Intersections of sets, diophantine equations and Fourier analysis

Suresh Eswarathasan, Alex Iosevich, and Krystal Taylor

Abstract. A classical theorem due to Mattila (see [23]; see also [26], Chapter 13) says that if $A, B \subset \mathbb{R}^d$ of Hausdorff dimension $s_A, s_B$, respectively, with $s_A + s_B \geq d$, $s_B > \frac{d+1}{2}$ and $\dim_H(A \times B) = s_A + s_B \geq d$, then

$$\dim_H(A \cap (z + B)) \leq s_A + s_B - d$$

for almost every $z \in \mathbb{R}^d$, in the sense of Lebesgue measure.

In this paper, we replace the Hausdorff dimension on the left hand side of the first inequality above by the upper Minkowski dimension, replace the Lebesgue measure of the set of translates by a Hausdorff measure on a set of sufficiently large dimension and replace the translation and rotation group by a more general variable coefficient family of transformations. Interesting arithmetic issues arise in the consideration of sharpness examples. These results are partly motivated by those in [4] and [17] where in the former the classical regular value theorem from differential geometry was investigated in a fractal setting and in the latter discrete incidence theory is explored from an analytic standpoint. Fourier Integral Operator bounds and other techniques of harmonic analysis play a crucial role in our investigation.

We also consider, in the spirit of the Furstenberg conjecture, inverse problems for intersections by asking how small a dimension of a set can be given that the dimension of its intersections with a suitably well-curved family of manifolds is bounded from below by a given threshold.

Finally, we shall discuss applications of our estimates to the problem of estimating the number of solutions of systems of diophantine equations over integers.

Contents

1. Introduction 2
2. Main results of this paper 4
3. Examples and sharpness 10
4. Some connections with number theory 12
5. Proof of Theorem 2.2 13
6. Proof of Theorem 2.6 14
7. Proof of Theorem 2.8 16
8. Proof of Theorem 2.13 17
9. Proof of Theorem 2.14 22
10. Proof of Theorem 2.17 27
11. Proof of Theorem 2.18 29
12. Proof of Theorem 2.20 35

The work of the second listed author was partially supported by the NSF Grant DMS10-45404.
1. Introduction

A series of results due to Mattila (see [23], [24], [25]; see also [26], Chapter 13) give lower and upper bounds on the Hausdorff dimension of the intersection of subsets of the Euclidean space of a given Hausdorff dimension.

**Theorem 1.1.** Suppose that $A, B \subset \mathbb{R}^d$ of Hausdorff dimension $s_A, s_B$, respectively, with $s_A + s_B > d$, $s_B > \frac{d+1}{2}$, then for almost every $g \in O(d)$, the group of orthogonal $d$ by $d$ matrices,

\[
\mathcal{L}^d(\{(z \in \mathbb{R}^d : \dim_H(A \cap (gB + z)) \geq s_A + s_B - d)\}) > 0.
\]

This means that for a set of $z$s of positive Lebesgue measure and almost every rotation $g$, the Hausdorff dimension of $A \cap (gB + z)$ is at least $s_A + s_B - d$. The converse does not in general hold, but the following result gives a partial description.

**Theorem 1.2.** With the notation of Theorem 1.1, suppose in addition that

\[\dim_H(A \times B) = s_A + s_B \geq d.\]

Then ([26])

\[\dim_H(A \cap (z + B)) \leq s_A + s_B - d\]

for almost every $z \in \mathbb{R}^d$, in the sense of Lebesgue measure.

This tells us that if $s_A + s_B \geq d$, $s_B > \frac{d+1}{2}$, and (1.1) holds, then the Hausdorff dimension of $A \cap (gB + z)$ is at most $s_A + s_B - d$ for almost every $g \in O(d)$ and almost every $z \in \mathbb{R}^d$.

A more general question, described in [26] and the references contained therein is the following.

**Problem 1.3.** To understand the Hausdorff dimension of $A \cap T(B)$, where $A, B$ are subsets of $\mathbb{R}^d$ of suitable Hausdorff dimension and $T$ ranges contained a suitable set of transformations of $\mathbb{R}^d$.

Before we give a detailed description of the goals of this papers, we wish to illustrate a simple motivating point by considering $(x - A) \cap B$, where $A, B \subset \mathbb{R}^d$. In order for the intersection to be non-empty, $x$ must be an element of the sum set $A + B$. If $A$ and $B$ are both sets of a given Hausdorff dimension $< d$, the Hausdorff dimension of $A + B$ is also quite often $< d$ and this naturally leads us to consider translates $x$ in $(x - A) \cap B$ belonging to a set of a given Hausdorff dimension and exploring thresholds on the size of this set for the natural inequalities involving the dimension of $(x - A) \cap B$ to hold. This simple point of view also indicates that arithmetic properties of $A$ and $B$ play an important role and we explore this idea below, particularly in the construction of sharpness examples in Section 3 and the investigation of number theoretic implications of our results in Section 4.1.

We now describe in some detail the goals of this paper:

- Under structural assumptions on the set $B$, with $T_zB = B + z$, prove that the set of translates $z$ for which the upper Minkowski dimension of $A \cap T_zB$ is larger than $\dim_H(A) + \dim_H(B) - d$ does not only have Lebesgue measure 0 but also a small Hausdorff dimension.
This would be an analog of Theorem 1.2 above where finer information on the exceptional set of translates and replacing the Hausdorff dimension on the left hand side with upper Minkowski dimension is obtained at the expense of additional assumptions on the set $B$.

- Without any additional assumptions on $B$, beyond Ahlfors-David regularity, to replace the Hausdorff dimension by the upper Minkowski dimension in Theorem 1.2.

- To obtain the same type of results for arbitrary sets $A, B$ with $TB = gB + z$, where $g \in O(d)$ and $z \in \mathbb{R}^d$. We shall see that for almost every $g \in O(d)$, the set of translates $z$ for which upper Minkowski dimension of the set $A \cap TB$ is greater than $\dim_H(A) + \dim_H(B) - d$ has small Hausdorff dimension. Here the regularity afforded by averaging in $\theta \in O(d)$ compensates for the lack of regularity of $B$.

- To use Fourier Integral Operator regularity to obtain variable coefficient variants of the previous two items under a variety of assumptions. In particular, we shall estimate the upper Minkowski dimension of the set

$$\{ y \in A : \hat{\Phi}(x^1, \ldots, x^k, y) = \hat{t}, \ x^j \in E_j \subset \mathbb{R}^{n_j}, \ y \in A \subset \mathbb{R}^d, \ t \in \mathbb{R}^m \},$$

where $A, E_1, \ldots, E_k$ are sets of given Hausdorff dimension and $\hat{\Phi}$ is a suitably regular mapping. A suitable conversion mechanism will then be employed to study the corresponding system of diophantine equations over integers.

- To use maximal operators to study an inverse problem of the following type. Suppose that a lower bound on the the upper Minkowski dimension of the intersection of an Ahlfors-David regular set and each member of a suitably well-curved family of smooth manifolds is given. How small can the Hausdorff dimension of the original set be?

The main results of this paper are described in Section 2 below. The remainder of the paper is dedicated to proofs and remarks.

1.1. Notation. The following notions shall be used throughout this paper:

- $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$.

- $X \lesssim_{R} Y$ with the controlling parameter $R$ means that given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon R^\epsilon Y$.

- Given a Borel measure $\mu$, $\mu^c(x) = \mu * \rho_c(x)$, where $\rho$ is a smooth cut-off function, supported in the ball of radius two, identically equal to one in the ball of radius one, $\int \rho(x)dx = 1$ and $\rho_c(x) = \epsilon^{-d} \rho \left( \frac{x}{\epsilon} \right)$.

- Given $A \subset \mathbb{R}^d$, $A^c$ denotes the $c$-neighborhood of $A$.

- Given $A \subset \mathbb{R}^d$, $s_A$ shall denote the Hausdorff dimension of $A$ and $\mu_A$ shall denote a probability measure on $A$. When $A$ is assumed to be Ahlfors-David regular, $\mu_A$ shall denote the restriction of the $s$-dimensional Hausdorff measure to $A$. 
• Given a compactly supported measure $\mu$ on $\mathbb{R}^d$, $I_s(\mu)$ is the energy integral given by $\int \int |x - y|^{-s} d\mu(x) d\mu(y)$. By elementary properties of the Fourier transform, this expression is equivalent to $\int |\hat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi$.

• Given $A \subset \mathbb{R}^d$, let $M(A)$ denote the set of Radon measures $\mu$ with compact support such that the support of $\mu$ is contained in $A$ and $0 < \mu(A) < \infty$.

2. Main results of this paper

The results of this paper fall into three rough categories. First, in the subsection 2.1 we present results about intersections of general translated, dilated and rotated sets. In the subsection 2.2 we study equations over fractals, which amounts to the intersection of sets of given Hausdorff dimension with families of smooth surfaces satisfying appropriate curvature assumptions. In the subsection 2.3 we obtain a lower bound on the dimension of the set given information about the dimension of its intersection with certain families of surfaces. Various examples illuminating the results and establishing their degree of sharpness are presented in Section 3. Number theoretic implications of our investigations are explored in Section 4.1.

We shall primarily work with Ahlfors-David regular sets defined as follows.

**Definition 2.1.** We say that $E \subset \mathbb{R}^d$ is Ahlfors-David regular if for any $x \in E$ there exists $C > 0$ such that

$$C^{-1} \delta^{s_E} \leq \mu(B(x, \delta)) \leq C \delta^{s_E},$$

where $s_E$ is the Hausdorff dimension of $E$, $\mu$ is the Hausdorff measure restricted to $E$ and $B(x, \delta)$ is the ball of radius $\delta$ centered at $x$.

2.1. Intersections of translated, rotated and dilated sets. We begin with the following variant of Theorem 1.2, where translation by $x \in \mathbb{R}^d$ is replaced by translation by $s(x)$, a local diffeomorphism and the Hausdorff dimension on the left hand side is replaced by the upper Minkowski dimension at the expense of assuming that the sets being intersected are Ahlfors-David regular.

**Theorem 2.2.** Let $A, B \subset \mathbb{R}^d$, $d \geq 1$, be compact, Borel and Ahlfors-David regular, with Hausdorff dimension $s_A, s_B$ respectively. Suppose that $s_A + s_B > d$. Let $\gamma(x)$ denote the upper Minkowski dimension of $A \cap (s(x) - B)$, where $s$ is a local $(C^\infty)$ diffeomorphism. Let $\mu_A, \mu_B$ denote restrictions of the Hausdorff measure to $A, B$, normalized so that $\int d\mu_A = \int d\mu_B = 1$. Let $N(x, \epsilon)$ denote the minimal number of $\epsilon$-balls needed to cover $A \cap (s(x) - B)$. Then for any smooth compactly supported smooth function $\psi$, with $\int \psi = 1$, and every $\epsilon > 0$, there exists a universal constant $C$ such that

$$\int N(x, \epsilon) \psi(x) dx \leq C(\epsilon^{-1})^{s_A + s_B - d},$$

and, consequently,

$$\int \gamma(x) \psi(x) dx \leq s_A + s_B - d,$$

from which it follows that $\gamma(x) \leq s_A + s_B - d$ for almost every $x \in \mathbb{R}^d$ in the sense of Lebesgue.
Remark 2.3. In the case when \( \tilde{s}(x) \equiv x \), Theorem 2.2 can be deduced from results in [8]. We include the result because our proof sets up the arguments in the remainder of the paper and due to its somewhat greater applicability.

Combining Theorem 2.2 with Theorem 1.1, we deduce that the upper Minkowski dimension and the Hausdorff dimension of the intersection of an Ahlfors-David regular set with a rotated copy of a Borel set quite frequently coincide.

Corollary 2.4. Suppose that \( A, B \subset \mathbb{R}^d \) compact, Borel, Ahlfors-David regular of Hausdorff dimension \( s_A, s_B \), respectively, positive \( s_A, s_B \)-dimensional Hausdorff measure, \( s_A + s_B \geq d, s_B > \frac{d+1}{2} \) and \( s_A > 1 \). Then for almost every \( g \in O(d) \),
\[
\mathcal{L}^d \{ z \in \mathbb{R}^d : \text{dim}_H(A \cap (z - gB)) = \text{dim}_M(A \cap (z - gB)) \} > 0.
\]

Remark 2.5. It is reasonable to conjecture that under the assumptions of Corollary 2.4, \( A \cap (z - gB) \) is Ahlfors-David regular, but this does not follow from the equality of the Hausdorff and upper Minkowski dimensions. This can be seen by taking a Cantor construction and changing the dissection ratio at each stage. The second listed author is grateful to Pertti Mattila for pointing this construction in the context of Ahlfors-David regularity.

If we are willing to rotate \( B \) before translating it, we discover that the exceptional set, which was found to have Lebesgue measure zero in Theorem 2.2 above, has a small Hausdorff dimension.

Theorem 2.6. Suppose that \( A, B \subset \mathbb{R}^d, d \geq 2, \) are compact, Borel, with Hausdorff dimension \( s_A, s_B \), respectively and assume, in addition, that \( A \) and \( B \) are Ahlfors-David regular and that \( s_A + s_B > d \). Let \( \mu \) be a compactly supported probability measure such that \( I_\alpha(\mu) < \infty \), where \( \alpha + s_A > d + 1 \).

Let \( \gamma_g(x) \) denote the upper Minkowski dimension of \( A \cap (x - gB) \), where \( g \in O(d) \), the orthogonal group of \( d \) by \( d \) matrices. Let \( N(x, g, \epsilon) \) denote the minimal number of \( \epsilon \)-balls needed to cover \( A \cap (x - gB) \). Then there exists \( C > 0 \) such that for every \( \epsilon > 0 \)
\[
\int \int N(x, g, \epsilon)d\theta(g)d\mu(x) \leq C \sqrt{I_\alpha(\mu)I_B(\mu_A)} \cdot (\epsilon^{-1})^{s_A + s_B - d},
\]
where \( a, b > 0, a + b = d + 1 \), and, consequently,
\[
\int \int \gamma_g(x)d\theta(g)d\mu(x) \leq s_A + s_B - d,
\]
where \( d\theta(g) \) denotes the Haar measure on \( O(d) \). It follows that for almost every \( g \in O(d) \),
\[
\text{dim}_H(\{ x : \gamma_g(x) > s_A + s_B - d \}) \leq d + 1 - s_A.
\]

We are also able to obtain a good upper bound on the Hausdorff dimension of the exceptional set if we put additional structural assumptions on one of the sets being intersected.

