On Some Properties of the Beta Inverse Rayleigh Distribution

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Abstract

We study with some details a lifetime model of the class of beta generalized models, called the beta inverse Rayleigh distribution, which is a special case of the Beta Fréchet distribution. We provide a better foundation for some properties including quantile function, moments, mean deviations, Bonferroni and Lorenz curves, Rényi and Shannon entropies and order statistics. We fit the proposed model using maximum likelihood estimation to a real data set to illustrate its flexibility and potentiality.

Keywords

Beta-Generated class, Entropy, Generalized distribution, Maximum likelihood estimation, Moment.

1 Introduction

After its inception by Treier [1964], the inverse Rayleigh (IR) distribution was championed by Voda [1972] and Iliescu and Voda [1973] during the 1970s. In Voda [1972] several of its statistical properties were addressed, in particular, maximum likelihood estimation, confidence intervals, and hypothesis tests. An early application involved lifetime modeling of experimental units. More recently, Gharraph [1993] provided closed-form expressions for the mean, harmonic mean, geometric mean, mode and the median of this distribution. In Mohsin and Shabnaz [2005] the negative moment estimator for the IR distribution was investigated. Moreover, different methods of estimation have been numerically compared in Gharraph [1993] and Soliman et al. [2010]. Acceptance sampling techniques also received a treatment based on the IR distribution Rosaiah and Kanta [2005]. In 2010, a model for lower record value based on the IR distribution was proposed in Soliman et al. [2010] and a Bayesian approach for its associate parameter estimation.

Distribution generalization theory has received considerable attention in the past decades, Amoroso [1925], Good [1953], Hosking and Wallis [1987] and McDonald [1984]. A particular prominent generalization model is the class of beta generalized distributions, first introduced in Eugene et al. [2002]. In this seminal work, the authors introduced the new class of distributions from the logit of the beta random variable, and obtained as a special case the beta normal (BN) distribution. This distribution could provide flexible shapes including...
bimodality, being therefore a candidate for a wide range of applications. Additional properties of the BN distribution have been studied in detail by Gupta and Nadarajah [2004] and Rege et al. [2012]. In a similar manner, other beta generalizations have been proposed taking into account several baseline distributions. To cite a few, we identify the beta Gumbel by Nadarajah and Kotz [2004], beta Fréchet by Nadarajah and Gupta [2004], beta exponential by Nadarajah and Kotz [2005], beta Weibull by Lee et al. [2007], beta Pareto by Akinsete et al. [2008], beta generalized exponential by Barreto-Souza et al. [2010], beta generalized normal by Cintra et al. [2011] and beta generalized half-normal by Pescim et al. [2010] distributions.

In this paper, we study the beta generalized distribution based on the IR distribution, called the beta inverse Rayleigh (BIR) distribution. The BIR distribution is a special case of the beta Fréchet (BF) distribution, which was introduced by Nadarajah and Gupta [2004] and studied by Barreto-Souza et al. [2011]. These two papers provide some mathematical properties for the BF distribution, which in turn can be easily adapted for the BIR distribution. We provide a better foundation for these and other mathematical properties. An application to a real life data set is presented. The BIR distribution is expected to have immediate application in reliability and survival studies.

The rest of the paper unfolds as follows. In Section 2, we present the BIR distribution, derive its density and some expressions for the cumulative distribution function (cdf), and provide an analytical study of the unimodality region. In Section 3, we give the hazard rate function and its asymptotic behavior. In Section 4, we derive the formulae for the moments. Further, in Sections 5–8, we derive quantile function, skewness and kurtosis, mean deviations, Rényi entropy, Shannon entropy and order statistics. In Section 9, we discuss maximum likelihood estimation and present the elements of the observed information matrix. An application to real data is performed in Section 10. Finally, in Section 11, we offer some concluding remarks.

2 The BIR distribution

Let \( G(x) \) be a baseline cumulative distribution function (cdf). Then, the associated beta generalized distribution \( F(x) \) based on the logit of the beta random variable is given by [Eugene et al., 2002]

\[
F(x) = I_{G(x)}(a, b),
\]

(1)

where \( a > 0, b > 0 \), \( I_{\cdot}(a, b) \) is the incomplete beta function ratio

\[
I_{\cdot}(a, b) = \frac{1}{B(a, b)} \int_0^z \omega^{a-1}(1-\omega)^{b-1}d\omega,
\]

and \( B(\cdot, \cdot) \) denotes the beta function. The extra shape parameters \( a \) and \( b \) control skewness, kurtosis and tail weights.

