On Geometric Properties of Passive Random Advection

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We study geometric properties of a random Gaussian short-time correlated velocity field by considering statistics of a passively advected metric tensor. That describes universal properties of fluctuations of tensor objects frozen into the fluid and passively advected by it. The problem of one-point statistics of co- and contravariant tensors is solved exactly, provided the advected fields do not reach dissipative scales, which would break the symmetry of the problem. Asymptotic \((t \to \infty)\) duality of the problem is established, which in the three-dimensional case relates the probabilities of the volume deformations into “tubes” and into “sheets”.

I. INTRODUCTION

A problem of passive advection in a turbulent medium attracts considerable attention as a solvable model of turbulence. Exact solutions can be found in a simplified case, when the velocity field is chosen to be a random, short-time correlated Gaussian process. Statistics of density, concentration, passive vectors advected by such a field were investigated by many authors (see, e.g., [1–6]), where intermittent nature of the fluctuations, non-trivial scalings of structure functions and anomalous role of the dissipation were discovered. All these features are very common in the general picture of turbulence and, therefore, the problem of passive advection can serve as a model for developing corresponding analytical tools.

In the present paper we consider passive advection (in the Lie sense) of a second-rank covariant tensor in \(d\)-dimensional space. Though our master equation for the probability density function (PDF) of the tensor (Eq. (1) below) is very general, we concentrate mainly on statistics of a symmetric (metric) tensor \(g_{ij}\). One-point statistics of any tensor object frozen into the fluid can be related to statistics of such a tensor. We do not impose any restrictions (such as incompressibility) on the velocity field, and, therefore, statistics in both Eulerian and Lagrangian frames are studied. Also, we are only interested in the “initial stage” of the advection, when the advected field does not reach dissipative scales. This allows us to explore the symmetries of the problem, which are broken when dissipation is included.

We show that the probability-density function of the eigenvalues of the metric is governed by a \(d\)-particle Hamiltonian that can be split into two non-interacting parts. Its non-universal part describes the motion of the center of mass (the determinant \(g\) of the metric) and can be separated from the motion relative to the center of mass, i.e. dynamics of the metric’s eigenvalues normalized to their geometrical mean, \(\lambda_i/g^{1/d}\). The Hamiltonian of the latter motion is of the Calogero-Sutherland type, remains the same in both Lagrangian and Eulerian frames of reference, and therefore describes the universal properties of the advection. These properties are dictated by the symmetry of the problem. The exact integrability of the Calogero-Sutherland Hamiltonian is known to be related to \(SL(d)\) symmetry: the Hamiltonian can be represented as a quadratic polynomial in terms of the generators of the corresponding algebra \(\mathfrak{sl}(d)\). The eigenfunctions of this Hamiltonian are the so-called Jack polynomials, which are symmetric homogeneous functions of the eigenvalues. This allows us to find exactly all moments \((\hat{T}^m)\) of any tensor \(\hat{T}\) advected by the fluid. Indeed, calculating any such moment reduces to averaging expressions of the type \(\text{Tr}^k(\hat{g}^n)\), which are symmetric polynomials in terms of the metric’s eigenvalues, and can therefore be expanded in Jack polynomials of degree \(nk\). We illustrate this method by calculating exactly all moments of passively advected vectors and covectors, in particular, of the magnetic field in kinematic regime and of the passive-scalar gradient. We also demonstrate how this approach works in the general case of a passively advected tensor of any rank.

Calculating the moments requires knowing the statistics of the metric \(\hat{g}\) with special initial conditions, \(g_{ij}(t = 0) = \delta_{ij}\). However, it is also interesting to consider the evolution of the PDF of the symmetric tensor \(g_{ij}\) subject to arbitrary initial conditions. In this context, we show that a beautiful dual picture exists: the time-dependent PDF of the tensor becomes asymptotically \((t \to \infty)\) invariant under the inversion of the eigenvalues with respect to their geometrical mean. For example, in three dimensions, that means that if a magnetic field advected by ideally conducting fluid develops flux tubes, it must develop magnetic sheets with the same probability.

The paper is organized as follows. In Section [1] we derive the master equation for the PDF of the metric’s eigenvalues, and analyze the symmetry properties of

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II. MASTER EQUATION

A covariant second-rank tensor field \( \varphi_{ij}(t, \mathbf{x}) \) passively advected by the velocity field \( \xi^{k}(t, \mathbf{x}) \) evolves according to the following equation:

\[
\partial_{t} \varphi_{ij} + \xi^{k} \partial_{x} \varphi_{ij,k} + \xi_{i}^{j} \varphi_{k,j} + \xi_{j}^{k} \varphi_{i,k} = 0,
\]

(1)

where \( \xi^{k}_{i} = \partial \xi^{k}/\partial x^{i} \), and \( \varphi_{ij,k} = \partial \varphi_{ij}/\partial x^{k} \). Let \( \xi^{i}(t, \mathbf{x}) \) be a Markovian Gaussian field:

\[
\langle \xi^{i}(t, \mathbf{x})\xi^{j}(t', \mathbf{x}') \rangle = \kappa^{ij}(\mathbf{x} - \mathbf{x}') \delta(t - t'),
\]

(2)

\[
k^{ij}(y) \simeq \kappa_{0} \delta^{ij} - \kappa_{2} (y^{2} \delta^{ij} + 2ay^i y^j), \quad y \to 0,
\]

where \( a \) is the compressibility parameter, and \( \kappa_{2} = 1 \) for simplicity. Here \( a \) can vary between \(-1/(d + 1)\) for the incompressible flow and 1 for the fully compressible flow.

