Asymptotic stability of solitons to 1D nonlinear Schrödinger equations in subcritical case

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Abstract We prove the asymptotic stability of solitary waves to 1D nonlinear Schrödinger equations in the subcritical case with symmetry and spectrum assumptions. One of the main ideas is to use the vector fields method developed by S. Cuccagna, V. Georgiev, and N. Visciglia [Comm. Pure Appl. Math., 2013, 6: 957–980] to overcome the weak decay with respect to $t$ of the linearized equation caused by the one dimension setting and the weak nonlinearity caused by the subcritical growth of the nonlinearity term. Meanwhile, we apply the polynomial growth of the high Sobolev norms of solutions to 1D Schrödinger equations obtained by G. Staffilani [Duke Math. J., 1997, 86(1): 109–142] to control the high moments of the solutions emerging from the vector fields method.

Keywords Nonlinear Schrödinger equation (NLS), solitons, weak nonlinearity, asymptotic stability

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1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation (NLS):

\[
\begin{aligned}
&i u_t + \partial_x^2 u = F(|u|^2)u, \\
&u(0, x) = u_0(x),
\end{aligned}
\]

where $u: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. The nonlinear Schrödinger equation is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides, and to Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps in the mean-field regime.
NLS has a family of localized solutions among which the so-called solitons are the best understood. We are interested in the dynamics of NLS around solitons. The orbital stability of solitons and multi-solitons, was considered by many authors for various models, for instance, Grillakis et al. [18,19], Weinstein [43], Martel et al. [29]. The other main problem on solitons is the asymptotic stability which states that any solution of NLS initiated near the family of solitons decomposes into a moving soliton and a radiation part with an asymptotically vanishing remainder as $t \to \infty$. This is best known for completely integrable equations for instance the one dimensional cubic NLS by using the inverse scattering method. For general nonlinearities, the first asymptotic stability result was obtained by Soffer and Weinstein [38] in context of the equation

$$iu_t + \Delta_x u + V(x)u = F(|u|^2)u. \tag{2}$$

There have been a lot of works in the study of asymptotic stability of solitons for (2) especially for attractive potentials, for instance, [14,16,17,20,22,23,28,31].

The asymptotic stability of solitons for (1) started from Buslaev and Perelman [7] where the one dimension NLS was considered and further refined in [8,9]. Their work was extended to high dimensions in Cuccagna [12]. Later, Perelman [33] and Rodnianski et al. [35] proved the asymptotic stability for multi-solitons in high dimensions. When blow up solution exists such as the super-mass critical and mass critical NLS equations, instead of asymptotic stability, stable and center stable manifolds are introduced to describe the dynamics of NLS near solitons, see Krieger and Schlag [26,27], Beceanu [1–3], Cuccagna [13], and Schlag [36]. More recently, Pusateri and Soffer [34] developed bilinear estimates method for global stability problems of critical NLS in 3D.

The idea in this paper to prove asymptotic stability of solitons for (1) is to use the vector field method to obtain more decay of the solution to the linearized equation with respect to $t$. This appeared originally in Klainerman [24,25] where the small data global well-posedness of quasilinear wave equations was solved. One of the key idea of the vector field method is to gain the decay with respect to time by paying more decay and regularity with respect to the spatial variables. The same idea works for Schrödinger equations, see McKean and Shatah [30] for NLS and Cuccagna et al. [15] for NLS with a potential.

In order to state our theorem, we give the definition of solitary waves.

**Definition 1** We call the periodic localized solution to (1),

$$w(x; \sigma) = \exp\left(-i\beta + i \frac{v}{2} x\right) \varphi(x - b; \alpha),$$

solitary wave, if $\varphi(x; \alpha)$ is a radial positive function and the time dependent parameters $\sigma(t) := (\beta(t), \omega(t), b(t), v(t))$ satisfy

$$\partial^2_x \varphi = \frac{\alpha^2}{4} \varphi + F(\varphi^2)\varphi,$$

$$\beta' = \omega, \quad \omega' = 0, \quad b' = v, \quad v' = 0, \quad \omega = \frac{1}{4}(v^2 - \alpha^2). \tag{3}$$
It is known that when $F(s) = |s|^{(p-1)/2}$, $p$ is mass-subcritical, $\varphi$ exists and decays exponentially at infinity. (See Berestycki and Lions [4] where a larger class of nonlinearity was considered.)

### 1.1 Linearized operator

As in [7], the linearization operator of (1) around the solitary wave $w(x; \sigma)$ is given by

$$i\partial_t \chi = -\partial_x^2 \chi + F(|w|^2)\chi + F'(|w|^2)w(w\chi + w\overline{\chi}).$$

If we denote

$$\chi(x, t) = e^{i\Phi}f(y, t), \quad \Phi = -\beta(t) + \frac{1}{2}vx, \quad y = x - b,$$

then the function $f$ satisfies

$$i\partial_t f = L(\alpha)f,$$

where

$$L(\alpha)f = -\partial_x^2 f + \frac{\alpha^2 f}{4} + F(\varphi^2)f + F'(\varphi^2)\varphi^2(f + \overline{f}), \quad \varphi = \varphi(y; \alpha). \quad (4)$$

Consider the complexification of (4):

$$i\partial_t f = H(\alpha)f, \quad f = (f, \overline{f})^T,$$

$$H(\alpha) = H_0(\alpha) + V(\alpha), \quad H_0(\alpha) = \left(-\Delta_y + \frac{\alpha^2}{4}\right)\theta_3,$$

$$V(\alpha) = [F(\varphi^2) + F'(\varphi^2)\varphi^2]\theta_3 + iF'(\varphi^2)\varphi^2\theta_2,$$

where $\theta_2$ and $\theta_3$ are the Pauli matrices:

$$\theta_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

There are four known generalized eigenfunctions, denoted by $\{\xi_1, \xi_2, \xi_3, \xi_4\}$, for the zero eigenvalue to $H(\alpha)$. Among them the two radial functions are

$$\xi_1 = \begin{pmatrix} v_1 \\ \overline{v}_1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} v_2 \\ \overline{v}_2 \end{pmatrix}, \quad v_1 = -i\varphi(y; \alpha), \quad v_2 = -\frac{2}{\alpha} \varphi_\alpha(y; \alpha).$$

Moreover,

$$H(\alpha)\xi_1 = 0, \quad H(\alpha)\xi_2 = i\xi_1, \quad \langle \xi_1, \xi_2 \rangle = 0.$$

### 1.2 Main results

We have the following assumptions about the nonlinearity $F$.

**Assumption A** (i) There exists a solitary wave solution to (1) and it is of exponential decay.
(ii) There exist $m, n \geq 4$ such that for $k \in \{0, 1, 2, 3, 4\}$,

\[
\left| \frac{d^k}{ds^k} F(s) \right| \leq \begin{cases} 
|s|^{\frac{m-1}{2}-k}, & |s| \leq 1, \\
|s|^{\frac{n-1}{2}-k}, & |s| \geq 1.
\end{cases}
\]

(iii) The linearized operator $H(\alpha)$ has zero as its only eigenvalue with generalized eigenfunction space spanned by $\{\xi_1, \xi_2, \xi_3, \xi_4\}$, no resonance and no imbedded eigenvalues in its continuous spectrum.

(iv) (1) is globally well-posed in $H^1$ and the solution $u(t)$ satisfies

\[
\|u(t)\|_{H^1} \leq C(\|u_0\|_{H^1}).
\]

Remark 1 If we restrict ourself to the linear combination of power functions, roughly speaking, (ii) means that the lowest degree of $F$ is $3/2$, which improves the result in [7] where the degree of $F$ is assumed to be at least four.

Remark 2 For $n < 5$, it is proved by Weinstein [42,43] that the generalized eigenfunction space to zero of $H(\alpha)$ is four dimensional. For cubic nonlinearity in three dimensions, Costin et al. [11] proved that $H(\alpha)$ has no imbedded eigenvalues. However, whether there exists imbedded eigenvalue or eigenvalues in the gap between zero and the continuous spectrum in the one dimension case for subcritical power nonlinearity is unknown.

Remark 3 Although (iii) is widely assumed in the papers studying asymptotic stability, the rationality of (iii) is unknown when $F(x) = x^p$ and $0 < p < 2$. However, there are some possible remedies to deal with the case when more than one discrete spectrum occurs, such as the so-called ‘Fermi Golden Rule’ hypothesis (FGR) introduced by Sigal [37]. See also Cuccagna [14] and Soffer and Weinstein [39,40] for applications of normal forms and Fermi Golden Rule in the study of dynamics near solitons.

Remark 4 If we assume $n < 5$, then (iv) is naturally satisfied (e.g., [10]). Meanwhile, (iv) may hold for combined nonlinearities.

Our main theorem is as follows. Let

\[
\|f\|_\Sigma = \|f\|_{H^2_x} + \|xf\|_{L^2_x} + \|x^2 f\|_{L^2_x}.
\]

**Theorem 1** Let $F$ satisfy Assumption A. Assume that $w(x; \sigma_0(t))$ is a solitary wave solution to (1) with $\sigma_0(t) = (\beta_0(t), \omega_0(t), 0, 0)$ and satisfies $\frac{d}{d\alpha}\|\varphi\|_{L^2_x}^2 \neq 0$ at $\alpha_0$, where $-\frac{1}{4} \alpha_0^2 = \omega_0$. If the radial initial data $u_0$ satisfies

\[
\|u_0 - w(x; \sigma_0(0))\|_\Sigma \ll 1,
\]

then there exists a modulated solitary wave $w(x; \sigma_+ (t))$ such that as $t \to \infty$ $u(x, t)$ decomposes into

\[
u(x, t) = w(x; \sigma_+(t)) + \chi(t),
\]
where
\[ \| \chi(t) \|_\infty \leq C t^{-s + 1}, \quad s = \left( \frac{7}{4} \right)^+. \]

1.3 Main ideas

The key ingredient of our proof is to establish the decay estimate with some proper vector field operator \( |J_V(t)|^s \):

\[ \| \chi \|_\infty \leq C t^{-s} \| |J_V(t)|^s \chi \|_{L^\infty_t L^2_x}. \]  

(6)

Let \( \mathcal{H}(\alpha) = P_c(H(\alpha))H(\alpha) \) be the projection of \( H(\alpha) \) onto its continuous spectrum space. We will choose \( |J_V(t)|^s \) to be

\[ U(t + h)t^s \mathcal{H}(\alpha(t))^{s/2}U(-t - h), \]

see Section 2 below. The importance of (6) is that it fills the gap between the weak decay resulting from dispersive effects in the one dimension and the desired decay to put the quadratic remainder in \( L^1_t \).

There are two main difficulties in the establishment of (6). The first is how to give a proper definition of \( \mathcal{H}(\alpha)^{s/2} \). We notice that it is not trivial since \( \mathcal{H}(\alpha) \) is not selfadjoint. The second difficulty is how to prove (6). A similar estimate for Schrödinger operators was given in [15] by the inverse scattering theory. This may not work for our \( \mathcal{H}(\alpha) \) since it is not selfadjoint and no inverse scattering is available for it. To overcome aforementioned difficulties, we rely heavily on the functional calculus techniques. Specifically speaking, the fractional power of \( \mathcal{H}(\alpha) \) is defined via Dunford-Schwartz type integral with a properly chosen integral contour (see formula (9) below). We emphasize again that the fractional power is defined for \( \mathcal{H}(\alpha) \) rather than \( H(\alpha) \) itself. Thus, we need to separate the discrete spectrum part from the solution and study the modulation equations. To get (6), we translate the problem to the corresponding estimate of the inverse of \( \mathcal{H}(\alpha)^{s/2} \) which finally reduces to estimates of resolvent \( (H(\alpha) - \lambda)^{-1} \). When \( \lambda \) is sufficiently large, we apply a perturbation technique to obtain the decay of the resolvent \( (\lambda - H(\alpha))^{-1} \).

It is important that (6) remains valid only for \( \chi \) satisfying \( U(-t - h)\chi \in P_c(H(\alpha)) \). Hence, orthogonal conditions should be added to \( \chi \). As soon as we establish (6), the main theorem follows by the study of modulation equation and Strichartz estimates.

This paper is organized as follows. In Section 2, we define the fractional power of the linearized operator. In Section 3, we define the vector field operator \( |J_V|^s \) and introduce corresponding orthogonal conditions. In Section 4, we set up the bootstrap argument and study the modulation equation. In Section 5, we derive the equation of \( |J_V|^s u \). In Section 6, we close the bootstrap by nonlinear estimates and prove Theorem 1.

Notations \( A \lesssim B \) means that there exists some positive constant \( C \) such that \( A \leq CB \). \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \). Without confusion, \( \| \cdot \|_{L^p_x} \) will
always be denoted as $\| \cdot \|_p$. The spacetime norm

$$\left( \int_{s_1}^{s_2} \left( \int_{\mathbb{R}} |f(t,x)|^p \, dx \right)^{r/p} \, dt \right)^{1/r}$$

is denoted by $\| f \|_{L^r_{[s_1,s_2]} L^p_x}$. We denote constants by $C$, and they will change from line to line. Given $b > 0$, let $b^+$ be some constant slightly larger than $b$ and $b^−$ be some constant slightly smaller than $b$.

