CANONICAL BASES AND KLR-ALGEBRAS

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Abstract. We prove a recent conjecture of Khovanov-Lauda concerning the categorification of one-half of the quantum group associated with a simply laced Cartan datum.

0. Introduction and notation.

In [KL1], [KL2] Khovanov and Lauda have introduced a new family of algebras and formulated some conjecture predicting a connection between the representation theory of these algebras and Lusztig’s geometric construction of canonical bases. The goal of this paper is to prove part of this conjecture.

The construction in loc. cit. is as follows. Let $\mathcal{A}f$ be Lusztig’s integral form of the negative half of the quantum universal enveloping algebra associated with a quiver $\Gamma$. Let $I$ be the set of vertices of $\Gamma$. One construct a family of graded rings $R_\nu$ over $\nu \in \mathbb{N}I$. These rings are defined in a combinatorial way. One consider the Grothendieck group $K(R_\nu)$ of the category of finitely generated graded projective $R_\nu$-modules. Set $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. The direct sum

$$K(\mathcal{R}) = \bigoplus_{\nu \in \mathbb{N}I} K(R_\nu)$$

has a natural structure of a free $\mathcal{A}$-bialgebra under induction and restriction. In loc. cit. an $\mathcal{A}$-algebra isomorphism $\gamma_\mathcal{A} : \mathcal{A}f \to K(\mathcal{R})$ is given. It is conjectured that $\gamma_\mathcal{A}^{-1}$ maps the classes of the indecomposable projective modules to the canonical basis. We prove this here.

Before to go on let us make a few historical remarks. The KLR-algebras were first introduced by Khovanov and Lauda in [KL1], [KL2] with some restrictions on the quiver. There were independently discovered by Rouquier, in a more general version. See Remark 3.3 below for details and [R] for the definition and the first properties. After our paper was written Rouquier informed us he has also obtained the same result as ours, independently.

Now we give some notation we’ll use in this paper. By a canonical isomorphism in a given category we’ll mean an explicit isomorphism. We’ll identify two objects with a canonical isomorphism whenever needed. Unless specified otherwise, by a graded object of an additive category $\mathcal{C}$ we’ll always mean a $\mathbb{Z}$-graded object, i.e., an object of the form $M = \bigoplus_{i \in \mathbb{Z}} M_i$ where each $M_i$ is an object of $\mathcal{C}$. Then, given an integer $r$, we’ll write $M[r]$ for the shift of the grading on $M$ up by $r$, i.e., the degree $i$ component of $M[r]$ is equal to $M_{i+r}$.

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Given a positive integer \( m \) and a tuple \( \mathbf{m} = (m_1, m_2, \ldots, m_r) \) of positive integers we write \( \mathfrak{S}_m \) for the symmetric group and \( \mathfrak{S}_m \) for the group \( \prod_{i=1}^r \mathfrak{S}_{m_i} \). Set

\[
|\mathbf{m}| = \sum_{i=1}^r m_i, \quad \ell_m = \sum_{i=1}^r \ell_{m_i}, \quad \ell_m = m(m-1)/2.
\]

We use the following notation for \( q \)-numbers

\[
[m]_q = \sum_{l=1}^m q^{m+1-2l}, \quad [m]_q! = \prod_{l=1}^m [l]_q, \quad [\mathbf{m}]_q! = \prod_{l=1}^r [m_l]_q!.
\]

If no confusion is possible we’ll abbreviate

\[
[m] = [m]_q, \quad [m]_q! = [m]_q^1, \quad [\mathbf{m}]_q! = [\mathbf{m}]_q^1.
\]

Given two tuples \( \mathbf{m} = (m_1, m_2, \ldots, m_r), \mathbf{m}' = (m'_1, m'_2, \ldots, m'_r) \) we define the tuple

\[
\mathbf{mm}' = (m_1, m_2, \ldots, m_r, m'_1, m'_2, \ldots, m'_r).
\]

Let \([\mathcal{C}]\) denote the Grothendieck group of an exact category \( \mathcal{C} \). Let \( k \) be a field and \( R = \bigoplus_i R_i \) be a \( \mathbb{Z} \)-graded \( k \)-algebra. Let \( R\Mod \) be the category of finitely generated graded left \( R \)-modules, \( R\Mod^f \) be the full subcategory of finite-dimensional modules and \( R\Proj \) be the full subcategory of projective objects. We’ll abbreviate

\[
K(R) = [R\Proj], \quad G(R) = [R\Mod^f].\]

Assume that the \( k \)-vector spaces \( R_i \) are finite dimensional for each \( i \). Then \( K(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in \( R\Proj \). Further \( G(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in \( R\Mod^f \). Given an object \( M \) of \( R\Proj \) or \( R\Mod^f \) let \( [M] \) denote its class in \( K(R) \), \( G(R) \) respectively. Both \( K(R), G(R) \) are \( \mathcal{A} \)-modules, where \( \mathcal{A} \) acts on \( G(R), K(R) \) as follows

\[
q[M] = [M[1]], \quad q^{-1}[M] = [M[-1]], \quad \forall M.
\]

If the grading of \( R \) is bounded below then the \( \mathcal{A} \)-modules \( K(R), G(R) \) are free.

From now on the symbol \( k \) will denote a field of characteristic zero.

1. Reminder on quivers, extensions and convolution algebras.

We give a few notation on quivers and equivariant homology. Given a quiver \( \Gamma \) we first recall the definition of the semisimple complexes on the moduli stack of representations of \( \Gamma \) introduced by Lusztig in [L1]. Next we define their Ext-algebras (with respect to the Yoneda product). Finally we recall a lemma of Ginzburg which relates Ext-algebras to convolution algebras in equivariant homology.
1.1. Representations of quivers. Let $\Gamma$ be a finite nonempty quiver such that no arrow may join a vertex to itself. Recall that $\Gamma$ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where $I$ is the set of vertices, $H$ is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of $h$ respectively. Although this hypothesis is not necessary, we’ll always assume that the set $I$ is finite in the rest of the paper.

For each $i, j \in I$ we write

$$H_{i,j} = \{ h \in H; h' = i, h'' = j \}.$$ 

We’ll abbreviate $i \to j$ for $H_{i,j} \neq \emptyset$, $i \not\to j$ for $H_{i,j} = \emptyset$, and $h : i \to j$ for $h \in H_{i,j}$.

Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$ 

Let $\mathcal{V}$ be the category of finite-dimensional $I$-graded $\mathbb{C}$-vector spaces $V = \bigoplus_{i \in I} V_i$ with morphisms being linear maps respecting the grading. For each $\nu = \sum_i \nu_i i \in \mathbb{N}I$ let $\mathcal{V}_\nu$ be the full subcategory of $\mathcal{V}$ whose objects are those $V$ such that $\dim(V_i) = \nu_i$ for all $i$. We call $\nu$ the dimension vector of $V$. Given $V \in \mathcal{V}$ let

$$E_V = \bigoplus_{h \in H} \text{Hom}(V_{h'}, V_{h''}).$$ 

The algebraic group $G_V = \prod_i \text{GL}(V_i)$ acts on $E_V$ by $(g, x) \mapsto y$ where $y_h = g_{h'}x_hg_{h''}^{-1}$, $g = (g_i)$, $x = (x_h)$, and $y = (y_h)$.

Fix a nonzero element $\nu$ of $NI$. Let $Y_\nu$ be the set of all pairs $y = (i, a)$ where $i = (i_1, i_2, \ldots i_k)$ is a sequence of elements of $I$ and $a = (a_1, a_2, \ldots a_k)$ is a sequence of positive integers such that $\sum_i a_i i_l = \nu$. The set $Y_\nu$ is finite. Let $I^\nu \subset Y_\nu$ be the set of all pairs $y = (i, a)$ such that $a_l = 1$ for all $l$. We’ll abbreviate $i$ for $y = (i, a)$ if $y \in I^\nu$. Note that the assignment

$$y \mapsto (a_1 i_1, a_2 i_2, \ldots a_k i_k)$$ 

identifies $Y_\nu$ with a set of sequences

$$(1.1) \quad \nu^1, \nu^2, \ldots, \nu^k \in \mathbb{N}I \quad \text{with} \quad \nu = \sum_{i=1}^k \nu^i.$$ 

For each $\nu = \sum_i \nu_i i \in \mathbb{N}I$ we set $|\nu| = \sum_i \nu_i$, an integer $\geq 0$. Given $m \in \mathbb{N}$ we have $\bigcup_\nu I^\nu = I^m$, where $\nu$ runs over the subset $\mathbb{N}_m I \subset NI$ of elements with $|\nu| = m$.

From now on we assume that $V \in \mathcal{V}_\nu$. For any sequence $y = (\nu^1, \nu^2, \ldots, \nu^k)$ as in (1.1), a flag of type $y$ in $V$ is a sequence

$$\phi = (V = V^k \supset V^{k-1} \supset \cdots \supset V^0 = 0)$$

of $I$-graded subspace of $V$ such that for any $l$ the $I$-graded subspace $V^l/V^{l-1}$ belongs to $\mathcal{V}_\nu$. Let $F_y$ be the variety of all flags of type $y$ in $V$. The group $G_V$ acts transitively on $F_y$ in the obvious way, yielding a smooth projective $G_V$-variety structure on $F_y$. 

We’ll say that the sequence $y = (\nu^1, \nu^2, \ldots, \nu^k)$ as in (1.1) is multiplicity free if we have $\nu^l = \sum_{i \in I} \nu^1_i$ with $\nu^l_i = 0$ or 1 for each $l$. Note that $I^\nu$ consists of multiplicity free sequences. Given a finite dimensional $\mathbb{C}$-vector space $W$ let $F(W)$ be the variety of all complete flags in $W$. If $y$ is a multiplicity free sequence then we have

$$
\dim(F_y) = \ell_y = \sum_{i \in I} \ell_{\nu_i}, \quad F_y \simeq \prod_{i \in I} F(V_i), \quad \phi \mapsto (\phi \cap V_i).
$$

If $x \in E_V$ we say that the flag $\phi$ is $x$-stable if $x_h(V^l_{h^l}) \subset V^l_{h^l}$ for all $h, l$. Let $\tilde{F}_y$ be the variety of all pairs $(x, \phi)$ such that $\phi$ is $x$-stable. Set

$$
d_y = \dim(\tilde{F}_y).
$$

The group $G_V$ acts on $\tilde{F}_y$ by $g : (x, \phi) \mapsto (gx, g\phi)$. The first projection

$$
\pi_y : \tilde{F}_y \to E_V
$$

is a $G_V$-equivariant proper morphism.

For each $l = 1, 2, \ldots, [\nu]$ we define $O_{\tilde{F}_y}(l)$ to be the $G_V$-equivariant line bundle over $\tilde{F}_y$ whose fiber at the flag $\phi$ is equal to $V^l/V^{l-1}$.

1.2. Constructible sheaves. Given an action of a complex linear algebraic group $G$ on a quasiprojective algebraic variety $X$ over $\mathbb{C}$ we write $D_G(X)$ for the bounded $G$-equivariant derived category of complexes of sheaves of $k$-vector spaces on $X$. Objects of $D_G(X)$ are referred to as complexes. If $G = \{e\}$, the trivial group, we abbreviate $D(X) = D_G(X)$. For each complexes $L, L'$ we’ll abbreviate

$$
\text{Ext}^*_G(L, L') = \text{Ext}^{\bullet}_{D_G(X)}(L, L'), \quad \text{Ext}^*(L, L') = \text{Ext}^{\bullet}(L, L')
$$

if no confusion is possible.

There is a lot of literature on the category $D_G(X)$, see [BL], [J], [L3, sec. 1], [L4, sec. 1] for instance. Although we’ll only use standards properties of $D_G(X)$ let us recall a few definitions for the comfort of the reader. We’ll use the notation of [BBD] for sheaves and Deligne’s definition of $D_G(X)$, see [BL, sec. 1.2, app. B], [J, sec. 1.2.3]. More precisely, let $[G \setminus X]$ be the usual simplicial topological set whose $n$-skeleton is $G^n \times X$ for each $n$. We put the transcendental topology on $X, G$. Let $\mathcal{S}h_G(X)$ be the full subcategory of the category of simplicial sheaves on $[G \setminus X]$ for which all structure morphisms are isomorphisms. It is an Abelian category which is equivalent to the category of $G$-equivariant sheaves on $X$ by [D, (6.1.2)]. We define $D_G(X)$ as the full subcategory of the bounded derived category of simplicial sheaves on $[X \setminus G]$ consisting of the complexes whose cohomology sheaves belong to $\mathcal{S}h_G(X)$.

Let us recall a few simple facts. If $H \subset G$ is a closed subgroup then the pull back by the obvious morphism of simplicial topological spaces $[H \setminus X] \to [G \setminus X]$ is the forgetful functor $D_G(X) \to D_H(X)$. An object of $D_G(X)$ is perverse if the corresponding object in $D(X)$ is perverse.

The constant sheaf $\mathbf{k}$ on $X$ will be denoted $\mathbf{k}$. For any object $L$ of $D_G(X)$ let $H^*_G(X, L)$ be the space of $G$-equivariant cohomology with coefficients in $L$. Let
\( \mathcal{D} \in \mathcal{D}_G(X) \) be the \( G \)-equivariant dualizing complex, see [BL, def. 3.5.1]. For each \( \mathcal{L} \) we'll abbreviate
\[
\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{D})
\]
(the internal Hom). Recall that
\[
(\mathcal{L}^\vee)^\vee = \mathcal{L}, \quad \text{Ext}_G^*(\mathcal{L}, \mathcal{D}) = H^*_G(X, \mathcal{L}^\vee), \quad \text{Ext}_G^*(\mathcal{k}, \mathcal{L}) = H^*_G(X, \mathcal{L}).
\]
We define the space of \( G \)-equivariant homology by
\[
H^*_G(X, \mathcal{k}) = H^*_G(X, \mathcal{D}).
\]
Note that we have not chosen the grading in the most usual way, compare [L4, sec. 1.17] for instance. Note also that \( \mathcal{D} = \mathcal{k}[2d] \) if \( X \) is a smooth \( G \)-variety of pure dimension \( d \). Consider the following graded \( \mathcal{k} \)-algebra
\[
\mathcal{S}_G = H^*_G(\bullet, \mathcal{k}).
\]
The graded \( \mathcal{k} \)-vector space \( H^*_G(X, \mathcal{k}) \) has a natural structure of a graded \( \mathcal{S}_G \)-module. We have
\[
H^*_G(\bullet, \mathcal{k}) = \mathcal{S}_G
\]
as graded \( \mathcal{S}_G \)-module. There is a canonical graded \( \mathcal{k} \)-algebra isomorphism
\[
\mathcal{S}_G \simeq \mathcal{k}[\mathfrak{g}]^G.
\]
Here the symbol \( \mathfrak{g} \) denotes the Lie algebra of \( G \) and a \( G \)-invariant homogeneous polynomial over \( \mathfrak{g} \) of degree \( d \) is given the degree \( 2d \) in \( \mathcal{S}_G \).

