HIGHER INTEGRABILITY FOR
DOUBLY NONLINEAR PARABOLIC SYSTEMS

VERENA BÖGELEIN, FRANK DUZAAR, JUHA KINNUNEN, AND CHRISTOPH SCHEVEN

ABSTRACT. This paper proves a local higher integrability result for the spatial gradient of weak solutions to doubly nonlinear parabolic systems. The new feature of the argument is that the intrinsic geometry involves the solution as well as its spatial gradient. The main result holds true for a range of parameters suggested by other nonlinear parabolic systems.

1. INTRODUCTION

This paper studies regularity of the spatial gradient of weak solutions to doubly nonlinear parabolic equations (systems) of the type

\[ \partial_t (|u|^{p-2} u) - \text{div} \left( |Du|^{p-2} Du \right) = \text{div} \left( |F|^{p-2} F \right) \]

with \(1 < p < \infty\) in a space-time cylinder \(\Omega_T := \Omega \times (0, T)\), where \(\Omega \subset \mathbb{R}^n\) is a bounded domain, \(n \geq 1\), and \(T > 0\). Equation (1.1) is a special case of the general doubly nonlinear parabolic equation

\[ \partial_t (|u|^{m-1} u) - \text{div} \left( |Du|^{p-2} Du \right) = \text{div} \left( |F|^{p-2} F \right), \]

with \(p > 1\) and \(m > 0\). This includes the parabolic \(p\)-Laplacian and the porous medium equation. Note that with the choice \(m = p - 1\) we recover (1.1). Equation (1.2) has a different behavior when \(m < p - 1\) and \(m \geq p - 1\). The first range is called the slow diffusion case, since disturbances propagate with a finite speed and free boundaries occur, while in the second range disturbances propagate with infinite speed and extinction in finite time may occur. This is called the fast diffusion case. In this sense, equation (1.1) represents the borderline case between the slow and fast diffusion ranges.

One might expect that the regularity theory for the doubly nonlinear equation (1.1) is similar to the one for the heat equation. In fact, the equation is homogeneous, in the sense that solutions are invariant under multiplication by constants. In addition, a scale and location invariant parabolic Harnack’s inequality holds true for non-negative weak solutions, see [23, 15]. However, in this case Harnack’s inequality does not immediately imply Hölder continuity of solutions, which indicates that there is a difference compared to the heat equation. The main difficulty with (1.1) is that adding a constant to a solution destroys the property of being a solution. The general doubly nonlinear equation (1.2) is non-homogeneous and an intrinsic geometry is used in the regularity theory, i.e. the space-time scaling of cylinders depends either on the solution or the spatial gradient of the solution. The idea that the inhomogeneous behavior of a nonlinear parabolic equation can be compensated by an intrinsic geometry goes back to the pioneering work of DiBenedetto and Friedman, see for example the monograph [4]. The regularity theory of weak solutions of (1.1) and (1.2) is reasonably developed, at least in the scalar case for non-negative solutions; see [23, 10, 15, 5] for Harnack’s inequality, [24, 17, 18] for Hölder regularity results, and finally [22] for Lipschitz regularity with respect to the spatial variable for
solutions bounded from below by a positive constant. However, little is known about signed solutions, regularity of the gradient of a weak solution and systems.

The primary purpose of this paper is to establish a local higher integrability result for the spatial gradient of weak solutions to parabolic equations and systems of the type (1.1). We show that there exists a constant $\varepsilon > 0$, such that

$$|Du|^{p(1+\varepsilon)} \in L^1_{\text{loc}}(\Omega_T),$$

whenever $u$ is a weak solution to the equation or the system. In particular, our result ensures that weak solutions of (1.1) belong to a slightly better Sobolev space than the natural energy space and therefore obey a self-improving property of integrability. Our result comes with a reverse Hölder type estimate, see Theorem 2.2. The higher integrability for the doubly nonlinear equation (1.1) has been an open problem for a long time. Here we give an answer to this question in the range

$$\max \left\{ \frac{2n}{n+2}, 1 \right\} < p < \frac{2n}{(n-2)n}.$$

This range may seem unexpected, but the lower bound also appears in the higher integrability for the parabolic $p$-Laplace system [16], while the upper bound is exactly the expected one for the porous medium system in the fast diffusion range. For $n = 1$ and $n = 2$ our result applies whenever $1 < p < \infty$. It remains an open question whether the corresponding result holds true when $n \geq 3$.

The key ingredient in the proof of our main result is a suitable intrinsic geometry. By now, variants of this idea have been successfully used in establishing the higher integrability for the parabolic $p$-Laplace system [16] and very recently for the porous medium equation [9] and system [2]. Our idea is to consider space-time cylinders $Q_{r,s}(z_o) := B_r(x_o) \times (t_o - s, t_o + s)$, with $z_o = (x_o, t_o)$, such that the quotient $\frac{|u|}{|Du|}$ satisfies

$$\frac{8}{r_p} = \mu^{p-2} \quad \text{with} \quad \mu^p \approx \frac{\iiint_{Q_{r,s}(z_o)} \frac{|u|^p}{r^p} \, dx \, dt}{\iiint_{Q_{r,s}(z_o)} [|Du|^p + |F|^p] \, dx \, dt}.$$  

(1.3)

This geometry involves the solution as well as its spatial gradient and therefore allows to balance the mismatch between $|u|$ and $|Du|$ in the equation. To our knowledge this is the first time that such a geometry is used. On these cylinders we are able to prove Sobolev-Poincaré and reverse Hölder type inequalities. The construction of the cylinders is quite involved, since the cylinders on the right-hand side of (1.3) also depend on the parameter $\mu$. In the course of the construction we modify the argument in [9]; see also [2].

In the stationary elliptic case the higher integrability was first observed by Elcrat & Meyers [19], see also the monographs [11, Chapter 11, Theorem 1.2] and [13, Section 6.5]. The first higher integrability result, in the context of parabolic systems, can be found in [12, Theorem 2.1]. The higher integrability for the gradient of solutions for general parabolic systems with $p$-growth has been established by Kinnunen & Lewis [16]. This local interior result has been generalized in the meantime in various directions, e.g. global results, higher order parabolic systems (interior and at the boundary); see [20, 1, 3]. For the porous medium equation, i.e. equation (1.2) with $p = 2$, the question of higher integrability turned out to be more challenging than for the parabolic $p$-Laplace equation, i.e. equation (1.2) with $m = 1$. The problem was solved only recently by Gianazza & Schwarzacher [9]. They proved that non-negative weak solutions to the porous medium equation possess the higher integrability for the spatial gradient. Their proof, however, uses the method of expansion of positivity and therefore cannot be extended to signed solutions and porous medium type systems. A simpler and more flexible proof, which does not rely on the expansion of positivity, is given in [2], where higher integrability for porous medium type systems is achieved. As special case, signed solutions are included in this result.
2. Notation and the main result

2.1. Notation. Throughout the paper we use space-time cylinders of the form

\[ Q^{(\mu)}_G(z_o) := \begin{cases} B_G(x_0) \times \Lambda^{(\mu)}_G(t_o), & \text{if } p < 2, \\ B^{(\mu)}_G(x_0) \times \Lambda_G(t_o), & \text{if } p \geq 2, \end{cases} \]

with center \( z_o = (x_o, t_o) \in \mathbb{R}^n \times \mathbb{R} \), radius \( \rho > 0 \) and scaling parameter \( \mu > 0 \), where

\[ B^{(\mu)}_G(x_0) := \{ x \in \mathbb{R}^n : |x - x_o| < \mu \frac{2}{p-2} \frac{\rho}{g} \}, \quad B_G(x_0) := B^{(1)}_G(x_0). \]

and

\[ \Lambda^{(\mu)}_G(t_o) := (t_o - \mu^{p-2} \rho^p, t_o + \mu^{p-2} \rho^p), \quad \Lambda_G(t_o) := \Lambda^{(1)}_G(t_o). \]

Note that in both cases the cylinders (2.1) admit the scaling property (1.3)1. Moreover, they satisfy the inclusion

\[ Q^{(\mu_1)}_G(z_o) \subseteq Q^{(\mu_2)}_G(z_o) \quad \text{whenever } \mu_1 \leq \mu_2. \]

In the case that \( \mu = 1 \), we omit the scaling parameter in our notation and instead of \( Q^{(1)}_G(z_o) \) we write \( Q_G(z_o) \). For a map \( u \in L^1((0, T); L^1(\Omega; \mathbb{R}^N)) \) and a given measurable set \( A \subset \Omega \) with positive Lebesgue measure the slicewise mean \( \langle u \rangle_{A} := \langle u \rangle_{A}((0, T) \to \mathbb{R}^N) \) of \( u \) on \( A \) is defined by

\[ \langle u \rangle_{A}(t) := \int_{A} u(\cdot, t) \, dx, \quad \text{for a.e. } t \in (0, T). \]

Note that if \( u \in C^0([0, T]; L^p(\Omega; \mathbb{R}^N)) \) the slicewise means are defined for any \( t \in [0, T] \). If the set \( A \) is a ball \( B_G^{(\mu)}(x_0) \), then we abbreviate \( \langle u \rangle_{B_G^{(\mu)}(x_0)}(t) := \langle u \rangle_{B_G}^{(\mu)}(x_0)(t) \) and \( \langle u \rangle_{B_G}^{(1)}(t) := \langle u \rangle_{B_G}^{(1)}(t) \) for \( \mu = 1 \). Similarly, for a given measurable set \( E \subset \Omega_T \) of positive Lebesgue measure the mean value \( \langle u \rangle_{E} \in \mathbb{R}^N \) of \( u \) on \( E \) is defined by

\[ \langle u \rangle_{E} := \int_{E} u \, dx \, dt. \]

If \( E \equiv Q^{(\mu)}_G(z_o) \), we abbreviate \( \langle u \rangle_{Q^{(\mu)}_G(z_o)} := \langle u \rangle_{Q^{(\mu)}_G(z_o)} \). Moreover, we often write \( u(t) := u(\cdot, t) \) for notational convenience. For the power of a vector \( u \in \mathbb{R}^N \), we use the short-hand notation

\[ u^\alpha := |u|^{\alpha-1} u, \quad \text{for } \alpha > 0, \]

which we interpret as \( u^\alpha = 0 \) in the case \( u = 0 \) and \( \alpha \in (0, 1) \). Finally, we let \( p := \max\{p, 2\} \).

2.2. Assumptions and the main result. We consider general systems of the type

\[ \partial_t \left( |u|^{p-2}u \right) - \text{div} \ A(x, t, u, Du) = \text{div} \left( |F|^{p-2} F \right) \quad \text{in } \Omega_T \]

where the vector-field \( A : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N} \) is a Carathéodory function satisfying the standard \( p \)-growth and coercivity conditions

\[ \begin{cases} A(x, t, u, \xi) \cdot \xi \geq \nu |\xi|^p, \\ |A(x, t, u, \xi)| \leq L |\xi|^{p-1}, \end{cases} \]

for a.e. \( z = (x, t) \in \Omega_T \) and any \( (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times N} \), where \( 0 < \nu \leq L < \infty \) are positive constants. In order to formulate our main result, we need to introduce the concept of weak solution.

**Definition 2.1.** Assume that the vector field \( A : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N} \) satisfies (2.3). A measurable function \( u : \Omega_T \to \mathbb{R}^N \) in the class

\[ u \in C^0([0, T]; L^p(\Omega; \mathbb{R}^N)) \cap L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N)) \]
is a weak solution to the doubly non-linear parabolic system (2.2) if and only if the identity
\begin{equation}
\int_{\Omega_T} |u|^p - 2 u \cdot \varphi_t - A(x, t, u, Du) \cdot D\varphi \, dx \, dt = \int_{\Omega_T} |F|^p - 2 F \cdot D\varphi \, dx \, dt
\end{equation}
holds true, for every testing function \( \varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N) \).
\( \square \)

The following theorem is our main result.

**Theorem 2.2.** Let
\begin{equation}
\max \left\{ \frac{2n}{n+2}, 1 \right\} < p < \frac{2n}{(n-2)+},
\end{equation}
where the right-hand side is interpreted as \( \infty \) for the dimensions \( n = 1 \) and \( n = 2 \), and assume that \( \sigma > p \). Then, there exists \( \varepsilon_0 = \varepsilon_0(n, p, \nu, L) \in (0, 1] \) such that whenever \( F \in L^p(\Omega_T, \mathbb{R}^N) \) and \( u \) is a weak solution to (2.2) in the sense of Definition 2.1, then there holds
\( Du \in L^{p(1+\varepsilon)}_{\text{loc}}(\Omega_T, \mathbb{R}^{Nn}) \),
where \( \varepsilon_1 := \min \{ \varepsilon_0, \frac{\sigma}{p} - 1 \} \). Moreover, for every \( \varepsilon \in (0, \varepsilon_1] \) and every cylinder \( Q_{2R} \subseteq \Omega_T \), we have
\begin{align*}
\int_{Q_{2R}} |Du|^{(1+\varepsilon)p} \, dx \, dt &\leq c \left[ 1 + \int_{Q_{2R}} \left( \frac{|u|^p}{(2R)^p} + |Du|^p \right) \, dx \, dt \right]^\varepsilon \int_{Q_{2R}} |Du|^p \, dx \, dt \\
&\quad + c \int_{Q_{2R}} |F|^{(1+\varepsilon)p} \, dx \, dt,
\end{align*}
where \( c = c(n, p, \nu, L) \).
\( \square \)

Although Theorem 2.2 is proved for exponents \( p \) in the range (2.5), we indicate in each sub-step of the proof what are the exact restrictions on \( p \) that are needed in the particular step. In this way, the reader can easily retrace where restriction (2.5) occurs.

### 3. Auxiliary Material

In order to “re-absorb” certain terms, we will use the following iteration lemma, cf. [13, Lemma 6.1].

**Lemma 3.1.** Let \( 0 < \theta < 1 \), \( A, C \geq 0 \) and \( \alpha > 0 \). Then there exists a constant \( c = c(\alpha, \theta) \) such that whenever \( \phi: [r, \theta] \to [0, \infty) \), with \( 0 < r < \theta \), is a non-negative bounded function satisfying
\( \phi(t) \leq \theta \phi(s) + \frac{A}{(s-t)^\alpha} + C \quad \text{for all } r \leq t < s \leq \theta \),
then
\( \phi(r) \leq c \left[ \frac{A}{(\theta-r)^\alpha} + C \right] \).

The next lemma can be deduced as in [13, Lemma 8.3].

**Lemma 3.2.** For any \( \alpha > 0 \), there exists a constant \( c = c(\alpha) \) such that, for all \( a, b \in \mathbb{R}^N \), \( N \in \mathbb{N} \), we have
\( \frac{1}{\alpha} |b^\alpha - a^\alpha| \leq (|a| + |b|)^{\alpha-1} |b-a| \leq c |b^\alpha - a^\alpha| \).

The following lemma is a simple consequence of Lemma 3.2.

**Lemma 3.3.** For any \( \alpha \geq 1 \), there exists a constant \( c = c(\alpha) \) such that, for all \( a, b \in \mathbb{R}^N \), \( N \in \mathbb{N} \), we have
\( |b-a|^\alpha \leq c |b^\alpha - a^\alpha| \).

The next lemma provides useful estimates for the boundary term
\begin{equation}
b[u, v] := \frac{1}{\theta} |v|^p - \frac{1}{\theta} |u|^p - (v-u) \cdot (v-u), \quad \text{for } u, v \in \mathbb{R}^N.
\end{equation}
Lemma 3.4. For any $p \geq 1$ there exists a constant $c = c(p)$ such that for any $u, v \in \mathbb{R}^N$, $N \in \mathbb{N}$, we have

$$\frac{1}{p} b[u, v] \leq \|u^\alpha - v^\alpha\|_p^2 \leq c b[u, v].$$

Proof. The case $1 < p \leq 2$ follows from [2, Lemma 2.3 (i)] applied with $m = \frac{1}{p^\alpha - 1}$. Therefore it remains to consider the case $p > 2$. In the following we denote $\phi_p(u) := \frac{1}{p} |u|^p$. With the abbreviations $a = u^{p - 1}$ and $b = v^{p - 1}$ we compute

$$b[u, v] = \phi_p(v) - \phi_p(u) - D\phi_p(u)(v - u)$$

$$= \frac{1}{p} |b|^{\frac{p}{p - 1}} - \frac{1}{p} |a|^{\frac{p}{p - 1}} - a \cdot (b^{\frac{1}{p - 1}} - a^{\frac{1}{p - 1}})$$

$$= \frac{p - 1}{p} |a|^{\frac{p}{p - 1}} - \frac{p - 1}{p} |b|^{\frac{p}{p - 1}} - b^{\frac{1}{p - 1}} \cdot (a - b)$$

$$= \phi_p(a) - \phi_p(b) - D\phi_p(b)(a - b).$$

Since $\frac{p}{p - 1} < 2$ we may apply Lemma 3.4 in the subquadratic case. In this way we obtain

$$b[u, v] \leq c |a^{\frac{p}{p - 1}} - b^{\frac{p}{p - 1}}|^2 = c \|u^\alpha - v^\alpha\|_p^2$$

and

$$b[u, v] \geq \frac{1}{c} |a^{\frac{p}{p - 1}} - b^{\frac{p}{p - 1}}|^2 \geq \frac{1}{c} \|u^\alpha - v^\alpha\|_p^2.$$

This finishes the proof of the lemma. \qed

It is well known that mean values over subsets $A \subset B$ are quasi-minimizers in the integral $a \mapsto \int_B |u - a|^p dx$. The following statement shows that mean values over subsets are still quasi-minimizing for $u^\alpha$ with $\alpha \geq \frac{1}{p}$. For $p = 2$ and $A = B$, the lemma has been proved in [6, Lemma 6.2]; see also [2, Lemma 2.6]. Here, we state a general version for powers. As expected, the quasi-minimality constant depends on the ratio of the measures of the set and the subset.

