On generalized length spectrum in quotients of SL4

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Abstract. In this paper we consider a generalized length spectrum in the case of compact symmetric spaces generated as quotients of the special linear group of order four over real numbers. While the classical length spectrum is given as an estimate for a yes function counting prime geodesics of appropriate length, its generalized form is usually represented by a higher order counting function of Chebyshev type. Our goal is to prove that the error term that appears in the classical case in this setting can be significantly improved when derived via analogous, generalized apparatus.

1. Introduction
Recently, we proved that in the case of compact symmetric spaces obtained as quotients of the Lie group $SL_4(\mathbb{R})$, the corresponding length spectrum is given by

$$\pi(x) = 2\left(\text{li}(x) + O \left( x^{1-\frac{1}{D}} (\log x)^{-\frac{1}{2}} \right) \right)$$

as $x \to +\infty$, where $\pi(x)$ is a yes function counting prime geodesics of the length not larger that $\log x$, $\text{li}(x) = \int_{2}^{x} \frac{dt}{\log t}$ is the integral logarithm, and $D$ is the degree of the polynomial that appears in the functional equation of the corresponding Selberg zeta function.

It is well known that the length spectrum stated above easily follows from the corresponding equation for the function $\psi_k(x)$, $k \in \mathbb{N} \cup \{0\}$ (these functions are defined below). Such reasoning can be found in [19], as well as [2]-[4], [9], [11]-[13], [21], [24], etc.

Wa also refer to [5], [14]-[16], [18], [22], [6].

Thus, for example, the length spectrum stated above follows from the relation

$$\psi_0(x) = 2x + O \left( x^{1-\frac{1}{D}} \right)$$

as $x \to +\infty$.

In other words, the knowledge about $\pi(x)$ is equivalent to the knowledge about $\psi_0(x)$.

Note that $\psi_1(x) = \int_{\frac{1}{2}}^{x} \psi_0(t) \, dt$. This means that, in some sense, the knowledge about $\psi_0(x)$ (and hence about $\pi(x)$) can be compared to the knowledge about $\frac{\psi_1(x)}{x}$. 

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However, any estimate for the function $\frac{\psi_1(x)}{x}$ (or more generally for the function $\frac{\psi_k(x)}{x^k}$) is known in literature as a generalized length spectrum (or a weighted prime geodesic theorem).

Our goal is to prove the generalized length spectrum

$$\frac{\psi_1(x)}{x} = \sum_I + O \left( x^{\frac{1}{2} - \frac{1}{2k}} \right)$$

as $x \to +\infty$, where the sum $\sum_I$ is indexed over singularities of the corresponding Selberg zeta functions inside $\left( \frac{1}{2} - \frac{1}{2k}, \frac{3}{4} \right]$. The generalized length spectrum can be also found in [1], [10], etc., and not necessarily for the same underlying symmetric space.

For the background on the representation theory of semi-simple groups, we refer to [17] (see also, [23]). For the Selberg trace formula in this setting, see [25].

2. Preliminaries

Let $k \in \mathbb{Z}$, and $I_k = \{0, -1, \ldots, -k\}$.

Put $I_q, q \in \{0, 1, \ldots, 4\}$ to be the set of all $j \in I_k$ such that $j$ is a singularity of the Selberg zeta function $Z_{P,A^n\bar{n}}(s + \frac{q}{4})$, and $I'_q = I_k \setminus I_q$.

The set of the remaining singularities $s^q$ of $Z_{P,A^n\bar{n}}(s + \frac{q}{4})$ will be denoted by $S^q$.

Note that by Theorem 3.2.1 in [8], the function $Z_{P,A^n\bar{n}}(s)$ extends to a meromorphic function on the whole of $\mathbb{C}$, and its singularities lie in $\mathbb{R} \cup \left( \frac{1}{2} + i \mathbb{R} \right)$ (see also, [7]).

Precise locations an the orders of the singularities of $Z_{P,A^n\bar{n}}(s)$ are described by Proposition 3.4.3 in [8] and the discussion afterwards.

Thus, $Z_{P,1}(s)$ has a double zero at $s = 1$, and, apart from that, all poles and zeros of $Z_{P,1}(s)$ lie in the strip $0 \leq \text{Re} (s) \leq \frac{3}{4}$.

We also put $S^q_{\mathbb{R}} = S^q \cap \mathbb{R}$, and $S^q_{\frac{1}{2}-\frac{3}{4}} = S^q \setminus S^q_{\mathbb{R}}$.

$\bar{n}$ is the complexified Lie algebra of

$$\bar{N} = \begin{pmatrix} I_2 & 0 \\ \text{Mat}_2(\mathbb{R}) & I_2 \end{pmatrix}.$$

$P = MAN$ is a parabolic subgroup of $G$ with Levi component $MA$ and the unipotent radical $N$, where

$$M \cong \{ (x, y) \in \text{Mat}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{R}) : \det (x), \det (y) = \pm 1, \det (x) \det (y) = 1 \},$$

$$A = \left\{ \begin{pmatrix} a & a \\ a & a^{-1} \\ \end{pmatrix} : a > 0 \right\},$$

$$N = \begin{pmatrix} I_2 & \text{Mat}_2(\mathbb{R}) \\ 0 & I_2 \end{pmatrix}.$$
We point out that in this paper the underlying symmetric space $X_\Gamma$ is given by $X_\Gamma = \Gamma \backslash G / K$, where $G = SL_4(\mathbb{R})$, $K$ is the maximal compact subgroup of $G$ (thus, $K = SO(4)$), and $\Gamma$ is a discrete, co-compact subgroup of $G$.

