Bulk to Boundary Relation Between Topological Models
and Supergravity Theories

I.V. Lavrinenko†

Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843

ABSTRACT

We establish a direct correspondence between certain higher-rank $p$-form Chern-Simons topological type theories in the bulk of a manifold with boundary and particular sectors of supergravity models on the boundary, provided that certain boundary conditions are satisfied. The cases we investigate include eleven-dimensional supergravity and both of the type II theories in ten dimensions.

† e-mail: lavrik@rainbow.physics.tamu.edu
1 Introduction

Superstring models and their low-energy supergravity limits are among the most complex and exciting of physical theories, and they are also among the most promising as far as phenomenological expectations are concerned. Even though so far no concrete predictions have been made, the intrinsic mathematical consistency and beauty of these theoretical constructions have been attracting much attention. In this paper we will uncover one more rather interesting aspect of the mathematical conspiracies that make these theories so unique. It is well known that symmetries play an important role in all physical theories. It is a simple observation that if one is able to devise a formulation where these symmetry properties are manifest, then previously hidden aspects of the theory can become more transparent.

In this paper we shall demonstrate that there is a direct relation between certain unusual Chern-Simons (CS) type theories in higher-dimensional spacetimes with boundaries and particular sectors of supergravity models on the boundary of the spacetime. In order to make this relation explicit shall develop a certain first-order formalism. An almost identical approach, called the doubled-field formalism, has been used in the past [6] to unify the duality symmetries and gauge invariances of supergravity theories, and later on the formalism has been successfully applied to obtain certain, rather general, relations among $p$-brane tensions [7]. However, to prevent a confusion, we should emphasize that the approach used in this paper is not identical to that introduced in [6]. For instance, our definitions for the field-strengths are different, and all fields in the theory are put on an equal footing, including the dilatons. As for the general structure, it is strikingly reminiscent of the situation in an ordinary CS-theory in a three-dimensional spacetime with boundary, where it is well known that in the bulk the theory is topological, and all the dynamics takes place only on the boundary, where the degrees of freedom are described by the chiral Wess-Zumino-Witten-Novikov (WZWN) model [1]. However, now the CS-theories are replaced by some higher $p$-form analogs, and instead of a WZWN-model we obtain various supergravity theories.

Boundary conditions play a rather important role in the construction [2–5]. One needs to impose these conditions in order for the variational principle to make sense, and to maintain gauge invariance in the theory on a manifold with boundary. The CS-type theories we obtain have much in common with the so-called higher-rank BF theories [9–11], although in our approach these theories acquire a non-Abelian generalization. Even though many issues have been left unaddressed, the results obtained may indicate that there are hidden topological sectors in supergravity theories, or even that these theories may be described by pure
topological models on manifolds with boundaries. It is also conceivable that the topological theories we describe may have applications of their own, related to some properties of higher-dimensional manifolds without boundaries [8]. However, the fact that these models preserve metric-independence after quantization still remains to be proved.

2 D=11 Supergravity

In this section we shall consider the example of $D = 11$ supergravity. It is a simplest case possible, however it already includes many characteristic features. We would like to start with the following Lagrangian in $D = 12$

$$\mathcal{L} = \frac{1}{2}(A_{(4)} \wedge dA_{(7)} + A_{(7)} \wedge dA_{(4)}) + \frac{1}{6}A_{(4)} \wedge A_{(4)} \wedge A_{(4)},$$

(2.1)

where $A_{(4)}$ and $A_{(7)}$ are four-form and seven-form gauge fields.

It is easy to see that this Lagrangian changes by total derivative under the following gauge transformations

$$\delta A_{(4)} = d\omega_{(3)},$$

$$\delta A_{(7)} = -\omega_{(3)} \wedge A_{(4)} + d\omega_{(6)},$$

(2.2)

where $\omega_{(3)}$ and $\omega_{(6)}$ are arbitrary three-forms and six-forms.

