Proof of a Conjecture of Reiner-Tenner-Yong on Barely Set-valued Tableaux

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Abstract

The notion of a barely set-valued semistandard Young tableau was introduced by Reiner, Tenner and Yong in their study of the probability distribution of edges in the Young lattice of partitions. Given a partition $\lambda$ and a positive integer $k$, let $\text{BSSYT}(\lambda, k)$ (respectively, $\text{SYT}(\lambda, k)$) denote the set of barely set-valued semistandard Young tableaux (respectively, ordinary semistandard Young tableaux) of shape $\lambda$ with entries in row $i$ not exceeding $k + i$. In the case when $\lambda$ is a rectangular staircase partition $\delta_{(a,b)}$, Reiner, Tenner and Yong conjectured that
\[
|\text{BSSYT}(\lambda, k)| = \frac{kab}{(a+b)}|\text{SYT}(\lambda, k)|.
\]
In this paper, we establish a connection between barely set-valued tableaux and reverse plane partitions with designated corners. We show that for any shape $\lambda$, the expected jaggedness of a subshape of $\lambda$ under the weak probability distribution can be expressed as
\[
2\frac{|\text{BSSYT}(\lambda, k)|}{k|\text{SYT}(\lambda, k)|}.
\]
On the other hand, when $\lambda$ is a balanced shape with $r$ rows and $c$ columns, Chan, Haddadan, Hopkins and Moci proved that the expected jaggedness of a subshape in $\lambda$ under the weak distribution equals $2rc/(r+c)$. Hence, for a balanced shape $\lambda$ with $r$ rows and $c$ columns, we establish the relation that
\[
|\text{BSSYT}(\lambda, k)| = \frac{krc}{(r+c)}|\text{SYT}(\lambda, k)|.
\]
Since a rectangular staircase shape $\delta_{(b^a)}$ is a balanced shape, we confirm the conjecture of Reiner, Tenner and Yong.

1 Introduction

The structure of a set-valued semistandard Young tableau was introduced by Buch \cite{Buch} in his study of the Littlewood-Richardson rule for stable Grothendieck polynomials. A flagged set-valued semistandard Young tableau, defined by Knutson, Miller and Yong \cite{KMY}, is a set-valued semistandard Young tableau such that each value in row $i$ does not exceed a positive integer $\phi_i$. The notion of flagged barely set-valued semistandard Young tableaux arose in the work of Reiner, Tenner and Yong \cite{RTY} on the probability distribution of the edges in the Young lattice of partitions.
The main objective of this paper is to prove a conjecture of Reiner, Tenner and Yong [9] concerning the enumeration of barely set-valued tableaux. A barely set-valued semistandard Young tableau is a set-valued semistandard Young tableau such that exactly one square is assigned two integers and each of the remaining squares is occupied by a single integer. For an example, see Figure 1.1.

Figure 1.1: A barely set-valued semistandard Young tableau.

For a partition $\lambda$ and a positive integer $k$, let $\text{BSSYT}(\lambda, k)$ (respectively, $\text{SYT}(\lambda, k)$) denote the set of barely set-valued semistandard Young tableaux (respectively, ordinary semistandard Young tableaux) of shape $\lambda$ such that the every integer in row $i$ does not exceed $k+i$. When $\lambda$ is a rectangular staircase shape $\delta_d(a^b)$, namely, the Young diagram obtained from the staircase shape $\delta_d = (d-1, d-2, \ldots, 1)$ by replacing each square by an $a \times b$ rectangle, Reiner, Tenner and Yong [9] posed the following conjecture.

**Conjecture 1.1** (Reiner, Tenner and Yong [9]). For any positive integers $a, b, d$ and $k$,

$$|\text{BSSYT}(\delta_d(a^b), k)| = \frac{kab(d-1)}{(a+b)}|\text{SYT}(\delta_d(a^b), k)|. \quad (1.1)$$

For $d = 2$, Reiner, Tenner and Yong showed that the above conjecture is true by employing the RSK algorithm as well as Stanley’s hook content formula for semistandard Young tableaux.

