Steady-state Analysis of an $M/M/2$ Queueing System Operating in a Multi-phase Random Environment Subject to Disaster and Repair

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Abstract

In this research article, an $M/M/2$ queueing system operating in a multi-phase random environment subject to disaster and repair is studied. The random environment discussed here has $N$ phases and the $k$-th phase is exponentially distributed with mean $1/\eta_k$, $k = 1, 2, \ldots, N$. The queueing system behaves like $M(\lambda_k)/M(\mu_k)/2$ while in the $k$-th phase. At the end of the $k$-th phase, a disaster occurs where the servers are taken for repair and all the customers in the system are wiped out. Both the servers are repaired jointly and the repair time is exponentially distributed with mean $1/\eta_0$. Immediately after the repair completions, the system goes to phase $k$ with probability $q_k$. During repair time, Customers are permitted to enter the queueing system. The probabilistic behaviour of the system is studied in steady state by using probability generating function technique. Some performance measures of this queueing model are also obtained.

Key-words: Disaster, Wash-Out, Repair, Random Environment, Steady State Analysis.

AMS Subject Classification: 60K20, 60K25, 90B22

1. Introduction

Several researchers have studied many-server queueing systems (see, for example Karlin and McGregor [4], Takacs [11], Kleinrock [5], Natarajan [9], and Dharmaraja and Rakesh Kumar [3]). A few researchers have obtained transient solution of many-server queueing systems (see Parthasarathy
and Sharafali [10], Krishna Kumar and Arivudainambi [6], Krishna Kumar and Madheswari [7] and Seedy et al. [1]). On the other hand, many authors have paid their attention to study some special class of queueing systems subject to impatience of customers and/or randomly occurring disasters (see for example, Sengupta [12], Chakravarthy [2], Krishna Kumar et al. [8]). The steady-state behavior of an $M/M/1$ queue operating in random environment subject to disasters where the underlying environment is described by a $n$ phase continuous-time Markov chain have been analyzed. The time-dependent behaviour of the above said model has also been analyzed. However, to the best of our knowledge, $M/M/2$ queue systems operating in a random environment subject to disasters and repair have not been studied so far in literature. In this research paper, we block this gap by obtaining a steady-state analysis of a M/M/2 queueing system operating in an uncertain random environment subject to disasters and repairs. The sections in this paper are arranged in the following manner: Section 2 briefly describes the model. In section 3, the time-dependent probabilities of the system are discussed and its the integral equations are derived. In section 4, explicit expressions of the steady-state probabilities of the above mentioned queueing system are obtained.

2. Model Description

Let us consider a two server queueing system working in a random environment. We assume that the environment is in any one of the $m+1$ states $0, 1, 2, \ldots, N$. The environmental state 0 represents the state that the servers are jointly undergoing repair. The random repair time is an exponential random variable distributed with mean $\frac{1}{\eta_0}$. In the course of repair time, customers join the system according to Poisson process with arrival rate $\lambda_0$. Immediately after the repair, the system goes to phase $k, k = 1, 2, \ldots, N$ with probability $q_k$; such that $\sum_{k=1}^{N} q_k = 1$. The system resides in phase $k$ for a random interval of time that is exponentially distributed with mean $1/\eta_k$ and at the end of the residing period, all customers in the system are washed out and the system moves to phase 0: When the environment is in phase $k$, the system behaves like an $M(\lambda_k)/M(\mu_k)/2$ queue with arrival rate $\lambda_k$ and service rate $\mu_k$.

