Spinning Braid Group Representation
and the Fractional Quantum Hall Effect

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Abstract

The path integral approach to representing braid group is generalized for particles with spin. Introducing the notion of charged winding number in the super-plane, we represent the braid group generators as homotopically constrained Feynman kernels. In this framework, super Knizhnik-Zamolodchikov operators appear naturally in the Hamiltonian, suggesting the possibility of spinning nonabelian anyons. We then apply our formulation to the study of fractional quantum Hall effect (FQHE). A systematic discussion of the ground states and their quasi-hole excitations is given. We obtain Laughlin, Halperin and Moore-Read states as exact ground state solutions to the respective Hamiltonians associated to the braid group representations. The energy gap of the quasi-excitation is also obtainable from this approach.

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1. Introduction

The fractional quantum Hall effect (FQHE) [1] is a collective phenomenon of $N$ electrons living in an effectively 2-dimensional plane. Under suitable conditions, the Hall conductivity is “quantized” as $\frac{p}{q} \frac{e^2}{h}$, i.e. plateaux pegged at these values for some integers $p$ and $q$ are observable along the axis of the strength $B$ of the external magnetic field. This macroscopic quantum behaviour has been successfully captured by Laughlin’s theory when $p = 1$ and $q$ is an odd number [2]. Essentially, the ground states are that of an incompressible quantum liquid. The particle-like excitations, called quasi-particles and quasi-holes respectively, are some gap away from the ground state in the spectrum. They do not contribute to the transport coefficients because of localization effect. What is more interesting is that they are fractionally charged anyons. The reason why the filling fractions have odd denominators is that the many-body ground states proposed by Laughlin must pick up a minus sign whenever any two electrons swap positions; afterall, electrons are fermions. These Laughlin states constitute the corner stones of the theory of FQHE.

All the key ingredients such as the existence of a finite energy gap, and the fractional statistics of quasi-excitations follow from the ans"atz.

Nevertheless, nature vouchsafes more pleasant surprises. FQHE with even-denominator filling fractions was discovered [3][4]. If not for this discovery, Laughlin’s theory would have been more or less adequate. To account for these even filling fractions, spin-unpolarized states have been proposed [5]. Despite some disagreements, the consensus is that the electron’s spin, which is totally frozen out in Laughlin’s picture plays a role in the occurrence of even-denominator states. Experimental evidence of an unpolarized state even for odd denominator filling fraction [6] makes it all the more imperative to scrutinize the role of spin in FQHE.

With this in mind, we propose here a microscopic $N$-body Hamiltonian obtained from the path integral representation of the braid group [7]. When this Hamiltonian is minimally coupled to the background gauge potential of the uniform external magnetic field in the symmetric gauge, one finds that Laughlin states are exact ground states. This was done for electrons carrying the representation of $U(1)$ [8]. Furthermore, since our formulation is a non-abelian generalization of Y. S. Wu’s [9], it is possible to proceed directly to consider the case where the representation carried by the electrons is $SU(2) \times U(1)$; presumably, spin may be regarded as isospin in the non-relativistic regime [10]. Thus, we obtain Halperin state as exact ground state solution. We also extend our previous works

*Additional ideas, though, are needed to account for those plateaux with $p \neq 1$. At any rate, it is not unfair to say that they all build upon the conceptual foundation of Laughlin’s theory.
by switching on the spin degree of freedom in an alternative fashion. Using Grassmannian variables to formulate spin as dynamical variable, we study the path integral of free spinning particles on a super-plane. In this manner, we obtain non-trivial results generalizing the spinless case. With this approach, we get a wavefunction which is the exact ground state of the spinning Hamiltonian in the external magnetic field. It turns out to be the same as the one constructed by Moore and Read which is a product of some conformal blocks of the Ising model and rational torus [11]. From these analyses, we conclude that (super-)Knizhnik-Zamolodchikov operator minimally coupled to the background gauge field is the microscropic ground state equation of FQHE.

In section 2, we review the basic ideas leading to the path integral representation of braid group. After proposing the quantization procedure for the spinning quantum mechanics, we proceed to construct an analogous representation with the path integral of free particles with spin in section 3. We then consider the link between braid group statistics and FQHE. The ground state equations are solved for polarized FQHE states, followed by the spinning states in section 4. Since the key issue of FQHE is its incompressibility, we feature the topological origin of the quasi-excitations in section 5 and suggest how the energy gap may be obtained from this perspective. In particular, a novel formula to calculate the ratio of the energy gaps of Laughlin states is presented. In section 6, we see how Halperin’s state is obtained as the exact solution of the ground state equation of spin singlets. In section 7, we discuss the connection with WZW models and the crucial role played by the external magnetic field. The main results are summarized in the last section.

2. Path Integral Representation

Artin’s braid group $B_N$ is intrinsically a 3-dimensional object which comprises of $N$ ambient isotopic classes of curves in $\mathbb{R}^3$. An intuitive representation of the elements of the group is to use a number of threads and weave them. Given $N$ threads, the elements of $B_N$ can be constructed from $N - 1$ basic weaves $\sigma_i, i = 1, \ldots, N - 1$. Here $\sigma_i$ is used to denote a pattern in Figure 1, where the $i$-th thread crosses over the $i + 1$-th thread. It is worth remarking that the pictorial representation of $\sigma_i$ is faithful and irreducible.

These braid group generators satisfy the following algebraic relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2,$$  \hspace{1cm} (2.1)
\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n - 2.
\] (2.2)

The word \(\sigma_i \sigma_j\), for instance, has been represented as putting one diagram on top of the other as shown in the left-hand side of Figure 2. The meanings of (2.1) and (2.2) are explicit from the weave patterns depicted in Figure 2 and Figure 3 respectively. When stacking the diagrams, one has to exchange the labels to ensure that the glued world lines carry the same representations throughout. This is implicitly carried out in the figures.

The basic idea of the path integral representation \[\text{[9]}\] is to see the threads as non-relativistic world lines of point particles. In this light, the path parametrized by time \(t\) of \(i\)-th particle in the 2-dimensional plane is conceivable. By definition, the number of threads \(N\) is a constant of motion; at all times, no two threads can fuse together and become one. In the language of the configuration space \(M_N\) of \(N\) particles, it means
that the topology is multiply connected. Each particle sees the rest as punctures. As opposed to higher dimensions, the fundamental group of the configuration space $\pi_1(M_N)$ is an infinite non-abelian group, by virtue of which particles that are neither bosons nor fermions are theoretically allowed. It turns out that $\pi_1(M_N)$ is isomorphic to the pure braid group if all the particles do not carry the same representation, and $\pi_1(M_N) \cong B_N$ if they do.

Because the configuration space is multiply connected, the paths are homotopically classified and those of different classes cannot be smoothly deformed from one to the other. When one considers the Feynman kernel for a particle to move from point $z_{a(0)}$ at time $t_0$ to point $z_{a(1)}$ at time $t_1$, one has to organize the paths according to their homotopical classes. Now, the homotopy class of a path in $M_N$ is determined by the winding numbers with respect to the punctures. The number of times a path goes around a puncture is well-defined and non-trivial only when the path is in 2-dimensional space. It is this peculiarity of the spatial dimension being two that gives rise to the possibility of anyonic statistics.

Earlier, we have generalized these ideas to particles carrying representations of a non-abelian group \cite{7}. We shall briefly review the work here. To construct a non-abelian representation of the braid group, we introduce the notion of charged winding number $w$. 

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Fig. 3: Graphical representation of $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. 

for a path:

\[
    w = \frac{1}{2\pi i} \int_C \frac{dz_a}{z_a - z_b} T_a \otimes T_b.
\]  

(2.3)

Here, \( T_a \) and \( T_b \) are the representations carried by the particles. In this manner, the threads are more than merely worldlines; they have become Wilson lines. The charged winding angle \( \Theta \) can now be defined as:

\[
    \Theta = \text{sign}(C) \left| \Theta_{a(1)} - \Theta_{a(0)} \right| + 2\pi w.
\]  

(2.4)

We choose the convention that a path going counterclockwise about the puncture \( z_b \) has positive sign, namely \( \text{sign}(C) = 1 \), and denote \( \vartheta = \text{sign}(C) \left| \Theta_{a(1)} - \Theta_{a(0)} \right| \). With this convention, for the homotopically equivalent paths corresponding to \( \sigma_i \), which cross over from the left, the change in the azimuthal angle \( \vartheta \) is non-negative.

The constrained Feynman kernel of homotopy class \( l \) for particle \( a \) with mass \( m \) can be expressed formally as:

\[
    K_l(z_a(1), t_1, z_a(0), t_0) = \int D_l z_a(t) D_l \pi_a(t) \exp \left( i \int_{t_0}^{t_1} \frac{1}{2} m \left| \dot{z}_a(t) \right|^2 dt \right) \delta^2(2\pi l T_a \otimes T_b - \Theta).
\]  

(2.5)

With the path ordering determined by that in the definition of charged winding angle (2.4), the matrix-valued Dirac delta function can be represented by the following path-ordered Fourier transform:

\[
    \delta^2(2\pi l T_a \otimes T_b - \Theta) = \iint \frac{dk \, d\overline{k}}{2\pi \, 2\pi} e^{-i(k\vartheta + \overline{k}\vartheta)} \mathcal{P} \exp \left( i \left[ 2\pi k(l T_a \otimes T_b - w) + \text{c.c.} \right] \right). \]  

(2.6)

This expression is nothing but a functional integral description of the topological properties of the configuration space. It is the main ingredient of our representation. Technically, the way we formulate the homotopic constraint via (2.6) is quite different from Wu’s [9]. Here, the change of azimuthal angle \( \vartheta \) is fixed by the initial and final positions of particle \( a \). Substituting (2.6) into the Feynman kernel (2.5), we obtain the Fourier transform:

\[
    K_l(z_a(1), t_1, z_a(0), t_0) = \iint \frac{dk \, d\overline{k}}{2\pi \, 2\pi} e^{-i(k\vartheta + \overline{k}\vartheta)} \overline{K}_l(z_a(1), t_1, z_a(0), t_0; k, \overline{k}),
\]  

(2.7)

where

\[
    \overline{K}_l(z_a(1), t_1, z_a(0), t_0; k, \overline{k}) = \int D_l z_a(t) D_l \pi_a(t) \mathcal{P} \exp \left( i \int_{t_0}^{t_1} \frac{1}{2} m \left| \dot{z}_a(t) \right|^2 dt \right) \times \exp \left( i \int_{t_0}^{t_1} \left( k \left( \frac{i \dot{z}_a}{z_a - z_b} + 2\pi l \right) T_a \otimes T_b + \text{c.c.} \right) dt \right).
\]  