Definition 2.7. We say that \( E \subset \mathbb{R}^d \) has Fourier dimension \( \beta > 0 \) (see e.g. [34]) if \( \beta \) is supremum of the set of numbers \( \alpha \) such there exists a Borel measure \( \mu \) supported in \( E \) such that
\[
|\hat{\mu}_E(\xi)| \leq C|\xi|^{-\frac{\alpha}{2}}.
\]
Theorem 2.8. Suppose that $A \subset \mathbb{R}^d$, $d \geq 2$, is compact, Borel and Ahlfors-David regular, with Hausdorff dimension $s_A$. Suppose that $B \subset \mathbb{R}^d$ is compact, Borel, Ahlfors-David regular and has Fourier dimension $\beta > 0$. Let $\mu_A, \mu_B$ denote the probability measures on $A, B$ as before. Suppose that $\mu$ is a compactly supported probability measure with $I_\alpha(\mu) < \infty$ such that

$$\frac{\alpha + s_A}{2} > d - \frac{\beta}{2}.$$  

Let $\gamma_{A,B}(x)$ denote the upper Minkowski dimension of $A \cap (x - B)$. Let $N(x, \epsilon)$ denote the minimal number of $\epsilon$-balls needed to cover $A \cap (x - B)$. Then there exists $C > 0$ such that for every $\epsilon > 0$

$$(2.7) \quad \int N(x, \epsilon) d\mu(x) \leq C \sqrt{I_a(\mu) I_b(\mu_A)} \cdot (\epsilon^{-1})^{s_A + s_B - d},$$

where $a, b > 0$, $a + b = 2d - \beta$.

It follows that

$$(2.8) \quad \int \gamma_{A,B}(x) d\mu(x) \leq s_A + s_B - d.$$  

It follows that

$$(2.9) \quad \dim_H(\{x : \gamma_{A,B}(x) > s_A + s_B - d\}) \leq 2d - \beta - s_A.$$  

The extent to which this result is sharp is discussed in Example 3.1 and Example 3.2 below.

Remark 2.9. Observe that Theorem 2.8 cannot be used to obtain the conclusion of Corollary 2.19 because the Fourier transform of the Lebesgue measure on the intersection of two spheres does not decay in the direction orthogonal to the lower dimension sphere where the two original spheres intersect.

Remark 2.10. Observe that if $B$ is a convex body with a smooth boundary and non-vanishing Gaussian curvature, then Theorem 2.8 follows from Theorem 2.16 by taking $\phi(x, y) = \phi_0(x - y)$, where $\phi_0$ is the Minkowski functional of $B$.

Definition 2.11. We say that a compact set $B \subset \mathbb{R}^d$ of Hausdorff dimension $s_B$ satisfies the hyperplane size condition of order $h$ for some $h > 0$ if there exists a Borel measure $\mu_B$ supported in $B$ such that

$$\mu_B \{ H_\omega^\delta \} \leq C \delta^{s_B - h},$$

where $H_\omega = \{ x \in \mathbb{R}^d : x \cdot \omega = 0 \}$.

Remark 2.12. Note that if $\mu_B$ is a Frostman measure, then the hyperplane size condition with $h = d - 1$ always holds. This is because the intersection of $B$ with a hyperplane can be decomposed into $\approx \delta^{-(d-1)}$ $\delta$-cells, and the measure of each cell is $\leq C \delta^{s_B}$ by the Frostman property. One should think of $h$ as an upper bound on the dimension of the intersection of $B$ with a $(d - 1)$-dimensional hyperplane.

Theorem 2.13. Suppose that $A, B \subset \mathbb{R}^d$, $d \geq 2$, are compact, Borel and Ahlfors-David regular, with Hausdorff dimension $s_A$ and $s_B$ respectively, with $s_A + s_B > d$. Let $\mu$ be a compactly supported probability measure with $I_\alpha(\mu) < \infty$. Suppose that $B$ satisfies the hyperplane size condition of order $\beta$ and

$$\frac{\alpha + s_A}{2} > d - (s_B - h).$$
Let $\gamma_t(x)$ denote the upper Minkowski dimension of $A \cap (x - tB)$. Let $N(x,t,\epsilon)$ denote the minimal number of $\epsilon$-balls needed to cover $A \cap (x - tB)$. Then

$$
\int_1^2 \int N(x,t,\epsilon) dtd\mu(x) \leq C(\epsilon^{-1})^{s_A + s_B - d}.
$$

It follows that

$$
\int_1^2 \gamma_t(x) dtd\mu(x) \leq s_A + s_B - d.
$$

It follows that for almost every $t$,

$$
dim_H(\{x: \gamma_t(x) > s_A + s_B - d\}) \leq 2(d - (s_B - h)) - s_A.
$$

2.2. Equations over fractals. In this subsection we present a series of results pertaining to the general setup described in (1.3) in the introduction. We need a few preliminaries on generalized Radon transforms. Given $f: \mathbb{R}^d \to \mathbb{R}$, define

$$
T_{\phi_t} f(x) = \int_{\{\phi_t(x,y) = t_l; 1 \leq l \leq m\}} f(y) \psi(x,y)d\sigma_{x,t}(y),
$$

where $d\sigma_{x,t}$ is the Lebesgue measure on the set $\{y: \phi_t(x,y) = t_l; 1 \leq l \leq m\}$ and $\psi$ is a smooth cut-off function. Here $\phi = (\phi_1, \ldots, \phi_m)$ and $t = (t_1, \ldots, t_m)$. We shall assume throughout that

$$
\{\langle \nabla_x \phi_l(x,y) \rangle \}_{l=1}^m \text{ and } \{\nabla_y \phi_l(x,y) \}_{l=1}^m
$$

form two linearly independent sets of vectors in $\mathbb{R}^d$ in a neighborhood of the sets

$$
\{x: \phi_t(x,y) = t_l; 1 \leq l \leq m\}, \{y: \phi_t(x,y) = t_l; 1 \leq l \leq m\},
$$

respectively. This can be justified by details in [27] and is meant to provide an underlying manifold structure. We call $T_{\phi_t}$ the Radon transform associated to $\phi_t$. Technically,

$$
T_{\phi_t} : C^\infty(\mathbb{R}_y^n) \to C^\infty(\mathbb{R}_x^n \times \mathbb{R}_t^m).
$$

Our first result is closely related to the main result in [4] where the classical regular value theorem was studied in a fractal setting.

THEOREM 2.14. Let $A \subset \mathbb{R}^d$, $d \geq 2$, be compact, Borel and Ahlfors-David regular, of Hausdorff dimension $s_A$. Let $\mu$ be a compactly supported probability measure with $I_\alpha(\mu) < \infty$. Let $\phi_t$ be a smooth function and let $T_{\phi_t}$ be as in (2.13) above. Suppose that

$$
T_{\phi_t} : L_1(\mathbb{R}^d) \to L_1^s(\mathbb{R}^d)
$$

with constants uniform in $t \in T = T_1 \times T_2 \times \cdots \times T_m$, $T_j$ an interval in $\mathbb{R}$, for some $s > 0$ and assume that

$$
\frac{\alpha + s_A}{2} > d - s; \ s_A > m.
$$

Let $\gamma_t(x)$ denote the upper Minkowski dimension of

$$
\left\{y \in A: \phi_t(x,y) = \tilde{t}\right\}.
$$

Then for $t \in T$,

$$
\int \gamma_t(x) d\mu(x) \leq s_A - m.
$$
It follows that for \( t \in T \),
\[
\dim_H(\{ x : \gamma_t(x) > s_A - m \}) \leq 2d - 2s - s_A.
\]

**Definition 2.15.** (See [27]) We say that \( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies the Phong-Stein rotational curvature condition if \( \phi \) is smooth away from the origin and
\[
\det \left( \begin{array}{cc}
0 & \nabla_x \phi \\
-(\nabla_y \phi)^T & \frac{\partial^2 \phi}{\partial x \partial y}
\end{array} \right) \neq 0
\]
on the set \( \{ (x, y) : \phi(x, y) = t \} \).

It is known that if \( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies the Phong-Stein rotational curvature condition, then
\[
T_{\phi, t} : L^2(\mathbb{R}^d) \to L^2_{k}(\mathbb{R}^d)
\]
with constants uniform in \( t \) and \( k = \frac{d-1}{2} \).

**Corollary 2.16.** Suppose that \( A \subset \mathbb{R}^d, d \geq 2, \) is compact, Borel and Ahlfors-David regular, with Hausdorff dimension \( s_A \). Suppose that \( \mu \) is a compactly supported measure such that
\[
I_\alpha(\mu) < \infty \text{ such that } \alpha + s_A > d + 1.
\]
Let \( \gamma_t(x) \) denote the upper Minkowski dimension of
\[
\{ y \in A : \phi(x, y) = t \},
\]
where \( \phi \) satisfies the Phong-Stein rotational curvature assumption. Then
\[
\int \gamma_t(x) d\mu(x) \leq s_A - 1.
\]

It follows that,
\[
\dim_H(\{ x : \gamma_t(x) > s_A - 1 \}) \leq d + 1 - s_A.
\]

**Theorem 2.17.** Suppose that \( A \subset \mathbb{R}^d, d \geq 2, \) is compact, Borel and Ahlfors-David regular, with Hausdorff dimension \( s_A \). Suppose that \( \mu \) is a compactly supported probability measure with
\[
I_\alpha(\mu) < \infty \text{ such that } \alpha + s_A > d + 2.
\]
Let \( \gamma(x) = \sup_{t \in [1,2]} \gamma_t(x) \), where \( \gamma_t(x) \) is as in Corollary 2.16 above. Then
\[
\int \gamma(x) d\mu(x) \leq s_A - 1,
\]
and thus
\[
\dim_H(\{ x : \gamma(x) > s_A - 1 \}) \leq d + 2 - s_A.
\]

Our next result deals with the system of two equations over a fractal set and does not fall in the realm of the classical generalized Radon transform theory of Theorem 2.14 and Corollary 2.16.
Theorem 2.18. Suppose that $A \subset \mathbb{R}^d$, $d \geq 2$, is compact, Borel and Ahlfors-David regular, with Hausdorff dimension $s_A > 2$. Suppose that $\mu_j$, $j = 1, 2$, are compactly supported probability measures with $I_{\alpha_j}(\mu) < \infty$, such that both
\[ \alpha_1 + s_A > d + 1 \]
and
\[ \alpha_2 + s_A > d + 1. \]
Let $\gamma_\ell(x^1, x^2)$ denote the upper Minkowski dimension of
\[ \{ y \in A : \phi_1(x^1, y) = t_1, \phi_2(x^2, y) = t_2 \}, \]
where $\phi_j$ satisfies the Phong-Stein rotational curvature condition. Then
\begin{equation}
\int \int \gamma_\ell(x^1, x^2) d\mu_1(x^1) d\mu_2(x^2) \leq s_A - 2.
\end{equation}

Corollary 2.19. Suppose that $A \subset \mathbb{R}^d$, $d \geq 2$, is compact, Borel and Ahlfors-David regular, with Hausdorff dimension $s_A > 2$. Suppose that $\mu_j$, $j = 1, 2$, are compactly supported probability measures with $I_{\alpha_j}(\mu) < \infty$, such that both
\[ \alpha_1 + s_A > d + 1 \]
and
\[ \alpha_2 + s_A > d + 1. \]
Let $\gamma_\ell(x^1, x^2)$ denote the upper Minkowski dimension of
\[ A \cap \{ y : |x^1 - y| = t_1, |x^2 - y| = t_2 \}. \]
Then
\begin{equation}
\int \int \gamma_\ell(x^1, x^2) d\mu_1(x^1) d\mu_2(x^2) \leq s_A - 2.
\end{equation}

2.3. Inverse problems. In this subsection we discuss to what extent it is possible to obtain a lower bound on the dimension of a set from a lower bound on the dimension of the intersection of this set with a suitably large collection of manifolds. This is strongly related to the generalization of the Besicovitch-Kakeya conjecture known as the Falconer conjecture. See, for example, [33] and the references contained therein. In this paper we only discuss the case of intersection of a set with families of $(d - 1)$-dimensional manifolds satisfying suitably non-degeneracy assumptions. These results can be viewed as "inverse" problems with the respect to the theorems presented above.

Theorem 2.20. Let $\phi$ be as in the Definition 2.15 above and let $E$ be a compact, Borel subset of $\mathbb{R}^d$. Suppose that there exists $U \subset \mathbb{R}^d$ such that the Lebesgue measure of $U$ is positive and for each $x \in U$ there exists $t(x) \neq 0$ such that
\[ \overline{\dim}_\mathcal{M}(E \cap \{ y : \phi(x, y) = t(x) \}) \geq \gamma > 0. \]
Then
\[ \overline{\dim}_\mathcal{M}(E) \geq \frac{d\gamma}{d - 1}. \]
We show in the Example 3.3 below that we cannot in general expect $\dim_{\mathcal{M}}(E) \geq \gamma + 1$ under the assumptions of Theorem 2.20.

There are a couple of special cases worth noting. If $\phi(x, y) = |x - y|$, the Euclidean distance, the result says that if the intersection of $E$ with at least one dilate of many translates of $S^{d-1}$ is large, then the dimension of $E$ is also suitably large. Another interesting special case is when $\phi(x, y) = x \cdot y$. In this case, we ask for $E$ to have a large intersection with at least one hyper-plane with a given unit normal. An interesting additional feature of this case is that it suffices to take $U \subset S^{d-1}$ of positive Lebesgue measure since if the intersection condition is satisfied for a given $x$, it is also satisfied for any multiple of $x$.

3. Examples and sharpness

In this section we construct a series of examples indicating the extent to which the results stated above are best possible. An example of two sets $A$ and $B$, of Hausdorff dimension $s_A$ and $s_B$, respectively, such that the Hausdorff dimension of $A \cap (x - B)$ is "generically" $s_A + s_B - d$ is easily constructed by taking $A$ and $B$ to be smooth surfaces in $\mathbb{R}^d$. A simple example in the non-integer case is obtained by considering

$$A = \{ r \omega : \omega \in S^{d-1}; \ r \in U \},$$

where $U$ is an Ahlfors-David regular set of Hausdorff dimension $s_U$. It is not difficult to check that the Hausdorff dimension of $A$ is $d - 1 + s_U$. It is also straightforward to verify that the Hausdorff dimension of $A$ and every line that intersects $A$ is at most $s_U = d - 1 + s_U + 1 - d$. Modifying this construction yields examples of this type for arbitrary $s_A, s_B > \frac{d+1}{2}$, $s_A + s_B > d$.

A more complicated matter is to construct examples showing that our results proving that the upper bound on the intersection dimensions can only fail on a set of relatively small Hausdorff dimension, i.e estimates (2.6), (2.9), (2.23), (2.11) and (2.12). We are only able to do that with respect to the first two listed estimates. Our main tool is a paraboloid construction, previously employed in [4] and [21] in a related context. See also [1], [31] and [20] where the paraboloid example arises in an analogous context.

