DENSITY OF ORBITS IN LAMINATIONS AND THE SPACE OF CRITICAL PORTRAITS

ALEXANDER BLOKH

Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170, USA

CLINTON CURRY

Department of Mathematics
1500 East Fairview Avenue
Montgomery, AL 36106, USA

LEX OVERSTEEGEN

Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170, USA

(Communicated by the associate editor Sebastian van Strien)

ABSTRACT. Thurston introduced $\sigma_d$-invariant laminations (where $\sigma_d(z)$ coincides with $z^d : S \to S,$ $d \geq 2$). He defined wandering $k$-gons as sets $T \subset S$ such that $\sigma^n_d(T)$ consists of $k \geq 3$ distinct points for all $n \geq 0$ and the convex hulls of all the sets $\sigma^n_d(T)$ in the plane are pairwise disjoint. Thurston proved that $\sigma_2$ has no wandering $k$-gons and posed the problem of their existence for $\sigma_d, \ d \geq 3.$ Call a lamination with wandering $k$-gons a WT-lamination. Denote the set of cubic critical portraits by $A_3.$ A critical portrait, compatible with a WT-lamination, is called a WT-critical portrait; let $WT_3$ be the set of all of them. It was recently shown by the authors that cubic WT-laminations exist and cubic WT-critical portraits, defining polynomials with condense orbits of vertices of order three in their dendritic Julia sets, are dense and locally uncountable in $A_3.$ Here we show that $WT_3$ is a dense first category subset of $A_3$, that critical portraits, whose laminations have a condense orbit in the topological Julia set, form a residual subset of $A_3$, and that the existence of a condense orbit in the Julia set $J$ implies that $J$ is locally connected.

1. Introduction. Let $\mathbb{C}$ be the complex plane and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the complex sphere. The following result is a special case of a theorem due to Thurston [24].

**Theorem 1.1 (No Wandering Vertices for Quadratics).** Let $P(z) = z^2 + c$ be a polynomial with connected Julia set $J_P.$ If $z_0 \in J_P$ is a point such that $J_P \setminus \{z_0\}$ has at least three components, then $z_0$ is either preperiodic or eventually maps to the critical point 0.

2000 Mathematics Subject Classification. Primary 37F20; Secondary 37B45, 37F10, 37F50.

Key words and phrases. Complex dynamics, locally connected, Julia set, lamination, wandering gaps.

The first named author is supported by NSF grant DMS-0901038. The third named author is supported by NSF grant DMS-0906316.
In [7] we construct an uncountable family of cubic polynomials $P$ with $z_0 \in J_P$ such that $J_P \setminus \{z_0\}$ has three components and $z_0$ is neither preperiodic nor precritical; such a point is called a wandering vertex. In [2], we improve on these results by finding a collection of polynomials, dense in the appropriate parameter space, with wandering vertices whose orbits have a property that we call condensity.

**Definition 1.2.** For a topological space $X$ a set $A \subset X$ is *continuumwise dense* (abbreviated *condense*) in $X$ if $A \cap Z \neq \emptyset$ for each non-degenerate continuum $Z \subset X$. A map $f : X \to X$ is also called condense if there exists $x_0 \in X$ such that $\{f^n(x_0) \mid n \geq 0\}$ is condense in $X$.

It is not hard to see that condensity is much stronger than density. For example, if $f$ is a Julia set from the real quadratic family which is not homeomorphic to an interval, the set of endpoints is dense in $J$, but not condense. Moreover, in this case the set of transitive points (i.e., points with dense orbit in $J$) is a subset of the endpoints of $J$, so such maps are not condense.

To state the results of [3] precisely, we must indicate in which parameter space we are working. Polynomials are naturally associated to critical portraits, introduced by Yuval Fisher in his Ph.D. thesis [11]. Let $\sigma_d : \mathbb{S} \to \mathbb{S}$ be the angle $d$-tupling map $\sigma_d(z) = z^d$. A degree $d$ critical portrait, loosely speaking, is a maximal collection $\Theta = \{\Theta_1, \ldots, \Theta_n\}$ of sets of angles in $\mathbb{S}$ which are pairwise disjoint, pairwise unlinked (i.e., have disjoint convex hulls in $\mathbb{S}$ when angles are interpreted as points in $\mathbb{S}$), and such that $\sigma_d(\Theta_i)$ is a point for each $\Theta_i \in \Theta$ (it is easy to see that $\sum(|\Theta_i| - 1) = d - 1$).

This notion is used to capture the location of critical points. The set of all critical portraits of degree $d$ is denoted $A_d$, and is naturally endowed with a topology (see Definition 2.4 for details). We say that a critical portrait $\Theta$ corresponds to a polynomial $P$ with dendritic Julia set if for each $\Theta_i \in \Theta$ there is a distinct critical point $c_i \in J_P$ such that the external rays whose angles are in $\Theta_i$ land at $c_i$ (see Section 2.3 for more information). Now we state the main theorem of [2].

**Theorem 1.3** ([2]). $A_3$ contains a dense locally uncountable set $\{\Theta_\alpha \mid \alpha \in A\}$ of critical portraits such that for each $\alpha \in A$ the following holds:

- $\Theta_\alpha$ corresponds to a polynomial $P_\alpha$ with dendritic Julia set $J_{P_\alpha}$.
- $\{P_\alpha | J_{P_\alpha}\}$ are pairwise non-conjugate, and
- $J_{P_\alpha}$ contains a wandering vertex with condense orbit.

The aim of this paper is to further investigate the notions and objects studied in Theorem 1.3, such as condensity and the set of critical portraits which correspond to polynomials with wandering vertices. To explain our results, we recall constructions from [14, 3]: given a polynomial $P$ with connected Julia set $J_P$, one can construct a corresponding locally connected continuum $J \subset \mathbb{C}$ (called a topological Julia set) and branched covering map $f : \mathbb{C} \to \mathbb{C}$ (called a topological polynomial) so that $P$ is monotonically semiconjugate to $f$ (i.e., there exists a monotone map $m : \mathbb{C} \to \mathbb{C}$ such that $m \circ P = f \circ m$) and $J = m(J_P)$. We refer to $f|_J$ as the locally connected model of $P$. It is known [6] that in some cases $J$ is a single point.

Let us describe the organization of the paper and the main results. After discussing preliminary notions and history in Section 2, we study properties of condense maps in Section 3. In particular we show in Theorem 3.6 that polynomials which admit condense orbits either in their Julia sets (or in some circumstances their locally connected models) have locally connected Julia sets. In Section 4 we prove that the set of cubic critical portraits corresponding to polynomials with condense orbits in their Julia sets is residual in $A_3$.
(Theorem 4.1), while the set of critical portraits which correspond to polynomials with wandering vertices is meager (Theorem 4.3).

