Allometric Scaling of Countries

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Abstract

As huge complex systems consisting of geographic regions, natural resources, people and economic entities, countries follow the allometric scaling law which is ubiquitous in ecological, urban systems. We systematically investigated the allometric scaling relationships between a large number of macroscopic properties and geographic (area), demographic (population) and economic (GDP, gross domestic production) sizes of countries respectively. We found that most of the economic, trade, energy consumption, communication related properties have significant super-linear (the exponent is larger than 1) or nearly linear allometric scaling relations with GDP. Meanwhile, the geographic (arable area, natural resources, etc.), demographic(labor force, military age population, etc.) and transportation-related properties (road length, airports) have significant and sub-linear (the exponent is smaller than 1) allometric scaling relations with area. Several differences of power law relations with respect to population between countries and cities were pointed out. Firstly, population increases sub-linearly with area in countries. Secondly, GDP increases linearly in countries but not super-linearly as in cities. Finally, electricity or oil consumptions per capita increases with population faster than cities.

Keywords: Allometry, Power Law Relation, Macro-Economy

1. Introduction

Scientists always look for universal patterns of complex systems. As huge complex systems consisting of geographic regions, natural resources, people
and economic entities, countries also share some common features despite their different latitudes, climates and cultures. For example, bigger countries in area always have greater population; richer countries always provide more healthy lives for people[1]. Nevertheless, these features are just qualitative, vague and subjective judgements. The quantitative universal patterns which are reflected by the mathematical relations between macro-variables such as GDP, population, total oil consumptions etc. are very necessary for macroeconomics of countries[1, 2].

Scaling property is one of the most important universal quantitative laws governing different dynamics of complex systems[3–5]. Allometric scaling particularly describes the power law relation between two variables[6]. In biology, the allometric scaling relation between metabolism and body size is also known as Kleiber’s law[5, 7, 8]. Biologists found not only metabolism but also other important biological variables including heart beat frequency, life span, fertility rate and so on all exhibit power law relationships with the body size of species[5, 9].

Allometric scaling law can be also extended to the larger complex systems such as cities [10–14]. The power law relation between area and population is one of the earliest allometries found in cities [14]. Recently, more scaling relations were discovered due to more data of cities are available. The scaling relation between energy consumption and population which is a parallel of Kleiber’s law is discovered for cities [10, 13, 15]. Not only energy consumption but also GDP has a power law relation with population of cities with an exponent around 1.20. In [12], authors systematically studied allometric scaling relations between various properties including inventors, total wages, GDP, total housing, etc. and population of cities. All of these properties fall into three groups according to the exponents. The economic and innovation related variables have super-linear relationship with population (i.e., the exponent is larger than 1); the household properties have nearly linear relationship (i.e., the exponent is approaching 1), and the energy consumption and infrastructure-related properties have sub-linear relationship (i.e., the exponent is smaller than 1). Allometries between population and GDP for cities in a province have been also studied[16]. The exponent in this special allometric scaling relation is larger than 1.

Although less attention was paid on the allometric scaling of countries than cities, some existing studies should be noted. Roehner has studied the power law relationship between the export value and population of countries[17]. He found a 3/4 power law relation by grouping countries in different cate-
categories according to GDP per capita. Miguel [18] studied the $CO_2$ emission of countries as the metabolism of the whole country and compared to the biological allometries.

This paper will try to find the possible universal allometric scaling of countries by systematic exploring. It is organized as follows. The first part introduces the methods of our systematic exploring. The results are discussed in section 3. We will show different allometries between variables and area as the geographic size, population as the demographic size and GDP as the economic size of countries in section 3.1, 3.2 and 3.3 respectively. Section 3.4 mainly discusses the power law relation between population and GDP. In section 3.5 we also investigate the allometric scaling relationships between intensive variables. We discussed the major axis regression method and the comparison between cities and countries in section 4.

2. Methods

We investigated the possible allometric scaling relationships between macro-variables of countries by empirical data. The main source of the data is from the software Mathematica. An online database with hundreds of properties for various countries around the world is accessible through the embedded command "CountryData" in Mathematica. These data is from the authoritative organizations, magazines, year books and websites such as Encyclopaedia Britannica, Britannica Book of the Year, Encyclopaedia Britannica, United States National Geospatial-Intelligence Agency, GeoHive: Global Statistics, World Health Organization, etc. [19]. The carbon emission data is from United Nations Framework Convention on Climate Change web site [20].

