Superfluidity of trapped dipolar Fermi gases.

M.A. Baranov$^{1,2}$, L. Dobrek$^1$, and M. Lewenstein$^1$

(1) Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany
(2) Russian Research Center Kurchatov Institute, Kurchatov sq. 1, 123183 Moscow, Russia

(Dated: June 7, 2018)

We derive the phase diagram for ultracold trapped dipolar Fermi gases. Below the critical value of the dipole-dipole interaction energy, the BCS transition into a superfluid phase ceases to exist. The critical dipole strength is obtained as a function of the trap aspect ratio. The order parameter exhibits a novel behavior at the criticality.

PACS numbers: 03.75.Kk, 42.50.Vk

The quest for the superfluid (BCS) transition in trapped Fermi gases is one of the most challenging goals of modern atomic, molecular and optical physics. The possibility of the BCS transition for trapped gases with attractive short range interactions has been predicted in Ref. [9], and has been a subject of extensive experimental research since then. In typical experiments evaporative cooling is used to cool fermions. However, as the Pauli principle forbids the s-wave scattering for fermions in the same internal state, Fermi-Fermi or Fermi-Bose mixtures have to be used to assure collisional thermalization of the sample. Such a combination of evaporation and sympathetic cooling allows to reach temperatures $T \approx 0.1 T_F$, where $T_F$ is the Fermi temperature at which the gas exhibits quantum degeneracy. Unfortunately, predicted critical temperatures for the BCS transition, $T_c$, are much smaller than $T_F$. One way to circumvent this difficulty is to increase $T_c$: this may be achieved by increasing the strength of the atomic interactions employing a Feshbach resonance, which allows to make the atomic scattering length $a_s$ large negative. Such “resonance superfluidity” should lead to superfluid transition at $T_c \approx 0.1 T_F$. Another way is to use the cooling scheme that can overcome the effects of Pauli blocking, such as appropriately designed laser cooling. Yet another promising route is to locate the ultracold gas in an optical trap and enter the strongly correlated regime by controlling the lattice potential.

The temperature of the BCS transition in a two-component Fermi gas, however, depends also drastically on the difference of the concentrations of the two components, which presents another experimental obstacle. This problem is not relevant for a polarized Fermi gas with long-range interactions, such as dipole-dipole interactions, and that is why there has been a considerable interest recently in studying the BCS transition in dipolar Fermi gases. The possibility of the Cooper pairing has been predicted in Refs. [8]. The critical temperature (including many-body corrections) and the other parameters have been obtained in Ref. [8]. At this point it is worth mentioning that possible realizations of dipolar gases include ultracold heteronuclear molecular gases [10], atomic gases in a strong DC electric field [11], atomic gases with laser-induced dipoles [12], or with magnetic dipoles [13]. For dipolar moments $d$ of the order of one Debye and densities $n$ of $10^{12} \text{cm}^{-3}$, $T_c$ should be in the range of $100 \text{nK}$, i.e. experimentally feasible.

Dipole-dipole interaction is not only of long-range, but also anisotropic, i.e. partially attractive and partially repulsive. Thus for trapped gases, the nature of the interaction may be controlled by the geometry of the trap. For a dipolar Bose gas in a cylindrical trap with the axial (radial) frequency $\omega_z (\omega_r)$, there exist a critical aspect ratio $\lambda = (\omega_z / \omega_r)^{1/2}$, above which the Bose-condensed gas collapses if the atom number is too large [13], and below which the condensate exhibits the roton-maxon instability [13]. The effects of trap geometry are expected to dominate also the physics of trapped dipolar Fermi gases [14]. So far, we have only been able to derive analytic corrections to $T_c$ in ”loose” traps, and to consider the case of an infinite ”slab“ ($\omega_r = 0$, $\omega_z$ is finite). In this case there exists a critical frequency above which the dipole interaction is predominantly repulsive, and the superfluid phase ceases to exist.

In this Letter we present the ultimate solution of the fundamental problem of the effect of trap geometry on the BCS transition in trapped dipolar Fermi gases. We calculate the phase diagram in the plane $\Gamma - \lambda^{-1}$, where $\Gamma \sim nd^2 / \mu$ is the dipole-dipole interaction energy per particle in the units of the chemical potential $\mu$. Below the critical value $\Gamma < \Gamma_c$, the BCS transition does not take place. We determine dependence of $\Gamma_c$ on $\lambda$, and calculate the order parameter at the criticality. The order parameter exhibits a novel oscillatory behavior in strongly elongated cylindrical traps.

