QUANTUM STOCHASTIC CONVOLUTION COCYCLES III

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Abstract. The theory of quantum Lévy processes on a compact quantum group, and more generally quantum stochastic convolution cocycles on a $C^*$-bialgebra, is extended to locally compact quantum groups and multiplier $C^*$-bialgebras. Strict extension results obtained by Kustermans, and automatic strictness properties developed here, are exploited to obtain existence and uniqueness for coalgebraic quantum stochastic differential equations in this setting. Working in the universal enveloping von Neumann bialgebra, the stochastic generators of Markov-regular, completely positive, respectively *-homomorphic, quantum stochastic convolution cocycles are characterised. Every Markov-regular quantum Lévy process on a multiplier $C^*$-bialgebra is shown to be equivalent to one governed by a quantum stochastic differential equation, and the generating functionals of norm-continuous convolution semigroups on a multiplier $C^*$-bialgebra are characterised. Applying a recent result of Belton’s, we give a thorough treatment of the approximation of quantum stochastic convolution cocycles by quantum random walks.

Introduction

The principal aim of this paper is the extension of the theory of quantum Lévy processes, and more generally quantum stochastic evolutions with tensor-independent identically distributed increments on a $C^*$-bialgebra, to the context of locally compact quantum semigroups—in other words multiplier $C^*$-bialgebras. The notion of quantum Lévy process generalises that of classical Lévy process on a semigroup. It was first introduced by Accardi, Schürmann and von Waldenfels, in the purely algebraic framework of $*$-bialgebras ([ASW]), and was further developed by Schürmann and others ([Sch], [Fra]) who, in particular, extended it to other quantum notions of independence (free, boolean and monotone), still in the algebraic context. Inspired by Schürmann’s reconstruction theorem, which states that every quantum Lévy process on a $*$-bialgebra can be equivalently realised on a symmetric Fock space, we first showed how the algebraic theory of quantum Lévy processes can be extended to the natural setting of quantum stochastic convolution cocycles ([LS1]). These are families of linear maps $(l_t)_{t \geq 0}$ from a $*$-bialgebra $B$ to operators on the symmetric Fock space $F$, over a Hilbert space of the form $L^2(\mathbb{R}_+; k)$, satisfying the following cocycle identity with respect to the amplified CCR flow $(\sigma_t)_{t \geq 0}$:

$$l_{s+t} = l_s \ast (\sigma_s \circ l_t), \quad s, t \geq 0,$$

together with regularity and adaptedness conditions. Our approach enabled us to then establish a theory of quantum Lévy processes on compact quantum groups and, more generally, quantum stochastic convolution cocycles on operator space coalgebras ([LS3]). The recent development of a satisfactory theory of locally compact quantum groups ([KuV]) provides the challenge which is addressed in the current work, namely to extend our analysis to the locally compact realm.

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On the algebraic level the theories of quantum stochastic convolution cocycles on compact and locally compact quantum semigroups look similar, however their analytic aspects have a rather different nature. Whereas the coproduct on a compact quantum semigroup $B$ takes values in the spatial tensor product $B \otimes B$, which led us to an operator-space theoretic development of the theory and enabled us to establish the main results in the corresponding natural category of operator-space coalgebras, a noncompact locally compact quantum semigroup $B$ is a nonunital $C^*$-algebra whose coproduct takes values in the multiplier algebra of $B \otimes B$. Consequently, $C^*$-algebraic methods come more to the fore, with the strict topology ([Lan]), strict maps ([Ku]) and enveloping von Neumann algebras all playing crucial roles. The modern approach to quantum stochastics involves matrix spaces as a natural tool for combining $C^*$-algebraic quantum state spaces with von Neumann algebraic quantum noise ([LW]), and this was successfully exploited in [LS]. In the context of the present paper, the strict topology on the initial $C^*$-algebra has to be harnessed to the matrix-space technology, so that both may be exploited in tandem. For this reason the first part of the paper (Section 1) is devoted to a careful analysis of relations between the strict topology and the matrix-space topology, compatibility between extensions of maps continuous with respect to the different topologies, automatic strictness of certain completely bounded maps and connections with the enveloping von Neumann algebra and ultraweak extension.

The structure of the rest of the paper is as follows. In Section 2 basic properties of multiplier $C^*$-bialgebras are described and a universal enveloping construction is given; in particular, the $R$-maps which play a central role are introduced. Section 3 contains a brief summary of the background quantum stochastics needed here. Weak and strong coalgebraic quantum stochastic differential equations are treated in Section 4 where an automatic strictness result is used to establish uniqueness of weak solutions. Quantum stochastic convolution cocycles are analysed in Section 5 where Markov-regular completely positive contraction cocycles are shown to satisfy quantum stochastic differential equations, and the form of the stochastic generator is given — also for $^*$-homomorphic cocycles. Markov-regularity means that the Markov semigroup of the cocycle is norm-continuous. In Section 6 quantum Lévy processes are defined in our setting and are shown to be realisable as Fock-space convolution cocycles when they are Markov-regular. This leads to two characterisations of the generating functionals of norm-continuous convolution semigroups of states on a locally compact quantum semigroup. The final section contains a thorough analysis of a natural discrete approximation scheme for the type of cocycles treated in Section 5 founded on a recent approximation theorem of Belton ([Bel]). The method of approximation closely mirrors the structure of the solution of the corresponding quantum stochastic differential equation via a (sophisticated) form of Picard iteration. The results of Sections 4 to 6 incorporate and extend corresponding results for unital $C^*$-bialgebras in [LS], and those of Section 7 do likewise for [FSS].

Our most satisfactory results are obtained in the case of Markov-regular quantum Lévy processes, where the Markov (convolution) semigroup of the process is norm-continuous. A general theory of weakly continuous convolution semigroups of functionals on multiplier $C^*$-bialgebras is initiated in [LS4]. In that paper every such semigroup of states on a multiplier $C^*$-bialgebra of discrete type is shown to be norm-continuous so that all the results of this paper apply directly in that case. Also Theorem 6.3 of this paper is used in [LS] to derive a classical result on conditionally positive-definite functions on a compact group.
**General notations.** In this paper the multiplier algebra and universal enveloping von Neumann algebra of a $C^*$-algebra $A$ are denoted by $\hat{A}$ and $\overline{A}$ respectively. The symbols $\otimes$, $\otimes_\square$ and $\boxtimes$ are used respectively for linear/algebraic, spatial/minimal and ultraweak tensor products, of spaces and respectively, linear, completely bounded and ultraweakly continuous completely bounded maps. For any Hilbert space $h$, we have the ampliation and Hilbert space, given respectively by

$$t_h : B(H; K) \to B(H \otimes h; K \otimes h), \quad T \mapsto T \otimes I_h, \quad \text{and} \quad \widehat{h} := C \oplus h.$$  

(0.1)

where context determines the Hilbert spaces $H$ and $K$.

1. **Strict extensions, tensor products and $\chi$-structure maps.**

In this section we recall some definitions and relevant facts about Hilbert $C^*$-modules ([La]), strict topologies ([Ku]), tensor products and $h$-$k$-matrix spaces ([LV]). We establish an automatic strictness result and show how strict tensor product constructions compare with $h$-matrix space constructions over a multiplier $C^*$-algebra. The section ends by recalling a central concept for quantum Lévy processes, namely that of $\chi$-structure maps.

**Hilbert $C^*$-modules and multiplier algebras.** For Hilbert $C^*$-modules $E$ and $F$ over a $C^*$-algebra $C$, $\mathcal{L}(E; F)$ denotes the space of adjointable operators $E \to F$. Hilbert $C^*$-modules are endowed with a natural operator space structure under which $M_n(\mathcal{L}(E; F))$ is identified with $\mathcal{L}(E^n; F^n)$, where the column direct sums $E^n$ and $F^n$ are also Hilbert $C$-modules, and $\mathcal{L}(E; F) \subset BC(E; F)$, the space of right $C$-linear completely bounded maps $E \to F$ (BM). The strict topology on $\mathcal{L}(E; F)$ is the locally convex topology generated by the seminorms $T \mapsto \|Te\| + \|T^*f\|$ ($e \in E, f \in F$); it is Hausdorff and complete. The closed subspace of $\mathcal{L}(E; F)$ generated by the elementary maps $[f]_e := x \mapsto f(e, x)$ ($e \in E, f \in F$) is denoted $\mathcal{K}(E; F)$. The unit ball of $\mathcal{K}(E; F)$ is strictly dense in that of $\mathcal{L}(E; F)$, $\mathcal{K}(E)$ is a $C^*$-algebra and $\mathcal{L}(E)$ is a model for its multiplier algebra. In particular, viewing a $C^*$-algebra $A$ as a Hilbert $C^*$-module over itself, $\mathcal{K}(A) = A$ so that $\mathcal{L}(A)$ is a model for the multiplier algebra $\hat{A}$. A net of positive contractions $(e_\alpha)$ in $A$ is an approximate identity for $A$ if and only if $e_\alpha \to 1_\hat{A}$ strictly. When $A$ is unital the strict topology coincides with the norm topology. For Hilbert spaces $h$ and $k$, $|h⟩ := B(C; h)$ and $|k⟩$ are Hilbert $C^*$-modules over $C$, $\mathcal{L}(|h⟩; |k⟩)$ and $\mathcal{K}(|h⟩; |k⟩)$ are naturally identified with $B(h; k)$ and $K(h; k)$ respectively, and the strict topology on $\mathcal{L}(|h⟩; |k⟩)$ corresponds to the strong*-topology on $B(h; k)$. When a $C^*$-algebra $A$ acts nondegenerately on a Hilbert space $h$, the multiplier algebra $\widehat{A}$ is realised as the double centraliser of $A$ in $B(h)$: $\{x \in B(h) : \forall a \in A \quad xa, ax \in A\}$, the inclusion $A \subset A''$ holds and bounded strictly convergent nets in $\widehat{A}$ converge (strong*- and thus) $\sigma$-weakly.

Two elementary classes of strictly continuous maps that feature below are component maps $\varepsilon_{kl} : \mathcal{L}(E_1 \oplus E_2) \to \mathcal{L}(E_k; E_l)$, $T \mapsto [T_{ij}] \mapsto T_{kl}$, where the column direct sum $E_1 \oplus E_2$ is a Hilbert $C$-module, and multiplication operators $\mathcal{L}(E; F) \to \mathcal{L}(E'; F')$, $T \mapsto X^*TY$, where $X \in \mathcal{L}(E'; F)$ and $Y \in \mathcal{L}(E'; E)$ for Hilbert $C$-modules $E'$ and $F'$.

**Strict maps and extensions.** There is a more prevalent notion in the theory than strict continuity. A bounded operator $\varphi$ from $\mathcal{K} = \mathcal{K}(E; F)$ to $\mathcal{L}' = \mathcal{L}(E'; F')$ where $E'$ and $F'$ are Hilbert $C^*$-modules over a $C^*$-algebra $C'$, is called strict if it is strictly continuous on bounded sets; the collection of such maps, denoted $B_\beta(\mathcal{K}; \mathcal{L}')$, is a closed subspace of $B(\mathcal{K}; \mathcal{L}')$; we describe some of its contents below. Here we are particularly interested in the classes $B_\beta(A; B)$ for $C^*$-algebras $A$ and spaces $B$ of the form $B(h; k)$ where $h$ and $k$ are Hilbert spaces. An important general class of
strict maps is the set of *-homomorphisms \( \varphi : A \to \mathcal{L}(E) \), for a C*-algebra \( A \) and Hilbert C*-module \( E \), which are nondegenerate in the sense that \( \overline{\text{im} \, \varphi(A)} = E \).

For a C*-algebra \( A \), let \( \overline{A} \) denote its universal enveloping von Neumann algebra, let \( \rho \) be the embedding \( A \to \overline{A} \) and let \( \iota \) be the inclusion/natural map \( \overline{A} \to \overline{\overline{A}} = (\overline{A})'' \). The map \( \iota^* \circ \rho^* : A'' \to (\overline{A})'' = \overline{A} \) is a *-isomorphism for the common Arens product on \( A'' \) and a weak*-\( \sigma \)-weak homeomorphism. Since \( A \) acts nondegenerately in the universal representation, \( \overline{A} \) may be viewed as a subalgebra of \( \overline{\overline{A}} \). All of this is well-known. For ease of reference we collect together some extension properties which will play an important role here. The notation \( B_\sigma \) stands for bounded ultraweakly continuous.

**Theorem 1.1.** Let \( A \) be a C*-algebra with multiplier algebra \( \overline{A} \) and universal enveloping von Neumann algebra \( \overline{\overline{A}} \).

(a) Let \( \varphi \in B_3(A; \mathcal{L}) \) where \( \mathcal{L} = \mathcal{L}(E; F) \) for C*-modules \( E \) and \( F \) over a C*-algebra \( C \). Then \( \varphi \) has a unique strict extension \( \overline{\varphi} : \overline{A} \to \mathcal{L} \), moreover \( \overline{\varphi} \) is bounded and \( \| \overline{\varphi} \| = \| \varphi \| \).

(b) Let \( \psi \in B(A; B) \) where \( B = B(h; k) \) for Hilbert spaces \( h \) and \( k \). Then \( \psi \) has a unique normal extension \( \overline{\psi} \in B_\sigma(\overline{A}; B) \), moreover \( \| \overline{\psi} \| = \| \psi \| \).

