ON THE HARDY-LITTLEWOOD MAXIMAL FUNCTION FOR THE CUBE

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Abstract. It is shown that the Hardy-Littlewood maximal function associated to the cube in \( \mathbb{R}^n \) obeys dimensional free bounds in \( L^p \) for \( p > 1 \). Earlier work only covered the range \( p > \frac{3}{2} \).

(Dedicated to J. Lindenstrauss)

1. Introduction

Let \( B \) be a convex centrally symmetric body in \( \mathbb{R}^n \) and define the corresponding maximal function

\[
Mf(x) = M_B f(x) = \sup_{t > 0} \frac{1}{|B|} \int_B |f(x + ty)| dy; f \in L^1_{\text{loc}}(\mathbb{R}^n)
\]

(1.1)

where \(|B|\) denotes the volume of \( B \). For \( 1 < p < \infty \), let \( C_p(B) \) be the best constant in the inequality

\[
\|M_B f\|_p \leq C_p(B) \|f\|_p
\]

(1.2)

while \( C_1(B) \) is taken to satisfy the weak-type inequality

\[
\|M_B f\|_{1,\infty} \leq C_1(B) \|f\|_1.
\]

(1.3)

Using the theory of spherical maximal functions, Stein [St1] established the remarkable fact that for \( B = B_2 \), the Euclidean ball, \( C_p(B_2) \) may be bounded independently of the dimension \( n \), for all \( p > 1 \). The author obtained the boundedness of \( C_2(B) \) by an absolute constant, independently of \( B \) as above (cf [B1]) and this statement was generalized to \( C_p(B), p > \frac{3}{2} \) in [B2] and [C]. On the other hand, it is shown in [S-S] that \( C_1(B) \lesssim n \log n \) (see also [N-T]). Note that the constants \( C_p(B) \)

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are invariant under linear transformation, i.e. \( C_p(B) = C_p(u(B)) \) for \( u \in CL_n(\mathbb{R}) \). It is convenient to choose \( u \) as to make \( B \) isotropic, meaning that

\[
\int_B |\langle x, \xi \rangle|^2 dx = L(B)^2 \text{ for all unit vectors } \xi \in \mathbb{R}^n. \tag{1.4}
\]

If \( B \) is in isotropic position, then all \((n-1)\)-dimensional central sections of \( B \) are approximately of the same volume \( \sim L(B)^{-1} \), up to absolute constants, see [H1] for details. Recall at this point that \( L(B) \) is known to be bounded from below by an absolute constant. Conversely, the uniform bound from above is a well-known open problem with several equivalent formulations. While such bounds were obtained for various classes of convex symmetric bodies (in particular zonoids), the best currently available general estimate on \( L(B) \) is \( O(n^{1/4}) \). Interestingly, this issue did not impact the proofs of the dimension free bounds obtained in [H1], [H2], [C]. Next, following [M], denote by \( Q(B) \) the maximum volume of an orthogonal \((n-1)\)-dimensional projection of the isotropic position \( S(B) \) of \( B \). That is

\[
Q(B) = \max_{\xi} |\pi_\xi(S(B))| \tag{1.5}
\]
denoting \( \pi_\xi \) the orthogonal projection on \( \xi^\perp, \xi \in \mathbb{R}^n \) a unit vector. It is proven in [M] that for all \( p > 1 \), one may estimate \( C_p(B) \) in terms of \( L(B) \) and \( Q(B) \). Consequently, [M] obtains dimension free maximal bounds in the full range \( p > 1 \) for \( B = B_q = \) the unit ball with respect to the \( \ell^q \)-norm in \( \mathbb{R}^n \), provided \( 1 \leq q < \infty \). For \( q = \infty \), one gets

\[
Q(B_\infty) = \sqrt{n} \tag{1.6}
\]
resulting in no further progress for the Hardy-Littlewood maximal function for the cube.

Still for the cube, a brakethrough was made recently in the works of Aldaz [Al] and Aubrun [Au], that disprove a dimension free weak \((1, 1)\) maximal inequality for \( B_\infty \). More specifically, it is shown in [Au] that

\[
C_1(B_\infty^{(n)}) > c(\varepsilon)(\log n)^{1-\varepsilon} \text{ for all } \varepsilon > 0. \tag{1.7}
\]
The purpose of this work is to prove that on the other hand

**Theorem.** $C_p(B_\infty) < C_p$ for all $p > 1$.

While it is reasonable to believe that this statement holds in general, our argument is based on a very explicit analysis which does not immediately carry over to other convex symmetric bodies. But the results of [Al] and [Au] are certainly inviting to a further study of $M_{B_\infty}$ which after all, together with $M_{B_2}$, is the most natural setting of the Hardy-Littlewood maximal operator.

Let us next give a brief description of our approach.

Denote in the sequel $B = B^{(n)}_\infty = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbb{R}^n$ with Fourier transform

$$m(\xi) = \hat{1}_B(\xi) = \prod_{j=1}^n \frac{\sin \pi \xi_j}{\pi \xi_j}. \quad (1.8)$$

Then

$$\int_B f(x + ty)dy = \int_{\mathbb{R}^n} \hat{f}(\xi)m(t\xi)e(x.\xi)d\xi \quad (1.9)$$

(with notation $e(y) = e^{2\pi i y}$) which reduces matters to the study of the Fourier multiplier $m(\xi)$. It satisfies in particular the estimates

$$|m(\xi)| < \frac{C}{|\xi|} \text{ for } |\xi| \to \infty \quad (1.10)$$

and

$$|\langle \nabla m(\xi), \xi \rangle| < C. \quad (1.11)$$

In fact, (1.10), (1.11) hold in general for isotropic convex symmetric $B$, $|B| = 1$, replacing (1.10) by

$$|\hat{1}_B(\xi)| < \frac{C}{L(B)|\xi|}. \quad (1.12)$$

See[B1].