The IR distribution is a single-parameter distribution defined over the semi-infinite interval \([0, \infty)\). Its cdf is given by

\[
G(x; \theta) = \exp \left( -\frac{\theta}{x^2} \right), \quad x > 0, \theta > 0.
\]
Inserting $G(x;\theta)$ into (1), we obtain the BIR cumulative distribution

$$F(x) = I_{\exp\left(-\frac{\theta}{x^2}\right)}^{\exp\left(-\frac{\theta}{x^2}\right)} (a,b) = \frac{1}{B(a,b)} \int_0^{\exp\left(-\frac{\theta}{x^2}\right)} \omega^{a-1} (1-\omega)^{b-1} d\omega,$$

for $x > 0$, $a > 0$, $b > 0$ and $\theta > 0$. Note that if we take Fréchet cdf $G(x,\sigma,\lambda) = \exp\left\{-(\frac{x}{\sigma})^\lambda\right\}$, where $\sigma > 0$ and $\lambda > 0$ are the scale and shape parameters, respectively, into (1), we obtain the BF distribution. Thus, the BIR model is obtained for $\sigma^2 = \theta$ and $\lambda = 2$. Note also that for the special case $a = b = 1/2$, the BIR cumulative function has a closed-form expression given by

$$F(x) = \frac{2}{\pi} \arcsin\left\{\exp\left(-\frac{\theta}{2x^2}\right)\right\}.$$

The BIR probability density function (pdf) can be expressed as (for $x > 0$)

$$f(x) = \frac{2\theta}{B(a,b)x^3} \exp\left(-\frac{a\theta}{x^2}\right) \left[1 - \exp\left(-\frac{\theta}{x^2}\right)\right]^{b-1}.$$

The BIR random variable $X$ is denoted by $X \sim \text{BIR}(a,b,\theta)$. The parameters $a$ and $b$ affect the skewness of $X$ by changing the relative tail weights. Figure 1 displays the BIR pdf for several choices of parameter values. Simulating the BIR random variable is relatively simple. Let $Y$ be a random variable distributed according to the usual beta distribution with parameters $a$ and $b$. Thus, by means of the inverse transformation method, the random variable $X$ given by

$$X = \sqrt{-\frac{\theta}{\log(Y)}}$$

follows (3).

### 2.1 General expansion

Although the cdf and pdf of $X$ require mathematical functions that are widely available in contemporary statistical packages, Eaton et al. [2002] and R Development Core Team [2011] often further analytical and numerical derivations take advantage of power series expansions for the cdf. From the BIR density function (3), the cdf of $X$ can be expressed after usual integration as

$$F(x) = \frac{1}{B(a,b)} \int_0^\infty \frac{2\theta}{y^3} \exp\left(-\frac{a\theta}{y^2}\right) \left[1 - \exp\left(-\frac{\theta}{y^2}\right)\right]^{b-1} dy.$$

Setting $u = \theta y^{-2}$, it follows that

$$F(x) = \frac{1}{B(a,b)} \int_0^\infty \exp(-au) (1-\exp(-u))^{b-1} du.$$
Figure 1: Plots of the BIR pdf for $\theta = 0.5$ (solid line), $\theta = 1.0$ (dashed line), $\theta = 3.0$ (dotted line) and $\theta = 5.0$ (bold line).
Notice that for $|z| < 1$ and $b > 0$ a real non-integer number, we have the power series expansion

$$(1 - z)^{b-1} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(b)}{\Gamma(b-n)n!} z^n,$$  

where $\Gamma(\cdot)$ is the gamma function. Applying this identity into (4) yields

$$F(x) = \frac{1}{B(a,b)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(b)}{\Gamma(b-n)n!} \int_{\theta x^2}^{\infty} \exp(-(a+n)u) du$$

and then

$$F(x) = \frac{1}{B(a,b)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(b)}{(a+n) \Gamma(b-n)n!} \exp \left\{ -\frac{(a+n)\theta}{x^2} \right\}.$$  

Now, considering the following quantity,

$$c_n(a,b) = \frac{(-1)^n \Gamma(a+b)}{(a+n) \Gamma(a) \Gamma(b-n)n!},$$

we can write the BIR cdf as a linear combination of IR cdfs. Indeed, we obtain

$$F(x) = \sum_{n=0}^{\infty} c_n(a,b) F_G(x; (a+n) \theta).$$

In a similar way, the BIR pdf can be expressed according to the following linear combination

$$f(x) = \sum_{n=0}^{\infty} c_n(a,b) g(x; (a+n) \theta),$$

where $g(x; (a+n) \theta)$ denotes the IR density function with parameter $(a+n) \theta$.