During time $t = 0$, we presume that a catastrophe has just occurred so that the system is in phase 0 (repair state). Let $E(t)$ denote the phase of the environment at time $t$ and at time $t$, let $X(t)$ denotes
the number of customers in the queueing system. Then the joint process \( \{X(t), E(t)\}_{t \geq 0} \) is a continuous time parameter Markov process whose state space is
\[
\{(j, k) \mid j = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots, N\}
\]

We define the state probability as follows:
\[
\{p(j, k, t) = \Pr[X(t) = j, E(t) = k | X(0) = 0, E(0) = 0] \} = \{0, 1, 2, \ldots, N; j = 0, 1, 2, \ldots\} \quad (1)
\]

In the next section, we derive the integral equations for \( p(j, k, t) \)

3. To Obtain the Integral Equations in Steady-State

Using the convolution operation
\[
f(t) \Theta g(t) = \int_0^t f(u)g(t-u)du \quad (2)
\]
and renewal theoretic arguments, we find the following integral equations:

Case 1: \( j = 0, k = 0 \)
\[
p(0, 0, t) = e^{-(\lambda_0 + \eta_0)t} + \sum_{k=1}^{N} \sum_{j=0}^{k} p(j, k, t) \eta_k \Theta e^{-(\lambda_0 + \eta_0)t} \quad (3)
\]

Case 2: \( j = 1, 2, \ldots, k = 0 \)
\[
p(j, 0, t) = p(j-1, 0, t) \lambda_0 \Theta e^{-(\lambda_0 + \eta_0)t} \quad (4)
\]

Case 3: \( j = 0, k = 1, 2, \ldots, N \)
\[
p(0, k, t) = p(0, 0, t) \eta_0 q_k \Theta e^{-(\lambda_k + \eta_k)t} + p(1, k, t) \mu_k \Theta e^{-(\lambda_k + \eta_k)t} \quad (5)
\]

Case 4: \( j = 1, k = 1, 2, \ldots, N \)
\[
p(1, k, t) = p(1, 0, t) \eta_0 q_k \Theta e^{-(\lambda_k + \mu_k + \eta_k)t} + p(2, k, t) 2\mu_k \Theta e^{-(\lambda_k + \mu_k + \eta_k)t} \quad (6)
\]

Case 5: \( j = 2, 3, \ldots, k = 1, 2, \ldots, N \)
\[
p(j, k, t) = p(j, 0, t) \eta_0 q_k \Theta e^{-(\lambda_k + 2\mu_k + \eta_k)t} + p(j-1, k, t) \lambda_k \Theta e^{-(\lambda_k + 2\mu_k + \eta_k)t} + p(j+1, k, t) 2\mu_k \Theta e^{-(\lambda_k + \mu_k + \eta_k)t} \quad (7)
\]
4. **Steady-state Solution**

The limiting state probabilities are defined by

$$\pi(j, k) = \lim_{t \to \infty} p(j, k, t)$$

(8)

Using the Laplace transform’s final value theorem, equation (8) gives

$$\pi(j, k) = \lim_{s \to 0} sp^*(j, k, s),$$

(9)

Where \( p^*(j, k, t) \) is the Laplace transform of \( p(j, k, t) \) Applying Laplace transform on both sides of equations (3)-(7), we obtain

\[
(s + \lambda_0 + \eta_0)p^*(0,0,s) = 1 + \sum_{k=0}^{N} \sum_{j=0}^{\infty} p^*(j,k,t)\eta_k,
\]

(10)

\[
(s + \lambda_0 + \eta_0)p^*(j,0,s) = p^*(j-1,0,s)\lambda_0, j=1,2,\ldots
\]

(11)

\[
(s + \lambda_k + \eta_k)p^*(0,k,s) = p^*(0,0,s)\eta_0q_k + p^*(1,k,s)\mu_k, k=1,2,\ldots, N;
\]

(12)

\[
(s + \lambda_k + \mu_k + \eta_k)p^*(1,k,s) = p^*(1,0,s)\eta_0q_k + p^*(0,k,s)\lambda_k + p^*(2,k,s)2\mu_k, k=1,2,\ldots, N;
\]

(13)

\[
(s + \lambda_k + 2\mu_k + \eta_k)p^*(j,k,s) = p^*(j,0,s)\eta_0q_k + p^*(j-1,k,s)\lambda_k + p^*(j+1,k,s)2\mu_k, j=2,3,: k=1,2,\ldots, N.
\]

(14)