2. Preliminaries.

2.1. Laminations. In what follows, we parameterize the circle as \( \mathbb{S} = \mathbb{R}/\mathbb{Z} \), so the total arclength of \( \mathbb{S} \) is 1. The positive direction on \( \mathbb{S} \) is the counterclockwise direction, and by the arc \((p, q)\) in the circle we mean the positively oriented arc from \( p \) to \( q \). A (strictly) monotone map \( g : (p, q) \to \mathbb{S} \) is a map (strictly) monotone at each point of \((p, q)\) in the sense of positive direction on \( \mathbb{S} \). By \( \text{Ch}(A) \) we denote the convex hull of a set \( A \subset \mathbb{C} \) and by \(|B|\) we denote the cardinality of the set \( B \).

Laminations are combinatorial structures on the unit circle, introduced by Thurston [24] as a tool for studying individual complex polynomials \( P : \mathbb{C}_\infty \to \mathbb{C}_\infty \) and the space of all of them. In what follows, we use a restricted formulation of this concept. Specifically, [24] defines a lamination as a collection of chords in the unit disk, satisfying certain dynamical properties. These collections are intended to reflect the pattern of external rays landing in the Julia set. We instead use the formulation contained in [14], explained in detail below, based on pinched disk models (see [8] for more information in this vein, and also [5]).

Let \( P \) be a degree \( d \) polynomial with a locally connected (and hence connected) Julia set \( J_P \); we will recall how to associate an equivalence relation \( \sim_P \) on \( \mathbb{S} \) to \( P \), called the \( d \)-invariant lamination generated by \( P \). The filled-in Julia set \( K_P \) is compact, connected, and full, so its complement \( C_\infty \setminus K_P \) is conformally isomorphic to the open unit disk \( \mathbb{D} \). By [18, Theorem 9.5], there is a particular conformal isomorphism \( \Psi : \mathbb{D} \to C_\infty \setminus K_P \) so that \( \Psi \) conjugates \( \sigma_d(z) = z^d \) on \( \mathbb{D} \) to \( P|_{C_\infty \setminus K_P} \) (i.e., \( \Psi(z^d) = (P|_{C_\infty \setminus K_P} \circ \Psi)(z) \) for \( z \in \mathbb{D} \)). When \( J_P \) is locally connected, \( \Psi \) extends to a continuous map \( \overline{\Psi} : \overline{\mathbb{D}} \to \overline{C_\infty \setminus K_P} \) which semiconjugates \( z \mapsto z^d \) on \( \overline{\mathbb{D}} \) to \( P|_{\overline{C_\infty \setminus K_P}} \). Let \( \psi : \mathbb{S} \to J_P \) denote the restriction \( \overline{\Psi}|_\mathbb{S} \). Define the equivalence relation \( \sim_P \) on \( \mathbb{S} \) so that \( x \sim_P y \) if and only if \( \psi(x) = \psi(y) \); this equivalence relation is the aforementioned \( d \)-invariant lamination generated by \( P \). The quotient space \( \mathbb{S}/\sim_P = J_{\sim_P} \) is homeomorphic to \( J_P \) and the induced map \( f_{\sim_P} : J_{\sim_P} \to J_{\sim_P} \), defined by \( f_{\sim_P} = \psi \circ \sigma_d \circ \psi^{-1} \) is conjugate to \( P|_{J_P} \).

Kiwi [14] extended this construction to polynomials \( P \) with no irrationally neutral cycles and introduced a similar \( d \)-invariant lamination \( \sim_P \). Then \( J_{\sim_P} = \mathbb{S}/\sim_P \) is locally connected and \( P|_{J_P} \) is semi-conjugate to \( f_{\sim_P} \) by a monotone map \( m : J_P \to J_{\sim_P} \), i.e., a map \( m \) whose preimages are connected. This was extended in [3] to all polynomials \( P \) with connected \( J_P \). The lamination \( \sim_P \) combinatorially describes the dynamics of \( P|_{J_P} \).

One can introduce abstract laminations (frequently denoted by \( \sim \)) as equivalence relations on \( \mathbb{S} \) having properties in common with laminations generated by polynomials as above. Consider an equivalence relation \( \sim \) on the unit circle \( \mathbb{S} \). Equivalence classes of \( \sim \) will be called (\( \sim \)-)classes and will be denoted by boldface letters. A \( \sim \)-class consisting of two points is called a leaf; a class consisting of at least three points is called a gap (this is more restrictive than Thurston’s definition in [24]). Fix an integer \( d > 1 \). Then \( \sim \) is said to be a \( d \)-invariant lamination if:

(E1) \( \sim \) is closed: the graph of \( \sim \) is a closed set in \( \mathbb{S} \times \mathbb{S} \);

(E2) \( \sim \)-classes are pairwise unlinked: if \( g_1 \) and \( g_2 \) are distinct \( \sim \)-classes, then their convex hulls \( \text{Ch}(g_1), \text{Ch}(g_2) \) in the unit disk \( \mathbb{D} \) are disjoint;

(E3) \( \sim \)-classes are either totally disconnected (and hence \( \sim \) has uncountably many classes) or the entire circle \( \mathbb{S} \) is one class;

(D1) \( \sim \) is forward invariant: for a class \( g \), the set \( \sigma_d(g) \) is also a class;
(D2) \( \sim \) is backward invariant: for a class \( g \), its preimage \( \sigma_d^{-1}(g) = \{ x \in S : \sigma_d(x) \in g \} \) is a union of classes; and

(D3) for any gap \( g \), the map \( \sigma_d|_g : g \to \sigma_d(g) \) is a covering map with positive orientation, i.e., for every connected component \((s, t)\) of \( S \setminus \sigma_d(g) \) the arc \((\sigma_d(s), \sigma_d(t))\) is a connected component of \( S \setminus \sigma_d(g) \).

Notice that (D2) and (E3) follow from (D1).

Call a class \( g \) critical if \( \sigma_d|_g : g \to \sigma_d(g) \) is not one-to-one, and precritical if \( \sigma_d^j(g) \) is critical for some \( j \geq 0 \). Call \( g \) preperiodic if \( \sigma_d^i(g) = \sigma_d^j(g) \) for some \( 0 \leq i < j \). A gap \( g \) is wandering if \( g \) is neither preperiodic nor precritical. Let \( J_\sim = S/ \sim \), and let \( \pi_\sim : S \to J_\sim \) be the corresponding quotient map. The map \( f_\sim : J_\sim \to J_\sim \) defined by \( f_\sim = \pi_\sim \circ \sigma_d \circ \pi_\sim^{-1} \) is the map induced on \( J_\sim \) by \( \sigma_d \). Then we call \( f_\sim \) a topological polynomial, and \( J_\sim \) a topological Julia set.

2.2. Bounds for wandering classes. J. Kiwi [13] extended the No Wandering Triangles Theorem by showing that a wandering gap in a \( d \)-invariant lamination is at most a \( d \)-gon. Thus all infinite \( \sim \)-classes (and Jordan curves in \( J_\sim \)) are preperiodic. In [17] G. Levin showed that laminations with one critical class have no wandering gaps. For a lamination \( \sim \), let \( k_\sim \) be the size of a maximal collection of non-degenerate \( \sim \)-classes whose \( \sigma_d \)-images are points and whose orbits are infinite and pairwise disjoint. Also, let \( N_\sim \) be the number of cycles of infinite \( \sim \)-classes plus the number of cycles of Jordan curves in \( J_\sim \).

