On Discrete Quasiprobability Distributions

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Abstract

We analyse quasiprobability distributions related to the discrete Heisenberg-Weyl group. In particular, we discuss the relation between the Discrete Wigner and Q-functions.

1 Introduction

Due to the recent development of hardware facilities, the description of the state and evolution of the wave processes has also evolved. In addition to the configuration space description and Fourier analysis, it has now become possible to use the Quasiprobability distributions that describe the state of the wave process in phase space of the corresponding mechanical system. By wave process we mean any acoustical, optical or quantum mechanical process, where, correspondingly, time and frequency, the coordinate of the point where the ray of light intersects the screen and the ray direction, the coordinate and momentum of the quantum mechanical particle play the role the phase space coordinates (see, e.g. [1-6]). Until quite recently, the practical use of the Quasiprobability distributions was strongly limited by the speed and memory restrictions of the computers, but now the situation is changing. The possibility to use the Quasiprobability distributions is highly attractive for it allows us to visualize the process in a intuitively clear way: to analyze the signal (e.g. to discriminate between its different components, possibly originated from different sources) and to process the signal (say, removing noise) at the level which is much higher than the one that can be achieved when working with the signal itself or its Fourier transform.

The theory of Quasiprobability distributions in its most complete form was developed in the framework of quantum mechanics (cf. [1-4,6]). In its original version, it refers to continuous variables and is intrinsically related to the Heisenberg-Weyl group of translations of phase space. However, any numerical work involves the discretization, which by itself is not a trivial procedure, as is clear from the example of the approximation of the Fourier
integral by the Finite Fourier transform (see, e.g. [7]). Therefore, it is desirable to develop a systematic theory of the Quasi-Probablity distributions in discrete phase spaces. The basis of such a theory was given in the work of Wootters, [8] (see also [9], [10], [11], [12], [13]). The Hilbert space of system states was chosen to be a space of periodic functions with a finite number of Fourier harmonics. This is obviously a finite-dimensional space. This type of quasi-probability distributions is naturally related to the discrete Heisenberg-Weyl group of translations of discrete phase space. (It should not be confused with the quasi-probability distributions for spin systems, which are related to the SU(2) group [3, 14]; in this case, the finite-dimensional Hilbert space includes spherical harmonics on the sphere.)

Our goal in this work is to establish the correspondence between the two most important discrete quasi-probability distributions: the Wigner-Wootters (“W-function”) distribution and the time dependent spectrum (“Q-function”), as it exists for the continuous case, where the Q-function can be produced by smoothing the W-function with an appropriate Gaussian. A natural periodic counterpart of the Gaussian is the Jacobi Theta-function and therefore, they appear in our construction.

The paper is organized as follows. After the presentation of the discrete coherent states (which naturally involves the discrete Jacobi Theta-functions) and the Q-function, we define the discrete Wigner function (which is made in terms of the discrete Heisenberg-Weyl group and is complementary to the original work [8]). Finally, we establish the relation between them.

We would like to note that we do have in mind possible applications to signal processing. However, at this stage of theoretical development, we found it reasonable to confine ourselves within the language of Quantum mechanics.

2 The model

We consider periodic functions $f(x + L) = f(x)$. Therefore, the Fourier series coefficients contain the frequencies, $\omega_k = k2\pi/L$, where $k$ is an integer,

$$f(x) = \sum_k f_k e^{i2\pi k/L}, \quad f_k = \frac{1}{L} \int_0^L dx f(x)e^{-ix2\pi k/L}.$$  

Let us consider functions for which Fourier series contain only $M$ coefficients different from zero: $k = 0, 1, \ldots, M - 1$. Then all the information is stored in the values of the function in $M$ points on the circle:

$$x_m = \frac{Lm}{M}, \quad m = 0, 1, \ldots, M - 1;$$

$$f(m) \equiv f(x_m) = \sum_{k=0}^{M-1} f_k e^{i2\pi km/M}.$$  

The values of the function $f(x)$ at arbitrary points can be recovered by the sampling theorem.

