Statistical Inference and Probabilistic Modeling for Constraint-Based NLP

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Abstract

In this paper we present a probabilistic model for constraint-based grammars and a method for estimating the parameters of such models from incomplete, i.e., unparsed data. Whereas methods exist to estimate the parameters of probabilistic context-free grammars from incomplete data ([2]), so far for probabilistic grammars involving context-dependencies only parameter estimation techniques from complete, i.e., fully parsed data have been presented ([8]). However, complete-data estimation requires labor-intensive, error-prone, and grammar-specific hand-annotating of large language corpora. We present a log-linear probability model for constraint logic programming, and a general algorithm to estimate the parameters of such models from incomplete data by extending the estimation algorithm of [8] to incomplete data settings.

1 Introduction

Probabilistic grammars are of great interest for computational natural language processing (NLP), e.g., because they allow the resolution of structural ambiguities by a probabilistic ranking of competing analyses. A prerequisite for such applications is parameter estimation, i.e., a method to adapt the model parameters to best account for a given language corpus. Clearly, an estimation technique similar to the well-known maximization technique of [2] for context-free models would be desirable also for constraint-based models. Baum’s maximization technique permits model parameters to be efficiently estimated from incomplete, i.e., unparsed data rather than from complete, i.e., fully parsed data. Recently, an attempt to apply this estimation technique to a probabilistic version of the constraint logic programming (CLP) scheme of [12] has been presented by [11]. As recognized by Eisele, there is a context-dependence problem associated with applying this technique to constraint-based systems. That is, incompatible variable bindings can lead to failure derivations, which cause a loss of probability mass in the estimated probability distribution over derivations. This probability leakage prevents the estimation procedure from yielding the desired maximum likelihood values in the general case. A similar problem troubles every attempt to embed Baum’s maximization technique into an estimation procedure for probabilistic analogues of a constraint-based processing systems (see, e.g., [5], [4], [3], [16], or [17]). From a mathematical point of view, all such constraint-based
approaches contradict the inherent assumptions of Baum’s maximization technique which require that the derivation steps are mutually independent and that the set of licensed derivations is unconstrained. Only recently, [1] has shown how to overcome this problem by using the algorithm of [8] for estimation. This method, however, applies only to complete data.

Unfortunately, the need to rely on large samples of complete data is impractical. For parsing applications, complete data means several person-years of hand-annotating large corpora with specialized grammatical analyses. This task is always labor-intensive, error-prone, and restricted to a specific grammar framework, a specific language, and a specific language domain. Clearly, flexible techniques for parameter estimation of probabilistic constraint-based grammars from incomplete data are desirable.

The aim of this paper is to solve the problem of parameter estimation from incomplete data for probabilistic constraint-based grammars. For this aim, we present a log-linear probability model for CLP. CLP is used here to provide an operational treatment of purely declarative grammar frameworks such as PATR, LFG or HPKG. A probabilistic CLP scheme then yields a formal basis for probabilistic versions of various constraint-based grammar formalisms. The probabilistic model defines a probability distribution over the proof trees of a constraint logic program on the basis of weights assigned to arbitrary properties of the trees. In NLP applications, such properties could be, e.g., simply context-free rules or context-sensitive properties such as subtrees of proof trees or non-local head-head relations. The algorithm we will present is an extension of the estimation method for log-linear models of [8] to incomplete-data settings. Furthermore, we will present a method for automatic property selection from incomplete data.

The rest of this paper is organized as follows. Section 2 introduces the basic formal concepts of CLP. Section 3 presents a log-linear model for probabilistic CLP. Parameter estimation and property selection of log-linear models from incomplete data is treated in Sect. 4. Concluding remarks are made in Sect. 5.

2 Constraint Logic Programming for NLP

In the following we will quickly report the basic concepts of the CLP scheme of [12]. A constraint-based grammar is encoded by a constraint logic program $P$ with constraints from a grammar constraint language $L$ embedded into a relational programming constraint language $R(L)$.

