Numerical study of renormalization group flows of nuclear effective field theory without pions on a lattice

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We formulate the next-to-leading order nuclear effective field theory without pions in the two-nucleon sector on a spatial lattice, and investigate nonperturbative renormalization group flows in the strong coupling region by diagonalizing the Hamiltonian numerically. The cutoff (proportional to the inverse of the lattice constant) dependence of the coupling constants is obtained by changing the lattice constant with the binding energy and the asymptotic normalization constant for the groundstate being fixed. We argue that the critical line can be obtained by looking at the finite-size dependence of the groundstate energy. We determine the relevant operator and locate the nontrivial fixed point, as well as the physical flow line corresponding to the deuteron in the two-dimensional plane of dimensionless coupling constants. It turns out that the location of the nontrivial fixed point is very close to the one obtained by the corresponding analytic calculation, but the relevant operator is quite different.

I. INTRODUCTION

Since the seminal work by Weinberg [1–3], the low-energy effective field theory of nucleons (and other low-energy excitations, such as pions), the so-called nuclear effective field theory (NEFT), has been investigated extensively; see Refs. [4, 5] for the reviews. In NEFT, the “fundamental” degrees of freedom are low-lying hadrons so that NEFT is applicable only up to a certain momentum scale, the physical cutoff \( \Lambda_{\text{phys}} \). The effects of heavier degrees of freedom than \( \Lambda_{\text{phys}} \), the processes with momenta higher than \( \Lambda_{\text{phys}} \), and the internal structure of the hadrons are integrated out and have been encoded in the coupling constants of local interactions. For example the effects of heavy-meson exchange processes between two nucleons are represented by four-nucleon operators. Note that even if pions are included in NEFT, the exchange of the pion with momentum transfer higher than \( \Lambda_{\text{phys}} \) is represented as local four-nucleon (and 2nucleon, in general) operators.

Although the early investigations exclusively employed continuous, semi-analytic approach based on the Lippmann-Schwinger (LS) equation, the Faddeev equation, etc., the methods of numerical simulation on a lattice have been developed recently [6–19]; see Ref. [20] for the review. The inverse of the lattice constant provides the cutoff in momentum. It should not exceed the physical cutoff. Note that, unlike lattice QCD, we should not take the continuum limit in lattice NEFT.

Lattice simulation of NEFT is very interesting since it has several advantages. First of all, it allows us to calculate many-nucleon quantities without suffering from complications due to the increase of the number of nucleons. Remember that the Faddeev equation for three nucleons is more complex than the two-nucleon LS equation, and the Faddeev-Yakubovsky equation for four nucleons is even more complex. Lattice formulation does not have this kind of complication, except for the construction of many-nucleon operators, which are necessary when the correlators are calculated. By now, considerably large nuclei have already been investigated on a lattice [11, 12, 21]. Second, arbitrarily complicated pion interactions can be included in lattice simulations, just as arbitrarily complicated gluon interactions can be included in lattice QCD. It thus provides the possibility of the calculations with the exactly chiral symmetric interactions of pions with nucleons that are nonlinearly realized. This direction of investigation is now in progress [22]. Note that the truncation of pion interactions at a finite order inevitably breaks chiral symmetry. Third, it is straightforward to make the system contact to a heat reservoir, and also to a particle reservoir.

In order to perform lattice simulation of NEFT effectively, it is essential to understand how the operators behave on a lattice. That is, we need to know how the coupling constants depend on the lattice constant, hence on the momentum cutoff. This is a typical renormalization group (RG) problem.

It is critically important that the two-nucleon systems in the S waves are finely tuned: the scattering lengths are unnaturally long. From the RG point of view, this may be viewed as an evidence of the fact that the system is near a nontrivial fixed point. The nontrivial fixed point is located on a critical surface, which is the bound-
ary between the strong and weak coupling phases. The scattering length would be infinite if the system were on the critical surface.

Near the nontrivial fixed point, operators gain anomalous dimensions and hence their behavior is quite different from that in the perturbative region near the trivial fixed point where all the interactions are turned off. Anomalous dimensions change the importance of the operators. It has been shown that there is a relevant operator near the nontrivial fixed point [2]. (Remember that relevant operators are the ones with positive scaling dimensions and the most important at low energies.) The RG flow tells us what the relevant operator is. Pictorially it is the direction of the flow going out from the nontrivial fixed point. A good example is the dashed line in Fig. 9 shown later.

While the scaling dimensions are universal, the location of the nontrivial fixed point is not, i.e., it depends on how the cutoff is implemented. The scaling dimensions are obtained in the literature [23–25] by using a continuum formulation, and they must be the same for lattice regularization. The RG flow and hence the relevant operator are not universal. They on a lattice should be determined by explicit calculations. It is our purpose of the present paper to numerically determine the location of the nontrivial fixed point and the relevant operator of NEFT without pions on a lattice.

The determination of the location of the nontrivial fixed point and the RG flow are of direct physical importance. From the information, we know the relevant operator that dominates physics. It is also important to know which flow line corresponds to the physical system in order to perform numerical simulations since it provides the input parameters.

Even if pions are included, the results do not drastically change from those without pions. The strong short-distance part of pion exchange interactions is cutoff on a lattice. See Ref. [25] for the effects of pions in the EFT with a finite cutoff. The study here therefore is an important step toward the chirally-symmetric NEFT with pions on a lattice.

NEFT without pions with a lattice regularization has been considered by Seki and van Kolck [27] based on the analytic approach. Starting with the continuum S-wave LS equation, they replace the momentum integrals with the ones over the first Brillouin zone and the momentum squares in the integrands with the corresponding discretized expressions to imitate the theory defined on a lattice, and determine the dependence of the scattering length on the lattice constant.

Note that the method of Seki and van Kolck does not yield a genuine lattice result. Theory defined on a lattice has explicit rotational invariance breaking so that the amplitude is an admixture of “partial waves.” On the other hand, the starting point of Seki and van Kolck is the LS equation in a specific partial wave, derived in the continuum theory.