Theorem 2.1 ([5]). Let \( \sim \) be a \( d \)-invariant lamination and let \( \Gamma \) be a non-empty collection of wandering \( d_j \)-gons (\( j = 1, 2, \ldots \)) with distinct grand orbits. Then \( \sum_j (d_j - 2) \leq k_\sim - 1 \) and \( \sum_j (d_j - 2) + N_\sim \leq d - 2 \). In particular, in the cubic case if \( \Gamma \) is non-empty, then it must consist of one non-precritical \( \sim \)-class with three elements, all \( \sim \)-classes are finite, \( J_\sim \) is a dendrite, and both critical classes are leaves with disjoint forward orbits.

2.3. Critical portraits. Following [24] and [9, 10] we look at the set \( C_d \) from infinity and consider the shift locus, which is the set \( \mathcal{S}_d \) of polynomials whose critical points escape to infinity. The set \( \mathcal{S}_d \) is the unique hyperbolic component of \( \mathcal{P}_d \) consisting of polynomials with all cycles repelling. It is not known if all polynomials with all cycles repelling belong to the set \( \overline{\mathcal{S}_d} \). Looking at \( C_d \) from infinity means studying locations of polynomials in \( \overline{\mathcal{S}_d} \) depending on their dynamics and using this to describe the polynomials belonging to \( \overline{\mathcal{S}_d} \cap C_d \). A key tool in studying \( C_d \) is critical portraits, introduced in [11], and widely used afterward (see, e.g., [1, 21, 12] and [15]). We now recall some standard material; here we closely follow [15, Section 3]. Call a chord with endpoints \( a, b \in S \) critical if \( \sigma_d(a) = \sigma_d(b) \).

Definition 2.2. A critical portrait is a collection \( \Theta = \{ \Theta_1, \ldots, \Theta_n \} \) of finite subsets of \( S \) such that the following hold:

1. the boundary of the convex hull \( \text{Ch}(\Theta_i) \) of every set \( \Theta_i \) consists of critical chords (under \( \sigma_d \));
2. the sets \( \Theta_1, \ldots, \Theta_n \) are pairwise unlinked (that is, convex hulls of the sets \( \Theta_i \) are pairwise disjoint); and
3. \( \sum_i (|\Theta_i| - 1) = d - 1 \).

The sets \( \Theta_1, \ldots, \Theta_n \) are called the initial sets of \( \Theta \) (or \( \Theta \)-initial sets). Denote by \( A(\Theta) \) the union of all angles from the initial sets of \( \Theta \). The convex hulls of the \( \Theta \)-initial sets divide the rest of the open unit disk into components. In Definition 2.3, points of \( S \setminus A(\Theta) \) are declared equivalent if they belong to the boundary of one such component; we do not assume that \( \Theta \) is a critical portrait because we need this equivalence later in a more general situation.
Definition 2.3. Let $\Theta$ be a finite collection of pairwise unlinked finite subsets of $S$. Angles $\alpha, \beta \in S \setminus A(\Theta)$ are $\Theta$-unlinked equivalent if $\{\alpha, \beta\}, \Theta_1, \ldots, \Theta_n$ are pairwise unlinked. The equivalence classes $L_1(\Theta), \ldots, L_d(\Theta)$ are called $\Theta$-unlinked classes. Each $\Theta$-unlinked class $L$ is the intersection of $S \setminus A(\Theta)$ with the boundary of a component of $D \setminus \bigcup \operatorname{Ch}(\Theta)$. In the degree $d$ case, each $\Theta$-unlinked class of a critical portrait $\Theta$ is the union of finitely many open arcs of total length $1/d$. Thus, there are $d$ $\Theta$-unlinked classes.

Definition 2.4 (compact-unlinked topology [15]). Define the space $A_d$ as the set of all critical portraits endowed with the compact-unlinked topology generated by the subbasis $V_X = \{\Theta \in A_d : X \subset L_\Theta\}$ where $X \subset S$ is closed and $L_\Theta$ is a $\Theta$-unlinked class.

Note for example that $A_2$ is the quotient of $S$ with antipodal points identified, so it is homeomorphic to the unit circle. For a critical portrait $\Theta$, a lamination $\sim$ is called $\Theta$-compatible if all $\Theta$-initial sets are contained in $\sim$-classes; if there is a $\Theta$-compatible WT-lamination, $\Theta$ is said to be a WT-critical portrait. The trivial lamination, identifying all points of $S$, is compatible with any critical portrait.

To define critical portraits with aperiodic kneading, let us introduce the notion of a one-sided itinerary for $t \in S$ (see [15]). Given a critical portrait $\Theta = \{\Theta_1, \ldots, \Theta_d\}$ with $\Theta$-unlinked classes $L_1(\Theta), \ldots, L_d(\Theta)$ and $\theta \in S$, define $i^+(\theta)$ (respectively, $i^-(\theta)$) as the sequence $(i_0, i_1, \ldots)$, with $i_j \in \{1, \ldots, d\}$ such that there are $y_n \nearrow \theta$ (respectively, $y_n \searrow \theta$) with $\sigma^j(y_n) \in L_{i_j}(\Theta)$ for $n$ sufficiently large. Also, define the itinerary $i(\theta)$ as a sequence $I_0 I_1 \ldots$ such that each $I_j$ is the set from $\Theta \cup \{L_1(\Theta), \ldots, L_d(\Theta)\}$ to which $\sigma^j(\theta)$ belongs. An angle $\theta \in S$ is said to have a periodic kneading if $i^+(\theta)$ or $i^-(\theta)$ is periodic. A critical portrait $\Theta$ is said to have aperiodic kneading if no angle from $A(\Theta)$ has periodic kneading. The family of all degree $d$ critical portraits with aperiodic kneading is denoted by $\mathcal{AP}_d$.

Definition 2.5 ([14, 15]). The lamination $\sim_\Theta$ is the smallest closed equivalence relation identifying any pair of points $x, y \in S$ where $i^+(x) = i^-(y)$. By Kiwi [14, 15], for any critical portrait $\Theta$ the relation $\sim_\Theta$ is a $\Theta$-compatible lamination; it is said to be generated by $\Theta$.

Critical portraits reflect the landing patterns of the external rays at the critical points. By Kiwi [15], a nice correspondence between critical portraits of degree $d$ and the set $S_d \cap C_d$ associates to each critical portrait $\Theta \in A_d$ a connected set $I(\Theta) \subset S_d \cap C_d$, called the impression of $\Theta$, such that the dynamics of a polynomial in $I(\Theta)$ is closely related to the properties of $\Theta$. The relation is especially nice when $\Theta$ has aperiodic kneading. The following fundamental result of Kiwi [14, 15] explicitly lists properties of critical portraits with aperiodic kneading and their connections to polynomials.