Lemma 3.5. Let $p \geq 1$ and $\alpha \geq \frac{1}{p}$. Then, there exists a constant $c = c(\alpha, p)$ such that whenever $A \subset B \subset \mathbb{R}^k$, $k \in \mathbb{N}$, are two bounded domains of positive measure, then for any function $u \in L^{\alpha p}(B, \mathbb{R}^N)$ and any constant $a \in \mathbb{R}^N$, we have

$$\int_B |u^\alpha - (u)^\alpha_A|^p dx \leq \frac{c |B|}{|A|} \int_B |u^\alpha - a^\alpha|^p dx.$$

Proof. The key step in the proof is the estimate of the difference $|(u)^\alpha_A - a^\alpha|$. In the case $\alpha \geq 1$, we use Lemmas 3.2 and 3.3 in order to obtain for a constant $c = c(\alpha, p)$ that

$$|(u)^\alpha_A - a^\alpha|^p \leq c \left( |(u)^\alpha_A|^{(\alpha - 1)p} + |a|^{(\alpha - 1)p} \right) |(u)^\alpha_A - a|^p$$

$$\leq c \left( |(u)^\alpha_A - a|^{(\alpha - 1)p} + |a|^{(\alpha - 1)p} \right) |(u)^\alpha_A - a|^p$$

$$\leq c \int_A |u - a|^{\alpha p} dx + c \int_A |u^\alpha - a^\alpha|^p dx$$

$$\leq c \int_A |u^\alpha - a^\alpha|^p dx. \quad (3.2)$$

Our next goal is to derive the same bound in the case $\frac{1}{p} \leq \alpha < 1$. We begin by applying Lemma 3.2 to obtain

$$|(u)^\alpha_A - a^\alpha| \leq c \left( |(u)^\alpha_A| + |a| \right)^{\alpha - 1} |(u)^\alpha_A - a|$$

$$\leq c \int_A \left( |(u)^\alpha_A| + |a| \right)^{\alpha - 1} |u - a| dx \quad (3.3)$$
and distinguish between two cases. In points \( x \in A \) with \( |u(x)| < 2\hat{c}|a| \), we use the elementary bound
\[
|(u)_A| + |a| \geq |a| \geq c(\alpha)(|u(x)| + |a|),
\]
the fact \( \alpha - 1 < 0 \) and Lemma 3.2 in order to estimate the above integrand by
\[
\left( |(u)_A| + |a| \right)^{\alpha-1}|u(x) - a| \leq c \left( |u(x)| + |a| \right)^{\alpha-1}|u(x) - a|
\]
(3.4)
In the remaining case \( |u(x)| \geq 2\hat{c}|a| \), we have \( \frac{1}{2}|a(x)|^\alpha \leq |u(x)|^\alpha - |a|^\alpha \), which we use for the estimate
\[
|u(x) - |a|| \leq 2\hat{c}|u(x)|^\alpha \leq 2\hat{c}|u(x)|^\alpha - |a|^\alpha \leq 2\hat{c}|u(x)|^\alpha - |a|^\alpha.
\]
We use this and the fact \( \left( |(u)_A| + |a| \right)^{\alpha-1}\left|\right|u(x) - |a|| \leq c \left|\right|u(x)|-|a|\right|^{\alpha} \) to estimate the integrand in (3.3) by
\[
\left( |(u)_A|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \leq c \int_A |u|^\alpha - |a|^\alpha |^p dx + c \left( |u|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \int_A \left|\right|u|^\alpha - |a|^\alpha \right|^p dx.
\]
Now we join the two cases. In view of (3.4) and (3.5), the estimate (3.3) yields the bound
\[
\left( |(u)_A|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \leq c \int_A |u|^\alpha - |a|^\alpha |^p dx + c \left( |u|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \int_A \left|\right|u|^\alpha - |a|^\alpha \right|^p dx.
\]
We multiply this inequality by \( |(u)_A|^\alpha - \alpha |(\frac{\alpha}{p})|\), apply Young’s inequality with exponents \( \frac{1}{\alpha}, \frac{1}{\alpha} \) to the first term on the right-hand side, and Hölder’s inequality with exponents \( \alpha p, \frac{\alpha}{\alpha p} \) to the second term. Note that both is possible in the case \( \frac{1}{p} < \alpha < 1 \), while in the case \( \alpha = \frac{1}{p} \) the application of Hölder’s inequality is not necessary. This procedure results in the estimate
\[
\left( |(u)_A|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \leq c \left( |u|_A - |a|^\alpha \right)^{\frac{\alpha}{p}} \int_A |u|^\alpha - |a|^\alpha |^p dx + c \left( |u|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \int_A \left|\right|u|^\alpha - |a|^\alpha \right|^p dx.
\]
The second-last term can be re-absorbed into the left-hand side, which leads us to
\[
\left( |(u)_A|^\alpha - |a|^\alpha \right)^{\frac{\alpha}{p}} \leq c \int_A |u|^\alpha - |a|^\alpha |^p dx.
\]
This is the estimate (3.2) now also for the case \( \frac{1}{p} \leq \alpha < 1 \). In any case, we can apply either (3.2) or (3.6) to conclude
\[
\int_B |u|^\alpha - \left( |u|_A \right)^\alpha |^p dx \leq 2^{p-1} \int_B |u|^\alpha - |a|^\alpha |^p dx + 2^{p-1} \left( |u|_A - |a|^\alpha \right)^p
\]
\[
\leq 2^{p-1} \int_B |u|^\alpha - |a|^\alpha |^p dx + c \int_A |u|^\alpha - |a|^\alpha |^p dx
\]
\[
\leq c \frac{|B|}{|A|} \int_B |u|^\alpha - |a|^\alpha |^p dx,
\]
which proves the claim.

Finally, we state Gagliardo-Nirenberg’s inequality in the form we will use in the sequel.

\textbf{Lemma 3.6.} Let \( 1 \leq p, q, r < \infty \) and \( \theta \in (0, 1) \) such that \( \frac{\alpha}{p} \leq \theta(1 - \frac{\alpha}{q}) - (1 - \theta) \frac{\alpha}{r} \).

Then there exists a constant \( c = c(n, p) \) such that for any ball \( B_\theta(x_o) \subset \mathbb{R}^n \) with \( \varrho > 0 \) and any function \( u \in W^{1,q}(B_\varrho(x_o)) \), we have
\[
\int_{B_\varrho(x_o)} \frac{|u|^p}{\varrho^p} dx \leq c \left[ \int_{B_\varrho(x_o)} \left[ \frac{|u|^q}{\varrho^q} + |Du|^q \right] dx \right]^{\frac{\alpha}{q}} \left[ \int_{B_\varrho(x_o)} \left[ \frac{|u|^r}{\varrho^r} \right] dx \right]^{\frac{(1-\theta)p}{q}}.
\]
4. Energy bounds

In this section we exploit the doubly nonlinear system (2.2) in order to deduce an energy estimate and a gluing lemma. These are the only points in the proof where the fact that \( u \) is a solution of (2.2) is used.

**Lemma 4.1.** Let \( p > 1 \) and \( a \) be a weak solution to (2.2) in \( \Omega_T \) in the sense of Definition 2.1. Then, on any cylinder \( Q_{R,S}(z_0) := B_R(x_o) \times \Lambda_S(t_o) \subseteq \Omega_T \) with \( R, S > 0 \), and for all \( r \in [R/2, R) \), \( s \in [S/2^p, S) \) and \( a \in \mathbb{R}^N \), we have

\[
\sup_{t \in \Lambda_s(t_o)} \int_{B_r(x_o)} \frac{|u^a_\eta(t) - a^a_\eta|^2}{S} \, dx + \iint_{Q_{r,s}(z_0)} |Du|^p \, dz \, dt \leq c \iint_{Q_{R,S}(z_0)} \left[ \frac{|u^a_\eta - a^a_\eta|^2}{S - s} + \frac{|u - a|^p}{(R - r)^p} + |F|^p \right] \, dz \, dt,
\]

where \( c = c(p, \nu, L) \).

**Proof.** For \( v \in L^1(\Omega_T, \mathbb{R}^N) \), we define the following mollification in time

\[
\|v\|_b(x,t) := \frac{1}{\pi} \int_0^t e^{-\frac{(x-s)^2}{\pi}} v(x,s) \, ds.
\]

From the weak form (2.4) of the differential equation we deduce the mollified version

\[
\iint_{\Omega_T} \left[ \partial_t [u^{p-1}]_h \cdot \varphi + [A(x,t,u,Du)]_h \cdot D\varphi \right] \, dx \, dt
\]

(4.1)

\[
= \iint_{\Omega_T} [F]^p \cdot D\varphi \, dx \, dt + \frac{1}{\pi} \int_{\Omega} u^{p-1}(0) \cdot \int_0^T e^{-\frac{t^2}{\pi}} \varphi \, ds \, dx,
\]

for any \( \varphi \in L^p(0,T; W^{-1,p}_0(\Omega, \mathbb{R}^N)) \). Let \( \eta \in C_0^1(B_R(x_o), [0,1]) \) be a cut-off function with \( \eta \equiv 1 \) in \( B_r(x_0) \) and \( |D\eta| \leq \frac{2}{\pi^2} \) and \( \zeta \in W^{1,\infty}(\Lambda_S(t_o), [0,1]) \) defined by

\[
\zeta(t) := \begin{cases} \frac{t - t_o + S}{S - s} & \text{for } t \in (t_o - S, t_o - s), \\ 1 & \text{for } t \in [t_o - s, t_o + S). \end{cases}
\]

Furthermore, for \( \varepsilon > 0 \) small enough and \( t_1 \in \Lambda_s(t_o) \) we define the function \( \psi_\varepsilon \in W^{1,\infty}(\Lambda_S(t_o), [0,1]) \) by

\[
\psi_\varepsilon(t) := \begin{cases} 1 & \text{for } t \in (t_o - S, t_1], \\ 1 - \frac{1}{\varepsilon}(t - t_1) & \text{for } t \in (t_1, t_1 + \varepsilon), \\ 0 & \text{for } t \in [t_1 + \varepsilon, t_o + S). \end{cases}
\]

In (4.1) we choose the testing function

\[
\varphi(x,t) = \eta^p(x)\zeta(t)\psi_\varepsilon(t)(u(x,t) - a).
\]

In the following we abbreviate \( w^{p-1} := [u^{p-1}]_h \) and omit in the notation the reference to the center \( z_o = (x_o, t_o) \). For the integral in (4.1) containing the time derivative we compute

\[
\iint_{\Omega_T} \partial_t [u^{p-1}]_h \cdot \varphi \, dx \, dt = \iint_{Q_{R,S}} \eta^p \zeta_\psi \partial_t w^{p-1} \cdot (w - a) \, dx \, dt + \iint_{Q_{R,S}} \eta^p \zeta_\psi \partial_t [w^{p-1}]_h \cdot (u - w) \, dx \, dt \\
\geq - \iint_{Q_{R,S}} \eta^p \zeta_\psi \partial_t b(w, a) \, dx \, dt
\]

\[
= - \iint_{Q_{R,S}} \eta^p \zeta_\psi \partial_t b[w, a] \, dx \, dt
\]
\[ = \int_{Q_{n,s}} \eta^p (\zeta \psi' + \psi \zeta') b[w, a] dx dt, \]

where we used the identity \( \partial_t u^{p-1} = -\frac{1}{p}(u^{p-1} - u^{p-1}) \) and recall the definition of the boundary term \( b \) in (3.1). Since \( [u^{p-1}]_h \to u^{p-1} \) in \( L^\infty(\Omega_T) \) we can pass to the limit \( h \downarrow 0 \) in the integral on the right-hand side. We therefore get

\[
\lim \inf_{h \downarrow 0} \int_{Q_{n,s}} \partial_t [u^{p-1}]_h \cdot \varphi \, dx dt \geq \int_{Q_{n,s}} \eta^p (\zeta \psi' + \psi \zeta') b[u, a] \, dx dt =: I_\varepsilon + \Pi_\varepsilon.
\]

We now pass to the limit \( \varepsilon \downarrow 0 \). For the term \( I_\varepsilon \) we obtain for any \( t_1 \in \Lambda_s \) that

\[
\lim_{\varepsilon \downarrow 0} I_\varepsilon = \int_{B_R} \eta^p b[u(t_1), a] \, dx.
\]

Taking into account that the boundary term \( b[u, a] \) is non-negative, the term \( \Pi_\varepsilon \) can be estimated independently from \( \varepsilon \), since

\[
|\Pi_\varepsilon| \leq \int_{Q_{n,s}} \zeta' b[u, a] \, dx dt \leq \int_{Q_{n,s}} b[u, a] \, dx dt.
\]

Next, we consider the diffusion term. After passing to the limit \( h \downarrow 0 \), we use the ellipticity and growth assumption (2.3) for the vector-field \( A \), and subsequently Young’s inequality. In this way, we obtain

\[
\lim \sup_{h \downarrow 0} \int_{\hat{\Omega}_T} [A(x, t, u, Du)]_h \cdot D\varphi \, dx dt
\]

\[
= \int_{Q_{n,s}} \zeta \psi A(x, t, u, Du) \cdot [\eta^p Du + mp^{-1}(u - a) \otimes D\eta] \, dx dt
\]

\[
\geq \nu \int_{Q_{n,s}} \eta^p [\zeta' \psi] Du^p \, dx dt - Lp \int_{Q_{n,s}} \eta^p \zeta' \psi \, D\eta \, |u - a| |Du|^{p-1} \, dx dt
\]

\[
\geq \frac{\nu}{2} \int_{Q_{n,s}} \eta^p \zeta' \psi \, Du^p \, dx dt - c \int_{Q_{n,s}} \frac{|u - a|^p}{(R - r)^p} \, dx dt.
\]

Next, we consider the right-hand side term involving the inhomogeneity \( F \). With the help of Young’s inequality we find that

\[
\lim_{h \downarrow 0} \int_{\hat{\Omega}_T} [F|^{p-2} F]_h \cdot D\varphi \, dx dt
\]

\[
= \int_{Q_{n,s}} \zeta \psi |F|^{p-2} F \cdot [\eta^p Du + mp^{-1}(u - a) \otimes D\eta] \, dx dt
\]

\[
\leq \frac{\nu}{2} \int_{Q_{n,s}} \eta^p \zeta' \psi \, Du^p \, dx dt + c \int_{Q_{n,s}} \left[ \frac{|u - a|^p}{(R - r)^p} + |F|^p \right] \, dx dt.
\]

Finally, for the last integral in (4.1), the convergence of the mollifications and the fact \( \varphi(0) = 0 \) imply

\[
\lim_{h \downarrow 0} \frac{1}{\hat{T}} \int_{\hat{\Omega}} u^{p-1}(0) \cdot \int_0^T e^{-\frac{s}{\hat{T}}} \varphi \, ds \, dx = \int_\Omega u^{p-1}(0) \cdot \varphi(0) \, dx = 0.
\]

Combining the preceding results and passing to the limit \( \varepsilon \downarrow 0 \) we obtain for almost every \( t_1 \in \Lambda_s \) that

\[
\int_{B_R} b[u(t_1), a] \, dx + \int_{t_0 + s}^{t_1} \int_{B_R} |Du|^p \, dx dt \leq c \int_{Q_{n,s}} \left[ \frac{|u - a|^p}{(R - r)^p} + \frac{b[u, a]}{S - s} + |F|^p \right] \, dx dt,
\]

for a constant \( c = c(p, \nu, L) \). Here we pass to the supremum over \( t_1 \in \Lambda_s \) in the first term on the left-hand side. In the second one we let \( t_1 \uparrow t_0 + s \). Finally we take mean values on both sides and apply Lemma 3.4 twice. This leads to the claimed energy estimate. \( \square \)
Next, we deduce a gluing lemma for the doubly nonlinear system.

Lemma 4.2. Let $p > 1$ and $u$ be a weak solution to (2.2) in $\Omega_T$ in the sense of Definition 2.1. Then, on any cylinder $Q_{R,S}(z_0) := B_R(x_0) \times \Lambda_{S}(t_0) \subseteq \Omega_T$ with $R, S > 0$ there exists $\hat{r} \in [\frac{R}{2}, R]$ such that for all $t_1, t_2 \in \Lambda_{S}(t_0)$, we have

$$\left| \langle (u^{p-1})_{x_0, t_1} - (u^{p-1})_{x_0, t_2} \rangle \right| \leq \frac{cS}{R} \iint_{Q_{R,S}(z_0)} \left| Du \right|^{p-1} + \left| F \right|^{p-1} \, dx \, dt,$$

where $c = c(L)$.

