Duality Invariance of Cosmological Solutions with Torsion

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Abstract

We show that for a string moving in a background consisting of maximally symmetric gravity, dilaton field and second rank antisymmetric tensor field, the $O(d) \otimes O(d)$ transformation on the vacuum solutions gives inequivalent solutions that are not maximally symmetric. We then show that the usual physical meaning of maximal symmetry can be made to remain unaltered even if torsion is present and illustrate this through two toy models by determining the torsion fields, the metric and Killing vectors. Finally we show that under the $O(d) \otimes O(d)$ transformation this generalised maximal symmetry can be preserved under certain conditions. This is interesting in the context of string related cosmological backgrounds.
I. Introduction

Some time back it was shown that the low energy string effective action possesses, for time dependent metric \( G_{\mu\nu} \), torsion \( B_{\mu\nu} \) and dilaton \( \Phi \) background fields \( (\mu, \nu = 1, 2, \ldots d) \) a full continuous \( O(d,d) \) symmetry (a generalisation of T-duality in string theory) under which "cosmological" solutions of the equations of motion are transformed into other inequivalent solutions\(^1\). Subsequently, a generalisation to this was obtained\(^2\). These transformations are conjectured to be a generalisation of the Narain construction\(^3\) to curved backgrounds.

Here we investigate the consequences of this \( O(d) \otimes O(d) \) transformation on the space-time symmetries of the theory. We consider a string propagating in a gravity, dilaton and second rank antisymmetric tensor background and show that if the full metric corresponding to a given background is maximally symmetric then under the \( O(d) \otimes O(d) \) twist this symmetry is not preserved. However, in a generalised definition of maximal symmetry when torsion is present, we show that this generalised maximal symmetry can be preserved under certain conditions.

We first discuss the meaning of maximal symmetry and the \( O(d) \otimes O(d) \) symmetry. We then show that an approximate maximally symmetric solution with \( B_{\mu\nu} \neq 0 \) and non-zero curvature is possible with a linear dilaton background. However, this symmetry is destroyed under \( O(d) \otimes O(d) \) twist. Finally, we give a generalisation to the meaning of maximal symmetry when torsion is present and show that this generalised maximal symmetry can be preserved under the \( O(d) \otimes O(d) \) twist if the torsion fields satisfy certain conditions.
For a maximally symmetric space-time the curvature

\[ R_{iklm} = K (g_{im} g_{kl} - g_{il} g_{km}) \]  

(1)

K is the curvature constant proportional to the scalar curvature \( R_i^i \). Two maximally symmetric metrics with the same K and the same number of eigenvalues of each sign, are related by a coordinate diffeomorphism.

II. The O(d) \( \otimes \) O(d) Invariance of the String Effective Action

Now consider the low energy effective action of string theory in D space-time dimensions. This is

\[ S = -\int d^D X \sqrt{\det G} e^{-\phi} \left[ \Lambda - R^{(D)}(G) + \left( \frac{1}{12} \right) H_{\mu\nu\rho}H^{\mu\nu\rho} - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] \]  

(2)

where \( H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic perm.} \). \( R^{(D)} \) is the D-dimensional Ricci scalar and \( \Lambda \) is the cosmological constant equal to \( \frac{(D-26)}{3} \) for the bosonic string and \( \frac{(D-10)}{2} \) for the fermionic string. It has been shown that for (a) \( X \equiv (\hat{Y}^m, \hat{Y}^\alpha), 1 \leq m \leq d, 1 \leq \alpha \leq D - d \). \( \hat{Y}^m \) having Euclidean signature, (b) background fields independent of \( \hat{Y}^m \), and (c) \( G = \left( \begin{array}{cc} \hat{G}_{mn} & 0 \\ 0 & \hat{G}_{\alpha\beta} \end{array} \right) ; \quad B = \left( \begin{array}{cc} \hat{B}_{mn} & 0 \\ 0 & \hat{B}_{\alpha\beta} \end{array} \right) \) the action (1) can be recast into

\[ S = -\int d^D \hat{Y} \int d^{D-d} \hat{Y} \sqrt{\det \hat{G}} e^{-\chi} \left[ \Lambda - \tilde{G}^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi - \left( \frac{1}{8} \right) \tilde{G}^{\alpha\beta} Tr (\partial_\alpha M L \partial_\beta M L) - \tilde{R}^{(D-d)}(G) + \left( \frac{1}{12} \right) \tilde{H}_{\alpha\beta\gamma} \tilde{H}^{\alpha\beta\gamma} \right] \]  

(3)

where

\[ L = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \]  

(4)

\[ \chi = \Phi - ln \sqrt{\det \hat{G}} \]  

(5)
\[ M = \begin{pmatrix} \hat{G}^{-1} & -\hat{G}^{-1}\hat{B} \\ \hat{B}\hat{G}^{-1} & \hat{G} - \hat{B}\hat{G}^{-1}\hat{B} \end{pmatrix} \]  

(6)

