Incidence-free sets and edge domination in incidence graphs

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Abstract
A set of edges $\Gamma$ of a graph $G$ is an edge dominating set if every edge of $G$ intersects at least one edge of $\Gamma$, and the edge domination number $\gamma_e(G)$ is the smallest size of an edge dominating set. Expanding on work of Laskar and Wallis, we study $\gamma_e(G)$ for graphs $G$ which are the incidence graph of some incidence structure $D$, with an emphasis on the case when $D$ is a symmetric design. In particular, we show in this latter case that determining $\gamma_e(G)$ is equivalent to determining the largest size of certain incidence-free sets of $D$. Throughout, we employ a variety of combinatorial, probabilistic and geometric techniques, supplemented with tools from spectral graph theory.

KEYWORDS
design, edge domination, incidence-free sets, incidence structure, matching

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1 | INTRODUCTION

Roughly speaking, the area of extremal combinatorics centres around problems which ask how large or small a given combinatorial object can be under certain restrictions. For example, Mantel’s theorem states that the maximum number of edges of an \( n \)-vertex triangle-free graph is \( \frac{n^2}{4} \). In this paper, the extremal questions we consider are domination-type problems, which roughly involve studying the minimum number of vertices or edges one needs to “cover” a graph \( G \). Below we recall the basic definitions for domination, and we refer the interested reader to the book by Haynes, Hedetniemi, and Slater [12] for more on this subject.

A dominating set in a graph \( G = (V, E) \) is a set of vertices \( S \subseteq V \) such that each vertex is either contained in \( S \), or has a neighbour in \( S \). The size of the smallest dominating set is called the domination number of \( G \) and is denoted as \( \gamma(G) \). A large body of work is dedicated to studying \( \gamma(G) \) as well as its many variants, such as Roman domination [3] and total domination [13].

A related concept is an edge dominating set in a graph \( G = (V, E) \). This is a subset \( \Gamma \subseteq E \) of edges, such that for each edge \( e \in E \), there exists an edge \( e' \in \Gamma \) with \( e \cap e' \neq \emptyset \). We note that this could be equivalently defined as a dominating set in the line graph of \( G \).

The edge domination number of \( G \) is defined as the size of the smallest edge dominating set, and will be denoted by \( \gamma_e(G) \). Again a large body of work is dedicated to studying \( \gamma_e(G) \), see, for example, [16, 33].

In this paper we study domination-type problems for graphs which come from incidence structures. An incidence structure is a pair \( D = (\mathcal{P}, \mathcal{B}) \) such that \( \mathcal{B} \) is a collection of subsets of \( \mathcal{P} \). We say that \( P \in \mathcal{P} \) and \( B \in \mathcal{B} \) are incident if \( P \in B \). The elements of \( \mathcal{P} \) and \( \mathcal{B} \) are called points and blocks, respectively. We say that \( D \) is of type \( (v, b, k, r, \lambda) \) if

- \( |\mathcal{P}| = v \), and \( |\mathcal{B}| = b \),
- every block is incident with \( k \) points, every point is incident with \( r \) blocks,
- \( \lambda \geq 1 \) is the maximum number such that there exist two distinct points \( P \) and \( Q \) which have \( \lambda \) blocks incident with both \( P \) and \( Q \).

The convention \( \lambda \geq 1 \) is somewhat nonstandard and is adopted to exclude trivial incidence structures. From these conditions, we have \( vr = bk \) and \( (v - 1)\lambda \geq r(k - 1) \). If \( v = b \) (which implies \( r = k \)), and if \( \lambda \) is also the maximum number of points incident with two distinct blocks, then we call \( D \) a symmetric incidence structure, abbreviated SIS, of type \( (v, k, \lambda) \).

If any two distinct points of an incidence structure are incident with exactly \( \lambda \) blocks, then we call \( D \) a \((v, k, \lambda)\)-design, and in this case

\[
vr = bk, \quad (v - 1)\lambda = r(k - 1).
\]

Fisher’s inequality states that in this case we always have \( v \leq b \). In case of equality, that is, if \( D \) is a design with \( v = b \), then any two blocks of \( D \) intersect in exactly \( \lambda \) points, and \( D \) is called a symmetric design. We refer the reader to [4] for proofs of these statements and other background on designs.

The incidence graph \( I(D) \), also called the Levi graph, of an incidence structure \( D = (\mathcal{P}, \mathcal{B}) \) is the bipartite graph with bipartition \( \mathcal{P} \) and \( \mathcal{B} \), where a point \( P \) and a block \( B \) are adjacent if and only if they are incident.

It turns out that the problem of determining the domination number of \( I(D) \) is closely related to studying the smallest blocking sets and covers in \( D \). Partially because of this, there have been several papers in recent years investigating the domination number of incidence graphs of block
designs and other incidence structures. See, for example, [10, 28] for incidence graphs of designs, [15] for incidence graphs of projective planes and [14] for incidence graphs of generalised quadrangles.

Some work for edge domination numbers of incidence graphs of projective planes was done by Laskar and Wallis [21], but outside of this very little is known about edge domination numbers of incidence graphs. Our goal for this paper is to expand upon this literature by focusing on the following problem:

**Problem 1.1.** Given an incidence structure $D$, determine the edge domination number of its incidence graph $I(D)$.

For convenience’s sake, we will denote this quantity by $\gamma_e(D)$ instead of $\gamma_e(I(D))$.

We prove a number of general bounds for $\gamma_e(D)$ for incidence structures throughout this paper. We highlight one such result below, which shows that for symmetric designs $D$, determining $\gamma_e(D)$ is equivalent to determining the maximum size of certain incidence-free sets.

**Definition 1.2.** Let $X \subseteq \mathcal{P}$ and $Y \subseteq \mathcal{B}$ be such that there are no incidences between points in $X$ and blocks in $Y$. In this case the pair $(X, Y)$ will be called incidence-free. If $|X| = |Y|$ we will say that the pair is equinumerous and refer to $|X|$ as its size.

We note that the problem of determining the size of large incidence-free subsets of incidence structures is a well-studied problem, see, for example, [6, 22, 23, 27]. As such, the following connection to $\gamma_e(D)$ and incidence-free sets is of particular interest.

**Theorem 1.3.** Let $D = (\mathcal{P}, \mathcal{B})$ be a symmetric $(v, k, \lambda)$-design with $k \geq 36$. Then

$$\gamma_e(D) = v - \alpha,$$

where $\alpha$ is the maximum size of an equinumerous incidence-free pair $(X, Y)$.

We note that Equation (1) implies there exist only a finite number of symmetric $(v, k, \lambda)$-designs with $k < 36$, so this theorem applies to all but finitely many symmetric designs. It is easy to show that Theorem 1.3 cannot be strengthened to hold for arbitrary SISs, but a somewhat analogous statement does hold if one assumes some extra mild conditions on an SIS; see Theorem 5.8 for more on this.

1.1 | Organisation

The rest of the paper is organised as follows. The first half of the paper is dedicated to general techniques and results, with preliminaries established in Section 2, lower bounds in Section 3, upper bounds in Section 4 and a proof of Theorem 1.3 in Section 5.

The last half of the paper, Section 6, establishes bounds on specific classes of incidence structures, both by applying results from the first half of the paper and by utilising algebraic structures and symmetries particular to the individual incidence structures. Some specific classes of designs we look at include those coming from projective planes, from Hadamard matrices, and from certain strongly regular graphs, often related to finite geometries. In many of these cases we establish asymptotically sharp bounds. Our examples cover almost all known basic constructions of symmetric designs (see [4,
Part II Chap. 6.8], with the notable exception of symmetric designs coming from difference sets, as there seems to exist no uniform way of establishing effective bounds in this case. The problem of solving this remaining case, as well as a number of other open problems, is discussed in Section 7.

2 | PRELIMINARIES

The following key observation allows us to translate upper bounds for sizes of incidence-free sets into lower bounds for $\gamma_e(D)$. This will be the driving force behind almost all of the lower bounds throughout this paper.

**Lemma 2.1.** If $D = (\mathcal{P}, \mathcal{B})$ is an incidence structure with $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$, then there is an incidence-free set $(X, Y)$ with $|X| \geq v - \gamma_e(D)$ and $|Y| \geq b - \gamma_e(D)$.

**Proof.** Let $\Gamma$ be an edge dominating set of size $\gamma_e(D)$. Let $X \subseteq \mathcal{P}$ denote the set of points which are not contained in any edge of $\Gamma$, and similarly define $Y \subseteq \mathcal{B}$. Note that $(X, Y)$ is incidence-free by definition of $\Gamma$ being an edge dominating set. Since there are at most $|\Gamma|$ points of $\mathcal{P}$ in an edge of $\Gamma$, we have

$$|X| \geq |\mathcal{P}| - |\Gamma| = v - \gamma_e(D),$$

and the same argument gives the desired bound for $Y$. □

To construct small edge dominating sets, we rely on matchings in $I(D)$. Recall that a matching in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each vertex of $G$ lies in at most one edge of $M$. A matching is called maximal if it cannot be extended to a larger matching. It is not hard to see that a matching is maximal if and only if it is an edge dominating set, and an observation of Yannakakis and Gavril [33] implies that $\gamma_e(G)$ is equal to the smallest size of a maximal matching of $G$.

We say that a matching $M$ of $G = (V, E)$ covers a subset $S$ of vertices if each element of $S$ is contained in an element of $M$. A perfect matching is a matching that covers all vertices exactly once. Note that perfect matchings are maximal matchings and edge dominating sets. A standard but important fact about matchings that we need is the following easy consequence of Hall’s theorem, where we recall that a graph is biregular if it has a bipartition $U \cup V$ such that every vertex within $U$ has the same degree and every vertex within $V$ has the same degree.

**Lemma 2.2.** If $G$ is a biregular graph with bipartition $U \cup V$ such that $|U| \leq |V|$, then $G$ has a matching which covers $U$.

We can use this to show when a trivial upper bound on $\gamma_e(D)$ is tight for designs.

**Lemma 2.3.** If $D = (\mathcal{P}, \mathcal{B})$ is an incidence structure with $|\mathcal{P}| = v$, then $\gamma_e(D) \leq v$.

If $D$ is a $(v, k, \lambda)$-design, then equality holds if and only if $r \geq v$.

**Proof.** To prove the upper bound, we can assume without loss of generality that every point $P \in \mathcal{P}$ is incident to at least one block $B_\rho \in \mathcal{B}$. Then $\{(P, B_\rho)\}_{\rho \in \mathcal{P}}$ is an edge dominating set of $I(D)$ of size $v$, which shows $\gamma_e(D) \leq v$. From now on we suppose $D$ is a design.
First consider the case \( r \geq v \). Let \( \Gamma \) be a smallest edge dominating set of \( I(D) \). If every point in \( \mathcal{P} \) is contained in an edge of \( \Gamma \), then \( \gamma_e(D) = |\Gamma| \geq |\mathcal{P}| = v \). Thus we may assume there exists some point \( P \) not contained in an edge of \( \Gamma \). Since \( \Gamma \) is an edge dominating set, this implies that all of the \( r \) blocks incident to \( P \) are contained in an edge of \( \Gamma \). Thus \( \gamma_e(D) \geq r \geq v \), proving the “if” statement.

Now consider the case \( r < v \), and let \( B_P \) denote the set of \( r \) blocks incident to a point \( P \in \mathcal{P} \). The incidence graph of \( (\mathcal{P} \setminus \{P\}, B_P) \) is biregular since every point in \( \mathcal{P} \setminus \{P\} \) is incident to \( \lambda \) blocks in \( B_P \) by the design property and conversely every block in \( B_P \) contains \( k - 1 \) points more points after removing \( P \). By the biregularity and Lemma 2.2 it has a matching \( \Gamma' \) covering \( B_P \). Since \( |\Gamma'| = r < v \), one can find a set of edges \( \Gamma \) which contains \( \Gamma' \) and is such that each point of \( \mathcal{P} \setminus \{P\} \) is contained in exactly one edge of \( \Gamma \) (namely, by arbitrarily adding to \( \Gamma \) an edge containing each point not covered by \( \Gamma' \)). This is an edge dominating set of size \( v - 1 \), completing the proof of the if and only if statement.

We note that the equality case of Lemma 2.3 partially motivates our focus on studying symmetric incidence structures where the number of points is comparable to the number of blocks. However, even in this setting the trivial upper bound of Lemma 2.3 is often close to the true value. This is perhaps not too surprising given that the incidence graphs of designs are “expanding” in the sense of Thomason [30, Section 3.3]. Roughly speaking, this means that small sets of points are incident with a relatively large number of blocks and vice versa. This implies that edges are “well-spread” and do not concentrate on small sets of vertices. It is thus reasonable to think that to dominate all edges, we need a fairly large number of them. This is indeed the case as we will see.

