Unipotent monodromy and arithmetic $\mathcal{D}$-modules

Abstract. In the framework of Berthelot’s theory of arithmetic $\mathcal{D}$-modules, we introduce the notion of arithmetic $\mathcal{D}$-modules having potentially unipotent monodromy. For example, from Kedlaya’s semistable reduction theorem, the overconvergent isocrystals with Frobenius structure have potentially unipotent monodromy. We construct some coefficients stable under Grothendieck’s six operations, containing overconvergent isocrystals with Frobenius structure and whose objects have potentially unipotent monodromy. On the other hand, we introduce the notion of arithmetic $\mathcal{D}$-modules having quasi-unipotent monodromy. These objects are overholonomic, contain the isocrystals having potentially unipotent monodromy and are stable under Grothendieck’s six operations and under base change.

Let $\mathcal{V}$ be a complete discrete valued ring of mixed characteristic $(0, p)$, $K$ its field of fractions, $k$ its residue field which is supposed to be perfect. Let $X$ be a smooth quasi-projective $k$-variety, $Z := X - Y$ be a simple normal crossing divisor of $X$, let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of $Z$ into irreducible components, and $\prod_{i=1}^r \Sigma_i$ be a subset of $(\mathbb{Z}_p/\mathbb{Z})^r$. Let $E$ be an overconvergent isocrystal on $(Y, X/K)$. Atsushi Shiho defined (see the end of the definition [23, 3.9]) the notion of overconvergent isocrystals having $\prod_{i=1}^r \Sigma_i$-unipotent monodromy. When for any $i$ the sets $\Sigma_i$ are equal to a set $\Sigma_i$, we will say for short “overconvergent isocrystals having $\Sigma$-unipotent monodromy”. When $\Sigma_i = \{0\}$, we retrieve Kedlaya’s unipotent monodromy (see [17]). Without non Liouvilleness conditions, these isocrystals have no finite cohomology and in particular they are not (over)holonomic.

From now, suppose $\Sigma$ is a subgroup of $\mathbb{Z}_p/\mathbb{Z}$ with $p$-adically non Liouville numbers, then it follows from [15, 2.3.13] that overconvergent isocrystals with $\Sigma$-unipotent monodromy are overholonomic. We recall that with Frobenius structures, we already know the stability under Grothendieck’s six operations of the overholonomicity (see [15]) but, without Frobenius structures, the stability under tensor products is still an open question. In this paper, in the framework of Berthelot’s arithmetic $\mathcal{D}$-modules, we introduce the notion of arithmetic $\mathcal{D}$-modules having potentially $\Sigma$-unipotent monodromy (see 3.2.5). For example an overconvergent isocrystal has potentially $\Sigma$-unipotent monodromy if by definition it gets
\( \Sigma \)-unipotent monodromy after some generically etale alteration. By descent from this alteration, we check that they are overholonomic. Moreover, a reformulation of Kedlaya’s semistable reduction theorem (see [17–20]) is that overconvergent isocrystals with some Frobenius structure have potentially unipotent monodromy. We also introduce the notion of arithmetic \( D \)-modules having quasi-\( \Sigma \)-unipotent monodromy (see 3.3.1). These coefficients are overholonomic, contain isocrystals having potentially \( \Sigma \)-unipotent monodromy and are stable under Grothendieck’s six operations and base change. Finally, we construct some coefficients stable under Grothendieck’s six operations and base change, containing overconvergent isocrystals with Frobenius structure and whose objects have potentially unipotent monodromy.

**Notation and convention**

In this paper, we fix \( \Sigma \) a subgroup of \( \mathbb{Z}_p/\mathbb{Z} \) with \( p \)-adically non Liouville numbers. We choose \( \tau : \mathbb{Z}_p/\mathbb{Z} \to \mathbb{Z}_p \) a section of the canonical extension \( \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z} \) such that \( \tau(0) = 0 \).

We also fix \( \mathcal{V} \) a complete discrete valued ring of mixed characteristic \((0, p)\). We denote by \( K \) the field of fractions of \( \mathcal{V} \), \( k \) its residue field which is supposed to be perfect. A formal scheme over \( \mathcal{V} \) means a formal scheme for the \( p \)-adic topology. By convention, our formal schemes are always separated. The special fiber of a formal scheme over \( \mathcal{V} \) will be denoted by the corresponding capital roman letter.

1. Stability under cohomological operations of data of coefficients

1.1. Data of coefficients

**Definition 1.1.1.** We denote by \( \text{DVR} (\mathcal{V}) \) the full subcategory of the category of \( \mathcal{V} \)-algebras whose objects are complete discrete valued rings of mixed characteristic \((0, p)\) with perfect residue field.

1.1.2. Let \( \mathcal{W} \) be an object of \( \text{DVR} (\mathcal{V}) \), and \( \mathcal{X} \) be a smooth formal \( \mathcal{W} \)-scheme. If there is no possible confusion (some confusion might arise if for example we do know that \( \mathcal{V} \to \mathcal{W} \) is finite and etale), for any integer \( m \in \mathbb{N} \), we denote \( \widehat{\mathcal{D}}_\mathcal{X}/\text{Spf}(\mathcal{W})^{(m)} \) (resp. \( \mathcal{D}_\mathcal{X}/\text{Spf}(\mathcal{W}), \mathbb{Q} \)) simply by \( \widehat{\mathcal{D}}_\mathcal{X} \) (resp. \( \mathcal{D}_\mathcal{X}^{\dagger} \)). Berthelot checked the following equivalence of categories (see [4, 4.2.4]):

\[
\lim \to : L\mathcal{D}^{b}_{\mathcal{X}, \text{coh}}(\widehat{\mathcal{D}}_\mathcal{X}^{(\bullet)}) \cong D^{b}_{\mathcal{X}, \text{coh}}(\mathcal{D}_\mathcal{X}^{\dagger}). \tag{1.1.2.1}
\]

The category \( D^{b}_{\mathcal{X}, \text{coh}}(\mathcal{D}_\mathcal{X}^{\dagger}) \) is endowed with its usual t-structure. Via 1.1.2.1, we get a t-structure on \( L\mathcal{D}^{b}_{\mathcal{X}, \text{coh}}(\widehat{\mathcal{D}}_\mathcal{X}^{(\bullet)}) \) whose heart is \( LM_{\mathcal{X}, \text{coh}}(\widehat{\mathcal{D}}_\mathcal{X}^{(\bullet)}) \) (see Notation [14, 2.2.4]). In fact, from [14, 1.2.7] and [14, 2.5.1], we have canonical explicit cohomological functors \( \mathcal{H}^{n} : L\mathcal{D}^{b}_{\mathcal{X}, \text{coh}}(\widehat{\mathcal{D}}_\mathcal{X}^{(\bullet)}) \to LM_{\mathcal{X}, \text{coh}}(\widehat{\mathcal{D}}_\mathcal{X}^{(\bullet)}) \). The equivalence of categories 1.1.2.1 commutes with the cohomological functors \( \mathcal{H}^{n} \) (where the cohomological functors \( \mathcal{H}^{n} \) on \( D^{b}_{\mathcal{X}, \text{coh}}(\mathcal{D}_\mathcal{X}^{\dagger}) \) are the obvious ones), i.e. \( \lim \mathcal{H}^{n}(\mathcal{E}^{(\bullet)}) \) is canonically isomorphic to \( \mathcal{H}^{n}(\lim \mathcal{E}^{(\bullet)}) \).
Last but not least, via Theorem [14, 2.5.7] and Lemma [14, 2.5.2], we remind the equivalence of categories \( \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}}) \cong D_{\text{coh}}^{b}(\mathcal{L}M_{\mathcal{Q}}(\hat{\mathcal{D}}_{\mathcal{X}})) \) which is also compatible with t-structures (the t-structure on \( D_{\text{coh}}^{b}(\mathcal{L}M_{\mathcal{Q}}(\hat{\mathcal{D}}_{\mathcal{X}})) \) is the canonical one as the derived category of an abelian category).

**Definition 1.1.3.** A data \( \mathcal{C} \) of coefficients over \( \mathcal{V} \) will be the data of a full subcategory \( \mathcal{C}(\mathcal{X}/\mathcal{W}) \) of \( \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}})_{/\mathcal{W}} \) for any object \( \mathcal{W} \) of \( \text{DVR}(\mathcal{V}) \), and for any smooth formal scheme \( \mathcal{X} \) over \( \mathcal{W} \). When \( \mathcal{W} \) is understood, we simply write \( \mathcal{C}(\mathcal{X}) \) as a full subcategory of \( \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}}) \). If there is no ambiguity with \( \mathcal{V} \), we simply say a data of coefficients.

**Examples 1.1.4.** 1. We define the data of coefficients \( \mathcal{B}_g \) as follows: for any object \( \mathcal{W} \) of \( \text{DVR}(\mathcal{V}) \), for any smooth formal scheme \( \mathcal{X} \) over \( \mathcal{W} \), the category \( \mathcal{B}_g(\mathcal{X}) \) is the full subcategory of \( \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}}) \) whose unique object is \( \mathcal{O}^{(\bullet)}_{\mathcal{X}} \) (where \( \mathcal{O}^{(\bullet)}_{\mathcal{X}} \) is the constant object \( \mathcal{O}^{m}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \)).

2. We will need the larger data of coefficients \( \mathcal{B}_{\text{div}} \) defined as follows: for any object \( \mathcal{W} \) of \( \text{DVR}(\mathcal{V}) \), for any smooth formal scheme \( \mathcal{X} \) over \( \mathcal{W} \), the category \( \mathcal{B}_{\text{div}}(\mathcal{X}) \) is the full subcategory of \( \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}}) \) whose objects are of the form \( \mathcal{B}^{(\bullet)}_{\mathcal{X}}(T) \), where \( T \) is any divisor of the special fiber of \( \mathcal{X} \) (the sheaf \( \mathcal{B}^{(\bullet)}_{\mathcal{X}}(T) \) is defined in [3, 4.2.4]). From Corollary [14, 3.5.3], we have \( \mathcal{B}^{(\bullet)}_{\mathcal{X}}(T) \in \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}}) \).

3. We define \( \mathcal{B}_{\text{cst}} \) as follows: for any object \( \mathcal{W} \) of \( \text{DVR}(\mathcal{V}) \), for any smooth formal scheme \( \mathcal{X} \) over \( \mathcal{W} \), the category \( \mathcal{B}_{\text{cst}}(\mathcal{X}) \) is the full subcategory of \( \mathcal{L}D_{\mathcal{Q}, \text{coh}}^{b}(\hat{\mathcal{D}}_{\mathcal{X}}) \) whose objects are of the form \( \mathcal{R}\Gamma_{\mathcal{Y}}^{1} \mathcal{O}^{(\bullet)}_{\mathcal{X}} \), where \( \mathcal{Y} \) is a subvariety of the special fiber of \( \mathcal{X} \) and the functor \( \mathcal{R}\Gamma_{\mathcal{Y}}^{1} \) is defined in [6, 3.2.1] (to see that these objects are coherent, we proceed as in the proof of 1.4.5 this is a consequence of Corollary [14, 3.5.3]).

**Definition 1.1.5.** Let \( \mathcal{W} \) be an object of \( \text{DVR}(\mathcal{V}) \) (see Definition 1.1.1). Let \( f : \mathcal{P} \to \mathcal{P} \) be a morphism of smooth formal \( \mathcal{W} \)-schemes. We say that \( f \) is realizable if there exist an immersion \( u : \mathcal{P} \to \mathcal{P}'' \) of smooth formal \( \mathcal{W} \)-schemes, a proper morphism \( \pi : \mathcal{P}'' \to \mathcal{P} \) of smooth formal \( \mathcal{W} \)-schemes such that \( f = \pi \circ u \). When \( \mathcal{P} = \text{Spf} \mathcal{W} \), we say that \( \mathcal{P} \) is a realizable smooth formal \( \mathcal{W} \)-scheme. We remark that any morphism of realizable smooth formal \( \mathcal{W} \)-schemes is a realizable morphism.

Because of the relative duality isomorphism of the form 1.3.12, which is not known in a more general case, we will need to focus on pushforwards by realizable morphisms.

**Definition 1.1.6.** In order to be precise, let us fix some terminology. Let \( \mathcal{C} \) and \( \mathcal{D} \) be two data of coefficients over \( \mathcal{V} \).

1. We will say that the data \( \mathcal{C} \) of coefficients is stable under pushforwards (resp. realizable pushforwards) if for any object \( \mathcal{W} \) of \( \text{DVR}(\mathcal{V}) \), for any morphism (resp. realizable morphism) \( g : \mathcal{X} \to \mathcal{X} \) of smooth formal schemes over \( \mathcal{W} \), for any object \( \mathcal{E}'^{(\bullet)} \) of \( \mathcal{C}(\mathcal{X}') \) with proper support over \( \mathcal{X} \) via \( g \) (i.e., if \( Z' \) is the
support of $\mathcal{E}'(\mathcal{I})$ then the composition $Z' \hookrightarrow X' \xrightarrow{g} X$ is proper), the complex $g_+(\mathcal{E}'(\mathcal{I}))$ is an object of $\mathcal{C}(\mathcal{X})$.

2. We will say that the data $\mathcal{C}$ of coefficients is stable under extraordinary pullbacks (resp. under smooth extraordinary pullbacks) if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any morphism (resp. smooth morphism) $f: \mathcal{Y} \to \mathcal{X}$ of smooth formal schemes over $\mathcal{W}$, for any object $\mathcal{E}(\mathcal{I})$ of $\mathcal{C}(\mathcal{X})$, we have $f^!(\mathcal{E}(\mathcal{I})) \in \mathcal{C}(\mathcal{Y})$.

3. We will say that the data $\mathcal{C}$ of coefficients satisfies the first property (resp. the second property) of Berthelot–Kashiwara theorem or satisfies $BK^1$ (resp. $BK_+$) for short if the following property is satisfied: for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any closed immersion $u: \mathcal{Z} \hookrightarrow \mathcal{X}$ of smooth formal schemes over $\mathcal{W}$, for any object $\mathcal{E}(\mathcal{I})$ of $\mathcal{C}(\mathcal{X})$ with support in $\mathcal{Z}$, we have $u^!(\mathcal{E}(\mathcal{I})) \in \mathcal{C}(\mathcal{Z})$ (resp. for any object $\mathcal{G}(\mathcal{I})$ of $\mathcal{C}(\mathcal{Z})$, we have $u_+(\mathcal{G}(\mathcal{I})) \in \mathcal{C}(\mathcal{X})$). Remark that $BK^1$ and $BK_+$ hold if and only if the data $\mathcal{C}$ of coefficients satisfies (an analogue of) Berthelot–Kashiwara theorem, which justifies the terminology.

4. We will say that the data $\mathcal{C}$ of coefficients is stable under base change if for any morphism $\mathcal{W} \to \mathcal{W}'$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, for any object $\mathcal{E}(\mathcal{I})$ of $\mathcal{C}(\mathcal{X})$, we have $\mathcal{W}' \hat{\otimes}_{\mathcal{W}} \mathcal{E}(\mathcal{I}) \in \mathcal{C}(\mathcal{X} \times_{\text{Spf} \mathcal{W}'} \text{Spf} \mathcal{W'})$ (see Notation 1.4.2).

5. We will say that the data $\mathcal{C}$ of coefficients is stable under tensor products (resp. duals) if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, for any objects $\mathcal{E}(\mathcal{I})$ and $\mathcal{F}(\mathcal{I})$ of $\mathcal{C}(\mathcal{X})$ we have $\mathcal{F}(\mathcal{I}) \hat{\otimes}_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}(\mathcal{I}) \in \mathcal{C}(\mathcal{X})$ (resp. $\mathbb{D}_{\mathcal{X}}(\mathcal{E}(\mathcal{I})) \in \mathcal{C}(\mathcal{X})$).

6. We will say that the data $\mathcal{C}$ of coefficients is stable under local cohomological functors (resp. under localizations outside a divisor), if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, for any object $\mathcal{E}(\mathcal{I})$ of $\mathcal{C}(\mathcal{X})$, for any subvariety $Y$ (resp. for any divisor $T$) of the special fiber of $\mathcal{X}$, we have $\mathbb{R}\Gamma_{\mathcal{Y}}^Y \mathcal{E}(\mathcal{I}) \in \mathcal{C}(\mathcal{X})$ (resp. $(\hat{\mathcal{I}} T) \mathcal{E}(\mathcal{I}) \in \mathcal{C}(\mathcal{X})$), where we use the notation of [6, 3.2.1]).

7. We will say that the data $\mathcal{C}$ of coefficients is stable under shifts if, for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, for any object $\mathcal{E}(\mathcal{I})$ of $\mathcal{C}(\mathcal{X})$, for any integer $n$, $\mathcal{E}(\mathcal{I})[n]$ is an object of $\mathcal{C}(\mathcal{X})$.

8. We will say that the data $\mathcal{C}$ of coefficients is stable by devissages if $\mathcal{C}$ is stable by shifts and if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, for any exact triangle $\mathcal{E}_1(\mathcal{I}) \to \mathcal{E}_2(\mathcal{I}) \to \mathcal{E}_3(\mathcal{I}) \to \mathcal{E}_1(\mathcal{I})[1]$ of $LD^b_{\mathbb{Q}_{\text{qc}}}(\hat{\mathcal{D}}_{\mathcal{X}})$, if two objects are in $\mathcal{C}(\mathcal{X})$, then so is the third one.

9. We will say that the data $\mathcal{C}$ of coefficients is stable under direct factors if, for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$ we have the following property: any direct factor in $LD^b_{\mathbb{Q}_{\text{coh}}}(\hat{\mathcal{D}}_{\mathcal{X}})$ of an object of $\mathcal{C}(\mathcal{X})$ is an object of $\mathcal{C}(\mathcal{X})$.

10. We say that $\mathcal{C}$ contains $\mathcal{D}$ (or $\mathcal{D}$ is contained in $\mathcal{C}$) if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$ the category $\mathcal{D}(\mathcal{X})$ is a full subcategory of $\mathcal{C}(\mathcal{X})$.

11. We say that the data $\mathcal{C}$ of coefficients is local if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, for any open covering $(\mathcal{X}_i)_{i \in I}$ of $\mathcal{X}$, for any object $\mathcal{E}(\mathcal{I})$ of $LD^b_{\mathbb{Q}_{\text{coh}}}(\hat{\mathcal{D}}_{\mathcal{X}})$, we have $\mathcal{E}(\mathcal{I}) \in \text{Ob}\mathcal{C}(\mathcal{X})$ if and only if
\[ E^{(\bullet)}|_{\mathcal{X}_i} \in \text{Ob}\mathcal{C}(\mathcal{X}_i) \text{ for any } i \in I. \] For instance, it follows from Theorem [14, 2.5.7] (see the localness in Definition [14, 2.4.1]) that the data of coefficients \( \mathcal{L}_D^{b} \to \mathbb{Q}^{\text{coh}} \) is local.

We finish the subsection with some notation.

1.1.7. (Duality) Let \( \mathcal{C} \) be a data of coefficients. We define its dual data of coefficients \( \mathcal{C}^\vee \) as follows: for any object \( W \) of \( \text{DVR} (V) \), for any smooth formal scheme \( \mathfrak{X} \) over \( W \), the category \( \mathcal{C}^\vee (\mathfrak{X}) \) is the subcategory of \( \mathcal{L}_D^{b} \to \mathbb{Q}^{\text{coh}} (\hat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \) of objects \( \mathcal{E}^{(\bullet)} \) such that \( \mathcal{D}_{\mathfrak{X}} (\mathcal{E}^{(\bullet)}) \in \mathcal{C}(\mathfrak{X}) \).

1.2. Data of coefficients with potentially Frobenius structure over \( V \)

The definition 1.2.2 introduced in this section will be used later in the Sect. 3.3.

Notation 1.2.1. We assume that the absolute Frobenius homomorphism \( k \to k \) sending \( x \) to \( x^p \) lifts to an automorphism \( \sigma : V \to V \). We denote by \( \text{DVR} (V, \sigma) \) the full subcategory of \( \text{DVR} (V) \) whose objects \( \alpha : V \to W \) are such that the absolute Frobenius homomorphism of the residue field of \( W \) has a lifting of the form \( \sigma_W : W \to W \) commuting with \( \sigma \), i.e. such that \( \sigma_W \circ \alpha = \alpha \circ \sigma \).

Similarly to 1.1.3, we introduce the following definition.

Definition 1.2.2. We keep the hypothesis of 1.2.1. A data \( \mathcal{C} \) of coefficients with potentially Frobenius structure over \( V \) will be the data of a full subcategory \( \mathcal{C}(\mathfrak{X}) \) of \( \mathcal{L}_D^{b} \to \mathbb{Q}^{\text{coh}} (\hat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \), for any object \( W \) of \( \text{DVR} (V, \sigma) \), and for any smooth formal scheme \( \mathfrak{X} \) over \( W \). If there is no ambiguity with \( V \), we simply say a data \( \mathcal{C} \) of coefficients with potentially Frobenius structure.

As in 1.1.6, we define the notion of local data of coefficients with potentially Frobenius structure over \( V \), of its stability under shifts, devissages, direct factors, extraordinary pullbacks, pushforwards, base change (of course, we restrict here to morphism in \( \text{DVR} (V, \sigma) \)), tensor products, dual functors, local cohomological functors, localisation outside a divisor etc.

Remark 1.2.3. We notice that by definition, a data of coefficients over \( V \) induces by restriction a data \( \mathcal{C} \) of coefficients with potentially Frobenius structure over \( V \).

1.3. Overcoherence, overholonomicity (after any base change) and complements

Definition 1.3.1. Let \( \mathcal{C} \) and \( \mathcal{D} \) be two data of coefficients.

1. We denote by \( S_0(\mathcal{D}, \mathcal{C}) \) the data of coefficients defined as follows: for any object \( W \) of \( \text{DVR} (V) \), for any smooth formal scheme \( \mathfrak{X} \) over \( W \), the category \( S_0(\mathcal{D}, \mathcal{C})(\mathfrak{X}) \) is the full subcategory of \( \mathcal{L}_D^{b} \to \mathbb{Q}^{\text{coh}} (\hat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \) of objects \( \mathcal{E}^{(\bullet)} \) satisfying the following properties:
1. We get by definition the equality $\mathcal{E}(\bullet) \in \mathcal{D}(\mathcal{Y})$, we have $\mathcal{F}(\bullet) \cong f^!(\mathcal{E}(\bullet)) \in \mathcal{C}(\mathcal{Y})$.