Let us consider a simple non-linguistic example. The program of Fig. 1 consists of five definite clauses with embedded $L$-constraints from a language of hierarchical types. The

1 Moreover, even approaches such as that of [18], where the derivation steps are in fact context-free, must be characterized as constraint-based and exhibit a similar problem because discarding derivations incompatible with the bracketing of a training corpus from the estimation procedure also induces a problem of loss of probability mass.

2 An example for an embedding of feature-based constraint languages into the CLP scheme of [12] is the formalism CUF ([10]).
ordering on the types is defined by the operation of set inclusion on the denotations (·′) of the types and \( a' \subseteq c' \subseteq e' \), \( b' \subseteq d' \subseteq e' \), and \( c' \cap d' = \emptyset \).

\[
\begin{align*}
   s(Z) & \leftarrow p(Z) \& q(Z), \\
p(Z) & \leftarrow Z = a, \\
p(Z) & \leftarrow Z = b, \\
q(Z) & \leftarrow Z = a, \\
q(Z) & \leftarrow Z = b.
\end{align*}
\]

Figure 1: Simple constraint logic program

seen from a parsing perspective, an input string corresponds to an initial goal or query \( G \) which is a possibly empty conjunction of \( \mathcal{L} \)-constraints and \( \mathcal{R}(\mathcal{L}) \)-atoms. Parses of a string (encoded by \( G \)) as produced by a grammar (encoded by \( \mathcal{P} \)) correspond to \( \mathcal{P} \)-answers of \( G \). A \( \mathcal{P} \)-answer of a goal \( G \) is defined as a satisfiable \( \mathcal{L} \)-constraint \( \phi \) s.t. the implication \( \phi \rightarrow G \) is a logical consequence of \( \mathcal{P} \). The operational semantics of conventional logic programming, SLD-resolution (\[14\]), is generalized by performing goal reduction only on the \( \mathcal{R}(\mathcal{L}) \)-atoms and solving conjunctions of collected \( \mathcal{L} \)-constraints by a given \( \mathcal{L} \)-constraint solver. An example for queries and proof trees for the program of Fig. 1 is given in Fig. 3.

In the following it will be convenient to view the search space determined by this derivation procedure as a search of a tree. Each derivation from a query \( G \) and a program \( \mathcal{P} \) corresponds to a branch of a derivation tree, and each successful derivation to a subtree of a derivation tree, called a proof tree, with \( G \) as root note and a \( \mathcal{P} \)-answer as terminal node. We assume each parse of a sentence to be associated with a single proof tree. In order to rank parses in terms of their likelihood, we define a probability distribution over proof trees. To this end we propose a log-linear model.

### 3 A Log-Linear Probability Model for CLP

Log-linear models are powerful exponential probability distributions which define the probability of an event as being proportional to the product of weights assigned to selected properties of the event\[^3\]. For our application, the special instance of interest is a log-linear distribution over the countably infinite set of proof trees for a set of queries to a program. Log-linear distributions take the following form.

**Definition 1.** A log-linear probability distribution \( p_\lambda \) on a set \( \Omega \) is defined s.t. for all \( \omega \in \Omega \):

\[
p_\lambda(\omega) = Z_\lambda^{-1} e^{\lambda \cdot \nu(\omega)} p_0(\omega),
\]

\[
Z_\lambda = \sum_{\omega \in \Omega} e^{\lambda \cdot \nu(\omega)} p_0(\omega) \text{ is a normalizing constant},
\]

\[^3\] Log-linear models emerged in statistical physics as Gibbs- or Boltzmann-distribution and can be interpreted also as maximum entropy distributions \[13\].
\[ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \] is a vector of log-parameters,
\[ \nu = (\nu_1, \ldots, \nu_n) \] is a vector of property-functions s.t. for each \( \nu_i : \Omega \rightarrow \mathbb{N} \), \( \nu_i(\omega) \) is the number of occurrences of the \( i \)-th property in \( \omega \),
\[ \lambda \cdot \nu(\omega) \] is a weighted property-function s.t. \( \lambda \cdot \nu(\omega) = \sum_{i=1}^{n} \lambda_i \nu_i(\omega) \),
\( p_0 \) is a fixed initial distribution.

