CONICAL CALABI-YAU METRICS ON TORIC AFFINE VARIETIES AND
CONVEX CONES

ROBERT J. BERMAN,

Abstract. It is shown that any affine toric variety $Y$, which is $\mathbb{Q}$-Gorenstein, admits a
conical Ricci flat Kähler metric, which is smooth on the regular locus of $Y$. The correspond-
ing Reeb vector is the unique minimizer of the volume functional on the Reeb cone of $Y$.
The case when the vertex point of $Y$ is an isolated singularity was previously shown by
Futaki–Ono–Wang. The proof is based on an existence result for the inhomogeneous Monge-
Ampère equation in $\mathbb{R}^m$ with exponential right hand side and with prescribed target given
by a proper convex cone, combined with transversal a priori estimates on $Y$.

1. Introduction

The problem of finding canonical metrics on a (polarized) complex projective algebraic
manifold $X$ has a long and rich history leading up to the Yau–Tian–Donaldson conjecture,
which reduces the existence problem to verifying a purely algebroid-geometric property called
K-stability. In the case of Fano manifolds the conjecture was settled rather recently (see the
survey [18]). The projective framework has also been generalized to the setting of singular
affine varieties $Y$ [12, 13, 19, 20, 38]. Apart from the complex-geometric motivations - coming,
in particular, from the study of tangent cones at log terminal singularities and connections to
the Minimal Model Program (as discussed in the survey [38]) - an important motivation for
this generalization comes from the AdS/CFT correspondence in theoretical physics, relating
geometry to conformal field theory, or more precisely superconformal gauge theories [32, 31, 24].

1.1. Background. We will be mainly concerned with the case of toric affine Calabi–Yau
varieties, but we start by recalling some general background. Let $Y$ be a complex affine
normal variety of dimension $m$, which is $\mathbb{Q}$-Gorenstein (i.e. a reflexive tensor power of the
canonical sheaf of $Y$ is a well-defined line bundle on $Y$). Assume that $Y$ is endowed with a good
action by a compact torus $T$. This means that the complexification $T_\mathbb{C}$ acts holomorphically
and effectively on $Y$ and there is a unique fixed point $y_0$ such that that the closure of any orbit
of $T_\mathbb{C}$ in $Y$ contains $y_0$ [38]. The point $y_0$ in $Y$ is called the vertex point of $(Y,T)$. An element
$\xi$ in the Lie algebra of $T$ is said to be a Reeb vector if the weights $\lambda_i$ of the action by $\xi$ on
the coordinate ring $\mathcal{R}(Y)$ of $Y$ (or the ring of holomorphic functions on $Y$) are non-negative
and vanish only on constant functions. An affine variety $(Y,\xi)$ decorated with a Reeb vector
$\xi$ is called a polarized affine variety in [12]. More generally, a vector field $\xi$ on $Y$ is said to be
a Reeb vector field if it is the infinitesimal generator of some Reeb vector. We will denote by
$T_\xi$ the corresponding minimal torus (in other words, the orbits of $T_\xi$ in $Y$ coincide with the
closure of the orbits of the flow of $\xi$). The rank of $\xi$ is defined as the rank of the corresponding
minimal torus $T_\xi$ and $\xi$ is said to be quasi-regular if it has rank one and otherwise $\xi$ is said to
be irregular. Equivalently, $\xi$ is quasi-regular if the leaf space of the holomorphic foliation of
$Y$, defined by the complexification of $\xi$, is Hausdorff.
In the case when $y_0$ is an isolated singularity in $Y$ it was shown in [13] that there exists a Calabi–Yau metric on $Y_{\text{reg}}(= Y - \{y_0\})$, i.e. a Ricci flat Kähler metric $\omega$, which is conical with respect to $\xi$, i.e.,

\begin{equation}
\text{Ric } \omega = 0, \quad \mathcal{L}_{-J\xi} \omega = 2\omega \tag{1.1}
\end{equation}

iff $(Y, T, \xi)$ is $K$-stable for any torus $T$ containing $\xi$ in its Lie algebra. Here $J$ denotes the complex structure on $Y_{\text{reg}}$ and $\mathcal{L}_{-J\xi}$ denotes the Lie derivative along the vector field on $Y_{\text{reg}}$ defined by $-J\xi$, which generates a $\mathbb{R}^*$-action on $Y$ with repulsive fixed point $y_0$. The K-stability of $(Y, T, \xi)$ is the affine analog of the notion of K-stability of projective Fano manifolds, as appearing in the ordinary projective Yau–Tian–Donaldson conjecture. It amounts to a positivity condition for all special test configurations (i.e. equivariant deformations) of $(Y, T, \xi)$. We also recall that the conical Calabi–Yau condition (1.1) equivalently means that $\omega$ is a metric cone over the link $(Y - \{0\})/\mathbb{R}^*$ of the affine singularity $Y$, endowed with a smooth Sasaki–Einstein metric [13]. Remarkably, already in the case when the link is the topological five-sphere, there is an infinite family of Sasaki–Einstein metrics [13, 9].

The conical condition on the Kähler metric $\omega$ equivalently means that $\omega$ may be expressed as

$$\omega = dd^c(r^2), \quad dd^c := \frac{i}{2\pi} \partial \bar{\partial}$$

for a function $r$ which is radial with respect to the corresponding $\mathbb{R}^*$-action on $Y$, i.e. $r$ is positive and 1-homogeneous with respect to $\mathbb{R}^*$. As a consequence, when $y_0$ is an isolated singularity the function $r^2$ is automatically bounded (and even continuous) in a neighborhood of $y_0$. The condition that $\omega$ be Ricci flat equivalently means that $r^2$ solves the Calabi–Yau equation

\begin{equation}
(dd^c(r^2))^m = i^{m^2} \Omega \wedge \bar{\Omega}, \tag{1.2}
\end{equation}

where $\Omega$ is a nowhere vanishing $T$-equivariant (multi-valued) holomorphic top form on $Y_{\text{reg}}$. Any $(Y, T)$ admits such a form $\Omega$ and it is uniquely determined, up to multiplication by a complex constant. As a consequence, if there exists a Ricci flat Kähler metric on $Y$ which is conical with respect to the Reeb vector $\xi$, then $\xi$ has to satisfy the normalization condition

$$\mathcal{L}_\xi \Omega = im\Omega.$$

As first shown in [31, 32], motivated by the AdS/CFT correspondence, another more subtle condition on $\xi$ is that $\xi$ minimizes the volume functional $V$ on the space of all normalized Reeb vectors:

\begin{equation}
V(\xi) := \lim_{t \to 0^+} t^n \sum_1^\infty e^{-t\lambda_i}, \tag{1.3}
\end{equation}

where the sum runs over the infinite number strictly positive weights $\lambda_i$ of $\xi$, including multiplicity. As shown in [12] this volume condition is also directly implied by the K-stability of $(Y, T, \xi)$ (the corresponding test configuration is a product test configuration).

**Example 1.1.** Let $X$ be an $(m - 1)$-dimensional Fano manifold, i.e. a compact complex manifold whose anticanonical line bundle $-K_X$ is positive. Then the $m$-dimensional variety $Y$ obtained by blowing down the zero-section in the total space of $K_X \to X$ is an affine Gorenstein variety endowed with a Reeb vector $\xi$, which is regular (generating the standard $\mathbb{C}^*$–action along the fibers of $K_X$). Moreover, a conical Calabi–Yau metric $\omega_Y$ on $(Y, \mathbb{C}^*)$
corresponds to a Kähler metric $\omega_X$ on $X$ with constant positive Ricci curvature, defined as the “horizontal” part of $\omega_Y$ with respect to the fibration $Y \to X$. However, even when $X$ does not admit any Kähler metric $\omega_X$ with constant positive Ricci curvature, it frequently happens that there is another Reeb vector $\xi$ which does admit a conical Calabi–Yau metric. This is always the case when $X$ (and hence $Y$) is toric, in which case $\xi$ is obtained by minimizing the volume functional on the Reeb cone of $Y$ [22, 11]. For example, when $X$ is $\mathbb{P}^2$ blown up in one or two points the corresponding “minimal” Reeb vector $\xi$ on $Y$ is irregular. Irregular conical Calabi–Yau metrics were first constructed in the physics literature as special cases of a family $Y^{p,q}$ of explicit non-homogeneous metrics [23]. In fact, as shown in [29], these are all toric and $Y^{2,1}$ corresponds to $\mathbb{P}^2$ blown up in one point.

In the general case, where the vertex point $y_0$ of $Y$ is not assumed to be an isolated singularity, $\omega$ is said to be a conical Calabi–Yau metric for $(Y,T,\xi)$ if $\omega$ is a smooth conical Calabi–Yau metric on $Y_{\text{reg}}$ and moreover the corresponding Kähler potential $r^2$ is bounded in a neighborhood of $y_0$ (see [20, 39, 38]). Such metrics arise, in particular, on the tangent cones of Kähler–Einstein metrics with log terminal (klt) singularities [20]. The conjectural extension of the result in [13] to a non-isolated singularity $y_0$ can thus be formulated as the following generalized affine Yau–Tian–Donaldson conjecture:

**Conjecture 1.2.** $(YTD)$ Let $Y$ be a normal affine variety which is $\mathbb{Q}$-Gorenstein, endowed with a good action by a compact torus $T$. Let $\xi$ be a Reeb vector in the Lie algebra of $T$. Then there exists a Calabi–Yau metric which is conical with respect to $\xi$ if and only if $(X,T,\xi)$ is K-stable.

By [20, Prop 4.8] such a Calabi–Yau metric is uniquely determined modulo the action of the group $\text{Aut} (X,T)_0$ of automorphisms of $Y$, commuting with $T$ and homotopic to the identity element. A version of the Yau-Tian-Donaldson for singular projective Fano varieties, involving the stronger uniform version of K-stability, has recently been established in [40, 41].

Note that we are adopting the terminology for K-stability used in [13], which corresponds to the notion of K-polystability in [39, 38].

**1.2. Main results.** The main result in the present work establishes the previous conjecture in the case when the $Y$ is an affine toric variety (recall that a toric variety has $\mathbb{Q}$–Gorenstein singularities iff it has klt singularities [16]):

**Theorem 1.3.** Let $Y$ be an affine toric variety which is $\mathbb{Q}$-Gorenstein. Then the following is equivalent:

- $(Y,T,\xi)$ admits a conical Calabi–Yau metric $\omega$.
- $\xi$ is the unique minimizer of the volume functional on the space of normalized Reeb vectors in the Lie algebra of the maximal torus $T_m$.
- $(Y,T,\xi)$ is K-stable.

In fact, the Calabi-Yau metric $\omega$ can be taken to be $T_m$–invariant with locally bounded $T_m$–invariant potential $r^2$ (see Theorem 3.4) and hence, by local results [14, Prop 4.1], $r^2$ is automatically continuous on $Y$.

By [13] and [37, Remark 2.27] the implication “K-stability $\implies$ volume minimization” holds for any polarized affine variety $Y$ with klt singularities, but the converse implication, resulting from the previous theorem, is a special feature of the toric setting (but see [36, 38] for a generalized notion of volume minimization, applying to general affine varieties, where the role of Reeb vectors is played by general valuations). Combining the previous theorem with [13],
Thm 7.1] and [39, Thm 4.1, Cor 4.2] thus yields an analytic proof of the following purely algebro-geometric result (recall that $T(\xi)$ denotes the minimal torus determined by $\xi$):

**Corollary 1.4.** Let $Y$ be an $m$-dimensional affine toric variety which is $\mathbb{Q}$-Gorenstein and denote by $T_m$ the corresponding torus of maximal rank $m$. If $(Y, T_m, \xi)$ is K-stable then $(X, T(\xi))$ is K-stable with respect to all weak special test configurations and Ding polystable with respect to all $\mathbb{Q}$-Gorenstein test configurations.

