String theory, $\mathcal{N} = 4$ SYM and Riemann hypothesis

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Abstract

We discuss new relations among string theory, four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) and the Riemann hypothesis. It is known that the Riemann hypothesis is equivalent to an inequality for the sum of divisors function $\sigma(n)$. Based on previous results in literature, we focus on the fact that $\sigma(n)$ appears in a problem of counting supersymmetric states in the $\mathcal{N} = 4$ SYM with $SU(3)$ gauge group: the Schur limit of the superconformal index plays a role of a generating function of $\sigma(n)$. Then assuming the Riemann hypothesis gives bounds on information on the 1/8-BPS states in the $\mathcal{N} = 4$ SYM. The AdS/CFT correspondence further connects the Riemann hypothesis to the type IIB superstring theory on $AdS_5 \times S^5$. In particular, the Riemann hypothesis implies a miraculous cancellation among Kaluza-Klein modes of the supergravity multiplet and D3-branes wrapping supersymmetric cycles in the string theory. We also discuss possibilities to gain new insights on the Riemann hypothesis from the physics side.
1 Introduction

The Riemann hypothesis is the conjecture that the Riemann zeta function $\zeta(s)$ has nontrivial zeroes only along the line $\text{Re}(s) = 1/2$, where the trivial zeroes are at $s = -2n$ with $n \in \mathbb{Z}_+$. This is an important problem especially in number theory as the zeroes of the zeta function have significant information on prime number distribution, and indeed one of the millennium prize problems by Clay Mathematics Institute. While this is originally a purely mathematical problem, its physical realization – since the Hilbert-Pólya conjecture \[1, 2\] – has been discussed in various problems of physics (see e.g. review \[3–5\]), including quantum mechanics, statistical mechanics, random matrix theory and string theory \[6–10\]. In this paper, we point out new relations among the Riemann hypothesis, string theory and gauge theory.

There are various equivalent statements of the Riemann hypothesis. Here we focus on one of them \[11\] in terms of the sum of divisors function $\sigma(n)$:

$$\sigma(n) := \sum_{d|n} d, \quad (1.1)$$

where $\sum_{d|n}$ stands for the sum over the integers $d$ which divide $n$. It was proven in \[11\] that the Riemann hypothesis is equivalent to the following inequality\[1\] (c.f. fig. \(1\)) \[11\]:

$$\sigma(n) \leq H_n + e^{H_n} \log H_n \quad \text{for } \forall n \in \mathbb{Z}_+, \tag{1.2}$$

where the equality holds only for $n = 1$ and $H_n$ is the harmonic number defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k}. \quad (1.3)$$

In this paper we discuss implications of the inequality (1.2) to physics: gauge theory and string theory. We start with pointing out that there is a physical quantity in the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) which takes a form of a generating function of the sum of divisors function $\sigma(n)$ according to previous results in the literature \[13–16\]. The quantity is a supersymmetric (SUSY) index called the Schur index in the $\mathcal{N} = 4$ SYM with gauge group $SU(3)$. The Schur index is a special limit of a four-dimensional superconformal index and counts the difference between the numbers of bosonic and fermionic SUSY states annihilated by two Poincare supercharges:

$$I_{\text{Schur}}(q) = \text{Tr}_{\text{SUSY}} \left[ (-1)^F q^{2(E-R)} \right], \quad (1.4)$$

where $F$ is fermion number, $E$ is energy and $R$ is an $R$-charge. The Schur index of the $SU(3)$ $\mathcal{N} = 4$ SYM is known to take the following form \[13–17\]

$$I_{SU(3)}(q) = \sum_{n=1}^{\infty} \sigma(n) q^{2(n-1)}, \quad (1.5)$$

which is the generating function of $\sigma(n)$. Therefore $\sigma(n)$ corresponds to the difference between the numbers of bosonic and fermionic 1/8-BPS states with the quantum number $E - R = n - 1$. This enables us to rephrase the Riemann hypothesis in the language of the $\mathcal{N} = 4$ SYM. In particular the Riemann hypothesis claims bounds on information on the 1/8-BPS states in the $SU(3)$ $\mathcal{N} = 4$ SYM. Furthermore the AdS/CFT correspondence \[18\] gives an interpretation of the Schur index from the dual string theory \[19\] and then leads us to relations between the Riemann hypothesis and string theory. We argue that the Riemann hypothesis implies a huge cancellation among Kaluza-Klein modes of the supergravity multiplet and D3-branes wrapping supersymmetric cycles in the type IIB superstring theory on $AdS_5 \times S^5$. We also discuss possibilities to gain some insights on the Riemann hypothesis from the physics side.