In order to determine the statistics of the tensor, we follow a standard procedure \([9,10]\) and introduce the characteristic function of \( \varphi(t, \mathbf{x}) \):

\[
Z(t, \tilde{\sigma}) = \langle \exp \{i \sigma^{ij} \varphi_{ij}(t, \mathbf{x}) \} \rangle.
\]

(3)

This function is a Fourier transform of the PDF of the matrix elements \( \varphi_{ij} \). Clearly, \( Z \) is independent of \( \mathbf{x} \) due to spacial homogeneity. We find that \( Z \) satisfies

\begin{align*}
\partial_{t} Z & = -[1 + a(d + 1)] \sigma^{ij} \frac{\partial Z}{\partial \sigma^{ij}} + 2a \sigma^{ij} \frac{\partial^{2} P}{\partial \sigma^{im} \partial \sigma^{mn}} \frac{\partial Z}{\partial \sigma^{mn}} \\
& + \frac{1}{2} \left( \sigma^{ij} \frac{\partial}{\partial \sigma^{kl}} + \sigma^{ji} \frac{\partial}{\partial \sigma^{lk}} \right) \left( \sigma^{il} \frac{\partial}{\partial \sigma^{kl}} + \sigma^{li} \frac{\partial}{\partial \sigma^{kl}} \right) Z, \quad \text{(4)}
\end{align*}

where \( d \) is the dimensionality of space. This equation was derived by taking the time derivative of \( Z \), using Eq. (3), and splitting Gaussian averages. We obtain the equation for the probability density function of \( \varphi \) by Fourier-transforming (4):

\[
P(\varphi) = \int \exp \{-i \sigma^{ij} \varphi_{ij} \} Z(\sigma) \prod_{m,n} d\sigma_{mn}.
\]

(5)

The original equation (4) preserves symmetry properties of the tensor \( \varphi_{ij} \), which means that we may restrict our consideration either to advection of symmetric or antisymmetric tensors. Both reductions can be done in a similar fashion. For our present purposes we only consider fluctuations of a symmetric covariant tensor. The corresponding results for a contravariant tensor are summarized in Appendix C. We will use both (co- and contravariant) pictures when discussing statistics of passive vectors in Section V.

In the symmetric case, the PDF (5) can be factorized as follows:

\[
P(\varphi) = \hat{P}(\hat{g}) \prod_{m < n} \delta(\varphi_{mn} - \varphi_{nm}), \quad \text{(6)}
\]

where \( \hat{g} \) is the symmetric part of \( \varphi \). One may think of the tensor \( \hat{g} \) as of a metric associated with the medium. Due to spacial isotropy, \( \hat{g} \) depends only on the eigenvalues \( \lambda_{1}, \ldots, \lambda_{d} \) of \( \hat{g} \). After rather cumbersome but essentially simple calculations, we establish the following master equation for the PDF of the eigenvalues of the metric:

\begin{align*}
\partial_{t} P & = (a + 1) \sum_{i} \lambda_{i}^{2} \frac{\partial^{2} P}{\partial \lambda_{i}^{2}} + 2a \sum_{i \neq j} \lambda_{i} \lambda_{j} \frac{\partial^{2} P}{\partial \lambda_{i} \partial \lambda_{j}} \\
& + \frac{1}{2} \frac{d}{d(1 + a)(d + 2)} \prod_{i < j} |\lambda_{i} - \lambda_{j}| = 1.
\end{align*}

(7)

Clearly, the original stochastic equation (4) preserves the signature of the metric. We will restrict ourselves to the case of all positive \( \lambda \)'s. Since there is no means of distinguishing between different orderings of the eigenvalues, the PDF must be a symmetric function with respect to all permutations of \( \lambda_{1}, \ldots, \lambda_{d} \).

We should now notice that in logarithmic variables \( z_{i} = \log(\lambda_{i}) \), the master equation (6) describes the dynamics of \( d \) pair-wise interacting particles on the line. Furthermore, we can consider these dynamics in the reference frame associated with the center of mass of the particles \( z = \frac{1}{d} \sum z_{i} \). Denoting the coordinates of the particles in this frame \( \zeta_{i} = z_{i} - z \), and noticing that \( \det(\hat{g}) = g = \exp(\zeta d) \), we find that \( P \) now satisfies

\[
\partial_{t} P = d[a + a(d + 1)] \left( 2g^{2} \frac{\partial^{2} P}{\partial g^{2}} + (2d + 5) g \frac{\partial P}{\partial g} \right).
\]
\[ + \frac{1}{2} (d + 1)(d + 2) P \right) + 2(1 + a) \left[ - \frac{1}{d} \sum_{i,j} \frac{\partial^2 P}{\partial \zeta_i \partial \zeta_j} \right. \\
+ \sum_{i=1}^{d} \frac{\partial^2 P}{\partial \zeta_i^2} + \frac{1}{2} \sum_{i<j} \tanh \frac{1}{2}(\zeta_i - \zeta_j) \left( \frac{\partial P}{\partial \zeta_i} - \frac{\partial P}{\partial \zeta_j} \right) \right], \quad (9)\]