The previous corollary is closely related to results in [39] comparing K-(semi-)stability with $T$-equivariant K-(semi-)stability, shown using purely algebro-geometric techniques from the MMP.

The case of Theorem [1.3] when $y_0$ is an isolated singularity was first shown in [22] (using a method of continuity, similar to the projective Fano manifold case in [14]). The extension to general toric affine varieties $Y$ was conjectured in the last section of [13] (see also page 357-358 for relations to toric tangent cones). As discussed in [13], general toric affine varieties $Y$ are expected to appear as degenerations of certain K-unstable non-toric affine varieties. See also [35] for a discussion about a more general conjecture concerning general klt singularities $(Y, y_0)$ (the Stable Degeneration Conjecture).

As stressed in [31] toric affine varieties also play a prominent role in the AdS/CFT correspondence (mainly for $m = 3$ and $m = 4$), since the corresponding supersymmetric gauge theories may be constructed explicitly, using toric quivers, encoded by brane tilings and dimers (see [15] for an exposition aimed at mathematicians when $m = 3$ and [30] for $m = 4$).

**Example 1.5.** Non-isolated toric singularities also play an important role in the AdS/CFT correspondence. This is illustrated by the simple orbifold case $Y_0 = \mathbb{C}^3/\mathbb{Z}_2$, where $-1 \in \mathbb{Z}_2$ acts as $(z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3)$ and thus $Y$ is singular over the complex line $(\mathbb{C}, 0, 0)$. The affine variety $Y_0$ may be embedded as the hypersurface in $\mathbb{C}^4$ defined by $w_2 w_3 = w^4$ and corresponds to a gauge theory with $\mathcal{N} = 2$ supersymmetry. The non-isolated singularity locus of $Y_0$ is “resolved” by the deformation of $Y$ defined by the conifold $Y_\epsilon$, $\epsilon w_1^2 + w_2 w_3 = w^4$, which corresponds to a gauge theory with merely $\mathcal{N} = 1$ supersymmetry (see [26] pages 13-17). The AdS/CFT correspondence for general affine (log terminal) singularities is discussed in [33].

The proof of Theorem [1.3] also yields a new variational proof of the existence results in [22]. The starting point is the basic observation that the restriction of a toric conical Calabi–Yau metric $\omega_Y$ to the open orbit of $T_\mathbb{C}$ in $Y$ corresponds to the solution $f(x)$ of an inhomogeneous real Monge–Ampère equation on $\mathbb{R}^m$ with exponential right hand side, using the standard fibration $T_\mathbb{C} \to \mathbb{R}^m$, where $\mathbb{R}^m$ is identified with the Lie algebra of $T$. The image of the gradient of $f$ is prescribed to be the convex convex $\mathbb{C}^*$ defined by the moment polytope of the toric variety $Y$. The existence of the metric $\omega_Y$ and its regularity on the open orbit of $T_\mathbb{C}$ in $Y$, as well as an a priori $L^\infty$-bound, then follows from a general result about such Monge–Ampère equations on $\mathbb{R}^m$, which may be of independent interest (see Theorem [2.2]). The latter result is obtained by applying [7, Thm 1.1], concerning the second boundary value problem for the Monge–Ampère equation with an exponential non-linearity associated to a convex body $P$, to the $(m - 1)$-dimensional linear space $\mathbb{R}^m/\mathbb{R} \xi$ and a compact (possible irrational) polytope obtained by intersecting the convex cone $\mathbb{C}^*$ with a hyperplane determined by $\xi$. This leads, in particular, to a variational construction of the conical Calabi–Yau metric $\omega_Y$. Finally, in order to show that $\omega_Y$ is smooth on all of $Y_{reg}$ we employ a transversal generalization of the Laplacian a priori estimates in [6, Appendix B].
by constructing an appropriate barrier and exploiting a generalization of the $L^\infty$-estimate in Theorem 2.2.

As will be shown in [5] the toric conical Calabi-Yau metric appearing in Theorem 1.3 can be constructed probabilistically by sampling certain explicitly defined random point processes (which are tropicalizations of the canonical point processes on Fano varieties and Sasaki varieties introduced in [1] and [2], respectively).

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2. Convex cones and real Monge–Ampère equations

According to the classical Jörgens–Calabi–Pogorelov theorem [43] a smooth convex function $f$ on Euclidean space $\mathbb{R}^m$ solves the Monge–Ampère equation

$$\det \nabla^2 f = g$$

with $g$ constant iff $f$ is quadratic. Here we will be concerned with the case when the right hand side $g$ is of the form

$$g(x) = C e^{\langle l, x \rangle}$$

for a given non-zero vector $l$ and some non-zero constant $C$ (which after a scaling could as well have be taken to be equal to one). In this case it turns out that the solution space is, in general, infinite dimensional (if $m \geq 3$). Some extra conditions thus need to be imposed to get a finite dimensional space of solutions. Motivated by the setting of toric conical Calabi–Yau metrics we will impose that the gradient image

$$\nabla f(\mathbb{R}^m) = C^*,$$

for a given proper convex cone $C^*$ in $\mathbb{R}^m$, i.e. $C$ is a closed convex cone of dimension $m$ such that $C$ does not contain a line. Moreover, we demand that the convex solution $f$ is strictly positive and satisfies the following homogeneity property: there exists a vector $\xi \in \mathbb{R}^m$ such

$f$ is one-homogeneous with respect to $\xi$, i.e. for any $t \in \mathbb{R}$

$$(2.1) f(x + t\xi) = e^t f(x).$$

Recall that a proper convex cone is a closed convex cone which does not contain an entire line.

**Example 2.1.** When $l = (1, \ldots, 1)$ and $C^*$ is the “non-negative quadrant” a solution (with $C = 1$) is obtained by setting

$$f(x) = \sum_{i \leq m} e^{x_i}$$

(and any translation of $f$ yields a new solution).

Denote by $C$ the dual convex cone of the convex cone $C^*$, i.e. the set of all $x \in \mathbb{R}^m$ such that the corresponding linear function $\langle x, \cdot \rangle$ is non-negative on $C^*$. Then $C^*$ is a proper convex cone iff $C$ is (using that, conversely, the dual of $C$ is $C^*$). An element $\xi$ in the interior of $C$ will be called a *Reeb vector*. Since the determinant of an $m \times m$ matrix is homogeneous of degree $m$.
a necessary condition for the existence of a solution $f$, which is one-homogeneous with respect to $\xi$, is that $\xi$ is $l$-normalized in the sense that
$$\langle l, \xi \rangle = m.$$  
According to the next theorem a sufficient condition is that $\xi$ minimizes the volume $V(\xi)$ defined as
\begin{equation}
V(\xi) := \text{Vol} \left( C^* \cap \{ \langle \xi, \cdot \rangle \leq 1 \} \right) \in \mathbb{R}_+ \cup \{ \infty \},
\end{equation}
computed with respect to Lebesgue measure on $\mathbb{R}^m$.

**Theorem 2.2.** Let $C^*$ be a proper convex cone in $\mathbb{R}^m$, for $m \geq 2$, and $l$ a non-zero vector in the interior of $C^*$. Given a Reeb vector $\xi$ the following is equivalent:
\begin{itemize}
  \item There exists a smooth strictly positive convex function $f : \mathbb{R}^m \to \mathbb{R}$ which is one-homogeneous with respect to $\xi$ and satisfies
    $$\det \nabla^2 f = Ce^{\langle l, x \rangle}, \quad (df)(\mathbb{R}^m) = C^*$$
    for some non-zero constant $C$.
  \item $\xi$ minimizes the volume $V(\xi)$ among all $l$-normalized Reeb vectors.
\end{itemize}
Moreover, there exists a unique such minimizer $\xi$ and the corresponding solution $f$ is uniquely determined modulo translations, i.e. the additive group $\mathbb{R}^m$ acts transitively on the solution space.

In fact, the proof yields the following property of $f$ which is stronger than the gradient property above: there exists a constant $C$ such that
\begin{equation}
f(x) \leq Ce^{\phi_{P^*_\xi}(x)},
\end{equation}
where $\phi_P(x)$ denotes the support function of a given convex set $P \subseteq \mathbb{R}^n$, i.e.
\begin{equation}
\phi_P(x) = \sup_{p \in P} \langle x, p \rangle.
\end{equation}
and $P_\xi$ is the convex bounded set defined by
\begin{equation}
P_\xi := C^* \cap \{ \langle \xi, \cdot \rangle = 1 \}.
\end{equation}
Note that it follows directly from the definition of $P_\xi$, that $e^{\phi_{P^*_\xi}(x)}$ is one-homogeneous with respect to $\xi$.

2.1. **Proof of Theorem 2.2** It will be convenient to reformulate the theorem in a linearly invariant manner. Thus starting with the real vector space $W := \mathbb{R}^m$ we let $C^*$ be a proper convex cone in the space $W^*$ of linear functions on $W$. We identify $l$ with a non-zero element in $W^*$ and $C$ with a cone in $W$. In particular, a Reeb vector $\xi$ defines an element in $W$. A function $f : W \to \mathbb{R}$ as in Theorem 2.2 then satisfies
\begin{equation}
MA(f) = Ce^l dx, \quad (df)(W) = C^*,
\end{equation}
where $dx$ denotes a Lebesgue measure on $W$ and $MA(f)$ is the real Monge--Ampère measure,
\begin{equation}
MA(f) := \det(\nabla_\xi^2 f) dx
\end{equation}
(the equation is independent of the choice of linear coordinates $x$ on $W$, up to rescaling the constant $C$). Assume that $\xi$ is $l$-normalized. Writing $f = e^\Phi$ the homogeneity property 2.1
for $f$ holds iff the function $\Phi$ is $\xi$-equivariant, i.e. $\Phi(x + t\xi) = \Phi(x) + t$ for any $t \in \mathbb{R}$. Since $d(e^\Phi) = e^\Phi d\Phi$ this equivalently means that

$$
(2.8) \quad (d\Phi)(W) = P_\xi,
$$

where $P_\xi$ is defined in formula (2.5). Next, setting

$$
\phi := \Phi - l/m,
$$
gives a $\xi$-invariant function, i.e $\phi(x + t\xi) = \phi(x)$. Equivalently, one may regard $\phi$ as a function defined on $W/\mathbb{R}\xi$. The condition in formula (2.8) is equivalent to

$$
(2.9) \quad (d\phi)(W) = P_\xi - l/m.
$$

Now consider the linear function

$$
t := l/m \in W^*
$$

and take $m - 1$ linearly independent linear functions $s_1, \ldots, s_{m-1}$ on $W$ such that $s_i$ are $\xi$-invariant, i.e. $(s_i, \xi) = 0$. We then obtain an invertible linear map

$$
(2.10) \quad W \to \mathbb{R}^m, \quad x \mapsto (s_1(x), \ldots, s_{m-1}(x), t(x))
$$

The Monge–Ampère equation (2.6) for $e^\Phi$ can thus be expressed as

$$
(\det \nabla^2_\phi) \left( e^{t + \phi(s)} \right) = ae^{mt},
$$

for some non-zero constant $a$ (coming from the Jacobian of the linear map (2.10)). By a direct computation this equivalently means that on $W/\mathbb{R}\xi \cong \mathbb{R}^{m-1}$ with coordinates $s$ we have (after perhaps rescaling $s$)

$$
(2.11) \quad (\det \nabla^2_\phi) \left( \phi(s) \right) = e^{-m\phi(s)}
$$

This means that a $\xi$-homogeneous smooth convex function $e^\Phi$ solves the equation (2.6) subject to (2.11) iff the smooth convex function $\phi$ on $W/\mathbb{R}\xi$ solves the equation (2.11) subject to

$$
(2.12) \quad (\nabla \phi)(W) = Q_\xi \subset \xi^\perp,
$$

where $\xi^\perp \subset W^*$ is the subset of linear functions on $W$ that vanishes on $\xi$, $Q_\xi$ is the convex body defined as the $P_\xi - l/m$ regarded as a subset of $\xi^\perp$. By [7, Thm 1.1] there exists a solution $\phi$ iff the barycenter of $Q_\xi$ is equal to the origin in $\xi^\perp$. Equivalently, this condition means that the barycenter of $P_\xi$ is equal to $l/m$. Moreover, any other solution $\phi$ is of the form

$$
\phi(s + a) + c
$$

for some $a \in \xi^\perp$ and $c \in \mathbb{R}$. Furthermore, by the general $L^\infty$-estimate in [7, Thm 1.1], detailed in the following lemma, a solution $\phi$ satisfies the following global bound

$$
\sup_{W/\mathbb{R}\xi} |\phi - \phi_0| \leq C, \quad \phi_0(x) := \sup_{p \in Q_\xi} \langle x, p \rangle
$$

for some constant $C$, using that

$$
(2.13) \quad \mu := e^{-m\phi(s)} ds \leq Ae^{-|s|/A} ds
$$

for some constant $A$, since 0 is an interior point of $(\nabla \phi)(W/\mathbb{R}\xi)$. 