By [20, p. 63, (3.7)],

$$\psi_k(x) = \sum_{q=0}^{4} (-1)^q \sum_{\alpha \in S_{k,q}} c_\alpha(q,k),$$

where $S_{k,q}$ is the set of poles of

$$\frac{Z'_{P,A}^q(s + \frac{q}{4})}{Z_{P,A}^q(s + \frac{q}{4})} s^{-1} (s + 1)^{-1} \ldots (s + k)^{-1} x^{s+k},$$

and $c_\alpha(q,k)$ is the residue at $s = \alpha$.

It is not so hard to check that

$$c_{s^q}(q,k) = o_{s^q}^q(s^q)^{-1} (s^q + 1)^{-1} \ldots (s^q + k)^{-1} x^{s^q+k},$$

for $s^q \in S^q$,

$$c_{-j}(q,k) = o_{-j}^q \prod_{\substack{l=0 \\lnot l=j}}^{k} (-j + l)^{-1} x^{-j+k} \log x -$$

$$o_{-j}^q \prod_{\substack{l=0 \\lnot l=j}}^{k} (-j + l)^{-1} \left( -\sum_{\substack{l=0 \\lnot l=j}}^{k} (-j + l)^{-1} + a_{1,-j}^q \right) x^{-j+k}$$

for $-j \in I_q$, and

$$c_{-j}(q,k) = \frac{Z'_{P,A}^q(-j + \frac{q}{4})}{Z_{P,A}^q(-j + \frac{q}{4})} \prod_{\substack{l=0 \\lnot l=j}}^{k} (-j + l)^{-1} x^{-j+k}$$

for $-j \in I'_q$, where $o_z^q$ is the order of the singularity $z$ of $Z_{P,A}^q(s + \frac{q}{4})$, and $a_{i,z}^q$’s are the coefficients in the expansion

$$\frac{Z'_{P,A}^q(s + \frac{q}{4})}{Z_{P,A}^q(s + \frac{q}{4})} = \frac{o_z^q}{s - z} \left( 1 + \sum_{i=1}^{+\infty} a_{i,z}^q(s - z)^i \right).$$

We note that the counting function $\psi_k(x)$ is defined by $\psi_j(x) = \int_0^x \psi_{j-1}(t) \, dt$, $j \in \mathbb{N}$, where

$$\psi_0(x) = \sum_{[\gamma] \in E_p(\Gamma)} \chi_1(\Gamma \gamma) L_{\gamma_0}.$$
Here, $\mathcal{E}_P (\Gamma)$ is the set of all $\Gamma$-conjugacy classes $[\gamma]$, such that the element $\gamma \in \Gamma$ is conjugate in $G$ to an element $a_\gamma b_\gamma \in A^- B$, where

$$A^- = \left\{ \begin{pmatrix} a & a^{-1} \\ a^{-1} & a^{-1} \end{pmatrix} : 0 < a < 1 \right\}$$

is the negative Weyl chamber in $A$ with respect to the root system given by the choice of parabolic, and

$$B = \begin{pmatrix} SO(2) \\ SO(2) \end{pmatrix}$$

is a compact Cartan subgroup of $M$.

If $\gamma$ is a conjugate to $a_\gamma b_\gamma$, we define the length $l_\gamma$ of $\gamma$ to be $l_\gamma = b (\log a_\gamma, \log b_\gamma)^{1/2}$, where $b$ is an invariant bilinear form on the complexified Lie algebra $\mathfrak{g} = sl_4(\mathbb{C})$ of $G$.

Furthermore, $\chi_1 (\Gamma)$ is the first higher Euler characteristics of the symmetric space $X_{\Gamma_\gamma} = \Gamma_\gamma \backslash G_{\gamma} / K_\gamma$, where $G_\gamma$ and $\Gamma_\gamma$ are the centralizers of $\gamma$ in $G$ and $\Gamma$, respectively, and $K_\gamma = K \cap G_\gamma$.

In this research, we apply the following, well known differential operator (and its properties)

$$\Delta_k^+ f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x + (k - i) h),$$

where $h$ is a constant which will be fixed in the sequel.

Finally, we apply the fact that the number $N(t)$ of singularities of $Z_{P,A^q \bar{n}}(s)$ at points $\frac{1}{2} + i x$, $0 < x < t$ is $O(t^D)$, where $D$ is the degree of the polynomial that appears in the functional equation for $Z_{P,A^q \bar{n}}(s)$.

### 3. Main result
The following theorem represents the main result of our research.

**Theorem 1.** Let $X_\Gamma$ be as above. Then,

$$\frac{\psi_1(x)}{x} = x + \sum_{q=0}^{2} (-1)^q \sum_{s^q \in S_q^q} \alpha_{s^q}^q (s^q)^{-1} (s^q + 1)^{-1} x^{s^q} + O\left( x^{1-\frac{1}{2D}} \right)$$

as $x \to +\infty$.

