Regularization of ill-posed problems involving constant-coefficient pseudo-differential operators

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Abstract

This paper deals with the wavelet regularization for ill-posed problems involving linear constant-coefficient pseudo-differential operators. We concentrate on solving ill-posed equations involving these operators, which are behaving badly in theory and practice. Since a wide range of ill-posed and inverse problems in mathematical physics can be described and rewritten by the language of these operators, it has gathered significant attention in the literature. Based on a general framework, we classify ill-posed problems in terms of their degree of ill-posedness into mildly, moderately, and severely ill-posed problems in a certain Sobolev scale. Using wavelet multi-resolution approximations, it is shown that wavelet regularizers can achieve order-optimal rates of convergence for pseudo-differential operators in special Sobolev space both for the a priori and the a posteriori choice rules. Our strategy, however, turns out that both schemes yield comparable convergence rates. In this setting, ultimately, we provided some prototype examples for which our theoretical results correctly predict improved rates of convergence.

Keywords: pseudo-differential operators, degree of ill-posedness, Meyer wavelet, multi-resolution analysis, convergence rates

1. Introduction

The theory of pseudo-differential operators (ΨDOs) was launched by Kohn and Nirenberg around 1964. This theory has its roots in the theory of singular integrals and Fourier analysis.

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It was subsequently clarified and extended by many prominent mathematicians, notably Hörmander. In connection with the modern theory of differential equations, the theory of \( \Psi \)DOs plays a vital role and proposes a powerful and flexible way of applying Fourier analysis to the study of constant and variable-coefficient operators and singularities of distributions [12]. It turns out that the process of computing the \( \Psi \)DOs leads to an ill-posed problem and thereby provides a general framework for studying a much wider class of inverse and ill-posed problems. For instance, many classical and non-classical ill-posed problems can be classified by the language of \( \Psi \)DOs, including numerical differentiation [13], the Cauchy problems associated with the Laplace and Helmholtz equations [20, 32], ill-posed analytic continuation problem [10, 24], inverse and backward problems [7, 21–23, 33–35] and so on. During the last forty years, much progress has been made on numerical \( \Psi \)DOs including wavelet approximation methods, Fourier method and Galerkin methods. A suitable method that has turned out to be particularly successful in numerical experiments is the wavelet method. One central aim of this manuscript is to point out some of the underlying mathematical frameworks for this method.

Wavelet theory came out in the early 1980s as a new rigorous theory of mathematics for analyzing the non-stationary behavior of signals. Wavelets are just fashionable new waves that will soon come to rest. What has made the recent explosion of mathematical activities centered around wavelet theory are certainly a bandwagon effect, describing local features of physical phenomenon, and of course mathematical beauty. The potential of the wavelets will be one of the central themes of our discussions. Wavelet techniques are in some ways much more powerful tools than other conventional or classical techniques. It will be pointed out that the wavelet-based regularization scheme facilitates our analysis. During our analysis, we will take advantage of a certain kinds of band-limited wavelets called Meyer wavelets. They are easy to compute with fast decreasing property in the time domain and suitable for representing band-limited signals with sharp cut-off. These wavelets have compact support in the frequency domain as well as a closed form representation therein. In fact, they have good localization in the frequency domain but relatively poor localization in the time domain [5, 28]. This property enables us to filter high-frequency noise injected into the signal from destructing the true solution. Such features can be employed to study some ill-posed problems that are well-understood in the frequency domain. In contrast to Daubechies wavelets, the Meyer wavelets are of special importance, mainly because they have the so-called oversampling property for which the coefficients can be calculated by sampling procedures [41, 42]. Meyer wavelets are frequently used for studying a wide range of inverse and ill-posed problems including numerical analytic continuation problem [10, 24], Cauchy problem for the Helmholtz and Laplace equations [20, 32], backward problem in heat propagation [7, 21–23, 33–35], some inverse problems involving parabolic PDEs [26] and so on.

To the best of our knowledge, the first progress in connection with \( \Psi \)DO equations has been achieved by Dahmen \textit{et al} [3, 4] using wavelet approximation methods. In these papers, they comprehensively discussed and characterized stability and convergence of their method for the general case of variable symbols in terms of simple conditions on the Fourier transform of the generating refinable function. But in these papers, the estimates were established for the exactly given right-hand side of the operator equation, and the influence of the data errors was not investigated. This is, however, the substantial part when someone dealing with ill-posed problems. An introductory investigation on the ill-posed problems and \( \Psi \)DOs has been achieved in [16]. In the year 2009, Fu \textit{et al} [14] have proposed a Fourier regularization scheme for recovering the stability of numerical \( \Psi \)DOs. In the year 2016, Cheng \textit{et al} [2] have presented a wavelet regularization method to solve this problem. However, both methods were applied for one-dimensional \( \Psi \)DOs and took advantage of an \textit{a priori} approach.
All estimates were worked out in Hilbert space $L^2(\mathbb{R})$ which in some sense do not produce rigid qualitative analysis for ill-posed problems. Moreover, in the year 2017, Feng et al. [11] have introduced an a posteriori wavelet regularization scheme to attack the problem. The most recent article in connection with $\Psi$DOs is due to Hofmann et al. [27], where they studied the regularization analysis of linear ill-posed equations involving certain kinds of operators called multiplication operators in the case in which the multiplier function is positive almost everywhere and zero is an accumulation point of the range of this function. Our ultimate goal is a systematic analysis of multiscale techniques, the so-called wavelet techniques for the numerical $\Psi$DOs, which in particular includes rates of convergence for the solutions of corresponding problems.

During our investigation, it will be seen that the a priori wavelet regularization takes advantage of the regularization parameter depending on the solution smoothness and the level of noise. Although our strategy provides an order optimal rates of convergence, it asymptotically takes advantage of the minimal amount of information need to achieve the best possible error estimates. However dealing with practical problems one cannot use the a priori regularization methods, mainly because the choice of regularization parameter actually depends upon the a priori information and also the level of noise. Unfortunately, the a priori choice of information is in practice challenging, simply because the degree of smoothness is in general not precisely known. This fact motivates us to search for an a posteriori choice rule in which the regularization parameter no longer depends on the a priori information and is constructed during the algorithm. To derive some a posteriori rates of convergence we will link our strategy with the well-known Morozov’s discrepancy principle.

The main goal of this manuscript is to contribute to the general mathematical analysis of the wavelet method. Some prototype examples are also provided. We give mainly a general framework to obtain stability estimates. The greater part of numerical experiments can be found in the corresponding references.

The layout of this paper is as follows. In section 2, we set off our mathematical problem as a general framework for unifying some different classes of ill-posed problems into one package by means of pseudo-differential operators. Indeed, depending on how much is the degree of ill-posedness, we classify ill-posed problems into mildly, moderately, and severely ill-posed problems. In section 3, we describe Meyer wavelet systems and their general multiresolution analysis as powerful tool to study our mathematical problem. Moreover, we introduce some auxiliary lemmas which will be used for figuring out some theoretical consequences. In sections 4 and 5, we separately discuss the wavelet regularization analysis for classified ill-posed problems, deriving some order optimal rates of convergence both for the a priori and a posteriori parameter choice rules. Finally, we will represent some applicable prime examples to support our qualitative analysis in section 6.

2. Mathematical setting

Let $\mathcal{M}: X \to Y$ be a linear injective bounded operator between Hilbert spaces $X$ and $Y$ with non-closed range $R(\mathcal{M})$. We study the numerical solution of inverse problems formulated as linear ill-posed operator equations

$$\mathcal{M}f^\dagger = u^\dagger,$$

where $f^\dagger \in D(\mathcal{M}) \subset X$ is the unknown true solution and $u^\dagger \in R(\mathcal{M}) \subset Y$ is the exact data which is observed approximately by $u^{obs} \in Y$ and subsequently is controlled through the
following deterministic noise model
\[ \|u^1 - u^{\text{obs}}\|_Y \leq \delta. \]  
(2)

Here, \( \delta > 0 \) is a constant and stands for the level of noise. In order to give a qualitative picture, we will restrict our attention to constant coefficient \( \Psi \)DO operators between Sobolev spaces. Basically, we assume that the forward operator \( \mathcal{M} \) has a bounded inverse as a mapping between
\[ \mathcal{M} : \mathcal{H}^0(\mathbb{R}^N) \to \mathcal{H}^{\nu + 1}(\mathbb{R}^N), \quad \nu > 0, \]  
(3)

which means \( \mathcal{M} \) is a smoothing operator of the order \( \nu \) in the scale of Sobolev spaces. By \( \| \cdot \|_{\mathcal{H}^\nu} \) we denote the norm of Sobolev space \( \mathcal{H}^\nu(\mathbb{R}^N) \) with smoothness order \( \nu \in \mathbb{R} \) as
\[ \|u\|_{\mathcal{H}^\nu} = \left( \int_{\mathbb{R}^N} |(\mathcal{F}u)(\xi)|^2(1 + \|\xi\|^2)^q d\xi \right)^{1/2}, \]  
(4)

where \( \mathcal{F} : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N) \) is called the unitary Fourier operator on Schwartz space \( \mathcal{S}(\mathbb{R}^N) \). For some cross connections between the wavelets and general function spaces including ordinary Sobolev spaces, Besov spaces, Bessel potential spaces etc one can refer to [38, 39]. There are many well-known examples of type (3) including the Radon transform (with \( \nu = \frac{N-1}{2} \)), Symm’s operator (with \( \nu = 1 \)), and more popular \( \Psi \)DOs with the symbol \( m(\cdot) \). In this paper, we consider the constant-coefficient \( \Psi \)DOs described by the following form
\[ \mathcal{M}f(x) \equiv (M(D)f)(x) := \mathcal{F}^{-1} (m(\xi)(\mathcal{F}f)(\xi))(x), \quad \mathcal{M}(0) = 1, \]  
(5)

where \( D \) stands for differential operator and the complex-valued function \( m(\cdot) \in C^\infty(\mathbb{R}^N \setminus \{0\}) \) is called the symbol of order \( r \in \mathbb{R} \) satisfying the so-called Hörmander’s growth condition [3]
\[ |\partial^\alpha m(\xi)| \leq C_\alpha (1 + \|\xi\|^2)^{-r-|\alpha|}. \]  
(6)

for all multi-index \( \alpha \in \mathbb{N}_0^N \) and some constant \( C_\alpha \). In section 4, we shall confine our attention to functions \( m(\xi) \) that behave qualitatively like polynomials or homogeneous functions of \( \xi \) as \( \xi \to \infty \)—that is, they grow or decay like powers of \( \|\xi\| \), and differentiation in \( \xi \) lowers the order of growth. From (1) and (5) we have
\[ \mathcal{M}f(\xi) = \hat{u}(\xi), \quad \hat{\mathcal{M}f}(\xi) = m(\xi)\hat{f}(\xi), \]  
(7)

where for the sake of simplicity we used \( \hat{f}(\xi) := \mathcal{F}(f)(\xi) \). Obviously, when the function \( (m(\cdotp))^{-1} \) is unbounded the equation (7) behaves badly and the process of calculating the solution \( f \) being very complicated, mainly because the high frequency components in the given data \( u(\cdot) \) are uncontrollably amplified through the multiplication function \( m(\cdotp) \) and thereby it causes the blow-up in the solution. However, the equation (7) is ill-posed and naturally suffers from the so-called Hadamard instability. It will be pointed out that wide ranges of ill-posed problems can be formulated as equation (7). Indeed, different symbols, produce different kinds of ill-posed problems.

It is well-known that if \( A : X \to Y \) is a linear compact operator between Hilbert spaces \( X \) and \( Y \), then there is a non-increasing infinite sequence \( (\sigma(A))_{n \in \mathbb{N}} \) of positive singular values which is infinite and approaches to zero when the range \( R(A) \) of \( A \) is infinite dimensional. Then the first kind linear operator equation
is ill-posed and one can associate the nonnegative number $\nu$ as the degree of ill-posedness for such problems in the following sense \cite{18}

$$\nu := \sup \{ \mu : \sigma_j(A) = O(j^{-\mu}) \text{ as } j \to \infty \}.$$  \hfill (9)

Due to Wahba \cite{40}, the linear problem (8) is distinguished as the mildly ill-posed problem if $0 < \nu < 1$, the moderately ill-posed problem if $1 \leq \nu < \infty$, and the severely ill-posed problem if $\nu = \infty$ (i.e., no such $\mu$ exists such that $\sigma_j(A) = O(j^{-\mu})$ as $j \to \infty$). We know that the linear operator equation (8) is ill-posed if and only if $R(A) \neq R(A)$ \cite{8}. Note that $R(A) \neq R(A)$ cannot occur if dim($R(A)$) < $\infty$, simply because all finite dimensional subspaces are closed. Hence, the ill-posedness only occur if the $R(A)$ is an infinite dimensional subspace of $Y$. However, due to Nashed \cite{30}, not all ill-posed problems are related to compact operators and that the distinction between the ill-posedness of linear operator (8) is of type I and type II. In this sense, the linear ill-posed operator equation (8) is called of type I if $A$ is not compact and of type II if $A$ is compact. Note that the difference between the type I and type II of ill-posed problems does not reflect the fact that how much difficult it is to solve an ill-posed problem in a realistic situation. As the operator $M$ in equation (7) is not compact, the ill-posedness of its corresponding operator equation is of type I. Unfortunately, in this case it is not quite easy to characterize the degree of ill-posedness based on the approach (9) by a single constant $\nu$. However, we make a cross connection between these traditional approaches and our idea concerning the degree of ill-posedness.