If one of the coordinates \( \hat{Y}^1 \) is time-like, then the action (2) is invariant under an \( O(d - 1, 1) \otimes O(d - 1, 1) \) transformation on \( \hat{G}, \hat{B} \) and \( \Phi \) given by

\[
M \rightarrow \left( \frac{1}{4} \right) \begin{pmatrix} \eta(S + R)\eta & \eta(R - S) \\ (R - S)\eta & (S + R) \end{pmatrix} M \begin{pmatrix} \eta(S^T + R^T)\eta & \eta(R^T - S^T) \\ (R^T - S^T)\eta & (S^T + R^T) \end{pmatrix}
\]

(7)

with \( \eta = \text{diag} \ (-1, 1, \ldots, 1) \); \( S, R \) some \( O(d-1, 1) \) matrices satisfying \( S\eta S^T = \eta, R\eta R^T = \eta \); \( S_{11} = \cosh \theta = R_{11}, S_{21} = -\sinh \theta = -R_{21} \), and \( S_{11} = R_{i1} = 0, \) for \( i \geq 3 \).

In component form the transformed fields are given by 2 :

\[
\begin{align*}
(\hat{G}'^{-1})_{ij} &= \left( \frac{1}{4} \right) \left[ \eta(S + R)\eta\hat{G}^{-1}\eta \left( S^T + R^T \right) \eta \\
+ \eta(R - S) \left( \hat{G} - \hat{B}\hat{G}^{-1}\hat{B} \right) \left( R^T - S^T \right) \eta \\
- \eta(S + R)\eta\hat{G}^{-1}\hat{B} \left( R^T - S^T \right) \eta \\
+ \eta(R - S)\hat{B}\hat{G}^{-1}\eta \left( S^T + R^T \right) \eta \right]_{ij} 
\end{align*}
\]

(8a)

\[
(\hat{B}')_{ij} = \left( \frac{1}{4} \right) \left[ \left( S + R \right) \eta\hat{G}^{-1}\eta \left( S^T + R^T \right) \eta \\
+ \left( R - S \right) \eta\hat{G}^{-1}\hat{B} \left( R^T - S^T \right) \eta \\
+ \left( S + R \right)\hat{B}\hat{G}^{-1}\eta \left( S^T + R^T \right) \eta \\
- \left( R - S \right)\eta\hat{G}^{-1}\hat{B} \left( R^T - S^T \right) \eta \right]_{ij} 
\]

(8b)

\[
\Phi' = \Phi - \left( \frac{1}{2} \right) \ln \det \hat{G} + \left( \frac{1}{2} \right) \ln \det \hat{G}'
\]

(8c)

The equations of motion obtained from (2) are

\[
R_{\mu\nu} = D_{\mu}D_{\nu}\Phi + \left( \frac{1}{4} \right) H_{\lambda\rho}^{\mu}H_{\nu\lambda\rho}
\]

(9a)
\begin{equation}
D_{\mu} \Phi D^{\mu} \Phi - 2 D_{\mu} D^{\mu} \Phi + R - \left( \frac{1}{12} \right) H_{\mu\nu\rho} H^{\mu\nu\rho} = 0 \tag{9b}
\end{equation}

\begin{equation}
D_{\lambda} H_{\mu\nu}^{\lambda} - (D_{\lambda} \Phi) H_{\mu\nu}^{\lambda} = 0 \tag{9c}
\end{equation}

III. Construction of Maximally Symmetric Solutions

Maximal symmetry implies

\begin{equation}
R_{\mu\nu} = K(1 - D)g_{\mu\nu} \tag{10}
\end{equation}

i.e.

\begin{equation}
R = K(1 - D)D \tag{11}
\end{equation}

We can show that for $B_{\mu\nu} = 0$, the only maximally symmetric solution is that for which the dilaton background is a constant and the curvature constant $K = 0$ \(^6\). Here we shall discuss in detail the more general case where $B_{\mu\nu} \neq 0$.

For $B_{\mu\nu} \neq 0$, maximally symmetric solutions to (9) are for $D = 3$ \(^6\)

\begin{equation}
H_{01r}^2 = \frac{K(1 - D)}{2} g_{00} \ g_{11} \tag{12a}
\end{equation}

i.e.

\begin{equation}
B_{01} = \left[ \frac{K(1 - D)}{2} \right]^{\frac{1}{2}} \int dr \left[ g_{00} \ g_{11} \right]^{\frac{1}{2}} \tag{12b}
\end{equation}

and

\begin{equation}
\partial_r^2 \Phi = 0 \ i.e. \ \Phi = \alpha \ r + \beta \tag{12c}
\end{equation}

Now consider the case of $B_{\mu\nu} \neq 0$. We take the metric as

\begin{equation}
ds^2 = -f_0(r) dt^2 + f_1(r) (dx_1)^2 + dr^2 \tag{13a}
\end{equation}

\begin{equation}
f_0(r) = \cos^2 \left( \sqrt{K_1 r} \right) \ , \ f_1(r) = \sin^2 \left( \sqrt{K_1 r} \right) \ K_1 > 0 \tag{13b}
\end{equation}
\[ f_0(r) = \cosh^2\left(\sqrt{K_1}r\right), f_1(r) = \sinh^2\left(\sqrt{K_1}r\right), K_1 < 0 \quad (13c) \]

and assume that the curvature is small. Why we assume this will be evident shortly. The equations (12) imply

\[ H^2_{01r} = \frac{K(D - 1)}{2} f_0 f_1 \quad (14) \]

and

\[ \partial^2 r \Phi = 0 \text{ i.e. } \Phi = \alpha r + \beta \quad (15) \]

