On the entropy power inequality for the Rényi entropy of order $[0, 1]$

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Abstract

Using a sharp version of the reverse Young inequality, and a Rényi entropy comparison result due to Madiman and Wang, the authors are able to derive a Rényi entropy power inequality for log-concave random vectors when Rényi parameters belong to $(0, 1)$. Furthermore, the estimates are shown to be somewhat sharp.

Keywords. Entropy power inequality, Rényi entropy, log-concave.

1 Introduction

Let $r \in [0, \infty]$. The Rényi entropy \cite{30} of parameter $r$ is defined for continuous random vectors $X \sim f_X$ as

$$h_r(X) = \frac{1}{1-r} \log \left( \int_{\mathbb{R}^n} f_X \right).$$

We take the Rényi entropy power of $X$ to be

$$N_r(X) = e^{\frac{2}{n} h_r(X)} = \left( \int_{\mathbb{R}^n} f_X(x)^r \, dx \right)^{-\frac{2}{n} \frac{1}{1-r}}.$$

Three important cases are handled by continuous limits,

$$N_0(X) = \text{Vol}^\frac{2}{n} \left( \text{supp}(X) \right),$$

$$N_\infty(X) = \|f_X\|_{\infty}^{-\frac{2}{n}},$$

and $N_1(X)$ corresponds to the usual Shannon entropy power $N_1(X) = N(X) = e^{-\frac{2}{n} \int f \log f}$.

Here, Vol$(A)$ denotes the volume\footnote{More precisely the Lebesgue measure of a measurable set $A$.} of $A$, and supp$(X)$ denotes the support of $X$.

The entropy power inequality (EPI) is the statement that Shannon entropy power of independent random vectors $X$ and $Y$ is super-additive

$$N(X + Y) \geq N(X) + N(Y).$$

In this language we interpret the Brunn-Minkowski inequality of Convex Geometry, classically stated as the fact that

$$\text{Vol}^\frac{2}{n}(A + B) \geq \text{Vol}^\frac{2}{n}(A) + \text{Vol}^\frac{2}{n}(B)$$

for any pair of compact sets of $\mathbb{R}^n$ (see \cite{21} for an introduction to the literature surrounding this inequality), as a Rényi-EPI corresponding to $r = 0$. That is, the Brunn-Minkowski inequality is equivalent to the fact that for $X$ and $Y$ independent random vectors, the square root of the 0-th Rényi entropy is super-additive,

$$N_0^\frac{1}{2}(X + Y) \geq N_0^\frac{1}{2}(X) + N_0^\frac{1}{2}(Y).$$
The parallels between the two famed inequalities had been observed in the 1984 paper of Costa and Cover [14], and a unified proof using sharp Young’s inequality was given in 1991 by Dembo, Cover, and Thomas [16]. Subsequently, analogs of further Shannon entropic inequalities and properties in Convex Geometry have been pursued. For example the monotonicity of entropy in the central limit theorem (see [1] [24] [33]), motivated the investigation of quantifiable convexification of a general measurable set on repeated Minkowski summation with itself (see [18] [17]). Motivated by Costa’s EPI improvement [13], Costa and Cover conjectured that the volume of general sets when summed with a dilate of the Euclidean unit ball should have concave growth in the dilation parameter [14]. Though this was disproved for general sets in [20], open questions of this nature remain.

Conversely, V. Milman’s reversal of the Brunn-Minkowski inequality (for symmetric convex bodies under certain volume preserving linear maps) [28] inspired Bobkov and Madiman to ask and answer whether the entropy power inequality could be reversed for log-concave random vectors under analogous mappings [6]. In [5] The authors also formulated an entropic version of Bourgain’s slicing conjecture [10], a longstanding open problem in convex geometry that has attracted a lot of attention.

A further example of an inequality at the interface of geometry and information theory can be found in [2], where Ball, Nayar, and Tkocz conjectured the existence of an entropic Busemann’s inequality [12] for symmetric log-concave random variables and prove some partial results, see [34] for an extension to “s-concave” random variables.

We refer to the survey [25] for further details on the connections between convex geometry and information theory.

Recently the super-additivity of more general Rényi functionals has seen significant activity, starting with Bobkov and Chistyakov [8] [9] where it is shown (the former focusing on $r = \infty$ the latter on $r \in (1, \infty)$) that for $r \in (1, \infty]$ there exist universal constants $c(r) \in (\frac{1}{e}, 1)$ such that for $X_i$ independent random vectors

$$N_r(X_1 + \cdots + X_m) \geq c(r) \sum_{i=1}^{m} N_r(X_i).$$

This was followed by Ram and Sason [29] who used optimization techniques to sharpen bounds on the constant $c(r)$, which should more appropriately be written $c(r, m)$ as the authors were able to clarify the dependency on the number of summands as well as the Rényi parameter $r$. Bobkov and Marsiglietti [7] showed that for $r \in (1, \infty)$, there exist an $\alpha$ modification of the Rényi entropy power that preserved super-additivity. More precisely taking $\alpha = \frac{r+1}{2}$, $r \in [1, \infty)$, and $X, Y$ independent random vectors

$$N_r^\alpha(X + Y) \geq N_r^\alpha(X) + N_r^\alpha(Y).$$

This was sharpened by Li [23] who optimized the argument of Bobkov and Marsiglietti. The infinity case was studied using functional analytic tools by Madiman, Melbourne, and Xu [26] [36] who showed that the $N_\infty$ functional enjoys an analog of the matrix generalizations of Brunn-Minkowski and the Shannon-Stam EPI due to Feder and Zamir [37] [38] and began investigation into discrete versions of the inequality in [35].

