Characterization of SDP Designs That Yield Certain Spin Models

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Abstract

We characterize the SDP designs that give rise to four-weight spin models with two values. We prove that the only such designs are the symplectic SDP designs. The proof involves analysis of the cardinalities of intersections of four blocks.
Keywords: symmetric difference property, design, spin model, symplectic.
1 Introduction

The concept of a spin model was introduced by Jones [9]. The concept was generalised to two-weight spin models by Kawagoe, Munemasa and Watatani [13], and further generalised to four-weight spin models by Bannai and Bannai [1].

Guo and Huang [8] considered certain types of four-weight spin models, which they called “four-weight spin models with exactly two values on $W_2$” (we will explain this further in section 2). They showed a connection with symmetric designs. Bannai and Sawano [2] showed that the existence of a four-weight spin model with exactly two values on $W_2$ is equivalent to the existence of a quasi-3 design with certain properties (see Theorem 1 below), strengthening a result of Guo-Huang [8]. They (implicitly) raised the question of whether SDP designs, which are known to be quasi-3, would satisfy these properties. In this paper we will answer this question. Our main result is as follows.

**Theorem.** An SDP design $D$ satisfies all the conditions of Theorem 1, and thus corresponds to a four-weight spin model, if and only if $D$ is equivalent to the symplectic SDP design.

In section 2 we give some background and definitions. In section 3 we will prove some preliminary results about SDP designs. In section 4 we prove that the only SDP designs satisfying condition 2 of Theorem 1 are (up to design equivalence) the symplectic designs. In section 5 we prove that the symplectic SDP designs satisfy condition 3 of Theorem 1, and then our main result follows.

2 Background

We first give the definition of a four-weight spin model. Matrices will be indexed by the elements of a finite set $X$. We use $A \circ B$ to denote the Hadamard product of matrices $A = (A(\alpha, \beta))_{\alpha, \beta \in X}$ and $B = (B(\alpha, \beta))_{\alpha, \beta \in X}$, which is the matrix whose $(\alpha, \beta)$ entry is equal to $A(\alpha, \beta)B(\alpha, \beta)$. Let $I$ denote the identity matrix, let $J$ denote the all-1 matrix, and let $A^T$ denote the transpose of $A$.

**Definition.** Let $X$ be a finite set with $n$ elements, and let $D$ be a real number satisfying $D^2 = n$. We say that $(X, W_1, W_2, W_3, W_4)$ is a four-weight spin model of size $n$ if each $W_i$ ($1 \leq i \leq 4$) is an $n$-by-$n$ matrix with complex entries and the following conditions hold:
1. \( W_1^T \circ W_3 = W_2^T \circ W_4 = J \)

2. \( W_1 W_3 = W_2 W_4 = nI \)

3. (a) \( W_1 Y_{a\beta}^{41} = DW_4(\alpha, \beta) Y_{a\beta}^{41} \) for all \( \alpha, \beta \in X \)
   
   (b) \( W_1^T Y_{\beta\alpha}^{14} = DW_4(\beta, \alpha) Y_{\beta\alpha}^{14} \) for all \( \alpha, \beta \in X \)

where \( Y_{ij}^{\alpha\beta} \) is the \( n \)-dimensional column vector whose entry in position \( \gamma \) is \( W_i(\alpha, \gamma) W_j(\gamma, \beta) \).

The apparent lack of symmetry in 3(a) and 3(b) is explained in [1], where they show that each of these equations is equivalent to seven others.

If \( W_1 = W_2 \) and \( W_3 = W_4 \) then the definition above reduces to the definition of a two-weight spin model in [13]. If in addition we assume that \( W_1 \) and \( W_3 \) are symmetric, the definition reduces to the definition of a spin model in [1].

The papers [8] and [2] consider the case that \( W_2 \) has only two distinct entries, with each entry appearing the same number of times in each row and column. They refer to this case as a four-weight spin model with exactly two values on \( W_2 \). This is the case under consideration in this paper.

A block design with parameters 2-(\( v, k, \lambda \)) is often called a symmetric design if \( b = v \). Following [3] we shall use the term “square” design for a symmetric design. We refer the reader to [3] for the basic properties of block designs. A square design is said to be quasi-3 for points if the number of blocks incident with three distinct points takes only two values. Such designs seem to have been first considered in Cameron [3]; see also [4] for a survey of quasi-3 designs. We shall say that a square design is quasi-3 for blocks if the number of points in the intersection of any three distinct blocks takes only two values. A design is quasi-3 for blocks if and only if the dual design is quasi-3 for points.

We index the rows of an incidence matrix of a design by the blocks, and the columns by the points. When we speak of the sum of a number of blocks, we mean the sum modulo 2 of the rows of the incidence matrix corresponding to those blocks.

