $t > z'$, then $t$ is comparable to $y$.

**Proof of Claim 6:** It follows from Claim 4 that $F' := \{x = f_0, f_1, ..., f_{n-2}, y, z'\}$, $n \geq 2$, is a fence of maximum length among those fences starting at $x$. It follows from (i) of Lemma 10 applied to $F'$ that the smallest element of $U(z') \cap U(f_{n-2})$ must be less or equal to $y$. Claims 3 and 5 imply that $y$ is the smallest element of $U(z') \cap U(f_{n-2})$. Applying (ii) of Lemma 10 to $F'$ with $f = z'$ gives the required conclusion. The proof of Claim 6 is now complete.

Let $z' \leq z$ and $t \geq z'$. From Claim 6 we deduce that $t \sim y$. Since $P$ is diamond-free $\{y : z' \leq t \leq y\}$ must be a chain. It follows from our assumption that $U(y)$ is a chain that $U(z')$ is a chain. It follows from Claim 4 that $F' = \{x = f_0, f_1, ..., f_{n-2}, y, z'\} = (F \setminus \{f_{n-1}, f_n\}) \cup \{y, z'\}$ is a fence of maximum length among those fences starting at $x$. Choosing $z'$ to be minimal in $P$ it becomes now apparent that the fence $F'$ satisfies the required conditions of the lemma and we are done.

**Corollary 14** Let $P = (V, \leq)$ be an ordered set which is not a chain and whose cover graph is a tree and let $F := \{f_0, f_1, ..., f_n\}$, $n \geq 2$, be a fence of maximum length in $P$. If $f_0$ and $f_n$ are minimal elements in $P$, then $P$ has a very good pair.
Proof: We notice at once that $F$ is a fence of maximum length among those fences that start at $f_0$, respectively that start at $f_n$. Hence, if $f_0$ or $f_n$ is a minimal element in $P$, and hence minimal in $F$, then Lemma 13 applies. Assume that $f_0$ and $f_n$ are minimal elements in $P$. If $n = 2$, then it follows from Claim 1 of the proof of Lemma 13 and symmetry that $\{x : f_2 < x \text{ and } x \sim m_{0,2}\} = \emptyset$ where $m_{0,2}$ is the unique minimal element of $U(f_0) \cap U(f_2)$. Hence, $U(f_0) \setminus U(f_2)$ and $U(f_2) \setminus U(f_1)$ are chains proving that $(f_0, f_2)$ is a very good pair and we are done. Now assume $n \geq 4$. If $U(f_0)$ and $U(f_n)$ are chains, then $(f_0, f_n)$ is a very good pair and we are done. Suppose $U(f_n)$ is not a chain. Applying Lemma 13 to the fence $F$ with $x = f_0$ we deduce that either $P$ has a very good pair in $U(f_n)$ or there exists a fence $F' := (F \setminus \{f_{n-1}, f_n\}) \cup \{f_{n-1}', f_n'\}$ (of maximum length) such that $f_n'$ is minimal in $P$, $f_{n-2} < f_{n-1}' > f_n'$ and $U(f_n')$ is a chain. If $U(f_0)$ is a chain, then the pair $(f_0, f_n')$ is a very good pair and we are done. Else if $U(f_0)$ is not a chain, then applying Lemma 13 to the fence $F'$ with $x = f_n'$ we deduce that either $P$ has a very good pair in $U(f_0)$ or there exists a fence $F'' := (F \setminus \{f_0, f_1\}) \cup \{f_0', f_1'\}$ (of maximum length) such that $f_0'$ is minimal in $P$, $f_0' < f_1' > f_2$ and $U(f_0')$ is a chain. It follows then that the pair $(f_0', f_n')$ is a very good pair and we are done.

We now proceed to the proof of Theorem 6.

Proof: Let $P = (V, \leq)$ be an ordered set not totally ordered and whose cover graph is a forest. If all connected components of $P$ are chains, then any two distinct minimal elements of $P$ form a very good pair. Otherwise $P$ has a connected component which is not a chain. Clearly, a very good pair in this connected component remains very good in $P$. Hence, we lose no generality by assuming that $P$ is connected, that is, its cover graph is tree.

