Abstract

We determine the class of \( p \)-forms \( F \) that possess vanishing scalar invariants (VSIs) at arbitrary order in an \( n \)-dimensional spacetime. Namely, we prove that \( F \) is a VSI if and only if it is of type \( N \), its multiple null direction \( \ell \) is ‘degenerate Kundt’, and \( \ell \cdot F = 0 \). The result is theory-independent. Next, we discuss the special case of Maxwell fields, both at the level of test fields and of the full Einstein–Maxwell equations. These describe electromagnetic non-expanding waves propagating in various Kundt spacetimes. We further point out that a subset of these solutions possesses a universal property, i.e. they also solve (virtually) any generalized (non-linear and with higher derivatives) electrodynamics, possibly also coupled to Einstein’s gravity.

Keywords: exact solutions, higher dimensions, einstein-maxwell spacetimes

Contents

1. Introduction 2
   1.1. Background 2
   1.2. Main result 4

2. Explicit form of VSI electromagnetic fields and adapted coordinates 6
   2.1. Electromagnetic field and spacetime metric 6
   2.2. Maxwell’s equations 7
      2.2.1. The special case \( p = 1 \) 7
      2.2.2. The special cases \( n = 4, 3 \) 8
   2.3. Maxwell–Chern–Simons equations 8
      2.3.1. Generic case \( k \geq 2 \) 8
      2.3.2. Special case \( k = 1 \) 8

1 Author to whom any correspondence should be addressed.
2.4. Universal solutions of generalized electrodynamics (test fields) ........... 9

3. Einstein–Maxwell solutions 9
3.1. General equations ........................................... 9
3.1.1. VSI spacetimes with VSI Maxwell fields .................. 11
3.1.2. pp-waves with VSI Maxwell fields ....................... 12
3.2. Universal Einstein–Maxwell solutions ......................... 13

Appendix A. Some of the Newman–Penrose equations for Kundt spacetimes 14
A.1. General Kundt spacetimes .................................... 14
A.2. Kundt spacetimes of aligned Riemann type II ................... 15

Appendix B. Proof of theorem 1.5 16
B.1. Proof of 2. ⇒ 1 .............................................. 16
B.1.1. Preliminaries .................................................. 16
B.1.2. Proof .......................................................... 17
B.2. Proof of 1. ⇒ 2 .............................................. 17
B.2.1. Preliminaries .................................................. 17
B.2.2. Proof .......................................................... 19

Appendix C. VSI_{1} and VSI_{2} p-forms 20

1. Introduction

1.1. Background

Synge [1] called electromagnetic null fields those characterized by the vanishing of the two Lorentz invariants, i.e.

\[ F_{ab} F^{ab} = 0, \quad F_{ab} * F^{ab} = 0. \quad (1) \]

From a physical viewpoint, null Maxwell fields characterize electromagnetic plane waves [1] and the asymptotic behaviour of radiative systems (cf [2] and references therein). Fields satisfying (1) single out a unique null direction \( \ell \) such that the corresponding energy-momentum tensor can be written as \( T_{ab} = M \ell_{a} \ell_{b} \) [3] (cf also, e.g., [1, 4]). They also possess the unique property that their field strength at any spacetime point can be made as small (or large) as desired in a suitably boosted reference frame [1, 5]. Moreover, null solutions to the source-free Maxwell equations can be associated with shearfree congruences of null geodesics (and vice versa) via the Mariot–Robinson theorem [2, 4] and are therefore geometrically privileged.

There are further reasons that motivate the interest in fields with the property (1). On the gravity side, an analogue of null electromagnetic fields is given by metrics of Riemann type III and N, for which all the zeroth-order scalar invariants constructed from the Riemann tensor vanish identically (see [4] in 4D and [6] in higher dimensions\(^2\)). These have remarkable properties. For example, all type N Einstein spacetimes are automatically vacuum solutions of quadratic [8] (in particular, Gauss–Bonnet [9]) and Lovelock gravity [10]. Furthermore, it has been known for some time [11–13] that certain type N pp-wave solutions in general relativity (in vacuum or with dilaton and form fields) are classical solutions to string theory to all orders.

\(^2\) Hereafter, by ‘zeroth-order’ invariants of a certain tensor we refer to the algebraic ones, i.e. those not involving covariant derivatives of the given tensor. Additionally, we will restrict ourselves to polynomial scalar invariants (cf definition 5.1 of [7]).
In $\sigma$-model perturbation theory—in fact, they are also solutions to any gravity theory in which the ‘corrections’ to the field equations can be expressed in terms of scalars and tensors constructed from the field strengths and their covariant derivatives (see [11, 13, 14] for details), and are in this sense universal [15]. Apart from being of Riemann type N and universal, the metrics considered in [11–13] have the special property that all the scalar invariants constructed from the Riemann tensor and its covariant derivatives vanish and thus belong to the vanishing scalar invariant (VSI) class of spacetimes [6, 16] (cf definition 1.2 below).

In view of these results for gravity, one may wonder whether certain null (or VSI) Maxwell fields possess a similar ‘universal’ property and thus solve also generalized theories of electrodynamics. In fact, it was already known to Schrödinger [5, 19] that all null Maxwell fields solve the equations for the electromagnetic field in any non-linear electrodynamics (NLE; cf, e.g., [20]). This was later extended to the full Einstein–Maxwell equations, including the electromagnetic backreaction on the spacetime geometry [21–23]. However, the case of theories more general than NLE (including also derivatives of the field strength in the Lagrangian; cf, e.g., [24, 25]) seems not to have been investigated systematically from this viewpoint. As a first step in this direction, it is the purpose of the present paper to determine the class of electromagnetic fields for which all the scalar invariants constructed from the field strength and its derivatives vanish identically (VSI), which is obviously a subset of null fields. We will also point out a few examples possessing the universal property, while a more detailed study of the latter will be presented elsewhere.

Various extensions of Einstein–Maxwell gravity exist in which the number of spacetime dimensions $n$ may be greater than four, and the electromagnetic field is represented by a rank-$p$-form (cf, e.g., [26, 27]). These theories have attracted interest in recent years, motivated, in particular, by supergravity and string theory. We will therefore consider null fields with arbitrary $n \geq 3$ and $p$ (with $1 \leq p \leq n - 1$) to avoid trivial cases. The relevant generalization of the concept of null fields is straightforward: all the zeroth-order scalar polynomial invariant constructed out of a $p$-form $F$ vanish (thus generalizing (1)) iff [28] $F$ is of type N in the null alignment classification of [29] (cf corollary 1.4 below; for the case $p = 2$ this was proven earlier in [6]). For this reason, in this paper we shall use the terminology ‘null’ and ‘type N’ interchangeably when referring to $p$-forms.

It will be also useful to observe that the type N condition of [29] for $p$-forms can be easily rephrased in a manifestly frame-independent way, which we thus adopt as a definition here (see also [30] when $p = 2$):

**Definition 1.1.** ($p$-forms of type N.) At a spacetime point, a $p$-form $F$ is of type N if it satisfies

$$\ell^{a}F_{ab_{1}\ldots b_{p-1}} = 0, \quad \ell_{[a}F_{b_{1}\ldots b_{p}]} = 0, \quad (2)$$

where $\ell$ is a null vector (this follows from (2) and need not be assumed). The second condition can be equivalently replaced by $\ell^{a_{1}}F_{a_{1}b_{1}\ldots b_{p-1}} = 0$ (cf [1, 31] for $n = 4$, $p = 2$).

---

3 To be precise, in the terminology of [15] the universal property refers only to certain Einstein metrics, whereas here we clearly use the term universal in a broader sense (which applies not only to the metric—not necessarily Einstein—but also to the full solution, including possible matter fields).

4 In the vacuum case, recent analysis [17, 18] has extended the results of [11–13] in various directions. In particular, it is now clear that the VSI property is neither a sufficient nor a necessary condition for universality (however, universal spacetimes must be CSI [17], i.e. with constant scalar invariants). Moreover, certain non-pp-wave spacetimes of Weyl type III, N [17] and II (or D) [18] can also be universal.
The most general algebraic form of a null $p$-form $F$ is thus known (equation (6) below). We will determine what are the necessary and sufficient conditions for a $p$-form $F$ living in a certain spacetime to be VSI, in the sense of the following definition:

**Definition 1.2.** (VSI tensors.) A tensor in a spacetime with metric $g_{ab}$ is VSI if the scalar polynomial invariants constructed from the tensor itself and its covariant derivatives up to order $I$ ($I = 0, 1, 2, 3, \ldots$) vanish. It is VSI if all its scalar polynomial invariants of arbitrary order vanish. As in [6, 32], if the Riemann tensor of $g_{ab}$ is VSI (or VSI$_I$), the spacetime itself is said to be VSI (or VSI$_I$).

For the purposes of the present paper, it will be convenient to recall corollary 3.2 of [28], which can be expressed as:

**Theorem 1.3.** (Algebraic VSI theorem [28].) A tensor is VSI$_0$ iff it is of type III (or more special).

Again, the tensor type refers to the algebraic classification of [29] (see also the review [7]). Now, recalling that a non-zero $p$-form $F$ can only be of type G, II (D) or N (cf [6, 33, 34])$^5$, it immediately follows from theorem 1.3 that:

**Corollary 1.4.** (VSI$_0$ $p$-forms.) A $p$-form $F$ is VSI$_0$ iff it is of type N.

1.2. Main result

The main result of this paper (proven in appendix B) is the following.

**Theorem 1.5.** (VSI $p$-forms.) The following two conditions are equivalent:

1. A non-zero $p$-form field $F$ is VSI in a spacetime with metric $g_{ab}$.
2. (a) $F$ possesses a multiple null direction $\ell$, i.e. it is of type N.
   (b) $\mathcal{L}_\ell F = 0$.
   (c) $g_{ab}$ is a degenerate Kundt metric, and $\ell$ is the corresponding Kundt null direction.

We can already make a few observations about the implications of theorem 1.5.

**Remark 1.6.** (Degenerate Kundt metrics.) The definition of degenerate Kundt metrics [37, 38] is reproduced in appendix A.2 (definition A.1). By proposition A.2 (with propositions 7.1 and 7.3 of [7]), it follows that, e.g., all VSI spacetimes, all $pp$-waves, and all Kundt Einstein (in particular, Minkowski and (A)dS) or aligned pure radiation spacetimes necessarily belong to this class. As a consequence, when considering VSI $p$-forms coupled to gravity, the ‘degenerate’ part of condition 2c above becomes automatically trivially true (since a null $p$-form gives rise to an aligned pure radiation term in the energy-momentum tensor, cf section 3). It is also worth remarking that, in four dimensions, the degenerate Kundt

---

$^5$ However, it is well-known that for $p = 2$ and $n = 4$ the only possible types are D and N [4, 31]. Note also that the type G does not occur for $p = 2$ and $n$ even [30, 35, 36]. For $p = 1$ the type is G, D or N when the vector $F$ is timelike, spacelike or null, respectively (cf, e.g., [7]; in particular, $p = 1$ is the only case of interest when $n = 3$).
spacetimes are the only metrics not determined by their curvature scalar invariants [37], and
they are thus of particular relevance for the equivalence problem [37, 38].