### 3 | LOWER BOUNDS

To start this section, we will give a lower bound on the number of blocks incident with at least one point of a certain point set. For designs, this can be done using eigenvalue techniques, as was done by Haemers [11, Corollary 5.3]. We will revisit this idea in Section 6.4.

This same result can also be proved using simple counting techniques. Stinson [27, Theorem 3.1] and De Winter, Schillewaert and Verstraëte [6, Theorem 3] independently bound the largest number of nonincident points and blocks in a projective plane. Later, Elvey Price, Adib Surani and Zhou [9, Lemma 3] gave a combinatorial proof for Haemers’ bound for symmetric designs. However, their arguments can be generalised in a straightforward way to hold for all incidence structures of type \((v, b, k, r, \lambda)\). For the sake of completeness, we give such a proof.

**Definition 3.1.** A maximal arc (of order \( n \)) in an incidence structure of type \((v, b, k, r, \lambda)\) is a nonempty set of points \( S \) such that every block intersects \( S \) in either 0 or \( n \) points, for some integer \( n \). We call a maximal arc trivial if it consists of one point, all points, or the complement of a block.*

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*The complement of a block \( B \) is a maximal arc if and only if every other block intersects \( B \) in a constant number of points. This is the case in symmetric designs.
Lemma 3.2. Let $D$ be a $(v, k, \lambda)$-design, and $S$ be a maximal arc of order $n$ in $D$. Then $S = 1 + r\frac{n-1}{\lambda}$ and every point not in $S$ is incident with $r - \frac{r-\lambda}{n}$ blocks containing $n$ points of $S$.

Proof. Fix a point $P \in S$. Then there are $r$ blocks incident to $P$, each containing $n - 1$ other points of $S$. Since each point of $S \setminus \{P\}$ is incident with $\lambda$ blocks incident to $P$, this implies that $S = 1 + r\frac{n-1}{\lambda}$, or equivalently that the order of a maximal arc $S$ equals $1 + \frac{|S| - 1}{r-\lambda}$.

Now take a point $Q \not\in S$ and denote the number of blocks through $Q$ containing $n$ points of $S$ by $n_Q$. Counting the pairs in the set $\{(P, B) \in S \times B \parallel P, Q \in B\}$ in two ways, we see that $\lambda \sum_i n_i = S \lambda n_Q$. This left-hand side follows by first taking $P \in S$ and using that every two distinct points in a design are incident with $\lambda$ common blocks, which in this case necessarily intersect $S$ in $n$ points (as they already contain at least one point). We conclude that $x_Q = r - \frac{r-\lambda}{n}$ and moreover observe that $x_Q$ is independent of the choice of $Q$. $\square$

This implies that if $T$ denotes the set of blocks not incident with any point of $S$, then any point is incident with either 0 or $\frac{r-\lambda}{n}$ blocks of $T$. We call $T$ the dual arc of $S$.

Lemma 3.3. Let $D = (P, B)$ be an incidence structure of type $(v, b, k, r, \lambda)$. For any set $S \subseteq P$, there are at least $\sum_{i>0} i^2 n_i \leq |S| (r + (|S| - 1)\lambda)$ blocks which intersect $S$. Equality holds in designs if and only if $S$ is a maximal arc of order $1 + \frac{|S| - 1}{r-\lambda}$.

Proof. Let $x$ be the number of blocks intersecting $S$, and let $n_i$ be the number of blocks intersecting $S$ in exactly $i$ points. Then the so-called standard equations tell us that

$$\sum_{i>0} n_i = x, \quad \sum_{i>0} in_i = r|S|, \quad \sum_{i>0} i^2 n_i \leq |S| (r + (|S| - 1)\lambda),$$

by performing a double count on $\{(P, B) \in S \times B \parallel P \in B\}$ and $\{(P, Q, B) \in S \times S \times B \parallel P, Q \in B\}$. Using these equations we find

$$0 \leq \sum_{i>0} \left(i - \left(1 + \frac{|S| - 1}{r-\lambda}\right)\right)^2 n_i$$

$$= \sum_{i>0} i^2 n_i - 2 \left(1 + \frac{|S| - 1}{r-\lambda}\right) \sum_{i>0} i n_i + \left(1 + \frac{|S| - 1}{r-\lambda}\right)^2 \sum_{i>0} n_i$$

$$\leq S(r + (|S| - 1)\lambda) - 2 \left(1 + \frac{|S| - 1}{r-\lambda}\right) rS + \left(1 + \frac{|S| - 1}{r-\lambda}\right)^2 x$$

$$\iff \left(1 + \frac{|S| - 1}{r-\lambda}\right)^2 x \geq 2 \left(1 + \frac{|S| - 1}{r-\lambda}\right) rS - S(r + (|S| - 1)\lambda)$$

$$\iff x \geq \frac{rS}{1 + \frac{|S| - 1}{r-\lambda}} = \frac{r^2 S}{r + (|S| - 1)\lambda}.$$
Equality holds if and only if \( n_i \) is only nonzero for \( i = 0 \) and \( i = 1 + \frac{|S|-1}{r} \lambda \) and any two points of \( S \) determine \( \lambda \) blocks, which is always true in designs. This is equivalent to \( S \) being a maximal arc of order \( 1 + \frac{|S|-1}{r} \lambda \).

We can do a bit better for small values of \( S \), which we will need later on.

**Lemma 3.4.** Let \( D = (\mathcal{P}, \mathcal{B}) \) be an incidence structure of type \((v, b, k, r, \lambda)\). For any set \( S \subseteq \mathcal{P} \), there are at least \( r|S| - \lambda \binom{|S|}{2} \) blocks which intersect \( S \). Equality holds in designs if and only if no three (or more) points of \( S \) are incident with the same block.

**Proof.** We can recycle the definition of \( x, n_i \) and the inequalities above to compute

\[
0 \leq \sum_{i>0} (i-1)(i-2)n_i \\
= \sum_{i>0} i^2n_i - 3 \sum_{i>0} in_i + 2 \sum_{i>0} n_i \\
\leq |S|(r + (|S| - 1)\lambda) - 3r|S| + 2x \\
\Rightarrow x \geq r|S| - \lambda \binom{|S|}{2}.
\]

In case of equality, we see that only \( n_0, n_1 \) and \( n_2 \) can be nonzero, which implies the last statement.

One can check that Lemma 3.4 yields a better bound than Lemma 3.3 if and only if \( |S| < 1 + \frac{r}{\lambda} \). In fact, a set of points with no three in a block can have at most \( 1 + \frac{r}{\lambda} \) points, and equality holds if and only if it is a maximal arc of order 2. In that case, both bounds coincide.

We note that in Section 5 we will similarly want to estimate the number of blocks incident with a small set of points. There we will use a greedy approach, which would work equally well to prove Lemma 3.4.

We will now use the first bound combined with Lemma 2.1 to obtain lower bounds on the edge domination number of designs. We restrict ourselves to designs instead of general incidence structures both for ease of presentation and the fact that we do not need the general bounds in the remainder of this text.

**Proposition 3.5.** Let \( D = (\mathcal{P}, \mathcal{B}) \) be a \((v, k, \lambda)\)-design and let \((X, Y)\) be an incidence-free pair satisfying \( v - |X| = b - |Y| \). Then

1. \(|X| \leq r^{\frac{k-r-2+\sqrt{(r-k)^2+4(r-\lambda)}}{2\lambda}} + 1\).
   
   Equality holds if and only if \( X \) is a maximal arc of order \( \frac{k-r+\sqrt{(r-k)^2+4(r-\lambda)}}{2\lambda} \) and \( Y \) is its dual arc.

2. If \( D \) is a symmetric design, then \(|X| \leq \frac{k\sqrt{\lambda-1}}{\lambda} + 1\).
   
   Equality holds if and only if \( X \) is a maximal arc of order \( \sqrt{\lambda-1} \) and \( Y \) is its dual arc.
Proof. (1) Since $Y$ contains no blocks incident to $X$, there are at most $b - |Y| = v - |X|$ blocks incident with a point in $X$. By Lemma 3.3, this implies that

$$\frac{r^2|X|}{r + (|X| - 1)\lambda} \leq v - |X|.$$  

This gives us an equation of the form $f(|X|) \leq 0$, where $f$ is a quadratic polynomial with positive leading coefficient. Therefore, $|X|$ is at most the largest root of $f$. Calculating this root is tedious, but straightforward, and yields the bound from the proposition, relying on Equation (1).

Now suppose that equality holds. Then we attain equality in the bound from Lemma 3.3, which implies that $X$ is a maximal arc. In this case, one can calculate from the size of $X$ that it is a maximal arc of order $(k - r + \sqrt{(r - k)^2 + 4(r - \lambda)})/2$. Necessarily, $Y$ must consist of all blocks missing $X$, that is, $Y$ must be the dual arc of $X$.

(2) This follows immediately from (1) using $k = r$. \qed

Corollary 3.6. Let $D = (\mathcal{P}, \mathcal{B})$ be a $(v, k, \lambda)$-design, then

(1) $\gamma_e(D) \geq k\left(\frac{r + k - \sqrt{(r - k)^2 + 4(r - \lambda)}}{2}\right)$.  

Equality holds if and only if $D$ has a maximal arc of order $\frac{k - r + \sqrt{(r - k)^2 + 4(r - \lambda)}}{2}$.  

(2) If $D$ is a symmetric design, then $\gamma_e(D) \geq k\frac{k - \sqrt{k - \lambda}}{\lambda}$.  

Equality holds if and only if $D$ has a maximal arc of order $\sqrt{k - \lambda}$. \qed

Proof. By Lemma 2.1, there exists an incidence-free pair $(X, Y)$ with $|X| \geq v - \gamma_e(D)$, and $|Y| \geq b - \gamma_e(D)$. Hence, we can find subsets $X' \subseteq X$ and $Y' \subseteq Y$ of size $v - \gamma_e(D)$ and $b - \gamma_e(D)$, respectively. Note that $(X', Y')$ is necessarily also incidence-free. Plugging this into Proposition 3.5 yields the desired bound.

Vice versa, suppose that $D$ has a maximal arc $X$ of order $n = (k - r + \sqrt{(r - k)^2 + 4(r - \lambda)})/2$. Let $Y$ denote its dual arc. Consider the induced subgraph $H$ of $I(D)$ on $(\mathcal{P}\backslash X) \cup (\mathcal{B}\backslash Y)$. Then every vertex in $\mathcal{P}\backslash X$ has degree $r - \frac{r - \lambda}{n}$, and every vertex in $\mathcal{B}\backslash Y$ has degree $k - n$, that is, $H$ is biregular. Furthermore, we have $|\mathcal{P}\backslash X| = |\mathcal{B}\backslash Y|$. Indeed, by Lemma 3.2 we have that $|X| = 1 + r\frac{n - 1}{\lambda}$ and hence $|\mathcal{P}\backslash X| = v - 1 - r\frac{n - 1}{\lambda}$ and also $|\mathcal{B}\backslash Y| = \frac{r(\lambda + r(n - 1))}{n\lambda}$ by Lemma 3.3. One can now check that these two expressions are equal, by clearing denominators and making use of the facts that $(v - 1)\lambda = r(k - 1)$ and that $n$ is a solution to the quadratic $X^2 + (r - k)X - (r - \lambda) = 0$. Thus, $H$ is a regular bipartite graph, and hence has a perfect matching by Lemma 2.2. Since there are no edges between $X$ and $Y$ in $I(D)$, this matching is an edge dominating set of size $v - |X|$. It is straightforward to check that $v - |X|$ equals the right-hand side of the above bound. \qed
Remark 3.7.

1. Recall that for a symmetric \((v, k, \lambda)\)-design we have \(r = k\) and \(v = k(k - 1)/\lambda + 1\), so (2) can be restated as

\[
\gamma_e(D) \geq v - \frac{k\sqrt{k - \lambda} - k}{\lambda} - 1.
\]

This indeed shows that \(\gamma_e(D)\) is equal to \(v\) up to lower order terms.

2. Laskar and Wallis [21] showed that if \(D\) is a symmetric \((n^2 + n + 1, n + 1, 1)\) design one has \(\gamma_e(D) > \frac{1}{2}(n^2 + 3n)\). Our results improve this to \(\gamma_e(D) \geq n^2 - n\sqrt{n} + 2n - \sqrt{n} + 1\).

