OBSTRUCTIONS TO DEFORMING MAPS FROM CURVES TO SURFACES

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Abstract. This paper studies the obstruction to deforming a map from a complex curve to a complex surface, under the condition that the map is locally an embedding. We extend the classical notion of semiregularity of embedded curves to maps, and show that it largely extends the applicability. As an example, we apply it to K3 surfaces, and show that a generic complex polarized K3 surface contains infinitely many $g$ dimensional families of curves of geometric genus $g$ for any positive integer $g$. Also, we show that any complex projective K3 surface is dominated by families of elliptic curves in infinitely many different ways.

1. Introduction

In this paper, we study deformations of a map from a complex curve to a smooth complex surface, under the condition that the map is locally an embedding. The domain curve is assumed to be reduced but otherwise it can have any singularity.

The starting point of this study is the notion of semiregularity which goes back to Severi [14, 15]. Here a smooth complex curve $C$ on a smooth complex surface $S$ is called semiregular if the canonical linear system of the surface cuts out on $C$ a complete linear system (see Subsection 2.1). Severi proved that when $C$ is semiregular, then its deformation on $S$ is unobstructed.

The notion of semiregularity and the associated result were generalized by Kodaira and Spencer [9] to smooth divisors on higher dimensional complex manifolds. Later, Bloch [3] extended the notion of semiregularity to local complete intersection subvarieties, and related it to the smoothness of the Hilbert scheme at the corresponding point as well as to variation of Hodge structures. In particular, the semiregularity of a local complete intersection subvariety guarantees the vanishing of the obstruction to the deformation. More recently, these ideas have been generalized from multiple points of view, see [4, 7, 8] to name a few.

Although these results are striking, often it is not easy to check whether a given subvariety is semiregular or not, and even if we start from semiregular subvarieties, standard construction such as taking coverings tends to break the condition of semiregularity. On the other hand, since the obstruction to deformations is in principle determined by the information of a neighborhood of the subvariety, it would not be too optimistic to expect that we can extend the notion of semiregularity from the original cohomological (in other words, global) condition to a more local one.

In this paper, we show that this is in fact the case, and we extend the notion of semiregularity to maps rather than subvarieties (such an extension was also considered in [7] for maps with smooth domains, from a different perspective).

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Theorem 1. Let \( \varphi : C \to X \) be a map from a reduced complete complex curve to a smooth complex surface, which is locally an embedding. Assume that \( \varphi \) is semiregular, that is, the natural map \( H^0(X, K_X) \to H^0(C, \varphi^*K_X) \) is surjective. Then the map \( \varphi \) is unobstructed in the sense that any first order deformation can be extended to arbitrary higher order.

This largely extends the applicability of semiregularity. Namely, starting from a few embedded curves on a surface for which the classical semiregularity condition holds, we construct new curves by putting these together in a simple combinatorial way. Then the constructed curves, seen as the images of suitable maps, often satisfy the (extended) semiregularity again, and we can deform such curves on the surface.

For example, since curves on a surface whose associated invertible sheaves are very ample are semiregular in the classical sense, such curves can be the building blocks of the map \( \varphi \). Note that although \( \varphi \) is locally an embedding, the image \( \varphi(C) \) need not be reduced.

The case where Theorem 1 is most effective would be when the target \( X \) has the trivial canonical sheaf. In this case, any reduced curve on \( X \) is semiregular. Thus, any map \( \varphi \) from a reduced curve which is locally an embedding is unobstructed. Based on this observation, we can prove the following.

Corollary 2. A generic complex polarized K3 surface contains infinitely many \( g \) dimensional families of irreducible nodal curves of geometric genus \( g \), for any positive integer \( g \).

Here we say that a property for polarized K3 surfaces holds for generic ones if it holds for all members of a Zariski open subset of the moduli space, see [6, Chapter 13]. There is another corollary which holds for any projective K3 surface.

Corollary 3. Let \( X \) be a complex projective K3 surface. Then there are infinitely many different dominant maps \( p_i : \mathcal{E}_i \to X, i \in \mathbb{N} \), from one dimensional families \( \mathcal{E}_i \) of curves of geometric genus one. In other words, any complex projective K3 surface is dominated by families of elliptic curves in infinitely many different ways.

There are many directions to which Theorem 1 should be extended. In view of Bloch’s result, the method of this paper might well generalize to maps which are locally embeddings but the domain can be general local complete intersection. Also, again in view of [3], one can also consider the case where the target \( X \) deforms. For some special types of deformations of the target, the argument in this paper applies with little change, see [12]. The proof of Theorem 1 in this paper partially depends on transcendental method. It will also be important to pursue a purely algebraic proof and investigate the positive characteristic case, see [10]. We will study some of these issues in a forthcoming paper.

2. Obstruction to curves on surfaces

We think varieties in the complex analytic category. Let \( X \) be a smooth surface, not necessarily compact. Let \( C \) be a connected compact curve without embedded points. We assume all irreducible components of \( C \) are reduced, but otherwise \( C \) can have any singularity. Let

\[ \varphi : C \to X \]
be a map which is locally an embedding. Namely, at each point of $C$, there is a neighborhood of it such that the restriction of $\varphi$ to that neighborhood is an isomorphism onto the image. We will study deformations of $\varphi$.

**Remark 4.** Actually, we do not need to assume the variety $X$ to be smooth. To study the deformation theory of the map $\varphi$, information of a neighborhood of the image will suffice. Similarly, the curve $C$ need not be compact or without boundary. See also Remark 24.

Deformation of the map $\varphi$ is controlled by the hypercohomology groups

$$\text{Ext}^i(\varphi^*\Omega_X \to \Omega_C, \mathcal{O}_C).$$

In the case mentioned above, the map $\varphi^*\Omega_X \to \Omega_C$ is surjective and the kernel of it is a locally free sheaf known as the conormal sheaf $\nu^\vee$ of the map $\varphi$. In particular, in this case the deformation of $\varphi$ is controlled by the usual sheaf cohomology groups.

**Definition 5.** We write the locally free sheaf $\text{Hom}(\nu^\vee, \mathcal{O}_C)$ by $\nu_\varphi$ and call it the normal sheaf of the map $\varphi$.

By definition of $\nu_\varphi$, we have the following.

**Proposition 6.** The set of first order deformations of the map $\varphi$ is isomorphic to $H^0(C, \nu_\varphi)$. The obstruction to the existence of (the second and higher order as well) lifts lives in $H^1(C, \nu_\varphi)$.

Now, let $\omega_C$ be the dualizing sheaf of $C$ defined by

$$\omega_C = \varphi^*K_X \otimes \nu_\varphi.$$  

Here $K_X$ is the canonical sheaf of $X$. By the Serre duality, the group $H^1(C, \nu_\varphi)$ is isomorphic to the dual of $H^0(C, \nu^\vee \otimes \omega_C)$. In particular, we have the following.

**Proposition 7.** The obstruction to deform $\varphi$ belongs to the dual of the cohomology group $H^0(C, \varphi^*K_X)$.

**2.1. Relation to semiregularity.** In the space $H^0(C, \varphi^*K_X)$, there is a distinguished subspace $V$ composed by those coming from elements of $H^0(X, K_X)$. When this subspace coincides with $H^0(C, \nu^\vee \otimes \omega_C)$, the calculation of the obstruction becomes particularly simple.

