Abstract. We describe a model for $m$ vertex reinforced interacting random walks on complete graphs with $d \geq 2$ vertices. The transition probability of a random walk to a given vertex depends exponentially on the proportion of visits made by all walks to that vertex. The individual proportion of visits is modulated by a strength parameter that can be set equal to any real number. This model covers a large variety of interactions including different vertex repulsion and attraction strengths between any two random walks as well as self-reinforced interactions. We show that the process of empirical vertex occupation measures defined by the interacting random walks converges (a.s.) to the limit set of the flow induced by a smooth vector field. Further, if the set of equilibria of the field is formed by isolated points, then the vertex occupation measures converge (a.s.) to an equilibrium of the field. These facts are shown by means of the construction of a strict Lyapunov function. We show that if the absolute value of the interaction strength parameters are smaller than a certain upper bound, then, for any number of random walks ($m \geq 2$) on any graph ($d \geq 2$), the vertex occupation measure converges toward a unique equilibrium. We provide two additional examples of repelling random walks for the cases $m = d = 2$ and $m = 3$, $d = 2$. The latter is used to study some properties of three exponentially repelling random walks on $\mathbb{Z}$.

1. Introduction

Let $G = (E, V)$ be a finite complete graph with $d \geq 2$ vertices and let $W = \{(W^1(n), \ldots, W^m(n))\}_{n \geq 1}$ be a process described by $m \geq 2$ interacting random walks on $G$. The process $W$, defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is constructed as follows. For each $v \in [d] = \{1, 2, \ldots, d\}$ and $i \in [m]$, let $X^i_v(0) = 1$, and then for $n \geq 1$ let $X^i_v(n)$ be the empirical occupation measure of the vertices by the $i$-th walk, that is,

$$X^i_v(n) = \frac{1}{d + n} \left( 1 + \sum_{k=1}^{n} 1\{W^i(k) = v\} \right).$$

Define $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_n = \sigma(W(k) : 1 \leq k \leq n)$ as the filtration generated by $W$ up to time $n \geq 1$. Then, for all $v \in [d]$, $i, j \in [m]$ and $n \geq 0$ define the transition probability of $W$ as

$$P(W^i(n + 1) = v \mid \mathcal{F}_n) = \frac{\exp \left( \sum_{j=1}^{m} \alpha_{ij} X^j_v(n) \right)}{\sum_{u=1}^{d} \exp \left( \sum_{j=1}^{m} \alpha_{ij} X^j_u(n) \right)},$$
where
\begin{equation}
\alpha^{ij}_v \in \mathbb{R} \text{ is such that } \alpha^{ij}_v = \alpha^{ji}_v.
\end{equation}

By using (1), set \( X^i(n) = (X_1^i(n), \ldots, X_d^i(n)) \) and then denote by \( X \) the process of vertex occupation measures defined as
\begin{equation}
X = \{X(n)\}_{n \geq 0}, \quad \text{where } \ X(n) = (X^1(n), X^2(n), \ldots, X^m(n)).
\end{equation}

Notice that \( \{X(n)\}_{n \geq 1} \) belongs to the compact convex set \( \mathcal{D} = \Delta^m = \Delta \times \cdots \times \Delta \), which equals the \( m \)-fold Cartesian product of the \((d - 1)\)-simplex \( \Delta = \{ x = (x_v) \in \mathbb{R}^d \mid x_v \geq 0, \sum_v x_v = 1 \} \) with itself. In these terms, the process \( W \) is completely defined by specifying the initial condition \( X(0) \) and the smooth map \( \pi = (\pi^1_1, \ldots, \pi^d_d, \ldots, \pi^m_m) : \mathcal{D} \to \mathcal{D} \) which at \( x = (x_1^1, \ldots, x_1^d, \ldots, x_m^m) \) takes the value
\begin{equation}
\pi^i_v(x) = \frac{\exp \left( \sum_{j=1}^{m} \alpha^{ij}_v x^j_v \right)}{\sum_{v=1}^{d} \exp \left( \sum_{j=1}^{m} \alpha^{ij}_v x^j_v \right)},
\end{equation}
with \( \alpha^{ij}_v \) as the constants satisfying (3). In fact, rewriting (2) in terms of \( \pi \) gives
\[ P(W^i(n + 1) = v \mid \mathcal{F}_n) = \pi^i_v(X(n)). \]

It is worth mentioning that (2) and (3) cover a large variety of interactions which include different vertex repulsion and attraction strengths between \( m \geq 2 \) interacting walks on \( G \). In this context, \( \alpha^{ij}_v \) stands for the strength of reinforced repulsion (when \( \alpha^{ij}_v < 0 \)) or attraction (when \( \alpha^{ij}_v > 0 \)) between walks \( i \) and \( j \) at vertex \( v \). This may also include self-reinforced interactions (repulsion or attraction) when \( \alpha^{ii}_v \neq 0 \).

According to (2) and (3), the probability of a transition of walk \( i \) to a given vertex \( v \) at time \( n + 1 \) depends on the proportions \( X^i_1(n), X^i_2(n), \ldots, X^i_m(n) \) of the visits to vertex \( v \) made by all \( m \) walks up to time \( n \). The process \( X = \{X(n)\}_{n \geq 0} \) studied throughout belongs therefore to a family of processes known as vertex reinforced random walks. There is an extensive literature devoted to self-attracting reinforced random walks on graphs, see for instance Pemantle (1992), Benaim & Tarrès (2011), Volkov (2001), and self-repelling walks, see Toth (1995) and references therein. Several models for interacting generalised Pólya urn models have been considered more recently, see Aletti et al. (2020), Crimaldi et al. (2019a), Benaim et al. (2015), and van der Hofstad et al. (2016). Apart from Chen (2014) and Crimaldi et al. (2019b), there are relatively few studies of interacting vertex-reinforced random walks. Chen (2014) considers two repelling random walks on finite complete graphs and focuses on the asymptotic properties of their overlap measure. Crimaldi et al. (2019b) presents a model for several cooperative walks on the two vertex graph and describes their synchronisation toward a common limit.

This article is principally concerned with the asymptotic properties of the process of vertex occupation measures \( X = \{X(n)\}_{n \geq 0} \). A first step to characterise the long-term behaviour of \( X \) consists in identifying this process with a stochastic approximation. Stochastic approximations have been rather effective while dealing with several self-reinforced processes such as vertex reinforced random walks, generalised Pólya urns and population games, see Pemantle (2007) for a survey and further references. In particular, the identification with a stochastic approximation allows to study the asymptotic behaviour of the interacting walks by following the dynamical system approach described in Benaim (1996); Benaim (1999). Let \( \mathcal{T} \mathcal{D} = \{ x \in \mathbb{R}^{dm} \mid \sum_v x^i_v = 0 \text{ for each } i \in [m] \} \) be the tangent space of \( \mathcal{D} \). We show that the process of vertex
occupation measures $X = \{X(n)\}_{n \geq 0}$ can be understood if we know the asymptotic behaviour of the vector field $F : \mathfrak{D} \to T\mathfrak{D}$ defined by

$$F(x) = -x + \pi(x),$$

where $\pi$ is given in (5). To study the long-term behaviour of the vector field $F$, we adapt the arguments of Budhiraja et al. (2015a) and Budhiraja et al. (2015b) to construct an explicit strict Lyapunov function for the vector field. A key observation for the construction of this function is based on the fact that the relative entropy between two solutions of the ordinary differential equation

$$\dot{x} = F(x)$$

is strictly decreasing outside the set of equilibria of the vector field $F$, namely $F^{-1}(0) = \{x \in \mathfrak{D} \mid F(x) = 0\}$. This argument was put forward in Budhiraja et al. (2015a,b) while considering non-linear Markov processes with Gibbsian type interactions. These ideas were also considered and further extended in Benaïm (2015), in order to construct Lyapunov functions for several examples of self-reinforced processes arising in population games and vertex reinforced random walks.

The main contribution of this article is to extend the ideas in Budhiraja et al. (2015b,a) to the case of a vector of measures that represent the vertex occupation defined by many interacting reinforced random walks.

## 2. Statement of the results

### 2.1. Main results.

Our first result, stated as Theorem 1, shows that the vector field $F$ defined by (6) has a strict Lyapunov function. This result is crucial to understand the long-term behaviour of the vector field $F$ and the process $X = \{X(n)\}_{n \geq 0}$ defined by (4).

To state our first result, denote by $\Phi = \{\phi_t\}_{t \geq 0}$ the semi-flow associated with $F$ (see Lemma 1) and by $F^{-1}(0) \subset \mathfrak{D}$ the set of equilibrium points of $F$. We will need the following definition.

**Definition 1** (strict Lyapunov function). A continuous function $L : \mathfrak{D} \to \mathbb{R}$ is a strict Lyapunov function for the vector field $F : \mathfrak{D} \to T\mathfrak{D}$ if the function $t \in [0, \infty) \mapsto L(\phi_t(x_0))$ is strictly decreasing for all $x_0 \in \mathfrak{D} \setminus F^{-1}(0)$.

Now we are able to state our first result.

**Theorem 1.** The continuous function $L : \mathfrak{D} \to \mathbb{R}$ defined by

$$L(x) = \sum_{i,v} x_i^v \log(x_i^v) - \frac{1}{2} \sum_{i,j,v} \alpha_{ij}^v x_i^v x_j^v,$$

with $0 \log(0) = 0,$

is a strict Lyapunov function for the vector field $F$ defined by (6).

**Definition 2** (linearly stable/unstable equilibria). Let $\sigma(JF(x)) \subset \mathbb{C}$ be the set of eigenvalues of the Jacobian matrix of the vector field $F$ at a point $x \in \mathfrak{D}$. We say that an equilibrium point $x$ of $F$ is hyperbolic if $\sigma(JF(x))$ contains no eigenvalue with zero real part. An hyperbolic equilibrium point $x$ of $F$ is linearly stable if $\sigma(JF(x))$ contains only eigenvalues with negative real parts, otherwise we say that the hyperbolic equilibrium point $x$ is linearly unstable.

Our second result, stated below, characterises the convergence and non-convergence of $X$ toward the equilibria of the vector field $F$. The almost sure convergence described by the last item in this theorem is a consequence of Theorem 1.
Theorem 2. Let \( X = \{X(n)\}_{n \geq 0} \) be the process defined in (4) and \( F \) the vector field defined in (6). The following statements hold

(i) For each linearly stable equilibrium point \( x \) of \( F \),
\[
\mathbb{P}\left( \lim_{n \to \infty} X(n) = x \right) > 0;
\]
(ii) For each linearly unstable equilibrium point \( x \) of \( F \),
\[
\mathbb{P}\left( \lim_{n \to \infty} X(n) = x \right) = 0;
\]
(iii) If the equilibrium points of \( F \) are isolated, then
\[
\sum_{x \in F^{-1}(0)} \mathbb{P}\left( \lim_{n \to \infty} X(n) = x \right) = 1.
\]

2.2. Examples. This section presents a few results concerning several specific instances of the general model described in the Introduction. The first two results consider the asymptotic behaviour of any (i.e. \( m \geq 2 \)) “weakly interacting” random walks on any finite complete graph with \( d \geq 2 \) vertices. First, by using the global injectivity result in Gale & Nikaidô (1965), we show that if the absolute value of the constants \( \alpha_{ij}^v \) are smaller than a certain positive upper bound, then the vector field \( F \) has a unique equilibrium point \( x^* \) in \( \mathcal{D} \), see Theorem 4. A direct application of Theorem 2 in this case shows that the process \( X = \{X(n)\}_{n \geq 0} \) converges almost surely toward \( x^* \). We also present a sharp result for the case in which the constants \( \{\alpha_{ij}^v : i \neq j\} \) are all equal to each other, see Theorem 3.