Conspicuously absent from the discussion above, and mentioned as an open problem in [9] [29] [23] [24] are super-additivity properties of the Rényi entropy power when $r \in (0, 1)$. In this paper, we address this problem, and provide a solution in the log-concave case (see Definition 3). Our main result is the following.

**Theorem 1.** Let $r \in (0, 1)$. Let $X, Y$ be log-concave random vectors in $\mathbb{R}^n$. Then,

$$N_r(X + Y)^\alpha \geq N_r(X)^\alpha + N_r(Y)^\alpha,$$

(1)
where

\[
\alpha \triangleq \alpha(r) = \left[ 1 + \frac{1}{(1-r)\log(2)} \left( (r+1) \log \left( \frac{r+1}{2r} \right) + \log(r) \right) \right]^{-1} = \frac{(1-r)\log 2}{(1+r)\log(1+r) + r \log \frac{1}{4r}}.
\]

What is more, and in contrast to some previous optimism (see, e.g., [23]), these estimates are somewhat sharp even for log-concave random vectors. Letting \( \alpha_{opt} \) denote the infimum over all \( \alpha \) satisfying the inequality (1) for log-concave random vectors, we have

\[
\frac{(1-r)\log 2}{2\log \Gamma(1+r) + 2r \log \frac{1}{r}} \leq \alpha_{opt}.
\]

Observe that this is in stark contrast to the \( r = 0 \) case where \( \alpha_{opt} = 1/2 \) by the Brunn-Minkowski inequality. Indeed we have \( \alpha_{opt} r^{1-\varepsilon} \to \infty \) with \( r \to 0 \) for any \( \varepsilon > 0 \). However, in the case that the random vectors are uniformly distributed we do have better behavior.

**Theorem 2.** Let \( r \in (0, 1) \). Let \( X, Y \) be uniformly distributed random vectors. Then,

\[
N_r(X + Y)^\beta \geq N_r(X)^\beta + N_r(Y)^\beta,
\]

where

\[
\beta \triangleq \beta(r) = \left[ 1 - \frac{1}{(1-r)\log(2)} \left( (r+1) \log \left( \frac{r+1}{2r} \right) + \log(r) \right) \right]^{-1} = \frac{2\log 2 + r \log r - (r+1) \log(r+1)}{2 + r \log r - (r+1) \log(r+1)}.
\]

Observe that \( \lim_{r \to 0} \beta = 1/2 \) and we recover the Brunn-Minkowski inequality, while \( \lim_{r \to 1} \beta = 1 \) gives the usual entropy power inequality in the special case that the random vectors are uniform distributions. Thus Theorem 2 can be understood as a family of Rényi entropy power inequalities for uniform distributions, interpolating between the Brunn-Minkowski and entropy power inequalities.

### 2 Preliminaries

For \( p \in [0, \infty] \), we denote by \( p' \) the conjugate of \( p \),

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

For a non-negative function \( f : \mathbb{R}^n \to [0, +\infty) \) we introduce the notation

\[
\|f\|_p = \left( \int_{\mathbb{R}^n} f^p(x)dx \right)^{1/p}.
\]

**Definition 3.** A random vector \( X \) in \( \mathbb{R}^n \) is log-concave if it possess a log-concave density \( f_X : \mathbb{R}^n \to [0, +\infty) \) with respect to Lebesgue measure. That is that for all \( \lambda \in (0, 1) \) and \( x, y \in \mathbb{R}^n \),

\[
f_X((1-\lambda)x + \lambda y) \geq f_X^{1-\lambda}(x)f_X^{\lambda}(y).
\]

Equivalently \( f_X \) can be written in the form \( e^{-V} \), where \( V \) is a proper convex function.
Log-concave random vectors and functions are important classes in many disciplines. In the context of information theory, several nice properties involving entropy of log-concave random vectors were recently established (see, e.g., [5], [3], [31], [32], [15], [27]). Significant examples are Gaussian and exponential distributions as well as any uniform distribution on a convex set.