In the case where $\varphi$ is an embedding, this condition is nothing but the semiregularity of the subvariety $\varphi(C) \subset X$ studied in [13, 15] and extended to higher dimensional cases in [9, 3]. In [3], the definition of semiregularity is extended to local complete intersection subvarieties in any dimensional ambient space. Namely, let $X$ be a smooth complex projective variety of dimension $n$, and $Z \subset X$ be a local complete intersection of codimension $p$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the defining ideal of $Z$. Then the normal sheaf

$$\mathcal{N}_{Z/X} = \text{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

is locally free. Let $K_X = \Omega^n_X$ be the canonical sheaf, and define the invertible sheaves $K_{Z/X}$ and $\omega_Z$ on $Z$ by

$$K_{Z/X} = \wedge^p N_{Z/X}^\vee, \quad \omega_Z = K_{Z/X}^\vee \otimes K_X.$$
The natural inclusion $\varepsilon: N'_{Z/X} \to \Omega^1_X \otimes \mathcal{O}_Z$ gives rise to an element 
\[ \wedge^{p-1} \varepsilon \in \text{Hom}_{\mathcal{O}_Z} \left( \wedge^{p-1} N'_{Z/X}, \Omega^p \otimes \mathcal{O}_Z \right) = \Gamma \left( \left( \Omega^{n-p+1}_X \otimes K_X \otimes K^\vee_{Z/X} \otimes N'_{Z/X} \right) \right) = \text{Hom}_{\mathcal{O}_X} \left( \Omega^{n-p+1}_X, \omega_Z \otimes N'_{Z/X} \otimes K \otimes \omega \otimes K^\vee_{Z/X} \right). \]

This induces a map on cohomology 
\[ \wedge^{p-1} \varepsilon : H^{n-p-1}(X, \Omega^{n-p+1}_X) \to H^{n-p-1}(Z, \omega_Z \otimes N'_{Z/X}). \]

In \cite{3}, the dual of this map
\[ \pi: H^1(Z, N_{Z/X}) \to H^{p+1}(X, \Omega^p) \]
is called the *semiregularity map* and the subvariety $Z$ is called *semiregular* if the semiregularity map is injective.

One of the main theorems of \cite{3} is the following, which generalizes results of \cite{9, 14, 15}.

**Theorem 8.** When $Z$ is semiregular in $X$, then the Hilbert scheme $\text{Hilb}(X/\mathbb{C})$ is smooth at the point corresponding to $Z$. $\square$

In the case where $Z$ is a hypersurface so that $p = 1$, the above map $\wedge^{p-1} \varepsilon$ on cohomology becomes the restriction $H^{n-2}(X, K_X) \to H^{n-2}(Z, i^* K_X)$, where $i: Z \to X$ is the inclusion. In our situation where $n = 2$, while the map $\varphi: C \to X$ need not be an inclusion, the natural map $H^0(X, K_X) \to H^0(C, \varphi^* K_X)$ plays the similar role.

Although Theorem 8 is strikingly general, in explicit situations it is sometimes difficult to check the semiregularity, and divisors with higher multiplicities are usually not semiregular.

In this paper, we remedy this point by local and direct calculation of obstructions. In spite of the advances of the notions and technicalities concerning semiregularity in \cite{3} above and in even more sophisticated \cite{4, 7, 8}, the study of obstructions requires us to go back to the classical method of \cite{9}.

### 3. Calculation of Obstruction

As we mentioned in Subsection 2.1, when the natural map $H^0(X, K_X) \to H^0(C, \varphi^* K_X)$ is surjective, then the map $\varphi$ will have a good property from the point of view of deformation theory.

**Definition 9.** We call the map $\varphi: C \to X$ semiregular if the natural map $H^0(X, K_X) \to H^0(C, \varphi^* K_X)$ is surjective.

However, to see this is a legitimate extension of the notion of classical semiregularity, the cohomological method of previous works does not suffice, and we need to develop a way to study obstruction more directly.

#### 3.1. Meromorphic differential forms and cohomological pairings

We begin with giving a presentation of a Čech cocycle in a way suited to our purpose. Let $C$ be a reduced curve and $\mathcal{L}$ be an invertible sheaf on it. Take an open covering $\{U_1, \ldots, U_m\}$ of $C$ in the following way:

- Each $U_i$ is a disk or an open subset which contains exactly one singular point of $C$. In the latter case, the normalization of $U_i$ is a union of disks where the number of the disks is the same as the number of the branches of $C$ at the singular point.
For each singular point of $C$, there is a unique open subset from $\{U_1, \ldots, U_m\}$ containing it.

Here a disk means an analytic subset which is analytically isomorphic to the set $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ for some positive real number $r$.

If $U_i$ is the open subset containing a singular point $p_i$ of $C$, let $U_i = U_{1,i} \cup \cdots \cup U_{k,i}$ be the decomposition into branches. The subsets $U_{i,j}$ and those $U_i$ which are disks form a covering of $C$ by locally closed subsets. We write it by $\{V_j\}$.

**Remark 10.** Although the normalization of $U_{i,j}$ is a disk, in general $U_{i,j}$ itself is not necessarily a disk. However, we call it (and corresponding $V_j$) a disk for notational simplicity.

By construction, we can naturally associate with each $V_j$ the unique open subset from the covering $\{U_i\}$ above. We write this open subset by $\tilde{V}_j$. Namely, $\tilde{V}_j = U_i$ if $V_j = U_{a,i}$ for some $a$, and $\tilde{V}_j = U_i$ if $V_j = U_i$ is a disk.

We associate a local meromorphic section $\xi_j$ of $L|_{V_j}$ with each $V_j$ in a way that if $V_i \cap V_j$ is an open subset, then $\xi_i - \xi_j$ is a holomorphic section of $L|_{V_i \cap V_j}$.

Let $V_1$ and $V_2$ be two of these locally closed subsets whose intersection is an open subset (note that in this case $\tilde{V}_1 \neq \tilde{V}_2$). Then associate the section $\xi_1 - \xi_2$ of $L$ with the open subset $V_1 \cap V_2$ (the order also matters, that is, we associate $\xi_2 - \xi_1$ with $V_2 \cap V_1$). Taking each $U_i$ small enough, we can assume

$$V_1 \cap V_2 = \tilde{V}_1 \cap \tilde{V}_2.$$  

Therefore, by associating $\xi_1 - \xi_2$ with $\tilde{V}_1 \cap \tilde{V}_2$, the set of sections $\{\xi_i\}$ determines a Čech 1-cocycle $\{\xi_{ij}\}$ with values in $L$ for the covering $\{U_i\}$ above.

Conversely, any class in $H^1(C, L)$ can be represented in this way (see Proposition 11 below).

