Off-Diagonal Geometric Phases

Nicola Manini\textsuperscript{a,†} and F. Pistolesi\textsuperscript{b,*}

\textsuperscript{a}European Synchrotron Radiation Facility, B.P. 220, F-38043 Grenoble Cedex, France
\textsuperscript{b}Institut Laue Langevin, B.P. 156, F-38042 Grenoble Cedex 9, France

(September 7, 2018)

We investigate the adiabatic evolution of a set of non-degenerate normalized eigenstates of a parameterized Hamiltonian. Their relative phase change can be related to geometric measurable quantities that extend the familiar concept of Berry phase to the evolution of more than one state. We present several physical systems where these concepts can be applied, including an experiment on microwave cavities for which off-diagonal phases can be determined from published data.

Consider the adiabatic evolution of a set of nondegenerate normalized eigenstates \(|\psi_j(s)\rangle\) of a parameterized Hamiltonian \(H(s)\). The idea that, with a suitable definition, the phase of the scalar product \(\langle \psi_j(s_1)|\psi_j(s_2)\rangle\) contains a geometric, measurable contribution dates back to Pancharatnam’s pioneering work \([4]\). In particular, when \(s_1 = s_2\) and the state \(|\psi_j(s)\rangle\) is transported adiabatically along a closed loop, the existence of a nontrivial phase factor was discovered and put on a firm basis by Berry \([2]\) Since then, considerable work has been devoted to interpretation \([3]\), generalization \([4][5]\), and experimental determination \([6][7]\) of these geometric phase factors. Surprisingly, for \(s_1 \neq s_2\), the phase relation of \(\langle \psi_j(s_1)|\psi_k(s_2)\rangle\) between two different eigenstates has not been equally well investigated so far \([8]\).

This is even more surprising if one considers that, for some pair of points \(s_1\) and \(s_2\), it may occur that \(|\psi_k(s_2)\rangle = e^{i\alpha}|\psi_j(s_1)\rangle\) \((k \neq j)\). This implies that both scalar products \(\langle \psi_j(s_1)|\psi_j(s_2)\rangle\) and \(\langle \psi_k(s_1)|\psi_k(s_2)\rangle\) vanish, and, as well known, the usual Pancharatnam-Berry phase on any path connecting \(s_1\) to \(s_2\) is undefined for the states \(k\) and \(j\). The only phase information left is thus contained in the cross scalar products \(\langle \psi_j(s_1)|\psi_k(s_2)\rangle\).

In this Letter we determine the measurable and geometric phase factors associated to the off-diagonal matrix elements \(\langle \psi_j(s_1)|\psi_k(s_2)\rangle\) of the operator describing the evolution along a general open path in the parameter space that connect \(s_1\) to \(s_2\). We find a set of independent off-diagonal phase factors that exhaust the geometrical phase information carried by the basis of eigenstates along the path. Analogously to the familiar Berry phase, the values of these phases depend on the presence of degeneracies of the energy levels in the parameters space. The formalism is then applied to an experiment on quantum billiards \([7]\), where the off-diagonal phase factors can be extracted directly from published experimental data.

In order to introduce the off-diagonal geometric phases, it is convenient to consider the usual definition of the geometric phase of one normalized state \(|\psi_j(s)\rangle\) in terms of parallel transport \([4][6][8]\). Given any path \(\Gamma\) that joins \(s_1\) to \(s_2\), the state parallel-transported along it is defined by:

\[|\psi_j(s_2)\rangle = \exp \left\{-\int_{\Gamma} ds \cdot \langle \nabla_s \psi_j(s) | \psi_j(s) \rangle\right\} |\psi_j(s_2)\rangle. \tag{1}\]

This fixes the phase of the state along the path in the unique way satisfying \(\langle \psi_j(s_1)|\psi_j(s + \delta)\rangle = 1 + O(\delta^2)\) for \(\delta \to 0\), i.e. having maximal projection on the “previous” state. The geometric phase factor is then defined simply in terms of the scalar product along the parallel evolution:

\[\gamma_{jk}^\Gamma = \Phi(U_{jk}^\Gamma) = \Phi\left(\langle \psi_j(s_1)|\psi_j(s_2)\rangle\right), \tag{2}\]

where \(\Phi(z) = z/|z|\) for complex \(z \neq 0\). \(\gamma_{jk}^\Gamma\) is univocally determined by the sequence \(\Gamma_j\) of states \(|\psi_j(s)\rangle\), with \(s\) varying along \(\Gamma_j\). Indeed, \(\gamma_{jk}^\Gamma\) is unchanged by a local “gauge” transformation:

\[|\psi_j(s)\rangle \to |\psi_j(s)\rangle \exp[i\varphi_j(s)]\tag{3}\]

and by any reparametrization of the sequence of states \(\Gamma_j\). It is thus a geometric, measurable quantity.