When we search for a proper probability distribution over given training data in a maximum likelihood estimation framework, we want to find a distribution reflecting the statistics of the training corpus. This means, we have to choose useful properties (property selection) and appropriate corresponding log-parameters (parameter estimation). A definition of properties convenient for our application is a subtrees of proof trees.

Suppose we have a training corpus of ten queries, consisting of three tokens of query \( y_1 : s(\mathbf{Z}) \& \mathbf{Z} = a \), four tokens of \( y_3 : s(\mathbf{Z}) \& \mathbf{Z} = c \), and one token each of query \( y_2 : s(\mathbf{Z}) \& \mathbf{Z} = b \), \( y_4 : s(\mathbf{Z}) \& \mathbf{Z} = d \), and \( y_5 : s(\mathbf{Z}) \& \mathbf{Z} = e \). The corresponding proof trees generated by the program in Fig. 1 are given in Fig. 2. Note that queries \( y_1 \), \( y_2 \), \( y_3 \) and \( y_4 \) are unambiguous, being assigned a single proof tree, while \( y_5 \) is ambiguous.

A first useful distinction between the proof trees of Fig. 2 can be obtained by selecting the two subtrees \( t_1 : [\mathbf{Z} = a] \) and \( t_2 : [\mathbf{Z} = b] \) as properties. This allows us to cluster the proof trees into two disjoint sets on the basis of their similar statistical qualities. Since in our training corpus seven out of ten queries come unambiguously with a proof tree including property \( t_1 \), we would expect the maximum likelihood parameter value corresponding to property \( t_1 \) to be higher than the parameter value of property \( t_2 \). However, we cannot simply recreate the proportions of the training data from the corresponding proof trees because we do not know the frequency of the possible proof trees of query \( y_5 \). A solution to this incomplete-data problem is presented in the next section.

### 4 Inducing Log-Linear Models from Incomplete Data

As shown in the foregoing example, statistical inference for log-linear models involves two problems: parameter estimation and property selection. In the following, we will present the details of an algorithm to solve to these problems in the presence of incomplete data.
4.1 Parameter Estimation

The “improved iterative scaling” algorithm presented by [8] solves a maximum likelihood estimation problem for log-linear models with respect to complete data\(^4\). This algorithm itself is an extension of the “generalized iterative scaling” algorithm of [7] especially tailored to estimating models with large parameter spaces. We present a version of the first algorithm specifically designed for incomplete data problems. A proof of monotonicity and convergence of the algorithm is given in the appendix, i.e., we show that successive steps of the algorithm increase the incomplete-data log-likelihood and eventually lead to convergence to a (local) maximum.

Applying an incomplete-data framework to a log-linear probability model for CLP, we can assume the following to be given:

- observed, incomplete data \(y \in \mathcal{Y}\), corresponding to a given, finite sample of queries for a constraint logic program \(\mathcal{P}\),
- unobserved, complete data \(x \in \mathcal{X}\), corresponding to the countably infinite sample of proof trees for queries \(\mathcal{Y}\) from \(\mathcal{P}\),
- a many-to-one function \(Y : \mathcal{X} \to \mathcal{Y}\) s.t. \(Y(x) = y\) corresponds to the unique query labeling proof tree \(x\), and its inverse \(X : \mathcal{Y} \to \mathcal{X}\) s.t. \(X(y) = \{x | Y(x) = y\}\) is the countably infinite set of proof trees for query \(y\) from \(\mathcal{P}\),
- a complete-data specification \(p_\lambda(x)\), which is a log-linear distribution on \(\mathcal{X}\) with given initial distribution \(p_0\), fixed property-functions vector \(\nu\), and depending on parameter vector \(\lambda\),
- an incomplete-data specification \(p_\lambda(y)\), which is related to the complete-data specification by \(p_\lambda(y) = \sum_{x \in X(y)} p_\lambda(x)\).