In fact

$$\delta \mathcal{L} = \frac{1}{2}d(\omega_{(3)} \wedge A_{(7)} - \omega_{(6)} \wedge A_{(4)}).$$

(2.3)

The following brackets hold

$$\{\delta(\omega_{(3)}^{1}), \delta(\omega_{(3)}^{2})\} = \delta(\omega_{(3)}^{2} \wedge \omega_{(3)}^{1}), \quad \{\delta(\omega_{(3)}^{1}), \delta(\omega_{(6)}^{2})\} = 0$$

(2.4)

If the manifold has no boundary, the action obtained by integrating this Lagrangian over the whole spacetime is invariant with respect to these gauge transformations. In fact the Lagrangian (2.1) can be obtained by following the canonical procedure for CS-theories. Let us define the following field-strengths

$$F_{(5)} = dA_{(4)},$$

$$F_{(8)} = dA_{(7)} + \frac{1}{2}A_{(4)} \wedge A_{(4)}.$$  

(2.5)

It can be checked that under the gauge transformations (2.2) fields (2.5) transform covariantly

$$\delta F_{(5)} = 0,$$

$$\delta F_{(8)} = \omega_{(3)} \wedge F_{(5)}.$$  

(2.6)
Out of these field-strengths one can construct the analog of the Chern class

\[ F_{(5)} \wedge F_{(8)}. \]  

(2.7)

It is easy to see that the class (2.7) is invariant with respect to the gauge transformations (2.2), and also that the following identity holds

\[ F_{(5)} \wedge F_{(8)} = d(\frac{1}{2} A_{(4)} \wedge dA_{(7)} + A_{(7)} \wedge dA_{(4)}) + \frac{1}{6} A_{(4)} \wedge A_{(4)} \wedge A_{(4)}) = d\mathcal{L}. \]  

(2.8)

Thus everything is very reminiscent of ordinary Chern-Simons theory, and what we have said so far, with the exception of gauge invariance, is valid whether we have a boundary or not. In the presence of a boundary, things change in a rather interesting way. First of all, it seems that the theory is no longer gauge invariant, due to the total derivative which, upon integration, gives rise to a non-vanishing term on the boundary. However, we know how to deal with this problem. The solution again comes from the conventional CS-theory, where the same problem arises; one imposes some boundary conditions in order to restore gauge invariance [2–5]. The trivial condition that the gauge transformations vanish on the boundary is not interesting, since it appears to be too strong. The only other choice is to impose the following relations

\[ d\omega_{(3)} = \ast d\omega_{(6)}|_{\partial M_{12}}, \]  

\[ A_{(7)} = \ast A_{(4)}|_{\partial M_{12}}, \]  

(2.9)

where the Hodge * is taken with respect to the eleven-dimensional metric on the boundary. If the relations (2.9) hold then the boundary term vanishes, and gauge invariance is restored. It can also be checked that the variational principle is well defined in this case. Normally, if there is a boundary and the Lagrangian changes by a total derivative, problems may arise owing to the fact that the functional derivative is not well defined unless one adds some extra terms on the boundary to cancel terms coming from the bulk. In our situation the boundary contribution from the bulk has the form

\[ \int \delta\mathcal{L} = \frac{1}{2} \int_{\partial M_{12}} (A_{(4)} \wedge \delta A_{(7)} - A_{(7)} \wedge \delta A_{(4)}), \]  

(2.10)

and it vanishes if boundary conditions (2.9) are satisfied.

This is not the end of story though. The action of twelve-dimensional theory (2.1) can be written as

\[ S = \int_{\partial M_{12}} \frac{1}{2} A_{(4)} \wedge A_{(7)} + \int_{M_{12}} (A_{(7)} \wedge dA_{(4)} + \frac{1}{6} A_{(4)} \wedge A_{(4)} \wedge A_{(4)}), \]  

(2.11)

\[ = \int_{\partial M_{12}} \frac{1}{2} A_{(4)} \wedge \ast A_{(4)} + \int_{M_{12}} (A_{(7)} \wedge dA_{(4)} + \frac{1}{6} A_{(4)} \wedge A_{(4)} \wedge A_{(4)}), \]
where in the second line we used the boundary conditions introduced in (2.9).

