Kondo signature in heat transfer via a local two-state system

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(Dated: May 11, 2014)

We study the Kondo effect in heat transport via a local two-state system. This system is described by the spin-boson Hamiltonian with Ohmic dissipation, which can be mapped onto the Kondo model with anisotropic exchange coupling. We calculate thermal conductance by the Monte Carlo method based on the exact formula. Thermal conductance has a scaling form \(\kappa = (k_B T_K / h) f(\alpha, T / T_K)\), where \(T_K\) and \(\alpha\) indicate the Kondo temperature and dimensionless coupling strength, respectively. Temperature dependence of conductance is classified by the Kondo temperature as \(\kappa \propto (T / T_K)^{3}\) for \(T \ll T_K\) and \(\kappa \propto (k_B T / \hbar \omega_c)^{2\alpha - 1}\) for \(T \gg T_K\). Similarities to the Kondo signature in electric transport are discussed.

PACS numbers: 44.10.+i, 72.10.Fk, 05.60.Gg

Heat and electric transport are ubiquitous phenomena. Both phenomena have several similarities as well as dissimilarities. Fourier’s law in heat transport corresponds to Ohm’s law in electric transport, and these laws are commonly categorized as diffusive transport. We also note that heat transport shows unique anomaly in low dimensions. Ballistic transport leads to the quantization of conductance in electric transport as well as heat transport. The conductance quantum was measured in mesoscopic electric conduction in 1988, and much later, the version of heat transport was also measured. Recently, the concept of thermal diode has also been discussed, and an experiment has been conducted for demonstrating this. Recent progress in transport studies strongly indicates that heat transport analogue exists for many categories of electric transport.

In this paper, we consider heat transfer between phononic reservoirs via a local two-state system, and aim to clarify the signature of the Kondo effect in heat transport. This setup is analogous to electric transport via quantum dots, which is a typical and simplest example of quantum transport through a zero-dimensional physical object. In quantum-dot systems, the Kondo effect is an interesting and famous phenomenon induced by electron correlation. The Coulomb blockade for electron tunneling is overcome by the formation of the Kondo singlet between a localized electric spin and conduction electrons. Because of this effect, electric conduction is nontrivially enhanced and can eventually reach conductance quantum.

We consider heat transfer via a two-level system modeled by the spin-boson system with Ohmic dissipation. In the equilibrium situation, low-energy physics of this model is understood in terms of Kondo physics through mapping onto the anisotropic Kondo model. Let \(\alpha\) be a dimensionless coupling strength between the spin and bosonic reservoirs in the spin-boson Hamiltonian (exact definition is given below in Eq. (2)). The weak coupling region of \(\alpha < 1\) is mapped onto the antiferromagnetic parameter regime in the Kondo model, where the Kondo singlet is formed between a quantum dot and conduction electrons at sufficiently low temperatures. On the other hand, the region of \(\alpha > 1\) is mapped onto the ferromagnetic parameter regime, in which only a trivial spin-doublet state is realized. At zero temperature, quantum states in the spin-boson model are clearly separated by quantum phase transition of the Kosterlitz-Thouless type at \(\alpha = 1\), where the local two-state forms one bonding state for \(\alpha < 1\) and two degenerate states for \(\alpha > 1\). The Kondo...
temperature, a unique temperature scale characteristic of the Kondo effect, is defined as a function of $\alpha$, and in the regime below the Kondo temperature (referred to as the Kondo regime), the local two-state is strongly and coherently correlated with the phononic environment. See Fig. 11 for schematic explanation.

The spin-boson model is a minimal model that describes molecular junctions [14], a superconducting circuit [12], and a photonic waveguide with a local two-level system [17], etc.; heat transport has also been studied intensively [18–21]. These include thermal rectification effects [19], the cotunneling process [20], and fluctuations in current [21]. From the underlying Kondo physics in the equilibrium situation, it is also of general interest to analyze low-temperature properties in heat transport.

We note that the cotunneling mechanism has been studied [20]. However, so far, no systematic studies have yet been conducted to get the Kondo signature induced by higher-order processes beyond cotunneling. This is the first study that shows the universal aspects arising from the underlying Kondo physics based on the exact formula of heat current. We indicate similarities and dissimilarities of the Kondo signature between heat and electric transport via a zero-dimensional system.

