Abstract—Reconfigurable intelligent surfaces (RISs) allow controlling the propagation environment in wireless networks by tuning multiple reflecting elements. RISs have been traditionally realized through single connected architectures, mathematically characterized by a diagonal scattering matrix. Recently, beyond diagonal RISs (BD-RISs) have been proposed as a novel branch of RISs whose scattering matrix is not limited to be diagonal, which creates new benefits and opportunities for RISs. Efficient BD-RIS architectures have been realized based on group and fully connected reconfigurable impedance networks. However, a closed-form solution for the global optimal scattering matrix of these architectures is not yet available. In this paper, we provide such a closed-form solution proving that the theoretical performance upper bounds can be exactly achieved for any channel realization. We first consider the received signal power maximization in single-user single-input single-output (SISO) systems aided by a BD-RIS working in reflective or transmissive mode. Then, we extend our solution to single-user multiple-input multiple-output (MIMO) and multi-user MIMO systems. We show that our algorithm is less complex than the iterative optimization algorithms employed in the previous literature. The complexity of our algorithm grows linearly (resp. cubically) with the number of RIS elements in the case of group (resp. fully) connected architectures.

Index Terms—Beyond diagonal reconfigurable intelligent surface (BD-RIS), fully connected, group connected, optimal closed-form design.

I. INTRODUCTION

Reconfigurable intelligent surfaces (RISs) are an emerging technology that will enhance the performance of future wireless communications [1]–[4]. This technology relies on large planar surfaces comprising multiple reflecting elements, each of them capable of inducing a certain amplitude and phase change to the incident electromagnetic wave. Thus, an RIS can steer the reflected signal toward the intended direction by smartly coordinating the reflection coefficients of its elements. RIS-aided communication systems benefit from several advantages. RISs with passive elements are characterized by ultra-low power consumption and do not cause any active additive thermal noise or self-interference phenomena. Furthermore, RIS is a low-profile and cost-effective solution since it does not include expensive radio frequency (RF) chains. In conventional RIS architectures, denoted as single connected architectures, each RIS element is independently controlled by a tunable impedance connected to ground [5]. As a result, conventional RISs are characterized by a diagonal scattering matrix, also known as phase shift matrix.

Conventional RISs have been optimized with several objectives, such as transmit power minimization [6], weighted sum-power minimization [7], and weighted sum-rate maximization [8]. In [9], RISs have been designed to optimally support wide-band communications. Recently, RISs have been also applied to improve the efficiency of wireless power transfer (WPT) [10] and simultaneous wireless information and power transfer (SWIPT) systems [11]. Multi-RIS aided systems have been studied in [12]–[14], where the inter-RIS signal reflections are exploited to fully unveil the potential of this technology. Path-loss models for RISs considering both near-field and far-field propagation have been developed in [15], [16]. Since continuous phase shifts are hard to realize in practice, RISs have been designed based on discrete phase shifts [17], [18]. In [19], [20], the authors addressed the problem of low-overhead channel estimation in RIS-aided systems. In [21], [22], practical reflection models capturing the phase-dependent amplitude variation in the reflection coefficients have been developed. Finally, prototypes of discrete phase shift RISs have been designed in [23], [24].

Differently from conventional RISs, beyond diagonal RISs (BD-RISs) have been proposed as a novel branch of RISs in which the scattering matrix is not limited to be diagonal [25]. Several BD-RIS architectures have been introduced, as shown in the classification tree in Fig. 1. In [5], the authors generalized the single connected architecture by connecting all or a subset of RIS elements through a reconfigurable impedance network, resulting in the fully and group connected architecture, respectively. Group and fully connected RISs have been designed with discrete reflection coefficients in [26]. In [27], the concept of simultaneously transmitting and reflecting RIS (STAR-RIS), or intelligent omni-surface (IOS), has been introduced. This BD-RIS architecture is able to reflect and transmit the impinging signal, differently from conventional RISs working only in reflective mode [28]–[30]. In [25], a general RIS model has been proposed to unify different modes (reflective/transmissive/hybrid) and different architectures (single/group/fully connected). The authors also propose the novel cell-wise group/fully connected BD-RIS architecture. In [31], BD-RISs supporting multi-sector mode have been proposed to achieve full-space coverage. In multi-
sector BD-RISs, the antennas are divided into multiple sectors, with each sector covering a limited region of space. In [32], dynamically group connected RISs are optimized based on a dynamic grouping strategy. In [33], a BD-RIS architecture with a non-diagonal phase shift matrix is proposed, able to achieve a higher rate than conventional RISs. Several benefits of BD-RISs over conventional RISs can be identified. Since BD-RISs can adjust not only the phases but also the magnitudes of the impinging waves, the received signal power is consequently improved [5]. In group connected RISs, the grouping strategy can be properly optimized to further increase the received signal power [26], [32]. When discrete reflection coefficients are considered, BD-RISs achieve the performance upper bound with fewer resolution bits than conventional RISs [26]. Finally, BD-RISs enable efficient hybrid transmissive and reflective mode [25], and highly directional full-space coverage [31].

The fully connected architecture enables the best performance gain with respect to all other RIS models proposed to date [5]. This is due to the additional degrees of freedom provided by the complex architecture. Besides, the group connected architecture has been proposed to achieve a good trade-off between performance enhancement and complexity. Depending on the group size, this architecture bridges between the single and the fully connected ones. However, a closed-form solution for the global optimal scattering matrix of group and fully connected architectures is not yet available. The scattering matrix has been optimized in recent literature by employing costly iterative optimization algorithms [5], [26]. For this reason, it was possible to show that the theoretical performance upper bounds are tight only experimentally.

In this paper, driven by the success of these novel BD-RIS architectures, we provide a closed-form solution to design the global optimal scattering matrix in the case of group and fully connected RISs. The resulting scattering matrix is proved to exactly achieve the received signal power upper bounds derived in [5] for single-input single-output (SISO) systems. Thus, we mathematically prove that these upper bounds are tight. Furthermore, we show that our algorithm is less complex than the iterative optimization methods applied to design the scattering matrix in the recent literature [5]. The complexity of our algorithm grows linearly with the number of RIS elements in the case of group connected architectures, while it grows cubically in fully connected architectures. The contributions of this paper are summarized as follows:

First, we provide a low-complexity closed-form global optimal solution for the scattering matrix of BD-RISs working in reflective mode applied to single-user SISO systems. In these systems, group and fully connected RISs designed with our solution exactly achieve their performance upper bounds. The upper bound-achieving property of our solution is valid for any channel realization, with no assumptions on its distribution.

Second, we consider BD-RISs working in transmissive mode, enabled by the cell-wise group connected architecture proposed in [25]. We show that our optimal solution can be also applied to design these BD-RISs. Also in the case of transmissive mode, the performance upper bounds are always exactly achieved by BD-RISs designed through our solution.

Third, we extend our optimal design strategy to single-user multiple-input multiple-output (MIMO) systems, including multiple-input single-output (MISO) systems as a special case. For systems aided by a fully connected RIS and with negligible direct link, we derive a tight upper bound on the received signal power. We show that such an upper bound can be always exactly achieved with our optimal design strategy. In addition, we also propose an efficient sub-optimal solution for the case in which the direct link not negligible. In this case, a tight upper bound on the received signal power is not known.

Fourth, we study the weighted sum power maximization problem in multi-user MIMO systems. In the case of systems aided by a fully connected RIS and with negligible direct links, we provide a tight performance upper bound and an optimal solution to achieve it. Also for multi-user MIMO systems, we provide a sub-optimal solution to design the RIS in the case the direct links are not negligible. In fact, a tight performance upper bound is not available in this case.

Organization: In Section II, we define the system model and the problem formulation. In Section III, we derive the upper bound-achieving closed-form solution for the scattering matrix in single-user SISO systems. In Section IV, we show that our solution can be also applied to optimally design BD-RISs working in transmissive mode. In Sections V and VI, we extend our closed-form solution to single-user MIMO, and multi-user MIMO systems, respectively. In Section VII, we assess the obtained performance through numerical simulations. Finally, Section VIII contains the concluding remarks.

