Conformal bootstrap for the BFKL Pomeron

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Abstract

We calculate the interaction vertex of three BFKL states including the bare triple BFKL Pomeron coupling and discuss its relation with the correlation functions in two-dimensional conformal field theory. We construct the operator algebra of the fields interpolating the BFKL states and show that in the multi-color limit the vertex satisfies the constraints imposed by the conformal bootstrap on the structure constants of the operator product expansion in conformal field theory.

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1. Introduction

Recent studies of the high-energy scattering of heavy onium states within the BFKL approach indicate an intrinsic relation between the Regge asymptotics of scattering amplitudes in perturbative QCD and two-dimensional conformal field theories [1].

The BFKL pomeron appears in the leading logarithmic approximation as a color-singlet compound state of two interacting reggeized gluons, or reggeons, propagating in the t-channel. The BFKL equation for the compound reggeon states takes the form of the Schrödinger equation with the hamiltonian \( H_{\text{BFKL}} \) acting on the 2-dimensional transverse coordinates of the reggeons \( \rho = (\rho_x, \rho_y) \). One of the remarkable properties of the BFKL kernel \( H_{\text{BFKL}} \) is that it is invariant under projective \( SL(2, \mathbb{C}) \) transformations of the holomorphic and antiholomorphic reggeon coordinates

\[
    z = \rho_x + i\rho_y, \quad \bar{z} = z^*, \quad z \rightarrow az + b \\
    c\bar{z} + d, \quad ad - bc = 1.
\]  

(1.1)

This property allows to classify the solutions of the BFKL equation according to the (unitary) principal series representations of the \( SL(2, \mathbb{C}) \) group and express the wave function \( E_{h,\bar{h}}(\rho_{10}, \rho_{20}) \) of the compound state of two reggeons with the coordinates \( \rho_1 \) and \( \rho_2 \) in the form of three-point correlation function in 2-dimensional conformal field theory [1]

\[
    E_{h,\bar{h}}(\rho_{10}, \rho_{20}) = \left( \frac{z_{12}}{z_{10}z_{20}} \right)^h \left( \frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}} \right)^{\bar{h}} \langle \phi_{0,0}(z_1, \bar{z}_1)\phi_{0,0}(z_2, \bar{z}_2)O_{h,\bar{h}}(z_0, \bar{z}_0) \rangle,
\]

(1.2)

with \( \rho_{jk} = \rho_j - \rho_k \) and \( \rho_0 \) being the center-of-mass coordinate of the compound state. The operators \( \phi_{0,0}(z_1, \bar{z}_1) \) and \( \phi_{0,0}(z_2, \bar{z}_2) \) represent reggeized gluons and have zero conformal weights. One associates the 2-reggeon compound state with the local operator \( O^{(2)} = O_{h,\bar{h}}(z_0, \bar{z}_0) \) carrying the quantum numbers of the state. The conformal weights \( h \) and \( \bar{h} \) of the BFKL state take the form

\[
    h = \frac{1+n}{2} + i\nu, \quad \bar{h} = 1 - h^*, \quad n = \mathbb{Z}, \quad \nu = \mathbb{R},
\]

(1.3)

where \( h - \bar{h} = n \) and \( h + \bar{h} = 1 + 2i\nu \) are integer conformal spin and scaling dimension of the state, respectively. It follows from (1.2) that under \( SL(2, \mathbb{C}) \) transformations of the coordinates (1.1) the operator \( O_{h,\bar{h}}(z, \bar{z}) \) is transformed as

\[
    O_{h,\bar{h}}(z_0, \bar{z}_0) \rightarrow (cz_0 + d)^{2h}(\bar{c}\bar{z}_0 + \bar{d})^{2\bar{h}}O_{h,\bar{h}}(z_0, \bar{z}_0)
\]

(1.4)

and it can be identified [2] as a quasiprimary field in two-dimensional conformal field theory [3]. The BFKL pomeron appears as the state (1.2) with the quantum numbers \( h = \bar{h} = \frac{1}{2} \) corresponding to the ground state energy of the hamiltonian \( -H_{\text{BFKL}} \).

The Regge asymptotics of the scattering amplitude of two onia with mass \( M \) and large center-of-mass energy \( s \) is controlled by the propagator of 2-reggeon compound states, which has the following form in the impact parameter space [1]

\[
    f_Y(\rho_1, \rho_2; \rho'_1, \rho'_2) = \sum_{h \neq 0,1} e^{Y\omega(h)} \left| \frac{h - \frac{1}{2}}{h(h - 1)} \right|^2 G_{h,\bar{h}}(\rho_1, \rho_2; \rho'_1, \rho'_2)
\]

(1.5)

with \( Y = \ln \frac{s}{4\pi M} \gg 1 \). Here, the summation goes over all BFKL states (1.2) except of two degenerate states with the conformal weights \( (h = 1, \bar{h} = 0) \) and \( (h = 0, \bar{h} = 1) \). The sum over
\( h \) is understood as an integral over continuous \(-\infty < \nu < \infty\) and sum over integer spins \( n \). The energy of the BFKL state is defined as an eigenvalue of the BFKL Hamiltonian \( \mathcal{H}_{\text{BFKL}} \)

\[
\omega(h) = \frac{2\alpha_s N_c}{\pi} \text{Re} [\psi(1) - \psi(h)] ,
\]

with \( \psi(h) = \frac{d}{dh} \ln \Gamma(h) \). The Green function \( G_{h,h}(\rho_1, \rho_2; \rho_1', \rho_2') \) depends on the anharmonic ratios of the reggeon coordinates \( x = \frac{z_{12}\nu z_{23}}{z_{13}^2\nu^2} \) and \( \bar{x} = x^* \) and it can be expressed in terms of the hypergeometric functions as

\[
G_{h,h}(\rho_1, \rho_2; \rho_1', \rho_2') = \int d^2 \rho_0 E_{h,h}^*(\rho_1', \rho_2') E_{h,h}(\rho_1, \rho_2) = \frac{x^h \bar{x}^h}{B(1 - h)} \Gamma(h, \bar{h}; 2\bar{h}; \bar{x}) + (h \leftrightarrow 1 - h)
\]

where \( \bar{h} = 1 - h^* \) and the function \( B(h) \) is defined below in (2.7).

The scattering amplitude \( f_N(\rho_1, \rho_2; \rho_1', \rho_2') \) is invariant under projective transformations of the reggeon coordinates \( \mathbb{C}_{1,3} \) and apart from the \( Y \)-dependent factor it resembles an expansion of the 4-point correlation function of reggeon fields \( \phi_{0,0} \) over the conformal blocks \([3]\) in the channel \( 1 + 2 \rightarrow 1' + 2' \). However, this identification is not valid since \( f_N(\rho_1, \rho_2; \rho_1', \rho_2') \) does not satisfy the condition of the crossing symmetry imposed on the 4-point correlation functions in conformal field theory.

To preserve unitarity of the scattering amplitudes at very large energies one has to supplement the BFKL pomeron with unitarity corrections associated with the diagrams containing an arbitrary number of reggeized gluons in the \( t \)-channel \([4]\). These corrections can be implemented in two steps.

At the first step, one restores the unitarity of the scattering amplitudes only in the direct channels by taking into account the diagrams with a conserved number \( n \geq 2 \) of reggeized gluons in the \( t \)-channel. In this case, similar to the BFKL pomeron, the high-energy asymptotics is controlled by the \( n \)-reggeon compound states, which satisfy the Schrödinger equation with the Hamiltonian describing a pair-wise interaction of \( n \) reggeons through the BFKL kernel \([3]\). The \( n \)-reggeon Schrödinger equation inherits the \( SL(2, \mathbb{C}) \) symmetry of the BFKL equation and, moreover, in the multicolor limit, \( N_c \to \infty \), it acquires additional hidden symmetry due to the fact that the \( n \)-reggeon Hamiltonian becomes equivalent to the completely integrable XXX Heisenberg magnet model \([3, 4]\). This symmetry becomes large enough for the Schrödinger equation to be solved exactly by means of the Quantum Inverse Scattering Method \([3, 2]\). In particular, similar to the BFKL states, one can construct the quasiprimary fields \( \phi_{h_k, \zeta_k}(z_k, \bar{z_k}) \) interpolating the \( n \)-reggeon states and define their wave function as the \((n + 1)\)-point correlation function of this field with \( n \) additional reggeon fields \( \phi(z_k, \bar{z_k}) \).[2]

At the second step, one restores unitarity of the scattering amplitudes in the subchannels by including the interaction vertices which change the number of reggeons. The simplest example of such vertex is the transition kernel from 2 to 4 reggeized gluons \( V_{(2,4)} \) calculated in \([4, 5]\). Remarkably enough, \( V_{(2,4)} \) vertex is also invariant under projective transformations \([5]\) and it is natural to expect that the \( SL(2, \mathbb{C}) \) invariance is the general feature of all reggeon transition vertices \([10]\). Inclusion of reggeon transition vertices turns quantum mechanical description of the \( n \)-reggeon compound states into interacting quantum field theory.

The reggeon number changing transition kernels induce the interaction between \( n \)-reggeon compound states and the corresponding interaction vertices can be calculated by projecting.
the transition kernels on the wave functions of the compound states \([11]\). The resulting expressions depend on the center-of-mass coordinates and the conformal weights of the states. The \(SL(2,\mathbb{C})\) invariance of the transition kernels implies that the properties of the interaction vertices under projective transformations \([11]\) are determined by the properties of the interacting states. This suggests to identify the interaction vertices with the correlation functions \([11]\), as follows:

\[
V(\alpha \to \beta, \gamma) = \int \prod_{j=1',2',3',4'} d^2 \rho_j \, V_{(2,4)}(\rho_1', \rho_2', \rho_1, \rho_2, \rho_3, \rho_4) \times E_{h_\alpha \bar{h}_\alpha}(\rho_{\alpha'}, \rho_{\alpha''}) E_{h_\beta \bar{h}_\beta}(\rho_{\beta'}, \rho_{\beta''}) E_{h_\gamma \bar{h}_\gamma}(\rho_{\gamma'}, \rho_{\gamma''}),
\]

(1.8)

where integration goes over the coordinates of reggeons entering the BFKL states and the result should be compared with the following 3-point correlation function

\[
V_{\text{CFT}}(\alpha, \beta, \gamma) = \langle O_{h_\alpha \bar{h}_\alpha}(z_\alpha, \bar{z}_\alpha) O_{h_\beta \bar{h}_\beta}(z_\beta, \bar{z}_\beta) O_{h_\gamma \bar{h}_\gamma}(z_\gamma, \bar{z}_\gamma) \rangle.
\]

(1.9)

The same transition kernel \(V_{(2,4)}\) can be also projected on the wave functions of \(n = 2\) (or BFKL state) and \(n = 4\)-reggeon compound states defining the nondiagonal 2-point correlation function

\[
\langle O^{(2)}_{h_\alpha \bar{h}_\alpha}(z_\alpha, \bar{z}_\alpha) O^{(4)}_{h_\beta \bar{h}_\beta}(z_\beta, \bar{z}_\beta) \rangle.
\]

It is well known that in conformal field theory the functions \([11]\) determine the structure constants of the operator algebra of quasiprimary fields and they satisfy the set of constraints followed from the associativity condition of the operator algebra \([12, 3]\). Therefore, identification of the transition vertices \([1.8]\) as 3-point correlation functions \([1.9]\) becomes nontrivial since it imposes severe restrictions on the possible form of the vertex \(V(\alpha \to \beta, \gamma)\). In the present paper we study the relation between two different representations \([1.8]\) and \([1.9]\) of the transition vertex of three BFKL states. We calculate the explicit form of the vertex \(V(\alpha \to \beta, \gamma)\) and show that in the multi-color limit, \(N_c \to \infty\) and \(\alpha_s N_c = \text{fixed}\), it satisfies the constraints imposed by the conformal bootstrap \([12, 3]\).