The estimates (1.10), (1.11) set the limitation $p > \frac{3}{2}$ in bounding $\|M_B\|_p$. Following [B2] for instance, a faster decay in (1.10) would allow to reach smaller values of $p$. Now a quick inspection of (1.8) shows, roughly speaking, that most of the time $m(\xi)$ decays much faster.
and the worst case scenario (1.10) only occurs for \( \xi \) confined to narrow conical regions along the coordinate axes. Thus our strategy will consist in making suitable localizations in fourier space which contributions will be treated using different arguments.

For \( \Omega \in L^1(\mathbb{R}^n) \), denote for \( t > 0 \) the scaling \( \Omega_t(x) = \frac{1}{t^n} \Omega \left( \frac{x}{t} \right) \) satisfying \( \hat{\Omega}_t(\xi) = \hat{\Omega}(t\xi) \). Denote \( H \) the Gaussian distribution on \( \mathbb{R}^n \), \( \hat{H}(\xi) = e^{-|\xi|^2} \). We make a decomposition

\[
1_B = (1_B * H) + \sum_{s=1}^{\infty} \Omega^{(s)} \quad \text{with} \quad \Omega^{(s)} = 1_B * H_{2^{-s}} - 1_B * H_{2^{-s+1}} \quad (1.13)
\]

and consider the maximal function associated to each \( \Omega^{(s)} \).

Recall the following simple \( L^2 \)-estimate (Lemma 3 in [B1]).

**Lemma 1.** Consider a kernel \( K \in L^1(\mathbb{R}^n) \) and introduce the quantities

\[
\alpha_j = \max_{|\xi| \sim 2^j} |\hat{K}(\xi)| \quad \text{and} \quad \beta_j = \max_{|\xi| \sim 2^j} |\langle \nabla \hat{K}(\xi), \xi \rangle| \quad (j \in \mathbb{Z}).
\]

Then

\[
\| \sup_{t > 0} |f * K_t| \|_2 \leq C \Gamma(K) \|f\|_2 \quad (1.14)
\]

with

\[
\Gamma(K) = \sum_{j \in \mathbb{Z}} \alpha_j^{1/2} (\alpha_j + \beta_j)^{1/2}. \quad (1.15)
\]

Since

\[
|\hat{\Omega}^{(s)}(\xi)| = |m(\xi)| \left| e^{-4^{-s}|\xi|^2} - e^{-4^{-s+1}|\xi|^2} \right|
\]

it follows that

\[
\| \sup_{t > 0} |f * (\Omega^{(s)})_t| \|_2 \leq C 2^{-s/2} \|f\|_2. \quad (1.16)
\]

Taking \( 1 < p < 2 \), our aim is to interpolate (1.16) with an estimate

\[
\| \sup_{t > 0} |f * (\Omega^{(s)})_t| \|_p \leq A(p, s) \|f\|_p
\]

or

\[
\| \sup_{t > 0} |f * (1_B * H_{2^{-s}})_t| \|_p \leq A(p, s) \|f\|_p. \quad (1.17)
\]

In order to establish (1.17), we follow the approach in [M].
Lemma 2. Assume the multiplier operator with multiplier

$$|\xi|m(\xi)e^{-4|\xi|^2}$$

acts on $L^p(\mathbb{R}^n)$ with operator norm bounded by $A(p,s)$. Then (1.17) holds with a proportional constant.

The statement follows from the argument in [M], based on analytic interpolation and a suitable admissible family of Fourier multiplier operators.

In the present situation, rather than taking $K = 1_B$ in (9) of [M], we let $K = 1_B * H_{2-s}$. It is important to note that in the crucial Lemma 2 from [M], only the bound on $L(B)$ is required but not on $Q(B)$ (which enters at a later stage). In fact, the essential input in [M], Lemma 2 are bounds on

$$\sup_{|\xi|=1} \int_B |\langle x, \xi \rangle|^k dx$$

for fixed $k \geq 1$. In our setting, we obtain

$$\int |\langle x, \xi \rangle|^k (1_{B_\infty} * H_{2-s})(x) dx$$

which is easily evaluated. Indeed, since the distribution $1_{B_\infty} * H_{2-s}$ is symmetric in each coordinate $x_i$, application of Khintchine’s inequality implies for $|\xi| = 1$

$$$(1.19) < C_k \int \left( \sum_{i=1}^n x_i^2 \xi_i^2 \right)^{\frac{k}{2}} (1_B * H_{2-s})(x) dx$$

$$\leq C_k \int \left( \left( \sum (x_i - y_i)^2 \xi_i^2 \right)^{\frac{1}{2}} + |\xi| \right)^k 1_B(y) H_{2-s}(x - y) dxdy$$

$$< C_k + C_k \int \left( \sum x_i^2 \xi_i^2 \right)^{\frac{k}{2}} H_{2-s}(x) dx$$

$$= C_k + C_k 2^{-sk} \int \left( \sum x_i^2 \xi_i^2 \right)^{\frac{k}{2}} H(x) dx$$

$$< C_k$$

for $s \geq 0$. 