Notation: Vectors and matrices are denoted with bold lower and bold upper letters, respectively. Scalars are represented with letters not in bold font. \( |a| \), and \( \arg(a) \) refer to the modulus and phase of a complex scalar \( a \), respectively. \( [a]_i \), and \( \|a\| \) refer to the \( i \)th element and \( l_2 \)-norm of vector \( a \), respectively. \( A^\ast, A^T, A^H, |A|_{i,j}, \) and \( \|A\| \) refer to the conjugate, transpose, conjugate transpose, \( i,j \)th element, and \( l_2 \)-norm of a matrix \( A \), respectively. \( u_{\max}(A) \) and \( v_{\max}(A) \) denote the dominant left and right singular vectors of a matrix \( A \), respectively. \( \mathbb{R} \) and \( \mathbb{C} \) denote the real and complex number sets, respectively. \( j = \sqrt{-1} \) denotes imaginary unit. \( 0 \) and \( \mathbf{I} \) denote an all-zero matrix and an identity matrix, respectively, with appropriate dimensions. \( \mathcal{CN}(0, \mathbf{I}) \) denotes the distribution of a circularly symmetric complex Gaussian random vector with mean vector \( \mathbf{0} \) and covariance matrix \( \mathbf{I} \) and \( \sim \) stands for “distributed as”. \( \text{diag}(a_1, \ldots, a_N) \) refers to
a diagonal matrix with diagonal elements being $a_1, \ldots, a_N$. \( \text{diag}(A_1, \ldots, A_N) \) refers to a block diagonal matrix with blocks being $A_1, \ldots, A_N$. 

II. BD-RIS Model

Let us consider a single-user SISO scenario in which the communication is aided by an $N_I$ antenna RIS. The $N_I$ antennas of the RIS are connected to a $N_I$-port reconfigurable impedance network, with scattering matrix \( \Theta \in \mathbb{C}^{N_I \times N_I} \). Defining $x \in \mathbb{C}$ as the transmitted signal and $y \in \mathbb{C}$ as the received signal, we have $y = hx + n$, where $n$ is the additive white Gaussian noise (AWGN) at the receiver. The channel $h$ can be written as

$$
    h = h_{RT} + h_{RI} \Theta h_{IT},
$$

where $h_{RT} \in \mathbb{C}$, $h_{RI} \in \mathbb{C}^{1 \times N_I}$, and $h_{IT} \in \mathbb{C}^{N_I \times 1}$ refer to the channels from the transmitter to receiver, from the RIS to the receiver, and from the transmitter to the RIS, respectively. According to network theory [24], denoting with $Z_I \in \mathbb{C}^{N_I \times N_I}$ the impedance matrix of the $N_I$-port reconfigurable impedance network, \( \Theta \) can be expressed as \( \Theta = (Z_I + Z_0I)^{-1}(Z_I - Z_0I) \). The $N_I$-port reconfigurable impedance network is constructed with passive elements which can be adapted to properly reflect the incident signal. To maximize the power reflected by the RIS, $Z_I$ must be purely reactive and we can write $Z_I = jX_I$, where $X_I \in \mathbb{R}^{N_I \times N_I}$ denotes the reactance matrix of the $N_I$-port reconfigurable impedance network. Hence, \( \Theta \) is given by

$$
    \Theta = (jX_I + Z_0I)^{-1}(jX_I - Z_0I). \tag{2}
$$

Furthermore, the reconfigurable impedance network is also reciprocal so that we have $X_I = X_I^T$ and $\Theta = \Theta^T$. Depending on the topology of the reconfigurable impedance network, three different BD-RIS architectures have been identified in [5], which are described in the following.

A. Single Connected RIS Architecture

The single connected RIS architecture is the conventional architecture adopted in the literature [1], [2]. Here, each port of the reconfigurable impedance network is connected to ground with a reconfigurable impedance and is not connected to the other ports. The reactance matrix $X_I$ is a diagonal matrix given by $X_I = \text{diag}(X_{1I}, X_{2I}, \ldots, X_{NI})$, where $X_{nI}$ is the reactance connecting the $n$th port to ground, for $n_I = 1, \ldots, N_I$. According to (2), the scattering matrix $\Theta$ is also a diagonal matrix written as

$$
    \Theta = \text{diag}(e^{j\theta_1}, \ldots, e^{j\theta_{N_I}}), \tag{3}
$$

where $e^{j\theta_{nI}} = \frac{jX_{nI} - Z_0}{jX_{nI} + Z_0}$ is the reflection coefficient of the reactance $X_{nI}$, for $n_I = 1, \ldots, N_I$.

B. Fully Connected RIS Architecture

The fully connected RIS architecture is obtained by connecting every port of the reconfigurable impedance network to all other ports. Therefore, the reactance matrix $X_I$ can be an arbitrary symmetric matrix. According to (2), $\Theta$ is a complex symmetric unitary matrix

$$
    \Theta = \Theta^T, \; \Theta^H \Theta = I. \tag{4}
$$

C. Group Connected RIS Architecture

The group connected RIS architecture has been proposed as a trade-off between the single connected and the fully connected to achieve a good balance between performance and complexity. In the group connected architecture, the $N_I$ elements are divided into $G$ groups, each having $N_G = \frac{N_I}{G}$ elements. Each element of the $N_I$-port is connected to all other elements in its group, while there is no connection inter-group. Thus, $X_I$ is a block diagonal matrix given by

$$
    X_I = \text{diag}(X_{1I,G}, \ldots, X_{I,G}), \tag{5}
$$

where $X_{I,g} \in \mathbb{R}^{N_G \times N_G}$ is the reactance matrix of the $N_G$-port fully connected reconfigurable impedance network for the $g$th group. According to (2), the following constraints can be found for the scattering matrix in the group connected architecture

$$
    \Theta = \text{diag}(\Theta_1, \ldots, \Theta_G), \quad \Theta_g = \Theta_g^T, \quad \Theta_g^H \Theta_g = I, \quad \forall g, \tag{6}
$$

which show that $\Theta$ is a block diagonal matrix with each block $\Theta_g$ being a complex symmetric unitary matrix, for $g = 1, \ldots, G$.

III. Optimal Design for BD-RIS-Aided Single-User SISO Systems: Reflective Mode

In this section, the BD-RIS is assumed to work in reflective mode, as typically considered in the literature [6–24]. Our goal is to design $\Theta$ for group and fully connected RISs to maximize the received signal power given by

$$
    P_R = P_T |h_{RT} + h_{RI} \Theta h_{IT}|^2, \tag{7}
$$

where $P_T = \mathbb{E}[|x|^2]$ is the transmitted signal power. In a practical development, $\Theta$ can assume a finite number of discretized values due to hardware constraints [26]. However, this is beyond the scope of this paper, where the constraint considered for group connected architectures are given in [6]. Since the entries of $\Theta$ are not constrained to assume discrete values, the term $h_{RI} \Theta h_{IT}$ can be always combined in phase with $h_{RT}$. Thus, we first maximize (7) by omitting $h_{RT}$ and then we adjust the phase of the resulting $\Theta$ depending on $\arg(h_{RT})$. We assume unitary $P_T$ and introduce the normalized channels $\hat{h}_{RI} = h_{RI}/||h_{RI}||$ and $\hat{h}_{IT} = h_{IT}/||h_{IT}||$ such that our problem becomes to maximize

$$
    \hat{P}_R = |\hat{h}_{RI} \Theta \hat{h}_{IT}|^2. \tag{8}
$$

For traditional single connected architectures, it is known that the scattering matrix can be simply optimized in closed-form. The maximum normalized received signal power is

$$
    \hat{P}_R^\text{Single} = \left( \sum_{n_I=1}^{N_I} |\hat{h}_{RI,n_I}| \hat{h}_{IT,n_I} |^2 \right), \tag{9}
$$

which is achieved by designing $\Theta$ as in (3) with

$$
    \theta_{n_I} = - \arg \left( \hat{h}_{RI,n_I} \right) = - \arg \left( \hat{h}_{IT,n_I} \right), \quad \forall n_I. \tag{10}
$$
However, an exact solution for the optimal $\Theta$ in group and fully connected RISs is not known. In the following, we first consider the optimal design of fully connected RISs. Second, we generalize our approach to group connected RISs.