We systematically investigate totally 104 properties of 273 countries or regions. Firstly, these properties are classified as extensive variables (87 variables) and intensive variables (17 variables). Extensive ones are the gross quantities of countries such as total area, GDP etc. Intensive variables are the rates or ratios of several extensive variables such as GDP per capita, birth rate etc. All variables are classified into 11 different categories including geographical properties, natural resources related properties etc. The variables with their categories, features, demonstrations are listed in Appendix A.

The allometric scaling law always indicates power law relationship between macro-variable and the size of the system. For living organism, the size of the system is its body size or body mass [9]. For cities, population is
always used as the measurement of city size \[12,21\]. However, it is not clear what property is the size of a country. We can say the size of a country in three different senses, area, population or GDP. Thus, we studied the allometric scalings of countries between the given extensive macro-variables and the geographic size(area), demographic size(population) and economic size(GDP) of countries separately. We suppose the following equation holds:

\[ X = cM^\alpha \]  

(1)

Where \( X \) is one of the extensive macro-variables, \( M \) is the area or population or GDP of a country. \( c \) and \( \alpha \) are numbers we should estimate. Where, \( \alpha \) is the power law exponent of the allometry which is the most important parameter. We suppose that the equation holds for all countries. Thus we use linear regression method to find the best fitting line on log-log coordinates, and estimate the parameters \( c \) and \( \alpha \). From equation 1, another equation can be derived:

\[ \frac{X}{M} = cM^{\alpha-1} \]  

(2)

Note that the left hand of the equation \( X/M \) is the average value of \( X \) per size. For example, if \( X \) is GDP, \( M \) is population, then \( X/M \) is GDP per capita. So if \( \alpha > 1 \), the property \( X \) has a super linear relation with respect to \( M \), i.e. GDP per capita increases with the size \( M \). On the other hand, \( \alpha < 1 \) means \( X \) has a sub linear relation with respect to \( M \), the average value \( X/M \) decreases as the size of country \( X \) increases. And the linear relationship can be got with \( \alpha \approx 1 \[12\].

We adopt major axis (MA) approach instead of ordinary least square (OLS) regression to make linear regression on log-log coordinate\[22–24\]. MA method is preferable because our main purpose in this paper is to summarize the relationship between \( M \) and \( X \) but not predict \( X \) according to \( M \)\[22,23\]. And OLS always give biased estimations. From figure 1 we can see the obvious difference between slopes estimated by different methods. The line regressed by MA method locates in the middle of the data band because this method tries to minimize the total distances from data points to the regression line both from \( x \) and \( y \) directions, but not only the \( y \) direction as OLS method does \[22,23\]. However, we both list exponent (\( \alpha \)) estimated by MA method and \( \alpha' \) by OLS method in tables \[124\] and \[5\] to compare these two methods and their predicted exponents.

We systematically investigate the power law relations between any given property of the 87 extensive variables and three kinds of sizes. The \( 87 \times 3 = \)
Figure 1: Different Regression Methods Applied on Population and GDP Power Law Relation, $\alpha = 1.04$ and $\alpha' = 0.79$
Table 1: Allometric Scaling Relations with respect to Area (Exponent $\alpha$ is estimated by MA method and $\alpha'$ is estimated by OLS method. Only the allometries with $r^2 > 0.6$ are shown here)

| Property               | $\alpha$     | $r^2$  | $\alpha'$ | Observations |
|------------------------|---------------|--------|------------|--------------|
| BoundaryLength         | 0.52 ± 0.02   | 0.88   | 0.50       | 237          |
| ArableLandArea         | 1.06 ± 0.07   | 0.81   | 0.95       | 218          |
| NaturalResources       | 0.20 ± 0.02   | 0.62   | 0.20       | 237          |
| Population             | 0.77 ± 0.07   | 0.71   | 0.67       | 212          |
| MilitaryAgePopulation  | 0.75 ± 0.09   | 0.64   | 0.63       | 162          |
| LaborForce             | 0.84 ± 0.08   | 0.68   | 0.72       | 226          |
| AgriculturalValueAdded | 0.79 ± 0.07   | 0.70   | 0.69       | 207          |
| Airports               | 0.57 ± 0.04   | 0.73   | 0.52       | 231          |
| RoadLength             | 0.78 ± 0.05   | 0.78   | 0.71       | 227          |

261 regressions are computed. The regressions with significant goodness (the coefficient of determination $r^2$ is higher than 0.6, where $r$ is calculated as the correlation coefficient between two variables $X$ and $M$ [24]) are shown in tables in next section. We also used the similar method to investigate the regressions of intensive variables with respect to GDP per capita, the results with $r^2$ larger than 0.4 are kept. Because some of the properties of some countries may be absent, the number of observations is smaller than 273 in some regressions.