Fix a morphism of quasi-projective algebraic \( G \)-varieties \( f : X \to Y \). If \( f \) is a proper map there is a direct image homomorphism
\[
f_* : H^*_G(X, \mathcal{k}) \to H^*_G(Y, \mathcal{k}).
\]
If \( f \) is a smooth map of relative dimension \( d \) there is an inverse image homomorphism
\[
f^* : H^*_i(Y, \mathcal{k}) \to H^*_{i-2d}(X, \mathcal{k}), \quad \forall i.
\]
If \( X \) has pure dimension \( d \) there is a natural homomorphism
\[
H^*_G(X, \mathcal{k}) \to H^*_{i-2d}(X, \mathcal{k}).
\]
It is invertible if \( X \) is smooth. The image of the unit is called the fundamental class of \( X \) in \( H^*_G(X, \mathcal{k}) \). We denote it by \( [X] \). If \( f : X \to Y \) is the embedding of a \( G \)-stable closed subset and \( X' \subset X \) is the union of the irreducible components of maximal dimension then the image of \( [X'] \) by the map \( f_* \) is the fundamental class of \( X \) in \( H^*_G(Y, \mathcal{k}) \). It is again denoted by \( [X] \).
1.3. Ext-algebras. In the rest of this section we assume that \( \nu \in \mathbb{N}I \) and \( V \in \mathcal{V}_\nu \).

We abbreviate

\[
S_V = S_{G_V}.
\]

For each sequence \( y \) as above we have the following semisimple complexes in \( \mathcal{D}_{G_V}(E_V) \)

\[
L_y = (\pi_y)_!(k), \quad L_y^\vee = L_y[2d_y], \quad \delta L_y = L_y[d_y].
\]

Here the integer \( d_y \) is as in (1.2). Given \( y, y' \in Y_\nu \) we consider the graded \( S_V \)-module

\[
Z_{y, y'} = \text{Ext}^*_G(L_y^\vee, L_{y'}^\vee).
\]

Given \( y'' \in Y_\nu \) the Yoneda composition is a homogeneous \( S_V \)-bilinear map of degree zero

\[
*: Z_{y, y'} \times Z_{y', y''} \to Z_{y, y''}.
\]

The map \( * \) equips the \( k \)-vector spaces

\[
Z_V' = \bigoplus_{y, y' \in Y_\nu} Z_{y, y'}, \quad Z_V = \bigoplus_{i, i' \in I'} Z_{i, i'}
\]

with the structure of unital associative graded \( S_V \)-algebras. Our first goal is to compute \( Z_V \), because it is the most relevant for Lusztig’s construction of canonical bases. Due to the following basic fact it is indeed enough to consider only \( Z_V \), which is smaller.

1.4. Proposition. The categories \( Z_V^a \text{-Mod} \) and \( Z_V^g \text{-Mod} \) are equivalent.

Proof: Let \( y = (i, a) \in Y_\nu \). We have \( a_l > 0 \) for all \( l \). If \( a_l > 1 \) for some \( l \), let \( y' \in Y_\nu \) be obtained from \( y \) by replacing the single entry \( i_l \) by \( a_l \) entries equal to \( i_l \) and the single entry \( a_l \) by \( a_l \) entries equal to 1. By [L1, sec. 2.4] we have \( L_{y'} \simeq \sum_{l=1}^r L_y[d_l] \) for some integer \( d_1, d_2, \ldots, d_r \) with \( r \geq 1 \). Let us do this simultaneously for all \( l \) such that \( a_l > 1 \). Let \( i' \in I'' \) be the sequence obtained by expanding the pair \( y \).

So we have

\[
i' = (i_1, \ldots, i_k), \quad i = (i_1, \ldots, i_k), \quad a = (a_1, \ldots, a_k),
\]

where the entry \( i_l \) have the multiplicity \( a_l \) in \( i' \). There are integers \( \ell_1, \ell_2, \ldots, \ell_s \), with \( s \geq 1 \), such that \( L_{i'} \simeq \sum_{l=1}^s L_y[\ell_l] \). Thus the algebras \( Z_V' \), \( Z_V \) are Morita equivalent.

\[ \square \]

1.5. Remark. The integers \( \ell_1, \ldots, \ell_s \) in the proof of Proposition 1.4 can be computed explicitly. Recall that \( y = (i, a) \in Y_\nu \) with \( i = (i_1, \ldots, i_k) \), \( a = (a_1, \ldots, a_k) \) and that \( i' \in I'' \) is the sequence obtained by expanding the pair \( y \). In the category \( \mathcal{D}_{G_V}(E_V) \) we have an isomorphism of complexes

\[
L_{i'} \simeq \bigoplus_{w \in G_n} L_y[-2\ell(w)].
\]

This isomorphism is not canonical. It depends of the choice of a partition of the variety \( \tilde{F}_{i'} \) into cells as in [L2, sec. 8.1.6].
1.6. The canonical module over $Z_V$. For each $y \in Y_\nu$ we have the graded $S_V$-modules

\begin{equation}
F_y = \text{Ext}^*_G(V)(L_y^\nu, D), \quad F_V = \bigoplus_{i \in I^\nu} F_i.
\end{equation}

The Yoneda product gives a graded $S_V$-bilinear map

$$Z_{y,y'} \times F_{y'} \to F_y.$$ 

This yields a left graded representation of $Z_V$ on $F_V$. For each $i \in I^\nu$ let $1_i \in Z_{1_i}$ denote the identity of $L_i$. The elements $1_i$ form a complete set of orthogonal idempotents of $Z_V$ such that

$$Z_{1_i} = 1_i \ast Z_V \ast 1_i, \quad F_i = 1_i \ast F_V.$$

1.7. Convolution algebras. For $i, i' \in I^\nu$ we set

$$Z_{i,i'} = \bar{F}_i \times_{F_V} \bar{F}_{i'},$$

the reduced fiber product relative to the maps $\pi_i, \pi_{i'}$. We set also

$$Z_V = \coprod_{i, i' \in I^\nu} Z_{i, i'}, \quad F_V = \prod_{i \in I^\nu} F_i.$$

Consider also the obvious projections

$$p : \bar{F}_V \to F_V, \quad q : Z_V \to F_V \times F_V.$$ 

Note that (1.3), (1.5) yield

$$F_y = \text{Ext}^*_G(V)(k, L_y) = H^*_G(V)(E_V, L_y) = H^*_G(\bar{F}_V, k).$$

Recall the following isomorphism, see (1.4)

\begin{equation}
H^*_G(V)(\bar{F}_V, k) = H^*_G(V)(\bar{F}_y, D)[-2d_y] = H^*_G(\bar{F}_V, k)[-2d_y].
\end{equation}

We obtain a graded $S_V$-module isomorphism

\begin{equation}
F_y = H^*_G(\bar{F}_V, k)[-2d_y].
\end{equation}

Next, we equip the $S_V$-module $H^*_G(V, k)$ with the convolution product $\ast$ relative to the closed embedding $Z_V \subset \bar{F}_V \times \bar{F}_V$. See [CG, sec. 8.6] for details. We obtain a $S_V$-algebra with 1 which acts on the $S_V$-module $H^*_G(\bar{F}_V, k)$ from the left. The unit of $H^*_G(V, k)$ is the fundamental class of the closed subvariety $Z_V \subset Z_V$. See Section 2.8 below for the notation.

1.8. Lemma. (a) The left $H^*_G(V, k)$-module $H^*_G(\bar{F}_V, k)$ is faithful.

(b) There is a canonical $S_V$-algebra isomorphism $Z_V = H^*_G(V, k)$ such that (1.9) intertwines the $Z_V$-action on $F_V$ and the $H^*_G(V, k)$-action on $H^*_G(\bar{F}_V, k)$.

Proof: Part (b) is proved as in [CG, Prop. 8.6.35]. Note that in loc. cit. both claims are proved for non-equivariant homology. However the proof uses only standard tools and generalizes to the equivariant setting. Part (a) is standard. A proof of (a) is given in Remark 2.21 below for the comfort of the reader.

$\square$
1.9. Shift of the grading. For each $y,y’ \in Y_\nu$ we set

$$(1.10) \quad ^{\delta}Z_{y,y’} = \text{Ext}_G^\nu(\delta L_y, \delta L_{y’}), \quad ^{\delta}F_y = \text{Ext}_G^\nu(\delta L_y, D).$$

Consider the graded $S_\nu$-modules

$$^{\delta}Z = \bigoplus_{i \in I^\nu} \delta Z_i, \quad ^{\delta}F = \bigoplus_{i \in I^\nu} \delta F_i,$$

Thus we have

$$^{\delta}Z = \bigoplus_{i’, y \in I^\nu} \delta Z_{i,y’}, \quad ^{\delta}F = \bigoplus_{i \in I^\nu} \delta F_i,$$

$$(1.11) \quad ^{\delta}Z_i = Z_i[d_i - d_i], \quad ^{\delta}F_i = F_i[d_i].$$

1.10. Notation. The symbol $*$ will denote both the multiplication in the algebras $Z_\nu, H_\nu^G(Z_\nu, k)$ and the left action on $F_\nu, H_\nu^G(\bar{F}_\nu, k)$. Let 1 denote the unit in $Z_\nu$ and $H_\nu^G(Z_\nu, k)$. In this paper we’ll never consider the homological grading on $H_\nu^G(Z_\nu, k)$ because the isomorphism $Z_\nu = H_\nu^G(Z_\nu, k)$ in Lemma 1.8(b) is not homogeneous of degree 0. The gradings on $Z_\nu$ and $^{\delta}Z_\nu$ will always be the ones given in Sections 1.3 and 1.9.

2. Computation of the algebra $Z_\nu$.

This section contains background material needed to compute the graded Ext-algebras $Z_\nu, ^{\delta}Z_\nu$ introduced in the first section. More precisely we identify $Z_\nu, F_\nu$ with $H_\nu^G(Z_\nu, k), H_\nu^G(\bar{F}_\nu, k)$ via Lemma 1.8 and (1.9) and we compute explicitly the $H_\nu^G(Z_\nu, k)$-action on $H_\nu^G(\bar{F}_\nu, k)$. We also consider the grading in Corollary 2.25.

2.1. Notation. In this section we fix once for all a nonzero element $\nu \in N$ and an object $V \in \mathcal{Y}_\nu$. Fix also a maximal torus $T_\nu \subset G_\nu$. We’ll abbreviate $G = GL(V)$ and $F = F(V)$. It is convenient to see $F_\nu$ as the closed subvariety of $F$ consisting of all flags which are fixed under the action of the center of $G_\nu$.

Let $\mathfrak{S}_\nu, \mathfrak{S}$ be the Weyl groups of the pairs $(G_\nu, T_\nu), (G,T_\nu)$. There is a canonical embedding $\mathfrak{S}_\nu \subset \mathfrak{S}$. The group $\mathfrak{S}$ acts freely transiitively on the set $F^\nu_0$ of the flags which are fixed by the $T_\nu$-action. We’ll write $e$ for the unit in $\mathfrak{S}$.

2.2. The $\mathfrak{S}$-action on $T_\nu$-fixed flags. Fix once for all a flag $\phi_\nu \in F^T_\nu$. Write $\phi_\nu,w = w(\phi_\nu)$ for each $w \in \mathfrak{S}$. Let $i_w$ be the unique sequence such that $\phi_\nu,w \in F_{i_w}$. There is bijection

$$\mathfrak{S}_\nu \setminus \mathfrak{S} \rightarrow I^\nu, \quad \mathfrak{S}_\nu w \mapsto i_w.$$

For each $i \in I^\nu$ we have

$$\bar{F}_i^T \simeq F_i^T = \{ \phi_\nu,w; w \in \mathfrak{S}_i \},$$

where $\mathfrak{S}_i$ is the right $\mathfrak{S}_\nu$-coset

$$\mathfrak{S}_i = \{ w \in \mathfrak{S}; i_w = i \}.$$
Let $B_{V,w}$ be the stabilizer of the flag $\phi_{V,w}$ under the $G_V$-action. It is a Borel subgroup of $G_V$ containing $T_V$. Let $N_{V,w}$ be the unipotent radical of $B_{V,w}$. We'll abbreviate

$$F_w = F_{i_w}, \ \mathcal{E}_w = \mathcal{E}_{i_w}, \ \pi_w = \pi_{i_w}, \ d_w = d_{i_w}, \ B_V = B_{V,e}, \ N_V = N_{V,e}.$$ 

Note that $\mathcal{E}_w = \mathcal{E}_Vw$ and that we have an isomorphism of $G_V$-varieties

$$G_V/B_{V,w} \twoheadrightarrow F_w, \ \ g \mapsto g\phi_{V,w}.$$ 

### 2.3. Identification of $\mathcal{E}$ with the symmetric group.

Set $m = |\nu|$, a positive integer. Recall that $i_e$ is the unique sequence such that the flag $\phi_V$ belongs to $F_{i_e}$. Write $i_e = (i_1, i_2, \ldots, i_m)$. Fix one-dimensional $T_V$-submodules $D_1, D_2, \ldots, D_m \subset V$ such that

$$\phi_V = (V \supset D_1 \oplus D_2 \oplus \cdots \oplus D_m) = (V \supset D_1 \oplus D_2 \oplus \cdots \oplus D_m \supset D_1 \supset 0).$$ 

Then $D_i \subset V_{i_l}$ for each $l$. There is a canonical group isomorphism $\mathcal{E} = \mathcal{E}_m$ such that $w(D_i) = D_{w(i)}$ for each $w, l$. Let $\mathcal{E}_\nu$ be the image in $\mathcal{E}_m$ of the subgroup $\mathcal{E}_V$ of $\mathcal{E}$. The symmetric group $S_m$ acts on the set $I'$ by permutations: view a sequence $i$ as the map $\{1, 2, \ldots, m\} \to I$, $l \mapsto i_l$ and set $w(i) = i \circ w^{-1}$ for $w \in S_m$. Then we have

$$\mathcal{E}_\nu = \{w \in \mathcal{E}_m; w(i_e) = i_e\}.$$ 

Under the canonical isomorphism $\mathcal{E} = \mathcal{E}_m$ we have $w(i_e) = i_{w^{-1}}$ for each $w$.

