SOME SUFFICIENT CONDITIONS ON IMPULSIVE AND INITIAL VALUE FRACTIONAL ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we will develop a definition of mild solution for impulsive fractional differential equation of order $\alpha \in (1, 2)$ with the help of solution operator and study the existence results of mild solution for impulsive fractional differential equations with state dependent delay by using fixed point theorems. Finally, we present an example of partial fractional derivative to illustrate the existence and uniqueness result.

1. Introduction

For the past few decades, the study of the theory of fractional differential equations in infinite dimensional spaces have become an important role due to their numerous application in several fields of science and engineering. One can see the monographs [3,14,4,5,6,7] for more detail of the fractional calculus. On the other side differential equations with impulsive effects are paid attention by many researchers because the model processes which are subjected to abrupt changes are not described by ordinary differential equations, so such type equations can be modeled in term of impulses. The most important applications of these equations are in the ecology, mechanics, electrical, medicine and biology. Fractional differential equations with state-dependent delay normally arise in the remote control, implicit equations like Wheeler-Feynman equations, structured populations model which involve threshold phenomena etc. For more details one can see the papers [1,20,21,26] and references therein.

Feckan et al. [8] introduced a correct formula of solutions for a impulsive Cauchy problem with Caputo fractional derivative and some sufficient conditions for existence of the solutions are established by applying fixed point methods. Wang et al. [10] studied the existence of $PC$-mild solutions for Cauchy problems and nonlocal problems for impulsive fractional evolution equations involving Caputo fractional derivative by utilizing the theory of operators semigroup, probability density functions via impulsive conditions. Dabas and Chauhan [7] obtained the existence, uniqueness and continuous dependence of mild solution for an impulsive neutral fractional order differential equation of order $\alpha \in (0, 1)$ with infinite delay by using the fixed point technique and solution operator on a complex Banach space. Wang [9] extended the problem consider in [8] for order $\alpha \in (1, 2)$. Shu et al. [11] determined the definition of mild solution for fractional differential equations with nonlocal conditions of order $\alpha \in (1, 2)$ without impulse. However, it should be pointed out that no work has been reported in the existing literature regarding the existence of mild solution for impulsive fractional differential equation of order $\alpha \in (1, 2)$.

Functional differential equations originate in several branches of engineering, applied mathematics and science. Recently, fractional functional differential equations with state dependent delay seems frequently in many fields for modeling of equations such as panorama of natural phenomena and porous media. See for more details of relevant developments theory in the cited papers [12,15,13,2,18,16,17,19,20,21,23,24,25].

Inspired above mention works, in this paper we develop the definition of mild solution for impulsive fractional differential equation of order $\alpha \in (1, 2)$ and show the existence results for impulsive differential equation with state dependent delay of the form:

$$C^\alpha D^\alpha u(t) = Au(t) + f(t, u(t, ut)), \quad t \in J = [0, T], \quad t \neq t_k, \quad (1.1)$$

$$u(t) = \phi(t), \quad u'(t) = \varphi(t), \quad t \in [-d, 0], \quad (1.2)$$

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\[ \Delta u(t_k) = I_k(u(t_k^-)), \quad \Delta u'(t_k) = Q_k(u(t_k^-)), \quad k = 1, 2, \ldots, m, \quad (1.3) \]

where \( C^\alpha \) is the Caputo’s fractional derivative of order \( \alpha \in (1, 2) \) and \( J \) is operational interval. \( A : D(A) \subset X \to X \) is the sectorial operator defined on a complex Banach space \( X \). The functions \( f : J \times PC_0 \to [-d, T] \) and \( \phi, \varphi \in PC_0 \) are given and satisfies some assumptions, where \( PC_0 \) introduced in section 2. The history function \( u_t : [-d, 0] \to X \) is defined by \( u_t(\theta) = u(t + \theta), \ \theta \in [-d, 0] \) belongs to \( PC_0 \). Here \( 0 \leq t_0 < t_1 < \ldots < t_m < t_{m+1} \leq T < \infty \), the functions \( I_k, Q_k \in C(X, X), \ k = 1, 2, \ldots, m, \) are bounded. We have \( \Delta u(t_k) = u(t_k^+ - u(t_k^-)) \) where \( u(t_k^+ \) and \( u(t_k^-) \) represent the right and left-hand limits of \( u(t) \) at \( t = t_k \) respectively, also we take \( u(t_k^-) = u(t_k) \) and \( \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) \) where \( u'(t_k^+) \) and \( u'(t_k^-) \) represent the right and left-hand limits of \( u(t) \) at \( t = t_k \), also we take \( u'(t_k^-) = u'(t_k) \) respectively.

For further details, this work has four sections, second section provides some basic definitions, preliminaries, theorems and lemmas. Third section is equipped with main results for the considered problem \((1.1)-(1.3)\) and fourth section has an example.

2. Preliminaries and Back ground Martigials

Let \((X, ||\cdot||_X)\) be a complex Banach space of functions with the norm \( ||u||_X = \sup_{t \in J} \{ |u(t)| : u \in X \} \) and \( L(X) \) denotes the Banach space of bounded linear operators from \( X \) into \( X \) equipped with norm denoted by \( ||\cdot||_{L(X)} \).

