Integral Transform Approach to Generalized Tricomi Equation

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Abstract

In the paper is presented some integral transform that allows to obtain solutions of the generalized Tricomi equation with $x$-dependent coefficients from solutions of a simpler equation. The particular version of this transform was used in a series of papers [13, 14], [41]-[46] to investigate in a unified way several equations such as the linear and semilinear Tricomi equations, Gellerstedt equation, the wave equation in Einstein-de Sitter spacetime, the wave and the Klein-Gordon equations in the de Sitter and anti-de Sitter spacetimes.

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1 Introduction

In this paper we construct solution operators for the generalized Tricomi equation. The construction is based on the extension of the approach suggested in [41]; this extension allows to write solution of the generalized Tricomi equation with $x$-dependent coefficients via solutions of a simpler equation. The particular version of this approach was used in a series of papers [13, 14], [41]-[46] to investigate in a unified way several equations such as the linear and semilinear Tricomi equations, Gellerstedt equation, the wave equation in Einstein-de Sitter spacetime, the wave and the Klein-Gordon equations in the de Sitter and anti-de Sitter spacetimes. The listed equations play an important role in the gas dynamics, elementary particle physics, quantum field theory in curved spaces, and cosmology.

Consider for the smooth function $f = f(x,t)$ the solution $w = w_{A,f}(x,t;b)$ to the problem

$$w_{tt} - A(x,\partial_x)w = 0, \quad w(x,0;0) = f(x), \quad w_t(x,0;0) = 0, \quad t \in [0,T_1] \subseteq \mathbb{R}, \quad x \in \tilde{\Omega} \subseteq \mathbb{R}^n,$$

with the parameter $b \in I = [t_{in},T] \subseteq \mathbb{R}, \quad 0 \leq t_{in} < T \leq \infty,$ and with $0 < T_1 \leq \infty$. Here $\tilde{\Omega}$ is a domain in $\mathbb{R}^n$, while $A(x,\partial_x)$ is the partial differential operator $A(x,\partial_x) = \sum_{|\alpha| \leq m} a_\alpha(x)\partial_x^\alpha$ with smooth coefficients, $a_\alpha \in C^\infty(\tilde{\Omega})$. We are going to present the integral operator

$$K[w](x,t) = \int_{t_{in}}^t db \int_0^{[\phi(t) - \phi(b)]} K(t;r,b)w(x,r;b) \, dr, \quad x \in \tilde{\Omega}, \quad t \in I,$$

which maps the function $w = w(x,r;b)$ into the solution $u = u(x,t)$ of the generalized Tricomi equation

$$u_{tt} - a^2(t)A(x,\partial_x)u = f, \quad x \in \tilde{\Omega}, \quad t \in I,$$

where $a^2(t) = t^\ell, \quad \ell \in \mathbb{C}$. In fact, the function $u = u(x,t)$ takes initial values as follows

$$u(x,t_{in}) = 0, \quad u_t(x,t_{in}) = 0, \quad x \in \tilde{\Omega}.$$

In (2), $\phi = \phi(t)$ is a distance function produced by $a = a(t)$, that is $\phi(t) = \int_{t_{in}}^t a(\tau) \, d\tau$. Moreover, we also introduce the corresponding operators, which generate solutions of the source-free equation and takes
non-vanishing initial values. These operators are constructed in \[41\] in the case of \(\ell > 0\), \(A(x, \partial_x) = \Delta\), \(\Omega = \mathbb{R}^n\), where \(\Delta\) is the Laplace operator on \(\mathbb{R}^n\), and, consequently, equation (4) is the wave equation. In the present paper we restrict ourselves to the smooth functions, but it is easily seen that similar formulas, with the corresponding interpretations, are applicable to the distributions as well.

In order to motivate our approach, we consider the solution \(w = w(x, t; b)\) to the Cauchy problem

\[
\begin{align*}
  w_{tt} - \Delta w &= 0, & (t, x) &\in \mathbb{R}^{1+n}, & w(x, 0; b) &= \varphi(x, b), & w_t(x, 0; b) &= 0, & x &\in \mathbb{R}^n, \\
\end{align*}
\]

(4)

with the parameter \(b \in I \subseteq \mathbb{R}\). We denote that solution by \(w = w(x, t; b)\); if \(\varphi\) is independent of the second time variable \(b\), then we write simply \(w_\varphi(x, t)\). There are well-known explicit representation formulas for the solution of the problem (4). (See, e.g., \[19\], \[23\], \[35\].)

The starting point of the approach suggested in \[41\] is the Duhamel’s principle (see, e.g., \[35\], Ch.4), which has been revised in order to prepare the ground for generalization. Our first observation is that the function

\[
  u(x, t) = \int_{t_{1n}}^t \int_0^{t-b} w_f(x, r; b) \, dr,
\]

(5)

is the solution of the Cauchy problem \(u_{tt} - \Delta u = f(x, t)\) in \(\mathbb{R}^{n+1}\), and \(u(x, t_{1n}) = 0, u_t(x, t_{1n}) = 0\) in \(\mathbb{R}^n\), if the function \(w_f = w_f(x; t; b)\) is a solution of the problem (4), where \(\varphi = f\). The second observation is that in (5) the upper limit \(t - b\) of the inner integral is a distance function. Our third observation is that the solution operator \(G : f \mapsto u\) can be regarded as a composition of two operators. The first one

\[
  \mathcal{WE} : f \mapsto w
\]

is a Fourier Integral Operator, which is a solution operator of the Cauchy problem for wave equation. The second operator

\[
  \mathcal{K} : w \mapsto u
\]

is the integral operator given by (5). We regard the variable \(b\) in (5) as a “subsidiary time”. Thus, \(G = \mathcal{K} \circ \mathcal{WE}\) and we arrive at the diagrams of Figure 1.

\[
  \begin{array}{ccc}
  f & \xrightarrow{\mathcal{WE}} & w \\
  \downarrow & & \\
  G & \xrightarrow{\mathcal{K}} & u \\
  \downarrow & & \\
  f & \xrightarrow{\mathcal{EE}_A} & w \\
  \downarrow & & \\
  \mathcal{G}_A & \xrightarrow{\mathcal{K}} & u
  \end{array}
\]

Figure 1: (a) Case of wave equation \(A(x, \partial_x) = \Delta\) (b) Case of general \(A(x, \partial_x)\)

Based on the first diagram, we have generated in \[44\] a class of operators for which we have obtained explicit representation formulas for the solutions, and, in particular, the representations for the fundamental solutions of the partial differential operator.

In the present paper, by varying the first mapping, we extend the class of the equations for which we can obtain explicit representation formulas for the solutions. More precisely, consider the diagram (b) of Figure 1, where \(w = w_{A,\varphi}(x; t; b)\) is a solution to the problem (4) with the parameter \(b \in I \subseteq \mathbb{R}\). If we have a resolving operator of the problem (4), then, by applying (2), we can generate solutions of another equation. Thus, \(\mathcal{G}_A = \mathcal{K} \circ \mathcal{EE}_A\). The new class of equations contains operators with \(x\)-depending coefficients, and those equations are not necessarily hyperbolic.

In this paper we restrict ourselves to the generalized Tricomi equation, that is \(a(t) = t^\ell, \, \ell \in \mathbb{C}\). This class includes, among others, equations of the wave propagating in the so-called Einstein-de Sitter (EdeS) universe and in the radiation dominated universe with the spatial slices of the constant curvature. We believe that the integral transform and the representation formulas for the solutions that we derive in this article fill up the gap in the literature on that topic.
The transform linking to the generalized Tricomi operator is generated by the kernel
\[
K(t; r, b) = E(r, t; b; \gamma) := c_\ell \left((\phi(t) + \phi(b))^2 - r^2\right)^{-\gamma} F\left(\gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2}\right),
\]  
with the distance function \(\phi = \phi(t)\) and the numbers \(\gamma, c_\ell\) defined as follows
\[
\phi(t) = \frac{2}{\ell + 2} t^{\frac{\ell + 2}{2}}, \quad \gamma := \frac{\ell}{2(\ell + 2)}, \quad \ell \in \mathbb{C} \setminus \{-2\}, \quad c_\ell = \left(\frac{\ell + 2}{4}\right)^{-\frac{\ell + 2}{2}},
\]  
while \(F(a, b; c; \zeta)\) is the Gauss’s hypergeometric function. Here \(t_{\text{in}} = 0\).

According to Theorem 2.1 the function \(E(r, t; b; \gamma)\) solves the following Tricomi-type equation:
\[
E_{tt}(r, t; b; \gamma) - t^4 E_{rr}(r, t; b; \gamma) = 0, \quad 0 < b < t.
\]  
The proof of Theorem 2.1 which is given in Section 2 is straightforward. In fact, that proof is applicable to the different distance functions \(\phi = \phi(t)\), see, for instance, [47], where the case of \(a(t) = e^{-t}\) is discussed.

There are four important examples of the equations which are amenable to the integral transform approach, when \(\ell = 3, 1, -1, -4/3\); these are the small disturbance equation for the perturbation velocity potential of a two-dimensional near sonic uniform flow of dense gases in a physical plane (see, e.g., [21], [38]), the Tricomi equation (see, e.g., [2, 4, 8, 15, 11, 16, 20, 22, 24, 25, 26, 29, 30, 31, 33, 37, 39] and bibliography therein), the equation of the waves in the radiation dominated universe (see, e.g., [10] [17] and bibliography therein) and in the EdeS spacetime (see, e.g., [10] [17] [18] [32] and bibliography therein), respectively.

To introduce the integral transform we need some special geometric structure of the domains of functions.

**Definition 1.1** The set \(\Omega \subseteq \mathbb{R}^{\ell+1}_+\) is said to be backward time line-connected to \(t = 0\), if for every point \((x, t) \in \Omega\) the line segment \(\{(x, s) \mid s \in (0, t)\}\) is also in \(\Omega\); that is \(\{(x, s) \mid s \in (0, t)\} \subseteq \Omega\).

Henceforth we just write ”backward time connected” for such sets. Similarly, if \(\Omega \subseteq [0, T] \times \mathbb{R}^n, T > 0\), then one can define a forward time line-connected to \(t = T\) set. The union and intersection of the backward time connected sets is also a backward time connected. The interior and the closure of the backward time connected set are also a backward time connected sets. For every set there exists its minimal backward time connected covering. The domain of the dependence for the wave equation is backward time connected, while domain of influence is forward time connected.

**Definition 1.2** Let \(\phi \in C^1(\mathbb{R}_+^\ell)\) be positive, and \(\Omega\) be a backward time connected set. The backward time connected set \(\Omega_\phi \subseteq \mathbb{R}^{\ell+1}_+\) defined by
\[
\Omega_\phi := \bigcup_{(x, t) \in \Omega} \{(x, \tau) \mid \tau \in (0, \phi(t)]\}
\]
is said to be a \(\phi\)-image of \(\Omega\).

On Figures 2, 3 we illustrate the dependence domains for hyperbolic equations with \(\ell = 0, -\frac{4}{3}, 1, 3\) and \(A(x, \partial_x) = \Delta\).

Figure 4 illustrates the part of the domain in the hyperbolic region of the Tricomi problem (see, e.g., [24] and references therein) that has the form \(\Omega := \{(x, t) \mid |x| < x_0 - \frac{2}{3} t^2, -x_0 \leq x \leq x_0, t > 0\}\) for \(x_0 = 1/2\) and \(x_0 = 100\). Corresponding \(\phi\)-images of \(\Omega\) are \(\Omega_\phi := \{(x, t) \mid |x| < x_0 - \left(\frac{3}{4}\right)^{5/2} t^2, -x_0 \leq x \leq x_0\}\) with \(x_0 = 1/2\) and \(x_0 = 100\), respectively.
The next theorem describes the main property of the integral transform \(2\).

