Detecting q-Gaussian distributions and the normalization effect

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We show that whenever data are gathered using a device that performs a normalization-preprocessing, the ensuing normalized input, as recorded by the measurement device, will always be q-Gaussian distributed if the incoming data exhibit elliptical symmetry. As a consequence, great care should be exercised when “detecting” q-Gaussians. As an example, Gaussian data will appear, after normalization, in the guise of q-Gaussian records. Moreover, we show that the value of the resulting parameter q can be deduced from the normalization technique that characterizes the device.

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INTRODUCTION

Systems statistically described by power-law probability distributions (PLD) are rather ubiquitous [1] and thus of perennial interest [2, 3, 4]. Indeed, many objects that come in different sizes have a self-similar power-law distribution of their relative abundance over large size-ranges, from cities to words to meteorites [1]. Now, PLDs under variance constraint maximize a non-logarithmic information measure, often called Tsallis’ entropy or q-entropy $H_q [2, 3, 4]$.

$$H_q (x) = \frac{1}{1-q} \int_{\mathbb{R}^n} dx \left[ f^q(x) - f(x) \right]; \quad q \in \mathbb{R}. \quad (1)$$

This measure tends to the celebrated Shannon entropy in the limit $q \rightarrow 1$ [2, 3, 4]. Systems for whose description $H_q$ is relevant have received intense attention in the last years, with more than 1200 papers extant and hundreds of authors [5]. To a large extent, the relevance of the concomitant treatments relies on the fact that one often confronts a particular scenario: measuring real data distributed according to a q-Gaussian probability law, a special kind of power-law probability distribution function (PDF). Consider a system $S$ described by a vector $X$ with $d$ components whose covariance matrix reads

$$K = \langle XX^t \rangle \equiv EXX^t, \quad (2)$$

the superscript $t$ indicating transposition. We say that $X$ is $q$-Gaussian distributed if its probability distribution function writes in one of the two forms to be found below [2, 3, 4], for $1 < q < \frac{d+2}{d+1}$:

$$f_{X,q} (X) = \frac{\Gamma \left( \frac{1}{q-1} \right)}{\Gamma \left( \frac{1}{q-1} - \frac{d}{2} \right) |\pi \Lambda|^{1/2}} \left( 1 + X^t \Lambda^{-1} X \right)^{-1/q}.$$  

Matrix $\Lambda$ is related to the covariance matrix $K$ according to $\Lambda = (m - 2)K$, where the number of degrees of freedom $m$ is defined [3] in terms of the dimension $d$ of $X$ as $m = \frac{2}{q-1}d$. Instead, in the case $q < 1$ the $q$-Gaussian distribution is

$$f_{X,q} (X) = \frac{\Gamma \left( \frac{2-q}{2-q} + \frac{d}{2} \right)}{\Gamma \left( \frac{2-q}{2-q} \right) |\pi \Sigma|^{1/2}} \left( 1 - X^t \Sigma^{-1} X \right)^{-1/q}_+,$$  

where the matrix $\Sigma$ is related to the covariance matrix via $\Sigma = pK$. We introduce here a parameter $p$ defined as $p = 2^{2-q}/(1-q) + d$ and use the notation $(x)_+ = \max(x, 0)$. The focus of the present endeavor revolves around the influence of the measurement device on the distribution of these data. The many authors cited above together with their readers should find the considerations developed below, concerning the influence of the so-called normalization stage on the performance of a measurement device, of great interest [12].

Most measurement devices consist of a preprocessing stage that prevents the rest of the device to be provided with data of exceedingly large amplitude that would cause damage to the hardware. Since most of measured data are of stochastic nature, the concomitant most natural and common technique is to statistically normalize these input data. Since quite often the relevant statistical properties are of unknown character, the normalization process consists of two steps: the data are first centered by substraction of their estimated mean, and then scaled by division by their estimated standard deviation. In what follows, we detail these operations, their statistical consequences, and the rather surprising impact the procedure may have with regards to non-extensive q-considerations.

THE CASE OF MULTIVARIATE GAUSSIAN DATA

Both mean and variance unknown

Assume that we have $n$ observations $\{X_i\}_{1 \leq i \leq n}$ of identically distributed data, each $X_i$ being a vector in $\mathbb{R}^d$, and that neither the mean $\mu$ nor the covariance matrix $\Sigma$ are known. A first step of the normalization process consists in centering the data by substraction, from each $X_i$, of an estimate $\mu$ of its mean.

