Horizon area–angular momentum inequality in higher-dimensional spacetimes

Stefan Hollands

School of Mathematics, Cardiff University, Cardiff, UK

E-mail: HollandsS@Cardiff.ac.uk

Received 31 October 2011
Published 24 February 2012
Online at stacks.iop.org/CQG/29/065006

Abstract

We consider \( n \)-dimensional spacetimes which are axisymmetric—but not necessarily stationary—in the sense of having isometry group \( U(1)^{n-3} \) and which satisfy the Einstein equations with a non-negative cosmological constant. We show that any black hole horizon must have area \( A \geq 8\pi |J_+ J_-|^{\frac{1}{2}} \), where \( J_\pm \) are distinguished components of the angular momentum corresponding to linear combinations of the rotational Killing fields that vanish somewhere on the horizon. In the case of \( n = 4 \), where there is only one angular momentum component \( J_+ = J_- \), we recover an inequality of Acena et al. Our work can hence be viewed as a generalization of this result to higher dimensions. In the case of \( n = 5 \) with horizon of topology \( S^1 \times S^2 \), the quantities \( J_\pm \) are the same angular momentum component (in the \( S^2 \)-direction). In the case of \( n = 5 \) with horizon topology \( S^3 \), the quantities \( J_+ \), \( J_- \) are the distinct components of the angular momentum. We also show that, in all dimensions, the inequality is saturated if the metric is a so-called near horizon geometry. Our argument is entirely quasi-local, and hence also applies e.g. to any stably outer marginally trapped surface.

PACS numbers: 04.70.−s, 04.50.−h, 97.60.Lf

1. Statement of the result and basic definitions

In [1–3], a remarkable inequality was shown to hold between the area \( A \) of a black hole horizon (or more generally, stably outer marginally trapped horizon) and the angular momentum \( J \). The inequality holds in any axisymmetric—but not necessarily stationary—spacetime of dimension \( n = 4 \) satisfying the vacuum Einstein equations, possibly including a non-negative cosmological constant, and it states that

\[
A \geq 8\pi |J|.
\]  

(1.1)

Although it is not clear whether such an inequality will hold in general, non-axisymmetric spacetimes, it is still of considerable interest, because it is a universal bound for a very wide class of—possibly highly dynamical—spacetimes, which are very difficult to analyse within any existing scheme. Furthermore, it gives a new perspective upon the Kerr solution, which is in this class and, in addition, stationary. In fact, it was shown in [3] that the unique vacuum
spacetime saturating the bound is the so-called extremal Kerr near horizon geometry, which can be viewed as the close-up view of the geometry near the horizon of an extremal Kerr black hole.

The purpose of this paper is to provide an analogue of this statement in higher dimensions. For this, we will consider a subclass of \( n \)-dimensional spacetimes with a ‘comparable amount of symmetry’ as axisymmetric spacetimes in \( n = 4 \) spacetime dimensions. The class we consider consists of those spacetimes \((\mathcal{M}, g)\) which satisfy the vacuum Einstein equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}, \quad \Lambda \geq 0,
\]

with a non-negative cosmological constant, and which additionally admit an action of \( U(1)^{n-3} \) by isometries. We refer to these spacetimes again as axisymmetric. Thus, loosely speaking, the metric \( g \) can depend non-trivially on three coordinates, one of which is time. The commuting Killing vector fields generating the isometry group are denoted by \( \xi_i, i = 1, \ldots, n-3 \). One then defines

\[
J_i = \frac{1}{8\pi} \int \ast d\xi_i,
\]

where the integral is over any closed \((n-2)\)-dimensional surface co-bordant to \( \mathcal{B} \). If the spacetime is asymptotically Kaluza–Klein (or asymptotically flat in \( n = 4, 5 \)), then this surface can be taken to be at infinity, and the \( J_i \) then coincide with the usual ADM-type conserved quantities with the physical interpretation of angular momenta. Although our analysis is entirely ‘quasi-local’, i.e. involves only the geometry in a vicinity of \( \mathcal{B} \), not the asymptotics of the spacetime, one of course has this case in mind in physical situations.

We next explain what we mean by a stably outermost (marginally) trapped surface (‘horizon’). Consider an \((n - 1)\)-dimensional embedded null hypersurface \( \mathcal{H} \) ruled by affinely parameterized null geodesics with future directed tangent \( n \), and expansion \( \theta_n \) on \( \mathcal{H} \). If we have a cross section \( \mathcal{B} \) of \( \mathcal{H} \) transverse to these geodesics, we can define a second future directed null vector \( l \), perpendicular to \( \mathcal{B} \), having the property that \( g(n, l) = -1 \) on \( \mathcal{B} \). For \( \mathcal{B} \) to be a stably outermost (marginally) trapped surface it is required that

\[
\theta_n = 0, \quad \text{and} \quad \mathcal{L} \theta_n \leq 0, \quad \text{on} \ \mathcal{B},
\]

and it is required that \( \mathcal{B} \) is a compact closed submanifold of dimension \( n - 2 \), whose area is denoted \( A \). These conditions express that, on \( \mathcal{B} \), light rays tangent to \( \mathcal{H} \) are non-focussing, and become expanding ‘slightly outside of \( \mathcal{H} \)’, and contracting ‘slightly inside of \( \mathcal{H} \)’, see the following picture.
A special case of this setup is furnished by the horizon of a stationary black hole, but we emphasize that our spacetimes are not assumed to be stationary in general. Our proof still works if the first condition is relaxed to \( \theta_1 \geq 0 \) provided we additionally have \( \theta_1 \leq 0 \), but it would not work for a negative cosmological constant \( \Lambda < 0 \).

The cross section \( \mathcal{B} \) can be chosen so that it is invariant under the action of \( G = U(1)^{(n-3)} = T^{n-3} \), i.e. that the mutually commuting Killing vectors \( \xi_i \) are all tangent to \( \mathcal{B} \). Thus, \( \mathcal{B} \) is a compact \((n-2)\)-dimensional manifold with an action of an \((n-3)\)-dimensional torus. It is not too difficult to classify such actions and \( \mathcal{B} \)'s topologically \cite{4}; the only possibilities are as follows:

\[
\mathcal{B} \cong \begin{cases} 
S^3 \times T^{n-5}, \\
S^2 \times T^{n-4}, \\
L(p, q) \times T^{n-5}, \\
T^{n-2},
\end{cases}
\tag{1.4}
\]

The symbol \( L(p, q) \), with \( p, q \) mutually being prime integers, denotes a lens space. Furthermore, the orbit space \( \mathcal{B} : = \mathcal{B}/G \), consists of a closed interval \( \mathcal{B} \cong [-1, +1] \) in the first three cases, whereas it is \( \mathcal{B} \cong T^1 = S^1 \) in the last case. The last topology type is actually inconsistent with the the vacuum field equations and our geometric conditions on \( \mathcal{B} \).