(c) Let \( \phi \in B_3(A; B) \) where \( B = B(h; k) = \mathcal{L}([h]; [k]) \) for Hilbert spaces \( h \) and \( k \). Then \( \phi = \overline{\phi} = \overline{\phi} \).

In (a), \( \overline{\varphi}^1 = \overline{\varphi}^1 \) where \( \varphi^1 : A \to \mathcal{L}(F; E) \) is defined by \( \varphi^1(a^*) = \varphi(a)^* \); similarly, in (b) \( \overline{\psi}^1 = \overline{\varphi}^1 \). When \( E = F \), \( \overline{\varphi} \) is positive/completely positive/multiplicative if \( \varphi \) is, and likewise for \( \psi \) and \( \overline{\psi} \) when \( k = h \).

**Proof.** (a) is proved in [Ku11] in the case \( E = F = C \). The general case is obtained by applying this case with \( C = K(E \oplus F) \) and composing with the strict map \( \mathcal{L}(E \oplus F) \to \mathcal{L}(E; F) \), \( T = [T_{i,j}] \to T_{21} \). (b) is well-known: set \( \overline{\psi} := \iota^* \circ \psi^* \circ j \) where \( \iota \) is the natural map/inclusion \( B_* \to (B_*)^* = B^* \) and \( j \) is the natural isometric isomorphism \( \overline{A} \to A^* \). Since the unit ball of \( A \) is strictly dense in that of \( \overline{A} \) ([Lan], Proposition 1.4), (c) follows from the fact that strictly convergent bounded nets converge \( \sigma \)-weakly. The last part follows from Kaplansky’s density theorem and its Hilbert C*-module counterpart (just used), and the separate strict (respectively \( \sigma \)-weak) continuity of multiplication and corresponding continuity of the adjoint operation, in a multiplier algebra (respectively von Neumann algebra).

**Remarks.** (i) The extensions commute with matrix liftings:

\[
\overline{\varphi}^{(n)} = \overline{\varphi}^{(n)} : M_n(\overline{A}) = M_n(\overline{\overline{A}}) \to M_n(\mathcal{L}) = \mathcal{L}(E^n; F^n)
\]

\[
\overline{\psi}^{(n)} = \overline{\psi}^{(n)} : M_n(\overline{A}) = M_n(\overline{\overline{A}}) \to M_n(B) = B(h^n; k^n),
\]

so \( \| \overline{\varphi} \|_{cb} = \| \varphi \|_{cb} \) when \( \varphi \in CB_3(A; \mathcal{L}) \) and \( \| \overline{\psi} \|_{cb} = \| \psi \|_{cb} \) when \( \psi \in CB(A; B) \).

(ii) Clearly the range of \( \overline{\varphi} \) is contained in the strict closure of the range of \( \varphi \), and the range of \( \overline{\psi} \) is contained in the \( \sigma \)-weak closure of the range of \( \psi \).

(iii) As a consequence of (a), strict maps may be composed in the following sense: if \( \varphi_1 \in B_3(A_1; A_2) \) and \( \varphi_2 \in B_3(A_2; A_3) \), for C*-algebras \( A_1, A_2 \) and \( A_3 \), then \( a \mapsto \overline{\varphi_2} \circ \overline{\varphi_1} \) is strict with unique strict extension \( \overline{\varphi_2} \circ \overline{\varphi_1} \) following the widely adopted convention (e.g. [Lan]), it is denoted \( \varphi_2 \circ \varphi_1 \). For a nondegenerate *-homomorphism \( \varphi : A \to \mathcal{B} \), \( \overline{\varphi} \) is a unital *-homomorphism, and conversely every nondegenerate *-homomorphism \( \varphi : A \to \mathcal{B} \) is the restriction of a strict unital *-homomorphism \( \overline{\varphi} : \overline{A} \to \mathcal{B} \).

**Warning.** We now write \( B_3(\overline{A}; \mathcal{L}) \) for the class of strict maps \( \overline{A} \to \mathcal{L} \) where, for us, \( \mathcal{L} \) will always be either of the form \( B(h; k) = \mathcal{L}([h]; [k]) \) for Hilbert spaces \( h \) and \( k \).
or of the form $\tilde{C}$ for a $C^*$-algebra $C$ (or both: $B(h) = \mathcal{L}(h) = \tilde{K}(h)$). Use of this notation then always needs to reflect the algebras of which the source (and target) multiplier algebras are.

We note that the theorem delivers a commutative diagram of isometric isomorphisms:

$$
\begin{array}{ccc}
B_\beta(A; B) & \longrightarrow & B_\sigma(\tilde{A}; B) \\
\text{for any } C^*-\text{algebra } A \text{ and space } B \text{ of the form } B(h; k) \text{ for Hilbert spaces } h \text{ and } k.
\end{array}
$$

**Definition.** Let $\varphi \in B_\beta(A; \mathcal{L}(E))$ for a $C^*$-algebra $A$ and Hilbert $C^*$-module $E$. We call $\varphi$ *preunital* if its strict extension is unital: $\tilde{\varphi}(1) = I$.

**Remark.** For $*$-homomorphisms this is equivalent to nondegeneracy: in general it is equivalent to $\varphi(e_\lambda) \to 1$ for some/every $C^*$-approximate identity $(e_\lambda)$ for $A$, but is stronger than the condition $\lim \varphi(A)E = E$ ([Lan], Proposition 2.5; Corollary 5.7).

**Automatic strictness and strict tensor products.** In the next theorem we establish an automatic strictness property and identify a natural class of maps for strict tensoring. First some notation. When $A$ is the spatial tensor product $A_1 \otimes A_2$, for $C^*$-algebras $A_1$ and $A_2$, $\tilde{A}$ is denoted $A_1 \tilde{\otimes} A_2$. Note the relation

$$
\tilde{A}_1 \tilde{\otimes} \tilde{A}_2 \subset A_1 \tilde{\otimes} A_2 \quad (1.2)
$$

**Theorem 1.2.** Let $A$, $A_1$ and $A_2$ be $C^*$-algebras.

(a) Let $\varphi \in CB(A; B)$ where $B = B(h; k)$ for a Hilbert spaces $h$ and $k$. Then $\varphi$ is strict and $\tilde{\varphi} = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}$.

(b) Let $\varphi_i \in \text{Lin} CP_{13}(\tilde{A}_i; \tilde{C}_i)$ for $C^*$-algebras $C_1$ and $C_2$. Then, there is a unique map $\varphi_1 \otimes \varphi_2 \in \text{Lin} CP_{13}(A_1 \otimes A_2; C_1 \otimes C_2)$ extending the algebraic tensor product map $\varphi_1 \otimes \varphi_2$.

**Proof.** (a) In view of Theorem 1.1 it suffices to prove that $\varphi$ is strict. It follows from the Wittstock-Paulsen-Haaggup Decomposition Theorem ([LRR], Theorem 5.3.3) that $\varphi = \psi \circ \pi$ where $\pi$ is a $*$-homomorphism $A \to B(H)$ for some Hilbert space $H$ and $\psi : B(H) \to B$ is of the form $X \mapsto R^* XS$, for some operators $R \in B(k; H)$ and $S \in B(h; H)$. Moreover, replacing $H$ by $H' := \overline{\text{Lin}} \pi(A)H$, $\pi$ by its compression to $H'$ and $R$ by $RP$ and $SP$ where $P$ is the orthogonal projection $H \to H'$, if necessary, we may suppose that $\pi$ is nondegenerate and therefore strict, when viewed as a map into $\mathcal{L}(H)$. Since $\psi$ is strict $\mathcal{L}(H) \to \mathcal{L}(h; |k|)$, $\varphi$ is too. (b) By linearity we may suppose that $\varphi_1$ and $\varphi_2$ are completely positive. The result then follows easily from Kasparov’s extension of the Stinespring Decomposition Theorem ([Lan], Theorem 5.6).

**Remarks.** It follows from Part (a) and the remarks after Theorem 1.1 that (1.1) restricts to a commutative diagram of completely isometric isomorphisms

$$
\begin{array}{ccc}
CB(A; B) & \longrightarrow & CB_\sigma(\tilde{A}; B) \\
\text{for } A \text{ and } B \text{ are as in (1.4). In particular, we have complete isometries}
\end{array}
$$

$$
A^* \cong \tilde{A}_1^* \cong \tilde{A}_2^*.
$$
In the terminology of \([GL_2]\), the theorem implies that strict completely positive maps are **completely strict**.

By operator space considerations \(\text{Ran}(\varphi_1 \otimes \varphi_2) \subset \tilde{C}_1 \otimes \tilde{C}_2\). Part (b) leads to the following useful notation. For \(\varphi_i \in \text{Lin} \, CP_{\beta}(A_i; \tilde{C}_i)\) \((i = 1, 2)\) we denote the unique strict extension of \(\varphi_1 \otimes \varphi_2\) by \(\varphi_1 \tilde{\otimes} \varphi_2\). Thus
\[
\varphi_1 \tilde{\otimes} \varphi_2 \in \text{Lin} \, CP_{\beta}(A_1 \otimes A_2; C_1 \tilde{\otimes} C_2).
\] (1.5)

Note the following consequence of Part (a) and its proof, which provides a source for tensoring as in Part (b).

**Corollary 1.3.** For a \(C^*\)-algebra \(A\) and Hilbert space \(h\),
\[
CB(A; B(h)) = \text{Lin} \, CP_{\beta}(A; B(h)).
\] (1.6)

**Remark.** This ‘strict decomposability’ property is very useful. For a general multiplier algebra target space, completely bounded maps need neither be strict nor be linear combinations of completely positive maps.

**h-k-Matrix spaces.** Let \(V\) be an operator space in \(B(H; K)\), let \(B = B(h; k)\) for two further Hilbert spaces \(h\) and \(k\) with total subsets \(S\) and \(T\), and let \(Z \in B(H \otimes h; K \otimes k) = B(H; K) \tilde{\otimes} B\). Then the following are equivalent:
\[
E^\xi Z E_\eta \in V \quad \text{for all} \quad \xi \in T, \eta \in S; \quad (\text{id}_B(H; K) \tilde{\otimes} \omega)(Z) \in V \quad \text{for all} \quad \omega \in B_\ast;
\]
where, for a Hilbert space vector \(\xi\),
\[
E_\xi := I \otimes |\xi\rangle : u \mapsto u \otimes \xi \quad \text{and} \quad E^\xi := (E_\xi)^* = I \otimes \langle \xi|,
\] (1.7)
and \(I\) is the identity operator on the appropriate Hilbert space. The collection of operators \(Z\) enjoying this property is an operator space which is denoted \(V \otimes_M B\) and called the (right) \(h\)-\(k\)-**matrix space** over \(V\). It is situated between the norm-spatial and ultraweak-spatial tensor products:
\[
V \otimes B \subset V \otimes_M B \subset \tilde{\nabla} \otimes B
\]
and the latter inclusion is an equality if and only if \(V\) is \(\sigma\)-weakly closed. If \(\varphi \in CB(V; V')\) for another concrete operator space \(V'\) then there is a unique map, its h-k-**lifting**, denoted \(\varphi \otimes_M \text{id}_B\), from \(V \otimes_M B\) to \(V' \otimes_M B\) satisfying \(E^\xi (\varphi \otimes_M \text{id}_B)(Z) E_\eta = \varphi(E^\xi Z E_\eta)\) for all \(\xi \in k, \eta \in h\) and \(Z \in V \otimes_M B\); it is completely bounded, with \(\|\varphi \otimes_M \text{id}_B\|_{cb} = \|\varphi\|_{cb}\) and completely isometric if \(\varphi\) is (unless \(B = \{0\}\)), moreover it extends \(\varphi \tilde{\otimes} \text{id}_B\), and it coincides with \(\varphi \tilde{\otimes} \text{id}_B\) when \(V\) is \(\sigma\)-weakly closed and \(\varphi\) is \(\sigma\)-weakly continuous. The next proposition confirms the compatibility of h-k-matrix spaces and h-k-liftings on the one hand, and strict tensor products of algebras and strict maps on the other. First note the identity
\[
\|\eta\|^2 R E_\eta X = R(X \otimes |\eta\rangle \langle \eta|) E_\eta,
\] (1.8)
for Hilbert space operators \(R \in B(H \otimes h; H)\) and \(X \in B(H)\) and vectors \(\eta \in h\).