### 2.2 Unimodality

The BIR distribution is unimodal for all values of $a, b, \theta > 0$. In order to investigate the critical points of its density function, the first derivative of $f(x)$ with respect to $x$ is given by

$$\frac{d}{dx} f(x) = \frac{\theta^2}{B(a,b) x^6} \exp \left( -\frac{a \theta}{x^2} \right) \left[ 1 - \exp \left( -\frac{\theta}{x^2} \right) \right]^{b-1} \times \left\{ 4a - \frac{6 \theta^2}{\theta} + \frac{4(b-1)}{1 - \exp \left( \frac{\theta}{x^2} \right)} \right\}, \quad x > 0.$$  

The signal of this derivative is determined by the expression in the last square brackets, since the remaining terms are all positive. Considering the substitution $y = \theta x^{-2}$, the expression in square brackets becomes

$$4a - \frac{6}{y} + 4 \frac{b-1}{1 - \exp(y)}.$$  

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Now, we demonstrate that this expression is a monotonic function; therefore, (7) has a single zero, which implies a unique mode. Indeed, the derivative of (8) becomes
\[ t(y) = \frac{6 y^2 + 4(b-1) \exp(y)}{(1-\exp(y))^2}. \]
For \( b \geq 1 \), this derivative is clearly positive. For \( 0 < b < 1 \), Figure 2 displays the numerical results that illustrate the positiveness of the derivative of (8).

Moreover, let \( y_0 \) be the zero of (8). The BIR mode location is then given by \( \sqrt{\theta / y_0} \). Since \( y_0 \) is independent of \( \theta \), the mode location is an increasing function of \( \theta \).

## 3 Hazard rate function

The survival and hazard rate functions are given by \( S(x) = 1 - F(x) \) and \( h(x) = f(x)/S(x) \), where \( F(x) \) and \( f(x) \) are the BIR cdf and pdf, respectively. Thus, the hazard rate function of the random variable \( X \) is
\[
 h(x) = \frac{2\theta}{B(a,b)x^3} \left[ 1 - \exp\left( -\frac{\theta}{x^2} \right) \right]^{b-1} \frac{\exp\left( -\frac{a\theta}{x^2} \right)}{I_{1-\exp\left( -\frac{a\theta}{x^2} \right)}(b,a)}. \]
Notice that we applied in (2) the symmetry property of the incomplete beta function \( 1 - I_x(a,b) = I_{1-x}(b,a) \).

We now examine the asymptotic behavior of \( h(x) \) when \( x \to \infty \) or \( x \to 0 \). First, we prove that \( h(x) \sim 1/x \) as \( x \to \infty \). To establish this result, we verify that \( \lim_{x \to \infty} h(x)/x^{-1} \) is a constant. Indeed, we have
\[
 \lim_{x \to \infty} \frac{h(x)}{1/x} = \lim_{x \to \infty} \frac{2\theta}{B(a,b)x^3} \left[ 1 - \exp\left( -\frac{\theta}{x^2} \right) \right]^{b-1} \frac{\exp\left( -\frac{a\theta}{x^2} \right)}{I_{1-\exp\left( -\frac{a\theta}{x^2} \right)}(b,a)} x. \]
Since \( \exp \left( \frac{-a^2 \theta}{x^2} \right) \to 1 \) as \( x \to \infty \), we can write

\[
\lim_{x \to \infty} \frac{h(x)}{1/x} = \frac{2\theta}{B(a, b)} \lim_{x \to \infty} \left( 1 - \exp \left( \frac{-\theta}{x^2} \right) \right)^{b-1} \times \frac{1/x^2}{1 - \exp(\theta/x^2)(b, a)}.
\]

For any value of \( b > 0 \), the last expression gives rise to an indeterminate form. Invoking L'Hôpital's rule and again considering that \( \exp \left( \frac{-a^2 \theta}{x^2} \right) \to 1 \) as \( x \to \infty \), we obtain

\[
\lim_{x \to \infty} \frac{h(x)}{1/x} = 2\theta(b-1) \lim_{x \to \infty} \left( \frac{1/x^2}{1 - \exp \left( \frac{-\theta}{x^2} \right)} \right) - 2.
\]

Applying the L'Hôpital rule again we note that the above limit is well-defined and is equal to \(-2b\).

Similarly, let us show that \( h(x) \sim \exp(-a\theta/x^2)/x^3 \) as \( x \to 0 \). In fact, we have immediately that

\[
\lim_{x \to 0} \frac{h(x)}{\exp(-a\theta/x^2)/x^3} = \frac{2\theta}{B(a, b)} \lim_{x \to 0} \left( 1 - \exp \left( \frac{-\theta}{x^2} \right) \right)^{b-1} \times \frac{1/x^2}{1 - \exp(\theta/x^2)(b, a)}.
\]

Notice also that \( \lim_{x \to 0} \exp(-a\theta/x^2)/x^3 = 0 \). Figure 3 displays the behavior of \( h(x) \) for selected values of the model parameters.