**Remark 1.7.** (VSI vector field.) Condition 2c of theorem 1.5 implies that any affinely
parametrized principal null vector $\ell$ of a VSI $F$ is itself VSI. Indeed, in the special case $p = 1$,
thanks to condition 2c, condition 2a is trivially satisfied and condition 2b simply means that $\ell$
is affinely parametrized; i.e. for $p = 1$, theorem 1.5 reduces to: a vector field $\ell$ is VSI in a
spacetime with metric $g_{\alpha\beta}$ iff $\ell$ is Kundt and affinely parameterized, and $g_{\alpha\beta}$ is a degenerate
Kundt metric w.r.t. $\ell$. 

**Remark 1.8.** (Theory independence of the result.) In theorem 1.5, the $p$-form $F$ is not
assumed to satisfy any particular field equations and the result is thus rather general. On the
other hand, if $F$ is taken to be closed (i.e. $dF = 0$) then condition 2b automatically follows
from the type N condition 2a, and need not be assumed. Otherwise, condition 2b is needed to
ensure (together with 2a and 2c) that $\nabla F$ is of type III (or more special). Note that
$\ell \iota F = \nabla \ell F$ if $F$ is of type N and $\ell$ is Kundt.

**Remark 1.9.** (VSI $\Rightarrow$ VSI for a $p$-form.) From the proof of theorem 1.5 (appendix B) it
follows, in fact, that if $F$ is VSI$_3$ then it is necessarily VSI. Condition 1 could thus be
accordingly relaxed in the proof ‘1. $\Rightarrow$ 2.’. By contrast, recall that in the case of the Riemann
tensor one has VSI$_1 \Rightarrow$ VSI [6, 16, 32]. For completeness, necessary and sufficient conditions
for $F$ to be VSI$_1$ or VSI$_2$ are given in appendix C.

**Remark 1.10.** ($\epsilon$-property.) From lemma B.4 (appendix B), it is easy to see that a tensor is
VSI if and only if, given an arbitrary non-negative integer $N$, there exists a reference frame in
which its components and those of its covariant derivatives up to order $N$ can be made as
small as desired (the proof of this statement is essentially the same as the one given in [32] for
the Riemann tensor). This applies, in particular, to a $p$-form $F$ and is an extension of the early
observations of [1, 5] for the case $N = 0$.

In the rest of the paper we discuss further implications of theorem 1.5 from a physical
viewpoint. In section 2 we give the explicit form of VSI $p$-forms, the associated (degenerate
Kundt) background metric and the corresponding Maxwell equations in adapted coordinates.
We also observe that null (and thus VSI) Maxwell fields are ‘immune’ to adding a Chern–
Simons term to the Maxwell equations (except when it is linear, i.e. for $n = 2p - 1$). More
generally, we point out that certain VSI Maxwell fields solve any generalized electrodynamics.
In section 3 we consider VSI $p$-forms in the full Einstein–Maxwell theory (which solve also certain
supergravities due to the vanishing of the Chern–Simons term). We discuss consequences of the Einstein equations, also mentioning a few examples, and we comment on
various subclasses of exact solutions (such as VSI spacetimes and $pp$-waves) with arbitrary $n$
and $p$. We observe that certain VSI Maxwell fields also solve any generalized electrodynamics when the coupling to gravity is kept into account. Appendix A contains a summary
of properties of Kundt spacetimes which are useful in this paper. Appendix B gives the proof
of theorem 1.5. For completeness, in appendix C we present the necessary and sufficient
conditions for a $p$-form to be VSI$_1$ or VSI$_2$, which can be seen as an ‘intermediate’ result
between corollary 1.4 and theorem 1.5.

**Notation.** In an $n$-dimensional spacetime we set up a frame of $n$ real vectors $m_{(0)}$, which
consists of two null vectors $\ell \equiv m_{(0)}$, $n \equiv m_{(1)}$ and $n - 2$ orthonormal spacelike vectors.
Explicit form of VSI electromagnetic fields and adapted coordinates

In this section we study VSI Maxwell test fields, i.e. without taking into account their backreaction on the spacetime geometry (which will be discussed in section 3).

2.1. Electromagnetic field and spacetime metric

It is useful to express explicitly the conditions 2 of theorem 1.5 in a null frame adapted to \( \ell \) (but otherwise arbitrary), as defined above. Condition 2a reads (cf, e.g., [33])

\[
F_{ab} = p!F_{i_1 \ldots i_{p-1}} \ell_{a m(i)} \ldots \ell_{b m(j-1)}.
\]

The Kundt part of condition 2c means (A1), i.e.

\[
L_{00} = 0, \quad L_{ij} = 0.
\]

Condition 2b and the remaining part of condition 2c are more conveniently represented in a null frame parallelly transported along \( \ell \) (i.e. such that (A2) holds), where they take the form (cf also appendix A, and recall that a Kundt metric for which the Riemann tensor and its first covariant derivative are of aligned type II is necessarily degenerate Kundt [37, 38])

\[
DF_{i_1 \ldots i_{p-1}} = 0, \quad R_{0000} = 0, \quad DR_{0101} = 0, \quad \text{with (A2)}.
\]

In adapted coordinates, degenerate Kundt metrics are described by [37, 38]

\[
d\sigma^2 = 2dr [\mathrm{d}x + H(u, r, x) \mathrm{d}u + W_\alpha(u, r, x) \mathrm{d}x^\alpha] + g_{\alpha\beta}(u, x) \mathrm{d}x^\alpha \mathrm{d}x^\beta,
\]

where \( \ell = \partial_r \) is the Kundt vector, \( \alpha, \beta = 2 \ldots n - 1 \), \( x \) denotes collectively the set of coordinates \( x^\alpha \), and \( W_{\alpha rr} = 0 = H_{rr} \) (thanks to which the second and third of (8) are identically satisfied), i.e.

\[
W_\alpha(u, r, x) = rW_\alpha^{(1)}(u, x) + W_\alpha^{(0)}(u, x),
\]

\[
H(u, r, x) = r^2 H^{(2)}(u, x) + rH^{(1)}(u, x) + H^{(0)}(u, x).
\]

In these coordinates one has (cf the coordinate-independent expression (A3))

\[
\ell_{a \beta} \mathrm{d}x^a \mathrm{d}x^b = (2rH^{(2)} + H^{(1)}) \mathrm{d}u^2 + \frac{1}{2} W_\alpha^{(1)}(\mathrm{d}u \mathrm{d}x^\alpha + \mathrm{d}x^\alpha \mathrm{d}u).
\]
The corresponding VSI $p$-form (6) reads
\[ F = \frac{1}{(p - 1)!} f_{\alpha_1\ldots\alpha_{p-1}}^\ell (u, x) du \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{p-1}}, \] (13)
where $f_{\alpha_1\ldots\alpha_{p-1}}^\ell \equiv f_{\alpha_1\ldots\alpha_{p-1}}^\ell$ is $r$-independent due to the first of (8). In these coordinates we have (with the definition (B9))
\[ F^2 = f_{\alpha_1\ldots\alpha_{p-1}}^\ell f^{\alpha_1\ldots\alpha_{p-1}}, \] (14)
where, from now on, it is understood that the indices of $f^{\alpha_1\ldots\alpha_{p-1}}$ are raised using the transverse metric $g^{\alpha\beta}$. $F^2$ parametrizes the field strength of $F$ and is invariant under Lorentz transformations preserving $\ell$, as well as under transformations of the spatial coordinates $x \mapsto x'(x)$.

At this stage, $g_{\alpha\beta}$, $W_0^{(1)}$, $W_1^{(0)}$, $H_0^{(2)}$, $H_1^{(1)}$, $H_2^{(0)}$ and $f_{\alpha_1\ldots\alpha_{p-1}}^\ell$ are all arbitrary functions of $u$ and $x$ (recall that the invariant condition $W_0^{(1)} = 0 \Leftrightarrow L_{ij} = 0$ defines a special subfamily of Kundt metrics, cf appendix A.1). In general, the associated Weyl and Ricci tensor are both of aligned type II (as follows from definition A.1 in appendix A.2). The b.w. 0 components of all curvature tensors (and thus also their curvature invariants, of all orders) of the metric (9) are independent of the functions $W_0^{(0)}$, $H_0^{(1)}$ and $H_0^{(2)}$ [40] (as summarized in proposition 7.2 of [7]). Restrictions coming from the Einstein equations are described below in section 3. We emphasize here that although the $p$-form (13) is VSI, the spacetime (9) (with (10), (11)) in general is not (not even VSI0) [6, 16, 41], i.e. it may admit some non-zero invariants constructed from the Riemann tensor and its derivatives. However, all mixed invariants (i.e. those involving Riemann and $F$ together, along with their derivatives) are necessarily zero (since $F$ possesses only negative b.w. s and Riemann only non-positive ones, and similarly for their derivatives).

The metric (9) includes, in particular, spacetimes of constant curvature (Minkowski and (A)dS). Therefore, the $p$-form (13) can be used to describe VSI test fields in such backgrounds, as a special case.

2.2. Maxwell’s equations

The construction of VSI $p$-forms has been, so far, purely geometric. Using (9) and (13), the source-free Maxwell equations $dF = 0$ and $d^* F = 0$ reduce, respectively, to
\[ f_{\alpha_1\ldots\alpha_{p-1},\alpha_1} = 0, \quad (\sqrt{\bar{g}} f^{\alpha_1\ldots\alpha_{p-1}})_{,\beta} = 0, \] (15)
where $\bar{g} \equiv \det g_{\alpha\beta} = -\det g_{ab} = -g$. Effectively, these are Maxwell’s equations for the $(p - 1)$-form $f$ in the $(n - 2)$-dimensional Riemannian geometry associated with $g_{\alpha\beta}$. In other words, the most general VSI $F$ that solves Maxwell’s equations is given by (13) in the spacetime (9) with (10), (11), where $f$ is harmonic w.r.t. $g_{\alpha\beta}$\(^6\). Recall, however, that $f$ can also depend on $u$.