Let $F := \{f_0, f_1, ..., f_n\}$, $n \geq 2$, be a fence of maximum length in $P$. It follows from Lemma 9 that we may assume that all the $f_i$’s are minimal or maximal in $P$ and by duality we may assume without loss of generality that $f_0$ is a minimal element in $P$. It follows from Lemma 13 that we can assume $U(f_0)$ to be a chain. By duality and symmetry it then follows that we can assume that either $D(f_n)$ is a chain if $f_n$ is maximal or $U(f_n)$ is a chain if $f_n$ is minimal. It follows from Corollary 14 that we can assume $f_n$ to be maximal (hence $n$ is odd). We now define

$$\mathcal{F} := \{x : \text{ there exist } 1 \leq i, j \leq n \text{ such that } f_i \leq x \leq f_j\}$$

and

$$D := \{x \in \mathcal{F} : \text{ there exists a fence } F_x \text{ of length at least 2 starting at } x \text{ so that } \mathcal{F} \cap F_x = \{x\}\}.$$

We consider two cases.

Case 1: $D \neq \emptyset$.

Let $x \in D$ and let $F_x = \{x = e_0, e_1, ..., e_k\}$, $k \geq 2$, be a fence of maximum length at least 2 (among those fences starting at $x$ and satisfying $\mathcal{F} \cap F_x = \{x\}$). We notice at once that $f_0 \sim e_k \sim f_n$ (this follows from our assumption that $P$ is crown-free and diamond-free). Assume that $e_k$ is minimal in $F_x$. Let $m_{k-2,k}$ be the unique minimal element of $U(e_{k-2}) \cap U(e_k)$.

Claim: Let $u$ be such that $e_k < u$. Then $u$ is comparable to $m_{k-2,k}$.

Let $u$ be such that $e_k < u$ and assume for a contradiction that $u$ is incomparable to $m_{k-2,k}$. A similar argument as in the proof of (ii) of Lemma 10 yields that $F'_x = \{x = e_0, e_1, ..., e_k, u\}$ is a fence. Since $F_x$ is a fence of maximum length at least 2 among those fences starting at $x$ and satisfying $\mathcal{F} \cap F_x = \{x\}$ we infer that $u \in \mathcal{F}$. Observe then that $P$ has a crown which is impossible by assumption. This completes the proof of the claim.

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Similar to (ii) of Lemma 10 it follows then that every element $f$ such that $e_k \leq f < m_{k-2,k}$ has a unique upper cover and this upper cover is comparable to $m_{k-2,k}$. In particular, every element larger or equal to $f$ is comparable to $m_{k-2,k}$. Consequently, $U(m_{k-2,k})$ is a chain if and only if $U(e_k)$ is a chain (this is similar to Claims 1 and 2 of Lemma 13). Considering the set $T_{m_{k-2,k}}$ and applying similar arguments as in the proof of Lemma 13 we obtain that if $U(e_k)$ is not a chain, then either $P$ has a very good pair or we can find a new fence $F'_k = \{e_0 = x, \ldots, e'_{k-1}, e'_k\}$ such that $e'_k$ is minimal in $P$ and $U(e'_k)$ is a chain. If the former holds then we are done. Else if the latter holds, then it follows from $f_0$ is minimal in $P$ and $e_0 \sim f_0$ that $e'_k \sim f_0$. Hence, $(f_0, e'_k)$ is a very good pair.

If $e_k$ is maximal in $F_k$, then we apply the previous argument to the dual of $P$ and to the dual of the fence $F_k$ that either $P$ has a very good pair or we can find a new fence $F''_k = \{e_0 = x, \ldots, e''_{k-1}, e''_k\}$, $k \geq 2$, such that $e''_k$ is maximal in $P$ and $D(e''_k)$ is a chain. If the former holds then we are done. Else if the latter holds, then it follows from $f_n$ is maximal in $P$ and $e_k \sim f_n$ that $e''_k \sim f_n$. Hence $(f_n, e''_k)$ is a very good pair.