This paper is organized as follows. In sec. \[2\] we discuss the Schur index of the $\mathcal{N} = 4$ SYM and its relation to the Riemann hypothesis. In sec. \[3\] we discuss its interpretations from the string theory via the AdS/CFT correspondence. Sec. \[4\] is devoted to discussions.

\[1\]This inequality was derived from the Robin’s inequality \[12\]: $\sigma(n) \leq e^{\gamma n} \log \log n$ for $n \geq 5041$ with the Euler constant $\gamma$. 

In this paper we study a SUSY index called the Schur index which is a limit of a four-dimensional superconformal index. In general, SUSY indices can be defined through partition functions of SUSY theories on manifolds including $S^1$. The most elementary one is the Witten index \cite{20} of SUSY quantum mechanics given by

$$I_W = \text{Tr} \left[ (-1)^F e^{-\beta H} \right], \quad (2.1)$$

where the trace is over the entire Hilbert space, $\beta$ is a radius of $S^1$ and $F$ is the fermion number operator. While it apparently depends on $\beta$, it is known that there are typically the same numbers of bosonic and fermionic non-SUSY states, which always have non-zero energies, and only SUSY states with zero energy give net contributions to the Witten index:

$$I_W = \text{Tr}_{\text{SUSY}} \left[ (-1)^F \right], \quad (2.2)$$

where the trace is now over the SUSY states. Therefore the Witten index is independent of $\beta$ and gives a difference between the numbers of bosonic and fermionic SUSY states. More generally one can consider a refinement by chemical potentials of global symmetries which can be both internal symmetries and isometries of space, and generic SUSY index is schematically represented as

$$I_{\text{SUSY}} (\{\mu_i\}) = \text{Tr}_{\text{SUSY}} \left[ (-1)^F \prod_i e^{\mu_i Q_i} \right], \quad (2.3)$$

where $Q_i$’s are the conserved charges of global symmetries commuting with a part of supercharges. Expanding this leads us to

$$I_{\text{SUSY}} (\{\mu_i\}) = \sum_{\{Q_i\}} \left( d_B(\{Q_i\}) - d_F(\{Q_i\}) \right) \prod_i e^{\mu_i Q_i}, \quad (2.4)$$

where $d_{B/F}(\{Q_i\})$ denotes the number of bosonic/fermionic SUSY states with the quantum numbers $\{Q_i\}$. One advantage to study SUSY index is that it is SUSY invariant and can be often computed exactly. Therefore SUSY index has been used to explore strong coupling behavior of SUSY theories.

Superconformal index is a Witten index \cite{20} of a radially quantized SUSY theory (i.e. on $\mathbb{R} \times S^{d-1}$) refined by chemical potentials of global symmetries. It is also related to counting of SUSY operators via the state/operator correspondence in conformal field theory as $\mathbb{R} \times S^{d-1}$ is conformally flat\footnote{Strictly speaking, superconformal index can be defined even for non-conformal theory and the state/operator correspondence holds only for the superconformal cases.}. In the case of 4d
\( \mathcal{N} = 2 \) SUSY theory, the superconformal index is defined as\(^3\)

\[
I_{SCI} = \text{Tr}_{\text{SUSY}} \left[ (-1)^F p^{j_1 + j_2 - r} q^{2(-j_1 + j_2 - r)} e^{2(R + r)} \right],
\]

(2.5)

where \((j_1, j_2)\) is the Cartan part of the \(S^3\) isometry \(SO(4) \simeq SU(2) \times SU(2)\), \(R\) is the Cartan part of \(SU(2)_R\) and \(r\) is the \(U(1)_c\) charge. The trace is over the SUSY states annihilated by one Poincare supercharge with

\[
E - 2j_2 - 2R + r = 0.
\]