*Model and exact current formula.* — The local two-level system attached to two phononic reservoirs is described by the following spin-Boson Hamiltonian:

$$
H = \frac{\hbar \Delta}{2} \sigma_z + \sum_{\nu=L,R} \sum_{k} \frac{\hbar \sigma_z}{2} \lambda_{\nu k} (b_{\nu k} + b_{\nu k}^\dagger) + \hbar \omega_{\nu k} b_{\nu k}^\dagger b_{\nu k},
$$

where $\Delta$ is the tunneling frequency between the two states, i.e., the up-spin state $|\uparrow\rangle$ and the down-spin state $|\downarrow\rangle$. The operator $\sigma_{\mu} (\mu = x, y, z)$ is the Pauli matrix, and $b_{\nu k}$ is the annihilation operator of phonons with the wavenumber $k$ in the $\nu$th reservoir. The two reservoirs are characterized by the spectral density defined as $I_{\nu}(\omega) = \sum_k \lambda_{\nu k}^2 \delta(\omega - \omega_{\nu k})$. We assume the Ohmic dissipation for both reservoirs as follows:

$$
I_{\nu}(\omega) = 2\hbar \omega \tau \theta(\omega - \omega_{\nu}) \theta(\omega),
$$

where $\alpha_{\nu}$ is the dimensionless coupling strength between the system and the $\nu$th reservoir. The cutoff energy $\hbar \omega_c$ is assumed to be sufficiently large compared to the system’s energy scale. The Hamiltonian [11] can be mapped onto the Kondo Hamiltonian with anisotropic exchange coupling [12]. The in-plane and out-of-plane exchange parameters $J_{\|}$ and $J_{\perp}$ are related to the parameters in the spin-Boson model as $\alpha = [1 - (2/\pi) \arctan(\pi \rho_0 J_{\|}/4)]^2$ and $\Delta = \rho_0 \omega_c J_{\perp}$ respectively, where $\rho_0$ is a density of state (See the supplementary material [12]). The Kondo temperature $T_K$ is defined by renormalization group analysis [12]:

$$
T_K = \left\{ \begin{array}{ll}
(\hbar/k_B) g_0 \Delta (\Delta/\omega_c)^{\alpha/(1-\alpha)} & \text{for } 0 \leq \alpha < 1, \\
0 & \text{for } \alpha \geq 1,
\end{array} \right.
$$

where $k_B$ is the Boltzmann constant. The factor $g_0$ is a nonuniversal constant, and throughout the paper, we take $g_0 = [\Gamma(1-2\alpha)\cos(\pi\alpha)]^{1/2(1-\alpha)}$ with the Gamma function $\Gamma(x)$ [13].

The exact formula of heat current is derived with the standard procedure of the Keldysh formalism. The initial density matrix is prepared as the product form of equilibrium states of the system and left and right reservoirs. The temperatures of the left and right reservoirs are $T_L$ and $T_R$, respectively. The heat current operator from the $\nu$th reservoir into the system is given by

$$
J_{\nu} = i \frac{\sigma_z}{2} \sum_{k} \lambda_{\nu k} \hbar \omega_{\nu k} (-b_{\nu k} + b_{\nu k}^\dagger).
$$

Expressing this with the Keldysh green’s function [18–22], and noting the Gaussian properties for bosonic variables, we derive a formal expression for current [23]:

$$
\langle J_{\nu} \rangle = \frac{\hbar^2}{2} \int_0^\infty d\omega \chi''(\omega) I(\omega) [n_L(\omega) - n_R(\omega)],
$$

where $\gamma = 4\alpha L\alpha R/\alpha^2$ is an asymmetric factor. In the linear response regime, thermal conductance defined by $\kappa = d\langle J_{\nu} \rangle /dT L_{\nu} |_{T_{\nu}=T}$ is given by

$$
\kappa = \frac{k_B \hbar \alpha}{4} \int_0^\infty d\omega S_{\alpha}(\omega) \omega^2 \left[ \frac{\beta \omega / 2}{\sinh(\beta \omega / 2)} \right]^2,
$$

where we substitute the Ohmic spectral density for $\tilde{I}(\omega)$, and $S_{\alpha}(\omega)$ is the spectral function defined as