2 Fractional power of $H(\alpha)$ in continuous spectrum space

In this section, we give the definition of the fractional power of $H(\alpha)$. And some distorted Sobolev embedding and equivalence lemmas are proved. Given $\alpha > 0$, recall the linearized operator $H(\alpha)$. We will work on $L^2_{\text{rad}}$, and the domain of $H(\alpha)$ is taken as $D(H(\alpha)) = H^2_{\text{rad}}$. In the following, we use $L^2$ and $H^2$ to denote $L^2_{\text{rad}}$ and $H^2_{\text{rad}}$, respectively, for simplicity. Define $\tau = \alpha^2/4$ and

$$H_0 = \text{diag}(-\Delta + \tau, \Delta - \tau).$$

With these notations, the potential term in $H(\alpha) = H_0 + V$ is of the form

$$\begin{pmatrix} V_1 & -V_2 \\ V_2 & -V_1 \end{pmatrix}.$$

The explicit formula for the Green function of $H_0$ and Young’s inequality yield the following result.

**Lemma 1** Let $\lambda \in \mathbb{C}\backslash(-\infty, -\tau] \cup [\tau, \infty)$. Then, for $1 \leq p \leq \infty$,

$$\| (H_0 - \lambda)^{-1} \|_{p \to p} \leq c_1 [(\text{Re}\sqrt{\tau - \lambda})^{-2} + (\text{Re}\sqrt{\lambda + \tau})^{-2}] ;$$

and for $2 \leq p \leq \infty$,

$$\| (H_0 - \lambda)^{-1} \|_{2 \to p} \leq c_1 [(\text{Re}\sqrt{\tau - \lambda})^{-\frac{3}{2} - \frac{1}{p}} + (\text{Re}\sqrt{\lambda + \tau})^{-\frac{3}{2} - \frac{1}{p}}].$$

The following lemma is the high frequency estimate.

**Lemma 2** (High frequency resolvent estimate) Let $\lambda \in \mathbb{C}\backslash(-\infty, -\tau] \cup [\tau, \infty)$. Suppose that $\lambda$ is chosen such that

$$\text{Re}\sqrt{\tau - \lambda})^2 \geq 4c_1 \| V \|_{\infty}, \quad \text{Re}\sqrt{\lambda + \tau})^2 \geq 4c_1 \| V \|_{\infty}. \quad (7)$$

Then, for $1 \leq p \leq \infty$,

$$\| (H(\alpha) - \lambda)^{-1} \|_{p \to p} \leq c [(\text{Re}\sqrt{\tau - \lambda})^{-2} + (\text{Re}\sqrt{\lambda + \tau})^{-2}], \quad (8)$$

where $c$ is independent of $V$. 

Proof. By Lemma 1,
\[ \| (H_0 - \lambda)^{-1} V \|_{p \to p} \leq \| (H_0 - \lambda)^{-1} \|_{p \to p} \| V \|_{\infty} \]
\[ \leq c_1 [(\text{Re} \sqrt{\tau - \lambda})^{-2} + (\text{Re} \sqrt{\lambda + \tau})^{-2}] \| V \|_{\infty}. \]
Hence, by (7), we have
\[ \| (H(\alpha) - \lambda)^{-1} \|_{p \to p} \leq 2 \| (H_0 - \lambda)^{-1} \|_{p \to p} \]
\[ \leq c [(\text{Re} \sqrt{\tau - \lambda})^{-2} + (\text{Re} \sqrt{\lambda + \tau})^{-2}]. \]
Thus,
\[ \| (H_0 - \lambda)^{-1} V \|_{p \to p} < \frac{1}{2}. \]
Then by Neumann series,
\[ (H(\alpha) - \lambda)^{-1} = (I + (H_0 - \lambda)^{-1} V)^{-1} (H_0 - \lambda)^{-1}, \]
and (8) follows. \qed

Denote \( P_c(H(\alpha)) \) the projection onto continuous spectrum space of \( H(\alpha) \) in \( L^2_{\text{rad}} \), and let \( \mathcal{H}(\alpha) = H(\alpha)P_c(H(\alpha)) \). Define
\[ \mathcal{H}(\alpha)^{s/2} y = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{s-1} (\lambda - \mathcal{H}(\alpha))^{-1} \mathcal{H}(\alpha) y d\lambda, \quad 0 < s < 2, \, y \in H^2, \quad (9) \]
where
\[ \Gamma = \gamma_{1,+} \cup \gamma_{2,+} \cup \gamma_{1,-} \cup \gamma_{2,-}, \]
with
\[ \gamma_{1,\pm}(t) = t + a \pm i \varepsilon t, \quad \gamma_{2,\pm}(t) = -t - a \pm i \varepsilon, \quad t \geq 0. \]
We emphasize that \( \mathcal{H}(\alpha)^{s/2} \) is not \( H(\alpha)^{s/2} \) in general.

See Fig. 1 for the shape of \( \Gamma \) and we make the convention that the two connected branches of \( \Gamma \) are anticlockwise oriented. Here, \( a > 0 \) is some appropriate constant close to \( \tau \) excluding the discrete spectrum of \( H(\alpha) \) in the interior of \( \Gamma \), and \( \varepsilon > 0 \) is to ensure \( \gamma_{1,\pm}(t) \) and \( \gamma_{2,\pm}(t) \) lie in some single-value branch of \( z^{s/2} \).

The integration in (9) is well defined for each \( y \in H^2 \) because of Lemma 2. In fact, the analyticity of \( (H(\alpha) - \lambda)^{-1} \) in the resolvent set and the fact that \( \Gamma \) is strictly away from the discrete spectrum \( \{0\} \) show that (9) is well defined for \( \lambda \) contained in a ball. Meanwhile, Lemma 2 shows the integrand in (9) decays fast as \( \lambda \) goes to infinity along \( \Gamma \). Thus, (9) is well defined for \( y \in H^2 \).
For \( \beta < 0 \), define

\[
[[\mathcal{H}(\alpha)]^\beta]y = \frac{1}{2\pi i} \int_\Gamma \lambda^\beta (\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda.
\]  

(10)

For each \( y \in L^2 \), the integration is well defined due to Lemma 2. The notation \([[]\) in (10) does not mean brackets, we use it to distinguish \([[[\mathcal{H}(\alpha)]^\beta] \) from \( \mathcal{H}(\alpha)^s \) in (9).

Similarly, we define \( \mathcal{H}_0^{s/2} \) with \( 0 < s < 2 \) by

\[
\mathcal{H}_0^{s/2} y = \frac{1}{2\pi i} \int_\Gamma \lambda^{\frac{s}{2}-1}(\lambda - H_0)^{-1} H_0 y d\lambda, \quad y \in H^2,
\]

(11)
and \([\mathcal{H}_0]^{\beta}\) with \(\beta < 0\) to be

\[
[\mathcal{H}_0]^{\beta} y = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{\beta}(\lambda - H_0)^{-1} y d\lambda, \quad y \in L^2. \tag{12}
\]

Notice that since \(H_0\) has only continuous spectrum, there is no need to use \(P_c(H_0)\) in the definition.

**Lemma 3** For \(0 < s < 2\), there holds that

\[
\mathcal{H}(\alpha)^{s/2} \xi_i(\alpha) = 0, \quad i = 1, 2. \tag{13}
\]

**Proof** It follows by \(\mathcal{H}(\alpha) = H(\alpha)P_c(H(\alpha))\) and \(\{\xi_i(\alpha)\}_{i=1,2} \subset H^2\) are generalized eigenfunctions.

**Remark 5** Let us explain the logic behind the integral contours in Fig. 1. In order to make \(f(z) = z^\theta\) be single-valued analytic functions, we remove the negative \(y\)-axis. That is why we choose integral contours like (a), (c), and (d). However, in the special case \(f(z) = z^{-1}\) which is analytic outside 0, integral contours like (b) are reasonable.

**Lemma 4** Denote \(\text{Range}(P_c(H(\alpha)))\) the continuous spectrum space of \(H(\alpha)\) in \(L^2_{\text{rad}}\). Then we have

\[
\mathcal{H}(\alpha)^{s/2} g = ([\mathcal{H}(\alpha)]^{-s/2})^{-1} g, \quad \forall g \in H^2 \cap \text{Range}(P_c(H(\alpha))).
\]

**Proof** Step 1 We prove that \([\mathcal{H}(\alpha)]^{-1}\) is exactly the inverse of \(\mathcal{H}(\alpha)\) in \(\text{Range}(P_c(H(\alpha)))\).

It suffices to prove

\[
\frac{1}{2\pi i} \int_{\Gamma} \mathcal{H}(\alpha)(\lambda - \mathcal{H}(\alpha))^{-1}\lambda^{-1} y d\lambda = y, \quad \forall y \in H^2 \cap \text{Range}(P_c(H(\alpha))). \tag{14}
\]

Since \(\mathcal{H}(\alpha) = P_c(H(\alpha))H(\alpha)\) only has continuous spectrum in \(X := \text{Range}(P_c(H(\alpha)))\), there exists a small ball \(B(0, r)\) near zero such that \(B(0, r)\) belongs to resolvent set of \(\mathcal{H}(\alpha)\) in \(X\). Fix \(n \in \mathbb{Z}_+\), and define the curve \(\mathcal{C}_n\) to be

\[
\mathcal{C}_n = \mathcal{C}_o \cup \mathcal{C}_s \cup \mathcal{C}_n^*,
\]

where

\[
\mathcal{C}_o = \partial B(0, r), \quad \mathcal{C}_s = \mathcal{C}_{++} \cup \mathcal{C}_{+-} \cup \mathcal{C}_{-+} \cup \mathcal{C}_{--}, \quad \mathcal{C}_n^* = \mathcal{C}_+ \cup \mathcal{C}_-,
\]

with

\[
\mathcal{C}_{\pm} = \{\pm(a + t) + i\varepsilon t: t \in [0, n]\}, \quad \mathcal{C}_{\pm} = \{\pm(a + t) - i\varepsilon t: t \in [0, n]\},
\]

\[
\mathcal{C}_\pm = \{t \pm i\varepsilon n: t \in [-a - n, a + n]\}.
\]

Using the Cauchy integral formula on \(\mathcal{C}_n\), one obtains

\[
\frac{1}{2\pi i} \int_{\Gamma} \mathcal{H}(\alpha)(\lambda - \mathcal{H}(\alpha))^{-1}\lambda^{-1} y d\lambda = \lim_{n \to \infty} \frac{1}{2\pi i} \left( \int_{\mathcal{C}_o} \cdots - \int_{\mathcal{C}_n} \cdots \right).
\]
The resolvent decay in Lemma 2 shows that the integral on $C^*_n$ vanishes as $n \to \infty$. Recalling that $B(0,r)$ belongs to the resolvent set of $\mathcal{H}(\alpha)$ in $X$, the integral on $C^*\diamond$ now writes as

$$\frac{1}{2\pi i} \int_{C^*} \mathcal{H}(\alpha)(\lambda - \mathcal{H}(\alpha))^{-1} \lambda^{-1} y d\lambda$$

$$= \int_{\partial B(0,r)} \lambda^{-1} y d\lambda + \int_{\partial B(0,r)} (\lambda - \mathcal{H}(\alpha))^{-1} \mathcal{H}(\alpha) y d\lambda$$

$$= y,$$

provided $y \in H^2 \cap X$. Hence, (14) has been verified for arbitrary $y \in H^2 \cap X$.

**Step 2** Fix $0 < a < a' < \tau$, and let the curve $\Gamma'$ be (see Fig. 1 (c))

$$\Gamma' = \gamma_{1,+} \cup \gamma_{2,+} \cup \gamma_{1,-} \cup \gamma_{2,-},$$

where

$$\gamma_{1,\pm}(t) = t + a' \pm i\varepsilon t, \quad \gamma_{2,\pm}(t) = -t - a' \pm i\varepsilon t, \quad t \geq 0.$$

We prove that for $\beta < 0$,

$$\int_{\Gamma} \lambda^{\beta}(\lambda - \mathcal{H}(\alpha))^{-1} d\lambda = \int_{\Gamma'} \lambda^{\beta}(\lambda - \mathcal{H}(\alpha))^{-1} d\lambda. \quad (15)$$

Denote

$$\Gamma_{1,n} = \{-t - a + it\varepsilon: 0 \leq t \leq n\} \cup \{-t - a - it\varepsilon: 0 \leq t \leq n\},$$

$$\Gamma'_{1,n} = \{-t - a' + it\varepsilon: 0 \leq t \leq n\} \cup \{-t - a' - it\varepsilon: 0 \leq t \leq n\},$$

$$\gamma_{1,n} = \{-n + it: -\varepsilon(n + a) \leq t \leq -\varepsilon(n + a')\}$$

$$\cup \{-n + it: \varepsilon(n + a') \leq t \leq \varepsilon(n + a)\},$$

where $\gamma_{1,n}$ is oriented downward, $\Gamma_{1,n}$ and $\Gamma'_{1,n}$ are oriented anticlockwise (see Fig. 1 (d)). Then, by the analyticity of the resolvent, for $y \in X$,

$$\int_{\Gamma_{1,n}} \lambda^{\beta}(\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda - \int_{\Gamma'_{1,n}} \lambda^{\beta}(\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda$$

$$+ \int_{\gamma_{1,n}} \lambda^{\beta}(\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda = 0.$$

Since $\Gamma$ and $\Gamma'$ are symmetric with respect to the imaginary axis, in order to prove (15), it suffices to prove that for $y \in X$,

$$\lim_{n \to \infty} \int_{\gamma_{1,n}} \lambda^{\beta}(\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda = 0. \quad (16)$$
By the decay of resolvent, for \( n \) sufficiently large, we have
\[
\left\| \int_{\gamma_{1,n}} \lambda^\beta (\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda \right\|_p \\
\leq \int_{-\varepsilon(n+a')}^{-\varepsilon(n+a)} \|(-n + it - \mathcal{H}(\alpha))^{-1} y\|_p | - n + it|^\beta dt \\
+ \int_{\varepsilon(n+a')}^{\varepsilon(n+a)} \|(-n + it - \mathcal{H}(\alpha))^{-1} y\|_p | - n + it|^\beta dt \\
\leq \int_{-\varepsilon(n+a')}^{-\varepsilon(n+a')} (\text{Re}\sqrt{\lambda \pm (it - n)})^{-2} | - n + it|^\beta dt \|y\|_p \\
+ \int_{\varepsilon(n+a')}^{\varepsilon(n+a')} (\text{Re}\sqrt{\lambda \pm (it - n)})^{-2} | - n + it|^\beta dt \|y\|_p.
\]

Since \( \text{arg}(\lambda \pm (it - n)) \sim \pm \arctan \varepsilon \) as \( n \to \infty \), it is easy to see that
\[
(\text{Re}\sqrt{\lambda \pm (it - n)})^{-2} \sim c |n|^{-1},
\]
which immediately leads to (16). Thus, (15) follows.