Let \( PC_0 = C([-d, 0], X) \) with \([-d, 0] \subset \mathbb{R} \) denotes the space formed by all the continuous functions defined from \([-d, 0] \) into \( X \), endowed with the norm \( ||u(t)||_{C([-d, 0], X)} = \sup_{t \in [-d, 0]} ||u(t)||_X \).

To study the impulsive conditions, we consider \( PC_0^2 = PC([-d, T]; X), 0 < t < T < \infty \) be a Banach space of all functions \( u : [-d, T] \to X \), which have 2-times continuously differentiable on \([0, T]\) except for a finite number of points \( t_i \in (0, T), \ i = 1, 2, \ldots, N \), at which \( u(t'_i^-) \) and \( u(t'_i^+) = u(t_i) \) exists and endowed with the norm \( ||u||_{PC_0^2} = \sup_{t \in [-d, T]} \sum_{j=0}^2 \{ ||u(t)||_X, u \in PC_0^2 \} \).

**Definition 2.1.** Caputo’s derivative of order \( \alpha > 0 \) with lower limit \( a \), for a function \( f : [a, \infty) \to \mathbb{R} \) such that \( f \in C^n([a, \infty), X) \) is defined as

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} f^{(n)}(s) ds = aJ_t^{n-\alpha} f^{(n)}(t), \]

where \( a \geq 0, \ n - 1 < \alpha < n, \ n \in \mathbb{N} \).

**Definition 2.2.** The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \) with lower limit \( a \), for a continuous function \( f : [a, \infty) \to \mathbb{R} \) is defined by

\[ aJ_t^\alpha f(t) = f(t), \quad aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s) ds, \quad t > 0, \]

where \( a \geq 0 \) and \( \Gamma(\cdot) \) is the Euler gamma function.

**Definition 2.3.** ([11]) Let \( A : D(A) \subset X \to X \) be a densely defined, closed and linear operator in \( X \). \( A \) is said to be sectorial of the type \((M, \theta, \alpha, \mu)\) if there exist \( \mu \in \mathbb{R}, \ \theta \in (\frac{\pi}{2}, \pi), \ M > 0 \), such that the \( \alpha \)-resolvent of \( A \) exists outside the sector and following two conditions are satisfied:

1. \( \mu + S_\theta = \{ \mu + \lambda^\alpha : \lambda \in \mathbb{C} \}-\{|\text{Arg}(\lambda^\alpha)| < \theta\}, \)
2. \( ||(\lambda^\alpha I - A)^{-1}||_{L(X)} \leq \frac{M}{|\lambda^\alpha - \mu|}, \ \lambda \notin \mu + S_\theta, \)

where \( X \) is the complex Banach space with norm denoted \( ||\cdot||_X \).

**Definition 2.4.** A two parameter functional of the Mittag-Leffler type is defined by the series expansion

\[ E_{\alpha, \beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{c} \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^\alpha - y} d\mu, \ \alpha, \ \beta > 0, \ y \in \mathbb{C}, \]

where \( c \) is a contour which starts and ends at \(-\infty\) and encircles the disc \(|\mu| \leq |y|^\frac{1}{\alpha}\) counter clockwise. The Laplace integral of this function given by

\[ \int_0^\infty e^{-\lambda t^{\beta-1}} E_{\alpha, \beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \ \text{Re} \lambda > \frac{1}{\omega^\frac{1}{\beta}}, \ \omega > 0. \]
Let $A$ be positive definite operator which is linear and closed then Laplace integral of Mittag-Leffler function

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(At^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}I - A}, \text{ Re}\lambda > A^{\frac{1}{\alpha}}.$$  

**Definition 2.5.** Let $A : D(A) \subset X \rightarrow X$ be a closed and linear operator and $\alpha, \beta > 0$. We say that $A$ is the generator of $(\alpha, \beta)$ operator function if there exists $\omega \geq 0$ and a strongly continuous function $W_{\alpha,\beta} : \mathbb{R}^+ \rightarrow L(X)$ such that $\{\lambda^{\alpha} : \text{ Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-\beta}(\lambda^{\alpha}I - A)^{-1}u = \int_0^\infty e^{-\lambda t}W_{\alpha,\beta}(t)udt, \text{ Re}\lambda > \omega, u \in X.$$  

Here $W_{\alpha,\beta}(t)$ is called the operator function generated by $A$.

**Remark 2.6.** The operator function $W_{\alpha,\beta}(t)$ is general case of $\alpha$-resolvent family and solution operator. In case $\beta = 1$, operator function correspond to solution operator $S_{\alpha}(t)$ by definition 2.1 in [21], whereas in the case $\beta = \alpha$, operator function correspond to $\alpha$-resolvent family defined in [22] in definition (2.3) and operator function correspond to $K_{\alpha}(t)$ in [11] if case $\beta = 2$.