2.2.1. The special case $p = 1$. The case $p = 1$ (or $p = n - 1$ by duality), for which $F = \ell$ is a vector field, is special. Indeed by theorem 1.5 (see also remark 1.7), if $\ell$ is VSI then it must be degenerate Kundt and affinely parametrized. Further, by a boost, $\ell$ can always be rescaled (still remaining VSI) such that $L_{ij} = L_{ij}$ (cf., e.g., [6, 17]). It is then easy to see that

\(^6\) Note that equations (15) apply to type N Maxwell fields $F$ in any Kundt spacetime, so also to type N $F$ that are not VSI (i.e. we have not used (10) and (11) to obtain (15)). In the case $p = 2$, this agrees with the results of [42], once specialized to Maxwell fields of type N.
Maxwell’s equations are satisfied (i.e. $\ell^{\mu\nu} = 0 = \ell_{[\mu,\nu]}$, cf (A3)). Therefore, for $p = 1$ to any VSI $F$ it can always be associated a solution to the source-free Maxwell equations (in this statement, VSI can also be relaxed to VSI1, since only the Kundt property of $\ell$ has been employed, cf also proposition C.1).

2.2.2. The special cases $n = 4, 3$.

Case $n = 4$. We have seen (theorem 1.5) that, for any $n$ and $p$, if a $p$-form $F$ is VSI then the multiply aligned null direction $\ell$ is necessarily (degenerate) Kundt and thus, in particular, geodesic and shearfree. In the case $n = 4, p = 2$, the Robinson theorem [2, 4, 43] then implies that the family of null bivectors associated with $\ell$ includes a solution to the source-free Maxwell equations. Namely, from a VSI $F$ one can always obtain a solution $F'$ to the Maxwell equations by means of a Lorentz transformation (spin and boost) preserving the null direction defined by $\ell$ (of course, it may happen that $F$ is already a solution and no transformation is needed). The 2-form $F'$ will thus be also multiply aligned with $\ell$ and therefore it will be still VSI (condition 2b of theorem 1.5 will hold due to $dF' = 0$, cf remark 1.8). Along with the observation in section 2.2.1 for the case $p = 1$ (or $p = 3$), one concludes that for $n = 4$, given a VSI field, one can always associate with it a VSI solution to the source-free Maxwell equations (as above, it suffices to assume that $F$ is VSI1). This does not seem to be true when $n > 4$, in general.

Case $n = 3$. In three dimensions the situation is even simpler, since the only case of interest is $p = 1$ (dual to $p = 2$). The observation in section 2.2.1 thus immediately implies that also for $n = 3$ to any VSI (or VSI1) $F$ it can always be associated a solution to the source-free Maxwell equations.

2.3. Maxwell–Chern–Simons equations

Maxwell’s equations (15) for a $p$-form field admit a generalization that includes a Chern–Simons term, i.e. $dF = 0$ and $d^2F + \alpha F \wedge \ldots \wedge F = 0$, where $\alpha \neq 0$ is an arbitrary constant. The second term in the latter equation contains $k$ factors $F$, and the corresponding number of spacetime dimensions is given by $n = p(k + 1) - 1$. Such modifications of Maxwell’s equations appear, for example, in (the bosonic sector of) minimal supergravity in five and eleven dimensions (with $n = 5$, $p = 2$, $k = 2$ and $n = 11$, $p = 4$, $k = 2$, respectively; cf, e.g., [44] and references therein).

2.3.1. Generic case $k \geq 2$. Now, let us note that any $p$-form of type N (6) satisfies

$$F \wedge F = 0,$$

so that for an $F$ of type N, the Chern–Simons term vanishes identically, provided $k \geq 2$. Therefore, a type N solution $F$ of Maxwell’s theory (15) is automatically also a solution of the Maxwell–Chern–Simons theory. This applies, in particular, to VSI solutions. This has been also noticed, e.g., in [45] in the case of certain VSI $pp$-waves in $n = 11$ supergravity.

2.3.2. Special case $k = 1$. The special case $k = 1$ results in a linear theory

$$dF = 0, \quad d^2F + \alpha F = 0 \quad (n = 2p - 1).$$

It is clear that here the Maxwell equations are modified non-trivially also for type N fields. The second of (15) now has to be replaced by

$$(\sqrt{g} f^{\alpha_1 \ldots \alpha_{p-2}})_{\beta} = \alpha \sqrt{g} \star f^{\alpha_1 \ldots \alpha_{p-2}} = 0,$$
where \( * \) is the Hodge dual in the transverse geometry \( g_{\alpha \beta} \) (not to be confused with the \( n \)-dimensional Hodge dual \( * \) in the full spacetime \( g_{ab} \)). Recall that a linear Chern–Simon term appears, e.g., in topological massive electrodynamics in three dimensions \((n = 3, p = 2, k = 1)\) [46, 47] (and references therein).

2.4. Universal solutions of generalized electrodynamics (test fields)

Theories of electrodynamics described by a Lagrangian also depending on the derivatives of the field strength were proposed long ago in [24, 25]. More recently, interest in higher-derivative theories has been also motivated by string theory, cf, e.g., [48–50] and references therein. In this context, the electromagnetic field is typically represented by a closed 2-form \( F \) whose field equations contain correction terms constructed in terms of \( F \) and its covariant derivatives.

The class of VSI Maxwell fields defined in this paper can be employed to identify a subset of VSI solutions that are ‘universal’, i.e. solving simultaneously any electrodynamics whose field equations can be expressed as \( * \) \( F \) \( d \) \( F \) = 0, \( * \) \( \Phi \) \( d \) \( \Phi \) = 0, where \( \Phi \) can be any \( p \)-form constructed from \( F \) and its covariant derivatives.\(^7\) It can be shown, for example, that any VSI Maxwell \( F \) is universal if the background is a Kundt spacetime of Weyl and traceless-Ricci type III (aligned) with \( DR = 0 = \delta_i R \). In particular, Ricci flat and Einstein Kundt spacetimes of Weyl type III/N/O can occur (an explicit example is given in section 3.2), the latter including Minkowski and (A)dS. Details and more general examples (also of Weyl type II) will be presented elsewhere.

It should be emphasized that the above definition of universality includes terms with arbitrary higher-order derivative corrections. If one restricts oneself to theories in which \( \Phi \) is constructed algebraically from \( F \), i.e. without taking derivatives of \( F \) (like NLE, for which \( \Phi = \Phi \) and the modified Maxwell equations are of the form \( d(f^* F + f_2 F) = 0 \), then these admit as solutions all null solutions of Maxwell’s equations (not necessarily VSI), without any restriction on the background geometry. This has been known for a long time in NLE [5, 19] (see also, e.g., [51]).

3. Einstein–Maxwell solutions

3.1. General equations

What discussed so far applies to VSI test fields, since we have not considered the consequences of the backreaction on the spacetime geometry. In the full Einstein–Maxwell theory this is described by the energy-momentum tensor associated with \( F \)

\[
T_{ab} = \frac{\kappa_0}{8\pi} \left( F_{ac_1\ldots c_{p-1}} F_{b}^{\alpha_1\ldots\alpha_{p-1}} - \frac{1}{2p} g^{ab} F^2 \right),
\]

(19)

where \( F^2 = F_{\alpha_1\ldots\alpha_p} F^{\alpha_1\ldots\alpha_p} \). With (6), \( T_{ab} \) in (19) takes the form of aligned pure radiation, and Einstein’s equations with a cosmological constant \( \Lambda_{ab} \) \( = \frac{1}{2} R_{ab} \) \( + \Lambda g_{ab} \) \( = 8\pi T_{ab} \) reduce to

\(^7\) We assume that \( F \) is constructed polynomially from these quantities (however, the scalar coefficients appearing in such polynomials need not be polynomials of the scalar invariants of \( F \) and its covariant derivatives; cf, for instance, Born–Infeld’s theory). The same type of construction will be assumed for the energy-momentum tensor of generalized theories in section 3.2 (but see also footnote 12).
The Ricci tensor of \((9)\) thus must satisfy \(R_{01} = \frac{2}{n-2} \Lambda, R_{ij} = \frac{2}{n-2} \Lambda \delta_{ij}, R_{ii} = 0\) and \(R_{ij} = \kappa_0 F^2\) (with \(R = 2n \Lambda/(n - 2)\), while \(R_{00} = 0 = R_{0i}\) identically since the Riemann type is II by construction). Using these, the b.w. 0 components of the Einstein equations imply that in \((9)-\(11)\) (as follows readily from \([42]\))

\[
\mathcal{R}_{\alpha\beta} = \frac{2\Lambda}{n - 2} g_{\alpha\beta} + \frac{1}{2} W^{(1)}_{\alpha} W^{(1)}_{\beta} - W^{(1)}_{[\alpha|\beta]} - \frac{2\Lambda}{n - 2} g_{\alpha\beta},
\]

\[
2 H^{(2)} = \frac{\mathcal{R}}{2} - \frac{n - 4}{n - 2} \Lambda + \frac{1}{4} W^{(1)}_{\alpha} W^{(1)}_{\alpha},
\]  

where \(\mathcal{R}_{\alpha\beta}, \mathcal{R}\) and \(||\) denote, respectively, the Ricci tensor, the Ricci scalar and the covariant derivative associated with \(g_{\alpha\beta}\), and \(W^{(1)}_{\alpha} \equiv g^{\alpha\beta} W^{(1)}_{\beta}\). The first of these is an effective Einstein equation for the transverse metric, while the second one determines the function \(H^{(2)}\). Note that the functions \(W^{(0)}_{\alpha}, H^{(1)}\) and \(H^{(0)}\) do not appear here.