Lemma 2.3. Let $\phi$ be a convex function on $\mathbb{R}^n$ such that
$$MA(\phi) = \mu, \quad (\partial^2 \phi)(\mathbb{R}^n) = P$$
for a convex body $P \subset \mathbb{R}^n$ and a measure $\mu$ on $\mathbb{R}^n$, where $MA(\phi)$ denotes the Monge–Ampère measure of $\phi$ (defined by formula [27] when $\phi$ has a bounded Hessian). Set
$$\phi_P(x) := \sup_{p \in P} \langle x, p \rangle$$
and assume that $\phi$ is normalized so that $\sup_{\mathbb{R}^n} (\phi - \phi_P) = 0$. Given $q > n$ the following inequality holds
$$\sup_{\mathbb{R}^n} |\phi - \phi_P| \leq \frac{d(P)}{V(P)} \int_{\mathbb{R}^n} |x| \mu + C_{n,q} \frac{d(P)^{(1+n(1-1/q))}}{V(P)} \left( \int_{\mathbb{R}^n} |x|^q \mu \right)^{1/q},$$
where $d(P)$ denotes the diameter of $P$, $V(P)$ its volume and the constant $C_{n,q}$ only depends on $n$ and $q$.

Proof. Following the argument in the proof of [7 Prop 2.2] we denote by $v := \phi^*$ is the Legendre transform of $\phi$, which defines a convex function on the interior of $P$. By the Sobolev inequality for the embedding $W^{1,q}(P) \subseteq L^\infty(P)$ we have, since the interior of $P$ is a bounded convex domain,
$$\sup_P |v| \leq \frac{1}{V(P)} \int_P |v(y)| dy + C_{n,q} \frac{d(P)^{(1+n(1-1/q))}}{V(P)} \left( \int_P |\nabla v(y)|^q dy \right)^{1/q}$$
(see [17, Lemma 1.7.3] or [42 Thm 4.4]). In general, as explained in [7 Prop 2.2], $\inf_P v = -\sup(\phi - \phi_P)$ and hence, by assumption, $\inf_P v = 0$. Assume that the infimum is attained at $y_0 \in P$, i.e. $v(y_0) = 0$. By convexity $|v(y)| = v(y) - v(y_0) \leq \nabla v(y) \cdot (y - y_0)$. Thus the Cauchy-Schwartz inequality yields
$$\int_P |v(y)| dy \leq d(P) \int_P |\nabla v| dy.$$
Finally, the proof is concluded by observing that $\int |\nabla v(y)|^\alpha dy = \int |x|^\alpha MA(\phi)$ for any $\alpha > 0$ and $\sup_P |v| = \sup_{\mathbb{R}^n} |\phi - \phi_P|$. \hfill \qed

The proof of Theorem [2.2] is now concluded by invoking the following

Proposition 2.4. A Reeb vector $\xi$ minimizes the volume among all $l$-normalized Reeb vectors iff the barycenter $b_{P_\xi}$ of $P_\xi$, i.e. the element of $P_\xi$ defined by
$$b_{P_\xi} := \int_{p \in P_\xi} p d\lambda_{m-1} / \int_{p \in P_\xi} d\lambda_{m-1},$$
(where $d\lambda_{m-1}$ denotes any choice of Lebesgue measure on the $(m-1)$-dimensional affine space $\langle \xi, \cdot \rangle = 1$) is given by
$$b_{P_\xi} = l/m.$$
Moreover, there always exists a unique minimizer $\xi$.

To prove the proposition first observe that $V(\xi) \to \infty$ as $\xi$ approaches a non-zero point $\xi_0$ in the boundary of $C$. Indeed, there exist non-zero elements $p_0, p_1$ in the interior of $C^*$ such that $\langle \xi_0, p_1 \rangle = 1$ and $\langle \xi_0, p_0 \rangle = 0$. As a consequence, the corresponding convex set $P_{\xi_0}$ is unbounded (since it contains $p_1 + cp_0$ for any $c > 0$) and hence $V(\xi_0) = \infty$. Thus, the restriction of $V$ to the
convex bounded set \( C^* \cap \{ (l, \cdot) = m \} \) admits a minimizer \( \xi_* \) in the interior, i.e. a minimizing Reeb vector field.

**Lemma 2.5.** The function \( \log V(\xi) \) is smooth and strictly convex on the interior of \( C \) and its differential is given by

\[
d \log V|_{\xi} = -mbP_\xi
\]

**Proof.** First observe that

\[
V(\xi) = m! \int_{C^*} e^{-(\xi,p)} dp.
\]

Indeed, setting \( s := \langle \xi, p \rangle \) we have

\[
\int_{C^*} e^{-(\xi,p)} dp = \int_{[0,\infty)} e^{-s} s_*(1_{C^*} dp) = \int_{[0,\infty)} e^{-s} \frac{d}{ds} V(s) ds, \quad V(s) := \int_{\{\xi<s\}} 1_{C} dp
\]

But since \( V \) is homogeneous of degree \( m \), we have

\[
V(s) = s^m V(\xi)
\]

and hence formula 2.14 follows from computing

\[
m \int_{0,\infty} e^{-s} s^{m-1} ds = m!
\]

The first statement of the lemma then follows from standard convex analysis. Finally, we have

\[
-d\xi \log \int_{C^*} e^{-(\xi,p)} dp = \frac{\int_{C^*} p e^{-(\xi,p)} dp}{\int_{C^*} e^{-(\xi,p)} dp} = mbP_\xi,
\]

where the last equality follows from setting \( s = \langle \xi, p \rangle \) and changing the order of integration to get

\[
\int_{C^*} p e^{-(\xi,p)} dp = \left( \int_{C^* \cap \{(\xi,p)=s\}} pd\lambda_n \right) e^{-s} ds = \left( \int_{C^* \cap \{(\xi,p)=1\}} pd\lambda_{m-1} \right) \int_0^\infty s^m e^{-s} ds
\]

and

\[
\int_{C^*} e^{-(\xi,p)} dp = \left( \int_{C^* \cap \{(\xi,p)=s\}} d\lambda_{m-1} \right) e^{-s} ds = \left( \int_{C^* \cap \{(\xi,p)=1\}} d\lambda_{m-1} \right) \int_0^\infty s^n e^{-s} ds.
\]

Let now \( \xi \) be the unique minimizer in question. Since, as explained above, \( \xi \) is an interior point of the convex set \( C \cap \{ (l, \cdot) = m \} \) it follows from the previous lemma that

\[
b_\xi = c l
\]

for a non-zero constant \( c \). Finally, since \( 1 = \langle \xi, b_\xi \rangle = \langle \xi, l \rangle / m \) it must be that \( c = 1/m \), which concludes the proof of Prop 2.4. \( \square \)

### 2.2. Variational construction of the solution \( f \).

Given a proper convex cone \( C^* \) in \( (\mathbb{R}^m)^* \) and an \( l \)-normalized Reeb vector \( \xi \) denote by \( \mathcal{E}_\xi(\mathbb{R}^m) \) the space of all convex functions \( \Phi \) on \( \mathbb{R}^m \) such that \( \Phi \) is equivariant with respect to \( \xi \) and \( (\partial \Phi)(\mathbb{R}^m) = P_\xi \), where \( P_\xi \) is the convex body defined by formula 2.5 and \( \partial \Phi \) denotes the sub-gradient of \( \Phi \), viewed as a multivalued function. Denote by \( \mathcal{E}_\xi^1(\mathbb{R}^m) \) the subspace of \( \mathcal{E}_\xi(\mathbb{R}^m) \) consisting of all \( \Phi \) such that \( \Phi^* \in L^1(P_\xi) \), where \( \Phi^* \) is the Legendre–Fenchel transform of \( \Phi \), i.e. the convex function on \( P_\xi \) defined by

\[
\Phi^*(p) := \sup_{x \in \mathbb{R}^m} \langle x, p \rangle - \Phi(x).
\]
Consider the following functionals on $\mathcal{E}_\xi^1(\mathbb{R}^m)$:

\begin{equation}
(2.15) \quad \mathcal{D}(\Phi) := \frac{1}{m} \log \int_{\mathbb{R}^m/\mathbb{R}\xi} e^{-m\Phi} e^{(l,x)i_\xi} dx - \mathcal{E}(\Phi)
\end{equation}

where $i_\xi dx$ denotes the $(n - 1)$-form on $\mathbb{R}^m$ obtained by contracting $dx$ with $\xi$, which yields a well-defined measure on $\mathbb{R}^m/\mathbb{R}\xi$ and

\[ \mathcal{E}(\Phi) := -\frac{1}{V(P_\xi, d\lambda_{m-1})} \int_{P_\xi} \Phi^* d\lambda_{m-1}, \]

where $d\lambda_{m-1}$ denotes any choice of Lebesgue measure on the hyperplane $\{ \xi = 1 \}$ and $V(P_\xi, d\lambda_{m-1})$ denotes the volume of $P_\xi$ with respect to $d\lambda_{m-1}$.

**Theorem 2.6.** The following holds.

- The functional $\mathcal{D}$ is bounded from below on $\mathcal{E}_\xi^1(\mathbb{R}^m)$ iff $\xi$ is the unique minimizer of the volume $V(\xi)$ on the Reeb cone.
- $\Phi$ minimizes $\mathcal{D}$ on $\mathcal{E}_\xi^1(\mathbb{R}^m)$ iff $e^{\Phi}$ is a solution to the equation in Theorem 2.2.

Moreover, if $\Phi_j$ is a minimizing sequence for $\mathcal{D}$ in $\mathcal{E}_\xi^1(\mathbb{R}^m)$, i.e.

\[ \lim_{j \to \infty} \mathcal{D}(\Phi_j) = \inf_{\mathcal{E}_\xi^1(\mathbb{R}^m)} \mathcal{D} > -\infty, \]

then there exist sequences $a_j \in \mathbb{R}^m$ such that the translated sequence $\Phi_j(a_j + x)$ converges in $C_0^1(\mathbb{R}^m)$ to $\Phi \in \mathcal{E}_\xi^1(\mathbb{R}^m)$, where $e^{\Phi}$ is a solution to the equation in Theorem 2.2.