**Proof.** Let $z \in S_q^q$. We have,

\[ ... \]
Thus,

\[ h^{-1} \Delta_{k-1}^+ c_z (q, k) \]

\[ = h^{-1} \sum_{i=0}^{k-1} (-1)^i \left( \begin{array}{c} k - 1 \\ i \end{array} \right) a^q z \times (z + 1)^{-1} ... (z + k)^{-1} (x + (k - 1 - i) h)^{z+k} \]

\[ = h^{-1} a^q z \times (z + 1)^{-1} ... (z + k)^{-1} \sum_{i=0}^{k-1} (-1)^i \left( \begin{array}{c} k - 1 \\ i \end{array} \right) (x + (k - 1 - i) h)^{z+k} \]

\[ = O \left( h^{-1} |z|^{-1} x^{\frac{1}{2}} \right) = O \left( h^{-1} |z|^{-1} x^{\frac{1}{2}} \right) . \]

Furthermore,

\[ h^{-1} \Delta_{k-1}^+ c_z (q, k) \]

\[ = h^{-1} \int \int \int ... \int \left( a^q z \times (z + 1)^{-1} ... (z + k)^{-1} \frac{1}{t_1^{k-1}} \right) dt_1 ... dt_{k-1} \]

\[ = h^{-1} a^q z \times (z + 1)^{-1} \int \int \int ... \int t_1^{k-1} dt_1 ... dt_{k-1} . \]

Thus,

\[ |h^{-1} \Delta_{k-1}^+ c_z (q, k)| \]

\[ \leq h^{-1} a^q |z|^{-1} |z + 1|^{-1} \int \int \int ... \int t_1^{k-1} dt_1 ... dt_{k-1} \]

\[ = h^{-1} a^q |z|^{-1} |z + 1|^{-1} \int \int \int ... \int h(t_2 + h_2)^{\frac{1}{2}} dt_2 ... dt_{k-1} \]

\[ = h^{-1} a^q |z|^{-1} |z + 1|^{-1} \int \int \int ... \int h(t_3 + h_3 + h_2)^{\frac{1}{2}} dt_3 ... dt_{k-1} \]

\[ = ... \]

\[ = |a^q| |z|^{-1} |z + 1|^{-1} (x + h_k + h_{k-1} + ... + h_3)^{\frac{1}{2}} . \]

Now, the fact that \( h_j \in [0, h] \) for \( j \in \{2, 3, ..., k\} \), and the fact that \( h \leq \frac{x}{2} \), yield

\[ h^{-1} \Delta_{k-1}^+ c_z (q, k) = O \left( |z|^{-2} x^{\frac{1}{2}} \right) = O \left( |z|^{-2} x^{\frac{3}{2}} \right) . \]

First, we estimate the following sum
We have,

$$\sum_{z \in S_k^q} h^{-(k-1)} \Delta_{k-1}^+ c_z (q, k).$$

Now, we apply the estimate (1) if $|z| > M$.

Otherwise, we apply the estimate (2).

We obtain,

$$\sum_{z \in S_k^q} h^{-(k-1)} \Delta_{k-1}^+ c_z (q, k) = \sum_{z \in S_k^q \setminus M} h^{-(k-1)} \Delta_{k-1}^+ c_z (q, k) + \sum_{|z| > M} h^{-(k-1)} \Delta_{k-1}^+ c_z (q, k).$$

We obtain,

$$\sum_{z \in S_k^q \setminus M} h^{-(k-1)} \Delta_{k-1}^+ c_z (q, k) \approx O \left( x^2 \sum_{z \in S_k^q \setminus M} |z|^{-2} \right) + O \left( h^{-(k-1)} x^{\frac{1}{2}+k} \sum_{|z| > M} |z|^{-k-1} \right)$$

$$= O \left( x^2 \int_{|z| \geq M} t^{-2} dN (t) \right) + O \left( h^{-(k-1)} x^{\frac{1}{2}+k} \int_{|z| > M} t^{-k-1} dN (t) \right)$$

$$= O \left( x^2 M^{D-2} \right) + O \left( h^{-(k-1)} x^{\frac{1}{2}+k} M^{D-k-1} \right).$$

Now, we estimate $h^{-(k-1)} \Delta_{k-1}^+ c_1 (0, k)$.

We obtain,

$$h^{-(k-1)} \Delta_{k-1}^+ c_1 (0, k) = h^{-(k-1)} \Delta_{k-1}^+ \frac{2}{(k+1)!} x^{1+k} = \frac{2}{(k+1)!} \left( \tilde{x}^{1+k} \right)^{(k-1)} = \tilde{x}^2$$

for some $\tilde{x} \in [x, x + (k-1) h]$.