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We will use the quantum mechanics notation, introducing the basis $|m_0\rangle = \delta_{mm_0}$,

$$|f\rangle = \sum_{m=0}^{M-1} |m\rangle \langle m|f\rangle, \quad \langle m|f\rangle = f(x_m) = f(m).$$

One can use the matrix representation with

$$|f\rangle = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{M-1} \end{pmatrix}.$$

The Discrete Fourier Transform of the vector $f(m)$ gives us the wave function in momentum representation:

$$f(p) = \langle p|f\rangle = \sum_{m=0}^{M-1} \langle p|m\rangle \langle m|f\rangle, \quad p = 0, 1, \ldots, M - 1, \quad \langle p|m\rangle = \frac{1}{\sqrt{M}} e^{-i2\pi pm/M}.$$

Using the discrete orthogonality relation (1),

$$\sum_{m=0}^{M-1} e^{i2\pi m(n-k)/M} = M\delta_{n,k (\text{mod } M)},$$

one finds the DFT coefficients,

$$f(p) = \sqrt{M} f_{p (\text{mod } M)}.$$

Thus, the periodicity also appears in the momentum.

It is natural to introduce one-step translations in coordinate,

$$\hat{T}_x f(x_m) = f(x_{m-1}), \quad \hat{T}_x = \exp \left(-i \frac{L}{M} \hat{p}_x \right), \quad \hat{p}_x = -i\partial_x, \quad (\hbar = 1),$$

and in momentum,

$$\hat{T}_p = \exp \left(i \frac{2\pi}{L} \hat{x} \right), \quad \hat{x} f(x) = xf(x), \quad \hat{T}_p f(x_m) = r^{-m} f(x_m), \quad r = e^{i2\pi/M}.$$

It is clear, that

$$\hat{T}_p \hat{T}_x = \sqrt{r} \hat{T}(1,1) = r \hat{T}_x \hat{T}_p, \quad \hat{T}(1,1) = \exp \left[i \left(\frac{2\pi}{L} \hat{x} - \frac{L}{M} \hat{p}_x \right) \right].$$

Note, that in the course of the discretization of the initial continuous model, $L$ completely disappears from all of the formulas.
3 Vacuum state

The periodic analogue to Gaussian functions is the Jacobi $\Theta$-function defined as follows (see [15] and Appendix A),

$$
\Theta_3(z, \mu) = \sum_{n=\infty}^{\infty} e^{-i2\pi n - \mu n^2} = \sqrt{\frac{\pi}{\mu}} \sum_{k=\infty}^{\infty} \exp \left\{ -\frac{(z - \pi k)^2}{\mu} \right\}.
$$

(Here and below we often use the Poisson transformation.) Generally speaking, $\Re \mu > 0$ but we consider only real values of $\mu$. Recall, that $\Theta_3(z + \pi, \mu) = \Theta_3(z, \mu)$, and hence $x = zL/\pi$.

Considering the $\Theta$-function on a discrete set of points, $x_m = mL/M$ and $z_m = m\pi/M$, we come to a discrete periodic Gaussian,

$$
\Theta(m) = \langle m|\Theta_\mu \rangle = \Theta_3 \left( \frac{\pi m}{M} | \mu \right) = \sum_{n=-\infty}^{\infty} e^{-i2\pi mn/M - \mu n^2} = \sqrt{\frac{\pi}{\mu}} \sum_{k=-\infty}^{\infty} \exp \left\{ -\frac{\pi^2 (m - kM)^2}{M^2 \mu} \right\}.
$$

This function will play a role of the vacuum squeezed state (with squeezing parameter $\mu$). The choice

$$
\mu = \pi/M
$$

corresponds to the vacuum Coherent State. Thus, $\mu \ll 1, \mu M^2 \gg 1$ will be of interest. Using the discrete orthogonality relation (1) one can check the normalization,

$$
N = \langle \Theta_\mu | \Theta_\mu \rangle = \sum_{m=0}^{M-1} |\Theta(\phi_m | \mu)|^2 = M \sum_{n=0}^{\infty} \sum_k e^{-\mu n^2 - \mu(n-kM)^2}
$$