### 4 Moments

The moments play a crucial role in any statistical analysis. The \( r \)th moment of \( X \) is

\[
E(X^r) = \frac{2\theta}{B(a, b)} \int_0^\infty x^{r-3} \exp \left( \frac{-a\theta}{x^2} \right) \left( 1 - \exp \left( \frac{-\theta}{x^2} \right) \right)^{b-1} \, dx.
\]

Now, we simplify the above integral. First, letting \( y = \theta x^{-2} \), we have

\[
E(X^r) = \frac{\theta^{r/2}}{B(a, b)} \int_0^\infty y^{-r/2} \exp(-ay)(1 - \exp(-y))^{b-1} \, dy.
\]

We refer to the last integral as \( S_r(a, b) \). Applying the series expansion (5), for any real \( r \), we obtain

\[
S_r(a, b) = \int_0^\infty y^{-r/2} \exp(-ay) \sum_{n=0}^\infty (-1)^n \frac{\Gamma(b)}{\Gamma(b-n)n!} \exp(-ny) dy
\]

\[= \sum_{n=0}^\infty (-1)^n \frac{\Gamma(b)}{\Gamma(b-n)n!} \int_0^\infty y^{-r/2} \exp(-(a+n)y) \, dy.
\]
Figure 3: Plots of the BIR hazard rate function for $\theta = 0.5$ (solid line), $\theta = 1.0$ (dashed line), $\theta = 1.5$ (dotted line) and $\theta = 3.0$ (bold line).
This integral has a closed-form expression by means of a direct application of the gamma function integral, [Abramowitz and Stegun, 1972]. Since \(a + n > 0\), some manipulations yield

\[
\int_0^\infty y^{-r/2} \exp\{- (a + n)y\} \, dy = \frac{\Gamma\left(1 - \frac{r}{2}\right)}{(a + n)^{1-r/2}}, \quad r < 2. \tag{10}
\]

Therefore, we can rewrite (9) as

\[
S_r(a, b) = \Gamma(b) \Gamma\left(1 - \frac{r}{2}\right) \sum_{n=0}^\infty \frac{(-1)^n}{(a + n)^{1-r/2} \Gamma(b - n)n!}, \quad r < 2.
\]

If \(b > 0\) is an integer, we obtain

\[
S_r(a, b) = \Gamma\left(1 - \frac{r}{2}\right) \sum_{n=0}^b \frac{(-1)^n}{(a + n)^{1-r/2} \Gamma(b - n)n!}, \quad r < 2.
\]

We can write the \(r\)th moment of \(X\) as

\[
E(X^r) = \frac{\theta^{r/2}}{B(a, b)} S_r(a, b), \quad r < 2.
\]

In particular, for \(r = 1\) and integer an \(b\), we obtain

\[
E(X) = \frac{\sqrt{\pi} \theta}{B(a, b)} \sum_{n=0}^b \frac{(-1)^n}{n!} \left(\frac{b-1}{(a+n)^{3/2}}\right).
\]

Negative moments can also be evaluated. For example, considering \(r = -1\) and for an integer \(b\), we have

\[
E(X^{-1}) = \frac{\sqrt{\pi} \theta}{2 B(a, b)} \sum_{n=0}^b (-1)^n \left(\frac{b-1}{n!} \frac{1}{(a+n)^{3/2}}\right).
\]

Notice that attempting to compute (10) outside \(r < 2\) gives undefined forms. For instance, if \(r = 2\), we have

\[
\int_0^\infty \frac{\exp\{- (a + n)y\}}{y} \, dy = E_1(0),
\]

where \(E_1(\cdot)\) is the exponential integral function [Abramowitz and Stegun, 1972], which tends to \(-\infty\) as its argument goes to zero. As a consequence, the second moment of \(X\) does not exist, as well as all remaining higher order moments. It is known that the second and higher order moments of IR distribution are inexistent [Voda, 1972]. As shown above, the BIR distribution inherits this characteristic.

5 Quantile function and quantile measures

The quantile function of \(X\) is given by

\[
Q(u) = F^{-1}(u) = \sqrt{-\frac{\theta}{\log(I^{-1}_u(a, b))}}, \quad 0 < u < 1,
\]
where $I_u^{-1}(a, b)$ is the inverse of the incomplete beta function. The function $I_u^{-1}(a, b)$ can be written as a power series expansion [Wolfram|Alpha 2011]

$$I_u^{-1}(a, b) = \sum_{i=1}^{\infty} q_i [a B(a, b)]^{i/a},$$

where $q_1 = 1$ and the remaining coefficients satisfy the following recursion

$$q_i = \frac{1}{i^2 + (a-2)i + (1-a)} \left\{ (1-\delta_{i,2}) \sum_{r=2}^{i-1} q_r q_{i+1-r} [r(1-a)(i-r) - r(r-1)] + \sum_{r=1}^{i-1} \sum_{s=1}^{i-r} q_r q_s q_{i+1-r-s} [r(r-a) + s(a+b-2)(i+1-r-s)] \right\},$$

where $\delta_{i,2} = 1$ if $i = 2$ and $\delta_{i,2} = 0$ if $i \neq 2$.