$$
S_{\alpha}(\omega) = \chi''(\omega)/\omega.
$$

The formulas [13] and [6] are the basis of our calculation. Before discussing the Kondo signature in heat transport, it is helpful to examine the current formula derived by the quantum master equation approach by Segal and Nitzan [19]. By utilizing their expression of current, thermal conductance is obtained as

$$
\kappa_{WC} = \frac{k_B \alpha}{4} \frac{\pi \Delta}{2n(\Delta) + 1} \left[ \frac{\beta \Delta / 2}{\sinh(\beta \Delta / 2)} \right]^2.
$$
data represent results for $\alpha = 0.1, 0.2, 0.3, 0.5, 0.7$ from bottom to top. The solid line is an exact result for the Toulouse point $\alpha = 1/2$. The figure clearly shows scaling form $\kappa(T) \sim (k_B^2 \gamma T_K/h) f(\alpha, T/T_K)$ and the $T^2$-law in the low-temperature regime. The dotted line indicates $\kappa_{WC}$ for $\alpha = 0.1$. For $T \gg T_K$, $\kappa$ depends on $T^{2\alpha-1}$.

where $n(\omega)$ denotes the Bose-Einstein distribution of temperature $T$. This expression is reproduced from the exact formula (9) in the weak coupling limit ($\alpha \to 0$). This is checked by substituting the zeroth order of the spectral function, namely the expression for the isolated system: $S_0(\omega) = \pi \delta(\omega - \Delta)/[\omega(2n(\omega) + 1)]$.

We note that at low temperatures $k_B T \ll \hbar \Delta$, the weak coupling approximation predicts the Schottky-type temperature dependence as $\kappa_{WC} \propto e^{-\Delta/k_B T}$, leading to the exponential suppression of heat current. This property is analogous to the Coulomb-blockade phenomenon in electric conduction, where electric conductance is exponentially suppressed at low temperatures because of the excess electrostatic energy of electrons in a quantum dot. However, as shown below, finite coupling to reservoirs remarkably changes the transport properties showing nontrivial universal properties.

Numerical method.— In the subsequent sections, we focus on thermal conductance. In the exact formula (9), the spectral function $S_0(\omega)$ includes the entire information on the many-body effect. We calculate $S_0(\omega)$ by the Monte Carlo method, as follows. (i) We note that the path-integral representation of the equilibrium partition function for the Hamiltonian (11) is mapped onto a one-dimensional Ising model with long-range interaction (for details, see the supplementary material [15]). (ii) For this Ising model, the Monte Carlo simulation with the Wolff algorithm [26] is performed to obtain the spin-spin correlation, which is equivalent to Matsubara’s green function $G(u) = \langle e^{iH} \sigma_z e^{-iH} \sigma_z \rangle_{eq}$, where $\langle \cdots \rangle_{eq}$ implies an equilibrium average. (iii) From the Fourier transformation of Matsubara’s green function $G(i\xi)$, we calculate $\chi(\omega)$ using numerical analytic continuation $\chi(\omega) = G(i\xi \to \omega + i\delta)$ by the Padé approximation [27, 28].

Kondo signature in thermal conductance.— In Fig. 2 we show temperature dependence of the thermal conductance calculated with the abovementioned numerical procedure. The data indicate results for $\alpha < 1$, which corresponds to the antiferromagnetic coupling regime ($J_1 < 0$) in the mapped Kondo model. The horizontal axis indicates the temperature scaled by the Kondo temperature $T_K$, and the vertical axis indicates the conductance scaled by $k_B^2 \gamma T_K/h$. To confirm numerical accuracy, we note that $\alpha = 1/2$ is an exactly soluble point, called the Toulouse point, where the spectral function $S_{1/2}$ is given by [13]

$$S_{1/2}(\omega) = \text{Im} \left[ \frac{4}{\hbar \pi \omega^2} \frac{k_B T_K}{2 + \beta k_B T_K/(4\pi) - i\beta \hbar \omega/(2\pi)} \psi(z') - \psi(z'') \right],$$

where $\psi(z)$ is the digamma function with the variables $z' = 1/2 + \beta k_B T_K/(4\pi)$ and $z'' = 1/2 + \beta k_B T_K/(4\pi) - i\beta \hbar \omega/(2\pi)$. In Fig. 2 we also show the exact values for the Toulouse point by a solid line. Our numerical results clearly agree with the exact results. Another evidence of accuracy is given by a validation of the Shiba relation [29] for arbitrary coupling strength [13].