The paper is organized as follows. In Sect. 2 we define the vertex \(V(\alpha \to \beta, \gamma)\) and consider its general properties. The results of calculations are summarized at the end of the section. In Sect. 3 we construct the operator product expansion of the operators \(O_{h, \bar{h}}(z, \bar{z})\) and identify \(V(\alpha \to \beta, \gamma)\) in the multi-color limit as the structure constants of the operator algebra. Appendices A and B contain the details of the Feynman diagram techniques used for calculation of the vertex in Sect. 2.

## 2. Transition vertex

Let us consider the vertex \([1.8]\) and replace the transition kernel \(V_{(2,4)}\) by its explicit expression in the configuration space \([3, 11]\). Straightforward calculation gives the following result \([11]\)

\[
V(\alpha \to \beta, \gamma) = \left( \frac{\alpha_s N_c}{\pi} \right)^2 16 h_\alpha (1 - \bar{h}_\alpha) \bar{h}_\alpha (1 - \bar{h}_\alpha) \times \left[ V_0(\alpha, \beta, \gamma) - \frac{2\pi}{N_c^2} V_1(\alpha, \beta, \gamma) \Re \{\psi(1) + \psi(h_\alpha) - \psi(h_\beta) - \psi(h_\gamma)\} \right],
\]

(2.1)
where the conformal weights of three BFKL states, \( h_\alpha, h_\beta \) and \( h_\gamma \), are of the form (1.3). Each pair of the reggeons coming out of three BFKL wave functions in (1.8) is in the color singlet state and according to the color flow inside the transition kernel one gets planar, \( V_0 \), and nonplanar, \( V_1 \), contributions.

In the multi-color limit, only the first term survives in (2.1) and it has the following form

\[
V_0(\alpha, \beta, \gamma) = \int \frac{d^2 \rho_0 d^2 \rho_1 d^2 \rho_2}{|\rho_{01} \rho_{12} \rho_{20}|^2} E_{h_\alpha \bar{h}_\alpha}(\rho_{0\alpha}, \rho_{1\alpha}) E_{h_\beta \bar{h}_\beta}(\rho_{1\beta}, \rho_{2\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{2\gamma}, \rho_{0\gamma}), \tag{2.2}
\]

where in the l.h.s. the label \( \alpha \) denotes the set of conformal weights \((h_\alpha, \bar{h}_\alpha)\) and center-of-mass coordinate \( \rho_\alpha \) of the BFKL state. Replacing the BFKL wave functions by their explicit expressions (1.2) one can represent \( V_0 \) as a planar 2-dimensional Feynman diagram shown in Fig. 1a. It is interesting to notice that the same integral contributes to the triple-dipole vertex [13] in the QCD dipole model [14]. The nonplanar term in (2.1) is suppressed by the color factor \( 1/N_c^2 \). It is given by

\[
V_1(\alpha, \beta, \gamma) = \int \frac{d^2 \rho_0 d^2 \rho_1}{|\rho_{01}|^4} E_{h_\alpha \bar{h}_\alpha}(\rho_{0\alpha}, \rho_{1\alpha}) E_{h_\beta \bar{h}_\beta}(\rho_{0\beta}, \rho_{1\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{0\gamma}, \rho_{1\gamma}) \tag{2.3}
\]

and it can be represented as a nonplanar 2-dimensional Feynman diagram shown in Fig. 1b.

Figure 1: The Feynman diagrams corresponding to the planar (a) and nonplanar (b) contribution to the vertex \( V(\alpha \to \beta, \gamma) \). Solid lines represent two-dimensional propagators \( \frac{1}{z_{h \bar{h}}} \) with the exponents \( h \) and \( \bar{h} \) depending on the line and the vertices do not bring any additional factors.

One checks that the functions (2.2) and (2.3) are transformed under \( SL(2, \mathbb{C}) \) transformations in the same way as the 3-point correlation function of quasiprimary fields, (1.9) and (1.4). This property allows to restore the coordinate dependence of \( V_0 \) and \( V_1 \)

\[
\begin{align*}
V_0(\alpha_1, \alpha_2, \alpha_3) &= \Omega(h_1, h_2, h_3) \times \prod_{i<j}(z_i - z_j)^{-\Delta_{ij}} (\bar{z}_i - \bar{z}_j)^{-\bar{\Delta}_{ij}} \\
V_1(\alpha_1, \alpha_2, \alpha_3) &= \Lambda(h_1, h_2, h_3) \times \prod_{i<j}(z_i - z_j)^{-\Delta_{ij}} (\bar{z}_i - \bar{z}_j)^{-\bar{\Delta}_{ij}} \tag{2.4}
\end{align*}
\]
where $\Omega$ and $\Lambda$ depend only on the conformal weights $h_j$ and $\bar{h}_j$ ($j = 1, 2, 3$) and
\begin{equation}
\Delta_{12} = h_1 + h_2 - h_3, \quad \Delta_{12} = \bar{h}_1 + \bar{h}_2 - \bar{h}_3,
\end{equation}
etc. Here and in what follows, we indicate explicitly the dependence of $\Omega$ and $\Lambda$ only on the holomorphic conformal weights $h$, $\bar{h}$ and $\gamma$, keeping in mind the relation between holomorphic and antiholomorphic conformal weights, $\bar{h} = 1 - h^*$. Substituting (2.4) into (2.1) one finds that the vertex $V(\alpha \to \beta, \gamma)$ has similar dependence on the center-of-mass coordinates and its dependence on the conformal weights is given by
\begin{equation}
V(h_\alpha; h_\beta, h_\gamma) = 16 \left( \frac{\alpha \cdot N_c}{\pi} \right)^2 |h_\alpha(h_\alpha - 1)|^2 \times \left( \Omega(h_\alpha, h_\beta, h_\gamma) - \frac{2\pi}{N_c} \Lambda(h_\alpha, h_\beta, h_\gamma) \right),
\end{equation}
in this expression, $h_\alpha$ is the conformal weight of the incoming BFKL state and the outgoing states have the conformal weights $h_\beta$ and $h_\gamma$.

### 2.1. Symmetry properties of the vertex

Let us consider some general properties of the vertex $V(\alpha \to \beta, \gamma)$ which become useful for its calculation. It follows from the definitions (2.2) and (2.3) as well as from the symmetry of the diagrams in Fig. 1 that the function $V_0$ is invariant under cyclic permutations of its arguments, while $V_1$ is a completely symmetric function. Then, using the symmetry property of the BFKL wave function (1.2)
\begin{equation}
E_{h_\alpha, h_\alpha}(\rho_0, \rho_1) = (-)^{n_\alpha} E_{h_\alpha, h_\alpha}(\rho_1, \rho_0)
\end{equation}
it is easy to show from (2.4) that $\Omega$ and $\Lambda$ are symmetric functions of the conformal weights of the BFKL states
\begin{equation}
\Omega(h_\alpha, h_\beta, h_\gamma) = \Omega(h_\beta, h_\alpha, h_\gamma) = \Omega(h_\gamma, h_\beta, h_\alpha).
\end{equation}
The function $\Lambda$ satisfies similar relations plus additional selection rule on the conformal spins of three BFKL states, $n_\alpha$, $n_\beta$ and $n_\gamma$,
\begin{equation}
\Lambda(h_\alpha, h_\beta, h_\gamma) = 0 \quad \text{for} \quad n_\alpha + n_\beta + n_\gamma = \text{odd},
\end{equation}
which one obtains by changing the integration variables in (2.3) as $\rho_0 \leftrightarrow \rho_1$ and applying the identity (2.7).

Another set of identities follows from the intertwining relation for the BFKL wave function (1)
\begin{equation}
(E_{h, h}(\rho_{10}, \rho_{20}))^* = E_{1-h, 1-h}(\rho_{10}, \rho_{20}) = B(h) \int d^2 \rho E_{h, h}(\rho_{10'}, \rho_{20'}) z_{00}^{2(h-1)} z_{00'}^{2(h-1)},
\end{equation}
which expresses the fact that the principal series representations of the $SL(2, \mathbb{C})$ group labelled by the conformal weights $(h, \bar{h})$ and $(1-h, 1-\bar{h})$ are unitary equivalent. Here, the prefactor is given by
\begin{equation}
B(h) = \frac{(-)^n}{\pi} \frac{\Gamma^2(h)}{\Gamma^2(1-h)} \frac{\Gamma(2 - 2h)}{\Gamma(2h - 1)} = \frac{(-)^n}{\pi} 2^{-4i\nu(n - 2i\nu)} \frac{\Gamma(\frac{1+n}{2} + i\nu) \Gamma(\frac{n}{2} - i\nu)}{\Gamma(\frac{1+n}{2} - i\nu) \Gamma(\frac{n}{2} + i\nu)}.
\end{equation}
As a result, one gets from (2.2) and (2.10)

$$(\Omega(h_\alpha, h_\beta, h_\gamma))^* = \Omega(1 - h_\alpha, 1 - h_\beta, 1 - h_\gamma)$$

(2.12)

and similar relation for the function $\Lambda$. Let us multiply the both sides of (2.2) by $z_{\gamma\gamma'}^{2(h_\gamma-1)} z_{\gamma\gamma'}^{2(h_\gamma-1)}$ and integrate with respect to the center-of-mass coordinate $\rho_\gamma$. Then, one applies the identity (2.10) in the r.h.s., performs 2-dimensional integration in the l.h.s. and replaces the function $V_0$ by its expression (2.3) to arrive at the following relation

$$\frac{\Omega(h_\alpha, h_\beta, 1 - h_\gamma)}{\Omega(h_\alpha, h_\beta, h_\gamma)} = (-)^{n_\gamma+n_\beta} \frac{\Gamma^2(1-h_\gamma^*) \Gamma(1+h_\alpha-h_\beta-h_\gamma)\Gamma(1-h_\alpha+h_\beta-h_\gamma)}{\Gamma^2(1-h_\gamma) \Gamma(1+h_\alpha^*-h_\beta^*-h_\gamma^*)\Gamma(1-h_\alpha^*+h_\beta^*-h_\gamma^*)}.$$  

(2.13)

Here, the ratio of $\Gamma$-functions is a pure phase and it vanishes for real values of the conformal weights. Let us consider two special cases of (2.13)

$$\frac{\Omega(h_\alpha, h_\beta, 1 - h_\gamma)}{\Omega(h_\alpha, h_\alpha, h_\gamma)} = 1, \quad \frac{\Omega(h_\alpha, h_\beta, 1)}{\Omega(h_\alpha, h_\beta, 0)} = \frac{h_\alpha - h_\beta}{h_\alpha^* - h_\beta^*}.$$

The same transformations been applied to $V_1$ lead to the same set of identities for the function $\Lambda$.