### 2.4. The root system.

Let $B$ be the stabilizer of the flag $\phi_V$ in $G$. Note that $B_V = B \cap G_V$. Let $\Delta$ be the set of roots of $(G, T_V)$ and $\Delta^+ \subset \Delta$ be the set of positive roots relative to the Borel subgroup $B \subset G$. We abbreviate $\Delta^- = -\Delta^+$. Let $\Pi \subset \mathcal{E}$ be the set of simple reflections. Let $\leqslant$ and $w \mapsto \ell(w)$ denote the Bruhat order and the length function on $\mathcal{E}_m$ or $\mathcal{E}$. The Weyl group $S$ acts on $\Delta$. Let $\Delta_V$ be the set of roots of $(G_V, T_V)$. Write $\Delta^+_V = \Delta^+ \cap \Delta_V$ and $\Delta^-_V = -\Delta^+_V$. The action of $\mathcal{E}_V$ on $\Delta$ preserves the subset $\Delta_V$. Let $\chi_1, \chi_2, \ldots, \chi_m \in \mathfrak{t}'_V$ be the weights of the lines $D_1, D_2, \ldots, D_m$ respectively. The set of simple roots in $\Delta^+$ is

$$\{\chi_l - \chi_{l+1}; \ l = 1, 2, \ldots, m - 1\}.$$ 

Let $s_l \in \Pi$ denote the reflection with respect to the simple root $\chi_l - \chi_{l+1}$. Under the identification $\mathcal{E} = \mathcal{E}_m$ the reflection $s_l$ is taken to the simple transposition $(l, l + 1)$.

### 2.5. The stratification of $F_V \times F_V$.

The group $G$ acts diagonally on $F \times F$. The action of the subgroup $G_V$ preserves the subset $F_V \times F_V$. For $x \in \mathcal{E}$ let $O^x_V$ be the set of all pairs of flags in $F_V \times F_V$ which are in relative position $x$. More precisely, write

$$\phi_{V,w',w} = (\phi_{V,w'}, \phi_{V,w}), \ \forall w, w' \in \mathcal{E}.$$ 

Then we set

$$O^x_V = (F_V \times F_V) \cap G\phi_{V,e,x}.$$ 

Let $\overline{O}^x_V$ be the Zariski closure of $O^x_V$. It is the set of pairs of flags in $F_V \times F_V$ which are in relative position $\leqslant x$. For any $\phi,w \in \mathcal{E}$ we write also

$$O^x_{w',w} = O^x_V \cap (F_{w'} \times F_w), \ \overline{O}^x_{w',w} = \overline{O}^x_V \cap (F_{w'} \times F_w).$$ 

Fix a simple reflection $s \in \Pi$. Put

$$P^x_{V,w} = B_{V,w} \{ws^{-1}, 1\} B_{V,w}.$$ 

It is a parabolic subgroup of $G_V$ containing $B_{V,w}$. The following lemma is standard. Its proof is left to the reader.
2.6. Lemma. (a) The set of elements of $O^s_{\nu}$ which are fixed under the diagonal action of $T_{\nu}$ is \( \{ \phi_{\nu,v,w,x}; w \in \mathcal{S} \} \).

(b) The variety $O^s_{\nu}$ is smooth and is equal to $O^s_{\nu} \cup O^e_{\nu}$.

(c) We have $O^s_{w,w} = \emptyset$ unless $w' = w$ or $w_\nu$.

(d) Assume that $w_\nu \notin \mathcal{S}_\nu$. We have

\[
F_{w,x} \neq F_w, \quad B_{\nu,w,x} = B_{\nu,w}, \quad \bar{O}_{w,w}^s = O_{w,w}^s, \quad \bar{O}_{w,w}^e = O_{w,w}^e.
\]

(e) Assume that $\nu \in \mathcal{S}_\nu$. We have

\[
O_{\nu,\nu}^s = O_{\nu,\nu}^s, \quad F_{w,x} = F_w, \quad B_{\nu,w,x} \neq B_{\nu,w}.
\]

There is an isomorphism of $G_{\nu}$-varieties

\[
G_{\nu} \times_{B_{\nu,w}} (P_{\nu,w,x}/B_{\nu,w}) \to \bar{O}_{\nu,\nu}^s, \quad (g,h) \mapsto (g\phi_{\nu,w}, gh\phi_{\nu,w}).
\]

2.7. Example. Set $I = \{i,j\}, \nu = i + 2j$, $V_i = D_1$, $V_j = D_2 \oplus D_3$. So $I' = \{i_1, i_2, i_3\}$ where $s = s_1$, $t = s_2 s_3$, and $i = (i, j), i_1 = (j, j), i_2 = (j, j)$, $i_3 = (j, j)$.

We have

\[
\phi_{\nu} = (V \supset D_1 \oplus D_2 \supset D_1 \supset 0),
\]

\[
\phi_{\nu,s} = (V \supset D_1 \oplus D_2 \supset D_2 \supset 0),
\]

\[
\phi_{\nu,t} = (V \supset D_2 \oplus D_3 \supset D_3 \supset 0).
\]

We have also

\[
F_i = \{ V \supset D_1 \oplus D \supset D_1 \supset 0; D \in \mathbb{P}(V_j) \},
\]

\[
F_s = \{ V \supset D_1 \oplus D \supset D \supset 0; D \in \mathbb{P}(V_j) \},
\]

\[
F_t = \{ V \supset V_j \supset D \supset 0; D \in \mathbb{P}(V_j) \}.
\]

Finally we have

\[
\bar{O}_{s,t}^s = \bar{O}_{s,t}^e = \emptyset, \quad \bar{O}_{i,t}^s = F_i \times F_t, \quad \bar{O}_{i,s}^s = \Delta_{F_s}, \quad \bar{O}_{i,e}^s = \Delta_{F_e},
\]

where $\Delta$ is the diagonal, and

\[
\bar{O}_{e,s}^s = \bar{O}_{e,s}^e = \{(V \supset D_1 \oplus D \supset D \supset 0, V \supset D_1 \oplus D \supset D_1 \supset 0); D \in \mathbb{P}(V_j) \}.
\]

2.8. The stratification of $Z_{\nu}$ and $Z_{\nu}$. Recall the notation in Section 1.7. For $x \in \mathcal{S}$ let $Z_{\nu}^x \subset Z_{\nu}$ be the Zariski closure of the locally closed subset $q^{-1}(O^s_{\nu})$. Put

\[
Z_{\nu,v}^x = \bigcup_{w \leq x} Z_{\nu,w}^x, \quad Z_{\nu,i}^{x} = Z_{\nu,v}^x \cap Z_{\nu,i}, \quad \forall i, i' \in I'.
\]

Hence $Z_{\nu,v}^x$, $Z_{\nu}^x$, $Z_{\nu,i}^{x}$ are closed $G_{\nu}$-subvarieties of $Z_{\nu}$. Lemma 1.8(b) yields $k$-vector space isomorphisms

\[
Z_{\nu,i} = H^0_{\nu}(Z_{\nu,i}, k), \quad \forall i, i' \in I'.
\]

We set

\[
Z_{\nu,v}^x = H^0_{\nu}(Z_{\nu,v}^x, k), \quad Z_{\nu}^x = Z_{\nu,v}^x, \quad Z_{\nu,i}^{x} = Z_{\nu,v}^x \cap Z_{\nu,i}.\]
2.9. Lemma. (a) The direct image by the closed embedding $\mathbb{Z}^{<x} \subset \mathbb{Z}_V$ gives an injective left graded $S_V$-module homomorphism $\mathbb{Z}^{<x}_V \rightarrow \mathbb{Z}_V$.

(b) For each $x, y \in \mathcal{S}$ such that $\ell(xy) = \ell(x) + \ell(y)$ the convolution product in $\mathbb{Z}_V$ yields an inclusion $\mathbb{Z}^{<x}_V * \mathbb{Z}^{<y}_V \subset \mathbb{Z}^{<xy}_V$.

(c) We have $1_i \in \mathbb{Z}_i$ for each $i \in I$. For each $x, y \in \mathbb{S}$ such that $\ell(xy) = \ell(x) + \ell(y)$ the convolution product in $\mathbb{Z}_V$ yields an inclusion $\mathbb{Z}^{<x}_V * \mathbb{Z}^{<y}_V \subset \mathbb{Z}^{<xy}_V$.

Proof: Parts (a), (b) are standard, see [CG, chap. 6] for the case of the equivariant K-theory of the Steinberg variety. The proof is the same in our case. Part (c) is obvious: since the convolution product by $1_i$ is the identity $\mathbb{Z}_i$, we must have $1_i = [\mathbb{Z}_i]$.

\[ \square \]

2.10. Euler classes in $P_V$. Consider the $k$-algebra

\[ P_V = S_{T_V} \]

Assume that $M$ is a finite dimensional representation of the Lie algebra $t_V$. For each linear form $\lambda \in t_V^* \subset M$ be the weight subspace associated with $\lambda$. Let $\text{ch}(M) = \sum_{x \in \mathbb{S}} \dim(M) \lambda$ be the character of $M$. Let $\text{eu}(M)$ be the determinant of $M$, viewed as a homogeneous polynomial of degree $\dim(M)$ on $t_V$, i.e., an element of degree $2\dim(M)$ in $P_V$. If $M$ is a finite dimensional representation of $T_V$ we write $\text{eu}(M)$ again for the polynomial associated to the differential of $M$, a module over $t_V$. Now, assume that $X$ is a quasi-projective $T_V$-variety and that $x \in X$ is a smooth point of $X$. The cotangent space $T^*_x X$ at $x$ is equipped with a natural representation of $T_V$. We’ll abbreviate $\text{eu}(X, x) = \text{eu}(T^*_x X)$. We’ll be particularly interested by the following element

\[ \Lambda_w = \text{eu}(\tilde{F}_V, \phi_{V, w}), \quad \forall w \in \mathcal{S}. \]

Note that $\Lambda_w$ is an element of $P_V$ of degree $2d_w$.

2.11. Notation. For each $w \in \mathcal{S}$ let

\[ \mathfrak{c}_{V, w} = \{ x \in E_V; \phi_{V, w} \text{ is } x\text{-stable} \}. \]

Under restriction the natural $G_V$-action on $E_V$ yields a representation of $B_{V, w}$ on $\mathfrak{c}_{V, w}$. There is an isomorphism of $G_V$-varieties

\[ G_V \times_{B_{V, w}} \mathfrak{c}_{V, w} \rightarrow \tilde{F}_w, \quad (g, x) \mapsto (g \phi_{V, w}, gx). \]

Under this isomorphism the map $\pi_w : \tilde{F}_w \rightarrow E_V$ is identified with the map

\[ G_V \times_{B_{V, w}} \mathfrak{c}_{V, w} \rightarrow E_V, \quad (g, x) \mapsto gx. \]

As a $T_V$-module $\mathfrak{c}_{V, w}$ is the sum of the weight subspaces of $E_V$ whose weights belong to $w(\Delta^+)$. For $w, w' \in \mathcal{S}$ we write

\[ \mathfrak{c}_{V, w, w'} = \mathfrak{c}_{V, w} \cap \mathfrak{c}_{V, w'}, \quad \mathfrak{d}_{V, w, w'} = \mathfrak{c}_{V, w} / \mathfrak{c}_{V, w, w'}. \]
2.12. Fix a simple reflection $s \in \Pi$. Let $\mathfrak{g}_V, \mathfrak{t}_V, n_{V,w}, p_{V,w,ws}$ be the Lie algebras of $G_V$, $T_V$, $N_{V,w}$, $F_{V,w,ws}$ respectively. Let $n_{V,w,ws}$ be the nilpotent radical of $p_{V,w,ws}$. We have

$$n_{V,w,ws} = n_{V,w} \cap n_{V,ws}.$$ 

Let $m_{V,w,ws} = n_{V,w}/n_{V,w,ws}$. So we have the following $T_V$-module isomorphisms

$$n_{V,w} = n_{V,w,ws} \oplus m_{V,w,ws}, \quad m_{V,w,ws} \simeq m_{V,ws}^*.$$ 

Note that $n_{V,w} \subset \mathfrak{g}_V$ is the sum of the root subspaces whose weights belong to $\omega(\Delta^+) \cap \Delta_V$.

2.13. Reduction to the torus. Recall the graded $k$-algebra $P_V = S_{T_V}$. The canonical action of $\mathcal{G}$ on $T_V$ yields a $\mathcal{G}$-action on $P_V$. The restriction of functions from $\mathfrak{g}_V$ to $\mathfrak{t}_V$ gives an isomorphism of graded $k$-algebras

$$S_V = (P_V)^{\mathcal{G}_V}.$$ 

Now, recall the notation in Sections 2.3-4. So $\chi_1, \chi_2, \ldots, \chi_m$ are the weights of the lines $D_1, D_2, \ldots, D_m$ respectively. These weights generate the algebra $P_V$ and they have the degree 2. Under the identification $\mathcal{G} \simeq \mathcal{G}_m$ the action of $w$ on $P_V$ is given by

$$f = f(\chi_1, \ldots, \chi_m) \mapsto w(f) = f(\chi_{w(1)}, \ldots, \chi_{w(m)}).$$

Next, we recall here a standard result for a future use. If $X$ is a quasi-projective $G_V$-variety then the $P_V$-module $H^*_G(X, \mathcal{O}_X)$ is equipped with a canonical $P_V$-skewlinear action of the group $\mathcal{G}_V$. It is well-known that the forgetful map gives a $S_V$-module isomorphism

$$H^*_G(X, \mathcal{O}_X) \rightarrow H^*_G(X, \mathcal{O}_X)^{\mathcal{G}_V}.$$ 

2.14. The $k$-algebra structure on $F_V$. Let $i \in I'$. Assigning to a formal variable $x_i(l)$ of degree 2 the first equivariant Chern class of the $G_V$-equivariant line bundle $\mathcal{O}_{\widetilde{F}_i}(l)$ in Section 1.1 for each $l = 1, 2, \ldots, m$ yields a graded $k$-algebra isomorphism

$$k[x_i(1), x_i(2), \ldots, x_i(m)] = H^*_G(\widetilde{F}_i, \mathcal{O}_X).$$

Composing this map by the isomorphisms (1.8), (1.9) we get a canonical isomorphism of graded $k$-vector spaces

$$k[x_i(1), x_i(2), \ldots, x_i(m)] = H^*_G(\widetilde{F}_i, \mathcal{O}_X)[-2d_i] = F_i.$$ 

The multiplication of polynomials equip each $F_i$ with an obvious structure of graded $k$-algebra. Thus $F_V$ is also a graded $k$-algebra by (1.7).

