Convergence of volume forms on a family of log Calabi–Yau varieties to a non-Archimedean measure

Sanal Shivaprasad

Received: 15 January 2021 / Accepted: 3 April 2022 / Published online: 23 May 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We study the convergence of volume forms on a degenerating holomorphic family of log Calabi–Yau varieties to a non-Archimedean measure, extending a result of Boucksom and Jonsson. More precisely, let $(X, B)$ be a holomorphic family of sub log canonical, log Calabi–Yau complex varieties parameterized by the punctured unit disk. Let $\eta$ be a meromorphic form on $X$ with poles along $B$ such that the restriction of $\eta$ is a top-dimensional form on each of the fibers. We show that the (possibly infinite) measures induced by the restriction of $|\eta \wedge \overline{\eta}|$ to a fiber converge to a measure on the Berkovich analytification of $X \setminus B$ as we approach the puncture. The convergence takes place on a hybrid space, which is obtained by filling in the space $X \setminus B$ with the aforementioned Berkovich space over the puncture.

1 Introduction

Let $Y$ be an irreducible, normal and compact complex analytic space. Let $\eta$ be a top-dimensional meromorphic form on the smooth locus, $Y^{\text{reg}}$ of $Y$, and let $D \subset Y$ be a (possibly not reduced or not effective) divisor such that $\eta$ is holomorphic, does not vanish on $Y^{\text{reg}} \setminus D$, and has poles (and zeroes) given exactly by $D$. Then the pair $(Y, D)$ is called log Calabi–Yau. Any two such forms $\eta$ and $\eta'$ on $Y^{\text{reg}}$ which have poles given by $D$ will be equal up to a scalar factor. Let $|D|$ denote the support of $D$. The form $\eta$ gives rise to a volume form $|\eta \wedge \overline{\eta}| = i^{(\dim Y)^2} \eta \wedge \overline{\eta}$ on $Y^{\text{reg}} \setminus |D|$, and thus a positive Radon measure on $Y^{\text{reg}} \setminus |D|$. For a log Calabi–Yau pair $(Y, D)$, this measure is unique up to scaling. Note that locally near $|D|$ and $Y^{\text{sing}}$, it is possible for the mass to be infinite. When $D = 0$ and $Y$ is smooth, $Y$ is said to be Calabi–Yau. More generally, we get such a canonical measure if we assume that $K_Y + D$ is $\mathbb{Q}$-Cartier and $K_Y + D \sim_{\mathbb{Q}} 0$. See Sect. 4 for details.

Families of (log) Calabi–Yau varieties appear in many settings, for example in geometry and mirror symmetry [3]. For example, all toric varieties are log Calabi–Yau. It is natural to ask how this canonical measure varies along families of log Calabi–Yau varieties. Boucksom and Jonsson studied this canonical measure along families of Calabi–Yau varieties [10]. Here we extend some of the analysis to the logarithmic setting.

1 Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Let $X \to \mathbb{D}^*$ be a proper flat family of irreducible normal complex analytic spaces. Let $B$ be a horizontal $\mathbb{Q}$-Weil divisor on $X$ such that $K_{X/\mathbb{D}^*} + B$ is $\mathbb{Q}$-Cartier and is $\mathbb{Q}$-linearly equivalent to 0. We don’t need to assume that $B$ is effective. Then, $(X_t, B_t)$ is log Calabi–Yau, where $B_t := B|_{X_t}$. As above, using a trivializing section $\eta$ of $O_X(m(K_{X/\mathbb{D}^*} + B))$ for some sufficiently divisible integer $m$, we can obtain Radon measures $\mu_t$ on each of the fibers $X_t^{\text{reg}} \setminus |B_t|$ for $|t| \ll 1$ (see Sect. 4 for more details on how to handle the $\mathbb{Q}$-divisor case). This measure $\mu_t$ remains unchanged if we replace $m$ by $m_1m$ and $\eta$ by $\eta^{\otimes m_1}$. Two such families $\mu_t$ and $\mu'_t$ obtained by picking $\eta, \eta' \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ differ by a factor of $|h(t)|^{2/m}$, where $h$ is a holomorphic function on $\mathbb{D}^*$. Our goal is to understand if the measures $\mu_t$ converge in some sense as $t \to 0$.

One way to study the convergence would be to think of the measures $\mu_t$ as being Radon measures on $X^{\text{reg}} \setminus |B|$ with support $X_t^{\text{reg}} \setminus |B_t|$. However, since there is no fiber over the origin, the measures $\mu_t$ converge weakly to the zero measure on $X^{\text{reg}} \setminus |B|$ as $t \to 0$, which is not very interesting. This is where non-Archimedean geometry comes in handy.

To be able to deal with the problem in the non-Archimedean setting, we impose some restrictions on the pair $(X, B)$. Firstly, we assume that the pair $(X, B)$ is projective i.e. $X$ is a closed subset of $\mathbb{P}^N \times \mathbb{D}^*$ for some $N \in \mathbb{N}$ and $X$ and $B$ are cut out by homogeneous polynomials whose coefficients are holomorphic functions on $\mathbb{D}^*$ and meromorphic on $\mathbb{D}^*$.

This guarantees that there exists a proper flat family $\mathcal{X}$ over $\mathbb{D}$ with $\mathcal{X}$ normal and $\mathcal{X}|_{\mathbb{D}^*} \simeq X$, and a $\mathbb{Q}$-Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ extending $K_{X/\mathbb{D}^*} + B$ such that $\mathcal{D} \sim_{\mathbb{Q}} 0$. (Such an $\mathcal{X}$ is called a model of $X$).

Secondly, we assume that there exists a section $\psi$ of $O_{\mathcal{X}}(m\mathcal{D})$ which extends the section $\eta$ of $m(K_{X/\mathbb{D}^*} + B)$. In this case, we say that $\eta$ admits a meromorphic extension. Recall that two families of measures $\mu_t$ and $\mu'_t$ obtained by picking two trivializing sections $\eta, \eta' \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ differ by a factor of $|h(t)|^{2/m}$, where $h$ is a holomorphic function on $\mathbb{D}^*$. If we also assume that $\eta$ and $\eta'$ admit meromorphic extensions, then we further get that $h$ is meromorphic at 0, and thus $h(t) \sim Ct^\alpha$ as $t \to 0$ for some $\alpha \in \mathbb{Z}$.

For $(X, B)$ satisfying the above conditions, we can construct projective varieties $X_{C(\mathbb{Q})}$ and $|B|_{C(\mathbb{Q})}$ over the non-Archimedean field $C(\mathbb{Q})$ by considering the coefficients of the polynomials cutting out $X$ and $|B|$ as elements of $C(\mathbb{Q})$.

The Berkovich analytification of a variety $Y$ over the field $C(\mathbb{Q})$, denoted $Y^{\text{an}}$, is a topological space whose points are valuations on the residue fields of (scheme) points in $Y$ that extend the $t$-adic valuation on $C(\mathbb{Q})$ [4, 5]. By considering the Berkovich analytifications, we obtain locally compact Hausdorff spaces $(X_{C(\mathbb{Q})})^{\text{an}}$ and $|B|^{\text{an}}_{C(\mathbb{Q})}$.

The main tool that we use to study the asymptotics of $\mu_t$ is a hybrid space. Various hybrid spaces, i.e. spaces which are obtained by gluing complex analytic spaces with non-Archimedean spaces, have been constructed in the literature. They have been used to study compactifications [21] and degenerations [10, 14, 22]. Hybrid spaces were used in [10] to prove Theorem A below for sub klt pairs $(X, B)$. Following [8, 10, 18], we construct a locally compact hybrid topological space $(X, |B|)^{\text{hyb}}$, which as a set is a disjoint union of $X^{\text{reg}} \setminus |B|$ and $(X_{C(\mathbb{Q})})^{\text{reg}}_{C(\mathbb{Q})} \setminus |B|^{\text{an}}_{C(\mathbb{Q})}$. The topology on the hybrid space is given by the logarithmic rate of convergence (see Sect. 3 for more details).

We have the following convergence theorem for the measures $\mu_t$ on $(X, |B|)^{\text{hyb}}$.

**Theorem A** Suppose $(X, B)$ is a projective log Calabi–Yau pair over $\mathbb{D}^*$ and let $\eta \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ be a generating section that admits a meromorphic extension. Let $\mu_t$ be the Radon measure on $X^{\text{reg}} \setminus |B_t|$ defined by $i^{(\dim(X_t))^2} |\eta|_{X_t} \wedge \overline{|\eta|_{X_t}}^{1/m}$. In addition, assume that the pair $(X, B)$ is sub log canonical in the sense of the minimal model program. Then, there exists a non-zero measure $\mu_0$ on $(X_{C(\mathbb{Q})})^{\text{an}} \setminus |B|_{C(\mathbb{Q})}^{\text{an}}$ and constants $d \in \mathbb{N}$ and...
\( \kappa \in \mathbb{Q} \) such that the measures \( \frac{\mu_t}{|t|^{2+2\pi i(2\pi /|t| - 1)}} \) converge weakly to \( \mu_0 \), when viewed measures on \((X, |B|)^{\text{hyb}}\).

In the above theorem, sub log canonical (sometimes just called log canonical in the literature) is in the sense of the minimal model program i.e. \( \text{discrep}(X, B) \geq -1 \) (see [17, Section 2.3]) and we don’t assume that \( B \) is effective.

The measure \( \mu_0 \) is easy to describe when \((X, B)\) is log-smooth i.e. when \( X \) is smooth and \( B \) has snc support. In this case, the support of \( \mu_0 \) is the locus where a certain weight function associated to \((X, B, \eta)\), constructed in [20] and [11], is minimized (see also [23]). The minimizing locus of the weight function is called the essential skeleton in the literature, and thus we have that the measure \( \mu_0 \) is the Lebesgue measure on the top-dimensional faces of the essential skeleton of the triple \((X, B, \eta)\). In general, the support of \( \mu_0 \) is the image of a skeleton under a birational map \((X', B') \rightarrow (X, B)\).

If the pair \((X, B)\) is not sub log canonical, then there is no reasonable convergence in this non-Archimedean setting (see Example 4.1). This is consistent with the observation that the essential skeleton of \((X, B, \eta)\) is empty when \((X, B)\) is not sub log canonical.

As an example of Theorem A, we get a convergence result for a torus \( T = (\mathbb{C}^*)^n \). We have a canonical embedding \( \mathbb{R}^n \hookrightarrow T_{\mathbb{Q}}(0)^{\mathbb{Q}} \) given by sending \( r \in \mathbb{R}^n \) to the valuation \( \sum_{m \in \mathbb{Z}^n} a_m z^m \rightarrow \max_m \{|a_m| e^{-(r, m)}\} \). Consider the constant family \( T \times \mathbb{D}^* \) and the associated hybrid space \((T \times \mathbb{D}^*) \cup T_{\mathbb{Q}}(0)^{\mathbb{Q}} \). Then by applying Theorem A to a smooth projective toric compactification of \( T \) we get that as \( t \rightarrow 0 \), the Haar measure on \( T \times \{t\} \) scaled by a factor of \( \frac{1}{(2\pi i|t| - 1)^n} \) converges weakly to the Lebesgue measure on \( \mathbb{R}^n \). See Examples 4.7 and 5.13 for more details.

The motivation for this problem comes from [10], where the case for regular \( X \) and \( B = 0 \) [10, Theorem A] and the case for sub klt pairs [10, Theorem 8.4] are studied. The essential difference in our scenario is that the measures \( \mu_t \) are no longer finite measures when we drop the assumption that \( B \) is sub klt. This creates some technical difficulties.

It is enough to prove Theorem A in the case when the pair \((X, B)\) is log-smooth i.e. when \( X \) is regular and \( B \subset X \) is an snc divisor. Indeed, we can then prove Theorem A for a general pair \((X, B)\), by taking a log resolution \((X', B') \rightarrow (X, B)\), and using Theorem A for \((X', B')\) (see Sect. 5.5 for details). So, for the remainder of the introduction (and for the majority of the paper) we will assume that the pair \((X, B)\) is log-smooth.

For a regular model \( \mathcal{X} \) of \( X \), there is an associated CW complex \( \Delta(\mathcal{X}) \) given by the dual intersection complex of the central fiber \( \mathcal{X}_0 \). In [10], Boucksom and Jonsson construct a locally compact Hausdorff hybrid space \( \mathcal{X}^{\text{hyb}} \) over \( \mathbb{D} \), whose fiber over \( \mathbb{D}^* \) is \( X \) and the fiber over \( 0 \) is \( \Delta(\mathcal{X}) \). Then, they show that the measures \( \mu_t \), scaled appropriately, converge to a weighted Lebesgue measure \( \mu_0 \) on a subcomplex of \( \Delta(\mathcal{X}) \). Using this, they show a convergence of the measures to a measure on \( X^{\text{hyb}} \).

We will employ a similar approach. To prove Theorem A, we first prove Theorem B below, which shows the convergence on certain skeletal subsets of \( X_{\mathbb{Q}}(\mathbb{Q}) \setminus |B|_{\mathbb{Q}}(\mathbb{Q}) \). Since our measures are no longer finite, we would have to allow for the limit measures to be infinite and this would not be possible if we use Lebesgue measure on a compact simplicial complex. The solution is to allow our simplices to have unbounded faces. Pick a model \( \mathcal{X} \) such that \( \mathcal{X}_0 + \overline{B} \) is an snc divisor, where \( \overline{B} \) denotes the component-wise closure of \( B \) in \( \mathcal{X} \). A good candidate for this is \( \Delta(\mathcal{X}, |B|) \), the dual intersection complex of a pair, introduced in [1, 2, 24] in the one-dimensional case and in [11, 15] for higher dimensions.
We briefly explain the construction of $\Delta(\mathcal{X}, |B|)$ here. Let $E_i$ denote the irreducible components of $\mathcal{X}_0$ and let $\mathcal{X}_0 = \sum_{i=1}^{N} b_i E_i$. A connected component $Y$ of $\bigcap_{i \in I} (\mathcal{X}_0)$ is called a stratum where $\{E_i \mid i \in I\}$ denotes a non-empty collection of irreducible components of $\mathcal{X}_0$ and $\{B_j \mid j \in J\}$ denotes a possibly empty collection of irreducible components of the support of $B$. Associated to every such stratum $Y$ is a face $\sigma_Y = \{(s, \bar{s}) \in \mathbb{R}_{\geq 0}^{[I]+[J]} \mid \sum_{i \in I} b_i r_i = 1\}$ of $\Delta(\mathcal{X}, |B|)$. These faces are then glued together via some attaching maps to get $\Delta(\mathcal{X}, |B|)$. See Sect. 2 for more details.

Associated to such a model $\mathcal{X}$, we construct a hybrid space $(\mathcal{X}, |B|)^{hyb} = (\mathcal{X} \setminus |B|) \cup \Delta(\mathcal{X}, |B|)$, where the topology is given by logarithmic rate of convergence. We prove the following convergence theorem on this hybrid space.

**Theorem B** Let $X \to \mathbb{D}^*$ be a holomorphic family of proper complex manifolds. Let $B$ be an snc $\mathbb{Q}$-divisor such that $K_X/\mathbb{D}^* + B \sim_{\mathbb{Q}} 0$ and the pair $(X, B)$ is sub log canonical. Let $\mathcal{X}$ be a regular model of $X$ such that $\mathcal{X}_0 \sim_{\mathbb{Q}} 0$. Let $\psi \in H^0(\mathcal{X}, m\mathcal{X})$ be a generating section for sufficiently divisible $m$ and let $\mu_t = i^{(\dim X)^2} (\psi|_{X_t} \wedge \overline{\psi}|_{X_t})^{1/m}$ be the Radon measure induced on $X_t \setminus |B_t|$ by $\psi$. Then, there exists a non-zero measure $\mu_0$ supported on a subcomplex $\Delta(\mathcal{X})$ of $\Delta(\mathcal{X}, |B|)$ and explicit constants $d \in \mathbb{N}, \kappa_{\min} \in \mathbb{Q}$ such that

$$\frac{\mu_t}{|t|^{2\kappa_{\min}(2\pi \log |t|^{-1})^d}} \to \mu_0$$

converges weakly as measures on $(\mathcal{X}, |B|)^{hyb}$.

On each top-dimensional faces of $\Delta(\mathcal{X}), \mu_0$ is a suitably normalized Lebesgue measure while on all other faces of $\Delta(\mathcal{X}, |B|), \mu_0$ is zero. For a precise description of $\mu_0$, see Sect. 4.

