ÉTALE COHOMOLOGY OF A DM CURVE-STACK
WITH COEFFICIENTS IN $\mathbb{G}_m$

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Abstract. We compute étale cohomology groups $H^i_{\text{ét}}(X, \mathbb{G}_m)$ in several cases, where $X$ is a smooth tame Deligne-Mumford stack of dimension 1 over an algebraically closed field. We have complete results for orbicurves (and, more generally, for twisted nodal curves) and in the case all stabilizers are cyclic; we give partial results and examples in the general case. In particular, we show that if the stabilizers are abelian then $H^2_{\text{ét}}(X, \mathbb{G}_m)$ does not depend on $X$ but only on the underlying orbicurve and on the generic stabilizer.

1. Introduction

A classical theorem of Tsen states that a function field $K$ of dimension 1 over an algebraically closed field is quasi-algebraically closed (Theorem 2.4). As a consequence, the étale cohomology groups $H^r_{\text{ét}}(\text{Spec } K, \mathbb{G}_m)$ vanish for $r \geq 1$ (Corollary 2.3). Combining this result with étale cohomology techniques, Milne showed that, for a connected smooth curve $C$ over an algebraically closed field, $H^r_{\text{ét}}(C, \mathbb{G}_m)$ vanishes for $r \geq 2$ ([8], III.2.22(d)). Étale cohomology in low degree with coefficients in $\mathbb{G}_m$ can be interpreted geometrically. It is well-known for a scheme $X$ that $H^1_{\text{ét}}(X, \mathbb{G}_m) \cong \text{Pic}(X)$ ([8], III.4.9). Moreover, for a smooth variety $X$ over a field, $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is isomorphic to the Brauer group $\text{Br}(X)$ ([8], IV.2.15). It follows that computing the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ contributes to a better understanding of geometric properties of $X$. As an example, the vanishing of $H^2_{\text{ét}}(X, \mathbb{G}_m)$ implies that every gerbe over $X$ banded by a finite group (of order not divided by the characteristic of the base field) is obtained as a finite number of root constructions ([4]). More in general, once we know $H^r_{\text{ét}}(X, \mathbb{G}_m)$, we can use Kummer sequence to compute $H^r_{\text{ét}}(X, \mu_n)$, for all $n$ not divided by the characteristic of the base field.

In this paper we study the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ where $X$ is a connected smooth tame Deligne-Mumford stack of dimension 1 over an algebraically closed field $k$. Such an $X$ admits a natural map $X \to C$ to its coarse moduli space, which is a connected smooth curve over $k$; moreover the map $X \to C$ factors via an étale gerbe $X \to Y$, where $Y$ is an orbicurve ([1], Appendix A).

We generalize to algebraic stacks the étale cohomology techniques used by Milne and we obtain a description of the groups $H^r_{\text{ét}}(X, \mathbb{G}_m)$ (Theorem 4.10). In particular, we show that if the stabilizers are abelian then $H^2_{\text{ét}}(X, \mathbb{G}_m)$ depends only on the orbicurve $Y$ and the generic stabilizer, not on the structure of the gerbe $X \to Y$ (Proposition 4.12). This result suggests to investigate two special cases: when $X = Y$ is an orbicurve (Corollary 4.15) and when the gerbe $X \to Y$ is trivial (Proposition 4.14). The cohomology of orbicurves, and more in general of twisted nodal curves, has been described in [3], Theorem 3.2.3. We include it for completeness, since we give a different proof (Proposition 5.2). We show that, for a twisted nodal curve $Y$, the vanishing of $H^2_{\text{ét}}(Y, \mathbb{G}_m)$ implies that every $G$-gerbe over $Y$ banded by a finite group $G$ (of order not divided by $\text{char } k$), can be obtained as a finite number of root constructions (Section 5.1). This fact has been used in [3] and [5] to study the moduli stack of twisted stable maps and the associated Gromov-Witten invariants.
Finally, we show with two examples that, in general, the higher cohomology groups \( H^r_{\text{ét}}(X, \mathbb{G}_m) \) cannot be computed knowing only the base of the gerbe \( X \to Y \) and the banding group (Section 7).

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Notations. We write \( \text{Br}(K) \) for the Brauer group of a field \( K \). With the word stack we always mean a Deligne-Mumford algebraic stack in the sense of [6]. All stacks are assumed to be separated and of finite type over the base field. For the notion of tame stack see [1]. An orbicurve is a stack of dimension 1 with trivial generic stabilizer. We denote by \( \mathcal{O}^{\text{sh}}_{X,\varpi} \) the strictly Henselian local ring of a scheme (or an algebraic stack) \( S \) at a geometric point \( \varpi \to S \). We denote by \( \mu_n \) the sheaf defined by the group scheme \( \text{Spec} \mathbb{Z}[t]/(t^n - 1) \).

2. Preliminaries

We recall a few classical theorems about quasi-algebraically closed fields and cohomology of finite abelian groups (for more details see [10], chapter IV, and [11], chapter VI).

2.1. Definition. A field \( K \) is quasi-algebraically closed if every non constant homogeneous polynomial \( f(x_1, \ldots, x_N) \) of degree \( d < N \) with coefficients in \( K \) has a non-trivial zero in \( K \).

2.2. Proposition ([10], IV.3 Corollary 1). Let \( K \) denote a quasi-algebraically closed field and let \( G \) denote the absolute Galois group \( \text{Gal}(K^\text{sep}/K) \), where \( K^\text{sep} \) is a separable closure of \( K \). Then

1. \( \text{Br}(K) = 0 \);
2. \( H^r(G, T) = 0 \) for \( r \geq 2 \) and for all discrete torsion \( G \)-modules \( T \);
3. \( H^r(G, M) = 0 \) for \( r \geq 3 \) and for all discrete \( G \)-modules \( M \).

2.3. Corollary. Let \( K \) be a quasi-algebraically closed field. Then \( H^r_{\text{ét}}(\text{Spec} K, \mathbb{G}_m) = 0 \) for \( r \geq 1 \).

Proof. By [8], III.1.7 and Proposition 2.2.

\[
H^r_{\text{ét}}(\text{Spec} K, \mathbb{G}_m) = \begin{cases} 
\text{Pic}(\text{Spec} K) = 0 & \text{if } r = 1 \\
H^2(G, K^*_s) = \text{Br}(K) = 0 & \text{if } r = 2 \\
H^r(G, K^*_s) = 0 & \text{if } r \geq 3.
\end{cases}
\]

2.4. Theorem ([10], IV.3 Theorem 24). Let \( K \) be a function field of dimension 1 over an algebraically closed field \( k \). Then \( K \) is quasi-algebraically closed.

2.5. Theorem ([10], IV.3 Theorem 27). Let \( K \) be the field of fractions of an Henselian discrete valuation ring \( R \) with algebraically closed residue field. Let \( \hat{K} \) be the completion of \( K \), and assume that \( K \subset \hat{K} \) is a separable field extension. Then \( K \) is quasi-algebraically closed.

2.6. Remark. The separability condition in Theorem 2.5 holds in the case \( R = \mathcal{O}^{\text{sh}}_{X,\varpi} \), with \( X \) a scheme of finite type over a field ([8, III.2.22(b)]).

