Chiral Rings Do Not Suffice: $\mathcal{N}=(2,2)$ Theories with Nonzero Fundamental Group

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ABSTRACT

The Kähler moduli space of a particular non-simply-connected Calabi-Yau manifold is mapped out using mirror symmetry. It is found that, for the model considered, the chiral ring may be identical for different associated conformal field theories. This ambiguity is explained in terms of both A-model and B-model language. It also provides an apparent counterexample to the global Torelli problem for Calabi-Yau threefolds.
1 Introduction

One of the very first Calabi-Yau spaces to be considered for a superstring compactification was defined as follows \[1\]. Take the complex projective space \( \mathbb{P}^4 \) with homogeneous coordinates \([z_0, z_1, \ldots, z_4]\) and construct a quintic hypersurface within it from the condition
\[
p = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0. \tag{1}
\]
Let us denote this space by \( Q \). Now define a Calabi-Yau space as \( X = Q/G \) where \( G \) is the freely acting group of identifications isomorphic to \( \mathbb{Z}_5 \times \mathbb{Z}_5 \) generated by
\[
g_1 : [z_0, z_1, z_2, z_3, z_4] \mapsto [z_0, \zeta z_1, \zeta^2 z_2, \zeta^3 z_3, \zeta^4 z_4], \quad \zeta = e^{2\pi i/5},
\]
\[
g_2 : [z_0, z_1, z_2, z_3, z_4] \mapsto [z_1, z_2, z_3, z_4, z_0]. \tag{2}
\]
The smooth Calabi-Yau manifold \( X \) has Euler characteristic \(-8\) and can thus be used to build a four generation model. Although these days, four generations is not considered an attractive feature of a model, it will still prove interesting to study the space \( X \) and, in particular, the moduli space of \( N=(2,2) \) superconformal field theories containing a point corresponding to this manifold.

\( N=(2,2) \) superconformal field theories are used to represent a Calabi-Yau compactification with the “spin connection embedded in the gauge group” \[1\]. The moduli space of such conformal field theories, i.e., the moduli space of string vacua of this type is actually quite simple to analyze in many cases. One of the reasons for this is the existence of the “chiral ring” (see, for example, \[2\]). To analyze a generic conformal field theory it is usual to concentrate on the set of primary fields, which unfortunately are infinite in number.\(^1\) However, in the case of \( N=(2,2) \) superconformal field theories we may look at two distinguished subsets of the primary fields, the chiral primary fields and the antichiral primary fields. (The difference between the definitions is simply a change in the sign of the \( U(1) \) charge). The advantage of this is that the number of (anti)chiral primary fields is finite, and remains constant over the moduli space. This makes analysis of the set of such fields much easier. Furthermore, the “naïve product”
\[
(\phi \chi)(z) = \lim_{z' \to z} \phi(z') \chi(z) \tag{3}
\]
of two chiral primary fields is again a chiral primary field, giving the set of such fields the structure of a ring. The structure constants of this ring can be found by calculating two point and three point functions.

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\(^1\)If the conformal field theory is rational, the number of primary fields will be finite and the analysis may not be so difficult. However, it is believed that such rational conformal field theories do not deform and thus can, at best, form only a countable dense subset of the space of conformal field theories (like the set of rational numbers in the space of reals).
An $N=(2,2)$ superconformal field theory may be twisted to form a topological field theory. This may be done in two inequivalent ways, yielding the “A-model” or “B-model”. The observables in each of these models correspond to the (anti)chiral primary fields of the original $N=(2,2)$ theory, the difference amounting to a choice of sign of one of the $U(1)$ charges in the $N=(2,2)$ superconformal algebra. Deformations of the $N=(2,2)$ superconformal field theory can generated by (anti)chiral primary fields and thus map into deformations of the A-model and B-model, where they are detected by the ring structures. It is tempting to speculate that analyzing the two point and three point functions of the observables in the two topological field theories gives sufficient information to classify completely the $N=(2,2)$ theory. That is, the two chiral rings of the conformal field theory might contain sufficient information to obtain all other correlation functions from them.