3. One could try to improve the bounds above by studying the existence of maximal arcs in designs. This is however not feasible to do in general: even for the class of projective planes defined over a finite field \(\mathbb{F}_q\), which are a special type of symmetric \((q^2 + q + 1, q + 1, 1)\)-designs, the existence of maximal arcs depends on the parity of \(q\) as we will see later. Improvements are thus only possible by restricting oneself to particular classes of designs.

4 | CONSTRUCTING LARGE INCIDENCE-FREE SETS

In view of Theorem 1.3, upper bounding \(\gamma_e(D)\) when \(D\) is a symmetric design is equivalent to constructing large incidence-free pairs. Here we discuss two general techniques for achieving this goal.

4.1 | A probabilistic construction

In a symmetric \((v, k, \lambda)\)-design, one can greedily construct sets \(X \subseteq \mathcal{P}, Y \subseteq \mathcal{B}\) such that no point in \(X\) is contained in a block of \(Y\) and such that \(|X| = |Y| \geq v/(k + 1) \approx k/\lambda\). One can improve this bound somewhat through a simple probabilistic argument, which is based on a similar proof by De Winter, Schillewaert and Verstraëte [6, Theorem 6].

**Proposition 4.1.** Let \(D = (\mathcal{P}, \mathcal{B})\) be an SIS of type \((v, k, \lambda)\) with \(k \geq 3\) and \(\lambda \leq \frac{k}{64 \log k}\). Then there exists an equinumerous incidence-free pair whose size is at least \(\frac{k \log k}{8\lambda}\).

**Proof.** Let \(X \subseteq \mathcal{P}\) be a random set of points obtained by including each point of \(\mathcal{P}\) in \(X\) independently with probability \(p = \frac{\log k}{2k}\), and let \(Y \subseteq \mathcal{B}\) consist of the blocks which do not contain any points of \(X\). Note that by construction no point in \(X\) is contained in a block of \(Y\), so it suffices to show that with positive probability we have \(|X|, |Y| \geq \frac{k \log k}{8\lambda}\).

First observe that

\[
\mathbb{E}[|X|] = p \cdot v \geq \frac{\log k}{2k} \cdot \frac{k(k - 1)}{\lambda} \geq \frac{k \log k}{4\lambda}.
\]
where the second inequality used $k \geq 2$, the third used $\lambda \leq \frac{k}{64 \log k}$, and the last used $\log k \geq 1$ (since $k \geq 3$). Since $|X|$ is a binomial random variable, by the Chernoff bound and (3) we have

$$\Pr\left[|X| \leq \frac{1}{2} \mathbb{E}[|X|]\right] \leq e^{-\mathbb{E}[|X|]/8} < \frac{1}{2}. \tag{4}$$

For $B \in \mathcal{B}$, let $A_B$ denote the event that $B \in Y$. Then

$$\Pr[A_B] = (1 - p)^k \geq e^{-\frac{k p}{1-p}} = k^{-\frac{1}{2-2p}}, \tag{5}$$

where the inequality follows from exponentiating the inequality $\log(1 + x) \geq \frac{x}{1+x}$, and the last equality used $p = \frac{\log k}{2k}$. Using $\log x/x \leq 1/e$, we find

$$1 - p = 1 - \frac{\log k}{2k} \geq 1 - \frac{1}{2e} \geq \frac{4}{5}.$$ Combining this with (5) and $x^{3/8} \geq \log x$, we find

$$\Pr[A_B] \geq k^{-5/8} \geq \frac{\log k}{k}.$$ Since $v \geq \frac{k(k-1)}{\lambda} + 1 \geq \frac{k^2}{2\lambda}$, we conclude that

$$\mathbb{E}[|Y|] = \sum_{B \in \mathcal{B}} \Pr[A_B] \geq \frac{k^2}{2\lambda} \cdot \frac{\log k}{k} = \frac{k \log k}{2\lambda}. \tag{6}$$

Note that for any $B, B' \in \mathcal{B}$ (possibly nondistinct), we have

$$\Pr[A_B \cap A_{B'}] \leq (1 - p)^{2k - \lambda} = (1 - p)^{-\lambda} \Pr[A_B] \Pr[A_{B'}] \leq e^{\lambda p/(1-p)} \Pr[A_B] \Pr[A_{B'}] \leq e^{1/64} \Pr[A_B] \Pr[A_{B'}],$$

where the second inequality used $\log(1 + x) \geq \frac{x}{1+x}$ as before, and the last used $\lambda p = \frac{\lambda \log k}{2k} \leq \frac{1}{128}$ and $1 - p \geq 1/2$ by hypothesis on $\lambda$ and $k$. Combining these two results gives

$$\text{Var}(|Y|) = \sum_{B, B' \in \mathcal{B}} \Pr[A_B \cap A_{B'}] - \Pr[A_B] \Pr[A_{B'}] \leq (e^{1/64} - 1)\mathbb{E}[|Y|]^2 \leq 0.04 \mathbb{E}[|Y|]^2.$$

Using this, Chebyshev's inequality and denoting $Z = |Y|$ for readability, we have
By (2), (4), (6) and (7) we conclude with positive probability that \( X, Y \) of at least this size such that no point in \( X \) is contained in a block of \( Y \). Taking subsets of \( X, Y \) of size \( \min \{ |X|, |Y| \} \geq \frac{k \log k}{8\lambda} \) gives the result.

For symmetric designs, this probabilistic construction is suboptimal for proving asymptotically sharp bounds on the edge domination number, and it is reasonable to think that more advanced techniques might provide better constructions. However, [6, Problem 7] suggests that improving this result to \( \frac{k}{\lambda} + \varepsilon \) for any \( \varepsilon > 0 \) might be hard when \( \lambda \) is small, as it is related to an open and difficult conjecture by Erdős on \( C_4 \)-free graphs and its generalisation to \( K_{t,2} \)-free graphs.

This result (and its limitations mentioned above) motivates the need to look at particular classes of symmetric designs to do better, which is exactly what we will do in Section 6.

### 4.2 From polarities

In this section we describe a technique to construct equinumerous incidence-free pairs in symmetric designs with special kinds of symmetries, such as the Hadamard designs and those corresponding to projective planes over finite fields.

**Definition 4.2.** A **polarity** of a symmetric design \( D = (P, B) \) is a bijection \( \rho : P \cup B \to P \cup B \), such that

1. \( \rho \) maps points to blocks and blocks to points,
2. \( \rho \) preserves incidence: for each pair \( (P, B) \in P \times B \) it holds that \( P \in B \) if and only if \( \rho(B) \in \rho(P) \),
3. \( \rho^2 \) is the identity map.

A point or block is **absolute** if it is incident with its image under \( \rho \).

Given a symmetric design \( D = (P, B) \) with a polarity \( \rho \), define its **polarity graph** \( R(D, \rho) \) as the graph with \( P \) as vertices, where two vertices \( P \) and \( Q \) are adjacent if and only if \( P \in \rho(Q) \). Note that each absolute point will give rise to a loop in this graph. With this we can translate the problem of finding equinumerous incidence-free pairs to finding cocliques in the polarity graph, which we record in the following observation.

**Lemma 4.3.** If \( C \) is a coclique in \( R(D, \rho) \), then \( (C, \rho(C)) \), where \( \rho(C) := \{ \rho(P) | P \in C \} \), is an equinumerous incidence-free pair.

We count loops as edges, which implies that a coclique in the polarity graph contains no absolute points.
In this section we prove Theorem 1.3, which we recall says that for symmetric designs $D$, the edge domination number $\gamma_e(D)$ is equal to $v - \alpha$ where $\alpha$ is the size of a largest equinumerous incidence-free pair $(X, Y)$. The fact that $\gamma_e(D)$ is at least this quantity follows from Lemma 2.1, so it remains to construct an edge dominating set of this size. We do this through the following.

**Proposition 5.1.** Let $D = (\mathcal{P}, \mathcal{B})$ be a symmetric $(v, k, \lambda)$-design with $k \geq 36$. If $(X, Y)$ is an equinumerous incidence-free pair, then there is a perfect matching between $\mathcal{P}\backslash X$ and $\mathcal{B}\backslash Y$ in $I(D)$.

The perfect matching guaranteed by Proposition 5.1 gives an edge dominating set of size $v - |X|$, so choosing a largest equinumerous incidence-free pair gives the upper bound of Theorem 1.3. Thus to prove Theorem 1.3, it only remains to prove Proposition 5.1. We do this in a general way, so that we can reuse a lot of the arguments to give a variant of Proposition 5.1 in SISs.

For the remainder of this section, we fix the following assumptions and notation:

- $D = (\mathcal{P}, \mathcal{B})$ is an SIS of type $(v, k, \lambda)$,
- $(X, Y)$ is an equinumerous incidence-free pair in $D$;
- $G$ will denote the induced subgraph of $I(D)$ obtained by removing $X$ and $Y$ from its vertex set;
- $\alpha := |X| = |Y|$.

We will use the following easy consequence of Hall’s theorem to show the existence of perfect matchings in $G$; see, for example, [8] for details of its proof. Here $N_{G'}(S)$ denotes the set of vertices which are adjacent to some vertex of $S$ in a graph $G'$.

**Lemma 5.2.** Let $G'$ be a bipartite graph on $U \cup V$ with $|U| = |V|$. If every $S \subseteq U$ and $T \subseteq V$ with $|S| \leq \left\lfloor \frac{1}{2}|U| \right\rfloor$ satisfies $|N_{G'}(S)| \geq |S|$ and $|N_{G'}(T)| \geq |T|$, then $G'$ has a perfect matching.

We first show that Lemma 5.2 is satisfied whenever $S \subseteq \mathcal{P}\backslash X$ is relatively large. For this, we observe that by definition of $D$ being a symmetric $(v, k, \lambda)$-design, the graph $I(D)$ is $K_{2,\lambda+1}$-free, and hence all of its subgraphs are as well. By using the famous result by Kővári, Sós, and Turán [20], which upper bounds the maximum number of edges that a $K_{s,t}$-free bipartite graph can have, we immediately get the following.

**Lemma 5.3.** For any $S \subseteq \mathcal{P}$ and $T \subseteq \mathcal{B}$, we have

$$e(S, T) \leq \sqrt{\lambda}(|S| - 1)\sqrt{|T|} + |T|,$$

where $e(S, T)$ denotes the number of edges in $I(D)$ between $S$ and $T$.

Now define the function

$$f(s) := \left(k - 1 - \frac{\alpha}{s}\right)^2 - \lambda(s + \alpha).$$
Lemma 5.4. If $S \subseteq \mathcal{P}\setminus X$ satisfies $f(S) \geq 0$, then $|N_G(S)| \geq \mathfrak{B}$.

Proof. Let $S$ be as in the lemma statement and set $T = N_{f(D)}(S)$. If $|T| \geq \mathfrak{B} + \alpha$, then

$$|N_G(S)| = |T\setminus Y| \geq |T| - |Y| = |T| - \alpha \geq \mathfrak{B}. $$

Thus we may assume for contradiction that $|T| < \mathfrak{B} + \alpha$. By the preceding lemma, we find

$$k\mathfrak{B} = e(S, T) \leq \sqrt{\lambda}(\mathfrak{B} - 1)\sqrt{|T|} + |T| < \sqrt{\lambda}\mathfrak{B}\sqrt{\mathfrak{B} + \alpha} + \mathfrak{B} + \alpha. $$

By dividing both sides by $\mathfrak{B}$, we see that the above is equivalent to

$$\sqrt{\lambda}(\mathfrak{B} + \alpha) > k - 1 - \frac{\alpha}{\mathfrak{B}}. $$

This contradicts $f(\mathfrak{B}) \geq 0$, so we conclude the result. □

Our aim now is to show $f(s) \geq 0$ for “most” values of $s$. We begin with the following, which reduces our problem of checking $f(s) \geq 0$ to only two values of $s$.

Lemma 5.5. Suppose that $k \geq 4$ and that $f(s) \geq 0$ for $s = \frac{2\alpha}{k}$ and $s = \frac{v - \alpha + 1}{2}$. Then $f(s) \geq 0$ for all $s \in \left[\frac{2\alpha}{k}, \frac{v - \alpha + 1}{2}\right]$.

Proof. Note that

$$f'(s) = 2\alpha\frac{(k - 1)s - \alpha}{s^3} - \lambda, $$

$$f''(s) = 2\alpha\frac{3s - 2(k - 1)s}{s^4}. $$

Define $s_0 = \frac{3}{2} - \frac{\alpha}{k - 1}$. Then $f''(s) < 0$ for all $s > s_0$. This means that $f'(s)$ is decreasing on the interval $[s_0, \infty)$. As a consequence, given any subinterval $[s_1, s_2]$ of $[s_0, \infty), f$ reaches its minimum on $[s_1, s_2]$ at one of the endpoints. Since $k \geq 4$, we have that $\frac{2\alpha}{k} \geq s_0$. The lemma follows. □

We now show $f(s) \geq 0$ for these two specified values of $s$ provided $\alpha$ is not too large.