**Proposition 1.4.** Let \(A\) be a \(C^*\)-algebra and let \(B = B(h)\) and \(K = K(h)\) for a Hilbert space \(h\). Then, in any faithful nondegenerate representation of \(A\) (such as its universal representation),
\[
A \tilde{\otimes} K \subset \tilde{A} \otimes_M B,
\]
for the induced concrete realisations of \(\tilde{A}\) and \(A \tilde{\otimes} K\). Moreover, if \(\psi \in \text{Lin} \, CP_{\beta}(A; \tilde{C})\) for another \(C^*\)-algebra \(C\), also faithfully and nondegenerately represented, then
\[
\psi \tilde{\otimes} \text{id}_K \subset \tilde{\psi} \otimes_M \text{id}_B.
\]
Proof. Let $T \in A \hat{\otimes} K$ and $\xi, \eta \in \mathfrak{h}$. First note that (1.8) implies that
\begin{equation}
\|\eta\|^2 E^\xi T E_\eta a = E^\xi T (a \otimes |\eta\rangle \langle \eta|) E_\eta, \quad a \in A; \tag{1.9}
\end{equation}
similarly,
\begin{equation}
\|\xi\|^2 a E^\xi T E_\eta = E^\xi (a \otimes |\xi\rangle \langle \xi|) T E_\eta, \quad a \in A; \tag{1.10}
\end{equation}
and so $E^\xi T E_\eta \in \tilde{A}$. Thus $T \in \tilde{A} \otimes_M B$. This proves that $A \hat{\otimes} K \subset \tilde{A} \otimes_M B$, and (1.9) and (1.10) now imply that the map
\begin{equation}
T \in A \hat{\otimes} K \mapsto E^\xi T E_\eta \in \tilde{A}
\end{equation}
is strictly continuous. Therefore the maps
\begin{align*}
E^\xi (\tilde{\psi} \otimes_M id_B)(\cdot)E_\eta &= \tilde{\psi}(E^\xi \cdot E_\eta) \quad \text{and} \quad E^\xi (\psi \otimes id_K)(\cdot)E_\eta
\end{align*}
are strictly continuous $A \hat{\otimes} K \to \tilde{C}$ and agree on the strictly dense subspace $A \otimes K$. They therefore agree on $A \hat{\otimes} K$ and the result follows. \hfill \Box

Remark. In the universal representation of $A$ we have the further compatibility relations,
\begin{equation}
\tilde{A} \otimes_M B(\mathfrak{h}) \subset \tilde{A} \otimes B(\mathfrak{h}) \quad \text{and} \quad \tilde{\psi} \otimes_M id_B(\mathfrak{h}) \subset \tilde{\psi} \otimes id_B(\mathfrak{h}).
\end{equation}

C*-algebras with character. The following notion plays an important role in the theory. Recall the notation (0.1).

Definition. A $\chi$-structure map on a C*-algebra with character $(A, \chi)$ is a linear map $\varphi : A \to B(\mathfrak{h})$, for some Hilbert space $\mathfrak{h}$, satisfying
\begin{equation}
\varphi(a^* b) = \varphi(a)^* \chi(b) + \chi(a)^* \varphi(b) + \varphi(a)^* \Delta \varphi(b), \tag{1.11}
\end{equation}
where $\Delta := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in B(\mathfrak{h})$ (no relation to coproducts).

The following result is established in [LS] 3.

Theorem 1.5. Every $\chi$-structure map $\varphi$ is implemented by a pair $(\pi, \xi)$ consisting of a *-homomorphism $\pi : A \to B(\mathfrak{h})$ and vector $\xi \in \mathfrak{h}$, that is $\varphi$ has block matrix form
\begin{equation}
\begin{bmatrix} [\xi] \\ \mathbb{I}_\mathfrak{h} \end{bmatrix} \nu(\cdot) \begin{bmatrix} [\xi] \\ \mathbb{I}_\mathfrak{h} \end{bmatrix} \quad \text{where} \quad \nu := \pi - \iota_\mathfrak{h} \circ \chi,
\end{equation}
in other words,
\begin{equation}
\begin{bmatrix} \gamma \\ \nu(\cdot) \xi \end{bmatrix} \begin{bmatrix} [\xi] \nu(\cdot) \\ \nu(\cdot) \xi \end{bmatrix}, \quad \text{where} \quad \gamma := \omega_\xi \circ \nu. \tag{1.12}
\end{equation}
Moreover $\varphi$ is necessarily strict, and $\pi$ is nondegenerate if and only if $\tilde{\pi}(1) = 0$.

Proof. The first part is Theorem A6 of [LS]. It implies that $\varphi$ is completely bounded and so, by Theorem [1.2] $\varphi$ is strict. After strict extension, the last part now follows by inspection. \hfill \Box

Remarks. It is easily seen that conversely, any map with such a block matrix representation is a $\chi$-structure map.

By the separate strict/\sigma-weak continuity of multiplication, it follows that if $\varphi$ is a $\chi$-structure map then $\tilde{\varphi}$ is a $\tilde{\chi}$-structure map and $\tilde{\varphi}$ is a $\tilde{\chi}$-structure map.

We shall need the following result in Section [6]

Lemma 1.6. Let $(A, \chi)$ be a C*-algebra with character. Then, for any functional $\gamma \in A^*$, if $\gamma$ is positive on $\text{Ker} \chi$ then $\gamma$ is positive on $\text{Ker} \tilde{\chi}$ and $\tilde{\gamma}$ is positive on $\text{Ker} \chi$. 

Proof. It suffices to prove that $A_+ \cap \text{Ker} \chi$ is strictly dense in $\widetilde{A}_+ \cap \text{Ker} \chi$ and σ-weakly dense in $\bar{A}_+ \cap \text{Ker} \chi$. Let $a \in \bar{A}_+ \cap \text{Ker} \chi$. The Kaplansky Density Theorem for multiplier algebras ([Lam], Proposition 1.4) implies that there is a bounded net $(c_i)_{i \in I}$ of selfadjoint elements in $A$ converging strictly to $a^{1/4}$. Set $a_i = b_i^* b_i$ where $b_i := c_i (c_i - \chi(c_i)) \in \text{Ker} \chi$.

Then $a_i \in A_+ \cap \text{Ker} \chi$ and separate strict continuity of multiplication on bounded subsets of $A$, and strictness of $\chi$, imply that $(a_i)_{i \in I}$ converges strictly to $a$. The ultraweak density of $A_+ \cap \text{Ker} \chi$ in $\bar{A}_+ \cap \text{Ker} \chi$ is proved similarly, by appealing to the standard Kaplansky Density Theorem for von Neumann algebras. \hfill \Box

2. Multiplier $C^*$-bialgebras

It is convenient to consider bialgebras in both the $C^*$- and $W^*$- categories and a universal enveloping operation linking the two.

Definition. A (multiplier) $C^*$-bialgebra is a $C^*$-algebra $B$ with coproduct, that is a nondegenerate $*$-homomorphism $\Delta : B \to B \otimes B$ satisfying the coassociativity conditions

$$(\text{id}_B \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_B) \circ \Delta.$$ 

A counit for $(B, \Delta)$ is a character $\epsilon$ on $B$ satisfying the counital property:

$$(\text{id}_B \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}_B) \circ \Delta = \text{id}_B.$$

Remarks. The above definitions extend those for unital $C^*$-bialgebras, for which $B = B$ and $B \otimes B = B \otimes B$. The strict extension of a coproduct is a unital $*$-homomorphism and the strict extension of a counit is a character on $\bar{B}$. Note however that, in general, $(\bar{B}, \widetilde{\Delta})$ is not itself a $C^*$-bialgebra as the inclusion $\bar{B} \otimes \bar{B} \subset B \otimes B$ is usually proper.

Examples of counital $C^*$-bialgebras include locally compact quantum groups in the universal setting ([Ku2]), in particular all coamenable locally compact quantum groups are included. If the assumptions on the coproduct $\Delta$ are weakened to it being completely positive, strict and preunital then the resulting structure is called a (multiplier) $C^*$-hyperbialgebra (cf. [ChV]).

Let $B$ be a $C^*$-bialgebra. The convolute of $\phi_1 \in \text{Lin} CP_\beta(B; \bar{A}_1)$ and $\phi_2 \in \text{Lin} CP_\beta(B; \bar{A}_2)$ for $C^*$-algebras $A_1$ and $A_2$ is defined by

$$\phi_1 \ast \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta \in \text{Lin} CP_\beta(B; A_1 \bar{\otimes} A_2).$$

We denote its strict extension by $\phi_1 \bar{\ast} \phi_2$. Associativity of both of these convolutions follows from associativity of $\bar{\otimes}$ and coassociativity of $\Delta$. For each $C^*$-algebra $A$ define a map

$$R_A : \text{Lin} CP_\beta(B; \bar{A}) \to CB_\beta(B; \bar{A} \bar{\otimes} A), \quad \phi \mapsto \text{id}_B \ast \phi = (\text{id}_B \otimes \phi) \circ \Delta.$$ 

In case $A = \mathbb{C}$, $\text{Lin} CP_\beta(B; \bar{A})$ is simply $B^*$ and we have

$$R_\mathbb{C}(\varphi_1 \ast \varphi_2) = R_\mathbb{C}\varphi_1 \circ R_\mathbb{C}\varphi_2, \quad \varphi_1, \varphi_2 \in B^*.$$ 

When $B$ is counital each $R_A$ has left-inverse

$$E_A : CB_\beta(B; \bar{A} \bar{\otimes} A) \to CB_\beta(B; \bar{A}), \quad \psi \mapsto (\epsilon \otimes \text{id}_B) \circ \psi. \quad (2.1)$$

Remarks. By the complete positivity and strictness of the coproduct $R_A(CP_\beta(B; \bar{A})) \subset CP_\beta(B; \bar{A} \bar{\otimes} A)$ for any $C^*$-algebra $A$. In particular, by ([Lam]),

$$R_K(h)(CB(B; B(h))) \subset \text{Lin} CP_\beta(B; \bar{A} \bar{\otimes} K(h)).$$
Note also that, when \( \varphi_1 \in CB(B; B(h_1)) \) and \( \varphi_2 \in CB(B; B(h_2)) \) for Hilbert spaces \( h_1 \) and \( h_2 \),
\[
\varphi_1 \ast \varphi_2 \in CB(B; B(h_1 \otimes h_2)).
\]

For convenience we summarise useful properties of the \( R \)-maps next.

**Proposition 2.1.** Let \( B \) be a \( C^\star \)-bialgebra and let \( A \) be a \( C^\star \)-algebra. Then \( R_A \)

is a completely contractive map into \( CB_\beta(B; B \otimes A) \) with image in the subspace \( \Lin CP_\beta(B; B \otimes A) \) and, after strict extension, \( R_C \) is furthermore a homomorphism of Banach algebras: \( (B^\star, \ast) \cong ((B)_\beta^\star, \tilde{\ast}) \to CB_\beta(B) \). When \( B \) is counital, \( R_A \) is completely isometric with completely contractive left-inverse \( E_A \) and \( R_C \) is furthermore a unital algebra morphism.

We now turn briefly to the \( W^\star \)-category.

**Definition.** A von Neumann bialgebra is a von Neumann algebra \( M \) with coproduct, that is a normal unital *-homomorphism \( \Delta : M \to M \otimes M \) which is coassociative:

\[
(id_M \otimes \Delta) \circ \Delta = (\Delta \otimes id_M) \circ \Delta.
\]

A counit for \( (M, \Delta) \) is a normal character \( \epsilon \) on \( M \) satisfying

\[
(id_M \otimes \epsilon) \circ \Delta = (\epsilon \otimes id_M) \circ \Delta = id_M.
\]

Convolution in this category is straightforward. Let \( \phi_1 \in CB_\sigma(M; Z_1) \) and \( \phi_2 \in CB_\sigma(M; Z_2) \) for \( \sigma \)-weakly closed concrete operator spaces \( Z_1, Z_2 \) and \( Z_2 \), then

\[
\phi_1 \ast \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta \in CB_\sigma(M; Z_1 \otimes Z_2),
\]

so that we may define a map

\[
R_\sigma^C : CB_\sigma(M; Z) \to CB_\sigma(M; M \otimes Z), \quad \phi \mapsto id_M \ast \phi = (id_M \otimes \phi) \circ \Delta.
\]

In particular,

\[
R_\sigma^C(\phi_1 \ast \phi_2) = R_\sigma^C(\phi_1) \circ R_\sigma^C(\phi_2) \in CB_\sigma(M), \text{ for } \phi_1, \phi_2 \in M_\ast,
\]

When \( M \) is counital \( R_\sigma^C \) has left-inverse

\[
E_\sigma^C : CB_\sigma(M; M \otimes Z) \to CB_\sigma(M; Z), \quad \psi \mapsto (\epsilon \otimes id_Z) \circ \psi.
\]

**Proposition 2.2.** Let \( (B, \Delta) \) be a \( C^\star \)-bialgebra. Then \( (\overline{B}, \overline{\Delta}) \) is a von Neumann bialgebra. Moreover, if \( \epsilon \) is a counit for \( B \) then \( \overline{\epsilon} \) is a counit for \( \overline{B} \).

**Proof.** The map \( \overline{\Delta} \) is a normal, unital \( * \)-homomorphism and the normal maps

\[
(id_{\overline{B}} \otimes \overline{\Delta}) \circ \overline{\Delta} \text{ and } (\overline{\Delta} \otimes id_{\overline{B}}) \circ \overline{\Delta}
\]

give agree on \( B \), which is \( \sigma \)-weakly dense in \( \overline{B} \), and so coincide. In the counital case, \( \overline{\epsilon} \)

is a normal character on \( \overline{B} \) and the normal maps

\[
(id_{\overline{B}} \otimes \overline{\epsilon}) \circ \overline{\Delta}, \quad (\overline{\epsilon} \otimes id_{\overline{B}}) \circ \overline{\Delta} \text{ and } id_{\overline{B}}
\]

agree on \( B \) and so coincide.

\[ \square \]

Naturally, we refer to \( (\overline{B}, \overline{\Delta}) \), respectively \( (\overline{B}, \overline{\Delta}, \overline{\epsilon}) \) as the *universal enveloping von Neumann bialgebra* (resp. counital von Neumann bialgebra) of \( B \).