Apart from these examples, we carry out a complete study for the case of two repelling random walks on the two-vertex graph. Depending on the strength of the repulsion, we show that there may be multiple hyperbolic equilibrium points, see Theorem 5. In our last example, we describe some asymptotic properties of three repelling random walks defined on \( \mathbb{Z} \), in which the repulsion is determined by the full previous history of the joint process, see Theorem 6. These processes are defined so that the probability that a random walk makes a transition in one direction decreases with the number of times that the other walks made a transition in that direction.

2.2.1. Weakly interacting random walks. The next two theorems show that if the interaction strength parameters \( \alpha_{ij}^v \) have absolute value less than some positive upper bound, then the process \( X = \{X(n)\}_{n \geq 0} \) converges almost surely to the unique equilibrium point of the vector field \( F \).

Let \( \delta_{ij} \) denote Kronecker’s delta, i.e., \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

Theorem 3. Let \( \alpha_{ij}^v = -\beta(1 - \delta_{ij}) \) for some \( \beta > 0 \) and all \( v \in [d] \) and \( i, j \in [m] \). Suppose that at least one of the following conditions is satisfied:

(C1) \[ d = 2, \ m \geq 2 \quad \text{and} \quad \beta \leq 2, \]
(C2) \[ d \geq 2, \ m \geq 2 \quad \text{and} \quad \beta < \frac{4}{d(m-1)}. \]

Then the process \( X = \{X(n)\}_{n \geq 0} \) converges almost surely to \( \left(\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}\right) \).

In Theorem 3, the case \( d = 2 \) can be reduced to an one-variable problem. The case \( d \geq 2, m \geq 2 \) is a particular case of the following general result.
Theorem 4. Suppose that $\alpha_{ij}^v$, $v \in [d]$ and $i, j \in [m]$, are real numbers satisfying (3) and the following condition holds

\[(C3) \quad \alpha_{v}^{ii} = 0 \quad \text{and} \quad \sum_{i=1}^{d} \sum_{j=1 \atop j \neq i}^{m} |\alpha_{ij}^v| < 4 \quad \text{for each} \quad (v, i) \in [d] \times [m].\]

Then the process $X = \{X(n)\}_{n \geq 0}$ converges almost surely to the unique equilibrium point of $F$.

2.2.2. An example of planar dynamics: two repelling walks on the two-vertex graph. The following example considers a relatively simple model consisting of two exponentially repelling walks, $W^1$ and $W^2$, on the two-vertex complete graph. These two processes are defined according to (2) by setting for all $v \in [d] = \{1, 2\}$ and $i, j \in [m] = \{1, 2\},$

\[
\alpha_{ij}^v = \begin{cases} -\beta, & \text{if } i \neq j; \\ 0, & \text{if } i = j, \end{cases}
\]

with $\beta \geq 0$. In this case, for $x = (x_1^v, x_2^v, x_1^2, x_2^2) \in D$, the coordinate functions of the vector field $F = (F_1^1, F_1^2, F_2^1, F_2^2)$ defined by (6) are explicitly given by

\[
F_i^v(x) = -x_i^v + \frac{e^{-\beta x_i^v}}{e^{-\beta x_1^v} + e^{-\beta x_2^v}}, \quad v \in [d], \quad i \in [m], \quad j = 3 - i.
\]

The following theorem provides a complete description for the asymptotic behaviour for the occupation measure process $X = \{X(n)\}_{n \geq 0}$, depending on the strength of the repulsion between the walks $W^1$ and $W^2$.

Theorem 5. If $\beta \in [0, 2)$, then the vertex occupation measure process $X = \{X(n)\}_{n \geq 0}$ converges almost surely toward the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. If $\beta > 2$, the vertex occupation measure process $X = \{X(n)\}_{n \geq 0}$ converge almost surely to

\[
(a, 1 - a, 1 - a, a) \quad \text{or} \quad (1 - a, a, a, 1 - a),
\]

where $a \in (0, \frac{1}{2})$ is uniquely determined by $\beta$.

2.2.3. Three repelling random walks on $\mathbb{Z}$. Our next example considers the dynamics defined by three repelling random walks on the two-vertex graph in order to study the asymptotic behaviour of three random walks defined on $\mathbb{Z}$, reinforced to repel each other according to the model described in the Introduction.

Let $\{S_i^n; i = 1, 2, 3\}_{n \geq 0}$ be the process described by three random walks on $\mathbb{Z}$ defined as follows. Assume that, for all $i \in \{1, 2, 3\}$, $S_i^0$ are fixed. Let $A_0$ be the trivial $\sigma$-algebra and for $n \geq 1$, let $A_n = \sigma\{S_i^1, S_i^2, S_i^3; 1 \leq k \leq n\}$ be the natural filtration generated by these three processes. For $n = 0$, the transition probability for each random walk is set to $P(S_i^1 = S_i^0 + 1 \mid A_0) = \frac{1}{2}$. For $n \geq 1$, the transition probability is defined as

\[
P\left(S_n^{i+1} = S_n^i + 1 \mid A_n\right) = \mu\left((S_n^i - S_0^i)/n + (S_n^j - S_0^j)/n\right)
\]

\[
= 1 - P\left(S_{n+1}^i = S_n^i - 1 \mid A_n\right),
\]

where $\{i, j, k\} = \{1, 2, 3\}$, and $\mu : \mathbb{R} \to [0, 1]$ is given by the following decreasing function

\[
\mu(y) = \frac{1}{1 + \exp(\beta y)}, \quad \beta \geq 0.
\]
The following theorem shows that this model has a phase transition at $\beta = 2$. When $\beta < 2$, the three random walks behave asymptotically as three independent symmetric simple random walks on $\mathbb{Z}$. When $\beta > 2$, there are always two random walks such that one of them diverges to $-\infty$ while the other one diverges to $+\infty$. The third walk may behave asymptotically as a simple symmetric walk.

**Theorem 6.** If $\beta < 2$, for any $i \in \{1, 2, 3\}$, then, with probability one, it holds that

$$\lim_{n \to \infty} P\left( S_{n+1}^i - S_n^i = 1 \mid A_n \right) = \frac{1}{2}. \tag{12}$$

For sufficiently large $\beta$, with positive probability, there are two random walks $i, j$ such that

$$\lim_{n \to \infty} S_n^i = - \lim_{n \to \infty} S_n^j = \infty, \tag{13}$$

and for $S_n^k, k \notin \{i, j\}$, it holds that $S_n^k$ behaves asymptotically as a simple symmetric random walk, that is, (12) holds for $i = k$.

### 3. Proof of Theorem 1: The Lyapunov Function

This section presents the proof of Theorem 1. Throughout let $\pi : \mathcal{D} \to \mathcal{D}$ be the smooth map defined in (5), $F : \mathcal{D} \to T\mathcal{D}$ the smooth vector field defined in (6) and $L : \mathcal{D} \to \mathbb{R}$ the continuous function defined in (8). The boundary and the interior of $\mathcal{D}$ are respectively, the sets

$$\partial \mathcal{D} = \left\{ x \in \mathcal{D} \mid \prod_{i,v} x_{i,v} = 0 \right\}, \quad \mathcal{D} = \mathcal{D} \setminus \partial \mathcal{D}.$$

**Lemma 1.** There exists a uniquely defined one-parameter family $\Phi = \{\phi_t\}_{t \geq 0}$ of self-maps of $\mathcal{D}$, called the semi-flow associated with $F$, such that the map $(t, x) \mapsto \phi_t(x)$ is smooth, and the following holds for each $x_0 \in \mathcal{D}$:

(i) $\phi_0(x_0) = x_0$ and $\phi_t(x_0) \in \mathcal{D}$ for all $t > 0$,

(ii) $\frac{d}{dt} \phi_t(x_0) = F(\phi_t(x_0))$ for all $t \geq 0$.

**Proof.** Let $x_0 \in \partial \mathcal{D}$. By (5), $\pi(x_0) \in \mathcal{D}$. By (6), $F(x_0)$ is the displacement vector from $x_0 \in \partial \mathcal{D}$ to $\pi(x_0) \in \mathcal{D}$. Hence, by the convexity of $\mathcal{D}$, we have that $F(x_0)$ points towards the interior of $\mathcal{D}$, i.e, $\mathcal{D}$ is invariant by $F$. Moreover, since $F$ is smooth, we have that $F$ is locally Lipschitz. A widely known result of the theory of ordinary differential equations (see (Khalil, 1992, Theorem 3.3)) now asserts that every locally Lipschitz vector field defined on an invariant compact set admits a uniquely defined semi-flow. \(\square\)

For each $x \in \mathcal{D}$, let $\Gamma(x)$ be the $md \times md$ matrix

$$\Gamma(x) = - I + \Pi(x),$$

where $I$ denotes the $md \times md$ identity matrix and $\Pi(x)$ is defined as

$$\Pi(x) = \begin{bmatrix}
\Pi^1(x) & 0 & \cdots & 0 \\
0 & \Pi^2(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Pi^m(x)
\end{bmatrix}, \tag{14}$$
where $0$ is the $d \times d$ zero matrix and for each $i \in [m]$, the $d \times d$ block matrices $\Pi_i(x)$ are given by

\begin{equation}
\Pi_i(x) = \begin{bmatrix}
\pi^i_1(x) & \pi^i_2(x) & \cdots & \pi^i_d(x) \\
\pi^i_1(x) & \pi^i_2(x) & \cdots & \pi^i_d(x) \\
\vdots & \vdots & \ddots & \vdots \\
\pi^i_1(x) & \pi^i_2(x) & \cdots & \pi^i_d(x)
\end{bmatrix}.
\end{equation}

In the next lemma, $\pi(x)$ denotes the vector $\pi(x) = (\pi^1_1(x), \ldots, \pi^1_d(x), \ldots, \pi^m_1(x), \ldots, \pi^m_d(x))$.

\textbf{Lemma 2.} $\pi(x)\Gamma(x) = 0$ for all $x \in \Delta$.

\textit{Proof.} The $v,i$-entry of $\pi(x)\Gamma(x)$ is

$$\pi^i_v \pi^i_1 + \ldots + \pi^i_{v-1} \pi^i_v + \pi^i_v (\pi^i_v - 1) + \pi^i_{v+1} \pi^i_v + \ldots + \pi^i_d \pi^i_v = \left(\sum_v \pi^i_v\right) \pi^i_v - \pi^i_v = \pi^i_v - \pi^i_v = 0,$$

where we have omitted $x$ in $\pi^i_u(x)$ to save space. \qed

In the next lemma, $x(t)$ denotes the vector $x(t) = (x^1_1(t), \ldots, x^1_d(t), \ldots, x^m_1(t), \ldots, x^m_d(t))$.