(2.6)

Then the Schur index is defined as the following limit of the superconformal index\(^4\)

\[
p \to 1, \quad q = t = \text{fixed},
\]

(2.7)

which amounts to counting the SUSY states annihilated by two Poincare supercharges with

\[
E + j_1 - j_2 - 2R = 0,
\]

(2.8)

in addition to (2.6). The trace representation of the Schur index is\(^5\)

\[
I_{\text{Schur}} = \text{Tr}_{\text{SUSY}} \left[ (-1)^F q^{2(-j_1 + j_2 + R)} \right] = \text{Tr}_{\text{SUSY}} \left[ (-1)^F q^{2(E - R)} \right].
\]

(2.9)

From now on, we focus on the four-dimensional \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory which is the low energy worldvolume theory of D3-branes in the type IIB string theory. The \(\mathcal{N} = 4\) SYM consists of an \(\mathcal{N} = 2\) vector multiplet and one adjoint hypermultiplet in 4d \(\mathcal{N} = 2\) language. It is known that the superconformal index of any \(\mathcal{N} = 1\) Lagrangian theory has a finite dimensional integral representation with dimensions \(\text{rank}(G)\)\(^6\), which is physically an integration over gauge holonomy along \(S^1\). For the case of the Schur index in the \(\mathcal{N} = 4\) SYM with gauge group \(G\), it is given by

\[
I_G(q) = \frac{1}{|W(G)|} \left( \prod_{n=0}^{\infty} \frac{1 - q^{2n + 2}}{1 - q^{2n + 1}} \right)^{2\text{rank}(G)} \left( \prod_{j=1}^{\text{rank}(G)} \oint_{\alpha \in \text{root} \neq 0} \frac{dz_j}{2\pi i z_j} \right) \prod_{\alpha \neq 0} \theta(z^\alpha; q^2),
\]

(2.10)

where \(|W(G)|\) denotes the rank of the Weyl group of \(G\) and

\[
\theta(z; q^2) := \prod_{n=0}^{\infty} \left( 1 - z q^{2n} \right) \left( 1 - z^{-1} q^{2(n + 1)} \right).
\]

(2.11)

There are various works which wrote down a closed form of the Schur index of the \(\mathcal{N} = 4\) SYM for various gauge groups\(^7\). Here we are particularly interested in the \(G = SU(3)\) case:

\[
I_{SU(3)}(q) = \frac{q^{-2}}{24} (1 - E_2(q)),
\]

(2.12)

where \(E_2(q)\) is the Eisenstein series \(E_{2k}(q)\) with \(k = 1\) defined by

\[
E_{2k}(q) := \frac{1}{2 \zeta(2k)} \sum_{(m,n) \neq (0,0) \atop (m+n \tau)^{2k}} \frac{1}{(m+n \tau)^{2k}} \quad \text{with} \quad q^2 := e^{2\pi i \tau}.
\]

\(3\)This representation is associated with certain supercharge in the notation of\(^2\) (up to rescaling of the fugacities). In 4d \(\mathcal{N} = 1\) language, it is written as

\[
I_{SCI} = \text{Tr}_{\text{SUSY}} \left[ (-1)^F p^{j_1 + j_2 + \frac{R}{2} \tau N_1 = 1/q - j_1 + j_2 + \frac{3}{2} \tau N_1 = 1/q + \frac{3}{2} \tau N_1 = 1 - 3R} \right],
\]

where \(r = (pq)^{2/3} t^{-1}\).

\(4\)More precisely, minimally we need \(q = t\). Although \(p\) can be arbitrary for this case, the index is known to be independent of \(p\).

\(5\)It has been conjectured\(^8\) that the Schur index can be also computed from BPS spectrum on Coulomb branch captured by the wall-crossing formula in\(^9\).
The Eisenstein series is also known to have the following representation
\[ E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{2n}. \] (2.14)

Plugging this into (2.12), we find
\[ I_{SU(3)}(q) = \sum_{n=1}^{\infty} \sigma(n)q^{2(n-1)}. \]

Thus the Schur index of the \( SU(3) \) \( \mathcal{N} = 4 \) SYM is the generating function of the sum of divisors function \( \sigma(n) \). The sum of divisors function \( \sigma(n) \) directly corresponds to the difference between the numbers of bosonic and fermionic 1/8-BPS states with the quantum number \( E - R = n - 1 \). Therefore the Riemann hypothesis (1.2) is physically rephrased as follows: the difference of the numbers of the bosonic and fermionic 1/8-BPS states with \( E - R = n - 1 \) in the \( SU(3) \) \( \mathcal{N} = 4 \) SYM is less than or equal to \( H_n + e^{H_n} \log H_n \).