Example 3.1. Let $A_{q,s}$ denote the $q^{-\frac{d}{q}}$-neighborhood of

$$q^{-1} \left\{ \mathbb{Z}^d \cap \bigotimes_{j=1}^d [0, q^{\frac{d}{q^j+1}}] \times [0, q^{\frac{d}{q^j+1}}] \right\},$$

$s \in \left[ \frac{d}{2}, d \right]$, where $q$ is an asymptotically large positive integer. Let $\mu_{q,s}$ denote the Lebesgue measure restricted to $A_{q,s}$, normalized to make $\epsilon d\mu_{q,s} = 1$. It is not difficult to check (see e.g. [21]) that

$$I_s(\mu_{q,s}) = \int \int |x - y|^{-s} d\mu_{q,s}(x)d\mu_{q,s}(y) \approx 1.$$

If one takes $q_1 = 2$, $q_{i+1} > q_i^s$, then the set $A = \bigcap_i A_{q_i,s}$ is Ahlfors-David regular and has Hausdorff dimension $s$. See, for example, [7], Chapter 8.

Let $B = \{ x \in [0,1]^d : x_d = x_1^2 + \cdots + x_{d-1}^2 \}$. By the classical method of stationary phase (see e.g. [29]), the natural surface measure $\sigma_B$ on $B$ satisfies

$$|\tilde{\sigma}_B(\xi)| \leq C|\xi|^{-\frac{d-1}{2}},$$

which implies that $\beta = d - 1$ in Theorem 2.8 above. It follows that the left hand side of (2.8) is $\leq s_A - 1$, from which we conclude that the Hausdorff dimension of the set of $x$ such that the
upper Minkowski dimension of $A \cap (x - B)$ exceeds $s_A - 1$ is less than or equal to $d + 1 - s_A$. This establishes the sharpness of Theorem 2.6 and Theorem 2.8 up to the endpoint in the situation when $\beta = d - 1$. It also establishes the sharpness of Corollary 2.16.

The construction in the case of other $\beta$s is different in nature and is handled in Example 3.2 below.

**Example 3.2.** We now address the degree of sharpness of $\beta$s other than $\beta = d - 1$ in Theorem 2.8. Let $\mathcal{C}_\alpha$, a Cantor type set of Hausdorff dimension $0 < \alpha_0 < 1$. Let $F_\alpha = \mathcal{C}_\alpha \cup (\mathcal{C}_\alpha + 1)$. Let $A = [0,1]^{d-1} \times F_{\alpha_0}$. Then $A$ is an Ahlfors-David regular set with $dim_H(A) = d - 1 + \alpha_0$. The construction and the Ahlfors-David property also guarantee that

$$
\mu_A \{(A \cap (x - S_m))^\varepsilon\} \approx \mu_A \{y : 1 \leq ||x - y||_{l_m} \leq 1 + \varepsilon\},
$$

where $S_m = \{x \in \mathbb{R}^d : |x_1|^m + |x_2|^m + \cdots + |x_d|^m = 1\}$, with $m$ an even integer and

$$
||x||_{l_m} = (|x_1|^m + |x_2|^m + \cdots + |x_d|^m)^{\frac{1}{m}}.
$$

Let $\sigma_{S_m}$ denote the surface measure on $S_m$. It is not difficult to check (see e.g. [19]) that

$$
|\sigma_{S_m}(\xi)| \leq C|\xi|^{-\frac{d-1}{m}}.
$$

The right hand side of (3.2) is bounded from below by

$$
\mu_A \{y \in R : 1 \leq ||x - y||_{l_m} \leq 1 + \varepsilon\},
$$

where $R$ is an $\varepsilon^{\frac{1}{m}} \times \cdots \times \varepsilon^{\frac{1}{m}} \times \varepsilon$ box centered at $x + (0,0,\ldots,0,1)$. By construction and the Ahlfors-David property we see that if $x \in A$, this quantity is bounded from below by

$$
C\varepsilon^{d-1}\cdot \varepsilon^\alpha.
$$

On the other hand, the left hand side in (3.2) is bounded from above by

$$
C\varepsilon^{s_A}\cdot N(x,\varepsilon),
$$

where $s_A$ is, as usual, the Hausdorff dimension of $A$ and $N(x,\varepsilon)$ is the minimal number of balls needed to cover $A \cap (x - S_m)$. Plugging in $s_A = d - 1 + \alpha$, we see that

$$
N(x,\varepsilon) \gtrsim (\varepsilon^{-1})^{(d-1)(1-\frac{1}{m})},
$$

which implies that the lower Minkowski dimension of $A \cap (x - S_m)$ is bounded from below by $(d - 1) \left(1 - \frac{1}{m}\right)$ for any $x \in A$.

Thus we have shown that with $s_A = d - 1 + \alpha_0$ and $\beta = \frac{d-1}{m}$, $A \cap (x - S_m)$ has lower Minkowski dimension at least $(d - 1) \left(1 - \frac{1}{m}\right)$ for $x \in A$. Observe that

$$
(d - 1) \left(1 - \frac{1}{m}\right) > dim_H(A) + dim_H(S_m) - d = d - 1 + \alpha_0 + d - 1 - d = d - 2 + \alpha_0
$$

precisely when $1 - \alpha_0 \geq \frac{d-1}{m}$. This means that the expected threshold is exceeded on the set of dimension $d - 1 + \alpha_0$ and this quantity matches (2.9) if $1 - \alpha_0 = \frac{d-1}{m}$. This gives us sharpness for this range of exponents. A more flexible example can be built by taking $A = F_{\alpha_1} \times F_{\alpha} \times \cdots \times F_{\alpha_d}$.
Our next examples pertain to Theorem 2.20.

Example 3.3. It is important to note that under the assumptions of Theorem 2.20 we cannot in general conclude that the upper Minkowski dimension of $E$ is greater than or equal to $\gamma - 1$. To see this we simply modify Wolff’s example ([32]) for Furstenberg sets in the plane. Indeed, let $F \subset \mathbb{R}^2$ be a set constructed in the paper which has Hausdorff dimension $\frac{1}{2} + \frac{3}{2} \alpha_0$ such that its intersection with at least one line in every direction has Hausdorff dimension at least $\alpha_0$, $\alpha_0 \in (0, 1)$. Let $E = F \times \mathbb{R}^{d-2}$. Then $E$ has Hausdorff dimension $d - \frac{3}{2} + \frac{3}{2} \alpha_0$ and its intersection with at least one $(d - 1)$-dimensional plane corresponding to every unit normal on $S^{d-1}$ is at least $\alpha_0 + d - 2$. Observe that

$$d - \frac{3}{2} + \frac{3}{2} \alpha_0 < \alpha_0 + d - 2 + 1$$

since $\alpha_0 \in (0, 1)$.

4. Some connections with number theory

The sharpness examples above already suggests a strategy, developed and exploited in ([18]), ([15]), ([16]), ([12]), ([11]). The idea is that a fractal can be built via appropriate scaling and thickening of the integer lattice. If one has arithmetic information, this mechanism can be used to establish sharpness of analytic estimates as is done using the paraboloid example in Section 3 above. Conversely, if one is able to estimate the size of the solution set of an equation of fractals of a given Hausdorff dimension, this information can be recycled into an estimate on the number of solutions of a diophantine equation over the integers. See, for example, [2] for a background on the discrete analogs of the problems under consideration here.

The basic strategy can be described as follows. Suppose that we have an estimate on the size of the solution set of an equation, or a system of equations over a set of a given Hausdorff dimensions, as in Theorem 2.18 for instance. We then construct a sequence of positive integers $q_i$ such that $q_1 = 2$ and $q_{i+1} = q_i^j$. Define $E_i$ to be the $q_i^{-\frac{d}{2}}$-neighborhood of $[0, q_i]^d \cap \mathbb{Z}^d$, where $s \in (0, d)$. It is known (see [7]) that the Hausdorff and Minkowski dimensions of $E = \cap_i E_i$ is equal to $s$. We note that $\mathbb{Z}^d$ can be easily replaced by $T \mathbb{Z}^d$, where $T$ is an element of $SL_d(\mathbb{R})$. This construction allows us to transfer results proved over fractal to arithmetic results over the integer lattice. Moreover, it is typically sufficient to work with the iterates $E_i$ instead of taking the infinite intersection. This is because if one takes the normalized Lebesgue measure on $E_i$, denoted by $\mu_i$, the energy integral $I_s(\mu_i) \approx 1$ and it is this, rather than the dimensionality per se, is what we actually use in the estimates above.

We shall focus on applications of Theorem 2.18 to solving systems of diophantine equations. More precisely, let $\phi_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote smooth functions, homogeneous of degree one in both $(d\text{-dimensional})$ variables. Suppose that we wish to solve the system of equations

$$(4.1) \begin{cases} |\phi_1(n^1, n) - \lambda_1| \leq q^{-\frac{d}{2}+1} \\ |\phi_2(n^2, n) - \lambda_2| \leq q^{-\frac{d}{2}+1} \end{cases}$$

for $n$, where $n^j = (n^j_1, \ldots, n^j_d), n = (n_1, \ldots, n_d); n, n^j \in \mathbb{Z}^d \cap [0, q]^d, q$ is a large positive integer and $\lambda_j \approx q, j = 1, 2$ are real numbers. We have the following result.
Corollary 4.1. Let \( \nu_q(n^1, n^2) \) denote the number of solutions of the system of equations in (4.1). Then

\[
q^{-2d} \sum_{0 \leq |n^i| \leq q} \nu_q(n^1, n^2) \leq Cq^{d - \frac{d}{s}}
\]

for any \( s > \frac{d+1}{2} \).

To put it simply, this says that the typical size of \( \nu_q \) is \( \lesssim q^{d - \frac{d}{s}} \), with \( s > \frac{d+1}{2} \), i.e

\[
\nu_q \lesssim q^{d - \frac{d}{s}} = q^{d - 1 + \frac{1}{s+1}}.
\]

To get a clearer idea of what this means, consider the case when \( \phi_j(x, y) = ||x - y||_B \), where \( || \cdot ||_B \) is the norm induced by a symmetric convex body \( B \) with a smooth boundary and everywhere non-vanishing curvature. In this case, the result is about the number of lattice points that lie close to the intersection of two dilates of \( \partial B \), of diameter \( \approx q \), centered at integer lattice points in the grid \([0, q]^d\). If \( \partial B \) is the unit sphere in \( \mathbb{R}^d \), then the intersection is a sphere of one dimension lower. The number of lattice points on that sphere cannot exceed \( Cq^{d-3} \) by classical number theory. See, for example, [13]. Our result shows that even in the case when this sphere arises as an intersection of spheres centered at integer lattice points, the actual number of lattice points in the \( q^{-\frac{d+1}{2}} \) neighborhood is considerably less. Since the intersections of two spheres is a sphere contained in a “skew” plane, it is not very surprising that its intersection with \( \mathbb{Z}^d \) is smaller than the worst case. Our result makes this heuristic quantitative, not only in the case of the sphere, but also in a more general setup where standard analytic number theoretic techniques may be difficult to employ.

5. Proof of Theorem 2.2

Let \( \mu_A \) and \( \mu_B \) be the measures supported on \( A \) and \( B \) respectively defined as in definition (2.1). Observe the following which holds by the Ahlfors-David regularity of the set \( A \) and by the definition of upper Minkowski dimension:

\[
\left( \frac{1}{\epsilon} \right)^{\gamma(x)} \epsilon^{s_A} \lesssim \mu_A(\{x \in X^\epsilon \mid s(x) - B\}),
\]

where \( X^\epsilon \) denotes the \( \epsilon \)-neighborhood of the set \( X \).

Since \( (A \cap (s(x) - B))^\epsilon \subset \{y \in A^\epsilon : s(x) - y \in B^\epsilon\} \), it follows that

\[
\left( \frac{1}{\epsilon} \right)^{\gamma(x)} \epsilon^{s_A} \lesssim \mu_A(\{y \in A^\epsilon : s(x) - y \in B^\epsilon\}).
\]

Let \( \psi \) be a smooth cut-off function with \( \int \psi = 1 \), and consider

\[
\int \mu_A(\{y \in A^\epsilon : s(x) - y \in B^\epsilon\})\psi(x)dx.
\]

By (5.2), this expression is bounded below by a constant times

\[
\int \left( \frac{1}{\epsilon} \right)^{\gamma(x)} \epsilon^{s_A}\psi(x)dx.
\]
We now obtain an upper bound on the expression in (5.3). Let $J_s$ denote the Jacobian of the change of variables $x \to \bar{s}(x)$. We have

$$
\int \mu_A\{y \in A^c : \bar{s}(x) - y \in B^c\} \psi(x) dx 
\lesssim \epsilon^{d-s_B} \int \int \mu_B(\bar{s}(x) - y) d\mu_A(y) \psi(x) dx 
\lesssim \epsilon^{d-s_B} \int |\hat{\mu}_A(\xi)| \cdot |\hat{J_s} \circ s^{-1}(\xi)| \cdot |\hat{\mu}_B(\xi)| d\xi 
\leq C_N \epsilon^{d-s_B} \int |\hat{\mu}_A(\xi)| \cdot |\hat{\mu}_B(\xi)| \cdot (1 + |\xi|)^{-N} d\xi
$$

for any $N > 1$. This quantity is easily $\lesssim \epsilon^{d-s_B}$.

Comparing the upper and lower bounds, it follows that

$$
\epsilon^A \int (\epsilon^{-1})^{\gamma(x)} \psi(x) dx \lesssim \epsilon^{d-s_B},
$$

which implies that

$$
\int (\epsilon^{-1})^{\gamma(x)} \psi(x) dx \lesssim (\epsilon^{-1})^{s_A + s_B - d}.
$$

It follows by convexity that

$$
(\epsilon^{-1}) \int \gamma(x) \psi(x) dx \lesssim (\epsilon^{-1})^{s_A + s_B - d},
$$

which implies that

$$
\int \gamma(x) \psi(x) dx \leq s_A + s_B - d.
$$

Since this holds for every smooth $\psi$ with $\int \psi = 1$ it follows that

$$
\gamma(x) \leq s_A + s_B - d
$$

for almost every $x \in \mathbb{R}^d$, as desired.