**Remark.** The two forms of \( R \)-map enjoy an easy compatibility: if \( \phi \in \Lin CP_\beta(B; \overline{A}) \)

for a \( C^\star \)-algebra \( A \) then

\[
\overline{R_A \phi} = R_\sigma^C \overline{\phi},
\]

and similarly for the \( E \) maps in the counital case.

*From now on* we shall denote all maps of the form \( R_\sigma^C \), respectively \( E_\sigma^C \), by \( R_\sigma \),

respectively \( E_\sigma \), and similarly abbreviate all maps of the form \( R_A \) and \( E_A \) to \( R \) and \( E \).
3. Quantum stochastics

Fix now, and for the rest of the paper, a complex Hilbert space $k$ referred to as the noise dimension space. For $c \in k$ define $\hat{c} := (\frac{1}{c}) \in \hat{k}$; and for any function $g$ with values in $k$ let $\hat{g}$ denote the corresponding function with values in $\hat{k}$, defined by $\hat{g}(s) := g(s)$. Let $F$ denote the symmetric Fock space over $L^2(\mathbb{R}_+; k)$, let $S$ denote the linear span of $\{d_{[0,t]}^f : d \in k, t \in \mathbb{R}_+ \}$ in $L^2(\mathbb{R}_+; k)$ (for purposes of evaluating, we always take these right-continuous versions) and let $E$ denote the linear span of $\{\varepsilon(g) : g \in S \}$ in $F$, where $\varepsilon(g)$ denotes the exponential vector $((n!)^{-\frac{1}{2}}g^{\otimes n})_{n \geq 0}$.

(There will be no danger of confusion with the inverse of an $R$-map!) Also define

$$e_0 := \left(\frac{1}{0}\right) \in \hat{k} \text{ and } \Delta^{QS} := P_{(0)\oplus k} = \begin{bmatrix} 0 & \hat{I}_k \end{bmatrix} \in B(\hat{k}). \quad (3.1)$$

Quantum stochastic processes, differential equations and cocycles. A detailed summary of the relevant results from QS analysis ([LW1–4], [LS2]) is given in [LS3]. We shall therefore be brief here.

For operator spaces $V$ and $W$, with $W$ concrete, $\mathbb{P}(V \rightarrow W)$ denotes the space of adapted processes $k = (k_t)_{t \geq 0}$, thus $k_t \in L(E; L(V \otimes_M |F\rangle))$ for $t \in \mathbb{R}_+$, written $\varepsilon \mapsto k_{t,\varepsilon}$. As in [LS3], we abbreviate to $\mathbb{P}_*(V)$ when $W = \mathbb{C}$. Its associated maps $\kappa_t^{f,g} : V \rightarrow W$ (for $f, g \in S, s, t \in \mathbb{R}_+$) are defined by

$$\kappa_t^{f,g}(x) = (\text{id}_W \otimes_M (\varepsilon'))k_{t,\varepsilon}(x), \quad x \in V,$$

where $\varepsilon = \varepsilon(f_{[0,t]})$ and $\varepsilon' = \varepsilon(f_{[0,t]})$. For us here the pair $(V, W)$ will be either $(M, \mathbb{C})$ or $(M, M)$ for a von Neumann algebra $M$, or $(A, C)$ or $(\hat{A}, C)$ for a $C^*$-algebra $A$ with multiplier algebra $\hat{A}$. Thus $W \otimes_M |F\rangle$ is either $|F\rangle$ or $M \otimes |F\rangle$. The process $k$ is weakly initial space bounded if each $\kappa_t^{f,g}$ is bounded, and weakly regular if further $\sup \{||\kappa_t^{f,g}|| : s \in [0, T]\} < \infty$, for all $T \geq 0$. Here ‘column-boundedness’ usually obtains: $k_{t,\varepsilon} \in B(V; W \otimes_M |F\rangle)$ or $CB(V; W \otimes_M |F\rangle)$, for each $t \in \mathbb{R}_+$ and $\varepsilon \in E$.

For von Neumann algebras $M$ and $N$, a process $k \in \mathbb{P}(M \rightarrow N)$ is called normal if each $\kappa_t^{f,g}$ is normal. It follows that if $k \in \mathbb{P}(M \rightarrow N)$ is bounded (meaning that each $k_t$ is bounded) and normal then each map $k_t$ is normal $M \rightarrow N \otimes B(\mathcal{F})$. Note that, by Theorem [12], any completely bounded process $l \in \mathbb{P}_*(A)$ on a $C^*$-algebra $A$ is necessarily strict in the sense that each map $l_t : A \rightarrow B(\mathcal{F})$ is strict.

For $\phi \in L(\hat{k}; CB(V; V \otimes_M |\hat{k}\rangle))$ and $\kappa \in CB(V; W)$, where both $V$ and $W$ are concrete operator spaces, $k^{\phi,\kappa}$ denotes the unique weakly regular process $k \in \mathbb{P}(V \rightarrow W)$ satisfying the quantum stochastic differential equation

$$dk_t = k_t \cdot d\Lambda_\phi(t), \quad k_0 = \iota_\mathcal{F} \circ \kappa, \quad (3.2)$$

where $\iota_\mathcal{F}$ denotes the ampliation $W \rightarrow W \otimes_M B(\mathcal{F})$ and $x \mapsto x \otimes \iota_\mathcal{F}$. The solution is given by

$$k_{t,\varepsilon} = \sum_{n \geq 0} \Lambda^n_{t,\varepsilon} \circ (\kappa \circ \phi_n)$$

where $\phi_n$ is an $n$-fold composition of matrix liftings of $\phi$ and the sum is norm-convergent in $CB(V; W \otimes_M |\mathcal{F}\rangle)$.

When $W = V$ and $\kappa = \text{id}_V$, $k$ is a weak quantum stochastic cocycle on $V$ (denoted $k^\phi$), that is it satisfies $k_0 = \iota_\mathcal{F}$ and for $s, t \in \mathbb{R}_+$ and $f, g \in S$,

$$k_0^{f,g} = \text{id}_V, \quad k_{s+t}^{f,g} = k_s^{f,g} \circ k_t^{S_gf, S_gg} \quad (f, g \in S, s, t \in \mathbb{R}_+) \quad (3.3)$$

where $(S_t)_{t \geq 0}$ is the isometric semigroup of right-shifts on $L^2(\mathbb{R}_+; k)$. Let $(\sigma_t)_{t \geq 0}$ denote the induced endomorphism semigroup on $B(\mathcal{F})$, amplituated to $W \otimes_M B(\mathcal{F})$. Then, when $k$ is a completely bounded process, the cocycle relation simplifies to

$$k_{s+t} = k_s \cdot \sigma_s \circ k_t,$$
where the extended composition notation (which we do not need to go into here) is explained in \([\text{LS}_3]\).

4. Coalgebraic quantum stochastic differential equations

For this section we fix a C*-bialgebra \(\mathcal{B}\), which we do not assume to be counital, and consider the coalgebraic quantum stochastic differential equation

\[
dLe = l_t \ast d\Lambda_\varphi(t), \quad l_0 = \iota_x \circ \eta,
\]

where \(\varphi \in SL(\hat{k}, k; \mathcal{B}^*)\) and \(\eta \in \mathcal{B}^*\).

**Definition.** By a form solution of (4.1) is meant a family \(\{\lambda_t^{f,g} | f, g \in \mathcal{S}, t \in \mathbb{R}_+\}\) in \(\mathcal{B}^*\) satisfying

(i) the map \(s \mapsto (\lambda_s^{f,g} \ast \varphi_{\tilde{f}(s),\tilde{g}(s)})\) is locally integrable;

(ii) \(\lambda_t^{f,g}(b) - e^{(f,g)}\eta(b) = \int_0^t ds \lambda_s^{f,g} \ast \varphi_{\tilde{f}(s),\tilde{g}(s)}(b)\)

for all \(f, g \in \mathcal{S}\) and \(b \in \mathcal{B}\).

**Remarks.** Let \(f, g \in \mathcal{S}\) and \(b \in \mathcal{B}\). By automatic strictness of bounded linear functionals on \(\mathcal{B}\), (i) makes sense. By (ii) it follows that \(\lambda_t^{f,g}(b)\) is continuous in \(t\), and so is locally bounded. Therefore, by the Banach-Steinhaus Theorem, \(\lambda_t^{f,g}\) is locally bounded in \(t\) and (ii) therefore implies that (i) refines to

(i)' the map \(s \mapsto \lambda_s^{f,g}\) is continuous,

which in turn implies that (ii) refines to

(ii)' \(\lambda_s^{f,g} - e^{(f,g)}\eta = \int_0^s ds \lambda_s^{f,g} \ast \varphi_{\tilde{f}(s),\tilde{g}(s)}\),

the integrand being piecewise norm-continuous \(\mathbb{R}_+ \to \mathcal{B}^*\).

The following automatic strictness property is needed to establish uniqueness for form solutions.

**Lemma 4.1.** Let \(\varphi \in SL(\hat{k}, k; \mathcal{B}^*)\) and \(\eta \in \mathcal{B}^*\). Then every form solution \(\{\lambda_t^{f,g} | f, g \in \mathcal{S}, t \in \mathbb{R}_+\}\) of (4.1) is strict in the sense that it satisfies

(ii) \(\lambda_t^{f,g} - e^{(f,g)}\eta = \int_0^t ds \lambda_s^{f,g} \ast \varphi_{\tilde{f}(s),\tilde{g}(s)}\) for all \(f, g \in \mathcal{S}\) and \(t \in \mathbb{R}_+\),

where \(\lambda_t^{f,g} := (\lambda_t^{f,g})^\sim\).

Note that the integrand in (ii)' is piecewise continuous \(\mathbb{R}_+ \to (\mathcal{B})_0^*\).

**Proof.** Let \(\{\lambda_t^{f,g} | f, g \in \mathcal{S}, t \in \mathbb{R}_+\}\) be a form solution of (4.1) and let \(f, g \in \mathcal{S}\) and \(t \in \mathbb{R}_+\). Define bounded linear functionals

\[
\Phi := \int_0^t ds \lambda_s^{f,g} \ast \varphi_{\tilde{f}(s),\tilde{g}(s)} \quad \text{and} \quad \Psi := \int_0^t ds \lambda_s^{f,g} \ast \varphi_{\tilde{f}(s),\tilde{g}(s)}
\]
on \(\mathcal{B}\) and \(\mathcal{B}\) respectively. Note that each Riemann approximant \(\Psi_P\) of \(\Phi\) equals \((\Phi_P)^\sim\) where \(\Phi_P\) is the corresponding Riemann approximant of \(\Phi\). The extension map \(\mathcal{B}^* \to (\mathcal{B})_0^*\) is (isometric and thus) continuous therefore

\[
\Psi = \lim \Psi_P = (\lim \Phi_P)^\sim = \Phi^\sim.
\]

Since \(\Phi = \lambda_t^{f,g} - e^{(f,g)}\eta\) it follows that \(\Psi = (\lambda_t^{f,g})^\sim - e^{(f,g)}\eta\). Thus form solution is strict. \(\square\)

With this we have uniqueness as well as existence for form solutions.

**Theorem 4.2.** Let \(\varphi \in SL(\hat{k}, k; \mathcal{B}^*)\) and \(\eta \in \mathcal{B}^*\), for a C*-bialgebra \(\mathcal{B}\). Then the quantum stochastic differential equation (4.1) has a unique form solution.
Proof. For each \(c,d \in k\) let \((p_{t^c,d}^{c,d})_{t \geq 0}\) denote the norm-continuous one-parameter semigroup generated by \(\varphi_{c,d} \in B^*\), in the unitisation of the Banach algebra \((B^*, \ast)\). For \(f, g \in \mathbb{S}\) and \(t \in \mathbb{R}_+\), the prescription
\[
\lambda^f_t \triangleq \eta \ast p_{t_1-t_0}^{c_1,d_0} \ast \cdots \ast p_{t_n-1-t_{n-1}}^{c_n,d_n}
\]
(in which \(t_0 = 0, t_{n+1} = t, \{t_1 < \cdots < t_n\}\) is the (possibly empty) set of points in \([0,t]\) where \(f\) or \(g\) is discontinuous and \((c_i, d_i) = (f(t_i), g(t_i))\) for \(i = 0, \cdots, n\), defines an element \(\lambda_t^f g\) of \(B^*\). It is easily verified that the resulting family \(\{\lambda_t^f g \mid f, g \in \mathbb{S}, t \in \mathbb{R}_+\}\) is a form solution of (4.1).

Suppose now that \(\mu\) is the difference of two form solutions, and let \(f, g \in \mathbb{S}\) and \(t \in \mathbb{R}_+\). Then Lemma 4.1 yields the identity
\[
(\mu_t^f g)^\ast = \int_0^t ds \mu_s^f g \varphi(f(s), g(s))
\]
which may be iterated. Estimating after repeated iteration (and using the isometry \(F \ast \varphi, \mu \ast \varphi \in \mathbb{R}_+\)) we have
\[
\|\mu_t^f g\| \leq \frac{t^n}{n!} \sup_{x \in [0,t]} \|\mu_s^f g\| \max \{\|\varphi_{c,d}\| : c \in \text{Ran} f, d \in \text{Ran} g\}^n
\]
which tends to 0 as \(n \to \infty\). Thus \(\mu = 0\), proving uniqueness. \(\square\)

We now show how stronger forms of solution are obtained when the coefficient of the quantum stochastic differential equation is a bounded mapping rather than just a form. Below the following natural inclusions are invoked:
\[
B(B; B(h; h')) \cong B(h', h; B^*) \subset SL(h', h; B^*),
\]
\[
\varphi \mapsto ((\zeta, \eta) \mapsto \varphi_{\zeta, \eta} := \langle \zeta, \varphi(\cdot) \eta \rangle)
\]
for Hilbert spaces \(h\) and \(h'\). Recall the notation for the solution of a QS differential equation introduced above equation (3.2).