In this paper, we give a representation of a barely set-valued tableau of shape $\lambda$ in terms of a reverse plane partition of shape $\lambda$ together with a designated corner of a subshape of $\lambda$. This representation enables us to establish a connection between the enumeration of barely set-valued tableaux of shape $\lambda$ and the expected jaggedness of a subshape of $\lambda$ under the weak distribution. The notion of the jaggedness of a subshape of a Young diagram was introduced by Chan, Haddadan, Hopkins and Moci [3]. More precisely, we show that the expected jaggedness of a subshape of $\lambda$ under the weak distribution can be expressed as

$$\frac{2|\text{BSSYT}(\lambda, k)|}{k|\text{SYT}(\lambda, k)|}.$$ 

On the other hand, when $\lambda$ is a balanced shape with $r$ rows and $c$ columns, Chan, Haddadan, Hopkins and Moci [3] showed that the expected jaggedness of a subshape
of $\lambda$ under the weak distribution equals

$$\frac{2rc}{r+c}.$$ 

Hence, for a balanced shape $\lambda$ with $r$ rows and $c$ columns, the following relation holds:

$$|\text{BSSYT}(\lambda, k)| = \frac{krc}{(r+c)}|\text{SYT}(\lambda, k)|. \tag{1.2}$$

Chan, Haddadan, Hopkins and Moci [3] observed that a rectangular staircase partition $\delta_d(b^a)$ is a balanced shape. Restricting to a rectangular staircase shape $\delta_d(b^a)$, (1.2) yields (1.1), and this leads to a proof of Conjecture 1.1.

Let us proceed with some terminology and notation. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of a nonnegative integer $n$, that is, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a sequence of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The Young diagram of $\lambda$ is a left-justified array of squares with $\lambda_i$ squares in row $i$. If no confusion arises, we do not distinguish a partition and its Young diagram. A Young diagram is also called a shape. A set-valued semistandard Young tableau of shape $\lambda$ is an assignment of finite sets of positive integers into the squares of $\lambda$ such that the sets in each row (respectively, column) are weakly (respectively, strictly) increasing, see Buch [1]. For two sets $A$ and $B$ of positive integers, we write $A \leq B$ if $\max A \leq \min B$ and $A < B$ if $\max A < \min B$. When the set in each square contains a single integer, a set-valued semistandard Young tableau becomes an ordinary semistandard Young tableau. In the case when exactly one square receives two integers and each of the remaining squares receives only one integer, such a set-valued tableau is called a barely set-valued semistandard Young tableau, see Reiner, Tenner and Yong [9].

Given a vector $\phi = (\phi_1, \phi_2, \ldots)$ of positive integers, we say that a set-valued semistandard Young tableau is flagged by $\phi$ if every entry in row $i$ cannot exceed $\phi_i$, see Knutson, Miller and Yong [5]. In particular, when $\phi = (k+1, k+2, \ldots)$, that is, $\phi_i = k+i$, we use $\text{BSSYT}(\lambda, k)$ (respectively, $\text{SYT}(\lambda, k)$) to represent the set of barely set-valued semistandard Young tableaux (respectively, ordinary semistandard Young tableaux) flagged by $\phi$.

It is worth mentioning that Conjecture [11] can be reformulated in terms of the polynomials $FK(w, \ell)$ defined on 0-Hecke words of length $\ell$ for a permutation $w$ [9]. A 0-Hecke word of a permutation $w$ on $\{1, 2, \ldots, n\}$ can be constructed recursively as follows. As usual, we use $s_i$ ($1 \leq i \leq n-1$) to denote the simple transposition that swaps $i$ and $i+1$. An expression of $w$ as a product of simple transpositions is called reduced if it consists of a minimum number of simple transpositions. The length of $w$, denoted $\ell(w)$, is the number of simple transpositions in a reduced expression of $w$. The length $\ell(w)$ of $w$ can also be interpreted as the number of inversions of $w = w_1w_2\cdots w_n$, that is,

$$\ell(w) = |\{(i, j) \mid 1 \leq i < j \leq n, w_i > w_j\}|.$$
Given a sequence $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})$ of simple transpositions, we construct a permutation, denoted $H(S)$, by a recursive procedure as follows. If $\ell = 1$, set $H(S) = s_{i_1}$. If $\ell > 1$, let $S' = (s_{i_1}, s_{i_2}, \ldots, s_{i_{\ell-1}})$ and set

$$H(S) = \begin{cases} H(S'), & \text{if } \ell(H(S') \cdot s_{i_\ell}) < \ell(H(S')), \\ H(S') \cdot s_{i_\ell}, & \text{if } \ell(H(S') \cdot s_{i_\ell}) > \ell(H(S')). \end{cases}$$

If $H(S) = w$, then $S$ is called a 0-Hecke word of $w$ of length $\ell$. It is easily seen that a 0-Hecke word of $w$ of length $\ell(w)$ is a reduced expression of $w$.

