THE CHROMATIC NUMBER OF ALMOST STABLE KNESER HYPERGRAPHS

FRÉDÉRIC MEUNIER

Abstract. Let \( V(n, k, s) \) be the set of \( k \)-subsets of \([n]\) such that for all \( i, j \in S \), we have \(|i - j| \geq s\).

We define almost \( s \)-stable Kneser hypergraph \( KG^r([n])_{2\text{-stab}} \) to be the \( r \)-uniform hypergraph whose vertex set is \( V(n, k, s) \) and whose edges are the \( r \)-uples of disjoint elements of \( V(n, k, s) \).

With the help of a \( Z_p \)-Tucker lemma, we prove that, for \( p \) prime and for any \( n \geq kp \), the chromatic number of almost 2-stable Kneser hypergraphs \( KG^r([n])_{2\text{-stab}} \) is equal to the chromatic number of the usual Kneser hypergraphs \( KG^r([n]) \), namely that it is equal to \( \left\lfloor \frac{n-(k-1)r}{p-1} \right\rfloor \).

Defining \( \mu(r) \) to be the number of prime divisors of \( r \), counted with multiplicities, this result implies that the chromatic number of almost \( 2^{\mu(r)} \)-stable Kneser hypergraphs \( KG^r([n])_{2^{\mu(r)}\text{-stab}} \) is equal to the chromatic number of the usual Kneser hypergraphs \( KG^r([n]) \) for any \( n \geq kr \), namely that it is equal to \( \left\lfloor \frac{n-(k-1)r}{p-1} \right\rfloor \).

1. Introduction and main results

Let \([a]\) denote the set \( \{1, \ldots, a\} \). The Kneser graph \( KG^2([n])_k \) for integers \( n \geq 2k \) is defined as follows: its vertex set is the set of \( k \)-subsets of \([n]\) and two vertices are connected by an edge if they have an empty intersection.

Kneser conjectured [6] in 1955 that its chromatic number \( \chi \left( KG^2([n])_k \right) \) is equal to \( n - 2k + 2 \).

It was proved to be true by Lovász in 1979 in a famous paper [7], which is the first and one of the most spectacular application of algebraic topology in combinatorics.

Soon after this result, Schrijver [11] proved that the chromatic number remains the same when we consider the subgraph \( KG^2([n])_{2\text{-stab}} \) of \( KG^2([n])_k \) obtained by restricting the vertex set to the \( k \)-subsets that are 2-stable, that is, that do not contain two consecutive elements of \([n]\) (where 1 and \( n \) are considered to be also consecutive).

Let us recall that an hypergraph \( H \) is a set family \( H \subseteq 2^V \), with vertex set \( V \). An hypergraph is said to be \( r \)-uniform if all its edges \( S \in H \) have the same cardinality \( r \). A proper coloring with \( t \) colors of \( H \) is a map \( c : V \to [t] \) such that there is no monochromatic edge, that is such that in each edge there are two vertices \( i \) and \( j \) with \( c(i) \neq c(j) \). The smallest number \( t \) such that there exists such a proper coloring is called the chromatic number of \( H \) and denoted by \( \chi(H) \).

In 1986, solving a conjecture of Erdős [4], Alon, Frankl and Lovász [2] found the chromatic number of Kneser hypergraphs. The Kneser hypergraph \( KG^r([n])_k \) is a \( r \)-uniform hypergraph which has the \( k \)-subsets of \([n]\) as vertex set and whose edges are formed by the \( r \)-uple of disjoint \( k \)-subsets of \([n]\). Let \( n, k, r, t \) be positive integers such that \( n \geq (t-1)(r-1)+rk \). Then \( \chi \left( KG^r([n])_k \right) > t \). Combined with a lemma by Erdős giving an explicit proper coloring, it implies that \( \chi \left( KG^r([n])_k \right) = \left\lfloor \frac{n-(k-1)r}{t-1} \right\rfloor \).