Let us show that the functions $V_0$ and $V_1$ satisfy certain bilinear relations. The latter follow from the completeness condition for the BFKL states [4]

$$\sum_h d(h) \int d^2 \rho_0 E_{h,h}(\rho_{10}, \rho_{20}) E_{-h,-h}(\rho_{10'}, \rho_{20'}) = |\rho_{12}|^2 |\rho_{12'}|^2 \delta^2(\rho_{12}) \delta^2(\rho_{22'}) ,$$

(2.14)

where integration goes over the center-of-mass coordinate and the summation over conformal weights, (1.3), goes as in (1.3). The prefactor $d(h)$ is related to the norm of the BFKL state and it is given by

$$d(h) = \frac{1}{(2\pi)^4} |h - \frac{1}{2}|^2 = \frac{1}{(2\pi)^2} |B(h)|^2 = \frac{1}{(2\pi)^2} B(h) B(1 - h) ,$$

(2.15)

with the function $B(h)$ defined in (2.11). To apply (2.14) one identifies two BFKL wave functions entering the l.h.s. of (2.14) as belonging to two different functions $V_0$. This leads to the following expression for the product of two functions $V_0$ summed over the quantum numbers of the “intermediate” BFKL states [4]

$$\sum_{h_{\gamma'}} d(h_{\gamma'}) \int d^2 \rho_\gamma V_0(\alpha, \beta, \gamma) V_0(1 - \gamma, \beta', \alpha')$$

(2.16)

$$= \int \frac{d^2 \rho_0 d^2 \rho_1 d^2 \rho_2 d^2 \rho_3}{|\rho_{01} \rho_{12} \rho_{23} \rho_{30}|^2} E_{h_\gamma h_\alpha}(\rho_{00}, \rho_{1\alpha}) E_{h_\beta h_\beta}(\rho_{1\beta}, \rho_{2\beta}) E_{h_\gamma' h_\gamma'}(\rho_{3\gamma'}, \rho_{3\gamma'}) E_{h_\alpha' h_\alpha'}(\rho_{3\alpha'}, \rho_{0\alpha'}) .$$

Here, the label $1 - \gamma$ denotes the BFKL state with the conformal weights $(1 - h_\gamma, 1 - \bar{h}_\gamma)$ and the center-of-mass coordinate $\rho_\gamma$. We notice that the r.h.s. of this relation is invariant under cyclic permutations of the BFKL states $\alpha$, $\beta$, $\beta'$ and $\alpha'$ leading to the following crossing symmetry property

$$\sum_{h_{\gamma'}} d(h_{\gamma'}) \int d^2 \rho_\gamma V_0(\alpha, \beta, \gamma) V_0(1 - \gamma, \beta', \alpha') = \sum_{h_{\gamma'}} d(h_{\gamma'}) \int d^2 \rho_\gamma V_0(\alpha', \beta, \gamma) V_0(1 - \gamma, \beta, \beta').$$

(2.17)
The relations (2.16) and (2.17) can be depicted as shown in Fig. 2. Repeating similar analysis one can show that the function $V_1$ also satisfies the same relation and at the same time it does not hold for the cross product of the functions $V_0$ and $V_1$. As we will show in Sect. 3.2 the bilinear relations (2.17) give rise to the associativity condition of the operator algebra of the quasiprimary fields $O_{h,h}$.

2.2. Nonplanar diagram

Let us start with the nonplanar contribution to the interaction vertex given by (2.3). Its calculation is simpler than that of the planar diagram, (2.2). Using the $SL(2,C)$ symmetry one chooses the center-of-mass coordinates of three BFKL states at

$$z_\alpha = \bar{z}_\alpha = 0, \quad z_\beta = \bar{z}_\beta = 1, \quad z_\gamma = \bar{z}_\gamma = \infty$$

(2.18)

and obtains from (2.3) and (2.4) the function $\Lambda$ as the following 2-loop integral

$$\Lambda(h_\alpha, h_\beta, h_\gamma) = \int \frac{d^2z_0d^2z_1}{|z_0|^4} \left( \frac{z_{01}}{z_0 \bar{z}_1} \right)^{h_\alpha} \left( \frac{\bar{z}_{01}}{1-z_0}(1-z_1) \right)^{\bar{h}_\alpha} \left( \frac{z_{01}}{1-z_0}(1-\bar{z}_1) \right)^{h_\beta} \left( \frac{\bar{z}_{01}}{(1-\bar{z}_0)(1-\bar{z}_1)} \right)^{\bar{h}_\beta} z_{01}^{h_\gamma} \bar{z}_{01}^{\bar{h}_\gamma}$$

(2.19)

where $h_\alpha = \frac{1+n_\alpha}{2} + i\nu_\alpha$, $\bar{h}_\alpha = 1 - h_\alpha^*$ etc. In this expression, $z = \rho_x + i\rho_y$ and $\bar{z} = \rho_x - i\rho_y$ are complex valued holomorphic and antiholomorphic reggeon coordinates and $d^2z = dx dy = \frac{i}{2}dz d\bar{z}$.

We present here the final result of the calculation of the integral (2.19) and refer to the Appendix A for the details. Applying the techniques described in Appendix A, one can express $\Lambda$ as a sum of two terms each given by the product of two contour integrals, $J$ and $\bar{J}$, over holomorphic and antiholomorphic coordinates, respectively. These two terms differ only by a sign factor $(-)^{n_\alpha + n_\beta + n_\gamma}$ and their sum can be represented as

$$\Lambda(h_\alpha, h_\beta, h_\gamma) = \pi^2 \frac{[1 + (-)^{n_\alpha + n_\beta + n_\gamma}]}{\Gamma^2(h_\alpha)\Gamma^2(h_\beta)\Gamma(2-h_\alpha-h_\beta-h_\gamma)} \times J(h_\alpha, h_\beta, h_\gamma) \times \bar{J}(\bar{h}_\alpha, \bar{h}_\beta, \bar{h}_\gamma).$$

(2.20)

1The integral entering the r.h.s. of this relation can be interpreted as the quadruple dipole vertex in the QCD dipole model [13].
In this form, \( \Lambda \) explicitly satisfies the selection rule (2.9). As shown in Appendix A, the integrals \( J \) and \( \bar{J} \) are given by the ratio of \( \Gamma \)-functions leading to the following result

\[
\Lambda(h_\alpha, h_\beta, h_\gamma) = \frac{\pi^2 \cos\left(\frac{\pi}{2}(n_\alpha + n_\beta + n_\gamma)\right)}{2^{h_\alpha + \bar{h}_\alpha + h_\beta + \bar{h}_\beta + h_\gamma - 4} \frac{\Gamma(1 - h_\alpha) \Gamma(1 - \bar{h}_\alpha) \Gamma(1 - \bar{h}_\beta) \Gamma(1 - h_\beta) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \bar{h}_\gamma + \bar{h}_\gamma)}{\Gamma(h_\beta) \Gamma(h_\gamma) \Gamma(\frac{3}{2} - \frac{1}{2}(h_\alpha + h_\beta + h_\gamma))} \times \frac{\Gamma(\frac{1}{2}(h_\alpha + \bar{h}_\beta - \bar{h}_\gamma)) \Gamma(\frac{1}{2}(-h_\alpha + \bar{h}_\beta + \bar{h}_\gamma)) \Gamma(\frac{1}{2}(h_\alpha - \bar{h}_\beta + \bar{h}_\gamma))}{\Gamma(1 - \frac{1}{2}(h_\alpha + h_\beta - h_\gamma)) \Gamma(1 - \frac{1}{2}(-h_\alpha + h_\beta + h_\gamma)) \Gamma(1 - \frac{1}{2}(h_\alpha - h_\beta + h_\gamma))}.
\]

(2.21)

One verifies that this expression satisfies the relations (2.8), (2.9) and (2.13). We recall that \((h_\alpha, \bar{h}_\alpha), (h_\beta, \bar{h}_\beta)\) and \((h_\gamma, \bar{h}_\gamma)\) are the conformal weights of three BFKL states defined in (1.3).

Let us consider the general expression (2.21) in two special cases. In the first case, one takes \( h_\alpha = h \) and \( h_\beta = \bar{h}_\gamma = 1 \). We notice that the BFKL states with the conformal weights \( h = 1 \) and \( \bar{h} = 0 \) do not contribute to the scattering amplitude (1.5). As we will argue in Sect. 3.1, according to their conformal properties, the same states can be formally associated with the states of reggeized gluons. Under this identification, the function \( \Lambda(h, 1, 1) \) measures the nonplanar coupling of two reggeized gluons to the BFKL state

\[
\Lambda(h, 1, 1) = \frac{2\pi^2}{h(h - 1)}, \quad \text{for } n = \text{even}
\]

(2.22)

and it vanishes for odd spins \( n \). In the second case, \( h_\alpha = h_\beta = h_\gamma = \frac{1}{2} \) (and as a consequence \( \bar{h}_\alpha = \bar{h}_\beta = \bar{h}_\gamma = \frac{1}{2} \)) corresponding to the coupling of three BFKL Pomeron one obtains from (2.21)

\[
\Lambda\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{2\pi^6}{\Gamma(\frac{5}{2})} = 378.145
\]

(2.23)

This expression determines the nonplanar contribution to the bare triple BFKL Pomeron vertex (2.9).

### 2.3. Planar diagram

Calculation of the planar diagram follows the same steps as that of the nonplanar diagram. Choosing the center-of-mass coordinates in (2.2) and (2.4) according to (2.18) one writes the function \( \Omega \) as the following 3-loop integral

\[
\Omega = \int \frac{d^2z_0 d^2z_1 d^2\bar{z}_2 (z_{01} \bar{z}_{01})^{h_\alpha} (z_{12} \bar{z}_{12})^{h_\beta} (z_{20} \bar{z}_{20})^{h_\gamma}}{|z_{01}z_{12}z_{20}|^2} \frac{z_{01}}{z_{01} z_{12}} \frac{z_{12}}{(1 - z_1)(1 - z_2)} \frac{z_{20}}{z_{12} z_{20}} \frac{\bar{z}_{01}}{z_{01} \bar{z}_{01}} \frac{\bar{z}_{12}}{(1 - \bar{z}_1)(1 - \bar{z}_2)} \frac{\bar{z}_{20}}{z_{12} z_{20}} \frac{\bar{h}_\alpha}{\bar{h}_\beta} \frac{\bar{h}_\gamma}{\bar{h}_\gamma}.
\]

(2.24)

Its calculation is performed in Appendix B. Separating integrals over holomorphic and antiholomorphic coordinates one can express \( \Omega \) as a sum of three terms each given by the product of two contour integrals depending separately on the holomorphic and antiholomorphic conformal weights

\[
\Omega = \pi^3 \left[ \Gamma^2(h_\alpha) \Gamma^2(h_\beta) \Gamma(1 - h_\alpha) \Gamma(1 - h_\beta) \Gamma(1 - h_\gamma) \right]^{-1} \sum_{a=1}^{3} J_a(h_\alpha, h_\beta, h_\gamma) \times \bar{J}_a(h_\alpha, h_\beta, h_\gamma).
\]

(2.25)
Explicit expressions for the integrals $J_a$ and $\bar{J}_a$ can be found in Appendix B. For general values of the conformal weights, the functions $J_a$ and $\bar{J}_a$ can be calculated in terms of the Meijer’s $G_{41}^{pq}$−function, which in turn can be expanded over $_4F_3$−hypergeometric series and its derivatives with respect to indices $[\beta]$. Instead of presenting the general expression for the function $\Omega$, we consider two special physically most interesting cases.