(9b) then yields

\[ \alpha^2 = K(1 - D)D + \left(\frac{1}{4}\right) K(1 - D) \quad (16) \]

This means that \( \alpha^2 \) is of the order of \( K \). The equation of motion for \( H_{\mu\nu\lambda} \) (i.e.(9c)) is satisfied for the derived solutions (14) and (15) except when \( \mu = 1, \nu = 0 \) and this non-vanishing part is

\[ \alpha \left[ \frac{K(D - 1)}{2} f_0 f_1 \right]^{\frac{1}{2}} \quad (17) \]

For \( B_{\mu\nu} \neq 0 \), these solutions are valid and can be made compatible with the equations of motion as follows. In the light of (13b, c), (16) and the assumption of small \( K \), (17) is of the order of \( K^{5/2} \). Retaining upto terms linear in the curvature, (17) may be ignored and the equations of motion satisfied. So in this approximation of small curvature, we can have maximally symmetric solutions with \( f_0 \) and \( f_1 \), in conjunction with \( \Phi = \alpha r + \beta \) and a non-vanishing \( B_{\mu\nu} \). We confine ourselves to the case of positive curvature. Identical conclusions also hold for negative curvature. (13a) and (12b) give the solution for \( B_{\mu\nu} \) as

\[ \hat{B}_{01} = - \left(\frac{1}{4}\right) \cos \left[ 2\sqrt{K_1}r \right] \quad (18) \]
so our starting solution with a maximally symmetric metric is (13), (15) and (18). Using (8), the twisted solutions are

\[
\hat{G}' = \begin{pmatrix} -F_0 & 0 \\ 0 & F_1 \end{pmatrix} \quad (19a)
\]

\[
F_0 = \frac{f_0}{1 + (1 - f_0 f_1) \sinh^2 \theta + \hat{B}_{01} \left( \hat{B}_{01} \sinh^2 \theta + \sinh 2 \theta \right)}
\]

\[
F_1 = \frac{f_1}{1 + (1 - f_0 f_1) \sinh^2 \theta + \hat{B}_{01} \left( \hat{B}_{01} \sinh^2 \theta + \sinh 2 \theta \right)}
\]

\[
\hat{B}' = \frac{\left( \frac{1}{2} \right) \left( 1 - f_0 f_1 + \hat{B}_{01}^2 \right) \sinh 2 \theta + \hat{B}_{01} \cosh 2 \theta}{1 + (1 - f_0 f_1) \sinh^2 \theta + \hat{B}_{01} \left( \hat{B}_{01} \sinh^2 \theta + \sinh 2 \theta \right)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (19b)
\]

\[
\Phi' = \alpha r + \beta - \ln \left( 1 + (1 - f_0 f_1) \sinh^2 \theta + \hat{B}_{01} \left( \hat{B}_{01} \sinh^2 \theta + \sinh 2 \theta \right) \right) \quad (19c)
\]

The analogues of (12a) is now

\[
H''_{01r} = K_1 \left( \frac{D - 1}{2} \right) F_0 F_1 \quad (20)
\]

and this may be solved to get the antisymmetric tensor field as :

\[
\hat{B}'_{01} = - \left[ \frac{2 \left[ \frac{K_1}{K_1} \right]^{\frac{1}{2}}}{\sinh \theta \left( 12 + 11 \sinh^2 \theta \right)^{\frac{1}{2}}} \right] \tan^{-1} \left[ \frac{5 \sinh \theta \cos 2 \sqrt{K_1 r} - 4 \cosh \theta}{2 \left( 12 + 11 \sinh^2 \theta \right)^{\frac{1}{2}}} \right] \quad (21)
\]

(19b) and (21) can never be matched to be identical for any value of \( \theta \). So maximal symmetry is not preserved under the \( 0(d) \otimes 0(d) \) transformation.

### IV. The Meaning of Maximal Symmetry when torsion is present

We now give a generalisation of the meaning of maximal symmetry in presence of torsion. In \( N \) dimensions, a metric that admits the maximum
number \( N(N + 1)/2 \) of Killing vectors is said to be maximally symmetric. A maximally symmetric space is homogeneous and isotropic about all points. Such spaces are of natural interest in the general theory of relativity as they correspond to spaces of globally constant curvature which in turn is related to the concepts of homogeneity and isotropy. The requirement of isotropy and homogeneity leads to maximally symmetric metrics in the context of standard cosmologies - the most well known being the Robertson-Walker cosmology. However, in the presence of torsion there is a drastic change in the scenario and one needs to redefine maximal symmetry itself. Here we do this in a way such that \textit{the usual physical meaning of maximal symmetry remains the same.} The only requirement is that the torsion fields satisfy some mutually consistent constraints. We shall also give examples of toy models where these ideas can be realised by determining the torsion fields, the metric and the Killing vectors.