(1.20)
Returning to (1.18), we proceed further as in \([\mathbb{M}]\), writing

\[
|\xi| m(\xi) e^{-4^{-\frac{s}{2}}} \hat{f}(\xi) = \sum_{i=1}^{n} \hat{R}_{i} f(\xi) \hat{\mu}_{i}(\xi) \tag{1.21}
\]

with \(\mathcal{R}_i\) the \(i^{th}\) Riesz transform and \(\mu_i = \partial_{x_i}(1_B * H_{2^{-s}})\).

Arguing by duality, take \(g \in L^{p'}(\mathbb{R}^n), \frac{1}{p} + \frac{1}{p'} = 1, \|g\|_{p'} \leq 1\) and estimate

\[
\int |\xi| m(\xi) e^{-4^{-\frac{s}{2}}} \hat{f}(\xi) \hat{g}(\xi) d\xi = \sum_{i=1}^{n} \langle R_{i} f, g \ast u_{i} \rangle 
\leq \left\| \left( \sum |R_i f|^2 \right)^{\frac{1}{2}} \right\|_{p'} \left( \sum |g \ast \mu_i|^2 \right)^{\frac{1}{2}}_{p'}.
\tag{1.22}
\]

For the first factor in (1.22), use Stein’s dimensional free bound on the Riesz transform (see \([\text{St}2]\))

\[
\left\| \left( \sum |R_i f|^2 \right)^{\frac{1}{2}} \right\|_{p} \leq A_p \|f\|_p \quad \text{for } 1 < p < \infty.
\tag{1.23}
\]

This reduces the issue to an estimate on

\[
\left\| \left( \sum |g \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_{p} \tag{1.24}
\]

for \(2 \leq p < \infty\).

Note that since \(\left( \sum |\hat{\mu}_i(\xi)|^2 \right)^{\frac{1}{2}} \leq |\xi| |m(\xi')| < C\) by (1.10),

\[
\left\| \left( \sum |g \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_{2} \leq C \|g\|_2.
\tag{1.25}
\]

Bounding (1.24) for \(p = \infty\) amounts to an estimate on

\[
\sup_{|\eta|=1} \left\| \sum_{i=1}^{n} \eta_i \mu_i \right\|_1 = \|\nabla_{\eta}(1_B * H_{2^{-s}})\|_1. \tag{1.26}
\]

Clearly

\[
(1.26) \leq \|\nabla_{\eta}(H_{2^{-s}})\|_1 = 2^s \|\langle \nabla H, \eta \rangle\|_1 \lesssim 2^s \tag{1.27}
\]

implying

\[
\left\| \left( \sum |g \ast \mu_i|^2 \right)^{\frac{1}{6}} \right\|_{\infty} \leq C 2^s \|g\|_{\infty}
\]
and
\[ \left\| \left( \sum |g * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq C 2^{s(1-\frac{1}{p})} \|g\|_p \text{ for } 2 \leq p \leq \infty. \] (1.28)

In particular, (1.17) holds with \( A(p, s) < C_p 2^s \) and interpolation with (1.16) is conclusive for \( p > \frac{3}{2} \).

In order to prove the Theorem, it will suffice to establish an inequality of the form

Lemma 3. For \( R > 1 \) and \( \mu_i = \partial_i(1_{B_\infty} * H_{\frac{1}{R}}) \), there is an inequality
\[ \left\| \left( \sum_{i=1}^n |f * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p(\varepsilon) R^\varepsilon \|f\|_p \] (1.29)

for all \( 2 \leq p < \infty \) and \( \varepsilon < 0 \).

The proof of Lemma 3 will occupy the remainder of the paper. As mentioned before, the explicit form of \( m(\xi) \) is essential in the argument. In the next section, we introduce a new collection of Fourier multiplier operators that will enable to perform certain localizations in Fourier space. The proof of (1.29) will then proceed by analyzing the expression \( \left( \sum_i |f * \mu_i|^2 \right)^{\frac{1}{2}} \) on each of these regions.

2. Localization in Fourier space

The following statement is a particular instance of Pisier’s holomorphic semi-group theorem in \( B \)-convex spaces ([P]).

Lemma 4. Denote \( E_j \) a conditional expectation operator acting on the \( j^{th} \) variable of \( \mathbb{R}^n \). Then, for \( 1 < p < \infty \), the semi-group
\[ S_t = \prod_{j=1}^n (E_j + e^{-t}(1 - E_j)) \quad (t \geq 0) \] (2.1)
acting on \( L^p(\mathbb{R}^n) \) admits a holomorphic extension. Hence, for \( 0 \leq k \leq n \) the operator
\[ \sum_{S \subseteq \{1, \ldots, n\}} \left( \prod_{j \not\in S} E_j \right) \prod_{j \in S} (1 - E_j) \] (2.2)
acts on $L^p(\mathbb{R}^n)$ with norm bounded by $C^k_p$.

We may replace the expectation operators $E_j$ by convolution operators using a standard averaging procedure over translations. Let $E$ be the expectation operator corresponding to the partition of $\mathbb{R}$ in the intervals $[k, k+1]$, $k \in \mathbb{Z}$. Thus its kernel is given by

$$
\Phi(x, y) = \sum_{k \in \mathbb{Z}} 1_{[k,k+1]}(x)1_{[k,k+1]}(y).
$$

Averaging over translations, the operator \( \int_0^1 (\tau_\theta E \tau_{-\theta}) d\theta \) is the convolution by $\eta = 1_{[0,1]} * 1_{[0,1]}$, i.e.

$$
\eta(x) = (1 - |x|)_+.
$$

(2.3)

Lemma 4 convexity therefore implies

**Lemma 5.** Let $\eta$ be as in (2.3) and $(t_j)_{1 \leq j \leq n}$ positive numbers. Denote $T_j$ the convolution operator by $\eta_{t_j}$ in the $j$-variable. Then, for $0 \leq k \leq n$, the operator

$$
\sum_{S \subset \{1, \ldots, n\}} \prod_{j \in S} T_j \prod_{j \in S} (1 - T_j)
$$

acts on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with norm bounded by $C^k_p$.