Theorem 2.6. Let $\Theta \in \mathcal{AP}_d$. Then $\sim_\Theta$ is the unique $\Theta$-compatible invariant lamination. The quotient $J_{\sim_\Theta}$ is a non-degenerate dendrite, and all $\sim$-classes are finite. Furthermore, there exists a polynomial $P$ whose Julia set $J_P$ is a non-separating continuum in the plane and $P |_{J_P}$ is monotonically semiconjugate to $f_{\sim_\Theta} |_{J_{\sim_\Theta}}$. The semiconjugating map $n_{\Theta, P} : J_P \to J_{\sim_\Theta}$ maps impressions of external angles to points and maps the set of $P$-preperiodic points in $J_P$ bijectively to the set of $f_{\sim_\Theta}$-preperiodic points. Moreover, $J_P$ is locally connected at all $P$-preperiodic points.

In the situation of Theorem 2.6 polynomials $P$ such that $P |_{J_P}$ is monotonically semiconjugate to $f_{\sim_\Theta} |_{J_{\sim_\Theta}}$ are said to be associated to the critical portrait $\Theta$.

2.4. Monotone models for connected Julia sets. As was explained in Section 1, the main results of [14, 3] yield a locally connected model for the restriction of a polynomial to
its connected Julia set. We will need a detailed version of these results stated below in Theorem 2.7.

**Theorem 2.7** ([14, 3]). Let \( P \) be a degree \( d \) polynomial with connected Julia set \( J_P \). Then there exists a \( d \)-invariant lamination \( \sim \) and a monotone onto map \( M_P : \mathbb{C} \to \mathbb{C} \) with the following properties.

1. \( J_\sim = M_P(J_P) \) and \( J_P \subset M_P^{-1}(J_\sim) \subset K_P \).
2. \( M_P \) sends impressions of \( J_P \) to points.
3. \( m_P = M_P|_{J_P} \) is the finest monotone map of \( J_P \) onto a locally connected continuum (i.e., if \( \psi : J_P \to T \) is a monotone map onto a locally connected continuum \( T \), then there is a monotone map \( \psi' : J_\sim \to T \) such that \( \psi = \psi' \circ m_P \)).
4. \( M_P \) semiconjugates \( P \) to a branched covering map \( g_P : \mathbb{C} \to \mathbb{C} \) under which \( J_\sim \) is fully invariant so that \( g_P|_{J_\sim} \) is conjugate to the topological polynomial \( f_\sim \).

**Remark 1.** Suppose that \( \Theta \in \mathcal{A}_d \) is associated to the polynomial \( P \); let us show that the lamination \( \sim_{\Theta} \) defined in Theorem 2.6 and the lamination \( \sim_P \) defined in Theorem 2.7 coincide. Indeed, by Theorem 2.7 there exists a monotone map \( \psi' : J_\sim_P \to J_{\sim_{\Theta}} \). If this map is not a homeomorphism, it will collapse a non-degenerate subcontinuum \( Q \subset J_{\sim_P} \) to a point \( x \in J_{\sim_{\Theta}} \). Since impressions map to points of \( J_{\sim_P} \), infinitely many distinct impressions of external rays are contained in the fiber \( m_{\Theta,P}^{-1}(x) \) which by Theorem 2.6 implies that the \( \sim_{\Theta} \)-class corresponding to \( x \) is infinite. This contradicts Theorem 2.6, which states that \( \sim_{\Theta} \)-classes are finite.

Theorem 2.7 establishes the semiconjugacy \( m_P \) on the entire complex plane, so that \( m_P \)-images of external rays to \( J_P \) are curves in \( \mathbb{C} \) accumulating on points of \( J_{\sim_P} \). For \( x \in J_{\sim_P} \), the set \( m_P^{-1}(x) \cap J_P \) is the union of impressions of angles \( \alpha \) such that \( m_P(R_\alpha) \) lands on \( x \). The order of \( x \) in \( J_{\sim_P} \) is the number of components of \( J_{\sim_P} \setminus \{x\} \) and can be either a finite number or infinity. By Theorem 2.7 if the order of \( x \) in \( J_{\sim_P} \) is finite then it equals the number of angles with impressions in \( m_P^{-1}(x) \) (or equivalently the number of angles whose impressions intersect \( m_P^{-1}(x) \)). If the order of \( x \) in \( J_{\sim_P} \) is infinite, then there are infinitely many angles with impressions in \( m_P^{-1}(x) \).

3. **Condensity.** We begin with a few lemmas concerning the dynamics of a condense topological polynomial. If \( J \) is a dendrite, by \( [a, b],_J \) we mean the unique arc in \( J \) connecting the points \( a, b \in J \). A continuum \( X \subset \mathbb{C} \) is called unshielded if it coincides with the boundary of the unique unbounded component of \( \mathbb{C} \setminus X \). Note that all connected Julia sets of polynomials and all topological Julia sets are unshielded continua. A point \( x \in X \) is called a cutpoint of \( X \) if \( X \setminus \{x\} \) is not connected. In what follows a lamination \( \sim \) such that \( f_\sim \) is condense is called condense; also, a critical portrait compatible with a condense lamination is said to be condense.

**Lemma 3.1.** If \( X \subset \mathbb{C} \) is an unshielded locally connected continuum and \( A \subset X \) is connected and dense in \( X \), then \( A \) is condense in \( X \) and contains all cutpoints of \( X \).

**Proof.** If \( Z \subset X \) is a closed set with \( X \setminus Z \) disconnected, then all components of \( X \setminus Z \) are open. Hence all such components intersect \( A \). Since \( A \) is connected, this implies that \( A \cap Z \neq \emptyset \). Suppose that \( A \) is not condense in \( X \). Then there exists an arc \( I \subset X \) disjoint from \( A \). Note that \( X \setminus I \) is open and connected (by virtue of containing \( A \)). Therefore \( X \setminus I \) is path connected and locally path connected. It follows that there exists a simple closed curve \( S \subset X \) which contains a non-degenerate subsegment \( I' \) with endpoint \( a', b' \) of \( I \). The curve \( S \) encloses a topological disk \( U \). Clearly, any two-point set \( \{a, b\} \subset S \) separates
Proof. Suppose \( J_{\sim} \) is non-degenerate and let \( x \in J_{\sim} \) be a point with condense orbit. If \( J_{\sim} \) is not a dendrite, then it contains a Jordan curve. By [5] it follows that \( J_{\sim} \) contains a periodic Jordan curve \( B \) of period, say, \( k \). Since \( x \) must enter \( B \), it follows that the union of \( \bigcup_{i=1}^k f_i(B) = J_{\sim} \). Since \( J_{\sim} \) is a topological Julia set, it is easy to see that then \( J_{\sim} \) is the unit circle and the lamination \( \sim \) is trivial. \hfill \square

**Lemma 3.3.** Suppose that \( K \subset J_{\sim} \) is a continuum with dense orbit and that \( f^n(K) \cap K \neq \emptyset \). If \( t \geq 0 \) is an integer, the union \( \bigcup_{j=0}^t f_j^n(K) \) is a condense connected subset of \( J \) containing all cutpoints of \( J_{\sim} \). Further, if \( f^n(K) \subset K \), then \( K = J_{\sim} \).

Observe that in this lemma we do not assume that \( f \) is condense.