In this paper, we employ the numerical diagonalization of the NLO Hamiltonian of NEFT without pions defined on a spatial cubic lattice with periodic boundary condition in order to investigate RG flows in the strong coupling phase where a single two-nucleon boundstate appears as the groundstate. Of course, the RG flows may be obtained by using a Monte-Carlo simulation, but the diagonalization is much more accurate and numerically simpler. We confine ourselves to the strong coupling phase because in the weak coupling phase, where boundstates are absent physically, but the groundstate is found to have negative energy due to the periodic boundary condition [25].

We argue that at the phase boundary the finite-size effect is maximal. By looking at the finite-size dependence of the groundstate energy, we can infer the location of the phase boundary. It turns out that, with this information on the phase boundary, the location of the nontrivial fixed point can be determined quite accurately. We emphasize that this is the first determination of the location of the nontrivial fixed point and the RG flow of the NEFT genuinely defined on a lattice.

We consider the improvement of the discretized representation of the derivatives to reduce the finite lattice constant errors and the rotational symmetry breaking effects.

We compare the numerical results with those obtained by the analytic method with a lattice regularization, which is a generalization of the method of Seki and van Kolck. They treat the NLO coupling perturbatively, but we deal with it nonperturbatively by considering full dependence of the couplings on the scattering amplitude.

The structure of the paper is the following: In Sec. II we recapitulate the continuum approach to the NLO NEFT. The RG equations and the flow [24] are obtained by solving the LS equation for the scattering amplitude. We also consider the Schrödinger equation and the boundstate, and show that the asymptotic normalization constant (ANC) [29] can be used as a low-energy physical constant. In Sec. III we evaluate the integrals appearing in the analysis by using the lattice regularization in the same spirit of Seki and van Kolck [27], and draw the RG flows with and without the improvement of the discretization of the derivatives. We then switch to the numerical diagonalization of the Hamiltonian defined on a lattice and obtain the RG flow in the strong coupling phase by requiring the binding energy and the ANC to be independent of the cutoff in Sec. IV. We consider the lattice finite-size effect of the groundstate energy and argue that it is maximal at the phase boundary. With this information, together with the RG flow, we can determine the location of the nontrivial fixed point. In Sec. V the summary of the results and the discussions are given. In Appendix we outline the evaluation of the integral that is necessary to draw the RG flow with the improved discretization.
II. NLO NEFT WITHOUT PIONS

A. Renormalization group equations and the flow

In order to set up an effective field theory, one needs to choose the relevant degrees of freedom and the accuracy to be achieved. In the NEFT without pions, we consider only nucleons. There are an infinite number of local operators that represent interactions among nucleons. The leading order (LO) interaction is represented by the momentum independent four-nucleon operator, and the next-to-leading order (NLO) one by the four-nucleon operator with two spatial derivatives. In the following, we will concentrate on the S waves.

The isospin SU(2) symmetric Lagrangian of our effective field theory is given by

\[
\mathcal{L} = \frac{i}{2M} \nabla^2 - N C_0(N^T P_k^1 N) N^T P_k^N + \text{h.c.},
\]

where \( N \) is the nucleon operator, \( M \) represents the mass, and \( \nabla^2 = \nabla \cdot \nabla - 2 \nabla \cdot \nabla + \nabla \cdot \nabla \). The terms higher than NLO are omitted. \( P_k \) is a projection operator; for the \( 3S_1 \) (spin triplet) channel, it is \( P_k = \sigma^a \tau^2 / \sqrt{8} \), with \( \sigma^a \) and \( \tau^a \) being spin and isospin Pauli matrices respectively.

The LS equation for the off-shell center-of-mass nucleon-nucleon (NN) scattering amplitude is given by

\[
-iA(p^0, \mathbf{p}_1, \mathbf{p}_2) = -iV(\mathbf{p}_1, \mathbf{p}_2) + \int \frac{d^3k}{(2\pi)^3} \frac{i}{p^0 - k^0 - \frac{i\epsilon}{M}} A(p^0, \mathbf{p}_1, k),
\]

where \( V(\mathbf{p}_1, \mathbf{p}_2) = C_0 + 4C_2 (\mathbf{p}_1^2 + \mathbf{p}_2^2) \),

(2.2)

and 

\[
I_n = -M \int \frac{d^3k}{(2\pi)^3} \frac{|k|^{2n}}{|k|^2 + \mu^2}, \quad \mu = \sqrt{-M p^0 - i\epsilon}.
\]

The integrals \( I_n \) are divergent and require regularization. If we impose a sharp momentum cutoff \( \Lambda \), they are given as

\[
I_n = -M \frac{1}{2\pi^2} \int_0^\Lambda dk \frac{k^{2n+2}}{k^2 + \mu^2}, \quad \mu = \sqrt{-M p^0 - i\epsilon}.
\]

The Wilsonian RG analysis of this system is formulated elegantly by introducing the (energy-dependent) redundant operators, which can be eliminated by the use of equations of motion [23–25]. However, for our present purpose, it is simpler to consider the on-shell formulation: we require the scattering length \( a_0 \) and the effective range \( r_0 \) to be independent of the cutoff. See Ref. [24] for the relation between the two formulations. At low energies, the on-shell amplitude can be written as

\[
A^{-1}_{\text{on-shell}} = -M \frac{1}{4\pi} \left[ \frac{1}{a_0} + \frac{1}{r_0} p^0 + \mathcal{O}(p^2) - ip \right],
\]

(2.11)

with \( p = \sqrt{M p^0} = |\mathbf{p}_1| = |\mathbf{p}_2| \).