**Lemma 2.7.** Let the function $f$ continuous and $A$ is a sectorial operator of the type $(M, \theta, \alpha, \mu)$. Consider differential equation of order $\alpha \in (1, 2)$

$$C D_t^\alpha u(t) = Ay(t) + f(t), \quad t \in J = [0,T], t \neq t_k,$$

$$u(0) = u_0 \in X, \quad u'(0) = u_1 \in X, \quad \Delta u(t_k) = I_k(u(t^-_k)), \quad \Delta u'(t_k) = Q_k(u(t^-_k)), \quad t \neq t_k, \quad k = 1, 2, \ldots, m. \quad (2.3)$$

Then a function $u(t) \in PC^2([0,T],X)$ is a solution of the system (2.1)-(2.3) if it satisfies the following integral equation

$$u(t) = \begin{cases} 
S_{\alpha}(t)u_0 + K_{\alpha}(t)u_1 + \int_0^t T_{\alpha}(t-s)f(s)ds, & t \in (0, t_1] \\ 
S_{\alpha}(t)u_0 + K_{\alpha}(t)u_1 + \sum_{i=1}^k S_{\alpha}(t - t_i)Q_i(u(t^-_i)) + \int_0^t T_{\alpha}(t-s)f(s)ds, & t \in (t_k, t_{k+1}], \end{cases} \quad (2.2)$$

where $S_{\alpha}(t), K_{\alpha}(t), T_{\alpha}(t)$ are operators generated by $A$ and defined as

$$S_{\alpha}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}d\lambda, \quad K_{\alpha}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-2}(\lambda^{\alpha}I - A)^{-1}d\lambda,$$

$$T_{\alpha}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}d\lambda$$

with $\Gamma$ is a suitable path such that $\lambda^{\alpha} \notin \mu + S_{\theta}$ for $\lambda \in \Gamma$.

**Proof.** If $t \in (0, t_1]$, we have following problem

$$C D_t^\alpha u(t) = Au(t) + f(t), \quad (2.4)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (2.5)$$

By Lemma 3.1 in [9], the solution of Eq. (2.4)-(2.5) can be written as

$$u(t) = u_0 + u_1 t + \int_0^t (t-s)^{\alpha-1} \frac{\Gamma(t)}{\Gamma(\alpha)} \frac{A(u(s))}{\Gamma(\alpha)} + \int_0^t f(s)ds. \quad (2.6)$$

If $t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m$, we have the following equations

$$C D_t^\alpha u(t) = Au(t) + f(t), \quad (2.7)$$

$$u(t^+_k) = u(t^-_k) + I_k(u(t^-_k)), \quad (2.8)$$

$$u'(t^+_k) = u'(t^-_k) + Q_k(u(t^-_k)). \quad (2.9)$$

By Lemma 3.1 in [9], the solution of Eq. (2.7)-(2.9), can be written as

$$u(t) = u_0 + u_1 t + \sum_{i=1}^k I_i(u(t^-_i)) + \sum_{i=1}^k Q_i(u(t^-_i))(t-t_i) + \int_0^t (t-s)^{\alpha-1} \frac{\Gamma(t)}{\Gamma(\alpha)} A(u(s)) + \int_0^t f(s)ds. \quad (2.10)$$
Summarizing Eq. (2.3) and Eq. (2.10) to \( t \in (0, T] \), we get

\[
    u(t) = u_0 + u_t + \sum_{i=1}^{m} \chi_{i}(t) I_i(u(t_i^-)) + \sum_{i=1}^{m} \chi_{i}(t) Q_i(u(t_i^-))(t-t_i)
    + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,
\]

where \( \chi_{i}(t) = \begin{cases} 
    0 & t \leq t_i \\
    1 & t > t_i.
\end{cases} \)

By taking the Laplace transformation on Eq. (2.11), we have

\[
    L\{u(t)\} = \frac{u_0}{s} + \frac{u_1}{s^2} + \sum_{i=1}^{m} \frac{e^{-\lambda i}}{s} I_i(u(t_i^-)) + \sum_{i=1}^{m} \frac{e^{-\lambda i}}{s^2} Q_i(u(t_i^-))
    + \frac{1}{s^\alpha} L\{f(t)\}.
\]

On simplifying Eq. (2.12), we get

\[
    L\{u(t)\} = \frac{\lambda^{\alpha-1}(u_0)}{(\lambda^\alpha I - A)} + \frac{\lambda^{\alpha-2}(u_1)}{(\lambda^\alpha I - A)} + \sum_{i=1}^{m} \frac{\lambda^{\alpha-1}}{(\lambda^\alpha I - A)} e^{-\lambda i} I_i(u(t_i^-))
    + \sum_{i=1}^{m} \frac{\lambda^{\alpha-2}}{(\lambda^\alpha I - A)} e^{-\lambda i} Q_i(u(t_i^-)) + \frac{1}{(\lambda^\alpha I - A)} L\{f(t)\}.
\]

Now, taking the inverse Laplace transformation of Eq. (2.13), we have

\[
    u(t) = S_\alpha(t) u_0 + K_\alpha(t) u_1 + \sum_{i=1}^{m} S_\alpha(t-t_i) \chi_{i}(t) I_i(u(t_i^-))
    + \sum_{i=1}^{m} K_\alpha(t-t_i) \chi_{i}(t) Q_i(u(t_i^-)) + \int_0^t T_\alpha(t-s) f(s) ds,
\]

for \( t \in J. \)

This complete the proof of the lemma. \( \square \)

Now, we state the definition of mild solution of problem (1.1)-(1.3).