3 String theory and Riemann hypothesis

The AdS/CFT correspondence tells us that the \( SU(N) \) \( \mathcal{N} = 4 \) SYM is dual to the type IIB superstring theory on \( AdS_5 \times S^5 \) [18]. A natural question is whether the relation between the \( \mathcal{N} = 4 \) SYM and the Riemann hypothesis can be translated to the string theory language. Here we provide such a relation based on a recent proposal on an interpretation of the Schur index of the \( \mathcal{N} = 4 \) SYM from the gravity side for finite \( N \) [19] (see also [30–32]).

Before that, let us recall the dictionary of the AdS/CFT correspondence [18]. The plank length \( \ell_p \) and the string length \( \ell_s \) in the unit of the AdS radius \( R \) are related to \( N \) and the ‘t Hooft coupling \( \lambda \) by
\[ \ell_p R = \left( \frac{4\pi N}{1/4} \right), \quad \ell_s R = \lambda^{-1/4}. \] (3.1)

In the large \( N \)-limit and strong ‘t Hooft coupling limit, the \( \mathcal{N} = 4 \) SYM should be approximated by the type IIB supergravity on \( AdS_5 \times S^5 \). Correspondingly, it is known that the superconformal index of the \( \mathcal{N} = 4 \) SYM in the large-\( N \)-limit is described by Kaluza-Klein (KK) modes of the supergravity multiplet [24]. When \( N \) is finite, the dictionary tells us that quantum gravity effect becomes important. In particular, rewriting the dictionary in terms of the D3-brane tension \( T_{D3} \) as
\[ \frac{T_{D3}}{R^4} = \frac{N}{2\pi^2}, \] (3.2)

it is clear that the D3-branes have finite tensions for finite \( N \) and physical quantities likely get effects of extended D3-branes.

The proposal in [19] manifests the above intuitions for the Schur index. The authors [19] first decomposed the Schur index of the \( U(N) \) \( \mathcal{N} = 4 \) SYM as
\[ I_{U(N)}(q) = I_{KK}(q) \sum_{n=0}^{\infty} I_{BDF}^{n}(q; N), \] (3.3)

where \( I_{KK}(q) \) is the Schur index for the \( U(\infty) \) case, equivalent to the contribution from the KK modes:
\[ I_{KK}(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n-1})}. \] (3.4)

---

6 The ‘t Hooft coupling is irrelevant in this discussion since the superconformal index is insensitive to \( \lambda \).

7 In terms of plethystic exponential, it is written as \( I_{KK}(q) = \text{Pexp} \left( \frac{2q}{1-q} - \frac{q^2}{1-q} \right) \).
The second factor $I_n^{BDF}(q; N)$, based on the exact computation in [14], is given by

$$I_n^{BDF}(q; N) = \sum_{n=0}^{\infty} (-1)^n (n + nC_N + N + n - 1C_N) q^{nN + n^2}. \quad (3.5)$$

The proposal in [19] is about an interpretation of the object $I_n^{BDF}(q; N)$ from the string theory side: it comes from contributions from D3-branes wrapping SUSY cycles. Concretely, there are 1/8-BPS brane configurations on the string theory side [33] which are given by the intersection of a holomorphic surface $h(X, Y, Z) = 0$ and $S^5 = \{(X, Y, Z)| |X|^2 + |Y|^2 + |Z|^2 = 1\}$ [34]. Then the authors in [19] proposed

$$I_n^{BDF}(q; N) = \sum_{k=0}^{n} I_{(n-k,k)}(q; N), \quad (3.6)$$

where $I_{(n_1,n_2)}$ denotes the contribution from $n_1$ D3-branes wrapped on $X = 0$ and $n_2$ D3-branes wrapped on $Y = 0$ (see [19] for details). To get the result for the $SU(N)$ case, we can simply use the following relation

$$I_{SU(N)}(q) = I_{U(N)}(q) \prod_{n=0}^{\infty} \frac{(1 - q^{2n+1})^2}{(1 - q^{2n+2})^2}. \quad (3.7)$$