Proof. Let $t_1, t_2 \in \Lambda_{S}(t_0)$ with $t_1 < t_2$ and assume that $r \in [\frac{R}{2}, R)$. For $0 < \varepsilon, \delta \ll 1$, we define $\xi_{\varepsilon} \in W^{1,\infty}(\Lambda_{S}(t_0))$ by

$$\xi_{\varepsilon}(t) := \begin{cases} 0, & \text{for } t_0 - S \leq t \leq t_1 - \varepsilon, \\ \frac{t - t_1 + \varepsilon}{\varepsilon}, & \text{for } t_1 - \varepsilon < t < t_1, \\ 1, & \text{for } t_1 \leq t \leq t_2, \\ \frac{t - t_2 - \varepsilon}{\delta}, & \text{for } t_2 \leq t < t_2 + \varepsilon, \\ 0, & \text{for } t_2 + \varepsilon \leq t \leq t_0 + S \end{cases}$$

and a radial function $\Psi_{\delta} \in W^{1,\infty}(B_{r+\delta}(x_0))$ by $\Psi_{\delta}(x) := \psi_{\delta}(\|x - x_0\|)$, where

$$\psi_{\delta}(s) := \begin{cases} 1, & \text{for } 0 \leq s \leq r, \\ \frac{r + s - \delta}{\delta}, & \text{for } r < s < r + \delta, \\ 0, & \text{for } r + \delta \leq s \leq R. \end{cases}$$

For fixed $i \in \{1, \ldots, N\}$ we choose $\varphi_{\varepsilon, \delta} = \xi_{\varepsilon} \psi_{\delta} e_i$ as testing function in the weak formulation (2.4), where $e_i$ denotes the $i$-th canonical basis vector in $\mathbb{R}^N$. In the limit $\varepsilon, \delta \downarrow 0$ we obtain

$$\int_{B_r(x_0)} [u^{p-1}(t_2) - u^{p-1}(t_1)] \cdot e_i \, dx$$

$$= \int_{t_1}^{t_2} \int_{\partial B_r(x_0)} [A(x, t, u, Du) + |F|^{p-2} F] \cdot e_i \otimes \frac{x - x_0}{|x - x_0|} \, dH^{n-1}(x) \, dt,$$

for a.e. $r \in [\frac{R}{2}, R)$. Multiplying the preceding inequality by $e_i$ and summing over $i = 1, \ldots, N$ yields

$$\int_{B_r(x_0)} [u^{p-1}(t_2) - u^{p-1}(t_1)] \, dx$$

$$= \int_{t_1}^{t_2} \int_{\partial B_r(x_0)} [A(x, t, u, Du) + |F|^{p-2} F] \frac{x - x_0}{|x - x_0|} \, dH^{n-1}(x) \, dt.$$
there exists a radius \( \hat{r} \in \left[ \frac{R}{2}, R \right] \) with
\[
\int_{A_\hat{r}(t_o)} \int_{B_{\hat{r}}(x_o)} [L|Du|^{p-1} + |F|^{p-1}]d\mathcal{H}^{n-1}dt \\
\leq \frac{c}{R} \int_{A_{\hat{r}}(t_o)} \int_{B_{\hat{r}}(x_o)} [L|Du|^{p-1} + |F|^{p-1}]dxdt.
\]
In the above inequality, we choose \( r = \hat{r} \) and then take means on both sides of the resulting estimate. This implies
\[
\left| \langle u^{p-1}_{x_o, \hat{r}}(t_2) - u^{p-1}_{x_o, \hat{r}}(t_1) \rangle \right| \leq \frac{c}{R} \int_{A_{\hat{r}}(t_o)} \int_{B_{\hat{r}}(x_o)} [L|Du|^{p-1} + |F|^{p-1}]dxdt \\
= \frac{cS}{R} \int_{Q_{R,S}(x_o)} [L|Du|^{p-1} + |F|^{p-1}]dxdt,
\]
for any \( t_1, t_2 \in A_{\hat{r}}(t_o) \) and with a constant \( c = c(L) \).

5. Parabolic Sobolev-Poincaré Type Inequalities

One of the difficulties in the parabolic setting is that weak solutions are not necessarily differentiable with respect to time. As a consequence, the Sobolev-Poincaré inequality on \( \mathbb{R}^n \) is not applicable. Since such an inequality is indispensable in the proof of the higher integrability we will derive some type of Poincaré and Sobolev-Poincaré inequality which is valid for weak solutions. The idea is to use the Gluing Lemma 4.2 in order to manage the lack of differentiability with respect to time.

Throughout this section we consider scaled cylinders \( Q_{\varrho, \mu}^{(\varrho)}(z_o) \subseteq \Omega_T \) as defined in (2.1) on which certain intrinsic, respectively sub-intrinsic couplings with respect to \( u \) and its spatial gradient \( Du \) hold true. For \( \varrho, \mu > 0 \) we assume that
\[
(5.1) \quad \frac{\int_{Q_{\varrho}^{(\varrho)}(z_o)} \frac{|u|^p}{\mu^{2-p} \varrho^p} dx dt}{\int_{Q_{\varrho}^{(\varrho)}(z_o)} [L|Du|^p + |F|^p] dx dt} \leq K \mu^p
\]
holds true for a constant \( K \geq 1 \). Recall that \( p = \max\{2, p\} \). Such cylinders are termed \( \mu \)-sub-intrinsic. Furthermore, we assume that either
\[
(5.2) \quad \mu^p \leq K \frac{\int_{Q_{\varrho}^{(\varrho)}(z_o)} \frac{|u|^p}{\mu^{2-p} \varrho^p} dx dt}{\int_{Q_{\varrho}^{(\varrho)}(z_o)} [L|Du|^p + |F|^p] dx dt} \quad \text{or} \quad \mu^p \leq K
\]
holds true. A cylinder \( Q_{\varrho}^{(\varrho)}(z_o) \) satisfying (5.2) is called \( \mu \)-super-intrinsic. Finally, a cylinder which is \( \mu \)-sub- and \( \mu \)-super-intrinsic is called \( \mu \)-intrinsic. In the following we distinguish the cases whether the growth exponent \( p \) is sub- or superquadratic. In order to emphasize the stability of the proof when \( p \to 2 \), we include the quadratic case \( p = 2 \) in both subsections.

5.1. The case \( \max\{\frac{2n}{n+2}, 1\} < p \leq 2 \). As a first preliminary result, we compare the first and the second term on the right-hand side of the energy inequality in Lemma 4.1. It turns out that for \( p \in (1, 2] \) on \( \mu \)-sub-intrinsic cylinders the second term can easily be bounded in terms of the first one.
Lemma 5.1. Let $1 < p \leq 2$ and $u$ be a weak solution to (2.2) in $\Omega_T$ in the sense of Definition 2.1. Then, on any cylinder $Q^\mu_{\tilde{r}}(z_0) \subseteq \Omega_T$ with $\tilde{r}, \mu > 0$ satisfying the $\mu$-sub-intrinsic coupling (5.1), we have
\[
\iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u - (u)_{\tilde{r}, \mu}^\mu|}{\tilde{r}^p} \, dx \, dt \leq c \left[ \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u^\mu - [(u)_{\tilde{r}, \mu}^\mu]^\mu|}{\tilde{r}^p} dx \, dt \right]^2 \left[ \iint_{Q^\mu_{\tilde{r}}(z_0)} [|Du|^p + |F|^p] \, dx \, dt \right]^{\frac{2-p}{2}}
\]
where $c = c(p, K)$.

Proof. For simplicity in notation, we omit the reference point $z_0$. Due to Lemma 3.2 applied with $\alpha = \frac{p}{2}$ and Hölder’s inequality, we obtain
\[
\iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u - (u)_{\tilde{r}, \mu}^\mu|}{\tilde{r}^p} \, dx \, dt \leq c \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u^\mu - [(u)_{\tilde{r}, \mu}^\mu]^\mu|}{\tilde{r}^p} \, dx \, dt \leq c \left[ \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u^\mu - [(u)_{\tilde{r}, \mu}^\mu]^\mu|}{\tilde{r}^p} \, dx \, dt \right]^2 \left[ \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u|^p + |(u)_{\tilde{r}, \mu}^\mu|^p}{\tilde{r}^p} \, dx \, dt \right]^{\frac{2-p}{2}}.
\]
For the last integral, hypothesis (5.1) yields
\[
\iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u|^p + |(u)_{\tilde{r}, \mu}^\mu|^p}{\tilde{r}^p} \, dx \, dt \leq 2^p \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u|^p}{\tilde{r}^p} \, dx \, dt \leq 2^p K^p \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|Du|^p + |F|^p}{\tilde{r}^p} \, dx \, dt.
\]
Inserting this above proves the claim with a constant $c$ depending only on $p$ and $K$. □

The next lemma should be interpreted as a parabolic Poincaré inequality for solutions on $\mu$-sub-intrinsic cylinders. The fact that weak solutions do not necessarily possess a weak time derivative is compensated by the Gluing Lemma 4.2. However, the gluing lemma provides an estimate for time differences of slice-wise means of $u^{p-1}$ rather than $u$. Therefore, mean values of $u^{p-1}$ and $u$ have to be estimated very carefully against each other.

Lemma 5.2. Let $1 < p \leq 2$ and $u$ be a weak solution to (2.2) in $\Omega_T$ in the sense of Definition 2.1. Then, on any cylinder $Q^\mu_{\tilde{r}}(z_0) \subseteq \Omega_T$ satisfying the $\mu$-sub-intrinsic coupling (5.1), the inequality
\[
\iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u - (u)_{\tilde{r}, \mu}^\mu|^q}{\tilde{r}^q} \, dx \, dt \leq c \left[ \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|Du|^q + |F|^q}{\tilde{r}^q} \, dx \, dt \right]^{p-1} \left[ \iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|Du|^p + |F|^p}{\tilde{r}^p} \, dx \, dt \right]^{\frac{q(2-p)}{2}}
\]
holds true for any $q \in [1, p]$ and a constant $c = c(n, p, L, K)$.

Proof. In the proof we renounce again to consider the center $z_0$ in the notation. With $\tilde{r} \in [\frac{1}{2}, \rho]$ we denote the radius from Lemma 4.2. We start by estimating the left-hand side with the help of the quasi-minimality of the mean value as follows
\[
\iint_{Q^\mu_{\tilde{r}}(z_0)} \frac{|u - (u)_{\tilde{r}, \mu}^\mu|^q}{\tilde{r}^q} \, dx \, dt \leq c \iint_{Q^\mu_{\rho}(z_0)} \frac{|u - [(u^{p-1})_{\rho}^{\rho}]_{\rho}^{\rho}|}{\rho^q} dx \, dt \leq c \, [I + II],
\]
where we abbreviated
\[
I := \iint_{Q^p_\varepsilon} \frac{|u - (u_{p-1})^{\frac{1}{p-1}}(t)|^q}{q^q} \, dx dt,
\]
\[
II := \int_{\Lambda_{\varepsilon}^{(p)}} \frac{|(u_{p-1})^{\frac{1}{p-1}}(t) - (u_{p-1})^{\frac{1}{p-1}}(t)|^q}{q^q} dt.
\]

Next, we treat the terms I and II of the right-hand side. For the term I we first recall that \( \hat{\theta} \in [\frac{q}{p}, q] \). Therefore, the application of Lemma 3.5 with \( \alpha = \frac{1}{p-1} \geq \frac{q}{p} \) and subsequently Poincaré’s inequality leads to
\[
I \leq c \iint_{Q_{\varepsilon}^p} \frac{|u - (u_{\hat{\theta}}(t))|}{q^q} \, dx dt \leq c \iint_{Q_{\varepsilon}^p} |Du|^q \, dx dt,
\]
for a constant \( c = c(n, p) \). Note that \( q \in [1, p) \) and the constant in Poincaré’s inequality depends continuously on \( q \). Now we will treat II. An application of Lemma 3.2 with \( \alpha = \frac{1}{p-1} \geq 1 \) and subsequently Hölder’s inequality yields
\[
II \leq c \sup_{t, \tau \in \Lambda_{\varepsilon}^{(p)}} \left| \left( u_{p-1}^{(\hat{\theta})}(t) - u_{p-1}^{(\hat{\theta})}(\tau) \right) \right|^q \left( \int_{\Lambda_{\varepsilon}^{(p)}} \left| (u_{p-1}^{(\hat{\theta})}(t)) - (u_{p-1}^{(\hat{\theta})}(\tau)) \right|^q dt \right)^{\frac{p-2}{p}}.
\]
for a constant \( c(p) \). We continue estimating the right-hand side with the help of the Gluing Lemma 4.2, the \( \mu \)-sub-intrinsic coupling (5.1) and Hölder’s inequality. In this way we find
\[
II \leq c \mu^{\theta(p-2)} \left( \int_{Q_\varepsilon^{(p)}} |Du|^q + |F|^q \, dx dt \right)^{\frac{p-1}{p}} \left( \int_{Q_{\varepsilon}^p} |u_{\hat{\theta}}|^p \, dx dt \right)^{\frac{2(p-\mu)}{p}},
\]
for a constant \( c(p, L, K) \). Joining the preceding estimates for I and II finally proves the claim.

Our next aim is to derive a Sobolev-Poincaré type inequality. It has to be understood in the following way. Lemma 5.1 allows to bound the second term on the right-hand side of the energy inequality in terms of the first one. Therefore, in our Sobolev-Poincaré type inequality we will derive an upper bound for this term. In this bound we would like to have the integral of \( |Du|^q \) for some \( q < p \) on the right-hand side. However, due to the nonhomogeneous behavior of the underlying differential equation some extra terms show up. Fortunately they have exactly the form of the left-hand side of the energy estimate so that they can be re-absorbed later on. Note that the estimate of the term II in the proof of Lemma 5.3 is the only point in the paper where the condition \( p > \frac{2n}{n+2} \) is needed.

Lemma 5.3. Let \( \max \{ \frac{2n}{n+2}, 1 \} < p \leq 2 \) and \( u \) be a weak solution to (2.2) in \( \Omega_T \) in the sense of Definition 2.1. Then, on any cylinder \( Q_{\varepsilon}^{(\mu)}(z_0) \subseteq \Omega_T \) with \( q, \mu > 0 \) satisfying (5.1) and (5.2) and for any \( \varepsilon \in (0, 1] \), we have
\[
\iint_{Q_{\varepsilon}^{(\mu)}(z_0)} \frac{|u_{\hat{\theta}}|^q - (u_{\hat{\theta}}(\xi_{\varepsilon, \theta}))_{\hat{\theta}}^q}{\mu^{p-2} \hat{\theta}^q} \, dx dt.
\]
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\[ \leq \varepsilon \left( \sup_{t \in \Lambda_T^{(\mu)}} \int_{B_{r_0}(x_0)} \frac{|u^\tau(t) - \left( u \right)^{\mu}(z_0,x)|^2}{\mu^{p-2} \rho^p} \, dx + \iint_{Q_T^{(\rho)}} |Du|^p \, dx \right) \]

\[ + \frac{c}{\varepsilon^{1-p/(p-1)}} \left( \iint_{Q_T^{(\rho)}} |Du|^q \, dx \right)^{\frac{p}{q}} + \iint_{Q_T^{(\rho)}} |F|^p \, dx \right] \]

with \( q = \max \{ \frac{2n}{n+2}, 1 \} \) and \( c = c(n, p, L, K) \).

