Periodic integral operators over Cayley-Dickson algebras and spectra

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25 March 2012

Abstract

Periodic integral operators over Cayley-Dickson algebras related with integration of PDE are studied. Their continuity and spectra are investigated.

1 Introduction

Integral operators over Cayley-Dickson algebras are useful for integration of linear and non-linear partial differential equations [26, 27].

Sections 2 and 3 are devoted to spectra of periodic integral operators over Cayley-Dickson algebras, that can be used for analysis of solutions in a bounded domain or of periodic solutions on the Cayley-Dickson algebra of PDE including that of non-linear. This is actual, because spectra of operators are used for solutions of partial differential equations, for example, with the help of the inverse scattering problem method (see also [1]). Moreover, hypercomplex analysis is fast developing and also in relation with problems of theoretical and mathematical physics and of partial differential equations [4, 9, 11]. Cayley-Dickson algebras are used not only in mathematics, but also in applications [6, 12, 10, 15, 16].

1 key words and phrases: non-commutative functional analysis, hypercomplex numbers, Cayley-Dickson algebra, integral operator, spectra, non-commutative integration

Mathematics Subject Classification 2010: 30G35, 17A05, 17A70, 47A10, 47L30, 47L60
Analysis over Cayley-Dickson algebras was developed as well [18, 19, 20, 21, 22]. This paper continuous previous articles and uses their results [23, 24, 25, 26, 27]. Notations and definitions of papers [18, 19, 20, 21, 22] are used below. The main results of this paper are obtained for the first time.

2 Periodic integral operators over Cayley-Dickson algebras

1. Notation. Let $X$ be a Banach space over the Cayley-Dickson algebra $A_v$ with $2 \leq v, \ v \in \mathbb{N}$. Let also $A_w$ be the Cayley-Dickson subalgebra $A_w \subseteq A_v$, where $2 \leq w \leq v$.

The Cayley-Dickson algebra $A_v$ has the real shadow $\mathbb{R}^{2w}$. On this real shadow take the Lebesgue measure $\mu$ so that $\mu(\prod_{j=1}^{2w} [a_j, a_j + 1]) = 1$ for each $a_j \in \mathbb{R}$. This measure induces the Lebesgue measure on $A_w$ denoted also by $\mu$. A subset $A$ in $A_w$ is called $\mu$-null if a Borel subset $B$ in $A_w$ exists so that $A \subseteq B$ and $\mu(B) = 0$. The Lebesgue measure is defined on the completion $B_\mu(A_w)$ of the Borel $\sigma$-algebra $B(A_w)$ on $A_w$ by $\mu$-null sets.

If $1 \leq p < \infty$, $L^p(A_w, X)$ will denote the norm completion of a space of all $\mu$-measurable step functions $f : A_w \rightarrow X$ for which the norm

$$\|f\|_p := \sqrt[p]{\int_{A_w} \|f(z)\|_X^p \mu(dz)}$$

is finite, where

$$f = \sum_{k=1}^{n} b_k \chi_{B_k}$$

is a step function so that $b_k \in X$ and $B_k \in B_\mu(A_w)$ for each $k = 1, \ldots, n$; $B_k \cap B_j = \emptyset$ for each $j \neq k$; $\|x\|_X$ denotes the norm of a vector $x$ in $X$; certainly, the norm is non-negative, $0 \leq \|x\|_X$, $n$ is a natural number. If $p = \infty$, the norm is given by the formula:

$$\|f\|_\infty := \text{ess sup}_{z \in A_w} \|f(z)\|_X.$$

Then a Banach space $L_q(L^p(A_w, X))$ of all bounded $\mathbb{R}$-homogeneous $A_w$ additive operators $T : L^p(A_w, X) \rightarrow L^p(A_w, X)$ is considered. Let $K : A_w^2 \rightarrow L_q(X)$ be a strongly measurable operator valued mapping, that is a mapping
\[g(t, s) := K(t, s)y : A^2_v \to X \text{ is } (\mathcal{B}^2_v(A^2_v), \mathcal{B}(X)) \text{ measurable for each vector } y \in X, \text{ i.e. } g^{-1}(Q) \in \mathcal{B}^2_v(A^2_v) \text{ for each Borel subset } Q \in \mathcal{B}(X), \text{ where } \mu^2 \text{ is the Lebesgue measure on } A^2_w.\]

In the paper [27] the following theorem about first order partial differential operators with variable \(A_v\) coefficients was demonstrated.

**Theorem.** Suppose that a first order partial differential operator \(\Upsilon\) is given by the formula

\[(i) \quad \Upsilon f = \sum_{j=0}^n \left( \partial f / \partial z_j \right) \phi_j^* (z),\]

where \(\phi_j(z) \neq \{0\} \text{ for each } z \in U \text{ and } \phi_j(z) \in C^0(U, A_v) \text{ for each } j = 0, \ldots, n \text{ such that } \text{Re} \left( \phi_j(z) \phi_k^* (z) \right) = 0 \text{ for each } z \in U \text{ and each } 0 \leq j \neq k \leq n, \text{ where a domain } U \text{ satisfies Conditions 2.1.1(D1, D2), } a \text{ is a marked point in } U, 1 < n < 2^v, 2 \leq v. \text{ Suppose also that a system } \{\phi_0(z), \ldots, \phi_n(z)\} \text{ is for } n = 2^v - 1, \text{ or can be completed by Cayley-Dickson numbers } \phi_{n+1}(z), \ldots, \phi_{2^v-1}(z), \text{ such that (a) } \text{alg}_{\mathbb{R}} \{\phi_j(z), \phi_k(z), \phi_l(z)\} \text{ is alternative for all } 0 \leq j, k, l \leq 2^v - 1 \text{ and (b) } \text{alg}_{\mathbb{R}} \{\phi_0(z), \ldots, \phi_{2^v-1}(z)\} = A_v \text{ for each } z \in U. \text{ Then a line integral } \mathcal{I}_\Upsilon : C^0(U, A_v) \to C^1(U, A_v), \mathcal{I}_\Upsilon f(z) := \Upsilon \int_0^z f(y)dy \text{ on } C^0(U, A_v) \text{ exists so that}

\[(ii) \quad \mathcal{I}_\Upsilon f(z) = f(z)\]

for each \(z \in U; \text{ this anti-derivative is } \mathbb{R}\text{-linear (or } \mathbb{H}\text{-left-linear when } v = 2). \text{ If there is a second anti-derivative } \mathcal{I}_{\Upsilon, 2} f(z), \text{ then } \mathcal{I}_\Upsilon f(z) - \mathcal{I}_{\Upsilon, 2} f(z) \text{ belongs to the kernel } \ker(\Upsilon) \text{ of the operator } \Upsilon.\]

For a first order partial differential operator \(\sigma = \Upsilon\) over \(A_w\) with constant or variable coefficients consider the antiderivative operator \(\sigma f\) on \(A_w\). Put

\[
(3) \quad (Bx)(t) := \sigma \int_{-\infty}^{\infty} K(t, s)x(s)ds
\]

whenever this integral converges in the weak sense as

\[
(4) \quad \sigma \int_{-\infty}^{\infty} u[K(t, s)x(s)ys] := \lim_{a \to a_0, b \to b_0} \sigma \int_{\gamma^u[a, b]} u[K(t, s)x(s)ys]ds \in A_v
\]

for each \(y \in X\) and right \(A_v\) linear continuous functional \(u \in L_r(X, A_v) = X_r^*, \text{ where } x \in L_q(L^p(A_w, X), L^p(A_w, X)) \text{ so that } (xf)(s) = x(s)f(s) \text{ for each } f \in L^p(A_w, X), \text{ } x(s) \in L_q(X) \text{ for every } s \in A_w,

\[
\lim_{a \to a_0} \gamma^a(t) = \infty \text{ and } \lim_{b \to b_0} \gamma^b(t) = \infty,
\]

\[3\]
\(a_a < b_\alpha\), \(\hat{A}_w\) is the one-point (Alexandroff) compactification of the Cayley-Dickson algebra as the topological space, \(\infty = \hat{A}_w \setminus A_w\), \(\alpha \in \Lambda\). The integral in Formula (4) reduces to the integral described in \(\S 4.2.5\) [22]. Consider a periodic integral kernel

\[
(5) \ K(t, s) = K(t + p_j\omega ji_j, s + p_j\omega ji_j),
\]
also \(\phi_j(s + p_j\omega ji_j) = \phi_j(s)\) for \(\mu\) almost every Cayley-Dickson numbers \(t, s \in A_w\) for all integers \(p_j\), where \(\omega_j > 0\) is a period by \(z_j\), \(z = z_0i_0 + \ldots + z_{2^w-1}i_{2^w-1} \in A_w\), \(z_0, \ldots, z_{2^w-1} \in \mathbb{R}\); \(i_0, \ldots, i_{2^w-1}\) denote the standard generators of \(A_w\). Suppose that a foliation of \(A_w\) by paths \(\gamma^\alpha\) is so that

\[
(6) \ K(t, s) = K(\gamma^\alpha(t + p^\alpha_\omega^\alpha), \gamma^\alpha(\kappa + p^\alpha_\omega^\alpha)) = K(\gamma^\alpha(t), \gamma^\alpha(\kappa)),
\]
also \(\phi_j(\gamma^\alpha(\kappa + p^\alpha_\omega^\alpha)) = \phi_j(\gamma^\alpha(\kappa))\)
for \(\mu\) almost all \(t = \gamma^\alpha(t)\) and \(s = \gamma^\alpha(\kappa)\) in \(A_w\) for each \(\alpha\) with \(\omega^\alpha > 0\) and all integers \(p^\alpha\). Let \(S(\omega)\) denote the shift operator

\[
(7) \ S(\nu)x(t) = x(t + \nu)
\]
on a Cayley-Dickson number \(\nu \in A_w\), where \(t \in A_w\).

For example, a foliation \(\{\gamma^\alpha : \alpha\}\) of \(A_w\) may be done by straight lines parallel to \(\mathbb{R}i_0\) indexed by \(\alpha \in \Lambda = \{z \in A_w : Re(z) = 0\}\).

2. Definition. An operator \(x \in L_q(L^p(A_w, X), L^p(A_w, X))\) so that \((xf)(t) = x(t)f(t)\) and \(x(t) \in L_q(X)\) for every \(t \in A_w\) we shall call periodic, if

\[
(1) \ S(\omega ji_j)x(t)f(t) = x(t)S(\omega ji_j)f(t)
\]
for each \(t \in A_w\) and \(f \in L^p(A_w, X)\) and every \(j\), where \(X\) is a Banach space over the Cayley-Dickson algebra \(A_w\) with \(2 \leq v\). A set \(\{\omega_j : j = 0, \ldots, 2^w-1\}\) will be called a net of periodic values. If for \(x\) exists a set of positive minimal periodic values, then it will be call a set of periods.

3. Definition. Let \(Y\) be a Banach space over the Cayley-Dickson algebra \(A_w\), where \(2 \leq v\). Put \(l_\infty(\mathbb{Z}, Y) := \{x : x : \mathbb{Z} \rightarrow Y, \|x\|_\infty := \sup_{k \in \mathbb{Z}} \|x(k)\|_Y < \infty\}\) and \(l_p(\mathbb{Z}, Y) := \{x : x : \mathbb{Z} \rightarrow Y, \|x\|_p := [\sum_{k \in \mathbb{Z}} \|x(k)\|_Y^p]^{1/p} < \infty\}\) to be the Banach spaces of norm \(\|\cdot\|_p\) bounded sequences and with values in \(Y\), where \(\mathbb{Z}\) denotes the ring of all integers, \(1 \leq p < \infty\).

A sequence \(\{x_n : n \in \mathbb{N}\} \subset l_\infty(\mathbb{Z}, Y)\) is called \(c\)-convergent to an element
if for each integer \( k \in \mathbb{Z} \) the limit is zero:

\[
\lim_{n \to \infty} \| x_n(k) - x(k) \| = 0.
\]

4. Definition. An operator \( B \in L_q(\ell_\infty(\mathbb{Z}, Y)) \) will be called \( c \)-continuous, if an image \( \{ Bx_n : n \in \mathbb{N} \} \) of each \( c \)-convergent sequence \( \{ x_n : n \in \mathbb{N} \} \) is a \( c \)-convergent sequence, where \( L_q(X) \) is written shortly for \( L_q(X, X) \), in particular for \( X = \ell_\infty(\mathbb{Z}, Y) \), \( Y \) is a Banach space over the Cayley-Dickson algebra \( \mathcal{A}_v \), \( 2 \leq v \). The family of all \( c \)-continuous operators will be denoted by \( L_c^q(\ell_\infty(\mathbb{Z}, Y)) \).

5. Lemma. The family \( L_c^q(\ell_\infty(\mathbb{Z}, Y)) \) from Definition 4 is a closed sub-algebra over the Cayley-Dickson algebra \( \mathcal{A}_v \) in \( L_q(\ell_\infty(\mathbb{Z}, Y)) \) relative to the operator norm topology.

Proof. Evidently, the unit operator \( I \) is \( c \)-continuous. Definitions 3 and 4 imply that \( L_c^q(\ell_\infty(\mathbb{Z}, Y)) \) is a subalgebra in \( L_q(\ell_\infty(\mathbb{Z}, Y)) \). Take an arbitrary sequence \( B_n \) of \( c \)-continuous operators converging to an operator \( B \in L_q(\ell_\infty(\mathbb{Z}, Y)) \) relative to the operator norm. Let \( x_n \) be an arbitrary \( c \)-converging sequence to \( x \) in \( \ell_\infty(\mathbb{Z}, Y) \). From the \( c \)-continuity of an operator \( B_n \) it follows that for each \( \epsilon > 0 \) and \( j \in \mathbb{Z} \) there exists a natural number \( m \) so that \( \|(B_n(x_k - x))(j)\| < \epsilon/2 \) for each \( k > m \). On the other hand, the triangle inequality gives:

\[
\|(B(x_k - x))(j)\| \leq \|(B_n - B)(x_k - x)))(j)\| + \|(B_n(x_k - x))(j)\| \\
\leq \|(B_n - B)(x_k - x)))(j)\| + \epsilon/2.
\]

The sequence \( B_n \) is norm convergent, hence there exists a natural number \( l \in \mathbb{N} \) such that \( \|(B(x_k - x))(j)\| < \epsilon \) for each \( k > l \), since the limit \( \lim_{k \to \infty} x_k(j) = x(j) \) exists for each \( j \) with \( x \in \ell_\infty(\mathbb{Z}, Y) \) and \( \sup_k \|(x_k - x)(j)\|_Y < \infty \). Thus the sequence \( Bx_k \) is \( c \)-convergent:

\[
\lim_k Bx_k(j) = Bx(j) \quad \text{for each } j,
\]

hence this operator \( B \) is \( c \)-continuous.

6. Definitions. Let \( e_k \in l_p(\mathbb{Z}, \mathcal{A}_v) \) be basic elements so that \( e_k(j) = \delta_{k,j} \), where \( \delta_{k,j} = 0 \) for each \( j \neq k \in \mathbb{Z} \), while \( \delta_{j,j} = 1 \) for every \( j \in \mathbb{Z} \). For an operator \( B \in L_q(l_p(\mathbb{Z}, Y)) \) with \( 1 \leq p \leq \infty \) and a Banach space \( Y \) over the Cayley-Dickson algebra \( \mathcal{A}_v \), let
for each vector $y \in Y$, where $e_k y \in l_p(Z, Y)$ with $(e_k y)(j) = \delta_{k,j} y$ for each $j$.

