Relaxation-time approximation and relativistic viscous hydrodynamics from kinetic theory

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Abstract

Using the iterative solution of Boltzmann equation in the relaxation-time approximation, the derivation of a third-order evolution equation for shear stress tensor is presented. To this end we first derive the expression for viscous corrections to the phase-space distribution function, \( f(x,p) \), up to second-order in derivative expansion. The expression for \( \delta f(x,p) \) obtained in this method does not lead to violation of the experimentally observed \( 1/\sqrt{mT} \) scaling of the femtoscopic radii, as opposed to the widely used Grad’s 14-moment approximation. Subsequently, we present the derivation of a third-order viscous evolution equation and demonstrate the significance of this derivation within one-dimensional scaling expansion. We show that results obtained using third-order evolution equations are in excellent accordance with the exact solution of Boltzmann equation as well as with transport results.

Keywords: Relativistic hydrodynamics, Kinetic theory, Boltzmann equation

1. Introduction

Hydrodynamics is an effective theory used to describe the long-wavelength, low-frequency limit of the microscopic dynamics of a many-body system, which is close to equilibrium. Relativistic hydrodynamics is formulated as derivative expansion where ideal hydrodynamics is of zeroth-order. The first-order dissipative theories (collectively known as relativistic Navier-Stokes theory) suffer from acausality and numerical instability. Although the acausality problem is rectified in the second-order Israel-Stewart (IS) theory, stability is not guaranteed. The IS hydrodynamics has been used quite extensively to study the collective behaviour of hot and dense, strongly interacting matter created in high-energy heavy-ion collision experiments at the Relativistic Heavy-Ion Collider (RHIC) and at the Large Hadron Collider (LHC).

Despite its success in explaining a wide range of collective phenomena observed in ultra-relativistic heavy-ion collisions, the formulation of IS theory is based on several approximations and assumptions. Israel and Stewart assumed Grad’s 14-moment approximation for the non-equilibrium distribution function and obtained the dissipative evolution equations from the second moment of the Boltzmann equation. It was shown recently that iterative solution of the Boltzmann equation can be used instead of 14-moment approximation and the dissipative equations can be derived directly from their definitions without resorting to an arbitrary choice of moment of the Boltzmann equation. Apart from these problems in the theoretical formulation, IS theory suffers from several other shortcomings. In one-dimensional scaling expansion, IS theory results in unphysical effects such as negative longitudinal pressure and reheating of the expanding medium. Moreover, comparison of results obtained using IS evolution equation with transport results show disagreement for \( \eta/s > 0.5 \), indicating the breakdown of second-order theory. Furthermore, inclusion of viscous corrections to the phase-space distribution function, \( f(x,p) \), via 14-moment approximation results in the violation of experimentally observed and ideal hydrodynamic prediction of \( 1/\sqrt{mT} \) scaling of the longitudinal Hanbury Brown-Twiss (HBT) radii.
In order to widen the range of applicability of the IS theory, second-order dissipative equations were derived from the Boltzmann equation where the collision term was generalized to incorporate nonlocal effects through gradients of $f(x, p)$ \cite{11}. Moreover, it was also shown that inclusion of third-order corrections to shear evolution equation led to an improved agreement with the transport results \cite{12,13}. Furthermore, to improve Grad’s 14-moment approximation beyond its current scope, a general moment method was devised by introducing orthogonal basis in momentum series expansion \cite{13}. The accurate and consistent formulation of the theory of relativistic viscous hydrodynamics is not yet conclusively resolved and is presently a topic of intense investigation \cite{4,5,11,12,13,14,15,16,17}.