Fig. 2 shows that all the data collapse onto one curve for each value of $\alpha$ regardless of the tunneling frequency $\Delta$. This implies emergence of the scaling form in $\kappa$:

$$\kappa(T) \propto (k_B^2 \gamma T_K/h) f(\alpha, T/T_K).$$ (9)

This scaling form is an indication of the Kondo effect in heat transfer. In addition, conductance is proportional to $T^3$ at sufficiently low temperatures $T \ll T_K$. At high temperatures $T \gg T_K$, conductance depends on temperature as $T^{2\alpha-1}$, which intriguingly implies that the coupling strength determines monotonicity of temperature dependence. These numerical findings are the first main results of this study.

For comparison, $\kappa_{WC}$ given by Eq. (8) is shown for $\alpha = 0.1$ in Fig. 2 by a dotted line. Although $\kappa_{WC}$ is quantitatively good around the Kondo temperature, it deviates from the numerical results at lower temperatures, showing exponential reduction. Enhanced heat transport from $\kappa_{WC}$ is analogous to enhanced electronic transport via quantum dots in the Kondo regime. However, note that conductance does not reach the universal quantum of thermal conductance, $g(T) = \pi k_B^3 T/(6h)$, which is linear in $T$. Heat transfer is generally sensitive to the scattering mechanism, and hence conductance tends to be reduced. This aspect is dissimilar to the Kondo signature in electric transport, where electric conductance can reach the conductance quantum.

To explain the universal $T^3$-law for $T \ll T_K$, we first consider the spectral function at the Toulouse point. The spectral function $S_{1/2}(\omega)$ is expanded with respect to temperature and frequency as in the form $S_{1/2}(\omega) =
Theoretical analysis and numerical simulations reveal the strong coupling limit where conductance is dominated by the Kondo effect. For small temperatures, conductance is given by a series expansion in the coupling strength. In the high-temperature regime, conductance decreases exponentially as the temperature increases. This behavior is quantified by the scaling form

\[ S_\alpha(0) \sim (k_B T_K)^{2(1 - \alpha)} / h, \]

where \( S_\alpha(0) \) is the zeroth-order conductance, \( k_B \) is the Boltzmann constant, and \( T_K \) is the Kondo temperature. The prefactor \( C \) weakly depends on the coupling strength. This explains the numerical observation on temperature dependence in the high-temperature region as shown in Fig. 4. Conductance is found to be maximum around \( T = T_K / 2 \), indicating a significant coupling strength. When the temperature is much higher than the Kondo temperature, the spectral function is insensitive to temperature, leading to an incoherent tunneling approximation. Clear exponential suppression is observed as the temperature increases, consistent with the theoretical predictions.
Acknowledgement
We are grateful to Mikio Eto, Teemu Ojansen, Rui Sakano and Dvira Segal for useful comments. We also thank Ulrich Weiss for showing the derivation of $g_a$. KS was supported by MEXT (23740289). TK was supported by JSPS KAKENHI Grant Number 24540316.