The main tool in establishing Theorems 1 and 2 is the reverse form of the sharp Young inequality. The reversal of Young’s inequality for parameters in [0,1] is due to Leindler [22], while sharp constants were obtained independently by Beckner, and Brascamp and Lieb:

**Theorem 4 ([11]).** Let 0 ≤ p, q, r ≤ 1 such that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). Then,

\[
\| f \ast g \|_r \geq C_{r}^{\frac{p}{q}} \| f \|_p \| g \|_q,
\]

where

\[
C = C(p, q, r) = \frac{c_p c_q}{c_r}, \quad c_m = \frac{m^{1/m}}{|m'|^{1/m'}}.
\]

Let us recall the information theoretic interpretation of Young’s inequality. Given independent random vectors \( X \) with density \( f \) and \( Y \) with density \( g \), the random vector \( X + Y \) will be distributed according to \( f \ast g \). Observe that the \( L_p \) “norms” have the following expression as Rényi entropy powers, \( \| f \|_r = N_r(X)^{-\frac{1}{2r}} = N_r(X)^{-\frac{1}{2m'}} \). Hence, we can rewrite (2) as follows,

\[
N_r(X + Y)^{\frac{n}{2m}} \geq C^\frac{n}{2m} N_p(X)^{\frac{n}{2m'}} N_q(Y)^{\frac{n}{2m'}}.
\]

Equivalently,

\[
N_r(X + Y)^{\frac{n}{2m}} \geq C_{n} N_p(X)^{\frac{n}{2m'}} N_q(Y)^{\frac{n}{2m'}}.
\]

We also need a Rényi comparison result for log-concave random vectors.

**Lemma 5** (Madiman-Wang). Let 0 < p < q. Then, for every log-concave random vector \( X \),

\[
N_q(X) \leq N_p(X) \leq \frac{p^{1/q}}{q^{1/p}} N_q(X).
\]

The first inequality is classical and holds for general \( X \), a resultant of the fact that \( N_p(X) \) can be expressed as the reciprocal of a \( p-1 \) norm, explicitly \( h_p(X) = (E f^{p-1}(X))^{-1/(p-1)} \). The increasingness of norms (which follows from Jensen’s inequality) implies the decreasingness of Rényi entropy powers. The content of Madiman and Wang’s result is thus the second inequality, that this decrease is not too rapid for log-concave random vectors, and its proof will be given for completeness of exposition in the appendix.

We will also have use for a somewhat technical but elementary Calculus result.

**Lemma 6.** Let \( c > 0 \). Let \( L, F : [0, c] \to [0, \infty) \) be twice differentiable on \( (0, c] \), continuous on \( [0, c] \), such that \( L(0) = F(0) = 0 \) and \( L'(c) = F'(c) = 0 \). Let us also assume that \( F(x) > 0 \) for \( x > 0 \), that \( F \) is strictly increasing, and that \( F' \) is strictly decreasing. Then \( \frac{L''}{F''} \) increasing on \( (0, c) \) implies that \( \frac{L}{F} \) is increasing on \( (0, c) \) as well. In particular,

\[
\max_{x \in [0,c]} \frac{L(x)}{F(x)} = \frac{L(c)}{F(c)}.
\]

The proof is an exercise in Cauchy’s mean value theorem.

**Proof.** For \( 0 < u < v < c \), by Cauchy’s mean value theorem

\[
\frac{L'(v) - L'(u)}{F'(v) - F'(u)} = \frac{L''(c_0)}{F''(c_0)}, \quad \frac{L'(c) - L'(v)}{F'(c) - F'(v)} = \frac{L''(c_1)}{F''(c_1)}.
\]
for some $c_0 \in (u, v)$ and $c_1 \in (v, c)$. Thus by the assumed monotonicity of $\frac{L'}{F'}$ and the fact that $L'(c) = F'(c) = 0$, the above implies

$$\frac{L'(v)}{F'(v)} \geq \frac{L'(v) - L'(u)}{F'(v) - F'(u)}.$$ 

Since $F'$ is non-negative and strictly decreasing on $(0, c)$, we have by algebra

$$-L'(v)F'(u) \leq -L'(u)F'(v),$$

which implies $L'(v)/F'(v) \geq L'(u)/F'(u)$ since $F' \geq 0$. Thus $L'/F'$ is non-decreasing on $(0, c)$. Now we can apply a similar argument to show that $L/F$ is non-decreasing. Again Cauchy’s mean value theorem, for $0 < u < v < c$ we have

$$\frac{L(v) - L(u)}{F(v) - F(u)} \geq \frac{L(u)}{F(u)},$$

Since $F$ is non-negative and strictly increasing on $(0, c)$, we have

$$L(v)F(u) \geq L(u)F(v).$$

Thus it follows that $L/F$ is indeed non-decreasing.

\[\square\]

3 Proof of Theorem

We first combine the information theoretic formulation of reverse Young’s inequality and Lemma to obtain,

$$N_r(X + Y)^{\frac{1}{|r'|}} \geq C \left( \frac{p^{\frac{2}{p-2}}}{r^{\frac{r}{r-1}}} \right)^{\frac{1}{|r'|}} \left( \frac{q^{\frac{2}{q-2}}}{r^{\frac{r}{r-1}}} \right)^{\frac{1}{|r'|}} N_r(X)^{\frac{1}{|p'|}} N_r(Y)^{\frac{1}{|q'|}}. \quad (4)$$

