A PHRAGMÉN-LINDELÖF PROPERTY OF VISCOSITY SOLUTIONS TO A CLASS OF NONLINEAR, POSSIBLY DEGENERATE, PARABOLIC EQUATIONS

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Abstract. We study Phragmén-Lindelöf properties of viscosity solutions to a class of doubly nonlinear parabolic equations in $\mathbb{R}^n \times (0, T)$. We also include an application to some doubly nonlinear equations.

1. Introduction

In this work, we discuss Phragmén-Lindelöf type results for a class of nonlinear parabolic equations. This is a follow-up of the work in [3] where we stated similar results for viscosity solutions of Trudinger’s equation in $\mathbb{R}^n \times (0, T)$, where $n \geq 2$ and $0 < T < \infty$.

We introduce notations for our discussion. Let $n \geq 2$, $g : \mathbb{R}^n \to (0, \infty)$ and $h : \mathbb{R}^n \to \mathbb{R}$ be two continuous functions. We impose that

$$\max \left( \sup_x |\log g(x)|, \sup_x |h(x)| \right) < \infty$$

Let $0 < T < \infty$ and define $\mathbb{R}^n_T = \mathbb{R}^n \times (0, T)$.

Our motivation for the work arises from the study of viscosity solutions of doubly nonlinear equations of the kind

(1.2) $H(Du, D^2 u) - f(u)u_t = 0$, in $\mathbb{R}^n_T$, $u(x, t) > 0$ and $u(x, 0) = g(x)$, $\forall x$ in $\mathbb{R}^n$,

where $H$ satisfies certain homogeneity conditions and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function, see Section 2 for more details. As shown in [5], if $f$ satisfies certain conditions then a change of variable $u = \phi(v)$ transforms (1.2) to

(1.3) $H(Dv, D^2 v + Z(v)Dv \otimes Dv) - v_t = 0$, in $\mathbb{R}^n_T$, and $v(x, 0) = \phi^{-1}(g(x))$, $\forall x$ in $\mathbb{R}^n$,

where $Z : \mathbb{R} \to \mathbb{R}^+$ is a non-increasing function. As observed in [2, 5], one can conclude a comparison principle for (1.3), and hence, for (1.2).

An example of such an equation is the well-known Trudinger’s equation [9]:

$$\text{div} \left( |Du|^{p-2} Du \right) - (p - 1)u^{p-2}u_t = 0$$

in $\Omega \times (0, T)$, where $u > 0$.

The works in [2, 4] address the existence and uniqueness of viscosity solutions $u$, for $p \geq 2$, in cylindrical domains $\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and [3] includes Phragmén-Lindelöf type results.
A related but somewhat more general equation is to consider, in $\mathbb{R}^n_T$,
\[
\text{div } (|Du|^{p-2}Du) + \chi(t)|Du|^\sigma - (p-1)u^{p-2}u_t = 0,
\]
and $u > 0$, $u(x,0) = g(x)$, $\forall x$ in $\mathbb{R}^n$,

where $\sigma \geq 0$ and $\chi(t)$ is continuous on $[0,T]$. Employing the change of variables $u = e^v$ (see [2]), we obtain the equation
\[
\text{div } (|Dv|^{p-2}Dv) + (p-1)|Dv|^p + \chi(t)e^{(\sigma-(p-1))v}|Dv|^\sigma - (p-1)v_t = 0, \quad \text{in } \mathbb{R}^n_T,
\]
and $v(x,0) = \log g(x)$, $\forall x$ in $\mathbb{R}^n$.

Writing $H(Dw, D^2w) = \text{div}(|Dw|^{p-2}|Dw|)$, the above equation may be written as
\[
H(Dv, D^2v + Dv \otimes Dv) + \chi(t)e^{(\sigma-(p-1))v}|Dv|^\sigma - (p-1)v_t = 0, \quad \text{in } \mathbb{R}^n_T,
\]
\[
v(x,0) = \log g(x), \quad \forall x \text{ in } \mathbb{R}^n.
\]

At this time, it is not clear to us as to how to address the above equation. Nonetheless, the above discussion provides motivation for addressing the following related question of studying Phragmén-Lindelöf results for equations of the kind
\[
H(Dv, D^2v + Z(v)Dv \otimes Dv) + \chi(t)|Dv|^\sigma - v_t = 0, \quad \text{in } \mathbb{R}^n_T
\]
\[
v(x,0) = h(x), \quad \forall x \text{ in } \mathbb{R}^n.
\]

Here $\chi$, $h$ can have any sign.

We will show that if $v$ satisfies certain growth conditions, for large $|x|$, then $v$ satisfies a maximum principle. A similar conclusion follows for the equation in [12]. We assume $\inf_{\mathbb{R}} Z(s) > 0$ for the main results and this strongly influences our work. It is clear that $Z(v)Dv \otimes Dv$ and $\chi(t)|Dv|^\sigma$ are dueling terms and the analysis will bear this out. Moreover, it will also show how the imposed growth rates and solutions are influenced by the power $\sigma$.

We do not address existence and uniqueness issues in this work. It would be interesting to know if the growth rates stated in this work would imply such results. Omitted also from this work is the question of optimality of the growth rates.

We have divided our work as follows. In Section 2, we present some notations, assumptions and main results. In Sections 3 and 4, we present comparison principles, a change of variables result and calculations for some of the auxiliary functions we use. Sections 5 and 6 address the super-solutions and sub-solutions respectively. Finally, Section 7 presents proofs of the main results.

For additional discussion and motivation, we direct the reader to the works [1, 6, 7, 8].
We state that throughout this work sub-solutions or super-solutions or solutions are understood in the viscosity sense, see [5, 6] for definitions. We \textit{usc} (\textit{lsc}) for upper(lower) semicontinuous functions.

We introduce notations that will be used throughout this work. We take \( n \geq 2 \). Let \( 0 < T < \infty \) and set \( \mathbb{R}^n_T = \mathbb{R}^n \times (0, T) = \{(x, t) : x \in \mathbb{R}^n \text{ and } 0 < t < T\} \). The functions \( g \) and \( h \) will always satisfy (1.1)

By \( o \), we denote the origin in \( \mathbb{R}^n \) and \( e \) denotes a unit vector in \( \mathbb{R}^n \). The letters \( x, y \) will denote points in \( \mathbb{R}^n \). Let \( S^{n \times n} \) be the set of all symmetric \( n \times n \) real matrices, \( I \) be the \( n \times n \) identity matrix and \( O \) the \( n \times n \) zero matrix.

We now describe the conditions placed on \( H \).

\textbf{Condition A (Monotonicity):} The operator \( H : \mathbb{R}^n \times S^{n \times n} \to \mathbb{R} \) is continuous for any \((q, X) \in \mathbb{R}^n \times S^{n \times n}\). We assume that

\begin{enumerate}
\item[(i)] \( H(q, X) \leq H(q, Y) \), for any \( q \in \mathbb{R}^n \) and for any \( X, Y \in S^{n \times n} \) with \( X \leq Y \),
\item[(ii)] \( H(q, O) = 0 \), for any \( q \in \mathbb{R}^n \).
\end{enumerate}

Clearly, for any \( q \in \mathbb{R}^n \) and \( X \in S^{n \times n} \), \( H(q, X) \geq 0 \) if \( X \geq O \).

\textbf{Condition B (Homogeneity):} There is a constant \( k_1 \geq 0 \) such that for any \((q, X) \in \mathbb{R}^n \times S^{n \times n}\),

\begin{enumerate}
\item[(i)] \( H(\theta q, X) = |\theta|^{k_1} H(q, X) \), for any \( \theta \in \mathbb{R} \), and
\item[(ii)] \( H(q, \theta X) = \theta H(q, X) \), for any \( \theta > 0 \).
\end{enumerate}

Our results in this work can be adapted to include the case \( H(q, \theta X) = \theta^{k_2} H(q, X) \) where \( k_2 \) is an odd natural number. However, in this work, \( k_2 = 1 \). We note that if \( k_1 = 0 \) then \( H(q, X) = H(q/\theta, X) \), \( \forall \theta > 0 \). Hence, \( H(q, X) = H(X) \).

Before stating the next condition, we introduce additional notation. Let \( \rho \in \mathbb{R}^n \) be a vector and we write its component form as \((\rho_1, \rho_2, \ldots, \rho_n)\). Recall that \((\rho \otimes \rho)_{ij} = \rho_i \rho_j, \; i, j = 1, \ldots, n\). Clearly, \( \rho \otimes \rho \in S^{n \times n} \) and \( \rho \otimes \rho \geq O \).

Recalling that \( e \in \mathbb{R}^n \) is a unit vector, define, for every \( \lambda \in \mathbb{R} \),

\begin{align}
(2.3) \quad \Lambda_{\min}(\lambda) &= \min_{e} H(e, \lambda e \otimes e - I) \quad \text{and} \quad \Lambda_{\max}(\lambda) = \max_{e} H(e, \lambda e \otimes e + I).
\end{align}

By Condition A, \( \Lambda_{\min}(\lambda) \) and \( \Lambda_{\max}(\lambda) \) are both non decreasing functions of \( \lambda \).

\textbf{Condition C(Growth at Infinity):} Firstly, we require that

\[ \max_{e} H(e, -I) < 0 < \min_{e} H(e, I). \]
Next, we assume that $H$ satisfies
\begin{equation}
\Lambda_{\min}(\lambda_0) = \min_e H(e, \lambda_0 e \otimes e - I) > 0, \text{ for some } \lambda_0 > 1.
\end{equation}
We require $\lambda_0 > 1$ since, by Condition A, $e \otimes e - I \leq O$.

We state some simple implications of Condition C. By Condition A, $\Lambda_{\min}(\lambda) \geq \Lambda_{\min}(\lambda_0) > 0$, for any $\lambda \geq \lambda_0$. By Condition B, for $\lambda \geq \lambda_0$,
\begin{equation}
\Lambda_{\min}(\lambda) = (\frac{\lambda}{\lambda_0}) \min_e H \left( e, \lambda_0 e \otimes e - \frac{\lambda_0}{\lambda} I \right) \geq \frac{\lambda \Lambda_{\min}(\lambda_0)}{\lambda_0},
\end{equation}

since $\lambda_0 e \otimes e - I \leq \lambda_0 e \otimes e - (\lambda_0/\lambda)I$. Clearly,
\begin{equation}
\frac{\Lambda_{\min}(\lambda)}{\lambda} \geq \frac{\Lambda_{\min}(\lambda_0)}{\lambda_0} \quad \text{and} \quad \sup_{\lambda > 0} \Lambda_{\min}(\lambda) = \infty.
\end{equation}

Thus, under Conditions A, B and C, (2.4) implies (2.5). Clearly, (2.5) implies (2.4).

Next, by Conditions A, B and (2.5), for $\lambda \geq \lambda_0$,
\begin{equation}
\min_e H(e, e \otimes e) \geq \frac{\Lambda_{\min}(\lambda)}{\lambda} = \min_e H \left( e, e \otimes e - \frac{I}{\lambda} \right) \geq \frac{\Lambda_{\min}(\lambda_0)}{\lambda_0} > 0.
\end{equation}

If $\min_e H(e, e \otimes e) > 0$ then by the continuity of $H$, Conditions A and B, $\min_e H(e, \lambda_0 e \otimes e - I) > 0$ for some $\lambda_0 > 1$. See Section 3 for further discussion.

Examples of operators that satisfy Conditions A, B and C are the $p$-Laplacian, pseudo $p$-Laplacian, for $p \geq 2$, infinity-Laplacian and the Pucci operators, see [5] for a more detailed discussion. It is easily seen that they all satisfy (2.6). We remark that some of the conditions here differ from those in [5].

For the rest of this work, we set
\begin{equation}
k = k_1 + 1 \quad \text{and} \quad \gamma = k_1 + 2 = k + 1.
\end{equation}
Also, $\chi : [0, T] \to \mathbb{R}$ is a continuous function and $Z : \mathbb{R} \to \mathbb{R}^+$ is a non-increasing continuous function with $0 < \inf Z \leq \sup Z < \infty$. Let $h : \mathbb{R}^n \to \mathbb{R}$, continuous, satisfy (1.1).

We now state the main results of this work. For Theorems 2.1 and 2.2, we assume that Conditions A, B and C hold. We set
\begin{equation}
\mu = \inf_{x \in \mathbb{R}^n} h(x), \quad \nu = \sup_{x \in \mathbb{R}^n} h(x), \quad \ell = \inf_{s} Z(s), \quad \mathcal{H} = \min_{|e|=1} H(e, e \otimes e) \quad \text{and} \quad \alpha = \sup_{0 \leq t \leq T} |\chi(t)|.
\end{equation}
Theorem 2.1. (Maximum Principle) Let $0 < T < \infty$, and $\nu$ and $\alpha$ be as in (2.8). Assume that $\sup_{\mathbb{R}^n} \nu < \infty$. Let $u \in \mathrm{usc}(\mathbb{R}_T^n)$ solve
\[ H(Du, D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t \geq 0, \quad \text{in } \mathbb{R}_T^n \]
and $u(x) \leq h(x)$, $\forall x \in \mathbb{R}^n$.

Suppose that there is $\delta > 0$ such that $\sup_{0 \leq |x| \leq R, 0 \leq t \leq T} u(x,t) \leq o(R^\delta)$ as $R \to \infty$.

The following hold.