The proof found by Alon, Frankl and Lovász used tools from algebraic topology.

In 2001, Ziegler gave a combinatorial proof of this theorem [13], which makes no use of homology, simplicial approximation,... He was inspired by a combinatorial proof of the Lovász theorem found by Matoušek [9]. A subset \( S \subseteq [n] \) is \( s \)-stable if any two of its elements are at least “at distance \( s \)
apart” on the $n$-cycle, that is, if $s \leq |i - j| \leq n - s$ for distinct $i, j \in S$. Define then $K^r\binom{n}{k}_{s,\text{stab}}$ as the hypergraph obtained by restricting the vertex set of $K^r\binom{n}{k}$ to the $s$-stable $k$-subsets. At the end of his paper, Ziegler made the supposition that the chromatic number of $K^r\binom{n}{k}_{r,\text{stab}}$ is equal to the chromatic number of $K^r\binom{n}{k}$ for any $n \geq kr$. This supposition generalizes both Schrijver’s theorem and the Alon-Frankl-Lovász theorem. Alon, Drewnowski and Lucsak make this supposition an explicit conjecture in [1].

**Conjecture 1.** Let $n, k, r$ be non-negative integers such that $n \geq rk$. Then

$$\chi\left(K^r\binom{n}{k}_{r,\text{stab}}\right) = \left\lceil \frac{n - (k - 1)r}{r - 1} \right\rceil.$$ 

We prove a weaker form of this statement, but which strengthens the Alon-Frankl-Lovász theorem. Let $V(n, k, s)$ be the set of $k$-subsets $S$ of $[n]$ such that for all $i, j \in S$, we have $|i - j| \geq s$ We define the almost $s$-stable Kneser hypergraphs $K^r\binom{n}{k}_{s,\text{stab}}$ to be the $r$-uniform hypergraph whose vertex set is $V(n, k, s)$ and whose edges are the $r$-uples of disjoint elements of $V(n, k, s)$.

**Theorem 1.** Let $p$ be a prime number and $n, k$ be non negative integers such that $n \geq pk$. We have

$$\chi\left(K^p\binom{n}{k}_{2,\text{stab}}\right) \geq \left\lceil \frac{n - (k - 1)p}{p - 1} \right\rceil.$$

Combined with the lemma by Erdős, we get that

$$\chi\left(K^p\binom{n}{k}_{2,\text{stab}}\right) = \left\lceil \frac{n - (k - 1)p}{p - 1} \right\rceil.$$

Moreover, we will see that it is then possible to derive the following corollary. Denote by $\mu(r)$ the number of prime divisors of $r$ counted with multiplicities. For instance, $\mu(6) = 2$ and $\mu(12) = 3$. We have

**Corollary 1.** Let $n, k, r$ be non-negative integers such that $n \geq rk$. We have

$$K^r\binom{n}{k}_{2p(r),\text{stab}} = \left\lceil \frac{n - (k - 1)r}{r - 1} \right\rceil.$$ 

2. **Notations and tools**

$Z_p = \{\omega, \omega^2, \ldots, \omega^p\}$ is the cyclic group of order $p$, with generator $\omega$.

We write $\sigma^{n-1}$ for the $(n - 1)$-dimensional simplex with vertex set $[n]$ and by $\sigma^{n-1}_{k-1}$ the $(k - 1)$-skeleton of this simplex, that is the set of faces of $\sigma^{n-1}$ having $k$ or less vertices.