\textbf{Lemma 3.} Given $x_0 \in \Delta$, let $x(t) = \phi(t,x_0)$ for all $t \geq 0$. Then

$$\frac{d}{dt} x(t) = x(t)\Gamma(x(t)) \quad \text{for all} \quad t \geq 0.$$

\textit{Proof.} By the item (ii) of Lemma 1, we have that the $v,i$-entry of $\frac{d}{dt} x(t)$ is

\begin{equation}
\frac{d}{dt} x^i_v(t) = F^i_v(x(t)) = -x^i_v(t) + \pi^i_v(x(t)).
\end{equation}

On the other hand, by Lemma 2, we have the $v,i$-entry of the vector

$$x(t)\Gamma(x(t)) = \left(x(t) - \pi(x(t))\right)\Gamma(x(t))$$

is given by (the expressions $t$ and $x(t)$ were omitted to save space)

\begin{align*}
(x^i_1 - \pi^i_1)\pi^i_v + \ldots + (x^i_{v-1} - \pi^i_{v-1})\pi^i_v + \pi^i_v (\pi^i_v - 1) \\
+ (x^i_{v+1} - \pi^i_{v+1})\pi^i_v + \ldots + (x^i_d - \pi^i_d)\pi^i_v \\
= (x^i_v - \pi^i_v)(-1) + \sum_u (x^i_u - \pi^i_u)\pi^i_v \\
= -x^i_v + \pi^i_v + \pi^i_v \sum_u F^i_u(x) = -x^i_v + \pi^i_v,
\end{align*}

where $\sum_u F^i_u(x) = 0$ because $F(x) \in T\Delta$.

Putting it all together, we have that

$$\frac{d}{dt} x^i_v(t) = -x^i_v(t) + \pi^i_v(x(t)) = [x(t)\Gamma(x(t))]^i_v,$$

which concludes the proof. \qed

\textbf{Definition 3.} Given two vectors of probability measures $x, y \in \Delta$, we write $x = (x^1, \ldots, x^m)$ and $y = (y^1, \ldots, y^m)$, where $x^i = (x^i_1, \ldots, x^i_d)$, $y^i = (y^i_1, \ldots, y^i_d)$, $i \in [m]$. The relative entropy of $x$ with respect to $y$ is defined as

$$\text{Ent} \left( \frac{x}{y} \right) = \sum_{i=1}^{m} \text{Ent} \left( \frac{x^i}{y^i} \right), \quad \text{where} \quad \text{Ent} \left( \frac{x^i}{y^i} \right) = \sum_v x^i_v \log \left( \frac{x^i_v}{y^i_v} \right)$$
and log is the natural logarithm.

The following lemmas will be used in the proof of Theorem 1.

**Lemma 4.** Given $x_0 \in \mathcal{D}$, let $x(t) = \phi_t(x_0)$ for all $t > 0$. Then

$$
\left. \frac{d}{dt} L(x(t)) \right|_{t=\tau} = \frac{d}{dt} \text{Ent} \left( \frac{x(t)}{\pi(x(\tau))} \right) \Bigg|_{t=\tau} \quad \text{for all } \tau > 0.
$$

**Proof.** By Lemma 1, it follows that $x(t) \in \hat{\mathcal{D}}$ for all $t > 0$, hence $\text{Ent}(x(t)/\pi(x(\tau)))$ is well-defined for all $t, \tau > 0$. Computing the spatial derivatives of $L$ at $x = (x_\tau^i) \in \hat{\mathcal{D}}$ and using (3) lead to

$$
\frac{\partial L}{\partial x_\tau^i}(x) = \log(x_\tau^i) + 1 - \sum_j \alpha_{ij} x_\tau^j.
$$

Hereafter, let $w = x(\tau)$ for some arbitrary but fixed $\tau > 0$. Computing the derivative of $t \mapsto L(x(t))$ at $\tau$ yields

$$
\left. \frac{d}{dt} L(x(t)) \right|_{t=\tau} = \sum_{i,v} \left( \log(w_i^v) + 1 - \sum_j \alpha_{ij} w_j^v \right) \frac{d}{dt} x_\tau^i(t) \bigg|_{t=\tau} = \sum_{i,v} \left( \log(w_i^v) - \sum_j \alpha_{ij} w_j^v \right) \frac{d}{dt} x_\tau^i(t) \bigg|_{t=\tau} = \sum_{i,v} \log(w_i^v) \frac{d}{dt} x_\tau^i(t) \bigg|_{t=\tau} - \sum_{i,j,v} \alpha_{ij} w_j^v \frac{d}{dt} x_\tau^j(t) \bigg|_{t=\tau},
$$

where the third equality above holds because, since $F(w) \in T\mathcal{D}$, we have that

$$
\sum_{i,v} \frac{d}{dt} x_\tau^i(t) \bigg|_{t=\tau} = \sum_{i,v} F^i_v(w) = 0.
$$

On the other hand, the derivative of the entropy between $x(t)$ and $\pi(w)$ gives

$$
\left. \frac{d}{dt} \text{Ent} \left( \frac{x(t)}{\pi(x(\tau))} \right) \right|_{t=\tau} = \frac{d}{dt} \sum_{i,v} x_\tau^i(t) \log x_\tau^i(t) \bigg|_{t=\tau} - \frac{d}{dt} \sum_{i,v} x_\tau^i(t) \log \pi^i_v(w) \bigg|_{t=\tau}.
$$

Using (18), the first term at the right-hand side of (19) equals

$$
\frac{d}{dt} \sum_{i,v} x_\tau^i(t) \log x_\tau^i(t) \bigg|_{t=\tau} = \sum_{i,v} \log(w_i^v) \frac{d}{dt} x_\tau^i(t) \bigg|_{t=\tau}.
$$

To analyse the second term at the right-hand side of (19), note that $\pi^i_v(w)$ can be written as

$$
\pi^i_v(w) = e^{\sum_j \alpha_{ij} w_j^v} / Z_i(w),
$$

where $Z_i(w)$ is the normalising factor, that is,

$$
Z_i(w) = \sum_u e^{\sum_j \alpha_{ij} w_j^u}.
$$
By (21), the second term at the right-hand side of (19) becomes
\[
-\frac{d}{dt} \sum_{i,v} x^i_v(t) \log \pi^i_v(w) \bigg|_{t=\tau} = -\frac{d}{dt} \sum_{i,v} x^i_v(t) \log \left( \frac{\sum_{j} \alpha_{ij}^d w_{ij}^d}{Z_i(w)} \right) \bigg|_{t=\tau} \\
= -\sum_{i,j,v} \alpha_{ij}^d w_{ij}^d \frac{d}{dt} x^i_v(t) \bigg|_{t=\tau} + \sum_{i,v} \log(Z_i(w)) \frac{d}{dt} x^i_v(t) \bigg|_{t=\tau} \\
= -\sum_{i,j,v} \alpha_{ij}^d w_{ij}^d \frac{d}{dt} x^i_v(t) \bigg|_{t=\tau},
\]
where the last equality holds because \( \frac{d}{dt} x^i_v(t) \bigg|_{t=\tau} = F^i_v(w) \) and \( F(w) \in T\Phi \).

Comparing (17) with (19), (20), and (22) concludes the proof. \( \square \)

The following lemma is an adapted version of Lemma 3.1 in Budhiraja et al. (2015b).

**Lemma 5.** Given \( x_0 \in \Phi \), let \( x(t) = \phi_t(x_0) \) for all \( t > 0 \). Let \( x^i(t) = (x^i_1(t), \ldots, x^i_d(t)) \) and \( \pi^i(x(t)) = (\pi^i_1(x(t)), \ldots, \pi^i_d(x(t))) \). The following inequality holds
\[
\frac{d}{dt} \text{Ent} \left( \frac{x^i(t)}{\pi^i(x(\tau))} \right) \bigg|_{t=\tau} \leq 0, \quad \forall \tau > 0, \quad \forall i \in [m].
\]

Moreover,
\[
\exists \tau > 0, \quad \forall i \in [m], \quad \frac{d}{dt} \text{Ent} \left( \frac{x^i(t)}{\pi^i(x(\tau))} \right) \bigg|_{t=\tau} = 0 \quad \text{if and only if} \quad x_0 \in F^{-1}(0).
\]

**Proof.** By Lemma 1, \( x(t) \in \Phi \) for all \( t > 0 \), thus \( \text{Ent} \left( x^i(t)/\pi^i(x(\tau)) \right) \) is well-defined for all \( t, \tau > 0 \). Hereafter, let \( w = x(\tau) \in \Phi \) for some arbitrary but fixed \( \tau > 0 \). By Lemma 3 and (14),
\[
\frac{d}{dt} x^i(t) \bigg|_{t=\tau} = x^i(\tau) \Gamma^i(w), \quad i \in [m],
\]
where
\[
\Gamma^i(w) = -I_d + \Pi^i(w),
\]
\( I_d \) is the \( d \times d \) identity matrix and \( \Pi^i(w) \) is the matrix in (15) with \( x \) replaced by \( w \).

Let \( \ell : (0, \infty) \to [0, \infty) \) be the continuous function defined as \( \ell(z) = z \log z - z + 1 \).

The inequality (23) will be shown by assuming, and proving later, that
\[
\frac{d}{dt} \text{Ent} \left( \frac{x^i(t)}{\pi^i(w)} \right) \bigg|_{t=\tau} = -\sum_{u,v:u \neq v} \ell \left( \frac{x^i_u(\tau) \pi^i_u(w)}{x^i_v(\tau) \pi^i_v(w)} \right) x^i_u(\tau) \frac{\pi^i_v(w)}{\pi^i_u(w)} \Gamma^i_{vu}(w),
\]
where \( \Gamma^i_{vu}(w) \) is the \( v, u \)-entry of the matrix \( \Gamma^i(w) \) defined in (26).

Assuming (27), inequality (23) is an immediate consequence of the following three facts:
1. \( \pi^i_v(w) > 0 \) for all \( v \in [d] \),
2. \( x^i_v(\tau) > 0 \) for all \( v \in [d] \), and
3. \( \ell((0, \infty)) \subset [0, \infty) \).

The first assertion follows from the definition of \( \pi^i_v \). The second holds because, \( x(t) = \phi_t(x_0) \in \Phi \) for all \( t > 0 \). The third is trivial. Next we verify (24). If \( x_0 \in F^{-1}(0) \), then \( x(t) = x_0 \) and \( F(x(t)) = 0 \) for all \( t \geq 0 \). In particular, \( x^i(t) = \pi^i(w) \), for all \( t > 0 \), then \( \frac{d}{dt} \text{Ent} \left( x^i(t)/\pi^i(w) \right) \bigg|_{t=\tau} = 0 \), for all \( t > 0 \), because the argument of
the function $\ell$ in (27) equals 1 for all $u$, $v$, and $\ell(1) = 0$. Conversely, assume that
\[
\frac{d}{dt} \text{Ent}(x^i(t)/\pi^i(w)) \big|_{t=\tau} = 0.
\]
By the facts $a)$ and $b)$ above and also because $\Gamma^i_{vu}(w) > 0$ for all $u \neq v$, we have that $\sum_{u,v:u \neq v} \ell(x^i_{vu}) = 0$, where
\[
x^i_{vu} = \frac{x^i_u(\tau)\pi^i_u(w)}{x^i_u(\tau)\pi^i_v(w)}.
\]
Now the fact c) implies that $\ell(x^i_{vu}) = 0$ for all $v \neq u$. This implies that $z^i_{vu} = 1$ for all for all $v, u \in V$. As a consequence, $x^i_u(\tau)\pi^i_u(w) = x^i_u(\tau)\pi^i_v(w)$ for all vertices $v$ and $u$. Summing both sides of the previous equality over all $u \in [d]$ yields $x^i_v(\tau) = \pi^i_v(w)$ for all $v$, that is, $x^i(\tau) = \pi^i(w) = \pi^i(x(\tau))$, for all $i \in [m]$. This implies that $x(\tau) = \pi(x(\tau))$, and hence that $F(x(\tau)) = 0$, i.e., $x(\tau)$ is an equilibrium point of $F$. Hence, $x(t) = x(\tau) = x_0$ for all $t > 0$.