2. We denote by $S(\mathcal{O}, \mathcal{C})$ the data of coefficients defined as follows: for any object $\mathcal{W}$ of $\text{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, the category $S(\mathcal{O}, \mathcal{C})(\mathcal{X})$ is the full subcategory of $\mathcal{L}D_{\mathcal{O}, \text{coh}}^b(\mathcal{D}(\mathcal{X}))$ of objects $\mathcal{E}(\bullet)$ satisfying the following property:

$\mathcal{E}(\bullet)$ for any morphism $\mathcal{W} \to \mathcal{W}'$ of $\text{DVR}(\mathcal{V})$, we have $\mathcal{W}^\sim \mathcal{O}_{\mathcal{W}} \mathcal{E}(\bullet) \in S_0(\mathcal{O}, \mathcal{C})(\mathcal{X} \times_{\text{Spf} \mathcal{W}} \text{Spf} \mathcal{W}')$.

Examples 1.3.2. 1. We get by definition the equality $\mathcal{L}D_{\mathcal{O}, \text{ovcoh}}^b = S_0(\mathcal{B}_{\text{div}}, \mathcal{L}D_{\mathcal{O}, \text{coh}}^b)$ (this notion of overcoherence is defined in [14, 5.4]). Moreover, $S(\mathcal{B}_{\text{div}}, \mathcal{L}D_{\mathcal{O}, \text{coh}}^b)$ corresponds to the notion of overcoherence after any base change as defined in [13].

2. We put $\mathcal{S}_0 := S(\mathcal{B}_{\text{div}}, \mathcal{L}D_{\mathcal{O}, \text{coh}}^b)$ and by induction on $i \in \mathbb{N}$, we put $\mathcal{S}_i := \mathcal{S}_i \cap S(\mathcal{B}_{\text{div}}, \mathcal{S}_i^\vee)$ (see Notation 1.1.7). The coefficients of $\mathcal{S}_i$ are called $i$-overholonomic after any base change. We get the data of coefficients $\mathcal{L}D_{\mathcal{B}_{\text{div}}, \mathcal{V}}^b := \cap_{i \in \mathbb{N}} \mathcal{S}_i$ whose objects are called overholonomic after any base change.

3. Replacing $S$ by $S_0$ in the definition of $\mathcal{L}D_{\mathcal{B}_{\text{div}}, \mathcal{V}}^b$, we get a data of coefficients that we will denote by $\mathcal{L}D_{\mathcal{O}, \text{ovhol}}^b$.

Remark 1.3.3. 1. Let $\mathcal{C}$ be a data of coefficients. The data $\mathcal{C}$ of coefficients is stable under smooth extraordinary inverse image, localizations outside a divisor (resp. under smooth extraordinary inverse image, localizations outside a divisor, and base change) if and only if $S_0(\mathcal{B}_{\text{div}}, \mathcal{C}) = \mathcal{C}$ (resp. $S(\mathcal{B}_{\text{div}}, \mathcal{C}) = \mathcal{C}$).

2. By construction, we remark that $\mathcal{L}D_{\mathcal{B}_{\text{div}}, \mathcal{V}}^b$ is the biggest data of coefficients which contains $\mathcal{B}_{\text{div}}$, is stable by devissage, dual functors and the operation $S_0(\mathcal{B}_{\text{div}}, -)$. Moreover, $\mathcal{L}D_{\mathcal{O}, \text{h}}^b$ is the biggest data of coefficients which contains $\mathcal{B}_{\text{div}}$, is stable by devissage, dual functors and the operation $S(\mathcal{B}_{\text{div}}, -)$.

3. Let $\mathcal{W}$ be an object of $\text{DVR}(\mathcal{V})$, and $\mathcal{X}$ be a smooth formal $\mathcal{W}$-scheme. We denote by $\mathcal{D}_{\text{ovcoh}}^b(\mathcal{D}(\mathcal{X}))$ (resp. $\mathcal{D}_{\text{ovhol}}^b(\mathcal{D}(\mathcal{X}))$, resp. $\mathcal{D}_{\text{h}}^b(\mathcal{D}(\mathcal{X}))$) the category of overcoherent complexes (resp. overholonomic complexes, resp. overholonomic complexes after any base change) of $\mathcal{D}(\mathcal{X})$-modules as defined in [13, 3.2.1] (resp. [7, 3], resp. [13, 3.2.1]). We recall that, from the proposition [14, 5.4.3], we have as in 1.1.2.1 the following equivalence of categories 1.1.2.1

$$\lim: \mathcal{L}D_{\mathcal{O}, \text{ovcoh}}^b(\mathcal{D}(\mathcal{X})) \cong \mathcal{D}_{\text{ovcoh}}^b(\mathcal{D}(\mathcal{X})).$$

(1.3.3.1)

Hence, as in 1.1.2, we get from 1.3.3.1 a canonical t-structure on $\mathcal{L}D_{\mathcal{O}, \text{ovcoh}}^b(\mathcal{D}(\mathcal{X}))$. Moreover, we check that the functor $\lim$ of 1.3.3.1 induces an equivalence between $\mathcal{L}D_{\mathcal{O}, \text{ovhol}}^b(\mathcal{D}(\mathcal{X}))$ (resp. $\mathcal{L}D_{\mathcal{B}_{\text{div}}, \mathcal{V}}^b(\mathcal{D}(\mathcal{X}))$) and $\mathcal{D}_{\text{ovhol}}^b(\mathcal{D}(\mathcal{X}))$ (resp. $\mathcal{D}_{\text{h}}^b(\mathcal{D}(\mathcal{X}))$). Finally, recall from [13, 3.4.2] that $\mathcal{D}_{\text{h}}^b(\mathcal{D}(\mathcal{X}))$ has a canonical t-structure induced by that of $\mathcal{D}_{\text{ovhol}}^b(\mathcal{D}(\mathcal{X}))$, i.e. a coherent complex is overholonomic (after any base change) if and only if so are its cohomological spaces. Hence, we get a canonical t-structure on and $\mathcal{L}D_{\mathcal{O}, \text{ovhol}}^b(\mathcal{D}(\mathcal{X}))$. 

$$\mathcal{L}D_{\mathcal{O}, \text{ovhol}}^b(\mathcal{D}(\mathcal{X})) \cong \mathcal{D}_{\text{ovhol}}^b(\mathcal{D}(\mathcal{X})).$$
1.3.4. We already know that the data of coefficients \( LD_{b,ovcoh}^D \) is stable under direct factors and extraordinary pullbacks (see [14, 5.4.3]), or remark that this is a consequence of Lemmas 1.4.3, 1.4.5 and 1.4.9, 4–5. Concerning the stability under pushforwards, this is the purpose of Proposition 1.3.7 (in the literature, we only knew the stability under pushforwards by a proper morphism: see [14, 5.4.8]). First, we need to recall some properties and notations concerning the devissability of overcoherent isocrystals.

1.3.5. (Isocrystals and notation) Let \( \mathcal{W} \) be an object of DVR(\( \mathcal{V} \)) (see Definition 1.1.1). Let \( \mathcal{P} \) be a smooth formal scheme over \( \mathcal{W} \), and \( X \) be a closed subscheme of \( P \), and \( T \) be a divisor of \( P \) such that \( Y := X \setminus T \) is smooth (over the residue field of \( \mathcal{W} \)). We denote by \( \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) \) the category of partially overcoherent isocrystals on \( (\mathcal{P}, T, X/\mathcal{W}) \) (see [12, 1.4.2]), but we replaced in the notation \( \mathcal{W} \) by its field of fraction). This is a full subcategory of that of overcoherent \( D_{\mathcal{P}}^\dagger \)-modules with support in \( X \). On the other hand, we denote by \( \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) \), Berthelot’s category of overconvergent isocrystals on \( (\mathcal{P}, T, X/\mathcal{W}) \) (i.e. \( \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) := \text{Isoc}^\dagger(Y, X/\mathcal{W}) \)), more precisely their objects are the realization over \( \mathcal{P} \) of overconvergent isocrystals on \( (Y, X/\mathcal{W}) \). From [9, 5.4.6.1], we have the equivalence of categories of the form

\[
\text{sp}_+: \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) \cong \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}),
\]

which explains the terminology of the right hand side. We also denote by \( D_{\text{iso}}(\mathcal{P}, T, X/\mathcal{W}) \) the full subcategory of \( \text{D}_{ovcoh}^b(\mathcal{D}_{\mathcal{P}}^\dagger(T)_\mathcal{Q}) \) whose cohomological spaces are objects of \( \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) \).

Following [12, 1.4.3], we denote by \( \text{Isoc}^{(\bullet)}(\mathcal{P}, T, X/\mathcal{W}) \) the full subcategory of \( \text{LM}_{\mathcal{Q},\text{coh}}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^\bullet(T)) \) (see Notation [14, 2.2.4]) of objects \( \mathcal{E}^{(\bullet)} \) such that \( \text{lim} \mathcal{E}^{(\bullet)} \in \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) \). Following [12, 4.1.4], we denote by \( \text{LD}_{\mathcal{Q},\text{iso}, X}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^\bullet(T)) \) the full subcategory of \( \text{LD}_{\mathcal{Q},\text{ovcoh}}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}(T)) \) of objects \( \mathcal{E}^{(\bullet)} \) such that \( \mathcal{H}^j(\mathcal{E}^{(\bullet)}) \in \text{Isoc}^{(\bullet)}(\mathcal{P}, T, X/\mathcal{W}) \) for any integer \( j \in \mathbb{Z} \). Following [12, 4.1.5], we have the equivalence of categories \( \text{lim} \mathcal{LD}_{\mathcal{Q},\text{iso}, X}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}(T)) \cong \mathcal{D}_{iso}^b(\mathcal{P}, T, X/\mathcal{W}) \). Following [12, 4.1.4], the category \( \text{LD}_{\mathcal{Q},\text{iso}, X}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}(T)) \) does not depend on the choice of the closed scheme \( X \) of \( P \) and on the divisor \( T \) of \( P \) such that \( Y = X \setminus T \). Hence, the category \( \text{LD}_{\mathcal{Q},\text{iso}, X}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}(T)) \) will simply be denoted by \( \text{LD}_{\mathcal{Q},\text{iso}, X}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}) \).

1.3.6. (Devissage in overcoherent isocrystals) Let \( \mathcal{W} \) be an object of DVR(\( \mathcal{V} \)). Let \( \mathcal{P} \) be a smooth formal \( \mathcal{W} \)-scheme. Let \( \mathcal{E}^{(\bullet)} \in \text{LD}_{\mathcal{Q},\text{ovcoh}}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}) \). Let \( X \) be the support of \( \mathcal{E}^{(\bullet)} \). There exists a smooth \( d \)-stratification \( (Y_i)_{i=1, \ldots, r} \) of \( X \) in \( P \) (see Definition [12, 4.1.2.2]) such that we have \( \mathbb{R} \mathcal{L}_Y^+(\mathcal{E}^{(\bullet)}) \in \text{LD}_{\mathcal{Q},\text{iso}, Y_i}^b(\widehat{\mathcal{D}}_{\mathcal{P}}^{\bullet}) \), for any \( i = 1, \ldots, r \) (see Theorem [9, 6.2.3] and Definition [9, 6.2.2]).

Proposition 1.3.7. Let \( \mathcal{W} \) be an object of DVR(\( \mathcal{V} \)). Let \( g : X' \to X \) be a morphism of smooth formal \( \mathcal{W} \)-schemes. For any \( \mathcal{E}^{(\bullet)} \in \text{LD}_{\mathcal{Q},\text{ovcoh}}^b(\widehat{\mathcal{D}}_{\mathcal{X}}^{\bullet}) \) with proper support over \( X \), the object \( g_+(\mathcal{E}^{(\bullet)}) \) belongs to \( \text{LD}_{\mathcal{Q},\text{ovcoh}}^b(\widehat{\mathcal{D}}_{\mathcal{X}}^{\bullet}) \).
Proof. This is an analogue of Theorem 11.2.3.2 (see the remark 1.3.8 below which explain why this is not a straightforward consequence of Theorem 11.2.3.2) and its proof can be adapted. For the comfort of the reader, a complete detailed proof is given as follows: Let \( Z' \) be the support of \( \mathcal{E}'(\bullet) \). Following 1.3.6, since \( \mathcal{E}'(\bullet) \) is overcoherent, there exists a smooth \( d \)-stratification \( (U'_i)_{i=1,...,r} \) of \( Z' \) in \( X' \) such that \( \mathbb{R}^{\Gamma_+}_U(E_i(\bullet)) \in LD_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \), for any \( i = 1, \ldots, r \). Since \( LD_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \) is a triangle subcategory of \( LD_{\mathbb{Q}, qc}(\hat{\mathcal{D}}(\bullet)) \), we reduce by devisage to check that \( g_+\mathbb{R}^{\Gamma_+}_U(E_i(\bullet)) \in LD_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \). We can suppose \( U'_i \) integral. Again by devisage, we reduce to check \( g_+\mathbb{R}^{\Gamma_+}_U(E_i(\bullet)) \in LD_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \) for any integer \( j \in \mathbb{Z} \). We have \( E_{i,j}(\bullet) := \mathbb{H}^j(\mathbb{R}^{\Gamma_+}_U(E_i(\bullet))) \in \text{Isoc}(\mathcal{X}', T'_i, Z'_i/\mathcal{W}) \), where \( Z'_i \) is the closure of \( U'_i \) in \( X' \) and \( T'_i \) is some divisor of \( X'_i \).

Following [9, 5.3.1.1], there exists a commutative diagram of the form

\[
\begin{array}{cccc}
Z''_i & \xrightarrow{u''} & \mathbb{P}^N_{X'} & \xrightarrow{\mathbb{P}^N_{X'}} \\
\downarrow{a'} & & \downarrow{g} & \downarrow{q} \\
Z'_i & \xrightarrow{u'} & X' & \xrightarrow{g} X,
\end{array}
\tag{1.3.7.1}
\]

where \( Z''_i \) is smooth over the residue field of \( \mathcal{W} \), \( q \) and \( q' \) are the canonical projections, \( u'' \) is a closed immersion, \( T''_i := a'^{-1}(T'_i \cap Z'_i) \) is a strict normal crossing divisor of \( Z''_i \), \( a' \) is proper, surjective, generically finite and etale. Put \( E''_{i,j}(\bullet) := \mathbb{R}^{\Gamma_+}_{Z''_i} q'^!(E_{i,j}(\bullet)) \in \text{Isoc}(\mathbb{P}^N_{X'}, T''_i, Z''_i/\mathcal{W}) \cap LM_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \). By copying word by word the proof of [9, 5.3.1.1], we check that \( E''_{i,j}(\bullet) \) is a direct factor of \( q'_+(E_{i,j}(\bullet)) \).

By construction (see the beginning of the proof of [9, 5.3.1.1]), the morphism \( Z''_i \rightarrow \mathbb{P}^N_{X'} \) is an immersion (indeed, this is the composition of the graph of \( Z''_i \rightarrow X \) with the immersion \( Z''_i \times X \xleftarrow{\mathbb{P}^N_{X'}} \mathbb{P}^N_X \)) induced by an immersion of the form \( Z''_i \hookrightarrow \mathbb{P}^N_{X'} \). Since \( Z''_i \) is proper over \( X \) then \( Z''_i \rightarrow \mathbb{P}^N_{X'} \) is more precisely a closed immersion. Since \( E''_{i,j}(\bullet) \in LM_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \) has in support in \( Z''_i \) which is smooth, since overcoherence is a local notion, we check similarly to [9, 5.1.4] that \( q'_+(E_{i,j}(\bullet)) \in LM_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \). Since \( q \) is proper, then \( q_+ \) preserves the overcoherence and then \( g_+q'_+(E_{i,j}(\bullet)) \sim q_+ q'_+(E_{i,j}(\bullet)) \in LD_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \). Since \( E_{i,j}(\bullet) \) is a direct factor of \( q'_+(E_{i,j}(\bullet)) \), then \( g_+(E_{i,j}(\bullet)) \) is a direct factor of the overcoherent complex \( g_+q'_+(E_{i,j}(\bullet)) \). Hence, we have done. \( \square \)

Remark 1.3.8. With the notation of 1.3.7, for any complex \( E(\bullet) \in LD_{\mathbb{Q}, coh}(\hat{\mathcal{D}}(\bullet)) \) such that \( \lim(E(\bullet)) \in D_{\mathbb{Q}, coh}(\hat{\mathcal{D}}(\bullet)) \) we have \( E(\bullet) \in LD_{\mathbb{Q}, ovcoh}(\hat{\mathcal{D}}(\bullet)) \) (with notation 1.1.2.1, this is Proposition 14.5.4.3). Hence, when \( g \) is proper, since \( g_+(E(\bullet)) \) belongs to \( LD_{\mathbb{Q}, coh}(\hat{\mathcal{D}}(\bullet)) \), it follows that Proposition 1.3.7 is a straightforward consequence of Theorem 11.2.3.2.
But, when $g$ is not proper, this is not a clear consequence. Indeed, since $g$ is not proper, we only know without effort that $g_+(\mathcal{E}'(\bullet))$ belongs to $\underline{L}D_{\mathbb{Q},\text{qc}}^b(\overline{\mathcal{D}}_X^\gamma)$ (the check of the coherence seems as hard as the check of the overcoherence). Without coherence hypothesis, we still have the functor $\lim: \underline{L}D_{\mathbb{Q},\text{qc}}^b(\overline{\mathcal{D}}_X^\gamma) \to D^b_{\text{coh}}(\overline{\mathcal{D}}_X^\gamma)$. But, for any complex $\mathcal{E}(\bullet) \in \underline{L}D_{\mathbb{Q},\text{coh}}^b(\overline{\mathcal{D}}_X^\gamma)$ such that $\lim(\mathcal{E}(\bullet)) \in D^b_{\text{coh}}(\overline{\mathcal{D}}_X^\gamma)$ (resp. $\lim(\mathcal{E}(\bullet)) \in D^b_{\text{ovcoh}}(\overline{\mathcal{D}}_X^\gamma)$), it seems false that this implies that $\mathcal{E}(\bullet) \in \underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_X^\gamma)$ (resp. $\mathcal{E}(\bullet) \in \underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_X^\gamma)$).

We will need later the following base change isomorphism.

**Proposition 1.3.9.** Let $\mathcal{W}$ be an object of DVR($\mathcal{V}$) (see Definition 1.1.1). Let $f: \mathcal{Y} \to \mathcal{X}$ and $g: \mathcal{X}' \to \mathcal{X}$ be two morphisms of smooth formal $\mathcal{W}$-schemes. We suppose $f$ smooth. Let $f': \mathcal{Y} \times_\mathcal{X} \mathcal{X}' \to \mathcal{X}'$, and $g': \mathcal{Y} \times_\mathcal{X} \mathcal{X}' \to \mathcal{Y}$ be the structural projections. For any $\mathcal{E}(\bullet) \in \underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_X^\gamma)$ with proper support over $X$, we have the base change isomorphism in $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_Y^\gamma)$ of the form

$$f^!g_+(\mathcal{E}(\bullet)) \sim g'_+f'^!(\mathcal{E}(\bullet)).$$

(1.3.9.1)

**Proof.** This is analogue to the proof [14, 5.4.6]: let $\mathcal{E}(\bullet) \in \underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_X^\gamma)$ with proper support over $X$. First, we remark that by using 1.3.7, both objects of 1.3.9.1 belongs to $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_Y^\gamma)$. The morphism $f$ is the composition of its graph $\gamma: \mathcal{Y} \leftarrow \mathcal{X} \times \mathcal{Y}$ with the projection $\pi: \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$. Let $g'': \mathcal{X}' \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$, and $\pi': \mathcal{X}' \times \mathcal{Y} \to \mathcal{X}'$ be the canonical projections. Let $\gamma': \mathcal{X}' \times_\mathcal{X} \mathcal{Y} \leftarrow \mathcal{X}' \times \mathcal{Y}$ be the closed immersion induced by base change via $g''$ of $\gamma$. In the second part of the proof of [14, 5.4.6], we have proved the isomorphism $\pi'_+g_+(\mathcal{E}(\bullet)) \sim g''_+\pi'^!(\mathcal{E}(\bullet))$. This yields the second isomorphism $\gamma'_+f^!g_+(\mathcal{E}(\bullet)) \sim \gamma'_+\gamma''_+\pi'^!(\mathcal{E}(\bullet)) \sim \gamma'_+\gamma''_+\pi''!(\mathcal{E}(\bullet))$. Using Theorem [14, 5.2.8.2] and Corollary [14, 5.3.8], we get the first isomorphism $\gamma'_+f^!g_+(\mathcal{E}(\bullet)) \sim g''_+\gamma''_+\pi''!(\mathcal{E}(\bullet)) \sim g''_+\gamma''_+\pi''!(\mathcal{E}(\bullet))$. Hence, by composition, we get the isomorphism $\gamma'_+f^!g_+(\mathcal{E}(\bullet)) \sim \gamma'_+\gamma''_+\pi''!(\mathcal{E}(\bullet)) \sim \gamma'_+\gamma''_+\pi''!(\mathcal{E}(\bullet))$. Since $f^!g_+(\mathcal{E}(\bullet))$ and $g'_+f'^!(\mathcal{E}(\bullet))$ are coherent (this is a consequence of Proposition 1.3.7), then we can use Berthelot–Kashiwara theorem in the form [14, 5.3.7.1]. In other words, by applying $\gamma!$ to the isomorphism $\gamma'_+f^!g_+(\mathcal{E}(\bullet)) \sim \gamma'_+\gamma''_+\pi''!(\mathcal{E}(\bullet))$ we get the isomorphism 1.3.9.1. $\square$

**Notation 1.3.10.** Let $\mathcal{W}$ be an object of DVR($\mathcal{V}$). Let $\mathcal{P}$ be a smooth formal $\mathcal{W}$-scheme, and $Y$ be a subscheme of $P$. We denote by $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(Y, \mathcal{P}/\mathcal{W})$ the full subcategory of $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(\overline{\mathcal{D}}_\mathcal{P}^\gamma)$ of complexes $\mathcal{E}(\bullet)$ such that there exists an isomorphism of the form $\underline{R}\Gamma^\gamma_Y \mathcal{E}(\bullet) \sim \mathcal{E}(\bullet)$.