We consider a dipolar single-component Fermi gas in a cylindrically symmetric trap described by the Hamiltonian

$$\hat{H} = \int \hat{\psi}^\dagger (\mathbf{r}) \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}} (\mathbf{r}) - \mu \right] \hat{\psi} (\mathbf{r}) + \frac{1}{2} \int_{\mathbf{r}, \mathbf{r}'} \hat{\psi}^\dagger (\mathbf{r}) \hat{\psi}^\dagger (\mathbf{r}') V_{\text{dip}} (\mathbf{r} - \mathbf{r}') \hat{\psi} (\mathbf{r}) \hat{\psi} (\mathbf{r}') ,$$

where $m$ is the atoms mass, $V_{\text{trap}} (\mathbf{r}) = m [\omega_z^2 x^2 + \omega_r^2 y^2] + \omega_z^2 z^2$ is the trapping potential, $\mu$ the chemical potential,
and \( V_{\text{dip}}(r) = (d^2/r^3)(1 - 3z^2/r^2) \). The dipoles are assumed to be polarized along the \( z \)-direction, \( \hat{\psi}(r) \) and \( \hat{\nu}(r) \) are the atomic creation and polarization operators that fulfill canonical anticommutation relations. Our aim is to apply the BCS theory (see, e.g., [1]) to the system described by Eq. (1), calculate the critical temperature, and the superfluid order parameter \( \Delta(r, r') = V_{\text{dip}}(r - r') \left( \hat{\psi}(r)\hat{\nu}(r') \right) \). In particular, we are interested in the critical value of the aspect ratio \( \lambda \) below which the BCS pairing does not take place for a given strength of the dipole interaction. The BCS theory leads to the gap equation which is a linear integral equation for \( \Delta(r, r') \) (a function of 6 variables) with a kernel that depends on 12 coordinates. Numerical solution of this equation is impossible without further analytical simplifications. As shown in Ref. [9], the dominant contribution to the BCS pairing comes from the \( p \)-wave scattering (contributions of higher angular momentum, although present due to the long-range character of the dipole-dipole interaction, are numerically small) with zero projection of the angular momentum on the \( z \)-axis, \( l_z = 0 \), in which the interaction is attractive (see also Ref. [8]). In the \( p \)-wave channels with \( l_z = \pm 1 \) the interaction is repulsive and, therefore, results only in renormalizations of a Fermi-liquid type. The latter are controlled by the small parameter of the theory \( \Gamma \) and will be neglected. Therefore, for the considered pairing problem we can model the dipole-dipole interaction by the following (off-shell) amplitude

\[
\Gamma_1(p, p', E) = p_2p'_2\tilde{\gamma}_1(E),
\]

(2)

where \( p \) is the incoming momentum, \( p' \) the outgoing one, and \( \tilde{\gamma}_1(E) \) some function of the energy \( E \). The amplitude \( \Gamma_1 \) describes anisotropic scattering only in the \( p \)-wave channel with the projection of the angular momentum \( l_z = 0 \). The function \( \tilde{\gamma}_1(E) \) obeys the equation

\[
\tilde{\gamma}_1(E) - \tilde{\gamma}_1(E') = \frac{1}{\Lambda} \int \frac{dp}{(2\pi)^3} \tilde{\gamma}_1(E) \left\{ \frac{p_z^2}{p^2 - E + i0} - \frac{p_z^2}{p'^2 - E' + i0} \right\} \tilde{\gamma}_1(E'),
\]

(3)

that follows from the Lipmann-Schwinger equation for the off-shell scattering amplitude, and is assumed to be negative, \( \tilde{\gamma}_1(E) < 0 \), in order to guarantee the BCS pairing. The cut-off parameter \( \Lambda \) ensures the convergence of the integral and, in fact, can be expressed in terms of the observable scattering data corresponding to on-shell scattering with \( p = p' \) and \( E = p^2/m \). It follows from the above equation that \( \tilde{\gamma}_1(E) \) is inversely proportional to \( E \), \( \tilde{\gamma}_1(E) = \tilde{\gamma}_1(2mE)^{-1} \), with some coefficient \( \tilde{\gamma}_1 \). Therefore, the on-shell amplitude is energy independent, as it should be for low-energy scattering on the dipole-dipole potential (see Ref. [17]).