Now, variation of the $A_{(7)}$-field (which simply plays the role of a constraint) in the bulk gives the condition $dA_{(4)} = 0$, which implies that (at least locally) $A_{(4)} = dA_{(3)}$, and by continuation the same must be true on the boundary. Thus if one substitutes this relation back into the Lagrangian the result is a theory on the boundary whose Lagrangian is identical to the one for $D = 11$ supergravity (without Einstein-Hilbert term). There are also some subtleties with the gauge symmetry of the theory. The generators are simply constraints coming from the variation over the components of $A_{(4)}$ and $A_{(7)}$ gauge fields with a time index, since these have no conjugate momenta. They are

$$
G_m(\omega_{(6)}) = \int_{\mathcal{M}_{11}} \omega_{(6)} \wedge dA_{(4)}, \quad G_e(\omega_{(3)}) = \int_{\mathcal{M}_{11}} \omega_{(3)} \wedge (dA_{(7)} + \frac{1}{2}A_{(4)} \wedge A_{(4)}),
$$

(2.12)

where $\omega_{(3)}$ and $\omega_{(6)}$ are arbitrary three-forms and six-forms, and the integration is over only the spatial coordinates.

All states in the theory must satisfy the constraints $G_e(\cdot) = 0$, and $G_m(\cdot) = 0$. If there is no boundary, the algebra of these constraints can be easily calculated using the fact that the $A_{(4)}$ and $A_{(7)}$ fields are canonically conjugate to each other. The algebra gives rise to the following brackets

$$\{G_e(\omega^1_{(3)}), G_e(\omega^2_{(3)})\} = G_m(\omega^1_{(3)} \wedge \omega^2_{(3)}), \quad \{G_e(\omega_{(3)}), G_m(\omega_{(6)})\} = 0.
$$

(2.13)

However, if the spacetime has a boundary, one runs into a problem. Functional derivatives of the constraints are not well defined, and one needs to add some boundary terms to the generators to cancel these contributions, and these extra terms will play the role of boundary symmetries. Namely one needs to add $g_m = -\int_{\partial\mathcal{M}_{11}} \omega_{(6)} \wedge A_{(4)}$ to $G_m$, and $g_e = -\int_{\partial\mathcal{M}_{11}} \omega_{(3)} \wedge A_{(7)} = -\int_{\partial\mathcal{M}_{11}} \omega_{(3)} \wedge *A_{(4)}$ to $G_e$, and also the commutator between the (modified) $G_e$ and $G_m$ generators changes, becoming $\{G_e(\omega_{(3)}), G_m(\omega_{(6)})\} = \int_{\partial\mathcal{M}_{11}} \omega_{(3)} \wedge d\omega_{(6)}$. After we have imposed the constraints $G_e(\cdot) = 0$, and $G_m(\cdot) = 0$ the gauge symmetry in the bulk therefore gives rise to the symmetry of a theory on the boundary

$$\{g_e(\omega^1_{(3)}), g_e(\omega^2_{(3)})\} = g_m(\omega^1_{(3)} \wedge \omega^2_{(3)}), \quad \{g_e(\omega_{(3)}), g_m(\omega_{(6)})\} = \int_{\partial\mathcal{M}_{11}} \omega_{(3)} \wedge d\omega_{(6)}.
$$

(2.14)

A similar idea has been employed in CS-theory in $D = 3$, where it gave rise to certain conformal models on the boundary with affine symmetries [3–5].

As a nontrivial consistency check one can verify that Jacobi-identity is satisfied. The only interesting one is
\[
\{ g_e(\omega^1_3), \{ g_e(\omega^2_3), g_e(\omega^3_3) \} \} + \{ g_e(\omega^3_3), \{ g_e(\omega^1_3), g_e(\omega^2_3) \} \} + \\
\{ g_e(\omega^2_3), \{ g_e(\omega^3_3), g_e(\omega^1_3) \} \}\]
\[
\int_{\partial M_{11}} (\omega^1_3 \wedge d(\omega^3_3 \wedge \omega^3_3) + \omega^2_3 \wedge d(\omega^3_3 \wedge \omega^1_3) + \omega^3_3 \wedge d(\omega^3_3 \wedge \omega^2_3)) = \\
-2 \int_{\partial M_{11}} d(\omega^1_3 \wedge \omega^2_3 \wedge \omega^3_3) = 0.
\]