where the \( d \) variables \( \zeta_1, \ldots, \zeta_d \) are not independent, \( \sum \zeta_i = 0 \). The Hamiltonian remaining after the dynamics of the center of mass are separated, is translationally invariant, therefore the total momentum of the particles \( \sum (\partial P/\partial \zeta_i) \) is conserved. The normalization rule now is:

\[ \int d\zeta_1 \cdots d\zeta_d \delta (\zeta_1 + \cdots + \zeta_d) |J(\zeta)| \int dg \, g^{(d-1)/2} P = 1, \]

\[ J(\zeta) = \frac{2d(d-1)/2}{d} \prod_{i<j} \sinh \frac{1}{2} (\zeta_i - \zeta_j), \quad (10) \]

where by \( \zeta \) we denote the set \( \{ \zeta_1, \ldots, \zeta_d \} \). The operator in the square brackets is a Sutherland Hamiltonian \( H_S \), which is exactly solvable (see, e.g., [12, 13, 14]; this Hamiltonian appeared in a similar context in [7]). The Hamiltonian \( H_S \) is the same for co- and contravariant tensors, and in both Eulerian and Lagrangian frames.

It is important that \( H_S \) is self-adjoint with respect to the measure \([8] \). Its eigenfunctions are the so-called Jack polynomials, that are homogeneous polynomials in \( \exp(\zeta) \) and are symmetric with respect to all permutations of \( \zeta \). Their construction is discussed in Appendix \([8] \). We will use particular eigenfunctions of this operator in Sec. \([7\).

We see that if \( P \) is initially chosen in a factorized form, \( P = P_1(g) P_2(\zeta_1, \ldots, \zeta_d) \), it will remain so factorized at all times. Thus, the statistics of \( g \) are independent of the statistics of the \( \zeta \)'s at all times if they are initially independent. In particular, this property of Eq. \([8\) allows us to consider separately the PDFs for the determinant of the metric and for the logarithmic quantities \( \zeta_i = \log(\lambda_i g^{-1/d}) \):

\[ S(t, g) = \int d^d \zeta |J(\zeta)| P(g, \zeta), \]

\[ F(t, \zeta) = \int dg \, g^{(d-1)/2} P(g, \zeta). \quad (11) \]

An additional symmetry emerges in this context: \([8\) and \([10\) remain invariant if the coordinates \( z_i \) of all particles are simultaneously reflected with respect to their center of mass. Such reflection leaves the center of mass intact and reverses the signs of all \( \zeta_i \), i.e., transforms all \( \lambda_i \) into \( g^{2/d} \lambda_i \). The origin of this symmetry can be understood if we notice that the master equations for the PDFs of the dimensionless quantities \( G_{ik} = g^{-1/d} g_{ik} \) and \( G^{ik} = g^{1/d} g^{ik} \) are the same, although the initial stochastic equations are different. This symmetry leads to nontrivial results for \( d \geq 3 \), and will be considered in Section \([7\).

### III. Eulerian and Lagrangian PDF's

The equation for the metric-determinant PDF \( S(t, g) \) follows from Eq. \([8\):

\[ \partial_t S = 2g^2 \frac{\partial^2 S}{\partial g^2} + (2d + 5) g \frac{g^2}{\partial g} + \frac{1}{2} (d + 1)(d + 2) S, \quad (12) \]

where we have rescaled time by the factor of \( \gamma = d(1 + a/(d + 1)) \). This factor is always non-negative and vanishes if the velocity field is incompressible, \( a = -1/(d + 1) \), in which case any time-independent function \( S(g) \) is a solution. Note that the right-hand side of Eq. \([12\) becomes a full derivative when multiplied by the Jacobian \( g^{(d-1)/2} \).

The solution of this equation is a log-normal distribution:

\[ S(t, g) = \frac{g^{-\gamma t}}{\sqrt{8\pi \gamma t}} \exp \left\{ - \frac{(\log(g) + \gamma t)^2}{8\gamma t} \right\}, \quad (13) \]

where we took the initial distribution in the form \( S(0, g) = \delta(g - 1) \).

This result can be simply understood if we note that the determinant \( g \) obeys the same equation as \( \rho^2 \), the squared density of the medium. The density satisfies the continuity equation, which can be written in logarithmic form:

\[ \partial_t \rho + \xi k \partial_k \log \rho + \xi_k = 0. \quad (14) \]

Since the time increments of \( \xi_k \) are independent identically distributed random variables, the Central Limit Theorem implies the normal distribution of \( \log \rho \). Indeed, either from Eq. \([12\) or directly from Eq. \([14\), one can easily establish that the density PDF \( R(t, \rho) = 2\rho^2 S(t, \rho^2) \) satisfies \( \partial_t R = (\gamma/2)(\rho^2 R)'' \).