Note that for Theorem B, we don’t need to assume that $(X, B)$ is projective. The projectivity assumption is needed in Theorem A in order to define $X_{C^{an}(\mathcal{Y})}$ and $|B|_{C^{an}(\mathcal{Y})}$. In this case, we can view $\Delta(\mathcal{X})$ and $\Delta(\mathcal{X}, |B|)$ as subsets of the Berkovich analytification, $X_{C^{an}(\mathcal{Y})}$. Moreover, $\Delta(\mathcal{X})$ is a strong deformation retract of $X_{C^{an}(\mathcal{Y})}$. In the case when $B = 0$, Theorem A follows from Theorem B by using the following result. The collection of $\Delta(\mathcal{X})$ for all snc models $\mathcal{X}$ is a directed system and $X_{C^{an}(\mathcal{Y})} \simeq \lim_{\mathcal{X} \to \mathcal{Y}} \Delta(\mathcal{X})$ (see [18, Theorem 10], [9, Corollary 3.2]).

We prove a similar result (see Theorem 5.1) that

**Theorem C** Let $(X, B)$ be a log-smooth pair. Then there exists a canonical homeomorphism.

$$X_{C^{an}(\mathcal{Y})} \setminus |B|_{C^{an}(\mathcal{Y})} \simeq \lim_{\mathcal{X} \to \mathcal{Y}} \Delta(\mathcal{X}, |B|).$$

Theorems B and C together prove Theorem A.

As a corollary of Theorem C, we also get the following result. Suppose that $U \to \mathbb{D}^*$ is a smooth meromorphic family of quasi-projective complex manifolds. Consider an snc compactification of $U \subset X$ with snc boundary divisor $B$. Then, $U^{hyb} = (X, |B|)^{hyb} = \lim_{\mathcal{X} \to \mathcal{Y}} (\mathcal{X}, |B|)^{hyb}$ is independent of the choice of $(X, B)$. The space $U^{hyb}$ can also be seen as the Berkovich analytification of $U$ with respect to a suitable Banach ring [8], [10, Appendix]. The hybrid spaces $(\mathcal{X}, |B|)^{hyb}$ can be thought of as approximations of $U^{hyb}$ and we hope that they will be useful in future applications.

It would interesting to see if the application of Theorem A to various examples of log Calabi–Yau varieties in the literature [19] yields any interesting results.
In [16], Jonsson and Nicaise prove a \( p \)-adic version of [10], where they consider the measure induced by a pluricanonical form \( \eta \) on a smooth proper variety \( X \) over a local field \( K \). They show that the measures induced by \( \{ \eta \otimes_{K} K' \}_{K'} \) for all finite tame extensions \( K' \) of \( K \) converge to a Lebesgue type measure on the Berkovich analytification. The measures considered in [16] are finite and it would be interesting to see whether it would be possible to generalize Theorem A to this \( p \)-adic setting to extend the result to a family of infinite measures as well.

**Structure of the paper**

The paper is structured as follows. In Sect. 2, we recall the definition of a model and the construction of the dual intersection complex associated to a model \( X \) and use this to realize the central fiber of a non-Archimedean space and prove Theorem A. In Sect. 3, we construct the hybrid space \( (X, |B|)^{hyb} \), associated to a model \( X \). In Sect. 4, we prove Theorem B. In Sect. 5, we construct the space \( (X, |B|)^{hyb} \), realize it and its the central fiber as a non-Archimedean space and prove Theorem A.

**Notations and conventions**

We use \( \mathbb{D} \) to denote the (open) unit complex disc, \( \{ t \in \mathbb{C} \mid |t| < 1 \} \), and \( \mathbb{D}^* = \mathbb{D} \setminus \{0\} \).

We will use \( X, Y \) etc to denote families of complex analytic spaces parametrized by \( \mathbb{D}^* \) and use \( \mathcal{X}, \mathcal{Y} \) etc to denote extensions of these families to \( \mathbb{D} \). We will use \( B \) to denote horizontal divisors in \( X \) and \( \overline{B} \) will denote its component-wise closure in \( \mathcal{X} \). We will denote the irreducible components of \( \mathcal{X}_0 \) by \( E_i \)'s and their multiplicities by \( b_i \)'s. We will denote the irreducible components of the support of \( B \) as \( B_j \)'s and their multiplicities by \( \beta_j \)'s. The support of a divisor \( D \) will be denoted by \( |D| \).

**2 The dual simplicial complex associated to an snc model**

In this section, we recall the definition of a model and the construction of the dual intersection complex associated to an snc model of a log-smooth pair \((X, B)\). Let \( X \) be a holomorphic flat family of compact complex manifolds parametrized by \( \mathbb{D}^* \) i.e. \( X \) is a smooth complex manifold with a proper smooth map \( X \to \mathbb{D}^* \). Let \( B \) be a horizontal snc \( \mathbb{Q} \)-divisor in \( X \). Write \( B = \sum_j \beta_j B_j \), where \( \beta_j \in \mathbb{Q} \) and \( B_j \) are prime divisors. A model \( \mathcal{X} \) of \( X \) is a a normal complex analytic space \( \mathcal{X} \) which is proper and flat over \( \mathbb{D} \) such that \( \mathcal{X} \mid_{\mathbb{D}^*} = X \). Let \( \mathcal{X}_0 \) denote the central fiber of \( \mathcal{X} \) i.e. the fiber over \( 0 \in \mathbb{D} \).

We say that \((X, B)\) has removable singularities at the origin if there exists a model \( \mathcal{X} \) of \( X \) such that the topological closure of \( |B| \) in \( \mathcal{X} \) is a divisor in \( \mathcal{X} \). Throughout this paper, we will assume that all pairs have removable singularities at the origin. This is automatic if we assume that \((X, B)\) is projective.

**Remark 2.1** (Non-removable singularities) Let \( X = \mathbb{P}^1 \times \mathbb{D}^* \) and \( B = \{(e^{1/t}, t) \mid t \in \mathbb{D}^* \} \) with \( \pi : X \to \mathbb{D}^* \) defined by \((z, t) \mapsto t\). Then, \( \mathcal{X} = \mathbb{P}^1 \times \mathbb{D} \) is a model of \( X \) and the topological closure of \( B \) in \( \mathcal{X} \) is \( \overline{B} = B \cup (\mathbb{P}^1 \times \{0\}) \), which is not even a divisor in \( \mathcal{X} \) ! To avoid such pathologies, we need to assume that \((X, B)\) has removable singularities.

In this section, we don’t need to assume that \((X, B)\) is projective or that the pair \((X, B)\) is log Calabi–Yau.
2.1 Snc models of \((X, B)\)

We say that a model \(\mathcal{X}\) of \((X, B)\) is an snc model of \((X, B)\) if \(\mathcal{X}\) is regular and \((\mathcal{X}_0 + B)_{\text{red}}\) is an snc divisor in \(\mathcal{X}\), where \(\mathcal{B} = \sum b_j \mathcal{B}_j\) denotes the component-wise closure of \(B\) in \(\mathcal{X}\).

Let \(\mathcal{X}_0 = \sum b_i E_i\), where \(E_i\) are the irreducible components of the central fiber.

Since \((X, B)\) has removable singularities at the origin, using Hironaka’s theorem, we can always find an snc model of \((X, B)\). Thus, the existence of an snc model of \((X, B)\) is equivalent to \((X, B)\) having removable singularities at the origin.

Given an snc model \(\mathcal{X}\) of \((X, B)\), we can obtain new snc models of \((X, B)\) by blowing up \(\mathcal{X}\) at any smooth subvariety of \(\mathcal{X}_0\).

Example 2.2 If \(X = \mathbb{P}^1 \times \mathbb{D}^s\), then \(\mathcal{X} = \mathbb{P}^1 \times \mathbb{D}\) is an snc model of \(X\). The blowup of \(\mathcal{X}\) at the point \((0, 0)\) is also an snc model of \(X\).

2.2 The dual complex

Let \(\mathcal{X}\) be an snc model of \((X, B)\). Let \(E_{i_0}, \ldots, E_{i_p}\) for \(p \geq 0\) denote a non-empty collection of irreducible components of \(\mathcal{X}_0\). Let \(B_{j_0}, \ldots, B_{j_q}\) for \(q \geq 0\) denote a (possibly empty) collection of irreducible components of \(|B|\).

A non-empty connected component \(Y\) of \(E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}\) is called a stratum in \(\mathcal{X}_0 + \mathcal{B}\) and we write \(Y \subset \text{conn. comp.} \ E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}\).

Note that in our definition of a stratum, we insist that we start with a non-empty collection \(\mathcal{X}\) of irreducible components of \(|B|\).

Given a stratum \(Y \subset \text{conn. comp.} \ E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}\), let \(b_{i_k}\) denote the multiplicity of \(E_{i_k}\) in \(\mathcal{X}_0\). Then the face associated to \(Y\) is the (open) simplex \(\sigma_Y\) defined as follows.

\[
\sigma_Y = \left\{ (x_0, \ldots, x_p, y_1, \ldots, y_q) \in \mathbb{R}_{\geq 0}^{p+1} \times \mathbb{R}_{\geq 0}^q \mid \sum b_{i_k} x_k = 1 \right\}.
\]

We define the dual complex \(\Delta(\mathcal{X}, |B|)\) associated to the snc model \(\mathcal{X}\) of \((X, B)\) to be the CW complex (with possibly open faces) obtained by gluing the faces \(\sigma_Y\) for all possible strata \(Y\).

More precisely,

\[
\Delta(\mathcal{X}, |B|) = \left( \bigcup_{Y \text{strata}} \sigma_Y \right) / \sim
\]

where \(\sim\) is an equivalence relation generated by the following identification. If \(Y'\) and \(Y\) are strata with \(Y \subset Y'\), then without loss of generality we can write \(Y' \subset \text{conn. comp.} \ E_{i_0} \cap \cdots \cap E_{i_{p'}} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}\) and \(Y \subset \text{conn. comp.} \ E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}\) for some \(p' \leq p\) and \(q' \leq q\), and we can identify \(\sigma_Y'\) as a subset of \(\sigma_Y\) via

\[
(x_0, \ldots, x_{p'}, y_1, \ldots, y_{q'}) \mapsto (x_0, \ldots, x_{p'}, 0, \ldots, 0, y_1, \ldots, y_{q'}, 0, \ldots, 0).
\]

For example, if \(\dim(X) = 1\), then \(\Delta(\mathcal{X}, |B|)\) is the dual graph of \(\mathcal{X}_0 + \mathcal{B}\) with the vertices associated to \(|\mathcal{B}|\) as well as the edges with both endpoints in \(|\mathcal{B}|\) removed. The dual complex of a pair was introduced in [11, 15].

\(
\text{Springer}
\)
Example 2.3 (The dual complex associated to \( \mathbb{P}^1 \times \mathbb{D} \)) Let \( X = \mathbb{P}^1 \times \mathbb{D}^* \), with projection to \( \mathbb{D}^* \), and \( B = \{0\} \times \mathbb{D}^* + \{\infty\} \times \mathbb{D}^* \) is a horizontal divisor in \( X \). Consider the model \( \mathcal{X} = \mathbb{P}^1 \times \mathbb{D} \). Then, the dual complex \( \Delta(\mathcal{X}, |B|) \) is homeomorphic to \( \mathbb{R} \), with 0 being the vertex \( \sigma_{\mathbb{P}^1 \times \{0\}} \), the positive axis being identified with \( \sigma_{(0,0)} \) and the negative axis with \( \sigma_{(\infty,0)} \). See Fig. 1.

### 2.3 Integral piecewise affine structure on the dual intersection complex

We briefly discuss some results related to the natural integral piecewise affine structure on \( \Delta(\mathcal{X}, |B|) \). The reader can take a look at [6, 7, 12] and [10, Section 1.3] for more details. Let \( \sigma = \{(x_0, \ldots, x_p)\} = \sum_{i=0}^p b_i x_i = 1 \times \mathbb{R}^{q}_{\geq 0} \) be a simplex. Let \( \lambda_{\mathcal{X}} \) denote the abelian group of integral affine functions on \( \mathbb{R}^{p+1+q} \) restricted to \( \sigma \) (two such functions are identified if they are equal on \( \sigma \)). Let \( (\lambda_{\mathcal{X}})_{\mathbb{R}} := \lambda_{\mathcal{X}} \otimes_{\mathbb{Z}} \mathbb{R} \) and let \( (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \) be its \( \mathbb{R} \)-dual. Denote \( b_\sigma := \gcd(b_0, \ldots, b_p) \).

Let \( I_{\sigma} \in \lambda_{\mathcal{X}} \) denote the constant function 1 on \( \sigma \). The evaluation map \( \sigma \rightarrow (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \) realizes \( \sigma \) as a simplex of codimension one in \( (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \) contained in the affine plane \( \{v \mid v(I_{\sigma}) = 1\} \). So, the tangent space of \( \sigma \) in \( (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \) can be realized as \( (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \), where

\[
\overrightarrow{M_{\sigma}} = \lambda_{\mathcal{X}} / (\mathbb{Q} I_{\sigma} \cap \lambda_{\mathcal{X}})
\]

and \( (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \) is the \( \mathbb{R} \)-dual of \( \overrightarrow{M_{\sigma}} \otimes_{\mathbb{Z}} \mathbb{R} \).

Consider the Lebesgue measure on \( (\lambda_{\mathcal{X}})_{\mathbb{R}}^\vee \) for which the lattice \( \text{Hom}_{\mathbb{Z}}(\overrightarrow{M_{\sigma}}, \mathbb{Z}) \) has covolume 1. This gives rise to a measure on \( \sigma \). This is called the normalized Lebesgue measure \( \lambda_{\sigma} \) of \( \sigma \). The following remark, stated with a typo in [10, Remark 1.3], gives an explicit description of the normalized Lebesgue measure, which will be useful for computations. We provide a quick proof here for the convenience of the reader.

**Proposition 2.4** [10, Remark 1.3] Let \( b_0, \ldots, b_p \in \mathbb{N}_+ \) and let

\[
\sigma = \left\{ (x_0, \ldots, x_p, y_1, \ldots, y_q) \in \mathbb{R}^{p+q+1}_{\geq 0} \mid \sum_{i=0}^p b_i x_i = 1 \right\}
\]

be a simplex. Then, we have a homeomorphism

\[
\sigma \rightarrow \left\{ (x_1, \ldots, x_p, y_1, \ldots, y_q) \in \mathbb{R}^{p+q}_{\geq 0} \mid \sum_{i=1}^p b_i x_i \leq 1 \right\},
\]
where we can recover $x_0$ by $x_0 = b_0^{-1}(1 - \sum_{i=1}^P b_i x_i)$. Under this homeomorphism, the normalized Lebesgue measure is given by

$$\lambda_\sigma = b_\sigma b_0^{-1}|dx_1 \wedge \cdots \wedge dx_p \wedge dy_1 \wedge \cdots \wedge dy_q|$$

**Proof** Note that $1, X_1, \ldots, X_p, Y_1, \ldots, Y_q$ is an $\mathbb{R}$-basis for $(M_\sigma)_\mathbb{R}$, where $X_i$ and $Y_j$ denote projection to the $x_i$ and $y_j$ coordinates. Let $1^*_\sigma, X^*_1, \ldots, Y^*_q$ denote its dual basis.

Then, $X^*_1, \ldots, Y^*_q$ is a $\mathbb{R}$-basis for the $(M_\sigma)^*_\mathbb{R}$ and $\text{Hom}(M_\sigma, \mathbb{Z})$ is a sub lattice of $\Lambda = \mathbb{Z}X^*_1 + \cdots + \mathbb{Z}Y^*_q$.

Now that we can view $\text{Hom}_{\mathbb{Z}}(M_\sigma, \mathbb{Z})$ as the kernel of the map $\phi : \Lambda \to \mathbb{Z}/b_0\mathbb{Z}$ given by $\alpha_1 X^*_1 + \cdots + \alpha_p X^*_p + \gamma_1 Y^*_1 + \cdots + \gamma_q Y^*_q \mapsto b_1 \alpha_1 + \cdots + b_p \alpha_p + b_0 \mathbb{Z}$. The image of $\phi$ is generated by $b_\sigma$ and the size of the image is $b_0/b_\sigma$. Thus, the index of $\text{Hom}_{\mathbb{Z}}(M_\sigma, \mathbb{Z})$ in $\Lambda$ is $b_0/b_\sigma$, and thus $b_\sigma b_0^{-1}|dx_1 \wedge \cdots \wedge dx_p \wedge dy_1 \wedge \cdots \wedge dy_q|$ is the normalized Lebesgue measure on $\sigma$. \qed

### 3 The hybrid space associated to a dual complex

Let $X$ be a holomorphic flat family of compact complex manifolds parametrized by $\mathbb{D}^*$ i.e. $X$ is a smooth complex manifold with a proper smooth map $X \to \mathbb{D}^*$. Let $B$ be a horizontal snc $\mathbb{Q}$-divisor in $X$. Write $B = \sum_j \beta_j B_j$, where $\beta_j \in \mathbb{Q}$ and $B_j$ are prime divisors. We don’t need to assume that $(X, B)$ is projective or that $(X, B)$ is log Calabi–Yau. The constructions in this section only depend on $|B|$ i.e. they are independent of $\beta_j$’s. Let $\mathcal{X}$ be an snc model of $(X, B)$ and write $\mathcal{X}_0 = \sum_i b_i E_i$.