In the first case, we put $h_\alpha = h_\beta = 1$ and $h_\gamma = h$. The function $\Omega(1, 1, h)$ measures the planar coupling of two reggeized gluons to the BFKL state with the conformal weight $(h, \bar{h})$. Each of the integrals entering (2.23) is singular at $h_\alpha = h_\beta = 1$ (or equivalently $\bar{h}_\alpha = \bar{h}_\beta = 0$) and one regularizes them by replacing $h_\alpha = 1 + i\nu_\alpha$ and $h_\beta = 1 + i\nu_\beta$ as $\nu_\alpha, \nu_\beta \to 0$. In this limit, the integrals $J_a$ and $\bar{J}_a$ are given by (B.7). Substituting them into (2.23) one finds that the poles in $\nu_\alpha$ and $\nu_\beta$ are cancelled in the sum of integrals leading to the finite expression for the interaction vertex:

$$\Omega(h, 1, 1) = \frac{4\pi^3}{h(1-h)} \text{Re} [\psi(1) - \psi(h)] ,$$

which satisfies the relation (2.13). Taking $h = \frac{1}{2}$ one calculates the planar coupling of the BFKL Pomeron to two reggeized gluons as

$$\Omega\left(\frac{1}{2}, 1, 1\right) = 32\pi^3 \ln 2 .$$

Let us consider the limit $h = 1 + i\nu$ as $\nu \to 0$ corresponding to the coupling of three reggeized gluons. One finds from (2.26) that the function $\Omega$ vanishes as

$$\Omega(1 + i\nu, 1, 1) = 4i\pi^3 \zeta(3)\nu + O(\nu^2) .$$

Moreover, this case corresponds to the values of conformal spins, $n_\alpha = n_\beta = n_\gamma = 1$, and according to the selection rules (2.9), the nonplanar contribution (2.22) vanishes identically for any $\nu$. Therefore, substituting (2.22) into (2.1) one finds that the triple reggeon vertex vanishes as

$$V(1 + i\nu; 1, 1) \sim \nu^3$$

with two additional powers of $\nu$ coming from the prefactor in (2.6). This property of the transition vertex is in agreement with the Gribov’s signature conservation rule which prohibits the existence of the vertices with odd number of interacting reggeized gluons. Two reggeons could couple however to the BFKL states with $h \neq 1, 0$. The corresponding transition vertex can be calculated from (2.7), using (2.22) and (2.26), as

$$V(h; 1, 1) = \left(\frac{\alpha_s N_c}{\pi}\right)^2 (4\pi)^3 h(1-h) \left(1 - \frac{1}{N_c^2}\right) \text{Re} [\psi(1) - \psi(h)] ,$$

where the nonplanar $1/N_c^2$−term is absent for odd spin $n$.

In the second case, one chooses the conformal weights as $h_\alpha = h_\beta = h$ and $h_\gamma = \frac{1}{2}$ and obtains from (2.23) the coupling of the BFKL Pomeron to two BFKL states with the conformal weights $(h, \bar{h})$. The values of integrals $J_a$ and $\bar{J}_a$ are given by (B.6) and their substitution into (2.23) yields the following result

$$\Omega\left(\frac{1}{2}, h, h\right) = -\pi^5 \frac{\Gamma(2h - \frac{1}{2})\Gamma(2\bar{h} - \frac{1}{2})}{\Gamma^2(h)\Gamma^2(\bar{h})} \frac{\cot^3(\pi h)}{(h - \frac{1}{2})^2(\bar{h} - \frac{1}{2})} _4F_3(\bar{h}; 6F_5(h) + \left(h \leftrightarrow \bar{h}\right) ,$$

Here we used the symmetry property (2.8), $\Omega(1, 1, h) = \Omega(h, 1, 1)$. 

where the notations were introduced for the following combinations of the generalized hypergeometric series

\[ _4F_3(h) = _4F_3\left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, h, 1 - h \\ 1, \frac{1}{2} + h, \frac{3}{2} - h \end{array} | 1 \right) \]

\[ _6F_5(h) = \left[ 1 + 4 \frac{d}{dx} \right] _6F_5\left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, h, h, 1 - h, 1 - h \\ 1, \frac{1}{2} + h, \frac{1}{2} + h, \frac{3}{2} - h, \frac{3}{2} - h \end{array} | x = 1 \right) \]

\[ = _6F_5\left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, h, h, 1 - h, 1 - h \\ 1, \frac{1}{2} + h, \frac{1}{2} + h, \frac{3}{2} - h, \frac{3}{2} - h \end{array} | 1 \right) \]

\[ + _6F_5\left( \begin{array}{c} \frac{3}{2}, \frac{3}{2}, h + 1, h + 1, 2 - h, 2 - h \\ 2, \frac{3}{2} + h, \frac{3}{2} + h, \frac{5}{2} - h, \frac{5}{2} - h \end{array} | 1 \right) \]

\[ \frac{(1 - h)^2 h^2}{(\frac{1}{2} + h)^2(\frac{3}{2} - h)^2} . \]

One verifies that this expression satisfies the relation \((2.12)\) and it reproduces \((2.27)\) for \(h = 1\).

Taking \(h = \frac{1}{2}\) one calculates the planar contribution to the bare triple BFKL Pomeron vertex as

\[ \Omega\left( \frac{3}{2}, \frac{3}{2} \right) = 2\pi^7 _4F_3\left( \frac{1}{2} \right) _6F_5\left( \frac{1}{2} \right) = 7766.679 \]  

\[ (2.30) \]

This exact value is close to the one \([10]\) obtained by numerical Monte Carlo integration of \((2.24)\).

Finally, one substitutes \((2.23)\) and \((2.30)\) into \((2.6)\) to obtain the bare triple BFKL Pomeron interaction vertex as

\[ V\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = \left( \frac{\alpha_s N_c}{\pi} \right)^2 \left[ \Omega\left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) - \frac{4\pi \ln 2}{N_c^2} \Lambda\left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \right] = \left( \frac{\alpha_s N_c}{\pi} \right)^2 2\pi^7 \left( 1.286 - \frac{0.545}{N_c^2} \right) . \]

We observe that the nonplanar contribution is negative and for \(N_c = 3\) it is much smaller than the planar one.

### 3. Conformal bootstrap

Let us interpret the transition vertex calculated in the previous section as a 3-point correlation function of the quasiprimary fields in two-dimensional conformal field theory. Following the bootstrap approach \([12, 8]\) we will specify the set of local operators and construct their operator algebra. The latter will allow us to calculate the correlation function of the operators \((1.9)\) and compare it with the transition vertex \((1.8)\).

#### 3.1. Operator algebra

Let us associate the quasiprimary operator \(O_{h,\bar{h}}(z, \bar{z})\) with the BFKL state having the conformal weights \((h, \bar{h})\) and the center-of-mass coordinate \(\rho\). According to the possible values of the conformal weights, \((1.3)\), the number of the operators is infinite. Similarly, one associates the operator \(\phi = \phi_{0,0}(z, \bar{z})\) with the reggeized gluons. Since the BFKL state is built from two reggeized gluons one has to include the operators \(\phi(z, \bar{z})\) into the operator algebra of the fields \(O_{h,\bar{h}}(z, \bar{z})\). The operator \(\phi(z, \bar{z})\) has to carry the color charge of gluon and in order to simplify the color structure of the correlation functions involving the reggeon fields one considers the multi-color limit, \(N_c \to \infty\). In this limit, the color flow becomes simple and one can treat \(\phi(z, \bar{z})\) as a scalar operator.

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It follows from (1.2) that \( \phi_{0,0}(z, \bar{z}) \) has vanishing conformal weights while in the unitary conformal field theory the only operator having this property is the identity operator \( \mathbb{1} \). To avoid this problem one notices that taking derivative of the both sides of (1.2) with respect to \( z_1 \) and \( z_2 \) one can express the BFKL wave function as

\[
E_{h,\bar{h}}(z_{10}, z_{20}) = \frac{1}{h(h-1)} z_{12}^2 \langle \partial \phi(z_1, \bar{z}_1) \partial \phi(z_2, \bar{z}_2) O_{h,\bar{h}}(z_0, \bar{z}_0) \rangle \tag{3.1}
\]

where \( \partial \phi(z, \bar{z}) = \frac{\partial}{\partial z} \phi(z, \bar{z}) \). This identity allows us to associate the reggeized gluon with the operator \( \partial \phi(z, \bar{z}) \) having the conformal weights \( h = 1 \) and \( \bar{h} = 0 \). These values fit into the spectrum of the conformal weights of the fields \( O_{h,\bar{h}}(z, \bar{z}) \) at \( n = 1 \) and \( \nu = 0 \) and one has to distinguish between the operators \( O_{1,0}(z, \bar{z}) \) and \( \partial \phi_{0,0}(z, \bar{z}) \) as corresponding to the BFKL state and reggeized gluon, respectively.

However, examining the BFKL state with \( h = 1 \) and \( \bar{h} = 0 \) one finds that its wave function (1.2) is given by the sum of two terms, \( E_{1,0}(\rho_01, \rho_02) = \frac{1}{z_{10}} - \frac{1}{z_{20}} \), each does not depending on one of the reggeon coordinates. This implies that the BFKL state \( E_{1,0}(\rho_01, \rho_02) \) cannot couple to the gauge invariant physical states like onium. Indeed, the coupling is proportional to \( \int d^2 \rho_1 d^2 \rho_2 \Phi(\rho_1, \rho_2) E_{1,0}(\rho_01, \rho_02) \) with \( \Phi \) being the onium wave function and it vanishes due to the condition of gauge invariance (1.7). The same property allows us to identify the corresponding two “unphysical” operators, \( O_{1,0}(z, \bar{z}) \) and \( O_{0,1}(z, \bar{z}) = (O_{1,0}(z, \bar{z}))^* \), as the fields \( \partial \phi(z, \bar{z}) \) and \( \bar{\partial} \phi(z, \bar{z}) \), respectively. Therefore, once we will construct the operator algebra of the quasiprimary fields \( O_{h,\bar{h}}(z, \bar{z}) \), the reggeized gluons will be automatically included into it as the special case \( h = 1, \bar{h} = 0 \) and \( h = 0, \bar{h} = 1 \).