**Proof.** As before, we omit the reference point \( z_0 \) in our notation. Moreover, we abbreviate \( a := (u)_q^{(\mu)} \) and define

\[ I := \sup_{t \in \Lambda_T^{(\mu)}} \int_{B_{r_0}} \frac{|u^\tau(t) - a^\tau|^2}{\mu^{p-2} \rho^p} \, dx, \]

and

\[ II := \left( \int_{\Lambda_T^{(\rho)}} \left( \int_{B_{r_0}} \frac{|u^\tau - a^\tau|^2}{\rho^p} \, dx \right)^{\frac{2}{\rho}} \, dt \right)^{\frac{q}{2}}. \]

Then, we obtain

\[ \iint_{Q_T^{(\rho)}} \frac{|u^\tau - a^\tau|^2}{\mu^{p-2} \rho^p} \, dx \, dt \leq 1^{\frac{2}{\rho}} \cdot [\mu^{2-p} II]^{\frac{q}{2}}. \]

In the following, it remains to consider the second term on the right-hand side. For the estimate of \( \mu \) we use hypothesis (5.2). If (5.2) is satisfied, we first apply Lemma 5.2 with \( q = p \) to obtain

\[ \left[ \iint_{Q_T^{(\rho)}} \frac{|u|^p}{\rho^p} \, dx \, dt \right]^{\frac{q}{p}} \leq \left[ \iint_{Q_T^{(\rho)}} \frac{|u - a|^p}{\rho^p} \, dx \, dt \right]^{\frac{q}{p}} + \frac{|a|}{\rho} \]

\[ \leq c \left[ \iint_{Q_T^{(\rho)}} \left( |Du|^p + |F|^p \right) \, dx \, dt \right]^{\frac{q}{p}} + \frac{|a|}{\rho}. \]

Together with the \( \mu \)-super-intrinsic coupling (5.2) this yields

\[ \mu \leq c + \frac{c |a|}{\rho \left[ \iint_{Q_T^{(\rho)}} \left( |Du|^p + |F|^p \right) \, dx \, dt \right]^{\frac{q}{p}}}, \]

with a constant \( c \) depending on \( n, p, L \) and \( K \). On the other hand, if (5.2) is satisfied, then (5.3) holds true with \( c = K^{1/p} \). Consequently, we have inequality (5.3) in any case and therefore obtain

\[ \mu^{2-p} II \leq c \left( I_1 + I_2 \right), \]

where

\[ I_1 := \left( \int_{\Lambda_T^{(\mu)}} \left( \int_{B_{r_0}} \frac{|u^\tau - a^\tau|^2}{\rho^p} \, dx \right)^{\frac{2}{\rho}} \, dt \right)^{\frac{q}{2}}, \]

and

\[ I_2 := \left[ \iint_{Q_T^{(\rho)}} \left( |Du|^p + |F|^p \right) \, dx \, dt \right]^{\frac{q}{2}}. \]
For the estimate of $II_1$ we apply Lemma 3.3 with $\alpha = \frac{2}{p}$, Sobolev’s inequality and Lemma 5.2. In this way we find

$$II_1 \leq c \left[ \int_{\mathcal{A}_{\nu}^{[\varphi]}} \left[ \int_{B_{\varphi}} \frac{|u - a|^p}{\varrho^p} \, dx \right]^\frac{q}{q-1} \, dt \right]^\frac{q}{q-1}$$

$$\leq c \left[ \int_{\mathcal{A}_{\nu}^{[\varphi]}} \left[ \int_{B_{\varphi}} \frac{|Du|^q + |u - a|^q}{\varrho^q} \, dx \right] \, dt \right]^\frac{q}{q-1}$$

$$\leq c \left[ \int_{Q_{\nu}^{[\varphi]}} |Du|^q + \frac{|u - a|^q}{\varrho^q} \, dx dt \right]^\frac{2(p-1)}{q} \left[ \int_{Q_{\nu}^{[\varphi]}} |Du|^p + |F|^p \, dx dt \right]^{2 - \frac{p}{p+1}},$$

where $c = c(n, p, L, K)$. Now we turn our attention to the second term. With the help of Lemma 3.2 applied with $\alpha = \frac{2}{p}$ and Sobolev’s inequality, we find that

$$II_2 \leq c \left[ \int_{\mathcal{A}_{\nu}^{[\varphi]}} \left[ \int_{B_{\varphi}} \frac{|u - a|^2}{\varrho^2} \, dx \right]^\frac{q}{q-1} \, dt \right]^\frac{q}{q-1} \leq c \left[ \int_{Q_{\nu}^{[\varphi]}} |Du|^q + \frac{|u - a|^q}{\varrho^q} \, dx dt \right]^\frac{q}{q-1}$$

$$\leq c \left[ \int_{Q_{\nu}^{[\varphi]}} |Du|^p + |F|^p \, dx dt \right]^{2 - \frac{p}{p+1}}$$

for a constant $c = c(n, p)$. The term involving $|u - a|^q$ is now treated as above with Lemma 5.2, so that

$$II_2 \leq c \left[ \int_{Q_{\nu}^{[\varphi]}} |Du|^q + |F|^q \, dx dt \right]^\frac{q(p-1)}{q} \left[ \int_{Q_{\nu}^{[\varphi]}} |Du|^p + |F|^p \, dx dt \right]^{2 - \frac{p}{p+1}}$$

holds true with a constant $c$ depending only on $n, p, L$, and $K$. Inserting the preceding estimates above and applying Young’s inequality, we derive the desired inequality.

5.2. The case $p \geq 2$. Now, we turn our attention to the superquadratic case $p \geq 2$. We emphasize that all results of this section hold true for the full range $p \geq 2$. The restriction $p < \frac{2n}{n + 2}$ will be necessary later on in the covering argument. Contrary to the sub-quadratic case in Lemma 5.1, we find in the superquadratic case a straightforward bound on $\mu$-sub-intrinsic cylinders for the first term on the right-hand side of the energy inequality in Lemma 4.1 in terms of the second one.

**Lemma 5.4.** Let $p \geq 2$ and $u$ be a weak solution to (2.2) in $\Omega_T$ in the sense of Definition 2.1. Then, on any cylinder $Q_{\nu}^{[\varphi]}(z) \subset \Omega_T$ with $\varrho, \mu > 0$ satisfying the $\mu$-sub-intrinsic coupling (5.1), we have

$$\left\| \left( \frac{u}{\varrho^p} - \left( \frac{u}{\varrho^p} \right)_{\nu}^{(\mu)} \right) \right\|_{L^2(Q_{\nu}^{[\varphi]}(z), \varrho^p)} \leq c \left\| \left( \frac{u - (u)_{\nu}^{(\mu)}}{\mu^2 - \varrho^p} \right)^p \right\|_{L^2(Q_{\nu}^{[\varphi]}(z), \varrho^p)} \left\| \int_{Q_{\nu}^{[\varphi]}(z)} |Du|^p + |F|^p \, dx dt \right\|^{\frac{p-2}{p}}$$

with $c = c(p, K)$.

**Proof.** As before, we omit the reference point $z_0$ in our notation. Applying Lemma 3.2 with $\alpha = \frac{2}{p}$, Hölder’s inequality, and finally hypothesis (5.1), we obtain

$$\left\| \left( \frac{u}{\varrho^p} - \left( \frac{u}{\varrho^p} \right)_{\nu}^{(\mu)} \right) \right\|_{L^2(Q_{\nu}^{[\varphi]}(z), \varrho^p)} \leq c \left\| \left( \frac{u - (u)_{\nu}^{(\mu)}}{\mu^2 - \varrho^p} \right)^p \right\|_{L^2(Q_{\nu}^{[\varphi]}(z), \varrho^p)} \left\| \int_{Q_{\nu}^{[\varphi]}(z)} |Du|^p + |F|^p \, dx dt \right\|^{\frac{p-2}{p}}$$
which proves the claim. 

The next lemma is the analogue of Lemma 5.2 for the superquadratic case and should be interpreted as a parabolic Poincaré type inequality.

**Lemma 5.5.** Let \( p \geq 2 \) and \( u \) be a weak solution to (2.2) in \( \Omega_T \) in the sense of Definition 2.1. Then, on any cylinder \( Q_\varrho(z_o) \subseteq \Omega_T \) with \( \varrho, \mu > 0 \) satisfying (5.1) and (5.2), where

\[
\iint_{Q_\varrho(z_o)} \frac{|u - (u)_{\varrho}^{(\mu)}|^q}{\varrho^{q/2-p}} \, dx \, dt \leq c \iint_{Q_\varrho(z_o)} |Du|^q + |F|^q \, dx \, dt,
\]

for any \( q \in [p-1, p] \) and \( c = c(n, p, L, K) \).

**Proof.** Throughout the proof we omit the reference to the center \( z_o \) in our notation. Similar to the proof of Lemma 5.2, we find that

\[
\iint_{Q_\varrho(z_o)} \frac{|u - (u)_{\varrho}^{(\mu)}|^q}{\varrho^{q/2-p}} \, dx \, dt \leq \iint_{Q_\varrho(z_o)} \frac{|u - [(u_{\varrho}^{(\mu)})_{(\mu)}]|^{p-1}}{\varrho^{p-2}} \, dx \, dt \leq 2^{p-1} [I + II],
\]

where we abbreviated

\[
I := \iint_{Q_\varrho(z_o)} \frac{|u - [(u_{\varrho}^{(\mu)})_{(\mu)}]|^{p-1}}{\varrho^{p-2}} \, dx \, dt,
\]

\[
II := \mu^{\frac{p-2}{p}} \int_{\Lambda_\varrho} \left[ \frac{|(u_{\varrho}^{(\mu)})_{(\mu)}(t)|^{p-1}}{\varrho^{p-2}} - \frac{|(u_{\varrho}^{(\mu)})_{(\mu)}(\tau)|^{p-1}}{\varrho^{p-2}} \right] \, dt,
\]

with \( \varrho \in [\frac{q_1}{2}, \varrho] \) denoting the radius from the Gluing Lemma 4.2. Recall the abbreviation \( \langle u \rangle_{\varrho}^{(\mu)}(t) := \langle u \rangle_{\varrho}^{(\mu)}(t) \) for the slice-wise mean. For the estimate of I we in turn apply Lemma 3.5 with \( \alpha = \frac{1}{p-1} \geq \frac{1}{q} \) and \( \varrho \in [\frac{q_1}{2}, \varrho] \) and subsequently Poincaré’s inequality. This leads to

\[
I \leq c \iint_{Q_\varrho(z_o)} \frac{|u - (u)_{\varrho}^{(\mu)}(t)|^{q}}{\varrho^{q/2-p}} \, dx \, dt \leq c \iint_{Q_\varrho(z_o)} |Du|^q \, dx \, dt,
\]

for a constant \( c = c(n, p) \). Now we will treat II. We start with an application of Lemma 3.3 with \( \alpha = p-1 \geq 1 \) and subsequently the Gluing Lemma 4.2 with \( R = \mu^{\frac{p-2}{p}} \) and \( S = \varrho^p \). This gives

\[
II \leq \mu^{\frac{q(p-2)}{p}} \int_{\Lambda_\varrho} \frac{|(u_{\varrho}^{(\mu)})_{(\mu)}(t) - (u_{\varrho}^{(\mu)})_{(\varrho)}(t)|^{p-1}}{\varrho^{p-2}} \, dt
\]

\[
\leq \mu^{\frac{q(p-2)}{p}} \int_{\Lambda_\varrho} \int_{\Lambda_\varrho} |(u_{\varrho}^{(\mu)})_{(\mu)}(t) - (u_{\varrho}^{(\mu)})_{(\varrho)}(\tau)|^{p-1} \, dt \, d\tau
\]

\[
\leq c \mu^{\frac{q(p-2)}{p}} \iint_{Q_\varrho(z_o)} |Du|^{p-1} + |F|^{p-1} \, dx \, dt,
\]
Lemma II

If hypothesis (5.2) is satisfied, the lemma is proven. Therefore, it remains to consider the case where (5.2) is in force. Here, we apply Lemma 3.5 to deduce that

\[
\left(\int_Q |u|^p \, dx \right)^{\frac{1}{p}} \leq \left(\int_Q |u - (u)^{\mu}_{\theta}|^p \, dx \right)^{\frac{1}{p}} \leq c \left(\int_Q |u - (u)^{\mu}_{\theta}|^p \, dx \right)^{\frac{1}{p}} + \left(\int_Q |(u)^{\mu}_{\theta}|^p \, dx \right)^{\frac{1}{p}}.
\]

Having arrived at this point, we take the last inequality to the power \( q(p - 2) \) and obtain

\[
\Pi = \left(\int_Q |u|^p \, dx \right)^{\frac{q(p-2)}{p}} \left(\int_Q |u - (u)^{\mu}_{\theta}|^p \, dx \right)^{\frac{q(p-2)}{p}} \cdot \Pi = c [\Pi_1 + \Pi_2],
\]

with the abbreviations

\[
\Pi_1 := \left(\int_Q |u - (u)^{\mu}_{\theta}|^p \, dx \right)^{\frac{q(p-2)}{p}} \cdot \Pi,
\]

and

\[
\Pi_2 := \frac{\left(\int_Q |u - (u)^{\mu}_{\theta}|^p \, dx \right)^{\frac{q(p-2)}{p}}}{\left(\int_Q |u|^p \, dx \right)^{\frac{q(p-2)}{p}}} \cdot \Pi.
\]

For the estimate of \( \Pi_2 \), we proceed as follows. We first insert the definition of \( \Pi \), then use Lemma 3.2 with \( \alpha := p - 1 \geq 1 \) and finally apply the Gluing Lemma 4.2. This leads to

\[
\Pi_2 \leq c \mu \frac{\frac{q(p-2)}{p}}{\theta^q} \int_{\Lambda_{\theta}} |(u)^{\mu}_{\theta}(t) - (u)^{\mu}_{\theta}(t)|^q \, dt
\]

\[
\leq c \mu \frac{\frac{q(p-2)}{p}}{\theta^{q(2-p)}} \left(\int_Q |u|^p \, dx \right)^{\frac{q(p-2)}{p}} \left[\int_Q |Du|^p + |F|^p \, dx \right]^{\frac{q(p-2)}{p}}.
\]

With Hölder’s inequality and hypothesis (5.2), we finally obtain

\[
\Pi_2 \leq c \left(\int_Q |Du|^q + |F|^q \, dx \right)^{\frac{p-1}{p}} \leq c \left(\int_Q |Du|^p + |F|^p \, dx \right)^{\frac{q(p-2)}{p}} \leq c \left(\int_Q |Du|^q + |F|^q \, dx \right)^{\frac{p-1}{p}}.
\]

Inserting the preceding estimates above, we have shown that

\[
\left(\int_Q |u - (u)^{\mu}_{\theta}|^q \, dx \right)^{\frac{1}{q}} \mu \frac{\frac{q(p-2)}{p}}{\theta^q} \leq c \left(\int_Q |Du|^q + |F|^q \, dx \right)^{\frac{p-1}{p}} + c \Pi_1.
\]
For the estimate of $\Pi_1$, we use (5.4) and hypothesis (5.2) to obtain

$$\Pi_1 \leq c \mu^{\frac{p-2}{p-1}} \left( \frac{\iint_{Q_1^\mu} \left| u - (u)^\mu \right|^p \, dx \, dt}{\iint_{Q_1^\mu} |Du|^q + |F|^q \, dx \, dt} \right)^{\frac{p-2}{p-1}} \frac{\iint_{Q_1^\mu} \left| u - (u)^\mu \right|^p \, dx \, dt}{\iint_{Q_1^\mu} |Du|^q + |F|^q \, dx \, dt} \leq c \frac{\iint_{Q_1^\mu} \left| u - (u)^\mu \right|^p \, dx \, dt}{\iint_{Q_1^\mu} |Du|^q + |F|^q \, dx \, dt} \left( \frac{\iint_{Q_1^\mu} \left| u - (u)^\mu \right|^p \, dx \, dt}{\iint_{Q_1^\mu} |Du|^q + |F|^q \, dx \, dt} \right)^{\frac{p-2}{p-1}}.$$  

(5.6)

In the case $q = p$, we use Young’s inequality with exponents $\frac{p-1}{p-2}$ and $p-1$ and obtain

$$c \Pi_1 \leq \frac{1}{p-1} \iint_{Q_1^\mu} \frac{|u - (u)^\mu|^p}{\mu^{2-p} \theta^p} \, dx \, dt + c \iint_{Q_1^\mu} |Du|^p + |F|^p \, dx \, dt.$$

Inserting this into (5.5) and reabsorbing the first term of the right-hand side into the left yields the desired Poincaré type inequality in the case $q = p$. At this point it remains to consider the case $q \in [p-1, p)$. Here, we use in (5.6) the Poincaré type inequality for $q = p$ to conclude

$$\Pi_1 \leq c \iint_{Q_1^\mu} |Du|^q + |F|^q \, dx \, dt.$$  

Together with (5.5) this finishes the proof in the remaining case $q \in [p-1, p)$.

As final result of this section we derive a Sobolev-Poincaré type inequality, which should be seen as the analogue of Lemma 5.3 for the superquadratic case.

**Lemma 5.6.** Let $p \geq 2$ and $u$ be a weak solution to (2.2) in $\Omega_T$ in the sense of Definition 2.1. Then, on any cylinder $Q_1^\mu(z_0) \subseteq \Omega_T$ with $\theta, \mu > 0$ satisfying (5.1) and (5.2) and for any $\epsilon \in (0, 1]$, we have

$$\iint_{Q_1^\mu(z_0)} \frac{|u - (u)^\mu|^p}{\mu^{2-p} \theta^p} \, dx \, dt \leq \epsilon \sup_{t \in \Lambda_e(z_0)} \int_{B^\mu_t(z_0)} \frac{|u_t^\mu - [(u)^\mu]_{\Lambda_e(z_0)}|^2}{\theta^p} \, dx + c \frac{1}{p-1} \iint_{Q_1^\mu(z_0)} |Du|^q + |F|^q \, dx \, dt \left( \frac{\iint_{Q_1^\mu} \left| u - (u)^\mu \right|^p \, dx \, dt}{\iint_{Q_1^\mu} |Du|^q + |F|^q \, dx \, dt} \right)^{\frac{p-2}{p-1}}$$

with $q = \max\{\frac{np}{p+2}, p-1\}$ and $c = c(n, p, L, K)$.