Multiplying both sides of equations (10)-(14) with \( s \) and applying equation (9), we obtain

\[
(\lambda_0 + \eta_0)\pi(0,0) = 1 + \sum_{k=1}^{N} \sum_{j=0}^{\infty} \pi(j,k)\eta_k,
\]

(15)

\[
(\lambda_0 + \eta_0)\pi(j,0) = \pi(j-1,0)\lambda_0, j=1,2,\ldots;
\]

(16)

\[
(\lambda_k + \eta_k)\pi(0,k) = \pi(0,0)\eta_0q_k + \pi(1,k)\mu_k, k=1,2,\ldots, N
\]

(17)

\[
(\lambda_k + \mu_k + \eta_k)\pi(1,k) = \pi(1,0)\eta_0q_k + \pi(0,k)\lambda_k + \pi(2,k)2\mu_k, k=1,2,\ldots, N
\]

(18)

\[
(\lambda_k + 2\mu_k + \eta_k)\pi(j,k) = \pi(j,0)\eta_0q_k + \pi(j-1,k)\lambda_k + \pi(j+1,k)2\mu_k,
\]

\[
 j = 2,3,: k=1,2,\ldots, N
\]

(19)
The system of equations (16)-(19) together with the total probability law can be explicitly solved. The result is given in the following theorem:

4.1 Theorem. The limiting state probabilities are given by

\[ \pi(0,0) = \frac{1}{(\lambda_0 + \eta_0) \left[ \frac{1}{\eta_0} + \sum_{k=1}^{N} \frac{q_k}{\eta_k} \right]} ; \]

\[ \pi(j,0) = \left( \frac{\lambda_0}{\lambda_0 + \eta_0} \right)^j \pi(0,0), j = 1, 2, \ldots ; \]

\[ \pi(0,k) = \frac{\eta_0 q_k \beta_k \left[ \frac{\lambda_0 + \eta_0}{\lambda_0 + \eta_0 - \lambda_0 \beta_k} + (1 - \beta_k) \right]}{(1 - \beta_k) \left( \lambda_0 + \eta_0 - \lambda_0 \beta_k + 2 \mu_k \right)} \pi(0,0), k = 1, 2, \ldots N ; \]

\[ \pi(j, k) = \eta_0 q_k \left\{ \sum_{i=0}^{j} \sum_{r=0}^{\infty} \pi(j - 1 + r, 0) \frac{\lambda_k^i \beta_k^{l+r+1}}{\mu_k^{l+r+1}} \right\} \]

+ \pi(0,k) \left[ \frac{\lambda_k^0 \beta_k^{l+1}}{\mu_k^{l+1}} + \left\{ 2 \mu_k - (\lambda_k + \eta_k)(1 - \beta_k) \right\} \frac{\lambda_k^j \beta_k^{l+1}}{\mu_k^{l+1}} \right] \]

+ \pi(0,0) \eta_0 q_k \left[ \frac{(1 + \beta_k) \lambda_k^j \beta_k^{l+1}}{\mu_k^{l+1}} - \frac{\lambda_k^{l+1} \beta_k^j}{\mu_k^{l+1}} \right], j = 1, 2, \ldots ; k = 1, 2, \ldots, N , \]

where \( \beta_k = \frac{(\lambda_k + 2 \mu_k + \eta_k) - \sqrt{(\lambda_k - 2 \mu_k + \eta_k)^2 + 8 \mu_k \eta_k}}{2 \lambda_k} \).

Proof:

From equation (16), we get \( \pi(j,0) = \left( \frac{\lambda_0}{\lambda_0 + \eta_0} \right)^j \pi(0,0), j = 1, 2, \ldots ; \)

(20)

\[ G_k(\theta) = \sum_{j=0}^{\infty} \pi(j, k) \theta^{j-k} = 0, 1, 2, \ldots, N. \]

define

From equation (20), we get \( G_0(\theta) = \frac{(\lambda_0 + \eta_0)}{(\lambda_0 + \eta_0 - \lambda_0 \theta)} \pi(0,0) \).