This set of operators $\{B_{j,k} : j, k \in Z\}$ is called a matrix of an operator $B$.

An $A_v$ Banach subspace in $l_\infty(Z, Y)$ of all two-sided sequences converging to zero $\lim_{|k|\to \infty} x(k) = 0$ is denoted by $c_0(Z, Y)$.

**7. Lemma.** Suppose that $B \in L^c_q(l_\infty(Z, Y))$ is a $c$-continuous non-zero operator or $B \in L_q(l_p(Z, Y))$, where $1 \leq p < \infty$, $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then its matrix is non-zero and bounded. Moreover, $Bc_0(Z, Y) \subset c_0(Z, Y)$ for each $B \in L_q(l_\infty(Z, Y))$.

**Proof.** For each $B \in L_q(l_p(Z, Y))$ we have the estimate:

$$\|B_{j,k} y\|_Y = \|(Be_k y)(j)\|_Y \leq \|Be_k y\|_p \leq \|B\|\|l_q(l_p(Z, Y))\|_p\|y\|_Y$$

for each integers $j, k \in Z$ and every vector $y \in Y$, hence $\sup_{j,k} \|B_{j,k}\| \leq \|B\|l_q(l_p(Z, Y))$. Particularly, for $B \in L_q(l_\infty(Z, Y))$ and $x \in c_0(Z, Y)$ this implies that

$$\|(B - \sum_{|k| \leq n} e_k x(k))(j)\|_Y \leq \|B\|l_q(l_\infty(Z, Y))\|\sum_{|k| \leq n} e_k x(k))(j)\|_Y.$$ 

But for each $\epsilon > 0$ the set $\{k : \|x(k)\|_Y > \epsilon\}$ is finite, since $x \in c_0(Z, Y)$, consequently,

$$\lim_{n \to \infty} \|x - \sum_{|k| \leq n} e_k x(k)\|_Y = 0$$

and hence $Bx \in c_0(Z, Y)$.

Suppose that $B \in L^c_q(l_\infty(Z, Y))$ has a zero matrix $\{B_{j,k} : j, k \in Z\}$. Take an arbitrary vector $x \in l_\infty(Z, Y)$ and a sequence $\{x_n : n \in Z\} \subset l_\infty(Z, Y) \cap c_0(Z, Y)$ so that it $c$-converges to $x$. In the Banach space $c_0(Z, Y)$ with norm $\|x_n\|\infty$ the set of vectors $\{e_k y : y \in Y\}$ is everywhere dense. Since $\{B_{j,k} : j, k \in Z\} = 0$, the restriction $B|_{c_0(Z, Y)}$ is zero and $Bx_n = 0$ for each $n$. Thus the sequence $Bx_n$ does not converge to $Bx$. This produces the contradiction. Therefore, the matrix $\{B_{j,k} : j, k \in Z\}$ is non-zero.

In the space $l_p(Z, Y)$ with $1 \leq p < \infty$ a subset of finite two-sided sequences in $Y$ is dense, hence a non-zero operator $B \in L_q(l_p(Z, Y))$ has a non-zero matrix $\{B_{j,k} : j, k \in Z\}$. 

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8. Definition. Let Λ be a Banach algebra with unit over the Cayley-Dickson algebra $A_v$, $2 \leq v$. A subalgebra Ξ over $A_v$ with unit is called saturated if each element of Ξ invertible in Λ is invertible in Ξ as well.

9. Remark. Henceforth in this section it will be denoted for short by $l(Z,Y)$ each of the Banach spaces $c_0(Z,Y)$ and $l_p(Z,Y)$ with $p \in [1, \infty]$ if something other will not specified. Henceforward operators from $L_q^c(l_\infty(Z,Y))$ will be considered.

Let $g \in l_\infty(Z,A_v)$, then an operator $D(g)$ is defined by the formula:

$$(1) \ (D(g)x)(k) := g(k)x(k)$$

for each $x \in l_\infty(Z,Y)$. The family of all such operators is denoted by $\mathcal{E}$.

The family of all operators $A \in L_q^c(l_\infty(Z,Y))$ quasi-commuting with each $D(g) \in \mathcal{E}$ is denoted by $L_q^s(l_\infty(Z,Y))$, that is

$$(2) \ jA^k D(g) = (-1)^{\kappa(j,k)} k D(g) jA$$

for each $j, k = 0, 1, 2, ...$ (see §§II.2.1 [28], 2.5 and 2.23 [29]).

Evidently, this subalgebra $L_q^s(l_\infty(Z,Y))$ is saturated in the operator algebra $L_q^c(l_\infty(Z,Y))$ with unit operator $I$ as the unit of these algebras.

10. Definition. Take a Cayley-Dickson number of absolute value one $M \in S_v := \{z : z \in A_v, |z| = 1\}$ and put $M = \{M^n : n \in \mathbb{Z}\} \in l_\infty(Z,A_v)$, where $2 \leq v$.

An operator $B \in L_q^c(l_\infty(Z,Y))$ is called diagonal if it quasi-commutes with $D(M) \in \mathcal{E}$ for each $M \in S_v$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. The family of all diagonal operators we denote by $L_q^d(l_\infty(Z,Y))$.

11. Proposition. The family of all diagonal operators $L_q^d(l_\infty(Z,Y))$ is a saturated subalgebra in the algebra $L_q^c(l_\infty(Z,Y))$.

Proof. If $A, B \in L_q^d(l_\infty(Z,Y))$, then equations 9(2) for $A$ and $B$ and $D(M)$ imply

$$(1) \ (jA+jB)^k D(M) = jA^k D(M) + jB^k D(M) = (-1)^{\kappa(j,k)} k D(M) (jA+jB)$$

for each $j, k = 0, 1, 2, ...$ and every $M \in S_v$ due to distributivity of the operator multiplication (see also §3.3.1). If $z \in A_v$, then $zA$ has components:

$$(2) \ j(zA) = \sum_{s,l; i_s i_l = i_j} (z_s i_s^l A + (-1)^{\kappa(s,l)} z_l i_l^s A)$$

7
for each \( j \), where as usually \( z = z_0 i_0 + z_1 i_1 + \ldots \) with \( z_0, z_1, \ldots \in \mathbb{R} \). Therefore, \( zA \) quasi-commutes with each \( D(\mathcal{M}) \) for each \( z \in \mathcal{A}_v \). Analogously it can be demonstrated for \( Az \). Thus \( L^d_q(l_\infty(Z, Y)) \) is an algebra.

An inverse \( A^{-1} \in L^d_q(l_\infty(Z, Y)) \) of an operator \( A \in L^d_q(l_\infty(Z, Y)) \) quasi-commuting with \( D(\mathcal{M}) \) also quasi-commutes with \( D(\mathcal{M}) \), since \( D(\mathcal{M}) \) is invertible and \( (AD(\mathcal{M}))^{-1} = (D(\mathcal{M}))^{-1}A^{-1} = D(\mathcal{M}^* )A^{-1} \) and \( (D(\mathcal{M})A)^{-1} = A^{-1}D(\mathcal{M}^* ) \). \( Ax = y \) implies \( x = A^{-1}y \),

\[
I = (AD(\mathcal{M}))^{-1}(AD(\mathcal{M})) =
\]

\[
(AD(\mathcal{M}))^{-1} \sum_{j; \: i_\ast t_j = i_j} [^* A \: ^t D(\mathcal{M}) + (-1)^{s(l,j)} \: ^t A \: ^s D(\mathcal{M})]
\]

\[
= (AD(\mathcal{M}))^{-1} \sum_{j; \: i_\ast t_j = i_j} [^s D(\mathcal{M}) \: ^t A + (-1)^{s(l,j)} \: ^t D(\mathcal{M}) \: ^s A]
\]

\[
= \sum_{j; \: i_p i_q = i_l; \: i_s t_l = i_j} \{[^p D(\mathcal{M}^* ) \: ^s A^{-1} + (-1)^{\kappa(p,q)} \: ^q D(\mathcal{M}^* ) \: ^p A^{-1}]}
\]

\[
[^* D(\mathcal{M}) \: ^t A + (-1)^{s(l,j)} \: ^t D(\mathcal{M}) \: ^s A] \},
\]

where \( \mathcal{M}^* \) corresponds to \( M^* = \hat{M} \).

12. **Definition.** A sequence \( g \in l_\infty(Z, \mathcal{A}_v) \) is called periodic of period \( k \in \mathbb{N} \) if \( g(n+k) = g(n) \) for each integer \( n \in \mathbb{Z} \), where \( Y \) is a Banach space over the Cayley-Dickson algebra \( \mathcal{A}_v \), \( 2 \leq v \).

13. **Lemma.** Diagonal operators from \( L^d_q(l_\infty(Z, Y)) \) quasi-commute with operators of multiplication on periodic sequences \( g \in l_\infty(Z, \mathcal{A}_v) \), where \( 2 \leq v \).

**Proof.** Let \( A \in L^d_q(l_\infty(Z, Y)) \) be a diagonal operator and \( g \in l_\infty(Z, \mathcal{A}_v) \) be a periodic sequence of period \( k \in \mathbb{N} \). Put \( \theta_j := \exp(2\pi i j/k) \), where \( j = 0, 1, \ldots, k-1 \), \( i \) is an additional purely imaginary generator so that \( i^2 = -1, \: i_l = i i_l \) for each \( l \geq 0 \).

A minimal real algebra with basis of generators \( i_0, i_1, \ldots, i_{2v-1}, i, i i_1, \ldots, i i_{2v-1} \) and their relations as above is the complexification \( (\mathcal{A}_v)_C \) of the Cayley-Dickson algebra \( \mathcal{A}_v \), where \( C_1 = R \oplus Ri \). Then \( g \) can be presented in the form:

\[
(1) \quad g(n) = \sum_{j=0}^{k-1} c_j \theta_j^n, \quad \text{where}
\]

\[
(2) \quad c_j = \frac{1}{k} \sum_{n=0}^{k-1} g(n) \theta_j^n
\]
are \((A_v)_C\) Fourier coefficients for each \(j = 0, 1, \ldots, k - 1\). Indeed, \(i(x_ji_j) = -x_ji_j\) for each \(j\) and \(x_j \in Y_j\), while \((a + b)c = ac + bc\) and \(c(a + b) = ca + cb\) for each \(c \in (A_v)_C\) and \(a, b \in Y \oplus Yi\), where we put \(ix = xi\) for each \(x \in Y\).

Put \(\|x + yi\|^2 = \|x\|^2 + \|y\|^2\) for each \(x, y \in Y\) and \(x + yi \in Y \oplus Yi\). Therefore,

\[
\sum_{j=0}^{k-1} c_j \theta_j^m = \frac{1}{k} \sum_{j=0}^{k-1} \left[ \sum_{n=0}^{k-1} g_l(n)i_l \hat{\theta}_j^n \right] \theta_j^m
\]

\[
= \sum_l g_l(n) \frac{1}{k} \sum_{j=0}^{k-1} \sum_{n=0}^{k-1} i_l \theta_j^m - n = g(m),
\]

since the multiplication in the Cayley-Dickson algebra is distributive, where \(g(m) = \sum_l g_l(m)i_l\) with \(g_l(m) \in \mathbb{R}\) for each \(l = 0, 1, 2, \ldots,\)

\[
\frac{1}{k} \sum_{j=0}^{k-1} \exp(2\pi ji(m - n)/k) = \delta_{n,m}.
\]

Therefore, the diagonal operator has the decomposition

\[
(D(g)x)(n) = g(n)x(n) = \sum_{j=0}^{k-1} (c_j I)D(\hat{\theta}_j)x(n)
\]

that is

\[
(D(g) = \sum_{j=0}^{k-1} (c_j I)D(\hat{\theta}_j)
\]

again due to distributivity of the multiplication in the Cayley-Dickson algebra \(A_v\) (see §2.5 [29]). On the other hand, \(A\) and \(D(\hat{\theta}_j)\) quasi-commute:

\[
^jA^kD(\hat{\theta}_j) = (-1)^{\alpha(j,k)} D(\hat{\theta}_j) ^jA
\]

for each \(j, k\) and from Proposition 11 it follows that \(A\) and \(D(g)\) quasi-commute, where \(k D(\hat{\theta}_j)(n) \in C_i \ell_k\) for each \(n \in \mathbb{Z}\).

\textbf{14. Corollary.} Diagonal operators from \(L_q^d(l_\infty(\mathbb{Z}, Y))\) quasi-commute with multiplication operators \(D(g)\) on periodic sequences \(g \in l_\infty(\mathbb{Z}, A_v)\), where \(2 \leq v\).

\textbf{15. Definition.} If \(x \in l_p(\mathbb{Z}, Y)\), where \(p \in [1, \infty]\), \(Y\) is a Banach space over the Cayley-Dickson algebra \(A_v\), \(2 \leq v\), then its support is \(\text{supp } x := \{n : n \in \mathbb{Z}; x(n) \neq 0\}\).

\textbf{16. Lemma.} Let \(B \in L_q^d(l_\infty(\mathbb{Z}, Y))\) be a diagonal operator and let \(x \in l_\infty(\mathbb{Z}, Y)\) be with finite support, where \(Y\) is a Banach space over the Cayley-Dickson algebra \(A_v\), \(2 \leq v\). Then \(\text{supp } Bx \subseteq \text{supp } x\).
Proof. If the support of \( x \) is finite, then there exists a natural number \( N \in \mathbb{N} \) so that \( \text{supp} \ x \subseteq [−N,N] \). Consider natural numbers \( m > N \) and \( n \in [−N,N] \). Put \( g(n + 2mk) = 1 \) if \( n \in \text{supp} \ x \), while \( g(n + 2mk) = 0 \) for \( n \in [−N,N] \setminus \text{supp} \ x \), where \( k \in \mathbb{Z} \), hence \( g \in l_{∞}(\mathbb{Z},A) \) is of period \( 2m \). Therefore, \( D(g)x = x \) by the construction of \( g \). In view of Corollary 14 \( G(g) \) and \( B \) quasi-commute. On the other hand, \( g(l) = 0 \) for each \( l \in [−m,m] \setminus \text{supp} \ x \), consequently, \( (Bx)(l) = 0 \), since \( Bx = BD(g)x \). Thus \( (Bx)(l) = 0 \) for each \( l \notin \text{supp} \ x \), since \( m > N \) is arbitrary.

17. Theorem. The algebras \( L^d_q(l_{∞}(\mathbb{Z},Y)) \) and \( L^d_q(l_{∞}(\mathbb{Z},Y)) \) over the Cayley-Dickson algebra \( A \), coincide, where \( 2 \leq v \).

Proof. From the definitions it follows that the algebra \( L^d_q(l_{∞}(\mathbb{Z},Y)) \) is a subalgebra of the algebra \( L^d_q(l_{∞}(\mathbb{Z},Y)) \) of diagonal continuous operators. Therefore, it remains to prove the inclusion \( L^d_q(l_{∞}(\mathbb{Z},Y)) \subseteq L^d_q(l_{∞}(\mathbb{Z},Y)) \).