So far, we have worked in the Eulerian frame, considering statistics at an arbitrary fixed point \( x \). Now we show how the one-point joint Eulerian and Lagrangian PDF's are related. Let us assume that initially Lagrangian particles are uniformly distributed in space. We denote the Eulerian PDF \( P_E(\rho, \zeta; t, x) \), the Lagrangian PDF \( P_L(\rho, \zeta; t, y) \), where \( y \) is the Lagrangian label (initial coordinate of the Lagrangian particle), and \( \rho = |\det(\partial y/\partial x)| \) (the density of the medium). The relation between \( P_E \) and \( P_L \) can be established from the following:

\[ P_E(\rho, \zeta; t, x) = \langle \delta (\rho - \rho(t, x)) \delta (\zeta - \zeta(t, x)) \rangle = \int \frac{dy}{\rho} \langle \delta (x - x(t, y)) \delta (\rho - \rho(t, y)) \delta (\zeta - \zeta(t, y)) \rangle. \quad (15) \]

Since the one-point PDF \( P_E(\rho, \zeta; t, x) \) is independent of position (due to spacial homogeneity), we can integrate \([15\) with respect to \( x \). Also noting that the one-point PDF \( P_L(\rho, \zeta; t, y) \) is independent of \( y \), we get:
\[ P_E(\rho, \zeta) = \frac{1}{\rho} P_L(\rho, \zeta). \] (16)

Transformation to the Lagrangian frame can also be performed on the level of the original stochastic equations such as \(\ref{eq:eigen1} \) with the aid of the stochastic calculus (see, e. g., \(\ref{eq:eigen3} \)).

In our considerations, if we choose initially \(S(0, g) \propto \delta(g - \gamma^2)\), we may substitute \(\rho = \sqrt{g}\) from formula \(\ref{eq:eigen2} \). We see therefore that only the PDF of \(g\) is affected by the transformation between Eulerian and Lagrangian frames. The Lagrangian version of \(S(g)\) is:

\[
S(t, g) = \frac{g^{-(d+1)/2}}{\sqrt{8\pi \gamma t}} \exp \left\{ -\frac{(\log(g) - \gamma t)^2}{8\gamma t} \right\}. \tag{17}
\]

Analogous results for the contravariant case are presented in Appendix C.

The log-normal statistics such as \(\ref{eq:eigen11} \) and \(\ref{eq:eigen17} \) are a signature of this problem, and they will also be present for fluctuations of the eigenvalue ratios in asymptotically-free régimes, i. e. where different ratios do not interact with each other \(\ref{eq:eigen1} \).

### IV. PDF’S OF EIGENVALUE RATIOS IN TWO AND THREE DIMENSIONS

We saw in the previous section that \(F(t, \zeta)\), the PDF of the ratios \(\lambda_i/g^{1/d}\), would remain the same in both Eulerian and Lagrangian frames. In this section we analyze the equations for these PDF’s in two- and three-dimensional cases. Having in mind numerical simulations, we will write these equations using \(d - 1\) independent variables. In the general case such reduction is done in Appendix A.

Let us start with the two-dimensional case. It is now convenient to integrate the \(\delta\)-function in \(\ref{eq:eigen10} \) and work with the logarithm of the eigenvalue ratio as a new variable: \(x = \frac{1}{2} \log(\lambda_1/\lambda_2) = \frac{1}{2}(\zeta_1 - \zeta_2)\). The equation for \(F(t, x)\) then becomes

\[
\partial_t F = (1 + a) \left[ F''_{xx} + \frac{1}{\tanh(x)} F'_x \right]. \tag{18}
\]

As expected, the rhs of Eq. \(\ref{eq:eigen18} \) becomes a full derivative when multiplied by the Jacobian \(J(x) = 2\sinh(x)\). Note that the differential operator in the right-hand side of Eq. \(\ref{eq:eigen18} \) becomes a Legendre operator under the change of variables \(\hat{x} = \cosh(x)\). This property is a consequence of integrability of the initial Hamiltonian \(H_S\) (Eq. \(\ref{eq:eigen5} \)), and will be of use in Sec. B when we calculate the moments of passive vectors.

The nature of the solution can be easily understood if we first consider only the advective term \(F'_x/\tanh(x)\). The characteristic of Eq. \(\ref{eq:eigen18} \) then satisfies \(\dot{x} = 1/\tanh(x)\), which implies that \(F\) is advected to regions where \(|x| \gg 1\), and, for \(t \to \infty\), the asymptotic solution can be found from \(\ref{eq:eigen18} \) by approximating \(\tanh(x) \approx 1\). The asymptotic is log-normal as expected.

Note that the reflection symmetry \(x \to -x\) of Eq. \(\ref{eq:eigen18} \) is just a consequence of the previously mentioned general symmetry \(\lambda_1 \leftrightarrow \lambda_2\), and does not add anything new. The function \(F\) must be initially chosen in such symmetric form. This is not so in the three-dimensional case that we now consider in more detail.

In three dimensions, integrating the \(\delta\)-function in \(\ref{eq:eigen10} \) as before and introducing new variables, \(x = \frac{1}{2} \log(\lambda_1/\lambda_3)\) and \(y = \frac{1}{2} \log(\lambda_2/\lambda_3)\), we obtain the equation for \(F(t, x, y)\):

\[
\partial_t F = (1 + a) \left\{ F''_{xx} + F''_{xy} + F''_{yy} \right. \\
+ \left. \left( \frac{1}{\tanh(x)} + \frac{1}{\tanh(y)} \right) F'_x \right. \\
+ \left. \left( \frac{1}{\sinh(y)} + \frac{1}{\sinh(x)} \right) F'_y \right\}. \tag{19}
\]

The normalization Jacobian for this PDF is \(J(x, y) = \frac{2\pi}{\sinh(x) \sinh(y) \sinh(x - y)}\).