2.7. Theorem ([11], Theorem 6.2.2). Let \( G = \mathbb{Z}/r\mathbb{Z} \) and let \( A \) be a discrete \( G \)-module. Then

\[
H^r(G, A) = \begin{cases} 
A^G & r = 0 \\
\ker N_G/I_G A & r \equiv 1 \pmod{r} \\
A^G/\text{im} N_G & r \equiv 0 \pmod{r}, r > 0,
\end{cases}
\]

where \( N_G : A \to A \) is the norm \( N_G(a) = \sum_{g \in G} ga \), \( A^G \) is the set of elements in \( A \) fixed by the \( G \)-action, and \( I_G A \) is the subgroup of \( A \) generated by elements \((ga-a)\), with \( a \in A \) and \( g \in G \).
2.8. **Corollary.** Let $G$ be a finite cyclic group acting trivially on $\mathbb{Z}$, then

$$H^r(G, \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & r = 0 \\
0 & r \equiv 1 \pmod{2}, r > 0.
\end{cases}$$

2.9. **Lemma.** Let $G$ be a finite group and let $(J^\bullet, d)$ be a cochain complex of abelian groups on which $G$ acts trivially. Then, for $p \geq 0$, $q \geq 1$ and $r = 2, \ldots, q + 1$, the following map is zero

$$\Phi_{r}^{p,q} : H^p(G, H^q(J^\bullet)) \to H^{p+r}(G, H^{q-r+1}(J^\bullet)).$$

**Proof.** Let assume $r = 2$. The map $\Phi_{2}^{p,q}$ is defined as follows. Let $\xi \in H^p(G, H^q(J^\bullet))$, then $\xi$ is the class of a cycle $f : G^p \to H^q(J^\bullet)$. Let $\tilde{f} : G^p \to Z^q(J^\bullet)$ be a lifting of $f$, (we write $Z^q(J^\bullet)$ for cycles in $J^q$). If we denote by $d_G$ the boundary map for the group cohomology, then we have $d_G(f) = 0$, hence $d_G(\tilde{f}) : G^{p+1} \to d(J^{q-1})$. In particular, there exists a map $\nu_f : G^{p+1} \to J^{q-1}$ such that $d \circ \nu_f = d_G(\tilde{f})$. Then $\Phi_{2}^{p,q}(\xi) = \pi \circ d_G(\nu_f)$, where $\pi : Z^q(J^\bullet) \to H^q(J^\bullet)$ is the projection. It is easy to show that $d_G(\tilde{f}) = 0$, because then $\im \nu_f \subset Z^{q-1}(J^\bullet)$ and therefore $\pi \circ d_G(\nu_f) = d_G(\pi \circ \nu_f) = 0$. Since the action of $G$ is trivial, for all $(g_1, \ldots, g_{p+1}) \in G^{p+1}$,

$$d_G(\tilde{f})(g_1, \ldots, g_{p+1}) = \tilde{f}(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i \tilde{f}(g_1, \ldots, g_ig_{i+1}, \ldots, g_{p+1}) + (-1)^{p+1} \tilde{f}(g_1, \ldots, g_p).$$

If we require $d_G(\tilde{f}) = 0$, then we get a linear system of $n^{p+1}$ equations of the form

$$(1) \quad x(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i x(g_1, \ldots, g_ig_{i+1}, \ldots, g_{p+1}) + (-1)^{p+1} x(g_1, \ldots, g_p) = 0,$$ in the indeterminates $\{ x_g \mid g = (g_1, \ldots, g_p) \in G^p \}$, where $n$ is the order of $G$. Since $d_G(f) = 0$, there exists a solution of these equations in $H^q(J^\bullet)$. In particular there is a set $I$ of indeterminates whose values can be chosen freely, and the values of the others are determined by $\Pi$. If $I = \emptyset$ then there is only the trivial solution. Notice that $\pi$ preserves relations $\Pi$. Hence we can assign values $z_g \in Z^q(J^\bullet)$ to the indeterminates in $I$ so that $\pi(z_g) = f(g)$. Then, using equations $\Pi$, we find a solution $z_g \in Z^q(J^\bullet)$ of the system $\Pi$. It follows that the map $\tilde{f}$ defined by $\tilde{f}(g) = z_g$ is a lifting of $f$ such that $d_G(\tilde{f}) = 0$. Hence $\Phi_{2}^{p,q} = 0$, for all $p \geq 0$ and $q \geq 1$.

The maps $\Phi_{r}^{p,q}$ are defined recursively. Let $\xi \in H^p(G, H^q(J^\bullet))$, and consider $\Phi_{r}^{p,q}(\xi)$. Let $\varphi : G^{p+r} \to H^{q-r+1}(J^\bullet)$ be a cycle that represents $\Phi_{r}^{p,q}(\xi)$. We know that

$$\Phi_{r}^{p+r,q-r+1}(\Phi_{r}^{p,q}(\xi)) = d_G(\pi \circ \nu_{\varphi}) = 0,$$

with $\nu_{\varphi}$ defined as above, then we set $\Phi_{r}^{p,q}(\xi) = \pi \circ \nu_{\varphi}$. Now we prove the statement by induction on $r$. Assume that $\Phi_{r}^{p,q} = 0$. Let $\xi \in H^p(G, H^q(J^\bullet))$, then $\varphi = \Phi_{r}^{p,q}(\xi) = 0$. It follows that there exists $\rho : G^{p+r} \to Z^{q-r}(J^\bullet)$ which is a lifting of $\varphi$. Hence we can take $\nu_{\varphi} = d_G(\rho)$ and we have

$$\pi \circ \nu_{\varphi} = \pi \circ d_G(\rho) = d_G(\pi \circ \rho) = 0.$$

$\square$

3. **Setting**

Let $X$ be a connected smooth tame Deligne-Mumford stack of dimension 1 over an algebraically closed field $k$. Let $\eta$ be the generic point of $X$ and let $g : \eta \to X$ be the inclusion. We denote by $G_\emptyset$ the generic stabilizer. Moreover if $\sigma$ is a closed point of $X$ with stabilizer $G_\sigma$, we write $\sigma : \sigma \to X$ for the inclusion and we denote by $X(k)$ the set of closed points of $X$. Let $C$ be the coarse moduli space of $X$, we denote by $\pi : X \to C$ the natural map.

We begin with some general remarks on the stack $X$ and its coarse moduli space.
3.1. Proposition. The coarse moduli space $C$ is a connected smooth curve over $k$.

Proof. Since $X$ is connected and $\pi$ is surjective, also $C$ is connected. Moreover, there exists an étale cover of $C$ consisting of schemes of the form $U/G$, where $U$ is a smooth affine scheme of dimension 1 over $k$ and $G$ is a finite group of order not divided by char $k$ ([1], Theorem 3.2). By [8], Proposition I.3.24, locally in the étale topology, there exists a linearization of the action of $G$ on $U$. In particular $G \subset k^*$ is a finite subgroup, hence $G = \mu_n$. It follows that, locally in the étale topology, $U/G = \Spec (k[t])^{\mu_n} = \Spec k[t^p]$, which is smooth of dimension 1. □

3.2. Theorem ([8], III.2.22(d)). Let $C$ a smooth curve over an algebraically closed field. Then

$$H^r_\text{ét}(C, \mathbb{G}_m) = \begin{cases} \Gamma(C, \mathbb{G}_m) & r = 0 \\
\Pic(C) & r = 1 \\
0 & r \geq 2. \end{cases}$$

3.3. Theorem ([1], A.1). Let $M$ be a regular Deligne-Mumford stack over a field $k$. Let $\mathcal{G}$ be the closure of the fibers of the inertia stack $I(M)$ over the generic points of irreducible components of the moduli space of $M$. Then there exist a regular Deligne-Mumford stack $Y$ with trivial generic stabilizer and an étale morphism $M \rightarrow Y$, such that $M$ is a $\mathcal{G}$-gerbe banded by $\mathcal{G}$ over $Y$.

3.4. By Theorem 3.3 $X$ is an étale gerbe over an orbicurve $Y$. It is known that $Y$ is obtained from its coarse moduli space $C$ by applying a finite number of root constructions. Explicitly, there exist distinct points $p_1, \ldots, p_N \in C(k)$ and integers $d_1, \ldots, d_N \geq 2$, with $N \geq 0$, such that $X = X_1 \times_C \cdots \times_C X_N$, where $X_l = \sqrt[d_l]{C}$, for $l = 1, \ldots, N$ (for details on the root construction see [1]).

Let $\Sigma = \{\sigma_1, \ldots, \sigma_N\}$ be the set of closed points of $X$ corresponding to $p_1, \ldots, p_N \in C(k)$. Then $G_{\sigma} = G_0$ for $\sigma \in X(k)$, $\sigma \notin \Sigma$, and $G_{\sigma}/G_0 = \mathbb{Z}/d_l\mathbb{Z}$ for $l = 1, \ldots, N$.