A simple counterexample to this proposition is the complex 3-torus. In this case the A-model (having no instantons at tree-level) is too trivial and contains no local information. One possible explanation for this is that in the case of the torus, the local supersymmetry is actually $N=(4,4)$. Whenever this happens the moduli space takes on a significantly different form which no longer splits naturally into A-model and B-model part (see, for example, [3]). One might thus modify the proposal to apply to theories with only $N=(2,2)$ symmetry. We shall see from our example that this too fails—for global rather than local reasons.

In section 2 we will discuss the A-model on $X$. In section 3 the moduli space will be formulated by considering the B-model on the mirror of $X$. To help understand the form of the moduli space we will consider rational and elliptic curves on $X$ in section 4 and finally in section 5 we will present concluding remarks.

2 A-model Analysis

In terms of the geometric data of the target space, the B-model captures the information concerned with the complex structure. The A-model moduli however are concerned with variation of the complexified Kähler form. By expressing the constraint in the form (1) we have effectively fixed the complex structure and thus frozen the B-model data. Thus we will concentrate only on the A-model.

In the conventional approach to the A-model one describes a non-linear $\sigma$-model whose target space is equipped with a Kähler metric given by a Kähler form $J$ and a real 2-form, $B$. These may be combined to form a complex 2-form $B + iJ$ upon whose cohomology class the A-model depends. This description does not capture all the possibilities however.

For a map $\phi : \Sigma \rightarrow X$ from the world-sheet into the target space, the A-model correlation functions vary as a function of

$$\xi = \exp(2\pi i \int_{\Sigma} \phi^*(B + iJ)),$$

(4)
where $\phi^*$ is the pull-back. Thus the degree of freedom represented by $B + iJ$ can be thought of as an element of $\text{Hom}(H_2(X), \mathbb{C}^*)$. That is, it associates some non-zero complex number, $\xi$, to each homology class of the image of $\Sigma$ in $X$. The group $\text{Hom}(H_2(X), \mathbb{C}^*)$ may contain more freedom than that described by the complex 2-form $B + iJ$. It is possible that the singular homology group $H_2(X)$ contains torsion, i.e., there is an element $\tau \in H_2(X)$ such that $N\tau \sim 0$ for some integer $N$. Using only de Rham cohomology in the form of $B + iJ$ will then miss the corresponding torsion elements of $\text{Hom}(H_2(X), \mathbb{C}^*)$. Since it would seem natural to allow for torsion elements of $H_2(X)$ to be associated with a non-trivial $\xi$ it would appear that the A-model moduli space is better described as $\text{Hom}(H_2(X), \mathbb{C}^*)$ rather than the potentially smaller space of $B + iJ$'s. This is closely related to issues studied in [7]. This form of the moduli space can also be justified by looking at fundamental properties of maps of the world-sheet into the target space [4, 5, 6].

It is important to note that the A-model picture presented here depends on which “phase” of moduli space we are in in the sense of [4, 6, 11]. We will assume that we are in some neighbourhood of the large radius limit of $X$ and thus in the Calabi-Yau phase. When one leaves this phase, the moduli space will no longer appear to be in the form $\text{Hom}(H_2(X), \mathbb{C}^*)$.

For the quintic threefold, $Q$, $\dim H^2_{\text{DR}}(Q) = 1$ and there is no torsion in $H_2(Q)$. Thus, in the neighbourhood of the large radius limit (where the A-model is well-defined) the moduli space is locally isomorphic to $\text{Hom}(H_2(Q), \mathbb{C}^*) \cong \mathbb{C}^*$. Since $Q$ is a simply-connected non-ramified cover of $X$, we have that $\pi_2(X) \cong \pi_2(Q) \cong H_2(Q) \cong \mathbb{Z}$ [12]. Since $X = Q/G$ we also have that $\pi_1(X) \cong G \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. Given $\pi_1(X)$ and $\pi_2(X)$ we may calculate $H_2(X)$ and consequently $\text{Hom}(H_2(X), \mathbb{C}^*) \cong H^2(X, \mathbb{C}^*)$ by the method of Eilenberg and MacLane [13]. According to [13] there is an exact sequence

$$0 \to \pi_2(X) \to H_2(X) \to H_2(G) \to 0$$

which relates the homotopy and homology groups of $X$ to the group homology $H_2(G)$. In the present case, the group homology can be calculated as $H_2(\mathbb{Z}_5 \times \mathbb{Z}_5) \cong \mathbb{Z}$, and the exact sequence (5) becomes

$$0 \to \mathbb{Z} \to H_2(X) \to \mathbb{Z}_5 \to 0.$$  \hspace{1cm} (6)