### A. Closed-Form Solution for Optimal Fully Connected RIS

Since $X_I$ is real symmetric, we can use the eigenvalue decomposition to write $X_I = V \Lambda V^T$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{N_I}) \in \mathbb{R}^{N_I \times N_I}$ is a diagonal matrix containing the eigenvalues of $X_I$ ordered in decreasing order and $V \in \mathbb{R}^{N_I \times N_I}$ is orthonormal. Applying \eqref{eq:Theta}, the scattering matrix $\Theta$ is given by

$$\Theta = (jVAV^T + Z_0I)^{-1} (jVAV^T - Z_0I) = VDV^T,$$

where $D = \text{diag}(\mathbf{e}^{j\delta_1}, \ldots, \mathbf{e}^{j\delta_{N_I}}) \in \mathbb{C}^{N_I \times N_I}$ is a diagonal matrix with $\mathbf{e}^{j\delta_{N_I}} = \frac{\mathbf{e}^{j\lambda_{N_I}} - Z_0}{\lambda_{N_I} + Z_0}$. Note that the complex diagonal elements of the matrix $D$ have unit modulus by construction. Using the decomposition of $\Theta$ given by \eqref{eq:Theta}, the normalized received signal power $P_R$ can be expressed as

$$P_R = \left| \hat{h}_{RI} \right|^2 \left| D \hat{h}_{IT} \right|^2 = \left| \hat{h}_{RI} \right|^2 \left| D \hat{h}_{IT} \right|^2,$$

where $\hat{h}_{RI} = \hat{h}_{RI} V$ and $\hat{h}_{IT} = V^T \hat{h}_{IT}$. Note that \eqref{eq:PR} is the squared modulus of the dot product between $\hat{h}_{RI}$ and $D \hat{h}_{IT}$. Thus, using the Cauchy-Schwarz inequality, we have

$$P_R \leq \left| \hat{h}_{RI} \right|^2 \left| D \hat{h}_{IT} \right|^2 = 1,$$

where the equality $P_R = 1$ is achieved if and only if $\hat{h}_{RI} \mid_{n_I} = D \hat{h}_{IT} \mid_{n_I}$, $\forall n_I$.

Since we are interested in achieving the received signal power upper bound, our goal is now to find a real orthonormal matrix $V = [v_1, \ldots, v_{N_I}]$ such that condition \eqref{eq:hRI=0} is satisfied.

It is easy to recognize that if the channels $h_{RI}$ and $h_{IT}$ are linearly dependent, the optimal $V$ is $V = I$. Consequently, $D$ can be designed according to \eqref{eq:Theta} and the matrix $\Theta = VDV^T$ is readily obtained. For this reason, in the following discussion, we assume that the channels $h_{RI}$ and $h_{IT}$ are linearly independent.

Noting that $\hat{h}_{RI} \mid_{n_I} = \hat{h}_{RI} V n_I$ and $\hat{h}_{IT} \mid_{n_I} = V^T \hat{h}_{IT}$, condition \eqref{eq:hRI=0} becomes equivalent to

$$\hat{h}_{RI} V n_I = \hat{h}_{IT} V^T$$

which can be in turn rewritten as

$$v_T n_I R_{RI} v_{n_I} = v_T n_I R_{IT} v_{n_I},$$

where $R_{RI} = \hat{h}_{RI}^H \hat{h}_{RI} \in \mathbb{C}^{N_I \times N_I}$ and $R_{IT} = \hat{h}_{IT} \hat{h}_{IT}^H \in \mathbb{C}^{N_I \times N_I}$. The left and right hand sides of \eqref{eq:RI=IT} are quadratic forms. Since $v_T n_I R_{RI} v_{n_I} = v_T n_I R_{RI} ^H v_{n_I}$ and $v_T n_I R_{IT} v_{n_I} = v_T n_I R_{IT} ^H v_{n_I}$, we can replace in \eqref{eq:RI=IT} the matrices $R_{RI}$ and $R_{IT}$ with their symmetric parts $A_{RI} = 1/2 (R_{RI} + R_{RI} ^H) \in \mathbb{R}^{N_I \times N_I}$ and $A_{IT} = 1/2 (R_{IT} + R_{IT} ^H) \in \mathbb{R}^{N_I \times N_I}$, respectively, without changing the two quadratic forms. Thus, \eqref{eq:RI=IT} is equivalent to

$$v_T n_I A_{RI} v_{n_I} = v_T n_I A_{IT} v_{n_I},$$

which in turn becomes

$$v_T n_I A v_{n_I} = 0,$$

where the symmetric matrix $A = A_{RI} - A_{IT} \in \mathbb{R}^{N_I \times N_I}$ has been introduced. To solve \eqref{eq:RI=IT}, let us consider the eigenvalue decomposition $A = U \Lambda U^T$, where $\Lambda = \text{diag}(\delta_1, \ldots, \delta_{N_I})$ is a diagonal matrix containing the eigenvalues of $A$ ordered in decreasing order and $U \in \mathbb{R}^{N_I \times N_I}$ is orthonormal. By introducing the orthonormal vectors $t_{n_I} = U^T v_{n_I} \in \mathbb{R}^{N_I \times 1}$, for $n_I = 1, \ldots, N_I$, \eqref{eq:RI=IT} can be reformulated as a diagonal quadratic form

$$t_{n_I}^T \Lambda t_{n_I} = 0.$$ 

In other words, we need to solve $s_{n_I} \delta = 0$, where $s_{n_I} = [t_{n_I}_1]^2, \ldots, [t_{n_I}_{N_I}]^2 \in \mathbb{R}^{1 \times N_I}$ and $\delta = [\delta_1, \ldots, \delta_{N_I}]^T \in \mathbb{R}^{N_I \times N_I}$. Note that we need to find $N_I$ orthonormal vectors $t_{n_I}$ which are solutions of \eqref{eq:delta}. This task is hard in general since the solution space of \eqref{eq:delta} is a non-linear space. However, in our case, we can rely on the special structure of the vector $\delta$. As proved in the following, $\delta$ contains only two, three, and four non-zero elements when $N_I = 2$, $N_I = 3$, and $N_I \geq 4$, respectively. Thus, we solve \eqref{eq:delta} by separately studying these three cases. The following proposition is introduced to simplify \eqref{eq:delta} in the cases $N_I \in \{2, 3\}$.

**Proposition 1.** For any linearly independent $h_{RI} \in \mathbb{C}^{1 \times N_I}$ and $h_{IT} \in \mathbb{C}^{N_I \times 1}$, with $N_I \in \{2, 3\}$, the matrix $A$ has rank $r(A) = N_I$ and trace $\text{Tr}(A) = 0$.

**Proof.** Please refer to Appendix A. 

1) $N_I = 2$: In the case of fully connected RISs with $N_I = 2$, $A$ has two eigenvalues, both non-zero and one opposite of the other, as a consequence of Proposition 1. Denoting the two eigenvalues of $A$ as $\delta_1$ and $\delta_2$, we have that the vector $\delta$ writes as $\delta = [\delta_1, \delta_2]^T$, where $\delta_2 = -\delta_1$. Applying Proposition 1, we simplify \eqref{eq:delta} as

$$\delta_1 [t_{n_I}_1]^2 = \delta_1 [t_{n_I}_2]^2 = 0.$$ 

Thus, we need to solve

$$\begin{cases}
[t_{n_I}_1]^2 = [t_{n_I}_2]^2 = 0
\end{cases},$$

where the first equation is derived from \eqref{eq:delta} and the second equation is the unitary norm constraint on $t_{n_I}$, $\forall n_I \in \{1, 2\}$. Solving by substitution, we obtain $[t_{n_I}_1]^2 = [t_{n_I}_2]^2 = 1/2$. Finally, we choose $t_1 = \sqrt{1/2}$, $\sqrt{1/2}^T$ and $t_2 = \left[\sqrt{1/2}, -\sqrt{1/2}\right]^T$ to guarantee orthonormality between $t_1$ and $t_2$.