In this section, we construct the hybrid space $(\mathcal{X}, |B|)_{hyb}$, associated to the snc model $\mathcal{X}$ of $(X, B)$; this is a topological space over $\mathbb{D}$ such that the fiber over $\mathbb{D}^*$ is isomorphic to $X \setminus |B|$ and the central fiber is isomorphic to $\Delta(\mathcal{X}, |B|)$. This construction closely follows [10, Section 2.2], where the construction for $B = 0$ was done.

#### 3.1 Local Log function

To construct the hybrid space, we will first construct a Log function on $\mathcal{X}$ and glue $X \setminus |B|$ and $\Delta(\mathcal{X}, |B|)$ using this Log function. To do this, we first construct a local version of the Log function. Let $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ denote a stratum of $\mathcal{X}_0 + \overline{B}$ (see Sect. 2.2 for the definition of a stratum) and let $b_i$ denote the multiplicity of $E_i$ in $\mathcal{X}_0$. For an open set $U \subset \mathcal{X}$ and for local coordinates $(z, w, y)$ on $U$ where $z = (z_0, \ldots, z_p)$, $w = (w_1, \ldots, w_q)$ and $y = (y_1, \ldots, y_r)$, we say that $(U, (z, w, y))$ is adapted to the stratum $Y$ if

- The only irreducible components of $\mathcal{X}_0 + \overline{B}$ intersecting $U$ are $E_0, \ldots, E_p$, and $\overline{B}_1, \ldots, \overline{B}_q$.
- $U \cap (E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q) = U \cap Y$.
- We have $|z_i|, |w_j|, |y_k| < 1$ on $U$ and $E_i \cap U = \{z_i = 0\}$ and $\overline{B}_j \cap U = \{w_j = 0\}$.

If there exists a choice of coordinates that make $U$ adapted to some stratum of $\mathcal{X}_0 + \overline{B}$, we say that $U$ is an adapted coordinate chart in $\mathcal{X}$.

Suppose that $(U, (z, w, y))$ is a coordinate chart adapted to a stratum $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$. Then, we can define $\text{Log}_U : U \setminus (\mathcal{X}_0 + \overline{B}) \to \sigma_Y$. Let $f_U := \zeta_0^b \cdots \zeta_p^b$. Then there exists invertible holomorphic function $u$ on $U$ such that the
projection $U \to \mathbb{D}$ is given by $t = u \cdot f_U$. Define

$$\text{Log}_U(z, w, y) := \left( \frac{\log |z_0|}{\log |f_U|}, \ldots, \frac{\log |z_p|}{\log |f_U|}, \frac{\log |w_1|}{\log |f_U|}, \ldots, \frac{\log |w_q|}{\log |f_U|} \right).$$

Note that $\text{Log}_U$ depends on the choice of the coordinates on an adapted coordinate chart $U$, however the following lemma tells us that the difference goes off to 0 for a different choice of coordinates on $U$.

**Remark 3.1** [10, Prop 2.1] If $(U, (z, w, y))$ and $(U', (z', w', y'))$ are adapted to a stratum $Y$, then

$$\text{Log}_U - \text{Log}_{U'} = O \left( \frac{1}{\log |t|^{-1}} \right)$$

as $t \to 0$ uniformly on compact subsets of $U \cap U'$ where $t$ denotes the coordinate on $\mathbb{D}$.

Here, we view $\text{Log}_U$ and $\text{Log}_{U'}$ as maps with image $\sigma_Y \subset \mathbb{R}^{p+1+q}$ and $\text{Log}_U - \text{Log}_{U'} = O \left( \frac{1}{\log |t|^{-1}} \right)$ just means that the equality is true coordinate-wise on $\mathbb{R}^{p+1+q}$.

### 3.2 Constructing the global Log function

Here, we globalize the Log construction by patching up the local Log functions and to do so, we will have to find a ‘nice’ open covering of $\mathcal{X}_0 + B$. The following construction, as well as Proposition 3.2 is similar to [10, Proposition 2.1], but we provide some more details.

For a non-empty collection $\{E_i \mid i \in I\}$ of the irreducible components of $\mathcal{X}_0$, denote $E_I = \cap_{i \in I} E_i$. Similarly, for a (possibly empty) collection $\{B_j \mid j \in J\}$ of irreducible components of $B$, denote $B_J = \cap_{j \in J} B_j$.

Following [13, Theorem 5.7], we can find tubular neighborhoods $U_{1,J}$ of $D_{1,J} := E_I \cap \overline{B}_J$ and a smooth projection $\pi_{1,J} : U_{1,J} \to D_{1,J}$ satisfying $U_{1,J} \cap U_{I,J'} = U_{I \cup J, J' \cup J}$. In particular, if $U_{1,J}$ and $U_{I,J'}$ intersect, then $D_{1,J} \cap D_{I,J'} \neq \emptyset$. Also, note that $U_{1,J}$ has as many connected components as $D_{1,J}$ and each connected component $U_Y$ of $U_{1,J}$ corresponds to a stratum $Y \subset \text{conn.compl. } E_I \cap \overline{B}_J$.

Pick $x \in \mathcal{X}_0$. Suppose that $Y_x$ is the smallest stratum containing $x$. Around $x$, pick an open neighborhood $U_x$ that is adapted to $Y_x$ and lies in $U_{Y_x}$. The union of all such $U_x$ for $x \in \mathcal{X}_0$ covers $\mathcal{X}_0$. Since $\mathcal{X}_0$ is compact, we only need finitely many of these to cover $\mathcal{X}_0$. Call these open sets $U_1, \ldots, U_l$ and let their corresponding strata be $Y_1, \ldots, Y_l$ respectively. Let $\chi_1, \ldots, \chi_l$ be a partition of unity with respect to $U_1, \ldots, U_l$ and let $V = \bigcup_{\lambda=1}^l U_\lambda$. Then, $V$ is a neighborhood of $\mathcal{X}_0$.

**Proposition 3.2** The function $\text{Log}_V : V \setminus (\mathcal{X}_0 + \overline{B}) \to \Delta(\mathcal{X}, |B|)$ given by $\text{Log}_V = \sum_{\lambda=1}^l \chi_\lambda \text{Log}_{U_\lambda}$ is well defined. Here, addition in $\Delta(\mathcal{X}, |B|)$ means that the sum makes sense in a face of $\Delta(\mathcal{X}, |B|)$.

**Proof** Note that $\text{Log}_{U_\lambda}$ and $\text{Log}_{U_\lambda}$ are maps with image $\sigma_{Y_\lambda_1}$ and $\sigma_{Y_\lambda_2}$ respectively. A priori, there might not be a face of $\Delta(\mathcal{X}, |B|)$ that contains both $\sigma_{Y_\lambda_1}$ and $\sigma_{Y_\lambda_2}$ in which case there is no way for us to make sense of the sum $\sum_{\lambda=1}^l \chi_\lambda \text{Log}_{U_\lambda}$ at a point $x \in V \setminus (\mathcal{X}_0 + \overline{B})$ where $\chi_{\lambda_1}(x), \chi_{\lambda_2}(x) \neq 0$. To show the well-definedness of the map, we need to show that such a scenario does not happen.

Pick a point $x \in V \setminus (\mathcal{X}_0 + \overline{B})$. After a possible re-indexing, suppose $x \in (U_1 \cap \cdots \cap U_a) \setminus (U_{a+1} \cup \cdots \cup U_l)$. Then, we would like to define $\text{Log}_V(x) = \chi_1(x) \text{Log}_{U_1}(x) + \cdots + \chi_l(x) \text{Log}_{U_l}(x).$
\(\chi_a(x) \log_{U_a}(x)\). For this to make sense, it is enough to find a face \(\sigma'\) of \(\Delta(\mathcal{X}', |B|)\) such that \(\sigma_1, \ldots, \sigma_a \subset \sigma'\). Note that \(U_1 \cap \cdots \cap U_a \subset U_{Y_1} \cap \cdots \cap U_{Y_a}\). Each connected component of \(\bigcap_{a=1}^a U_{Y_a}\) corresponds to a stratum of \(\bigcap_{a=1}^a Y_a\). Let \(Y'\) be the stratum corresponding to the connected component of \(\bigcap_{a=1}^a U_{Y_a}\) containing \(x\). Then, \(\sigma' := \sigma_Y\) contains \(\sigma_Y\) for all \(\lambda = 1, \ldots, a\) and \(\log_{U}(x) = \chi_1(x) \log_{U_1}(x) + \cdots + \chi_a(x) \log_{U_a}(x)\) makes sense in \(\sigma'\).

**Proposition 3.3** Let \(U\) be an open set adapted to a stratum \(Y\). Then, \(\log_{U} - \log_{U} = O(\frac{1}{|x|^{-1}})\) locally uniformly as \(t \to 0\), where the equality is interpreted as being true coordinate-wise on some faces of \(\Delta(\mathcal{X}', |B|)\) containing \(\sigma_Y\).

**Proof** We may replace \(U\) by \(U \cap U_Y\) and assume that \(U \subset U_Y\). It is enough to prove the result in a small neighborhood of every point \(x \in U \cap \mathcal{X}_0\).

Suppose \(x \in U \cap \mathcal{X}_0\) such that \(x \in (U_1 \cap \cdots \cap U_a) \setminus (U_{a+1} \cup \cdots \cup U_i)\). Then, from the previous proof, we know that there exists a stratum \(Y'\) such that \(x \in U_Y\) and \(\log_{U}(x) = \chi_1(x) \log_{U_1}(x) + \cdots + \chi_a(x) \log_{U_a}(x)\) makes sense in \(\sigma_Y\). Since \(x \in U_Y \cap U_{Y'}\), we get that \(Y \cap Y' \neq \emptyset\). Let \(Z\) be the stratum corresponding to the connected component of \(U_Y \cap U_{Y'}\) containing \(x\). Then, \(\sigma_Y, \sigma_{Y'} \subset \sigma_Z\) and we can think of \(\log_{U} \) and \(\log_{U}\) as maps with image contained in \(\sigma_Z \subset \mathbb{R}^N\). We now need to show that \(\log_{U} - \log_{V} = O(\frac{1}{|x|^{-1}})\) coordinate-wise on \(\sigma_Z\).

Suppose \(x \in E_i\), \(z_i = 0\) defines \(E_i\) in \(U\), and \(z'_i = 0\) defines \(E_i\) in \(U_1\). Then, \(z_i\) and \(z'_i\) differ by the factor of a unit in a neighborhood of \(x\) and we get that \(\frac{\log|z_i|}{\log|z'_i|} = O(\frac{1}{|x|^{-1}})\) in a neighborhood of \(x\).

Suppose \(x \notin E_i\) and \(z'_i = 0\) defines \(E_i\) in \(U_1\). Then, \(\log|z'_i|\) is bounded near \(x\) and we get that \(\frac{\log|z'_i|}{\log|z_i|} = O(\frac{1}{|x|^{-1}})\) in a neighborhood of \(x\).

Using a similar argument for \(B_i\)'s as well gives us that \(\log_{U} - \log_{U_1} = O(\frac{1}{|x|^{-1}})\) in a neighborhood of \(x\). Repeating the argument for all \(U_k\) for \(k = 1, \ldots, a\), we get that \(\log_{U} - \log_{V} = O(\frac{1}{|x|^{-1}})\) in a neighborhood of \(x\).

**3.3 The hybrid space**

The hybrid space of an snc model \(\mathcal{X}\) of \((X, B)\), as a set, is defined as \((\mathcal{X}', |B|)^{hyb} := (X \setminus |B|) \cup \Delta(\mathcal{X}', |B|)\). The topology on the hybrid space is defined to be the coarsest one satisfying the following.

- \(X \setminus |B| \hookrightarrow (\mathcal{X}', |B|)^{hyb}\) is an open immersion.
- The projection map \(\pi : (\mathcal{X}', |B|)^{hyb} \to \mathbb{D}\), given by the projection \(X \setminus |B| \to \mathbb{D}\) and by sending \(\Delta(\mathcal{X}', |B|)\) to the origin, is continuous.
- \(\log^{hyb} : (V \setminus |\mathcal{X}_0 + B|) \cup \Delta(\mathcal{X}', |B|) \to \Delta(\mathcal{X}', |B|)\) defined by \(\log_V\) on \(V \setminus |\mathcal{X}_0 + B|\) and identity on \(\Delta(\mathcal{X}', |B|)\) is continuous.

Note that the hybrid space does not contain \(|B|\). It follows from Proposition 3.3 that the topology of the hybrid space does not depend on the global log function we pick. Also note that the fiber of \(\pi : (X, |B|)^{hyb} \to \mathbb{D}\) over \(t \in \mathbb{D}^*\) is \(X_t \setminus |B_t|\).

**Example 3.4** (Hybrid space of \(\mathbb{P}^1 \times \mathbb{D}\)) The hybrid space \((\mathcal{X}', |B|)^{hyb}\) for Example 2.3 is given by \(\mathbb{C}^* \times \mathbb{D}\) with the identification \((re^{i\theta_1}, 0) \sim (re^{i\theta_2}, 0)\) for all \(r \in \mathbb{R}, \theta_i \in [0, 2\pi]\). Over any \(t \in \mathbb{D}^*\) the fiber is \(\mathbb{P}^1 \setminus [0, \infty)\), which is topologically a cylinder. Over \(t = 0\), the fiber is homeomorphic to \(\mathbb{R}\). See Fig. 2.
The hybrid space $\mathcal{X}^{hyb} := (\mathcal{X}, 0)^{hyb}$, constructed in [10], is compact over a compact neighborhood of the origin. But the hybrid space $(\mathcal{X}, |B|)^{hyb}$ that we construct is not always compact over a compact neighborhood of the origin, as can be seen from Example 3.4. However, the following proposition tells us that it is not too bad. In particular, it implies that the hybrid space is locally compact.

**Proposition 3.5** The map $\Log_{V^{hyb}} : (V \setminus |B₀ + B|) \cup \Delta(\mathcal{X}, |B|) \to \Delta(\mathcal{X}, |B|)$ is proper near the central fiber, in the sense that there exists an $r > 0$ such that for any compact set $K \subset \Delta(\mathcal{X}, |B|)$, $\Log^{-1}_{V^{hyb}}(K) \cap \pi^{-1}(r \overline{\mathbb{D}})$ is a compact subset of $(\mathcal{X}, |B|)^{hyb}$.

**Proof** By rescaling the coordinate $t$, we may without loss of generality assume that $V = \mathcal{X}$. Let $V = \bigcup_a U_a$ such that $\Log_V = \sum_a \chi_a \Log_{U_a}$ for adapted coordinate charts $U_a$.

Pick a compact set $K \subset \Delta(\mathcal{X}, |B|)$. It is enough to show that $L = \Log^{-1}_{V^{hyb}}(K) \cap \pi^{-1}(1/2 \overline{\mathbb{D}})$ is compact. Let $\bigcup_{j \in J} V_\lambda$ be an open cover of $L$. Since $K \subset L$ is compact, there exists a finite subset $J' \subset J$ such that $K \subset \bigcup_{\lambda \in J'} V_\lambda$. For each point $P \in \Delta(\mathcal{X}, |B|) \subset (\mathcal{X}, |B|)^{hyb}$, the sets of the form $\Log^{-1}_{V^{hyb}}(W) \cap \pi^{-1}(1/\lambda \overline{\mathbb{D}})$, where $W \subset \Delta(\mathcal{X}, |B|)$ is an open neighborhood of $P$ in $(\mathcal{X}, |B|)$ and $\lambda \in \mathbb{N}$, form basic open neighborhoods of $P$ in $(\mathcal{X}, |B|)^{hyb}$. For every point $P \in K$, we pick such a neighborhood contained in $\bigcup_{\lambda \in J'} V_\lambda$. Since finitely many such neighborhoods cover $K$, we can pick an $r > 0$ such that $K \subset \bigcup_{j=1}^m \Log^{-1}_{V^{hyb}}(W_j) \cap \pi^{-1}(1/\lambda \overline{\mathbb{D}}) \subset \bigcup_{\lambda \in J'} V_\lambda$. Thus, $\Log^{-1}_{V^{hyb}}(K) \subset \bigcup_{\lambda \in J'} V_\lambda$.