The scattering length and the effective range are given by

\[
M \frac{1}{4\pi} a_0 = \frac{M\Lambda}{2\pi^2} \left[ \theta_1 + (1 + \theta_5 Y)^2 \right],
\]

\[
M \frac{1}{4\pi} r_0 = \frac{M}{2\pi^2} \left[ -R(0) + \frac{Y(2 + \theta_3 Y)(1 + \theta_5 Y)^2}{(X - \theta_5 Y^2)^2} \right],
\]

(2.12)

(2.13)

where we have introduced dimensionless coupling constants \( X \) and \( Y \) defined by

\[
C_0 = \frac{2\pi^2}{M\Lambda X}, \quad C_2 = \frac{2\pi^2}{M\Lambda^2 Y}.
\]

(2.14)

as well as the constants \( \theta_n \) \( (n = 1, 3, 5) \) and the function \( R(x) \) defined by

\[
I_0 = -\frac{M\Lambda}{2\pi^2} \left[ \theta_1 + \left( \frac{p^2}{X^2} \right) R \left( \frac{p^2}{X^2} \right) \right] - i\frac{M}{4\pi} p^0,
\]

\[
I_3 \equiv -M \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} = -\frac{MA^3}{2\pi^2} \theta_3,
\]

\[
I_5 \equiv -M \int \frac{d^3k}{(2\pi)^3} |k|^2 = -\frac{MA^5}{2\pi^2} \theta_5,
\]

(2.15)

(2.16)

(2.17)

according to Seki and van Kolck [27]. For the regularization with the sharp momentum cutoff \( \Lambda \), we have

\[
\theta_1 = 1, \quad \theta_3 = \frac{1}{3}, \quad \theta_5 = \frac{1}{5}, \quad R(0) = -1.
\]

(2.18)

Note that Seki and van Kolck disregard the terms higher than linear in \( Y \) of Eqs. (2.12) and (2.13).

By requiring that \( a_0 \) and \( r_0 \) are independent of \( \Lambda \), we obtain the following RG equations:

\[
D = 1 - C_0 I_0 - 8C_2 I_1 + 16C_2^2 I_2 - 16C_2^3 I_0 I_2,
\]

(2.19)
\[
\frac{\lambda}{d\lambda} = X (1 + 6\theta_3 Y) + Y^2 (5\theta_3 + 3\theta_3^2 X) + 3\theta_3 \theta_5 Y
\]
\[
+ \frac{X - \theta_3 Y^2}{(1 + \theta_3 Y)^2} \left[ -R(0)(\theta_3 X + \theta_5 Y)(X - \theta_3 Y^2) + \theta_1 \left\{ \theta_3 Y^2 (3 + 2\theta_3 Y) + X (1 + 2\theta_3 Y (2 + \theta_3 Y)) \right\} \right],
\] (2.19)
\[
\frac{\lambda}{d\lambda} = 3Y \left( 1 + \frac{\theta_3 Y}{2} \right) (1 + \theta_3 Y) + \frac{X - \theta_3 Y^2}{(1 + \theta_3 Y)^2} \left[ -R(0)X + 4\theta_1 Y + (R(0)\theta_5 + 2\theta_1 \theta_3) Y^2 \right].
\] (2.20)

From these RG equations, we obtain the nontrivial fixed point as

\[
(X_*, Y_*) = \left( \frac{3}{5} (4 - 3\sqrt{3}), \frac{3}{2} (2 + \sqrt{3}) \right)
\]
\[
= (-0.717691 \ldots, -0.401924 \ldots), \quad (2.21)
\]

which is responsible for the “unnaturally” large scattering length in the \(^3S_1\) channel. The flow and the nontrivial fixed point (as well as the trivial one) is depicted in Fig. 1.

![FIG. 1. The flow and the fixed points of the NLO NEFT in the X-Y plane obtained by using a sharp momentum cut-off in the continuum formulation. The arrows indicate the directions of the smaller values of the cutoff.](image)

Although the existence of the nontrivial fixed point and the scaling dimensions, which are the eigenvalues of the linearized RG equations in the vicinity of the nontrivial fixed point, are universal, the location of the fixed point and the flow are not universal: they depend on the details of the regularization. In the following sections, we will see how they vary as the regularization is changed.

**B. Groundstate wavefunction and the ANC**

For the comparison with the results obtained in Sec. IV, let us consider the (stationary) Schrödinger equation for the relative motion of the two-nucleon boundstate in momentum space,

\[
E\psi(p) = \frac{p^2}{M} \psi(p) + \int_{\Lambda} \frac{d^3q}{(2\pi)^3} \left[ C_0 + 4C_2 (p^2 + q^2) \right] \psi(q),
\] (2.22)

where \(\psi(p)\) is the cutoff wavefunction satisfying \(\psi(p) = 0\) for \(|p| > \Lambda\) and \(E\) is the (negative) energy eigenvalue. The integration is over the region \(|q| \leq \Lambda\).

The Schrödinger equation can be solved as

\[
\psi(p) = \frac{-M}{p^2 + \mu^2} \left[ (C_0 + 4C_2 p^2) \alpha + 4C_2 \beta \right],
\] (2.23)

where \(\mu = \sqrt{M|E|}\), and \(\alpha\) and \(\beta\) are constants defined by

\[
\alpha = \int_{\Lambda} \frac{d^3q}{(2\pi)^3} \psi(q), \quad \beta = \int_{\Lambda} \frac{d^3q}{(2\pi)^3} q^2 \psi(q). \quad (2.24)
\]

By multiplying Eq. (2.23) by \((2\pi)^{-3}\) and \(p^2(2\pi)^{-3}\), and integrating over \(p\), we obtain

\[
\alpha = (C_0 \alpha + 4C_2 \beta) I_0 + 4C_2 \alpha I_1, \quad (2.25)
\]
\[
\beta = (C_0 \alpha + 4C_2 \beta) I_1 + 4C_2 \alpha I_2, \quad (2.26)
\]

respectively. These equations have a nonzero solution if

\[
\begin{vmatrix}
1 - C_0 I_0 - 4C_2 I_1 & -4C_2 I_0 \\
-C_0 I_1 - 4C_2 I_2 & 1 - 4C_2 I_1
\end{vmatrix} = 0.
\] (2.27)

This determinant is equal to \(D\) given in Eq. (2.25). This condition (the vanishing of the denominator of the scattering amplitude) determines \(\mu\), hence the energy eigenvalue \(E\), and the ratio \(\beta/\alpha\),

\[
\frac{\beta}{\alpha} = \frac{1 - C_0 I_0 - 4C_2 I_1}{4C_2 I_0}. \quad (2.28)
\]

The wavefunction is written as

\[
\psi(p) = A + \frac{B}{p^2 + \mu^2}, \quad (2.29)
\]
where
\begin{align}
A &= -4MaC_2, \quad (2.30) \\
B &= 4MC_2(\mu^2\alpha - \beta) - M\alpha C_0 \\
&= M\alpha \left[4C_2\mu^2 - (1 - 4C_2I_1)/I_0\right]. \quad (2.31)
\end{align}

Note that, after Fourier transformation, the coordinate-space wavefunction is written as a sum of the regularized delta function and the regularized Yukawa function.