If $A$ and $B$ are two sets, we write $A \uplus B$ for the set $(A \times \{1\}) \cup (B \times \{2\})$. For two simplicial complexes, $K$ and $L$, with vertex sets $V(K)$ and $V(L)$, we denote by $K \ast L$ the join of these two complexes, which is the simplicial complex having $V(K) \uplus V(L)$ as vertex set and

$$\{F \uplus G : F \in K, G \in L\}$$

as set of faces. We define also $K^n$ to be the join of $n$ disjoint copies of $K$.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$. We denote by $\operatorname{alt}(X)$ the size of the longest alternating subsequence of non-zero terms in $X$. A sequence $(j_1, j_2, \ldots, j_m)$ of elements of $Z_p$ is said to be alternating if any two consecutive terms are different. For instance (assume $p = 5$) $\operatorname{alt}(\omega^2, \omega^3, 0, \omega^4, 0, \omega^2) = 4$ and $\operatorname{alt}(\omega^2, \omega^4, \omega^1, 0, 0, \omega^4) = 2$.

Any element element $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$ can alternatively and without further mention be denoted by a $p$-uple $(X_1, \ldots, X_p)$ where $X_j := \{i \in [n] : x_i = \omega^j\}$. Note that the $X_j$ are then necessarily disjoint. For two elements $X, Y \in (Z_p \cup \{0\})^n$, we denote by $X \subseteq Y$ the fact
that for all $j \in [p]$ we have $X_j \subseteq Y_j$. When $X \subseteq Y$, note that the sequence of non-zero terms in $(x_1, \ldots, x_n)$ is a subsequence of $(y_1, \ldots, y_n)$.

The proof of Theorem 1 makes use of a variant of the $Z_p$-Tucker lemma by Ziegler [13].

**Lemma 1** ($Z_p$-Tucker lemma). Let $p$ be a prime, $n, m \geq 1$, $\alpha \leq m$ and let

$$
\lambda : (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\} \to Z_p \times [m]
$$

be a $Z_p$-equivariant map satisfying the following properties:

- for all $X^{(1)} \subseteq X^{(2)} \subseteq (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then $\lambda_1(X^{(1)}) = \lambda_1(X^{(2)})$;
- for all $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \subseteq (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1$, then the $\lambda_1(X^{(i)})$ are not pairwise distinct for $i = 1, \ldots, p$.

Then $\alpha + (m - \alpha)(p - 1) \geq n$.

We can alternatively say that $X \mapsto \lambda(X) = (\lambda_1(X), \lambda_2(X))$ is a $Z_p$-equivariant simplicial map from $\text{sd}(Z_p^n)$ to $\left( (Z_p^n)^* \ast (\sigma_{p-2}^{p-1})^{*m-\alpha}\right)$, where $\text{sd}(K)$ denotes the fist barycentric subdivision of a simplicial complex $K$.

**Proof of the $Z_p$-Tucker lemma.** According to Dold’s theorem [3, 8], if such a map $\lambda$ exists, the dimension of $\left( (Z_p^n)^* \ast (\sigma_{p-2}^{p-1})^{*m-\alpha}\right)$ is strictly larger than the connectivity of $Z_p^n$, that is $\alpha + (m - \alpha)(p - 1) > n - 2$.

It is also possible to give a purely combinatorial proof of this lemma through the generalized Ky Fan theorem from [5].

3. Proof of the main results

**Proof of Theorem 1.** We follow the scheme used by Ziegler in [13]. We endow $2^{[n]}$ with an arbitrary linear order $\preceq$.

Assume that $K^{(n)}_{2, \text{stab}}$ is properly colored with $C$ colors $\{1, \ldots, C\}$. For $S \in V(n, k, 2)$, we denote by $c(S)$ its color. Let $\alpha = p(k - 1)$ and $m = p(k - 1) + C$.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. We can write alternatively $X = (X_1, \ldots, X_p)$.