Now it is enough to show that we need finitely many $V_\lambda$’s to cover $L \cap \pi^{-1}((r \leq |t| \leq 1/2))$. To do this, it is enough to show that $L \cap \pi^{-1}((r \leq |t| \leq 1/2))$ is relatively compact in $X \setminus |B|$. Suppose to the contrary that the closure of $L$ in $\mathcal{X}$ intersects $B$. Then, there exists a sequence $b_n \in L$ with limit $b \in B$. By the assumption $V = \mathcal{X}$, $b$ lies in an adapted coordinate chart $U_a$ for some $a$. Let $U_a$ be adapted to the stratum $Y_a \subset_{\text{conn. comp.}} E_{I_a} \cap \overline{B}_{J_a}$. As $b \in U_a$, $U_a \cap \overline{B} \neq \emptyset$ and since $U_a$ is an adapted coordinate chart, it follows that $J_a$ is non-empty. But this implies that $\Log_{U_a}(b_n)$ is an unbounded sequence in $\sigma_{Y_a}$, which is a contradiction. \qed

### 4 Convergence of measure

In this section, we prove Theorem B by imitating the proof of [10, Theorem A]. Since $(\mathcal{X}, |B|)^{hyb}$ is not compact, we can no longer use Stone-Weierstrass as done in [10]. Instead, we use Lemma 4.3. Let $(X, B)$ be as in the previous section. Further assume that $K_{\mathcal{X}/\mathbb{D}^n} + B \sim_{Q} 0$ and that $(X, B)$ is sub log canonical. Here, sub log canonical means that $(X, B)$ is log canonical in the sense of the minimal model program (i.e. $\text{discrep}(X, B) \geq -1$) but we are not necessarily assuming that $B$ is effective.
Write $B = \sum_j \beta_j B_j$ for irreducible components $B_j$ of $B$. Since $(X, B)$ is log-smooth, being sub log canonical is equivalent to requiring that $\beta_j \leq 1$ for all $j$. Fix an snc model $\mathcal{X}$ of the pair $(X, B)$. Note that we don’t yet need to assume that $X$ is projective. Let $n + 1$ denote the complex dimension of $X$ i.e. each of the fibers $X_t$ has dimension $n$.

### 4.1 The subcomplex $\Delta(\mathcal{D})$ of $\Delta(\mathcal{X}, |B|)$

Suppose $\mathcal{D}$ is a $\mathbb{Q}$-divisor on $\mathcal{X}$ that extends $K_X/B + B$ such that $\mathcal{D} \sim_{\mathbb{Q}} 0$. Denote $K_{\mathcal{X}/\mathbb{D}} := K_{\mathcal{X}/\mathbb{D}}^- + \mathcal{D}_0 + |(\mathcal{X}_0)_{\text{red}}|$. Then, $\mathcal{D}$ differs from $K_{\mathcal{X}/\mathbb{D}} + B$ by only divisors supported in $\mathcal{X}_0$. Recall that we write $\mathcal{X}_0 = \sum_i b_i E_i$, where $E_i$ are the irreducible components of $\mathcal{X}_0$. Thus, we can write $\mathcal{D} = K_{\mathcal{X}/\mathbb{D}} - \sum_i a_i E_i + \sum_j \beta_j B_j$ for some $a_i \in \mathbb{Q}$. Let $\kappa_i := \frac{a_i}{b_i}$ and $\kappa_{\min} = \min \kappa_i$.

Define the subcomplex $\Delta(\mathcal{D}) \subset \Delta(\mathcal{X}, |B|)$ as follows. If $Y \subset_{\text{conn. comp.}} E_i \cap B_j$ is a stratum, then $\sigma_Y \in \Delta(\mathcal{D})$ if $\kappa_i = \kappa_{\min}$ for all $i \in I$ and if $\beta_j = 1$ for all $j \in J$. In the case when $\dim(X_t) = 1$, this just means that we pick the subgraph generated by vertices corresponding to irreducible components with minimal $\kappa$-value and the rays corresponding to intersections $E_i \cap B_j$ with $\kappa_i = \min \kappa_k$ and $\beta_j = 1$.

For a stratum $Y \subset_{\text{conn. comp.}} E_i \cap B_j$, define $b_{\sigma_Y} = \gcd(b_i)_i \in I$ and let $\lambda_{\sigma_Y}$ be the normalized Lebesgue measure on $|\sigma_Y|$ (see Sect. 2.3). Define $d := \dim(\Delta(\mathcal{D}))$, the maximum of the dimensions of the faces of $\Delta(\mathcal{D})$.

### 4.2 The residual measure

Let $m$ be a sufficiently divisible integer. Given a section $\psi \in H^0(\mathcal{X}, m \mathcal{D})$ and a stratum $Y \subset_{\text{conn. comp.}} E_i \cap B_j$, we can get a section $\text{Res}_Y(\psi) \in H^0(Y, m(\mathcal{D} - \sum_{j \in J} B_j - \sum_{i \in I} E_i)|_Y)$. Suppose that $z_0 = 0, \ldots, z_p = 0, w_1 = 0, \ldots, w_q = 0$ define $Y$ locally. Thinking of $\psi$ as a relative $m$-canonical section, we can write $\psi = f \left( \frac{dz_0}{\phi} \wedge \cdots \wedge \frac{dz_p}{\phi} \wedge \frac{dw_1}{\phi} \wedge \cdots \wedge \frac{dw_q}{\phi} \right)^{\otimes m}$ locally for some local meromorphic function $f$. Then, $\text{Res}_Y(\psi) := f \cdot \phi|_Y^{\otimes m}$.

Note that $\dim(Y) = n - p - q$ and $|\text{Res}_Y(\psi)|^2/m$ gives rise to a $(n - p - q, n - p - q)$-form on $Y \setminus (\cup_{i \notin I} E_i \cup \cup_{j \notin J} B_j)$ of $|\text{Res}_Y(\psi)|^{2/m}$ gives rise to a positive measure on $Y$.

### 4.3 The convergence theorem

Let $m$ be sufficiently divisible integer. Let $\eta \in H^0(X, m(K_X/B + B))$ be a generating section and suppose there exists a generating section $\psi \in H^0(\mathcal{X}, m \mathcal{D})$ that extends $\eta$. Let $\psi_t$ denote the restriction $\psi|_{X_t}$ for $t \neq 0$. If $\psi_t = \alpha \cdot (dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$ on a local chart, then $i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m}$ given locally by

$$i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m} = |\alpha|^{2/m} (idx_1 \wedge d\overline{x_1}) \wedge \cdots (idx_n \wedge d\overline{x_n})$$

is a well-defined positive continuous volume form on $X_t \setminus |B_t|$. Define a measure

$$\mu_t = \frac{i^{n^2}}{|t|^{2\kappa_{\min}(2\pi \log|t|)^{-1}}} (\psi_t \wedge \overline{\psi_t})^{1/m}$$
on $X_t \setminus |B_t|$, and a measure

$$\mu_0 := \sum_{\sigma \in \text{face}(\Delta(\mathcal{D}), \dim(\sigma)) = d} \left( \int_{Y_\sigma} |\text{Res}_{Y_\sigma}(\psi)|^{2/m} \right) b_\sigma^{-1} \lambda_\sigma$$

on $\Delta(\mathcal{D}, |B|)$, where $Y_\sigma$ denotes the stratum associated to the face $\sigma$. We will see in the proof of Lemma 4.2 that the measure $|\text{Res}_{Y_\sigma}(\psi)|^{2/m}$ is a finite measure on $Y_\sigma$ and thus $\int_{Y_\sigma} |\text{Res}_{Y_\sigma}(\psi)|^{2/m}$ is well defined.

**Example 4.1** This example illustrates the importance of the sub log canonical assumption. For simplicity, assume that $X$ has relative dimension 1. Let $E_0$ be an irreducible component of $\mathcal{D}_0$ and let $B_0$ be an irreducible component of $|B|$ occurring with multiplicity $\beta_0 > 1$. Let $\sigma \simeq \mathbb{R}_{\geq 0}$ be the face corresponding to $E_0 \cap B_0$. Let $z$ and $w$ denote the functions that define $E_0$ and $B_0$ in an open neighborhood $U$ of $E_0 \cap B_0$ such that $|z|, |w| < 1$ on $U$. We may assume that $t = z_0$.

We have $\log_U : (U \setminus (E_0 + B_0)) \to \mathbb{R}_{\geq 0}$ given by $(z, w) \mapsto \frac{\log |w|}{\log |t|}$. Suppose we had that $(\log_U)_* (\alpha(t) \mu_t)$ weakly converged to a measure $\mu_0$ on $\mathbb{R}_{\geq 0}$ for some positive scaling function $\alpha(t)$. By scaling by a suitable power of $|t|$, we may assume that $\mu_t = i|w|^{-2\beta_0} dw \wedge d\overline{w}$. Pick a compactly supported continuous function $f$ on $\mathbb{R}_{\geq 0}$. Then,

$$\int_{U_t} (f \circ \log_U) d\mu_t = \int_{U_t} f \left( \frac{\log |w|}{\log |t|} \right) i|w|^{-2\beta_0} dw \wedge d\overline{w}.$$  

Making a change of variable $w = |t|^u e^{i\theta}$, we get

$$\int_{U_t} (f \circ \log_U) d\mu_t = \frac{2\pi}{(\log |t|)^{-1}} \int_0^\infty f(u) |t|^{-2(\beta_0 - 1)u} du.$$  

If we pick a function $f$ that is close to the indicator function of $[0, N]$, then $\alpha(t) \int_{U_t} (f \circ \log_U) d\mu_t = O\left( \frac{\alpha(t)}{\log |t|} |t|^{-2(\beta_0 - 1)N} \right)$ as $t \to 0$. If we require that this expression converge for all values of $N$ as $t \to 0$, then it is easy to see that this is only possible if $\mu_0$ is the zero measure and $\frac{1}{\alpha(t)}$ is growing super-polynomially as $t \to 0$. Thus, we see that the convergence in this hybrid space setting is not very interesting if don’t assume that $(X, B)$ is sub log canonical.

To prove Theorem B, we first prove a local version for functions that are pulled-back from a face $\sigma_Y$ via a local Log map.

**Lemma 4.2** Let $(U, (z, w, y))$ be a coordinate chart adapted to a stratum $Y$ of $\mathcal{D}_0$. Let $f$ be a compactly-supported continuous (real-valued) function on $\sigma_Y$ and let $\chi$ be a compactly supported continuous function on $U$. If a maximal face of $\Delta(\mathcal{D})$ is contained in $\sigma_Y$, let $\sigma_Y'$ denote this (unique) maximal face and let $Y'$ be the stratum associated to $\sigma_Y'$.

If a maximal face of $\Delta(\mathcal{D})$ is contained in $\sigma_Y$, then

$$\int_{U_t \cap X_t} (f \circ \log_U) \chi d\mu_t \to \left( \int_{Y'} \chi |\text{Res}_{Y'}(\psi)|^{2/m} \right) \int_{\sigma_Y'} f b_{\sigma_Y'}^{-1} \lambda_{\sigma_Y'}$$

as $t \to 0$. If $\sigma_Y$ does not contain a maximal face of $\Delta(\mathcal{D})$, then the above limit is 0.

**Proof** By replacing $\mathcal{D}$ by $\mathcal{D} - \kappa_{\text{min}} \mathcal{D}_0$ and $\psi$ by $t^{m_{\text{min}} \psi}$, we may assume that $\kappa_{\text{min}} = 0$. Suppose $Y = E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ locally. The proof for the case $q = 0$ can be found in [10, Lemma 3.5]. The new estimate we need is Equation (2). Let $(z, w, y)$ be coordinates on $U$ such that $E_i = \{ z_i = 0 \}$ and $\overline{B}_j = \{ w_j = 0 \}$ on $U$. To simplify notation, denote $z^a_\sigma := z_0^a \cdots z_p^a$ and $w^\beta := w_1^{\beta_1} \cdots w_q^{\beta_q}$. Then, we can write $\psi$ locally in $U$ as
\[
\psi \otimes \left( \frac{dt}{t} \right)^{\otimes m} = u \cdot z^{m_a} \cdot w^{-m_B} \cdot \left( \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dw_1 \wedge \cdots \wedge dw_q \wedge dy \right)^{\otimes m}
\]

\[
= u \cdot z^{m_a} \cdot w^{m(1-\beta)} \left( \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dw_1 \wedge \cdots \wedge dw_q \wedge dy \right)_{X_t} \left( \frac{dt}{t} \right)^{\otimes m}
\]

for some invertible function \( u \) on \( U \). Here, the second equality follows from \( \frac{dt}{t} = \sum_{i=0}^p b_i \frac{dz_i}{z_i} \).

Thus, we have the following expression for \( \psi_t \),

\[
\psi_t = u \cdot z^{m_a} \cdot w^{m(1-\beta)} \left( \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dw_1 \wedge \cdots \wedge dw_q \wedge dy \right)_{X_t} \left( \frac{dt}{t} \right)^{\otimes m}
\]

(1)

and

\[
i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m} = \left| u \right|^{2/m} \left| z \right|^{2a} \left| w \right|^{2(1-\beta)} \left( \frac{\left| dz_1 \wedge \cdots \wedge dz_p \right|}{b_0^m} \left( \frac{\left| dw_1 \wedge \cdots \wedge dw_q \wedge dy \right|}{\left| z_1 \right|^2 \cdots \left| z_p \right|^2 \left| w \right|^2 \left| y \right|^2} \right) \right).
\]

For \( t \in \mathbb{D}^n \), denote by \( \text{Log}_t : ((X_t \cap U) \setminus \{ B \}) \rightarrow \sigma_Y \times Y \) the map given by \( \text{Log}_t(z, w, y) = (\text{Log}_U(z, w), y) \).

We now apply a change of variables using log-polar coordinates. Let \( u_i = b_i \frac{\log|z_i|}{\log|t|} \) and \( v_j = \frac{\log|w_j|}{\log|t|} \), \( (\kappa, u) := \sum_{i=0}^p \kappa_i u_i, (\nu, -\beta + 1) := \sum_{j=1}^q v_j (-\beta_j + 1) \). Then, we can write

\[
\int_{X_t \cap U} (f \circ \text{Log}_U) d\mu_t
\]

\[
= C \left( \log|t|^{-1} \right)^{p+q-d} \int_{\sigma_p \times \mathbb{R}_{\geq 0} \times Y} \left| t \right|^{2((\kappa_0, u) + (\nu, -\beta + 1))} \left( \int_{\text{Log}_t^{-1}(u, v, y)} \phi \rho_{t, u, v, y} \right) dudv|dy|^2,
\]

where \( \phi = f \circ \text{Log}_U \), \( \rho_{t, u, v, y} \) is the Haar measure on the torsor \( \text{Log}_t^{-1}(u, v, y) \) for the (possibly disconnected) Lie-group \( \{ (\theta_0, \ldots, \theta_p) \in (S^1)^{p+1} | e^{i\theta_0} \cdots e^{i\theta_p} = 1 \} \times (S^1)^q \) and \( C \) is a constant.

First, let us try to figure out the order of magnitude of the expression on the right-hand side. After re-indexing, assume that \( \kappa_0 = \min_{i=1}^p \kappa_i \). Note that

\[
\int_{\sigma} \left| t \right|^{2(\kappa_0, u)} du = O \left( \frac{\left| t \right|^{2\kappa_0}}{\left( \log|t|^{-1} \right)^{\#\{ t | \kappa_i > \kappa_0 \}}} \right),
\]

and for a fixed \( N \) such that \( \text{supp}(f) \subset \{ \sum_{i=0}^p u_i = 1 \} \times [0, N]^q \),

\[
\int_{[0, N]^q} \left| t \right|^{\sum_{j=1}^q (-2\beta_j + 2)v_j} dv = O \left( \frac{1}{\left( \log|t|^{-1} \right)^{\#\{ j | \beta_j < 1 \}}} \right).
\]

(2)

Thus, we see that

\[
\int_{U \cap X_t} (f \circ \text{Log}_U) \chi d\mu_t = O \left( \frac{\left| t \right|^{2\kappa_0}}{\left( \log|t|^{-1} \right)^{d-p-q+\#\{ t | \kappa_i > \kappa_0 \} + \#\{ j | \beta_j < 1 \}}} \right).
\]

Note that the right hand side in the above expression goes off to 0, unless \( \kappa_0 = 0 \) and \( d = \#\{ t | \kappa_i = 0 \} + \#\{ j | \beta_j = 1 \} \). This corresponds exactly to the case when there exists a face \( \sigma_Y' \subset \sigma_Y \) such that \( \sigma_Y' \subset \Delta(\mathcal{P}) \) and \( \sigma_Y' \) has dimension \( d \).

After a possible re-indexing, assume that \( \kappa_0 = \cdots = \kappa_p = 0 \) and \( \kappa_i > 0 \) for all \( i > p' \), and \( \beta_1 = \cdots = \beta_q = 1 \) and \( \beta_j < 1 \) for all \( j > q' \), and \( p' + q' = d \). Then, \( Y' \subset \text{conn. comp.} \ E_0 \cap \cdots \cap E_{p'} \cap \overline{B}_{1} \cap \cdots \cap \overline{B}_{q'} \).