**Step 3** We prove that for \( \gamma, \beta < 0 \),
\[
[[\mathcal{H}(\alpha)]\gamma + \beta] = [[[\mathcal{H}(\alpha)]\gamma \mathcal{H}(\alpha)]\beta]. \tag{17}
\]

By (15), for \( y \in X \), one has
\[
[[\mathcal{H}(\alpha)]\gamma][[\mathcal{H}(\alpha)]\beta] y \\
= \frac{1}{(2\pi i)^2} \int_{\Gamma} \mu^\gamma (\tau - \mathcal{H}(\alpha))^{-1} d\mu \int_{\Gamma'} \lambda^\beta (\lambda - \mathcal{H}(\alpha))^{-1} y d\lambda \\
= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} \lambda^\beta \mu^\gamma (\mu - \mathcal{H}(\alpha))^{-1}(\lambda - \mathcal{H}(\alpha))^{-1} y d\mu d\lambda \\
= \frac{1}{(2\pi i)^2} \int_{\Gamma} (\mu - \mathcal{H}(\alpha))^{-1} d\mu \int_{\Gamma'} \frac{\lambda^\beta \mu^\gamma}{\mu - \lambda} y d\lambda \\
- \frac{1}{(2\pi i)^2} \int_{\Gamma'} (\lambda - \mathcal{H}(\alpha))^{-1} d\lambda \int_{\Gamma} \frac{\lambda^\beta \mu^\gamma}{\mu - \lambda} y d\mu,
\]
which, by the Cauchy formula, equals
\[
\frac{1}{2\pi i} \int_{\Gamma'} (\lambda - \mathcal{H}(\alpha))^{-1} \lambda^{\beta+\gamma} y d\lambda = [[[\mathcal{H}(\alpha)]\gamma + \beta] y.
\]

The following two steps are standard (see Pazy [32]), but for completeness, we give a sketch.

**Step 4** We define \( [[\mathcal{H}(\alpha)]\gamma \mathcal{H}(\alpha)]\beta \) for \( \gamma \in \mathbb{R} \), and extend (17) to \( \gamma, \beta \in \mathbb{R} \).
From Step 1, we see that \([\mathcal{H}(\alpha)]^{-1}\) is one-to-one in \(X\). Therefore, for any integer \(n \geq 1\), \([\mathcal{H}(\alpha)]^{-n}\) is one-to-one. Let \(\gamma < 0\), suppose \([\mathcal{H}(\alpha)]^\gamma y = 0\), and take \(n \geq |\alpha|\). Then, from Step 2, we have

\[
[\mathcal{H}(\alpha)]^{-n} y = \left[\mathcal{H}(\alpha)\right]^{-n-\gamma} [\mathcal{H}(\alpha)]^\gamma y = 0,
\]

which implies \(y = 0\). Thus, \([\mathcal{H}(\alpha)]^\gamma\) is one-to-one and we can define \([\mathcal{H}(\alpha)]^{-\gamma}\) as \((\left[\mathcal{H}(\alpha)\right]^\gamma)^{-1}\).

We claim

\[
[\mathcal{H}(\alpha)]^\gamma \left[\mathcal{H}(\alpha)\right]^\beta y = \left[\mathcal{H}(\alpha)\right]^\gamma+\beta y
\]  

(18)

for \(\gamma, \beta \in \mathbb{R}, y \in C_c^\infty(\mathbb{R}; \mathbb{C}^2) \cap X\). Indeed, for example, when \(\beta < 0 < \gamma + \beta\), for \(y \in C_c^\infty(\mathbb{R}; \mathbb{C}^2) \cap X\), in order to verify (18), it suffices to prove

\[
[\mathcal{H}(\alpha)]^\gamma y = [\mathcal{H}(\alpha)]^{-\gamma} [\mathcal{H}(\alpha)]^\gamma+\beta y.
\]

Let \(y = [\mathcal{H}(\alpha)]^\gamma+\beta y\). Then it is equivalent to prove

\[
[\mathcal{H}(\alpha)]^{-\gamma} y = \left[\mathcal{H}(\alpha)\right]^\beta \left[\mathcal{H}(\alpha)\right]^{-\gamma-\beta} y,
\]

which follows immediately from Step 2.

**Step 5**  We finish our proof.

By Step 1, \([\mathcal{H}(\alpha)]^1 = H(\alpha)\) in \(H^2 \cap X\). From Step 4,

\[
[\mathcal{H}(\alpha)]^{s/2} = \left[\mathcal{H}(\alpha)\right]^1 \left[\mathcal{H}(\alpha)\right]^{\frac{s}{2}-1} = H(\alpha)[\mathcal{H}(\alpha)]^{\frac{s}{2}-1}.
\]

Thus, since \(\frac{s}{2} - 1 < 0\), by the definition of \(\mathcal{H}(\alpha)^{s/2}\), we have \([\mathcal{H}(\alpha)]^{s/2} = \mathcal{H}(\alpha)^{s/2}\) in \(H^2 \cap X\). Since

\[
[\mathcal{H}(\alpha)]^{s/2} \left[\mathcal{H}(\alpha)\right]^{-s/2} = I,
\]

we obtain our lemma. \(\square\)

**Lemma 5**  Recall \(X := \text{Range}(P_c(H(\alpha)))\). For \(2 \leq p \leq \infty\), \(f \in X \cap H^2\), and \(0 < s < 2\), we have

\[
\|\mathcal{H}(\alpha)^{s/2} f - \mathcal{H}_0^{s/2} f\|_{L^2_{\lambda}} \leq C \|f\|_{L^p_{\lambda}},
\]

(19)

**Proof**  Equation (9) gives that, in the space \(L(H^2 \cap X; L^2)\), there holds that

\[
\mathcal{H}(\alpha)^{s/2} = \int_{\Gamma} \mathcal{H}(\alpha) \lambda^{-1+s/2} (\lambda - \mathcal{H}(\alpha))^{-1} d\lambda
\]

\[
= \int_{\Gamma} \mathcal{H}(\alpha) \lambda^{-1+s/2} ((\lambda - \mathcal{H}(\alpha))^{-1} - (\lambda - H_0)^{-1}) d\lambda
\]

\[+ \mathcal{H}_0^{s/2} + V \int_{\Gamma} \lambda^{-1+s/2} (\lambda - H_0)^{-1} d\lambda.\]
It is easy to see that, for \( y \in X \) (\( \mathcal{H}(\alpha) = H(\alpha) \) in \( X \)),
\[
(\lambda - \mathcal{H}(\alpha))^{-1} y - (\lambda - H_0)^{-1} y = (\lambda - \mathcal{H}(\alpha))^{-1} V(\lambda - H_0)^{-1} y.
\]

Then we have
\[
\mathcal{H}(\alpha)^{s/2} = - \int_{\Gamma} (\lambda - \mathcal{H}(\alpha))^{1+\frac{s}{2}} (\lambda - \mathcal{H}(\alpha))^{-1} V(\lambda - H_0)^{-1} d\lambda
\]
\[
+ V \int_{\Gamma} \lambda^{1+\frac{s}{2}} (\lambda - H_0)^{-1} d\lambda
\]
\[
+ \int_{\Gamma} \lambda^{s/2} (\lambda - \mathcal{H}(\alpha))^{-1} V(\lambda - H_0)^{-1} d\lambda + \mathcal{H}_0^{s/2}
\]
\[= \int_{\Gamma} \lambda^{s/2} (\lambda - \mathcal{H}(\alpha))^{-1} V(\lambda - H_0)^{-1} d\lambda + \mathcal{H}_0^{s/2}.
\]

Define
\[
\Gamma_1 = \Gamma \cap \{(\text{Re} \sqrt{\tau - \lambda})^2 \geq 4c_1 \|V\|_\infty, (\text{Re} \sqrt{\lambda + \tau})^2 \geq 4c_1 \|V\|_\infty\},
\]
\[
\Gamma_2 = \Gamma \cap \{(\text{Re} \sqrt{\tau - \lambda})^2 \leq 4c_1 \|V\|_\infty, (\text{Re} \sqrt{\lambda + \tau})^2 \geq 4c_1 \|V\|_\infty\},
\]
\[
\Gamma_3 = \Gamma \cap \{(\text{Re} \sqrt{\tau - \lambda})^2 \geq 4c_1 \|V\|_\infty, (\text{Re} \sqrt{\lambda + \tau})^2 \leq 4c_1 \|V\|_\infty\},
\]
\[
\Gamma_4 = \Gamma \cap \{(\text{Re} \sqrt{\tau - \lambda})^2 \leq 4c_1 \|V\|_\infty, (\text{Re} \sqrt{\lambda + \tau})^2 \leq 4c_1 \|V\|_\infty\}.
\]

Then, from Lemma 2, when \( \frac{1}{q} + \frac{1}{p} = \frac{1}{2} \), one deduces that
\[
\left\| \int_{\Gamma_1} \lambda^{s/2} (\lambda - H(\alpha))^{-1} V(\lambda - H_0)^{-1} f d\lambda \right\|_2
\]
\[
\lesssim \int_{\Gamma_1} |\lambda|^{s/2} (\lambda - H(\alpha))^{-1} \|V\|_q \|H(\alpha)\|_p \|f\|_p d\lambda,
\]
which, by Lemma 1, is further dominated by
\[
\int_{\Gamma_1} |\lambda|^{s/2} [(\text{Re} \sqrt{\tau - \lambda})^{-2} + (\text{Re} \sqrt{\lambda + \tau})^{-2}]^2 \|V\|_q \|f\|_p d\lambda.
\]  \hspace{1cm} (20)

The only singular point for (20) is infinity. But at infinity, the integrand behaves like \( |\lambda|^{\frac{s}{2} - 2} \), which is integrable. Notice that
\[
(\text{Re} \sqrt{\tau - \lambda})^{-2} \sim c|\lambda|^{-1}, \quad (\text{Re} \sqrt{\lambda + \tau})^{-2} \sim c|\lambda|^{-1},
\]
as \( \lambda \to \infty \) in \( \Gamma \), the curves \( \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) are actually bounded. Hence, the remaining three terms can be estimated by
\[
\int_{\{\lambda: |\lambda| \leq R\} \cap \Gamma} |\lambda|^{s/2} (\lambda - H(\alpha))^{-1} \|V\|_q \|H(\alpha)\|_p \|f\|_p d\lambda
\]
\[
\leq C \|V\|_q \|f\|_p.
\] \hspace{1cm} \square
The same arguments give the following result.

**Lemma 6** For any \( s \in (0, 2) \) and \( f \in H^2 \), one has

\[
\|\mathcal{H}(\alpha_1)^{s/2} f - \mathcal{H}(\alpha)^{s/2} f\|_2 \leq C(\alpha - \alpha_1)\|f\|_2,
\]

(21)

**Proof** If \( f \in \text{Range}(I - P_c(H(\alpha))) \) or \( \text{Range}(I - P_c(H(\alpha_1))) \), then (21) holds naturally because the generalized eigenfunctions \( \{\xi_i\} \) are Schwartz functions and depends smoothly on parameter \( \alpha \).