Then we find the Schur index for the $SU(N)$ case as

$$I_{SU(N)}(q) = \left[ \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n}} \right] \sum_{n=0}^{\infty} I_n^{BDF}(q; N). \quad (3.8)$$

Note that the first factor is the same as the generating function for the number of $M$-colored partitions with $M = 3$:

$$\prod_{n=1}^{\infty} \frac{1}{1 - z^n} M = \sum_{\tilde{Y}} z^{|\tilde{Y}|}, \quad (3.9)$$

where $\tilde{Y}$ denotes the set of $M$ Young diagrams of $U(M)$ and $|\tilde{Y}|$ is the total number of boxes of $\tilde{Y}$.

To see a relation to the Riemann hypothesis, let us go back to the $SU(3)$ case:

$$I_{SU(3)}(q) = \left[ \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n}} \right] \sum_{n=0}^{\infty} I_n^{BDF}(q; 3) = \left[ \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n}} \right] \sum_{n=0}^{\infty} (-1)^n \frac{(n + 1)(n + 2)(2n + 3)}{6} q^{nN + n^2}. \quad (3.10)$$

Compared with the expression (1.5), the RHS in (3.10) must generate the sum of divisors function $\sigma(n)$ and the Riemann hypothesis claims that it is bounded by $H_n + e^{H_n} \log H_n$, which is asymptotically $O(n \log \log n)$ for large $n$. This implies that we have a kind of miraculous cancellations between the first and second factors in (3.10); the first factor generates factorially growing coefficients with positive definite sign while the second one gives $O(n^3)$ coefficients with alternate signs. Let us explicitly demonstrate it. The first factor in (3.10) is expanded as

$$\prod_{n=1}^{\infty} \frac{1}{1 - q^{2n}} \quad = \quad 1 + 3q^2 + 9q^4 + 22q^6 + 51q^8 + 108q^{10} + 221q^{12} + 429q^{14} + 810q^{16} + 1479q^{18} + 2640q^{20} + 4599q^{22} + 7868q^{24} + 13209q^{26} + 21843q^{28} + 35581q^{30} + \mathcal{O}(q^{31}) \quad (3.11)$$

It may be interesting to note that this factor is formally the same as the Nekrasov instanton partition function [35] of $U(M)$ $\mathcal{N} = 2^*$ theory with vanishing equivariant mass [36][37] although the parameter $q$ here is not apparently related to complex gauge coupling.
while the second factor is expanded as

\[ \sum_{n=0}^{\infty} I_n^{BDF}(q; 3) = 1 - 5q^4 + 14q^{10} - 30q^{18} + 55q^{28} + O(q^{40}). \]  

(3.12)

Multiplying the two factors, we find

\[ I_{SU(3)}(q) = 1 + 3q^2 + 4q^4 + 7q^6 + 6q^8 + 12q^{10} + 8q^{12} + 15q^{14} + 13q^{16} + 18q^{18} + 12q^{20} + 28q^{22} + 14q^{24} + 24q^{26} + 24q^{28} + 31q^{30} + O(q^{31}), \]  

(3.13)

whose coefficients are indeed the sum of divisors function \( \sigma(n) \). The cancellations may imply that there is a more appropriate string language which describes the net contributions to the Schur index more directly. This might be something like bound states of the KK modes and the wrapping D3-branes.

### 4 Discussions

In this paper we have pointed out the new relations among the \( \mathcal{N} = 4 \) SYM, string theory and Riemann hypothesis. We started with noting the fact that the Schur index of the \( SU(3) \mathcal{N} = 4 \) SYM is the generating function of the sum of divisors function \( \sigma(n) \) [13–17], which appears in the statement (1.2) equivalent to the Riemann hypothesis. In this identification, \( \sigma(n) \) corresponds to the difference between the numbers of bosonic and fermionic \( 1/8 \)-BPS states with the quantum number \( E - R = n - 1 \), and the Riemann hypothesis claims the upper bound on that. Then the AdS/CFT correspondence allows us to interpret it from the viewpoint of the string theory. According to the proposal in [19], the Schur index has contributions from the KK modes of the supergravity multiplet and wrapping D3-branes, and their complicated combinations give the sum of divisors function after the miraculous cancellations.