Since

$$\frac{1}{|p'|} = -\frac{1}{p'} = \frac{1}{p} - 1 = \frac{1 - p}{p},$$

we deduce that

$$\frac{1}{|p'|(|p - 1|)} = \frac{1}{p}.$$ 

Also, we have

$$\frac{1}{|p'|} + \frac{1}{|q'|} = \frac{1}{|r'|}.$$ 

Hence, we can rewrite (4) as,

$$N_r(X + Y)^{\frac{1}{|r'|}} \geq C \frac{p^{\frac{2}{p-2}}q^{\frac{2}{q-2}}}{r^{\frac{r}{r-1}}} N_r(X)^{\frac{1}{|p'|}} N_r(Y)^{\frac{1}{|q'|}} = A(p, q, r)N_r(X)^{\frac{1}{|p'|}} N_r(Y)^{\frac{1}{|q'|}},$$
where
\[ A(p, q, r) = \frac{c_p c_q}{c_r} \frac{r^{\frac{1}{2}}}{p^{\frac{1}{2}} q^{\frac{1}{2}}}. \]

Equivalently,
\[ N_r(X + Y) \geq A(p, q, r)^{|r'\rangle} N_r(X)^{|r'\rangle} N_r(Y)^{|r'\rangle}. \]  

Thus to complete our proof of Theorem 1 it suffices to obtain for a fixed \( r \in (0, 1) \), an \( \alpha > 0 \) such that for any given pair of independent log-concave random vectors \( X \) and \( Y \), there exist \( 0 \leq p, q \leq 1 \) such that
\[ \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'} \]
and
\[ A(p, q, r)^{\alpha |r'\rangle} N_r(X)^{\alpha |r'\rangle} N_r(Y)^{\alpha |r'\rangle} \geq N_r^\alpha(X) + N_r^\alpha(Y). \]  

Let us observe that there is nothing probabilistic about equation (6). If we write
\[ x = N_r(X)^\alpha, \quad y = N_r(Y)^\alpha, \]
our Rényi entropy power inequality is implied by the following algebraic inequality.

**Proposition 7.** Given \( r \in (0, 1) \) and taking
\[ \alpha = \frac{(1 - r) \log 2}{(1 + r) \log(1 + r) + r \log \frac{1}{4r}}, \]  
then for any \( x, y > 0 \) there exist \( 0 < p, q < 1 \) satisfying \( \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'} \) such that
\[ A(p, q, r)^{\alpha |r'\rangle} N_r(X)^{\alpha |r'\rangle} N_r(Y)^{\alpha |r'\rangle} \geq x + y. \]  