Returning to Lemma 3, set $t_j = t = R^{-\varepsilon}$ for each $j = 1, \ldots n$, with $\varepsilon > 0$ and a fixed small constant. Denote $A_k$ the corresponding convolution operator (2.4), which satisfies

$$
\|A_k\|_p < C^k_p \quad \text{for} \quad 1 < p < \infty.
$$

(2.5)

Let $K = K(\varepsilon, p) \in \mathbb{Z}_+$ and decompose $f$

$$
f = (A_0 + \cdots + A_K)f + g.
$$

(2.6)

Going back to the $L^2$-inequality (1.25), we obtain from Parseval that

$$
\| \left[ \sum_{i=1}^n |g * \mu_i|^2 \right]^{\frac{1}{2}} \|_2 \leq \rho \|f\|_2
$$

(2.7)
with \( \rho \) an upper bound on
\[
|m(\xi)| e^{-4^{-s}|\xi|^2} |1 - \hat{A}_0(\xi) - \cdots - \hat{A}_K(\xi)| = \\
\prod_{j=1}^n \left| \frac{\sin \pi \xi_j}{\pi \xi_j} \right| e^{-4^{-s}|\xi|^2} \sum_{|S| > K} \prod_{j \in S} \hat{n}(t\xi_j) \prod_{j \notin S} (1 - \hat{n}(t\xi_j)).
\]  

(2.8)

**Lemma 6.** For all \( \delta > 0 \) and \( k \geq 1 \),
\[
|m(\xi)| < C_k \left( 1 + \sum_{|\xi_j| < R^\delta} \xi_j^2 \right)^{-\frac{k}{2}} R^{6k} 
\]  

(2.9)

**Proof.** Denoting \( I_0 = \{j = 1, \ldots, n; |\xi_j| > 1\} \), clearly
\[
\prod_{j \notin I_0} \left| \frac{\sin \pi \xi_j}{\pi \xi_j} \right| < e^{-c \sum_{j \notin I_0} \xi_j^2}
\]
while
\[
\prod_{j \in I_0} \left| \frac{\sin \pi \xi_j}{\pi \xi_j} \right| < e^{-c|I_0|}.
\]

Estimating
\[
\sum_{|\xi_j| < R^\delta} \xi_j^2 < R^{2\delta} |I_0| + \sum_{j \notin I_0} \xi_j^2
\]
\[(2.3)\] follows. \( \square \)

Returning to (2.8), set \( I_1 = \{j = 1, \ldots, n; |\xi_j| > R^\frac{\delta}{2}\} \). If \( |I_1| > K \),
\[
(2.8) \leq |m(\xi)| < R^{-\frac{6k}{10}}.
\]  

(2.10)

Assume \( |I_1| \leq \frac{K}{2} \) and bound (2.8) by
\[
|m(\xi)| \left\{ \sum_{S \cap I_1 = \phi} \prod_{j \notin S} \hat{n}(t\xi_j) \prod_{j \in S} (1 - \hat{n}(t\xi_j)) \right\} = \\
|m(\xi)| \left| \partial_r \left[ \prod_{1 \leq j \leq n} \left( \hat{n}(t\xi_j) + r(1 - \hat{n}(t\xi_j)) \right) \right] \right|_{r=1}
\]
\[
|m(\xi)| \left[ \sum_{j \notin I_1} (1 - \hat{n}(t\xi_j)) \right]^{\frac{K}{2}} \leq
\]

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\[ |m(\xi)| r^{2|\xi|} \left( \sum_{j \notin I} \xi_j^2 \right)^{|\xi|} \lesssim R^{-2\varepsilon|\xi|} R^{2 \varepsilon|\xi|} \lesssim R^{-\varepsilon|\xi|}. \] (2.11)

Since \( t = R^{-\varepsilon} \), \( 1 - \hat{\eta}(x) < cx^2 \) for \(|x| < 1\) and (2.9).

Combining (2.10), (2.11), it follows that we may take \( \rho = R^{-\varepsilon K} \) in (2.7).

Since by (1.28), certainly
\[
\left\| \left( \sum_{i=1}^{n} |g * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p R \|f\|_p \quad \text{for } 1 < p < \infty
\] (2.12)
interpolation between (2.7), (2.12) implies that
\[
\left\| \left( \sum_{i=1}^{n} |g * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq \|f\|_p
\] (2.13)
provided we choose \( K = K(\varepsilon, p) \) appropriately.

Thus we are left with estimating (1.24) for \( g = A_k f, k \leq K \), that will be done using different arguments.

Write
\[
\left( \sum_{i=1}^{n} |A_k f * \mu_i|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{n} \left| \sum_{|S|=k \atop i \notin S} \Gamma_S f * \mu_i \right|^2 \right)^{\frac{1}{2}} \] (2.14)
\[
+ \left( \sum_{i=1}^{n} \left| \sum_{|S|=k \atop i \in S} \Gamma_S f * \mu_i \right|^2 \right)^{\frac{1}{2}} \] (2.15)
denoting
\[ \Gamma_S = \prod_{j \in S} (1 - T_j), \prod_{j \notin S} T_j \]
and \( T_j \) the convolution by \( \eta_k \) in \( x_j \).