The construction of the operator algebra goes as follows. Let us define the 2-point correlation function of the operators \( O_{h,\bar{h}}(z, \bar{z}) \) as

\[
\langle O_{h_1,\bar{h}_1}(z_1, \bar{z}_1) O_{h_2,\bar{h}_2}(z_2, \bar{z}_2) \rangle = D(h_1) \delta_{h_1, h_2} z_{12}^{-2h_1} z_{12}^{-2\bar{h}_1} \tag{3.2}
\]

where \( \delta_{h_1, h_2} = \delta_{n_1, n_2} \delta(\nu_1 - \nu_2) \) and \( D(h) \) is some function to be determined later on. The choice of \( D(h) \) fixes normalization of the operators. Let us define the 3-point correlation function as

\[
\langle O_{h_1,\bar{h}_1}(z_1, \bar{z}_1) O_{h_2,\bar{h}_2}(z_2, \bar{z}_2) O_{h_3,\bar{h}_3}(z_3, \bar{z}_3) \rangle = \Omega(h_1, h_2, h_3) \prod_{i<j}(z_i - z_j)^{-\Delta_{ij}} (\bar{z}_i - \bar{z}_j)^{-\bar{\Delta}_{ij}} \tag{3.3}
\]

where the function \( \Omega \) determines the planar contribution to the interaction vertex (2.6) and the exponents \( \Delta_{ij} \) and \( \bar{\Delta}_{ij} \) are given by (2.3). Combining together (3.2) and (3.3) we obtain the following operator algebra for the infinite set of local operators \( \{ O_{h,\bar{h}}(z, \bar{z}), \mathbb{1} \} \)

\[
O_{h_1,\bar{h}_1}(z_1, \bar{z}_1) O_{h_2,\bar{h}_2}(z_2, \bar{z}_2) = \mathbb{1} \times D(h_1) \delta_{h_1, h_2} z_{12}^{-2h_1} z_{12}^{-2\bar{h}_1} \tag{3.4}
\]

Here, the sum goes over the quasiprimary operators with conformal weights satisfying (1.3). Two operators entering the l.h.s. of (1.4) are transformed under projective transformations according to the principal series representations of \( SL(2, \mathbb{C}) \) denoted as \( t^{(n_1, \nu_1)} \) and \( t^{(n_2, \nu_2)} \). The operator
algebra (3.4) can be interpreted as the decomposition of the tensor product of two representations over irreducible components, 
\[ t^{(n_1,\nu_1)} \otimes t^{(n_2,\nu_2)} = \bigoplus_{\nu_3,n_3} t^{(n_3,\nu_3)} . \]

The coefficient in front of the operator \( O_{1-h_3,1-h_3}(z_3, \bar{z}_3) \) in (3.4) is chosen in such a way that substitution of (3.4) into (3.3) reproduces the 3-point function. To perform this check one applies (3.4) and integrates over \( z_3 \) using the representation (2.4) and (2.2) for the function \( \Omega \) and by taking into account the intertwining relation (2.10).

Two terms entering the r.h.s. of (3.4) have a simple physical meaning. The first term describes the propagation of the BFKL state while the second one corresponds to the transition of two BFKL states into a single BFKL state. The origin of the operator algebra can be traced back to the projective invariance of the transition vertex \( V_{2,4} \) and assuming that the same property holds for all transition vertices \( V_{n,m} \) one can generalize the relation (3.4) to take into account the transitions between the BFKL states and higher \( n > 2 \) reggeon compound states. This can be achieved by including additional quasiprimary operators \( O^{(n)}_{h,\bar{h}} \) interpolating the \( n \)-reggeon compound states into the operator algebra (3.4). Their contribution to the r.h.s. of (3.4) will be however suppressed in the multi-color limit by the additional factor \((\alpha_s N_c)^{n-2}\) and can be considered as higher order correction to (3.4).

### 3.2. Crossing symmetry

The associativity condition of the operator algebra (3.4) which is the main dynamical principle of the conformal bootstrap approach [12], imposes restrictions on the possible form the function \( \Omega \) and it is not obvious that this function defined before as a multi-color limit of the interaction vertex of three BFKL states, (2.6), satisfies them. The same condition can be expressed as the crossing symmetry of the 4-point function [3]

\[ V_{\text{CFT}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle O_{h_1,\bar{h}_1}(z_1, \bar{z}_1)O_{h_2,\bar{h}_2}(z_2, \bar{z}_2)O_{h_3,\bar{h}_3}(z_3, \bar{z}_3)O_{h_4,\bar{h}_4}(z_4, \bar{z}_4) \rangle . \] (3.5)

Applying (3.4) together with (3.3) one can write the 4-point function as a product of two 3-point functions summed over all intermediate states \( \alpha \)

\[ V_{\text{CFT}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{h} \int d^2 \rho_\alpha \; V_0(\alpha_1, \alpha_2, \alpha) \frac{B(h)}{D(1-h)} V_0(1-\alpha, \alpha_3, \alpha_4) , \]

where we used the notations introduced in (2.4) and (2.16). Comparing this expression with (2.16) one notices that the 4-point function \( V_{\text{CFT}}(\alpha, \beta, \beta', \alpha') \) coincides with the r.h.s. of (2.16) and, as a consequence, obeys the crossing symmetry (2.17) provided that

\[ \frac{B(h)}{D(1-h)} = \text{const} \times d(h) . \]

Substituting (2.15) into this relation one finds that in order to satisfy the associativity condition, the 2-point function (3.2) should be fixed (up to an overall constant factor) as

\[ D(h) = \left( \frac{2\pi}{B(h)} \right)^2 \] (3.6)

with the function \( B(h) \) defined in (2.11).
At short distances the operator algebra (3.4) can be represented in the form of the local operator product expansion. Namely, expanding the operator \( O_{1-h_3,1-h_3}(z_3, \bar{z}_3) \) in powers of \( z_3 \) and \( \bar{z}_3 \) and performing \( z_3 \)-integration in (3.4) one gets

\[
O_{\bar{h}_1, \bar{h}_2}(z, \bar{z}) O_{h_2, \bar{h}_2}(0, 0) = 1 \times D(h_1) \delta_{h_1, h_2} z^{-2h_1} \bar{z}^{-2h_1} + \sum_{h_3} \frac{\Omega(h_1, h_2, h_3)}{D(h_3)} z^{-\Delta_{12}} \bar{z}^{-\Delta_{12}} (3.7)
\]

\[
\times F(h_1 - h_2 + h_3; 2h_3; z\partial) F(h_1 - \bar{h}_2 + \bar{h}_3; 2\bar{h}_3; z\bar{\partial}) O_{h_3, \bar{h}_3}(0, 0),
\]

where \( F(a; b; x) = 1 + \frac{a}{b} x + \frac{a+1}{2(b+1)} x^2 + \ldots \) denotes the degenerate hypergeometric function. This relation coincides with the operator expansion of quasiprimary operators in conformal field theory [3].

### 3.3. Conformal blocks

The 4-point function of quasiprimary fields, (3.5), is given by the 4-fold integral entering the r.h.s. of (2.16) and depicted in Fig. 2. Let us apply the operator product expansion (3.7) to expand it over the conformal blocks. The function \( V_{\text{CFT}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) depends on two anharmonic ratios

\[
x = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad \bar{x} = x^*. \quad (3.8)
\]

In the standard way one chooses the coordinates as \( z_1 = \bar{z}_1 = \infty, z_2 = \bar{z}_2 = 1, z_3 = x, \bar{z}_3 = \bar{x} \) and \( z_4 = \bar{z}_4 = 0 \) and defines the following function [3]

\[
G_{43}^{12}(x, \bar{x}) = \lim_{z_1, \bar{z}_1 \to \infty} z_1^{2h_1} \bar{z}_1^{2\bar{h}_1} \langle O_{h_1, \bar{h}_1}(z_1, \bar{z}_1) O_{h_2, \bar{h}_2}(1, 1) O_{h_3, \bar{h}_3}(x, \bar{x}) O_{h_4, \bar{h}_4}(0, 0) \rangle.
\]

One replaces the product of the last two operators according to (3.7), calculates the resulting 3-point correlator using (3.3) and applies the identity

\[
F(a; b; x\partial \xi)(1 - \xi)^{-c} \bigg|_{\xi=0} = F(a; b; x)
\]

to express the function \( G_{43}^{12} \) in the following form

\[
G_{43}^{12} = \sum_{h} \Omega(h_1, h_2, h) \Omega(h, h_3, h_4) D^{-1}(h) \mathcal{F}^{12}_{43}(h|x) \mathcal{F}^{12}_{43}(|\bar{h}|\bar{x}),
\]

where the notation was introduced for the conformal block

\[
\mathcal{F}^{12}_{43}(h|x) = x^{h-h_3-h_4} F(h_3-h_4+h, h_2-h_1+h; 2h; x).
\]

The associativity of the operator algebra (3.7) implies the following crossing symmetry condition [3]

\[
G_{43}^{12}(x, \bar{x}) = G_{23}^{14}(1-x, 1-\bar{x}) = x^{-2h_3} \bar{x}^{-2\bar{h}_3} G_{13}^{12} \left( \frac{1}{x}, \frac{1}{\bar{x}} \right). \quad (3.11)
\]

Expression (3.3) gives the \( s \)-channel expansion of the 4-point correlation function over the partial waves corresponding to the principal series representation of the \( SL(2, \mathbb{C}) \) group labelled by the conformal weight \( (h, \bar{h}) \). We recall however that the representations \( (h, \bar{h}) \) and \( (1-h, 1-\bar{h}) \) are unitary equivalent and one has to combine together two terms in the sum (3.3) having the
conformal weights $h$ and $1 - h$. The $t$– and $u$–channel expansions of the same function follow immediately from the duality property (3.11).

Let consider the special case, $h_1 = h_2 = 1$ and $h_3 = h_4 = 0$, corresponding to the correlator of 4 reggeized gluon fields. \[ G(x, \bar{x}) = \lim_{z, \bar{z} \to \infty} z^2 \langle O_{1,0}(z, \bar{z})O_{1,0}(1,1)O_{0,1}(x, \bar{x})O_{0,1}(0,0) \rangle \]

\[ = \sum_{h} \Omega(h, 1, 1)\Omega(h, 0, 0)D^{-1}(h)\mathcal{F}(h|x)\mathcal{F}(\bar{h}|\bar{x}) , \]

with the conformal blocks $\mathcal{F}(h|x) = x^{h-2}F(h, h; 2h; x)$ and $\mathcal{F}(\bar{h}|\bar{x}) = x^{\bar{h}}F(\bar{h}, \bar{h}; 2\bar{h}; \bar{x})$. Following our interpretation of the transition amplitudes of the BFKL states as correlators of quasiprimary operators and the reggeon Green function one obtains the following relation between the 4-point correlation function of quasiprimary operators and the 4-point correlation function of the reggeized gluons defined in (1.7). We realize that the r.h.s. of (1.7) is given by the product of conformal blocks

\[ G_{h,\bar{h}}(\rho_1, \rho_2; \rho_3, \rho_4) = \frac{x^2}{B(1-h)}\mathcal{F}(h|x)\mathcal{F}(\bar{h}|\bar{x}) + (h \leftrightarrow 1-h) . \]

Finally, using (2.26), (2.12), (3.6) and (2.15) to calculate the product $\Omega(h, 1, 1)\Omega(h, 0, 0)D^{-1}(h)$, one obtains the following relation between the 4-point correlation function of quasiprimary operators and the reggeon Green function

\[ G(x, \bar{x}) = \frac{8\pi^2}{x^2} \sum_{h} (\text{Re} \psi(1) - \psi(h))^2 \left| \frac{h - \frac{1}{2}}{h(h-1)} \right|^2 G_{h,\bar{h}}(\rho_1, \rho_2; \rho_3, \rho_4) . \quad (3.12) \]

Although each term in this sum does not obey the crossing symmetry, the symmetry is restored in their total sum due to (3.11).