- if $\text{alt}(X) \leq p(k - 1)$, let $j$ be the index of the $X_j$ containing the smallest integer ($\omega^l$ is then the first non-zero term in $(x_1, \ldots, x_n)$), and define
  $$
  \lambda(X) := (j, \text{alt}(X)).
  $$

- if $\text{alt}(X) \geq p(k - 1) + 1$: in the longest alternating subsequence of non-zero terms of $X$, at least one of the elements of $Z_p$ appears at least $k$ times; hence, in at least one of the $X_j$ there is an element $S$ of $V(n, k, 2)$; choose the smallest such $S$ (according to $\preceq$). Let $j$ be such that $S \subseteq X_j$ and define
  $$
  \lambda(X) := (j, c(S) + p(k - 1)).
  $$

$\lambda$ is $Z_p$-equivariant map from $(Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$ to $Z_p \times [m]$.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \subseteq (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then the longest alternating subsequences of non-zero terms of $X^{(1)}$ and $X^{(2)}$ have same size. Clearly, the first non-zero terms of $X^{(1)}$ and $X^{(2)}$ are equal.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \subseteq (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1$, then for each $i \in [p]$ there is $S_i \in V(n, k, 2)$ and $j_i \in [p]$ such that we have $S_i \subseteq X^{(i)}_{j_i}$ and $\lambda_2(X^{(i)}) = c(S_i)$. If all $\lambda_1(X^{(i)})$ would be distinct, then it would mean that all $j_i$
would be distinct, which implies that the $S_i$ would be disjoint but colored with the same color, which is impossible since $c$ is a proper coloring.

We can thus apply the $Z_p$-Tucker lemma (Lemma 1) and conclude that $n \leq p(k-1) + C(p-1)$, that is

$$C \geq \left\lceil \frac{n - (k-1)p}{p-1} \right\rceil.$$

\[
\square
\]

To prove Corollary 1 we prove the following lemma, both statement and proof of which are inspired by Lemma 3.3 of [1].

**Lemma 2.** Let $r_1, r_2, s_1, s_2$ be non-negative integers $\geq 1$, and define $r = r_1 r_2$ and $s = s_1 s_2$.

Assume that for $i = 1, 2$ we have $\chi \left(KG^{r_i}([n])_{k, s_{stab}}\right) = \left\lceil \frac{n - (k-1)r_i}{r_i - 1} \right\rceil$ for all integers $n$ and $k$ such that $n \geq r_i k$.

Then we have $\chi \left(KG^{r}([n])_{k, s_{stab}}\right) = \left\lceil \frac{n - (k-1)r}{r - 1} \right\rceil$ for all integers $n$ and $k$ such that $n \geq rk$.

**Proof.** Let $n \geq (t-1)(r-1)+rk$. We have to prove that $\chi \left(KG^{r}([n])_{k, s_{stab}}\right) > t$. For a contradiction, assume that $KG^{r}([n])_{k, s_{stab}}$ is properly colored with $C \leq t$ colors. For $S \in V(n, k, p)$, we denote by $c(S)$ its color. We wish to prove that there are $S_1, \ldots, S_t$ disjoint elements of $V(n, k, s)$ with $c(S_1) = \ldots = c(S_t)$.

Take $A \in V(n, n_1, s_1)$, where $n_1 := r_1 k + (t-1)(r_1 - 1)$. Denote $a_1 < \ldots < a_{n_1}$ the elements of $A$ and define $h : V(n_1, k, s_2) \to [t]$ as follows: let $B \in V(n_1, k, s_2)$; the $k$-subset $S = \{a_i : i \in B\} \subseteq [n]$ is an element of $V(n, k, s)$, and gets as such a color $c(S)$; define $h(B)$ to be this color. Since $n_1 = r_1 k + (t-1)(r_1 - 1)$, there are $B_1, \ldots, B_{r_1}$ disjoint elements of $V(n_1, k, s_2)$ having the same color by $h$. Define $\hat{h}(A)$ to be this common color.