Any significant simplification is obtained by decoupling the variables in (2.14), (2.15). We recall the procedure.
Let $\gamma_i$ be independent $\{0, 1\}$-valued random variables of mean $\frac{1}{k}$ say and for $S \subset \{1, \ldots, n\}$, $|S| = k$ and $i \not\in S$, let

$$\sigma_{S,i} = \gamma_i \prod_{j \in S} (1 - \gamma_j). \quad (2.16)$$

By construction

$$E_\omega [\sigma_{S,i}] = \frac{1}{k} \left( 1 - \frac{1}{k} \right)^k = c_k.$$ 

Hence, by convexity

$$E_\omega \left[ (\sum_{i=1}^{n} \left| \sum_{|S|=k, i \not\in S} \sigma_{S,i}(\omega)(\Gamma_S f \ast \mu_i) \right|^2 \right]^{\frac{1}{2}} \leq c_k^{-1} \left( \sum_{i=1}^{n} \left| \sum_{|S|=k, i \not\in S} \sigma_{S,i}(\omega)(\Gamma_S f \ast \mu_i) \right|^2 \right)^{\frac{1}{2}} \|p \leq c_k^{-1} \left( \sum_{i=1}^{n} \left| \sum_{|S|=k, i \not\in S} \sigma_{S,i}(\omega)(\Gamma_S f \ast \mu_i) \right|^2 \right)^{\frac{1}{2}} \|p \quad (2.17)$$

for some $\omega$. Denoting $I = \{1 \leq i \leq n; \gamma_i(\omega) = 1\}$, $(2.17)$ can be rewritten as

$$\left\| \left( \sum_{i \in I} \left( \sum_{|S|=k, i \not\in S} \Gamma_S f \ast \mu_i \right)^2 \right)^{\frac{1}{2}} \right\|_p \leq c_k^{-1} \left( \sum_{i=1}^{n} \left| \sum_{|S|=k, i \not\in S} \sigma_{S,i}(\omega)(\Gamma_S f \ast \mu_i) \right|^2 \right)^{\frac{1}{2}} \|p \quad (2.18)$$

Let

$$F = \sum_{|S|=k, S \cap I = \phi} \prod_{j \in S} (1 - T_j) \prod_{j \not\in I \cup S} T_j \quad (2.19)$$

which, applying Lemma 5 in the variable $(x_j)_{j \not\in I}$, satisfies

$$\|F\|_p \leq C_p^k \|f\|_p \quad (2.20)$$

and

$$\left( \sum_{i \in I} \left( \prod_{i \in I} T_i \right) F \ast \mu_i \right)^2 \|^\frac{1}{2} \|_p \quad (2.21)$$

Assuming we dispose of an estimate

$$\left\| \left( \sum_{i \in I} \left( \prod_{i \in I} T_i \right) g \ast \mu_i \right)^2 \right\|_{L^p(\otimes_{i \in I} dx_i)} \leq b_0 \|g\|_{L^p(\otimes_{i \in I} dx_i)} \quad (2.22)$$

for $g \in L^p(\otimes_{i \in I} dx_i)$, it will follow from $(2.20)$ that $(2.18)$ is bounded by $b_0 C_p^k \|f\|_p$.

For $(2.15)$ we proceed similarly, taking $S = \{i\} \cup S'$, $|S'| = k - 1$. 

\[ \]
Instead of (2.18), we get
\[
\left\| \left( \sum_{i \in I} \left( \sum_{|S'| = k-1, S' \cap I = \emptyset} \Gamma_{(i) \cup S'} f \right) \ast \mu_i \right)^2 \right\|_{L^2}^{\frac{1}{2}},
\] (2.23)
Let
\[
F = \sum_{|S'| = k-1, S' \cap I = \emptyset} \prod_{j \in S'} (1 - T_j) \prod_{j \notin I \cup S'} T_j
\] (2.24)
satisfying by Lemma 5
\[
\|F\|_p \leq C_{k-1}^{k-1} \|f\|_p.
\] (2.25)
Then
\[
(2.23) = \left\| \left( \sum_{i \in I} \left| \Gamma_i f \ast \mu_i \right|^2 \right)^{\frac{1}{2}} \right\|_p
\] (2.26)
where \( \Gamma_i = (1 - T_i) \left( \prod_{j \notin I} T_j \right) \). Assuming an inequality
\[
\left\| \left( \sum_{i \in I} \left| \Gamma_i g \ast \mu_i \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\bigotimes_{i \in I} dx_i)} \leq b_1 \|g\|_{L^p(\bigotimes_{i \in I} dx_i)}
\] (2.27)
will imply that (2.23) may be bounded by \( b_1 \cdot C_{k-1}^{k-1} \|f\|_p \).

Summarizing, in view of (2.22) and (2.27), we are finally reduced to establishing inequalities
\[
\left\| \left( \sum_{i=1}^{n} |A_0 f \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq b_0 \|f\|_p
\] (2.28)
and
\[
\left\| \left( \sum_{i=1}^{n} |\Gamma_i f \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq b_1 \|f\|_p \quad \text{with} \quad \Gamma_i = (1 - T_i) \prod_{j \neq i} T_j
\] (2.29)
for suitable \( b_0 = b_0(R), b_1 = b_1(R) \). From the preceding, this will permit to estimate
\[
\left\| \left( \sum_{i} |\mu_i \ast f|^2 \right)^{\frac{1}{2}} \right\|_p \leq A_p(R) \|f\|_p
\] (2.30)
with
\[ A_p(R) < C(p, K)(1 + b_0(R) + b_1(R)) \]
\[ = C(p, \varepsilon)(1 + b_0 + b_1). \]  
(2.31)

Bounds on \( b_0, b_1 \) will be obtained in Section 4.