$\begin{array}{cccc}
\begin{array}{cccc}
X_i & \rightarrow & Y_i & \rightarrow \\
\text{normalization} & \text{measurement} & \\
& \rightarrow & z_i & \\
\end{array}
\end{array}$

Figure 1: a measurement device
devised as follows
\[ \hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} X_j. \]  
(5)

We note that this estimate coincides with the maximum likelihood one if the data are of Gaussian nature. Let us denote, with \( 1 \leq i \leq n \), the residuals
\[ \epsilon_i = X_i - \hat{\mu} \]  
so that
\[ E\epsilon_i = (\epsilon_i) = 0. \]  
(6)

The next step is the scaling of these residuals by the unbiased estimate of the \((p \times p)\) covariance matrix
\[ \hat{\Sigma} = \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \epsilon_j^t \]  
(7)

Then the normalized version of vector \( X_i \), denoted as \( Y_i \), writes
\[ Y_i = \hat{\Sigma}^{-\frac{1}{2}} \epsilon_i = \left( \frac{1}{n} \sum_{j=1}^{n} \epsilon_j \epsilon_j^t \right)^{-\frac{1}{2}} \epsilon_i. \]  
(8)

We assume here that \( n \geq p + 1 \) so that the estimated covariance matrix \( \hat{\Sigma} \) is positive definite with probability one [9]. This procedure is called internal Studentization, "internal" referring to the fact that the estimated covariance matrix \( \hat{\Sigma} \) is used on all available data \( X_j \), \( 1 \leq j \leq n \), including the vector \( X_i \) to which it is applied. Another procedure is the so-called "external Studentization": it consists in normalizing the data \( X_i \) using an estimate of the covariance matrix that involves all data except \( X_i \), according to
\[ \hat{\Sigma}_{(i)} = \frac{1}{n-1} \sum_{j \neq i}^{n} \epsilon_j \epsilon_j^t, \]  
(9)

so that we denote by
\[ Z_i = \hat{\Sigma}_{(i)}^{-\frac{1}{2}} \epsilon_i = \left( \frac{1}{n-1} \sum_{j \neq i}^{n} \epsilon_j \epsilon_j^t \right)^{-\frac{1}{2}} \epsilon_i \]  
(10)

the resulting normalized data. It turns out that the distribution of normalized data \( Y_i \) and \( Z_i \) can be explicitly computed when the measured data \( X_i \) are Gaussian, as follows from the theorem below due to Diaz-Garcia et al. [6].

**Theorem 1.** [Diaz-Garcia] suppose that matrix \( X = [X_1, \ldots, X_n] \) is Gaussian \( \mathcal{N}(\mu \otimes 1_n, I_n \otimes \Sigma) \) distributed with \( 1_n = [1, \ldots, 1] \in \mathbb{R}^n \). Then the normalized data \( Y_i \) and \( Z_i \) are q-Gaussian distributed
\[ f_{Y_i}(V) = \frac{\Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-p-1}{2} \right)}{(\pi (n-1))^\frac{p}{2} \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-p-1}{2} \right)} \left( 1 + \frac{V^t V}{n-1} \right)^{-\frac{n+p-1}{2}}. \]  
(11)

and
\[ f_{Z_i}(Z) = \frac{\Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-p-1}{2} \right)}{(\pi (n-1))^\frac{p}{2} \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-p-1}{2} \right)} \left( 1 + \frac{Z^t Z}{n-1} \right)^{-\frac{n+p-1}{2}}. \]  
(12)

We remark that in the case of internal Studentization, the normalized data have bounded support. This corresponds to a "hard normalization" strategy that ensures the boundedness of the data that feed the measurement device. On the other hand, external Studentization corresponds to a "soft normalization" strategy in which boundedness of the data is not crucial to the proceedings.

Moreover, a third strategy, that may be called "full external Studentization", is considered in [7]: in order to normalize vector \( X_i \), the estimate of the covariance matrix \( \Sigma \) and of the mean \( \mu \) are built with all available data but \( X_i \). This can be the case whenever the measurement device builds estimates on a batch basis. As a consequence, the estimated mean is
\[ \hat{\mu}_{(i)} = \frac{1}{n-1} \sum_{j \neq i}^{n} X_j \]  
and the estimated covariance matrix is
\[ \hat{\Sigma}_{(i)} = \frac{1}{n-1} \sum_{j \neq i}^{n} (X_j - \hat{\mu}_{(i)})(X_j - \hat{\mu}_{(i)})^t. \]  

In such a scenario the following result holds [7].