**Theorem 4.3.** Let \(\varphi \in CB(B; B(\hat{k}))\) and \(\eta \in B^*\), for a \(C^*-\)bialgebra \(B\). Set
\[
\tilde{\varphi}_{\varphi, \eta} := k \tilde{\varphi}_{\varphi, \eta} \text{ and } \tilde{\eta} := k \tilde{\eta} \text{ where } \phi := R \varphi.
\]
Thus \(\tilde{\varphi} \in CB_{\beta}(B; B \otimes K(\hat{k}))\) and \(\tilde{\eta} = \sigma R \varphi \in CB_{\sigma}(B; B \otimes B(\hat{k}))\).

(a) Abbreviating \(\tilde{\varphi}_{\varphi, \eta}\) to \(\tilde{\varphi}\) and \(\tilde{\eta} = \tilde{\varphi} \in \mathbb{S}\) we have, for all \(\varepsilon \in \mathcal{E}\) and \(t \in \mathbb{R}_+\),
\begin{itemize}
  \item[(i)] \(\tilde{\varphi}_{\varepsilon, \eta} \in \text{CB}(B; |\varphi|)\);
  \item[(ii)] \(\tilde{\varphi}_{\varepsilon, \eta} = \tilde{\varphi}_{\varepsilon, \eta} |B|\);
  \item[(iii)] \(\tilde{\varphi}_{\varepsilon, \eta} \in \text{CB}(B; |\varphi|)\).
\end{itemize}

(b) For all \(f, g \in \mathbb{S}\) and \(t \in \mathbb{R}_+\), setting
\[
\tilde{\lambda}_{\varepsilon}^f g := \omega_{\varepsilon(f(0, t), \varepsilon(g(0, t)))} \tilde{\varphi}_{\varepsilon, \eta} \text{ and } \kappa_{\varepsilon}^f g := E_{\varepsilon(f(0, t))} \tilde{\varphi}_{\varepsilon, \eta} E_{\varepsilon(g(0, t))},
\]
\[
\begin{align*}
(i) & \{\tilde{\lambda}_{\varepsilon}^f g | f, g \in \mathbb{S}, t \in \mathbb{R}_+\} \text{ is the unique form solution of (4.1)};
(ii) & \tilde{\lambda}_{\varepsilon}^f g = \kappa_{\varepsilon}^f g \text{ and } R \sigma \tilde{\lambda}_{\varepsilon}^f g = (R \sigma \tilde{\varphi}) \circ \kappa_{\varepsilon}^f g.
\end{align*}
\]

**Proof.** Fix \(\varepsilon \in \mathcal{E}\) and \(t \geq 0\). By linearity we may assume that \(\varepsilon = \varepsilon(g)\).

(a) (i) The operator \(\tilde{\varphi}_{\varepsilon, \eta} \) is a norm-convergent sum, in \(\text{CB}(B; |\varphi|)\), of terms of the form \(\Lambda_{\varepsilon}^n \circ (\eta \varphi \tilde{\varphi}^n)\) \((n \in \mathbb{Z}_+)\), and each map \(\eta \varphi \tilde{\varphi}^n\) is \(\sigma\)-weakly continuous. Since \(\text{CB}_{\varphi}(B; |\varphi|)\) is a norm-closed subspace of \(\text{CB}(B; |\varphi|)\), it remains only to show that the bounded operator \(\Lambda_{\varepsilon}^n : B(\hat{k}^\otimes n) \to |\varphi|\) is \(\sigma\)-weakly continuous. By the
Krein-Smulian Theorem it suffices to prove this on bounded sets. This follows from the following identity for multiple QS integrals:

\[
\langle \varepsilon(f), \Lambda^\pi_n(A)(g) \rangle = \int_{\Delta^n} ds \left( \pi_f(s), A\pi_g(s) \right) e^{(f,g)}, \quad A \in B(\hat{\mathcal{O}}^{\otimes n}),
\]

since the integrand is a step function on \(\Delta^n := \{ s \in \mathbb{R}^n : 0 \leq s_1 \leq \cdots \leq s_n \leq t \} \).

(ii) Since \( \tilde{t}_t \) is a norm-convergent sum, in \( CB(\mathcal{B}; |\mathcal{F}|) \), of terms of the form \( \Lambda^\pi_t \circ (\eta \times \varphi)^n \), this follows from (i) and the identity \( \eta \tilde{\varphi} \varphi^n |_{\tilde{B}} = \tilde{\varphi} \varphi^n (n \in \mathbb{Z}_+) \).

(iii) This follows from (i) and (ii) since, for any map \( \alpha \in CB_\sigma(\mathcal{B}; |\mathcal{F}|) \), \( \alpha|_{\tilde{B}} = \alpha|_{\tilde{B}} \) (see (1.3)).

(b) (i) This follows from the identity

\[
\omega_{\varepsilon,\varepsilon'} \circ \kappa \phi \circ \nu_{\varepsilon,\varepsilon'} = \omega_{\varepsilon,\varepsilon'} \circ \kappa \phi \circ \nu_{\varepsilon,\varepsilon'}, \quad \varepsilon,\varepsilon' \in \mathcal{E}, c, d \in k, s \in \mathbb{R}_+,
\]

where \( \nu_{\varepsilon,\varepsilon'} := (\text{id} \otimes \omega_{\varepsilon,\varepsilon'}) \circ \phi \).

(ii) The first identity expresses the general relation between \( \kappa \phi \) and \( \kappa \phi \) (LS2). By (i), it follows from the proof of Theorem 4.2 that \( l^{\varepsilon,\eta}_t \) may be written in the form

\[
\eta \varphi \varphi^n |_{\tilde{B}} \varphi^{n-1} \varphi^{n-2} \cdots \varphi_1 \varphi_2 \cdots \varphi_{n+1} \varphi_{n+2} = \eta \varphi \varphi^{n+1} |_{\tilde{B}} \varphi^{n+2} \cdots \varphi_{n+1} \varphi_{n+2} \cdots \varphi_1 \varphi_2 \cdots \varphi_{n+1} \varphi_{n+2},
\]

where \( \nu_{\varepsilon,\varepsilon'} \) denotes the normal extension of \( \nu_{\varepsilon,\varepsilon'} \). Thus \( R_\sigma l^{\varepsilon,\eta}_t \) equals

\[
R_\sigma \nu_{\varepsilon,\varepsilon'} \circ \nu_{\varepsilon,\varepsilon'} \circ \cdots \circ \nu_{\varepsilon,\varepsilon'},
\]

where \( \nu_{\varepsilon,\varepsilon'} := \exp t(\nu_{\varepsilon,\varepsilon'}) = R_\sigma \nu_{\varepsilon,\varepsilon'} \). (ii) therefore now follows from the semigroup representation of the standard QS cocycle \( k \phi \) (LW2). \( \square \)

Notation. Setting \( l^{\varepsilon,\eta}_t = \tilde{l}^{\varepsilon,\eta}_t |_{\mathcal{B}} \) \( (t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}) \) defines a process \( l^{\varepsilon,\eta} \in \mathbb{P}_*(\mathcal{B}) \), which we denote by \( l^{\varepsilon} \) when \( \mathcal{B} \) is unital and \( \eta = \varepsilon \). This extends the notation introduced in [LS4] for the unital case.

Remarks. (i) In view of the identity

\[
(k \phi \varphi \phi)_{z(f),z(s)} = l^{\varepsilon,\eta}_t \varphi^{n+1} \varphi_1 \varphi_2 \cdots \varphi_{n+1} \varphi_{n+2},
\]

\( l^{\varepsilon,\eta} \) satisfies

\[
l^{\varepsilon} = l^{\varepsilon} + \int_0^t l^{\varepsilon} d\Lambda(s), \quad t \in \mathbb{R}_+.
\]

In this sense, \( l^{\varepsilon,\eta} \) is a strong solution of (4.1).

(ii) The proper hypothesis for the theorem above is \( \varphi \in L(\mathcal{B}; CB(\mathcal{B}; |\mathcal{B}^\ast|)) \), in which case, \( \phi \in L(\mathcal{B}; CB_\sigma(\mathcal{B}; L(\mathcal{B}; \mathcal{B} \otimes |\mathcal{B}^\ast|))) \) and \( \phi \subseteq R_\sigma \phi \) \( (c \in k) \). However, we have no need of this generality here.

Remark. Only the coalgebraic structure of \( \mathcal{B} \) has been used so far, not its algebraic structure.

We end this section by noting some correspondence between convolution processes and associated standard processes. Recall the notation for QSDE solutions introduced above equation (3.3).

**Proposition 4.4.** Let \( l = l^{\varphi} \) and \( \overline{l} = \overline{l}^{\varphi} \), where \( \varphi \in CB(\mathcal{B}; |\mathcal{B}|) \) for a unital \( C^* \)-bialgebra \( \mathcal{B} \), and set \( k = k^{\varphi} \) where \( \phi := R_\sigma \varphi \). Then

(a) \( \overline{l} \) is unital if and only if \( k \) is.
(b) $l$ is completely bounded (respectively, completely positive or *-homomorphic) if and only if $k$ is, in which case

$$k_t = R_\sigma \bar{I}_t, \quad \bar{I}_t = E_\sigma k_t \text{ and } \|l_t\|_{cb} = \|k_t\|_{cb}, \quad t \in \mathbb{R}^+.$$  

Proof. In the notations (12), Theorem (b)(i) implies that,

$$\lambda_t^{f,g} = E_\sigma k_t^{f,g} \text{ and } k_t^{f,g} = R_\sigma \lambda_t^{f,g}, \quad f, g \in \mathcal{S}, t \in \mathbb{R}^+.$$  

Thus (a) follows from the unitality of the maps $\tau$ and $\lambda$. Moreover, if $k$ is completely bounded then, since

$$\omega_{\varepsilon,\varepsilon'} \circ \tau_t^{f,g} = \lambda_t^{f,g} = E_\sigma k_t^{f,g} = \omega_{\varepsilon,\varepsilon} \circ \sigma_\alpha k_t,$$$$

where $\varepsilon = \varepsilon(f_{[0,t]}, t \in \mathbb{R}^+$ and $\varepsilon' = \varepsilon(f_{[t,t]}', t \in \mathbb{R}^+$, for all $f, f' \in \mathcal{S}$ and $t \in \mathbb{R}^+$, it follows that $l_t = E_\sigma k_t (t \in \mathbb{R}^+)$, in particular $l$ is completely bounded. Conversely, if $l$ is completely bounded then $\bar{I}_t = I_t (t \in \mathbb{R}^+)$ and

$$(\text{id}_E \otimes \omega_{\varepsilon,\varepsilon'}) \circ R_\sigma \bar{I}_t = R_\sigma \lambda_t^{f,g} = k_t^{f',g} = (\text{id}_E \otimes \omega_{\varepsilon,\varepsilon'}) \circ k_t$$

for all $f, f' \in \mathcal{S}$ and $t \in \mathbb{R}^+$, so $k_t = R_\sigma \bar{I}_t (t \in \mathbb{R}^+)$, therefore $k$ is completely bounded. The rest follows from the fact that $\lambda$ and $\tau \otimes \text{id}_{B(F)}$ are *-homomorphisms. \qed

5. Quantum stochastic convolution cocycles

For this section we fix a counital $C^*$-bialgebra $B$.

Definition. A family $\{\lambda_{t}^{f,g} \mid f, g \in \mathcal{S}, t \in \mathbb{R}^+\}$ in $B^*$ is a form quantum stochastic cocycle on $B$ if it satisfies

$$\lambda_0^{f,g} = \varepsilon, \quad \lambda_{s+t}^{f,g} = \lambda_s^{f,g} \ast \lambda_t^{S^*_s f, S^*_g}, \quad f, g \in \mathcal{S}, s, t \in \mathbb{R}^+,$$$$

where $(S_t)_{t \geq 0}$ is the isometric shift semigroup on $L^2(\mathbb{R}^+; k)$.

Note that for such a cocycle

$$p_t^{c,d} := \lambda_t^{c[0,t], d[0,t]}$$

defines one-parameter semigroups $\{p_t^{c,d} \}_{c,d \in \mathbb{K}}$ in the unital Banach algebra $(B^*, \ast)$ which we refer to as the associated convolution semigroups of the cocycle. The cocycle is said to be Markov-regular if each of its associated semigroups is norm-continuous.

Definition. A process $l \in \mathcal{P}_*(B)$ is a (weak) QS convolution cocycle on $B$ if its associated family $\{\omega_{\varepsilon(f_{[0,t]}, t \in \mathbb{R}^+)} \circ l_t | f, f' \in \mathcal{S}, t \in \mathbb{R}^+\}$ is a form QS cocycle on $B$.