2.15. The $\mathcal{G}_m$-action and the $S_V$-action on $F_V$. For each $i \in I'$ and each $w \in \mathcal{G}_i$ the pull-back by the closed embedding $\{\phi_{V,w}\} \subset \widetilde{F}_i$ yields a graded $k$-algebra isomorphism

$$F_i \rightarrow P_V, \quad f(x_i(1), \ldots, x_i(m)) \mapsto f(\chi_{w(1)}, \ldots, \chi_{w(m)}).$$

We’ll write $w(f)$ for the right hand side. This isomorphism is not canonical, because it depends on the choice of $w$. 
Consider the $\mathfrak{S}_m$-action on $F_V$ given by
\[ wF_i = F_{w(i)}, \quad wf(x_1(1), \ldots, x_1(m)) = f(x_{w(i)}(w(1)), \ldots, x_{w(i)}(w(m))). \]

Next, consider the canonical $S_V$-action on $F_V$ coming from the $G_V$-equivariant cohomology. It can be regarded as a $S_V$-action on $\bigoplus_k k[x_1(1), x_1(2), \ldots, x_1(m)]$ which is described in the following way. The composition of the obvious projection $F_V \to F_i$ with the map (2.5) identifies the graded $k$-algebra of the $\mathfrak{S}_m$-invariant polynomials in the $x_1(l)$'s with $(P_V)^{\mathfrak{S}_V} = S_V$. This isomorphism is canonical. The $S_V$-action on $F_V$ is the composition of this isomorphism and of the multiplication by $\mathfrak{S}_m$-invariant polynomials.

2.16. Examples. (a) Fix a simple reflection $s_l \in \Pi$. For each $w \in \mathfrak{S}$ the $T_V$-module $n_{V,w}$ is the sum of the root subspaces of weight $\chi_{w(k)} - \chi_{w(k')}$. which are contained in $g_V$. The following hold

- if $w_{s_l} \in \mathfrak{S}_V w$ then we have
  \[ \text{eu}(n_{V, w_{s_l}}) = -\text{eu}(n_{V, w}), \quad \text{eu}(m_{V, w, w_{s_l}}) = -\text{eu}(m_{V, w_{s_l}, w}) = \chi_{w(l)} - \chi_{w(l+1)}, \]

- if $w_{s_l} \notin \mathfrak{S}_V w$ then we have
  \[ n_{V, w_{s_l}} = n_{V, w}, \quad \text{eu}(m_{V, w, w_{s_l}}) = \text{eu}(m_{V, w_{s_l}, w}) = 0. \]

(b) Let $s_l$ be as above. Given $w \in \mathfrak{S}$ we write
\[ i_w = (i_1, i_2, \ldots, i_m), \quad V_w^k = D_{w(1)} \oplus D_{w(2)} \oplus \cdots \oplus D_{w(k)}, \quad \forall k. \]

Note that in Section 2.3 we used the (different) notation $i_e = (i_1, i_2, \ldots, i_m)$. The following properties are straightforward
\[ \phi_{V, w} = (V = V_w^m \supset V_w^{m-1} \supset \cdots \supset V_w^0 = 0), \]
\[ D_{w(k)} \subset V_{i_k}, \]
\[ V_w^k = V_{w_{s_l}}^k \text{ if } k \neq l, \]
\[ x(V_{i_k}^w) \subset V_{i_k}^{w-1}, \quad \forall x \in \mathfrak{e}_V. \]

The last claim follows from the inclusion $x(V_{i_k}^w) \subset V_{i_k}^w$, because else $x$ would give a non-zero map $D_{w(k)} \to D_{w(k)}$ and this is impossible because $i_k \neq i_{k'}$. So we have
\[ \mathfrak{e}_{V, w} = \{ x \in \mathfrak{e}_V; x(V_{i_k}^w) \subset V_{i_k}^{w-1}, \forall k \}, \]
\[ \mathfrak{e}_{V, w, w, s} = \{ x \in \mathfrak{e}_{V, w}; x(D_{w(t+1)}) \subset V_{w}^{t-1} \}. \]

Therefore, composing the map
\[ \mathfrak{e}_{V, w} \to \bigoplus_{h:i_{i+1} \to i_i} \text{Hom}(D_{w(t+1)}, V_{w}^l), \quad x \mapsto (x_h|D_{w(t+1)}; h \in H_{ii+1, i_i}), \]
with the isomorphism $V_{w}^l/V_{w}^{l-1} = D_{w(l)}$ we get a $T_V$-module isomorphism
\[ \mathfrak{d}_{V, w, w, s} = \bigoplus_{h:i_{i+1} \to i_i} D_{w(t+1)} \otimes D_{w(l)}. \]

Hence we have
\[ \text{eu}(\mathfrak{d}_{V, w, w, s}) = (\chi_{w(l)} - \chi_{w(l+1)})^{h_{ii+1, i_i}}. \]
2.17. Localization to the $T_V$-fixed points. For each $w \in \mathcal{S}$ we set $\psi_w = \{\phi_{V,w}\}$, a class in $H^T_V(\tilde{F}_V, k)$. For each $w, w' \in \mathcal{S}$ we set also $\psi_{w,w'} = \{\phi_{V,w,w'}\}$, a class in $H^T_V(Z_V, k)$. Let $K_V$ be the fraction field of $P_V$. For each quasi-projective $T_V$-variety $X$ we'll abbreviate

$$\mathcal{H}_*(X, k) = H^T_V(X, k) \otimes_{P_V} K_V.$$ 

We'll also abbreviate $\psi_w, \psi_{w,w'}$ for the elements

$$\psi_w \otimes 1 \in \mathcal{H}_*(\tilde{F}_V, k), \quad \psi_{w,w'} \otimes 1 \in \mathcal{H}_*(Z_V, k).$$

The following lemma is standard. Its proof is left to the reader.

2.18. Lemma. (a) The $P_V$-modules $H^T_V(\tilde{F}_V, k), H^T_V(Z_V, k)$ are free.

(b) We have $K_V$-vector spaces isomorphisms

$$\mathcal{H}_*(\tilde{F}_V, k) = \bigoplus_{x \in \mathcal{S}} K_V \psi_x, \quad \mathcal{H}_*(Z_V, k) = \bigoplus_{x,y \in \mathcal{S}} K_V \psi_{x,y}.$$ 

(c) The canonical $\mathcal{S}_V$-action on the $P_V$-modules $H^T_V(\tilde{F}_V, k), H^T_V(Z_V, k)$ (see Section 2.13) is given by $w(\psi_x) = \psi_{wx}$ and $w(\psi_{x,y}) = \psi_{wx,wy}$ for each $w \in \mathcal{S}_V$, $x, y \in \mathcal{S}$.

(d) Fix a sequence $i \in P'$. Composing (2.3), (2.4) and the obvious map

$$H^T_V(\tilde{F}_i, k) \to \mathcal{H}_*(\tilde{F}_i, k)$$

we get a canonical embedding

$$k[x_1, \ldots, x_m] \to \bigoplus_{w \in \mathcal{S}_i} K_V \psi_w, \quad f(x_1, \ldots, x_m) \mapsto \sum_{w \in \mathcal{S}_i} w(f) \Lambda^{-1}_w \psi_w.$$ 

2.19. Fix a simple reflection $s \in \Pi$. Recall the closed $G_V$-subvariety $Z'_V \subset Z_V$ from Section 2.8. There are unique rational functions $\Lambda^*_w, w' \in K_V$, $w, w' \in \mathcal{S}$, such that the image of $[Z'_V]$ in $\mathcal{H}_*(Z_V, k)$ is

$$[Z'_V] = \sum_{w,w'} \Lambda^*_w, w' \psi_w, w'.$$

Note that $\Lambda^*_w, w' = 0$ unless $w' = w$ or $w$s by Lemma 2.6(c). Let $\star$ denote also the convolution products

$$H^T_V(Z_V, k) \times H^T_V(Z_V, k) \to H^T_V(Z_V, k),$$

$$H^T_V(Z_V, k) \times H^T_V(\tilde{F}_V, k) \to H^T_V(\tilde{F}_V, k)$$

relative to the closed embedding $Z_V \subset \tilde{F}_V \times \tilde{F}_V$. Compare Section 1.7.
2.20. Lemma. (a) The forgetful maps below commute with $*$
\[ Z_V \to H^*_{TV}(Z_V, k), \quad F_V \to H^*_{TV}(F_V, k). \]
(b) For $w, w', w'' \in \mathcal{S}$ we have
\[ \psi_{w, w} \ast \psi_{w'} = \Lambda_{w} \psi_{w' \ast w}, \quad \psi_{w', w'} \ast \psi_{w'' \ast w} = \Lambda_{w''} \psi_{w'' \ast w}. \]
(c) For $w \in \mathcal{S}$ we have
\[
\begin{align*}
\Lambda_w &= \operatorname{eu}(\epsilon^*_w, n_V, w), \\
\Lambda^s_{w, w'} &= \operatorname{eu}(\epsilon^*_w, n_V, w + m_V, w, w')^{-1} \\
\Lambda^s_{w, ws} &= \operatorname{eu}(\epsilon^*_w, n_V, w + m_V, w, ws)^{-1} \\
\Lambda^s_{w, ws} &= \Lambda^s_{w, w} = \operatorname{eu}(\epsilon^*_w, n_V, w + m_V, w; w')^{-1} \text{ if } ws \not\in \mathcal{S}. \end{align*}
\]
Proof: Parts (a), (b) are well-known. Let us concentrate on (c). The fiber at $\phi_{V, w}$ of the vector bundle $p : F_V \to V$ is isomorphic to $\epsilon_{V, w}$ as a $T_V$-module. Thus the cotangent space to $F_V$ at the point $\phi_{V, w}$ is isomorphic to $\epsilon_{V, w} \oplus n_V, w$ as a $T_V$-module. This yields the first formula. The variety $Z^s_V$ is smooth. A standard computation yields the following
\[ \Lambda^s_{w, w'} = \operatorname{eu}(Z^s_V, \phi_{V, w, w'})^{-1}. \]
The fiber at $\phi_{V, w, w'}$ of the vector bundle $q : Z^s_V \to F_V \times F_V$ is isomorphic to $\epsilon_{V, w, ws}$ if $w' = ws, w$ and to 0 else, as a $T_V$-module. Therefore, we have
\[ \Lambda^s_{w, w'} = \operatorname{eu}(\epsilon^*_w, n_V, w, ws)^{-1} \operatorname{eu}(\bar{O}_V, \phi_{V, w, w'})^{-1}. \]
If $ws \in \mathcal{S}$ we have that $\phi_{V, w, ws}$ are isomorphic to $n_V, w \oplus m_V, w, ws$ respectively as $T_V$-modules, by Lemma 2.6(c). If $ws \not\in \mathcal{S}$ we have both isomorphic to $n_V, w$ by Lemma 2.6(d). This yields the remaining formulas.

\[ \square \]

2.21. Remark. We can now prove Lemma 1.8(a). By Lemma 2.20(a) and (2.3) it is enough to prove that the convolution product yields a faithful representation of $H^*_{TV}(Z_V, k)$ on $H^*_{TV}(F_V, k)$. By Lemma 2.18(a) it is thus enough to prove that $*$ yields indeed a faithful representation of $\mathcal{H}_s(Z_V, k)$ on $\mathcal{H}_s(F_V, k)$. This is obvious by Lemma 2.18(b), 2.20(b).

2.22. Description of the $Z_V$-action on $F_V$. Fix a simple reflection $s = s_t \in \Pi$. The fundamental class of $Z_V$ yields an element $\sigma(l) \in Z_V^{s}$. For each sequences $i, i' \in I^r$ we write
\[ \sigma_{i, i'}(l) = 1_V \ast \sigma(l) \ast 1_i \in Z_V^{s}. \]
Next, let $k$ be any positive integer $\leq m$. The pull-back of the first equivariant Chern class of the line bundle $\mathcal{O}_{I^r}(k)$ by the obvious map
\[ Z_V \subset F_V \times F_V \to F_V \]
belongs to $H^*_{TV}(Z_V, k)$. It yields an element $\varphi(k) \in Z_V$ as in Section 1.7. For each sequences $i, i' \in I^r$ we write
\[ \varphi_{i, i'}(k) = 1_V \ast \varphi(k) \ast 1_i \in Z_V. \]
Now, recall that $Z_V^s, Z_V \subset Z_V$ by Lemma 2.9(a) and that $Z_V$ acts on $F_V$ by Section 1.6. The action of $\sigma_{i, i'}(l), \varphi_{i, i'}(k)$ on $F_V$ is given by the following formulas.
2.23. Proposition. Let \( k, l \) be as above. Fix sequences \( \mathbf{i}, \mathbf{i}', \mathbf{i}'' \in I' \) and fix an element \( f \in F_1 \). Write \( \mathbf{i} = (i_1, \ldots, i_m) \).