**Theorem 1.3** Let \(f = f(x,t)\) be a smooth function defined in the backward time connected domain \(\Omega\). Suppose that for a given \((x_0,t_0) \in \Omega\) the smooth function \(w(x,r;b)\) satisfies the relations

\[
\begin{align*}
    w_{rr} - A(x, \partial_x)w &= 0 \quad \text{at} \quad x = x_0 \quad \text{for all} \quad r \in (0, |\phi(t_0) - \phi(b)|), \\
    w(x_0,0;b) &= f(x_0,b) \quad \text{at} \quad x = x_0 \quad \text{for all} \quad b \in (0,t_0).
\end{align*}
\]

Then for \(\ell > -2\) the function

\[
u(x,t) = \ell \int_0^t db \int_0^r ((\phi(t) + \phi(b))^2 - r^2)^{\gamma - 1} F \left( \gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2} \right) w(x,r;b) \, dr,
\]

defined on the past \(\{x_0\} \times [0,t_0)\) of the point \((x_0,t_0)\), satisfies the equation

\[
u_{tt} - \ell^2 A(x, \partial_x)\nu = f(x,t) + c\ell((\phi'(t))^2 \int_0^t ((\phi(t) + \phi(b))^2)^{-\gamma} F \left( \gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2}{(\phi(t) + \phi(b))^2} \right) w_r(x,0;b) \, db,
\]

defined on the past \(\{x_0\} \times [0,t_0)\) of the point \((x_0,t_0)\), satisfies the equation

\[
u_{tt} - \ell^2 A(x, \partial_x)\nu = f(x,t) + c\ell((\phi'(t))^2 \int_0^t ((\phi(t) + \phi(b))^2)^{-\gamma} F \left( \gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2}{(\phi(t) + \phi(b))^2} \right) w_r(x,0;b) \, db,
\]

at \(x = x_0\) and for all \(t \in (0,t_0)\). Furthermore, it has the vanishing initial values

\[
u(x_0,0) = 0, \quad \nu_t(x_0,0) = 0.
\]

We stress here that the integral transform \(w \mapsto \nu\) is point-wise in \(x\) and non-local in time. Let \(\pi_x: \Omega \longrightarrow \mathbb{R}^n\) be a projection \(\pi_x: \Omega \longrightarrow \mathbb{R}^n\) of the backward time connected domain \(\Omega\), and denote \(\tilde{\Omega} := \pi_x(\Omega)\).

**Corollary 1.4** Let \(f = f(x,t)\) be a smooth function defined in the backward time connected domain \(\Omega\). Suppose that the smooth function \(w(x,r;b)\) satisfies

\[
\begin{align*}
    w_{rr} - A(x, \partial_x)w &= 0 \quad \text{for all} \quad (x,r) \in \Omega_{\phi} \quad \text{and for all} \quad (x,b) \in \Omega, \\
    w(x,0;b) &= f(x,b) \quad \text{for all} \quad (x,b) \in \Omega.
\end{align*}
\]
Then for $\ell > -2$ the function

$$u(x, t) = c_\ell \int_0^t \int_0^{\gamma - \phi(b)} \left((\phi(t) + \phi(b))^2 - r^2\right)^{-\gamma} F\left(\gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2}\right) w(x, r; b) \, dr,$$

defined on $\Omega$, satisfies the equation

$$u_t - t^\ell A(x, \partial_x) u = f(x, t) + c_\ell (\phi'(t))^2 \int_0^t \left((\phi(t) + \phi(b))^2 - r^2\right)^{-\gamma} F\left(\gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2}\right) w_r(x, 0; b) \, db,$$  \hspace{1em} (12)

in $\Omega$. Furthermore, it has the vanishing initial values

$$u(x, 0) = 0, \quad u_t(x, 0) = 0 \quad \text{for all} \quad x \in \Omega.$$

If the initial value problem for the operator $\partial_t^2 - A(x, \partial_x)$ admits two initial conditions, then we can eliminate the function $w$ from the right-hand side of equation (12). For instance, that holds for the Cauchy problem for the second order hyperbolic equation. Thus, we derive the following representation formula for the solution of the initial value problem for the generalized Tricomi equation.

**Theorem 1.5** Let $f = f(x, t)$ be a smooth function defined in $\mathbb{R}^n \times [0, T]$. Suppose that the function $w(x, r; b) \in C^{m,2,0}_{x,r,t}(\mathbb{R}^n \times [0, \phi(T)] \times [0, T])$ solves the Cauchy problem

$$w_{rr} - \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha w = 0, \quad \text{for all} \quad x \in \mathbb{R}^n, \quad b \in [0, T], \quad r \in [0, \phi(T)],$$

for the equation with smooth coefficients $a_\alpha \in C^\infty(\mathbb{R}^n \times [0, T])$.

Then for $\ell > -2$ the function

$$u(x, t) = c_\ell \int_0^t \int_0^{\gamma - \phi(b)} \left((\phi(t) + \phi(b))^2 - r^2\right)^{-\gamma} F\left(\gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2}\right) w(x, r; b) \, dr,$$  \hspace{1em} (13)

defined on $\mathbb{R}^n \times [0, T]$, solves the equation

$$u_t - t^\ell \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha u = f(x, t), \quad x \in \mathbb{R}^n, \quad t \in (0, T),$$

and has the vanishing initial values

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad x \in \mathbb{R}^n.$$

We remind here that for the weakly hyperbolic operators $\partial_t^2 - \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$, which satisfy the Levi conditions (see, e.g., [10]), the Cauchy problem can be solved for the smooth initial data. On the other
hand, the Cauchy-Kowalewski theorem guarantees solvability of the problem in the real analytic functions category for the partial differential equation (13) with any positive $\ell$ and $m = 2$. Furthermore, the operator $\sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$ can be replaced with an abstract operator $A$ acting on some linear topological space of functions.

**Example 1.** Consider equations of the gas dynamics. (a) For the Tricomi equation in the hyperbolic domain,

$$u_{tt} - t \Delta u = f(x,t), \quad (14)$$

$\phi(t) = \frac{2}{3} t^2$ and $A(x, \partial_x) = \Delta$. Then for every $f \in C(\mathbb{R}^n \times [0,T])$ we can solve the Cauchy problem for the wave equation

$$w_{tt} - \Delta w = 0, \quad w(x,0;b) = f(x,b), \quad w_t(x,0;b) = 0, \quad x \in \mathbb{R}^n, \quad t \in \left[0, \frac{2}{3} T^{2/3}\right]$$

in $\mathbb{R}^n \times \left[0, \frac{2}{3} T^{2/3}\right] \times [0,T]$. (For the explicit formula see, e.g., (24), (25).) The solution to the Cauchy problem for (14) with vanishing initial data is given as follows

$$u(x,t) = 3^{-1/3} 2^{2/3} \int_0^t db \int_0^t \left( \frac{2}{3} t - \frac{2}{3} r \right)^2 - r^2 \right)^{-\frac{1}{3}} \times F \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}; \frac{2}{3} t - \frac{2}{3} r \right) \frac{2}{3} t + \frac{2}{3} r \right)^2 - r^2 \right) w(x,r;b)dr, \quad t \in [0,T].$$

For the Tricomi equation in the elliptic domain,

$$u_{tt} + t \Delta u = f(x,t), \quad t > 0, \quad (15)$$

we have $A(x, \partial_x) = -\Delta$ and, since the Cauchy problem is not well posed, Theorem 1.3 gives representation of the solutions only for some specific functions $f$.

(b) The small disturbance equation for the perturbation velocity potential of a two-dimensional near sonic uniform flow of dense gases in a physical plane, has been derived by Kluwick [21], Tarkenton and Cramer [38]. It leads to the equation

$$u_{tt} - t^3 \Delta u = f(x,t), \quad (16)$$

with $\ell = 3$ and $\phi(t) = \frac{2}{3} t^2$, and $A(x, \partial_x) = \Delta$. The solution to the Cauchy problem for (14) with vanishing initial data is given as follows

$$u(x,t) = \frac{3}{10} \int_0^t db \int_0^t \left( \frac{2}{3} t - \frac{2}{3} r \right)^2 - r^2 \right)^{-\frac{1}{3}} \times F \left( \frac{3}{10}, \frac{3}{10}, \frac{3}{10}; \frac{2}{3} t - \frac{2}{3} r \right) \frac{2}{3} t + \frac{2}{3} r \right)^2 - r^2 \right) w(x,r;b)dr, \quad t > 0.$$

**Example 2.** Consider the wave equation in the Einstein-de Sitter (EdS) spacetime with hyperbolic spatial slices. The metric of the Einstein & de Sitter universe (EdS universe) is a particular member of the Friedmann-Robertson-Walker metrics

$$ds^2 = -dt^2 + a_{sc}^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right], \quad (17)$$

where $K = -1, 0, +1$, for a hyperbolic, flat or spherical spatial geometry, respectively. For the EdS the scale factor is $a_{sc}(t) = t^{2/3}$. The covariant d’Alambert’s operator,

$$\Box g \psi = \frac{1}{\sqrt{|g|}} \partial_{x^i} \left( \sqrt{|g|} g^{ik} \frac{\partial \psi}{\partial x^k} \right),$$
in the spherical coordinates is

\[ \Box_{EdeS} \psi = -\left( \frac{\partial}{\partial x^\mu} \right)^2 \psi - \frac{2}{t} \left( \frac{\partial \psi}{\partial x^\mu} \right) + t^{-\frac{4}{3}} \sqrt{1 - \frac{K}{r^2}} \frac{\partial}{\partial t} \left( r^2 \sqrt{1 - \frac{K}{r^2}} \frac{\partial \psi}{\partial r} \right) \]

\[ + t^{-\frac{2}{3}} \frac{1}{r^4 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + t^{-\frac{2}{3}} \frac{1}{r^4 \sin^2 \theta} \left( \frac{\partial}{\partial \phi} \right)^2 \psi. \]

The change \( \psi = t^{-1}u \) of the unknown function leads the equation \( \Box_{EdeS} \psi = g \) to the equation

\[ u_{tt} - t^{-4/3} A(x, \partial_x) u = f, \]

where

\[ A(x, \partial_x) u = \frac{\sqrt{1 - K r^2}}{r^2} \frac{\partial}{\partial r} \left( r^2 \sqrt{1 - K r^2} \frac{\partial u}{\partial r} \right) + \frac{1}{r^4 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^4 \sin^2 \theta} \left( \frac{\partial}{\partial \phi} \right)^2 u. \tag{18} \]

The spatial part \( X \) of the spacetime \( \mathbb{R}^1 \times (\mathbb{R}^\ell \times \{0\}) \) has a constant curvature \( 6K \). Operator \( A(x, \partial_x) \) is the Laplace-Beltrami operator on \( X \). The explicit formulas for the solutions of the Cauchy problem for the wave operator on the spaces with constant negative curvature are known, see, for instance, \( [19, 23] \). Thus, Theorem 1.5 gives explicit representation for the solution of the Cauchy problem with vanishing initial data for the wave equation in the EdeS spacetime with the negative constant curvature \( K < 0 \). In order to keep down the length of this paper, we postpone applications of Theorem 1.5 to the derivation of the Strichartz estimates and to global well-posedness of the nonlinear generalized Tricomi equation in the metric \( [17] \).

