Partial result of Yau’s Conjecture of the first eigenvalue in unit sphere $S^{n+1}(1)$

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Abstract: In this paper, we partially solve Yau’ Conjecture of the first eigenvalue of an embedded compact minimal hypersurface of unit sphere $S^{n+1}(1)$, i.e., Corollary 1.2. In particular, Corollary 1.3 proves that the condition $\int_{\Omega} |\nabla u|^2 = (n+1) \int_{\Omega} u^2$ is naturally true and meaningful in Corollary 1.2.

Keywords: Minimal hypersurface; Riemannian manifold; The first eigenvalue; Unit sphere

1 Introduction and main results

Let $M^n$ be an embedded compact orientable minimal hypersurface in an $(n+1)$-dimensional compact orientable Riemannian manifold $N$.

In 1982, Yau [1] proposed the following conjecture.

Yau’ Conjecture. If $\lambda_1(M^n)$ is the first eigenvalue of an embedded compact minimal hypersurface $M^n$ of unit sphere $S^{n+1}(1)$, the standard $(n+1)$-sphere of sectional curvature 1, then $\lambda_1(M^n) = n$.

In 1983, Choi-Wang [2] showed that if $M^n$ is an embedded compact minimal hypersurface of unit sphere $S^{n+1}(1)$, then $\lambda_1(M^n) \geq \frac{n}{2}$, where $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$.

In the paper, in order to prove Yau’ Conjecture, first we obtain the following result in Riemannian manifold $N$.

Theorem 1.1. Let $M^n$ be an embedded compact orientable minimal hypersurface in an $(n+1)$-dimensional compact orientable Riemannian manifold $N$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. If the Ricci curvature of $N$ is bounded below by a positive constant $k$, then

$$\frac{k}{2} + \frac{n}{2} \delta_1(u) \leq \lambda_1(M^n) \leq \frac{k}{2} + \frac{n}{2} \delta_2(u),$$

where

$$0 < \delta_1(u) = \frac{\int_{\Omega_1} |\nabla u|^2 \left( 1 - \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2} \right)^2}{(n+1) \int_{\Omega_1} u^2} \leq \frac{k}{n},$$

$$\delta_2(u) = \frac{\int_{\Omega_1} |\nabla u|^2 \left( 1 + \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2} \right)^2}{(n+1) \int_{\Omega_1} u^2} \geq \frac{k}{n}.$$
and $u$ is the solution of the Dirichlet problem such that

$$
\begin{align*}
\Delta u &= 0, & \text{in } \Omega_1; \\
u &= f, & \text{in } \partial \Omega_1 = M^n,
\end{align*}
$$

which is called the Dirichlet problem (1).

We see that $0 < \delta_1(u) \leq \frac{k}{n}$, $\delta_2(u) \geq \frac{k}{n}$ in Theorem 1.1. In particular, we have $\delta_1(u) = \delta_2(u) = \frac{k}{n}$ iff $\int_{\Omega_1} |\nabla u|^2 = \frac{(n+1)k}{n} \int_{\Omega_1} u^2$. When we take $\int_{\Omega_1} |\nabla u|^2 = \frac{(n+1)k}{n} \int_{\Omega_1} u^2$ in Theorem 1.1, we have $\lambda_1(M^n) = k$. Note that the Ricci curvature of unit sphere $S^{n+1}(1)$ is $n$ and an embedded compact hypersurface of unit sphere $S^{n+1}(1)$ is already orientable. We obtain Corollary 1.2 which partially solves Yau Conjecture.

**Corollary 1.2.** Let $M^n$ be an embedded compact minimal hypersurface of unit sphere $S^{n+1}(1)$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. If the solution $u$ of Dirichlet problem (1) satisfies $\int_{\Omega_1} |\nabla u|^2 = (n + 1) \int_{\Omega_1} u^2$, then

$$\lambda_1(M^n) = n.$$

According to (3.3) in the proof of Theorem 1.1, we know that $\int_{\Omega_1} |\nabla u|^2 \geq \frac{(n+1)k}{n} \int_{\Omega_1} u^2$ is naturally true and meaningful in Theorem 1.1. Note that the Ricci curvature of unit sphere $S^{n+1}(1)$ is $n$. Hence, we get the following corollary.