There are two possibilities for $H_2(X)$ compatible with (6), depending on whether the exact sequence splits: either $H_2(X) \cong \mathbb{Z} \times \mathbb{Z}_5$, or $H_2(X) \cong \mathbb{Z}$. In either case there will be a homology class $e$ which is not represented by a sphere; in the former case, $e$ may be chosen so that $5e \sim 0$ and in the latter so that $5e$ generates $\pi_2(X) \cong H_2(Q)$. (These two possibilities are mutually exclusive.)

In fact, it is the second possibility $H_2(X) \cong \mathbb{Z}$ which occurs for our example. We will show this in section 4 by exhibiting an elliptic curve $E$ on $X$ whose inverse image $\pi^{-1}(E)$ on $Q$ is an irreducible elliptic curve of degree 5. The homology class $e$ of $E$ cannot lie in
\( \pi_2(X) \), since for every rational curve \( \Gamma \) on \( X \), the inverse image \( \pi^{-1}(\Gamma) \) consists of 25 disjoint rational curves, all of the same degree \( d \), so that the degree of \( \pi^{-1}(\Gamma) \) is a multiple of 25.

The group \( H^2(X, \mathbb{C}^*) \cong \text{Hom}(H_2(X), \mathbb{C}^*) \) which describes the degree of freedom represented by \( B + iJ \) fits in an exact sequence of its own

\[
0 \to \mathbb{Z}_5 \to H^2(X, \mathbb{C}^*) \xrightarrow{\alpha} \mathbb{C}^* \to 0, \tag{7}
\]

whose structure is easily deduced from that of (6). In particular, \( H^2(X, \mathbb{C}^*) \cong \mathbb{C}^* \) and the map \( \alpha \) is a five-fold cover.

This occurrence of \( H^2(G, \mathbb{C}^*) \) when modding out a space by a group action \( G \) was first discussed in the context of string theory in \([9]\) where it was given the name “discrete torsion”. Note that in our example this name is somewhat misleading since \( H^2(X, \mathbb{C}^*) \) is torsion free.

### 3 The B-model of the Mirror

Given \( X \) we now hope to find another space \( Y \), which is the “mirror” of \( X \), such that the A-model on \( X \) is equivalent to the B-model on \( Y \). To be more precise, \( Y \) will actually be in the Landau-Ginzburg phase rather than Calabi-Yau phase but the phase picture is not important for the B-model and we may imagine \( Y \) to be a Calabi-Yau manifold for all practical purposes. In the case of the quintic and many of its quotients the mirror is given by the method of \([14]\). \( X \) does not quite fall into this class but we may use an extension of this method \([13]\) to find the mirror. Define \( G_1 \), isomorphic to \( \mathbb{Z}_5 \), to be the group generated by the element \( g_1 \) defined in eq. (2). By the method of \([14]\), the manifold \( Q/G_1 \) is known to be mirror to \( Q/\widetilde{G}_1 \), where \( \widetilde{G}_1 \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \) is generated by \( g_1 \) and another element \( g_3 \) defined by

\[
g_3 : [z_0, z_1, z_2, z_3, z_4] \mapsto [z_0, \zeta z_1, \zeta^3 z_2, \zeta z_3, z_4], \quad \zeta = e^{2\pi i/5}. \tag{8}
\]

\( g_2 \) now acts on the mirror pair of theories corresponding to the spaces \( Q/G_1 \) and \( Q/\widetilde{G}_1 \) in precisely the same manner (once the mirror map is taken into account). Thus we may divide both spaces (i.e., orbifold both conformal field theories) by the group generated by this action to yield another mirror pair. This pair consists of \( X \) and \( Y \cong Q/G \) where \( G \cong \mathbb{Z}_5 \ltimes (\mathbb{Z}_5)^2 \) and is generated by \( g_1, g_2 \) and \( g_3 \).