In a similar way, we define \([2]\) the phase factors associated to the off-diagonal elements of the parallel-evolution operator \(U_{jk}^\Gamma\):

\[\sigma_{jk}^\Gamma = \Phi\left(U_{jk}^\Gamma\right) = \Phi\left(\langle \psi_j(s_1)|\psi_j(s_2)\rangle\right). \tag{4}\]

Like \(\gamma_{jk}^\Gamma\), the phase factor \(\sigma_{jk}^\Gamma\) is independent of the path parametrization. However, \(\sigma_{jk}^\Gamma\) depends on the relative phase of the two vectors \(|\psi_j\rangle\) and \(|\psi_k\rangle\) at \(s_1\). Indeed, under the gauge transformation \([3]\), \(\sigma_{jk}^\Gamma\) transforms as follows:

\[\sigma_{jk}^\Gamma \to \sigma_{jk}^\Gamma \exp[i(\varphi_j(s_1) - \varphi_j(s_1))]. \tag{5}\]

This shows that \(\sigma_{jk}^\Gamma\) is arbitrary, thus non-measurable. In order to define a gauge-invariant quantity, we combine two \(\sigma\)’s in the following product:

\[\gamma_{jk}^\Gamma = \sigma_{jk}^\Gamma \sigma_{kj}^\Gamma. \tag{6}\]

This new phase factor \(\gamma_{jk}^\Gamma\) is determined uniquely by the trajectories \(\Gamma_j\) and \(\Gamma_k\) of \(|\psi_j\rangle\) and \(|\psi_k\rangle\) in the Hilbert space. The finding of the measurable geometric quantity \(\gamma_{jk}^\Gamma\) is the central result of this Letter.
A simple geometric interpretation for $\gamma_{jk}^\Gamma$ can be obtained in analogy with that for the Pancharatnam phase. Consider the path of state $j$ in the space of rays (where two states differing only for a complex factor are identified). If $|\psi_j(s_1)\rangle$ is not orthogonal to $|\psi_j(s_2)\rangle$, there exists a unique geodesic path $G_{jj}$ going from $|\psi_j(s_2)\rangle$ to $|\psi_j(s_1)\rangle$, along which the geometric phase factor is unity. Then, trivially, the open-path geometric factor $\gamma_{jk}^\Gamma$ equals the phase factor on the circuit composed by $\Gamma_j$ and $G_{jj}$ (see Fig. 1). Once reduced to a closed path, using Stokes’ theorem, one can write $\gamma_{jk}^\Gamma$ in terms of the integral of Berry’s local-gauge-invariant 2-form on any surface $S_j$ bounded by $\Gamma_j + G_{jj}$.

Consider now two states $j$ and $k$ evolving along $\Gamma_j$ and $\Gamma_k$ in the space of rays. We generate all possible oriented loops by connecting the extremal points with geodesics. As Fig. 1 shows, only the three loops $\Gamma_j + G_{jj}$, $\Gamma_k + G_{kk}$, and $G_{kj}$ (dashed) lead back from the evolved states $|\psi_j(s_2)\rangle$ to the initial ones $|\psi_j(s_1)\rangle$ and $|\psi_k(s_1)\rangle$. Integration of Berry’s 2-form over the shaded surface $S_{jk}$ yields the off-diagonal phase $\gamma_{jk}^\Gamma$.

Interference experiments have measured the non-cyclic Pancharatnam-Berry phases $\gamma_j$ in the spin-$\frac{1}{2}$ system. In a similar way, one can envisage a spin-rotation experiment to measure by interference $\sigma_{12}$ and $\sigma_{21}$ for an arbitrary fixed gauge at the starting point. The dependence on the gauge chosen cancels out in the product $\gamma_{12}$, which, for this simple system, must equal $-1$ for any rotation angle $\theta \neq 2\pi$. Essentially any experiment sensitive to open-path diagonal geometric phases can be generalized to observe off-diagonal phases. In systems of larger dimensionality, several off-diagonal phase factors can be defined, and they may assume different values on different paths.