In fact, as shown in \cite{6, 7}, these conditions imply that \( \mathcal{B} \) can carry off metric of positive scalar curvature, which \( T^{n-2} \) cannot. Thus, we only need to focus on the first three topology types, in which case we denote by \( x \in [-1, +1] \) a coordinate parameterizing the orbit space. One then shows \cite{4, 5} that there are integer vectors \( a_{\pm} \in \mathbb{Z}^{n-3} \) such that

\[
a_{\pm}^i \xi_i \rightarrow 0, \quad \text{at } x = \pm 1.
\tag{1.5}
\]

Stated differently, the Gram matrix,

\[
f_{ij} = g(\xi_i, \xi_j),
\tag{1.6}
\]

is non-singular (positive definite) in the interior of the interval and it has a one-dimensional null space at each of the two end points, spanned by, respectively, \( a_{\pm}^i \):

\[
f_{ij} (x) a_{\pm}^i \rightarrow 0 \quad \text{at } x = \pm 1.
\tag{1.7}
\]

The integers \( a_{\pm}^i \), which may without loss of generality be assumed to satisfy \( \text{g.c.d.}(a_{\pm}^i) = 1 = \text{g.c.d.}(a_{\pm}^j) \), determine the topology of \( \mathcal{B} \) (i.e. which of the first three cases we are in). Up to a globally defined redefinition\(^1\) of the Killing fields of the form

\[
\xi_i \mapsto \sum A_{pj}^i \xi_i, \quad A \in \text{SL}(\mathbb{Z}, n-3),
\tag{1.8}
\]

we have

\[
a_{+} = (1, 0, 0, \ldots, 0), \quad a_{-} = (q, p, 0, \ldots, 0), \quad p, q \in \mathbb{Z}, \quad \text{g.c.d.}(p, q) = 1.
\tag{1.9}
\]

It is then simple to see that the topology of \( \mathcal{B} \) is, respectively \cite{4, 5}

\[
\mathcal{B} \cong \begin{cases} 
S^3 \times T^{n-5} \quad \text{if } p = \pm 1, q = 0, \\
S^2 \times T^{n-4} \quad \text{if } p = 0, \quad q = \pm 1, \\
L(p, q) \times T^{n-5} \quad \text{otherwise},
\end{cases}
\tag{1.10}
\]

We are now in a position to state our inequality.

**Theorem 1.** Let \( (\mathcal{M}, g) \) be a spacetime satisfying the vacuum Einstein equations with a cosmological constant \( \Lambda \geq 0 \) having isometry group of at least \( T^{n-3} \). Define

\[
J_{\pm} := J_{a_{\pm}^i},
\tag{1.11}
\]

\(^1\) This corresponds to an automorphism of \( G = T^{n-3} \) and is hence merely a redefinition of the group action on \( \mathcal{M} \).
Then:

1. The area of any stably outer marginally trapped surface (e.g. event horizon cross section of a black hole) satisfies

\[ A \geq 8\pi |J_+ J_-|^2. \]  

(1.12)

2. Furthermore, if \( \Lambda = 0 \), and if \((\mathcal{M}, g)\) is a ‘near horizon geometry’, then the inequality is saturated. Conversely, if the inequality is saturated, then the tensor fields \( \alpha, \beta, \gamma \) determining the induced geometry on \( \mathcal{B} \) (see equation (3.1)) are equal to those of a near horizon geometry.

The near horizon metrics referred to in (2) have isometry group \( O(2, 1) \times G \) and were given explicitly in [8]. Their definition and properties are recalled in the appendix for convenience.

The proof of the theorem is given in section 3. It consists of the following steps: first, the stably outer marginally trapped condition on \( \mathcal{B} \) is used in combination with Einstein’s equations to derive an inequality between various geometric quantities on \( \mathcal{B} \), which are integrated over \( \mathcal{B} \) against a testfunction, see lemma 1. A specific choice of this testfunction is then made in order that the inequality takes a particularly simple form (see lemma 2) in terms of the harmonic energy of a map into the coset \( SL(n-2, \mathbb{R})/O(n-2, \mathbb{R}) \). The inequality can be stated as saying that the area is greater than or equal to a certain functional. By minimizing this functional over all possible maps (see lemma 3), we then obtain the desired inequality stated in (1) of theorem 1. The rigidity statement (2) follows from the same reasoning.

2. Special cases

Before we give the proof, we consider various special cases of the theorem for the sake of illustration.

1. In \( n = 4 \) dimensions, we evidently only have one angular momentum component \( J = J_\perp \), so the inequality reduces to that of [1–3], namely (1.1). In this sense, our results generalize those in four dimensions.

2. In \( n = 5 \) dimensions and topology type \( \mathcal{B} \cong S^1 \), the quantities \( J_+ , J_- \) correspond to different angular momentum components. By making a suitable redefinition of \( \xi_1 , \xi_2 \) as in (1.8), one can achieve that \( J_+ = J_1 \), and \( J_- = J_2 \), so the inequality becomes

\[ A \geq 8\pi |J_1 J_2|^2. \]  

(2.1)

An illustrative example is provided by the Myers–Perry black hole [9]. The area and angular momenta of this solution can be written in parametric form as

\[ A = 2\pi^2 r_0 \left( \frac{r^2_0 + a^2_1}{r^2_0} \right) \left( \frac{r^2_0 + a^2_2}{r^2_0} \right), \quad J_1 = \frac{(r^2_0 + a^2_1)(r^2_0 + a^2_2)\pi}{4r^2_0} a_i, \]  

(2.2)

where \( r_0 > 0 \) is the location of the horizon in standard coordinates, related to the surface gravity by \( 0 \leq \kappa = (r^2_0 - a^2_1 a^2_2) [r_0 (r^2_0 + a^2_1)(r^2_0 + a^2_2)]^{-1} \). One verifies explicitly that these expressions are compatible with the bound (2.1), with equality for the extremal black hole \( (r^2_0 = |a_1 a_2|) \).