**Proof.** In terms of the coordinates $(s, t)$ on $\mathbb{R}^m$, appearing in formula 2.10 we can write $\Phi = \phi(s) + t$, where $\phi(s)$ is a convex function on $\mathbb{R}^{m-1}$ such $(\partial \phi)(\mathbb{R}^{m-1}) = Q_\xi$ and $\phi^* \in L^1(Q_\xi)$. Denote by $\mathcal{E}^1(\mathbb{R}^{m-1})$ the space of all such functions $\phi$. Then

\[ \mathcal{D}(\Phi) = D(\phi) := -\frac{1}{m} \log \int_{\mathbb{R}^n} e^{-m\phi(s)} ds + \frac{1}{V(Q_\xi, dq)} \int_{Q_\xi \subset \mathbb{R}^n} \phi^*(q)dq. \]

Recall that, as shown in the proof of Theorem 2.2, $\xi$ minimizes $V(\xi)$ iff 0 is the barycenter of the convex body $Q_\xi$. By [7, Thm 2.16] the latter condition equivalently means that $D$ is bounded from below on $\mathcal{E}^1(\mathbb{R}^{m-1})$. Moreover, as shown in Section 2.8 and Section 2.9 of [7] $\phi$ minimizes $D$ on $\mathcal{E}^1(\mathbb{R}^{m-1})$ iff $\phi$ is a smooth solution of the equation (2.11) which, as shown in the proof of Theorem 2.2 equivalently means that $e^\Phi$ is a solution to the equation in Theorem 2.2. Finally, the convergence of a $D$-minimizing sequence $\Phi_j$ follows from the convergence of a $D$-minimizing sequence $\phi_j$ established in the proof of [7 Thm 1.1].

In fact, setting

\[ J(\Phi) := -\mathcal{E}(\Phi) + \sup_{\mathbb{R}^m} (\Phi - \Phi_{P_\xi}) \]

the proof of the previous theorem shows (thanks to the results in [7] alluded to above) that $\xi$ minimizes the volume function on the Reeb cone iff $\mathcal{D}$ is coercive, modulo translations or more precisely: iff there exists a constant $C > 0$ such that

\[ \mathcal{D}(\Phi) \geq \frac{1}{C} \inf_{a \in \mathbb{R}^m} J(a^* \Phi) - C, \]

where $a^* \Phi(x) := \Phi(x + a)$ is the action on $\Phi$ by a translation of $\mathbb{R}^m$. \(\square\)
3. Toric varieties and Conical Calabi–Yau metrics

3.1. The toric setup. Fixing the rank $m$ we will denote by $T_C$ the complex torus $(\mathbb{C}^*)^m$, endowed with its standard group structure and by $T$ the corresponding compact torus. The group of characters on $T_C$ and the group of 1-parameters subgroups of $T_C$ are denoted, as usual, by $M$ and $N$, respectively:

$$M := \text{Hom}(T_C, \mathbb{C}^*), \quad N := \text{Hom}(\mathbb{C}^*, T_C)$$

Composing the corresponding homomorphisms yields a character on $\mathbb{C}^*$, which defines a non-degenerate pairing between $M$ and $N$. The real vector spaces $N \otimes \mathbb{R}$ and $M \otimes \mathbb{R}$ are naturally identified with the Lie algebra $\mathfrak{t}$ of $T$ and its dual $\mathfrak{t}^*$ respectively.

Let now $Y$ be an $m$–dimensional normal affine toric variety, i.e. $Y$ is a normal affine variety endowed with a faithful action by the complex torus $T_C$ with an open dense orbit. An extensive exposition of general toric varieties is given in the book [15]. Anyway, we will recall the basic properties of affine toric varieties that we shall need.

We will identify $T_C$ with the corresponding open orbit in $Y$. Decompose the ring of holomorphic functions $\mathcal{R}(Y)$ on $Y$ with respect to the induced $T_C$–action on $\mathcal{R}(Y)$:

$$\mathcal{R}(Y) = \bigoplus_{p \in G} \mathbb{C} F_p,$$

where $G$ denotes the corresponding set of characters in $M$. Restricting $F_p$ to the dense $T_C$–orbit, endowed with its standard complex coordinates $(z_1, ..., z_m)$, we can identify a character $F_p$ with a multinomial:

$$F_p := z^p := z_1^{p_1} \cdots z_m^{p_m},$$

where $G$ has been identified with $\mathbb{Z}^m$. Since $\mathcal{R}$ is a finitely generated ring the set $G$ forms a commutative semi-group in $M$, which is finitely generated (i.e. $G$ is an affine semi-group). Denote by $\mathcal{C}^*$ the convex hull of the set $G$:

$$\mathcal{C}^* := \text{Conv}(G) \subset M \otimes \mathbb{R}(= \mathfrak{t}^*)$$

Then $\mathcal{C}$ is a proper convex cone, in the sense of Section 2 which is polyhedral and the action of $T$ on $Y$ is good, in the sense of Section 1.1 (see [15] Thm 1.3.5 and [15] Prop 1.2.12; proper convex cones are called strongly convex cones in [15]).

Denote by $\mathcal{C}$ the dual convex cone defined by all elements in the linear dual of $M \otimes \mathbb{R}$ which are non-negative on $\mathcal{C}^*$. Given an element $\xi$ in $N \otimes \mathbb{R}$ its weights are defined by the real numbers

$$\lambda_\xi(p) := \frac{1}{t} \mathcal{L}_\xi F_p = \langle \xi, p \rangle, \quad p \in G$$

Hence, $\xi$ is a Reeb vector (in the terminology recalled in Section 1.1) iff $\xi$ is an interior point of the cone $\mathcal{C}$. Accordingly, the interior of $\mathcal{C}$ will be called the Reeb cone. Denote by $\xi_1, ..., \xi_d$ the non-zero primitive vectors in the lattice $N$ such that

$$\mathcal{C}^* = \langle \xi_i, \cdot \rangle \geq 0, \quad i = 1, ..., d$$

Equivalently, this means that $\mathcal{C}$ is the convex hull of $0, \xi_1, ..., \xi_d$.

Given a Reeb vector $\xi$ and a positive number $C$ we let $N(\xi, C)$ be the number of lattice points $p$ in $\mathcal{C}$ satisfying $\langle \xi, p \rangle \leq C$. Fixing an enumeration of the points in question we have the following basic
Proposition 3.1. Given a Reeb vector $\xi$ and $C > 0$. For $C$ sufficiently large the map
$$(\mathbb{C}^*)^m \to \mathbb{C}^N, \ z \mapsto Z := (z^{p_1}, ..., z^{p_N})$$
with $N := N(\xi, C)$ is injective and $T_\mathbb{C}$–equivariant and its closure is isomorphic to the affine variety $Y$ corresponding to the convex cone $\mathbb{C}^*$. 

Proof. Since the group $G$ is finitely generated we have that $\mathbb{C}^* = \mathbb{N}\{p_1, ..., p_r\}$ for a finite number of $p_i \in \mathbb{C}^*$. For $C \geq \min \{\xi, p_i\}$ we thus get $\mathbb{C}^* = \mathbb{N}\{\mathbb{C}^* \cap (\xi, \cdot) \leq C\}$ and hence the proposition follows from [15] Prop 1.1.14].

Given a Reeb vector $\xi$ exponentiating the vector field on $T_\mathbb{C}$ defined by $-J\xi$ yields an $\mathbb{R}^*$–action on $Y$. Accordingly, the embedding in the previous proposition is $\mathbb{R}^*$–equivariant when $\mathbb{C}^N$ is endowed with the linear action
$$(c, Z_i) \mapsto c^{(\xi, p_i)}Z_i, \ \mathbb{R}^* \times \mathbb{C}^N \to \mathbb{C}^N$$
(the space $\mathbb{C}^N$ endowed with such an $\mathbb{R}^*$–action is called weighted $\mathbb{C}^N$). We also recall the following

Proposition 3.2. An affine toric variety $Y$ is $\mathbb{Q}$–Gorenstein iff there exists an element $l \in M \otimes \mathbb{Q}$ such that
$$\langle \xi, l \rangle = 1$$
In particular, $l$ is an interior point of the cone $\mathbb{C}^*$.

Proof. This is well-known [16], but for the convenience of the reader we outline a proof. First assume that $Y$ is $\mathbb{Q}$–Gorenstein (which is the direction that we shall need). Then it admits a (manyvalued) holomorphic top form $\Omega$ (defined and non-vanishing on the regular locus) which is equivariant with respect to the action of the torus $T_\mathbb{C}$. Hence, the restriction of $\Omega$ to $T_\mathbb{C} \subset Y$ may be expressed as
$$\Omega = \chi \Omega_0, \ \Omega_0 := z_1^{-1}dz_1 \wedge ... \wedge z_m^{-1}dz_m$$
for some many valued character $\chi$, i.e. $\chi^r$ is a well-defined character for some integer $r$. The holomorphicity condition and the non-vanishing condition implies that $\nu_{F_1}(\Omega) = 0$, where $\nu_{F_1}$ denotes the order of vanishing along a given prime divisor on $Y$ and $F_1$ denote the toric prime divisors whose union give $Y - T_\mathbb{C}$. It follows readily from its definition that $\Omega_0$ defines a rational (meromorphic) top form on $Y$ such that $\nu_{F_1}(\Omega_0) = 1$. Hence, $\nu_{F_1}(\Omega) = 0$ iff $\nu_{F_1}(\chi) = 1$. But writing $\chi^p = z^p$ for some $p \in \mathcal{C} \cap M$ this means that $\langle \xi, l/p/r \rangle = 1$ (by the “orbit-cone correspondence” [15]). Hence, we can take $l := p/r$. Conversely, if an element $l$ as in the proposition exists we can simply define $\Omega$ by formula (3.1). Then the previous argument shows that $\nu_{F_1}(\Omega) = 0$, which shows that $\Omega$ defines a non-vanishing equivariant (manyvalued) holomorphic top form on $Y$ with positive weight under $\xi$. By general facts, this implies that $Y$ is $\mathbb{Q}$–Gorenstein. \hfill $\Box$

3.1.1. Conical Kähler potentials $r^2$ and $\xi$–equivariant functions $\Phi$. A function $r$ on $Y$ will be said to be radial with respect to $\xi$ if it is non-negative homogeneous of degree one under the corresponding $\mathbb{R}^*$–action and $\xi$–invariant. A function $f$ on $Y$ will be called a conical Kähler potential with respect to $\xi$ if it may be expressed as $f = r^2$ for a $\xi$–radial function $r$ on $Y$ and $f$ is smooth and strictly plurisubharmonic (psh) on $Y^*$ and continuous on $Y$ (i.e. $f$ is the restriction of a smooth strictly plurisubharmonic function on some local embedding of $Y^*$ into $\mathbb{C}^M$).
Lemma 3.3. Let $Y$ be an affine variety endowed with the action of a good torus $T$. For any given Reeb vector $\xi$ there exists a conical Kähler potential $f_\xi := r_\xi^2$ on $Y$ which is $T$–invariant. Moreover, $f_\xi$ may be chosen so that the set $\{f_\xi = 1\}$ is independent of $\xi$.

Proof. In the case when $Y - \{y_0\}$ is smooth this is the content of [35, Lemma 2.2]. In particular, when $Y$ is weighted $\mathbb{C}^N$ we can define $f_\xi$ so that $\{f_\xi = 1\}$ is the unit-sphere. The general case when $Y$ is singular then follows by embedding $Y$ equivariantly into weighted $\mathbb{C}^N$ (as in Prop 3.1) and restricting $f_\xi$ on $\mathbb{C}^N$ to the image of $Y$ (under this embedding the action of $T$ on $Y$ is intertwined by the linear action of a good torus on $\mathbb{C}^N$).

Note that if $r^2$ is a conical Kähler potential, then the function $\Phi := \log r^2$ is $\xi$–equivariant in the following sense:

$$L_\xi \Phi = 0, \quad L_{-J_\xi} \Phi = 2.$$  

3.2. The YTD conjecture for toric affine varieties. In order to prove Theorem 1.3 stated in the introduction, it will be enough to prove the following

Theorem 3.4. Let $Y$ be an affine toric variety which is $\mathbb{Q}$–Gorenstein and denote by $T_m$ the corresponding maximal torus and $\xi$ a Reeb vector in the Lie algebra of $T_m$. Then the following is equivalent:

- $(Y, T_m, \xi)$ admits a conical Calabi-Yau metric, which is $T_m$–invariant (and uniquely determined modulo the action of $T$)
- $\xi$ is the unique minimizer of the volume functional on the space of normalized Reeb vectors

Indeed, if there exists a conical Calabi-Yau metric associated to a Reeb vector $\xi$, then it follows from [39, Cor 4.2] that $(X, T, \xi)$ is K-stable for any torus $T$ containing $\xi$ in its Lie algebra. Conversely, if $(X, T, \xi)$ is K-stable, then it follows from [12, Thm 6.1] that $\xi$ minimizes the volume functional on the space of normalized Reeb vectors in the Lie algebra of the maximal torus $T_m$.