**Definition 2.8.** A function \( u : (-\infty, T] \to X \) such that \( u \in PC^2, u(0) = \phi(0), u'(0) = \varphi(0) \), is called a mild solution of problem (1.1)-(1.3) if it satisfies the following integral equation

\[
    u(t) = \begin{cases} 
        S_\alpha(t) \phi(0) + K_\alpha(t) \varphi(0) + \int_0^t T_\alpha(t-s) f(s, u(s, u_i)) ds, & t \in (0, t_1) \\
        S_\alpha(t) \phi(0) + K_\alpha(t) \varphi(0) + \sum_{i=1}^{k} S_\alpha(t-t_i) I_i(u(t_i^-))
        + \sum_{i=1}^{k} K_\alpha(t-t_i) Q_i(u(t_i^-)) + \int_0^t T_\alpha(t-s) f(s, u(s, u_i)) ds, & t \in (t_k, t_{k+1}].
    \end{cases}
\]

For the analysis and to prove the results, we use following fixed point theorems.

**Theorem 2.9** (Banach fixed point theorem). Let \( C \) be a closed subset of a Banach space \( X \) and let \( f \) be contraction mapping from \( C \) to \( C \), i.e.

\[
    \| f(y) - f(z) \| \leq \delta \| y - z \| \quad \forall \, y, z \in C; \quad 0 < \delta < 1.
\]

Then there exists a unique \( z \in C \) such that \( f(z) = z \).

**Theorem 2.10** (Krasnosel’s fixed point theorem). Let \( B \) be a closed convex and nonempty subset of a Banach space \( X \). Let \( P \) and \( Q \) be two operators such that

(i) \( Px + Qy \in B \), whenever \( x, y \in B \).

(ii) \( P \) is compact and continuous.

(iii) \( Q \) is a contraction mapping.

Then there exists \( z \in B \) such that \( z = Px + Qz \).

**Theorem 2.11** (Nonlinear Leray-Schauder Alternative). Let \( D \) be a closed convex subset of Banach space \( X \) and assume that \( 0 \in D \). Let \( P : D \to D \) be a completely continuous mapping. Then, either the set \( \{ y \in D : y = \lambda P(y), \lambda \in (0, 1) \} \) is unbounded or the map \( P \) has a fixed point in \( D \).
3. Existence Result of mild solution

In this section, we prove the existence of mild solutions for (1.1)-(1.3) with a non-convex valued right-hand side. If $A$ is sectorial operator then the strongly continuous function $\|W_{a,b}(t)\|_{L^1} \leq M$. We have $\|S_\alpha(t)\| \leq M$; $\|K_\alpha(t)\| \leq M$; $\|T_\alpha(t)\| \leq M$. To prove our results, we shall assume the function $\rho$ is continuous. Our result is based on contraction fixed point theorem for this we have following assumptions

(H$_1$) $f$ is continuous and there exists $l_f \in L^1(J, \mathbb{R}^+)$ such that
\[
\|f(t, \psi) - f(t, \xi)\|_X \leq l_f(t)\|\psi - \xi\|_{PC_0} \text{ for every } \psi, \xi \in PC_0.
\]

(H$_2$) The functions $I_k, Q_k : X \to X$, are continuous and there exists $l_i, l_j \in L^1(J, \mathbb{R}^+)$ such that
\[
\|I_k(x) - I_k(y)\|_X \leq l_i(t)\|x - y\|_X; \|Q_k(x) - Q_k(y)\|_X \leq l_j(t)\|x - y\|_X,
\]

for all $x, y \in X$ and $k = 1, \ldots, m$.

**Theorem 3.1.** Let the assumption (H$_1$) and (H$_2$) hold and the constant
\[
\Delta = M\left[m\|l_i\|_{L^1(J, \mathbb{R}^+)} + m\|l_j\|_{L^1(J, \mathbb{R}^+)} + \int_0^T l_f(s)ds\right] < 1.
\]

Then problem (1.1)-(1.3) has a unique mild solution $u \in PC_0^2$ on $J$.

**Proof.** In order to prove the main result, first we convert the problem (1.1)-(1.3) into fixed point problem. Consider the Banach space $PC_0^2 = \{u \in PC_0^2 : u(0) = \phi(0), u'(0) = \varphi(0)\}$ and defined the operator $P : PC_0^2 \to PC_0^2$ by
\[
P(u) = \begin{cases}
S_\alpha(t)\phi(0) + K_\alpha(t)\varphi(0) + \int_0^t T_\alpha(t-s)f(s, u_{\rho(s, u_x)})ds, & t \in [0, t_1] \\
S_\alpha(t)\phi(0) + K_\alpha(t)\varphi(0) + \sum_{i=1}^k S_\alpha(t - t_i)I_i(u(t_i^-)) + \sum_{i=1}^k K_\alpha(t - t_i)Q_i(u(t_i^-)) + \int_0^t T_\alpha(t-s)f(s, u_{\rho(s, u_x)})ds, & t \in (t_k, t_{k+1}].
\end{cases}
\]

(3.1)

It is clear that $u$ is unique mild solution of the problem (1.1)-(1.3) if and only if $u$ is a solution of the operator equation $Pu = u$. Let $u, u^* \in PC_0^2$, for $[0, t_1]$ we have
\[
\|Pu - Pu^*\|_X \leq \int_0^t \|T_\alpha(t-s)\|_{L^1} \|f(s, u_{\rho(s, u_x)}) - f(s, u_{\rho(s, u_x)})\|_X ds,
\]

using the assumptions (H$_1$) we get
\[
\|Pu - Pu^*\|_{PC_0^2} \leq M\left[\int_0^T l_f(s)ds\right] \|u - u^*\|_{PC_0^2}.
\]