Consider arbitrary continuous diagonal operator \( B \in L^d_q(l_{∞}(\mathbb{Z},Y)) \) and an element \( g \in l_{∞}(\mathbb{Z},A) \). Take \( x \in l_{∞}(\mathbb{Z},Y) \) so that \( x_n = 0 \) for each \( |n| > N \), where \( N \) is a natural number. We have \( jD(g)Bx = (-1)^{\kappa(j,k)} k B^j D(g)x \) for each \( j, k \). Extend the sequence \( g \) periodically to \( h \) so that \( h(−N) = g(N+1) \) and \( h(m) = g(m) \) for each \( m \in [−N,N] \), consequently, \( (D(g)x)(m) = (D(h)x)(m) \) for each \( m \in [−N,N] \). In view of Corollary 14

\[
(iB^k D(g)x)(m) = (iB^k D(h)x)(m) = (-1)^{\kappa(j,k)} (kD(h)^j Bx)(m)
\]

\[
= (-1)^{\kappa(j,k)} (kD(g)^j Bx)(m)
\]

for each \( j, k \) and \( m \in [−N,N] \). Applying Lemma 16 for each \( |m| > N \) we get \( (Bx)(m) = 0 \) and hence

\[
(iB^k D(g)x)(m) = (-1)^{\kappa(j,k)} (kD(g)^j Bx)(m)
\]

for each \( j, k = 0,1,2,... \) and every integer number \( m \in \mathbb{N} \). Thus for sequences with finite supports this theorem is accomplished. But a set of all sequences with finite support is dense in \( c_0(\mathbb{Z},Y) \) and in \( l_p(\mathbb{Z},Y) \) for every \( 1 \leq p < ∞ \). Therefore, the statement of this theorem is valid on these spaces. In accordance with Lemma 7 the conjecture spreads on a \( c \)-continuous operator \( B \) from \( c_0(\mathbb{Z},Y) \) on the entire Banach space \( l_{∞}(\mathbb{Z},Y) \).
18. **Definition.** Let \( g = (G^m : m \in \mathbb{Z}) \) be a sequence belonging to the Banach space \( l_\infty(\mathbb{Z}, L_q(Y)) \), where \( Y \) is a Banach space over the Cayley-Dickson algebra \( \mathcal{A}_v \), \( 2 \leq v \). An operator \( D(g) \in L^c_\infty(l_\infty(\mathbb{Z}, Y)) \) will be defined by the formula: \( (D(g)x)(m) = G^m x(m) \) for each integer number \( m \). A set of all (left) multiplication operators on bounded operator valued sequences forms an algebra over the Cayley-Dickson algebra \( \mathcal{A}_v \), which will be denoted by \( L^b_q(l_\infty(\mathbb{Z}, Y)) \).

An operator \( B \in L^c_q(l_\infty(\mathbb{Z}, Y)) \) is called a \((k, n)\) ribbon operator with \( k \in \mathbb{N} \) and \( n \in \mathbb{Z} \) if \( B_{s,l} = 0_Y \) is the zero operator from \( L_q(Y) \) for each \(|s - l + n| \geq k\), where \( s, l \in \mathbb{Z} \). Their family is denoted by \( L^{(k, n)}_q(l_\infty(\mathbb{Z}, Y)) \).

A \((k, 0)\) ribbon operator is called \( k\)-ribbon (single ribbon for \( k = 1 \)).

Two propositions follow immediately from the latter definition.

19. **Proposition.** An operator \( B \in L^c_q(l_\infty(\mathbb{Z}, Y)) \) is \((1, 0)\) ribbon if and only if it is a multiplication operator on operator valued sequence.

20. **Proposition.** The algebra \( L^{(k, 0)}_q(l_\infty(\mathbb{Z}, Y)) \) is the saturated subalgebra of the algebra \( L^c_q(l_\infty(\mathbb{Z}, Y)) \) over the Cayley-Dickson algebra \( \mathcal{A}_v \).

21. **Theorem.** Let \( B \in L^c_q(l_\infty(\mathbb{Z}, Y)) \), where \( Y \) is a Banach space over the Cayley-Dickson algebra \( \mathcal{A}_v \), \( 2 \leq v \). An operator \( B \) is \((1, 0)\) ribbon if and only if \( B \) is diagonal.

**Proof.** Suppose that \( B \) is a diagonal operator. Take a vector \( x \in Y \) and an element \( y^* = e_s x \in l_\infty(\mathbb{Z}, Y) \), where \( y^*(k) = \delta_{s,k} x \), hence \( B_{m,s}(By)(m) \in Y \), where \( s, m \in \mathbb{Z} \), \( B_{s,m} \) are elements of the matrix of \( B \). For \( M \in S_v \) matrix elements of the operator \( D(\mathcal{M}^*)BD(\mathcal{M}) \) are prescribed by the formula:

\[
(D(\mathcal{M}^*)BD(\mathcal{M})y^*)(m) = M^{-m}(BM^s y^*)(m)
\]

for each \( s, m \in \mathbb{Z} \).

Take any purely imaginary generator \( i_p \) of the Cayley-Dickson algebra and put \( M = i_p \) with \( p \geq 1 \). As an operator \( B \) is diagonal, the equalities follow:

\[
(kBy^*)(m) = (-1)^{\kappa(p,k)} \eta(s) i_p^{s-m}(kBy^*)(m)
\]

for each \( m, s \), where \( \eta(s) = 0 \) for \( s \) even, while \( \eta(s) = 1 \) for \( s \) odd. This implies that \( B_{m,s} = 0_Y \) for \( s \neq m \) and \( B_{m,s} = B_{m,m} \) for \( m = s \). Thus the operator \( B \) is \((1, 0)\) ribbon.
The inverse conjecture follows from Lemma 7 and Definition 18.

22. **Corollary.** An operator \( B \in L_q^c(l_\infty(Z,Y)) \) is diagonal if and only if \( B \) is an operator of multiplication on a bounded operator valued sequence in \( L_q(Y) \).

23. **Definition.** An operator \( B \in L_q(l_\infty(Z,Y)) \) is called uniformly \( c \)-continuous, if a mapping \( \bar{B} : S^1 \to L_q(l_\infty(Z,Y+Yi)) \) is continuous relative to the operator norm topology on \( L_q(l_\infty(Z,Y+Yi)) \) and a topology on \( S^1 := \{ z : z \in C_i; \ |z| = 1 \} \) induced by the norm on the complex field \( C_i \), where \( 2 \leq v, \ \bar{B}(M) := D(M)BD(M^*) \) for each \( M \in S^1 \), \( D(M) \) is a diagonal (left) multiplication operator on a sequence \( M(k) = M^k, \ Y \) is a Banach space over \( A_v \). The family of all uniformly \( c \)-continuous operators is denoted by \( L_q^{uc}(l_\infty(Z,Y)) \) and is supplied with the uniform operator norm topology

\[
\|B\|_u := \sup_{M \in S^1} \|\bar{B}(M)\|.
\]

The real field is the center of the Cayley-Dickson algebra \( A_v \) with \( v \geq 2 \), hence the generator \( i \) can be realized as the real matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). If \( X \) and \( Y \) are two \( A_v \) vector spaces and \( B : X \to Y \) is a real homogeneous \( A_v \) additive operator, then it has a natural extension \( B : X_1 \to Y_1 \) so that \( B(a+bi) = (Ba)+(Bb)i \) for each vectors \( a,b \in X \), where \( X_1 \) is obtained form \( X \) by extending the algebra \( A_v \) to \( (A_v)_{C_i} \) and \( X_1 \) can be presented as the direct sum \( X_1 = X \oplus Xi \) of two \( A_v \) vector spaces \( X \) and \( Xi \). It is convenient to denote \( B \) also by \( B \) on \( X_1 \).

24. **Proposition.** The family \( L_q^{uc}(l_\infty(Z,Y)) \) is a normed algebra over the Cayley-Dickson algebra \( A_v \) (see §23). If an operator \( B \in L_q^c(l_\infty(Z,Y)) \) is \( k \)-ribbon, then this operator \( B \) is uniformly \( c \)-continuous.

**Proof.** In algebra \( L_q(X) \) for a Banach space \( X \) over the Cayley-Dickson algebra \( A_v \), the operator norm satisfies the inequality: \( \|AB\| \leq \|A\|\|B\| \), particularly for \( X = l_\infty(Z,Y) \) or \( X = l_\infty(Z,Y+Yi) \). Therefore, for each uniformly \( c \)-continuous operators \( A \) and \( B \) the inequality follows:

\[
\|AB\|_u = \sup_{M \in S^1} \|D(M)ABD(M^*)\| = \sup_{M \in S^1} \|D(M)AD(M^*)D(M)BD(M^*)\|
\leq \sup_{M \in S^1} \|D(M)AD(M^*)\| \sup_{M \in S^1} \|D(M)BD(M^*)\| = \|A\|_u \|B\|_u,
\]

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since set theoretic composition of operators is associative and $D(M)D(M^*) = I$ for each $M \in S^1$ (see [32]). On the other hand, $\|B\|_u \geq \|B\|$ for each $B \in L^u_q(l_\infty(Z,Y))$, since $D(I) = I$ and $1 \in S^1$, where $I = (... ,1,1,...)$ corresponds to 1.

For a $k$ ribbon operator $B$ a finite sequence $nB$ of single ribbon operators exists with $|n| \leq k$ so that

$$B = \sum_{n=-k}^{k} nBS(n),$$

where $S(n)$ denotes a shift operator on $n$, $(S(n)g)(m) := g(m+n)$ for each $n, m \in Z$ and $g \in l_\infty(Z,Y)$. Therefore, the equality follows:

$$(\tilde{B}(M)x)(j) = \sum_{n=-k}^{k} [(M^{j+n}I)_nB((M^*)^nI)x](j+n),$$

but $\|[(M^{j+n}I)_nB((M^*)^nI)y]\| \leq \|nBy\|$ for each $y \in Y$ and $M \in S^1$, consequently, $\|B\|_u \leq \sum_{n=-k}^{k} \|nB\| < \infty$.

**25. Lemma.** Let $B \in L^u_q(l_\infty(Z,Y))$ be a uniformly $c$ continuous operator and $(B_{s,p})$ be its matrix, where $s, p \in Z$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then for each $M \in S^1$ matrix elements of an operator $\tilde{B}(M)$ have the form:

$$\tilde{B}(M)_{s,p} = (M^sI)B_{s,p}(M^{-p}I).$$

**Proof.** In accordance with Definition 6 the equalities are valid:

$$(\tilde{B}(M)_{s,p}) = ((D(M)BD(M^*))_{s,p}x = ((D(M)BD(M^*))e_px)(s) = ((M^sI)B(M^{-p}I)e_px)(s)$$

for all integer numbers $s, p \in Z$ and for each vector $x \in Y$, since $M \in S^1$ implies $|M|^2 = MM^* = M^*M = 1$ and hence $M^{-1} = M^*$. 

**26. Proposition.** Let $A, B \in L^u_q(l_\infty(Z,Y))$ be two uniformly $c$-continuous operators, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then $(AB)(M) = \tilde{A}(M)\tilde{B}(M)$ for each $M \in S^1$.

**Proof.** The algebra of operators relative to the set-theoretic composition is evidently associative (see also [32]), hence

$$(\tilde{A}\tilde{B})(M) = D(M)\tilde{A}(M)\tilde{B}(M) = D(M)AD(M^*)BD(M^*)$$
since \( MM^* = M^*M = 1 \) and hence \( D(M)D(M^*) = D(M^*)D(M) \).

27. Proposition. Let \( B \in L_q^{uc}(l_\infty(Z,Y)) \) be a uniformly \( c \)-continuous invertible operator, where \( Y \) is a Banach space over the Cayley-Dickson algebra \( A_v \), \( 2 \leq v \). Then an operator \( \tilde{B}(M) \) is invertible so that \( \tilde{B}^{-1}(M) = (\tilde{B})^{-1}(M) \) for each \( M \in S^1 \).

Proof. Applying Proposition 26 with \( A = B^{-1} \) one gets \( (\tilde{B})^{-1}(M) = (D(M)BD(M^*))^{-1} = D(M^*)^{-1}B^{-1}D(M)^{-1} = D(M)B^{-1}D(M^*) = B^{-1}(M) \).

28. Lemma. Let \( B \in L_q^{uc}(l_\infty(Z,Y)) \) be a uniformly \( c \)-continuous operator, where \( Y \) is a Banach space over the Cayley-Dickson algebra \( A_v \), \( 2 \leq v \). Then there exists an equivalent norm on \( Y \) relative to an initial one so that \( \|B\| \geq \|\tilde{B}(M)\| \) for each \( M \in S^1 \).

Proof. The norm on the Cayley-Dickson algebra \( A_v \) satisfies the inequality \( |ab| \leq |a||b| \) for each \( a, b \in A_v \) with \( 2 \leq v \). Particularly, for \( v \leq 3 \) the norm on \( A_v \) is multiplicative.

Two norms \( \|\cdot\| \) and \( \|\cdot\|' \) on a Banach space \( Y \) are called equivalent if two positive constants \( 0 < c_1 \leq c_2 < \infty \) exist so that \( c_1\|x\| \leq \|x\|' \leq c_2\|x\| \) for each vector \( x \in Y \). Then \( \|ax\| \leq |a|\|x\| \) for each \( a \in A_v \) and \( x \in Y \) up to a topological isomorphism of Banach spaces, i.e. up to an equivalence of norms on \( Y \), since \( \|tx_ji_j\| = |t|\|x_j\| = \|tx_j\| \) for each \( x_j \in Y_j \) and \( j = 0, 1, 2, \ldots \). Indeed, the multiplication of vectors on numbers \( A_v \times Y \ni (a, x) \mapsto ax \in Y \) is continuous relative to norms on \( A_v \) and \( Y \). Therefore, \( \|\tilde{B}(M)x\| = \|D(M)BD(M^*)x\| \leq \|D(M)\|\|\tilde{B}\|\|D(M^*)\|\|x\| \), consequently, \( \|B\| \geq \|\tilde{B}(M)\| \) for each \( M \in S^1 \), since \( |M|^n = |M^n| = 1 \) for each integer \( n \) and hence \( \|D(M)\| = 1 \).

29. Definition. Let \( C_s(S^1, L_q^{uc}(l_\infty(Z, Y \oplus Y))) \) denote a Banach space of continuous bounded mappings from \( S^1 \) into \( L_q^{uc}(l_\infty(Z, Y \oplus Y)) \), where \( Y \) is a Banach space over the Cayley-Dickson algebra \( A_v \), \( 2 \leq v \).

30. Corollary. There exists an equivalent norm on a Banach space \( Y \) over the Cayley-Dickson algebra \( A_v \) such that the mapping \( F : L_q^{uc}(l_\infty(Z, Y)) \rightarrow C_s(S^1, L_q^{uc}(l_\infty(Z, Y \oplus Y))) \) given by the formula \( F(B)(M) = \tilde{B}(M) \) for each \( M \in S^1 \) is \( R \)-linear and \( A_v \)-additive and isometric operator.

Proof. This follows by combining Proposition 24 and Lemma 28.
31. Lemma. Let $B \in L^uc_q(l_\infty(Z,Y))$. Then an operator valued mapping $\tilde{B} : S^1 \rightarrow L^uc_q(l_\infty(Z,Y))$ has the Fourier series of the form:

$$\tilde{B} \sim \sum_{n=-\infty}^{\infty} M^n n \tilde{B}$$

for each $M \in S^1$, where

$$n \tilde{B} = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-nt} \tilde{B}(e^{it})dt$$

are Fourier coefficients. Moreover, each operator $n \tilde{B} S(-n)$ is diagonal.