Remark 3.5. More precisely, in [12, Thm 6.1] it is assumed that $Y$ is an isolated Gorenstein singularity, but, as pointed out in [37, Remark 2.27], the result holds for a general $\mathbb{Q}$–Gorenstein affine variety $Y$ with klt singularities (and thus, in particular, for any toric affine $\mathbb{Q}$–Gorenstein variety, since any such variety has klt singularities [16]).

The proof of Theorem 3.4 is divided into two parts. First, in the next section, we deduce from Theorem 1.3 that $\xi$ minimizes the volume functional iff there exists a conical Calabi-Yau metric on $T_C \subset Y$ with locally bounded potential $r^2$. Finally, the regularity of $r^2$ on all of $Y_{reg}$ is established in Section 4.

3.3. Existence of a conical Calabi-Yau metric on $T_C \subset Y$ with locally bounded potential $r^2$. Let $\text{Log}$ be the standard map from $T_C$ onto $\mathbb{R}^m$ defined by

$$\text{Log} : T_C \to \mathbb{R}^m, \quad z \mapsto x := (\log(|z_1|^2), \ldots, \log(|z_n|^2))$$

so that the compact torus $T$ acts transitively on its fibers. We will refer to $x$ as the logarithmic real coordinates on $T_C$.

By Prop 3.2 we can write

$$i^{m^2} \Omega \wedge \bar{\Omega} = \text{Log}^* \left( e^{(l,x)} dx \right) \wedge d\theta,$$

where $\supp L = \{x \in \mathbb{R}^m \mid \text{Log}(x) \in Y_{reg}\}$.
where $d\theta$ denotes the invariant measure on $T$. Accordingly, if $r$ is conical with respect to $\xi$, then, writing $r^2 = \text{Log}^* f$, the Calabi-Yau equation on $T_C \subset Y$

$$
(dd^c(r^2))^m = r^m \Omega \wedge \bar{\Omega}, \quad dd^c(r^2) > 0
$$

is equivalent to the real MA-equation for a smooth strictly convex function $f$ in Theorem 2.2. Next, note that in the toric setting the volume of $\xi$ defined by formula 1.3 coincides with the functional $V(\xi)$ defined by formula 2.2 in the convex-geometric setting (as follows from formula 2.14). Moreover, the bound 2.3 on $f$ translates into the bound

$$
r^2 \leq C \exp \log^* \left( \sup_{p \in P_\xi} \langle x, p \rangle \right)
$$

As we will be explained next this bound implies that $r^2$ is locally bounded in a neighborhood of the vertex point $y_0$ of $Y$. Indeed, it will be enough to show that the rhs in the previous inequality extends from $\mathbb{C}^m$ to a continuous function on $Y$. Now, since $P_\xi$ is a compact and convex polyhedron the sup in the rhs in the formula in question can be written as the maximum of the corresponding vertices of $\partial P_\xi$. Since the maximum of a finite number of continuous functions is continuous it will thus be enough to show that that for a fixed $p \in P_\xi$ the function $F_p$ on $Y$ defined by $\exp \log^* (\langle x, p \rangle)$ is continuous on $Y$. But since $p \in \mathbb{C}^*$ and $\mathbb{C}^*$ is the convex hull of $\mathbb{C}^* \cap M$ there exists a finite number of points $p_1, ..., p_L$ in $\mathbb{C}^* \cap M$ that such that

$$
p = a_1 p_1 + \cdots + a_L p_L
$$

Hence, $F_p$ is the pull-back to $Y$ under the embedding in Prop 3.1 (for $C$ sufficiently large) of the following continuous function on $\mathbb{C}^N$ with coordinates $Z_{i_1}, ..., Z_{i_L}$:

$$
|Z_{i_1}|^{2a_1} \cdots |Z_{i_L}|^{2a_L}
$$

and thus continuous, as desired.

### 3.4. Intermezzo: A variational construction of $r^2$

We next make a short digression to point out that $r^2$, whose existence was established in the precious section, can be constructed by minimizing the Ding functional $D$ (but this fact will not be used in the sequel).

Denote by $\mathcal{H}(Y)$ the space of all functions $\Phi$ on $Y$ such that $e^\Phi$ is a $\xi-$conical Kähler potential on $Y$. First recall that exists a functional $E$ on $\mathcal{H}(Y)$ determined, up to an additive constant, by the property that along any affine curve $\Phi_t$ in $\mathcal{H}(Y)$

$$
\frac{dE(\Phi_t)}{dt} = \frac{1}{mV(\xi)} \int_{Y^*/\mathbb{R}^*} \frac{d\Phi_t}{dt} (dd^c\Phi_t)^{m-1} \wedge \bar{d} \Phi_t
$$

where the basic form $\frac{d\Phi_t}{dt} (dd^c\Phi_t)^{m-1} \wedge \bar{d} \Phi_t$ on $Y$ has been identified with a top form on the compact space $Y^*/\mathbb{R}^*$ (compare Section 4.3) and $V(\xi)$ denotes the volume of $\xi$ (this functional is denoted by $I$ in the appendix of [20]). More generally, we denote by $E$ the smallest usc extension of the functional $\tilde{E}$ to the space $PSH(Y, \xi)$ of all $\xi-$equivariant psh functions $\Phi$ on $Y$. Set

$$
E^1(Y) := \{ \Phi \in PSH(Y, \xi) : E(\Phi) > -\infty \}
$$

Now consider the Ding functional on $E^1(Y)$ defined by

$$
D(\Phi) := -\frac{1}{m} \log \int_{Y^*/\mathbb{R}^*} i_{-J_\xi} e^{-m\Phi} r^2 \Omega \wedge \bar{\Omega} - E(\Phi)
$$
where the basic $(m-1)$–form $i_{-J\xi}e^{-(m+1)\Phi}i^m\Omega\wedge\bar{\Omega}$ on $Y$ has been identified with a measure on $Y^*/\mathbb{R}^*$ (the functional $\mathcal{D}$ essentially coincides with the Ding functional appearing in [20, 13, 39]). Next, we specialize to the toric setting and denote by $E^1(Y)_T$ the subspace of all $T$–invariant functions in $E^1(Y)$, where $T$ denotes the maximal torus. Then it is not hard to see that, under the Log map $3.3$, the space $E^1(Y)_T$ corresponds to the space $E^1(\mathbb{R}^m)$ introduced in section 2.2. Moreover, the Ding functional $D(\Phi)$ corresponds to the functional $D(\Phi)$ in formula 2.15. Thus theorem 2.6 yields a variational construction of the potential $r^2$ of the conical Calabi-Yau metric on $Y$.

4. Regularity of the Calabi-Yau metric on the regular locus of $Y$

Let us first come back to the general setting of a good $T$–action on a $\mathbb{Q}$–Gorenstein affine variety $(Y, T)$ (recalled in section 1.1). Let $r$ be locally bounded psh solution to the Calabi-Yau equation 1.2 on $Y$, in the usual sense of local pluripotential theory. As pointed out in [39] one would expect that this implies that $r$ is, in fact, smooth on the regular locus $Y_{\text{reg}}$. Here we will provide a proof in the toric setting. In fact, the proof can be generalized to the non-toric setting, but since this would require a detour into pluripotential theory (see remark 4.10) the development of the necessary pluripotential theory is left for the future.

4.1. The general setup. Let $r^2$ be a locally bounded psh $\xi$–conical function on $Y$ which satisfies the following equation on $Y$:

\[(dd^c(r^2))^m = i^{m^2} \Omega \wedge \bar{\Omega},\]

in the usual sense of pluripotential theory.

Remark 4.1. If there exists a locally bounded psh solution $r^2$ as above then it follows that $Y$ has log terminal (klt) singularities. Indeed, by local pluripotential theory the integral of $(dd^c(r^2))^m$ over any neighborhood $V_0$ of $y_0$ with compact closure is finite. Hence, $i^{m^2} \Omega \wedge \bar{\Omega}$ gives finite volume to $V_0$, which, by general principles, implies that $Y$ has klt singularities [27].

In fact, in the toric setting, where $T$ is has rank $m$, we already know from section 3.3 that $r$ is smooth on $T_C \subseteq Y_{\text{reg}}$ and it will be enough to consider the equation there. Take a $T$–equivariant smooth resolution $Y'$ of $Y$ (as constructed in the proof of lemma 4.2 below) and denote by $p$ the corresponding projection:

\[p : Y' \to Y\]

and by $D'$ the corresponding discrepancy divisor, i.e. the effective $p$–exceptional $\mathbb{Q}$–divisor defined by the relation

\[(4.2) \quad p^*K_Y = K_{Y'} + D' \]

(since $Y$ has Kawamata Log Terminal (klt) singularities the divisor $D'$ is subklt, i.e the coefficients of $D'$ are $< 1$). We may assume that the divisor $D'$ is $T$–invariant and has simple normal crossings. Denote by $U$ the Zariski open subset of $Y'$ defined by

\[U := p^*Y_{\text{reg}}\]

Since the Reeb vector field $\xi$ may be identified with an element in the Lie algebra of $T$ and $T$ acts on $Y'$ we can identify $\xi$ with a vector field on $Y'$. We will use a prime to indicate pullbacks
of functions and forms to $Y'$. Accordingly, the equation (4.3) induces the following equation on $U$:

\[(dd^c(r')^2)^m = im^2 \Omega' \wedge \overline{\Omega'},\]

Let $\varphi$ be the function on $Y$ defined by

\[r^2 = f_\xi e^{\varphi},\]

where $f_\xi$ is a fixed conical Kähler potential on $Y$ (as furnished by Lemma 3.3) and denote by $\theta$ the following semi-positive smooth form on $Y^*$:

\[\theta := dd^c \log f_\xi.\]

Note that the function $\varphi$ is invariant under both $\xi$ and $J\xi$.

### 4.2. A transversal Kähler metric $\omega_B$ on $Y' - \pi^{-1}(y_0)$ and the corresponding $\xi$–equivariant barrier $\Phi_B$ on $U$

Let us first recall some general terminology in the context of foliations. Let $S$ be a manifold and $\mathcal{F}$ a foliation on $S$ (i.e. an integrable subbundle of $TS$). The sheaf $\Omega_B$ of basic forms on $S$ is defined as follows. Given an open subset $U \subset S$ the space $\Omega_B(U)$ is defined as the space of all differential forms $\alpha$ on $U$ such that

\[L_V \alpha = 0, \quad i_V \alpha = 0\]

for any local vector field $V$ tangent to $\mathcal{F}$ (the definition is made so that if the set theoretic quotient $B := S/\mathcal{F}$ is a manifold, then $\Omega_B$ is the pull-back to $S$ of the sheaf of smooth forms on $B$). In particular, $\Omega_B^1$ is naturally isomorphic to the sheaf of sections of the dual of the normal bundle $TS/\mathcal{F}$ of $\mathcal{F}$. The exterior derivative $d$ on $S$ preserves $\Omega_B$.

Now consider the case when $S := Y' - \pi^{-1}(y_0)$ and $\mathcal{F}$ is the foliation generated by the commuting vector fields $\xi$ and $J\xi$. Since $T\mathcal{F}$ is closed under the complex structure $J$ on $S$ there is an induced complex structure on $TS/\mathcal{F}$. A closed real $(1,1)$–form $\omega_B$ on $S$ will be called a transversal Kähler form if

\[\omega_B > 0 \text{ on } TS/\mathcal{F},\]

i.e. if the symmetric form $\omega(\cdot, J\cdot)$ is positive definite on $TS/\mathcal{F}$ (note that, since by assumption, $\omega_B$ is basic it descends to the quotient $TS/\mathcal{F})$. Given a point $p \in S$ there exists, by the inverse function theorem, local holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ on $S$ centered at $p$ and such that the derivations with respect to the imaginary and real parts of $w$ are given by $\xi$ and $-J\xi$ respectively. In particular,

\[\xi^{1,0}w = 1, \quad \xi^{1,0}z_i = 0, \quad \xi^{1,0} := \frac{1}{2}(J\xi + i\xi)\]

We will say that $(z, w)$ are holomorphic coordinates adapted to $\xi$ and call $z$ for the corresponding transversally holomorphic coordinates.