$$
= M \begin{cases} 
\frac{[\theta_3(2\mu)\theta_3(2\mu M^2) + \theta_2(2\mu)\theta_2(2\mu M^2)]}{\theta_3(2\mu)\theta_3(\mu M^2/2)} , & M = \text{odd} \\
\frac{[\theta_3(2\mu)\theta_3(\mu' M^2/2) + \theta_2(2\mu)\theta_4(\mu' M^2)]}{\theta_3(2\mu)\theta_3(2\mu')} , & M = \text{even}
\end{cases}
$$

$$
= \sqrt{\frac{2\pi}{\mu}} \begin{cases} 
\frac{[\theta_3(2\mu)\theta_3(\mu'/2) + \theta_2(2\mu)\theta_4(\mu'/2)]}{\theta_3(2\mu)\theta_3(2\mu')} , & M = \text{odd} \\
\frac{[\theta_3(2\mu)\theta_3(\mu) + \theta_2(2\mu)\theta_4(\mu)]}{\theta_3(2\mu)\theta_3(2\mu')} , & M = \text{even}
\end{cases}
$$

Here

$$
\mu' = \frac{\pi}{\mu M^2};
$$

for unsqueezed vacuum CS, $\mu = \mu' = \pi/M$. We will assume, that $\mu = \mu_0 \pi/M$, $\mu_0 \sim 1$, and thus, $\mu, \mu' \ll 1$ simultaneously with $\mu M^2, \mu' M^2 \gg 1$.

In Eq. (3) we use the notation,

$$
\theta_k(\mu) = \Theta_k(0 | \mu), \quad k = 1, 2, 3, 4;
$$

(see Appendix A). For instance,

$$
\theta_3(\mu) = \sum e^{-\mu n^2} = \sqrt{\frac{\pi}{\mu}} \sum e^{-k^2 \pi^2 / \mu} = \sqrt{\frac{\pi}{\mu}} + O\left(e^{-\pi^2 / \mu}\right),
$$
\[ \theta_2(\mu) = \sum e^{-\mu(n+1/2)^2} = \sqrt{\frac{\pi}{\mu}} \sum (-1)^k e^{-k^2\pi^2/\mu} = \sqrt{\frac{\pi}{\mu}} + O\left(e^{-\pi^2/\mu}\right). \]

In passing from the second to the third line of Eq. (3) we again used the Poisson transformation, \( \theta_3(\mu M^2/2) = \sqrt{2\mu'/\pi} \theta_3(2\mu') \), and \( \theta_2(2\mu M^2) = \sqrt{\mu'/2\pi} \theta_4(\mu'/2) \). If \( M \gg 1 \), the normalization constant has a simple asymptotic form. Indeed, in this case only the terms \( k = 0 \) are important in the Poisson-transformed expressions for \( \theta_2, \theta_3, \theta_4 \), and neglecting the terms \( O\left(e^{-\mu M^2/2}\right), O\left(e^{-\mu' M^2/2}\right) \), we have,

\[ \langle \Theta_\mu | \Theta_\mu \rangle \approx M \sqrt{\frac{\pi}{2\mu}}. \quad (4) \]

The wave function of the vacuum state in the momentum representation is given by the Discrete Fourier transform:

\[ \langle p | \Theta_\mu \rangle = \sqrt{M} \sum_{k=-\infty}^{\infty} e^{-\mu(p-kM)^2} = \sqrt{\frac{\pi}{\mu M}} \sum_{n=-\infty}^{\infty} e^{-i2\pi pn/M-M' n^2} = \sqrt{\frac{\pi}{\mu M}} \Theta(\frac{\pi p}{M} | \mu'). \]

4 Discrete Heisenberg-Weyl group

Let us return to the one-step translations in coordinate and momentum, which are defined in the framework of the initial continuous model as \( \hat{T}_x = e^{-i\theta/L/M}, \hat{T}_p = e^{i\theta M/2/L} \),

\[ \hat{T}_x |m\rangle = |m+1\rangle, \quad \hat{T}_x f(m) = \langle m | \hat{T}_x | f \rangle = f(m-1), \quad \hat{T}_x |p\rangle = r^{-p} |p\rangle, \]

\[ \hat{T}_p |m\rangle = r^m |m\rangle, \quad \hat{T}_p |p\rangle = |p+1\rangle, \quad \hat{T}_p f(p) = f(p-1). \]

