Analysis of the Knight shift data on Li and Zn substituted YBa$_2$Cu$_3$O$_{6+x}$

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The Knight shift data on Li and Zn substituted YBa$_2$Cu$_3$O$_{6+x}$ are analysed using an itinerant model with short-range antiferromagnetic correlations. The model parameters, which are determined by fitting the experimental data on the transverse nuclear relaxation rate $T_2^{-1}$ of pure YBa$_2$Cu$_3$O$_{6+x}$, are used to calculate the Knight shifts for various nuclei around a nonmagnetic impurity located in the CuO$_2$ planes. The calculations are carried out for Li and Zn impurities substituted into optimally doped and underdoped YBa$_2$Cu$_3$O$_{6+x}$. The results are compared with the $^7$Li and $^{89}$Y Knight shift measurements on these materials.

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I. INTRODUCTION

The substitution of nonmagnetic impurities into cuprates serves as a probe of the electronic correlations in these materials. In particular, it has been found that the nonmagnetic impurities significantly influence the magnetic correlations in the normal state [1–10]. The 89Y NMR experiments [11,12] find that when Zn is substituted into YBa$_2$Cu$_3$O$_{6+x}$, the Knight shifts of the Y sites near the impurity are strongly enhanced as the temperature is lowered. It has been also found that the Li impurities induce changes in the local magnetic environment similar to that of Zn [2]. A Curie-like $T$ dependence is found in the SQUID measurements of the uniform susceptibility for Zn substituted YBa$_2$Cu$_3$O$_{6+x}$ [3] as well. It has been also shown that the Zn impurities induce a Curie-like $T$ dependence in the broadening of the $^{65}$Cu(2) spectral line in YBa$_2$Cu$_3$O$_{6.7}$ [4] and by the inelastic neutron scattering experiments [13,14]. The $T_1^{-1}$ measurements have been also carried out for Cu(2) in Zn substituted YBa$_2$Cu$_3$O$_{6.7}$ [15] and YBa$_2$Cu$_3$O$_8$ [4] systems. The effects of the nonmagnetic impurities on the magnetic correlations were also studied in La$_{1.85}$Sr$_{0.15}$CuO$_4$ with Al impurities [16].

Numerous theoretical studies [17,21] have been carried out for exploring the effects of the nonmagnetic impurities on the magnetic properties of the cuprates. The effects of a nonmagnetic impurity embedded into the 2D Heisenberg lattice [18,19] as well as in gapped Heisenberg antiferromagnets [10] have been studied. The exact diagonalization calculations have been used to study the effects of a nonmagnetic impurity in the $t$-$J$ model [17]. Theoretical studies have been also carried out for the underdoped YBa$_2$Cu$_3$O$_{6+x}$, where there is a spin pseudogap [19]. Calculations of the neutron scattering intensity for Zn substituted YBa$_2$Cu$_3$O$_{6.5}$ have been also carried out [13,22].

In this paper, a simple framework [23] of one nonmagnetic impurity embedded into the lattice of the two-dimensional (2D) Hubbard model will be used to make comparisons with the $^7$Li and $^{89}$Y Knight shift measurements on the Li and Zn substituted YBa$_2$Cu$_3$O$_{6+x}$. The purpose of this paper is to study the effects of the nonmagnetic impurities on the magnetic correlations in the normal state of the cuprates. In the following, the 2D Hubbard model and the random-phase approximation (RPA) will be used for modelling the antiferromagnetic correlations in pure YBa$_2$Cu$_3$O$_{6+x}$. The 2D Hubbard model is defined by

$$H = -t \sum_{\langle i,j \rangle,\sigma} (c^\dagger_i c_j + c^\dagger_j c_i) + U \sum_i c^\dagger_i c^\dagger_i c_i c_i$$

$$-\mu \sum_{i,\sigma} c^\dagger_i c_i \sigma.$$  (1)

Here $c^\dagger_i$ ($c^\dagger$) annihilates (creates) an electron with spin $\sigma$ at site $i$, $t$ is the near-neighbor hopping matrix element, $U$ is the onsite Coulomb repulsion, and $\mu$ is the chemical potential. The model parameters for the pure system will be determined by fitting the experimental data [22,23] on the transverse nuclear relaxation rate $T_2^{-1}$ of pure YBa$_2$Cu$_3$O$_{6+x}$.

Next, a static extended impurity potential will be used to incorporate the effects of one nonmagnetic impurity embedded into the 2D Hubbard model. There is considerable support for using an extended impurity potential for modelling the effective impurity potential [26,27]. Here, it will be assumed that the effective interaction between an impurity located at site $r_0$ and a quasiparticle at $r$ can be approximated by the static potential

$$V_{\text{eff}}(r_0,r) = V_0 \delta(r_0,r) + V_1 \sum_{\alpha=1}^4 \delta(r,r_0 + \rho_{\alpha}),$$  (2)

which has a range of one lattice spacing. In Eq. (2), $\alpha$ sums over the four near-neighbor sites of the impurity as illustrated in Fig. 1. Using this simple form of $V_{\text{eff}}$, the Knight shift of various nuclei around the impurity will be calculated, and the results will be compared with the experimental data.
In the following, since the Zn and Li impurities are considered to have closed outer electronic shells with $S = 0$, the onsite component $V_0$ will be set to a large negative value. On the other hand, the near-neighbor component $V_1$ will be used as a free model parameter. Clearly, this form of $V_{\text{eff}}$, Eq. (2), is a simple approximation. For instance, the range of $V_{\text{eff}}$ could be more than one lattice spacing. Furthermore, $V_{\text{eff}}$ could have scattering in the magnetic channel (3). In fact, any magnetic scattering could drastically change the results presented here. Nevertheless, it is interesting to explore the consequences of this form of $V_{\text{eff}}$ on the Knight shifts within this simple framework. Within this model at the level of RPA, the fitting of the $^7$Li and $^{89}$Y Knight shift data will require that $V_1$ is weakly attractive.

The comparisons will be first carried out with the $^7$Li Knight shift, $^7K$, measurements in Li substituted YBa$_2$Cu$_3$O$_{6.97}$ by Bobroff et al. [2]. Here, $^7K$ will be calculated for various values of $V_1$. It will be seen that the temperature dependence of $^7K$ can be fitted by using a $V_1 \approx -0.15t$. The effects of the nonmagnetic impurity on the $^{89}$Y and the $^{63}$Cu(2) Knight shifts for the sites near the impurity will also be given for this compound.

According to this simple model [2], the anomalous $T$ dependence observed in the Knight shift data is due to the enhancement of the antiferromagnetic correlations in the local environment of the impurity. In this model, the scattering of the antiferromagnetic spin fluctuations by the impurity with large momentum transfers near $2k_F$, where $k_F$ is the fermi wave vector, plays a particularly important role [2]. These scatterings locally enhance the antiferromagnetic correlations, and, in addition, allow for the Knight shift of the nuclear sites near the impurity to be coupled to the locally enhanced antiferromagnetic correlations.

Because of the presence of the spin pseudogap in underdoped YBa$_2$Cu$_3$O$_{6+x}$, it is indeed difficult to extend this analysis to this compound. However, the Knight shift experiments on underdoped YBa$_2$Cu$_3$O$_{6+x}$ yielded interesting results on the real-space structure of the induced magnetic correlations around the impurity. These measurements have been carried out for $^7$Li in Li substituted YBa$_2$Cu$_3$O$_{6.6}$ [3] and for $^{89}$Y in Zn or Li substituted YBa$_2$Cu$_3$O$_{6.64}$ [4]. For $^{89}$Y, two magnetic resonance satellites in addition to the main line are found in the presence of the Zn or Li impurities in underdoped YBa$_2$Cu$_3$O$_{6+x}$. These satellites have been identified as belonging to the first and the second neighbour Y sites of the impurity. The availability of the Knight shift data on $^7$Li and the two $^{89}$Y satellites gives valuable information on the real-space structure of the magnetic correlations in the local environment of the impurity [3]. The $^7$Li and $^{89}$Y Knight shift measurements by Mahajan et al. [4] and the data on the Knight shift of $^{63}$Cu(2), $^{63}K_\text{Cu}$, by Julien et al. [5] point out that a staggered polarisation cloud forms around the impurity when a uniform magnetic field is applied. Hence, here, using simple assumptions, the results of the calculations will be compared with the $^7$Li and $^{89}$Y Knight shifts in order to see whether this simple model has some of the features observed experimentally. Results on $^{63}K_\text{Cu}$ will also be presented. It is important to keep in mind that the purpose in this paper will not be to present a theory of the magnetic correlations in the pseudogap state within the presence of nonmagnetic impurities. This is clearly beyond the simple model used here. Rather, the purpose is to study the real-space structure of the deformations induced by the impurity by making simple assumptions. It would have been more desirable to carry out such a study for optimally doped YBa$_2$Cu$_3$O$_7$, where this model is more applicable, but in that case the $^{89}$Y lines are not resolved. Here, the model parameters determined by fitting the $T_{\text{2}}^{-1}$ data on pure YBa$_2$Cu$_3$O$_{6.63}$ will be used to calculate the $^7$Li and $^{89}$Y Knight shifts. Initially, the Knight shifts will be calculated by neglecting the opening of the pseudogap in the $q \to 0$ component of the magnetic susceptibility of the pure system. Even in this case, it will be found that for $V_1$ between $-0.125t$ and $-0.15t$, the real-space pattern of the magnetic correlations around the impurity is similar to what is seen experimentally. Later on, a pseudo gap in the $q \to 0$ component of the magnetic susceptibility will be introduced artificially. In this case, comparisons with the data require $V_1 \approx -0.15t$. It has to be re-emphasized that the results on the underdoped YBa$_2$Cu$_3$O$_{6+x}$ needs to be interpreted cautiously. The treatment of the pseudogap is clearly not rigorous. An especially important point is that the presence of the pseudogap can introduce new physics which is quite different than the simple scenario discussed here using a weak-coupling model.

In Section II below, the model will be introduced. After comparisons with the experimental data in Sections III.A and III.B, the pattern of the magnetic correlations around the impurity will be shown in Section III.C. The implications of these calculations and the role of the antiferromagnetic correlations will be discussed in Section IV. In addition, here, the dependence of the results on the effective bandwidth will be studied, and the hyperfine couplings used in the calculations will be compared with the experimental estimates. In Section V, the summary and the conclusions will be given.

II. MODEL

Here, the framework for calculating the Knight shifts will be introduced. In Section II.A, the modelling of the magnetic susceptibility for pure YBa$_2$Cu$_3$O$_{6+x}$ will be presented. The model parameters, which will be used to parameterise the antiferromagnetic correlations of pure YBa$_2$Cu$_3$O$_{6+x}$, will be determined by fitting the experimental data on the transverse nuclear relaxation rate $T_{\text{2}}^{-1}$.
of YBa$_2$Cu$_3$O$_{6+x}$ [22, 25]. In Section II.B, the static extended impurity potential used for approximating the interaction between the impurity and the electrons will be discussed. In the following section, the magnetic susceptibility for the case of one impurity will be presented. In Section II.D, the hyperfine interactions for the $^7$Li and $^{89}$Y nuclear spins will be introduced.