3. Results

3.1. Allometric Scaling with respect to Area

Countries are geometric regions that have relatively clear boundaries in the geographic spaces, therefore, area is an important property which can be treated as the geographic size of the countries. We denote area as $M$, and other properties as $X$ to regress on log-log coordinate with the methods mentioned in the previous section. The results are shown in table 1.

Notice that all exponents in table 1 are smaller than 1 except Arable Land Area which should be proportional to areas of countries. That means
all properties have sub-linear allometries with respect to geometric size. As the area increases, the average value of population and number of natural sources etc. per area decreases because more barren places can be observed in countries. The sub-linear allometries with respect to area implies the expansion of territory can not give countries proportional returns.

The first property, boundary length of the countries has an about 1/2 power law relation with area. This is the obvious geometric fact which can prove the validity of our method. The allometry between the number of different natural resources and area can be compared to the area-species relationship in ecology [25]. The number of different natural resources in a country which corresponds to the number of different species found in an island indicates the diversity of resources. And the area of country corresponds to the area of island. Therefore, this power law relationship is the extension of area-species relation in country scale. Ecologists found that the power law exponent of species-area relation is from 0.15 to 0.4 [26] which is coincident to the number of natural resource-area power law here.

Another interesting fact is the allometric scaling relation between population and area of countries. The exponent of this allometry is approaching 3/4 which implies a fractal pattern of human being dwelling. This allometry reminds us the allometric scaling of river basins [27–29]. It is interesting to compare the population-area allometry in cities and countries. Studies on cities showed that the exponent was larger than 1 [14], which means the population density (population per area) increases with city area. Nevertheless, the allometry with the exponent being smaller than 1 suggests that the population density decreases with areas of countries. This conclusion is coincident to our observations that big countries always have less dense population.

3.2. Allometric Scaling with respect to Population

Population is an important property for allometries in cities. We also studied the allometric scaling relations between population and other properties of countries. The results are shown in table 2.

One of the obvious facts is all properties (except boundary length) have super linear or linear allometric scaling relationships with respect to population. We know that the geographic, communication related and energy related properties have super linear relation as well as the economic, trade related and transportation properties have approximately linear relations with
Table 2: Allometric Scaling Relations with respect to Population (Exponent $\alpha$ is estimated by MA method and $\alpha'$ is estimated by OLS method). Only the allometries with $r^2 > 0.6$ are shown here.

| Property             | $\alpha$    | $r^2$  | $\alpha'$ | Observations |
|----------------------|-------------|--------|-----------|--------------|
| Area                 | 1.30 ± 0.11 | 0.71   | 1.05      | 212          |
| BoundaryLength       | 0.61 ± 0.07 | 0.61   | 0.53      | 212          |
| ArableLandArea       | 1.31 ± 0.08 | 0.82   | 1.16      | 205          |
| IrrigatedLandArea    | 1.66 ± 0.19 | 0.63   | 1.20      | 173          |
| MilitaryAgePopulation| 1.00 ± 0.02 | 0.99   | 1.00      | 161          |
| GDPAtParity          | 1.03 ± 0.08 | 0.75   | 0.89      | 212          |
| LaborForce           | 1.02 ± 0.03 | 0.97   | 1.00      | 205          |
| AgriculturalValueAdded| 1.01 ± 0.05 | 0.88   | 0.95      | 207          |
| HouseholdConsumption | 1.03 ± 0.11 | 0.64   | 0.82      | 206          |
| TotalConsumption     | 1.01 ± 0.11 | 0.62   | 0.79      | 208          |
| CarbonEmission       | 1.30 ± 0.17 | 0.60   | 0.95      | 153          |
| CellularPhones       | 1.17 ± 0.09 | 0.78   | 1.02      | 211          |
| InternetUsers        | 1.16 ± 0.13 | 0.61   | 0.88      | 208          |
| RoadLength           | 1.00 ± 0.06 | 0.82   | 0.90      | 210          |
respect to population. Many economic-related properties such as GDP, government debt are filtered out because their $r^2$s are smaller than 0.6. However, we know that the economic-related properties such as GDP, number of new patents always have super-linear relation with population in cities [12]. To make the differences between cities and countries clearer, we list the comparisons of several properties with respect to population between cities and countries in table 3.