We recognize the striking similarity of the expression (3.13) and the BFKL scattering amplitude (1.5). Namely, the second derivative of $f_Y(\rho_1, \rho_2; \rho_3, \rho_4)$ with respect to rapidity coincides up to $Y$–dependent factor with the 4-point correlation function $x^2G(x, \bar{x})$. To introduce the dependence on the rapidity into the sum over conformal weights in (3.12) one notices that the Green function $G_{h,\bar{h}}(\rho_1, \rho_2; \rho_3, \rho_4)$ diagonalizes the quadratic Casimir operator of the $SL(2,\mathbb{C})$–group,

\[ \mathbf{L}_{12}^2 = z_{12}^2 \partial_1 \partial_2 = h(h-1) , \]

as well as the BFKL Hamiltonian $\mathcal{H}_{BFKL} = \mathcal{H}_{BFKL}(\mathbf{L}_{12}^2) = \omega(h)$ with $\omega(h)$ being the energy of the BFKL states, (1.6). This allows us to write the relation between the BFKL scattering amplitude and the 4-point correlation function of reggeized gluon operators as

\[ \partial_Y f_Y(\rho_1, \rho_2; \rho_3, \rho_4) = \left( \frac{\alpha_s N_c}{2\pi} \right)^2 e^{\mathcal{H}_{BFKL}(\mathbf{L}_{12}^2)} x^2 G(x, \bar{x}) , \quad (3.13) \]

where the energy of the BFKL state was replaced in (1.5) by the BFKL Hamiltonian and anharmonic ratios, $x$ and $\bar{x}$, were defined in (3.8).

The following remarks are in order. The factor $e^{\mathcal{H}_{BFKL}}$ plays the role of the evolution operator with the rapidity interval $Y$ defining the $t$–channel evolution time of the system of two reggeons. Although the function $G(x, \bar{x})$ is crossing symmetric and it can be expanded over the conformal blocks in different channel using (3.9) and (3.11), the BFKL kernel $\mathcal{H}_{BFKL}(\mathbf{L}_{12}^2)$ picks up the diagonal contribution only in the channel $1 + 2 \to 3 + 4$. As a result, the crossing symmetry of the BFKL scattering amplitude $f_Y(\rho_1, \rho_2; \rho_3, \rho_4)$ is broken in the high-energy limit, $Y \gg 1$.

\[ ^3 \text{To avoid singular contribution of the diagonal term in (3.7) one has to consider the limit } h_{1,2} \to 1 \text{ with } h_1 \neq h_2 \text{ rather than put } h_1 = h_2 = 1. \]
4. Summary

Studying the high-energy asymptotics of the scattering amplitude of the perturbative hadronic states like heavy quarkonia one is trying to construct the effective $(1 + 2)$-theory which will describe the effective QCD dynamics in the Regge limit. The main objects of the effective theory are the $n = 2, 3, \ldots$ reggeon compound states which propagate in time defined as rapidity of the scattered particles and interact on the 2-dimensional plane of impact parameters. The corresponding bare interaction vertices are local in time and they can be calculated by projecting the reggeon number changing kernels on the wave functions of the compound states entering the vertex.

The reggeon transition kernels exhibit remarkable $SL(2, \mathbb{C})$ symmetry suggesting that the interaction vertices of the reggeon compound states in the Regge effective theory can be evaluated as correlators of interpolating quasiprimary fields in two-dimensional conformal field theory. In the present paper we have discussed this possibility by calculating the interaction vertex of three BFKL states, $V(\alpha \to \beta, \gamma)$.

Projecting the reggeon transition kernel $V_{(2,4)}$ on the wave functions of three BFKL states entering the interaction vertex and performing 2-dimensional integrations over the reggeon coordinates we have obtained the analytical expression for the vertex $V(\alpha \to \beta, \gamma)$ valid for arbitrary values of the conformal weights of the BFKL states. For special values of the conformal weights, $h_\alpha = h_\beta = h_\gamma = \frac{1}{2}$, we have found the expression for the bare triple BFKL Pomeron coupling which is of some importance for Regge phenomenology as it enters into perturbative QCD description of the inclusive cross sections of the diffractive dissociation of the deep inelastic photon \[8\].

We have shown that in the multi-color limit the vertex $V(\alpha \to \beta, \gamma)$ satisfies bilinear relations which were interpreted as associativity conditions of the operator algebra of the operators $O_{h,\bar{h}}(z, \bar{z})$ interpolating the BFKL states and defined as quasiprimary fields in two-dimensional conformal field theory. We have constructed the operator algebra of the fields $O_{h,\bar{h}}(z, \bar{z})$ and demonstrated that their three-point correlation function coincides with the interaction vertex of three BFKL states in the multi-color limit.

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Note added on November 14, 1997

Recently I became aware that the same numerical expression for the planar contribution to the triple BFKL Pomeron vertex, \[2,30\], was obtained independently by different technique in the paper \[18\].
Appendix A. Calculation of the nonplanar diagram

In this appendix we describe the calculation of the nonplanar diagram (2.3). Using the property (2.13) one can restrict analysis only to the positive values of the conformal spins \( n_\alpha, n_\beta, n_\gamma > 0 \) and rewrite (2.19) as

\[
\Lambda = \int d^2z_0 \int d^2z_1 \frac{\alpha^n}{\alpha_0!} (z_0 \bar{z}_1)^n (1 - z_0)(1 - \bar{z}_1)^n \beta \\
x |z_0|^\alpha |\bar{z}_1|^\gamma (1 - z_0)(1 - \bar{z}_1)^n \beta.
\]

The calculation of the integral is based on the following integral representation

\[
|z^2|^{-h} = \frac{1}{\Gamma(h)} \int_0^\infty d\alpha \alpha^{h-1} e^{-sz\bar{z}}.
\]

Repeatedly applying this formula to all \( |z^2|^{-h} \) factors we get additional 5-dimensional integral over the \( \alpha \)-parameters weighted with \( (\alpha_1 \alpha_2)^{\beta_1-1}(\alpha_3 \alpha_4)^{\beta_2-1} \alpha_5^{-1} h_\alpha - h_\beta - h_\gamma \) and move all \( |z^2|^{-h} \) terms into the exponent

\[
\exp\left(-\alpha_1 |z_0|^2 - \alpha_2 |z_1|^2 - \alpha_3 |(1 - z_0)|^2 - \alpha_4 |(1 - z_1)|^2 - \alpha_5 |(z_0 - z_1)|^2\right).
\]

We note that integration goes over complex \( z = x + iy \) and \( \bar{z} = x - iy \) with the measure \( d^2z = dx dy = \frac{1}{2} dz d\bar{z} \). One rotates the integration contour over \( y \) as \( y \to iy \) without encountering any singularities and integrates over \( z \) and \( \bar{z} \) along the real axis. Then, the integration by parts over the holomorphic coordinates \( z_0 \) and \( z_1 \) gives the product of two \( \delta \)-functions

\[
\delta(\alpha_1 z_0 - \alpha_3 (1 - z_0) + \alpha_5 (z_0 - z_1)) \delta(\alpha_2 z_1 - \alpha_4 (1 - z_1) - \alpha_5 (z_0 - z_1)).
\]

Taking into account that the \( \alpha \)-parameters can be positive, one can satisfy the \( \delta \)-function constraints only for \( 0 \leq \bar{z}_0 \leq \bar{z}_1 \leq 1 \) or \( 0 \leq \bar{z}_1 \leq \bar{z}_0 \leq 1 \). In the second case, one changes the integration variables \( \bar{z} \to 1 - \bar{z} \) and finds that the integral takes the same form as in the first case up to a sign factor \( (-)^{n_\alpha + n_\beta + n_\gamma} \). Finally, one rescales \( \alpha \)-parameters to get rid of \( \bar{z} \)-dependent prefactors in the arguments of \( \delta \)-functions and writes \( \Lambda \) in the form (2.20) with the integrals over \( \alpha \) and \( \bar{z} \) factorized into

\[
J = \int_0^\infty d\alpha_1 \ldots d\alpha_5 (\alpha_1 \alpha_2)^{\beta_1-1}(\alpha_3 \alpha_4)^{\beta_2-1} \alpha_5^{-1} h_\alpha - h_\beta - h_\gamma \delta(\alpha_1 - \alpha_3 + \alpha_5) \delta(\alpha_2 - \alpha_4 - \alpha_5) e^{-\alpha_3 - \alpha_4},
\]

\[
\bar{J} = \int_0^1 \int_0^\infty d\bar{z}_0 d\bar{z}_1 \bar{z}_0^{\alpha_1} \bar{z}_1^{\alpha_2} (\bar{z}_0 - \bar{z}_1)^{-h_\alpha} (1 - \bar{z}_0)(1 - \bar{z}_1)^{-h_\beta}.
\]

Changing the integration variables in \( \bar{J} \) as \( \alpha_4 = \lambda(1 - v) \) and \( \alpha_5 = \lambda(1 - u + v) \) with \( 0 \leq \lambda < \infty \) and \( 0 \leq v \leq u \leq 1 \) one gets

\[
J = \Gamma(h_\alpha + h_\beta - h_\gamma) \int_0^1 \int_0^u dv (u(1 - u))^\alpha - 1 (v(1 - v))^\beta - 1 (u - v)^{1 - h_\alpha - h_\beta - h_\gamma}.
\]

Applying the identity

\[
(1 - v)^{\beta - 1} = \sum_{k=0}^\infty v^k \frac{\Gamma(1 + k - \beta)}{\Gamma(1 + k) \Gamma(1 - \beta)} \tag{A.2}
\]
one can express \( J \) as an infinite sum of the ratio of \( \Gamma \)–functions which can be summed into \( 3F_2 \)–hypergeometric series

\[
J = \frac{\Gamma(h_\alpha + h_\beta - h_\gamma)\Gamma(h_\beta)\Gamma(2 - h_\alpha - h_\beta - h_\gamma)\Gamma(h_\alpha)\Gamma(1 - h_\gamma)}{\Gamma(2 - h_\alpha - h_\gamma)\Gamma(1 + h_\alpha - h_\gamma)} \times 3F_2 \left( \begin{array}{c} h_\beta, 1 - h_\beta, 1 - h_\gamma \\ 2 - h_\alpha - h_\gamma, 1 + h_\alpha - h_\gamma \end{array} \right) .
\]

Calculating the integral \( \tilde{J} \) one applies the same identity to \( (1 - \bar{z}_1)^{-h_\beta} \) and obtains in similar way the following result

\[
\tilde{J} = \frac{\Gamma(1 - \bar{h}_\alpha)\Gamma(1 - \bar{h}_\beta)\Gamma(-1 + \bar{h}_\alpha + \bar{h}_\beta + \bar{h}_\gamma)\Gamma(-\bar{h}_\alpha + \bar{h}_\beta + \bar{h}_\gamma)}{\Gamma(h_\beta + h_\gamma)\Gamma(1 - h_\alpha + h_\gamma)} \times 3F_2 \left( \begin{array}{c} \bar{h}_\beta, 1 - \bar{h}_\alpha, -\bar{h}_\alpha + \bar{h}_\beta + \bar{h}_\gamma \\ \bar{h}_\beta + \bar{h}_\gamma, 1 - \bar{h}_\alpha + \bar{h}_\gamma \end{array} \right) .
\]

Thanks to the Whipple and Dixon identities [7] the \( 3F_2 \)–series entering \( J \) and \( \tilde{J} \) can be calculated as the ratio of \( \Gamma \)–functions leading to the final result (2.21) for the function \( \Lambda \). We would like to stress that the same result can be obtained by using the conventional technique of calculating 2-dimensional integrals in conformal field theories [17].