A. Parameterisation of $\chi(q, \omega)$ for pure YBa$_2$Cu$_3$O$_{6+x}$

In this paper, it will be assumed that the magnetic correlations of pure YBa$_2$Cu$_3$O$_{6+x}$ can be approximated by the two-dimensional Hubbard model. The magnetic susceptibility of the pure system is defined by

$$\chi_{\text{pure}}(q) = \int_0^\beta d\tau \langle m^{-}(q, \tau)m^{+}(q, 0) \rangle. \quad (3)$$

Here, $\tau$ is the Matsubara time, $m^{\pm}(q, 0) = N^{-1/2} \sum_p c_{p+q} c_{p}$, $m^{-}(q, \tau) = e^{iH\tau}m^{-}(q, 0)e^{-iH\tau}$, and $m^{-}(q, 0) = N^{-1/2} \sum_p c_{p+q} c_{p}$. The RPA form for $\chi_{\text{pure}}$ is

$$\chi_{\text{pure}}(q) = \frac{\chi_0^L(q)}{1 - U\chi_0^L(q)}. \quad (4)$$

Here, $U$ is the renormalized Coulomb repulsion and the static Lindhard susceptibility $\chi_0^L(q)$ is given by

$$\chi_0^L(q) = \frac{1}{N} \sum_p f(\varepsilon_{p+q} - f(\varepsilon_p)}{\varepsilon_{p+q} - \varepsilon_p}, \quad (5)$$

where $f(\varepsilon_p)$ is the fermi factor and $\varepsilon_p = -2t(\cos q_x + \cos q_y) - \mu$. Using this form for $\chi_{\text{pure}}(q)$ with the renormalised values of $U$ and the hopping matrix element $t$, the longitudinal and the transverse relaxation rates $T_2^{-1}$ and $T_2^{-1}$ for YBa$_2$Cu$_3$O$_{7}$ have been calculated [23, 31]. In this model, the renormalization of $U$ is due to the particle-particle correlations and the single-particle self-energy corrections, and the renormalization of the bandwidth is due to the Coulomb correlations [11]. The value of $U$ will be determined by fitting the experimental data on the $^{63}$Cu(2) transverse relaxation rate $T_2^{-1}$ in pure YBa$_2$Cu$_3$O$_{6+x}$. In the following, the effective bandwidth $W$ will be taken to be 1 eV as in Refs. [23, 30]. Later, in Section IV.C the dependence of the results on $W$ will be studied.

The transverse nuclear relation rate $T_2^{-1}$ of $^{63}$Cu(2) for an orienting field along the $c$-axis is given by [22]

$$\left(\frac{1}{T_2}\right)^2 = \frac{0.69}{32\hbar^2} \left[\frac{1}{N} \sum_q |A(q)|^4 \chi_{\text{pure}}(q) - \left(\frac{1}{N} \sum_q |A(q)|^2 \chi_{\text{pure}}(q)\right)^2\right]. \quad (6)$$

where

$$A(q) = A_c + 2B(\cos q_x + \cos q_y). \quad (7)$$

Here, the following hyperfine interaction [32] has been assumed for the $^{63}$Cu nuclear spin at site $i$,

$$\sum_{\alpha=x,y,z} A_{\alpha\alpha} I^\alpha_i S^\alpha_i + B \sum_{\delta=1}^4 I_i \cdot S_{i+\delta}, \quad (8)$$

where $\delta$ sums over the four nearest neighbours of site $i$. For the hyperfine couplings, $A_c = -4B = -2.45 \times 10^{-18}$ erg, corresponding to 164 kOe/$\mu_B$, will be used as in Ref. [30]. In fitting the $T$ dependence of $T_2^{-1}$ with the RPA form of Eq. (4), $U$ will be adjusted as $T$ is varied. Clearly, this is a simple, approximate procedure for determining the strength of the antiferromagnetic correlations in the pure system. In fact, the model is so simple that electron filling $\langle n \rangle$ will be taken to be 0.86 for both the underdoped and optimally doped systems. Hence, the only feature which differentiates between the underdoped and the optimally doped systems in this model is the strength of the antiferromagnetic correlations of the pure system which is set by adjusting $U$ to fit the $T_2^{-1}$ data. As discussed in Ref. [24], the antiferromagnetic correlations of the impure system can get significantly enhanced by the impurity scattering as compared to those in the pure system.

The fitting of $T_2^{-1}$ will be first carried out for optimally doped YBa$_2$Cu$_3$O$_{6+x}$. The transverse relaxation rate $T_2^{-1}$ of $^{63}$Cu(2) has been measured by Pennington and Slichter [23] in YBa$_2$Cu$_3$O$_7$, and by Imai et al. [24] in YBa$_2$Cu$_3$O$_{6.9}$. The data by Imai et al. [24] are represented by the open circles in Fig. 2(a). Here, it is seen that $T_2^{-1}$ increases linearly as $T$ is lowered from 300 K to 100 K. The dashed line in this figure has been obtained by using the RPA form of Eq. (4) in Eq. (6) and by adjusting $U$ to fit the experimental data. The resultant values of $U(T)$ are given by the dashed line in Fig. 2(b). Here one observes that as $T$ is lowered from 300 K to 100 K, $U(T)$ decreases by 5%. Note also that the dashed curve in Fig. 2(a) has been extrapolated up to 400 K, since in Section III the Knight shifts will be calculated up to this temperature.

The filled circles in Fig. 2(b) represent the $T_2^{-1}$ data for $^{63}$Cu(2) in YBa$_2$Cu$_3$O$_{6.63}$ measured by Takigawa [25]. Here it is seen that $T_2^{-1}$ varies almost linearly for $T$ between 160 K and 300 K. Below 160 K, $T_2^{-1}$ saturates. Because of this, the fitting of $T_2^{-1}$ for $T < 160$ K will require more attention. This saturation might be due to two reasons: (1) the suppression of $\chi_{\text{pure}}(q \sim 0)$ because of the opening of the magnetic pseudogap, or (2) the saturation of $\chi_{\text{pure}}(q)$ for $q \sim (\pi, \pi)$. In the first case, $\chi_{\text{pure}}(q \sim (\pi, \pi))$ would continue to increase as $T$ is lowered and the saturation of $T_2^{-1}$ would be due directly to the suppression of $\chi_{\text{pure}}(q \sim 0)$. In the other case,
the saturation of $T_2^{-1}$ would be due to the saturation of the antiferromagnetic correlations. In order to take into account these two possibilities, the fitting of $T_2^{-1}$ below 160 K will be done in two ways. In the first case, the saturation of $T_2^{-1}$ will be attributed to the suppression of $\chi_{\text{pure}}(q \sim 0)$ and it will be assumed that without this effect $T_2^{-1}$ would have continued to increase linearly as $T$ is lowered below 160 K. Hence, in this case, $T_2^{-1}$ will be extrapolated for $T < 160$ K as shown by the solid line in Fig. 2(a). The resultant values of $U(T)$ are shown by the solid curve in Fig. 2(b). In the second case, the $T_2^{-1}$ data below 160 K will be fitted directly as shown by the dotted curve in Fig. 2(a). The resultant $U(T)$ is shown by the dotted curve in Fig. 2(b).

Further results on the nature of the antiferromagnetic correlations are shown in Fig. 3(b). The dashed line in Fig. 3(c) shows similar results for $\chi_{\text{pure}}(q)$ used here, Eqs. (4) and (5), the peak in $\chi_{\text{pure}}(q)$ occurs at an incommensurate wave vector $Q^*$ away from $Q = (\pi, \pi)$. The neutron scattering experiments [33,34] on YBa$_2$Cu$_3$O$_{6.6+\delta}$ find that in the superconducting state, the spin-fluctuation spectral weight $\text{Im} \chi_{\text{pure}}(q, \omega)$ peaks away from $(\pi, \pi)$, but with an incommensuration which is more than what is observed in Fig. 3(a). The form of $\chi_{\text{pure}}(q, \omega)$ used here is clearly a crude approximation. The results shown below will not depend sensitively on the amount of the incommensuration, rather they will depend on the total integrated weight in the $q \sim (\pi, \pi)$ region of the $q$-space. Further results on the nature of the antiferromagnetic correlations are shown in Fig. 3(b). The dashed line in Fig. 3(b) shows $\chi_{\text{pure}}(Q^*)$ versus $T$ for optimally doped YBa$_2$Cu$_3$O$_{6+\delta}$. Also shown in this figure by the solid line are the results for $\chi(q^*)$ of the impure system with 0.5% randomly distributed impurities which will be discussed below in Section IV.A. Fig. 3(c) shows similar results for $\chi(q^*)$ of underdoped YBa$_2$Cu$_3$O$_{6+\delta}$. Here, it is seen that $\chi(q^*)$ gets significantly enhanced by the substitution of the nonmagnetic impurities, especially for underdoped YBa$_2$Cu$_3$O$_{6+\delta}$. By comparing Fig. 2(a) with Figs. 3(b) and (c), it is also seen that the temperature dependence of $T_2^{-1}$ and $\chi_{\text{pure}}(q^*)$ are closely related, as expected.

### B. Effective Impurity Potential

The effects of one nonmagnetic impurity located at site $r_0$ will be taken into account by adding to Eq. (1) the term

$$\sum_{i,\sigma} V_{\text{eff}}(r_0, r_i)c_{i,\sigma}^\dagger c_{i,\sigma}$$

where $V_{\text{eff}}(r_0, r)$ is given by Eq. (3). The importance of using an extended impurity potential has been previously noted [26,27]. In Ref. [27], it has been pointed out that the Coulomb correlations of the host could cause the extended nature of the effective interaction. The fact that the defects generated by the electron irradiation of the samples have similar effects on the $T_c$ suppression [35] and on the magnetic properties [36] also supports using an impurity potential with only a potential scattering term.

In the following, the onsite component of the impurity potential, $V_0$, will be set to a large negative value, $-100t$, in order to model the closed electronic shell of the nonmagnetic impurity. As long as $|V_0|$ has a large value, its exact magnitude or its sign does not play an important role. For instance, using $V_0 = -200t$ instead of $-100t$ does not change the results. The near-neighbour component $V_1$, on the other hand, will be used as a free parameter in fitting the Knight shift measurements on Li and Zn substituted YBa$_2$Cu$_3$O$_{6+\delta}$. In the next section, the effects of this static extended impurity potential on the magnetic susceptibility will be calculated.

### C. Magnetic susceptibility within the presence of one nonmagnetic impurity

The calculation of $\chi$ within the presence of a nonmagnetic impurity used here follows that given in Ref. [20]. First, the effects of the impurity on the single-particle Green’s function will be calculated. The single-particle Green’s function in Matsubara frequency space is defined by

$$G(r_i, r_j, i\omega_n) = -\int_0^\beta d\tau e^{i\omega_n \tau} \langle c_{i,\sigma}(\tau) c_{j,\sigma}^\dagger(0) \rangle,$$

where $\omega_n = (2n+1)\pi T$. For the pure system with $U = 0$, one has

$$G_0(r_i, r_j, i\omega_n) = \frac{1}{N} \sum_p e^{i p \cdot (r_i - r_j)} G_0(p, i\omega_n)$$

where

$$G_0(p, i\omega_n) = \frac{1}{i\omega_n - \varepsilon_p}.$$

If an impurity is introduced at site $r_0$, then one gets
\[ G(\mathbf{r}, \mathbf{r}', i\omega_n) = G_0(\mathbf{r}, \mathbf{r}', i\omega_n) + \sum_{\mathbf{r}'', i\omega_n} G_0(\mathbf{r}, \mathbf{r}'', i\omega_n) T(\mathbf{r}'', \mathbf{r}', i\omega_n) G(\mathbf{r}'', \mathbf{r}', i\omega_n), \]  

(13)

where the \( T \)-matrix for the impurity scattering is given by

\[ T(\mathbf{r}, \mathbf{r}', i\omega_n) = \delta(\mathbf{r}, \mathbf{r}') V_{\text{eff}}(\mathbf{r}_0, \mathbf{r}) + \sum_{\mathbf{r}''} V_{\text{eff}}(\mathbf{r}_0, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}'', i\omega_n) T(\mathbf{r}'', \mathbf{r}', i\omega_n). \]  

(14)

The calculation of \( G(\mathbf{r}, \mathbf{r}', i\omega_n) \) is illustrated diagrammatically in Fig. 4(a).