The overall normalization is determined by the condition \( \int d^3p/(2\pi)^3 |\psi(p)|^2 = 1 \).

It is natural to define the ANC as \( B/4\pi \), since the Yukawa function governs the asymptotic behavior of the wave function in the limit of \( \Lambda \to \infty \).

In Sec. [IV] we take the binding energy and the ANC as low-energy physical quantities to be fixed to obtain the RG flow, instead of the scattering length and the effective range. We will numerically show that fixing the binding energy and the ANC is equivalent to fixing the scattering length and the effective range.

Given the values of the scattering length and the effective range, we determine the coupling constants \( X \) and \( Y \) by solving Eqs. (2.12) and (2.13) for each value of the cutoff \( \Lambda \). Then, solving Eq. (2.27) numerically, we obtain the binding energy and the ANC. The results for the deuteron scattering length and the effective range are shown in Fig. 2. We see that the binding energy and the ANC are constant (approximately equal to 2.19 MeV and 0.244 fm\(^{-1/2}\), respectively) for a wide range of the cutoff.

Note also that both the binding energy and the ANC vanish at \( \Lambda \approx 57.2 \) MeV, corresponding to 3.4 fm in length scale, or \( \Lambda^2/M \approx 3.5 \) MeV in energy scale. Physically it means that the resolution there is too low to see the deuteron. Remember that deuteron binding energy is 2.22 MeV, and the mean-square radius is 1.97 fm.

Incidentally, it is interesting to note that the value of the ANC we obtain in our NLO NEFT, \( B/4\pi = 0.244 \) fm\(^{-1/2}\), is very close to the recommended value \( 0.8845(8)/\sqrt{4\pi} \) fm\(^{-1/2}\) = 0.245(2) fm\(^{-1/2}\), obtained with a completely different NN potential. (The factor \( \sqrt{4\pi} \) comes from the normalization of the spherical harmonics.)

III. ANALYTIC RESULTS WITH THE LATTICE-REGULARIZED INTEGRALS

In this section, we consider the lattice regularization of the integrals, following Seki and van Kolck [27]. Consider an infinitely large lattice with a finite lattice constant \( a \). We replace the integrals \( I_0 \) by the corresponding ones integrated over the first Brillouin zone,
\[-\pi/a \leq k_i \leq \pi/a \quad (i = 1, 2, 3), \quad (3.1)\]

and the momentum square \( |k|^2 \) coming from the Laplacian \( \nabla^2 \) in the continuum by the corresponding discretized one from the finite difference representation.

The three-point formula corresponding to the replacement,
\[ |k|^2 \to \frac{4}{a^2} \sum_{i=1}^{3} \sin^2 \left( \frac{k_i a}{2} \right), \quad (3.2) \]

is widely used. We also consider the five-point formula,
\[ |k|^2 \to \frac{4}{a^2} \sum_{i=1}^{3} \left[ \sin^2 \left( \frac{k_i a}{2} \right) + \frac{1}{3} \sin^4 \left( \frac{k_i a}{2} \right) \right], \quad (3.3) \]

which has higher-order discretization errors than the three-point formula. Note that we use the same difference formula for both the interaction term and the kinetic term. With the three-point formula, for example, the integral \( I_0 \) is defined by
\[ I_0 = \frac{M}{a} \prod_{i=1}^{3} \int_{-\pi/a}^{\pi/a} \frac{dk_i}{2\pi} \left[ p^2 - 4 \sum_{i=1}^{3} \sin^2 (k_i/2) + i\epsilon \right], \quad (3.4) \]

where the change of variables \( k_i \to k_i/a \) has been performed so that the integration variables are now dimensionless. We have also introduced a dimensionless quantity \( p = \sqrt{(Ma)(p^2/a)} \).

It is important to note that the prescription described above does not produce a genuine lattice result. On a lattice the rotational invariance is explicitly broken so that the notion of “partial waves” is not good. In the
above procedure, however, we start with the continuum, rotational invariant theory, derive the LS equation for the S waves, solve it formally without specifying the regularization of the integrals, and finally invoke the lattice regularization. Although this prescription is not fully consistent, the analytic results are a very useful guide for the genuine lattice study, as shown later in Sec. IV C.

Seki and van Kolck [27] obtained the values of the constants,

$$\theta_1 = 1.58796 \ldots, \quad \theta_3 = \frac{2}{\pi}, \quad R(0) = 0.754330 \ldots,$$

with the three-point difference formula (with $\Lambda = \pi/a$), and $\theta_5$ is easily evaluated as $\theta_5 = 12/\pi^3$. (The integral $I_0$ in (3.4) can be calculated in a closed form. See Refs. [33, 34].) With these parameters, we see that the nontrivial fixed point is now located at $(X^\star, Y^\star) = (-0.76602 \ldots, 0.17501 \ldots)$. The fixed points and the flow are depicted in Fig. 3.

Note that the flow is very different from the one in the continuum, especially in the strong-coupling phase, i.e., the left-hand part of the figure. This shows precisely a non-universal feature of the flow.

As we will show in the next section, effects of the rotational symmetry breaking by the discretization with the three-point formula are large. We therefore use the five-point difference formula in the RG analysis. In this case, the values of the constants are

$$\theta_1 = 1.37619 \ldots, \quad \theta_3 = \frac{2}{\pi}, \quad \theta_5 = \frac{15}{\pi^3},$$

$$R(0) = -0.41278 \ldots.$$  

(3.6)

Here we have obtained the constants $\theta_1$ and $R(0)$ by a method similar to that of Appendix of Ref. [27]; see Appendix [ ] for the detail. In this case, the nontrivial fixed point is located at $(X^\star, Y^\star) = (-0.63338 \ldots, -0.098805 \ldots)$. The fixed points and the flow are depicted in Fig. 4.