For the rest of this section we will refer to a given vector \(\nu\) of property functions, which is assumed to result from the property selection procedure presented below. If the incomplete-data log-likelihood function \(L\) is defined over a sample \(\mathcal{Y}\) of tokens of queries \(y\) s.t. \(L(\lambda) = \ln \prod_{y \in \mathcal{Y}} p_\lambda(y)\), then the problem of maximum-likelihood estimation for log-linear models from incomplete data can be stated as follows. Given a fixed \(\mathcal{Y}\)-sample and a set \(\Lambda = \{\lambda | p_\lambda(x)\) is a log-linear distribution on \(\mathcal{X}\) with fixed \(p_0\), fixed \(\nu\) and \(\lambda \in \mathbb{R}^n\}\), we want to find a maximum likelihood estimate \(\lambda^*\) of \(\lambda\) s.t. \(\lambda^* = \arg \max_{\lambda \in \Lambda} L(\lambda)\).

The key idea of the proposed method is to iteratively maximize a strictly concave auxiliary function when the non-concave log-likelihood function cannot be maximized analytically. Following [5], we can define an auxiliary function \(A\) directly as a lower bound on \(L(\gamma + \lambda) - L(\lambda)\), i.e., as a conservative estimate of the difference in log-likelihood when

\[A(\gamma, \lambda) = \ln \prod_{y \in \mathcal{Y}} p_{\gamma + \lambda}(y) - \ln \prod_{y \in \mathcal{Y}} p_\lambda(y)\]

For complete data, this is equivalent to solving a constrained maximum entropy problem.
going from a basic model $p_\lambda$ to an extended model $p_{\gamma + \lambda}$. The specific design of $A$ for incomplete data can be derived from the complete data case, in essence, by replacing an expectation of complete, but unobserved, data by a conditional expectation given the observed data and the current parameter values. Let $\lambda \in \Lambda, \gamma \in \mathbb{R}^n$. Then

$$A(\gamma, \lambda) = \sum_{y \in Y} (1 + k_\lambda [\gamma \cdot \nu] - p_\lambda [\sum_{i=1}^n \bar{\nu}_i e^{\gamma_i \nu_i}]).$$

$A$ is maximized in $\gamma$ at the unique point $\hat{\gamma}$ satisfying for each $\hat{\gamma}_i$:

$$\sum_{y \in Y} k_\lambda [\nu_i] = \sum_{y \in Y} p_\lambda [\nu_i e^{\hat{\gamma}_i \nu_i}].$$

An iterative algorithm for maximizing $L$ is constructed from $A$ as follows. For the want of a name, we will call this the “Iterative Maximization (IM)” algorithm.

**Definition 2 (Iterative maximization).** Let $\mathcal{M} : \Lambda \rightarrow \Lambda$ be a mapping defined by $\mathcal{M}(\lambda) = \hat{\gamma} + \lambda$ with $\hat{\gamma} = \arg \max_{\gamma \in \mathbb{R}^n} A(\gamma, \lambda)$. Then each step of the IM algorithm is defined by $\lambda^{(k+1)} = \mathcal{M}(\lambda^{(k)})$.

As shown in the appendix, this procedure stepwise increases the log-likelihood function $L$ and eventually converges to a (local) maximum of $L$. For large configuration spaces $\mathcal{X}$ the expectations to be calculated can get intractable. Here approximations by conditional models or Monte Carlo methods have to be used.

### 4.2 Property Selection

A further problem is that exhaustive sets of properties can get unmanageably large. Let properties of proof trees be defined as connected subgraphs of proof trees, and suppose that properties can incrementally be constructed by selecting from an initial set of goals and from subtrees built by performing a resolution step at a terminal node of a subtree already in the model. Clearly, the exponentially growing set of possible properties must be pruned by some quality measure. An appropriate measure can then be used to define an algorithm for automatic property selection.