The polynomials $FK(w, \ell)$ are a generalization of the following polynomials defined by Fomin and Kirilov [4]:

$$\sum_{(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})} (x + i_1)(x + i_2) \cdots (x + i_\ell),$$

where the sum ranges over reduced expressions of the longest permutation $w_0 = n(n - 1) \cdots 1$ of length $\ell_0 = \ell(w_0) = n(n - 1)/2$. Using a counting formula for the monomials in a Schubert polynomial due to Macdonald [6] and a formula on the number of reverse plane partitions of a staircase shape found by Proctor [8], Fomin and Kirilov [4] established the following relation:

$$\sum_{(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})} (x + i_1)(x + i_2) \cdots (x + i_\ell) = \left(\binom{n}{2}\right)! \prod_{1 \leq i < j \leq n} \frac{x + i + j - 1}{i + j - 1}. \quad (1.4)$$

Equating the leading coefficients on both sides, (1.4) gives the number of reduced expressions of the longest permutation $w_0$, as proved by Stanley [11]. Reiner, Tenner and Yong [9] defined the polynomial $FK(w, \ell)$ as follows:

$$FK(w, \ell) = \sum_{(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})} (x + i_1)(x + i_2) \cdots (x + i_\ell),$$

where the sum ranges over 0-Hecke words of $w$ of length $\ell$. For the case $w = w_0$ and $\ell = \ell(w_0)$, $FK(w, \ell)$ reduces to the polynomial in (1.3).

Reiner, Tenner and Yong [9] showed that Conjecture 1.1 is equivalent to a relation on $FK(w, \ell)$, where $w$ is a dominant permutation whose Lehmer code is a rectangular staircase shape. Recall that $w$ is called a dominant permutation if it is 132-avoiding, that is, if there are no indices $i_1 < i_2 < i_3$ such that $w_{i_1} < w_{i_3} < w_{i_2}$. There is an alternative characterization of a dominant permutation, that is, $w$ is dominant if and only if the Lehmer code $(c_1(w), c_2(w), \ldots, c_n(w))$ of $w$ is a nonincreasing sequence, where, for $1 \leq i \leq n$, $c_i(w)$ is the number of inversions of $w$ at position $i$, namely,

$$c_i(w) = |\{j \mid i < j, w_i > w_j\}|.$$

Employing properties of Grothendieck polynomials, Reiner, Tenner and Yong [9] showed that Conjecture 1.1 is equivalent to the following assertion.
**Conjecture 1.2** (Reiner, Tenner and Yong [9]). Let \( w \) be a dominant permutation whose Lehmer code is a rectangular staircase shape \( \lambda = \delta_d(b^a) \). Then

\[
\frac{FK(w, \ell(w) + 1)}{FK(w, \ell(w))} = \left( \frac{\ell(w) + 1}{2} \right) \left( \frac{4x}{d(a + b)} + 1 \right).
\] (1.5)

In this paper, we obtain an extension of (1.5) to a dominant permutation whose Lehmer code is a balanced shape.

## 2 The formula of Chan-Haddadan-Hopkins-Moci

In this section, we shall give an overview of a formula of Chan, Haddadan, Hopkins and Moci [3] for the expected jaggedness of a subshape in a Young diagram under a toggle-symmetric distribution. This formula is needed in our proof of the conjecture of Reiner, Tenner and Yong. It is a far reaching generalization of a formula derived by Chan, Lópezmartín, Pflueger and Teixidor i Bigas [2]. While the formula is quite involved for a general shape \( \lambda \), as far as this paper is concerned, we only need the case when \( \lambda \) is a balanced shape. In this case it admits a closed form. Moreover, we shall restrict our attention to a special distribution, namely, the weak distribution on the subshapes of a balanced shape. This is possible because as shown in [3], the weak distribution is indeed a toggle-symmetric distribution.