Let $\omega_C$ be the dualizing sheaf of $C$. When $C$ is a reduced curve (as we are assuming in this paper), it is known that sections of $\omega_C$ are given by Rosenlicht differentials ([13]), see also [12]). Let $\nu: \tilde{C} \to C$ be the normalization. Let $M_{\tilde{C}}$ be the sheaf of meromorphic 1-forms on $\tilde{C}$. By definition, a section of $\nu_*M_{\tilde{C}}$ is called a meromorphic differential on $C$. Then a meromorphic differential $\sigma$ is called a Rosenlicht differential if for all $x \in C$ and $g \in \mathcal{O}_{C,x}$,

$$\sum_{y_k \in \nu^{-1}(x)} \text{res}(y_k, g\sigma) = 0$$

holds. Here $\text{res}$ denotes the ordinary residue on $\tilde{C}$.

Now let $\psi$ be a section of $\mathcal{L}^r \otimes \omega_C$. Then $\{\xi_{ij}\}$ and $\psi$ make a natural pairing. The value of this pairing is given as follows. Namely, on a locally closed subset $V_i$, the fiberwise pairing between $\xi_i$ and $\psi$ gives a meromorphic section $\langle \psi, \xi_i \rangle$ of $\omega_C|_{V_i}$. Let $p \in V_i$ be a point at which $\langle \psi, \xi_i \rangle$ has a pole, and let $r_p$ be its residue. If $p$ is contained in another $V_j$ and $V_i \cap V_j$ is an open subset (in this case, $p$ is not a singular point of $C$ and $\psi$ is a holomorphic 1-form there), then since $\xi_i - \xi_j$ is holomorphic, the residue of $\langle \psi, \xi_i \rangle$ at $p$ is the same as that of $\langle \psi, \xi_j \rangle$. On the other hand, let $\{p_1, \ldots, p_l\}$ be the set of singular points of $C$. Let $V_{i_1}, \ldots, V_{i_k}$ be the disks on the branches at $p_i$ as above, and let $r_{i_k}$ be the residue of $\langle \psi, \xi_{i_k} \rangle$ at $p_i$ on $V_{i_k}$ (see Remark 11 for our convention of the terminology).
Note that, by definition, the residue of $\langle \psi, \xi_i \rangle$ at $p_i$ on $V_i$ is the residue of the pull back of $\langle \psi, \xi_i \rangle$ to the normalization of $V_i$ at the inverse image of $p_i$.

Then we have the following.

**Proposition 11.** (1) Any cohomology class in $H^1(C, \mathcal{L})$ can be represented by some set $\{\xi_i\}$ of local sections on locally closed subsets $\{V_i\}$ as above.

(2) The pairing between $\{\xi_{ij}\}$ and $\psi$ is given by

$$\langle \psi, \{\xi_{ij}\} \rangle = \sum_{i=1}^l \sum_{j=1}^{a_i} r_{ij} + \sum_p r_p,$$

where in the second summation, $p$ runs over the poles of $\langle \psi, \xi_i \rangle$ for some $i$ which is not a singular point of $C$. This gives the natural nondegenerate pairing between $H^1(C, \mathcal{L})$ and its dual space $H^0(C, \mathcal{L}^\vee \otimes \omega_C)$.

**Proof.** We begin with proving that the given pairing is well-defined. This contains the following two claims.

First, we prove that the sum on the right hand side is independent of the choice of sections $\{\xi_i\}$ which gives the same $\{\xi_{ij}\}$. To see this, take another set of sections $\{\xi'_i\}$ which defines the same Čech 1-cocycle. Take an irreducible component $C_a$ of $C$. Then, if two locally closed subsets $V_i$ and $V_j$ contained in $C_a$ intersect and $V_i \cap V_j$ is open, the differences $\xi_i - \xi'_i$ and $\xi_j - \xi'_j$ coincide on $V_i \cap V_j$. Thus, these differences define a global section $\xi''_i$ of $\pi_a^* \mathcal{L}|_{C_a}$, where $\pi_a : \tilde{C}_a \to C_a$ is the normalization. Then the fiberwise pairing between $\xi''_i$ and the pull back of $\psi$ gives a meromorphic 1-form on $\tilde{C}_a$, and by the residue theorem the sum of its residues is zero. Doing this calculation on each component of $C$, we see that the pairing $\langle \psi, \{\xi_{ij}\} \rangle$ does not depend on the representative $\{\xi_i\}$.

Second, we need to show that the sum depends only on the cohomology class of $\{\xi_{ij}\}$. Let $\{\xi_i\}$ and $\{\xi'_i\}$ be the sets of local sections defining the same cohomology class. That is, $[\{\xi_{ij}\}] = [\{\xi'_{ij}\}]$, where the indices $i, j$ belong to $\{1, \ldots, m\}$, which is the set of the indices of the open covering $\{U_i\}$. Then there are local sections $\nu_i$ on $\mathcal{L}|_{U_i}$ such that

$$\xi_{ij} - \xi'_{ij} = \nu_i - \nu_j$$

on $U_i \cap U_j$. The sections $\nu_i$ determine sections on locally closed subsets $\{V_i\}$ by restriction. By the equality $\xi_{ij} - \xi'_{ij} = \nu_i - \nu_j$ above, defining $\xi''_i = \xi_i - \nu_i$, we have $\xi''_i = \xi'_i$. Then by the above argument, the classes defined by $\{\xi''_i\}$ and $\{\xi'_i\}$ give the same value of the pairing. On the other hand, adding restrictions of sections of $\mathcal{L}$ does not change the sum of residues at the singular points since $\omega_C$ is the sheaf of Rosenlicht differentials. Thus, $\{\xi_i\}$ and $\{\xi'_i\}$ give the same value of the pairing.

Now, we prove the rest of the claims of the proposition. Assume that $C$ has at least one singular point. Assume $\psi$ has an $|a_1|$-th order zero or pole (when $a_1$ is non-negative or negative, respectively) at some singular point $p_1$ of $C$. Let $V_1$ be one of the disks containing $p_1$ and take a section $\xi_1$ of $\mathcal{L}|_{V_1}$ so that it has an exactly $|a_1| + 1$-th order pole or zero at $p_1$, and take all the other $\xi_i$ to be zero. Then the pairing between $\psi$ and $\{\xi_{ij}\}$ gives a nonzero value, showing that the formula $\langle \psi, \{\xi_{ij}\} \rangle = \sum_{i=1}^l \sum_{j=1}^{a_i} r_{ij} + \sum_p r_p$ actually gives a nondegenerate pairing. This also shows that any cohomology class in $H^1(C, \mathcal{L})$ can be represented by some $\{\xi_i\}$. When $C$ does not contain a singular point, we can take the open cover $\{U_i\}$ in the following way. Namely, given any finite set (in
fact, \( n = 1 \) suffices) of points \( \{ p_1, \ldots, p_n \} \), we can take \( \{ U_{i} \} \) so that each \( p_j \) is contained in the unique open subset belonging to \( \{ U_{i} \} \) (we call it \( U_{j} \)), and \( U_{j} \neq U_{k} \) if \( p_j \neq p_k \). Then applying the same argument as above, we see the same conclusion holds. □

3.2. Expression of the obstruction. Now let us return to the calculation of the obstruction to deform \( \varphi : C \to X \), where \( X \) is a smooth complex surface. We write by \( X_k \) the product space \( X \times \text{Spec} \mathbb{C}[t]/t^{k+1} \).