3. An auxiliary class of operators

The key inequality is (2.29) and we will deal with it using classical techniques from martingale theory. This will require us to introduce some additional convolution operators that are approximately stable under small translation (note that the function \( \eta(x) = (1 - |x|)_+ \) introduced earlier does not have this property.)

Denote
\[
\varphi(x) = \frac{c}{1 + x^4} \text{ normalized s.t. } \int_{-\infty}^{\infty} \varphi(x) dx = 1. 
\]  
(3.1)

Note that
\[
\varphi \lesssim \varphi * \varphi \lesssim \varphi 
\]  
(3.2)

and
\[
|\hat{\varphi}(\lambda)| < O(e^{-c|\lambda|}) \text{ for } |\lambda| \to \infty 
\]  
(3.3)

\[
|1 - \hat{\varphi}(\lambda)| < O(\lambda^2). 
\]  
(3.4)

Let \( 0 < t_0 \ll t = R^{-\varepsilon} \) be another parameter (to specify) and denote \( L_j \) the convolution in \( x_j \) by \( \varphi_{t_0}, \varphi_{t_0}(x) = \frac{1}{t_0} \varphi(\frac{x}{t_0}). \) Hence the \( \{L_j\} \) are contractions on \( L^p(\mathbb{R}^n), 1 \leq p \leq \infty. \)

**Lemma 7.** Assume \( q \in \mathbb{Z}_+ \) a power of 2 and \( f_1, \ldots, f_n \in L^q(\mathbb{R}^n) \) positive functions. Then
\[
\|L_2 \ldots L_n f_1 + \cdots + L_1 \ldots L_{n-1} f_n\|_q \leq \]
\[
C_q\{\|(L_1 \ldots L_n)(f_1 + \cdots + f_n)\|_q + \|(L_1 \ldots L_n)(f_1^2 + \cdots + f_n^2)\|^{\frac{1}{2}}_q + \cdots + (\|f_1\|_q^q + \cdots \|f_n\|_q^q)^{\frac{1}{q}}\} 
\]  
(3.5)

\[
\leq C_q\|f_1 + \cdots + f_n\|_q 
\]  
(3.5')
Proof. The statement is obvious for $q = 1$.

In general, proceed by direct calculation of

\[
\int_{\mathbb{R}^n} \left( \sum_j L^{(j)} f_j \right)^q \sim \sum_{j_1 \leq j_2 \leq \ldots \leq j_q} \int (L^{(j_1)} f_{j_1}) \cdots (L^{(j_q)} f_{j_q}) \tag{3.6}
\]
denoting $L_1 \ldots L_j \ldots L_n = L^{(j)}$.

Using Hölder’s inequality, the contribution of $j_1 = j_2$ in (3.6) is bounded by

\[
\int \left[ \sum_j (L^{(j)} f_j)^2 \right]^{(q-2)/q} \| \sum_j L^{(j)} f_j \|_q^q \leq \| \sum_j L^{(j)} f_j \|_q^{q-2}
\]
reducing $q$ to $\frac{q}{2}$. For the $j_1 < j_2$ contribution, proceed as follows.

We can assume $j_1 = 1$ and rewrite the integral in the r.h.s of (3.6) as

\[
\int_{\mathbb{R}^n} g_1(L_1 g_2) \cdots (L_1 g_q) \tag{3.7}
\]
with $g_1 = L^{(1)} f_1$ etc. Integration in $x_1$ gives

\[
\int g_1(x_1) g_2(x_1 - y_2) \cdots g_q(x_1 - y_q) \varphi_{t_0}(y_2) \cdots \varphi_{t_0}(y_q) dx_1 dy_2 \cdots dy_q. \tag{3.8}
\]
Perform a translation $x_1 \mapsto x_1 + \tau$ with $|\tau| < t_0$ and use the property that $\varphi_{t_0}(y + \tau) \sim \varphi_{t_0}(y)$ for $|\tau| \leq t_0$. This gives that

\[
(3.8) \sim \int g_1(x_1 + \tau) g_2(x_1 - y_2) \cdots g_q(x_1 - y_q) \varphi_{t_0}(y_2) \cdots \varphi_{t_0}(y_q) dx_1 dy_2 \cdots dy_q
\]
and averaging over $|\tau| \leq t_0$

\[
(3.7) \lesssim \int_{\mathbb{R}^n} (L_1 g_1)(L_1 g_2) \cdots (L_1 g_q). \tag{3.7}
\]
Thus the $j_1 < j_2$ contribution in (3.6) may be estimated by

\[
\int (L_1 \ldots L_n) \left( \sum f_j \right) \left( \sum L^{(j)} f_j \right)^{q-1} \lesssim \| (L_1 \ldots L_n) \left( \sum f_j \right) \|_q || \sum L^{(j)} f_j ||_{q-1}^{q-1}
\]
proving the Lemma. \(\square\)
Recall the definition of $\mu_i = \partial x_i (1_B \ast H_{1_R})$. Using Lemma 7 we prove

**Lemma 8.** For $q$ as above and $f_1, \ldots, f_n \in L^q(\mathbb{R}^n)$

$$\left\| \left[ \sum_{i=1}^n |(\partial x_i 1_B) \ast L^{(i)} f_i|^2 \right]^{\frac{1}{2}} \right\|_q \leq c_q R^{8e} \left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{\frac{1}{2}} \right\|_q.$$  (3.9)

**Proof.** Note that

$$|\partial x_i 1_B| \leq (\delta_i + \delta_{-\frac{1}{2}})(x_i) 1_{B^{(i)}}$$

with $B^{(i)} = \prod_{j \neq i} \left[ -\frac{1}{2}, \frac{1}{2} \right] \subset \mathbb{R}^{n-1}$.