Because the second, third, and fourth moments of the BIR distribution are nonexistent, usual skewness and kurtosis are not defined. However, quantile based measures, such as Bowley skewness [Kenney and Keeping 1962] and Moors kurtosis [Moors 1998], can quantify asymmetry and the peakedness of a given distribution. These measures exist even when moments are not available. Bowley skewness and Moors kurtosis are expressed according to

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)},$$

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Plots of the Bowley skewness and Moors kurtosis for selected values of $a$ and $b$ are displayed in Figure 4(a).

The parameter $\theta$ was set to one.

6 Mean deviations and inequality measures

The amount of scatter in $X$ is measured to some extent by the totality of deviations from the mean ($\mu$) and median ($m$). These are known as the mean deviation about the mean and the mean deviation about the median given by

$$\delta_1(X) = 2\mu F(\mu) - 2\mu + 2 \int_{\mu}^{\infty} xf(x)dx \quad \text{and} \quad \delta_2(X) = 2 \int_{m}^{\infty} xf(x)dx - \mu,$$

respectively, where $\mu = E(X)$ and $m = Q(1/2)$.

Defining the integral $J(z) = \int_{x}^{\infty} xf(x)dx$, the measures $\delta_1(X)$ and $\delta_2(X)$ are given by

$$\delta_1(X) = 2\mu F(\mu) - 2J(\mu) \quad \text{and} \quad \delta_2(X) = \mu - 2J(m),$$

where $F(\mu)$ and $F(m)$ are easily obtained from (2).
Figure 4: Plots of the Bowley skewness and Moors kurtosis in terms of (a) $a$ for $b = 1.0$ (solid curve) and $b = 1.5$ (dashed curve), $b = 3.5$ (dotted line) and $b = 4.5$ (bold line); (b) $b$ for $a = 1.0$ (solid curve), $a = 1.5$ (dashed curve), $a = 3.5$ (dotted line) and $a = 4.5$ (bold line); (c) $a$ for $b = 1.0$ (solid curve) and $b = 1.5$ (dashed curve), $b = 3.5$ (dotted line) and $b = 4.5$ (bold line); and (d) $b$ for $a = 1.0$ (solid curve), $a = 1.5$ (dashed curve), $a = 3.5$ (dotted line) and $a = 4.5$ (bold line).
We now determine $J(z)$. Substituting $y = \theta x^{-2}$ in equation (3), we obtain

\[ J(z) = \int_{0}^{z} xf(x)dx = \frac{\sqrt{\theta}}{B(a,b)} \int_{0}^{\infty} y^{-1/2} \exp(-ay)|1 - \exp(-y)|^{b-1} dy. \]

Considering the power series (5), we have

\[ J(z) = \frac{\sqrt{\theta}}{B(a,b)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(b-n)n!} \int_{0}^{\infty} y^{-1/2} \exp(-(a+n)y)dy = \frac{\sqrt{\theta}}{B(a,b)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(b-n)n!} \frac{1}{\sqrt{\theta \sqrt{a+n}} \erfc(\sqrt{\theta / (a+n)})}, \]

where $\erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt$ is the complementary error function.

Bonferroni [1930] and Lorenz [1905] curves are inequality measures which have applications in economics, reliability, demography, actuarial sciences, and medicine, among others. They are defined by

\[ B(p) = \frac{1}{p\mu} \int_{0}^{Q(p)} xf(x)dx = \frac{1}{p\mu} J(Q(p)) \quad \text{and} \quad L(p) = \frac{1}{\mu} \int_{0}^{Q(p)} xf(x)dx = \frac{1}{\mu} J(Q(p)), \]

respectively, for $0 < p < 1$, see Pundir et al. [2005] for details.