With \((21)\), the contracted Bianchi identity in the transverse geometry, i.e. \(2 \mathcal{R}_{\alpha\beta} || = \mathcal{R}_{\alpha\beta}\), tells us (after simple manipulations with the Ricci identity) that \(W^{(1)}_{\alpha} \) is constrained by

\[
W^{(1)}_{\alpha} || = \frac{1}{2} \left( W^{(1)}_{\alpha} \left( 3 W^{(1)}_{\alpha} - W^{(1)}_{\alpha} || + W^{(1)}_{\alpha} \left( W^{(1)}_{\alpha} || - \frac{1}{2} W^{(1)}_{\alpha} W^{(1)}_{\beta} - \frac{2\Lambda}{n - 2} \right) \right) \right).
\]  

Finally, the Einstein equations of negative b.w., which can be used to determine the functions \(H^{(1)}\) and \(H^{(0)}\), reduce to \([42]^8\)

\[
2 H^{(1)}_{\alpha} = - g_{\alpha\beta} || + 2 W^{(0)}_{\alpha\beta} || - 2 W^{(0)}_{\alpha\beta} W^{(1)}_{\alpha\beta} + (W^{(0)})_{\alpha\beta} W^{(1)}_{\alpha\beta} + W^{(1)}_{\alpha,\alpha} + 2 (\ln \sqrt{s})_{\alpha\alpha} + W^{(1)}_{\alpha} \left[ W^{(0)}_{\alpha\beta} W^{(1)}_{\alpha\beta} - W^{(0)}_{\alpha\beta} || + (\ln \sqrt{s})_{\alpha\beta} \right] + \frac{4\Lambda}{n - 2} W^{(1)}_{\alpha},
\]

\[
\Delta H^{(0)} + W^{(1)}_{\alpha} H^{(0)}_{\alpha} + W^{(1)}_{\alpha} H^{(0)}_{\alpha} = W^{(0)}_{\alpha\beta} W^{(1)}_{\alpha\beta} \left( \frac{1}{2} W^{(1)}_{\alpha} || - \frac{2\Lambda}{n - 2} \right) + H^{(1)} \left[ W^{(0)}_{\alpha\beta} - (\ln \sqrt{s})_{\alpha\beta} \right] - \frac{1}{2} (W^{(0)}_{\alpha\beta} W^{(1)}_{\alpha\beta})^2 + W^{(0)}_{\alpha\beta} W^{(0)}_{\alpha\beta} || + W^{(0)} || - W^{(0)}_{\alpha\beta} (W^{(0)}_{\alpha\beta} W^{(1)}_{\alpha\beta} + W^{(1)}_{\alpha\beta} - 2 H^{(1)}_{\alpha} ||) - (\ln \sqrt{s})_{\alpha\beta} + \frac{1}{4} s^{\alpha\beta} g_{\alpha\beta\gamma}, \kappa_0 F^2,
\]

where \(\Delta\) is the Laplace operator in the geometry of the transverse metric \(g_{\alpha\beta}\) (not to be confused with the symbol \(\Delta\) defined in \((5)\)). In addition to \((21)-(25)\), the equations for the electromagnetic field \((15)\) must also be satisfied (note that the only metric functions entering...
there are the \( g_{\alpha\beta} \). When \( \mathcal{F}^2 = 0 \), equations (21)–(25) represent the vacuum Einstein equations for the most general Kundt spacetime.

Let us observe that the functions \( \mathcal{F}^2(u, x) \) and \( H^{(0)}(u, x) \) enter only (25), which is linear in \( H^{(0)}(u, x) \). It follows, e.g., that given any Kundt Einstein spacetime (necessarily of the form (9) with (10), (11)), one can add to it an electromagnetic (and gravitational) wave by just appropriately choosing a new function \( H^{(0)}(u, x) \), and leaving the other metric functions unchanged (amounting, in fact, to a generalized Kerr–Schild transformation; cf theorem 31.1 and section 31.6 of [4] in four dimensions). The resulting solution will describe a wave-like VSI \( p \)-form field propagating in the chosen Kundt Einstein spacetime. If the Einstein ‘seed’ is VSI (or CSI), so will be the corresponding Einstein–Maxwell spacetime (see [40] and the comments in section 2.1).

The simplest of such examples one can construct are electromagnetic and gravitational ‘plane-fronted’ waves (with \( W^{(0)} = 0 \)) propagating in a constant curvature background, giving rise to Kundt waves of Weyl type N. These are well-known in four dimensions for any value of \( \Lambda \) [53, 54] (and include, e.g., the Siklos waves when \( \Lambda < 0 \); see also [4, 55, 56]), and have been considered also in arbitrary dimensions [57] (for \( p = 2 \), but a generalization to any \( p \) is straightforward). Similarly, one can construct, e.g., electrovac waves of Weyl type II (in (anti-)Nariai product spaces for \( n = 4 \) [58] and higher [52]. All these examples are CSI spacetimes, and for \( \Lambda = 0 \) they become VSI (pp- or Kundt waves) spacetimes (cf sections 3.1.1 and 3.1.2). More general (e.g., with \( W^{(0)} = 0 \)) degenerate Kundt metrics with null Maxwell fields are also known (see [4, 56, 59] and references therein for \( n = 4 \) and, e.g., [52, 60] in higher dimensions —there are more references in section 3.1.1 below in the case of VSI spacetimes).

In general, for all Einstein–Maxwell solutions with a VSI \( p \)-form \( \mathcal{F} \), since all the mixed invariants are zero (cf section 2.1) and since the Ricci tensor is constructed out of \( \mathcal{F} \) (equation (20)), the only possible non-zero scalar invariants are those constructed from the Weyl tensor (and its derivatives), and the Ricci scalar \( R \). We also remark that, thanks to the observations of section 2.3, all such Einstein–Maxwell solutions having \( n = 5 \), \( p = 2 \) or \( n = 11 \), \( p = 4 \) are also solutions (with the same \( \mathcal{F} \)) of the bosonic sector of 5D minimal supergravity (ungauged or gauged) and 11D supergravity, respectively (further comments and references in section 3.1.1).

3.1.1. VSI spacetimes with VSI Maxwell fields. The special case when the spacetime metric is VSI is of particular interest. All VSI metrics in HD (in particular, with Ricci type N) are given in [41] (see also [61]). In a VSI spacetime, one can always coordinate such that the metric is given by (9) with [41]

\[
g_{\alpha\beta} = \delta_{\alpha\beta}, \quad W^{(1)} = -\delta_{\alpha, 2} \frac{2\epsilon}{x_2}, \quad H^{(2)} = \frac{\epsilon}{2(x^2)} \quad (\epsilon = 0, 1).
\]  

(26)

The VSI assumption implies \( \Lambda = 0 \), so that the Einstein equations (20) give \( R_{ab} = \kappa_{\alpha} \mathcal{F}^2 \delta_a \delta_b \), i.e. the Ricci tensor is of aligned type N (and (21) and (22) are satisfied identically). This constrains the functions \( W^{(0)} \), \( H^{(1)} \) and \( H^{(0)} \), as detailed in [41]. The Weyl tensor is of type III aligned with \( \ell = \partial_u \) (it becomes of type N for special choices of \( W^{(0)} \), including \( W^{(0)} = 0^9 \), and of type O under further conditions on \( H^{(0)} \) [41]). The Maxwell field is still given by (13). Some VSI spacetimes (more general than VSI pp-waves) coupled to

\[9\] We note that, when the Weyl type is N, the metric takes the Kerr–Schild form—the argument given in section 4.2.2 of [62] in the vacuum case also holds for Ricci type N. When the Weyl type is III, they are of the more general ‘extended Kerr–Schild’ form [63].
null $p$-forms have been discussed in [61] in the context of type IIB supergravity (but note that a few of those are of Ricci type III due to the presence of an additional non-trivial dilaton\(^{10}\).)

As a special subcase, for $\epsilon = 0 = H^{(1)}$ one obtains VSI $pp$-waves with a null $p$-form, for which the Weyl type can only be III(a) or more special (since the Ricci type is N [41]). In 4D, these are well-known and necessarily of Weyl type N (which coincides with the type III(a) for $n = 4$ [7]) or O (cf section 24.5 of [4] and references therein). In higher dimensions, their role in the context of supergravity and string theory has been known for some time, see, e.g., [11–13, 65–68]. More recently, some of these (with $n$ arbitrary, $p = 2, 3$ and $W_{\alpha} = W^{(0)}_{\alpha} (u, x) \neq 0$) have been interpreted as ‘charged gyratons’ [69, 70]\(^{11}\). From the viewpoint of supersymmetry, it is also worth recalling that a VSI spacetime admitting a timelike or null Killing vector field must necessarily be a VSI $pp$-wave (see the appendix of [61]). Supersymmetric VSI $pp$-waves coupled to null $p$-forms are indeed well-known in various supergravities (see, e.g., [11, 65, 66, 68, 71, 72] and references therein).

### 3.1.2. $pp$-waves with VSI Maxwell fields

As mentioned above, spacetimes admitting a null Killing vector field are of special interest for supersymmetry. A null Killing vector field aligned with the Ricci tensor is necessarily Kundt (cf, e.g., Proposition 8.21 of [7]). Now, in the metric (9), the Kundt vector $\ell = \partial_{\ell}$ is Killing iff

$$W^{(1)}_{\alpha} = 0, \quad H^{(2)} = 0 = H^{(1)},$$

which is equivalent to requiring (9) to be a $pp$-wave [73], i.e. $\ell_{\alpha \beta} = 0$ (cf (12)).

For $pp$-waves the Einstein equations (20) imply $\Lambda = 0$ (cf, e.g., proposition 7.3 of [7]), so that $R_{ab} = c_{0} F^{2} \ell_{a} \ell_{b}$ is of type N. By (21), it follows that the transverse metric $g_{\alpha \beta} (u, x)$ must be Ricci flat (if it is flat, as happens necessarily for $n = 4, 5$, one has VSI $pp$-waves, cf section 3.1.1), and (22) is then identically satisfied. Constraints on the functions $W^{(0)}_{\alpha}$ and $H^{(0)}$ follow from the remaining Einstein equations (24), (25), i.e.

$$2W^{(0)}_{\alpha \beta \gamma \alpha} = g_{\beta \gamma \alpha} || \beta = 2 \left( \ln \sqrt{g^{\gamma}} \right)_{\mu \alpha},$$

$$\Delta H^{(0)} = W^{(0)}_{\alpha \\\beta \gamma \alpha} + W^{(0)}_{\beta \alpha \mu} || \alpha = \left( \ln \sqrt{g^{\gamma}} \right)_{\mu \alpha} + \frac{1}{4} g_{\alpha \beta \gamma \alpha} g_{\beta \gamma \alpha} - c_{0} F^{2}.$$

Similar to Ricci flat $pp$-waves (cf table 2 of [74] and proposition 7.3 of [7]), here the Weyl type is II’(abd) (which reduces to III(a) for $n = 5$ and to N for $n = 4$) or more special. The Maxwell field is given by (13).

Some CSI (non-VSI) $pp$-waves coupled to null forms in Ricci flat direct products arise as special cases of the solutions of [75] (where $n = 11$ and $p = 4$).