When the translational invariance is broken, the magnetic susceptibility is defined in real space as

\[ \chi(\mathbf{r}, \mathbf{r}', i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} (m^-(\mathbf{r}, \tau) m^+(\mathbf{r}', 0)), \]  

(15)

where \( m^+(\mathbf{r}) = c_{\mathbf{r}0}^\dagger c_{\mathbf{r}1} \) and \( m^-(\mathbf{r}) = c_{\mathbf{r}1}^\dagger c_{\mathbf{r}0} \). The effects of a single impurity will be first calculated for \( U = 0 \), giving the irreducible susceptibility \( \chi_0(\mathbf{r}, \mathbf{r}', i\omega_m) \), and then the effects of the Coulomb correlations will be included. The diagrams representing the effects of the impurity on \( \chi_0 \) are shown in Figs. 4(b) and (c). Both the self-energy and the vertex corrections need to be included [37], and the resulting expression for \( \chi_0(\mathbf{r}, \mathbf{r}', i\omega_m) \) is given by

\[ \chi_0(\mathbf{r}, \mathbf{r}', i\omega_m) = -T \sum_{i\omega_n} \left[ G(\mathbf{r}, \mathbf{r}', i\omega_n + i\omega_m) \times G_0(\mathbf{r}', \mathbf{r}, i\omega_n) + G_0(\mathbf{r}, \mathbf{r}', i\omega_n + i\omega_m) G(\mathbf{r}', \mathbf{r}, i\omega_n) \right. \]

\[ \left. -G_0(\mathbf{r}, \mathbf{r}', i\omega_n + i\omega_m) G(\mathbf{r}', \mathbf{r}, i\omega_n) \right] \]

\[ -T \sum_{i\omega_n} \sum_{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4} G_0(\mathbf{r}_1, \mathbf{r}_2, i\omega_n + i\omega_m) T(\mathbf{r}_1, \mathbf{r}_2, i\omega_n + i\omega_m) \times G_0(\mathbf{r}_3, \mathbf{r}_4, i\omega_n) G_0(\mathbf{r}_3, \mathbf{r}_4, i\omega_n). \]  

(16)

Note that the irreducible impurity-scattering vertex has been used in calculating the vertex corrections to \( \chi_0 \) instead of the reducible one. This is necessary in order to prevent double counting, since here the effects of only one impurity are calculated.

Next, the Coulomb correlations are included by solving the RPA equation

\[ \chi(\mathbf{r}, \mathbf{r}', i\omega_m) = \chi_0(\mathbf{r}, \mathbf{r}', i\omega_m) + U \sum_{\mathbf{r}''} \chi_0(\mathbf{r}, \mathbf{r}'', i\omega_m) \chi(\mathbf{r}'', \mathbf{r}', i\omega_m) \]  

(17)

for \( \chi(\mathbf{r}, \mathbf{r}', i\omega_m) \). Here \( \chi \) is calculated for an \( N \)-site square lattice with one nonmagnetic impurity located at the center using periodic boundary conditions. The calculations are carried out on sufficiently large lattices so that the finite size effects are small. In Ref. [27], the finite size effects on \( \chi \) have been studied in detail.

The Knight shift is determined by \( \chi(\mathbf{r}, \mathbf{r}') = \chi(\mathbf{r}, \mathbf{r}', i\omega_m = 0) \). In the following section, the hyperfine interactions, which relate \( \chi(\mathbf{r}, \mathbf{r}') \) to the Knight shifts, will be discussed for \( ^7\text{Li} \) and \(^{89}\text{Y} \).

### D. Hyperfine interactions for \(^7\text{Li} \) and \(^{89}\text{Y} \)

The hyperfine interaction between the \(^7\text{Li} \) nuclear spin with \( I = 3/2 \) substituted into a \( \text{Cu}(2) \) site and the electronic spins will be modelled by the following hyperfine coupling [2]

\[ C \sum_\delta^4 \vec{\gamma}_I \cdot \vec{S}_\delta. \]  

(18)

Here \( \delta \) sums over the four \( \text{Cu}(2) \) sites neighbouring the \( \text{Li} \) impurity. The nature of this interaction is similar to that of the transferred hyperfine coupling of the \(^{63}\text{Cu}(2) \) nuclear spins to the electronic spins at the neighbouring \( \text{Cu}(2) \) sites given by the second term in Eq. (8) [32]. In Section III, the magnitude of the \(^7\text{Li} \) hyperfine coupling \( C \) will be taken to be \( 1.8 \times 10^{-20} \text{ erg} \), corresponding to 0.85 kOe/\( \mu_B \). This choice for the value of \( C \) will be discussed later in Section IV.C. As a result of this coupling, Eq. (18), the Knight shift of \(^7\text{Li} \) is (see Appendix I)

\[ 7K = \frac{1}{2} \left( \frac{\gamma_e}{\gamma_n} \right) C \delta k(\mathbf{r} = (1, 0)), \]  

(19)

where \( k(\mathbf{r}) \) is defined by

\[ k(\mathbf{r}) = \sum_{\mathbf{r}'} \chi(\mathbf{r}, \mathbf{r}'), \]  

(20)

and \( \gamma_e \) and \( \gamma_n \) are the gyromagnetic ratios of the electron and the \(^7\text{Li} \) nuclear spin. The factor of four in Eq. (19) is because the \( \text{Li} \) impurity has four \( \text{Cu}(2) \) neighbours.

The hyperfine interaction for the \(^{89}\text{Y} \) nuclear spin with \( ^{89}I = 1/2 \) is

\[ D \sum_{\alpha=1}^8 \vec{\gamma}_\alpha \cdot \vec{S}_\alpha \]  

(21)

where \( \alpha \) sums over the eight \( \text{Cu}(2) \) sites neighbouring the \(^{89}\text{Y} \) nuclear spin. In Section III, the hyperfine interaction \( D \) will be assumed to be \( -2.2 \times 10^{-20} \text{ erg} \), corresponding to \( -1.0 \text{ kOe}/\mu_B \), in order to fit the normal state value of \( ^{89}K \) in pure \( \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \) using a bandwidth of \( W = 1 \text{ eV} \). Later in Section IV.C, the dependence of the results on \( W \) and on the values of the hyperfine couplings \( C \) and \( D \) will be discussed. Here, it will also be assumed that the \(^{89}\text{Y} \) hyperfine couplings do not change upon the substitution of the nonmagnetic impurity. Then the Knight shift for \(^{89}\text{Y} \) at a particular \( \text{Y} \) lattice site is given by
\[ 89K = \frac{1}{2} \left( \frac{\gamma_e}{89\gamma_n} \right) D \sum_{\alpha=1}^{8} k(r_\alpha), \quad (22) \]

where \( r_\alpha \) are the locations of the eight nearest-neighbour Cu(2) sites of the Y site in the CuO\(_2\) bilayer. For dilute Li or Zn impurities, the \( ^{89}\text{Y} \) nuclear spin will be affected by an impurity located in one of the layers of the CuO\(_2\) bilayer in YBa\(_2\)Cu\(_3\)O\(_{6+x}\). Here, it will be assumed that the changes induced in \( ^{89}K \) are due only to the changes in the magnetic correlations in the layer which contains the impurity as it was done in Ref. 4. Also, in the following, the quantity of interest for \( ^{89}Y \) will be the change in the Knight shift of \( ^{89}Y \) due to the presence of the impurity. Hence, the calculations will be carried out for the impurity in the layer which contains the impurity. In Fig. 5, a sketch of the various Y sites with respect to the impurity site is given. Note that the measurement of \( ^{89}K \) at the Y sites near the impurity, along with that of \( {^7}K \), determines the real-space structure of \( k(r) \) in the environment of the impurity. For instance, \( ^{89}K \) for Y(1) is determined by \( k(r = (1, 0)) \) and \( k(r = (1, 1)) \), while \( {^7}K \) is set by \( k(r = (1, 0)) \). In Section III, the results of the calculations on \( {^7}K \) and \( ^{89}K \) for various Y sites will be shown and compared with the experimental data.

### III. RESULTS OF THE CALCULATIONS

The comparison of the results with the experimental data will be carried out first for the optimally doped and then for the underdoped YBa\(_2\)Cu\(_3\)O\(_{6+x}\).

#### A. Results on optimally doped YBa\(_2\)Cu\(_3\)O\(_{6+x}\)

The filled circles in Fig. 6(a) denote the experimental data by Bobroff et al. 24 on the \( {^7}\text{Li} \) Knight shift \( {^7}K \) for YBa\(_2\)Cu\(_3\)O\(_{6,97}\) with dilute Li impurities. The curves in this figure represent the results of the calculations of \( {^7}K \) for various values of \( V_1 \) as indicated next to the curves. The model parameters other than \( V_1 \) were already set by fitting the \( T_2^{-1} \) data on pure YBa\(_2\)Cu\(_3\)O\(_{6,9}\) by Imai et al. 24 as discussed in Section II.A. Here, it is seen that for \( V_1 \approx -0.15t \) the calculated values of \( {^7}K \) are in agreement with the experimental data. It is useful to compare these results with what is expected for the \( {^7}\text{Li} \) Knight shift if the magnetic correlations had not changed upon the substitution of the impurity and had remained the same as in the pure material. In Fig. 6(b), \( {^7}K_0 \) defined by

\[ {^7}K_0 = \frac{1}{2} \left( \frac{\gamma_e}{89\gamma_n} \right) C4\chi_{\text{pure}}, \quad (24) \]

where \( \chi_{\text{pure}} \) is the uniform susceptibility of the pure system, is plotted as a function of the temperature. The solid line has been obtained from \( \chi_{\text{pure}}(q \rightarrow 0) \) given by Eq. (4) for optimally doped YBa\(_2\)Cu\(_3\)O\(_{6+x}\). The dashed line in Fig. 6(b) has been obtained by rescaling the \( ^{89}\text{Y} \) Knight shift \( ^{89}K \) of pure YBa\(_2\)Cu\(_3\)O\(_7\) 38 with

\[ {^7}K_0 = \frac{1}{2} \left( \frac{89\gamma_n C}{7\gamma_n D} \right) ^{89}K. \quad (25) \]

The factor of 1/2 in this expression is because Y has eight Cu(2) near neighbours while Li has four. In these figures, it is also seen that for \( V_1 = 0 \), corresponding to an onsite impurity potential in the unitary limit, \( {^7}K \) is enhanced with respect to \( {^7}K_0 \) but this enhancement is not sufficient to explain the experimental data. If positive values of \( V_1 \) are used, then \( {^7}K \) gets suppressed with respect to the \( V_1 = 0 \) case.