The next theorem represents the integral transforms for the case of the equation without source term. In that theorem the transformed function has non-vanishing initial values. For \( \gamma \in \mathbb{C}, \Re \gamma > 0, \) that is for \( \ell \in \mathbb{C} \setminus D_1(-1, 0) = \{ z \in \mathbb{C} \mid |z - 1| > 1 \} \), and, in particular, for \( \ell \in (-\infty, -2) \cup (0, \infty) \), we define the integral operator

\[ (K_0 v)(x, t) := 2^{2 - 2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 v(x, \phi(t)s)(1 - s^2)^{\gamma - 1} ds \]

\[ = \phi(t)^{1 - 2\gamma} 2^{2 - 2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^{\phi(t)} v(x, \tau)(\phi^2(t) - \tau^2)^{\gamma - 1} d\tau. \]

For \( \gamma \in \mathbb{C}, \Re \gamma < 1, \) that is for \( \ell \in \mathbb{C} \setminus D_1(-3, 0) = \{ z \in \mathbb{C} \mid |z + 3| > 1 \} \), and, in particular, for \( \ell \in (-\infty, -4) \cup (-2, \infty) \), we define the integral operator

\[ (K_1 v)(x, t) := t^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 v(x, \phi(t)s)(1 - s^2)^{-\gamma} ds \]

\[ = t\phi(t)^{2\gamma - 1} 2^{2\gamma - 2} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^{\phi(t)} v(x, \tau)(\phi^2(t) - \tau^2)^{-\gamma} d\tau. \]

Thus, both operators are defined simultaneously for \( \gamma \in \mathbb{C}, 0 < \Re \gamma < 1, \) and, in particular, for \( \ell \in (-\infty, -4) \cup (0, \infty) \). Denote

\[ a_\ell := 2^{1 - 2\gamma} \frac{\ell \Gamma(2\gamma)}{2\gamma \Gamma^2(\gamma)}, \quad b_\ell := (\ell + 2)2^{2\gamma - 1} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)}. \]

The next theorem describes the properties of the integral transforms \( K_0 \) and \( K_1 \) in the case when \( \ell \) is a positive number.

**Theorem 1.6** Let \( \ell \) be a positive number and let \( \Omega \subset \mathbb{R}^{\ell+1}_+ \) be a backward time connected domain. Suppose that the function \( v \in C^\infty(\Omega) \) for given \( (x_0, t_0) \in \Omega \) solves the equation

\[ \partial_t^2 v - A(x, \partial_x) v = 0 \quad \text{at} \quad x = x_0 \quad \text{and all} \quad t \in (0, \phi(t_0)). \tag{19} \]
Then the functions $K_0v$ and $K_1v$ satisfy equations
\[
(\partial_t^2 - t^\ell A(x, \partial_x)) K_0v = a\ell t^{\ell + 1} \partial_t v(x, 0) \quad \text{at } x = x_0 \quad \text{for all } t \in (0, t_0),
\]
and
\[
(\partial_t^2 - t^\ell A(x, \partial_x)) K_1v = b\ell t^{\ell + 1} \partial_t v(x, 0) \quad \text{at } x = x_0 \quad \text{for all } t \in (0, t_0),
\]
respectively. They have at $x = x_0$ the following initial values
\[
(K_0v)(x_0, 0) = v(x_0, 0), \quad (K_0v)_t(x_0, 0) = 0,
\]
and
\[
(K_1v)(x_0, 0) = 0, \quad (K_1v)_t(x_0, 0) = v(x_0, 0).
\]
Thus, the value $v(x_0, 0)$ of the solutions of (19) is invariant under operation $K_0$, while the operator $K_1$ acts similarly to the Dirichlet-to-Neumann map.

**Corollary 1.7** Let $\ell$ be a positive number and $\Omega \subset \mathbb{R}^{n+1}$ be a backward time connected domain. Suppose that the function $v \in C^\infty(\Omega_\phi)$ solves the equation
\[
\partial_t^2 v - A(x, \partial_x)v = 0 \quad \text{for all } \ (x, t) \in \Omega_\phi.
\]
Then the functions $K_0v$ and $K_1v$ satisfy equations
\[
(\partial_t^2 - t^\ell A(x, \partial_x)) K_0v = a\ell t^{\ell + 1} \partial_t v(x, 0) \quad \text{for all } x \in \Omega,
\]
and
\[
(\partial_t^2 - t^\ell A(x, \partial_x)) K_1v = b\ell t^{\ell + 1} \partial_t v(x, 0) \quad \text{for all } x \in \Omega,
\]
respectively. They have the following initial values
\[
(K_0v)(x, 0) = v(x, 0), \quad (K_0v)_t(x, 0) \quad \text{for all } x \in \tilde{\Omega},
\]
and
\[
(K_1v)(x, 0) = 0, \quad (K_1v)_t(x, 0) = v(x, 0) \quad \text{for all } x \in \tilde{\Omega}.
\]

For the Cauchy problem with the full initial data, like the Cauchy problem for the hyperbolic equation, we have the following result for the generalized Tricomi equation in the hyperbolic domain.

**Theorem 1.8** Let $\ell$ be a positive number. Suppose that the functions $v_0, v_1 \in C^2((0, T); C^m(\mathbb{R}^n)) \cap C^1([0, T); C^{m-1}(\mathbb{R}^n))$ solve the problem
\[
\partial_t^2 v_i - A(x, \partial_x) v_i = 0, \quad v_i(x, 0) = \delta_{ik} \varphi_k(x), \quad \partial_t v_i(x, 0) = 0, \quad i, k = 0, 1.
\]
Then the function $u = K_0v_0 + K_1v_1$ solves the problem
\[
(\partial_t^2 - t^\ell A(x, \partial_x)) u = 0 \quad u(x, 0) = \varphi_0(x), \quad \partial_t u(x, 0) = \varphi_1(x),
\]
\[
\partial_t v_\varphi(x, t) := \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^\frac{n-2}{2} \frac{t^{n-2}}{\omega_{n-1} c_0^{n-1}} \int_{S^{n-1}} \varphi(x + ty) dS_y,
\]
while for \( x \in \mathbb{R}^n, n = 2m, m \in \mathbb{N} \),
\[
v_\varphi(x, t) := \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right) \frac{2^{\frac{n}{2}}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^R(0)} \frac{1}{\sqrt{1 - |y|^2}} \varphi(x + ty) \, dV_y.
\] (25)

The last formulas can be also written in terms of Radon transform; for details, see [13, 23].

The case of negative \( \ell \) requires some modifications in the setting of the initial conditions at \( t = 0 \). For the EdeS spacetime these modifications are suggested in [13]; they are the so-called weighted initial conditions.

In the forthcoming paper we also will apply integral transform approach to the maximum principle (see, e.g., [24, 34]) for the generalized Tricomi equation, to the derivation of the \( L_p - L_q \) estimates (see [20, 43]), to the mixed problem for Friedlander model (see [14 and references therein]), global existence problem for the semilinear generalized Tricomi equations on the hyperbolic space (for wave equation see [11, 6, 27]), and Price’s law for the corresponding cosmological models (see, e.g., [28] and references therein).

This paper is organized as follows. In Section 2 we prove the main property (8) of the kernel function \( E \). In Section 3 we prove Theorem 1.3. In Section 4 we present some properties of operators \( K_0 \) and \( K_1 \) and prove Theorem 1.5.

## 2 The kernel function \( E \)

The function \( E \) defined by (9) contains the hypergeometric function \( F(\gamma, \gamma; 1; z) \), where \( \gamma = \frac{\ell}{2(\ell + 2)} \). The hypergeometric function is defined by the Gauss series on the disk \( |z| < 1 \), and by analytic continuation over the whole complex \( z \)-plane and for all complex \( \ell \in \mathbb{C} \setminus \{-2\} \) (see, e.g., [35]).

According to the next theorem, the function \( E(r, t; b; \gamma) \) solves the Tricomi-type equation. The similar property (see Theorem 1.12 [17]) possesses another kernel function, which is used to solve Klein-Gordon equation in the de Sitter spacetime, that is, the equation with \( a(t) = e^{-t} \) and the mass term.

**Theorem 2.1** The kernel function

\[
E(r, t; b; \gamma) = c_\ell \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} F \left( \gamma; \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2} \right),
\]

where \( \ell \in \mathbb{C} \setminus \{-2\}, \phi(t) = \frac{2}{\ell + 2} t^{\frac{\ell + 2}{2}}, \gamma = \frac{\ell}{2(\ell + 2)}, \) solves the equation

\[
E_{tt}(r, t; b; \gamma) - (\phi'(t))^2 E_{rr}(r, t; b; \gamma) = 0, \quad \text{for all} \quad t > b > 0, \quad r > 0, \quad r^2 \neq (\phi(t) + \phi(b))^2.
\]

Theorem 2.1 generalizes corresponding statement from [2], which is due to Darboux [9]. In fact, in [2] the Tricomi equation is considered, that is, \( \ell = 1, n = 1, A(x, \partial x) = \partial^2_x \), and the statement is made for the reduced hyperbolic Tricomi operator, which is the Tricomi operator written in the characteristic coordinates. For the case of reduced hyperbolic Tricomi operator with \( \ell \in \mathbb{R}_+ \), see Lemma 2.1 [17]. The proof of Theorem 2.1, which is given below, is straightforward. It works out for different distance functions \( \phi = \phi(t) \), see, for instance, [17], where the case of \( a(t) = e^{-t} \) is discussed.
The case of $\gamma = 0$ is trivial, and thereafter we set $\gamma \neq 0$. In order to prove Theorem 2.1, we need the following technical lemma. Denote

\[
\alpha(t, b, r) := ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma}, \quad \beta(t, b, r) := \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2},
\]

then the next identity is evident

\[
1 - \beta(t, b, r) = \frac{4\phi(t)\phi(b)}{(\phi(t) + \phi(b))^2 - r^2}.
\]

Lemma 2.2 The derivatives of the functions $\alpha = \alpha(t, b, r)$ and $\beta = \beta(t, b, r)$ are as follows

- $\partial_t \alpha(t, b, r) = -2\gamma\phi'(t)(\phi(t) + \phi(b))\alpha(t, b, r)\frac{\alpha(t, b, r)}{r^2},$
- $\partial_t^2 \alpha(t, b, r) = -2\gamma\phi''(t)(\phi(t) + \phi(b))\alpha(t, b, r)\frac{\alpha(t, b, r)}{r^2} - 2\gamma\phi'(t)^2\alpha(t, b, r)\frac{\alpha(t, b, r)}{r^2} + 4\gamma(\gamma + 1)(\phi'(t))^2(\phi(t) + \phi(b))^2\frac{\alpha(t, b, r)}{r^2} + 4\gamma(\gamma + 1)(\phi'(t))^2\alpha(t, b, r)$,
- $\partial_r \beta(t, b, r) = 4\alpha\frac{\phi(b)}{\phi(t)}\phi'(t)\phi(b)(\phi(t) - \phi(b) + r^2),$
- $\partial_r^2 \beta(t, b, r) = 4\phi(b)\left((\phi(t) + \phi(b))^2 - r^2\right)^{-2}\left[\phi''(t)(\phi^2(t) - \phi^2(b) + r^2) - 4(\phi'(t))^2(\phi(t) - \phi(b) + r^2)\left((\phi(t) + \phi(b))^2 - r^2\right)^{-1}\right] + 2\phi(t)(\phi'(t))^2$

and

- $\partial_r \alpha(t, b, r) = 2\gamma r\left(\alpha(t, b, r)\right)^{\frac{\gamma+1}{\gamma}},$
- $\partial_r^2 \alpha(t, b, r) = 2\gamma\left(\alpha(t, b, r)\right)^{\frac{\gamma+1}{\gamma}} + 4\gamma(\gamma + 1)r^2\left(\alpha(t, b, r)\right)^{\frac{\gamma+2}{\gamma}},$
- $\partial_r \beta(t, b, r) = -8r\frac{\phi(t)\phi(b)}{(\phi(t) + \phi(b))^2 - r^2} = -2r(1 - \beta(t, b, r))(\alpha(t, b, r))^{\frac{1}{2}},$
- $\partial_r^2 \beta(t, b, r) = -8\frac{\phi(t)\phi(b)}{(\phi(t) + \phi(b))^2 - r^2}\left[(\phi(t) + \phi(b))^2 + 3r^2\right] = 2(\beta(t, b, r) - 1)\alpha\frac{\phi(b)}{\phi(t)}\left[(\phi(t) + \phi(b))^2 + 3r^2\right].$