Similarly to [2, 1.2.1.5], there is a canonical t-structure on $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(Y, \mathcal{P}/\mathcal{W})$ defined as follows: choose $\mathfrak{U}$ an open set of $\mathcal{P}$ such that $Y$ is closed in $\mathfrak{U}$. We denote by $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^{\leq n}(Y, \mathcal{P}/\mathcal{W})$ and $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^{\geq n}(Y, \mathcal{P}/\mathcal{W})$ the full subcategory of $\underline{L}D_{\mathbb{Q},\text{ovcoh}}^b(Y, \mathcal{P}/\mathcal{W})$ of complexes $\mathcal{E}$ such that $\mathcal{E}|\mathfrak{U} \in \underline{L}D_{\mathbb{Q},\text{ovcoh}}^{\leq n}(\overline{\mathcal{D}}_\mathfrak{U})$ (resp.
\(E|\mathcal{U} \in LD_{\mathcal{Q},\text{ovcoh}}^n(\mathcal{D}_\mathcal{U})\), where the t-structure on \(LD_{\mathcal{Q},\text{ovcoh}}^n(\mathcal{D}_\mathcal{U})\) is the canonical one (see 1.3.3.3). The heart of this t-structure, the category of overcoherent modules on \((Y, \mathcal{P}/\mathcal{W})\), will be denoted by \(LM_{\mathcal{Q},\text{ovcoh}}(Y, \mathcal{P}/\mathcal{W})\). Finally, we denote by \(H^i\) the \(i\)th space of cohomology with respect to this canonical t-structure.

**Theorem 1.3.11.** (Independence) Let \(\mathcal{W}\) be an object of DVR(\(\mathcal{V}\)). Let \(f: \mathcal{P}\to \mathcal{P}\) be a realizable morphism of smooth formal \(\mathcal{W}\)-schemes. Let \(X'\) be a closed subscheme of \(P\), and \(X\) be a closed subscheme of \(P\) such that \(f(X') \subset X\) and the induced morphism \(X' \to X\) of schemes is proper. Let \(Y\) be an open subscheme of \(X\) such that the composition \(Y \to X' \to X\) is an open immersion.

1. For any \(E(\bullet) \in LM_{\mathcal{Q},\text{ovcoh}}(Y, \mathcal{P}/\mathcal{W})\), for any \(E'(\bullet) \in LM_{\mathcal{Q},\text{ovcoh}}(Y, \mathcal{P}'/\mathcal{W})\), for any \(n \in \mathbb{Z} \setminus \{0\}\), we have
   \[H^n_f(E(\bullet)) = 0, H^n_{f+}(E'(\bullet)) = 0.\]

2. For any \(E(\bullet) \in LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}/\mathcal{W})\), for any \(E'(\bullet) \in LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}'/\mathcal{W})\), we have canonical isomorphisms of the form \(\mathbb{R} \Gamma_Y^+ f^!(E(\bullet)) \sim E'(\bullet)\) and \(f^+ \mathbb{R} \Gamma_Y^+ f^!(E'(\bullet)) \sim E'(\bullet)\). In particular, the functors \(\mathbb{R} \Gamma_Y^+ f^!\) and \(f^+\) induce quasi-inverse equivalences of categories between \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}/\mathcal{W})\) and \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}'/\mathcal{W})\).

**Proof.** With the first part of the Remark 1.3.8, the first statement is a consequence of [11, 4.2.3.2]. Let us check the second one.

First, we reduce to the case where \(f\) is proper. Let \(u: \mathcal{P}' \inj \mathcal{P}''\) be an immersion of smooth formal \(\mathcal{W}\)-schemes, and \(\pi: \mathcal{P}'' \to \mathcal{P}\) be a proper morphism of smooth formal \(\mathcal{W}\)-schemes such that \(f = \pi \circ u\). Let \(v: \mathcal{P}' \to \mathcal{U}\) be a closed immersion, and \(j: \mathcal{U}' \inj \mathcal{U}''\) be an open immersion such that \(u = j \circ v\). Since \(X' \to P\) is proper, then \(X' \to U''\) and \(X' \to P''\) are closed immersions (because they are proper immersions). Since the objects of \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}'/\mathcal{W})\) and of \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{U}'/\mathcal{W})\) have their support in \(X'\), since the functors \(j_*\) and \(j^!\) are quasi-inverse equivalences of categories between complexes over \(\mathcal{U}'\) (resp. \(\mathcal{P}''\)) with support in \(X'\), since \(j_+ = j_*\) and \(j^! = j^*\) preserve overcoherence (use 1.3.7 for \(j_+\)), then \(j_+\) and \(j^!\) induce an equivalence of categories between \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}'/\mathcal{W})\) and \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{U}'/\mathcal{W})\) (remark that \(\mathbb{R} \Gamma_Y^+ j^! = j^!\) over \(LD_{\mathcal{Q},\text{ovcoh}}^b(Y, \mathcal{P}'/\mathcal{W})\)). Hence, we reduce to the case where \(f\) is proper.

Finally, when \(f\) is proper, with the first part of the Remark 1.3.8, the second statement is a consequence of [11, 4.2.3.4]. \(\square\)

**Theorem 1.3.12.** (Relative duality isomorphism) Let \(\mathcal{W}\) be an object of DVR(\(\mathcal{V}\)). Let \(g: \mathcal{P}' \to \mathcal{P}\) be a realizable morphism of smooth formal \(\mathcal{W}\)-schemes. For any \(E(\bullet) \in LD_{\mathcal{Q},\text{ovcoh}}^b(\mathcal{D}_{\mathcal{P}'})\) with proper support over \(P\), we have the isomorphism of \(\mathbb{R} \Gamma_{\mathcal{P}'}^+\) of the form

\[g_+ \circ \mathbb{D}(E(\bullet)) \sim \mathbb{D} \circ g_+(E(\bullet)).\]
Proof. Let $X'$ be a (closed) subscheme of $P'$ which is proper over $P$ via $g$. Let $E' \in L_{Q,ovcoh}(X', P'/W)$. Let $u : P' \hookrightarrow P''$ be an immersion of smooth formal $\mathcal{W}$-schemes, and $\pi : P'' \rightarrow P$ be a proper morphism of smooth formal $\mathcal{W}$-schemes such that $g = \pi \circ u$. Let $v : P' \hookrightarrow P''$ be a closed immersion, and $j : P'' \hookrightarrow P'$ be an open immersion such that $u = j \circ v$.

From the relative duality isomorphism in the proper case (see [24]), $D_{\mathcal{W}}(E') \sim u_+ D(E')$. Set $\mathcal{F} := u_+ (E')$. Since $j_+ \mathcal{F} \in \mathcal{L}D_{Q,ovcoh}(X', P''/W)$ has its support in $X'$, then $\mathcal{D}j_+ \mathcal{F} \in \mathcal{L}D_{Q,coh}(\widehat{\mathcal{D}}_{P''})$ and has its support in $X'$. Hence, $\mathcal{D}j_+ \mathcal{F} \sim j_+ j^! \mathcal{D}j_+ \mathcal{F}$. Moreover, this is obvious that $j^! \mathcal{D}j_+ \mathcal{F} \sim \mathcal{D}j_+ \mathcal{F}$. Hence, $\mathcal{D}j_+ \mathcal{F} \sim j_+ \mathcal{D} \mathcal{F}$. By composition we get $\mathcal{D}u_+ (E') \sim u_+ D(E')$. Since $\pi$ is proper, from the relative duality isomorphism in the proper case (see [24]), we obtain the first isomorphism $\mathcal{D}u_+ (E') \sim \pi_+ \mathcal{D}u_+ (E') \sim \pi_+ \mathcal{D}(E')$. Hence, we are done. \(\square\)

1.4. Constructions of stable data of coefficients

1.4.1. Let $\mathcal{W}$ be an object of $\text{DVR}(\mathcal{V})$ (see Definition 1.1.1). Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of smooth formal $\mathcal{W}$-schemes. Following [5, 2.1.4], for any $E' \in \mathcal{L}D_{Q,\text{qc}}(\widehat{\mathcal{D}}_{\mathcal{X}})$, $F' \in \mathcal{L}D_{Q,\text{qc}}(\widehat{\mathcal{D}}_{\mathcal{Y}})$ we have the isomorphism

$$f_+ \left( F' \otimes_{O_{\mathcal{Y}}} f^!(E') \right) \sim f_+ (F') \otimes_{O_{\mathcal{X}}} E' [d_{Y/X}],$$

(1.4.1.1)

where $d_{Y/X} := \dim Y - \dim X$.

Let $U$ be a subscheme of $X$. Since this is not explicitly written in the literature, let us clarify the following isomorphism. Using [5, 2.2.6.1, 2.2.8, 2.2.14] (or for a wider version with more details, use [14, 4.3.6.1, 4.4]) for any $E' \in \mathcal{L}D_{Q,\text{qc}}(\widehat{\mathcal{D}}_{\mathcal{X}})$, $E'' \in \mathcal{L}D_{Q,\text{qc}}(\widehat{\mathcal{D}}_{\mathcal{X}})$, we have the isomorphisms

$$\mathbb{R} \Gamma_U^+ (E' \otimes_{O_{\mathcal{X}}} E') \sim \mathbb{R} \Gamma_U^+ (E' \otimes_{O_{\mathcal{X}}} E' \otimes_{O_{\mathcal{X}}} \mathcal{E}(\mathcal{X})) \sim \mathbb{R} \Gamma_U^+ (E' \otimes_{O_{\mathcal{X}}} \mathcal{E}(\mathcal{X})) \sim \mathbb{R} \Gamma_U^+ (E') \otimes_{\mathcal{O}_{\mathcal{X}}} \mathbb{R} \Gamma_U^+ (E').$$

(1.4.1.2)

1.4.2. (Base change and their commutation with cohomological operations)

Let $\alpha : \mathcal{W} \rightarrow \mathcal{W}'$ be a morphism of $\text{DVR}(\mathcal{V})$, let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{W}$, $E' \in \mathcal{L}D_{Q,\text{qc}}(\widehat{\mathcal{D}}_{\mathcal{X}})$, $\mathcal{X}' := \mathcal{X} \times_{\text{Spf}(\mathcal{W})} \text{Spf} \mathcal{W}'$, and $\pi : \mathcal{X}' \rightarrow \mathcal{X}$ be the projection. The base change of $E'$ by $\alpha$ is the object $\pi_+^{\mathcal{W}'}(E') = \pi_+^{\mathcal{W}'}(E')$ of $\mathcal{L}D_{Q,\text{qc}}(\widehat{\mathcal{D}}_{\mathcal{X}'})$ (see [4, 2.2.2]). Similarly to [4, 2.2.2], it will simply be denoted by $\mathcal{W}' \otimes_{\mathcal{W}} \mathcal{E}(\mathcal{X})$.

From [4, 2.4.2], push forwards commute with base change. The commutation of base change with extraordinary pullbacks, local cohomological functors, duals functors (for coherent complexes), and tensor products is straightforward.

We will need later the following Lemmas.
Lemma 1.4.3. Let $\mathcal{E}$ be a data of coefficients stable under local cohomological functors. Then the data of coefficients $\mathcal{E}$ is stable under smooth extraordinary pullbacks and satisfies $BK^1$ if and only if $\mathcal{E}$ is stable under extraordinary pullbacks (see Definitions 1.1.6).

Proof. Since the converse is obvious, let us check that if $\mathcal{E}$ is stable under smooth extraordinary pullbacks and satisfies $BK^1$ then $\mathcal{E}$ is stable under extraordinary pullbacks. Let $\mathcal{W}$ be an object of DVR $(\mathcal{V})$, $f : \mathcal{W} → \mathfrak{X}$ be a morphism of smooth formal schemes over $\mathcal{W}$, and $\mathcal{E}(\bullet) ∈ \mathcal{C}(\mathfrak{X})$. Since $f$ is the composition of its graph $\mathcal{W} ↪ \mathcal{W} \times \mathfrak{X}$ followed by the projection $\mathcal{W} \times \mathfrak{X} → \mathfrak{X}$ which is smooth, using the stability under smooth extraordinary pullbacks, we reduce to the case where $f$ is a closed immersion. From the stability under local cohomological functors, $\mathcal{W}_Y\mathcal{E}(\bullet) ∈ \mathcal{C}(\mathfrak{X})$. Since $\mathcal{E}$ satisfies $BK^1$, then $f^!\mathcal{W}_Y\mathcal{E}(\bullet) ∈ \mathcal{C}(\mathcal{W})$. We conclude using the isomorphism $f^!\mathcal{W}_Y\mathcal{E}(\bullet) ≃ f^!(\mathcal{E}(\bullet))$ (use [14, 2.5.8]).

Lemma 1.4.4. Let $\mathcal{D}$ be a data of coefficients over $\mathcal{V}$. If $\mathcal{D}$ contains $\mathcal{B}_{\text{div}}$ (see the second example of 1.1.4), and if $\mathcal{D}$ is stable under tensor products, then $\mathcal{D}$ is stable under localizations outside a divisor.

Proof. This is a consequence of the isomorphisms 1.4.1.2 (use the case where $\mathcal{E}(\bullet) = \mathcal{O}_{\mathfrak{X}}(\bullet)$).

Lemma 1.4.5. Let $\mathcal{E}$ be a data of coefficients stable under devissage. Then the data $\mathcal{E}$ of coefficients is stable under local cohomological functors if and only if it is stable under localizations outside a divisor.

Proof. This is checked by using exact triangles of localisation (see [5, 2.2.6] or [14, 4.4.3] for a more precise and general version), Mayer-Vietoris exact triangles (see [5, 2.2.16] or [14, 4.5.2]).

Remark 1.4.6. Let $\mathcal{E}$ be a data of coefficients stable under devissage which contains $\mathcal{B}_{\text{div}}$ (see Notation 1.3.2). Then using the arguments of the proof of 1.4.5, we check $\mathcal{E}$ contains $\mathcal{B}_{\text{cst}}$ (see Notation 1.3.2). Similarly, we check that $S_0(\mathcal{B}_{\text{div}}, \mathcal{E}) = S_0(\mathcal{B}_{\text{cst}}, \mathcal{E})$ and $S(\mathcal{B}_{\text{div}}, \mathcal{E}) = S(\mathcal{B}_{\text{cst}}, \mathcal{E})$.

Lemma 1.4.7. Let $\mathcal{E}$ be a data of coefficients. If the data $\mathcal{E}$ of coefficients satisfies $BK^1$, then so is $\mathcal{E}^\vee$ (see Notation 1.1.7).

Proof. Let $\mathcal{W}$ be an object of DVR $(\mathcal{V})$, $u : Z ↪ \mathfrak{X}$ be a closed immersion of smooth formal schemes over $\mathcal{W}$, and $\mathcal{E}(\bullet) ∈ \mathfrak{X}$ be an object of $\mathfrak{X}$ with support in $Z$. From Berthelot–Kashiwara theorem (see [14, 5.3.6]), there exists $\mathcal{G}(\bullet) ∈ LD^{b}_{X, \text{coh}}(\mathfrak{X})$ such that $u_+(\mathcal{G}(\bullet)) → \mathcal{E}(\bullet)$. Since $\mathcal{D}_X(\mathcal{E}(\bullet)) ∈ \mathcal{C}(\mathfrak{X})$ has its support in $Z$, since $BK^1$ property holds, we get $u_+ \mathcal{D}_X(\mathcal{E}(\bullet)) ∈ \mathcal{C}(\mathfrak{Z})$. From the relative duality isomorphism (see [24]), we get $\mathcal{D}_X(\mathcal{E}(\bullet)) → \mathcal{D}_X u_+(\mathcal{G}(\bullet)) → u_+(\mathcal{D}_Z(\mathcal{G}(\bullet)))$. Hence, $u_+ \mathcal{D}_X(\mathcal{E}(\bullet)) → \mathcal{D}_X u_+(\mathcal{G}(\bullet)) → u_+(\mathcal{D}_Z(\mathcal{G}(\bullet))) ∈ \mathcal{C}(\mathfrak{Z})$. From Berthelot–Kashiwara theorem (see [14, 5.3.6]), we have $u_+ \mathcal{D}_X(\mathcal{E}(\bullet)) → \mathcal{D}_Z(\mathcal{G}(\bullet))$. This yields $\mathcal{D}_Z(\mathcal{G}(\bullet)) ∈ \mathcal{C}(\mathfrak{Z})$. Since $u_+ \mathcal{D}_X(\mathcal{E}(\bullet)) → u_+ \mathcal{D}_X(\mathcal{E}(\bullet)) → \mathcal{G}(\bullet)$, this implies that $u_+ \mathcal{D}_X(\mathcal{E}(\bullet)) ∈ \mathcal{C}(\mathfrak{Z})$.□
Lemma 1.4.8. Let $\mathcal{C}$ be a data of coefficients which is included in $LD_{\mathbb{Q}, \text{ovcoh}}^b$. If the data $\mathcal{C}$ of coefficients is stable under realizable pushforwards, then so is $\mathcal{C}'$.

Proof. This is a straightforward consequence of the relative duality isomorphism of the form 1.3.12. $\square$

Lemma 1.4.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two data of coefficients. With the notation of 1.3.1, we have the following properties.

1. With Notation 1.1.4, if $\mathcal{D}$ contains $\mathcal{B}_\emptyset$ (resp. $\mathcal{B}_{\text{div}}$) then $S(\mathcal{D}, \mathcal{C})$ is contained in $\mathcal{C}$ (resp. and in $LD_{\mathbb{Q}, \text{ovcoh}}^b$).

2. Suppose that $\mathcal{D}$ is stable under smooth extraordinary pullbacks, base change and tensor products and that $\mathcal{C}$ contains $\mathcal{D}$. Then $S(\mathcal{D}, \mathcal{C})$ contains $\mathcal{D}$. If $\mathcal{C}$ is moreover stable under shifts then $S(S(\mathcal{D}, \mathcal{C}), S(\mathcal{D}, \mathcal{C}))$ contains $\mathcal{D}$.

3. If the data $\mathcal{C}$ of coefficients is local (resp. stable under devissages, resp. stable under direct factors), then so is $\mathcal{C}'$ and $S(\mathcal{D}, \mathcal{C})$.

4. The data of coefficients $S(\mathcal{D}, \mathcal{C})$ is stable under smooth extraordinary pullbacks and under base change.

5. If $\mathcal{D}$ is stable under local cohomological functors (resp. localizations outside a divisor), then so is $S(\mathcal{D}, \mathcal{C})$.

6. Suppose that $\mathcal{C}$ is stable under realizable pushforwards and shifts. Suppose that $\mathcal{D}$ contains $\mathcal{B}_{\text{div}}$, and is stable under extraordinary pullbacks. Then the data of coefficients $S(\mathcal{D}, \mathcal{C})$ is stable under realizable pushforwards.

7. Suppose that $\mathcal{C}$ is stable under shifts, and satisfies $BK^+$. Moreover, suppose that $\mathcal{D}$ satisfies $BK^+$. Then the data of coefficients $S(\mathcal{D}, \mathcal{C})$ satisfies $BK^+$.

Proof. (a) The respective case of 1) is a consequence of the equality $LD_{\mathbb{Q}, \text{ovcoh}}^b = S_0(\mathcal{B}_{\text{div}}, LD_{\mathbb{Q}, \text{coh}}^b)$. The non respective case of 1), the assertions 3) and 4) are obvious.

(b) Let us prove 2). Using every hypotheses on $\mathcal{C}$, we check easily that $S(\mathcal{D}, \mathcal{C})$ contains $\mathcal{D}$. Let us suppose moreover $\mathcal{C}$ stable under shifts. Since $\mathcal{D}$ is stable under base change, it remains to check $\mathcal{D}$ is included in $S_0(S(\mathcal{D}, \mathcal{C}), S(\mathcal{D}, \mathcal{C}))$. Let $W$ be an object of DVR$(\mathcal{V})$, $X$ be a smooth formal scheme over $W$, and $\mathcal{C}''(\bullet) \in \mathcal{D}(\mathcal{X})$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a smooth morphism of smooth formal $W$-schemes, and $\mathcal{F}''(\bullet) \in S(\mathcal{D}, \mathcal{C})(\mathcal{Y})$. We have to check that $\mathcal{F}''(\bullet) \otimes_{\mathcal{O}[y]} f'^!(\mathcal{C}''(\bullet)) \in S(\mathcal{D}, \mathcal{C})(\mathcal{Y})$. Since $\mathcal{D}$ is stable under base change, since tensor products and extraordinary inverse images commute with base change, we reduce to establish that $\mathcal{F}''(\bullet) \otimes_{\mathcal{O}[y]} f'^!(\mathcal{C}''(\bullet)) \in S_0(\mathcal{D}, \mathcal{C})(\mathcal{Y})$. Let $g : \mathcal{Z} \to \mathcal{Y}$ be a smooth morphism of smooth formal $W$-schemes, and $\mathcal{G}''(\bullet) \in \mathcal{D}(\mathcal{Z})$. We have the isomorphisms

$$
\begin{align*}
\mathcal{G}''(\bullet) \otimes_{\mathcal{O}[z]} g'^! & \left( \mathcal{F}''(\bullet) \otimes_{\mathcal{O}[y]} f'^!(\mathcal{C}''(\bullet)) \right) \\
& \sim_{[12, 2.19.1]} \left( \mathcal{G}''(\bullet) \otimes_{\mathcal{O}[z]} g'^! \mathcal{F}''(\bullet) \right) \otimes_{\mathcal{O}[z]} (f \circ g)'(\mathcal{C}''(\bullet))[−d_{Z/Y}] \\
& \sim (\mathcal{G}''(\bullet) \otimes_{\mathcal{O}[z]} (f \circ g)'(\mathcal{C}''(\bullet))[−d_{Z/Y}]) \otimes_{\mathcal{O}[z]} g'^! \mathcal{F}''(\bullet).
\end{align*}
$$

(*
Since $\mathcal{D}$ is stable under smooth extraordinary pullbacks, shift and tensor products, then $\mathcal{G}^{(\bullet)} \boxtimes_{\mathcal{O}_Z} (f \circ g)^{\ast}(\mathcal{E}^{(\bullet)})[-d_{Z/Y}] \in \mathcal{D}(Z)$. Since $\mathcal{F}^{(\bullet)} \in S(\mathcal{D}, \mathcal{C})(y)$, then $\left(\mathcal{G}^{(\bullet)} \boxtimes_{\mathcal{O}_Z} (f \circ g)^{\ast}(\mathcal{E}^{(\bullet)})[-d_{Z/Y}]\right) \boxtimes_{\mathcal{O}_Z} g^{\ast}\mathcal{F}^{(\bullet)} \in \mathcal{C}(Z)$. Hence, using (\ast) we conclude.

(c) Let us check 5). From the commutation of the base change with local cohomological functors, we reduce to check that $S_0(\mathcal{D}, \mathcal{C})$ is stable under local cohomological functors (resp. localisations outside a divisor). Using 1.4.1.2 and the commutation of local cohomological functors with extraordinary inverse images (see [14, 5.2.8]), we check the desired properties.