Here, once again, \( r = e^{i2\pi/M} \). These operators generate a discrete group. It is clear that there is a periodicity in momentum, as well as in coordinate, \( \hat{T}_p^M = \hat{T}_x^M = 1 \). These translations do not commute,

\[ \hat{T}_p \hat{T}_x = r^{nm} \hat{T}_x \hat{T}_p. \]

Therefore, an arbitrary group element must include the multiplication by a phase factor,

\[ g(s, m, n) = r^s \hat{T}_x^m \hat{T}_p^n. \quad (5) \]

The naive way to introduce the displacement operator in phase space is (compare [10], [11]),

\[ \hat{T}(m, n) = r^{nm/2} \hat{T}_x^m \hat{T}_p^n = r^{-nm/2} \hat{T}_x^{-m} \hat{T}_p^{m} = \exp \left[ i \left( \frac{2\pi}{L} x - \frac{L}{M} p \right) \right]. \]

However, such an operator does not belong to the group (5). Fortunately, there exists a way to improve the situation. Let us now consider the case of odd \( M \). Then, in the set of numbers
\( \{n (\text{mod} M)\} = \{1, 2, \ldots, M - 1\} \) there exists a unique solution to the equation \( 2n = m \), which we denote as

\[
\left[ \frac{m}{2} \right] = \frac{m}{2 + \frac{M}{2} \quad m = 2k, \right.
\]

\[
\left. \frac{m}{2} \quad m = 2k + 1. \right]
\]

(More generally, in the case of prime \( M \) there always exists a unique solution to the equation \( sn = m \), and the numbers \( \{n (\text{mod} M)\} \) then form a field.) It is clear, that

\[
\left[ \frac{mn}{2} \right] \left[ \frac{m}{2} \right] n = \left[ \frac{m}{2} \right] n = \left[ \frac{m}{2} \right] + \left[ \frac{n}{2} \right].
\]

We now introduce the displacement operator as follows,

\[
\hat{T}(m, n) = r^{\left\lfloor \frac{-nm}{2} \right\rfloor} \hat{T}_x \hat{T}_p = r^{-\left\lfloor \frac{-nm}{2} \right\rfloor} \hat{T}_p \hat{T}_x \cdot (-1)^{\left\lfloor \frac{nm}{2} \right\rfloor} \hat{T}(m, n).
\]

\( \hat{T} \) is unitary, \( \hat{T}^\dagger (m, n) = \hat{T}^{-1}(m, n) = \hat{T}(-m, -n) \), and periodic, \( \hat{T}(m + M, n) = \hat{T}(m, n + M) = \hat{T}(m, n) \). It is easy to check the multiplication formula,

\[
\hat{T}(m_2, n_2) \hat{T}(m_1, n_1) = r^{\left\lfloor \frac{(m_1 n_2 - n_1 m_2)}{2} \right\rfloor} \hat{T}(m_1 + m_2, n_1 + n_2).
\]

One can introduce the adjoint action,

\[
\hat{T}(q, p) \hat{T}(m, n) \hat{T}^\dagger (q, p) = r^{qm - pn} \hat{T}(m, n).
\]

The matrix elements in the coordinate basis,

\[
\langle k| \hat{T}(m, n)|l \rangle = r^{nl + \left\lfloor \frac{nm}{2} \right\rfloor} \delta_{k,l+m(\text{mod} M)}.
\]

These matrix elements are orthogonal,

\[
\sum_{m,n=0}^{M-1} \langle a| \hat{T}(m, n)|b \rangle \langle d| \hat{T}(m, n)|c \rangle = M \delta_{ad} \delta_{bc}. \quad (9)
\]

Finally,

\[
\text{Tr} \hat{T}(m, n) = M \delta_{m,0} \delta_{n,0}.
\]