**Proof.** Under the hypotheses, \( A_0 = \bigcup f_i^n(K) \) is a connected subset of \( J_{\sim} \), and so are the sets \( A_i = \bigcup f_i^n+1(K) \) where \( 1 \leq l \leq n-1 \). By the assumption, the union \( A = \bigcup_{l=0}^{n-1} A_l \) is dense in \( J \). Observe that \( f_{\sim}(A_l) \subset A_{l+1} \), where indices are interpreted modulo \( n \).

Since \( \bigcup_{l=0}^{n-1} A_l = J_{\sim} \) it follows from the Baire Category Theorem that some \( A_l \) contains an open subset of \( J_{\sim} \). Since \( f_{\sim} \) eventually maps any open set onto \( J_{\sim} \), it follows that \( f_{\sim}^r(A_l) = J_{\sim} \) for some \( r \geq 0 \). Hence, for all \( i \geq 0 \), \( f_{\sim}^{n+i}(A_l) = A_l \cup J_{\sim} \), and so for any \( t \) the set \( A_t \) is connected and dense in \( J_{\sim} \). Then Lemma 3.1 implies that \( A_t \) is condense and contains all cutpoints of \( J_{\sim} \).

In the case that \( f^n(K) \subset K \), it follows that \( A_0 \subset K \); that \( K \) is closed and \( A_0 \) is dense implies that \( K = J_{\sim} \). \hfill \square

The next lemma shows that condense maps resemble transitive maps. Recall that any topological polynomial on a dendrite must have fixed cutpoints (see, e.g., [24, 4]).

**Lemma 3.4.** For any topological polynomial \( f_{\sim} \), the following claims are equivalent.

1. \( f_{\sim} \) is condense.
2. The orbit of every continuum \( K \subset J_{\sim} \) is dense.
3. The orbit of every interval \( I \subset J_{\sim} \) is dense.
4. There are no proper periodic continua in \( J_{\sim} \).

Moreover, if these conditions are satisfied, then the set of all points with condense orbits is residual in every interval \( I \subset J_{\sim} \).

**Proof.** Since every subcontinuum of \( J_{\sim} \) contains an interval, it is clear that (3) and (2) are equivalent. If a point \( x \in J_{\sim} \) has condense orbit and \( K \subset J_{\sim} \) is a continuum, then \( x \) must enter \( K \), and the orbit of \( K \) is dense. This shows that (1) implies (2). Moreover, by Lemma 3.3, (1) implies (4).

Let us show that (2) and (4) are equivalent. Suppose that (2) holds and let \( K \) be a periodic continuum \( K \). Then \( K \) has to have a dense orbit which by Lemma 3.3 implies that \( K = J_{\sim} \). Suppose that (4) holds and let \( L \subset J_{\sim} \) be a continuum. By [5] there exist \( m \) and \( n > 0 \) such that \( f_i^{m+n}(L) \cap f_i^{m+n}(L) \neq \emptyset \). Then the set \( \bigcup_{i=0}^\infty f_i^{m+n}(L) = T \) is a
periodic continuum which by the assumption coincides with \( J_{\sim} \). Hence \( L \) has a dense orbit as desired.

Let us show that (2) implies (1). If \( J_{\sim} \) has a bounded complementary domain \( U \), then we may assume that \( \text{Bd}(U) \) is periodic. By Lemma 3.3 we conclude that \( \text{Bd}(U) = J_{\sim} \), so \( f_{\sim} \) is conjugate to \( z \mapsto z^d \) and condense. Therefore we may assume that \( J_{\sim} \) is a dendrite. Let \( \{ A_i \mid i \geq 0 \} \) be a countable collection of closed arcs such that any continuum \( K \subset J_{\sim} \) contains some \( A_i \). For convenience, we choose the sequence \( \{ A_i \} \) so that no element of the sequence contains an endpoint of \( J_{\sim} \).

Let \( I \subset J_{\sim} \) be an arc; we will show for each \( s \geq 0 \) that \( B_s = \{ x \in I \mid f^k_s(x) \in A_s \text{ for some } k \} \) is an open and dense subset of \( I \). Let \( \alpha \) denote a fixed cutpoint of \( J_{\sim} \). It follows that, for \( i \) sufficiently large, \( \alpha \in f^i(I) \). This is because no subcontinuum of \( J_{\sim} \) is wandering, i.e., there exists \( s, n \) such that \( f^s_\sim(I) \cap f^{s+n}_\sim(I) \neq \emptyset \). By Lemma 3.3, for some \( M \geq 0 \) we have \( \alpha \in f^{s+Mn}(I) \); since \( \alpha \) is fixed, \( \alpha \in f^i(I) \) for all \( i \geq s + Mn \).

There exist components \( K \) of \( J_{\sim} \setminus A_s \) such that every arc intersecting \( K \) and containing \( \alpha \) also contains a subinterval of \( A_s \). Since every continuum in \( J \) has a dense orbit, there exists \( k \geq 0 \) such that \( \alpha \in f^K_\sim(I) \) and \( f^K_\sim(I) \cap K \neq \emptyset \). Hence, \( f^K_\sim(I) \) intersects \( A_s \) in an open subset. Since \( f^K_\sim \) is finite-to-one, this implies that an open subset of \( I \) maps into \( A_s \).

By the Baire Category Theorem, \( \bigcap_{s \geq 0} B_s \) is then a residual (and hence non-empty) subset of \( I \); this is the set of points in \( I \) which eventually map into each \( A_s \), and therefore into every subcontinuum of \( J_{\sim} \) as desired.

Powers of condense maps are condense, too.

**Lemma 3.5.** If \( f_{\sim} \) is condense and \( s \geq 1 \), then \( f^s_{\sim} \) is condense.

**Proof.** By Lemma 3.4 we need to show that any continuum \( K \subset J_{\sim} \) has dense \( f^s_{\sim} \)-orbit in \( J_{\sim} \). By Lemma 3.2 we only need to consider the case that \( J_{\sim} \) is a dendrite. Let \( \alpha \in J_{\sim} \) be a fixed cutpoint. By Lemma 3.3 there exists \( i \geq 0 \) such that \( \alpha \in f^i_\sim(K) \); since \( \alpha \) is fixed we may assume that \( i = ks \) for some integer \( k \). Clearly, \( (f^K_\sim)^k(K) \cap (f^K_\sim)^k(K) \neq \emptyset \), since it contains \( \alpha \). By Lemma 3.3, \( \bigcup_{s=0}^{\infty} (f^K_\sim)^s(K) \) is a connected condense subset of \( J \), so the \( f^K_\sim \)-orbit of \( K \) is condense. Since \( K \) was an arbitrary continuum in \( J_{\sim} \), \( f^K_\sim \) is condense by Lemma 3.4.

**Theorem 3.6.** Let \( P \) be a polynomial with connected Julia set. Then the following claims hold.

1. Suppose that the finest model \( J_{\sim} \) of \( J_P \), given by a lamination \( \sim \), is non-degenerate, all points of \( J_{\sim} \) are of finite order, and \( f_{\sim} \) is condense. Then \( J_P \) is locally connected and \( P|_{J_P} \), is conjugate to \( f_{\sim} \).
2. Suppose that \( P|_{J_P} \) is condense. Then \( P \) has no proper periodic subcontinua (in particular, \( P \) is non-renormalizable). \( J_P \) is locally connected and \( P \) is conjugate to \( g_P \) from Theorem 2.7.