**Corollary 1.3.** Let $M^n$ be an embedded compact minimal hypersurface of unit sphere $S^{n+1}(1)$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. Then

$$\int_{\Omega_1} |\nabla u|^2 \geq (n + 1) \int_{\Omega_1} u^2,$$

where $u$ is the solution of Dirichlet problem (1).

**Remark 1.4.** Corollary 1.3 proves that the condition $\int_{\Omega_1} |\nabla u|^2 = (n + 1) \int_{\Omega_1} u^2$ is naturally true and meaningful in Corollary 1.2.

From Lemma 2.1 and $\int_{\partial \Omega_1} h(\nabla u, \nabla u) \geq 0$ in the proof of Theorem 1.1, we can conclude the following corollary.

**Corollary 1.5.** Let $M^n$ be an embedded compact orientable minimal hypersurface in an $(n + 1)$-dimensional compact orientable Riemannian manifold $N$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. If the Ricci curvature of $N$ is bounded below by a positive constant $k$, then

$$\lambda_1(M^n) > \frac{k}{2}.$$

Note that the Ricci curvature of unit sphere $S^{n+1}(1)$ is $n$. Hence, from Corollary 1.5 we obtain Corollary 1.6 which improves the result $\lambda_1(M^n) \geq \frac{n}{2}$ of [1].

**Corollary 1.6.** Let $M^n$ be an embedded compact minimal hypersurface of unit sphere $S^{n+1}(1)$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. Then

$$\lambda_1(M^n) > \frac{n}{2}.$$
Finally, by Corollaries 1.2 and 1.3, what we want to know is whether the following problem is true.

**Problem 1.7.** Let $M^n$ be an embedded compact minimal hypersurface of unit sphere $S^{n+1}(1)$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. If the solution $u$ of the Dirichlet problem \( \Delta u = f \) satisfies $\int_{\Omega_1} |\nabla u|^2 > (n+1) \int_{\Omega_1} u^2$, then $\lambda_1(M^n) = n$?

**Remark 1.8.** If Problem 1.7 is true, combining Corollary 1.2, Corollary 1.3 and Problem 1.7, then Yau’s Conjecture is true.

## 2 Preliminaries

Let $\Omega_1$ be a Riemannian manifold of dimensional $(n+1)$ with smooth boundary $\partial \Omega_1 = M^n$. Let $u$ be a function defined on $\Omega_1$ which is smooth up to $\partial \Omega_1$. The symbols $\Delta u$ and $\nabla u$ will be respectively the Laplacian and the gradient of $u$ with respect to the induced Riemannian metric on $\Omega_1$ while $\Delta u$ and $\nabla u$ will be the Laplacian and the gradient of $u$ (defined on $\partial \Omega_1$) with respect to the induced Riemannian metric on $\partial \Omega_1$. For $x \in \Omega_1$ and $Y \in T_x \Omega_1$, we can define the Hessian tensor $(D^2u)(X,Y) = X(Yu) - (\nabla_X Y)u$, where $\nabla_X Y$ is the covariant derivative of the Riemannian connection of $\Omega_1$. The covariant derivative of the Riemannian connection of $\partial \Omega_1 = M^n$ is given by $\nabla_X Y$.

Suppose that $\{e_1, e_2, \cdots, e_{n+1}\}$ is a local orthonormal frame such that at $x \in \partial \Omega_1$, $e_1, e_2, \cdots, e_n$, are tangent to $\partial \Omega_1$ and $e_{n+1}$ is the outward normal vector. Let $h$ be the second fundamental form, $h(\nu, \omega) = \langle \nabla_\nu e_n, \omega \rangle$, where $\nu, \omega$ are vectors tangent to $\partial \Omega_1 = M^n$, and $H$ be the mean curvature, i.e., $H = \frac{\sum_{i=1}^n h(e_i, e_i)}{n}$.