Lemma 5.6. Suppose that $\alpha \leq \frac{k^2}{6\lambda}$ and $k \geq 16$. Then $f\left(\frac{2\alpha}{k}\right) \geq 0$.

Proof.

$$f\left(\frac{2\alpha}{k}\right) = \left(k - 1 - \frac{k}{2}\right)^2 - \lambda\left(\frac{2}{k} + 1\right)\alpha \geq \left(\frac{k}{2} - 1\right)^2 - \left(\frac{2}{k} + 1\right)^2 \frac{k^2}{6}. $$
The right-hand side is nonnegative whenever \( k \geq 16 \). □

The next lemma requires an additional assumption on \( v \) which is automatically satisfied if our SIS is a symmetric design.

**Lemma 5.7.** Suppose that \( \alpha \leq \frac{k^2}{6\lambda}, k \geq 8, \) and \( v = 1 + \frac{k(k-1)}{\lambda} \). Then \( f\left(\frac{v - \alpha + 1}{2}\right) \geq 0. \)

**Proof.**

\[
f\left(\frac{v - \alpha + 1}{2}\right) = \left( k - 1 - \frac{2\alpha}{v - \alpha + 1} \right)^2 - \frac{v + \alpha + 1}{2}.
\]

Note that this expression is decreasing in \( \alpha \). Using \( \alpha \leq \frac{k^2}{6\lambda}, \lambda \leq k, \) and \( v = 1 + \frac{k(k-1)}{\lambda} \), we obtain

\[
f\left(\frac{v - \alpha + 1}{2}\right) \geq \left( k - 1 - \frac{2\frac{k^2}{6\lambda}}{1 + \frac{k(k-1)}{\lambda} - \frac{k^2}{6\lambda} + 1} \right)^2 - \frac{1 + \frac{k(k-1)}{\lambda} + \frac{k^2}{6\lambda} + 1}{\lambda}
\]
\[
\geq \left( k - 1 - \frac{\frac{k^2}{2\lambda} \frac{k^2}{\lambda} - \frac{k^2}{6\lambda}}{\lambda} \right)^2 - \frac{k}{2} \left( k - 1 + \frac{k}{6} \right)
\]
\[
\geq \left( k - 1 - \frac{2k}{5k - 6} \right)^2 - \frac{k}{2} \left( \frac{7k}{6} + 1 \right).
\]

The last expression is nonnegative whenever \( k \geq 8 \). □

**Proof of Proposition 5.1.** If \( \alpha = 0 \) then \( G = I(D) \) is a regular bipartite graph, which has a perfect matching by Lemma 2.2. Thus we may assume \( \alpha > 0 \). We now show that \( G \) has a perfect matching by using Lemma 5.2, and by the symmetry of the problem, it suffices to prove \( |N_G(S)| \geq B \) whenever \( S \subseteq \mathcal{P}\setminus X \) with \( B \leq \left\lfloor \frac{1}{2} |\mathcal{P}\setminus X| \right\rfloor \leq \frac{v - \alpha + 1}{2}. \)

First consider the case \( \frac{2\alpha}{k} \leq B \leq \frac{v - \alpha + 1}{2}. \) Using Lemmas 5.4 and 5.5, it suffices to show that \( f\left(\frac{2\alpha}{k}\right) \) and \( f\left(\frac{v - \alpha + 1}{2}\right) \) are nonnegative. By Proposition 3.5(2) and the fact that \( k \geq 36, \) we have \( \alpha \leq \frac{k\sqrt{k}}{\lambda} \leq \frac{k^2}{6\lambda}. \) Moreover, since \( D \) is a design, \( v = 1 + \frac{k(k-1)}{\lambda}. \) Hence, Lemmas 5.6 and 5.7 finish the proof in this case.

It remains to deal with the case \( B < \frac{2\alpha}{k}, \) for which we utilise the following observation. Fix any vertex \( P \in \mathcal{P}\setminus X. \) Observe that each vertex in \( X \) has \( \lambda \) common neighbours with \( P \) in \( I(D), \) and that these neighbours necessarily lie in \( B\setminus Y \) since \( X \) and \( Y \) are incidence-free. Thus
\[ \lambda \alpha = \lambda |X| \leq e_{1(D)}(N_G(P), X) \leq k|N_G(P)|, \]

where this last step used that each vertex of \( I(D) \) has degree \( k \). We conclude that \( |N_G(P)| \geq \frac{2\alpha}{k} \) for all \( P \in \mathcal{P} \setminus X \). In particular, this proves \( |N_G(S)| \geq \frac{2\alpha}{k} \) if \( \lambda \geq 2 \). Therefore, \( |N_G(S)| \geq \mathcal{B} \) whenever \( \lambda \geq 2 \) and \( \mathcal{B} \leq \frac{2\alpha}{k} \). Similarly \( |N_G(S)| \geq \mathcal{B} \) if \( \mathcal{B} = \lambda = 1 \).

Now suppose that \( \lambda = 1 \) and \( 2 \leq \mathcal{B} < \frac{2\alpha}{k} \). Then \( \mathcal{B} \leq \left\lfloor \frac{2\alpha}{k} - 1 \right\rfloor \). Take 2 points \( P, R \in S \). They both have at least \( \frac{\alpha}{k} \) neighbours in \( G \), of which they share at most 1, hence

\[ |N_G(S)| \geq \frac{2\alpha}{k} - 1. \]

Since \( |N_G(S)| \) is an integer,

\[ |N_G(S)| \geq \left\lceil \frac{2\alpha}{k} - 1 \right\rceil \geq \mathcal{B}, \]

completing the proof.

Finally, let us state a version of Proposition 5.1 for general symmetric incidence structures.

**Theorem 5.8.** Let \( D = (\mathcal{P}, \mathcal{B}) \) be an SIS of type \((v, k, \lambda)\). Suppose that \( k \geq 36 \) and \( v \leq 1.65 k^2 \). If \((X, Y)\) is an equinumerous incidence-free pair with \(|X| \leq \frac{k^2}{\lambda} \) and every point or block not in \( X \cup Y \) is incident with at least \( \frac{2|X|}{k} \) blocks or points not in \( X \cup Y \), then

\[ \gamma_e(D) \leq v - |X|. \]

**Proof.** The proof is very similar to that of Proposition 5.1. Again it suffices to check that \( |N_G(S)| \geq \mathcal{B} \) for all \( S \subset \mathcal{P} \setminus X \) with \( \mathcal{B} \leq \frac{v - \alpha + 1}{2} \). By assumption, every \( P \in \mathcal{P} \setminus X \) satisfies \( |N_G(P)| \geq \frac{2\alpha}{k} \), hence \( |N_G(S)| \geq \mathcal{B} \) whenever \( \mathcal{B} \leq \frac{2\alpha}{k} \). Thus, by Lemmas 5.4 and 5.5, it suffices to check that \( f\left(\frac{2\alpha}{k}\right) \geq 0 \), which holds by Lemma 5.6, and that \( f\left(\frac{v - \alpha + 1}{2}\right) \geq 0 \).

Recall, as stated in Section 1, that \((v - 1)\lambda \geq r(k - 1)\) in an incidence structure of type \((v, b, r, k, \lambda)\). In particular, this implies that \( v \geq 1 + \frac{k(k - 1)}{\lambda} \) in any SIS. Using \( \alpha \leq \frac{k^2}{6\lambda} \) and \( v \geq 1 + \frac{k(k - 1)}{\lambda} \), one can check that \( \frac{v - \alpha + 1}{2} \geq \frac{3\alpha}{2k - 1} \). Indeed, we need to check that \((v - \alpha + 1)(k - 1) \geq 3\alpha \), or equivalently that \((v + 1)(k - 1) \geq (k + 2)\alpha \). Plugging in the bounds on \( v \) and \( \alpha \), we find that

\[ (v + 1)(k - 1) > \frac{k(k - 1)^2}{\lambda} > \frac{(k + 2)k^2}{6\lambda} \geq (k + 2)\alpha. \]

The inequality in the middle holds whenever \( k \geq 3 \).

Using the arguments of Lemma 5.5 and \( 1 + \frac{k(k - 1)}{\lambda} \leq v \leq 1.65 k^2 / \lambda \), it follows that

\[ f\left(\frac{v - \alpha + 1}{2}\right) \geq \min_{u \in \left\{1 + \frac{k(k - 1)}{\lambda}, 1.65 k^2 / \lambda\right\}} f\left(\frac{u - \alpha + 1}{2}\right). \]
We know that \( f\left(\frac{u-\frac{\alpha+1}{2}}{2}\right) \geq 0 \) for \( u = 1 + \frac{k(k-1)}{\lambda} \) from Lemma 5.7.

\[
f\left(\frac{1.65 \frac{k^2}{\lambda} - \alpha + 1}{2}\right) = \left(k - 1 - \frac{2\alpha}{1.65 \frac{k^2}{\lambda} - \alpha + 1}\right)^2 - \frac{1.65 \frac{k^2}{\lambda} + \alpha + 1}{2}
\geq \left(k - 1 - \frac{2}{1.65 \frac{k^2}{\lambda} - \frac{k^2}{6\lambda}}\right)^2 - \frac{1.65 \frac{k^2}{\lambda} + \frac{k^2}{6\lambda}}{2}
\geq \left(k - 1 - \frac{2}{6 \cdot 1.65 - 1}\right)^2 - \frac{k}{2} - \frac{6 \cdot 1.65 + 1}{12} k^2,
\]

where this first inequality used once again that this expression is decreasing in \( \alpha \). This last expression is nonnegative whenever \( k \geq 32 \). \( \square \)

We note that one can improve upon the upper bound of \( v \) in the hypothesis of Theorem 5.8 at the cost of increasing the value of \( k \) for which it holds.

6 | APPLICATION TO PARTICULAR CLASSES OF DESIGNS

6.1 | Projective point-subspace designs

In this section we will discuss the edge domination number of the design of points and \( k \)-spaces in the projective space \( \text{PG}(n, q) \), with \( 0 < k < n \). By \( k \)-spaces we refer to subspaces of projective dimension \( k \). The 0-, 1-, and \( (n - 1) \)-spaces are referred to as points, lines and hyperplanes. This design is symmetric only when \( kn = -1 \). Denote the number of points in \( \text{PG}(n, q) \) by

\[
\theta_n = \frac{q^{n+1} - 1}{q - 1}.
\]

The smallest case is the design of points and lines in the Desarguesian projective plane. In this setting, we can construct large incidence-free sets from polarities.

Assign coordinates \((x_0, ..., x_n) \in F_q^{n+1}\) to the points in \( \text{PG}(n, q) \), where \((x_0, ..., x_n)\) and \((y_0, ..., y_n)\) are considered as the same point if they are scalar multiples of each other. For each projective point \( P = (a_0, ..., a_n) \), let \( P^\perp \) denote the hyperplane with equation \( a_0X_0 + \cdots + a_nX_n = 0 \). Then mapping a point \( P \) to the hyperplane \( P^\perp \) and vice versa, determines a polarity \( \perp \) of \( \text{PG}(n, q) \). When \( n = 2 \), the following lower bounds on the independence number of the polarity graph \( R(D, \perp) \) are known.

**Theorem 6.1** (Mattheus et al. [22] and Mubayi and Williford [23]). Let \( D \) be the design of points and lines in \( \text{PG}(2, q), q = p^h \) where \( p \) prime and \( h \geq 1 \), and let \( \perp \) be the polarity described above. Then the polarity graph \( R(D, \perp) \) has a coclique of size at least \( cq\sqrt{q} - O(q) \), where
Corollary 6.2. Let $D$ be the design of points and lines in $\text{PG}(2, q)$, $q \geq 37$. Then
\[
q^2 - q\sqrt{q} - O(q) \leq \gamma_e(D) \leq q^2 - cq\sqrt{q} - O(q),
\]
where $c$ is defined as in Theorem 6.1.

Proof. The lower bound follows from Corollary 3.6. The upper bound follows from Lemma 4.3 and Theorem 6.1, giving us the incidence-free pair, and Theorem 1.3 turning it into an edge dominating set of the wanted size. \qed

Going via the polarity graph is actually a detour when $p = 2$. It is known that $\text{PG}(2, q)$ has a maximal arc of order $n$ if and only if $q = 2^h$ and $n$ divides $q$ [1, 7]. So starting from a maximal arc of order $\sqrt{q}$ or $\sqrt{q}/2$, depending on the parity of $h$, we can combine our earlier observations on arcs and the corresponding dual arcs with Lemma 2.2 to improve the result in Corollary 6.2 for even $q$. The improvement only lies in the omission of the restriction on $q$, the actual value of $\gamma_e(D)$ will be the same. We state the sharp result for even $h$ for completeness.