Let us begin with the necessary terminology. A toggle-symmetric probability distribution is a probability distribution on the order ideals of a finite poset subject to certain symmetry conditions. Given a finite poset \((P, \leq)\), an order ideal \( I \) of \( P \) is a subset of \( P \) such that if \( p \in I \) and \( q \in P \) with \( q \leq p \), then \( q \in I \). Let \( J(P) \) denote the set of order ideals of \( P \). We say that an element \( p \in P \) can be toggled into \( I \) if \( p \) is a minimal element not in \( I \), and that \( p \) can be toggled out of \( I \) if \( p \) is a maximal element in \( I \). To be more specific, we say that \( p \) can be toggled into \( I \) if \( p \notin I \) and \( I \cup \{p\} \) is an order ideal of \( P \), and that \( p \) can be toggled out of \( I \) if \( p \in I \) and \( I \setminus \{p\} \) is an order ideal of \( P \). For each \( p \in P \), the indicator random variables \( T_p^+ \) and \( T_p^- \) on \( J(P) \) are defined as follows. For an order ideal \( I \) of \( P \), set \( T_p^+(I) = 1 \) if \( p \) can be toggled into \( I \), and \( T_p^-(I) = 0 \) otherwise. Similarly, set \( T_p^-(I) = 1 \) if \( p \) can be toggled out of \( I \), and \( T_p^+(-I) = 0 \) otherwise.

Given a probability distribution on \( J(P) \) and an element \( p \in P \), the distribution is called toggle-symmetric at \( p \) if the probability that \( p \) can be toggled into an order ideal \( I \) equals the probability that \( p \) can be toggled out of an order ideal \( I \). We say that a distribution on \( J(P) \) is toggle-symmetric if it is toggle-symmetric at every \( p \in P \). In other words, a probability distribution on \( J(P) \) is toggle-symmetric if for every \( p \in P \), the expected value of the random variable \( T_p^+ \) equals the expected value of the random variable \( T_p^- \).

When \( P \) is the poset corresponding to a skew Young diagram, Chan, Haddadan,
Hopkins and Moci [3] found a formula for the expected jaggedness of an order ideal of $P$ for a toggle-symmetric distribution. The jaggedness of an order ideal $I$ of a poset $P$, denoted $\text{jag}(I)$, is defined to be the total number of elements in $P$ which can be toggled into $I$ or toggled out of $I$. In this paper, we shall be concerned only with the posets corresponding to Young diagrams.

To a Young diagram $\lambda$, one can associate a poset structure on the squares of $\lambda$. For two squares $B$ and $B'$ of $\lambda$, we say that $B$ is less than or equal to $B'$ if $B$ occurs northwest of $B'$. More precisely, assume that $B$ is in row $i$ and column $j$, and $B'$ is in row $i'$ and column $j'$. Then $B \leq B'$ if and only if $i \leq i'$ and $j \leq j'$. It is readily seen that a subset of squares of $\lambda$ forms an order ideal with respect to the above poset structure if and only if it is a subshape of $\lambda$, namely, a Young diagram contained in $\lambda$.

By the definition of the jaggedness of an order ideal, it is easily seen that the jaggedness of a subshape $\mu$ of $\lambda$ equals the total number of corners and proper outside corners of $\mu$ [3]. A corner of a shape $\mu$ is a square in $\mu$ such that the squares immediately below and to the right are not in $\mu$. While an outside corner of $\mu$ is a square out of $\mu$ such that the squares immediately above and to the left are in $\mu$, see, for example, the survey of Pak [7]. We assume that the square just to the right of the first row and the square just below the first column are also outside corners. It should be noticed that the outside corners of $\mu$ are also called outer boxes of $\mu$, see Stanley [10] Chapter 7, Appendix 1. By a proper outside corner of $\mu$ we mean an outside corner of $\mu$ contained in $\lambda$.

Clearly, a square of $\lambda$ can be toggled out of $\mu$ if and only if it is a corner of $\mu$, while a square of $\lambda$ can be toggled into $\mu$ if and only if it is an outside corner of $\mu$. Thus the jaggedness $\text{jag}(\mu)$ equals the total number of corners and proper outside corners of $\mu$. For example, the jaggedness of the subshape $(3, 3, 2, 1)$ of the diagram $(4, 4, 3, 2)$ in Figure 2.2 equals 5, since it has three corners and two proper outside corners, which are depicted by solid squares and open squares respectively.

Figure 2.2: Corners and proper outside corners.