Remarks. (i) Let $l \in \mathcal{P}_*(B)$ be a completely bounded QS convolution cocycle on $B$. Then $l$ is a QS convolution cocycle in the full sense:

$$l_{s+t} = l_s \ast (\sigma_s \circ l_t), \quad l_0 = l_F \circ \varepsilon, \quad s, t \in \mathbb{R}^+,$$$$

where $(\sigma_s)_{s \geq 0}$ is the injective *-homomorphic semigroup of right shifts on $B(F)$ and the identification

$$B(F) = B(F_{[0,s]}) \otimes \sigma_s (B(F))$$

is invoked.

(ii) It follows from the proof of Theorem [12] that, for $\varphi \in SL(\hat{\mathbb{K}}; \hat{\mathbb{k}}; B^*)$, the unique form solution of the QS differential equation

$$dl_t = l_t \ast d\Lambda_\varphi (t), \quad l_0 = l_F \circ \varepsilon,$$$$

is a Markov-regular weak QS convolution cocycle on $B$. 

\[5.1\]
(iii) Form-cocycles may equally be defined on $\mathfrak{B}$ and $\mathfrak{E}$ with the requirement of strictness/normality, and $\epsilon$ replaced by $\overline{\epsilon}$, respectively $\overline{\epsilon}$. From the correspondence (L3), it follows that any one of these uniquely determines the others.

Our essential strategy for analysing QS convolution cocycles is to work in the universal enveloping von Neumann bialgebra $\mathfrak{E}$ and, by transferring between convolution and standard QS cocycles using the maps $R_\sigma$ and $E_\tau$, to apply the theory developed in [LW$_{1-4}$], and [LS$_2$].

We first establish a converse to Remark (ii) above.

**Proposition 5.1.** Let $l$ be a Markov-regular, completely positive, contractive quantum stochastic convolution cocycle on $\mathfrak{B}$. Then there is a unique map $\varphi \in CB(\mathfrak{B}; B(k))$ such that $l = l^\varphi$.

*Proof.* Set $k := (R_\sigma \Xi)$, where $\Xi_t := \bar{\Xi}_t (t \geq 0)$. Then $k$ is a standard quantum stochastic cocycle on $\mathfrak{B}$ which is Markov-regular, completely positive, contractive and normal. Therefore, by Theorem 5.10 of [LW$_2$] and Theorem 5.3 of [LW$_1$], $k$ has a stochastic generator $\hat{\varphi} \in CB_0(\mathfrak{B}; B(\mathfrak{B})), \sigma$ for $c, d \in k$, its associated semigroup $P_{c,d}$ has generator $\pi(\hat{\varphi}_{c,d}) \circ \hat{\varphi}$. Set $\Pi := E_\tau \Xi \in CB_0(\mathfrak{B}; B(k))$. Since $\Xi_t = E_\tau \Xi_t$, the associated convolution semigroup $p_{c,d}$ of $\Pi$ has generating functional $\tau \circ (\hat{\varphi}_{c,d}) \circ \hat{\varphi} = \hat{\varphi}_{c,d} \circ \hat{\varphi}$

which equals the generating functional of the associated convolution semigroup of the QS convolution cocycle $\bar{\Pi}$. It follows that $\bar{\Pi} = \bar{\Pi}^\varphi$ where $\varphi := \Pi_{|\mathfrak{B}}$ and so $l = l^\varphi$. \hfill $\Box$

We refer to $\varphi$ as the *stochastic generator* of the QS convolution cocycle $l$. The proof of the next result now proceeds similarly to those of Theorems 5.1 and 6.1 in [LS$_3$].

**Theorem 5.2.** Let $l$ be Markov-regular quantum stochastic convolution cocycle on $\mathfrak{B}$. Then the following equivalences hold:

(a) (i) $l$ is completely positive and contractive;
(ii) there is $\psi \in CP(\mathfrak{B}; B(k))$ and $\zeta \in \mathfrak{k}$ such that $l = l^\varphi$ where

$$\varphi = \psi - \epsilon(\cdot) \left( \Delta_{\text{QS}} + |\zeta\rangle \langle \zeta| + |e_0\rangle \langle e_0| \right)$$

and $\tilde{\varphi}(1) \leq 0$.

In this case, $l$ is preunital if and only if $\tilde{\varphi}(1) = 0$.

(b) (i) $l$ is completely positive and preunital;
(ii) there is a nondegenerate $*$-representation $(\rho, K)$ of $\mathfrak{B}$ (as $C^*$-algebra), an isometry $D \in B(k; K)$ and vector $\xi \in K$ such that $l = l^\varphi$ where

$$\varphi = \left[ \begin{matrix} \xi \\ D^\ast \end{matrix} \right] \nu(\cdot) \left[ \begin{matrix} \xi \\ D \end{matrix} \right]$$

for $\nu := \rho - \iota_K \circ \epsilon$,

(c) (i) $l$ is $*$-homomorphic;
(ii) $l = l^\theta$ where $\theta$ is an $\epsilon$-structure map;
(iii) there is a $*$-homomorphism $\pi : \mathfrak{B} \rightarrow B(k)$ and vector $c \in k$ such that

$$\theta = \frac{\left[ \begin{matrix} c \\ I_k \end{matrix} \right] \nu(\cdot) \left[ \begin{matrix} c \\ I_k \end{matrix} \right]}{\left[ \begin{matrix} c \\ I_k \end{matrix} \right]}$$

where $\nu := \pi - \iota_k \circ \epsilon$.

In this case, the cocycle $l$ is nondegenerate if and only if the $*$-representation $\pi$ is.
Proof. In case (i) of (a), (b) and (c) we let ϕ be the stochastic generator of l, let \( \mathcal{I} = \mathcal{I}^\circ = (\mathcal{I})_{t \geq 0} \), and set \( k = k^\circ \) where \( \overline{\phi} = R_\sigma \phi \in CB_\sigma (\mathcal{B}, \mathcal{B} \otimes B(k)) \). Thus \( k \) is a Markov-regular standard QS cocycle on \( \mathcal{B} \) and \( \overline{\phi} = E_\sigma \phi \).

(a) If (i) holds then \( k \) is completely positive and contractive, by Proposition 4.3 and normal. Therefore, by Theorem 5.10 of [LW_2], there is a map \( \Phi \in CP_\sigma (\mathcal{B}, \mathcal{B} \otimes B(k)) \) and operator \( Z \in \mathcal{B} \otimes (\mathcal{H}_k) \) such that

\[
\overline{\phi}(x) = \Phi(x) - \langle \Delta^\mathcal{Q}S + (x \otimes |e_0\rangle + (x \otimes \langle e_0\rangle)Z \rangle \quad (x \in \mathcal{B})
\]

and \( \overline{\phi}(1) \leq 0 \). It follows that \( \overline{\phi}(1) \leq 0 \) and

\[
\overline{\phi} = \Psi - \tau \cdot (\Delta^\mathcal{Q}S + |\zeta\rangle \langle e_0| + |e_0\rangle \langle \zeta|)
\]

where \( \Psi = E_\zeta \Phi \) and \( \langle \zeta| = (\tau \otimes \text{id}_{\mathcal{H}_k})(Z) \). Thus (ii) holds with \( \psi = \Psi|_{\mathcal{B}} \), moreover if \( l \) is preunital then \( \overline{\phi}(1) \) is unital and so \( k \) is too, therefore \( \overline{\phi}(1) = 0 \) so \( \overline{\phi}(1) = 0 \) also.

Conversely, if (ii) holds then, taking normal extensions,

\[
\overline{\phi} = \overline{\psi} - \tau \cdot (\Delta^\mathcal{Q}S + |\zeta\rangle \langle e_0| + |e_0\rangle \langle \zeta|)
\]

and so (5.5) holds with \( \Phi = R_\zeta \psi \) and \( Z = 1_{\mathcal{B}} \otimes |\zeta| \). Therefore, by [LW_1] Theorem 5.3, \( k \) is completely positive and contractive and so, by Proposition 4.3 \( l \) is too. Similarly, if \( \overline{\phi}(1) = 0 \) then \( \overline{\phi}(1) = 0 \) so \( k \) is unital, thus \( \mathcal{I} \) is too, and therefore \( l \) is preunital. This proves (a).

(b) If (i) holds then, choosing \( \psi \) and \( \zeta \) as in (a), let

\[
\begin{bmatrix}
\|\xi\|_2 \\
\rho(\cdot) \|\xi\| \\
\end{bmatrix}
\begin{bmatrix}
D^* \\
\rho(\cdot) \|\xi\|D \\
\end{bmatrix} = D^* \rho(\cdot) \|\xi\|D \\
\phi(y) = \psi(y) - \langle \Delta^\mathcal{Q}S + |\zeta\rangle \langle e_0| + |e_0\rangle \langle \zeta| \rangle \quad (x, y \in \mathcal{B})
\]

so \( D \) is isometric and (ii) holds. Conversely, suppose that (ii) holds then \( \phi \) has the form \( 5.2 \) with \( \zeta = (\frac{\mathcal{I}}{D^* \xi}) \) and \( \overline{\phi}(1) = 0 \) so (i) holds, by (a). This proves (b).

(c) If (i) holds then \( k \) is *-homomorphic so, by [LW_1] Proposition 6.3, \( \overline{\phi} \) is a structure map:

\[
\overline{\phi}(x^*y) = \overline{\phi}(x)^* \iota(y) + \iota(x)^* \overline{\phi}(y) + \overline{\phi}(x)^* (1_{\mathcal{B}} \otimes \Delta^\mathcal{Q}S) \overline{\phi}(y) \quad (x, y \in \mathcal{B})
\]

where \( \iota \) denotes the ampliation map \( x \mapsto x \otimes I_\mathcal{F} \). Since \( \tau \otimes \text{id}_{\mathcal{H}_k} \) is a unital *-homomorphism this implies that

\[
\overline{\phi}(x^*y) = \overline{\phi}(x)^* \overline{\phi}(y) + \overline{\phi}(x)^* \iota(y) + \overline{\phi}(x)^* \Delta^\mathcal{Q}S \overline{\phi}(y) \quad (x, y \in \mathcal{B})
\]

and so (ii) holds. Suppose conversely that (ii) holds. By separate \( \sigma \)-weak continuity of multiplication in \( \mathcal{B} \) it follows that (5.7) holds and a brief calculation confirms the identity

\[
\Omega(u^*v) = (\text{id}_{\mathcal{B}} \otimes \overline{\phi})(u)^* \Omega(v) + \Omega(u)^* (\text{id}_{\mathcal{B}} \otimes \overline{\phi})(v) + \Omega(u)^* (1_{\mathcal{B}} \otimes \Delta^\mathcal{Q}S) \Omega(v),
\]

where \( \Omega := (\text{id}_{\mathcal{B}} \otimes \overline{\phi}) \), for simple tensors \( u, v \) in \( \mathcal{B} \otimes \overline{\mathcal{B}} \). Since both sides are separately \( \sigma \)-weakly continuous the identity is valid for all \( u \) and \( v \) in \( \mathcal{B} \otimes \overline{\mathcal{B}} \). Substituting in \( u = \overline{x}x \) and \( v = \overline{y}y \) we see that \( \overline{\phi} \) satisfies (5.6). Therefore, by Corollary 4.2 of [LW_2], \( k \) is *-homomorphic thus, by Proposition 4.3 \( l \) is too and therefore (ii) holds. The equivalence of (ii) and (iii) is the general form of an \( \epsilon \)-structure map (see (1.12)). In view of (a), the last part is easily seen from the representation (iii). This completes the proof. □
Remark. The proper hypothesis for Parts (a) and (b) above is that $\mathcal{B}$ be a (multiplier) $C^*$-bialgebra, since the multiplicative property of $\Delta$ is not used in their proof. The above result therefore generalises Theorems 5.1 and 6.2 of [LS3] to the locally compact category.

6. Quantum Lévy processes on multiplier $C^*$-bialgebras

In this section we extend the definition of weak quantum Lévy process to multiplier $C^*$-bialgebras and establish a reconstruction theorem which is analogous to Schüermann’s for purely algebraic bialgebras ([Sch]) and extends ours, proved for unital $C^*$-bialgebras in [LS3].

Throughout this section $\mathcal{B}$ denotes a fixed counital $C^*$-bialgebra.

**Definition.** A weak quantum Lévy process on $\mathcal{B}$ over a $C^*$-algebra-with-a-state $(\mathcal{A}, \omega)$ is a family $(\lambda_{s,t}: \mathcal{B} \to \mathcal{A})_{0 \leq s \leq t}$ of nondegenerate $*$-homomorphisms for which the functionals $\lambda_{s,t} := \omega \circ j_{s,t}$ satisfy the following conditions, for $0 \leq r \leq s \leq t$:

(i) $\lambda_{r,t} = \lambda_{r,s} \ast \lambda_{s,t}$;
(ii) $\lambda_{t,t} = \epsilon$;
(iii) $\lambda_{s,t} = \lambda_{0,t-s}$;
(iv) $\tilde{\omega} \left( \prod_{i=1}^{n} j_{s_i,t_i}(x_i) \right) = \prod_{i=1}^{n} \lambda_{s_i,t_i}(x_i)$

whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{B}$ and the intervals $[s_1, t_1], \ldots, [s_n, t_n]$ are disjoint;

(v) $\lambda_{0,t} \to \epsilon$ pointwise as $t \to 0$.