The coefficient \( \gamma_1 \) determines the value of the critical temperature \( T_c \) of the BCS transition in a spatially homogeneous gas and hence, using the results of Ref. [9], can be expressed through the dipole moment \( d \) in this case the order parameter has the form \( \Delta(p) = p_2\Delta_0 \) with some constant \( \Delta_0 \), and the linearized gap equation results in the equation for the critical temperature \( T_c \):

\[
\frac{1}{\gamma_1(\mu)} = -\int \frac{dp}{(2\pi)^3} \frac{p_z^2}{2\xi_p} \left[ \tanh \frac{\xi_p}{2T_c} - 1 \right],
\]

(4)

where \( \xi_p = p^2/2m - \mu \) and the bare interaction is renormalized in terms of the scattering amplitude with \( \tilde{\gamma}_1(\mu) = \tilde{\gamma}_1/p^2_F \) at the Fermi energy \( \varepsilon_F = \mu = p^2_F/2m \) along the lines of Ref. [17] (\( p^2_F \) is the Fermi momentum).

After integrating over \( p \) we obtain the following equation for \( T_c \):

\[
1 = \frac{1}{3} |\gamma_1| \nu_F \left[ \ln \frac{2\mu}{T_c} - \frac{8}{3} - \ln \frac{\pi}{4} + C \right],
\]

(5)

where \( \nu_F = mp_F/2\pi^2 \) is the density of states at the Fermi energy and \( C = 0.5772 \) the Euler constant, and therefore \( \gamma_1 = -24d^2/\pi \) in order to reproduce the result of Ref. [9] for \( T_c \).

In the ordinary space, the scattering amplitude \( \Gamma_1 \) is

\[
\Gamma_1(r, r', E) = \partial_z \delta(r) \partial_z \delta(r') \tilde{\gamma}_1(E),
\]

(6)

where \( r \) and \( r' \) are the relative distances between the two incoming and two outgoing particles, respectively. Therefore, the order parameter in a trapped gas, \( \Delta(r_1, r_2) = \langle \psi(r_1)\psi(r_2) \rangle \), has the form \( \Delta(r_1, r_2) = \partial_z \delta(r) \Delta_0(R) \), where \( r = r_1 - r_2 \) and \( R = (r_1 + r_2)/2 \), and the corresponding equation for the critical temperature is

\[
\frac{\Delta_0(R)}{\tilde{\gamma}_1(\mu)} = -\int R' \sum_{n_1, n_2} M_{n_1n_2}(R)M_{n_1n_2}(R') \times \frac{2(\xi_1 + \xi_2)}{\tanh(\xi_1/2T) + \tanh(\xi_2/2T)} - \int \frac{dp}{(2\pi)^3} \int \frac{dq}{(2\pi)^3} \frac{p_z^2}{2\xi_p + q^2/4m} \exp(iq(R - R')) \Delta_0(R'),
\]

(7)

where \( \xi_i = \xi(n_i) \), \( n = (n_x, n_y, n_z) \) are the harmonic oscillator quantum numbers, \( \xi(n) = \hbar [\omega_x(n_x + 1/2) + \omega_y(n_y + 1/2) + \omega_z(n_z + 1/2)] - \mu \), and

\[
M_{n_1n_2}(R) = M_{n_1n_2}(z)M_{n_1n_2}(x)M_{n_1n_2}(y) \text{ with } M_{n_1n_2}(z) = \frac{1}{2} \left[ \varphi_{n_1}(z)\partial_z \varphi_{n_2}(z) - \varphi_{n_2}(z)\partial_z \varphi_{n_1}(z) \right], \quad M_{n_1n_2}(x) = \varphi_{n_1}(x)\varphi_{n_2}(x), \quad \text{and similar for } M_{n_1n_2}(y), \varphi_n \text{ are the harmonic oscillator wave functions.}
\]
and hence, the chemical potential \( \mu \) is much larger than the trap frequencies, \( \mu \gg \omega_0, \omega_i, \omega_F \). Then, for the most important for the BCS pairing states near the Fermi energy \( \varepsilon_F = \mu \) we can use the WKB approximation for the wave functions \( \varphi_n \) while calculating the functions \( M_{n,mz}(R) \). Another simplification is due to the fact that the BCS order parameter \( \Delta_0(R) \) varies slowly on an interparticle distance scale \( n^{-1/3} \sim h/p_F \), where \( p_F = \sqrt{2m\mu} \) is the Fermi momentum in the center of the trap in the Thomas-Fermi approximation. As a result, the pairing correlations are pronounced only between close in energy states. Therefore, one can neglect \( q^2/4m \) in the denominator of the second term in Eq. (7), as well as rapidly oscillating contributions in the functions \( M_{n,mz}(R) \). Under these assumptions these functions become proportional to the Chebyshev polynomials \( U_{n-1}(z/l_{N}) \), \( T_n(y/l_{N}) \), and \( T_n(y/l_{N}) \), where \( l_{iN} = \sqrt{2Nh/\omega_i} = l_0\sqrt{2N} \).