2) $N_I = 3$: Considering fully connected RISs with $N_I = 3$, we still rely on proposition 1 to simplify \eqref{eq:delta}. As a consequence of Proposition 1, $A$ has three eigenvalues, all non-zero. Denoting the three eigenvalues of $A$ as $\delta_1$, $\delta_2$, and $\delta_3$, we have that the vector $\delta$ writes as $\delta = [\delta_1, \delta_2, \delta_3]^T$. Applying Proposition 1, we simplify \eqref{eq:delta} as

$$\delta_1 [t_{n_I}_1]^2 + \delta_2 [t_{n_I}_2]^2 + \delta_3 [t_{n_I}_3]^2 = 0.$$
We choose the vector $t_1$ with only the first and the third entries non-zero. Such a vector always exists since Proposition 1 implies $\delta_1 > 0$ and $\delta_3 < 0$. Thus, we need to solve

\[
\begin{align*}
\delta_1 \|t_1\|^2 + \delta_3 \|t_1\|^2 &= 0 \\
\|t_1\|^2 + \|t_1\|^2 &= 1,
\end{align*}
\]

where the first equation is derived from (22) and the second equation is the unitary norm constraint. Solving by substitution, we obtain

\[
\begin{align*}
\|t_1\|^2 &= \frac{-\delta_3}{\delta_1 - \delta_3} - 1 - \frac{\delta_3}{\delta_1 - \delta_3},
\end{align*}
\]

(23)
giving $t_1 = \sqrt{\frac{-\delta_3}{\delta_1 - \delta_3}}, 0, \frac{\delta_3}{\delta_1 - \delta_3}]^T$. Now, we select the two remaining vectors in the form

\[
\begin{align*}
t_2 &= \left[ \frac{1}{K} \sqrt{\frac{-\delta_3}{\delta_1 - \delta_3}}, \sqrt{\frac{1 - 1}{K} - \frac{-\delta_3}{\delta_1 - \delta_3}} \right]^T, \\
t_3 &= \left[ -\frac{1}{K} \sqrt{\frac{-\delta_3}{\delta_1 - \delta_3}}, \sqrt{\frac{1 - 1}{K} + \frac{-\delta_3}{\delta_1 - \delta_3}} \right]^T,
\end{align*}
\]

(24)
where $K$ is a positive constant. It is easy to recognize that $t_1$, $t_2$, and $t_3$ are an orthonormal basis of $\mathbb{R}^3$ for any $\delta \neq 1$. Thus, $K$ must be designed such that $t_2$ and $t_3$ satisfy (22), that is

\[
\frac{\delta_1^3}{K^2 (\delta_1 - \delta_3)} + \frac{\delta_2^3}{K^2 (\delta_1 - \delta_3)} - \frac{\delta_3^2}{K^2 (\delta_1 - \delta_3)} = 0.
\]

(27)
Equation (27) can be simplified by substituting $\delta_2 = -\delta_1 - \delta_3$, which is always valid according to Proposition 1. Eventually, (27) gives $K = \sqrt{2}$, yielding $t_2 = \left[ \frac{1}{\sqrt{2(\delta_1 - \delta_3)}}, \frac{1}{\sqrt{2(\delta_1 - \delta_3)}} \right]^T$ and $t_3 = \left[ -\frac{1}{\sqrt{2(\delta_1 - \delta_3)}}, \frac{1}{\sqrt{2(\delta_1 - \delta_3)}} \right]^T$.

In the case of fully connected RISs with $N_I \geq 4$, we introduce the following proposition to simplify (19).

**Proposition 2.** For any linearly independent $h_{IJ} \in \mathbb{C}^{1 \times N_I}$ and $h_{IT} \in \mathbb{C}^{N_I \times 1}$, with $N_I \geq 4$, the matrix $A$ has rank $r(A) = 4$ and trace Tr $(A) = 0$. Furthermore, among its four non-zero eigenvalues, two are positive and two are negative.

**Proof.** Please refer to Appendix B. □

Denoting the first two eigenvalues of $A$ as $\delta_1$ and $\delta_2$, and the last two as $\delta_N$, we have that the vector $\delta$ writes as $\delta = [\delta_1, \delta_2, 0, \ldots, 0, \delta_N, -\delta_N]$. Applying Proposition 2, we simplify (19) as

\[
\delta_1 \|t_{nI}\|^2 + \delta_3 \|t_{nI}\|^2 + \delta_N \|t_{N-I}\|^2 = 0,
\]

(28)
We notice that $N_I - 4$ orthonormal solutions to (28) are given by the vectors $e_3, \ldots, e_{N_I-2}$, where $e_i \in \mathbb{R}^{N_I \times 1}$ denotes the vector with the $i$th entry being 1 and the others being 0, for $i = 3, \ldots, N_I - 2$. Thus, we now want to find the remaining four orthonormal vectors $t_1, t_2, t_3, t_4 \in \mathbb{R}^{N_I}$ solutions of (28), all orthogonal to $e_3, \ldots, e_{N_I-2}$. To make them orthogonal to $e_3, \ldots, e_{N_I-2}$, it is sufficient to set $t_i|_{nI} = 0$ for $i = 1, 2, 3, 4$ and $n_I = 3, \ldots, N_I - 2$.

We choose the first vector $t_3$ with only the first and the $(N_I - 1)$th entries non-zero. Note that such a vector always exists since $\delta_1 > 0$ and $\delta_N, -\delta_N < 0$. Thus, we need to solve

\[
\begin{align*}
\delta_1 \|t_1\|^2 + \delta_N \|t_1\|^2 &= 0, \\
\|t_1\|^2 + \|t_1\|^2 &= 1,
\end{align*}
\]

(29)
where the first equation is derived from (28) and the second equation is the unitary norm constraint. Solving by substitution, we obtain

\[
\begin{align*}
\|t_1\|^2 &= \frac{-\delta_N}{\delta_1 - \delta_N}, \\
\|t_1\|^2 &= \frac{-\delta_N}{\delta_1 - \delta_N},
\end{align*}
\]

(30)
which gives $t_1 = \left[ \sqrt{\frac{-\delta_N}{\delta_1 - \delta_N}}, 0, \ldots, 0 \right]^T$. Similarly, we choose the second vector $t_2$ with only the second and $N_I$th entries non-zero. Also this vector always exists since $\delta_2 > 0$ and $\delta_N, -\delta_N < 0$. With a similar procedure, we obtain

\[
\begin{align*}
\|t_2\|^2 &= \frac{-\delta_N}{\delta_2 - \delta_N}, \\
\|t_2\|^2 &= \frac{-\delta_N}{\delta_2 - \delta_N},
\end{align*}
\]

(31)
giving $t_2 = \left[ 0, \sqrt{\frac{-\delta_N}{\delta_2 - \delta_N}}, \ldots, 0, \sqrt{\frac{-\delta_2}{\delta_2 - \delta_N}} \right]^T$. Note that these first two vectors $t_1$ and $t_2$ are orthonormal by construction.

The remaining two vectors $t_3$ and $t_4$ must be linear combinations of a basis of the null space of the matrix $M = [t_1, t_2]^T$. Such a basis is readily given by two vectors $b_1$ and $b_2$ in the form

\[
\begin{align*}
b_1 &= [t_1]_{N-I-1}, 0, \ldots, -[t_1]_1, 0]^T, \\
b_2 &= [0, [t_2]_1, 0, \ldots, -[t_2]_2]^T,
\end{align*}
\]

(32)
whose $n_I$th entry is zero, for $n_I = 3, \ldots, N_I - 2$, in addition to the vectors $e_3, \ldots, e_{N-I-2}$. Thus, $t_3$ and $t_4$ can be expressed as a generic linear combination $c = a_1 b_1 + a_2 b_2$ given by

\[
\begin{align*}
c &= [a_1 [t_1]_{N-I-1}, a_2 [t_2]_1, \ldots, -a_1 [t_1]_1, -a_2 [t_2]_2]^T.
\end{align*}
\]

(34)
Now, our objective is to find $a_1$ and $a_2$ satisfying (33) and the unitary norm constraint, that is

\[
\begin{align*}
\delta_1 a_1^2 [t_1]_{N-I-1}^2 + \delta_2 a_2^2 [t_2]_1^2 + \delta_N - \delta_N a_1^2 [t_1]_1^2 + \delta_N, a_2^2 [t_2]_2^2 = 0, \\
a_1^2 [t_1]_{N-I-1}^2 + a_2^2 [t_2]_1^2 + a_1^2 [t_1]_1^2 + a_2^2 [t_2]_2^2 = 1.
\end{align*}
\]

(35)
Substituting in (35) the entries of the vectors $t_1$ and $t_2$ given by (30) and (31), respectively, we obtain

\[
\begin{align*}
\delta_1 a_1^2 + (\delta_2 + \delta_N) a_2^2 &= 0, \\
\delta_2 a_2^2 &= \frac{1}{2}.
\end{align*}
\]

(36)
Solving by substitution, we have

\[
\begin{align*}
a_1^2 &= \frac{-\delta_2 - \delta_N}{\delta_1 - \delta_N}, \\
a_2^2 &= \frac{1 - \delta_2 - \delta_N}{\delta_1 - \delta_N - \delta_2 - \delta_N}.
\end{align*}
\]

(37)
Note that this always means $a_1^2 = a_2^2 = 1/2$ since Proposition 2 gives $\delta_1 + \delta_N = -\delta_2 - \delta_N$. Finally, we choose $t_3 = \sqrt{1/2} b_1 + \sqrt{1/2} b_2$ and $t_4 = \sqrt{1/2} b_1 - \sqrt{1/2} b_2$ to guarantee orthonormality.
Algorithm 1: Optimal fully connected RIS design for single-user SISO systems.