A straightforward measure to take would be the improvement in log-likelihood when extending a model by a single candidate property $c$ with corresponding parameter $\alpha$. This would require iterative maximization for each candidate property and is thus infeasible. Following [8], we could instead approximate the improvement due to adding a single property by adjusting only the parameter of this candidate and holding all other parameters of the model fixed. Unfortunately, the incomplete-data log-likelihood $L$ is not concave in

\[ k_\lambda(x|y) = p_\lambda(x)/\sum_{x \in \mathcal{X}(y)} p_\lambda(x) \] is the conditional probability of the complete data $x$ given the observed data $y$ and the current fit of the parameter $\lambda$. Furthermore, $p[f] = \sum_{\omega \in \Omega} p(\omega) f(\omega)$ is the expectation of a function $f : \Omega \rightarrow \mathbb{R}$ with respect to a probability distribution $p$ on a set $\Omega$, $\nu_\#(x) = \sum_{i=1}^n \nu_i(x), \nu_i(x) = \nu_i(x)/\nu_\#(x)$.\[ \text{Another possibility to arrive at the same auxiliary function is to use the complete-data auxiliary function of Section 5 in the M-step of a generalized EM algorithm [1]. This guarantees monotonicity of the resulting algorithm, but convergence yet has to be proved. Our approach views the incomplete-data auxiliary function directly as a lower bound on the improvement in incomplete-data log-likelihood, which enables an intuitive and elegant proof of convergence.} \]

\[ 6 \]
the parameters and thus cannot be maximized directly. However, we can instantiate the auxiliary function $A$ used in parameter estimation to the extension of a model $p_\lambda$ by a single property $c$ with log-parameter $\alpha$, i.e., we can express an approximate gain $G_c(\alpha, \lambda)$ of adding a candidate property $c$ with log-parameter value $\alpha$ to a log-linear model $p_\lambda$ as a conservative estimate of the true gain in log-likelihood as follows.

$$G_c(\alpha, \lambda) = \sum_{y \in Y} (1 + k_\lambda[c] - p_\lambda[e^{\alpha c}]).$$

$G_c(\alpha, \lambda)$ is maximized in $\alpha$ at the unique point $\hat{\alpha}$ satisfying

$$\sum_{y \in Y} k_\lambda[c] = \sum_{y \in Y} p_\lambda[e^{\hat{\alpha} c}].$$

Property selection will incorporate that property out of the set of candidates that gives the greatest improvement to the model at the property’s best adjusted parameter value. Since we are interested only in relative, not absolute gains, a single, non-iterative maximization of the approximate gain will be sufficient to choose from the candidates.

4.3 Combined Statistical Inference

A combined algorithm for statistical inference for log-linear models from incomplete data is as follows.

**Input** Initial model $p_0$, multiple incomplete-data sample $Y$.

**Output** Log-linear model $p^*$ on complete-data sample $X$ with selected property function vector $\nu^*$ and estimated log-parameter vector $\lambda^* = \arg \max_{\lambda \in \Lambda} L(\lambda)$ where $\Lambda = \{\lambda | p_\lambda$ is a log-linear model on $X$ based on $p_0, \nu^*$ and $\lambda \in \mathbb{R}^n\}$.

**Procedure**

1. $p^{(0)} := p_0$ with $C^{(0)} := \emptyset$,

2. Property selection: For each candidate property $c \in C^{(t)}$, compute the gain $G_c(\lambda^{(t)}) := \max_{\alpha \in \mathbb{R}} G_c(\alpha, \lambda^{(t)})$, and select the property $\hat{c} := \arg \max_{c \in C^{(t)}} G_c(\lambda^{(t)})$.

3. Parameter estimation: Compute a maximum likelihood parameter value $\hat{\lambda} := \arg \max_{\lambda \in \Lambda} L(\lambda)$ where $\Lambda = \{\lambda | p_\lambda(x)$ is a log-linear distribution on $X$ with initial model $p_0$, property function vector $\nu := \nu^{(t)} + \hat{c}$, and $\lambda \in \mathbb{R}^n\}$.