If \( f \in \text{Range}(P_c(H(\alpha))) \cap \text{Range}(P_c(H(\alpha_1))) \), then the same arguments as Lemma 5 and the smooth dependence of potential \( V \) on \( \alpha \) give (21). \( \square \)

**Lemma 7** For \( 1/2 < s < 2 \), \( 2 \leq p \leq \infty \), and \( \forall f \in H^2 \), it holds that

\[
\|f\|_p + \|f\|_{H^s} \leq C\|\mathcal{H}_0^{s/2} f\|_2.
\]

**Proof** By Lemma 2, the integrand in (11) is absolutely integrable in \( H^2 \times \). Hence, the Fourier transform denoted by \( \mathcal{F} \) on \( \mathbb{R} \) can go across the integral symbol and act directly on the integrand in (11) by Plancherel theorem. Thus, the Cauchy integral formula yields

\[
\mathcal{F}\left[ \int_{\Gamma} \lambda^{\frac{s}{2} - 1}(\lambda - H_0)^{-1}H_0f d\lambda \right](k) = \text{diag}(\tau + |k|^2, -\tau - |k|^2)
\]

\[
= \int_{\Gamma} \lambda^{\frac{s}{2} - 1}\text{diag}([\lambda - (\tau + |k|^2)]^{-1}, (\lambda + \tau + |k|^2)^{-1})\mathcal{F}(f)(k)d\lambda
\]

\[
= \text{diag}((\tau + |k|^2)^{s/2}, (-\tau - |k|^2)^{s/2})\mathcal{F}(f)(k).
\]

Thus, \( \mathcal{H}_0^{s/2} \) is roughly \( (\tau - \Delta)^{s/2} \). Then standard Sobolev embedding yields

\[
\|f\|_p \leq C\|\mathcal{H}_0^{s/2} f\|_2.
\]

\( \square \)

**Lemma 8** For \( f \in H^2 \) in the continuous spectrum space of \( H(\alpha) \), \( 1/2 < s < 2 \), and \( 2 \leq p \leq \infty \), we have

\[
\|f\|_p \leq C\|\mathcal{H}(\alpha)^{s/2} f\|_2.
\]

**Proof** Due to Lemma 4, it suffices to prove

\[
\|[\mathcal{H}(\alpha)]^{-s/2} f\|_p \leq C\|f\|_2.
\]

Similar arguments as Lemma 5 yield

\[
\|[\mathcal{H}(\alpha)]^{-s/2} f - [\mathcal{H}_0]^{-s/2} f\|_p 
\leq C\left\| \int_{\Gamma} \lambda^{-\frac{s}{2} + 1}(\lambda - H_0)^{-1}V(\lambda - \mathcal{H}(\alpha))^{-1}f d\lambda \right\|_p
\leq C\int_{\Gamma} |\lambda|^{-\frac{s}{2} + 1}\|\lambda - H_0\|^{-1}_2f_2 \|V\|_\infty \|\lambda - \mathcal{H}(\alpha)\|^{-1}_2 \|f\|_2 d\lambda + \Pi,
\]

\( \Pi \)
where II denotes the remaining $\Gamma_2$, $\Gamma_3$, and $\Gamma_4$ parts. Applying Lemmas 2, 1, and 7, and using similar arguments as Lemma 5, we obtain Lemma 8.

\[\square\]

### 3 Orthogonal conditions

For vector-valued function $\phi$, define

\[|J_V(\alpha)|^s\phi = \text{diag}(M(t+h), M(-t-h))((t+h)^2\mathcal{H}(\alpha))^{s/2}\]

\[\cdot \text{diag}(M(-t-h), M(t+h))\phi,\]

where $M(t) = e^{i|x|^2/(4t)}$ and $h > 0$. $\mathcal{H}(\alpha)^{s/2}$ has been defined in Section 2. Here, we recall the facts that

$\mathcal{H}(\alpha)^{s/2}\xi_i = 0$, $ (\mathcal{H}^*(\alpha))^{s/2}\theta_3 = \theta_3\mathcal{H}(\alpha)^{s/2}$.

Assume that the solution to (1) is of the following form:

\[u(x, t) = w(x; \sigma(t)) + \chi(x, t), \quad w(x; \sigma(t)) = e^{-i\beta(t)}\varphi(x; \alpha(t)),\]  

(22)

where $\sigma(t) = (\beta(t), \omega(t), 0, 0)$ is not a solution of (3) in general. Define

$\chi(x, t) = e^{-i\beta(t)}f(x, t)$, $\quad f = (f, f)^T$, $\quad U(t) = \text{diag}(M(t), M(-t))$.

**Lemma 9** For any $\phi \in H^2$, one has

\[\langle |J_V(\alpha)|^s\phi, U(t+h)\theta_3\xi_i(\alpha) \rangle = 0, \quad i = 1, 2.\]  

(23)

**Proof** Since

\[ (\mathcal{H}^*(\alpha))^{s/2}\theta_3 = \theta_3\mathcal{H}(\alpha)^{s/2},\]

it holds that

\[\langle |J_V(\alpha)|^s\phi, U(t+h)\theta_3\xi_i \rangle = (t+h)^s\langle U(-t-h)\phi, (\mathcal{H}^*(\alpha))^{s/2}\theta_3\xi_i \rangle\]

\[= (t+h)^s\langle U(-t-h)\phi, \theta_3(\mathcal{H}(\alpha))^{s/2}\xi_i \rangle.\]

Since $\mathcal{H}(\alpha)^{s/2}\xi_i = 0$ (by Lemma 3), we obtain

\[\langle U(-t-h)\phi, \theta_3(\mathcal{H}(\alpha))^{s/2}\xi_i \rangle = 0.\]  

We impose two orthogonal conditions to $f$:

\[\langle U(-t-h)f, \theta_3\xi_i(\alpha(t)) \rangle = 0, \quad i = 1, 2.\]  

(24)

The existence of $\sigma(t)$ follows by the following lemma.

**Lemma 10** If $h$ is sufficiently large and $\|\chi(0, x)\|_{L^2}$ is sufficiently small, then there exists $\sigma(t) = (\beta(t), \omega(t), 0, 0)$ such that (24) holds.
Proof We first prove the result for \( t = 0 \), namely, there exist appropriate \( \alpha \) and \( \beta \) such that

\[
\left\langle \left( \frac{u_0(x) - e^{-i\beta \varphi(x; \alpha)}}{u_0(x) - e^{i\beta \varphi(x; \alpha)}} \right), \text{diag}(e^{-i\beta}, e^{i\beta})U(-h)\theta_3\xi_i(\alpha) \right\rangle = 0.
\]

This solvability is a consequence of the non-singularity of the corresponding Jacobian. Indeed, since \( |M(t) - 1| \leq C|x|^2/t \) and \( \chi(0, x) \) is small in \( L^2 \), for \( h \) sufficiently large, the leading term of the Jacobian is

\[
\begin{pmatrix}
  e & 0 \\
  0 & s
\end{pmatrix},
\]

where

\[
e = -i \frac{d}{d\alpha} \| \varphi(x; \alpha) \|^2_2, \quad s = -\frac{2i}{\alpha} \frac{d}{d\alpha} \| \varphi(x; \alpha) \|^2_2.
\]

At \( \alpha_0 \), \( e \) and \( s \) are nonzero by the assumption in Theorem 1. Thus, we have proved our lemma when \( t = 0 \).

Second, we show the existence of \( \sigma(t) \) for \( t > 0 \), but this follows by the standard arguments and the orbital stability, see [7] for instance. □

Lemma 8 and the definition of \( |J_V(\alpha)|^s u \) immediately yield the following result.

**Corollary 1** For \( \phi \in H^2 \), \( 1/2 < s < 2 \), and \( 2 \leq p \leq \infty \), we have

\[
\| P_c(H(\alpha))(U(-t - h)\phi) \|_p \leq t^{-s}\| |J_V(\alpha)|^s \phi \|_{L^2_x},
\]

provided that the right-hand side is finite.

**Proof** Decompose \( U(-t - h)\phi \) into

\[
U(-t - h)\phi = P_c(H(\alpha))[U(-t - h)\phi] + \mu_1\xi_1(\alpha) + \mu_2\xi_2(\alpha).
\]

Applying Lemma 3 shows the discrete part \( \sum \mu_i \xi_i \) vanishes in \( |J_V(\alpha)|^s \phi \). And then using Lemma 8 yields our result. □

4 Set up of bootstrap argument

Let \( f = (f, \overline{f})^T \) be the remainder term in decomposition (22) such that (24) holds (see Lemma 10). For any given \( t_1 > 0 \), define

\[
\mathcal{M}_{t_1} = \sup_{0 \leq \tau \leq t_1} (\| f \|_m + \| f \|_n + \| f \|_\infty)\langle \tau \rangle^s.
\]

Let \( \mathcal{A} \subset [0, \infty) \) be the set of \( t_1 \) such that \( \mathcal{M}_{t_1} \) is sufficiently small, i.e.,

\[
\mathcal{A} = \{ t_1 \in [0, \infty) : \mathcal{M}_{t_1} < \kappa \ll 1 \},
\]
where $\kappa > 0$ is sufficiently small to be determined in Section 6. $\mathcal{A}$ is nonempty due to (5) and Sobolev embedding. Moreover, $\mathcal{A}$ is closed by the global well-posedness theory and Sobolev embeddings.

### 4.1 Modulation equation

In this subsection, we study the modulation equation. Define

$$
\beta(t) = \int_0^t \omega(\lambda) d\lambda + \gamma(t).
$$

Then we rewrite (1) in terms of $f$:

$$
i f_t = L(\alpha(t)) f + N(\varphi, f) - \gamma' f - \gamma' \varphi + i\omega^2 \varphi \alpha,
$$

where

$$
N(\varphi, f) = F(|\varphi + f|^2)(\varphi + f) - F(\varphi^2)\varphi - F(\varphi^2) f - F'(\varphi^2)\varphi^2(f + \overline{f}).
$$

By (24), we have

$$
\text{Im} \langle f, M(t + h) v_i \rangle = 0. \tag{27}
$$

**Lemma 11** If $t_1 \in \mathcal{A}$, then for $t \in [0, t_1]$, $\|f(t)\|_\infty$ is sufficiently small, and

$$
|\gamma'(t)| + |\omega'(t)|
\leq C \left( \|f||_\infty^2 + \|f||_m^{m-1} \|f||_m + \|f||_\infty^{n-1} \|f||_n + \frac{1}{t^2} \|f||_\infty + \frac{1}{t} \|f||_\infty \right).
$$

**Proof** Since $t_1 \in \mathcal{A}$ and $t \in [0, t_1]$, $\|f(t)\|_\infty$ is sufficiently small by (25). Differentiating (27) with respect to $t$, we obtain

$$
\text{Im} \left\langle i\gamma' \varphi + \omega' \varphi \alpha, M(t + h) v_i \right\rangle + \text{Im} \left\langle f, M(t + h) \partial_\alpha v_i \right\rangle \omega' \frac{2}{\alpha}
+ \text{Im} \langle -i L(\alpha(t)) f, M(t + h) v_i \rangle + \text{Im} \langle -i N(\varphi, f), M(t + h) v_i \rangle
+ \text{Im} \langle i \gamma' f, M(t + h) v_i \rangle + \text{Im} \left\langle f, \frac{d}{dt} M(t + h) v_i \right\rangle = 0. \tag{28}
$$

Direct calculations give

$$
\left| \left\langle f, \frac{d}{dt} M(t + h) v_i \right\rangle \right| \leq C t^{-2} \|f\|_\infty. \tag{29}
$$

By the orthogonal condition (24), the identity $H(\alpha) \xi_2 = i\xi_1$, and the obvious commutator inequality, we have

$$
|\text{Im} \langle -i L(\alpha(t)) f, M(t + h) v_i \rangle|
= |\langle H(\alpha(t)) f, U(t + h) \theta_3 \xi_i \rangle|
= |\langle f, U(t + h) H^*(\alpha(t)) \theta_3 \xi_i \rangle| + |\langle f, [U(t + h), H^*(\alpha(t))] \theta_3 \xi_i \rangle|
= |\langle f, U(t + h) \theta_3 H(\alpha(t)) \xi_i \rangle| + |\langle f, [U(t + h), H^*(\alpha(t))] \theta_3 \xi_i \rangle|
= |\langle f, [U(t + h), H^*(\alpha(t))] \theta_3 \xi_i \rangle|
\leq C (t + h)^{-1} \|f\|_\infty. \tag{30}
$$
Combining (30) with (28) and (29) yields
\[
\left\| \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \left( \begin{pmatrix} \gamma' \\ \omega' \end{pmatrix} \right) \right\| \lesssim \| f \|_{\infty} \left\| \begin{pmatrix} \gamma' \\ \omega' \end{pmatrix} \right\| + O_2(f) + t^{-1} \| f \|_{\infty} + t^{-2} \| f \|_{\infty},
\]
(31)
where
\[
O_2 \leq C(|f|^m + |\varphi|m^{-2}|f|^2 + |f|^n + |\varphi|n^{-2}|f|^2).
\]
Thus, our lemma follows by the smallness of $\| f \|_{\infty}$. □

For $s = (7/4)^+$, Lemma 11 implies the following result.

**Corollary 2** For any $s = (7/4)^+$, any $t_1 \in \mathcal{A}$, and arbitrary $t \in [0, t_1]$, we have
\[
|\gamma'(t)| + |\omega'(t)| \leq CW(\mathcal{M}_{t_1})t^{-1-s}\mathcal{M}_{t_1}^2,
\]
(32)
where $W(x)$ is some bounded function around zero.

### 4.2 Reduction to time-independent linearized operator

Fix any time $t_1 \in \mathcal{A}$. Suppose
\[
\omega(t_1) = \omega_1, \quad \beta(t_1) = \beta_1.
\]
Define
\[
\omega_1 = -\frac{1}{4} \alpha_1^2, \quad \gamma_1 = \beta_1 - \omega_1 t_1, \quad \Phi_1 = -\omega_1 t - \gamma_1.
\]
Let $\chi = e^{i\Phi_1}g(x, t)$. Then $g$ satisfies
\[
ig_t = L(\alpha_1)g + D.
\]
(33)

The function $D$ is given by
\[
D = D_0 + D_1 + D_2 + D_3 + D_4,
\]
with
\[
D_0 = e^{-i\Omega}\left[ -\gamma'\varphi(\alpha) + \frac{2i}{\alpha} \omega'\varphi_\alpha \right],
\]
\[
D_1 = [F(\varphi^2(\alpha)) + F'(\varphi^2(\alpha))\varphi^2(\alpha) - F(\varphi^2(\alpha_1)) - F'(\varphi^2(\alpha_1))\varphi^2(\alpha_1)]g,
\]
\[
D_2 = F'(\varphi^2(\alpha))\varphi^2(\alpha)(e^{-2i\Omega} - 1)\bar{g},
\]
\[
D_3 = [F'(\varphi^2(\alpha))\varphi^2(\alpha) - F'(\varphi^2(\alpha_1))\varphi^2(\alpha_1)]\bar{g},
\]
\[
D_4 = e^{-i\Omega}N(\varphi(\alpha), e^{i\Omega}g),
\]
where
\[
\Omega = \Phi_1 - \Phi.
\]
The equation for $g = (g, \bar{g})^T$ is
\[
ig_t = H(\alpha_1)g + D.
\]
(34)
For the given time \( t_1 \in \mathcal{A} \), denote \( \alpha(t_1) \) by \( \alpha_1 \). Recall

\[
H(\alpha_1) = \text{diag}\left(\frac{\Delta^2}{4}, \Delta - \frac{\alpha_1^2}{4}\right) + \left(\begin{array}{cc}
V_1(\alpha_1) & V_2(\alpha_1) \\
V_2(\alpha_1) & -V_1(\alpha_1)
\end{array}\right).
\]

Recall that \( g \) is the remainder in the above decomposition of \( u(x,t) \) (see (33)). Moreover, let \( s = (7/4)^+ \). Then, by Corollary 2 and the Newton-Leibnitz formula, for \( t_1 \in \mathcal{A}, t \in [0,t_1] \),

\[
\sup_{0 \leq \tau \leq t_1} |\alpha(t) - \alpha_0| \lesssim W(\mathcal{M}_1, \mathcal{M}_1^2),
\]

\[
|\alpha'| \lesssim W(\mathcal{M}_1, t^{-1-s}) \cdot \mathcal{M}_1^2 ,
\]

\[
|\alpha(t) - \alpha(t_1)| + |\Omega| \lesssim W(\mathcal{M}_1, \mathcal{M}_1^2(t)^{2-s}).
\]

Section 1 implies the discrete spectral part of \( H(\alpha) \) is spanned by \( \{\xi_1(\alpha), \xi_2(\alpha)\} \) in the radial case. Moreover, it is known that \( \xi_1(\alpha) \) depends continuously with respect to \( \alpha \). Denote the projection to the continuous part of \( H(\alpha_1) \) as \( P_2 \), i.e., \( P_2 = P_c(H(\alpha_1)) \).