An SDP (symmetric difference property) design is a square (\( v, k, \lambda \)) design with the property that the symmetric difference (or sum) of any three blocks is either a block or a block complement. It follows immediately from the definition that SDP designs are quasi-3 for blocks. For, the identity

\[
|\mathcal{B} + \mathcal{B}' + \mathcal{B}''| = |\mathcal{B}| + |\mathcal{B}'| + |\mathcal{B}''| - 2|\mathcal{B} \cap \mathcal{B}'| - 2|\mathcal{B} \cap \mathcal{B}''| - 2|\mathcal{B}' \cap \mathcal{B}''| \\
+ 4|\mathcal{B} \cap \mathcal{B}' \cap \mathcal{B}''| \quad (\clubsuit)
\]

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shows that there are only two possibilities for $|B \cap B' \cap B''|$. It is shown in [4] that all SDP designs are also quasi-3 for points.

We now state the result of [2] on the case that $W_2$ has only two distinct entries.

**Theorem 1** [2] Let $W_2 = \alpha A + \beta(J - A)$, where $\alpha, \beta$ are distinct nonzero complex numbers, and $A$ is a $(0,1)$-matrix with the property that each row and column has exactly $k$ ones, where $2 \leq k \leq n - 2$. Let $X$ be a finite set with $n$ elements, and let $D$ be a real number satisfying $D^2 = n$. Then $W_2$ defines a four-weight spin model if and only if $A$ is the incidence matrix of a square $(n,k,\lambda)$ design $D(X,B)$ which satisfies the following three properties:

1. $D(X,B)$ is quasi-3 for blocks with triple intersection sizes
   \[ x = \frac{k\lambda + \lambda - (k - \lambda)\sqrt{k - \lambda}}{n}, \quad y = \frac{k\lambda + \lambda + (k - \lambda)\sqrt{k - \lambda}}{n}. \]

2. For any set $S \subseteq B$ of four blocks, an even number of the four 3-subsets of $S$ have triple intersection size $x$.

3. There exists a 1-1 correspondence $\phi : X \rightarrow B$ with the property that for any three points $a, b, c \in X$, the number of blocks containing $\{a, b, c\}$ is $|\phi(a) \cap \phi(b) \cap \phi(c)|$.

Moreover, if conditions 1, 2 and 3 hold, then $\alpha, \beta$ and $W_1$ are determined by $D$ and $k$. In particular, $\alpha = -\beta$ if and only if $n = 4q^2$ where $q$ is an even integer.

Guo and Huang [8] point out that the $(16,6,2)$ SDP design satisfies conditions 1, 2 and 3 of Theorem [4], and thus gives an example of a four-weight spin model with exactly two values on $W_2$. Bannai and Sawano [3] showed that the other (non-SDP) $(16,6,2)$ designs do not satisfy conditions 1, 2 and 3 of Theorem [4]. They state that it is known that SDP designs are quasi-3 for blocks (satisfy condition 1 of Theorem [4]), but as we said above this follows from the definition. They appear to be wondering whether all SDP designs satisfy conditions 1, 2 and 3 of Theorem [4]. We investigate this question in this paper. We will show that, although the number of nonisomorphic SDP designs grows exponentially with $m$, there is one and only one SDP design (up to isomorphism) satisfying the three conditions.
3 On SDP Designs

It was shown by Kantor [11] that any SDP design must have parameters

\[(v, k, \lambda) = (2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}).\]

There is one particular SDP design of interest to us, which is called the symplectic SDP design. It is constructed using the \(2^{2m}\) quadratic forms that polarise to a given nondegenerate symplectic bilinear form on a \(2m\)-dimensional vector space over \(GF(2)\), see [7], [6] or [11]. The symplectic design has a 2-transitive automorphism group. Kantor [12] showed that the number of nonisomorphic SDP designs grows exponentially with \(m\).

We recall that a regular Hadamard matrix is a Hadamard matrix with constant rowsums. Such a matrix of size \(4u^2\) gives rise to a square 2-(\(4u^2, 2u^2 - u, u^2 - u\)) design (replacing \(-1\) by 0 and perhaps complementing). When \(u = 2^{m-1}\) these parameters are the same as the SDP parameters. Taking Kronecker products of the 4-by-4 matrix \(J - 2I\) with itself results in the symplectic SDP designs (this description is due to Block [3]).

Theorem 2 Let \(D(X, B)\) be the 2-(\(4u^2, 2u^2 - u, u^2 - u\)) design induced by a regular Hadamard matrix of size \(4u^2\). Then \(D\) is an SDP design if and only if the sum of any four blocks is a vector of weight 0, \(2u^2\), or \(4u^2\).