It remains to show that (27) is true. First note that since $\ell(x^i_{vu}) = 0$, we can replace $\sum_{u,v:u \neq v}$ by $\sum_{u,v}$ in the right-hand side of (27). Applying the definition of $\ell$ and rearranging terms, the right-hand side of (27) equals
\[
\sum_{v,u \in V} \left( x^i_u(\tau) + x^i_v(\tau) \log \left( \frac{x^i_u(\tau)}{\pi^i_u(w)} \right) - x^i_u(\tau)\pi^i_v(w) \right) \Gamma^i_{vu}(w)
\]
\[
- \sum_{v,u \in V} x^i_v(\tau) \log \left( \frac{x^i_v(\tau)}{\pi^i_v(w)} \right) \Gamma^i_{vu}(w).
\]
(28)

Since for each $v \in V$, $\sum_{u \in V} \Gamma^i_{vu}(w) = 0$, the second line of (28) equals zero. In addition, by Lemma 2, it follows that $\sum_{v \in V} \pi^i_v(w)\Gamma^i_{vu}(w) = 0$ for each $u \in V$. Taking these two facts into account together with (25) shows that (28) reduces to
\[
\sum_{u,v \in V} \left[ x^i_u(\tau) + x^i_v(\tau) \log \left( \frac{x^i_u(\tau)}{\pi^i_u(w)} \right) \right] \Gamma^i_{vu}(w)
\]
\[
= \sum_{u \in V} \left[ \frac{d}{dt} x^i_u(t) \bigg|_{t=\tau} + \log \left( \frac{x^i_u(t)}{\pi^i_u(w)} \right) \frac{d}{dt} x^i_u(t) \bigg|_{t=\tau} \right]
\]
\[
= \frac{d}{dt} \left[ 1 + \sum_{u \in V} x^i_u(t) \log \left( \frac{x^i_u(t)}{\pi^i_u(w)} \right) \right] \bigg|_{t=\tau} = \frac{d}{dt} \text{Ent} \left( \frac{x^i(t)}{\pi^i(w)} \right) \bigg|_{t=\tau}.
\]

Proof of Theorem 1. To prove that $L$ is a strict Lyapunov function, we will show that $t \in [0, \infty) \rightarrow L(\phi_t(x_0))$ is strictly decreasing. See Definition 1. First we show how to combine Definition 3 with Lemmas 4 and 5 to prove the following claim:
\[
\frac{d}{dt} L(\phi_t(x_0)) \bigg|_{t=\tau} < 0, \quad \forall x_0 \in \mathcal{D} \setminus F^{-1}(0), \quad \forall \tau \in (0, \infty).
\]
(29)

In fact, let $x_0 \in \mathcal{D} \setminus F^{-1}(0)$. By Definition 3 and Lemma 4 we have that
\[
\frac{d}{dt} L(\phi_t(x_0)) \bigg|_{t=\tau} = \sum_{i=1}^m \frac{d}{dt} \text{Ent} \left( \frac{\phi^i_t(x_0)}{\pi^i(\phi^i_t(x_0))} \right) \bigg|_{t=\tau} \quad \text{for all } \tau > 0.
\]
Hence, using (24) gives
\[
\frac{d}{dt} L(\phi_t(x_0)) \bigg|_{t=\tau} \neq 0, \quad \forall x_0 \in \mathcal{D} \setminus F^{-1}(0), \quad \forall \tau \in (0, \infty),
\]
and by (23) we have
\[ \frac{d}{dt} L(\phi_t(x_0)) \bigg|_{t=0} \leq 0 \quad \forall \tau \in (0, \infty). \]
These two assertions combined prove the claim.

Let \( x_0 \in \mathcal{D}\backslash F^{-1}(0) \). By (29) and by the continuity of \( t \mapsto L(\phi_t(x_0)) \) at 0, we have that the function \( t \in [0, \infty) \mapsto L(\phi_t(x_0)) \) is strictly decreasing, showing that \( L \) is a Lyapunov function for the vector field \( F \).

\[ \square \]

4. Proof of Theorem 2: Stochastic approximations

In this section, we show how the asymptotic behaviour of the process of empirical vertex occupation measures \( X = \{X(n)\}_{n \geq 0} \) defined in (4) is related to the asymptotic behaviour of the ordinary differential equation (7) where \( F \) is the vector field defined in (6). A formulation based on dynamical systems theory that makes precise the connection between the process \( X = \{X(n)\}_{n \geq 0} \) and the semi-flow \( \Phi = \{\phi_t\}_{t \geq 0} \) induced by the vector field \( F \) has been developed in [1996], [1999].

This connection will be established by Lemma 7 stated below.

For each \( n \geq 1 \) define
\[ \xi(n) = (\xi^i_v(n); 1 \leq i \leq m, 1 \leq v \leq d) \quad \text{where} \quad \xi^i_v(n) = 1\{W^i(n + 1) = v\}. \]

The following lemma allows to identify \( X = \{X(n)\}_{n \geq 0} \) with a specific process known as a stochastic approximation. This step is key to the general approach followed throughout.

**Lemma 6.** The process \( X = \{X(n)\}_{n \geq 0} \) satisfies the recursion
\[ X(n + 1) - X(n) = \gamma_n(F(X(n)) + U_n), \]
where
\[ \gamma_n = \frac{1}{n + d + 1}, \quad U_n = \xi(n) - \mathbb{E}[\xi(n) \mid \mathcal{F}_n], \]
and \( F \) is the vector field defined in (6).

**Proof.** The increment of the occupation measure for the vertex \( v \in V \) at time \( n + 1 \) by the \( i \)-th random walk is given by
\[
X^i_v(n + 1) - X^i_v(n) = \frac{1 + \sum_{k=0}^{n-1} \xi^i_v(k) + \xi^i_v(n)}{d + n + 1} - \frac{1 + \sum_{k=0}^{n-1} \xi^i_v(k)}{d + n} \\
= \frac{1}{d + n + 1} \left( - \frac{1 + \sum_{k=0}^{n-1} \xi^i_v(k)}{d + n} + \xi^i_v(n) \right) \\
= \frac{1}{d + n + 1} \left( - X^i_v(n) + \xi^i_v(n) \right).
\]
Setting \( \gamma_n = (n + d + 1)^{-1} \) and using \( \xi \) as defined in (30) leads to
\[
X(n + 1) - X(n) = \gamma_n(-X(n) + \xi(n)) \\
= \gamma_n \left\{ \left( -X(n) + \mathbb{E}[\xi(n) \mid \mathcal{F}_n] \right) + \left( \xi(n) - \mathbb{E}[\xi(n) \mid \mathcal{F}_n] \right) \right\} \\
= \gamma_n \left\{ \left( -X(n) + \mathbb{E}[\xi(n) \mid \mathcal{F}_n] \right) + U_n \right\}.
\]
Now, according to (5),
\[
\mathbb{E}[\xi(n) \mid \mathcal{F}_n] = (\mathbb{P}(W^i_{n+1} = v \mid \mathcal{F}_n); 1 \leq i \leq m, 1 \leq v \leq d) = \pi(X(n)).
\]
Substituting this into the expression for the increments of $X$ gives $X(n+1) - X(n) = \gamma_n \{ -X(n) + \pi(X(n)) \} + U_n$, which, by using the definition of $F$ in (6), concludes the proof.

The following two definitions are necessary to state Lemma 7.

**Definition 4** (Chain-recurrent set). Let $\delta > 0$, $T > 0$. A $(\delta, T)$-pseudo orbit from $x \in \mathcal{D}$ to $y \in \mathcal{D}$ is a finite sequence of partial orbits $\{\phi_i(y_k) : 0 \leq t \leq t_i\}; i = 0, \ldots, k-1$; $t_i \geq T$ of the semi-flow $\Phi = \{\phi_t\}_{t \geq 0}$ such that

$$||y_0 - x|| < \delta, \quad ||\phi_i(y_k) - y_{i+1}|| < \delta, \quad i = 0, \ldots, k - 1, \quad \text{and} \quad y_k = y.$$ A point $x \in \mathcal{D}$ is chain-recurrent if for every $\delta > 0$ and $T > 0$ there is a $(\delta, T)$-pseudo orbit from $x$ to itself. The set of chain-recurrent points of $\Phi$ is denoted by $\mathcal{R}(\Phi)$.

It follows that $\mathcal{R}(\Phi)$ is closed, positively invariant and such that $F^{-1}(0) \subset \mathcal{R}(\Phi)$.

**Definition 5.** Let $\mathcal{L}(\{X(n)\})$ be the limit set of the stochastic approximation process $\{X(n)\}_{n \geq 0}$. That is, for any point $\omega \in \Omega$, the value of $\mathcal{L}(\{X(n)\})$ at $\omega$ is given by the set of points $x \in \mathbb{R}^{md}$ for which $\lim_{k \rightarrow \infty} X(n_k, \omega) = x$, for some strictly increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$.

**Lemma 7.** Let $X = \{X(n)\}_{n \geq 0}$ be the occupation measure process satisfying the recursion in (31). The following hold

(i) $\{X(n)\}_{n \geq 0}$ is bounded,

(ii) $\lim_{n \rightarrow \infty} \gamma_n = 0$, $\sum_{n \geq 0} \gamma_n = \infty$, and $\sum_{n \geq 0} \gamma_n^2 < \infty$,

(iii) For each $T > 0$, almost surely it holds that

$$\lim_{n \rightarrow \infty} \left( \sup_{\{r: 0 \leq \tau_r - \tau_{r-1} \leq T\}} \left\| \sum_{k=n}^{r-1} \gamma_k U_k \right\| \right) = 0,$$

where $\tau_0 = 0$ and $\tau_n = \sum_{k=0}^{n-1} \gamma_k$.

(iv) The set $\mathcal{L}(\{X(n)\})$ is almost surely connected and included in $\mathcal{R}(\Phi)$, the chain-recurrent set of the semi-flow induced by the vector field $F$ in (6).