The flow changes considerably from the case of the three-point formula, and gets more similar to the flow in the continuum, as one might expect.

The flow line corresponding to the deuteron is drawn in Fig. 5.

FIG. 3. The flow and the fixed points of the NLO NEFT in the $X$-$Y$ plane obtained by using a lattice regularization with the 3-point formula.

FIG. 4. The flow and the fixed points of the NLO NEFT in the $X$-$Y$ plane obtained by using a lattice regularization with the 5-point formula.

FIG. 5. The flow line corresponding to the deuteron obtained by using a lattice regularization with the 5-point formula.
IV. DIAGONALIZATION OF LATTICE HAMILTONIAN

A. Lattice Hamiltonian

In this section, we consider the Hamiltonian diagonalization of the NLO NEFT without pions on a spatial cubic lattice of a finite lattice constant \( a \) and a finite size \( L = N_s a \) with the periodic boundary condition. The three-dimensional position vector \( \mathbf{x} \) is replaced by \( n a \), where \( n \) is a three-dimensional vector with integer components \( n = (n_1, n_2, n_3) \). The periodic boundary condition identifies \( n \) with \( n + N_s \mathbf{e}_i \), where \( \mathbf{e}_i \) \((i = 1, 2, 3)\) is the unit vector in the \( i \)-th direction.

The Hamiltonian in the continuum,

\[
H = \int d^3x \left[ N^4 \left( \frac{\nabla^2}{2M} \right) N + C_0 (N^T P_k N)^4 (N^T P_k N) \right. \\
- C_2 \left\{ (N^T P_k N)^4 (N^T P_k \nabla^2 N) + \text{h.c.} \right\}, \tag{4.1}
\]

is given by

\[
H_L = \sum_n \left[ \frac{1}{2ML} N_n^4 \nabla^2 L N_n \right. \\
+ C_0^L (N_n^T P_k N_n)^4 (N_n^T P_k N_n) \right. \\
- C_2^L \left\{ (N_n^T P_k N_n)^4 (N_n^T P_k \nabla^2 L N_n) + \text{h.c.} \right\}. \tag{4.2}
\]

where \( \nabla^2 L \) represents the discretization of the (dimensionless) Laplacian. It is given by

\[
\nabla^2_L N_n = \sum_{i=1}^3 \left( -\frac{1}{12}N_{n+2e_i} + \frac{4}{3}N_{n+e_i} - \frac{5}{2}N_n \right. \\
+ \frac{4}{3}N_{n-e_i} - \frac{1}{12}N_{n-2e_i} \right) \tag{4.3}
\]

for the three-point formula and

\[
N_n^T \nabla^2_L N_n = \sum_{i=1}^3 \left[ (N_{n+2e_i}^T - 2N_{n+e_i}^T + N_{n-e_i}^T) P_k N_n \right. \\
- (N_{n+e_i}^T - N_{n+2e_i}^T) P_k (N_{n+e_i} - N_n) \right. \\
- (N_n^T - N_{n-e_i}^T) P_k (N_n - N_n) \right. \\
\left. + N_{n+e_i}^T P_k (N_n + e_i - 2N_n + N_{n-e_i}) \right] \tag{4.4}
\]

for the five-point formula. Similarily, the \( N_n^T \nabla^2_L N_n \) is given by

\[
\text{for the three-point formula and}
\]

\[
N_n^T \nabla^2_L N_n = \sum_{i=1}^3 \left[ (N_{n+2e_i}^T - 16N_{n+e_i}^T + 30N_n^T - 16N_{n-e_i}^T + N_{n-2e_i}^T) P_k N_n \right. \\
- \frac{1}{2} (N_{n+e_i}^T - N_{n+2e_i}^T) P_k (N_{n+e_i} - 15N_n + 15N_{n-e_i} - N_{n-2e_i}) \right. \\
- \frac{1}{2} (N_n^T - N_{n-e_i}^T) P_k (N_n + 15N_{n+e_i} - 15N_{n-e_i} - N_{n-2e_i}) \right. \\
- \frac{1}{2} (N_{n+2e_i}^T - 15N_{n+e_i}^T + 15N_n^T - N_{n-2e_i}^T) P_k (N_{n+e_i} - N_n) \right. \\
- \frac{1}{2} (N_{n+e_i}^T - 15N_n^T + 15N_{n-2e_i}^T) P_k (N_n - N_{n-e_i}) \right. \\
\left. + N_{n+e_i}^T P_k (N_{n+2e_i} - 16N_{n+e_i} + 30N_n - 16N_{n-e_i} + N_{n-2e_i}) \right] \tag{4.6}
\]

for the five-point formula.

It is easier to work in momentum space. We then

\[
N_n = \frac{1}{N_s} \sum_p e^{ip\cdot n} a_p, \tag{4.7}
\]

Fourier transform the nucleon operator as

\[
N_n = \frac{1}{N_s^{3/2}} \sum_p e^{ip\cdot n} a_p, \tag{4.7}
\]
where \( p_i = \frac{2\pi}{N_s} \hat{p}_i \),

\[ \hat{p}_i = N_s \sum \delta_{p_1+p_2+p_3-p_i} \]

where integers \( \hat{p}_i \) \((i = 1, 2, 3)\) satisfy

\[ -\frac{N_s}{2} < \hat{p}_i \leq \frac{N_s}{2} \].

The creation and annihilation operators satisfy the canonical anti-commutation relations,

\[ \{ a_p, a_{p'} \} = \{ a^\dagger_p, a^\dagger_{p'} \} = 0, \quad \{ a_p, a^\dagger_{p'} \} = \delta_{p,p'} \].