3 Type IIA-Theory in D=10

Now we would like to consider the case of the Type IIA theory in $D = 10$. Here things are slightly more complicated, owing to the increase in the number of fields, but main ideas stay the same. The ten-dimensional Lagrangian has the form

\[
\mathcal{L}_{10} = R \mathbb{1} - \frac{1}{2} d\phi \wedge d\phi - \frac{1}{8} e^{\frac{1}{2}\phi} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \frac{1}{2} e^{-\phi} * F_{(3)} \wedge F_{(3)} \\
- \frac{1}{2} e^{\frac{3}{2}\phi} * F_{(4)} \wedge F_{(4)} - \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)},
\]

where $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge A_{(1)}$, $F_{(3)} = dA_{(2)}$ and $\mathcal{F}_{(2)} = dA_{(1)}$. From this, it follows that the equations of motion for the antisymmetric tensor and scalar fields are:

\[
d(e^{\frac{1}{2}\phi} * F_{(4)}) = - F_{(4)} \wedge F_{(3)},
\]
\[
d(e^{-\phi} * F_{(3)}) = - \mathcal{F}_{(2)} \wedge (e^{\frac{1}{2}\phi} * F_{(4)}) - \frac{1}{2} F_{(4)} \wedge F_{(4)},
\]
\[
d(e^{\frac{3}{2}\phi} * \mathcal{F}_{(2)}) = - F_{(3)} \wedge (e^{\frac{3}{2}\phi} * F_{(4)}),
\]
\[
d* d\phi = - \frac{1}{4} F_{(4)} \wedge (e^{\frac{1}{2}\phi} * F_{(4)}) - \frac{1}{2} F_{(3)} \wedge (e^{-\phi} * F_{(3)}) - \frac{3}{4} \mathcal{F}_{(2)} \wedge (e^{\frac{3}{2}\phi} * \mathcal{F}_{(2)}).
\]

Let us make the following redefinitions

\[
e^{\frac{1}{2}\phi} F_{(4)} = B_{(4)}, \quad e^{-\phi} F_{(3)} = B_{(3)}, \quad e^{\frac{1}{2}\phi} \mathcal{F}_{(2)} = B_{(2)}, \quad d\phi = B_{(1)},
\]

and introduce the following new fields

\[
B_{(6)} = * B_{(4)}, \quad B_{(7)} = * B_{(3)}, \quad B_{(8)} = * B_{(2)}, \quad B_{(9)} = * B_{(1)}.
\]

In terms of these new fields equations (3.2) can be written as follows

\[
H_{(2)} = dB_{(1)} = 0,
\]
\[
H_{(3)} = dB_{(2)} - \frac{1}{4} B_{(1)} \wedge B_{(2)} = 0,
\]
\[
H_{(4)} = dB_{(3)} + \frac{1}{2} B_{(1)} \wedge B_{(3)} = 0,
\]
where we have also introduced new field-strengths $H_2$, $H_3$, $H_4$, $H_5$, $H_7$, $H_8$, $H_9$, and $H_{10}$.

At this stage we would like to emphasize that there is a one-to-one correspondence between original equations of motion (3.2) and the system (3.5), provided that relations (3.4) are satisfied. Indeed, one can solve for $B_1$ first (locally), and then for $B_2$, $B_3$, and $B_4$, and after that, using relations (3.4) for $B_6$, $B_7$, $B_8$, and $B_9$, so eventually one recovers the original system (3.2).