(2.8)
Expressions (2.5) and (2.8) can be easily generalized to $N$ particles at $z_1, z_2, \cdots, z_N$, with $\text{Re} \, z_1 < \text{Re} \, z_2 < \cdots < \text{Re} \, z_N$. Let particle $i$ make a trip from $z_i(0) = z_i(t_0) = z_i$ to $z_i(1) = z_i(t_1)$, $\text{Re} \, z_i(1) > \text{Re} \, z_i+1$. Denoting the difference in the initial and the final angle of the paths of particle $i$ with respect to particle $j$ as $\vartheta_{ij}$, $\vartheta_{ij} = \text{sign}(C_i) |\Theta_{ij}(1) - \Theta_{ij}(0)|$, the constrained Feynman kernel of homotopy class $(l_1, \cdots, l_i-1, l_{i+1}, \cdots, l_n)$ for particle $i$ carrying representation $T_i$ is

$$K_{l_i}(z_i(1), t_1, z_i(0), t_0) =$$

$$\int \int \frac{dk \, d\overline{k}}{2\pi \, 2\pi} \exp \left( -i \sum_{j=1, j \neq i}^{n} (k \vartheta_{ij} + \overline{k} \vartheta_{ij}) \right) \overline{K}_{l_i}(z_i(1), t_1, z_i(0), t_0; k, \overline{k}),$$

(2.9)

where

$$\overline{K}_{l_i}(z_i(1), t_1, z_i(0), t_0; k, \overline{k}) = \int \mathcal{D}_i z_i(t) \mathcal{D}_i \overline{z}_i(t) \text{P} \exp \left( i \int_{t_0}^{t_1} \frac{1}{2} m_i |\dot{z}_i(t)|^2 \, dt \right) \times \exp \left( i \int_{t_0}^{t_1} \left( k \sum_{j=1, j \neq i}^{n} \left( \frac{i \dot{z}_i}{z_i - z_j} + 2\pi l_j \right) T_i \otimes T_j + c.c. \right) \, dt \right).$$

(2.10)

Given these initial and final conditions, $\sigma_i$ can be represented by the positively oriented Feynman kernel of class $(0, \cdots, 0, \cdots, 0)$, the $i$-th 0 is omitted as we do not consider self-linking. The self-linking problem does not arise here because Feynman kernels are defined for $t \geq 0$ only. Writing,

$$A_{z_i} = T_i^a A_{z_i}^a = ik \sum_{j=1, j \neq i}^{N} \frac{T_i \otimes T_j}{z_i - z_j},$$

(2.11)

$$A_{\overline{z}_i} = T_i^a \overline{A}_{z_i}^a = i\overline{k} \sum_{j=1, j \neq i}^{N} \frac{T_i \otimes T_j}{z_i - \overline{z}_j},$$

(2.12)

the proposed representation $D(\sigma_i)$ is $K_i(t_1, t_0; \vartheta_{i1}, \cdots, \vartheta_{ii-1}, \vartheta_{i+1}, \cdots, \vartheta_{iN})$ given below:

$$\int \mathcal{D}_i \mathcal{D}_i \mathcal{D}_i \mathcal{D}_i \int \frac{dk \, d\overline{k}}{2\pi \, 2\pi} \text{P} \exp \left( i \int_{C_i} \frac{1}{2} m_i |dz_i|^2 + A_{z_i} dz_i + A_{\overline{z}_i} d\overline{z}_i \right) \times \exp \left( -i \sum_{j=1, j \neq i}^{N} (k \vartheta_{ij} + \overline{k} \vartheta_{ij}) \right),$$

(2.13)

followed by an exchange operation $\Pi_{i+1}$,

$$D(\sigma_i) = \Pi_{i+1} K_i(t_1, t_0; \vartheta_{i1}, \cdots, \vartheta_{ii-1}, \vartheta_{i+1}, \cdots, \vartheta_{iN}).$$

(2.14)
$\Pi_{i+1}$ is to make every world line stick to the same representation space it has started with. The multiplication rule for braid group generators is realised as the usual multiplication of kernels.

It remains to verify that

$$D(\sigma_i)D(\sigma_j) = D(\sigma_j)D(\sigma_i), \quad |i - j| \geq 2, \quad (2.15)$$

$$D(\sigma_i)D(\sigma_{i+1})D(\sigma_i) = D(\sigma_{i+1})D(\sigma_i)D(\sigma_{i+1}), \quad i = 1, \cdots, N - 2. \quad (2.16)$$

One can first look at the paths in the plane corresponding to the space-time diagrams of Figure 2 and 3. It is obvious that (2.16) holds; the two paths in the plane are disjoint by definition (Figure 4). Similarly, the proof of (2.16) is readily seen from Figure 5.

Fig 4: The paths of particles $i$ and $j$ in the $x$-$y$ plane.

Upon careful examination of the overall changes in the azimuthal angles before ($t_0$) and after ($t_3$), one finds that the two figures give the same results; it does not matter whether particle $a$ moves first as in (A) of Figure 5 or particle $b$ in (B). The expressions in terms of Feynman kernels for the proof of (2.16) were given in [7].

Now, the effective Lagrangian of particle $i$ can be readily read from (2.13).

$$L = \frac{1}{2} m_i |\dot{z}_i|^2 + A_{z_i} \dot{z}_i + A_{\overline{z}_i} \dot{\overline{z}}_i. \quad (2.17)$$

It is amusing that $A_{z_i}, A_{\overline{z}_i}$, together with $A_{0_i} = 0$ may be seen as the components of some gauge field in the temporal gauge. In fact, $A^\alpha_{z_i}$ and $A^\alpha_{\overline{z}_i}$ satisfy Gauss’ law:

$$\frac{k}{2\pi} F^\alpha_{z_i \overline{z}_i} = - \sum_{j=1, j\neq i}^{N} T^\alpha_j \delta^2(z_i - z_j), \quad \alpha = 1, \cdots, \dim G, \quad (2.18)$$

where $F^\alpha_{\overline{z}_i \overline{z}_i}$ are the components of the field strength. In a sense, this result furnishes an interpretation to Witten’s Chern-Simons theory [12, 13, 14]: The topological quantum
field theory of pure Chern-Simons action can be embedded in a non-relativistic, quantum mechanical system of free particles. To see this, we consider the Schrödinger equation associated to the Feynman kernel of particle $i$:

$$i \frac{\partial}{\partial t} \psi = -\frac{1}{m_i} \left[ (\partial_{z_i} - iA_{z_i}) (\partial_{\bar{z}_i} - iA_{\bar{z}_i}) + (\partial_{\bar{z}_i} - iA_{\bar{z}_i}) (\partial_{z_i} - iA_{z_i}) \right] \psi. \quad (2.19)$$

In the limit $m_i \to 0$, a class of solutions of (2.19) consists of those wavefunctions $\psi$ satisfying

$$\left( \partial_{z_i} - iA_{z_i} \right) \psi = \left( \frac{\partial}{\partial z_i} + k \sum_{j=1,j\neq i}^{N} \frac{T_i \otimes T_j}{z_i - z_j} \right) \psi = 0, \quad (2.20)$$

$$\left( \partial_{\bar{z}_i} - iA_{\bar{z}_i} \right) \psi = \left( \frac{\partial}{\partial \bar{z}_i} + k \sum_{j=1,j\neq i}^{N} \frac{T_i \otimes T_j}{\bar{z}_i - \bar{z}_j} \right) \psi = 0. \quad (2.21)$$

These are precisely the Knizhnik-Zamolodchikov equations if we set $k = T_i = -2/(l + c_V)$, where $l$ is the level of the WZW model and $c_V$ is the quadratic Casimir of the adjoint representation of the group $G$. It is interesting to note that the wavefunctions, though non-normalizable, are the parallel transport sections of a complex vector bundle over the base manifold $M_N$.

In the context of particle statistics, (2.19) can be interpreted as the Schrödinger equation for “non-Abelian” anyons. When $T_i = T_j = 1$, $i = 1, \cdots, N$, it is the (abelian) 1-dimensional irreducible representation constructed by Wu. Therefore our construction is a non-Abelian generalization of the general theory of quantum statistics in two dimensions.

3. Spinning Path Integral Representation

The braid group representation constructed in the previous section can be generalized for particles with spin. In this section, we first propose a spinning quantization rule suitable for such purpose. The little difference with the usual supersymmetric quantum mechanics is that the eigen-wavefunction of the spinning Hamiltonian describing the dynamics in the super plane can be found before integrating out the anti-commuting axes $\theta, \bar{\theta}$. Then, using the definition of a super winding number and its charged version, we construct the spinning path integral representation of Artin’s braid group.
3.1. Spinning Quantum Mechanics

The spin degree of freedom may be described in terms of the Grassmannian variables. For a single spinning electron in the world of flat-land, the configuration space is $\mathbb{R}^2 \times Gr_2$. The non-relativistic quantum mechanics of a free, spinning particle in the flat-land can be formulated as the sum over all possible paths in the super-plane. The real commuting variables $x$ and $y$ denote the coordinates of the plane, and $\theta, \bar{\theta}$ the anti-commuting “axes” for the spin degree of freedom. Now, the dynamical variables of the particle in the configuration space can be specified by:

$$\phi(t) = z(t) + i\bar{\theta}\xi(t),$$
$$\bar{\phi}(t) = \bar{z}(t) + i\theta\bar{\xi}(t),$$

(3.1)

where $z = x + iy, \bar{z} = x - iy$ and $\xi, \bar{\xi}$ are the Grassmannian variables for the components of the spin degree of freedom in flat-land. Thus, we regard the dynamical degrees of freedom of the particle as a pair of chiral superfields. (Quantum mechanics can be seen as 1-dimensional “field” theory, the dimension being time $t$.) Notice that $\phi$ and $\bar{\phi}$ have even Grassmannian parity. The Lagrangian of a free spinning particle is

$$L = \frac{1}{2} |\dot{z}|^2 + \frac{i}{2} (\dot{\xi} \bar{\xi} - \bar{\xi} \dot{\xi}).$$

(3.2)

In terms of superfields, we have

$$L = \int d\theta d\bar{\theta} \mathcal{L},$$

(3.3)

where

$$\mathcal{L} = \frac{1}{2} (\theta \bar{\phi})(\bar{\theta} \dot{\phi}) + \frac{i}{2} (\dot{\phi} \bar{\phi} - \bar{\phi} \dot{\phi}).$$

(3.4)

In the calculation, we have adopted the following convention for the Berezin integral:

$$\int d\theta = \int d\bar{\theta} = 0,$$
$$\int d\theta \theta = \int d\bar{\theta} \bar{\theta} = 1.$$  

(3.5)

Though the form of the spinning Lagrangian is exactly the same as (3.2), there is a difference between them. The first term in (3.4) is now a product of odd variables and the second term is composed of even variables. This is the reverse of (3.2), where $z$ *In this subsection, we set all the universal constants $\hbar = c = e = 1$, as well as the mass of the particle and the magnetic field strength equal to 1.*
is even and $\xi, \bar{\xi}$ odd. The spinning Hamiltonian can be obtained from the Legendre transformation, with the canonical momenta $\mathcal{P}, \mathcal{P}, \pi, \bar{\pi}$ defined and calculated as follows.