The stoichiometry of the samples is a factor which could affect the comparisons with the data. Note that \( U(T) \) used in calculating \( {^7}K \) for YBa\(_2\)Cu\(_3\)O\(_{6,97}\) was determined by fitting the \( T_2^{-1} \) data on YBa\(_2\)Cu\(_3\)O\(_{6,9}\). For YBa\(_2\)Cu\(_3\)O\(_7\), \( T_2^{-1} \) was measured at 100 K, and its value is 7.7 ± 0.6 msec\(^{-1}\) compared to ~ 9.5 msec\(^{-1}\) for YBa\(_2\)Cu\(_3\)O\(_{6,9}\). The value of 7.7 msec\(^{-1}\) for \( T_2^{-1} \) requires a slightly smaller value for \( U \) and, consequently, the fitting of \( {^7}K \) is done using larger values of \( V_1 \). If one assumes that the \( T \) dependence of \( T_2^{-1} \) is given by a line which passes through 7.7 msec\(^{-1}\) at 100 K and which is parallel to the dashed line in Fig. 2(a), then the \( {^7}K \) data can be fitted by using \( V_1 \approx -0.275t \)...

Based on these results and considering the simplicity of the model, the conclusions which can be reached are limited. Probably, the best thing to say is that the analysis of the \( {^7}K \) data on optimally doped YBa\(_2\)Cu\(_3\)O\(_{6+x}\) within this model at the level of RPA requires an impurity potential which is weakly attractive at the nearest-neighbor Cu(2) sites.

The results of the calculations for \( ^{89}\text{K} \) are shown in Fig. 7(a). Here, the temperature dependence of \( \delta^{89}\text{K} \) is shown for the first four nearest-neighbor Y sites which are indicated in Fig. 5. In obtaining these results, \( V_1 = -0.15t \) was used. Note that in Fig. 7(a) the expected values of the Knight shifts are close to each other. Experimentally, for optimally doped YBa\(_2\)Cu\(_3\)O\(_{6+x}\), the expected resonance lines for the \( ^{89}\text{Y} \) nuclei near the Zn impurity are not resolved.

Next, results on how the nonmagnetic impurity affects the \( ^{63}\text{Cu}(2) \) Knight shifts in this model are presented.
The $^{63}$Cu(2) Knight shift at site $\mathbf{r}_i$ with the orienting magnetic field $\mathbf{H}||\mathbf{c}$ is given by

$$63K_e(\mathbf{r}_i) = \frac{1}{2}(\gamma_e^{63}n)\left[A_i k(\mathbf{r}_i) + B \sum_{\alpha=1}^{4} k(\mathbf{r}_\alpha)\right], \quad (26)$$

where $\alpha$ sums over the four Cu(2) neighbors of $\mathbf{r}_i$. Figure 7(b) shows the $T$ dependence of $^{63}K_e(\mathbf{r}_i)$ for $V_1 = -0.15t$ for the first four neighboring Cu(2) sites of the impurity, which correspond to $\mathbf{r}_i = (1,0), (1,1), (2,0),$ and $(2,1)$. For the pure system, $^{63}K_e$ vanishes since $A_c + 4B = 0$. Here, it is seen that $^{63}K_e$ has a staggered pattern; it is negative at the sublattice of the $(1,0)$ site and it is positive at the sublattice of the impurity. This spatial pattern of $^{63}K_e(\mathbf{r}_i)$ is due to the staggered pattern of $k(\mathbf{r})$, which will be discussed in Sections III.C and IV.A. The measurements of $^{63}K_e$ have been carried out in YBa$_2$Cu$_3$O$_{6.7}$ by Julien et al. [7]. While the individual $^{63}$Cu(2) lines are not resolved, it has been observed that the broadening of the linewidth has a Curie-like $T$ dependence. This has been attributed to the development of a staggered polarization cloud around the impurity [8].

### B. Results on underdoped YBa$_2$Cu$_3$O$_{6+x}$

The analysis of the Knight shift data on the underdoped YBa$_2$Cu$_3$O$_{6+x}$ is considerably more complicated because of the magnetic pseudogap. This is so because the origin of the magnetic pseudogap in this material is currently an unresolved issue. Hence, it is necessary to note that the purpose here is not to develop a theory for the magnetic susceptibility of the underdoped YBa$_2$Cu$_3$O$_{6+x}$ with impurities. Instead, the purpose is to explore any possible role of the antiferromagnetic correlations in producing the anomalous $T$ dependence of the Knight shift data in underdoped YBa$_2$Cu$_3$O$_{6+x}$ within the presence of nonmagnetic impurities. The emphasis here will be on the real-space structure of the magnetic correlations around the impurity.

The $^7$Li Knight shift measurements by Bobroff et al. [9] on underdoped YBa$_2$Cu$_3$O$_{6+x}$ with Li impurities found that $^{7}K$ increases rapidly as $T$ is lowered. In Fig. 8(a), the experimental data on $^{57}K$ defined by

$$^{57}K = ^{7}K - ^{7}K_0$$

are compared with the results of the calculations. The comparisons are carried out for $^{57}K$ rather than for $^{7}K$ in order to compensate for the effect of the magnetic pseudogap which is not taken into account here. The data points shown by the filled circles in Fig. 8(a) were obtained by subtracting from the $^{7}K$ data of Bobroff et al. [9] the quantity

$$^{7}K_0 = \frac{1}{2}\left(\frac{89\gamma_{n}C}{\gamma_{n}D}\right)89K(\text{main}), \quad (28)$$

where $89K(\text{main})$ is the Knight shift of the main $^{89}$Y spectral line measured by Mahajan et al. [10] in Zn substituted YBa$_2$Cu$_3$O$_{6.6}$. The solid points in Fig. 8(b) indicate $^{7}K_0$ estimated this way. The dashed curve in Fig. 8(b) shows the estimate for $^{7}K_0$ obtained by using the $^{57}K$ measurements on pure YBa$_2$Cu$_3$O$_{6.63}$ [11]. In the Zn substituted samples, $^{89}K(\text{main})$ is slightly shifted with respect to $^{89}K$ of the pure samples. This might be due to the difference in the stoichiometry of the Zn substituted and the pure samples, as discussed in Ref. [11].

The results of the calculations on $^{57}K$ are represented by the various curves in Fig. 8(a). The term subtracted from $^{7}K$ to obtain these results is shown by the solid line in Fig. 8(b). Note that the solid line in Fig. 8(b) lies slightly above the solid line in Fig. 6(b), since larger values of $U$ were used for the underdoped YBa$_2$Cu$_3$O$_{6+x}$. These calculations are shown for four different values of $V_2$: $-0.125t$ (short-dashed), $-0.15t$ (solid), $-0.175t$ (long-dashed), and $0.0$ (dot-dashed). These results have been obtained by using $U(T)$ represented by the solid curve in Fig. 2(b). Hence, they correspond to the first scenario described above where $T_2^{-1}$ has been linearly extrapolated for $T < 160$ K. For each value of $V_1$, the calculations of $^{57}K$ have been repeated according to the second scenario, where the saturation of $T_2^{-1}$ below 160 K is attributed to the saturation of the antiferromagnetic correlations. In this case, the values of $U(T)$ shown by the dotted curve in Fig. 2(b) were used. In Fig. 8(a), these results are shown by the dotted curves for each value of $V_1$ as indicated. Comparing the results of the calculations with the experimental data, one sees that the values of $V_1$ between $-0.125t$ and $-0.15t$ would produce a fit of the experimental data on $^{57}K$.

Next, the results for $^{89}$Y are discussed. The experiments by Mahajan et al. [11] found that Zn substitution in underdoped YBa$_2$Cu$_3$O$_{6+x}$ strongly modifies the nuclear resonance spectrum of $^{89}$Y. In this case, in addition to the main $^{89}$Y resonance line, two satellite peaks are observed. These outer and middle resonance lines have been identified as belonging to the $^{89}$Y nuclear spins which are respectively the first and the second near-neighbours of the Zn impurity, Y(1) and Y(2) in Fig. 5. The fact that the magnetic resonance spectrum of $^{89}$Y in Li substituted YBa$_2$Cu$_3$O$_{6+x}$ is nearly the same as in Zn substituted samples confirms this identification [11]. In the following, the data on $^{89}K$ will be compared with the results of the calculations which were outlined in Section III. However, the comparisons will not be carried out directly for $^{89}K$. Rather, the shift of the satellite lines with respect to the main line will be used in order to compensate for the opening of the pseudogap in the pure system. Hence, the comparisons will be carried out for the following quantities,

$$^{57}K(\text{outer}) = ^{89}K(\text{outer}) - ^{89}K(\text{main})$$

$$^{57}K(\text{middle}) = ^{89}K(\text{middle}) - ^{89}K(\text{main}). \quad (29)$$
Since $^{89}\text{K}(\text{main})$ in Zn substituted material nearly follows the $T$ dependence of $^{89}\text{K}$ in the pure system, $\delta^{89}\text{K}(\text{outer})$ and $\delta^{89}\text{K}(\text{middle})$ represent the change in the Knight shifts of the nuclear spins at the Y(1) and Y(2) sites, respectively, due to the substitution of the Zn impurity.

In Figures 9(a) and (b), the points represent the experimental data on $-\delta^{89}\text{K}(\text{outer})$ and $-\delta^{89}\text{K}(\text{middle})$, respectively, for underdoped YBa$_2$Cu$_3$O$_{6+x}$ measured by Mahajan et al. $^{15}$ In these figures, the data points are shown for two different orientations of the magnetic field $\mathbf{H}$. It is estimated that the $^{89}\text{Y}$ hyperfine coupling has an anisotropy of about 15%, which could be the cause of the anisotropy of the data seen in Figs. 9(a) and (b).$^{15}$ In Fig. 9(a), the error bars shown for the lowest and the highest temperature measurements are representative of the error bars for the other data points. In Fig. 9(b), the data points are shown up to 130 K, since at higher temperatures the middle satellite is not resolved.

The curves in Figs. 9(a) and (b) represent the results of the calculations for $-\delta^{89}\text{K}(\text{outer})$ and $-\delta^{89}\text{K}(\text{middle})$, respectively. These calculations have been carried out for the same values of $V_1$ as in Fig. 8(a) for $\delta^{7}\text{K}$. For $T < 160$ K, these calculations were carried out in two ways by using $U(T)$ shown by the solid and the dotted curves in Fig. 2(b). In Figs. 9(a) and (b), it is seen that for $V_1$ between $-0.125t$ and $-0.15t$, the data can be fitted. Finally, in Fig. 10 the $T$ dependence of $^{63}\text{K}_c(r_i)$ in underdoped YBa$_2$Cu$_3$O$_{6+x}$ is shown for various Cu(2) sites near the impurity. Here, $V_1 = -0.15t$ was used. In order to interpret these, in the next section the real-space structure of the changes induced in the magnetic correlations by the impurity will be shown. The $^{63}\text{K}_c$ measurements in Zn substituted YBa$_2$Cu$_3$O$_{6.7}$ imply that the polarization around the impurity is staggered.$^{7}$

C. Pattern of the magnetic correlations around the impurity

In Fig. 11(a), $k(\mathbf{r})$ defined by Eq. (31), is shown as a function of the lattice distance $r = |\mathbf{r}|$ away from the impurity at $T = 100$ K. Here, $r$ is given in units of the lattice constant, and in obtaining these results $V_1 = -0.15t$ was used. The horizontal long-dashed line denotes what is expected for $k(\mathbf{r})$ within this model for pure optimally doped YBa$_2$Cu$_3$O$_{6+x}$, and the open circles represent $k(\mathbf{r})$ when an impurity is introduced into this material at $r_0 = 0$. The results for underdoped YBa$_2$Cu$_3$O$_{6.6+x}$ are also shown in Fig. 11(a). The dotted horizontal line denotes $k(\mathbf{r})$ for pure underdoped YBa$_2$Cu$_3$O$_{6.6+x}$, while the filled circles show $k(\mathbf{r})$ within the presence of the impurity for this material. The results on underdoped YBa$_2$Cu$_3$O$_{6+x}$ were obtained using the value of $U$ given by the solid curve in Fig. 2(b). Note that, for underdoped YBa$_2$Cu$_3$O$_{6+x}$ in this graph, the physically more relevant quantity is the change in $k(\mathbf{r})$ induced by the impurity, since the presence of the pseudogap in the uniform susceptibility of the pure system is not included in the model.