(a) We have \( 1_{\mathbf{i}'} \star f = f \) if \( \mathbf{i} = \mathbf{i}' \) and 0 else.
(b) We have \( \sigma \mathbf{i}', \mathbf{i}(k) \star f = 0 \) unless \( \mathbf{i}' = \mathbf{i} = \mathbf{i} \) and \( \sigma \mathbf{i}, \mathbf{i}(k) \star f = x_1(k)f \).
(c) We have \( \sigma \mathbf{i}', \mathbf{i}(l) \star f = 0 \) unless \( \mathbf{i}' = \mathbf{i} \) and \( \mathbf{i}' \in \{s_l(\mathbf{i}), \mathbf{i}\} \). More precisely

- if \( s_l(\mathbf{i}) = \mathbf{i} \) then
  \[
  \sigma \mathbf{i}(l) \star f = (f - s_l(f))(x_1(l) - x_1(l + 1)),
  \]
- if \( s_l(\mathbf{i}) \neq \mathbf{i} \) then
  \[
  \sigma s_l(\mathbf{i}), \mathbf{i}(l) \star f = (x_{s_l(\mathbf{i})}(l + 1) - x_{s_l(\mathbf{i})}(l)) h_{s_l(\mathbf{i}), l+1} s_l(f),
  \]
  \[
  \sigma \mathbf{i}, \mathbf{i}(l) \star f = (x_1(l + 1) - x_1(l)) h_{1, l+1} f.
  \]

Proof: We'll abbreviate \( s = s_l \). Parts (a), (b) are standard and are left to the reader. Let us concentrate on (c). The first claim is obvious, because \( Z_{w', w} = 0 \) unless \( w' = w \) or \( ws \) and \( \mathbf{i}' \in \mathbf{w} \). Now, given \( \mathbf{i}' = \mathbf{i} \) or \( s_l(\mathbf{i}) \) we must compute the linear operator

\[
(2.10) \quad F_1 \rightarrow F_1', \quad f \mapsto \sigma \mathbf{i}', \mathbf{i}(l) \star f.
\]

By (2.4) we have a \( k \)-vector space isomorphism

\[
F_1 = k[x_1(1), \ldots, x_1(m)].
\]

So Lemma 2.18(d) yields an embedding

\[
F_1 \rightarrow \bigoplus_{w \in \mathcal{S}_1} K \psi_w, \quad f(x_1(1), \ldots, x_1(m)) \mapsto f_1, \quad f_1 = \sum_{w \in \mathcal{S}_1} w(f) \Lambda_w^{-1} \psi_w, \quad f \in P_\chi.
\]

Under this inclusion the map (2.10) is given by

\[
f_1 \mapsto \sum_{w' \in \mathcal{S}_{\chi}} g_{w'} \psi_{w'}, \quad g_{w'} = \sum_{w \in \mathcal{S}_1} w(f) \Lambda_w^s \psi_w.
\]

See (2.9) and Lemma 2.20(b). We claim that the rhs is equal to \( g_{\chi} \) for some polynomial \( g \in P_\chi \) that we'll compute explicitly. So we must find \( g \) such that

\[
(2.11) \quad g_{w'} = w'(g) \Lambda_w^{-1}, \quad \forall w' \in \mathcal{S}_{\chi}.
\]

In the rest of the proof we'll assume that the following hold

\[
w \in \mathcal{S}_1, \quad w' \in \mathcal{S}_{\chi}, \quad w' = w \text{ or } ws, \quad \mathbf{i}' = \mathbf{i} \text{ or } s_l(\mathbf{i}).
\]

As in Example 2.16(a) we'll write

\[
\mathbf{i} = \mathbf{i}_w = (i_1, i_2, \ldots, i_m).
\]
(i) Assume that \( s(i) = i \). Then \( i' = i \), \( ws \in \mathcal{G}_V w \) and we have
\[
g_{w'} = w'(f)\Lambda_{w',w'}^s + w'(f)\Lambda_{w',w'}^s.
\]
Further there is no arrow in \( H \) joining \( i_l \) and \( i_{l+1} \). Thus by (2.7) we have
\[
\epsilon_{V,w} = \epsilon_{V,ws},
\]
and by Lemma 2.16(a) we have
\[
eu(n_{V,ws}) = -eu(n_{V,w}), \quad eu(m_{V,w,ws}) = -eu(m_{V,ws,w}) = \chi_w(l) - \chi_w(l+1).
\]
By Lemma 2.20(c) we have also
\[
\Lambda_w = eu(\epsilon_{V,w} \oplus n_{V,w}) = eu(\epsilon_{V,ws} + n_{V,ws}) = -\Lambda_{ws},
\]
\[
\Lambda_{w,w}^s = eu(\epsilon_{V,ws,w} \oplus n_{V,w} \oplus m_{V,ws,w})^{-1} = \Lambda_{ws}^{-1}eu(m_{V,ws,w})^{-1},
\]
\[
\Lambda_{w,ws}^s = eu(\epsilon_{V,ws,w} \oplus n_{V,w} \oplus m_{V,ws,w})^{-1} = \Lambda_{ws}^{-1}eu(m_{V,ws,w})^{-1}.
\]
Therefore we obtain
\[
g_{w'} = w'(f)eu(m_{V,w',w'}^s)^{-1}\Lambda_{w',w'}^{-1} + w'(f)eu(m_{V,w',w'}^s)^{-1}\Lambda_{w',w'}^{-1},
\]
\[
= (w'(f)\Lambda_{w'}^{-1} + w'(f)\Lambda_{w'}^{-1})/(\chi_{w'}(l) - \chi_{w'}(l+1)),
\]
\[
= w'(g)\Lambda_{w'}^{-1},
\]
where \( g = (f - s(f))/(\chi_l - \chi_{l+1}) \).

(ii) Assume that \( s(i) \neq i \), i.e., that \( ws \notin \mathcal{G}_V w \). Then one of the two following alternatives holds:

- \( i' = s(i) \), \( w' = ws \) and \( g_{w'} = w's(f)(\Lambda_{ws,w}^s\Lambda_{ws})\Lambda_{w'}^{-1} \),
- \( i' = i \), \( w' = w \) and \( g_{w'} = w'(f)(\Lambda_{w,w}^s\Lambda_{w})\Lambda_{w'}^{-1} \).

Now, by (2.8) we have
\[
eu(\epsilon_{V,ws,w}) = (\chi_w(l+1) - \chi_w(l))^{h_{l+1}},
\]
\[
eu(\epsilon_{V,ws,w}) = (\chi_w(l) - \chi_w(l+1))^{h_{l+1}}.
\]
By Lemma 2.16(a) we have
\[
eu(n_{V,ws}) = eu(n_{V,w}).
\]
By Lemma 2.20(c) we have
\[
\Lambda_w = eu(\epsilon_{V,w} \oplus n_{V,w}),
\]
\[
\Lambda_{ws} = eu(\epsilon_{V,ws} \oplus n_{V,ws}),
\]
\[
\Lambda_{w,w}^s = \Lambda_{ws,w}^s = eu(\epsilon_{V,ws,w} \oplus n_{V,w})^{-1},
\]
\[
\Lambda_{w,ws}^s = \Lambda_{ws,ws}^s = eu(\epsilon_{V,ws,ws} \oplus n_{V,ws})^{-1}.
\]
We have also
\[
eu(\epsilon_{V,w})eu(\epsilon_{V,ws,w}) = eu(\epsilon_{V,ws})eu(\epsilon_{V,ws,w}),
\]
\[
eu(\epsilon_{V,w,ws})eu(\epsilon_{V,w,ws}) = eu(\epsilon_{V,w}).
\]
Thus (2.13) yields
\begin{align}
\Lambda_w \text{eu}(\delta_{V,w,s,w}) &= \Lambda_{ws} \text{eu}(\delta_{V,w,s,w}), \\
\Lambda_{w,w}^s &= \Lambda_{ws,w}^s = \Lambda_{ws,w}^{-1} \text{eu}(\delta_{V,w,s,w}) = \Lambda_{ws,w}^s = \Lambda_{ws,w}^s.
\end{align}
Therefore we get the following alternatives
\begin{itemize}
  \item $i' = s(i), w' = ws$ and $g_{w'} = w's(f) \text{eu}(\delta_{V,w,s,w}) \Lambda_{ws}^{-1}$. Thus (2.11) holds with $g = s(f)(\chi_{i+1} - \chi_i)^{h_{i+1}-i}.$
  \item $i' = i, w' = w$ and $g_{w'} = w'(f) \text{eu}(\delta_{V,w,s,w}) \Lambda_{ws}^{-1}$. Thus (2.11) holds with $g = f(\chi_{i+1} - \chi_i)^{h_{i+1}-i}.$
\end{itemize}

\[
\square
\]

\textbf{2.24. Description of the grading on $Z_V$ and $\delta Z_V$.} For each sequence $i = (i_1,i_2,\ldots,i_m)$ and each integers $k, l$ as above we abbreviate
\[
\begin{align}
\nu_i(k) &= \nu_{i,i}(k), & \sigma_i(l) &= \sigma_{s(i),i}(l), \\
h_i(l) &= h_{i_{i+1}} & h_i(l) &= -1 & \text{if } s_i(i) &\neq i, \\
a_i(l) &= h_i(l) + h_{s_i(i)}(l), & a_i(l) &= -i & \text{if } s_i(i) &\neq i.
\end{align}
\]

The action of the elements $1_i, \nu_i(k), \sigma_i(l)$ of $Z_V$ on $F_V$ yields linear operators in $\text{End}(F_V)$. Let us denote them by $1_i, \nu_i(k), \sigma_i(l)$ respectively. Recall that $F_V$ is a faithful left graded $Z_V$-module by Section 1.6 and Lemma 1.8, and that the isomorphism in (2.4)

\[
F_V = \bigoplus_i k[x_i(1), x_i(2), \ldots x_i(m)]
\]
is a graded $k$-vector space isomorphism. Hence Proposition 2.23 and (2.4) imply the following.

\textbf{2.25. Corollary.} The grading on the $k$-algebra $Z_V$ is uniquely determined by the following rules. For each $i \in I$ and each $k, l = 1, 2, \ldots, m, l \neq m$, we have : $1_i$ has the degree 0, $\nu_i(k)$ has the degree 2, and $\sigma_i(l)$ has the degree $2h_i(l)$.

By (1.11) we have

\[
\delta Z_{s_i(i),i} = Z_{s_i(i),i}[d_{s_i(i)} - d_i].
\]

Further [L1, lem. 1.6(c)] yields

\[
d_i = \ell_i + \sum_{l \leq l'} h_{i,i'},
\]

(note that our notation for flags is opposite to the one in loc. cit.). Thus we have

\[
d_{s_i(i)} = d_i - h_{s_i(i)}(l) + h_i(l).
\]

In particular the grading in $\delta Z_V$ obeys the following rules : $1_i$ has the degree 0, $\nu_i(k)$ has the degree 2, and $\sigma_i(l)$ has the degree $a_i(l)$. 

2.26. The PBW theorem for $Z_V$. View $F_V$ as a graded (commutative) $k$-algebra as in Section 2.14. The graded $S_V$-algebra $Z_V$ is also a left graded $F_V$-module: we let $f(x_1(1), \ldots, x_1(m)) \in F_i$ act on $Z_V$ as the operator

$$z \mapsto f(\zeta_1(1), \ldots, \zeta_1(m)) \star z.$$ 

By Lemma 2.9(b) the left graded $S_V$-submodule $Z_{\leq x} V \subset Z_V$ is indeed a left graded $F_V$-submodule for each $x \in \mathcal{G}$.

2.27. Lemma. We have $Z_{\leq x} V = \bigoplus_{w \leq x} F_V + [Z_v^w]$ for each $w \in \mathcal{G}$. In particular $Z_V$ is a free left graded $F_V$-module of rank $m!$.

Proof: The proof is the same as in [CG, sec. 7.6.11] which considers the equivariant $K$-theory of the Steinberg variety. It uses a decomposition of $Z_{\leq x} V$ as a disjoint union of affine cells. Details are left to the reader.

2.28. Examples. (a) Set $I = \{i\}$ and $v = mi$. So $I' = \{i\}$ with $i = (i, i, \ldots, i)$, $G_V = GL(C^m)$, $E_V = \{0\}$, $\bar{F}_V = F_V = F$ and $Z_V = F \times F$. We have $F_V \simeq F_V$ as a left graded $S_V$-module. We have also $Z_V \simeq \text{End}_{S_V}(F_V)$ as a graded $S_V$-algebra. It is the nil-Hecke ring. Note that $F_V$ is a projective left graded $Z_V$-module such that $Z_V = \bigoplus_{w \in \mathcal{G}} F_V[2\ell(w)]$ as a left graded $Z_V$-module.

(b) Set $I = \{i_1, i_2, \ldots, i_m\}$, $H = \emptyset$ and $v = i_1 + i_2 + \cdots + i_m$. So $I' \simeq \mathcal{G}_m$ and $F_i \simeq \{\bullet\}$ for all $i$. We have also $G_V \simeq (\mathbb{C}^\times)^d$, $E_V = \{0\}$, $\bar{F}_V = F_V$ and $Z_V = F_V \times F_V$. Thus $F_V$ is a free left graded $S_V$-module of rank $m!$ and $Z_V = \text{End}_{S_V}(F_V)$ as a graded $S_V$-algebra.

(c) Set $I = \{i, j\}$ with $j \to i$ and $v = i + j$. So $I' = \{i, i'\}$ with $i = (i, i)$, $i' = (j, i)$ and $F_i \simeq F_{i'} \simeq \{\bullet\}$, $E_V \simeq \mathbb{C}$, $\bar{F}_i \simeq \mathbb{C}$, $\bar{F}_{i'} \simeq \{\bullet\}$, $Z_{i,1} \simeq \mathbb{C}$ and $Z_{i, i'} \simeq Z_{i', i} \simeq \mathbb{C}$. In particular $F_V$ is a free left graded $S_V$-module of rank 2.

3. KLR-algebras.

In this section we compute the Ext-algebras introduced in the first section.