[1] F. Bonetto, J. L. Lebowitz and L. Rey-Bellet, in Mathematical Physics 2000, edited by A. Fokas et. al. (Imperial College Press, London, 2000), p. 128.
[2] S. Lepri, R. Livi and A. Politi, Phys. Rep. 377, 1 (2003).
[3] A. Dhar, Adv. Phys. 57, 457 (2008).
[4] S. Datta, Quantum Transport: Atom to Transistor, (Cambridge, 2005).
[5] L. G. C. Rego and G. Kirczenow, Phys. Rev. Lett. 81, 232 (1998).
[6] B. J. Wees et al., Phys. Rev. Lett. 60, 848 (1988).
[7] K. Schwab et al., Nature (London) 404, 974 (2000); H.-Y. Chiu et al., Phys. Rev. Lett. 95, 226101 (2005).
[8] N. Li, J. Ren, L. Wang, G. Zhang, P. Hänggi and B. Li, Rev. Mod. Phys. 84, 1045 (2012).
[9] C. W. Chang, D. Okawa, A. Majumdar and A. Zettl, Science 314, 1121 (2006).
[10] D. Goldhaber-Gordon et al., Nature 391, 156 (1998); W. van der Wiel et al., Science 289, 2105 (2000).
[11] A. C. Hewson, The Kondo Problem to Heavy Fermions, (Cambridge University Press, Cambridge, 1997).
[12] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[13] U. Weiss, Quantum Dissipative Systems, (World Scientific, Singapore, 1999).
[14] F. Guinea, V. Hakim and A. Muramatsu, Phys. Rev. B 32, 4410 (1985); F. Guinea, Phys. Rev. B 32, 4486 (1985).
[15] In the supplementary material, we present details on mapping onto the anisotropic Kondo model, calculation of heat current from the quantum master equation approach, relation to an Ising model with long-range interaction, the Shiba relation, and the cotunneling formula.
[16] A. Nitzan, Chemical Dynamics in Condensed Phases: Relaxation, Transfer, and Reactions in Condensed Molecular Systems, (Oxford, 2006).
[17] K. Le Hur, Phys. Rev. B 85, 145056 (2012).
[18] K. A. Melzhanin, M. Thoss and H. Wang, J. Chem. Phys. 133 084503 (2010).
[19] D. Segal and A. Nitzan, Phys. Rev. Lett. 94, 034301 (2005); J. Chem. Phys. 122, 194704 (2005).
[20] T. Ruokola and T. Ojansen, Phys. Rev. B 83, 045417 (2011).
[21] L. Nicolin and D. Segal, Phys. Rev. B 84, 161414 (2011).
[22] J.-S. Wang, J. Wang and N. Zeng, Phys. Rev. B 74, 033408 (2006).
[23] K. Saito, Europhys Lett. 83, 50006 (2008).
[24] Y. Meir and N. S. Wingreen, Phys. Rev. Lett. 68, 2512 (1992).
[25] T. Ojansen and A. -P. Jauho, Phys. Rev. Lett. 100, 155902 (2008).
[26] U. Wolff, Phys. Rev. Lett. 62, 361 (1989).
[27] K. Völker, Phys. Rev. B 58, 1862 (1998).
[28] C. Brezinski and Z. M. Redivo, Extrapolation Methods. Theory and Practice. (North-Holland, 1991).
[29] H. Shiba, Prog. Theor. Phys. 54, 967 (1975).
[30] The $T^3$ law in the Kondo regime differs from a recent result by renormalization analysis based on cotunneling picture $\alpha \sim T^{1-2\alpha}$ [24]. We note that only cotunneling process also leads to the $T^3$ law, but fails in reproducing a correct energy scale. Detailed discussion including correction of renormalization procedure is given in the supplementary material [15].
[31] H. Grabert and U. Weiss, Phys. Rev. Lett. 54, 1605 (1985).
[32] M. P. A. Fisher and A. T. Dorsey, Phys. Rev. Lett. 54, 1609 (1985).
[33] K. A. Melzhanin, H. Wang and M. Thoss, Chem. Phys. Lett. 460, 325 (2008).
[34] L. Nicolin and D. Segal, J. Chem. Phys. 135, 164106 (2011).
[35] T. Kato and K. Saito, in preparation.
[36] D. Segal, arXiv:1303.5815.
[37] J. Ren, P. Hänggi and B. Li, Phys. Rev. Lett. 104, 170601 (2010). T. Chen, X.B. Wang and J. Ren, Phys. Rev. B 87, 144303 (2013).

Anisotropic Kondo model.— The Hamiltonian of the anisotropic Kondo model is given by

\[
H_{\text{AK}} = \sum_{k} \sum_{\sigma=\uparrow,\downarrow} \epsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + J_\perp \sum_{k,k'} (c_{k\uparrow}^\dagger c_{k'\downarrow} S^- + c_{k\downarrow}^\dagger c_{k'\uparrow} S^+) \\
+ J_\parallel \sum_{k,k'} \frac{1}{2} (c_{k\uparrow}^\dagger c_{k'\downarrow} - c_{k\downarrow}^\dagger c_{k'\uparrow}) S_z, \tag{12}
\]

where $c_{k,\sigma}$ is an annihilation operator of the fermion of the wave number $k$ and spin $\sigma$. Operators $S_{\mu}(\mu = x, y, z)$ is the spin operator by localized electron inside quantum-dot and $S^\pm = S_x \pm i S_y$. The parameters $J_{\parallel}$ and $J_\perp$, respectively, represent the in-plane and out-of-plane exchange parameters. When $J_{\parallel} > 0$, the Hamiltonian describes the antiferromagnetic Kondo (AF-Kondo) model, while $J_{\parallel} < 0$ corresponds to the ferromagnetic Kondo (F-Kondo) model. The correspondence between the anisotropic Kondo model and the spin-boson model via
obtained from the quantum master equation: We first express Segal and Nitzan’s result [3], which is a Hamiltonian, i.e.,