7 Shannon and Rényi entropies

The entropy of a random variable quantifies its associated uncertainty [Song, 2001]. Two important entropy measures are the Shannon entropy and its generalization known as the Rényi entropy. For the BIR distribution, the Shannon entropy is

\[ H(X) = -E[\log(f(X))] = -\int_{0}^{\infty} f(x) \log(f(x))dx \]

\[ = -\int_{0}^{\infty} f(x) \log \left( \frac{2\theta}{B(a,b)x^\alpha} \exp \left( -\frac{a\theta}{x^\alpha} \right) |1 - \exp\left( -\frac{\theta}{x^\alpha} \right)|^{b-1} \right) dx \]

\[ = -\int_{0}^{\infty} \log \left( \frac{2\theta}{B(a,b)} \right) f(x)dx + 3 \int_{0}^{\infty} \log(x)f(x)dx + a\theta \int_{0}^{\infty} \frac{1}{x^\beta} f(x)dx - (b-1) \int_{0}^{\infty} \log \left( 1 - \exp \left( -\frac{\theta}{x^\alpha} \right) \right) f(x)dx, \]

where the first of the last four integrals is equal to $-\log(2\theta/B(a,b))$. The second integral can be calculated as follows.
where \( \gamma \) is the Euler-Mascheroni constant. The third integral can be expressed by

\[
\int_0^\infty \frac{1}{x^2} f(x) \, dx = \frac{2\theta}{B(a,b)} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(b)}{(b-n)n!} \int_0^\infty \exp\left(\frac{-a+n\theta}{x^2}\right) \, dx
\]

Setting \( t = \frac{(a+n)\theta}{x^2} \), we obtain

\[
\int_0^\infty \frac{1}{x^2} f(x) \, dx = \frac{2\theta}{B(a,b)} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(b)}{(b-n)n!} \int_0^\infty \exp(t) \frac{1}{[2(a+n)\theta]^2} \, dt
\]

\[
= \frac{1}{\theta B(a,b)} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(b)}{(b-n)n!} \frac{1}{(a+n)^2}.
\]

Considering the fourth integral, let \( u = \theta x^2 \). From the power series expansion \( \log(1+z) = z + \frac{1}{2} z^2 - \frac{1}{3} z^3 - \cdots \), we can write

\[
\int_0^\infty \log(1 - \exp(-\theta x^2)) f(x) \, dx = \frac{1}{B(a,b)} \int_0^\infty \log(1 - \exp(-u)) \exp(-au) [1 - \exp(-u)]^{b-1} \, du
\]

\[
= - \frac{1}{B(a,b)} \int_0^\infty \sum_{k=1}^\infty \frac{\exp(-u(k+a))[1 - \exp(-u)]^{b-1}}{k} \, du
\]

\[
= \frac{1}{B(a,b)} \sum_{k=1}^\infty \frac{(-1)^{k+1} \Gamma(b)}{k(a+k)\Gamma(b-n)n!} \int_0^\infty \exp(-u(k+a+n)) \, du
\]

\[
= \frac{1}{B(a,b)} \sum_{k=1}^\infty \sum_{n=0}^\infty \frac{(-1)^{n+1} \Gamma(b)}{k(a+k+n)\Gamma(b-n)n!}.
\]

Finally, we obtain

\[
H(X) = -\log \left\{ \frac{2\theta}{B(a,b)} \right\} + \frac{\Gamma(b)}{B(a,b)} \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(b-n)n!} \left[ \frac{3 \log((a+n)\theta) + \gamma}{2(a+n)} + \frac{a}{(a+n)^2} + (b-1) \sum_{k=1}^\infty \frac{1}{k(a+k+n)} \right].
\]
Now, the Rényi entropy can be expressed as

\[ H_\alpha(X) = \frac{1}{1-\alpha} \log \left( \int_0^\infty f(x)^\alpha \, dx \right), \quad \alpha > 0, \alpha \neq 1, \tag{11} \]

where \( \alpha > 0 \) and \( \alpha \neq 1 \). Notice that when \( \alpha \to 1 \), the Rényi entropy converges to the Shannon entropy. For calculating (11), we apply (3) and consider the power series expansion (5) yielding

\[
\int_0^\infty f(x)^\alpha \, dx = \left[ \frac{2\theta}{B(a, b)} \right]^\alpha \Gamma(a(b-1) + 1) \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(a(b-1) + 1 - n)n!} \times \int_0^\infty x^{-3\alpha} \exp \left\{ -(a\alpha + n) \frac{\theta}{x^2} \right\} \, dx.
\]

The last integral can be evaluated as follows. Let \( u = \theta x^{-2} \). Then, we have

\[
\int_0^\infty x^{-3\alpha} \exp \left\{ -(a\alpha + n) \frac{\theta}{x^2} \right\} \, dx = \int_0^\infty u^{\frac{3\alpha-1}{2}} \exp \left\{ -(a\alpha + j)u \right\} \, du = \left( \frac{1}{aa + n} \right)^{\frac{3\alpha-1}{2}} \Gamma \left( \frac{3\alpha - 1}{2} \right).
\]