The definition of the off-diagonal phase factors $\gamma_{jk}^\Gamma$ can be generalized to the simultaneous evolution of more than two orthonormal states. Consider for example $n$ orthonormal eigenstates $|\psi_j(s)\rangle$ (ordered by increasing energy) of a parameterized Hermitian Hamiltonian matrix $H(s)$, representing a physical system. Observing the effect of a gauge change on the $\sigma_{jk}^l$ phase factors, we note that any cyclic product of $\sigma_l$s is gauge-invariant. It is then natural to generalize Eq. (8) by defining

$$\gamma_{j_1j_2j_3\ldots j_l}^{(l)} = \sigma_{j_1j_2}^{G_{12}} \sigma_{j_2j_3}^{G_{23}} \cdots \sigma_{j_{l-1}j_l}^{G_{l-1,l}} \sigma_{j_lj_1}^{G_{l1}} \cdots$$

(8)

For $l = 1$, Eq. (8) reduces to the familiar definition of the Pancharatnam-Berry diagonal phase factor $\gamma_{jk} = \gamma_{j}^{(1)} = \gamma_{jk}^{G_{jk}}$. The 2-indexes $\gamma_{jk}^{(2)}$ phase factors coincide with those introduced by Eq. (8). Larger $l$ describe more complex phase relations among off-diagonal components of the eigenstates at the endpoints of $\Gamma$. The same geometrical construction of a closed path done for $\gamma_{jk}^{(2)}$ extends to $\gamma_{jk}^{(l)}$ with $l > 2$.

We note that any cyclic permutation of all the indexes $j_1j_2j_3\ldots j_l$ is immaterial. Moreover, if one index is repeated, the associated $\gamma_{j_k}^{(l)}$ can be decomposed into the product $\gamma_{j_k}^{(l_1)}\gamma_{j_k}^{(l_2)}$’s with $l_1 + l_2 = l$. We can thus reduce to consider the $\gamma_{j_k}^{(l)}$’s with no repeated indexes, which means in particular $l \leq n$.
One can readily verify that the number of $\gamma^{(i)}$'s left grows with $n$ faster than $n^2$. Since $n^2$ is the number of the constituent $\sigma_{jk}$'s, not all the $\gamma^{(i)}$'s can be independent. We shall now find a complete set of independent $\gamma^{(i)}$'s, under the condition that $U^\Gamma_{jk} \neq 0$ for all $j$ and $k$. Clearly, the $n$ Pancharatnam-Berry diagonal phase factors $\gamma_{(i)}^{(1)}$ are all independent, since any diagonal $\sigma_{jj}$ enters only $\gamma_{(i)}^{(1)}$. On the other hand, the off-diagonal $\gamma_{(i)}^{(l)}$'s are interrelated by the following exact equalities [they can be verified substituting explicitly the definition (3)]:

$$\gamma_{(j)k)(m)}^{(l)} = \gamma_{(j)k}^{(l')}(^{(l''})\gamma_{km}^{(2)} (l \geq 4) \quad (9)$$

$$\gamma_{jkm}^{(3)} = \gamma_{km}^{(2)}\gamma_{jm}^{(2)} \quad (10)$$

$$\gamma_{ijm}^{(3)}\gamma_{km}^{(2)} = \gamma_{ijk}\gamma_{km}^{(2)} \quad \gamma_{ijm}^{(2)}\gamma_{km}^{(3)} = \gamma_{ijkm}^{(3)} \quad (11)$$

In Eq. (3), $(ij)$ indicates a set of one or more indexes, and $l', l'' \ (l < l''$) count the indexes in the corresponding $\gamma$. Combining relations (3), any $\gamma^{(l)}$'s may be expressed in terms of three categories: the diagonal phases $\gamma^{(1)}$, the $n(n-1)/2$ quadratic $\gamma_{j<k}^{(2)}$, and the $(n-1)(n-2)/2$ cubic $\gamma_{j<k}^{(3)}$. These $n^2 - n + 1$ factors are indeed functionally independent combinations of the $\sigma$'s: we verified that the Jacobian determinant $|\partial\gamma^{(j)}/\partial\sigma_{km}|$ is nonzero. The number of independent phases can be easily understood: it amounts to the $n^2$ phases of $U^\Gamma_{jk}$ minus the arbitrary $n - 1$ relative phases among the $n$ eigenstates at a given point s.