Assume that we already have a \( k \)-th order deformation
\[
\varphi_k : C_k \to X_k,
\]
for some natural number \( k \), where \( C_k \) is a flat family of curves over \( \mathbb{C}[t]/t^{k+1} \) whose central fibre is \( C \), and \( \varphi_k \) restricts to \( \varphi \) over \( \mathbb{C}[t]/t \).

Let \( I, J \) be sets such that \( I \subset J \). Take an open covering \( \{ U_{i,k} \}_{i \in I} \) of \( C_k \) and a covering \( \{ W_{j} \}_{j \in J} \) of \( X \) by coordinate neighborhoods so that for each \( i \), the following conditions hold:

- The image \( \varphi_k(U_{i,k}) \) is contained in \( W_{k,i} := W_{i} \times \text{Spec} \mathbb{C}[t]/t^{k+1} \).
- The intersection \( W_{k,i} \cap \varphi_k(U_{k,i}) \) is represented by an equation \( f_{k,i} = 0 \), where \( f_{k,i} \) is an analytic function on \( W_{j,k} \).

On the intersection \( W_{k,i} \cap W_{k,j} \) such that \( U_{k,i} \cap U_{k,j} \neq \emptyset \), the functions \( f_{k,i} \) and \( f_{k,j} \) are related by
\[
f_{k,i} = g_{k,ij} f_{k,j},
\]
where \( g_{k,ij} \) is a holomorphic function on \( W_{k,i} \cap W_{k,j} \) whose reduction over \( \mathbb{C}[t]/t \) is a non-vanishing function on \( W_{i} \cap W_{j} \). Regarding these functions as defined over \( \mathbb{C}[t]/t^{n+2} \), we obtain the difference
\[
t^{n+1} \nu_{ij,n+1} = f_{k,i} - g_{k,ij} f_{k,j}
\]
on \( W_{k+1,i} \cap W_{k+1,j} \). Here \( \nu_{ij,n+1} \) can be regarded as a holomorphic function on \( W_{i} \cap W_{j} \).

Assume that there is another \( W_{k,l} \) containing \( \varphi_k(U_{k,i}) \), and define \( f_{k,l} \), \( g_{k, jl} \) and \( g_{k,il} (= g_{k, il}^{-1}) \) as above. These determine functions \( \nu_{k+1, jl}, \nu_{k+1,il} \) etc. on the relevant intersections.

Lemma 12. On \( W_{i} \cap W_{j} \cap W_{l} \cap \varphi(U_{i}) \), the identity
\[
\nu_{k+1, il} = \nu_{k+1, ij} + g_{0,ij} \nu_{k+1, jl}
\]
holds (see also [9]).

Remark 13. Note that \( W_{i} \cap W_{j} \cap W_{l} \cap \varphi(U_{i}) = W_{i} \cap W_{j} \cap W_{l} \cap \varphi(U_{j}) = W_{i} \cap W_{j} \cap W_{l} \cap \varphi(U_{l}) \) holds. However, in general this is not equal to \( W_{i} \cap W_{j} \cap W_{l} \cap \varphi(C) \).

Proof. By definition, we have
\[
t^{n+1} \nu_{k+1, il} = f_{k, i} - g_{k, il} f_{k, l} = f_{k, i} - g_{k, il} f_{k, j} + g_{k, il} f_{k, j} - g_{k, il} f_{k, l} = t^{k+1} \nu_{k+1, ij} + g_{k, ij} f_{k, j} - g_{k, il} f_{k, l} + (g_{k, ij} g_{k, jl} - g_{k, il}) f_{k, l} = t^{k+1} \nu_{k+1,ij} + g_{k, ij} g_{k, il} f_{k, l} + (g_{k, ij} g_{k, jl} - g_{k, il}) f_{k, l}.
\]
Also, note that the identity
\[
g_{k, il} \equiv g_{k, ij} g_{k, jl} \mod t^{n+1}
\]
holds. Therefore, we have
\[
(g_{k, ij} g_{k, jl} - g_{k, il}) f_{k, l} = (g_{k, ij} g_{k, jl} - g_{k, il}) f_{0, l} \mod t^{n+2}.
\]
Since the identity $f_{0,i} = 0$ holds on $W_i \cap W_j \cap W_l \cap \varphi(U_i)$, we have the claim. \hfill \Box

Note that the restriction of the set of functions $\{g_{0,ij}\}$ to $\varphi(C)$ is the set of transition functions for the normal sheaf of $\varphi$. Thus, the lemma shows that the set of functions $\{\nu_{k+1,ij}\}$, when restricted to $\varphi(C)$, behaves as a Čech 1-cocycle with values in the normal sheaf of $\varphi$. By construction, we have the following (see also [9]).

Lemma 14. The Čech 1-cocycle $\{\nu_{k+1,ij}\}$ is the obstruction cocycle to deform $\varphi_k$ to a map over $\mathbb{C}[t]/t^{k+2}$. \hfill \Box

However, it is difficult to calculate this class in this form. So we develop another way to express it. Recall we fixed a function $f_{k,i}$ on $W_{k,i}$. Regard it as a function on $W_{k+1,i}$ as above. Then we can express it in the form

$$f_{k,i} = f_{0,i} \exp(h_i(k + 1)),$$

where $h_i(k + 1)$ is a function on $W_{k+1,i}$ which can have poles along $\{f_{0,i} = 0\}$ (here $f_{0,i}$ is also regarded as a function on $W_{k+1,i}$), and is zero when reduced over $\mathbb{C}[t]/t$. Similarly, we can write $f_{k,j}$, regarded as a function on $W_{k+1,j}$, in the form $f_{k,j} = f_{0,j} \exp(h_j(k + 1))$. Let $h_{k+1,i}$ and $h_{k+1,j}$ be the coefficients of $t^{k+1}$ in $h_i(k + 1)$ and $h_j(k + 1)$, respectively. These can be naturally considered as meromorphic functions on $W_i$ and $W_j$. Then we have the following.

Lemma 15. The identity

$$h_{k+1,i} - h_{k+1,j} = \frac{\nu_{k+1,ij}}{f_{0,i}} + \kappa$$

holds on $W_i \cap W_j$, here $\kappa$ is a holomorphic function.

Proof. By definition, we have

$$t^{k+1} \nu_{k+1,ij} = f_{0,i} \exp(h_i(k + 1)) - g_{k,ij} f_{0,j} \exp(h_j(k + 1)).$$

Dividing this equation by $f_{0,i} \exp(h_j(k + 1))$, we have

$$t^{k+1} \frac{\nu_{k+1,ij}}{f_{0,i}} = \exp(h_i(k + 1) - h_j(k + 1)) - g_{k,ij} \frac{f_{0,j}}{f_{0,i}}.$$

Note that since the function $\exp(h_j(k + 1))$ is of the form $1 + t(\cdots)$, dividing by it does not affect the left hand side because the equation is defined over $\mathbb{C}[t]/t^{k+2}$. Also, note that the term $g_{k,ij} \frac{f_{0,j}}{f_{0,i}}$ is holomorphic on $W_{k+1,i} \cap W_{k+1,j}$. Therefore, the divergent terms of the function $\exp(h_i(k + 1) - h_j(k + 1))$ coincide with the divergent terms of $t^{k+1} \frac{\nu_{k+1,ij}}{f_{0,i}}$. It follows that the function $h_i(k + 1) - h_j(k + 1)$ does not have a divergent term when it is reduced over $\mathbb{C}[t]/t^{k+1}$. Thus, the divergent terms of $h_i(k + 1) - h_j(k + 1)$ coincide with those of $t^{k+1} \frac{\nu_{k+1,ij}}{f_{0,i}}$. This proves the claim. \hfill \Box