**Lemma 12** Let \( p \in [2, \infty] \), \( h \gg 1 \), and \( t_1 \in \mathcal{A} \). Then, for \( 0 \leq t \leq t_1 \), we have

\[
\|U(-t-h)g(t)\|_p \lesssim \|P_2 U(-t-h)g\|_p .
\]

**Proof** First of all, by (36) and \( t_1 \in \mathcal{A} \), we have

\[
|\alpha_1 - \alpha(t)| = |\alpha(t_1) - \alpha(t)| \ll 1.
\]

Recall \( g = e^{-i\Omega}f \). Then, by (24), we have

\[
0 = \langle f, U(-t-h)\xi_i(\alpha(t)) \rangle \\
= \langle g, e^{-i\Omega}U(-t-h)\xi_i(\alpha(t)) \rangle \\
= \langle U(-t-h)g, e^{-i\Omega}U(-2t-2h)\xi_i(\alpha_1) \rangle \\
+ \langle U(-t-h)g, e^{-i\Omega}U(-2t-2h)(\xi_i(\alpha(t)) - \xi_i(\alpha_1)) \rangle .
\]

Using (38) and that \( \xi_i \) is of exponential decay as \( x \to \infty \), we have

\[
|\langle U(-t-h)g, e^{-i\Omega}U(-t-h)(\xi_i(\alpha(t)) - \xi_i(\alpha_1)) \rangle| \ll \|U(-t-h)g\|_p .
\]

Splitting \( U(-t-h)g \) into continuous spectral part and discrete part:

\[
U(-t-h)g = P_c(H(\alpha_1))U(-t-h)g + \mu_1 \xi_1(\alpha_1) + \mu_2 \xi_2(\alpha_1) ,
\]

by (39) and (40), we have

\[
\mu_1 \langle \xi_1(\alpha_1), e^{-i\Omega}U(-t-h)\xi_i(\alpha_1) \rangle + \mu_2 \langle \xi_2(\alpha_1), e^{-i\Omega}U(-t-h)\xi_i(\alpha_1) \rangle \\
= O(\nu(|\mu_1| + |\mu_2|)) + O(\|P_2 U(-t-h)g\|_p)
\]

\[
\mu_1 \langle \xi_1(\alpha_1), e^{-i\Omega}U(-t-h)\xi_i(\alpha_1) \rangle + \mu_2 \langle \xi_2(\alpha_1), e^{-i\Omega}U(-t-h)\xi_i(\alpha_1) \rangle \\
= O(\nu(|\mu_1| + |\mu_2|)) + O(\|P_2 U(-t-h)g\|_p).
\]
with some positive constant \( \nu \ll 1 \). And (36) gives

\[
\| e^{-i\Omega U(-t-h) - I} \| \leq W(\mathcal{M}_{t_1}) \mathcal{M}_{t_1}^{2}(t)^{1-s} + \frac{1}{t+h} |x|^2.
\]

Hence, by the exponential decay of \( \xi_i \), for \( \mathcal{M}_{t_1} \) sufficiently small and \( h \gg 1 \), we obtain

\[
\|(\mu_1, \mu_2)\| \lesssim \|P_2 U(-t-h)\|^p,
\]

which proves (37). \( \square \)

**Lemma 13** Let \( s = (7/4)^+ \). If \( t_1 \in \mathcal{A} \) and \( 0 \leq t \leq t_1 \), then there exists some universal constant \( C > 0 \) such that

\[
\|\|J_V(\alpha_1)|^s g(t)\|_2 \leq C \|P_2|J_V(\alpha_1)|^s g\|_2,
\]

\[
\|g\|_p \leq C(t+h)^{-s} \|\|J_V(\alpha_1)|^s g\|_2. \tag{41}
\]

**Proof** Suppose

\[
|J_V(\alpha_1)|^s g = P_2|J_V(\alpha_1)|^s g + k_1 \xi_1(\alpha_1) + k_2 \xi_2(\alpha_1).
\]

Then (23) gives

\[
0 = \langle |J_V(\alpha(t))|^s g, U(t+h)\theta_3 \xi_i(\alpha(t)) \rangle \\
= \langle |J_V(\alpha_1)|^s g, U(t+h)\theta_3 [\xi_i(\alpha(t)) - \xi_i(\alpha_1)] \rangle \\
+ \langle |J_V(\alpha_1)|^s g, U(t+h)\theta_3 \xi_i(\alpha_1) \rangle \\
+ \langle (|J_V(\alpha(t))|^s - |J_V(\alpha_1)|^s) g, U(t+h)\theta_3 \xi_i(\alpha(t)) \rangle \\
= : \text{III}_1 + \text{III}_2 + \text{III}_3.
\]

**Step 1** \( \text{III}_2 + \text{III}_3 \) is bounded by \( \|(|J_V(\alpha(t))|^s - |J_V(\alpha_1)|^s) g\|_2 \), which, by Lemma 6, has an upper bound of \( |\alpha_1 - \alpha| \|g\|_2 \). Hence, Lemma 12 shows that \( \text{III}_2 + \text{III}_3 \) is bounded by \( \nu \|P_2 U(-t-h)g\|_2 \) with some positive constant \( \nu \ll 1 \) (notice that the matrix \( U(t) \) is unitary). Then Corollary 1 implies

\[
\|\text{III}_2 + \text{III}_3\| \leq \nu \|\|J_V(\alpha_1)|^s g\|_2.
\]

**Step 2** Meanwhile, \( \text{III}_1 \) is bounded by

\[
\langle |J_V(\alpha_1)|^s g, \theta_3 \xi_i(\alpha_1) \rangle + O(|\alpha_1 - \alpha| \|\|J_V(\alpha_1)|^s g\|_2 \\
+ \|(U(t+h) - I)\xi_i\|_{L^\infty} \|\|J_V(\alpha_1)|^s g\|_2.
\]

Therefore, substituting the spectrum decomposition of \( |J_V|^s g \) into the above formula yields

\[
\|(k_1, k_2)\| \lesssim \|P_2|J_V(\alpha_1)|^s g\|_2 + O(|\alpha_1 - \alpha| \|\|J_V(\alpha_1)|^s g\|_2 + h^{-1} \|\|J_V(\alpha_1)|^s g\|_2 \\
\lesssim \|P_2|J_V(\alpha_1)|^s g\|_2 + O(|\alpha_1 - \alpha| \|(k_1, k_2)\| + h^{-1} \|P_2|J_V(\alpha_1)|^s g\|_2 \\
+ h^{-1} \|(k_1, k_2)\|.
\]
Thus, by (38) and \( h \gg 1 \), we have
\[
\|(k_1, k_2)\| \leq C\|P_2|J_V(\alpha_1)|^s g\|_2.
\]
Finally, notice that (41) is a direct corollary of (37) and Corollary 1. \( \square \)

Therefore, Corollary 1 and Lemma 13 yield the following result.

**Corollary 3** Suppose \( t_1 \in A \) and \( s = (7/4)^+ \). Then
\[
\mathcal{M}_{t_1} \leq C\|P_2|J_V(\alpha_1)|^s g\|_{L^\infty_t[0,t_1]}L^2_x.
\] (42)

5 Equation for \(|J_V(\alpha_1)|^s g\)

We will prove Theorem 1 by bootstrap. As a preparation, we derive the equation of \(|J_V(\alpha_1)|^s g\) first.

Denote
\[
\partial_t E_2 = \text{diag}(\partial_t, \partial_t).
\]
Then direct calculations imply the following results.

**Lemma 14**
\[
[i\partial_t E_2 - \text{diag}(-\Delta + \tau, \Delta - \tau), \text{diag}(M(t + h), M(-t - h))] = \text{diag} \left( M(t + h) \left( -\frac{i}{2(t + h)} - \frac{ix \cdot \nabla}{t + h} \right), M(-t - h) \left( -\frac{i}{2(t + h)} + \frac{ix \cdot \nabla}{t + h} \right) \right).
\]

**Lemma 15**
\[
[i\partial_t E_2 - \text{diag}(-\Delta + \tau, \Delta - \tau), \text{diag}(M(-t - h), M(t + h))] = \text{diag} \left( \frac{ix \cdot \nabla}{2(t + h)} - \frac{x^2}{2(t + h)^2}, \frac{ix \cdot \nabla}{2(t + h)} + \frac{x^2}{2(t + h)^2} \right)
\]
\[
= : \mathcal{M}.
\]

For the potential term, we have the following result.

**Lemma 16**
\[
[V, U(t)] = \begin{pmatrix} 0 & -V_2[M(-t - h) - M(t + h)] \\ V_2[M(-t - h) - M(t + h)] & 0 \end{pmatrix}.
\]

**Lemma 17** Write \( \mathcal{H}(\alpha_1) \) as \( K \), and denote
\[
A := -U(t + h)((t + h)^2 K)^{s/2}[V, U(-t - h)] - [V, U(t + h)]((t + h)^2 K)^{s/2}U(-t - h).
\] (43)
Then there holds that
\[
[i\partial_t E_2 - K, |J_V|^s] = \frac{is}{t+h} |J_V|^s + \frac{i}{t+h} U(t+h)[x \cdot \nabla, ((t+h)^2K)^{s/2}]U(-t-h) + A.
\]

Proof Since $K$ is independent of $t$, Lemma 14 yields
\[
[i\partial_t E_2 - K, |J_V|^s]
\]
\[
= [i\partial_t E_2 - K, U(t+h)]((t+h)^2K)^{s/2}U(-t-h)
+ U(t+h)[i\partial_t E_2 - K, ((t+h)^2K)^{s/2}U(-t-h)]
\]
\[
= \frac{i}{2(t+h)} |J_V|^s + U(t+h)((t+h)^2K)^{s/2}[i\partial_t E_2 - K, U(-t-h)]
- [V, U(t+h)]((t+h)^2K)^{s/2}U(-t-h)
+ U(t+h)[i\partial_t E_2 - K, (t^2K)^{s/2}]U(-t-h) + \Psi((t+h)^2K)^{s/2}U(-t-h),
\]
where the matrix $\Psi$ is
\[
\Psi := \text{diag}\left(M(t+h)\frac{ix \cdot \nabla}{t+h}, M(-t-h)\frac{ix \cdot \nabla}{t+h}\right).
\]
Furthermore, by Lemmas 15, 16, and (43), we get
\[
[i\partial_t E_2 - K, |J_V|^s]
\]
\[
= \frac{i}{2(t+h)} |J_V|^s + \frac{is}{t+h} |J_V|^s + A
+ U(t+h)((t+h)^2K)^{s/2}\Phi + U(t+h)((t+h)^2K)^{s/2}\Upsilon
\]
Part of the terms in above formula can be cancelled. Thus, we obtain
\[
[i\partial_t E_2 - K, |J_V|^s]
\]
\[
= \frac{is}{t+h} |J_V|^s + A + \Psi((t+h)^2K)^{s/2}U(-t-h)
- U(t+h)((t+h)^2K)^{s/2}\Phi + U(t+h)((t+h)^2K)^{s/2}\Upsilon
\]
\[
= \frac{is}{t+h} |J_V|^s + A + \frac{i}{t+h} U(t+h)[x \cdot \nabla, ((t+h)^2K)^{s/2}U(-t-h)]
+ U(t+h)((t+h)^2K)^{s/2}\Upsilon,
\]
where we denote
\[
\Phi := \text{diag}\left(M(-t-h)\frac{ix \cdot \nabla}{t+h}, M(t+h)\frac{ix \cdot \nabla}{t+h}\right),
\]
\[
\Upsilon := \text{diag}\left(M(-t-h)\left(-\frac{x^2}{2(t+h)^2}\right), M(t+h)\frac{x^2}{2(t+h)^2}\right).
\]
Then our lemma follows by
\[
[x \cdot \nabla, ((t + h)^2 K)^{s/2} U(-t - h)] = [x \cdot \nabla, ((t + h)^2 K)^{s/2} U(-t - h) + ((t + h)^2 K)^{s/2} [x \cdot \nabla, U(-t - h)]
\]
\[= [x \cdot \nabla, ((t + h)^2 K)^{s/2} U(-t - h) - ((t + h)^2 K)^{s/2} \Upsilon]. \]

Denote
\[B(s) = s(K)^{s/2} + [x \cdot \nabla, K^{s/2}].\] (44)

Then Lemma 17 yields
\[i \partial_t E_2 - (J_V)^s = i(t + h)^{s-1} U(t + h) B(s) U(-t - h) + A.\] (45)

For \(B(s)\), we have the following result.