Proof. From the definition of the uniformly $c$-continuous operator it follows that the restriction of $\tilde{B}$ on $S^1$ is continuous, since a mapping $\tilde{B} : S^1 \rightarrow L^uc_q(l_\infty(Z,Y))$ is continuous. The function $e^{it}$ has the period $2\pi$. Therefore, integrals (2) exist for every $n$ and a formal Fourier series (1) can be written. On the other hand, the algebra $alg_{R}(i_k,i_l)$ is associative for every $k$ and $l = 0,1,2,\ldots$. Therefore, using the distributivity law in the algebra $(A_v)_C$, we deduce that

$$M^{\pm k} B_{k,l} M^{\pm l} \sum_{p} x_p i_p = M^{\pm k} B_{k,l} \sum_{p} M^{\pm l} x_p i_p,$$

where $M \in S^1$, $x_p \in Y_p$ for each $p = 0,1,2,\ldots$. The algebra $alg_{R}(i,i_p,i_k)$ is associative for each $p,k$, consequently, the inversion formula (1) is valid, since

$$\frac{1}{2\pi} \int_{0}^{2\pi} e^{-nt} e^{niti} m \tilde{B} x dt = \delta_{m,n} m \tilde{B} x$$

for each vector $x \in Y$. Then one gets

$$D(M) \ n \tilde{B} S(-n) D(M^*) = D(M) \ n \tilde{B} D(M^*) D(M) S(-n) D(M^*)$$

$$= (M^n I) \ n \tilde{B} (M^{-n} I) (M^n I) S(-n) (M^{-n} I) = (M^n I) \ n \tilde{B} S(-n) (M^{-n} I)$$

for each $M \in S^1$ and every integer $n$, since the product of diagonal operators is diagonal.

32. Theorem. The algebra $L^uc_q(l_\infty(Z,Y))$ is the saturated subalgebra in $L^c_q(l_\infty(Z,Y))$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$.

Proof. The algebraic $R$ linear $A_v$ additive embedding $L^uc_q(l_\infty(Z,Y)) \hookrightarrow L^c_q(l_\infty(Z,Y))$ follows from the definitions. If a uniformly $c$-continuous operator $B$ on $l_\infty(Z,Y)$ is invertible in the algebra $L^c_q(l_\infty(Z,Y))$, then the mapping
$\tilde{B}^{-1}$ is continuous from $S^l$ into $L_q^c(l_\infty(Z, Y))$ due to Proposition 27. Thus $B^{-1} \in L_q^{c,\text{per}}(l_\infty(Z, Y))$.

33. Definition. An operator $B \in L_q^c(l_\infty(Z, Y))$ is called periodic of period $n$ on the Banach space $l_\infty(Z, Y)$ over the Cayley-Dickson algebra $A_v$ with $2 \leq v$ if $S(n)B = BS(n)$, where $n$ is a natural number, $(S(n)x)(k) = x(n+k)$ for each vector $x \in l_\infty(Z, Y)$ and every integer $k$. A set of $n$ periodic operators will be denoted by $L_n^{\text{per}}(l_\infty(Z, Y))$.

34. Proposition. A set $L_n^{\text{per}}(l_\infty(Z, Y))$ of $n$ periodic operators on $l_\infty(Z, Y)$ is a closed saturated subalgebra in the algebra $L_q^c(l_\infty(Z, Y))$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$.

Proof. If operators $A, B \in L_q^c(l_\infty(Z, Y))$ commute with $S(n)$, then

$$[(\alpha I)A + B(\beta I)]S(n) = (\alpha I)S(n)A + BS(n)(\beta I)$$

$$= S(n)(\alpha I)A + S(n)B(\beta I) = S(n)[(\alpha I)A + B(\beta I)]$$

for each Cayley-Dickson numbers $\alpha, \beta \in A_v$ and

$$ABS(n) = AS(n)B = S(n)AB.$$ 

Thus $L_n^{\text{per}}(l_\infty(Z, Y))$ is an algebra over the Cayley-Dickson algebra $A_v$.

From Definition 33 the algebraic $R$ linear $A_v$ additive embedding $L_n^{\text{per}}(l_\infty(Z, Y)) \hookrightarrow L_q^c(l_\infty(Z, Y))$ follows.

The relation $S(n)B - BS(n) = 0$ defines a closed subset in $L_q^c(l_\infty(Z, Y))$, since $S(n)$ is the bounded continuous operator on $l_\infty(Z, Y)$ and the mapping $f(B) := S(n)B - BS(n)$ is continuous from $L_q^c(l_\infty(Z, Y))$ into itself $L_q^c(l_\infty(Z, Y))$. Thus $L_n^{\text{per}}(l_\infty(Z, Y))$ is the closed subalgebra.

If an $n$ periodic operator $B \in L_n^{\text{per}}(l_\infty(Z, Y))$ is invertible in $L_q^c(l_\infty(Z, Y))$, then $B^{-1}S(n) = S(n)B^{-1}$, since $BS(n) = S(n)B \iff S(n) = B^{-1}S(n)B \iff S(n)B^{-1} = B^{-1}S(n)$. Thus $B^{-1} \in L_n^{\text{per}}(l_\infty(Z, Y))$ and hence the subalgebra $L_n^{\text{per}}(l_\infty(Z, Y))$ is saturated in $L_q^c(l_\infty(Z, Y))$.

35. Lemma. An operator $B \in L_q^c(l_\infty(Z, Y))$ is $n$ periodic if and only if its matrix satisfies the condition $B_{k+n,l+n} = B_{k,l}$ for each integers $k$ and $l$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$.

Proof. Suppose that $B \in L_n^{\text{per}}(l_\infty(Z, Y))$ and $(B_{k,l})$ is its matrix. Then $B_{k,l}x = (Be_lx)(k) = (S(-n)BS(n)e_lx)(k) = (S(-n)Be_{l+n}x)(k) = (Be_{l+n})(k + n) = B_{k+n,l+n}x$ for each vector $x \in Y$ and integers $k$ and $l$.

Vise versa if $B \in L_q^c(l_\infty(Z, Y))$ is a $c$-continuous operator, then it has
a matrix \((B_{k,l})\) by Lemma 7. Then the condition \(B_{k+n,l+n} = B_{k,l}\) for each integers \(k\) and \(l\) implies \(B_{k+n,l+n}x = (Be_{l+n})x(k+n) = (S(-n)Be_{l+n})x(k) = (S(-n)BS(n)e_{l}x)(k) = (Be_{l})x(k)\), consequently, \(S(-n)BS(n) = B\) and hence \(BS(n) = S(n)B\). Thus the operator \(B\) is \(n\)-periodic.

36. **Corollary.** Suppose that \(B\) is a \(c\)-continuous operator \(B \in L^c_q(l_\infty(Z,Y))\). Then \(B\) is \(n\)-periodic and diagonal if and only if it is a (left) multiplication operator on a stationary \(n\)-periodic sequence in \(L_q(Y)\).

**Proof.** This follows from Theorem 17 and Lemma 35.

37. **Definition.** A function \(\hat{B} : S^1 \rightarrow L_q(Y \oplus Y)\) prescribed by the formula \(\hat{B}(M)x := \bigoplus_{j=0}^{\infty} B(D(M)x)(j)\) will be called the Fourier transform of an \(n\)-periodic operator \(B \in L^q_{n,\text{per}}(l_\infty(Z,Y))\), where \(Y\) is a Banach space over the Cayley-Dickson algebra \(A_n\), \(2 \leq v\). We put

\[
\|\hat{B}(M)x\| := \max_{j=0}^{\infty} \|B(D(M)x)(j)\|.
\]

By \(C_s(S^1, (L_q(Y \oplus Y))^{\infty})\) will be denoted the Banach space of all bounded continuous mappings \(G : S^1 \rightarrow L_q(Y \oplus Y)\) supplied with the norm

\[
\|G\| := \sup_{M \in S^1} \max_{j=0}^{\infty} \|jG(D(M))\|,
\]

where \(\|A\|\) denotes a norm of an operator \(A \in L_q(Y)\), \(Y\) is a Banach space over the Cayley-Dickson algebra \(A_v\), \(2 \leq v\), \(G = \bigoplus_{j=0}^{\infty} jG\) with \(jG \in L_q(Y \oplus Y)\) for every \(j\).

38. **Lemma.** Let \(B \in L^q_{n,\text{per}}(l_\infty(Z,Y))\) be a periodic operator, where \(Y\) is a Banach space over the Cayley-Dickson algebra \(A_v\), \(2 \leq v\). Then \(\hat{B} \in C_s(S^1, (L_q(Y \oplus Y))^{\infty})\).

**Proof.** A uniform space \(C_s(S^1, L_q(Y \oplus Y))\) is complete for a Banach space \(Y\) over the Cayley-Dickson algebra \(A_v\). Take an arbitrary vector \(x \in Y\) and a complex number \(K \in S^1\) and a sequence \(\varepsilon M \in S^1\) converging to \(K\). The Banach spaces \((L_q(Y \oplus Y))^{\infty}\) and \(\bigoplus_{j=0}^{n-1} L_q(Y \oplus Y)\) are isometrically isomorphic when supplied with the corresponding norms, since \(n\) is a natural number, where

\[
\|A\| = \sup_{0 \leq j \leq n-1} \|jA\|
\]

for each \(A = (0A, \ldots, n-1A) \in \bigoplus_{j=0}^{n-1} L_q(Y \oplus Y)\), also
for each \(x = (0, x, \ldots, n-1) \in \bigoplus_{j=0}^{n-1}(Y \oplus Y)\). Then a sequence \(\{B(D(k,M)x) : k \in \mathbb{N}\}\) \(c\)-converges to \(B(D(K)x)\), since an operator \(B\) is \(c\)-continuous. By Definitions 3 and 37 this means that the limit exists
\[
\lim_{k \to \infty} \|\hat{B}(k,M)x - \hat{B}(K)x\| = \lim_{k \to \infty} \max_{j=0}^{n-1} \|B(D(k,M)x)(j) - B(D(K)x)(j)\| = 0
\]
and hence \(\|F\| \leq 1\), where \(F\) denotes the Fourier transform operator on \(L_{q}^{n,\text{per}}(l_{\infty}(Z,Y))\) with values in \(C_{s}(\mathbb{S}, (L_{q}(Y \oplus Y))^{n})\).

39. Corollary. If a sequence \(\{B_{p} : p \in \mathbb{N}\}\) of \(n\)-periodic operators converges to an \(n\)-periodic operator \(B\) relative to the norm on \(L_{q}^{n,\text{per}}(l_{\infty}(Z,Y))\), then a sequence of their Fourier transforms \(F(B_{p})\) converges to \(F(B)\) in \(C_{s}(\mathbb{S}, (L_{q}(Y \oplus Y))^{n})\).

40. Corollary. If \(B\) is an \(n\)-periodic operator and \(F(B) = \hat{B}\) its Fourier transform, then \(\|B\| \geq \sup_{M \in \mathbb{S}} \|\hat{B}(M)\|\).

41. Notation. A family of all \(n\)-periodic operators \(B \in L_{q}^{n,\text{per}}(l_{\infty}(Z,Y))\) such that its Fourier transform \(F(B) = \hat{B}\) has an absolutely converging Fourier series
\[
\hat{B}(M) = \sum_{k=-\infty}^{\infty} \left( \bigoplus_{j=0}^{n-1} M^{(k-1)n+j} \right) B,
\]
i.e. \(\sum_{k=-\infty}^{\infty} \max_{j=0}^{n-1} \|(k-1)n+jB\| < \infty\), will be denoted by \(L_{q}^{n,1}(l_{\infty}(Z,Y))\).

42. Lemma. Let \(B \in L_{q}^{n,\text{per}}(l_{\infty}(Z,Y))\) be an \(n\)-periodic operator and its Fourier transform \(F(B) = \hat{B}\) has the form:
\[
\hat{B}(M) = \sum_{l=-\infty}^{\infty} M^{l}B,
\]
where \(Y\) is a Banach space over the Cayley-Dickson algebra \(\mathbb{A}_{v}\), \(2 \leq v\). Then
\[
(1) \quad B = \sum_{k=-\infty}^{\infty} k\hat{B}S(kn),
\]
where \(k\hat{B} = \bigoplus_{j=0}^{n-1} (k-1)n+jB \in \bigoplus_{j=0}^{n-1} L_{q}^{c}(l_{\infty}(Z,Y))\) is an operator of (left) multiplication on stationary operator valued sequence \(k\hat{B} \in \bigoplus_{j=0}^{n-1} L_{q}(Y)\), moreover,
\[
(2) \quad \|k\hat{B}\| = \sup_{0 \leq j \leq n-1} \|k-1)\| B\| \text{ for each } k.
\]

Proof. In view of Lemma 7 the Banach spaces \(\bigoplus_{j=0}^{n-1} L_{q}^{c}(l_{\infty}(Z,Y))\) and \(L_{q}^{c}(l_{\infty}(Z,Y))\) are isometrically isomorphic, that follows from using the block
form of matrices $(B_{(k-1)n+j,(m-1)n+l})$ of operators $B$, where $k, m \in \mathbb{Z}$ and $j, l = 0, ..., n - 1$, $n \geq 1$. From Lemma 5 it follows that an operator
\[ \sum_{k=-\infty}^{\infty} B_{(k-1)n+j}(kn) \] is $c$-continuous for each $j$, consequently, \[ \sum_{k=-\infty}^{\infty} k\bar{B}_{(k-1)n+j}(kn) \] is also $c$-continuous. A matrix of the operator $B$ coincides with that of \[ \sum_{k=-\infty}^{\infty} k\bar{B}_{(k-1)n+j}(kn) \] by Lemma 35 and Definition 37. Therefore, Formula (1) is satisfied in accordance with Lemma 7. The natural isometric embedding $Y \hookrightarrow Y \oplus Y_l$ induces isometric embeddings $L_q(Y) \hookrightarrow L_q(Y \oplus Y_l)$ and $L_q(l_\infty(Z, Y)) \hookrightarrow L_q(l_\infty(Z, Y \oplus Y_l))$ of normed spaces over the Cayley-Dickson algebra $A_v$.

From the definition of the operator norms on $L_q(Y \oplus Y_l)$ and $\bigoplus_{j=0}^{n-1} L_q(Y \oplus Y_l)$ (see Formulas 38(1, 2)) and $L_q^c(l_\infty(Z, Y \oplus Y_l))$. Equality (2) follows, where
\[ (3) \|A\| = \sup_{0 \leq j \leq n-1} \|jA\| \]
for each $A = (0A, ..., n-1A) \in \bigoplus_{j=0}^{n-1} L_q^c(l_\infty(Z, Y \oplus Y_l))$.

43. Corollary. If $B, D \in L_q^{n,per}(l_\infty(Z, Y))$, then
\[ (1) \quad (Bx)(l) = \sum_{s=-\infty}^{\infty} sBx(s + l) \]
\[ = \sum_{s=-\infty}^{\infty} s-lBx(s) =: (b \ast x)(l) \]
and $(BD)(x) = b \ast (d \ast x)$ for each $x \in l_\infty(Z, Y)$, where $b = \{ sB : s \in \mathbb{Z} \} \in l_1(Z, L_q(Y))$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$. Particularly, as $0 \leq v \leq 2$ the convolution is associative $b \ast (d \ast x) = (b \ast d) \ast x$.

Proof. The equalities follow
\[ (Bx)(l) = \sum_{k=-\infty}^{\infty} \sum_{j=0}^{n-1} B_{(k-1)n+j}(kn) \]
\[ = \sum_{k=-\infty}^{\infty} \sum_{j=0}^{n-1} (k-1)n+jBx((k-1)n + j + l) \]
from Lemma 42. Putting $s = (k-1)n + j$ one gets Formula (1).

44. Remark. If $n = 1$, the Fourier transform of an operator valued function $b : \mathbb{Z} \rightarrow L_q(Y)$ with $b \in l_1(Z, L_q(Y))$ coincides with the Fourier series for a mapping $\hat{B}$.