**Proof.** The proof of (iv) follows from Theorem 1.2 in Benaïm (1996) together with Corollary 1.2 to Theorem 1.1 in Benaim & Hirsch (1995), and relies on the items (i)-(iii). The assertions in (i) and (ii) are immediate. We will prove (iii). Let $M_n = \sum_{k=0}^{n} \gamma_k U_k$. The process $\{M_n\}_{n \geq 0}$ is a martingale with respect to $\{\mathcal{F}_n, n \geq 0\}$, that is

$$\mathbb{E}[M_{n+1} | \mathcal{F}_{n+1}] = \sum_{k=0}^{n} \gamma_k U_k + \gamma_{n+1} \mathbb{E}[U_{n+1} | \mathcal{F}_{n+1}] = M_n.$$ Observe that

$$\mathbb{E}[\| M_{n+1} - M_n \|^2 | \mathcal{F}_{n+1}] = \gamma_{n+1}^2 \mathbb{E}[\| U_{n+1} \|^2 | \mathcal{F}_{n+1}] \leq \gamma_{n+1}^2 \left( \sum_{i=1}^{\infty} \xi_i(n+1) \right)^2 \leq (md)^2 \gamma_{n+1}^2.$$ Using Doob’s decomposition for the sub-martingale $M_n^2$, let $\{A_n, n \geq 1\}$ be a predictable increasing sequence defined by $A_{n+1} = M_{n+1}^2 - M_n$ with $A_1 = 0$. The conditional variance formula for the increment $M_{n+1} - M_n$ gives

$$A_{n+2} - A_{n+1} = \mathbb{E}[M_{n+1}^2 | \mathcal{F}_n] - M_n^2 = \mathbb{E}[\| M_{n+1} - M_n \|^2 | \mathcal{F}_{n+1}],$$
and hence for any $n$,

$$A_{n+2} = \sum_{k=0}^{n} \mathbb{E} \left[ \| M_{k+1} - M_k \|^2 \right] \mathcal{F}_{n+1} \leq (md)^2 \sum_{k=0}^{n} \gamma_{k+1}^2.$$  

Passing to the limit $n \to \infty$ shows that almost surely $A_\infty < \infty$. According to Theorem 5.4.9 in Durrett (2010), this in turn implies that $M_n$ converges almost surely to a finite limit and hence that $\{M_n\}_{n \geq 0}$ is a Cauchy sequence. This is sufficient to conclude the proof. \hfill $\square$

The proof of the second item of Theorem 2, concerning the non-convergence toward linearly unstable equilibria, makes use of the following lemma. For $w \in \mathbb{R}$, let $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$.

**Lemma 8.** Let $x^*$ be a linearly unstable equilibrium of the vector field $F$ defined by (6). There is a neighborhood $\mathcal{B}(x^*)$ of $x^*$ and a constant $c > 0$

$$\mathbb{E} \left[ \langle \theta, U_n \rangle^+ \bigg| X(n) = x, \mathcal{F}_n \right] \geq c$$

for every $n > 0$, every $x \in \mathcal{B}(x^*)$, and every $\theta \in \mathcal{T}_D$.

**Proof.** It is sufficient to show that, for all $n > 0$, $x \in \mathcal{D}$, and $\theta \in \mathcal{T}_D$, we have that

$$\mathbb{E} \left[ \langle \theta, U_n \rangle^+ \bigg| X(n) = x, \mathcal{F}_n \right] \geq s(x),$$

where $s : \mathcal{D} \to \mathbb{R}$ is a continuous function such that $s(x^*) > 0$.

Let

$$s(x) = \frac{1}{2md} \left( \min_{i,v} \pi_x^i (x) \right)^{m+1}.$$  

Clearly $s$, as defined in (35), is continuous. Since $F(x) = -x + \pi(x)$ and since $F(x^*) = 0$, we have that $\pi(x^*) = x^*$. As a consequence, $s(x^*) > 0$, where the previous inequality holds because $x^*$ belongs to the interior of $\mathcal{D}$.

It remains show (34). Let $\theta \in \mathcal{T}_D$. For each walk $i \in [m]$, choose a vertex $v^i \in \{1, 2, \ldots, d\}$, such that

$$\theta_{v^i} = \max_v \theta_v^i.$$  

Now, define the event $A = \bigcap_{i \in [m]} \left\{ \xi_{v^i} (n) = 1 \right\}$, with $\xi$ as defined by (30). That is, $A$ is the event in which each walk $i$ makes a transition to vertex $v^i$ at time $n + 1$, $i = 1, 2, \ldots, m$. For all $n \geq 0$, we have that for all $\theta \in \mathcal{T}_D$,

$$\mathbb{E} \left[ \langle \theta, U_n \rangle^+ \bigg| X(n) = x, \mathcal{F}_n \right] = \mathbb{E} \left[ \langle \theta, U_n \rangle^+ \bigg| X(n) = x \right] \geq q(x, \theta)$$

where

$$q(x, \theta) = \mathbb{E} \left[ \langle \theta, U_n \rangle^+ A, X(n) = x \right] \mathbb{P} (A \mid X(n) = x).$$  

To see that (36) holds, note that the first equality follows because the distribution of $U_n$ is uniquely determined by $X(n)$ according to (32). The inequality in (36) holds because $\langle \theta, U_n \rangle^+$ is non-negative. Now, to show (34), it is sufficient to prove that for all $\theta \in \mathcal{T}_D$ and $x \in \mathcal{D}$

$$q(x, \theta) \geq s(x).$$
To show \( (38) \), we show first that
\[
q(\theta, x) = \left[ \sum_i \max_v \theta^i_v - \sum_i \langle \theta^i, \pi^i(x) \rangle \right] + \prod_{i=1}^m \pi^i_v(x).
\]

To show \( (39) \), note that, given \( X(n) = x \), the transitions of the walks are independent, and therefore,
\[
\mathbb{P}(A \mid X(n) = x) = \prod_{i=1}^m \pi^i_v(x).
\]

To conclude the proof of \( (39) \), we show that, given \( X(n) = x \) and \( A \), we have that \( \langle \theta, U_n \rangle = \sum_i \max_v \theta^i_v - \sum_i (\langle \theta^i, \pi^i(x) \rangle) \). According to \( (32) \), we have \( (U_n)_v = \xi^i_v(n) - \mathbb{E}[\xi^i_v(n) \mid \mathcal{F}_n] = \xi^i_v(n) - \pi^i_v(X(n)) \), where, by \( (30) \), \( \xi^i_v(n) = 1\{W^i(n + 1) = v\} \). Let \( \delta_{v,v'} = 1 \) if \( v = v' \) and zero otherwise. So, given \( X(n) = x \) and \( A \), it follows that \( (U_n)_v = \delta_{v,v} - \pi^i_v(x) \) and therefore
\[
\langle \theta, U_n \rangle = \sum_i \theta^i_v \left( \delta_{v,v} - \pi^i_v(x) \right)
= \sum_i \theta^i_v - \sum_i \langle \theta^i, \pi^i(x) \rangle
= \sum_i \max_v \theta^i_v - \sum_i \langle \theta^i, \pi^i(x) \rangle.
\]

Next we use \( (39) \) to show \( (38) \). For \( \theta^i \in \mathbb{R}^d \), we set \( (\theta^i)^+ = ((\theta^i_1)^+, \ldots, \theta^i_d)^+ \), \( (\theta^i)^- = ((\theta^i_1)^-, \ldots, \theta^i_d)^- \), and \( \mathcal{D}_1 = \{ \theta \in \mathcal{D} : \sum_{i,v} |\theta^i_v| = 1 \} \). To save notation, we set \( y = \pi(x) \). Now observe that
\[
q(\theta, x) = \left[ \sum_i \max_v \theta^i_v - \sum_i \langle \theta^i, y^i \rangle \right] + \prod_{i=1}^m y^i_v
\geq \left[ \sum_i \max_v \theta^i_v - \sum_i \langle \theta^i, y^i \rangle \right] + \left( \min_{i,v} y^i_v \right)^m
= \left[ \sum_i \max_v \theta^i_v - \sum_i \langle \theta^i, y^i \rangle \right] + \left( \min_{i,v} y^i_v \right)^m
\geq \sum_i \langle (\theta^i)^-, y^i \rangle + \sum_i \langle (\theta^i)^+, y^i \rangle \left( \min_{i,v} y^i_v \right)^m
\geq \frac{1}{2md} \left( \min_{i,v} y^i_v \right)^m
= \frac{1}{2md} \left( \min_{i,v} y^i_v \right)^{m+1},
\]
which shows \( (38) \) as claimed. Above, the first inequality holds because \( 0 \leq \min_{i,v} y^i_v \leq y^i_v \leq 1 \) for all \( i \). The second inequality holds because \( \max_v \theta^i_v \geq (\theta^i)^_v \) all \( i \) and \( v \), and because \( y^i \) is a probability measure for all \( i \), and therefore, \( \max_v \theta^i_v - \langle (\theta^i)^+, y^i \rangle \geq 0 \).
for all $i$. To show the last inequality, it is sufficient to show that

$$\sum_{i=1}^{m} \langle (\theta^i)^-, y^i \rangle \geq \frac{1}{2md} \min_{i,v} \{ y_v^i \}. \tag{41}$$

To verify (41), observe that

$$\sum_{i} \langle (\theta^i)^-, y^i \rangle = \sum_{i,v} y_v^i (\theta_v^i)^- \geq \min_{i,v} \{ y_v^i \} \sum_{i,v} (\theta_v^i)^-$$

$$\geq \min_{i,v} \{ y_v^i \} \max_{i,v} (\theta_v^i)^- \geq \min_{i,v} \{ y_v^i \} \frac{1}{2md}$$

The last inequality is justified by observing that $\max_{i,v} (\theta_v^i)^- \geq \frac{1}{2md}$. To check this, we show that $1 \leq 2md \max_{i,v} (\theta_v^i)^-$. Since $\theta \in T\Omega_1$, it follows that $1 = \sum_{i,v} |\theta_v^i|$ and therefore

$$1 = \sum_{i,v} |\theta_v^i| = \sum_{i=1}^{m} \left( \sum_{v=1}^{d} (\theta_v^i)^+ + \sum_{v=1}^{d} (\theta_v^i)^- \right)$$

$$= \sum_{i=1}^{m} \left( \sum_{v=1}^{d} (\theta_v^i)^- \right) \leq 2md \max_{i,v} (\theta_v^i)^-. \quad \square$$

We will use the following definitions and lemma for the proof of the first item in Theorem 2.

**Definition 6 (Attractor).** A subset $A \subset \Omega$ is an attractor for the semi-flow $\Phi = \{ \phi_t \}_{t \geq 0}$ if the following conditions hold:

(i) $A$ is non-empty, compact and invariant by $\Phi$, that is, $\phi_t (A) = A, \forall t \geq 0$;

(ii) $A$ has a neighborhood $W \subset \Omega$ such that $\text{dist}(\phi_t(x), A) \to 0$ as $t \to \infty$ uniformly in $x \in W$,

where $\text{dist}(p, A) = \inf_{a \in A} \| p - a \|$. The basin of $A$, $B(A)$, is the positively invariant open set formed by the points $x \in \Omega$ such that $\text{dist}(\phi_t(x), A) \to 0$ as $t \to \infty$.

**Lemma 9 (Hirsch & Smale, 1974, Theorem, (b), p. 181)).** Let $x^*$ be a linearly stable equilibrium point of the vector field $F$ defined in (6). Then $A = \{ x^* \}$ is an attractor for the semi-flow $\Phi$ induced by $F$.