Make the same definition for all $A \in V(n, n_1, s_1)$. The map $\hat{h}$ is a coloring of $KG^{r}([n])_{k, s_{stab}}$ with $t$ colors. Now, note that

$$(t-1)(r-1)+rk = (t-1)(r_1 r_2 - r_2 + r_2 - 1) + r_1 r_2 k = (t-1)(r_2 - 1) + r_2(t-1)(r_1 - 1) + r_1 k$$

and thus that $n \geq (t-1)(r_2 - 1) + r_2 n_1$. Hence, there are $A_1, \ldots, A_{r_2}$ disjoint elements of $V(n, n_1, s_1)$ with the same color. Each of the $A_i$ gets its color from $r_1$ disjoint elements of $V(n, k, s)$, whence there are $r_1 r_2$ disjoint elements of $V(n, k, s)$ having the same color by the map $c$. \[
\square
\]

**Proof of Corollary 1.** Direct consequence of Theorem 1 and Lemma 2.

\[
\square
\]

4. SHORT COMBINATORIAL PROOF OF SCHRIJVER’S THEOREM

Recall that Schrijver’s theorem is

**Theorem 2.** Let $n \geq 2k$. $\chi \left(KG([n])_{k, 2_{stab}}\right) = n - 2k + 2$.

When specialized for $p = 2$, Theorem 1 does not imply Schrijver’s theorem since the vertex set is allowed to contain subsets with 1 and $n$ together. Anyway, by a slight modification of the proof, we can get a short combinatorial proof of Schrijver’s theorem. Alternative proofs of this kind – but not that short – have been proposed in [10, 13]

For a positive integer $n$, we write $\{+, -, 0\}^n$ for the set of all signed subsets of $[n]$, that is, the family of all pairs $(X^+, X^-)$ of disjoint subsets of $[n]$. Indeed, for $X \in \{+, -, 0\}^n$, we can define $X^+ := \{i \in [n] : X_i = +\}$ and analogously $X^-$. We define $X \subseteq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. By alt$(X)$ we denote the length of the longest alternating subsequence of non-zero signs in $X$. For instance: alt$(+0-+0-) = 4$, while alt$((-++-+0+-) = 5$. 

4
The proof makes use of the following well-known lemma see \cite{2, 12, 13} (which is a special case of Lemma 1 for \( p = 2 \)).

**Lemma 3** (Tucker’s lemma). Let \( \lambda : \{-,0,+,+\}^n \setminus \{(0,0,\ldots,0)\} \to \{-1,+1,\ldots,-n,+n\} \) be a map such that \( \lambda(-X) = -\lambda(X) \). Then there exist \( A, B \in \{-,0,+,+\}^n \) such that \( A \subseteq B \) and \( \lambda(A) = -\lambda(B) \).

**Proof of Schrijver’s theorem.** The inequality \( \chi(KG^2(\frac{n}{k})_{2-\text{stab}}) \leq n - 2k + 2 \) is easy to prove (with an explicit coloring) and well-known. So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.

Let us assume that there is a proper coloring \( c \) of \( KG^2(\frac{n}{k})_{2-\text{stab}} \) with \( n - 2k + 1 \) colors. We define the following map \( \lambda \) on \( \{-,0,+,+\}^n \setminus \{(0,0,\ldots,0)\} \).

- if \( \text{alt}(X) \leq 2k - 1 \), we define \( \lambda(X) = \pm \text{alt}(X) \), where the sign is determined by the first sign of the longest alternating subsequence of \( X \) (which is actually the first non zero term of \( X \)).
- if \( \text{alt}(X) \geq 2k \), then \( X^+ \) and \( X^- \) both contain a stable subset of \( [n] \) of size \( k \). Among all stable subsets of size \( k \) included in \( X^- \) and \( X^+ \), select the one having the smallest color. Call it \( S \). Then define \( \lambda(X) = \pm (c(S) + 2k - 1) \) where the sign indicates which of \( X^- \) or \( X^+ \) the subset \( S \) has been taken from. Note that \( c(S) \leq n - 2k \).