6. Proof of Theorem 2.6

Let $\mu_A$ and $\mu_B$ be the measures supported on $A$ and $B$ respectively defined as in definition (2.1). Observe the following which holds by the Ahlfors-David regularity of the set $A$ and by the definition of upper Minkowski dimension:

$$
(\frac{1}{\epsilon})^{\gamma(x)} \epsilon^{s_A} \lesssim \mu_A(A \cap x - gB)^c,
$$

where $X^c$ denotes the $\epsilon$-neighborhood of the set $X$. 
Since \( \{ A \cap x - gB \}^c \subset \{ y \in A^c : x - y \in gB^c \} \), it follows that

\[
(6.2) \quad \left( \frac{1}{\epsilon} \right)^{\gamma_g(x)} e^{s_A} \lesssim \mu_A(\{ y \in A^c : x - y \in gB^c \}) .
\]

Consider

\[
(6.3) \quad \int \int \mu_A \{ y \in A^c : x - y \in gB^c \} d\mu(x)d\theta(g).
\]

By (6.2), this expression is bounded below by a constant times

\[
\int \int \left( \frac{1}{\epsilon} \right)^{\gamma_g(x)} e^{s_A} d\mu(x)d\theta(g).
\]

We now obtain an upper bound on the expression in (6.3). First observe, by the Ahlfors-David regularity of \( B \), that this quantity is comparable to

\[
(6.4) \quad e^{d-s_B} \int \int \hat{\mu}_B (g(x-y)) d\mu(x) d\mu_A (y) d\theta(g)
= e^{d-s_B} \int \left( \int \hat{\mu}_B (g(x) - y) d\theta(g) \right) \hat{\mu}_E (\xi) \hat{\mu}_A (\xi) d\xi,
\]

where, as before, \( \mu_B (x) = \mu_B * \rho_s (x) \), where \( \rho_s (x) = \epsilon^{-d} \rho (x/\epsilon) \), \( \rho \geq 0 \), smooth, supported in the ball of radius 2 and \( \int \rho = 1 \).

**Lemma 6.1.** With the notation above,

\[
\left| \int \hat{\mu}_B (g(x) - y) d\theta(g) \right| \leq C |\xi|^{-\frac{d}{a+b}}.
\]

With Lemma 6.1 in tow, we see that the expression in (6.4) is

\[
\leq C e^{d-s_B} \int |\xi|^{-\frac{d}{a+b}} |\hat{\mu}_E (\xi)| \cdot |\hat{\mu}_A (\xi)| d\xi
\leq C e^{d-s_B} \sqrt{I_a (\mu) I_b (\mu_A)}
\]

with \( a + b = d + 1 \) by the Cauchy-Schwartz inequality. If \( a < s_E \) and \( b < s_A \), this quantity is bounded.

Thus we may conclude that this quantity is bounded by a constant times \( e^{d-s_B} \) if \( s_E + s_A > d + 1 \). Combining the upper and lower bounds, we conclude that

\[
\int \int (\epsilon^{-1})^{\gamma_g(x)} e^{s_A} d\theta(g) d\mu(x) \leq C e^{d-s_B},
\]

which implies that

\[
\int \int (\epsilon^{-1})^{\gamma_g(x)} d\theta(g) d\mu(x) \leq C (\epsilon^{-1})^{s_A + s_B - d}.
\]

Applying convexity, as in the proof of Theorem 1.2 implies that

\[
\int \int \gamma_g (x) d\theta(g) d\mu(x) \leq s_A + s_B - d
\]

and the conclusion of Theorem 2.6 follows.
We are left to prove Lemma 6.1. We have

$$\int \hat{\mu}_B(g\xi) d\theta(g)$$

$$= \int \int e^{-2\pi i g^{-1} x \cdot \xi} d\theta(g) d\mu_B(x)$$

$$= \int \hat{\sigma}(|x||c(x)| d\mu_B(x),$$

where $\sigma$ is the Lebesgue measure on $S^{d-1}$.

The modulus of this quantity is $\leq C |\xi|^{-\frac{d+1}{2}}$, so the proof of Lemma 6.1 and thus of Theorem 2.6 is complete.

7. Proof of Theorem 2.8

Let $\mu_A$ and $\mu_B$ be the measures supported on $A$ and $B$ respectively defined as in definition (2.1). Observe the following which holds by the Ahlfors-David regularity of the set $A$ and by the definition of upper Minkowski dimension:

$$\left(\frac{1}{\epsilon}\right)^{\gamma_{A,B}(x)} \epsilon^{s_A} \lesssim \mu_A(\{A \cap x - B\}^\prime),$$

where $X^\prime$ denotes the $\epsilon-$neighborhood of the set $X$.

Since $\{A \cap (x - B)\}^\prime$ is contained in the set

$$\{y \in A^\prime : x - y \in B^\prime\},$$

it follows that

$$\left(\frac{1}{\epsilon}\right)^{\gamma_{A,B}(x)} \epsilon^{s_A} \lesssim \mu_A(\{y \in A^\prime : x - y \in B^\prime\}).$$

Consider

$$\int \mu_A\{y \in A^\prime : x - y \in B^\prime\} d\mu(x).$$

By (7.2), this quantity is bounded below by a constant times

$$\int \left(\frac{1}{\epsilon}\right)^{\gamma_{A,B}(x)} \epsilon^{s_A} d\mu(x).$$

We now obtain an upper bound on (7.3). Observe that this expression is comparable to

$$\epsilon^{d-s_B} \int \mu_B(x-y) d\mu(x) d\mu_A(y),$$

where $\mu_B(x) = \mu_B * \rho_\epsilon(x), \rho_\epsilon(x) = \epsilon^{-d} \rho(x/\epsilon)$, $\rho \in C_0^\infty(\mathbb{R}^d)$, with $\int \rho(x) dx = 1$.

The right hand side of (7.5) equals

$$\epsilon^{d-s_B} \int \hat{\mu}_B(\xi) \overline{\hat{\rho}(\xi)} \hat{\mu}_E(\xi) |\hat{\mu}_A(\xi)| d\xi$$

$$\leq C \epsilon^{d-s_B} \int |\xi|^{-\frac{d}{2}} |\hat{\mu}_E(\xi)| |\hat{\mu}_A(\xi)| d\xi.$$
\[ (7.6) \quad C \epsilon^{d-s_B} \left( \int |\hat{\mu}_E(\xi)|^2 |\xi|^{-d+a} d\xi \right)^{\frac{1}{2}} \cdot \left( \int |\hat{\mu}_A(\xi)|^2 |\xi|^{-d+b} d\xi \right)^{\frac{1}{2}} = C \epsilon^{d-s_B} \sqrt{I_a(\mu)I_b(\mu_A)}, \]

where

\[ \frac{a + b}{2} = d - \beta \]

and \( I_a(\mu) \) is the standard energy integral given by

\[ \int \int |x-y|^{-a} d\mu(x)d\mu(y) = c \int |\hat{\mu}_E(\xi)|^2 |\xi|^{-d+a} d\xi. \]

It follows that the quantity in (7.6) is bounded by \( C \epsilon^{d-s_B} \) due to our assumption that \( \frac{a + b}{2} > d - \beta \) and elementary properties of energy integrals. See, for example, [26].

Comparing the upper and lower bounds, we see that

\[ \int (\epsilon^{-1})^\gamma_{A,B}(x) d\mu(x) \leq C(\epsilon^{-1})^{s_B + s_A - d}. \]

By convexity, this implies that

\[ (\epsilon^{-1}) \int \gamma_{A,B}(x) d\mu(x) \leq C(\epsilon^{-1})^{s_B + s_A - d}, \]

from which we conclude that

\[ \int \gamma_{A,B}(x) d\mu(x) \leq s_B + s_A - d, \]

as claimed.

8. **Proof of Theorem 2.13**

Let \( \mu_A \) and \( \mu_B \) be the measures supported on \( A \) and \( B \) respectively defined as in definition (2.1). Observe the following which holds by the Ahlfors-David regularity of the set \( A \) and by the definition of upper Minkowski dimension:

\[ (8.1) \quad \left( \frac{1}{\epsilon} \right)^{\gamma_t(x)} \epsilon^{s_A} \lesssim \mu_A(\{ A \cap x - tB \}^c), \]

where \( X^\epsilon \) denotes the \( \epsilon \)-neighborhood of the set \( X \).

Since \( \{ A \cap x - tB \}^c \subset A^c \cap (x - tB^c) \), it follows that

\[ (8.2) \quad \left( \frac{1}{\epsilon} \right)^{\gamma_t(x)} \epsilon^{s_A} \lesssim \mu_A(A^c \cap (x - tB^c)). \]

Consider

\[ (8.3) \quad \int \int_{1}^{2} \mu_A(A^c \cap (x - tB^c)) dtd\mu(x). \]

By (8.2), this expression is bounded below by a constant times

\[ \int \int_{1}^{2} \left( \frac{1}{\epsilon} \right)^{\gamma_t(x)} \epsilon^{s_A} dtd\mu(x). \]
We now obtain an upper bound on the expression in (8.3). Let \( \rho : \mathbb{R}^d \to \mathbb{R} \) be a non-negative and smooth function which is greater or equal to one on the ball of radius two centered at the origin. Assume that \( \rho \) has the additional property that its Fourier transform is a positive function which is greater or equal to one on the unit ball and has compact support. Let \( \rho_\epsilon(x) = \frac{1}{\epsilon^d} \rho(\frac{x}{\epsilon}) \).

Re-write the integrand in (8.2) as

\[
\mu_A(\Lambda \cap (x - tB^c)) \sim (\epsilon)^{d - s_B} \int \mu_B * \rho_\epsilon \left( \frac{x - y}{t} \right) d\mu_A(y).
\]

Integrating in \( x \) with respect to the measure \( \mu \) and in \( t \in [1, 2] \) we bound the expression in (8.2) by a constant times

\[
\left( \frac{1}{\epsilon} \right)^{s_B - d} \int \int \int \mu_B * \rho_\epsilon \left( \frac{x - y}{t} \right) \psi(t) d\mu_A(y) dtd\mu(x),
\]

where \( \psi \) is a translated smooth bump function equal to one on \([1, 2] \).

**Lemma 8.1.** With the notation above,

\[
\int \int \int \mu_B * \rho_\epsilon \left( \frac{x - y}{t} \right) \psi(t) d\mu_A(y) dtd\mu(x) \lesssim 1,
\]

whenever \( \frac{s_A + s_E}{2} > d - (s_B - h) \).

Assuming Lemma 8.1 for the moment, it follows that (8.2) is bounded above by a constant times

\[
\left( \frac{1}{\epsilon} \right)^{s_B - d},
\]

whenever \( \frac{s_A + s_E}{2} > d - (s_B - h) \).

Combining the upper and lower bounds for (8.2), we conclude that

\[
\int \int (\frac{1}{t})^{\gamma(x)} \epsilon^{s_A} dtd\mu(x) \lesssim \left( \frac{1}{\epsilon} \right)^{s_B - d}.
\]

Now (2.11) follows by convexity (use Jensen’s inequality).

To demonstrate (2.12), we observe that (2.11) implies that

\[
\mu(\{ x : \gamma(x) > s_A + s_B - d \}) = 0,
\]

for a.e. \( t \in [1, 2] \), and we conclude that (2.12) holds whenever \( \frac{s_A + s_E}{2} > d - (s_B - h) \).

To finish the proof of the theorem, it remains to prove Lemma 8.1. Using elementary properties of the Fourier transform, we bound the right-hand-side of (8.6) by

\[
\int \int \hat{\mu}_B(t\xi) \hat{\rho}_\epsilon(t\xi) \hat{\mu}_A(\xi) \overline{\hat{\mu}(\xi)} t^d \psi(t) dt d\xi.
\]

The presence of the smooth cut-off function, \( \hat{\rho}_\epsilon \), implies that we need only consider \( |\xi| < 1/\epsilon \). Additionally, the left-hand-side of (8.8) restricted to \( |\xi| < 1 \) is bounded since the integrand is finite.

It remains to see that the left-hand-side of (8.8) restricted to the region where \( 1 < |\xi| < 1/\epsilon \) is bounded. Fix \( 1 < |\xi| < 1/\epsilon \), and consider

\[
\int \hat{\mu}_B(t\xi) \hat{\rho}_\epsilon(t\xi) t^d \psi(t) dt.
\]
Let $\tilde{\psi} = td\psi$, let $\mu_B' = \mu_B * \rho_\epsilon$, and use the definition of the Fourier transform to re-write (8.9) as

$$\int \mu_B(t\xi)\hat{\rho}_{2t}(t\xi)\tilde{\psi}(t)dt = \int \hat{\psi}(x \cdot \xi)\mu_B'(x)dx.$$  

(8.10)

We break this integral into dyadic pieces:

$$\int \hat{\psi}(x \cdot \xi)\mu_B'(x)dx + \sum_{m=0}^\infty \int_{\{x: 2^m \leq |x \cdot \xi| < 2^{m+1}\}} \hat{\psi}(x \cdot \xi)\mu_B'(x)dx.$$  

(8.11)

Since for all integers $N > 1$, there exists $c_N$ so that $\hat{\psi} \lesssim \min\{1, |\cdot|^{-N}\}$, we may bound the first piece as follows:

$$\left| \int_{\{x: |x \cdot \xi| \leq 1\}} \hat{\psi}(x \cdot \xi)\mu_B'(x)dx \right| \lesssim \int_{\{x: |x \cdot \xi| \leq 1\}} \mu_B'(x)dx = \frac{1}{\epsilon^s} \int \int_{\{x: |x \cdot \xi| \leq 1\}} \rho\left(\frac{x - y}{\epsilon}\right) dx d\mu_B(y).$$

Recall that $\rho$ is smooth, non-negative, and equal to one on the unit ball. We fix $y$ and break the integral in $x$ to consider $\{x: |x - y| < \epsilon\}$ and $\{x: |x - y| > \epsilon\}$. Observe that

$$\frac{1}{\epsilon^s} \int \int_{\{x: |x \cdot \xi| \leq 1\} \cap \{x: |x - y| < \epsilon\}} \rho\left(\frac{x - y}{\epsilon}\right) dx d\mu_B(y) \lesssim \frac{1}{\epsilon^s} \int \int_{\{x: |x \cdot \xi| \leq 1\} \cap \{x: |x - y| < \epsilon\}} dx d\mu_B(y) \lesssim \mu_B\left(\{y : |y \cdot \xi| \leq 1 + \epsilon|\xi|\}\right) \lesssim \mu_B\left(\left\{y : \left|y \cdot \frac{\xi}{|\xi|}\right| \leq \frac{1}{1|\xi|}\right\}\right).$$

where the last line follows because we have fixed $|\xi| < \frac{1}{\epsilon}$.