\( \copyright \) Springer
In this case, the Poincaré residue of $\psi$ at $Y'$ is given by

$$|\text{Res}_{Y'}(\psi)|^{2/m} = |u|^{2/m}|z_{p'+1}|^{2a_{p'+1}} \cdots |z_p|^{2a_p} |w_{q'+1}|^{2(1-\beta_{q'+1})} \cdots |w_q|^{2(1-\beta_q)}.$$  

Writing $\sigma = (z_{p'+1}, \ldots, z_p)$ and $w = (w_{q'+1}, \ldots, w_q)$, we can view $\chi(z_i, \sigma)$ as coordinates on $Y' \cap U$. Let $S = \{(u, v) \in \mathbb{P}^{P+q} \mid \sum_{i=1}^P b_i u_i \leq 1\}$. Write

$$S := \left\{(u, v, z', w', y) \in \tilde{S} \times (Y' \cap U) \mid \sum_{i=1}^P b_i u_i + \sum_{i=p'+1}^P b_i \log|z_i|/\log|t| \leq 1 \right\}$$

and let $\mathbf{1}_S$ denote its indicator function. Applying the change of variables, we get

$$\frac{1}{(2\pi \log|t|)^d} \int_{U \cap X_t} (f \circ \log U) \chi \left| \frac{1}{b_0} \frac{dz_1}{z_1} \cdots \frac{dz_p}{z_p} \frac{dw_1}{w_1} \cdots \frac{dw_q}{w_q} \right|^2 |\text{Res}_{Y'}(\psi)|^{2/m} = \frac{1}{b_0^2 (2\pi)^d} \int_{\tilde{S} \times [0,2\pi]^{p'+q} \times (Y' \cap U)} \sum_{(z_0:b_0 = \pi^{p'+q}:1)} f \cdot \chi \cdot \mathbf{1}_S \, du \, dv \, d\theta \cdot d\bar{\theta} \cdot |\text{Res}_{Y'}(\psi)|^{2/m}.$$  

The integral on the right hand side is taken over $\tilde{S} \times [0,2\pi]^{p'+q} \times (Y' \cap U)$, where we view $(u, v) \in \tilde{S}$, $\theta_i \in [0,2\pi]$ for $1 \leq i \leq p'$, $\bar{\theta}_j \in [0,2\pi]$ for $1 \leq j \leq q'$ and $(z', w', y) \in (Y' \cap U)$.

Let us analyze the pointwise limit of each of the factors appearing in the right hand side of the previous expression. We have that

$$f \left(1 - \sum_{i=1}^{p'} b_i u_i - \sum_{i=p'+1}^P b_i \frac{\log|z_i|}{\log|t|}, u, \frac{\log|z'|}{\log|t|}, v, \frac{\log|w|}{\log|t|}\right) \to f \left(1 - \sum_{i=1}^{p'} b_i u_i, u, 0, v, 0\right)$$

pointwise on $\tilde{S} \times (Y' \cap U)$ as $t \to 0$.

As for $\chi$, note that $z_0 \to 0$ as $t \to 0$ for a fixed $(u, v, z', w', y) \in \tilde{S} \times (Y' \cap U)$. So,

$$\chi(z_0, |t|^j e^{i\theta} z', |t|^j e^{i\bar{\theta}} w', y) \to \chi(0, z', 0, w', y)$$

as $t \to 0$.

It is easy to check that $\mathbf{1}_S \to 1$ a.e on $\tilde{S} \times (Y' \cap U)$ as $t \to 0$, and from our analysis in Proposition 2.4, we have that $b_0^{-1} \lambda_\sigma y = \frac{1}{b_0} - d u d v$ under the homeomorphism $\sigma \to \tilde{S}$ given by $(u_0, \ldots, u_{p'}, v_1, \ldots, v_{q'}) \to (u_1, \ldots, u_{p'}, v_1, \ldots, v_{q'})$. The remaining factor of $\frac{1}{b_0}$ is taken care of by the fact that the number of solutions $z_0$ to the equation $z_0^{b_0} = \frac{t}{\prod_{i=1}^P z_i}$ is exactly $b_0$.

Using Lebesgue’s dominated convergence theorem, we have the result. \hfill $\Box$

The following lemma helps to ‘glue’ to the result of the previous lemma to obtain a global version.
Lemma 4.3 Let $L$ be a compact subset of $(\mathcal{X},|B|)_{\text{hyb}}$. Then, $\limsup_{t \to 0} \int_{X_t \cap L} d\mu_t < \infty$.

**Proof** Consider a global log function $\Log_V$ on $\mathcal{X}$. Without loss of generality, assume that $V = \mathcal{X}$. We may further rescale $t$ to assume that $(\Log_{1\text{hyb}}^{-1}(K) \cap \pi^{-1}(\frac{1}{2}\mathbb{D}))_{K \subset \text{cpt} \Delta(\mathcal{X},|B|)}$ forms a compact exhaustion of $(\mathcal{X},|B|)_{\text{hyb}} \cap \pi^{-1}(\frac{1}{2}\mathbb{D})$ (see Proposition 3.5). So, we may enlarge $L$ to assume that $L = \Log_{1\text{hyb}}^{-1}(K) \cap \pi^{-1}(\frac{1}{2}\mathbb{D})$ for some compact $K \subset \Delta(\mathcal{X},|B|)$.

We wish to show that $\limsup_{t \to 0} \int_{X_t} 1_K \circ \Log_V d\mu_t < \infty$. Let $V = \bigcup_{i \in I} U_i$ for adapted coordinate charts $U_i$ and let $(\chi_i)_{i \in I}$ be a partition of unity on $\{U_i\}_{i \in I}$ such that $\Log_V = \sum_{i} \chi_i \Log_{U_i}$. It is enough to show that $\limsup_{t \to 0} \int_{U_i \cap X_t} \chi_i(1_K \circ \Log_V) d\mu_t < \infty$ for all $i$.

Since $\Log_V - \Log_{U_i} = O(\frac{1}{\log|t|})$ on the support of $\chi_i$, we can find a compactly supported continuous function $f$ on $\Delta(\mathcal{X},|B|)$ such that $f \circ \Log_{U_i} \geq 1_K \circ \Log_V$ on $(U_i \setminus (\mathcal{X}_0 + |B|)) \cap \text{supp}(\chi_i)$.

Then,

$$\limsup_{t \to 0} \int_{U_i \cap X_t} \chi_i(1_K \circ \Log_V) d\mu_t \leq \limsup_{t \to 0} \int_{U_i \cap X_t} \chi_i(f \circ \Log_{U_i}) d\mu_t,$$

and the right hand side exists and is finite by Lemma 4.2. \hfill $\Box$

We now prove the statement of Theorem B for functions that are pulled back from compactly-supported continuous functions on $\Delta(\mathcal{X},|B|)$ via a global Log map.

Lemma 4.4 Let $f$ be a continuous compactly supported function on $\Delta(\mathcal{X},|B|)$ and let $\Log_V$ be a global log function on $\mathcal{X}$. Then, $\int_{X_t}(f \circ \Log_V) d\mu_t \to \int_{\Delta(\mathcal{X},|B|)} f d\mu_0$ as $t \to 0$.

**Proof** Let $V = \bigcup_{i \in I} U_i$ and let $\chi_i$ be a partition of unity on $U_i$ so that $\Log_V = \sum_{i} \chi_i \Log_{U_i}$. Let $Y_i$ be the stratum associated to $U_i$.

Then, we can write $\int_{X_t}(f \circ \Log_V) d\mu_t = \sum_{i} \int_{U_i \cap X_t} \chi_i(f \circ \Log_{U_i}) d\mu_t$. It follows from Lemma 4.3 and from Proposition 3.3 that

$$\lim_{t \to 0} \int_{U_i \cap X_t} \chi_i(f \circ \Log_{U_i}) d\mu_t - \int_{U_i \cap X_t} \chi_i(f \circ \Log_{U_i}) d\mu_t = 0. \quad (3)$$

If $\sigma_{Y_i}$ contains a maximal face $\sigma_{Y_i'}$ of $\Delta(\mathcal{X})$, it follows from Lemma 4.2 that

$$\lim_{t \to 0} \int_{U_i \cap X_t} \chi_i(f \circ \Log_{U_i}) d\mu_t = \left(\int_{Y_i} \chi_i|\text{Res}_{Y_i}(\psi)|^{2/m}\right) \int_{\sigma_{Y_i'}} f b_{\sigma_{Y_i'}}^{-1} \lambda_{\sigma_{Y_i'}}.$$

If $\sigma_{Y_i}$ does not contain a maximal face of $\Delta(\mathcal{X})$, then the above limit is $0$. Note that any $\sigma_{Y_i}$ contains at most one maximal face $\sigma_{Y_i'}$ of $\Delta(\mathcal{X})$ and this happens if and only if $Y_i$ intersects $U_i$. Thus, for all $i \in I$, we have

$$\lim_{t \to 0} \int_{U_i \cap X_t} \chi_i(f \circ \Log_{U_i}) d\mu_t = \sum_{\sigma \subset \Delta(\mathcal{X}), \dim(\sigma) = d} \left(\int_{Y_\sigma} \chi_i|\text{Res}_{Y_\sigma}(\psi)|^{2/m}\right) \left(\int_{\sigma} f b_{\sigma}^{-1} \lambda_{\sigma}\right). \quad (4)$$

Combining Equations (3) and (4), we are done. \hfill $\Box$

Now, we are ready to prove Theorem B.

**Proof of Theorem B** Let $f$ be a continuous compactly supported function on $(\mathcal{X},|B|)_{\text{hyb}}$. Fix a global log function $\Log_V$ and let $\chi$ be a continuous function on $(\mathcal{X},|B|)_{\text{hyb}}$ that is 1 in a neighborhood of $\Delta(\mathcal{X}_0,B)$ and is supported in $\pi^{-1}(\frac{1}{2}\mathbb{D})$. By replacing $f$ by $(f|_{\Delta(\mathcal{X},|B|)} \circ \Log_{V_{\text{hyb}}}) \cdot \chi - f$, we may assume that $f|_{\Delta(\mathcal{X},|B|)} = 0$. \hfill $\bigcirc$ Springer
Let $K = \text{supp}(f)$ and pick $\epsilon > 0$. Since $f$ is continuous and compactly supported, there exists $t_0 \ll 1$ such that $|f| \leq \epsilon$ on $\pi^{-1}(t_0 \mathbb{D})$. Then, $\lim sup_{t \to 0} |\int_{\mathcal{X}_t} f d\mu_t| \leq \epsilon \lim sup_{t \to 0} \int_{K \cap \mathcal{X}_t} d\mu_t$, which goes to 0 as $\epsilon \to 0$ by Lemma 4.3. □

Remark 4.5 (Independence of $m$) Note that Theorem B seems to involve the choice of a sufficiently divisible integer $m$. However, it is easy to see that both the measures $\mu_t$ and $\mu_0$ remain invariant if we replace $m$ by $km$ for some positive integer $k$.

Example 4.6 (Convergence of Haar measure on $(\mathbb{P}^1, 0+\infty)^{\text{hyb}}$) In the setting of Example 3.4, let $\mu_t$ denote the Haar measure on $(\mathbb{P}^1 \setminus \{0, \infty\}) \times \{t\}$. Then, $\frac{1}{2\pi \log |t|^{-1}} \mu_t$ weakly converges to the Lebesgue measure on $\mathbb{R} \cong \Delta((\mathcal{X}^-, [B]))$ as measures on the hybrid space $(\mathbb{P}^1, 0+\infty)^{\text{hyb}}$.

More generally, we can prove a similar result for toric varieties.

Example 4.7 (Convergence for the Haar measure on a torus) Let $N$ be a free abelian group of rank $n$. Let $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $T = \text{Spec}(\mathbb{C}[M])$ be the associated torus. Let $Y$ be a smooth projective toric variety associated to a regular fan in $N_{\mathbb{R}}$ (for example, $Y = \mathbb{P}^n$). Let $\omega$ be a torus invariant meromorphic $n$-form on $Y$. Note that there is a canonical choice of such an $\omega$ up to a sign and $\omega$ has poles of order one along all boundary divisors. Let $D$ be the reduced divisor given by the sum of the boundary divisors. Then, $K_Y + D$ is linearly equivalent to 0 and $\omega \in H^0(K_Y + D)$ is a trivializing section.

Consider the constant family $Y \times \mathbb{D}^* \times \mathbb{D}$ over $\mathbb{D}$. Then $(Y \times \mathbb{D}^*, D \times \mathbb{D}^*)$ is log smooth and consider the projective snc model $\mathcal{Y} = Y \times \mathbb{D}$ of $(Y \times \mathbb{D}^*, D \times \mathbb{D}^*)$. Then, $\Delta(\mathcal{Y}, D \times \mathbb{D}^*)$ is canonically isomorphic to $N_{\mathbb{R}}$, with the faces given by the cones in the fan defining $Y$. Thus we have a hybrid space given by $(\mathcal{Y}, D \times \mathbb{D}^*)^{\text{hyb}} = (T \times \mathbb{D}^*) \cup N_{\mathbb{R}}$. We also get a top-dimensional meromorphic form $\eta$ on $Y \times \mathbb{D}^*$ whose restriction to each fiber gives the measure $\omega$. Let $\mu_t$ denote the measure given by $\frac{|t|^2 \omega \wedge \Delta}{(2\pi \log |t|^{-1})^n}$ on the fiber $T \times \{t\}$.

Applying Theorem B to this setting, we get that the measures $\mu_t$ converge to the Lebesgue measure on each of the cones. The Lebesgue measures on each of the cones is exactly the Lebesgue measure on $N_{\mathbb{R}}$ (normalized such that the lattice $N$ has unit covolume) restricted to that cone. Thus, $\mu_t$ converges weakly to the Lebesgue measure on $N_{\mathbb{R}}$ as $t \to 0$.

5 Convergence on the limit hybrid model

Let $(X, B)$ be a log-smooth pair over $\mathbb{D}^*$ with removable singularities at the origin (i.e. there exists an snc model of $(X, B)$). The choice of a hybrid space $(\mathcal{X}, [B])^{\text{hyb}}$ depends on the choice of an snc model $\mathcal{X}$ of $(X, B)$. We construct a canonical hybrid space $(X, [B])^{\text{hyb}}$ that does not depend on a choice of a model. Such a space is obtained by an inverse limit $(X, [B])^{\text{hyb}} = \lim_{\leftarrow} (\mathcal{X}, [B])^{\text{hyb}}$. Theorem 5.1 implies that this definition matches with the definition in the introduction when $(X, B)$ is a projective over $\mathbb{D}^*$. We also explain how the space $(X, [B])^{\text{hyb}}$ can itself be viewed as a Berkovich analytic space when $(X, B)$ is projective over $\mathbb{D}^*$.

5.1 The limit hybrid model

Given two models $\mathcal{X}', \mathcal{X}$ of $(X, B)$, there is always a bimeromorphic map $\mathcal{X}' \dashrightarrow \mathcal{X}$ induced by the given isomorphism with $X$ over $\mathbb{D}^*$. We say that $\mathcal{X}'$ dominates $\mathcal{X}$ when this
bimeromorphic map extends to a morphism. More precisely, we say that $\mathcal{X}'$ dominates $\mathcal{X}$ if we have a proper holomorphic map $\mathcal{X}' \to \mathcal{X}$ which is compatible with the isomorphisms $\mathcal{X}'|_{\mathcal{D}_0} \cong X \cong \mathcal{X}|_{\mathcal{D}_0}$.

When $\mathcal{X}$ and $\mathcal{X}'$ are snc models of $(X, B)$ such that $\mathcal{X}'$ dominates $\mathcal{X}$ via a map $\pi : \mathcal{X}' \to \mathcal{X}$, we also have an integral affine map $\pi_\ast : \Delta(\mathcal{X}', |B|) \to \Delta(\mathcal{X}, |B|)$ and also a continuous surjective map $(\mathcal{X}', |B|)^{\text{hyb}} \to (\mathcal{X}, |B|)^{\text{hyb}}$ as in Sections 4.2 and 4.8 of [10]. If $\sigma_{\mathcal{Y}}$ is a face of $\Delta(\mathcal{X}', |B|)$, associated to a stratum $Y'$ of $\mathcal{X}'_0$, let $Y$ be the smallest stratum that contains $\pi(Y')$. Then, $\pi_\ast(\sigma_{\mathcal{Y}}) \subset \sigma_Y$. We describe these maps in detail in the projective case in the following subsection.