The moduli space of the B-model consists of varying the complex structure of \( Y \) which may be done by varying the defining equation (1) of \( Q \). There is only one deformation compatible with the group \( \widetilde{G} \) and we follow \([10]\) by using the following parametrization

\[
p_\psi = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0. \tag{9}
\]

\(^2\)In that paper only the \( B \)-field is discussed leading to the equivalent group \( H^2(G, U(1)) \).
Further analysis of the moduli space and the mirror map between this B-model and the A-model of section 2 is now very close to the analysis of [16] where the A-model considered was that associated to $Q$. The only difference between the B-model considered here and that of [16] is that a different 125 element group of symmetries (call it $\hat{G}$) is used to divide $Q$, although both lead to the same general form of defining equation (9). Little of the analysis of [16] depends on the exact form of the group dividing $Q$ and so can be copied over to the case considered here.

In [16] it was observed that the family of Calabi-Yau manifolds described by (9) admits a symmetry $R$ defined by $z_0 \to \zeta z_0$, $z_i \to z_i$ for $i > 0$, and $\psi \to \zeta^{-1} \psi$. This symmetry establishes an isomorphism between the Calabi-Yau manifolds at $\psi$ and at $\zeta \psi$, and shows that the correct parameter for this family is in fact $\psi^{-5}$ rather than $\psi$. (This parameter can be seen directly by changing coordinates via $z_0 = \psi^{-1} \tilde{z}$ to give a new defining equation

$$\psi^{-5} \tilde{z}^5 + \tilde{z}_1^5 + \tilde{z}_2^5 + \tilde{z}_3^5 + \tilde{z}_4^5 - 5 \tilde{z} z_1 z_2 z_3 z_4 = 0$$

in which the parameter is visibly $\psi^{-5}$.) When forming a quotient of this family by $\hat{G}$ to obtain the mirror family of $Q$, $R$ remains a symmetry of the quotient—in fact, $\hat{G}$ acts on (10) equally well as on (9). However, when forming the quotient by our group $\tilde{G}$ which includes a permutation, $R$ is no longer a symmetry—it does not normalize the group $\tilde{G}$, nor does it preserve the alternate form (10) of the defining equation.

This immediately tells us that the chiral ring of $X$ has “lost” some information concerning the conformal field theory. The chiral ring as calculated in [16] is a function of $\psi^5$. However since $R$ is not a symmetry of $Y$ we expect the points given by $\psi, \zeta \psi, \zeta^2 \psi, \ldots$ to correspond to different conformal field theories.$^3$

In order to construct the mirror map between the A- and B-models we first require a set of “flat” coordinates on the B-model moduli space (see [17] for a full discussion of this issue). This is obtained by considering the variation of Hodge structure on $Y$. That is, we consider the periods $\omega_i = \int_{\gamma_i} \Omega$ where $\Omega$ is a (3,0)-form and $\gamma_i$ are elements of $H_3(Y)$.

Let us use $q$ to denote the image of $e$, the fundamental generator of $H_2(X)$, under the action of an element of $\text{Hom}(H_2(X), \mathbb{C}^*)$. The mirror map then relates the A- and B-model moduli spaces by relating $q$ to $\psi$. The local geometry of the $N=(2,2)$ moduli space tells us that [17]

$$q = \exp \left( \frac{\varpi_1(\psi)}{\varpi_0(\psi)} \right),$$

for two suitably chosen periods $\varpi_0$ and $\varpi_1$. To find exactly which periods to use one must look at global considerations of $H_3(Y)$ as discussed in [18]. Rather than use this method