A weak quantum Lévy process is called Markov-regular if $\lambda_{0,t} \to \epsilon$ in norm, as $t \to 0$.

**Remarks.** In the case of unital $C^*$-bialgebras we did not insist that the $*$-algebra $\mathcal{A}$ was a $C^*$-algebra.

As in the unital case, we refer to the weakly continuous convolution semigroup $(\lambda_t := \lambda_{0,t})_{t \geq 0}$ on $\mathcal{B}$ as the one-dimensional distribution of the process, and call the process Markov-regular if this is norm-continuous, in which case we refer to the convolution semigroup generator as the generating functional of the process ([LS3]). Moreover, as in the unital case, we call two weak quantum Lévy processes equivalent if their one-dimensional distributions coincide.

The generating functional $\gamma$ of a Markov-regular weak quantum Lévy process, being the generator of a norm-continuous convolution semigroup of states, is real, that is $\gamma = \gamma^\dagger$ where $\gamma^\dagger(a) := \gamma(a^*)$, conditionally positive, that is positive on the ideal $\text{Ker} \, \epsilon$, and its strict extension satisfies $\tilde{\gamma}(1) = 0$. Note that if $l \in \mathbb{P}_{s}(\mathcal{B})$ is a QS convolution cocycle on $\mathcal{B}$, with noise dimension space $\mathcal{K}$, which is $*$-homomorphic and preunital then, setting $\mathcal{A} := K(\mathcal{F})$, $\omega := \omega_{(0)}$, and $j_{s,t} := \sigma_s \circ l_{t-s}$ for all $0 \leq s \leq t$, we obtain a weak quantum Lévy process on $\mathcal{B}$, called a Fock space quantum Lévy process, which is Markov-regular if $l$ is. Our goal now is to establish a converse, in other words to extend the reconstruction theorem of [LS3] to the nonunital case. We give an elementary self-contained proof, independent of automatic implementability/complete boundedness properties of $\lambda$-structure maps. Recall Lemma [1.6].

**Theorem 6.1.** Let $\gamma \in \mathcal{B}^*$ be real, conditionally positive and satisfy $\tilde{\gamma}(1) = 0$. Then there is a (Markov-regular) Fock space quantum Lévy process with generating functional $\gamma$. 

process extends naturally to the nonunital case, with the assumption of unitality.

The definition of a product system quantum Lévy process and each product system quantum Lévy process on a unital and counital finite-dimensional distribution. The definition of a product system quantum Lévy process on \( B \) is implemented (see Theorem 1.5). For all \( a, b \in B \), there are bounded operators \( \hat{\delta} \) and \( \hat{\psi} \) such that

\[
\delta(a) \hat{\delta}(b) = \gamma(a^* b - \epsilon(a) b - a^* \epsilon(b)) \quad \text{and} \quad \psi(a) \hat{\psi}(b) = \gamma(a^* b - \epsilon(a) b).
\]

Now

\[
\|d(ab) - \epsilon(b) d(a)\|^2 = \gamma_0(\psi(b)^* a^* a \psi(b)) \leq \|a\|^2 \|a\|^2 \|d(b)\|^2, \quad a, b \in B,
\]

so there are bounded operators \( \hat{\pi}(a) \) on \( k \) satisfying

\[
\pi(a) d(b) = d(ab) - \epsilon(b) d(a), \quad a, b \in B.
\]

Using the density of \( d(B) \) it is straightforward to verify that the map \( a \mapsto \pi(a) \) defines a \( * \)-representation of \( B \) on \( k \). From (6.3), \( \delta \) is a \( (\pi, \epsilon) \)-derivation and so, from (6.2), \( \varphi := \begin{bmatrix} \gamma & \delta^t \\ \delta & \pi - \epsilon \end{bmatrix} \) defines an \( \epsilon \)-structure map \( B \to B(k) \), and therefore it only remains to prove that \( \varphi(1) = 0 \). Since \( \gamma_0(1) = 0 \), this follows from the identities

\[
\delta(a) \delta(b) = \gamma(a^* b - \epsilon(a) b - a^* \epsilon(b)) \quad \text{and} \quad \pi(a) \delta(b) = \delta(ab) - \delta(a) \epsilon(b), \quad a, b \in B,
\]

and the density of \( \bigcup \{ \text{Ran} \delta(b) : b \in B \} = d(B) \) in \( k \).

This has two significant consequences.

**Corollary 6.2.** Every Markov-regular weak quantum Lévy process is equivalent to a Fock space quantum Lévy process.

The second consequence uses the deeper fact that every \( \epsilon \)-structure map is implemented (see Theorem 1.5).

**Theorem 6.3.** Let \( \gamma \in B^* \). Then the following are equivalent:

(i) \( \gamma \) is the generating functional of a (necessarily norm-continuous) convolution semigroup of states on \( B \);

(ii) \( \gamma \) is real, conditionally positive and satisfies \( \gamma_0(1) = 0 \);

(iii) There is a nondegenerate representation \( (\pi, h) \) of \( B \) and vector \( \eta \in h \) such that \( \gamma = \omega_\eta \circ \varphi \).

In [LS3] we also introduced a stronger notion of product system quantum Lévy processes on a unital and counital \( C^* \)-bialgebra \( B \) and established the following two facts: each Fock space quantum Lévy process on \( B \) is in particular a product system quantum Lévy process and each product system quantum Lévy process determines in a natural way a weak quantum Lévy process on \( B \) with the same finite-dimensional distribution. The definition of a product system quantum Lévy process extends naturally to the nonunital case, with the assumption of unitality.
of \(\ast\)-homomorphisms constituting the process replaced by nondegeneracy, and the proofs of the above two facts remain valid.

7. Approximation by discrete evolutions

For this section \(B\) is again a fixed counital \(C^\ast\)-bialgebra. We show that any Markov-regular, completely positive, contractive quantum stochastic convolution cocycle on \(B\) may be approximated in a strong sense by discrete completely positive evolutions, and that the discrete evolutions may be chosen to be preunital and/or \(\ast\)-homomorphic, if the cocycle is. This extends and strengthens results of [FrS] for quantum Lévy processes on compact quantum semigroups.

We first note that Belton’s condition for discrete approximation of standard Markov-regular QS cocycles ([Bel]) readily translates to the convolution context using the techniques of this paper. We denote by \(\mathcal{I}^{(h)}\) \((h > 0, n \in \mathbb{N})\) the injective \(\ast\)-homomorphism

\[
B(\hat{k}^\otimes n) = B(\hat{k})^\otimes n \to B(\mathcal{F}_{[0,hn]}) \otimes I_{\mathcal{F}_{[hn,\infty]}} = \left(\bigotimes_{j=1}^{n} B(\mathcal{F}_{[(j-1)h,jh]}扫\right) \otimes I_{\mathcal{F}_{[hn,\infty]}}
\]

arising from the discretisation of Fock space. Thus

\[
\mathcal{I}^{(h)}: A \mapsto D^{(h)} A D^{(h)\ast} \otimes I_{\mathcal{F}_{[hn,\infty]}}
\]

where

\[
D^{(h)} := \bigotimes_{j=1}^{n} D^{(h)}_{n,j}, \text{ for the isometries}
\]

\[
D^{(h)}_{n,j}: A \otimes I_{\mathcal{F}_{[hn,\infty]}} \mapsto A D^{(h)\ast} \otimes I_{\mathcal{F}_{[hn,\infty]}}
\]

Also write \(\mathcal{I}^{(h)}_{n,\varepsilon}\) for the complete contraction

\[
\mathcal{I}^{(h)}_{n,\varepsilon}(\cdot|\varepsilon): B(\hat{k}^\otimes n) \to \mathcal{F}, \quad h > 0, n \in \mathbb{N}, \varepsilon \in \mathcal{E}.
\]

For a map \(\Psi \in CB(\mathcal{V}; \mathcal{V} \otimes M B(\hat{k}))\), where \(\mathcal{V}\) is a concrete operator space, its composition iterates are defined by

\[
\Psi_0 := \text{id}_\mathcal{V}, \quad \Psi_n := (\Psi_{n-1} \otimes M \text{id}_B(\hat{k})) \circ \Psi \in CB(\mathcal{V}; \mathcal{V} \otimes M B(\hat{k}^\otimes n)), \quad n \in \mathbb{N}.
\]

Similarly, for a map \(\psi \in CB(B; B(\hat{k}))\), its convolution iterates are defined by

\[
\psi_0 := \epsilon, \quad \psi_n = \psi_{n-1} \ast \psi \in \epsilon \in CB(B; B(\hat{k}^\otimes n)) \quad (n \in \mathbb{N}).
\]

As usual we are viewing \(B(\hat{k}^\otimes n)\) as the multiplier algebra of \(K(\hat{k}^\otimes n)\) here, and automatic strictness is being invoked to ensure meaning for \(\psi_{n-1} \ast \psi\). In short, \(\psi_n = \psi^*\) for \(n \in \mathbb{Z}^+\). Note the easy compatibility between the two methods of iteration: for \(\psi \in CB(B; B(\hat{k}))\),

\[
\mathcal{R}\psi_n = \Psi_n \quad \Psi := \mathcal{R}\psi.
\]

(7.1)

We need the following block matrices, on a Hilbert space of the form \(\mathcal{H}\):

\[
\mathcal{S}_h := \begin{bmatrix} h^{-1/2} & I_h \end{bmatrix}, \quad h > 0;
\]

we set \(\Sigma_h\) equal to the conjugation by \(\mathcal{S}_h\): \(X \mapsto \mathcal{S}_h X \mathcal{S}_h\). Conjugation provides the correct scaling for quantum random-walk approximation ([LjP]).
Theorem 7.1. Let $\varphi \in CB(\mathcal{B}; B(\hat{k}))$. Suppose that there is a family of maps 
$(\psi^{(h)}_{\varphi})_{0 < h \leq C}$ in $CB(\mathcal{B}; B(\hat{k}))$ for some $C > 0$, satisfying 
$$
\Sigma_h \circ (\psi^{(h)} - \iota_k \circ \epsilon) \to \varphi \text{ in } CB(\mathcal{B}; B(\hat{k})) \text{ as } h \to 0^+.
$$
Then the convolution iterates \{\psi^{(h)}_{n} : n \in \mathbb{Z}_+, 0 < h < C\} satisfy 
$$
\sup_{t \in [0, T]} \left\| \int_{[t/h, t]} \psi^{(h)}_{[t/h]} - l^h_{t, \epsilon} \right\|_{CB(\mathcal{B}; \mathcal{F})} \to 0 \text{ as } h \to 0^+, 
$$
for all $T \in \mathbb{R}_+$ and $\epsilon \in \mathcal{E}$.

Proof. Let $\overline{\varphi}, \overline{\psi^{(h)}} \in CB_\sigma(\mathcal{B}; B(\hat{k}))$ denote respectively the normal extensions of $\varphi$ 
and $\psi^{(h)}$ to the universal enveloping von Neumann bialgebra of $\mathcal{B}$ and set $\overline{\varphi} := \mathcal{R}_\sigma \overline{\varphi}$ 
and $\overline{\psi^{(h)}} := \mathcal{R}_\sigma \psi^{(h)}$ in $CB_\sigma(\mathcal{B}; B(\hat{k}))$. Then, by the complete isometry of the 
map $\mathcal{R}_\sigma$ and the complete isometry expressed in (1.3), 
$$
\| (id_{\mathcal{B}} \otimes \Sigma_h) \circ (\psi^{(h)} - \iota_k) - \overline{\varphi} \|_{cb} = \| \Sigma_h \circ (\psi^{(h)} - \iota_k \circ \epsilon) - \varphi \|_{cb}
$$
which tends to 0 as $h \to 0^+$. Therefore, by Theorem 7.6 of [Be], it follows that 
$$
\sup_{t \in [0, T]} \left\| (id_{\mathcal{B}} \otimes \Sigma_h) \circ (\psi^{(h)} - \iota_k) - k^h_{t, \epsilon} \right\|_{CB(\mathcal{B}; \mathcal{F})} \to 0 \text{ as } h \to 0^+.
$$
Now recall that $l^h_{t, \epsilon} = T^\varphi_{t, \epsilon} |_{\mathcal{B}}$ where $T^\varphi_{t, \epsilon} = k^\varphi_{t, \epsilon} \overline{\varphi} = \mathcal{E}_\varphi k^\varphi_{t, \epsilon}$ ($t \in \mathbb{R}_+, \epsilon \in \mathcal{E}$). The result 
therefore follows from the fact that $\mathcal{E}_\varphi \psi^{(h)}_{n} |_{\mathcal{B}} = \psi^{(h)}_{n} (n \in \mathbb{Z}_+)$, which is evident 
from (7.3). \hfill \Box

Remarks. (i) Since multiplicativity of the coproduct plays no role in the above proof, 
the proper hypothesis for the theorem is that $\mathcal{B}$ be a (multiplier) $C^*$-hyperbialgebra.

(ii) Belton’s result allows the hypothesis to be weakened to a condition on columns (like that of the conclusion), namely $\varphi \in L(\hat{k}; CB(\mathcal{B}; |\hat{k}|))$ and, for all 
c $\in \mathcal{K},$
$$
\Sigma_h \circ (\psi^{(h)} - \iota_k \circ \epsilon)(\cdot)|\hat{c} \to \varphi|\hat{c} \text{ in } CB(\mathcal{B}; |\hat{k}|) \text{ as } h \to 0^+.
$$
The proof requires only minor modification, however the result given is adequate for our purposes here.