A similar approach is mentioned in \cite{6} where a pair of characteristics including smoothing order and smoothing potential are introduced to quantify the smoothing properties of a convolution operator $A$. In this framework, a further approach based on both the smoothing properties of the operator $A$ and the solution smoothness for classifying linear ill-posed problems is used. In the sequel, motivated by the framework defined in \cite{6} we introduce a quantitative measure of the degree of ill-posedness to classify some different kinds of ill-posed problems which come out from the operator equation (7) in the following sense:

**Definition 1.** Let $M$ be a $\Psi$DO with the symbol $m(\cdot)$. Then the quantity $\mu(M)$ is called the degree of ill-posedness of the operator $M$ and is defined by

$$\mu(M) := \sup \{ \nu : |m(\xi)| = O(\|\xi\|^{-\nu}) \text{ as } \|\xi\| \to \infty \}.$$  \hfill (10)

Note that this quantity can also be measured as a lower limit as

$$\mu(M) = \liminf_{\|\xi\| \to \infty} \left( -\frac{\log |m(\xi)|}{\log \|\xi\|} \right).$$  \hfill (11)

The meaning of $\mu(M)$ is the slowest asymptotic decay rate of the symbol $m(\cdot)$ associated with the operator $M$ in the frequency space. Based on this definition, we present the following definition in which classifies the ill-posedness in three cases:

**Definition 2.** Let $M$ be a $\Psi$DO with symbol $m(\cdot)$. Then the ill-posedness of the operator equation (7) is called:

- **Case 1:** mildly ill-posed if $0 < \mu(M) < 1$.
- **Case 2:** moderately ill-posed if $1 \leq \mu(M) < +\infty$.
- **Case 3:** severely ill-posed if $\mu(M) = +\infty$. 

It can be inferred from this definition that the smoother the forward operator, the higher the degree of ill-posedness of associated problem. Thus, the smoother the symbol $m(\cdot)$, the greater losses of information in the solution process and errors in the solution.

**Remark 1.** In Hilbert scales there is another way to describe the quantitative measure of ill-posedness [8]. To be more precise, the operator equation $Af = u$ with bounded operator $A$ between Hilbert spaces $X$ and $Y$ is ill-posed with a degree $\nu > 0$ in the sense that there exists two constants $0 < c \leq C < \infty$ such that for all $q \in \mathbb{R}$, we have the following

$$c\|f\|_{H^q(\mathbb{R}^N)} \leq \|Af\|_{H^{q+\nu}(\mathbb{R}^N)} \leq C\|f\|_{H^q(\mathbb{R}^N)}, \quad \text{for all } f \in H^q(\mathbb{R}^N).$$

(12)

In this way, the corresponding operator equation is called *mildly* ill-posed if $0 < \nu < 1$, *moderately* ill-posed if $1 \leq \nu < \infty$, and *severely* ill-posed if $\nu = \infty$.

Our mathematical tool to treat these different ill-posed problems in one mathematical framework is the method based on the Meyer wavelet systems. The mathematical introductory to the Meyer wavelet systems is discussed in the next section.

### 3. Meyer wavelet systems

In late 1986, Meyer and Mallat recognized that the construction of different wavelet systems can elegantly be described by the so-called multi-resolution analysis (MRA) [28]. This is a general framework in which functions $f \in L^2(\mathbb{R}^N)$ can be traced as a limit of successive approximations where each one is a smoother version of the function $f$. A natural way to construct $N$-dimensional MRA is the tensor product [5, 25, 28, 43]. To work out this construction, we first introduce one-dimensional Meyer’s wavelet and scaling functions. The Meyer scaling function is described by the dilation equation as

$$\phi(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} \gamma_n \phi(2x - n).$$

(13)

It is also used to construct the mother wavelet of Meyer

$$\psi(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} d_n \phi(2x - n),$$

(14)

where $d_n = \gamma_{1-a}(-1)^n$. The most natural way to pass from the one-dimensional setting to the multi-dimensional setting is to use the tensor product. Using this general concept, the $N$-dimensional Meyer scaling function can be defined as

$$\Phi(x) = \prod_{k=1}^{N} \phi(x_k), \quad x \in \mathbb{R}^N.$$

(15)

Similarly, the corresponding Meyer wavelet function is defined by

$$\Psi(x) = \prod_{k=1}^{N} \psi(x_k), \quad x \in \mathbb{R}^N.$$

Let $V_J$ be the closed linear span defined by

$$V_J := \text{Span} \{ \Phi_{J,k}(\cdot) : \Phi_{J,k}(x) = 2^{N/2} \Phi(2^Jx - k), k \in \mathbb{Z}^N \}.$$

(16)
This space is also called the *scale space*. Similarly, we denote by $W_J$ the closed linear span, called the *detailspace*, and is defined by

$$W_J := \text{Span}\left\{ \Psi_{J,k}(\cdot) : \Psi(2^J \mathbf{x} - k), k \in \mathbb{Z}^N \right\}. \quad (17)$$

In frequency space, the Meyer scaling function $\Phi(\cdot)$ has compact support in the following form

$$\text{Supp} \hat{\Phi} = \left[ -\frac{4\pi}{3}, \frac{4\pi}{3} \right]^N, \quad (18)$$

and

$$\hat{f}(\xi) = 0 \quad \text{for} \quad \|\xi\|_\infty \leq \frac{2}{3} 2^J, \quad f \in W_J, \quad J \in \mathbb{N}. \quad (19)$$

The orthogonal projection operators $P_J : L^2(\mathbb{R}^N) \rightarrow V_J$ and $Q_J : L^2(\mathbb{R}^N) \rightarrow W_J$ are respectively defined by the following relations,

$$P_J f := \sum_{k \in \mathbb{Z}^N} \langle f, \Phi_{J,k} \rangle \Phi_{J,k}, \quad f \in L^2(\mathbb{R}^N), \quad (20)$$

and

$$Q_J f := \sum_{k \in \mathbb{Z}^N} \langle f, \Psi_{J,k} \rangle \Psi_{J,k}, \quad f \in L^2(\mathbb{R}^N), \quad (21)$$

where $\langle \cdot, \cdot \rangle$ indicates the $L^2$-inner product. The basic tenet of MRA is that whenever a set of nested closed subspaces like $V_J$ fulfills the principles of MRA, then there exists an orthonormal wavelet basis for $L^2(\mathbb{R}^N)$. To be more precise, the family of parametrized functions $\{\Psi_{J,k}(\mathbf{x}) := 2^{J/2} \Psi(2^J \mathbf{x} - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^N\}$ constitute an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^N)$. Furthermore the $\Psi_{J,k}(\mathbf{x})$ are entire functions, since their Fourier transforms have compact support. For $J \in \mathbb{N}$, suppose $W_J$ to be the orthogonal complement of $V_J$ in $V_{J+1}$. Then we have

$$V_{J+1} = V_J \oplus W_J, \quad \text{for all} \quad J \in \mathbb{Z}. $$

For the convenient purposes, we set

$$\Lambda_J := 2^J \left[ -\frac{2}{3} \pi, \frac{2}{3} \pi \right]^N. \quad (22)$$

Utilizing (19), for $J \in \mathbb{N}$, we get

$$\hat{P}_J f(\xi) = 0, \quad \text{for} \quad \xi \in \Gamma_{J+1}, \quad (23)$$

where $\Gamma_J := \mathbb{R}^N \setminus \Lambda_J$. Also, we have

$$(I - P_J) f(\xi) = \hat{Q}_J f(\xi), \quad \text{for} \quad \xi \in \Lambda_{J+1}. \quad (24)$$

Another important operator to develop our approach is $M_J : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ which is defined by

$$M_J f := (1 - \chi_J) \hat{f}, \quad J \in \mathbb{N}, \quad (25)$$
where \( \chi_J \) denotes the characteristic function of the cube \( \Lambda_J \). We also recall some useful properties as

\[
\langle f, \Psi \rangle = \langle \hat{f}, \hat{\Psi} \rangle = \langle (1 - \chi_J) \hat{f}, \hat{\Psi} \rangle = \langle M_J, \Psi \rangle, 
\]

\( Q_J = Q_J M_J \),

\[
I - P_J = (I - P_J) M_J. 
\]

Moreover, it can be seen that from equation (19) each function \( \Psi \in W_j \) holds

\[
\hat{\Psi}(\xi) = 0, \quad \xi \in \Lambda_J, \quad \text{for all } j \geq J. 
\]

**Definition 3 ([28, 43]).** For \( m \in \mathbb{N} \), a MR A \( \{V_j\}_{j \in \mathbb{Z}} \) is called \( m \)-regular, if there exists a function \( f \in V_0 \) such that

\[
|\partial^\kappa f(x)| \leq C_N (1 + \|x\|)^{-N}, \quad \forall N \in \mathbb{N}, 
\]

where the multi-index \( \kappa \) satisfies \( |\kappa| \leq m \) and \( C_N \) depends only on \( N \).

In what follows, we recall the well-known Jackson and Bernstein inequalities.

**Theorem 4 (Jackson’s inequality) ([17, 28]).** Let \( \{V_j\}_{j \in \mathbb{Z}} \) be an \( m \)-regular MR A. Then for \( J \in \mathbb{N} \) and \( \varphi \in H^q(\mathbb{R}^N) \), the following inequality holds:

\[
\|\varphi - P_J \varphi\|_{H^p} \leq C(p, q) 2^{-j(p-q)} \|\varphi\|_{H^p}, 
\]

where \( C = C(p, q) \) is a positive constant depending on real constants \( p, q \) satisfying \(-m < q < p < m\).

**Theorem 5 (Bernstein’s inequalities) ([17, 28]).** Let \( \{V_j\}_{j \in \mathbb{Z}} \) be an \( m \)-regular Meyer’s MR A, and suppose \(-m < q < q + r < m\). Then

(a) If \( \varphi \in V_0 \), then

\[
\|D_{x_i}^r \varphi\|_{H^q} \leq C_0 \|\varphi\|_{H^q}, \quad i = 1, \ldots, N, \quad r \in \mathbb{N}, 
\]

where \( D_{x_i} \) is the differential operator and \( C_0 \) is a positive constant.

(b) If \( f \in W_J, J \in \mathbb{N} \), then

\[
\|D_{x_i}^r \varphi\|_{H^q} \leq C_1 2^J \|\varphi\|_{H^q}, \quad i = 1, \ldots, N, \quad r \in \mathbb{N}, 
\]

where \( C_1 \) is a positive constant.

**Lemma 6 ([17]).** Suppose \( \{V_j\}_{j \in \mathbb{Z}} \) is a Meyer’s MR A. Then there exists a positive constant \( \tilde{C} \) such that for all \( q \in \mathbb{R}, J \in \mathbb{N} \) and \( \varphi \in V_J \), the following inequality is satisfied

\[
\|D_{x_i}^r \varphi\|_{H^q} \leq \tilde{C} 2^{(J-r)q} \|\varphi\|_{H^q}, \quad i = 1, \ldots, N, \quad r \in \mathbb{N}, 
\]

where \( \tilde{C} \) is a positive constant.

In the next section, we introduce our regularization strategy based on the *a priori* and the *a posteriori* choice rules. These rules are multi-level approaches in terms of wavelet systems.
4. Wavelet regularization analysis: mildly and moderately cases

In this section, we first analyze the cases mildly and moderately ill-posed problems in one scenario. Here, we give some theoretical consequences based on the a priori and a posteriori choice rules.

4.1. The a priori choice rule

Consider the pseudo-differential operator \( M : H^q(\mathbb{R}^N) \rightarrow H^{q+\nu}(\mathbb{R}^N) \) with the assignment

\[
Mf^\dagger = u^\dagger,
\]

where \( \nu > 0 \) is the degree of smoothness for the \( \Psi \)DO \( M \). We assume that the following conditions are hold:

C1 There exists \( q \leq 0 \) such that \( \| u^\dagger - u^{obs} \|_{H^q} \leq \delta \). This condition is replaced by \( \| u^\dagger - u^{obs} \|_{L^2} \leq \delta \) if \( q > 0 \). Since the observed data \( u^{obs} \) are generally in \( L^2(\mathbb{R}^N) \).

C2 There exists \( p > q \) such that \( -m < q < q + \nu < p < m \) and \( \| f^\dagger \|_{H^p} \leq M \), where \( M \) is a positive non-dimensional a priori bound. This condition sometimes is called the smoothness condition.

C3 Assume that the symbol of the operator \( M \) is continuous and there exist two positive constants \( \alpha \) and \( \beta \) such that

\[
\alpha \| \xi \|^{-\nu} \leq |m(\xi)| \leq \beta \| \xi \|^{-\nu},
\]

or

\[
\alpha (1 + \| \xi \|)^{-\nu} \leq |m(\xi)| \leq \beta \| \xi \|^{-\nu},
\]

where \( \nu > 0 \) and \( \xi \neq 0 \).

These conditions describe the standard setup for studying ill-posed problems. According to definition 1, \( \mu(M) = \nu \) for which \( 0 < \nu < 1 \) yields the mildly and for \( 1 \leq \nu < +\infty \) the moderately ill-posed case. Based on equation (7), we define the operator \( R := M^{-1} \) by \( Ru^\dagger := f^\dagger \). Then the following lemma shows that the solution of the pseudo-differential operator equation (35) in the case of mildly and moderately ill-posedness is stable in the sense that it depends continuously on the data within the Meyer MRA.