Proof. These identities can be verified by straightforward calculations. Lemma is proven. \[\square\]

Proof of Theorem 2.1 It is sufficient to consider the function

\[
E_{\epsilon_f}(r; t, b; \gamma) := \alpha(t, b, r)F(\gamma, \gamma; 1; \beta(t, b, r)),
\]

since $E(r; t, b; \gamma) = \alpha \epsilon F(r, t; b; \gamma)$. We have

\[
\partial_r E_{\epsilon_f}(r; t, b; \gamma) = (\partial_r \alpha(t, b, r))F(\gamma, \gamma; 1; \beta(t, b, r)) + \alpha(t, b, r)F_{\epsilon_f}(\gamma, \gamma; 1; \beta(t, b, r)) \partial_r \beta(t, b, r).
\]

Hence,

\[
\partial_r^2 E_{\epsilon_f}(r; t, b; \gamma) = \alpha \epsilon (t, b, r)F(\gamma, \gamma; 1; \beta(t, b, r)) + 2\alpha(t, b, r)\beta_r(t, b, r)F_{\epsilon_f}(\gamma, \gamma; 1; \beta(t, b, r)) + \alpha(t, b, r)F_{\epsilon_f} (\gamma, \gamma; 1; \beta(t, b, r))\beta_{rr}(t, b, r) + \alpha(t, b, r)F_{\epsilon_f} (\gamma, \gamma; 1; \beta(t, b, r))\beta_{rr}(t, b, r),
\]

Proof. These identities can be verified by straightforward calculations. Lemma is proven. \[\square\]
while

\[ \partial_t^2 E_{cl}(r, t; b; \gamma) = \alpha_{tt}(t, b, r) F(\gamma, \gamma; 1; \beta(t, b, r)) + 2\alpha_t(t, b, r) \beta_t(t, b, r) F_\gamma(\gamma, \gamma; 1; \beta(t, b, r)) \]

Then we use the identities

\[ \text{We can write} \]

Thus, we obtain

\[ \partial_t^2 E_{cl}(r, t; b; \gamma) = t^f \partial_t^2 E_{cl}(r, t; b; \gamma) \]

It follows

\[ \partial_t^2 E_{cl}(r, t; b; \gamma) = t^f \partial_t^2 E_{cl}(r, t; b; \gamma) \]

where

\[ I(t, b, r) = \alpha_{tt}(t, b, r) - t^f \alpha_{tttt}(t, b, r), \]

\[ J(t, b, r) = 2\alpha_t(t, b, r) + \alpha_{tt}(t, b, r) + \alpha_{ttt}(t, b, r) - t^f (2\alpha_{ttt}(t, b, r) \beta_t(t, b, r) + \alpha_{ttt}(t, b, r) \beta_{ttt}(t, b, r)) \]

We have due to Lemma 2.2

\[ I(t, b, r) = -2\gamma \phi''(t) (\phi(t) + \phi(b)) \alpha(t, b, r) \frac{1}{\gamma^2 - 1} - 2\gamma (\phi'(t))^2 \alpha(t, b, r) \frac{1}{\gamma^2 - 1} \]

We can write

\[ I(t, b, r) = (\phi(t) + \phi(b))^2 \left\{ -2\gamma \phi''(t) (\phi(t) + \phi(b)) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} - 2\gamma (\phi'(t))^2 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} \right\} \]

Then we use the identities

\[ t^f = (\phi'(t))^2, \quad 1 + \frac{1}{2\gamma} \phi''(t) \phi(t) = (\phi'(t))^2, \quad (28) \]
and derive

\[
I(t, b, r) = \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right)^{-\gamma - 2} \times \left\{ -2\gamma \phi''(t) \phi(t) + \phi(b) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) - \phi''(t) \phi(t) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) \\
+ 2(\gamma + 1) \phi''(t) \phi(t) \phi(t) + \phi(b)^2 \\
- \frac{1}{2\gamma} \phi''(t) \phi(t) \left[ 2\gamma \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) + 4\gamma(\gamma + 1)r^2 \right] \right\}.
\]

It follows

\[
I(t, b, r) = \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right)^{-\gamma - 2} \times \left\{ -2\gamma \phi''(t) \phi(t) + \phi(b) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) - \phi''(t) \phi(t) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) \\
+ 2\gamma \phi''(t) \phi(t) \phi(t) + 2\phi''(t) \phi(t) \phi(t) + \phi(b)^2 \\
- \phi''(t) \phi(t) \left[ \left( \phi(t) + \phi(b) \right)^2 - r^2 \right] + 2(\gamma + 1)r^2 \right\} = \phi''(t) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right)^{-\gamma - 2} \times \left\{ -2\gamma \phi''(t) \phi(t) + \phi(b) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) - 2\phi(t) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right) \\
+ 2\phi(t) \phi(t) + \phi(b)^2 - 2\phi(t)r^2 \right\}.
\]

Thus,

\[
I(t, b, r) = -2\gamma \phi''(t) \phi(b) \left( \left( \phi(t) + \phi(b) \right)^2 - r^2 \right)^{-\gamma - 1}.
\] (29)

Further, we consider the coefficient \( J(t, b, r) \):

\[
J(t, b, r) = 2 \left[ -2\gamma \phi''(t) \phi(b) \phi(t) + \phi(b) \phi(t) \phi(b) \phi(t) (\gamma + 1)^{\gamma + 1} \right] \left[ 4\alpha^2 (t, b, r) \phi''(t) \phi(b) \phi(t) - \phi^2(b) + r^2 \right] \\
+ \alpha(t, b, r) 4\phi(b) \left( \phi(b) + \phi(t) \right)^{-\gamma - 2} \left[ \phi''(t) \phi(t) \phi(b) - \phi^2(b) + r^2 \right] \\
- 4 \phi''(t) \phi(b) \phi(t) \phi(b) \phi(t) \phi(b) \phi(t) \phi(t) r^2 \left( \phi(b) + \phi(t) \right)^{-1} \\
+ 2\phi(t) \phi(t) \phi(t) + t^2 \left[ 2\gamma r \phi(b)(t, b, r) \left[ -2r(1 - \beta(t, b, r)) \phi(t)(t, b, r) \right] \\
- t^2 \alpha(t, b, r) \left[ -8 \frac{\phi(t) \phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^3} \left[ (\phi(t) + \phi(b))^2 + 3r^2 \right] \right].
\]
Next we use the identity and definition of $\alpha(t, b, r)$ to rewrite $J$ as follows:

\[
J(t, b, r) = -16\gamma \frac{1}{2\gamma} \phi''(t)\phi(t)\phi(b)(\phi(t) + \phi(b)) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} (\phi^2(t) - \phi^2(b) + r^2)
\]
\[+ \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma} 4\phi(b) \left((\phi(b) + \phi(t))^2 - r^2\right)^{2} \left(\phi''(t) (\phi^2(t) - \phi^2(b) + r^2)\right)
\]
\[-4\frac{1}{2\gamma} \phi''(t)(\phi(t) (\phi(b) + \phi(t))) (\phi^2(t) - \phi^2(b) + r^2) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-1}
\]
\[+2\phi(t) \frac{1}{2\gamma} \phi''(t)\phi(t)\]
\[-\frac{1}{2\gamma} \phi''(t)\phi(t) 2 \left[2\gamma r \left((t, b, r)\right)^{-\gamma+1}\right] \left[-2r(1 - \beta(t, b, r)) (\alpha(t, b, r))^{\gamma}\right]
\]
\[-\frac{1}{2\gamma} \phi''(t)\phi(t)\phi(t)\alpha(t, b, r) \left[ -8\frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^{3}} [((\phi(t) + \phi(b))^2 + 3r^2)\right].
\]

Hence

\[
J(t, b, r) = -8\phi''(t)(\phi(t)\phi(b)(\phi(t) + \phi(b)) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} (\phi^2(t) - \phi^2(b) + r^2)
\]
\[+\left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-2} 4\phi(b)\phi''(t) \left((\phi^2(t) - \phi^2(b) + r^2)\right)
\]
\[-2\frac{1}{\gamma} \phi(t) (\phi(b) + \phi(t)) (\phi^2(t) - \phi^2(b) + r^2) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-1}
\]
\[+\phi(t) \frac{1}{\gamma} \phi(t)\]
\[+16\phi''(t)\phi(t)r^2 \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} \phi(t)\phi(b)
\]
\[+4\frac{1}{\gamma} \phi''(t)\phi(t) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} \phi(t)\phi(b) \left[(\phi(t) + \phi(b))^2 + 3r^2\right].
\]

Then

\[
J(t, b, r) = \phi''(t)\phi(b) \left\{ -8\phi(t)(\phi(t) + \phi(b)) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} (\phi^2(t) - \phi^2(b) + r^2)
\]
\[+\left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-2} 4 \left((\phi^2(t) - \phi^2(b) + r^2)\right)
\]
\[-2\frac{1}{\gamma} \phi(t) (\phi(b) + \phi(t)) (\phi^2(t) - \phi^2(b) + r^2) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-1} + \phi^2(t) \frac{1}{\gamma}\]
\[+16\phi(t)r^2 \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} \phi(t)
\]
\[+4\frac{1}{\gamma} \phi(t) \left((\phi(b) + \phi(t))^2 - r^2\right)^{-\gamma-3} \phi(t) \left[(\phi(t) + \phi(b))^2 + 3r^2\right]\right\},
\]
First we consider the second factor of the right-hand side of the last expression and apply Lemma 2.2 and, consequently,

\[ J(t, b, r) = \phi''(t)\phi(b) \left( (\phi(b) + \phi(t))^2 - r^2 \right)^{-\gamma - 3} \left\{ -8\phi(t)(\phi(t) + \phi(b))(\phi^2(t) - \phi^2(b) + r^2) \\
+4 \left( (\phi(b) + \phi(t))^2 - r^2 \right) (\phi^2(t) - \phi^2(b) + r^2) \\
-8\frac{1}{\gamma}\phi(t) (\phi(b) + \phi(t)) (\phi^2(t) - \phi^2(b) + r^2) + \phi^2(t) \frac{1}{\gamma} 4 \left( (\phi(b) + \phi(t))^2 - r^2 \right) \\
+16\phi^2(t)r^2 + 4\frac{1}{\gamma}\phi^2(t) \left[ (\phi(t) + \phi(b))^2 + 3r^2 \right] \right\}. \]

The terms containing \( \gamma \) in the denominators are

\[ -8\frac{1}{\gamma}\phi(t) (\phi(b) + \phi(t)) (\phi^2(t) - \phi^2(b) + r^2) + \phi^2(t) \frac{1}{\gamma} 4 \left( (\phi(b) + \phi(t))^2 - r^2 \right) \\
+4\frac{1}{\gamma}\phi^2(t) \left[ (\phi(t) + \phi(b))^2 + 3r^2 \right] \\
= \frac{8}{\gamma}\phi(t)\phi(b) (\phi^2(t) - \phi^2(b) + r^2). \]