The collection of all snc models of $(X, B)$ is a directed system. See [10, Lemma 4.1] for more details. We can then define $(X, |B|)^{\text{hyb}} := \varprojlim (\mathcal{X}, |B|)^{\text{hyb}}$. It is easy to see that we have a projection map $(X, |B|)^{\text{hyb}} \to \mathbb{D}$ such that $\pi^{-1}(\mathbb{D}^*) \cong X \setminus |B|$, and the central fiber, $(X, |B|)^{\text{hyb}}_0$, is $\varprojlim (\mathcal{X}, |B|)$. Here the inverse limit runs over all snc models $\mathcal{X}$ of $(X, B)$, and the inverse limit is taken in the category of topological spaces. Theorem 5.1 tells us why this definition of $(X, |B|)^{\text{hyb}}$ matches with the one in the introduction when $(X, B)$ is projective.

Suppose now that $(X, B)$ is projective over $\mathbb{D}^*$, i.e., we can view $X$ as a closed subset of $\mathbb{P}^N \times \mathbb{D}^*$ for some $N$ such that $X$ and $B$ are cut out by polynomials whose coefficients are holomorphic on $\mathbb{D}^*$ and meromorphic on $\mathbb{D}$. Thus, we can view the coefficients of the defining equations as elements of $\mathbb{C}(\mathcal{O})$. Using the same defining equations in $\mathbb{P}^N_{\mathbb{C}(\mathcal{O})}$, we get varieties $X_{\mathbb{C}(\mathcal{O})}$ and $B_{\mathbb{C}(\mathcal{O})}$ over $\text{Spec} \mathbb{C}(\mathcal{O})$. A projective snc model $\mathcal{X}$ of $(X, B)$ gives rise to an snc model $\mathcal{X}_{\mathbb{C}[[t]]}$ over $\text{Spec} \mathbb{C}[[t]]$ whose generic fiber is $X_{\mathbb{C}(\mathcal{O})}$ and special fiber is $\mathcal{X}_0$, and $\mathcal{X}_0 + B_{\mathbb{C}[[t]]}$ is an snc divisor in $\mathcal{X}_{\mathbb{C}[[t]]}$. Then, we can define $\Delta(\mathcal{X}_{\mathbb{C}[[t]]}, B_{\mathbb{C}(\mathcal{O})})$ similar to the construction in Sect. 2.2, and we have a canonical identification $\Delta(\mathcal{X}_{\mathbb{C}[[t]]}, B_{\mathbb{C}(\mathcal{O})}) \cong \Delta(\mathcal{X}, |B|)$.

The following theorem, analogous to [18, Theorem 10], [9, Cor 3.2], realizes the central fiber $(X, |B|)^{\text{hyb}}_0$ as a non-Archimedean space.

**Theorem 5.1** Let $(X, B)$ be a projective log-smooth pair over $\mathbb{D}^*$. We have an isomorphism $X_{\mathbb{C}(\mathcal{O})} \setminus |B|_{\mathbb{C}(\mathcal{O})} \cong \lim_{\to} \Delta(\mathcal{X}, |B|)$ where $(\_)^{\text{an}}$ denotes the Berkovich analytification with respect to the $t$-adic norm on $\mathbb{C}(\mathcal{O})$ and, the inverse limit is taken over all projective snc models $(\mathcal{X}, B)$ of $(X, B)$.

We will prove the above theorem in the following subsection, after setting up some preliminaries.

### 5.2 The central fiber of the limit hybrid model as a non-Archimedean space

In this section, we will work over the field $\mathbb{C}(\mathcal{O})$ instead of $\mathbb{D}^*$. So, let $X$ be a smooth projective variety over the discretely valued field $K = \mathbb{C}(\mathcal{O})$, $B$ be an snc divisor on $X$ and, $\mathcal{X}$ be a smooth projective integral scheme over $R = \mathbb{C}[[t]]$ along with a specified isomorphism $\mathcal{X}_K \cong X$ such that $\mathcal{X}$ is an snc model of $(X, B)$ (that is, $\mathcal{X}_0 + B$ is an snc divisor in $\mathcal{X}$). Then, $\Delta(\mathcal{X}, |B|)$ is the dual intersection complex defined similar to the construction in Sect. 2.2. We also have a CW complex $\Delta(\mathcal{X}) := \Delta(\mathcal{X}, 0)$, which can be viewed as a subcomplex of $\Delta(\mathcal{X}, |B|)$.

Let $X^{\text{an}}$ and $|B|^{\text{an}}$ denote the Berkovich analytification of $X$ and $|B|$, respectively, with respect to the $t$-adic norm on $K$. We recall a few definitions related to the Berkovich analytification that will be useful in this section. Recall that points in $X^{\text{an}}$ correspond to valuations on the residue field at (scheme) points of $X$ that extend the valuation on $\mathbb{C}(\mathcal{O})$. This gives us
a continuous map \( \ker : X^{an} \to X \) which sends a valuation to the underlying point i.e a point \( x \in X^{an} \) is a valuation on the residue field of its kernel \( \ker(x) \in X \). We also have a map \( \text{red}_{\mathcal{X}} : X^{an} \to \mathcal{X}_0 \) defined as follows.

Let \( x \in X^{an} \) and let \( k(\ker(x)) \) denote the residue field at \( \ker(x) \). Then \( x \) is a valuation on \( k(\ker(x)) \). Let \( k(\ker(x))^\circ \) denote the valuation ring in \( k(\ker(x)) \) with respect to the valuation \( x \). By the valuative criteria of properness, the map \( \text{Spec}(k(\ker(x))) \to X \) lifts to a unique map \( \text{Spec}(k(\ker(x))^\circ) \to \mathcal{X} \). The image of the closed point of \( \text{Spec}(k(\ker(x))^\circ) \) is denoted as \( \text{red}_{\mathcal{X}}(x) \) and is called the center of the valuation \( x \). The map \( \text{red}_{\mathcal{X}} : X^{an} \to \mathcal{X}_0 \) is anti-continuous in the sense that the inverse image of an open set is closed.

We have an inclusion \( i_{\mathcal{X}} : \Delta(\mathcal{X}) \to X^{an} \) and a retraction \( r_{\mathcal{X}} : X^{an} \to \Delta(\mathcal{X}) \) as constructed in [20]. We would like to do a similar construction for \( \Delta(\mathcal{X}, \{B\}) \) and \( X^{an} \setminus \{B\}^{an} \).

Let \( \mathcal{X} \) be a snc model of \((X, B)\). Then, we have an inclusion map \( i_{\mathcal{X}, \{B\}} : \Delta(\mathcal{X}, \{B\}) \to X^{an} \setminus \{B\}^{an} \), which is given as follows. Let \( Y \subset \text{conn. comp.} \ E_0 \cap \cdots \cap E_{p} \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q \) denote a stratum of \( \mathcal{X}_0 \). Pick a point \((r_0, \ldots, r_p, s_1, \ldots, s_q) \in \sigma_Y \). Let \( z_i \) and \( w_j \) locally define \( E_i \) and \( \overline{B}_j \) near \( Y \) for \( 0 \leq i \leq p \) and \( 1 \leq j \leq q \). Then, we have an isomorphism \( \mathcal{O}_{\mathcal{X}_{0,Y}} \simeq \mathbb{C}[z_0, \ldots, z_p, w_1, \ldots, w_q] \). Pulling back the valuation defined by \( v(\sum_{\alpha \in \mathbb{N}^{p+1}, \beta \in \mathbb{N}} c_{\alpha, \beta} z^\alpha w^\beta) = \min_{c_{\alpha, \beta} \neq 0}(\alpha \cdot r + \beta \cdot s) \), we get an element of \( X^{an} \setminus \{B\}^{an} \). It is clear that \( i_{\mathcal{X}, \{B\}} \) is injective, and it follows from [20, Proposition 3.1.4] that \( i_{\mathcal{X}, \{B\}} \) is continuous. We will often identify \( \Delta(\mathcal{X}, \{B\}) \) with its image under \( i_{\mathcal{X}, \{B\}} \).

We also have a continuous retraction map \( r_{\mathcal{X}, \{B\}} : X^{an} \setminus \{B\}^{an} \to \Delta(\mathcal{X}, \{B\}) \), which is a left inverse to the map \( i_{\mathcal{X}, \{B\}} \), defined as follows. Pick \( x \in X^{an} \setminus \{B\}^{an} \) and let \( \text{red}_{\mathcal{X}}(x) \) be the center of the valuation \( x \). Pick the smallest stratum \( Y \subset \text{conn. comp.} \ E_0 \cap \cdots \cap E_{p} \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q \) containing \( \text{red}_{\mathcal{X}}(x) \). Then, we define

\[
\text{red}_{\mathcal{X}, \{B\}}(x) = (v_x(E_0), \ldots, v_x(E_p), v_x(\overline{B}_1), \ldots, v_x(\overline{B}_q))
\]

in \( \sigma_Y \).

To see why \( r_{\mathcal{X}, \{B\}} \) is continuous, recall that the map \( X^{an} \to \mathcal{X}_0 \) taking any valuation to its center is anti-continuous (i.e. the inverse image of a closed set is open). For any stratum \( Y \subset \text{conn. comp.} \ E_0 \cap \cdots \cap E_{p} \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q \) of \( \mathcal{X}_0 \), the subset \( r_{\mathcal{X}, \{B\}}^{-1}(\sigma_Y) \subset X^{an} \setminus \{B\}^{an} \) is a closed set as it corresponds to a subset of \( X^{an} \) whose center lies on an open set of \( \mathcal{X}_0 \). Therefore, it is enough to prove that \( r_{\mathcal{X}, \{B\}}^{-1}(\sigma_Y) \) is \( \text{comp} \mathcal{X}_0 \annthat \{B\} \subset \mathcal{X}_0 \annthat \{B\} \) for all possible strata. But this is clear from the description of the map above.

We also have a continuous retraction map \( \phi_{\mathcal{X}} : \Delta(\mathcal{X}, \{B\}) \to \Delta(\mathcal{X}) \), which we obtain from the composition.

\[
\Delta(\mathcal{X}, \{B\}) \xrightarrow{i_{\mathcal{X}, \{B\}}} X^{an} \setminus \{B\}^{an} \xleftarrow{r_{\mathcal{X}}} X^{an} \xrightarrow{\phi_{\mathcal{X}}} \Delta(\mathcal{X}).
\]

Explicitly, if \( Y \subset \text{conn. comp.} \ E_0 \cap \cdots \cap E_{p} \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q \), let \( Y' \subset \text{conn. comp.} \ E_0 \cap \cdots \cap E_{p} \) be the stratum containing \( Y \). Then, \( \phi_{\mathcal{X}}(\sigma_Y) \subset \sigma_Y' \) and

\[
\phi_{\mathcal{X}}(r_0, \ldots, r_p, s_1, \ldots, s_q) = (r_0, \ldots, r_p).
\]

If \( \mathcal{X} \) and \( \mathcal{X}' \) are two snc models of \((X, B)\) such that \( \mathcal{X}' \) dominates \( \mathcal{X} \), then there is a surjective map \( r_{\mathcal{X}', \mathcal{X}, \{B\}} : \Delta(\mathcal{X}', \{B\}) \to \Delta(\mathcal{X}, \{B\}) \) given by

\[
\Delta(\mathcal{X}', \{B\}) \xrightarrow{i_{\mathcal{X}, \{B\}}} X^{an} \setminus \{B\}^{an} \xrightarrow{r_{\mathcal{X}, \{B\}}} \Delta(\mathcal{X}, \{B\}).
\]

The surjectivity of the map follows from [20, Proposition 3.17].

We have an explicit description of \( r_{\mathcal{X}', \mathcal{X}, \{B\}} \) similar to [10, Section 4.2] as follows. Let \( \rho : \mathcal{X}' \to \mathcal{X} \) denote the proper map between \( \mathcal{X}' \) and \( \mathcal{X} \), let \( Y' \subset \text{conn. comp.} \ E_0' \cap \cdots \cap \overline{B}_q' \)
Proposition 5.2 We have a commutative diagram

\[
\begin{array}{ccc}
\Delta(\mathcal{X}', |B|) & \xrightarrow{\phi_{X'}} & \Delta(\mathcal{X}') \\
\downarrow r_{\mathcal{X}', \mathcal{X}', |B|} & & \downarrow r_{\mathcal{X}', \mathcal{X}} \\
\Delta(\mathcal{X}, |B|) & \xrightarrow{\phi_{X}} & \Delta(\mathcal{X})
\end{array}
\]

which gives rise to a continuous map \( \phi : \lim_{\leftarrow \mathcal{X}} \Delta(\mathcal{X}, |B|) \to \lim_{\to \mathcal{X}} \Delta(\mathcal{X}) \).

Proof To see that the diagram commutes, it enough to use the fact that \( r_{\mathcal{X}', \mathcal{X}} \circ r_{\mathcal{X}', |B|} = r_{\mathcal{X}} \) [20, Proposition 3.1.7] and show that \( \phi_{X} \circ r_{\mathcal{X}, |B|} = r_{\mathcal{X}} \) on \( X^{\text{an}} \setminus |B|^{\text{an}} \). Pick \( v \in X^{\text{an}} \setminus |B|^{\text{an}} \).

Let \( Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \widetilde{B}_1 \cap \cdots \cap \widetilde{B}_q \) be the minimal stratum of \( \mathcal{X}_0 + B \) containing the center of \( v \). Then,

\[
r_{(\mathcal{X}, |B|)}(v) = (v(E_0), \ldots, v(E_p), v(\widetilde{B}_1), \ldots, v(\widetilde{B}_q))
\]

in \( \sigma_Y \).

Let \( Y' \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \) be the stratum containing \( Y \). Then, \( Y' \) is the minimal stratum in \( \mathcal{X}_0 \) containing the center of \( v \) and \( r_{\mathcal{X}}(v) = (v(E_0), \ldots, v(E_p)) \) in \( \sigma_{Y'} \). It follows from the description of \( \phi_{X} \) that \( \phi_{X}(r_{(\mathcal{X}, |B|)}(v)) = r_{\mathcal{X}}(v) \).

Proportion 5.3 If \( \mathcal{X}' \) is a blowup of \( \mathcal{X} \) along a stratum \( Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \widetilde{B}_1 \cap \cdots \cap \widetilde{B}_q \), then \( r_{\mathcal{X}, \mathcal{X}, |B|} : \Delta(\mathcal{X}', |B|) \to \Delta(\mathcal{X}, |B|) \) is a homeomorphism obtained by a subdivision.

Proof This follows from a local blowup computation. Let \( E' \) denote the exceptional divisor in \( \mathcal{X}' \). Then, the maximal strata of \( \mathcal{X}' \) that map down to \( Y \) are of the form \( E' \cap \widetilde{E}_I \cap \widetilde{B}_1 \cap \cdots \cap \widetilde{B}_q \) and \( E' \cap \widetilde{E}_0 \cap \cdots \cap \widetilde{E}_p \cap \widetilde{B}_j \), where \( I \) and \( J \) denote subsets of \( \{0, \ldots, p\} \) and \( \{1, \ldots, q\} \) of size \( p \) and \( q - 1 \) respectively and \( \widetilde{E}_I \) and \( \widetilde{B}_j \) denote the strict transforms of \( E_I \) and \( B_j \).

First, let’s compute the image of \( \sigma_{E'} \) in \( \Delta(\mathcal{X}, |B|) \). Note that \( \text{div}_{\mathcal{X}, |B|}(t) = \sum_{i=0}^p b_i \widetilde{E}_i + (\sum_{i=0}^p b_i) E' \). Let \( v_{E'} \) denote the divisorial valuation corresponding to \( \sigma_{E'} \). Then,

\[
v_{E'}(E_i) = v_{E'}(\widetilde{E}_i + E') = v_{E'}(E') = \frac{1}{\text{ord}_{E'}(t)} = \frac{1}{\sum_{i=0}^p b_i}
\]
for all \( i = 0, \ldots, p \). Similarly, \( v_{E_i}(B_j) = \frac{1}{\sum_{i=0}^{p} b_i} \) for all \( j = 1, \ldots, q \). Thus, the image of \( \sigma_{E_i} \) in \( \Delta(\mathcal{X}, |B|) \) is \( \frac{1}{\sum_{i=0}^{p} b_i}(1, \ldots, 1) \in \sigma_Y \).

It is easy to check that the \( \Delta(\mathcal{X}', |B|) \) is a subdivision obtained by adding the vertex \( \sigma_{E_i} \) to \( \sigma_Y \). For example, let’s compute the image of \( \sigma_{Y'} \) for \( Y' = E' \cap \tilde{E}_1 \cap \cdots \cap \tilde{E}_q \cap \tilde{B}_q \). Note that

\[
\sigma_{Y'} = \left\{ (x_0, \ldots, x_p, y_1, \ldots, y_q) \mid \left( \sum_{i=0}^{p} b_i \right) x_0 + \sum_{i=1}^{p} b_i x_i = 1 \right\}.
\]

Suppose \( v \) is a valuation represented by \( (x_0, \ldots, x_p, y_1, \ldots, y_q) \in \sigma_{Y'} \). Then, \( v(E_0) = v(\tilde{E}_0 + E') = v(E') = x_0 \) and \( v(E_i) = v(\tilde{E}_i + E') = x_i + x_0 \) for \( i = 1, \ldots, p \). Similarly, \( v(B_j) = y_j + x_0 \) for \( j = 1, \ldots, q \).