**Theorem 2.** [Eaton] Suppose that matrix \( X = [X_1, \ldots, X_n] \) is Gaussian \( \mathcal{N}(\mu \otimes 1_n, I_n \otimes \Sigma) \) distributed. Then, the random vector
\[ V_i = \hat{\Sigma}_{(i)}^{-\frac{1}{2}} (X_i - \hat{\mu}_{(i)}) \]  
(13)

has probability distribution function (pdf)
\[ f_{V_i}(V) = \frac{\Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-p-1}{2} \right)}{(n \pi)^\frac{p}{2} \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n-p-1}{2} \right)} \left( 1 + \frac{V^t V}{n} \right)^{-\frac{n+p-1}{2}}. \]  
(14)

**Proof.** See the proof in [7], where it is qualified as a "routine multivariate calculation". \( \square \)

**Unknown variance**

In some contexts one may assume that the mean of the data is known. Thus, by replacing \( X_i \) by \( X_i - \mu \) one may state, without loss of generality, that the mean equals zero. The two strategies - internal or external Studentization - remain possible, except that \( \hat{\mu} \) should now be replaced by 0 in (8), (10), and (6). The distributions of the normalized data are expressed as follows.
Theorem 3. Under the same hypotheses as in Th[7] and assuming that the expectation of the data vanishes, the normalized data $Y_i$ and $Z_i$ are $q$-Gaussian distributed

$$f_{Y_i}(Y) = \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi n)^{\frac{n}{2}}} \left(1 - \frac{Y_i^t Y_i}{n}\right)^{-\frac{n}{2}}$$

and

$$f_{Z_i}(Z) = \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi n)^{\frac{n}{2}}} \left(1 + \frac{Z_i^t Z_i}{n}\right)^{-\frac{n}{2}}.$$ (16)

We remark that the known mean distributions (15) and (16) can be recovered from the unknown mean ones (11) and (12) by replacing the parameter $n$ by $n - 1$.

EXTENSION TO ELLIPTICAL DISTRIBUTIONS

The class of elliptically distributed random matrices plays an important role in statistics [8]. Let us devote a few words to the concept of elliptical symmetry, a generalization of the celebrated spherical symmetry to which the multi-normal distribution belongs. Spherical symmetry, that is invariance against rotations, found in the fundamental laws of nature, constitutes one of the most powerful principles in elucidating the structure of individual atoms, complicated molecules, entire crystals, and many other systems. Elliptical distributions have recently gained a lot of attention in financial mathematics, being of use particularly in risk management. In what follows, we restrict our attention to the subset of elliptical matrices which have absolutely continuous distributions. The pertinent definition reads as follows [10].

Definition. A $(p \times n)$ random matrix $X$ has a matrix variate, elliptical contoured distribution, denoted as $E_{p,n}(M, \Sigma \otimes \Phi, h)$, if its probability distribution function writes

$$f_X(X) = |\Sigma|^{-\frac{n}{2}}|\Phi|^{-rac{p}{2}} \times h\left[tr\left((X - M)^t \Sigma^{-1} (X - M)\right)\Phi^{-1}\right],$$  (17)

where the dimensions of the involved matrices are, respectively, $T: (p \times n), M: (p \times n), \Sigma: (p \times p), \Phi: (n \times n)$. Moreover, matrices $\Sigma$ and $\Phi$ are definite positive and function $h: [0, \infty] \rightarrow \mathbb{R}^+$ is called the density generator of $X$. In what follows, we restrict our attention to the case $\Phi = I_n$.

A very important result of this paper can be phrased as follows:

Theorem 4. Theorem[7][2] and[3] still hold under the general assumption that $X \sim E_{p,n}(M, \Sigma \otimes I_n, h)$.

Proof. By proper scaling, it suffices to consider $X \sim E_{p,n}(0, I_{p,n}, h)$: in this case, a stochastic representation of $X$ is, by lemma [1] below

$$X = rU$$  (18)

with

$$U = \frac{N}{||N||},$$  (19)

$N$ being a Gaussian $N_{n,p}(0, I_{n,p})$ matrix. Since any of the vectors $Y_i, Z_i$ and $V_i$ given by equations (8), (10) and (13) are homogeneous functions of order 0 of the data $X_i$, we deduce that Gaussian data can be replaced by uniform data according to (19), and thus by elliptical data according to (18).

Note that the proof can also deduced from the more general Thm. 5.3.1 in [11].