Table 3: Comparison of Countries and Cities in Allometric Scaling Relations with respect to Population. ($\alpha$, observations and $r^2$ are parameters of countries. $\alpha$ in cities are from [12] and [14].)

| Property                  | $\alpha$ in cities | $\alpha$ | $r^2$ | Observations |
|---------------------------|--------------------|---------|------|-------------|
| Area                      | 0.33-0.91          | 1.30±0.11 | 0.71 | 212         |
| GDP                       | 1.13-1.26          | 1.04±0.12 | 0.59 | 212         |
| Electricity Consumption   | 1.07               | 1.45±0.19 | 0.54 | 202         |
| HIV AIDS Population       | 1.23(New cases)    | 1.88±0.37 | 0.40 | 163         |
| Household Consumption     | 1.00-1.05(Household electricity/water consumption) | 1.03±0.11 | 0.64 | 206         |
| Oil Consumption           | 0.79(Gasoline sales) | 1.05±0.13 | 0.56 | 201         |

We can see that the exponents of countries are very different from cities. Especially, the exponents of GDP (which is not appear in table 2 because its $r^2$ is smaller than 0.6) is around 1. That means the economic activities per capita doesn’t increase with population. The electricity consumption and oil consumption have larger exponents than cities which means the energy consumption per capita increases with population in countries.

Another significant phenomenon in table 3 is the variances (the width of credit interval) of $\alpha$ are very large, as well as the coefficients of determination are small in countries. (The variances of exponents in cities are much more
smaller than countries according to \[12\]). That means there are large uncertainties on exponent estimations in countries. Therefore, population may be not an appropriate proxy of size of countries as cities. We will further discuss this problem in sub-sections 3.3 and section 4.

### 3.3. Allometric Scaling with respect to GDP

We study the allometric scaling relations with respect to economic sizes of the countries by the same method, and the results are shown in table 4.

Table 4: Allometric Scaling Relations with respect to GDP (Exponent $\alpha$ is estimated by MA method and $\alpha'$ is estimated by OLS method). Only the allometries with $r^2$ larger than 0.6 are shown. GDP data in 1998 is used to show the relations of Radio Stations and Television Stations with respect to GDP because we have only the data of these two properties in 1998 (See appendix)

| Property                        | $\alpha$  | $r^2$  | $\alpha'$ | Observations |
|---------------------------------|-----------|--------|-----------|--------------|
| GDPAtParity                     | 1.02 ± 0.03| 0.96   | 0.99      | 230          |
| GovernmentDebt                  | 1.10 ± 0.09| 0.85   | 1.00      | 120          |
| GovernmentExpenditures          | 1.02 ± 0.04| 0.93   | 0.98      | 224          |
| GovernmentReceipts              | 1.04 ± 0.04| 0.93   | 1.00      | 225          |
| GovernmentSurplus               | 1.06 ± 0.07| 0.78   | 0.93      | 217          |
| LaborForce                      | 1.02 ± 0.09| 0.70   | 0.85      | 223          |
| NationalIncome                  | 0.99 ± 0.01| 1.00   | 0.99      | 208          |
| AgriculturalValueAdded          | 0.96 ± 0.08| 0.73   | 0.83      | 207          |
| ConstructionValueAdded          | 0.99 ± 0.03| 0.96   | 0.97      | 208          |
| FixedInvestment                 | 0.98 ± 0.02| 0.97   | 0.97      | 207          |
| GovernmentConsumption           | 1.00 ± 0.04| 0.91   | 0.96      | 207          |
| HouseholdConsumption            | 0.98 ± 0.02| 0.98   | 0.97      | 206          |
| IndustrialValueAdded            | 1.18 ± 0.03| 0.96   | 1.15      | 208          |
| InventoryChange                 | 1.26 ± 0.13| 0.67   | 0.99      | 193          |
| ManufacturingValueAdded         | 1.22 ± 0.04| 0.95   | 1.19      | 208          |
| MiscellaneousValueAdded         | 1.05 ± 0.03| 0.96   | 1.03      | 208          |
| TotalConsumption                | 0.97 ± 0.02| 0.98   | 0.96      | 208          |
| TradeValueAdded                 | 1.00 ± 0.02| 0.97   | 0.98      | 207          |
| TransportationValueAdded        | 0.99 ± 0.03| 0.96   | 0.97      | 207          |