(21)

Putting \( \theta = 1 \) in equation (21), we get \( G_0(1) = \frac{(\lambda_0 + \eta_0)}{\eta_0} \pi(0,0) \).

(22)

We Multiply equation (19) by \( \theta^j \) on both sides and summing from \( \theta = 2 \) to \( \infty \), we get...
(\lambda_k + 2\mu_k + \eta_k) \sum_{j=2}^{\infty} \pi(j,k) \theta^j = \eta_0 q_k \sum_{j=2}^{\infty} \pi(j,0) \theta^j + \lambda_k \sum_{j=2}^{\infty} \pi(j-1,k) \theta^j + 2\mu_k \sum_{j=2}^{\infty} \pi(j+1,k) \theta^j. \hspace{1cm} (23)

Simplifying equation (23) by using equations (17) and (18), we get

\begin{align*}
G_k(\theta) &= \frac{\eta_0 q_k \theta [G_k(\theta) + \pi(0,0)(1-\theta)] - \pi(0,k)[1-\theta][\lambda_k + \eta_k \theta + 2\mu_k]}{\lambda_k + 2\mu_k + \eta_k \theta - \lambda_k \theta^2 - 2\mu_k} k=1,2,\ldots,N. \hspace{1cm} (24)
\end{align*}

Putting \theta=1 in (24) and using (22), we get

\begin{align*}
G_k(1) = \frac{(\lambda_0 + \eta_0) \pi(0,0)}{\eta_k} k=1,2,\ldots. \hspace{1cm} (25)
\end{align*}

By total probability law, we get \sum_{k=0}^{N} \sum_{j=0}^{\infty} \pi(j,k) = 1. \hspace{1cm} (26)

From equation (26), we get \sum_{k=0}^{N} G_k(1) = 1 \hspace{1cm} (27)

Substituting equations (22) and (25) in equation (27), we obtain

\begin{align*}
\pi(0,0) = \frac{1}{(\lambda_0 + \eta_0) \left[ \frac{1}{\eta_0} + \sum_{k=1}^{N} \frac{q_k}{\eta_k} \right]} \hspace{1cm} (28)
\end{align*}

It is evident that \( G_k(\theta) \) is analytic inside the unit disk \(|\theta|<1\). Consequently, the numerator of the right hand side of (24) disappears at the zeros of the denominator of the right hand side of (24). The zeros of the denominator of the right hand side of (24) are given by

\begin{align*}
(\lambda_k + 2\mu_k + \eta_k) \theta - \lambda_k \theta^2 - 2\mu_k = 0. \hspace{1cm} (29)
\end{align*}

The discriminant of the quadratic equation (29) is

\begin{align*}
(\lambda_k + 2\mu_k + \eta_k)^2 - 8\lambda_k \mu_k = (\lambda_k + 2\mu_k + \eta_k)^2 + 8\lambda_k \mu_k.
\end{align*}

Hence both the roots of equation (29) are real, and they are given by

\begin{align*}
\alpha_k &= \frac{(\lambda_k + 2\mu_k + \eta_k) + \sqrt{(\lambda_k + 2\mu_k + \eta_k)^2 + 8\lambda_k \mu_k}}{2\lambda_k}, \hspace{1cm} (30)
\beta_k &= \frac{(\lambda_k + 2\mu_k + \eta_k) - \sqrt{(\lambda_k + 2\mu_k + \eta_k)^2 + 8\lambda_k \mu_k}}{2\lambda_k}. \hspace{1cm} (31)
\end{align*}

The product of the roots of equation (29) is \( \alpha_k \beta_k = \frac{2\mu_k}{\lambda_k} \). For stable solution, we assume that
$$2\mu_k > \lambda_k.$$ Then $2\mu_k + \eta_k > 2\mu_k > \lambda_k$. It is easy to establish that $0 < \beta_k < 1$ Further, we find that $\alpha_k > 1$. Using the roots $\alpha_k$ and $\beta_k$, equation (24) leads to

$$G_k(\theta) = \eta_0 q_k \theta [G_0(\theta) + \pi(0,0)(1-\theta)] - \pi(0,k)(1-\theta)[(\lambda_k + \eta_k)\theta + 2\mu_k]$$

$$\lambda_k(\alpha_k - \theta)(\theta - \beta_k)$$

$k = 1, 2, \ldots, N$ (32)