3.5. Proposition. Let $\Spec K$ be the generic point of $C$, then $\eta = \Spec K \times \text{B}G_0$.

Proof. By Theorem 3.3 the stack $X$ is a gerbe over $Y$ banded by $\mathcal{G}$. Let $\mathcal{G}_\eta$ be the sheaf over $\Spec K$ induced by $\mathcal{G}$. Notice that $\eta = \Spec K \times_Y X$, hence $\eta \rightarrow \Spec K$ is a $\mathcal{G}_\eta$-gerbe banded by $\mathcal{G}_\eta$. Recall that such gerbes are classified by $H^2_\text{ét}(\Spec K, \mathcal{G})$, where $\mathcal{G}$ is the center of $\mathcal{G}_\eta$ ([7], IV.5.2). Let $\text{Gal}(K^*/K)$ be the absolute Galois group of $K$, then there is an isomorphism

$$H^r_\text{ét}(\Spec K, \mathcal{G}) \cong H^r(\text{Gal}(K^*/K), M),$$

with $M = \lim_{\rightarrow \downarrow} \mathcal{G}(\Spec L)$, where the limit is taken over all subfields $L \subset K_s$ that are finite over $K$ ([8], III.1.7). Notice that $M$ is a torsion module, hence, by Proposition 2.2 $H^r(\text{Gal}(K^*/K), M) = 0$ for $r \geq 2$. In particular $H^2_\text{ét}(\Spec K, \mathcal{G}) = 0$. □

3.6. Remark. The action of $G_0$ over $K^*$ is trivial, since the sheaf $\mathbb{G}_{m,\eta}$ comes from the sheaf $\mathbb{G}_{m,\Spec K}$ (which is equivariant under the action of $G_0$). It follows that $\pi_\eta^* \mathbb{G}_{m,\eta} = \mathbb{G}_{m,\Spec K}$, where $\pi_\eta: \eta \rightarrow \Spec K$ is the natural morphism. Similarly, for $\sigma \in X(k)$, the action of $G_\sigma$ over $\mathbb{Z}$ is trivial and $\pi_\sigma^* \mathbb{Z} = \mathbb{Z}$, where $\pi_\sigma: \sigma \rightarrow \Spec k$.

4. The Weil-divisor exact sequence

Throughout this section we will use the notations of Section 3

4.1. Proposition. There exists an exact sequence of sheaves on $X$ (with the étale topology)

$$0 \rightarrow \mathbb{G}_{m,X} \overset{\alpha}{\rightarrow} g_* \mathbb{G}_{m,\eta} \overset{\beta}{\rightarrow} \bigoplus_{\sigma \in X(k)} \sigma_* \mathbb{Z}_{\sigma} \rightarrow 0;$$
α and β are defined as follows: if \( W \xrightarrow{f} X \) is an étale morphism from a connected scheme and \( R(W) \) is the ring of rational functions on \( W \), then \( \Gamma(W, \mathbb{G}_m) \xrightarrow{\alpha(W,f)} R(W)^* \) is induced by the morphism \( W \times_X \eta \to W \) and, for all \( \lambda \in R(W)^* \),

\[
\beta(W, f)(\lambda) = \sum_{w \in W(k)} v_w(\lambda),
\]

where \( v_w \) is the discrete valuation on \( R(W) \) induced by \( \mathcal{O}_{W,w} \).

Proof. If \( W \) is a scheme as in the statement, then it is normal and regular. Moreover

\[
\mathbb{G}_{m,X}(W, f) = \Gamma(W, \mathbb{G}_m),
\]

\[
g_\ast \mathbb{G}_{m,Y}(W, f) = \Gamma(W \times_X \eta, \mathbb{G}_m) = R(W)^*,
\]

\[
\bigoplus_{\sigma \in \mathcal{X}(k)} \sigma_\ast \mathbb{Z}_\sigma(W, f) = \bigoplus_{\sigma \in \mathcal{X}(k)} \Gamma(W \times_X \sigma, \mathbb{Z}) = \bigoplus_{w \in W(k)} \mathbb{Z}.
\]

Therefore \( \alpha(W, f) \) and \( \beta(W, f) \) are well-defined. If \( Y \xrightarrow{\beta} X \) is another étale morphism from a scheme, we form the fiber product \( W \times \mathcal{X} Y \). Then the maps induced by restrictions on \( Y \times_X W \) by \( \alpha(W, f) \) and \( \alpha(Y, g) \) coincide with \( \alpha(Y \times_X W, h) \). A similar argument holds for \( \beta \). It follows that \( \alpha \) and \( \beta \) are well-defined.

Recall that the exactness of a sequence of sheaves on a stack can be checked on stalks at geometric points. Let \( \overline{x} \xrightarrow{x} X \) be a geometric point and let \( f : U \to X \) be an étale morphism from a smooth connected curve, such that \( x \) factors through \( f \). Then

\[
\left( \mathbb{G}_{m,U} \right)_x = \left( \mathbb{G}_{m,U} \right)_{\overline{x}} = A^*,
\]

\[
\left( g_\ast \mathbb{G}_{m,Y} \right)_x = \left( g_\ast \mathbb{G}_{m,Y} \right)_{\overline{x}} = Q(A)^*,
\]

where \( A = \mathcal{O}_{U,\overline{x}} \) (note that \( A \) is a discrete valuation ring), \( Q(A) \) is its field of fractions, the morphism \( g_\ast : \eta_U \to U \) is the inclusion of the generic point of \( U \) (since \( g \) is an open immersion and \( U \) is irreducible, we obtain \( U \times_X \eta = \eta_U \)). Moreover, for every \( \sigma \in \mathcal{X}(k) \), \( \sigma \times_X U \) is a set of closed points of \( U \) whose image in \( X \) is \( \sigma \). Hence \( (\sigma_\ast \mathbb{Z}_\sigma) = \bigoplus_{u \in \sigma \times_X U} u_\ast \mathbb{Z} \) and \( (u_\ast \mathbb{Z})_{\overline{x}} \) is non zero if and only if \( u = x \), in which case \( (u_\ast \mathbb{Z})_{\overline{x}} = \mathbb{Z} \). Therefore the sequence of stalks is

\[
0 \to A^* \to Q(A)^* \xrightarrow{v_A} \mathbb{Z} \to 0,
\]

where \( v_A \) is the discrete valuation on \( Q(A) \) defined by \( A \); this sequence is exact by [8], II.3.9. □

4.2. The short exact sequence [2] induces a long exact sequence of étale cohomology groups

\[
\cdots \to H^{\text{ét}}_{\alpha}(X, \mathbb{G}_{m,X}) \to H^{\text{ét}}_{\alpha}(X, g_\ast \mathbb{G}_{m,Y}) \to \bigoplus_{\sigma \in \mathcal{X}(k)} H^{\text{ét}}_{\alpha}(X, \sigma_\ast \mathbb{Z}_\sigma) \to H^{\text{ét}}_{\alpha+1}(X, \mathbb{G}_{m,X}) \to \cdots
\]

where we used the fact that étale cohomology commutes with arbitrary direct sums of sheaves on a quasi-compact Deligne-Mumford stack (see [8], chapter III, Remark 3.6(d)).