In order to construct a transversal Kähler form $\omega_B$ on $Y' - \pi^{-1}(y_0)$ and an associated $\xi$–equivariant potential $\Phi_B$ (in the sense of formula 3.2) on $U \subset Y' - \pi^{-1}(y_0)$ (to be used as a barrier in the transversal a priori estimates) we start with the following

**Lemma 4.2.** There exists a $T$–invariant Kähler form $\omega$ on $Y'$ and a smooth $T$–invariant function $\Phi_\omega$ on $U$ such that $\Phi_\omega \to -\infty$ at $\partial U$ and

\[dd^c \Phi_\omega = \omega\]

on $U$. 
Proof. Take an equivariant embedding of \((Y,T)\) into \(\mathbb{C}^N\), so that \(T\) corresponds to a linear torus action on \(\mathbb{C}^N\). Denote by \(\tilde{Y}\) the Zariski closure of \(Y\) in \(\mathbb{P}^N\) under the standard embedding of \(\mathbb{C}^N\) into \(\mathbb{P}^N\). There exists a \(T\)--equvariant resolution \(\pi : \tilde{Y}^t \to \tilde{Y}\) such that the inverse image of the singular locus of \(\tilde{Y}\) coincides with the support of an effective divisor \(E\) on \(X\) (\(\pi\) can be taken as a strong functorial resolution, as explained in [28, Paragraphs 8,9]). Denote by \(E_i\) the irreducible components of \(E\). Since \(\pi\) is relatively ample there exist positive numbers \(a_i\) such that the \(\mathbb{Q}\)--line bundle \(A := \pi^*O(1) - \sum a_i E_i\) over \(\tilde{Y}^t\) is ample [34]. In other words, we get a decomposition as \(\mathbb{Q}\)--line bundles

\[
\pi^*O(1) = A + \sum_{i} a_i E_i
\]

with \(A\) ample and \(a_i > 0\). Fix a \(T\)--invariant metric \(h_A\) on \(A\) whose curvature form defines a Kähler metric \(\omega\) on \(\tilde{Y}^t\) and denote by \(h\) the corresponding singular metric on \(\pi^*O(1)\), induced from the decomposition [44]. Now set \(Y' := \pi^{-1}(Y)\). Since the restriction of \(\pi^*O(1)\) to \(Y'\) is trivial - trivialized by the section \(s\) obtained by pulling back the standard trivialization of \(O(1)\) over \(\mathbb{C}^N\) - the function function \(\Phi_\omega := -\log h(s)\) has the required properties. \(\square\)

We can now construct \(\omega_B\) and the barrier \(\Phi_B\) by mimicking the construction of a Kähler metric on the symplectic quotient of a Symplectic Kähler manifold \((S,\omega)\):

**Proposition 4.3.** There exists a transversal basic Kähler metric \(\omega_B\) on \(Y' - \pi^{-1}(y_0)\) and an \(\xi\)--equivariant function \(\Phi_B\) on \(\mathcal{U}\) such that \(\Phi_B \to -\infty\) at \(\partial \mathcal{U}\) and

\[
dd^c \Phi_B = \omega_B
\]

on \(\mathcal{U}\).

**Proof.** First observe that the Reeb vector field \(\xi\) is Hamiltonian with respect to the symplectic form defined by \(\omega\) in the previous lemma. This equivalently means that there exists a function \(H\) on \(Y' - \pi^{-1}(y_0)\) such that

\[
\nabla H = -J\xi,
\]

where \(\nabla\) denotes the gradient with respect to the Kähler metric \(\omega\). Indeed, the existence of the Hamiltonian function \(H\) follows from the fact that \(\omega\) is the curvature of a Hermitian metric on a line bundle \(A\) over \(Y'\) and that \(\xi\) lifts to \(A\), preserving the Hermitian metric (and hence \(H\) can be explicitly defined as the derivative of the logarithm of the metric on \(A\) along \(-J\xi\)). Since \(-J\xi\) generates the \(\mathbb{R}^+\)--action on \(Y' - \pi^{-1}(y_0)\) the equation [4.5] implies that \(H\) is strictly increasing along the \(\mathbb{R}^+\)--action. Consequently, for \(\lambda\) a sufficiently large generic positive number,

\[
M_\lambda := \{H = \lambda\}
\]

is a compact smooth submanifold of \(Y' - \pi^{-1}(y_0)\), which is diffeomorphic to \((Y' - \pi^{-1}(y_0))/\mathbb{R}^+\). Denote by \(\pi_\lambda\) the submersion from \(Y' - \pi^{-1}(y_0)\) onto \(M_\lambda\) (whose fibers are the \(\mathbb{R}^+\)--orbits) and by \(i_\lambda\) the embedding of \(M_\lambda\) into \(Y' - \pi^{-1}(y_0)\):

\[
\pi_\lambda : Y' - \pi^{-1}(y_0) \to M_\lambda, \ i_\lambda M_\lambda \hookrightarrow Y' - \pi^{-1}(y_0)
\]

Now, given a point \(p\) in \(M_\lambda\) fix holomorphic coordinates \((z, w) \in \mathbb{C}^n \times \mathbb{C}\) adapted to \(\xi\) and defined on a neighborhood \(U_p\) of \(p\) in \(\mathcal{U}\). Consider the following \(\xi\)--invariant function on \(M_\lambda \cap \mathcal{U}\):

\[
\Phi_\lambda := i_\lambda^*(\Phi - \lambda \Re w)
\]
and the following $\xi$–invariant function on $U_p$:

$$\Psi_B := \pi_\lambda^* \Phi + \lambda \Im w$$

In fact, this yields a well-defined function in a neighborhood of $M_\lambda \cap U$. Indeed, if $(z', w')$ are $\xi$–adapted holomorphic coordinates on $U_{p'}$ for $p' \in M_\lambda$, then $w' - w = f(z)$ for some holomorphic function $f(z)$ of $z$, defined on on $U_p \cap U_{p'}$ (since $\xi^{1,0}(w' - w) = 0$). As a consequence,

$$\Im w - i_\lambda^* (\Im w) = \Im w' - i_\lambda^* (\Im w'),$$

showing that $\Psi_B$ is well-defined in a neighborhood of $M_\lambda \cap U$. Next, observe that the Lie derivative of $\Psi_B$ along the generator of the $\mathbb{R}^*$–action is given by

$$(4.7) \quad \mathcal{L}_{-J\xi} \Psi_B = \lambda.$$

Indeed, by construction, $\Psi_\lambda$ is $\mathbb{R}^*$–invariant and $\mathcal{L}_{-J\xi} \Im w = 1$. As a consequence, the function $\Psi_B$ admits a unique smooth extension to all of $Y' - \pi^{-1}(y_0)$ preserving the property (4.7). This means that the function

$$(4.8) \quad \Phi_B := 2\lambda^{-1} \Psi_B$$

is $\xi$–invariant and satisfies $\mathcal{L}_{-J\xi} \Phi_B = 2$. Next, observe that the two-form

$$\omega_B := dd^c \Phi_B$$

defines a transversal Kähler metric on $U$. This follows from the fact that, locally, $\omega_B$ can be expressed as the pull-back of a local Kähler metric $\omega_\lambda$ on $\mathbb{C}^n$, under the local map defined by the transversal holomorphic coordinates $z$. Indeed, a direct calculation, using the chain rule, reveals that the restrictions of $\omega$ and $\omega_B$ to the hypersurface $M_\lambda$ coincide. In fact, this is the same local computation used in the construction of Kähler metrics on symplectic quotients, where the local function $\Phi_\lambda$ appears as the local Kähler potential of the reduced Kähler metric $\omega_\lambda$ (see the proof of [10] Formula 9.3) and note that our local coordinate $w$ corresponds to the logarithm of the coordinate on $\mathbb{C}^*$ used in [10]).

Finally, we note that $\omega_B$, originally defined on $U$, extends to a transversal Kähler metric on $Y' - \pi^{-1}(y_0)$. Indeed, this is shown by, locally, replacing the Kähler potential $\Psi$ of $\omega$, used in the previous argument, with $\lambda^{-1}$ times any local $\xi$–invariant Kähler potential $\Phi_{U_p}$ for $\omega$ on a given open subset $U_p$ for $p \in M'$. The proof is now concluded by observing that $\Phi_B \rightarrow -\infty$ at $\partial U$, as follows from the fact that $\lambda^{-1} \Phi - \Phi_B$ is bounded on $M_\lambda \cap U$ (since $\lambda^{-1} \Phi - \Phi_B = \Im w - i_\lambda^* (\Im w)$ on $U \cap U_p$ for any given $p \in M_\lambda$ and using that $M_\lambda$ is compact. \qed

### 4.3. Transversal a priori estimates

We are now in a position to apply a transversal version of the a Laplacian estimates in [6] Appendix B) to the compact manifold

$$M' := \left(Y' - \pi^{-1}(y_0)\right)/\mathbb{R}^*,$$

where $\mathbb{R}^*$ denotes, as before, the $\mathbb{R}^*$–action generated by $-J\xi$. Denote by $\pi_{M'}$ the corresponding submersion from $Y' - \pi^{-1}(y_0)$ onto $M'$. The Reeb vector field descends to $M'$ and thus induces a foliation $\mathcal{F}$ on $M'$. Using the terminology recalled in the previous section we denote by $\Omega_B$ the corresponding sheaf of basic forms on $M'$. Hence, a form $\alpha$ on $M'$ is basic iff

$$\mathcal{L}_\xi \alpha = 0, \quad i_\xi \alpha = 0.$$

Moreover, the pull-back $\pi_{M'}^*$ induces a one-to-one correspondence between basic forms on $Y' - \pi^{-1}(y_0)$ and $M'$. Next note that the complex structure $J$ on $Y'$ induces a complex structure on the normal bundle of $\mathcal{F}$ in $M'$, i.e. on $TM'/\mathbb{R}\xi$ (since the pullback of $TM'/\mathbb{R}\xi$ to
The hypersurface $\{\xi, J\xi\}$ in $Y' - \pi^{-1}(y_0)$ and the action of $\mathbb{R}^*$ on $Y' - \pi^{-1}(y_0)$ preserves $J$. By duality we thus obtain a complex structure on $\Omega^1_B(M')$ that we shall denote by the same symbol $J$. Accordingly, we decompose

$$\Omega^1_B \otimes \mathbb{C} = \Omega^1_B \otimes \mathbb{C}^0 + \Omega^0_B \otimes \mathbb{C}^1,$$

as a sum of the eigenspaces corresponding to the eigenvalues $i$ and $-i$ of $J$, respectively. The corresponding decomposition of the exterior derivative yields

$$d =: \partial_B + \bar{\partial}_B, \quad \partial_B: \Omega^{p,q}_B := \left( \Omega^{0,0}_B \right)^p \wedge \left( \Omega^{0,1}_B \right)^q \rightarrow \Omega^{p+1,q}_B.$$ 

The pullback $\pi_{M'}^*$ induces a one-to-one correspondence between basic $(p, q)-$forms on $Y' - \pi^{-1}(y_0)$ and the sheaf $\Omega^{p,q}_B$ on $M'$. Moreover, under this correspondence the operators $\partial_B$ and $\bar{\partial}_B$ on $M'$ correspond to the ordinary operators $\partial$ and $\bar{\partial}$ on $Y'$.