We restrict now to the particular case of a path joining a pair of points $s_1^P$ $s_2^P$ such that the $n$ eigenstates at the final point are a permutation $P$ of the initial eigenstates, i.e.

$$\begin{align*}
H(s_1^P) &= \sum_j E_j \langle \psi_j | H(s_2^P) = \sum_j E_j \langle \psi_{P_j} | \psi_j \rangle,
\end{align*}$$

where $E_j$ and $E_{P_j}$ are in increasing order as usual. The only well-defined $\sigma^\Gamma$'s are the $n$ phase factors $\sigma^\Gamma_{j,P_j}$. When the permutation is nontrivial ($P_j \neq j$) the familiar Berry-Pancharatnam phase factor associated to state $j$ is undefined. For this special case the only well-defined geometric phases are the off-diagonals. One can classify them according to standard group theory. Any permutation $P$ can be decomposed univocally into $c$ cycles of lengths $l_1, l_2, \ldots, l_c$. To each cycle $i$, it is possible to associate one $\gamma^{(i)}_{(j)l_i}^{(l_i)}$ $\gamma_{(j)l_i}^{(l_i)}$ follows the corresponding cycle. These phase factors involve only nonzero $U^\Gamma_{jk}$ and are thus well defined. In contrast, all other $\gamma^{(i)}_{(j)l_i}$ are undefined. In Table I, for each permutation $P$ of the eigenstates we report the corresponding well-defined $\gamma^{(i)}$ for $n \leq 4$.

For these paths permuting the eigenvectors, the determinant $|U^\Gamma|$ of the overlap matrix is related to the product of the $\sigma$'s. The equality $|U^\Gamma| = 1$ becomes therefore

$$\prod_{j=1}^n \sigma^\Gamma_{j,P_j} = (-1)^P. \quad (13)$$

The third column of Table I summarizes this condition in terms of the $\gamma^{(i)}$'s. In the special case of a real symmetric Hamiltonian $H(s)$, all $\sigma$'s, and thus all $\gamma^{(i)}$'s either equal +1 or −1. For this simple but relevant situation, the last column of Table I reports the number of combinations of values that the $\gamma^{(i)}$'s may take, as allowed by the condition (13).

| n | P | geometric phase factors | $|U^\Gamma|$ | condition | # of cases |
|---|---|-------------------------|-------------|-----------|-----------|
| 1 | 1 | $\gamma_1$ | $\gamma_1 = 1$ | 1 |
| 2 | 1 1 | $\gamma_1 \gamma_2$ | $\gamma_1 \gamma_2 = 1$ | 2 |
| 2 | 1 2 | $\gamma_1 \gamma_2$ | $\gamma_1 \gamma_2 = 1$ | 1 |
| 3 | 1 2 3 $\gamma_1 \gamma_2 \gamma_3$ | $\gamma_1 \gamma_2 \gamma_3 = 1$ | 4 |
| 3 | 2 1 3 | $\gamma_1 \gamma_2 \gamma_3$ | $\gamma_1 \gamma_2 \gamma_3 = 1$ | 2 |
| 3 | 2 3 1 | $\gamma_1 \gamma_2 \gamma_3$ | $\gamma_1 \gamma_2 \gamma_3 = 1$ | 2 |
| 3 | 3 1 2 | $\gamma_1 \gamma_2 \gamma_3$ | $\gamma_1 \gamma_2 \gamma_3 = 1$ | 2 |
| 4 | 1 2 3 4 | $\gamma_1 \gamma_2 \gamma_3 \gamma_4$ | $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$ | 8 |
| 4 | 2 3 1 4 | $\gamma_1 \gamma_2 \gamma_3 \gamma_4$ | $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$ | 4 |
| 4 | 3 2 1 $\gamma_1 \gamma_2 \gamma_3 \gamma_4$ | $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$ | 2 |
| 4 | 2 3 1 4 | $\gamma_1 \gamma_2 \gamma_3 \gamma_4$ | $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$ | 2 |
| 4 | 3 2 1 4 | $\gamma_1 \gamma_2 \gamma_3 \gamma_4$ | $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$ | 2 |