The real-space structure of $k(\mathbf{r})$ is important since it determines the Knight shift at various sites. For instance, $k(\mathbf{r})$ at $\mathbf{r} = (1,0)$ determines the value of $7K$. On the other hand, $\delta^{89}\text{K}(\text{outer})$, corresponding to the Y(1) site in Fig. 5, is given by the change in

$$k(0,0) + 2k(1,0) + k(1,1)$$

induced by the impurity. Similarly, $\delta^{89}\text{K}(\text{middle})$, corresponding to the Y(2) site, probes the change induced in

$$k(1,0) + k(1,1) + k(2,0) + k(2,1)$$

by the impurity. Hence, it is seen that by the fitting of $\delta^{7}\text{K}$ in Fig. 8(a), the value of $k(1,0)$ is fixed. Since it is known that $k(0,0)$ vanishes when the impurity is introduced, the fitting of $\delta^{89}\text{K}(\text{outer})$ in Fig. 9(a) fixes the value of $k(1,1)$.

In Fig. 11(a), it is seen that while $k(1,0)$ is enhanced strongly by the impurity, $k(1,1)$ is strongly suppressed and it even becomes negative. The fact that both $\delta^{7}\text{K}$ and $\delta^{89}\text{K}(\text{outer})$ can be fitted at the same time means that the induced changes in $k(\mathbf{r})$ of the actual system must have a staggered pattern at the sites (1,0) and (1,1). Since $k(1,0)$ and $k(1,1)$ are now fixed, the fact that $\delta^{89}\text{K}(\text{middle})$ can also be fitted means that the change in $k(2,0) + k(2,1)$ is fixed. In Fig. 11(a), it is seen that $k(2,0)$ gets suppressed by the impurity and $k(2,1)$ gets enhanced while the total effect on $k(2,0) + k(2,1)$ is small. This in turn means that the induced changes in $k(\mathbf{r})$ of the actual system must also be staggered at sites (2,0) and (2,1). Hence, the fits obtained in Figs. 8(a), 9(a) and (b) imply that $k(\mathbf{r})$ has a staggered pattern in the vicinity of the impurity. This conclusion was reached by the analysis of Mahajan et al. $^{15}$ Here it is seen that this simple model produces this pattern as well.

It is useful to compare the staggered pattern of $k(\mathbf{r})$ with the pattern of $\chi(\mathbf{r},\mathbf{r})$ near the impurity. Fig. 11(b) shows $\chi(\mathbf{r},\mathbf{r})$ versus $r$ plotted in the same way as $k(\mathbf{r})$ is plotted in Fig. 11(a). Here one sees that $\chi(\mathbf{r},\mathbf{r})$ is strongly modified near the impurity. It is also seen that $\chi(\mathbf{r},\mathbf{r})$ does not have a staggered pattern as, for instance, one would have had for antiferromagnetically ordered spins. The comparisons shown in Figs. 6, 8 and 9 mean that the experimental data on $7\text{K}$ and $^{89}\text{K}$ require a staggered pattern for $k(\mathbf{r})$ in the vicinity of the impurity. But a staggered structure for $\chi(\mathbf{r},\mathbf{r})$ is not necessary.

In Figs. 11(a) and (b), it is seen that while $\chi(\mathbf{r},\mathbf{r})$ is positive for all $\mathbf{r}$, $k(\mathbf{r})$ can become negative, for instance at $\mathbf{r} = (1,1)$. This is possible because $\chi(\mathbf{r},\mathbf{r}')$ for $\mathbf{r} = (1,0)$ and $\mathbf{r}' = (1,1)$ has a large negative value after the impurity is introduced, meaning that the impurity is inducing strong antiferromagnetic correlations in its...
local environment in this model. These points will be discussed further in Section IV.A.

At this point it is also useful to present results on how the impurity potential affects the single-particle properties. For this purpose, in Fig. 11(c) the electron occupation number \( n(r_i) = \sum_q \langle \sigma \rangle \) for sites near the impurity is shown as a function of the distance \( r = |r_i| \) from the impurity for various values of \( V_1/t \). Since \( V_0 \) is strongly attractive, the impurity site which is not shown in this figure, is nearly doubly occupied, \( n(r_i = 0) \approx 2.0 \). For \( V_1 = 1 \), it is seen that \( n(r_i) \) is enhanced at \( r_i = (1,1) \) with respect to its value at \( r_i = (1,0) \). Away from the impurity, these oscillation in \( n(r_i) \) decay, and \( n(r_i) \) goes to its value in the pure system, \( \langle n \rangle = 0.86 \), which is indicated by the horizontal long-dashed line. Here, it is also seen that this structure in \( n(r_i) \) holds for weak attractive values of \( V_1 \). But as \( V_1 \) becomes more attractive, for instance for \( V_1 = -0.15t \), the pattern in \( n(r_i) \) changes. In this case, \( n(r_i = (1,0)) \) gets enhanced over \( n(r_i = (1,1)) \). Hence, in Fig. 11(c) one observes that the structure in \( n(r_i) \) in the vicinity of the impurity does not necessarily reflect the sign of \( V_1 \). For instance, a finite attractive \( V_1 \) is required before \( n(r_i = (1,0)) \) gets enhanced with respect to \( n(r_i = (1,1)) \). This has to do with the presence of the strongly attractive \( V_1 \) term in the impurity potential.

In Section III.B, it was seen that the data on \( \delta^7K \) and \( \delta^{89}K \) can be fitted reasonably well over the whole temperature range by using a simple form for the effective impurity potential without any \( T \) dependence. On the other hand, because of the various approximations employed in this model, the quantitative agreement obtained in the fits might actually be misleading. However, what is significant is the fact that \( \delta^7K \), \( \delta^{89}K \) (outer) and \( \delta^{89}K \) (middle) can be fitted all at the same time, which means that the real-space structure of the changes induced in the magnetic correlations near the impurity appears to be described by this simple model.

IV. DISCUSSION

A. Role of the antiferromagnetic correlations

It is useful to discuss the connection between the antiferromagnetic correlations of the system which is composed of the impurity and the host, and the anomalous enhancement of the Knight shift at sites around the impurity. This connection becomes more clear if the Fourier transform \( \chi(q, q') \) is introduced,

\[
\chi(q, q') = \sum_{r, r'} e^{i(q \cdot r - q' \cdot r')} \chi(r, r').
\]

Note that \( \chi(r, r') \) is the susceptibility for an \( N \)-site square lattice with one impurity at the center and periodic boundary conditions.

An especially important quantity is the off-diagonal susceptibility \( \chi(q, 0) \equiv \chi(q, q') = 0 \), since its Fourier transform with respect to \( q \) gives \( k(r) \). Particularly, at sites \( r = (1,0) \) and \( (1,1) \) with respect to the impurity, one has

\[
k(1,0) = \sum_q \frac{1}{2}(\cos q_x + \cos q_y) \chi(q, 0)
\]

\[
k(1,1) = \sum_q \cos(q_x + q_y) \chi(q, 0).
\]

In Figs. 12(a) and (b), the \( q \) dependence of \( -\chi_0(q, 0) \), which is for \( U = 0 \), and \( -\chi(q, 0) \) are shown at 100 K for optimally doped \( \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \) obtained by using \( V_1 = -0.15t \). Here, the \( \delta \)-function component of \( \chi(q, q') = 0 \) at \( q = 0 \) has been omitted. In these figures, it is seen that \( -\chi(q, 0) \) is enhanced with respect to \(-\chi_0(q, 0), \) and they both peak at \( q \) near \( (\pi, \pi) \). Note that for \( q \sim (\pi, \pi) \), the form factors entering Eq. (33), \( \frac{1}{2}(\cos q_x + \cos q_y) \sim -1 \) and \( \cos(q_x + q_y) \sim 1 \). Because of the \( q \) structure of \( \chi(q, 0) \), \( k(1,0) \) gets enhanced and \( k(1,1) \) gets suppressed. Hence, within this model, the staggered pattern of \( k(r) \) near the impurity is a consequence of the peaking of \(-\chi(q, 0) \) near \( (\pi, \pi) \).

The enhancement of \(-\chi(q, 0) \) with respect to \(-\chi_0(q, 0) \) is understood better if the RPA equation, Eq. (17), is written in momentum space,

\[
\chi(q, q') = \chi_0(q, q') + U \sum_{q''} \chi_0(q, q'') \chi(q'', q'),
\]

where \( \chi_0(q, q') \) is for \( U = 0 \). For the pure system, \( \chi(q, q' = 0) \) is given by

\[
\chi(q, 0) = N \delta_{q,0} \chi_{\text{pure}}(q \to 0)
\]

where \( \chi_{\text{pure}} \) is defined by Eq. (11), and within the presence of the impurity \( \chi(q, q' = 0) \) is obtained by solving

\[
\chi(q, 0) = \chi_0(q, 0) + U \sum_{q''} \chi_0(q'', 0) \chi(q'', q).
\]

This expression shows that when \( q'' \sim (\pi, \pi), \chi(q \sim (\pi, \pi), q' = 0) \) couples to the antiferromagnetic correlations determined by \( \chi(q, q) \) with \( q \sim (\pi, \pi) \) of the system which is composed of the host and the impurity. This is the reason for the strong enhancement of \( \chi(q, q' = 0) \) with respect to \( \chi_0(q, q' = 0) \).

It is useful to discuss the physical meaning of the off-diagonal susceptibility \( \chi_0(q, q') \) where \( q \neq q' \). During the scattering of the spin fluctuations by the impurity potential, the momentum is not conserved and \( \chi_0(q, q' \neq q) \) acts as the vertex for the scattering of the spin fluctuations by the impurity with \( Q^* = q - q' \) momentum transfers. The peaking of \( \chi_0(q, q' = 0) \) near \( q \sim (\pi, \pi) \) means that the scattering of the antiferromagnetic spin fluctuations with momentum transfers near
The real-space pattern of $^{63}K_c(r_i)$ seen in Fig. 7(b) also reflects the $q$ dependence of $\chi(q,q' = 0)$. Equation (26) for $^{63}K_c(r_i)$ can be rewritten as

$$^{63}K_c(r_i) = \frac{1}{2} \left( \frac{\gamma_c}{63 \gamma_n} \right) \frac{1}{N} \sum_q e^{i q \cdot r_i} (A_c + 4B\gamma_q) \chi(q,0),$$

(37)

where $\gamma_q = (\cos q_x + \cos q_y)/2$. Since $\chi(q,q' = 0)$ has most of its weight at $q \sim (\pi, \pi)$ for which $\gamma_q \sim -1$, one gets

$$^{63}K_c(r_i) \approx -\frac{1}{2} (|A_c| + 4B) \left( \frac{\gamma_c}{63 \gamma_n} \right) \frac{1}{N} \sum_{q \sim (\pi, \pi)} e^{i q \cdot r_i} \chi(q,0).$$

(38)

Here one sees that $|A_c| + 4B$ has a large value and $^{63}K_c(r_i)$ directly couples to $\chi(q \sim (\pi, \pi), q' = 0)$. It has been noted in Ref. [21] that the anomalous broadening of the $^{63}Cu(2)$ linewidth reflects the staggered polarization of the magnetic correlations near the impurity.