The critical aspect ratio \( \lambda_c \) corresponds to vanishing critical temperature. We therefore calculate the gap equation in the limit \( T \ll \omega_i \). After very tedious, but fully analytical calculations with the substitution \( \Delta_0(R) \rightarrow \Delta\rho(r) = \Delta_0(z)R_{TF}(r) = \Delta_0(z)R_{TF}(y)R_{TF}(z) \), where \( x, y, z \) are now dimensionless \( |x|, |y|, |z| \leq 1 \), and \( R_{TF} \) denote the radius of the gas cloud in the i-direction, calculated in the Thomas-Fermi approximation, \( R_{TF} = p_F/m\omega_i \), we arrive at the gap equation

\[
\frac{3}{\Gamma}(1 - \mu^2)\Delta_0(r) = (1 - \mu^2)^{3/2}\left(\ln \frac{2\mu(r)}{\omega_z} - \frac{2}{3}(4 - \ln 4)\right) \Delta_0(r) - \frac{3\pi^2}{2} \int_0^1 K(r, r') \Delta_0(r') ,
\]

where \( \Gamma = |\gamma_1|\nu_F, \mu(r) = \mu - \nu_T \) (r), and

\[
K(r, r') = \sum_{n_x, n_y \geq 0} \delta_{n_x n_y} \ln[n_x + \omega_0(n_x + n_y)]
\]

\[
\times \int_{M_z} d\zeta \int_{M_z} d\alpha \int_{M_y} d\beta \delta(1 - \zeta - \alpha - \beta) \times \frac{4}{\pi^2} \sqrt{(\zeta - z^2)(\zeta - z^2)} \frac{U_{n_z-1}(\zeta \sqrt{\zeta})}{\zeta} U_{n_z-1}(\zeta \sqrt{\zeta})
\]

\[
\times \frac{4}{\pi^2} \sqrt{(\alpha - x^2)(\alpha - x^2)} T_{n_x}(\frac{x}{\sqrt{\alpha}}) T_{n_x}(\frac{x}{\sqrt{\alpha}})
\]

\[
\times \frac{4}{\pi^2} \sqrt{(\beta - y^2)(\beta - y^2)} T_{n_y}(\frac{y}{\sqrt{\beta}}) T_{n_y}(\frac{y}{\sqrt{\beta}}) ,
\]

with \( \delta_n = 2 \) for \( n > 0 \), \( \delta_0 = 1 \), and \( M_s = \max(s^2, s^2) \) for \( s = x, y, z \). The above equation can be viewed as the equation for an extremum of the quadratic form

\[
F[\Delta_0] = \frac{1}{2} \int_{r, r'} \Delta_0(r)[L(r)\delta(r - r') - K(r, r') \Delta_0(r')] ,
\]

where \( L(r) = 3(1 - r^2)/\Gamma - (1 - r^2)^{3/2}\left[\ln[2\mu(r)/\omega_z] - 2(4 - \ln 4)/3\right] . \) The extremum can be found numerically using the ansatz

\[
\Delta_0(r) = (1 - r^2)^{3/2} \sum_{mz, m\rho} c_{mz, m\rho} U_{mz}(z^2)T_{m\rho}(x^2 + y^2) ,
\]