4.3. Remark. By [8], III.2.22, there is a short exact sequence in cohomology

\[
0 \to \Gamma(C, \mathbb{G}_m) \to K^* \xrightarrow{\delta_0} \bigoplus_{x \in C(k)} \mathbb{Z} \to \text{Pic}(C) \to 0,
\]

obtained from the Weil-divisor exact sequence for \( C \) ([8], II.3.9).
4.4. **Proposition.** Let $L$ be a quasi-algebraically closed field and let $G$ be a finite group of order not divided by the characteristic of $L$, acting trivially on $L$. Let $F$ be a sheaf over the quotient stack $[\text{Spec } L/G]$. If either $L$ is algebraically closed or $F = \mathbb{G}_m$ then, for $r \geq 0$,

$$H^r_{\text{ét}}([\text{Spec } L/G], F) \cong H^r(G, \Gamma(\text{Spec } L, F)).$$

**Proof.** The natural map $\text{Spec } L \rightarrow [\text{Spec } L/G]$ is a Galois covering with group $G$, then we can consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{\text{ét}}(\text{Spec } L, F)) \Rightarrow H^{p+q}([\text{Spec } L/G], F).$$

We can view the groups $H^r_{\text{ét}}(\text{Spec } L, F)$ as Galois cohomology groups of the absolute Galois group of $L$ ([8], III.1.7). If $L$ is algebraically closed then $H^r_{\text{ét}}(\text{Spec } L, F) = 0$ for $r \geq 1$. If $F = \mathbb{G}_m$ then, by Corollary 2.3, $H^r_{\text{ét}}(\text{Spec } L, \mathbb{G}_m) = 0$ for $r \geq 1$. In both cases the sequence degenerates and we get the statement. $\square$

4.5. **Lemma.** Let $\sigma \in X(k)$ with stabilizer $G_\sigma$. Then $H^r_{\text{ét}}(X, \sigma_* \mathbb{Z}_\sigma) = H^r(G_\sigma, \mathbb{Z})$, for $r \geq 0$.

**Proof.** Consider the Leray spectral sequence for the inclusion $\sigma \rightarrow X$,

$$E_2^{p,q} = H^p_{\text{ét}}(X, R^q \sigma_* \mathbb{Z}_\sigma) \Rightarrow H^{p+q}(\sigma, \mathbb{Z}).$$

Since $\sigma$ is a closed embedding, the functor $\sigma_*$ is exact ([8], II.3.6), hence $R^q \sigma_* \mathbb{Z}_\sigma = 0$ for $q \geq 1$. Therefore the spectral sequence degenerates and $H^r_{\text{ét}}(X, \sigma_* \mathbb{Z}_\sigma) \cong H^r_{\text{ét}}(\sigma, \mathbb{Z})$ for $r \geq 0$. By Theorem 3.3 $\sigma$ is a gerbe over $\text{Spec } k$ banded by $\mathcal{G}_\sigma$, and recall that such gerbes are classified by $H^2_{\text{ét}}(\text{Spec } k, \mathcal{G}_\sigma)$, which vanishes since $k$ is algebraically closed. It follows that $\sigma$ is the trivial $G_\sigma$-gerbe over $\text{Spec } k$ and, by Proposition 4.4 $H^r_{\text{ét}}(\sigma, \mathbb{Z}) \cong H^r(G_\sigma, \mathbb{Z})$ for $r \geq 0$. $\square$

4.6. **Lemma.** Let $G_0$ denote the stabilizer of $\eta$. Then $H^r_{\text{ét}}(X, g_* \mathbb{G}_m, \eta) = H^r(G_0, K^*)$, for $r \geq 0$.

**Proof.** Consider the Leray spectral sequence for $\eta \rightarrow X$,

$$E_2^{p,q} = H^p_{\text{ét}}(X, R^q g_* \mathbb{G}_m, \eta) \Rightarrow H^{p+q}(\eta, \mathbb{G}_m).$$

Let $\overline{x} \rightarrow X$ be a geometric point and let $f : U \rightarrow X$ be an étale morphism from a connected curve, such that $\overline{x}$ factors through $f$. Then $\eta \times_X U$ is the generic point $\eta_U$ of $U$. Therefore

$$(R^q g_* \mathbb{G}_m, \eta)_{\overline{x}} = (R^q g_{U, \eta} \mathbb{G}_m, \eta)_{\overline{x}} = H^q_{\text{ét}}(\text{Spec } O_{U, \overline{x}}^{\text{sh}}, \eta_U, \mathbb{G}_m) = H^q_{\text{ét}}(\text{Spec } Q(O_{U, \overline{x}}^{\text{sh}}), \mathbb{G}_m) = 0$$

for $q \geq 1$, where $Q(O_{U, \overline{x}}^{\text{sh}})$ is the field of fractions of $O_{U, \overline{x}}^{\text{sh}}$ and the last equality follows from the fact that $Q(O_{U, \overline{x}}^{\text{sh}})$ is quasi-algebraically closed (Theorem 2.5). Hence $R^q g_* \mathbb{G}_m, \eta = 0$ for $q \geq 1$ and $H^r_{\text{ét}}(X, g_* \mathbb{G}_m, \eta) \cong H^r_{\text{ét}}(\eta, \mathbb{G}_m)$ for $r \geq 0$. Since $\eta = [\text{Spec } K/G_0]$ is the trivial gerbe over $\text{Spec } K$ (Proposition 3.5), the statement follows by Proposition 4.4. $\square$

4.7. By Lemma 4.5 and Lemma 4.6 the sequence (3) becomes

$$\cdots \rightarrow H^r_{\text{ét}}(X, \mathbb{G}_m) \rightarrow H^r(G_0, K^*) \xrightarrow{\beta^{(r)}} \bigoplus_{\sigma \in X(k)} H^r(G_\sigma, \mathbb{Z}) \rightarrow H^{r+1}_{\text{ét}}(X, \mathbb{G}_m) \rightarrow \cdots$$

4.8. **Lemma.** For every $r \geq 0$, the map $\beta^{(r)}$ in (5) factors as

$$H^r(G_0, K^*) \xrightarrow{\beta^{(r)}_0} \bigoplus_{\sigma \in X(k)} H^r(G_\sigma, \mathbb{Z}) \xrightarrow{\tau^{(r)}} \bigoplus_{\sigma \in X(k)} H^r(G_\sigma, \mathbb{Z}),$$

where $\beta^{(r)}_0$ is induced by $\beta_0$ in (4) and $\tau^{(r)}$ is the transfer map (II), 6.7.16) on each component.
Proof. Let us notice that $H^1(G_{\sigma}, \mathbb{Z}) = \text{Hom}(G_{\sigma}, \mathbb{Z}) = 0$, since $G_{\sigma}$ acts trivially on $\mathbb{Z}$ and $\mathbb{Z}$ does not contain non-trivial finite subgroups. From (3.6) and (3.7), we get the following commutative diagram with exact rows (by Remark 3.8, $\pi^* \mathbb{G}_m, C = \mathbb{G}_m, X$ and $\pi_{\sigma}^* \mathbb{Z} = \mathbb{Z}$)

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Gamma(C, \mathbb{G}_m) & \rightarrow & K^* & \rightarrow & \bigoplus_{\sigma \in X(k)} \mathbb{Z} & \rightarrow & \text{Pic}(C) & \rightarrow & 0 \\
& & \downarrow{\pi^*} & & \downarrow{\pi_{\sigma}^*} & & \downarrow{(\pi_{\sigma}^*)_0} & & \downarrow{\pi^*} & \\
0 & \rightarrow & H^0_\text{ét}(X, \mathbb{G}_m) & \rightarrow & K^* & \rightarrow & \bigoplus_{\sigma \in X(k)} \mathbb{Z} & \rightarrow & H^1_\text{ét}(X, \mathbb{G}_m) & \rightarrow & H^1(G_0, K^*) & \rightarrow & 0
\end{array}
\]

Elements in $\Gamma(\text{Spec } K, \mathbb{G}_m)$ corresponds to rational maps $C \rightarrow \mathbb{P}^1$. Since $C$ is smooth, we obtain $\Gamma(\text{Spec } K, \mathbb{G}_m) \subset \text{Mor}(C, \mathbb{P}^1)$. Similarly, $H^0_\text{ét}(\eta, \mathbb{G}_m) \subset \text{Mor}(X, \mathbb{P}^1)$. Because $C$ is the coarse moduli space, $\text{Mor}(C, \mathbb{P}^1) \cong \text{Mor}(X, \mathbb{P}^1)$ and under this identification $\pi_{\eta}^* = \text{id}$.