**Lemma 7 (Stability).** Let \( \{V_j\}_{j \in \mathbb{Z}} \) be an \( m \)-regular Meyer MRA. Then for all \( \varphi \in V_j \), with \( J \in \mathbb{N} \) the following stability estimate holds

\[
\| R\varphi \|_{H^q} \leq \frac{N}{\alpha} C 2^{J/\nu} \| \varphi \|_{H^q}, \quad q \in \mathbb{R},
\]

where \( \alpha \) and \( C \) are the same constant appeared in condition C3 and lemma 6, respectively.

**Proof.** For all \( \varphi \in V_j \), we have
\[ \| \mathcal{R} \varphi \|_{H^q} = \left( \int_{\mathbb{R}^N} |(\widehat{\mathcal{R} \varphi})(\xi)|^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2} \]

\[ \leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} \|\xi\|^{2\nu} |\hat{\varphi}(\xi)|^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2} \]

\[ \leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} |(i\xi)^{2\nu} \hat{\varphi}(\xi)|^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2} \]

\[ = \frac{1}{\alpha} \sum_{i=1}^{N} \| D_{\nu}^i \varphi \|_{H^q} \]

\[ \leq \frac{N}{\alpha} C 2^{J} \| \varphi \|_{H^q}, \quad (39) \]

where we have used the lemma 6. \( \Box \)

In what follows, we introduce a regularizer based on the Meyer wavelet as

\[ \mathcal{R}_J := \mathcal{P}_J \mathcal{R}_J, \quad (40) \]

where \( J \) stands for the regularization parameter. This parameter plays an essential role in recovering the regularized solution. It controls the compromise between stability and approximation. The next theorem provides an asymptotic order optimal error estimate for the solution of pseudo-differential operator equation (1).

**Theorem 8 (Convergence rate of the a priori rule).** Suppose that the conditions C1, C2 and C3 hold true. Then with

\[ J_{\text{opt}} := \left\lfloor \log_2 \left( \frac{(p-q)M}{\nu \delta} \right) \right\rfloor \quad (41) \]

we have

\[ \| \mathcal{R} u^\dagger - \mathcal{R}_{J_{\text{opt}}} u^\text{obs} \|_{H^q} = O \left( \delta^{\frac{p-q}{r+q}} M^{\frac{p-q}{r+q}} \right), \quad (42) \]

where \( u^\dagger \in V_{J_{\text{opt}}} \) and \( \left\lfloor a \right\rfloor \) denotes the largest integer not exceeding \( a \).

**Proof.** The fundamental or total error can be estimated as follows

\[ \| \mathcal{R} u^\dagger - \mathcal{R}_{J_{\text{opt}}} u^\text{obs} \|_{H^q} \leq \| \mathcal{R} u^\dagger - \mathcal{R}_J u^\dagger \|_{H^q} + \| \mathcal{R}_J u^\dagger - \mathcal{R}_{J_{\text{opt}}} u^\text{obs} \|_{H^q} \]

\[ = A1 + B1, \quad (43) \]

where \( A1 := \| \mathcal{R} u^\dagger - \mathcal{R}_J u^\dagger \|_{H^q} \) is called the approximation error and \( B1 := \| \mathcal{R}_J u^\dagger - \mathcal{R}_{J_{\text{opt}}} u^\text{obs} \|_{H^q} \) is called the propagated error. For quantity \( A1 \) we have

\[ A1 = \| \mathcal{R} u^\dagger - \mathcal{R}_J u^\dagger \|_{H^q} \leq \| \mathcal{R} u^\dagger - \mathcal{P}_J \mathcal{R} u^\dagger \|_{H^q} + \| \mathcal{P}_J \mathcal{R} u^\dagger - \mathcal{P}_J \mathcal{R}_J u^\dagger \|_{H^q} \]

\[ = A2 + B2, \quad (44) \]
where $A_2 := \|R u^d - \mathcal{P}_J R u^d\|_{\mathcal{H}^r}$ and $B_2 := \|\mathcal{P}_J R u^d - \mathcal{P}_J R \mathcal{P} J R u^d\|_{\mathcal{H}^r}$. Applying (28) and Jackson’s inequality, we have

$$
A_2 = \|R u^d - \mathcal{P}_J R u^d\|_{\mathcal{H}^r} = \|(I - \mathcal{P}_J)M_J R u^d\|_{\mathcal{H}^r} \\
\leq C 2^{-\mu p - \rho q} \|M_J R u^d\|_{\mathcal{H}^r} \\
= C 2^{-\mu p - \rho q} \|R u^d\|_{\mathcal{H}^r(\Gamma_J)} \\
= C 2^{-\mu p - \rho q} M.
$$

(45)

For quantity $B_2$ we have

$$
B_2 = \|\mathcal{P}_J R u^d - \mathcal{P}_J R \mathcal{P} J R u^d\|_{\mathcal{H}^r} \leq \|R (I - \mathcal{P}_J) u^d\|_{\mathcal{H}^r} \\
\leq \|R u^d\|_{\mathcal{H}^r(\Gamma_{J+1})} + \|R \mathcal{Q}_J u^d\|_{\mathcal{H}^r(\Lambda_{J+1})} \\
= A_3 + B_3,
$$

(46)

where $A_3 := \|R u^d\|_{\mathcal{H}^r(\Gamma_{J+1})}$ and $B_3 := \|R \mathcal{Q}_J u^d\|_{\mathcal{H}^r(\Lambda_{J+1})}$. Simple calculations reveal that

$$
A_3 = \|R u^d\|_{\mathcal{H}^r(\Gamma_{J+1})} \leq 2^{-\mu p - \rho q} M.
$$

(47)

Also, for quantity $B_3$ we get

$$
B_3 = \|R \mathcal{Q}_J u^d\|_{\mathcal{H}^r(\Lambda_{J+1})} \leq \frac{N}{\alpha} C 2^{2\nu} \|\mathcal{Q}_J u^d\|_{\mathcal{H}^r(\Lambda_{J+1})} \\
= \frac{N}{\alpha} C 2^{2\nu} \|\mathcal{Q}_J M_J u^d\|_{\mathcal{H}^r(\Gamma_J)} \\
\leq \frac{N}{\alpha} C 2^{2\nu} \|M_J u^d\|_{\mathcal{H}^r(\Gamma_J)} \\
\leq \frac{N}{\alpha} C 2^{2\nu} \|u^d\|_{\mathcal{H}^r(\Gamma_J)} \\
\leq N C \frac{\beta}{\alpha} 2^{-\mu p - \rho q} M.
$$

(48)

Combining $A_3$ and $B_3$ with $B_2$, we arrive at

$$
B_2 \leq \left(1 + N C \frac{\beta}{\alpha}\right) 2^{-\mu p - \rho q} M.
$$

(49)

Adding up quantities $A_1$ and $B_1$, one can estimate quantity $A_1$ as

$$
A_1 \leq \left(1 + C + N C \frac{\beta}{\alpha}\right) 2^{-\mu p - \rho q} M.
$$

(50)

Now, we turn back to quantity $B_1$. Using condition $C_1$ and lemma 6, we have

$$
B_1 = \|\mathcal{R}_J u^d - \mathcal{R}_J u^{\text{obs}}\|_{\mathcal{H}^r} \leq \frac{N}{\alpha} C 2^{2\nu} \delta.
$$

(51)

Consequently, the fundamental error can be estimated as
\[
\| \mathcal{R}u^\dagger - \mathcal{R}_{\text{opt}}u^\text{obs} \|_{\mathcal{H}^q} \leq \frac{N}{\alpha} \bar{C} 2^{J^*} \delta + \left(1 + C + N \bar{C} \frac{\beta}{\alpha} \right) 2^{-J(p-q)M} \\
= \bar{C} \left(2^{J^*} \delta + 2^{-J(p-q)M} \right),
\]

where \( \bar{C} := \max \left\{ \frac{\beta}{\alpha}, 1 + C + N \bar{C} \frac{\beta}{\alpha} \right\} \). Now, it is necessary to minimize the right-hand side of inequality (52). To this end, we first suppose that \( J \in \mathbb{R} \) and setting

\[
\vartheta(J) := 2^{J^*} \delta + 2^{-J(p-q)M}.
\]

It is clear that \( J_{\text{opt}} := J_{\text{min}} = \log_2 \left( \frac{(p-q)M}{\nu \delta} \right)^{-\frac{1}{p-q}} \) is a minimizer for the function \( \vartheta(\cdot) \). As \( J \in \mathbb{N} \) we take

\[
J_{\text{opt}} := \left\lfloor \log_2 \left( \frac{(p-q)M}{\nu \delta} \right)^{-\frac{1}{p-q}} \right\rfloor.
\]

It follows,

\[
J_{\text{opt}} \leq \log_2 \left( \frac{(p-q)M}{\nu \delta} \right)^{-\frac{1}{p-q}} < J_{\text{opt}} + 1.
\]

Therefore,

\[
\vartheta(J_{\text{opt}}) = 2^{J_{\text{opt}}^*} \delta + 2^{-J_{\text{opt}}(p-q)M} \\
= 2^{J_{\text{opt}}^*} \delta + 2^{J_{\text{opt}}+1}(p-q)2^{-p-q}M \\
\leq \delta \left( \frac{(p-q)M}{\nu \delta} \right)^{-\frac{1}{p-q}} \nu^{p-q}M \left( \frac{(p-q)M}{\nu \delta} \right)^{-\frac{1}{p-q}} \\
= \delta M^{-\frac{1}{p-q}} \nu^{p-q} \left\{ \left( \frac{p-q}{\nu} \right)^{-\frac{1}{p-q}} + 2^{-p-q} \left( \frac{p-q}{\nu} \right)^{-\frac{1}{p-q}} \right\} \\
= C(p,q,\nu) \delta M^{-\frac{1}{p-q}} \nu^{p-q},
\]

where \( C(p,q,\nu) := \left( \frac{\nu^2}{\nu} \right)^{-\frac{1}{p-q}} + 2^{-p-q} \left( \frac{\nu^2}{\nu} \right)^{-\frac{1}{p-q}} \nu^{p-q} \). Eventually, setting \( C^1 := C(p,q,\nu) \) we get the desired result. \( \square \)

As mentioned in condition C1, in the case \( q > 0 \) it is not consistent to demand \( \| u^\dagger - u^\text{obs} \|_{\mathcal{H}^q} \leq \delta \), simply because the measured data \( u^\text{obs} \) are belong to \( L^2(\mathbb{R}^N) \). Therefore, from theorem 8 we have the following consequence

**Corollary 9.** Let the conditions C1, C2 and C3 be fulfilled for \( 0 < \nu < p < m \). Then with

\[
J_{\text{opt}} := \left\lfloor \log_2 \left( \frac{pM}{\nu \delta} \right)^{-\frac{1}{p-q}} \right\rfloor
\]

we have

\[
\| \mathcal{R}u^\dagger - \mathcal{R}_{\text{opt}}u^\text{obs} \|_{L^2} = O \left( \delta M^{-\frac{1}{p-q}} \nu^{p-q} \right),
\]

where \( u^\dagger \in V_{\text{opt}} \) and \( \lfloor a \rfloor \) denotes the largest integer not exceeding \( a \).
This corollary states that when \( q = 0 \), then \( p \) must be positive which in turn is equivalent to the raise of demand for the smoothness degree of the exact solution \( f' \). In fact, the larger \( p \) is, the more restrictive is the condition C3.

**Remark 2.** It is important to estimate \( \| R u^I - R_{J_0} u^{ab} \|_{H^q} \) in the \( H^q \)-scale for \( \tilde{q} \leq q \). But in this case we obtain nothing new as \( \| u^I - u^{ab} \|_{H^q} \leq \| u^I - u^{ab} \|_{H^\tilde{q}} \leq \delta \).

**Remark 3.** If \( 0 < q < \frac{N}{2} \) and \( q \geq \frac{N}{2} - \frac{n}{p} \) for \( 2 < p < \infty \), then it follows from the continuity of the embedding \( H^q(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \) that under the assumption of theorem 8 we also have an error bound with respect to the \( L^p \)-norm

\[
\| R u^I - R_{J_0} u^{ab} \|_{L^p} = O \left( \frac{\delta^{\frac{q}{p} - \tilde{q}}}{\beta^{\frac{q}{p} - \tilde{q}} M^{\frac{n}{p} - \tilde{q}}} \right).
\] (59)

### 4.2. The a posteriori choice rule

As is known from the previous subsection, the regularization parameter generally depends upon the noise level and the a priori information. But precisely determining the a priori information is generally impossible in practice. Thus processing data with wrong a priori information will result in an unwanted regularized solution. In this subsection, we will introduce the a posteriori approach where the regularization parameter no longer depends on the a priori information and the solution smoothness but in turn is certainly constructed during the algorithm. The structure of this approach is based on the Morozov discrepancy principle. Based on this principle, we first introduce the following consequence.