**Appendix B. Calculation of the planar diagram**

The calculation of the planar diagram is similar to that of the nonplanar diagram. We choose the conformal spins of three BFKL states to be positive and rewrite (2.24) as

\[
\Omega = \int d^2z_0 \int d^2z_1 \int d^2z_2 \left( z_{01}\bar{z}_0\bar{z}_1 \right)^{n_\alpha} \left( z_{12}(1 - \bar{z}_1)(1 - \bar{z}_2) \right)^{n_\beta} \tilde{z}_{20}^{n_\gamma} \times |z_{01}^{2\bar{h}_\alpha - 1}|z_{12}^{2\bar{h}_\beta - 1}|z_{20}^{2\bar{h}_\gamma - 1}|z_0^2|z_1^2|z_2^2|\alpha - h_\alpha|z_{12}^2|\alpha - h_\beta|z_{20}^2|\alpha - h_\gamma|.
\]

Using the representation (A.11) one moves \( |z^2| \)–factors into the exponent

\[
\exp \left( -\alpha_1|z_0^2| - \alpha_2|z_1^2| - \alpha_3|(1 - z_1)^2| - \alpha_4|(1 - z_2)^2| - \alpha_5|z_{01}^2| - \alpha_6|z_{12}^2| - \alpha_7|z_{20}^2| \right)
\]

and integrates over the corresponding seven \( \alpha \)–parameters with the weight \( (\alpha_1\alpha_2)^{\alpha_1\alpha_2 - 1}(\alpha_3\alpha_4)^{\alpha_3\alpha_4 - 1} \alpha_5^{\alpha_5 - \alpha_6\alpha_7} \). Rotating the integration contours in complex \( z \)–plane and integrating by parts over the holomorphic coordinates \( z_0, z_1 \) and \( z_2 \) along the real axis one gets the product of three \( \delta \)–functions

\[
\delta((\alpha_1\bar{z}_0 + \alpha_5\bar{z}_{01} + \alpha_7\bar{z}_{02}) \delta((\alpha_2\bar{z}_1 + \alpha_3(\bar{z}_1 - 1) + \alpha_5\bar{z}_{10} + \alpha_6\bar{z}_{12}) \delta((\alpha_4\bar{z}_2 - 1) + \alpha_6\bar{z}_{21} + \alpha_7\bar{z}_{20}) , \quad (B.3)
\]

which restrict the possible values of antiholomorphic coordinates \( \bar{z}_0, \bar{z}_1 \) and \( \bar{z}_2 \). Namely, the \( \delta \)–function constraints define three different regions:

\[
1st : \quad 0 \leq \bar{z}_0 \leq \bar{z}_1 \leq \bar{z}_2 \leq 1, \quad 2nd : \quad 0 \leq \bar{z}_0 \leq \bar{z}_2 \leq \bar{z}_1 \leq 1, \quad 3rd : \quad 0 \leq \bar{z}_1 \leq \bar{z}_0 \leq \bar{z}_2 \leq 1. \quad (B.4)
\]
In each of these regions one rescales the \( \alpha \)–parameters to get rid of \( \bar{z} \)–dependent prefactors in \( \delta \) and factorizes \( \alpha \)– and \( \bar{z} \)– integrations into two integrals denoted as \( J_a \) and \( \tilde{J}_a \), respectively, with \( a = 1, 2, 3 \) referring to the particular region \( \delta \). This leads to the expression \( 2.25 \) for the function \( \Omega \) as a sum of three terms, in which the integrals over \( \alpha \)–parameters corresponding to three different regions are given by

\[
\begin{pmatrix}
J_1 \\
J_2 \\
J_3
\end{pmatrix} = \int_0^\infty \mathcal{D} \alpha \ e^{-\alpha_3 - \alpha_4} \times
\begin{pmatrix}
\delta(\alpha_1 - \alpha_5 - \alpha_7)\delta(\alpha_2 - \alpha_3 + \alpha_5 - \alpha_6)\delta(-\alpha_4 + \alpha_6 + \alpha_7) \\
\delta(\alpha_1 - \alpha_5 - \alpha_7)\delta(\alpha_2 - \alpha_3 + \alpha_5 + \alpha_6)\delta(-\alpha_4 - \alpha_6 + \alpha_7) \\
\delta(\alpha_1 + \alpha_5 - \alpha_7)\delta(\alpha_2 - \alpha_3 - \alpha_5 - \alpha_6)\delta(-\alpha_4 + \alpha_6 + \alpha_7)
\end{pmatrix}
\]

with the integration measure

\[
\int_0^\infty \mathcal{D} \alpha \equiv \int_0^\infty \prod_{j=1}^7 d\alpha_j \ (\alpha_1\alpha_2)^{h_a-1}(\alpha_3\alpha_4)^{h_\beta-1}\alpha_5^{-h_\alpha}\alpha_6^{-h_\beta}\alpha_7^{-b_\gamma}.
\]

The corresponding \( \bar{z} \)–integrals are given by

\[
\begin{pmatrix}
\tilde{J}_1 \\
\tilde{J}_2 \\
\tilde{J}_3
\end{pmatrix} = \int_0^1 \mathcal{D} \bar{z} \times
\begin{pmatrix}
\theta(0 \leq \bar{z}_0 \leq \bar{z}_1 \leq \bar{z}_2 \leq 1) (-)^{n_\alpha + n_\beta} \\
\theta(0 \leq \bar{z}_0 \leq \bar{z}_2 \leq \bar{z}_1 \leq 1) (-)^{n_\alpha} \\
\theta(0 \leq \bar{z}_1 \leq \bar{z}_0 \leq \bar{z}_2 \leq 1) (-)^{n_\beta}
\end{pmatrix}
\]

where the \( \theta \)–functions define the integration region according to \( \delta \) and the integration measure is

\[
\int_0^1 \mathcal{D} \bar{z} = \int_0^1 d\bar{z}_0 \int_0^1 d\bar{z}_1 \int_0^1 d\bar{z}_2 \ |\bar{z}_0|^{h_a-1}|\bar{z}_1|^{h_\beta-1}|\bar{z}_2|^{h_\gamma-1}(\bar{z}_0\bar{z}_1)^{-h_\alpha}(1-\bar{z}_1)(1-\bar{z}_2)^{-h_\beta}.
\]

The integrals corresponding to the regions 2 and 3 are related to each other as

\[
J_2(h_\alpha, h_\beta, h_\gamma) = J_3(h_\beta, h_\alpha, h_\gamma), \quad \tilde{J}_2(h_\alpha, h_\beta, h_\gamma) = \tilde{J}_3(h_\beta, h_\alpha, h_\gamma).
\]

To obtain these relations one replaces the integration variables \( (\alpha_1, \alpha_2, \alpha_5) \equiv (\alpha_4, \alpha_3, \alpha_6) \) and \( (\bar{z}_0, \bar{z}_1, \bar{z}_2) \equiv (1-\bar{z}_2, 1-\bar{z}_1, 1-\bar{z}_0) \) in the expressions for \( J_2 \) and \( \tilde{J}_2 \), respectively. In similar way, one can show that \( J_1 \) and \( \tilde{J}_1 \) are symmetric functions of \( h_\alpha \) and \( h_\beta \)

\[
J_1(h_\alpha, h_\beta, h_\gamma) = J_1(h_\beta, h_\alpha, h_\gamma), \quad \tilde{J}_1(h_\alpha, h_\beta, h_\gamma) = \tilde{J}_1(h_\beta, h_\alpha, h_\gamma).
\]

The calculation of the integrals goes as follows. One first performs the calculation of the functions \( J_a \) and \( \tilde{J}_a \) for zero values of the conformal spins \( n_\alpha = n_\beta = n_\gamma = 0 \), at which the integrals entering these functions are convergent and then analytically continues the final expressions to the general values of the conformal weights.

Integral \( J_1 \). One first replaces \( \alpha_1 = \alpha_5 + \alpha_7, \alpha_2 = \alpha_3 + \alpha_6 - \alpha_5 \) and \( \alpha_4 = \alpha_6 + \alpha_7 \) using the \( \delta \)–functions and then changes the remaining integration variables as

\[
\alpha_3 + \alpha_6 + \alpha_7 = \lambda, \quad \alpha_3 + \alpha_6 = \lambda x, \quad \alpha_3 = \lambda y, \quad \alpha_5 = (\alpha_3 + \alpha_6)(1 - x)
\]

with \( 0 \leq \lambda < \infty \) and \( 0 \leq x, y, z \leq 1 \). Integration over \( y \) and \( z \) gives two \( {}_2F_1 \)–hypergeometric functions leading to

\[
J_1 = \Gamma(h_\alpha + h_\beta - h_\gamma)\Gamma(1 - h_\alpha)\Gamma(1 - h_\beta)\Gamma(h_\beta)
\times \int_0^1 dx \ (1 - x)^{-h_\gamma} {}_2F_1(h_\alpha, 1 - h_\alpha; 1; x) {}_2F_1(h_\beta, 1 - h_\beta; 1; x),
\]
which explicitly obeys (B.3). For general values of the conformal weights this integral can be expressed in terms of the Meijer’s $G$–function as follows. One uses the Mellin-Barnes representation for one of the hypergeometric functions as an integral of $(1 - x)^s$ with certain $\Gamma$–function prefactor along the $s$–contour parallel to the imaginary axis and interchanges the order of $s$– and $x$–integration to obtain

$$J_1 = \Gamma(h_\alpha + h_\beta - h_\gamma) \frac{\Gamma(h_\alpha)\Gamma(1 - h_\alpha)}{\Gamma(h_\beta)\Gamma(1 - h_\beta)} \sum_{k,n=0}^{\infty} \frac{(1 - \bar{h}_\gamma)k+n}{k!n!} \bar{z}_0^k (1 - \bar{z}_2)^n,$$