Recall that $\pi$ factors through an étale gerbe $X \rightarrow Y$ over an orbicurve $Y$. Let $\beta_Y$ be the analogous of $\beta$ for $Y$, then $\beta = \beta_Y$, since the valuation at a point doesn’t change for étale morphisms. Hence we can assume $X = Y$. As in the proof of Proposition 3.3, étale locally $Y = [\text{Spec } k[t]/\mu_{\ell}]$ and $C = \text{Spec } k[t^\ell]$. Considering the étale cover $\text{Spec } k[t] \rightarrow Y$, we see that $\pi_{\sigma}^*$ is the multiplication by the order of the stabilizer of $\sigma \in Y(k)$ in $Y$. Let $d_{\sigma}$ be the order of $G_{\sigma}$ and $d$ the order of $G_0$, then $\beta = (d/d_{\sigma}) \circ \beta_0$ and $\beta(r)$ is induced by $\beta_0$ and the inclusion $G_0 \hookrightarrow G_{\sigma}$. □

4.9. REMARK. By Lemma 4.8 and snake lemma, we have an exact sequence

$$0 \rightarrow \text{ker } \beta_0^{(r)} \rightarrow \text{ker } \beta(r) \rightarrow \text{ker } \tau(r) \rightarrow \text{coker } \beta_0^{(r)} \rightarrow \text{coker } \beta(r) \rightarrow \text{coker } \tau(r) \rightarrow 0.$$

Moreover, the sequence (3.6) induces exact sequences in cohomology (notice that $\text{im } \beta_0$ is a free $\mathbb{Z}$-module)

$$\begin{cases}
0 \rightarrow H^r(G_0, \Gamma(C, \mathbb{G}_m)) \rightarrow H^r(G_0, K^*) \xrightarrow{\beta_0^{(r)}} H^r(G_0, \text{im } \beta_0) \rightarrow 0, \\
\cdots \rightarrow H^r(G_0, \text{im } \beta_0) \rightarrow \bigoplus_{x \in C(k)} H^r(G_0, \mathbb{Z}) \rightarrow H^r(G_0, \text{Pic}(C)) \rightarrow \cdots.
\end{cases}$$

By snake lemma, we get $\text{coker } \beta_0^{(r)} \subset H^r(G_0, \text{Pic}(C))$ and $H^r(G_0, \Gamma(C, \mathbb{G}_m)) \subset \text{ker } \beta_0^{(r)}$.

4.10. Theorem. With the notations of Section 3, we have $H^0_\text{ét}(X, \mathbb{G}_m) = \Gamma(C, \mathbb{G}_m)$ and, for $r \geq 1$, there exists a short exact sequence

$$0 \rightarrow \text{coker } \gamma^{(r-1)} \rightarrow H^r_\text{ét}(X, \mathbb{G}_m) \rightarrow \text{ker } \beta(r) \rightarrow 0,$$

where $\text{ker } \beta(r)$ fits in the following exact sequence

$$0 \rightarrow H^r(G_0, \Gamma(C, \mathbb{G}_m)) \rightarrow \text{ker } \beta(r) \rightarrow \text{ker } \gamma^{(r)} \rightarrow 0$$

and $\gamma^{(r)}: H^r(G_0, \text{im } \beta_0) \rightarrow \bigoplus_{x \in X(k)} H^r(G_0, \mathbb{Z})$ is induced by $G_0 \hookrightarrow G_{\sigma}$ and the natural inclusion $\text{im } \beta_0 \subset \bigoplus_{x \in X(k)} \mathbb{Z}$. In particular there is a short exact sequence

$$0 \rightarrow \text{Pic}(Y) \rightarrow H^1_\text{ét}(X, \mathbb{G}_m) \rightarrow \text{Hom}(G_0, K^*) \rightarrow 0.$$

Proof. By sequence (3.6), we have short exact sequences

$$0 \rightarrow \text{coker } \beta^{(r-1)} \rightarrow H^r_\text{ét}(X, \mathbb{G}_m) \rightarrow \text{ker } \beta(r) \rightarrow 0$$

for $r \geq 1$. By Remark 4.9, $H^0_\text{ét}(X, \mathbb{G}_m) = H^0_\text{ét}(C, \mathbb{G}_m)$ and $\beta(r) = \gamma^{(r)} \circ \beta_0^{(r)}$, with $\gamma^{(r)}$ as described in the statement. Applying snake lemma we get $\text{coker } \beta^{(r-1)} = \text{coker } \gamma^{(r-1)}$ and

$$0 \rightarrow H^r(G_0, \Gamma(C, \mathbb{G}_m)) \rightarrow \text{ker } \beta(r) \rightarrow \text{ker } \gamma^{(r)} \rightarrow 0.$$
As noticed in the proof of Lemma 4.8, \( \ker \beta = \text{Pic}(Y) \), hence we obtain

\[
0 \to \text{Pic}(Y) \to H^1_{\text{ét}}(X, \mathcal{G}_m) \to \text{Hom}(G_0, K^*) \to 0,
\]

where we used that \( H^1(G_\sigma, \mathbb{Z}) = \text{Hom}(G_\sigma, \mathbb{Z}) = 0 \), since \( G_\sigma \) acts trivially on \( \mathbb{Z} \). \( \square \)

4.11. **Remark.** In the case \( G_0 \) is abelian and \( X \) is a \( G_0 \)-gerbe banded over \( Y \), we recover the description of \( H^1_{\text{ét}}(X, \mathcal{G}_m) \) given in [4], Corollary 3.2.1 (see also Section 5.1).

4.12. **Proposition.** If \( G_\sigma \) is abelian for every \( \sigma \in X(k) \) then \( \ker \beta^{(r)} \) depends only on \( Y \) and \( G_0 \), not on the structure of the gerbe \( X \to Y \). In particular \( H^2_{\text{ét}}(X, \mathcal{G}_m) = H^2_{\text{ét}}(Y \times BG_0, \mathcal{G}_m) \).

**Proof.** By Theorem 4.10 \( H^2_{\text{ét}}(X, \mathcal{G}_m) = \ker \beta^{(2)} \). Consider the restriction map

\[
\text{res}^{(r)} : H^r(G_{\sigma_1}, \mathbb{Z}) \to H^r(G_0, \mathbb{Z}),
\]

where \( G_0 \) and \( G_{\sigma_1} \) act trivially on \( \mathbb{Z} \). By Hochschild-Serre spectral sequence ([11], 6.8.2)

\[
H^p(\mathbb{Z}/d_1 \mathbb{Z}, H^q(G_0, \mathbb{Z})) \Rightarrow H^{p+q}(G_{\sigma_1}, \mathbb{Z})
\]

and Lemma 2.9 we see that \( \text{res}^{(r)} \) is surjective. Recall that \( \tau^{(r)} \circ \text{res}^{(r)} = d_1 \) ([11], Lemma 6.7.17), where \( H^r(G_{\sigma_1}, \mathbb{Z}) \xrightarrow{d_1} H^r(G_{\sigma_1}, \mathbb{Z}) \) is the multiplication by \( d_1 \) which is induced by \( \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \). It follows that \( \ker \tau^{(r)} = \text{im} \text{res}^{(r)} |_{\ker d_1} \). The same argument applies to \( Y \times BG_0 \). Notice that

\[
\tau^{(r)}_Y : H^r(G_0, \mathbb{Z}) \to H^r(G_0 \times \mathbb{Z}/d_1 \mathbb{Z}, \mathbb{Z}),
\]

is induced by composition with the projection \( G_0 \times \mathbb{Z}/d_1 \mathbb{Z} \to G_0 \) and the multiplication by \( d_1 \), therefore \( \ker d_1 \subset \ker \tau^{(r)}_Y \). As a consequence \( \ker \tau^{(r)} \subset \ker \tau^{(r)}_Y \) and hence \( \ker \beta^{(r)} \subset \ker \beta^{(r)}_Y \), where \( \beta^{(r)}_Y = \tau^{(r)}_Y \circ \beta^{(r)}_0 \). By Remark 4.9 we get the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & \ker \beta^{(r)}_0 & \to & \ker \beta^{(r)} & \to & \ker \tau^{(r)} & \to & \text{coker} \beta^{(r)}_0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker \beta^{(r)}_0 & \to & \ker \beta^{(r)}_Y & \to & \ker \tau^{(r)}_Y & \to & \text{coker} \beta^{(r)}_0 \\
& & \downarrow & & \uparrow & & \downarrow & & \\
& & & \bigoplus_{x \in C(k)} H^r(G_0, \mathbb{Z}) & \to & H^r(G_0, \text{Pic}(C))
\end{array}
\]

which implies \( \ker \beta^{(r)} = \ker \beta^{(r)}_Y \). \( \square \)

4.13. **Remark.** In general, the groups \( H^r_{\text{ét}}(X, \mathcal{G}_m) \) depend on the gerbe \( X \to Y \); if the stabilizers are not abelian, also \( H^2_{\text{ét}}(X, \mathcal{G}_m) \) may depend on \( X \to Y \) (see Section 7).