The fact that for any \( X \in \{-,0,+,+\}^n \setminus \{(0,0,\ldots,0)\} \) we have \( \lambda(-X) = -\lambda(X) \) is obvious. \( \lambda \) takes its values in \( \{-1,+1,\ldots,-n,+n\} \). Now let us take \( A \) and \( B \) as in Tucker’s lemma, with \( A \subseteq B \) and \( \lambda(A) = -\lambda(B) \). We cannot have \( \text{alt}(A) \leq 2k - 1 \) since otherwise we will have a longest alternating in \( B \) containing the one of \( A \), of same length but with a different sign. Hence \( \text{alt}(A) \geq 2k \). Assume w.l.o.g. that \( \lambda(A) \) is defined by a stable subset \( S_A \subseteq A^- \). Then the stable subset \( S_B \) defining \( \lambda(B) \) is such that \( S_B \subseteq B^+ \), which implies that \( S_A \cap S_B = \emptyset \). We have moreover \( c(S_A) = |\lambda(A)| = |\lambda(B)| = c(S_B) \), but this contradicts the fact that \( c \) is proper coloring of \( KG^2(\frac{n}{k})_{2-\text{stab}} \).

5. **Concluding remarks**

We have seen that one of the main ingredients is the notion of alternating sequence of elements in \( \mathbb{Z}_p \). Here, our notion only requires that such an alternating sequence must have \( x_i \neq x_{i+1} \). To prove Conjecture 1 we need probably something stronger. For example, a sequence is said to be alternating if any \( p \) consecutive terms are all distinct. Anyway, all our attempts to get something through this approach have failed.

Recall that Alon, Drewnowski and Luczak \cite{1} proved Conjecture 1 when \( r \) is a power of 2. With the help of a computer and \texttt{ipsolve}, we check that Conjecture 1 is moreover true for

- \( n \leq 9, \ k = 2, \ r = 3 \).
- \( n \leq 12, \ k = 3, \ r = 3 \).
- \( n \leq 14, \ k = 4, \ r = 3 \).
- \( n \leq 13, \ k = 2, \ r = 5 \).
- \( n \leq 16, \ k = 3, \ r = 5 \).
- \( n \leq 21, \ k = 4, \ r = 5 \).

**References**

1. N. Alon, L. Drewnowski, and T Luczak, *Stable Kneser hypergraphs and ideals in \( \mathbb{N} \) with the Nikodým property*, Proceedings of the American mathematical society 137 (2009), 467–471.
2. N. Alon, P. Frankl, and L. Lovász, *The chromatic number of Kneser hypergraphs*, Transactions Amer. Math. Soc. 298 (1986), 359–370.
3. A. Dold, *Simple proofs of some Borsuk-Ulam results*, Contemp. Math. 19 (1983), 65–69.
4. P. Erdős, *Problems and results in combinatorial analysis*, Colloquio Internazionale sulle Teorie Combinatorie (Rome 1973), Vol. II, No. 17 in Atti dei Convegni Lincei, 1976, pp. 3–17.

5. B. Hanke, R. Sanyal, C. Schultz, and G. Ziegler, *Combinatorial stokes formulas via minimal resolutions*, Journal of Combinatorial Theory, series A (to appear).

6. M. Kneser, *Aufgabe 360*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abteilung, vol. 50, 1955, p. 27.

7. L. Lovász, *Kneser's conjecture, chromatic number and homotopy*, Journal of Combinatorial Theory, Series A 25 (1978), 319–324.

8. J. Matoušek, *Using the Borsuk-Ulam theorem*, Springer Verlag, Berlin–Heidelberg–New York, 2003.

9. ———, *A combinatorial proof of Kneser's conjecture*, Combinatorica 24 (2004), 163–170.

10. F. Meunier, *Combinatorial Stokes formulae*, European Journal of Combinatorics 29 (2008), 286–297.

11. A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wiskd., III. Ser. 26 (1978), 454–461.

12. A. W. Tucker, *Some topological properties of disk and sphere*, Proceedings of the First Canadian Mathematical Congress, Montreal 1945, 1946, pp. 285–309.

13. G. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. 147 (2002), 671–691.

Université Paris Est, LVMT, ENPC, 6-8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-la-Vallée cedex 2, France.

E-mail address: frederic.meunier@enpc.fr