The terms without \( \gamma \) are

\[ -8\phi(t)(\phi(t) + \phi(b))(\phi^2(t) - \phi^2(b) + r^2) + 4 \left( (\phi(b) + \phi(t))^2 - r^2 \right) (\phi^2(t) - \phi^2(b) + r^2) + 16\phi^2(t)r^2 \]

\[ = -4 \left( (\phi(b) + \phi(t))^2 - r^2 \right) ((\phi(t) - \phi(b))^2 - r^2). \]

Thus we have

\[ J(t, b, r) = \phi''(t)\phi(b) \left( (\phi(b) + \phi(t))^2 - r^2 \right)^{-\gamma - 3} \times \left\{ \frac{1}{\gamma}\phi(t)\phi(b) (\phi^2(t) - \phi^2(b) + r^2) - 4 \left( (\phi(b) + \phi(t))^2 - r^2 \right) ((\phi(t) - \phi(b))^2 - r^2) \right\}. \]

Finally

\[ J(t, b, r) = -4\phi''(t)\phi(b)((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} \left[ \frac{2}{\gamma}\phi(b)\phi(t) + (\phi(t) - \phi(b))^2 - r^2 \right]. \quad (30) \]

Next we turn to the coefficient \( Y(t, b, r) \):

\[ Y(t, b, r) = \alpha(t, b, r) \left\{ (\beta_l(t, b, r))^2 - t \ell (\beta_r(t, b, r))^2 \right\}. \]

First we consider the second factor of the right-hand side of the last expression and apply Lemma 2.2

\[ (\beta_l(t, b, r))^2 - t \ell (\beta_r(t, b, r))^2 \]

\[ = 4\alpha \hat{\beta}(t, b, r)(\phi'(t))^2 \left\{ 4\alpha \hat{\beta}(t, b, r)(\phi' (t))^2 (\phi^2(t) - \phi^2(b) + r^2) - r^2 (1 - \beta(t, b, r))^2 \right\}. \]

Then we use 26:

\[ (\beta_r(t, b, r))^2 - t \ell (\beta_r(t, b, r))^2 \]

\[ = 4\alpha \hat{\beta}(t, b, r)(\phi'(t))^2 \left\{ 4((\phi(t) + \phi(b))^2 - r^2)^{-2} \phi^2(b)(\phi^2(t) - \phi^2(b) + r^2) - r^2 \frac{16\phi^2(t)\phi^2(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right\} \]

\[ = 16\alpha \hat{\beta}(t, b, r)(\phi'(t))^2 \phi^2(b) (((\phi(t) + \phi(b))^2 - r^2)^{-1} \left( (\phi(t) - \phi(b))^2 - r^2 \right). \]

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Hence

\[ Y(t, b, r) = \alpha(t, b, r) 16\alpha(z) (\phi(t) + \phi(b))^2 (\phi(t) - \phi(b))^2 \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-1} \left( (\phi(t) - \phi(b))^2 - r^2 \right) \]

\[ = 16\alpha(z) (\phi(t))^2 (\phi(t) + \phi(b))^2 (\phi(t) - \phi(b))^2 \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-1} \left( (\phi(t) - \phi(b))^2 - r^2 \right). \]

Finally

\[ Y(t, b, r) = 16\phi^2(b)(\phi(t))^2 \left( (\phi(b) - \phi(t))^2 - r^2 \right)^{-1} \left( (\phi(b) + \phi(t))^2 - r^2 \right)^{-\gamma-3}. \]

We denote

\[ G(t, b, r) := 2\gamma^{-1}\phi''(t)\phi(b) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1}. \]

Lemma 2.3 Let \( z := \beta(t, b, r) \), then

\[ I(t, b, r) = -\gamma^2 G(t, b, r), \quad J(t, b, r) = (1 - (2\gamma + 1)z) G(t, b, r), \quad Y(t, b, r) = z(1-z) G(t, b, r). \]

Proof. The first equation is evident. For the second one we calculate the ratio

\[ \frac{J(t, b, r)}{G(t, b, r)} = 2 \left[ -2\phi(b)\phi(t) + \gamma(\phi(t) - \phi(b))^2 - \gamma r^2 \right] \left( (\phi(t) + \phi(b))^2 - r^2 \right) . \]

On the other hand, the following identity is easily seen

\[ 1 - (2\gamma + 1)z = -\frac{2}{(\phi(b) + \phi(t))^2 - r^2} \left[ -2\phi(b)\phi(t) + \gamma(\phi(t) - \phi(b))^2 + \gamma r^2 \right] . \]

Hence the second equation is proven. For the third one we consider the ratio \( Y/G \) and reduce it to the following

\[ \frac{Y(t, b, r)}{G(t, b, r)} = \frac{4\phi(b)\phi(t) \left( (\phi(b) - \phi(t))^2 - r^2 \right)}{\left( (\phi(t) + \phi(b))^2 - r^2 \right)^2} . \]

On the other hand, the following identity is evident

\[ z(1-z) = \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2} \frac{4\phi(t)\phi(b)}{\left( (\phi(t) + \phi(b))^2 - r^2 \right)^2} . \]

Lemma is proven. \( \square \)

Completion of the proof of Theorem 2.1. We use Lemma 2.3 in equation (27):

\[ \partial_t^2 E_{c_{\ell}}(r, t; b; \gamma) - t^\ell \partial_t^2 E_{c_{\ell}}(r, t; b; \gamma) = G(t, b, r) \left( -\gamma^2 F(\gamma, \gamma; 1;z) + (1 - (2\gamma + 1)z) F_z(\gamma, \gamma; 1;z) + z(1-z) F_z(\gamma, \gamma; 1;z) \right) . \]

The second factor of the right-hand side vanishes due to the equation for the hypergeometric function \( F(\gamma, \gamma; 1;z) \). Theorem 2.1 is proven. \( \square \)

3 The problem with vanishing initial data. Proof of Theorem 1.3

The proof of Theorem 2.1 given in the previous section is instructive in its own right. More exactly, the proofs which are used in [12, 19, 41] are based on the transformation of equation (35) with \( A(x, \partial_x) = \Delta \) and \( \ell = 1 \), due to the choice of the characteristic coordinates. For Theorem 1.3, where \( x = x_0 \) is frozen, and even for Theorem 1.5, with \( x \) running over \( \mathbb{R}^n \), and with the general \( A(x, \partial_x) \), that approach does not work, while, as it will be shown in this section, the straightforward check of the solution formula can be carried out. It is important that the proof of Theorem 1.3 presented in this section contains the key calculations inherited
from the proof of Theorem 2.1. The proof of Theorem 1.3 also uses Lemma 2.3 which summarizes those calculations.

In this section we consider the case of the vanishing initial values of \( u \). Assume that \( \gamma < 1 \) and consider the function \( u = u(x, t) \) written in terms of the auxiliary functions \( \alpha(t, b, r) \) and \( \beta(t, b, r) \) as follows

\[
\begin{align*}
\frac{\partial}{\partial t} u (x, t) &= c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} \alpha(t, b, r) F (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr, \quad t > 0,
\end{align*}
\]

where \( w(x, r; b) \) is defined by (9) and (10), and where we skip a subindex 0 of \( x_0 \). Then, for the derivative of the function \( u = u(x, t) \) we obtain

\[
\begin{align*}
\frac{\partial}{\partial t} u (x, t) &= c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} 2 \phi'(t)(-\gamma)(\phi(t) + \phi(b)) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \\
&\quad \times F (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
&\quad + c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} (\partial_t \beta(t, b, r)) F_z (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
&= c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} (4 \phi'(t) \phi(b))^{-\gamma} w(x, \phi(t) - \phi(b); b) db \\
&\quad + c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} 2 \phi'(t)(-\gamma)(\phi(t) + \phi(b)) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \\
&\quad \times F (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
&\quad + c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} \\
&\quad \times (\partial_t \beta(t, b, r)) F_z (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr.
\end{align*}
\]

The second order derivative can be represented as follows:

\[
\frac{\partial^2}{\partial t^2} u (x, t) = A + B + C,
\]

where we denoted

\[
\begin{align*}
A &:= \partial_t \left\{ c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} (4 \phi'(t) \phi(b))^{-\gamma} w(x, \phi(t) - \phi(b); b) db \right\}, \\
B &:= \partial_t \left\{ c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} 2 \phi'(t)(-\gamma)(\phi(t) + \phi(b)) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \\
&\quad \times F (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \right\}, \\
C &:= \partial_t \left\{ c_{\ell} \int_{0}^{t} db \int_{0}^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} \\
&\quad \times (\partial_t \beta(t, b, r)) F_z (\gamma; \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \right\}.
\end{align*}
\]
Then

\[
\frac{A}{c_t} = 4^{-\gamma} \phi''(t) \phi(t)^{-\gamma} \int_0^t \phi(b)^{-\gamma} w(x, (\phi(t) - \phi(b)); b) \, db \\
+ 4^{-\gamma} (-\gamma) (\phi'(t))^2 \phi(t)^{-\gamma-1} \int_0^t \phi(b)^{-\gamma} w(x, (\phi(t) - \phi(b)); b) \, db \\
+ 4^{-\gamma} \phi'(t) \phi(t)^{-\gamma} \phi(t)^{-\gamma} w(x, 0; t) \\
+ 4^{-\gamma} (\phi'(t))^2 \phi(t)^{-\gamma} \int_0^t \phi(b)^{-\gamma} w_r(x, (\phi(t) - \phi(b)); b) \, db .
\]

The choice of \( w(x, 0; t) = f(x, t) \) implies

\[
\frac{A}{c_t} = 4^{-\gamma} \phi''(t) \phi(t)^{-\gamma} \int_0^t \phi(b)^{-\gamma} w(x, \phi(t) - \phi(b); b) \, db \\
- 4^{-\gamma} \gamma (\phi'(t))^2 \phi(t)^{-\gamma-1} \int_0^t \phi(b)^{-\gamma} w(x, \phi(t) - \phi(b); b) \, db \\
+ \frac{1}{c_t} f(x, t) + 4^{-\gamma} (\phi'(t))^2 \phi(t)^{-\gamma} \int_0^t \phi(b)^{-\gamma} w_r(x, \phi(t) - \phi(b); b) \, db ,
\]

since

\[
4^{-\gamma} c_t \phi'(t) \phi(t)^{-2\gamma} = 1 . \tag{33}
\]

Further,

\[
\frac{B}{c_t} = 2(-\gamma) \phi''(t) \int_0^t db \int_0^{\phi(t)-\phi(b)} (\phi(t) + \phi(b)) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
+ 2(-\gamma) (\phi'(t))^2 \int_0^t db \left( \phi(t) + \phi(b) \right) \left( (\phi(t) + \phi(b))^2 - (\phi(t) - \phi(b))^2 \right)^{-\gamma-1} w(x, \phi(t) - \phi(b); b) \, db \\
+ 2(-\gamma) \phi'(t)^2 \int_0^t db \int_0^{\phi(t)-\phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
+ 2(-\gamma) (-\gamma - 1) (\phi'(t))^2 \int_0^t db \int_0^{\phi(t)-\phi(b)} 2(\phi(t) + \phi(b))^2 \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-2} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
+ 2(-\gamma) \phi'(t) \int_0^t db \int_0^{\phi(t)-\phi(b)} \left( \phi(t) + \phi(b) \right) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} \\
\times (\partial_t \beta(t, b, r)) F_z(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr.
\]
Application of Lemma 2.2 yields

\[
\frac{B}{c_\ell} = -2\gamma \phi''(t) \int_0^t db \int_0^{\phi(t) - \phi(b)} (\phi(t) + \phi(b)) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
-2\gamma (\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - (\phi(t) - \phi(b))^2)^{-\gamma-1} w(x, \phi(t) - \phi(b); b) db \\
-2\gamma (\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
+2\gamma (\gamma + 1)(\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} 2(\phi(t) + \phi(b))^2 \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-2} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
-2\gamma (\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \\
\times \left(4\phi'(t)\phi(b) - \phi^2(b) + r^2 \right) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} F_z(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr.
\]