Input: \( h_{RI} \in \mathbb{C}^{1 \times N_I}, h_{IT} \in \mathbb{C}^{N_I \times 1} \)  
Output: \( \Theta \)

1. \( \bar{h}_{RI} = \frac{h_{RI}}{\|h_{RI}\|} \);  
2. \( R_{RI} = h_{RI}^H \bar{h}_{RI}, R_{IT} = h_{IT}^H \bar{h}_{IT} \);  
3. \( A_{RI} = \frac{R_{RI} + R_{RI}^T}{2} \), \( A_{IT} = \frac{R_{IT} + R_{IT}^T}{2} \);  
4. \( \Delta \triangleq U \Delta U^T = A_{RI} - A_{IT} \);  
5. \( \delta \triangleq [\delta_1, \ldots, \delta_N] \text{T} = \text{diag}(\Delta) \);  
6. if \( N_I = 2 \) then  
   7. \( T = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{bmatrix} \);  
   8. else if \( N_I = 3 \) then  
      9. \( T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \);  
   10. else  
      11. \( t_1 = \left[ \frac{1}{\sqrt{2N_I - 1}}, \ldots, \frac{1}{\sqrt{2N_I - 1}}, 0 \right]^T \);  
      12. \( t_2 = \left[ 0, \frac{1}{2N_I - 1}, \frac{1}{2N_I - 1}, \ldots, 0 \right]^T \);  
      13. \( t_3 = \frac{1}{\sqrt{2}} [t_1]_{N_I - 1} \) \( \text{,...,} \) \( [t_2]_{N_I - 1} \);  
      14. \( t_4 = \frac{1}{\sqrt{2}} [t_1]_{N_I - 1} \) \( \text{,...,} \) \( [t_2]_{N_I - 1} \);  
      15. \( T = [t_1, t_2, t_3, t_4, e_3, \ldots, e_{N-2}] \);  
   16. end  
17. \( V = UT \);  
18. \( d_{ni} = -\text{arg} \left( \bar{h}_{RI} V \right)_{ni} \) \( \text{arg} \left( V^T \bar{h}_{IT} \right)_{ni} \), \( \forall n_I \);  
19. \( D = \text{diag}(e^{jd_1}, \ldots, e^{jd_{N_I}}) \);  
20. \( \Theta = VDV^T \);  
21. return \( \Theta \)

In conclusion, we construct an orthonormal matrix \( T \in \mathbb{R}^{N_I \times N_I} \) depending on the number of RIS elements \( N_I \). If \( N_I = 2 \), \( T = [t_1, t_2] \); if \( N_I = 3 \), \( T = [t_1, t_2, t_3] \); and if \( N_I \geq 4 \), \( T = [t_1, t_2, t_3, t_4, e_3, \ldots, e_{N-2}] \). Note that the columns of \( T \) are orthogonal with each other, have unitary norm, and solve (19). At this stage, all the building elements of the optimal scattering matrix, denoted as \( \Theta \), are available. Applying (11), we can write \( \Theta = VDV^T \), where \( V = UT \) by definition of the columns of \( T \), and \( D \) is designed according to (10). We summarize the necessary steps to construct the optimal \( \Theta \) in Alg. 1. To maximize \( P_R \) in the presence of the direct link \( h_{RT} \), the scattering matrix can be adjusted as

\[
\Theta^* = e^{j \text{arg}(h_{RT})} \Theta,
\]

such that the term \( h_{RI} \Theta h_{IT} \) is made in phase with \( h_{RT} \).

B. Closed-Form Solution for Optimal Group Connected RIS

Now, we extend our closed-form strategy to design fully connected architectures to group connected ones. As previously discussed, we initially omit the direct link \( h_{RT} \) and assume unitary transmitted signal power. Thus, the received signal power for group connected architectures writes as

\[
P_R = \left| \sum_{g=1}^{G} h_{RI,g} \Theta_g h_{IT,g} \right|^2
\]  (39)

where \( h_{RI} = [h_{RI,1}, \ldots, h_{RI,G}] \) with \( h_{RI,g} \in \mathbb{C}^{1 \times N_G} \) and \( h_{IT} = [h_{IT,1}, \ldots, h_{IT,G}]^T \) with \( h_{IT,g} \in \mathbb{C}^{N_G \times 1} \). It is easy to recognize that (39) is maximized when the terms \( h_{RI,g} \Theta_g h_{IT,g} \) are all individually maximized in absolute value and they are all co-phased. Recalling the constraint on \( \Theta_g \) given by (6), the optimal \( \Theta_g \) that maximizes \( h_{RI,g} \Theta_g h_{IT,g} \) is given by Alg. 1 if applied to the truncated channels \( h_{RI,g} \text{ and } h_{IT,g} \). Note that \( \Theta_g \) constructed by Alg. 1 ensures that the complex number \( h_{RI,g} \Theta_g h_{IT,g} \) has phase zero. Thus, (39) is maximized when the matrices \( \Theta_g \) are constructed by Alg. 1 since all the terms \( h_{RI,g} \Theta_g h_{IT,g} \) are co-phased. The block diagonal matrix \( \Theta \) is finally obtained from the matrices \( \Theta_g \) applying (6). To maximize \( P_R \) in the presence of the direct link \( h_{RT} \), also for group connected architectures \( \Theta \) can be adjusted as in (38).

IV. OPTIMAL DESIGN FOR BD-RIS-AIDED SINGLE-USER SISO SYSTEMS: TRANSMISSIVE MODE

In Section III, we assumed the BD-RIS to work in reflective mode. This implies that both the transmitter and the receiver are covered by all the RIS elements. In other words, all the entries of \( h_{RI} \) and \( h_{IT} \) are non-zero in general. In this section, we study the case in which the BD-RIS works in transmissive mode, as modeled in (25). Following (25), we consider a BD-RIS made of \( N_I = 2M_I \) elements, where \( M_I \) is the number of RIS cells. Each RIS cell is formed by two RIS elements placed back to back and connected to each other through a reconfigurable impedance. Specifically, we assume that the \( m_I \)th cell is formed by the \((2m_I - 1)\)th and \((2m_I)\)th RIS elements, for \( m_I = 1, \ldots, M_I \). With this notation, the RIS elements can be partitioned into two sectors, where sector 1 is formed by the odd RIS elements and sector 2 is formed by the even RIS elements. Thus, the whole space is divided into two sides, respectively covered by the two sectors. When the BD-RIS is working in transmissive mode, the transmitter and the receiver are located in opposite sectors. In the following, we assume the transmitter to be in sector 1 and the receiver in sector 2. Denoting as \( h_{RI} \in \mathbb{C}^{1 \times 2M_I} \) the channel from the RIS to the receiver, and as \( h_{IT} \in \mathbb{C}^{2M_I \times 1} \) the channel from the transmitter to the RIS, the received signal power writes as

\[
P_R = P_T \left| h_{RT} + h_{RI} \Theta h_{IT} \right|^2
\]  (40)

where the odd entries of the channel \( h_{RI} \) are zero, as well as the even entries of the channel \( h_{IT} \).

Transmissive mode is enabled by the cell-wise single/group/fully connected BD-RIS architectures presented in
of the equivalent channel \( H \) precoder and combiner are given by the dominant eigenvectors reflected by the RIS. This assumption reflects real scenarios where half of the entries of \( N \) are negligible compared to the channel energy of the reflected link. In this case, at the first iteration of the optimization process, it is possible to design \( \Theta \) through our optimal solution to achieve a received signal power \( P_{R} \), exactly solved for the SISO setting in Section III-A.