We need the following lemmas will play a crucial role in the proof of Theorem 1.1.

**Lemma 2.1.** Let $M^n$ be an embedded compact orientable minimal hypersurface in an $(n+1)$-dimensional compact orientable Riemannian manifold $N$. Suppose that $\lambda_1(M^n)$ is the first eigenvalue of the Laplacian of $M^n$. If the Ricci curvature of $N$ is bounded below by a positive constant $k$, then

\[
(2\lambda_1(M^n) - k) \int_{\Omega_1} |\nabla u|^2 - \int_{\partial \Omega_1} h(\nabla u, \nabla u) > 0,
\]

where $u$ is the solution of Dirichlet problem (1).

**Proof.** Since the Ricci curvature of $N$ is strictly positive, the first Betti number of $N$ must be zero. Since $M^n$ and $N$ are orientable, by looking at the exact sequences of homology groups, we can see that $M^n$ divides $N$ into components $\Omega_1$ and $\Omega_2$ such that $\partial \Omega_1 = \partial \Omega_2 = M^n$.

Let $f$ be the first eigenfunction of $M^n$, i.e.,

\[
\Delta f + \lambda_1(M^n)f = 0.
\]

Let $u$ be the solution of the Dirichlet problem such that

\[
\begin{cases}
\Delta u = 0, & \text{in } \Omega_1; \\
u = f, & \text{in } \partial \Omega_1 = M^n,
\end{cases}
\]

which is called Dirichlet problem (1).
So \( u \) is a function defined on \( \Omega_1 \) smooth up to \( \partial \Omega_1 \). Then we have
\[
\Delta u = \sum_{i=1}^{n+1} D^2 u(e_i, e_i) = \sum_{i=1}^{n+1} u_{ii},
\]
(2.3)
where \( u_{ij} = D^2 u(e_i, e_j) \), \( i, j = 1, \ldots, n + 1 \).

When \( i \neq n + 1 \) and \( x \in \partial \Omega_1 \), we get
\[
\nabla_e e_i = \nabla_e e_i - h_{ii} e_{n+1},
\]
(2.4)
where \( h_{ij} = h(e_i, e_j) \).

Hence when \( x \in \partial \Omega_1 = M^n \), we can conclude from (2.2), (2.3) and (2.4) that
\[
\Delta u = u_{(n+1)(n+1)} + \nabla f + \sum_{i=1}^{n} h_{ii} e_{n+1} = u_{(n+1)(n+1)} + \nabla f + nHu_{n+1},
\]
(2.5)
where \( u_{n+1} = \langle \nabla u, e_{n+1} \rangle \) and \( H \) is the mean curvature of \( M^n \).

For \( x \in \partial \Omega_1 \), we know from (2.1) and (2.5) that
\[
u_{(n+1)(n+1)} = \lambda_1(M^n) f - nHu_{n+1}.
\]
(2.6)
For \( x \in \Omega_1 \), note that the fact \( \Delta |\nabla u|^2 = 2 \sum_{i,j=1}^{n+1} u_{ij}^2 + 2 \sum_{i,j=1}^{n+1} R_{ij} u_i u_j + 2 \sum_{i=1}^{n+1} u_i (\Delta u)_i \) can be found in [2]. Since \( \Delta u = 0 \), we have
\[
\Delta |\nabla u|^2 = 2 \sum_{i,j=1}^{n+1} u_{ij}^2 + 2 \sum_{i,j=1}^{n+1} R_{ij} u_i u_j,
\]
(2.7)
where \( R_{ij} = \text{Ric}(e_i, e_j) \).