(a) Let $\sigma = 0$. Either (i) $k = 1$ i.e., $\gamma = 2$ and $\delta = 2 - \varepsilon$, for any fixed and small $\varepsilon > 0$, or (ii) $k > 1$ and $\delta = \gamma/k$. In both cases,
\[ \sup_{\mathbb{R}_T^n} u(x,t) \leq \nu + \alpha t. \]

(b) Let $0 < \sigma \leq \gamma$. Either (i) $k = 1$ i.e., $\gamma = 2$ and $\beta = 2 - \varepsilon$, for any fixed and small $\varepsilon > 0$, or (ii) $k > 1$ and $\beta = \gamma/k$. In both cases,
\[ \sup_{\mathbb{R}_T^n} u(x,t) \leq \nu. \]

(c) Let $\sigma > \gamma$ and $\delta = \sigma/(\sigma - 1)$. Then
\[ \sup_{\mathbb{R}_T^n} u(x,t) \leq \nu. \]
□

Theorem 2.2. (Minimum Principle) Let $0 < T < \infty$ and $\mu$, $\alpha$, $\ell$ and $\mathcal{H}$ be as in (2.8). Assume that $\mu > -\infty$. Let $u \in \mathrm{lsc}(\mathbb{R}_T^n)$ solve
\[ H(Du, D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t \leq 0, \quad \text{in } \mathbb{R}_T^n, \]
and $u(x) \geq h(x)$, $\forall x \in \mathbb{R}^n$.

(a) If $\sigma = 0$ then $\inf_{\mathbb{R}_T^n} u(x,t) \geq \mu - \alpha t$.

(b) If $0 < \sigma < \gamma$ then $\inf_{\mathbb{R}_T^n} u(x,t) \geq \mu - (\alpha^\gamma / (\ell \mathcal{H})^\sigma)^{1/(\gamma - \sigma)} t$.

If $\chi(t) \geq 0$ then, for any $0 \leq \sigma < \infty$, $\inf_{\mathbb{R}_T^n} u(x,t) \geq \mu$.

(c) If $\sigma = \gamma$ and $\alpha < \ell \mathcal{H}$ then $\inf_{\mathbb{R}_T^n} u(x,t) \geq \mu$.

(d) Let $\sigma = \gamma$ and $\alpha \geq \ell \mathcal{H}$. Assume that either (i) $k = 1$ ($\gamma = 2$) and, for any fixed small $\varepsilon > 0$, we have
\[ \sup_{0 \leq |x| \leq R, 0 \leq t \leq T} (-u(x,t)) \leq o(R^{2-\varepsilon}), \quad \text{as } R \to \infty, \]
or (ii) $k > 1$ ($\gamma > 2$) and
\[
\sup_{0 \leq |x| \leq R, 0 \leq t \leq T} (-u(x, t)) \leq o(R^{\gamma/k}), \text{ as } R \to \infty.
\]
Then $\inf_{\mathbb{R}^n_T} u(x, t) \geq \mu$.

(e) If $\sigma > \gamma$ and
\[
\sup_{0 \leq |x| \leq R, 0 \leq t \leq T} (-u(x, t)) \leq o(R^{\sigma/(\sigma-1)}), \text{ as } R \to \infty,
\]
then $\inf_{\mathbb{R}^n_T} u(x, t) \geq \mu$.

As an observation, if $H$ is quasilinear (the $p$-Laplacian, for instance) then
\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) = H(Dw, D^2w) + Z(w)|Dw|^\gamma H(e, e \otimes e).
\]
The above holds for both $Dw \neq 0$ and $Dw = 0$ since $H(q, O) = 0$, for any $q \in \mathbb{R}^n$. If $H$ is the $p$-Laplacian, $p \geq 2$, $\chi(t) = 1 - p$ and $Z(w) = 1$ then $k_1 = p - 2$, $k = p - 1$, $\gamma = p$ and $H(e, e \otimes e) = p - 1$. Clearly,
\[
H(Dw, D^2w + Dw \otimes Dw) - (p-1)|Dw|^p
= H(Dw, D^2w) + (p-1)|Dw|^p - (p-1)|Dw|^p = H(Dw, D^2w).
\]
Thus, the above results also apply to equations of the kind $\Delta_p u - u_t = 0$ in $\mathbb{R}^n_T$.

Finally, we obtain the following theorem for a class of doubly nonlinear equations. We apply parts (a) of Theorems 2.1 and 2.2 with $\alpha = \sigma = 0$.

**Theorem 2.3.** Let $k \geq 1$, $f : [0, \infty) \to [0, \infty)$ be a $C^1$ non-decreasing function, and $g : \mathbb{R}^n \to (0, \infty)$ be such that $0 < \inf_x g(x) \leq \sup_x g(x) < \infty$.

Let $k > 1$. We assume that $f^{1/(k-1)}$ is concave and
\[
0 < \inf_{0 \leq s < \infty} \frac{d}{ds} f^{1/(k-1)}(s) \leq \sup_{0 \leq s < \infty} \frac{d}{ds} f^{1/(k-1)}(s) < \infty.
\]
Select $\phi : \mathbb{R} \to [0, \infty)$, a $C^2$ increasing function, such that
\[
\phi'(\tau) = f(\phi(\tau))^{1/(k-1)}.
\]

(a) Suppose that $u \in \text{usc}(\mathbb{R}^n_T)$, $u > 0$, solves
\[
H(Du, D^2u) - f(u)u_t \geq 0, \text{ in } \mathbb{R}^n_T \text{ and } u(x, 0) \leq g(x), \forall x \in \mathbb{R}^n.
\]
Suppose that $\sup_{|x| \leq R, 0 \leq t \leq T} u(x, t) \leq \phi(o(R^{\gamma/k}))$, as $R \to \infty$. Then $\sup_{\mathbb{R}^n_T} u(x, t) \leq \sup_x g(x)$.

(b) Suppose that $u \in \text{lsc}(\mathbb{R}^n_T)$, $u > 0$, solves
\[
H(Du, D^2u) - f(u)u_t \leq 0, \text{ in } \mathbb{R}^n_T \text{ and } u(x, 0) \geq g(x), \forall x \in \mathbb{R}^n.
\]
Then $\inf_{\mathbb{R}^n_T} u(x, t) \geq \inf_x g(x)$. 
If \( k = 1 \), we take \( f \equiv 1 \) and \( \phi(\tau) = e^\tau \). The conclusion in part (a) holds provided that we assume that, for any \( \varepsilon > 0 \), \( \sup_{|x| \leq R, 0 \leq t \leq T} u(x, t) \leq \exp(o(R^{2-\varepsilon})) \), as \( R \to \infty \). The conclusion in part (b) holds without any modifications.

The condition placed on \( f^{1/(k-1)} \) implies that \( \phi''(\tau)/\phi'(\tau) \) is positive and non-increasing in \( \tau \in (-\infty, \infty) \). Moreover, this quotient is bounded from above and its lower bound is positive. See Section 3.

3. Preliminaries

In this section, we present some calculations important for our work, a comparison principle and a change of variable result useful for our work. We also present additional discussion about the condition in (2.4).

For definitions and a discussion of viscosity solutions, we direct the reader to [6] and Section 2 in [3]. For additional discussion and motivation, see [1, 2, 7, 8].

Recall that \( Z : \mathbb{R} \to \mathbb{R}^+ \) is continuous and non-increasing. We assume that

\[
0 < \inf R Z \leq \sup R Z < \infty.
\]

We present now some elementary but important calculations. Let \( z \in \mathbb{R}^n \) and \( r = |x - z| \). Suppose that \( v(x) = v(r) \) is a \( C^2 \) function. Set \( e = (e_1, e_2, \cdots, e_n) \) where \( e_i = (x - z)_i / r, \ \forall i = 1, 2, \cdots, n \). For \( x \neq z \),

\[
\begin{cases}
Dv = v'(r)e, & Dv \otimes Dv = (v'(r))^2e \otimes e, \ \text{and} \\
D^2v = (v'(r)/r)(I - e \otimes e) + v''(r)e \otimes e.
\end{cases}
\]

Remark 3.1. Let \( \kappa : [0, T] \to (0, \infty) \) be a \( C^1 \) function and \( Z \) be as in (3.1). Fix \( z \in \mathbb{R}^n \) and set \( r = |x - z| \). Suppose that \( 0 < R \leq \infty \) and \( w : B_R(z) \times [0, T] \to \mathbb{R} \) is a \( C^1 \) function with \( w(x, t) = w(r, t) \). Assume also that \( w \) is \( C^2 \) in \( x \) in \( B_R(z) \setminus \{z\} \).

Using (3.2) in \( x \neq z \), we get that

\[
H(Dw, D^2w + Z(w)Dw \otimes Dw)
\]

\[
= H \left( w_re, \frac{w_r}{r}(I - e \otimes e) + (w_{rr} + Z(w)(w_r)^2) e \otimes e \right).
\]

Recall Condition B in (2.2) and (2.7) i.e, \( k = k_1 + 1 \) and \( \gamma = k_1 + 2 \).

Case (a) \( w_r > 0 \): Let \( a \) be any scalar, \( b \geq 0 \) and \( R = \infty \). Suppose that \( w(x, t) = (a + bv(r))\kappa(t) \), where \( v'(r) > 0 \) and \( \kappa \geq 0 \).
In (3.3), factor \( w_r \) from the first entry, \( w_r/r \) from the second, use (2.2) and \( k = k_1 + 1 \) to get

\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) = \frac{w_r^k}{r} H \left( e, I + \left( \frac{rw_{rr}(r)}{w_r} + rw_rZ(w) - 1 \right) e \otimes e \right)
\]

(3.4)

\[
= \left( \frac{bw'(r)\kappa(t)}{r} \right)^k H \left( e, I + \left( \frac{rv''(r)}{v'(r)} - 1 + b\kappa(t)rv'(r)Z(w) \right) e \otimes e \right).
\]

This version will be used for small \( r \).

For the second version, in (3.3) we factor \( w_r \) from the first entry, \( w_r^2 \) from the second entry of \( H \), use (2.2) and \( \gamma = k_1 + 2 \) to get, in \( r > 0 \),

\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) = w_r^\gamma H \left( e, \frac{I - e \otimes e}{rw_r} + \left( \frac{w_{rr}}{w_r^2} + Z(w) \right) e \otimes e \right)
\]

(3.5)

\[
= \left( \frac{bv'(r)\kappa(t)}{\kappa(t)rv'(r)} \right)^\gamma H \left( e, I - e \otimes e \frac{v''(r)}{\kappa(t)(v'(r))^2} + bZ(w) \right) e \otimes e).
\]

This version will be used for large \( r \).

In this work, we take \( 0 < b < 1 \). By factoring \( 1/b \) from the second entry in \( H \), using Condition B and \( \gamma = k + 1 \), the above may be rewritten as

\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) = b^k(\kappa(t)v'(r))^{\gamma} H \left( e, \frac{I - e \otimes e}{\kappa(t)rv'(r)} + \left( \frac{v''(r)}{\kappa(t)(v'(r))^2} + bZ(w) \right) e \otimes e \right).
\]

(3.6)

**Case (b) \( w_r < 0 \):** Using (2.2), (3.3) and arguing as in part (a), we get

\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) = \left| w_r \right|^k H \left( e, \frac{I - e \otimes e}{rw_r} + \left( \frac{w_{rr}}{w_r^2} + Z(w) \right) e \otimes e \right)
\]

(3.7)

Set \( w(x, t) = v(r) - \kappa(t) \), where \( v'(r) < 0 \). Using (3.7) we get

\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) = \left| v'(r) \right|^k H \left( e, \frac{I - e \otimes e}{rv'(r)} + \left( \frac{v''(r)}{(v'(r))^2} + Z(w) \right) e \otimes e \right).
\]

(3.8)

\[\square\]

We now state a comparison principle that will be used in this work. See [6] and Section 4 in [5].
Lemma 3.2. (Comparison principle) Let \( F \) satisfy (3.9), \( g : \mathbb{R} \to \mathbb{R} \) be a bounded non-increasing continuous function and \( \hat{f} : \mathbb{R}^+ \to \mathbb{R}^+ \) be continuous. Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded domain and \( T > 0 \). Let \( u \in \text{usc}(\Omega_T \cup P_T) \) and \( v \in \text{lsc}(\Omega_T \cup P_T) \) satisfy in \( \Omega_T \),

\[
F(t, u, Du, D^2u + g(u)Du \otimes Du) - \hat{f}(t)u_t \geq 0 \quad \text{and} \\
F(t, v, Dv, D^2v + g(v)Dv \otimes Dv) - \hat{f}(t)v_t \leq 0.
\]

If \( \sup_{P_T} v < \infty \) and \( u \leq v \) on \( P_T \) then \( u \leq v \) in \( \Omega_T \). \( \Box \)

We now discuss a change of variables result in the context of doubly nonlinear equations of the kind:

\[
H(Du, D^2u) - f(u)u_t = 0, \quad \text{where} \ u > 0.
\]

This is shown in Lemma 2.3 in \([5]\). An earlier version appears in \([3]\).

Recall that \( k = k_1 + 1 \) and \( \gamma = k_1 + 2 \). Let \( f : [0, \infty) \to [0, \infty) \) be a \( C^1 \) increasing function. For \( k > 1 \), the function \( \phi : \mathbb{R} \to [0, \infty) \) be a \( C^2 \) solution of

\[
\frac{d\phi}{d\tau} = \{(f \circ \phi)(\tau)\}^{1/(k-1)}, \quad \phi \geq 0.
\]

Thus, \( \phi \) is increasing. We will assume further that

\[
f^{1/(k-1)} \text{ is concave, i.e, } \{f^{1/(k-1)}\}'(s) \text{ is non-increasing in } s.
\]

For example, if \( f(s) = s^\alpha, \ \alpha \geq 0 \), then \( f^{1/(k-1)} \) is concave if \( \alpha \leq k - 1 \). This condition ensures that the comparison principle in Lemma 3.2 holds.