**Proof.** Using the homogeneity of equation (8), we may assume without loss of generality that
\[ x + y = \frac{1}{|r'\rangle}. \]
We then choose admissible \( p, q \) by selecting \( \frac{1}{p'} = -x \) and \( \frac{1}{q'} = -y \). Hence, equation (8) becomes
\[ A(p, q, r)^{\alpha |r'\rangle} N_r(X)^{\alpha |r'\rangle} N_r(Y)^{\alpha |r'\rangle} \geq x + y. \]
Hence, taking logarithm, we can choose
\[ \alpha = \sup \frac{\log \left( \frac{(x+y)^{x+y}}{x^x y^y} \right)}{\log(A(p, q, r))}, \]
where the sup runs over all \( x, y > 0 \) satisfying \( x + y = \frac{1}{|r'\rangle} \) (recall that \( r \in (0, 1) \) is fixed). We claim that this is exactly the \( \alpha \) defined in (7). To establish this fact, let us first rewrite \( A(p, q, r) \) in terms of \( x \) and \( y \). Notice that,
\[ p = \frac{1}{x + 1}, \quad q = \frac{1}{y + 1}, \quad r = \frac{1}{x + y + 1}. \]
We deduce that,
\[ c_p = \frac{p^{1/p}}{|p'\rangle^{1/p'}} = \frac{1}{x^{x+1}}, \quad c_q = \frac{1}{y^{y+1}(y + 1)^{y+1}}, \quad c_r = \frac{1}{(x + y)^{x+y} (x + y + 1)^{x+y+1}}. \]
It follows that
\[
\frac{c_p c_q}{c_r} = \frac{(x + y)^{x+y}(x + y + 1)^{x+y+1}}{x^y y^y (x + 1)^{x+1} (y + 1)^{y+1}}.
\]
Also,
\[
\frac{r^2}{p^r q^r} = \frac{(x + y)^{2(x+1)} (y + 1)^{2(y+1)}}{(x + y + 1)^{2(x+y+1)}}.
\]
Finally,
\[
A(p, q, r) = \frac{c_p c_q}{c_r} \frac{r^2}{p^r q^r} = \frac{(x + y)^{x+y}(x + y + 1)^{x+y+1}}{x^y y^y (x + 1)^{x+1} (y + 1)^{y+1}} \frac{(x + y)^{2(x+1)} (y + 1)^{2(y+1)}}{(x + y + 1)^{2(x+y+1)}}.
\]
Let us denote
\[
F(x) \triangleq \log \left( \frac{(x + y)^{x+y}}{x^y y^y} \right) = \frac{1}{|r|} \log \left( \frac{1}{|r|} \right) - x \log(x) - \left( \frac{1}{|r|} - x \right) \log \left( \frac{1}{|r|} - x \right),
\]
and
\[
G(x) \triangleq \log \left( \frac{(x + y)^{(x+1)y+1}(y + 1)^{y+1}}{x^y y^y (x + y + 1)^{x+y+1}} \right) = F(x) - L(x),
\]
where
\[
L(x) \triangleq \left( \frac{1}{|r'|} + 1 \right) \log \left( \frac{1}{|r'|} + 1 \right) - (x + 1) \log(x + 1) - \left( \frac{1}{|r'|} - x + 1 \right) \log \left( \frac{1}{|r'|} - x + 1 \right).
\]
Our claim is that
\[
\alpha = \sup F = \sup \frac{F}{F - L} = \left( 1 - \sup \frac{L}{F} \right)^{-1}.
\]
We invoke Lemma 6 to prove that the ratio \( L/F \) is increasing on \([0, 1/2|r'|] \). Indeed, taking derivatives it is easy to see that \( F \) is positive and increasing on \((0, 1/2|r'|]\), and its derivative \( F' \) is strictly decreasing on the same interval. What is more, \( \frac{L'}{F'} \) is non-decreasing on \((0, \frac{1}{2|r'|}]\). Indeed,
\[
\frac{L''(x)}{F''(x)} = \frac{1}{|r'|} + 2 \frac{x(\frac{1}{|r'|} - x)}{(x + 1)(\frac{1}{|r'|} - x + 1)},
\]
and one can see that this is non-decreasing when \(x \in (0, \frac{1}{2|r'|})\) again, by taking the derivative. Now by Lemma 6 applied to \( F, L, \) and \( c = \frac{1}{2|r'|} \) we have
\[
\sup \left( 1 - \frac{L(x)}{F(x)} \right)^{-1} = \left( 1 - \frac{L(c)}{F(c)} \right)^{-1} = \left( 1 - \frac{L(1/2|r'|)}{F(1/2|r'|)} \right)^{-1}.
\]
Let us compute \( F(c) \) and \( L(c) \), with \( c = \frac{1}{2|r'|} \). We have
\[
F(c) = 2c \log 2c - 2c \log(c) = 2c \log 2 = \frac{(1 - r) \log 2}{r}.
\]
and
\[
\begin{align*}
L(c) &= (2c + 1) \log(2c + 1) - 2(c + 1) \log(c + 1) \\
&= (2c + 1) \log \left( \frac{2c + 1}{c + 1} \right) - \log(c + 1) \\
&= \log \left( \frac{2}{1+r} \right) - \log \left( \frac{r + 1}{2r} \right).
\end{align*}
\]

Thus
\[
\alpha = \left( 1 - \frac{L(c)}{F(c)} \right)^{-1} = \left( 1 - \frac{\log \left( \frac{2}{1+r} \right) - \log \left( \frac{r + 1}{2r} \right)}{(1-r) \log 2} \right)^{-1} = \left( \frac{(1+r) \log (1+r) + r \log \left( \frac{1}{2r} \right)}{(1-r) \log 2} \right)^{-1}.
\]

\[
4 \text{ Proof of Theorem } [2]
\]

The proof is very similar to the proof of Theorem [1]. The improvement is by virtue of the fact that for $U$ a random vector uniformly distributed on a set $A \subseteq \mathbb{R}^n$, the Rényi entropy is determined entirely by the volume of $A$, and is thus independent of parameter. Indeed,
\[
N_r(U) = \left( \int_{\mathbb{R}^n} (1_A(x)/\text{Vol}(A))^r \, dx \right)^{2/n(1-r)} = \text{Vol}(A)^{2/n}.
\]

We again use the information-theoretic version of the sharp Young inequality (see [3]):
\[
N_r(X + Y)^{\frac{p}{|p|}} \geq C N_p(X)^{\frac{p}{|p|}} N_q(Y)^{\frac{q}{|q|}}.
\]

Now, since $X$ and $Y$ are uniformly distributed, we have
\[
N_p(X) = N_r(X), \quad N_q(Y) = N_r(Y).
\]

Hence,
\[
N_r(X + Y) \geq C^{\frac{|r|}{|p|}} N_r(X)^{\frac{p}{|p|}} N_r(Y)^{\frac{q}{|q|}}. \tag{9}
\]