Consequently,

\[
\frac{B}{c_\ell} = -2\gamma \phi''(t) \int_0^t db \int_0^{\phi(t) - \phi(b)} (\phi(t) + \phi(b)) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
-2\gamma (\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - (\phi(t) - \phi(b))^2)^{-\gamma-1} w(x, \phi(t) - \phi(b); b) db \\
-2\gamma (\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
+2\gamma (\gamma + 1)(\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} 2(\phi(t) + \phi(b))^2 \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-2} \\
\times F(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr \\
-8\gamma (\phi'(t))^2 \int_0^t db \int_0^{\phi(t) - \phi(b)} (\phi(t) + \phi(b)) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-3} \\
\times \phi(b)(\phi^2(t) - \phi^2(b) + r^2) F_z(\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr.
\]
Finally, we derive
\[
\frac{B}{c_\ell} = -2\gamma\phi''(t) \int_0^t db \int_0^{\phi(t)-\phi(b)} (\phi(t) + \phi(b)) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma - 1} \\
\times F \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr \\
-2\gamma(\phi'(t))^2 \int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{\gamma - 1} w(x, \gamma(t) - \phi(b); b) db \\
-2\gamma(\phi'(t))^2 \int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} \\
\times F \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr \\
+2\gamma(\gamma + 1)(\phi'(t))^2 \int_0^t db \int_0^{\phi(t)-\phi(b)} 2(\phi(t) + \phi(b))^2 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} \\
\times F \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr \\
-8\gamma(\phi'(t))^2 \int_0^t db \int_0^{\phi(t)-\phi(b)} (\phi(t) + \phi(b)) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 3} \\
\times \phi(b)(\phi^2(t) - \phi^2(b) + r^2) F_{\gamma} \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr.
\]

Next we consider \( C/c_\ell \) and by means of Lemma 2.2 we obtain
\[
\frac{C}{c_\ell} = \phi'(t) \int_0^t db \left( (\phi(t) + \phi(b))^2 - (\phi(t) - \phi(b))^2 \right)^{-\gamma} \\
\times \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - 2\phi(t)\phi(b)}{(4\phi(t)\phi(b))^2} \right) F_{\gamma} \left( \gamma, \gamma; 1; 0 \right) w(x, r; b) dr \\
+(-\gamma)\phi'(t) \int_0^t db \int_0^{\phi(t)-\phi(b)} 2(\phi(t) + \phi(b))^2 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} \\
\times \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) F_{\gamma} \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr \\
+\int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} \\
\times \left[ \partial_t \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) \right] F_{\gamma} \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr \\
+\int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} \\
\times \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right)^2 F_{\gamma} \left( \gamma, \gamma; 1; \beta(t, b, r) \right) w(x, r; b) dr.
\]

If we take into account the following simply verified identity
\[
\partial_t \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) \\
= 4\phi''(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \\
+ 8(\phi'(t))^2\phi(b) \frac{\phi^3(t) + 3\phi(t)\phi(b)^2 + 2\phi^3(b) - 3\phi(t)r^2 - 2\phi(b)r^2}{((\phi(t) + \phi(b))^2 - r^2)^3},
\]
then we can write

\[
\frac{C}{c_t} = \phi'(t) \int_0^t db \left( 4\phi(t)\phi(b) \right)^{-\gamma} \left( \phi'(t) \frac{\phi(t) - \phi(b)}{2\phi(t)\phi(b)} \right) F_z (\gamma, \gamma; 1; 0) w(x, (\phi(t) - \phi(b)); b)
\]

\[- \gamma \phi'(t) \int_0^t db \int_0^{\phi(t) - \phi(b)} 2(\phi(t) + \phi(b)) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} \times \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) F_z (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr
\]

\[
+ \int_0^t db \int_0^{\phi(t) - \phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} \left[ 4\phi''(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right]
\]

\[
x F_{zz} (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) dr.
\]

Next we consider

\[
\frac{1}{c_t} A(x, \partial_x) u = \int_0^t db \int_0^{\phi(t) - \phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} \times F (\gamma, \gamma; 1; \beta(t, b, r)) A(x, \partial_x) w(x, r; b) dr
\]

\[
= \int_0^t db \int_0^{\phi(t) - \phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} F (\gamma, \gamma; 1; \beta(t, b, r)) w_r(x, r; b) dr,
\]

where we have used the relation (49). Integrating by parts we obtain

\[
\frac{1}{c_t} A(x, \partial_x) u = \int_0^t \left( 4\phi(t)\phi(b) \right)^{-\gamma} w_r(x, \phi(t) - \phi(b); b) db
\]

\[- \int_0^t \left( (\phi(t) + \phi(b))^2 \right)^{-\gamma} F (\gamma, \gamma; 1; \beta(t, b, r)) w_r(x, 0; b) db
\]

\[- \int_0^t db \int_0^{\phi(t) - \phi(b)} \left[ \partial_r \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} F (\gamma, \gamma; 1; \beta(t, b, r)) \right] w_r(x, r; b) dr.
\]

The first term of the last equation will be canceled with the similar term of \( A/c_t \). We consider now the last term and transform it as follows

\[
II := - \int_0^t db \int_0^{\phi(t) - \phi(b)} \left[ \partial_r \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} F (\gamma, \gamma; 1; \beta(t, b, r)) \right] w_r(x, r; b) dr
\]

\[- \int_0^t db \int_0^{\phi(t) - \phi(b)} (2r\gamma) \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} F (\gamma, \gamma; 1; \beta(t, b, r)) w_r(x, r; b) dr
\]

\[- \int_0^t db \int_0^{\phi(t) - \phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} \left[ \partial_r, \beta(t, b, r) \right] F_z (\gamma, \gamma; 1; \beta(t, b, r)) w_r(x, r; b) dr.
\]

Then we use Lemma 22 and the integration by parts and obtain

\[
II = -2\gamma \int_0^t db \int_0^{\phi(t) - \phi(b)} r \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-1} F (\gamma, \gamma; 1; \beta(t, b, r)) w_r(x, r; b) dr
\]

\[+ 8\phi(t) \int_0^t db \phi(b) \int_0^{\phi(t) - \phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma-2} r F_z (\gamma, \gamma; 1; \beta(t, b, r)) w_r(x, r; b) dr.
\]
One more integration by parts gives

\[
II = -2\gamma \int_0^t (\phi(t) - \phi(b)) (4\phi(t)\phi(b))^2 \, w(x, (\phi(t) - \phi(b)); b) \, db \\
+ 2\gamma \int_0^t db \int_0^t (\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} F(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
+ 8\phi(t) \int_0^t \phi(b) (4\phi(t)\phi(b))^{-\gamma - 2} (\phi(t) - \phi(b)) F(z, \gamma; 1; 0) \, w(x, \phi(t) - \phi(b); b) \, db \\
- 8\phi(t) \int_0^t db \phi(b) \int_0^t (\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} r F(z, (\gamma, \gamma; 1; \beta(t, b, r))) \, w(x, r; b) \, dr.
\]

It is easily seen that

\[
II = -2\gamma \int_0^t (\phi(t) - \phi(b)) (4\phi(t)\phi(b))^2 \, w(x, (\phi(t) - \phi(b)); b) \, db \\
+ 2\gamma \int_0^t db \int_0^t ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} F(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
+ 4\gamma(\gamma + 1) \int_0^t db \int_0^t r^2 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} F(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
+ 2\gamma \int_0^t db \int_0^t r ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} \left( -8r \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) F_z(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
+ 8\phi(t) \int_0^t \phi(b) (4\phi(t)\phi(b))^{-\gamma - 2} (\phi(t) - \phi(b)) F_z(\gamma, \gamma; 1; 0) \, w(x, \phi(t) - \phi(b); b) \, db \\
- 8\phi(t) \int_0^t db \phi(b) \int_0^t ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} F_z(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
- 8\phi(t) \int_0^t db \phi(b) \int_0^t 2(\gamma + 2) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 3} r^2 \\
- 8\phi(t) \int_0^t db \phi(b) \int_0^t ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} r \left( -8r \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) F_z(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
- 8\phi(t) \int_0^t db \phi(b) \int_0^t ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} \left( -8r \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) \times F_{zz}(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr.
\]

Then we take into account the properties \( F(\gamma, \gamma; 1; 0) = 1, F_z(\gamma, \gamma; 1; 0) = \gamma^2 \) of the hypergeometric function and denote

\[
III = III + IV,
\]

where

\[
III := -2\gamma \int_0^t (\phi(t) - \phi(b)) (4\phi(t)\phi(b))^{-\gamma - 1} w(x, (\phi(t) - \phi(b)); b) \, db \\
+ 2\gamma \int_0^t db \int_0^t ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} F(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
+ 4\gamma(\gamma + 1) \int_0^t db \int_0^t r^2 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 2} F(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr \\
+ 2\gamma \int_0^t db \int_0^t r ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma - 1} \left( -8r \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) F_z(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr.
\]

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and

\[
IV := +8\gamma^2\phi(t)\int_0^t \phi(b) (4\phi(t)\phi(b))^{-\gamma-2} (\phi(t) - \phi(b))w(x, \phi(t) - \phi(b); b) \, db \\
-8\phi(t)\int_0^t db \phi(b) \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-2} x F_z (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr \\
-8\phi(t)\int_0^t db \phi(b) \int_0^{\phi(t)-\phi(b)} 2(\gamma + 2) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-3} r^2 x F_z (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr \\
-8\phi(t)\int_0^t db \phi(b) \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - 2)^{-\gamma-2} r \left( -8r \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right) \\
\times x F_{zz} (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr.
\]

Thus

\[
II = \int_0^t (\phi(t) - \phi(b)) (4\phi(t)\phi(b))^{-\gamma-1} \left\{ (-2\gamma) + 2\gamma^2 \right\} w(x, \phi(t) - \phi(b); b) \, db \\
+ \int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-2} \left\{ 2\gamma ((\phi(t) + \phi(b))^2 - r^2) + 4\gamma(\gamma + 1)r^2 \right\} \times F (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr \\
+ \int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \left\{ -16\gamma r^2 \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right\} \\
-8\phi(t)\phi(b) ((\phi(t) + \phi(b))^2 - r^2)^{-1} - 8\phi(t)\phi(b) (\gamma + 2) ((\phi(t) + \phi(b))^2 - r^2)^{-2} r^2 \times F_z (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr \\
+ \int_0^t db \int_0^{\phi(t)-\phi(b)} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-2} \frac{64(\phi(t))^2(\phi(b))^2r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \\
\times x F_{zz} (\gamma, \gamma; 1; \beta(t, b, r)) w(x, r; b) \, dr.
\]
Consequently, for \(-\frac{1}{c_t} t^t A(x, \partial_x)u\) we obtain

\[
-\frac{1}{c_t} t^t A(x, \partial_x)u = -\left(\phi'(t)\right)^2 \int_0^t \left(4\phi(t)\phi(b)^2\right)^2 w_r(x, \phi(t) - \phi(b); b) \, db \\
+ \left(\phi'(t)\right)^2 \int_0^t \left((\phi(t) + \phi(b))^2 \right)^2 F \left(\gamma, \gamma; 1; \frac{\phi(t) - \phi(b)^2}{(\phi(t) + \phi(b))^2}\right) w_r(x, 0; b) \, db \\
+ \left(\phi'(t)\right)^2 \int_0^t (\phi(t) - \phi(b)) \left(4\phi(t)\phi(b)\right)^{2\gamma - 1} 2\gamma(\gamma - 1) w(x, \phi(t) - \phi(b); b) \, db \\
\int_0^t db \int_0^t \phi(t) - \phi(b) \left((\phi(t) + \phi(b))^2 - r^2\right)^{2\gamma - 2} \left(\phi'(t)\right)^2 \left(2\gamma \left((\phi(t) + \phi(b))^2 - r^2\right) + 4\gamma(\gamma + 1)r^2\right) \\
\times F \left(\gamma, \gamma; 1; \beta(t, b, r)\right) w(x, r; b) \, dr \\
\int_0^t db \int_0^t \phi(t) - \phi(b) \left((\phi(t) + \phi(b))^2 - r^2\right)^{2\gamma - 2} \left(\phi'(t)\right)^2 \left(\frac{64\phi(t)^2\phi(b)^2 r^2}{(\phi(t) + \phi(b))^2 - r^2)^2}\right) \\
\times F_z \left(\gamma, \gamma; 1; \beta(t, b, r)\right) w(x, r; b) \, dr .
\]