Hence, $h \leq \frac{\tilde{x}}{2}$ yields that

$$h^{-(k-1)} \Delta_{k-1}^+ c_1 (0, k) = O (x^2).$$

Having in mind the fact that

$$h^{-(k-1)} \Delta_{k-1}^+ c_1 (0, k) = h^{-(k-1)} \Delta_{k-1}^+ \frac{2}{(k+1)!} x^{1+k}$$

$$= h^{-(k-1)} \sum_{i=0}^{k-1} (-i)^k \frac{2}{(k+1)!} (x + (k - 1 - i) h)^{1+k}$$

$$= 2 h^{-(k-1)} \sum_{i=0}^{k-1} \frac{1}{i!} \sum_{j=0}^{1+k} \binom{1+k}{j} x^{1+k-j} ((k - 1 - i) h)^{j},$$
We conclude, as we already derived, we may write
\[ h^{-(k-1)} \Delta_{k-1}^+ 0, k) = P x^2 + Q x + R \]
for some \( P, Q \) and \( R \).

We fix some \( k \).
We put \( h = mD \) for some even \( m \).
Thus, \( P = 2h^{-(mD-1)} \frac{1}{(mD + 1)!} \sum_{i=0}^{mD-1} (-1)^i \left( \frac{mD - 1}{i} \right) \left( \frac{1 + mD}{mD - 1} \right) ((mD - 1 - i) h)^{mD-1} \)
\[ = 2h^{-(mD-1)} \frac{1}{(mD + 1)!} \sum_{i=0}^{mD-1} (-1)^i \left( \frac{mD - 1}{i} \right) (mD - 1 - i)^{mD-1} = 1, \]
\[ Q = 2h^{-(mD-1)} \frac{1}{(mD + 1)!} \sum_{i=0}^{mD-1} (-1)^i \left( \frac{mD - 1}{i} \right) (mD - 1 - i)^{mD} = (mD - 1) h, \]
\[ R = 2h^{-(mD-1)} \frac{1}{(mD + 1)!} \sum_{i=0}^{mD-1} (-1)^i \left( \frac{mD - 1}{i} \right) (mD - 1 - i)^{mD+1} \]
\[ = 2h^2 \frac{1}{(mD + 1)!} \sum_{i=0}^{mD-1} (-1)^i \left( \frac{mD - 1}{i} \right) (mD - 1 - i)^{mD+1} . \]

We conclude,
\[ h^{-(mD-1)} \Delta_{mD-1}^+ 0, mD) = x^2 + O (hx) + O \left( h^2 \right) \]
As we already derived,
\[ \sum_{q=0}^{4} (-1)^q \sum_{z \in S^3_{\frac{1}{2} - \frac{q}{2}}} h^{-(mD-1)} \Delta_{mD-1}^+ (q, mD) \]
\[ = O \left( x^{\frac{3}{2} M^{D-2}} \right) + O \left( h^{-(mD-1)} x^{\frac{1}{2} + mD} M^{D-mD-1} \right) . \]

Now, we are interested to determine \( h \) and \( M \) such that the error terms \( O (hx), O \left( h^2 \right) \), \( O \left( x^{\frac{3}{2} M^{D-2}} \right) \), and \( O \left( h^{-(mD-1)} x^{\frac{1}{2} + mD} M^{D-mD-1} \right) \) be dominated by the same error term.

First, we consider the error terms \( O \left( h^2 \right) \), \( O \left( x^{\frac{3}{2} M^{D-2}} \right) \) and \( O \left( h^{-(mD-1)} x^{\frac{1}{2} + mD} M^{D-mD-1} \right) \).

We put \( h = x^\alpha, M = x^\beta \), and require that
\[ h^2 = x^{\frac{3}{2} M^{D-2}} = h^{-(mD-1)} x^{\frac{1}{2} + mD} M^{D-mD-1} . \]
We obtain,

\[ 2\alpha = \frac{3}{2} + \beta D - 2\beta = -\alpha mD + \alpha + \frac{1}{2} + mD + \beta D - \beta mD - \beta, \]
i.e.,

\[ \frac{3}{2} + \beta D - 2\beta = -\left(\frac{3}{4} + \frac{1}{2}\beta D - \beta\right) mD + \frac{3}{4} + \frac{1}{2}\beta D - \beta + \frac{1}{2} + mD + \beta D - \beta mD - \beta. \]

Hence, \( \beta = \frac{1}{2D} \), and \( \alpha = \frac{3}{4} + \frac{1}{2}\beta D - \beta = \frac{1}{2} - \frac{1}{2D} \), i.e., \( h = \frac{x^{1/2}}{\beta} \), and \( M = x^{\frac{1}{2}} \).

This means that the error terms \( O(h^2), O\left(x^{\frac{3}{2}}M^{D-2}\right) \) and \( O\left(h^{-(mD-1)}x^{\frac{1}{2}+mD}M^{D-mD-1}\right) \) are \( O(h^2) = O\left(x^{1-\frac{1}{2}}\right) \).

Moreover, the error term \( O(hx) \) is \( O\left(x^{\frac{3}{2}}\right) \).

Note that \( 1 - \frac{1}{2D} \leq \frac{3}{2} - \frac{1}{2D} \) since \(-1 \leq \frac{1}{2D} \) holds true.

Hence, in this scenario, the error terms \( O(hx), O\left(h^2\right), O\left(x^{\frac{3}{2}}M^{D-2}\right), \) and \( O\left(h^{-(mD-1)}x^{\frac{1}{2}+mD}M^{D-mD-1}\right) \) are all dominated by \( O\left(x^{\frac{3}{2}}\right) \).