\[
\mathcal{P} \equiv \frac{\partial \mathcal{L}}{\partial (\dot{\theta} \dot{\phi})} = \frac{1}{2} \dot{\theta} \dot{\phi},
\]

\[
\mathcal{P} \equiv \frac{\partial \mathcal{L}}{\partial (\dot{\theta} \dot{\phi})} = -\frac{1}{2} \dot{\theta} \dot{\phi},
\]

\[
\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{i}{2} \dot{\phi},
\]

\[
\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{i}{2} \dot{\phi}.
\]

The minus sign of the second expression in (3.6) is a property of the chain rule for differentiating a product of Grassmannian odd variables. The consistency of the formulation can be checked by examining whether the spinning Hamiltonian thus obtained reproduces the usual Hamiltonian after integrating over $\theta$ and $\bar{\theta}$.

\[
\mathcal{H} = (\theta \dot{\phi}) \mathcal{P} + (\bar{\theta} \dot{\phi}) \mathcal{P} + \dot{\phi} \pi + \dot{\bar{\phi}} \bar{\pi} - \mathcal{L}
\]

\[
= \frac{1}{2} (\theta \dot{\phi})(\bar{\theta} \dot{\phi}).
\]

Since

\[
\int d\theta d\bar{\theta} \mathcal{H} = \frac{1}{2} |\dot{z}|^2,
\]

we see that the spinning formulation is correct; in the absence of magnetic field, the spin degree of freedom is hidden and the energy spectrum of a free spinning particle is determined exclusively by the kinetic energy. Now we introduce the differential operators whose Grassmannian parity is odd:

\[
D_z \equiv \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z},
\]

\[
D_{\bar{z}} \equiv \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \frac{\partial}{\partial \bar{z}}.
\]

The usual quantization rule $[q_i, p_j] = i \delta_{ij}$ for pairs of canonical variables $q_i, p_i, i = 1, 2, \cdots$ takes the following form in the spinning formalism:

\[
\{\theta \dot{\phi}, \mathcal{P}\} = i \theta \dot{\phi} = -i \theta \bar{\theta},
\]

\[
\{\bar{\theta} \dot{\phi}, \mathcal{P}\} = i \bar{\theta} \dot{\phi},
\]

(3.10)

where $\{, \}$ is anticommutator since all the operators entering the bracket in (3.9) are odd. Using the definitions of $\phi, \bar{\phi}$ and $D_z, D_{\bar{z}}$, it is straightforward to calculate that

\[
\{D_z, \theta \dot{\phi}\} = \theta \bar{\theta},
\]

\[
\{D_{\bar{z}}, \bar{\theta} \dot{\phi}\} = \bar{\theta} \theta,
\]

\[
\{D_z, \bar{\theta} \dot{\phi}\} = -\theta \theta,
\]

\[
\{D_{\bar{z}}, \theta \dot{\phi}\} = -\bar{\theta} \bar{\theta},
\]

\[
\{D_z, D_{\bar{z}}\} = 0.
\]
\{D_\tau, \theta \phi \} = -\theta \theta, \quad (3.11)

Therefore we can represent the coordinates and momenta operators as \( \bar{\theta} \phi \rightarrow \bar{\theta} \phi, \theta \bar{\phi} \rightarrow \theta \bar{\phi}, \) \( \mathcal{P} \rightarrow -iD_z, \) and \( \mathcal{P} \rightarrow -iD_\tau. \) This representation is the spinning analogue of the usual Schrödinger representation.

The spinning eigen-wavefunction \( \Psi \) of \( \mathcal{H} \) can be defined with respect to the eigen-wavefunction \( \psi \) of \( H \) as follows.

\[
H \psi = \left( \int d\theta d\theta \mathcal{H} \right) \psi \\
\overset{\text{def}}{=} \int d\theta d\theta \mathcal{H} \Psi. \quad (3.12)
\]

Since if \( E \) is the eigenvalue of both \( H \) and \( \mathcal{H}, \) i.e. \( H \psi = E \psi \) and \( \mathcal{H} \Psi = E \Psi, \) we have

\[
\psi = \int d\theta d\theta \Psi. \quad (3.13)
\]

As an example of this formalism, let us consider the quantum mechanics of a spin-\( \frac{1}{2} \) particle moving in an external magnetic field which is uniform, constant and perpendicular to the plane. The minimally coupled covariant derivatives are

\[
D \equiv D_z + i\theta A_z, \\
\overline{D} \equiv D_\tau + i\bar{\theta} A_\tau, \quad (3.14)
\]

where \( B_z, B_\tau \) denote the components of the gauge field of the magnetic field. In the symmetric gauge \( B_z = -i \frac{\tau}{4}, B_\tau = i \frac{\tau}{4}, \) we have

\[
D = \frac{\partial}{\partial \theta} + \theta \left( \frac{\partial}{\partial z} + \frac{\tau}{4} \right), \\
\overline{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \left( \frac{\partial}{\partial \bar{z}} - \frac{\tau}{4} \right). \quad (3.15)
\]

One finds that

\[
\{D, \overline{D} \} = -\frac{\theta \bar{\theta}}{2}. \quad (3.16)
\]

So in the Schrödinger representation the spinning Hamiltonian is

\[
\mathcal{H} = \overline{D}D - D\overline{D} + \frac{1}{2}[\phi, \phi] \\
= 2\overline{D}D + \frac{1}{2}\theta \bar{\theta} + \frac{1}{2}[\bar{\phi}, \phi], \quad (3.17)
\]
\[[,]\] being commutator. The quantization rule for \(\phi, \bar{\phi}\) before integrating out the Grassmannian axes is the usual one: \(\{\bar{\phi}, \phi\} = \theta \bar{\theta}\). Representing \(\bar{\phi}\) and \(\phi\) as

\[
\bar{\phi} \rightarrow \sigma_+ \theta \bar{\theta} \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \theta \bar{\theta}, \\
\phi \rightarrow \sigma_- \theta \bar{\theta} \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \theta \bar{\theta},
\]

(3.18)

we have \([\bar{\phi}, \phi] = -\sigma_3 \theta \bar{\theta} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \theta \bar{\theta}\) and the Hamiltonian is diagonalized. Denoting the 2-component Pauli spinor \(\Phi\) in this basis:

\[
\Phi \equiv \Psi_{up} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Psi_{down} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(3.19)

the ground state \(\Psi_0\) is polarized: \(\Psi_0 = \Psi_{up} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). For \(\Psi_0\), the ground state energy is zero. To get an analytic form of \(\Psi_{up}\), one considers the following ground state equation:

\[
\mathcal{X} \Psi_{up} \equiv \left[ \frac{\partial}{\partial \theta} + \theta \left( \frac{\partial}{\partial z} + \frac{z}{4} \right) \right] \Psi_{up} = 0,
\]

(3.20)

Notice that the ground state \(\Psi_{up}\) is “chiral” with respect to the Grassmannian axes in the sense that

\[
X \psi_{up} = \int d\theta \mathcal{X} \Psi_{up}.
\]

(3.21)

where \(X = \frac{\partial}{\partial z} + \frac{z}{4}\) as it should, and hence \(\psi_{up} = \Psi_{up}\) in this case. One readily finds that \(\Psi_{up} = e^{-|z|^2/4}\) satisfies (3.20), for \(\frac{\partial}{\partial \theta} \Psi_{up} = 0, \theta \left( \frac{\partial}{\partial z} + \frac{z}{4} \right) \Psi_{up} = 0\). The result agrees with the standard supersymmetric quantum mechanics of a particle in the superpotential \(W_z = -i\frac{\bar{z}}{2}, W_{\bar{z}} = i\frac{z}{2}\):

\[
Q = (\sqrt{2} P_z + \frac{1}{\sqrt{2}} W_z) \sigma_+,
\]

(3.22)

\[
\overline{Q} = (\sqrt{2} P_{\bar{z}} + \frac{1}{\sqrt{2}} W_{\bar{z}}) \sigma_-,
\]

(3.23)

\[
H = Q \overline{Q} + \overline{Q} Q.
\]

(3.24)

Indeed, our spinning formalism is a variation of the same theme. The only difference is that it allows us to find some non-trivial ground states before integrating out the Grassmannian axes, as will be seen in the case of spinning fractional quantum Hall effect.
3.2. Spinning Representation

With this formulation, one can proceed to generalize the braid group representation discussed in section 2. As will be explicit from the wavefunctions to be calculated later on in section 5.2, this generalization, though straightforward, is non-trivial because the spin degree of freedom is incorporated.

For a start, let us consider two spinning particles moving freely in a super-plane. The winding number for a path going about a point \((z_0, \theta_0)\) in the super-plane is

\[
\frac{1}{2\pi i} \int dz \int d\theta \frac{\theta - \theta_0}{z - z_0 - \theta \theta_0}.
\] (3.25)

Notice that \(z - z_0 - \theta \theta_0\) is even and \(\theta - \theta_0\) odd. They are respectively the even and odd intervals of the super-plane. Following [16], denote a point in the super plane as \(Z\), we can formally write the super intervals as

\[
Z - Z_0 \equiv z - z_0 - \theta \theta_0
\]

\[
(Z - Z_0)^{\frac{1}{2}} \equiv \theta - \theta_0,
\]

which can be conveniently expressed in the following way:

\[
(Z - Z_0)^k = \begin{cases} (z - z_0 - \theta \theta_0)^k, & k \in \mathbb{Z} \\ (\theta - \theta_0)(z - z_0 - \theta \theta_0)^{k - \frac{1}{2}}, & k \in \mathbb{Z} + \frac{1}{2}. \end{cases}
\] (3.26)

In this notation, the integrand \(\frac{\theta - \theta_0}{z - z_0 - \theta \theta_0}\) can be seen as \((Z - Z_0)^{-\frac{1}{2}}\), and (3.25) is formally \(\frac{1}{2\pi i} \int dZ (Z - Z_0)^{-\frac{1}{2}}\) which looks more like the expression for the usual winding number integral \(\frac{1}{2\pi i} \int dz (z - z_0)^{-1}\). The “reason” for \((Z - Z_0)^{-\frac{1}{2}}\) instead of \((Z - Z_0)^{-1}\) is that \(\int dZ\) is odd and we need an odd integrand to make the whole integral even. In our setup, it may be rewritten as

\[
\frac{1}{2\pi i} \int d\bar{\theta} (\bar{\theta} d\phi) \int d\theta \frac{\theta - \theta_0}{z - z_0 - \theta \theta_0}.
\] (3.28)