In the double integrals the terms with \(F \left(\gamma, \gamma; 1; \beta(t, b, r)\right)\) in the expression of \(u_{tt} - t^t A(x, \partial_x)u\) are:

\[
\int_0^t db \int_0^t \phi(t) - \phi(b) \left((\phi(t) + \phi(b))^2 - r^2\right)^{2\gamma - 2} \left(\phi'(t)\right)^2 \left(2\gamma \left((\phi(t) + \phi(b))^2 - r^2\right) + 4\gamma(\gamma + 1)r^2\right) \\
\times F \left(\gamma, \gamma; 1; \beta(t, b, r)\right) w(x, r; b) \, dr \\
\int_0^t db \int_0^t \phi(t) - \phi(b) \left((\phi(t) + \phi(b))^2 - r^2\right)^{2\gamma - 2} \left(\phi'(t)\right)^2 \left(\frac{64\phi(t)^2\phi(b)^2 r^2}{(\phi(t) + \phi(b))^2 - r^2)^2}\right) \\
\times F_z \left(\gamma, \gamma; 1; \beta(t, b, r)\right) w(x, r; b) \, dr .
\]

In the double integrals the terms with \(F_z \left(\gamma, \gamma; 1; \beta(t, b, r)\right)\) in the expression of \(u_{tt} - t^t A(x, \partial_x)u\) are:

\[
\int_0^t db \int_0^t \phi(t) - \phi(b) \left((\phi(t) + \phi(b))^2 - r^2\right)^{2\gamma - 2} \left(\phi'(t)\right)^2 \left(2\gamma \left((\phi(t) + \phi(b))^2 - r^2\right) + 4\gamma(\gamma + 1)r^2\right) \\
\times F \left(\gamma, \gamma; 1; \beta(t, b, r)\right) w(x, r; b) \, dr \\
\int_0^t db \int_0^t \phi(t) - \phi(b) \left((\phi(t) + \phi(b))^2 - r^2\right)^{2\gamma - 2} \left(\phi'(t)\right)^2 \left(\frac{64\phi(t)^2\phi(b)^2 r^2}{(\phi(t) + \phi(b))^2 - r^2)^2}\right) \\
\times F_z \left(\gamma, \gamma; 1; \beta(t, b, r)\right) w(x, r; b) \, dr .
\]
In the double integrals the terms with $F_{zz}(\gamma, \gamma; 1; \beta(t, b, r))$ in the expression of $u_{tt} - t^\ell A(x, \partial_x) u$ are:

$$
\int_0^t \int_0^{\phi(t) - (\phi(b))} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} \left( 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right)^2 \times F_{zz}(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr
$$

$$
\int_0^t \int_0^{\phi(t) - (\phi(b))} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma} ((\phi'(t))^2) \frac{64(\phi(t))^2(\phi(b))^2r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \times F_{zz}(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr
$$

$$
= \int_0^t \int_0^{\phi(t) - (\phi(b))} ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-4} \times 16(\phi'(t))^2(\phi(b))^2 \left\{ (\phi^2(t) - \phi^2(b) + r^2)^2 - 4(\phi(t))^2 r^2 \right\} F_{zz}(\gamma, \gamma; 1; \beta(t, b, r)) \, w(x, r; b) \, dr.
$$

Now we denote by $I, J, Y$ the coefficients in the integrands in the corresponding to $F, F_z, F_{zz}$ terms. Then

$$
I := ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} (2(\phi'(t))((\phi(t) + \phi(b))
$$

$$
+ ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} (2(\phi'(t))^2
$$

$$
+ ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-2} 4\gamma(\gamma + 1)(\phi'(t))^2(\phi(t) + \phi(b))^2
$$

$$
- ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-2} (\phi'(t))^2 \left\{ 2\gamma ((\phi(t) + \phi(b))^2 - r^2) + 4\gamma(\gamma + 1)r \right\}
$$

$$
= -2\gamma \phi'(t)\phi(b) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1},
$$

which coincides with (29). Further, after simple calculations we obtain

$$
J := ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-3} \phi'(t)((\phi(t) + \phi(b))16(\phi'(t))^2(\phi(b)(\phi^2(t) - \phi^2(b) + r^2)
$$

$$
+ ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} \left[ 4\phi'(t)\phi(b) \frac{\phi^2(t) - \phi^2(b) + r^2}{((\phi(t) + \phi(b))^2 - r^2)^2} \right.
$$

$$
+ 8(\phi'(t))^2\phi(b) \frac{-\phi^3(t) + 3\phi(t)\phi(b) + 2\phi^3(b) - 3\phi(t)r^2 - 2\phi(b)r^2}{((\phi(t) + \phi(b))^2 - r^2)^3}
$$

$$
- ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-1} (\phi'(t))^2 \left\{ -2\gamma r^2 \frac{\phi(t)\phi(b)}{((\phi(t) + \phi(b))^2 - r^2)^2} \right.
$$

$$
- \phi(t)\phi(b) ((\phi(t) + \phi(b))^2 - r^2)^{-1} - 2\phi(t)\phi(b)(\gamma + 2) ((\phi(t) + \phi(b))^2 - r^2)^{-2} r^2 \right\}
$$

$$
= -4\phi'(t)\phi(b) ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-2} \left[ \left\{ -2\phi(b)\phi(t) \frac{\phi(t)}{\gamma} + (\phi(t) - \phi(b))^2 - r^2 \right\} ,
$$

which coincides with (30). Similarly, we derive

$$
Y := 16 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-4} (\phi'(t))^2(\phi(b))^2 \left\{ (\phi^2(t) - \phi^2(b) + r^2)^2 - 4(\phi(t))^2 r^2 \right\}
$$

$$
= 16\phi^2(b)(\phi'(t))^2 ((\phi(t) + \phi(b))^2 - r^2)^{-\gamma-3} ((\phi(t) - \phi(b))^2 - r^2)
$$

that coincides with (31). Lemma 23 implies

$$
Y(t, b; r; z)F_{zz} + J(t, b; r; z)F_z + I(t, b; r; z)F
$$

$$
= \frac{G(t, b; r; \gamma)}{\gamma} \left\{ \left\{ z(1 - z)F_{zz}(\gamma, \gamma; 1; z) + [1 - (2\gamma + 1)z]F_z(\gamma, \gamma; 1; z) - \gamma^2 F(\gamma, \gamma; 1; z) \right\} = 0 ,
$$

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where \( z := \beta(t, b, r) \).

In the expression of \( u_{tt} - t^\ell A(x, \partial_x)u \), we split into two parts the terms which contain only single integral (without factor \( c_\ell \)):

\[
\frac{1}{c_\ell} f(x, t) + (\phi'(t))^2 \int_0^t ((\phi(t) + \phi(b))^2)^{-\gamma} F \left( \gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2}{(\phi(t) + \phi(b))^2} \right) w_r(x, 0; b) \, db
\]

and

\[
4^{-\gamma} \phi''(t) \phi(t)^{-\gamma} \int_0^t \phi(b)^{-\gamma} w(x, \phi(t) - \phi(b); b) \, db
\]

\[
-4^{-\gamma} \gamma (\phi'(t))^2 \phi(t)^{-\gamma-1} \int_0^t \phi(b)^{-\gamma} w(x, \phi(t) - \phi(b); b) \, db
\]

\[
-2\gamma (\phi'(t))^2 \int_0^t (\phi(t) + \phi(b)) (4\phi(t)\phi(b))^{-\gamma-1} w(x, \phi(t) - \phi(b); b) \, db
\]

\[
+\phi'(t) \int_0^t db (4\phi(t)\phi(b))^{-\gamma} \left( \phi'(t) \frac{\phi(t) + \phi(b)}{2\phi(t)\phi(b)} \right)^\gamma w(x, (\phi(t) - \phi(b)); b)
\]

\[
+ t^{\ell k} \left[ 2\gamma \int_0^t (\phi(t) - \phi(b)) (4\phi(t)\phi(b))^{-\gamma-1} w(x, (\phi(t) - \phi(b)); b) \, db
\]

\[
-8\gamma^2 \phi(t) \int_0^t \phi(b) (4\phi(t)\phi(b))^{-\gamma-2} (\phi(t) - \phi(b)) w(x, (\phi(t) - \phi(b)); b) \, db \right]
\]

In fact, it is easily seen that the sum of the second group vanishes:

\[
4^{-\gamma} \phi''(t) \phi(t)^{-\gamma} \phi(b)^{-\gamma} - 4^{-\gamma} \gamma (\phi'(t))^2 \phi(t)^{-\gamma-1} \phi(b)^{-\gamma}
\]

\[
-2\gamma (\phi'(t))^2 (\phi(t) + \phi(b)) (4\phi(t)\phi(b))^{-\gamma-1} + \phi'(t) (4\phi(t)\phi(b))^{-\gamma} \left( \phi'(t) \frac{\phi(t) - \phi(b)}{2\phi(t)\phi(b)} \right)^\gamma
\]

\[
+ t^{\ell k} \left[ 2\gamma (\phi(t) - \phi(b)) (4\phi(t)\phi(b))^{-\gamma-1} - 8\gamma^2 \phi(t) (4\phi(t)\phi(b))^{-\gamma-2} (\phi(t) - \phi(b)) \right]
\]

\[
= 0.
\]

Theorem 1.3 is proven. \( \square \)

4 Operators \( K_0, K_1 \). Problem without source term. Proof of Theorem 1.6

First we check the initial values of the images of smooth functions. In the following lemma we consider operator \( K_0 \), which is defined for all \( \ell \in (\infty, -2) \cup (0, \infty) \).