**Proof.** As before, we omit the reference point $z_0$ in our notation. Moreover, we abbreviate $a := (u)^\mu$. Applying Gagliardo-Nirenberg’s inequality in Lemma 3.6 with $(p, q, r, \theta)$ replaced by $(p, q, 2, \frac{2}{p})$ and Lemma 5.5, we find that

$$\iint_{Q_1^\mu} \frac{|u - a|^p}{\mu^{2-p} \theta^p} \, dx \, dt \leq c \sup_{t \in \Lambda_e} \int_{B^\mu_t} \frac{|u(t) - a|^2}{\mu^{\frac{2-p}{p}} \theta^2} \, dx \iint_{Q_1^\mu} \frac{|u - a|^q}{\mu^{\frac{2-p}{p} \theta^2}} + |Du|^q \, dx \, dt.$$


\begin{equation}
\leq c \sup_{t \in \Lambda_\theta} \left[ \mu \frac{2(\rho - 2)}{\rho^2} \int_{B_\rho(t)} \frac{|u(t) - a|^2}{\rho^2} dx \right]^{\frac{\rho}{\rho - 2}} \iint_{Q_\rho(t)} \left[ |Du|^q + |F|^q \right] dx dt.
\end{equation}

We now exploit assumption (5.2) in order to obtain an upper bound for \( \mu \). If (5.2) is satisfied we have \( \mu \leq K^{1/p} \). On the other hand, if (5.2)_1 is in force we apply Lemma 5.5 to infer that

\[ \left[ \iint_{Q_\rho(t)} \frac{|u|^p}{\mu^{2 - p} \rho^2} dx dt \right]^{\frac{1}{p}} \leq \left[ \iint_{Q_\rho(t)} \frac{|u - a|^p}{\mu^{2 - p} \rho^2} dx dt \right]^{\frac{1}{p}} + \frac{\mu^{\frac{p - 2}{\rho}} |a|}{\rho} \]

\[ \leq c \left[ \iint_{Q_\rho(t)} \left[ |Du|^p + |F|^p \right] dx dt \right]^{\frac{1}{p}} + \frac{\mu^{\frac{p - 2}{\rho}} |a|}{\rho}, \]

which in combination with the \( \mu \)-super-intrinsic coupling \((5.2)_1 \) yields

\[ \mu \leq c + \frac{\mu^{\frac{p - 2}{\rho}} |a|}{\rho} \left[ \iint_{Q_\rho(t)} \left[ |Du|^p + |F|^p \right] dx dt \right]^{\frac{1}{p}} \]

\[ \leq \frac{1}{2} \mu + c + \frac{|a|^{\frac{2}{p}}}{\rho^{\frac{2}{p}}} \left[ \iint_{Q_\rho(t)} \left[ |Du|^p + |F|^p \right] dx dt \right]^{\frac{1}{p}}. \]

This shows that

\[ \mu \leq c + \frac{2|a|^{\frac{2}{p}}}{\rho^{\frac{2}{p}}} \left[ \iint_{Q_\rho(t)} \left[ |Du|^p + |F|^p \right] dx dt \right]^{\frac{1}{p}} \]

holds true in any case. Inserting this upper bound for \( \mu \) into (5.7) yields

\[ \iint_{Q_\rho(t)} \frac{|u - a|^p}{\mu^{2 - p} \rho^2} dx dt \leq c \left[ I_1 + I_2 \right], \]

with the obvious abbreviations

\[ I_1 := c \sup_{t \in \Lambda_\theta} \left[ \int_{B_\rho(t)} \frac{|u(t) - a|^2}{\rho^2} dx \right]^{\frac{2}{p - 2}} \iint_{Q_\rho(t)} \left[ |Du|^q + |F|^q \right] dx dt. \]

and

\[ I_2 := \sup_{t \in \Lambda_\theta} \left[ \int_{B_\rho(t)} \frac{|a|^2|u(t) - a|^2}{\rho^2} dx \right]^{\frac{2}{p - 2}} \iint_{Q_\rho(t)} \left[ |Du|^q + |F|^q \right] dx dt \]

\[ \left[ \iint_{Q_\rho(t)} \left[ |Du|^p + |F|^p \right] dx dt \right]^{\frac{(p - 2)p - 2}{p - 2}}. \]

For the first term, we use Hölder’s inequality and Lemma 3.3 to infer that

\[ I_1 \leq c \sup_{t \in \Lambda_\theta} \left[ \int_{B_\rho(t)} \frac{|u(t) - a|^2}{\rho^2} dx \right]^{\frac{2}{p - 2}} \iint_{Q_\rho(t)} \left[ |Du|^q + |F|^q \right] dx dt \]

\[ \leq c \sup_{t \in \Lambda_\theta} \left[ \int_{B_\rho(t)} \left| u + \frac{a}{\rho} \right|^2 dx \right]^{\frac{2}{p - 2}} \iint_{Q_\rho(t)} \left[ |Du|^q + |F|^q \right] dx dt, \]

with a constant \( c = c(n, p, L, K) \). Now we turn our attention to the second term. With the help of Lemma 3.2 applied with \( \alpha = \frac{2}{p} \) and Hölder’s inequality, we find that

\[ I_2 \leq c \sup_{t \in \Lambda_\theta} \left[ \int_{B_\rho(t)} \left| u + \frac{a}{\rho} \right|^2 dx \right]^{\frac{2}{p - 2}} \iint_{Q_\rho(t)} \left[ |Du|^q + |F|^q \right] dx dt \]

\[ \left[ \iint_{Q_\rho(t)} \left[ |Du|^p + |F|^p \right] dx dt \right]^{\frac{(p - 2)p - 2}{p - 2}}, \]
for a constant \( c = c(n, p) \). We add the resulting inequalities for \( I_1 \) and \( I_2 \) and apply Young’s inequality. This yields the desired result. \( \square \)

6. Reverse Hölder Inequality

Our aim in this section is to derive a reverse Hölder type inequality for weak solutions of (2.2). It will be a consequence of the energy estimate in Lemma 4.1 and the Sobolev-Poincaré type inequality in Lemma 5.3, respectively Lemma 5.6.

In contrast to Section 5 we now consider two concentric cylinders \( Q^{(\mu)}_2(z_0) \subset Q^{(\mu)}_2(z_0) \subseteq \Omega_T \) with \( \varrho, \mu > 0 \). We suppose that a \( \mu \)-sub-intrinsic coupling of the type

\[
\mathcal{I} = \frac{\|u\|_p}{\mu^{2-p}(2\varrho)^p} \leq K \mu^p
\]

is satisfied for some \( K \geq 1 \). Furthermore, we assume that either

\[
\mu^p \leq K \frac{\|u\|_p}{\mu^{2-p}(2\varrho)^p} \quad \text{or} \quad \mu^p \leq K
\]

holds true. Then, we obtain the following reverse Hölder type inequality.

**Proposition 6.1.** Let \( p > \max\{\frac{2n}{n+2}, 1\} \) and \( u \) be a weak solution to (2.2) in \( \Omega_T \) in the sense of Definition 2.1. Then, on any cylinder \( Q^{(\mu)}_2(z_0) \subseteq \Omega_T \) with \( \varrho, \mu > 0 \) satisfying (6.1) and (6.2), we have

\[
\mathcal{I} = \frac{\|u\|_p}{\mu^{2-p}(2\varrho)^p} \leq c \left( \frac{\|u\|_p}{\mu^{2-p}(2\varrho)^p} \right)^q + c \frac{\|F\|_p}{\mu^{2-p}(2\varrho)^p}
\]

with the exponent \( q := \max\{\frac{2n}{n+2}, \frac{np}{n+p+2}, 1, p - 1\} \) and a constant \( c = c(n, p, \nu, L, K) \).

**Proof.** Once again, we omit the reference to the center \( z_0 \) in the notation. We consider radii \( r, s \) with \( \varrho \leq r < s \leq 2\varrho \) and let

\[
R_{r,s} := \frac{s}{s-r}
\]

Note that hypothesis (6.1) and (6.2) imply that the coupling conditions (5.1) and (5.2) are satisfied on \( Q^{(\mu)}_2 \) with the constant \( 2^{2n+3p}K \) in place of \( K \). From now on we distinguish between the cases \( \max\{\frac{2n}{n+2}, 1\} < p \leq 2 \) and \( p \geq 2 \).

**The case** \( \max\{\frac{2n}{n+2}, 1\} < p \leq 2 \). Here the energy estimate from Lemma 4.1 reads as

\[
\sup_{t \in A(t)_{\alpha}} \int_{B_\varrho} \left| u^2 (t) - \left[ (u)_{\alpha}^{(\mu)} \right] ^2 \right| \mu^{p-2} \, dx + \int_{Q^{(\mu)}_2} |Du|^p \, dx dt
\]

\[
\leq c \int_{Q^{(\mu)}_2} \left| u^2 - \left( (u)_{\alpha}^{(\mu)} \right) ^2 \right| \mu^{p-2} \, dx dt + c \int_{Q^{(\mu)}_2} \left| u - (u)_{\alpha}^{(\mu)} \right| ^p \, dx dt
\]

\[
+ c \int_{Q^{(\mu)}_2} |F|^p \, dx dt
\]

(6.4)

\[= I + II + III,\]
with the obvious meaning of I–III. The constant \( c \) depends only on \( p, \nu, \) and \( L \). We estimate II with the help of Lemma 3.5, Lemma 5.1 and Young's inequality with the result that

\[
II \leq c \mathcal{R}_{r,s}^p \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

\[
\leq c \mathcal{R}_{r,s}^p \left[ \int_{Q_{r,s}^p} |Du|^p + |F|^p \, dx \, dt \right] ^{\frac{2}{2p}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

\[
\leq c \mathcal{R}_{r,s}^p \left[ \int_{Q_{r,s}^p} |Du|^p + |F|^p \, dx \, dt \right] ^{\frac{2}{2p}} \frac{1}{\delta^{\frac{2}{2p}}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

holds true for any \( \delta \in (0, 1] \). Taking into account that \((s - r)^p \leq s^p - r^p\), we obtain due to Lemma 3.5 that

\[
I \leq c \mathcal{R}_{r,s}^p \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt.
\]

We add both inequalities and apply Lemma 5.3 on \( Q_{r,s}^p \) with \( \varepsilon = \delta^{\frac{2}{2p}} \). In this way we obtain

\[
I + II \leq c \mathcal{R}_{r,s}^p \left[ \int_{Q_{r,s}^p} |Du|^p + |F|^p \, dx \, dt \right] ^{\frac{2}{2p}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

\[
\leq c \mathcal{R}_{r,s}^p \left[ \int_{Q_{r,s}^p} |Du|^p + |F|^p \, dx \, dt \right] ^{\frac{2}{2p}} \frac{1}{\delta^{\frac{2}{2p}}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

\[
\leq c \mathcal{R}_{r,s}^p \left[ \sup_{t \in \Lambda^p_{r,s}} \int_{B_r} \frac{|u|^p(t) - [u]_{r}^{(\mu)}|^p}{s^p} \, dx + \int_{Q_{r,s}^p} |Du|^p \, dx \right] ^{\frac{2}{2p}} \frac{1}{\delta^{\frac{2}{2p}}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

\[
+ c \mathcal{R}_{r,s}^p \left[ \int_{Q_{r,s}^p} |Du|^q \, dx \right] ^{\frac{2}{2q}} \frac{1}{\delta^{\frac{2}{2p}}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

where \( q = \max \left\{ \frac{2n}{n+2}, 1 \right\} \). We insert this inequality into (6.4). Then, we choose

\[
\delta = \frac{1}{2c \mathcal{R}_{r,s}^p}
\]

and apply the Iteration Lemma 3.1 to re-absorb the term \( \frac{1}{2}[\ldots] \) from the right-hand side into the left. This leads to the claimed reverse Hölder type inequality, i.e. to

\[
\sup_{t \in \Lambda^p_{r,s}} \int_{B_r} \frac{|u|^p(t) - [u]_{r}^{(\mu)}|^p}{s^p} \, dx + \int_{Q_{r,s}^p} |Du|^p \, dx \]

\[
\leq c \left[ \int_{Q_{r,s}^p} |Du|^q \, dx \right] ^{\frac{2}{2q}} \frac{1}{\delta^{\frac{2}{2p}}} \int_{Q_{r,s}^p} \frac{|u - (u)_r^{(\mu)}|^p}{s^p} \, dx \, dt
\]

and finishes the proof of Proposition 6.1 in the case \( \max \left\{ \frac{2n}{n+2}, 1 \right\} < p \leq 2 \).

The case \( p \geq 2 \). In this case, the energy estimate from Lemma 4.1 yields

\[
\sup_{t \in \Lambda^p_{r,s}} \int_{B_r} \frac{|u|^p(t) - [u]_{r}^{(\mu)}|^p}{r^p} \, dx + \int_{Q_{r,s}^p} |Du|^p \, dx \]

\[
\leq c \int_{Q_{r,s}^p} |u - (u)_r^{(\mu)}|^p \, dx \, dt + c \int_{Q_{r,s}^p} |u - (u)_r^{(\mu)}|^p \, dx \, dt
\]

\[
+ c \int_{Q_{r,s}^p} |F|^p \, dx \, dt
\]

(6.6) \( =: I + II + III \)
with the obvious meaning of I–III. Now, we estimate the term I by using the fact that 
\((s-r)^p \leq s^p - r^p\), Lemma 3.5, Lemma 5.4 and Young’s inequality. In this way we obtain

\[I \leq c R_{r,s}^p \left( \int_{Q_s(r)} |u(t) - (u)_{s}^{(r)}|^2 \right)^{\frac{2}{p}} \]

\[\leq c R_{r,s}^p \left( \int_{Q_s(r)} |Du|^p + |F|^p \right) \left( \int_{Q_s(r)} \frac{|u - (u)_{s}^{(r)}|^p}{\mu^{2-\frac{p}{s}}} \right) \]

\[\leq c R_{r,s} \left( \int_{Q_s(r)} |Du|^p + |F|^p \right) \left( \int_{Q_s(r)} \frac{|u - (u)_{s}^{(r)}|^p}{\mu^{2-\frac{p}{s}}} \right),\]

for any \(\delta \in (0, 1] \). Moreover, from Lemma 3.5 we know that

\[II \leq c R_{r,s}^p \left( \int_{Q_s(r)} |u - (u)_{s}^{(r)}|^p \right) \]

We combine the preceding estimates and apply Lemma 5.6 with \(\varepsilon = \delta \tilde{\tau}\) in order to obtain

\[I + II \leq c R_{r,s}^p \left[ \sup_{t \in [0,1]} \left( \int_{B_s^{(r)}} \frac{|u(t) - (u)_{s}^{(r)}|^2}{s^p} \right) \right] \]

\[+ \frac{c R_{r,s}^p}{\delta} \left[ \int_{Q_s(r)} |Du|^q \, dx + \int_{Q_s(r)} |F|^p \, dx \right],\]

where \(q = \max\{\frac{np}{2}, p-1\} \). As before, we insert this inequality into (6.6), choose \(\delta \in (0, 1] \) of the form (6.5) and apply the iteration Lemma 3.1. This allows to re-absorb the term \(\frac{1}{\delta} \) into the left-hand side and yields the desired reverse H"older type inequality in the remaining case \(p \geq 2\). This finishes the proof of the proposition. \(\square\)

7. Higher integrability: Proof of Theorem 2.2

In this section we finally prove the higher integrability result of Theorem 2.2. We consider a fixed cylinder \(Q_{4R} = Q_{4R,(4R)^p}(z_0) \subset \Omega_T \) with \(R > 0\) and let

\[\lambda \geq \lambda_0 \geq 1 + \left( \int_{Q_{4R}} \frac{|u|^p}{(4R)^p} \right)^{\frac{1}{p}}.\]

We recall the notation (2.1) for the scaled cylinders \(Q_{\phi}^{(r)}(z_0)\), and observe that \(Q_{\phi}^{(r)}(z_0) \subset Q_{\phi}^{(\kappa)}(z_0) \) whenever \(\kappa \leq \mu\). Moreover, we recall the abbreviation \(p = \max\{2, p\}\).

7.1. Construction of a non-uniform system of cylinders. The main difficulty now is to construct a covering of the \(\lambda\)-superlevel set of \(|Du|\) by cylinders on which the reverse Hölder type inequality from Proposition 6.1 is applicable. This means that the scaled cylinders have to satisfy hypothesis (6.1) and (6.2). The following construction of a non-uniform system of cylinders is inspired by the one in [21, 9]. Let \(z_0 \in Q_{2R}\). For a radius \(\phi \in (0, R]\) we now define

\[\tilde{\mu}_{\lambda;\phi}^{(r)} := \inf \left\{ \mu \in [1, \infty) : \frac{1}{|Q_{\phi}|} \int_{Q_{\phi}^{(r)}(z_0)} |u|^p \, dxdy \leq \mu^{p-\beta} \lambda^p \right\},\]

where \(\beta := 2 - p + (p-2)(2 + \frac{n}{p})\). Note that

\[\mu^{p-\beta} = \left\{ \begin{array}{ll} \mu^{2(p-1)} & \text{if } p \leq 2, \\ \mu^{2\frac{2n-\beta(n-2)}{p}} & \text{if } p > 2. \end{array} \right.\]
In particular, the restriction $p < \frac{2n}{n-2}$ for $n > 2$ ensures that $p - \beta > 0$ in any case. However, we note that in dimensions $n > 2$, the exponent of $\mu$ tends to zero in the limit $p \uparrow \frac{2n}{n-2}$. This is the only point where the restriction $p < \frac{2n}{n-2}$ enters the proof. If $z_0$ and $\lambda$ are fixed and if the meaning is clear from the context we write $\tilde{\mu}_g$ instead of $\tilde{\mu}_g^{(\lambda)}$. Observe that the set of those $\mu \geq 1$ for which the condition in the infimum is satisfied is not empty. In fact, in the limit $\mu \uparrow \infty$ the integral on the left-hand side converges to zero (note that the measure of $Q_o(\mu_o)(z_0)$ shrinks to 0), while the right-hand side blows up with speed $\mu_p^{p-\beta}$ (recall that $p - \beta > 0$). We point out that the condition in the infimum is equivalent to

$$\iint_{Q_o(\mu_o)(z_0)} \frac{|u|^p}{\mu^{2-p} g^p} \, dx \, dt \leq \mu^p \lambda^p.$$ 

Therefore, we either have

$$\tilde{\mu}_g = 1 \quad \text{and} \quad \iint_{Q_o(\mu_o)(z_0)} \frac{|u|^p}{\tilde{\mu}_g^{2-p} g^p} \, dx \, dt \leq \tilde{\mu}_g^p \lambda^p = \lambda^p,$$

or otherwise

$$\tilde{\mu}_g > 1 \quad \text{and} \quad \iint_{Q_o(\mu_o)(z_0)} \frac{|u|^p}{\tilde{\mu}_g^{2-p} g^p} \, dx \, dt = \tilde{\mu}_g^p \lambda^p.$$ 