**Lemma 18** Let \(r = 1^+\). Then, for any \(g \in L^2\), we have
\[\|B(s)g\|_r \leq C \|g\|_2.\]

**Proof** A small modification of the proof of [15, Lemma 7.1] leads to
\[B(s) = c \int_{\Gamma} \tau^{s/2} (\tau - K)^{-1} V_3(\tau - K)^{-1} d\tau,
\]
where
\[V_3 = 2V + x \frac{d}{dx} V, \quad K = \mathcal{H}(\alpha_1).
\]

The detailed proof is given in Appendix. Similar arguments as Lemma 5 with additional efforts yield our lemma. In fact, as presented in the proof of Lemma 5, we split \(\Gamma\) into four parts. Define
\[B_j g = \int_{\Gamma_j} \lambda^{s/2} (\lambda - K)^{-1} V_3(\lambda - K)^{-1} g d\lambda.
\]

Let \(\frac{1}{r} = \frac{1}{2} + \frac{1}{q}\). For \(B_1\), Lemma 2 and the Hölder inequality give
\[\|B_1 g\|_r \leq \left\| \int_{\Gamma_1} \lambda^{s/2} (\lambda - K)^{-1} V_3(\lambda - K)^{-1} g d\lambda \right\|_r
\]
\[\leq \int_{\Gamma_1} \| (\lambda - K)^{-1} \|_{r \to r} |\lambda|^{s/2} \| V_3 \|_q \| (\lambda - K)^{-1} \|_{2 \to 2} d\lambda \| g \|_2
\]
\[\leq C \|g\|_2 \int_{\Gamma_1} |\lambda|^{\frac{s}{2} - 2} d\lambda
\]
\[\leq C \|g\|_2.
\]

In the following, all \(L^p\) norms refer to \(L^p_{rad}\), i.e., we only consider radial functions. Denote the resolvent set of \(K\) in \(L^2_{rad}\) by \(\rho_{L^2}(K)\). The resolvent set
and the spectrum of $K$ in $L^2 \cap L^r$ are denoted by $\rho_{L^2 \cap L^r}(K)$ and $\sigma_{L^2 \cap L^r}(K)$, respectively. The domain of $K$ in $L^2 \cap L^r$ is taken as $W^{2,r} \cap W^{2,2}$. We claim

$$\rho_{L^2}(K) \subseteq \rho_{L^2 \cap L^r}(K).$$

In fact, let $g \in L^r \cap L^2$ and $\lambda \in \rho_{L^2}(K)$. Then the definition of resolvent set indicates that there exists $f \in L^2$ such that

$$g = (K - \lambda)f,$$

i.e.,

$$
\begin{cases}
-\Delta f_1 + \tau_1 f_1 - \lambda f_1 + V_1(\alpha_1)f_1 - V_2(\alpha_1)f_2 = g_1, \\
-\Delta f_2 + \tau_1 f_2 - \lambda f_2 + V_1(\alpha_1)f_1 - V_1(\alpha_1)f_2 = g_2.
\end{cases}
$$

The Hölder inequality implies

$$g_1 - V_1(\alpha_1)f_1 + V_2(\alpha_1)f_2 \in L^r.$$

Therefore, we have

$$-\Delta f_1 + (\tau_1 - \lambda)f_1 \in L^r.$$

Since $\tau_1 - \lambda \notin (-\infty, 0]$, $-\Delta + (\tau_1 - \lambda)$ is invertible in $L^r$, which can be directly proved by checking the integral kernel of the resolvent of $\Delta$. Hence, $f_1 \in L^r$. Again by the first equation in (46), we have $\Delta f_1 \in L^r$, and thus, $f_1 \in W^{2,r}$. The same arguments show $f_2 \in W^{2,r}$, by which we have proved our claim. Thus, we have $\sigma_{L^2 \cap L^r}(K) \subseteq \sigma_{L^2}(K)$, which implies for $\lambda \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$,

$$\|{(\lambda - K)^{-1}}\|_{L^r \cap L^2 \to L^r \cap L^2} \leq C.$$

Hence, by the Hölder inequality and the boundedness of $\Gamma_j$, $j \in \{2, 3, 4\}$, for $\frac{1}{q} + \frac{1}{2} = \frac{1}{r}$, we obtain

$$\|B_{\Gamma_j} g\|_r \leq \left\| \int_{\Gamma_j} \lambda^{s/2}(\lambda - K)^{-1}V_3(\lambda - K)^{-1}g d\lambda \right\|_r \leq \int_{\Gamma_j} |\lambda|^{s/2} \|{(\lambda - K)^{-1}}\|_{L^r \cap L^2 \to L^r \cap L^2} \|V_3\|_{L^q \cap L^\infty} \|{(\lambda - K)^{-1}}\|_{2 \to 2} d\lambda \|g\|_2 \leq C\|g\|_2.$$

**Lemma 19** Recall $K = \mathcal{H}(\alpha_1)$ and $K^{s/2}$ defined via (9). Let $r = 1^+$. Then, $\forall g \in L^2$, we have

$$\|K^{s/2}g - \mathcal{H}_0^{s/2}g\|_r \leq C\|g\|_2.$$

**Proof** From the proof of Lemma 5,

$$\|K^{s/2}g - \mathcal{H}_0^{s/2}g\|_r = \left\| \int_{\Gamma} \lambda^{s/2}(\lambda - K)^{-1}V(\lambda - H_0)^{-1}g d\lambda \right\|_r.$$

The same arguments in the proof of Lemma 18 yield our lemma. □
6 Close bootstrap via nonlinear estimates

Let $0 < \epsilon \ll h^{-4} \ll 1$. $h$ is determined first, $\epsilon$ is then determined according to $h$. Define

$$\|u\|_\Sigma = \|u\|_{H^2} + \|x|\partial_x u\|_2 + \|x^2 u\|_2.$$  \hspace{1cm} (47)

And assume $\|g(0,x)\|_\Sigma \leq \epsilon$.

6.1 Nonlinear estimates

Lemma 20 Recall $f$ and $g$ defined by (24) and (33), respectively, and $g = (g, \tilde{g})^T$. Recall the definition of $A$ in (43). Let $r = 1^+$, and let $(p', r')$ be a Strichartz admissible pair. Then, for $t_1 \in \mathcal{A}$, we have

$$\|Ag\|_{L^p_t L^r_x} \leq C(h^{-1})\|P_2|J_V(\alpha_1)|^s g\|_{L^\infty_t L^2_x}.$$  

Proof Denote

$$\tilde{V}(t) = \begin{pmatrix} 0 & -V_2 \\ V_2 & 0 \end{pmatrix}, \quad \tilde{V}(t) = [M(t+h) - M(-t-h)]\tilde{V}(t).$$

Let $r = \frac{1}{2} + \frac{1}{q}$. Lemma 19, the fractional Leibnitz formula ([21, Appendix]), Lemmas 5, 12, and 13 imply

$$\|Ag\|_r \leq \|(t+h)^s \mathcal{H}_0^{s/2}[U(t+h)\tilde{V}(t)U(-t-h)g]\|_r$$

$$+ \|\tilde{V}(t)|J_V|^s g\|_r + (t+h)^s \|\tilde{V}(t)g\|_2$$

$$\leq \|(t+h)^s U(t+h)\tilde{V}(t)U(-t-h)g\|_{W^{s,r}}$$

$$+ \|\tilde{V}(t)|x|^2\|_q \|J_V|^s g\|_2(t+h)^{-1} + (t+h)^s\|g\|_2 \|\tilde{V}(t)|x|^2\|_\infty$$

$$\lesssim \|(t+h)^s U(t+h)\tilde{V}(t)\|_{W^{s,2}} \|g\|_q + \|(t+h)^s U(-t-h)g\|_{W^{s,2}} \|\tilde{V}(t)\|_q$$

$$+ \|J_V(\alpha_1)|^s g\|_2(t+h)^{-1} + (t+h)^s\|g\|_2$$

$$\lesssim (t+h)^{s-1}\|g\|_q + (t+h)^{-1}\|J_V(\alpha_1)|^s g\|_2 + (t+h)^s\|g\|_2$$

$$\lesssim (t+h)^{-1}\|J_V(\alpha_1)|^s g\|_2$$

$$\leq C(t+h)^{-1}\|P_2|J_V(\alpha_1)|^s g\|_{L^\infty_t L^2_x}. \hspace{1cm} \square$$

Recall $B(s)$ defined in (44). Denote

$$\tilde{B}(s)g = \text{i}(t+h)^{s-1}U(t+h)B(s)U(-t-h)g.$$  

Lemma 21 Let $r = 1^+$, and let $(p', r')$ be an admissible pair. Then, for $t_1 \in \mathcal{A}$,

$$\|\tilde{B}(s)g\|_{L^p_t L^r_x} \leq C(h^{-1})\|P_2|J_V(\alpha_1)|^s g\|_{L^\infty_t L^2_x}.$$  

Proof Thanks to Lemmas 18, 12, and 13, it is easy to see that

$$\|\text{i}(t+h)^{s-1}U(t+h)B(s)U(-t-h)g\|_{L^p_t L^2_x}$$

$$\leq \|\text{i}(t+h)^{s-1}g\|_{L^p_t L^2_x}$$

$$\leq \|(t+h)^{s-1}|L^r_t|J_V(\alpha_1)|^s g\|_{L^\infty_t L^2_x}$$

$$\leq C(h^{-1})\|P_2|J_V(\alpha_1)|^s h(t)\|_{L^\infty_t L^2_x}. \hspace{1cm} \square$$
Let us estimate the $\{D_i\}_{i=0}^4$ terms in (34). The main tools are Lemmas 19, 12, and the fractional Leibnitz formula ([21, Appendix]).

**Lemma 22** Let $r = 1 +$, and let $(p', r')$ be an admissible pair. Then, for $t_1 \in \mathcal{A}$, we have (all the spacetime norms used below are restricted in $[0, t_1] \times \mathbb{R}$)

$$\| J_V(\alpha_1) s D_0 \|_{L^p_t L^\infty_x} \leq \| P_2 |J_V(\alpha_1)|^s g \|_{L^\infty_t L^2_x}^2,$$

$$\| J_V(\alpha_1) s D_4 \|_{L^p_t L^\infty_x} \leq \| P_2 |J_V(\alpha_1)|^s g \|_{L^\infty_t L^2_x}^2 + \| P_2 |J_V(\alpha_1)|^s g \|_{L^\infty_t L^2_x}^{m-1}
+ \| P_2 |J_V(\alpha_1)|^s g \|_{L^\infty_t L^2_x}^{m-2},$$

$$\| J_V(\alpha_1) s D_j \|_{L^p_t L^\infty_x} \leq \| P_2 |J_V(\alpha_1)|^s g \|_{L^\infty_t L^2_x}^3, \quad j = 1, 2, 3.$$

**Proof** By (32), we have

$$\| J_V(\alpha_1) s D_0 \|_{L^p_t L^\infty_x} \leq C \| (t + h)^s + |\omega'| (t + h)^s \|_{L^p_t}
\leq C \mathcal{M}^2 \| t^{-1} \|_{L^p_t}
\leq C \mathcal{M}^2.$$

We can write $D_4$ as

$$D_4 = e^{-i\Omega t} \int_0^1 (A_1 + A_2 + \cdots + A_5)(1 - r) d\tau,$$

where

$$A_1 = \bar{g}^2 (\varphi + \tau \bar{g})^2 (\varphi + \tau \bar{g}) F''(|\varphi + \tau \bar{g}|^2),$$
$$A_2 = 2 \bar{g}^2 (\varphi + \tau \bar{g})^2 \varphi + \tau \bar{g} F''(|\varphi + \tau \bar{g}|^2),$$
$$A_3 = (\bar{g})^2 (\varphi + \tau \bar{g})^3 F''(|\varphi + \tau \bar{g}|^2),$$
$$A_4 = 2 \bar{g}^2 \varphi + \tau \bar{g} F'(|\varphi + \tau \bar{g}|^2),$$
$$A_5 = 4 \bar{g}^2 (\varphi + \tau \bar{g}) F'(|\varphi + \tau \bar{g}|^2),$$

with

$$\bar{g} = g e^{i\Omega}.$$

We give the detailed proof for $A_3$, the proofs of $A_1, A_2, A_4,$ and $A_5$ are almost the same. Let

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{q}; \quad r = 1^+, \quad \{\eta\} := (\eta, \bar{\eta})^T.$$