(d) Let us check 6). Let $\mathcal{W}$ be an object of DVR$(\mathcal{V})$. Let $g : \mathcal{X} \to \mathfrak{X}$ be a morphism of smooth formal $\mathcal{W}$-schemes. Let $\mathcal{E}^{(\bullet)} \in S(\mathcal{D}, \mathcal{C})(\mathfrak{X})$ with proper support over $\mathfrak{X}$. From the commutation of the base change with realizable pushforwards (see 1.4.2), we deduce that $g_+(\mathcal{E}^{(\bullet)}) \in S_0(\mathcal{D}, \mathcal{C})(\mathfrak{X})$. Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a smooth morphism of smooth formal $\mathcal{W}$-schemes. Let $f' : \mathfrak{Y} \times_{\mathfrak{X}} \mathcal{X} \to \mathfrak{X}'$, and $g' : \mathfrak{Y} \times \mathfrak{X} \to \mathfrak{Y}$ be the structural projections. Let $\mathcal{F}^{(\bullet)} \in \mathcal{D}(\mathfrak{Y})$. We have to check $\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} f'_+g_+(\mathcal{E}^{(\bullet)}) \in \mathcal{C}(\mathfrak{Y})$. Since $\mathcal{D}$ contains $\mathcal{B}_{\text{div}}$, then $\mathcal{E}^{(\bullet)} \in LD^b_{\mathcal{O}_{\mathfrak{Y},\text{cogr}}}([\mathcal{D}^{(\bullet)}, \mathfrak{Y}])$. Hence from 1.3.9.1 we get $f'_+g_+(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} g'_+f'^{\ast}(\mathcal{E}^{(\bullet)})$. Using the hypotheses on $\mathcal{C}$ and $\mathcal{D}$, via the isomorphisms

$$
\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} f'_+g_+(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} g'_+f'^{\ast}(\mathcal{E}^{(\bullet)})
$$

$$
\xrightarrow{\sim} g'_+\left((\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} f'^{\ast}(\mathcal{E}^{(\bullet)}))[d_{\mathfrak{X}'/\mathfrak{X}]},
$$

we check that $\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} f'_+g_+(\mathcal{E}^{(\bullet)}) \in \mathcal{C}(\mathfrak{Y})$.

(e) Let us check 7) (we might remark the similarity with the proof of [5, 3.1.7]). Let $\mathcal{W}$ be an object of DVR$(\mathcal{V})$, and $u : \mathfrak{X} \to \mathfrak{P}$ be a closed immersion of smooth formal schemes over $\mathcal{W}$. Let $\mathcal{E}^{(\bullet)} \in S(\mathcal{D}, \mathcal{C})(\mathfrak{P})$ with support in $\mathfrak{X}$. We have to check that $u'_!(\mathcal{E}^{(\bullet)}) \in S(\mathcal{D}, \mathcal{C})(\mathfrak{X})$. We already know that $u'_!(\mathcal{E}^{(\bullet)}) \in LD^b_{\mathcal{O}_{\mathfrak{Y},\text{cogr}}}([\mathcal{D}^{(\bullet)}, \mathfrak{Y}])$ (thanks to Berthelot–Kashiwara theorem [14, 5.3.6]). Since extraordinary pullbacks commute with base change, we deduce to check that $u'_!(\mathcal{E}^{(\bullet)}) \in S_0(\mathcal{D}, \mathcal{C})(\mathfrak{X})$. Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a smooth morphism of smooth formal $\mathcal{W}$-schemes, and $\mathcal{F}^{(\bullet)} \in \mathcal{D}(\mathfrak{Y})$. We have to check $\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} f'_!(u'_!(\mathcal{E}^{(\bullet)})) \in \mathcal{C}(\mathfrak{Y})$. The morphism $f$ is the composition of its graph $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \times \mathfrak{X}$ with the projection $\mathfrak{Y} \times \mathfrak{X} \to \mathfrak{X}$. We denote by $\nu$ the composition of $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \times \mathfrak{X}$ with $id \times u : \mathfrak{Y} \times \mathfrak{X} \to \mathfrak{Y} \times \mathfrak{P}$. Let $g : \mathfrak{Y} \times \mathfrak{P} \to \mathfrak{P}$ be the projection. Set $\mathfrak{U} := \mathfrak{Y} \times \mathfrak{P}$. Since $\mathcal{D}$ satisfies $BK_+$, then $v_+(\mathcal{F}^{(\bullet)}) \in \mathcal{D}(\mathfrak{U})$. Since $\mathcal{E}^{(\bullet)} \in S_0(\mathcal{D}, \mathcal{C})(\mathfrak{P})$ and $g$ is smooth, this yields $v_+(\mathcal{F}^{(\bullet)}) \boxtimes_{\mathcal{O}_U} g'^{\ast}(\mathcal{E}^{(\bullet)}) \in \mathcal{C}(\mathfrak{U})$.

Since $\mathcal{C}$ satisfies $BK'_+$, this implies $v'_!(\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_U} g'^{\ast}(\mathcal{E}^{(\bullet)})) \in \mathcal{C}(\mathfrak{Y})$. Since $u'_!(\mathcal{F}^{(\bullet)}) \boxtimes_{\mathcal{O}_U} g'^{\ast}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} v'_!v_+(\mathcal{F}^{(\bullet)}) \boxtimes_{\mathcal{O}_Y} v'^{\ast}(\mathcal{E}^{(\bullet)})[r]$ with $r$ an integer (see [12, 2.1.9.1]), since $v'_!v_+(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ (see Berthelot–Kashiwara theorem [14, 5.3.6]), since $\mathcal{C}$ is stable under shifts, since by transitivity $v'_!g'^{\ast} \xrightarrow{\sim} f'_u\ast$, we get $\mathcal{F}^{(\bullet)} \boxtimes_{\mathcal{O}_Y} f'_u(\mathcal{E}^{(\bullet)}) \in \mathcal{C}(\mathfrak{Y})$. \qed
Definition 1.4.10. Let $\mathcal{D}$ be a data of coefficients over $\mathcal{V}$. We say that $\mathcal{D}$ is almost stable under dual functors if the following property holds: for any data $\mathcal{C}$ of coefficients over $\mathcal{V}$ which is stable under devissages, direct factors and realizable pushforwards, if $\mathcal{D} \subset \mathcal{C}$ then $\mathcal{D}^\vee \subset \mathcal{C}$. Remark from the biduality isomorphism that the inclusion $\mathcal{D}^\vee \subset \mathcal{C}$ is equivalent to the following one $\mathcal{D} \subset \mathcal{C}^\vee$.

Notation 1.4.11. Let $\mathcal{C}, \mathcal{D}$ be two data of coefficients. We put $T_0(\mathcal{D}, \mathcal{C}) := S(\mathcal{D}, \mathcal{C})$. By induction on $i \in \mathbb{N}$, we set $T_i(\mathcal{D}, \mathcal{C}) := T_{i-1}(\mathcal{D}, \mathcal{C}) \cap \mathcal{C}^\vee$, $T_i(\mathcal{D}, \mathcal{C}) := S(\mathcal{D}, U_i(\mathcal{D}, \mathcal{C}))$ and $T_{i+1}(\mathcal{D}, \mathcal{C}) := S(T_i(\mathcal{D}, \mathcal{C}), \mathcal{C})$. We put $T(\mathcal{D}, \mathcal{C}) := \cap_{i \in \mathbb{N}} T_i(\mathcal{D}, \mathcal{C})$.

Theorem 1.4.12. Let $\mathcal{D}$ be a data of coefficients which contains $\mathfrak{B}_{\text{div}}$, which satisfies $BK_+$, which is stable under extraordinary pullbacks, base change, tensor products and which is almost stable under dual functors. Let $\mathcal{C}$ be a data of coefficients containing $\mathcal{D}$, which satisfies $BK^+$, is stable under devissages, direct factors and realizable pushforwards. Then, the data of coefficients $T(\mathcal{D}, \mathcal{C})$ (see Definition 1.4.11) is included in $\mathcal{C}$, contains $\mathcal{D}$, is stable by devissages, direct factors, local cohomological functors, realizable pushforwards, extraordinary pullbacks, base change, tensor products and duals.

Proof. (I) First, we check by induction on $i \in \mathbb{N}$ that the data of coefficients $T_i(\mathcal{D}, \mathcal{C})$ contains $\mathcal{D}$, is contained in $\mathcal{C}$, is stable under devissages, direct factors, local cohomological functors, realizable pushforwards, extraordinary pullbacks, base change (which implies such stability properties for $T(\mathcal{D}, \mathcal{C})$).

(a) Let us verify that $T_0(\mathcal{D}, \mathcal{C})$ satisfies these properties. From 1.4.9.1 (resp. 1.4.9.2), $T_0(\mathcal{D}, \mathcal{C})$ is included in $\mathcal{C}$ (resp. contains $\mathcal{D}$). From 1.4.9.3, $T_0(\mathcal{D}, \mathcal{C})$ is stable under devissages, and under direct factors. From 1.4.9.4, $T_0(\mathcal{D}, \mathcal{C})$ is stable under smooth extraordinary pullbacks and under base change. From 1.4.4 and 1.4.9.5, $T_0(\mathcal{D}, \mathcal{C})$ is stable under localizations outside a divisor. Since $T_0(\mathcal{D}, \mathcal{C})$ is stable under devissage, then from 1.4.5 $T_0(\mathcal{D}, \mathcal{C})$ is stable under local cohomological functors. From 1.4.9.6 (resp. 1.4.9.7), $T_0(\mathcal{D}, \mathcal{C})$ is stable under realizable pushforwards (resp. satisfies $BK^+$). Hence, from 1.4.3, this implies that $T_0(\mathcal{D}, \mathcal{C})$ is stable under extraordinary pullbacks.

(b) Suppose that this is true for $T_i(\mathcal{D}, \mathcal{C})$ for some $i \in \mathbb{N}$.

(i) Since $\mathcal{D}$ is almost stable under duals, then $U_i(\mathcal{D}, \mathcal{C})$ contains $\mathcal{D}$. Since $\mathcal{D}$ is stable by tensor products, extraordinary pullbacks, and base change then, using 1.4.9.2 (where $\mathcal{C}$ is replaced by $U_i(\mathcal{D}, \mathcal{C})$), this implies that $\tilde{T}_i$ is contained in $\tilde{T}_i(\mathcal{D}, \mathcal{C})$ and $T_{i+1}(\mathcal{D}, \mathcal{C})$. Using 1.4.9.1, we get that $\tilde{T}_i(\mathcal{D}, \mathcal{C})$ and $T_{i+1}(\mathcal{D}, \mathcal{C})$ are included in $\mathcal{C}$.

(ii) From Lemmas 1.4.7 (resp. 1.4.8, resp. 1.4.9.3), $U_i(\mathcal{D}, \mathcal{C})$ satisfies $BK^+$ (resp. is stable under realizable pushforwards, resp. is stable under devissages, and direct factors). Hence, using the step Ia) in the case where $\mathcal{C}$ is replaced by $U_i(\mathcal{D}, \mathcal{C})$, we get that $\tilde{T}_i(\mathcal{D}, \mathcal{C})$ is stable under devissages, direct factors, local cohomological functors, realizable pushforwards, extraordinary pullbacks, base change. From Lemma 1.4.9.3 (resp. Lemma 1.4.9.4, resp. Lemma 1.4.9.5, resp.
Lemma 1.4.9.6, resp. Lemma 1.4.9.7), this yields that $T_{i+1}(\mathcal{D}, \mathcal{E})$ is stable under devisorizations, and direct factors (resp. smooth extraordinary pullbacks and base change, resp. local cohomological functors, resp. realizable pushforwards, resp. satisfies $BK^{-1}$). Using 1.4.3, this implies that $T_{i+1}(\mathcal{D}, \mathcal{E})$ is stable under extraordinary pullbacks.

(II) From 1.4.9.1, $T_{i+1}(\mathcal{D}, \mathcal{E})$ is contained in $\widetilde{T}_i(\mathcal{D}, \mathcal{E})$ and $\widetilde{T}_i(\mathcal{D}, \mathcal{E})$ is contained in $\widetilde{T}_i(\mathcal{D}, \mathcal{E}) \cap T_i(\mathcal{D}, \mathcal{E})$. Hence, by construction, the tensor product of two objects of $T_{i+1}(\mathcal{D}, \mathcal{E})$ is an object of $T_i(\mathcal{D}, \mathcal{E})$ and the dual of an object of $T_{i+1}(\mathcal{D}, \mathcal{E})$ is an object of $T_i(\mathcal{D}, \mathcal{E})$. □

Remark 1.4.13. We keep the notation and hypothesis of 1.4.12.

1. From 2.2.6, the proposition 1.4.12 gives a formalism of Grothendieck’s six operations and base change on couples. From Remark 1.3.3.1–2, we get that the data of coefficients $T$ of coefficients is local, then so is $T(\mathcal{D}, \mathcal{E})$.

2. If the data $\mathcal{C}$ of coefficients is local, then so is $T(\mathcal{D}, \mathcal{E})$.

3. Let $\Omega(\mathcal{D}, \mathcal{E})$ be the collection of data of coefficients which contains $\mathcal{D}$, which are included in $\mathcal{E}$, and which are stable by devisorizations, direct factors, local cohomological functors, realizable pushforwards, extraordinary pullbacks, base change, tensor products, and duals. A translation of Theorem 1.4.12 is that $\Omega(\mathcal{D}, \mathcal{E})$ is not empty.

Notation 1.4.14. Let $\mathcal{D}$ and $\mathcal{E}$ be two data of coefficients satisfying the hypotheses of the proposition 1.4.12. Let $\Omega(\mathcal{D}, \mathcal{E})$ be as defined in the last remark of 1.4.13.

We define the data of coefficients $T_{\text{max}}(\mathcal{D}, \mathcal{E})$ (resp. $T_{\text{min}}(\mathcal{D}, \mathcal{E})$) as follows: for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, the category $T_{\text{max}}(\mathcal{D}, \mathcal{E})(\mathcal{X})$ (resp. $T_{\text{min}}(\mathcal{D}, \mathcal{E})(\mathcal{X})$) is the full subcategory of $L\mathcal{D}_b^{\text{coh}}(\mathcal{X})$ of objects $\mathcal{E}(\bullet)$ satisfying the following max property (resp. min property):

(max) there exists a data of coefficients $\mathcal{B}$ of $\Omega(\mathcal{D}, \mathcal{E})$ such that $\mathcal{E}(\bullet) \in \mathcal{B}(\mathcal{X})$.

(min) for any data of coefficients $\mathcal{B}$ of $\Omega(\mathcal{D}, \mathcal{E})$ we have $\mathcal{E}(\bullet) \in \mathcal{B}(\mathcal{X})$.

Theorem 1.4.15. Let $\mathcal{D}$ and $\mathcal{E}$ be two data of coefficients satisfying the hypotheses of the proposition 1.4.12.

The data of coefficients $T_{\text{max}}(\mathcal{D}, \mathcal{E})$ and $T_{\text{min}}(\mathcal{D}, \mathcal{E})$ (see the definition in 1.4.14) belongs to $\Omega(\mathcal{D}, \mathcal{E})$. Moreover, they satisfy the following universal property: for any data of coefficients $\mathcal{B}$ of $\Omega(\mathcal{D}, \mathcal{E})$, the data of coefficients $T_{\text{max}}(\mathcal{D}, \mathcal{E})$ contains $\mathcal{B}$ and the data of coefficients $T_{\text{min}}(\mathcal{D}, \mathcal{E})$ is included in $\mathcal{B}$.

2. Formalism of Grothendieck six operations for arithmetic $\mathcal{D}$-modules over couples

Let $\mathcal{W}$ be an object of DVR($\mathcal{V}$), and $l$ be its residue field.

2.1. Data of coefficients over frames

Definition 2.1.1. 1. We define the category of frames over $\mathcal{W}$ as follows. A frame $(Y, X, \mathcal{P})$ over $\mathcal{W}$ means that $\mathcal{P}$ is a realizable smooth formal scheme over
\( \mathcal{W}, X \) is a closed subscheme of the special fiber \( P \) of \( \mathcal{P} \) and \( Y \) is an open subscheme of \( X \). Let \( (Y', X', \mathcal{P}') \) and \( (Y, X, \mathcal{P}) \) be two frames over \( \mathcal{W} \). A morphism \( \theta = (b, a, f): (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P}) \) of frames over \( \mathcal{W} \) is the data of a morphism \( f: \mathcal{P}' \rightarrow \mathcal{P} \) of realizable smooth formal schemes over \( \mathcal{W} \), a morphism \( a: X' \rightarrow X \) of \( l \)-schemes, and a morphism \( b: Y' \rightarrow Y \) of \( l \)-schemes inducing the commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{b} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{a} & X \\
\downarrow & & \downarrow f \\
\mathcal{P}' & \xrightarrow{f} & \mathcal{P}.
\end{array}
\]

If there is no ambiguity with \( \mathcal{W} \), we simply say frame or morphism of frames.

2. A morphism \( \theta = (b, a, f): (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P}) \) of frames over \( \mathcal{W} \) is said to be complete (resp. strictly complete) if \( a \) is proper (resp. \( f \) and \( a \) are proper).

**Definition 2.1.2.** 1. We define the category of couples over \( \mathcal{W} \) as follow. A couple \((Y, X)\) over \( \mathcal{W} \) means the two first data of a frame over \( \mathcal{W} \) of the form \((Y, X, \mathcal{P})\). A frame of the form \((Y, X, \mathcal{P})\) is said to be enclosing \((Y, X)\). A morphism of couples \( u = (b, a): (Y', X') \rightarrow (Y, X) \) over \( \mathcal{W} \) is the data of a morphism of \( l \)-schemes of the form \( a: X' \rightarrow X \) such that \( a(Y') \subset Y \) and \( b: Y' \rightarrow Y \) is the induced morphism.

2. A morphism of couples \( u = (b, a): (Y', X') \rightarrow (Y, X) \) over \( \mathcal{W} \) is said to be complete if \( a \) is proper.

**Remark 2.1.3.** 1. Let \( u = (b, a): (Y', X') \rightarrow (Y, X) \) be a complete morphism of couples over \( \mathcal{W} \). Then there exists a strictly complete morphism of frames over \( \mathcal{W} \) of the form \( \theta = (b, a, f): (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P}) \). Indeed, by definition, there exist some frames over \( \mathcal{W} \) of the form \((Y', X', \mathcal{P}'')\) and \((Y, X, \mathcal{P})\). There exists an immersion \( \mathcal{P}' \rightarrow \mathcal{P}'' \) with \( \mathcal{P}'' \) a proper and smooth formal \( \mathcal{W} \)-scheme. Hence, put \( \mathcal{P}' := \mathcal{P}'' \times \mathcal{P} \) and let \( f: \mathcal{P}' \rightarrow \mathcal{P} \) be the projection. Since \( a \) is proper, \( X \xleftarrow{a} \mathcal{P} \) is proper, and \( f \) is proper, then the immersion \( X' \xleftarrow{a} \mathcal{P}' \) is also proper.

2. Let \( u = (b, a): (Y', X') \rightarrow (Y, X) \) be a morphism of couples over \( \mathcal{W} \). Similarly, we check that there exists a morphism of frames over \( \mathcal{W} \) of the form \( \theta = (b, a, f): (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P}) \).

**Notation 2.1.4.** Let \( \mathcal{C} \) be a data of coefficients over \( \mathcal{V} \). Let \((Y, X, \mathcal{P})\) be a frame over \( \mathcal{W} \). We denote by \( \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \) the full subcategory of \( \mathcal{E}(\mathcal{P}) \) of objects \( \mathcal{E} \) such that there exists an isomorphism of the form \( \mathcal{E} \xrightarrow{\sim} \mathcal{R}_{Y}^{\mathcal{V}}(\mathcal{E}) \).

**Notation 2.1.5.** Let \((Y, X, \mathcal{P})\) be a frame over \( \mathcal{W} \). The full subcategory of \( D_{\text{coh}}^{b}(\mathcal{D}_{\mathcal{P}}, \mathcal{Q}) \) which is the essential image of \( \mathcal{L}D_{h}^{b}(Y, \mathcal{P}/\mathcal{W}) \) via the equivalence \ref{1.1.2.1} will be denoted by \( D_{h}^{b}(Y, \mathcal{P}/\mathcal{W}) \). Recall that \( D_{h}^{b}(\mathcal{D}_{\mathcal{P}}, \mathcal{Q}) \) is endowed with a canonical t-structure induced by that of \( D_{\text{coh}}^{b}(\mathcal{D}_{\mathcal{P}}, \mathcal{Q}) \) (see Remark \ref{1.3.3.3}). Similarly to \ref{1.3.10}, there is a canonical t-structure on \( D_{h}^{b}(Y, \mathcal{P}/\mathcal{W}) \) defined as follows: choose \( \mathfrak{U} \) an open set of \( \mathcal{P} \) such that \( Y \) is closed in \( \mathfrak{U} \). Then \( D_{h}^{\mathfrak{U}}(Y, \mathcal{P}/\mathcal{W}) \) and \( D_{h}^{\mathfrak{U}^{\mathcal{V}}}(Y, \mathcal{P}/\mathcal{W}) \) is the full subcategory of \( D_{h}^{b}(Y, \mathcal{P}/\mathcal{W}) \) of complexes \( \mathcal{E} \) such that
\( \mathcal{E}|_{\mathcal{U}} \in D^{\geq n}_{h}(\mathcal{D}^{\dagger}_{\mathcal{U}, \mathbb{Q}}) \) (resp. \( \mathcal{E}|_{\mathcal{U}} \in D^{\geq n}_{h}(\mathcal{D}^{\dagger}_{\mathcal{U}, \mathbb{Q}}) \)), where the t-structure on \( D^{b}_{h}(\mathcal{D}^{\dagger}_{\mathcal{U}, \mathbb{Q}}) \) is the canonical one. The heart of this t-structure, the category of overholonomic modules on \( (Y, \mathcal{P}/\mathcal{W}) \) after any base change, will be denoted by \( \mathcal{H}(Y, \mathcal{P}/\mathcal{W}) \).

Finally, we denote by \( \mathcal{H}^{i}_{t} \) the \( i \)th space of cohomology with respect to this canonical t-structure.

**Definition 2.1.6.** Let \((Y, X, \mathcal{P})\) be a frame over \( \mathcal{W} \) with \( Y \) smooth.

1. Choose \( \mathcal{U} \) an open set of \( \mathcal{P} \) such that \( Y \) is closed in \( \mathcal{U} \). Let \( \mathcal{E} \in \mathcal{H}(Y, \mathcal{P}/\mathcal{W}) \) (see the notation of 2.1.5). We say that \( \mathcal{E} \) is an overholonomic after any base change isocrystals on \((Y, \mathcal{P}/\mathcal{W})\) if \( \mathcal{E}|_{\mathcal{U}} \in \text{Isoc}^{\dagger\dagger}(Y, \mathcal{U}/\mathcal{W}) \) (see the notation of 1.3.5, remark we use the case where the divisor is empty). We denote by \( \mathcal{H}-\text{Isoc}^{\dagger\dagger}(Y, \mathcal{P}/\mathcal{W}) \) the full subcategory of \( \mathcal{H}(Y, \mathcal{P}/\mathcal{W}) \) whose objects are overholonomic after any base change isocrystals on \((Y, \mathcal{P}/\mathcal{W})\).