Using (3.10) and (3.11) we get that

\[
\frac{\phi''(\tau)}{\phi'(\tau)} = \left( \frac{d}{d\phi} f(\phi)^{1/(k-1)} \right)(\phi(\tau)) \text{ is non-increasing in } \tau.
\]

Note that if \( f(s) = s^\alpha, 0 < \alpha < k - 1 \), then \( (\phi''/\phi')(s) = C/s \), for an appropriate \( C = C(\alpha, k) \). Our work, however, excludes such cases as the quotient becomes small for large \( s \). If \( \alpha = k - 1 \) then \( \phi(\tau) = Ae^\tau \) and \( (\phi''/\phi')(\tau) = 1 \). The latter is included in our work and is addressed in Theorem 2.3.

We now state the following change of variables lemma which is a simplified version of Lemma 2.3 in \([5]\).

Lemma 3.3. Let \( H \) satisfy Conditions A and B, see (2.1) and (2.2) and \( f : [0, \infty) \to [0, \infty) \) be a \( C^1 \) increasing function that satisfies (3.11). Suppose that \( \phi : \mathbb{R} \to [0, \infty) \) is a positive \( C^2 \) increasing function.
Case (i): Suppose that $k > 1$ and $\phi$ is as in (3.10). We assume that $f$ is non-constant, $u > 0$ and $v = \phi^{-1}(u)$.

Then $u \in \text{usc}(\text{lsc})(\Omega_T)$ solves $H(Du, D^2u) \geq (\leq) f(u)u_t$, in $\Omega_T$, if and only if $v \in \text{usc}(\text{lsc})(\Omega_T)$ and

$$
H \left( Du, D^2u + \frac{\phi''(v)}{\phi'(v)} Du \otimes Du \right) \geq (\leq) v_t, \text{ in } \Omega_T.
$$

Case (ii): Let $k = 1$, i.e., $k_1 = 0$. If $f \equiv 1$ then the claims in (a) and (b) hold if $\phi(\tau)$ is any increasing positive $C^2$ function. In particular, if $\phi(\tau) = e^\tau$ and $u \in \text{usc}(\text{lsc})(\Omega_T)$ then $H(D^2u) \geq (\leq) u_t$ if and only if $v \in \text{usc}(\text{lsc})(\Omega_T)$ and $H(D^2u + Du \otimes Du) \geq (\leq) v_t$.

Finally, we make further comments on the dependence of $H(\cdot, e \otimes e \pm I)$ on $\lambda$.

**Remark 3.4.** Recall (2.1), (2.2), (2.3) and (2.7). As observed in Section 2 (see discussion immediately following Condition C), $H(\cdot, e \otimes e \pm I)$ is non-decreasing in $\lambda$, and, for $\lambda \geq 0$ and any $e$,

$$
\lim_{\lambda \to \infty} \frac{H(\cdot, e \otimes e \pm I)}{\lambda} = \lim_{\lambda \to \infty} H \left( e, e \otimes e \pm \frac{I}{\lambda} \right) = H(e, e \otimes e) \geq 0.
$$

This follows since $e \otimes e \geq 0$ and $H(e, e \otimes e) \geq 0$.

The $p$-Laplacian, the pseudo $p$-Laplacian, the infinity-Laplacian and the Pucci type operators all satisfy $\sup_\lambda \Lambda_{\min}(\lambda) = \infty$ and $H(e, e \otimes e) > 0$ (note that eigenvalues of $e \otimes e$ are 1 and 0 (0 has multiplicity $n - 1$)). Our current work applies to these operators. See Section 3 in [5].

Note that the condition $\min_e H(e, e \otimes e) > 0$ implies that $\sup_\lambda \Lambda_{\min}(\lambda) = \infty$. Clearly, if $\sup_\lambda \Lambda_{\max}(\lambda) < \infty$ then $H(e, e \otimes e) = 0$. Moreover, if $H$ is quasilinear then

$$
H(e, \lambda e \otimes e \pm I) = H(e, \lambda e \otimes e) + H(e, \pm I) = H(e, \pm I), \forall \lambda \geq 0.
$$

An example of such an operator is

$$
H(p, X) = |p|^{k_1} \left( |p|^2 \text{Trace}(X) - \sum_{i,j=1}^n p_i p_j X_{ij} \right), \forall (p, X) \in \mathbb{R}^n \times \mathbb{S}^{n \times n}, \forall k_1 \geq 0,
$$

i.e., $H(Du, D^2u) = |Du|^{k_1+2} \Delta u - |Du|^{k_1} \Delta_{\infty} u$. Clearly, $H$ is elliptic and $\forall e$,

$$
H(e, e \otimes e) = 0, \text{ and } H(e, \lambda e \otimes e \pm I) = \pm (n-1), \forall \lambda.
$$

Our current work omits such operators.

Note that the condition $\max_e H(e, e \otimes e) = 0$ does not imply the boundedness of $H(e, \lambda e \otimes e \pm I)$. An example is $H(e, X) = \text{det}(X)$. The eigenvalues of $I + \lambda e \otimes e$ are $1 + \lambda$ and 1, the latter has multiplicity $n - 1$. Thus,

$$
H(e, \lambda e \otimes e + I) = 1 + \lambda = o(\lambda^n), \text{ as } \lambda \to \infty. \quad \square
$$
4. Auxiliary Functions

In this section, we record observations about auxiliary functions that are used in the proofs of the theorems in this work. We recall that \( k = k_1 + 1 \) and \( \gamma = k_1 + 2 = k + 1 \).

**Lemma 4.1.** Let \( 1 < \bar{\beta} < \beta \) and \( y \in \mathbb{R}^n \). Set \( r = |x - y| \) for all \( x \in \mathbb{R}^n \). For \( r \geq 0 \), define

\[
v(r) = \int_0^r (1 + \tau^p)^{-1} d\tau, \quad \text{where} \quad p = \frac{\beta - \bar{\beta}}{\beta}.
\]

Then (i) \( 0 < p < 1 \), (ii) \( (1 - p)\beta = \bar{\beta} \), and

(iii) \( r^\beta(1 + R^{\bar{\beta}p})^{-1} \leq v(r) \leq r^\beta, \ \forall 0 \leq r \leq R \), and, for any \( R > 0 \).

Set \( c_p = [2(1-p)]^{-1} \). If \( R > 1 \) then

(iv) \( c_p (r^\beta - R^\beta) \leq v(r) - v(R) \leq (2c_p)(r^\beta - R^\beta) \), for \( r \geq R \).

Moreover, in \( r > 0 \), we have

(v) \( v'(r) = \frac{\beta r^{\beta-1}}{1 + r^{\bar{\beta}p}} \leq \beta \min\left(\frac{r^{\bar{\beta}-1}}{\beta^{\bar{\beta}-1}} \right) \),

(vi) \( rv'(r) = \frac{\beta r^\beta}{1 + r^{\bar{\beta}p}} \leq \beta \min\left(\frac{r^\beta}{\beta^\beta} \right), \)

and (vii) \( v''(r) = \beta r^{\beta-2} \left[ \frac{\beta - 1 + (\bar{\beta} - 1)r^{\bar{\beta}p}}{r^{\beta} + \bar{\beta}} \right] \).

Next, we have

(viii) \( \frac{(v'(r))^k}{r} = \left( \frac{\beta}{1 + r^{\bar{\beta}p}} \right)^k \leq \beta^k \min\left(\frac{r^{k\beta-\gamma}}{\beta^k r^{k\beta-\gamma}} \right), \ \forall r > 0, \)

and (ix) \( \bar{\beta} - 1 \leq \frac{rv''(r)}{v'(r)} = \frac{\beta - 1 + (\bar{\beta} - 1)r^{\bar{\beta}p}}{1 + r^{\bar{\beta}p}} \leq \beta - 1 \).

Finally,

(x) \( \frac{v''(r)}{(v'(r))^2} \leq \frac{\beta - 1}{\beta r^\beta} + \frac{\bar{\beta} - 1}{\beta r^\beta}, \ \forall r > 0, \) and (xi) \( \frac{\bar{\beta} - 1}{\beta r^\beta} \leq \frac{v''(r)}{(v'(r))^2} \leq \frac{2(\beta - 1)}{\beta r^\beta}, \ \forall r \geq 1. \)

**Proof.** Parts (i) and (ii) follow easily. Part (iii) is a consequence of the bound \( 1 + \tau^p \leq 1 + R^{\beta}, \ \forall \tau \leq R^\beta \). Part (iv) follows by noting that \( \tau^p \leq 1 + \tau^p \leq 2\tau^p, \ \tau \geq 1 \), (ii) and writing

\[
v(r) = v(R) + \int_{R^\beta}^r (1 + \tau^p)^{-1} d\tau.
\]

Parts (v), (vi) and (viii) are easily obtained by the estimate \( 1 + r^{\bar{\beta}p} \geq \max(1, r^{\beta p}) \) and noting that \( \gamma = k + 1 \) and \( \beta - \bar{\beta} = p\beta \).
To see (vii), we differentiate (v) and use (ii) to find
\[ v''(r) = \beta \left[ \frac{(\beta - 1)r^{\beta-2} - p\beta r^{\beta^2 + \beta - 2}}{1 + r^{\beta^2}} \right] = \beta r^{\beta - 2} \left[ \frac{(\beta - 1)(1 + r^{p\beta}) - p\beta r^{p\beta}}{(1 + r^{p\beta})^2} \right] \]

Applying (v), (vii) and using \( \bar{\beta} < \beta \), (ix) follows. To see (x) and (xi), use (ii), (v) and (vii) to get
\[ \frac{v''(r)}{(v'(r))^2} = \frac{\beta - 1 + (\bar{\beta} - 1)r^{p\beta}}{\beta r^\beta} = \frac{\beta - 1 + \bar{\beta} - 1}{\beta r^\beta}. \]

Since \( \bar{\beta} < \beta \) and \( r > 1 \), the estimates in (xi) hold. \( \square \)

**Remark 4.2.** We now list observations based on Lemma 4.1. These arise from the various cases described in Theorems 2.1 and 2.2.

Recall that \( k = k_1 + 1, \gamma = k + 1 = k_1 + 2 \) and \( \sigma \) is as in Theorems 2.1 and 2.2. Set \( \gamma^* = \gamma/k \). We discuss the following three cases.

- **Case (A)** \( \beta = \gamma^* = 2 \) and \( \bar{\beta} = 2 - \varepsilon \), where \( 0 < \varepsilon < 1 \) and \( k = 1 \).
- **Case (B)** \( \beta = \bar{\beta} = \gamma^* \) and \( k > 1 \).
- **Case (C)** \( \beta = \gamma^* \) and \( \bar{\beta} = \sigma/\sigma - 1 \), where \( \sigma > \gamma \) and \( k \geq 1 \).

**Case (A) \( k = 1 \):** Take \( \beta = 2 \) and \( \bar{\beta} = 2 - \varepsilon \). From Lemma 4.1, \( \bar{p} = \varepsilon/2 \) and
\[ v(r) = \int_0^r (1 + \tau^{\varepsilon/2})^{-1} d\tau. \]

Let \( 0 < \varepsilon < 1 \), then \( 1 - p = (2 - \varepsilon)/2 > 0 \). We apply Lemma 4.1 (iii), (iv), (vi), (vii), (viii), (ix) and (x). Thus,
\[ (iii) \quad \min \left( \frac{r^{2-\varepsilon} - r^2}{2}, \frac{r^2}{1 + r^\varepsilon} \right) \leq v(r) \leq \min \left( \frac{r^{2-\varepsilon}}{r^2}, \frac{r^{2-\varepsilon}}{1 + r^\varepsilon} \right), \quad \forall r \geq 0, \]
\[ (iv) \quad \frac{r^{2-\varepsilon} - R^{2-\varepsilon}}{2} \leq v(r) - v(R) \leq 2 \left( r^{2-\varepsilon} - R^{2-\varepsilon} \right), \quad \forall r \geq R \text{ and } \forall R > 1. \]

Next,
\[ (vi) \quad \min \left( \frac{r^{2-\varepsilon}}{r^2}, \frac{r^{2-\varepsilon}}{r^2} \right) \leq r v'(r) \leq 2 \min \left( \frac{r^{2-\varepsilon}}{r^2}, \frac{r^{2-\varepsilon}}{r^2} \right), \quad (viii) \quad \frac{v'(r)}{r} \leq 2 \min(1, r^{-\varepsilon}). \]

Finally,
\[ (ix) \quad 1 - \varepsilon \leq \frac{r v''(r)}{v'(r)} \leq 1, \quad \forall r > 0, \quad (xi) \quad \frac{1 - \varepsilon}{2 r^{2-\varepsilon}} \leq \frac{v''(r)}{(v'(r))^2} \leq \frac{1}{r^{2-\varepsilon}}, \quad \forall r > 1. \]

**Case (B) \( k > 1 \):** Set \( \beta = \bar{\beta} = \gamma^* \) and \( v(r) = r^{\gamma^*} \).
Using that $\gamma = k + 1$ and $k(\gamma^* - 1) = 1$, we have

1. $rv'(r) = \gamma^*_r^\gamma$,  
2. $\frac{(v'(r))^k}{r} = (\gamma^*)_k$,  
3. $rv''(r) = \gamma^* - 1 = \frac{1}{k}$,

4. $\frac{v''(r)}{(v'(r))^2} = \left(\frac{\gamma^* - 1}{\gamma^*}\right)r^{-\gamma^*} = \frac{1}{\gamma^*r^{\gamma^*}}$.