3. In \( n = 5 \) dimensions and for the black ring topology type \( \mathcal{B} \cong S^2 \times S^1 \), \( J_+ = J_- = J_1 \) is the angular momentum component in the \( S^2 \)-direction, and the inequality reduces to

\[ A \geq 8\pi |J_1|. \]  

(2.3)

In particular, the lower bound is independent of the angular momentum \( J_2 \) in the \( S^1 \)-direction so that the area remains bounded below no matter how large \( J_2 \) becomes. An
The first step of the proof is to take advantage of the stably outer marginally trapped condition on $\mathcal{B}$. In this case, the area and $S^2$-angular momentum are given in parametric form by

$$A = \frac{32\pi^2 k^3 (1 + \lambda + v) \lambda}{(y_0 - 1/y_0)(1 - v)^2}, \quad J_1 = \frac{4\pi k^3 \lambda \sqrt{v} \sqrt{(1 + v)^2 - \frac{\lambda^2}{(1 - v)^2}}}{1 - \lambda + v},$$

(2.4)

where $0 \leq v < 1$, $2\sqrt{v} \leq \lambda < 1 + v$ and $y_0 = (2v)^{-1}(-\lambda + \sqrt{\lambda^2 - 4v})$ is the coordinate location of the horizon. (Here we refer to the coordinates and notations of [11].) Some algebra confirms that these expressions are indeed compatible with the inequality (2.3), with equality for the extremal black ring ($\lambda = 2\sqrt{v}$).

(4) For $n = 5$ dimensions and the lens space topology $\mathcal{B} \cong L(p, q)$, we may achieve that $J_+ = J_1$ and $J_- = qJ_1 + pJ_2$ by a redefinition of the Killing fields as in equation (1.8). The inequality then states that

$$A \geq 8\pi \left|qJ_1^2 + pJ_1J_2\right|^\frac{1}{2}. \quad (2.5)$$

We are not aware of the existence of regular stationary black lenses, but it is possible to construct initial data for a dynamical asymptotically Kaluza–Klein (or asymptotically flat) axisymmetric spacetime having a stable outer trapped surface $\mathcal{B}$ of lens space topology. Our inequality would hence apply to such a spacetime.

3. Proof of theorem 1

The first step of the proof is to take advantage of the stably outer marginally trapped condition on $\mathcal{B}$, in combination with the field equations, $R_{\mu\nu} = \frac{1}{8\pi} \Lambda g_{\mu\nu}$. There are several well known and essentially equivalent ways to do this, see e.g. [12–14] and many other references. Here we use a method [6] based on a special set of coordinates near $\mathcal{N}$. These ‘Gaussian null coordinates’ $u, r, y^a$ are defined as follows [15, 16]. First, we choose arbitrarily local2 coordinates $y^a$ on $\mathcal{B}$, and we Lie-transport them along the flow of $n$ to other places on $\mathcal{N}$, denoting $u$ the affine parameter, and by $\mathcal{B}(u)$ the transported cross sections. Then, at each point of each $\mathcal{B}(u)$ we shoot off transversally null geodesics tangent to $l$ with affine parameter $r$, where $g(n, l) = -1$. Then by definition $n = \partial/\partial u$, $l = \partial/\partial r$, and it can be shown that the metric then takes the Gaussian null form

$$g = -2du(dr - r^2\alpha du - r\beta_a dy^a) + \gamma_{ab} dy^a dy^b, \quad (3.1)$$

where the function $\alpha$, the 1-form $\beta = \beta_a dy^a$, and the tensor field $\gamma = \gamma_{ab} dy^a dy^b$ are invariantly defined tensor fields on each of the copies $\mathcal{B}(u, r)$ of $\mathcal{B} = \mathcal{B}(0,0)$ of constant $r, u$. The horizon $\mathcal{N}$ is located by definition at $r = 0$. The Ricci tensor in these coordinates is given e.g. in [16]. The $ab$-components of the field equations (see equation (82) in [16]) then give

$$R(\gamma) - \frac{1}{2} \beta_a\beta^a - D_a\beta^a = -2 \theta_\alpha \theta_\beta - 2 \mathcal{L}\theta_\alpha + 2 \Lambda \geq 0, \quad (3.2)$$

where $R(\gamma)$ and $D$ are the scalar curvature and connection of $\gamma$ on $\mathcal{B}$ and where in the last step we used the conditions $\theta_\alpha = 0$, $\mathcal{L}\theta_\alpha \leq 0$, $\Lambda \geq 0$. (The conditions $\theta_\alpha \geq 0$, $\theta_\beta \leq 0$, $\mathcal{L}\theta_\alpha \leq 0$, $\Lambda \geq 0$ would clearly also be sufficient.)

The next step is to use the fact that $g$, and hence $u, r, \alpha, \beta, \gamma$ are invariant under the isometry group $G$. This is exploited in the following way. We first choose an axisymmetric

2 Of course, it will take more than one patch to cover $\mathcal{B}$, but the fields $\gamma, \beta, \alpha$ on $\mathcal{B}$ below in equation (3.1) are globally defined and independent of the choice of coordinate systems.

3 Note that, if $\mathcal{N}$ is an extremal stationary Killing horizon, then $n$ coincides with the Killing field tangent to the horizon. For non-extremal stationary Killing horizons, $n$ differs from the Killing field.
testfunction $\psi \in C^\infty(\mathcal{B})$, multiply (3.2) by $\psi^2$, and then integrate over $\mathcal{B}$ with the integration element $dS$ coming from $\gamma$. Then we obtain with $R \equiv R(\gamma)$:

$$
0 \leq \int_{\mathcal{B}} \left( R - \frac{1}{2} \beta_\alpha \beta^\alpha - D_\alpha \beta^\alpha \right) \psi^2 \, dS \\
= \int_{\mathcal{B}} \left( 2 \psi \beta^\alpha D_\alpha \psi - \frac{1}{2} \beta_\alpha \beta^\alpha \psi^2 + R \psi^2 \right) \, dS \\
= \int_{\mathcal{B}} \left( 2 (\psi \beta_\alpha N^\alpha) N^\beta D_\beta \psi - \frac{1}{2} (\psi N^\alpha \beta_\alpha)^2 - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_\alpha \beta_\beta \psi^2 + R \psi^2 \right) \, dS \\
\leq \int_{\mathcal{B}} \left( 2 (N^b D_\beta \psi)^2 - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_\alpha \beta_\beta \psi^2 + R \psi^2 \right) \, dS \\
= \int_{\mathcal{B}} \left( 2 (D_\beta \psi) D^b \psi - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_\alpha \beta_\beta \psi^2 + R \psi^2 \right) \, dS.
$$

(3.3)

Here, we denote by $N$ a unit norm vector field\footnote{$N^\alpha = D^\alpha \chi / (\partial^\alpha D_\alpha \chi)$ is not defined at $x = \pm 1$ as $D_\alpha \chi = 0$ there, but the expressions in the above integral are nevertheless well defined.} on $\mathcal{B}$ perpendicular to the $\xi_i$, and in the third line we have used the elementary inequality $2ab \leq 2a^2 + \frac{1}{8}b^2$. Thus, we have shown the following lemma (compare lemma 1 of [1]).