Let $\psi|_C$ be an element of the group $H^0(C, \varphi^*K_X)$ and assume it is the pull back of an element $\psi$ of $H^0(X, K_X)$. The Poincaré residue of the 2-form $\frac{\nu_{k+1,ij}}{f_{0,i}} \psi$ along $\{f_{0,i} = 0\} \cap W_i \cap W_j$ is defined by the pullback to $U_i \cap U_j$ of the 1-form $\zeta_{ij}$ on $W_i \cap W_j$ satisfying

$$\frac{\nu_{k+1,ij}}{f_{0,i}} \psi = \zeta_{ij} \wedge \frac{df_{0,i}}{f_{0,i}}.$$

(1)
From this definition, it is clear that the pullback of $\zeta_{ij}$ to $U_i \cap U_j$ coincides with the fiberwise pairing between $\nu_{k+1,ij}$ and $\psi|_C$ (recall that $\nu_{k+1,ij}$ is naturally considered as a local section of the normal sheaf $N_\varphi$ on $C$). We write $\zeta_{ij}$ and its pull back to $U_i \cap U_j$ by the same letter. These $\zeta_{ij}$ constitute a Čech 1-cocycle on $C$ with values in $\omega_C$, and according to Lemma\ref{lemma14} this is essentially (one of) the obstruction to deform $\varphi_k$. Precisely, we have the following.

**Proposition 16.** If the cohomology class of $\{\zeta_{ij}\}$ in $H^1(C, \omega_C)$ associated to each $\psi \in H^0(C, \varphi^* K_X)$ vanishes, and $\varphi$ is semiregular, then the obstruction to deform $\varphi_k$ vanishes.

Now let us consider the meromorphic two form $h_{k+1,i} \psi$ on $W_i$. We would like to define its natural analogue of the Poincaré residue along $\varphi(U_i)$ as above. However, we need to take care of the facts that $h_{k+1,i} \psi$ may have poles worse than logarithmic ones and also that the hypersurface $\varphi(U_i)$ is singular. In the following subsections, we study these points.

### 3.3. Residues of meromorphic differential forms.

We have constructed a meromorphic two form $h_{k+1,i} \psi$ on each open subset $W_i$ of $X$. On the intersection $W_i \cap W_j$, the difference $h_{k+1,i} \psi - h_{k+1,j} \psi$ equals to $\frac{\nu_{k+1,ij}}{f_{0,i}} \psi$ modulo holomorphic terms. The logarithmic two form $\frac{\nu_{k+1,ij}}{f_{0,i}} \psi$ essentially represents the obstruction (see Equation \ref{equation11}). We will associate a Čech representative in the form described in Subsection \ref{subsection3.1} with this datum. In the argument below, we use the same notation as in Subsection \ref{subsection3.1}.

Recall that the cohomology class of the Čech representative is determined by the residues at the poles of suitable pairings. First, we introduce a quantity which will play the role of the residues in the present context. Let $\{U_i\}$ be the open covering of $C$ as before. Each $U_i$ is the union of disks $U_{1,i}, \ldots, U_{k,i}$ attached at one point $p_i$, when $U_i$ contains a singular point of $C$. Since $\varphi$ is locally an embedding, we identify $U_{a,i}$ with its image in $X$.

Fix an arbitrary Riemannian metric on $X$. Let $\gamma$ be any smooth simple closed curve on $U_{a,i}$ which generates the fundamental group of the punctured disk $U_{a,i} \setminus \{p\}$. On any open subset $U_{a,i}$ of $U_{a,i}$ away from $p$ (that is, the closure of $U_{a,i}$ does not contain $p$), the exponential map gives a diffeomorphism from a small neighborhood of the zero section of the normal bundle of $U_{a,i}$ to a tubular neighborhood of $U_{a,i}$ in $X$.

Let $S_\delta U_{a,i}$ be the circle bundle of radius $\delta$ in the normal bundle. Here, $\delta$ is a small positive number. By taking $\delta$ small enough, we can assume the image of $S_\delta U_{a,i}$ by the exponential map is disjoint from $\varphi(U_i)$. Then, the meromorphic 2-form $h_{k+1,i} \psi$ is pulled back to a smooth 2-form on $S_\delta U_{a,i}$. We write the pull back by the same notation $h_{k+1,i} \psi$ to save letters. Let $T_\gamma$ be the union of the fibers of $S_\delta U_{a,i}$ over $\gamma$, which is diffeomorphic to the two dimensional torus. Note that the fiber of $T_\gamma$ has a natural (positive) orientation induced from the natural complex orientation of the normal bundle of $U_{a,i}$. This and the orientation of $\gamma$ determine an orientation of $T_\gamma$ by ordering a basis of a tangent space as $\{\text{base, fiber}\}$. This is chosen so that it will be compatible with Equation \ref{equation11} in the following calculation.

**Lemma 17.** The integration of $h_{k+1,i} \psi$ over $T_\gamma$ does not depend on the metric, $\delta$, or $\gamma$. Therefore, it only depends on $p$ and $a$. 
Lemma 17. by associating $\mu$ once in the positive direction does not depend on the contour and given by

$$\int_{a,i} \mu$$

contour integral of $\mu$ assume that the intersection

$U$ path circles around $p$ only on the homotopy class of the path when $U$ does not contain a marked point, and the integral along a closed path in $\mu$ on $U$ can also be seen as an open subset of $C$ the form $U$ can also be seen as an open subset of $C$, and we assume $\tilde{C}_i$ is covered by open subsets of the form $U_{a,i}$. On such an open subset $\tilde{C}_i$, the above construction glues and gives a circle bundle $S_{\delta C_i}$ of radius $\delta$ inside the normal bundle of $\tilde{C}_i$.

On each intersection $U_{a,i} \cap \tilde{C}_i$, the 2-form $h_{k+1,i}\psi$ is pulled back to $S_{\delta \tilde{C}_i}|_{U_{a,i} \cap \tilde{C}_i}$ and gives a smooth 2-form, which again we write by the same notation. Then since the fiber integration commutes with the exterior derivative, that of $h_{k+1,i}\psi$ gives a closed 1-form on $U_{a,i} \cap \tilde{C}_i$, which we write by $\int_{\delta} h_{k+1,i}\psi$.

When $U_{a,i}$ does not contain a singular point of $C$ (so that $U_i = U_{a,i}$ is really a disk), then the form $\int_{\delta} h_{k+1,i}\psi$ is defined on the whole $U_{a,i}$. When $U_{a,i}$ contains a singular point $p$, $\int_{\delta} h_{k+1,i}\psi$ is defined only on $U_{a,i} \cap \tilde{C}_i$. But taking $\tilde{C}_i$ larger by taking $\delta$ smaller, we can assume that the intersection $U_i \cap U_j \cap \tilde{C}_i$ is contained in $\tilde{C}_i$ for any distinct $i$ and $j$. Since $\int_{\delta} h_{k+1,i}\psi$ is a closed form, the value of the integration of it along a contour which circles $p$ once in the positive direction does not depend on the contour and given by $r(p,a)$ by Lemma 17.