Thus, we see that \( r_{\mathcal{X}', \mathcal{X}, |B|}|_{\sigma_{Y'}} \) is given by

\[
(x_0, \ldots, x_p, y_1, \ldots, y_q) \mapsto (x_0, x_1 + x_0, \ldots, x_p + x_0, y_1 + x_0, \ldots, y_q + x_0)
\]

\( \square \)

In general, the map \( \Delta(\mathcal{X}') \rightarrow \Delta(\mathcal{X}) \) is not a homeomorphism, as illustrated by the following example.

**Example 5.4** (Blowup of \( \mathbb{P}^1 \times \mathbb{D} \)) Let the notation be the same as in Example 2.3. Let \( E_0 = \mathbb{P}^1 \times \{0\}, \tilde{B}_1 = \{0\} \times \mathbb{D}, \tilde{B}_2 = \{\infty\} \times \mathbb{D} \). Let \( \mathcal{X}' \) denote the blowup of \( \mathcal{X} \) at \( E_0 \cap \tilde{B}_1 \) and let \( \mathcal{X}'' \) denote the blowup of \( \mathcal{X} \) at some point in \( E_0 \) that is different from 0 and \( \infty \). Then \( \Delta(\mathcal{X}'', |B|) \) is obtained from \( \Delta(\mathcal{X}', |B|) \) by adding a vertex along the ray \( E_0 \cap \tilde{B}_1 \) and \( \Delta(\mathcal{X}'', |B|) \) is obtained from \( \Delta(\mathcal{X}, |B|) \) by adding an extra vertex and joining it to \( \sigma_{E_0} \). The retraction \( r_{\mathcal{X}'', \mathcal{X}, |B|} : \Delta(\mathcal{X}', |B|) \rightarrow \Delta(\mathcal{X}, |B|) \) is an isomorphism, while \( r_{\mathcal{X}'', \mathcal{X}, |B|} : \Delta(\mathcal{X}'', |B|) \rightarrow \Delta(\mathcal{X}, |B|) \) is given by collapsing the newly added edge and vertex to \( \sigma_{E_0} \) (Fig. 3).

**Lemma 5.5** Let \( \mathcal{X} \) be a snc model of \((X, B)\) and let \( K \subseteq \Delta(\mathcal{X}, |B|) \) be a compact set. Then there exists a snc model \( \mathcal{X}' \) of \((X, B)\) dominating \( \mathcal{X} \) such that \( r_{\mathcal{X}'', \mathcal{X}, |B|}^{-1}(K) \subseteq \Delta(\mathcal{X}') \).

**Proof** For a valuation \( v \in X^{an} \) and a divisor \( D \subseteq \mathcal{X} \) not contained in \( \ker v \), set \( v(D) := v(f) \), where \( f \) defines \( D \) in an open neighborhood of the red \( \mathcal{X} \) (\( v \)). We identify \( \Delta(\mathcal{X}, |B|) \) with its image under \( i_{\mathcal{X}, |B|} \) and think of points in \( \Delta(\mathcal{X}, |B|) \) as valuations.

Since it is enough to prove the result for some small enough compact neighborhoods of all points in \( K \), we may assume without loss of generality that there exists an irreducible component \( E \) of \( \mathcal{X}_0 \) and an \( \epsilon > 0 \) such that \( v(E) \geq \epsilon \) for all \( v \in K \). Let \( \tilde{B}_1, \ldots, \tilde{B}_q \) be the irreducible components of \( |B| \) containing the centers of all \( v \in K \). It is enough to show that there exists an snc model \( \mathcal{X}' \) of \((X, B)\) such that red \( \mathcal{X'}(v') \) is not contained in the closures of \( \tilde{B}_1, \ldots, \tilde{B}_q \) in \( \mathcal{X}' \) for all \( v' \in \Delta(\mathcal{X}, |B|) \) such that \( r_{\mathcal{X}', \mathcal{X}, |B|}(v') \in K \). Note that if \( q = 0 \), we are done. We will prove the result by induction on \( q \).
Pick $N > 0$ large enough so that $Nv(E) \geq v(B_1)$ for all $v \in K$. Let $\mathcal{I}_E$ and $\mathcal{I}_{B_1}$ be the ideal sheaf defining $E$ and $B_1$ respectively. Let $\widetilde{\mathcal{X}}$ be the blowup of $\mathcal{X}$ along the ideal sheaf $\mathcal{I}_E + \mathcal{I}_{B_1}$. Then, $\widetilde{\mathcal{X}}$ is a model of $X$ although it may not necessarily be regular. Pick $v \in K$ and let $U$ be an affine open neighborhood of red $\mathcal{X}(v)$. If $E$ is defined by $z = 0$ and $B_1$ is defined by $w_1 = 0$ on $U$, then $\widetilde{U} = \text{Spec} \mathcal{O}_X(U)[\frac{z^N}{w_1}]$ is a chart of the blowup. Let $\mathcal{X}'$ be a resolution of singularities of $\widetilde{\mathcal{X}}$ such that $\mathcal{X}'$ is a snc model for $(X, B)$. Pick $v' \in \Delta(\mathcal{X}', |B|)$ such that $r_{X', \mathcal{X}, |B|}(v') = v$. Then, $v'(\frac{z^N}{w_1}) = v(\frac{z^N}{w_1}) \geq 0$. Thus, the center of $v'$ in $\widetilde{\mathcal{X}}$ is contained in $\widetilde{U}$. But $\widetilde{U}$ misses the strict transform of $B_1$, and thus the center of $v'$ in $\mathcal{X}'$ is not contained in $B_1$. Thus, the irreducible components of $B$ in $\mathcal{X}'$ containing the centers of any valuations $v' \in r_{\mathcal{X}', \mathcal{X}, |B|}(K)$ can only be $B_2, \ldots, B_q$. Thus we are done by induction.

To simplify the discussion, for the remainder of this subsection we will identify $\Delta(\mathcal{X}', |B|)$ with its image under $i_{\mathcal{X}', |B|}$.

**Corollary 5.6** Let $v \in \varinjlim \Delta(\mathcal{X}', |B|)$ be defined by a sequence of valuations $v_{\mathcal{X}'} \in \Delta(\mathcal{X}, |B|)$ for each snc model $\mathcal{X}$ of $(X, B)$. Then, given a snc model $\mathcal{X}$ of $(X, B)$, there exists a snc model $\mathcal{X}'$ of $(X, B)$ dominating $\mathcal{X}$ such that the center of $v_{\mathcal{X}'}$ in $\mathcal{X}'$ does not intersect $|B|$.

**Proof** This easily follows Lemma 5.5. Once we find a model $\mathcal{X}'$ of $\mathcal{X}$ such that red $\mathcal{X}'(v_{\mathcal{X}'})$ is not contained in the closure of $|B|$, we can further blowup to assume that the two become disjoint.

**Proposition 5.7** The map $\phi : \varprojlim \Delta(\mathcal{X}, |B|) \rightarrow \varinjlim \Delta(\mathcal{X})$ is open and injective, where $\mathcal{X}$ ranges over all snc models $\mathcal{X}$ of $(X, B)$.

**Proof** Let $v, v'$ be two distinct elements in $\varprojlim \Delta(\mathcal{X}, |B|)$ defined by sequences $v_{\mathcal{X}}, v'_{\mathcal{X}} \in \Delta(\mathcal{X}, |B|)$ respectively. Once again, we identify the elements of $\Delta(\mathcal{X}, |B|)$ with its image under $i_{\mathcal{X}, |B|}$ and think of them as valuations. Let $\mathcal{X}$ be an snc model of $(X, B)$ such that $v_{\mathcal{X}} \neq v'_{\mathcal{X}}$ in $\Delta(\mathcal{X}, |B|)$. From Corollary 5.6, we can find a model $\mathcal{X}'$ such that $\phi_{\mathcal{X}'}(v_{\mathcal{X}'}) = v_{\mathcal{X}'}$, and $\phi_{\mathcal{X}'}(v'_{\mathcal{X}'}) = v'_{\mathcal{X}'}$. Note that $v_{\mathcal{X}'} \neq v'_{\mathcal{X}'}$, as $r_{\mathcal{X}', \mathcal{X}, |B|}(v_{\mathcal{X}'}) \neq r_{\mathcal{X}', \mathcal{X}, |B|}(v'_{\mathcal{X}'})$. Thus, $\phi$ is injective.

To see that $\phi$ is open, it is enough to show that given an snc model $\mathcal{Y}$ of $(X, B)$ and an open set $U \subset \Delta(\mathcal{Y}, B)$

\[
\phi \left( \left\{ v \in \varprojlim \Delta(\mathcal{X}, |B|) \mid v_{\mathcal{Y}} \in U \right\} \right)
\]

is an open set. We may further assume that $U$ is small enough and has compact closure. Using Lemma 5.6, we can find a model $\mathcal{Y}'$ such that $U' := r^{-1}_{\mathcal{Y}', \mathcal{Y}, B}(U) \subset \Delta(\mathcal{Y}')$. Then, it is easy to check that
\[ \phi \left( \left\{ v \in \lim_{\mathcal{X}} \Delta(\mathcal{X}', |B|) | v_{\mathcal{X}'} \in U \right\} \right) = \left\{ v \in \lim_{\mathcal{X}} \Delta(\mathcal{X}) | v_{\mathcal{X}'} \in U' \right\}. \]

To prove Theorem 5.1, we exploit the isomorphism \( X^{an} \xrightarrow{\sim} \lim_{\mathcal{X}} \Delta(\mathcal{X}') \) (see [18, Theorem 10], [9, Cor. 3.2]).

**Remark 5.8** The homeomorphism \( X^{an} \xrightarrow{\sim} \lim_{\mathcal{X}} \Delta(\mathcal{X}') \) in [18, Theorem 10] is stated when the inverse limit runs over all snc models \( \mathcal{X}' \) of \( X \). However, we may as well take the inverse limit over all snc models \( \mathcal{X}' \) of \( (X, B) \) because such models form a cofinal system.

**Proof of Theorem 5.1** We obtain a map \( r_{(X, |B|)} : X^{an} \setminus |B|^{an} \rightarrow \lim_{\mathcal{X}} \Delta(\mathcal{X}', |B|) \) by considering the inverse limit over the retraction map \( r_{(\mathcal{X}', |B|)} : X^{an} \setminus |B|^{an} \rightarrow \Delta(\mathcal{X}', |B|) \).

Observe that we have the following commutative diagram where the bottom map is a homeomorphism.

\[
\begin{array}{ccc}
X^{an} \setminus |B|^{an} & \xrightarrow{r_{(X, |B|)}} & \lim_{\mathcal{X}} \Delta(\mathcal{X}', |B|) \\
\downarrow \phi & & \downarrow \phi \\
X^{an} & \xrightarrow{r_{X}} & \lim_{\mathcal{X}} \Delta(\mathcal{X})
\end{array}
\]

Therefore, it is enough to show that the image of \( |B|^{an} \) in \( \lim_{\mathcal{X}} \Delta(\mathcal{X}') \) does not intersect with the image of \( \phi \). Let \( v \) be an element of \( \lim_{\mathcal{X}} \Delta(\mathcal{X}', |B|) \) defined by a sequence \( v_{\mathcal{X}'} \in \Delta(\mathcal{X}', |B|) \). Let \( v_1 := r_{X}^{-1}(\phi(v)) \). Without loss of generality, assume to the contrary that \( v_1 \in |B|^{an} \).

Using Corollary 5.6, we can find a model \( \mathcal{X}' \) such that the center of \( v_{\mathcal{X}'} \) in \( \mathcal{X}' \) does not intersect \( |B| \). Then, \( \phi_{\mathcal{X}'}(v_{\mathcal{X}'} \) = \( v_{\mathcal{X}'} \). We also have that \( r_{\mathcal{X}'}(v_1) = \phi_{\mathcal{X}'}(v_{\mathcal{X}'}) = v_{\mathcal{X}'} \) and the center of \( v_1 \) in \( \mathcal{X}' \) is contained in the center of \( v_{\mathcal{X}'} \) in \( \mathcal{X} \). But the center of \( v_1 \) is contained in the closure of \( B_1 \), which is a contradiction.

### 5.3 The limit hybrid space as a Berkovich analytic space

Let \( (X, B) \) be a log-smooth pair of projective varieties over \( \mathbb{D}^* \). In this section, for any \( 0 < r < 1 \), we realize \( (X, |B|)^{hyb} := (X, |B|)^{hyb} |_{r\mathbb{D}} \) as the analytification of a scheme over a Banach ring, \( A_r \).

As in [8], consider the Banach ring

\[
A_r = \left\{ \sum_i c_i t^i \in \mathbb{C}(\!(t)\!) \left| c_i \in \mathbb{C} \text{ and } \sum_{i \in \mathbb{Z}} ||c_i||_{hyb} t^i < \infty \right. \right\},
\]

where \( ||c_i||_{hyb} = \max\{||c_i||, 1\} \) if \( c_i \neq 0 \) and \( ||0||_{hyb} = 0 \). Then, its Berkovich spectrum \( \mathcal{M}(A_r) \) is homeomorphic to \( r\mathbb{D} \). For more details, see [8], [10, Appendix 1]. Note that any function that is holomorphic in open neighborhood of \( r\mathbb{D} \setminus \{0\} \) and meromorphic at \( 0 \) gives an element of \( A_r \).

Given a projective family \( X \rightarrow \mathbb{D}^* \), we can think of \( X \) as a finite type scheme over \( \text{Spec} \ A_r \) because the coefficients of the homogeneous equations cutting out \( X \) in \( \mathbb{P}^N \times \mathbb{D}^* \) can be viewed as elements of \( A_r \). We denote this scheme as \( X_{A_r} \). Similarly, we get \( |B|_{A_r} \subset X_{A_r} \). Let \( (\_)^{An} \) denote the Berkovich analytification functor on the category of finite type
schemes over \( \text{Spec } A_r \). The map \( X_{A_r} \setminus |B|_{A_r} \to \text{Spec } A_r \) gives rise to the canonical map \( X_{A_r}^{\text{An}} \setminus |B|_{A_r}^{\text{An}} \to \mathcal{M}(A_r) \simeq r\overline{D} \). The following proposition tells us how this analytic space is related to \((X, |B|)^{\text{hyb}}\).

**Proposition 5.9** We have a homeomorphism \( X_{A_r}^{\text{An}} \setminus |B|_{A_r}^{\text{An}} \overset{\sim}{\to} (X, |B|)^{\text{hyb}}_r \) as spaces over \( r\overline{D} \).

**Proof** Let \( \pi_r : (X_{A_r} \setminus |B|_{A_r}) \to r\overline{D} \simeq \mathcal{M}(A_r) \) be the canonical projection map. From [10, Lemma A.6] we have the following homeomorphisms:

\[
\pi_r^{-1}(r\overline{D}^*) \simeq (X \setminus |B|)_{r\overline{D}^*} \quad \text{and} \quad \pi_r^{-1}(0) \simeq (X^{\text{an}}_{\Delta_1} \setminus |B|^{\text{an}}_{\Delta_1}).
\]

Moreover, the first homeomorphism is compatible with the projections to \( r\overline{D}^* \).

The above homeomorphisms let us define a bijection \( X_{A_r}^{\text{An}} \setminus |B|_{A_r}^{\text{An}} \to (X, |B|)^{\text{hyb}}_r \). It remains to check that this map is continuous. To do this, first note that we have an embedding \((X, B)^{\text{hyb}} \hookrightarrow X^{\text{hyb}}\), where \(X^{\text{hyb}} := \lim_{\rightarrow} X^{\text{reg}}_r\), given by the canonical inclusion over \( \overline{D}^* \) and by Proposition 5.2 over the central fiber. We also have a homeomorphism \( X_{A_r}^{\text{An}} \to X_r^{\text{hyb}} \) as topological spaces over \( r\overline{D} \) [10, Proposition 4.12]. It is straightforward to check that the following diagram of topological spaces over \( r\overline{D} \) commutes.

\[
\begin{array}{ccc}
X_{A_r}^{\text{An}} \setminus |B|_{A_r}^{\text{An}} & \longrightarrow & (X, |B|)^{\text{hyb}}_r \\
\downarrow & & \downarrow \\
X_{A_r}^{\text{An}} & \simeq & X_r^{\text{hyb}}
\end{array}
\]

Since the map at the bottom is a homeomorphism, the vertical maps are open immersions, and the top map is a bijection, the top map is also a homeomorphism. \( \square \)

Now, we can define the hybrid space associated to a (not necessarily log-smooth) projective pair \((X, B)\) over \( \overline{D}^* \) as \((X, |B|)^{\text{hyb}} := \lim_{\rightarrow} (X^{\text{reg}} \setminus |B|)^{\text{an}}_{\Delta_1} \). Proposition 5.9 tells us that this matches with our previous definition when \((X, B)\) is log-smooth.