Corollary 6.3. For all $q = 2^{2h}$ we have
\[
\gamma_e(D) = q^2 - q\sqrt{q} + \sqrt{q} + 1.
\]

A similar result holds in the more general design of points and hyperplanes in $\text{PG}(n, q)$, $n \geq 2$. A large incidence-free set was found by De Winter, Schillewaert and Verstraëte [6, Section 4.2].

Theorem 6.4. There exists an equinumerous incidence-free pair $(X, Y)$ of sets of points and hyperplanes, respectively, in $\text{PG}(n, q)$ of size $\Theta(q^{n+1})$.

Corollary 6.5. Let $D$ be the design of points and hyperplanes in $\text{PG}(n, q)$, $n \geq 2$. Then
\[
\gamma_e(D) = \theta_n - \Theta_q\left(q^{n+1}\right).
\]

Proof: This follows from Corollary 3.6 and the preceding result combined with Theorem 1.3. \qed

In other projective point-subspace designs, it is not difficult to determine the edge domination number exactly.
**Proposition 6.6.** Let $D$ be the design of points and $k$-spaces in $PG(n, q)$, $1 \leq k < n - 1$, and $n \geq 3$. Then

$$\gamma_e(D) = \begin{cases} \vartheta_n - q & \text{if } k = 1, \\ \vartheta_n & \text{otherwise.} \end{cases}$$

**Proof.** If $1 < k < n - 1$, then the number of $k$-spaces each point is incident to equals

$$r = \prod_{i=1}^{k} \frac{q^{n-k+i} - 1}{q^i - 1} > \frac{q^{n+1} - 1}{q - 1} = v.$$

By Lemma 2.3, $\gamma_e(D) = v = \vartheta_n$.

Now suppose $k = 1$. First we show that $\gamma_e(D) \geq \vartheta_n - q$. One can see that

$$\vartheta_n - (q + 1) < (q + 1)\vartheta_{n-1} - \binom{q + 1}{2}$$

if $n \geq 3$. Therefore, by Lemma 3.4 every set $S$ of $q + 1$ points is incident with a set of lines $T$, where $|T| > \vartheta_n - (q + 1)$. Now suppose for the sake of contradiction that we have an edge dominating set $\Gamma$ with $|\Gamma| \leq \vartheta_n - (q + 1)$. Then there are at least $q + 1$ points not covered by an edge of $\Gamma$, so that the lines with which they are incident should all be covered by an edge. However, the number of such lines is more than $\vartheta_n - (q + 1) = |\Gamma|$, so this is impossible.

We can describe a construction that yields edge dominating sets of $I(D)$ of size $\vartheta_n - q$. Consider an incident point-line pair $(P, \ell)$. Let $S$ denote the points not on $\ell$ and $T$ denote the set of lines intersecting $\ell \setminus \{P\}$ in a point (so not $\ell$ itself). Then $B = |\Gamma| = \vartheta_n - (q + 1)$. The subgraph of $I(D)$ induced on $(S, T)$ is $q$-regular and hence has a perfect matching due to Lemma 2.2. By adding the edge $(P, \ell)$, we find a maximal matching in $I(D)$ of size $\vartheta_n - q$. $\square$

### 6.2 Symmetric designs from Hadamard matrices

A **Hadamard matrix** is an $n \times n$-matrix $M$ with only 1 and $-1$ as entries such that $M'M = nI_n$. We refer the reader to [18, Chap. 4] for a treatise of the subject. There are several constructions of symmetric designs from Hadamard matrices.

#### 6.2.1 Hadamard designs and Paley matrices

Let $\mathbf{1}$ denote the all-one column vector. If a $4u \times 4u$ Hadamard matrix $M$ is normalised, that is, of the form

$$M = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & N \end{pmatrix},$$
then we can replace every $-1$ in $N$ with a 0. Equivalently, if we define $J$ to be the all-one matrix of the appropriate size (which will be clear from context), then we consider the matrix $\frac{1}{2}(N + J)$. This yields the incidence matrix of a symmetric $2 - (4u - 1, 2u - 1, u - 1)$ design. Such a design is called a Hadamard design. The next bound follows directly from Corollary 3.6(2).

**Lemma 6.7.** Let $D$ be a $(4u - 1, 2u - 1, u - 1)$ Hadamard design. Then

$$\gamma_e(D) \geq 4u - 2\sqrt{u} - \frac{1}{\sqrt{u} + 1}.$$  

We consider the following construction due to Paley [25] of a Hadamard matrix. Consider the field $\mathbb{F}_q$ with $q$ odd. Define the quadratic character of $\mathbb{F}_q$ as

$$\chi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \text{ is a nonzero square}, \\ -1 & \text{if } x \text{ is not a square}. \end{cases}$$

Let $Q$ denote the Jacobstahl matrix of $\mathbb{F}_q$, that is, the rows and columns of $Q$ are indexed by the elements of $\mathbb{F}_q$, and $Q_{x,y} = \chi(x - y)$.

Suppose that $q \equiv 1 \pmod{4}$. Then the matrix

$$M = \begin{pmatrix} 0 & I \\ I & Q \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + I \otimes \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

is a Hadamard matrix, where $\otimes$ denotes the tensor product. We can interpret this matrix as follows. Say that the first row and column of $\begin{pmatrix} 0 & I \\ I & Q \end{pmatrix}$ are indexed by $\infty$. The other rows and columns are naturally indexed by $\mathbb{F}_q$. Say that the rows and columns of the $2 \times 2$-matrices are indexed by 1 and $-1$, in that order. Define $\infty - \infty = 0$, $\infty - x = x - \infty = \infty$ for all $x \in \mathbb{F}_q$, and $\chi(\infty) = 1$. Then

$$M((x, i), (y, j)) = \begin{cases} 1 & \text{if } x = y \text{ and } (i, j) = (1, 1), \\ -1 & \text{if } x = y \text{ and } (i, j) \neq (1, 1), \\ \chi(x - y) & \text{if } x \neq y \text{ and } (i, j) = (-1, -1), \\ -\chi(x - y) & \text{if } x \neq y \text{ and } (i, j) = (-1, 1). \end{cases}$$

If we scale the second row and the second column with a factor $-1$, then we obtain a Hadamard matrix of the form $\begin{pmatrix} 1 & I \\ I & N \end{pmatrix}$, and therefore $\frac{1}{2}(N + J)$ is the incidence matrix of a Hadamard design. We denote this design as $HD(q)$. It is a $(2q + 1, q, \frac{q-1}{2})$ design. The point set is $(\mathbb{F}_q \times \{1, -1\}) \cup \{\infty\}$. For each point $P$ there is a block $B_P$, with
\[ B_{\infty} = \mathbb{F}_q \times \{ -1 \}, \]
\[ B_{(x,1)} = \{(y, i) \mid \chi'(x - y) = 1, i = \pm 1\} \cup \{(x, 1)\}, \]
\[ B_{(x,-1)} = \{(y, i) \mid \chi'(y - x) = i\} \cup \{\infty\}. \]

**Lemma 6.8.** Let \( C \) be a clique in the Paley graph of order \( q \geq 37 \). Then \( I(HD(q)) \) has an edge dominating set of size \( 2q + 1 - |C| \).

**Proof.** Consider the map \( \rho \) which maps a point \( P \) to the block \( B_P \), and vice versa the block \( B_P \) to the point \( P \). As \( N \) is symmetric, for any two points \( P \) and \( Q \), it holds that \( P \in Q^\rho \) if and only if \( Q \in P^\rho \). In other words, \( \rho \) is a polarity of the design. The cocliques in the polarity graph of \( HD(q) \) are exactly the sets \( K \times \{-1\} \), with \( K \) a clique in the Paley graph of order \( q \), and \( \{\infty\} \).

Hence, if \( C \) is a clique in the polarity graph, then the statement of the lemma follows from Theorem 1.3 and Lemma 4.3. \( \square \)

If \( q \) is a square, then it is known that \( \mathbb{F}_{\sqrt{q}} \) is a clique of maximal size in the Paley graph of order \( q \). We thus obtain the following corollary.

**Corollary 6.9.** If \( q \geq 49 \) is a square, then \( \gamma_c(HD(q)) = 2q - \Theta(\sqrt{q}) \).

### 6.2.2 Menon designs and symmetric Bush-type Hadamard matrices

If a \( 4u \times 4u \) Hadamard matrix \( M \) has constant row sum, then it is called regular. In that case \( u \) must be square, say \( u = h^2 \), and \( \frac{1}{2}(J + M) \) and \( \frac{1}{2}(J - M) \) are incidence matrices of symmetric designs, called *Menon designs*. These designs have parameters \( 2(4h^2, 2h^2 + \varepsilon h, h^2 + \varepsilon h) \) with \( \varepsilon = \pm 1 \). By Lemma 3.3(2), the edge domination number of such a design is at least \( 4h^2 - 2h \) if \( \varepsilon = -1 \), and \( 4h^2 - 2h + 2 - \frac{2}{h+1} \) if \( \varepsilon = 1 \).

A regular \( 4h^2 \times 4h^2 \) Hadamard matrix \( M \) is of *Bush type* if it is a block matrix where all the blocks are \( 2h \times 2h \)-matrices, all the diagonal blocks are all-one matrices, and all the other blocks have constant row and column sum 0. If such a matrix \( M \) is symmetric, then \( \frac{1}{2}(J - M) \) is the incidence matrix of Menon design with \( \varepsilon = -1 \), which has a polarity without absolute points, and where the vertices of the polarity graph can be partitioned into cocliques of size \( 2h \). Note that each such coclique consists of \( 2h \) points, with \( 2h \) blocks not incident with these points (namely, their image under the polarity).

**Proposition 6.10.** If \( D \) is a Menon design associated with a symmetric Hadamard matrix of Bush type, it is a \((4h^2, 2h^2 - h, h^2 - h)\)-design and \( \gamma_c(D) = 4h^2 - 2h \).

**Proof.** Applying Corollary 3.6(2) to this set of parameters gives the lower bound, while the incidence-free set of size \( 2h \) described above provides the upper bound. We note that this set is a maximal arc by Proposition 3.5(2) and so indeed gives rise to an edge dominating set as seen before. \( \square \)
There are constructions of infinite families of symmetric Bush-type Hadamard matrices. One such family can be found in [24]. Another interesting family is the symplectic symmetric designs, which Kantor [19] proved to be one of the only two infinite families of 2-transitive symmetric designs with \( v > 2k \).

### 6.3 Symmetric designs admitting polarities without absolute points

We call a (loopless, undirected) graph \( G \) strongly regular with parameters \( (v, k, \lambda, \mu) \) and denote it as an SRG\((v, k, \lambda, \mu)\) if it is a \( k \)-regular graph on \( v \) vertices where any two distinct vertices share \( \lambda \) (resp., \( \mu \)) neighbours if they are adjacent (resp., not adjacent).

Of particular interest to us are \( \text{SRG}(v, k, \lambda, \lambda) \). These are exactly the polarity graphs of symmetric \((v, k, \lambda)\) designs admitting a polarity without absolute points. Therefore, any \( \text{SRG}(v, k, \lambda, \lambda) \) admitting a coclique of size \( k \frac{\sqrt{k-\lambda} - 1}{\lambda} + 1 \) gives rise to a symmetric design attaining equality in the bound of Corollary 3.6(2).