Proof: Let \(v = 4u^2\). First suppose \(D(X, B)\) is an SDP design. Let \(B_1, B_2, B_3 \in B\) be three distinct blocks of \(D\). Then \(B_1 + B_2 + B_3 = B\) or \(B + j\), where \(j\) denotes the all-1 vector and \(B \in B\). If \(B_4\) is a block not equal to any of \(B_1, B_2, B_3\), then

\[B_1 + B_2 + B_3 + B_4 = B + B_4 \text{ or } B + B_4 + j.\]

Since the sum of any two distinct blocks has weight \(v/2\), the weight of \(B + B_4\) (and \(B + B_4 + j\)) will be one of \(0, v/2, \text{ or } v\) (as \(B\) could equal \(B_4\)).

Conversely, suppose \(D(X, B)\) is not an SDP design. Then there exist \(B_1, B_2, B_3 \in B\) such that \(B_1 + B_2 + B_3\) is not a block or a block complement. Let

\[\mathcal{H} = \{B_2 + B_3\} \cup \{B_1 + B_i : 2 \leq i \leq v\}\]

and let

\[\overline{\mathcal{H}} = \{x + j : x \in \mathcal{H}\}.\]

Then \(|\mathcal{H}| = |\overline{\mathcal{H}}| = v\), and \(\mathcal{H} \cap \overline{\mathcal{H}} = \emptyset\).

For the sake of contradiction, assume that the sum of any four blocks of \(D\) has weight 0, \(v/2\), or \(v\). Then the sum of any two distinct elements of
$H$ has weight $v/2$, and the same applies to any two distinct elements of $\overline{H}$. Also, if $x \in H$ and $y+j \in \overline{H}$, then $x+y+j$ has weight $v/2$ (unless $x = y$ in which case $x+y+j = j$ has weight $v$).

It follows that any two distinct elements of $H \cup \overline{H}$ have Hamming distance at least $v/2$. Adding the all-0 vector to these vectors yields a binary $(v, 2v+1, v/2)$ code, which violates the Plotkin bound (see [14] chapter 2). This contradiction completes the proof.

We shall use Theorem 2 to calculate the possible quadruple intersection sizes of blocks in an SDP design.

**Theorem 3** Let $D(X, B)$ be a $2$-$(4u^2, 2u^2 - u, u^2 - u)$ SDP design, where $u = 2^{m-1}$. Then the cardinality of the intersection of four distinct blocks takes one of the following seven values:

1. $0$
2. $u^2/2 - u$
3. $u^2/4$
4. $u^2/4 - u/4$
5. $u^2/4 - u/2$
6. $u^2/4 - 3u/4$
7. $u^2/4 - u$.

**Proof:** Let $B_1, B_2, B_3, B_4 \in B$ be four distinct blocks of $D$. Let

$$\alpha = |B_1 \cap B_2 \cap B_3|$$
$$\beta = |B_1 \cap B_2 \cap B_4|$$
$$\gamma = |B_1 \cap B_3 \cap B_4|$$
$$\delta = |B_2 \cap B_3 \cap B_4|,$$

and let

$$q = |B_1 \cap B_2 \cap B_3 \cap B_4|.$$

It follows easily from (♠) that each of $\alpha, \beta, \gamma, \delta, \text{is equal to either } x = u^2/2 - u \text{ or } y = u^2/2 - u/2$. Let $w$ be the weight of the vector $B_1 + B_2 + B_3 + B_4$. Then (by the obvious generalisation of (♠) to four blocks)

$$w = 4(2u^2 - u) - 12(u^2 - u) + 4(\alpha + \beta + \gamma + \delta) - 8q. \quad (\Diamond)$$
By Theorem 2, \( w \) must be one of 0, 2\( u^2 \) or 4\( u^2 \). The case \( w = 4u^2 \) corresponds to \( B_1 + B_2 + B_3 + B_4 = j \), which clearly implies \( \alpha = \beta = \gamma = \delta = y = u^2/2 - u/2 \). In this case (\( \diamond \)) gives \( q = 0 \).

The case \( w = 0 \) corresponds to \( B_1 + B_2 + B_3 + B_4 = 0 \), which clearly implies \( \alpha = \beta = \gamma = \delta = x = u^2/2 - u \). In this case (\( \diamond \)) gives \( q = u^2/2 - u \).

Finally, suppose \( w = 2u^2 \). Let \( N_x \) be the number of \( \alpha, \beta, \gamma, \delta \) that are equal to \( x \), so \( N_x \in \{0, 1, 2, 3, 4\} \). Then (\( \diamond \)) gives \( q = u^2/4 - N_x u/4 \), so each of the five possibilities for \( N_x \) gives the remaining five possibilities for \( q \).

\[ \square \]

4 SDP Designs and Condition 2

We now consider the question of which SDP designs \( D(X, \mathcal{B}) \) satisfy condition 2 of Theorem 1, which states: for any set \( S \subseteq \mathcal{B} \) of four blocks, an even number of the four 3-subsets of \( S \) have triple intersection size \( x \).