**Lemma 10.** Suppose that the conditions C1, C2, and C3 hold true. Moreover, suppose that the regularization parameter, \( J \), is selected such that

\[
\| (I - P_J) u^{ab} \|_{H^q} \leq \theta \delta \leq \| (I - P_{J-1}) u^{ab} \|_{H^q},
\] (60)

where \( \theta > 1 \) is a constant. Then there holds the following inequality

\[
2^{J(p-q+\nu)} \leq \frac{2\beta M}{\delta(\theta - 1)}.
\] (61)

**Proof.** It follows from the triangle inequality

\[
\| (I - P_{J-1}) u^I \|_{H^q} \leq \| u^I \|_{H^q(I_{J-1})} + \| Q_{J-1} u^I \|_{H^q(I_{J-1})}
= N1 + N2,
\] (62)

where \( N1 := \| u^I \|_{H^q(I_{J-1})} \) and \( N2 := \| Q_{J-1} u^I \|_{H^q(I_{J-1})} \). These quantities can separately be evaluated as

\[
N1 = \left( \int_{I_{J-1}} \left| m(\xi) f^I(\xi) \right|^2 (1 + \| \xi \|^2) d\xi \right)^{1/2}
\leq \beta \sup_{\xi \in I_{J-1}} \| \xi \|^{-(p-q+\nu)} \left( \int_{I_{J-1}} \left| f^I(\xi) \right|^2 (1 + \| \xi \|^2) d\xi \right)^{1/2}
\leq \beta M 2^{-J(p-q+\nu)}.
\] (63)
From (27), it follows
\[
N^2 = \| Q_{JJ}^{-1} u^\dagger \|_{H^q(\Omega_{J,-1})} = \| Q_{JJ}^{-1} M_{JJ}^{-1} u^\dagger \|_{H^q(\Omega_{J,-1})} \\
= \| M_{JJ}^{-1} u^\dagger \|_{H^q(\Omega_{J,-1})} \\
\leq \| u^\dagger \|_{H^q(\Omega_{J,-1})} \\
\leq \beta M^2 \cdot J^{p-q+\nu},
\]
(64)

Therefore,
\[
\| (I - P_{JJ}) u^\dagger \|_{H^q} \leq 2 \beta M^2 \cdot J^{p-q+\nu}. \tag{65}
\]

On the other hand,
\[
\| (I - P_{JJ}) u^\dagger \|_{H^q} \geq \| (I - P_{JJ}) u^{\text{obs}} \|_{H^q} - \| (I - P_{JJ}) (u^\dagger - u^{\text{obs}}) \|_{H^q} \\
\geq (\theta - 1) \delta. \tag{66}
\]

Combining (65) with (66), the desired consequence is derived. \(\square\)

Note that according to [8] in the \textit{a posteriori} choice rule, the \textit{a priori} information bound \(M\) is no longer required for the choice of the regularization parameter. It is just required for the theoretical analysis of the convergence rates of the regularized solution. The next theorem is the main result and provides the \textit{a posteriori} convergence rate of the Hölder-type.

**Theorem 11 (Convergence rate of the \textit{a posteriori} rule).** Suppose that the conditions C1, C2, and C3 are hold. Moreover, suppose that the regularization parameter, \(J\), is selected such that
\[
\| (I - P_J) u^{\text{obs}} \|_{H^p} \leq \theta \delta \leq \| (I - P_{JJ}) u^{\text{obs}} \|_{H^p}, \tag{67}
\]
where \(\theta > 1\) is a constant. Then there holds
\[
\| R u^\dagger - R_{I_{JJ}} u^{\text{obs}} \|_{H^p} = \mathcal{O} \left( \delta^{\frac{p-q+\nu}{p-\nu+\nu}} M^{\frac{p-q+\nu}{p-\nu+\nu}} \right). \tag{68}
\]

**Proof.** From triangle inequality we have
\[
\| R u^\dagger - R_{I_{JJ}} u^{\text{obs}} \|_{H^p} \leq \| R u^\dagger - R_{JJ} u^\dagger \|_{H^p} + \| R_{JJ} u^\dagger - R_{I_{JJ}} u^{\text{obs}} \|_{H^p} \\
= N^3 + N^4, \tag{69}
\]
where \(N^3 := \| R u^\dagger - R_{JJ} u^\dagger \|_{H^p}\) and \(N^4 := \| R_{JJ} u^\dagger - R_{I_{JJ}} u^{\text{obs}} \|_{H^p}\). From lemma 6 and (61), we have
Using Hölder inequality quantity

\[ N4 = \| R J u^1 - R J u^{\text{obs}} \|_{H^\vartheta} \leq \| R \mathcal{P}_J (u^1 - u^{\text{obs}}) \|_{H^\vartheta} \]
\[ \leq \frac{N}{\alpha} C 2^{\nu \vartheta} \| \mathcal{P}_J (u^1 - u^{\text{obs}}) \|_{H^\vartheta} \]
\[ \leq \frac{N}{\alpha} C (2^{\nu \vartheta - q + \nu}) \frac{1}{\delta^{\vartheta - 1}} \]
\[ = \frac{N}{\alpha} C \left( \frac{2\beta}{\vartheta} \right)^{\frac{1}{\vartheta - 1}} \delta^{\nu \vartheta - q + \nu} M^{\frac{1}{\vartheta - 1}}. \]  

For quantity \( N3 \), we have

\[ N3 = \| R u^1 - R J u^1 \|_{H^\vartheta} \leq \| (I - \mathcal{P}_J) R u^1 \|_{H^\vartheta} + \| R (I - \mathcal{P}_J) u^1 \|_{H^\vartheta} \]
\[ =: N5 + N6, \]  

where \( N5 := \| (I - \mathcal{P}_J) R u^1 \|_{H^\vartheta} \) and \( N6 = \| R (I - \mathcal{P}_J) u^1 \|_{H^\vartheta} \). Using (28) and Jackson’s inequality, we have

\[ N5 = \| (I - \mathcal{P}_J) R u^1 \|_{H^\vartheta} = \| (I - \mathcal{P}_J) M J R u^1 \|_{H^\vartheta} \]
\[ \leq C 2^{-J(p - q)} \| M J R u^1 \|_{H^\vartheta} \]
\[ = C 2^{-J(p - q)} \| R u^1 \|_{H^\vartheta(I, \gamma)} \]
\[ = C 2^{-J(p - q)} M. \]  

If we take \( J = \left[ \log_2 \left( \frac{2\beta}{\delta(\theta - 1)} \right) \right] \), then after some calculations quantity \( N5 \) can be estimated as

\[ N5 \leq C 2^{p - q} \left( \frac{2\beta}{\theta - 1} \right)^{-\frac{1}{\vartheta - 1}} \delta^{\nu \vartheta q} M^{\frac{1}{\vartheta - 1}}. \]  

Using Hölder inequality quantity \( N6 \) is estimated as

\[ N6 = \| R (I - \mathcal{P}_J) u^1 \|_{H^\vartheta} \]
\[ = \left( \int_{\mathbb{R}^N} |m(\xi)|^{-1} ((I - \mathcal{P}_J) u^1) (\xi) (1 + |\xi|^2)^{\nu/2} \right)^{1/2} \]
\[ \leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} |\xi|^\nu ((I - \mathcal{P}_J) u^1) (\xi) (1 + |\xi|^2)^{\nu/2} \right)^{1/2} \]
\[ = \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} |\xi|^{\nu q + \nu} ((I - \mathcal{P}_J) u^1) (\xi) (1 + |\xi|^2)^{\nu/2} \right)^{1/2} \]
\[ \times \left( (I - \mathcal{P}_J) u^1) (\xi) (1 + |\xi|^2)^{\nu/2} \right)^{1/2} \]
\[ \leq \frac{2}{\alpha} (p - q) \delta^{\nu q + \nu} M^{\frac{1}{\vartheta - 1}} \].
The equation
\[ \leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^d} \frac{||\xi||^{p-q+\nu_0}}{m(\xi)} (1 + ||\xi||^2)^{\eta/2} \right)^{1/2} \]
\[ \times \left| \left( I - \mathcal{P}_J \right) u^1 \right| (\xi) (1 + ||\xi||^2)^{\eta/2} \frac{2\nu_0}{p-q} \right)^{1/2} \]
\[ \leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^d} \frac{\beta}{m(\xi)} ||\xi||^{p-q+\nu_0} (1 + ||\xi||^2)^{\eta/2} \right)^{1/2} \]
\[ \times \left| \left( I - \mathcal{P}_J \right) u^1 \right| (\xi) (1 + ||\xi||^2)^{\eta/2} \frac{2\nu_0}{p-q} \right)^{1/2} \]
\[ \leq \frac{\beta}{\alpha} M^{\frac{p-q}{p-q+\nu_0}} \left( \left| \left( I - \mathcal{P}_J \right) u^1 \right| \right)^{\frac{p-q}{p-q+\nu_0}}. \] (74)

It can readily be seen that
\[ \left| \left( I - \mathcal{P}_J \right) u^1 \right|_{H^\nu} \leq \left| \left( I - \mathcal{P}_J \right) u^\text{obs} \right|_{H^\nu} + \left| \left( I - \mathcal{P}_J \right) (u^1 - u^\text{obs}) \right|_{H^\nu} \leq (\theta + 1)\delta. \] (75)

Therefore,
\[ N6 \leq \frac{\beta}{\alpha} M^{\frac{p-q}{p-q+\nu_0}} (\theta + 1)\delta. \] (76)

Plugging (73) and (76) into (71), we obtain
\[ N3 \leq \left\{ C 2^{p-q} \left( \frac{2\beta}{\theta - 1} \right)^{-\frac{\nu_0}{p-q+\nu_0}} + \frac{\beta}{\alpha} (\theta + 1)^{\frac{p-q}{p-q+\nu_0}} \right\} \delta^{\frac{p-q}{p-q+\nu_0}} M^{\frac{p-q}{p-q+\nu_0}}. \] (77)

Combining now quantities (70) and (77) with (69), we ultimately get
\[ \left| \mathcal{R}u^1 - \mathcal{R}_J u^\text{obs} \right|_{H^\nu} \leq C^* (p, q, \nu, \alpha, \beta, \theta, \overline{C}) \delta^{\frac{\nu_0}{p-q+\nu_0}} M^{\frac{p-q}{p-q+\nu_0}}, \] (78)

where \( C^* := C(p, q, \nu, \alpha, \beta, \theta, \overline{C}) = \left( C 2^{p-q} + \frac{2\beta}{\theta - 1} \right) \left( \frac{2\beta}{\theta - 1} \right)^{-\frac{\nu_0}{p-q+\nu_0}} + \frac{\beta}{\alpha} \left( \theta + 1 \right)^{\frac{p-q}{p-q+\nu_0}}. \)

This completes the proof. \( \square \)

**Remark 4.** Obviously, the function \( J \mapsto \left| \left( I - \mathcal{P}_J \right) u^\text{obs} \right|_{H^\nu} \) is continuous and decreasing and satisfies
\[ \lim_{J \to \infty} \left| \left( I - \mathcal{P}_J \right) u^\text{obs} \right|_{H^\nu} = 0 \] (79)

and
\[ \lim_{J \to \infty} \left| \left( I - \mathcal{P}_J \right) u^\text{obs} \right|_{H^\nu} = ||u^\text{obs}||_{H^\nu}, \quad q \leq 0. \] (80)

The equation \( \left| \left( I - \mathcal{P}_J \right) u^\text{obs} \right|_{H^\nu} = \theta \delta \) for \( \theta > 1 \) is uniquely solvable, provided \( ||u - u^\text{obs}||_{H^\nu} \leq \delta < ||u^\text{obs}||_{H^\nu} \) with \( q \leq 0 \). In practice, the condition \( ||u^\text{obs}||_{H^\nu} > \delta \) certainly makes sense since otherwise the right-hand side would be less than the noise level \( \delta \), and \( \mathcal{R}_J u^\text{obs} = 0 \) would be
an acceptable approximation to $Ra$ [8]. However, if the equation has multiple solutions, we will understand $J$ as its minimal solution.

5. Wavelet regularization analysis: severely ill-posed case

In this section, we present both the a priori and a posteriori wavelet regularization analysis for solving the pseudo-differential operator equation in the severely ill-posed case.

5.1. The a priori choice rule

For $\sigma > 0$, consider the problem of solving pseudo-differential operator equation associated with the parameter $y \in [0, \sigma]$ by

$$M_y f_y^\dagger = u^\dagger,$$  \hspace{1cm} (81)

where $M_y : H^q(\mathbb{R}^N) \rightarrow H^q(\mathbb{R}^N)$ and in terms of the Fourier transform is described by

$$M_y f_y^\dagger(\xi) \equiv \{M_y (D f_y^\dagger)(\xi)\}(\xi).$$  \hspace{1cm} (82)

Here, $D$ stands for differential operator, and $m_y(\cdot) : \mathbb{R}^N \rightarrow \mathbb{C}$ is called the symbol of $M_y$. This symbol will behave like an exponential function as $\|\xi\| \rightarrow \infty$. However, the situations where the severely and exponentially ill-posed problems can play a role are not the same and do not describe a similar concept. For more information about the exponentially ill-posed problems one can see [19]. We assume that the following conditions hold true:

H1 There exists $q \leq 0$ such that $\|u^\dagger - u^{obs}\|_{H^q} \leq \delta$. This condition is later replaced by $\|u^\dagger - u^{obs}\|_{L^2} \leq \delta$ if $q > 0$. Since the observed data $u^{obs}$ are generally in $L^2(\mathbb{R}^N)$.

H2 There exists $p > q$ such that $-m < q < p < m$ and $\|f_y^\dagger\|_{H^p} \leq M$, where $M$ is a positive non-dimensional a priori bound.