By Lemma 19, we have

$$\| J_V(\alpha_1)|^s \{ (\bar{g})^2 (\varphi + \tau \bar{g})^3 F''(|\varphi + \tau \bar{g}|^2) \} \|_r
\leq (t + h)^s \| U(-t - h) \{ (\bar{g})^2 (\varphi + \tau \bar{g})^3 F''(|\varphi + \tau \bar{g}|^2) \} \|_{W^{s,r}}
+ (t + h)^s \| (\bar{g})^2 (\varphi + \tau \bar{g})^3 F''(|\varphi + \tau \bar{g}|^2) \|_2
\leq (t + h)^s \| M(-3(t + h))(\varphi + \tau \bar{g})^3 F''(|\varphi + \tau \bar{g}|^2)(M(t + h)\bar{g})(M(t + h)\bar{g}) \|_{W^{s,r}}
+ (t + h)^s \| g \|_\infty.$$
Moreover, Lemma 19, the fractional Leibnitz formula ([21, Appendix]) and proof of \( D_2 \), we obtain

\[
\| |J_V|^s \{ (\tilde{g})^2 (\varphi + \tau \tilde{g}) F''(|\varphi + \tau \tilde{g}|) \} \|_r \\
\lesssim (t + h)^s \| M(-3(t + h)) (\varphi + \tau \tilde{g})^3 F''(|\varphi + \tau \tilde{g}|^2) \|_{H^2} g_2^2 \|_q + (t + h)^s \| g \|_q^2
\]

\[
+ (t + h)^s \| M(-3(t + h)) (\varphi + \tau \tilde{g}) F''(|\varphi + \tau \tilde{g}|^2) \|_q \| M(t + h) \tilde{g} \|_{H^2} \| g \|_q^\infty \| g \|_q
\]

Thus, Lemma 12 gives

\[
\| |J_V|^s \{ (\tilde{g})^2 (\varphi + \tau \tilde{g}) F''(|\varphi + \tau \tilde{g}|^2) \} \|_r \\
\lesssim (t + h)^{-s-2} \| x^2 (\varphi + \tau \tilde{g}) F''(|\varphi + \tau \tilde{g}|^2) \|_2 \| g \|_q \| g \| q
\]

Then the assumption of \( F \) shows that the left-hand side of (48) is bounded by

\[
(t + h)^{-s-2}(\| g \|_q \| g \| \| g \|_q + \| x^2 g \|_2 \| g \|_q (\| g \|_\infty^{m-2} + \| g \|_\infty^{n-2}))
\]

\[
+ (t + h)^{-s-1}(\| g \|_q \| g \|_q + \| x \|g \|_2 \| g \|_q (\| g \|_\infty^{m-3} + \| g \|_\infty^{n-3}))
\]

\[
+ \| |J_V|^s g \|_2 \| g \|_q + (t + h)^s \| g \|_q^2 + (t + h)^s \| g \|_q^2
\]

which, by Lemma 23, is further dominated by

\[
(t + h)^{-s} \| |J_V|^s g \|_2^2 \| J_V|^s g \|_2^{m-1}
\]

\[
+ (t + h)^{s+2-s(m-1)} \| |J_V|^s g \|_2^{m-1}
\]

\[
+ (t + h)^{s-(m-2)s} \| |J_V|^s g \|_2^{m-2} + (t + h)^{s+2-s(n-1)} \| |J_V|^s g \|_2^{n-1}
\]

Since \( s = (7/4)^+ \) and \( m, n \geq 4 \), we get the desired estimates of \( D_4 \) in Lemma 22.

Since the proofs of \( D_1 \), \( D_2 \), and \( D_3 \) are almost the same, we only give the proof of \( D_3 \). By the definition of \( D_3 \),

\[
\| |J_V|^s D_3 \|_r \lesssim (t + h)^s \| K^{s/2} U(-t - h) D_3 \|_r.
\]

Moreover, Lemma 19, the fractional Leibnitz formula ([21, Appendix]), and
Lemma 12 yield
\[
\|K^{s/2}U(-t-h)D_3\|_r \\
\lesssim \int_{\alpha_1}^{\alpha(t)} \|U(-t-h)[F''(\varphi^2(\tau))\varphi^3(\tau)\varphi_\alpha(\tau) + F'(\varphi^2(\tau))\varphi(\tau)\varphi_\alpha(\tau)]\|_{W^{s,r}}d\tau \\
+ \int_{\alpha_1}^{\alpha(t)} \|U(-t-h)[F''(\varphi^2(\tau))\varphi^3(\tau)\varphi_\alpha(\tau) + F'(\varphi^2(\tau))\varphi(\tau)\varphi_\alpha(\tau)]\|_{L^2}d\tau \\
\lesssim \int_{\alpha_1}^{\alpha(t)} \|M(-2(t+h))[F''(\varphi^2(\tau))\varphi^3(\tau)\varphi_\alpha(\tau) + F'(\varphi^2(\tau))\varphi(\tau)\varphi_\alpha(\tau)] \cdot M(t+h)\|_{W^{s,r}}d\tau + |\alpha(t) - \alpha_1|\|g\|_{L^2},
\]
which, by the Hölder inequality, is further dominated by
\[
\int_{\alpha_1}^{\alpha(t)} \|M(-2(t+h))[F''(\varphi^2(\tau))\varphi^3(\tau)\varphi_\alpha(\tau) + F'(\varphi^2(\tau))\varphi(\tau)\varphi_\alpha(\tau)]\|_{W^{s,2}}d\tau \\
\cdot \|g\|_{L^2}d\tau + \int_{\alpha_1}^{\alpha(t)} \|M(-2(t+h))[F''(\varphi^2(\tau))\varphi^3(\tau)\varphi_\alpha(\tau) + F'(\varphi^2(\tau))\varphi(\tau)\varphi_\alpha(\tau)]\|_{L^2}d\tau \\
\cdot M(t+h)\|_{W^{s,2}}d\tau + |\alpha(t) - \alpha_1|\|g\|_{L^2}.
\]
Thus, by Lemmas 12 and 13, we conclude that
\[
\|J_V|sD_3\|_r \\
\lesssim (t+h)^s|\alpha(t) - \alpha_1|\|g\|_q + |\alpha(t) - \alpha_1|\|J_V|s\|_2 + (t+h)^s|\alpha(t) - \alpha_1|\|g\|_2 \\
\lesssim |\alpha(t) - \alpha_1|\|J_V|s\|_2 \\
\lesssim t^{-s+1}\|J_V|s\|_{L^3_tL^\infty_x}^3 \\
\lesssim t^{-s+1}\|P_2J_V|s\|_{L^3_tL^\infty_x}^3,
\]
where we have used (36) in the last inequality. \qed

6.2 Proof of Theorem 1

Recall (47), \(P_2 = P_2(H(\alpha_1))\), and \(K = H(\alpha_1)\). Applying \(J_V(\alpha_1)|s\) and \(P_2\) to (34), by the commutator relation (45), we obtain
\[
i\partial_tP_2|J_V(\alpha_1)|s\beta - KP_2\|J_V(\alpha_1)|s\beta - P_2\|J_V(\alpha_1)|sD = P_2(\overline{B}(s)g + Ag).
\]
Let \((p', r')\) be an admissible pair with \(4 < p \leq \infty\) and \(r = 1^+\). Let \(t_1 \in \mathcal{A}\). By Lemmas 20–22 and the Strichartz estimates (see [26, Corollary 7.3]), we have
\[
\|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x} \\
\lesssim \|J_V(\alpha_1)|sD\|_{L^p_t[0,t_1]L^r_x} + \|\overline{B}(s)g + Ag\|_{L^p_t[0,t_1]L^r_x} + h^s\|g(0, x)\|_\Sigma \\
\lesssim \|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x}^3 + \|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x}^2 \\
+ \|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x}^{m-1} + \|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x}^{m-2} \\
+ \|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x}^{n-1} + \|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x}^{n-2} \\
+ C(h^{-1})\|P_2\|J_V(\alpha_1)|s\|_{L^\infty_t[0,t_1]L^2_x} + h^s\|g(0, x)\|_\Sigma.
Thus, choosing $h$ to be sufficiently large and $\|g(0)\|_\Sigma$ sufficiently small, for any $t_1 \in \mathcal{A}$, we obtain

$$\mathcal{M}_{t_1} \lesssim \|P_2 J_V(\alpha_1)g\|_{L^\infty_t[0,t_1]L^2_x} < C\epsilon + C\kappa^2. \quad (49)$$

By $\epsilon \ll \kappa \ll 1$, (49) shows that $\mathcal{A}$ is open. Since $\mathcal{A}$ is defined to be closed, we have $\mathcal{A} = [0, \infty)$ and thus for all $t \in (0, \infty)$,

$$\mathcal{M}_t \leq \kappa. \quad (50)$$

It is standard to deduce our main theorem (Theorem 1) from (50) (e.g., [7]). In fact, (32) implies that as $t \to \infty$, $\gamma(t)$ and $\omega(t)$ have limits $\gamma_\infty$ and $\omega_\infty$, respectively. Consequently, we can introduce the limit trajectory $\sigma_+(t)$:

$$\beta(t) = \omega(t) + \gamma(t), \quad \omega(t) = \omega_\infty, \quad \gamma(t) = \int_0^\infty (\omega(\tau) - \omega_\infty)d\tau.$$  

Obviously,

$$\sigma(t) - \sigma_+(t) = O(t^{-s+1}), \quad t \to \infty.$$  

Hence, we obtain the limit soliton $w(x; \sigma_+(t))$ and

$$\|w(x; \sigma(t)) - w(x; \sigma_+(t))\|_{L^2_t L^\infty_x} = O(t^{-s+1}).$$

Introduce the transformation

$$\chi = e^{i\Phi_\infty}g(x, t), \quad \Phi_\infty = -\beta_+(t).$$

Repeat the construction in Section 2 to Section 5 with the operators $K$ and $P_2$ replaced by $K_+$ and $P_c(K_+)$, respectively, where

$$K_+ = \text{diag}(-\Delta - \omega_+, \Delta + \omega_+) + \begin{pmatrix} V_1(\alpha_+) & -V_2(\alpha_+) \\ V_2(\alpha_+) & -V_1(\alpha_+) \end{pmatrix}.$$  

Then we can also prove

$$\|\chi\|_\infty \leq C t^{-s}.$$  

Therefore, we have obtained

$$u = w(x; \sigma_+(t)) + \chi + O(t^{-s+1}),$$

which combined with the estimate of $\chi$ yields Theorem 1.

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Appendix

In the following lemma, we deal with the growth of Sobolev norms and moments of \( u \).

**Lemma A1** Let \( F \) satisfy Assumption A. If the initial data \( u_0 \in H^2 \) satisfies \( \|u_0(1 + |x|)^2\|_2 < \infty \), then there exists a unique solution \( u(t) \) to (1), and

\[
\|u(t)\|_{H^2} \leq C(1 + t), \quad \|u(t)(1 + |x|)\|_2 \leq C(1 + t), \quad \|u(t)(1 + |x|^2)\|_2 \leq C(1 + t^3).
\]

**Proof**

**Step 1** Growth of Sobolev norms.

First of all, we prove the Bourgain-Staffilani bound for the growth of \( \|u\|_{H^2} \):

\[
\|u\|_{H^2} \leq C(1 + t). \tag{A1}
\]

(A1) is indeed contained in the proof of [41, Theorem 2.2]. Although Staffilani considered the nonlinearities of the form \( \sum_{\alpha_1 + \alpha_2 = m} u^{\alpha_1} \overline{u}^{\alpha_2} \) with \( m \geq 4 \) in [41, Theorem 2.2], her proof can be extended to our case with very small modifications. For reader’s convenience, we write down the proof in Lemma A2 below. The originality of the proof belongs to Staffilani [41].

**Step 2** Growth of moments.

In the following, \( u' \) denotes \( \partial_x u \) and \( u'' \) denotes \( \partial_x^2 u \). First, we prove \( \|u|x|\|_2 \leq C(1 + t) \). Note that \( ux \) satisfies

\[
i\partial_t(ux) = -\Delta u x - F(|u|^2)ux.
\]

(A2)

Multiplying (A2) by \( \overline{ux} \), then taking the imaginary part, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|x|u|\|^2_2 = 2\text{Im} \int \nabla u x \overline{u}.
\]

Hence, one has

\[
\frac{d}{dt} \|x|u|\|^2_2 \leq C \|x|u|\|_2.
\]

Gronwall’s inequality yields \( \|u|x|\|_2 \leq C(1 + t) \). Second, we prove

\[
\|u'|x|\|_2 \leq C(1 + t^2). \tag{A3}
\]

The equation for \( u'x \) is

\[
i\partial_t(xu') = -\Delta(u')x - F'(|u|^2)(u' \overline{u} + \overline{u'} u)x - F(|u|^2)u'x.
\]

Multiplying the above formula with \( \overline{u'x} \), and taking the imaginary part, we have

\[
\frac{d}{dt} \|x|u'|\|^2_2 \leq \|u''\|_2 \|x|u'|\|_2 + \|u'|\|_\infty (\|u||^{m-2}_\infty + \|u||^{n-2}_\infty) \|x|u'|\|_2 \|x|u|\|_2.
\]
Since \( \|u\|_{H^2} \) is bounded by \( C(1+t) \) and \( \|u\|_{H^1} \leq C \), from the Sobolev imbedding theorem and (A3), it follows that

\[
\frac{d}{dt} \|x|u'|_2^2 \leq C \|x|u'|_2(1+t).
\]

Thus, Gronwall’s inequality implies

\[
\|x|u'|_2 \leq C(1+t^2).
\]

(A4)

Third, we prove \( \|u|x|^2\|_2 \leq C(1+t^3) \). Similar arguments as the first step, we can show that

\[
\frac{d}{dt} \|x|u|^2\|_2 \leq C \left| \int_{\mathbb{R}} u' \overline{ux^3} \right| \leq C \|x|u|^2\|_2 \|x|u'|_2 \leq C(1+t^2)\|x|^2u\|_2.
\]

Again by Gronwall’s inequality, we get the desired result. \( \square \)

**Lemma A2** Let \( F \) satisfy Assumption A. If the initial data \( u_0 \in H^2 \) satisfies \( \|u_0\|_{H^2} < \infty \), then there exists a unique solution \( u(t) \) to (1) and

\[
\|u(t)\|_{H^2} \leq C(1+t).
\]

(A5)