Recall that any SDP design has parameters

\[
\left(2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}\right)
\]

and 2-rank \( 2m + 2 \), and the derived design with respect to any block has parameters

\[
\left(2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}, 2^{2m-2} - 2^{m-1} - 1\right) \quad (\dagger)
\]

and 2-rank \( 2m + 1 \). We will use the following result from McGuire and Ward [15] (Corollary 4 and Theorem 8 there), which characterises the derived designs of the symplectic SDP designs by their triple intersection sizes.

**Theorem 4** [15] Let \( D \) be a design with parameters (\( \dagger \)) and 2-rank \( 2m + 1 \). Then \( D \) is equivalent to a derived design of the symplectic SDP design if and only if all sizes of intersections of three blocks are divisible by \( 2^{m-2} \).

We now prove our main result.

**Theorem 5** Let \( D(X, \mathcal{B}) \) be a \( (2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}) \) SDP design. Then \( D \) satisfies condition 2 of Theorem 1 if and only if \( D \) is equivalent to the symplectic SDP design.

Proof: We continue the notation of the proof of Theorem 4. Condition 2 of Theorem 1 states that \( N_x \) can only equal 0, 2, or 4, for all choices of four blocks \( B_1, B_2, B_3, B_4 \in \mathcal{B} \). By the proof of Theorem 3, this is equivalent
to saying that only five quadruple intersection sizes are allowed; the two quadruple intersection sizes that are forbidden are (since \( u = 2^{m-1} \))
\[
\frac{u^2}{4} - \frac{u}{4} = 2^{2m-4} - 2^{m-3} \quad \text{and} \quad \frac{u^2}{4} - 3\frac{u}{4} = 2^{2m-4} - 3 \cdot 2^{m-3},
\]
corresponding to \( N_x = 1 \) and \( N_x = 3 \).

We further observe that the five allowable quadruple intersection sizes are divisible by \( 2^{m-2} \), and the two forbidden quadruple intersection sizes are not divisible by \( 2^{m-2} \). Therefore, \( D \) will satisfy condition 2 of Theorem 1 if and only if all quadruple intersection sizes are divisible by \( 2^{m-2} \).

Note that the quadruple intersection sizes of blocks in \( D \) are the same as the triple intersection sizes of blocks in a derived design of \( D \). Therefore, \( D \) will satisfy condition 2 of Theorem 1 if and only if all triple intersection sizes of blocks in a derived design of \( D \) are divisible by \( 2^{m-2} \). Theorem 4 implies that \( D \) will satisfy condition 2 of Theorem 1 if and only if any derived design of \( D \) is equivalent to a derived design of the symplectic SDP design. By a result of Jungnickel and Tonchev [11], non-isomorphic SDP designs have non-isomorphic derived designs, so a derived design of \( D \) is equivalent to a derived design of the symplectic design if and only if \( D \) itself is equivalent to the symplectic design.

\[ \Box \]

5 Symplectic Designs and Condition 3

We now show that the symplectic designs satisfy condition 3 of Theorem 1, which states: there exists a 1-1 correspondence \( \phi : X \rightarrow B \) with the property that for any three points \( a, b, c \in X \), the number of blocks containing \( \{a, b, c\} \) is \(|\phi(a) \cap \phi(b) \cap \phi(c)|\).

Recall that a polarity of a square design \( D(X, B) \) is a bijection \( \sigma : X \rightarrow B \) such that \( \sigma \circ \sigma \) is the identity and \( p \in \sigma(q) \) if and only if \( q \in \sigma(p) \), for all \( p, q \in X \). A square design has a polarity if and only if it has a symmetric incidence matrix, with respect to some ordering of the points and blocks.

**Lemma 6** Let \( D(X, B) \) be a square design with a polarity. Then there exists a 1-1 correspondence \( \phi : X \rightarrow B \) with the property that for any three points \( a, b, c \in X \), the number of blocks containing \( \{a, b, c\} \) is \(|\phi(a) \cap \phi(b) \cap \phi(c)|\).

Proof: Let \( \phi \) be the polarity of \( D \). Then, for \( p, a, b, c \in X \), it follows from the definition of a polarity that
\[
p \in (\phi(a) \cap \phi(b) \cap \phi(c)) \iff \{a, b, c\} \subseteq \phi(p).
\]
\[ \Box \]
Theorem 7  The symplectic SDP designs satisfy condition 3 of Theorem 1.

The proof follows from Lemma 6 and the fact that the symplectic SDP designs have a polarity (see [11], or [6] page 78).

We now combine Theorems 3 and 7 to give our characterisation of the SDP designs satisfying all of conditions 1, 2, and 3, of Theorem 1.

Theorem 8  Let \( D \) be an SDP design. Then \( D \) satisfies conditions 1, 2 and 3 of Theorem 1 if and only if \( D \) is the symplectic SDP design.

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