**Lemma 1.** For any axisymmetric testfunction $\psi$ on $\mathcal{B}$ there holds

$$
\int_{\mathcal{B}} \left( 2 (D_\beta \psi) D^b \psi - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_\alpha \beta_\beta \psi^2 + R \psi^2 \right) \, dS \geq 0,
$$

(3.4)

where $D, R, dS$ are the intrinsic derivative operator, scalar curvature and measure on $\mathcal{B}$.

To continue our discussion, it is useful to express the above inequality in terms of the functions $f_{ij} = g(\xi_i, \xi_j)$ and certain potentials. These potentials are defined as follows. First, we pass to the quotient $\hat{\mathcal{H}} = \mathcal{H}/G$. The global structure of this quotient was described in detail in [4]; here, we only need to consider a small open neighborhood of $\mathcal{B}$. In such a neighborhood, $\hat{\mathcal{H}}$ is simply parameterized by the coordinates $u, r, x$ defined above. We complement these coordinates by $2\pi$-periodic coordinates $\varphi^1, \ldots, \varphi^{n-3}$ in such a way that $\xi_i = \partial / \partial \varphi^i$. Because $f_{ij}$ is positive definite (except at the points $x = \pm 1$), $\hat{\mathcal{H}}$ inherits a Riemannian metric from the spacetime metric $g$. We write this metric as $(\det f)^{-1} \, d\hat{s}_1^2$, and the spacetime metric can then be written in the usual Kaluza–Klein form as

$$
g = f_{ij} (d\varphi^i + \hat{A}^i) (d\varphi^j + \hat{A}^j) + (\det f)^{-1} \, d\hat{s}_1^2,
$$

(3.5)

for 1-forms $\hat{A}^i$ on $\hat{\mathcal{H}}$. The condition $R_{\mu\nu} = \frac{2}{n-2} \Lambda g_{\mu\nu}$ implies the ‘Maxwell equation’

$$
d(\det f \cdot f_{ij} \, d\hat{A}^i) = 0,
$$

(3.6)

implying locally the existence of potentials $\chi_i$ on $\hat{\mathcal{H}}$ satisfying

$$
d\chi_i = 2 \det f \cdot f_{ij} \, d\hat{A}^i.
$$

(3.7)

These so-called twist potentials are in fact defined globally [4], and can be viewed as axisymmetric functions on $\hat{\mathcal{H}}$, or alternatively on $\mathcal{B}$. In the latter case, they are functions of $x$ only, and from (1.3) they satisfy

$$
\frac{1}{8} (2\pi)^{n-4} \chi_i (x) \bigg|^{x+1}_{x-1} = J_i.
$$

(3.8)

\footnote{$N^\alpha = D^\alpha \chi / (\partial^\alpha D_\alpha \chi)$ is not defined at $x = \pm 1$ as $D_\alpha \chi = 0$ there, but the expressions in the above integral are nevertheless well defined.}
So far, the coordinate \( x \in [-1, +1] \) has only been an arbitrarily chosen parameter of the orbit space \( \hat{\mathcal{B}} = [-1, +1] \), but now we make a more specific choice. Our choice is fixed by requiring the metric \( \gamma \) on \( \mathcal{B} \) to take the form

\[
\gamma = \frac{dx^2}{C^2 \det f} + f_{ij} d\varphi^i d\varphi^j,
\]

where \( C > 0 \) is some constant. We also set \( \beta_i := \imath \xi_i \beta \). Then a straightforward calculation delivers the following expressions:

\[
R = -\frac{1}{2} C^2 f^{ij} \partial_i (\det f \partial_j f_{ij}) + \frac{1}{4} C^2 \det f f^{ij} f^{kl} \partial_i \partial_j f_{kl} - \frac{1}{4} C^2 \det f f^{ij} \partial_i \partial_j f_{ij},
\]

\[
\beta_i = C \partial_i \chi_i,
\]

\[
dS = C^{-1} dx \prod_i d\varphi^i.
\]

We insert this into lemma 1 with the choice \( \psi = [(1 - x^2)(\det f)^{-1}]^{1/2} \in C^\infty(\mathcal{B}) \). A longer but entirely straightforward calculation then gives the following beautifully simple relation.

**Lemma 2.** The functions \( f_{ij}, \chi_j \) on \( \mathcal{B} \) satisfy

\[
0 \geq \int_{-1}^{+1} \left( \frac{1}{8} (1 - x^2) \text{Tr}(\Phi^{-1} \partial_i \Phi)^2 - \frac{1}{1 - x^2} \right) dx,
\]

where the matrix \( \Phi \) is defined by

\[
\Phi = \begin{pmatrix}
(\det f)^{-1} & - (\det f)^{-1} \chi_i \\
-(\det f)^{-1} \chi_i & f_{ij} + (\det f)^{-1} \chi_i \chi_j
\end{pmatrix}.
\]

The matrix \( \Phi \) is symmetric, positive definite, and \( \det \Phi = 1 \).

**Remark.** As is well known, the matrix \( \Phi \) can be thought of as parameterizing the coset space \( SL(n - 2, \mathbb{R}) / O(n - 2, \mathbb{R}) \), and the quantity \( ds^2 = \text{Tr}(\Phi^{-1} d\Phi)^2 \) gives the natural Riemannian metric on that space, which can be shown to have negative curvature. It is also well known that Einstein’s equations, with the symmetry group \( G \) imposed, reduce to that of a gravitating sigma-model on this space [17] (for a concise review of sigma-model approaches to higher-dimensional gravity, see e.g. [18]). But it is not clear to us why this implies that precisely the combination (3.13) satisfies the beautifully simple inequality of lemma 2! In particular, note that the inequality holds true for \( \Lambda > 0 \), in which case the Einstein equations do not reduce to that of a sigma-model.