$^3$It might be objected that there could be some other symmetry, not manifest in the present description, which produces an isomorphism between the theory at $\psi$ and the theory at $\zeta \psi$. The calculations in the next section will demonstrate that this is not the case.
directly we may use a simple argument as follows. By mapping the correlation functions of the A-models of $X$ and $Q$ to each other we obtain a five-fold cover of the moduli space of $Q$ by the moduli space of $X$. This cover is branched at $q = 0$, which represents the large radius limit. A path once around this point in the $X$ moduli space corresponds to changing $B$ by the generator of $H^2(X, \mathbb{Z})$. But since $e_Q$, the generator of $H_2(Q)$, descends to $5e$ on $X$, a path which winds once around the large radius limit of the $Q$ moduli space (changing that space’s $B$ by the generator of $H^2(Q, \mathbb{Z})$) will wind 5 times around $q = 0$ in the $X$ moduli space. Thus, denoting by $q_Q$ the image of $e_Q$ for the $Q$ moduli space, we obtain

$$q^5 = q_Q.$$  

(12)

From [10] it then follows that

$$\varphi_0 = \sum_{N=0}^{\infty} \frac{(5N)!}{(N!)^5} (5\psi)^{-5N}$$

$$\varphi_1 = -\varphi_0 \log(5\psi) + \sum_{N=1}^{\infty} \frac{(5N)!}{(N!)^5} [\Psi(5N + 1) - \Psi(N + 1)] (5\psi)^{-5N}.$$  

(13)

4 Some curve counting

We may now proceed and count rational curves on $X$. The suitably normalized three-point function for the A-model is

$$\langle O^3 \rangle = 25 + 14375q^5 + 24384375q^{10} + \ldots ,$$  

(14)

It follows [10, 19] that, for $n_i$ the number of rational curves of degree $i$ on a generic manifold diffeomorphic to $X$ and $n_i(Q)$ the same quantity for $Q$, we have

$$n_i = 0, \quad \text{when } 5 \nmid i,$$

$$n_{5i} = n_i(Q)/25.$$  

(15)

This is exactly what we would expect from geometry. The group $G$ is of order 25 and acts freely on $Q$. Since rational curves do not admit a freely acting symmetry, $G$ must identify the rational curves on $Q$ in groups of 25 (as observed earlier). A curve of degree $i$ on $Q$ maps into a curve of degree $5i$ on $X$ because of the relationship between $H_2$ of the two spaces.\footnote{The degree in both cases refers to the number of intersection points with a generator of $H^2$.}

The form of the expression (14) shows that the chiral ring of $X$ does not contain enough information to classify the conformal field theory. This series may be considered as an
instanton expansion. Since this is a tree-level computation, the instantons are spheres. Spheres correspond to elements of $\pi_2(X)$ and thus can only represent homology classes which are a multiple of $5e$. Thus (14) is a power series in $q^5$ and cannot fully distinguish between conformal field theories.

In order to measure any quantity which depends properly on $q$ rather than $q^5$ we must therefore go beyond tree level. Non trivial information is obtained beyond genus 0 when the topological A-model is coupled to gravity [20, 17]. In this case one may consider a partition function $F_1$ defined for one-loop world-sheets. This partition function contains information concerning elliptic curves on the target space and, with luck, may be used to count them as follows [20]. The holomorphic anomaly dictates that $F_1$ is of the form

$$F_1 = \log \left[ \left( \frac{\psi}{\omega_0} \right)^w f(\psi) \frac{d\psi}{dq} \right] + \text{const},$$

where $w = 3 + h^{1,1} - \chi/12$ (which is $\frac{14}{3}$ for $X$), and $f(\psi)$ is an unknown holomorphic function of $\psi$. The relationship between $F_1$ and the number, $n_i$, of rational curves of degree $i$ and elliptic curves, $d_i$, of degree $i$ on $X$ is given by

$$F_1 = -\frac{1}{12} (c_2, e) \log q - \sum_i \left\{ 2d_i \log \eta(q^i) + \frac{1}{6} n_i \log(1 - q^i) \right\} + \text{const},$$

where

$$\eta(q) = \prod_{n=1}^{\infty} (1 - q^n),$$

and $(c_2, e)$ is obtained by wedging the second Chern class of $X$ with the 2-form dual to $e$ and integrating over $X$.