The possible implications of torsion have been discussed extensively by G. Esposito\(^8\). He studied a model of gravity (cast in hamiltonian form) with torsion in a closed Friedmann-Robertson-Walker universe and obtained the full field equations. He showed that the torsion leads to a primary constraint linear in the momenta and a secondary constraint quadratic in the momenta (1989). Subsequently, using the generalised Raychaudhuri equation he showed that Hawking’s Singularity Theorem can be generalised if the torsion satisfies some conditions (1990). Esposito also studied the geometry of complex spacetimes with torsion (1993). Alternative approaches involving torsion have been discussed by F. Hehl\(^9\).

Presence of torsion implies that the affine connections \( \Gamma^\alpha_{\mu\nu} \) are asymmetric and contain an antisymmetric part \( H^\alpha_{\mu\nu} \) in addition to the symmetric term
\[ \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + H^\alpha_{\mu\nu} \]  

(22)

\[ H^\alpha_{\mu\nu} = \partial_{(\alpha} B_{\mu\nu)} \], the background torsion, is completely arbitrary to start with. \( B_{\mu\nu} \) is the second rank antisymmetric tensor field.

Defining covariant derivatives with respect to \( \bar{\Gamma}^\alpha_{\mu\nu} \) we have for a vector field \( V_\mu \):

\[ V_{\mu;\nu;\beta} - V_{\mu;\beta;\nu} = -\bar{R}^\lambda_{\mu\nu\beta} V_\lambda + 2H^\alpha_{\beta\nu} V_{\mu;\alpha} \]  

(23)

\[ \bar{R}^\lambda_{\mu\nu\beta} = R^\lambda_{\mu\nu\beta} + \tilde{R}^\lambda_{\mu\nu\beta} \]  

(24a)

\[ R^\lambda_{\mu\nu\beta} = \Gamma^\lambda_{\mu\nu,\beta} - \Gamma^\lambda_{\mu\beta,\nu} + \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\beta} - \Gamma^\alpha_{\mu\beta} \Gamma^\lambda_{\alpha\nu} \]  

(24b)

\[ \tilde{R}^\lambda_{\mu\nu\beta} = H^\lambda_{\mu\nu,\beta} - H^\lambda_{\mu\beta,\nu} + H^\alpha_{\mu\nu} H^\lambda_{\alpha\beta} - H^\alpha_{\mu\beta} H^\lambda_{\alpha\nu} \]  

(24c)

The generalised curvature \( \bar{R}^\lambda_{\mu\nu\beta} \) does not have the usual symmetry (antisymmetry) properties. The last term on the right hand side of (23) is obviously a tensor. Hence \( \bar{R}^\lambda_{\mu\nu\beta} \) is also a tensor.

Let \( \xi_\mu \) be a Killing vector defined through the Killing condition:

\[ \xi_\mu ; \nu + \xi_\nu ; \mu = 0 \]  

(25)

This condition is preserved also in the presence of \( H^\alpha_{\mu\nu} \).

Equation (23) for a Killing vector hence takes the form:

\[ \xi_{\mu;\nu;\beta} - \xi_{\mu;\beta;\nu} = -\bar{R}^\lambda_{\mu\nu\beta} \xi_\lambda + 2H^\alpha_{\beta\nu} \xi_{\mu;\alpha} \]  

(26)

The \( H^\alpha_{\beta\nu} \) are arbitrary to start with and so we may choose them to be such that

\[ H^\alpha_{\beta\nu} \xi_{\mu;\alpha} = 0 \]  

(27a)
This is a constraint on the $H_{\beta\nu}^\alpha$ and not the $\xi_\nu$ and is essential for the existence of maximal symmetry in presence of torsion as we shall shortly see.

Therefore:

$$\xi_{\mu\nu;\beta} - \xi_{\mu;\beta\nu} = -\bar{R}_{\mu\nu\beta}^\lambda \xi_\lambda$$

(28)

We now impose the cyclic sum rule on $\bar{R}_{\mu\nu\beta}^\lambda$:

$$\bar{R}_{\mu\nu\beta}^\lambda + \bar{R}_{\nu\beta\mu}^\lambda + \bar{R}_{\beta\mu\nu}^\lambda = 0$$

(29a)

The constraint (29a) implies

$$\bar{R}_{\mu\nu\beta}^\lambda + \bar{R}_{\nu\beta\mu}^\lambda + \bar{R}_{\beta\mu\nu}^\lambda = 0$$

i.e.

$$H_{\mu\nu,\beta}^\lambda + H_{\mu\nu}^\alpha \bar{\Gamma}_{\alpha\beta}^\lambda + H_{\nu,\beta\mu}^\lambda + H_{\nu\beta}^\alpha \bar{\Gamma}_{\alpha\mu}^\lambda + H_{\beta\mu,\nu}^\alpha + H_{\beta\mu}^\alpha \bar{\Gamma}_{\alpha\nu}^\lambda = 0$$

(29b)