H3 There exists two positive constants $\alpha$ and $\beta$ such that

$$\alpha \exp(-\gamma y\|\xi\|^\nu) \leq |m_y(\xi)| \leq \beta \exp(-\gamma y\|\xi\|^\nu),$$  \hspace{1cm} (83)

where $\gamma$ and $\nu$ are two positive parameters.

Note that in the case of severely ill-posed problems the symbol $m_y(\cdot)$ cannot belong to Hörmander’s symbol class (see e.g., [3]), mainly because the symbol $m_y(\cdot)$ does not satisfy the Hörmander’s growth condition (6). In this case the corresponding operator cannot be considered as a constant-coefficient $\Psi$DO but rather a Fourier multiplier operator. The operator equation (81) can be formulated in the frequency space as

$$m_y(\xi)f_y^\dagger(\xi) = \hat{u}^\dagger(\xi).$$  \hspace{1cm} (84)

According to definition 1, the degree of ill-posedness $\mu(M_y) = +\infty$. Because no finite power $\nu$ exists. So the problem (81) is severely ill-posed and the smoother the symbol of the linear pseudo-differential operator $M_y(D)$, the higher the degree of ill-posedness of the associated problem (81). Therefore, the smoother the symbol, the greater losses of information in the process of finding the solution and creeping errors into the solution. However, the drawbacks coming up here can be treated within the language of wavelet analysis. It turns out that for this general class of ill-posed problems there are many order optimal convergence rates. Similar to
previous section and to dig these results, we consider the operator equation $R_y u^\dagger = f^\dagger$, where $R_y := M_y^{-1}$. Then the following stability estimate holds true within the Meyer MRA.

**Lemma 12 (Stability).** Let $\{V_j\}_{j \in \mathbb{Z}}$ be an m-regular Meyer MRA in $L^2(\mathbb{R}^N)$. Then for all $\varphi \in V_j$, with $J \in \mathbb{N}$ the following stability estimate holds

$$\|M_y \varphi\|_{H^q} \leq \frac{N}{\alpha} \tilde{C} \exp(\gamma y^{2^{J^\nu}}) \|\varphi\|_{H^q}, \quad q \in \mathbb{R},$$

(85)

where $\alpha$ and $\tilde{C}$ are the same constant appeared in condition H3 and lemma 6, respectively.

**Proof.** Let $\varphi \in V_j$,

$$\|R_y \varphi\|_{H^q} = \left( \int_{\mathbb{R}^N} |(R_y \varphi)(\xi)|^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2}$$

$$= \left( \int_{\mathbb{R}^N} |m_y(\xi)^{-1}\hat{\varphi}(\xi)|^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2}$$

$$\leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} \exp(\gamma y^{2^{J^\nu}}) \|\xi\|^2 \hat{\varphi}(\xi)^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2}$$

$$= \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} \sum_{k=0}^{+\infty} \frac{(\gamma y)^k}{k!} \|\xi\|^2 \hat{\varphi}(\xi)^2 (1 + \|\xi\|^2)^q d\xi \right)^{1/2}$$

$$= \frac{1}{\alpha} \sum_{k=0}^{+\infty} \frac{(\gamma y)^k}{k!} \sum_{i=1}^{N} \|D^k_{\nu_i^*}\|_{H^q}$$

$$\leq \frac{N}{\alpha} \tilde{C} \exp(\gamma y^{2^{J^\nu}}) \|\varphi\|_{H^q}.$$  

(86)

**Remark 5.** Lemma 12 shows that for the general class of severely ill-posed problems formulated in terms of pseudo-differential operators there exists a certain type of stability within the Meyer MRA. That means the solution of the problem continuously depends upon the data.

To establish some order optimal convergence rates, we introduce the wavelet regularizer as follows:

$$R_y J := \mathcal{P}_J R_y \mathcal{P}_J, \quad R_y := M_y^{-1},$$

(87)

for all $J \in \mathbb{N}$ and $y \in [0, \sigma]$. To work out some order optimal convergence rates, we state the following auxiliary lemma.

**Lemma 13 ([37]).** Let $\rho : [0, a] \to \mathbb{R}$ be defined by

$$\rho(\eta) = \eta^b \left( d \ln \frac{1}{\eta} \right)^{-c},$$

(88)
where \(0 < a < 1, \ b, d \in \mathbb{R}_+\) and \(c \in \mathbb{R}\). Then the function \(\rho(\cdot)\) is invertible and has the following inverse form

\[
\rho^{-1}(\eta) = \eta^\frac{1}{\beta} \left( \frac{d}{b} \ln \frac{1}{\eta} \right)^\frac{\beta}{\alpha} (1 + o(1)) \quad \text{for} \ \eta \to 0.
\]  

(89)

In detail, we will show that how the Meyer wavelet theory can recover the exact solution in such away that the stability property being preserved. The next theorem will show this argument.

**Theorem 14 (Convergence rate of the a priori rule).** Let \(u^1 \in V_J\) with \(J \in \mathbb{N}\). Suppose also the conditions H1, H2 and H3 hold true. Then with

\[
J_{\text{opt}} := \left[ \left( \frac{1}{\nu} \log_2 \left( \left( \frac{M}{\delta} \right)^\frac{\alpha}{\beta} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^\frac{\beta}{\alpha} \right) \right) \right],
\]  

(90)

the following order optimal convergence rate is satisfied

\[
\| R_j u^1 - R_{j,J_{\text{opt}}} u^\text{obs} \|_{H^W} = O \left( \delta^{\frac{1}{\nu} M^\frac{\alpha}{\beta} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^\frac{\beta}{\alpha} } \right).
\]  

(91)

where \([a]\) denotes the largest integer not exceeding \(a\).

**Proof.** By the triangle inequality the fundamental error can be estimated as follows

\[
\| R_j u^1 - R_{j,J_{\text{opt}}} u^\text{obs} \|_{H^W} \leq \| R_j u^1 - R_{j,J} u^1 \|_{H^W} + \| R_{j,J} u^1 - R_{j,J_{\text{opt}}} u^\text{obs} \|_{H^W}
\]  

= \(A1 + B1\),

(92)

where \(A1 := \| R_j u^1 - R_{j,J} u^1 \|_{H^W}\) and \(B1 := \| R_{j,J} u^1 - R_{j,J_{\text{opt}}} u^\text{obs} \|_{H^W}\). For quantity \(A1\), we have

\[
A1 = \| R_j u^1 - R_{j,J} u^1 \|_{H^W} \leq \| R_j u^1 - P_J R_j u^1 \|_{H^W} + \| P_J R_j u^1 - P_J R_{J_{\text{opt}}} u^1 \|_{H^W}
\]  

= \(A2 + B2\),

(93)

where \(A2 := \| R_j u^1 - P_J R_{J_{\text{opt}}} u^1 \|_{H^W}\) and \(B2 := \| P_J R_{J_{\text{opt}}} u^1 - P_J R_{J} u^1 \|_{H^W}\). Applying Jackson’s inequality and (28), we get

\[
A2 = \| R_j u^1 - P_J R_{J_{\text{opt}}} u^1 \|_{H^W} = \| (I - P_J) M_J R_j u^1 \|_{H^W}
\]  

\[
\leq C 2^{-\lambda (p+q)} \| M_J R_j u^1 \|_{H^W (\Gamma_J)}
\]  

\[
= C 2^{-\lambda (p+q)} \| R_j u^1 \|_{H^W (\Gamma_J)}
\]  

\[
= C 2^{-\lambda (p+q)} \left( \int_{\Gamma_J} \frac{m_\sigma (\xi)}{m_\sigma (\xi)} f_\sigma (\xi) |(1 + \| \xi \|^2)^{\nu} d\xi \right)^{1/2}
\]  

\[
\leq C \frac{\alpha}{\lambda} 2^{-\lambda (p+q)} \sup_{\xi \in \Gamma_J} |(\gamma (y - \sigma))|^{\nu} M
\]  

\[
\leq C \frac{\alpha}{\lambda} \exp \left( \gamma (y - \sigma)^2 \right) 2^{-\lambda (p+q)} M.
\]  

(94)

To evaluate \(B2\), we use (24) and obtain...
\[ B2 = \| \mathcal{P}_j \mathcal{R}_3 u^1 - \mathcal{P}_j \mathcal{R}_2 \mathcal{P}_j u^1 \|_{H^1} \]
\[ \leq \| \mathcal{R}_2 (I - \mathcal{P}_j) u^1 \|_{H^1} \]
\[ \leq \| \mathcal{R}_2 Q_j \mathcal{R}_j u^1 \|_{H^1(\Lambda_{j+1})} + \| \mathcal{R}_j u^1 \|_{H^1(\Gamma_{j+1})} \]
\[ =: A3 + B3, \quad (95) \]

where \( A3 := \| \mathcal{R}_2 Q_j \mathcal{R}_j u^1 \|_{H^1(\Lambda_{j+1})} \) and \( B3 := \| \mathcal{R}_j u^1 \|_{H^1(\Gamma_{j+1})} \). First we estimate quantity \( A3 \) by applying lemma 6 as

\[ A3 = \| \mathcal{R}_2 Q_j \mathcal{R}_j u^1 \|_{H^1(\Lambda_{j+1})} \]
\[ \leq \frac{N C}{\alpha} \exp(\gamma_2^{2^\nu}) \| Q_j \mathcal{R}_j u^1 \|_{H^1(\Lambda_{j+1})} \]
\[ = \frac{N C}{\alpha} \exp(\gamma_2^{2^\nu}) \| Q_j M_j u^1 \|_{H^1(\Lambda_{j+1})} \]
\[ \leq \frac{N C}{\alpha} \exp(\gamma_2^{2^\nu}) \| M_j u^1 \|_{H^1(\Lambda_{j+1})} \]
\[ = \frac{N C}{\alpha} \exp(\gamma_2^{2^\nu}) \| u^1 \|_{H^1(\Gamma_j)} \]
\[ = \frac{N C}{\alpha} \exp(\gamma_2^{2^\nu}) \left( \int_{\Gamma_j} |m_\sigma(\xi) \hat{r}_\sigma(\xi)|^2 \left( 1 + \| \xi \|^2 y^\nu d\xi \right) \right)^{1/2} \]
\[ \leq \frac{N C}{\alpha} \beta \exp(\gamma_2^{2^\nu}) \sup \left( -\gamma \sigma \| \xi \|^p \right) \| \xi \|^{-(p-\eta)} M \]
\[ \leq \frac{N C}{\alpha} \beta \exp(\gamma_2^{2^\nu}) 2^{2^\nu} 2^{-\eta(p-\eta)} M, \quad (96) \]

where we have used the fact that \( Q_j u^1 \in V_{j+1} \). On the other hand, we have for quantity \( B3 \)

\[ B3 = \| \mathcal{R}_j u^1 \|_{H^1(\Gamma_{j+1})} \]
\[ = \left( \int_{\Gamma_{j+1}} |m_\gamma(\xi) \hat{r}_\gamma(\xi)|^2 \left( 1 + \| \xi \|^2 y^\nu d\xi \right) \right)^{1/2} \]
\[ = \left( \int_{\Gamma_{j+1}} \frac{m_\gamma(\xi) \hat{r}_\gamma(\xi)}{m_\gamma(\xi) \hat{r}_\gamma(\xi)} \left( 1 + \| \xi \|^2 y^\nu d\xi \right) \right)^{1/2} \]
\[ \leq \frac{\beta}{\alpha} \sup_{\xi \in \Gamma_{j+1}} \exp(\gamma_2^{2^\nu}) \| \xi \|^p \| \xi \|^{-(p-\eta)} M \]
\[ \leq \frac{\beta}{\alpha} \exp(\gamma_2^{2^\nu}) 2^{2^\nu} 2^{-\eta(p-\eta)} M. \quad (97) \]

Combining now quantities \( A3 \) and \( B3 \) with \( B2 \), we arrive at

\[ B2 \leq (1 + N C) \frac{\beta}{\alpha} \exp(\gamma_2^{2^\nu}) 2^{2^\nu} 2^{-\eta(p-\eta)} M. \quad (98) \]
Also, inserting quantities $A2$ and $B2$ into $A1$, we get

\[ A1 \leq (1 + C + NC) \frac{\beta}{\alpha} \exp \left( \gamma(y - \sigma)2^{J^\nu} \right) 2^{-J(p-q)M}. \]  

Finally, quantity $B1$ can simply be estimated through lemma 6 as follows

\[ B1 = ||R_{\nu}^T u^1 - R_{\nu}^T u^{\text{obs}}||_{H^\nu} \leq \frac{N}{\alpha} \exp(\gamma y 2^{J^\nu}) \delta. \]  

Consequently, the fundamental error is worked out as

\[
\| R_{\nu}^T u^1 - R_{\nu}^T u^{\text{obs}} \|_{H^\nu} \leq \frac{N}{\alpha} \bar{C} \exp(\gamma y 2^{J^\nu}) \delta + (1 + C + \bar{C}) \times \frac{\beta}{\alpha} \exp(\gamma(y - \sigma)2^{J^\nu}) 2^{-J(p-q)M} = C_1 \left\{ \exp(\gamma y 2^{J^\nu}) \delta + \exp(\gamma(y - \sigma)2^{J^\nu}) 2^{-J(p-q)M} \right\},
\]

where $C_1 := \max \left\{ \frac{\alpha}{\bar{C}}, (1 + C + \bar{C}) \frac{\beta}{\alpha} \right\}$. Setting $\kappa(J) := \exp(\gamma y 2^{J^\nu}) \delta + \exp(\gamma(y - \sigma)2^{J^\nu}) 2^{-J(p-q)M}$ and $\eta := e^{-2^{J^\nu}}$ such that $\eta \in (0, 1)$, we can find a minimizer for the function $\kappa(J)$. In fact, by minimizing $\kappa(J)$, we have

\[ \delta \eta^{-\nu} = \eta^{-\gamma} \left( \frac{1}{\gamma} \ln \frac{1}{\eta} \right)^{\frac{\nu}{\gamma}} M. \]  