$$\alpha = \left[1 - \frac{2}{\pi} \arctan(\pi \rho_0 J_{\parallel}/4)\right]^2, \quad (13)$$

$$\Delta = \rho_0 \omega_c J_{\perp}, \quad (14)$$

where $\rho_0$ is the density of the state of conduction electrons. These relations indicate that the regime $-\infty < \rho J_{\parallel} < \infty$ is mapped onto $4 > \alpha > 0$. The phase transition point $\alpha = 1$ in the spin-boson model corresponds to the transition point between the AF-Kondo and F-Kondo regimes (see Fig. 1 in the main text).

**Exact formula of heat transfer.** — The formal current formula is given by

$$\langle J_\nu \rangle = \frac{\hbar^2}{2} \int_0^\infty d\omega \omega [\chi''(\omega) I_\nu(\omega)n_\nu(\omega) - iC(\omega)I_\nu(\omega)/2]. \quad (15)$$

We define a quantity

$$r_\nu = \frac{\int_0^\infty d\omega \hbar \omega C(\omega) I_\nu(\omega)}{\sum_{\nu' = -L, R} \int_0^\infty d\omega \hbar \omega C(\omega) I_{\nu'}(\omega)}. \quad (16)$$

From the conservation law $\langle J_L \rangle + \langle J_R \rangle = 0$, the current formula is rewritten as

$$\langle J_L \rangle = r_R \langle J_R \rangle - r_L \langle J_L \rangle = \frac{\hbar}{2} \int_0^\infty d\omega \hbar \omega \chi''(\omega) \times \left[r_R I_L(\omega)n_L(\hbar\omega) - r_L I_R(\omega)n_R(\hbar\omega)\right]. \quad (17)$$

Note that if $\hat{I}_L(\omega) = \hat{I}_R(\omega) = \hat{I}(\omega)$, $r_\nu$ is simplified as $r_\nu = a_\nu/(a_L + a_R)$. Hence, the above formula is reduced to (5) in the main text.

**Weak coupling limit ($\alpha \to 0$).** — Let $H_x$ be the system’s Hamiltonian, i.e., $H_x = \hbar \Delta \sigma_z/2$. It is diagonalized as

$$e^{i\pi \sigma_y/4} H_x e^{-i\pi \sigma_y/4} = \hbar \Delta \sigma_z/2. \quad (18)$$

We first express Segal and Nitzan’s result [3], which is obtained from the quantum master equation:

$$\langle J \rangle = \frac{\hbar \pi}{2} \frac{\Delta I_L(\Delta) I_R(\Delta) [n_L(\Delta) - n_R(\Delta)]}{I_L(\Delta)(2n_L(\Delta) + 1) + I_R(\Delta)(2n_R(\Delta) + 1)}. \quad (19)$$

Note that the above expression is the first-order expression with respect to the coupling strength between the system and reservoirs. Then, we find the zeroth-order expression for $\chi''$, which implies the response function at the weak coupling limit. For the case of thermal conductance, we use the equilibrium value of $\chi''$:

$$\chi''(\omega) \rightarrow \pi/[\hbar(2n(\omega) + 1)] \delta(\omega - \Delta). \quad (20)$$

We note that contribution arises only from $\omega = \Delta$, which immediately implies $r_\nu = I_\nu(\Delta)/(I_L(\Delta) + I_R(\Delta))$.