Since Ricci curvature of \( N \) is bounded by \( k \), by (2.7) we have
\[
\Delta |\nabla u|^2 \geq 2 |D^2 u|^2 + 2k|\nabla u|^2.
\]
(2.8)
Then integrating (2.8) we have
\[
\int_{\Omega_1} \Delta |\nabla u|^2 \geq 2 \int_{\Omega_1} |D^2 u|^2 + 2k \int_{\Omega_1} |\nabla u|^2.
\]
(2.9)
When \( i \neq n + 1 \), we have
\[
\nabla_i u_{(n+1)} = D^2 u(e_i, e_{n+1}) = e_i (e_{n+1} u) - (\nabla e_i e_{n+1}) u = e_i (u_{n+1}) - \sum_{j=1}^{n} h_{ij} u_j,
\]
(2.10)
By using the Stokes theorem, we can conclude from (2.1), (2.6) and (2.10) that
\[
\int_{\Omega_1} \Delta |\nabla u|^2 = \int_{\partial \Omega_1} \nabla f \cdot \nabla u_{n+1} - 2 \int_{\partial \Omega_1} \sum_{i,j=1}^{n} h_{ij} u_i u_j + 2 \int_{\partial \Omega_1} u_{n+1} u_{(n+1)(n+1)}
\]
\[
= \int_{\partial \Omega_1} \nabla f \cdot \nabla u_{n+1} - 2 \int_{\partial \Omega_1} \sum_{i,j=1}^{n} h_{ij} u_i u_j + 2 \int_{\partial \Omega_1} u_{n+1} u_{(n+1)(n+1)}
\]
\[
= -2 \int_{\partial \Omega_1} u_{n+1} \nabla f - 2 \int_{\partial \Omega_1} h(\nabla u, \nabla u) + 2 \int_{\partial \Omega_1} u_{n+1} u_{(n+1)(n+1)}
\]
\[
= 4\lambda_1(M^n) \int_{\partial \Omega_1} u_{n+1} f - 2 \int_{\partial \Omega_1} h(\nabla u, \nabla u) - 2n \int_{\partial \Omega_1} H u_{n+1}^2.
\]
(2.11)
Combining (2.2) and Stokes theorem, we have
\[ \int_{\Omega_1} |\nabla u|^2 = -\int_{\Omega_t} u \Delta u + \int_{\partial \Omega_1} u u_{n+1} = \int_{\partial \Omega_1} u_{n+1} f. \quad (2.12) \]

From (2.11) and (2.12), we have
\[ \int_{\Omega_1} \Delta |\nabla u|^2 = 4 \lambda_1 (M^n) \int_{\Omega_1} |\nabla u|^2 - 2 \int_{\partial \Omega_1} h(\nabla u, \nabla u) - 2 n \int_{\partial \Omega_1} H u^2_{n+1}. \quad (2.13) \]

Since \( M^n \) is minimal, by (2.9) and (2.13), we have
\[ (2 \lambda_1 (M^n) - k) \int_{\Omega_1} |\nabla u|^2 \geq \int_{\partial \Omega_1} h(\nabla u, \nabla u) + \int_{\Omega_1} |D^2 u|^2. \quad (2.14) \]

We claim that
\[ \int_{\Omega_1} |D^2 u|^2 \neq 0. \quad (2.15) \]

Otherwise, for all \( 1 \leq i, j \leq n + 1 \), we have \( u_{ij} = 0 \) on \( \Omega_1 \). Since \( u \) is smooth up to \( \partial \Omega_1 \), for all \( 1 \leq i, j \leq n \), we have \( f_{ij} = 0 \) on \( M^n \), which implies that \( \nabla f = 0 \) which is impossible since \( f \) is the first eigenfunction of \( M^n \). Thus, our claim is true. Hence from (2.15) we get
\[ \int_{\Omega_1} |D^2 u|^2 > 0. \quad (2.16) \]

From (2.14) and (2.16), we have
\[ (2 \lambda_1 (M^n) - k) \int_{\Omega_1} |\nabla u|^2 - \int_{\partial \Omega_1} h(\nabla f, \nabla f) > 0. \quad (2.17) \]

This completes the proof of Lemma 2.1. \( \square \)

**Lemma 2.2.** Let \( M^n \) be an embedded compact orientable minimal hypersurface in an \((n + 1)\)-dimensional compact orientable Riemannian manifold \( N \). Suppose that \( \lambda_1 (M^n) \) is the first eigenvalue of the Laplacian of \( M^n \). If the Ricci curvature of \( N \) is bounded below by a positive constant \( k \), then
\[ Q(t) = \left[ (2 \lambda_1 (M^n) - k) \int_{\Omega_1} |\nabla u|^2 - \int_{\partial \Omega_1} h(\nabla f, \nabla f) \right] \cdot t^2 + 2 \lambda_1 (M^n) \int_{\Omega_1} u^2 \cdot t + \frac{n}{n + 1} \int_{\Omega_1} u^2 \geq 0, \quad \forall \ t \in \mathbb{R}, \quad (2.18) \]

where \( u \) is the solution of Dirichlet problem (1).