By substituting Eq. (4.7) into Eq. (4.2), we obtain the Hamiltonian in terms of creation and annihilation operators,

\[ H_L = \sum_p \frac{\Delta_p}{2M_L} a^\dagger_p a_p + \frac{1}{N_s^3} \sum \delta_{p_1+p_2+p_3-p_i,0} \]

\[ \times \left[ C_0^L + C_2^L \left( \Delta_{p_1} + \Delta_{p_2} + \Delta_{p_3} \right) \right] \]

\[ \times \left( a^\dagger_{p_1} a_{p_3} a^\dagger_{p_2} \right) (a_{p_1} a_{p_2} a^\dagger_{p_3}) \],

\[ \Delta_p = \begin{cases} 
4 \sum_{i=1}^3 \sin^2 \left( \frac{p_i}{2} \right), & \text{(three-point formula)} \\
4 \sum_{i=1}^3 \left[ \sin^2 \left( \frac{p_i}{2} \right) + \frac{1}{3} \sin^4 \left( \frac{p_i}{2} \right) \right], & \text{(five-point formula)} 
\end{cases} \]

\[ \Delta_{p,q} = \begin{cases} 
4 \sum_{i=1}^3 \left( \sin^2 \left( \frac{p_i}{2} \right) + \sin^2 \left( \frac{q_i}{2} \right) \right) \\
-2 \cos \left( \frac{p_i + q_i}{2} \right) \sin \left( \frac{p_i}{2} \right) \sin \left( \frac{q_i}{2} \right), & \text{(three-point formula)} \\
4 \sum_{i=1}^3 \left( \sin^2 \left( \frac{p_i}{2} \right) - \sin^2 \left( \frac{q_i}{2} \right) \right) \\
+ \sin^2 \left( \frac{q_i}{2} \right) + \frac{1}{3} \sin^4 \left( \frac{q_i}{2} \right) \\
- \cos \left( \frac{q_i}{2} \right) \sin \left( \frac{p_i}{2} \right) \sin \left( \frac{q_i}{2} \right) \sin \left( \frac{q_i}{2} \right), & \text{(five-point formula)} 
\end{cases} \]

Note that \( \Delta_{p,q} = \Delta_{q,p} \) and \( \Delta_{p,-p} = 4\Delta_p \).

### B. Schrödinger equation for two-nucleon states

Now we consider the lattice version of the (stationary) Schrödinger equation,

\[ H_L |\Psi^k\rangle = E_L |\Psi^k\rangle \]

where \( |\Psi^k\rangle \) is the two-nucleon state with the zero total momentum and the spin-triplet isospin-singlet projection,

\[ |\Psi^k\rangle = \sum_p \psi(p) a^\dagger_{p,k} a_{-p} |0\rangle \]

and \( E_L = Ea \) is the dimensionless energy eigenvalue. In terms of the discretized “momentum-space wavefunction” \( \psi(p) \) of relative motion, the Schrödinger equation can be written as

\[ E_L \psi(p) = \frac{\Delta_p}{ML} \psi(p) + \frac{1}{N_s^3} \sum_q \left[ C_0^L + 4C_2^L \left( \Delta_{p} + \Delta_{q} \right) \right] \psi(q) \]

which is nothing but the discretized version of Eq. (2.22).

We numerically diagonalize the eigenvalue equation (4.16). The physical length of the lattice constant is determined by giving \( M_L \) through the relation \( M_L = M a \), where \( M \) is the physical nucleon mass which we set \( M = 938.9 \) MeV. (Note that there is no self-energy contribution in our theory.) We typically consider the case \( a = 5 \) fm, which corresponds to the momentum cutoff \( \Lambda = \pi/a \approx 124 \) MeV. Most of the calculations are done with \( N_s = 16 \), which corresponds to a cube with the edge of length \( L = 80 \) fm.

We are interested only in the groundstate. In the strong coupling phase, it is a boundstate. In the weak coupling phase, there is no boundstate physically, but the periodic boundary condition makes the groundstate have negative energy [25].

### C. RG analysis

In the following analysis, we use the dimensionless coupling constants \( X \) and \( Y \) defined in Eq. (2.14), but with \( \Lambda = \pi/a \).

We choose the binding energy and the ANC as low energy physical quantities and require them to be independent of the lattice constant. The RG flow can be numerically obtained by first calculating the binding energy and the ANC for a set of \((X,Y)\) and then changing the lattice constant a bit from \( a \) to \( a + \delta a \) and searching numerically the new set of \((X + \delta X, Y + \delta Y)\) that gives the same binding energy and the ANC.

The ANC is most easily obtained by fitting the numerically obtained (normalized) “momentum-space wave-
function $\psi(p)$ to the expression

$$
\psi(p) = A + \frac{B}{M_L|E_L| + \Delta p},
$$

(4.17)

and determining the constants $A$ and $B$. Note that this form of the wavefunction is implied by the Schrödinger equation (4.16), and corresponds to the continuum wavefunction, Eq. (2.29). We thus identify $B/4\pi$ with the ANC.

In Fig. 6, we show the RG flow calculated with the five-point formula. We draw the change $(\delta X, \delta Y)$ for $a = 5$ fm and $\delta a = 0.25$ fm. FIG. 6. The flow of the NLO NEFT in the strong coupling phase in the $X$-$Y$ plane obtained by numerical diagonalization of the Hamiltonian defined on a lattice with the 5-point formula.

The right upper part of the figure corresponds to the weak coupling phase. Because of the fictitious feature of the groundstate energy in the weak coupling phase, we do not calculate the flow.

It is difficult to calculate the flow near the phase boundary. Near the phase boundary, the groundstate energy becomes small, and the effects of the periodic boundary condition becomes noticeable [35]. The wavefunction with the radius $\sim L/2 = 40$ fm is affected by the boundary condition. This radius corresponds to the binding energy 0.03 MeV. The finite-size effect however brings about useful information, as shown below.

The $L$ dependence of the groundstate is shown in Fig. 7, where the difference of the calculated groundstate energies with $N_s = 14$ and $N_s = 16$ is larger in the $Ns$ than in the $N_s$ with $\delta a = 0.25$ [35] for the both sides of the ridge line. FIG. 7. The difference of the calculated groundstate energies with $N_s = 14$ and $N_s = 16$ (upper surface), and that with $N_s = 16$ and $N_s = 18$ (lower surface) are shown as functions of $X$ and $Y$. This side of the mountain range is the weak coupling phase, the other side the strong coupling phase.