5 Discrete CS and Q-dunction

One can generate the complete set of (squeezed) CS by the action of the HW group (6) to the vacuum state (2),

\[
|m_0, n_0, \mu \rangle = \hat{T}(m_0, n_0)|\Theta_{\mu} \rangle,
\]

\[
\Theta_{m_0n_0}(m) = \langle m|m_0, n_0, \mu \rangle = e^{i2\pi n_0 (m - m_0)/M} \Theta(m - m_0)
\]

\[
= e^{i2\pi n_0 (m - m_0)/M} \sum_{n=-\infty}^{\infty} \exp \left[ i \frac{2\pi}{M} (m - m_0)n - \mu n^2 \right].
\]

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and in the momentum representation,

$$\Theta_{m_0n_0}(p) = \langle p|m_0,n_0,\mu \rangle = \sqrt{M} e^{-i2\pi(p-n_0/2)m_0/M} \sum_{k=-\infty}^{\infty} \exp \left[ -\mu(p-n_0-kM)^2 \right].$$

The normalization constant $\langle m_0,n_0,\mu|m_0,n_0,\mu \rangle = N$ is the same as in (3).

*The completeness relation* holds (for any $\mu$):

$$\frac{1}{N} \sum_{m_0,n_0=0}^{M-1} |m_0,n_0,\mu \rangle \langle m_0,n_0,\mu| = \hat{1}. \quad (12)$$

**Scalar products of Coherent States.** It is enough to consider the vacuum matrix element of the displacement operator,

$$\langle \Theta_\mu|m_0,n_0,\mu \rangle = \langle \Theta_\mu|\hat{T}(m_0,n_0)|\Theta_\mu \rangle.$$  

If $M$ is even, it is equal to

$$\langle \Theta_\mu|m_0,n_0,\mu \rangle = M \sqrt{\frac{2\mu'}{\pi}} \Theta_\alpha \left( \frac{\pi m_0}{M} \big| 2\mu \right) \Theta_\beta \left( \frac{\pi n_0}{M} \big| 2\mu' \right),$$

where,

$$\begin{cases} 
\alpha = 3, & n_0 \text{ even}; \\
\alpha = 2, & n_0 \text{ odd}; \\
\beta = 3, & m_0 \text{ even}; \\
\beta = 2, & m_0 \text{ odd}.
\end{cases}$$

If $M$ is odd, $n_0$ is even,

$$\langle \Theta_\mu|m_0,n_0,\mu \rangle = M \sqrt{\frac{\mu'}{2\pi}} \left[ \Theta_3 \left( \frac{\pi m_0}{M} \big| 2\mu \right) \Theta_3 \left( \frac{\pi n_0}{2M} \big| \frac{\mu'}{2} \right) + (-1)^m_0 \Theta_2 \left( \frac{\pi m_0}{M} \big| 2\mu \right) \Theta_4 \left( \frac{\pi n_0}{2M} \big| \frac{\mu'}{2} \right) \right],$$

and if $M$ is odd, $n_0$ is odd,

$$\langle \Theta_\mu|m_0,n_0,\mu \rangle = M \sqrt{\frac{\mu'}{2\pi}} \left[ (-1)^m_0 \Theta_3 \left( \frac{\pi m_0}{M} \big| 2\mu \right) \Theta_4 \left( \frac{\pi n_0}{2M} \big| \frac{\mu'}{2} \right) + \Theta_2 \left( \frac{\pi m_0}{M} \big| 2\mu \right) \Theta_3 \left( \frac{\pi n_0}{2M} \big| \frac{\mu'}{2} \right) \right],$$

Asymptotic for large $M$ is

$$\frac{\langle \Theta_\mu|m_0,n_0,\mu \rangle}{\langle \Theta_\mu|\Theta_\mu \rangle} \approx \exp \left\{ -\frac{\mu' m_0^2 + \mu m_0^2}{2} \right\}.$$  