The symmetry with respect to all permutations of eigenvalues \(\lambda_1, \lambda_2, \lambda_3\), leads to the following two symmetries of the solutions of Eq. \(\ref{eq:eigen19} \):

\[ x \to -x, \quad y \to y - x; \quad \text{and} \quad x \leftrightarrow y. \tag{20} \]

Eq. \(\ref{eq:eigen19} \) possesses another (reflection) symmetry as well:

\[ x \to -x, \quad y \to -y, \tag{21} \]

which corresponds to the inversion of \(\lambda_1/\lambda_3\), \(\lambda_2/\lambda_3\), and does not follow from \(\ref{eq:eigen20} \). Therefore a general initial distribution should contain both symmetric, \(F^s\), and antisymmetric, \(F^a\), parts with respect to this reflection. The symmetries \(\ref{eq:eigen21} \) act as reflections \(\ref{eq:eigen22} \) on the points of the plane located on the lines \(y = 2x, \ y = x/2\), and \(y = -x\); hence the antisymmetric part of the PDF \(F^a\) must vanish on these lines.

Characteristic trajectories of Eq. \(\ref{eq:eigen19} \) are presented in Fig. \[1\] The lines \(y = \pm x, \ y = 2x, \ y = x/2\), \(x = 0\), and \(y = 0\) are combined in groups that are transformed by the symmetries \(\ref{eq:eigen21} \) independently. Those groups correspond to sheet, tube, and strip volume deformations as shown. Let us concentrate our attention on the sector \(x \geq 0, \ y \leq 0\). Due to the symmetries \(\ref{eq:eigen22} \) and \(\ref{eq:eigen21} \), this allows us to understand the behavior of the PDF in the entire plane \((x, y)\). Considering the characteristic trajectories (they advect \(F\) towards the line \(y = -x\) from both sides), or the flux of the conserved function \(F(x, y)J(x, y)\) (calculated on the line \(y = -x\), it is found to be directed from the semisector with positive \(F^a\) to that with negative \(F^a\)), one can show that the antisymmetric part of the PDF decays with time. The symmetry of the solution with respect to the sheet and tube configurations thus emerges asymptotically as \(t \to \infty\).
of the determinant. Therefore, for all $g$, the distribution of the matrix $\hat{g}$ exactly the same formula (22), with $\hat{g}$ now the contravariant tensor.

\[ A_n = f_d(n,t) \left( g^{n/d} \right), \]
\[ f_d(n,t) = \left( g^{-n/d} (a_0 \cdot \hat{g} \cdot a_0)^n \right), \]  
(23)

where the functions $f_d(n,t)$ do not depend on the statistics of the determinant, and are, therefore, universal. These functions are the same in the co- and contravariant cases, and in both Eulerian and Lagrangian frames. The only parts of the moments $A_n$ that are non-universal are the averages of the determinant. These averages can be calculated exactly using formulas (13), (17), and (23):

\[ \langle g^s \rangle_{E}^{\text{co}} = \langle g^s \rangle_{L}^{\text{contra}} = e^{s(2s-1)\gamma t}, \]  
(24)
\[ \langle g^s \rangle_{E}^{\text{contra}} = e^{s(2s+1)\gamma t}. \]  
(25)

The universal functions $f_d(n,t)$ can, in fact, be easily calculated directly (cf. (14,15,16)), if one starts from the equation for advection of a passive vector $a^i(t,x)$:

\[ \partial_t a^i + \xi^k a^i_k - \xi^k a^k = 0, \]  
(26)

where statistics of $\xi^i(t,x)$ are given by (2). However, for methodological purposes, we prefer to rederive this result using the technique of Jack polynomials. While it is also quite simple, it illustrates the general method that can be applied to finding moments of any passively advected tensor. At the end of this section, we show, e. g., how moments of a bilinear form $a^i b^k$ can be calculated.

Formula (23) can be further simplified if we do the average with respect to the distribution of $a_0$. Introducing the generating function

\[ Z(\beta) = \exp \left\{ \beta g^{-1/d} (a_0 \cdot \hat{g} \cdot a_0) \right\}, \]  
(27)

we represent $f_d$ as follows:

\[ f_d(n,t) = \left[ \frac{\partial^n Z(\beta)}{\partial \beta^n} \right]_{\beta=0}. \]  
(28)

The Gaussian average with respect to the initial distribution of the vector can now be easily done, resulting in

\[ Z(\beta) = \prod_{i=1}^{d} \left( 1 - \beta \exp(\zeta_i) \right)^{-1/2}, \]  
(29)

where the remaining averaging is with respect to the statistics of $\zeta_i$. The PDF of the $\zeta_i$’s is $F(\zeta|J(\zeta)|\delta(\sum \zeta_i)$ with the initial condition $\delta(\zeta_1) \cdots \delta(\zeta_d)$. It is important that the function that is being averaged in (29) is the generating function for a particular class of Jack polynomials, that are eigenfunctions of the self-adjoint Sutherland operator $H_S$ in (1). Therefore, all functions (28) can be found exactly in the general case. The appropriate calculation is carried out in Appendix $\text{C}$. The answer is:

\[ f_d(n,t) = \left( \frac{d}{2} \right)_n \exp \left\{ \frac{d-1}{d} n(2n+d)(1+a)t \right\}, \]  
(30)
where we denote: \((d/2)_{n} = (d/2)(d/2+1) \cdots (d/2+n-1)\).