**Case 2:** $D = \emptyset$.

**Claim 1:** Let $x \in F$. Then every element of $U(x) \setminus F$ has a unique lower cover and this lower cover is comparable to $x$. Dually, every element of $D(x) \setminus F$ has a unique upper cover and this upper cover is comparable to $x$.

**Proof of Claim 1:** Suppose there exists $y \in U(x) \setminus F$ that has two distinct lower covers $y_1$ and $y_2$ and note that $y_1 \sim y_2$. Then $y_1$ or $y_2$ is incomparable to $x$. Indeed, suppose not. If $x \notin \{y_1, y_2\}$, then $\{x, y_1, y_2, y\}$ is a diamond in $P$ which is not possible. Else if, say $x = y_2$, then $x \prec y_2$. Hence $y_1 \in F$ because otherwise $\{x, y, y_1\}$ is a fence of length at least 2 starting at $x$ and verifying $F \cap \{x, y, y_1\} = \{x\}$ contradicting $D = \emptyset$. Let $k', k$ be nonnegative integers such that $f_{k'} \leq y_1 \leq f_k$ and $|k' - k| = 1$. Since $x \in F$ there are nonnegative integers $i$ and $j$ such that $f_i \leq x \leq f_j$ and $|i - j| = 1$. If $y_1$ is comparable to $f_i$, that is $k' = i$, then $y_1 \neq f_i$ (this is because $y_1$ is a lower cover of $y$ and $f_i < x < y$) and since $f_i$ is minimal in $P$ we have $f_i < y_1$. Hence, $(f_i, x, y_1, y)$ is a diamond in $P$. Else if $y_1$ is incomparable to $f_i$, then $\{y_1, f_k, \ldots, f_i, y\}$ is a crown. In both cases we obtain a contradiction since $P$ is diamond-free and crown-free. This proves Claim 1.

**Claim 2:** If there exists $x \in F$ such that $U(x) \setminus F$ or $D(x) \setminus F$ is not a chain, then $P$ has a very good pair.

**Proof of Claim 2:** Let $x \in F$ be such that $U(x) \setminus F$ is not a chain. Since $P$ is diamond-free $(U(x) \setminus F)$ has at least two maximal elements. It follows from Claim 1 of Case 2 that every element of $U(x) \setminus F$ has a unique lower cover and that this lower cover is comparable to $x$. It becomes now apparent that any pair of distinct maximal elements of $U(x) \setminus F$ is a very good pair and we are done.

It follows from Claim 2 that we can assume that for every element $x \in F$ the sets $U(x) \setminus F$ and $D(x) \setminus F$ are chains.

For integers $0 \leq i, j \leq n$ with $|i - j| = 1$ and $i$ even, set

$$D_{i,j} := \{x : f_i < x < f_j\} \text{ and there exists } t \notin F \text{ such that } t \text{ covers } x\}.$$ 

**Claim 3:** If $D_{0,1} \neq \emptyset$, then $P$ has a very good pair.

**Proof of Claim 3:** Assume that $D_{0,1} \neq \emptyset$ and let $x$ be such that $f_0 < x < f_1$ and let $t \notin F$ be a cover of $x$. From $U(f_0)$ is a chain it follows that $t < x$ and hence $t$ is a lower cover of $x$. We claim that $t \sim f_0$. Since $f_0$ is minimal, if $t$ is comparable to $f_0$ (and $t \neq f_0$ since $t \notin F$), then we must have $f_0 < t < x < f_1$. But then $t \in F$, which is a contradiction. Our claim is then proved. Now let $t' \leq t$ be a minimal element. It follows from Claim 1 of Case 2 that every element of $D(x) \setminus F$ has a unique upper cover and this upper cover is
comparable to $x$. Hence, $U(t') = \{z : t' < z \leq x\} \cup U(x)$. Since $P$ is diamond-free the set $\{z : t' < z \leq x\}$ is a chain. From our assumption $U(x) \setminus F$ is a chain it follows then that $U(t')$ is a chain. Hence the pair $(f_0, t')$ is a very good pair and we are done.