Continued on next page
### Table:

| Property                              | Exponent ± Standard Deviation | Exponential Coefficient | Allometric Coefficient | Value |
|---------------------------------------|-------------------------------|-------------------------|------------------------|-------|
| ExportValue                          | 1.09 ± 0.05                   | 0.90                    | 1.03                   | 208   |
| ImportValue                           | 0.95 ± 0.04                   | 0.92                    | 0.92                   | 209   |
| CurrentAccountBalance                 | 0.97 ± 0.07                   | 0.84                    | 0.89                   | 163   |
| ForeignExchangeReserves               | 1.00 ± 0.08                   | 0.80                    | 0.89                   | 155   |
| ExternalDebt                          | 1.17 ± 0.07                   | 0.86                    | 1.07                   | 201   |
| ElectricityConsumption                | 1.25 ± 0.09                   | 0.80                    | 1.09                   | 215   |
| ElectricityProduction                 | 1.16 ± 0.07                   | 0.85                    | 1.05                   | 213   |
| OilConsumption                        | 0.95 ± 0.05                   | 0.89                    | 0.90                   | 211   |
| OilImports                            | 0.92 ± 0.07                   | 0.77                    | 0.81                   | 200   |
| CarbonEmission                        | 1.10 ± 0.06                   | 0.89                    | 1.03                   | 153   |
| PhoneLines                            | 0.98 ± 0.05                   | 0.88                    | 0.92                   | 229   |
| CellularPhones                        | 1.14 ± 0.07                   | 0.83                    | 1.03                   | 223   |
| RadioStations*                        | 0.66 ± 0.06                   | 0.69                    | 0.58                   | 205   |
| TelevisionStations*                  | 0.75 ± 0.08                   | 0.65                    | 0.64                   | 197   |
| InternetUsers                         | 1.06 ± 0.06                   | 0.85                    | 0.97                   | 221   |
| RoadLength                            | 0.95 ± 0.09                   | 0.69                    | 0.80                   | 223   |

Interestingly, most economic related properties have exponents around 1, they increase proportionally with GDP. However, some properties such as government debt, industrial value added, inventory change, manufacturing added values have super-linear relations with respect to GDP because their exponents are larger than 1 significantly.

In trade-related properties, export value and external debt have super-linear relations, while import value has sub-linear relation with respect to GDP. If we treat the economy of a whole country as a money flow system, then GDP is the body size of this system, and import value dedicating money out-flow is just the metabolism of the whole system. Hence the allometric scaling relationship between import value and GDP is the correspondence of Kleiber law in money flow\[7\]. However, the fact that the exponent of this allometry is 0.95 needs more careful explanations.

For the energy-related properties, the electricity consumption, production and the carbon emission which are important measures of the metabolism of countries grow faster than GDP. However, the oil consumption and import all have sub-linear allometric scaling relations with respect to GDP. That
means the economy depends more on electricity power rather than gas or oil consumptions as GDP increases.

Similarly, in the category of communication, radio stations, television stations, phone lines have sub-linear relation with respect to GDP, nevertheless, cellular phones, internet hosts and users have super linear or linear relations with respect to GDP which also means the variables with super linear relations grows faster than those with sub linear relations.

3.4. Population and GDP

Among the allometries we have discussed in previous sections, the most important one is the allometry between population and GDP because this relation links not only natural properties and economic characteristics of countries but also demographic size and economic size of countries. However, this allometry does not appear in table 2 because the value of $r^2$ between these two variables is very small. Is it possible to improve the fitting goodness? We will discuss this problem in this sub-section.

At first, we can use GDP at parity instead of GDP because GDP at parity measures the real purchasing power of one economy. More improvement can be made by using labor force property instead of the population property because the labor force fraction of population can really contribute to the macro-economy. And further improvement can be made by calculating the employed labor force (labor force $\times (1-unemployment\ fraction)$) rather than labor force. These results are shown in figure 2.

As shown in figure 2 from (a) to (d), data become more concentrated with larger $r^2$ values when population and GDP are replaced by more appropriate properties employed labor force and GDP at parity. Meanwhile, the exponent also decreases and converges to 1 when suitable properties are used. We can conclude that GDP increases proportional with population. This is very different from the super-linear relations in cities [12, 16].