Second, we consider the error terms \( O(hx), O\left(x^{\frac{3}{2}}M^{D-2}\right) \) and \( O\left(h^{-(mD-1)}x^{\frac{1}{2}+mD}M^{D-mD-1}\right) \).

We put \( h = x^\alpha, M = x^\beta \), and require that

\[ hx = x^{\frac{3}{2}}M^{D-2} = h^{-(mD-1)}x^{\frac{1}{2}+mD}M^{D-mD-1}. \]

Hence,

\[ \alpha + 1 = \frac{3}{2} + \beta D - 2\beta = -\alpha mD + \alpha + \frac{1}{2} + mD + \beta D - \beta mD - \beta, \]
i.e.,

\[ \frac{3}{2} + \beta D - 2\beta = -\left(\frac{1}{2} + \beta D - 2\beta\right) mD + \frac{1}{2} + \beta D - 2\beta + \frac{1}{2} + mD + \beta D - \beta mD - \beta. \]

We obtain, \( \beta = \frac{1}{2D-2}, \alpha = \frac{1}{2} + \beta D - 2\beta = \frac{1}{2} + \frac{D}{2D-2} - \frac{1}{D-1} \), i.e., \( h = x^{\frac{3}{2} + \frac{D}{2D-2} - \frac{1}{D-1}}, M = x^{\frac{1}{2D-2}} \).

In this case, the error terms \( O(hx), O\left(x^{\frac{3}{2}}M^{D-2}\right) \) and \( O\left(h^{-(mD-1)}x^{\frac{1}{2}+mD}M^{D-mD-1}\right) \) are \( O(hx) = O\left(x^{\frac{3}{2} + \frac{D}{2D-2} - \frac{1}{D-1}}\right) \).

Furthermore, the error term \( O(h^2) \) is \( O\left(x^{1+\frac{D}{2} - \frac{1}{D-1}}\right) \).

Note that the inequality \( 1 + \frac{D}{2D-2} - \frac{2}{D-1} \leq \frac{3}{2} + \frac{D}{2D-2} - \frac{1}{D-1} \) holds true since it is equivalent to the inequality \(-1 \leq 0. \)

In other words, in this scenario, the error terms \( O(hx), O\left(h^2\right), O\left(x^{\frac{3}{2}}M^{D-2}\right) \) and \( O\left(h^{-(mD-1)}x^{\frac{1}{2}+mD}M^{D-mD-1}\right) \) are all dominated by \( O\left(x^{\frac{3}{2} + \frac{D}{2D-2} - \frac{1}{D-1}}\right) \).

If we compare the first and the second scenario, we can see that the error term \( O\left(x^{\frac{3}{2} - \frac{1}{2D}}\right) \) in the first case is smaller than the error term \( O\left(x^{\frac{3}{2} + \frac{D}{2D-2} - \frac{1}{D-1}}\right) \). Namely, this becomes obvious if we note that
\[
O \left( x^{\frac{3}{2} + \frac{D-2}{2D-2}} \right) = O \left( x^{\frac{3}{2} + \frac{D-2}{2D-2}} \right),
\]

and remember that \( D \geq 2 \).

This means that the first scenario gives a better result than the second one.

Moreover, note that the remaining possible scenarios

\[
O (hx) = O \left( h^2 \right) = O \left( x^{\frac{3}{2} M D^{-2}} \right)
\]

and

\[
O (hx) = O \left( h^2 \right) = O \left( h^{-(mD-1)} x^{\frac{1}{2} + mD M D^{-mD-1}} \right)
\]
do not yield a better error term the error term \( O \left( x^{\frac{3}{2} - \frac{1}{2D}} \right) \).

Namely, \( h = x^\alpha \) and \( O (hx) = O \left( h^2 \right) \) in both of them, yield the error term \( O \left( x^2 \right) \) which is worse than the error term \( O \left( x^{\frac{3}{2} - \frac{1}{2D}} \right) \).

Thus, we are interested in the error term \( O \left( x^{\frac{3}{2} - \frac{1}{2D}} \right) \) achieved for the choice \( h = x^{\frac{1}{2} - \frac{1}{2D}} \), \( M = x^{\frac{1}{2D}} \). Now, we estimate

\[
\sum_{q=0}^{4} (-1)^q \sum_{\frac{s^q}{2D} \leq s^q \leq \frac{3}{4}} h^{-(k-1)} \Delta^+_{k-1} c_{s^q} (q, k).
\]

Note that \( \frac{1}{4} \leq \frac{1}{2} - \frac{1}{2D} < \frac{1}{2} \). Hence,

\[
\sum_{q=0}^{4} (-1)^q \sum_{\frac{s^q}{2D} \leq s^q \leq \frac{3}{4}} h^{-(k-1)} \Delta^+_{k-1} c_{s^q} (q, k)
= \sum_{q=0}^{2} (-1)^q \sum_{\frac{s^q}{2D} \leq s^q \leq \frac{3}{4}} h^{-(k-1)} \Delta^+_{k-1} c_{s^q} (q, k).
\]

As earlier, we know that

\[
h^{-(k-1)} \Delta^+_{k-1} c_{s^q} (q, k) = h^{-(k-1)} \Delta^+_{k-1} o_{s^q} (s^q)^{-1} (s^q + 1)^{-1} \ldots (s^q + k)^{-1} x^{s^q + k}
= o_{s^q} (s^q)^{-1} (s^q + 1)^{-1} \ldots (s^q + k)^{-1} \left( \bar{x}_{s^q, q, k} \right)^{k-1}
= o_{s^q} (s^q)^{-1} (s^q + 1)^{-1} \bar{x}_{s^q, q, k}
\]

for some \( \bar{x}_{s^q, q, k} \in [x, x + (k - 1) h] \).