It turns out that the system (3.5) is invariant under the following gauge transformations

$$
\begin{align*}
\delta B_1 &= d\omega, \\
\delta B_2 &= d\omega_1 + \frac{2}{3}\omega_1 \wedge B_{(1)} + \frac{2}{5}\omega B_{(2)}, \\
\delta B_3 &= d\omega_2 + \frac{1}{3}\omega_2 \wedge B_{(1)} - \frac{1}{5}\omega B_{(3)}, \\
\delta B_4 &= d\omega_3 + \frac{1}{3}\omega_3 \wedge B_{(1)} + \omega_2 \wedge B_{(2)} + \omega_1 \wedge B_{(3)} + \frac{1}{2}\omega B_{(4)}, \\
\delta B_6 &= d\omega_5 - \frac{1}{5}\omega_5 \wedge B_{(1)} - \omega_3 \wedge B_{(3)} - \omega_2 \wedge B_{(4)} - \frac{1}{2}\omega B_{(6)}, \\
\delta B_7 &= d\omega_6 - \frac{1}{3}\omega_6 \wedge B_{(1)} - \omega_5 \wedge B_{(2)} - \omega_4 \wedge B_{(3)} - \omega_1 \wedge B_{(6)} + \frac{1}{2}\omega B_{(7)}, \\
\delta B_8 &= d\omega_7 - \frac{2}{3}\omega_7 \wedge B_{(1)} - \omega_5 \wedge B_{(3)} - \omega_2 \wedge B_{(6)} - \frac{3}{4}\omega B_{(8)}, \\
\delta B_9 &= d\omega_8 - \frac{2}{3}\omega_8 \wedge B_{(2)} + \frac{1}{2}\omega_6 \wedge B_{(3)} - \frac{1}{2}\omega_5 \wedge B_{(9)} - \frac{1}{2}\omega_7 \wedge B_{(1)} \wedge B_{(8)}, \\
&- \frac{1}{3}\omega_3 \wedge B_{(6)} - \frac{1}{2}\omega_2 \wedge B_{(7)} - \frac{2}{3}\omega_1 \wedge B_{(9)},
\end{align*}
$$

where $\omega$, $\omega_1$, $\omega_2$, $\omega_3$, $\omega_5$, $\omega_6$, $\omega_7$, and $\omega_8$ are arbitrary $p$-forms of the appropriate degree. This is actually an infinitesimal form of the transformations, but it is good enough for our purposes. Another way to say that transformations (3.6) leave the system (3.5) invariant is to claim that field-strengths $H_i$ transform covariantly, namely that they transform into each other, which can be checked by a direct calculation. Now we would like to apply the same idea as we did in $D = 11$ supergravity to the type IIA theory. Let us consider the following Lagrangian in eleven-dimensional spacetime.
As a nice consistency check, one can derive the following for the identity

\[ d \mathcal{L} = H_{(7)} \wedge H_{(5)} - H_{(10)} \wedge H_{(2)} - H_{(8)} \wedge H_{(4)}, \]  

where we are using the definitions (3.5). Again it is evident that the Lagrangian (3.7) can be obtained from the descent process from a thirteen-dimensional spacetime.

First of all under the gauge transformations (3.6) the Lagrangian (3.7) transforms as a total derivative; for example the \( \omega \) and \( \omega_(8) \) transformations give rise to the following term

\[ \delta \mathcal{L} = \frac{1}{2} d(d\omega_{(8)} \wedge B_{(1)} + d\omega B_{(9)}). \]  

All other transformations produce similar results, namely it is always a sum of two terms each of which is a wedge product of the derivative of a gauge parameter and a field of complementary degree.

Therefore if there is no boundary in the eleven-dimensional spacetime we have gauge invariance; otherwise it is lost. Of course, we can restore gauge invariance by imposing appropriate boundary conditions on the fields and gauge parameters. Not too surprisingly, they turn out to be the same as equations (3.4), plus corresponding conditions for the gauge-parameters, namely \( d\omega_{(8)} = *d\omega, \ d\omega_{(7)} = *d\omega_{(1)}, \) and so on. Also following the procedure described in the second section, namely through integrating certain terms by parts, we get the action in eleven-dimensional bulk plus boundary terms

\[ S = -\frac{1}{2} \int_{\partial M_{11}} (\ast B_{(4)} \wedge B_{(4)} + \ast B_{(3)} \wedge B_{(3)} + \ast B_{(2)} \wedge B_{(2)} + \ast B_{(1)} \wedge B_{(1)}) + \int_{M_{11}} (B_{(6)} \wedge dB_{(4)} - B_{(7)} \wedge dB_{(3)} + B_{(8)} \wedge dB_{(2)} - B_{(9)} \wedge dB_{(1)} - \frac{1}{2} B_{(6)} \wedge B_{(1)} \wedge B_{(4)} - B_{(7)} \wedge B_{(1)} \wedge B_{(3)} - \frac{3}{2} B_{(8)} \wedge B_{(1)} \wedge B_{(2)} - \frac{1}{2} B_{(4)} \wedge B_{(4)} \wedge B_{(3)}). \]  