Applying the hyperplane size condition of order $h$ on the set $B$, we may bound this quantity by:

$$\lesssim |\xi|^{-(s_B - h)}.$$
Observe that
\[
\frac{1}{\epsilon^d} \int \int \left( \left\{ x : |x \cdot \xi| \leq 1 \right\} \cap \left\{ x : |x-y| > \epsilon \right\} \right) \rho \left( \frac{x-y}{\epsilon} \right) \, dx \, d\mu_B(y) \\
\lesssim \frac{1}{\epsilon^d} \int \int \left( \left\{ x : |x \cdot \xi| \leq 1 \right\} \cap \left\{ x : |x-y| > \epsilon \right\} \right) |x-y|^{-N} \, dx \, d\mu_B(y) \\
\sim \epsilon^{N-d} \sum_{j=0}^{\infty} \int \int \left( \left\{ x : |x \cdot \xi| \leq 1 \right\} \cap \left\{ x : |\frac{x-y}{\epsilon}| \sim 2^j \right\} \right) 2^{-jN} \, dx \, d\mu_B(y) \\
\lesssim \epsilon^N \sum_{j=0}^{\infty} 2^{jd} 2^{-jN} \mu_B \left( \left\{ y : |y \cdot \xi| \leq 1 + 2^j \epsilon |\xi| \right\} \right). \\
= \epsilon^N \sum_{j=0}^{\infty} 2^{jd} 2^{-jN} \mu_B \left( \left\{ y : \left| \frac{y \cdot \xi}{|\xi|} \right| \leq \frac{1}{|\xi|} + 2^j \epsilon \right\} \right),
\]
where we may choose $N$ arbitrarily large.

Applying the hyperplane size condition of order $h$ on the set $B$, we may bound this quantity by:
\[
\lesssim \epsilon^N \sum_{j=0}^{\infty} 2^{jd} 2^{-jN} \max \left\{ |\xi|^{-s_B-h}, (2^j \epsilon)^{s_B-h} \right\}.
\]
Recalling that we have restricted our attention to the region where $1 < |\xi| < \frac{1}{\epsilon}$ and recalling that we may choose $N$ arbitrarily large, we conclude that this quantity is bounded by:
\[
\lesssim |\xi|^{-(N+s_B-h)}.
\]

In conclusion,
\[
(8.12) \quad \left| \int_{\{x : |x \cdot \xi| \leq 1\}} \hat{\psi}(t) \, d\mu_B(x) \right| \lesssim |\xi|^{-s_B+h}.
\]

We bound the second part of the sum in (8.11) with similar estimates. Observe
\[
\sum_{m=0}^{\infty} \left| \int_{\{x : 2^m \leq |x \cdot \xi| \leq 2^{m+1}\}} \hat{\psi}(x \cdot \xi) \mu_B^M(x) \, dx \right| \\
\lesssim \sum_{m=0}^{\infty} \int_{\{x : 2^m \leq |x \cdot \xi| \leq 2^{m+1}\}} |x \cdot \xi|^{-N} \mu_B^M(x) \, dx \\
\lesssim \epsilon^{-d} \sum_{m=0}^{\infty} 2^{-mN} \int_{\{x : 2^m \leq |x \cdot \xi| \leq 2^{m+1}\}} \rho \left( \frac{x-y}{\epsilon} \right) \, dx \, d\mu_B(y).
\]
Recall that $\rho$ is smooth, non-negative, and equal to one on the unit ball. We fix $y$ and break the integral in $x$ to consider $\{x : |x-y| < \epsilon\}$ and $\{x : |x-y| > \epsilon\}$. 


Observe that
\[
\lesssim \epsilon^{-d} \sum_{m=0}^{\infty} 2^{-mN} \int \int_{\{x: 2^m \leq |x| \leq 2^{m+1}\} \cap \{x: |x-y| < \epsilon\}} \rho \left( \frac{x-y}{\epsilon} \right) \, dx \, d\mu_B(y).
\]
\[
\lesssim \epsilon^{-d} \sum_{m=0}^{\infty} 2^{-mN} \int \int_{\{x: 2^m \leq |x| \leq 2^{m+1}\} \cap \{x: |x-y| < \epsilon\}} \, dx \, d\mu_B(y).
\]
\[
\lesssim \sum_{m=0}^{\infty} 2^{-mN} \mu_B \left( \left\{ y : 2^m |\xi|^{-1} - \epsilon \leq \left| y \cdot \frac{\xi}{|\xi|} \right| \leq 2^{m+1} |\xi|^{-1} + \epsilon \right\} \right).
\]
\[
\lesssim \sum_{m=0}^{\infty} 2^{-mN} 2^{m(sB-h)} |\xi|^{-(sB-h)},
\]
\[
\lesssim |\xi|^{-(sB-h)}.
\]
for \(N\) chosen sufficiently large.

Observe that
\[
\lesssim \epsilon^{-d} \sum_{m=0}^{\infty} 2^{-mN} \int \int_{\{x: 2^m \leq |x| \leq 2^{m+1}\} \cap \{x: |x-y| > \epsilon\}} \rho \left( \frac{x-y}{\epsilon} \right) \, dx \, d\mu_B(y).
\]
\[
\lesssim \epsilon^{-d} \sum_{m=0}^{\infty} 2^{-mN} \int \int_{\{x: 2^m \leq |x| \leq 2^{m+1}\} \cap \{x: |x-y| > \epsilon\}} |\frac{x-y}{\epsilon}|^{-N} \, dx \, d\mu_B(y).
\]
\[
\lesssim \epsilon^{N-d} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} 2^{-mN} 2^{-jN} \int \int_{\{x: 2^m \leq |x| \leq 2^{m+1}\} \cap \{x: |\frac{y}{x}| < 2j\}} \, dx \, d\mu_B(y).
\]
\[
\lesssim \epsilon^{N} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} 2^{jd} 2^{-mN} 2^{-jN} \mu_B \left( \left\{ y : 2^m |\xi|^{-1} - 2^j \epsilon \leq \left| y \cdot \frac{\xi}{|\xi|} \right| \leq 2^{m+1} |\xi|^{-1} + 2^j \epsilon \right\} \right).
\]
\[
\lesssim \epsilon^{N} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} 2^{jd} 2^{-mN} 2^{-jN} \max \left\{ \left(2^m |\xi|^{-1}\right)^{(sB-h)}, \left(2^j \epsilon\right)^{(sB-h)} \right\}.
\]

Since \(N\) may be taken arbitrarily large and since \(|\xi| < (\epsilon)^{-1}\), we may bound this quantity by:
\[
\lesssim |\xi|^{-(sB-h)}.
\]

These observations demonstrate that the left-hand-side of (8.9) is bounded by a constant times \(|\xi|^{-(sB-h)}\). It follows that the left-hand-side of (8.8) is bounded by a constant times the following:
\[
(8.13) \quad \left| \int |\hat{\mu}_A(\xi)| \cdot |\hat{\nu}(\xi)| \cdot |\xi|^{-sB+h} \, d\xi \right|
\]

To handle the integral in (8.13), use the Cauchy-Schwartz inequality to bound the right-hand-side of (8.13) by
\[
(8.14) \quad \left( \int |\hat{\mu}_A(\xi)|^2 |\xi|^{-(2(sB-h)+\gamma)/2} \right)^{1/2} \left( \int |\hat{\nu}(\xi)|^2 |\xi|^{-(2(sB-h)-\gamma)/2} \right)^{1/2},
\]
where \(\gamma\) will be chosen momentarily.
Observe that (see [6], pg. 208 and [7], section 6.2)
\begin{equation}
\int |\hat{\mu}(\xi)|^2 |\xi|^{-2(s_B-h)+\gamma)/2} \lesssim 1,
\end{equation}
whenever \((-2(s_B-h)+\gamma)/2 < s_A - d\). Similarly
\begin{equation}
\int |\hat{\mu}(\xi)|^2 |\xi|^{-2(s_B-h)-\gamma)/2} \lesssim 1,
\end{equation}
whenever \((-2(s_B-h)-\gamma)/2 < \alpha - d\).
Both of these conditions are satisfied when \(\gamma\) is chosen such that
\(-\frac{s_B-h}{2} < \gamma < \frac{2(s_A-d)}{2} + (s_B-h)\).
Such a choice of \(\gamma\) is possible whenever \(\frac{s_A+\alpha}{2} > d - (s_B-h)\). We conclude that (8.14) is bounded by a positive constant which does not depend on \(\epsilon\) whenever \(\frac{s_A+\alpha}{2} > d - (s_B-h)\). This completes the proof of Lemma 8.1.

9. Proof of Theorem 2.14

First notice that the \(\epsilon\)-neighborhood of \(A \cap \{y : \vec{\phi}(x,y) = \vec{t}\}\) is contained in the set \(\{y \in A^\epsilon : |\vec{\phi}(x,y) - \vec{t}| \leq \epsilon\}\).

9.1. A reduction. Assume that we have, after rewriting our quantities by using the above containment,
\begin{equation}
\int \mu_A \{y \in A^\epsilon : |\vec{\phi}(x,y) - \vec{t}| \leq \epsilon\} \, d\mu(x) \lesssim \epsilon^m.
\end{equation}
The Ahlfors-David regularity of the sets \(A\) and \(E\) then shows that (9.1) is bounded below by
\begin{equation}
C \int (\epsilon^{-1})^{\gamma_t(x)} \epsilon^{s_A} \, d\mu(x),
\end{equation}
where \(\gamma_t(x)\) denotes the upper Minkowski dimension of \(A \cap \{y : \vec{\phi}(x,y) = \vec{t}\}\) and \(C > 0\) is a universal constant. Comparing the upper and lower bounds, we see that
\begin{equation}
\int (\epsilon^{-1})^{\gamma_t(x)} \, d\mu(x) \leq C(\epsilon^{-1})^{s_A-m}.
\end{equation}
Convexity implies
\begin{equation}
(\epsilon^{-1}) \int \gamma_t(x) \, d\mu(x) \leq C(\epsilon^{-1})^{s_A-m},
\end{equation}
from which we conclude that
\begin{equation}
\int \gamma_t(x) \, d\mu(x) \leq s_A - m,
\end{equation}
where
\(\int \gamma_t(x) d\mu(x) \leq s_A - m\), hence completing the proof of Theorem 2.14.

Now, we continue with a slight modification of Proposition 2.2 in [4] to prove estimate (9.1). The actual proof is almost identical to that of the referenced proposition, but we provide it below, with the necessary changes, for the sake of completeness.
9.2. Decomposition of integral in (9.1). We start by taking Schwartz the class functions $\eta_0(\xi)$ supported in the ball $\{|\xi| \leq 4\}$ and $\eta(\xi)$ supported in the annulus

\begin{equation}
1 < |\xi| < 4
\end{equation}

with

\begin{equation}
\eta_0(\xi) + \sum_{j=1}^{\infty} \eta_j(\xi) = 1.
\end{equation}

Define the Littlewood-Paley piece of $\mu_j$ by the relation

\begin{equation}
\hat{\mu}_j(\xi) = \hat{\mu}(\xi) \eta_j(\xi).
\end{equation}

Consider the integral in (9.1). Using the Littlewood-Paley decomposition, this integral is bounded above by a constant times

\begin{equation}
\sum_{j,k} \int \int_{\{t_1 \leq \phi(x,y) \leq t_1 + \varepsilon: 1 \leq l \leq m\}} \psi(x,y)d\mu_{A,j}(y) d\mu_k(x) = \sum_{j,k} \langle \mu_{A,j}, T^\varepsilon \mu_k \rangle
\end{equation}

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R}^d)$ inner product and

\begin{equation}
T^\varepsilon \mu_k(y) = \int_{\{t_1 \leq \phi(x,y) \leq t_1 + \varepsilon: 1 \leq l \leq m\}} \psi(x,y)d\mu_k(x)
= \int_{t_1}^{t_1 + \varepsilon} \cdots \int_{t_m}^{t_m + \varepsilon} \int_{\phi(x,y) = r} \psi(x,y)\mu_k(x)d\sigma_{y,r}(x)dr_1 \cdots dr_m,
\end{equation}

where $d\sigma_{y,r}$ is the Lebesgue measure on the set $\{x : \phi(x,y) = r\}$ and $r = (r_1, \ldots, r_m)$. It should be noted that the innermost integral on the right side of (9.7) is just $T^\varepsilon_{\phi_r}$ applied to $\mu_k$. It follows that the right hand side of (9.6) becomes

\begin{equation}
\sum_{j,k} \int_{t_1}^{t_1 + \varepsilon} \cdots \int_{t_m}^{t_m + \varepsilon} \langle \mu_{A,j}, T^\varepsilon_{\phi_r}(\mu_k) \rangle dr_1 \cdots dr_m.
\end{equation}

We will now use the mapping properties of $T^\varepsilon_{\phi_r}$ to prove $\langle \mu_A, T^\varepsilon_{\phi_r} \mu \rangle$ is uniformly bounded in $r$ over the domain of integration. This, in turn, will prove our desired upper bound.

We have

\begin{equation}
\langle \mu_A, T^\varepsilon_{\phi_r}(\mu) \rangle = \sum_{j,k} \langle \mu_{A,j}, T^\varepsilon_{\phi_r}(\mu_k) \rangle
\end{equation}

\begin{equation}
= \sum_{|j-k| \leq K} \langle \mu_{A,j}, T^\varepsilon_{\phi_r}(\mu_k) \rangle + \sum_{|j-k| > K} \langle \mu_{A,j}, T^\varepsilon_{\phi_r}(\mu_k) \rangle
\end{equation}

for $K$ large enough; the choice of $K$ will be justified later.