4.14. **Corollary.** If \( G_\sigma \) is cyclic for every \( \sigma \in X(k) \) then, for every \( r \geq 1 \),

\[
H^r_{\text{ét}}(X, \mathcal{G}_m) = H^r_{\text{ét}}(Y \times BG_0, \mathcal{G}_m)
\]

and there is a short exact sequence

\[
0 \to H^2(G_0, \text{Pic}(Y)) \to H^{2r+1}_{\text{ét}}(X, \mathcal{G}_m) \to Q \to 0,
\]

where \( Q \) fits in the following exact sequence

\[
0 \to \bigoplus_{i=1}^N \mathbb{Z}/d_i \mathbb{Z} \to Q \to \text{Hom}(G_0, K^*) = G_0 \to 0.
\]
Proof. By Corollary 2.8 and Theorem 2.7, the sequence (5) induces exact sequences for \( r \geq 1 \),

\[
0 \to H^2_{\text{ét}}(X, \mathbb{G}_m) \to K^*/K^* \xrightarrow{\beta(2)} \bigoplus_{\sigma \in X(k)} G_\sigma \to H^2_{\text{ét}}(Y, \mathbb{G}_m) \to G_0 \to 0.
\]

In particular \( H^2_{\text{ét}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(X, \mathbb{G}_m) \) and \( H^2_{\text{ét}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(Y \times B G_0, \mathbb{G}_m) \), by Proposition 4.12. Moreover, by Lemma 4.8, the map \( \beta(2) \) factors as

\[
H^2(G_0, K^*) \xrightarrow{\beta(2)} \bigoplus_{\sigma \in X(k)} H^2(G_0, \mathbb{Z}) = G_0 \xrightarrow{} \bigoplus_{\sigma \in X(k)} G_\sigma = H^2(G, \mathbb{Z}),
\]

where \( \beta = (d*/d) \circ \beta_0(2) \) is the map induced by \( \beta \). By Remark 4.9, there is an exact sequence

\[
H^2(G_0, \text{im} \beta) \to \bigoplus_{\sigma \in X(k)} H^2(G_0, \mathbb{Z}) \to H^2(G_0, \text{Pic}(Y)) \to H^3(G_0, \text{im} \beta) = 0,
\]

which implies \( \text{coker} \beta = H^2(G_0, \text{Pic}(Y)) \). By snake lemma, we get the following exact sequence

\[
0 \to H^2(G_0, \text{Pic}(Y)) \to \text{coker} \beta(2) \to \bigoplus_{l=1}^n \mathbb{Z}/d_l \mathbb{Z} \to 0. \quad \square
\]

4.15. Corollary. Let \( Y \) be a smooth tame orbicurve over an algebraically closed field, then

\[
H^r_{\text{ét}}(Y, \mathbb{G}_m) = \begin{cases} 
\Gamma(C, \mathbb{G}_m) & r = 0 \\
\text{Pic}(Y) & r = 1 \\
0 & r \equiv 0 \pmod{2}, \ r \geq 2 \\
\bigoplus_{l=1}^n \mathbb{Z}/d_l \mathbb{Z} & r \equiv 1 \pmod{2}, \ r \geq 3.
\end{cases}
\]

Moreover, there is a short exact sequence

\[
0 \to \text{Pic}(C) \to \text{Pic}(Y) \to \bigoplus_{l=1}^n \mathbb{Z}/d_l \mathbb{Z} \to 0.
\]

Proof. By the proof of Lemma 4.8, we get the description of \( H^1_{\text{ét}}(Y, \mathbb{G}_m) \). Moreover, applying Theorem 4.10 with \( G_0 = 0 \), we have \( H^r_{\text{ét}}(Y, \mathbb{G}_m) = \bigoplus_{l=1}^n H^{r-1}(\mathbb{Z}/d_l \mathbb{Z}, \mathbb{Z}) \), for \( r \geq 2 \), and the statement follows from Corollary 2.8. \( \square \)

5. Twisted nodal curves

Let \( Y \) be a twisted nodal curve over an algebraically closed field \( k \) (2, Definition 4.1.2). In particular \( Y \) is a connected tame Deligne-Mumford stack over \( k \) with trivial generic stabilizer. If \( C \) is the coarse moduli space of \( Y \), then \( C \) is a connected nodal curve over \( k \). Let \( \pi: Y \to C \) be the natural morphism; we denote by \( S \) the set of singular points of \( C \) and we write \( S_Y = S \times_C Y \).

5.1. Proposition. We have \( H^r_{\text{ét}}(C, \mathbb{G}_m) = 0 \) for \( r \geq 2 \).

Proof. Let \( \nu: \hat{C} \to C \) be the normalization of \( C \), then, by Theorem 3.2, \( H^r_{\text{ét}}(\hat{C}, \mathbb{G}_m) = 0 \) for \( r \geq 2 \). Moreover \( \nu \) is a finite morphism, therefore \( \nu_* \) is an exact functor (8, II.3.6), and by Leray spectral sequence for \( \nu \), we get \( H^r_{\text{ét}}(C, \nu_* \mathbb{G}_m) \cong H^r_{\text{ét}}(\hat{C}, \mathbb{G}_m) \) for \( r \geq 0 \). There is a natural injective morphism of sheaves \( \mathbb{G}_m, C \to \nu_* \mathbb{G}_m, \hat{C} \), whose cokernel is concentrated in the singular locus of \( C \). Equivalently we have the following exact sequence

\[
0 \to \mathbb{G}_m, C \to \nu_* \mathbb{G}_m, \hat{C} \to \bigoplus_{x \in S} x_* \mathbb{Q}_x \to 0,
\]
where $Q_x$ is a sheaf over $x = \text{Spec } k$. Consider the long exact sequence in cohomology
\[
\cdots \to H^r_{\text{ét}}(C, \mathbb{G}_m) \to H^r_{\text{ét}}(C, \nu_* \mathbb{G}_m, C) \to \bigoplus_{x \in S} H^r_{\text{ét}}(C, x_* Q_x) \to \cdots.
\]
Since $x: \text{Spec } k \to C$ is a closed embedding, the functor $x_*$ is exact \cite{S, II.3.6} and by Leray spectral sequence for $x$, we get $H^r_{\text{ét}}(C, x_* Q_x) \cong H^r_{\text{ét}}(\text{Spec } k, Q_x)$ for $r \geq 0$. Finally, the groups $H^r_{\text{ét}}(\text{Spec } k, Q_x)$ vanish for $r \geq 1$, since $k$ is algebraically closed. 