*Q-function.* It is natural to introduce the Q-function as the diagonal matrix element of the density matrix between the coherent states,

$$Q(m,p) = \langle m,p,\mu|\rho|m,p,\mu \rangle, \quad (13)$$

or, for a pure state, $\rho = |\Psi \rangle \langle \Psi |$,

$$Q(m,p) = |\langle m,p,\mu]|^2.$$  

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6 Wigner function

By analogy with the continuous case, let us introduce the Wigner operator as a two-dimensional discrete Fourier transform of the displacement operator,

\[ \hat{W}(q, p) = \frac{1}{M} \sum_{m,n=0}^{M-1} e^{-i p m - q n} \hat{T}(m, n), \quad q, p = 0, 1, \ldots, M - 1. \] (14)

Therefore, it is an Hermitian operator valued function on the phase space, \( \hat{W}(q, p) = \hat{W}^\dagger(q, p) \). One can notice that,

\[ \hat{W}(q, p) = \hat{T}(q, p) \hat{W}(0, 0) \hat{T}^\dagger(q, p). \]

One immediately calculates its matrix elements,

\[ \langle k | \hat{W}(q, p) | l \rangle = \exp \left\{ i \frac{2\pi}{M} p (k - l) \right\} \delta_{2q = k + l \text{(mod} M)}. \]

These are precisely the matrix elements of the "Phase Point Operators" considered by Wooters [8], with the property

\[ \text{Tr} \left( \hat{W}(q, p) \hat{W}(p_1, q_1) \right) = M \delta_{q,q_1} \delta_{p,p_1}. \] (15)

This property was used to define these operators in [8].

The Wigner function (quasi-probability distribution) on the discrete phase space for the state \( |f\rangle \) is the mean value of the Wigner operator in this state,

\[ W_f(p, q) = \langle f | \hat{W}(q, p) | f \rangle. \]

To an arbitrary Hermitian operator \( \hat{A} \), corresponds the Wigner function

\[ W_f(p, q) = \text{Tr} \left( \hat{W}(q, p) \hat{A} \right). \]

If \( \hat{A} \rightarrow \rho \) is a density matrix and \( \text{Tr} \rho = 1 \), then

\[ \sum_{p,q} \hat{W}(q, p) = 1. \]

For a pure state, \( \text{Tr} \rho^2 = 1, \)

\[ \sum_{p,q} \hat{W}^2(q, p) = 1. \]

The orthogonality of the Wigner operator at different phase points (15), has a simple physical sense: the Wigner function of the Wigner operator itself is a \( \delta \)-function,

\[ W_{\hat{W}(p_0, q_0)}(p, q) = M \delta_{q,q_0} \delta_{p,p_0}. \]
Covariance under HW group. Using the adjoint action of the discrete HW group, (8), one can show that

\[ \hat{W}(q,p) = \hat{T}(p,q)W(0,0)\hat{T}^\dagger(p,q). \]

The Wigner operator at the origin, \( W(0,0) \) has a simple form, which we will explain, considering examples of odd and even \( M \). For \( M = 5 \),

\[
W(0,0) = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{vmatrix},
\]

thus the spectrum of Wigner operator for odd \( M \) consists of \( M/2 \) eigenvalues \(-1\) and \( M/2 + 1 \) eigenvalues \(+1\). For \( M = 4 \),

\[
W(0,0) = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{vmatrix},
\]

the spectrum for even \( M \) consists of \((M-1)/2\) eigenvalues \(-1\) and \((M-1)/2 + 1\) eigenvalues \(+1\). Therefore,

\[
\text{Tr} \hat{W}(q,p) = \sum_{m=0}^{M-1} \langle m|\hat{W}(q,p)|m\rangle = 1, = \begin{cases} 1, & M = \text{odd}, \\ 2, & M = \text{even}, \end{cases}
\]

As in the continuous case, \( \hat{W}(0,0) \) is the inversion operator with respect to the origin. In the case of odd \( M \), there is only one stable point \((m = 0)\) under this inversion, while for even \( M \) there are two stable points \( m = 0 \), and \( m = M/2 \). In the rest of the paper we will restrict ourselves with the case when \( M \) is odd. (In the opposite case the information is lost and one cannot reconstruct the state from its Wigner function.)