TABLE I. All possible geometric phase factors $\gamma^{(i)}$ defined in Eq. (3), for an arbitrary path joining a point $s_1$ to $s_2$, such that the eigenvectors of $H(s_2)$ are permuted according to $P$ with respect to those of $H(s_1)$. The last column lists the number of the possible combinations of values (±1) that the $\gamma^{(i)}$ factors can take in the special case of a real $H(s)$. The stars mark the permutations induced by relation (4), observed at the half-loop of Ref. [17] for $n = 2$ and 3.

The above arguments on the permutational symmetry remain valid even if Eq. (12) is only approximate, provided that $|U^\Gamma_{j,P_j}| \gg n \max_{k \neq P_j} |U^\Gamma_{jk}|$ for all $j$. This extends the interest of the permutational case to a finite domain of the parameters’ space around the point where Eq. (12) holds exactly or, more in general, to any region where the inequality on $U^\Gamma_{jk}$ holds. For example, an approximate permutation occurs when the energy levels of an Hamiltonian $H(s)$ undergo a sequence of sharp avoided crossings along the path. At each avoided crossing, the two involved eigenstates, to a good approximation, exchange. As a result, there exist sizable regions between two avoided crossings where the eigenvectors are an approximate permutation of the starting ones.

Probably the simplest example of a nontrivial permutation of the Hamiltonian eigenstates occurs when the
relation

\[ H(s_1) = -H(s_2) \tag{14} \]

holds at the ends of the path. This symmetry is verified exactly by the spin-\frac{1}{2} system, where it determines the swap of the eigenstates between \( \theta = 0 \) and \( \theta = \pi \). Relation (14) holds also, approximately, in very common situations. Suppose, for example, that a point, say \( s = 0 \), locates an \( n \)-fold degeneracy, and consider the perturbative expansion around there:

\[ H(s) = s \cdot H^{(1)} + \ldots. \tag{15} \]

[\(H^{(1)}\) is a vector of Hermitian numerical matrices.] In the sufficiently small neighborhood of the degeneracy, where the linear term accounts for the main contribution to the energy shifts, pairs of opposite points \((s_1, s_2 = -s_1)\) satisfy the relation (14). The permutation of the eigenstates associated to (14) is composed by \(n/2\) 2-cycles for even \(n\), or by \((n-1)/2\) 2-cycles plus one 1-cycle for odd \(n\): the corresponding \(\gamma\)'s are marked by stars in Table I.

In the final part of this Letter, we examine the deformed microwave resonators experiment of Ref. [17]. In a recent work [17], the diagonal, closed-path Berry phases were calculated for that system. Here we analyze the experiment of Ref. [17] as a transparent example of how off-diagonal \(\gamma_{jk}\)'s can be measured for open paths. For these systems, \(s = (s \cos \theta, s \sin \theta)\) parameterizes the displacement of one corner of the resonator away from the position of a conical intersection of the energy levels. Lauber et al. [17] investigate the Berry phase of these nearly degenerate states, when the distortion is driven through a loop \(\theta = 0\) to \(2\pi\) around the degenerate point. The distortion path is traced in small steps in \(\theta\), following adiabatically the real eigenfunctions. In Fig. 3 we report the initial \((\theta = 0)\), half-way \((\theta = \pi)\) and final \((\theta = 2\pi)\) parallel-transported eigenfunctions from the original pictures of Ref. [17].

The first case considered is that of a triangular cavity deformed around a twofold degeneracy: for small distortions, the system behaves similarly to a spin \(\frac{1}{2}\). In particular, the Berry phases \(\gamma_{ij}^{(1)}\) at the end of the loop both equal \(-1\) as expected for such a situation (cf. in Fig. 2 the recurrence of the pattern with changed sign at \(\theta = 0\) and \(2\pi\)). Due to the well approximate symmetry (13) at half path \((\theta = \pi)\), the diagonal Berry phases are undefined there, but it is instead possible to determine the experimental value of \(\gamma_{12}^{(2)}\) for this path. From inspection of Fig. 3 we determine \(\sigma_{12} = 1, \sigma_{21} = -1\). This is consistent with the only possible value \(\gamma_{12}^{(2)} = -1\) allowed in this spin-\(\frac{1}{2}\)–like case (see Table I). The same holds for the path going from \(\theta = \pi\) to \(2\pi\).