If $z$ is a local transversally holomorphic coordinate centered at $\tilde{z} \in Y' - \pi^{-1}(y_0)$, then $z_i$ descends to a local $\xi-$invariant function on $M'$ that we shall call a transversally holomorphic coordinate on $M'$. We note the following elementary

**Lemma 4.4.** Let $\alpha$ be a basic $(1, 1)-$form on $Y' - \pi^{-1}(y_0)$. Then the corresponding basic form on $M'$ has the property that, locally,

$$\alpha = \pi^{\ast}_{C, n} \tilde{\alpha},$$

where $\pi_{C, n}$ denotes the local map to $\mathbb{C}^n$ defined by $z$ and $\tilde{\alpha}$ is a local closed $(1, 1)-$form on $\mathbb{C}^n$. Conversely, any form $\alpha$ on $M'$ which locally can be expressed in terms of $\tilde{\alpha}$, as above, is basic.

**Proof.** First note that the local functions $z_1, ... z_n$ descend to define local functions on $M'$. Moreover, the corresponding local differentials $dz_1, ..., dz_N$ span the sheaf $\Omega^1_B$ (since, $\Omega^0_B$ has rank $n$ and, by construction, $dz_i \in \Omega^1_B$). Hence, if $\alpha$ is basic we can locally express

$$\alpha = \sum f_{i_j} dz_i \wedge d\bar{z}_j.$$ 

By using $L_\xi \alpha = 0 = L_\xi dz_j$ it follows that $L_\xi f_{i_j} = 0$. Since $(z, \Omega)$ yield local holomorphic coordinates on $M'$ and $\xi$ is the derivation with respect to the imaginary part $\Im w$ of $w$ this shows that $\alpha$ can be expressed in terms of a form $\tilde{\alpha}$ as in the statement of the lemma. The converse statement follows directly from the definitions.\[\square\]

We will say that a basic function $\psi: M' \rightarrow [\infty, \infty]$ is quasi-psh if the function $\psi$ can be expressed as the restriction to $M'$ of a local quasi-psh function on $Y'$, when $M'$ is identified with the hypersurface $\{f_\xi = 1\}$ in $Y'$.

**Lemma 4.5.** There exist two basic quasi-psh functions $\Psi_\pm$ on $M'$ which are smooth on $U$ such that

$$\left( f_\xi \right)^{-m'} \frac{i}{m^2} \Omega^1 \wedge \Omega^1(\xi, J\xi, \cdot) = e^{m(\Psi_+ - \Psi_-)} \Omega^1$$

on $M'$. Moreover, there exists a constant $A$ such that

$$\frac{i}{2\pi} \partial_B \bar{\partial}_B \Psi_+ \geq -A \Omega_B$$

holds in the sense of currents on $M'$.

**Proof.** Decompose the discrepancy divisor $D'$ in formula \[\ref{discrepancy} as $D' = -D_+ + D_-$ where $D_\pm$ are effective $T-$invariant $\mathbb{Q}-$divisors (since $Y$ has klt singularities we can assume that the coefficients of $D_-$ are $< 1$). It follows from formula \[\ref{discrepancy} that there exists a $T-$invariant volume
form $dV$ on $Y'$, (multivalued) $T$–equivariant holomorphic sections $s_\pm$ of the $\mathbb{Q}$–line bundles over $Y'$ defined by $D_\pm$ and $T$–invariant metrics $\|\cdot\|$ on $D_\pm$ such that

$$i\Omega' \wedge \bar{\Omega}' = \|s_+\|^2 \|s_-\|^{-2} dV$$

Now identify $M'$ with the hypersurface $\{f_z = 1\}$ in $Y'$. Then the formula stated in the lemma holds for some functions $\Psi_\pm$ such that $\Psi_\pm - \log \|s_\pm\|_{M'}^2 \in C^\infty(M')$. All that remains is thus to verify that there exists a constant $C$ such that

$$(4.9) \quad \frac{i}{2\pi} \partial_B \bar{\partial}_B \log \|s_\pm\|^2_{M'} \geq -C \omega_B$$

on $M'$. To this end fix a point $p$ in $M$ and take holomorphic coordinates $(z,w)$ adapted to $\xi$ and centered at $p$. We can then locally express the $T$–invariant functions $\|s_\pm\|^2$ as $\|s_\pm\|^2 = |F_\pm(z,w)|^2 e^{-\phi_\pm(z,w)}$ for local holomorphic functions $F_\pm$ and smooth $T$–invariant functions $\phi_\pm$. Since $F_\pm(z,w)$ are $T$–invariant there exist $\lambda_\pm \in \mathbb{R}$ such that $\partial F_\pm / \partial \theta = i\lambda_\pm F_\pm$, where $\theta$ denotes the imaginary part of $w$. After perhaps replacing $F_\pm$ with $F_\pm e^{-i\lambda w}$ we may thus assume that $F_\pm$ is $\xi$–invariant and hence its restriction to $M'$ defines a local basic function on $M$, which is annihilated by $\partial_B$. Hence, locally, we have

$$\frac{i}{2\pi} \partial_B \bar{\partial}_B \log \|s_\pm\|^2_{M'} \geq -\frac{i}{2\pi} \partial_B \bar{\partial}_B \phi_\pm$$

and since $\omega_B$ is transversely Kähler on $M'$ and $M'$ is compact this proves the inequality $4.9$.

Next, we observe that the function $\varphi'$ satisfies the following “transversal Monge-Ampère equation” on $M' \cap \mathcal{U}$:

**Lemma 4.6.** A conical smooth function $r^2$ satisfies the equation $4.3$ on $Y_{reg}$ iff $\varphi'$ satisfies the following equation on $M' \cap \mathcal{U}$

$$\left(\theta' + \frac{i}{2\pi} \partial_B \bar{\partial}_B \varphi'\right)^n = e^{-(n+1)\varphi'} e^{(n+1)(\Psi_+ - \Psi_-)} \omega_B^n$$

where $\Psi_\pm$ are the functions appearing in the previous lemma.

**Proof.** Setting $\Phi := \log r^2$ gives $dd^c(r^2) = e^\Phi (dd^c \Phi + d\Phi \wedge d^c \Phi)$ and hence the equation $4.3$ is equivalent to

$$(4.10) \quad (dd^c \Phi)^n \wedge d\Phi \wedge d^c \Phi = e^{-(n+1)\Phi} i\Omega' \wedge \bar{\Omega}'$$

Next fix local holomorphic coordinates $(z,w)$ on $\mathcal{U}$ adapted to $\xi$. Note that the local function $\phi := \Phi - 2\Im w$ is basic, i.e. satisfies $\xi \phi = (J\xi) \phi = 0$, using that $r^2$ is conical with respect to $\xi$. Hence, we can identify $\phi$ with a function of $z$ and write $\phi = \phi(z)$, abusing notation slightly. Since $dd^c \Im w = 0$ we can thus locally express

$$(dd^c \Phi)^n \wedge d\Phi \wedge d^c \Phi = (d_z d^c_z \phi(z))^n \wedge d(\Im w) \wedge d^c(\Im w) = (dd^c \Phi)^n \wedge d(\Im w) \wedge d^c(\Im w)$$

As a consequence, contracting the equation $4.10$ with first $-J\xi$ and then $\xi$ yields

$$(dd^c \Phi)^n = (d_z d^c_z \Phi)^n = e^{-(n+1)\varphi'} (f_\xi)^{-2m_i \xi^2} (\Omega \wedge \bar{\Omega})(\xi, J\xi, \cdot)$$

on $\mathcal{U}$. Since $dd^c \Phi = \theta + dd^c \varphi$ on $Y$ the previous argument reveals that $\theta$ is basic on $\mathcal{U}$ and that the equation $4.3$ implies the transversal equation stated in the lemma. The converse statement follows in a similar way, using that the measure appearing in the rhs of formula $4.10$ is invariant under both $\xi$ and $J\xi$. \qed
Remark 4.7. More generally, the previous lemma holds if $\phi'$ is merely assumed bounded (and $\theta'$-psh) if one defines the basic current $(\theta_B + \frac{i}{2\pi} \partial B \bar{\partial} B \phi')^n$ on $M'$ to be the one corresponding by pull-back to the $(n, n)$-current defined in the usual sense of local pluripotential theory on $Y'$ (i.e. in the local sense of Bedford-Taylor).

Setting

$$\text{tr}_{\omega_B} \alpha := \frac{\alpha \wedge \omega_B^{n-1}}{\omega_B^n},$$

for a given basic $(1, 1)$-form $\alpha$, we have the following transversal generalization of the Aubin-Yau Laplacian inequality in Kähler geometry (in Siu’s form, as stated in [6, Appendix B]):

**Lemma 4.8.** Let $\omega$ be a Kähler form on a neighborhood of $M'$ and denote by $\kappa$ the sup over $M'$ of the absolute value of the holomorphic bisectional curvatures of $\omega$. If $\bar{\omega}_B$ is a transversal Kähler form on $M'$, then

$$\text{tr}_{\omega_B} \left( \frac{i}{2\pi} \partial B \bar{\partial} B \log(\text{tr}_{\omega_B} \bar{\omega}_B) \right) \geq -\frac{\text{tr}_{\omega_B} (\text{Ric} \; \bar{\omega}_B)}{\text{tr}_{\omega_B} \omega_B} - \kappa \text{tr}_{\omega_B} \omega_B$$

**Proof.** Thanks to Lemma 4.4 this follows from the usual Laplacian estimate in Kähler geometry (as stated in [6, Appendix B]), since the proof of the latter inequality is purely local. In fact, this shows that $\kappa$ can, more precisely, be taken to be a lower bound on the “transversal” bisectional curvatures of $\omega_B$. \hfill \Box

With these preparations in place we are now ready to repeat the argument in [6, Appendix B]. Set

$$\psi_{\pm} := \Psi_{\pm}, \quad \psi_{-} := \Psi_{-} + \phi'_{\pm},$$

The functions $\psi_{\pm}$ still satisfy an inequality as in Lemma 4.5, since $\theta'$ is smooth and basic. The equation appearing in Lemma 4.6 can thus be expressed as

$$\left( \theta' + \frac{i}{2\pi} \partial B \bar{\partial} B \varphi' \right)^n = e^{(n+1)(\psi_{+} - \psi_{-})} \omega_B^n.$$

Now take two sequences of smooth functions $\psi_{\pm,j}$ on $M'$ decreasing to $\psi_{\pm}$, respectively, and such that

$$\frac{i}{2\pi} \partial B \bar{\partial} B \psi_{\pm,j} \geq -C \omega_B,$$

on $M'$ (the existence of such sequences follows, for example, from the regularization procedure introduced in [3], adapted to the present setting, by solving transversal Monge–Ampère equation on $(M', \omega_B)$ involving a large parameter $\beta$).

Next, fix $\epsilon > 0$ and consider the following equation on $M'$ where $\omega$ is the Kähler form in Lemma 4.2

$$\left( \theta' + \epsilon \omega_B + \frac{i}{2\pi} \partial B \bar{\partial} B \varphi_{j,\epsilon} \right)^n = e^{(n+1)(\psi_{+,j} - \psi_{-,j})} \omega_B^n,$$

for a smooth $\xi$–invariant function $\varphi_{j,\epsilon}$, satisfying

$$\theta' + \epsilon \omega_B + \frac{i}{2\pi} \partial B \bar{\partial} B \varphi_{j,\epsilon} > 0$$
By the transversal generalization of the Calabi-Yau theorem in [21] such a function exists and is uniquely determined modulo an additive constant that we fix by requiring that the sup of \( \varphi_{j,\epsilon} \) over \( M' \) vanishes. Using the \( \mathbb{R}^{+} \)-action generated by \(-J_\xi \) we can identify \( \varphi_{j,\epsilon} \) with a smooth function on \( U \) which is invariant under both \( \xi \) and \( J_\xi \).