4. Until the model converges, set $p^{(t+1)} := p_{\hat{\lambda}, \hat{\nu}}$, $t := t + 1$, go to 2.

Let us return to the example of Sect. 3 and apply the IM algorithm to the incomplete-data problem stated there. For the selected properties $t_1$ and $t_2$, we have $\nu_{\#}(x) = \nu_1(x) + \nu_2(x) = 1$ for all possible proof trees $x$ for the sample of Fig. 2. Thus the parameter updates $\hat{\gamma}_i$ can be calculated from a particularly simple closed form as follows.

$$\hat{\gamma}_i = \ln \frac{\sum_y k_\lambda[\nu_i]}{\sum_y p_\lambda[\nu_i]}.$$
A sequence of IM iterates up to stability in the third place after the decimal point of the incomplete-data log-likelihood is given in Table 1. Probabilities of proof trees involving property \( t_i \) are denoted by \( p_i \). Starting from an initial uniform probability of \( 1/6 \) for each proof tree, this estimation sequence converges to the desired accuracy after three iterations and yield probabilities \( p_1 \approx .259 \) and \( p_2 \approx .074 \) for the respective proof trees.

| Iteration | \( \lambda_1^{(t)} \) | \( \lambda_2^{(t)} \) | \( p_1^{(t)} \) | \( p_2^{(t)} \) | \( L(\lambda^{(t)}) \) |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0         | 0               | 0               | 1/6             | 1/6             | -17.224448     |
| 1         | ln 1.5          | ln .5           | .25             | .083            | -15.772486     |
| 2         | ln 1.55         | ln .45          | .2583           | .075            | -15.753678     |
| 3         | ln 1.555        | ln .445         | .25916          | .07416          | -15.753481     |

Table 1: Estimation using the IM algorithm

5 Conclusion

We have presented a probabilistic model for CLP, coupled with an algorithm to induce the parameters and properties of log-linear models from incomplete data. This algorithm is applicable to log-linear probability distributions in general, and has been shown here to be useful to estimate the parameters of probabilistic context-sensitive NLP models. In contrast to related approaches such as that of [1] or [15], our statistical inference algorithm provides the means for automatic and reusable training of probabilistic constraint-based grammars from unparsed corpora. Furthermore, heuristic search algorithms for finding the most probable analysis in the CLP model can be based upon this probability model. For example, a combination of the dynamic-programming techniques of Earley deduction [19] and Viterbi-searching [20] could be employed. Depending on the class of constraint-based grammars under consideration, a considerable gain in search efficiency can be obtained.

The statistical inference algorithm presented is fully implemented and has already been tested empirically with simple examples. Clearly, the performance of the presented techniques in real-world NLP problems has to be thoroughly investigated. Unfortunately, the current availability of broad-coverage constraint-based grammars limited so far the empirical evaluation of the presented techniques for the area of constraint-based parsing. In future work we also will investigate the applicability of the statistical methods here described to NLP problems other than constraint-based parsing.

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7 However, note that the choice of a particular class of constraint-based grammars also influences the behaviour of the algorithm in finding the optimal analysis. For example, in grammars where variable-bindings are ignored in Viterbi-searching in order to avoid the overhead of storing each variable binding separately the problem of pursuing a non-optimal path arises. In such cases only approximate heuristic searching can be done (see [1] for a similar approach to a Viterbi-like heuristic search procedure for unification-based grammars).
A Iterative Maximization: Propositions and Proofs

In the following, we assume that for each property function \( \nu \), some proof tree \( x \in X \) with \( \nu(x) > 0 \) exists, and require \( p_\lambda \) to be strictly positive on \( X \), i.e., \( p_\lambda(x) > 0 \) for all \( x \in X \). Furthermore, \( p_{\gamma + \lambda}(x) = Z^{-1}_{\gamma + \lambda} \exp(\nu(x)) p_\lambda(x) \) denotes an extended log-linear model with \( Z_{\gamma + \lambda} = p_\lambda(\exp(\nu)) \).