#### 6.3.1 Hyperbolic hyperplanes of a parabolic quadric

As a first example, we consider a graph whose description can be found in [2, pp. 82–83]. Let \( \mathcal{Q}(2n, q) \) be a nondegenerate parabolic quadric in \( \text{PG}(2n, q) \). Construct a graph \( G \) whose vertices are the hyperplanes of \( \text{PG}(2n, q) \) intersecting \( \mathcal{Q}(2n, q) \) in a hyperbolic quadric, and let two distinct hyperplanes \( \pi \) and \( \rho \) be adjacent if \( \pi \cap \rho \cap \mathcal{Q}(2n, q) \) is a degenerate quadric. As stated in [2, p. 83], this graph admits cliques of size \( q^n \). If \( q = 3 \), the complement \( \overline{G} \) of \( G \) is an \( \text{SRG}(v, k, \lambda, \lambda) \) with parameters

\[
\begin{align*}
v &= 3^n \frac{3^n + 1}{2}, \\
k &= 3^{n-1} \frac{3^n - 1}{2}, \\
\lambda &= 3^{n-1} \frac{3^{n-1} - 1}{2}.
\end{align*}
\]

Since cliques in \( G \) become cocliques in \( \overline{G} \), \( \overline{G} \) has cocliques of size \( 3^n \). Note that \( k \frac{\sqrt{k-\lambda} - 1}{\lambda} + 1 = 3^n \). Thus, if \( D \) denotes the symmetric \((v, k, \lambda)\) design associated to \( \overline{G} \), then

\[
\gamma_e(D) = v - 3^n = 3^n \frac{3^n - 1}{2}.
\]

#### 6.3.2 Nets

**Definition 6.11.** Let \( D = (\mathcal{P}, \mathcal{B}) \) be an incidence structure. Two points \( P \) and \( R \) are said to be collinear if there exists a block incident with both \( P \) and \( R \). The collinearity graph of \( D \) is the graph whose vertices are the points \( \mathcal{P} \), and where two distinct points \( P \) and \( R \) are adjacent if there exists a block incident with both \( P \) and \( R \).

A net of type \((q, r)\) is an incidence structure \( D = (\mathcal{P}, \mathcal{B}) \) of type \((q^2, rq, q, r, 1)\), where \( \mathcal{B} \) can be partitioned into \( r \) sets \( \mathcal{B}_1, ..., \mathcal{B}_r \), each of size \( q \), such that each \( \mathcal{B}_i \) partitions the points. Nets are equivalent to other combinatorial objects, such as mutually orthogonal Latin squares,
transversal designs and orthogonal arrays. The collinearity graph \( G \) of the net \( D \) is an SRG\((q^2, r(q - 1), q - 2 + (r - 1)(r - 2), r(r - 1))\), see, for example, [2, p. 193]. If \( r = q/2 \), then we obtain an SRG\((v, k, \lambda, \lambda)\) with parameters

\[ v = q^2, \quad k = q\frac{q - 1}{2}, \quad \lambda = q\frac{q - 2}{4}. \]

In that case, \( k\frac{\sqrt{k - \lambda - 1}}{\lambda} + 1 = q \). Does \( G \) admit cocliques of size \( q \)? Suppose that \( D \) can be extended to a net \( D_2 \) of type \( (q, \frac{q}{2} + 1) \). Then we can add \( q \) extra blocks \( B_1, ..., B_q \), each intersecting all the previous blocks in exactly one point. In particular, every new block contains \( q \) points which were not collinear in \( D \). This gives rise to the desired cocliques in \( G \). Hence, every net of type \( (q, r) \) with \( q \) even and \( r > q/2 \) gives rise to (possibly many nonisomorphics) symmetric \((v, k, \lambda)\) designs \( D' \), with \( v, k, \lambda \) as above, and with

\[ \gamma_e(D') = q^2 - q. \]

Since an affine plane of order \( q \) is a net of type \((q, q + 1)\), every affine plane of even order gives rise to such symmetric designs.

6.3.3  | Generalised quadrangles

**Definition 6.12.** A generalised quadrangle of order \((s, t)\) is an incidence structure satisfying the following conditions.

- Every block is incident with \( s + 1 \) points.
- Every point is incident with \( t + 1 \) blocks.
- Given a point \( P \) and a block \( B \not\ni P, B \) contains a unique point collinear with \( P \).

We denote such an incidence structure as a \( GQ(s, t) \).

For more background on generalised quadrangles, we refer to the monograph [26]. Analogous to the collinearity graph of a \( GQ(s, t) \), we can define its block graph (which is just the collinearity graph of its dual).

**Definition 6.13.** Let \( D = (\mathcal{P}, \mathcal{B}) \) be an incidence structure. The block graph of \( D \) is the graph which has the blocks \( \mathcal{B} \) as vertices, and where two distinct blocks are adjacent if there exists a point incident with both of the blocks.

The collinearity and block graphs of a \( GQ(s, t) \) are always strongly regular. If in addition \( s = q - 1 \) and \( t = q + 1 \) for some integer \( q \), then the block graph of the \( GQ(q - 1, q + 1) \) is an SRG\((v, k, \lambda, \lambda)\) with

\[ v = (q + 2)q^2, \quad k = q(q + 1), \quad \lambda = q. \]
In this case,

\[ k \sqrt{\frac{k - \lambda}{\lambda}} - 1 + 1 = q^2. \]

A coclique of this size is equivalent to a spread in the \( GQ(q - 1, q + 1) \), that is, a subset \( S \subseteq B \) of blocks such that every point is incident with a unique block of \( S \). There are several families of \( GQ(q - 1, q + 1) \) admitting spreads. These include the Tits \( T^*_2(\mathcal{O}) \) generalised quadrangles (see, e.g., [26, Section 3.1.3]), the Ahrens–Szekeres generalised quadrangle (see, e.g., [26, Section 3.1.5]), and certain generalised quadrangles of order \( (q - 1, q + 1) \) derived from a \( GQ(q, q) \) (see, e.g., [26, Section 3.1.4]).

These \( GQ(q - 1, q + 1) \)'s give rise to symmetric \( ((q + 2)q^2, q(q + 1), q) \) designs \( D \) satisfying

\[ \gamma_c(D) = (q + 2)q^2 - q^2 = (q + 1)q^2. \]

### 6.4 | Semibiplanes

There are only a very limited number of explicit symmetric designs with \( \lambda = 2 \). These designs are known as biplanes. However, we can relax the definition to obtain the so-called semibiplanes, of which there are several known infinite families. They were introduced by Hughes [17].

**Definition 6.14.** A semibiplane is a symmetric incidence structure \( D = (\mathcal{P}, \mathcal{B}) \) such that

1. for every two distinct points \( P \) and \( Q \), there are either 0 or 2 blocks incident with \( P \) and \( Q \),
2. for every two distinct blocks \( B \) and \( C \), \( |B \cap C| \) equals either 0 or 2,
3. the incidence graph \( I(D) \) is connected.

Given a semibiplane, there exist numbers \( v \) and \( k \) such that \( |\mathcal{P}| = |\mathcal{B}| = v \), and every point or block is incident with exactly \( k \) blocks or points, respectively. The parameters \( (v, k) \) are called the order of the semibiplane, and a semibiplane of order \( (v, k) \) is denoted as \( \text{sbp}(v, k) \).

The collinearity graph of a \( \text{sbp}(v, k) \) is regular of degree \( \binom{k}{2} \). If we denote by \( N \) the point-block incidence matrix of an \( \text{sbp}(v, k) \), and by \( A \) the adjacency matrix of its collinearity graph, then

\[ NN^t = kI_v + 2A. \]

### 6.4.1 | Divisible semibiplanes

We call a semibiplane \( D = (\mathcal{P}, \mathcal{B}) \) divisible if its collinearity graph is a complete multipartite graph. Another way to formulate is that the relation \( \sim \) defined on \( \mathcal{P} \) by
is an equivalence relation. It is not difficult to show that every equivalence class contains the same number of elements, say \( d \), and that \( v = \binom{k}{2} + d \).

We would like to give a lower bound on the edge domination number of the incidence graph of a divisible semibiplane. As before, such a bound follows from an upper bound on the size of an equinumerous incidence-free pair. One way to bound this size is by using Lemma 3.3. Another way is by using an eigenvalue bound from spectral graph theory. For designs, this yields the same bound, but for divisible semibiplanes, this is no longer the case. Both approaches yield lower bounds on \( \gamma_e(D) \) that look roughly like \( v - \frac{k\sqrt{k}}{2} \), but the spectral approach yields a significantly simpler expression.

**Lemma 6.15** (Haemers [11, Theorem 5.1, Expander mixing lemma]). Let \( G \) be a \( k \)-regular bipartite graph, with bipartition \( L \) and \( R \). Take sets \( S \subseteq L \) and \( T \subseteq R \). Suppose that the second largest eigenvalue of the adjacency matrix of \( G \) is \( \lambda_2 \). Then

\[
\left( e(S, T) - \frac{k}{|R|} |S||T| \right)^2 \leq \lambda_2^2 |S||T| \left( 1 - \frac{|S|}{|L|} \right) \left( 1 - \frac{|T|}{|R|} \right).
\]

**Proposition 6.16.** Let \( D \) denote an SIS of type \((v, k, \lambda)\). Let \( \lambda_2 \) denote the second largest eigenvalue of \( I(D) \). Then

\[
\gamma_e(D) \geq \frac{k}{\lambda_2 + k} v.
\]

**Proof.** Take an edge dominating set of size \( \gamma := \gamma_e(D) \). Then by Lemma 2.1 there exist sets \( S \) and \( T \) of blocks and points, respectively, of size \( v - \gamma \) with no incidences between them. Apply the expander mixing lemma in \( I(D) \) to \( S \) and \( T \). This tells us that

\[
\left( \frac{k}{v} (v - \gamma)^2 \right)^2 \leq \lambda_2^2 (v - \gamma)^2 \left( \frac{\gamma}{v} \right)^2.
\]

Taking square roots, this reduces to

\[
k(v - \gamma) \leq \lambda_2 \gamma,
\]

which gives the desired equality. \( \square \)

**Remark 6.17.** Observe that if \( N \) is the incidence matrix of an SIS of type \((v, k, \lambda)\), then \( vk = \text{Tr}(NN^t) \leq k^2 + (v - 1)\lambda_2 \). This shows that \( \lambda_2 = \Omega(\sqrt{k}) \). On the other hand, we can have \( \lambda_2 = k \) when the SIS is the union of two disjoint symmetric designs. This shows that the maximum size of an equinumerous incidence-free pair could lie anywhere between \( O(v/\sqrt{k}) \) and \( v/2 \). Even when we restrict ourselves to SISs with connected incidence graph, the size of such a pair can get close to \( v/2 \), see Remark 6.27. For the symmetric designs we have seen before, we observe that the truth is closer \( O(v/\sqrt{k}) \). This is again
related to the fact that incidence graphs of symmetric designs are “expanding”, this time expressed by the fact that $\lambda_2$ is small compared to $k$.

**Lemma 6.18.** Let $D$ be a divisible sbp$(v, k)$ and write $d = v - \binom{k}{2}$. Then the spectrum of the adjacency matrix of $I(D)$ equals

$$\left\{ \pm k, \pm\sqrt{k} \left( \frac{v}{d-1} \right), \pm \sqrt{k - 2d} \left( \frac{v}{d-1} \right) \right\}.$$ 

**Proof.** Let $N$ denote the incidence matrix of $D$. It suffices to compute the spectrum of $NN^t$. The adjacency matrix of the collinearity graph of $D$ equals $A = (J - I)_{\frac{d}{2}} \otimes J_d$. Since $(J - I)_{\frac{d}{2}}$ has spectrum $(\frac{d}{2} - 1)(1), -1(\frac{d}{2} - 1)$, and $J_d$ has spectrum $d(1), 0(d-1)$, the spectrum of $NN^t = kI + 2A$ equals

$$\left\{ k^2, k \left( \frac{v}{d-1} \right), (k - 2d) \left( \frac{v}{d-1} \right) \right\}.$$ 

It is well known that if $N$ is a square matrix, then $\lambda$ is an eigenvalue of $NN^t$ with multiplicity $m$ if and only if $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ are eigenvalues of $\begin{pmatrix} O & N \\ N^t & O \end{pmatrix}$, both with multiplicity $m$. □

**Remark 6.19.** The lemma implies that $d \leq k/2$ and thus $v \leq k^2/2$. This also has an easy combinatorial proof, see Wild [31, Result 2]. In particular this implies that the condition $v \leq 1.65 \frac{k^2}{\lambda}$ from Theorem 5.8 holds in any divisible semiplane.