The jaggedness of a subshape $\mu$ of $\lambda$ can also be described in terms of the total number of left turns and right turns of the lattice path corresponding to $\mu$. By a lattice path in $\lambda$ we mean a lattice path in $\lambda$ from the bottom left corner to the top right corner consisting of unit east steps and unit north steps. Clearly, a subshape of $\lambda$ is determined by a lattice path in the Young diagram of $\lambda$. For example, the thick line
in Figure 2.3 is a lattice path in \((4, 4, 3, 1)\) corresponding to the subshape \((3, 3, 2, 1)\).

Figure 2.3: Left turns and right turns of a lattice path in a Young diagram.

The notion of left turns and right turns of a lattice path in \(\lambda\) was introduced by Chan, López Martín, Pflueger and Teixidor i Bigas [2] for the computation of the genera of the Brill-Noether curves. To be more specific, a left turn of a lattice path in \(\lambda\) is an east step immediately followed by a north step, and a right turn is a north step immediately followed by an east step with the additional requirement that these two consecutive steps are borders of a square of \(\lambda\). In Figure 2.3 the solid circles and the open circles represent the left turns and the right turns of the path in \((4, 4, 3, 1)\), respectively. It is evident that for a subshape \(\mu\) of \(\lambda\), a corner (respectively, a proper outside corner) of \(\mu\) corresponds to a left turn (respectively, a right turn) of the lattice path determined by \(\mu\). It follows that the jaggedness of a subshape of \(\lambda\) also equals the total number of left turns and right turns of the corresponding lattice path in \(\lambda\), see [3].

Chan, Haddadan, Hopkins and Moci [3] found a formula for the expected jaggedness of a subshape for a general skew Young diagram, which turns out to have a closed form when it is a balanced Young diagram. A balanced shape is defined in terms of the positions of outward corners of a Young diagram \(\lambda\). An outward corner of \(\lambda\) is a north step immediately followed by an east step along the southeast boundary of \(\lambda\). A Young diagram \(\lambda\) is called a balanced shape if the turning point of each outward corner of \(\lambda\) lies on the main anti-diagonal of \(\lambda\), that is, the straight line connecting the starting point and the terminating point of a lattice path in \(\lambda\). For example, Figure 2.4 illustrates two balanced shapes, where the solid dots stand for the turning points of outward corners and the dashed lines represent the main anti-diagonals. From the largest part and the number of parts of \(\lambda\), it is easy to determine whether the turning point of an outward corner lies on the anti-diagonal.

When \(\lambda\) is a balanced shape, Chan, Haddadan, Hopkins and Moci [3, Corollary 3.8] obtained the following formula for any toggle-symmetric distribution.

**Theorem 2.1.** For a balanced Young diagram \(\lambda\) with \(r\) rows and \(c\) columns and for any toggle-symmetric distribution, the expected jaggedness of a subshape of \(\lambda\) equals

\[
\frac{2rc}{r + c}.
\]
We conclude this section with a description of a specific toggle-symmetric distribution, called the weak distribution, see [3, Definition 2.2], which is closely related to the enumeration of barely set-valued tableaux. The weak distribution is defined on reverse plane partitions. Recall that a reverse plane partition of shape $\lambda$ is an assignment of nonnegative integers into the squares of $\lambda$ such that the integers in each row and each column are weakly increasing, see Stanley [10, Chapter 7]. Given a positive integer $k$, let $\text{RPP}(\lambda, k)$ denote the set of reverse plane partitions of shape $\lambda$ with every entry not exceeding $k$.

To define the weak distribution, consider the pairs $(P, i)$ with $P \in \text{RPP}(\lambda, k)$ and $i \in \{1, 2, \ldots, k\}$. A pair $(P, i)$ determines a subshape of $\lambda$, denoted $\alpha(P, i)$, which consists of squares of $P$ occupied by the entries strictly less than $i$. The subshape $\alpha(P, i)$ is also called an induced subshape. Let

$$Q(\lambda, k) = \{ (P, i) | P \in \text{RPP}(\lambda, k), 1 \leq i \leq k \}.$$ 

Assume that the pairs $(P, i)$ in $Q(\lambda, k)$ are generated uniformly. Then we are led to a distribution of subshapes of $\lambda$. More precisely, among all the subshapes $\alpha(P, i)$ generated by the pairs in $Q(\lambda, k)$, a subshape $\mu$ occurs with probability

$$\frac{|\{(P, i) \in Q(\lambda, k) | \alpha(P, i) = \mu\}|}{|Q(\lambda, k)|}.$$ 

(2.1)

The distribution defined in (2.1) is called the weak distribution on the set of subshapes of $\lambda$.