It is important to note that, within this model, the antiferromagnetic correlations also get enhanced due to the presence of the impurity [22]. If $N_i$ randomly distributed nonmagnetic impurities are considered in the dilute limit, then the $q$-dependent magnetic susceptibility of this system is given by

$$\chi(q) = \chi_{\text{pure}}(q) + n_i N \left[ \chi(q,q) - \chi_{\text{pure}}(q) \right],$$

(39)

where $n_i = N_i/N$ is the impurity concentration. The results on $\chi(Q^*)$ for 0.5% nonmagnetic impurities are compared with $\chi_{\text{pure}}(Q^*)$ in Figs. 3(b) and (c), where it is seen that the impure system has stronger antiferromagnetic correlations than the pure system. Here, $Q^*$ is the incommensurate wave vector where $\chi$ peaks at low temperatures. The enhancement of $\chi(Q^*)$ over $\chi_{\text{pure}}(Q^*)$ is especially significant for the underdoped system. Hence, it needs to be noted that the enhancement of the Knight shifts near the impurity in this model is a result of the coupling to $\chi(q,q)$ for $q \sim (\pi, \pi)$ of the whole system which is composed of the impurity and the host rather than to $\chi_{\text{pure}}(q \sim (\pi, \pi))$ of the pure host.

Experimentally, Im $\chi(Q, \omega)$ is the inelastic neutron scattering spectral weight. The enhancement of Im $\chi(Q, \omega)$ in the normal state by the impurity scattering is clearly seen for dilute Zn impurities in YBa$_2$Cu$_3$O$_{7-\delta}$ [23,24]. The low frequency part of Im $\chi$ is also probed by the measurement of the longitudinal relaxation rate $T^{-1}_L$ at sites near the impurity. The measurements of $T^{-1}_L$ at sites near the impurity have found interesting results [24,25]. The calculations presented here have been extended to obtain the $^7Li T^{-1}_L$ using the Pade approximation for analytic continuation to the real-frequency axis of the results calculated in terms of the Matsubara frequencies. While it is difficult to obtain reliable results on Im $\chi(Q, \omega)$ for general $\omega$ using the Pade approximation, it is possible to obtain control on the calculation of $T^{-1}_L$, which requires only the $\omega \rightarrow 0$ limit. Since the analytic continuation procedure requires special attention, the calculations of the $^7Li T^{-1}_L$ will be presented elsewhere [26,27].

B. Effects of the pseudogap in underdoped YBa$_2$Cu$_3$O$_{6+x}$

The analysis of the data on underdoped YBa$_2$Cu$_3$O$_{6+x}$ needs to be interpreted carefully because of the presence of the pseudogap in this system. The uniform susceptibility of pure YBa$_2$Cu$_3$O$_{6.3}$ starts to decrease below 300 K, with a $T$ dependence proportional to that given by the dashed curve in Fig. 8(b). This has been ignored in the calculations presented above. In order to explore the effects of the pseudogap on the Knight shift results, the following simple calculation has been carried out. The suppression of the diagonal irreducible susceptibility $\chi_0(q,q)$ for $q \sim 0$ has been artificially incorporated into the model by multiplying $\chi_0(q,q)$ by the factor

$$F(q) = 1 - ae^{-|q|^2/\kappa^2},$$

(40)

which has been chosen in order to cause a suppression for $q \sim 0$. After artificially suppressing the diagonal irreducible susceptibility for $q \sim 0$, the Fourier transform is taken to obtain $\chi_0(r,r')$ of the pure system, which is then used in calculating $\chi_0(r,r')$ of the impure system. Hence, in this procedure, the diagonal terms $\chi_0(q,q)$ for $q \sim 0$ of the impure system have the effects of the pseudogap, but $\chi_0(q,q)$ for $q \sim (\pi, \pi)$ and the off-diagonal terms $\chi_0(q,q')$, where $q \neq q'$, are not affected by this artificial opening of the pseudogap. Next, $\chi_0(r,r')$ calculated with this procedure was used in solving Eqs. (17) for $\chi(r,r')$ and in determining the Knight shifts. The parameter $\kappa$ entering Eq. (10) was arbitrarily chosen to be $\pi/2$. In addition, $a$ has been chosen such that the ratio of
Section III. For this reason, here, the Knight shifts are effective bandwidth necessary to check the dependence of the results on the effective bandwidth should be in an RPA framework. Hence, it is necessary for fitting the Knight shift data. In Section III.B, the data on $\delta K$, $\delta ^{89} K$(outer) and $\delta ^{89} K$(middle) were fitted by using $V_1$ between $-0.125 t$ and $-0.15 t$. Here, if $V_1 = -0.15 t$ is used along with the above mentioned values of $\kappa$ and $a$, then at $T = 100 K$ one obtains the following results for the Knight shifts: $\delta K = 650$ ppm, $\delta ^{89} K$(outer) = $-180$ ppm, and $\delta ^{89} K$(middle) = $-75$ ppm. On the other hand, at 300 K, it is necessary to use $a = 0.25$ and in this case one obtains: $\delta K = 250$ ppm, $\delta ^{89} K$(outer) = $-35$ ppm, and $\delta ^{89} K$(middle) = $-20$ ppm. These values for the Knight shifts are comparable to the data seen in Figs. 8(a) and 9. Hence, when the pseudogap is introduced in $\chi_0(q, q)$ for $q \sim 0$ in this artificial way, the Knight shift data can be still fitted, but by using a slightly larger value for $V_1$. However, it must be kept in mind that the way the pseudogap is introduced here is not rigorous, and in fact the fitting of the data in this way can be considered as superfluous. For this reason, the fitting of the data on the underdoped compound will not be pursued further. Rather, here it is only pointed out that the staggered nature of the induced magnetic correlations seen in this model appears to be consistent with the measurements of the $^7 Li$ and $^{89} Y$ Knight shifts in underdoped $YBa_2Cu_3O_{6+x}$.

C. Dependence of the results on the effective bandwidth

The results seen in Section III were obtained by using an effective bandwidth $W$ of 1 eV. For a one-band model of the cuprates, the bare hopping matrix element is estimated to be of order 0.45 eV leading to a bare bandwidth of 3.6 eV. While in principle one expects that the Coulomb correlations act to reduce the bandwidth, it is not clear what the precise value of the effective bandwidth should be in an RPA framework. Hence, it is necessary to check the dependence of the results on the effective bandwidth $W$, which was assumed to be 1 eV in Section III. For this reason, here, the Knight shifts are calculated for $W = 3$ eV. In this case, larger values of $U$ are required for fitting the $T_2^{-1}$ data and, consequently, the pure system has a stronger Stoner enhancement of the antiferromagnetic correlations. For $W = 3$ eV, the value of the $^{89} Y$ hyperfine coupling $D$ was chosen to be $-2$ kOe/$\mu_B$ in order to fit the value of $^{89} K$ for pure $YBa_2Cu_3O_7$ in the normal state. The experimental estimate for $D$ is also about $-2$ kOe$/\mu_B$.

In Fig. 13, $7K$ versus $T$ is shown for optimally doped $YBa_2Cu_3O_{6+x}$ calculated using $W = 3$ eV. Here, the $^7 Li$ hyperfine coupling $C$ was taken to be 1.4 kOe$/\mu_B$. The open circles represent $7K_0$ deduced from the $^{89} Y$ data on pure $YBa_2Cu_3O_7$. The dotted curve represents the theoretical results for $7K_0$. Here, it is seen that a smaller value of $V_1$ is required for fitting the $7K$ data as compared to that for $W = 1$ eV. This is because in this case the antiferromagnetic correlations are stronger. It is also necessary to compare the value of $C$ used here with the experimental estimate. In Ref. 2, $C$ was estimated to be 2.4 kOe$/\mu_B$. This value was deduced by comparing the enhancement of the uniform susceptibility $\Delta \chi$ with the enhancement of $7K$ for the underdoped $YBa_2Cu_3O_{6+x}$ by assuming that the contribution to $\Delta \chi$ is arising from the changes in the magnetic correlations at only the four nearest-neighbour Cu$(2)$ sites of the impurity. Note, however, that $\Delta \chi$ is given by

$$\Delta \chi = \sum_j k(r_j) - \chi_{\text{pure}}(q \to 0),$$

where the sum over $j$ is carried over the whole lattice, and the Knight shift data and the numerical results presented in this paper indicate that the changes induced in $k(r_j)$ by the impurity is extended and not restricted to the four nearest-neighbor sites of the impurity. In fact, by limiting the changes in $k(r_j)$ to be only at the nearest-neighbor sites, one would overestimate $\Delta \chi$. Similarly, when one calculates $C$ by using the experimental data on $\Delta \chi$ and the $7K$ data, one would overestimate $C$. For instance, in Fig. 11(a), it is seen that at 100 K the enhancement of $k(1,0)$ by the impurity, $\Delta k(1,0)$, is about 20 states/eV for underdoped $YBa_2Cu_3O_{6+x}$. If $\Delta \chi$ is estimated by just using the change in $k(1,0)$, then one would obtain $\Delta \chi = 0.4$ states/eV in the dilute limit for an impurity concentration of 0.5%. On the other hand, the calculated value of $\Delta \chi$ at $T = 100 K$ and $V_1 = -0.15 t$ for underdoped $YBa_2Cu_3O_{6+x}$ in this model is $\sim 0.2$ states/eV. Hence, by using $\Delta k(1,0)$ and assuming that the changes in $k(r)$ occur only at the nearest-neighbor sites of the impurity, one would overestimate $\Delta \chi$ by about a factor of two. Similarly, if $\Delta k(1,0)$ and $\Delta \chi$ are used to calculate $C$ one would overestimate $C$ by the same amount. When this is taken into account, it is seen that the values of $C$ used here, 0.85 kOe$/\mu_B$ in Section III and 1.4 kOe$/\mu_B$ in Section IV.C, are comparable to the experimental estimate which was obtained by using the data on $\Delta \chi$ and $7K$.

In Figs. 14(a)–(c), the $T$ dependence of $\delta K$, $-\delta ^{89} K$(outer), and $-\delta ^{89} K$(middle) are shown for underdoped $YBa_2Cu_3O_{6+x}$. Here, it is seen that for $V_1 = -0.03 t$, the magnitudes of the calculated Knight shifts agree with the experimental data. However, the fitting of the $T$ dependence of the Knight shifts is not as good as that seen in Section III for $W = 1$ eV.
The results presented in this section show that the value of \( V_1 \) required for fitting the Knight shift data depends on the effective bandwidth. For \( W = 3 \) eV, the system has stronger enhancement of the AF correlations compared to that for \( W = 1 \) eV. Here, it has been shown that as \( W \) increases \( V_1 \) required for fitting the data decreases. However, even if a \( W \) of 4 eV is used, at the level of RPA, an impurity potential with \( V_1 = 0 \) does not produce sufficient \( T \) variation for fitting the \(^7K\) data on optimally doped YBa\(_2\)Cu\(_3\)O\(_{6+x}\), and a weakly attractive \( V_1 \) is necessary.