3.5. Allometric scaling of intensive variables

Although we have investigated allometric scaling of various variables with different types of body size of countries, all variables are just extensive ones. According to statistical physics, the intensive variables can reflect the state of systems in average. Here, we also show the allometries of intensive variables.

We investigate the allometric scaling of intensive variables with respective to GDP per capita. Some intensive properties have scaling relation with
Figure 2: Improvements of Allometric Scaling Relations between Population and GDP

Table 5: Allometric Scaling Relations between Intensive Variables and GDPPerCapita

| Property               | $\alpha$  | $r^2$  | $\alpha'$ | Observations |
|------------------------|-----------|--------|-----------|--------------|
| LiteracyFraction       | 0.11 ± 0.02 | 0.40   | 0.11      | 204          |
| BirthRateFraction      | -0.24 ± 0.03 | 0.58   | -0.23     | 208          |
| LifeExpectancy         | 0.08 ± 0.01 | 0.48   | 0.08      | 207          |
| MedianAge              | 0.14 ± 0.02 | 0.63   | 0.14      | 208          |
| TotalFertilityRate     | -0.21 ± 0.03 | 0.49   | -0.20     | 208          |
| InfantMortalityFraction| -0.58 ± 0.05 | 0.71   | -0.53     | 206          |
The relations between birth rate fraction, median age and GDP per capita deserve more attention. In [30], the authors predicted that the power law relation between fertility and energy consumption per capita for U.S. in different years has the exponent -1/3 according to the Kleiber’s law. Energy consumption per capita was regarded as the proxy of metabolism of humans in their work [30]. In fact, we didn’t find the relevant fact by using energy consumption per capita (we’ve tested Oil consumption, electrical consumption and carbon emission per capita). However, GDP per capita as the important economic index may have a deep connection with metabolism of individuals.
4. Discussion

4.1. Comparisons between Regression Methods

In this paper, the main conclusions are drawn according to the major axis regression method. However, we also listed the estimated allometric scaling power law exponents in tables 1, 2, 4 and 5. Because MA method tries to minimize the total distances from data to the regression line in both directions but not only the $y$ direction, it considers the uncertainties both from independent variable and dependent variable\[^22\]. The difference between $\alpha$ and $\alpha'$ decrease with the correlation degree between two variables which is measured by $r^2$. Sometimes the super linear or linear allometric scaling relations may be underestimated by OLS method as sub linear relations (see table 2) when $r^2$ is large.

Another advantage of MA method is it always gives compatible exponent estimations when independent variable and dependent variable are interchangeable. For example, the power law relation between area and population in table 1 is $Y \sim X^{0.77}$, where $Y$ stands for population and $X$ stands for area. In table 2, we have another power law relation, $X \sim Y^{1.30}$. These two allometric scaling relations are compatible because the product of exponents $0.77 \times 1.30 = 1.001 \approx 1$. However, we know that $\alpha'$ is 0.67 in table 1 and 1.05 in 2. But, the product of these two exponents $0.67 \times 1.05 = 0.7035 \approx 0.71 = r^2$. Therefore, we can not get compatible exponents by OLS method.

As we know, the OLS method assumes that the dependent variable, i.e., $\log(M)$ is measured without errors to give a reasonable estimation of $\alpha$. However, the purpose of this paper is to find possible allometric scaling relations between macro-variables of countries but not to predict some properties according to a given allometric scaling relation, so MA method is more suitable than OLS method. Second, MA method is symmetric and regardless of which variable is independent variable and which is dependent one. In this paper, it is necessary to compare one pair of variables with different orders. So MA method is preferable\[^22\]. However, we should also point out that MA method can not be abused anywhere because it may have its own limitation that is the variances of dependent and independent variables should be equal\[^31\]. In this paper, we assume this condition is satisfied, that is most variables have similar measurement errors. Therefore, MA method can be applied to give more appropriate estimations.
4.2. Similarities and Differences between Cities and Countries

Allometric scaling is a very ubiquitous law in complex systems. Lots of macro-variables exhibit power law relations both in cities and countries. It implies that some common mechanisms govern the development of these complex systems. There are several possible explanations for universal power law relations in the literatures. First, self-affine fractals may explain the allometric scaling between two variables. As pointed by [32, 33], cities are self-affine fractals, therefore, the allometric scaling exponent between two variables is the ratio of two fractal dimensions. This conclusion may also suitable for countries.