2. Let \( D^{b}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) \) be the full subcategory of \( D^{b}_{h}(Y, \mathcal{P}/\mathcal{W}) \) of the objects \( \mathcal{E} \) such that, for any integer \( i \), the module \( \mathcal{H}^{i}_{t}(\mathcal{E}) \in \mathcal{H}-\text{Isoc}^{\dagger\dagger}(Y, \mathcal{P}/\mathcal{W}) \), where \( \mathcal{H}^{i}_{t} \) means the \( i \)th spaces of cohomology with respect to the canonical t-structure (see the notation of 2.1.5). The canonical t-structure on \( D^{b}_{h}(Y, \mathcal{P}/\mathcal{W}) \) induces canonically another one on \( D^{b}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) \). For any integer \( n \in \mathbb{Z} \), we get the subcategories \( D^{\leq n}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) := D^{b}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) \cap D^{\leq n}_{h}(Y, \mathcal{P}/\mathcal{W}) \) and \( D^{\geq n}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) := D^{b}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) \cap D^{\geq n}_{h}(Y, \mathcal{P}/\mathcal{W}) \).

3. We denote by \( LD^{b}_{\mathbb{Q}, \text{coh}}(Y, \mathcal{P}/\mathcal{W}) \) the full subcategory of \( LD^{b}_{\mathbb{Q}, \text{coh}}(\mathcal{D}(\mathcal{P})) \) of objects \( \mathcal{E}(\bullet) \) such that \( \lim \mathcal{E}(\bullet) \in D^{b}_{h-\text{iso}}(Y, \mathcal{P}/\mathcal{W}) \).

**2.1.7.** Let \((Y, X, \mathcal{P})\) be a frame over \( \mathcal{W} \). Let \( \mathcal{E}(\bullet) \in LD^{b}_{\mathbb{Q}, \text{coh}}(Y, \mathcal{P}/\mathcal{W}) \). Then there exists a smooth \( d \)-stratification \((Y_{i})_{i=1,...,r}\) of \( Y \) in \( P \) (see Definition [12, 4.1.2.2]) such that we have \( \mathcal{E}(\bullet) \in LD^{b}_{\mathbb{Q}, \text{iso}}(Y_{i}, \mathcal{P}/\mathcal{W}) \), for any \( i = 1, \ldots, r \) (see [9, 6.2.3]).

2.2. Formalism of Grothendieck six operations over couples

**Theorem 2.2.1.** (Independence of the frame enclosing a couple) Let \( \mathcal{C} \) be a data of coefficients over \( \mathcal{V} \) which contains \( \mathfrak{B}_{\text{div}} \), which is stable under de vissage, realizable pushforwards, extraordinary pullbacks, and under local cohomological functors. Let \( \theta = (id, a, f) : (Y, X', \mathcal{P}') \to (Y, X, \mathcal{P}) \) be a complete morphism of frames over \( \mathcal{W} \).

The functors \( \mathbb{R}f_{!}^{\dagger} \) and \( f_{+} \) induce quasi-inverse equivalences of categories between \( \mathcal{C}(Y, \mathcal{P}/\mathcal{W}) \) and \( \mathcal{C}(Y, \mathcal{P}'/\mathcal{W}) \) (recall notation 2.1.4).

**Proof.** Using the stability properties that \( \mathcal{C} \) satisfies, we check that the functors \( \mathbb{R}f_{+} : \mathcal{C}(Y, \mathcal{P}'/\mathcal{W}) \to \mathcal{C}(Y, \mathcal{P}/\mathcal{W}) \) and \( \mathbb{R}f_{!} : \mathcal{C}(Y, \mathcal{P}/\mathcal{W}) \to \mathcal{C}(Y, \mathcal{P}'/\mathcal{W}) \) are well defined. Since \( \mathcal{C} \) is included in \( LD^{b}_{\mathbb{Q}, \text{ovcoh}} \), then this is a straightforward consequence of Theorem 1.3.11. \( \square \)

**Lemma 2.2.2.** Let \( \mathcal{C} \) be a data of coefficients over \( \mathcal{V} \) which contains \( \mathfrak{B}_{\text{div}}, \) which is stable under de vissage, realizable pushforwards, extraordinary pullbacks, and
local cohomological functors. Let \( Y := (Y, X) \) be a couple over \( \mathcal{W} \). Choose a frame of the form \( (Y, X, \mathcal{P}) \). The category \( \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \) does not depend, up to a canonical equivalence of categories, on the choice of the frame \( (Y, X, \mathcal{P}) \) over \( \mathcal{W} \) enclosing \( (Y, X) \). Hence, we can simply write \( \mathcal{E}(\mathcal{Y}/\mathcal{W}) \) instead of \( \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \) without ambiguity (up to canonical equivalence of categories).

**Proof.** Let \((Y, X, \mathcal{P}_1)\) and \((Y, X, \mathcal{P}_2)\) be two frames over \( \mathcal{W} \) enclosing \((Y, X)\). The closed immersions \( X \hookrightarrow \mathcal{P}_1 \) and \( X \hookrightarrow \mathcal{P}_2 \) induce \( X \hookrightarrow \mathcal{P}_1 \times \mathcal{P}_2 \). Denoting by \( \pi_1 : \mathcal{P}_1 \times \mathcal{P}_2 \twoheadrightarrow \mathcal{P}_1 \) and \( \pi_2 : \mathcal{P}_1 \times \mathcal{P}_2 \twoheadrightarrow \mathcal{P}_2 \) the structural projections, we get two morphisms of frames over \( \mathcal{W} \) of the form \((id, id, \pi_1) : (Y, X, \mathcal{P}_1 \times \mathcal{P}_2) \twoheadrightarrow (Y, X, \mathcal{P}_1) \) and \((id, id, \pi_2) : (Y, X, \mathcal{P}_1 \times \mathcal{P}_2) \twoheadrightarrow (Y, X, \mathcal{P}_2)\). From 2.2.1, the functors \( \pi_{2+} \mathcal{R}_{\mathcal{P}_1} \mathcal{G}_Y^\dagger \) and \( \pi_{1+} \mathcal{R}_{\mathcal{P}_2} \mathcal{G}_Y^\dagger \) are quasi-inverse equivalences of categories between \( \mathcal{E}(Y, \mathcal{P}_1/\mathcal{W}) \) and \( \mathcal{E}(Y, \mathcal{P}_2/\mathcal{W}) \).

**Lemma 2.2.3.** Let \( \mathcal{E} \) be a data of coefficients over \( \mathcal{V} \) which contains \( \mathcal{B}_{\text{div}} \), which is stable under de vissage, realizable pushforwards, extraordinary pullbacks, local cohomological functors, and duals. Let \( \mathcal{Y} := (Y, X) \) be a couple over \( \mathcal{W} \). Choose a frame of the form \((Y, X, \mathcal{P})\). The functor \( \mathcal{R}_{\mathcal{P}} \mathcal{G}_Y^\dagger \mathcal{D}_\mathcal{P} : \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \to \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \) does not depend, up to canonical isomorphism of 2.2.2 (more precisely, we have the commutative diagram 2.2.3.1 up to canonical isomorphism), on the choice of the frame enclosing \((Y, X)\). Hence, we will denote by \( \mathcal{D}_\mathcal{Y} : \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \to \mathcal{E}(Y, \mathcal{P}/\mathcal{W}) \) the functor \( \mathcal{R}_{\mathcal{P}} \mathcal{G}_Y^\dagger \mathcal{D}_\mathcal{P} \).

**Proof.** As in the beginning of the proof, 2.2.2, let \((Y, X, \mathcal{P}_1)\) and \((Y, X, \mathcal{P}_2)\) be two frames over \( \mathcal{W} \) enclosing \((Y, X)\). Let \( \pi_1 : \mathcal{P}_1 \times \mathcal{P}_2 \twoheadrightarrow \mathcal{P}_1 \) and \( \pi_2 : \mathcal{P}_1 \times \mathcal{P}_2 \twoheadrightarrow \mathcal{P}_2 \) be the structural projections. We have to check that the diagram

\[
\begin{array}{ccc}
\mathcal{E}(Y, \mathcal{P}_1/\mathcal{W}) & \xrightarrow{\sim} & \mathcal{E}(Y, \mathcal{P}_1 \times \mathcal{P}_2/\mathcal{W}) \\
\mathcal{R}_{\mathcal{P}_1} \mathcal{G}_Y^\dagger & \downarrow & \mathcal{R}_{\mathcal{P}_1 \times \mathcal{P}_2} \mathcal{G}_Y^\dagger \\
\mathcal{E}(Y, \mathcal{P}_1/\mathcal{W}) & \xrightarrow{\pi_{2+}} & \mathcal{E}(Y, \mathcal{P}_2/\mathcal{W})
\end{array}
\]

is commutative, up to canonical isomorphism. Let \( \mathcal{E}(\bullet) \in \mathcal{E}(Y, \mathcal{P}_1 \times \mathcal{P}_2/\mathcal{W}) \). From 1.3.12, we have the isomorphism \( \mathcal{D}_{\mathcal{P}_2} \pi_{2+} (\mathcal{E}(\bullet)) \sim \pi_{2+} \mathcal{D}_{\mathcal{P}_1 \times \mathcal{P}_2} (\mathcal{E}(\bullet)) \). Hence, by applying the functor \( \mathcal{R}_{\mathcal{P}} \mathcal{G}_Y^\dagger \) to this isomorphism, we get the first one

\[
\begin{aligned}
\mathcal{R}_{\mathcal{P}_2} \mathcal{G}_Y^\dagger \mathcal{D}_{\mathcal{P}_2} \pi_{2+} (\mathcal{E}(\bullet)) & \sim \mathcal{R}_{\mathcal{P}_1 \times \mathcal{P}_2} \mathcal{D}_{\mathcal{P}_1 \times \mathcal{P}_2} (\mathcal{E}(\bullet)) \\
& \sim \pi_{2+} \mathcal{R}_{\mathcal{P}_1 \times \mathcal{P}_2} \mathcal{G}_Y^\dagger (\mathcal{E}(\bullet)) \\
\end{aligned}
\]

(\( \mathcal{E}(\bullet) \)). Since \( Y \hookrightarrow \pi_{2-1}(Y) \) is a closed immersion (recall formal schemes are separated by convention), then \( Y = \overline{Y} \cap \pi_{2-1}(Y) \), where \( \overline{Y} \) is the closure of \( Y \) in \( \mathcal{P}_1 \times \mathcal{P}_2 \). Since \( \mathcal{D}_{\mathcal{P}_1 \times \mathcal{P}_2} (\mathcal{E}(\bullet)) \) has in support in \( \overline{Y} \), then \( \mathcal{R}_{\mathcal{P}_1 \times \mathcal{P}_2} \mathcal{G}_Y^\dagger (\mathcal{E}(\bullet)) \sim \mathcal{R}_{\mathcal{P}_1 \times \mathcal{P}_2} \mathcal{G}_Y^\dagger (\mathcal{E}(\bullet)) \). Hence, we have checked the commutativity, up to commutative isomorphism, of the right square of 2.2.3.1. From 2.2.1, \( \pi_{1+} \) is canonically a quasi-inverse of the equivalence of categories \( \mathcal{R}_{\mathcal{P}_1} \mathcal{G}_Y^\dagger : \mathcal{E}(Y, \mathcal{P}_1 \times \mathcal{P}_2/\mathcal{W}) \cong \mathcal{E}(Y, \mathcal{P}_1/\mathcal{W}) \) (we means that we have canonical
isomorphisms $\pi_{1+} \mathbb{R} \Gamma_Y^+ \pi_1^+ \xrightarrow{\sim} id$ and $id \xrightarrow{\sim} \mathbb{R} \Gamma_Y^+ \pi_1^+ \pi_1^+$). Hence, we get the commutativity, up to canonical isomorphism, of the left square of 2.2.3.1. \hfill \Box

**Lemma 2.2.4.** Let $\mathcal{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathcal{B}_{\text{div}}$, which is stable under devissage, realizable pushforwards, extraordinary pullbacks, and local cohomological functors. Let $u = (b, a): (Y', X') \rightarrow (Y, X)$ be a morphism of couples over $\mathcal{W}$. Put $\mathcal{Y} := (Y, X)$ and $\mathcal{Y}' := (Y', X')$. Let us choose a morphism of frames $\theta = (b, a, f): (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P})$ over $\mathcal{W}$ enclosing $u$.

1. The functor $\theta^!: := \mathbb{R} \Gamma_Y^+ b \circ f^! : \mathcal{C}(Y, \mathcal{P}/\mathcal{W}) \rightarrow \mathcal{C}(Y', \mathcal{P}'/\mathcal{W})$ does not depend on the choice of such $\theta$ enclosing $u$ (up to canonical equivalences of categories). Hence, it will be denoted by $u^! : \mathcal{C}(\mathcal{Y}/\mathcal{W}) \rightarrow \mathcal{C}(\mathcal{Y}'/\mathcal{W})$.

2. Suppose that $u$ is complete, i.e. that a frame of the form $(Y, X, \mathcal{P})$. The bifunctor $+ := \mathbb{R} \Gamma_Y^+ a \circ \mathbb{R} \Gamma_X^+ f \circ \mathbb{R} \Gamma_{\mathcal{P}/\mathcal{W}}^+ (\mathcal{E}(\mathcal{\bullet}))$ does not depend on the choice of such $\theta$ enclosing $u$, we proceed as in the proof of 2.2.3. \hfill \Box

**Lemma 2.2.5.** Let $\mathcal{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathcal{B}_{\text{div}}$, which is stable under devissage, realizable pushforwards, extraordinary pullbacks, and tensor products. Let $\mathcal{Y} := (Y, X)$ be a couple over $\mathcal{W}$. Choose a frame of the form $(Y, X, \mathcal{P})$. The bifunctor $- \otimes_{\mathcal{O}_Y} [- \dim P] : \mathcal{C}(Y, \mathcal{P}/\mathcal{W}) \times \mathcal{C}(Y, \mathcal{P}/\mathcal{W}) \rightarrow \mathcal{C}(Y, \mathcal{P}/\mathcal{W})$ does not depend, up to the canonical equivalences of categories of 2.2.2, on the choice of the frame enclosing $(Y, X)$. It will be denoted by $\otimes_{\mathcal{Y}} : \mathcal{C}(\mathcal{Y}/\mathcal{W}) \times \mathcal{C}(\mathcal{Y}/\mathcal{W}) \rightarrow \mathcal{C}(\mathcal{Y}/\mathcal{W})$.

**Proof.** From Lemmas 1.4.4 and 1.4.5, the data of coefficients $\mathcal{C}$ is also stable under local cohomological functors. From [12, 2.1.9.1] (resp. 1.4.1.2), extraordinary inverse images (resp. local cohomological functors) commute with tensor products (up to a shift). Proceeding as in the proof of 2.2.3 with its notation, $\mathbb{R} \Gamma_Y^+ \pi_1^+$ and $\mathbb{R} \Gamma_Y^+ \pi_2^+$ commute with tensor products and then so are $\pi_{1+}$ and $\pi_{2+}$. \hfill \Box

2.2.6. (Formalism of Grothendieck six operations) Let $\mathcal{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathcal{B}_{\text{div}}$, which is stable under devissage, realizable pushforwards, extraordinary pullbacks, duals, and tensor products. To sum-up the above Lemmas we can define a formalism of Grothendieck operations on couples as follows. Let $u = (b, a): (Y', X') \rightarrow (Y, X)$ be a morphism of couples over $\mathcal{W}$. Put $\mathcal{Y} := (Y, X)$ and $\mathcal{Y}' := (Y', X')$. 


1. We have the dual functor \( D_Y : \mathcal{E}(\mathcal{Y}/\mathcal{W}) \to \mathcal{E}(\mathcal{Y}/\mathcal{W}) \) (see 2.2.3).

2. We have the extraordinary pullback \( u' : \mathcal{E}(\mathcal{Y}/\mathcal{W}) \to \mathcal{E}(\mathcal{Y}'/\mathcal{W}) \) (see 2.2.4). We get the pullbacks \( u^+ := D_Y \circ u' \circ D_Y \).

3. Suppose that \( u \) is complete. Then, we have the functor \( u'_+ : \mathcal{E}(\mathcal{Y}'/\mathcal{W}) \to \mathcal{E}(\mathcal{Y}/\mathcal{W}) \) (see 2.2.4). We denote by \( u'_! := D_Y \circ u'_+ \circ D_Y' \), the extraordinary pushforward by \( u \).

4. We have the tensor product \( \otimes \mathcal{Y} : \mathcal{E}(\mathcal{Y}/\mathcal{W}) \times \mathcal{E}(\mathcal{Y}/\mathcal{W}) \to \mathcal{E}(\mathcal{Y}/\mathcal{W}) \) (see 2.2.5).

Example 2.2.7. We recall the data of coefficients \( L D_{\mathbb{Q}_{\text{ovhol}}}^b \) and \( L D_{\mathbb{Q}_{\text{h}}}^b \) are defined respectively in 1.3.2.2 and 1.3.2.3. Using Lemmas 1.4.3 and 1.4.9 (and Remark 1.4.6) are stable under local cohomological functors, realizable pushforwards, extraordinary pullbacks, and duals. Hence, with the notation 2.2.2, using Lemmas 2.2.4, 2.2.5, and 2.2.3, for any frame \((Y, X, \mathcal{P})\) over \( \mathcal{W} \), we get the categories of the forms \( L D_{\mathbb{Q}_{\text{h}}}^b (Y, \mathcal{P}/\mathcal{W}), L D_{\mathbb{Q}_{\text{h}}}^b (\mathcal{Y}/\mathcal{W}) \), \( L D_{\mathbb{Q}_{\text{ovhol}}}^b (Y, \mathcal{P}/\mathcal{W}) \) or \( L D_{\mathbb{Q}_{\text{ovhol}}}^b (\mathcal{Y}/\mathcal{W}) \) endowed with five of Grothendieck cohomological operations (the tensor product is a priori missing). We keep in this context the notation 2.2.6.1–3 concerning these five functors.

Notation 2.2.8. Let \((Y, X, \mathcal{P})\) be a frame over \( \mathcal{W} \).

1. With the notation of 2.1.5, the category \( D_{\mathbb{h}}^b (Y, \mathcal{P}/\mathcal{W}) \) does not depend on the choice of the frame \((Y, X, \mathcal{P})\) enclosing the couple \( \mathcal{Y} := (Y, X) \) (up to canonical equivalences of categories). Hence, it will be denoted by \( D_{\mathbb{h}}^b (\mathcal{Y}/\mathcal{W}) \) without any ambiguity.

2. From 2.1.5, there is a canonical t-structure on \( D_{\mathbb{h}}^b (Y, \mathcal{P}/\mathcal{W}) \). Using 1.3.11, this t-structure is independent on the choice of the frame \((Y, X, \mathcal{P})\) enclosing \( \mathcal{Y} := (Y, X) \). Hence, we get a canonical t-structure on \( D_{\mathbb{h}}^b (\mathcal{Y}/\mathcal{W}) \), whose heart, the category of overholonomic modules on \( \mathcal{Y}/\mathcal{W} \) after any base change, is denoted by \( H(\mathcal{Y}/\mathcal{W}) \). Finally, we denote by \( H_i^b \) the \( i \)-th space of cohomology with respect to this canonical t-structure. With this canonical t-structure, for any integer \( n \in \mathbb{Z} \), we get the subcategories \( D_{\mathbb{h}}^{<n} (\mathcal{Y}/\mathcal{W}) \) and \( D_{\mathbb{h}}^{\leq n} (\mathcal{Y}/\mathcal{W}) \).

3. From 2.1.6, we have a canonical t-structure on \( D_{\mathbb{h}_{\text{isoc}}}^b (Y, \mathcal{P}/\mathcal{W}) \) such that the inclusion \( D_{\mathbb{h}_{\text{isoc}}}^b (Y, \mathcal{P}/\mathcal{W}) \subset D_{\mathbb{h}}^b (Y, \mathcal{P}/\mathcal{W}) \) preserves t-structures. Using Lemma [9, 5.4.1.1], this t-structure is independent (up the canonical equivalence of categories of the type of Theorem 1.3.11) on the choice of the frame \((Y, X, \mathcal{P})\) enclosing \( \mathcal{Y} := (Y, X) \). Hence, we get a canonical t-structure on \( D_{\mathbb{h}_{\text{isoc}}}^b (\mathcal{Y}/\mathcal{W}) \), whose heart, the category of overholonomic after any base change isocrystals on \( \mathcal{Y}/\mathcal{W} \), is denoted by \( H_{\text{Isoc}}^{\mathbb{h}}(\mathcal{Y}/\mathcal{W}) \). With the notation of 2.2.8.2, for any integer \( n \in \mathbb{Z} \), we get the subcategories \( D_{\mathbb{h}_{\text{isoc}}}^{\leq n} (\mathcal{Y}/\mathcal{W}) := D_{\mathbb{h}_{\text{isoc}}}^b (\mathcal{Y}/\mathcal{W}) \cap D_{\mathbb{h}_{\text{isoc}}}^{< n} (\mathcal{Y}/\mathcal{W}) \) and \( D_{\mathbb{h}_{\text{isoc}}}^{\geq n} (\mathcal{Y}/\mathcal{W}) := D_{\mathbb{h}_{\text{isoc}}}^b (\mathcal{Y}/\mathcal{W}) \cap D_{\mathbb{h}_{\text{isoc}}}^{\geq n} (\mathcal{Y}/\mathcal{W}) \). Finally, we denote by \( L D_{\mathbb{Q}_{\text{h}_{\text{isoc}}}}^b (\mathcal{Y}/\mathcal{W}) \) the full subcategory of \( L D_{\mathbb{Q}_{\text{coh}}}^b (\mathcal{Y}/\mathcal{W}) \) of objects \( E(\mathbb{h}) \) such that \( \lim E(\mathbb{h}) \in D_{\mathbb{h}_{\text{isoc}}}^b (\mathcal{Y}/\mathcal{W}) \).

2.3. Formalism of Grothendieck six operations over realizable varieties

Definition 2.3.1. (Proper compactification)
1. A frame \((Y, X, \mathcal{P})\) over \(\mathcal{W}\) is said to be proper if \(\mathcal{P}\) is proper. The category of proper frames over \(\mathcal{W}\) is the subcategory of the category of frames over \(\mathcal{W}\) whose objects are proper frames over \(\mathcal{W}\).