Thus, the equality (43) is intended up to a phase shift since the complex singular vectors of a matrix are only defined up to a phase shift. Thus, condition (43) is satisfied when the cosine similarity

\[
\rho = |v_{RI}^H \Theta u_{IT}|^2
\]

is maximized, i.e., \( \rho = 1 \). Maximizing (44) is similar to the problem of maximizing the normalized received signal power in (8), exactly solved for the SISO setting in Section III-A.

Therefore, the optimal \( \Theta \) satisfying (43) can be found by applying Alg. 1 to the vectors \( v_{RI}^H \) and \( u_{IT} \) and the upper bound (42) is tight. In the MISO setting, the optimal \( \Theta \) is given by Alg. 1 applied to the vectors \( h_{RI} \) and \( u_{IT} \).

### B. Optimizing Fully/Group Connected RIS-Aided Systems With Direct Link

In single-user RIS-aided MIMO systems, tight upper bounds on the received signal power are not available in general, i.e., when group connected RISs are considered or the direct link is not negligible. For these cases, we propose a suboptimal solution to maximize the received signal power in which the matrix \( \Theta \) and the beamforming vectors \( w \) and \( g \) are alternatively optimized, as established in the literature on single connected RISs [4], [6]. After \( w \) and \( g \) are initialized to feasible values, this optimization process alternates between the two following steps until convergence is reached. With fixed \( w \) and \( g \), we update \( \Theta \) by optimally maximizing the objective \( |gH_{RT}^H w + gH_{RI} \Theta H_{IT} w|^2 \) as proposed for SISO systems.

The optimal \( \Theta \) is obtained by applying the strategy proposed in Section III-B to the channels \( h_{RT}^H = gH_{RT}^H w \), \( h_{eff}^H = gH_{RI} \), and \( h_{IT}^H = H_{IT} w \). With fixed \( \Theta \), we update \( w \) and \( g \) as the dominant right and left singular vectors of \( H_{RT} + H_{RI} \Theta H_{IT} \), respectively. The convergence is considered reached when the fractional increase of the objective \( P_{R} \) in a full iteration is below a certain parameter \( \epsilon \).

Depending on the direct link strength, we consider two possible initializations for the beamforming vectors \( w \) and \( g \). In the following, we define the left and right dominant singular vectors of the matrices \( H_{ij} \) as \( u_{ij} \) and \( v_{ij} \), respectively, for \( i, j \in \{ RT, RI, IT \} \). If the direct link is particularly strong, we set \( w = v_{RT} \) and \( g = u_{RT}^H \) to capture the energy in the direct channel dominant eigenmode. In this case, at the first iteration of the optimization process, it is possible to design \( \Theta \) through our optimal solution to achieve a received signal power

\[
P_{R}^{\text{dir}} = P_{T} \left( \|H_{RT}\|^2 + \sum_{g=1}^{G} \|h_{RG}\| \|h_{TG}\|^2 \right),
\]

where \( h_{RG} = u_{RG}^H H_{RI} \) and \( h_{TG} = H_{IT} v_{RT} \). Conversely, if the direct link is weak or a high number of RIS elements is employed, we set \( w = v_{IT} \) and \( g = u_{IT}^H \) to capture the energy of the reflected link. In this case, at the first iteration of the optimization process, \( \Theta \) can be optimized to achieve a
Algorithm 2: BD-RISs design for single-user MIMO systems.

Input: \( H_{RT} \in \mathbb{C}^{N_R \times N_T}, H_{RI} \in \mathbb{C}^{N_R \times N_I}, H_{IT} \in \mathbb{C}^{N_I \times N_T}, N_G, \epsilon \)

Output: \( \Theta \)

1. if \( P_R^{\text{ref}} \geq P_R^{\text{ref}} \) then
2. \( w = v_{RT}, g = u_{RT}^H; \)
3. else
4. \( w = v_{IT}, g = u_{RI}^H; \)
5. end
6. repeat
7. \( h_{RT}^\text{eff} = gH_{RT}w, h_{RI}^\text{eff} = gH_{RI}, h_{IT}^\text{eff} = H_{IT}w; \)
8. Compute \( \Theta \) by applying Alg. 1 to \( h_{RI}^\text{eff}, h_{IT}^\text{eff}; \)
9. \( \Theta = e^{i \arg(H_{RT}^\text{eff})} \Theta; \)
10. \( w = v_{\max}(H_{RT} + H_{RI} \Theta H_{IT}); \)
11. \( g = u_{\max}^H(H_{RT} + H_{RI} \Theta H_{IT}); \)
12. until The fractional increase of the objective \( P_R^{\text{MIMO}} \)
13. return \( \Theta \)

The fractional increase of the objective \( P_R^{\text{MIMO}} \) is below \( \epsilon \).

received signal power

\[
P_R^{\text{eff}} = P_T \left( \left| u_{RI}^H H_{RT} v_{IT} \right| + \| H_{RI} \| \| H_{IT} \| \sum_{g=1}^{G} \left| v_{IT,g}^H \right| \| u_{IT,g} \| \right)^2.
\]

Because of the initialization strategy, a lower bound on the received signal power achieved in MIMO systems is given by

\[
P_R^{\text{MIMO}} = \max \{ P_R^{\text{ref}}, P_R^{\text{ref}} \},
\]

which is the received signal power obtained after the first iteration of our optimization process. Note that this is a lower bound since the objective function \( P_R^{\text{MIMO}} \) is non-decreasing over iterations.

We summarize the steps necessary to optimize \( \Theta \) in single-user MIMO systems in Alg. 2. The convergence of Alg. 2 is guaranteed by the following two facts. First, at each iteration, the objective \( P_R^{\text{MIMO}} \) is non-decreasing. Second, the objective function is bounded from above by \( P_T (\| H_{RT} \|^2 + \| H_{RI} \|^2 \| H_{IT} \|)^2 \) because of the triangle inequality, and the sub-multiplicativity of the spectral norm. Note that Alg. 2 can be readily applied to the MISO setting, as it is a special case of the MIMO setting, in which \( N_R = 1 \).

VI. BD-RIS-AIDED MULTI-USER MIMO SYSTEMS

In this section, we study the weighted sum power maximization in multi-user MIMO systems, which is a problem particularly relevant in WPT applications [35]. Let us consider an \( N_T \) antenna transmitter serving \( K \) single-antenna receivers through the support of a BD-RIS working in reflective mode.

We denote the channel from the transmitter to the \( k \)th receiver and from the RIS to the \( k \)th receiver as \( h_{RT,k} \in \mathbb{C}^{1 \times N_T} \) and \( h_{RI,k} \in \mathbb{C}^{1 \times N_I} \), respectively. Consequently, the equivalent channel seen by the \( k \)th receiver is denoted as \( h_k = h_{RT,k} + h_{RI,k} \Theta H_{IT} \).

In general, the transmitted signal writes as

\[
x = \sqrt{P_T} \sum_{k=1}^{K} w_k s_k,
\]

where the precoding vectors \( w_k \) are subject to the constraints \( \| w_k \|^2 = 1 \) and \( s_k \) are the energy-carrying signals subject to \( E[\| s_k \|^2] = 1 \). We denote by \( \alpha_k > 0 \) the power weight of the \( k \)th receiver, where a larger \( \alpha_k \) indicates a higher power requirement for the \( k \)th receiver. The weighted sum power is given by

\[
S_R = \sum_{k=1}^{K} \alpha_k P_{R,k},
\]

where \( P_{R,k} \) is the received signal power at the \( k \)th receiver, which is written as

\[
P_{R,k} = P_T \sum_{j=1}^{K} | h_k w_j|^2.
\]

Substituting [50] into [49], we obtain

\[
S_R = \sum_{k=1}^{K} \alpha_k P_{R,k} \sum_{j=1}^{K} | h_k w_j|^2 = \sum_{j=1}^{K} P_T w_j^H S w_j,
\]

where we introduced \( S = \sum_{k=1}^{K} \alpha_k h_k^H h_k \). From [51], we notice that the optimal precoding vectors \( w_j \) should be all aligned with the dominant eigenvector of \( S \), denoted as \( v_{\max}(S) \). As in [35], we consider a single-stream precoding given by \( w = v_{\max}(S) \), with no loss of optimality. With this optimal precoding, the weighted sum power is

\[
S_R = P_T \| S \| = P_T \| H^H H \| = P_T \| H \|^2,
\]

where we introduced \( H = [\sqrt{\alpha_1} h_1^H, \ldots, \sqrt{\alpha_K} h_K^H]^H \). To maximize \( S_R \), it is convenient to rewrite [52] by explicitly highlighting the role of \( \Theta \). Defining the matrices \( G_{RT} = [\sqrt{\alpha_1} h_{RT,1}^H, \ldots, \sqrt{\alpha_K} h_{RT,K}^H]^H \) and \( G_{RI} = [\sqrt{\alpha_1} h_{RI,1}^H, \ldots, \sqrt{\alpha_K} h_{RI,K}^H]^H \), we have that \( H \) can be expressed as \( H = G_{RT} + G_{RI} \Theta H_{IT} \).