9.3. Proof of estimate (9.1). We will estimate each of the above sums separately.
9.3.1. Near-diagonal terms. For the first sum,

\[ \sum_{|j-k| \leq K} \langle \mu_{A,j}, T_{\varphi_r} (\mu_k) \rangle \lesssim \sum_{|j-k| \leq K} 2^{j-d-s_A} 2^{k-d-s_E} 2^{-k} \lesssim 1 \]

provided that \( \frac{s_E + s_A}{2} > d - s \). Indeed, as \( \eta_j \sim \eta_j^2 \),

\[ \sum_{|j-k| \leq K} \langle \mu_{A,j}, T_{\varphi_r} (\mu_k) \rangle = \sum_{|j-k| \leq K} \langle \hat{\mu}_{A,j}, \hat{T}_{\varphi_r} (\mu_k) \rangle \lesssim \sum_{|j-k| \leq K} \| \mu_j \|_2 \| \hat{T}_{\varphi_r} (\mu_k) \eta_j \|_2 \]

\[ \lesssim 2^{j-d-s_A} \]

where we use the Cauchy-Schwartz inequality. Since \( \mu \) is an Ahlfors-David regular measure on a set of Hausdorff dimension \( \alpha \), we have that

\[ \| \mu_{A,j} \|_2 \lesssim 2^{d-s_A}. \]

Indeed,

\[ \| \mu_{A,j} \|_2^2 = \int |\hat{\mu}(\xi)|^2 \eta(2^{-j} \xi) d\xi \]

\[ = \int \int e^{2\pi i (x-y) \cdot \xi} \eta(2^{-j} \xi) d\xi d\mu(x) d\mu(y) \]

\[ = 2^{dj} \int \int \hat{\eta}(2^j (x-y)) d\mu(x) d\mu(y). \]

The absolute value of this quantity is bounded, for every \( N > 0 \), by

\[ C_N 2^{dj} \int \int (1 + 2^j |x-y|)^{-N} d\mu(x) d\mu(y) \]

\[ = C_N 2^{dj} \int \int_{|x-y| \leq 2^{-j}} (1 + 2^j |x-y|)^{-N} d\mu(x) d\mu(y) \]

\[ + C_N 2^{dj} \sum_{l=0}^{\infty} \int \int_{2^l \leq |x-y| \leq 2^{l+1}} (1 + 2^j |x-y|)^{-N} d\mu(x) d\mu(y) \]

\[ = I + II. \]

By the Ahlfors-David property,

\[ I \lesssim C_N 2^{dj} 2^{-j s_A}. \]

Since \( \mu \) is compactly supported, there exists \( M > 0 \) such that

\[ II = C_N 2^{dj} \sum_{l=0}^{j+M} \int \int_{2^l \leq |x-y| \leq 2^{l+1}} (1 + 2^j |x-y|)^{-N} d\mu(x) d\mu(y). \]

This expression is

\[ \lesssim C_N 2^{dj} \sum_{l=0}^{j+M} 2^{-j s_A} 2^{l s_A} 2^{-l N} \lesssim C_N 2^{(d-s_A)}. \]
It follows that $I + II \lesssim 2^{(d-s_A)}$ and (9.12) is established.

We also have that
\begin{equation}
\|T_{\phi_r}^{-1}(\mu_k)\eta_j\|_2 \lesssim 2^{-ks}2^{\left(\frac{k(d-s_A)}{2}\right)}
\end{equation}
by the mapping properties of the operator $T_{\phi_r}^{-1}$ in the regime of $|j - k| < K$.

9.3.2. Off-diagonal terms. We use the following lemma to get a bound on the second sum.

**Lemma 9.1.** For any $M > 2d + m + 1$ there exists a constant $C_M > 0$ such that for all indices $j,k$ with $|j - k| > K$ with $K$ large enough,
\[
\langle \mu_{A,j}, T_{\phi_r}^{-1}(\mu_k) \rangle \leq C_M 2^{-M \max\{j,k\}}.
\]
To prove the lemma, for simplicity, we replace $T_{\phi_r}^{-1}$ by $T$ and write
\[
T\mu_k(y) = \int_{\{x : \phi(x,y) = r\}} \psi(x,y)\mu_k(x)ds_{y,r}(x),
\]
where $ds_{y,r}$ is the Lebesgue measure on the set $\{x : \phi(x,y) = r\}$. It follows from our upcoming arguments that as long as $t_1 \leq r_1 \leq t_1 + \varepsilon$, the estimates hold uniformly in $r$.

As $\phi_1$ satisfies the property that $\{\nabla_x \phi_i(x,y)\}_{i=1}^m$ are linearly independent on a relatively open, bounded subset of $\{x : \phi_i(x,y) = t\}$ from (2.14), we can assume that $|\nabla_y \phi_i(x,y)| \approx 1$ on this set by making the support of $\psi$ small enough. Next, we use an approximation argument on $T$ by letting
\begin{equation}
T_n\mu_k(y) = n^m \int \psi(x,y)\Pi_1(\chi_i(n(\phi_1(x,y) - r_1)))\mu_k(x)dx
\end{equation}
where $\{\chi_i\}_{i=1}^m$ is a family of smooth cutoffs supported near 0 and equal to 1 near 0. It is shown in [10] that
\begin{equation}
n^m\Pi_1(\chi_i(n(\phi_1(x,y) - r_1))) dx
\end{equation}
converges to the measure that appears in $T_{\phi_r}^{-1}$ as $n \to \infty$. Therefore, proving the estimate in the case where $T_{\phi_r}^{-1}$ is replaced by $T_n$ is sufficient by convergence theorems found in [9] which in turn shows the uniformity in $r$. We will drop the domains of integration in the upcoming calculations for brevity.

By Fourier inversion, we have
\[
T_n\mu(y) = \int e^{ix\cdot\xi}e^{i\phi(x,y)-r}\psi(x,y)\Pi_1(\chi_i(n^{-1}s_i))\tilde{\mu}(\xi)d\xi ds
\]
and therefore
\begin{equation}
\widehat{T_n\mu}(\eta) = \int e^{-ix\cdot\eta}e^{i\phi(x,y)-r}\psi(x,y)\Pi_1(\chi_i(n^{-1}s_i))\tilde{\mu}(\xi)d\xi dy ds d\xi.
\end{equation}
Invoking the properties of the Fourier transform on $L^2$, we see that
\begin{equation}
\langle T_n\mu_{A,j}, \tilde{\mu_k} \rangle = \langle \widehat{T_n\mu_{A,j}}, \widehat{\tilde{\mu_k}} \rangle
= \int e^{-ix\cdot\eta}e^{i\phi(x,y)-r}\psi(x,y)\Pi_1(\chi_i(n^{-1}s_i))\mu_{A,j}(\xi)\tilde{\mu}(\eta)d\xi dy ds d\eta
= \int \mu_{A,j}(\xi)\tilde{\mu}(\eta)\Pi_1(\chi_i(n^{-1}s_i))I_{jk}(\xi,\eta, s)d\eta d\xi ds
\end{equation}

(9.18)
where

\begin{equation}
I_{jk}(\xi, \eta, s) = \psi_0(2^{-j}|\xi|)\psi_0(2^{-k}|\eta|) \int e^{i s \cdot (\vec{\phi}(x,y) - r)} e^{iy \cdot \xi} e^{-ix \cdot \eta} \psi(x,y) dxdy
\end{equation}

and \( \psi_0 \) is smooth cutoff equal to 1 on \(|z| \leq 10\) and vanishing in the ball of radius 1/2. The justification of such cutoffs comes from the support of \( \hat{\mu}_{A,j}(\xi) \) and \( \hat{\mu}_k(\eta) \) and again that \( \eta_j \approx \eta_j^2 \).

We will show that

\begin{equation}
|I_{jk}(\xi, \eta, s)| \leq C M^{2 - M_{\max}(j,k)}
\end{equation}

when \(|j - k| > K\) for a large enough \(K\).

Computing the critical points of the phase function in (9.19), we see that

\begin{equation}
\sum_l |s| \tilde{s}_l \nabla_x \phi_l(x,y) = \eta \quad \text{and} \quad \sum_l |s| \tilde{s}_l \nabla_y \phi_l(x,y) = -\xi,
\end{equation}

where \(s = |s| (\tilde{s}_1, ..., \tilde{s}_m)\) and \((\tilde{s}_1, ..., \tilde{s}_m) \in S^{m-1}\), the unit sphere. The compactness of the support of \(\psi\) and the domain of the variable \((\tilde{s}_1, ..., \tilde{s}_m)\) along with the linear independence condition from (2.14) implies that

\begin{equation}
\left| \sum_l \tilde{s}_l \nabla_x \phi_l(x,y) \right| \approx \left| \sum_l \tilde{s}_l \nabla_y \phi_l(x,y) \right| \approx 1.
\end{equation}

More precisely, the upper bound follows from smoothness and compact support. The lower bound follows from the fact that a continuous non-negative function achieves its minimum on a compact set. This minimum is not zero because of the linear independence condition (2.14).

It follows that

\begin{equation}
|\xi| \approx |\eta|
\end{equation}

when we are near the critical points in \((x,y)\). The support of the cutoffs \(\psi_0\), when \(|j - k| > K\), tell us that we are supported away from critical points in \((x,y)\) since (9.23) no longer holds. This condition implies that for some \(h\) or \(h'\) in \(\{1, 2, ..., d\}\),

\begin{equation}
\left( \sum_l s_l \frac{\partial \phi_l}{\partial x_h} - \eta_h \right) \neq 0 \quad \text{or} \quad \left( \sum_l s_l \frac{\partial \phi_l}{\partial y_{h'}} + \xi_{h'} \right) \neq 0.
\end{equation}

Without loss of generality, assume the former holds and that \(k > j\). It is immediate that \(e^{-ix \cdot \eta} e^{is \cdot (\vec{\phi}(x,y) - r)}\) is an eigenfunction of the differential operator

\begin{equation}
L = \frac{1}{i(\sum_l s_l \frac{\partial \phi_l}{\partial x_h} - \eta_h)} \frac{\partial}{\partial x_h};
\end{equation}

We integrate by parts in (9.19) using this operator. The expression that we get after performing this procedure \(M > 2d + m + 1\) times is

\begin{equation}
I(\xi, \eta, s) \lesssim \sup_{x,y} \left| \sum_l s_l \frac{\partial \phi_l}{\partial x_h} - \eta_h \right|^{-M}.
\end{equation}
Now, suppose that we are in the region \( \{ |s| < |\eta| \} \) (i.e. \( |s| \leq c|\eta| \) with a sufficiently large constant \( c > 0 \)). Since \( |\sum_i s_i \nabla_x \phi_i| \approx |s| \) it follows, after possibly changing our initial choice of \( h \), that

\[
(9.27) \quad \left| \sum_i s_i \frac{\partial \phi_i}{\partial x_h} - \eta_h \right| \gtrsim \left| \sum_i s_i \frac{\partial \phi_i}{\partial x_h} \right| - |\eta| \approx |\eta|.
\]

Similarly, if \( \{ |s| >> |\eta| \} \) then, after possibly changing our initial choice of \( h \),

\[
(9.28) \quad \left| \sum_i s_i \frac{\partial \phi_i}{\partial x_h} - \eta_h \right| \gtrsim \left| \sum_i s_i \frac{\partial \phi_i}{\partial x_h} \right| - |\eta| \approx |s|.
\]

In either region,

\[
(9.29) \quad |I_{jk}(\xi, \eta, s)| \lesssim \sup(|s|, |\eta|)^{-M} \lesssim 2^{-Mk}.
\]

Considering (9.18), the integrand \( (\Pi_0 \tilde{\chi}(n^{-1}s_i))I_{jk}(\xi, \eta, s) \) is integrable in \( s \) as the first term is at most 1 and \( I_{jk} \) is bounded above by \( |s|^{-M} \). Performing the remaining integrations and keeping in mind the support properties of \( \tilde{\mu}_{A,j} \) and \( \hat{\mu}_k \), it follows that

\[
(9.30) \quad \sum_{|j-k| > K} \langle \mu_{A,j}, T_{\phi_r}^*(\mu_k) \rangle \lesssim \sum_{|j-k| > K} C_M 2^{-(M-2d)\max(j,k)} \lesssim 1.
\]

This completes the proof of Lemma 9.1.

Since both sums in (9.9) are bounded by 1, this implies that the left hand side of (9.6) is bounded above by \( \varepsilon^m \) after completing the integrations in (9.7).

### 10. Proof of Theorem 2.17

The following proof is a slight modification of that of Theorem 2.14. For the remaining details, please see Section 9.

In a similar vein as the other proofs of this article, we first notice that the \( \varepsilon \)-neighborhood of \( A \cap \{ y : \phi(x, y) = t \} \) is contained in the set \( \{ y \in A^\varepsilon : |\phi(x, y) - t| \leq \varepsilon \} \).

#### 10.1. A reduction

Now, the estimate we would like to prove, as a result of the above containment, is

\[
(10.1) \quad \sup_{t \in [1,2]} \int A \mu_A \{ y \in A^\varepsilon : |\phi(x, y) - t| \leq \varepsilon \} \, d\mu(x) \lesssim \varepsilon.
\]

Notice that, as we are dealing only with finite-measures over compact sets with \( t \in [1,2] \), there exists \( t(x) \in L^1(\mathbb{R}^d) \) such that \( \sup_{t \in [1,2]} \int A \mu_A \{ y \in A^\varepsilon : |\phi(x, y) - t| \leq \varepsilon \} \, d\mu(x) = \int \mu_A \{ y \in A^\varepsilon : |\phi(x, y) - t(x)| \leq \varepsilon \} \, d\mu(x) \). We can rewrite the integral in the lefthand side of (10.1) as

\[
(10.2) \quad \int (\rho \ast \mu_A)_t(x) \, d\mu(x),
\]

where \( (\rho \ast \mu_A)_t(x) = \varepsilon^{-d} \int_{y \in A^\varepsilon : |\phi(x, y) - t(y)| \leq \varepsilon} \rho\left(\frac{x-y}{\varepsilon}\right) \, d\mu_A(z) \) and \( \rho \in C^\infty_0(\mathbb{R}^d) \).

Assuming that we have the estimate

\[
(10.3) \quad \int (\rho \ast \mu_A)_t(x) \, d\mu(x) \lesssim \varepsilon,
\]

the rest of the proof follows identically to that of Theorem 2.14, except in two places for the proof analogue of the estimate (9.1). We briefly digress in order to give some background.
10.2. Maximal functions. Unlike in Theorem 2.14, the main operator of interest is now the maximal averaging operator

\[(Mf)(x) = \sup_{t \in [1,2]} \left| \int_{\{y : \phi(x,y) = t\}} f(y)\psi(x,y) \, d\sigma_{x,t}(y) \right|,\]

where \(\psi\) is a smooth cut-off. Clearly, the Schwarz kernel of this operator is same as the generalized Radon transforms described in (2.13), except we take the additional step of taking a supremum in \(t\).

Under the same non-degeneracy conditions as those for generalized Radon transforms, for all \(t \in [1,2]\), we have the following Sobolev mapping property:

\[(10.5) \quad \|M(f)\|_{L^2} \leq \|f\|_{L^2}.\]

For some standard \(L^p\) estimates for maximal functions of this kind, please see [29].

A key example of a maximal function of this type is the spherical maximal operator,

\[(10.6) \quad \sup_{t \in (0,\infty)} \left| \int_{S^{d-1}} f(x-ty) \, d\sigma(y) \right|,\]

which was introduced in the 1970s by Elias Stein. In particular, Stein was able to establish the optimal range of exponents \(p\) for the mapping properties of \(M\) on \(L^p\), in dimensions \(d \geq 3\); see [30]. The \(d = 2\) was, however, settled by Bourgain [3]. It is this operator, and more particularly the example of the sphere, which provides many of the intuitions in this article.