Now, let us evaluate the equations of motion for \( B_{(9)}, B_{(8)}, B_{(7)}, \) and \( B_{(6)} \) in the bulk, which imply that the rest of the fields are pure gauge (everywhere, including the boundary). After integrating some terms by parts, we obtain a Lagrangian on the boundary identical to the
one for the type IIA theory in $D = 10$, where pure gauge degrees of freedom from the bulk become dynamical. We should remark that presumably this procedure can be carried out even at the quantum (functional integral) level, since the $B(9), B(8), B(7)$, and $B(6)$ fields simply play the role of Lagrange multipliers. Thus the integrations over these fields produce delta-functions which impose that the $B(1), B(2), B(3)$, and $B(4)$ gauge fields are flat, or pure gauge. The associated redefinitions of variables may give rise to non-trivial Jacobians in the functional integral measure, and in turn may produce some dynamics for the metric on the boundary. It is very tempting to conjecture that these Jacobians will eventually produce an Einstein-Hilbert curvature term on the boundary, and that the theory we are describing is all that is needed to produce the complete supergravity action on the boundary, including gravity. Clearly these issues require a more careful investigation.

4 Type IIB-Theory in D=10

There is no covariant Lagrangian for type IIB supergravity, since it includes a self-dual 5-form field strength. However one can write down covariant equations of motion. In order to make manifest their global $SL(2,\mathbb{R})$ symmetry, it is useful first to assemble the dilaton $\phi$ and axion $\chi$ into a $2 \times 2$ matrix:

$$
\mathcal{M} = \begin{pmatrix}
e^\phi & \chi e^\phi \\
\chi e^\phi & e^{-\phi} + \chi^2 e^\phi
\end{pmatrix}
$$

Also, define the $SL(2,\mathbb{R})$-invariant matrix

$$
\Xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
$$

and the two-component column vector of 2-form potentials

$$
A_{(2)} = \begin{pmatrix} A_{(2)}^1 \\ A_{(2)}^2 \end{pmatrix}.
$$

Here $A_{(2)}^1$ is the R-R potential, and $A_{(2)}^2$ is the NS-NS potential. The bosonic matter equations of motion can then be written as

$$
d*dH_{(5)} = -\frac{1}{2} \epsilon_{ij} F_{(3)}^i \wedge F_{(3)}^j,
$$

$$
d(\mathcal{M}*H_{(3)}) = H_{(5)} \wedge \Xi H_{(3)},
$$

$$
d(e^{2\phi}*d\chi) = e^{\phi} F_{(3)}^2 \wedge *F_{(3)}^1,
$$

$$
d*d\phi = e^{2\phi} d\chi \wedge *d\chi + \frac{1}{2} e^\phi F_{(3)}^1 \wedge *F_{(3)}^1 - \frac{1}{2} e^{-\phi} F_{(3)}^2 \wedge *F_{(3)}^2,
$$

(4.4)
where $F_{(3)}^{1} = dA_{(2)}^{1} - \chi dA_{(2)}^{2}$, $F_{(3)}^{2} = dA_{(2)}^{2}$, $H_{(3)} = dA_{(2)}$, and $H_{(5)} = dB_{(4)} - \frac{1}{2} \epsilon_{ij} A_{(2)}^{i} \wedge dA_{(2)}^{j}$.