Overlap relation. Let us consider the following operator acting in the tensor product of two spaces,

\[ \hat{\sigma} = \frac{1}{M} \sum_{p,q} \hat{W}(q,p) \times \hat{W}(q,p). \]

Using the orthogonality of the displacement operator matrix elements (9), one can prove that \( \hat{\sigma} \) is an exchange operator,

\[ \langle a|_1\langle c|_2\hat{\sigma}|d|_2|b\rangle_1 = \delta_{b,c}\delta_{a,d}, \quad \hat{\sigma}|f\rangle_1|g\rangle_2 = |g\rangle_1|f\rangle_2. \]

From here the overlap relation follows directly. Let \( W_A(p,q) \) and \( W_B(p,q) \) be the Wigner symbol of the operators \( \hat{A} \) and \( \hat{B} \). Then

\[
\frac{1}{M} \sum_{p,q} W_A(p,q)W_B(p,q) = \text{Tr} \hat{A}\hat{B}. \tag{16}
\]
Reconstruction of the initial state. If we know the Wigner function \(W_\rho(p, a)\), then the density matrix of this state can be reconstructed as
\[
\hat{\rho} = \sum_{p, q} \hat{W}(q, p) W_\rho(p, q).
\]

7 Relation between W and Q-functions

We assume now that \(M\) is odd, and \(\mu = \mu' = \pi/M\). Let us start with the kernel
\[
\langle k|\hat{Q}(q, p)|l\rangle = \langle k|q, p\rangle \langle q, p|\Theta_\mu\rangle \langle \Theta_\mu|\hat{T}(q, p)^\dagger|l\rangle,
\]
and find its Fourier transform,
\[
\sum_{q, p=0}^{M-1} r^{qn-\mu n} \langle k|\hat{Q}(q, p)|l\rangle = M^2 \langle k|\hat{T}(m, n)|l\rangle F(m, n),
\]
where the function \(F(m, n)\) is defined as
\[
F(m, n) = \sqrt{\frac{\mu'}{2\pi}} \left[ \Theta_3 \left( \frac{\pi m}{M} \mid 2\mu \right) \Theta_3 \left( \frac{\pi n}{2 M} \mid \frac{\mu'}{2} \right) + (-1)^m \Theta_2 \left( \frac{\pi m}{M} \mid 2\mu \right) \Theta_4 \left( \frac{\pi}{M} \left[ \frac{n}{2} \right] \mid \frac{\mu'}{2} \right) \right].
\]

(17)

In other words,
\[
\hat{Q}(q, p) = \sum_{m, n=0}^{M-1} \hat{T}(m, n) F(m, n) r^{pm-qn} = \sum_{m, n=-\infty}^{\infty} \hat{T}(m, n) f(m, n) r^{pm-qn}.
\]

(18)

To find the function \(f(m, n)\), let us note that for any periodic discrete function \(\phi(m+M) = \phi(m)\) the following formulas hold:
\[
\sum_{n=-\infty}^{\infty} \phi(n) e^{-\mu n^2} = \sqrt{\frac{\mu'}{\pi}} \sum_{m=0}^{M-1} \phi(m) \Theta_3(\pi n/M | \mu'),
\]
\[
\sum_{n=-\infty}^{\infty} (-1)^n \phi(n) e^{-\mu n^2} = \sqrt{\frac{\mu'}{\pi}} \sum_{m=0}^{M-1} (-1)^m \phi(m) \Theta_2(\pi n/M | \mu'),
\]
\[
\sum_{n=-\infty}^{\infty} \phi(n) e^{-\mu(n-M/2)^2} = \sqrt{\frac{\mu'}{\pi}} \sum_{m=0}^{M-1} \phi(m) \Theta_4(\pi n/M | \mu'),
\]
and moreover:
\[
\sum_{n=-\infty}^{\infty} \phi([2n]) e^{-\mu n^2} = \sqrt{\frac{\mu'}{\pi}} \sum_{m=0}^{M-1} \phi(m) \Theta_3 \left( \frac{\pi}{M} \left[ \frac{m}{2} \right] \mid \mu' \right)
\]
\[
\sum_{n=-\infty}^{\infty} \phi([2n]) e^{-\mu(n-M/2)^2} = \sqrt{\frac{\mu'}{\pi}} \sum_{m=0}^{M-1} \phi(m) \Theta_4 \left( \frac{\pi}{M} \left[ \frac{m}{2} \right] \mid \mu' \right).
\]