Therefore, $\frac{1}{\delta} = \eta^{-\gamma} \left( \frac{1}{\gamma} \ln \frac{1}{\eta} \right)^{\frac{\nu}{\gamma}}$. According to lemma 13 we conclude $b = \sigma, d = \frac{1}{\gamma}$ and $c = \frac{\nu}{\gamma}$. Therefore, we have

\[ \eta = \left( \frac{\delta}{M} \right)^{\frac{1}{\gamma}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{\frac{\nu}{\gamma}} (1 + o(1)), \]  

for $\frac{M}{\delta} \to 0$. Eventually, the regularization parameter $J_{\text{min}}$ is worked out as

\[ J_{\text{min}} = \frac{1}{\nu} \log_2 \left( \left( \frac{M}{\delta} \right)^{\frac{1}{\gamma}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu}{\gamma}} \right). \]  

Since $J = J_{\text{min}} \in \mathbb{N}$, we set $J_{\text{opt}} := \lfloor J_{\text{min}} \rfloor$ as

\[ J_{\text{opt}} := \left\lfloor \frac{1}{\nu} \log_2 \left( \left( \frac{M}{\delta} \right)^{\frac{1}{\gamma}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu}{\gamma}} \right) \right\rfloor. \]  

Therefore, the optimal value for the function $\kappa(\cdot)$ is worked out as
Corollary 15. Let $u^i$ be fulfilled for $0 < p < m$. Then with

$$J_{\text{opt}} := \left[ \frac{1}{\nu} \log_2 \left( \frac{M}{\delta} \right) \right] \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{\frac{\nu - \sigma \gamma}{\sigma}} \right]$$

the following asymptotic order optimal convergence rate holds true

$$\| \mathcal{R}_\nu u^i - \mathcal{R}_\nu J_{\text{opt}} u^{\text{obs}} \|_{L^2} = O \left( \delta^{1 - \frac{\nu}{2}} M^{\frac{1}{2}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}} \right),$$

where $\lfloor a \rfloor$ denotes the largest integer not exceeding $a$. 

for $\delta \to 0$. The fundamental error is ultimately estimated as

$$\| \mathcal{R}_\nu u^i - \mathcal{R}_\nu J_{\text{opt}} u^{\text{obs}} \|_{H^1} \leq C |\delta|^{1 - \frac{\nu}{2}} M^{\frac{1}{2}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}} (1 + o(1)) \quad (107)$$

for $\delta \to 0$. 

According to condition H1, it is not appropriate to ask $\| u - u^{\text{obs}} \|_{H^1} \leq \delta$ with $q > 0$ since generally the measured data $u^{\text{obs}}$ live in Hilbert space $L^2(\mathbb{R}^N)$. Hence if we replace $\| u - u^{\text{obs}} \|_{H^1}$ by $\| u - u^{\text{obs}} \|_{L^2} \leq \delta$, we have the following result

Corollary 15. Let $u^i \in V_J$ with $J \in \mathbb{N}$ and the conditions H1, H2, and H3 be fulfilled for $0 < p < m$. Then with

$$\kappa(J_{\text{opt}}) = \delta \eta^{-\gamma} + \eta^{\sigma - \gamma} \left( \frac{1}{\gamma} \ln \frac{1}{\eta} \right)^{\frac{\nu - \sigma \gamma}{\sigma}} M$$

$$= \delta \left( \frac{\delta}{M} \right)^{\frac{1}{\sigma \gamma}} \left( \frac{1}{\sigma \gamma} \frac{\ln M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}} + M \left( \frac{\delta}{M} \right)^{\frac{1}{\sigma \gamma}} \left( \frac{1}{\sigma \gamma} \frac{\ln M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}}$$

$$= \delta^{1 - \frac{\nu}{2}} M^{\frac{1}{2}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}} + \delta^{1 - \frac{\nu}{2}} M^{\frac{1}{2}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}}$$

$$= \delta^{1 - \frac{\nu}{2}} M^{\frac{1}{2}} \left( \frac{1}{\sigma \gamma} \ln \frac{M}{\delta} \right)^{-\frac{\nu - \sigma \gamma}{\sigma}} \left( 1 + o(1) \right) \quad (106)$$
Especially, at $y = 0$ the fundamental error, $\|R_y u^\dagger - R_y J_{opt} u^{obs}\|_{L^2}$, is of order optimal $O(\delta)$.

**Remark 6.** In practice, the *a priori* information $M$ is not known exactly. Hence if one take

$$J_{opt} := \left[\frac{1}{\nu} \log_2 \left(\frac{1}{\delta} \left(\frac{1}{\sigma \gamma} \ln \frac{1}{\delta} \right)^{\frac{\nu}{2}}\right)\right],$$

then there holds

$$\|R_y u^\dagger - R_y J_{opt} u^{obs}\|_{H^q} = O\left(\delta^{1-\frac{\nu}{2}} \left(\frac{1}{\sigma \gamma} \ln \frac{1}{\delta}\right)^{-\frac{p-q}{2}}\right).$$

### 5.2. The a posteriori choice rule

As is known from the previous section, the regularization parameter generally depends upon the noise level and the *a priori* information. But precisely determining the *a priori* information is generally impossible in practice. Thus a special strategy involving the *a posteriori* rule is necessary to achieve a better error estimates in this case. However, in this section, we will introduce an *a posteriori* approach where the regularization parameter no longer depends upon the *a priori* information and the solution smoothness. The structure of this approach is based on the Morozov discrepancy principle.

**Lemma 16.** Suppose that the conditions $H1$, $H2$, and $H3$ hold true. Moreover, suppose that the regularization parameter, $J$, is selected such that

$$\| (I - P_J) u^{obs} \|_{H^q} \leq \theta \delta \leq \| (I - P_{J-1}) u^{obs} \|_{H^q},$$

where $\theta > 1$ is a constant. Then there holds the following inequality

$$\exp(\gamma \sigma 2^{J/\nu} 2^{J(p-q)}) \leq \frac{2\beta M}{\delta(\theta - 1)}.$$  

**Proof.** It follows from the triangle inequality

$$\| (I - P_{J-1}) u^\dagger \|_{H^q} \leq \| u^\dagger \|_{H^q(\Gamma_{J-1})} + \| Q_{J-1} u^\dagger \|_{H^q(\Lambda_{J-1})} \leq N1 + N2,$$

where $N1 := \| u^\dagger \|_{H^q(\Gamma_{J-1})}$ and $N2 := \| Q_{J-1} u^\dagger \|_{H^q(\Lambda_{J-1})}$. These quantities can separately be evaluated as

$$N1 = \left(\int_{\Gamma_{J-1}} \left| m_\sigma(\xi) j_{\delta}^\dagger(\xi) \right|^2 (1 + \| \xi \|^2)^p d\xi\right)^{1/2} \leq \beta \sup_{\xi \in \Gamma_{J-1}} \exp(-\gamma \sigma \| \xi \|^{\nu}) \left(\int_{\Gamma_{J-1}} \left| j_{\delta}^\dagger(\xi) \right|^2 (1 + \| \xi \|^2)^p d\xi\right)^{1/2} \leq \beta M \exp(-\gamma \sigma 2^{J/\nu} 2^{J(p-q)}).$$

From (27), it follows
\[ N_2 = \|Q_{J-1}u^\dagger\|_{H^q(\Lambda_{J-1})} = \|Q_{J-1}M_{J-1}u^\dagger\|_{H^q(\Lambda_{J-1})} \]
\[ = \|M_{J-1}u^\dagger\|_{H^q(\Gamma_{J-1})} \leq \|u^\dagger\|_{H^q(\Gamma_{J-1})} \]
\[ = \left( \int_{\Gamma_{J-1}} |m_\sigma(\xi) \hat{f}_\sigma(\xi)|^2 (1 + \|\xi\|^2)^p d\xi \right)^{1/2} \leq \beta \sup_{\xi \in \Gamma_{J-1}} \exp(-\gamma \sigma \|\xi\|^p) (1 + \|\xi\|^p)^{1/2} \]
\[ \leq \beta M \exp(-\gamma \sigma 2^{J(p-q)}) 2^{-(p-q)}. \quad (116) \]

Therefore,
\[ \|(I - P_{J-1})u^\dagger\|_{H^q} \leq 2\beta M \exp(-\gamma \sigma 2^{J(p-q)}) \quad (117) \]

On the other hand,
\[ \|(I - P_{J-1})u^\dagger\|_{H^q} \geq \|(I - P_{J-1})u^\text{obs}\|_{H^q} - \|(I - P_{J-1})(u^\dagger - u^\text{obs})\|_{H^q} \geq (\theta - 1)\delta. \quad (118) \]

Finally, relations (117) and (118) will give us the main consequence.

The next theorem is the main result and provides the \textit{a posteriori} convergence rate of the Hölder-logarithmic type.

\textbf{Theorem 17 (Convergence rate of the \textit{a posteriori} rule).} \quad \textbf{Let the conditions H1, H2, and H3 be fulfilled. Moreover, suppose that the regularization parameter, } J, \text{ is selected such that}
\[ \|(I - P_{J-1})u^\text{obs}\|_{H^q} \leq \theta \delta \leq \|(I - P_{J-1})u^\text{obs}\|_{H^q}. \quad (119) \]

where \( \theta > 1 \) is a constant. \textbf{Then there holds}
\[ \|\hat{R}_{\gamma}u^\dagger - \hat{R}_{\gamma}u^\text{obs}\|_{H^q} = \mathcal{O} \left( \delta^{1 - \beta M} \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} \right)^{-\frac{p}{2}} \right). \quad (120) \]

\textbf{Proof.} \quad \textbf{From triangle inequality, we have}
\[ \|\hat{R}_{\gamma}u^\dagger - \hat{R}_{\gamma}u^\text{obs}\|_{H^q} \leq \|\hat{R}_{\gamma}u^\dagger - \hat{R}_{\gamma}u^\text{obs}\|_{H^q} + \|\hat{R}_{\gamma}u^\dagger - \hat{R}_{\gamma}u^\text{obs}\|_{H^q} \]
\[ = Q_1 + Q_2, \quad (121) \]

where \( Q_1 := \|\hat{R}_{\gamma}u^\dagger - \hat{R}_{\gamma}u^\text{obs}\|_{H^q} \) and \( Q_2 := \|\hat{R}_{\gamma}u^\dagger - \hat{R}_{\gamma}u^\text{obs}\|_{H^q} \). \textbf{From lemma 6 and (113), we have}
\[ Q^2 = \| R_y u^j - R_y u^j \|^\mathbb{H} \leq || R_y P_J (u^j - u^{obs}) ||^\mathbb{H} \]
\[ \leq \frac{N}{\alpha} C \exp(\gamma y^{2v}) \| P_J (u^j - u^{obs}) \| \]
\[ \leq \frac{N}{\alpha} C \exp(\gamma y^{2v}) \delta \]
\[ \leq \frac{N}{\alpha} C \left( \frac{2M}{\alpha \delta (\theta - 1)} \right)^{\frac{1}{\beta}} 2^{-J(p-q) \frac{\delta}{M}} \]
\[ Q^2 \leq \frac{N}{\alpha} C \left( \frac{2\beta}{\theta - 1} \right) \delta^{1 - \frac{\beta}{M}} \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta (\theta - 1)} \right)^{\frac{\beta}{M}} \]
\[ \times \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta (\theta - 1)} + \ln \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta (\theta - 1)} \right) \right)^{\frac{\beta}{M}}. \quad (123) \]

For quantity \( Q_1 \), we arrive at
\[ Q_1 = \| R_y u^j - R_y u^j \|^\mathbb{H} \leq || (I - P_J) R_y u^j \|^\mathbb{H} + || R_y (I - P_J) u^j \|^\mathbb{H} \]
\[ = Q_3 + Q_4, \quad (124) \]

where \( Q_3 := || (I - P_J) R_y u^j \|^\mathbb{H} \) and \( Q_4 = || R_y (I - P_J) u^j \|^\mathbb{H} \). Using (28) and the Jackson inequality, we obtain
\[ Q_3 = || (I - P_J) R_y u^j \|^\mathbb{H} = || (I - P_J) M_J R_y u^j \|^\mathbb{H} \]
\[ \leq C 2^{-J(p-q)} \| M_J R_y u^j \|^\mathbb{H} \]
\[ = C 2^{-J(p-q)} \| R_y u^j \|^\mathbb{H} \]
\[ \leq C \frac{\beta}{\alpha} \exp \left( \gamma (y - \sigma) \| \xi \|^p \right) M \]
\[ \leq C \frac{\beta}{\alpha} \exp \left( \gamma (y - \sigma) 2^j \right) 2^{-J(p-q)} M. \quad (125) \]