Finally, we obtain

\[
\int_0^\infty f(x)^\alpha \, dx = \left[ \frac{2\theta}{B(a, b)} \right]^\alpha \Gamma \left( \frac{3\alpha-1}{2} \right) \sum_{n=0}^\infty \frac{(-1)^n}{(aa + n)\frac{3\alpha-1}{2} \Gamma(a(b-1) + 1) \Gamma(a(b-1) + 1 - n)n!} \Gamma(a(b-1) + 1 - n)n!.
\]

### 8 Order statistics

Here, we present an explicit expression for the density function \( f_{i:n}(x) \) of the \( i \)th order statistic \( X_{i:n} \) in a random sample of size \( n \) from the BIR distribution. Consider the well-known result

\[
f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1}(1-F(x))^{n-i},
\]

for \( i = 1, \ldots, n \). Applying the binomial expansion in the above equation, we obtain

\[
f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l F(x)^{i+l-1}.
\]

Inserting (3) and (6) in the last equation, \( f_{i:n}(x) \) can be expressed as

\[
f_{i:n}(x) = \frac{2\theta}{B(i, n-i+1)x^3} \exp \left\{ -\frac{a\theta}{x^2} \right\} \left[ 1 - \exp \left\{ -\frac{\theta}{x^2} \right\} \right]^{b-1} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l \frac{1}{B(a, b)^{i+l}} \sum_{j=0}^\infty \left( -\frac{(a+j)\theta}{x^2} \right)^j \exp \left\{ -\frac{(a+j)\theta}{x^2} \right\}^{l+j-1}, \tag{12}
\]

for \( b > 0 \) real non-integer.
Now, using the following identity
\[
\left( \sum_{i=0}^{\infty} a_i \right)^k = \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} a_{m_1} \cdots a_{m_k},
\]
for \( k \) positive integer, we can write \((12)\) as
\[
f_{i:n}(x) = \sum_{l=0}^{n-i} \sum_{m_1=0}^{\infty} \cdots \sum_{m_{i+l-1}=0}^{\infty} \delta_{i,l} f_{i,l}(x),
\]
\[(13)\]
where
\[
f_{i,l}(x) = \frac{2\theta \exp \left( -\frac{a \theta}{x^2} \right) \left[ 1 - \exp \left( -\frac{\theta}{x^2} \right) \right]^{b-1} \exp \left( -\frac{\theta}{x^2} \sum_{j=1}^{i+l-1} (a + m_j) \right)}{x^3 B(a(i+l) + \sum_{j=1}^{i+l-1} m_j, b)},
\]
and
\[
\delta_{i,l} = \frac{(-1)^{i+\sum_{j=1}^{i+l-1} m_j} (n-i) \Gamma(b)^{i+l-1} B(a(i+l) + \sum_{j=1}^{i+l-1} m_j, b)}{B(a,b)^{i+l} B(i,n-i+1) \prod_{j=1}^{i+l-1} (a + m_j) \Gamma(b - m_j) m_j!}.
\]

Note that \( f_{i,l}(x) \) is the density function of the BIR \((a(i+l) + \sum_{j=1}^{i+l-1} b, \theta)\) distribution. Also, the constants \( \delta_{i,l} \) are obtained given \( i, n, l \) and a sequence of indices \( m_1, \ldots, m_{i+l-1} \). The sums in \((13)\) extend over all \((i+l)\)-tuples \((l, m_1, \ldots, m_{i+l-1})\) of non-negative integers. These sums indicate that the density function of the BIR order statistics is a linear combination of BIR densities. So, several structural quantities of the BIR order statistics can be obtained from those of BIR distribution.

9 Maximum likelihood estimation and information matrix

Consider independent BIR distributed random variables \( X_1, \ldots, X_n \) with parameter vector \( \lambda = (a, b, \theta)^T \). The log-likelihood function \( \ell(\lambda) \) for the BIR model reduces to
\[
\ell(a, b, \theta) = n \left[ \log(2\theta) - \log[B(a, b)] \right] - 3 \sum_{i=1}^{n} \log(x_i) - a \theta \sum_{i=1}^{n} \frac{1}{x_i^2}
\]
\[+ (b - 1) \sum_{i=1}^{n} \log \left\{ 1 - \exp \left( -\frac{\theta}{x_i^2} \right) \right\}.
\]
The elements of the score vector are:

\[
U_a(\lambda) = \frac{\partial}{\partial a} \ell(a, b, \theta) = n[\psi(a + b) - \psi(a)] - \theta \sum_{i=1}^{n} \frac{1}{x_i^2},
\]

\[
U_b(\lambda) = \frac{\partial}{\partial b} \ell(a, b, \theta) = n[\psi(a + b) - \psi(b)] + \sum_{i=1}^{n} \log \left\{1 - \exp\left(-\frac{1}{x_i^2}\right)\right\},
\]