For the remainder of the proof we assume that $D_{0,1} = \emptyset$.

**Claim 4:** If $D_{2,1} \neq \emptyset$, then $P$ has a very good pair.

**Proof of Claim 4:** We recall that $U(f_0) \cap U(f_2)$ has a unique minimal element denoted $m_{0,2}$ and that $f_0 < m_{0,2} \leq f_1$. Let $x \in D_{2,1}$ and notice that since $D_{0,1} = \emptyset$ we have $f_2 < x < m_{0,2}$. Choose $x$ to be maximal in $D_{2,1}$. We argue on whether $x$ is a lower cover of $m_{0,2}$ or not. We first consider the case $x$ is a lower cover of $m_{0,2}$. Let $t$ be a cover of $x$ not in $F$. Suppose $t$ is a lower cover of $x$ and let $t' \leq t$ be a minimal element in $P$. We claim that $(f_0, t')$ is a very good pair. Indeed, by assumption $U(f_0)$ is a chain and hence $U(f_0) \setminus U(t')$ is a chain. Moreover, it follows from the maximality of $x$ and Claim 1 of Case 2 that $U(t') \setminus U(f_0)$ is also a chain. Since $f_0$ and $t'$ are both minimal in $P$ our claim follows. Now suppose that $t$ is an upper cover of $x$ and let $t'' \geq t$ be a maximal element in $P$. We claim that $(f_1, t'')$ is a very good pair. Indeed, $D(t'') \setminus D(f_1) = \{z : t \leq z < t''\}$ which is a chain (this follows from Claim 1 of Case 2 and our assumption that $D(x) \setminus F$ is a chain). Moreover, $D(f_1) \setminus D(t'') = \{z : f_0 \leq z < f_1\}$ which is also a chain (by assumption $D_{0,1} = \emptyset$). The required conclusion follows since $f_1$ and $t''$ are maximal in $P$. Now we consider the case $x$ is not a lower cover of $m_{2,1}$. From our choice of $x$ it follows that for all $u$ such that $x < u < m_{2,1}$ we have $u \notin D_{2,1}$, that is, every cover of $u$ is in $F$. From our assumption that $U(f_0)$ is a chain follows that $U(u)$ is a chain. Let $u$ be an upper cover of $x$ such that $x < u < m_{2,1}$. Then $D(u) = D(t) = \{x\} \cup D(x)$. Hence $(u, t)$ is a very good pair and we are done.

For the remainder of the proof we assume that $D_{2,1} = \emptyset$.

Now it becomes apparent that similar arguments as in the proof of Claim 4 lead to $P$ has a very good pair if $D_{2,3} \neq \emptyset$. Hence we may assume that $D_{2,3} = \emptyset$. Let $y_1$ and $y_2$ be two distinct lower covers of $m_{0,2}$ such that $f_0 \leq y_1 < m_{0,2}$ and $f_2 \leq y_2 < m_{0,2}$. We claim that $(y_1, y_2)$ is a very good pair if $y_2 \neq f_2$, or $(f_0, f_1)$ is a very good pair if $y_2 = f_2$. Indeed, $D(y_1)$ is a chain since $D_{0,1} = \emptyset$ and $D(y_2)$ is a chain since $D_{2,1} = \emptyset$ and $U(y_1) = U(m_{0,2}) \cup \{m_{0,2}\}$ is a chain since $U(f_0)$ is a chain. Moreover, if $y_2 \neq f_2$, then $U(y_2) = U(m_{0,2}) \cup \{m_{0,2}\}$ which is a chain. Else if $y_2 = f_2$, then $f_2$ is a lower cover of $m_{0,2}$ and $U(f_2) \setminus U(f_1)$ is a chain since by assumption $D_{2,3} = \emptyset$. This proves our claim and completes the proof of the theorem.

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