2. Relativistic hydrodynamics

The equation of motion governing the hydrodynamical evolution of a relativistic system with no net conserved charges is obtained from the local conservation of energy and momentum, $\partial_{\mu}T^{\mu\nu} = 0$. In terms of single-particle phase-space distribution function, the energy-momentum tensor of a macroscopic system can be expressed as \cite{18}

$$ T^{\mu\nu} = \int dp \; p^{\mu} p^{\nu} f(x, p) = \epsilon u^{\mu} u^{\nu} - P g^{\mu\nu} + \pi^{\mu\nu}, $$

where $dp \equiv gdp/((2\pi)^3|p|)$, $g$ being the degeneracy factor, $p^\mu$ is the particle four-momentum, and $f(x, p)$ is the phase-space distribution function. In the tensor decomposition, $\epsilon$, $P$, and $\pi^{\mu\nu}$ are energy density, thermodynamic pressure, and shear stress tensor, respectively. The projection operator $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is orthogonal to the hydrodynamic four-velocity $u^\mu$ defined in the Landau frame: $T^{\mu\nu} u_\nu = \epsilon u^\mu$. The metric tensor is Minkowskian, $g^{\mu\nu} \equiv \text{diag}(+, +, +, -)$. Here we restrict ourselves to a system of massless particles (ultrarelativistic limit) for which the bulk viscosity vanishes.

The conservation of the energy-momentum tensor, when projected along and orthogonal to $u^\mu$, leads to the evolution equations for $\epsilon$ and $u^\mu$:

$$ \dot{\epsilon} + (\epsilon + P) \dot{\theta} - \pi^{\mu\nu} \sigma_{\mu\nu} = 0, \quad (\epsilon + P) \dot{u}^{\mu} - \nabla^{\mu} P + \Delta_{\mu}^{\nu} \partial_{\mu} \pi^{\nu\sigma} = 0, $$

where we employ the standard notation $\dot{A} \equiv u^{\mu} \partial_{\mu} A$ for comoving derivative, $\theta \equiv \partial_{\mu} u^\mu$ for expansion scalar, $\sigma^{\mu\nu} \equiv (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - (\theta/3)\Delta^{\mu\nu}$ for velocity stress tensor, and $\nabla^{\mu} \equiv \Delta^{\mu\nu} \partial_{\nu}$ for space-like derivatives. For the massless case, the equation of state relating energy density and pressure is $\epsilon = 3P \approx \beta^4$. The matching condition $\epsilon \approx \epsilon_0$ is employed to fix the inverse temperature, $\beta \equiv 1/T$, where $\epsilon_0$ is the equilibrium energy density. The derivatives of $\beta$,

$$ \dot{\beta} = \frac{\beta}{T_0} \dot{T} - \frac{\beta}{12T^2} \pi^{\mu\nu} \sigma_{\mu\nu}, \quad \nabla^{\mu} \beta = -\beta \dot{u}^{\mu} - \frac{\beta}{4\nu} \Delta^{\mu}_{\nu} \partial_{\nu} \pi^{\nu\sigma}, $$

can be obtained from Eq. (2). When the system is close to local thermodynamic equilibrium, the distribution function can be written as $f = f_0 + \delta f$, where $\delta f \ll f_0$, $f_0 \equiv \exp(-\beta u \cdot p)$ is the equilibrium distribution function of Boltzmann particles at vanishing potential and $u \cdot p \equiv u_\mu p^\mu$. Projecting the traceless symmetric part of Eq. (1) using the operator $\Delta^{\mu\nu} \equiv (\Delta^{\mu}_{\lambda} \Delta_{\nu}^{\lambda} + \Delta^{\nu}_{\lambda} \Delta_{\mu}^{\lambda})/2 - (1/3)\Delta^{\mu\nu} \Delta^{\lambda\mu}$, we can write the shear stress tensor and its time evolution as,

$$ \pi^{\mu\nu} = \Delta^{\mu\nu}_{\lambda\rho} \int dp \; p^{\rho} p^{\delta} \delta f, \quad \pi^{(\mu\nu)} = \Delta^{\mu\nu}_{\lambda\rho} \int dp \; p^{\rho} p^{\delta} \delta f. $$

In the following we obtain $\delta f$ and derive evolution equation for shear stress tensor in terms of hydrodynamic variables.