Now consider the intersection $U_i \cap U_j \cap \tilde{C}_i$. According to Lemma 15 we have

$$\int_{\delta} h_{k+1,i}\psi - \int_{\delta} h_{k+1,j}\psi = \int_{\delta} \left( \frac{\nu_{k+1,i,j}}{\int_{\partial i} + \kappa} \right) \psi.$$ 

These naturally give a $C^1(\tilde{C}_i)$-valued Čech 1-cocycle on $\tilde{C}_i$ associated with a covering of $\tilde{C}_i$ induced by $\{U_i\}$. Here $C^1(\tilde{C}_i)$ is the sheaf of complex valued $C^\infty$ closed 1-forms on $\tilde{C}_i$. We will compare the cohomology class defined by this cocycle with the ones described in Subsection 3.3.1.

3.3.1. Cohomology classes defined by closed 1-forms and the pairing. In general, let $S$ be a compact nonsingular Riemann surface with finite number of marked points $\{p_i\}_{i=1}^b$, $b \geq 1$. Take an open covering $\{U_j\}$ of $S$ so that each $p_i$ is contained in a unique open subset belonging to $\{U_j\}$, which we write by $U_i$. We assume $U_i \neq U_j$ if $p_i \neq p_j$. When $U_j$ does not contain a marked point, then we associate with it a closed 1-form $\mu_j$ on it. When $U_j$ contains a marked point $p_j$, then we associate with it a closed 1-form $\mu_j$ on $U_j \setminus B(p_j)$, where $B(p_j)$ is a small disk around $p_j$. We take each disk $B(p_j)$ small enough so that the intersection $U_i \cap U_j$ for distinct $i$ and $j$ does not intersect any such disk. Note that since $\mu_i$ is closed, the contour integral along a closed path in $U_j$ is zero when $U_j$ does not contain a marked point, and the integral along a closed path in $U_j \setminus B(p_j)$ depends only on the homotopy class of the path when $U_j$ contains the marked point $p_j$. When a path circles around $p_j$ once in the positive direction, we write by $r(p_j)$ the value of the contour integral of $\mu_j$ along the path.

Such a family of closed 1-forms $\{\mu_i\}$ determines a Čech 1-cocycle with values in $C^1(S)$ by associating $\mu_{ij} := \mu_i - \mu_j$ with the intersection $U_i \cap U_j$. There is a standard resolution

Proof. This follows from the Stokes’ theorem since $h_{k+1,i}\psi$ is a closed 2-form. □

Definition 18. When $\gamma$ is positively oriented with respect to the complex orientation of $U_{a,i}$, then we write the value obtained in Lemma 17 by $r(p,a)$. 

Let $C_i$ be any irreducible component of $C$. Let $\pi_l: \tilde{C}_i \rightarrow C_i$ be its normalization. Let $s\tilde{C}_i$ be the set of points on $\tilde{C}_i$ which are mapped to singular points of $C$ by $\pi_l$. Also, let $\tilde{C}_i$ be an open subset of $\tilde{C}_i$ obtained by deleting small disks around $s\tilde{C}_i$. Note that $\tilde{C}_i$ can also be seen as an open subset of $C_i$, and we assume $\tilde{C}_i$ is covered by open subsets of the form $U_{a,i}$. On such an open subset $\tilde{C}_i$, the above construction glues and gives a circle bundle $S_{\delta C_i}$ of radius $\delta$ inside the normal bundle of $\tilde{C}_i$.
of the sheaf $\mathcal{C}^1(S)$, 
\[ 0 \rightarrow \mathcal{C}^1(S) \rightarrow \mathcal{A}^1(S) \rightarrow \mathcal{A}^2(S) \rightarrow 0 \]
by flabby sheaves. Here $\mathcal{A}^i(S)$ is the sheaf of complex valued $C^\infty$ $i$-forms on $S$. In particular, the cohomology group $H^i(S, \mathcal{C}^1(S))$ is isomorphic to $H^2(S, \mathcal{C}) \cong H^1(S, \omega_S)$, where $\omega_S$ is the canonical sheaf of $S$.

Since $H^2(S, \mathcal{C})$ is dual to $H^0(S, \mathcal{C})$, the Čech 1-cocycle $\{\mu_{ij}\}$ should give an element of the dual of $H^0(S, \mathcal{C})$ in a natural manner. This is an analogue of Proposition 11. We need to associate a scalar with a constant function $\alpha \in H^0(S, \mathcal{C})$ in a natural way. Namely, we associate the value
\[ \langle \{\mu_{ij}\}, \alpha \rangle = \alpha \sum_i r(p_i) \]
with it.

**Proposition 19.** This pairing between the classes $\{\mu_{ij}\} \in H^2(S, \mathcal{C})$ and $\alpha \in H^0(S, \mathcal{C})$ is well-defined and non-trivial.

**Proof.** We need to check that the value $\langle \{\mu_{ij}\}, \alpha \rangle$ does not depend on the choices of the 1-forms $\{\mu_i\}$ with $\{\mu_{ij}\}$ fixed, nor the representative $\{\mu_{ij}\}$ of the cohomology class.

Given $\{\mu_{ij}\}$, assume we have another set of singular 1-forms $\{\mu_i'\}$ satisfying $\mu_i' - \mu_i' = \mu_{ij}$. As in the proof of Proposition 11 the set of differences $\{\mu_i - \mu_i'\}$ gives a global closed 1-form $\mu$ on $\tilde{S} := S \setminus (\cup_j B(p_j))$. If $\{\gamma_i\}$ is the set of disjoint contours around $p_i$ encircling $p_i$ once in the positive direction, then by the Stokes’ theorem, we have
\[ \sum_i \int_{\gamma_i} \mu = 0. \]
Therefore, the pairing between $\mu$ and $\alpha$ is zero. Thus, the pairing does not depend on the choice of $\{\mu_i\}$ with $\{\mu_{ij}\}$ fixed.

Suppose there is another representative $\{\mu_{ij}'\}$ of the cohomology class $[\{\mu_{ij}\}]$ in $H^1(S, \mathcal{C}^1(S))$. Then there is a set of smooth closed 1-forms $\{\nu_i\}$ on $U_i$ such that $\mu_{ij} - \mu_{ij}' = \nu_i - \nu_j$ on $U_i \cap U_j$. Then defining $\mu_{ij}'' := \mu_i - \nu_i$, we have $\mu_{ij}'' = \mu_{ij}'$. From this, it is clear that the pairing between $\{\mu_{ij}\}$ and $\alpha$ is equal to that between $\{\mu_{ij}'\}$ and $\alpha$.

The proof of the fact that the paring is non-trivial is given by an obvious modification of the proof of Proposition 11. $\square$

By definition of the pairing above, we see the following.