Let $\tau_0$ and $\tau_n = \sum_{k=1}^{n} \gamma_k$ for $n \geq 1$ with $\gamma_k$ defined as in (32). Let $Z = \{ Z(t) \},$ $t \in [0, \infty)$, be a continuous-time affine and piecewise constant process defined by considering the linear interpolation of $X = \{ X(n) \}_{n \geq 0}$, that is

$$Z(\tau_n + s) = X(n) + s \frac{X(n+1) - X(n)}{\tau_{n+1} - \tau_n}, \quad 0 \leq s \leq \gamma_{n+1}, \quad n \geq 0. \tag{42}$$

**Definition 7.** A point $x \in \Omega$ is said to be attainable by $Z = \{ Z(t) \}$ if for each $t > 0$ and every open neighborhood $U$ of $x$

$$\mathbb{P} \left( \exists s \geq t : Z(s) \in U \right) > 0.$$ 

The set of attainable points of $Z$ is denoted by $\text{Att}(Z)$.

**Proof of Theorem 2.** Throughout let $X = \{ X(n) \}_{n \geq 0}$ be the vertex occupation measure process defined in (4) which satisfies the recursion (31). Let $F$ be the smooth vector field defined in (6).

(i) Let $x^*$ be a linearly stable equilibrium of the vector field $F$ and let $A = \{ x^* \}$. The proof of the first assertion follows from Theorem 7.3 in Benaïm (1999), provided
that \( Att(Z) \cap B(A) \neq \emptyset \), that is, provided the basin of \( A \) is attainable by the process \( Z \) defined in (42). It is sufficient to show that \( Att(X) \cap B(A) \neq \emptyset \) because \( \lim_{n \to \infty} \gamma_n = 0 \).

Here \( Att(X) \) refers to the set of points \( x \in \mathcal{D} \) attainable by \( X \), that is, such that, for each open neighborhood \( U \) of \( x \) and each \( n_\ast \in \mathbb{N} \), we have that \( \mathbb{P}(\exists n \geq n_\ast : X(n) \in U) > 0 \) or, equivalently, \( \exists n \geq n_\ast : \mathbb{P}(X(n) \in U) > 0 \). Since each equilibrium \( x^* \) of \( F \) is arbitrarily close to a rational point \( \bar{q} \) of \( \mathbb{Q} \), \( \mathbb{P}(\exists n_\ast \in \mathbb{N} : X(n) = \bar{q}) > 0 \) for each \( n_\ast \in \mathbb{N} \). To check this, let \( q \in \mathbb{Q} \) and consider the following sequence of vertices \( V(n) \in [d] \), \( n \in \mathbb{N} \), defined as follows. For each \( n \in \mathbb{N} \), \( v \in [d] \), and \( \bar{n} = 1, 2, \ldots \), we set

\[
v^i(\bar{n}) = v \quad \text{if and only if} \quad \bar{n} \in \bigcap_{\ell=1}^{\infty} N_{\ell,v}^i
\]

where, for each \( i, v, \) and \( \ell = 1, 2, \ldots \), the set \( N_{\ell,v}^i \) is defined as \( N_{\ell,v}^i = \{ n_{\ell,v}^i + 1, n_{\ell,v}^i + 2, \ldots, n_{\ell,v}^i + k_v^i \} \), \( n_{\ell,v}^i = (\ell - 1)k + k_v^i \) for \( v \geq 2 \), and \( n_{\ell,v}^i = (\ell - 1)k \) for \( v = 1 \). In other words, the sequence \( \{v^i(\bar{n})\}_{\bar{n} \geq 1} \) is a cycling sequence of vertices of \( G \) for which the cycle, of length \( k = k_1^i + k_2^i + \cdots + k_v^i \), contains \( k_v^i \) repetitions of vertex \( v \).

Now, for each \( n \geq 1 \), let \( A_n \) be the event, in which the process \( W(\bar{n}) \) follows the vertex sequences \( v(\bar{n}) \) up to time \( n \). That is

\[
A_n = \bigcap_{\bar{n}=1}^{n} \bigcup_{i=1}^{m} \left\{ W^i(\bar{n}) = v^i(\bar{n}) \right\}
\]

Choose \( n = Lk \), where \( L \) is a sufficiently large integer, such that \( n > n_\ast \) and \( n > md/\varepsilon \). Given \( A_n \), it holds that \( X^i_v(n) = (1 + Lk_v^i)/(Lk) = 1/Lk + q^i_v = 1/n + q_v^i \), in which case \( |X(n) - q| = \sum_{i,v} |X^i_v(n) - q_v^i| = md/n < \varepsilon \). Thus, assuming that \( \mathbb{P}(A_n) > 0 \), we have that

\[
\mathbb{P}(|X(n) - q| < \varepsilon) \geq \mathbb{P}(|X(n) - q| < \varepsilon | A_n) \mathbb{P}(A_n) = \mathbb{P}(A_n) > 0.
\]

It remains to show that \( \mathbb{P}(A_n) > 0 \). Let \( X^i_v(0) = 1 \), and for each \( \bar{n} \in \{1, 2, \ldots, n\} \), let \( x^i_v(\bar{n}) \) be the value of \( X^i_v(\bar{n}) \) computed according to (1) when \( W^i(\bar{n}) = v^i(\bar{n}) \), \( \bar{n} = 1, 2, \ldots, n \). Since, for each \( \bar{n} \in \{1, 2, \ldots, n\} \), we have that \( W^i(\bar{n}) \), \( i = 1, 2, \ldots, m \) are conditionally independent given \( \{X(\bar{n} - 1) = x(\bar{n} - 1)\} \), it follows, by (2) and (5) that

\[
\mathbb{P}(A_n) = \prod_{\bar{n}=1}^{n} \prod_{i=1}^{m} \pi^i_v(\bar{n})(x(\bar{n} - 1)).
\]

Since \( \pi^i_v(y) > 0 \) for all \( i, v \) and \( y \in \mathcal{D} \), it follows that \( \mathbb{P}(A_n) > 0 \). This concludes the proof of the first claim made in the theorem.

(ii) The proof of the second claim follows by Theorem 1 in Pemantle (1990). In order to use this result, we observe that all the required assumptions and hypotheses required by this theorem are easily verified for the vertex occupation measure process \( X = \{X_n\}_{n \geq 0} \) satisfying (31). Only the condition determined by (33) is more involved and deserves special attention. This condition is satisfied in our case by Lemma 8.

(iii) By Lemma 7, the limit set of \( X = \{X(n)\}_{n \geq 0} \) is connected and contained in \( \mathcal{R}(\Phi) \), where \( \Phi = \{\phi_t\}_{t \geq 0} \) is the semi-flow induced by the vector field \( F \). Moreover, since \( F \) is a continuous map on its compact domain \( \mathcal{D} \) with isolated zeros (equilibrium
points), we have that $F$ has finitely many equilibrium points. Hence, if $L$ is the strict Lyapunov function defined in Theorem 1, then $L(F^{-1}(0))$ is a finite set. In this case, by Proposition 3.2 in Bena"ım (1996), it follows that $\mathcal{R}(\Phi)$ is contained in the set of equilibrium points. Since $\Sigma(\{X(n)\})$ is connected, we have proved that $\Sigma(\{X(n)\})$ is an equilibrium point of $F$ that may depend on $\omega$. However, because $\omega$ is an arbitrary point of $\Omega$, we have that

$$\sum_{x \in F^{-1}(0)} \mathbb{P}(\lim_{n \to \infty} X(n) = x) = 1.$$ 

This concludes the proof of the theorem. \qed

5. Proofs of Theorems 3, 4, 5, and 6

5.1. Theorems 3 and 4. In order to prepare for the proof of Theorems 3 and 4, we will provide conditions on the constants $\alpha^{ij}_{uv} \in \mathbb{R}$ under which the map $\pi$ defined in (5) has a unique fixed point or, equivalently, the vector field $F$ defined in (6) has an unique equilibrium point.

We will need the following injectivity result.

**Theorem 7** (Gale-Nikaidô Gale & Nikaidô (1965)). Let $n \geq 2$, $\Lambda \subset \mathbb{R}^n$ be an (open or closed) convex set and $f : \Lambda \to \mathbb{R}^n$ be a differentiable map whose Jacobian matrix $JF(x)$ is positive quasi-definite for all $x \in \Lambda$. Then $F$ is injective on $\Lambda$.

To be more precise, the statement above is Theorem 6 of Gale & Nikaidô (1965) (see also Theorem 3 of Parthasarathy (1983)). We recall that an $n \times n$ matrix $A = (a_{ij})$ with real entries is positive quasi-definite if its symmetric part, namely $\frac{1}{2}(A + A^T)$ is positive-definite, i.e. $x^T(A + A^T)x > 0$ for all non-null $n \times 1$ column matrix $x$.

**Proof of Theorem 4.** Since $\pi$ is a continuous self-map of the compact convex set $\mathcal{D}$, by Brouwer’s Fixed Point Theorem, $\pi$ has at least one fixed point $x^\ast$ in $\mathcal{D}$. Moreover, $x \in \mathcal{D}$ is a fixed point of $\pi$ if and only if $x$ is a zero of $F : \mathcal{D} \to \mathbb{R}^{dm}$ defined in (6). Hence, to prove that $\pi$ has a unique fixed point in $\mathcal{D}$ it suffices showing that $F$ (as a map) is injective on $\mathcal{D}$.

Let $x = (x^i_u) \in \mathcal{D}$. By (6), the Jacobian of $F$ at $x$ is the $dm \times dm$ matrix

$$\frac{\partial F^i_v}{\partial x^u}(x) = \begin{cases} -1 & \text{if } u = v \text{ and } j = i, \\ \alpha^{ij}_{uv}\exp\left(\sum_{j=1}^{m} \alpha^{ij}_{uv} x^j_v\right) \sum_{u \neq v} \exp\left(\sum_{j=1}^{m} \alpha^{ij}_{uv} x^j_u\right) & \text{if } u = v \text{ and } j \neq i, \\ \frac{-\alpha^{ij}_{uv}\exp\left(\sum_{j=1}^{m} \alpha^{ij}_{uv} x^j_v\right) \exp\left(\sum_{j=1}^{m} \alpha^{ij}_{uv} x^j_u\right)}{\sum_{u=1}^{d} \exp\left(\sum_{j=1}^{m} \alpha^{ij}_{uv} x^j_u\right) \sum_{u \neq v} \exp\left(\sum_{j=1}^{m} \alpha^{ij}_{uv} x^j_u\right)} & \text{if } u \neq v \text{ and } j \neq i, \\ 0 & \text{if } u \neq v \text{ and } j = i. \end{cases}$$

where we used the hypothesis that $\alpha^{ij}_{uv} = 0$ if $i = j$.