Let us raise [3] to the power $\beta$, and put $x = N_r(X)^\beta$, $y = N_r(Y)^\beta$. As before, we can assume that $x + y = \frac{1}{|p|}$. Thus, it is enough to show that
\[
C^{\beta |r|} x^{\frac{|p|}{|p|}} y^{\frac{|q|}{|q|}} \geq \frac{1}{|r|},
\]

for some admissible $(p, q)$. Let us choose $p, q$ such that $x = \frac{1}{|p|}$ and $y = \frac{1}{|q|}$. The inequality is valid since
\[
\beta = \sup \frac{\log \left( \frac{(x+y)^x+y}{x^x y^y} \right)}{\log(C)} \geq \sup \frac{\log \left( \frac{(x+y)^x+y}{x^x y^y} \right)}{\log \left( \frac{(x+y)^x+y}{(x+1)^x+y+1} \right)}. \tag{10}
\]
where the sup runs over all \(x, y > 0\) satisfying \(x + y = \frac{1}{|r|}\) (recall that \(r \in (0, 1)\) is fixed). Indeed, as in Section 3 it is a consequence of Lemma 8 that the sup is attained at \(x = \frac{1}{2|r|}\) and from this the result follows.

5 Lower bound on the optimal exponent

Proposition 8. The optimal exponent \(\alpha_{opt}\) that satisfies (1) verifies,

\[
\max \left\{ 1, \frac{(1 - r) \log 2}{2 \log \Gamma(r + 1) + 2 r \log \frac{1}{r}} \right\} \leq \alpha_{opt} \leq \frac{2 (1 - r) \log 2}{(1 + r) \log(1 + r) + r \log \frac{1}{r}}.
\]

Let us remark that smooth interpolation of Brunn-Minkowski and the entropy power inequality as in Theorem 2 cannot hold for any class of random variables that contains the Gaussians. Indeed, let \(Z_1\) and \(Z_2\) be i.i.d. standard Gaussians. Hence, \(Z_1 + Z_2 \sim \sqrt{2} Z_1\), and

\[
N_r(\alpha_{opt})(Z_1 + Z_2) = 2^\alpha N_r(\alpha_{opt})(Z_1),
\]
while

\[
N_r(\alpha_{opt})(Z_1) + N_r(\alpha_{opt})(Z_2) = 2 N_r(\alpha_{opt})(Z_1).
\]

It follows that for a modified Rényi EPI to hold, even when restricted to the class of log-concave random vectors, we must have \(2^\alpha \geq 2\). That is, \(\alpha \geq 1\).

We now show by direct computation on the exponential distribution on \((0, \infty)\) the lower bounds on \(\alpha_{opt}\).

Let \(X \sim f_X\) be a random variable with exponential distribution, \(f_X(x) = \mathbb{1}_{(0, \infty)}(x)e^{-x}\). The computation of the Rényi entropy of \(X\) is an obvious change of variables,

\[
N_r(X) = \left( \int f_X^r \right)^{\frac{2}{1-r}} = \left( \int_0^\infty e^{-rx} dx \right)^{\frac{2}{1-r}} = \left( \frac{1}{r} \right)^{\frac{2}{1-r}}.
\]

Let \(Y\) be an independent copy of \(X\). The density of \(X + Y\) is

\[
f \ast f(x) = \int_{-\infty}^{\infty} \mathbb{1}_{(0, \infty)}(x - y)e^{-(x+y)} \mathbb{1}_{(0, \infty)}(y)e^{-y} dy
\]

\[
= \int_0^x e^{-x} dy = \mathbb{1}_{(0, \infty)}x e^{-x}.
\]

Hence,

\[
N_r(X + Y) = \left( \int \mathbb{1}_{(0, \infty)}(x)x^r e^{-rx} dx \right)^{\frac{2}{1-r}}
\]

\[
= \left( \frac{1}{r^{r+1}} \int_0^\infty x^r e^{-x} dx \right)^{\frac{2}{1-r}}
\]

\[
= \left( \frac{\Gamma(r + 1)}{r^{r+1}} \right)^{\frac{2}{1-r}}.
\]

Thus, the optimal exponent \(\alpha_{opt}\) satisfies

\[
N_r^{\alpha_{opt}}(X + Y) \geq 2 N_r^{\alpha_{opt}}(X).
\]
This is exactly
\[ \left( \frac{\Gamma(r+1)}{r^{r+1}} \right)^{\frac{2\alpha_{\text{opt}}}{1-r}} \geq 2 \left( \frac{1}{r} \right)^{\frac{2\alpha_{\text{opt}}}{1-r}}. \]

Canceling and taking logarithms, this rearranges to
\[ \log \Gamma(r+1) + r \log \frac{1}{r} \geq (1-r) \log 2 \frac{1}{2\alpha_{\text{opt}}}, \]
which implies that we must have
\[ \alpha_{\text{opt}} \geq \frac{(1-r) \log 2}{2(\log \Gamma(r+1) + r \log \frac{1}{r})}. \]

Note that by the log-convexity of \( \Gamma \) and the fact that \( \Gamma(1) = \Gamma(2) = 1 \), we have
\[ \log(\Gamma(1+r)) \leq 0, \]
which implies
\[ \alpha_{\text{opt}} \geq \frac{(1-r) \log 2}{2r \log \frac{1}{r}}. \]

In particular we must have \( \alpha_{\text{opt}} r^{1-\varepsilon} \to \infty \) with \( r \to 0 \), for any \( \varepsilon > 0 \).