The direction in which the RG flow goes out from the nontrivial fixed point is. It is ($X_*, Y_*) = (-0.65 \sim -0.63, -0.13 \sim -0.11)$, which is surprisingly close to the one obtained analytically with the five-point formula in Sec. 11

The direction in which the RG flow goes out from the nontrivial fixed point (the dashed line direction in Fig. 6) represents the relevant operator. The (unit) vector for the direction is $(-1/\sqrt{2}, -1/\sqrt{2})$ within the accuracy of the present analysis. This is very different from $(-0.933, \ldots, -0.359 \ldots)$ obtained by linearizing the RGEs (2.19) and (2.20) with the five-point formula around the nontrivial fixed point.
FIG. 8. Typical wavefunctions near the critical line. In the weak coupling phase (upper), the wavefunction spreads out over the whole space with a small peak at the center of potential. In the strong coupling phase (lower), it is sharply peaked at the center of the potential and decays exponentially.

We perform similar analysis with the three-point formula. In Fig. 10 we show the ridge line together with the RG flow. The RG flow is considerably different from that with the five-point formula. The location of the nontrivial fixed point is \((X^\star, Y^\star) = (-0.75 \sim -0.77, 0.12 \sim 0.14)\). It is again very close to the one analytically obtained. The relevant direction is now represented by a vector \((-1/2, -\sqrt{3}/2)\) within the accuracy of the present analysis. It should be compared with \((-0.935 \ldots, 0.353 \ldots)\) obtained from the linearized RGEs derived from Eqs. (2.19) and (2.20) with the three-point formula.

To summarize, the analytic results with the lattice regularization, which are not obtained on a lattice, are very accurate for the location of the nontrivial fixed point, but the direction of the relevant operator is considerably different from the one on a lattice.

It is instructive to see how the explicit rotational-symmetry breaking affects the shape of the “wavefunction.” In Figs. 11, we show \(\psi(r)/(e^{-\sqrt{M_L}E_L|r|}/r)\) in the \((1,0,0), (1,1,0),\) and \((1,1,1)\) directions as a function of \(r = |r|\), where \(\psi(r)\) is the Fourier transform of \(\psi(p)\). Precisely, for the function taken as vertical axis, we have taken into account the periodicity minimally, that is, the effect of the potentials within the distance \(L = N_s a\), while the potentials at larger distances give negligibly small corrections and are thus ignored. For example, the function we actually have plotted in the \((1,0,0)\) direction is \(\psi(na, 0, 0)/(e^{-\sqrt{M_L}E_L|na|}/na + e^{-\sqrt{M_L}E_L|(N_s a - na)|}/(N_s a - na))\) for integers \(n\) satisfying \(0 \leq n \leq N_s\). If the wavefunction were rotationally symmetric, they would coincide with each other and show a plateau (with the value of ANC) at long distances. We see that the calculation with the three-point formula shows rather large direction-dependence, but the use of the five-point formula reduces it largely.

D. The flow line corresponding to deuteron

Finally we draw a flow line that corresponds to deuteron. As input parameters, we use the binding energy \(E = 2.19\) MeV and the ANC = 0.244 fm\(^{-1/2}\), which are obtained in Sec. II. The flow line is shown in Fig. 12 for the five-point formula, and in Fig. 13 for the three-point formula, together with the RG flow, the nontrivial fixed point, the phase boundary, and the relevant direction.

The flow line is obtained as follows: (1) We look for a
point \((X_0, Y_0)\) for which the binding energy and the ANC take the values given above for a certain value of the lattice constant \(a_0\). (2) We then change the lattice constant 5% larger, \(a_1 = 1.05 a_0\), and search for a new set of coupling constants \((X_1, Y_1)\) for which the binding energy and the ANC take the same values. (3) We repeat the procedure; that is, starting with the set of coupling constants \((X_1, Y_1)\) and the lattice constant \(a_1\), we change the lattice constant 5% larger, \(a_2 = 1.05 a_1\), and search for a new set of the coupling constant \((X_2, Y_2)\) for which the binding energy and the ANC take the same values, and so on. Remember that our lattice Hamiltonian does not contain the lattice constant and its value is determined through the dimensionless nucleon mass \(M_L = Ma\), so that changing the value of \(M_L\) amounts to changing the value of \(a\). When drawing Figs. 12 and 13 we change \(M_L\) in the region \(9 \lesssim M_L \lesssim 80\), corresponding 2 fm \(\lesssim a \lesssim 17\) fm. Of course the lattice with \(a \sim 2\) fm is too fine for the present EFT, the calculation there should not take too seriously.

The part of the flow closest to the nontrivial fixed point corresponds to the lattice constant \(a\) in the range 5 to 10 fm, corresponding to the momentum scale 62 to 124 MeV, just in the region of validity of the EFT without pions.

V. SUMMARY AND DISCUSSIONS

In this paper, we diagonalize the Hamiltonian for the NLO NEFT without pions defined on a spatial lattice in order to obtain the two-nucleon boundstate, which is mainly in the S wave. We obtain the RG flows by chang-

![Graph](image-url)
FIG. 12. The flow line corresponding to deuteron is shown as a dotted line (magenta) against the RG flow given in Fig. 9. The calculations are done with the five-point formula. The nontrivial fixed point (blue bullet) phase boundary (red solid line), and the relevant direction (green dashed line) are also shown.

FIG. 13. The same as in Fig. 12, but the calculations are done with the three-point formula.

(deuteron).

The ANC however does not seem to work for deep boundstates. This is because of a finite cutoff imposed on the EFT, not specific to the lattice regularization. Remember that the wavefunction may be written as a sum of a regularized delta function and a regularized Yukawa function, and our definition of the ANC is the coefficient of the latter. The regularized Yukawa function damps exponentially and the damping depends on the binding energy; it damps more rapidly for larger binding energies. On the other hand, the regularized delta function damps independently of the binding energy. Thus, for deep boundstates, the asymptotic form of the wavefunction is dominated by the regularized delta function. Fixing the ANC there does not control physics any more. We think that this is the reason why the RG flows obtained numerically differ considerably from those obtained analytically in the region of deep boundstates.