**Corollary 6.20.** Let $D$ be a divisible sbp$(v, k)$. Then

$$\gamma_c(D) \geq v - \frac{v}{\sqrt{k} + 1}.$$ 

**Proof.** This follows directly from Lemma 6.18 and Proposition 6.16. □

### 6.4.2 | Divisible semiplanes from projective planes

A classical construction of a divisible sbp$(v, k)$ is given by Hughes, Leonard and Wilson [17]. Take an involution $\varphi$ of $PG(2, q)$, that is, an incidence-preserving map of order two. Then either $\varphi$ is a perspectivity, which means it fixes a point, called the centre of $\varphi$, and fixes a line pointwise, called the axis of $\varphi$; or $\varphi$ is a Baer involution, that is, it fixes a Baer subplane. Take as point set the points $P$ of $qPG(2, q)$ not fixed by $\varphi$, where we consider $P$ and $P^\varphi$ as equivalent, that is, $\mathcal{P} = \{|P, P^\varphi| \parallel P \in \mathcal{L}, P \neq P^\varphi \}$. For each line $l$ such that $l^\varphi \neq l$ define the block $B_l = \{|P, P^\varphi| \parallel P \in l, P \neq P^\varphi \}$. Then $B_l = B_l^\varphi$. The biplane is given by $(\mathcal{P}, B)$ with $B = \{B_l \parallel l \neq l^\varphi \}$. An easy counting argument shows that a perspectivity of order 2 in
PG(2, q) is an elation (i.e., centre and axis are incident) if q is even, and a homology (i.e., the centre and axis are not incident) if q is odd. It is not difficult to verify that two perspectivities of order 2 in PG(2, q) must be projectively equivalent, and all Baer involutions are projectively equivalent as well. This allows us to give a more concrete description of the semibiplanes described above. For more details on involutions of projective planes, we refer to [5, p. 30].

Case 1. q is even, \( \varphi \) is an elation.

Then we can give PG(2, q) coordinates such that \( \varphi : (x_0, x_1, x_2) \mapsto (x_0, x_1, x_2 + x_0) \). The axis of \( \varphi \) is the line \( X_0 = 0 \), and the centre is \( (0, 0, 1) \). Every point not on the axis has a unique representation \( (1, x_1, x_2) \). We denote this point as \( (x_1, x_2) \). Every line not through the centre has a unique equation of the form \( X_2 = mx_1 + b \). We denote this line as \( l_{m,b} \). Then \( (x_1, x_2)^P = (x_1, x_2 + 1) \) and \( l_{m,b}^P = l_{m,b+1} \). If we identify each point and line with its image under \( \varphi \), then \( (x_1, x_2) \) and \( l_{m,b} \) are incident if and only if \( x_2 + mx_1 + b \in \{0, 1\} \).

Inspired by Mubayi and Williford [23], we give the following construction of a small edge dominating set.

**Proposition 6.21.** Let \( q \geq 64 \) be an even prime power, and let \( D \) be the sbp(\( q^2/2 \), q) arising from an elation in PG(2, q). Then

\[
\frac{q^2}{2} - \frac{q^2}{2(\sqrt{q} + 1)} \leq \gamma_e(D) \leq \begin{cases} 
\frac{q^2}{2} - \frac{q\sqrt{q}}{4} & \text{if } q \text{ is a square,} \\
\frac{q^2}{2} - \frac{q\sqrt{q}}{4\sqrt{2}} & \text{otherwise.}
\end{cases}
\]

**Proof.** The lower bound follows from Corollary 6.20, so we focus on the upper bound.

Use the notation for points and lines of this semibiplane as described above. Suppose that \( q = 2^h \). Let \( \omega \) be a primitive element of \( F_q \). Then \( 1, \omega, \ldots, \omega^{h-1} \) is an \( F_2 \)-basis of \( F_q \). For each \( x \in F_q \), let \( (x^{(0)}, \ldots, x^{(h-1)}) \) denote its coordinate vector with respect to this basis, that is, \( x = \sum_{i=0}^{h-1} x^{(i)} \omega^i \). Define \( f = \lfloor \frac{h}{2} \rfloor - 1 \), and let \( F \) denote \( F_2 \)-span of \( 1, \omega, \ldots, \omega^f \). If we take \( x_1 \) and \( x_2 \) in \( F \), then \( (x_1, x_2)^{(h-1)} = 0 \), since \( 2f < h - 1 \). Let \( X \) denote the set of all points \( (x_1, x_2) \) with \( x_1 \in F \) and \( x_2^{(h-1)} = 0 \). Let \( Y \) denote the set of all blocks \( l_{m,b} \) with \( m \in F \) and \( b^{(h-1)} = 1 \). Then \( (x_2 + mx_1 + b)^{(h-1)} = 1 \). In particular, \( x_2 + mx_1 + b \not\in \{0, 1\} \), which implies that no point of \( X \) is incident with a block of \( Y \).

Since each point has two coordinate representatives, and likewise for the blocks, we find an equinumerous incidence-free pair of size

\[
|X| = |Y| = \frac{|F|^2 q}{2} = \frac{2^{f+1} q}{4} = \frac{2^{h+1} q}{4} = \begin{cases} 
\frac{q\sqrt{q}}{4} & \text{if } h \text{ is even,} \\
\frac{q\sqrt{q}}{4\sqrt{2}} & \text{if } h \text{ is odd,}
\end{cases}
\]

Thus, all that is left to show that \((X, Y)\) satisfies the conditions of Theorem 5.8 with \( v = q^2/2, k = q \) and \( \alpha = |X| \) as above.

First off, we have \( q \geq 64 \), which is larger than 36. Secondly, we have indeed that \( |X| \leq q^2/12 \) for \( q \geq 64 \) as \( |X| \leq q\sqrt{q}/4 \). Finally, take a point \((x_1, x_2)\). For every value of \( m \), there is a unique \( b \) with \((x_1, x_2) \in l_{m,b}, \) hence \((x_1, x_2)\) lies on at most \(|F| \leq \sqrt{q}\) lines in \( Y \). Equivalently, \((x_1, x_2)\) has at least \( q - \sqrt{q} \geq 2|X|/k \) neighbours in \( B \setminus Y \).
Case 2. $q$ is odd, $\varphi$ is a homology.

We can choose coordinates such that $\varphi : (x_0, x_1, x_2) \mapsto (-x_0, x_1, x_2)$. The axis of $\varphi$ is the line $X_0 = 0$, the centre of $\varphi$ is $(1, 0, 0)$. Similar to the previous case, we represent each point of the semiplane as $(x_1, x_2) \neq (0, 0)$, where $(x_1, x_2)$ and $(-x_1, -x_2)$ are considered to be the same point. Every line in $\text{PG}(2, q)$ distinct from $X_0 = 0$ that misses $(1, 0, 0)$ has a unique equation of the form $X_0 = aX_1 + bX_2$, $(a, b) \neq (0, 0)$. Denote this line as $l_{a,b}$. Then $l_{a,b} = l_{-a,-b}$. The lines in the biplane are of the form $l_{a,b}$ with $(x_1, x_2) \in l_{a,b}$ if and only if $ax_1 + bx_2 = \pm 1$.

**Proposition 6.22.** Let $q \geq 7$ be odd, and let $D$ be the sbp($(q^4 - 1)/2, q^2$) arising from a homology in $\text{PG}(2, q^2)$. Then

$$\frac{q^4 - 1}{2} - \frac{q^4 - 1}{2(q + 1)} \leq \gamma_e(D) \leq \frac{q^4 - 1}{2} - \frac{q^4 - 1}{4}.$$

**Proof.** The lower bound again follows from Corollary 6.20.

Let $\zeta$ denote a primitive $(q + 1)$st root of unity in $\mathbb{F}_q$. Consider the sets

$$X = \left\{(x_1, x_2) \mid x_1 \in \mathbb{F}_q, x_2^{q-1} \in \{\zeta^0, \ldots, \zeta^{(q-1)/2}\}\right\},$$

$$Y = \{l_{a,b} \mid a \in \mathbb{F}_q, b^{q-1} \in \{\zeta^1, \ldots, \zeta^{(q+1)/2}\}\}.$$

If you take a point $(x_1, x_2) \in X$ and a line $l_{a,b} \in Y$, then $ax_1 \in \mathbb{F}_q$ and $(bx_2)^{q-1} \in \{\zeta^1, \ldots, \zeta^q\}$. In particular, $(bx_2)^{q-1} \notin \{0, 1\}$, which implies that $bx_2 \notin \mathbb{F}_q$. Therefore, $ax_1 + bx_2 \notin \mathbb{F}_q$, hence $ax_1 + bx_2 \neq \pm 1$. Thus, $(X, Y)$ is an equinumerous incidence-free pair.

To calculate the size of $X$, note that there are $q$ choices for $x_1$, $\frac{q+1}{2}$ choices for $x_2^{q-1}$, so $\frac{(q-1)(q+1)}{2}$ choices for $x_2$. Since $(x_1, x_2)$ and $(-x_1, -x_2)$ are the same point and either both or neither are in $X$, we conclude that $|X| = |Y| = \frac{q(q^2 - 1)}{4}$.

To finish the proof, we check that the conditions of Theorem 5.8 are met. Using that $q \geq 7$, the only nontrivial condition to check is that every point outside of $X$ is incident with at least $2q\frac{q^2 - 1}{4} = \frac{q^2 - 1}{2q}$ lines outside of $Y$ and vice versa. By the symmetry of the situation, we only check the condition for points outside of $X$. So take a point $(x_1, x_2) \notin X$.

First suppose that $x_2 \neq 0$. Then for every $a \notin \mathbb{F}_q$, the lines $l_{a,\frac{x_1-a}{x_2}}$ are lines outside of $Y$ incident with $(x_1, x_2)$. So $(x_1, x_2)$ is incident with at least $q^2 - q > \frac{q^2 - 1}{2q}$ lines outside of $Y$.

Now suppose that $x_2 = 0$. Then $x_1 \neq 0$, and every line $l_{\frac{x_1}{x_2},b}$ is incident with $(x_1, x_2)$. There are $q^2 - \frac{q^2 - 1}{2} = \frac{q^2 + 1}{2}$ values of $b$ such that $b^{q-1} \notin \{\zeta^1, \ldots, \zeta^{(q+1)/2}\}$, so $(x_1, x_2)$ is incident with at least $\frac{q^2 + 1}{2} \geq \frac{q^2 - 1}{2q}$ lines outside of $Y$.

**Remark 6.23.** The semiplanes with $\varphi$ a homology and $q$ an odd nonsquare prime power are left untreated. If $\mathbb{F}_q$ has a large subfield, a similar construction yields a fairly large incidence-free pair. However, if $\mathbb{F}_q$ does not have a large subfield, one would need some new ideas.
Case 3. \( \varphi \) is a Baer involution in \( \text{PG}(2, q^2) \).

After choosing appropriate coordinates, \( \varphi : (x_0, x_1, x_2) \mapsto (x_0^q, x_1^q, x_2^q) \). The semiplane then consists of the points \((x_0, x_1, x_2)\) not fixed by \( \varphi \), which we identify with \( (x_0^q, x_1^q, x_2^q) \), and the blocks \( l_{a,b,c} \) with \( (a, b, c) \neq (a^q, b^q, c^q) \) containing the points \((x_0, x_1, x_2)\) satisfying \( ax_0 + bx_1 + cx_2 = 0 \) or \( ax_0^q + bx_1^q + cx_2^q = 0 \). Note that \( l_{a,b,c} = l_{a^q,b^q,c^q} \).

**Proposition 6.24.** Let \( q \geq 7 \) be a prime power, and let \( D \) denote the sbp \( \text{sbp}(q(q^3 - 1)/2, q^2) \) arising from a Baer involution in \( \text{PG}(2, q^2) \). Then

\[
\frac{q^4 - q}{2} - \frac{q^4 - q}{2(q + 1)} \leq \chi(D) \leq \begin{cases} 
\frac{q^4 - 1}{2} - \frac{q^4 - 2q^2 - 1}{2} & \text{if } q \equiv 3 \pmod{4}, \\
\frac{q^4 - 1}{2} - (q^2 - 1)\left\lfloor \frac{q}{4} \right\rfloor & \text{otherwise}.
\end{cases}
\]

**Proof.** The lower bound follows from Corollary 6.20.

Let \( \zeta \) be a primitive \( (q + 1)\)st root of unity in \( F_{q^2} \). Define \( f = \left\lfloor \frac{q+1}{4} \right\rfloor \). Consider the sets

\[
X = \{ (x_0, x_1, x_2) \parallel (x_0, x_1) \in \text{PG}(1, q), x_2^{q-1} \in \{ \zeta, ..., \zeta^f \} \},
\]

\[
Y = \{ l_{a,b,c} \parallel (a, b) \in \text{PG}(1, q), c^{q-1} \in \{ \zeta^{f+1}, ..., \zeta^{2f} \} \}.
\]

Take \((x_0, x_1, x_2) \in X\) and \(l_{a,b,c} \in Y\). Then \( ax_0 + bx_1 = ax_0^q + bx_1^q \in F_q, x_2^{q-1} = \zeta^i\) with \( 1 \leq i \leq f \), and \( c^{q-1} = \zeta^j\) with \( f + 1 \leq j \leq 2f \). Therefore, \( (c^q)^{q-1} = \zeta^{i+j} \) with \( f + 2 \leq i + j \leq 3f \). On the other hand, since \( \zeta^q = \zeta^{-1} \), \( (c^q)^{q-1} = \zeta^{i-j} \) with \( 1 \leq j - i \leq 2f - 1 \). Thus, \( (c^q)^{q-1} \) and \( (c^q)^{q-1} \) are not equal to 1. This means that \( c_2 \) and \( c_3 \) are not in \( F_q \), which implies that \( ax_0 + bx_1 + cx_2 \) and \( ax_0^q + bx_1^q + cx_2^q \) are not in \( F_q \), so definitely not equal to 0. Hence, no point of \( X \) lies on a block of \( Y \).