45. Proposition. Let $A$ and $B$ be two operators in $L_q^{n,per}(l_\infty(Z, Y))$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then
\[ (1) \quad \hat{AB}(M) = \hat{A}(M)\hat{B}(M) \]
for each $M \in S^1$.

**Proof.** With $M \in S^1$ we infer that

$$(2) \quad m(AB) = \sum_{p=0}^{m} pA_{m-p}B$$

and this implies Formula (1), since $\hat{AB}(M)x = A(D(M)B(D(M)x))$ for each $x \in Y$ and $M \in S^1$, since $\hat{i}i_j = i_j$ for each $j$.

46. **Proposition.** Let an operator $B \in L_q^{n,\text{per}}(l_\infty(Z,Y))$ be $n$-periodic, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then an operator $\hat{B}(M)$ is invertible and $(\hat{B}(M))^{-1}x = \hat{A}^{-1}(M)x$ for each $M \in S^1$ and $x \in Y$.

**Proof.** If $N \in S^1$, then an algebra $\text{alg}_R(N, i_s)$ is associative for each $s \geq 0$, since $N = N_0 + N_1i$ with $N_0, N_1 \in R$ and $i_s = i_4i$ for each $s \geq 0$. If $M \in S^1$ and $x \in Y$, one can take the algebra $\text{alg}_R(M)$ which is either the real or complex field. Therefore, $B(D(M)B^{-1}(D(M)x)) = AA^{-1}x = x$ with $Ax = B(D(M))$ by Proposition 45.

47. **Corollary.** Let $B, D \in L_q^{n,\text{per}}(l_\infty(Z,Y))$ be $n$-periodic operators and let $(\hat{B}(M))^{-1}x = \hat{D}(M)x$ for each $M \in S^1$ and $x \in Y$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then $D = B^{-1}$.

**Proof.** This follows from Proposition 46, since the $R$ linear span $\text{span}_R\{y = Mx : M \in S^1, x \in X\}$ of such set of vectors is isomorphic with $X \oplus Xi$.

48. **Lemma.** Let an $n$-periodic operator $B \in L_q^{n,\text{per}}(l_\infty(Z,Y))$ be uniformly $c$-continuous, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$. Then its Fourier transform $\hat{B}$ is uniformly $c$-continous, $\hat{B} \in L_q^{c}(l_\infty(Z,Y \oplus Yi))$.

**Proof.** From the conditions of this lemma it follows, that the mapping $\hat{B} : S^1 \ni M \mapsto D(M)B(M^*)$ is continuous from $S^1$ into $L_q^{n,\text{per}}(l_\infty(Z,Y \oplus Yi))$. Up to an $R$-linear continuous algebraic automorphism of the Cayley-Dickson algebra $A_v$ and the corresponding automorphism of a Banach space $Y$, the Fourier series

$$(1) \quad \hat{B}(M) \sim \sum_{k=-\infty}^{\infty} M^k \hat{B}$$

exists by Lemma 31. This series converges to $\hat{B}(M)$ by Cezaro, that is

$$(2) \quad \hat{B}(M) = \lim_{m \to \infty} \sum_{k=-m}^{m} (1 - \frac{|k|}{m+1})M^k \hat{B},$$
since \((1 - \frac{|k|}{m+1}) \in \mathbb{R}\) while the real field is the center of the Cayley-Dickson algebra \(\mathcal{A}_v\). Particularly, for \(M = 1\) one has \(M^k = 1\) and \(\hat{B}(1) = B\) (see also §20.2(743) [S]).

The operator \(B\) is \(n\)-periodic, so consider its Fourier transform and get
\[
(3) \quad \hat{B}(M) = \lim_{m \to \infty} \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) k \hat{B}(M).
\]

In view of Lemma 31 each operator \(k \hat{B}S(-k)\) is diagonal. Since \(B\) is \(n\)-periodic, this implies that every operator \(\bigoplus_{j=0}^{n-1} (k-1)n+j \hat{B}S(-(k-1)n-j)\) is \(n\)-periodic as well. Therefore, \(k \hat{B}(M)x = (k \hat{B}(D(M)x))(0) = (M^k I) kBx\) for each vector \(x \in Y \oplus Y_i\), since \(M^k(M^{-k}x_i l) = M^k(M^{-k}i x) = x_i l\) for each \(l \geq 0\) and \(x_i l \in Y_i\), consequently, \(k \hat{B}(M) = (M^k I) kB\) and hence
\[
(4) \quad \hat{B}(M) = \lim_{m \to \infty} \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) (M^k I) kB,
\]
where \((k \hat{B})_{s,p} = kB\) for each \(s - p = k\), \(s, p \in \mathbb{Z}\).

49. Notation. Let \(P\) be a Banach algebra over the Cayley-Dickson algebra \(\mathcal{A}_v\) with \(2 \leq v\). We denote by \(F(S^1, P)\) a Banach space of all continuous functions \(f : S^1 \to P\) with absolutely converging Fourier series
\[
(1) \quad f(M) = \sum_{k=-\infty}^{\infty} M^k k f
\]
relative to the norm:
\[
(2) \quad \|f\| := \sum_{k=-\infty}^{\infty} \|k f\|,
\]
where \(k f \in P\) for each \(k \in \mathbb{Z}\).

50. Corollary. Let \(B \in L_q^{n,per}(L_\infty(\mathbb{Z}, Y))\), where \(Y\) is a Banach space over the Cayley-Dickson algebra \(\mathcal{A}_v\), \(2 \leq v\). Then the following conditions are equivalent:

(1) \(\hat{B} \in F(S^1, L_q(Y \oplus Y_i))\) and
(2) \(\hat{B} \in F(S^1, L_q^c(l_\infty(\mathbb{Z}, Y \oplus Y_i))\).

Proof. This follows from Lemma 48, since \(\|k \hat{B}\| = \|kB\|\) for each \(k \in \mathbb{Z}\).

Indeed, generally \(\|k \hat{B}x\| = \|kBx\|\), since \(|ab| \leq |a||b|\) for each Cayley-Dickson numbers \(a, b \in \mathcal{A}_v\) and \(\|ax\| \leq |a||x|\) for each \(a \in \mathcal{A}_v\) and \(x \in Y\) (see §I.2.1 [28]). In particular, if \(x \in Y_0\) or \(x \in Y_i\), then \(\|ax\| = |a||x|\).
51. Theorem. The algebra $L_{n,\text{per}}^q(l_\infty(Z, Y))$ is the subalgebra of the algebra $L_{uc}^q(l_\infty(Z, Y))$, where $Y$ is a Banach space over the Cayley-Dickson algebra $A_v$, $2 \leq v$.

Proof. If $B \in L_{n,\text{per}}^q(l_\infty(Z, Y))$, then by Lemma 42 we have

$$B = \sum_{k=-\infty}^{\infty} k \bar{B} S(kn),$$

where $k \bar{B} = \bigoplus_{j=0}^{n-1} (k-1)n+j B \in \bigoplus_{j=0}^{n-1} L_q^c(l_\infty(Z, Y))$ is an operator of (left) multiplication on stationary operator valued sequence $k \tilde{B} \in \bigoplus_{n=0}^{\infty} L_q(Y)$, moreover,

$$\|k \bar{B}\| = \|k \tilde{B}\|$$

for each $k$. In view of Proposition 24 an 1-ribbon operator $k \bar{B}$ is uniformly $c$-continuous. On the other hand, each shift operator $S(n)$ is uniformly $c$-continuous. The algebra $L_{uc}^q(l_\infty(Z, Y))$ is complete as the uniform space. Therefore, the operator $B$ is uniformly $c$-continuous.

3 Fourier transform on algebras and spectra

52. Definitions. Let $G$ be a quasi-group, i.e. a set with one binary operation (multiplication) so that

1. there exists a unit element $e$ so that $eb = be = b$;
2. each element $b$ has an inverse $b^{-1}$, i.e. $b^{-1}b = bb^{-1} = e$;
3. a multiplication is alternative $(ab)b = a(ab)$ and $b(aa) = (ba)a$ and
4. $a^{-1}(ab) = b$ and $(ba)a^{-1} = b$ for each $a, b \in G$.

Let $R_a$ be a Banach algebra over the real field $\mathbb{R}$ for each $a \in G$ such that $R_a$ is isomorphic with $R_b$ for all $a, b \in G$. Put

$$R = \{B : B \in \bigoplus_{a \in G} aR_a, \|B\| < \infty\},$$

$$G_R = : \{x : x \in \bigoplus_{a \in G} aR, \ |x| \ < \infty\}$$

is a quasi-group ring over the real field so that $a\beta = \beta a$ for each $a \in G$ and $\beta \in \mathbb{R}$,

$$|x|^2 = \sum_{a \in G} |x_a|^2 \text{ for } x = \sum_{a \in G} x_a a$$

with $x_a \in \mathbb{R}$ for each $a \in G$;

$$\|A\|^2 := \sum_{b \in G} \|A_b\|^2$$
for $A = \sum_{b \in G} A_b b$ with $A_b \in R_b$ for each $b \in G$,

(8) $bA_a = A_a b$ for each $A_a \in R_a$ and $a, b \in G$. Suppose that $R_a$

(9) contains a unit element $I$ and that

(10) $\|I\| = 1$ and

(11) $\|AB\| \leq \|A\|\|B\|$ for each $A, B \in R_a$.

Denote by $L(R)$ the Banach space of all strongly integrable functions $f : R \to R$ supplied with the norm

(12) $\|f\| := \int_{-\infty}^{\infty} \|f(t)\| dt < \infty$.

Henceforward, we suppose that an algebra $R$

(13) is alternative $(AA)B = A(AB)$ and $B(AA) = (BA)A$ for each $A, B \in R$ and

(14) if $A$ is left invertible, then also $A^{-1}(AB) = B$, if $A$ is right invertible $(BA)A^{-1} = B$ for every $A, B \in R$.

The alternativity implies that $G$ and $R$ are power-associative that is

$b^m b^n = b^{m+n}$ and $B^m B^n = B^{m+n}$ for each $b \in G$ and $B \in R$ and natural numbers $n, m$, where $b^n = b(b(...(bb)...))$ denotes the $n$-fold product,

$b^0 = e$ for $b \neq 0$, $B^0 = I$ for $B \neq 0$.

We consider their complexifications $G_{C_1} := G_R \oplus G_R i$ and $R_{C_1} := R \oplus R i$, where $C_1 = R \oplus R i$ with $ai = ia \in G_{C_1}$,

(15) $|a + bi|^2 = |a|^2 + |b|^2$ for each $a, b \in G_R$ and

(16) $\|A + Bi\|^2 = \|A\|^2 + \|B\|^2$ for every $A, B \in R$.

Analogously a Banach space $L(R_{C_1})$ is defined with $f : R \to R_{C_1}$.

53. Lemma. Let $R$ be an algebra as in §52 and let an element $A \in R$
be of norm $\|A\| < 1$, then the series $C = I - A + A^2 - A^3 + ...$ is absolutely convergent and

(1) $C(I + A) = (I + A)C = I$.

Proof. From Formulas 52(7, 11, 16) it follows that $\|A^n\| \leq \|A\|^n$ for each natural number $n$, consequently, the sequence of partial sums $S_n := I - A + A^2 - A^3 + ... + (-1)^n A^n$ converges in $R$. A Banach algebra $R$ is power-associative and this implies Formula (1).

54. Lemma. Let $R$ be an algebra as in §52 and let an element $A \in R$
have a left inverse $Q$. If $B \in R$ is an element such that $\|B\|\|Q\| < 1$, then
A + B has a left inverse C so that

\[ C = Q(I - BQ + (BQ)^2 - (BQ)^3 + ...). \]

**Proof.** A Banach algebra R satisfies conditions 52(13, 14), hence \((A + B) = (I + BQ)A\), since \(QA = I\). The alternativity (13) implies the Moufang identities in the algebra R:

1. \((M1) (XYX)Z = X(Y(XZ))\),
2. \((M2) Z(XYX) = ((ZX)Y)X\),
3. \((M3) (XY)(ZX) = X(YZ)X\) for each \(X, Y, Z \in R\).

From Lemma 53 and Formulas \((M1, M2)\) it follows that \(C(A + B) = (Q(I - BQ + (BQ)^2 - ...))(I + BQ)(A + B) = I\).

**55. Corollary.** The set \(U_l\) of all left invertible elements \(A \in R\) is an open subset in \(R\).

**56. Notation.** Denote by \(R'\) the algebra over \(R\) of all periodic functions \(x : [0, 2\pi] \to R\) of the form

\[ x(t) = \sum_{n=-\infty}^{\infty} a_n e^{nti} \]

with coefficients \(a_n \in R\) such that

\[ \sum_{n} ||a_n|| < \infty \]

with point-wise addition and multiplication of functions

\[ (x + y)(t) = x(t) + y(t), \quad (xy)(t) = x(t)y(t) \text{ for each } t \in [0, 2\pi]. \]

**57. Lemma.** Suppose that \(x \in R'\) and \(x(0)\) has a left inverse in \(R\), then there exists an element

\[ y(t) = \sum_{n=-\infty}^{\infty} c_n e^{nti} \in R' \]

such that

1. \(c_0\) has a left inverse \(q_0\) in \(R\) and
2. \(||q_0|| \sum_{n=1}^{\infty} ||c_n + c_{-n}|| < 1\) and
3. there exists \(\epsilon > 0\) so that \(y(t) = x(t)\) for each \(t \in (-\epsilon, \epsilon)\).
Proof. Consider the following function given piecewise \( w_\varepsilon(t) = 1 \) for 
\(|t| < \varepsilon, \ w_\varepsilon(t) = 2 - |t|/\varepsilon \) for \( \varepsilon \leq |t| < 2\varepsilon, \ w_\varepsilon(t) = 0 \) for \( 2\varepsilon \leq |t| \), where 
\( 0 < \varepsilon \leq \pi/2 \). Then one defines the function

\[
y_\varepsilon(t) = w_\varepsilon(t)x(t) + [1 - w_\varepsilon(t)]x(0) = \sum_{n=-\infty}^{\infty} b_n(\varepsilon)e^{int}.
\]

This function satisfies Condition (4). It has the Fourier series with coefficients \( b_n = b_n(\varepsilon) \):

\[
b_n = \frac{3\varepsilon}{2\pi}a_n + \sum_{k=1}^{\infty} \frac{a_{n-k} + a_{n+k}}{\pi k^2\varepsilon} (\cos(\varepsilon k) - \cos(2\varepsilon k)) - \sum_{k=-\infty}^{\infty} a_k \frac{\cos(\varepsilon n) - \cos(2\varepsilon n)}{\pi n^2\varepsilon}
\]

for \( n \neq 0 \) and

\[
b_0 = a_0 + \sum_{k=1}^{\infty} (a_{-k} + a_k)[1 + \frac{\cos(\varepsilon k) - \cos(2\varepsilon k)}{\pi k^2\varepsilon} - \frac{3\varepsilon}{2\pi}].
\]

Therefore,

\[
\lim_{\varepsilon \downarrow 0} ||b_0|| = || \sum_{k=-\infty}^{\infty} a_k || = ||y(0)|| > 0 \quad \text{and} \quad \sum_{k=1}^{\infty} (||b_k|| + ||b_{-k}||) \leq \sum_{k=-\infty}^{\infty} ||a_k||A_k,
\]

where a positive number \( \delta > 0 \) exists such that \( 0 \leq A_k < \varepsilon^{1/2}[2|k|C + 9/\pi] \) for each \( 0 < \varepsilon < \delta \) and every \( k \in \mathbb{Z} \), where \( C = const > 0 \). Thus a positive number \( \epsilon_0 > 0 \) exists so that

\[
||b_0|| > \sum_{k=1}^{\infty} (||b_k|| + ||b_{-k}||)
\]

for each \( 0 < \epsilon < \epsilon_0 \) (see also [2, 33]). From Lemma 54 statements (2, 3) of this lemma follow.