Applying \( J \) and doing some calculations, we get
\[ Q_3 \leq C \frac{\beta}{\alpha} \left( \frac{\theta - 1}{2\beta} \right) \delta^{1 - \frac{\beta}{M}} \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta (\theta - 1)} \right)^{\frac{\beta}{M}} \]
\[ \times \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta (\theta - 1)} + \ln \left( \frac{1}{\sigma \gamma} \ln \frac{2\beta M}{\delta (\theta - 1)} \right) \right)^{\frac{\beta}{M}}. \quad (126) \]
Employing the Hölder inequality, quantity $Q4$ is estimated as

$$Q4 = \| R_J (I - P_J) u^1 \|_{H^\nu} $$

$$= \left( \int_{\mathbb{R}^N} |m_\delta(\xi)^{-1} ((I - P_J) u^1)(\xi)(1 + \|\xi\|^2)^{q/2} |^2 \, d\xi \right)^{1/2} $$

$$\leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} |\exp(\gamma\nu\|\xi\|^p) ((I - P_J) u^1)(\xi)(1 + \|\xi\|^2)^{q/2} |^2 \, d\xi \right)^{1/2} $$

$$= \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} |\exp(\gamma\sigma\|\xi\|^p) ((I - P_J) u^1)(\xi)(1 + \|\xi\|^2)^{q/2} |^{2\nu} \, d\xi \right)^{1/2} $$

$$\times \left( (I - P_J u^1)(\xi)(1 + \|\xi\|^2)^{q/2} \right)^{2\nu} \, d\xi \right)^{1/2} $$

$$\leq \frac{1}{\alpha} \left( \int_{\mathbb{R}^N} |\beta\sigma^{-1} \hat{u}^1(\xi)(1 + \|\xi\|^2)^{q/2} |^{2\nu} \, d\xi \right)^{1/2} $$

$$\times \left( (I - P_J u^1)(\xi)(1 + \|\xi\|^2)^{q/2} \right)^{2\nu} \, d\xi \right)^{1/2} $$

$$\leq \frac{\beta^p}{\alpha} 2^{-\beta(p - q) \over 2} M^{2\over 2} \| (I - P_J) u^1 \|_{H^\nu} . \quad (127)$$

It can readily be seen that

$$\| (I - P_J) u^1 \|_{H^\nu} \leq \| (I - P_J) w_{\text{obs}} \|_{H^\nu} + \| (I - P_J) (u^1 - w_{\text{obs}}) \|_{H^\nu} $$

$$\leq (\theta + 1) \delta. \quad (128)$$

Inserting $J$ into (127) and then simplifying, we arrive at

$$Q4 \leq \frac{\beta^p}{\alpha} (\theta + 1) \frac{2^{\nu} \delta}{2^{\nu} \delta} \frac{1}{\alpha} \left( \ln \frac{2\delta M}{\delta(\theta - 1)} \right)^{-\nu} \frac{2^{\nu} \delta}{2^{\nu} \delta} \times \left( \frac{1}{\sigma} \ln \frac{2\delta M}{\delta(\theta - 1)} \right)^{-\nu} \frac{2^{\nu} \delta}{2^{\nu} \delta} \quad (129)$$

Plugging (126) and (129) into (124), we obtain
\begin{equation}
Q_1 \leq \left\{ C\beta \alpha \left( \frac{\theta - 1}{2\beta} \right)^{\frac{\alpha - 1}{\alpha}} + \beta \alpha (\theta + 1)^{\frac{\alpha - 1}{\alpha}} \right\} \delta^{\frac{\alpha - 1}{\alpha}} M^{\frac{\alpha - 1}{\alpha}} \left( \frac{1}{\sigma^\gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} \right)^{-\frac{\alpha - 1}{\alpha}} \times \left( \frac{1}{\sigma^\gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} + \ln \left( \frac{1}{\sigma^\gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} \right)^{-\frac{\alpha - 1}{\alpha}} \right). \tag{130}
\end{equation}

Combining now quantities (123) and (130) with (121), we ultimately get
\begin{equation}
\| R\hat{u}^\theta - R_j \hat{u}^{\theta,\Delta} \|_{H^\nu} \leq C^\delta \delta^{\frac{\alpha - 1}{\alpha}} M^{\frac{\alpha - 1}{\alpha}} \left( \frac{1}{\sigma^\gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} \right)^{-\frac{\alpha - 1}{\alpha}} \times \left( \frac{1}{\sigma^\gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} + \ln \left( \frac{1}{\sigma^\gamma} \ln \frac{2\beta M}{\delta(\theta - 1)} \right)^{-\frac{\alpha - 1}{\alpha}} \right), \tag{131}
\end{equation}

where $C^\delta := C^\delta(\sigma, \gamma, \theta, \alpha, \beta, C, \bar{C}) = C^\delta \left( \frac{\beta}{2\alpha} \right)^{\frac{\alpha - 1}{\alpha}} + \frac{\beta}{\alpha} \sigma^\gamma (\theta + 1)^{\frac{\alpha - 1}{\alpha}} + \frac{\beta}{\alpha} \sigma^\gamma M^{\frac{\alpha - 1}{\alpha}} C \left( \frac{2\beta}{\delta(\theta - 1)} \right)^{\frac{\alpha - 1}{\alpha}}$. This completes the proof.

**Remark 7.** On comparing the convergence rates for all type of ill-posedness, one is thus come to the conclusion that all rates of convergence derived by the *a priori* and the *a posteriori* regularization approaches are generally of comparable accuracy and no meaningful discrepancy is observed. But it should be pointed out that evaluating and working with the *a posteriori* regularization approach is more economical in compared to the *a priori* approach; mainly because the regularized solution is independent of the *a priori* information, $M$, and the degree of smoothness, $p$.

6. **Applications**

In this section we discuss the capability of the presented method to solve inverse and ill-posed partial differential equation problems containing parabolic and elliptic problems, numerical fractional differentiation, the analytic continuation problem, and one source problem.

6.1. **Applications for mildly and moderately cases**

**Example 1 (Numerical fractional differentiation)** (see [15]). The problem of numerical fractional differentiation is described by
\begin{equation}
\mathbb{D}^\kappa \phi(t) := \frac{1}{\Gamma(1 - \kappa)} \int_0^t \frac{\phi'(s)}{(t - s)\kappa} \, ds := f(t), \quad 0 \leq t \leq 1, \tag{132}
\end{equation}

where $\Gamma(\cdot)$ is called the well-known Euler’s gamma function and $0 < \kappa < 1$ is called the fractional order. Here, we supposed that the function $\phi^\beta(\cdot) \in C^\beta([0, 1])$ is the exact data. This
function, however, can be extended to the Hilbert space $L^2(\mathbb{R})$ by defining it zero outside the interval $[0, 1]$. In terms of Fourier transform the fractional differentiation can be rewritten as

$$\hat{\mathcal{D}^\alpha \phi}(\xi) = (i\xi)^\alpha \hat{\phi}(\xi) = \hat{f}(\xi),$$

which is actually equivalent with the pseudo-differential operator equation

$$\hat{\mathcal{D}f}(\xi) = \hat{\phi}(\xi), \quad \hat{\mathcal{D}f}(\xi) = (i\xi)^{-\alpha} \hat{f}(\xi).$$

Here, the symbol of the operator $\mathcal{D}$ is $m(\xi) = (i\xi)^{-\alpha}$. We assume that the indirect observable data $\phi^{\text{obs}}$ satisfying the so-called deterministic noise model $||\phi^i - \phi^{\text{obs}}||_{H^s} \leq \delta$ with a fixed noise level $\delta > 0$. We also suppose that the exact solution, $f^i$, satisfies the smoothness condition $||f^i||_{H^s} \leq M$ for some positive constant $M$ and $q + \kappa < p$. Note that the symbol $m(\xi) = |\xi|^{-\kappa} \exp \left(-\frac{\kappa}{\kappa^\beta} \text{sign}(\xi)\right)$ fulfills the inequality (36) with constant $\alpha = \beta = 1$. According to definition 1, the degree of ill-posedness is $\mu(\mathcal{D}) = \kappa$ which in turn manifests that the process of numerical fractional differentiation is a mildly ill-posed problem. If we define the following wavelet regularizer the following regularization parameter

$$J_{op} := \left\lfloor \log_2 \left( \frac{(p-q)M}{\kappa \delta} \right) \right\rfloor,$$

we have the a priori convergence rate

$$||\mathcal{R}^{\text{NFD}} \phi^i - \mathcal{R}_{J_{op}}^{\text{NFD}} \phi^{\text{obs}}||_{H^s} \leq C^i \delta^\frac{\kappa}{p+q} M^{\frac{\kappa}{p+q}},$$

(137)

where $C^i := (1 + C + C) \left\{ \left( \frac{\kappa}{p-q} \right)^{\frac{\kappa}{p+q}} + 2^{p-q} \left( \frac{\kappa}{p-q} \right)^{\frac{\kappa}{p+q}} \right\}$. Moreover, the rate of convergence by a posteriori choice rule is derived as follows

$$||\mathcal{R}^{\text{NFD}} \phi^i - \mathcal{R}_J^{\text{NFD}} \phi^{\text{obs}}||_{H^s} \leq C^* \delta^\frac{\kappa}{p+q} M^{\frac{\kappa}{p+q}},$$

(138)

where $C^* := \left( C2^{p-q} + \frac{2C}{p-q} \right)^{\frac{\kappa}{p+q}} (\theta + 1)^{\frac{\kappa}{p+q}}$.

**Example 2 (Inverse source problem).** Let us consider the following inverse source problem[44]

$$\begin{cases}
\partial_t u - \Delta u = f(x), & \text{in } \mathbb{R}^N \times (0, 1), \\
u = 0, & \text{on } \mathbb{R}^N \times \{t = 0\}, \\
u = \phi^i, & \text{on } \mathbb{R}^N \times \{t = 1\},
\end{cases}$$

(139)

where $u(\cdot, t)$ and $f(\cdot) \in L^2(\mathbb{R}^N)$ respectively denote the state and source functions. This problem is of practical importance. For instance, in cities with a high populations a precise description of contamination source can help us in protecting the environment. Many investigations regarding this inverse problem have been conducted both in theoretical and numerical
aspects. The solution of this problem can be described by the language of Fourier analysis as follows

\[ \hat{f}(\xi) = \frac{\|\xi\|^2}{1 - e^{-||\xi||^2}} \hat{\phi}(\xi). \]  

(140)

The above equation can be rewritten as a pseudo-differential operator equation with symbol \( m(\xi) = \frac{1 - e^{-||\xi||^2}}{||\xi||^2} \) in the following sense

\[ \tilde{S}\hat{f}(\xi) = \hat{\phi}(\xi), \quad \tilde{S}f(\xi) = \frac{1 - e^{-||\xi||^2}}{||\xi||^2}f(\xi). \]  

(141)

Clearly, the symbol \( m(\xi) \) satisfies the inequality (36) with constants \( \alpha = \beta = 2 \) and \( \nu = 2 \). As \( m(\xi) = O(||\xi||^{-2}) \), the degree of ill-posedness is \( \mu(\mathcal{S}) = 2 \) which means that the problem of source identification is moderately ill-posed. To establish the rate of convergence, we suppose that for some \( q \leq 0 \) the deterministic noise model, \( ||\phi^j - \phi^{obs}||_W \leq \delta \), and the smoothness condition, \( ||f(\cdot)||_W \leq M \) hold true for some \( q + 2 < p \). We consider the regularization operator as

\[ \mathcal{R}^\text{SP}_j := \mathcal{P}_j \mathcal{R}^\text{SP}_j \mathcal{P}_j, \quad \mathcal{R}^\text{SP} = S^{-1}. \]  

(142)

If one take the following wavelet regularization parameter

\[ J_{opt} := \left[ \log_2 \left( \frac{(p-q)M}{2\delta} \right)^{\frac{1}{p-q}} \right]. \]  

(143)

then the following \textit{a priori} error estimate is satisfied

\[ ||\mathcal{R}^\text{SP} \phi^j - \mathcal{R}^\text{SP} \phi^{obs}||_W \leq C^1 \delta^{\frac{p-q}{p-q-2}} M^{\frac{1}{p-q-2}} (1 + o(1)), \quad \text{as } \delta \to 0, \]  

(144)

where \( C^1 := (1 + C + NC) \left\{ \left( \frac{p-q}{2} \right)^{\frac{p-q}{p-q+2}} + 2^{p-q} \left( \frac{p-q}{2} \right)^{\frac{p-q}{p-q+2}} \right\} \). Also, the following rate of convergence by the \textit{a posteriori} choice rule is established as follows

\[ ||\mathcal{R}^\text{SP} \phi^j - \mathcal{R}^\text{SP} \phi^{obs}||_W \leq C^* \delta^{\frac{p-q}{p-q-2}} M^{\frac{1}{p-q-2}}, \]  

(145)

where \( C^* := \left( C 2^{p-q} + \frac{2N}{2-p} \right) \left( \frac{\delta}{\nu} \right)^{\frac{p-q}{p-q+2}} + \left( \frac{\delta}{\nu} \right)^{\frac{p-q}{p-q+2}}. \)

6.2. Applications for severely ill-posed problems

\textbf{Example 3 (Analytic continuation problem)} (see [10, 24]). Let \( f(z) = f(x + iy) \) be a complex-valued analytic function on the high-dimensional infinite complex domain \( \Omega \) defined by

\[ \Omega := \{ x + iy \in \mathbb{C}^N : x \in \mathbb{R}^N, ||y|| \leq ||y_0||, y, y_0 \in \mathbb{R}^N \}, \]  