3.1. The algebra $R_V$ and its canonical representation on $F_V$. Let $\nu$ be a non zero element of $\mathbb{N}_m I$. Fix $V \in \nu$. Let $R_V$ be the KLR-algebra associated with $V$. It is a graded $k$-algebra with 1. By [KL1, sec. 2.3] there is a faithful left representation of $R_V$ on

$$Q_V = \bigoplus_{i \in I'} Q_i, \quad Q_i = k[x_1(1), \ldots x_1(m)].$$

Recall that (2.4) yields a $k$-vector space isomorphism

$$Q_V = F_V.$$ 

From now on we’ll write $F_V$ everywhere instead of $Q_V$. Next, Proposition 2.23 and Section 2.24 yield explicit linear operators

$$\hat{I}_i, \hat{\zeta}_i(k), \hat{\sigma}_i(l) \in \text{End}(F_V), \quad i \in I', \quad 1 \leq k, l \leq m, \quad k \neq l.$$
They are given as follows

\begin{itemize}
  \item $\hat{1}_i$ is the projection on $F_i$ relatively to $\bigoplus_{i' \neq i} F_{i'}$.
  \item $\hat{\tau}_i(k)$ acts by zero on $F_{i'}$, $i' \neq i$, and by multiplication by $x_i(k)$ on $F_i$.
  \item $\hat{\sigma}_i(l)$ acts by zero on $F_{i'}$, $i' \neq i$, and by the following rules
    \begin{align}
      f \mapsto (x_i(l + 1) - x_i(l))^{-1}(s_i(f) - f) & \text{ if } s_i(l) = i,
      \\
      f \mapsto (x_{s_i(i)}(l + 1) - x_{s_i(i)}(l))b_i(l)s_i(f) & \text{ if } s_i(i) \neq i.
    \end{align}
\end{itemize}

By loc. cit. the embedding $R_V \subset \text{End}(F_V)$ identifies $R_V$ with the $k$-subalgebra generated by $\hat{1}_i$, $\hat{\tau}_i(k)$ and $\hat{\sigma}_i(l)$ for all $i$, $k$, $l$ as above.

3.2. The grading on $R_V$ and $F_V$. The grading on $R_V$ is given by the following rules, see [KL1]: $\hat{1}_i$ has the degree 0, $\hat{\tau}_i(k)$ has the degree 2, and $\hat{\sigma}_i(l)$ has the degree $a_i(l)$. Given a sequence $i_0 \in I^\nu$ there is a unique grading on $F_{i_0}$ such that $F_V$ is a graded left $R_V$-module and the unit of $F_{i_0}$ has the degree 0. Compare [KL1, sec. 2.3]. This grading depends on the choice of the sequence $i_0$. By Corollary 2.25 there is a canonical graded $k$-vector space isomorphism

\begin{equation}
F_V = \delta F_V,
\end{equation}

up to a shift by an integer that we’ll ignore from now on. Here the lhs denotes the underlying graded $k$-vector space of the graded $R_V$-module $Q_V$ while the rhs is the underlying $k$-vector space of the graded $\delta Z_V$-module from Section 1.9.

3.3. Remarks. (a) The formulas in Proposition 2.23 differ indeed from the ones in [KL1, sec. 2.3] by a sign. However it is explained in [KL2] how to modify the algebra $R_V$ so that all the results of [KL1] remain true and so that this difference of sign disappear. From now on we’ll ignore this point, and we’ll refer only to [KL1] to simplify.

(b) In [KL1] the authors assume that two different vertices in $\Gamma$ may be joined by at most one arrow. The quivers and the algebras we consider are more general. They appear in [R, sec. 3.2.4]. Since this generalization does not affect the rest of the paper, from now on we’ll assume that [KL1] works in this greater generality. Indeed, we’ll only use the fact that $R_V$ admits a faithful representation on $F_V$ given by the same formulas as in Section 3.1 and the PBW-theorem. Both facts are proved for general quivers in [R, prop. 3.12, thm. 3.7]. Note that some algebra are also defined in [KL2] for arbitrary quivers, but they are different from the ones considered here and in [R].

(c) The $k$-algebra $R_V$ has the following presentation, see [R, def. 3.2.1] (we’ll not use this) : it is generated by $1_i$, $\tau_i(k)$, $\pi_i(l)$ with $i \in I^\nu$, $1 \leq k, l \leq m$ and $l \neq m$, modulo the following defining relations

\begin{itemize}
  \item $1_i 1_i = \delta_{i,i} 1_i$,
  \item $\pi_i(l) = 1_{s_i(i)} \pi_i(l) 1_i$,
  \item $\tau_i(k) = 1_{s_i(i)} \tau_i(k) 1_i$,
  \item $\tau_i(k) \tau_i(k') = \tau_i(k') \tau_i(k)$,
  \item $\tau_{s_i(i)}(l) \pi_i(l) = Q_{i,i} (\tau_i(l), \tau_i(l + 1))$,
  \item $\tau_{s_i(i)}(l') \pi_i(l) = \tau_{s_i(i)}(l) \pi_i(l')$ if $|l - l'| > 1$,
\end{itemize}
σ is the graded position of w

3.5. Proposition. We have

\[ Q_{i,l}(u, v) = \left\{ \begin{array}{ll} (-1)^{h_i(l)}(u-v)^{\sigma_i(l)} & \text{if } s_l(i) \neq 1, \\ 0 & \text{else.} \end{array} \right. \]

Here we have set \( s_{l, l+2} = s_l s_{l+1} s_l \) if \( l \neq m-1, m \) and

\[ Q_{i,l}(u, v) = \left\{ \begin{array}{ll} (-1)^{h_i(l)}(u-v)^{\sigma_i(l)} & \text{if } s_l(i) \neq 1, \\ 0 & \text{else.} \end{array} \right. \]

The element \( \tau_i(l) \) acts on \( F_\mathbf{V} \) as the operator \((-1)^{h_i(l)}\hat{\sigma}_i(l)\), while the element \( \zeta_l(k) \) acts by multiplication by \( x_l(k) \).

3.4. The PBW Theorem for \( R_\mathbf{V} \) and the isomorphism \( R_\mathbf{V} \cong \delta Z_\mathbf{V} \). Note that \( R_\mathbf{V} \) is a free left graded \( F_\mathbf{V} \)-module such that \( f(x_l(1), \ldots, x_l(m)) \) acts on \( R_\mathbf{V} \) by the left multipication with the element \( f(\hat{\zeta}_l(1), \ldots, \hat{\zeta}_l(m)) \). Let the symbol \( \star \) denote the left \( F_\mathbf{V} \)-action on \( R_\mathbf{V} \). One constructs a \( F_\mathbf{V} \)-basis of \( R_\mathbf{V} \) in the following way. For each permutation \( w \in S_m \) we choose a reduced decomposition \( w = s_{l_1} s_{l_2} \cdots s_{l_r} \) with \( r \) an integer \( \geq 0 \) and \( l_1, l_2, \ldots, l_r \in \{1, 2, \ldots, m-1\} \). For each \( i \in I^r \) let \( \hat{\sigma}_i(w) \in 1_i \star R_\mathbf{V} \) be given by \( \hat{\sigma}_i(w) = 1_i \) if \( r = 0 \), and

\[ \hat{\sigma}_i(w) = \hat{\sigma}_{s_{l_1}(i)}(l_1) \star \hat{\sigma}_{s_{l_2}(i)}(l_2) \star \cdots \star \hat{\sigma}_{s_{l_r}(i)}(l_r) \] else.

Observe that \( \hat{\sigma}(w) = \sum_i \hat{\sigma}_i(w) \) may depend on the choice of the reduced decomposition of \( w \). The following is proved in [KL1, thm. 2.5].

3.5. Proposition. We have \( R_\mathbf{V} = \bigoplus_{w \in S_m} F_\mathbf{V} \star \hat{\sigma}(w) \) as a left \( F_\mathbf{V} \)-module.

Restricting the \( F_\mathbf{V} \)-action on \( R_\mathbf{V} \) to the subalgebra \( S_\mathbf{V} \subset F_\mathbf{V} \) we get a structure of graded \( S_\mathbf{V} \)-algebra on \( R_\mathbf{V} \). See Sections 2.13-2.15 for details. The first main result of this paper is the following.

3.6. Theorem. There is an unique graded \( S_\mathbf{V} \)-algebra isomorphism

\[ \Psi_\mathbf{V} : R_\mathbf{V} \to \delta Z_\mathbf{V} \]

such that (3.3) intertwines the actions of \( R_\mathbf{V}, \delta Z_\mathbf{V} \).

Proof: First, recall that \( \delta F_\mathbf{V} \) is a faithful left graded \( \delta Z_\mathbf{V} \)-module and that \( R_\mathbf{V} \) is the graded \( k \)-subalgebra of \( \text{End}(F_\mathbf{V}) \) generated by the operators \( 1_i, \zeta_l(k) \) and \( \hat{\sigma}_i(l) \). Thus there is a unique injective graded \( k \)-algebra homomorphism

\[ \Psi_\mathbf{V} : R_\mathbf{V} \to \delta Z_\mathbf{V}, \quad \Psi_\mathbf{V}(1_i) = 1_i, \quad \Psi_\mathbf{V}(-1_i) = \zeta_l(k), \quad \Psi_\mathbf{V}(\hat{\sigma}_i(l)) = \sigma_i(l). \]

We must prove that \( \Psi_\mathbf{V} \) is a surjective map. The map \( \Psi_\mathbf{V} \) is a left graded \( F_\mathbf{V} \)-module homomorphism by Proposition 2.23, Corollary 2.25 and Sections 3.1, 3.2. For each \( x \in S \) we set

\[ R_\mathbf{V}^x = \bigoplus_{w \in x} F_\mathbf{V} \star \hat{\sigma}(w), \]
a left graded $F_V$-submodule of $R_V$. We abbreviate $R_V^{\leq e} = R_V^{\leq e}$. The proof of the theorem consists of two steps. First we prove that $\Psi_V(R_V^{\leq e}) \subset Z_V^{\leq e}$. Then we prove that this inclusion is an equality.

**Step 1:** Since $\Psi_V$ is a left $F_V$-module homomorphism it is enough to prove that $\Psi_V(\tilde{\sigma}(w)) \subset Z_V^{\leq e}$ for each $w$. By Lemma 2.9(b) and an easy induction on the length of $w$ it is enough to prove the following

$$\Psi_V(1) \subset Z_V, \quad \Psi_V(\tilde{\sigma}(s_l)) \subset Z_V^{\leq e}, \quad \forall l.$$  

This is obvious, because $1 \in Z_V$ and $\sigma_i(l) \in Z_V^{\leq e}$ for each $i$, see Section 2.22.

**Step 2:** Lemma 2.27 implies that $Z_V^{\leq e}$ is the free $F_V$-module of rank one generated by $[Z_V^{\leq e}]$. Therefore we have

$$\Psi_V(R_V^{\leq e}) = \Psi_V(F_V \ast 1) = F_V \ast \Psi_V(1) = F_V \ast [Z_V] = Z_V.$$  

Next, fix an integer $l = 1, 2, \ldots, m - 1$. We claim that $\Psi_V(R_V^{\leq e_l}) = Z_V^{\leq e_l}$. By Lemma 2.27, Proposition 3.5 we have

$$R_V^{\leq e_l} = (F_V \ast 1) \oplus (F_V \ast \tilde{\sigma}(s_l)),$$

$$Z_V^{\leq e_l} = (F_V \ast [Z_V]) \oplus (F_V \ast [Z_V^{\leq e}])$$.

Thus it is enough to observe that

$$\Psi_V(1) = [Z_V], \quad \Psi_V(\tilde{\sigma}(l)) = [Z_V^{\leq e}].$$  

See the definition of $\sigma(l)$ in Section 2.22 for the second identity. To complete the proof of Step 2 we are reduced to prove the following.

**3.7. Lemma.** If $\ell(s_l w) = \ell(w) + 1$ we have $[Z_V^{\leq e_l}] \ast [Z_V^{\leq e}] = [Z_V^{\leq e_l w}]$ in $Z_V^{\leq e_l w} / Z_V^{\leq e_l w}$.

**Proof:** Assume that $\ell(s_l w) = \ell(w) + 1$. We’ll abbreviate $s = s_l$ as above. By Lemma 2.9(b) and Lemma 2.27 there is an unique element $c \in F_V$ such that

$$[Z_V^{\leq e}] \ast [Z_V^{\leq e}] = c \ast [Z_V^{\leq e}]$$ in $Z_V^{\leq e_l w} / Z_V^{\leq e_l w}$.

We must prove that $c = 1$.

For each $x \in \mathcal{G}$ we abbreviate $[Z_V^{\leq e}] = [Z_V^{\leq e}] \otimes 1$, an element of $H^*(Z_V, k)$. For each $y, z \in \mathcal{G}$ there is a unique element $\Lambda^x_{y, z} \in K_V$ such that

$$[Z_V^{\leq e}] = \sum_{y, z} \Lambda^x_{y, z} \psi_{y, z}.$$  

Compare (2.9). Since $\phi_{V, y, z}$ is a smooth point of $Z_V^{\leq e}$ we have also

$$\Lambda^x_{y, z} = \text{eu}(Z_V^{\leq e}, \phi_{V, y, z})^{-1}.$$
Hence, in the expansion of the element $[Z_{sv}]$ in the $K_V$-basis $(\psi_{y,z})$ the coordinate
along the vector $\psi_{x,xsw}$ is equal to

$$\Lambda_{x,xsw}^w = eu(Z_{sv}^w, \phi_{V,x,xsw})^{-1}.$$ 

On the other hand, since $\Lambda_{x,xsw}^w = 0$ and

$$[Z_{sv}^w] = \sum_x (\Lambda_{x,x}^w \psi_{x,x} + \Lambda_{x,xs}^w \psi_{x,xs}),$$
the coordinate of $[Z_{sv}] \ast [Z_{sv}^w]$ along $\psi_{x,xsw}$ is equal to

$$\Lambda_{x,xs}^w \Lambda_{x,xs}^w \Lambda_{x,s} = eu(Z_{sv}^w, \phi_{V,x,xs})^{-1} eu(Z_{sv}^w, \phi_{V,x,xsw})^{-1} \Lambda_{x,s}.$$ 

Thus we must check that

$$eu(Z_{sv}^w, \phi_{V,x,xs}) eu(Z_{sv}^w, \phi_{V,x,xsw}) = eu(Z_{sv}^w, \phi_{V,x,xsw}) \Lambda_{x,s}.$$ 

This follows from Lemma 3.8 below.

\[\square\]

3.8. Lemma. (a) For each $x, y \in \mathcal{S}$ we have

$$eu(O_{V,x}, \phi_{V,x,xy}) = eu(n_{V,x} \oplus m_{V,xy,x}),$$

$$eu(Z_{sv}^w, \phi_{V,x,xy}) = eu(O_{V,x}^w, \phi_{V,x,xy}) eu(\epsilon_{V,x,xy}),$$

$$\Lambda_x = eu(Z_{sv}^w, \phi_{V,x,xy}) eu(F_{V,x}^w, \phi_{V,x,xy}) eu(\epsilon_{V,xy}).$$