2. The category of proper couples over \(\mathcal{W}\) is the full subcategory of the category of couples over \(\mathcal{W}\) whose objects \((Y, X)\) are such that \(X\) is proper. We remark that if \((Y, X)\) is a proper couple over \(\mathcal{W}\) then there exists a proper frame over \(\mathcal{W}\) of the form \((Y, X, \mathcal{P})\).

3. A realizable variety over \(\mathcal{W}\) is a \(l\)-scheme \(Y\) such that there exists a proper frame of the form \((Y, X, \mathcal{P})\). For such frame \((Y, X, \mathcal{P})\), we say that the proper frame \((Y, X, \mathcal{P})\) encloses \(Y\) or that the proper couple \((Y, X)\) encloses \(Y\).

2.3.2. (Formalism of Grothendieck six operations) Let \(\mathcal{C}\) be a data of coefficients over \(\mathcal{V}\) which contains \(\mathcal{B}_{\text{div}}\), which is stable under devissage, realizable pushforwards, extraordinary pullbacks, duals, and tensor products. Similarly to Lemma 2.2.2, we check using Theorem 2.2.1 that the category \(\text{pushforwards, extraordinary pullbacks, duals, and tensor products. Similarly to V}\) encloses \(Y\) where \(\mathcal{C}\) of constructibility in the context of overholonomic complexes after any base change.

2.4. Constructible t-structure for overholonomic complexes after any base change

For completeness (this will not be useful in this paper), we extend Abe’s definition of constructibility in the context of overholonomic complexes after any base change by introducing a new way of defining it (i.e. by devissage).

2.4.1. (Constructible t-structure)

Let \(\mathcal{Y} := (Y, X)\) be a couple. Choose a frame \((Y, X, \mathcal{P})\). If \(Y' \to Y\) is an immersion, then we denote by \(i_{Y'}: (Y', X', \mathcal{P}) \to (Y, X, \mathcal{P})\) the induced morphism where \(X'\) is the closure of \(Y'\) in \(X\). We define on \(D_{h}^{b}(\mathcal{Y}/\mathcal{W})\) the constructible t-structure as follows.

1. An object \(\mathcal{E} \in D_{h}^{b}(\mathcal{Y}/\mathcal{W})\) belongs to \(D_{h}^{c; \geq 0}(\mathcal{Y}/\mathcal{W})\) if there exists a smooth stratification (see Definition [2, 2.2.1]) \((Y_{i})_{i=1, \ldots, r}\) of \(Y\) such that for any \(i\), the complex \(i_{Y_{i}}^{+}(\mathcal{E})[d_{Y_{i}}]\) (see notation 2.2.7) belongs to \(D_{h-\text{isoc}}^{\leq 0}(Y_{i}, \mathcal{P}/\mathcal{W})\).

2. An object \(\mathcal{E} \in D_{h}^{b}(\mathcal{Y}/\mathcal{W})\) belongs to \(D_{h}^{c; \leq 0}(\mathcal{Y}/\mathcal{W})\) if there exists a smooth stratification \((Y_{i})_{i=1, \ldots, r}\) of \(Y\) such that for any \(i\), the complex \(i_{Y_{i}}^{+}(\mathcal{E})[d_{Y_{i}}]\) belongs to \(D_{h-\text{isoc}}^{\leq 0}(Y_{i}, \mathcal{P}/\mathcal{W})\).
Proposition 2.4.2. Let $\mathcal{Y} := (Y, X)$ be a couple.

1. Let $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}'[1]$ be an exact triangle in $D^b_h(\mathcal{Y}/\mathcal{W})$. If $\mathcal{E}'$ and $\mathcal{E}''$ are in $D^c_{k;0}(\mathcal{Y}/\mathcal{W})$ (resp. $D^c_{k;0}(\mathcal{Y}/\mathcal{W})$) then so is $\mathcal{E}$.

2. Suppose that $Y$ is smooth. Let $\mathcal{E} \in D^c_{h-isoc}(\mathcal{Y}/\mathcal{W})$. Then $\mathcal{E} \in D^c_{k;0}(\mathcal{Y}/\mathcal{W})$ (resp. $\mathcal{E} \in D^c_{k;0}(\mathcal{Y}/\mathcal{W})$) if and only if $\mathcal{E} \in D^c_{dX}(\mathcal{Y}/\mathcal{W})$ (resp. $\mathcal{E} \in D^c_{dX}(\mathcal{Y}/\mathcal{W})$).

Proof. The proof of the first part is similarly to 3.2.7. The second part is easy. □

Remark 2.4.3. Let $\mathcal{Y} := (Y, X)$ be a proper couple. Then, the categories $D^c_{k;0}(\mathcal{Y}/\mathcal{W})$ and $D^c_{k;0}(\mathcal{Y}/\mathcal{W})$ only depend on $Y$ and can be simply denoted by $D^c_{k;0}(\mathcal{Y}/\mathcal{W})$ and $D^c_{k;0}(\mathcal{Y}/\mathcal{W})$. This constructible t-structure is compatible with that defined by Abe in [1, 1.3.1] (more precisely, one can check that, if we restrict to the categories denoted there by $D^b_{hol}(\mathcal{Y}/\mathcal{W})$, we get Abe’s definition of constructibility). Indeed, let $\mathcal{E} \in D^c_{h-isoc}(\mathcal{Y}/\mathcal{W})$. For any immersion $i$ of realizable varieties, Abe’s definition of $D^c_{hol}$ and of $D^c_{hol}$ are stable by $i^+$ and under $i_!$. Since this property is obvious with the definition of 2.4.1, by devissage in overconvergent isocrystals, we reduce to the case where there exist a smooth subvariety $Z$ of $Y$ and an object $\mathcal{G} \in D^c_{h-isoc}(\mathcal{Z}/\mathcal{W})$ such that $\mathcal{E} = i_{Z!}(\mathcal{G})$. In that case, it is clear that both definitions of $D^c_{hol}(\mathcal{Y}/\mathcal{W})$ are the same. We proceed in the same way for $D^c_{h-isoc}(\mathcal{Y}/\mathcal{W})$.

3. Around unipotence

3.1. $\Sigma$-unipotent monodromy

Let $\mathcal{W}$ be an object of DVR($\mathcal{V}$), and $l$ be its residue field.

3.1.1. Let $(Y, X, \mathcal{P})$ be a frame over $\mathcal{W}$. We suppose that $X$ is $l$-smooth, $Z := X - Y$ is a simple normal crossing divisor of $X$ and that there exists a divisor $T$ of $\mathcal{P}$ such that $Z = X \cap T$. Let $Z = \bigcup_{i=1}^r Z_i$ be the decomposition of $Z$ into irreducible components. We denote by $\text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W})$, the full subcategory of Berthelot’s category of overconvergent isocrystals on $(\mathcal{P}, T, X/\mathcal{W})$ (see Notation 1.3.5) of isocrystals on $(Y, X/\mathcal{W})$ having $\Sigma^r$-unipotent monodromy according to Shiho’s definition [23, 3.9] (and its remark). We denote by $\text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W})$ the full subcategory of $\text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W})$ such that the equivalence of categories 1.3.5.1 induces the following one

$$\text{sp}_+ : \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}) \cong \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}).$$

We denote by $\text{Isoc}^{(*)}(\mathcal{P}, T, X/\mathcal{W})$ the full subcategory of $\text{Isoc}^{(*)}(\mathcal{P}, T, X/\mathcal{W}) \subset \text{LM}_{\text{coh}}(\widehat{D}^{(*)}(\mathcal{P}, T))$ (see Notation [14, 2.2.4]) of objects $\mathcal{E}^{(*)}$ such that $\text{lim} \ E^{(*)} \in \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W})$. Since the equivalence of categories 1.1.2.1 is still valid by adding overconvergent singularities along a divisor (i.e. see [14, 2.2.4.2]), then we get an equivalence of categories

$$\text{lim} : \text{Isoc}^{(*)}(\mathcal{P}, T, X/\mathcal{W}) \cong \text{Isoc}^\dagger(\mathcal{P}, T, X/\mathcal{W}).$$
Lemma 3.1.2. We keep the notation and hypotheses of 3.1.1. The category \( \text{Isoc}^{††}_\Sigma(\mathcal{P}, T, X/\mathcal{W}) \) is an abelian subcategory of \( \text{H-Isoc}^{††}(Y, \mathcal{P}/\mathcal{W}) \) (see Notation 2.1.6.1) stable under extension.

Proof. Let \( \mathcal{E} \in \text{Isoc}^{††}_\Sigma(\mathcal{P}, T, X/\mathcal{W}) \). Since the property that \( \mathcal{E} \) belongs to a category of the form \( \text{Isoc}^{††}_\Sigma(\mathcal{P}, T, X/\mathcal{W}) \) is stable under base change, then we reduce to check that \( \mathcal{E} \) is an overholonomic \( \mathcal{D}^{†}_{\mathcal{P}, \mathcal{Q}} \)-module. Let \( E \in \text{Isoc}^{†}_\Sigma(\mathcal{P}, T, X/\mathcal{W}) \) such that \( \text{sp}_+ (E) \sim \mathcal{E} \) (see 3.1.1.1). Since this local in \( \mathcal{P} \), we can suppose \( \mathcal{P} \) affine, that there exists a closed immersion of smooth formal schemes over \( \mathcal{W} \) of the form \( \mathcal{X} \hookrightarrow \mathcal{P} \) which is a lifting of \( X \hookrightarrow \mathcal{P} \), and that there exists a strict normal crossing divisor \( \mathcal{Z} \) of \( \mathcal{X} \) which lifts \( Z \). By using Berthelot–Kashiwara theorem (see [14, 5.3.6]), we reduce to the case where \( X = P \). Let \( \text{sp}: \mathcal{X}_K \rightarrow \mathcal{X} \) be the specialisation morphism from the rigid analytic space associated to \( \mathcal{X} \) (also called Raynaud generic fiber of \( \mathcal{X} \)) to \( \mathcal{X} \). From Theorem [23, 3.16] (or better Remark [23, 3.17]), there exists a convergent isocrystal \( G \) on the log scheme \( (X, M_Z) \) over \( \mathcal{W} \), where \( M_Z \) is the log-structure corresponding to the strict normal crossing divisor \( Z \) of \( X \), with exponents in \( \pi (\Sigma) \) such that \( j^{†}(G) \sim E \), where \( j: Y \rightarrow X \) is the open immersion. From [15, 2.3.13], since by hypothesis the elements of the group \( \Sigma \) are \( p \)-adically non Liouville numbers, then \( u_+ \text{sp}_a (G) \) is overholonomic, where \( u: (\mathcal{X}, M_Z) \rightarrow \mathcal{X} \) is the canonical morphism. Since \( \mathcal{E} \sim (\mathcal{Z}) u_+ \text{sp}_a (G) \), and since the overholonomicity is stable under \( (\mathcal{Z}) \) then \( \mathcal{E} \) is also overholonomic.

The stability under extension is clear by definition (see Definition [23, 1.3]) and the fact that \( \text{Isoc}^{††}_\Sigma(\mathcal{P}, T, X/\mathcal{W}) \) is an abelian subcategory of \( \text{H-Isoc}^{††}(Y, \mathcal{P}/\mathcal{W}) \) follows from [23, 1.17]. \( \square \)

Definition 3.1.3. Let \( (Y, X, \mathcal{P}) \) be a frame over \( \mathcal{W} \). We suppose that \( X \) is \( l \)-smooth, \( Z := X - Y \) is a simple normal crossing divisor of \( X \). We put \( \mathcal{Y} := (Y, X) \).

1. Let \( \mathcal{E} \in \text{H-Isoc}^{††}(Y, \mathcal{P}/\mathcal{W}) \) (see Notation 2.1.6.1). We will say that \( \mathcal{E} \) has “\( \Sigma \)-unipotent monodromy” if for any open set \( \mathcal{P}' \) of \( \mathcal{P} \) such that there exists a divisor \( T' \) of \( \mathcal{P}' \) satisfying \( Z \cap P' = X \cap T' \), we have \( \mathcal{E}|_{\mathcal{P}'} \in \text{Isoc}^{††}_\Sigma(\mathcal{P}', T', X \cap P'/\mathcal{W}) \) (see Notation 3.1.1).

2. We denote by \( \text{Isoc}^{††}_\Sigma(Y, \mathcal{P}/\mathcal{W}) \) the full subcategory of \( \text{H-Isoc}^{††}(Y, \mathcal{P}/\mathcal{W}) \) whose objects “have \( \Sigma \)-unipotent monodromy”. We remark that Lemma 3.1.2 justifies the fact that we remove “H” in the notation. Since the category \( \text{Isoc}^{††}_\Sigma(Y, \mathcal{P}/\mathcal{W}) \) is independent (up to canonical equivalences of categories appearing in 2.2.8 to define \( \text{H-Isoc}^{††}(\mathcal{Y}/\mathcal{W}) \)) of the choice of the frame \( (Y, X, \mathcal{P}) \) enclosing \( \mathcal{Y} \), we will denote it by \( \text{Isoc}^{††}_\Sigma(\mathcal{Y}/\mathcal{W}) \).

3. We denote by \( \text{Isoc}^{(\bullet)}_\Sigma(Y, \mathcal{P}/\mathcal{W}) \) or simply by \( \text{Isoc}^{(\bullet)}_\Sigma(\mathcal{Y}/\mathcal{W}) \) the full subcategory of \( \text{LM}_{\mathcal{Q}, \text{coh}}(\mathcal{D}^{(\bullet)}_{\mathcal{P}}) \) (see Notation [14, 2.2.4]) of objects \( \mathcal{E}^{(\bullet)} \) such that \( \lim \mathcal{E}^{(\bullet)} \in \text{Isoc}^{††}_\Sigma(\mathcal{Y}/\mathcal{W}) \). The functor \( \lim : \text{Isoc}^{(\bullet)}_\Sigma(\mathcal{Y}/\mathcal{W}) \cong \text{Isoc}^{††}_\Sigma(\mathcal{Y}/\mathcal{W}) \).

Proposition 3.1.4. We keep the notation and hypotheses of 3.1.3.
1. The property that an object $\mathcal{E}(\bullet)$ of $\text{LM}_{\text{Q,coh}}(\mathcal{D}_P^{\bullet})$ is in $\text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$ is local in $\mathcal{P}$.

2. The category $\text{Isoc}_\Sigma^+(\mathbb{Y}/\mathbb{W})$ (resp. $\text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$) is an abelian subcategory of $H\text{-Isoc}^+(\mathbb{Y}/\mathbb{W})$ (resp. $\text{LM}_{\text{Q,coh}}(\mathcal{D}_P^{\bullet})$) stable under extension.

3. The category $\text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$ is stable under base change in the following sense: for any morphism $\mathcal{W} \to \mathcal{W}'$ of DVR$(\mathcal{V})$, for any $\mathcal{E}(\bullet) \in \text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$, putting $(\mathcal{Y}', X', \mathcal{P}')$ the frame over $\mathcal{W}'$ induced by base change from $(Y, X, \mathcal{P})$ by $\mathcal{W} \to \mathcal{W}'$, we get $\mathcal{W}\mathcal{W}\mathcal{E}(\bullet) \in \text{Isoc}_\Sigma(\mathcal{Y}', X'/\mathcal{W}')$.

4. Let $\mathcal{E}(\bullet), \mathcal{F}(\bullet) \in \text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$. We have $\mathcal{E}(\bullet) \otimes_{\mathcal{O}_P(\mathcal{T})_Q} \mathcal{F}(\bullet)[d_{Y/P}] \in \text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$ (which means in particular that the complex is in fact isomorphic to a module).

Proof. Let $\mathcal{E}(\bullet)$ of $\text{LM}_{\text{Q,coh}}(\mathcal{D}_P^{\bullet})$. The fact that $\mathcal{E}(\bullet) \in \text{LM}_{\text{Q,coh}}(\mathcal{D}_P^{\bullet})$ is local in $\mathcal{P}$ (recall Definition [14, 2.2.1]). Hence, we get the first assertion. The second one is a consequence of 3.1.2. The assertion 3) is straightforward. Let us check 4). Let $\mathcal{E}(\bullet), \mathcal{F}(\bullet) \in \text{Isoc}(\mathbb{Y}/\mathbb{W})$. By using the assertion 1), we can suppose that there exists a divisor $T$ of $P$ such that $Y = X \setminus T$. Then, $\text{Isoc}_\Sigma(\mathbb{Y}/\mathbb{W})$ is a subcategory of $\text{Isoc}(\mathbb{Y}/\mathbb{W})$. From Lemma [12, 3.2.2.1] (in fact, [12, 3.1.5.1] is sufficient), we get $\mathcal{E}(\bullet) \otimes_{\mathcal{O}_P(\mathcal{T})_Q} \mathcal{F}(\bullet)[d_{Y/P}] \in \text{Isoc}(\mathbb{Y}/\mathbb{W})$. Since $\mathcal{E}(\bullet), \mathcal{F}(\bullet) \in \text{LM}_{\text{Q,coh}}(\mathcal{D}_P^{\bullet}) \cap \text{Isoc}(\mathbb{P}, T, X/\mathbb{W})$, then $\mathcal{E}(\bullet) \otimes_{\mathcal{O}_P(\mathcal{T})_Q} \mathcal{F}(\bullet)[d_{Y/P}] \in \text{Isoc}(\mathbb{P}, T, X/\mathbb{W})$. Hence, from [12, 2.1.5], we get $\mathcal{E}(\bullet) \otimes_{\mathcal{O}_P(\mathcal{T})_Q} \mathcal{F}(\bullet)[d_{Y/P}] \in \text{Isoc}(\mathbb{P}, T, X/\mathbb{W})$. Hence, we reduce to check that $E \otimes F$ has $\Sigma'$-unipotent monodromy according to Shiho’s definition [23, 3.9] (and its remark). Using [23, 3.16] (or better [23, 3.17]), we get that $E$ (resp. $F$) comes from a log convergent isocrystal $G_1$ (resp. $G_2$) with exponents in $\tau(\Sigma)$. If $\text{Exp}(G_1)$ and $\text{Exp}(G_2)$ are the exponents of respectively $G_1$ and $G_2$ then the exponents of $G_1 \otimes G_2$ are $\text{Exp}(G_1) + \text{Exp}(G_2)$. Hence, since $\Sigma$ is a group, since $E \otimes F$ comes from $G_1 \otimes G_2$, then using [23, 3.16], we get that $E \otimes F$ has $\Sigma'$-unipotent monodromy.

Definition 3.1.5. We keep the notation and hypotheses of 3.1.3.

1. Let $\mathcal{E} \in \mathcal{D}^b(\mathcal{Y}/\mathcal{W})$ (see Notation 2.2.8.1). We say that $\mathcal{E}$ “has $\Sigma$-unipotent monodromy” if, for any integer $i$, the module $\mathcal{H}_i(\mathcal{E}) \in \text{Isoc}_\Sigma^+(\mathbb{Y}/\mathbb{W})$. We will denote by $\mathcal{D}_h^{\text{isoc}, \Sigma}(\mathbb{Y}/\mathbb{W})$ the full subcategory of $\mathcal{D}_h(\mathbb{Y}/\mathbb{W})$ whose objects have $\Sigma$-unipotent monodromy.

2. We denote by $\mathcal{L} \mathcal{D}^b_{\text{Q, iso}, \Sigma}(\mathbb{Y}/\mathbb{W})$ the full subcategory of $\mathcal{L} \mathcal{D}^b_{\text{Q, coh}}(\mathcal{D}_P^{\bullet})$ of the objects $\mathcal{E}(\bullet)$ such that $\lim \mathcal{E}(\bullet) \in \mathcal{D}^b_{\text{isoc}, \Sigma}(\mathbb{Y}/\mathbb{W})$.

Remark 3.1.6. We keep the notation and hypotheses of 3.1.3. Let $\mathcal{E} \in \mathcal{D}^b(\mathcal{Y}/\mathcal{W})$. The fact that $\mathcal{E} \in \mathcal{D}^b_{\text{isoc}, \Sigma}(\mathbb{Y}/\mathbb{W})$ is local in $\mathcal{P}$.
**Proposition 3.1.7.** We keep the notation and hypotheses of 3.1.5.

1. The category $\mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$ (resp. $\mathcal{D}_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$) is a triangle subcategory of $\mathcal{L}D_{b,\text{coh},\Sigma}(\mathcal{Y}/\mathcal{W})$ (resp. $\mathcal{D}_{b,\text{coh},\Sigma}(\mathcal{Y}/\mathcal{W})$).

2. A direct factor in $\mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$ of an object of $\mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$ is an object of $\mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$.

3. The category $\mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$ is stable under base change, i.e. for any morphism $\mathcal{W} \to \mathcal{W}'$ of DVR($\mathcal{V}$), for any $\mathcal{E}(\bullet) \in \mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$, putting $(Y', X', \mathcal{P}')$ the frame over $\mathcal{W}'$ induced by base change from $(Y, X, \mathcal{P})$ by $\mathcal{W} \to \mathcal{W}'$, we get $\mathcal{W}' \otimes_{\mathcal{W}} \mathcal{E}(\bullet) \in \mathcal{L}D_{b,\text{isoc},\Sigma}(Y', X'/\mathcal{W}')$.

4. Let $\mathcal{E}(\bullet), \mathcal{F}(\bullet) \in \mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$. We have $\mathcal{E}(\bullet) \otimes_{\mathcal{O}_\mathcal{P}} \mathcal{F}(\bullet) \in \mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W})$.

**Proof.** The second assertion is straightforward. The other ones are consequences of 3.1.4. \hfill $\square$

**Proposition 3.1.8.** Let $\theta = (b, a, f): (Y', X', \mathcal{P}') \to (Y, X, \mathcal{P})$ be a morphism of frames over $\mathcal{W}$. We suppose that $X$ and $X'$ are l-smooth, and $Z := X - Y$ (resp. $Z' := X' - Y'$) is a simple normal crossing divisor of $X$ (resp. $X'$). We put $\mathcal{Y} := (Y, X)$ and $\mathcal{Y}' := (Y', X')$.

1. We have the exact functor

$$\mathbb{R}\Gamma_Y^f[d_Y - d_{Y'}]: \text{Isoc}^\Sigma(\mathcal{Y}/\mathcal{W}) \to \text{Isoc}^\Sigma(\mathcal{Y'}/\mathcal{W}). \tag{3.1.8.1}$$

2. We have the t-exact functor

$$\mathbb{R}\Gamma_Y^f[d_Y - d_{Y'}]: \mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y}/\mathcal{W}) \to \mathcal{L}D_{b,\text{isoc},\Sigma}(\mathcal{Y'}/\mathcal{W}), \tag{3.1.8.2}$$

and a similar one by replacing “$\mathcal{L}D_{b}$” by “$\mathcal{D}$”.