---

\(^{10}\) We did not consider a dilaton $\varphi$ in our discussion, but this can be easily included. The dilaton $\varphi$ itself cannot be VSI unless zero, being a scalar field. However, if we want $\varphi_{\alpha}$ to be VSI, then it must obviously be a null vector field. If we also require the mixed invariant $F_{\gamma \alpha} \varphi _{\gamma \alpha}$, $F_{\gamma \alpha, \beta} \varphi _{\gamma \alpha, \beta}$ to vanish, we immediately obtain $\varphi_{\alpha} \propto \ell_{\alpha}$, which implies $\varphi = \varphi (u)$ (as assumed in [11, 13, 61, 64]). This is also a sufficient condition for $\varphi_{\alpha}$ to be VSI (since $\varphi_{\alpha} = \varphi \ell_{\alpha}$ and $\ell$ is degenerate Kundt, cf remark 1.7) and for all mixed invariants (i.e. containing $\varphi$, the Riemann/Maxwell tensors and their derivatives) to vanish as well. However, $\varphi_{\alpha \beta} = \varphi \ell_{\alpha} \ell_{\beta}$ is a symmetric rank-2-tensor (of aligned type III or more special), which contains also non-zero components of b.w. $-1$ if $\varphi W^{(0)}_{\gamma \alpha} = 0$ (cf (12)) and appears on the r.h.s. of Einstein’s equations [61], thus modifying the Ricci type.

\(^{11}\) It was indeed observed in [69] that the ansatz used there leads to VSI spacetimes in which also all the electromagnetic scalar invariants vanish. This appears as a special subcase of the result given in theorem 1.5.
3.2. Universal Einstein–Maxwell solutions

In section 2.4 we commented on the role of a subset of the VSI Maxwell fields as universal solutions of all generalized electrodynamics on certain backgrounds (test fields). More generally, some of those can also be used to construct exact solutions of full general relativity, i.e. keeping into account the backreaction of the electromagnetic field on the spacetime geometry. This is described by Einstein’s equations in which, however, the \( T_{ab} \) associated with the electromagnetic field is determined in the generalized electrodynamics (in terms of \( F \) and its covariant derivatives, thus being generically different from (19); cf. e.g., [25]). The class of universal Einstein–(generalized-)Maxwell solutions deserves a more detailed study, which we will present elsewhere. Here we only point out some examples. Namely, it can be shown that all VSI spacetimes with \( L_{41} = 0 = L_{4i} \) (i.e. the recurrent ones) coupled to an aligned VSI p-form field that solve the standard Einstein–Maxwell equations (and are thus of Ricci type N) are also exact solutions of gravity coupled to generalized electrodynamics\(^{12} \), provided \( p > 1 \) and \( \delta_i F_{ih} \cdots j_{i-1} = 0 \) (in an ‘adapted parallely’ transported frame, i.e. such that \( \dot{M}_k = 0 \)). Within this family, metrics of Weyl type N are necessarily \( pp \)-waves, for which such a universal property was pointed out in [11, 13, 14], at least for certain values of \( p \) ([11] considered only plane waves, but included also Yang–Mills field). But metrics of Weyl type III are also permitted, including \( pp \)-waves \( (L_{41} = 0) \) and also genuinely recurrent \( (L_{41} \neq 0) \) spacetimes (for \( n = 4 \), \( p = 3 \) this was discussed in [64]). One explicit example of the latter solutions in 4D is given by\(^{13} \),

\[
\begin{align*}
\mathrm{d}s^2 &= 2 \mathrm{d}u \left[ \mathrm{d}r + \frac{1}{2}(x\mathrm{d}x - xe^x - 2\kappa_0 e^x c^2(u))\mathrm{d}u \right] + e^x (\mathrm{d}x^2 + e^{2u} \mathrm{d}y^2), \\
F &= e^{x/2} c(u) \mathrm{d}u \wedge \left(-\cos \frac{ye^u}{2} \mathrm{d}x + e^u \sin \frac{ye^u}{2} \mathrm{d}y \right).
\end{align*}
\]

Similar to section 2.4, the above discussion applies to generalized electrodynamics with arbitrary higher-order derivative corrections. As a special case, the fact that Einstein–Maxwell solutions with aligned null electromagnetic fields (not necessarily VSI) are also a solution of NLE coupled to gravity was previously demonstrated in [21–23].

Acknowledgments

This work has been supported by research plan RVO: 67985840 and research grant GAČR 13-10042S.

\(^{12} \)To be precise, we should exclude from the discussion possible peculiar theories admitting an energy-momentum tensor \( T_{ab} \), which vanishes for certain non-zero electromagnetic fields (or at least for the ‘universal ones’). One possible way to ensure this is, for example, to consider only theories for which \( T_{ab} \) is of the form \( T_{ab} \propto T_{ab}^{EM} + \text{‘corrections’} \), where \( T_{ab}^{EM} \) is the energy-momentum tensor of the standard Einstein–Maxwell theory, and the ‘corrections’ are terms that go to zero faster than \( T_{ab}^{EM} \) in the limit of weak fields. Additionally, it is also understood that a constant rescaling of a universal p-form \( F \) (or, alternatively, of the corresponding metric, and so also the Ricci tensor) may be necessary when going from one theory to another.

\(^{13} \)This solution has been obtained by adding a null \( F \) to a type III vacuum spacetime found by Petrov (equation (31.40) in [4]), cf section 3.1. A parallely transported frame satisfying \( \dot{M}_k = 0 \) is given by \( \ell = \partial_t, n = \partial_x, m_1 = e^{-1/2} \left( \cos \frac{ye^u}{x} \partial_t - e^x \sin \frac{ye^u}{x} \partial_x \right), \) \( m_2 = e^{-1/2} \left( \sin \frac{ye^u}{x} \partial_t + e^x \cos \frac{ye^u}{x} \partial_x \right) \). Note that by setting \( \kappa_0 = 0 \) in (29), this solution represents a universal test Maxwell field in a Petrov type III vacuum spacetime, relevant to the discussion in section 2.4.
Appendix A. Some of the Newman–Penrose equations for Kundt spacetimes

A.1. General Kundt spacetimes

Using the assumption that $\ell$ is geodesic and Kundt (i.e. expansionfree, shearfree and twist-free), so that (recall the definitions (3)) [6, 39, 76]

$$L_{00} = 0, \quad L_{ij} = 0. \quad (A1)$$

Without loss of generality we can use an affine parametrization and a frame parallely transported along $\ell$, such that, in addition to (A1), we also have [6, 39, 76]

$$L_{10} = 0, \quad \dot{M}_{0j} = 0, \quad N_{0i} = 0. \quad (A2)$$

Thanks to these, the covariant derivatives of the frame vectors take the form [6, 39]

$$\ell_{a;b} = L_{11} \ell_a \ell_b + L_{1i} \ell_a m_b^{(i)} + L_{1i} m_a^{(i)} \ell_b, \quad (A3)$$

$$n_{a;b} = -L_{11} n_a \ell_b - L_{1i} n_a m_b^{(i)} + N_{1i} m_a^{(i)} \ell_b + N_{ij} m_a^{(j)} m_b^{(i)}, \quad (A4)$$

$$m_{a;b} = -N_{11} m_a \ell_b - L_{ij} n_b \ell_a - N_{ij} \ell_a m_b^{(j)} + \dot{M}_{ij} m_a^{(j)} \ell_b + \dot{M}_{ij} m_a^{(k)} m_b^{(j)}. \quad (A5)$$

From the Ricci identities (11g) and (11k) of [76] with (A1) and (A2) it follows immediately

$$R_{00ij} = 0, \quad R_{0ijk} = 0, \quad (A6)$$

which implies that all Kundt spacetimes are of aligned Riemann type I, with the further restriction $R_{0ijk} = 0$ (cf [38, 42]). Furthermore, the Ricci identities (11b), (11e), (11n), (11a), (11j), (11m) and (11d) of [76] read

$$DL_{ij} = -R_{010i}, \quad DL_{i1} = -R_{010i}, \quad (A7)$$

$$DM_{jk} = 0, \quad (A8)$$

$$DL_{11} = -L_{1i} L_{11} - R_{0101}, \quad (A9)$$

$$DN_{ij} = -R_{00ij}, \quad (A10)$$

$$DM_{j1} = -\dot{M}_{jk} L_{k1} - R_{01ij}, \quad (A11)$$

$$DN_{i1} = -N_{ij} L_{i1} + R_{i101}, \quad (A12)$$

while the commutators [6] needed in this paper simplify to

$$\Delta D - D\Delta = L_{11} D + L_{11} \delta_i, \quad (A13)$$

$$\delta_i D - D\delta_i = L_{i1} D. \quad (A14)$$

In adapted coordinates, the general Kundt line-element can be written as (9), where $\ell = \partial_t$ and all the metric functions depend arbitrarily on their arguments. In those coordinates $L_{11} = L_{i1}$ (since $L_{i1} = (d\ell_a)\ell_b$, cf (12)). From (12) and (A3) it follows that $\ell$ is recurrent iff $W_1^{(1)} = 0 \Leftrightarrow L_{ij} = L_{1j} = 0$, which is equivalent to \( [\delta_i D] = 0 \); this condition can be used to invariantly characterize subfamilies of Kundt spacetimes.
A.2. Kundt spacetimes of aligned Riemann type II

The above results hold for any Kundt spacetime. If one now restricts to the Kundt spacetimes of aligned Riemann type II (i.e. we assume \( R_{010} = 0 \) in addition to (A6))\(^{14} \), using (A1) and (A2) the Bianchi identities (B3), (B5), (B12), (B1), (B6) and (B4) of [39] reduce to

\[
DR_{01i} = 0, \\
DR_{01j} = 0, \\
DR_{ijkl} = 0, \\
DR_{01i} - \delta_i R_{010} = -R_{010} L_{i1} - R_{01x} L_{x1} - R_{01x} L_{x1}, \\
DR_{kij} + \delta_j R_{01j} = R_{01j} L_{k1} + 2R_{0k1} L_{j1} + 2R_{0k1} L_{j1} - 2R_{0k1} L_{j1}, \\
DR_{0ij} - \delta_i R_{01i} = R_{01i} N_{ij} - R_{01x} N_{ij} + R_{01x} N_{ij} + R_{01x} N_{ij} + R_{01x} N_{ij} + R_{01x} N_{ij} + R_{01x} N_{ij}.
\]

Note that here (A7) reduces to \( DL_{i1} = 0 = DL_{a1} \), so that differentiation of (A9) gives

\[
D^2 L_{i1} = -DR_{01i1}.
\]

An important subset of Kundt spacetimes of Riemann type II is given by the degenerate Kundt metrics [37, 38], defined by:

**Definition A.1.** (Degenerate Kundt metrics [37, 38].) A Kundt spacetime is ‘degenerate’ if the Kundt null direction \( \ell \) is also a multiple null direction of the Riemann tensor and of its covariant derivatives of arbitrary order (which are thus all of aligned type II or more special).