(b) For each $w, x, y \in \mathcal{S}$ such that $\ell(xy) = \ell(x) + \ell(y)$ we have

$$eu(O_{V,x}^w, \phi_{V,w,wx,y}) eu(F_{V,x}^w, \phi_{V,w,wx,y}) eu(O_{V,x}^w, \phi_{V,w,wx,y}) eu(\epsilon_{V,wx,y}),$$

$$eu(\epsilon_{V,w,wx,y} \oplus \epsilon_{V,wx,y}) = eu(\epsilon_{V,w,wx} \oplus \epsilon_{V,wx,y}).$$

Proof: Part (a) is left to the reader. Let us prove (b). Set

$$\Delta(y)^- = y(\Delta^+) \cap \Delta^-, \quad \Delta(y)^+ = y(\Delta^-) \cap \Delta^+.$$ 

For each $x, y \in \mathcal{S}$ the $T_V$-module $m_{V,xy,x}$ is the sum of the root subspaces whose weight belong to the set $x(\Delta(y)^-) \cap \Delta_V$. Thus, by (a), the first claim is equivalent to the following equality

$$w(\Delta(xy)^-) \cap \Delta_V = w(\Delta(x)^- \cup x(\Delta(y)^-)) \cap \Delta_V.$$ 

This equality is a consequence of the following well-known formula

$$\ell(xy) = \ell(x) + \ell(y) \Rightarrow \Delta(xy)^- = \Delta(x)^- \cup x(\Delta(y)^-).$$ 

Now, let $\Xi_V \subset \Delta$ be the set of weights of the $T_V$-module $E_V$. Note that a weight subspace appear in $E_V$ with the multiplicity at most one. So the character of the $T_V$-module $\epsilon_{V,x,xy}$ is the sum of the roots $\alpha$ which belong to the set

$$x(\Delta^+ \setminus \Delta(y)^+) \cap \Xi_V.$$ 

Let

$$A = (\Delta^+ \setminus \Delta(xy)^+) \cup x(\Delta^+), \quad B = (\Delta^+ \setminus \Delta(x)^+) \cup x(\Delta^+ \setminus \Delta(y)^+).$$

The character of the $T_V$-module $\epsilon_{V,w,wx,y} \oplus \epsilon_{V,wx}$ is $\sum \alpha$ where $\alpha$ runs over the set $w(A) \cap \Xi_V$. The character of the $T_V$-module $\epsilon_{V,w,wx} \oplus \epsilon_{V,wx,y}$ is $\sum \beta$ where $\beta$ runs over the set $w(B) \cap \Xi_V$. Since $\ell(xy) = \ell(x) + \ell(y)$ we have

$$\Delta(xy)^+ = \Delta(x)^+ \cup x(\Delta(y)^+)$$

This implies that $A = B$. The second claim follows.

\[\square\]
3.9. From now on we’ll use freely the isomorphism $\Psi_\mathcal{V}$ to identify the graded $S_\mathcal{V}$-algebras $R_\mathcal{V}$ and $\delta Z_\mathcal{V}$. In particular we’ll abbreviate $\hat{1} = 1_\nu$, $\hat{\sigma}(k) = \sigma(k)$ and $\hat{\sigma}(l) = \sigma(l)$. When no confusion is possible we’ll also identify the graded $k$-vector spaces $F_\mathcal{V}$ and $\delta F_\mathcal{V}$ as in (3.3).

4. Canonical bases and projective graded $R_\nu$-modules.

In this section, using the computations of Section 3 we prove a conjecture of [KL1, sec. 2.5], because both (0.2) and the grading on $R_\nu$ contain new results. The graded $\mathcal{A}$-algebra $R_\nu$ is finite dimensional over its center $S_\nu$, a commutative graded $k$-subalgebra, and the quotient $R_\nu/S_\nu R_\nu$ is finitedimensional. Therefore any simple object of $R_\nu$ is a free Abelian group of finite rank with a basis formed by the classes of the simple objects. Both are free $\mathcal{A}$-modules where $q$ shifts the grading by 1. For each $\mathcal{V} \in \mathcal{V}_\nu$ there is a canonical isomorphism $R_\nu = R_\mathcal{V}$. Thus, from now on we’ll abbreviate $R_\nu = R_\mathcal{V}$. Consider the Abelian group

$$K(R) = \bigoplus_\nu K(R_\nu).$$

Recall that $K(R)$ is a $\mathcal{A}$-module by (0.2). The $\mathcal{A}$-action on $K(R)$ is the same as the one in [KL1, sec. 2.5], because both (0.2) and the grading on $R_\nu$ are opposite to the conventions in loc. cit. We equip $K(R)$ with an associative unital $\mathcal{A}$-algebra structure as follows. Fix $\nu, \nu' \in \mathbb{N}$. Set $\nu'' = \nu + \nu'$ and $m = |\nu|$. Given sequences $i \in I_\nu, i' \in I_\nu'$ we write $i'' = ii'$, see the notation in (0.1). There is an unique inclusion of graded $k$-algebras

$$R_\nu \otimes R_{\nu'} \rightarrow R_{\nu''}$$

such that for each $i, i', k, l$ we have

$$1_i \otimes 1_{i'} \mapsto 1_{i''}, \quad z_{k}(k) \otimes 1_{i'} \mapsto z_{k'}(k), \quad 1_i \otimes z_{l}(k) \mapsto z_{k'}(m + k),$$

$$\sigma_{l}(l) \otimes 1_{i'} \mapsto \sigma_{l'}(l), \quad 1_i \otimes \sigma_{l}(k) \mapsto \sigma_{k'}(m + l).$$

Let $1_{i'',i'}$ be the image of the identity element by the map (4.1). The induction yields an additive functor

$$\text{Ind}_{\nu,\nu'} : \begin{cases} R_{\nu} \otimes \text{Mod} \times R_{\nu'} \otimes \text{Mod} \rightarrow R_{\nu''} \otimes \text{Mod}, \\ (M, M') \mapsto R_{\nu,\nu'} 1_{i',i'} \otimes_{R_{\nu} \otimes R_{\nu'}} (M \otimes M'). \end{cases}$$

It takes projectives to projectives and it commutes with the shift of the grading. Thus it yields an $\mathcal{A}$-linear homomorphism

$$K(R_\nu) \otimes_\mathcal{A} K(R_\nu') \rightarrow K(R_{\nu''}).$$

Taking the sum over all $\nu, \nu'$ we get an associative unital $\mathcal{A}$-algebra structure on $K(R)$. 

4.2. The projective $R_\nu$-module $R_\nu$. Assume that $\nu \in \mathbb{N}I$. Given a pair $y = (i, a) \in Y_\nu$ we define an object $R_y$ in $R_\nu$-$\mathcal{P}roj$ as follows, see [KL1, sec. 2.5].

- If $I = \{i\}$, $\nu = m i$, $i = i^m$ and $a = m$ then $y = (i, m)$ and we set
  \[ R_y = R_{i,m} = F_{\nu}[\ell_m]. \]

As a left graded $R_\nu$-modules we have $R_\nu \simeq \bigoplus_{w \in \mathbb{S}_\nu} R_{i,m}[2\ell(w) - \ell_m]$, see Example 2.28(a). So $R_{i,m}$ is a direct summand of $R_\nu[\ell_m]$. We choose once for all an idempotent $1_{i,m} \in R_\nu$ such that
  \[ R_{i,m} = ([R_\nu \ast 1_{i,m}])[\ell_m]. \]

- If $i = (i_1, \ldots, i_k)$ and $a = (a_1, \ldots, a_k)$ we define the idempotent $1_y \in R_\nu$ as the image of the element $\bigotimes_{l=1}^k 1_{i_l a_l}$ by the inclusion of graded $k$-algebras $\bigotimes_{i=1}^k R_{i a_l} \subset R_\nu$ in (4.1). Then we set
  \[ R_y = ([R_\nu \ast 1_y])[\ell_a]. \]

4.3. The $R_\nu$-module $R_y$ satisfies the following properties, see loc. cit. for details.

- Let $i' \in I_\nu$ be the sequence obtained by expanding the pair $y = (i, a)$. We have the following formula in $R_\nu$-$\mathcal{P}roj$
  \[ R_{i'} \simeq \bigoplus_{w \in \mathbb{S}_\nu} R_y[2\ell(w) - \ell_a] \simeq: [a]! R_y. \]

Thus we have the following formula in $K(R)$
  \[ (4.2) \quad [R_{i'}] = [a]! [R_y]. \]

- If $I = \{i\}$, $\nu = m i$, $i = i^m$ then
  \[ R_{i,m} = R_m i = [m]! R_{i,m}, \quad 1_{i,m} = 1. \]

- Given $y = (i, a) \in Y_\nu$, $y' = (i', a') \in Y_\nu$ we set $yy' = (ii', aa')$. We have an isomorphism of left graded $R_{\nu';\nu}$-modules
  \[ (4.3) \quad \text{Ind}_{\nu';\nu}(R_y \otimes R_{y'}) \simeq R_{y'}. \]

4.4. The quantum group $f$. Set $\mathcal{K} = \mathbb{Q}(q)$. Let $f$ be the negative half of the quantum universal enveloping algebra associated with the quiver $(I, H)$. It is the $\mathcal{K}$-algebra generated by elements $\theta_i$, $i \in I$, with the defining relations
  \[ \sum_{a + b = 1 - i} (-1)^a \theta_i^a \theta_i^b = 0, \quad \forall i \neq j. \]

Here we have set $\theta_i^a = \theta_i^a / [a]!$ for each integer $a > 0$. Let $\mathcal{A}f \subset f$ be the $A$-lattice generated by all products of the elements $\theta_i^a$. We have the weight decomposition
  \[ \mathcal{A}f = \bigoplus_{\nu \in \mathbb{N}I} \mathcal{A}f_\nu, \quad f = \bigoplus_{\nu \in \mathbb{N}I} f_\nu. \]

For each pair $y = (i, a) \in Y_\nu$ with $i = (i_1, \ldots, i_k)$, $a = (a_1, \ldots, a_k)$ we write
  \[ \theta_y = \theta_i^{(a_1)} \theta_i^{(a_2)} \cdots \theta_i^{(a_k)} \in \mathcal{A}f_\nu. \]

Let $B$ be the canonical basis of $\mathcal{A}f$. We write
  \[ B_Z = \{ q^a b; b \in B, d \in \mathbb{Z} \}, \quad B_0 = B \cap \mathcal{A}f. \]

By [KL1, prop. 3.4, sec. 3.2] there is an unique $A$-algebra isomorphism
  \[ (4.4) \quad \gamma_A : \mathcal{A}f \rightarrow K(R), \quad \theta_y \mapsto [R_y], \quad y \in Y_\nu. \]

The following is the second main result of the paper.
4.5. Theorem. The map $\gamma_A$ takes $B_Z$ to the $\mathbb{Z}$-basis of $K(R)$ consisting of the indecomposable projective objects.

4.6. Definition of the canonical basis $B$. Before the proof let us recall the construction of $B$ in [L2]. Fix a non zero element $\nu \in \mathbb{N}I$ and fix $V \in \mathcal{V}_\nu$. Let $\mathcal{P}_V$ the set of isomorphism classes of simple perverse sheaves $L$ on $E_V$ such that $L[r]$ appears as a direct summand of $L_i$ for some $i \in I^\nu$, $r \in \mathbb{Z}$. Such a perverse sheaf belongs to $\mathcal{D}_{G_V}(E_V)$. Let $\mathcal{Q}_V$ the full subcategory of $\mathcal{D}_{G_V}(E_V)$ consisting of all complexes that are isomorphic to finite direct sums of complexes of the form $L[r]$ for various $r \in \mathbb{Z}$ and $L \in \mathcal{P}_V$.

Let $K(\mathcal{Q}_V)$ be the Abelian group with one generator $[L]$ for each isomorphism class of objects of $\mathcal{Q}_V$ and with relations $[L]+[L']=[L'']$ whenever $L''$ is isomorphic to $L \oplus L'$. It is a free $A$-module such that $qL=L[1]$ and $q^{-1}L=L[-1]$, where $L$ runs over $\mathcal{Q}_V$. An isomorphism $V \cong V'$ in $\mathcal{V}$ induces a canonical isomorphism $K(\mathcal{Q}_V) \cong K(\mathcal{Q}_V')$. Taking the direct limit over the groupoid consisting of the objects of $\mathcal{V}$ with their isomorphisms, we get

$$K(\mathcal{Q}) = \lim_{\longrightarrow \mathcal{V}} K(\mathcal{Q}_V).$$

Given $\nu, \nu' \in \mathbb{N}I$ and $V \in \mathcal{V}_\nu$, $V' \in \mathcal{V}_{\nu'}$ we set $\nu'' = \nu + \nu'$, $V'' = V \oplus V'$. Let

$$*: \mathcal{Q}_V \times \mathcal{Q}_{V'} \to \mathcal{Q}_{V''}$$

be Lusztig’s induction functor [L2, sec. 9.2.5]. There is an unique associative $A$-bilinear multiplication $\odot$ on $K(\mathcal{Q})$ such that for each $L \in \mathcal{Q}_V$, $L' \in \mathcal{Q}_{V'}$ we have

$$L \odot L' = (L \ast L')[m_{\nu,\nu'}], \quad m_{\nu,\nu'} = \sum_{h \in H} \nu_h \nu'_h + \sum_{i \in I} \nu_i \nu'_i.$$

There is also an unique $A$-algebra isomorphism

$$(4.5) \quad \lambda_A : K(\mathcal{Q}) \to A$, \quad [^{\delta}L_y] \mapsto \theta_y, \quad \forall y.$$