**Case (C) $k \geq 1$:** Set $\beta = \gamma^*$ and $\bar{\beta} = \sigma/(\sigma - 1)$, where $\sigma > \gamma$.

Since $\sigma > \gamma$, we have that $\beta > \bar{\beta}$. Using that $\gamma = k + 1$, we get

$$p = \frac{\beta - \bar{\beta}}{\beta} = \gamma^* - \sigma/(\sigma - 1) = \frac{\gamma(\sigma - 1) - k\sigma}{\gamma(\sigma - 1)} = \frac{\sigma - \gamma}{\gamma(\sigma - 1)} > 0.$$

Set

$$v(r) = \int_0^r (1 + \tau^p)^{-1} d\tau.$$

We list the observations obtained by applying parts (iii), (iv), (vii), (viii), (ix) and (xi) of Lemma 4.1.

Let $R > 1$. Parts (iii) and (iv) read

1. $\min\left(\frac{\gamma^*}{2}, \frac{r^\gamma}{\sigma/(\sigma - 1)}\right) \leq v(r) \leq \min\left(\frac{r^\gamma}{\gamma^*}, \frac{r^\gamma}{\sigma/(\sigma - 1)}\right), \forall r \geq 0,$
2. $c_p \left(\frac{r^\sigma}{\sigma - 1} - \frac{R^\sigma}{\sigma - 1}\right) \leq v(r) - v(R) \leq (2c_p) \left(\frac{r^\sigma}{\sigma - 1} - \frac{R^\sigma}{\sigma - 1}\right), \forall r \geq R.$

Next,

1. $\frac{\gamma^*}{2} \min\left(\frac{r^\sigma}{\sigma - 1}, \frac{r^\gamma}{\gamma^*}\right) \leq rv'(r) \leq \gamma^* \min\left(\frac{r^\sigma}{\sigma - 1}, \frac{r^\gamma}{\gamma^*}\right),$
2. $\frac{(v'(r))^k}{r} = \left(\frac{\gamma^*}{1 + r^p\gamma^*}\right)^k$, and

$$\left(\frac{\gamma^*}{2}\right)^k \min\left(1, \frac{1}{r^{(\sigma - \gamma)/(\sigma - 1)}}\right) \leq \frac{(v'(r))^k}{r} \leq (\gamma^*)_k^k \min\left(1, \frac{1}{r^{(\sigma - \gamma)/(\sigma - 1)}}\right).$$

The lower bounds in (iii), (vii) and (viii) have been obtained by considering the intervals $0 \leq r \leq 1$ and $r \geq 1$.

Finally, since $\sigma > \gamma \geq 2$, Lemma 4.1 (ix) and (xi) read

1. $\frac{1}{\sigma} \leq \frac{rv''(r)}{v'(r)} \leq \gamma^* - 1,$
2. $\frac{(\gamma^* \sigma)^{-1}}{r^\sigma/(\sigma - 1)} \leq \frac{v''(r)}{(v'(r))^2} \leq \frac{2}{r^\sigma/(\sigma - 1)}, \forall r \geq 1.$

We make an observation that applies to the various auxiliary functions we make use of in this work.
Remark 4.3. The sub-solutions and super-solutions in this work involve a $C^1$ function of $t$ and a $C^1$ function $v(r)$, see Remark 42. We verify that the expressions for the operator $H$, that arise from the use of these functions, hold in the sense of viscosity at $r = 0$. For $r > 0$, $v(r)$ is $C^\infty$. See Lemma 41.

Let $\kappa(t) \geq 0$ be a $C^1$ function in $t \geq 0$. Set $r = |x|$ and $w(x,t) = \kappa(t)v(r)$, where $v(r)$ is as in (B) and (C) in Remark 42. Note that in (A), $v$ is $C^2$. Thus, we discuss

$$v(r) = \begin{cases} \gamma_r, & \beta = \gamma^*, \\ \int_0^\gamma_r (1 + \tau^r)^{-1}d\tau, & \beta = \gamma^*, \beta = \sigma(\sigma - 1)^{-1}. \end{cases}$$

Here $\gamma^* = \gamma/k$. Since $k \geq 1$, we have that $1 \leq \gamma^* \leq 2$. The case of interest is $\gamma^* < 2$.

Recall (3.4) in Remark 31. Taking $r > 0$ and setting $e = x/r$ and $w = \kappa(t)v(r)$, we get with a slight rearrangement

$$H(Dw, D^2 w + Z(w) Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t = \chi(t)(\kappa(t))^\sigma |v'(r)|^\sigma - \kappa'(t)v(r)$$

(4.1) $$+ \frac{(v'(r)\kappa(t))^k}{r} H\left(e, I + \left(\frac{rv''(r)}{v'(r)} - 1 + \kappa(t)(r^\gamma(r))Z(w)\right)e \otimes e \right).$$

We now recall parts (viii) and (ix) in Lemma 41. See also Cases (B) and (C) in Remark 42. Thus, $v(0) = v'(0) = 0$ and $rv''(r)/v'(r) \to 2 - 1$ and $(v'(r))^k/r \to 2 - 1$ as $r \to 0$. It is clear that the right hand side of (4.1) may be extended continuously to $r = 0$. Set the limit (as $r \to 0$) of the right hand side of (4.1) as

$$\hat{H}(0) + \chi(t)L(\sigma), \text{ where } \hat{H}(0) = (\gamma^*\kappa(s))^k H\left(e, I + (\gamma^* - 2)e \otimes e \right),$$

and $L(\sigma) = 1$, if $\sigma = 0$, and $L(\sigma) = 0$, if $\sigma = 0$.

Note that $\hat{H}(0) \geq 0$ since $\gamma^* - 2 \geq -1$. Our goal is to show that

$$H(Dw, D^2 w + Z(w) Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t = \hat{H}(0) + \chi(s)L(\sigma).$$

holds at points $(0, s)$, i.e., at $r = 0$ and $s > 0$, in the viscosity sense.

Let $s > 0$. Suppose that $\psi$, $C^1$ in $t$ and $C^2$ in $x$, is such that $(w - \psi)(x,t) \leq (w - \psi)(o,s)$, for $(x,t)$ near $(o,s)$. Thus,

$$w(x,t) = \kappa(t)v(r) \leq \langle D\psi(o,s), x \rangle + \psi_t(o,s)(t-s) + o(|x| + |t-s|),$$

as $(x,t) \to (o,s)$. Since $\kappa(t)$, $v(r) \geq 0$ and $v'(0) = 0$, we have that $\psi_t(o,s) = 0$, $D\psi(o,s) = 0$.

Next, using $w(x,t) = \kappa(t)v(r) \leq \langle D^2\psi(o,s)x, x \rangle/2 + o(|x|^2 + |t-s|)$, as $(x,t) \to (o,s)$, and recalling $\beta < 2$ and Lemma 41(vii), it is clear that $D^2\psi(o,s)$ does not exist and $w$ is a sub-solution of (4.1).

Now, let $\psi$, $C^1$ in $t$ and $C^2$ in $x$, be such that $(w - \psi)(x,t) \geq (u - \psi)(o,s)$, for $(x,t)$ near $(o,s)$. Thus, $\kappa(t)v(r) \geq \langle D\psi(o,s), x \rangle + \psi_t(o,s)(t-s) + o(|x| + |t-s|)$, as
\((x, t) \to (o, s)\). As argued before, \(D\psi(o, s) = 0\) and \(\psi_t(o, s) = 0\). If \(D^2\psi(o, s)\) does not exist then \(w\) is a super-solution.

If \(D^2\psi(o, s)\) exists then

\[
H \left( D\psi, D^2\psi + Z(\psi) D\psi \otimes D\psi \right)(o, s) + \chi(s) |D\psi|^{\sigma}(o, s) - \psi_t(o, s) = \chi(s) L(\sigma).
\]

(4.3)

We now observe that since \(\beta = \gamma/k < 2\) and \(\gamma = k + 1\), we have \(k = k_1 + 1 > 1\) and hence, \(k_1 > 0\). Applying Condition B (see (2.2)), \(H(0, D^2\psi)(o, s) = 0\) and (4.3) reads

\[
H \left( D\psi, D^2\psi + Z(\psi) D\psi \otimes D\psi \right)(o, s) + \chi(s) |D\psi|^{\sigma}(o, s) - \psi_t(o, s) = \chi(s) L(\sigma).
\]

Since \(\hat{H}(0) \geq 0\), using (4.2), we see that \(w\) is a super-solution. \(\square\)

## 5. Super-solutions

Our goal in this section is to construct super-solutions whose growth rates, for large \(r\), are as stated in Theorem 2.1. The auxiliary functions discussed in Remark 4.2 are used to achieve our goal. The construction involves making separate estimates for small \(r\) and for large \(r\). For small \(r\), we employ (3.4) and, for large \(r\), we use (3.5), see Remark 3.1.

The section has been divided into two parts: (I) \(0 \leq \sigma \leq \gamma\) and (II) \(\sigma > \gamma\). The work in Part I is further divided into two sub-parts (i) \(k = 1\) and (ii) \(k > 1\). Part (II) provides a unified work for \(k \geq 1\).

The super-solutions we construct are of the kind

\[
w(x, t) = m + at + b(1 + t)v(r), \quad \text{in} \quad \mathbb{R}^n_T, \quad \text{where} \ a \geq 0 \ \text{and} \ 0 < b < 1,
\]

where \(v\) is \(C^1\) in \(\mathbb{R}^n\) and \(C^2\) in \(\mathbb{R}^n \setminus \{o\}\) and \(-\infty < m < \infty\). We choose \(v\) as

\[
either \ v(r) = \int_0^r \frac{1}{1 + \tau^p} d\tau \ 
or \ v(r) = r^\beta, \quad \text{in} \ r \geq 0,
\]

for some appropriate \(\beta\) and \(p\) (or \(\bar{\beta}\)), see Lemma 4.1. The scalars \(a\) and \(b\) are determined later.

In proving that \(w\) is a super-solution for appropriate \(a\) and \(b\), we also calculate the dependence of \(a\) on \(b\), thus, aiding our calculation of \(\lim_{b \to 0^+} a\). This is important in showing the claims in Theorem 2.1.

Throughout this section \(\beta = \gamma/k = \gamma^*\) regardless of the form of \(v(r)\), see (5.1) and Remark 4.2. The quantity \(\bar{\beta}\), however, depends on \(k\) and \(\sigma\), see (5.3) below.

We begin with some preliminary calculations before moving on to Parts I and II. Set

\[
\alpha = \sup_{[0, T]} |\chi(t)|, \quad \ell = \inf_{\mathbb{R}} Z, \quad L = \sup_{\mathbb{R}} Z \quad \text{and} \quad \gamma^* = \gamma/k.
\]

(5.2)
We assume that $0 < \ell \leq L < \infty$. We recall that

\begin{equation}
(5.3) \quad \beta = \gamma^* = \begin{cases} 2, & k = 1 \\ \gamma/k, & k > 1, \end{cases} \quad \text{and} \quad \bar{\beta} = \begin{cases} 2 - \varepsilon, & k = 1, \ 0 \leq \sigma \leq \gamma, \\ \gamma^*, & k > 1, \ 0 \leq \sigma \leq \gamma, \\ \sigma/(\sigma - 1), & k \geq 1, \ \sigma > \gamma. \end{cases}
\end{equation}

Moreover, we require that

\begin{equation}
(5.4) \quad \text{(i) if } \sigma = 0, \text{ take } 0 < \varepsilon < 1/8, \quad \text{and} \quad \text{(ii) if } \sigma > 0, \text{ take } 0 < \varepsilon < \sigma/8.
\end{equation}

Next, we provide upper bounds for $H$. These will be done for small $r$ and for large $r$ separately. Recall $w$ from (5.1).

**Step 1:** For small $r$, we use (3.4) with $\kappa(t) = 1 + t$ to obtain that

\begin{equation}
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t
= \frac{[b(1 + t)v'(r)]^k}{r} H(e, I + \left(\frac{rv''(r)}{v'(r)} - 1 + b(1 + t)(rv'(r))Z(w)\right) e \otimes e) + \chi(t)(b(1 + t)v'(r))^\sigma - a - bv(r).
\end{equation}

For large $r$, we use (3.5) (or (3.6)) to obtain that

\begin{equation}
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t
= b^k[(1 + t)v'(r)]^\gamma H\left(e, \frac{I - e \otimes e}{(1 + t)rv'(r)} + \left(\frac{v''(r)}{(1 + t)(v'(r))^2} + bZ(w)\right) e \otimes e\right) + \chi(t)(b(1 + t)v'(r))^\sigma - a - bv(r).
\end{equation}

**Step 2: Bounds for $H$.** We employ Remark 4.2 and use estimates for $v(r)$ (and its derivatives) in (5.5) and (5.6) to obtain upper bounds for $H$. Assume $R \geq 1$. A value will be chosen later.