In the two-dimensional case the corresponding result can be obtained in a rather simple manner, which nevertheless illustrates the main idea of the general derivation. In order to do this, we notice that the generating function \(Z(\beta)\) expressed in the two-dimensional case in terms of \(x = \frac{1}{2}(\zeta_{1} - \zeta_{2})\) (see Sec. IV), coincides with the generating function for the Legendre polynomials \(P_{n}(\cosh(x))\), and, therefore, \(f_{2}(n,t) = n! \langle P_{n}(\cosh(x)) \rangle\). The average can now be completed with the aid of Eq. (18). Multiplying it by \(|\mathbf{J}(x)|P_{n}(\cosh(x))\), integrating by parts twice, and using the equation for the Legendre polynomials, \(P_{n}^{\prime}(\mu)\mu - 1 + 2\mu P_{n}(\mu) = n(n+1)P_{n}(\mu)\), we get:

\[
\begin{align*}
\hat{f}_{2}(n,t) &= (1 + a)n(n+1) f_{2}(n,t) , \\
\hat{f}_{2}(n,t) &= n! \exp\{n(n+1)(1+a)t\},
\end{align*}
\]

which is in agreement with (30).

As an example, consider moments of a magnetic field advected by the fluid. The contravariant vector in this case is \(B^{i}/\rho\), where \(\rho\) is the density of the fluid. Let us denote the moments of \(B^{i}\) as \(H_{n} = \langle |\mathbf{B}|^{2n} \rangle\). Recalling that, in the contravariant case, \(g = 1/\rho^{2}\), we get from (23):

\[
\begin{align*}
H_{n} &= f_{d}(n,t) \langle g^{-n(d-1)/d} \rangle_{\text{contra}},
\end{align*}
\]

where for the \(g\) average we use the formula (25) in Eulerian frame, or (24) in Lagrangian frame.

An analogous derivation can be carried out for a covariant vector, e.g., gradient of a passive scalar \(\nabla \theta\). For its moments \(C_{n} = \langle |\nabla \theta|^{2n} \rangle\), we find:

\[
\begin{align*}
C_{n} &= f_{d}(n,t) \langle g^{n/d} \rangle_{\text{co}},
\end{align*}
\]

where for the \(g\) average we use formulas (24) or (25) depending on the frame of reference.

On passively advected tensors

We now briefly demonstrate how one can calculate exactly the moments of a passively advected higher-rank tensor \(\hat{T}\). Suppose that we are interested in some moment \(\langle \mathbf{T}^{m} \rangle\). After averaging with respect to the initial distribution of \(\hat{T}\), we are left with a combination of \(\text{Tr}^{k}(\hat{g}^{n})\), which are polynomials of degree \(nk\) in the eigenvalues of the metric \(\hat{g}\). But any symmetric polynomial of degree \(m\) can be expanded in Jack polynomials of degree \(m\), which can then be averaged exactly. The result will therefore be a linear combination of exponents growing at the rates given by (31).

For example, consider a contravariant bilinear form \(a^{i}b^{k}\), where \(a^{i}\) and \(b^{k}\) are initially independent Gaussian random vectors, \(\langle a_{i}a_{k} \rangle = \langle b_{i}b_{k} \rangle = \delta_{ik}\), and find its second moment \(B_{2} = \langle (a \cdot b)^{2} \rangle = \langle \text{Tr}(\hat{g}^{2}) \rangle\). Then

\[
\begin{align*}
B_{2} &= \left(\sum_{i}^{d} \lambda_{i}^{2}\right) \langle g^{2/d} \rangle \left[ \langle J_{(2,0)} \rangle - \frac{2}{3} \langle J_{(1,1)} \rangle \right],
\end{align*}
\]

where polynomials \(J_{(2,0)}\) and \(J_{(1,1)}\) are constructed in (B5). The corresponding eigenvalues are \(\hat{E}_{(2,0)}^{(2)} = (d + 4)(d - 1)/d\) and \(\hat{E}_{(1,1)}^{(2)} = (d^2 - 4)/d\), as follows from (B10). The answer is:

\[
\begin{align*}
B_{2} &= \exp\left\{ \left( \frac{8}{d^{2}} + \frac{2}{d} \right) \gamma t \right\} \\
&\times \left[ \frac{d^{2} + 2d}{3} \exp\left\{ 2\hat{E}_{(2,0)}^{(2)}(1 + a)t \right\} - \frac{d^{2} - d}{3} \exp\left\{ 2\hat{E}_{(1,1)}^{(2)}(1 + a)t \right\} \right].
\end{align*}
\]

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**APPENDIX A: PDF OF EIGENVALUE RATIOS**