Now let us define the following new fields

\begin{align*}
C_{(1)}^{1} &= d\phi, \quad C_{(1)}^{2} = e^{\phi} d\chi, \quad C_{(5)} = H_{(5)}, \\
C_{(3)}^{1} &= e^{\frac{1}{2} \phi} dA_{(2)}^{1} + \chi e^{\frac{1}{2} \phi} dA_{(2)}^{2}, \quad C_{(3)}^{2} = e^{-\frac{1}{2} \phi} dA_{(2)}^{2},
\end{align*}

(4.5)

and

\begin{align*}
C_{(7)}^{1} &= *C_{(3)}^{1}, \quad C_{(7)}^{2} = *C_{(3)}^{2}, \\
C_{(9)}^{1} &= *C_{(1)}^{1}, \quad C_{(9)}^{2} = *C_{(1)}^{2},
\end{align*}

(4.6)

where we are also assuming that $C_{(5)} = *C_{(5)}$.

It turns out that in terms of these fields the system (4.4) can be written as follows

\begin{align*}
G_{(2)}^{1} &= dC_{(1)}^{1} = 0, \\
G_{(2)}^{2} &= dC_{(1)}^{2} - C_{(1)}^{1} \wedge C_{(1)}^{2} = 0, \\
G_{(6)} &= dC_{(5)} - C_{(3)}^{2} \wedge C_{(3)}^{1} = 0, \\
G_{(4)}^{1} &= dC_{(3)}^{1} - \frac{1}{2} C_{(1)}^{1} \wedge C_{(3)}^{1} - C_{(1)}^{2} \wedge C_{(3)}^{2} = 0, \\
G_{(4)}^{2} &= dC_{(3)}^{2} + \frac{1}{2} C_{(1)}^{1} \wedge C_{(3)}^{2} = 0, \\
G_{(8)}^{1} &= dC_{(7)}^{1} + \frac{1}{2} C_{(1)}^{1} \wedge C_{(7)}^{1} - C_{(5)} \wedge C_{(3)}^{2} = 0, \\
G_{(8)}^{2} &= dC_{(7)}^{2} - \frac{1}{2} C_{(1)}^{1} \wedge C_{(7)}^{2} + C_{(1)}^{2} \wedge C_{(7)}^{1} + C_{(5)} \wedge C_{(3)}^{1} = 0, \\
G_{(10)}^{1} &= dC_{(9)}^{1} - C_{(1)}^{2} \wedge C_{(9)}^{2} - \frac{1}{2} C_{(3)}^{1} \wedge C_{(7)}^{1} + \frac{1}{2} C_{(3)}^{2} \wedge C_{(7)}^{2} = 0, \\
G_{(10)}^{2} &= dC_{(9)}^{2} + C_{(1)}^{1} \wedge C_{(9)}^{2} - C_{(3)}^{2} \wedge C_{(9)}^{1} = 0,
\end{align*}

(4.8)

where we have also introduced corresponding field-strengths $G_{j}^{i}$.

Again we would like to emphasize that the system (4.8) is completely equivalent to the original set of equations (4.4) provided that conditions (4.7) are satisfied.

One can easily check that this new system is invariant with respect to the following gauge transformations

\begin{align*}
\delta C_{(1)}^{1} &= d\omega^{1}, \\
\delta C_{(1)}^{2} &= d\omega^{2} - C_{(1)}^{1} \omega^{2} + \omega^{1} C_{(1)}^{2}, \\
\delta C_{(3)}^{1} &= d\omega^{1} - \frac{1}{2} C_{(1)}^{1} \wedge \omega^{2} + \frac{1}{2} \omega^{1} C_{(3)}^{1} + \omega^{2} C_{(3)}^{2} - \omega^{2} \wedge C_{(1)}^{2}, \\
\delta C_{(3)}^{2} &= d\omega^{2} + \frac{1}{2} C_{(1)}^{1} \wedge \omega^{2} - \frac{1}{2} \omega^{1} C_{(3)}^{2},
\end{align*}