Chan, Haddadan, Hopkins and Moci [3, Lemma 2.8] showed that the weak distribution is indeed a toggle-symmetric distribution. Hence, in the case when $\lambda$ is a balanced shape, the expected jaggedness under the weak distribution can be computed by the formula in Theorem 2.1, and so the following relation holds.

**Theorem 2.2.** For a balanced shape $\lambda$ with $r$ rows and $c$ columns, we have

$$\sum_{\mu} \frac{|\{(P, i) \in Q(\lambda, k) | \alpha(P, i) = \mu\}| \text{jag}(\mu)}{|Q(\lambda, k)|} = \frac{2rc}{r + c},$$

where $\mu$ ranges over the subshapes of $\lambda$. 

8
3 Proof of the conjecture

In this section, we present a proof of the conjecture of Reiner, Tenner and Yong. First, we establish the following relation on $|BSSYT(\lambda, k)|$ and $|SYT(\lambda, k)|$ for a balanced shape $\lambda$.

**Theorem 3.1.** For any positive integer $k$ and a balanced shape $\lambda$ with $r$ rows and $c$ columns, we have

$$|BSSYT(\lambda, k)| = \frac{krc}{(r+c)}|SYT(\lambda, k)|.$$  \hspace{1cm} (3.1)

As observed by Chan, Haddadan, Hopkins and Moci [3], a rectangular staircase shape is a balanced shape. Moreover, a rectangular staircase shape $\delta(d)(a b)$ has $a(d-1)$ rows and $b(d-1)$ columns. Thus Theorem 3.1 specializes to Conjecture 1.1.

To prove Theorem 3.1, we find the following representation of a barely set-valued tableau.

**Theorem 3.2.** A barely set-valued tableau in $BSSYT(\lambda, k)$ can be uniquely represented by a reverse plane partition $P$ in $RPP(\lambda, k)$ and an integer $i$ ($1 \leq i \leq k$) together with a designated corner of the induced subshape $\alpha(P, i)$.

**Proof.** Let $T$ be a barely set-valued tableau in $BSSYT(\lambda, k)$. We aim to construct a reverse plane partition $P \in RPP(\lambda, k)$, an integer $i$ ($1 \leq i \leq k$) and a corner $C$ in the induced subshape $\alpha(P, i)$. For each entry in $T$, if it is in the $t$-th row, then subtract it by $t$. This results in a tableau $T'$ with every entry not exceeding $k$ in which each row and each column are weakly increasing. Assume that $B$ is the square of $T$ containing two entries, say, $a$ and $b$ with $a < b$, and assume that $B$ is in the $r$-th row of $T$. By the above operation, the entries of $T'$ in the square $B$ are $a - r$ and $b - r$. Define $P$ to be the reverse plane partition in $RPP(\lambda, k)$ obtained from $T'$ by deleting the entry $b - r$ in $B$.

We next proceed to determine the integer $i$ and the corner $C$ in the induced subshape $\alpha(P, i)$, from which we can recover the deleted entry $b - r$ in the tableau $T'$. Notice that $r \leq a < b$. So we have $1 \leq b - r \leq k$. Set $i = b - r$.

We may choose the corner $C$ of $\alpha(P, b - r)$ to be the square $B$. This is feasible because it can be shown that the square $B$ is a corner of $\alpha(P, b-r)$. Keep in mind that the subshape $\alpha(P, b-r)$ consists of the squares of $P$ occupied by the entries smaller than $b - r$. Note that the entry in the square $B$ of $P$ is $a - r$. Since $a - r < b - r$, the square $B$ must be a square of the subshape $\alpha(P, b-r)$. To verify that $B$ is a corner of $\alpha(P, b-r)$, we need to check that if $B'$ is a square of $\lambda$ just to the right of $B$ or just below $B$, then $B'$ does not belong to $\alpha(P, b-r)$, or, equivalently, the entry of $P$ in $B'$ is bigger than or equal to $b - r$. This is obvious owing to the construction of $P$. Thus $C$ is indeed a corner of $\alpha(P, b-r)$.
To show that the above construction is reversible, we give a brief description of the reverse procedure. Given a reverse plane partition $P$ in $RPP(\lambda, k)$ together with an integer $1 \leq i \leq k$ and a corner $C$ of $\alpha(P, i)$, we shall recover a barely set-valued tableau $T$ in $BSSYT(\lambda, k)$ as follows. Let $T'$ be the tableau obtained from $P$ by joining the entry $i$ into the square $C$ so that the square $C$ has two entries. Increase each entry in $T'$ by $t$ if it is in the $t$-th row of $T'$. Let $T$ denote the resulting tableau. It is easily verified that $T$ is a barely set-valued tableau in $BSSYT(\lambda, k)$. This completes the proof.