It is worth emphasizing that, in fact, the degenerate condition is automatically met at all orders once it is satisfied by the Riemann tensor and its first derivative (theorem 4.2 and section 7 of [38]). For degenerate Kundt spacetimes we have the following:

**Proposition A.2.** (Conditions for degenerate Kundt metrics.) A Kundt spacetime is ‘degenerate’ if it is of aligned Riemann type II and \( \ell R_{a1} = 0 \) (using an affine parameter and a parallely transported frame, the latter condition is equivalent to any of the following: \( DR = 0, LR_{01} = 0 \) or (for \( n > 3 \)) \( DC_{0101} = 0 \))\(^{15} \). A Kundt spacetime for which the tracefree part of the Ricci tensor is of aligned type III is necessarily degenerate.

**Proof.** It is a result of [37, 38] that for Kundt spacetimes the degenerate condition is equivalent to the Riemann type II with \( DR_{0101} = 0 \). The first part of the proposition thus simply follows from \( DR_{0101} = 0 \) and the contracted Bianchi identities. The second part follows using proposition 2 of [76] (implying the Weyl type II; cf also proposition 7.1 of [7]) and, again, the contracted Bianchi identities (whose component of b.w. +1 gives \( DR = 0 \)).

\(^{14} \) This is equivalent to saying that \( \ell \) is a multiply aligned null direction of both the Weyl and the Ricci tensors.

\(^{15} \) Recall that for \( n = 3 \) the Weyl tensor vanishes identically and therefore the condition \( DC_{0101} = 0 \) is trivial.
For Kundt spacetimes of Riemann type II, the degenerate condition is also equivalent to
\[ D^2 L_{11} = 0 \] (cf (A21)). Note that the assumptions on the Ricci tensor in the second part of
proposition A.2 are of physical interest since they correspond to the case when the energy-
momentum tensor is triply aligned with \( \ell \). See remark 1.6 for further comments.

An alternative covariant characterization of degenerate Kundt metrics was given in
proposition 6.1 of [38] for \( n = 4 \). This result in fact holds for any \( n \) and we reproduce it here,
along with the sketch of a proof different from the one of [38] (i.e., not using the explicit form
of the Kundt metric in adapted coordinates). After defining the symmetric 2-tensor
\[ Q_{ab} \equiv R^{cdef} \ell^c \ell^d g_{ef}, \] (A22)
we can state:

Proposition A.3 (Covariant characterization of degenerate Kundt metrics [38]). A Kundt
spacetime is:

(i) of aligned Riemann type II (or more special) iff \( Q_{ab} = 0 \)  
(ii) degenerate iff \( Q_{ab} = 0 \) and \( \nabla_a \ell \nabla_b \ell g_{ab} = 0 \).

Proof. In a Kundt spacetime, using an affine parameter from (A3) one obtains
\[ \ell^i g_{ab} = 2L_{11}(\ell^d \ell^b) + (L_{1i} + L_{i1})(\ell^d m^b_d + m^b_d \ell^b). \] (A23)
Taking the Lie derivative of this expression and using (A5) and (A7), it is easy to see that
\( Q_{ab} = 0 \) \( \Rightarrow R_{010i} = 0 \), which (recalling (A6)) proves (i).

When \( Q_{ab} = 0 \), using a parallelly transported frame one easily finds that
\( \nabla_a \ell \nabla_b \ell g_{ab} = 0 \) \( \Rightarrow D^2 L_{11} = 0 \), which (recalling (A21)) proves (ii).

Appendix B. Proof of theorem 1.5

B.1. Proof of 2. \( \Rightarrow \) 1.  

B.1.1. Preliminaries. Before starting with the proof, let us make a few helpful observations
on the strategy we shall adopt. First, by assumption 2a, \( F \) is VSI. If we are able to show that
all its covariant derivatives are of aligned type III (or more special), then we are done with the
proof.

Now, by assumption 2c, \( \ell \) is Kundt. Using an affine parameter and a frame parallely
transported along \( \ell \), this implies that the covariant derivatives of the frame vectors do not
produce terms of higher b.w., cf (A3)–(A5) (i.e. \( \ell_{x, a} \ell \) has only components of b.w. \( -1 \) or less,
etc.). Together with assumption 2b (which here can be written as the first of (8)), this
immediately shows that \( \nabla_a F \) is of aligned type III, as required (cf also proposition C.1 in
appendix C). The problem is now to show that the same is true for covariant derivatives of \( F \)
of arbitrary order.

To this end, the balanced-scalar approach of [6, 16] will be useful. This approach can be
applied to various tensors (or spinors). In the context of VSI tensors, the main idea is to show
that (under proper assumptions) the covariant derivative of a tensor of type III is necessarily
of aligned type III and thus, by induction, the tensor under consideration is VSI. Let us thus
recall the relevant definition of [6, 16]:

Definition B.1. (Balanced scalars and tensors [6, 16]). In a frame parallely transported along
an affinely parameterized geodesic null vector field \( \ell \), a scalar \( \eta \) of b.w. \( b \) under a constant
boost is a ‘balanced scalar’ if $D^{-b\eta} = 0$ for $b < 0$ and $\eta = 0$ for $b \geq 0$. A tensor whose components are all balanced scalars is a ‘balanced tensor’.

Note, in particular, that balanced tensors are of type III (or more special), multiply aligned with $\ell$. Restating the inductive method mentioned above in more technical terms, this will thus consist in showing that the covariant derivative of a balanced tensor is again a balanced tensor (lemma B.3 below, which will then apply to $F$)\(^{16}\).

**B.1.2. Proof.** We are thus ready to prove the direction $2. \Rightarrow 1.$ of theorem 1.5.

By assumption 2c, $\ell$ is degenerate Kundt (and thus geodesic). Equation (A1) is satisfied and, employing an affine parameter and a parallelly transported frame, also (A2) and (A6) hold, along with $R_{0010} = 0$ and $\mathcal{D}R_{0010} = 0$. Using the Ricci and Bianchi identities and the commutators summarized in appendix A, one easily arrives at

$$DL_{11} = 0, \quad DL_{i1} = 0, \quad D\hat{M}_{ik} = 0,$$

$$D^2N_0 = 0, \quad D^2\hat{M}_0 = 0, \quad D^3L_{i1} = 0, \quad D^3N_1 = 0.$$  \hspace{1cm} (B1)

Together with the commutators (A13) and (A14), this suffices to readily extend lemma 4 of [6] (see [6, 16] for more technical details), i.e.

**Lemma B.2.** (Balanced scalars in degenerate Kundt spacetimes.) In a degenerate Kundt spacetime, employing an affine parameter and a parallelly transported frame, if $\eta$ is a balanced scalar of b.w. $b$, then all the following scalars (ordered by b.w.) are also balanced: $D\eta; L_{i1}\eta, L_{i1}\eta, \hat{M}_{ik}\eta, \hat{\delta}_{i}\eta; L_{i1}\eta, N_{0}\eta, \hat{M}_{1}\eta, \hat{\Delta}_{i}\eta, N_{1}\eta$.

Now, considering (A3)–(A5) and lemma B.2, one can also easily extend lemma 6 of [6] (cf also [6] for a spinorial version of in four dimensions), i.e.:

**Lemma B.3.** (Derivatives of balanced tensors in degenerate Kundt spacetimes.) In a degenerate Kundt spacetime, the covariant derivative of a balanced tensor is again a balanced tensor.

Next, by assumptions 2a and 2b, we also have that (6) and the first of (8) hold. But these two equations precisely mean that $F$ *is a balanced tensor*. By lemma B.3, the covariant derivatives of arbitrary order of $F$ are thus balanced tensors. In particular, they all possess only components of negative boost weight, which implies that $F$ is VSI, as we wanted to prove.

**B.2. Proof of 1. $\Rightarrow 2.$**

**B.2.1. Preliminaries.** Let us start by proving two useful lemmas (of some interest on their own) about general tensors with vanishing invariants\(^{17}\). The first of these (lemma B.4) follows

\(^{16}\) In the balanced-scalar approach, it is convenient to assign a b.w. to all the Newman–Penrose quantities, and this is why we consider only constant boosts in definition B.1 (so that, e.g., $L_{11}$ has b.w. $-1$ and, if $\eta$ has b.w. $b$, then $D\eta$ and $\hat{\Delta}_{i}\eta$ have, respectively, b.w. $(b + 1)$ and $(b - 1)$, etc; these quantities would not admit a b.w. under a general boost [33]). It is important to observe that there is no loss of generality here as far as our proof is concerned; cf also [6, 17, 18].

\(^{17}\) To avoid confusion, let us emphasize that in the present section B.2.1, and only here, $T$ can be any tensor and does not necessarily coincide with the energy-momentum tensor (19).
directly from theorem 2.1 and corollary 3.2 of [28] (since $T$ and its covariant derivatives up to order $I$ are clearly not characterized by their invariants). We nevertheless provide an independent simple proof.

**Lemma B.4.** (Alignment of VSI tensors.) If a tensor field $T$ is VSI$_I$, $T$ and its covariant derivatives up to order $I$ are of aligned type III (or more special).

**Proof.** The case $I = 0$ is contained in the algebraic VSI theorem (theorem 1.3), so we need to discuss only the case $I > 1$. Additionally, the lemma is trivially true in the case $\nabla T = 0$, so that we can assume hereafter $\nabla T \neq 0$. $T$ being VSI$_I$ implies that $T$ and its covariant derivatives up to order $I$ are all VSI$_0$ (definition 1.2). That all these tensors are of type III thus follows immediately from [28]. It remains to be proven that they are all aligned (i.e. for all of them the same null vector $\ell$ defines a multiple null direction such that all the non-negative b.w. components vanish).

Since $T$ is of type III, it admits a unique multiply aligned null direction such that all the non-negative b.w. components vanish. Furthermore, such a tensor cannot admit a distinct null direction with respect to which all positive b.w. components vanish. Let us work in a null frame such that $\ell$ is parallel to this unique null direction. In this frame, $T$ possesses only components of negative b.w.. Now, let us consider two possible cases separately. (i) First, if $T$ has only components of b.w. $\leq 2$ or less, then the components of $\nabla T$ have b.w. 0 or less (since the covariant derivative of a tensor can, at most, raise the b.w. of $+2$), i.e. $\nabla T$ is also multiply aligned to $\ell$. But $\nabla T$ must be of type III (as noticed above), therefore the only possibility is that the components of $\nabla T$ have, in fact, b.w. $\leq 1$ (or less) in the frame we are using, so $\nabla T$ is of aligned type III. (ii) On the other hand, if $T$ has some components of b.w. $\geq 1$, we cannot use the same argument, since some components of $\nabla T$ will have b.w. $+1$, in general. Let us thus assume this is indeed the case (if not, i.e. if components of $\nabla T$ have only b.w. 0 or less, then we can proceed as in case (i)) and let us consider, instead, the tensor product $T_\ell \equiv T \times \nabla T$. Obviously, the components of $T_\ell$ cannot have b.w. greater than 0 (cf. e.g. proposition A.11 of [34]), but since $T$ is VSI$_I$, then $T_\ell$ must be of type III, which thus implies that the components of $T_\ell$ can only have b.w. $\leq 1$ or smaller in a frame adapted to $\ell$. However, it is not difficult to see that this cannot be true if $\nabla T$ possesses some components of b.w. $+1$ (as we assumed), thus leading to a contradiction. In other words, if $T$ is VSI$_I$, then $\nabla T$ can only have components of b.w. 0 or less; but then, in fact, these can be only of b.w. $\leq 1$ or less (since $\nabla T$ must be of type III, similar to point (i)), so that, again, $\nabla T$ is of aligned type III.