The lemma is our main technical tool. The final step in the proof consists of a variational argument similar to that given in [2, 3]. For this, we first note that, since the metric \( \gamma \) on \( \mathcal{B} \) (see equation (3.9)) is free of any type of conical singularity, we must have the relation

\[
(1 - x^2)^2 \rightarrow C^2 \text{ as } x \rightarrow \pm 1.
\]

We also note that, again from equation (3.9), the area \( A \) of \( \mathcal{B} \) is given by

\[
A = 2(2\pi)^{n-1} C^{-1}.
\]
Lemma 3. Let \( s \) be smooth, axisymmetric such that conditions (3.8) and (3.18) are satisfied, and such that \( I[\Phi] \) (cf. (3.17)) is finite. Then \( I[\Phi] \geq I[\Phi_0] \), where \( \Phi_0 \) corresponds to the near horizon geometry metric \( g_0 \) with angular momenta \( J_i \), see the appendix. Moreover, if \( \Phi \) is a minimizer of \( I[\Phi] \), then \( \Phi = \Phi_0 \) where \( \Phi_0 \) corresponds to a near horizon metric.

Before we give the proof of the last lemma, let us finish the proof of theorem 1. The lemma and equation (3.17) immediately imply that

\[
A \geq 2(2\pi)^{n-3} C^2 e^{-\bar{H}[\Phi_0]},
\]

so we only need to calculate \( I[\Phi_0] \). As shown in [8], the near horizon geometries are characterized by a \( \Phi_0 \) satisfying the equations

\[
\frac{1}{\det f_0} \frac{1}{f_0} \frac{1}{\partial_i \Phi_0} \frac{1}{\partial_j \Phi_0} = 0,
\]

and the boundary conditions

\[
\frac{1}{\det f_0} \frac{1}{f_0} \frac{1}{\partial_i \Phi_0} \frac{1}{\partial_j \Phi_0} \rightarrow C_0^2 \quad \text{as} \quad x \rightarrow \pm 1,
\]

for some \( C_0 \) (not necessarily equal to \( C \)), as well as \( f_0 \partial_i \rightarrow 0 \) for \( x \rightarrow \pm 1 \), and satisfying also \( \partial_0 \rightarrow \pm 4(2\pi)^{4-n} J_i \) for \( x \rightarrow \pm 1 \). The corresponding metrics \( g_0 \) that follow from these conditions were determined in [8]; they are reviewed for completeness in the appendix. For us, the relevant consequence of the relations between the various parameters of the metric \( g_0 \) stated in theorem 2 and (A.7), (A.5), is that

\[
C_0 = \frac{2(2\pi)^{n-3}}{8\pi |J_+ J_-|^2}.
\]
contribution to I equations (3.20) for $\Phi_0$ and the boundary conditions are seen to imply [8] that
\[
(1 - x^2)\Phi_0^{-1} \partial_x \Phi_0 = 2 S \left( \begin{array}{cccc}
-1 & 0 & 0 & \ldots & 0 \\
0 & +1 & 0 & \ldots & 0 \\
& & & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0
\end{array} \right) S^{-1}
\]
for some constant unimodular matrix $S$. Hence, one immediately sees that the integral term in $I[\Phi_0]$ (see equation (3.17)) is precisely $= 0$, which in view of (3.22), (3.21) means
\[
A \geq 2(2\pi)^{n-3} e^{\frac{1}{2}/0} = 8\pi |J_0 J_1|^\frac{1}{2}.
\]
This is the inequality claimed in part (1) of theorem 1. To prove part (2), suppose first that $g = g_0$ is a near horizon geometry. Then, again from (3.22) and $A_0 = 2(2\pi)^{n-3} C_0^{-1}$, equality holds. Conversely, suppose that the inequality in the theorem is saturated. Then the corresponding matrix $\Phi$ must be a minimizer of $I[\Phi]$, and by lemma 3, must satisfy the equation (3.20) and must coincide with the $\Phi_0$ of a near horizon geometry. This implies that the metric functions $\alpha, \beta, \gamma$ in (3.1) must coincide with those of a near horizon geometry, since these are determined by $\Phi$ as shown in [8]. This finishes the proof of theorem 1. □

Proof of lemma 3: The proof of this lemma has the same structure as that of lemma 4.1 of [2], so we only outline the analogous steps and emphasize the new ideas that are needed. The main point is that, if $\Phi$ is a minimum of $I[\Phi]$ (cf equation (3.17)), then it is evident from the definition of $I[\Phi]$ that the matrix function $\Phi$ must satisfy the same Euler–Lagrange equations (3.20) that are satisfied by the $\Phi_0$ of a near horizon geometry. Since $\Phi$ must satisfy boundary conditions (3.8) and (1.7), it follows that a minimizer also satisfies the same boundary conditions as a near horizon geometry. Hence, it must be equal to a near horizon geometry. The only gap in the argument is to show that $I[\Phi]$ actually has minimizers. Note that the integral contribution to $I[\Phi]$ can be integrated further against $d\phi_1 \cdots d\phi_2^{-2}$, and thereby essentially becomes the ‘harmonic energy’ of maps $\Phi$ from $\mathbb{B}$ into the negatively curved target space $SL(n-2, \mathbb{R})/O(n-2, \mathbb{R})$. The strategy is therefore to appeal to existence and uniqueness properties of harmonic maps into negatively curved target spaces [19]. However, as stated, the result of [19] only applies to harmonic maps $\Phi : \Omega \to SL(n-2, \mathbb{R})/O(n-2, \mathbb{R})$ where $\Omega$ is a domain with non-empty boundary $\partial \Omega$, on which Dirichlet-type conditions are imposed. On the other hand, our maps $\Phi$ are defined on $\mathbb{B}$, and as $x \to \pm 1$, behave in a singular way. So, in order to apply the result of [19], one has to get around these problems. This was explained in [2] for the case $n = 4$, in which the target space is isometric to the two-dimensional hyperbolic space $SL(2, \mathbb{R})/O(2, \mathbb{R}) \cong \mathbb{H}$. The same type of argument also applies in the present context.