Now, without lose of generality we consider the subinterval $(t_k, t_{k+1}]$ to prove our result.
\[
\|Pu - Pu^*\|_X \leq \sum_{i=1}^k \|S_\alpha(t - t_i)\|_{L^1} \|I_i(u(t_i^-)) - I_i(u^*(t_i^-))\|_X
\]
\[
+ \sum_{i=1}^k \|K_\alpha(t - t_i)\|_{L^1} \|Q_i(u(t_i^-)) - Q_i(u^*(t_i^-))\|_X
\]
\[
+ \int_0^t \|T_\alpha(t-s)\|_{L^1} \|f(s, u_{\rho(s, u_x)}) - f(s, u_{\rho(s, u_x)})\|_X ds
\]

again using the assumptions (H$_1$) and (H$_2$) we get
\[
\|Pu - Pu^*\|_{PC_0^2} \leq M\left[m\|l_i\|_{L^1(J, \mathbb{R}^+)} + m\|l_j\|_{L^1(J, \mathbb{R}^+)} + \int_0^T l_f(s)ds\right] \|u - u^*\|_{PC_0^2}
\]
\[
\leq \Delta \|u - u^*\|_{PC_0^2}.
\]

Since $\Delta < 1$, which implies that $P$ is contraction map on Banach space. Hence $P$ has a unique fixed point, which is the mild solution of problem (1.1)-(1.3) on $J$. This completes the proof of the theorem.

To apply the Krasnoselkii’s fixed point theorem we have following assumptions.
(H₃) There exists a function \( m_f \in L^1(J, \mathbb{R}^+) \) such that
\[
\|(f(t, \psi) - f(t, \xi))\|_X \leq m_f(t)\|\psi - \xi\|_{PC_0}
\]
for every \( \psi \in PC_0 \).

(H₄) The functions \( I_k, J_k : X \to X \), are continuous and there exist positive constants \( C_i, C_j \), such that
\[
\|I_k(x)\|_X \leq C_i; \|Q_k(x)\|_X \leq C_j,
\]
for all \( x \in X \) and \( k = 1, \ldots, m \).

**Theorem 3.2.** Let the assumption \((H_1); (H_3); (H_4)\) hold and there exists a constant
\[
\Theta = M \int_0^t m_f(s)ds < 1.
\]
Then problem \((1.1)-(1.3)\) has a mild solution \( u \in PC^2_T \) on \( J \).

**Proof.** Let us choose a number \( r \) such that
\[
r \geq \frac{M(\|\phi(0)\| + \|\varphi(0)\| + mC_i + mC_j)}{1 - M \int_0^t m_f(s)ds}
\]
Define a set \( B_r = \{u \in PC^2_T : \|u\|_{PC^2_T} \leq r\} \) which is bounded, closed and convex subset \( PC^2_T \). Consider the operators \( P_1 : B_r \to B_r \) and \( P_2 : B_r \to B_r \) for the interval \((t_k, t_{k+1})\) by
\[
P_1u(t) = S_\alpha(t)\phi(0) + K_\alpha(t)\varphi(0) + \sum_{i=1}^k S_\alpha(t - t_i)I_i(u(t_i^-)) + \sum_{i=1}^k K_\alpha(t - t_i)Q_i(u(t_i^-)).
\]
\[
P_2u(t) = \int_0^t T_\alpha(t - s)f(s, u_{\rho(s,u_i)})ds.
\]
Let \( u, u^* \in B_r \), we have
\[
\|P_1u(t) + P_2u^*(t)\|_X \leq \|S_\alpha(t)\|_{L(X)}\|\phi(0)\| + \|K_\alpha(t)\|_{L(X)}\|\varphi(0)\|
\]
\[
+ \sum_{i=1}^k \|S_\alpha(t - t_i)\|_{L(X)}\|I_i(u(t_i^-))\|
\]
\[
+ \sum_{i=1}^k \|K_\alpha(t - t_i)\|_{L(X)}\|Q_i(u(t_i^-))\|
\]
\[
+ \int_0^t \|T_\alpha(t - s)\|_{L(X)}\|f(s, u_{\rho(s,u_i^*)})\|ds.
\]
using the assumptions \((H_1); (H_3)\) and \((H_4)\) we get
\[
\|P_1u(t) + P_2u^*(t)\|_{B_r} \leq \frac{M(\|\phi(0)\| + \|\varphi(0)\| + mC_i + mC_j + r \int_0^t m_f(s)ds)}{1 - M \int_0^t m_f(s)ds} \leq r
\]
It is obvious that \( B_r \) is closed with respect to the maps \( P_1 \) and \( P_2 \).