58. Corollary. If \( y \in R' \) and \( y(t) \) satisfies Properties (2, 3) of Lemma 57, then \( y \) has a left inverse \( z \) in \( R' \).

59. Theorem. If \( R' \) is an algebra of periodic functions as in §56. Then \( x(t) \) has a left inverse in \( R' \) if \( x(t_0) \) has a left inverse in \( R \) for each \( t_0 \).

Proof. In view of Lemma 57 and Corollary 58 for each \( \tau \in [0,2\pi] \) a positive number \( \epsilon > 0 \) and an element \( y_\tau \in R' \) exist such that \( y_\tau(t)x(t) = I \) for each \( t \in (\tau - \epsilon, \tau + \epsilon) \). The segment \([-\pi,\pi]\) is compact, that is, each its open covering has a finite subcovering, consequently, a finite number of functions \( y_\tau \) induces a function \( y \in R' \) so that \( y(t)x(t) = I \) for each \( t \).
60. Lemma. Suppose that $-\pi < \alpha < a < b < \beta < \pi$ and $x_1, x_2 \in R'$ and $x_2(t)$ has a left inverse for each $t \in (\alpha, \beta)$ and $x_1(t)$ vanishes for every $-\pi \leq t < a$ and $b < t \leq \pi$. Then an element $x_3 \in R'$ exists vanishing on $[-\pi, \alpha) \cup (\beta, \pi]$ such that

(1) $x_1(t) = x_3(t) x_2(t)$ for each $t \in [-\pi, \pi]$.

Proof. From Lemma 57 and Corollary 58 it follows, that to any $\tau \in [a, b]$ a positive number $\epsilon > 0$ and an element $y_\tau \in R'$ correspond such that $y_\tau(t) x_2(t) = I$ for each $t \in (\tau - \epsilon, \tau + \epsilon)$. As in §59 one gets that an element $z \in R'$ exists such that $z(t) x_2(t) = I$ for every $t \in [a, b]$. Put $x_3(t) = x_1(t) z(t)$, consequently, $x_3(t) = 0$ for each $t \in [-\pi, \alpha) \cup (\beta, \pi]$ and $x_3 \in R'$ and hence Assertion (1) is valid, since the algebra $R$ satisfies Conditions 52(13, 14) and $e^{n \tau i}$ commutes with $R$ and $(a_n e^{n \tau i})(b_k e^{k \tau i}) = (a_n b_k) e^{(n+k)\tau i}$ for each $a_n, b_k \in R$ and $n, k \in \mathbb{Z}$.

61. Lemma. Suppose that $x(t)$ is strongly integrable on $(-\pi, \pi)$ function with values in $R$ and vanishes on $(-\pi, -\pi + \epsilon) \cup (\pi - \epsilon, \pi)$ with $0 < \epsilon < \pi/2$ and

(1) $f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\tau) e^{-\tau t i} d\tau$,

(2) $a_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\tau) e^{-n \tau i} d\tau$,

then

(3) $\int_{-\infty}^{\infty} \|f(t)\| dt < \infty$

if and only if

(4) $\sum_{n=-\infty}^{\infty} \|a_n\| < \infty$.

Proof. Consider a positive number $0 < \delta < \pi/2$ so that $x(t)$ vanishes on $[-\pi, -\pi + 2\delta) \cup (\pi - 2\delta, \pi)$. Put $\phi(t) = 1$ for $|t| < \pi - \delta$, $\phi(t) = \frac{\pi - |t|}{\delta}$ for $|\pi - \delta| \leq |t| < \pi$, $\phi(t) = 0$ for $\pi \leq |t|$. If

(5) $x(t) = \sum_{n=-\infty}^{\infty} a_n e^{n \tau i}$,

then

(6) $x(t) = x(t) \phi(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{\tau t i} \left[ \sum_{n=-\infty}^{\infty} a_n \frac{\cos(\tau + n)(\pi - \delta) - \cos(\tau + n)\pi}{(\tau + n)^2 \epsilon} \right] d\tau$,\n
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since this integral and this sum are absolutely convergent. Therefore, the function

\[ f(t) := \sqrt{\frac{2}{\pi}} \sum_{n=-\infty}^{\infty} a_n \frac{\cos(\tau + n)(\pi - \delta) - \cos(\tau + n)\pi}{(\tau + n)^2\epsilon} \]

satisfies Conditions (1, 3), if (4) is fulfilled.

Vise versa, Condition (3) implies that

\[ \sum_{n=-\infty}^{\infty} \| \int_{n-1/2}^{n+1/2} f(t) dt \| < \infty, \]

consequently,

\[ \int_{n-1/2}^{n+1/2} f(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x(t) \frac{2\sin(t/2)}{t} e^{\pi i t} dt. \]

Thus the Fourier series of the function \( x(t)\frac{2\sin(t/2)}{t} \) converges absolutely. Moreover, the Fourier series of the mapping \( \frac{t}{2\sin(t/2)} \phi(t) \) also is absolutely convergent. Thus the Fourier series of \( x(t) = [x(t)\frac{2\sin(t/2)}{t}] [\frac{t}{2\sin(t/2)} \phi(t)] \) is absolutely convergent, since \( R' \) is an algebra over the real field \( R \) and \( i \) commutes with each \( y \in R' \).

62. Corollary. Let \( g \) and \( f \in L(R) \), let also

\[ x_1(t) = \int_{-\infty}^{\infty} g(\tau) e^{-\pi i t} d\tau \]

vanish outside some interval \( (a,b) \subset (-\pi, \pi) \). Suppose that

\[ x_2(t) = \int_{-\infty}^{\infty} f(\tau) e^{-\pi i t} d\tau \]

is zero outside an interval \( (\alpha, \beta) \subset (-\pi, \pi) \) with \( \alpha < a \) and \( b < \beta \) and \( x_2(t) \) has a left inverse for each \( \alpha < t < \beta \). Then an element \( y \in L(R) \) exists so that

\[ g(t) = \int_{-\infty}^{\infty} y(\tau) f(t-\tau) d\tau. \]

This follows immediately from Lemmas 60 and 61.

63. Lemma. If \( f \in L(R) \), then

\[ \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \| f(t + \epsilon) - f(t) \| dt = 0. \]
Proof. The Lebesgue measure is $\sigma$-finite and $\sigma$-additive, so the statement of this theorem for step functions is evident. In $L(R)$ the set of step functions

$$g(t) = \sum_{k=1}^{n} a_k \chi_{B_k}(t)$$

is dense with $a_k \in R$, $B_k \in \mathcal{B}(R)$, $n \in N$, where $\mathcal{B}(R)$ denotes the $\sigma$-algebra of all Borel subsets of $R$.

64. Lemma. Suppose that $f \in L(R)$ and $h \in L(R)$ so that $\text{supp}(h) \subset (-\epsilon, \epsilon)$ for some positive number $0 < \epsilon < \infty$. Then

$$\int_{-\infty}^{\infty} \| f(t) \int_{-\infty}^{\infty} h(\tau)d\tau - \int_{-\infty}^{\infty} f(t + \tau)h(\tau)d\tau \| dt \leq \left[ \int_{-\infty}^{\infty} \| h(\tau) \| d\tau \right] \sup_{|u| \leq \epsilon} \int_{-\infty}^{\infty} \| f(t + u) - f(t) \| dt.$$  

Proof. This follows from Fubini’s theorem

$$\int_{-\infty}^{\infty} \| f(t) \int_{-\infty}^{\infty} h(\tau)d\tau - \int_{-\infty}^{\infty} f(t + \tau)h(\tau)d\tau \| dt \leq \int_{-\infty}^{\infty} \| f(t) - f(t + \tau) \| \| h(\tau) \| d\tau dt \leq \int_{-\infty}^{\infty} \| h(\tau) \| d\tau \sup_{|u| \leq \epsilon} \int_{-\infty}^{\infty} \| f(t + u) - f(t) \| dt.$$  

65. Lemma. If $f \in L(R)$, then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \| f(t) - \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(n\tau)}{\tau^2} d\tau \| dt = 0.$$  

Proof. Put $h_0(t) = \frac{\sin^2(nt)}{t^2} = h_1(t) + h_2(t)$ with $h_1(t) = h_0(t)[1 - |t|\sqrt{n}]$ for $|t| \leq n^{-1/2}$, while $h_1(t) = 0$ for $|t| > n^{-1/2}$. An application of Lemmas 63 and 64 leads to

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \| f(t) - \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau)h_1(\tau)d\tau \| dt = 0$$

and

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{1}{\pi n} \| f(t + \tau)h_2(\tau)d\tau \| dt \leq \left[ \int_{-\infty}^{\infty} \| f(t) \| dt \right] \lim_{n \to \infty} \frac{1}{\pi n} \int_{-\infty}^{\infty} h_2(\tau)d\tau.$$  

On the other hand,

$$\frac{1}{\pi n} \int_{-\infty}^{\infty} h_2(t) dt =$$
\[
\frac{1}{\pi n} \left[ \int_{-\infty}^{\infty} \frac{\sin^2(nt)}{t^2} dt - \int_{-\infty}^{n^{-1/2}} \frac{\sin^2(nt)}{t^2} dt \right] = \frac{2}{\pi} \int_{n^{1/2}}^{\infty} \frac{\sin^2(t)}{t^2} dt + \frac{2}{\pi \sqrt{n}} \int_{0}^{\sqrt{n}} \frac{\sin^2(t)}{t^2} dt = O(n^{-1/2} \ln n),
\]
consequently,

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \left| \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) h_2(\tau) d\tau \right| dt = 0.
\]

66. Theorem. Let \( f \in L(R) \) and let the Fourier transform

\[
(1) \quad x(\tau) = \int_{-\infty}^{\infty} f(t) e^{\tau it} dt
\]

have a left inverse in \( R \) for each \( \tau \in [-\pi, \pi] \). Then the \( R \)-linear combinations

\[
(2) \quad \sum_{n} b_{n} f(t - \tau_{n}) \quad \text{with} \quad b_{n} \in R
\]

are dense in \( L(R) \), where \( \tau_{n} \in [-\pi, \pi] \).

Proof. Lemma 65 means that a function

\[
(3) \quad f_{\delta}(t) = \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(n\tau)}{\tau^2} d\tau
\]

exists so that

\[
(4) \quad \int_{-\infty}^{\infty} \| f(t) - f_{\delta}(t) \| dt < \delta,
\]

where \( 0 < \delta \). Consider its Fourier transform:

\[
(5) \quad h_1(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{uti} \left[ \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(n\tau)}{\tau^2} d\tau \right] dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{uti} \left[ \frac{1}{\pi n} \int_{-\infty}^{\infty} \frac{\sin^2(n\tau)}{\tau^2} e^{-urt} d\tau \right] dt,
\]

consequently,

\[
h_1(u) = (1 - |u|) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{uti} dt \quad \text{when} \quad |u| < 2n
\]

and \( h_1(u) = 0 \) for \( |u| \geq 2n \). Analogously the Fourier transform

\[
h_2(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{uti} \left[ \frac{1}{2\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(2n\tau)}{\tau^2} d\tau \right] dt
\]

vanishes for each \( u \) so that \( |u| > 4n \). The Fourier series of \( h_1 \) and \( h_2 \) over \((-8n, 8n)\) converge absolutely by Lemma 61. Then one can write \( h_1(u) = h_2(u) h_3(u) \), where

\[
h_3(u) = \int_{-\infty}^{\infty} \psi(t) e^{uti} dt
\]
with $\psi \in L(R)$, since the algebra $R$ is alternative and $e^{tu}$ commutes with any $y \in R'$ for each real numbers $t$ and $u$. Therefore, we deduce that

$$
(6) \quad \int_{-\infty}^{\infty} f_\delta(t)e^{tu}dt = \int_{-\infty}^{\infty} e^{tu}[f_\delta(t) - \frac{1}{2\pi n}] f(t + x)\left\{ \int_{-\infty}^{\infty} \frac{\sin^2(2\pi \tau)}{\tau^2} \psi(\tau - x)d\tau \right\} dx dt, \quad \text{consequently,}
$$

$$
\int_{-\infty}^{\infty} e^{tu}[f_\delta(t) - \frac{1}{2\pi n}] f(t + x)\left\{ \int_{-\infty}^{\infty} \frac{\sin^2(2\pi \tau)}{\tau^2} \psi(\tau - x)d\tau \right\} dx dt = 0
$$

for each $u$, hence

$$
(7) \quad f_\delta(t) = \frac{1}{2\pi n} \int_{-\infty}^{\infty} f(t + x)\Phi(x)dx, \quad \text{where}
$$

$$
\Phi(x) = \int_{-\infty}^{\infty} \frac{\sin^2(2\pi \tau)}{\tau^2} \psi(\tau - x)d\tau
$$

is absolutely integrable. Lemmas 63 and 64 imply that

$$
(8) \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} \| \int_{-\infty}^{\infty} f(t + x)\Phi(x)dx - \frac{n^2 - 1}{n} \sum_{k=-n^2}^{n^2-1} f(t + \frac{k}{n}) \int_{k/n}^{(k+1)/n} \Phi(x)dx \| dt = 0.
$$

From Formulas (5, 6) and Lemma 65 the assertion of this theorem follows.

67. **Lemma.** Let $R$ be an algebra with unit (see §52) and let $I$ be a maximal left ideal. Suppose that $X$ is an additive group of all cosets $R/I$ and $H$ is an algebra of homomorphisms of $X$ onto itself produced by multiplying by elements of $R$ from the left. Then $X$ is irreducible relative to $H$.

**Proof.** Consider the quotient mapping $\theta : R \to R/I$ (see also §I.2.39 [28]). A space $X$ is $R$-linear, since $R$ is an algebra over $R$. Therefore, $\theta(R)a =: V_a$ is an $R$-linear space for each nonzero element $a \in R \setminus \{0\}$. Put $S_a = \theta^{-1}(V_a)$. Evidently, $S_a$ is a left ideal in $R$ and $I \subset S_a, \quad S_a \not\subset I$. The ideal $I$ is maximal, hence $S_a = R$, consequently, $V_a = X$ and $Ha = X$ for each nonzero element $a \in R \setminus \{0\}$.

68. **Lemma.** Let $R, I, X$ and $H$ be the same as in Lemma 67. Suppose that for a marked element $x \in R$ and each maximal left ideal $I$ the corresponding element $\theta(x)$ is left invertible in $H$. Then $x$ is left invertible in $R$.

**Proof.** An algebra $R$ has the unit element $I \subset R$. Each left ideal is contained in a maximal left ideal. Therefore, an element $x \in R$ is left invertible if and only if this element $x$ is not contained in a maximal left
ideal. On the other hand, if \(yx = I\), then \(\theta(y)\theta(x) = \theta(I)\). Take cosets \(b_I\) and \(b_0\) in \(R/I\) so that \(I \in b_I\) and \(0 \in b_0\). Then \(\theta(y)\theta(x)b_I = \theta(I)b_I\). But \(\theta(I)b_I = b_I\), since \(II = I\). If \(x \in b_0 = I\), then \(\theta(x)b_I = b_0\) and \(\theta(y)\theta(x)b_I = b_0\), since \(x \in \theta(x)b_I\).