**Lemma 4.9.** There exists a uniform constant \( C \) such that
\[
\|\varphi_{j,\epsilon}\|_{L^\infty(M')} \leq C
\]

**Proof.** In the toric setting this follows from Lemma 2.3. To see this first note that if \( T \) is taken to be the compact maximal torus acting on the toric variety \( Y \), then \( Y' \) is a non-singular toric variety, which yields a toric resolution \( Y' \to Y \). We can thus identify \( T \) with an open subset of \( Y' \). Proceeding as in the proof of Theorem 2.2 we have a fibration
\[
(4.13) \quad \text{Log: } T_\mathbb{C}/(\mathbb{R}\xi + \mathbb{R}J_\xi) \to \mathbb{R}^{n+1}/\mathbb{R}\xi \simeq \mathbb{R}^n
\]
Indeed, we can take local \( \xi \)-adapted holomorphic coordinates \((z,w)\) on \( T_\mathbb{C} \) as in Lemma 4.4, which have the property that
\[
s := (\log |z_1|^2, \ldots, \log |z_n|^2) \in \mathbb{R}^n
\]
coinsides with \( s \), appearing in formula 2.10. Under this fibration we can express \( \theta'_B \) and \( \omega_B \) on \( T_\mathbb{C}/(\mathbb{R}\xi + \mathbb{R}J_\xi) \) in terms of smooth convex functions \( f \) and \( g \) on \( \mathbb{R}^n \) (with coordinates \( s \)):
\[
\theta' = d_Bd_B^*\text{Log}^*f, \quad \omega_B = d_Bd_B^*\text{Log}^*g
\]
As a consequence,
\[
\theta' + \epsilon\omega_B = d_Bd_B^*\text{Log}^*(f + \epsilon g)
\]
As will be shown below the closure of \((\nabla(f + \epsilon g))(\mathbb{R}^n)\) is a bounded convex body in \( \mathbb{R}^n \) of the form
\[
P_\epsilon := P_f + \epsilon P_g,
\]
where \( P_f \) and \( P_g \) are bounded convex polytopes arising as the closures of \((\nabla f)(\mathbb{R}^n)\) and \((\nabla g)(\mathbb{R}^n)\), respectively. We can thus express
\[
\varphi_{j,\epsilon} = \phi_{j,\epsilon} - (f + \epsilon g),
\]
for a smooth convex function \( \phi_{j,\epsilon} \) on \( \mathbb{R}^n \) satisfying
\[
(\det \nabla_{s}^2)(\phi_{j,\epsilon})ds = \mu, \quad \nabla\phi_{j,\epsilon}(\mathbb{R}^n) = P_\epsilon,
\]
where the measure \( \mu \) on \( \mathbb{R}^n \) is the volume form on \( \mathbb{R}^n \) corresponding to the basic form \( r^{n-2m}\Omega \wedge \Omega \) under the fibration 4.13 and thus has exponential decay (by formula 2.13). Hence, by the \( L^\infty \)-estimate in Lemma 2.3
\[
\left\| \phi_{j,\epsilon} - \phi_{P_\epsilon} - \sup_\mathbb{R}^n(\phi_{j,\epsilon} - \phi_{P_\epsilon}) \right\|_{L^\infty(\mathbb{R}^n)} \leq C'
\]
for a uniform constant \( C' \) (recall that the support function of a convex set \( P \) is denoted by \( \phi_P \)). Note that
\[
\phi_{P_\epsilon} = \phi_{P_f} + \epsilon \phi_{P_g}
\]
since, in general, the supporting function of a Minkowski sum of convex sets is the sum of the supporting functions. Since we have assumed that \( \sup_\mathbb{R}^n(\phi_{j,\epsilon} - (f + \epsilon g)) = 0 \), this means that in order to conclude that \( \|\phi_{j,\epsilon} - (f + \epsilon g)\|_{L^\infty(\mathbb{R}^n)} \) is uniformly bounded, all that remains is to verify that
\begin{equation}
\|\phi_{P_f} - f\|_{L^\infty(\mathbb{R}^n)} \leq C_f, \quad \|\phi_{P_g} - g\|_{L^\infty(\mathbb{R}^n)} \leq C_g.
\end{equation}

To this end first recall that, by construction, \(\exp(f(s) + t)\) is the convex function on \(\mathbb{R}^m\) corresponding to a \(\xi\)-conical Kähler potential on \(Y\). Moreover, as explained in Section 3.3, the function \(\exp(\phi_{P^\xi}(x))\) on \(\mathbb{R}^n\) (which may be expressed as \(\exp(\phi_{Q^\xi}(s) + t)\), where \(Q^\xi\) is defined below formula 2.12) corresponds to a \(\xi\)-conical continuous function on \(Y\). Hence, \(\exp(f(s) + t)/\exp(\phi_{Q^\xi}(s) + t)\) extends to a continuous function on the compact set \(Y/\mathbb{R}^s\) and is thus bounded. It follows that \(f - \phi_{Q^\xi}\) is bounded on \(\mathbb{R}^n\) and hence the first bound in formula [4.14] holds with \(P_f = Q^\xi\).

To prove the second bound in formula [4.14] recall that \(\omega_B\) is defined as the basic form in Prop 4.3 constructed from the \(T\)-invariant Kähler form \(\omega\) on \(Y'\) in Lemma 4.2. In turn, in the toric setting \(\omega\) is the restriction to \(Y'\) of the curvature form of a toric smooth metric \(h_A\) on a toric line bundle \(A \to Y'\) over toric projective manifold. To simplify the notation we will assume that \(\lambda = 1\) in formula 4.8 but the general case is shown in essentially the same way, up to a scaling. By standard projective toric geometry \(h_A\) corresponds to a convex function \(g_A(s,t)\) on \(\mathbb{R}^m\) such that \(\nabla g_A(\mathbb{R}^n) = P_A\), where \(P_A\) is a bounded convex polytope in \(\mathbb{R}^m\):

\[
\log^* g_A = \Phi := - \log h_A(s_0, s_0),
\]

where \(s_0\) is the standard invariant trivializing holomorphic section of \(A\) over \(T_C \subseteq Y\). Moreover, the Legendre-Fenchel transform \(g^*_A\) of \(g_A\) is bounded on \(P_A\) and equal to \(\infty\) on the complement of \(P_A\) [7 Prop 3.3]. All that remains is thus to verify the following

Claim: (i) \(P_g = P_A \cap \{\xi, \cdot\} = 1\}, \quad \text{(ii) } g^*(p) = (g_A(s,t))^*(p,1)

where \(\xi\) is identified with the unit-vector along the \(t\)-axis in \(\mathbb{R}^n \times \mathbb{R}\) and \(g^*(p)\) denotes the Legendre-Fenchel transform of \(g\) on \(\mathbb{R}^n\). Indeed, accepting this claim we get

\begin{equation}
\|\phi_{P_g} - g\|_{L^\infty(\mathbb{R}^n)} = \|g^*\|_{L^\infty(P_g)} \leq \|g^*_A\|_{L^\infty(P_A)} := C_A < \infty,
\end{equation}

proving the second bound in [4.14] with \(P_g\) equal to the polytope in Claim (i) and \(C_g = C_A\) (the first identity in formula [4.15] follows from the basic properties of the Legendre transforms; see [7 Prop 2.3]). Finally, to prove the Claim above, first note that in terms of the coordinates \((s,t)\) the Hamiltonian function appearing in formula 3.5 is given by

\[
H(s,t) = \partial_t g_A(s,t)
\]

(as follows directly from the definition of \(H\)). Hence, by the definition 4.6

\[
g(s) = g_A(s,t) - t, \quad t = \partial_t g_A(s,t)
\]

But this means that

\[
g(s) = \inf_t (g_A(s,t) - t)
\]

Hence,

\[
g^*(p) = \sup_{s,t} ((p,s) - (g_A(s,t) - t)) = \sup_{s,t} ((p,s) + 1 \cdot t - g_A(s,t)) =: g_A^*(p,1),
\]

which proves (ii) and then (i) follows from the fact that the closure of the gradient image of a convex function \(g\) coincides with the locus where \(g^* < \infty\) (by standard convex analysis).
Now set 
\[ \omega'_B := \theta' + \epsilon \omega_B, \quad \overline{\omega}_B := \theta' + \epsilon \omega_B + dd^c \varphi_{j, \epsilon} \]
and 
\[ \psi := \Phi_B - f_\xi, \]
which is a basic function on \( U \) which, by Prop 4.3, has the property that \( \psi \to -\infty \) at the boundary of \( M' \cap U \) and \( \frac{i}{2\pi} \partial B \overline{\partial} \psi + \theta' \geq 0 \). Consider the following smooth function on \( M' \cap U \):
\[ h := \log(\text{tr}_{\omega_B} \overline{\omega}_B) + (n + 1)\psi - j - A_1(\varphi_{j, \epsilon} - \psi) \]
for \( A_1 \) a constant (to be chosen sufficiently large). Since \( \varphi_{j, \epsilon} \) is bounded and \( \psi \to -\infty \) at the boundary of \( M' \cap U \) the maximum of \( h \) is attained in \( M' \cap U \). Combining the inequality in Lemma 4.8 with the transversal MA-equation 4.12 and the uniform \( L^\infty \)-estimate in Lemma 4.9 then yields, precisely as in [6, Appendix B],
\[ \sup_{M' \cap U} \text{tr}_{\omega_B} \overline{\omega}_B \leq A_2 e^{-\psi - j} e^{-A_1 \psi} \leq A_2 e^{-\psi} e^{-A_1 \psi} \]
for some constants \( A_1 \) and \( A_2 \) (only depending on the lower bound in formula 4.11, the upper bound on the \( L^\infty \)-norms in Lemma 4.9 and the bound \( \kappa \) on the bisectional curvatures of the reference Kähler form \( \omega \)).

4.4. Conclusion of the proof of the regularity of \( r^2 \) on \( Y_{reg} \). By construction the rhs in the estimate 4.16 is locally bounded on \( M' \cap U \). Hence, locally on \( U \), the functions \( \varphi_{j, \epsilon} \) have uniformly bounded Laplacians. As a consequence, there exists a subsequence \( \varphi_{j, \epsilon(j)} \) converging in the local \( C^1 \)-topology on \( U \) to a function \( \phi_0 \in L^\infty(U) \) such that \( dd^c \phi_0 \) is locally bounded on \( U \), invariant under both \( \xi \) and \( J\xi \) and satisfies the “inhomogeneous” MA-equation
\[ (\theta + dd^c \phi_0)^m = im^2 r^{-2m} \Omega \wedge \overline{\Omega}, \quad dd^c \Phi_0 \geq 0 \]
on \( U \), identified with \( Y_{reg} \) (using the same local computation as in the proof of Lemma 4.6). But there is a unique such solution, modulo the addition of a constant. Indeed, in the toric case this follows from the well-known uniqueness, modulo additive constants, of solution to the real Monge–Ampère equation in Lemma 2.3 (see [11, Lemma 3.1] for a simple proof). Hence, \( \varphi_0 = \varphi' + C \). This shows that \( \varphi' \) has a locally bounded Laplacian on \( U \). Thus, applying standard local Evans-Krylov-Trudinger theory to the fully non-linear PDE 4.3 on \( U \) shows that \( \varphi' \) is, in fact, smooth on \( U \) (by [8] Thm 2.6) it is enough to know that \( \Delta \varphi' \in L^p_{loc} \) for \( p > 2m(m - 1) \). This means that \( r^2 \) is smooth on \( Y_{reg} \) as desired.

Remark 4.10. The only place where the toric structure was used was in the \( L^\infty \)-bound in Lemma 4.9 and the uniqueness of bounded pluriharmonic potential solutions to the equation \((\theta + dd^c \phi_0)^{m-1} = \nu \), for \( \nu \) a given basic \((m - 1, m - 1)-\)form on \( Y_{reg} \) such that the quotient of \( \nu \) and \( f_\xi^{-m} im^2 \Omega \wedge \overline{\Omega}(\xi, J\xi, \cdot) \) is bounded.

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