With time \(t\) as the parameter for the path, the spinning analogue of the charged winding number is

\[
\frac{1}{2\pi i} \int_{t_0}^{t_1} dt \int d\bar{\theta} \int d\theta \frac{\theta - \theta_0}{z - z_0 - \theta \theta_0} (\bar{\theta} \dot{\phi}) T \otimes T_0,
\]

and following the same procedure, we arrive at the spinning Lagrangian:

\[
\mathcal{L} = \frac{1}{2}(\dot{\theta} \dot{\phi} (\bar{\theta} \dot{\phi}) + \mathcal{A} (\bar{\theta} \dot{\phi}) + (\dot{\theta} \bar{\phi}) \mathcal{A} + \frac{i}{2}(\bar{\phi} \dot{\phi} - \dot{\phi} \bar{\phi}),
\]

(3.30)
where, \( l, \overline{l} \) being any real numbers,

\[
\mathcal{A}_z = il \frac{\theta - \theta_0}{z - z_0 - \theta \theta_0} T \otimes T_0,
\]

\[
\overline{\mathcal{A}_z} = il \frac{\overline{\theta} - \overline{\theta}_0}{\overline{z} - \overline{z}_0 - \theta \overline{\theta}_0} \overline{T} \otimes \overline{T}_0.
\]  

(3.31)

As before, \( T \) and \( T_0 \) are the respective representations carried by the winding particle and its counterpart which appears as a puncture. In the Schrödinger representation, the spinning Hamiltonian becomes

\[
\mathcal{H} = \Pi \Pi - \Pi \Pi,
\]  

(3.32)

where

\[
\Pi = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} + i \mathcal{A}_z,
\]

\[
\Pi = \frac{\partial}{\partial \theta} + \overline{\theta} \frac{\partial}{\partial \overline{z}} + i \overline{\mathcal{A}_z}.
\]  

(3.33)

We can readily write down the Hamiltonian of \( N \) spinning particles. The zero-energy states of the \( N \)-body Hamiltonian can be easily found from the first order equations which are the supersymmetric generalization of the ones that appeared in [15]:

\[
\left[ \frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial z_i} - l \sum_{j=1, j \neq i}^{N} \frac{\theta_i - \theta_j}{z_i - z_j - \theta_i \theta_j} T_i \otimes T_j \right] \Psi = 0,
\]

\[
\left[ \frac{\partial}{\partial \overline{\theta}_i} + \overline{\theta}_i \frac{\partial}{\partial \overline{z}_i} - \overline{l} \sum_{j=1, j \neq i}^{N} \frac{\overline{\theta}_i - \overline{\theta}_j}{\overline{z}_i - \overline{z}_j - \overline{\theta}_i \overline{\theta}_j} \overline{T}_i \otimes \overline{T}_j \right] \Psi = 0.
\]  

(3.34)

These supersymmetric Knizhnik-Zamolodchikov equations have been discussed extensively in the literature [17]. They originate from the null vectors of the combined representation of Kac-Moody algebra and super Virasoro algebra. In the path integral approach, it is explicit that they give the covariant horizontality condition with respect to the flat connection

\[
\Omega = -l \sum_{k=1, k \neq j}^{N} \frac{\theta_j - \theta_k}{z_j - z_k - \theta_j \theta_k} T_i \otimes T_j \, dz_j d\theta_j,
\]  

(3.35)

which is the spinning analogue of the Kohno connection [18] of a holomorphic bundle.

We have therefore constructed a representation of Artin’s braid group \( B_N \) with Feynman kernels of spinning particles. The threads of \( B_N \) correspond to the spinning world lines. As in the spinless case, the representation space contains the space of correlation functions of super WZW theories. We have thus made an explicit link between spinning anyons and super WZW model. The factorizable ground states of spinning anyons are given by the exact solutions of super Knizhnik-Zamolodchikov equations (3.34).
4. Polarized Ground States of FQHE

The quantum Hall effect [19] is a rather unusual collective transport phenomenon of two-dimensional electron gas. When the external magnetic field is strong, the thermal fluctuation is suppressed at low temperature, and the mobility of the charge carriers is high etc., the Hall conductance has a staircase dependence on the magnetic field strength. Concomitantly, the longitudinal conductivity is practically zero at the centre of the plateau. To understand the peculiarity of the Hall effect at these extreme conditions, it is essential to find the many-body ground state of the quantum system. In the case of the integer quantum Hall effect, the system is a collection of simple harmonic oscillators. The Landau level provides the necessary energy gap that supports the plateaux of Hall conductivities at integral multiples of $\frac{e^2}{h}$. However, for the fractional Quantum Hall effect (FQHE), the incompressibility of the liquid is less straightforward. Additional ideas are needed to account for the experimental discoveries of FQHE.

4.1. Laughlin Ground State

The starting point of a plausible theory of FQHE is Laughlin’s ansatz [2]:

\[ |m\rangle = \prod_{j<k} (z_j - z_k)^m \exp\left(-\frac{1}{4l^2} \sum_i |z_i|^2\right), \tag{4.1} \]

where \( l = \sqrt{\frac{\hbar c}{eB}} \) is the magnetic length, \( \hbar, c \) being the usual universal constants, \( e \) is the charge of the electron and \( B \) is the strength of the magnetic field. It is postulated that \( |m\rangle \) is the ground state of the electrons exhibiting FQHE with fractional filling factor $\frac{1}{m}$. The reason why \( m \) is odd is because \( |m\rangle \) describes a system of electrons which have fermionic statistics. In [8], we have proposed a Hamiltonian \( H \) (4.3) for which \( |m\rangle \) is the exact ground state. The \( N \)-body Hamiltonian contains Kohno connection [18]

\[ A_{z_j} = i\hbar \sum_{k=1, k \neq j}^{N} T_j \otimes T_k \frac{z_j - z_k}{z_j - z_k}, \]

\[ A_{\overline{z_j}} = i\hbar \sum_{k=1, k \neq j}^{N} T_j \otimes T_k \frac{\overline{z_j} - \overline{z_k}}{\overline{z_j} - \overline{z_k}}, \tag{4.2} \]

which reflects the topological properties of the configuration space as mentioned in section 2. Let \( m^* \) be the effective mass of the electron, \( B_{z_j}, B_{\overline{z_j}} \) the components of the gauge field.

*For a quick review of quantum Hall effect, see appendix B of [20].
of the external magnetic field, the Hamiltonian is
\[
H = \frac{1}{m^*} \sum_{j=1}^{N} \left[ (-i\hbar \partial_z + \frac{e}{c} B_{z_j} + A_{z_j})(-i\hbar \partial_{\bar{z}} + \frac{e}{c} B_{\bar{z}_j} + A_{\bar{z}_j}) + (-i\hbar \partial_{\bar{z}} + \frac{e}{c} B_{\bar{z}_j} + A_{\bar{z}_j})(-i\hbar \partial_z + \frac{e}{c} B_{z_j} + A_{z_j}) \right]
\]
\[= \frac{\hbar^2}{m^*} \sum_{j=1}^{N} \left( D_{z_j} D_{\bar{z}_j} + D_{\bar{z}_j} D_z \right), \tag{4.3} \]

where
\[
D_{z_j} \equiv \partial_{z_j} + i \frac{e}{\hbar c} B_{z_j} + i \frac{\hbar}{\hbar} A_{z_j}, \tag{4.4} \\
D_{\bar{z}_j} \equiv -\partial_{\bar{z}_j} - i \frac{e}{\hbar c} B_{\bar{z}_j} - i \frac{\hbar}{\hbar} A_{\bar{z}_j}, \tag{4.5} \\
\]

Now, since all the particles are indistinguishable, they carry the same representation. Thus, for any two particles \(k, j\), we have \(T_j = T_k\) and \(T_j = T_j\), \(j = 1, \cdots, N\). One may use hermitian matrices to represent \(T_j^\alpha, \alpha = 1, \cdots, \dim G\). In the symmetric gauge,
\[
B_{z_j} = -i \frac{B}{4} \zeta_j, \\
B_{\bar{z}_j} = i \frac{B}{4} \zeta_j, \\
m = -m, \tag{4.6} \\
\]
one calculates the commutator of \(D_{z_i}\) and \(D_{z_j}\):
\[
\left[ D_{z_j}, D_{\bar{z}_j} \right] = \frac{eB}{2\hbar c} + 2\pi m \sum_{k=1, k \neq j}^{N} \delta^{(2)}(z_j - z_k) T_j \otimes T_k. \tag{4.7} \]

The term \(\frac{eB}{2\hbar c}\) is related to the zero-point energy of a simple harmonic oscillator, whereas the Dirac delta functions arise from the 2-dimensional Green function of the plane:
\[
\partial_z \frac{1}{z - w} = -\pi \delta^{(2)}(z - w), \tag{4.8} \\
\partial_{\bar{z}} \frac{1}{z - w} = -\pi \delta^{(2)}(z - w). \tag{4.9} \\
\]

With \(\omega \equiv \frac{eB}{m^* c}\), we can rewrite (4.3) as
\[
H = \frac{2\hbar^2}{m^*} \sum_j D_{z_j} D_{z_j} + \frac{N}{2} \hbar \omega + \frac{2\hbar^2}{m^*} \pi m \sum_j \sum_{k=1, k \neq j} \delta^{(2)}(z_j - z_k) T_j \otimes T_k. \tag{4.10} \]
Since this Hamiltonian is derived from the assumption that the underlying configuration space is not simply connected, the ground state of $H$ can be obtained by letting $z_j \neq z_k$ for all $j$ and $k$, and then consider the following first order equation for $j$-th electron:

$$D_{z_j} \psi_{0j} = \left[ \partial_{z_j} + \frac{eB}{4\hbar c} z_j - m \sum_{k=1, k \neq j}^{N} \frac{T_j \otimes T_k}{z_j - z_k} \right] \psi_{0j} = 0. \quad (4.11)$$

Physical considerations require $f_j$ to be holomorphic. As discussed by Laughlin [2], the many-body wavefunction comprises only of single-body wavefunctions lying in the lowest Landau level. This idealization is valid, in view of the facts that there are only enough electrons to fill the lowest Landau level and that the cyclotron energy $\hbar \omega$ is much greater than Coulomb interaction. Overlaps with contributions from higher Landau levels are practically negligible. Writing

$$\psi_{0j} = \exp(-\frac{1}{4l^2} |z_j|^2) f_j(z_1, \ldots, z_N), \quad (4.12)$$

equation (4.11) then becomes

$$\partial_{z_j} f_j(z_1, \ldots, z_N) - m \sum_{k=1, k \neq j}^{N} \frac{T_j \otimes T_k}{z_j - z_k} f_j(z_1, \ldots, z_N) = 0. \quad (4.13)$$

Thus, we see that *chiral* Knizhnik-Zamolodchikov equations are relevant in FQHE. (These equations have also been used to explore the possibility of non-abelian Aharanov-B"{o}hm effect [21].) For $T_j = 1, j = 1, \ldots, N$, the holomorphic function satisfying (4.13) is

$$f_j(z_1, \ldots, z_N) = \text{const} \prod_{k=1, k \neq j}^{N} (z_j - z_k)^m. \quad (4.14)$$

For $m > 0$, $f_j$ vanishes whenever $z_j$ coincides with any other $z_k$. In other words, particle $j$ is kept apart from the other electrons. This solution is consistent with the repulsive delta-function potential $\sum_{k=1, k \neq j} \delta^{(2)}(z_j - z_k)$, because for any $j$,

$$\int dz_j dz_j \left( \sum_{k=1, k \neq j} \delta^{(2)}(z_j - z_k) \right) |f_j|^2 = 0. \quad (4.15)$$

Though $f_j$ is not normalizable, $\psi_{0j}$ is, thanks to the factor $\exp(-\frac{1}{4l^2} |z_j|^2)$ contributed by the strong magnetic field. Solving $D_{z_j} \psi_{0j} = 0$ for arbitrary $j$, we find that the solution is exactly the Laughlin wavefunction:

$$\psi_0 = \text{const}. \prod_{j<k} (z_j - z_k)^m \exp\left(-\frac{1}{4l^2} \sum_i |z_i|^2 \right). \quad (4.16)$$
Because $\psi_0$ is the many-body wavefunction of electrons, $m$ is an odd number. From these results, one is able to identify the physical origin of FQHE with filling fractions $\frac{1}{p}$, $p = m$: Since the configuration space is multiply-connected, one has to consider the minimal coupling of the Kohno connection in addition to the electromagnetic gauge potential. The factor $\prod_{j<k}(z_j - z_k)^m$ bears testimony to the non-simply connected nature of the topology; Kohno connection arises as homotopical labels of the paths in terms of charged winding numbers \(^8\).