69. Lemma. A maximal left or right ideal \(I\) in \(R\) is closed.

Proof. If \(I\) is not closed, then its closure \(cl_R(I)\) in \(R\) contains an ideal \(I\). On the other hand, \(cl_R(I)\) is a left or right ideal in \(R\) respectively, since \(R\) is a topological algebra. Therefore, \(cl_R(I) = I\), since \(I\) is maximal.

70. Lemma. If \(R\) is a Banach algebra, and \(I, X\) and \(H\) have the same meaning as in Lemma 67, then \(X\) is a Banach space, \(H\) is a normed algebra with norm \(\|\theta(z)\|\) on \(H\) so that \(\|\theta(x)\|_H \leq \|x\|_R\) for each \(x \in R\). If moreover \(R\) is a Hilbert algebra over either the quaternion skew field or the octonion algebra \(A_v\) with \(2 \leq v \leq 3\), then \(X\) is a Hilbert space over \(A_v\).

Proof. The quotient algebra \(R/I\) is supplied with the quotient norm \(\|\theta(x)\|_X = \inf_{z \in \theta(x)} \|z\|_R\) (see also §I.2.39 [28]). Therefore,

\[
\|\theta(x)\|_H = \sup_{b \in X} \|\theta(x)b\|_X/\|b\|_X \leq \|x\|_R.
\]

If \(R\) is a Hilbert algebra over \(A_v\), then \(\|x\|_R = \sqrt{<x;x>}\), where a scalar \(A_v\)-valued product \(<x;y>\) on \(R\) satisfies conditions of §I.2.3 [28]. From the parallelogram identity and the polarization formula one gets that \(\|\theta(x)\|_X\) induces an \(A_v\)-valued scalar product on \(X\) (see Formulas I.2.3(1 – 3, SP) [28]).

71. Notation. Let \(R\) be a quasi-commutative \(C^*\)-algebra over either the quaternion skew field or the octonion algebra \(A_r\), \(2 \leq r \leq 3\), satisfying conditions of §52. Let also \(F\) be a normed algebra of functions either \(f, g : \Lambda \to A_r\) or \((A_r)_{C_1}\) with point-wise multiplication \(f(t)g(t)\) and addition \(f(t) + g(t)\) of functions and \(\phi : F \to A_r\) or \((A_r)_{C_1}\) respectively be a continuous \(R\) homogeneous additive multiplicative homomorphism, \(\phi(fg) = \phi(f)\phi(g)\).

Suppose that the unit function \(h(t) = 1\) for each \(t \in \Lambda\) belongs to \(F\), also \(R'\) is a family of functions \(x : \Lambda \to R\) or \(x : \Lambda \to R_{C_1}\) satisfying the following conditions:

1. \(R'\) is an algebra over the real field \(R\) under point-wise multiplication and addition;
(2) if \( x_1, \ldots, x_n \in R \) and \( f_1, \ldots, f_n \in F \), then \( (x_1 f_1 + \ldots + x_n f_n) \in R' \); 

(3) \( R' \) is a normed algebra so that \( \| x^f \| = \| x \| | f | \) in the case over \( A_r \) or \( \| x^f \| \leq \| x \| | f | \) over \( (A_r)_{C_1} \) for each \( x \in R \) and \( f \in F \), where \( x^f := xf \), \( |f| \) denotes a norm of \( f \) in \( F \); 

(4) the \( \mathbb{R} \)-linear combinations of Form (2) are dense in \( R' \); 

(5) if \( x = x_1 f_1 + \ldots + x_n f_n \) and \( \phi \) is a continuous homomorphism as above, then \( \| x_1 \phi(f_1) + \ldots + x_n \phi(f_n) \| \leq \| x \| \).

Each homomorphism \( \phi \) of \( F \) induces \( \hat{\phi}(x) \) with the property \( \hat{\phi}(x^f) = x \phi(f) \) and \( \hat{\phi} \) will be called a generated homomorphism.

72. Theorem. Let suppositions of §71 be satisfied. Then and element \( x \in R' \) has a left inverse if for each generated homomorphism \( \hat{\phi} \) the corresponding element \( \hat{\phi}(x) \) of \( R \) has a left inverse in \( R \).

Proof. Since a homomorphism \( \phi \) is \( \mathbb{R} \)-homogeneous and additive, then it is \( \mathbb{R} \)-linear. Take an arbitrary maximal ideal \( \mathcal{I} \) in \( R' \). It has the decomposition

\[
\mathcal{I} = \bigoplus_{j=0}^{2^r-1} \mathcal{I}_j i_j, 
\]

where \( \mathcal{I}_j \) is either a real or complex algebra isomorphic with \( \mathcal{I}_k \) for each \( 0 \leq j, k \leq 2^r - 1 \). Each \( x \in R' \) has the corresponding element \( \theta(x) \) of \( H \) (applying Lemma 68 to \( R' \) here instead of \( R \) in §68). The algebra \( R' \) has the decomposition \( R' = R'_{0i_0} \oplus \ldots \oplus R'_{mi_m} \) induced by that of \( R \) with pairwise isomorphic commutative algebras \( R'_{j} \) and \( R'_{k} \) either over \( \mathbb{R} \) or \( C_1 \) respectively for each \( k, j, \ m = 2^r - 1 \). Thus any two elements \( a, b \in R' \) quasi-commute and \( a = a_{0i_0} + \ldots + a_{mi_m} \) and \( b = b_{0i_0} + \ldots + b_{mi_m} \) with \( a_j, b_j \in R'_{j} \) for each \( j \). Particularly, elements \( I^f = I \phi \) of \( R' \) quasi-commute with each \( x^g \in R' \) and hence with each \( b \in R' \). In view of Theorem I.2.81 and Corollary I.2.84 \[28\] and Lemmas 67 and 70 above the mapping \( \theta(I^f) \) is the continuous algebraic homomorphism from \( F \) into \( A_r \) or \( (A_r)_{C_1} \) correspondingly. There exists a homomorphism \( \hat{\phi} \) so that \( \theta(I^f) = J \hat{\phi}(f) \), where \( J := \theta(I) \) is a unit of \( H \). Therefore, \( \hat{\phi}(x_1 f_1 + \ldots + x_n f_n) = \theta(x_1) \theta(I f_1) + \ldots + \theta(x_n) \theta(I f_n) = \theta(x_1 f_1) + \ldots + \theta(x_n f_n) \in R' \). From Condition 71(5) and Lemma 70 it follows that \( \theta(x) = \theta(\hat{\phi}(x) 1) \) for each \( x \in R' \).
If \( \hat{\phi}(x(t)) \) has a left inverse \( y \) in \( R \), then \( \theta(y1)\theta(x) = \theta(yx) \), consequently, \( \theta(yx) = \theta(\hat{\phi}(yx)1) = \theta(y\hat{\phi}(x)1) = \theta(I1) = J \). This means that \( \theta(x) \) has a left inverse, hence by Lemma 68 \( x \) has a left inverse in \( R' \).

73. Corollary. If suppositions of §71 are fulfilled and for each \( \phi(f) \) with \( f \in F \) a point \( t_0 \) exists so that \( \phi(f) = f(t_0) \), then Condition 71(5) can be replaced by \( \|x(t_0)\| \leq \|x\| \) for each \( t_0 \). Moreover, in the latter situation an element \( x \in R' \) has a left inverse in \( R' \), if \( x(t) \) has a left inverse in either \( R \) or \( R_{C_1} \) correspondingly for each \( t \).

74. Remark. If an algebra \( F \) has not a unit, then one can formally adjunct a unit \( 1 \) and consider an algebra \( \bar{F} := \{ c1 + f : c \in Q, f \in F \} \), where either \( Q = A_r \) or \( Q = (A_r)_{C_1} \) correspondingly, putting \( |c1 + f|^2 = |c|^2 + |f|^2 \) and \( \bar{R}' := \{ z = cI1 + x : x \in R', c \in Q \} \) with \( \|z\|^2 = |c|^2 + \|x\|^2 \). This standard construction induces an extended homomorphism either \( \bar{\phi}(c1 + f) = c + \phi(f) \) or an exceptional homomorphism \( \bar{\phi}(c1 + f) = c \). If \( F \) has not a unit, then statements above can be applied to \( \bar{F} \) and \( \bar{R}' \) so that an element \( \bar{x} = cI1 + x \) with \( c \neq 0 \) may have a left inverse of the form \( (bI1 + y) \).

75. Corollary. Suppose that \( \Gamma \) is an additive discrete group so that \( \Gamma = \Gamma_0i_0 \oplus \ldots \oplus \Gamma_mi_m \) with pairwise isomorphic commutative groups \( \Gamma_j \) and \( \Gamma_k \) for each \( 0 \leq j, k \leq m \) with \( m = 2^r - 1, \ 0 \leq r \leq 3 \), while \( G = G_0i_0 \oplus \ldots \oplus G_mi_m \) is an additive group so that \( G_j \) is dual to \( \Gamma_j \) with continuous characters \( \chi(\beta, t) = \prod_{k=0}^{m} \chi_k(\beta_k, t_k) \in S^1 \), where \( S^1 := \{ u \in C_1 : |u| = 1 \} \), \( \beta = \beta_0i_0 + \ldots + \beta_mi_m \) and \( t = t_0i_0 + \ldots + t_mi_m \) with \( \beta_k \in \Gamma_k \) and \( t_k \in G_k \) for each \( k \). Let

\[
(1) \quad x(t) = \sum_{\beta \in \Gamma} a_{\beta} \chi(\beta, t)
\]

with \( a_{\beta} \in R \) for each \( \beta \in \Gamma \) and

\[
(2) \quad \sum_{\beta \in \Gamma} \|a_{\beta}\| < \infty,
\]

then \( x \in R' \) has a left inverse in \( R' \) if \( x(t) \) has a left inverse in \( R_{C_1} \) for each \( t \in G \).

76. Corollary. Let suppositions of Corollary 75 be satisfied, but with (2) replaced by

\[
(1) \quad \sum_{\beta \in \Gamma} e^{\phi(\beta)}\|a_{\beta}\| < \infty,
\]

33
where \( q(\beta) \in \mathbb{R} \) and

\[
q(\alpha + \beta) \leq q(\alpha) + q(\beta) \quad \text{and} \quad q(0) = 0.
\]

Then \( x \in R' \) is invertible, if

\[
\sum_{\beta \in \Gamma} a_\beta e^{p(\beta)} \chi(\beta, t) \chi(\beta, t) \lambda(d\beta)
\]

has a left inverse in \( R_{C_1} \) for each \( t \in G \) with a system

of reals \( p(\beta) \in \mathbb{R} \) so that

\[
(2) \quad p(\alpha + \beta) = p(\alpha) + p(\beta) \quad \text{and} \quad p(0) = 0 \quad \text{and} \quad p(\beta) \leq q(\beta) \quad \text{for each} \beta \in \Gamma.
\]

77. Corollary. Let suppositions of Corollary 76 be satisfied, but let \( \Gamma \) be a locally compact group with a nontrivial nonnegative Haar measure \( \lambda \). If \( R' \) is formed by elements of the form:

\[
(1) \quad x(t) = \int_{\Gamma} a_\beta \chi(\beta, t) \lambda(d\beta)
\]

with \( a_\beta \in \mathbb{R} \) and

\[
(2) \quad \int_{\Gamma} e^{q(\beta)}\|a_\beta\|\lambda(d\beta) < \infty.
\]

If \( R' \) has the unit \( 1(t) = 1 \) for each \( t \in G \), then \( x \) is left invertible, if \( x(t) \) has a left inverse in \( R_{C_1} \), for each \( t \in \Gamma \). If \( R' \) has not a unit, but 1 is an adjoint unit as in §74, then an element \( \check{x} = c1 + x \) with \( c \neq 0 \) has a left inverse of the form \( b1 + y \), if \( [cI + \int_{\Gamma} a_\beta e^{p(\beta)} \chi(\beta, t) \lambda(d\beta)] \) has a left inverse for every \( t \in G \) and each continuous system \( p(\beta) \) satisfying Conditions 76(2,3).

78. Remark. Duality theory for locally compact groups is contained in [31, 13]. Particularly, \( A_r \) can be considered as the additive commutative group \( (A_r, +) \). As the additive group it is isomorphic with \( \mathbb{R}^{2^r} \). The group of characters of \( \mathbb{R}^n \) is isomorphic with \( \mathbb{R}^n \) for any natural number \( n \) (see §23.27(f) in Chapter 6 of the book [13]). The Lebesgue measure on the real shadow \( \mathbb{R}^{2^r} \) induces the Lebesgue measure \( \lambda \) on \( A_r \), which is the Haar measure on \( (A_r, +) \) (see also §1).

It is possible to consider a dense subgroup \( K \) of the total compact dual group \( G \), when \( \Gamma \) is discrete. It is sufficient an existence of a left inverse \( y(t) \) of \( x(t) \) for each \( t \in K \) and that \( \sup_{t \in K} \|y(t)\| < \infty \) due to the following lemma.

79. Lemma. Let \( x_n \) tend to \( x \) in \( R \), when a natural number \( n \) tends to the infinity, let also \( y_n \) be a left inverse of \( x_n \) for each \( n \) and \( \sup_n \|y_n\| < \infty \), then \( x \in R \) possesses a left inverse.

Proof. From the equality \( I - y_n x = I - y_n x_n + y_n x_n - y_n x \) it follows that

\[
\|I - y_n x\| \leq \|y_n\| \|x_n - x\|.
\]

Then Lemma 53 implies that a natural number \( k \)
exists so that $y_n x$ has a left inverse $z_n$ for each $n \geq k$, consequently, $z_n y_n$ is a left inverse of $x$ due to the alternativity of the algebra $R$ or using Moufang’s identities.

80. Corollaries. Suppose that an algebra $R$ is over the Cayley-Dickson algebra $A_v$ (see §52) and $0 \leq r \leq v$ and $2 \leq v \leq 3$, $\Gamma = (A_r, +)$ (see §§75-78).

1. If

$$x(t) = \sum_n a_n e^{(\beta(n), t)_1} \in R'$$

with $a_n \in R$ and $\sum_n \|a_n\| < \infty$, $(\beta, t) = \text{Re}(\beta t^*) = \beta_0 t_0 + \ldots + \beta_m t_m$, where $\beta = \beta_0 i_0 + \ldots + \beta_m i_m \in (A_v, +)$, $t = t_0 i_0 + \ldots + t_m i_m \in G = (A_v, +)$, $m = 2^r - 1$, moreover, a left inverse $z(t)$ exists for each $t$ and $\sup_t \|z(t)\| = C < \infty$, then a function

$$y(t) = \sum_n b_n e^{(\tau(n), t)_1} \in R'$$

exists with $\sum_n \|b_n\| < \infty$ such that $y(t) x(t) = I$ for each $t$.

2. If $a(\beta) \in R$ is a strongly integrable function with

$$\int_{A_v} \|a(\beta)\| \lambda(d\beta) < \infty$$

and if for a nonzero complex number $c \in C \setminus \{0\}$ a function

$$[cI + \int_{A_v} a(\beta) e^{(\beta, t)_1} \lambda(d\beta)]$$

has a left inverse for all $t$ (see $\lambda$ in §78), then a left inverse of the form

$$[qI + \int_{A_v} b(\beta) e^{(\beta, t)_1} \lambda(d\beta)]$$

exists with

$$\int_{A_v} \|b(\beta)\| \lambda(d\beta) < \infty.$$
(1) an operator $B$ is invertible;

(2) a Fourier transform operator $\hat{B}(M)$ is invertible for each $M \in S^1$ and $x \in Y$.