Now, we show that \( P_1 \) is continuous and compact map. Consider a sequence \( \{u^n\}_{n=1}^{\infty} \) in \( B_r \) with \( \lim u^n \to u \)
in \( B_r \). Then
\[
\|P_1u^n(t) - P_1u(t)\|_X \leq \sum_{i=1}^k \|S_\alpha(t - t_i)\|_{L(X)}\|I_i(u^n(t_i^-)) - I_i(u(t_i^-))\|_X
\]
\[
+ \sum_{i=1}^k \|K_\alpha(t - t_i)\|_{L(X)}\|Q_i(u^n(t_i^-)) - Q_i(u(t_i^-))\|_X.
\]
Since the functions \( I_i, Q_i, i = 1, 2, \ldots, m \), are continuous, hence \( \lim_{n \to \infty} P_1u^n = P_1u \) in \( B_r \), which show that \( P_1 \) is continuous map on \( B_r \).
Let \( u \in B_r \), we have
\[
\|P_t u(t)\|_X \leq \|S_\alpha(t)\|_{L(X)} \|\phi(0)\| + \|K_\alpha(t)\|_{L(X)} \|\varphi(0)\| + \sum_{i=1}^{k} \|S_\alpha(t-t_i)\|_{L(X)} \|L(u(t_i^-))\|
+ \sum_{i=1}^{k} \|K_\alpha(t-t_i)\|_{L(X)} \|Q_i(u(t_i^-))\|
\]
using the assumptions \((H_4)\) we get
\[
\|P_t u(t)\|_{B_r} \leq M \|\phi(0)\| + \|\varphi(0)\| + mC_i + mC_j = C^*.
\]
It proves that \( P_t \) maps bounded set into bounded sets in \( B_r \). Consider \( P_t(B_r) = \{ P_t u : u \in B_r \} \) is an equi-continuous family of functions. Next, we show that \( P_t \) maps bounded set into equi-continuous sets in \( P_t(B_r) \). Let \( l_1, l_2 \in (t_i, t_{i+1}], t_i \leq l_1 < l_2 \leq t_{i+1}, i = 0, 1, \ldots, m \), then we have
\[
\|P_u(l_2) - P_u(l_1)\|_X \leq \|S_\alpha(l_2) - S_\alpha(l_1)\|_{L(X)} \|\phi(0)\|_{PC_0} + \|K_\alpha(l_2) - K_\alpha(l_1)\|_{L(X)} \|\varphi(0)\|_{PC_0}
+ \sum_{i=1}^{k} \|S_\alpha(l_2-t_i) - S_\alpha(l_1-t_i)\|_{L(X)} \|L(u(t_i^-))\|_X
+ \sum_{i=1}^{k} \|K_\alpha(l_2-t_i) - K_\alpha(l_1-t_i)\|_{L(X)} \|Q_i(u(t_i^-))\|_X
\leq \|S_\alpha(l_2) - S_\alpha(l_1)\|_{L(X)} \|\phi(0)\|_{PC_0} + \|K_\alpha(l_2) - K_\alpha(l_1)\|_{L(X)} \|\varphi(0)\|_{PC_0}
+ \sum_{i=1}^{k} \|S_\alpha(l_2-t_i) - S_\alpha(l_1-t_i)\|_{L(X)} \|L(u(t_i^-))\|_X
+ \sum_{i=1}^{k} \|K_\alpha(l_2-t_i) - K_\alpha(l_1-t_i)\|_{L(X)} \|Q_i(u(t_i^-))\|_X
\]
Since \( K_\alpha(t), S_\alpha(t) \) are strongly continuous functions so \( K_\alpha(l_2) - K_\alpha(l_1); S_\alpha(l_2) - S_\alpha(l_1) \to 0 \) as \( l_1 \to l_2 \), which implies that \( \|P_t u(l_2) - P_t u(l_1)\|_X \to 0 \) as \( l_1 \to l_2 \). Hence \( P(B_r) \) is equi-continuous. We conclude that the operator \( P_t \) is compact by Arzela Ascoli’s theorem.

Let \( u, u^* \in B_r \), we have the following step
\[
\|P_t u(t) - P_t u^*(t)\| \leq \int_0^t T_\alpha(t-s) \|f(s, u_{\rho(s,u)}(s) - f(s, u^*_{\rho(s,u^*)})\|ds
\leq M \int_0^t m_f(s) ds \|u - u^*\|
= \Theta \|u - u^*\|
\]
Since \( \Theta < 1 \), it implies that \( P_t \) is a contraction map. By Krasnoselkii’s theorem set \( B_r \) has a fixed point \( u(t) \) which is the solution of system \((1.1)-(1.3)\). This completes the proof of the theorem. \( \square \)

To use Nonlinear Leray-Schauder Alternative fixed point theorem we have following assumptions.

\((H_3)\) The functions \( I_k, J_k : X \to X \) are continuous and there exists positive constants \( C_i, C_j \), such that
\[
\|I_k(x)\|_X \leq C_i; \|Q_k(x)\|_X \leq C_j,
\]
for all \( x \in X \) and \( k = 1, \ldots, m \).