**Proof.** Let us check 3.1.8.1. From 3.1.4.1, we can suppose that there exist a divisor $T$ of $P$ such that $Y = X \setminus T$ and a divisor $T'$ of $P'$ such that $Y' = X' \setminus T'$. Is this case, $\text{Isoc}^\Sigma(\mathcal{Y}/\mathcal{W}) = \text{Isoc}^\Sigma(\mathcal{P}, T, X/\mathcal{W})$ is a full subcategory of $\text{Isoc}^\Sigma(\mathcal{P}, T, X/\mathcal{W})$. Let $\mathcal{E}(\bullet) \in \text{Isoc}^\Sigma(\mathcal{P}, T, X/\mathcal{W})$. From [12, 1.4.5.3], we have $\mathbb{R}\Gamma_Y^f[d_Y - d_{Y'}](\mathcal{E}(\bullet)) \in \text{Isoc}^\Sigma(\mathcal{P'}, T', X'/\mathcal{W})$. Let $\mathcal{E} := \lim \mathcal{E}(\bullet)$.

From 3.1.1.1, there exists $E \in \text{Isoc}^\Sigma(\mathcal{P}, T, X/\mathcal{W})$ such that $\text{sp}_+(E) \sim \mathcal{E}$. Using [23, 3.17], the overconvergent isocrystal $E$ comes from a log convergent isocrystal $G$ with exponents in $\tau(\Sigma)$. Using the Remark [15, 1.1.3.1], we get that $a_\Sigma^+(G)$ is a log convergent isocrystal with exponents in $\tau(\Sigma)$, where $a_\Sigma: (X', M_2) \to (X, M_2)$ is the morphism of log-schemes induced by $a$. Hence, $a_\Sigma^+(E)$ has $\Sigma'$-unipotent monodromy. Using [12, 1.4.5.4], we get that $\lim(\mathbb{R}\Gamma_Y^f[d_Y - d_{Y'}](\mathcal{E}(\bullet))) \sim \mathbb{R}\Gamma_Y^f[d_Y - d_{Y'}](\mathcal{E}(\bullet)) \in \text{Isoc}^\Sigma(\mathcal{P'}, T', X'/\mathcal{W})$.

Hence, $(\mathbb{R}\Gamma_Y^f[d_Y - d_{Y'}](\mathcal{E}(\bullet))) \in \text{Isoc}^\Sigma(\mathcal{P'}, T', X'/\mathcal{W})$ has also $\Sigma'$-unipotent monodromy. \hfill $\square$
Definition 3.1.9. Let $\mathcal{P}$ be a smooth formal scheme over $\mathcal{W}$. We denote by 
$\mathcal{L}D^b_{Q, \Sigma}(\hat{\mathcal{D}})^{(\bullet)}_{\mathcal{P}}$ the smallest subcategory of $\mathcal{L}D^b_{Q, h}(\hat{\mathcal{D}})^{(\bullet)}_{\mathcal{P}}$ stable by devisor and containing the categories of the form $\mathcal{L}D^b_{Q, \text{isoc}, \Sigma}(Y, \overline{Y}/\mathcal{W})$ where $\overline{Y}$ is a closed $l$-smooth subvariety of $P$, $Y$ is an open subscheme of $\overline{Y}$ such that $\overline{Y} \setminus Y$ is a strict normal crossing divisor in $Y$ (thanks to the Proposition 3.1.7.1, this is concretely defined as in [6, 3.2.21]). We call the objects of $\mathcal{L}D^b_{Q, \Sigma}(\hat{\mathcal{D}})^{(\bullet)}_{\mathcal{P}}$ “having $\Sigma$-unipotent monodromy”. Finally, we denote by $D^b_{\Sigma}(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$ the essential image of the functor $\mathcal{L}D^b_{Q, \Sigma}(\hat{\mathcal{D}})^{(\bullet)}_{\mathcal{P}} \to D^b_{h}(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$ induced by 1.1.2.1.

Theorem 3.1.10. Let $\mathcal{P}$ be a smooth formal scheme over $\mathcal{W}$. The dual functor $\mathcal{D}_{\mathcal{P}}$ induces an autoequivalence of $\mathcal{L}D^b_{Q, \Sigma}(\hat{\mathcal{D}})^{(\bullet)}_{\mathcal{P}}$ (resp. of $D^b_{\Sigma}(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$).

Proof. Let $\mathcal{F} \in D^b_{\Sigma}(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$. By devisor, we can suppose that there exists a frame $(Y, X, \mathcal{P})$ where $X$ is $l$-smooth, $Z := X \setminus Y$ is a strict normal crossing divisor of $X$, and there exists $\mathcal{E} \in D^b_{\text{isoc}, \Sigma}(Y, X/\mathcal{W})$ such that $j_+\mathcal{E} \xrightarrow{\sim} \mathcal{F}$, where $j: Y \hookrightarrow P$ is the immersion (here $j_+$ means simply the inclusion of $D^b_h(Y, \mathcal{P}/\mathcal{W})$ in $D^b_h(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$ but we keep it in the notation to be precise). By devisor, we can suppose that $\mathcal{E} \in \text{Isoc}^{\dagger}_{\Sigma}(Y, X/\mathcal{W})$. From the remark [23, 3.17], there exists a convergent log isocrystals $\mathcal{G}$ on the log scheme $(X, M_Z)$, where $M_Z$ is the log-structure induced by $Z$, with exponents in $\tau(\Sigma)$ such that $j_+(\mathcal{E}) \sim \mathcal{G}$. 

First, suppose there exists a morphism of smooth formal $\mathcal{W}$-schemes $\mathcal{X} \to \mathcal{P}$, and a strict normal crossing divisor $Z$ of $\mathcal{X}$ which lifts $Z$. Using Berthelot–Kashiwara Theorem, we reduce to the case where $\mathcal{X} = \mathcal{P}$. Since $\tau(0) = 0$, from [15, 2.2.9] (or [10, 3.5.6.2]), we have $j_+(\mathcal{E}) = \alpha_+(\mathcal{G})$, where $\alpha: (\mathcal{X}, M_Z) \to (\mathcal{X})$ is the canonical morphism of log formal schemes. Then, and with [8, 5.24.(ii)] for the last isomorphism, we get $\mathcal{D}_{\mathcal{P}} \circ j_+(\mathcal{E}) \sim \alpha_!(\mathcal{G}) \sim \alpha_!(\mathcal{G}) \sim \alpha_!(\mathcal{G})$. From [10, 3.5.6], we get that $\alpha_!(\mathcal{G}) \sim \alpha_!(\mathcal{G}) \in D^b_{\Sigma}(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$ (with the remark that the exponents stay in $\Sigma$). Hence, $\mathcal{D}_{\mathcal{P}}(\mathcal{F}) \in D^b_{\Sigma}(\hat{\mathcal{D}})^{\dagger}_{\mathcal{P} Q}$.

In general, we remark that the devisor in overconvergent isocrystals having $\Sigma$-unipotent monodromy in the local situation (i.e. the paragraph above) from [10, 3.5.6] is given by a smooth stratification which is constructed from $X, D$. Hence, using Remark 3.1.6, we get by localness that the restriction of $\mathcal{D}_{\mathcal{P}}(\mathcal{F})$ to any strata of this smooth stratification has $\Sigma$-unipotent monodromy. \hfill \Box

3.2. Potentially $\Sigma$-unipotent monodromy

Let $\mathcal{W}$ be an object of DVR($\mathcal{V}$), and $l$ be its residue field.

Lemma 3.2.1. Let $\mathcal{P}$ be a smooth formal scheme over $\mathcal{W}$, $X$ be a closed subscheme of $P$ and $T$ be a divisor of $P$ such that $Y := X \setminus T$ is $l$-smooth (over the residue field of $\mathcal{W}$). Let $\mathcal{E} \in \text{Isoc}^{\dagger}(\mathcal{P}, T, X/\mathcal{W})$ (see Notation 1.3.5). If there exists a complete morphism of frames over $\mathcal{W}$ of the form $\theta = (b, a, f): (Y', X', \mathcal{P}') \to (Y, X, \mathcal{P})$ such that $a: X' \to X$ is a projective surjective generically finite and
etale morphism, $X'$ is $l$-smooth, $Z' := X' \setminus Y'$ is a simple normal crossing divisor of $X'$ and $\theta^j(\mathcal{E}) := \mathbb{R}\Gamma^+_Y f^!(\mathcal{E}) \in \text{Isoc}_\Sigma^+(Y', \mathcal{P}'/\mathcal{W})$ (see Notation 3.1.3.2), then $\mathcal{E} \in \text{H-Isoc}_{\Sigma}^+(Y, \mathcal{P}/\mathcal{W})$ (see Notation 2.1.6.1).

**Proof.** From Lemma 3.1.2, we get $\mathbb{R}\Gamma^+_Y f^!(\mathcal{E}) \in \text{H-Isoc}_{\Sigma}^+(Y', \mathcal{P}'/\mathcal{W})$. In particular we have $\theta^j(\mathcal{E}) \in \text{H}(Y', \mathcal{P}'/\mathcal{W})$. Since the overholonomicity after any base change is stable by realizable pushforwards, we get $f_*\mathbb{R}\Gamma^+_Y f^!(\mathcal{E}) \in \text{H}(Y, \mathcal{P}/\mathcal{W})$. Moreover, since $\mathcal{E}$ is a direct factor of $f_*\mathbb{R}\Gamma^+_Y f^!(\mathcal{E})$, then this yields that $\mathcal{E} \in \text{H}(Y, \mathcal{P}/\mathcal{W})$. \hfill $\square$

The above lemma 3.2.1 justifies why we restrict to overholonomic after any base change isocrystals in the definition 3.2.2 below:

**Definition 3.2.2.** Let $\mathcal{Y} := (Y, X)$ be a couple over $\mathcal{W}$ such that $Y$ is smooth over $l$. Choose a frame over $\mathcal{W}$ of the form $(Y, X, \mathcal{P})$.

1. Let $\mathcal{E} \in \text{H-Isoc}_{\Sigma}^+(Y, \mathcal{P}/\mathcal{W})$ (see Notation 2.1.6.1). We say that $\mathcal{E}$ is an isocrystal on $(Y, \mathcal{P}/\mathcal{W})$ (or simply on $\mathcal{Y}$) having “potentially $\Sigma$-unipotent monodromy” if, for any irreducible component $Y_1$ of $Y$, denoting by $X_1$ the closure of $Y_1$ in $X$, there exists a morphism of frames over $\mathcal{W}$ of the form $\theta = (b, a, f): (Y', X', \mathcal{P}') \to (Y_1, X_1, \mathcal{P})$ such that $a: X' \to X_1$ is a projective surjective generically finite and etale morphism, $Y' = a^{-1}(Y)$, $X'$ is $l$-smooth, $X' \setminus Y'$ is a simple normal crossing divisor of $X'$ and such that $\theta^j(\mathcal{E}) := \mathbb{R}\Gamma^+_Y f^!(\mathcal{E}) \in \text{Isoc}_\Sigma^+(Y', \mathcal{P}'/\mathcal{W})$ (see Notation 3.1.3.2).

We denote by $\text{Isoc}_{\Sigma}^+_{\text{pot}}(Y, \mathcal{P}/\mathcal{W})$ the full subcategory of $\text{H}(Y, \mathcal{P}/\mathcal{W})$ whose objects are isocrystals having potentially $\Sigma$-unipotent monodromy. Using 3.1.3.2, we check that the category $\text{Isoc}_{\Sigma}^+_{\text{pot}}(Y, \mathcal{P}/\mathcal{W})$ does not depend on the choice of the frame $(Y, X, \mathcal{P})$ enclosing $\mathcal{Y}$. Hence, we will also write $\text{Isoc}_{\Sigma}^+_{\text{pot}}(\mathcal{Y}/\mathcal{W})$ instead of $\text{Isoc}_{\Sigma}^+_{\text{pot}}(Y, \mathcal{P}/\mathcal{W})$.

2. Let $\mathcal{D}_{\text{isoc-pot-}\Sigma}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$ be the full subcategory of $\mathcal{D}_{\text{h}}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$ of the objects $\mathcal{E}$ such that, for any integer $i$, we have $\mathcal{H}_i^j(\mathcal{E}) \in \text{Isoc}_{\Sigma}^+_{\text{pot}}(Y, \mathcal{P}/\mathcal{W})$. Since the category $\mathcal{D}_{\text{isoc-pot-}\Sigma}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$ does not depend on the choice of the frame $(Y, X, \mathcal{P})$ enclosing $\mathcal{Y}$, we will also write $\mathcal{D}_{\text{isoc-pot-}\Sigma}^\mathbb{b}(\mathcal{Y}/\mathcal{W})$ instead of $\mathcal{D}_{\text{isoc-pot-}\Sigma}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$.

3. We denote by $\mathcal{L}\mathcal{D}_{\text{Q, isoc-pot-}\Sigma}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$ the full subcategory of $\mathcal{L}\mathcal{D}_{\text{Q, h-isoc}}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$ of objects $\mathcal{E}(\bullet)$ such that $\lim \mathcal{E}(\bullet) \in \mathcal{D}_{\text{isoc-pot-}\Sigma}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$. Since this does not depend on the choice of the frame $(Y, X, \mathcal{P})$ enclosing $\mathcal{Y}$, we can simply write $\mathcal{L}\mathcal{D}_{\text{Q, isoc-pot-}\Sigma}^\mathbb{b}(\mathcal{Y}/\mathcal{W})$. We denote by $\mathcal{L}\mathcal{M}_{\text{Q, isoc-pot-}\Sigma}(Y, \mathcal{P}/\mathcal{W})$ the full subcategory of $\mathcal{L}\mathcal{D}_{\text{Q, isoc-pot-}\Sigma}^\mathbb{b}(Y, \mathcal{P}/\mathcal{W})$ of complexes $\mathcal{E}(\bullet)$ such that $\mathcal{H}_i^j(\mathcal{E}(\bullet)) = 0$ for any $j \neq 0$.

**Remark 3.2.3.** The following remark should justify our notation above. Let $\mathcal{Y} := (Y, X)$ be a couple over $\mathcal{W}$ where $Y$ is $l$-smooth. Suppose in this remark that the absolute Frobenius homomorphism $l \xrightarrow{\sim} l$ sending $x$ to $x^p$ lifts to an automorphism of the form $\mathcal{W} \xrightarrow{\sim} \mathcal{W}$. In that case, in [2, 1.2.13], we have defined the
category $F$-Isoc$^{\dagger \dagger}(\mathcal{Y}/\mathcal{W})$, whose objects belong to H-Isoc$^{\dagger \dagger}(\mathcal{Y}/\mathcal{W})$. We can translate Kedlaya’s semistable reduction theorem of [20] as follows: if $\mathcal{E}$ is an object of $F$-Isoc$^{\dagger \dagger}(\mathcal{Y}/\mathcal{W})$ then $\mathcal{E} \in \text{Isoc}_{\text{pot}}^{\dagger \dagger}(\mathcal{Y}/\mathcal{W})$ (for $\Sigma = 0$ and then for any $\Sigma$ satisfying the convention of the paper).

**Proposition 3.2.4.** Let $(Y, X, \mathcal{P})$ be a frame over $\mathcal{W}$ with $Y$ smooth over $l$.

1. Let $\theta = (b, a, f) : (Y', X', \mathcal{P}') \rightarrow (Y, X, \mathcal{P})$ be a morphism of frames over $\mathcal{W}$. We suppose that $Y$ and $Y'$ are $l$-smooth. We have the $t$-exact functor

$$
\mathbb{R}\Gamma_{Y'}^\dagger, f^!\left( d_Y - d_{Y'} \right) : LD^{b}_{\mathcal{Q}, \text{isoc}, \text{pot}):(Y, \mathcal{P}/\mathcal{W}) \rightarrow LD^{b}_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y', \mathcal{P}'/\mathcal{W}).
$$

(3.2.4.1)

2. The category $LD^{b}_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y, \mathcal{P}/\mathcal{W})$ is a triangle subcategory of $LD^{b}_{\mathcal{Q}, \text{h}}(Y, \mathcal{P}/\mathcal{W})$, stable under direct factors and base change.

3. For any $\mathcal{E}(\bullet), \mathcal{F}(\bullet) \in LD^{b}_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y, \mathcal{P}/\mathcal{W})$, we have $\mathcal{E}(\bullet) \otimes_{\mathcal{Q}, \mathcal{P}} \mathcal{F}(\bullet) \in LD^{b}_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y, \mathcal{P}/\mathcal{W})$.

We have similar properties by replacing “$LD^{b}_{\mathcal{Q}}$” by “$D$”.

**Proof.** Let $\mathcal{E}(\bullet) \in LM_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y, \mathcal{P}/\mathcal{W})$. We have to check $\mathbb{R}\Gamma_{Y'}^\dagger, f^!\left( d_Y - d_{Y'} \right)(\mathcal{E}(\bullet)) \in LM_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y', \mathcal{P}'/\mathcal{W})$. We can suppose $Y$ and $Y'$ integral. By definition, there exists a morphism of frames over $\mathcal{W}$ of the form $(d, c, g) : (Y'', X'', \mathcal{P}'') \rightarrow (Y, X, \mathcal{P})$ such that $c : X'' \rightarrow X$ is a projective surjective generically finite and etale morphism, $Y'' = c^{-1}(Y)$, $X''$ is $l$-smooth, $X'' \setminus Y''$ is a simple normal crossing divisor of $X''$ and such that $\mathbb{R}\Gamma_{Y''}^\dagger, g^!\left( \mathcal{E} \right) \in LM_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y'', \mathcal{P}''/\mathcal{W})$.

Replacing $\mathcal{P}''$ by $\mathcal{P}' \times \mathcal{P}$ if necessary, we can suppose $g$ smooth. Let $Y_1$ be an irreducible component of $Y'' \times_Y Y'$, and $X_1$ be the closure of $Y_1$ in $X'' \times_X X'$.

Using de Jong desingularization theorem (see [16]), we get a morphism of frames over $\mathcal{W}$ of the form $(d', c', g') : (Y''', X''', \mathcal{P}''') \rightarrow (Y_1, X_1, \mathcal{P}'' \times \mathcal{P})$ such that $c' : X''' \rightarrow X_1$ is a projective surjective generically finite and etale morphism, $Y''' = (c')^{-1}(Y_1)$, $X'''$ is $l$-smooth, $X''' \setminus Y'''$ is a simple normal crossing divisor of $X'''$. Let $\pi_1 : \mathcal{P}'' \times \mathcal{P} \rightarrow \mathcal{P}''$, and $\pi_2 : \mathcal{P}'' \times \mathcal{P} \rightarrow \mathcal{P}$ be the canonical projections. By using 3.1.8, we get that

$$
\mathbb{R}\Gamma_{Y''}^\dagger, (\pi_2 \circ g')^!\left( \mathbb{R}\Gamma_{Y''}^\dagger, f^!\left( d_Y - d_{Y'} \right)(\mathcal{E}(\bullet)) \right)
$$

$$
\sim \mathbb{R}\Gamma_{Y''}^\dagger, \mathbb{R}\Gamma_{Y''}^\dagger, (\pi_1 \circ g')^!\left( d_Y - d_{Y'} \right),
$$

which yields that $\mathbb{R}\Gamma_{Y''}^\dagger, f^!\left( d_Y - d_{Y'} \right)(\mathcal{E}(\bullet)) \in LM_{\mathcal{Q}, \text{isoc}, \text{pot}}(Y', \mathcal{P}'/\mathcal{W})$.

Since the functor $\mathbb{R}\Gamma_{Y''}^\dagger, f^!\left( d_Y - d_{Y'} \right) : LD^{b}_{\mathcal{Q}, \text{h-isoc}}(Y, \mathcal{P}/\mathcal{W}) \rightarrow LD^{b}_{\mathcal{Q}, \text{h-isoc}}(Y', \mathcal{P}'/\mathcal{W})$ is t-exact, then so is 3.2.4.1, which completes the proof of 1).

Using 3.1.8.2, 3.1.7.1, (resp. 3.1.8.2, 3.1.7.4) by proceeding similarly to the proof of the part 1), we check that the category $LD^{b}_{\mathcal{Q}, \text{h-isoc}}(Y, \mathcal{P}/\mathcal{W})$ is a triangle subcategory of $LD^{b}_{\mathcal{Q}, \text{h}}(Y, \mathcal{P}/\mathcal{W})$, and that part 3) is valid.

The stability under direct factors and under base change are respectively a consequence of 3.1.7.2 and 3.1.7.3. \qed
Definition 3.2.5. Let $\mathcal{Y} := (Y, X)$ be a couple over $\mathcal{W}$. Choose a frame $(Y, X, \mathcal{P})$ over $\mathcal{W}$.

1. Let $\mathcal{E}^0 \in LD_{Q,h}^b(D_{\mathcal{P}}^\mathcal{Y})$. We say that $\mathcal{E}^0$ has “potentially $\Sigma$-unipotent monodromy” if there exist a smooth stratification (see Definition 2.2.1) $(P_0, \ldots, P_r)$ of $P$ such that $\mathcal{E}^0 \in LD_{Q,\text{isoc},\text{pot},\Sigma}^b(P_i, \mathcal{P}/\mathcal{W})$. We denote by $LD_{Q,\text{pot},\Sigma}^b(D_{\mathcal{P}}^\mathcal{Y})$ the full subcategory of $LD_{Q,h}^b(D_{\mathcal{P}}^\mathcal{Y})$ whose objects have potentially $\Sigma$-unipotent monodromy.

2. We denote by $LD_{Q,\text{pot},\Sigma}^b(\mathcal{Y}, \mathcal{P}/\mathcal{W})$ or by $LD_{Q,\text{pot},\Sigma}^b(\mathcal{Y}/\mathcal{W})$ the full category of $LD_{Q,\text{pot},\Sigma}^b(D_{\mathcal{P}}^\mathcal{Y})$ of objects $\mathcal{E}$ such that there exists an isomorphism of the form $\mathcal{E} \sim \mathcal{E}^0$. From 3.2.2.3, we check that $LD_{Q,\text{pot},\Sigma}^b(\mathcal{Y}/\mathcal{W})$ does not depend on the choice of the frame enclosing $\mathcal{Y}$, which justifies the notation.

3. We denote $D_{\text{pot},\Sigma}(\mathcal{Y}/\mathcal{W})$ by the essential image of the functor $LD_{Q,\text{pot},\Sigma}^b(\mathcal{Y}/\mathcal{W})$ to $D_{\text{coh}}^b(D_{\mathcal{P}}^\mathcal{Y})$ induced by 1.1.2.1. We say that $D_{\text{pot},\Sigma}(\mathcal{Y}/\mathcal{W})$ is the full subcategory of $D_{h}^b(\mathcal{Y}/\mathcal{W})$ of objects having “potentially $\Sigma$-unipotent monodromy”.