**A Proof of Lemma 5**

The proof is due to Madiman and Wang and is a corollary of the following elementary lemma.

**Lemma 9.** For convex function \( V : \mathbb{R}^n \to \mathbb{R} \), the associated function \( U : \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \) defined by
\[ U(x, t) = tV(x/t) \]
is convex.

**Proof.** The proof is a direct computation. Take points \( (x_1, t_1), (x_2, t_2) \) and \( \lambda \in (0, 1) \). We write \( t = (1-\lambda)t_1 + \lambda t_2 \) for brevity of notation. Hence,
\[ U((1-\lambda)(x_1, t_1) + \lambda(x_2, t_2)) = tV\left( \frac{(1-\lambda)x_1 + \lambda x_2}{t} \right) \]
\[ = tV\left( \left( \frac{(1-\lambda)t_1}{t} \right) \frac{x_1}{t_1} + \left( \frac{\lambda t_2}{t} \right) \frac{x_2}{t_2} \right) \]
\[ \leq (1-\lambda)t_1V(x_1/t_1) + \lambda t_2V(x_2/t_2) \]
\[ = (1-\lambda)U(x_1, t_1) + \lambda U(x_2, t_2). \]

\[ \blacksquare \]

**Corollary 10.** For a log-concave function \( f \) on \( \mathbb{R}^n \), the map
\[ \varphi(t) = t^n \int_{\mathbb{R}^n} f^t, \quad t > 0, \]
is log-concave as well.

**Proof.** Let us write \( f \) as \( e^{-V} \), with \( V \) convex. Lemma 9 implies that \( e^{-tV(x/t)} \) is log-concave on \( \mathbb{R}^n \times (0, \infty) \), and hence by the Prékopa-Leindler inequality,
\[ t \mapsto \int_{\mathbb{R}^n} e^{-tV(x/t)} \, dx \]
is log-concave. The change of variable \( z = x/t \) gives the corollary since
\[ t^n \int_{\mathbb{R}^n} f^t(z) \, dz = \int_{\mathbb{R}^n} e^{-tV(x/t)} \, dx. \]

\[ \blacksquare \]
Proof of Lemma. What remains is an algebraic computation. When $f$ is a density $\varphi(1) = 1$. Write $1, p, q$ in convex combination, and unwind the implication of $\varphi$ being log-concave. We will show the result in the case that we need $0 < p < q < 1$, the other arguments are similar. In this case, $\lambda p + (1 - \lambda)1 = q$ for $\lambda = \frac{1 - q}{1 - p} \in (0,1)$. By log-concavity,

$$\varphi(q) \geq \varphi^\lambda(p)\varphi^{1-\lambda}(1),$$

which is

$$q^n \int f^q \geq \left(p^n \int f^p \right)^{\frac{1-q}{1-p}}.$$ 

Since $1 - q > 0$ raising both sides to the power $2/n(1-q)$ preserves the inequality, and we have

$$q^{2/(1-q)}N_q(f) \geq p^{2/(1-p)}N_p(f).$$

which implies our result. 

References

[1] S. Artstein, K. M. Ball, F. Barthe, and A. Naor. Solution of Shannon’s problem on the monotonicity of entropy. J. Amer. Math. Soc., 17(4):975–982 (electronic), 2004.

[2] K. Ball, P. Nayar, and T. Tkocz. A reverse entropy power inequality for log-concave random vectors. Preprint, arXiv:1509.05926, 2015.

[3] K. Ball and V. H. Nguyen. Entropy jumps for isotropic log-concave random vectors and spectral gap. Studia Math., 213(1):81–96, 2012.

[4] W. Beckner. Inequalities in Fourier analysis. Ann. of Math. (2), 102(1):159–182, 1975.

[5] S. Bobkov and M. Madiman. The entropy per coordinate of a random vector is highly constrained under convexity conditions. IEEE Trans. Inform. Theory, 57(8):4940–4954, August 2011.

[6] S. Bobkov and M. Madiman. Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures. J. Funct. Anal., 262:3309–3339, 2012.

[7] S. Bobkov and A. Marsiglietti. Variants of entropy power inequality. Preprint, arXiv:1609.04897, 2016.

[8] S. G. Bobkov and G. P. Chistyakov. Bounds for the maximum of the density of the sum of independent random variables. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 408(Veroyatnost i Statistika. 18):62–73, 324, 2012.

[9] S. G. Bobkov and G. P. Chistyakov. Entropy power inequality for the Rényi entropy. IEEE Trans. Inform. Theory, 61(2):708–714, February 2015.