To obtain the expression at far-from-equilibrium [19], we replace the temperature in Eq. (19) with an effective temperature at the weak coupling limit between the reservoirs:

$$\chi''(\omega) \rightarrow \pi/[\hbar(2n_{\text{eff}}(\omega) + 1)] \delta(\omega - \Delta). \quad (21)$$

In order to obtain the effective temperature, we use the quantum master equation. We write the master equation in the representation diagonalizing the spin Hamiltonian:

$$\dot{\rho} = i [\rho, \Delta \sigma_z/2] - \sum_{\nu = L, R} \left\{[X, R_\nu \rho] + [X, R_\nu \rho]^{-1}\right\}, \quad (22)$$

$$X = -\sigma_x, \quad (23)$$

where the matrices $R$ are given by the $2 \times 2$ matrix:

$$R_\nu = -I_\nu(\Delta) \begin{pmatrix} 0 & n_\nu(\Delta) \\ -n_\nu(-\Delta) & 0 \end{pmatrix}. \quad (24)$$

In case of a single reservoir, the equilibrium is guaranteed by the detailed balance for the matrix $R_\nu$:

$$R_{\nu,1,2}/R_{\nu,2,1} = n_\nu(\Delta)/[-n_\nu(-\Delta)] = e^{-\beta_\nu \hbar \Delta}. \quad (25)$$

Then, by analyzing the effective detailed balance for the nonequilibrium situation with two reservoirs, the effective temperature is obtained as

$$e^{-\beta_\text{eff} \hbar \Delta} = \frac{I_L(\Delta)n_L(\Delta) + I_R(\Delta)n_R(\Delta)}{-I_L(\Delta)n_L(-\Delta) - I_R(\Delta)n_R(-\Delta)}. \quad (26)$$

From this, a simple manipulation yields

$$\chi''(\omega) = \frac{I_L(\Delta) + I_R(\Delta)}{I_L(\Delta)(2n_L(\Delta) + 1) + I_R(\Delta)(2n_R(\Delta) + 1)} \times \pi/\hbar \delta(\omega - \Delta). \quad (27)$$

Combining the ratio $r_\nu = I_\nu(\Delta)/(I_L(\Delta) + I_R(\Delta))$, one gets the result [19].

**Ising model with long-range interaction.** — The spin-boson model can be mapped onto an Ising model with long-range exchange interaction. We start with the derivation of the partition function for the Ising model with long-range exchange interaction. For simplicity, we set $\hbar = 1$. We first divide the Hamiltonian into two parts and create the following definition:

$$H = \Delta \sigma_z/2 + H_z, \quad (28)$$

$$H_z = \frac{\sigma_z}{2} \sum_{\nu = L, R, k} \lambda_{\nu k}(b_{\nu k} + b_{\nu k}^\dagger)$$

$$+ \sum_{\nu = L, R, k} \omega_{\nu k} b_{\nu k}^\dagger b_{\nu k}, \quad (29)$$

$$\bar{\sigma}_x(u) := e^{iH_z u} \sigma_x e^{-iH_z u}. \quad (30)$$

Then, the partition function $Z = \text{Tr} e^{-\beta H}$ is divided into two parts: $Z = Z_+ + Z_-$ as

$$Z = \langle \pm | \text{Tr}_{\text{boson}} e^{-\beta H} | \pm \rangle, \quad (31)$$
\[ \frac{\Delta/\omega_c}{\pi \chi_m^2/2} \]

where \(|\pm\rangle\) is the eigenstate of \(\sigma_z\), i.e., \(\sigma_z|\pm\rangle = \pm 1|\pm\rangle\), and \(\text{Tr}_{\text{boson}}\) implies the trace with respect to boson’s degrees of freedom. We expand \(Z_+\) as

\[
Z_+ = \text{Tr}_{\text{boson}} \left\{ (+)|e^{-\beta H_z} e^{-\beta \int_0^\beta du \Delta \sigma_x(u)/2}|+\rangle \right\} = \sum_{n=0}^{\infty} \text{Tr}_{\text{boson}} \left\{ (+)|e^{-\beta H_z} \int_0^\beta d\tau_1 \cdots \int_0^{\tau_{2n-1}-\tau_c} d\tau_2n \right. \\
\times \left. \left( \frac{\Delta}{2} \right)^{2n} \tilde{\sigma}_x(\tau_1) \cdots \tilde{\sigma}_x(\tau_{2n}) |+\rangle \right\},
\]

where \(\tau_c = 1/\omega_c\). By taking trace with respect to the boson part, we obtain a formal expression for \(Z_+\) [6, 7]:

\[
Z_+ = Z_0 \sum_{n=0}^{\infty} \left( \frac{\Delta \tau_c}{2} \right)^{2n} \int_0^{\beta} \frac{d\tau_1}{\tau_c} \cdots \int_0^{\tau_{2n-1}-\tau_c} \frac{d\tau_2n}{\tau_c} \\
\times \exp \left\{ 2\alpha \sum_{i<j} (-1)^{i+j} \ln \left( \frac{\beta}{\pi \tau_c} \sin(\pi(\tau_i - \tau_j)/\beta) \right) \right\},
\]

where \(Z_0\) is a partition function for the free boson part, and \(Z_-\) takes the same form. Finally, we connect this to the kink dynamics in the Ising model [8, 9]. The equivalent Ising dynamics is obtained when the imaginary time is considered to be a position of spin. By discretizing the space, the equivalent Ising Hamiltonian \(H_I\) reads

\[
\beta I H_I = -J_{nn} \sum_{i=1}^N \sigma_i \sigma_{i+1} - \frac{\alpha}{2} \sum_{j<i} (\pi/N)^2 \sigma_i \sigma_j \sin^2 \left[ \frac{\pi(j-i)/N}{\beta} \right],
\]

where \(\beta_I\) is the inverse temperature in the mapped Ising model. The nearest neighbor interaction coefficient \(J_{nn}\) is given by \(J_{nn} = -\alpha(1 + \gamma) - \ln(\Delta \tau_c/2)\), where \(\gamma\) is Euler’s constant. The lattice number \(N\) is determined by \(N = \beta \omega_c\), and hence the Monte Carlo simulation is possible only for \(\beta \omega_c \gg 1\).

The Shiba relation and the cotunneling formula.— The Shiba relation [7], which is a general identity equation valid at low temperatures well below the Kondo temperature, is written by

\[
S_\alpha(0) = \frac{\alpha \pi \chi_m^2}{2},
\]

where \(\chi_m\) is the susceptibility of the local spin. For the numerical validation, we calculate \(S_\alpha(0)\) and \(\chi_m\) at temperatures much lower than \(T_K\), and show the ratio \(S_\alpha(0)/(\alpha \pi \chi_m^2/2)\) as a function of \(\alpha\) for several sets of \((\Delta, T)\) in Fig. 5. The figure clearly demonstrates the validity of the Shiba relation, implying one of the evidences on the reliability of our numerical calculations.

By substituting the Shiba relation [7] into the thermal conductance (Eq. (6) in the main text), we obtain an approximation which is valid at low temperatures \((T \ll T_K)\)

\[
\kappa \sim \frac{\pi k_B^2 \chi_m^2}{8} \int_0^{\omega_c} d\omega I_L(\omega) I_R(\omega) \left[ \frac{\beta \omega/2}{\sinh(\beta \omega/2)} \right]^2.
\]

For weak coupling \((\alpha_L, \alpha_R \ll 1)\), the susceptibility \(\chi_m\) is approximately given by \(2/k_B \Delta\), which is the one for the isolated two-state system, and one can recover the cotunneling formula derived in Ref. [8]. For arbitrary coupling, however, the cotunneling formula of Ref. [8] becomes wrong, and needs consideration of strong renormalization by the Kondo effect. Actually, since the energy scale is renormalized from \(h \Delta\) to \(k_B T_K\), one should take \(\chi_m \sim 2/(k_B T_K)\) to obtain the correct result of thermal conductance at low temperatures \((T \ll T_K)\).

Finally, we point out that the same cotunneling formula can be derived even when the local system is a harmonic oscillator by replacing \(\Delta\) with the frequency of the local oscillator. This indicates that the low-temperature heat transport of the present model is governed by the fixed-point Hamiltonian of a local “harmonic oscillator” after nontrivial renormalization by the Kondo physics.

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[1] F. Guinea, V. Hakim and A. Muramatsu, Phys. Rev. B 32, 4410 (1985); F. Guinea, Phys. Rev. B 32, 4486 (1985).
[2] T. A. Costi, Phys. Rev. Lett. 80, 1038 (1998).
[3] D. Segal and A. Nitzan, Phys. Rev. Lett. 94, 034301 (2005); J. Chem. Phys. 122, 194704 (2005).
[4] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[5] K. Völlker, Phys. Rev. B 58, 1862 (1998).
[6] J. Cardy, J. Phys. A 14, 1407 (1981).
[7] H. Shiba, Prog. Theor. Phys. 54, 967 (1975).
[8] T. Ruokola and T. Ojanen, Phys. Rev. B 83, 045417 (2011).