Moreover, this result can be generalized to a wider class of distributions by noticing that the distribution of the normalized residuals does not depend on the covariance matrix $\Sigma$ of the data. Consequently, matrix $\Sigma$ can be randomly chosen within the set of positive definite matrices, independently of the data. We thus can state the following

Theorem 5. The result of theorem [3] extends to the general case $X \sim E_{p,n}(M, \Sigma \otimes I_n, h)$ with a random and positive definite matrix $\Sigma$.

DISCUSSION AND CONCLUSIONS

The following sketch illustrates the main result of this paper.

![Figure 2: the effect of Studentization](image)

In other words, any of the three scaling preprocessing operations described above maps the set of elliptically invariant distributions on the set of $q$-Gaussian distributed data. Depending on the scaling method used, the resulting value of the parameter $q$ is given as follows.

| Studentization method | internal | external | full external |
|-----------------------|----------|----------|---------------|
| value of $q$           | $\frac{p-n-5}{p-4}$ | $\frac{p+1}{p}$ | $\frac{p+1}{p-1}$ |

Table I: Values of parameter $q$ according to the normalization procedure: case where both mean and covariance are unknown

| Studentization method | internal | external |
|-----------------------|----------|----------|
| value of $q$           | $\frac{p-n}{p-1}$ | $\frac{p+1}{p}$ |

Table II: Values of parameter $q$ according to the normalization procedure: case where only the covariance is unknown

Two remarks are of interest at this point:
1. observation of the data after the normalization stage does not allow to infer the distribution of the data: in particular, putative Gaussian data are systematically transformed into \( q \)-Gaussian data.

2. in the same way, \( q \)-Gaussian data are transformed into \( q' \)-Gaussian data, with parameter \( q' \) given by one of the values in Table 1, depending on the normalization procedure: this means that the real value of parameter \( q \) is "erased" by the normalization process.

As a conclusion, the origin of \( q \)-Gaussian data should be carefully analyzed, since they may occur, for a very large set of recorded data (namely the set of elliptical ones), as a simple consequence of a statistical normalization step. In other words, the putative "\( q \)-Gaussianity" may be a mere artifact of the statistical normalization step: caution is to be exercised.

As an example, a measured value of the nonextensivity parameter \( q \) appears in [14, p.230] in the context of financial markets as follows:

\[ \ldots \text{returns (once demeaned and normalized by their standard deviation) have a distribution that is very well fit by } q \text{-Gaussians with } q \approx 1.4. \]

The estimated value \( q = \frac{3}{2} \) corresponds to an external Studentization with \( n = 6 \) data. Now, the general result from our theorem 5 above suggests that the "real" distribution of these data may well indeed differ from the estimated distribution.

ANNEX: STOCHASTIC REPRESENTATION

Lemma 1. If \( X \sim \mathcal{E}_{p,n}(M, \Sigma \otimes I_n, h) \) then the stochastic representation

\[ A = M + r \Sigma^{1/2} U \]

holds where \( U \) is a uniform matrix on the manifold of \((p \times n)\) matrices with unit Frobenius norm (see Footnote 2) [16], and \( r \) is a positive random variable independent of matrix \( U \).

Proof. Associate to \( A \) \((n \times p)\) the vector \( a = \text{vec}(A) \in \mathbb{R}^{np} \) so that \( \|a\|^2 = \|A\|^2 \) and

\[ f_a(a) = f \left( \|a\|^2 \right). \]

Thus a stochastic representation for vector \( a \) is [13] \( a = ru \) where \( u \) is uniformly distributed on the sphere in \( \mathbb{R}^{np} \) and \( r \) is positive and independent of \( u \); thus it writes

\[ u = \frac{g}{\|g\|} \]

where \( g \) is a \((n \times p)\) Gaussian vector; we deduce that

\[ A = rU \]

where \( U \) is obtained by stacking the columns of \( u \) in a \((n \times p)\) matrix form so that

\[ U = \frac{G}{\sqrt{\text{tr} \left( G^2 \right)}} \]

where \( G \) is a Gaussian \((n \times p)\) matrix, so that \( U \) is uniform on the set of \((n \times p)\) matrices with unit norm.

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[15] Footnote 1: equivalently, vectors \( X_i \) are independent and identically Gaussian \( \mathcal{N}(\mu, \Sigma) \) distributed
[16] Footnote 2: the Frobenius norm of matrix \( A \) is \( \|A\| = \sqrt{\text{tr} \left( A^T A \right)} \)