For every \( (q + 1)\)st root of unity \( \zeta^i \), there are \( q - 1 \) elements \( x_2 \) with \( x_2^{q-1} \) = \( \zeta^i \), which in fact form a multiplicative coset of \( F_{q^2}^\ast \). Hence, \(|X| = (q + 1)(q - 1)f\). Note that we do not count any points twice under the equivalence \((x_0, x_1, x_2) = (x_0^q, x_1^q, x_2^q)\), since \((x_0, x_1, x_2) \in \text{PG}(1, q)\) implies that \( (x_0^q, x_1^q) = (x_0, x_1) \), and \( x_2^{q-1} \in \{ \zeta, \zeta^2, ..., \zeta^f \} \) implies that \( (x_2^q)^{q-1} \in \{ \zeta^q, \zeta^{q-1}, ..., \zeta^{q-1} \} \) as \( \zeta^a = \zeta^{a+1} \) for all \( a \in \{1, ..., f\} \) follows from \( \zeta^{q+1} = 1 \).

For the set \( Y \), things are a little more delicate. If \( f = \frac{q+1}{4} \), then \( q \equiv 3 \pmod{4} \), and we must beware that if \( c^{q-1} = \zeta^2f \), then \( l_{a,b,c} \) and \( l_{a,b,c^q} \) define the same line, but \( (c^q)^{q-1} \) also equals \( \zeta^{2f} \) since \( (\zeta^q)^{q-1} = \zeta^{q+1-2f} = \zeta^{2f} \), where the first equality again follows from \( \zeta^{q+1} = 1 \) and the second as \( 2f = \frac{q+1}{2} \). Thus, in this case, the size of \( Y \) equals \( (q + 1)(f - 1)(q - 1) + \frac{q - 1}{2} = (q^2 - 1)\left\lfloor \frac{q}{4} \right\rfloor \). Consider the blocks \( l_{a,1,0} \) with \( a \not\in F_q \). If \((x_0, x_1, x_2) \in X \), then \( ax_0 + x_1 \) cannot be zero, since \( ax_0 \not\in F_q \) and \( x_1 \in F_q \). There are \( \frac{q^2 - q}{2} \) such blocks. Define \( Y' = Y \cup \{ l_{a,1,0} \parallel a \not\in F_q \} \). Then \(|Y'| = |X| - \frac{q - 1}{2} \). So we can delete \( \frac{q - 1}{2} \) points from \( X \) to obtain a subset \( X' \). Then \((X', Y')\) is an equinumerous incidence-free pair.
To finish the proof, one can again check that the conditions of Theorem 5.8 are satisfied. Apart from some elementary inequalities, one needs to observe that every point \((x_0, x_1, x_2)\) in \(\mathcal{P} \setminus \mathcal{X}\) is contained in at most \(q + 1\) blocks of \(Y\). First suppose that \(x_2 = 0\) so that we can write it as \((x_0, 1, 0)\) with \(x_0 \notin \mathbb{F}_q\). Then it cannot be incident with a block in \(Y\) as otherwise \(ax_0 + b = 0\) or \(ax_0^q + b = 0\) leads to the contradiction that \(x_0 \in \mathbb{F}_q\). Now suppose that \(x_2 \neq 0\). Choose \((a, b) \in \text{PG}(1, q)\), then \(ax_0 + bx_1 + cx_2 = 0\) has a unique solution for \(c\). Hence for this choice of \((a, b)\), there is at most one \(l_{a,b,c} \in Y\) incident with the point \((x_0, x_1, x_2)\).

\[\square\]

6.4.3 | Semibiplanes from binary affine spaces

In this subsection, we consider another large family of semibiplanes. They are no longer divisible, but we can still find lower bounds on the edge-domination number, emphasising the flexibility of the approach based on eigenvalues. In one particular member of this family of semibiplanes, we will construct a large equinumerous incidence-free pair in an SIS that will not give rise to an edge dominating set. This shows the necessity of some extra conditions in Theorem 5.8 when compared with Proposition 5.1.

Consider the \(n\)-dimensional affine space over \(\mathbb{F}_2\), denoted \(\text{AG}(n, 2)\). Give coordinates to the points. The weight of a point is the number of coordinate positions in which the point has a nonzero entry. Let \(W\) and \(\mathcal{P}\) denote the sets of points of odd and even weight, respectively. Consider a set \(S \subseteq W\) of size \(k\) such that

- the (affine) span of \(S\) is \(W\),
- \(S\) does not fully contain any (affine) plane of \(\text{AG}(n, 2)\).

Define \(B = \{y + S \parallel y \in W\}\). Then \(D = (\mathcal{P}, B)\) is an \(\text{spb}(2^{n-1}, k)\), see [32].

**Lemma 6.25.** Let \(\mathcal{H}\) denote the set of all affine hyperplanes of \(W\). Then the spectrum of the adjacency matrix of \(I(D)\) equals the multiset

\[\{\pm k\} \cup \{k - 2\mid I \cap S \parallel \mathcal{H} \in \mathcal{H}\}.\]

**Proof:** As in Lemma 6.18, let \(N\) denote the incidence matrix of the semibiplane, and \(A\) the adjacency matrix of its collinearity graph. Then \(NN^t = kI + 2A\), and the spectrum of the adjacency matrix of \(I(D)\) can be derived from the spectrum of \(NN^t\).

Denote \(S^+ = \{s + t \parallel s, t \in S, s \neq t\}\). The planes in \(\text{AG}(n, 2)\) are exactly the sets of four distinct points whose total sum is the zero vector. Since \(S\) does not contain any plane, \(s + t\) uniquely determines \(\{s, t\} \subseteq S\), and \(|S^+| = \binom{k}{2}\). Furthermore, two distinct points \(x, z \in \mathcal{P}\) are collinear if and only if \(x, z \in y + S\) for some \(y \in W\). This is equivalent to \(x = y + s\) and \(z = y + t\) for some \(y \in W\), and some \(s, t \in S\). This again is equivalent to \(x + s = z + t\) for some \(s, t \in S\), since this immediately implies that \(x + s = z + t \in W\). The last statement is equivalent to \(x + z \in S^+\). In conclusion, \(x\) and \(z\) are collinear if and only if \(x + z \in S^+\).

We can now use [2, Section 7.1] to conclude that the spectrum of \(A\) equals the multiset
Consider the standard inner product \( \mathbf{x} \cdot \mathbf{y} = \sum x_i y_i \) on \( \text{AG}(n, 2) \). Every hyperplane \( H \) in \( \mathcal{P} \) through \( 0 \) is of the form \( \mathbf{X} \cdot \mathbf{a} = 0 \) for some \( \mathbf{a} \in \mathcal{P} \setminus \{0, 1\} \). Then \( H \) is an \((n - 2)\)-space in \( \text{AG}(n, 2) \). There are three hyperplanes in \( \text{AG}(n, 2) \) through \( H \), namely, \( H_0 \) with equation \( \mathbf{X} \cdot \mathbf{a} = 0 \), \( H_1 \) with equation \( \mathbf{X} \cdot (\mathbf{a} + 1) = 0 \), and \( \mathcal{P} \). Then \( H_0 \cap W \) and \( H_1 \cap W \) are parallel hyperplanes of \( W \), partitioning the points. Take a point \( x \) in \( S^+ \). There exist unique \( s \) and \( t \) in \( S \) such that \( x = s + t \). Then

\[
x \notin H \iff x \cdot a = 1 \iff s \cdot a \neq t \cdot a.
\]

Thus, \(|S^+ \setminus H|\) equals the number of ways to choose an element of \( s \in S \cap H_0 \) and an element \( t \in S \cap H_1 \). If \(|S \cap H_0| = m \), then \(|S \cap H_1| = k - m \). Hence,

\[
2|H \cap S^+| - |S^+| = |S^+| - 2|H \setminus S^+| = \binom{k}{2} - 2m(k - m).
\]

This gives an eigenvalue

\[
k + 2 \left( \binom{k}{2} - 2m(k - m) \right) = k + k(k - 1) - 4m(k - m) = k^2 - 4m(k - m)
\]

\[
= (k - 2m)^2
\]

of \(NN'\). This gives us eigenvalues \( k - 2m \) and \(-(k - 2m)\) of the adjacency matrix of \( I(D) \). Note that if \( k - 2|S \cap H_0| = k - 2m \), then \( k - 2|S \cap H_1| = -(k - 2m) \). Lastly, the eigenvalue \( \binom{k}{2} \) of \( A \), gives us the eigenvalue \( k^2 \) of \( NN' \), hence the eigenvalues \( \pm k \) of the adjacency matrix of \( I(D) \). This proves the statement of the lemma.

The following lower bound on the edge domination number of \( I(D) \) follows directly from Proposition 6.16.

**Lemma 6.26.** Let \( W \) denote the hyperplane of odd-weight points in \( \text{AG}(n, 2) \), and let \( S \) be a subset spanning \( W \), not containing a plane. Denote the associated \( \text{sbp}(2^{n-1}, k) \) by \( D \). Let \( m \) denote the maximum number of points of \( S \) contained in a hyperplane of \( W \). Then

\[
\gamma_e(D) \geq \frac{k}{m} 2^{n-2}.
\]

**Remark 6.27.** Let \( S \) be the set of weight one vector. In this \( \text{sbp}(2^{n-1}, n) \), we can construct a large incidence-free set by taking \( X \) to be the set of even weight vectors with weight at most \( \lfloor n/2 \rfloor - 1 \) while we take \( Y \) to be the set of blocks \( y + S \), where \( y \) runs over the odd weight vectors with weight at least \( \lceil n/2 \rceil + 1 \). By taking subsets, we can easily find a large equinumerous incidence-free pair of sets, but it will not give rise to a small edge dominating set in its complement. This is easily seen, as a point corresponding to a
vector of even weight at least \([n/2] + 2\) has no neighbours in \(B \setminus Y\). This example reaffirms that an assumption like the minimum degree condition in Theorem 5.8 is necessary, even though \(I(D)\) is connected.

7 | CONCLUSION

In this paper, we studied the edge domination number \(\gamma_e(D)\) of incidence structures \(D\) through its connections with maximal matchings and incidence-free sets. In almost all families we studied, we saw that \(\gamma_e(D) = (1 - o(1))v\), so the interesting quantity to study is \(v - \gamma_e(D)\). Using a combination of probabilistic, combinatorial and geometric techniques, supplemented with tools from spectral graph theory, we made headway on bounding the edge domination number of various designs, often obtaining sharp bounds on \(v - \gamma_e(D)\) up to a constant factor. Nevertheless, many problems remain open, and we state a few of them here.

**Problem 7.1.** What is the edge domination number for non-Desarguesian projective planes? Can one construct large incidence-free pairs?

We remark that in the Lüneburg planes, which are non-Desarguesian projective planes of order \(q = 4^{2h+1}, h \geq 1\), Thas [29] constructed maximal arcs of order \(\sqrt{q}\). Hence, in these planes the edge domination number equals \(q^2 - q\sqrt{q} + \sqrt{q} + 1\), cf. Corollary 6.3.

Various classes of symmetric designs are constructed using difference sets [4, Section 6.8]. Quite a few of them are difference sets in cyclic groups.

**Problem 7.2.** Is it possible to construct incidence-free pairs in a “uniform” way for symmetric designs coming from difference sets? That is, without resorting to the specific structure of the difference set, but only using properties of the group?

If \(G\) is a group with difference set \(S\), one can translate this problem to finding so-called cross-intersecting independent sets in the (directed) Cayley graph \(C(G, S)\) where we have an arc between two distinct elements \(x\) and \(y\) if \(xy^{-1} \in S\).

The following problem asks if we can improve our general lower bound from Proposition 4.1.

**Problem 7.3.** Can we always find equinumerous incidence-free pairs of size \(k^{1+\varepsilon}/\lambda\) in a symmetric \((v, k, \lambda)\)-design for small \(\lambda\)?

It seems plausible that such incidence-free pairs should always exist for symmetric designs, but we do not necessarily think it should hold for SISs in general. Finding an example of such an SIS (if it exists) would also be of significant interest.

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