Integral $J_2$. One integrates over $\alpha_1$, $\alpha_3$ and $\alpha_7$ using the $\delta$–functions and replaces the integration variables as

$$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 = \lambda, \quad \alpha_4 + \alpha_5 + \alpha_6 = \lambda x_1, \quad \alpha_4 + \alpha_6 = \lambda x_1 x_2, \quad \alpha_4 = \lambda x_1 x_2 x_3,$$

with $0 \leq \lambda < \infty$ and $0 \leq x_{1,2,3} \leq 1$. The integrals over $x_i$ are factorized up to a factor $(1 - x_1 x_2 x_3)^{h_\beta - 1}$ which one expands using (A.2) to obtain the expression for $J_1$ as an infinite sum of the ratio $\Gamma$–functions. It can be summed into $_4F_3$–hypergeometric series as

$$J_2 = \frac{\Gamma(h_\alpha + h_\beta - h_\gamma)\Gamma(1 - h_\alpha)\Gamma(h_\alpha)\Gamma(1 - h_\beta)\Gamma(h_\beta)\Gamma(2 - h_\alpha - h_\gamma)}{\Gamma(1 + h_\alpha - h_\gamma)\Gamma(2 - h_\alpha - h_\gamma)} \times_4F_3\left(\left| \begin{array}{c} h_\beta, 1 - h_\beta, 1 - h_\gamma, 1 - h_\gamma \\ 1, 2 - h_\alpha - h_\gamma, 1 + h_\alpha - h_\gamma \end{array} \right| 1 \right)$$

Integral $\bar{J}_1$. The calculation of the integral is based on the identity

$$(\bar{z}_2 - \bar{z}_0)^{h_\gamma - 1} = \sum_{k,n=0}^{\infty} \frac{(1 - \bar{h}_\gamma)k+n}{k!n!} \bar{z}_0^k (1 - \bar{z}_2)^n,$$

where $(a)_k \equiv \Gamma(a + k)/\Gamma(a)$. It allows to separate $\bar{z}_0$– and $\bar{z}_2$–integrations and obtain after simple calculation the expression for $\bar{J}_1$ in the form of double series

$$\bar{J}_1 = (-)^{n_\alpha + n_\beta} \frac{\Gamma(h_\alpha)\Gamma(h_\beta)}{\Gamma(1 - h_\gamma)} \sum_{k,n=0}^{\infty} \frac{\Gamma(1 + k - \bar{h}_\alpha)\Gamma(1 + n - \bar{h}_\beta)\Gamma(1 + k + n - \bar{h}_\gamma)}{\Gamma(2 + k + n - h_\alpha - h_\beta) \Gamma(2 + k + n + \bar{h}_\gamma)}.$$

One can check that the same series can be represented as the following integral

$$\bar{J}_1 = (-)^{n_\alpha + n_\beta} \frac{\Gamma(h_\alpha)\Gamma(h_\beta)\Gamma(1 - \bar{h}_\gamma)}{\Gamma(1 - h_\gamma)} \Gamma(1 - h_\alpha - h_\beta + h_\gamma) \times \int_0^1 dx x^{-h_\gamma} (1 - x)^{-h_\alpha - h_\beta + h_\gamma} \quad F_1(1 - \bar{h}_\alpha, 1 - \bar{h}_\beta; 1; x) \quad F_1(1 - h_\beta, 1 - h_\beta; 1; x),$$

which obviously satisfies (B.5). For general values of the conformal weights this integral can be expressed in terms of the Meijer’s $G$–function. Repeating the same steps as for calculation of $J_1$ one obtains

$$\bar{J}_1 = (-)^{n_\alpha + n_\beta} \frac{\Gamma(1 - \bar{h}_\alpha)\Gamma(1 - \bar{h}_\beta)}{\Gamma(1 - h_\gamma)\Gamma(1 - h_\alpha - h_\beta + h_\gamma)} \times_4F_3\left(\left| \begin{array}{c} h_\beta, 1 - \bar{h}_\beta, \bar{h}_\gamma, \bar{h}_\gamma \\ 0, 0, -\bar{h}_\alpha + \bar{h}_\gamma, -1 + \bar{h}_\alpha + \bar{h}_\gamma \end{array} \right| 1 \right)$$
Integral $\tilde{J}_2$. Similar to the previous case, one applies the identity

$$(\tilde{z}_1 - \tilde{z}_0)^{\tilde{h}_\alpha - 1} = \sum_{k,n=0}^{\infty} \frac{(1 - \tilde{h}_\alpha)_{k+n}}{k! n!} \tilde{z}_0^k (1 - \tilde{z}_1)^n$$

to express the integral in the form of double series

$$\tilde{J}_2 = (-)^n a \frac{\Gamma(\tilde{h}_\beta)\Gamma(\tilde{h}_\gamma)}{\Gamma(1 - \tilde{h}_\alpha)} \sum_{k,n=0}^{\infty} \frac{\Gamma^2(1 + k - \tilde{h}_\alpha)\Gamma^2(1 + n - \tilde{h}_\beta)\Gamma(1 + k + n)}{\Gamma(1 + k)\Gamma(1 + n)\Gamma(2 + k + n - h_\alpha - h_\beta + h_\gamma)}.$$ 

This series is given by

$$\tilde{J}_2 = (-)^n a \frac{\Gamma(1 - \tilde{h}_\alpha)\Gamma(\tilde{h}_\gamma)}{\Gamma(\tilde{h}_\beta)\Gamma(1 - \tilde{h}_\alpha - h_\beta + h_\gamma)} \times \int_0^1 dx (1 - x)^{-\tilde{h}_\alpha - h_\beta + h_\gamma} 2F_1(1 - \tilde{h}_\alpha, 1 - \tilde{h}_\gamma; 1; x) 2F_1(1 - \tilde{h}_\beta, 1 - \tilde{h}_\gamma; 1; x),$$

Although this expression is not symmetric in $\tilde{h}_\alpha$ and $\tilde{h}_\beta$ its asymmetry is contained in the ratio of $\Gamma$–functions. Changing the integration variable as $y = \frac{x}{1-x}$ and using the Mellin-Barnes representation one obtains the following expression

$$\tilde{J}_2 = (-)^n a \frac{\Gamma(1 - \tilde{h}_\alpha)\Gamma(\tilde{h}_\gamma)}{\Gamma(\tilde{h}_\beta)\Gamma(1 - \tilde{h}_\alpha - h_\beta + h_\gamma)} G^{42}_{44} \left( \begin{array}{c} \tilde{h}_\beta, 1 - \tilde{h}_\beta, \tilde{h}_\gamma, \tilde{h}_\gamma \\ 0, 0, -\tilde{h}_\alpha + h_\gamma, -1 + \tilde{h}_\alpha + h_\gamma \end{array} \right)$$

The Meijer’s $G$–function entering into $J_2$ and $\tilde{J}_2$ can be expressed in terms of the $4F_3$–hypergeometric series and their derivative with respect to parameters $[7]$. We do not give here the explicit formulas but rather consider two special physically most interesting cases $h_\alpha = h_\beta = h$ and $h_\gamma = \frac{1}{2}$. This case corresponds to the coupling of two BFKL states with the conformal weights $(h, \tilde{h})$ and the BFKL Pomeron. Simplification occurs thanks to the Burchall-Chaundy identity $[7]$ which expresses $(2F_1)^2$ as a sum of $2F_1$. The values of integrals are

$$J_1 = \frac{\pi^2 \cos^2(\pi h)}{2 \sin^2(\pi h)} (2h - \frac{1}{2}) \, \frac{4F_3(h)}{4F_3(h)}$$

$$J_2 = -\frac{\pi^2 \cos(\pi h)}{\sin^2(\pi h)} \frac{(2h - \frac{1}{2})}{h - \frac{1}{2}} \, \frac{4F_3(h)}{4F_3(h)}$$

$$\tilde{J}_1 = -2\sqrt{\pi} \frac{\cos(\pi h)}{\sin(\pi h)} (1 - \tilde{h}) \frac{\Gamma(2\tilde{h} - \frac{1}{2})}{\tilde{h} - \frac{1}{2}} \, \frac{4\tilde{F}_3(\tilde{h})}{4\tilde{F}_3(\tilde{h})}$$

$$\tilde{J}_2 = (-)^n \frac{\sqrt{\pi} \cos^2(\pi h)}{2 \sin(\pi h)} (1 - \tilde{h}) \frac{\Gamma(2\tilde{h} - \frac{1}{2})}{(\tilde{h} - \frac{1}{2})^2} \, \frac{4\tilde{F}_3(\tilde{h})}{4\tilde{F}_3(\tilde{h})}$$

where we used the notations introduced in (2.29). Substitution of these integrals into (2.23) yields the expression (2.28) for the interaction vertex.

$h_\alpha = h_\beta = 1$ and $h_\gamma = \frac{1}{2}$

This case corresponds to the coupling of two reggeized gluons to the BFKL state with the conformal weight $(h, \tilde{h})$. Examining the integrals one finds that each of them is separately
divergent at $h_{\alpha} = h_{\beta} = 1$ (or equivalently, $\bar{h}_{\alpha} = \bar{h}_{\beta} = 0$). We consider instead the following values

$$h_{\alpha} = 1 + i\nu_{\alpha}, \quad h_{\beta} = 1 + i\nu_{\beta}$$

and take the limit $\nu_{\alpha}, \nu_{\beta} \to 0$. Due to simplification of the hypergeometric functions the integrals can be calculated as

$$J_1 = -\frac{\Gamma(1-h)}{\nu_{\alpha}\nu_{\beta}} \left[ 1 + i(\nu_{\alpha} + \nu_{\beta})\psi(1-h) + \mathcal{O}(\nu^2) \right]$$

$$J_2 = -\frac{\Gamma(1-h)}{\nu_{\alpha}\nu_{\beta}} \left[ 1 + i\nu_{\alpha}\psi(1-h) + i\nu_{\beta}\psi(1) + \mathcal{O}(\nu^2) \right]$$

$$\bar{J}_1 = \frac{(\bar{h}(\bar{h} - 1))^{-1}}{\nu_{\alpha}\nu_{\beta}} \left[ -i(\nu_{\alpha} + \nu_{\beta}) + 2\nu_{\alpha}\nu_{\beta}\left(\psi(\bar{h}) - \psi(1)\right) + \mathcal{O}(\nu^3) \right]$$

$$\bar{J}_2 = \frac{(\bar{h}(\bar{h} - 1))^{-1}}{\nu_{\beta}} \left[ i + (\nu_{\alpha} + \nu_{\beta})\left(\psi(1) - \psi(1 + \bar{h}) + \frac{1}{\bar{h} - 1}\right) + \mathcal{O}(\nu^2) \right]$$

Combining these integrals together in the expression for the interaction vertex we find that all terms singular at $\nu_{\alpha}, \nu_{\beta} \to 0$ cancel against each other giving a finite result (2.26) for the vertex $\Omega(1,1,h)$.

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