**Corollary 20.** The class defined by $\{\mu_{ij}\}$ can be identified with $\sum_i r(p_i)$ using the basis of $H^2(S, \mathcal{C})$ dual to the standard generator of $H^0(S, \mathbb{Z}) \subset H^0(S, \mathcal{C})$. $\square$

3.3.2. The main theorem. Applying the above construction to our situation, we have the following.

**Proposition 21.** The cohomology class of $H^1(\tilde{C}_i, \mathcal{C}^1(\tilde{C}_i)) \cong H^2(\tilde{C}_i, \mathcal{C})$ defined by the Čech 1-cocycle $\{\int_{\delta} h_{k+1,i} \psi - \int_{\delta} h_{k+1,j} \psi\}$ does not depend on $\delta$ and identified with $\sum_{a,p} r(a, p)$ in the sense of Corollary 20. Here $p$ runs through the singular points of $C$ on $\tilde{C}_i$ and $a$ indexes the local branches of $C$ at $p$ contained in $C_i$. $\square$

In particular, if the limit $\lim_{\delta \to 0} \{\int_{\delta} h_{k+1,i} \psi - \int_{\delta} h_{k+1,j} \psi\}$ exists, then it will give the same cohomology class as $\{\int_{\delta} h_{k+1,i} \psi - \int_{\delta} h_{k+1,j} \psi\}$. This is in fact the case by the equality
\[ \int_\delta h_{k+1,i} \psi - \int_\delta h_{k+1,j} \psi = \int_\delta \left( \frac{\nu_{k+1,j}}{h, i} + \kappa \right) \psi, \] since the integral \[ \int_\delta \left( \frac{\nu_{k+1,j}}{h, i} + \kappa \right) \psi \] converges to the usual Poincaré residue \( \zeta \) (see Equation (1)) as \( \delta \) goes to zero.

The set of local 1-forms \( \{ \int_\delta h_{k+1,i} \psi - \int_\delta h_{k+1,j} \psi \} \) determines a class of \( H^1(\bar{C}_i, C^1(\bar{C}_i)) \cong H^1(\bar{C}_i, \omega_{\bar{C}_i}) \) for each component \( \bar{C}_i \), while the set of holomorphic 1-forms \( \{ \zeta \} \) determines a class of \( H^1(C, \omega_C) \). Although these groups are a priori different, the restriction of a class in \( H^1(C, \omega_C) \) to the component \( C_i \) is naturally identified with the dual of \( H^0(C_i, C) \cong H^0(\bar{C}_i, \omega_{\bar{C}_i}) \), using the description in Subsection 3.1. By Proposition 11, the class \( \{ \zeta \} \) can be represented by local meromorphic 1-forms on open subsets \( U_{i,a} \).

Regarding these 1-forms as closed singular \( C^\infty \) 1-forms, it is clear that this identification is compatible with the one \( H^1(\bar{C}_i, C^1(\bar{C}_i)) \cong H^0(\bar{C}_i, \omega_{\bar{C}_i}) \) described above. Thus, the set of local 1-forms \( \{ \int_\delta h_{k+1,i} \psi - \int_\delta h_{k+1,j} \psi \} \) also determines a class of \( H^1(C, \omega_C) \). Therefore, we have the following.

**Corollary 22.** The cohomology classes of \( H^1(C, \omega_C) \) determined by \( \{ \int_\delta h_{k+1,i} \psi - \int_\delta h_{k+1,j} \psi \} \) and \( \{ \zeta \} \) coincide. \( \square \)

By this corollary, to prove the vanishing of the obstruction class \( \{ \zeta \} \), it suffices to see the vanishing of the class \( \{ \int_\delta h_{k+1,i} \psi - \int_\delta h_{k+1,j} \psi \} \) for all \( \psi \in H^0(X, K_X) \), when semiregularity holds. Recall that this is the same as the vanishing of the scalar \( \sum_{a,p} r(a, p) \). Thus, if the sum \( \sum_a r(a, p) \) vanishes for each fixed \( p \), the obstruction vanishes, too. Here, the sum runs over all the branches of \( C \) at \( p \). Recall that the scalar \( r(a, p) \) is the value of the integration of the two form \( h_{k+1,i} \) along the restriction of the circle bundle \( S_\delta \bar{C}_i \) over a loop \( \gamma_{a,p} \) encircling the singular point \( p \) (here \( p \) is contained in the open subset \( U_{a,i} \subset C_i \)). The circle bundle is embedded in \( X \) by the exponential map.

Now consider the boundary \( \partial B \cong S^3 \) of a small ball in \( X \) around \( p \). Let \( D_\delta \gamma_{a,p} \) be the disk bundle over \( \gamma_{a,p} \) whose boundary is \( S_\delta \bar{C}_i \mid \gamma_{a,p} \). Topologically it is a solid torus. By suitably isotoping, these disk bundles can be disjointly embedded in \( \partial B \) for all branches indexed by \( a \). Now by the Stokes’ theorem, we have

\[ \sum_a r(a, p) = \sum_a \int_{S_\delta \bar{C}_i \mid \gamma_{a,p}} h_{k+1,i} \psi = - \int_{\partial B \cup \bigcup_a D_\delta \gamma_{a,p}} d(h_{k+1,i} \psi) = 0. \]

Thus, we have proved the following theorem. Let \( \varphi: C \to X \) be a map from a reduced complete curve to a smooth complex surface which is locally an embedding.

**Theorem 23.** If the map \( \varphi: C \to X \) is semiregular, then it is unobstructed in the sense that any first order deformation can be extended to any higher order. \( \square \)

This enables us to construct large number (often infinitely many) of curves with specific geometric properties from a small number of data.

**Remark 24.** Note that the calculation of the obstruction to deform \( \varphi \) is local. Namely, whether \( \varphi \) is obstructed or not is determined by the data of a neighborhood of \( \varphi(C) \) in \( X \). In particular, we may replace \( X \) by a suitable open subset \( X' \) of \( X \). Even if \( \varphi \) is not semiregular as a map \( \varphi: C \to X \) (for example, when \( H^0(X, K_X) \) vanishes but \( H^0(C, \varphi^* K_X) \) does not), when some \( \varphi: C \to X' \), with the target suitably replaced, is semiregular, then the original \( \varphi \) is also unobstructed.

By the same reasoning, the curve \( C \) can be more general than the argument above, where we assumed \( C \) to be compact (and reduced). Namely, \( C \) can contain some punctures and
boundary components, and if we combine with the method in [11], we can use Theorem 23 to produce many disks with Lagrangian boundary conditions in various situations.

4. Deformation of maps into Calabi-Yau surfaces

Let \( \varphi : C \to X \) be a map from a reduced connected curve to a smooth surface which is locally an embedding as before. Theorem 23 applies to various situations. Among them, it is particularly effective when the canonical sheaf of \( X \) is trivial, because in this case any \( \varphi \) is semiregular, and thus unobstructed. Before applying this to actual problems, we introduce the following terminology.

Let us write \( C = \bigcup C_i \), where \( C_i \) is an irreducible component of \( C \). Let \( \tilde{C} = \bigcup \tilde{C}_i \) be the normalization of \( C \). Here \( \tilde{C}_i \) is the normalization of \( C_i \).

Definition 25. Let \( C' \) be a nodal curve. We call it an unchaining of \( C \) if the natural map \( p : \tilde{C} \to C \) factors through \( C' \).