(4.9)
\[ \delta C_{(5)} = d\omega_{(4)} - \omega^1_{(2)} \wedge C^2_{(3)} + \omega^2_{(2)} \wedge C^1_{(3)} , \]
\[ \delta C^1_{(7)} = d\omega^1_{(6)} + \frac{1}{2} C^1_{(1)} \wedge \omega^1_{(6)} - \frac{1}{2} \omega^1 C^1_{(7)} - \omega^2_{(2)} \wedge C^1_{(5)} , \]
\[ \delta C^2_{(7)} = d\omega^2_{(6)} - \frac{1}{2} C^1_{(1)} \wedge \omega^2_{(6)} + \frac{1}{2} \omega^1 C^2_{(7)} - \omega^2 C^1_{(7)} + \omega^1_{(2)} \wedge C_{(5)} + \omega^1_{(6)} \wedge C^2_{(1)}, \]
\[ \delta C^1_{(9)} = d\omega^1_{(8)} + \frac{1}{2} \omega^2_{(6)} \wedge C^2_{(3)} - \frac{1}{2} \omega^1_{(6)} \wedge C^1_{(3)} - \frac{1}{2} \omega^2_{(2)} \wedge C^2_{(7)} - \omega^1_{(8)} \wedge C_{(1)} + \omega^2 C_{(9)}, \]
\[ \delta C^2_{(9)} = d\omega^2_{(8)} + C^1_{(1)} \wedge \omega^1 - \omega^1_{(6)} \wedge C^2_{(3)} + \omega^2_{(2)} \wedge C_{(1)} - \omega^2 C_{(9)}, \]

where \( \omega^j \) are arbitrary p-form parameters.

Now, as always, we go one dimension higher, and introduce the following Lagrangian in eleven-dimensional spacetime

\[ L = \frac{1}{2} (C^1_{(1)} \wedge dC^1_{(9)} + C^1_{(9)} \wedge dC^1_{(1)}) + \frac{1}{2} (C^2_{(1)} \wedge dC^2_{(9)} + C^2_{(9)} \wedge dC^2_{(1)}) + \]
\[ \frac{1}{2} (C^1_{(3)} \wedge dC^1_{(7)} + C^1_{(7)} \wedge dC^1_{(3)}) + \frac{1}{2} (C^2_{(3)} \wedge dC^2_{(7)} + C^2_{(7)} \wedge dC^2_{(3)}) + \]
\[ \frac{1}{2} C^1_{(5)} \wedge dC^1_{(5)} + C^1_{(1)} \wedge C^2_{(3)} \wedge C^2_{(7)} \wedge C^1_{(1)} \wedge C^1_{(7)} + \]
\[ \frac{1}{2} C^1_{(1)} \wedge C^2_{(3)} \wedge C^2_{(7)} + C^2_{(3)} \wedge C^2_{(1)} \wedge C^1_{(7)} + C^1_{(5)} \wedge C^1_{(3)} \wedge C^2_{(3)}. \]

Now one has to go through the same sequence of steps, and observe that under the symmetry (4.9) this Lagrangian changes by a total derivative, so that if there is a boundary then gauge invariance is broken. To restore the symmetry one needs to impose some boundary conditions, and of course, not too surprisingly, they turn out to be the same as equations (4.7), including the self-duality constraint for 5-form field-strength, plus the similar relations for gauge-parameters. Let us just demonstrate how this comes about for the 5-form. Under the symmetry \( \delta C_{(5)} = d\omega_{(4)} \) the action transforms as follows

\[ \delta S = \int_{M_1} \delta L = \frac{1}{2} \int_{\partial M_1} d\omega_{(4)} \wedge C_{(5)} \]

Now if we impose the conditions \( C_{(5)} = *C_{(5)} \) and \( d\omega_{(4)} = *d\omega_{(4)} \) then remarkably this term vanishes, by virtue of the Lorentzian metric signature. Again one can easily check that the theory lives on the boundary only, where it coincides with \( D = 10 \) type IIB-Theory (without gravity, of course). In order to demonstrate this one needs to write out the action for the Lagrangian (4.10) in such a form that the bulk part does not have terms where the \( C^i \) and \( C^j \) fields are covered by derivatives. They will therefore play the role of bulk Lagrange multipliers which impose certain constraints on the rest of the fields, plus boundary terms with appropriate boundary conditions to have gauge invariance. After integrating out the \( C^i \) and \( C^j \) fields in the bulk and solving constraints they impose, and integrating certain terms by parts, one derives the type IIB action together with the extra equations for \( C_{(5)} \).
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