Invoking the analyticity of $G_k(\theta)$ in $|\theta| < 1$, the numerator on the right hand side of equation (32) vanishes at $\theta = \beta_k$. Consequently, we get

$$\eta_0 q_k \beta_k [G_0(\beta_k) + \pi(0,0)(1-\beta_k)] - \pi(0,k)(1-\beta_k)[(\lambda_k + \eta_k)\beta_k + 2\mu_k] = 0$$

Using equation (33), we get

$$\pi(0,k) = \frac{\eta_0 q_k \beta_k \left[ (\lambda_0 + \eta_0) (\lambda_0 + \eta_0 - \lambda_0 \beta_k) \right]}{(1-\beta_k) (\lambda_k + \eta_k) \beta_k + 2\mu_k}] \pi(0,0), k = 1, 2, \ldots, N$$

Using equation (33) in equation (32), and after much algebraic simplification, we obtain

$$G_k(\theta) = \eta_0 q_k \sum_{j=0}^{\infty} \left\{ \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \pi(j-1+r,0) \frac{\lambda_k^i \beta_k^{i+1}}{2^i \mu_k^{i+1}} \right\} \theta^j$$

$$+ \pi(0,k) \left[ (\lambda_k + \eta_k) \sum_{j=0}^{\infty} \frac{\lambda_k^j \beta_k^{j+1} \theta^j}{2^j \mu_k^{j+1}} + \sum_{i=1}^{\infty} \frac{\lambda_k^i \beta_k^i \theta^i}{2^i \mu_k^{i+1}} \right] + \pi(0,0) \eta_0 q_k \left[ (1 + \beta_k) \sum_{j=0}^{\infty} \frac{\lambda_k^j \beta_k^{j+1} \theta^j}{2^j \mu_k^{j+1}} - \sum_{i=1}^{\infty} \frac{\lambda_k^i \beta_k^i \theta^i}{2^i \mu_k^{i+1}} \right]$$

$k = 1, 2, \ldots, N$ (35)

Equating the coefficients of $\theta^0$ on both sides of equation (35), we get back equation (34).

Equating the coefficients of $\theta^j$, $j = 1, 2, \ldots$, on both sides of equation (35), we obtain

$$\pi(j,k) = \eta_0 q_k \left\{ \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \pi(j-1+r,0) \frac{\lambda_k^i \beta_k^{i+1}}{2^i \mu_k^{i+1}} \right\}$$

$$+ \pi(0,k) \left[ (\lambda_k + \eta_k) \frac{\lambda_k^j \beta_k^{j+1} \theta^j}{2^j \mu_k^{j+1}} + \sum_{r=1}^{\infty} \frac{\lambda_k^i \beta_k^i \theta^i}{2^i \mu_k^{i+1}} \right]$$

$$+ \pi(0,0) \eta_0 q_k \left[ (1 + \beta_k) \frac{\lambda_k^j \beta_k^{j+1} \theta^j}{2^j \mu_k^{j+1}} - \sum_{i=1}^{\infty} \frac{\lambda_k^i \beta_k^i \theta^i}{2^i \mu_k^{i+1}} \right], j = 1, 2, \ldots; k = 1, 2, \ldots, N$$

(36)