Figure 3.5 illustrates the construction of the representation of a barely set-valued tableau $T$ in $BSSYT(\lambda, k)$ with $\lambda = (4, 4, 1)$ and $k = 2$, where the subshape $\alpha(P, 2)$ is determined by the lattice path in $\lambda$ drawn with thick line.

In the spirit of Theorem 3.2, we have an alternative representation of a barely set-valued tableau involving a designated proper outside corner.

**Theorem 3.3.** A barely set-valued tableau in $BSSYT(\lambda, k)$ can be uniquely represented by a reverse plane partition $Q$ in $RPP(\lambda, k)$ and an integer $j$ $(1 \leq j \leq k)$ together with a designated proper outside corner of the induced subshape $\alpha(Q, j)$.

**Proof.** The proof is similar to that of Theorem 3.2 and so we only give a description of the construction from a barely set-valued tableau $T$ in $BSSYT(\lambda, k)$ to a reverse plane partition $Q$ in $RPP(\lambda, k)$ and an integer $j$ $(1 \leq j \leq k)$ together with a designated proper outside corner $C'$ of $\alpha(Q, j)$.

Let $T'$ be the tableau as constructed in the proof of Theorem 3.2. Define $Q$ to be the reverse plane partition in $RPP(\lambda, k)$ obtained from $T'$ by deleting the entry $a - r$ in $B$. Set $j = a - r + 1$. It can be verified that $B$ is a proper outside corner of $\alpha(Q, a - r + 1)$. Then choose $C'$ to be the proper outside corner $B$. This completes the proof.

Figure 3.6 is an illustration of Theorem 3.3, where $T$ is a barely set-valued tableau in $BSSYT(\lambda, k)$ with $\lambda = (4, 4, 1)$ and $k = 2$.

We are now ready to complete the proof of Theorem 3.1 based on the above two representations of a barely set-valued tableau and the formula in Theorem 2.2.

**Proof of Theorem 3.1.** Recall that the expected jaggedness of a subshape of $\lambda$ under

\[
\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 \\
2 & 4 & 4 & 4 \\
5 & 5 & & \\
6 & & & \\
\hline
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|}
0 & 0 & 1 \\
0 & 2 & 2 \\
2 & 2 & \\
2 & & \\
\hline
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|}
0 & 0 & 1 \\
0 & 2 & 2 \\
2 & 2 & \\
2 & & \\
\hline
\end{array}
\quad
(P, \alpha(P, 2), C)
\]