We first pick an arbitrary $\Phi_0$ corresponding to a near horizon geometry in the class of functions satisfying equation (3.18), with the same values of the $J_i$. Then, we consider a domain $\Omega_1 \subset \mathbb{B}$ which is a small neighborhood,
\[
\Omega_1 = \{ p \in \mathbb{B} \mid |x(p) - 1| < e^{-/(log e)^2} \quad \text{or} \quad |x(p) + 1| < e^{-/(log e)^2} \},
\]
of the points characterized by $x = \pm 1$ where the map $\Phi$ behaves in a singular way. We take a suitable cutoff function $\psi_\epsilon$ which is equal to 1 on $\Omega_1$, and which is equal to 0 on a region
\[
\Omega_{II} = \{ p \in \mathbb{B} \mid |x(p) - 1| > \epsilon \quad \text{and} \quad |x(p) + 1| > \epsilon \},
\]
and which interpolates between 0 and 1 in the remaining intermediate annular region $\Omega_{III}$. One then considers an interpolation (constructed using $\psi_\epsilon$) $\Phi_\epsilon$ so that $\Phi_\epsilon = \Phi_0$ on $\Omega_1$, and so
that $\Phi = \Phi$ on $\Omega_{III}$. The explicit form of this interpolation will be described below. Having defined the interpolation, one defines $I_k[\Phi] := I_k[\Phi_\epsilon]$. Now on $\Omega_{IV} = \Omega_{III} \cup \Omega_{IV}$, the function $\Phi_\epsilon : \Omega_{IV} \to SL(n-2, \mathbb{R})/O(n-2, \mathbb{R})$ by construction has the same boundary values as $\Phi_0$, and so on $\Omega_{IV}$ cannot have smaller harmonic energy by the result of [19], i.e. $I_k |_{\Omega_{IV}}$ (we mean restriction to maps defined on $\Omega_{IV}$ with prescribed boundary conditions), is not smaller than $I_k |_{\Omega_{IV}}$, evaluated on $\Phi_0$. On the other hand, on $\Omega_I$, we have by definition $I_k |_{\Omega_I} = I_k |_{\Omega_I} [\Phi_0]$, so one obtains that $I_k [\Phi] \geq I_k [\Phi_0] = I_k [\Phi_0]$. So, one is done if one can show that $\lim_{\epsilon \to 0} I_k = I$. This part requires a special choice of the cutoff function—we may chose the same one as in equation (71) of [2]—and a special interpolation.

To choose the interpolation, we parameterize the coset space (i.e. matrices $\Phi$) by $f_{ij}$, $\chi_i$ as above. But we further parameterize $f_{ij}$ by $\nu$, $\sigma$ (defined above in (3.18)), and by new quantities $c^{ij}$, $b^i$:

$$f^{ij} = \left( \frac{e^{-\nu}}{1 - e^{-\nu}} + c_{ij} b^j b^i \right) \alpha^a + 2\alpha^{(i} b^{j)} + c^{ij}.$$  
(3.25)

The functions $\alpha^i$ were defined above before equation (3.17). The quantities $b^i$ and $c^{ij}$ can be viewed as linear maps on covectors $\nu_i$. On covectors which satisfy $\nu_i \nu^i = 0$, the bilinear form $c^{ij}$ is positive definite and non-degenerate, everywhere on $\mathbb{R}$, including $x = \pm 1$. To make $c^{ij}$, $b^i$ unique, we choose an arbitrary but fixed covector function $\sigma_i$ such that $\alpha_i \sigma^i = 1$, and we demand that $c^{ij} \sigma^i = 0 = b^i \sigma^i$. It follows from these definitions that there is a $c^{ij}$ such that $c_{ij} \sigma^j = 0$ and $c_{ij} \nu_k \nu^k = \nu_i$ for all $\nu_i$ satisfying $\nu_i \sigma^i = 0$. We denote by $c$ the determinant of $c^{ij}$ when restricted to $\nu_i$'s of this form, and we also denote $b_i = c_{ij} b^j$. Combining these formulae with (3.18), we see that $c$ is constrained to be of the form $c = e^{\nu - \sigma}$. Thus, we parameterize $\Phi$ by the quantities $\sigma, \nu, b^i, c^{ij}, \chi_i$, all of which are smooth on $\mathbb{R}$, including at $x = \pm 1$, and axisymmetric. We can then simply linearly interpolate

$$\sigma_e = \sigma, \sigma_0 + (1 - \sigma_e) \sigma, \quad b^i_e = \sigma_e b^i_0 + (1 - \sigma_e) b^i,$$
(3.26)

$$\nu_e = \nu, \nu_0 + (1 - \nu_e) \nu, \quad \chi_e = \chi, \chi_0 + (1 - \nu_e) \chi,$$
(3.27)

and we interpolate

$$c^{ij}_e = [\exp(\sigma_e A_0) \exp((1 - \sigma_e) A)]^{ij},$$
(3.28)

where $A_0^i$, $A^i$ are the matrix logarithms of $c^{ij}_0$, $c^{ij}$. The last definition ensures that $c_e = e^{\nu - \sigma}$. We denote the interpolated matrix parameterized by $\sigma_e$, $\nu_e$, $b^i_e$, $c^{ij}_e$, $\chi_e$ by $\Phi_e$. Our parameters have been chosen in such a way that the functional $I[\Phi]$ takes a form in which we can analyse it in the same way as done in [2]. Namely, a longer, but entirely straightforward, calculation shows that $I[\Phi]$ is given by

$$I[\Phi] = \frac{1}{8} \int_{\mathcal{M}} \left( 8 + 4\nu + 4\sigma + (\partial_\theta \nu)^2 + (\partial_\theta \sigma)^2 + 2 \frac{d\alpha^i}{d\sigma} \frac{\partial_\theta A_0^i}{\sin^2 \theta} e^{-\nu - \sigma} \right) dS_0$$

$$+ \frac{1}{8} \int_{\mathcal{M}} \left( -b^k b^j \left[ \frac{2 e^{-\nu}}{\sin^2 \theta} + b^i b^j \right] \frac{(\partial_\theta (e^\nu \sin^2 \theta))^2 - 4 [1 + b^i b^j e^\nu \sin^2 \theta]^2}{\sin^2 \theta} \right)$$

$$\times \left[ \cos \theta \frac{\partial_\theta \nu}{\sin \theta} + \partial_\theta \nu - d^a \partial_\theta a_i - d^a \partial_\theta b_i \right] \left( a_i \partial_\theta a_i + b_i \partial_\theta b_i \right) \left( \frac{e^{-\nu}}{\sin^2 \theta} + b^k b^k \right)^2$$

$$\times \partial_\theta (d^a \partial_\theta e^\nu \sin^2 \theta) (e^\nu \sin^2 \theta) a_i a_i - 2 a_i b_i) (e^\nu \sin^2 \theta) a_i a_i - 2 a_i b_i)$$

$$+ 8 (1 + b^k b^k e^\nu \sin^2 \theta) \left[ \frac{\cos \theta}{\sin \theta} + \frac{1}{2} b^k \partial_\theta b_i - a_i b_i \right]$$

$$\cdot \left[ (e^\nu \sin^2 \theta) b_i \partial_\theta a_i - 2 \partial_\theta (e^\nu \sin^2 \theta) b_i b_i - 2 e^\nu \sin^2 \theta a_i b_i \partial_\theta (a_i b_i) - b_i \partial_\theta a_i \right]$$

where $\mathcal{M}$ is the manifold with boundary $\mathcal{B}$.