The proof is now complete.
5. Performance Measures

To illustrate, we assume the below mentioned values for the parameters of our model subject to the stability condition $2\mu_k > \lambda_k, k = 1,2,\ldots, N$:

$$\eta_0 = 1.5; \lambda_0 = 1.2;$$

| $q_1$ | $\eta_1$ | $\lambda_1$ | $\mu_1$ |
|------|---------|-------------|--------|
| 0.2  | 0.03    | 1.8        | 2.6    |
| $q_2$ | $\eta_2$ | $\lambda_2$ | $\mu_2$ |
| 0.3  | 0.04    | 1.6        | 2.4    |
| $q_3$ | $\eta_3$ | $\lambda_3$ | $\mu_3$ |
| 0.2  | 0.01    | 1.5        | 2.5    |
| $q_4$ | $\eta_4$ | $\lambda_4$ | $\mu_4$ |
| 0.2  | 0.02    | 1.7        | 2.7    |
| $q_5$ | $\eta_5$ | $\lambda_5$ | $\mu_5$ |
| 0.1  | 0.03    | 1.4        | 2.4    |

We have computed the steady-state probabilities $\pi(j,k)$ and obtained the following table:

| $j$ | $\pi(j,0)$ | $\pi(j,1)$ | $\pi(j,2)$ | $\pi(j,3)$ | $\pi(j,4)$ | $\pi(j,5)$ |
|-----|--------------|-------------|-------------|-------------|-------------|-------------|
| 0   | 0.0077       | 0.0672      | 0.0779      | 0.2235      | 0.1082      | 0.0379      |
| 1   | 0.0034       | 0.0465      | 0.0518      | 0.1341      | 0.068       | 0.0221      |
| 2   | 0.0015       | 0.0161      | 0.0173      | 0.0403      | 0.0215      | 0.0065      |
| 3   | 0.0007       | 0.0056      | 0.0058      | 0.0121      | 0.0068      | 0.0019      |
| 4   | 0.0003       | 0.0019      | 0.002       | 0.0037      | 0.0022      | 0.0006      |
| 5   | 0.0001       | 0.0007      | 0.0007      | 0.0011      | 0.0007      | 0.0002      |
| 6   | 0.0001       | 0.0002      | 0.0002      | 0.0003      | 0.0002      | 0.0001      |
| 7   | 0            | 0.0001      | 0.0001      | 0.0001      | 0.0001      | 0          |
| 8   | 0            | 0           | 0           | 0           | 0           | 0           |
| 9   | 0            | 0           | 0           | 0           | 0           | 0           |
| 10  | 0            | 0           | 0           | 0           | 0           | 0           |

5.1 Mean Queue Size in $k$-th phase, $k = 0,1,2,\ldots, N$

Let $E[L_k]$ denote the stationary mean number of customers in the $k$-phase. Then, we have

$$E[L_k] = \sum_{j=1}^{\infty} j \pi(j,k)k = 0,1,2,\ldots, N. \quad (37)$$

Then, the stationary mean total number of customers in the queueing system is obtained by

$$E[L] = \sum_{j=0}^{\infty} \sum_{k=0}^{N} j \pi(j,k) = \sum_{k=0}^{N} \sum_{j=0}^{\infty} j \pi(j,k) = \sum_{k=0}^{N} E[L_k]k = 0,1,2,\ldots, N. \quad (38)$$
5.2 Mean Number of Customers Washed Out by Disasters per Unit Time from the System

If \( C \) denote the number of customers wiped out from the queueing system per unit time, then

\[
E[C] = \sum_{k=1}^{N} \eta_i \sum_{j=1}^{\infty} j \pi(j, k) = \sum_{k=1}^{N} \eta_i E[L_k]. \tag{39}
\]

Using table 1, we obtain the following table of mean values:

\[
E[L_0] = 0.0114, E[L_1] = 0.1091, E[L_2] = 0.1173, E[L_3] = 0.2743
\]

\[
E[L_4] = 0.1455, E[L_5] = 0.0444, E[L] = 0.7017, E[C] = 0.0149
\]

6. Conclusion

Steady-state analysis of an \( M/M/2 \) queueing system is discussed which was operating on a multi-phase uncertain random environment subject to disaster and repair and its steady state probability is obtained. Further, we also seek the characteristics of \( M/M/2 \) queueing system which operates in a multi-phase uncertain environment subject to repair and disaster.

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