Combining (i) and (ii) we have proven the lemma for the case $I = 1$. Clearly the same argument extends to any higher $I$, i.e. the proof is complete.

In turn, this can be used to prove the following result for rank-2 tensors.

**Lemma B.5.** (Alignment of VSI$_I$ implies Kundt.) If a 2-tensor field $T_{ab}$ is VSI$_2$, then: (a) $T_{ab}$, $T_{abc}$, and $T_{abcd}$ are of aligned type III (or more special); (b) the corresponding multiple null direction $\ell$ is necessarily Kundt.

**Proof.** Point (a) follows immediately from lemma B.4. It will be used in the following. Now, thanks to (a), in an adapted null frame we can write

$$T_{ab} = T_{ij} \ell_i \ell_j + T_{ij} \ell_i m_a^i \ell_j + T_{ij} \ell_i \ell_j.$$  \hfill (B.3)
It is convenient to first prove (b) in the special case \( T_{ij} = 0 = T_{ij} \), i.e. \( T_{ab} = T_{i1} \ell_i \ell_b \) (of course with \( T_{i1} = 0 \)). Let us define a compact notation for the covariant derivatives
\[
T_{ab;\ldots;\ell_I} = T_{ab,\ldots;\ell_I} \quad (I = 1, 2, 3 \ldots).
\]
For our purposes, it will now suffice to require that certain components of b.w. 0 of \( T^{(1)} \) and \( T^{(2)} \) vanish (in view of (a)). First, requiring \( T_{i1}^{(1)} = 0 \) one obtains (using also the first of (3))
\[
L_{i0} = 0,
\]
i.e. \( \ell \) must be geodesic (note that the VSI property of \( T \) suffices to prove this). Next, the condition \( T_{01ij}^{(2)} = 0 \) is equivalent to \( L_{i0} L_{ij} = 0 \), which, by tracing, leads to
\[
L_{ij} = 0,
\]
i.e. (together with (B5)), \( \ell \) must be a Kundt null direction, which thus proves (b) in the case \( T_{ab} = T_{i1} \ell_i \ell_b \).

Finally, for a generic \( T_{ab} \) (equation (B3)) we can apply the same argument to the tensor \( \tilde{T}_{ab} = T_{ib} T_b^i = (T_{i1} \ell_i) \ell_b \) if \( T_{i1} \neq 0 \) (or to \( \tilde{T}_{ab} = T_{ia} T_a^i = (T_i \ell_1) \ell_b \) if \( T_i \neq 0 \)), so the proof of (b) is now complete (where we used the fact that if \( T_{ab} \) is VSI then \( \tilde{T}_{ab} \) must obviously also be VSI2).

**B.2.2. Proof.** Let us now prove the direction 1. \( \Rightarrow \) 2. of theorem 1.5. We assume that \( F \) is VSI. It is, in particular, VSI0 and thus, by corollary 1.4, \( F \) must be of type N, i.e. condition 2a is proven. In an adapted frame this means that (6) holds. It remains to prove that conditions 2b and 2c are also satisfied, i.e. we need to show (recall (7) and (8))

i. \( L_{i0} = 0 \), \( L_{ij} = 0 \)

ii. \( R_{0100} = 0 \), \( DR_{0100} = 0 \) (in a parallely transported frame)

iii. \( D F_{i1 \ldots j \ldots} = 0 \) (in a parallely transported frame).

It is convenient to define the following 2-tensor \( T^{18} \)
\[
T_{ab} = \frac{\kappa_0}{8\pi} \Gamma_{a1 \ldots c_{p-1}}^b \Gamma_{p \ldots c_{p-1}}^{i1 \ldots}.
\]

Since \( F \) is VSI, then \( T \) must also be VSI, thus condition (i) follows immediately from lemma B.5 applied to \( T \), so that \( \ell \) is Kundt. (Alternatively, condition (i) can also be proven using proposition C.1 in appendix C; note also that so far we used only the assumption that \( F \) is VSI2.)

For the next steps, it is useful to observe that, by (6), \( T \) is also of type N w.r.t. \( \ell \), i.e.
\[
T_{ab} = \frac{\kappa_0}{8\pi} \mathcal{F}^2 \ell_a \ell_b,
\]
where
\[
\mathcal{F}^2 = F_{i1 \ldots j_{p-1}} F_{i1 \ldots j_{p-1}} \geq 0,
\]
and has b.w. \( -2 \). Since the b.w. of \( \mathcal{F} \) and \( \nabla^{ij} \mathcal{F} \) is always \(-1\) or less (lemma B.4), also the b.w. of \( T^{(i)} \) (defined in (B4)) must always be \(-2\) (or less; cf proposition A.11 of [34]). This condition will be used below.

From now on, we employ an affine parameter along the Kundt null vector \( \ell \) and a frame parallely transported along it, so that (A2) holds. As observed in appendix A, this implies \( R_{0000} = 0 = R_{0ijk} = 0 \) (equation (A6)).

\footnote{For a VSI0 field \( F \), \( T \) equals the associated energy-momentum tensor (19), since \( F^2 = 0 \).}
Next, requiring $\nabla F$ to be of type III, $(\nabla F)_{01i_1...i_{p-1}} = 0$ is now equivalent to condition (iii), which is thus also proven (again, this alternatively follows from proposition C.1).

Condition (iii) in turn implies $D(F^2) = 0$, so that by a boost we can set $F^2 = 1$ in (B8) (while preserving the affine parametrization of $\ell$ and the parallel transport of the frame), i.e. from now on $T_{ab} = \frac{\kappa}{8\pi} \ell_a \ell_b$.

Now, imposing $T_{1100}^{(2)} = 0$ and $T_{1100}^{(2)} = 0$ (these components have b.w. $-1$ and thus must vanish, as observed above) gives, respectively,

$$DL_{i_1} = 0, \quad DL_{i_1} = 0. \quad \text{(B10)}$$

By (A7) this in turn implies

$$R_{0100} = 0, \quad \text{(B11)}$$

so that the Riemann type is II or more special (recall that we already obtained $R_{00ij} = 0 = R_{00ij}$ above).

Finally, requiring $T_{1100}^{(3)} = 0$ (b.w. $-1$) leads to

$$D^2L_{i_1} = 0, \quad \text{(B12)}$$

and thus, by (A9) (with (B10)),

$$DR_{0101} = 0, \quad \text{(B13)}$$

which completes the proof of condition (ii), and the proof is now complete. \(\square\)

Note that, in fact, we have used only up to the third derivatives of $F$ in the argument above (cf. remark 1.9).

### Appendix C. VSI\textsubscript{1} and VSI\textsubscript{2} $p$-forms

A bridge between corollary 1.4 and theorem 1.5 is provided by the following result (see also remark 1.9).

**Proposition C.1.** (VSI\textsubscript{1} and VSI\textsubscript{2} $p$-forms.) A $p$-form $F$ is VSI\textsubscript{1} iff it is is of type $N$, $\xi \ell F = 0$, $\ell$ is Kundt. It is VSI\textsubscript{2} iff it is is of type $N$, $\xi \ell F = 0$, $\ell$ is Kundt and (at least) doubly aligned with the Riemann tensor.

In particular, this means that for solutions of the Einstein–Maxwell theory, we have that VSI\textsubscript{1} $\Rightarrow$ VSI\textsubscript{2} since condition 2c of theorem 1.5 is automatically satisfied (cf. remark 1.6). Similarly, VSI\textsubscript{1} $\Rightarrow$ VSI also for a $F$ in the classes of spacetimes mentioned in remark 1.6 which are necessarily degenerate Kundt.

**Proof.** Let us first prove the ‘only if’ part. We assume that $F$ is VSI\textsubscript{1}. Then $T$ must have the same property. In particular, in an adapted frame, we have (B8). Further, requiring $T_{1100}^{(1)} = 0$ (recall definition (B4)) we obtain $L_{i_0} = 0$ (i.e. $\ell$ is geodesic). From now on we can thus employ an affine parameter and a frame parallelly transported along $\ell$. The condition $(\nabla F)_{0i_1...i_{p-1}} = 0$ gives $DF_{i_1...i_{p-1}} = 0$. From $(\nabla F)_{i_1...i_{p-1}} = 0$ and $(\nabla F)_{i_1...i_{p-2}} = 0$ we obtain, respectively,

$$F_{i_1...i_{p-1}}L_{i_1} = 0, \quad \text{(C1)}$$
\[ F_{l_{i_1}...l_{i_p}j} = 0. \] (C2)

Contracting (C1) with \( L_{i,j} \) and using (C2) leads to \( F_{l_{i_1}...l_{i_p}j} L_{i,j} = 0. \) Further contraction with \( L_{k,j} \) gives \( L_{k} L_{j} = 0 \) and thus \( L_{k} = 0 \), i.e. \( L \) is Kundt. This is all we needed to prove for the VSI1 statement.

If, additionally, \( F \) is VSI2, then from \( T_{1i10}^{(2)} = 0 \) and \( T_{1i10}^{(2)} = 0 \) we obtain \( DL_{i1} = 0 = DL_{i0} \). By (A7) this implies \( R_{0i10} = 0 \), so that (recall also (A6)) \( L \) is doubly aligned with the Riemann tensor, as we wanted to prove.