In this way, we have that

$$\frac{\partial F^i_v}{\partial x^u}(x) = \begin{cases} -1 & \text{if } u = v \text{ and } j = i, \\ \alpha^{ij}_{uv}\pi^i_v(x)(1 - \pi^i_v(x)) & \text{if } u = v \text{ and } j \neq i, \\ -\alpha^{ij}_{uv}\pi^i_v(x)\pi^i_u(x) & \text{if } u \neq v \text{ and } j \neq i, \\ 0 & \text{if } u \neq v \text{ and } j = i. \end{cases}$$

(44)
Since \( \sum_{u=1}^{d} \pi_u^i(x) = 1 \), we have that for all \( u \neq v \),

\[
\pi_u^i(x) \pi_u^v(x) = \pi_u^i(x) \left( 1 - \sum_{r \neq u} \pi_r(x) \right) \leq \pi_u^v(x) (1 - \pi_u^i(x)) \leq \frac{1}{4}.
\]

Putting it all together, we have that for each \( (v, i) \in [d] \times [m] \),

\[
\sum_{(u,j) \neq (v,i)} \left| \frac{\partial F_i^v}{\partial x_u^j} (x) \right| \leq \sum_{j=1}^{m} \frac{1}{4} |a_{ij}^v| + \sum_{u=1}^{d} \sum_{j=1}^{m} \frac{1}{4} |a_{ij}^v| = \sum_{u=1}^{d} \sum_{j=1}^{m} \frac{1}{4} |a_{ij}^v|.
\]

Hence, if Condition (C1) holds, then

\[
\sum_{(u,j) \neq (v,i)} \left| \frac{\partial F_i^v}{\partial x_u^j} (x) \right| \leq \frac{1}{4} \sum_{u=1}^{d} \sum_{j=1}^{m} |a_{ij}^v| < 1 = \left| \frac{\partial F_i^v}{\partial x_v^i} (x) \right|.
\]

This shows that the Jacobian matrix \( JF(x) = \left( \frac{\partial F_i^v}{\partial x_u^j} (x) \right) \) is strictly row diagonally dominant. By (3), \( JF(x) \) is also symmetric. Now we use a result from Linear Algebra that asserts that every symmetric strictly (row or column) diagonally dominant matrix with real entries and positive diagonal entries is positive-definite. By Theorem 7, we have that \( F \) is injective on \( \mathcal{O} \). We have proved that \( \pi \) has a unique fixed point in \( \mathcal{O} \). The proof is concluded by applying Theorem 2.

**Proof of Theorem 3.** Assume first that Condition (C2) is true. By definition, the constants \( \alpha_{ij}^v \), \( v \in [d] \) and \( i, j \in [m] \) satisfy (3). We claim that Condition (C3) is true. In fact, \( \alpha_{ij}^v = 0 \) for all \( v, i, \) and

\[
\sum_{u=1}^{d} \sum_{j=1}^{m} |\alpha_{ij}^v| = d(m - 1)\beta < 4 \quad \text{for each} \quad (v, i) \in [d] \times [m].
\]

By Theorem 4, the process \( X(n) = (X_1(n), \ldots, X_m(n)) \) converges almost surely to the unique equilibrium point of \( F \).

Now let us consider the case in which Condition (C1), rather than Condition (C2), is true. Since \( d = 2 \), we have that \( u, v \in \{1, 2\} \) and \( x_u^i = 1 - x_v^i \) for all \( u \neq v \) and \( j \in [m] \). In this way, the map \( \pi \) simplifies into

\[
\pi_u^i(x) = \frac{\exp \left( \sum_{j \neq i} -\beta x_v^j \right)}{\exp \left( \sum_{j \neq i} -\beta x_v^j \right) + \exp \left( \sum_{j \neq i} -\beta (1 - x_v^j) \right)}
\]

\[
= \frac{1}{1 + \exp \left( \sum_{j \neq i} -\beta (1 - 2x_v^j) \right)}
\]

Hence,

\[
\pi_u^i(x) = \psi \left( \sum_{j \neq i} x_v^j \right), \quad \text{where} \quad \psi(t) = \frac{1}{1 + \exp \left( -\beta (m - 1 - 2t) \right)}.
\]

Hence, \( x = (x_v^i) \) is a fixed point of \( \pi \) if and only if

\[
(45) \quad x_v^i = \psi \left( \sum_{j \neq i} x_v^j \right), \quad i = 1, 2, \ldots, m; \quad v = 1, 2.
\]
The function $\psi$ is monotone, hence invertible. Therefore, (45) is equivalent to
\begin{equation}
\psi^{-1}(x^1_v) = \sum_{j \neq i} x^j_v, \quad i = 1, 2, \ldots, m; \quad v = 1, 2.
\end{equation}
In this way, for all $1 \leq i, k \leq m$ with $i \neq k$ we have that
\[ \psi^{-1}(x^i_v) - \psi^{-1}(x^k_v) = x^k_v - x^i_v. \]
That is to say,
\[ \psi^{-1}(x^i_v) + x^i_v = \psi^{-1}(x^k_v) + x^k_v. \]
Defining $\varphi : \mathbb{R} \to \mathbb{R}$ as $\varphi(t) = \psi(t) + t$ leads to
\[ \varphi(x^i_v) = \varphi(x^k_v). \]
Since $\beta \leq 2$, we have that
\[ \varphi'(t) = \psi'(t) + 1 = -2\beta \frac{\exp \left( -\beta(m - 1 - 2t) \right)}{\left[ 1 + \exp \left( -\beta(m - 1 - 2t) \right) \right]^2} + 1 > 0, \]
implying that $\varphi$ is monotone, hence injective. Therefore,
\[ x^i_v = x^k_v, \quad i \neq k, \quad v = 1, 2. \]
We have proved that
\begin{equation}
\label{eq:47}
x^1_v = x^2_v = \ldots = x^m_v, \quad v = 1, 2.
\end{equation}
We claim that $x^1_v = \frac{1}{2}$ for all $v, i$. By way of contradiction, without loss of generality, suppose that $x^1_v > \frac{1}{2}$. Then, by (47), we have that $\sum_{j>1} x^j_v > \frac{m-1}{2}$. Replacing this in (45) and using the fact that $\psi$ is decreasing gives
\[ \frac{1}{2} < x^1_v = \psi\left( \sum_{j>1} x^j_v \right) < \psi\left( \frac{m-1}{2} \right) = \frac{1}{2}, \]
which is a contradiction. This shows that $x^* = \left( \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right)$ is the unique fixed point of $\pi$ and the unique equilibrium of $F$. The application of Theorem 2 concludes the proof. \hfill \Box

\subsection*{5.2. The proof of Theorem 6}

We present first a couple of lemmas that will be used in the proof of Theorem 6 and then conclude its proof. Throughout this section, we will use $p \in \mathcal{D}$ to denote the point
\[ p = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right). \]

\begin{lemma}
Let $X = \{X(n)\}_{n \geq 0}$ be the process defined in (4) and $F$ the vector field defined in (6). There is $\beta_0 > 2$ such that for all $\beta \geq \beta_0$ the following statements hold
\begin{itemize}
  \item[(i)] There exists a unique $w_\beta \in \left( 0, \frac{1}{\beta} \right)$ such that $w_\beta = 1/(1 + e^{2\beta(\frac{1}{2} - w_\beta)})$;
  \item[(ii)] The set $S$ defined as
    \[ S = \left\{(a, 1-a, b, 1-b, c, 1-c) : \{a, b, c\} = \left\{ \frac{1}{2}, w_\beta, 1-w_\beta \right\}, a \neq b \neq c \right\} \]
    consists of linearly stable equilibrium points of the vector field $F$
  \item[(iii)] $\mathbb{P}\left( \lim_{n \to \infty} X(n) = x \right) > 0$ for each $x \in S$.
\end{itemize}
\end{lemma}
Proof. (i) Since
\[
\lim_{\beta \to \infty} \frac{\beta^3}{1 + e^{(\beta - \frac{2}{3})^2}} = 0,
\]
there exists \( \beta_1 > 2 \) so large that
\[
\frac{1}{1 + e^{(\beta - \frac{2}{3})^2}} < \frac{1}{2\beta^3}.
\]
In what follows, we assume that \( \beta \geq \beta_1 \).

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be the function defined by \( \varphi(t) = 1/(1 + e^{2\beta(\frac{t}{3} - t)}) - t. \) By (48), we have that
\[
\varphi(0) > 0 \quad \text{and} \quad \varphi\left(\frac{1}{\beta^2}\right) = \frac{1}{1 + e^{(\beta - \frac{2}{3})^2}} = \frac{1}{\beta^3} < 0.
\]
In this way, by the continuity of \( \varphi \), we have that \( \varphi \) has a zero in the interval \( (0, \frac{1}{\beta^2}) \). Using (48) once more, we have that
\[
\varphi'(t) = 2\beta \frac{e^{2\beta(\frac{t}{3} - t)}}{1 + e^{2\beta(\frac{t}{3} - t)}} \cdot \frac{1}{1 + e^{2\beta(\frac{t}{3} - t)}} - 1 \leq \frac{2\beta}{1 + e^{(\beta - \frac{2}{3})^2}} - 1 < 0, \quad \forall t \in (0, \frac{1}{\beta^3}).
\]
Hence, \( \varphi \) is strictly decreasing on \( (0, \frac{1}{\beta^2}) \). Therefore, there exists a unique \( w_\beta \in (0, \frac{1}{\beta^2}) \) such that \( \varphi(w_\beta) = 0 \), i.e., \( w_\beta = 1/(1 + e^{2\beta(\frac{1}{3} - w_\beta)}) \).

(ii) We will prove now that every point of \( S \) is an equilibrium point of \( F \). By the definition of the vector field \( F \), we have that \( x = (a, 1 - a, b, 1 - b, c, 1 - c) \) is an equilibrium point of \( F \) if
\[
a = \psi(b + c), \quad b = \psi(a + c), \quad \text{and} \quad c = \psi(a + b),
\]
where \( \psi : [0, 2] \to \mathbb{R} \) is defined by
\[
\psi(t) = \frac{1}{1 + e^{2\beta(t - 1)}}.
\]
Since \( x \in S \), we have that \( a + b + c = \frac{3}{2} \). Hence, the following conditions are sufficient for \( x \in S \) to be an equilibrium point of \( F \):
\[
a = \psi\left(\frac{3}{2} - a\right) = \varphi(a), \quad b = \psi\left(\frac{3}{2} - b\right) = \varphi(b), \quad c = \psi\left(\frac{3}{2} - c\right) = \varphi(c),
\]
where \( \varphi \) is the function used in the definition of \( w_\beta \).