10.3. Proof of estimate (10.1). Returning to the proof, we make notice of the two places where the proof is different.

Again, we start by taking Schwartz the class functions \(\eta_0(\xi)\) supported in the ball \(\{|\xi| \leq 4\}\) and \(\eta(\xi)\) supported in the annulus

\[(10.7) \quad \{1 < |\xi| < 4\} \text{ with } \eta_j(\xi) = \eta(2^{-j}\xi) \text{ for } j \geq 1\]

with

\[(10.8) \quad \eta_0(\xi) + \sum_{j=1}^{\infty} \eta_j(\xi) = 1.\]

Define the Littlewood-Paley piece of \(\mu_j\) by the relation

\[(10.9) \quad \hat{\mu}_j(\xi) = \hat{\mu}(\xi)\eta_j(\xi).\]

Consider the integral in (10.1). Using the Littlewood-Paley decomposition, this integral is comparable to

\[(10.10) \quad \sup_{t \in [1,2]} \sum_{j,k} \int \int_{\{t \leq \phi(x,y) \leq t+\varepsilon\}} \psi(x,y)d\mu_k(y) \, d\mu_{A,j}(x) = \sup_{t \in [1,2]} \sum_{j,k} \langle \mu_{A,j}, M_{t}\mu_k \rangle,\]

where

\[(10.11) \quad M_{t}\mu_k(y) = \int_{\{t \leq \phi(x,y) \leq t+\varepsilon\}} \psi(x,y) d\mu_k(x).\]

Using the corresponding Littlewood-Paley decomposition, we get that
\[ \sup_{t \in [1, 2]} \langle \mu_A, M_t^\epsilon(\mu) \rangle = \sup_{t \in [1, 2]} \sum_{j, k} \langle \mu_{A, j}, M_t^\epsilon(\mu_k) \rangle \]

\[ \leq \sum_{|j-k| \leq K} \langle \mu_{A, j}, M_t^\epsilon(\mu_k) \rangle + \sup_{t \in [1, 2]} \sum_{|j-k| > K} \langle \mu_{A, j}, M_t^\epsilon(\mu_k) \rangle, \]

where

\[ M_t^\epsilon(\mu_k)(y) = \sup_{t \in [1, 2]} M_t^\epsilon_t(\mu_k)(y). \]

10.3.1. Near-diagonal terms. Similar to the case of near-diagonal terms in the proof of Theorem 2.14, we arrive at the estimate

\[ \sum_{|j-k| \leq K} |j - k|^s A_2 < M_2^d - s E^2 - k(d-2)/2 \lesssim 1 \]

provided that \( s_E + s_A > d + 2. \)

10.3.2. Off-diagonal terms. The estimate

\[ \sup_{t \in [1, 2]} \sum_{|j-k| > K} \langle \mu_{A, j}, M_t^\epsilon(\mu_k) \rangle \lesssim 2^{-\max(j,k)M}, \]

follows immediately from Lemma 9.1 for \( m = 1 \) and its independence of the value \( t \), for any \( M > 2d + 2. \)

The main estimate (10.1) is completed after performing the final integration in \( r \).

11. Proof of Theorem 2.18

Let \( \gamma_t(x^1, x^2) \) denote the upper Minkowski dimension of

\[ A \cap \{ y : \phi_1(x^1, y) = t_1 \} \cap \{ y : \phi_2(x^2, y) = t_2 \}. \]

Let \( \mu_A \) denote the measure inherit from the Ahlfors-David regularity of the set \( A \), and let \( \mu_1 \) and \( \mu_2 \) be as in the statement of the theorem.

11.1. A reduction. As in the proof of Theorem 2.14, we reduce the proof of the Theorem 2.18 to the following estimate:

\[ \int \int \mu_A \{ y \in A^c : |\phi_i(x^1, y) - t_i| \leq \epsilon, \ i = 1, 2 \} \ d\mu_1(x^1) d\mu_2(x^2) \gtrsim \epsilon^2. \]

To see why this estimate implies the statement of the theorem, first observe that the \( \epsilon \)-neighborhood of \( A \cap \{ y : \phi_1(x^1, y) = t_1 \} \cap \{ y : \phi_2(x^2, y) = t_2 \} \) is contained in the set \( \{ y \in A^c : |\phi_i(x^1, y) - t_i| \leq \epsilon, \ i = 1, 2 \}. \) Here, \( A^c \) denotes the \( \epsilon \)-neighborhood of the set \( A \).

Now, using the Ahlfors-David regularity of the set \( A \), the left-hand-side of (11.1) is bounded below by

\[ C \int \int (\epsilon^{-1})^{\gamma_t(x^1, x^2)} \epsilon^{s_A} d\mu_1(x^1) d\mu_2(x^2), \]
where $C > 0$ is a universal constant. Comparing the upper and lower bounds, we see that
\[ \int \int (\epsilon^{-1}) \gamma(x^1, x^2) \, d\mu_1(x^1) \, d\mu_2(x^2) \leq C(\epsilon^{-1})^{s_A - 2}. \]

Convexity implies
\[ (\epsilon^{-1}) \int \int \gamma(x^1, x^2) \, d\mu_1(x^1) \, d\mu_2(x^2) \leq C(\epsilon^{-1})^{s_A - 2}, \]
from which we conclude that
\[ \int \gamma(x^1, x^2) \, d\mu_1(x^1) \, d\mu_2(x^2) \leq s_A - 2, \]
hence completing the proof of Theorem 2.18.

The rest of this section is dedicated to proving estimate (11.1).

11.2. Approximation operators. Begin by re-writing equation (11.1) as
\[ (11.3) \quad \frac{1}{\epsilon^2} \int \int \int \{ x^1, x^2 \in E_1 \times E_2 : |\phi_i(x^1, y) - t_i| \leq \epsilon, \ i = 1, 2 \} \, d\mu_1(x^1) \, d\mu_2(x^2) \, d\mu_A(y) \lesssim 1. \]

Rather than working directly with operators $T_{\phi, t}$, introduced in (2.13), define
\[ (11.4) \quad T_{\phi, t}^\epsilon f(x) = \frac{1}{\epsilon} \int_{\{ y : t - \epsilon \leq \phi(x, y) \leq t + \epsilon \}} \psi(x, y) f(y) \, dy, \]
where $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denotes a positive, smooth, and compactly supported function.

Now, write the left-hand-side of equation (11.3) as:
\[ (11.5) \quad \int T_{\phi_1, t_1}^\epsilon \mu_1(y) T_{\phi_2, t_2}^\epsilon \mu_2(y) \, d\mu_A(y). \]
Therefore, in order to prove Theorem 2.18, it suffices to show that
\[ (11.6) \quad \int T_{\phi_1, t_1}^\epsilon \mu_1(y) T_{\phi_2, t_2}^\epsilon \mu_2(y) \, d\mu_A(y) \lesssim 1. \]

11.3. Dissection of range of the operators. For $i = 1, 2$, we consider the pre-image of $T_{\phi_i, t_i}^\epsilon \mu_i$ on a dyadic shell. Let $a$ be a non-negative integer. For $a \geq 1$, set
\[ S_a^i = \{ y : 2^a \leq T_{\phi_i, t_i}^\epsilon \mu_i(y) < 2^{a+1} \}. \]
For $a = 0$, set
\[ S_0^i = \{ y : T_{\phi_i, t_i}^\epsilon \mu_i(y) < 2 \}. \]
For $a, b$ fixed, set
\[ I_{a, b} = S_a^1 \cap S_b^2. \]

The left-hand-side of (11.6) can now be re-written as
\[ (11.7) \quad \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \int T_{\phi_1, t_1}^\epsilon \mu_1(y) T_{\phi_2, t_2}^\epsilon \mu_2(y) \chi_{I_{a, b}}(y) \, d\mu_A(y). \]
Notice that if $a = b = 0$, then the summand in (11.7) is trivally bounded since $\mu_A$ is a finite measure. Therefore we may assume that
\[ (11.8) \quad \max\{a, b\} \neq 0. \]
For simplicity of notation, we consider the case when \( a \geq b \). The other case is handled by re-writing the proof with roles of \( a \) and \( b \) reversed.

This implies that the proof of Theorem 2.18 is reduced to showing that:

\[
\sum_{a=1}^{\infty} \sum_{b=0}^{a} \int T_{\phi_1, t_1}^e \mu_1(y) T_{\phi_2, t_2}^e \mu_2(y) \chi_{I_{a,b}}(y) d\mu_A(y) \lesssim 1.
\]

The plan now is to bound the summand on the left-hand-side of (11.9) by \( 2^{2a} \mu_A(I_{a,b}) \) and to show that \( \mu_A(I_{a,b}) \) is sufficiently small.

The following Lemma gives an upper bound on \( \mu_A(I_{a,b}) \).

**Lemma 11.1.** Fix \( a \geq b \) non-negative integers with \( a \neq 0 \). Then for any real number \( \gamma \) satisfying

\[
d - s_A < \gamma < \alpha_1 - 1,
\]

the following estimate holds:

\[
\mu_A(I_{a,b}) \lesssim 2^{-2a} 2^{-\frac{2a}{2_a} (d - \gamma - s_a)}.
\]

Notice that such a \( \gamma \) exists whenever \( \alpha_1 + s_A > d + 1 \).

We use Lemma 11.1 to bound the summand on the left-hand-side of (11.9) by a quantity which decays in \( a \), and hence to finish the proof. In particular,

\[
\int T_{\phi_1, t_1}^e \mu_1(y) T_{\phi_2, t_2}^e \mu_2(y) \chi_{I_{a,b}}(y) d\mu_A(y) \lesssim 2^{2a} \mu_A(I_{a,b}) \lesssim 2^{2a} 2^{-\frac{2a}{2_a} (d - \gamma - s_a)}.
\]

Since \( \gamma > d - s_A \) and since we assume that \( a \geq b \), this quantity is summable in both \( a \) and \( b \).

This completes the proof of the theorem. It remains to show that Lemma 11.1 holds.

**11.4. Proof of Lemma 11.1.** We now demonstrate that:

\[
\mu_A(I_{a,b}) \lesssim 2^{-2a} 2^{-\frac{2a}{2_a} (d - \gamma - s_a)},
\]

where \( a > b \geq 0 \) and \( \gamma \in (d - s_A, \alpha_1 - 1) \).

Recall that \( S_a^1 = \{ y : 2^a \leq T_{\phi_1, t_1}^e \mu_1(y) < 2^{a+1} \} \). By elementary properties of the Fourier transform and the Cauchy-Schwarz inequality:

\[
2^a \mu_A(S_a^1) \leq \int T_{\phi_1, t_1}^e \mu_1(y) \chi_{S_a^1}(y) d\mu_A(y)
\]

\[
\leq \left( \int |\widehat{T_{\phi_1, t_1}^e \mu_1}(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} d\xi \right)^{\frac{1}{2}} \left( \int |\widehat{\mu_A \chi_{S_a^1}}(\xi)|^2 (1 + |\xi|^2)^{-\gamma/2} d\xi \right)^{\frac{1}{2}}.
\]

We use the following estimates to bound the equation in (11.13) from above:

**Lemma 11.2.** If \( d - s_A < \gamma < \alpha_1 - 1 \), then

\[
\int |\widehat{\mu_A \chi_{S_a^1}}(\xi)|^2 |\xi|^{-\gamma} d\xi \lesssim \mu_A(S_a^1) 2^{2a} 2^{-\frac{2a}{2_a} (d - \gamma - s_a)}.
\]

**Lemma 11.3.** If \( \gamma < \alpha_1 - 1 \), then

\[
\int |\widehat{T_{\phi_1, t_1}^e \mu_1}(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} d\xi \lesssim 1.
\]
Now
\[ 2^a \mu_A(S^1_a) \leq \left( \mu_A(S^1_a) 2^{2a_s(d-\gamma-s)} \right)^{1/2}, \]
whenever \( \gamma > d-s_A \) and \( \gamma < \alpha_1 - 1 \), and so
\[ \mu_A(S^1_a) \lesssim 2^{-2a_s} 2^{2a_s(d-\gamma-s)}. \]

It remains to prove Lemmas 11.2 and 11.3.

11.5. Proof of Lemma 11.2. The following proof relies on Lemma 11.3 which is proved next.

Set
\[ Q = \int |\hat{\mu_A} \chi_{S^1_a}(\xi)|^2 |\xi|^{-\gamma} d\xi. \]

Before going into the details, we outline the proof of this lemma in two steps.

**Step 1** First, we show that
\[ Q \lesssim \mu(S^1_a) \]
whenever \( d-s < \gamma < \alpha_1 - 1 \). Plugging this observations into (11.12) and assuming for now that Lemma 11.3 holds, we conclude that
\[ \mu(S^1_a) \leq 2^{-2a}. \]

**Step 2** Next, we use this information to show that
\[ Q \lesssim \mu(S^1_a) 2^{2a_s(d-\gamma-s)}, \]
whenever \( \gamma > d-s_A \) and \( \gamma < \alpha_1 - 1 \). The details follow.

**Proof of Step 1** By elementary properties of the Fourier transform,
\[ Q \sim \int \int |x-y|^{-d} \chi_{S^1_a}(x) \chi_{S^1_a}(y) d\mu_A(y) d\mu_A(x). \]

Recall that \( A \) is a compact subset of \( \mathbb{R}^d \), and so \( x, y \in A \) implies that \( |x-y| \lesssim 1 \). Therefore
\[ Q \sim \sum_{\eta=0}^{\infty} \int \int_{\{y:2^{-\eta+1}|x-y|<2^{-\eta}\}} |x-y|^{-d} \chi_{S^1_a}(x) \chi_{S^1_a}(y) d\mu_A(y) d\mu_A(x). \]
\[ \lesssim \mu_A(S^1_a) \sum_{\eta=0}^{\infty} 2(\gamma-d)\eta 2^{-\eta s_A}, \]
\[ \lesssim \mu_A(S^1_a), \]
when \( \gamma > d-s_A \). In the second line, we used the Ahlfors-David regularity of the set \( A \), introduced in definition 2.1, to assume that \( \mu_A(\{y:|x-y| \leq 2^{-\eta}\}) \leq 2^{-\eta s_A} \).