Hence, \( \bar{x}_{s^q, q, k} = x + \theta_{s^q, q, k} \) for some \( \theta_{s^q, q, k} \in [0, (k - 1) h] \).
We obtain,

\[ x^{s^q+1} = (x + \theta_{s^q, q, k})^{s^q+1} = \sum_{j=0}^{+\infty} \left( \begin{array}{c} s^q + 1 \\ j \end{array} \right) x^{s^q+1-j} \theta_{s^q, q, k} \]

\[ = x^{s^q+1} + \sum_{j=1}^{+\infty} \left( \begin{array}{c} s^q + 1 \\ j \end{array} \right) x^{s^q+1-j} \theta_{s^q, q, k}. \]

Note that \( \sum_{j=0}^{+\infty} \left( \begin{array}{c} s^q + 1 \\ j \end{array} \right) = (1 + 1)^{s^q+1} = 2^{s^q+1}. \) Hence,

\[ \sum_{j=1}^{+\infty} \left( \begin{array}{c} s^q + 1 \\ j \end{array} \right) = 2^{s^q+1} - \left( \begin{array}{c} s^q + 1 \\ 0 \end{array} \right) = 2^{s^q+1} - 1. \]

Moreover, \( \theta_{s^q, q, k} = O(h). \) Consequently,

\[ x^{s^q+1} = x^{s^q+1} + \sum_{j=1}^{+\infty} \left( \begin{array}{c} s^q + 1 \\ j \end{array} \right) x^{s^q+1-j} O(h^j) = x^{s^q+1} + \sum_{j=1}^{+\infty} \left( \begin{array}{c} s^q + 1 \\ j \end{array} \right) O \left( x^{s^q+1-j+\frac{1}{2}-\frac{1}{2\pi}} \right) \]

\[ = x^{s^q+1} + O \left( x^{s^q+1-\frac{1}{2}-\frac{1}{2\pi}} \right) = x^{s^q+1} + O \left( x^{s^q+1-\frac{1}{2}-\frac{1}{2\pi}} \right) \]

\[ = x^{s^q+1} + O \left( x^{\frac{3}{2}-\frac{1}{2\pi}} \right) = x^{s^q+1} + O \left( x^{\frac{3}{2}-\frac{1}{2\pi}} \right). \]

Thus,

\[ h^{-(k-1)} \Delta_{k-1}^+ c_{s^q} (q, k) = o_{s^q}^q (s^q)^{-1} (s^q + 1)^{-1} x^{s^q+1} + O \left( x^{\frac{3}{2}-\frac{1}{2\pi}} \right). \]

Furthermore,

\[ \sum_{q=0}^{4} (-1)^q \sum_{\frac{1}{2}-\frac{1}{2\pi} < s^q \leq 3/4} h^{-(k-1)} \Delta_{k-1}^+ c_{s^q} (q, k) \]

\[ = \sum_{q=0}^{2} (-1)^q \sum_{\frac{1}{2}-\frac{1}{2\pi} < s^q \leq 3/4} o_{s^q}^q (s^q)^{-1} (s^q + 1)^{-1} x^{s^q+1} + O \left( x^{\frac{3}{2}-\frac{1}{2\pi}} \right). \]

In particular,

\[ \sum_{q=0}^{4} (-1)^q \sum_{\frac{1}{2}-\frac{1}{2\pi} < s^q \leq 3/4} h^{-(mD-1)} \Delta_{mD-1}^+ c_{s^q} (q, mD) \]

\[ = \sum_{q=0}^{2} (-1)^q \sum_{\frac{1}{2}-\frac{1}{2\pi} < s^q \leq 3/4} o_{s^q}^q (s^q)^{-1} (s^q + 1)^{-1} x^{s^q+1} + O \left( x^{\frac{3}{2}-\frac{1}{2\pi}} \right). \]
Now, we estimate
\[
\sum_{q=0}^{4} (-1)^q \sum_{-j=-2}^{-k} h^{-(k-1)} \Delta^+_{k-1} c_{-j} (q, k).
\]

Obviously, \( h^{-(k-1)} \Delta^+_{k-1} c_{-j} (q, k) = 0 \) for \( j \in \{-2, -3, \ldots, -k\} \).