4.2. Spinning Analogue of the Laughlin State

While FQHE with odd $p$ stems from the braid group representation associated with the non-simply connected configuration space $M_N$, it is of interest to examine if the spinning braid group representation associated to particles with spin in the “puncture” phase will also yield FQHE. Put differently, when the spin degree of freedom is turned on, we want to know if there is an incompressible ground state exhibiting FQHE. For this purpose, we consider the Hamiltonian:

$$H = \sum_j \int d\theta_j d\theta_j \mathcal{H}_j, \quad (4.17)$$

$$\mathcal{H}_j = \frac{\hbar^2}{m^*} (\overline{\mathcal{D}}_j D_j - D_j \overline{\mathcal{D}}_j) - g\mu B \sigma_3 \theta_j \overline{\theta}_j. \quad (4.18)$$

Each $\mathcal{H}_j$ is the spinning Hamiltonian of particle $j$. $g\mu B$ is the Zeeman energy, $g$ the $g$-factor and $\mu$ denotes the magnetic moment of the electron. In the symmetric gauge (4.6), the covariant derivatives are

$$D_j = \partial_{\theta_j} + \theta_j \left( \frac{\partial}{\partial z_j} + \frac{eB}{4\hbar c} \overline{z}_j \right) - m \sum_{k=1, k\neq j}^N \frac{\theta_j - \theta_k}{z_j - z_k - \theta_j \theta_k} T_j \otimes T_k, \quad (4.19)$$

$$\overline{D}_j = \partial_{\overline{\theta}_j} + \overline{\theta}_j \left( \frac{\partial}{\partial \overline{z}_j} - \frac{eB}{4\hbar c} \overline{z}_j \right) - m \sum_{k=1, k\neq j}^N \frac{\overline{\theta}_j - \overline{\theta}_k}{\overline{z}_j - \overline{z}_k - \overline{\theta}_j \overline{\theta}_k} \overline{T}_j \otimes \overline{T}_k. \quad (4.20)$$

Now, write $D_j \equiv \partial_{\theta_j} + \theta_j \partial_{z_j}$, $\overline{D}_j \equiv \partial_{\overline{\theta}_j} + \overline{\theta}_j \partial_{\overline{z}_j}$; we have $D_j \overline{D}_j = -\overline{D}_j D_j$, because $D_j$ and $\overline{D}_j$ are odd differential operators. With this consideration, the Green functions of the super-plane are

$$D_j \left( \frac{1}{z_j - z_0 - \theta_j \theta_0} \right) = -\pi \delta^2(z_j - z_0 - \theta_j \theta_0), \quad (4.21)$$

$$\overline{D}_j \left( \frac{1}{z_j - z_0 - \theta_j \theta_0} \right) = +\pi \delta^2(z_j - z_0 - \theta_j \theta_0). \quad (4.22)$$

So in the symmetric gauge (4.6), and when all the particles are identical, the anticommu-
The operator is
\[ \{ \mathcal{D}_j, \mathcal{D}_j \} = -\frac{eB}{2\hbar c} \theta_j \mathcal{D}_j - 2\pi m \sum_{j=1, k \neq j} \delta^2(z_j - z_k - \theta_j \theta_k) T_j \otimes T_k. \] (4.23)

Again, we see that Dirac delta functions appear. They prevent two particles from occupying the same point at the same instance in the super-plane. The spinning Hamiltonian of particle \( j \) becomes
\[ H_j = \frac{2\hbar^2}{m^*} \mathcal{D}_j \mathcal{D}_j + \frac{1}{2} \hbar \omega \theta_j \mathcal{T}_j - g \mu B \sigma_3 \theta_j \mathcal{T}_j \]
\[ + \frac{2\hbar^2}{m^*} \pi m \sum_{j=1, k \neq j} \delta^2(z_j - z_k - \theta_j \theta_k) T_j \otimes T_k. \] (4.24)

It is implicit in the Hamiltonian that the spin of each electron is aligned either parallel (up) or anti-parallel (down) with respect to the external magnetic field. Only the spin components normal to the direction of the magnetic field enter as dynamical variables. To find the spin-polarized ground state with zero energy, we need to consider \( \Psi_{up} \) such that
\[ \text{for arbitrary } j, \mathcal{D}_j \Psi_{up} = 0. \]
In addition, due to the presence of the repulsive interaction term of infinitesimal range, \( \Psi_{up} \) must contain a factor which is some positive power of \( (z_j - z_k - \theta_j \theta_k) \). As before, we write \( T_j = 1, j = 1, \ldots, N, \) and
\[ \mathcal{F}_j = \text{const.}(-1)^{j-1} \prod_{1 \leq j < k \leq N} (z_j - z_k - \theta_j \theta_k)^m. \] (4.25)

It is easy to show that
\[ \Psi_{j up} = \int d\theta_1 \cdots d\theta_{j-1} d\theta_{j+1} \cdots d\theta_N \mathcal{T}_j \exp(-\frac{1}{4l^2} \sum_i |z_i|^2) \mathcal{F}_j \] (4.26)

satisfies the ground state equation:
\[ \mathcal{D}_j \Psi_{j up} = \left[ \frac{\partial}{\partial \theta_j} + \theta_j \left( \frac{\partial}{\partial z_j} + \frac{eB}{4\hbar c} \right) - m \sum_{k=1, k \neq j}^N \frac{\theta_j - \theta_k}{z_j - z_k - \theta_j \theta_k} \right] \Psi_{j up} = 0. \] (4.27)

In particular, it is worth remarking that \( \mathcal{F}_j \) is the conformal block of the super \( U(1) \) current algebra:
\[ \left( D_j - m \sum_{k=1, k \neq j}^N \frac{\theta_j - \theta_k}{z_j - z_k - \theta_j \theta_k} \right) \mathcal{F}_j = 0. \] (4.28)

Using the many-body analogue of (3.12), (3.13), namely
\[ H \Psi_{0 up}^j = \sum_j \left( \int d\theta_j d\theta_j \mathcal{H}_j \Psi_{j up} \right), \] (4.29)
we have

\[ \psi_{0}^{up} = N \text{const.} \int \prod_{j=1}^{N} d\theta_j \prod_{1 \leq j < k \leq N} (z_j - z_k - \theta_j \theta_k)^m \exp\left(-\frac{1}{4l^2} \sum_{i} |z_i|^2\right). \]  

(4.30)

Now, since \( \theta_j^2 = \theta_k^2 = 0 \) for all \( j \), we see that

\[ \prod_{1 \leq j < k \leq N} (z_j - z_k - \theta_j \theta_k)^m = \prod_{1 \leq j < k \leq N} \left[(z_j - z_k)^m \left(1 - \frac{\theta_j \theta_k}{z_j - z_k}\right)^m\right] \]

\[ = \prod_{1 \leq j < k \leq N} (z_j - z_k)^m \prod_{1 \leq j < k \leq N} \left(1 - m \frac{\theta_j \theta_k}{z_j - z_k}\right). \]  

(4.31)

In the expansion of (4.31), the terms that do not vanish under the operation \( \int \prod_{j=1}^{N} d\theta_j \) must contain \( \prod_{j=1}^{N} \theta_{\sigma(j)} \). This is possible only if \( N \) is an even number. In this case,

\[ \int d\theta_1 \cdots d\theta_N \prod_{1 \leq j < k \leq N} \left(1 - m \frac{\theta_j \theta_k}{z_j - z_k}\right) = \frac{m^N}{2^N N!} \sum_{\sigma} (-1)^{\sigma} \left(\frac{1}{z_{\sigma_1} - z_{\sigma_2}}\right) \cdots \left(\frac{1}{z_{\sigma_{N-1}} - z_{\sigma_N}}\right) \equiv m^N \text{Pf} \left(\frac{1}{z_j - z_k}\right) \]  

(4.32)

Here \( \sigma \) runs over permutations of the \( N \) indices, \((-1)^{\sigma}\) is the parity of the permutation. The expression \( \text{Pf}(M_{jk}) \) is called the Pfaffian of an antisymmetric \( N \times N \) matrix \( M \) with entries \( M_{jk} \). Now, \( \psi_{0}^{up}(z_1, \cdots, z_N) \) is the physical wavefunction describing an ensemble of electrons. Any interchange of arbitrary pair of coordinates must result in a negative sign as Pauli principle says. Consequently, \( m \) must be an even number since Pfaffian is antisymmetric. We remark that (4.30) is exactly the same as the Moore-Read ansatz for spin polarized FQHE states at even denominator filling fractions.

In this manner, we have unveiled the physical origin of the Laughlin states and the Moore-Read states. The non-trivial topology of the configuration space of \( N \) electrons in the “puncture” phase is manifested in the Hamiltonians (4.10) and (4.18). Respectively, they yield the Laughlin state and the Moore-Read state as exact non-degenerate ground state solutions.

5. Topological Excitations

*Throughout the paper, we only mention quasi-hole excitation. The quasi-particle is taken to be the particle-hole conjugate of the quasi-hole.
One of the necessary conditions for a many-body ground state to display FQHE is that its quantum excitations are massive. Among other things, it behooves the system to be non-degenerate across a sufficiently finite range of variation in the background magnetic field strength. In other words, the collection of electrons in the “puncture” phase must be capable of buffering a certain amount of excess or deficiency in the quantum flux tubes in the form of excited states in the energy spectrum. The crucial point is that these excited states must lie within the large gap of $\hbar\omega$ between two neighbouring Landau levels, if FQHE plateaux are to take shape. The existence of such a substratum structure superimposed over the Landau levels of a collection of oscillators is a key to the understanding of FQHE.