### 5.4 Convergence on limit hybrid model

Let \((X, B)\) be a projective log-smooth pair of varieties over \( \overline{D}^* \). The convergence described in Theorem B depends on the choice of a model \((\mathcal{X}, B)\) of \((X, B)\). We would like to remedy this by describing the convergence on \((X, |B|)^{\text{hyb}} = \lim_{\leftarrow} (\mathcal{X}, |B|)^{\text{hyb}}\), which is independent of the choice of a model.

Suppose we have two models \( \mathcal{X} \) and \( \mathcal{X}' \) of \((X, B)\) with \( \mathcal{X}' \) dominating \( \mathcal{X} \) via \( \rho : \mathcal{X}' \to \mathcal{X} \). Suppose that we have a \( \mathbb{Q}\)-Cartier divisor \( \mathcal{D} \) on \( \mathcal{X} \) extending \( K_{X/\Delta^*} + B \) and a generating section \( \psi \in H^0(\mathcal{X}, m\mathcal{D}) \) extending \( \eta \in H^0(X, m(K_{X/\Delta^*} + B)) \). Then, we can get a \( \mathbb{Q}\)-Cartier divisor \( \mathcal{D}' = \rho^* \mathcal{D} \) on \( \mathcal{X}' \) extending \( K_{X'/\Delta^*} + B \) and a section \( \psi' = \rho^* \psi \) extending \( \eta \). Applying Theorem B to both \( \mathcal{X} \) and \( \mathcal{X}' \), we get measures \( \mu_0^{\mathcal{X}} \) and \( \mu_0^{\mathcal{X}'} \) on \( \Delta(\mathcal{X}, |B|) \) and \( \Delta(\mathcal{X}', |B|) \) respectively which are the limits of \( \mu_1 \) on \((\mathcal{X}, |B|)^{\text{hyb}}\) and \((\mathcal{X}', |B|)^{\text{hyb}}\) respectively. Since the pushforward of Radon measures commutes with weak limits, we have that \( \mu_0^{\mathcal{X}'} \) is just the push-forward of the measure \( \mu_0^{\mathcal{X}} \) under the map \( r\mathcal{X}', \mathcal{X}, |B| \).

Thus, we get a compatible system of measure \( \mu_0^{\mathcal{X}'} \) on all models \( \mathcal{X}' \) dominating a fixed model \( \mathcal{X} \). This gives rise to a measure on \( \mu_0 \) on \((X, |B|)^{\text{hyb}}_r\), and thus we get the following convergence theorem.

\( \square \) Springer
Theorem 5.10 Let $(X, B)$ be a projective log-smooth pair over $\mathbb{D}^{*}$. Suppose that $K_{X/\mathbb{D}^{*}} + B \sim_{\mathbb{Q}} 0$ and let $\eta \in H^{0}(X, m(K_{X/\mathbb{D}^{*}} + B))$ admit a meromorphic extension (i.e. there exists a model $\mathcal{X}$ of $(X, B)$, a $\mathbb{Q}$-Cartier divisor $\mathcal{D}$ extending $K_{X/\mathbb{D}^{*}} + B$ and $\psi \in H^{0}(\mathcal{X}, m \mathcal{D})$ extending $\eta$). Then, there exists $\kappa_{\min} \in \mathbb{Q}$ and $d \in \mathbb{N}$ such that the measure $\mu_{t} = \frac{\mu^{2}(\eta \wedge \overline{\eta})^{1/m}}{|t|^{\kappa_{\min}(2\pi \log(t)^{1} - 1)^{d}}}$ converges weakly to a measure $\mu_{0}$ on $(X, |B|)^{\text{hyp}}$.

Moreover if we fix an snc model $\mathcal{X}$, a $\mathbb{Q}$-Cartier divisor $\mathcal{D}$ and a section $\psi \in H^{0}(\mathcal{X}, m \mathcal{D})$ extending $\eta$, then $\mu_{0}$ is supported on $\Delta(\mathcal{D}) \subset \Delta(\mathcal{X}) \subset \mathcal{X}(\mathbb{C}(\mathcal{D}))^{\text{an}} \setminus |B|_{\mathcal{C}(\mathcal{D})}^{\text{an}}$ and $d$, $\kappa_{\min}$ and $\mu_{0}$ have the same description as in Sect. 4.3.

Example 5.11 Following up Example 4.6, we see that the Haar measures on $\mathbb{P}^{1}$ converges to the Lebesgue measure on $\mathbb{R}$, which can be thought of as the unique line joining the type 1 points corresponding to 0 and $\infty$ in $(\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}}$. More generally, we could take $B_{t}$ to be given by $p(t) + q(t)$ for distinct functions $p, q$ which are meromorphic on $\mathbb{D}$ and holomorphic on $\mathbb{D}^{*}$. Then, there exists an isomorphism of pairs $(\mathbb{P}^{1} \times \mathbb{D}^{*}, \{(p(t), t)\} + \{(q(t), t)\}) \simeq (\mathbb{P}^{1} \times \mathbb{D}^{*}, [0] \times \mathbb{D}^{*} + [\infty] \times \mathbb{D}^{*})$. This extends to an isomorphism $(\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus \{p, q\} \simeq (\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus [0, \infty]$, where $p, q$ denote the type 1 points corresponding to $p(t)$ and $q(t)$. Thus, as $t \to 0$, the Haar measure on $\mathbb{P}^{1} \setminus \{p(t), q(t)\}$ converges to the Lebesgue measure on the unique line joining the points $p$ and $q$ in $(\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus \{\overline{p}, \overline{q}\}$.

Example 5.12 Similar to the above example, let $X = \mathbb{P}^{1} \times \mathbb{D}^{*}$ denote the constant family. Let $B = \{z^{2} + a_{1}z + a_{2} = 0\} \subset \mathbb{P}^{1} \times \mathbb{D}^{*}$, where $z$ denotes the coordinate on $\mathbb{P}^{1}$ and $a_{1}, a_{2}$ are functions that are meromorphic on $\mathbb{D}$ and holomorphic on $\mathbb{D}^{*}$. Then, $(X, B)$ is log Calabi–Yau. Also assume that the polynomial $z^{2} + a_{1}z + a_{2} \in \mathcal{C}(\mathcal{D})[z]$ is irreducible.

Fix a square root $u = \sqrt{t}$ and consider the field extension $\mathbb{C}(\mathcal{D}) \to \mathbb{C}(\mathcal{D})$. This corresponds to a degree two map $\mathbb{D}^{*} \to \mathbb{D}^{*}$ given by $u \mapsto u^{2}$.

The polynomial $z^{2} + a_{1}z + a_{2} \in \mathbb{C}(\mathcal{D})[z]$ splits into factors $(z - p)(z - q)$ in $\mathbb{C}(\mathcal{D})[z]$. By the previous example, as $u \to 0$, the Haar measure on $\mathbb{P}^{1} \setminus \{p(u), q(u)\}$ converges to the Lebesgue measure on the line joining $p$ and $q$ in $(\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus \{p, q\}$. Call this measure $\overline{\mu}_{0}$. We have a map $(\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus \{p, q\} \to (\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus \{|B|\}^{\text{an}}$.

To understand the convergence of the Haar measure on $\mathbb{P}^{1} \setminus |B_{t}|$, note that $\mathbb{P}^{1} \setminus |B_{t}| \simeq \mathbb{P}^{1} \setminus \{p(u), q(u)\}$. Thus, as $t \to 0$ the Haar measure on $\mathbb{P}^{1} \setminus |B_{t}|$ converges to the pushforward of $\overline{\mu}_{0}$ to $(\mathbb{P}^{1}_{\mathbb{C}(\mathcal{D})})^{\text{an}} \setminus \{|B|\}^{\text{an}}$.

Example 5.13 Following up Example 4.7, we get that the (scaled) Haar measure on the constant family of tori $T = N \otimes \mathbb{C}^{*}$ converges to the Lebesgue measure on $\mathbb{R}^{n}$. For any smooth projective toric compactification $Y$ of $T$ with boundary divisor $D$, the image of $\Delta(Y, D) \subset T(\mathbb{C})^{\text{an}}$ coincides with the image of $N_{\mathbb{R}} \hookrightarrow T(\mathbb{C})^{\text{an}}$ given by sending $\sum n_{i} \otimes r_{i} \in N_{\mathbb{R}}$ to the seminorm $|\sum_{j} a_{j} \chi^{m_{j}}| = \max_{j} \{|a_{j}|e^{-\sum_{i} r_{i}(m_{j}, n_{i})}\}$.

5.5 Convergence for general sub log canonical pairs $(X, B)$

In this subsection, we drop the assumption that $(X, B)$ is log-smooth and prove Theorem A in general.

Suppose that $(X, B)$ is a projective pair over $\mathbb{D}^{*}$ such that $(X, B)$ is a sub log canonical and log Calabi–Yau pair. Here, sub log canonical means that $(X, B)$ is log canonical in the sense of the minimal model program (i.e. $\text{discrep}(X, B) \geq -1$) but we are not necessarily assuming that $B$ is effective. Let $\eta \in H^{0}(X, m(K_{X/\mathbb{D}^{*}} + B))$ be a generating section that admits a meromorphic extension.
Let $\pi : (Y, B') \to (X, B)$ be a log resolution of singularities. Here, $B'$ is the divisor supported on the exceptional locus and the preimage of $B$ such that $K_Y + B' = \pi^*(K_X + B) \sim_\mathbb{Q} 0$. Moreover, $\pi$ gives an isomorphism $Y \setminus |B'| \cong \mathcal{X}^\text{reg} \setminus |B|$.

Since $(X, B)$ is sub log canonical, all the coefficients that show up in $B'$ are at most 1. Thus, the pair $(Y, B')$ is log-smooth, sub log canonical and log Calabi–Yau. Let $\eta' \in H^0(Y, m(K_Y + B'))$ denote the section $\eta' = \pi^*(\eta)$. Applying Theorem B to $Y$, we get that there exist $\kappa_{\min} \in \mathbb{Q}, d \in \mathbb{N}_+$ such that the measures $\mu'_t = \frac{i^n^2(\eta'_t \wedge \eta'_t)}{(2\pi |t|^{-1})^{d|t|^{2\kappa_{\min}}}}$ converge weakly to a measure $\mu'_0$ on the space $(Y, B')^\text{hyb}$ for any $0 < r < 1$.

Note that the map $\pi^\text{An}_{A_r} : (Y \setminus |B'|)^\text{An}_{A_r} \to (X^\text{reg} \setminus |B|)^\text{An}_{A_r}$ is a homeomorphism as the restriction of $\pi$ to $Y \setminus |B'|$ is an isomorphism. Taking $\lim_{t \to 0, r \leq 1}$, we get a homeomorphism $(Y, |B'|)^\text{hyb} \simeq (X, |B|)^\text{hyb}$. Then, it follows from the change of variables formula that $\mu_t := (\pi^\text{An}_{A_r})_*(\mu'_t) = \frac{i^{n^2}(\eta_t \wedge \eta_t)}{(2\pi |t|^{-1})^{d|t|^{2\kappa_{\min}}}}$. Since the pushforward of Radon measures under a continuous map commutes with weak limits, it follows that $\mu_t \to (\pi^\text{An}_{A_r})_*(\mu'_0)$, which finishes the proof of Theorem A.

Acknowledgements I thank my advisor, Mattias Jonsson, for suggesting this problem, and also for his support and guidance. I also thank the anonymous referee for their comments and suggestions. This work was supported by the NSF grants DMS-1600011 and DMS-1900025.

References

1. Baker, M., Payne, S., Rabinoff, J.: On the structure of non-Archimedean analytic curves. In: Tropical and Non-Archimedean Geometry, Contemp. Math., vol. 605, pp. 93–121. Amer. Math. Soc., Providence (2013). https://doi.org/10.1090/conm/605/12113
2. Baker, M., Payne, S., Rabinoff, J.: Non-archimedean geometry, tropicalization, and metrics on curves. Algebr. Geom. 3(1), 63–105 (2016). https://doi.org/10.14231/AG-2016-004
3. Batyrev, V.V.: Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. J. Algebr. Geom. 3(1), 493–535 (1994)
4. Berkovich, V.G.: Spectral theory and analytic geometry over non-Archimedean fields, Mathematical Surveys and Monographs, vol. 33. American Mathematical Society, Providence (1990). https://doi.org/10.1090/surv/033
5. Berkovich, V.G.: Étale cohomology for non-Archimedean analytic spaces. Publ. Math. l’IHÉS 78, 5–161 (1993). https://doi.org/10.1007/BF02712916
6. Berkovich, V.G.: Smooth p-adic analytic spaces are locally contractible. Invent. Math. 137, 1–84 (1999). https://doi.org/10.1007/s002220050323
7. Berkovich, V.G.: Smooth p-adic analytic spaces are locally contractible. II. In: Geometric Aspects of Dwork Theory, vol. I, II, pp. 293–370. Walter de Gruyter, Berlin (2004)
8. Berkovich, V.G.: A non-Archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures. In: Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin, vol. I. Progr. Math., vol. 269, pp. 49–67. Birkhäuser Boston, Inc., Boston (2009). https://doi.org/10.1007/978-0-8176-4745-2_2
9. Boucksom, S., Favre, C., Jonsson, M.: Singular semipositive metrics in non-Archimedean geometry. J. Algebr. Geom. 25(1), 77–139 (2016). https://doi.org/10.1090/jag/656
10. Boucksom, S., Jonsson, M.: Tropical and non-Archimedean limits of degenerating families of volume forms. Journal de l’École polytechnique-Mathématiques 4, 87–139 (2017). https://doi.org/10.5802/jep.39
11. Brown, M.V., Mazzon, E.: The essential skeleton of a product of degenerations. Compos. Math. 155(7), 1259–1300 (2019). https://doi.org/10.1112/s0010437x19007346
12. Chambert-Loir, A., Ducros, A.: Formes différentielles réelles et courants sur les espaces de Berkovich (2012). arXiv:1204.6277
13. Clemens, C.H.: Degeneration of Kähler manifolds. Duke Math. J. 44(2), 215–290 (1977). http://projecteuclid.org/euclid.dmj/1077312231
14. Favre, C.: Degeneration of endomorphisms of the complex projective space in the hybrid space. J. Inst. Math. Jussieu, 1–43 (2017). https://doi.org/10.1017/s147474801800035x
15. Gubler, W., Rabinoff, J., Werner, A.: Skeletons and tropicalizations. Adv. Math. 294, 150–215 (2016). https://doi.org/10.1016/j.aim.2016.02.022
16. Jonsson, M., Nicaise, J.: Convergence of $p$-adic pluricanonical measures to Lebesgue measures on skeleta in Berkovich spaces. J. Écol. Polytech. Math. 7, 287–336 (2020). https://doi.org/10.5802/jep.118
17. Kollár, J., Mori, S.: Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998). https://doi.org/10.1017/CBO9780511662560
18. Kontsevich, M., Soibelman, Y.: Affine structures and non-Archimedean analytic spaces. In: The Unity of Mathematics, pp. 321–385. Springer, New York (2006). https://doi.org/10.1007/0-8176-4467-9_9
19. Mandel, T.: Classification of rank 2 cluster varieties. SIGMA Symmetry Integrability. Geom. Methods Appl. 15, Paper 042, 32 (2019). https://doi.org/10.3842/SIGMA.2019.042
20. Mustaţă, M., Nicaise, J.: Weight functions on non-Archimedean analytic spaces and the Kontsevich–Soibelman skeleton. Algebr. Geom. 2(3):365–404 (2015). https://doi.org/10.14231/AG-2015-016
21. Odaka, Y.: Tropical geometric compactification of moduli, II: $A_g$ case and holomorphic limits. International Mathematics Research Notices, 01 (2018). https://doi.org/10.1093/imrn/rnx293
22. Pille-Schneider, L.: Hybrid convergence of Kähler–Einstein measures (2019). arXiv:1911.03357
23. Temkin, M.: Metrization of differential pluriforms on Berkovich analytic spaces. In: Non-Archimedean and Tropical Geometry, Simons Symp., pp. 195–285. Springer, New York (2016)
24. Tyomkin, I.: Tropical geometry and correspondence theorems via toric stacks. Math. Ann. 353(3), 945–995 (2012). https://doi.org/10.1007/s00208-011-0702-z

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.