In conclusion, when \( \gamma > d-s_A \), we have that
\[ Q \lesssim \mu_A(S^1_a). \]
Plugging this observations into (11.12) and assuming for now that Lemma 11.3 holds, we conclude that
\[ (11.15) \]
\[ \mu_A(S^1_a) \lesssim 2^{-2a}. \]

**Proof of Step 2** Next, we use (11.15) to improve the estimate on \( Q \). In step 1, we demonstrated that
\[ Q \sim \sum_{\eta=0}^{\infty} \int \int_{\{y:2^{-\eta+1}|x-y|<2^{-\eta}\}} |x-y|^{-d} \chi_{S^1_a}(x) \chi_{S^1_a}(y) d\mu_A(y) d\mu_A(x). \]
It follows by the Ahlfors-David regularity of the set $A$ that:
\[ Q \lesssim \mu_A(S^1_a) \sum_{\eta=0}^{\infty} 2^{\eta(d-\gamma)} \min\{2^{-\eta s_A}, \mu_A(S^1_a)\}. \]

By (11.15), it follows that:
\[ Q \lesssim \mu_A(S^1_a) \left( \sum_{\eta>2a/s_A} 2^{\eta(d-\gamma)} 2^{-\eta s_A} + \sum_{\eta \leq 2a/s_A} 2^{\eta(d-\gamma)} 2^{-2a} \right) \sim \mu_A(S^1_a) 2^{\frac{d}{2}(d-\gamma-s_A)}. \]

11.6. Proof of equation (11.3). We now show that if $\gamma < \alpha_1 - 1$, then

(11.16) \[ \int |\hat{T}^\epsilon_{\phi,t,1} f(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} d\xi \lesssim 1. \]

The proof of the lemma relies on Lemma 9.1 combined with the following observations. If $\phi$ satisfies the Phong-Stein rotational curvature condition at $t$, then the operator $T^\epsilon_{\phi,t}$ also satisfies (2.15). That is

(11.17) \[ T^\epsilon_{\phi,t} : L^2(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d) \]

with constants uniform in $t$ and $k = \frac{d-1}{2}$.

To prove (11.17), first notice that
\[ T^\epsilon_{\phi,t} f(x) = \frac{1}{\epsilon} \int_{t-\epsilon}^{t+\epsilon} T_{\phi,r} f(x) dr, \]
and so by Fubini’s Theorem we have
\[ \hat{T}^\epsilon_{\phi,t} f(\xi) = \frac{1}{\epsilon} \int_{t-\epsilon}^{t+\epsilon} \hat{T}_{\phi,r} f(\xi) dr. \]

Now
\[
\int |\hat{T}^\epsilon_{\phi,t} f(\xi)|^2 (1 + |\xi|^2)^k d\xi \\
= \int \hat{T}^\epsilon_{\phi,t} f(\xi) \overline{\hat{T}^\epsilon_{\phi,t} f(\xi)} (1 + |\xi|^2)^k d\xi \\
= \frac{1}{\epsilon^2} \int_{t-\epsilon}^{t+\epsilon} \int_{t-\epsilon}^{t+\epsilon} \hat{T}_{\phi,r} f(\xi) \overline{\hat{T}_{\phi,s} f(\xi)} (1 + |\xi|^2)^k dr ds d\xi \\
= \frac{1}{\epsilon^2} \int_{t-\epsilon}^{t+\epsilon} \int_{t-\epsilon}^{t+\epsilon} |\hat{T}_{\phi,r} f(\xi)|^2 (1 + |\xi|^2)^k d\xi dr ds.
\]
By (2.15):
\[
\lesssim \frac{1}{\epsilon^2} \int_{t-\epsilon}^{t+\epsilon} \int_{t-\epsilon}^{t+\epsilon} |\hat{f}(\xi)|^2 d\xi dr ds \\
\lesssim \int |f(x)|^2 dx.
\]

We now decompose the measure \(\mu_1\) as follows: Take positive Schwartz class functions \(\eta_0(\xi)\) supported in the ball \(\{|\xi| \leq 4\}\) and \(\eta(\xi)\) supported in the annulus

\[
\text{(11.18)} \quad \{1 < |\xi| < 4\} \text{ with } \eta_j(\xi) = \eta(2^{-j}\xi) \text{ for } j \geq 1
\]

with

\[
\text{(11.19)} \quad \eta_0(\xi) + \sum_{t=1}^{\infty} \eta_j(\xi) = 1.
\]

Define the Littlewood-Paley piece, \(\mu_j\), of the measure \(\mu_1\) by the relation

\[
\text{(11.20)} \quad \hat{\mu}_j(\xi) = \hat{\mu}_1(\xi) \eta_j(\xi).
\]

Let \(\eta_m\) be as in 11.18 and write \(\mu_1 = \sum_{j=0}^{\infty} \mu_j\), then the right-hand-side of 11.3 can be written as:

\[
\int |T_{\phi_1, t_1}^\epsilon \mu_1(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} d\xi = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \int T_{\phi_1, t_1}^\epsilon \mu_j(\xi) \overline{T_{\phi_1, t_1}^\epsilon \mu_l(\xi)} (1 + |\xi|^2)^{\gamma/2} d\xi \\
= \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \int T_{\phi_1, t_1}^\epsilon \mu_j(\xi) \overline{T_{\phi_1, t_1}^\epsilon \mu_l(\xi)} (1 + |\xi|^2)^{\gamma/2} \eta_m(\xi) d\xi.
\]

By Cauchy-Schwartz inequality, this expression is bounded by

\[
\text{(11.21)} \quad \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left( \int |T_{\phi_1, t_1}^\epsilon \mu_j(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} \eta_m(\xi) d\xi \right)^{1/2} \left( \int |T_{\phi_1, t_1}^\epsilon \mu_l(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} \eta_m(\xi) d\xi \right)^{1/2}.
\]

Consider

\[
\text{(11.22)} \quad \int |T_{\phi_1, t_1}^\epsilon \mu_j(\xi)|^2 (1 + |\xi|^2)^{\gamma/2} \eta_m(\xi) d\xi.
\]

Let \(K\) be as in Lemma 9.1 If \(|j - m| > K\), then the proof of Lemma 9.1 can be used to show that for any \(M\) sufficiently large, the expression in (11.22) is bounded by \(2^{-K \max\{j,m\}}\) times a constant dependent on \(M\).

When \(|j - m| < K\), we use the mapping properties of the operator \(T_{\phi_1, t_1}^\epsilon\) along with the hypothesis of the theorem that \(I_{\alpha_1}(\mu_1)\) is finite to demonstrate that the expression in (11.22) is bounded by
which is summable in $j$ when $\gamma < \alpha_1 - 1$. In more detail, if $|j - m| > K$ then

\[
\left( \int |T_{\phi_1, t_1} \mu_j(x) |^2 (1 + |\xi|^2)^{\gamma/2} \eta_m(\xi) d\xi \right)^{1/2} \sim 2^{m\gamma} 2^{-m(d-1)} \int |T_{\phi_1, t_1} \mu_j(x) |^2 (1 + |\xi|^2)^{(d-1)/2} \eta_m(\xi) d\xi \\
\lesssim 2^{m\gamma} 2^{-m(d-1)} \int |\mu_j(\xi)|^2 d\xi \\
\sim 2^{m\gamma} 2^{-m(d-1)} 2^m(d-\alpha_1) \int |\mu_j(\xi)|^2 |\xi|^\alpha_1^d d\xi \\
\sim 2^{m\gamma} 2^{-m(d-1)} 2^m(d-\alpha_1) I_{\alpha_1}(\mu_1) \\
\sim 2^{-j(\alpha_1 - 1 - \gamma)},
\]

which is summable in $j$ when $\gamma < \alpha_1 - 1$.

The second part of the summand in (11.21) is handled in the same way. Since the given bounds are summable in $j$, $m$, and $l$, the expression in (11.21) is bounded.

### 12. Proof of Theorem 2.20

We shall need the following result that follows from the Fourier Integral Operator regularity results due to Seeger, Sogge and Stein ([28]). See also [29].

**Theorem 12.1.** With $\phi$ as in Theorem 2.20, define

\[
\mathcal{M} f(x) = \sup_{t > 0} \left| \int_{\{y : \phi(x, y) = t\}} f(y) d\sigma_{x,t}(y) \right|
\]

where $d\sigma_{x,t}$ denotes the Lebesgue measure on $\{y : \phi(x, y) = t\}$ multiplied by a smooth cut-off function. Then if $d \geq 3$,

\[
||\mathcal{M} f||_{L^p(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)} \quad \text{for} \quad p > \frac{d}{d-1}.
\]

Observe that Theorem 12.1 still holds if we take a supremum over $t > t_0 > 0$ in the definition of the maximal function. We also linearize the operator by replacing the supremum over $t > t_0$ by $t(x)$, an arbitrary measurable function of $x$ taking on values in $(t_0, \infty)$. Denote the resulting linearized operator by $T$. Applying Theorem 12.1 in this form to $\chi_A^\delta$, we see that

\[
|A^\delta|^{\frac{d}{p}} \geq C_p^{-1} \left( \int_{\text{support}(T \chi_A^\delta)} |N(x, \delta)|^p \right)^{\frac{1}{p}}
\]

where $N(x, \delta)$ is minimal number of $\delta$ balls needed to cover $E \cap \{y : \phi(x, y) = t(x)\}$. By definition of upper Minkowski dimension, this is quantity is

\[
\gtrsim \delta^{d-1-\gamma}.
\]

It follows that
\[ |A^\delta|^{\frac{1}{p}} \gtrapprox \delta^{d-1}\delta^{-\gamma}. \]

On the other hand,

\[ |A^\delta| \lesssim \delta^{d-s_A}, \]

where \( s_A \), by Ahlfors-David regularity, is the Hausdorff dimension of \( A \). It follows that

\[ \frac{d-s_A}{p} \leq d - 1 - \gamma, \]

which implies that

\[ s_A \geq p(\gamma + 1) - d(p - 1) \]

for \( p > \frac{d}{d-1} \). The conclusion follows.
Intersections of Sets, Diophantine Equations and Fourier Analysis

References

[1] J. Barcelo, J. Bennett, A. Carbery, A. Ruiz, A and M. Vilela, Some special solutions of the Schrodinger equation, Indiana Univ. Math. J. 56 (2007), no. 4, 1581-1593.

[2] P. Brass, W. Moser and J. Pach, Research Problems in Discrete Geometry, Springer (2000).

[3] J. Bourgain, Averages in the plane over convex curves and maximal operators, J. Analyse Math. 47 (1986), 69-85.

[4] S. Eswarathasan, A. Iosevich and K. Taylor, Fourier integral operators, fractal sets and the regular value theorem, Advances in Mathematics, 228, (2011), 1, 7, 10, 22.

[5] B. Erdős, A bilinear Fourier extension theorem and applications to the distance set problem IMRN (2006).

[6] K. J. Falconer, On the Hausdorff dimensions of distance sets, Mathematika 32 (1986) 206-212.

[7] K. J. Falconer, The geometry of fractal sets, Cambridge Tracts in Mathematics, 85 Cambridge University Press, Cambridge, (1986). 10, 12, 22.

[8] K. J. Falconer, Sets with large intersections, JLMS 49 (1994), 267-280.

[9] G. Folland, Real analysis. Modern techniques and their applications Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1984).

[10] I. Gel’fand and G. Shilov, Generalized functions, Vol. 1, Academic Press, (1966).

[11] L. Grafakos, A. Greenleaf, A. Iosevich and E. Palsson, Multilinear generalized Radon transforms and point configurations, (submitted for publication), (http://arxiv.org/pdf/1204.4429.pdf), (2011).

[12] A. Greenleaf and A. Iosevich, On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry Anal. PDE 5 (2012), no. 2, 397-409.

[13] E. Grosswald, Representations of integers as sums of squares, Springer Verlag, New York, Berlin (1985).

[14] L. Hormander, Fourier integral operators. I Acta Math. 127 (1971), no. 1-2, 79-183.

[15] S. Hofmann and A. Iosevich, Circular averages and Falconer/Erds distance conjecture in the plane for random metrics, Proc. Amer. Math. Soc. 133 (2005), no. 1, 133173.

[16] A. Iosevich, Fourier analysis and geometric combinatorics Topics in mathematical analysis, 321175, Ser. Anal. Appl. Comput., 3, World Sci. Publ., Hackensack, NJ, (2008).

[17] A. Iosevich, H. Jorati and I. Laba, Geometric incidence theorems via Fourier analysis, Trans. Amer. Math. Soc. 361 (2009), no. 12, 6595-1711.

[18] A. Iosevich and I. Laba, K-distance sets, Falconer conjecture, and discrete analogs, Integers 5 (2005).

[19] A. Iosevich and E. Sawyer, Maximal averages over surfaces, Adv. Math. 132 (1997), no. 1, 4617.

[20] A. Iosevich and S. Senger, Sharpness of Falconer’s estimate in continuous and arithmetic settings, geometric incidence theorems and distribution of lattice points in convex domains, (submitted for publication) (2010).

[21] A. Iosevich and K. Taylor, Lattice points close to families of surfaces, non-isotropic dilations and regularity of generalized Radon transforms, New York J. Math. 17 (2011), 811178.

[22] J.P. Kahane, Sur la dimension des intersections. (French) [On the dimension of intersections] Aspects of mathematics and its applications. 419170, North-Holland Math. Library, 34, North-Holland, Amsterdam, (1986).

[23] P. Mattila, Hausdorff dimension and capacities of intersections of sets in n-space, Acta Math. 152 (1984), no. 1-2, 77-105.

[24] P. Mattila, On the Hausdorff dimension and capacities of intersections, Mathematika 32 (1985) 213-217.

[25] P. Mattila Spherical averages of Fourier transforms of measures with finite energy: dimensions of intersections and distance sets Mathematika, 34 (1987), 207-228.

[26] P. Mattila, Geometry of sets and measures in Euclidean spaces, Cambridge University Press, volume 44, (1995).

[27] D. Phong and E. Stein, Radon transforms and torsion, Inter. Math. Res. Not., 4, (1991). 7, 8.

[28] A. Seeger, C. D. Sogge and E. M. Stein, Regularity properties of Fourier integral operators, Ann. of Math. (2) 134 (1991), no. 2, 231171.

[29] C. Sogge, Fourier integrals in classical analysis, Cambridge University Press, (1993).

[30] E. M. Stein, Harmonic Analysis, Princeton University Press, (1993).

[31] P. Valtor, Strictly convex norms allowing many unit distances and related touching questions, manuscript 2005.

[32] T. Wolff, Recent work connected with the Kakeya problem. Prospects in mathematics (Princeton, NJ, 1996), 129-162, Amer. Math. Soc., Providence, RI, (1999).

[33] T. Wolff, Decay of circular means of Fourier transforms of measures, International Mathematics Research Notices 10 (1999) 547-567.
[34] T. Wolff, *Lectures on harmonic analysis* Edited by Laba and Carol Shubin. University Lecture Series, 29. American Mathematical Society, Providence, RI, (2003).

E-mail address: suresh@math.mcgill.ca
E-mail address: iosevich@math.rochester.edu
E-mail address: taylor@math.rochester.edu

Department of Mathematics, University of Rochester, Rochester, NY