In [2], Laughlin gave an ansatz of the wavefunction which is a 1-quasi-hole excitation of the ground state $|m\rangle$ (4.1): 

$$
\psi_m(w; z_1, \ldots, z_N) = \prod_{j=1}^{N} (z_j - w)|m\rangle,
$$

where $w$ is the position of the quasi-hole. The existence of the quasi-hole excitation is demonstrated in the gedanken experiment. An infinitesimally thin solenoid is pierced through the ground state $|m\rangle$ at position $w$. Adiabatically, a flux quantum $\hbar c/e$ is added; $|m\rangle$ evolves in such a way that it remains an eigenstate of the changing Hamiltonian. After the flux tube is completely installed, the resulting Hamiltonian is related to the initial one by a (singular) gauge transformation. To get back to the original Hamiltonian, the flux tube is gauged away, leaving behind an excited state $\psi_m(w; z_1, \ldots, z_N)$. This idea is strongly reminiscent of the Aharanov-Böhm effect.

The interesting and strange feature of FQHE is that the charge $q_h$ of the quasi-hole is fractional. The exact value can be determined via the plasma analogy. The square of the wavefunction $\psi_m$ can be interpreted as a probability distribution function of a plasma:

$$
|\psi_m(w; z_1, \ldots, z_N)|^2 = e^{-\beta E},
$$

where $\beta = m$ plays the role of inverse temperature, and the Gibbs energy $E(w; z_1, \ldots, z_N)$ is given by

$$
E(w; z_1, \ldots, z_N) = -2\sum_{j<k} \log|z_j - z_k| + \frac{1}{2ml^2} \sum_j |z_j|^2 - \frac{2}{m} \sum_{i=1}^{N} \log|z_i - w|.
$$

(5.3)
This is the total energy of a gas of $N$ classical particles each carrying charge $q = 1$ plus a particle of charge $q_h = \frac{1}{m}$ which repel each other through the 2-dimensional “Coulomb” potential $-2 \sum_{j<k} \log |z_j - z_k|$ in a uniform neutralizing background charge of density $\rho_0 = \frac{1}{2\pi ml^2}$. It is clear that the first two terms in (5.3) are contributed by the ground state $|m\rangle$. The charge $q$ being 1 is related to the fact that representation $T_j = 1$ is chosen for each electron.

Motivated by this physical picture, we can carry the plasma analogy further and consider the same Hamiltonian (4.3) for the 1-quasi-hole excitation but with a gauge transformed $A_{z_j}, A_{\bar{z}_j}$:

$$
A_{z_j} \to im\hbar \sum_{k=1,k \neq j}^N \left( \frac{1}{z_j - z_k} + \frac{\frac{1}{m}}{\bar{z}_j - w} \right),
$$

$$
A_{\bar{z}_j} \to -im\hbar \sum_{k=1,k \neq j}^N \left( \frac{1}{\bar{z}_j - z_k} + \frac{\frac{1}{m}}{z_j - w} \right). \tag{5.4}
$$

It can be easily verified that $\psi_m$ satisfy the ground state equations, $j = 1, \ldots, N$:

$$
\left[ \partial_{z_j} + \frac{eB}{4\hbar c} \bar{z}_j - m \sum_{k=1,k \neq j}^N \left( \frac{1}{z_j - z_k} + \frac{\frac{1}{m}}{z_j - w} \right) \right] \psi_m = 0. \tag{5.5}
$$

Remember that $A_{z_j}, A_{\bar{z}_j}$ comes from the charged winding number constraint of the paths of particle $j$ in the multiply-connected configuration space. Attaching an additional solenoid on $|m\rangle$ therefore results in a new configuration space. In other words, electron $j$ sees the quasi-hole as a puncture as well, but this time with charge $q_h = \frac{1}{m}$. The excitation is topological in nature. When a quasi-hole develops, the configuration space is topologically changed. It is no longer $M_N$, but $M_{N+1}$.

So far, we are only concerned with one quasi-hole excitation at $w$. What about the wavefunctions of two or more quasi-holes? As discussed by Halperin [22], these multi-excitation states should be an analytic function of the coordinates of the electrons $z_1, \ldots, z_N$, and of the quasi-holes $w_1, \ldots, w_{N_h}$ up to exponential factors. The analytic condition is to require that even the excitation wavefunctions should only come from the lowest Landau level. The Halperin ansatz is

$$
\psi_m(w_1, \ldots, w_{N_h}) = \prod_{1 \leq j<k \leq N_h} (w_j - w_k)^{\frac{1}{m}} \exp\left(-\frac{1}{4ml^2} \sum_i |z_i|^2 \right) \prod_{j,k}(w_j - z_k)|m\rangle. \tag{5.6}
$$

If we write

$$
|\frac{1}{m}\rangle = \prod_{1 \leq j<k \leq N_h} (w_j - w_k)^{\frac{1}{m}} \exp\left(-\frac{1}{4ml^2} \sum_i |w_i|^2 \right), \tag{5.7}
$$

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which is of the same form as Laughlin’s ground state $|m\rangle$, we find that

$$
\psi_m = \left| \frac{1}{m} \right| m \prod_{j,k} (w_j - z_k).
$$

(5.8)

Written in this form, the physical content of a collection of quasi-holes $|\frac{1}{m}\rangle$ is explicit: they are just “electrons” of (representation) charge $q_h = \frac{1}{m}$ each in the “puncture” phase! The quasi-holes are also under the influence of the external magnetic field, for the exponential factor $\exp\left(-\frac{1}{4\pi}\sum_i |w_i|^2\right)$ is required to make the wavefunction well defined under normalization. The Hamiltonian for two species of electrons labelled by $q = 1$ and $q_h = \frac{1}{m}$ is

$$
H = \frac{2\hbar^2}{m^*} \sum_j D_{z_j} D_{z_j} + \frac{N}{2} \hbar \omega + \frac{2\hbar^2}{m^*} \pi m \sum_j \sum_{k=1, k \neq j} \delta^{(2)}(z_j - z_k) \\
+ \frac{2\hbar^2}{m_h} \sum_j d_{w_j} d_{w_j} + \frac{N_h \hbar}{2} \hbar \omega_h + \frac{2\hbar^2}{m_h} \frac{1}{m} \sum_j \sum_{k=1, k \neq j} \delta^{(2)}(w_j - w_k) \\
+ 2\hbar^2 \pi \left( \frac{1}{m^*} + \frac{1}{m_h} \right) \sum_j \sum_k \delta^{(2)}(w_j - z_k).
$$

(5.9)

where

$$
D_{z_j} = \partial_{z_j} + \frac{eB}{4\hbar c} z_j - m \sum_{k=1, k \neq j}^N \frac{1}{z_j - z_k} - m \sum_{k=1}^{N_h} \frac{1}{z_j - w_k},
$$

$$
d_{w_j} = \partial_{w_j} + \frac{\pi B}{4\hbar c} w_j - m \sum_{k=1, k \neq j}^{N_h} \frac{1}{w_j - w_k} - m \sum_{k=1}^N \frac{1}{w_j - z_k},
$$

(5.10)

with similar expressions for $D_{\overline{z}_j}$ and $d_{\overline{w}_j}$. We have denoted the “mass” of a quasi-hole as $m_h$, and $\omega_h = \frac{eB}{m_h c}$ is the angular frequency of the cyclotron motion of the quasi-holes.

It is readily shown that $\psi_m(w_1, \cdots, w_{N_h}; z_1, \cdots, z_N)$ is the exact ground state solution of $H$ (5.9). If these many-quasi-hole wavefunctions describe real physics as Halperin suggested, so does $H$. Furthermore, the path integral representation approach which reveals the relevance and meaning of the Kohno connection allows one to see explicitly that quasi-holes behave as if they were spinless particles of charge $-\frac{e}{m}$ in the “puncture” phase. The picture which emerges from $H$ (5.9) may be captured in Figure 6. Our results show that the topological excitation also has a Landau level structure for its spectrum. The FQHE ground state thus comprises of two species of “punctures”, namely, electrons and quasi-holes. The ground state energy of the quasi-hole is the energy gap responsible for the incompressibility of the FQHE liquid.
In order to support this interpretation, one has to have an answer to the pressing question: What is the mass $m_h$ of the quasi-hole?

As we learn in nuclear physics, the binding energy of nucleons can be equated with $\delta mc^2$ if $\delta m$ is the mass difference between a nucleus and the total of its fission moitities. It is tempting to apply this popularly known $\delta E = \delta mc^2$ formula to FQHE as well:

$$\delta E = \hbar \omega_h = m_h c^2.$$  

(5.11)

The mass of the electron $m_e$ in the crystal lattice is not its rest mass in the vacuum but gets modified to $m^* = xm_e$ where $x$ is a dimensionless number. By the same token, since the quasi-hole excitation is treated as if it is some spinless electron with fractional charge, $m_h$ must also be modified to $m_h^* = ym_h$ for some empirical factor $y$. Then, we find that the energy gap of the quasi-hole excitation is

$$\delta E = y C \sqrt{B},$$  

(5.12)

with $C = \sqrt{\frac{\epsilon}{m^*hc}}$. It is interesting to note that this interpretation also leads to a square root dependence of the energy gap $\delta E$ on $B$. The proportional constant $C$ is determined solely by the absolute value of the fractional charge $\frac{\epsilon}{m}$ and the universal constants.

Except for the threshold\footnote{A possible origin of threshold is discussed in section 7.2.}, the $\sqrt{B}$ dependence is quite in line with experiments\cite{23}. Of course, the many-body quantum mechanics we have here is oversimplified in the sense that the imperfections of the GaAs-AlGaAs heterostructure, the thickness of the heterojunction, the mixing of higher Landau levels \emph{etc} are neglected. Nevertheless
the main characteristics of FQHE such as the quantum statistics, the exact value of the fractional charge are sufficiently robust even in the presence of those complications and the plausibility of a simple Hamiltonian like (5.9) is warranted.

One of the implications of expression (5.12) is that the ratio of the energy gaps of \( \nu_a = \frac{1}{m_a} \) and \( \nu_b = \frac{1}{m_b} \) FQHE is given by

\[
\sqrt{\frac{m_b B_a}{m_a B_b}} \tag{5.13}
\]

where \( m_a \) and \( m_b \) are both odd numbers and \( B_a \) and \( B_b \) are the magnetic field strengths at the centres of the respective FQHE plateaux.