Adding (28) and its two cyclic permutations and using (25),

$$\xi_{\mu\nu;\beta} = -\bar{R}_{\beta\mu\nu}^\lambda$$

(30)

Then following usual arguments

$$\xi_\mu^n(x) = A_\mu^n(x ; X) \xi_\lambda^n(X) + C_\mu^{\lambda\nu}(x ; X) \xi_{\lambda;\nu}^n(X)$$

(31)

where $A_\mu^n$ and $C_\mu^{\lambda\nu}$ are functions that depend on the metric and torsion and $X$, but not on the initial values $\xi_\lambda(X)$ and $\xi_{\lambda;\nu}(X)$, and hence are the same for all Killing vectors. Also note that the torsion fields present in $A_\mu^n(x ; X)$ and $C_\mu^{\lambda\nu}(x ; X)$ obey the constraint (29b). A set of Killing vectors $\xi_\mu^n(x)$ is said to be independent if they do not satisfy any relations of the form $\sum_n d_n \xi_\mu^n(x) = 0$, with constant coefficients $d_n$. It therefore follows that there
can be at most \( \frac{N(N+1)}{2} \) independent Killing vectors in N dimensions, even in the presence of torsion provided the torsion fields satisfy the constraints (27a) and (29b)\(^7\).

Consider the constraints (27a) and (29b). (27a) ensures that the generalised curvature \( \bar{R}^\lambda_{\mu\nu\beta} \) behaves in the same way as \( R^\lambda_{\mu\nu\beta} \) so far as the behaviour of the quantity \( V_{\mu\nu\beta} - V_{\mu\beta\nu} \) is concerned. We shall soon see that this constraint is essential for the existence of maximal symmetry in presence of torsion. (29b) follows by demanding the cyclicity property of \( \bar{R}^\lambda_{\mu\nu\beta} \) which again is the usual property of a curvature tensor \( R^\lambda_{\mu\nu\beta} \). These are the only two constraints necessary to have maximal symmetry in presence of torsion and follow from usual properties that a curvature tensor is expected to possess. All this therefore implies that the constraints (27a) and (29b) are unique.

\( \bar{R}^\lambda_{\mu\nu\beta} \) is antisymmetric in the indices \( \nu \) and \( \beta \) once the above constraints are satisfied. Proceeding as in ref.[4] we have :

\[
(N - 1)\bar{R}\lambda_{\mu\nu\beta} = \bar{R}_{\beta\mu}g_{\lambda\nu} - \bar{R}_{\nu\mu}g_{\lambda\beta}
\]
i.e.

\[
(N - 1)\bar{R}\lambda_{\mu\nu\beta} + (N - 1)\bar{R}_{\lambda\mu\nu\beta} = R_{\beta\mu}g_{\lambda\nu} - R_{\nu\mu}g_{\lambda\beta} + \bar{R}_{\beta\mu}g_{\lambda\nu} - \bar{R}_{\nu\mu}g_{\lambda\beta}
\]

(32)

\( R_{\lambda\mu\nu\beta}, R_{\beta\mu} \) are functions of the symmetric affine coefficients \( \Gamma \) only, whereas \( \bar{R}_{\lambda\mu\nu\beta}, \bar{R}_{\beta\mu} \) are functions of both \( \Gamma \) and \( H \). Moreover, to start with \( \Gamma \) and \( H \) are independent. Broadly, the solution space of equation (32) consists of (a) solutions with \( H \) determined by \( \Gamma \) or vice versa (b) solutions where \( H \) and \( \Gamma \) are independent of each other. All these solutions lead to maximally symmetric spaces even in the presence of torsion.
We shall now illustrate that the smaller subspace (b) of these solutions enables one to cast the definition of maximal symmetry in the presence of torsion in an exactly analogous way to that in the absence of torsion. A particular set of such solutions of (32) can be obtained by equating corresponding terms on both sides to get:

\[(N - 1)R_{\lambda\mu\nu\beta} = R_{\beta\mu}g_{\lambda\nu} - R_{\nu\mu}g_{\lambda\beta}\]  
(33a)

\[(N - 1)\tilde{R}_{\lambda\mu\nu\beta} = \tilde{R}_{\beta\mu}g_{\lambda\nu} - \tilde{R}_{\nu\mu}g_{\lambda\beta}\]  
(33b)

The above two equations lead to

\[R_{\lambda\mu\nu\beta} = \frac{R_\alpha^\alpha (g_{\lambda\nu}g_{\mu\beta} - g_{\lambda\beta}g_{\nu\mu})}{N(N - 1)}\]  
(34a)

\[\tilde{R}_{\lambda\mu\nu\beta} = \frac{\tilde{R}_\alpha^\alpha (g_{\lambda\nu}g_{\mu\beta} - g_{\lambda\beta}g_{\nu\mu})}{N(N - 1)}\]  
(34b)