Therefore, for any periodic function of two variables, \( \phi(m+M,n) = \phi(m,n+M) = \phi(m,n) \), one can prove,

\[
\sum_{m,n=0}^{M-1} \phi(m,n) F(m,n) = \sum_{m,n=-\infty}^{\infty} \phi(m,n) f(m,n),
\]

with

\[
f(m,n) = \sqrt{\frac{\pi}{2\mu}} (-1)^{mn} \exp \left( -\frac{\mu'}{2} m^2 - \frac{\mu}{2} n^2 \right).
\]

From here it follows that \( F(m,n) = F(n,m) \) (provided that \( \mu = \mu' \), i.e. the coordinate and momentum have the same rights.

8 Conclusions

The principal results of this paper are the relations (18),(17),(19) between the discrete Q-function (time-dependent spectrum) and the discrete Wigner function. As in the continuous case, the Q-function can be produced by smoothing the W-function by integrating it with the discrete Gaussian function.

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Appendix A: Theta functions

Jacobi Theta functions are the integral functions defined as follows [15]:

\[
\Theta_3(z|\mu) = \sum_{n=-\infty}^{\infty} \exp \left\{ -i2nz - \mu n^2 \right\} = \sqrt{\frac{\pi}{\mu}} \sum_{k=-\infty}^{\infty} \exp \left\{ -\frac{(z-\pi k)^2}{\mu} \right\},
\]

\[
\Theta_4(z|\mu) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \left\{ -i2nz - \mu n^2 \right\} = \sqrt{\frac{\pi}{\mu}} \sum_{k=-\infty}^{\infty} \exp \left\{ -\frac{(z-\pi(k+1/2))^2}{\mu} \right\},
\]

\[
\Theta_2(z|\mu) = \sum_{n=-\infty}^{\infty} \exp \left\{ -i2 \left( n + \frac{1}{2} \right) z - \mu n^2 \right\} = \sqrt{\frac{\pi}{\mu}} \sum_{k=-\infty}^{\infty} (-1)^k \exp \left\{ -\frac{(z-\pi k)^2}{\mu} \right\},
\]

\[
\Theta_1(z|\mu) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \left\{ -i2 \left( n + \frac{1}{2} \right) z - \mu n^2 \right\} = \sqrt{\frac{\pi}{\mu}} \sum_{k=-\infty}^{\infty} (-1)^k \exp \left\{ -\frac{(z-\pi(k+1/2))^2}{\mu} \right\}.
\]
Here $i\mu$ is quasiperiod, $\text{Re} \mu > 0$. Note, that in [15] Theta functions are written as $\Theta_k(z, q)$, where $r = e^{-\mu}$. $\Theta_{1,2,4}$ are shifts of $\Theta_3$:

$$
\begin{align*}
\Theta_4(z|\mu) &= \Theta_3(z + \pi/2|\mu), \\
\Theta_2(z|\mu) &= -ie^{iz-\mu/4}\Theta_3(z + i\mu/2|\mu), \\
\Theta_1(z|\mu) &= \Theta_2(z - \pi/2|\mu) = -ie^{iz-\mu/4}\Theta_3(z + \pi/2 + i\mu/2|\mu).
\end{align*}
$$

Note, that

$$
\Theta_{1,2}(z + \pi|\mu) = -\Theta_{1,2}(z|\mu), \quad \Theta_{3,4}(z + \pi|\mu) = \Theta_{3,4}(z|\mu),
$$

and

$$
\Theta_{1,4}(z + i\mu|\mu) = -g\Theta_{1,4}(z|\mu), \quad \Theta_{2,3}(z + i\mu|\mu) = g\Theta_{2,3}(z|\mu),
$$

where $g = e^{-iz+\mu}$. Finally,

$$
\Theta_1(-z|\mu) = -\Theta_1(z|\mu), \quad \Theta_{2,3,4}(-z|\mu) = \Theta_{2,3,4}(z|\mu).
$$

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