**Proof.** Let \( z = \frac{\partial u}{\partial \nu} = u_{\nu} \) be the normal outward derivative of \( u \). We know that the Reilly formula:
\[ \int_{\Omega_1} (\Delta u)^2 = \int_{\Omega_1} |D^2 u|^2 + \int_{\Omega_1} \text{Ric}(\nabla u, \nabla u) + \int_{\partial \Omega_1} 2 z \Delta u + \int_{\partial \Omega_1} h(\nabla u, \nabla u) + \int_{\partial \Omega_1} n Hz^2. \quad (2.19) \]

Case 1: \( t = 0 \), then (2.18) is naturally true. Case 2: \( t \neq 0 \), we will consider the following Dirichlet problem
\[ \begin{cases} \Delta g = u, & \text{in } \Omega_1; \\ g = tf, & \text{in } \partial \Omega_1 = M^n. \end{cases} \quad (2.20) \]

Combining Green formula and (2.20), we have
\[ \begin{cases} \int_{\partial \Omega_1} f \frac{\partial u}{\partial \nu} = \int_{\Omega_1} |\nabla u|^2; \\ t \int_{\partial \Omega_1} f \frac{\partial u}{\partial \nu} = \int_{\Omega_1} < \nabla u, \nabla g >; \\ \int_{\partial \Omega_1} f^2 \frac{\partial^2 u}{\partial \nu^2} = \int_{\Omega_1} u^2 + \int_{\Omega_1} < \nabla u, \nabla g >. \end{cases} \quad (2.21) \]
From (2.21), we have
\[ \int_{\Omega_1} \langle \nabla u, \nabla g \rangle = t \int_{\Omega_1} |\nabla u|^2. \tag{2.22} \]

From (2.22) and Cauchy-Schwarz inequality, we have
\[ \int_{\Omega_1} |\nabla g|^2 \geq t^2 \int_{\Omega_1} |\nabla u|^2. \tag{2.23} \]

From the third equation in (2.21) and (2.22), we get
\[ t \int_{\partial \Omega_1} f \frac{\partial g}{\partial \nu} = t \int_{\Omega_1} u^2 + t^2 \int_{\Omega_1} |\nabla u|^2. \tag{2.24} \]

Since \( M^n \) is minimal and \( |D^2 g|^2 \geq \frac{1}{n+1} (\Delta g)^2 \), applying (2.19) to \( g \), we obtain
\[ \frac{n}{n+1} \int_{\Omega_1} (\Delta g)^2 \geq k \int_{\Omega_1} |\nabla g|^2 + 2 \int_{\partial \Omega_1} \frac{\partial g}{\partial \nu} \Delta (f) + \int_{\partial \Omega_1} h(\nabla g, \nabla g). \tag{2.25} \]

On the other hand, combining (2.1), (2.23), (2.24), (2.25) and \( \Delta g = u \), we have
\[ \frac{n}{n+1} \int_{\Omega_1} u^2 \geq kt^2 \int_{\Omega_1} |\nabla u|^2 - 2 \lambda_1(M^n) \left[ t \int_{\Omega_1} u^2 + t^2 \int_{\Omega_1} |\nabla u|^2 \right] + t^2 \int_{\partial \Omega_1} h(\nabla f, \nabla f). \tag{2.26} \]

Hence, we have
\[
Q(t) = \left[ (2 \lambda_1(M^n) - k) \int_{\Omega_1} |\nabla u|^2 - \int_{\partial \Omega_1} h(\nabla f, \nabla f) \right] \cdot t^2 + 2 \lambda_1(M^n) \int_{\Omega_1} u^2 \cdot t
+ \frac{n}{n+1} \int_{\Omega_1} u^2 \geq 0, \quad \forall t \in \mathbb{R}.
\]