Using this observation for $g = R$, we have that either $\tilde{\mu}_R = 1$, or $\tilde{\mu}_R > 1$ and

$$\mu_p^{p-\beta} = \frac{1}{\lambda^p |Q|R} \iint_{Q_o(\mu_o)(z_0)} \frac{|u|^p}{R^p} \, dx \, dt \leq \frac{1}{\lambda^p} \iint_{Q_R(z_0)} \frac{|u|^p}{R^p} \, dx \, dt \leq \frac{4^{n+2} \lambda_R^p}{\lambda^p} \leq 4^{n+2p}.$$ 

Therefore, in any case we have the bound

$$\tilde{\mu}_R \leq 4^{n+2p}.$$ 

Our next aim is to ensure that the mapping $(0, R] \ni g \mapsto \tilde{\mu}_g$ is continuous. To this end, we consider $g \in (0, R]$ and $\varepsilon > 0$, and define $\mu_+ := \tilde{\mu}_g + \varepsilon$. Then, there exists $\delta = \delta(\varepsilon, g) > 0$ such that

$$\frac{1}{|Q|} \iint_{Q_o(\mu_+)(z_0)} \frac{|u|^p}{R^p} \, dx \, dt < \mu_+^{p-\beta} \lambda^p$$

for any $r \in (0, R]$ with $|r - g| < \delta$. In fact, due to the definition of $\tilde{\mu}_g$, the preceding strict inequality holds for $r = g$, since $\mu_+ > \tilde{\mu}_g$ and $Q_o(\mu_+)(z_0) \subset Q_o(\tilde{\mu}_g)(z_0)$. The claim now follows, since the left-hand side depends continuously on the radius $r$. Recalling the very definition of $\tilde{\mu}_g$, the last inequality implies $\tilde{\mu}_r \leq \mu_+ = \tilde{\mu}_g + \varepsilon$ for any $r \in (0, R]$ with $|r - g| < \delta$. It remains to prove $\tilde{\mu}_r \geq \mu_- := \tilde{\mu}_g - \varepsilon$ for $r$ close to $g$. If $\tilde{\mu}_g = 1$, then we have $\tilde{\mu}_r \geq 1 = \tilde{\mu}_g \geq \mu_-$. If $\tilde{\mu}_g > 1$, we get after diminishing $\delta = \delta(\varepsilon, g) > 0$ if necessary that

$$\frac{1}{|Q|} \iint_{Q_o(\mu_-)(z_0)} \frac{|u|^p}{R^p} \, dx \, dt > \mu_-^{p-\beta} \lambda^p$$

for all $r \in (0, R]$ with $|r - g| < \delta$. For $r = g$, this is a direct consequence of the definition of $\tilde{\mu}_g$, since $\tilde{\mu}_g > \mu_-$ and $Q_o(\tilde{\mu}_g)(z_0) \subset Q_o(\mu_-)(z_0)$. Due to the continuity of the left-hand side with respect to $r$, this implies the claim for $r$ with $|r - g| < \delta$ small enough. The preceding inequality implies that $\tilde{\mu}_r \geq \mu_- := \tilde{\mu}_g - \varepsilon$. This completes the proof of the continuity of $(0, R] \ni g \mapsto \tilde{\mu}_g$.

Unfortunately, the mapping $g \mapsto \tilde{\mu}_g$ might not be monotone. For this reason we modify $\tilde{\mu}_g$ in such a way that the modification – denoted by $\mu_g$ – becomes monotone. Therefore, we define

$$\mu_g \equiv \mu_{\tilde{\mu}_g, \mu} := \max_{r \in [g, R]} \tilde{\mu}_g^{(\lambda)}(z_o, r).$$

As before, we abbreviate $\mu_{\tilde{\mu}_g, \mu}$ by $\mu_g$ if $z_o$ and $\lambda$ are fixed, so that no confusion is possible. By construction the mapping $(0, R] \ni g \mapsto \mu_g$ is continuous and monotonically
In fact, the definition of $\mu_s$ and the monotonicity of $\mu_\theta$ imply $\tilde{\mu}_s \leq \mu_s \leq \mu_\theta$, so that $Q^{(\mu_s)}(z_0) \subset Q^{(\tilde{\mu}_s)}(z_0)$. This allows to estimate

$$
\iint_{Q^{(\mu_s)}(z_0)} |u|^p |x|^{2-p} \, dx \, dt \leq \left( \frac{\mu_\theta}{\mu_s} \right)^{2-p} \iint_{Q^{(\tilde{\mu}_s)}(z_0)} |u|^p \, dx \, dt
$$

for any $0 < \theta \leq s \leq R$.

In the last step we used the fact $p - \beta > 0$. We now define

$$
\tilde{\theta} := \begin{cases} 
R & \text{if } \mu_\theta = 1, \\
\inf \{ s \in [\theta, R] : \mu_s = \tilde{\mu}_s \} & \text{if } \mu_\theta > 1.
\end{cases}
$$

Note that $\mu_s = \tilde{\mu}_s$ for any $s \in (\theta, \tilde{\theta})$ and in particular $\mu_\theta = \tilde{\mu}_\theta$. Next, we claim that

$$
\mu_\theta \leq \left( \frac{\tilde{\theta}}{\theta} \right)^{n+2p} \mu_s \quad \text{for any } s \in (\theta, R].
$$

If $\mu_\theta = 1$, then also $\mu_s = 1$, so that (7.6) trivially holds. Therefore, it remains to consider the case $\mu_\theta > 1$. If $s \in (\theta, \tilde{\theta})$, then $\mu_\theta = \mu_s$, and (7.6) obviously holds true. Otherwise, if $s \in (\tilde{\theta}, R]$, the monotonicity of $s \mapsto \mu_s$ and (7.4) imply

$$
\mu_\theta^{p-\beta} = \tilde{\mu}_\theta^{p-\beta} = \frac{1}{\lambda^p |Q_\tilde{\theta}|} \iint_{Q^{(\tilde{\mu}_s)}(z_0)} \frac{|u|^p}{\theta^p} \, dx \, dt
$$

$$
\leq \left( \frac{\tilde{\theta}}{\theta} \right)^{n+2p} \frac{1}{\lambda^p |Q_\theta|} \iint_{Q^{(\mu_s)}(z_0)} \frac{|u|^p}{s^p} \, dx \, dt
$$

$$
\leq \left( \frac{\tilde{\theta}}{\theta} \right)^{n+2p} \mu_s^{p-\beta}.
$$

This proves the claim (7.6). We now apply (7.6) with $s = R$. Since $\mu_R = \tilde{\mu}_R$, the bound (7.3) for $\tilde{\mu}_R$ yields

$$
\mu_\theta \leq \left( \frac{R}{\theta} \right)^{n+2p} \mu_R \leq \left( \frac{4R}{\theta} \right)^{n+2p},
$$

In the following, we consider the system of concentric cylinders $Q^{(\mu_s)}_{Q_\theta}(z_0)$ with radii $\theta \in (0, R]$ and $z_0 \in Q_{2R}$. The cylinders are nested, in the sense that

$$
Q^{(\mu_s)}_{Q_\theta}(z_0) \subset Q^{(\mu_s)}(z_0) \subset Q_{4R} \quad \text{whenever } 0 < r < s \leq R.
$$

The inclusions hold true due to the monotonicity of the mapping $\theta \mapsto \mu_s$, and the fact that $\mu_s^{(\lambda)} \geq 1$. The disadvantage of using $\mu_s^{(\lambda)}$ instead of $\tilde{\mu}_s^{(\lambda)}$ is that the associated cylinders are in general only $\mu$-sub-intrinsic with $K = 1$, but not $\mu$-intrinsic.

### 7.2. Covering property

The system of cylinders $Q^{(\mu_s)}_{Q_\theta}(z_0)$ constructed above satisfies a Vitali-type covering property. This will be proven in the following lemma.

**Lemma 7.1.** There exists a constant $\hat{c} = \hat{c}(n, p) \geq 20$ such that whenever $\lambda \geq \lambda_0$ and $F$ is any collection of cylinders $Q^{(\mu_s)}_{Q_\theta}(z_0)$, where $Q^{(\mu_s)}_{Q_\theta}(z_0)$ is a cylinder of the form as

1 Note that later $\lambda^p \approx \iint_{Q^{(\mu_\theta)}(z_0)} |D u|^p + |F|^p \, dx \, dt$. This justifies the notion $\mu$-sub-intrinsic in the sense of (6.1).
constructed in Section 7.1 with radius $r \in (0, \frac{L}{n+1})$, then there exists a countable subfamily $\mathcal{G}$ of disjoint cylinders in $\mathcal{F}$ such that

\begin{equation}
\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{Q \in \mathcal{G}} \hat{Q},
\end{equation}

where $\hat{Q}$ denotes the $\frac{1}{2}c$-times enlarged cylinder $Q$, i.e. if $Q = Q_{4\epsilon r}^{(\mu)}(z)$, then $\hat{Q} = Q_{20c}^{(\mu)}(z)$.

**Proof.** Throughout the proof we abbreviate $\mu_{z:r} := \mu_{z;r}$. We let $\epsilon \geq 20$ be a parameter that will be chosen later. For $j \in \mathbb{N}$ we define

\[ \mathcal{F}_j := \left\{ Q_{4\epsilon r}^{(\mu_{z;r})}(z) \in \mathcal{F} : \frac{R}{2r c} < r \leq \frac{R}{2r c} \right\} \]

and select $\mathcal{G}_j \subset \mathcal{F}_j$ by the following procedure: We choose $\mathcal{G}_1$ to be any maximal disjoint collection of cylinders in $\mathcal{F}_1$. Note that $\mathcal{G}_1$ contains only finitely many cylinders, since by the definition of $\mathcal{F}_1$ and (7.7) the $L^{n+1}$-measure of each cylinder $Q \in \mathcal{G}_1$ is uniformly bounded from below. Now, assume that for some $k \in \mathbb{N}_{\geq 2}$ the collections $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{k-1}$ have already been inductively selected. Then, we choose a maximal disjoint sub-collection of cylinders from $\mathcal{F}_k$ which do not intersect any of the cylinders $Q^*$ from one of the collections $\mathcal{G}_j$, $j \in \{1, \ldots, k-1\}$. More precisely, we choose a maximal disjoint collection of cylinders in

\[ \bigcup_{Q \in \mathcal{F}_k : Q \cap Q^* = \emptyset} \text{ for any } Q^* \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \bigg]. \]

Note again that $\mathcal{G}_k$ is finite. Finally, we let

\[ \mathcal{G} := \bigcup_{j=1}^{\infty} \mathcal{G}_j. \]

By construction, $\mathcal{G} \subset \mathcal{F}$ is a countable subfamily of disjoint cylinders in $\mathcal{F}$.

At this point it remains to prove that for each $Q \in \mathcal{F}$ there exists a cylinder $Q^* \in \mathcal{G}$ such that $Q \cap Q^* \neq \emptyset$ and $Q \subset \hat{Q}$. To this aim we consider some arbitrary cylinder $Q = Q_{4\epsilon r}^{(\mu_{z;r})}(z) \in \mathcal{F}$. Then, there exists an index $j \in \mathbb{N}$ such that $Q \in \mathcal{F}_j$. The maximality of $\mathcal{G}_j$ ensures that there exists a cylinder $Q^* = Q_{4\epsilon r}^{(\mu_{z;r})}(z_*) \in \bigcup_{i=1}^{\infty} \mathcal{G}_i$ with $Q \cap Q^* \neq \emptyset$. Then, we have $r < 2r_*$, since $r \leq \frac{R}{2r c}$ and $r_*> \frac{R}{2r}$. The main difficulty now is to establish a bound for $\mu_{z;2r_*}$ in terms of $\mu_{z;r_*}$. We claim that the following estimate holds true:

\begin{equation}
\mu_{z;2r_*} \leq (4\eta)^{n+2p} \mu_{z;r_*},
\end{equation}

where $\eta := 13$. To prove the claim we denote by $\tilde{r}_* \in [r_*, R]$ the radius associated to the cylinder $Q_{4\epsilon r}^{(\mu_{z;r})}(z_*)$; see (7.5) for the construction. Recall that either $\mu_{z;2r_*} = 1$ and $\tilde{r}_* = R$ or $Q_{4\epsilon r}^{(\mu_{z;r})}(z_*)$ is intrinsic in the sense of (7.2). In the former case we have $\mu_{z;2r_*} = 1 \leq \mu_{z;r_*}$, so that (7.9) is satisfied. If $Q_{4\epsilon r}^{(\mu_{z;r})}(z_*)$ is intrinsic, we know that

\begin{equation}
\mu_{z;2r_*}^{p-\beta} = \frac{1}{\lambda^p |Q_{2r_*}|} \int_{Q_{4\epsilon r}^{(\mu_{z;r})}(z_*)} \frac{|u|^p}{r_*^p} \, dx dt.
\end{equation}

Now, we distinguish between the cases $\tilde{r}_* \leq \frac{R}{\eta}$ and $\tilde{r}_* > \frac{R}{\eta}$. We first consider the simpler case $\tilde{r}_* > \frac{R}{\eta}$. Here, we exploit (7.10) and (7.1) to conclude that

\[ \mu_{z;2r_*}^{p-\beta} \leq (4R/r_*)^p \frac{1}{\lambda^p |Q_{2r_*}|} \int_{Q_{4\epsilon r}^{(\mu_{z;r})}(z_*)} \frac{|u|^p}{r_*^p} \, dx dt \leq \left( \frac{4R}{r_*} \right)^p \frac{|Q_{4\epsilon r}|}{|Q_{2r_*}|} \leq (4\eta)^{n+2p}, \]
which implies
\[ \mu_{z, r} \leq \left( 4\eta \right)^{\frac{n+2p}{2p-n}} \leq 4\eta \mu_{z, r} \]
and proves (7.9) in this case. Therefore it remains to consider radii \( \tilde{r}_s \leq \frac{p}{\eta} \). Note that we can assume \( \mu_{z, r} \leq \mu_{z, r} \). Otherwise (7.9) trivially holds. Therefore, the monotonicity of \( \varrho \mapsto \mu_{z, r} \) and the fact that \( r \leq 2r_s \leq 2\tilde{r}, \eta \tilde{r}_s \) imply
\[ \mu_{z, \eta \tilde{r}} \leq \mu_{z, r} \leq \mu_{z, r_s} . \]

Next, we claim that
\[ Q^{(\mu_{z, r_s})}(z_s) \subset Q^{(\mu_{z, \eta \tilde{r}})}(z) . \]

For the proof of (7.12) a distinction must be made between the cases \( p \leq 2 \) and \( p \geq 2 \). We first consider exponents \( \max\left\{ \frac{2n}{n+2}, 1 \right\} < p \leq 2 \). Since \( \tilde{r}_s \geq r_s \) and \( |x - x| < 4r + 4r_s \leq 12r_s \), we know \( B_{r_s}(x) \subset B_{2\tilde{r}}(x) \). Moreover, due to (7.11) we may conclude that
\[ \mu^{p-2}_{z, r_s} \varrho^{p} + |t - t_s| \leq \mu^{p-2}_{z, r_s} \varrho^{p} + \mu^{p-2}_{z, r_s} (4r)^p + \mu^{p-2}_{z, r_s} (4r_s)^p \]
\[ \leq (1 + 4^p + 8^p) \mu^{p-2}_{z, r_s} (\eta \tilde{r}_s)^p , \]
and this immediately implies the inclusion
\[ \Lambda^{(\mu_{z, r_s})}(t_s) \subset \Lambda^{(\mu_{z, \eta \tilde{r}})}(t) , \]
so that (7.12) is proven for exponents \( \max\left\{ \frac{2n}{n+2}, 1 \right\} < p \leq 2 \). Otherwise, if \( 2 \leq p < \frac{2n}{n+2} \), we have \( |t - t_s| < (4r)^p + (4r_s)^p \leq (12r_s)^p \) and hence \( \Lambda_{r_s}(x) \subset \Lambda_{\eta \tilde{r}_s}(t) \).

Furthermore, (7.11) yields
\[ (\mu_{z, r_s})^{\frac{2-n}{p}} \varrho^{r_s} + |x - x_s| \leq (\mu_{z, r_s})^{\frac{2-n}{p}} \varrho^{r_s} + (\mu_{z, r_s})^{\frac{2-n}{p}} 4r + (\mu_{z, r_s})^{\frac{2-n}{p}} 4r_s \]
\[ \leq (\mu_{z, \eta \tilde{r}_s})^{\frac{2-n}{p}} \eta \tilde{r}_s , \]
which implies the inclusion
\[ B^{(\mu_{z, r_s})}(x) \subset B^{(\mu_{z, \eta \tilde{r}})}(x) . \]

This establishes the claim (7.12) also for the remaining case \( 2 \leq p < \frac{2n}{n+2} \). Now we can finish the proof of (7.9). Due to (7.10), (7.12), (7.4) applied with \( \varrho = s = \eta \tilde{r}_s \), and (7.11), we obtain
\[ \mu^{p-\beta}_{z, r_s} \leq \frac{\eta^p}{\lambda^p} \int_{Q^{(\mu_{z, \eta \tilde{r}})}(z)} |u|^p (\eta \tilde{r}_s)^p \lambda^p dt \leq \eta^{n+2p} \mu^{p-\beta}_{z, \eta \tilde{r}_s} \leq \eta^{n+2p} \mu^{p-\beta}_{z, \eta \tilde{r}} , \]
so that
\[ \mu_{z, r_s} \leq \eta^{n+2p} \mu_{z, r} . \]

This finishes the proof of (7.9).