D. Nature of the effective impurity potential

Because of the Coulomb correlations, the bare impurity potential which has only an onsite component \( V_0 \sum_x \epsilon_0 c_{x \sigma}^\dagger c_{x \sigma} \) could acquire an extended component through higher-order scattering processes \([27]\). How this could happen at lowest order in the Coulomb repulsion \( U \) is discussed in Appendix II. This extended component is modelled here by using a static \( V_1 \) as a free parameter. In this picture, the fact that the Knight shift data imply \( V_1 < 0 \) is giving information on the effective particle-particle interaction in the system. The real-space structure of the effective impurity interaction was studied at half-filling for the \( t-J \) model within the exact diagonalization calculations \([2]\). It is useful to calculate \( V_1 \) in exact numerical calculations away from half-filling. This would be a test of the model presented here. Furthermore, note that here \( V_1 \) is assumed to have no temperature dependence. However, \( V_1 \) could depend on \( T \), if indeed the Coulomb correlations play a role in inducing the extended component of the impurity potential. This could quantitatively affect the fits seen above.

These results carried out at the level of RPA imply that \( V_1 \) is weakly attractive. At this level, even though an onsite scattering potential (\( V_1 = 0 \)) yields large enhancements of the Knight shifts, it is still insufficient for fitting the \( T \) dependence, for instance, of \(^7K\) in optimally doped YBa\(_2\)Cu\(_3\)O\(_{6+x}\). However, note that corrections beyond RPA might change this result. For instance, the spin-fluctuation self-energy corrections to the single-particle Green’s functions, when taken into account self-consistently, could play an important role.

Within this model, the Knight shift experiments on the cuprates with nonmagnetic impurities are interesting especially because they probe the interplay of the correlations in the density and the magnetic channels. The effective impurity potential acts in the density channel while its spatially-resolved magnetic response is detected through the Knight shift measurements. Similarly, the recent \(^7\)Li Knight shift measurements \([41]\) in the \( d\)-wave superconducting state of YBa\(_2\)Cu\(_3\)O\(_{6+x}\) give valuable information about the interplay of the density, magnetic and the \( d\)-wave superconducting correlations in this material.

Another set of experiments which produce spatially resolved information on the effects of the nonmagnetic impurities are the STM measurements on Zn substituted BISCO \([44]\). These measurements have been carried out in the superconducting state. Clearly, similar STM measurements above \( T_c \) would be useful for understanding the effects of the nonmagnetic impurities in the normal state. This would allow for a direct comparison of the single-particle properties with this model.

It is also desirable to extend these model calculations to the superconducting state in order to make comparisons with the STM \([44]\) and the recent NMR \([46]\) measurements below \( T_c \). However, note that if a spin-fluctuation mediated mechanism is assumed for the \( d\)-wave pairing, then the effects of the impurity on the pairing potential need to be taken into account as well as the scattering of the quasiparticles by the impurity. This is because it is already known in the normal state that in the local environment of the impurity the spin-fluctuations are strongly modified \([15]\).

Here, the effects of the scatterings by a nonmagnetic impurity on the magnetic spectrum is studied by modelling the impurity as a potential scatterer. In this respect, disorder could have effects similar to those of nonmagnetic impurities, and it has been already pointed out that the anomalous line broadening of \(^{63}\)Cu(2) in La\(_{2-\delta}\)Sr\(_{\delta}\)CuO\(_4\) could be due to the intrinsic disorder in this compound \([17]\).

V. SUMMARY AND CONCLUSIONS

In this paper, the Knight shift data on YBa\(_2\)Cu\(_3\)O\(_{6+x}\) with nonmagnetic impurities have been analyzed within a rather simple model exhibiting short-range antiferromagnetic correlations. The antiferromagnetic correlations have been modelled within the framework of the 2D Hubbard model, and the effects of an impurity have been approximated by using a static extended impurity potential. The strength of the antiferromagnetic correlations in the pure system has been determined by fitting the \( T^{-1}\) data of pure YBa\(_2\)Cu\(_3\)O\(_{6+x}\). The impurity potential has been assumed to have a range of just one lattice spacing without any temperature dependence. The onsite component of the impurity potential was taken to be strongly attractive, and the near-neighbor component \( V_1 \) was treated as a free parameter in fitting the Knight shift data. The simplicity of the model and the differences in the stoichiometry of the samples along with the uncertainties in the hyperfine couplings are factors which limit the conclusions which can be drawn. Nevertheless, here, it has been found that the anomalous \( T \) dependence of \(^7K\) in YBa\(_2\)Cu\(_3\)O\(_7\) can be fitted, if a weakly attractive \( V_1 \) is used. The nature of the effective
impurity potential could be studied in the paramagnetic state of the Hubbard model, where there are short range antiferromagnetic correlations, by using exact numerical methods. This could be a test of one of the main assumptions of this model. Especially, the sign of $V_1$ could be tested. These calculations have been also extended to the case of underdoped YBa$_2$Cu$_3$O$_{6+x}$ after making various assumptions about the magnetic correlations in this compound, which has a magnetic pseudogap. Based on these assumptions, it has been found that the real-space structure of the magnetic correlations in the vicinity of the impurity is consistent with the Knight shift experiments. However, caution is necessary in interpreting the results on underdoped YBa$_2$Cu$_3$O$_{6+x}$.

The results presented in this paper depend on the nature of the effective impurity potential used. Any magnetic scattering component in $V_{\text{eff}}$ could significantly change the results. Furthermore, it needs to be kept in mind that the validity of these results depend on the weak-coupling approach used for calculating the magnetic correlations. Clearly, much remains to be understood about the effects of the Zn or Li impurities on the magnetic correlations of YBa$_2$Cu$_3$O$_{6+x}$ in the normal state.

\textbf{Appendix I: $^7$Li Knight shift}

In the presence of a uniform static magnetic field, $B = B_0 z$, and a hyperfine coupling to the electronic spins given by Eq. (48), the interaction energy of a $^7$Li nuclear moment $\hbar \gamma_n I$ is

$$- \hbar \gamma_n I^z B_0 + CI^z \sum_{i=1}^{4} \langle S^z(r_i) \rangle,$$

where $i$ sums over the four Cu(2) sites neighboring the $^7$Li impurity. This expression can be rewritten as

$$- \hbar \gamma_n I^z B_0 (1 + 7K),$$

where the $^7$Li Knight shift $7K$ defined by

$$7K = - \left( \frac{C}{\hbar \gamma_n B_0} \right) \sum_{i=1}^{4} \langle S^z(r_i) \rangle$$

gives the fractional change in the Zeeman frequency of the nuclear magnetic moment due to the hyperfine coupling. According to the Kubo linear-response theory [48], the expectation value at time $t$ of the electronic spin at site $r_i$, $\langle S^z(r_i, t) \rangle$, is given by

$$\langle S^z(r_i, t) \rangle = -i \int_{-\infty}^{t} dt' \langle [S^z(r_i, t), V_Z(t')] \rangle \theta(t - t'),$$

where $V_Z(t') = e^{iH_0t'}V_Z e^{-iH_0t}$ with the Zeeman term for the electronic spins

$$V_Z = -\hbar \gamma_e B_0 \sum_j S^z(r_j).$$

In Eq. (46), the expectation values are evaluated with respect to the eigenstates of

$$H_0 = H_{\text{Hubbard}} + V_{\text{imp}},$$

where $H_{\text{Hubbard}}$ is given by Eq. (1) and the impurity interaction $V_{\text{imp}}$ is given by Eq. (11). In the adiabatic limit and as $t \rightarrow \infty$, $\langle S^z(r_i) \rangle$ is related to the magnetic susceptibility [48], and in this limit Eq. (46) reduces to

$$\langle S^z(r_i) \rangle = \frac{1}{2} \hbar \gamma_e B_0 \sum_j \chi(r_i, r_j)$$

where the transverse susceptibility $\chi(r_i, r_j) = \chi(r_i, r_j, i\omega_n = 0)$ is defined by Eq. (13). Hence, the $^7$Li Knight shift is given by

$$7K = \frac{1}{2} \left( \frac{\gamma_n}{\gamma_n} \right) C \sum_{i=1}^{4} k(r_i)$$

with $j$ summing over the whole lattice. Note that for the pure system $k(r_i)$ reduces to $\chi_{\text{pure}}(q \rightarrow 0)$.

\textbf{Appendix II: $V_{\text{eff}}$ at lowest order in $U$}

It is possible that the effective impurity potential has an extended component because of the correlated nature of the host material [27]. At lowest order in the bare impurity potential $V_0$ and the Coloumb repulsion $U$, such a contribution originates from the scattering process shown in Fig. 15. This process leads to a momentum-dependent effective interaction which is given by

$$V_{\text{eff}}^{(1)}(q) = -V_0 U \chi^L_0(q),$$

where $\chi^L_0$ is the Lindhard susceptibility of the pure system, Eq. (3). Since $\chi^L_0(q)$ peaks at $q \sim (\pi, \pi)$, $V_{\text{eff}}^{(1)}$ is attractive at sites neighboring the impurity when $V_0 < 0$. For large $V_0$, it is necessary to replace the bare impurity potential in Fig. 15 by the impurity scattering $t$-matrix, in which case one obtains

$$V_{\text{eff}}^{(1)}(q) = U \frac{T}{N} \sum_{k, i\omega_n} G_0(k, i\omega_n)G_0(k + q, i\omega_n) \frac{V_0}{1 - V_0 F_0(i\omega_n)},$$

13
where
\[ F_0(i\omega_n) = \frac{1}{N} \sum_p G_0(p, i\omega_n). \]  

The Fourier transform
\[ V_{\text{eff}}^{(1)}(r) = \frac{1}{N} \sum_q e^{iqr} V_{\text{eff}}^{(1)}(q) \]
gives the real-space structure of the effective interaction. This calculation of \( V_{\text{eff}}^{(1)}(r) \) has been carried out for \( V_0 = -100t, \) \( T = 100 \text{ K} \) and \( W = 3 \text{ eV}, \) and it is found that \( V_{\text{eff}}^{(1)}(r) \) at \( r = (1,0) \) is \(-0.017t\). Even though \( V_{\text{eff}}^{(1)}(r = (1,0)) \) is found to be attractive in leading order in \( U, \) its magnitude is smaller by a factor of three from \( V_1 = -0.05t \) found by fitting the \( 7K \) data on optimally doped \( \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \).

This is the behavior expected for \( V_{\text{eff}} \) at lowest order in \( U. \) Clearly, an approximation which is first order in \( U \) would be insufficient. The actual structure of \( V_{\text{eff}}, \) which includes scattering processes to all orders in \( U, \) depends on the reducible particle-particle vertex in the singlet and the triplet channels. Hence, it would be useful to calculate \( V_{\text{eff}} \) using exact numerical techniques in the paramagnetic state of the Hubbard or the \( t-J \) models. However, it should be noted that the structure in the electron density \( n(r) \) does not necessarily reflect the structure in \( V_{\text{eff}}, \) as it was seen in Fig. 11(c). In Ref. [27], \( V_{\text{eff}} \) was calculated for the \( t-J \) model but at half-filling, where \( V_{\text{eff}} \) at the \((1,0)\) site was found to be repulsive. However, the insulating state might be different than the paramagnetic state, which is considered here. In Ref. [20], an effective impurity interaction which is repulsive at \( r = (1,0) \) was used for calculating the enhancement of the uniform susceptibility by dilute nonmagnetic impurities. Since both positive and negative values of \( V_1 \) act to enhance the uniform susceptibility, this does not determine the sign of \( V_1. \) On the other hand, the Knight shift data studied here require that, at the level of RPA, \( V_{\text{eff}} \) at the nearest-neighbor sites of the impurity is weakly attractive.