Note that

\[\tilde{R}_\alpha^\alpha_{\lambda\mu\nu\beta} = 0\]  
(35a)

\[\tilde{R}_\mu^\alpha_{\nu\mu} = H_\mu^\alpha_{\nu\mu,\alpha} - H_\mu^\alpha_{\nu\mu}\Gamma_{\alpha\gamma}^\gamma + H_\mu^\alpha_{\nu\mu}\Gamma_{\gamma\alpha}^\gamma - H_\mu^\alpha_{\nu\mu}\Gamma_{\alpha\gamma}^\gamma - H_\mu^\alpha_{\nu\mu}\Gamma_{\gamma\alpha}^\gamma\]  
(35b)

Consider the constraint (29b) with \(\beta = \lambda\). This gives:

\[H_\mu^\alpha_{\nu\mu,\alpha} + H_\mu^\alpha_{\nu\mu}\Gamma_{\alpha\gamma}^\gamma + H_\mu^\alpha_{\nu\mu}\Gamma_{\gamma\alpha}^\gamma - H_\mu^\alpha_{\nu\mu}\Gamma_{\alpha\gamma}^\gamma - H_\mu^\alpha_{\nu\mu}\Gamma_{\gamma\alpha}^\gamma = 0\]  
(36)

It is straightforward to verify that this implies \(\tilde{R}_{\mu\nu} = \tilde{R}_{\nu\mu}\). Hence (29b) implies that \(\tilde{R}_{\mu\nu}\) is symmetric. Under these circumstances

\[R_{\mu\nu} = \left(\frac{1}{N}\right) g_{\mu\nu}R_\alpha^\alpha\]  
(37a)

\[\tilde{R}_{\mu\nu} = \left(\frac{1}{N}\right) g_{\mu\nu}\tilde{R}_\alpha^\alpha = H_\mu^\alpha_{\nu\mu}H_\alpha^\beta\]  
(37b)

\[\tilde{R} = \tilde{R}_\alpha^\alpha = g^{\mu\nu}\tilde{R}_{\mu\nu} = g^{\mu\nu}H_\mu^\alpha_{\nu\mu}H_\alpha^\beta = H_\beta^\alpha H_\alpha^\beta\]  
(37c)
Now using arguments similar to those given in Ref.[4] for the Bianchi identities we can conclude that

\[ R_{\lambda \mu \nu} = R_{\lambda \mu \nu} + \tilde{R}_{\lambda \mu \nu} = (K + \tilde{K}) (g_{\lambda \nu} g_{\mu \beta} - g_{\lambda \beta} g_{\mu \nu}) = \tilde{K} (g_{\lambda \nu} g_{\mu \beta} - g_{\lambda \beta} g_{\mu \nu}) \]  

(38)

\[ R_\alpha^\alpha = constant = K N (1 - N) \]  

(39a)

\[ \tilde{R}_\alpha^\alpha = H_\beta^\mu H_\lambda^\beta = constant = \tilde{K} N (1 - N) \]  

(39b)

\[ \tilde{K} = K + \tilde{K} = constant \]  

(39c)

(In deriving the above results from the Bianchi identities we have used the fact that for a flat metric the curvature constant \( K = 0 \). Hence demanding \( \tilde{K} = 0 \) for a (globally) zero curvature space means that \( \tilde{K} = 0 \) which in turn means that the torsion must vanish.)

We now discuss two simple toy models where the torsion field which satisfies (39b) and is also consistent with (27a). First note that any non-vanishing torsion is always consistent with (27a) because

\[ H_\beta^\alpha \xi_{\mu;\alpha} = 0 \]  

(27a)

implies

\[ H_\beta^\alpha \xi_{\alpha;\mu} = 0 \]  

(27b)

through the Killing condition (25). Adding (27a) and (27b) gives

\[ H_\beta^\alpha (\xi_{\mu;\alpha} + \xi_{\alpha;\mu}) = 0 \]

Hence any non-zero torsion is consistent with (27a,b). The torsion is an antisymmetric third rank tensor obtained from a second rank antisymmetric tensor \( B_{\mu \nu} \) as follows:

\[ H_\mu^\alpha = \partial^\alpha B_{\mu \nu} + \partial_\mu B^\alpha_\nu + \partial_\nu B^\alpha_\mu \]
Consider dimension $D = 3$ and a general form for the metric as

$$ds^2 = -f_0(r)dt^2 + f(r)dr^2 + f_1(r)(dx^1)^2 \quad (40)$$

So the metric components are

$$g_{00} = -f_0(r), g_{rr} = f(r), g_{11} = f_1(r)$$

i.e. the metric components are functions of $r$ only. Further assume that all fields, including $B_{\mu\nu}$, depend only on $r$. Then the torsion is just $H_{01}^r$.

It is straightforward to verify that with the our chosen metric the only non-zero components of $\Gamma^\alpha_{\mu\nu}$ are $\Gamma^r_{00}$, $\Gamma^r_{11}$, $\Gamma^r_{rr}$, $\Gamma^0_{0r}$, and $\Gamma^1_{1r}$.