Another possible explanation is that self-similar transportation networks may exist widely in organisms, cities and countries [10, 13, 27]. Supply networks such as traffic networks and electricity grids resembling the vascular networks in organisms determine the power law relations between size of cities or countries and energy consumption [13, 34]. However, we find it is week to explain other ubiquitous allometric scaling relations, such as population and GDP, where the transportation networks are hard to be found.

Finally, we assert another kind of possible mechanism can explain a large varieties power law relations both in cities and countries. The ubiquitous power law relations imply different cities or countries may have similar structures which can be described by similar distributions. For example, there are common wealth distribution patterns in different countries [35]. The similar distribution curves may determine the power law relation between population and GDP. Some studies have found the relation between power law distributions and relations in language [36] and family names [37].

However, there are many differences between cities and countries. First, the allometric scaling relations in countries are less significant than cities because countries as units of research are more different and heterogenous. For example, some developing countries with large population are less urbanized than developed countries, therefore, their economic structures, cultural conventions and living styles are very different. However, in studies of cities, researchers always select cities with similar economic or geographic conditions [12, 38]. That is the reason why the allometries in countries are more uncertain and less significant than cities.

Second, the exponent of the same allometric scaling relations are different between countries and cities. For example, in table 3, the exponent of population and GDP is 1.04 for countries. However, it is about 1.20 for cities. One of the possible explanations is cities are more free for the flows
than countries. As we all know, people, money and materials can flow freely and quickly between different cities in one country. Consequently even the boundaries of cities can change dynamically\cite{33, 38}. However, people, money and goods flows are much weaker between countries. And the boundaries of countries are almost unchanged. Hence countries are rigid to self organize. This may explain the differences of exponents between cities and countries.

5. Concluding Remarks

In this paper, we used MA method to find the allometric scaling relationships among numerical macro-variables in different countries. By systematic studies, we found some interesting facts.

In general, the results reveal that economic, trade, energy consumption and communication related properties have significant allometric scalings with respect to GDP of countries (see table 4). And the geographic, demographic, and natural resource related properties have significant scaling relations with respect to area or population of countries (see table 1 and 2).

Many differences between countries and cities are revealed by the studies of allometric scaling relations with respect to population. One reason is countries are more heterogenous than cities. The other possible reason is the mobility between cities is much larger than countries.

We also studied the allometric scaling between intensive variables and GDP per capita. Some regularities and connections with metabolic theory are pointed out.

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Appendix A. Demonstration of Properties

Appendix Table A.6 lists properties of countries. The first column is the name of the property in Mathematica, the second column is the explanation of the property. The third column is its category of the property (Totally 11 categories). And the fourth column denoting whether the property is intensive (I) or extensive (E) one. The last column is the time (in which year)
that the property refers to. The stars behind the numbers indicate that most but not all data of this property are from one single year for all countries. All these properties can be obtained from Mathematica except property of "carbon emission". Actually, there are totally 225 different properties in "CountryData" command of Mathematica. However, most of properties, such as the flag, or the capital city of a country, are not numerical. So we selected only numerical properties to study. Furthermore, because some numerical properties are so similar that they can give identical allometric scaling relations (such as military age male population and military age female population, etc.), we only selected some of the representative properties to study. Finally, 92 properties which appeared in the main text are shown.

Table A.6: List of Country Properties

| Property               | Explanation                                      | Category                  | E/I | Year   |
|------------------------|--------------------------------------------------|---------------------------|-----|--------|
| Area                   | Total country area in square kilometers          | Geographical              | E   | -      |
| Boundary Length        | Total large-scale boundary length in kilometers  | Geographical              | E   | -      |
| ArableLand Area        | Arable land area in square kilometers            | Natural resources and features | E   | 2005   |
| IrrigatedLand Area     | Irrigated land area in square kilometers         | Natural resources and features | E   | 2003*  |
| Natural Resources Population | Estimated population                          | Natural resources and features | E   | -      |
| HIV AIDS Population    | Total population infected with HIV               | Public health             | E   | 2006   |
| Military Fit Population| Total population considered fit for military service | Military-related        | E   | 2005   |
| GDP                    | GDP in U.S. dollars at official exchange rate    | Economic-related         | E   | 2006   |
| GDP At Parity          | GDP in U.S. dollars at purchasing power parity   | Economic-related         | E   | 2007*  |
| Government Debt        | Outstanding government debt in U.S. dollars      | Economic-related         | E   | -      |
| Government Expenditures| Annual government expenditures in U.S. dollars   | Economic-related         | E   | 2005*  |