Let \( C'' \) be an unchingning of \( C \) and assume \( C \) is nodal. Then the map \( \varphi : C \to X \) gives a natural map \( \varphi' : C' \to X \) which is also locally an embedding. In general it is more difficult to deform \( \varphi' \) than \( \varphi \). However, as noted above, when the canonical sheaf \( K_X \) is trivial, they are both unobstructed.

This simple claim can be used to produce a lot of curves of prescribed genera on Calabi-Yau surfaces. For example, let us consider the following result, see [6, Chapter 13, Proposition 2.4].

Proposition 26. Let \( X \) be a K3 surface over an algebraically closed field of characteristic zero. Suppose there exists an integral nodal rational curve \( C \) in \( X \) of arithmetic genus \( g > 0 \). Then there is a one-dimensional family of nodal elliptic curves in the linear system \( |O(C)| \).

In [6], this result is proved by comparing the dimensions of \( |O(C)| \) and the moduli space of stable curves. Using Theorem 23 we can prove a stronger result with a simpler proof.

Proposition 27. In the situation of Proposition 26 and assuming \( X \) is a complex K3 surface, there is a \( k \)-dimensional family of nodal curves of geometric genus \( k \) for each \( 0 \leq k \leq g \).

Proof. Consider the curve \( C \) as an embedding \( \varphi : C \to X \). Take a connected unchaining \( C' \) of \( C \) so that the arithmetic genus of \( C' \) is one. By Theorem 23 the induced map \( \varphi' : C' \to X \) is unobstructed. It is easy to see that the dimension of \( H^0(C', \nu_{\varphi'}) \) is one. Thus, the map \( \varphi' \) has a one dimensional nontrivial deformation. Since a complex K3 surface cannot be dominated by a family of rational curves, the image of a deformed map of \( \varphi' \) must be an elliptic curve.

Next, consider a connected unchaining \( C'' \) of \( C \) so that the arithmetic genus of \( C'' \) is two. As before, the induced map \( \varphi'' : C'' \to X \) is unobstructed, and one sees that the dimension of \( H^0(C'', \nu_{\varphi''}) \) is two. Since there are only one dimensional families of elliptic curves by the above paragraph, the image of a general deformation of \( \varphi'' \) has geometric genus two. Repeating the argument in the same way, one obtains the proposition. \( \square \)
Note that in general this produces plenty of local families of curves since there are many different unchainings of the given curve even when the genus is fixed. Pursuing the idea of the proof of this proposition, we can prove a much stronger result.

**Theorem 28.** Let $\mathcal{M}$ be a moduli space of polarized complex K3 surfaces. Then there is a Zariski open subset which satisfies the following property. Namely, if $X$ is a member of the open subset, then it contains infinitely many $g$ dimensional families of irreducible nodal curves of geometric genus $g$, for any positive integer $g$.

**Proof.**

Let $\alpha \in H^2(X, \mathbb{Z})$ be the class of $X$ which gives the polarization. In the paper \[5\], Chen proved that for every positive integer $k$, there is a Zariski open subset $U_k$ of $\mathcal{M}$ whose member contains (finite number of) irreducible nodal rational curves of the class $k\alpha$. Moreover, for different $k$ and $k'$, the intersection $U_k \cap U_{k'}$ contains a Zariski open subset on which the intersection of the rational curves of degrees $k$ and $k'$ are ordinary nodes.

Using this, it is clear that there is a Zariski open subset of $\mathcal{M}$ whose member contains a (not necessarily irreducible) nodal rational curve of arithmetic genus at least two. As in the proof of Proposition 27, we first regard it as the image of a map $\varphi_1$ from a suitable unchaining which is a connected nodal rational curve of arithmetic genus one, see Figure 1.

![Figure 1](image)

**Figure 1.** A nodal rational curve in a K3 surface $X$ (the figure on the right) is seen as the image of a map from a nodal rational curve of arithmetic genus one. In this figure, the curve looks irreducible, but in general the curve need not be irreducible.

As in the proof of Proposition 27, this map $\varphi_1$ deforms and gives a one parameter family of nodal curves of geometric genus one.

Next, since the domain of the map $\varphi_1$ has the fundamental group isomorphic to $\mathbb{Z}$, we can take its covering of arbitrary degree, see Figure 3.

![Figure 3](image)

We write this curve as $C_k$ when the covering degree is $k$. The map $\varphi_1$ naturally gives a map $\varphi_k: C_k \to X$. As before, this curve deforms and the image is a one parameter family of nodal curves of geometric genus one. Note that under the deformation, all the nodes of the domain curve must be smoothed in order to acquire non-zero genus. In particular, the domain curve of the deformed map and its image as well are irreducible. This proves the theorem in the case $g = 1$. 
Figure 2. As the map $\varphi_1$ deforms, the image in $X$ deforms to a curve of geometric genus one.

Figure 3. The points $P$ and $Q$ are identified. The result is a nodal rational curve of arithmetic genus one.

Next, consider the case $g = 2$. Recall that we took a nodal rational curve of arithmetic genus at least two. Therefore, there is an unchaining $C'$ of it which is of arithmetic genus two such that the map $\varphi_1$ above splits through it, see Figure 4.

Then replacing one of the sheets of the covering constructed in the case $g = 1$ above by a partial normalization of $C'$, we obtain a prestable rational curve of arithmetic genus two, see Figure 5.

We write this curve as $C'_k$. As before, the map $\varphi_1$ induces a map $\varphi'_k: C'_k \to X$, and it has two dimensional deformations. As in the proof of Proposition 27, a general member of the deformed curves must be of geometric genus two. This proves the theorem for $g = 2$. For curves with higher genera, we can use the similar construction, replacing more sheets.
Figure 5. One of the sheets of the curve in Figure 3 is replaced by a partial normalization of $C'$. The points $P$ and $R$ are the inverse images of the node under the partial normalization. As in Figure 3, the points $P$ and $Q$ are identified so that the resulting curve has arithmetic genus two.

in Figure 2 by $C'$.

This theorem proves that any moduli space of polarized K3 surface contains a Zariski open subset, independent of $g$, whose members contain infinitely many $g$ dimensional families of curves of genus $g$. It is plausible that the same holds for any polarized K3 surface. According to the proof of the theorem, this is proved when one can prove the existence of a few (in fact, just one when the arithmetic genus is at least two) rational curves on that K3 surface which are at worst nodal.

On the other hand, assume that this does not hold. That is, suppose that there is a complex projective K3 surface $X$ such that for some $g \geq 1$, $X$ contains only finitely many $g$ dimensional families of nodal curves of geometric genus $g$. Then such $X$ must contain a $g$ dimensional family of curves of geometric genus $g' < g$ (general member of such a family must have singularity worse than the node). This follows from the theorem by considering a 1-parameter family of K3 surfaces such that its central fiber is $X$ and the others are contained in the open subset given in the theorem.

However, when $g$ is one, since rational curves on $X$ cannot deform, such a situation does not occur. Thus, we have the following (see also [6, Chapter 13]).

Corollary 29. Let $X$ be a complex projective K3 surface. Then there are infinitely many different dominant maps $p_i: E_i \to X$, $i \in \mathbb{N}$, from one dimensional families $E_i$ of curves of geometric genus one. In other words, any complex projective K3 surface is dominated by families of elliptic curves in infinitely many different ways.

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