\((H_4)\) There exist \( m_f \in C(J, [0, \infty)) \) and a continuous non-decreasing function \( \Omega_f : [0, \infty) \to (0, \infty) \) such that
\[
\|f(t, \varphi)\| \leq m_f(t) \Omega_f(\|\varphi\|), \forall (t, \varphi) \in J \times PC_0.
\]

**Theorem 3.3.** Let the assumption \((H_3)\) and \((H_4)\) hold and
\[
M \int_0^T m_f(s) ds < \int_0^\infty \frac{1}{\Omega_f(s)} ds,
\]
where \( C' = M \|\phi(0)\| + \|\varphi(0)\| + mC_i + mC_j \), then there exists a mild solution of problem \((1.1)-(1.3)\) on \( J \).
Proof. Consider the operator $P : PC^2_2 \rightarrow PC^2_2$ defined as in Theorem [3.1]. Now, we show that $P$ is completely continuous map for general subinterval $(t_k, t_{k+1}]$. For this purpose first, let $\{u^n\}_{n=1}^{\infty}$ be a sequence in $PC^2_2$ with $\lim u^n \rightarrow u$ in $PC^2_2$.

$$\|P_{u^n}(t) - Pu(t)\|_X \leq \sum_{i=1}^{k} \|S_\alpha(t - t_i)\|_{L(X)} \|I_i(u^n(t_i^-)) - I_i(u(t_i^-))\|_X$$

$$+ \sum_{i=1}^{k} \|K_\alpha(t - t_i)\|_{L(X)} \|Q_i(u^n(t_i^-)) - Q_i(u(t_i^-))\|_X$$

$$+ \int_{0}^{t} \|T_\alpha(t - s)\|_{L(X)} \|f(s, u^n_{\rho(s, u^n)}) - f(s, u_{\rho(s, u^n)})\|_X ds.$$ 

Since the functions $f, I_i, Q_i$, $i = 1, 2, \ldots, m$, are continuous, hence $\lim_{n \rightarrow \infty} P_{u^n} = Pu$ in $PC^2_2$, which implies that the map $P$ is continuous on $PC^2_2$. Consider set $B_r = \{u \in PC^2_2 : \|u\|_{PC^2_2} \leq r\}$. It is clear that $B_r$ is a bounded, closed convex subset in $PC^2_2$. Let $u \in B_r$, we have

$$\|Pu(t)\|_X \leq \|S_\alpha(t)\|_{L(X)} \|\phi(0)\| + \|K_\alpha(t)\|_{L(X)} \|\varphi(0)\| + \sum_{i=1}^{k} \|S_\alpha(t - t_i)\|_{L(X)} \|I_i(u(t_i^-))\|$$

$$+ \sum_{i=1}^{k} \|K_\alpha(t - t_i)\|_{L(X)} \|Q_i(u(t_i^-))\| + \int_{0}^{t} \|T_\alpha(t - s)\|_{L(X)} \|f(s, u_{\rho(s, u^n)})\| ds$$

using the assumptions $(H_3)$ and $(H_4)$ we get

$$\|Pu(t)\|_{B_r} \leq M \left[ \|\phi(0)\| + \|\varphi(0)\| + mC_i + mC_j + \Omega_f(r) \int_{0}^{t} m_f(s) ds \right] = C^*.$$ 

It proves that $P$ maps bounded set into bounded sets in $B_r$ for all sub interval $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, m$. Consider $P(B_r) = \{Pu : u \in B_r\}$ is an equi-continuous family of functions. Next, we show that $P$ maps bounded set into equi-continuous sets in $P(B_r)$. Let $l_1, l_2 \in (t_i, t_{i+1}], t_i \leq l_1 < l_2 \leq t_{i+1}, i = 0, 1, \ldots, m$, then we have

$$\|Pu(l_2) - Pu(l_1)\|_X \leq \|S_\alpha(l_2) - S_\alpha(l_1)\|_{L(X)} \|\phi(0)\|_{PC^0} + \|K_\alpha(l_2) - K_\alpha(l_1)\|_{L(X)} \|\varphi(0)\|_{PC^0}$$

$$+ \sum_{i=1}^{k} \|S_\alpha(l_2 - t_i) - S_\alpha(l_1 - t_i)\|_{L(X)} \|I_i(u(t_i^-))\|_X$$

$$+ \sum_{i=1}^{k} \|K_\alpha(l_2 - t_i) - K_\alpha(l_1 - t_i)\|_{L(X)} \|Q_i(u(t_i^-))\|_X$$

$$+ \int_{0}^{l_2} \|T_\alpha(l_2 - s) - T_\alpha(l_1 - s)\|_{L(X)} \|f(s, u_{\rho(s, u^n)})\|_X ds$$

$$+ \int_{0}^{l_1} \|T_\alpha(l_2 - s) - T_\alpha(l_1 - s)\|_{L(X)} \|f(s, u_{\rho(s, u^n)})\|_X ds$$

$$\leq \|S_\alpha(l_2) - S_\alpha(l_1)\|_{L(X)} \|\phi(0)\|_{PC^0} + \|K_\alpha(l_2) - K_\alpha(l_1)\|_{L(X)} \|\varphi(0)\|_{PC^0}$$

$$+ \sum_{i=1}^{k} \|S_\alpha(l_2 - t_i) - S_\alpha(l_1 - t_i)\|_{L(X)} \|I_i(u(t_i^-))\|_X$$

$$+ \sum_{i=1}^{k} \|K_\alpha(l_2 - t_i) - K_\alpha(l_1 - t_i)\|_{L(X)} \|Q_i(u(t_i^-))\|_X$$

$$+ \int_{0}^{l_2} \|T_\alpha(l_2 - s) - T_\alpha(l_1 - s)\|_{L(X)} \|f(s, u_{\rho(s, u^n)})\|_X ds$$

$$+ (l_2 - l_1)M \|f(s, u_{\rho(s, u^n)})\|_X.$$
Since $T_0(t)$, $K_0(t)$, $S_0(t)$ are strongly continuous functions so $T_0(t_2) - T_0(t_1); K_0(t_2) - K_0(t_1); S_0(t_2) - S_0(t_1) \to 0$ as $t_1 \to t_2$, which implies that $|Pu(t_2) - Pu(t_1)|_X \to 0$ as $t_1 \to t_2$. Hence $P(B_r)$ is equicontinuous.