An immediate question is whether it is just a coincidence or there is a deep physical origin. One might also ask why the \( SU(3) \mathcal{N} = 4 \) SYM is “selected”. At least the structure discussed in this paper does not seem unique for the \( SU(3) \mathcal{N} = 4 \) SYM because there is another theory called \( \hat{D}_4 \cdot SU(3) \mathcal{N} = 2 \) superconformal theory which has the same Schur index as the \( SU(3) \mathcal{N} = 4 \) SYM after the rescaling \( q \rightarrow q^2 \) [15]. Regardless of these questions, it would be extremely interesting if one can give some implications on the Riemann hypothesis from the physics side. One possible scenario would be that some constraints by physical requirements could (un)support the Riemann hypothesis. For example, while unitarity bound would be too obvious, we could consider constraints by quasi-modularity [16,38], information theoretic inequalities [39,40] and/or something else for the Schur index.

It may be also useful to look at the Schur index from different viewpoints. In this paper we have discussed the Schur index from the viewpoint of SUSY states counting. It is also known that the superconformal index is proportional to a SUSY partition function on a space with a topology of \( S^1 \times S^3 \):

\[ Z_{S^1 \times S^3} = e^{-E_{\text{SUSY}}} I_{\text{SCI}}, \]  

(4.1)

where \( E_{\text{SUSY}} \) is so-called supersymmetric Casimir energy [41]. Then the AdS/CFT correspondence tells us that this is dual to the partition function of the type IIB superstring theory on \( AdS_5 \times S^5 \). It might be helpful to interpret the relation to the Riemann hypothesis from this viewpoint.

The Schur index of the \( \mathcal{N} = 4 \) SYM is also known to correspond to physical quantities in other theories. First, it is known that the Schur index of the \( SU(N) \mathcal{N} = 4 \) SYM corresponds to the partition function of the 2d \( SU(N) \) \( q \)-deformed Yang-Mills theory on torus in the zero area limit [21,42] in a similar spirit to the AGT relation [43]. Because the \( q \)-deformed YM appears also in a problem of counting BPS black holes in type II superstring on a local Calabi-Yau threefold [44], this gives another connection between the string theory and Riemann hypothesis. Second, it is also expected that the Schur index of the \( \mathcal{N} = 4 \) SYM with gauge group \( G \) corresponds to a torus partition function of a chiral algebra called an \( \mathcal{N} = 4 \) super-W algebra.
with rank($G$) generators [45]. Because there is also an argument which derives a chiral algebra structure from the SUSY localization on $AdS_5$ [46], it may also imply a new relation between the string theory and Riemann hypothesis. These viewpoints may give some insights on the Riemann hypothesis.

There may be also a connection to black hole microstate counting as the superconformal index is known to account for SUSY black hole entropy on the string theory side [47-50], similar to the spirit of the seminal work by Strominger-Vafa [51]. Although the Schur index itself does not seem to capture the black hole entropy due to cancellations, the Macdonald index, which counts the same sector as the Schur index but with a refinement, has a limit to reproduce the black hole entropy [47]. To explore such a connection, it may be useful to consider the “Cardy limit” [52-54]. A connection between the black hole and the Riemann hypothesis might be expected also by noting the two intuitions that black holes are related to chaos (see e.g. [55,56]) and chaos is related to the Riemann hypothesis (see e.g. [3-5]).

In this paper we have focused on connections between string theory and the Riemann hypothesis for the Riemann zeta function. It would be illuminating to explore connections for other types of zeta functions. For example, while we have focused on the $SU(3)$ case of the $\mathcal{N}=4$ SYM, the Schur indices for other gauge groups may be related to more general zeta functions. One possible hint may be the fact that there appear Macmahon’s generalized sum of divisors function [57] and more general Eisenstein series for $SU(N)$ case [15,16].

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