5.2. Proposition. Let $\Sigma$ be the set of closed points of $Y$ with non trivial stabilizer. Then
\[
H^r_{\text{ét}}(Y, \mathbb{G}_m) = \begin{cases} 
\Gamma(C, \mathbb{G}_m) & r = 0 \\
\text{Pic}(Y) & r = 1 \\
0 & r \equiv 0 \pmod{2}, r \geq 2 \\
\bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma \mathbb{Z} & r \equiv 1 \pmod{2}, r \geq 3.
\end{cases}
\]
Moreover, there is a short exact sequence
\[
0 \to \text{Pic}(C) \to \text{Pic}(Y) \to \bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma \mathbb{Z} \to 0.
\]

Proof. With notations as in the proof of Proposition 5.1 let $\hat{Y} = Y \times_C \hat{C}$. Notice that the induced morphism $\hat{\nu}: \hat{Y} \to Y$ is finite because $\nu$ is finite, hence, by Leray spectral sequence for $\hat{\nu}$, we get $H^r_{\text{ét}}(Y, \hat{\nu}_* \mathbb{G}_m, \hat{Y}) = H^r_{\text{ét}}(\hat{Y}, \mathbb{G}_m)$ for $r \geq 0$. Moreover $\hat{Y}$ is smooth, hence Corollary 4.15 gives a description of the groups $H^r_{\text{ét}}(\hat{Y}, \mathbb{G}_m)$. We have the following exact sequence of sheaves
\[
0 \to \mathbb{G}_m, Y \to \hat{\nu}_* \mathbb{G}_m, Y \to Q \to 0,
\]
where $Q = \bigoplus_{\sigma \in S_Y} \sigma_* Q_\sigma$ is concentrated in $S_Y$. Consider the long exact sequence in cohomology
\[
\cdots \to H^r_{\text{ét}}(Y, \mathbb{G}_m) \to H^r_{\text{ét}}(\hat{Y}, \hat{\nu}_* \mathbb{G}_m, \hat{Y}) \to \bigoplus_{\sigma \in S_Y} H^r_{\text{ét}}(Y, \sigma_* Q_\sigma) \to \cdots.
\]
Since $\sigma$ is a closed embedding, by Leray spectral sequence for $\sigma$ and Proposition 4.4 we obtain $H^r_{\text{ét}}(Y, \sigma_* Q_\sigma) \cong H^r_{\text{ét}}(\sigma, Q_\sigma) = H^r(G, Q_\sigma)$ for $r \geq 0$, where $G_\sigma = \mathbb{Z}/d_\sigma$ is the stabilizer of $\sigma$ and $Q_\sigma$ is the strictly henselian local ring of $Q$ at $\sigma$. Notice that $Q_\sigma = \hat{A}/\hat{A}^\times$, where $A = Q_\sigma$ and Spec $\hat{A}$ is the normalization of Spec $A$. Since $Y$ is étale locally a nodal curve over $k$, we have that $\hat{A}^\times = k^\times \times k^\times$ and $A^\times = k^\times$, therefore $Q_\sigma = k^\times$. By Theorem 2.7 we have
\[
H^r(G, Q_\sigma) = \begin{cases} 
k^\times & r = 0 \\
\mathbb{Z}/d_\sigma \mathbb{Z} & r \equiv 1 \pmod{2} \\
0 & r \equiv 0 \pmod{2}, r \geq 2.
\end{cases}
\]
We substitute these results in (6) and obtain, for $r \geq 1$,
\[
0 \to H^{2r+1}_{\text{ét}}(Y, \mathbb{G}_m) \to \bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma \mathbb{Z} \xrightarrow{\rho} \bigoplus_{\sigma \in S_Y} \mathbb{Z}/d_\sigma \mathbb{Z} \to H^{2r+2}_{\text{ét}}(Y, \mathbb{G}_m) \to 0,
\]
where $\Sigma$ is the set of closed points of $\hat{Y}$ with non trivial stabilizer and the map $\rho$ is described as follows. If $\sigma \in S_Y$, with $d_\sigma > 0$, then $\sigma \times_Y \hat{Y}$ consists of two points $\sigma_1, \sigma_2 \in \hat{\Sigma}$; let $\rho_\sigma$ be the restriction of $\rho$ to $\mathbb{Z}/d_\sigma \mathbb{Z} \oplus \mathbb{Z}/d_\sigma \mathbb{Z}$, then $\rho_\sigma(a, b) = a - b \in \mathbb{Z}/d_\sigma \mathbb{Z}$. Otherwise, the restriction of $\rho$ to $\mathbb{Z}/d_\sigma \mathbb{Z}$ is zero. It follows that, for every $r \geq 3$,
\[
H^r_{\text{ét}}(Y, \mathbb{G}_m) = \begin{cases} 
\bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma \mathbb{Z} & r \equiv 1 \pmod{2} \\
0 & r \equiv 0 \pmod{2}.
\end{cases}
\]
Moreover we have the following commutative diagram with exact rows

$$
\begin{array}{cccccccccccc}
0 & \longrightarrow & \Gamma(C, \mathbb{G}_m) & \longrightarrow & \Gamma(\hat{C}, \mathbb{G}_m) & \longrightarrow & \bigoplus_{\sigma \in S_Y} k^* & \longrightarrow & \text{Pic}(C) & \longrightarrow & \text{Pic}(\hat{C}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0_{\text{ét}}(Y, \mathbb{G}_m) & \longrightarrow & \Gamma(\hat{C}, \mathbb{G}_m) & \longrightarrow & \bigoplus_{\sigma \in S_Y} k^* & \longrightarrow & \text{Pic}(Y) & \longrightarrow & \text{Pic}(\hat{Y}) & \longrightarrow & \bigoplus_{\sigma \in S_Y} \mathbb{Z}/d_\sigma \mathbb{Z} & \longrightarrow & H^2_{\text{ét}}(Y, \mathbb{G}_m) & \longrightarrow & 0
\end{array}
$$

which implies $H^0_{\text{ét}}(Y, \mathbb{G}_m) = \Gamma(C, \mathbb{G}_m)$ and $H^2_{\text{ét}}(Y, \mathbb{G}_m) = 0$. Finally, by snake lemma, the following sequence is exact

$$0 \rightarrow \text{Pic}(C) \rightarrow H^1_{\text{ét}}(Y, \mathbb{G}_m) \rightarrow \bigoplus_{\sigma \in \Sigma} \mathbb{Z}/d_\sigma \mathbb{Z} \rightarrow 0. \quad \square$$

5.1. **Banded gerbes over orbicurves.** Recall that, if $\mathcal{G}$ is a sheaf of abelian groups on a twisted nodal curve $Y$, then $H^2_{\text{ét}}(Y, \mathcal{G})$ classifies the $\mathcal{G}$-gerbes over $Y$ banded by $\mathcal{G}$ ([7], IV.3.5). Let $G_0$ be an abelian finite group of order not divided by char $k$ and let $G_0 = \bigoplus_{h=1}^M \mathbb{Z}/d_h \mathbb{Z}$ be a decomposition of $G_0$ as a direct sum of cyclic groups. We have a short exact sequence

$$0 \rightarrow G_0 \rightarrow \bigoplus_{h=1}^M \mathbb{Z}/d_h \mathbb{Z} \rightarrow \bigoplus_{h=1}^M \mathbb{Z}/d_h \mathbb{Z} \rightarrow 0,$$

deduced by Kummer sequence for each factor. From the induced long exact sequence in cohomology we get

$$\bigoplus_{h=1}^M \text{Pic}(Y) \xrightarrow{\psi} H^2_{\text{ét}}(Y, G_0) = \bigoplus_{h=1}^M H^2_{\text{ét}}(Y, \mathbb{Z}/d_h \mathbb{Z}) \rightarrow \bigoplus_{h=1}^M H^3_{\text{ét}}(Y, \mathbb{G}_m) = 0,$$

where, according to [4] (Section 2.4), the map $\psi$ associates to $(L_1, \ldots, L_M) \in \bigoplus_{h=1}^M \text{Pic}(Y)$ the $G_0$-gerbe $\left(\sqrt{d_1/L_1} \times \cdots \times \sqrt{d_M/L_M} \right)$. Therefore every $G_0$-gerbe over $Y$ banded by $G_0$ is obtained as a finite number of root constructions.