Finally, we establish a priori estimate for the solutions of the integral equation $u = \lambda Pu$ for $\lambda \in (0,1)$. Let $u$ be a solution of $z = \lambda Pz, \lambda \in (0,1)$, then we have

\[
\|u(t)\|_X \leq \|S_0(t)\|_{L(X)}\|\phi(0)\|_{PC_0} + \|K_0(t)\|_{L(X)}\|\varphi(0)\|_{PC_0} + \sum_{i=1}^k \|S_0(t - t_i)\|_{L(X)}\|I_i(u(t_i))\|_X \\
+ \sum_{i=1}^k \|K_0(t - t_i)\|_{L(X)}\|Q_i(u(t_i^+))\|_X + \int_0^t \|T_0(t - s)\|_{L(X)}\|f(s, u_{\rho(s,u,s)})\|_X \, ds \\
\leq M \left[ \|\phi(0)\|_{PC_0} + \|\varphi(0)\|_{PC_0} + mC_i + mC_j + \int_0^t m_f(s)\Omega_f(\|u_{\rho(s,u,s)}\|) \, ds \right]
\]

again using the assumptions $(H_1)$ and $(H_2)$ we get

\[
\|u(t)\|_{PC_0^2} \leq M \left[ \|\phi(0)\|_{PC_0} + \|\varphi(0)\|_{PC_0} + mC_i + mC_j + \int_0^t m_f(s)\Omega_f(\|u\|_{PC_0^2}) \, ds \right].
\]

If $\beta_\lambda(t) = \|u\|_{PC_0^2}$, we get that

\[
\beta_\lambda(t) \leq M \left[ \|\phi(0)\|_{PC_0} + \|\varphi(0)\|_{PC_0} + mC_i + mC_j + \int_0^t m_f(s)\Omega_f(\beta_\lambda(t)) \, ds \right].
\]

Then, we get

\[
\beta_\lambda'(t) \leq Mm_f(t)\Omega_f(\beta_\lambda(t)),
\]

and hence

\[
\int_{C'=\beta(0)}^{\beta_\lambda(t)} \frac{1}{\Omega_f(s)} \, ds \leq M \int_0^T m_f(s) \, ds.
\]

It is clear that set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded, which implies that $\{u : \lambda \in (0,1)\}$ is bounded. By nonlinear alternative of Leray-Schauder fixed theorem, we deduce that $P$ has a fixed point $u$, which is a mild solution of the problem (1.4) - (1.3) on $J$. $lacksquare$

4. Application

Consider the following impulsive fractional partial differential equation of the form

\[
\frac{\partial^\alpha}{\partial t^\alpha} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + \frac{u(t - \tau(t)\sigma_2(||u||), x)}{49}, \quad t \neq \frac{1}{2},
\]

\[
u(t, 0) = u(t, \pi) = 0; u'(t, 0) = u'(t, \pi) = 0 \quad t \geq 0,
\]

\[
u(t, x) = \phi(t, x), u'(t, x) = \varphi(t, x), \quad t \in [-d, 0], x \in [0, \pi],
\]

\[
\Delta u|_{t=\frac{1}{2}} = \frac{\|u(\frac{1}{2})\|}{25 + \|u(\frac{1}{2})\|}, \quad \Delta u|_{t=\frac{1}{2}} = \frac{\|u(\frac{1}{2})\|}{16 + \|u(\frac{1}{2})\|},
\]

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo’s fractional derivative of order $\alpha \in (1,2), 0 < \tau_1 < \tau_2 < \cdots < \tau_n < T$ are prefixed numbers and $\phi, \varphi \in PC_0$. Let $X = L^2[0, \pi]$ and define the operator $A : D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) := \{w \in X : w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = 0 = w(\pi)\}$. Then

\[
Aw = \sum_{n=1}^\infty n^2(w, w_n)w_n, \quad w \in D(A),
\]

where $w_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ in $X$ and is given by

\[
T(t)w = \sum_{n=1}^\infty e^{-n^2t}(\omega, w_n)\omega_n, \quad \text{for all } \omega \in X, \text{ and every } t > 0.
\]
We assume that \( \rho_i : [0, \infty) \to [0, \infty) \), \( i = 1, 2 \), are continuous functions.

Set \( u(t)(x) = u(t, x) \), and \( \rho(t, \phi) = \rho_1(t) \rho_2(\|\phi(0)\|) \) we have
\[
 f(t, \phi)(x) = \frac{\phi}{49} I_k(u) = \frac{\|u\|}{25 + \|u\|}, \quad J_k(u) = \frac{\|u\|}{16 + \|u\|}
\]
then with these settings the problem (4.1)-(4.2) can be written in the abstract form of equation (1.1)-(1.3).

It is obvious that the maps \( f, I_k, J_k \) follow the assumption \( H_1, H_2 \), this implies that there exists a unique mild solution of problem (4.1)-(4.2) on \([0, T]\).

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