In an analogous fashion, we can also study the quasi-hole excitation of the Moore-Read state. The spinning analogue of (5.9) is

\[
\mathcal{H} = \sum_j \left[ \frac{2\hbar^2}{m^*} D_{z_j} D_{\bar{z}_j} + \frac{1}{2} \hbar \omega_j \bar{\theta}_j - g \mu B \sigma_3 \theta \bar{\theta}_j + \frac{2\hbar^2}{m^*} \pi m \sum_{k=1, k \neq j} \delta^{(2)} (z_j - z_k - \theta_j \theta_k) \right. \\
+ 2\hbar^2 \pi \frac{1}{m^*} \sum_k^{N_h} \delta^{(2)} (z_j - w_k - \theta_j \eta_k) \left. \right] + \sum_j \left[ \frac{2\hbar^2}{m_h} \Delta_{\bar{w}_j} \Delta_{w_j} + \frac{1}{2} \hbar \omega_h \eta \bar{\eta}_j - g_h \mu_h B \sigma_3 \eta \bar{\eta}_j + \frac{2\hbar^2}{m_h} \pi \frac{1}{m} \sum_{k=1, k \neq j} \delta^{(2)} (w_j - w_k - \eta_j \eta_k) \right. \\
+ 2\hbar^2 \pi \frac{1}{m_h} \sum_k^{N} \delta^{(2)} (w_j - z_k - \eta_j \theta_k) \right]
\tag{5.14}
\]

where \( D_{z_j}, \Delta_{w_j} \) are the Grassmannian odd covariant derivatives:

\[
D_{z_j} = \frac{\partial}{\partial \theta_j} + \theta_j \left( \frac{\partial}{\partial z_j} + \frac{eB}{4\hbar c} \bar{z}_j \right) - m \sum_{k=1, k \neq j}^{N_h} \frac{\theta_j - \theta_k}{z_j - z_k - \theta_j \theta_k} - m \sum_{k=1}^{N_h} \frac{1}{m} (\theta_j - \eta_k) \\
\Delta_{w_j} = \frac{\partial}{\partial \theta_j} + \eta_j \left( \frac{\partial}{\partial w_j} + \frac{eB}{4\hbar c} \bar{w}_j \right) - m \sum_{k=1, k \neq j}^{N_h} \frac{1}{m} (\eta_j - \eta_k) \frac{w_j - w_k - \eta_j \eta_k}{w_j - w_k - \eta_j \theta_k} - m \sum_{k=1}^{N_h} \frac{1}{m} (\eta_j - \theta_k) 
\tag{5.15}
\]

The many-quasi-hole wavefunction that is the ground state solution of \( \mathcal{H} \) (5.14) is

\[
(N + N_h) \text{const.} \int \prod_{j=1}^{N} d\theta_j \prod_{j=1}^{N_h} d\eta_j \prod_{j < k} (z_j - z_k - \theta_j \theta_k)^m \prod_{j < k} (w_j - w_k - \eta_j \eta_k)^{\frac{m}{2}} \\
\times \prod_{j,k} (w_j - z_k - \eta_j \theta_k) \exp\left(-\frac{1}{4l^2} \sum_i |z_i|^2 - \frac{1}{4ml^2} \sum_i |w_i|^2 \right)
\]

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\[(N + N_h)m^{N - N_h} \text{ const.} \prod_{j<k} (z_j - z_k)^m \prod_{j<k} (w_j - w_k)^{1\over m} \prod_{j,k} (w_j - z_k)\]
\[
\times \exp \left(-\frac{1}{4l^2} \sum_i |z_i|^2 - \frac{1}{4ml^2} \sum_i |w_i|^2\right) \text{Pf} (M_{ij}) \tag{5.16}
\]

where \(M_{ij} = \frac{1}{u_i - u_j}\), \(u_i\) being the combined set of \(z_i, i = 1, \cdots, N\) and \(w_i, i = 1, \cdots, N_h\).

We emphasize that both \(N\) and \(N_h\) must be even numbers. Therefore in the spinning case, the quasi-hole excitations are paired.

6. FQHE Ground State of Spin Singlets

So far, we are only concerned with spin-polarized ground states, \textit{i.e.} all the spins align themselves parallel to the magnetic field normal to the plane. In view of the large Zeeman energy when the magnetic field is strong, it is justifiable to assume that the spins are fully polarized.

However, experimental data reveal that partially polarized FQHE ground states also exist. In particular, FQHE at a \textit{shared} filling factor of \(8\over 5\) was observed to transit from a spin-unpolarized state to a polarized one when the specimen was tilted with respect to the magnetic field [6]. This experimental result is quite in line with Halperin’s original suggestion [22]: The \(g\)-factor of GaAs is rather small; hence, when the magnetic field is not too strong, spin unpolarized states should be viable. In this scenario, Zeeman energy cost \(g\mu B\) is low enough for some spins to get reversed.

To accommodate the spin degree of freedom parallel or anti-parallel to the external field, it is useful to consider the Hamiltonian (4.3), or equivalently (4.10), with each \(T_j\) carrying the representation of \(U(2)\) which is isomorphic to \(SU(2) \times U(1)\). As discussed earlier, we ignore the Zeeman energy which is of the same order of magnitude as the static Coulomb energy at characteristic length (magnetic length \(l\)); for the time being, we just want to study the \textit{topological} “interaction”. Intuitively, it is not hard to realize that such Hamiltonian corresponds to the situation where each electron carries a spin-1\(\over 2\) (highest weight) representation of \(SU(2)\) and a \(U(1)\) charge. In the “puncture” phase, when one electron moves around the other, a \textit{non-abelian} charged winding number (2.3) furnishes a topological label for the path; not only does an electron see the \(U(1)\) charges, it also perceives the spins on other electrons.

In the non-abelian analogue, the ground state solution of such Hamiltonian is found
by solving the equation for all particles $j$:

$$
[\partial z_j + \frac{eB}{4\hbar c} z_j - \ell_{\text{spin}} \sum_{k=1, k \neq j}^{N} \frac{T_j \otimes T_k}{z_j - z_k} - \ell_{\text{charge}} \sum_{k=1, k \neq j}^{N} \frac{1}{z_j - z_k}] \psi_{0,j} = 0. \quad (6.1)
$$

From the physical viewpoint, the fundamental weights of the representations have to be chosen in such a way that the resulting wavefunction is a singlet. This is the non-abelian analogue of the neutrality condition in the Coulomb gas picture of 2-dimensional conformal field theory. Using the Fierz identity for the Hermitian generators $T^\alpha$, $\alpha = 1, \cdots, n^2 - 1$ in the fundamental representation of $SU(n)$,

$$
(T^\alpha)^b_a (T^\alpha)^d_c = \frac{1}{2} \left( \delta^d_c \delta^b_a - \frac{1}{n} \delta^b_a \delta^d_c \right), \quad (6.2)
$$

the ground state equation becomes

$$
[\partial z_j + \frac{eB}{4\hbar c} z_j - \ell_{\text{spin}} \left( \frac{-n+1}{2n} \right) \sum_{k=1, k \neq j}^{N} \frac{1}{z_j - z_k} - \ell_{\text{charge}} \sum_{k=1}^{N} \frac{1}{z_j - z_k}] \psi_{0,j} = 0, \quad (6.3)
$$

or

$$
[\partial z_j + \frac{eB}{4\hbar c} z_j - \ell_{\text{spin}} \frac{n^2 - 1}{2n} \sum_{k=1, k \neq j}^{N} \frac{1}{z_j - z_k} - \ell_{\text{spin}} \left( \frac{-n+1}{2n} \right) \sum_{k=1, k \neq j}^{N} \frac{1}{z_j - z_k}] \psi_{0,j} = 0, \quad (6.4)
$$

Now, if we let $\ell_{\text{spin}} = -\frac{2}{n+k}$ with $k = 1$, the spin portion of (6.3) and (6.4) can be identified with the bosonization of free fermions carrying the representation of $SU(n)$. With this choice and $\ell_{\text{charge}} = q + \frac{1}{2}$, the contribution of spin as $SU(2)$ in FQHE ground state combines with the $U(1)$ winding number label as follows.

$$
[\partial z_j + \frac{eB}{4\hbar c} z_j - (\ell_{\text{charge}} + \frac{1}{2}) \sum_{k=1, k \neq j}^{N} \frac{1}{z_j - z_k} - (\ell_{\text{charge}} - \frac{1}{2}) \sum_{k=1}^{N} \frac{1}{z_j - z_k}] \psi_{0,j} = 0, \quad (6.5)
$$

and a corresponding expression for (6.4). Setting $\ell_{\text{charge}} = q + \frac{1}{2}$, we find that Halperin state $|22\rangle$ given by

$$
\psi_{mmn}(z_1^{\uparrow}, \cdots, z_n^{\uparrow}; z_1^{\downarrow}, \cdots, z_n^{\downarrow}) = \prod_{j<k}^{N} (z_j^{\uparrow} - z_k^{\uparrow})^p (z_j^{\downarrow} - z_k^{\downarrow})^p \prod_{r,s}^{N} (z_r^{\uparrow} - z_s^{\uparrow})^q
$$

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\[
\times \exp \left( -\frac{1}{4l^2} \sum_i^{N} |z_i^\uparrow|^2 + |z_i^\downarrow|^2 \right) \tag{6.6}
\]

turns out to be the exact ground state solution with \( p \) constrained as \( p = q + 1 \). It is interesting to mention that the same constraint is discussed by Girvin using the Fock cyclic condition in the appendix of reference [13]. Also, the Halperin state has been constructed \textit{a priori} in terms of the conformal block of \( k = 1 \, SU(2) \) WZW theory and that of the rational torus at level \( 2q + 1 \) [11]. We have shown that, starting from an appropriate Hamiltonian which describes a system of electrons in the “puncture” phase, there is no mystery why a conformal field theory with \( SU(2)_{k=1} \) symmetry can be used to produce the wavefunction of a non-relativistic phenomenon.

Similar to what happened to the Laughlin’s state, the zero-range delta potential requires \( q \) to be positive. The filling fraction of Halperin state is \( \frac{2}{2q+1} \), with \( q \) an even number since electrons are fermions. It is likely that the unpolarized FQHE state with filling fraction \( 1 + \frac{3}{5} \) observed in the real world [8] is the particle-hole conjugate of Halperin state with \( q = 2 \). Following the same line of thought of the previous section, the quasi-hole excitation of the Halperin state can be ascertained to be characterized by a fractional (representation) charge of \( \frac{1}{2q+1} \) and spin \( \frac{1}{2} \).

7. Discussions

7.1. Connection with WZW models

From the quantum mechanics of a system of \( N \) particles in the collective “puncture” phase, the zero-energy equations of (2.19), namely (2.20) and (2.21) determine the factorizable \( N \)-body wavefunctions. In this special case, the outcome is the same as the 3-dimensional Chern-Simons gauge theory [12][13]. With a suitable value chosen for \( k \), solving the equation (2.20) gives \( \psi \) as the conformal blocks of the WZW theory. The quantum mechanics of \( N \) punctures give yet another 3-dimensional description of 2-dimensional conformal field theories. However, unlike the previous correspondence of Chern-Simons theory with the chiral moiety of the WZW theory, \( \psi \) has to satisfy (2.21) as well. In addition, since a quantum mechanical wavefunction must be invariant with respect to monodromy, \( \psi \) may be identified with the correlation function of a WZW theory. Analogously, the spinning version of the path integral representation of the braid group admits the space of the correlation functions of a super WZW theory as the representation space.
In the representation theory of current algebra, Knizhnik-Zamolodchikov equations originate from the existence of null vectors of the combined conformal and Kac-Moody algebras [15]. Though these first-order differential equations are not sufficient to determine the operator content of a WZW theory, they nevertheless provide a way to calculate the $N$-point function of the fields corresponding to the integrable representation of the theory [24]. The correlation function of a non-integrable field with any other fields vanishes, indicative of a selection rule in the theory. It follows that the Knizhnik-Zamolodchikov equations supplemented with a set of algebraic equations yield a solution space which is identical to the Hilbert space of the WZW theory [24] [25].