The \(\delta\)-function in (10) can be integrated over, and \(\zeta_{1}, \ldots, \zeta_{d}\) reduced to \(d - 1\) independent variables, viz. the logarithms of the eigenvalue ratios: \(x_{n} = \frac{1}{2}\log(\lambda_{n}/\lambda_{d}) = \frac{1}{2}(\zeta_{n} - \zeta_{d})\). In these variables, the equation for \(F\) becomes:

\[
\begin{align*}
\partial_{x} F &= (1 + a)\left\{ \sum_{n=1}^{d-1} \frac{\partial^{2} F}{\partial x_{n}^{2}} + \sum_{n=1}^{d-1} \frac{1}{\sinh(x_{n})} \frac{\partial F}{\partial x_{n}} + \frac{1}{2} \sum_{n \neq m}^{d-1} \frac{\partial^{2} F}{\partial x_{n} \partial x_{m}} + \frac{1}{4} \sum_{n \neq m}^{d-1} \frac{1}{\sinh(x_{n} - x_{m})} \left[ \frac{\sinh(x_{n})}{\sinh(x_{m})} \frac{\partial F}{\partial x_{n}} - \frac{\sinh(x_{m})}{\sinh(x_{n})} \frac{\partial F}{\partial x_{m}} \right] \right\}. \tag{A1}
\end{align*}
\]

The last two terms correspond to interactions between different \(x\)'s and only enter for \(d \geq 3\). The normalization rule now is:

\[
\begin{align*}
\frac{2^{(d+2)(d-1)/2}}{d} \int \prod_{n<m}^{d-1} |\sinh(x_{n} - x_{m})| \\
\times \prod_{n=1}^{d-1} |\sinh(x_{n})|dx_{n} F = 1. \tag{A2}
\end{align*}
\]

This form of the equation for \(F\) is most convenient for numerical solution and for geometric analysis such as that of Sec. IV.
APPENDIX B: JACK POLYNOMIALS

Jack polynomials $J_{\mu}(x_1, \ldots, x_d; \alpha)$ of degree $m$ are homogeneous (of degree $m$) polynomials, depending on $d$ variables $x_i$, and symmetric under all permutations of $x_i$. They depend on a parameter $\alpha$ and are labeled by partitions $\mu$ of an integer number $m$.

Partition $\mu$ of $m$ is a non-increasing sequence of integers: $\mu = (\mu_1 \geq \ldots \geq \mu_d) \in \mathbb{Z}^d_{\geq 0}$, such that $m = \mu_1 + \ldots + \mu_d$. The polynomials $J_{\mu}(x; \alpha)$ vanish if the number of parts $l(\mu)$ is greater than the number of variables $d$. Consider two partitions $\mu$ and $\lambda$ of the same length $l(\mu) = l(\lambda) = d$. One writes that $\lambda \geq \mu$ if $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$ for each $i \leq d$. This defines the so-called natural (or dominance) ordering on partitions.

In order to give a formal definition of the Jack polynomials, first define the monomial symmetric function $m_{\mu}$, corresponding to the partition $\mu$:

$$m_{\mu} = \sum_{\lambda \leq \mu} u_{\lambda\mu} m_{\mu},$$

where the summation is over all permutations of $\mu_1, \ldots, \mu_d$. The Jack polynomials must, by definition, be represented as:

$$J_\lambda(x; \alpha) = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_{\mu},$$

and be the eigenfunctions of the Calogero-Sutherland Hamiltonian:

$$H_\alpha^{(\infty)} = \sum_{i=1}^{d} \left( x_i \partial_{x_i} \right)^2 + \frac{2}{\alpha} \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \partial_{x_i} \partial_{x_j}.$$  \quad (B3)

All coefficients $u_{\lambda\mu}$ can be found recursively in terms of $u_{\lambda\lambda}$ with the aid of this definition [17].

Let us use this definition to construct the Jack polynomials for $m = 2$ and $\alpha = 2$. The corresponding partitions are $(2, 0, 0, \ldots)$ and $(1, 1, 0, \ldots)$. Using the first condition [22], we write:

$$J_{(2,0)}(x; 2) = \sum_{i=1}^{d} x_i^2 + A \sum_{i<j} x_i x_j,$$

$$J_{(1,1)}(x; 2) = \sum_{i<j} x_i x_j.$$  \quad (B4)

The coefficient $A$ must be found from the requirement that the polynomials be eigenfunctions of (B3), which gives $A = 2/3$.

The eigenvalues (energies) corresponding to Jack polynomials are:

$$E_\mu^{(\alpha)} = \sum_{i=1}^{d} \mu_i^2 + \frac{2}{\alpha} \sum_{i=1}^{d} (d - i) \mu_i.$$  \quad (B5)