In the next two propositions the coproduct plays no role. Recall Theorem 1.6 on the automatic implementability of $\chi$-structure maps.

Proposition 7.2. Let $(\mathcal{A}, \chi)$ be a $C^*$-algebra with character and let $\varphi : \mathcal{A} \to B(\hat{h})$ 
be a $\chi$-structure map. Set 
$$
U^{(h)}_{\chi} := \begin{pmatrix} c_{h, \xi} & -s_{h, \xi} \\
sl_{h, \xi} & c_{h, \xi} \xi + Q_{\xi} \end{pmatrix}, \text{ for } h > 0 \text{ such that } h\|\xi\|^2 \leq 1,
$$
where $(\pi, \xi)$ is an implementing pair for $\varphi$, and with $\xi' := \|\xi\|^{-1}\xi$ (or 0 if $\xi = 0$), 
$$
c_{h, \xi} = ch^{1/2} |\xi| \|\xi\|^2, \ s_{h, \xi} = h^{1/2} |\xi| \text{ and } Q_{\xi} := P_{\xi} \xi = |\xi'| \xi'.
$$
Then each $U^{(h)}_{\chi}$ is a unitary operator on $\hat{h}$ and the family of *-representations 
$\mathcal{B} \to B(\hat{h})$
$$
\left( \psi^{(h)} = \overline{\psi^{(h)}} := U^{(h)}_{\chi} \psi (\chi \otimes \pi)(\cdot) U^{(h)}_{\chi} \right)_{h > 0, h \|\xi\|^2 \leq 1}
$$
satisfies 
$$
\varphi - \Sigma_h \circ (\psi^{(h)} - \iota_k \circ \chi) = \frac{h}{1 + c_{h, \xi}} \varphi_1 - \frac{h^2}{(1 + c_{h, \xi})^2} \varphi_2
$$
for completely bounded maps $\varphi_1, \varphi_2 : \mathcal{A} \to B(\hat{h})$ independent of $h$. Moreover each *-representation $\psi^{(h)}$ 
is nondegenerate if (and only if) $\pi$ is.
Proof. Unitarity of $U^{(h)}_\xi$ is evident from the identities
\[
\nu^2 + s_{h,\xi}^* s_{h,\xi} = 1, \quad s_{h,\xi} Q^\perp_{\xi} = 0 \quad \text{and} \quad s_{h,\xi} s_{h,\xi} = (1 - \nu^2) Q_{\xi}.
\]
Set $\nu = \pi - t_h \circ \chi$ and $\gamma = \omega_{\xi} \circ \nu$ so that $\varphi$ has block matrix form \([12]\), and $d_{h,\xi} := c_{h,\xi} - 1$. Then, noting the identities
\[
d_{h,\xi} = -h(1 + c_{h,\xi})^{-1} \|\xi\|^2, \quad \|\xi\|^2 Q_{\xi} = |\xi\rangle \langle \xi|, \quad c_{h,\xi} Q_{\xi} + Q_{\xi}^\perp = d_{h,\xi} Q_{\xi} + I_h,
\]
we have
\[
\psi^{(h)}(a) - \chi(a) I_h
= \left[ \begin{array}{cc} 0 & \nu(a) \\ 0 & d_{h,\xi} Q_{\xi} + I_h \end{array} \right] \left[ \begin{array}{cc} h^{1/2} \nu(a) |\xi\rangle \langle \xi| & 0 \\ 0 & d_{h,\xi} \nu(a) Q_{\xi} + \nu(a) \end{array} \right]
= \left[ \begin{array}{cc} h^{1/2} \nu(a) \nu(a) & h^{1/2} \nu(a) \\ h^{1/2} [d_{h,\xi} Q_{\xi} + I_h] & d_{h,\xi} Q_{\xi} \nu(a) Q_{\xi} + d_{h,\xi} (Q_{\xi} \nu(a) + \nu(a) Q_{\xi}) + \nu(a) \end{array} \right]
= \left[ \begin{array}{cc} h^{1/2} I_h & 0 \\ 0 & X \end{array} \right] \left[ \begin{array}{cc} h^{1/2} I_h & 0 \\ 0 & X \end{array} \right]
\]
where
\[
\varphi_1 = \left[ \begin{array}{cc} 0 & \gamma(\cdot) \langle \xi| \\ \gamma(\cdot) |\xi\rangle & X \nu(\cdot) + \nu(\cdot) X \end{array} \right] \quad \text{and} \quad \varphi_2 = \gamma(\cdot) \left[ \begin{array}{cc} 0 & \gamma(\cdot) \langle \xi| \\ \gamma(\cdot) |\xi\rangle & X \nu(\cdot) + \nu(\cdot) X \end{array} \right]
\]
for $X = |\xi\rangle \langle \xi|$, from which the result follows. \qed

Note that $U^{(h)}_\xi$ and $\tilde{\pi}^{(h)}_{\xi}$ are norm-continuous in $h$ and converge to $I_\xi$ and $\chi \oplus \pi$ respectively as $h \to 0^+$. Note also, for the simplest class of $\chi$-structure map, namely
\[
\varphi = \left[ \begin{array}{cc} 0 & \nu \circ \chi \\ \nu \circ \chi & \nu \circ \chi \end{array} \right], \quad \text{where} \quad \nu = \pi - t_h \quad \text{for a} \quad *-\text{homomorphism} \quad \pi : A \to B(h),
\]
\[
\tilde{\pi}^{(h)}_{\xi} = (\chi \oplus \pi) \quad \text{and} \quad \tilde{\pi}^{(h)}_{\xi} - \iota_h \circ \chi = \Sigma_h \circ (\tilde{\pi}^{(h)}_{\xi} - \iota_h \circ \chi) = \varphi \quad \text{for all} \quad h.
\]

Remarks. (i) Unwrapping $\tilde{\pi}^{(h)}_{\xi}(a)$:
\[
\left[ \begin{array}{cc} \chi(a) + h \gamma(a) & s_{h,\xi}^*(\nu(a) - \frac{h \gamma(a)}{1 + c_{h,\xi}} I_h) \\ (\nu(a) - \frac{h \gamma(a)}{1 + c_{h,\xi}} I_h) s_{h,\xi} & \pi(a) - \frac{h \gamma(a)}{1 + c_{h,\xi}} X \nu(a) + \nu(a) X \end{array} \right]
\]
reveals the vector-state realisation
\[
\omega_{\xi_0} \circ \tilde{\pi}^{(h)}_{\xi} = \omega_{\Omega^{(h)}_\xi} \circ (\chi \oplus \pi) \quad \text{where} \quad \Omega^{(h)}_\xi := U^{(h)}_\xi \xi_0
\]
for the state $\chi + h \gamma = \chi + h \|\xi\|^2 (\omega_{\xi} \circ \pi - \chi)$. Indeed, finding such a representation was the strategy of proof in [14].

(ii) The next remark will be used in the proof of Theorem \[23\] If instead of being a $\chi$-structure map, $\varphi$ is given by
\[
\left[ \begin{array}{cc} \langle \xi| & D^* \nu(\cdot) \end{array} \right] \left[ \begin{array}{c} \xi \\ D \end{array} \right] \quad \text{where} \quad \nu := \pi - t_h \circ \chi, \quad (7.6)
\]
for a nondegenerate representation \( \pi : A \to B(H) \), vector \( \xi \in H \) and isometry \( D \in B(h; H) \) then, replacing the unitaries \( V^{(h)}_{\xi} \) by the isometries \( V^{(h)}_{\xi,D} := U^{(h)}_{\xi} [1_{D}] \in B(h; H) \) in the above proof yields a family of completely positive preunital maps 

\[
(\psi^{(h)}_{\xi,D}) := \left( V^{(h)}_{\xi,D} \ast (\chi \oplus \pi)(\cdot) V^{(h)}_{\xi,D} \right)_{0 < h, \|\xi\|^2 \leq 1}
\]

satisfying (7.4), with completely bounded maps 

\[
\varphi_1 = \begin{bmatrix} 0 & \gamma(\cdot)|\eta\rangle \\ \gamma^{\ast}(\cdot)\langle \eta | & Y^{\ast} \nu(\cdot) D + D^{\ast} \nu(\cdot) Y \end{bmatrix} \quad \text{and} \quad \varphi_2 = \gamma(\cdot) \begin{bmatrix} 0 & X \end{bmatrix}
\]

where \( Y = |\xi\rangle \langle \eta| \) and \( X = |\xi\rangle \langle \xi| \) for \( \eta = D^{\ast} \xi \in h \).

Recall the notations [GL]7.

**Proposition 7.3.** Let \((A, \chi)\) be a \( C^{\ast}\)-algebra with character and let \( \varphi \in CB(A; B(\hat{k})) \) satisfy \( \varphi(1) \leq 0 \) and be expressible in the form 

\[
\varphi_1 - \varphi_2 \quad \text{where} \quad \varphi_1 \in CP(A; B(\hat{k})) \quad \text{and} \quad \varphi_2 = \chi(\cdot)(\Delta^{QS} + |\zeta\rangle \langle \zeta| + |e_0\rangle \langle e_0|),
\]

for a vector \( \zeta \in \hat{k} \). Then there is a family of completely positive contractions 

\( (\psi^{(h)}_{\xi,D}) : A \to B(\hat{k})_{0 < h, \xi^2 \leq C} \)

such that \( \varphi \) is the compression of \( \theta \) to \( B(\hat{k}) \). By Proposition 7.2, there is a family of \( *\)-homomorphisms \( (\hat{\pi}^{(h)}_{\xi,D}) : A \to B(\hat{k})_{0 < h, \xi^2 \leq C} \) for some \( C > 0 \), satisfying 

\[
\Sigma_{h} \circ (\hat{\pi}^{(h)}_{\xi,D} - i_{h} \circ \chi) \to \varphi \quad \text{in} \quad CB(A; B(\hat{k})) \quad \text{as} \quad h \to 0^+.
\]

**Proof.** It follows from Proposition 4.3 and Theorem 4.4 of [S], and their proofs, that there is a Hilbert space \( h \) containing \( k \) and a \( \chi\)-structure map \( \theta : A \to B(h) \) such that \( \varphi \) is the compression of \( \theta \) to \( B(\hat{k}) \). By Proposition 7.2, there is a family of \( *\)-homomorphisms \( (\hat{\pi}^{(h)}_{\xi,D}) : A \to B(\hat{k})_{0 < h, \xi^2 \leq C} \) for some \( C > 0 \), satisfying 

\[
\Sigma_{h} \circ (\hat{\pi}^{(h)}_{\xi,D} - i_{h} \circ \chi) \to \theta \quad \text{in} \quad CB(A; B(\hat{k})) \quad \text{as} \quad h \to 0^+.
\]

It follows that (7.8) holds for the compressions \( \psi^{(h)}_{\xi,D} \) to \( B(\hat{k}) \), which are manifestly completely positive and contractive. \square

**Remark.** When \( B \) and \( k \) are assumed to be separable, there is an alternative proof via standard QS cocycles ([GL]7).

Combining the above results we obtain the following discrete approximation result for quantum stochastic convolution cocycles which, in particular, gives natural quantum random walk approximation for Markov-regular quantum Lévy processes on a locally compact quantum semigroup. For more details on quantum random walks on quantum groups, see [FrS] and references therein.

**Theorem 7.4.** Let \( l \) be a Markov-regular, completely positive, contractive quantum stochastic convolution cocycle on a counital \( C^{\ast}\)-bialgebra \( B \). Then there is a family of completely positive contractions \( (\psi^{(h)}_{\xi,D}) : B \to B(\hat{k})_{0 < h, \xi^2 \leq C} \) for some \( C > 0 \), whose convolution iterates satisfy 

\[
\sup_{t \in [0, T]} \left\| l^{(h)}_{t,h} \circ \psi^{(h)}_{t,h} - l^{(h)}_{t,h} \right\|_{CB(B; \mathcal{F})} \to 0 \quad \text{as} \quad h \to 0^+,
\]

for all \( T \in \mathbb{R}_+ \) and \( \varepsilon \in \mathcal{E} \). Moreover if \( l \) is preunital, and/or *-homomorphic, then each \( \psi^{(h)}_{\xi,D} \) may be chosen to be so too.

**Proof.** By Theorem 5.2 we know that \( l = l^\varphi \) for some \( \varphi \in CB(B; B(\hat{k})) \) which has a decomposition of the form (7.7). The first part therefore follows from Proposition 7.3 and Theorem 7.4. If \( l \) is preunital then \( \varphi \) may be expressed in the form (5.3) and so, by the remark containing (7.6), it follows that the completely positive maps \( \psi^{(h)}_{\xi,D} \) may be chosen to be preunital. Now suppose that \( l \) is *-homomorphic. Then, by Theorem 5.2 \( \varphi \) is an \( \varepsilon\)-structure map. By Theorem 7.5 \( \varphi \) has an implementing
pair \((\pi, \xi)\), with \(\pi\) nondegenerate if \(l\) is. By Proposition 7.2, the maps \(\psi^{(h)}\) may be chosen to be *-homomorphic—and also nondegenerate if \(l\) is. This completes the proof. □

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