Observe, that by this theorem \( P|_{J_P} \) satisfies Lemmas 3.2 - 3.5. Observe also, that by Theorem 2.6 (1) holds for polynomials associated with condense critical portraits having aperiodic kneading.

**Proof.** (1) Let \( m : J_P \to J_{\sim} \) be the finest monotone map to a locally connected continuum defined in Theorem 2.7. Since the order of any periodic point \( p \in J_{\sim} \) is finite, by [3, Lemma 37] the set \( m^{-1}(p) \) is a repelling or parabolic periodic point. Hence, \( P \) has no Cremer points: if \( U \) were a periodic Siegel domain of \( P \), then \( m(\text{Bd}(U)) \) would be a
periodic subcontinuum of $J_\omega$, homeomorphic to a circle on which the appropriate power of the map is an irrational rotation, and hence a proper subcontinuum.

Now we show that $P$ is non-renormalizable. Indeed, if $P$ is renormalizable, then there exists a polynomial-like connected Julia set $J' \subsetneq J_P$ which is a periodic continuum. If $m(J')$ is a point, then it is periodic and again by [3, Lemma 37] the set $m^{-1}(m(J'))$ is a point, a contradiction. Hence $m(J')$ is a periodic continuum in $J_\omega$. Clearly, $m(J') \neq J_\omega$. This contradicts Lemma 3.3 and Lemma 3.4 and shows that $P$ is non-renormalizable. Hence $J_P$ is locally connected [16]. By Theorem 2.7, $P|_{J_P}$ and $f_\omega$ are conjugate as required.

(2) Assume now that $P|_{J_P}$ is condense. Let us show that $J_P$ has no proper periodic subcontinua. Indeed, let $A \subset J_P$ be a periodic continuum. Then the (finite) union $B$ of its images must coincide with $J_P$ (because $P|_{J_P}$ is condense). As at least one of these images must have non-empty interior, $A$ must coincide with $J_P$.

This fact has several consequences. To begin with, let us show that $J_P$ cannot have Cremer points. Indeed, suppose that $z_0 \in J_P$ is a periodic Cremer point of period $p$. Then, for any small neighborhood $U$ of $z_0$, the component of the set $\{z \mid P^k(z) \in \mathcal{T} \text{ for all } k\}$ containing $z_0$, called a hedgehog, is a proper periodic subcontinuum of $J_P$ [20], contradicting that $P|_{J_P}$ has no proper periodic subcontinuum.

Now let us show that $P$ cannot have Siegel domains either. Since $J_P$ contains no proper periodic subcontinua, then any periodic Siegel domain $U$ of $P$ must be such that $\text{Bd}(U) = J_P$. By J. Rogers’ result [22], there are two cases. In the first case, $P|_{\text{Bd}(U)}$ is monotonically semiconjugate to an irrational rotation which contradicts the fact that $\text{Bd}(U) = J_P$. In the second case, $\text{Bd}(U)$ is an indecomposable continuum (i.e., cannot be represented as $A \cup B$ where $A$ and $B$ are proper subcontinua of $\text{Bd}(U)$). Then, given a point $x \in \text{Bd}(U)$, one can define the composant of $x$ in $\text{Bd}(U)$, that is the union of all proper subcontinua of $\text{Bd}(U)$ containing $x$. Then it is known [19, Theorem 11.15] that distinct non-degenerate composants of $\text{Bd}(U)$ are pairwise disjoint and there are uncountably many of them. Since the orbit of $x$ can only enter countably many composants of $\text{Bd}(U)$, we have a contradiction with the assumption that $P|_{J_P}$ is condense. Hence, $P$ does not have Siegel domains.

Since $J_P$ has no proper periodic subcontinua, $P$ is non-renormalizable. Thus, as before, all this implies that $J_P$ is locally connected [16]. The rest follows from Theorem 2.7. □

4. Family of critical WT-portraits. First we show that condense laminations are residual in $\mathcal{A}_3$.

**Theorem 4.1.** Let $\Theta \in \mathcal{A}_d$ be a critical portrait which consists of $d - 1$ critical chords whose orbits are dense in $\mathcal{S}$. Then $\Theta$ has aperiodic kneading, is condense, and any polynomial $P$ associated to $\Theta$ has locally connected Julia set $J_P$ so that $P|_{J_P}$ is conjugate to $f_\sim \mid_{J_\sim}$.

**Proof.** Let us show that $\Theta$ has aperiodic kneading. Indeed, the orbit of any critical leaf $\ell$ comes arbitrarily close to the fixed point 0. Hence, if $C$ is the $\Theta$-unlinked class of 0, then the itinerary of $\ell$ includes arbitrarily long segments consisting of $C$. This implies that $\Theta$ has aperiodic kneading, and Theorem 2.6 applies. Let $\sim$ denote the lamination generated by $\Theta$.

Let us show that $\Theta$ is condense. Take an arc $I \subset J_\omega$ and consider its orbit. By [5] there are positive numbers $m, k$ with $f^m(I) \cap f^{m+k}(I) \neq 0$. Consider the connected set $A_0 = \bigcup_{i=0}^{\infty} f^{-i}(I)$. Clearly, $A_0 = B \subset J_\omega$ is a subdendrite of $J_\omega$ and $f^k(B) \subset B$.

Let us show that $f^k|_B$ has a critical point $c$. Indeed, by Theorem 7.2.6 of [4] there are infinitely many periodic cutpoints of $f^k|_B$; let $Q \subset B$ be an arc joining some pair $x$ and
of such periodic cutpoints. If $f^k|_S$ has no critical points, then some power of $f^k|_Q$ is a homeomorphism and there must exist a point $z \in Q$ attracting for $g$ from at least one side, which is impossible. Hence, $B$ contains a critical point of $f^k$. By the assumptions on $\Theta$, $B$ contains a point with dense orbit, so $I$ has a dense orbit. Since $I$ was arbitrary, we conclude by Lemma 3.4 that $\Theta$ is condense.

Since $\Theta$ satisfies Theorem 2.6, and since $\sim = \sim_P$ by Remark 1, it follows from Theorem 3.6 (1) that $J_{\sim}$ is locally connected and that $P|_{J_{\sim}}$ is conjugate to $f_{\sim}$. □

Since the set of critical portraits consisting of $d \geq 1$ critical leaves with dense orbits in $S$ is residual in $A_d$, we obtain the following corollary.

**Corollary 1.** A residual subset of critical portraits in $A_d$ correspond to polynomials whose restrictions to their Julia sets are condense.

Recall that a lamination with wandering $k$-gons ($k \geq 3$) is called a WT-lamination. A critical portrait, compatible with a WT-lamination, is called a WT-critical portrait; $WT_3$ is the set of all cubic WT-critical portraits. By Theorem 1.3, $WT_3$ is a dense and locally uncountable subset of $A_3$.