Knowing $c_2(X)$ and the fact that $F_1$ should be finite for a good conformal field theory is often sufficient to determine $f(\psi)$. This was the case for all the examples studied in [20, 21, 22] but fails for our example. However, we may find the solution by working a little harder. The function $f(\psi)$ is generally of the form

$$f(\psi) = \prod_s (\psi_s - \psi)^{a_s},$$

where $s$ runs over the points (where $\psi = \psi_s$) in the moduli space where the conformal field theory is “bad”. The constants $a_s$ are to be determined. In our example, $Y$ is singular (and thus the associated conformal field theory is bad) whenever $\psi^5 = 1$. The fact that the series part of the the expansion in (17) has rational coefficients means that $F_1$ must be of the form

$$f(\psi) = (1 - \psi)^{a_0} (1 + \psi + \psi^2 + \psi^3 + \psi^4)^{a_1}.$$
The fact that \((c_2,e) = 10\) for \(X\) the tells us that \(a_0 + 4a_1 = \frac{-29}{6}\).

We can also directly count the number of degree one elliptic curves on \(X\) as follows. In contrast to the usual cases (see for example [23]) this may be done very explicitly. The inverse image of such a curve has total degree 5 on \(Q\). In principle this might split as 5 curves of degree 1, permuted by \(G\) (and each invariant under some \(\mathbb{Z}_5\) subgroup), but since there are no elliptic curves of degree 1 on \(Q\) this is impossible. Thus, the inverse image is an irreducible curve of degree 5, preserved by \(G\). Our task is to count those curves.

Any elliptic curve of degree 5 in \(\mathbb{P}^4\) is defined by an ideal generated by 5 quadrics. When the curve is preserved by \(G\), this space of quadrics must form a projective representation of \(G\), and in fact can be generated by 5 quadrics of the form

\[
\begin{align*}
\alpha z_0^2 + \beta z_1 z_4 + \gamma z_2 z_3 \\
\alpha z_1^2 + \beta z_2 z_0 + \gamma z_3 z_4 \\
\vdots \\
\alpha z_4^2 + \beta z_0 z_3 + \gamma z_1 z_2
\end{align*}
\]

(21)

for some constants \(\alpha\), \(\beta\), and \(\gamma\) (cf. [24]). Generic values of those constants lead to 5 quadrics which do not intersect; however, for any constants satisfying

\[
\alpha^2 + \beta \gamma = 0,
\]

(22)

the intersection is a curve of degree 5.\(^5\) A result from classical projective geometry (cf. [23]) says that the genus of this curve is at most 1. On the other hand, if the curve does not pass through the fixed points of \(G\) on \(\mathbb{P}^4\), i.e. if

\[
\alpha \neq 0, \text{ and } \alpha + \zeta \beta + \zeta^{-1} \gamma \neq 0 \text{ for all } \zeta^5 = 1,
\]

(23)

then \(G\) acts without fixed points so the genus cannot be 0 and must be 1.

When (22) is satisfied and \(\alpha \neq 0\), we can take \(\alpha = 1\), \(\beta = -1/a\), \(\gamma = a\) (as in [24]). The corresponding quadrics (21) form a Gröbner basis for the ideal of the curve, with leading monomials \(z_2 z_3\), \(z_3 z_4\), \(z_2^2\), \(z_3^2\), \(z_4^2\). Since this is so, it is easy to find the condition that the quintic \(Q\) defined by (1) contain the curve—it is simply

\[
a^{10} + 6a^5 - 1 = 0.
\]

(24)

Thus there are precisely 10 such curves. (Note that all solutions also satisfy (23).) These descend to 10 elliptic curves on \(X\), and we conclude that \(d_1 = \frac{5}{2}(a_1 - a_0) = 10\). This tells us that \(a_0 = \frac{-25}{6}\) and \(a_1 = \frac{-1}{6}\). It also shows, as asserted earlier, that (6) does not split.