(i) $0 \leq r \leq R$: Since $Z(w) \leq Z(m) \leq L$ (see (5.2)), define

\begin{equation}
(5.7) \quad M(b, r) = \max_{|e| = 1} H(e, I + b(1 + T)L^\gamma r^\gamma e \otimes e).
\end{equation}

By using the monotonicity in Condition A (see (2.1)(i)) and Condition B (see (2.2)), $M(b, r)$ is non-decreasing in $r$ and $b$, $M(b, r) \geq \max_{|e| = 1} H(e, I) > 0$ and

\begin{equation}
M(b, r) \leq M(1, R) \leq R^\gamma M(1, 1), \quad \forall R > 1.
\end{equation}

Recall parts (vii) and (ix) of Cases A, B and C in Remark 4.2. It is seen that

\begin{equation}
(5.8) \quad \frac{rv''(r)}{v'(r)} - 1 \leq \gamma^* - 2 = \frac{1 - k}{k} \leq 0 \quad \text{and} \quad rv' \leq \gamma^* r^\gamma, \quad \forall k \geq 1.
\end{equation}
We apply the above to \( (5.5) \) and use monotonicity to get
\[
H \left( e, I + \left( \frac{rv''(r)}{v'(r)} - 1 + b(1 + t)Z(w)rv'(r) \right) e \otimes e \right) \\
\leq H \left( e, I + \gamma^* b(1 + T)Z(w)r^\gamma e \otimes e \right).
\]
Since \( Z(w) \leq Z(m) \leq L \), using \( (5.7) \) and the bound for \( M(b, r) \) we obtain that for \( 0 \leq r \leq R \),
\[
(5.9) \quad H \left( e, I + \left( \frac{rv''(r)}{v'(r)} - 1 + b(1 + t)Z(w)rv'(r) \right) e \otimes e \right) \leq R^\gamma M(1, 1), \quad \forall R > 1.
\]

Next, we recall the upper bound \( (v'(r))^k/r \leq (\gamma^*)^k \) from part (viii) of Cases A, B and C in Remark 4.2. Thus, \( (5.5), (5.6) \) and \( (5.9) \) lead to the estimate
\[
H(Dw, D^2 w + Z(w) Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \\
\leq [b\gamma^* (1 + T)]^k M(1, 1) R^\gamma + \alpha[b(1 + T)v'(r)]^\sigma - a - bv(r).
\]

(ii) \( 1 \leq R \leq r \): We recall parts (vii) and (xi) of Cases A, B and C in Remark 4.2. Part (vii) of the Cases A, B and C show that
\[
rv'(r) \geq \begin{cases}
  r^{2-\varepsilon}, & k = 1, \quad 0 \leq \sigma \leq \gamma, \\
  \gamma^* r^{\gamma^*}, & k > 1, \quad 0 \leq \sigma \leq \gamma, \\
  (\gamma^*/2)r^{\sigma/(-1)}, & k \geq 1, \quad \sigma > \gamma.
\end{cases}
\]
In the last inequality, \( \sigma/(-1) < \gamma^* \), if \( \sigma > \gamma \). Thus, using the above and part (xi) of the Cases A, B and C, we obtain
\[
\max \left( \frac{1}{rv'(r)}, \frac{v''(r)}{(v'(r))^2} \right) \leq \begin{cases}
  2r^{-(2-\varepsilon)}, & k = 1, \quad 0 \leq \sigma \leq \gamma, \\
  2r^{-\gamma^*}, & k > 1, \quad 0 \leq \sigma \leq \gamma, \\
  2r^{-\sigma/(-1)}, & k \geq 1, \quad \sigma > \gamma.
\end{cases}
\]
Thus,
\[
(5.11) \quad \max \left( \frac{1}{rv'(r)}, \frac{v''(r)}{(v'(r))^2} \right) \leq 2, \quad \text{in } r \geq 1.
\]

Noting that both quantities on the left hand side of \( (5.11) \) are non-negative, using Condition A and \( (5.11) \), the term \( H \) in \( (5.6) \) yields in \( t \geq 0 \),
\[
H \left( e, \frac{I - e \otimes e}{(1 + t)rv'(r)} + \left( \frac{v''(r)}{(1 + t)(v'(r))^2} + bZ(w) \right) e \otimes e \right) \\
\leq H \left( e, \frac{I - e \otimes e}{rv'(r)} + \left( \frac{v''(r)}{(v'(r))^2} + bZ(w) \right) e \otimes e \right) \\
\leq H(e, 2(I - e \otimes e) + 2e \otimes e + bZ(w)I) \leq H(e, (2 + L)I),
\]
since \( I \geq e \otimes e, 0 < b < 1 \) and \( 0 < Z \leq L \).
Observing that \( w \geq m \), we define
\[
(5.13) \quad \bar{M} = \max_{|e|=1} H(e, (2 + L)I).
\]
Using Conditions A, B and C, \( \bar{M} \geq H(e, 2I) = 2H(e, I) > 0 \).

Thus, in \( r \geq R \geq 1 \), by using (5.13) in (5.12) we get
\[
H \left( e, \frac{I - e \otimes e}{(1 + t)rv'(r)} + \left( \frac{v''(r)}{(1 + t)(v'(r))^2} + Z(w)b \right) e \otimes e \right) \leq \bar{M}.
\]
Using (5.2) and the above upper bound in (5.6) we get
\[
H(Dw, D^2w + Z(w) Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t
\leq b^k[(1 + T)v'(r)]^\gamma \bar{M} + \alpha[b(1 + T)v'(r)]^\sigma - a - bv(r).
\]

**Step 3: Additional bounds:** We record the following bounds that would be useful for what follows. Refer to part (vii) of Cases A, B and C in Remark 4.2. In \( r \geq 0 \),
\[
(5.15) \quad v'(r) \leq \begin{cases} 
2 \min(r^{1-\varepsilon}, r), & k = 1, \ 0 \leq \sigma \leq \gamma, \\
\gamma^* r^{\gamma^*-1}, & k > 1, \ 0 \leq \sigma \leq \gamma, \\
\gamma^* \min(r^{1/(\sigma-1)}, r^{\gamma^*-1}), & k \geq 1, \ \sigma > \gamma.
\end{cases}
\]

**Constructions of Super-solutions:** We remind the reader that \( k_2 = 1, \gamma = k + 1 = k_1 + 2 \) and \( \gamma^* = \gamma/k \) throughout.

**Part I:** \( 0 \leq \sigma \leq \gamma \). In what follows we take \( R \geq 1 \), to be determined later.

**Sub-part (i):** \( k = 1 \). Thus, \( k_1 = 0 \). Let \( \varepsilon > 0 \) be small. Recall from (5.3) that \( \gamma = \gamma^* = 2 \). We take \( p = \varepsilon/2 \). Thus, using (5.1) we get
\[
(5.16) \quad w(x, t) = m + at + b(1 + t)v(r), \quad \text{in } \mathbb{R}^n_T,
\]
where
\[
v(r) = \int_0^{r^2} (1 + \tau^{\varepsilon/2})^{-1} d\tau,
\]
and \( a \geq 0 \) and \( 0 < b < 1 \) are to be determined.

We address the interval \( 0 \leq r \leq R \). Using (5.15) and \( 0 \leq r \leq R \), we get that \( v'(r) \leq 2R \). Employing this in the second term on the right hand side of (5.10) we get
\[
H(Dw, D^2w + Z(w) Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t
\leq 2b(1 + T)M(1, 1) R^2 + \alpha(2b(1 + T))^\sigma R^\sigma - a.
\]
We choose
\begin{align}
(5.17) \quad a &= \begin{cases} 
\alpha + 2b(1 + T)M(1, 1)R^2 + bR^{2-\varepsilon}/2, & \sigma = 0, \\
\alpha(2b(1 + T))^{\sigma} R^{\sigma} + 2b(1 + T)M(1, 1)R^2 + bR^{2-\varepsilon}/2, & 0 < \sigma \leq \gamma.
\end{cases}
\end{align}
This ensures that \( w \) is a super-solution in \( 0 \leq r \leq R \).

Next, we address \( r \geq R \). We use the estimate \( v'(r) \leq 2r^{1-\varepsilon} \) (see (5.15)) in the second term of the right hand side of (5.14) to obtain
\begin{align}
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t 
& \leq 4b((1 + T)^r 1^{1-\varepsilon})^2 \bar{M} + \alpha(2b(1 + T)r 1^{1-\varepsilon})^{\sigma} - a - bv(r).
\end{align}
We apply the lower bound in part (iv) of Case A in Remark 4.2, that is,
\begin{align}
v(r) & \geq \frac{r^{2-\varepsilon} - R^{2-\varepsilon}}{2}, \quad \forall r \geq R \geq 1.
\end{align}
Thus, we obtain from (5.18) that
\begin{align}
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t 
& \leq 4b((1 + T)^2 \bar{M} r^{2-2\varepsilon} + \alpha(2b(1 + T)r^{\sigma(1-\varepsilon)})^{\sigma} - a - b \left( \frac{r^{2-\varepsilon} - R^{2-\varepsilon}}{2} \right)
\leq 4b((1 + T)^2 \bar{M} r^{2-2\varepsilon} + \alpha(2b(1 + T)r^{\sigma(1-\varepsilon)})^{\sigma} - a - \frac{br^{2-\varepsilon}}{2},
\end{align}
where in the last inequality we have used the expression for \( \hat{a} = a - bR^{2-\varepsilon}/2 \), see (5.17).

(a): \( \sigma = 0 \). Using (5.17) and that \( \hat{a} \geq 0 \), the right hand side in (5.19) yields
\begin{align}
4b((1 + T)^2 \bar{M} r^{2-2\varepsilon} + \alpha - a - \left( \frac{b}{2} \right) r^{2-\varepsilon} \leq br^{2-2\varepsilon} \left( 4(1 + T)^2 \bar{M} - r^\varepsilon \right).
\end{align}
Choose \( R \) such that \( R^\varepsilon = \max (1, 8(1 + T)^2 \bar{M}) \). Clearly, \( w \) is a super-solution in \( \mathbb{R}_T^n \).

We record that the above choice for \( R \) and (5.17) yield that
\begin{align}
(5.20) \quad \lim_{b \to 0^+} a = \alpha, \quad \text{for } \sigma = 0.
\end{align}

(b): \( 0 < \sigma \leq 2 \). Note that \( \gamma = 2 \) and \( \hat{a} \geq 0 \). The right hand side of (5.19) yields
\begin{align}
4b((1 + T)^2 \bar{M} r^{2-2\varepsilon} + \alpha(2b(1 + T)r^{\sigma(1-\varepsilon)})^{\sigma} - \frac{br^{2-\varepsilon}}{2} - \hat{a}
\leq 4b((1 + T)^2 \bar{M} r^{2-2\varepsilon} + \alpha(2b(1 + T)r^{\sigma(1-\varepsilon)})^{\sigma} - \frac{br^{2-\varepsilon}}{2}
\leq br^{2-2\varepsilon} \left( 4(1 + T)^2 \bar{M} + \alpha(2b(1 + T)r^{\sigma(1-\varepsilon)})^{\sigma} \frac{br^{2-\varepsilon}}{2} - \frac{R^\varepsilon}{2} \right),
\end{align}
in \( r \geq R \geq 1 \). Also, \( (2 - \sigma)(1 - \varepsilon) \geq 0 \). Set
\begin{align}
(5.22) \quad P = 4(1 + T)^2 \bar{M} \quad \text{and} \quad Q = \alpha(2(1 + T))^{\sigma}.
\end{align}
Select
\[
R = \begin{cases} 
\max \left \{(2(1 + P))^{1/\varepsilon}, \frac{(2Qb^{\sigma-1})^{1/(2-\sigma)(1-\varepsilon)}}{1 - \sigma}, \frac{(2P + 2Q)^{1/\varepsilon}}{1 - \sigma}\right \}, & 0 < \sigma < 1, \\
\max \left \{1, \frac{(2P + 2Q)^{1/\varepsilon}}{1 - \sigma}\right \}, & 1 \leq \sigma \leq 2.
\end{cases}
\]

For \(0 < \sigma < 1\), we have set \(r = R\) in the second term of (5.21) and chosen \(R\), and for \(1 \leq \sigma \leq 2\), we have taken \(r = b = 1\) in the second term of (5.21).

Using (5.22) and (5.23) in (5.21) and recalling (5.19), \(w\) is a super-solution in \(\mathbb{R}^n_T\).

We recall the expression for \(a\) in (5.17) and claim that \(a \to 0\) as \(b \to 0\). This is clear for \(1 \leq \sigma \leq 2\) because of the choice in (5.23). The case of interest is \(0 < \sigma < 1\) since \(R \to \infty\) as \(b \to 0\). It suffices to show that \(bR^2 \to 0\) as \(b \to 0\) as this would imply the same of \(bR\) and \(bR^{2-\varepsilon}\). Taking \(b\) small in (5.23), one can write
\[
R = Kb^{(\sigma-1)/(2-\sigma)(1-\varepsilon)} \quad \text{and} \quad bR^2 = K^2b^{1+2(\sigma-1)/(2-\sigma)(1-\varepsilon)},
\]
for an appropriate \(K\) that is independent of \(b\). A simple calculation shows that
\[
1 + \frac{2(\sigma - 1)}{(2 - \sigma)(1 - \varepsilon)} = \frac{\sigma(1 + \varepsilon) - 2\varepsilon}{(2 - \sigma)(1 - \varepsilon)}.
\]

From (5.4), \(0 < \varepsilon < \sigma/8\) and, hence, \(\sigma(1 + \varepsilon) - 2\varepsilon > 0\). Recalling (5.16), (5.17) and (5.20) we obtain that, for any small \(\varepsilon > 0\),
\[
\lim_{b \to 0} a = \begin{cases} 
\alpha, & \sigma = 0, \\
0, & 0 < \sigma \leq 2.
\end{cases}
\]

Sub-part (ii): \(k > 1\). We set
\[
w(x, t) = m + at + b(1 + t)r^{\gamma^*}, \quad \text{in} \ \mathbb{R}^n_T.
\]

Recall that \(\gamma^* = \gamma/k = 1 + 1/k\), \(k = k_1 + 1\) and \(\gamma = k + 1 = k_1 + 2\).

Consider \(0 \leq r \leq R\), where \(R > 1\) is to be determined. We recall (5.10) and use \(v'(r) = \gamma^* r^{\gamma^* - 1}\) to obtain
\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \\
\leq [b\gamma^*(1 + T)]^k M(1, 1) R^{\gamma^*} + \alpha [b\gamma^*(1 + T) R^{\gamma^* - 1}]^\sigma - a - bv(r).
\]

Noting that \(\gamma^* - 1 = 1/k\), we choose,
\[
a = b^k [\gamma^*(1 + T)]^k M(1, 1) R^{\gamma^*} + \alpha b^\sigma [\gamma^*(1 + T)]^\sigma R^{\sigma/k}.
\]

Thus, \(w\) is a super-solution in \(\overline{B_R(o)} \times (0, T)\).