This completes the proof of Lemma 2.2. \( \Box \)

### 3 Proof of Theorem 1.1

**Proof of Theorem 1.1.** Combining (2.17) and (2.18), we know that the discriminant of \( Q(t) \) is non-positive, i.e.,
\[
\frac{n+1}{n} \lambda_1^2(M^n) \frac{\int_{\Omega_1} u^2}{\int_{\Omega_1} |\nabla u|^2} - 2 \lambda_1(M^n) + k + \frac{\int_{\partial \Omega_1} h(\nabla f, \nabla f)}{\int_{\Omega_1} |\nabla u|^2} \leq 0. \tag{3.1}
\]

Since \( \int_{\partial \Omega_1} h(\nabla u, \nabla u) = \int_{\partial \Omega_2} h(\nabla f, \nabla f) \) and the outward normal vector of \( \partial \Omega_2 \) is \(-e_n\), we have
\[
\int_{\partial \Omega_2} h(\nabla u, \nabla u) = - \int_{\partial \Omega_1} h(\nabla u, \nabla u).
\]

Hence we can assume that \( \int_{\partial \Omega_1} h(\nabla u, \nabla u) \geq 0 \); otherwise, we work with \( \Omega_2 \) rather with \( \Omega_1 \). Since \( \int_{\partial \Omega_1} h(\nabla f, \nabla f) \geq 0 \), from (3.1) we have
\[
\frac{n+1}{n} \lambda_1^2(M^n) \frac{\int_{\Omega_1} u^2}{\int_{\Omega_1} |\nabla u|^2} - 2 \lambda_1(M^n) + k \leq 0. \tag{3.2}
\]

Combining (2.17), \( \int_{\partial \Omega_1} h(\nabla f, \nabla f) \geq 0 \) and \( \int_{\Omega_1} |\nabla u|^2 > 0 \), we have \( \lambda_1(M^n) > \frac{k}{2} \). When we take \( t = -\frac{n}{n+1} k \) in (2.18), combining \( \lambda_1(M^n) > \frac{k}{2} \), \( \int_{\partial \Omega_1} h(\nabla f, \nabla f) \geq 0 \) and \( \int_{\Omega_1} |\nabla u|^2 > 0 \), we get
\[
1 - \frac{(n+1)k}{n} \frac{\int_{\Omega_1} u^2}{\int_{\Omega_1} |\nabla u|^2} \geq 0. \tag{3.3}
\]
From (3.2) and (3.3), we have
\[
\lambda_1(M^n) \geq \frac{n \int_{\Omega_1} |\nabla u|^2 - n \int_{\Omega_1} |\nabla u|^2 \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}}{(n+1) \int_{\Omega_1} u^2} \]
\[
= \frac{k(n+1) \int_{\Omega_1} u^2}{2n(n+1) \int_{\Omega_1} u^2}
+ \frac{2n \int_{\Omega_1} |\nabla u|^2 - k(n+1) \int_{\Omega_1} u^2 - 2n \int_{\Omega_1} |\nabla u|^2 \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}}{2(n+1) \int_{\Omega_1} u^2}
= \frac{k}{2} + n \delta_1(u)
\]
(3.4)
and
\[
\lambda_1(M^n) \leq \frac{n \int_{\Omega_1} |\nabla u|^2 + n \int_{\Omega_1} |\nabla u|^2 \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}}{(n+1) \int_{\Omega_1} u^2} \]
\[
= \frac{k(n+1) \int_{\Omega_1} u^2}{2n(n+1) \int_{\Omega_1} u^2}
+ \frac{2n \int_{\Omega_1} |\nabla u|^2 - k(n+1) \int_{\Omega_1} u^2 + 2n \int_{\Omega_1} |\nabla u|^2 \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}}{2(n+1) \int_{\Omega_1} u^2}
= \frac{k}{2} + n \delta_2(u),
\]
(3.5)
where
\[
\delta_1(u) = \frac{\int_{\Omega_1} |\nabla u|^2 \left(1 - \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}\right)^2}{(n+1) \int_{\Omega_1} u^2}
\]
(3.6)
and
\[
\delta_2(u) = \frac{\int_{\Omega_2} |\nabla u|^2 \left(1 + \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}\right)^2}{(n+1) \int_{\Omega_1} u^2}.
\]
(3.7)
Finally we will prove $0 < \delta_1(u) \leq \frac{k}{n}$, $\delta_2(u) \geq \frac{k}{n}$. Firstly, from (3.6) we have $\delta_1(u) \geq 0$. But $\delta_1(u) \neq 0$, otherwise, from (3.6) we can conclude that $1 - \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2} = 0$, which implies that $\int_{\Omega_1} u^2 = 0$ which is impossible. Hence we get $\delta_1(u) > 0$ and $1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2 \neq 1$. Combining (3.3), we know that
\[
0 \leq 1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2 < 1.
\]
(3.8)
Obviously, $\delta_1(u)$ can be written as
\[
\delta_1(u) = \frac{2 \int_{\Omega_1} |\nabla u|^2 \left(1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2\right) - \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2} + \frac{(n+1)k}{n} \int_{\Omega_1} u^2}{(n+1) \int_{\Omega_1} u^2}.
\]
(3.9)
From (3.8), we obtain
\[
1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2 - \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2} \leq 0.
\]
(3.10)
We can conclude from (3.9) and (3.10) that
\[ \delta_1(u) \leq \frac{k}{n}. \]