A. Optimizing Fully Connected RIS-Aided Systems Without Direct Links

We first consider a system aided by a fully connected RIS, and we assume that the direct channels between transmitter and receivers are negligible compared to the channels reflected by the RIS. Consequently, the equivalent channel seen by the \( k \)th receiver is given by \( h_k = h_{RT,k} \Theta H_{IT} \), yielding \( H = G_{RI} \Theta H_{IT} \). Thus, the maximum weighted sum power is given by \( P_T \| G_{RI} \Theta H_{IT} \|^2 \), which is upper bounded by

\[
S_R = P_T \| G_{RI} \|^2 \| H_{IT} \|^2,
\]

because of the sub-multiplicativity of the spectral norm. Using the discussion carried out for the single-user MIMO setting,
we introduce the vectors $\mathbf{t}_{RI}$ and $\mathbf{u}_{IT}$ as the dominant left singular vectors of $\mathbf{G}^H_{RI}$ and $\mathbf{H}_{IT}$, respectively. Thus, the global optimal $\Theta$ achieving (53) can be found by applying Alg. 1 to the vectors $\mathbf{t}^H_{RI}$ and $\mathbf{u}_{IT}$. This proves that the upper bound (53) is tight.

**B. Optimizing Fully/Group Connected RIS-Aided Systems With Direct Links**

For general systems in which group connected RISs are considered or the direct links are not negligible, tight performance upper bounds are not available. In this case, we notice that maximizing (52) is equivalent to maximizing

$$S_R = P_T |z_1 (\mathbf{G}_{RT} + \mathbf{G}_{RI} \Theta_{H_{IT}}) z_2|^2,$$

(54)

where $z_1 \in \mathbb{C}^{1 \times N_R}$ and $z_2 \in \mathbb{C}^{N_T \times 1}$ are auxiliary variables such that $\|z_1\| = 1$ and $\|z_2\| = 1$. Furthermore, maximizing (54) is similar to the problem of maximizing the received signal power in (41), solved through Alg. 2 in Section V-B. Thus, our sub-optimal strategy provided by Alg. 2 can be readily applied to solve also this maximization problem.

**VII. NUMERICAL RESULTS**

Let us consider a two-dimensional coordinate system, in which the $y$-axis represents the height above the ground in meters (m). The transmitter and the receiver are located at $(0, 0)$ and $(52, 0)$, respectively. The RIS is located at $(50, 2)$ and is equipped with $N_I$ antennas. The distance-dependent path loss is modeled as $L_{ij}(d_{ij}) = L_0 d_{ij}^{-\alpha_{ij}}$, where $L_0$ is the reference path loss at distance 1 m, $d_{ij}$ is the distance, and $\alpha_{ij}$ is the path loss exponent for $ij \in \{RT, RI, IT\}$.

We set $L_0 = -30$ dB, $\alpha_{RT} = 3.5$, $\alpha_{RI} = 2.8$, $\alpha_{IT} = 2$, and $P_T = 10$ W. For the small-scale fading, we assume that the channels from the transmitter to the receiver and from the transmitter to the RIS are both Rayleigh fading channels. The channel from the RIS to the receiver is modeled with both Rayleigh and Ricean fading, given by

$$\mathbf{h}_{RI} = \sqrt{L_{RI}} \left( \sqrt{\frac{K_F}{1 + K_F}} \mathbf{h}^{LoS}_{RI} + \sqrt{\frac{1}{1 + K_F}} \mathbf{h}^{NLoS}_{RI} \right),$$

(55)

where $K_F$ refers to the Ricean factor, while $\mathbf{h}^{LoS}_{RI}$ and $\mathbf{h}^{NLoS}_{RI} \sim \mathcal{CN}(0, 1)$ represent the small-scale line-of-sight (LoS) and non-line-of-sight (NLoS) (Rayleigh fading) components, respectively.

**A. RIS-Aided Single-User SISO Systems: Reflective Mode**

We start by analyzing the performance of SISO systems aided by single, group, and fully connected RISs working in reflective mode. In Fig. 2, we report the average received signal power given in (7) obtained by optimizing the scattering matrix $\Theta$ through our optimal strategy proposed in...
Section III-B for different group sizes. We compare these results with the average received signal power upper bound given by

$$\bar{P}_{\text{Group}} = P_T \left( |h_{RT}| + \sum_{g=1}^{G} \|h_{RI,g}\| \|h_{IT,g}\| \right)^2,$$  

(56)

as derived in [5]. Note that the upper bound (56), valid for group connected RISs, boils down to

$$\bar{P}_{\text{Single}} = P_T \left( |h_{RT}| + \sum_{n_I=1}^{N_I} |h_{RI}|_{n_I} |h_{IT}|_{n_I} \right)^2,$$  

(57)

for single connected RISs and to

$$\bar{P}_{\text{Fully}} = P_T \left( |h_{RT}| + \|h_{RI}\| \|h_{IT}\| \right)^2,$$  

(58)

for fully connected RISs. As expected, we observe that the upper bounds are exactly achieved by our closed-form solution. Fully connected RISs achieve the same performance in both Rayleigh and Ricean fading conditions. However, single and group connected RISs benefit from the LoS component, achieving higher power with Ricean fading, agreeing with [5].

### B. RIS-Aided Single-User SISO Systems: Transmissive Mode

We now consider SISO systems aided by a BD-RIS working in transmissive mode. The receiver is now located in (52, 4) and we set $\alpha_{RT} = 4$ to model a weak direct link. As previously discussed, we assume the transmitter to be in sector 1 and the receiver in sector 2. Thus, the odd entries of the channel $h_{RI}$ and the even entries of the channel $h_{IT}$ are forced to zero. In Fig. 3, we report the received signal power achieved by the optimal design strategy proposed in Section III-B and its upper bounds. We observe that the received signal power upper bounds are exactly achieved by our solution involving Alg. 1. Thus, our optimal design strategy can be successfully applied to BD-RIS working in both reflective and transmissive modes.

### C. RIS-Aided Single-User MIMO Systems

We analyze the performance of RIS-aided single-user MIMO and MISO systems, in which the RIS works in reflective mode. In Fig. 4, we report the received signal power achieved by a fully connected RIS, and with negligible direct link. The received signal power obtained by the exact solution provided by Alg. 1 is compared with its upper bound (42). We observe that the performance upper bounds are exactly achieved by our solution. Furthermore, higher performance is obtained by increasing the number of antennas $N_R$ and $N_T$. Rayleigh fading channels allow reaching a slightly higher received signal power since they offer richer scattering. In Fig. 5, we consider single-user MIMO and MISO systems aided by a single, group, or fully connected RIS. Alg. 2 is used to maximize the received
signal power \((41)\) by optimizing the RIS scattering matrix. As expected, fully connected RISs achieve higher received signal power than group connected RISs, which in turn outperform single connected RISs. The performance gap between fully connected and single connected architectures is slightly higher in Rayleigh fading conditions.

**D. RIS-Aided Multi-User MIMO Systems**

We now consider the weighted sum power maximization problem in RIS-aided multi-user MIMO systems, with the RIS working in reflective mode. In our numerical simulations, all \(K\) receivers are placed in \((52, 0)\), and we set \(N_T = 4\) and \(\alpha_k = 1\) for \(k = 1, \ldots, K\). In Fig. 6, we consider multi-user MIMO systems aided by a fully connected RIS, and with negligible direct links between transmitter and receivers. The received sum power \((49)\) is maximized by applying the optimal precoding \(w = v_{\text{max}}(S)\), and by designing \(\Theta\) through Alg. 1. We compare this received sum power with its upper bound \((53)\). As expected, the solution offered by Alg. 1 is optimal as it exactly achieves the performance upper bounds. In Fig. 7, we report the weighted sum power of multi-user MIMO systems aided by a single, group, or fully connected RIS, as given in \((49)\). In these systems, Alg. 2 is used to optimize the RIS scattering matrix. Fully connected RISs achieve the highest performance over group and single connected BD-RISs, while single connected RISs obtain the lowest weighted sum power. Besides, the weighted sum power increases with the number of receivers \(K\) because of the higher diversity offered.