10
The method of proof in [8] shows that the matrix referred to as the ‘extremal Kerr throat’. The generalization to \( n \) in \( M \) been grouped in the following way. The first integral, which we call horizon geometries. Note that the notion of near horizon geometry is defined for any extremal black hole with Killing horizon, which satisfies the equations (3.1). The ones we discuss here are thus special, ‘codimension-1’, near horizon geometries. The near horizon geometry of an extremal \( - \) dimensional vacuum stationary black hole with isometry group\(^5 \mathbb{R} \times G \) is defined by a certain scaling limit of the metric near the horizon. This scaling limit is a new Ricci-flat metric and automatically possesses an enhanced symmetry group of \( O(2, 1) \times G \), see [20, 21]. The classification of all such geometries was achieved first in \( n = 4 \) in [22, 23]. In \( n = 4 \), it had been known for a long time and is also sometimes referred to as the ‘extremal Kerr throat’. The generalization to \( n \) dimensions was performed in [8]. The method of proof in [8] shows that the matrix \( \Phi_0 \), formed from the quantities \( f_{00}/, \chi_0 \) associated with the near horizon geometry, satisfies the equations (3.20). Combining this with the boundary conditions, one arrives at the following classification [8].

\[
+ 2 \left[ \frac{e^{-\nu}}{\sin^2 \theta} + b' b_0 \right] \hat{e}^j [e^\nu \sin^2 \theta \partial_\theta a_k - 2a' \partial_\theta (e^\nu \sin^2 \theta a_i b_k) - c_k \partial_\theta a'_i] \\
\cdot [e^\nu \sin^2 \theta \partial_\theta a_i - 2a' \partial_\theta (e^\nu \sin^2 \theta a_i b_k) - e^{\nu} (d_j + 2b' a_l)] (e^j + 2b' a_l) \\
\times \partial_\theta [c_k + e^\nu \sin^2 \theta b_j b_k + e^\nu \sin^2 \theta a_i a_k - 2e^\nu \sin^2 \theta a_i b_k] \\
\cdot \partial_\theta [c_j + e^\nu \sin^2 \theta b_i b_j + e^\nu \sin^2 \theta a_i a_j - 2e^\nu \sin^2 \theta a_i b_j] + 2 \frac{e^{-\sigma}}{\sin^2 \theta} b_j b'_i a'_l (\partial_\theta \chi_j) \\
\times \partial_\theta \chi_k + 4 \frac{e^{-\sigma}}{\sin^2 \theta} b'_i a'_l (\partial_\theta \chi_j) \partial_\theta \chi_k + 2 \frac{e^{-\sigma}}{\sin^2 \theta} e^j (\partial_\theta \chi_j) \partial_\theta \chi_k \right] d\sigma_0. \tag{3.29}
\]

Here, we have defined \( x = \cos \theta \), and also \( d\sigma_0 = (2\pi)^{3-n} \sin \theta d\theta \prod d\varphi^i \), which coincides up to a constant factor with the measure \( d\mathcal{S} \) on \( \mathbb{R} \) coming from the metric. The terms have been grouped in the following way. The first integral, which we call \( M[\Phi] \), is analogous to the ‘mass functional’ of [2]. In \( n = 4 \), we have that \( \nu = \sigma \) is equal to the quantity \( \xi \) in that reference, \( a^1 = a^1 \), \( \chi_1 \) is equal to the quantity \( \omega \) in that reference, and \( \mathcal{R} \equiv S^2 \). Then \( M[\Phi] \) is actually exactly equal to the mass functional defined in that reference. Also in higher dimensions, the integral has a completely analogous structure and can therefore be handled in exactly the same way as in [2] to show that \( M[\Phi] \equiv M[\Phi_0] \to M[\Phi] \) as \( \epsilon \to 0 \). The second integral, which we call \( M'[\Phi] \), has only terms that are manifestly regular at \( \theta = 0 \, \text{or} \, \pi \) (here, we also use that \( \partial_\theta \chi_i = O(\sin \theta) \) due to \( 3.8 \), and that \( \partial_\theta \omega = O(\sin^2 \theta) \)). It is not present in \( n = 4 \), but because the integrand is regular, it is straightforward to see that it converges, \( M'[\Phi] \to M'[\Phi] \). Because the arguments are very similar, and no new ideas are required, we will not give the details here. □

**Conventions**: Our conventions for the signature of the metric and the curvature tensor are the same as in Wald’s textbook [24]. Our convention for the Hodge dual \( \star \) is such that \( \star^2 = -1 \).

**Acknowledgments**

It is a pleasure to thank S Dain, A Ishibashi and J Lucietti for their comments on the first draft of this paper.

**Appendix. Near horizon geometries in \( n \) dimensions**

The near horizon geometry of an extremal \( n \)-dimensional vacuum stationary black hole with isometry group\(^5 \mathbb{R} \times G \) is defined by a certain scaling limit of the metric near the horizon. This scaling limit is a new Ricci-flat metric and automatically possesses an enhanced symmetry group of \( O(2, 1) \times G \), see [20, 21]. The classification of all such geometries was achieved first in \( n = 4 \) in [22, 23]. In \( n = 4 \), it had been known for a long time and is also sometimes referred to as the ‘extremal Kerr throat’. The generalization to \( n \) dimensions was performed in [8]. The method of proof in [8] shows that the matrix \( \Phi_0 \), formed from the quantities \( f_{00}/, \chi_0 \) associated with the near horizon geometry, satisfies the equations (3.20). Combining this with the boundary conditions, one arrives at the following classification [8].