Now, when the group manifold $G$ is $U(1)$, WZW theory reduces to a boson compactified on a circle. In this case, the conformal field theory is a representation of a chiral algebra called rational torus or $U(1)$ current algebra. The null vectors of the purely Kac-Moody algebra do not tell much story except that the correlation functions must be singlets. Therefore, for $U(1)$ charges, Knizhnik-Zamolodchikov equations are sufficient to determine the operator content of the corresponding WZW theory with central charge $c = 1$.

Because of this connection, we can understand why it is possible to use the conformal blocks of rational torus [11] or the vertex operators of string theory [20] to construct the Laughlin wavefunctions. In our approach, the Knizhnik-Zamolodchikov equations are the ground state equations of the Hamiltonian (2.19) and they provide a microscopic description of the “puncture” phase. Solving these equations with a set of physical considerations is tantamount to finding the conformal blocks of a WZW theory.

For the spinning case, when $G = U(1)$, one also has the same correspondence with the $N = 1$ super WZW theory up to a boundary condition for the fermionic components of the superfields. Depending on the boundary condition, one can have either the Neveu-Schwarz sector or the Ramond sector. These possibilities follow from the fact that spinors can be double-valued on the local coordinate patches of the 2-dimensional manifold. It is known that even at the quantum level, super WZW theory is equivalent to the direct sum of a bosonic WZW theory and a system of free Majorana fermions in the adjoint representation of the gauge group [17]; the spectrum of supersymmetric WZW is just the bosonic WZW plus a number of free fermions. Consequently, it is possible to interpret the solutions of super Knizhnik-Zamolodchikov equations as the spinning non-abelian analogues of the Laughlin wavefunctions. In particular, super $U(1)$ WZW with central charge $c = \frac{3}{2}$ comprises of a compactified boson and a free Majorana fermion, alias Ising model at criticality. In this light, the significance of the correlator of Ising model’s energy

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operators in FQHE [11] becomes transparent. It ties up neatly with the spinning braid group representation approach presented in section 4.2 where the microscopic origin of the Moore-Read state was made manifest.

7.2. FQHE IS A MANIFESTATION OF THE “PUNCTURE” PHASE

Our path integral representation may leave an impression that the many-body system in two dimensions is necessarily in the strongly correlated phase. The derivative \( \frac{\partial}{\partial z} = \frac{\partial}{\partial z} - iA_z \), \( A_z = 0 \) is related to \( d_z \equiv \frac{\partial}{\partial z} - k \sum_j^{N} \frac{1}{z-w_j} \) by a singular gauge transformation:

\[
A_z \rightarrow A_z + \frac{\partial \varphi}{\partial z},
\]

(7.1)

where

\[
\varphi = -k \log \left[ (z - w_1) \cdots (z - w_N) \right],
\]

(7.2)

and \( k \neq 0 \). Except at a finite number of isolated points \( w_j \), the field strength is still zero (see (2.18)); \( A \equiv (A_0, A_z, A_{\bar{z}}) \) is still a flat connection of a bundle over \( R^2 - \{w_1, \cdots , w_N\} \). Mathematically, it seems that every free particle with a Hamiltonian in the Schrödinger representation of the form \( -\frac{2\hbar^2}{m} \frac{\partial^2}{\partial z^2} \) is gauge equivalent to \( -\frac{\hbar^2}{m} (d_z d_{\bar{z}} + d_{\bar{z}} d_z) \). If arbitrary singular gauge transformations are allowed, the supposedly simply-connected configuration space becomes riddled with punctures \( w_j \) and thus arbitrarily multiply-connected. Consequently, as section 2 shows, the wavefunction of the free particle belongs to the representation space of the braid group. In short, for arbitrary \( k \), every free particle or quasi-particle is always anyonic!

Certainly this is ostensible. It is not the picture we want to portray. Under ordinary circumstances, the statistics of 2-dimensional systems is still fermionic or bosonic. As we have discussed in [1] [3], the strongly correlated wavefunction is a result of the configuration space being multiply connected. Physically, this corresponds to the situation where the system of particles is in a peculiar type of quantum phase wherein each particle sees the others as punctures. Having understood its origin, the next question is: Why do the electrons see each other as punctures?

In the experimentally verified case of FQHE, plateaux develop only if the quality of the samples is good, the temperature is at the vicinity of absolute zero, and the magnetic field strength is strong. Then, according to Laughlin’s theory, the ground state of \( N \) electrons corresponding to a particular filling fraction is an incompressible fluid. The quasi-excitation at a finite energy gap from the ground state is characterized by fractional statistics. When these conditions are not met, the collective effect is absent and the
statistics of the excitations is just as usual; the Hall conductance is not quantized with fractional filling fraction\(^*\). In other words, one does not automatically have anyon (or braid group) statistics for the excitations.

To understand how the configuration space becomes multiply connected, we take the illustrative analogy of Aharanov-Böhm effect. When a 2-dimensional cross section is taken, the infinitesimally thin solenoid appears as a puncture in the plane. Hence, our formulation can be applied to the Aharanov-Böhm effect as well; (2.17) is the Lagrangian describing the phenomenon. In the case of FQHE, we suspect that the high-frequency cyclotron motion is the one that creates the puncture. As is well known, each electron is in circular motion in the presence of an uniform external magnetic field perpendicular to the plane. When the field strength gets stronger, the radius of the circular motion becomes smaller; the area enclosed by the circular motion vanishes when the magnetic field strength is infinitely large. Other electrons cannot stray into it anymore. Thus, the centre of the circular motion becomes a puncture. This is exactly the same as Aharanov-Böhm effect where the interior of the solenoid is not accessible. In some sense, the cyclotron motion with vanishing radius also chimes in with the heuristic procedure of localizing or attaching flux tubes onto the electrons. When the radius of the cyclotron motion is not sufficiently small, the location of the flux tube does not coincide with that of the electron. The flux tube will sit on the electron only if the magnetic field is strong enough to diminish the radius effectively to zero. Therefore we see that the strong external magnetic field is indispensable for fractional quantum Hall effect with anyonic excitations\(^†\).

On a more rigorous note, the physical significance of the external magnetic field is reflected in the mathematical requirement that any physical wavefunction must be normalizable. In this aspect, the gauge potential of the magnetic field results in an exponential damping term which renders the otherwise non-normalizable wavefunction of an anyon normalizable.

Having assigned a bigger role for the external magnetic field in FQHE, it is germane to speculate on the physical origin of the threshold \(B_0\) of incompressible excitation. The experimental evidence of \(B_0 > 0\) is built upon the result of a systematic study of activation energies of the \(\frac{p}{3}\) states on different specimens of comparable mobility, with \(p = 1, 2, 4, 5\) \[^{[23]}\]. The data show that the energy gaps vanish below 6 T. We propose that the existence of \(B_0\) may be understood in the following manner. Since the formation of punctures is

\(^*\)Of course, quantum Hall effect with integral filling fractions can still occur.

\(^†\)The crucial role played by the background magnetic field has also been discussed in \[^{[27]}\]. There, no-go theorems forbidding the existence of anyons with any statistics on a torus is circumvented by the presence of magnetic field.
due to the high-frequency cyclotron motion, there must be a minimum \( \omega_0 = \frac{eB_0}{m^*c} \) below which the cyclotron motion fails to hem in and excise a small region of the plane from being accessible to other electrons effectively. In other words, below \( \omega_0 \), punctures are not formed and the configuration space has trivial topology, and therefore no FQHE. From this perspective, the integral quantum Hall effect is physically distinct from the FQHE. In the former case, each electron as a single particle is very much indifferent to the existence of its counterparts in the heterojunction. The FQHE differs fundamentally in that it is the manifestation of strongly correlated “puncture” phase.

8. Summary

The braid group representation we have constructed is based on the sum over the homotopically equivalent paths in the punctured plane. The key element in our construction is to employ the charged winding numbers to label the homotopy classes. The homotopical constraint is then enforced through the path-ordered Fourier integral. Naturally, information about the non-simply connected configuration space is translated into the language of Hamiltonian associated with the path integral. In this way, we explicitly show the link between non-abelian anyon statistics and conformal field theory. It is also of interest to point out the close relationship between the braid group representation via path integral and gauged non-linear Schrödinger equations [28].

In this paper, we propose a quantization procedure suitable for the construction of spinning braid group representation. We find that super Knizhnik-Zamolodchikov equations are the zero-energy equations of free spinning particles when they see each other as punctures on the super plane. In other words, if a system of spinning particles condenses in the “puncture” phase, the many-body ground state will be characterized by the super Knizhnik-Zamolodchikov equations. This is analogous to the spinless case which we addressed in previous work [7]. In a nutshell, everything boils down to the quantum mechanical interpretation of (super) Kohno connection as the topological constraint in terms of charged winding numbers.

We have applied the “puncture” phase aspect of the representation theory to the FQHE [8]. Specifically, spin-polarized wavefunctions constructed a priori by Laughlin, and Moore and Read in the spinning case have been shown to be the exact ground states of the respective Hamiltonians (4.3) and (4.18). The repulsive zero-range delta potentials in these Hamiltonians are consequent upon the non-simply connected nature of the configuration space. This feature agrees with the established views as reviewed by Laughlin.
and Haldane in reference [19] where arguments involving numerical studies and pseudopotential method are presented. In addition, spin-singlet Halperin states describing unpolarized FQHE with filling fractions $\frac{2q+1}{2q^2}$, $q = 2, 4, \cdots$ are also accountable in this framework. The common theme of all these FQHE states is none other than the topology of the configuration space.

The phenomenological implications of the braid group approach have also been explored. We found that the energy gap of a FQHE state is the zero-point energy of the quasi-excitation. Within the interval of two Landau levels of the electrons, sub-levels corresponding to the spectral signature of the quasi-excitations exist (Figure 6). Indeed, the topological excitations behave very much like electrons in the sense that they also execute cyclotron motion under the influence of the magnetic field. We have presented an experimentally testable formula (5.13) giving the ratio of the excitation energy gaps of two Laughlin ground states $|m_a\rangle, |m_b\rangle$. Finally, the role of the magnetic field in the formation of “puncture” phase and hence the occurrence of threshold is emphasized.

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