It remains to show the inclusion
\[ Q = Q^{(\mu_{z, \eta \tilde{r}})}(z) \subset Q^{(\mu_{z, r})}(z) \]
for a constant \( \hat{c} = \hat{c}(n, p) \geq 20 \). If \( \max\left\{ \frac{2n}{n+2}, 1 \right\} < p \leq 2 \) we get with (7.9) that
\[ \mu^{p-2}_{z, r} (4r)^p + |t - t_s| \leq 2 \mu^{p-2}_{z, r} (4r)^p + \mu^{p-2}_{z, r_s} (4r_s)^p \]
\[ \leq \left[ 2^{p+1} (4\eta)^{\frac{n+2p}{2p-n}} (2^{p-n}) + 1 \right] \mu^{p-2}_{z, r_s} (4r_s)^p \]
\[ \leq \mu^{p-2}_{z, r_s} (\hat{c} \tilde{r}_s)^p , \]
where \( \hat{c} = \hat{c}(n, p) \) is chosen suitably. This proves that \( \Lambda_{4r}^{(\mu_{x,r})}(t) \subset \Lambda_{\hat{c}r}^{(\mu_{x,r})}(t) \). Moreover, if we choose \( \hat{c} \geq 20 \) we have the inclusion \( B_{2r}(x) \subset B_{\hat{c}r}(x) \). This implies (7.13). In the case \( 2 \leq p < \frac{2n}{(n-2)n} \), inequality (7.9) shows

\[
(\mu_{x,r}) \frac{2p}{p-2} 4r + |x - x_*| \leq 2(\mu_{x,r}) \frac{2p}{p-2} 4r + (\mu_{x,r}) \frac{2p}{p-2} 4r_*
\]

\[
\leq 4(4\eta) \frac{2p}{p-2} \frac{n+2p}{p} + 1 (\mu_{x,r}) \frac{2p}{p-2} 4r_*
\]

\[
\leq (\mu_{x,r}) \frac{2p}{p-2} r^* \epsilon
\]

for a suitable constant \( \hat{c} = \hat{c}(n, p) \), from which we deduce \( B_{2r}^{(\mu_{x,r})}(x) \subset B_{\hat{c}r}^{(\mu_{x,r})}(x) \). Moreover, if we choose \( \hat{c} \geq 20 \) we have the inclusion \( \Lambda_{4r}(t) \subset \Lambda_{\hat{c}r}(t) \). Again this implies (7.13). In any case we have thus established the claim (7.8). This completes the proof of the Vitali type covering property.

\[\square\]

7.3. Stopping time argument. We now let

\[
(7.14) \quad \lambda_0 := 1 + \left( \frac{\lambda}{2} \int_{Q_{4R}} \left[ \frac{|n|^p}{(AR)^p} + |Du|^p + |F|^p \right] dx \right)^\frac{1}{p}
\]

so that \( \lambda_0 \) satisfies the previously demanded requirement (7.1). For \( \lambda \geq \lambda_0 \) and \( r \in (0, 2R) \), we define the superlevel set of \( |Du| \) by

\[
E(r, \lambda) := \left\{ z \in Q_r : z \text{ is a Lebesgue point of } |Du| \text{ and } |Du|(z) > \lambda \right\}.
\]

Here, we mean Lebesgue points of \( |Du| \) with respect to the system of cylinders constructed in Section 7.1. For radii \( R \leq R_1 < R_2 \leq 2R \), we consider the concentric parabolic cylinders \( Q_R \subset Q_{R_1} \subset Q_{R_2} \subset Q_{2R} \). We fix \( z_0 \in E(R_1, \lambda) \) and write \( \mu_s \equiv \mu_{s, z_0}^{(\lambda)} \) for \( s \in (0, R) \) throughout this section. By Lebesgue’s Differentiation Theorem, cf. [8, §2.9.1], we have

\[
(7.15) \quad \liminf_{s \downarrow 0} \int_{Q_s^{(\mu_s)}(z_0)} \left[ |Du|^p + |F|^p \right] dx \geq |Du|^p(z_0) > \lambda^p.
\]

By \( \hat{c} = \hat{c}(n, p) \) we denote the constant from the Vitali type covering Lemma 7.1. From now on, we consider values of \( \lambda \) satisfying

\[
(7.16) \quad \lambda > B \lambda_0, \quad \text{where} \quad B := \left( \frac{4\hat{c}R}{R_2 - R_1} \right)^{\frac{p}{p-2}} > 1.
\]

For \( s \) with

\[
(7.17) \quad \frac{R_2 - R_1}{\hat{c}} \leq s \leq R
\]

we have, due to the definition of \( \lambda_0 \) in (7.14), (7.7) and (7.17) that

\[
\int_{Q_s^{(\mu_s)}(z_0)} \left[ |Du|^p + |F|^p \right] dx \leq \frac{|Q_{4R}|}{|Q_s^{(\mu_s)}(z_0)|} \int_{Q_{4R}} \left[ |Du|^p + |F|^p \right] dx
\]

\[
\leq \frac{|Q_{4R}|}{|Q_s^{(\mu_s)}(z_0)|} R_s^{\beta(p-2)} \lambda_0^p \leq \left( \frac{4R}{s} \right)^{\frac{p(n+2)}{p-2}} \lambda_0^p
\]

\[
\leq \left( \frac{4\hat{c}R}{R_2 - R_1} \right)^{\frac{p(n+2)}{p-2}} \lambda_0^p = (B \lambda_0)^p < \lambda^p.
\]

On the other hand, due to (7.15) we find a sufficiently small radius \( 0 < s < \frac{R_2 - R_1}{\hat{c}} \) such that the integral in (7.15) possesses a value larger than \( \lambda^p \). By the continuity of \( \rho \mapsto \mu_{\rho} \) and the absolute continuity of the integral, there exists a maximal radius \( 0 < z_0 < \frac{R_2 - R_1}{\hat{c}} \) such that

\[
(7.18) \quad \int_{Q_{z_0}^{(\mu_{z_0})}} \left[ |Du|^p + |F|^p \right] dx = \lambda^p.
\]
By the maximality of \( q_{z_0} \), we know that
\[
\iint_{Q^\ast_{(s)} (z_0)} |Du|^p + |F|^p \, dx \, dt < \lambda^p
\]
for any \( q_{z_0} < s \leq R \).
Moreover, due to the monotonicity of \( \varrho \mapsto \mu_\varrho \) and (7.6) we have
\[
\mu_s \leq \mu_{q_{z_0}} \leq \left( \frac{s}{q_{z_0}} \right)^{n+2p} \mu_s,
\]
so that
\[
\iint_{Q^\ast_{q_{z_0}} (z_0)} |Du|^p + |F|^p \, dx \, dt \leq \left( \frac{\mu_{q_{z_0}}}{\mu_s} \right)^{n+2p} \iint_{Q^\ast_{(s)} (z_0)} |Du|^p + |F|^p \, dx \, dt
\]
\[
< \left( \frac{s}{q_{z_0}} \right)^{(n+2p)(p-2)} \lambda^p,
\]
(7.19)
for any \( q_{z_0} < s \leq R \). Finally, since \( R_2^1 + (R_2 - R_1)^2 \leq R_2^p \) we have that \( Q^\ast_{(s)} (z_0) \subset Q_{d_{z_0}} (z_0) \subset Q_{d_{z_0}} \).

7.4. A Reverse Hölder Inequality. As before, we consider \( z_0 \in \mathcal{E}(R_1, \lambda) \) with \( \lambda \) as in (7.16). Since \( \lambda \) and \( z_0 \) are fixed, we once again use the abbreviation \( \mu_{q_{z_0}} := \mu_{(q_{z_0}, q_{z_0})} \). We keep in mind that by construction \( 0 < q_{z_0} < \frac{R_1 - R_2}{c} \). According to (7.5) we construct \( \tilde{q}_{z_0} \in [q_{z_0}, R] \) and recall that, at least in the case \( \tilde{q}_{z_0} < R \), the cylinder \( Q^\ast_{(q_{z_0})} (z_0) \) is \( \mu \)-intrinsic, while \( Q^\ast_{(q_{z_0})} (z_0) \) is possibly only \( \mu \)-sub-intrinsic. By construction we have \( \mu_s = \mu_{q_{z_0}} \) for any \( s \in [q_{z_0}, \tilde{q}_{z_0}] \). In particular, \( \mu_{q_{z_0}} = \mu_{q_{z_0}} \). Our aim now is to apply Proposition 6.1 on the cylinder \( Q^\ast_{(q_{z_0})} (z_0) \). To this aim we have to verify that hypotheses (6.1) and (6.2) are fulfilled on this cylinder. From (7.19) applied with \( s = 4q_{z_0} \) and (7.18) we first observe that
\[
c^{-1} \iint_{Q^\ast_{(q_{z_0})} (z_0)} |Du|^p + |F|^p \, dx \, dt \leq \lambda^p = \iint_{Q^\ast_{(q_{z_0})} (z_0)} |Du|^p + |F|^p \, dx \, dt
\]
(7.20)
for a constant \( c = c(n, p) > 1 \). Together with (7.4) applied with \( s = 4q_{z_0} \) this shows
\[
\iint_{Q^\ast_{(q_{z_0})} (z_0)} |u|^p \, dx \, dt \leq 2^{n+p} \iint_{Q^\ast_{(q_{z_0})} (z_0)} |Du|^p + |F|^p \, dx \, dt
\]
\[
\iint_{Q^\ast_{(q_{z_0})} (z_0)} |Du|^p + |F|^p \, dx \, dt \leq 2^{n+p} \mu_{q_{z_0}}^p,
\]
ensuring that (6.1) is satisfied for the cylinder \( Q^\ast_{(q_{z_0})} (z_0) \) with \( K = 2^{n+p} \). We now turn our attention to hypothesis (6.2). If \( \tilde{q}_{z_0} \leq 2q_{z_0} \) we use the fact that \( \mu_{q_{z_0}} = \mu_{q_{z_0}} \) and inequality (7.20) to infer that
\[
\mu_{q_{z_0}}^p = \frac{1}{\lambda^p} \iint_{Q^\ast_{(q_{z_0})} (z_0)} |u|^p \, dx \, dt \leq c \iint_{Q^\ast_{(q_{z_0})} (z_0)} \frac{|u|^p}{\mu_{q_{z_0}}^p} \, dx \, dt \leq \iint_{Q^\ast_{(q_{z_0})} (z_0)} \frac{|F|^p}{\mu_{q_{z_0}}^p} \, dx \, dt,
\]
for a constant \( c = c(n, p) \). This shows that \( Q^\ast_{(q_{z_0})} (z_0) \) satisfies (6.2) with \( K = c(n, p) \). It remains to consider the case \( \tilde{q}_{z_0} > 2q_{z_0} \). If \( \mu_{q_{z_0}} = 1 \), then (6.2) is satisfied with \( K = 1 \). If \( \mu_{q_{z_0}} > 1 \), then by construction \( Q^\ast_{(q_{z_0})} (z_0) \) is intrinsic. Using in turn Lemma 3.5, inequality (7.4) with \( (q, s) \) replaced by \( (q_{z_0}, \frac{1}{2} \tilde{q}_{z_0}) \) (note that this is possible since \( \frac{1}{2} \tilde{q}_{z_0} \geq
\)
Lemma 5.2, respectively Lemma 5.5 (for \( q = p \)) and (7.19) (applied with \( s = g_{z_0} \in (g_{z_0}, R) \)) we obtain

\[
\mu_{g_{z_0}} \lambda = \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} \frac{|u|^p}{\tilde{g}_{z_0}^{2-p} \tilde{g}_{z_0}^{p}} \, dx \, dt \right]^\frac{1}{p} 
\leq \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} \frac{|u - (u)_{\tilde{g}_{z_0}^{\lambda/2}}|^p}{\mu_{g_{z_0}}^{2-p} \tilde{g}_{z_0}} \, dx \, dt \right]^\frac{1}{p} + \left[ (u)_{\tilde{g}_{z_0}^{\lambda/2}} \right]_{\tilde{g}_{z_0}^{\lambda}} 
\leq c \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} \frac{|u|^p}{\tilde{g}_{z_0}^{2-p} \tilde{g}_{z_0}^{p}} \, dx \, dt \right]^\frac{1}{p} + \frac{1}{2} \mu_{g_{z_0}} \lambda 
\leq c \lambda + \frac{1}{2} \mu_{g_{z_0}} \lambda,
\]

with \( c = c(n, p, L) \). After re-absorbing \( \frac{1}{2} \mu_{g_{z_0}} \lambda \) into the left-hand side, we find that \( \mu_{g_{z_0}} \leq c(n, p, L) \). This ensures that (6.2) is satisfied with \( K = c(n, p, L) \). Therefore, we are allowed to apply Proposition 6.1 on the cylinder \( Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0) \) with a constant \( K = K(n, p, L) \) and thereby obtain the reverse Hölder inequality

\[
\iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} |Du|^p \, dx \, dt 
\leq c \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} |Du|^q \, dx \, dt \right]^\frac{p}{q} + c \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} |F|^p \, dx \, dt \right]^\frac{1}{p},
\]

(7.21)

with \( q := \max \left\{ \frac{2n}{n+2}, \frac{2n}{n+2} \frac{n}{2p}, 1, p - 1 \right\} < p \) and \( c = c(n, p, \nu, L) \).

### 7.5. Estimates on level sets.
We summarize what we have shown so far. If \( \lambda \) satisfies (7.16), then for any \( z_0 \in E(R_1, \lambda) \) there exists a cylinder \( Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0) \) such that the \( \tilde{c} \)-times enlarged cylinder \( Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0) \) is still contained in \( Q_{R_2} \), and such that (7.18), (7.19) and (7.21) hold on this cylinder. As before, we abbreviate \( \mu_{g_{z_0}} \equiv \mu_{g_{z_0}}^{(\lambda)} \). Moreover, we define the superlevel set of the inhomogeneity \( |F| \) by

\[
F(r, \lambda) := \left\{ z \in Q_r : \text{z is a Lebesgue point of } |F| \text{ and } |F|(z) > \lambda \right\}
\]

and let \( \eta \in (0, 1] \) to be specified later. Due to (7.18) and (7.21) we have

\[
\lambda^p = \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} |Du|^p + |F|^p \, dx \, dt 
\leq c \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} |Du|^q \, dx \, dt \right]^\frac{p}{q} + c \left[ \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)} |F|^p \, dx \, dt \right]^\frac{1}{p} 
\leq c \eta^p \lambda^p + \frac{1}{|Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)|} \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0) \cap E(R_2, \eta \lambda)} |Du|^q \, dx \, dt 
\leq c \eta^p \lambda^p + \frac{c}{|Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0)|} \iint_{Q_{\tilde{g}_{z_0}}^{(\mu_{g_{z_0}})}(z_0) \cap F(R_2, \eta \lambda)} |F|^q \, dx \, dt \cdot I
\]
where any \( \lambda > \lambda_0 \) with \( c = c(n, p, \nu, L) \) and

\[
I := \left[ \int_{Q_{\hat{c} \lambda_0} (z_0)} |Du|^q \, dx \, dt \right]^{\frac{1}{q-1}}.
\]

In view of Hölder’s inequality and (7.19) we find that

\[
1 \leq \left[ \int_{Q_{\hat{c} \lambda_0} (z_0)} |Du|^p \, dx \, dt \right]^{1-\frac{s}{p}} \leq c \lambda^{p-q}.
\]

We insert this inequality above. Then, we choose \( \eta = (\frac{1}{2c})^\frac{1}{p-1} \) and re-absorb \( \frac{1}{2} \lambda^p \) into the left-hand side. Multiplying the result by \( |Q_{\hat{c} \lambda_0} (z_0)| \) yields

\[
\lambda^p |Q_{\hat{c} \lambda_0} (z_0)| \leq \int_{Q_{\hat{c} \lambda_0} (z_0)} \lambda^{p-q} |Du|^q \, dx \, dt
\]

\[
+ c \int_{Q_{\hat{c} \lambda_0} (z_0) \cap F(R_2, \eta \lambda)} |F|^p \, dx \, dt,
\]

again with \( c = c(n, p, \nu, L) \). Now, (7.19) for the choice \( s = \hat{c} \delta_{z_0} \) allows us to estimate \( \lambda^p \) from below. In this way, we deduce

\[
\int_{Q_{\hat{c} \lambda_0} (z_0)} |Du|^p \, dx \, dt \leq c \int_{Q_{\hat{c} \lambda_0} (z_0) \cap E(R_2, \eta \lambda)} \lambda^{p-q} |Du|^q \, dx \, dt
\]

\[
+ c \int_{Q_{\hat{c} \lambda_0} (z_0) \cap F(R_2, \eta \lambda)} |F|^p \, dx \, dt,
\]