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| Category               | Description                                                                 | Relationship      | Year |
|------------------------|-----------------------------------------------------------------------------|-------------------|------|
| Government Receipts    | Annual government receipts in U.S. dollars                                   | Economic-related  | E    |
| Government Surplus     | Annual government surplus in U.S. dollars                                    | Economic-related  | E    |
| Labor Force            | Size of adult labor force, whether employed or not                          | Economic-related  | E    |
| National Income        | National income in U.S. dollars at official exchange rate                    | Economic-related  | E    |
| Agricultural Value Added| Value added by agricultural activities                                       | Economic-related  | E    |
| Construction Value Added| Value added by construction and real estate activities                     | Economic-related  | E    |
| Fixed Investment       | Investment in fixed capital                                                  | Economic-related  | E    |
| Government Consumption | Annual government consumption                                               | Economic-related  | E    |
| Household Consumption  | Annual household consumption                                                 | Economic-related  | E    |
| Industrial Value Added | Value added by all industrial activities                                   | Economic-related  | E    |
| Inventory Change       | Annual change in the value of inventories                                   | Economic-related  | E    |
| Manufacturing Value Added| Value added by manufacturing industries                                     | Economic-related  | E    |
| Miscellaneous Value Added| Value added by miscellaneous service and other activities                   | Economic-related  | E    |
| Total Consumption      | Total consumption expenditure                                               | Economic-related  | E    |
| Trade Value Added      | Value added by wholesale and retail trade                                    | Economic-related  | E    |
| Transportation Value Added| Value added by transportation and communications                             | Economic-related  | E    |
| Export Value           | Total estimated value of annual exports in U.S. dollars                     | Trade-related     | E    |

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| Category                  | Description                                                                 | Type                  | Year |
|---------------------------|-----------------------------------------------------------------------------|-----------------------|------|
| Import Value              | Total estimated value of annual imports in U.S. dollars                     | Trade-related         | 2006 |
| Current Account Balance   | Current account trade balance in U.S. dollars                               | Trade-related         | 2007*|
| Foreign Exchange Reserves | Government reserves of currency and gold in U.S. dollars                   | Trade-related         | 2007*|
| External Debt             | Total government foreign currency debt, in U.S. dollars                    | Trade-related         | 2007*|
| Electricity Consumption   | Annual electricity consumption in kilowatt hours                            | Energy-related        | 2005*|
| Electricity Production    | Annual electricity output in kilowatt hours                                 | Energy-related        | 2005*|
| Oil Consumption           | Oil consumption in barrels per day                                         | Energy-related        | 2005*|
| Carbon Emission           | Total anthropogenic carbon dioxide emissions(Gg)                           | Energy-related        | 2005  |
| Phone Lines               | Number of telephone land lines in use                                      | Communications-related| 2006*|
| Cellular Phones           | Number of cellular phones in use                                           | Communications-related| 2006*|
| Radio Stations            | Total number of public radio stations                                      | Communications-related| 1998*|
| Television Stations       | Number of broadcast television stations                                    | Communications-related| 1997*|
| Internet Users            | Estimated number of internet users                                         | Communications-related| 2006*|
| Airports                  | Total number of airports                                                   | Transportation-related| 2007  |
| Road Length               | Total length of all roads in kilometers                                     | Transportation-related| 2004*|
| Literacy Fraction         | Fraction of adult population able to read and write                       | Culture-related       | 2003*|
| Population Growth         | Annual fractional growth in estimated population                          | Demographic           | 2005*|

Continued on next page
| Birth Rate Fraction | Average number of births per person per year | Demographic | I | 2008* |
|---------------------|---------------------------------------------|-------------|---|------|
| Life Expectancy     | Average life expectancy in years            | Demographic | I | 2008* |
| Median Age          | Median age of population in years           | Demographic | I | 2006  |
| Total Fertility Rate| Lifetime total average number of births per woman | Demographic | I | 2008* |
| Infant Mortality    | Fraction of births with infant mortality    | Public health | I | 2008* |
| Fraction            |                                             | Economic-related | I | 2006  |
| GDP PerCapita       | GDP normalized by population               | Economic-related | I | 2007* |
| Unemployment Fraction| Fraction of labor force unemployed         | Economic-related | I | 2007* |

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