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[41] $\chi(q, q' = 0)$ seen in Fig. 12(b) is not a smooth curve, hence one could expect large finite size effects on the Knight shifts. This is not the case because of the way the calculation of $\chi(r, r')$ is carried out. As described in Ref. 22, $\chi_0(r, r')$ is calculated on a small lattice (28 × 28), however by using $G_0(r_1, r_2, \omega_n)$ and $\chi^L_0(q)$ calculated on large lattices. Hence, when the RPA equation is solved for $\chi(r, r')$ on the small lattice, the local quantities such as the Knight shifts for the sites near the impurity have small finite size effects, even though $\chi(q, q' = 0)$ obtained from $\chi(r, r')$ is not a smooth curve.
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FIG. 1. Illustration of the extended impurity potential due to a nonmagnetic impurity represented by the filled square at the center. Here, $V_0$ is the onsite component of the impurity potential and $V_1$ acts at the nearest-neighbour sites of the impurity.

FIG. 2. (a) Transverse nuclear relaxation rate $T_2^{-1}$ for $^{63}$Cu(2) versus $T$ in pure YBa$_2$Cu$_3$O$_{6+x}$. The open and the filled circles represent the experimental data by Imai et al. [24] and Takigawa [25] for YBa$_2$Cu$_3$O$_{6.9}$ and for YBa$_2$Cu$_3$O$_{6.63}$, respectively. The dashed line is a fit of the $T_2^{-1}$ data by Imai et al. [24] on YBa$_2$Cu$_3$O$_{6.9}$. For YBa$_2$Cu$_3$O$_{6.63}$, the fitting of the data below 160 K has been carried out in two ways. The solid line has been obtained by ignoring the saturation of $T_2^{-1}$ below $\sim$ 160 K, and the dotted curve has been obtained by fitting the data for 80 K $< T < 160$ K. (b) Temperature dependence of $U$. The dashed curve has been obtained by fitting the $T_2^{-1}$ measurements on pure YBa$_2$Cu$_3$O$_{6.9}$. The solid and the dotted curves have been obtained for pure YBa$_2$Cu$_3$O$_{6.63}$ by fitting the solid and the dotted curves seen in (a).
FIG. 3. (a) $\chi_{\text{pure}}(q)$ versus $q$ at 100 K for the pure system. The dashed curve represents results for optimally doped YBa$_2$Cu$_3$O$_{6+x}$. The solid and the dotted curves have been obtained for underdoped YBa$_2$Cu$_3$O$_{6+x}$. (b) $\chi(Q^*)$ versus $T$ for optimally doped YBa$_2$Cu$_3$O$_{6+x}$, where $Q^*$ is the wavevector at which $\chi(q)$ peaks at low temperatures. Here, the dashed curve is for optimally doped pure YBa$_2$Cu$_3$O$_{6+x}$, and the solid curve is for the case of 0.5% dilute impurities. (c) $\chi(Q^*)$ versus $T$ for underdoped YBa$_2$Cu$_3$O$_{6+x}$. The dashed curve is for the pure system and the solid curve is for the case with 0.5% impurities. The dotted curves were obtained by using the values of $U$ given by the dotted curves in Fig. 2(b). The results for the impure systems shown in (b) and (c) were calculated for $V_1 = -0.15t$ as described in Section IV.A.
FIG. 4. Feynman diagrams for (a) the dressed single-particle Green’s function $G$ and (b)-(c) the irreducible susceptibility $\chi_0$ within the presence of one impurity.

FIG. 5. Sketch of the neighbouring Y(i) sites of a non-magnetic impurity located in the CuO$_2$ plane. Here, the filled square indicates the impurity site.
FIG. 6. (a) $^7\text{Li}$ Knight shift $^7\mathit{K}$ versus $T$ in optimally doped YBa$_2$Cu$_3$O$_{6-x}$. The filled circles represent the Knight shift data by Bobroff et al. [2]. The curves represent the results of the calculations for various values of $V_1/t$ which are indicated next to the curves. (b) Estimated temperature dependence of $^7\mathit{K_0}$ for optimally doped YBa$_2$Cu$_3$O$_{6-x}$. Here, $^7\mathit{K_0}$ represents the value of the $^7\text{Li}$ Knight shift, if the substitution of the Li impurity had not induced any changes in the magnetic correlations around it. The dashed curve is the experimental estimate of $^7\mathit{K_0}$ obtained by using the $^{89}\text{K}$ data [38] on pure optimally doped YBa$_2$Cu$_3$O$_{6-x}$. The solid line is the result obtained by setting $V_0 = V_1 = 0$ in this model.

FIG. 7. (a) Temperature dependence of $-\delta^{89}\mathit{K}$ at the sites which are the 1st, 2nd, 3rd and the 4th nearest $^{89}\text{Y}$ neighbors of the impurity in optimally doped YBa$_2$Cu$_3$O$_{6-x}$. Here, $\delta^{89}\mathit{K}$ is the induced change in the $^{89}\text{Y}$ Knight shift up on the substitution of the impurity. (b) Temperature dependence of the $^{63}\text{Cu(2)}$ Knight shift for $H||c$, $^{63}\mathit{K_c}(r_i)$, at various sites $r_i$, which are indicated next to the curves, for the optimally doped YBa$_2$Cu$_3$O$_{6-x}$. The results shown in (a) and (b) were obtained by using $V_1 = -0.15t$. 
FIG. 8. (a) Temperature dependence of $\delta K$ for underdoped YBa$_2$Cu$_3$O$_{6+x}$. Here, $\delta K$ is defined as $7K - 7K_0$, where $7K$ is the $^7$Li Knight shift and $7K_0$ represents the value of the $^7$Li Knight shift, if the substitution of the Li impurity had not changed the magnetic correlations around it. The filled circles represent the data by Bobroff et al. [2] on YBa$_2$Cu$_3$O$_{6.6}$. The curves represent the results of the calculations for various values of $V_1/t$ which are indicated next to the curves. (b) Temperature dependence of $7K_0$ for underdoped YBa$_2$Cu$_3$O$_{6+x}$. The solid points have been obtained by using the $^{89}$K(main) data on underdoped YBa$_2$Cu$_3$O$_{6+x}$ with Zn impurities [1,4], while the dashed curve has been obtained by using the $^{89}$K data on pure underdoped YBa$_2$Cu$_3$O$_{6+x}$ [38]. The solid curve shows the results of the calculations for $V_0 = V_1 = 0$.

FIG. 9. (a) Temperature dependence of $-\delta K$(outer) in underdoped YBa$_2$Cu$_3$O$_{6+x}$ with Zn impurities. Here, $\delta K$(outer) is defined as the shift of the outer $^{89}$Y satellite with respect to the $^{89}$Y main resonance line. The open and the filled circles represent the experimental data on $-\delta K$(outer) in underdoped YBa$_2$Cu$_3$O$_{6+x}$ with Zn impurities for the magnetic field $\mathbf{H} || \mathbf{c}$ and $\mathbf{H} \perp \mathbf{c}$, respectively [1,4]. The curves represent the results obtained by using various values of $V_1/t$ which are indicated next to the curves for the Y(1) site, which is the nearest $^{89}$Y neighbour of the nonmagnetic impurity. (b) Results similar to those in (a) but for $-\delta K$(middle). Here, $\delta K$(middle) is defined as the shift of the middle $^{89}$Y resonance line with respect to the main line. The curves represent results for various values of $V_1/t$ for the Y(2) site, which is the second-nearest $^{89}$Y neighbour of the nonmagnetic impurity.
FIG. 10. Temperature dependence of the $^{63}$Cu(2) Knight shift for $\mathbf{H}_c \cdot \mathbf{K}_c$ at sites $\mathbf{r}_i$ which are indicated next to the curves. These results were obtained for $V_1 = -0.15t$ using the values of $U(T)$ given by the solid curve in Fig. 2(b).

FIG. 11. (a) $k(\mathbf{r})$ versus the distance $r$ away from the impurity in units of the lattice spacing $a$ for $T = 100$ K and $V_1 = -0.05t$. Here, the open and the filled circles represent the results for the optimally doped and the underdoped YBa$_2$Cu$_3$O$_{6+x}$, respectively. Also, the horizontal long-dashed and dotted lines represent $k(\mathbf{r})$ for the optimally doped and underdoped pure systems, respectively. (b) $\chi(\mathbf{r}, \mathbf{r})$ versus the distance $r$ away from the impurity presented in the same way as in (a) for $k(\mathbf{r})$. (c) Electron occupation $n(\mathbf{r}_i)$ at sites $\mathbf{r}_i$ near the impurity plotted as a function $r = |\mathbf{r}_i|$ at 100 K. These results were obtained for $V_1/t = -0.15$ with $W = 1$ eV (solid) and for $V_1/t = -0.05$ with $W = 3$ eV (dashed). Also shown are results for an onsite impurity potential ($V_1 = 0$) and $W = 1$ eV (dotted). In all of these cases, $V_0 = -100t$ was used.
FIG. 12. Off-diagonal susceptibilities (a) $-\chi_0(\mathbf{q},0)$ and (b) $-\chi(\mathbf{q},\mathbf{q}') = 0$ versus $\mathbf{q}$ obtained by using $V_1 = -0.15t$ for optimally doped YBa$_2$Cu$_3$O$_{6+x}$ at $T = 100$ K. Here, the $\delta$-function components at $\mathbf{q} = \mathbf{q}' = 0$ have been omitted.

FIG. 13. Temperature dependence of $^7K$ for optimally doped YBa$_2$Cu$_3$O$_{6+x}$, obtained for various values of $V_1/t$. These results are for an effective bandwidth of 3 eV, and here the $^7Li$ hyperfine coupling $C$ was taken to be 1.4 kOe/$\mu_B$. The filled circles represent the $^7K$ data on optimally doped YBa$_2$Cu$_3$O$_{6+x}$ by Bobroff et al. [2]. The open circles represent the estimate of $^7K_0$ obtained by using the $^{89}Y$ data [38] on pure optimally doped YBa$_2$Cu$_3$O$_{6+x}$. Here, the $^{89}Y$ hyperfine coupling was taken to be -2 kOe/$\mu_B$. The dotted curve has been obtained by using $V_0 = V_1 = 0$ and the remaining curves have been obtained by using $V_0 = -100t$ and the values of $V_1/t$ which are indicated next to the curves.
FIG. 14. Temperature dependence of (a) $\delta^7K$, (b) $-\delta^{89}K$ (outer) and (c) $-\delta^{89}K$ (middle) for underdoped YBa$_2$Cu$_3$O$_{6+x}$ obtained by using various values of $V_1/t$ which are indicated next to the curves. These results have been calculated by using an effective bandwidth $W$ of 3 eV, as compared to $W = 1$ eV used in Figs. 8 and 9. In addition, here the $^7$Li and the $^{89}$Y hyperfine couplings were taken to be 1.4 kOe/$\mu_B$ and -2 kOe/$\mu_B$, respectively. The experimental data in (a) are from Ref. [2], and the data in (b) and (c) are from Refs. [1,4].
FIG. 15. Feynman diagram for a scattering process which causes the effective impurity potential to be extended in real space. This diagram is lowest order in $U$ and the bare impurity potential $V_0$. Here, an electron with momentum $p$, Matsubara frequency $\omega_n$ and spin $\sigma$ scatters to a state with momentum $p + q$, Matsubara frequency $\omega_n$ and spin $\sigma$. 