(7.22)

where \( c = c(n, p, \nu, L) \). Since \( z_0 \in E(R_1, \lambda) \) was arbitrary, we have thus shown that for any \( \lambda > B \lambda_0 \) the associated super-level set \( E(R_1, \lambda) \) is covered by a family

\[
\mathcal{F} \equiv \left\{ Q_{\hat{c} \lambda_0}^{(\mu_{z_0, \hat{c} \lambda_0})} (z_0) \right\}
\]

of parabolic cylinders with center \( z_0 \in E(R_1, \lambda) \) which are contained in \( Q_{R_2} \), and such that (7.22) holds true on each of these cylinders. Recall, since \( \lambda \) is fixed we again write \( \mu_{z_0, \hat{c} \lambda_0} \equiv \mu_{z_0, \hat{c} \lambda_0}^{(\lambda)} \). The Vitali type covering Lemma 7.1 now ensures that there exists a countable subfamily

\[
\left\{ Q_{\hat{c} \lambda_0}^{(\mu_{z_i, \hat{c} \lambda_0})} (z_i) \right\}_{i \in \mathbb{N}} \subset \mathcal{F}
\]

of pairwise disjoint cylinders, such that the \( \frac{1}{2} \hat{c} \)-times enlarged cylinders \( Q_{\hat{c} \lambda_0}^{(\mu_{z_i, \hat{c} \lambda_0})} (z_i) \) cover the super-level set \( E(R_1, \lambda) \) and are still contained in \( Q_{R_2} \). More precisely, we have

\[
E(R_1, \lambda) \subset \bigcup_{i=1}^{\infty} Q_{\hat{c} \lambda_0}^{(\mu_{z_i, \hat{c} \lambda_0})} (z_i) \subset Q_{R_2}.
\]

Since the cylinders \( Q_{\hat{c} \lambda_0}^{(\mu_{z_i, \hat{c} \lambda_0})} (z_i) \) are pairwise disjoint we obtain with (7.22) that

\[
\int_{E(R_1, \lambda)} |Du|^p \, dx \, dt \leq \sum_{i=1}^{\infty} \int_{Q_{\hat{c} \lambda_0}^{(\mu_{z_i, \hat{c} \lambda_0})} (z_i)} |Du|^p \, dx \, dt
\]

\[
\leq c \sum_{i=1}^{\infty} \int_{Q_{\hat{c} \lambda_0}^{(\mu_{z_i, \hat{c} \lambda_0})} (z_i) \cap E(R_2, \eta \lambda)} \lambda^{p-q} |Du|^q \, dx \, dt.
\]
for a constant $c = c(n, p, \nu, L)$. On $E(R_1, \eta') \setminus E(R_1, \lambda)$ we have $|Du| \leq \lambda$, so that

\[
\int_{E(R_1, \eta')} |Du|^q \, dx \leq \int_{E(R_1, \eta')} \lambda^{p-q} |Du|^q \, dx + \int_{F(R_2, \eta')} |F|^p \, dx.
\]

Combining this with the second last inequality yields

\[
\int_{E(R_1, \eta')} |Du|^p \, dx \leq c \int_{E(R_1, \eta')} \lambda^{p-q} |Du|^q \, dx + c \int_{F(R_2, \eta')} |F|^p \, dx.
\]

We now replace $\eta'$ by $\lambda$ and recall that $n = n(p, \nu, L) < 1$. With this replacement we obtain for any $\lambda > \eta B\lambda_0 =: \lambda_1$ that

\[
\int_{E(R_1, \lambda)} |Du|^p \, dx \leq c \int_{E(R_1, \lambda)} \left( \frac{\lambda}{\eta} \right)^{p-q} |Du|^q \, dx + c \int_{F(R_2, \lambda)} |F|^p \, dx
\]

(7.23)

holds true with a constant $c = c(n, p, \nu, L)$. This is the reverse Hölder inequality on super-level sets we are looking for.

### 7.6. Proof of the gradient estimate

At this point the quantitative higher integrability estimate follows in a standard way from the reverse Hölder inequality on super-level sets by multiplying (7.23) by $\lambda^{p-1}$ and then integrating with respect to $\lambda$. For the sake of completeness we nevertheless provide the details. The just described procedure would lead on the left to an integral of $|Du|^{p(1+\varepsilon)}$ on $Q_{R_1}$, while on the right the same integral appears with factor $\frac{1}{\lambda}$ and $Q_{R_2}$ as domain of integration. If both integrals are finite the one on the right could be re-absorbed in view of Lemma 3.1. However, it is not clear in advance that these integrals are finite. For this reason we use a truncation argument in order to avoid powers of $|Du|$ that are larger than $p$. The rigorous argument is as follows: For $k > \lambda_1$ we define the truncation of $|Du|$ by

\[
|Du|_k := \min \{ |Du|, k \},
\]

and for $r \in (0, 2R]$ the corresponding super-level set by

\[
E_k(r, \lambda) := \{ z \in Q_r : |Du|_k > \lambda \}.
\]

Note that $|Du|_k \leq |Du|$ a.e., as well as $E_k(r, \lambda) = \emptyset$ for $k \leq \lambda$ and $E_k(r, \lambda) = E(r, \lambda)$ for $k > \lambda$. Therefore, (7.23) implies

\[
\int_{E_k(r, \lambda)} |Du|^{p-q} |Du|^q \, dx \leq c \int_{E_k(r, \lambda)} \lambda^{p-q} |Du|^q \, dx + c \int_{F(R_2, \lambda)} |F|^p \, dx.
\]

We now multiply this inequality by $\lambda^{p-1}$ with some $\varepsilon \in (0, 1]$ to be chosen later. Integrating the result with respect to $\lambda$ over the interval $(\lambda_1, \infty)$ leads to

\[
\int_{\lambda_1}^{\infty} \lambda^{p-1} \left[ \int_{E_k(r, \lambda)} |Du|^{p-q} |Du|^q \, dx \right] \, d\lambda \\
\leq c \int_{\lambda_1}^{\infty} \lambda^{p-q+\varepsilon} \left[ \int_{E_k(r, \lambda)} |Du|^q \, dx \right] \, d\lambda.
\]
\[+ \epsilon \int_1^\infty \lambda^{\epsilon p-1} \left[ \int_{F(R_2, \lambda)} |F|^p \, dx \right] d\lambda.\]

The idea now is to exchange the order of integration in each of the integrals by an application of Fubini’s theorem. For the integral on the left-hand side Fubini’s theorem shows

\[
\int_1^\infty \lambda^{\epsilon p-1} \int_{E_k(R_1, \lambda)} |Du|_{k}^{p-q} |Du|^q \, dx \, dt \, d\lambda
\]

\[
= \int_{E_k(R_1, \lambda)} |Du|_{k}^{\epsilon p} \int_1^\lambda \lambda^{\epsilon p-1} \, d\lambda \, dx \, dt
\]

\[
= \frac{1}{\epsilon p} \int_{E_k(R_1, \lambda)} \left[ |Du|_{k}^{p-q+\epsilon p} |Du|^q - \lambda^{\epsilon p} |Du|_{k}^{p-q} |Du|^q \right] \, dx \, dt,
\]

while for the first integral on the right we get

\[
\int_1^\infty \int_{R_2} |Du|^q \, dx \, dt \, d\lambda
\]

\[
= \int_{E_k(R_2, \lambda)} |Du|^q \int_1^\lambda \lambda^{\epsilon p-1} \, d\lambda \, dx \, dt
\]

\[
\leq \frac{1}{\epsilon p} \int_{E_k(R_2, \lambda)} |Du|_{k}^{p-q+\epsilon p} |Du|^q \, dx \, dt
\]

\[
\leq \frac{1}{\epsilon p} \int_{E_k(R_2, \lambda)} |Du|_{k}^{p-q+\epsilon p} \, dx \, dt.
\]

Finally, for the second integral on the right we find that

\[
\int_1^\infty \int_{F(R_2, \lambda)} |F|^p \, dx \, dt \, d\lambda
\]

\[
= \int_{F(R_2, \lambda)} |F|^p \int_1^\lambda \lambda^{\epsilon p-1} \, d\lambda \, dx \, dt
\]

\[
\leq \frac{1}{\epsilon p} \int_{F(R_2, \lambda)} |F|^{(1+\epsilon)p} \, dx \, dt
\]

\[
\leq \frac{1}{\epsilon p} \int_{Q_{2R}} |F|^{(1+\epsilon)p} \, dx \, dt.
\]

Inserting the preceding estimates above and multiplying by \(\epsilon p\) shows that

\[
\int_{E_k(R_1, \lambda)} \int_{E_k(R_2, \lambda)} |Du|_{k}^{p-q+\epsilon p} |Du|^q \, dx \, dt
\]

\[
\leq \lambda_{1}^{\epsilon p} \int_{E_k(R_1, \lambda)} |Du|^q \, dx \, dt
\]

\[
+ \frac{\epsilon p}{p-q} \int_{E_k(R_2, \lambda)} |Du|_{k}^{p-q+\epsilon p} |Du|^q \, dx \, dt + \epsilon \int_{Q_{2R}} |F|^{(1+\epsilon)p} \, dx \, dt.
\]

On the complement \(Q_{R_1} \setminus E_k(R_1, \lambda)\) we have \(|Du|_{k} \leq \lambda_1\) and hence

\[
\int_{Q_{R_1} \setminus E_k(R_1, \lambda)} |Du|_{k}^{p-q+\epsilon p} |Du|^q \, dx \, dt
\]

\[
\leq \lambda_{1}^{\epsilon p} \int_{Q_{R_1} \setminus E_k(R_1, \lambda)} |Du|^q \, dx \, dt.
\]

Joining the last two estimates and taking into account that \(|Du|_{k} \leq |Du|\), we obtain

\[
\int_{Q_{R_1}} |Du|_{k}^{p-q+\epsilon p} |Du|^q \, dx \, dt \leq \frac{C_{\epsilon} p}{p-q} \int_{Q_{R_2}} |Du|_{k}^{p-q+\epsilon p} |Du|^q \, dx \, dt
\]

\[
+ \lambda_{1}^{\epsilon p} \int_{Q_{2R}} |Du|^p \, dx \, dt + \epsilon \int_{Q_{2R}} |F|^{(1+\epsilon)p} \, dx \, dt,
\]
where $c_* = c_*(n, p, \nu, L) \geq 1$. Now, we choose
\[ 0 < \varepsilon \leq \varepsilon_* := \frac{p - q}{2pc_*}. \]

Note that $\varepsilon_*$ depends only on $n, p, \nu, $ and $L$. Furthermore, $\lambda_\ast \equiv (nB\lambda_0)^{\varepsilon} \leq B\lambda_0^\varepsilon$, since $B \geq 1$, $\eta < 1$ and $\varepsilon \leq 1$. With this choice the last inequality shows that for each pair of radii $R_1, R_2$ with $R \leq R_1 < R_2 \leq 2R$ the estimate
\[
\iint_{Q_{R_1}} |Du|^{p-q+\varepsilon p} |Du|^q \, dx \, dt \leq \frac{1}{2} \iint_{Q_{R_2}} |Du|^{p-q+\varepsilon p} |Du|^q \, dx \, dt
\] 
\[ + c \lambda_0^{\varepsilon} \left( \frac{2R}{R_2 - R_1} \right)^{\frac{q(n+2)}{q-n}} \iint_{Q_{2R}} |Du|^p \, dx \, dt + c \iint_{Q_{2R}} |F|^{(1+\varepsilon)p} \, dx \, dt
\]
holds true. In view of the Iteration Lemma 3.1 we conclude that
\[
\iint_{Q_{R}} |Du|^{(1+\varepsilon)p} \, dx \, dt \leq c \lambda_0^{\varepsilon} \iint_{Q_{2R}} |Du|^p \, dx \, dt + c \iint_{Q_{2R}} |F|^{(1+\varepsilon)p} \, dx \, dt.
\]
At this point we use Fatou’s lemma to pass to the limit $k \to \infty$ on the left-hand side. Subsequently we take means on both sides and infer that
\[
\mathbb{E} \left[ \iint_{Q_{R}} |Du|^{(1+\varepsilon)p} \, dx \, dt \right] \leq c \lambda_0^{\varepsilon} \mathbb{E} \left[ \iint_{Q_{2R}} |Du|^p \, dx \, dt \right] + c \mathbb{E} \left[ \iint_{Q_{2R}} |F|^{(1+\varepsilon)p} \, dx \, dt \right].
\]
In view of the definition of $\lambda_0$ from (7.14) the preceding inequality turns into
\[
\mathbb{E} \left[ \iint_{Q_{R}} |Du|^{(1+\varepsilon)p} \, dx \, dt \right] \leq c \left[ 1 + \mathbb{E} \left[ \iint_{Q_{4R}} \left[ \frac{|u|^p}{(4R)^p} + |Du|^p + |F|^p \right] \, dx \, dt \right] \right] \varepsilon \mathbb{E} \left[ \iint_{Q_{2R}} |Du|^p \, dx \, dt \right]
\] 
\[ + c \mathbb{E} \left[ \iint_{Q_{2R}} |F|^{(1+\varepsilon)p} \, dx \, dt \right].
\]
Note that $c = c(n, p, \nu, L)$. A straightforward covering argument now yields the claimed quantitative estimate. This completes the proof of Theorem 2.2.

REFERENCES

[1] V. Bögelein. Higher integrability for weak solutions of higher order degenerate parabolic systems. Ann. Acad. Sci. Fenn. Math. 33 (2008), no. 2, 387–412.
[2] V. Bögelein, F. Duzaar, R. Korte, and C. Scheven. The higher integrability of weak solutions of porous medium systems. Adv. Nonlinear Anal., DOI: https://doi.org/10.1515/anona-2017-0270.
[3] V. Bögelein and M. Parviainen. Self-improving property of nonlinear higher order parabolic systems near the boundary. NoDEA Nonlinear Differential Equations Appl. 17 (2010), no. 1, 21–54.
[4] E. DiBenedetto. Degenerate parabolic equations. Springer-Verlag, Universitext, 1993.
[5] E. DiBenedetto, U. Gianazza, and V. Vespri. Harnack’s inequality for degenerate and singular parabolic equations. Springer Monographs in Mathematics, 2011.
[6] L. Diening, P. Kaplický, and S. Schwarzacher. BMO estimates for the $p$-Laplacian, Nonlinear Anal. 75 (2012), no. 2, 637–650.
[7] S. Fornaro, M. Sosio, and V. Vespri. Harnack type inequalities for some doubly nonlinear singular parabolic equations. Discrete Contin. Dyn. Syst. 35 (2015), 5909–5926.
[8] H. Federer. Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag, New York, 1969.

[9] U. Gianazza and S. Schwarzacher. Self-improving property of degenerate parabolic equations of porous medium-type. Amer. J. Math., to appear.

[10] U. Gianazza and V. Vespri. A Harnack inequality for solutions of doubly nonlinear parabolic equations. J. Appl. Funct. Anal. 1 (2006), no. 3, 271–284.

[11] M. Giaquinta. Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton University Press, Princeton, 1983.

[12] M. Giaquinta and M. Struwe. On the partial regularity of weak solutions of nonlinear parabolic systems. Math. Z. 179 (1984), no. 4, 437–451.

[13] E. Giusti. Direct Methods in the Calculus of Variations. World Scientific Publishing Company, Tuck Link, Singapore, 2003.

[14] A. V. Ivanov, P. Z. Mkrtychyan, and W. Jäger. Existence and uniqueness of a regular solution of the Cauchy-Dirichlet problem for a class of doubly nonlinear parabolic equations. J. Math. Sci. (N. Y.) 84 (1997), 845–855.

[15] J. Kinnunen and T. Kuusi. Local behaviour of solutions to doubly nonlinear parabolic equations. Math. Ann. 337 (2007), no. 3, 705–728.

[16] J. Kinnunen and J. L. Lewis. Higher integrability for parabolic systems of p-Laplacian type. Duke Math. J. 102 (2000), no. 2, 253–271.

[17] T. Kuusi, R. Lazeoglu, J. Siljander and J. M. Urbano. Hölder continuity for Trudinger’s equation in measure spaces. Calc. Var. Partial Differential Equations 45 (2012), no. 1–2, 193–229.

[18] T. Kuusi, R. Lazeoglu, J. Siljander and J. M. Urbano. Local Hölder continuity for doubly nonlinear parabolic equations. Indiana Univ. Math. J. 61 (2012), no. 1, 399–430.

[19] N. G. Meyers and A. Elcrat. Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions. Duke Math. J. 42 (1975), 121–136.

[20] M. Parviainen. Global gradient estimates for degenerate parabolic equations in nonsmooth domains. Ann. Mat. Pura Appl. (4) 188 (2009), no. 2, 333–358.

[21] S. Schwarzacher. Hölder-Zygmund estimates for degenerate parabolic systems. J. Differential Equations 256 (2014), no. 7, 2423–2448.

[22] J. Siljander. Boundedness of the gradient for a doubly nonlinear parabolic equation. J. Math. Anal. Appl. 371 (2010), no. 1, 158–167.

[23] N. S. Trudinger. Pointwise estimates and quasilinear parabolic equations. Comm. Pure Appl. Math. 21 (1968), no. 7, 205–226.

[24] V. Vespri. On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations. Manuscripta Math. 75 (1992), no. 1, 65–80.

VERENA BÖGELEIN, FACHBEREICH MATHEMATIK, UNIVERSITÄT SALZBURG, HELLBRUNNER STR. 34, 5020 SALZBURG, AUSTRIA
E-mail address: verena.boegelein@sbg.ac.at

FRANK DUZAAR, DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN–NÜRNBERG, CAUER-STRASSE 11, 91058 ERLANGEN, GERMANY
E-mail address: duzaar@math.fau.de

JUHA KINNUNEN, AALTO UNIVERSITY, DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, P.O. BOX 11100, FI-00076 AALTO, FINLAND
E-mail address: juha.k.kinnunen@aalto.fi

CHRISTOPH SCHEVEN, FAHKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, 45117 ESPER, GERMANY
E-mail address: christoph.scheven@uni-due.de