with \( m_z, m_\rho \geq 0 \). The form \( F \) becomes now a quadratic form of the unknown coefficients \( c_{mz, m\rho} \), and the extremum corresponds to the eigenvector of the matrix \( M_{mz, m\rho, n_\rho, n_\rho} \) of the quadratic form with zero eigenvalue. Such an eigenvalue exists only if the interaction \( \Gamma \) and the trap frequencies \( \omega_i, \omega_F \) obey certain constraint. The parameter \( \Gamma \) can be written as \( \Gamma = 3\lambda \eta(0)d^2/\pi\mu \), where \( \eta(0) = (2m\mu)^{3/2}/6\pi^2 h^3 \) is the gas density in the center of the trap, and hence, for a given dipole moment \( d \), depends only on the chemical potential \( \mu \). On the other hand, the chemical potential \( \mu \) and the total number of particles \( N \) in the trap are related as \( 3\lambda = \mu^3/\omega_i^2 \). Therefore, after fixing \( \Gamma \) and \( M \), the product of the trap frequencies \( \omega_i, \omega_F^2 \) is also fixed, and the only free parameter left is the trap aspect ratio \( \lambda \), which critical value \( \lambda_c \) could be determined from the above constraint.

The problem is therefore reduced to finding the set of coefficients \( c_{mz, m\rho} \) and the aspect ratio \( \lambda \) such that for a given \( \Gamma \) and \( N \) the lowest eigenvalue of the matrix \( M_{mz, m\rho, n_\rho, n_\rho} \) of the quadratic form \( F \) is zero. The matrix \( M \) is symmetric, and hence, there exists a solution.

The calculation of the matrix elements \( M_{mz, m\rho, n_\rho, n_\rho} \) is naturally divided into two parts (see Eq. (8)). The local part with the kernel \( L(r) \) is just a 3-dimensional integral that can be easily computed using, for instance, the Monte Carlo integration routines, such as the VEGAS algorithm from the GSL library. The non-local part with the kernel \( K(r, r') \) is a triple sum, which elements are 8 dimensional integrals. The straightforward approach using the same numerical method as described above fails in this case, because it is too time consuming. To overcome this problem, we perform the integrations over \( r, r' \) and \( \zeta \) for fixed \( \alpha, \beta \) and \( n \)’s using the same method as before, and then use a 2-dimensional spline interpolation method to interpolate the integrand for the last integrations over \( \alpha \) and \( \beta \). In this way the time required to compute the matrix elements of \( M \) for a given set of physical parameters, was reduced to about 72 hours on a standard workstation.

The results of the calculations are presented in 2 figures. Fig.1 shows the desired relation between \( \Gamma \) and the aspect ratio \( \lambda \). The two curves correspond to two different numbers of particles. As it could be expected, for the larger number of particles, the critical aspect ratio \( \lambda_c \) is smaller because the interaction is stronger. For a pancake trap, \( \lambda^{-1} < 1 \), the interaction is predominantly repulsive, and higher values of \( \Gamma \) for fixed \( \lambda \) are required to achieve the BCS transition. On the other hand, for a cigar trap, \( \lambda^{-1} > 1 \), the interaction is predominantly
where $\Gamma$ measures the dipole strength and $\lambda$ is possible iff $\Gamma > 0$. The BCS transition at finite temperature $T$ goes beyond the scope of this paper.

FIG. 1: The critical $\Gamma_c$ as a function of the aspect ratio $\lambda$: above the depicted curves BCS takes place. The upper (lower) curve corresponds to $N = 10^6$ and $N = 2 \times 10^6$.

attractive and the occurrence of the BCS transition requires smaller values of $\Gamma$.

Fig. 2 shows the order parameter $\Delta_0(r)$ for the cigar trap with $\lambda^{-1} = 2.2$. Amazingly, the order parameter is a nonmonotonic function of the distance from the trap center, in contrast to the case of the BCS order parameter in a two component Fermi gas with short range interactions \[1\]. This effect persists, although being less pronounced, for the case of a pancake geometry. In the axial direction, the order parameter $\Delta(z, \rho = 0)$ develops a minimum at $\rho < 1$, whereas in the axial direction $\Delta(z = 0, \rho)$ becomes negative in the outer part of the cloud. This is a completely new behavior originated from the anisotropy of the interparticle interaction, that can have profound consequences for the form and spectrum of the elementary excitations. We expect an appearance of excitations with wave functions concentrated in the inner region of the atomic cloud, where $\Delta \approx 0$. This problem, however, goes beyond the scope of this paper.