**Lemma 4.1** Assume that \( \gamma > 0 \) and \( \phi(0) = \phi'(0) = 0 \). Let \( \Omega \) be a backward time connected domain. Then for every smooth function \( v = v(x, t) \in C^\infty(\Omega) \) we have

\[
(K_0 v)(x, 0) = v(x, 0), \quad (K_0 v)_{t}(x, 0) = 0.
\]

**Proof.** It is evident that

\[
(K_0 v)(x, 0) = 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 v(x, s)(1 - s^2)^{\gamma-1} ds = v(x, 0),
\]

and

\[
\partial_t K_0 v(x, t) = \phi'(t) 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_t v(x, r)|_{r=\phi(t), s}(1 - s^2)^{\gamma-1} ds.
\]

Lemma is proven. \( \square \)

Remind that the operator \( K_1 \) is defined for all \( \ell \in (\infty, -4) \cup (-2, \infty) \).
Lemma 4.2 Assume that $\gamma < 1$ and $\phi(0) = 0$, $\lim_{t \to 0} t \phi'(t) = 0$. Let $\Omega$ be a backward time connected domain. For every smooth function $v = v(x,t) \in C^\infty(\Omega)$ we have

$$(K_1v)(x,0) = 0, \quad (K_1v)_t(x,0) = v(x,0).$$

Proof. The first relation is evident. Next, we have

$$\partial_t K_1v(x,t) = 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 v(x, \phi(t)s)(1 - s^2)^{-\gamma} ds$$

$$+ t \phi'(t) 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r v(x,r)|_{r = \phi(t)s} s(1 - s^2)^{-\gamma} ds.$$ 

Hence

$$(K_1v)_t(x,0) = 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 v(x,0)(1 - s^2)^{-\gamma} ds = v(x,0).$$

since $\phi(0) = 0$ and $\lim_{t \to 0} t \phi'(t) = 0$. Lemma is proven. 

Now we turn to the equation.

Proposition 4.3 Assume that $\gamma > 0$ and the domain $\Omega$ is backward time connected. For $v \in C^\infty(\Omega)$, such that

$$(\partial_t^2 - A(x, \partial_x))v = 0 \quad \text{at} \quad x = x_0 \quad \text{for all} \quad t \in [0, \phi(T)],$$

we have

$$(\partial_t^2 - t^\gamma A(x, \partial_x))K_0v = 2^{1-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^{t^\frac{1}{\gamma}} \partial_t v(x,0) \quad \text{at} \quad x = x_0 \quad \text{for all} \quad t \in [0,T].$$

Proof. Consider [34], it follows:

$$\partial_t^2 K_0 v(x,t) = \phi''(t) 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_r v(x,r)|_{r = \phi(t)s} s(1 - s^2)^{\gamma - 1} ds$$

$$+ (\phi'(t))^{2^2 - 2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_r^2 v(x,r)|_{r = \phi(t)s} s^2(1 - s^2)^{\gamma - 1} ds.$$ 

On the other hand,

$$A(x, \partial_x)K_0v(x,t) = 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 A(x, \partial_x)v(x, \phi(t)s)(1 - s^2)^{\gamma - 1} ds$$

$$= 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_t^2 v(x,r)|_{r = \phi(t)s} (1 - s^2)^{\gamma - 1} ds.$$ 

Consequently,

$$(\partial_t^2 - t\gamma A(x, \partial_x))K_0v(x,t) = \phi''(t) 2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_r v(x,r)|_{r = \phi(t)s} s(1 - s^2)^{\gamma - 1} ds$$

$$+ (\phi'(t))^{2^2 - 2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_r^2 v(x,r)|_{r = \phi(t)s} s^2(1 - s^2)^{\gamma - 1} ds$$

$$- t\gamma 2^{2^2 - 2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 [\partial_r^2 v(x,r)|_{r = \phi(t)s}] (1 - s^2)^{\gamma - 1} ds.$$ 

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Integration by parts in the first term leads to

\[
(\partial_t^2 - t^\ell A(x, \partial_x))K_0v(x, t) = -\phi''(t)2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial_r v(x, r)\big|_{r=\phi(t)s} \frac{1}{2\gamma} \frac{d}{ds}(1 - s^2)^{\gamma-1} ds
\]

\[
+ (\phi'(t))2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial^2_r v(x, r)\big|_{r=\phi(t)s} s^2(1 - s^2)^{\gamma-1} ds
\]

\[
- t\ell^22^{-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial^2_r v(\phi(t)s)\big|_{r=\phi(t)s} (1 - s^2)^{\gamma-1} ds
\]

Then, (28) implies

\[
(\partial_t^2 - t^\ell A(x, \partial_x))K_0v(x, t) = \phi''(t)2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \partial_r v(x, 0) \frac{1}{2\gamma}
\]

\[
+ \frac{1}{2\gamma} \phi''(t)\phi(t)2^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial^2_r v(\phi(t)s)\big|_{r=\phi(t)s} (1 - s^2)^{\gamma-1} ds
\]

\[
- (\phi'(t))^22^{2-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 \partial^2_r v(x, r)\big|_{r=\phi(t)s} s^2(1 - s^2)^{\gamma-1} ds
\]

Thus,

\[
(\partial_t^2 - t^\ell A(x, \partial_x))K_0v(x, t) = 2^{1-2\gamma} \ell \Gamma(2\gamma) 
\]

\[
\frac{2\gamma}{\Gamma^2(\gamma)} \ell t^{\gamma-1} v(x, 0).
\]

Proposition is proven. \( \square \)

**Proposition 4.4** Assume that \( \gamma < 1 \) and \( \Omega \) is backward time connected. For \( v \in C^\infty(\Omega) \) such that

\[
(\partial_t^2 - A(x, \partial_x)) v = 0 \quad \text{at} \quad x = x_0 \quad \text{for all} \quad t \in [0, \phi(T)]
\]

we have

\[
(\partial_t^2 - t^\ell A(x, \partial_x)) K_1v = (\ell + 2)2^{2\gamma-1} \frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)} \ell \partial_t v(x, 0) \quad \text{at} \quad x = x_0 \quad \text{for all} \quad t \in [0, T].
\]

**Proof.** Consider the derivative \( \partial_t^2 K_1v(x, t) \). According to (35) we have:

\[
\partial_t^2 K_1v(x, t) = \partial_t \left[ 2^{2\gamma} \frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)} \int_0^1 v(x, \phi(t)s)(1 - s^2)^{-\gamma} ds \right] + t\phi'(t)2^\gamma \frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)} \int_0^1 \partial_r v(x, r)\big|_{r=\phi(t)s} s(1 - s^2)^{-\gamma} ds.
\]

27
Then

\[ \partial_t^2 K_1 v(x, t) = \phi'(t) 2^{2\gamma + 1} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r v(x, r)|_{r=\phi(t)s} s(1 - s^2)^{-\gamma} ds + t\phi''(t) 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r v(x, r)|_{r=\phi(t)s} s(1 - s^2)^{-\gamma} ds + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} s^2(1 - s^2)^{-\gamma} ds , \]

that is

\[ \partial_t^2 K_1 v(x, t) = -\phi'(t) 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r v(x, r)|_{r=\phi(t)s} \frac{1}{1 - \gamma} \frac{d}{ds}(1 - s^2)^{1-\gamma} ds + t\phi''(t) 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r v(x, r)|_{r=\phi(t)s} \frac{1}{2(1 - \gamma)} \frac{d}{ds}(1 - s^2)^{1-\gamma} ds + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} s^2(1 - s^2)^{-\gamma} ds . \]

Next we integrate by parts

\[ \partial_t^2 K_1 v(x, t) = \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \phi'(t) \partial_r v(x, 0) + \phi'(t) \phi(t) \frac{1}{1 - \gamma} 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} (1 - s^2)^{1-\gamma} ds + \frac{1}{2(1 - \gamma)} t\phi''(t) 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \partial_r v(x, 0) + t\phi''(t) \phi(t) \frac{1}{2(1 - \gamma)} 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} (1 - s^2)^{1-\gamma} ds + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} s^2(1 - s^2)^{-\gamma} ds . \]

It is easily seen that

\[ \phi'(t) \phi(t) \frac{1}{1 - \gamma} + t\phi''(t) \phi(t) \frac{1}{2(1 - \gamma)} = t(\phi'(t))^2 , \]

and, consequently,

\[ \partial_t^2 K_1 v(x, t) = \left[ \frac{1}{1 - \gamma} 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \phi'(t) + \frac{1}{2(1 - \gamma)} t\phi''(t) 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \right] \partial_r v(x, 0) + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} (1 - s^2)^{1-\gamma} ds + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} s^2(1 - s^2)^{-\gamma} ds = 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} t(\phi'(t))^2 \frac{1}{\phi(t)} \partial_r v(x, 0) + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} (1 - s^2)^{1-\gamma} ds + t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial_r^2 v(x, r)|_{r=\phi(t)s} s^2(1 - s^2)^{-\gamma} ds . \]
On the other hand
\[
\begin{align*}
\ell^t A(x, \partial_x) K_1 v(x, t) &= t^t A(x, \partial_x) t^2 t^2 \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 v(x, \phi(t)s)(1 - s^2)^{-\gamma} ds \\
&= t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 A(x, \partial_x) v(x, \phi(t)s)(1 - s^2)^{-\gamma} ds \\
&= t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial^2_x v(x, r)|_{r=\phi(t)s}(1 - s^2)^{-\gamma} ds.
\end{align*}
\]
Hence
\[
\partial^2_x K_1 v(x, t) - t^t A(x, \partial_x) K_1 v(x, t) = 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} t(\phi'(t))^2 \frac{1}{\phi(t)} \partial_x v(x, 0) \]
\[
+ t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial^2_x v(x, r)|_{r=\phi(t)s}(1 - s^2)^{-\gamma} ds \\
+ t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial^2_x v(x, r)|_{r=\phi(t)s}s^2(1 - s^2)^{-\gamma} ds \\
- t(\phi'(t))^2 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 \partial^2_x v(x, r)|_{r=\phi(t)s}(1 - s^2)^{-\gamma} ds \\
= 2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} t(\phi'(t))^2 \frac{1}{\phi(t)} \partial_x v(x, 0).
\]
Thus
\[
(\partial^2_x - t^t A(x, \partial_x)) K_1 v(x, t) = (\ell + 2) 2^{2\gamma - 1} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \phi'(t) \partial_x v(x, 0). \tag{39}
\]
Proposition is proven. \(\square\)

In particular, for \(\ell \in (0, \infty)\) both lemmas and propositions are applicable and we arrive at the following result.

**Corollary 4.5** Assume that \(\ell \in (0, \infty)\) and \(\Omega\) is backward time connected domain. For given \(\varphi = \varphi(x)\) let the function \(v_x = v_x(x, t)\) be a solution of the problem
\[
v_{tt} - A(x, \partial_x)v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0 \quad \text{for all} \quad (x, t) \in \Omega_x.
\]
Then for the given functions \(\varphi_0, \varphi_1 \in C^\infty(\overline{\Omega} \cap \{t = 0\})\) the function
\[
u(x, t) = 2^{2\gamma} \frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)} \int_0^1 v_{\varphi_0}(x, \phi(t)s)(1 - s^2)^{-\gamma} ds \]
\[
+ t2^{2\gamma} \frac{\Gamma(2 - 2\gamma)}{\Gamma^2(1 - \gamma)} \int_0^1 v_{\varphi_1}(x, \phi(t)s)(1 - s^2)^{-\gamma} ds, \quad (x, t) \in \Omega,
\]
is a solution of the problem
\[
u_{tt} - t^t A(x, \partial_x)\nu = 0 \quad \text{for all} \quad (x, t) \in \Omega, \tag{41}
\]
\[
u(x, 0) = \varphi_0(x), \quad \nu_t(x, 0) = \varphi_1(x) \quad \text{for all} \quad x \in \overline{\Omega} \cap \{t = 0\}. \tag{42}
\]

Finally, we remark that Theorem 1.3, Corollary 1.4 and Theorem 1.5 are applicable to the case of \(\ell \in \mathbb{C}\), provided that \(\ell \in \mathbb{C} \setminus \text{D}_1((-3, 0)) = \{z \in \mathbb{C} | |z + 3| > 1\}\). In order to carry out the continuation of the representation formulas with \(\ell\) in the complex plane we need some analyticity assumption on the solution of (9), (10). That question is out of the scope of this paper.
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