The case of the rectangular resonator is more interesting. Here, three states intersect conically at \(s = 0\). The three Berry phases \(\gamma_{ij}^{(1)}\) at the end of the loop \((-1, +1\) and \(-1)\) are compatible with the determinant requirement of Table I. Figure 3 shows that empirically also this system satisfies the symmetry relation \(H(\pi) = -H(0)\) at mid loop. Thus, for the path \(\theta = 0\) to \(\pi\) the only well defined Pancharatnam-Berry phase is that of the central state \(\gamma_2^{(1)} = -1\). The upper and lower states exchange, giving \(\sigma_{13} = 1, \sigma_{31} = 1\) thus \(\gamma_2^{(2)} = 1\). This is one of the two combinations of values allowed by the determinant rule \(\gamma_{13} \gamma_2 = -1\) of Table I.

In conclusion, we have identified novel off-diagonal geometric phase factors, generalizing the (diagonal) Berry phase. The two sets of diagonal and off-diagonal geometric phases together exhaust the number of independent observable phase relations among \(n\) orthogonal states evolved along a path. We show that, in many common situations, the off-diagonal factors carry the relevant geometric phase information on the basis of eigenstates.

We thank Prof. D. Dubbers, Dr. F. Faure, and Dr. A. F. Morpurgo for useful discussions.

\[ \text{FIG. 2. The observed initial (} \theta = 0\text{), intermediate (} \theta = \pi\text{) and final (} \theta = 2\pi\text{) eigenstates of the microwave cavities deformed following adiabatically the path of Ref. [17]. Left: the two eigenstates of the triangular resonator. Right: the three eigenstates of the rectangular resonator.} \]

† electronic address: manini@sissa.it
present address: International School for Advanced Studies (SISSA), via Beirut 4, I-34014 Trieste, Italy.
* electronic address: pistoles@ill.fr
[1] S. Pancharatnam, Proc. Ind. Acad. Sci. A 44, 247 (1956).
[2] M. V. Berry, Proc. R. Soc. London, Ser. A 392, 45 (1984).
[3] B. Simon, Phys. Rev. Lett. 51, 2167 (1983).
[4] J. Anandan and L. Stodolsky, Phys. Rev. D 35, 2597 (1987).
[5] J. Christian and A. Shimony, J. Phys. A: Math. Gen. 26, 5551 (1993).
[6] R. Resta, Rev. Mod. Phys. 66, 899 (1994).
[7] D. E. Manolopoulos and M. S. Child, Phys. Rev. Lett. 82, 2223 (1999).
[8] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
[9] J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).
[10] J. Zak, Europhys. Lett. 9, 615 (1989).
[11] R. Simon and N. Mukunda, Phys. Rev. Lett. 70, 880 (1993).
[12] N. Mukunda and R. Simon, Ann. Phys. 228, 205 (1993).
[13] E. M. Rabei, Arvind, N. Mukunda, and R. Simon, Phys. Rev. A 60, 3397 (1999).
[14] G. Delacrétaz, E. R. Grant, R. L. Whetten, L. Wöste, and J. W. Zwanziger, Phys. Rev. Lett. 56, 2598 (1986).
[15] R. Tycko, Phys. Rev. Lett. 58, 2281 (1987).
[16] H. Weinfurter and G. Badurek, Phys. Rev. Lett. 64, 1318 (1990).
[17] H.-M. Lauber, P. Weidenhammer, and D. Dubbers, Phys. Rev. Lett. 72, 1004 (1994).
[18] A. G. Wagh, V. C. Rakhecha, P. Fischer, and A. Ioffe, Phys. Rev. Lett. 81, 1992 (1998); R. Bhandari, ibid. 83, 2089 (1999); A. G. Wagh et al., ibid. 83, 2090 (1999).
[19] The Bargmann invariants of Ref. [11–13] are undefined for the orthogonal states we consider in Eq. (8).
[20] For a different purpose, similar overlaps where considered in Ref. [4].
[21] M. Hamermesh, Group Theory and its applications to physical problems (Addison-Wesley, London, 1962).