Lemma 2 shows that there is no estimated improvement in log-likelihood at the origin, and Lemma 3 shows that the critical points of interest are the same for \( A \) and \( L \).

Lemma 1. \( A(\gamma, \lambda) \leq L(\gamma + \lambda) - L(\lambda) \).

Proof.

\[
L(\gamma + \lambda) - L(\lambda) = \sum_{y \in Y} (\ln \frac{p_{\gamma + \lambda}(y)}{p_\lambda(y)}) = \sum_{y \in Y} \left( \sum_{x \in X(y)} \frac{p_{\gamma + \lambda}(x)}{p_\lambda(x)} \ln \frac{p_{\gamma + \lambda}(x)}{p_\lambda(x)} \right) = \sum_{y \in Y} \left( \sum_{x \in X(y)} \frac{p_\lambda(x)}{p_\lambda(y)} \ln \frac{p_{\gamma + \lambda}(x)}{p_\lambda(x)} \right) = \sum_{y \in Y} \left( \sum_{x \in X(y)} \frac{p_\lambda(x)}{p_\lambda(y)} \ln \frac{Z_{\gamma + \lambda}^{-1} \exp(\nu(x)) p_\lambda(x)}{p_\lambda(x)} \right) = \sum_{y \in Y} (k_{\lambda}(\gamma \cdot \nu) - \ln p_\lambda(\exp(\nu))) \geq \sum_{y \in Y} (k_{\lambda}(\gamma \cdot \nu) - \ln p_\lambda(\exp(\nu))) \text{ since } \ln x \leq x - 1 = \sum_{y \in Y} (k_{\lambda}(\gamma \cdot \nu) + 1 - \sum_{x \in X} (p_\lambda(x) e^{\sum_{i=1}^{n} \gamma_i \nu_i(x)})) = \sum_{y \in Y} (k_{\lambda}(\gamma \cdot \nu) + 1 - \sum_{x \in X} (p_\lambda(x) e^{\sum_{i=1}^{n} \gamma_i \nu_i(x)})) = \sum_{y \in Y} (k_{\lambda}(\gamma \cdot \nu) + 1 - \sum_{x \in X} (p_\lambda(x) e^{\sum_{i=1}^{n} \gamma_i \nu_i(x)})) \text{ by Jensen’s inequality} = \sum_{y \in Y} (k_{\lambda}(\gamma \cdot \nu) + 1 - p_\lambda(\sum_{i=1}^{n} \nu_i e^{\gamma_i \nu_i})) = A(\gamma, \lambda).
\]

Lemma 1 shows that there is no estimated improvement in log-likelihood at the origin, and Lemma 2 shows that the critical points of interest are the same for \( A \) and \( L \).

Lemma 2. \( A(0, \lambda) = 0 \).

Lemma 3. \( \frac{d}{dt} |_{t=0} A(t \gamma, \lambda) = \frac{d}{dt} |_{t=0} L(t \gamma + \lambda) \).

Theorem 4 shows the monotonicity of the IM algorithm.

Theorem 4. For all \( \lambda \in \Lambda: L(M(\lambda)) \geq L(\lambda) \) with equality iff \( \lambda \) is a fixed point of \( M \) or equivalently is a critical point of \( L \).
Proof.

\[ L(\mathcal{M}(\lambda)) - L(\lambda) \geq A(\mathcal{M}(\lambda)) \quad \text{by Lemma } 2 \]
\[ \geq 0 \quad \text{by Lemma } 2 \text{ and definition of } \mathcal{M}. \]

The equality \( L(\mathcal{M}(\lambda)) = L(\lambda) \) holds iff \( \lambda \) is a fixed point of \( \mathcal{M} \), i.e., \( \mathcal{M}(\lambda) = \hat{\gamma} + \lambda \) with \( \hat{\gamma} = 0 \). Furthermore, \( \lambda \) is a fixed point of \( \mathcal{M} \) iff \( \hat{\gamma} = \arg \max_{\gamma \in \mathbb{R}^n} A(\gamma, \lambda) = 0 \).