6. **Trivial gerbes**

Throughout this section we will use the notations of Section 3

6.1. **Proposition.** We have $H^1_{\text{ét}}(Y \times B G_0, \mathbb{G}_m) = \text{Pic}(Y) \oplus \text{Hom}(G_0, K^*)$ and, for $r \geq 2$, there are short exact sequences

$$0 \rightarrow Q^r \rightarrow H^r_{\text{ét}}(Y \times B G_0, \mathbb{G}_m) \rightarrow \bigoplus_{l=1}^N H^{r-1}(G_0 \times \mathbb{Z}/d_l \mathbb{Z})/H^{r-1}(G_0, \mathbb{Z}) \rightarrow 0,$$

where $Q^r$ fits in the following exact sequence

$$0 \rightarrow H^r(G_0, \Gamma(C, \mathbb{G}_m)) \rightarrow Q^r \rightarrow H^{r-1}(G_0, \text{Pic}(Y)) \rightarrow 0.$$

**Proof.** The map $Y \xrightarrow{\delta} Y \times B G_0$ obtained by Spec $k \rightarrow B G_0$ is a Galois cover with group $G_0$. Then we can consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_0, H^q_{\text{ét}}(Y, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(Y \times B G_0, \mathbb{G}_m).$$

By Corollary 4.15

$$E_2^{p,q} = \begin{cases} H^p(G_0, \Gamma C, \mathbb{G}_m) & q = 0 \\ H^p(G_0, \text{Pic}(Y)) & q = 1 \\ \bigoplus_{l=1}^N H^p(G_0, H^{q-1}(\mathbb{Z}/d_l \mathbb{Z})) & q \geq 2. \end{cases}$$
In particular, the groups $E_2^{p,q}$ and the maps between them coincide, for $q \geq 2$, with the groups $F_2^{p,q-1}$ and relative maps for the Hochschild-Serre spectral sequence

$$F_2^{p,q} = \bigoplus_{l=1}^{N} H^p(G_0, H^q(\mathbb{Z}/d, \mathbb{Z})) \Rightarrow \bigoplus_{l=1}^{N} H^{p+q}(G_0 \times \mathbb{Z}/d, \mathbb{Z}).$$

By Lemma 2.9 we have $E_\infty^{p,q} = E_2^{p,q}$. Comparing filtrations for (7) and (8), we get

$$0 \to H^r_{\text{ét}}(Y \times BG_0, \mathbb{G}_m) \to \bigoplus_{l=1}^{N} H^{r-1}(\mathbb{Z}/d \times G_0, \mathbb{Z}) \to 0,$$

for $r \geq 1$, where $H^r_{\text{ét}}$ fits in the following exact sequence

$$0 \to H^r(G_0, \Gamma(C, \mathbb{G}_m)) \to H^r_{\text{ét}} \to H^{r-1}(G_0, \text{Pic}(Y)) \to 0.$$

Consider the natural morphism $Y \times BG_0 \to Y$. We have the following commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Pic}(Y) \\
\downarrow & & \downarrow \text{(πσρ)} \downarrow \\
0 & \longrightarrow & \text{Pic}(Y) \\
\end{array}$$

where the second row is given by Theorem 4.10 and the first row is deduced from the argument above, after noticing that $\text{Hom}(G_0, \Gamma(C, \mathbb{G}_m)) = \text{Hom}(G_0, K^*)$. Hence we get

$$H^1_{\text{ét}}(Y \times BG_0, \mathbb{G}_m) = \text{Pic}(Y) \oplus \text{Hom}(G_0, K^*).$$

6.2. Remark. If $G_0$ is cyclic then, by Theorem 2.7 and Proposition 6.1, we have, for $r \geq 1$,

$$H^{2r-1}_{\text{ét}}(C \times BG_0, \mathbb{G}_m) = \text{Hom}(G_0, \Gamma(C, \mathbb{G}_m)) \oplus H^2(G_0, \text{Pic}(C)).$$

6.3. Corollary. If $C$ is projective and $G_0$ is cyclic, then

$$H^r_{\text{ét}}(C \times BG_0, \mathbb{G}_m) = \begin{cases} 
  k^* & r = 0 \\
  \text{Pic}(C) \oplus G_0 & r = 1 \\
  (G_0)^{2g} & r \equiv 0 \pmod{2}, r \geq 2 \\
  G_0 \oplus G_0 & r \equiv 1 \pmod{2}, r \geq 3.
\end{cases}$$

Proof. Let $d$ be the order of $G_0$. Recall that there is a short exact sequence

$$0 \to \text{Pic}^0(C) \to \text{Pic}(C) \to \mathbb{Z} \to 0.$$

Moreover, since $C$ is projective, $\Gamma(C, \mathbb{G}_m) = k^*$ and the sequence

$$0 \to (\mathbb{Z}/d\mathbb{Z})^{2g} \to \text{Pic}^0(C) \overset{d}{\to} \text{Pic}^0(C) \to 0$$

is exact, where the map $d$ is defined by $d(a) = a^d$, for all $a \in \text{Pic}^0(C)$ (see [9], Section IV.21, Lang’s Theorem). Therefore, applying Theorem 2.7, we obtain, for $r \geq 1$

$$\begin{cases} 
  H^{2r-1}(G_0, k^*) = G_0, \\
  H^{2r}(G_0, k^*) = 0, \\
  H^{2r}(G_0, \text{Pic}^0(C)) = (G_0)^{2g}, \\
  H^{2r}(G_0, \text{Pic}(C)) = (G_0)^{2g}, \\
  H^{2r}(G_0, \text{Pic}(C)) = G_0.
\end{cases}$$

Then the statement follows from Proposition 6.1 and Remark 6.2. \qed
7. Examples

In this section we present two examples: the first one shows that, in general, the groups $H_{\text{ét}}^r(X, \mathbb{G}_m)$ may depend on the gerbe $X \to Y$, for $r \geq 3$; the second one shows that $H_{\text{ét}}^2(X, \mathbb{G}_m)$ may depend on the gerbe $X \to Y$ if the stabilizers are not abelian.

Let $p$ be a prime integer and consider $\mathbb{A}^1_C$ with the action $\mathbb{C}[x] \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}[x]$ given by $(x, \lambda) \mapsto \lambda x$, for all $\lambda \in \mathbb{Z}/p\mathbb{Z}$. Then $Y = [\mathbb{A}^1_C/\mu_p]$ is an orbicurve.

7.1. Let $X = [\mathbb{A}^1_C/\mu_p^2]$, where the action is given by $(x, \lambda) \mapsto \lambda p x$, for $\lambda \in \mathbb{Z}/p^2\mathbb{Z}$. By Kummer sequence, $H_{\text{ét}}^2(Y, \mu_p) = \mathbb{Z}/p\mathbb{Z}$ and, since $p$ is prime, there are only two non isomorphic banded $\mu_p$-gerbes over $Y$ ([7], IV.3.5); in particular, by Theorem 3.3, $X$ is the only non trivial $\mu_p$-gerbe banded over $Y$ (up to isomorphism). By Corollary 4.14, $H_{\text{ét}}^{2r+1}(X, \mathbb{G}_m)$ is finite of order $p^3$ whereas, by Proposition 6.1, $H_{\text{ét}}^{2r+1}(Y \times B\mu_p, \mathbb{G}_m)$ has order $p^2$, for $r \geq 1$.

7.2. Set $p = 2$. Let $D_{2m}$ be the dihedral group with $2m$ elements, with $m$ odd, $m \geq 3$. We denote by $r \in \mathbb{Z}/m\mathbb{Z}$ and $s \in \mathbb{Z}/2\mathbb{Z}$ the generators of $D_{2m}$. Let $X = [\mathbb{A}^1_C/D_{2m}]$, where the action is given by $(x, r^i s^j) \mapsto (-1)^s x$, for $1 \leq i \leq m$ and $\epsilon = 0, 1$. By [11], 6.7.10, we have $H^2(D_{2m}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence $\tau(2) : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the zero map. Therefore, by Remark 4.9 and Proposition 6.1, $H_{\text{ét}}^{2r+1}(Y \times B\mu_p, \mathbb{G}_m) = \ker \beta(2) = \ker \tau(2) = \mathbb{Z}/m\mathbb{Z}$ whereas $H_{\text{ét}}^{2}(Y \times B\mu_p, \mathbb{G}_m) = 0$.

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