Now consider \(r \geq R\). Using (5.14) and \(v'(r) = \gamma^* r^{\gamma^* - 1}\), we get
\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \\
\leq b^k [\gamma^*(1 + T) r^{\gamma^* - 1}]^\sigma M + \alpha [b\gamma^*(1 + T) r^{\gamma^* - 1}]^\sigma - a - br^{\gamma^*}.
\]
Since γ* - 1 = 1/k, clearly, γ(γ* - 1) = γ*. Setting E = γ*(1 + T), the above reads
\[ H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \]
(5.27)
\[ \leq (E^\gamma M) b^k r^{\gamma*} + (\alpha E^\sigma) b^\sigma r^{\sigma/k} - a - br^{\gamma*}. \]

We analyze separately: (1) σ = 0, (2) 1 < σ ≤ γ, and (3) 0 < σ ≤ 1.

(1) σ = 0: Setting R = 1 and recalling (5.26), the right hand side of (5.27) yields
\[ (E^\gamma M) b^k r^{\gamma*} + \alpha - a - br^{\gamma*} \leq br^{\gamma*} (E^\gamma M b^{k-1} - 1) \leq 0, \]
if we choose 0 < b ≤ min \(1, (E^\gamma M)^{-1/(k-1)}\). Thus, w is a super-solution in \(R^n_1\) and
(5.28)
\[ \lim_{b \to 0} a = \alpha. \]

(2) 1 < σ ≤ γ: The right hand side of (5.27) is bounded above, in r ≥ R, by
(5.29)
\[ (E^\gamma M) b^k r^{\gamma*} + (\alpha E^\sigma) b^\sigma r^{\sigma/k} - a - br^{\gamma*} \leq br^{\gamma*} \left[ (E^k M) b^{k-1} + \frac{(\alpha E^\sigma) b^{\sigma-1}}{R(\gamma-\sigma)/k} - 1 \right]. \]

Setting R = 1 in the second term of the right hand side of (5.29), we get
\[ (E^\gamma M) b^k r^{\gamma*} + (\alpha E^\sigma) b^\sigma r^{\sigma/k} - br^{\gamma*} \leq br^{\gamma*} \left[ (E^k M) b^{k-1} + (\alpha E^\sigma) b^{\sigma-1} - 1 \right]. \]
Choosing 0 < b < 1, small enough, we get that w is super-solution in \(R^n_1\). Moreover, using (5.26) lim_{b \to 0} a = 0.

(3) 0 < σ ≤ 1: We recall (5.29) i.e.,
\[ (E^\gamma M) b^k r^{\gamma*} + (\alpha E^\sigma) b^\sigma r^{\sigma/k} - a - br^{\gamma*} \leq br^{\gamma*} \left[ (E^k M) b^{k-1} + \frac{(\alpha E^\sigma) b^{\sigma-1}}{R(\gamma-\sigma)/k} - 1 \right]. \]
We choose
\[ b < \min \left[ 1, \left( \frac{1}{4E^\gamma M} \right)^{(k-1)} \right] \quad \text{and} \quad R = \max \left[ 1, \left\{ \left(4\alpha E^\sigma b^{\sigma-1}\right)^{k/(\gamma-\sigma)} \right\} \right]. \]
It is clear that w is a super-solution in \(R^n_T\).

Our next task is to show that \(\lim_{b \to 0} a = 0\). Recalling (5.26) and comparing the terms \(b^k R^{\gamma*}\) and \((bR^{1/k})^\sigma\), we see that it is enough to show that \(b^k R^{\gamma*} \to 0\) as \(b \to 0\). This is clear if σ = 1. Assuming that σ < 1 and using the choice for R, we see that (use γ* = γ/k)
\[ b^k R^{\gamma*} = Kb^k \left[ b^{k(\sigma-1)/(\gamma-\sigma)} \right]^{\gamma*} = Kb^{k+\gamma(\sigma-1)/(\gamma-\sigma)}, \]
for some K independent of b. Using that γ = k + 1 = k_1 + 2, we calculate
\[ k + \frac{\gamma(\sigma - 1)}{\gamma - \sigma} = \frac{k_1 \gamma + \sigma}{\gamma - \sigma} > 0. \]
The claim holds.
Summarizing from Sub-Parts (i) (see (5.24)) and (ii) (see (1), (2) and (3)), we get
\[
\lim_{b \to 0} a = \begin{cases} 
\alpha, & \sigma = 0, \\
0, & 0 < \sigma \leq \gamma.
\end{cases}
\]

**Part II** $\sigma > \gamma, \ k \geq 1$: We set
\[
w(x, t) = m + at + b(1 + t)v(r), \text{ where}
\]
\[
v(r) = \int_0^{\gamma r} \frac{1}{1 + \tau p} d\tau \quad \text{and} \quad p = \frac{\sigma - \gamma}{\gamma(\sigma - 1)}.
\]
We recall estimates stated in Case C of Remark 4.2.

Take $R \geq 1$ and consider $0 \leq r \leq R$. We employ (5.10) i.e.,
\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \\
\leq [b\gamma^* (1 + T)]^k M(1, 1) R^{\gamma^*} + \alpha[b(1 + T)v'(r)]^\sigma - a - bv(r).
\]
Noting that $\sigma - 1 \geq \gamma - 1 = k$, using (5.15) ($v'(r) \leq \gamma^* r^{1/(\sigma - 1)}$) and setting $E = \gamma^*(1 + T)$, we get from above that
\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \\
\leq (bE)^k M(1, 1) R^{\gamma^*} + \alpha(bE) R^{\sigma/(\sigma - 1)} - a.
\]
Set $c_p = [2(1 - p)]^{-1}$ and select
\[
a = (bE)^k M(1, 1) R^{\gamma^*} + \alpha(bE) R^{\sigma/(\sigma - 1)} + c_p b R^{\sigma/(\sigma - 1)}.
\]
Thus, $w$ is a super-solution in $B_R(o) \times (0, T)$.

In $r \geq R$, we use (5.14) i.e.,
\[
H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \\
\leq b^k [(1 + T)v'(r)]^\gamma M + \alpha[b(1 + T)v'(r)]^\sigma - a - bv(r).
\]
From part (iv) of Case C in Remark 4.2 we have
\[
v(r) \geq c_p \left(r^{\sigma/(\sigma - 1)} - R^{\sigma/(\sigma - 1)}\right), \forall r \geq R.
\]
Using (5.15) ($v'(r) \leq \gamma^* r^{1/(\sigma - 1)}$), the lower bound for $v(r)$ stated above, $E = \gamma^*(1 + T)$ and (5.32) in the right hand side of (5.33), we get
\[
\begin{align*}
& b^k [(1 + T)v'(r)]^\gamma M + \alpha[b(1 + T)v'(r)]^\sigma - a - bv(r) \\
& \leq b^k (E^* M)^{\gamma/(\sigma - 1)} + b^\sigma (\alpha E^*) R^{\sigma/(\sigma - 1)} - a - c_p b \left(r^{\sigma/(\sigma - 1)} - R^{\sigma/(\sigma - 1)}\right) \\
& \leq b^k (E^* M)^{\gamma/(\sigma - 1)} + b^\sigma (\alpha E^*) R^{\sigma/(\sigma - 1)} - c_p b R^{\sigma/(\sigma - 1)} \\
& \leq b^{\sigma/(\sigma - 1)} \left[ \frac{b^{k-1}(E^* M)}{R^{(\sigma - \gamma)/(\sigma - 1)}} + b^{\sigma - 1}(\alpha E^*) - c_p \right],
\end{align*}
\]
where we have used $1 < \gamma < \sigma$ and $r \geq R$.

For $k > 1$, take $R = 1$ and $b > 0$ small enough (depending on $\sigma$, $\alpha$, $E$ and $M$) so that (5.34) is negative. If $k = 1$ we take

$$R = \max \left[ 1, \left( \frac{4E^\gamma M}{c_p} \right)^{(\sigma-1)/(\sigma-\gamma)} \right] \quad \text{and} \quad b \leq \min \left[ 1, \left( \frac{c_p}{4\alpha E^\gamma} \right)^{1/(\sigma-1)} \right].$$

With these selections, the right hand side of (5.34) is negative. Thus, (5.33) implies that $w$ is super-solution in $\mathbb{R}^n_T$. Recalling (5.32), we see that

$$\lim_{b \to 0} a = 0.$$ 

6. Sub-solutions

In this section, we construct sub-solutions. We place no restrictions on the growth rate if $0 \leq \sigma < \gamma$. This includes also the case when $\sup_{[0,T]} |\chi(t)|$ is small enough. However, in general, a lower bound in the case $\sigma \geq \gamma$ is needed for our work. We remark that the auxiliary functions employed are closely related to the functions used for super-solutions.

We achieve our goal by utilizing the expressions in Remark 3.1, in particular, the versions in (3.7) and (3.8). Thus, setting $w(x,t) = v(r) - \kappa(t)$ and assuming that $v'(r) \leq 0$, we get that

$$H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t$$

$$= \frac{|v'(r)|^k}{r} H(e, r|v'(r)|Z(w) + 1 - \frac{rv''(r)}{v'(r)} e \otimes e - I)$$

$$+ \chi(t)|v'(r)|^\sigma + \kappa'(t).$$

(6.1)

Next, we recall Condition C (see (2.4)), (2.6) and (3.1) and set

$$N = \min_{|e|=1} H(e, -I), \quad K_0 = \frac{\Lambda(\lambda_0)}{\lambda_0} \quad \text{and} \quad \ell = \inf \frac{s}{\bar{M}} Z(s).$$

Set $\mathcal{H}(\lambda) = \min_{|e|=1} H(e, e \otimes e - \lambda^{-1}I)$ and $\mathcal{H} = \min_{|e|=1} H(e, e \otimes e)$. We record that

$$0 < \ell < \infty, \quad N < 0, \quad \mathcal{H}(\lambda) \geq K_0 > 0, \quad \forall \lambda \geq \lambda_0, \quad \text{and} \quad \lim_{\lambda \to \infty} \mathcal{H}(\lambda) = \mathcal{H}.$$ 

An auxiliary function and preliminary calculations.

Fix $R > 1$. Let $p \geq 1$ and $E \geq 0$, to be determined later. In $0 \leq r < R$, set

$$\omega = r/R, \quad v(\omega) = E \int_0^{\omega^2} (\tau^p - 1)^{-1} d\tau = ER^{2(p-1)} \int_0^{\omega^2} (\tau^p - R^{2(p-1)})^{-1} d\tau.$$ 

Hence, $v$ is defined in $0 \leq \omega < 1$. We will often write $v(\omega)$ as $v(r)$.
Clearly,
\[ v \leq 0, \quad v(0) = 0, \quad v'(r) \leq 0 \quad \text{and} \quad v(r) \to -\infty \quad \text{as} \quad r \to R. \]

Set
\[ L = L(\omega) = \frac{2E}{1 - \omega^{2p}}, \quad \forall 0 \leq \omega < 1. \]

Differentiating \( v(r) \) in (6.1) and using (6.5), we get
\[
(i) \quad v'(r) = -\frac{L(\omega)\omega}{R} = -\frac{L(\omega)r}{R^2}, \quad (ii) \quad v''(r) = -\frac{L(\omega)}{R^2} \left( \frac{1 + (2p-1)\omega^{2p}}{1 - \omega^{2p}} \right), \\
(iii) \quad \frac{rv''(r)}{v'(r)} = \frac{1 + (2p-1)\omega^{2p}}{1 - \omega^{2p}}, \quad \text{and} \quad (iv) \quad \frac{rv''(r)}{v'(r)} - 1 = \frac{2p\omega^{2p}}{1 - \omega^{2p}}.
\]

Using \( k = k_1 + 1, \gamma = k_1 + 2 \) and (6.6)(i), we get
\[ \frac{|v'(r)|^k}{\ell} = \frac{L(\omega)^k\omega^k}{R^k(\omega R)} = \frac{L(\omega)^k\omega^{k_1}}{R^{\ell}}. \]

Next, recalling (6.1), (6.3), (6.5) and (6.6)(i) and (iv), we see that
\[
1 - \frac{rv''(r)}{v'(r)} + rZ(|v'(r)|) = Z(\cdot)L(\omega)\omega^2 - \frac{2p\omega^{2p}}{1 - \omega^{2p}} = \frac{2EZ(\cdot)\omega^2}{1 - \omega^{2p}} - \frac{2p\omega^{2p}}{1 - \omega^{2p}} \geq 2\omega^2 \left( \frac{\ell E - p\omega^{2(p-1)}}{1 - \omega^{2p}} \right).
\]

**Sub-solutions.**

We provide separate treatments for \( 0 \leq \sigma \leq \gamma \) and \( \sigma \geq \gamma \). The case \( \sigma = \gamma \) will be addressed in both situations.

**Case I:** \( 0 \leq \sigma \leq \gamma \). Let \( \mu \in (-\infty, \infty) \) and recall (6.4). Set in \( 0 \leq r < R, \omega = r/R, \)
\[ \bar{w}(x,t) = \mu + v(r) - Ft, \quad \text{where} \quad v(r) = E \int_0^\omega (\tau^p - 1)^{-1} d\tau, \]
where \( E, F \) and \( p \geq 2 \) are to be determined. Of importance is the limit \( \lim_{R \to \infty} F(R) \).