**Proof.** A real linear Banach subspace $X_k$ is considered, which is linearly isometrically isomorphic with $l_\infty(Z, Y_k)$ for each $k \geq 0$. On the other hand, the real span $\text{span}_R\{x \in X_k i_k : k \geq 0\}$ is dense in $X$. In view of Theorem 51 an operator $B$ is uniformly $c$-continuous, $B \in L_{uc}^q(l_\infty(Z, Y))$. Therefore, its invertibility on $\text{span}_R\{x \in X_k i_k : k \geq 0\}$ is equivalent to that of on $X$.

Let $B \in L_{n, \text{per}}^q(l_\infty(Z, Y))$ be an invertible operator with $D = B^{-1}$. In view of Corollary 81 the inverse operator $D$ is $n$-periodic as well and has an absolutely converging Fourier series. From Theorem 51 it follows that $D$ is uniformly $c$-continuous. Applying Corollary 47 and Proposition 45 we deduce that the Fourier transform operator $\hat{B}$ is invertible for each $M \in S^1$ and $x \in Y$, that is $\hat{B}(M)\hat{D}(M)x = \hat{B}\hat{D}(M)x = I_Y x = x$. Thus (1) $\Rightarrow$ (2).

Vise versa suppose that Condition (2) is fulfilled. Then by Corollary 81 the mapping $\psi : M \mapsto (\hat{B}(M))^{-1}$ has an absolutely converging Fourier series:

$$\psi(M) = \sum_{k=-\infty}^{\infty} M^k kD,$$

$$kD := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \psi(e^{it}) dt \in L_q(Y \oplus Y i).$$

Put $D = \sum_{k=-\infty}^{\infty} k\hat{D}S(k)$, where as usually $S(k)$ denotes the coordinatewise shift operator on $k$, $S(k)x(l) = x(l + k)$, while $k\hat{D}$ denotes an 1-ribbon operator, matrix elements of which on the $k$-th diagonal are equal to $kD$. Therefore, the operator $D$ is bounded with $\|D\| \leq \sum_{k=-\infty}^{\infty} \|kD\| = c < \infty$. In accordance with Corollary 47 the operator $D$ is inverse of $B$, i.e. $D = B^{-1}$.

**83. Corollary.** Let $B \in L_{q, \text{per}}^n(l_\infty(Z, Y))$, where $Y$ is a Banach space over either the quaternion skew field or the octonion algebra $A_v$, $2 \leq v \leq 3$. Then spectral sets of $B$ and $B(D(M))$ are related by the formula:

$$\sigma(B) = \bigcup_{M \in S^1} \sigma(B(D(M)))$$

where $B$ denotes the natural extension of $B$ from $l_\infty(Z, Y)$ onto $l_\infty(Z, Y \oplus Y i)$.

**Proof.** The spectral set $\sigma(B)$ is the complement of the resolvent set (see Definition I.2.6 [23]), where $Y \oplus Y i$ and $l_\infty(Z, Y \oplus Y i)$ have structures of $A_v$. 

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Banach spaces as well. In view of Proposition 45 and Theorem 82 one gets this corollary.

84. Theorem. Let a kernel $K$ of a periodic operator $B$ from §1 satisfy the condition:

$$\sup_{t,s} \|K(t, s)\| = c_1 < \infty,$$

where $2 \leq v \leq 3$. Then an operator $A = I - B$ is invertible if and only if a Fourier transform operator $\hat{A}(M)$ is invertible on $Y \oplus Y^i$ for each $M \in S^1$.

Proof. Condition (1) implies that an operator $B$ is bounded and the integral of the condition:

$$\int_{\mathbb{R}}^t \mathcal{F}(\hat{A})d\tau = 0$$

for each variable $\tau$. For a function $(x, \omega)$ where $\omega$ is invertible if and only if $\omega = \omega_j \in [0, \omega_j]$. Thus, this operator $\hat{A}$ is invertible on $Y \oplus Y^i$ for each $M \in S^1$.

Take an operator $A = I - B \in L_q(L^p(A_w, Y))$, where $p \in [1, \infty]$. Choose a domain $V$ in the Cayley-Dickson algebra $A_w$ so that $V = \{z : z \in A_w; \forall j z_j \in [0, \omega_j]; z = \sum_{j=0}^{2^{w-1}} z_j\}$. Then we define an operator $U : L^p(A_w, Y) \to l_p(\mathbb{Z}^{2^w}, L^p(V, Y))$ by the formula: $(Ux)(t) := y_m(\tau)$, where $x \in L^p(A_w, Y)$ and $y \in l_p(\mathbb{Z}^{2^w}, L^p(V, Y))$ are related by the equation: $y_m(\tau) = x(t - \sum_j m_j \omega_j)$ for each $0 \leq j \leq 2^w - 1$, where $m = (m_0, ..., m_{2^w-1}) \in \mathbb{Z}^{2^w}$. This definition implies that such an operator $U$ is an invertible isometry.

There exists an operator $Q = UAU^{-1}$, hence $Q \in L_q(l_p(\mathbb{Z}^{2^w}, L^p(V, Y)))$. Evidently $Q$ is invertible if and only if $A$ is such. If $S(\omega)x(t) = x(t + \omega)$ and $\hat{S}(\mathcal{F})y_m = y_{m+n}$ are shift operators, they satisfy the equation $\hat{S}(\mathcal{F}) = US(\sum_j m_j \omega_j)U^{-1}$. Therefore, the operator $Q$ commutes with each shift operator $\hat{S}(\mathcal{F})$, where $\mathcal{F} \in \mathbb{Z}^{2^w}$. Thus, this operator $Q$ is $1$-periodic by each $m_j \in \mathbb{Z}$ (see Definition 33).

For a function $x \in L^p(V, Y)$ put $\bar{x}_m(\mathcal{F}) := [\prod_{j=0}^{2^w-1} \delta_{m_j, k_j}]x$, hence $\bar{x}_m \in l_p(\mathbb{Z}^{2^w}, L^p(V, Y))$. By each variable $m_j$, a matrix of the operator $Q$ takes the form: $Q_{k_j, s_j}x = (Q\bar{x}_s)(\mathcal{F}) = (UAU^{-1}\bar{x}_s)(\mathcal{F}) = (UAy_s)(\mathcal{F})$, when $k_l = s_l$ for each $l \neq j$, where $y_s \in l_p(\mathbb{Z}^{2^w}, L^p(V, Y))$ is given by the formula: $y_s(\tau) = 0$ if there exists $j$ so that $\tau_j \notin [s_j \omega_j, (s_j + 1) \omega_j)$, while $y_s(\tau) = x(t - \sum_j s_j \omega_j)$ for each $j$, where $\tau = t - \sum_j s_j \omega_j$. This means that a tensor operator $Q^{\delta}_k$ is defined: $Q^{\delta}_k = (Q\bar{x}_s)(\mathcal{F})$. Then the function $(UAy_s)(\mathcal{F}) \in L^p(V, Y)$ takes the form:

$$2 \sup_{t,s} \|K(t, s)\| = c_1 < \infty,$$
a tensor operator takes the form:

\[-\sigma \int_{\gamma^n(b)|b_0 \in [x_0:0,(x_0+1)\omega_0]} \cdots \sigma \int_{\gamma^n(b)|b_u \in [x_u:0,(x_u+1)\omega_u]} K((\tau + \sum_j k_j \omega_j i_j), b) \]

\[x(b - \sum_j s_j \omega_j i_j) db_0 \cdots db_u\]

\[= x(\tau) - \sigma \int_{\gamma^n(b)|b_0 \in [x_0:0,(x_0+1)\omega_0]} \cdots \sigma \int_{\gamma^n(b)|b_u \in [x_u:0,(x_u+1)\omega_u]} K((\tau + \sum_j k_j \omega_j i_j), b) \]

\[(b + \sum_j (s_j - k_j) \omega_j i_j) x(b) db_0 \cdots db_u\]

in accordance with Conditions 1(5, 6), where \( u := 2^w - 1 \). This implies that a tensor operator takes the form:

\[(3) \quad Q_{\bar{s}}^\bar{k} = x(\tau) - \sigma \int_{\gamma^n(b)|b_0 \in [x_0:0,(x_0+1)\omega_0]} \cdots \sigma \int_{\gamma^n(b)|b_u \in [x_u:0,(x_u+1)\omega_u]} K((\tau + \sum_j k_j \omega_j i_j), \]

\[b + \sum_j (s_j - k_j) \omega_j i_j) x(b) db_0 \cdots db_u\]

for each \( \bar{s}, \bar{k} \in \mathbb{Z}^{2^w} \). Elements of this tensor depend only on the difference \( \bar{s} - \bar{k} \), so it is possible to put \( Q_{\bar{s}}^\bar{k} = Q_{\bar{s} - \bar{k}} \).

The operators \( U, A \) and \( U^{-1} \) are \( c \)-continuous by each \( s_j \), consequently, the operator \( Q_{\bar{s}} \) is also \( c \)-continuous by each \( s_j \), where \( j = 0, 1, \ldots, 2^w - 1 \).

Applying Theorem 82 we obtain the statement of this theorem.

85. Corollary. Let an operator \( B \) satisfy conditions of Theorem 84, then a spectral set is \( \sigma(Q) = \bigcup_{M \in \mathbf{M}} \sigma(Q(D(M))) \), where \( Q \) denotes the natural extension of \( Q \) from \( l_\infty(\mathbb{Z}^{2^w}, L_\infty(V,Y)) \) onto \( l_\infty(\mathbb{Z}^{2^w}, L_\infty(V,Y)) \).

Proof. This follows from Theorem 84 and Corollary 83 applying the Fourier transform by each variable, since \( Q \in L_{q,1}^{\text{per}}(l_\infty(\mathbb{Z}^{2^w}, L_\infty(V,Y))) \) due to Condition 84(1) and the latter Banach space over the Cayley-Dickson algebra \( \mathcal{A}_v \) is isomorphic with \( L_{q,1}^{\text{per}}(l_\infty(\mathbb{Z}^{2^w}, L_\infty(V,Y))) \).

References

[1] M. J. Ablowitz, H. Segur. "Solitons and the inverse scattering transform" (SIAM: Philadelphia, 1981).
[2] S. Bochner, R.S. Phillips. "Absolutely convergent Fourier expansions for non-commutative normed rings". Annals of Mathem. 43: 3 (1942), 409-418.

[3] J.C. Baez. ”The octonions”. Bull. Amer. Mathem. Soc. 39: 2 (2002), 145-205.

[4] F. Brackx, R. Delanghe, F. Sommen. "Clifford analysis" (London: Pitman, 1982).

[5] L.E. Dickson. ”The collected mathematical papers”. Volumes 1-5 (Chelsea Publishing Co.: New York, 1975).

[6] G. Emch. ”Mécanique quantique quaternionienne et Relativité restreinte”. Helv. Phys. Acta 36 (1963), 739-788.

[7] J.C. Ferrando, M. López Pellicer, L.M. Sánchez Ruiz. ”Metrizable barrelled spaces” (Longman Group Ltd: Harlow, 1995).

[8] G.M. Fihtengolz. ”Course of differential and integral calculus”, 8-th Edition, V. 1-3 (Moscow: Fizmatlit, 2003).

[9] J.E. Gilbert, M.A.M. Murray. ”Clifford algebras and Dirac operators in harmonic analysis”. Cambr. studies in advanced Mathem. 26 (Cambr. Univ. Press: Cambridge, 1991).

[10] P.R. Girard. ”Quaternions, Clifford algebras and relativistic Physics” (Birkhäuser: Basel, 2007).

[11] K. Gürlebeck, W. Sprössig. ”Quaternionic analysis and elliptic boundary value problem” (Birkhäuser: Basel, 1990).

[12] F. Gürsey, C.-H. Tze. ”On the role of division, Jordan and related algebras in particle physics” (World Scientific Publ. Co.: Singapore, 1996).

[13] E. Hewitt, K.A. Ross. ”Abstract harmonic analysis” (Berlin: Springer, 1979).
[14] I.L. Kantor, A.S. Solodovnikov. "Hypercomplex numbers" (Springer-Verlag: Berlin, 1989).

[15] R.S. Krausshar, J. Ryan. "Some conformally flat spin manifolds, Dirac operators and automorphic forms". J. Math. Anal. Appl. 325 (2007), 359-376.

[16] V.V. Kravchenko. "On a new approach for solving Dirac equations with some potentials and Maxwell’s system in inhomogeneous media". Operator Theory 121 (2001), 278-306.

[17] V.V. Kuznetzov. "Spectral properties of periodic integral operators". Prepr. 11 (2000), 1-32 (RAN DVO: Vladivostok, 2000).

[18] S.V. Ludkovsky, F. van Oystaeyen. "Differentiable functions of quaternion variables". Bull. Sci. Math. (Paris). Ser. 2. 127 (2003), 755-796.

[19] S.V. Ludkovsky. "Differentiable functions of Cayley-Dickson numbers and line integration". J. of Mathem. Sciences 141: 3 (2007), 1231-1298.

[20] S.V. Ludkovsky. "Algebras of operators in Banach spaces over the quaternion skew field and the octonion algebra". J. Mathem. Sciences 144: 4 (2008), 4301-4366.

[21] S.V. Ludkovsky. "Residues of functions of octonion variables". Far East Journal of Mathematical Sciences (FJMS), 39: 1 (2010), 65-104.

[22] S.V. Ludkovsky. "Analysis over Cayley-Dickson numbers and its applications" (LAP Lambert Academic Publishing: Saarbrücken, 2010).

[23] S.V. Ludkovsky, W. Sproessig. "Ordered representations of normal and super-differential operators in quaternion and octonion Hilbert spaces". Adv. Appl. Clifford Alg. 20: 2 (2010), 321-342.

[24] S.V. Ludkovsky, W. Sprössig. "Spectral theory of super-differential operators of quaternion and octonion variables", Adv. Appl. Clifford Alg. 21: 1 (2011), 165-191.
[25] S.V. Ludkovsky, W. Sprössig. “Spectral representations of operators in Hilbert spaces over quaternions and octonions”. Complex Variables and Elliptic Equations, online, DOI:10.1080/17476933.2010.538845, 24 pages (2011).

[26] S.V. Ludkovsky. ”Integration of vector hydrodynamical partial differential equations over octonions”. Complex Variables and Elliptic Equations, online, DOI:10.1080/17476933.2011.598930, 31 pages (2011).

[27] S.V. Ludkovsky. ”Line integration of Dirac operators over octonions and Cayley-Dickson algebras”. Computational Methods and Function Theory, 12: 1 (2012), 279-306.

[28] S.V. Ludkovsky. ”Operator algebras over Cayley-Dickson numbers” (LAP LAMBERT Academic Publishing AG & Co. KG: Saarbrücken, 2011).

[29] S.V. Ludkovsky. ”Unbounded normal operators in octonion Hilbert spaces and their spectra”, Los Alamos Nat. Lab., math.FA/1204.1554 (2012), 49 pages.

[30] F. van Oystaeyen. ”Algebraic geometry for associative algebras”. Series ”Lect. Notes in Pure and Appl. Mathem.” 232 (Marcel Dekker: New York, 2000).

[31] L.S. Pontrjagin. ”Continuous groups” (Moscow: Nauka, 1984).

[32] R.D. Schafer. ”An introduction to non-associative algebras” (Academic Press: New York, 1966).

[33] N. Wiener. ”Tauberian theorems”. Annals of Mathematics. 33: 1 (1932), 1-100.

[34] S. Zelditch. ”Inverse spectral problem for analytic domains, II: \( \mathbb{Z}_2 \)-symmetric domains”. Advances in Mathematics 170: 1 (2009), 205-269.