Secondly, from (3.7) and (3.8) we obtain
\[
\delta_2(u) = \frac{\int_{\Omega_1} |\nabla u|^2 \left(1 + \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2} \right)^2}{(n+1) \int_{\Omega_1} u^2}
\]
\[
= \frac{2 \int_{\Omega_1} |\nabla u|^2 \left(1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2 \right)^2 + \sqrt{1 - \frac{(n+1)k}{n} \int_{\Omega_1} u^2}}{(n+1) \int_{\Omega_1} u^2}
\]
\[
\geq \frac{k}{n}.
\]

So the proof of Theorem 1.1 is finished.

4 Applications

In 1980, Yang and Yau [3] obtained the following result.

**Theorem 4.1 ([3]).** If \( M^2 \) is an orientable Riemannian surface of genus \( g \) with area \( A \), then
\[ \lambda_1(M^2) \leq 8\pi(g + 1)A^{-1}. \]

From Corollary 1.5 and Theorem 4.1, we obtain the following theorem.

**Theorem 4.2.** Let \( M^2 \) be an embedded compact orientable minimal surface of genus \( g \) with area \( A \) in a 3-dimensional compact orientable Riemannian manifold \( N \). If the Ricci curvature of \( N \) is bounded below by a positive constant \( k \), then
\[ A < \frac{16\pi(g + 1)}{k}. \]

Note that the Ricci curvature of unit sphere \( S^{n+1}(1) \) is \( n \). From Theorem 4.2, we get the following corollary.

**Corollary 4.3.** Let \( M^2 \) be an embedded compact minimal surface of genus \( g \) with area \( A \) in unit sphere \( S^3(1) \). Then
\[ A < 8\pi(g + 1). \]

From Corollary 1.2 and Theorem 4.1, we get the following theorem.

**Theorem 4.4.** Let \( M^2 \) be an embedded compact minimal surface of genus \( g \) with area \( A \) in unit sphere \( S^3(1) \). If the solution \( u \) of Dirichlet problem (1) satisfies \( \int_{\Omega_1} |\nabla u|^2 = 3 \int_{\Omega_1} u^2 \), then
\[ A \leq 4\pi(g + 1). \]
Naturally, we propose the following conjecture.

**Conjecture 4.5.** Let $M^2$ be an embedded compact minimal surface of genus $g$ with area $A$ in unit sphere $S^3(1)$. Then

$$ A \leq 4\pi(g + 1). $$

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