Employing (6.1), (6.5), (6.7) and (6.8), we see that
\[
H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^{\sigma} - \bar{w}_t \geq \frac{L(\omega)^k\omega^{k_1}}{R^{\ell}} H(e, 2\omega^2 \left( \frac{\ell E - p\omega^{2(p-1)}}{1 - \omega^{2p}} \right) e \otimes e - I) - \alpha \left( \frac{L(\omega)\omega}{R} \right)^{\sigma} + F.
\]

The sub-solution we construct will depend on \( p \) and \( R \). Select
\[ E = \frac{p(p+1)}{\ell} \quad \text{and} \quad L(\omega) = \frac{2p(p+1)}{\ell(1 - \omega^{2p})}. \]
As \( 0 \leq \omega < 1 \) and \( p \geq 2 \), we get that \( 2\omega^2 (\ell E - p\omega^{2(p-1)}) \geq 2p^2\omega^2 \). Set

\[
J_p(\omega) = \frac{2p^2\omega^2}{1 - \omega^{2p}} = \left( \frac{p}{p+1} \right) \ell L(\omega)\omega^2,
\]

where we have used (6.3). Recalling (6.10) we see that

\[
H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^\sigma - \bar{w}_t
\geq \left( \frac{L(\omega)^k\omega^{k_1}}{R^\gamma} \right) H(e, J_p(\omega) e \otimes e - I) - \alpha \left( \frac{L(\omega)\omega}{R} \right)^\sigma + F.
\]

We fix \( \frac{1}{\sqrt{2}} \leq \omega_0 < 1 \) and consider separately: (i) \( 0 \leq \omega \leq \omega_0 \), and (ii) \( \omega_0 \leq \omega < 1 \).

(i) \( 0 \leq \omega \leq \omega_0 \): Recall (6.2), (6.3) and (6.12). We bound

\[
H(e, J_p(\omega) e \otimes e - I) \geq H(e, -I) \geq -|N|.
\]

Using the above in (6.13) we get that

\[
H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^\sigma - \bar{w}_t
\geq F - \left( \frac{L(\omega)^k|N|\omega^{k_1}}{R^\gamma} + \alpha \left( \frac{L(\omega)\omega}{R} \right)^\sigma \right).
\]

From (6.11), \( L(\omega) \) is increasing in \( \omega \). Since \( 0 \leq \omega \leq \omega_0 \), we choose

\[
F = \frac{L(\omega_0)^k|N|\omega_0^{k_1}}{R^\gamma} + \alpha \left( \frac{L(\omega_0)\omega_0}{R} \right)^\sigma.
\]

Thus (6.14) implies that \( \bar{w} \) is a sub-solution in \( B_{\omega_0R(o)} \times (0, T) \).

(ii) \( \omega_0 \leq \omega < 1 \): The work will lead to a determination of \( p \).

We estimate \( J_p(\omega) \) (recall (6.12)). Since \( J_p \) is increasing in \( \omega \), we get that

\[
J_p(\omega) \geq J_p(\omega_0) = \frac{2p^2\omega_0^2}{1 - \omega_0^{2p}} \geq p^2,
\]

since \( \omega_0^2 \geq 1/2 \). We note also that \( J_p(\omega_0) \to \infty \) if \( p \to \infty \).

Using the monotonicity and the homogeneity of \( H \) (Conditions A and B), \( \omega_0 \leq \omega < 1 \) and (6.16), we have that

\[
\min_{|e| = 1} H(e, J_p(\omega) e \otimes e - I) \geq J_p(\omega) \min_{|e| = 1} H\left(e, e \otimes e - \frac{I}{J_p(\omega_0)}\right)
\geq J_p(\omega) \mathcal{H}(p^2) \geq K_0 J_p(\omega_0) \geq K_0 p^2 > 0.
\]

Here we have used (6.3) and chosen \( p \geq p_0 \), where \( p_0 \geq 2 \) is large enough.
From here on we take \( p \geq p_0 \) such that (6.17) holds (see (6.16)). Next, using (6.17) in (6.13), we obtain

\[
H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^\sigma - \bar{w}_t \\
\geq \frac{L(\omega)^k\omega^{k_1}J_\nu(\omega)H(p^2)}{\bar{R}} - \alpha \left( \frac{L(\omega)\omega}{\bar{R}} \right)^\sigma + F \\
= \ell H(p^2) \left( \frac{p}{p+1} \right) \left( \frac{L(\omega)\omega}{\bar{R}} \right) - \alpha \left( \frac{L(\omega)\omega}{\bar{R}} \right)^\sigma + F.
\]

In the last inequality, we have used (6.12) and \( \gamma = k + 1 = k_1 + 2 \).

We factor \((\omega L(\omega)/\bar{R})^\sigma\) from (6.18) and use that \( \omega_0 \leq \omega < 1 \), to obtain that

\[
H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^\sigma - \bar{w}_t \\
\geq \left( \frac{\omega L}{\bar{R}} \right)^\sigma \left[ \ell H(p^2) \left( \frac{p}{p+1} \right) \left( \frac{L(\omega_0)\omega_0}{\bar{R}} \right) \right]^{\gamma-\sigma} - \alpha + F.
\]

We address \( 0 \leq \sigma \leq \gamma \). We make comments about \( \sigma = \gamma \) in Sub-Case (c).

**Sub-Case (a)** \( 0 \leq \sigma < \gamma \): As noted earlier, \( \bar{w} \) is a sub-solution in \( B_{\omega_0\bar{R}}(o) \times (0,T) \).

We refer to (6.19) and select \( \bar{R} \) such that

\[
\frac{L(\omega_0)}{\bar{R}} = \frac{1}{\omega_0} \left[ \left( \frac{\alpha}{\ell H(p^2)} \right) \left( \frac{1+p}{p} \right) \right]^{1/(\gamma-\sigma)}.
\]

With this choice, \( \bar{w} \) is a sub-solution in \( B_{\bar{R}}(o) \times (0,T) \).

Using (6.11) and (6.20), we get that for some \( K_1 = K_1(\alpha, \gamma, \ell, \omega_0, K_0) > 0 \),

\[
R = K_1 \left( p^{\gamma-\sigma+1}(p+1)^{k-\sigma} \right)^{1/(\gamma-\sigma)} = O(p^2), \quad \text{as } p \to \infty,
\]

where we have used \( \gamma = k + 1 \). Thus, \( R \to \infty \) if and only if \( p \to \infty \).

We now calculate \( \lim_{R \to \infty} F \). From (6.11) and (6.15), we write \( F \) as the sum of two terms \( X \) and \( Y \) as follows:

\[
F = \frac{|N|L(\omega_0)^k\omega_0^{k_1}}{\bar{R}^\gamma} + \alpha \left( \frac{L(\omega_0)\omega_0}{\bar{R}} \right)^\sigma = X + Y.
\]

We use (6.20), \( \gamma = k + 1 \) and \( k = k_1 + 1 \) to observe that

\[
\lim_{p \to \infty} X = \lim_{R \to \infty} X = \frac{|N|\omega_0^{k_1}}{\bar{R}} \left( \frac{L(\omega_0)}{\bar{R}} \right)^k = 0.
\]

Next, using (6.20), we get

\[
Y = \alpha \left( \frac{L(\omega_0)\omega_0}{\bar{R}} \right)^\sigma \left[ \frac{\alpha^\gamma}{(\ell H(p^2))^{\sigma}} \left( \frac{p+1}{p} \right)^{\sigma} \right]^{1/(\gamma-\sigma)}.
\]
Referring to (6.3), we see that

\[(6.21) \lim_{R \to \infty} F = \lim_{R \to \infty} Y = \lim_{p \to \infty} Y = \left( \frac{\alpha^\gamma}{(\ell H)^\sigma} \right)^{1/(\gamma - \sigma)}, \quad 0 \leq \sigma < \gamma.\]

From (6.21),

\[(6.22) \lim_{R \to \infty} F = \begin{cases} \alpha, & \sigma = 0, \\ (\alpha^\gamma/(\ell H)^\sigma)^{1/(\gamma - \sigma)}, & 0 < \sigma < \gamma. \end{cases}\]

**Sub-Case (b) $\chi \geq 0$:** An inspection of (6.13), (6.15) and (6.19) $(-\alpha$ is replaced by $+\alpha$) shows that $F = X$. Thus, $\tilde{w}$ is a sub-solution in $B_R(0) \times (0, T)$ for any $\sigma \geq 0$ and any $R > 0$, as there are no restrictions on $R$. Clearly, $\lim_{R \to \infty} F = 0$.

**Sub-Case (c) $\sigma = \gamma$:** An inspection of (6.19) shows that if \[\alpha < \ell H = (\inf_s Z(s))(\min_{|e| = 1} H(e, e \otimes e)),\]
then by selecting $p$, large enough, the right hand side of (6.18) may be written as

\[\left( \frac{\omega L}{R} \right)^\gamma \left[ \ell H(p^2) \left( \frac{p}{p + 1} \right) - \alpha \right] + F \geq 0.\]

For the chosen $p$, $\tilde{w}$ is a sub-solution in $B_R(0) \times (0, T)$ for any $R > 0$. Moreover, $R$ is independent of $p$ and $F(R) \to 0$ as $R \to \infty$. However, if $\alpha$ exceeds the above value then it is not clear if this conclusion holds. See Case II below.

**Case II $\gamma \leq \sigma < \infty$:** We assume a lower bound for $u$ and adapt the work in Section 5. See also the bounds on $H$ which appear in the beginning of Section 5.

Recall that $k_2 = 1$, $\gamma = k + 1 = k_1 + 2$ and $\gamma^* = \gamma/k$. We divide the work into two sub-cases.

**Sub-Case (i) $\sigma = \gamma$:** We assume that $\alpha \geq \ell H$ and refer to Sub-Parts (i) and (ii) of Part I in Section 5.

**(i1) $k = 1$:** Here $\gamma = \gamma^* = 2$. We assume that for any $\varepsilon > 0$, small, $\sup_{|x| \geq r}(-u(x, t)) \leq o(|r|^{2-\varepsilon})$ as $r \to \infty$. We take

\[\tilde{w}(x, t) = m - at - b(1 + t)v(r), \quad \text{where} \quad v(r) = \int_0^{r^2} (1 + \tau^{\varepsilon/2})^{-1} d\tau.\]

**(i2) $k > 1$:** Thus, $\gamma = k + 1 > 2$. We assume that $\sup_{|x| \geq r}(-u(x, t)) \leq o(|r|^{\gamma^*})$ as $r \to \infty$. We take

\[\tilde{w} = m - at - b(1 + t)r^{\gamma^*}.\]
Sub-Case (ii) $\sigma > \gamma$: We allow $k \geq 1$ and refer to Part II of Section 5. We assume that $\sup_{[x] \geq r}(-u(x,t)) \leq o(|r|^\sigma/(\sigma - 1))$ as $r \to \infty$. We take

$$\bar{w} = m - at - b(1 + t)v(r), \text{ where } v(r) = \int_0^r (1 + \tau^p)^{-1}d\tau \text{ and } p = \frac{\sigma - \gamma}{\gamma(\sigma - 1)}.$$

Since $\bar{w}_r \leq 0$ for all the cases described above, we recall the two versions in (3.7), i.e., for $R > 0$, to be determined,

$$H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w})$$

$$= \frac{|\bar{w}_r|^k}{r} H\left( e, \left( r|\bar{w}_r|Z(\bar{w}) + 1 - \frac{r\bar{w}_r}{\bar{w}_r} \right) e \otimes e - I \right), \forall 0 \leq r \leq R,$n

$$= |\bar{w}_r|^\gamma H\left( e, I - e \otimes e + \left( \frac{\bar{w}_r}{\bar{w}_r^2} + Z(\bar{w}) \right) e \otimes e \right), \forall r \geq R.$$

From parts (ix) of Cases A, B and C of Remark 4.2, we have that

$$\frac{r\bar{w}_r}{\bar{w}_r} = \frac{r\bar{w}_r}{\bar{w}_r} \leq \left\{ \begin{array}{ll}
1, & \sigma = \gamma = 2, k = 1, \\
\gamma^* - 1, & \sigma = \gamma > 2, k > 1, \\
\gamma^* - 1, & \sigma > \gamma \geq 2, k \geq 1.
\end{array} \right.$$

Using the first version in (6.23) and noting that $\gamma^* \leq 2, 1 - rw_{rr}/w_r \geq 0$, one estimates (see (6.2))

$$H\left( e, \left( r|\bar{w}_r|Z(\bar{w}) + 1 - \frac{r\bar{w}_r}{\bar{w}_r} \right) e \otimes e - I \right) \geq H(e, -I) \geq -|N|, \quad 0 \leq r \leq R.$$

Hence, in $0 \leq r \leq R$,

$$H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^\sigma - \bar{w}_t$$

$$\geq \frac{- [\alpha(b(1 + T)v'(r))k|N| + \alpha(b(1 + T)v'(r))^\sigma] - a - bv(r)}{r}.$$

Next, employing the estimate in (5.8), i.e., $v'(r) \leq \gamma^*r^{\gamma^*-1}$, and $(\gamma^* - 1)k = \gamma^*$, we get, in $0 \leq r \leq R$,

$$H(D\bar{w}, D^2\bar{w} + Z(\bar{w})D\bar{w} \otimes D\bar{w}) + \chi(t)|D\bar{w}|^\sigma - \bar{w}_t$$

$$\geq \frac{- [(\gamma^*b(1 + T)r^{\gamma^*-1})k|N| + \alpha(\gamma^*b(1 + T)r^{\gamma^*-1})^\sigma] - a}{r}$$

$$\geq \frac{- [(\gamma^*b(1 + T))^k|N| + \alpha(\gamma^*b(1 + T))^\sigma R^{\gamma^*(\gamma^*-1)} - a]}{r}.$$

As done in (5.32), we select an appropriate $a$. Thus, $\bar{w}$ is a sub-solution in $B_R(o) \times (0, T)$. 
Next, in $r \geq R$, one finds that (see (5.6))

$$|\tilde{w}_r|^\gamma H\left(e, \frac{I - e \otimes e}{r \tilde{w}_r} + \left(\frac{\tilde{w}_{rr}}{\tilde{w}_r^2} + Z(\tilde{w})\right) e \otimes e\right)$$

$$= (b(1 + t)v'(r))^\gamma H\left(e, \frac{e \otimes e - I}{b(1 + t)rv'(r)} + \left(Z(\tilde{w}) - \frac{v''(r)}{b(1 + t)(v'(r))^2}\right) e \otimes e\right)$$

(6.24)

$$\geq b^k((1 + t)v'(r))^\gamma H\left(e, \left(bZ(\tilde{w}) + \frac{1}{rv'(r)} - \frac{v''(r)}{(v'(r))^2}\right) e \otimes e - \frac{I}{rv'(r)}\right),$$

where we have factored out $1/b$ and used that $\gamma = k + 1$ and $e \otimes e - I \leq 0$.