The energies (B5) depend on particular partitions $\mu$. Any symmetric polynomial of degree $m$ can be expanded in Jack polynomials of the same degree $m$. Of all the other properties of the Jack polynomials we will need the following:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\mu} b_{\mu}(\alpha) J_{\mu}(x; \alpha) J_{\mu}(y; \alpha),$$  \quad (B6)

where the summation is performed over all possible partitions $\mu$ of all non-negative integers, and $b_{\mu}(\alpha)$ are some expansion coefficients that can be found in [17]. For our purposes we will need the formula (B6) with the set $\{y_j\}$ consisting of only one variable. In this case the expansion takes the form:

$$\prod_{i=1}^{d} \frac{1}{1 - y x_i} = \sum_{m=0}^{\infty} y^m Q_m(x; \alpha),$$  \quad (B7)

where $\mu = (m)$ is a partition consisting of only one element. $Q_m(x; \alpha)$ stand for the properly normalized Jack polynomials. The explicit expression for $Q_m(x; \alpha)$ is as follows:

$$Q_m(x; \alpha) = \frac{1}{q_1! \cdots q_l!} \left( \sum_{1 \leq i_1 \leq \ldots \leq i_m} \frac{\theta_{q_1} \cdots \theta_{q_l}}{x_{i_1} \cdots x_{i_m}} \right)$$

where $\theta = 1/\alpha$, $q_l = \#\{i | i_l = l\}$ is the multiplicity with which the number $l = 1, 2, \ldots, d$ appears in $i_1 \ldots i_m$, and $(\theta)_q = \theta(\theta + 1) \cdots (\theta + q - 1)$.

To use these results we need to transform our Hamiltonian (in the square brackets in (B5)) to the form (B3). Changing variables to $\tilde{\lambda}_i = \exp(\xi_i) = \lambda_i g^{-1/d}$, we get:

$$\tilde{H}_S = -\frac{1}{d} \sum_{i,j} \frac{\partial^2}{\partial \tilde{\xi}_i \partial \tilde{\xi}_j} + \frac{2}{\alpha} \sum_{i=1}^{d} \frac{\partial^2}{\partial \tilde{\xi}_i^2} + \frac{1}{2} \sum_{i<j} \frac{\tanh \frac{1}{2}(\xi_i - \xi_j)}{\xi_i - \xi_j} \left( \frac{\partial}{\partial \xi_i} - \frac{\partial}{\partial \xi_j} \right)$$

$$= H_\alpha^{(\infty)} - \frac{1}{d} \frac{\partial^2}{\partial \tilde{\xi}_i^2} - \frac{d-1}{2} \sum_{i=1}^{d} \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\xi}_i}.$$  \quad (B9)

For any Jack polynomials of degree $m$, the corresponding eigenvalues $E_\mu^{(\alpha)}$ of the Hamiltonian (B3) are:

$$E_\mu^{(\alpha)} = E_\mu^{(\alpha)} - \frac{m^2}{d} - \frac{(d - 1)}{2} m.$$  \quad (B10)

In particular, the energy of $Q_m(\tilde{\lambda}; 2)$ is

$$E_m = \frac{d-1}{d} m \left( m + \frac{d}{2} \right).$$  \quad (B11)

We now notice that the averaged Jack polynomials $Q_m(\tilde{\lambda}; 2)$ and the functions $f_d(n, t)/n!$ have the same generating function (see (20) and (B7)), whence
\[ f_d(n,t) = n! \langle Q(n) \tilde{\lambda}; 2 \rangle. \]  

(B12)

Since (Eq. (10)) \( \partial_t F = 2(1 + a) \tilde{H} S F \), where \( \tilde{H} S \) is self-adjoint with respect to the measure (10), \( f_d(n,t) \) satisfies: \( \dot{f}_d(n,t) = 2(1 + a) E_n f_d(n,t) \), the solution of which (with correct initial condition) is the expression (31).

APPENDIX C: PDF FOR CONTRAVARIANT TENSOR

The derivation of the main equations for the case of a contravariant tensor is quite similar to the case of the covariant tensor. Here we just explain the origin of the different coefficients in the determinant: \[ \dot{S}(\tilde{g}) \rightarrow \dot{S}(\tilde{g}). \]  Accordingly, the equation for the PDF of a contravariant tensor, \( \tilde{P}(\tilde{g}, \zeta) \), can be obtained from Eq. (10) by substituting \( \tilde{P}(\tilde{g}, \zeta) \). The \( \zeta \) part remains intact. This is not surprising, since the transition from \( \varphi_{ij} \) to \( \varphi^{ij} \) does not change the ratios of the eigenvalues, but results only in the inversion of the determinant: \( g \rightarrow 1/g \). Accordingly, the equation for the PDF of a contravariant tensor, \( \tilde{P}(\tilde{g}, \zeta) \), can be obtained from Eq. (10) by substituting \( \tilde{P}(\tilde{g}^{d+1}, g = 1/\tilde{g}) \). The resulting equation for the PDF of the determinant, \( \tilde{S}(\tilde{g}) \), is (dropping the overtilde):

\[ \partial_t S = 2g^2 \frac{\partial^2 S}{\partial g^2} + (2d + 3) g \frac{\partial S}{\partial g} + \frac{1}{2} d (d + 1) S, \]  

(C2)

where time has been rescaled by the factor of \( \gamma \) as in Sec. II. When using this equation, we should remember that \( g \) now satisfies the same equation as \( 1/\rho^2 \), where \( \rho \) is the density of the medium. Eq. (C2) is written in the Eulerian frame. For completeness, we write down the solution of Eq. (C2) with initial distribution \( S(0, g) = \delta(g - 1) \):

\[ S(t, g) = \frac{g^{-(d+1)/2}}{\sqrt{8\pi \gamma t}} \exp \left\{ - \frac{(\log(g) - \gamma t)^2}{8\gamma t} \right\}. \]  

(C3)

The Lagrangian analogue of (C3) is obtained via multiplication by \( \rho = 1/\tilde{g} \).

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