Thus

$$\sum_{i} |\partial_i 1_B \ast L^{(i)} f_i|^2 \leq \sum_{i} L^{(i)} \tau_i (|f_i|^2 \ast 1_{B^{(i)}})$$

with $\tau_i$ the shift $x_i \mapsto x_i \pm \frac{1}{2}$ and we evaluate the $L^\frac{2}{q}$-norm applying (3.3). This gives the expressions

$$\left\| \sum_{i} (L_1 \ldots L_n) [\tau_i (|f_i|^2 \ast 1_{B^{(i)}})]^{\frac{1}{2}} \right\|_{q^{2+\frac{4}{q}}} \leq c_R^{8e} 1_{B^{(i)}} \left( \sum_{i} L^{(i)} \tau_i (|f_i|^2 \ast 1_{B^{(i)}}) \right)^{\frac{4}{q^{2+\frac{4}{q}}}}.$$  (3.10)

with $1 \leq 2^* \leq \frac{q}{2}$ and

$$\left( \sum_{i} L_1 \ldots L_n [\tau_i (|f_i|^2 \ast 1_{B^{(i)}})]^{\frac{1}{2}} \right)^{\frac{4}{q^{2+\frac{4}{q}}}} \leq c_R^{8e} \left( \sum_{i} L^{(i)} \tau_i (|f_i|^2 \ast 1_{B^{(i)}}) \right)^{\frac{4}{q^{2+\frac{4}{q}}}}.$$  (3.11)

Next, observe that since $\varphi_{t_0} (x + \tau) \leq C t_0^{-4} \varphi_{t_0} (x)$, we have

$$L_i \tau_i < C R^{8e} L_i < C R^{8e} L_i (1_{[-\frac{1}{2}, \frac{1}{2}]})$$  (3.12)

with convolution in the $x_i$-variable. Therefore

$$\left( \sum_{i} (L_1 \ldots L_n) [\tau_i (|f_i|^2 \ast 1_{B^{(i)}})]^{\frac{1}{2}} \right)^{\frac{4}{q^{2+\frac{4}{q}}}} \leq c_R^{8e} \left( \sum_{i} |f_i|^2 \right)^{\frac{1}{2}} \left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{\frac{1}{2}} \right\|_q$$

proving Lemma 8. \qed

As a corollary
Lemma 9.

$$\left\| \left( \sum_{i=1}^{n} |\mu_i * L(i) f_i|^2 \right)^{\frac{1}{2}} \right\|_q \leq C_q R^{8\varepsilon} \left\| \left( \sum |f_i|^2 \right)^{\frac{1}{2}} \right\|_q$$  \hspace{1cm} (3.13)

4. Completion of the proof

Return to inequalities (2.28), (2.29). We may assume \( p \) a power of 2. Set

$$t_0 = R^{-3\varepsilon}$$  \hspace{1cm} (4.1)

and let \( \{L_j\} \) and \( \{L(i)\} \) be the operators introduced in Section 3.

Consider first (2.28) and estimate

$$\left\| \left( \sum_i |A_0 f * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \leq$$

$$\left\| \left( \sum_i |A_0 f * H_{t_0} * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p$$

$$+ \left\| \left( \sum_i |A_0(1 - H_{t_0}) f * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p.$$  \hspace{1cm} (4.2)

$$+ \left\| \left( \sum_i |A_0(1 - H_{t_0}) f * \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p.$$  \hspace{1cm} (4.3)

Since \( H_{t_0} * \mu_i = \partial_i H_{t_0} * H_{t_0} * 1_B \),

$$\left\| \left( \sum_i |\partial_i H_{t_0} * 1_B * f|^2 \right)^{\frac{1}{2}} \right\|_p < C t_0^{-1} \|f\|_p$$  \hspace{1cm} (4.4)

using interpolation between \( p = 2, p = \infty \) and recalling (1.25)-(1.28).

It follows from the definition of \( A_p \) in (2.30) that

$$\left(4.3\right) \leq A_p \|A_0(1 - H_{t_0})f\|_p$$  \hspace{1cm} (4.5)

and we estimate \( \|A_0(1 - H_{t_0})\|_p \) by interpolation.

Obviously

$$\|A_0(1 - H_{t_0})\|_{\infty} \lesssim \|A_0\|_{\infty}(1 + \|H_{t_0}\|_{\infty}) = 2$$
while for $p = 2$, we need to bound the multiplier
\[
\prod_{i=1}^{n} \hat{\eta}(t\xi_i)(1 - e^{-t_0^2|\xi|^2}). \tag{4.6}
\]

Since $|\hat{\eta}(\lambda)| < C\lambda^{-2}$, certainly
\[
(4.6) < CR^{2\epsilon}(\max|\xi_i|)^{-2} < CR^{-\epsilon}
\]
unless $\max|\xi_i| < R^{2\epsilon}$.