\[ \iff \text{ for all } \gamma \in \mathbb{R}^n : \hat{t} = \arg \max_{t \in \mathbb{R}} A(t \gamma, \lambda) = 0, \]
\[ \iff \text{ for all } \gamma \in \mathbb{R}^n : \frac{d}{dt} \bigg|_{t=0} A(t \gamma, \lambda) = 0, \]
\[ \iff \text{ for all } \gamma \in \mathbb{R}^n : \frac{d}{dt} \bigg|_{t=0} L(t \gamma + \lambda) = 0, \text{ by Lemma } 3 \]
\[ \iff \lambda \text{ is a critical point of } L. \]

Corollary 5 implies that a maximum likelihood estimate is a fixed point of the mapping \( \mathcal{M} \).

**Corollary 5.** Let \( \lambda^* = \arg \max_{\lambda \in \Lambda} L(\lambda) \). Then \( \lambda^* \) is a fixed point of \( \mathcal{M} \).

Theorem 6 discusses the convergence properties of the IM algorithm. In contrast to the improved iterative scaling algorithm, we cannot show convergence to a global maximum of a strictly concave objective function. Rather, similar to the EM algorithm, we can show convergence of a sequence of IM iterates to a critical point of the non-concave incomplete-data log-likelihood function \( L \).

**Theorem 6 (Convergence).** Let \( \{\lambda^{(k)}\} \) be a sequence in \( \Lambda \) determined by the IM Algorithm. Then all limit points of \( \{\lambda^{(k)}\} \) are fixed points of \( \mathcal{M} \) or equivalently are critical points of \( L \).

Proof. Let \( \{\lambda^{(k_n)}\} \) be a subsequence of \( \{\lambda^{(k)}\} \) converging to \( \bar{\lambda} \). Then for all \( \gamma \in \mathbb{R}^n \):

\[ A(\gamma, \lambda^{(k_n)}) \leq A(\hat{\gamma}^{(k_n)} + \lambda^{(k_n)}) \quad \text{by definition of } \mathcal{M} \]
\[ \leq L(\hat{\gamma}^{(k_n)} + \lambda^{(k_n)}) - L(\lambda^{(k_n)}) \quad \text{by Lemma } 2 \]
\[ = L(\lambda^{(k_n + 1)}) - L(\lambda^{(k_n)}) \quad \text{by definition of IM} \]
\[ \leq L(\lambda^{(k_n + 1)}) - L(\lambda^{(k_n)}) \quad \text{by monotonicity of } L(\lambda^{(k)}), \]

and in the limit as \( n \to \infty \), for continuous \( A \) and \( L \): \( A(\gamma, \bar{\lambda}) \leq L(\bar{\lambda}) - L(\bar{\lambda}) = 0 \). Thus \( \gamma = 0 \) is a maximum of \( A(\gamma, \bar{\lambda}) \), using Lemma 2, and \( \bar{\lambda} = \hat{\lambda} \) is a fixed point of \( \mathcal{M} \). Furthermore, \( \frac{d}{dt} \bigg|_{t=0} A(t \gamma, \bar{\lambda}) = \frac{d}{dt} \bigg|_{t=0} L(t \gamma + \bar{\lambda}) = 0 \), using Lemma 3, and \( \bar{\lambda} \) is a critical point of \( L \).

From this and Theorem 2 it follows immediately that each sequence of likelihood values, for which an upper bound exists, converges monotonically to a critical point of \( L \).

**Corollary 7.** Let \( \{L(\lambda^{(k)})\} \) be a sequence of likelihood values bounded from above. Then \( \{L(\lambda^{(k)})\} \) converges monotonically to a value \( L^* = L(\lambda^*) \) for some critical point \( \lambda^* \) of \( L \).

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