Now we show that $WT_3$ is a meager subset of $A_3$. We will do so by showing that the set of critical portraits in $WT_3$ compatible with a wandering triangle of area at least $\frac{1}{n}$ is disjoint from a particular dense subset of critical portraits. The dense subset we consider, called $K$, is the set of critical portraits consisting of two leaves $\{c, d\} \in A_3$ such that the orbits of $c$ and $d$ are dense, neither $c$ nor $d$ maps to an endpoint of the other, and $c$ and $d$ eventually map to the same point.

**Lemma 4.2.** The set $K$ is dense in $A_3$. All orbit portraits $\Theta \in K$ have aperiodic kneading. The critical classes of the lamination $\sim_\Theta$ generated by $\Theta$ are leaves.

**Proof.** The fact that $K$ is dense in $A_3$ is easy and left to the reader. Consider some $\Theta = \{c, d\} \in K$. By Theorem 4.1, $\Theta$ has aperiodic kneading. Let $g$ be the critical $\sim_\Theta$-class containing $c$, and $h$ the critical class containing $d$. It is easy to see that if $g$ contains at least three points, then $|\sigma(g)| \geq 2$. Indeed, consider two cases. If $g$ maps to its image in the two-to-one fashion, then $|\sigma(g)| \geq 2$ is obvious. If $g$ maps to its image in the three-to-one fashion then $g = h$ contains four endpoints of the leaves $c$ and $d$, so again $|\sigma(g)| \geq 2$. Similarly, if $|h| \geq 3$ then $|\sigma(h)| \geq 2$.

Suppose for contradiction that $g$ contains at least three points. We will first show that then all forward images of all critical classes of $\sim_\Theta$ are non-degenerate. Indeed, note that neither $g$ nor $h$ may eventually map onto itself, since the orbits of $c$ and $d$ are dense in $S$. This further implies that, if $g$ maps onto $h$, then $h$ cannot map onto $g$. We consider three cases.

1. Suppose that $g = h$. Since $|\sigma(g)| \geq 2$ and $g$ is not periodic, it is not precritical, so $|\sigma^l(g)| = |\sigma^l(h)| \geq 2$ for all $l \geq 0$.
2. Suppose that $\sigma^k(g) = h$ for some $k \geq 1$. Since $|\sigma(g)| \geq 2$ and $c$ never maps into $d$, we see that $h$ contains at least three points (the endpoints of $d$ and the point $\sigma^k(c)$). Therefore by the above $|\sigma(h)| \geq 2$. As noted before, $h$ is not precritical, so $|\sigma^k(g)|$ and $|\sigma^k(h)|$ are both at least two for all $k$.
3. If $g$ never maps onto $h$, then $|\sigma^k(g)| \geq 2$ for all $k$, since $g$ is not precritical and contains at least three points. Since $c$ and $d$ have a common image, so do $g$ and $h$, and $|\sigma^k(h)| \geq 2$ for all $k$.

We will use the metric where the distance between two points on $S$ is the length of the shortest arc in $S$ joining them. By the diameter of a chord we will mean the distance
between its endpoints. Let us show that \( \text{diam}(\sigma^k(g)) \) is bounded away from 0. It is easy to see that, for any chord \( \ell' \),

\[
\text{diam}(\sigma(\ell')) = \begin{cases} 
3 \text{diam}(\ell') & \text{if diam}(\ell') \leq 1/6 \\
3|\text{diam}(\ell') - 1/3| & \text{if } 1/6 \leq \text{diam}(\ell').
\end{cases}
\] (4.1)

This implies that \( \text{diam}(\sigma(\ell')) \geq \text{diam}(\ell') \) if and only if \( \text{diam}(\ell') \leq 1/4 \). Hence, every class of diameter less than 1/4 maps to a class of larger diameter. Let \( \ell \) be the chord on \( \text{Bd}(\text{Ch}(g)) \cup \text{Bd}(\text{Ch}(h)) \) of length closest to 1/3; since \( \sigma(g) \) and \( \sigma(h) \) are non-degenerate, \( \varepsilon = |\text{diam}(\ell) - 1/3| \) is positive. Since \( \sim \)-classes are unlinked, \( |\text{diam}(\ell') - 1/3| \geq \varepsilon \) for any chord \( \ell' \) from the boundary of the convex hull of a \( \sim \)-class. Hence, by Equation 4.1, no class of diameter at least 1/4 has an image of diameter less than \( 3\varepsilon \). In particular, \( \text{diam}(\sigma^k(g)) \geq 3\varepsilon \) for all \( k \).

Since the convex hulls of classes are dense in \( \mathbb{D} \), we can choose a class \( k \) such that there exists an element of \( S \setminus k \) of diameter less than \( \varepsilon \). Since \( \text{diam}(\sigma^k(g)) \geq 3\varepsilon \), the orbit of \( g \) can never enter \( A \). This contradicts the orbit of \( c \) is dense. We conclude that the classes \( g \) and \( h \) are leaves.

**Theorem 4.3.** The set \( \mathcal{W}_3 \) is of first category in \( \mathcal{A}_3 \).

**Proof.** Let \( \mathcal{W}_n \) be the set of critical portraits \( \Theta \in \mathcal{W}_3 \) such that there is a \( \sim \)-class \( T \) which is a wandering triangle and \( \text{Ch}(T) \) has area at least 1/n. We will show that \( \mathcal{W}_n \) is nowhere dense by showing that \( \mathcal{W}_n \cap \mathcal{K} = \emptyset \).

By Theorem 2.1, \( \mathcal{W}_n \) is disjoint from \( \mathcal{K} \) for every \( n \). Suppose that there is a sequence \((\Theta_i)_{i=1}^\infty\) of elements of \( \mathcal{W}_n \) which converges to a critical portrait \( \Theta = \{c, d\} \in \mathcal{K} \). By way of contradiction we will prove that \( \Theta \in \mathcal{K} \) is impossible. For each \( i \) set \( \sim_{\Theta_i} \), and let \( T_i \) be a wandering triangle in \( \sim_{\Theta_i} \) such that \( \text{Ch}(T_i) \) has area at least 1/n. We may assume that \( (T_i)_{i=1}^\infty \) converges to a triangle \( T = \{a, b, c\} \), with area of \( \text{Ch}(T) \) at least 1/n.

Let us prove that \( T \) is contained in some \( \sim_{\Theta_i} \)-class \( T' \). For any fixed \( k \) and for every \( i \), \( \sigma^k(T_i) \) is contained in a \( \sim_{\Theta_i} \)-unlinked class. Hence, we have the following.

**Claim A.** For every \( k \geq 0 \) we have that \( \lim_{i \to \infty} \sigma^k(T_i) = \sigma^k(T) \) is a subset of the closure of a \( \sim_{\Theta_i} \)-unlinked class.

Let us show that every pair of vertices of \( T \) are \( \sim_{\Theta_i} \)-equivalent. The crucial observation here is that the orbits of the vertices of \( T \) each cannot intersect \( \bigcup \Theta \) more than once (\( \bigcup \Theta \) denotes the collection of the endpoints of critical leaves in \( \Theta \)). Indeed, if an orbit intersects \( \bigcup \Theta \) twice, then an endpoint of one critical leaf eventually maps to an endpoint of either the same critical leaf or the other critical leaf, contradicting that \( \Theta \in \mathcal{K} \).