\(^5 \) Note that the notion of near horizon geometry is defined for any extremal black hole with Killing horizon, which does not have to possess the isometry group \( G \). The ones we discuss here are thus special, ‘codimension-1’, near horizon geometries.
**Theorem A.1.** All non-static near horizon metrics are parameterized by the real parameters $c_\pm$, $\mu_i$, $s_i$ and the integers $d_\pm^i$ where $I = 0, \ldots, n-5$ and $i = 1, \ldots, n-3$, and $c.d.(d_\pm^i) = 1$. The explicit form of the near horizon metric in terms of these parameters is

\[
g_0 = e^{-\lambda(2 \, du \, dR - C_0^2 R^2 \, du^2 + C_0^{-2} \, d\theta^2)} + e^{\lambda(2 \, (s_+ - s_-) \sin^2 \theta \, \Omega)} + (1 + \cos \theta)^2 c_+^2 \sum_I \left( \omega_I - \frac{s_I \cdot a_+}{\mu \cdot a_+} \right)^2 + (1 - \cos \theta)^2 c_-^2 \sum_I \left( \omega_I - \frac{s_I \cdot a_-}{\mu \cdot a_-} \right)^2 + c_+^2 \sin^2 \theta \sum_{I < J} \left( (s_I \cdot a_+) \omega_J - (s_J \cdot a_-) \omega_I \right)^2.
\]

(A.1)

Here, the sums run over $I, J$ from $0, \ldots, n - 5$, the function $\lambda(\theta)$ is given by

\[
\exp[-\lambda(\theta)] = c_+^2(1 + \cos \theta)^2 + c_-^2(1 - \cos \theta)^2 + \frac{c_+^2 \sin^2 \theta}{(\mu \cdot a_\pm)^2} \sum_I (s_I \cdot a_\pm)^2,
\]

(A.2)

$C_0$ is given by $C_0 = 4c_\pm[(c_+ - c_-)(\mu \cdot a_\pm)]^{-1}$, and we have defined the $1$-forms

\[
\Omega(R) = \mu \cdot d\phi + 4C_0R \frac{c_+ c_-}{c_+ - c_-} \, du \quad \text{and} \quad \omega_I(R) = s_I \cdot d\phi + \frac{\mu}{2} C_0 \sum_J (s_I \cdot a_+ + s_I \cdot a_-) \, du.
\]

(A.3, A.4)

We are also using the shorthand notations such as $s_I a_\pm^I = s_I \cdot a_\pm$ or $\mu \cdot d\phi = \mu_i d\phi^i$, etc. The parameters are subject to the constraints $\mu \cdot a_\pm \neq 0$ and

\[
\frac{c_+^2}{\mu \cdot a_+} = \frac{c_-^2}{\mu \cdot a_-} = \frac{c_+ (s_I \cdot a_+)}{\mu \cdot a_+} = \frac{c_- (s_I \cdot a_-)}{\mu \cdot a_-}, \quad \pm 1 = (c_+ - c_-) \epsilon^{ijk} m s_0 s_1 s_2 \cdots \mu_m.
\]

(A.5)

but they are otherwise free. When writing '±', we mean that the formulae hold for both signs.

**Remarks.**

1. The part $2 du \, dR - C_0^2 R^2 \, du^2$ of the metric is $AdS_2$ space with curvature $C_0^2$. This is the cause for the enhanced symmetry group of $O(2, 1) \times U(1)^{n-3}$, as explained in more detail in [22].

2. The coordinate $\theta \in [0, \pi]$ is related to the coordinate $x$ in the previous section by $x = \cos \theta$. The relation between $r$ and $R$ is more complicated and is explained in [8]. The constants $\mu_i$, $c_\pm$, $a_\pm^i$, hence $C_0$, are directly related to the horizon area by

\[
A_0 = 2(2\pi)^{n-2} C_0^{-1},
\]

(A.6)

and we also have

\[
8\pi J_i = (2\pi)^{n-2} \frac{c_+ - c_-}{2c_+ c_-} \mu_i.
\]

(A.7)
References

[1] Jaramillo J M, Reiris M and Dain S 2011 Black hole area–angular momentum inequality in non-vacuum spacetimes arXiv:1106.3743v1 [gr-qc]
[2] Acena A, Dain S and Gabrič Clement M E 2010 Horizon area–angular momentum inequality for a class of axially symmetric black holes Phys. Rev. Lett. 107 051101
[3] Dain S and Reiris M 2011 Area–angular momentum inequality for axisymmetric black holes Phys. Rev. Lett. 107 051101
[4] Hollands S and Yazadjiev S 2008 Uniqueness theorem for 5-dimensional black holes with two axial Killing fields Commun. Math. Phys. 283 749–68
[5] Hollands S and Yazadjiev S 2011 A uniqueness theorem for stationary Kaluza–Klein black holes Commun. Math. Phys. 302 631–74
[6] Racz I 2008 A simple proof of the recent generalisations of Hawking’s black hole topology theorem Class. Quantum Grav. 25 162001
[7] Galloway G J and Schoen R 2006 A generalization of Hawking’s black hole topology theorem to higher dimensions Commun. Math. Phys. 266 571–6
[8] Hollands S and Ishibashi A 2010 All vacuum near horizon geometries in arbitrary dimensions Ann. Henri Poincare 10 1537–57
[9] Myers R C and Perry M J 1986 Black holes in higher dimensional spacetimes Ann. Phys. 172 304
[10] Emparan R and Reall H S 2006 Black rings Class. Quantum Grav. 23 R169
[11] Pomeransky A A and Sen’kov R A 2006 Black ring with two angular momenta arXiv:hep-th/0612005v1
[12] Hayward S A 1994 Quasilocal gravitational energy Phys. Rev. D 49 831–9
[13] Hayward S A, Shiromizu T and Nakao K-I 1994 A cosmological constant limits the size of black holes Phys. Rev. D 49 5080–5
[14] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[15] Isenberg J and Moncrief V 1983 Symmetries of cosmological Cauchy horizons Commun. Math. Phys. 89 387–413
[16] Hollands S, Ishibashi A and Wald R M 2007 A higher dimensional stationary rotating black hole must be axisymmetric Commun. Math. Phys. 271 699–722
[17] Maison D 1979 Ehlers–Harrison-type transformations for Jordan’s extended theory of gravitation Gen. Rel. Grav. 10 717
[18] Clement G 2008 Sigma-model approaches to exact solutions in higher-dimensional gravity and supergravity arXiv:0811.0691 [hep-th]
[19] Hildebrandt S, Kaul H and Widmann Kjell-Ove 1977 An existence theorem for harmonic mappings of Riemannian manifolds Acta Math. 138 1–16
[20] Bardeen J M and Horowitz G T 1999 The extreme Kerr throat geometry: a vacuum analog of AdS(2) × S(2) Phys. Rev. D 60 104030
[21] Kunduri H K, Lucietti J and Reall H S 2007 Near-horizon symmetries of extremal black holes Class. Quantum Grav. 24 4169
[22] Kunduri H K and Lucietti J 2009 A classification of near-horizon geometries of extremal vacuum black holes J. Math. Phys. 50 082502
[23] Kunduri H K and Lucietti J 2009 Static near-horizon geometries in five dimensions Class. Quantum Grav. 26 245010
[24] Wald R M 1986 General Relativity (Chicago, IL: University of Chicago Press)