\[
U_\theta(\lambda) = \frac{\partial}{\partial \theta} \ell(a, b, \theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x_i^2} \left[a + \frac{b - 1}{1 - \exp\left(-\frac{\theta}{x_i^2}\right)} \right],
\]

where \(\psi(\cdot)\) is the digamma function, see [Abramowitz and Stegun, 1972]. The maximum likelihood equations can be solved numerically for \(a\), \(b\), and \(\theta\). Under standard regularity conditions [Cox and Hinkley, 1974] that are fulfilled for the proposed model whenever the parameters are in the interior of the parameter space, the observed information matrix \(I(\lambda)\) can be employed for interval estimation of the model parameters and for hypothesis tests. The BIR observed information matrix is given by

\[
I(\lambda) = - \begin{bmatrix}
U_{aa}(\lambda) & U_{ab}(\lambda) & U_{a\theta}(\lambda) \\
U_{ab}(\lambda) & U_{bb}(\lambda) & U_{b\theta}(\lambda) \\
U_{a\theta}(\lambda) & U_{b\theta}(\lambda) & U_{\theta\theta}(\lambda)
\end{bmatrix},
\]

whose elements are

\[
U_{aa}(\lambda) = \frac{\partial}{\partial a} U_a(\lambda) = n \left[\psi_1(a + b) - \psi_1(a)\right],
\]

\[
U_{ab}(\lambda) = \frac{\partial}{\partial b} U_a(\lambda) = n \psi_1(a + b),
\]

\[
U_{a\theta}(\lambda) = \frac{\partial}{\partial \theta} U_a(\lambda) = -\sum_{i=1}^{n} \frac{1}{x_i^2},
\]

\[
U_{bb}(\lambda) = \frac{\partial}{\partial b} U_b(\lambda) = n \left[\psi_1(a + b) - \psi_1(b)\right],
\]

\[
U_{b\theta}(\lambda) = \frac{\partial}{\partial \theta} U_b(\lambda) = -\sum_{i=1}^{n} \frac{1}{x_i^2} \left[\frac{1}{1 - \exp\left(-\frac{\theta}{x_i^2}\right)}\right],
\]

\[
U_{\theta\theta}(\lambda) = \frac{\partial}{\partial \theta} U_\theta(\lambda) = -\frac{n}{\theta^2} + (b - 1) \sum_{i=1}^{n} \frac{1}{x_i^2} \left[\frac{1}{1 - \exp\left(-\frac{\theta}{x_i^2}\right)} \cdot \frac{1}{1 - \exp\left(-\frac{\theta}{x_i^2}\right)}\right],
\]

and \(\psi_1(\cdot)\) is the polygamma function, which satisfies \(\psi_1(x) = \frac{d}{dx} \psi(x)\).

### 10 Application to real data

Bjerkedal [1960] considered guinea pigs as a model to study human tuberculosis. Indeed, guinea pigs are highly susceptible to the associate pathogen: tubercle bacillus. Bjerkedal generated several data sets with the survival times of infected guinea pigs. A particular set with 72 observations is listed in Table I. We analyze these data and fit them the BIR, exponentiated inverse Rayleigh (EIR) [Gupta et al., 1998], IR, Rayleigh (R)
and generalized Rayleigh (GR) distributions.

All these distributions are common models for lifetime data. Computational implementation was performed in Ox matrix programming language [Doornik, 2006].

Table 2 lists the maximum likelihood estimates (MLEs) of the model parameters (standard errors in parentheses) for each model. It is also shown the values for the Akaike information criterion (AIC) [Akaike, 1973], Bayesian information criterion (BIC) [Schwarz, 1978], bias-corrected Akaike information criterion (BAIC) [Hurvich and Tsai, 1989] and Hannan-Quinn information criterion (HQIC) [Hannan and Quinn, 1979]. These results indicate that the BIR distribution has the lowest AIC, BAIC, and HQIC values among the current models. Moreover, the GR distribution presents the lowest BIC value.

Plots of the estimated densities of the BIR, EIR, IR, Rayleigh and GR models fitted to these data are displayed in Figure 5. The overall results suggest that the BIR distribution is superior to the remaining distributions in terms of model fitting.

11 Conclusion

In this work, we study the beta inverse Rayleigh distribution as a generalization of the inverse Rayleigh distribution. We also provide a better foundation for some mathematical properties for this distribution, including the derivation of the hazard rate function, moments, quantile measures, mean deviations, entropy measures and order statistics. The model parameters are estimated by maximum likelihood. An application
Figure 5: Fitted densities for the BIR (black solid), EIR (black dashed), IR (gray dotted), Rayleigh (gray solid), and GR (gray dashed) models for the survival time data.

of the BIR distribution to a real data set indicates that the new distribution outperforms several distributions, including the IR and Rayleigh distributions.

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