The ‘if’ part of the proposition can be proven similarly by reversing the above steps (essentially showing that under the conditions given in proposition C.1 the first covariant derivatives (and for VSI2 also the second covariant derivatives) of \( F \) are b.w. negative. \( \square \)

References

[1] Synge J L 1955 *Relativity: the Special Theory* (Amsterdam: North-Holland)
[2] Penrose R and Rindler W 1986 *Spinors and Space-Time* vol 2 (Cambridge: Cambridge University Press)
[3] Ruse H S 1936 On the geometry of the electromagnetic field in general relativity *Proc. London Math. Soc.* 41 302–22
[4] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein’s Field Equations* 2nd edn (Cambridge: Cambridge University Press)
[5] Schrödinger E 1935 Contributions to Born’s new theory of the electromagnetic field *Proc. Roy. Soc. London Ser. A* 150 465–77
[6] Coley A, Milson R, Pravdová V and Pravdová A 2004 Vanishing scalar invariants of higher dimensions *Class. Quantum Grav.* 21 5519–42
[7] Ortaggio M, Pravda V and Pravdová A 2013 Algebraic classification of higher dimensional spacetimes based on null alignment *Class. Quantum Grav.* 30 013001
[8] Málek T and Pravda V 2011 Type III and N solutions to quadratic gravity *Phys. Rev. D* 84 024047
[9] Pravdová A and Pravdová V 2008 The Newman- Penrose formalism in higher dimensions: vacuum spacetimes with a non-twisting geodetic multiple weyl aligned null direction, *Class. Quantum Grav.* 25 235008
[10] Reall H S, Tanahashi N and Way B 2014 Causality and hyperbolicity of Lovelock theories *Class. Quantum Grav.* 31 205005
[11] Güven R 1987 Plane waves in effective theories of superstrings *Phys. Lett.* B 191 275–81
[12] Amati D and Klimčík C 1989 Nonperturbative computation of the Weyl anomaly for a class of nontrivial backgrounds *Phys. Lett.* B 219 443–7
[13] Horowitz G T and Steif A R 1990 Space–time singularities in string theory *Phys. Rev. Lett.* 64 260–3
[14] Horowitz G T 1990 Singularities in string theory (*Strings ’90* ed R Arnowitt, R Bryan, M J Duff, D Nanopoulos, C N Pope and E Sezgin (Singapore: World Scientific) pp 163–75
[15] Coley A A, Gibbons G W, Hervik S and Pope C N 2008 Metrics with vanishing quantum corrections *Class. Quantum Grav.* 25 145017
[16] Pravdová A, Pravdová V, Coley A and Milson R 2002 All spacetimes with vanishing curvature invariants *Class. Quantum Grav.* 19 6213–36
[17] Hervik S, Pravdová A and Pravdová A 2014 Type III and N universal spacetimes *Class. Quantum Grav.* 31 215005
[18] Hervik S, Málek T, Pravdová V and Pravdová A 2015 Type II universal spacetimes *Class. Quantum Grav.* 32 245012
[19] Schrödinger E 1943 A new exact solution in non-linear optics (two-wave-system) *Proc. Roy. Irish Acad. A* 49 59–66
[20] Plebański J 1970 *Lectures on Non-Linear Electrodynamics* (Copenhagen: Nordita)
[21] Kichenassamy S 1959 Sur le champ électromagnétique singulier en théorie de Born–Infeld, *C. R. Hebd. Seanc. Acad. Sci.* 248 3690–2
[22] Kremer H and Kichenassamy S 1960 Sur le champ électromagnétique singulier dans une théorie du type Born–Infeld *C. R. Hebd. Seanc. Acad. Sci.* 250 1192–4
[23] Peres A 1961 Nonlinear electrodynamics in general relativity Phys. Rev. 122 273–4
[24] Bopp F 1940 Eine lineare Theorie des Elektrons Ann. Physik 430 345–84
[25] Podolsky B 1942 A generalized electrodynamics. part I—non-quantum Phys. Rev. 62 68–71
[26] Teitelboim C 1986 Gauge invariance for extended objects Phys. Lett. B 167 63–8
[27] Henneaux M and Teitelboim C 1986 p-form electrodynamics Found. Phys. 16 593–617
[28] Hervik S 2011 A spacetime not characterized by its invariants is of aligned type II Class. Quantum Grav. 28 215009
[29] Milson R, Coley A, Pravda V and Pravdová A 2005 Alignment and algebraically special tensors in Lorentzian geometry Int. J. Geom. Meth. Mod. Phys. 2 41–61
[30] Sokolowski L M, Occhionero F, Litterio M and Amendola L 1993 Classical electromagnetic radiation in multidimensional space-times Ann. Physics 225 1–47
[31] Hall G 2004 Symmetries and Curvature Structure in General Relativity (Singapore: World Scientific)
[32] Pelavas N, Coley A, Milson R, Pravda V and Pravdová A 2005 VSI spacetimes and the c-property J. Math. Phys. 46 063501
[33] Durkee M, Pravda V, Pravdová A and Reall H S 2010 Generalization of the Geroch–Held–Penrose formalism to higher dimensions Class. Quantum Grav. 27 215010
[34] Hervik S, Ortaggio M and Wylleman L 2013 Minimal tensors and purely electric or magnetic spacetimes of arbitrary dimension Class. Quantum Grav. 30 165014
[35] Bergqvist G and Senovilla J 2001 Null cone preserving maps, causal tensors and algebraic Rainich theory Class. Quantum Grav. 18 5299–326
[36] Milson R Alignment and the classification of Lorentz-signature tensors arXiv:gr-qc/0411036
[37] Coley A, Hervik S and Pelavas N 2009 Spacetimes characterized by their scalar curvature invariants Class. Quantum Grav. 26 025013
[38] Coley A, Hervik S, Papadopoulos G O and Pelavas N 2009 Kundt spacetimes Class. Quantum Grav. 26 105016
[39] Pravda V, Pravdová A, Coley A and Milson R 2004 Bianchi identities in higher dimensions Class. Quantum Grav. 21 2873–97
[40] Pravda V, Pravdová A, Coley A and Milson R 2007 Class. Quantum Grav. 24 1691 (corrigendum).
[41] Coley A, Fuster A, Hervik S and Pelavas N 2010 Lorentzian manifolds and scalar curvature invariants Class. Quantum Grav. 27 102001
[42] Podolský J and Žitka M 2009 General Kundt spacetimes in higher dimensions Class. Quantum Grav. 23 7431–44
[43] Podolsky B and Žitka M 2009 General Kundt spacetimes in higher dimensions Class. Quantum Grav. 26 105008
[44] Ortín T 2015 Gravity and Strings 2nd edn (Cambridge: Cambridge University Press)
[45] Figueroa-O’Farrill J M and Papadopoulos G 2001 Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory J. High Energy Phys. JHEP08(2001)036
[46] Schönfeld J F 1981 A mass term for three-dimensional gauge fields Nucl. Phys. B 185 157–71
[47] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories Ann. Phys. 140 372–411
[48] Deser S and Jackiw R 1988 Erratum Ann. Phys. 185 406
[49] Andreev O D and Tseytlin A A 1988 Partition function representation for the open superstring effective action: cancellation of Möbius inﬁnities and derivative corrections to Born–Infeld lagrangian Nucl. Phys. B 311 205–52
[50] Thorlacius L 1998 Born–Infeld string as a boundary ﬂeld theory Phys. Rev. Lett. 80 1588–90
[51] Chemissany W, Kallosh R and Ortín T 2012 Born–Infeld with higher derivatives Phys. Rev. D 85 046002
[52] Bičák J and Slavík J 1975 Non-linear electrodynamics in the Newman–Penrose formalism Acta Phys. Polon. B6 489–508
[53] Krčouš P, Podolský J, Želňík A and Kadlecová H 2012 Higher-dimensional Kundt waves and gyratons Phys. Rev. D 86 044039
[54] García Díaz A and Plebański J F 1981 All nontwisting N’s with cosmological constant J. Math. Phys. 22 2655–8

22
[54] Oszváth I, Robinson I and Rózga K 1985 Plane-fronted gravitational and electromagnetic waves in spaces with cosmological constant J. Math. Phys. 26 1755–61
[55] Bičák J and Podolský J 1999 Gravitational waves in vacuum spacetimes with cosmological constant. I. Classification and geometrical properties of nontwisting type N solutions J. Math. Phys. 40 4495–505
[56] Griffiths J B and Podolský J 2009 Exact Space-Times in Einstein’s General Relativity (Cambridge: Cambridge University Press)
[57] Obukhov Y N 2004 Generalized plane-fronted gravitational waves in any dimension Phys. Rev. D 69 024013
[58] Podolský J and Ortaggio M 2003 Explicit Kundt type II and N solutions as gravitational waves in various type D and O universes Class. Quantum Grav. 20 1685–701
[59] Griffiths J, Docherty P and Podolský J 2004 Generalized Kundt waves and their physical interpretation Class. Quantum Grav. 21 207–22
[60] Caldarrelli M M, Klemm D and Zorzan E 2007 Supersymmetric gyratons in five dimensions Class. Quantum Grav. 24 1341–58
[61] Coley A, Fuster A, Hervik S and Pelavas N 2007 Vanishing scalar invariant spacetimes in supergravity J. High Energy Phys. JHEP05(2007)032
[62] Ortaggio M, Pravda V and Pravdová A 2009 Higher dimensional Kerr-Schild spacetimes Class. Quantum Grav. 26 025008
[63] Mälek T 2014 Extended Kerr–Schild spacetimes: general properties and some explicit examples Class. Quantum Grav. 31 185013
[64] Coley A 2002 A class of exact classical solutions to string theory Phys. Rev. Lett. 89 281601
[65] Kowalski-Glikman J 1984 Vacuum states in supersymmetric Kaluza–Klein theory Phys. Lett. B 134 194–6
[66] Hull C 1984 Exact pp-wave solutions of 11-dimensional supergravity Phys. Lett. B 139 39
[67] Tseytlin A A 1993 String vacuum backgrounds with covariantly constant null Killing vector and two-dimensional quantum gravity Nucl. Phys. B 390 153–72
[68] Bergshoeff E, Kallosh R and Ortin T 1993 Supersymmetric string waves Phys. Rev. D 47 5444–52
[69] Frolov V P and Zelnikov A 2006 Gravitational field of charged gyratons Class. Quantum Grav. 23 2119–28
[70] Frolov V P and Lin F-L 2006 String gyratons in supergravity Phys. Rev. D 73 104028
[71] Tod K P 1983 All metrics admitting super-covariantly constant spinors Phys. Lett. B 121 241–4
[72] Gauntlett J P, Gutowski J B, Hull C M, Pakis S and Reall H S 2003 All supersymmetric solutions of minimal supergravity in five dimensions Class. Quantum Grav. 20 4587–634
[73] Brinkmann H W 1925 Einstein spaces which are mapped conformally on each other Math. Ann. 94 119–45
[74] Ortaggio M 2009 Bel-Debever criteria for the classification of the Weyl tensor in higher dimensions Class. Quantum Grav. 26 195015
[75] Dereli T and Gürses M 1986 The generalized Kerr-Schild transform in eleven-dimensional supergravity Phys. Lett. B 171 209–11
[76] Ortaggio M, Pravda V and Pravdová A 2007 Ricci identities in higher dimensions Class. Quantum Grav. 24 1657–64