By Lemma 4.2, \( c \) and \( d \) are distinct classes of \( \sim_{\Theta_i} \). Let us show that there exists \( m \geq 1 \) such that \( \sigma^m(c) \) and \( \sigma^m(d) \) belong to distinct \( \Theta \)-unlinked classes. Indeed, otherwise \( \sigma_i(c) \) and \( \sigma_i(d) \) are distinct points of \( S \) (they are distinct because \( c \) and \( d \) are disjoint) which belong to the same \( \sim_{\Theta_i} \)-class; this contradicts the conclusion of Lemma 4.2 that the \( \sim_{\Theta_i} \)-class of \( \sigma^3(c) \) is a singleton.

We can now show that \( a \sim_{\Theta_i} b \) for any vertices \( a, b \in T \). Indeed, suppose first that for some \( l \geq 0 \) the points \( \sigma^l(a) \) and \( \sigma^l(b) \) are contained in the same critical class. By the previous observation that each orbit of a vertex of \( T \) can intersect \( \bigcup \Theta \) at most once, for every \( k \neq l \) we have that \( \sigma^k(a) \) and \( \sigma^k(b) \) are not in \( \bigcup \Theta \). We can therefore, by Claim A, choose two opposite (positive and negative) sides of \( a \) and \( b \) so that the corresponding one-sided itineraries of \( a \) and \( b \) coincide, so \( a \sim_{\Theta_i} b \).

On the other hand, suppose \( a \) and \( b \) never simultaneously map to the endpoints of a critical leaf. We now note that, for any \( l \geq 0 \), the points \( \sigma^l(a) \) and \( \sigma^l(b) \) cannot belong to different critical classes, or else \( \sigma^{l+m}(a) \) and \( \sigma^{l+m}(b) \) would belong to distinct \( \Theta \)-unlinked
classes, a contradiction with Claim A. This again implies that we can choose two distinct sides (positive and negative) such that the corresponding one-sided itineraries of \(a\) and \(b\) coincide. Thus, in any case \(a \sim_\Theta b\) and so \(T\) is contained in a \(\sim_\Theta\)-class \(T'\).

By Theorem 2.6, \(T'\) is finite. Since \(\Theta \in \mathcal{K}\), \(T'\) is not wandering by Theorem 2.1, and \(T'\) is not precritical by Lemma 4.2. Hence, \(T'\) is preperiodic, and either \(T\) itself is preperiodic or its future images cross each other inside \(D\). As the latter is impossible by continuity, we may assume that there exist powers \(s\) and \(t > 0\) such that \(\sigma_3^s(T) = \sigma_3^{s+t}(T)\). Again by continuity \(\sigma_3^s(T_i)\) and \(\sigma_3^{s+t}(T_i)\) approach \(\sigma_3^s(T)\) in the Hausdorff metric while the area of \(\text{Cl}(T)\) is at least \(1/n\). For geometric reasons this contradicts that \(\sigma_3^s(T_i)\) and \(\sigma_3^{s+t}(T_i)\) are disjoint for all \(i\). Therefore, \(\Theta \notin \mathcal{K}\).

We have established that \(W_n\) is nowhere dense in \(A_3\), so \(\bigcup_{n=1}^{\infty} W_n = W^T_3\) is a first category subset of \(A_3\).

\[\square\]

REFERENCES

[1] B. Bielefeld, Y. Fisher, J. H. Hubbard, The classification of critically preperiodic polynomials as dynamical systems, Journal AMS, 5 (1992), 721–762.

[2] A. Blokh, C. Curry, L. Oversteegen, Cubic Critical Portraits and Polynomials with Wandering Gaps, preprint arXiv:1003.4467.

[3] A. Blokh, C. Curry, L. Oversteegen, Locally connected models for Julia sets, Advances in Mathematics, 226 (2011), 1621–1661.

[4] A. Blokh, R. Fokkink, J. Mayer, L. Oversteegen, E. Tymchatyn, Fixed point theorems for plane continua with applications, preprint arXiv:1004.0214.

[5] A. Blokh, G. Levin, An inequality for laminations, Julia sets and “growing trees”, Erg. Th. and Dyn. Sys., 22 (2002), 63–97.

[6] A. Blokh and L. Oversteegen, Monotone images of Cremer Julia sets, Houston J. Math. 36 (2010), 469–476.

[7] A. Blokh, L. Oversteegen, Wandering gaps for weakly hyperbolic cubic polynomials, in: “Complex dynamics: Families and Friends” (ed. by D. Schleicher), A K Peters (2008), 139–168.

[8] A. Douady, Descriptions of compact sets in \(C\), in: “Topological methods in modern mathematics” (eds. L. R. Goldberg and A. V. Phillips), Publish or Perish (1993), 429–465.

[9] A. Douady, J. H. Hubbard, Étude dynamique des polynômes complexes I, Publications Mathématiques d’Orsay 84-02 (1984).

[10] A. Douady, J. H. Hubbard, Étude dynamique des polynômes complexes II, Publications Mathématiques d’Orsay 85-04 (1985).

[11] Y. Fisher, “The classification of critically preperiodic polynomials” Ph.D. thesis, Cornell University, 1989.

[12] L. Goldberg, J. Milnor, Fixed points of polynomial maps II: Fixed point portraits, Ann. Scient. Ecole Norm. Sup., 4e série, 26 (1993), 51–98.

[13] J. Kiwi, Wandering orbit portraits, Trans. Amer. Math. Soc. 354 (2002), 1473–1485.

[14] J. Kiwi, Real laminations and the topological dynamics of complex polynomials, Advances in Math. 184 (2004), 207–267.

[15] J. Kiwi, Combinatorial continuity, Proc. London Math. Soc. 91 (2005), 215–248.

[16] O. Kozlovski, S. van Strien, Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials, Proc. Lond. Math. Soc. (3) 99 (2009), 275–296.

[17] G. Levin, On backward stability of holomorphic dynamical systems, Fundamenta Mathematicae 158 (1998), 97–107.

[18] J. Milnor, “Dynamics in One Complex Variable”, 3rd edition, Princeton University Press, Princeton, 2006.

[19] S. B. Nadler, Jr., “Continuum theory”, Marcel Dekker Inc., New York, 1992.

[20] R. Perez-Marco, Fixed points and circle maps, Acta Math. 179 (1997), 243–294.

[21] A. Poirier, Critical portraits for postcritically finite polynomials, Fund. Math. 203 (2009), 107-163.

[22] J. Rogers, Singularities in the boundaries of local Siegel disks, Erg. Th. and Dyn. Syst., 12 (1992), 803–821.

[23] P. Roesch, Y. Yin, The boundary of bounded polynomial Fatou components, C. R. Math. Acad. Sci. Paris, 346 (2008), 877–880.

[24] W. Thurston, The combinatorics of iterated rational maps, in: “Complex dynamics: Families and Friends” (ed. by D. Schleicher), A K Peters (2008), 1–108.

Received xxxx 20xx; revised xxxx 20xx.
E-mail address: ablokh@math.uab.edu
E-mail address: ccurry@huntingdon.edu
E-mail address: overstee@math.uab.edu