See [L2, thm. 13.2.11]. The classes in $K(\mathcal{Q}_V)$ of the perverse sheaves of $\mathcal{P}_V$ form a $A$-basis of $K(\mathcal{Q}_V)$. We have

$$B_\nu = \{b_L; L \in \mathcal{P}_V\}, \quad b_L = \lambda_A([L]).$$

For any $\nu, \nu' \in \mathbb{N}I$ and $y \in Y_\nu$, $y' \in Y_{\nu'}$ we have [L2, sec. 9.2.6-7]

$$(4.6) \quad ^{\delta}L_y \odot ^{\delta}L_y' \simeq ^{\delta}L_{yy'}.$$

If $i' \in I^{\nu'}$ is the expansion of a pair $y = (i, a) \in Y_\nu$ then we have also

$$(4.7) \quad [^{\delta}L_y] = [a][^{\delta}L_y].$$
4.7. Proof of Theorem 4.5: Assume that \( \nu \in \mathbb{N}I \) and \( V \in \mathcal{V}_\nu \). Given \( \mathcal{L} \in \mathcal{Q}_V \) we consider the left graded \( \delta \mathcal{Z}_V \)-module given by
\[
\underline{Y}_\mathcal{L} = \text{Ext}^*_G(\delta \mathcal{L}, \mathcal{L}).
\]
We’ll view it as a left graded \( \mathbb{R}_\nu \)-module via the isomorphism \( \Psi_V \). For each complexes \( \mathcal{L}, \mathcal{L}' \in \mathcal{Q}_V \) and each integer \( d \) we have canonical isomorphisms
\[
\underline{Y}_{\mathcal{L} \oplus \mathcal{L}'} = \underline{Y}_\mathcal{L} \oplus \underline{Y}_\mathcal{L}', \quad \underline{Y}_{\mathcal{L}[r]} = \underline{Y}_\mathcal{L}[r].
\]
If \( \mathcal{L} \in \mathcal{P}_V \) there is a sequence \( i \in I^\nu \) and an integer \( r \) such that \( \mathcal{L}[r] \) is a direct summand of the semisimple complex \( \delta \mathcal{L}_i \). Further, we have
\[
(4.8) \quad \underline{Y}_{\mathcal{L}_i} = \text{Ext}^*_G(\delta \mathcal{L}_i, \delta \mathcal{L}_i) = \delta \mathcal{Z}_V \star 1_i \simeq R_i
\]
as left graded \( \mathbb{R}_\nu \)-modules. Therefore \( \underline{Y}_\mathcal{L} \) belongs to \( \mathbb{R}_\nu \)-\text{proj}. Since any element in \( \mathcal{Q}_V \) is a sum of shifts of elements of \( \mathcal{P}_V \) we have also \( \underline{Y}_\mathcal{L} \in \mathbb{R}_\nu \)-\text{proj} for each \( \mathcal{L} \in \mathcal{Q}_V \). In other words, we have constructed an additive functor
\[
\underline{Y}_V : \mathcal{Q}_V \to \mathbb{R}_\nu \text{-proj}, \quad \mathcal{L} \mapsto \underline{Y}_V(\mathcal{L}) = \underline{Y}_\mathcal{L}
\]
which commutes with the shift of the grading. Let \( \underline{Y} \) denote the \( \mathcal{A} \)-linear map
\[
\underline{Y} : K(\mathcal{Q}) \to K(\mathbb{R}), \quad [\mathcal{L}] \mapsto [\underline{Y}_\mathcal{L}].
\]
Since \( \mathcal{B}_\nu \) is a basis of the \( \mathcal{A} \)-module \( \mathcal{A} f_\nu \), there is a unique \( \mathcal{A} \)-linear map
\[
\mathcal{A} f_\nu \to K(\mathbb{R}), \quad b_\mathcal{L} \mapsto [\underline{Y}_\mathcal{L}], \quad \forall \mathcal{L} \in \mathcal{P}_V.
\]
Taking the limit over all \( V \in \mathcal{V} \) we get an \( \mathcal{A} \)-linear map
\[
(4.9) \quad \mu_\mathcal{A} : \mathcal{A} f \to K(\mathbb{R})
\]
such that \( \mu_\mathcal{A} \circ \lambda_\mathcal{A} = \underline{Y} \). We claim that we have
\[
\mu_\mathcal{A} = \gamma_\mathcal{A}.
\]
Indeed (4.4), (4.8) yield
\[
\mu_\mathcal{A} (\theta_i) = \mu_\mathcal{A} \lambda_\mathcal{A} ([\delta \mathcal{L}_i]) = [R_i] = \gamma_\mathcal{A} (\theta_i), \quad \forall i \in I^\nu.
\]
By base change the \( \mathcal{A} \)-linear maps \( \gamma_\mathcal{A}, \mu_\mathcal{A} \) yield \( \mathcal{K} \)-linear maps
\[
\gamma : f \to K(\mathbb{R})_\mathcal{K}, \quad \mu : f \to K(\mathbb{R})_\mathcal{K}, \quad K(\mathbb{R})_\mathcal{K} = K(\mathbb{R}) \otimes_\mathcal{A} \mathcal{K}.
\]
Since the \( \mathcal{A} \)-modules \( \mathcal{A} f, K(\mathbb{R}) \) are both torsion free it is enough to prove that the maps \( \mu, \gamma \) are the same. This is obvious because the monomials \( \theta_i \) span the \( \mathcal{K} \)-vector space \( f_\nu \) as \( i \) runs over \( I^\nu \).

Next we prove that \( \mu_\mathcal{A} \) takes the elements of \( \mathcal{B}_\nu \) to the classes in \( K(\mathbb{R}) \) of the indecomposable projective objects. We claim that it is enough to prove that \( \underline{Y}_\mathcal{L} \) is indecomposable in \( \mathbb{R}_\nu \text{-proj} \) for each \( \mathcal{L} \in \mathcal{P}_V \). Indeed the map \( \gamma_\mathcal{A} \) is invertible
and \( \mu_A = \gamma_A \). Thus \( \mu_A \) takes \( B_{\mathbb{Z}} \) to a \( \mathbb{Z} \)-basis of \( K(R) \). Since \( \mu_A \circ \lambda_A = Y \), if \( Y_L \) is indecomposable in \( R_\nu \text{-Mod} \) for each \( L \in \mathcal{P}_V \), then \( \mu_A \) takes \( B_{\mathbb{Z}} \) to a subset of the \( \mathbb{Z} \)-basis of \( K(R) \) consisting of the indecomposable projective objects. So we are done.

To prove the claim we’ll prove that the top of \( Y_L \) in \( R_\nu \text{-Mod} \) is a simple object. Fix a simple quotient \( Y_L \to L \) in \( R_\nu \text{-Mod} \). Note that \( L \) is finite dimensional, because \( R_\nu \) is finite dimensional over its center. The center of \( R_\nu \) is equal to \( S_\nu = S_V \). See [KL1, sec. 2.4]. Let \( S_\nu^+ \subset S_\nu \) be the unique graded maximal ideal. It acts by zero on each simple finite dimensional left graded \( R_\nu \)-module. So it acts by zero on \( L \). We set

\[
R_\nu^0 = (S_\nu/S_\nu^+) \otimes_{S_\nu} R_\nu, \quad Y_L^0 = (S_\nu/S_\nu^+) \otimes_{S_\nu} Y_L.
\]

Then \( L \) is a simple left graded \( R_\nu^0 \)-module, \( Y_L^0 \) is a projective left graded \( R_\nu^0 \)-module and, since taking the tensor product with \( S_\nu/S_\nu^+ \) is a right exact functor, the surjective map \( Y_L \to L \) factors to a surjective left \( R_\nu^0 \)-module homomorphism \( Y_L^0 \to L \).

To simplify the notation we’ll use the same symbol for a complex in \( D_{G\mathcal{V}}(E\mathcal{V}) \) and its image by the forgetful functor \( D_{G\mathcal{V}}(E\mathcal{V}) \to D(E\mathcal{V}) \). Write

\[
R_\nu^1 = \text{Ext}^*({}^4\mathcal{L}, {}^4\mathcal{L}), \quad Y_L^1 = \text{Ext}^*({}^4\mathcal{L}, L).
\]

The proof of the following lemma is postponed in Section 4.10.

**4.8. Lemma.** The forgetful map \( R_\nu \to R_\nu^1 \) factors to a graded \( k \)-algebra isomorphism \( R_\nu^0 \to R_\nu^1 \). The forgetful map \( Y_L \to Y_L^1 \) factors to a left graded \( R_\nu^0 \)-module isomorphism \( Y_L^0 \to Y_L^1 \).

For each simple perverse sheaf \( L' \in \mathcal{P}_V \) we define a finite dimensional \( \mathbb{C} \)-vector space \( M_{L'} \) by \( M_{L'} = \bigoplus_{r \in \mathbb{Z}} M_{L', r} \) with

\[
{}^4\mathcal{L} = \bigoplus_{L' \in \mathcal{P}_V} \bigoplus_{r \in \mathbb{Z}} M_{L', r} \otimes L'[r].
\]

We have

\[
R_\nu^0 = R_\nu^1 = \bigoplus_{L', L'' \in \mathcal{P}_V} \text{Hom}(M_{L''}, M_{L'}) \otimes \text{Ext}^*({}^4\mathcal{L}'', {}^4\mathcal{L}').
\]

Thus \( R_\nu^0 \) has a natural structure of finite dimensional \( \mathbb{Z}_{\geq 0} \)-graded \( k \)-algebra whose degree zero part is a semi-simple \( k \)-algebra isomorphic to \( \bigoplus_{L' \in \mathcal{P}_V} \text{End}(M_{L'}) \). See [CG, sec. 8] for more details. In particular each \( \mathbb{C} \)-vector space \( M_{L'} \) has a natural structure of simple left \( R_\nu^0 \)-module. Now, we have

\[
Y_L^0 = \bigoplus_{L' \in \mathcal{P}_V} M_{L'} \otimes \text{Ext}^*({}^4\mathcal{L}', {}^4\mathcal{L}),
\]

a \( \mathbb{Z}_{\geq 0} \)-graded \( R_\nu^0 \)-module whose degree zero part is isomorphic to \( M_L \). Observe that \( Y_L^0 \) is generated by its degree zero subspace as a \( R_\nu^0 \)-module, because

\[
R_\nu^0 \ast (Y_L^0)_0 = \left( \bigoplus_{L', L'' \in \mathcal{P}_V} \text{Hom}(M_{L''}, M_{L'}) \otimes \text{Ext}^*({}^4\mathcal{L}'', {}^4\mathcal{L}')) \ast M_L \otimes \text{Ext}^*({}^4\mathcal{L}, {}^4\mathcal{L}) \right)
= \bigoplus_{L' \in \mathcal{P}_V} M_{L'} \otimes \left( \text{Ext}^*({}^4\mathcal{L}', {}^4\mathcal{L}) \ast \text{Ext}^*({}^4\mathcal{L}, {}^4\mathcal{L}) \right)
= Y_L^0.
\]
Thus the $R_\nu$-module $Y_\nu^0$ has a unique simple quotient which is isomorphic to $M_\nu^0$. Hence we have $L = M_\nu^0$. Thus $Y_\nu$ has a unique simple quotient in $R_\nu$-Mod. We are done.

\[ \square \]

\textbf{4.9. Remarks.} \( a \) Fix a pair $y = (i, a)$ in $Y_\nu$. Let $I' \in I'$ be the sequence obtained by expanding the pair $y$. By (4.2), (4.7) and (4.8) we have

$$[a]! [R_y] = [a]! [Y_{i\nu}].$$

Since the $A$-module $K(R_\nu)$ is torsion free this implies that $[R_y] = [Y_{i\nu}]$. Now, recall that any object in $R_\nu$-Proj is a direct sum of a finite number of indecomposable ones, and that the indecomposable projective objects yield a basis of the Abelian group $K(R_\nu)$. Therefore we have $R_y \simeq Y_{i\nu}$ in $R_\nu$-Proj.

\( b \) By (4.3), (4.6) and the previous remark we have

$$[Y_{\nu \oplus \nu'}] = [\text{Ind}_{\nu \oplus \nu'}(Y_\nu \otimes Y_\nu')], \quad \forall L \in Q_{\nu'}, L' \in Q_{\nu'}. $$

Therefore, the same argument as above yields a non canonical isomorphism in $R_{\nu+\nu'}$-Proj

$$ Y_{\nu \oplus \nu'} \simeq \text{Ind}_{\nu \oplus \nu'}(Y_\nu \otimes Y_\nu'). $$

\( c \) Note that we don’t use the surjectivity of $\gamma_A$; on the contrary this surjectivity follows from our arguments. Namely, the image of $Y$ is the free abelian group spanned by the subset of the basis of indecomposable objects given by $[Y_L]$, with $L \in P_V$ for some $V$. On the other hand the image of $Y$ is the $A$-span of the $[R_i]$’s, with $i \in I^\nu$ and $\nu \in NI$. Since any indecomposable projective module is a direct summand of $R_i$ for some $i$ (up to a shift of the grading), the image of $Y$ must contain all the indecomposable projective objects.

\textbf{4.10. Proof of Lemma 4.8:} To prove the first claim we must check that the forgetful map yields an isomorphism

$$(S_V/S_V^\nu) \otimes_{S_V} H^*_V(Z_V, k) \rightarrow H_*(Z_V, k).$$

Let $P_V^+ \subset P_V$ be the unique graded maximal ideal. By (2.3) it is enough to prove that the forgetful map yields an isomorphism

$$(P_V/P_V^\nu) \otimes_{P_V} H^*_V(Z_V, k) \rightarrow H_*(Z_V, k).$$

This is well-known. It is proved using a decomposition of $Z_V$ into affine cells. See [CG, chap. 6] for similar results for the equivariant K-theory of the Steinberg variety, and [GKM, sec. 1] for details on equivariantly formal $T_V$-varieties.

The second claim follows from the first one. Indeed, the forgetful map

$$(S_V/S_V^\nu) \otimes_{S_V} \text{Ext}^*_V(\delta L_V, \delta L_V) \rightarrow \text{Ext}^*(\delta L_V, \delta L_V)$$

is invertible. Further $\text{Ext}^*_V(\delta L_V, L)$ is a direct summand of $\text{Ext}^*_V(\delta L_V, \delta L_V)$ and $\text{Ext}^*(\delta L_V, L)$ is a direct summand of $\text{Ext}^*(\delta L_V, \delta L_V)$. Therefore the forgetful map yields also an isomorphism

$$(S_V/S_V^\nu) \otimes_{S_V} \text{Ext}^*_V(\delta L_V, L) \rightarrow \text{Ext}^*(\delta L_V, L).$$

\[ \square \]
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