In other words, \( x = (a, 1 - a, b, 1 - b, c, 1 - c) \) is an equilibrium point of \( F \) if and only if each value \( u \in \{a, b, c\} = \{\frac{3}{2}, w_\beta, 1 - w_\beta\} \) is a fixed point of \( \varphi \). It is easy to verify that \( u = \frac{1}{2} \) is a fixed point of \( \varphi \). Moreover, \( w_\beta \) is a fixed point of \( \varphi \) by definition. Finally, \( u = 1 - w_\beta \) is a fixed point of \( \varphi \) because
\[
\varphi(1 - w_\beta) = \frac{1}{1 + e^{-2\beta(\frac{1}{3} - w_\beta)}} = \frac{e^{2\beta(\frac{1}{3} - w_\beta)}}{1 + e^{2\beta(\frac{1}{3} - w_\beta)}} = 1 - \frac{1}{1 + e^{2\beta(\frac{1}{3} - w_\beta)}} = 1 - \varphi(1 - w_\beta) = 1 - w_\beta.
\]
We have proved that every point in \( S \) is a fixed point of \( F \).
It remains to prove that if \( x = x(\beta) = (a, 1 - a, b, 1 - b, c, 1 - c) \in S \), then \( x \) is linearly stable for \( \beta \) big enough. In fact, the Jacobian matrix of \( F \) at \( x \) is given by

\[
\begin{pmatrix}
-1 & 0 & \psi'(b + c) & 0 & \psi'(b + c) & 0 \\
0 & -1 & \psi'(2 - b - c) & 0 & \psi'(2 - b - c) & 0 \\
\psi'(a + c) & 0 & -1 & \psi'(a + c) & 0 & 0 \\
0 & \psi'(2 - a - c) & 0 & -1 & \psi'(2 - a - c) & 0 \\
\psi'(a + b) & 0 & \psi'(a + b) & 0 & -1 & 0 \\
0 & \psi'(2 - a - b) & 0 & \psi'(2 - a - b) & 0 & -1
\end{pmatrix}
\]

Using the elementary facts that \( \psi'(t) = -2\beta\psi(t)(1 - \psi(t)) \), \( \psi(2 - t) = 1 - \psi(t) \), \( a = \psi(b + c) \), \( b = \psi(a + c) \), \( c = \psi(a + b) \) and further defining

\[
(49) \quad a = -2\beta a(1 - a), \quad b = -2\beta b(1 - b), \quad c = -2\beta c(1 - c),
\]

the Jacobian can be written as

\[
\begin{pmatrix}
-1 & a & 0 & a & 0 \\
0 & -1 & a & 0 & a \\
a & 0 & -1 & 0 & b \\
b & 0 & -1 & 0 & b \\
c & 0 & c & 0 & -1 \\
c & 0 & c & 0 & -1
\end{pmatrix}
\]

The characteristic polynomial of the Jacobian matrix is therefore

\[
(50) \quad p_x(\lambda) = \left(\frac{a b + a c + 2 a b c + a b \lambda + a c \lambda - (1 + \lambda)(-b c + (1 + \lambda)^2)}{\lambda} \right)^2.
\]

Since \( \{a, b, c\} = \left\{\frac{1}{2}, w_\beta, 1 - w_\beta\right\} \), \( w_\beta \in \left(0, \frac{1}{\beta}\right) \) and \( 1 - w_\beta \in (0, 1) \), we have that

\[
ab \leq 4\beta \max \left\{\frac{1}{4}, \frac{1}{\beta}\right\} \frac{1}{\beta} \leq \frac{1}{\beta}
\]

Likewise, we have that

\[
(51) \quad ab < \frac{1}{\beta}, \quad ac < \frac{1}{\beta}, \quad bc < \frac{1}{\beta} \quad \text{and} \quad abc < \frac{1}{\beta}.
\]

Each equilibrium point \( x = x(\beta) = (a, 1 - a, b, 1 - b, c, 1 - c) \in S \) depends on \( \beta \). In particular, when \( \beta \to \infty \), we obtain by (50) and (51) that

\[
\lim_{\beta \to \infty} p_{x(\beta)}(\lambda) = (1 + \lambda)^6.
\]

Hence, since the entries of \( JF(x(\beta)) \) depend smoothly on \( \beta \), we conclude that if \( \beta \) is big enough, say \( \beta \geq \beta_0 \), then all the eigenvalues of \( JF(x(\beta)) \) will lie in an open ball centered at \(-1 \in \mathbb{C} \) of radius \( \frac{1}{2} \). Therefore, they all will have negative real parts, that is, \( x(\beta) \in S \) will be a linearly stable equilibrium point.

(iii) This follows from the first item in Theorem 2. \( \square \)

**Lemma 11.** If \( \beta < 2 \), then \( p \) is linearly stable. If \( \beta > 2 \), then \( p \) is linearly unstable.
Proof. The proof consists in studying $\sigma(JF(p))$, the set of eigenvalues of the Jacobian matrix of $F$ at $p$. Relatively simple calculations show that the characteristic polynomial of $JF(p)$ equals

$$(1 + \lambda)^3 \left( -1 + \frac{\beta}{2} - \lambda \right)^2 (1 + \beta + \lambda).$$

The equilibrium $p$ is therefore hyperbolic. Further, up to algebraic multiplicity, the eigenvalues of $JF(p)$ are

$$-1, \quad -1 - \beta \quad \text{and} \quad -1 + \frac{\beta}{2}.$$ 

This shows that $p$ is linearly stable when $\beta < 2$ and linearly unstable when $\beta > 2$. \hfill \square

Proof of Theorem 6. Let $W = \{W(n)\}_{n \geq 0}$ with $W(n) = (W^1(n), W^2(n), W^3(n))$ be the process defined by $m = 3$ interacting random walks, taking values on the complete graph $G$ with vertices $V = \{1, 2\}$, such that for all $n \geq 1$ and any $i \in [m]$,

$$\{W^i(n) = 1\} = \{S^i_n - S^i_{n-1} = -1\},$$

$$\{W^i(n) = 2\} = \{S^i_n - S^i_{n-1} = +1\}.$$ 

Now, let $X^i(0) = 1$ for all $v \in [d] = \{1, 2\}$ and $i \in [m]$, and then, for $n \geq 1$ define $X^i(n)$ as in (1). Notice that $X^i_1(n)$ and $X^i_2(n)$ are the proportions of times the $i$-th walk, that is $S^i(n)$, makes a transition to the left and to the right, respectively. Finally, for $v \in [d]$ and $i, j \in [m]$, set

$$\alpha^i_j = \begin{cases} -\beta, & \text{if } i \neq j, \\ 0, & \text{if } i = j \end{cases}$$

and let $\beta \geq 0$. Using (10) and (11), it is readily seen that the transition probability for $W^i$ is given by (2). Indeed, since $X^i_1(n) = 1 - X^i_2(n)$,

$$\mathbb{P}(W^i(n+1) = 2 \mid \mathcal{F}_n) = \mathbb{P}(S^i_n - S^i_{n-1} = +1 \mid \mathcal{A}_n)$$

$$= \mu \left( (S^j_n - S^j_{n-1})/n + (S^k_n - S^k_{n-1})/n \right)$$

$$= \mu \left( 2X^j_2(n) - 1 + 2X^k_2(n) - 1 \right)$$

$$= \frac{\exp \left( -\beta(X^j_2(n) + X^k_2(n)) \right)}{\sum_{v=1}^2 \exp \left( -\beta(X^v_2(n) + X^v_2(n)) \right)} = \pi^i_2(X(n)).$$

Observe that Lemma 6 holds in this case. That is, $X$ is a stochastic approximation with $F(x) = -x + \pi(x)$ and with $\xi, \upsilon_n$, and $\gamma_n$ given as in Lemma 6.

To prove the first assertion of the theorem, let $\beta < 2$. By Theorem 3, $p$ is the only equilibrium, which by Lemma 11 is linearly stable. Using item (iii) of Theorem 2 we have that $X(n) \to p$ almost surely. As a consequence, almost surely it holds that

$$\lim_{n \to \infty} \mathbb{P}(S^i_n - S^i_{n-1} = +1 \mid \mathcal{A}_n) = \lim_{n \to \infty} \mathbb{P}(W^i(n+1) = 2 \mid \mathcal{F}_n)$$

$$= \lim_{n \to \infty} \pi^i_2(X(n)) = \pi^i_2(p) = \frac{1}{2}$$

where the last two equalities follow by continuity of $\pi$ and the fact that $p = (\frac{1}{2}, \ldots, \frac{1}{2})$ is a fixed point of $\pi$. This concludes the proof of the first part.

The second assertion of the theorem is proved as follows. For sufficiently large $\beta$, namely when $\beta \geq \beta_0 > 2$, Lemma 11 and Theorem 2(ii) rule out the possibility of
converging to $p$. Further, by the proof of Lemma 10, there is a $w > \frac{1}{2}$, such that $x = (w, 1-w, 1-w, w, \frac{1}{2}, \frac{1}{2})$ is a linearly stable equilibrium. By item (i) of Theorem 2, it follows that $X(n) \rightarrow x$ with positive probability. As shown previously we have that $\mathbb{P}(W^1(n+1) = v | S_n) = \pi_1^1(X(n))$. Again, by continuity of $\pi$ and using the fact that $x$ is a fixed point of $\pi$, we have, with positive probability, that

$$
\lim_{n \rightarrow \infty} \mathbb{P}(S_n^1 - S_{n-1}^1 = 1 | A_n) = \lim_{n \rightarrow \infty} \mathbb{P}(W^1(n+1) = 2 | S_n) = \lim_{n \rightarrow \infty} \pi_1^1(X(n)) = \pi_1^1(x) = 1 - w < \frac{1}{2}.
$$

Likewise,

$$
\lim_{n \rightarrow \infty} \mathbb{P}(S_n^2 - S_{n-1}^2 = 1 | A_n) = w > \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{P}(S_n^3 - S_{n-1}^3 = 1 | A_n) = \frac{1}{2}.
$$

This concludes the proof. \hfill \Box

5.3. The proof of Theorem 5. The following lemma will be used for the proof of Theorem 5. This lemma shows that the set of equilibria for the example of two repelling walks on the two vertex graph presented in Section 2.2.2, is finite for all $\beta \geq 0$. This lemma also identifies the form of the equilibria and characterises their stability.

**Lemma 12.** Let $\Phi$ be the semi-flow induced by the ODE (7) with $F$ given by (9). Then,

(i) The point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the only equilibrium of $\Phi$ when $\beta \in [0, 2]$.

(ii) When $\beta > 2$, there exist two further equilibria of the form $(a, 1-a, 1-a, a)$ and $(1-a, a, a, 1-a)$, where $a \in (0, \frac{1}{2})$ is uniquely determined by $\beta$.

(iii) $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is linearly stable when $\beta \in [0, 2)$ and linearly unstable when $\beta > 2$.

The equilibria in (ii) are linearly stable when $\beta > 2$.

**Proof.** See Lemma 8 in Coletti et al. (2020). \hfill \Box

**Proof of Theorem 5.** From Lemma 12 it follows that $F^{-1}(0)$ is formed by isolated points. The proof is therefore concluded by direct application of item (iii) in Theorem 2. \hfill \Box

**Remark 1.** The convergence of $X = \{X(n)\}_{n \geq 0}$ towards $F^{-1}(0)$ can be established in this example without using Theorems 1 and 2. An alternative proof is obtained from Theorem 6.12, Corollary 6.13 and Theorem 6.15 in Benaim & Hirsch (1999); see also Theorem 3.2 and Corollary 3.3 in Benaim (1999). In order to be able to use these results, the semi-flow $\Phi$ defined by vector field determined by the interacting random walks has to be planar. This is indeed the case in this example. To observe this, it suffices to identify $\mathcal{D}$ with $[0, 1]^2$ by using the map $\eta : (a, 1-a) \mapsto a$ for $a \in [0, 1]$, and then consider the projection of the field on $[0, 1]^2$ given by $F = (F_1, F_2)$. These steps cannot be carried out in the examples of Section 2.2.1 nor in the example presented in Section 2.2.3. More generally, the arguments presented in the mentioned literature cannot be used when $m \geq 3$. The main reason is that the dynamics induced by three or more interacting random walks cannot be identified with a subset of the plane. Indeed, when $m \geq 3$, the domain $\mathcal{D}$ may be identified via $\eta$ with the $m$ dimensional unit-cube $[0, 1]^m \subset \mathbb{R}^m$. 

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