4. When $\Sigma = 0$, we denote respectively $LD_{Q,\text{pot},\Sigma}^b(\mathcal{Y}/\mathcal{W})$ and $D_{\text{pot},\Sigma}(\mathcal{Y}/\mathcal{W})$ by $LD_{Q,\text{u}}^b(\mathcal{Y}/\mathcal{W})$ and $D_{\text{u}}^b(\mathcal{Y}/\mathcal{W})$.

5. We get some data of coefficients $LD_{Q,\text{pot},\Sigma}^b(\mathcal{P}) := LD_{Q,\text{pot},\Sigma}^b(D_{\mathcal{P}}^\mathcal{Y})$, $LD_{Q,\text{u}}^b(\mathcal{P}) := LD_{Q,\text{u}}^b(D_{\mathcal{P}}^\mathcal{Y})$.

Lemma 3.2.6. Let $(Y, X, \mathcal{P})$ be a frame over $\mathcal{W}$. For any $\mathcal{E}^0 \in LD_{Q,\text{pot},\Sigma}^b(D_{\mathcal{P}}^\mathcal{Y})$, we have $\mathcal{E}^0 \in LD_{Q,\text{pot},\Sigma}^b(\mathcal{Y}, \mathcal{P}/\mathcal{W})$. For any $\mathcal{P}^0 \in LD_{Q,h}^b(Y, \mathcal{P}/\mathcal{W})$, the property $\mathcal{P}^0 \in LD_{Q,\text{pot},\Sigma}^b(D_{\mathcal{P}}^\mathcal{Y})$ is equivalent to the one that there exists a smooth stratification $(Y_0, \ldots, Y_r)$ of $Y$ such that $\mathcal{E}^0 \in LD_{Q,\text{isoc},\text{pot},\Sigma}^b(P_i, \mathcal{P}/\mathcal{W})$.

Proof. Let $\mathcal{E}^0 \in LD_{Q,\text{pot},\Sigma}^b(D_{\mathcal{P}}^\mathcal{Y})$. Let $(P_1, \ldots, P_r)$ be a smooth stratification of the special fiber of $\mathcal{P}$ such that $\mathcal{E}^0 \in LD_{Q,\text{isoc},\text{pot},\Sigma}^b(P_i, \mathcal{P}/\mathcal{W})$. For each $i = 1, \ldots, r$, choose a smooth stratification $(Y_{i,1}, \ldots, Y_{i,r_i})$ of $Y \cap P_i$. Set $J = \{(i,j_i) : 1 \leq i \leq r, 1 \leq j_i \leq r_i\}$. By ordering the set $J$ with the lexicographic order, we get a smooth stratification $(Y_{i,j_i})_{(i,j_i) \in J}$ of $Y$. Set $\overline{Y}$ be the closure of $Y$ in $P$. Set $U := P \setminus \overline{Y}$. Then $U$ is an open subscheme of $P$ such that $Y$ is an open subscheme of $P \setminus U$. Hence, we get a smooth stratification $(U, (Y_{i,j_i})_{(i,j_i) \in J}, \overline{Y} \setminus Y)$ of $P$. We have $\mathcal{E}^0 \in LD_{Q,\text{pot},\Sigma}^b(P_i, \mathcal{P}/\mathcal{W})$, since $Y_{i,j_i} \subset P_i$, then we get $\mathcal{E}^0 \in LD_{Q,\text{isoc},\text{pot},\Sigma}^b(Y_{i,j_i}, \mathcal{P}/\mathcal{W})$. We check the second part of the Lemma in the same way.

Proposition 3.2.7. The data of coefficients $LD_{Q,\text{pot},\Sigma}^b(\mathcal{P}/\mathcal{W})$ is stable by devissages, direct factors and base change.

Proof. Let $\mathcal{P}$ be a smooth formal scheme over $\mathcal{W}$. Let $\mathcal{E} \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}'[1]$ be an exact triangle of $D_{h}^b(\mathcal{P}/\mathcal{W})$ with $\mathcal{E}', \mathcal{E}'' \in D_{\text{pot},\Sigma}^b(\mathcal{P}/\mathcal{W})$. Let $(P_1, \ldots, P_r)$
be a smooth stratification of the special fiber of $\mathcal{P}$ such that $\mathbb{R} \Gamma^+_p (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (P_i, \mathcal{P}/\mathcal{W})$. For each $i = 1, \ldots, r$, following 3.2.6, we have $\mathbb{R} \Gamma^+_p (\mathcal{E}''(\bullet)) \in L D_{Q, \text{pot-}}^b (P_i, \mathcal{P}/\mathcal{W})$. Hence, there exists a smooth stratification $(Y_{i, 1}, \ldots, Y_{i, r_i})$ of $P_i$ such that $\mathbb{R} \Gamma^+_p (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y_{i, 1}, \mathcal{P}/\mathcal{W})$. Set $I = \{(i, j_i) : 1 \leq i \leq r, 1 \leq j_i \leq r_i\}$. By ordering the set $I$ with the lexicographic order, we get a smooth stratification $(Y_{i, j_i}(i, j_i) \in I)$. On the other hand, using 3.2.4.1, we get $\mathbb{R} \Gamma^+_Y (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y_{i, j_i}, \mathcal{P}/\mathcal{W})$. From 3.2.4.2, $L D_{Q, \text{isoc, pot-}}^b (Y_{i, j_i}, \mathcal{P}/\mathcal{W})$ is a triangulated subcategory of $L D_{Q, h}^b (Y_{i, j_i}, \mathcal{P}/\mathcal{W})$. Hence, we get $\mathbb{R} \Gamma^+_Y (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y_{i, j_i}, \mathcal{P}/\mathcal{W})$, which gives the stability of $L D_{Q, \text{pot-}}^b$ by devissages. The rest of the proposition is a consequence of 3.2.4.2.

**Proposition 3.2.8.** Let $\theta = (b, a, f) : (Y', X', \mathcal{P}') \to (Y, X, \mathcal{P})$ be a morphism of frames over $\mathcal{W}$. We have the factorization

$$\theta^! = \mathbb{R} \Gamma^+_Y f^! : L D_{Q, \text{pot-}}^b (Y, \mathcal{P}/\mathcal{W}) \to L D_{Q, \text{pot-}}^b (Y', \mathcal{P}'/\mathcal{W}),$$

(3.2.8.1)

and a similar one by replacing “$L D_{Q}$” by “$D$”.

**Proof.** Let $\mathcal{E}''(\bullet) \in L D_{Q, \text{pot-}}^b (Y, \mathcal{P}/\mathcal{W})$. Using 3.2.7, we reduce by devissage to the case where $Y$ is smooth and $\mathcal{E}''(\bullet) \in L D_{Q, \text{isoc, pot-}}^b (Y, \mathcal{P}/\mathcal{W})$. Let $(Y_1', \ldots, Y_r')$ be a smooth stratification of $Y'$. Then, from 3.2.4.1, we get $\mathbb{R} \Gamma^+_Y f^! (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y_i', \mathcal{P}'/\mathcal{W})$ for any integer $i$. Hence, $\mathbb{R} \Gamma^+_Y f^! (\mathcal{E}''(\bullet)) \in L D_{Q, \text{pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$.

**Proposition 3.2.9.** The data of coefficients $L D_{Q, \text{pot-}}^b$ is stable under tensor products.

**Proof.** Using 3.2.7, we get by devissage the stability from 3.2.4.1 and 3.2.4.3. □

**Proposition 3.2.10.** Let $(b, a, f) : (Y', X', \mathcal{P}') \to (Y, X, \mathcal{P})$ be a complete morphism of frames such that $b : Y' \to Y$ is finite, étale, and surjective. Let $\mathcal{E}''(\bullet) \in L D_{Q, h}^b (Y, \mathcal{P}/\mathcal{W})$, $\mathcal{E}''(\bullet) \in L D_{Q, \text{pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$.

1. Suppose $Y$ is smooth. We have $\mathcal{E}''(\bullet) \in L D_{Q, \text{isoc, pot-}}^b (Y, \mathcal{P}/\mathcal{W})$ if and only if $\mathbb{R} \Gamma^+_Y f^! (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$. We have $\mathcal{E}''(\bullet) \in L D_{Q, \text{isoc, pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$ if and only if $f_+ (\mathcal{E}''(\bullet)) \in L D_{Q, h}^b (Y, \mathcal{P}/\mathcal{W})$.

2. We have $\mathcal{E}''(\bullet) \in L D_{Q, \text{pot-}}^b (Y, \mathcal{P}/\mathcal{W})$ if and only if $\mathbb{R} \Gamma^+_Y f^! (\mathcal{E}''(\bullet)) \in L D_{Q, \text{pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$. Moreover, we have $\mathcal{E}''(\bullet) \in L D_{Q, \text{pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$ if and only if $f_+ (\mathcal{E}''(\bullet)) \in L D_{Q, \text{pot-}}^b (Y, \mathcal{P}/\mathcal{W})$.

**Proof.** Let us check the part 1) of the Proposition. From 3.2.4.1, if $\mathcal{E}''(\bullet) \in L D_{Q, \text{isoc, pot-}}^b (Y, \mathcal{P}/\mathcal{W})$ then $\mathbb{R} \Gamma^+_Y f^! (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$. Conversely, suppose $\mathbb{R} \Gamma^+_Y f^! (\mathcal{E}''(\bullet)) \in L D_{Q, \text{isoc, pot-}}^b (Y', \mathcal{P}'/\mathcal{W})$. Since $b$ is finite and
étale, we get the functor \( f_+ : LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc(Y', \mathcal{P}/\mathcal{W}) \rightarrow LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc(Y, \mathcal{P}/\mathcal{W}) \) (see Notation 2.1.6.3). Since \( E'(\bullet) \) is a direct factor of \( f_+ \Gamma_Y^+ f_! E'(\bullet) \), then \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc(Y, \mathcal{P}/\mathcal{W}) \). Recalling \( \Gamma_Y^+ f_! (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \), this yields almost by definition \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y, \mathcal{P}/\mathcal{W}) \).

Suppose \( f_+ (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y, \mathcal{P}/\mathcal{W}) \). Then from the first part \( \Gamma_Y^+ f_! (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \). Since \( Y' \rightarrow Y \) is finite étale, \( Y' \) is an open connected component of \( Y' \times_Y Y' \) (see [21, I.3.12]). Hence, using the base change isomorphism 1.3.9.1, we check that \( E'(\bullet) \) is a direct factor of \( \Gamma_Y^+ f_! (E'(\bullet)) \). By using 3.1.7.2, this yields \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \). Conversely, suppose \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \). There exists a Galois finite, étale morphism \( b' : Y'' \rightarrow Y \) that factors through \( b : Y' \rightarrow Y \) (see the beginning of [21, I.5] or [22, V.4.g) and V.4.1 and V.7]). Denoting by \( b'' : Y'' \rightarrow Y' \) this factorization we get \( b' = b \circ b'' \). Since \( b'' \) is in particular projective, we get a morphism of frames over \( \mathcal{W} \) of the form \( (b'', a'', f'') : (Y'', a'', f'') : (Y'', X'', \mathcal{P}'') \rightarrow (Y', X', \mathcal{P}') \) where \( a'' \) is projective, \( f'' \) is projective and smooth, \( Y'' \) is dense in \( X'' \). Since \( E'(\bullet) \) is a direct factor of \( f'' \Gamma_{Y''}^+ f''_{\bullet} (E'(\bullet)) \), then \( f_+ (E'(\bullet)) \) is a direct factor of \( f_+ f''_{\bullet} \Gamma_{Y''}^+ f''_{\bullet} (E'(\bullet)) \). Hence, using 3.1.7.2 and 3.1.8.2, we can suppose that \( f_+ (E'(\bullet)) \) is a Galois morphism. In that case, \( Y' \times_Y Y' = \sqcup_{\sigma \in G} Y'_{\sigma}, \) where \( G = Aut_Y(Y') \) and \( Y_{\sigma} \) is a copy of \( Y \). Hence, using the base change isomorphism 1.3.9.1, we check that \( \Gamma_{Y'}^+ f_! (E'_{\sigma}) = \bigoplus_{\sigma \in G} E'_{\sigma} \), where \( E'_{\sigma} \) means a copy of \( E'(\bullet) \). Hence, \( \Gamma_{Y'}^+ f_! (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \). Then from the first part \( f_+ (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \).

Let us check the part 2) of the Proposition. Moreover, since \( E'(\bullet) \) is a direct factor of \( f_+ \Gamma_Y^+ f_! (E'_{\bullet}) \), this yields that if \( \Gamma_Y^+ f_! (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \) then \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y, \mathcal{P}/\mathcal{W}) \). From 3.2.8, the converse is known. Finally, we proceed as in the part 1) of the proof to check that \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y', \mathcal{P}/\mathcal{W}) \) if and only if \( f_+ (E'(\bullet)) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y, \mathcal{P}/\mathcal{W}) \).

**Lemma 3.2.11.** The data of coefficients \( LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma \) is almost stable by dual functors.

**Proof.** Let \( \mathcal{C} \) be a data of coefficients containing \( LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma \) and stable under devissages, under direct factors and under realizable pushforwards. Let \( \mathcal{W} \) be an object of DVR(\( \mathcal{V} \), \( \mathcal{P} \) be a smooth formal scheme over \( \mathcal{W} \). Let \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(\mathcal{P}) \). We have to check that \( \mathbb{D}(E'(\bullet)) \in \mathcal{C}(\mathcal{P}) \). Since \( \mathcal{C} \) is stable by devissages, we can suppose there exists an irreducible smooth subvariety \( Y \) of \( \mathcal{P} \) such that \( E'(\bullet) \in LD^b \rightarrow \mathbb{Q}, \hbox{-} isoc-pot-\Sigma(Y, \mathcal{P}/\mathcal{W}) \). Again by devissage, we can suppose that \( E := \lim \mathcal{E} \in \text{Isoc}^{\uparrow}_p \text{Y/W} \), where \( \text{lim} \) is the functor of 1.1.2.1. We denote by \( \overline{Y} \) the closure of \( Y \) in \( \mathcal{P} \). There exists a morphism of frames \( \theta = (b, a, f) : (Y', \overline{Y}, \mathcal{P}') \rightarrow (Y, \overline{Y}, \mathcal{P}) \) such that \( f \) is proper, \( a \) is generically finite and etale, \( Y' \) is smooth, \( Y' = a^{-1}(Y) \), \( \overline{Y}' \setminus Y' \) is a strict normal crossing divisor in \( \overline{Y} \) and \( \theta^{-1}(E) = \Gamma_{\overline{Y}'}^+ f_!(E) \in \text{Isoc}^{\uparrow}_p (Y', \overline{Y}/\mathcal{W}) \). From 3.1.10, we get
\(\mathbb{D}_{\mathcal{P}} \theta^!(\mathcal{E}) \in L_{D, b}^{\mathbb{Q}, \mathbb{Z}}(\mathcal{P}') \subset L_{D, b}^{\mathbb{Q}, \text{pot} \cdot \Sigma}(\mathcal{P}') \subset \mathcal{E}(\mathcal{P}')\). Since \(\mathcal{E}\) is a direct factor of \(\theta_+ \circ \theta^!(\mathcal{E})\) (where we set \(\theta_+ := f_+\)), by using the relative duality isomorphism (see 1.3.12), we check that \(\mathbb{D}_{\mathcal{P}}(\mathcal{E})\) is a direct factor of \(\theta_+ \circ \mathbb{D}_{\mathcal{P}} \theta^!(\mathcal{E})\). Since \(\mathcal{E}\) is stable by direct factors and realizable (in fact proper would have been sufficient here) pushforwards, this yields \(\mathbb{D}_{\mathcal{P}}(\mathcal{E}) \in \mathcal{E}(\mathcal{P})\).

\[\square\]

3.3. Quasi-unipotence, coefficients satisfying semi-stable reduction property

**Definition 3.3.1.** From 3.2.5.2, 3.2.7, 3.2.8, 3.2.9, 3.2.11, the data of coefficients \(L_{D, b}^{\mathbb{Q}, \text{pot} \cdot \Sigma}\) satisfies the hypotheses of 1.4.12 concerning the data \(\mathcal{D}\). Since \(L_{D, b}^{\mathbb{Q}, \text{ovcoh}}\) is stable by direct factors, devissages, extraordinary pullbacks, realizable pushforwards (see 1.3.4 and 1.3.7), then we can define \(L_{D, b}^{\mathbb{Q}, \text{qu} \cdot \Sigma} := T_{\text{min}}(L_{D, b}^{\mathbb{Q}, \text{pot} \cdot \Sigma} \cdot L_{D, b}^{\mathbb{Q}, \text{ovcoh}}).\) When \(\Sigma = 0\), we put \(L_{D, b}^{\mathbb{Q}, \text{qu}} := L_{D, b}^{\mathbb{Q}, \text{qu} \cdot \Sigma} \). The objects of the data of coefficients \(L_{D, b}^{\mathbb{Q}, \text{qu} \cdot \Sigma}\) (resp. \(L_{D, b}^{\mathbb{Q}, \text{div}}\)) are called “quasi-\(\Sigma\)-unipotent” (resp. “quasi-unipotent”). We also say that they have “quasi-\(\Sigma\)-unipotent monodromy”.

**Notation 3.3.2.** We define the data of coefficients with potentially Frobenius structure \(L_{D, b}^{\mathbb{Q}, F}\) as follows. Let \(\mathcal{W}\) be an object of DVR(\(\mathcal{V}, \sigma\)), and \(\mathcal{X}\) be a smooth formal scheme over \(\mathcal{W}\). The category \(L_{D, b}^{\mathbb{Q}, F}(\mathcal{X})\) is by definition the full subcategory of \(L_{D, b}^{\mathbb{Q}, \text{ovcoh}}(\mathcal{D}(\mathcal{X}))\) whose objects are equal to the essential image of the canonical functor which forgets Frobenius structures : \(F-L_{D, b}^{\mathbb{Q}, \text{ovcoh}}(\mathcal{D}(\mathcal{X})) \to L_{D, b}^{\mathbb{Q}, \text{ovcoh}}(\mathcal{D}(\mathcal{X}))\).

**Remark 3.3.3.** Let \(\mathcal{W}\) be an object of DVR(\(\mathcal{V}, \sigma\)), \(\mathcal{X}\) be a smooth formal scheme over \(\mathcal{W}\). From Kedlaya’s semistable reduction theorem (see 3.2.3), we remark that \(L_{D, b}^{\mathbb{Q}, F}(\mathcal{X})\) is contained in \(L_{D, b}^{\mathbb{Q}, u}(\mathcal{X})\) (recall the notation of 3.2.5), which is at its turn contained in \(L_{D, b}^{\mathbb{Q}, h}(\mathcal{X})\).

**Notation 3.3.4.** 1. We define by induction on \(n \in \mathbb{N}\) the data of coefficients with potentially Frobenius structure \(L_{D, b}^{\mathbb{Q}, u^+}(\mathcal{X})\) as follows. For \(n = 0\), we put \(L_{D, b}^{\mathbb{Q}, u^+}(\mathcal{X}) = 0\). Suppose \(L_{D, b}^{\mathbb{Q}, u^+}(\mathcal{X})\) constructed. Let \(\mathcal{W}\) be an object of DVR(\(\mathcal{V}, \sigma\)), and \(\mathcal{P}\) be a smooth formal scheme over \(\mathcal{W}\). The category \(L_{D, b}^{\mathbb{Q}, u^+}(\mathcal{P})\) is by definition the full subcategory of \(L_{D, b}^{\mathbb{Q}, u}(\mathcal{P})\) of complexes \(E(\mathcal{X})\) satisfying the following property: for any morphism \(f : \mathcal{X} \to \mathcal{P}\) of smooth formal \(\mathcal{W}\)-schemes, for any realizable morphism \(g : \mathcal{X} \to \mathcal{Y}\) of smooth formal \(\mathcal{W}\)-schemes, for any subscheme \(Z\) of \(X\) which is proper over \(Y\) (via \(g\)), we have \(g_* \mathbb{R} \bar{\mathcal{F}} \mathcal{X}(E(\mathcal{X})) \in L_{D, b}^{\mathbb{Q}, u}(\mathcal{Y})\).

2. We set \(L_{D, b}^{\mathbb{Q}, u^+}(\mathcal{X}) := \bigcap_{n \in \mathbb{N}} L_{D, b}^{\mathbb{Q}, u^+}(\mathcal{X})\).

3. The constructions of \(T_{\text{max}}\) and \(T_{\text{min}}\) in 1.4.14 are still valid if we restrict to data of coefficients with potentially Frobenius structure instead of data of coefficients. From [15], the data \(L_{D, b}^{\mathbb{Q}, F}\) satisfies the hypotheses of 1.4.12 concerning the data \(\mathcal{D}\). Moreover, the data \(L_{D, b}^{\mathbb{Q}, u^+}\) is stable by devissages, direct
factors, extraordinary pullbacks and realizable pushforwards. Hence, we get the following data of coefficients with potentially Frobenius structure by putting $\frac{LD_{Q, st}^b}{L D_{Q, F}^b, \frac{LD_{Q, u}^b}{L D_{Q, u}^b}} := T_{\text{max}} (LD_{Q, F}^b, \frac{LD_{Q, u}^b}{L D_{Q, u}^b})$.

3.3.5. Let $W$ be an object of DVR $(V, \sigma)$ and $\mathcal{P}$ be a smooth formal scheme over $W$. We denote by $D_{b}^q /_{Sigma1}(\mathcal{P})$ (resp. $D_{b}^{qu} /_{Sigma1}(\mathcal{P})$) the essential image of $\frac{LD_{Q, Sigma1}(\mathcal{P})}{LD_{Q, F}^b, \frac{LD_{Q, u}^b}{L D_{Q, u}^b}}$.

Remark 3.3.6. 1. The data of coefficients $\frac{LD_{Q, st}^b}{LD_{Q, F}^b}$ and is contained in $\frac{LD_{Q, u}^b}{L D_{Q, u}^b}$. In particular, the isocrystals in $\frac{LD_{Q, st}^b}{LD_{Q, F}^b}$ satisfy Kedlaya’s semistable reduction theorem.

2. From 3.2.7, 3.2.8, 3.2.9, 3.2.11, we remark that if the data of coefficients $\frac{LD_{Q, u}^b}{L D_{Q, u}^b}$ is stable under pushforwards (which is unlikely but this remains an open question), then we get $\frac{LD_{Q, u}^b}{L D_{Q, u}^b} = \frac{LD_{Q, st}^b}{LD_{Q, F}^b}$. 

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