Figure 3.5: An illustration of the representation in Theorem 3.2.
the weak distribution equals
\[
\frac{\sum_{\mu} |\{(P, i) \in Q(\lambda, k) \mid \alpha(P, i) = \mu\}| \text{jag}(\mu)}{|Q(\lambda, k)|},
\] (3.2)
where \(\mu\) ranges over subshapes of \(\lambda\). To compute the numerator of (3.2), note that
\[
\sum_{\mu} |\{(P, i) \in Q(\lambda, k) \mid \alpha(P, i) = \mu\}| \text{jag}(\mu) = \sum_{(P, i) \in Q(\lambda, k)} \text{jag}(\alpha(P, i)).
\]
Let \(C(P, i)\) denote the number of corners in the subshape \(\alpha(P, i)\), and let \(C'(P, i)\) denote the number of proper outside corners of \(\alpha(P, i)\). Then we have
\[
\text{jag}(\alpha(P, i)) = C(P, i) + C'(P, i).
\]
Recalling that
\[
Q(\lambda, k) = \{(P, i) \mid P \in \text{RPP}(\lambda, k), 1 \leq i \leq k\},
\]
we get
\[
\sum_{(P, i) \in Q(\lambda, k)} \text{jag}(\alpha(P, i)) = \sum_{P \in \text{RPP}(\lambda, k)} \sum_{i=1}^{k} \text{jag}(\alpha(P, i))
\]
\[
= \sum_{P \in \text{RPP}(\lambda, k)} \sum_{i=1}^{k} C(P, i) + \sum_{P \in \text{RPP}(\lambda, k)} \sum_{i=1}^{k} C'(P, i).
\]
By Theorem 3.2 and Theorem 3.3 both the first double sum and the second double sum in (3.3) are equal to \(|\text{BSSYT}(\lambda, k)|\). It follows that
\[
\sum_{\mu} |\{(P, i) \in Q(\lambda, k) \mid \alpha(P, i) = \mu\}| \text{jag}(\mu) = 2|\text{BSSYT}(\lambda, k)|. \quad (3.4)
\]
As to the denominator of (3.2), we notice that
\[
|Q(\lambda, k)| = k|RPP(\lambda, k)|.
\]
On the other hand, there is an obvious bijection between the set $\text{RPP}(\lambda, k)$ and the set $\text{SYT}(\lambda, k)$. Given a reverse plane partition $P \in \text{RPP}(\lambda, k)$, one can construct a semistandard Young tableau in $\text{SYT}(\lambda, k)$ from $P$ by increasing each entry in the $t$-th row of $P$ by $t$. Therefore,

$$|Q(\lambda, k)| = k|\text{SYT}(\lambda, k)|.$$  \hfill (3.5)

Substituting (3.4) and (3.5) into (3.2), the expected jaggedness in (3.2) can be rewritten as

$$\frac{2|\text{BSSYT}(\lambda, k)|}{k|\text{SYT}(\lambda, k)|},$$

which, together with Theorem 2.2 yields

$$\frac{2|\text{BSSYT}(\lambda, k)|}{k|\text{SYT}(\lambda, k)|} = \frac{2rc}{r+c}.$$  

This confirms (3.1), and hence the proof is complete. \hfill \Box

We conclude this paper with a formula on the polynomial $FK(w, \ell)$ with respect to a dominant permutation $w$ corresponding to a balanced shape. The proof is based on Theorem 3.1 and a relation on $FK(w, \ell)$ established by Reiner, Tenner and Yong [9].

Restricting to a dominant permutation corresponding to a rectangular staircase shape, this formula reduces to Conjecture 1.2. Bear in mind that $FK(w, \ell)$ is a polynomial in $x$ of degree $\ell$. For the reason that the proof of the following theorem involves the evaluation of the polynomials $FK(w, \ell)$ at $x = 1, 2, \ldots$, we shall write $FK_w,\ell(x)$ for $FK(w, \ell)$.

**Theorem 3.4.** Let $w$ be a dominant permutation whose Lehmer code is a balanced shape $\lambda$ with $r$ rows and $c$ columns, and let $\ell = \ell(w)$. Then we have

$$\frac{FK(w, \ell + 1)}{FK(w, \ell)} = \binom{\ell + 1}{2} + \binom{\ell + 1}{2} \frac{2rc}{\ell(r + c)} + 1.$$  \hfill (3.6)

**Proof.** In the proof of [9, Corollary 6.11], Reiner, Tenner and Yong established the following relation for any dominant permutation $w$ and any positive integer $k$:

$$\frac{FK_{w,\ell+1}(k)}{FK_{w,\ell}(k)} = \binom{\ell + 1}{2} + (\ell + 1)\frac{|\text{BSSYT}(\lambda, k)|}{|\text{SYT}(\lambda, k)|}.$$  \hfill (3.7)

Substituting (3.1) into (3.7), we obtain that

$$\frac{FK_{w,\ell+1}(k)}{FK_{w,\ell}(k)} = \binom{\ell + 1}{2} + (\ell + 1)\frac{krc}{r+c} = \binom{\ell + 1}{2} \left(\frac{2rc}{\ell(r + c)} + 1\right),$$

that is to say that (3.6) holds for any positive integer $k$. Since (3.6) can be recast as a relation on polynomials, it holds for $FK_{w,\ell}(x)$. This completes the proof. \hfill \Box

Notice that by (3.7), it is clear that Theorem 3.4 is equivalent to Theorem 3.1.

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