Hence,
\[
\sum_{q=0}^{4} (-1)^q \sum_{-j=-2}^{-k} h^{-(k-1)} \Delta^+_{k-1} c_{-j} (q, k) = 0.
\] (6)

Furthermore, we estimate
\[
\sum_{q=0}^{4} (-1)^q h^{-(k-1)} \Delta^+_{k-1} c_{-1} (q, k) = \sum_{q=0}^{3} (-1)^q h^{-(k-1)} \Delta^+_{k-1} c_{-1} (q, k) + h^{-(k-1)} \Delta^+_{k-1} c_{-1} (4, k).
\]

Since
\[
c_{-1} (q, k) = \frac{Z'_{P,A^\theta \bar{n}} (-1 + \frac{3}{4})}{Z_{P,A^\theta \bar{n}} (-1 + \frac{3}{4})} \prod_{l=0}^{k} (-j + l)^{-1} x^{k-1},
\]
it follows that
\[
h^{-(k-1)} \Delta^+_{k-1} c_{-1} (q, k) = \frac{Z'_{P,A^\theta \bar{n}} (-1 + \frac{3}{4})}{Z_{P,A^\theta \bar{n}} (-1 + \frac{3}{4})}
\]
for \( q \in \{0, 1, \ldots, 3\} \).

Suppose that \(-1 \in I_4'\).

We obtain,
\[
h^{-(k-1)} \Delta^+_{k-1} c_{-1} (4, k) = \frac{Z'_{P,A^\theta \bar{n}} (0)}{Z_{P,A^\theta \bar{n}} (0)}.
\]

Finally, suppose that \(-1 \in I_4\).

Now,
\[
c_{-1} (4, k) = a_{-1}^4 \prod_{l=0}^{k} (-1 + l)^{-1} x^{k-1} \log x - \sum_{l=0}^{k} (-1 + l)^{-1} x^{k-1} \log x - \sum_{l=0}^{k} (-1 + l)^{-1} a_{-1}^4
\]
\[
= a_{-1}^4 \prod_{l=0}^{k} (-1 + l)^{-1} \left( - \sum_{l=0}^{k} (-1 + l)^{-1} a_{-1}^4 \right) x^{k-1}.
\]

Since \((x^{k-1} \log x)^{(k-1)} = (k-1)! \log x + (k-1)! \sum_{l=1}^{k-1} \frac{1}{l!} \), it follows that
\[ h^{-(k-1)} \Delta^+_{k-1} c_{-1} (4, k) = o^4_{-1} (-1)^{-1} ((k - 1)!)^{-1} \left( (k - 1)! \log \bar{x}_{-1,4,k} + (k - 1)! \sum_{l=1}^{k-1} \frac{1}{l} \right) + \]

\[ o^4_{-1} (-1)^{-1} ((k - 1)!)^{-1} \left( - \sum_{l=0}^{k} (1 + l)^{-1} + a^4_{1,-1} \right) (k - 1)! \]

\[ = - o^4_{-1} \log \bar{x}_{-1,4,k} - o^4_{-1} \sum_{l=1}^{k-1} \frac{1}{l} + o^4_{-1} \sum_{l=0}^{k} (1 + l)^{-1} - o^4_{-1} a^4_{1,-1} \]

\[ = - o^4_{-1} \log \bar{x}_{-1,4,k} - o^4_{-1} a^4_{1,-1} = - o^4_{-1} \log \bar{x}_{-1,4,k} - o^4_{-1} (1 + a^4_{1,-1}) \]

for some \( \bar{x}_{-1,4,k} \in [x, x + (k - 1) h] \).

In other words,

\[ h^{-(k-1)} \Delta^+_{k-1} c_{-1} (4, k) = O (\log x) . \]

Therefore,

\[ \sum_{q=0}^{4} (-1)^q h^{-(k-1)} \Delta^+_{k-1} c_{-1} (q, k) = O (\log x) . \quad (7) \]

Finally, we estimate

\[ \sum_{q=0}^{4} (-1)^q h^{-(k-1)} \Delta^+_{k-1} c_0 (q, k) . \]

Suppose that \( 0 \in I_q' \) for some \( q \in \{0, 1, ..., 3\} \).

Now,

\[ c_0 (q, k) = \frac{Z'_{P,\Lambda^n} (q \frac{q}{4})}{Z_{P,\Lambda^n} (q \frac{q}{4})} (k!)^{-1} x^k . \]

We obtain,

\[ h^{-(k-1)} \Delta^+_{k-1} c_0 (q, k) = \frac{Z'_{P,\Lambda^n} (q \frac{q}{4})}{Z_{P,\Lambda^n} (q \frac{q}{4})} (k!)^{-1} \left( \bar{x}_{0,q,k}^k \right)^{(k-1)} = \frac{Z'_{P,\Lambda^n} (q \frac{q}{4})}{Z_{P,\Lambda^n} (q \frac{q}{4})} \bar{x}_{0,q,k} \]

for some \( \bar{x}_{0,q,k} \in [x, x + (k - 1) h] \).

This means that

\[ h^{-(k-1)} \Delta^+_{k-1} c_0 (q, k) = O (x) . \]

Suppose that \( 0 \in I_q \) for some \( q \in \{0, 1, ..., 4\} \).
We have,
\[
c_0(q, k) = o_p^q \prod_{l=0}^{k} (l)^{-1} x^k \log x - o_p^q \prod_{l=0 \atop l \neq 0}^{k} (l)^{-1} \left( - \sum_{l=0 \atop l \neq 0}^{k} (l)^{-1} + a_{q,0}^q \right) x^k.
\]

Note that \((x^k \log x)^{(k-1)} = k! x \log x + k! \sum_{l=2}^{k} \frac{1}{l} x, \) and \((x^k)^{(k-1)} = k! x.\)