Now (29b) with $\lambda = \beta$ gives (36) which in the case under consideration reduces to (using the antisymmetry of $H$)

$$H_{01,r}^r + H_{01}^r \Gamma_{rr}^r + H_{1r}^0 \Gamma_{00}^r - H_{0r}^1 \Gamma_{11}^r = 0$$

We may write

$$H_{0r}^0 = g^{00}g_{rr}H_{01}^r, \quad H_{0r}^1 = -g^{11}g_{rr}H_{01}^r$$

Therefore (36) becomes

$$H_{01,r}^r + H_{01}^r \Gamma_{rr}^r + H_{1r}^0 \Gamma_{00}^r + g^{11}g_{rr} \Gamma_{11}^r = 0 \quad (41a)$$

Using the values

$$\Gamma_{rr}^r = \partial_r \ln f^{1/2}; \quad \Gamma_{00}^r = (1/2)(1/f)\partial_r f_0; \quad \Gamma_{11}^r = -(1/2)(1/f)\partial_r f_1$$

leads to

$$H_{01,r}^r + H_{01}^r \partial_r [\ln (f/(f_0f_1))^{1/2}] = 0 \quad (41b)$$
whose solution is

\[ H_{01}^r = [f_0 f_1]/f \]^{1/2} \tag{42} \]

(Note that the torsion can be taken proportional to the completely antisymmetric \( \epsilon \) tensor in three dimensions as follows: \( H_{r01} = g_{rr} H_{01}^r \), and so can be written as \( H_{r01} = [f f_0 f_1]^{1/2} \epsilon_{r01} \) and this can be further integrated to give the "magnetic field" as \( B_{01} = \epsilon_{01} \int dr [f f_0 f_1]^{1/2} \), etc.)

It is immediately verified that

\[ H_{01}^r H_{r}^{01} = -1 \]

(i.e. a constant) thereby satisfying the constraint (39b).

If \( \xi \) denotes a Killing vector then the Killing equations are :

\[ \xi^r \partial_r f_0 + 2 f_0 \partial_0 \xi^0 = 0 \tag{43a} \]
\[ \xi^r \partial_r f + 2 f \partial_r \xi^r = 0 \tag{43b} \]
\[ \xi^r \partial_r f_1 + 2 f_1 \partial_1 \xi^1 = 0 \tag{43c} \]
\[ f_1 \partial_0 \xi^1 - f_0 \partial_1 \xi^0 = 0 \tag{43d} \]
\[ f \partial_0 \xi^r - f_0 \partial_r \xi^0 = 0 \tag{43e} \]
\[ f_1 \partial_1 \xi^1 + f \partial_1 \xi^r = 0 \tag{43f} \]

The most general solutions are :

\[ f_0 = \exp[-2\alpha \int dr f^{1/2}] \tag{44a} \]
\[ f_1 = \exp[-2\gamma \int dr f^{1/2}] \tag{44b} \]
\[ \xi^r = 1/f^{1/2} \tag{44c} \]
\[
\xi^0 = \alpha t + \eta(x^1) + \beta
\]
\[\xi^1 = \gamma x^1 + \psi(t) + \delta
\]  

One set of solution of these equations are:

\[f_0 = f_1 = \text{constant} \ (i.e. \gamma = \alpha = 0; \beta = -\delta)\]

so that \(\xi^0\) and \(\xi^1\) differ up to a sign. It can be readily verified that the finite non-zero torsion given by (42) satisfies all the constraints.

**Toy Model 2.**

Suppose the space is flat in the symmetric part of the connection, i.e. \(\Gamma^\lambda_{\mu\nu} = 0\). Then \(\bar{\Gamma}^\lambda_{\mu\nu} = H^\lambda_{\mu\nu}\). The metric coefficients depend on the symmetric part of the connection only and hence the metric may be taken as:

\[ds^2 = -dt^2 + dr^2 + (dx^1)^2\]  

The Killing equations now are:

\[
\partial_0 \xi^0 = 0; \partial_r \xi^r = 0; \partial_1 \xi^1 = 0; \partial_0 \xi^1 = \partial_1 \xi^0; \partial_0 \xi^r = \partial_r \xi^0; \partial_r \xi^1 = -\partial_1 \xi^r
\]

One set of solutions for the Killing vectors consist of constant vectors with the components \(\xi^0\) and \(\xi^1\) differing up to a sign. With this set, again a constant value for the torsion is consistent and satisfies all the constraints.

The motive of these illustrations is to show that one can construct scenarios with a finite non-zero value for the torsion and still have maximal symmetry.

Therefore, in the presence of torsion the criteria of maximal symmetry has been generalised through the equations (34b), (38) and (39). The physical meaning is still that of a globally constant curvature (which now also
has a contribution from the torsion). We emphasize that in (39c) $K$ and $\bar{K}$ are separately constants. The torsion fields are subject to the constraints (27), (29b), and (39b). We mention that relations exactly similar to eqs. (37) have been obtained in ref. [13] in a totally different context.