\(^5\)This is most easily seen by intersecting with \(z_0 = 0\) and explicitly solving for the points of intersection.
It is interesting to note that it appears that the exponents $a_s$ might be determined purely by the type of singularities of $Y$ when $\psi = \psi_s$. As noted above, $Y$ is singular when $\psi^5 = 1$. If $\psi \neq 1$ then the form of the singularity is that of a single isolated “simple” singularity, i.e., it is locally of the form of the hypersurface

$$x_1x_2 = x_3x_4$$

(25)

in $\mathbb{C}^4$ near the origin. For $\psi = 1$, there are 5 singularities locally of the form of an orbifold of a simple singularity by $\mathbb{Z}_5$. In all the cases considered in [20, 21, 22] the value of $a_s$ for a simple singularity is $-\frac{1}{6}$ and indeed we have found the same value in this case.

We now have

$$F_1 = -\frac{5}{6} \log q + \text{const} + 20q + 50q^2 + \frac{500}{3}q^3 + \ldots,$$

(26)

and we determine

$$d_1 = 10$$
$$d_2 = 10$$
$$d_3 = 70$$
$$d_4 = 280$$
$$d_5 = 888$$

(27)

We have extended this calculation through $d_{125}$ and find positive integers for every degree.

5 Conclusions

We have shown that the space of correlation functions for the A-model on $X$ does not faithfully represent the moduli space of conformal field theories on $X$. That is, the chiral ring is not sufficient to determine the conformal field theory. In this case for generic theories, there are 5 conformal field theories for each chiral ring. The exception the this occurs at $\psi = 0$ or $\infty$ where the chiral ring does uniquely determine the theory. This ambiguity is clearly shown in (14), valid for small $q$, where the correlation functions are a function of $q^5$. This occurs because the correlation function is at tree level and spheres do not span $H_2(X)$. One needs to go to a loop effect such as (26) to observe a faithful $q$ dependence. In our example, $H_2(X)$ is generated by tori and so a one-loop partition function suffices.
Our results can also be interpreted as providing a counterexample\(^6\) to the global Torelli problem for Calabi-Yau manifolds (contrary to assertions in \([26]\)). The Torelli problem asks whether the variation of Hodge structure determines the complex structure on the manifold. The variation of Hodge structure is completely determined by the chiral ring, and so depends only on \(\psi^{-5}\), but we have seen that the conformal field theory on the manifold (and so presumably the complex structure itself) actually depends on \(\psi\).

If a manifold is simply connected, we have \(\pi_2 = H_2\) and so \(H_2\) is generated by spheres. This suggests that the failure of the chiral ring to determine the conformal field theory is caused by non-trivial \(\pi_1\). Indeed it is precisely the (torsion) group \(H^2(\pi_1(X), \mathbb{C}^*)\) which led to the \(\mathbb{Z}_5\) ambiguity in the identification of the conformal field theory from the chiral ring. One is thus led to hypothesize that this is precisely the data required to supplement the chiral ring (cf. question 6 in \([1]\)). That is, given a theory corresponding to target space \(X\) with a mirror \(Y\), in the Calabi-Yau phase of both \(X\) and \(Y\) (i.e., near the large radius limits) we might have\(^7\)

\[
\text{Moduli Space of } N=(2,2) \text{ models} \cong \text{Moduli Space of A-ring} \times \text{Moduli Space of B-ring} \times H^2(\pi_1(X), \mathbb{C}^*) \times H^2(\pi_1(Y), \mathbb{C}^*), \quad (28)
\]

where by A-ring we mean the chiral ring of the N=(2,2) theory as determined by the A-model and similarly for the B-ring.

One of the motivations for studying this model was to hope to gain some understanding of the form of the moduli space when there is torsion in \(H_2(X)\) although this did not actually happen for the case we considered. This might have provided some insight into the situation when there is “discrete torsion” in orbifolds \([3]\). Although mirror symmetry has managed to give a complete picture of the blowing-up of an orbifold without discrete torsion, at least in the case where points are fixed by abelian groups (see, for example, \([27]\)), little is known about this same process when some discrete torsion is present. Therefore, although considerable progress has been made in recent years on the form of the moduli space of \(N=(2,2)\) theories there is still much to be understood.

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\(^6\)Further analysis of the B-model is likely needed to make this into a mathematically rigorous counterexample.

\(^7\)Modulo the \(\mathbb{Z}_2\) symmetry coming from complex conjugation.
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