3. Viscous evolution equations

We start from the relativistic Boltzmann equation with relaxation-time approximation for the collision term \cite{19},

$$ p^{\mu} \partial_{\mu} f = -(u \cdot p) \frac{\delta f}{\tau_R} \Rightarrow f = f_0 - \frac{\tau_R}{(u \cdot p)} p^{\mu} \partial_{\mu} f, $$

where $\tau_R$ is the relaxation time. Expanding the distribution function $f$ about its equilibrium value in powers of space-time gradients, i.e., $f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots$ and solving Eq. (5) iteratively, we obtain \cite{4,13}.

$$ \delta f^{(1)} = -\frac{\tau_R}{u \cdot p} p^{\mu} \partial_{\mu} f_0, \quad \delta f^{(2)} = \frac{\tau_R}{u \cdot p} p^{\mu} p^{\nu} \partial_{\mu} \left( \frac{\tau_R}{u \cdot p} \partial_{\nu} f_0 \right). $$
Substituting δf = δf(1) in the expression for n^µν in Eq. (3), performing the integrations, and retaining only first-order terms, we obtain n^µν = 2τ_β δ β^µν, where β_β = 4P/5.

To obtain the second-order evolution equation for shear stress tensor, we rewrite Eq. (5) in the form δf = −f_0 − p^γ ∇_γ f/(u·p) − δf/τ_R. Using this expression for δf in Eq. (4), we obtain
\[ \dot{\pi}^{(μν)} + \frac{\pi^{(μν)}}{τ_π} = -\Delta^{(μν)}_{σβ} \int dp \ p^β \ p^σ \left( f_0 + \frac{1}{u·p} p^γ ∇_γ f \right) \] (7)

Using Eq. (6) for δf(1) and Eq. (3) for derivatives of β, and keeping terms up to quadratic order in gradients, the second-order shear evolution equation is obtained as
\[ \dot{\pi}^{(μν)} + \frac{\pi^{(μν)}}{τ_π} = 2β_α δ^{μν}_α + 2π^{(μ}_ρ δ^{ρν)}_γ - \frac{10}{7} p^{(μ}_γ δ^{ρν)}_ρ - \frac{4}{3} \pi^{(μ}_γ δ^{ρν)}_ρ, \] (8)

where ω^{μν} = (∇^μ u^ν − ∇^ν u^μ)/2 is the vorticity tensor. Using Eqs. (3) and (8) in Eq. (9), we arrive at
\[ \dot{\delta} f = \frac{f_0 β}{2β_α (u·p)} p^μ p^ρ δ^{μν}_α - \frac{f_0 β}{β_π} \frac{p^μ}{u·p} \left( \frac{τ_π}{3(u·p)} p^ρ p^σ δ^{σρ}_α \right) + \frac{p^μ}{u·p} \left( \frac{τ_π}{70β_π} p^{(μ}_ρ δ^{ν)}_ρ + \frac{τ_π}{5} p^σ δ^{σρ}_ρ + \frac{τ_π}{5} p^σ \left( π^{(μ}_γ δ^{ρν)}_ρ \right) \right) \] (9)

where the first term on the right-hand side of the above equation corresponds to the first-order correction, δf_1, whereas the terms within square brackets are of second order, δf_2. It is straightforward to show that the form of δf_2 in Eq. (9) is consistent with the definition of the shear stress tensor, Eq. (6), and satisfies the matching condition ε = ε_0 and the Landau frame definition u_τ T^v = ε/u^v at each order [17]. It is important to note that the experimentally observed 1/√s scaling of the longitudinal HBT radii [20], also predicted by ideal hydrodynamics, is violated by incorporating viscous corrections through Grad's 14-moment approximation [10]. However, the form of δf given in Eq. (9) does not lead to such unphysical effects [17].

To obtain a third-order evolution equation for shear stress tensor, we substitute δf from Eq. (9) in Eq. (7). Keeping terms up to cubic order in derivatives, after a straightforward but tedious algebra, we finally obtain a third-order evolution equation for shear stress tensor [13]:
\[ \dot{\pi}^{(μν)} = -\frac{\pi^{(μν)}}{τ_π} + 2β_α δ^{μν}_α + 2π^{(μ}_ρ δ^{ρν)}_γ - \frac{10}{7} p^{(μ}_γ δ^{ρν)}_ρ - \frac{4}{3} \pi^{(μ}_γ δ^{ρν)}_ρ - \frac{1}{3} (p^{(μ}_γ δ^{ρν)}_ρ + \frac{25}{7β_π} \pi^{(μ}_ρ δ^{ν)}_ρ + \frac{25}{249β_π} \pi^{(μ}_ρ δ^{ν)}_ρ + \frac{24}{35} \pi^{(μ}_ρ δ^{ν)}_ρ + \frac{4}{35} \pi^{(μ}_ρ δ^{ν)}_ρ + \frac{24}{35} \pi^{(μ}_ρ δ^{ν)}_ρ + \frac{4}{35} \pi^{(μ}_ρ δ^{ν)}_ρ) - \frac{12}{7} \pi^{(μ}_γ δ^{ν)}_ρ - \frac{12}{7} \pi^{(μ}_γ δ^{ν)}_ρ - \frac{10}{63} \pi^{(μ}_γ δ^{ν)}_ρ + \frac{26}{21} \pi^{(μ}_γ δ^{ν)}_ρ \] (10)