[10] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. Amer. J. Math., 108(6):1467–1476, 1986.

[11] H. J. Brascamp and E. H. Lieb. Best constants in Young’s inequality, its converse, and its generalization to more than three functions. Advances in Math., 20(2):151–173, 1976.

[12] H. Busemann. A theorem on convex bodies of the Brunn-Minkowski type. Proc. Nat. Acad. Sci. U. S. A., 35:27–31, 1949.

[13] M. H. M. Costa. A new entropy power inequality. IEEE Trans. Inform. Theory, 31(6):751–760, 1985.
[14] M. H. M. Costa and T. M. Cover. On the similarity of the entropy power inequality and the Brunn-Minkowski inequality. *IEEE Trans. Inform. Theory*, 30(6):837–839, 1984.

[15] T. A. Courtade, M. Fathi, and A. Pananjady. Wasserstein stability of the entropy power inequality for log-concave densities. *Preprint, arXiv:1610.07969*, 2016.

[16] A. Dembo, T. M. Cover, and J. A. Thomas. Information-theoretic inequalities. *IEEE Trans. Inform. Theory*, 37(6):1501–1518, 1991.

[17] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. The convexification effect of minkowski summation. *Preprint, arXiv:1704.05486*, 2016.

[18] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. Do Minkowski averages get progressively more convex? *C. R. Acad. Sci. Paris Sér. I Math.*, 354(2):185–189, February 2016.

[19] M. Fradelizi, M. Madiman, and L. Wang. Optimal concentration of information content for log-concave densities. In C. Houdré, D. Mason, P. Reynaud-Bouret, and J. Rosinski, editors, *High Dimensional Probability VII: The Cargèse Volume*, Progress in Probability. Birkhäuser, Basel, 2016. Available online at *arXiv:1508.04093*.

[20] M. Fradelizi and A. Marsiglietti. On the analogue of the concavity of entropy power in the Brunn-Minkowski theory. *Adv. in Appl. Math.*, 57:1–20, 2014.

[21] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)*, 39(3):355–405 (electronic), 2002.

[22] L. Leindler. On a certain converse of Hölder’s inequality. II. *Acta Sci. Math. (Szeged)*, 33(3-4):217–223, 1972.

[23] J. Li. Rényi entropy power inequality and a reverse. *Preprint, arXiv:1704.02634*, 2017.

[24] M. Madiman and A. R. Barron. Generalized entropy power inequalities and monotonicity properties of information. *IEEE Trans. Inform. Theory*, 53(7):2317–2329, July 2007.

[25] M. Madiman, J. Melbourne, and P. Xu. Forward and reverse entropy power inequalities in convex geometry. *Convexity and Concentration*, pages 427–485, 2017.

[26] M. Madiman, J. Melbourne, and P. Xu. Rogozin’s convolution inequality for locally compact groups. *Preprint, arXiv:1705.00642*, 2017.

[27] A. Marsiglietti and V. Kostina. A lower bound on the differential entropy of log-concave random vectors with applications. *Preprint, arXiv:1704.07766*, 2017.

[28] V. D. Milman. Inégalité de Brunn-Minkowski inverse et applications à la théorie locale des espaces normés. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(1):25–28, 1986.

[29] E. Ram and I. Sason. On rényi entropy power inequalities. *IEEE Transactions on Information Theory*, 62(12):6800–6815, 2016.

[30] A. Rényi. On measures of entropy and information. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. I*, pages 547–561. Univ. California Press, Berkeley, Calif., 1961.

[31] G. Toscani. A Strengthened Entropy Power Inequality for Log-Concave Densities. *IEEE Trans. Inform. Theory*, 61(12):6550–6559, 2015.

[32] G. Toscani. A concavity property for the reciprocal of Fisher information and its consequences on Costa’s EPI. *Phys. A*, 432:35–42, 2015.
[33] A. M. Tulino and S. Verdú. Monotonic decrease of the non-gaussianness of the sum of independent random variables: A simple proof. IEEE Trans. Inform. Theory, 52(9):4295–7, September 2006.

[34] P. Xu, J. Melbourne, and M. Madiman. Reverse entropy power inequalities for s-concave densities. In Proc. IEEE Intl. Symp. Inform. Theory., pages 2284–2288, Barcelona, Spain, July 2016.

[35] P. Xu, J. Melbourne, and M. Madiman. A min-entropy power inequality for groups . In Proc. IEEE Intl. Symp. Inform. Theory., 2017.

[36] P. Xu, J. Melbourne, and M. Madiman. Infinity Entropy Power Inequalities. In Proc. IEEE Intl. Symp. Inform. Theory., 2017.

[37] R. Zamir and M. Feder. A generalization of the entropy power inequality with applications. IEEE Trans. Inform. Theory, 39(5):1723–1728, 1993.

[38] R. Zamir and M. Feder. On the volume of the Minkowski sum of line sets and the entropy-power inequality. IEEE Trans. Inform. Theory, 44(7):3039–3063, 1998.