Summarizing, we have determined the phase diagram for trapped dipolar Fermi gases in the $\Gamma - \lambda^{-1}$ plane, where $\Gamma$ measures the dipole strength and $\lambda$ is the trap aspect ratio. The BCS transition at finite temperature $T$ is possible iff $\Gamma > \Gamma_c(\lambda)$. We have calculated the critical line $\Gamma_c(\lambda)$, and the order parameter at criticality.

We acknowledge support from the DFG, the RTN Cold Quantum Gases, ESF PESC BEC2000+, Russian Foundation for Basic Research, and the Alexander von Humboldt Foundation.

[1] P.G. de Gennes, *Superconductivity of metals and alloys*, W.A. Benjamin Inc., New York, Amsterdam (1966).
[2] H.T.C. Stoof, M. Houbiers, C.A. Sackett, and R.G. Hulet, Phys. Rev. Lett. 76, 10 (1996); M.A. Baranov, Yu. Kagan, and M.Yu. Kagan, JETP Lett. 64, 301 (1996).
[3] B. deMarco and D.S. Jin, Science 285,1703 (1999); S.R. Granade, M.E. Gehm, K.M. O’Hara, and J.E. Thomas, Phys. Rev. Lett. 88, 120405 (2002).
[4] A.G. Truscott et al., Science 291, 2570 (2001); F. Schreck et al., Phys. Rev. Lett. 87, 080403 (2001); G. Roati, F. Riboli, G. Modugno, and M. Inguscio, Phys. Rev. Lett. 89, 150403 (2002); Z. Hadzibabic et al., Phys. Rev. Lett. 88, 160401 (2002); Z. Hadzibabic et al., e-print cond-mat/0306050.
[5] M. Holland, S.J.J.M.F. Kokkelmans, M.L. Chiofalo, and R. Walser, Phys. Rev. Lett. 87, 120406 (2001).
[6] Z. Idziaszek, L. Santos, M. Baranov, and M. Lewenstein, Phys. Rev. A 67, 041403 (2003).
[7] W. Hofstetter et al., Phys. Rev. Lett. 89, 220407 (2002).
[8] H.T.C. Stoof and M. Houbiers, in *Bose-Einstein Condensation in Atomic Gases*, Varenna School, 1999: L. You and M. Marinescu, Phys. Rev. A 60, 2324 (2000).
[9] M.A. Baranov, M.S. Mar’enko, Val. S. Rychkov, and G.V. Shlyapnikov, Phys. Rev. A 66, 013606 (2002).
[10] H.L. Bethlem et al., Phys. Rev. Lett. 84, 5744 (2000); H.L. Bethlem et al., Nature (London) 406, 491 (2000).
[11] M. Marinescu and L. You, Phys. Rev. Lett. 81, 4596 (1998).
[12] S. Giovannazzi, D. O’Dell, and G. Kurizki, Phys. Rev. Lett. 88, 130402-1 (2001).
[13] J.D. Weinstein et al., Nature (London) 395, 148 (1998); J. Stuhler et al., Phys. Rev. A 64, 031405 (2001); J.M. Doyle and B. Friedrich, Nature (London) 401, 749 (1999).
[14] L. Santos, G. V. Shlyapnikov, P. Zoller, and M. Lewenstein, Phys. Rev. Lett. 85, 1791 (2000).
[15] L. Santos, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 90, 250403 (2003).
[16] M. Baranov et al., Physica Scripta T102, 74 (2002).
[17] Yu. Kagan, I.A. Vartanyants, and G.V. Shlyapnikov, Sov. Phys. JETP 54, 590 (1981); J.M.V.A. Koelman, H.T.C. Stoof, B.J. Verhaar, and J.T.M. Walraven, Phys. Rev. Lett. 59, 676 (1987); M. Marinescu and L. You, Phys. Rev. Lett. 81, 4596 (1998).
[18] L.P. Gor’kov and T.K. Melik-Barkhudarov, Zh. Eksp. Teor. Fiz. 40, 1452 (1961) [Sov. Phys. JETP 13, 1018 (1961)].
[19] M Galassi et al., GNU Scientific Reference Manual (2003).
[20] M.A. Baranov and D.S Petrov, Phys. Rev. A 58, R801 (1998); G.M. Bruun, Y. Castin, R. Dum, and K. Burnett, Eur. Phys. J. D 7, 433 (1999).