**E. Computational Complexity**

Finally, we assess the computational complexity of our optimal design strategy. In fully connected architectures, the complexity growth of Alg. 1 as a function of \(N_I\) is given by the complexity of eigenvalue decomposition, that is \(O(N_I^3)\). This is less than the complexity of each iteration of the quasi-Newton optimization adopted in previous literature, which is given by \(O(N^2_I(N_I + 1)^2/4)\) \([5]\). In group connected architectures with group size \(N_G\), the block diagonal scattering matrix is designed by running \(G = N_I/N_G\) times Alg. 1 with complexity \(O(N_G^3)\). Thus, the complexity of our solution is \(O(N_G^3N_I)\) in this case, less than the complexity of quasi-Newton optimization \(O(N^2_I(N_G + 1)^2/4)\) \([5]\). In Fig. 8, the computational complexity of Alg. 1 is compared with the complexity of the quasi-Newton method adopted in \([5]\).

**VIII. Conclusion**

We provide a low-complexity closed-form solution to design the global optimal scattering matrix in the case of group and fully connected RISs. The resulting scattering matrix is proved to achieve exactly the received signal power upper
bounds derived in [5]. Our solution is upper bound-achieving for any channel realization since we do not pose assumptions on the channel distribution. We first present an optimal design strategy for single-user SISO systems. Subsequently, our strategy is extended to single-user MIMO and multi-user MIMO systems, where the weighted sum power maximization problem is considered. For systems aided by a fully connected RIS and with negligible direct links, we provide tight performance upper bounds. We show that such upper bounds can be exactly achieved with our optimal design strategy. Finally, we show that our algorithm is less complex than the iterative optimization methods applied to design the scattering matrix in recent literature. The complexity of our algorithm grows linearly (resp. cubically) with the number of RIS elements in the case of group (resp. fully) connected architectures.

Our optimal design strategy is expected to play a role in the solution of two problems related to the design of group and fully connected RISs. Firstly, a possible research direction is to consider our strategy to further improve the design of discrete-value group and fully connected RISs. Secondly, the optimal scattering matrix properties highlighted by our strategy could be exploited to enable efficient channel estimation for group and fully connected RISs.

APPENDIX

A. Proof of Proposition 1

The matrix $A$ is a linear combination of four outer products matrices, i.e.,

$$A = 2R_{RI} + 2R_{RT} - 2R_{IT} = BB^H,$$  \hspace{1cm} (59)$$

where the matrix $B \in \mathbb{C}^{N_I \times 4}$ is introduced as $B = \frac{1}{\sqrt{2}}[h_{RI}, h_{RT}, jh_{IT}, jh_{IT}]$. Since $B$ is full rank, we have $r(A) = r(B) = \min\{4, N_I\}$. Note that we assumed $h_{RI}$ and $h_{IT}$ to be independent in this discussion since the linearly dependent case has been trivially addressed. Thus, it holds $r(A) = N_I$ if $N_I \in \{2, 3\}$. The trace of $A$ can be readily computed from (59) by observing that $\text{Tr}(R_{RI}) = 1$ and $\text{Tr}(R_{IT}) = 1$ since $\|h_{RI}\| = 1$ and $\|h_{IT}\| = 1$. By applying the trace linearity property, we have $\text{Tr}(A) = 0$.

B. Proof of Proposition 2

To prove that $r(A) = 4$ and $\text{Tr}(A) = 0$, we can directly apply the Proof of Proposition 1. To prove that $A$ has two positive and two negative eigenvalues, we use the fact that $A = A_{RI} - A_{IT}$ is given by the sum of two symmetric matrices. This proof is carried out by trying differently the cases $N_I = 4$ and $N_I > 4$.

In the case $N_I = 4$, $A$ is a full rank matrix, i.e., $\det(A) \neq 0$. We denote the decreasingly ordered eigenvalues of $A_{RI}$ as $\delta_{RI,1}, \delta_{RI,2}, \delta_{RI,3}, \delta_{RI,4}$ and the decreasingly ordered eigenvalues of $A_{IT}$ as $\delta_{IT,1}, \delta_{IT,2}, \delta_{IT,3}, \delta_{IT,4}$. According to [36], $\det(A)$ can be lower bounded by

$$\min_P \prod_{n_I=1}^{N_I} (\delta_{RI,n_I} - \delta_{IT,n_I}) \leq \det(A),$$  \hspace{1cm} (60)$$

where the minimum is taken over all permutations of indices $1, \ldots, N_I$. Since $A_{RI}$ and $A_{IT}$ are rank-2, it is always possible to find a permutation $P$ such that

$$\min_P \prod_{n_I=1}^{N_I} (\delta_{RI,n_I} - \delta_{IT,n_I}) = 0.$$  \hspace{1cm} (61)$$

Thus, recalling that $\det(A) \neq 0$, we obtain $\det(A) > 0$. Since $A$ has four non-zero eigenvalues and $\text{Tr}(A) = 0$, $\det(A) > 0$ implies the presence of two positive and two negative eigenvalues. This concludes the proof for $N_I = 4$.

In the case $N_I > 4$, we begin by noticing that $A_{RI}$ and $A_{IT}$ have two non-zero eigenvalues, both positive. The reason is that the matrices are rank-2 by construction and positive semi-definite since both are the sum of two positive semi-definite matrices. This means that $\delta_{RI,1}, \delta_{IT,1} = 0$ if $n_i > 2$. Furthermore, since $A_{RI}$ and $A_{IT}$ are Hermitian, we can apply Weyl’s inequalities to study the eigenvalues of $A$. According to Weyl’s inequality, we have

$$\delta_i \leq \delta_{RI,i-j} - \delta_{IT,N_I-j},$$  \hspace{1cm} (62)$$

valid for $i = 1, \ldots, N_I$ and $j = 0, \ldots, i-1$. Considering (61) with $i = 3$ and $j = 0$, we have

$$\delta_3 \leq \delta_{RI,3} - \delta_{IT,N_I} = 0,$$  \hspace{1cm} (63)$$

since $\delta_{RI,3} = 0$ and $\delta_{IT,N_I} = 0$. Additionally, the dual Weyl’s inequality gives

$$\delta_{RI,i+k} - \delta_{IT,k} \leq \delta_i,$$  \hspace{1cm} (64)$$

valid for $i = 1, \ldots, N_I$ and $k = 1, \ldots, N_I - i + 1$. Considering (63) with $i = 3$ and $k = 3$, we have

$$0 = \delta_{RI,5} + \delta_{IT,3} \leq \delta_3,$$  \hspace{1cm} (65)$$

since $\delta_{RI,5} = 0$ and $\delta_{IT,3} = 0$, yielding $\delta_3 = 0$. Now, we consider (61) with $i = N_I - 2$ and $j = 0$ to obtain

$$\delta_{N_I-2} \leq \delta_{RI,N_I-2} - \delta_{IT,N_I} = 0,$$  \hspace{1cm} (66)$$

since $\delta_{RI,N_I-2} = 0$ and $\delta_{IT,N_I} = 0$. Additionally, (63) with $i = N_I - 2$ and $k = 3$ gives

$$0 = \delta_{RI,N_I} + \delta_{IT,3} \leq \delta_{N_I-2},$$  \hspace{1cm} (67)$$

since $\delta_{RI,N_I} = 0$ and $\delta_{IT,3} = 0$, yielding $\delta_{N_I-2} = 0$. Since $\delta_3 = 0$ and $\delta_{N_I-2} = 0$, it has to be $\delta_1, \delta_2 > 0$ and $\delta_{N_I-1}, \delta_{N_I} < 0$. This concludes the proof for $N_I > 4$. 

Fig. 8. Computational complexity versus the number of RIS elements. Alg. 1 is compared with the quasi-Newton method “QN” for group connected “GC” and fully connected “FC” BD-RISs.
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