Also $|\hat{\eta}(\lambda)| < e^{-C\lambda^2}$ for $|\lambda| < 1$ and hence
\[
\prod_{i=1}^{n} |\hat{\eta}(t\xi_i)| < e^{-ct^2(\sum_{|\xi_i|<R^\epsilon}\xi_i^2) - c|I_1|}
\]
with $I_1 = \{i \leq n; |\xi_i| \geq R^\epsilon\}$. Thus also $(4.6) < R^{-\epsilon}$ unless
\[
\sum_{|\xi_i|<R^\epsilon} \xi_i^2 < CR^{2\epsilon} \log R \text{ and } |I_1| < C \log R
\]
and we can assume
\[
|\xi|^2 < CR^{2\epsilon} \log R + |I_1|R^{4\epsilon} < CR^{4\epsilon} \log R.
\]
But then
\[
1 - e^{-t_0^2|\xi|^2} \lesssim t_0^2R^{4\epsilon} \log R < R^{-\epsilon}
\]
by $(4.1)$. This proves that
\[
(4.6) < CR^{-\epsilon}
\]
and consequently
\[
\|A_0(1 - H_{t_0})\|_2 < CR^{-\epsilon}. \tag{4.7}
\]
Interpolation with $p = \infty$ gives
\[
\|A_0(1 - H_{t_0})\|_p < CR^{2\epsilon - \frac{2\epsilon}{p}} \tag{4.8}
\]
and
\[
(4.5) \leq CA_pR^{-\frac{2\epsilon}{p}}\|f\|_p. \tag{4.9}
\]
From $(4.4)$, $(4.9)$, we find
\[
b_0 < CR^{3\epsilon} + CA_pR^{-\frac{2\epsilon}{p}}. \tag{4.10}
\]
Consider next (2.29) and estimate
\[
\left\| \left( \sum_i |\Gamma_i f \ast \mu_i|^2 \right)^{\frac{1}{p}} \right\|_p \leq \left\| \left( \sum_i |\Gamma_i L^{(i)} f \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p
\]
\[
+ \left\| \left( \sum_i |\Gamma_i (1 - L^{(i)}) f \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{4.12}
\]
Application of Lemma 9 with \( f_i = \Gamma_i f \) gives
\[
\text{(4.11)} \leq CR^8 \varepsilon \left\| \left( \sum_i |\Gamma_i f|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{4.13}
\]
Note that application of Lemma 5 to a subset of the variables implies that for any \( I \subset \{1, \ldots, n\}, \)
\[
\left\| \sum_{i \in I} \Gamma_i f \right\|_p \leq C_p \| f \|_p \quad (1 < p < \infty).
\]
Hence
\[
\left\| \left( \sum_i |\Gamma_i f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \| f \|_p \tag{4.14}
\]
and
\[
\text{(4.11)} \leq C_p R^{8\varepsilon}. \tag{4.15}
\]
Evaluate
\[
\text{(4.12)} \leq E_{\varepsilon} \left[ \left\| \left( \sum_i |F_{\varepsilon} \ast \mu_i|^2 \right)^{\frac{1}{2}} \right\|_p \right]. \tag{4.16}
\]
where \( \varepsilon \in \{1, -1\}^n \) and \( F_{\varepsilon} = \sum_i \varepsilon_i \Gamma_i (1 - L^{(i)}) f. \)
From (2.30)
\[
\text{(4.16)} \leq A_p E_{\varepsilon} \left[ \left\| F_{\varepsilon} \right\|_p \right]
\leq C_p A_p \left\| \left( \sum_i |\Gamma_i (1 - L^{(i)}) f|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{4.17}
\]
By (4.14) and (3.5'), it follows that
\[
\left\| \left( \sum_i |\Gamma_i (1 - L^{(i)}) f|^2 \right)^{\frac{1}{2}} \right\|_p \leq \left\| \left( \sum_i |\Gamma_i f|^2 \right)^{\frac{1}{2}} \right\|_p + \left\| \sum_i L^{(i)} |\Gamma_i f|^2 \right\|_p^{\frac{1}{2}}
\leq C_p \| f \|_p \tag{4.18}
\]
and we interpolate again with an $L^2$-bound. The latter is obtained by bounding the multiplier

$$\sum_i |\hat{\Gamma}_i(\xi)|^2 |1 - \hat{L}^{(i)}(\xi)|^2 =$$

$$\sum_i |1 - \hat{\eta}(t\xi_i)|^2 \prod_{j \neq i} |\hat{\eta}(t\xi_j)|^2 |1 - \prod_{j \neq i} \hat{\varphi}(t_0\xi_j)|^2 \leq$$

$$\max_i \prod_{j \neq i} |\hat{\eta}(t\xi_j)| |1 - \prod_{j \neq i} \hat{\varphi}(t_0\xi_j)|. \quad (4.19)$$

Since $|1 - \hat{\varphi}(\lambda)| < C\lambda^2$ by (3.4), (4.19) may be estimated as (4.6) and hence

$$\left\| \left( \sum_i |\Gamma_i(1 - L^{(i)}) f| \right)^2 \right\|_2 \leq CR^{-\xi} \|f\|_2 \quad (4.20)$$

$$\left\| \left( \sum_i |\Gamma_i(1 - L^{(i)}) f| \right|^2 \right\|_p \leq CR^{-\frac{\xi}{2}} \|f\|_p. \quad (4.21)$$

From (4.15), (4.21), we may take

$$b_1 < C_p R^{8\varepsilon} + C_p A_p R^{-\frac{2\varepsilon}{p}}. \quad (4.22)$$

Finally, from (2.31), (4.10), (4.22), we deduce

$$A_p(R) < C(p, \varepsilon)(R^{8\varepsilon} + A_p R^{-\frac{2\varepsilon}{p}})$$

$$A_p(R) < C_p(\varepsilon) R^{8\varepsilon}. \quad (4.23)$$

This completes the proof of Lemma 3 and the Theorem.

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