This is the main result of the present work. We compare the above equation with that obtained in Ref. [12] by invoking the second law of thermodynamics,
\[ \dot{\pi}^{(μν)} = -\frac{\pi^{(μν)}}{τ_π} + 2β_α δ^{μν}_α + \frac{4}{3} \pi^{(μ}_γ δ^{ν)}_ρ + \frac{5}{36β_π} \pi^{(μ}_ρ δ^{ν)}_ρ - \frac{16}{9β_π} \pi^{(μ}_ρ δ^{ν)}_ρ, \] (11)

where β_π = 2P/3 and τ_π = η/β_π. We notice that the right-hand-side of the above equation contains one second-order and two third-order terms compared to three second-order and fourteen third-order terms obtained in Eq. (10). This confirms the fact that the evolution equation obtained by invoking the second law of thermodynamics is incomplete.

4. Numerical results and conclusions

In the following, we consider boost-invariant Bjorken expansion of a massless Boltzmann gas [6]. We have solved the evolution equations with initial temperature T_0 = 300 MeV at initial time τ_0 = 0.25 fm/c and with T_0 = 500 MeV at τ_0 = 0.4 fm/c, corresponding to initial conditions of RHIC and LHC, respectively.
To summarize, we have derived a third-order evolution equation for the shear stress tensor from kinetic theory. We iteratively solved the Boltzmann equation in relaxation time approximation to obtain the non-equilibrium distribution function up to second-order in gradients. Using this form of the non-equilibrium distribution function, instead of Grad’s 14-moment approximation, the evolution equation for shear tensor was derived directly from its definition. Within one-dimensional scaling expansion we demonstrated that the third-order viscous hydrodynamic equation derived here provides a very good approximation to the exact solution of Boltzmann equation. We also showed that our results are in better agreement with the results of the parton cascade BAMPS (dots) \[12, 22\] as compared to second-order results (dashed line) suggesting the convergence of the derivative expansion. In Fig. 1(b) we notice that while the results from entropy derivation (dashed lines) overestimate the pressure anisotropy for $\eta/s > 0.2$, those obtained in the present work (solid lines) are in better agreement with the results of the parton cascade BAMPS (dots) \[12, 22\].

Figures 1(a) and (b) shows the proper time dependence of pressure anisotropy $P_\perp/P_T \equiv (P - \pi)/(P + \pi/2)$ where $\pi \equiv -T^2 \pi^\pi$. In Fig. 1(a), we observe an improved agreement of third-order results (solid lines) with the exact solution of Boltzmann equation (dotted line) \[21\] as compared to second-order results (dashed line) suggesting the convergence of the derivative expansion. In Fig. 1(b) we notice that while the results from entropy derivation (dashed lines) overestimate the pressure anisotropy for $\eta/s > 0.2$, those obtained in the present work (solid lines) are in better agreement with the results of the parton cascade BAMPS (dots) \[12, 22\].

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Figure 1. (a): Time evolution of $P_\perp/P_T$ obtained using exact solution of Boltzmann equation (dotted line), second-order equations (dashed lines), and third-order equations (solid lines). (b): Time evolution of $P_\perp/P_T$ in BAMPS (dots), third-order calculation from entropy method, Eq. (11) (dashed lines), and the present work (solid lines). Both figures are for isotropic initial pressure configuration ($n_0 = 0$) and various $\eta/s$. 