We now recall (5.11) i.e.,

$$0 < \min\left(\frac{1}{rv'(r)}, \frac{v''(r)}{(v'(r))^2}\right) \leq \max\left(\frac{1}{rv'(r)}, \frac{v''(r)}{(v'(r))^2}\right) \leq 2, \text{ in } r \geq R \geq 1.$$  

Employing this estimate in (6.24) and disregarding the term with $Z$, we get

$$|\tilde{w}_r|^\gamma H\left(e, \frac{I - e \otimes e}{r \tilde{w}_r} + \left(\frac{\tilde{w}_{rr}}{\tilde{w}_r^2} + Z(\tilde{w})\right) e \otimes e\right) \geq b^k((1 + t)v'(r))^\gamma S,$$

where

$$S = \min_{|e|=1} H(e, -2(I + e \otimes e)).$$

Clearly, by (6.2), $S \leq N < 0$ and we get that

$$H(D\tilde{w}, D^2\tilde{w} + Z(\tilde{w})D\tilde{w} \otimes D\tilde{w}) + \chi(t)|D\tilde{w}|^\sigma - \tilde{w}_t$$

$$\geq - \left[b^k((1 + T)v'(r))^\gamma|S| + \alpha(b(1 + T))^{\sigma}(v'(r))^{\sigma} - a - bv(r)\right],$$

which is analogous to (5.14). As done in Section 5, a choice for $b$ (see (5.35)) can now be made. Also, $\lim_{b \to 0} a = 0$. \qed

7. PROOFS OF THEOREMS 2.1-2.3

Let $T > 0$ and $(x,t) \in \mathbb{R}^n \times (0,T), \ n \geq 2$. Set

(i) $\mu = \inf_{\mathbb{R}^n} h, \ \nu = \sup_{\mathbb{R}^n} h$, and (ii) assume that $-\infty < \mu \leq \nu < \infty$.

Recall that $k = k_1 + 1, \ \gamma = k + 1$ and $\alpha = \sup_{(0,T]} |\chi(t)|$.

**Proof of Theorem 2.1**. Set $r = |x|$ and let $\eta > 0$ be small. Choose $\rho > \rho_0$, where $\rho_0$ is large enough so that

(7.1) \[ \sup_{[0,\rho] \times [0,T]} u(x,t) \leq \eta \rho^\beta, \ \forall \rho \geq \rho_0. \]

where $\beta$ is as described in the statement of Theorem 2.1.

**Proof of Theorem 2.1(a) $\sigma = 0$**. Recall from (5.1) that the super-solution $w(x,t)$, with $m = \nu$, is

(7.2) \[ w(x,t) = \nu + at + bv(r), \ \text{where } a > 0, b > 0 \text{ and } \lim_{b \to 0} a = \alpha. \]
For details, see Part I in Section 5, (5.20) in Sub-Part (i), (5.28) in Sub-Part (ii) and (5.30). Note that (7.3)

(a) If \( k = 1 \) then \( v(r) = \int_0^r (1 + \tau^{\varepsilon/2})^{-1}d\tau \), and (b) if \( k > 1 \) then \( v(r) = r^{\gamma^*} \).

See (5.16) and (5.25). Also, in (7.1) and (7.3), \( b \) where \( \beta = \gamma = \gamma/k \).

Recall that \( w, \) in (7.2), is a super-solution in \( \mathbb{R}^n_+ \) for any \( 0 < b < b_0 \), where \( b_0 \) is small enough, and for an appropriate \( a \) that depends on \( b \).

We observe that by part (iv) of Cases A, B and C of Remark 4.2, \( w \) is a super-solution such that (7.1) holds. Then \( u(x, 0) \leq h(x) \leq \nu, \forall x \in \mathbb{R}^n \). Clearly, \( w(x, 0) = \nu + bv(r) \geq u(x, 0) \), for \( |x| \leq \rho \). At \( |x| = \rho \), we have

\[
w(x, t) \geq bv(R) \geq 2\eta \rho^\beta \geq u(x, t).
\]

Thus, \( w \geq u \) on the parabolic boundary of \( B_{\rho}(o) \times (0, T) \) and Lemma 3.2 to conclude that \( u(x, t) \leq u(x, t) \) in \( B_{\rho}(o) \times (0, T) \) for any large \( \rho \), i.e.,

\[
u + at + bv(r), \forall |x| \leq \rho.
\]

Letting \( \rho \to \infty \), we see that \( u(x, t) \leq \nu + at + bv(r) \) in \( \mathbb{R}^n_+ \). Since this holds for any small \( b \), using (7.2), we obtain \( u(x, t) \leq \nu + at \). The claim holds.

**Proof of Theorem 2.1(b)** \( 0 < \sigma \leq \gamma \): The functions \( w, v \) and \( \beta \) are as in (7.2), (7.3) and (7.4). Refer to Part I in Section 5 and see Sub-Parts (i) and (ii). Arguing as in the proof of Theorem 2.10(a) above, we see that \( u(x, t) \leq \nu + at + bv(r), \) in \( \mathbb{R}^n_+ \), for any \( b > 0 \) small enough. Recalling (5.30), we get that \( u(x, t) \leq \nu \) and the claim holds.

**Proof of Theorem 2.1(c)** \( \sigma > \gamma \): Refer to Part II in Section 4. The quantity \( \beta = \sigma/(\sigma - 1) \) in (7.1). From (5.31)

\[
w(x, t) = \nu + at + b(1 + t)v(r), \quad \text{where} \quad v(r) = \int_0^r (1 + \tau^{p/2})^{-1}d\tau \quad \text{and} \quad p = \frac{\sigma - \gamma}{\gamma(\sigma - 1)},
\]

where \( a > 0 \) and \( b > 0 \). The function \( w \) is super-solution in \( \mathbb{R}^n_+ \) for any \( 0 < b < b_0 \), where \( b_0 \) is small enough, and an appropriate \( a \) that depends on \( b \). Moreover, by (5.36),

\[
\lim_{b \to 0} a = 0.
\]

The rest of the proof is similar to the proof of Theorem 2.1(a). \( \square \)

**Proof of Theorem 2.2**
We start with the proofs of parts (a)-(c).

**Proof of Theorem 2.2(a), (b) and (c)** $0 \leq \sigma \leq \gamma$: Fix $y \in \mathbb{R}^n$ and $R > 0$ and set $r = |x - y|$. Recall from (6.9)

$$\bar{w}(x, t) = \mu + v(r) - Ft, \text{ where } v(r) = E \int_0^\omega (\tau^p - 1)^{-1} d\tau \text{ and } \omega = r/R.$$ See Sub-Cases (a), (b) and (c) of Case I in Section 6. From (6.22) we see that

$$\lim_{R \to \infty} F = \begin{cases} \alpha, & \sigma = 0, \\ (\alpha^\gamma/(\ell H)^{\sigma})^{1/(\gamma - \sigma)}, & 0 < \sigma < \gamma, \\ 0, & \sigma = \gamma, \alpha < \ell H.\end{cases}$$

Let $u$ be as in the theorem. Clearly, $\bar{w}(x, 0) \leq h(x) \leq u(x, 0)$ in $\mathbb{R}^n$. Since $\sup |u| < \infty$ in $B_R(y) \times [0, T]$, $\bar{w}(x, t) \leq u(x, t)$ on $R' \leq |x - y| < R$ for some $R' < R$. By Lemma 3.2, $\bar{w} \leq u$ in $B_R(y) \times (0, T)$.

Thus, $w(y, t) \leq u(y, t)$ and $u(y, t) \geq \mu - Ft$. Letting $R \to \infty$, we get,

$$u(y, t) \geq \begin{cases} \mu - \alpha t, & \sigma = 0, \\ \mu - t(\alpha^\gamma/(\ell H)^{\sigma})^{1/(\gamma - \sigma)}, & 0 < \sigma < \gamma, \\ \mu, & \sigma = \gamma, \alpha < \ell H.\end{cases}$$

In order to show the claim for $\chi \geq 0$, take $\alpha = 0$ and refer to Sub-Part (b) in Part I in Section 5.

**Proof of Theorem 2.2(d) and (e)** $\sigma \geq \gamma$: If $\sigma = \gamma$ then we take $\alpha \geq \ell H$.

Assume that

$$(7.5) \sup_{B_R(0) \times [0, T]} (-u(x, t)) \leq o(R^\beta), \text{ as } R \to \infty.$$ Recall Sub-Cases (i) and (ii) in Case 2 in Section 6. We take

$$\bar{w}(x, t) = \mu - at - b(1 + t)v(r), \text{ in } \mathbb{R}_T^n.$$ Suppose that $\sigma = \gamma$ and $\alpha \geq \ell H$. If (a) $k = 1$ and $\gamma = 2$ then $\beta = 2 - \varepsilon$, for any small $\varepsilon > 0$, in (7.5), and we take

$$v(r) = \int_0^{\tau^2} (1 + \tau^2)^{-1} d\tau,$$

and (b) $k > 1$ and $\gamma > 2$ then $\beta = \gamma^*$, in (7.5), and we take $v(r) = r^{\gamma^*}$.

If $\sigma > \gamma$ and $k \geq 1$ then $\beta = \sigma/(\sigma - 1)$, in (7.5), and

$$v(r) = \int_0^{r^\gamma} (1 + \tau^p)^{-1} d\tau \text{ where } p = \frac{\sigma - \gamma}{\gamma(\sigma - 1)}.$$ It is to be noted that $\lim_{b \to 0} a = 0$ in the situations stated above. The rest of the proof is similar to that of Theorem 2.1 □
Proof of Theorem 2.3. We take $\alpha = \sigma = 0$ in Theorems 2.1. Let $u > 0$ be as in the statement of the theorem. Set $u > 0$ be as in the statement of the theorem. Set $u > 0$ be as in the statement of the theorem. Set $u > 0$ be as in the statement of the theorem. Set $u > 0$ be as in the statement of the theorem. Let $k \geq 1$.

Proof of Theorem 2.3(a): Since $v$ is a sub-solution we have that $\sup_{B_R(0) \times [0,T]} v(x,t) \leq o(R^{1/k})$ as $R \to \infty$. By Lemma 3.3, $v \in \text{usc}(\mathbb{R}^n_T)$ solves

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \geq 0, \text{ in } \mathbb{R}^n_T,$$

and $v(x,0) \leq \phi^{-1}(h(x))$, for all $x \in \mathbb{R}^n$.

By Theorem 2.1(a), $\sup_{\mathbb{R}^n_T} v \leq \sup_{\mathbb{R}^n} \phi^{-1}(h)$, and thus, $\sup_{\mathbb{R}^n_T} u \leq \sup_{\mathbb{R}^n} g$.

Proof of Theorem 2.3(b): In this case, $v \in \text{lsc}(\mathbb{R}^n_T)$ solves

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \geq 0, \text{ in } \mathbb{R}^n_T,$$

and $v(x,0) \leq \phi^{-1}(h(x))$, for all $x \in \mathbb{R}^n$.

By Theorem 2.2(a), $\inf_{\mathbb{R}^n_T} v \geq \inf_{\mathbb{R}^n} \phi^{-1}(h)$ and hence, $\inf_{\mathbb{R}^n_T} u \geq \inf_{\mathbb{R}^n} h$.

The case $k = 1$ also follows analogously.  □

REFERENCES

[1] G. Akagi, P. Juutinen and R. Kajikiya, *Asymptotic behavior of viscosity solutions for a degenerate parabolic equation associated with the infinity-Laplacian*, Math.Ann. 343 (2009), no 4, 921-953.

[2] T. Bhattacharya and L. Marazzi, *On the viscosity solutions to Trudinger’s equation*, Nonlinear Differential Equations and applications (NoDEA), Vol 22, No 5, 2015. DOI:10.1007/s00030-015-0315-4

[3] T. Bhattacharya and L. Marazzi, *Asymptotics of viscosity solutions of some doubly nonlinear parabolic eqns*. J. of Evolution. Eqns, vol 16, no 4, 2016, 759-788, DOI 10.1007/s00026-015-0319-x

[4] T. Bhattacharya and L. Marazzi, *Erratum to: On the viscosity solutions to Trudinger’s equation*, Nonlinear Differential Equations and Applications (NoDEA), Online (Nov 30, 2016), DOI 10.1007/s00030-016-0423-9

[5] T. Bhattacharya and L. Marazzi, *On the viscosity solution to a class of nonlinear degenerate Parabolic differential equations*, Revista Matematica Complutense, Vol 30, No 3, 2017, 621-656, DOI 10.1007/s13163-017-0299-2

[6] M. G. Crandall, H. Ishii and P. L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27(1992) 1-67.

[7] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer (1993)

[8] P. Juutinen and P. Lindqvist, *Pointwise decay for the solutions of degenerate and singular parabolic equations*, Adv. Differential Equations 14(2009), no. 7-8, 663-684.

[9] N. S. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. 21, 205-226 (1968)

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