V. The Question of Duality Invariance in Presence of Torsion

Let us investigate whether this generalised maximal symmetry can be preserved by the $0(d) \otimes 0(d)$ twist. For simplicity we take $D = 3$. It can be shown that for the metric of the type (13a) this is not possible. However, consider a more general form viz.

$$ds^2 = -f_0(r)dt^2 + f_1(r)(dx_1)^2 + f(r)dr^2$$

(46)

For $D = 3$, the torsion $H_{\mu\nu}^\alpha$ is just $H_{01}^r$. It is readily verified that such a torsion is consistent with (27) and satisfies (29b). (39b) now means

$$H_{r0}^{10}H_{01}^r = \text{constant} = \vartheta$$

(47)

Now, generalised maximal symmetry implies that

$$\bar{R}_{00} = (1 - N)\bar{K}G_{00}$$

(48a)

$$\bar{R}_{11} = (1 - N)\bar{K}G_{11}$$

(48b)

$$\bar{R}_{rr} = (1 - N)\bar{K}G_{rr}$$

(48c)

These equations lead to respectively

$$\partial_r P - \partial_r Q + P^2 - Q^2 + RQ - RP = 0$$

(49a)

$$\partial_r Q + Q^2 - QP - QR = 0$$

(49b)
\[ \partial_r P + P^2 - PQ - PR = 0 \]  
\[ \text{where } P = \left(\frac{1}{2}\right) \partial_r \ln f_1; Q = \left(\frac{1}{2}\right) \partial_r \ln f_1; R = \left(\frac{1}{2}\right) \partial_r \ln f \]

The solutions to eqs.(49a - 49c) can be formally written as:

\[ P = \left(\frac{1}{2}\right) \left[ A_1 \exp\left\{-\int dr (P + Q - R)\right\} + A_2 \exp\left\{-\int dr (P - Q - R)\right\}\right] \]  
\[ Q = \left(\frac{1}{2}\right) \left[ A_2 \exp\left\{-\int dr (P - Q - R)\right\} - A_1 \exp\left\{-\int dr (P + Q - R)\right\}\right] \]  
\[ R = \partial_r \ln (P - Q) + P + Q = \partial_r \ln (P + Q) + P - Q \]

There can be many possible solutions to the above equations. We reject the solution \( Q = 0 \) i.e. \( f_1 = \text{const.} \) as it can be shown that the generalised maximal symmetry cannot be preserved.

For generalised maximal symmetry to prevail we have the analogue of (12a) i.e. \( G_{00} = -f_0, G_{11} = f_1, G_{rr} = f \) etc.

\[ (H_{01r})^2 = \frac{\bar{K}(1 - D)}{2} G_{00} G_{11} G_{rr} \]  
\[ \text{i.e.} \]
\[ \hat{B}_{01} = \left[ \frac{\bar{K}(1 - D)}{2} \right]^{1/2} \int dr [G_{00} G_{11} G_{rr}]^{1/2} \]

The beautiful thing is that the relations (51) are exactly equivalent to (39b) under the identification of \( \bar{K} \) with \( \tilde{K} \). Actually it can be readily shown that \( K \) and \( \tilde{K} \) are proportional so that this identification is consistent.

Now, under the \( O(d) \otimes O(d) \) twist, the new fields are

\[ (H'_{01r})^2 = \frac{\bar{K}'(1 - D)}{2} G'_{00} G'_{11} G'_{rr} \]
i.e.

\[ \hat{B}'_{01} = \left( \frac{\hat{K}'(1 - D)}{2} \right)^{\frac{3}{2}} \int dr \left[ G_{00}' \ G_{11}' \ G_{rr} \right]^{\frac{3}{2}} \]  

(52b)

Here \( G_{00}' = -F_0 \), \( G_{11}' = F_1 \), \( G_{rr} = f \), \( F_0, F_1 \) are given by (19b) and \( G_{rr} = f \) remains unaffected by the \( O(d) \) twist. This gives the freedom of choosing the function \( f(r) \) to be such that the equations (19b), and (49 – 52) are satisfied. Hence under these circumstances the generalised maximal symmetry may be preserved.

VI. Conclusion

Maximally symmetric cosmological solutions have obvious physical implications for the observable universe. Such solutions in the context of low energy string effective theories already exist in the literature \(^{10,11}\). Tseytlin\(^{10}\) studied time dependent solutions of the leading order string effective equations for a non-zero central charge deficit and curved maximally space. Bento et al\(^{11}\) considered the effect of higher-curvature terms in the string low energy effective actions on the maximally symmetric cosmological solutions of the theory for various types of string theories.

Here we have shown that the \( O(d) \otimes O(d) \) twist on maximally symmetric vacuum solutions gives inequivalent solutions that are not maximally symmetric. However, a generalised definition of maximal symmetry can be given (taking into account torsion) and this may be preserved by the \( O(d) \otimes O(d) \) transformation under certain conditions.

Our work is therefore crucially important in the context of string related cosmology. We have seen that the background spacetime of string theory necessarily has torsion in the form of a second rank antisymmetric tensor field. Even if one starts with a torsion free background, duality transformation
automatically generates solutions with torsion. The concept of isotropic and homogenous universe in cosmology which is realized usually in the form of Robertson-Walker metric must therefore be changed when torsion is present. Our work is thus a step towards a consistent cosmological background in the context of string theory. The ideas and formalism introduced in this work requires further extension into various other aspects of observational cosmology. Such work is under progress.

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