(146)

here the symbol \( || \cdot || \) indicating the Euclidean norm. The central goal is to extend the function \( f(z) \) analytically from indirectly measured data of the known function \( \Phi^j(x) \) to the whole complex domain \( \Omega \) such that \( f(z)|_{y=0} = \Phi^j(x) \). The analytic continuation problem (ACP) has attracted many research activities including many practical physical applications, for instance, inverse Laplace integral transform [1], inverse scattering problems [29], medical
imaging [9, 31]. Applying the Fourier transformation about the variable $x$ in the frequency domain we get the operator equation $\hat{A}_x f(\cdot + iy)(\xi) = \hat{f}(\cdot + i0)(\xi)$, or equivalently, the following pseudo-differential operator equation

$$
\hat{A}_x f(\cdot + iy)(\xi) = \hat{\Phi}(\xi), \quad \hat{A}_y f(\cdot + iy)(\xi) = e^{\xi} f(\cdot + iy)(\xi),
$$

(147)

where $\xi \in \mathbb{R}^N$. Obviously, $m_x(\xi) = e^{i\gamma \|\xi\| \cos(\Theta)}$ for $\cos(\Theta) < 0$. According to condition H3, $y = \|\xi\| \in [0, \|\xi_0\|]$, $\gamma = -\cos(\Theta)$, $\nu = 1$, and $\alpha = \beta = 1$. The degree of ill-posedness, in this case, is $\mu(A_y) = +\infty$ and hence the ACP is severely ill-posed. We suppose that the measured data, $\Phi^{\text{obs}}$, satisfies $\|\Phi^{\text{obs}} - \Phi^{\text{obs}}\|_{\mathcal{H}^1} \leq \delta$ for some $q \leq 0$. Also, consider the smoothness condition $\|f(\cdot + iy_0)\|_{\mathcal{H}^1} \leq M$ for some $q \leq p$. We introduce the wavelet regularization operator in the following sense

$$
\mathcal{R}^{AC}_{y,J} \equiv \mathcal{P}_y \mathcal{R}^{AC}_{y,J}, \quad \mathcal{R}^{AC}_{y,J} \equiv A_y^{-1},
$$

(148)

where the regularization parameter is chosen as

$$
J_{\text{opt}} \equiv \left[ \log_2 \ln \left( \frac{M}{\delta} \right) \right] \cdot \left( \frac{1}{\gamma \|\xi_0\|} \ln \left( \frac{M}{\delta} \right) \right).
$$

(149)

Then there holds the following asymptotic a priori error estimate

$$
\|\mathcal{R}^{AC}_{y,J} \Phi^{\text{obs}} - \mathcal{R}^{AC}_{y,J} \Phi^{\text{obs}}\|_{\mathcal{H}^1} \leq (1 + C + N\mathcal{C}) \delta^{1 - \frac{|\gamma|}{\|\xi_0\|}} M \frac{|\xi|}{|\xi_0|} \left( \frac{1}{\gamma \|\xi_0\|} \ln \left( \frac{M}{\delta} \right) \right)
$$

(149)

for $\delta \to 0$. Also, there holds the following asymptotic a posteriori error estimate

$$
\|\mathcal{R}^{AC}_{y,J} \Phi^{\text{obs}} - \mathcal{R}^{AC}_{y,J} \Phi^{\text{obs}}\|_{\mathcal{H}^1} \leq C^1 \delta^{1 - \frac{|\gamma|}{\|\xi_0\|}} M \frac{|\xi|}{|\xi_0|} \left( \frac{1}{\gamma \|\xi_0\|} \ln \left( \frac{M}{\delta(\theta - 1)} \right) \right)^{-\left( p - q \right) \frac{|\xi_0|}{|\xi|} \left( 1 + o(1) \right)}, \quad \text{as } \delta \to 0,
$$

(150)

where $C^1 \equiv C \left( 1 + \frac{1}{2} \right)^{1 - \frac{|\gamma|}{\|\xi_0\|}} + \left( \theta + 1 \right)^{1 - \frac{|\gamma|}{\|\xi_0\|}} + N\mathcal{C} \left( \frac{\gamma}{\|\xi_0\|} \right)^{\frac{|\xi|}{|\xi_0|}}$.

**Example 4 (Backward heat conduction problem).** Let us consider the following backward heat conduction problem (BHCP)

$$
\begin{aligned}
\partial_t u - \kappa(t) \Delta u &= 0, &\text{in } \mathbb{R}^N \times (0, T), \\
u = g_T, &\text{on } \mathbb{R}^N \times \{ t = T \},
\end{aligned}
$$

(152)

where $k(t) \in C([0, T])$ and $g_T(\cdot) \in L^2(\mathbb{R}^N)$ respectively denote the positive thermal conductivity and the terminal distribution [21]. By the technique of Fourier transform we obtain in the frequency space the following operator equation $\hat{B}_t \hat{u}(\xi, t) = \hat{u}(\xi, T)$ or equivalently the pseudo-differential operator equation

$$
\hat{B}_t \hat{u}(\xi, t) = \hat{g}_T(\xi), \quad \hat{B}_t \hat{u}(\xi, t) = e^{-\partial_y(\xi) \|\xi\|^2} \hat{u}(\xi, t),
$$

(153)
where \( \vartheta(t) := \int_0^t \kappa(s) \, ds \). One can consider \( m_p(\xi) := e^{-\beta_{\vartheta}(t) |\xi|^2} \) with parameter \( \gamma = \vartheta(t) \in [0, \vartheta(T)] \). From inequality (83), we have \( \nu = 2 \) and \( \gamma = \alpha = \beta = 1 \). According to definition 1 the degree of ill-posedness is \( \mu(B_g) = \infty \), and based on our classification described in definition 2, the BHCP is a severely ill-posed problem. To extract some rates of convergence, we assume that the following conditions hold true:

- The terminal data \( g_{\text{obs}}(\cdot) \) are deterministic noise model \( \| g_{\text{obs}} \|_{\text{H}} \leq \delta \) for some \( q \leq p \) and noise level \( \delta > 0 \).
- The exact solution satisfies the solution smoothness condition \( \| u(\cdot, 0) \|_{\text{H}} \leq M \) for some \( q \leq p \) and a non-dimensional a priori bound \( M > 0 \).

Consider the following wavelet regularization operator

\[
\mathcal{R}_{i,j}^\text{BH} := \mathcal{P}_j \mathcal{R}_{i,j}^\text{BH} \mathcal{P}_i,
\]

\( \mathcal{R}_{i,j}^\text{BH} := B^{-1}_i \). If one take the regularization parameter

\[
J_{\text{opt}} := \left[ \frac{1}{2} \log_2 \left( \frac{M}{\delta} \right)^{\frac{2}{N}} \left( \frac{1}{\vartheta(T)} \ln \frac{M}{\delta} \right)^{-\frac{2}{2N}} \right].
\]

then there holds the following a priori convergence rate

\[
\| \mathcal{R}_{i,j}^\text{BH} \mathcal{P}_i g_{\text{obs}} - \mathcal{R}_{i,j}^\text{BH} g_{\text{obs}} \|_{\text{H}} \leq (1 + C + N\tilde{C}) \delta^{\frac{\theta(1)}{\theta + 1}} M^{-\frac{\theta(1)}{\theta + 1}}
\]

\[
\times \left( \frac{1}{\vartheta(T)} \ln \frac{M}{\delta} \right)^{-\frac{2M}{\theta - 1} \frac{\theta(1)}{\theta + 1} \left(1 + o(1)\right)},
\]

(156)

for \( \delta \to 0 \). Also, there holds the following a posteriori convergence rate

\[
\| \mathcal{R}_{i,j}^\text{BH} \mathcal{P}_i g_{\text{obs}} - \mathcal{R}_{i,j}^\text{BH} g_{\text{obs}} \|_{\text{H}} \leq C \delta^{\frac{\theta(1)}{\theta + 1}} M^{-\frac{\theta(1)}{\theta + 1}}
\]

\[
\times \left( \frac{1}{\vartheta(T)} \ln \frac{2M}{\delta} \right)^{-\frac{2M}{\theta - 1} \frac{\theta(1)}{\theta + 1} + N\tilde{C}(\frac{2M}{\theta - 1})^{\frac{\theta(1)}{\theta + 1}}},
\]

(157)

where \( C = C(\frac{\theta(1)}{\theta + 1})^{-\frac{\theta(1)}{\theta + 1}}(\theta + 1)^{\frac{\theta(1)}{\theta + 1}} + N\tilde{C}(\frac{2M}{\theta - 1})^{\frac{\theta(1)}{\theta + 1}} \).

Example 5 (Cauchy problem for the Helmholtz equation). Let us consider the following Cauchy problem for the Helmholtz equation (CPHE)

\[
\begin{aligned}
-\Delta u - k^2 u &= 0, \quad \text{in } (0, 1) \times \mathbb{R}^N, \\
\varphi &= \varphi, \quad \text{on } \{x = 0\} \times \mathbb{R}^N, \\
\partial_{\nu} u &= 0, \quad \text{on } \{x = 0\} \times \mathbb{R}^N,
\end{aligned}
\]

(158)

where \( -\Delta \) is known as \((N + 1)\)-dimensional Laplace operator and \( k > 0 \) is called the wavenumber [36]. Here, the function \( \varphi \in L^2(\mathbb{R}^N) \) is the exact data with approximation \( \varphi_{\text{obs}} \in L^2(\mathbb{R}^N) \) satisfying \( \| \varphi - \varphi_{\text{obs}} \|_H \leq \delta \) for some \( q \leq p \) and noise level \( \delta > 0 \). We also consider the solution smoothness condition as \( \| u(\cdot, 0) \|_{\text{H}} \leq M \) for some \( q \leq p \) and a non-dimensional a priori bound \( M > 0 \). By the method of Fourier transform, we obtain in the
frequency domain the following operator equation $\hat{H}_x \hat{u}(x, \xi) = \varphi(\xi)$, or equivalently, the following pseudo-differential operator equation

$$
\hat{H}_x \hat{u}(x, \xi) = \frac{1}{\cosh(x \sqrt{\|\xi\|^2 - k^2})} \hat{u}(x, \xi),
$$

(159)

where the symbol reads as $m_x(\xi) = \frac{1}{\cosh(x \sqrt{\|\xi\|^2 - k^2})}$. Obviously, the symbol $m_x(\cdot)$ satisfies the following inequality

$$
e^{-x\|\xi\|} \leq |m_x(\xi)| \leq 2e^{x\|\xi\|}, \quad \text{for } \|\xi\| \geq k.
$$

(160)

Therefore, according to inequality (83), we have $\alpha = 1, \beta = 2e^{x\xi}, \gamma = x, \sigma = 1, \text{and } \nu = 1$. As $\mu(H_x) = +\infty$, the CPHE is a severely ill-posed problem. Now, define the following wavelet regularization operator

$$
\mathcal{R}^{\text{HE}}_{x, j} := \mathcal{P}_j \mathcal{R}^{\text{HE}}_x \mathcal{P}_j, \quad \mathcal{R}^{\text{HE}}_x := \mathcal{H}^{-1}_x.
$$

(161)

If one takes the regularization parameter

$$
J_{\text{opt}} := \left[ \log_2 \ln \left( \frac{M}{\delta} \left( \ln \frac{M}{\delta} \right)^{-(p-q)} \right) \right].
$$

(162)

Then there holds the following a priori error estimates

$$
\| \mathcal{R}^{\text{HE}}_x \varphi^\dagger - \mathcal{R}^{\text{HE}}_{x, J_{\text{opt}}} \varphi^{\text{obs}} \|_{H^q} \leq 2e^{x\xi}(1 + C + NC)\delta^{1-x}M^\theta \left( \ln \frac{M}{\delta} \right)^{-(p-q)x} \times (1 + o(1)),
$$

(163)

for $\delta \to 0$. The a posteriori error estimates also is derived as

$$
\| \mathcal{R}^{\text{HE}}_x \varphi^\dagger - \mathcal{R}^{\text{HE}}_{x, J_{\text{opt}}} \varphi^{\text{obs}} \|_{H^q} \leq C\delta^{1-x}M^\theta \left( \ln \frac{2e^{xM}}{\delta(\theta - 1)} \right)^{-(p-q)x} (1 + o(1)),
$$

(164)

where $C = 2C e^{x\xi}(\theta - \frac{1}{4e^{xM}})^{1-x} + (2e^{x\xi})^x(\theta + 1)^x + NC \left( \frac{4e^{xM}}{\theta - 1} \right)^x$.

7. Conclusion

We have presented an error analysis of wavelet-based regularization for ill-posed problems involving constant-coefficient pseudo-differential operators in a Sobolev space setting. In our analysis, the degree of ill-posedness of the $\Psi$DO equation is characterized by the quantity $\mu(M)$ which in turn classifies ill-posed problems in terms of mildly, moderately, and severely ill-posed cases in one package. Systematically, we proposed a regularization scheme based on the Meyer wavelet to solve these ill-posed problems and subsequently obtained some order optimal rates of convergence in the presence of a deterministic noise model. These convergence rates were derived both by the a priori and a posteriori parameter choice rules. It turned out that the convergence rates produced by these approaches are all the same except for different constants. It would be interesting to extend this approach to nonlinear ill-posed problems described by $\Psi$DOs, which possibly enables us to treat a much wider class of ill-posed problems. This is an intractable problem to be solved in the future.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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