**Proof** The originality of the proof belongs to Staffilani [41]. The presentation here is a very small modification of [41, Theorem 2.2]. Recall the norms introduced by [41]:

\[
\nu_1^s(v) := \|\partial_x^{s+\frac{1}{2}} v\|_{L_x^\infty L_t^2}, \quad \nu_2^s(v) := \|\partial_x^{(s-\frac{1}{2})^-} v\|_{L_x^2 L_t^\infty},
\]

\[
\nu_3^s(v) := \|\partial_x^{s} v\|_{L_x^2 L_t^\infty}, \quad \Omega_T^s(v) := \max_i \nu_i(v) + \|v\|_{L_x^\infty L_t^2},
\]

and the Banach space

\[
X_T^s := \{ v \in C([0,T],H^s) : \Omega_T^s(v) < \infty \}.
\]

**Step 1 Claim A** For any \( s \geq 1 \), there exists a time \( T \) depending only on \( \|u_0\|_{H^1} \) and a unique solution to (1) such that

\[
\Omega_T^s(u) \leq C\|u_0\|_{H^2}.
\]

**Proof** Denoting the linear Schrödinger group by \( S(t) \), it suffices to apply contraction argument to the operator

\[
Lu(t) := S(t)u_0 + i \int_0^t S(t-\tau) F(|u|^2) u(\tau) d\tau.
\]

[41, Lemma 4.1] and the Minkowski inequality give

\[
\Omega_T(Lu) \leq C\|u_0\|_{H^2} + T^{1/2}\|F(|u|^2) u\|_{L^2_T H^2}.
\]
Then Assumption A (ii) and the fractional chain rule ([21]) imply
\[
\|F(|u|^2)u\|_{L^2_T L^\infty_x} \leq T^{1/2}\|F(|u|^2)u\|_{L^\infty_T L^\infty_x} \leq T^{1/2}(\|u\|_{L^\infty_T L^\infty_x}^{m} + T^{1/2}\|u\|_{L^\infty_T L^\infty_x}^{n}).
\]
Thus, (A6) yields
\[
\Omega_T(Lu) \leq C\|u_0\|_{H^2} + T(\Omega_T^s(u)^m + \Omega_T^c(u)^n). \tag{A7}
\]
Similarly, one has \(L\) is a contraction if \(T\) is small enough. Thus, Claim A follows.

\[\text{Step 2 Proof of (A5).}\]
Calculating \(\frac{d}{dt}\|u\|_{H^2_x}^2\) gives
\[
\|u(t)\|_{H^2_x}^2 \leq \|u_0\|_{H^2_x}^2 + \int_0^T \int_\mathbb{R} |D^2_x(F(|u|^2)u)D^2_xu|dxdt. \tag{A8}
\]
Meanwhile, Claim A shows
\[
\|u\|_{L^2_T L^\infty_x} \leq C\|u_0\|_{H^{1/2}} + \|D_xu\|_{L^2_T L^\infty_x} \leq C\|u_0\|_{H^{3/2}} + \|D_xu\|_{L^2_T L^\infty_x} \leq C\|u_0\|_{H^{1/2}}.
\]
Hence, by Assumption A (ii), one obtains
\[
\int_0^T \int_\mathbb{R} |D^2_x(F(|u|^2)u)D^2_xu|dxdt \\
\leq \|D^2_xu\|_{L^2_T L^\infty_x}^2 \|u\|_{L^2_T L^\infty_x}^{m-3} + \|u\|_{L^\infty_T L^\infty_x}^{n-3} \\
+ \|D^2_xu\|_{L^2_T L^\infty_x} \|D_xu\|_{L^\infty_T L^\infty_x} \|D_xu\|_{L^2_T L^\infty_x}^{2} \|u\|_{L^\infty_T L^\infty_x}^{m-3} + \|u_0\|_{L^\infty_T L^\infty_x}^{n-3} \\
\leq (\|u_0\|_{H^{3/2}}^2 + \|u_0\|_{H^{3/2}} + \|u_0\|_{H^{3/2}})(\|u_0\|_{H^{1/2}}^{m-1} + \|u_0\|_{H^{1/2}}^{n-1}).
\]
Hence, by the interpolation inequality,
\[
\|f\|_{H^s} \leq \|f\|_{H^1}^{2-s} \|f\|_{H^2}^{s-1}, \quad s \in [1, 2],
\]
we have
\[
\int_0^T \int_\mathbb{R} |D^2_x(F(|u|^2)u)D^2_xu|dxdt \leq C(\|u_0\|_{H^1_x})\|u_0\|_{H^2_x} + \|u_0\|_{H^1_x}^{1+\omega}
\]
for some \(\omega \in (0, 1)\). Thus, (A8) and the bound \(\|u\|_{H^2_x} \leq C(\|u_0\|_{H^1_x})\) yield
\[
\|u(t)\|_{H^2_x}^2 \leq C(\|u_0\|_{H^1_x})(\|u_0\|_{H^2_x} + 1).
\]
It is standard that if on any interval of the form \(t \in [\tau, \tau + T]\) with an fixed \(T\) depending only on \(\|u_0\|_{H^1_x}\), there holds
\[
\|u(t)\|_{H^2_x}^2 \leq C(\|u_0\|_{H^1_x})(\|u(\tau)\|_{H^2_x} + 1),
\]
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then one has the polynomial bound
\[ \|u(t)\|_{H_x^2} \leq C(\|u_0\|_{H_x^1})(1 + t). \]

One may see Bourgain [5,6] and Staffilani [41] for this. Therefore, (A5) follows due to the time invariance and the global \( \|u(t)\|_{H^1} \) norm. □

Lemma A3  Denote
\[ B(s) = s(K)^{s/2} + [x \cdot \nabla, K^{s/2}]. \]
Then
\[ B(s) = c \int_{\Gamma} \tau^{s/2}(\tau - K)^{-1} V_3(\tau - K)^{-1} d\tau. \]

Proof  The proof is adapted from [15, Lemma 7.1]. Recall \( K = \mathcal{H}(\alpha_1) \) and the definition of \( \mathcal{H}(\alpha_1)^{s/2} \) in Section 2. Then we have
\[ 2\pi i B(s)f = s \int_{\Gamma} \lambda^{s-1}(\lambda - K)^{-1} Kf d\lambda + \int_{\Gamma} \lambda^{s-1}[x \partial_x, (\lambda - K)^{-1} K] f d\lambda. \]
It is easy to verify that
\[ [x \partial_x, (\lambda - K)^{-1} K] = -2\lambda K(\lambda - K)^{-2} + \lambda(\lambda - K)^{-1} V_3(\lambda - K)^{-1}. \]
Then we obtain
\[ 2\pi i B(s)f = s \int_{\Gamma} \lambda^{s-1}(\lambda - K)^{-1} Kf d\lambda - 2 \int_{\Gamma} \lambda^{s/2} K(\lambda - K)^{-2} f d\lambda \]
\[ + \int_{\Gamma} \lambda^{s/2}(\lambda - K)^{-1} V_3(\lambda - K)^{-1} f d\lambda. \]
It remains to prove
\[ s \int_{\Gamma} \lambda^{s-1}(\lambda - K)^{-1} Kf d\lambda - 2 \int_{\Gamma} \lambda^{s-1} \lambda K(\lambda - K)^{-2} f d\lambda = 0. \quad (A9) \]
By the identity
\[ 2\lambda^{s/2} K(\lambda - K)^{-2} = -2 \frac{d}{d\lambda} (\lambda^{s/2} K(\lambda - K)^{-1}) + s \lambda^{s-1} K(\lambda - K)^{-1} \]
and Lemma 2, one easily obtains (A9), thus finishing our proof. □

References
1. Beceanu M. A centre-stable manifold for the focussing cubic NLS in \( \mathbb{R}^{1+3} \). Comm Math Phys, 2008, 280: 145–205
2. Beceanu M. New estimates for a time-dependent Schrödinger equation. Duke Math J, 2011, 159: 417–477
3. Beceanu M. A critical center-stable manifold for Schrödinger’s equation in three dimensions. Comm Pure Appl Math, 2012, 65: 431–507
4. Berestycki H, Lions P L. Nonlinear scalar field equations. I. Existence of a ground state. Arch Ration Mech Anal, 1983, 62: 313–345
5. Bourgain J. On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE. Int Math Res Not IMRN, 1996: 277–304
6. Bourgain J. Global Solutions of Nonlinear Schrödinger Equations. Amer Math Soc Colloq Publ, Vol 46. Providence: Amer Math Soc, 1999
7. Buslaev V, Perelman G S. Scattering for the nonlinear Schrödinger equations: states close to a soliton. St Petersburg Math J, 1993, 4(6): 1111–1142
8. Buslaev V, Perelman G S. On the stability of solitary waves for nonlinear Schrödinger equation. Amer Math Soc Transl Ser 2, 1995, 2(164): 75–99
9. Buslaev V, Sulem C. On asymptotic stability of solitary waves for nonlinear Schrödinger equations. Ann Inst H Poincaré Anal Non Linéaire, 2003, 20(3): 419–475
10. Cazenave T. Semilinear Schrödinger Equations. Courant Lect Notes Math, Vol 10. Providence: Amer Math Soc, 2003
11. Costin O, Huang M, Schlag W. On the spectral properties of $L_\pm$ in three dimensions. Nonlinearity, 2012, 25: 125–164
12. Cuccagna S. Stabilization of solutions to nonlinear Schrödinger equations. Comm Pure Appl Math, 2001, 4(9): 1110–1145
13. Cuccagna S. An invariant set in energy space for supercritical NLS in 1D. J Math Anal Appl, 2009, 352: 634–644
14. Cuccagna S. The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states. Comm Math Phys, 2011, 305(2): 279–320
15. Cuccagna S, Georgiev V, Visciglia N. Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential. Comm Pure Appl Math, 2013, 6: 957–980
16. Cuccagna S, Maeda M. On small energy stabilization in the NLS with a trapping potential. Anal PDE, 2015, 8(6): 1289–1349
17. Cuccagna S, Mizumachi T. On asymptotic stability in energy space of ground states for nonlinear Schrödinger equations. Comm Math Phys, 2008, 284: 51–77
18. Grillakis M, Shatah J, Strauss W. Stability theory of solitary waves in the presence of symmetry I. J Funct Anal, 1987, 74(1): 160–197
19. Grillakis M, Shatah J, Strauss W. Stability of solitary waves in presence of symmetry II. J Funct Anal, 1990, 94(2): 308–384
20. Gustafson S, Nakanishi K, Tsai T P. Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves. Int Math Res Not IMRN, 2004, 66: 3559–3584
21. Kenig C E, Ponce G, Vega L. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Comm Pure Appl Math, 1993, 46: 527–620
22. Kirr E, Mizraik O. Asymptotic stability of ground states in 3D nonlinear Schrödinger equation including subcritical cases. J Funct Anal, 2009, 257: 3691–3747
23. Kirr E, Zarnescu A. On the asymptotic stability of bound states in 2D cubic Schrödinger equation. Comm Math Phys, 2007, 272: 443–468
24. Klainerman S. Global existence for nonlinear wave equations. Comm Pure Appl Math, 1980, 33(1): 43–101
25. Klainerman S. Uniform decay estimates and the Lorentz invariance of the classical wave equation. Comm Pure Appl Math, 1985, 38(3): 321–332
26. Krieger J, Schlag W. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. J Amer Math Soc, 2006, 19: 815–920
27. Krieger J, Schlag W. Non-generic blow-up solutions for the critical focusing NLS in 1-D. J Eur Math Soc (JEMS), 2009, 11(1): 1–125
28. Maeda M. Stability of bound states of Hamiltonian PDEs in the degenerate cases. J Funct Anal, 2012, 263(2): 511–528
29. Martel Y, Merle F, Tsai T P. Stability in $H^1$ of the sum of $K$ solitary waves for some nonlinear Schrödinger equations. Duke Math J, 2006, 133(3): 405–466
30. McKean H P, Shatah J. The nonlinear Schrödinger equation and the nonlinear heat equation reduction to linear form. Comm Pure Appl Math, 1991, 44(8-9): 1067–1080
31. Mizumachi T. Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential. Kyoto J Math, 2008, 48: 471–497
32. Pazy A. Semigroups of Linear Operators and Applications to Partial Differential Equations. Berlin: Springer-Verlag, 1983
33. Perelman G S. Asymptotic stability of solitons for nonlinear Schrödinger equations. Comm Partial Differential Equations, 2004, 29: 1051–1095
34. Pusateri F, Soffer A. Bilinear estimates in the presence of a large potential and a critical NLS in 3d. arXiv: 2003.00312
35. Rodnianski I, Schlag W, Soffer A. Asymptotic stability of $N$-soliton states of NLS. arXiv: 0309114
36. Schlag W. Stable manifolds for an orbitally unstable nonlinear Schrödinger equation. Ann of Math, 2009, 169: 139–227
37. Sigal I M. Nonlinear wave and Schrödinger equations, I. Instability of periodic and quasi-periodic solutions. Comm Math Phys, 1993, 153: 297–320
38. Soffer A, Weinstein M I. Multichannel nonlinear scattering theory for nonintegrable equations. Comm Math Phys, 1989, 342: 312–327
39. Soffer A, Weinstein M I. Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. Invent Math, 1999, 136(1): 9–74
40. Soffer A, Weinstein M I. Selection of the ground state for nonlinear Schrödinger equations. Rev Math Phys, 2004, 16(8): 977–1071
41. Staffilani G. On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations. Duke Math J, 1997, 86(1): 109–142
42. Weinstein M I. Modulational stability of ground states of nonlinear Schrödinger equations. SIAM J Math Anal, 1985, 16: 472–491
43. Weinstein M I. Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm Pure Appl Math, 1986, 39: 51–67