Throughout these analyses, we confine ourselves to the strong coupling phase and look at the properties of the boundstates. Calculations in the weak coupling phase, on the other hand, will bring about information on physical quantities of scattering through so-called Luscher’s formula [36]. It can be used as inputs for the coupling constants of the operators in other channels.

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Appendix: Evaluation of $I_0$ with the five-point formula

In this section, we consider the integral

$$W(z) = \prod_{i=1}^{3} \left[ \int_{-\pi}^{\pi} \frac{dk_i}{2\pi} \right] \frac{1}{z + \sum_{i=1}^{3} \left[ \sin^2 \left( \frac{k_i}{2} \right) + \frac{1}{3} \sin^4 \left( \frac{k_i}{2} \right) \right]}$$

(A.1)

assuming $z > 0$. The integral $I_0$ with five-point formula is obtained by analytic continuation of the variable $z$. We are interested in the coefficients of the first few terms of the expansion of $W(z)$ in powers of $z$.

We employ the method similar to that Seki and van Kolck [27] used when they evaluated the Watson integral. We start with the identity,

$$\int_0^\infty d\alpha e^{-\alpha X} = \frac{1}{X}, \quad (X > 0),$$

(A.2)

and rewrite $W(z)$ as

$$W(z) = \int_0^\infty d\alpha e^{-\alpha z} [U(\alpha)]^3,$$

(A.3)

where we have introduced

$$U(\alpha) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \exp \left\{ -\alpha \left( \sin^2 \left( \frac{k}{2} \right) + \frac{1}{3} \sin^4 \left( \frac{k}{2} \right) \right) \right\}.$$

(A.4)

By making a change of variable from $k$ to $t = \sin(k/2)$, we rewrite it as

$$U(\alpha) = \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{1 - t^2}} e^{-\alpha (2t^2 + t^4/3)}.$$

(A.5)
We divide the integration region of $\alpha$ into two: the region $0 \leq \alpha \leq A$ and the region $A \leq \alpha < \infty$. In the former, we expand $e^{-\alpha z}$, 

$$\int_0^A d\alpha \, [U(\alpha)]^3 - z \int_0^A d\alpha \, [U(\alpha)]^3 + \cdots,$$  

(A.6)

and evaluate the integrals numerically. In the latter, the dominant contribution of the $U(\alpha)$ integral comes from the small $t$ region, so we approximate 

$$\frac{1}{\sqrt{1 - t^2}} = e^{-\frac{1}{2} \ln(1 - t^2) / t^2} \approx e^{\frac{1}{2} t^2 / t^4},$$  

(A.7)

and get 

$$U(\alpha) \approx \frac{2}{\pi} \int_0^\infty dt \, \exp \left[ - \left( \alpha - \frac{1}{2} \right) t^2 - \left( \frac{\alpha}{3} - \frac{1}{4} \right) t^4 \right]$$

$$= \frac{1}{\pi} \sqrt{\frac{a}{b}} e^{\frac{a^2}{2b}} K_2 \left( \frac{a^2}{2b} \right),$$  

(A.8)

where $K_\nu(z)$ is the modified Bessel function and we have introduced $a = \alpha - 1/2$ and $b = 4\alpha/3 - 1$. By using the asymptotic expansion of $K_\nu(z)$, 

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{\Gamma(\nu + n + \frac{1}{2})}{n! \Gamma(\nu - n + \frac{1}{2})(2z)^n},$$

where $|\text{arg}z| < 3\pi/2$, 

(A.9)

we obtain for a large value of $\alpha$ the following expansion, 

$$U(\alpha) = \frac{1}{\sqrt{\alpha \pi}} \left[ 1 + \left( \frac{1}{3} \right) \frac{1}{\alpha} - \frac{35}{48} \left( \frac{1}{\alpha} \right)^3 + \cdots \right].$$  

(A.10)

Substituting it into the integrand, we get the expansion 

$$\int_A^\infty d\alpha e^{-\alpha z} [U(\alpha)]^3$$

$$= \frac{1}{A^{1/2} \pi^{3/2}} \int_1^\infty ds \, s^{3/2} \frac{e^{-A z s}}{s^{1/2}} \left[ 1 + \frac{1}{A^2} \left( \frac{1}{s} \right)^2 - \frac{35}{16A^3} \left( \frac{1}{s} \right)^3 + \cdots \right]$$

$$= \frac{1}{A^{1/2} \pi^{3/2}} \left\{ \phi_{-\frac{1}{2}}(A z) + \frac{1}{A^2} \phi_{-\frac{3}{2}}(A z) - \frac{35}{16A^3} \phi_{-\frac{5}{2}}(A z) + \cdots \right\},$$  

(A.11)

where $\phi_m(z)$ is the incomplete Gamma function, 

$$\phi_m(z) = \int_0^\infty dt \, t^m e^{-zt} = z^{-(1+m)} \Gamma(1+m, z).$$  

(A.12)

Expanding Eq. (A.11) in powers of $z$, we have 

$$\int_A^\infty d\alpha e^{-\alpha z} [U(\alpha)]^3$$

$$= \frac{1}{A^{1/2} \pi^{3/2}} \left( 2 + \frac{2}{5A^2} - \frac{5}{8A^3} \right)$$

$$- \frac{2}{A^{1/2} \pi^{1/2}} \left( 2A - \frac{2}{3A} + \frac{7}{8A^2} \right) z$$

$$+ \frac{1}{A^{1/2} \pi^{1/2}} \left( -\frac{A^2}{3} + 1 - \frac{35}{48A} \right) z^2$$

$$- \frac{8}{15\pi} z^{5/2} + \cdots.$$  

(A.13)

Note that the coefficients of the half-odd-integer powers of $z$ do not depend on $A$.

We numerically evaluate the sum of Eqs. (A.6) and (A.13) for various values of $A$, we find that the sum is independent of $A$ for a wide range of $A$. The coefficient of the term of $O(z^0)$ is about 0.876111 and that of $O(z^1)$ is about 0.106502, which correspond to $\theta_1 = 1.37619\ldots$ and $R(0) = -0.412781\ldots$. 

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