We deduce,
\[
h^{-k+1} \Delta_{k-1}^+ c_0(q, k) = o_p^q (k!)^{-1} \left( k! x_0,q,k \log \bar{x}_{0,q,k} + k! \sum_{l=2}^{k} \frac{1}{l} \bar{x}_{0,q,k} \right) -
\]
\[
o_p^q (k!)^{-1} \left( - \sum_{l=0 \atop l \neq 0}^{k} (l)^{-1} + a_{q,0}^q \right) k! \bar{x}_{0,q,k}
\]
\[
= o_p^q \bar{x}_{0,q,k} \log \bar{x}_{0,q,k} + o_p^q \sum_{l=2}^{k} \frac{1}{l} \bar{x}_{0,q,k} + o_p^q \sum_{l=1}^{k} \frac{1}{l} \bar{x}_{0,q,k} - o_p^q a_{q,0}^q \bar{x}_{0,q,k}
\]
\[
= o_p^q \bar{x}_{0,q,k} \log \bar{x}_{0,q,k} + o_p^q \left( 2 \sum_{l=2}^{k} \frac{1}{l} + 1 - a_{q,0}^q \right) \bar{x}_{0,q,k}
\]

for some \(\bar{x}_{0,q,k} \in [x, x + (k-1) h].\)

Therefore,
\[
h^{-k+1} \Delta_{k-1}^+ c_0(q, k) = O(x \log x).
\]

We conclude,
\[
\sum_{q=0}^{4} (-1)^q h^{-k+1} \Delta_{k-1}^+ c_0(q, k) = O(x \log x).
\]

It remains to estimate
\[
\sum_{q=0}^{4} (-1)^q \sum_{s^{q} \in S_{q}^{\bar{q}}} h^{-k+1} \Delta_{k-1}^+ c_{s^{q}}(q, k).
\]

We have,
\[
h^{-k+1} \Delta_{k-1}^+ c_{s^{q}}(q, k) = o_{s^{q}}^p (s^{q})^{-1} (s^{q} + 1)^{-1} \bar{x}_{s^{q},q,k}^{s^{q}+1}
\]

for some \(\bar{x}_{s^{q},q,k} \in [x, x + (k-1) h].\)

Since \(s^{q} \leq \frac{1}{2} - \frac{1}{17\gamma},\) it follows that
\[
h^{-k+1} \Delta_{k-1}^+ c_{s^{q}}(q, k) = O \left( \frac{x^{3}}{2 - \frac{1}{17\gamma}} \right).
\]
Hence,

\[ \sum_{q=0}^{4} (-1)^q \sum_{s^q \in S^q_{k}} h^{-(k-1)}(q,k, s^q) = O\left( x^{\frac{3}{2} - \frac{1}{2m}} \right). \]  

\( (9) \)

Now, taking \( h = x^{\frac{1}{2} - \frac{1}{2m}}, M = x^{\frac{1}{2m}}, k = mD \) (m even), and combining (3)-(9), we obtain

\[ h^{-(mD-1)} \Delta^+_{mD-1} \psi_{mD}(x) \]

\[ = x^2 + \sum_{q=0}^{2} (-1)^q \sum_{s^q \in S^q_{k}} o_{s^q}(s^q)^{-1} \left( s^q + 1 \right)^{-1} x^{s^q+1} + O\left( x^{\frac{3}{2} - \frac{1}{2m}} \right). \]

Having in mind that \( \psi_1(x) \leq h^{-(mD-1)} \Delta^+_{mD-1} \psi_{mD}(x) \), we conclude

\[ \psi_1(x) \leq x^2 + \sum_{q=0}^{2} (-1)^q \sum_{s^q \in S^q_{k}} o_{s^q}(s^q)^{-1} \left( s^q + 1 \right)^{-1} x^{s^q+1} + O\left( x^{\frac{3}{2} - \frac{1}{2m}} \right). \]

Similarly,

\[ \psi_1(x) \geq x^2 + \sum_{q=0}^{2} (-1)^q \sum_{s^q \in S^q_{k}} o_{s^q}(s^q)^{-1} \left( s^q + 1 \right)^{-1} x^{s^q+1} + O\left( x^{\frac{3}{2} - \frac{1}{2m}} \right). \]

Consequently,

\[ \psi_1(x) = x^2 + \sum_{q=0}^{2} (-1)^q \sum_{s^q \in S^q_{k}} o_{s^q}(s^q)^{-1} \left( s^q + 1 \right)^{-1} x^{s^q+1} + O\left( x^{\frac{3}{2} - \frac{1}{2m}} \right). \]

Therefore,

\[ \frac{\psi_1(x)}{x} = x + \sum_{q=0}^{2} (-1)^q \sum_{s^q \in S^q_{k}} o_{s^q}(s^q)^{-1} \left( s^q + 1 \right)^{-1} x^{s^q} + O\left( x^{\frac{1}{2} - \frac{1}{2m}} \right) \]

as \( x \to +\infty \).

This completes the proof.

\[ \Box \]

4. Remarks

The error term \( O\left( x^{\frac{3}{2} - \frac{1}{2m}} \right) \), obtained in the generalized sense, is obviously better than the classical one \( O\left( x^{1 - \frac{1}{2m}} \right) \).
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