Monte-Carlo simulations for wall-bounded fluid flows via random vortex method

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Abstract

In this paper a Monte-Carlo method for simulating the motion of fluid flow moving along a solid wall is proposed. The random vortex method in the present paper is established by using the reflection technology and perturbation technique. The Monte-Carlo method based on this random vortex dynamic may be implemented, and several Monte-Carlo simulations are then carried out for the flows near the solid wall.

Key words: diffusion processes, incompressible fluid flow, Monte-Carlo simulation, random vortex method

MSC classifications: 76M35, 76M23, 60H30, 65C05, 68Q10.

1 Introduction

Our goal in this work is to present a new numerical method for computing solutions of the Navier-Stokes equations which describe viscous fluid flows passing a plate. The technology in the paper may be generalised to three dimensional case, while substantial modifications are required, hence in the present work only two dimensional case will be dealt with and the three dimensional case will be published in separate articles. The numerical schemes for solving fluid dynamics equations are based on the random vortex dynamics, which were developed in the past mainly for turbulent flows away from physical boundaries.

The key idea in the random vortex method proposed in Chorin [4] may be briefly described for two dimensional flows as the following, cf. [9], [18], [5] and [19] for details. A fluid flow may be described by its velocity vector field \( u(x,t) \), and the fundamental problem in fluid dynamics (and in the study of turbulence) is to extract information about the velocity \( u(x,t) \) from its equations of motion. It is well known that, regardless of the nature of fluid flows, determining the vector field \( u(x,t) \) is equivalent to determining its integral curves \( X_\xi \) of the dynamics

\[
\frac{d}{dt} X_\xi^t = u(X_\xi^t, t), \quad X_\xi^0 = \xi,
\]

where \( t \to X_\xi^t \) is considered as the trajectory of the “fluid” particle issued from location \( \xi \). In order to employ the idea of Monte-Carlo methods, one may consider \( u(x,t) \) as the velocity of “imaginary” Brownian fluid particles (this idea to the best knowledge of the present authors is due to G. I. Taylor [32]), whose trajectories, still denoted by \( X_\xi \) (while it is a random process on a probability space), may be determined by solving Itô’s stochastic differential equation (SDE, cf. [13])

\[
dx_\xi = u(X_\xi^t, t)dt + \sqrt{2}d\nu_B, \quad X_\xi^0 = \xi
\]
for $\xi \in \mathbb{R}^2$, where $\nu > 0$ is a constant (which will be the kinematic viscosity of the fluid). The key idea in any Monte-Carlo simulation is to average these trajectories of Brownian fluid particles and recover therefore the flow velocity $u(x, t)$. In order to implement this strategy, one first solves the closure problem: by utilising the motion equation for $u(x, t)$ and the distribution of $X$ to eliminate formally the vector field $u(x, t)$ in (1) in terms of the distribution of the random field $X(x, t)$, and therefore the SDE (1) is closed. In random vortex methods, this is achieved by using the motion equation of its vorticity $\omega = \nabla \times u$, a vector field with its components $\omega^i = \epsilon^{ijk} \partial_j^u$. This scheme can be explained best for two dimensional incompressible fluid flows. Let us explicate this point in more details below.

For a two dimensional fluid flow with viscosity $\nu > 0$, the velocity $u(x, t)$ satisfies the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla P = 0, \quad \text{and } \nabla \cdot u = 0,$$

so that its vorticity $\omega = \nabla \times u$, a scalar function, evolves according to the vorticity transport equation:

$$\frac{\partial}{\partial t} \omega + (u \cdot \nabla) \omega - \nu \Delta \omega = 0, \quad \text{and } \nabla \times u = \omega.$$

Hence $\omega$ has the following integral representation

$$\omega(x, t) = \int \rho_u(0, \xi, t, x) \omega_0(\xi) d\xi,$$

where $\omega_0$ is the initial vorticity, and $\rho_u(\tau, \xi, t, x)$ is the transition probability density function of Taylor’s diffusion defined by (1) (whose infinitesimal generator is $\nu \Delta + u \cdot \nabla$). $\rho_u(\tau, \xi, t, x)$ coincides with the Green function associated with the forward operator $\frac{\partial}{\partial t} - \nu \Delta + u \cdot \nabla$. Since $\nabla \cdot u = 0$ and $\nabla \times u = \omega$, the Biot-Savart law holds:

$$u(x, t) = -\int G(x, y) \nabla \times \omega = \int K(x, y) \omega(y, t) dy,$$

where $G$ is the Green function of the Laplacian on $\mathbb{R}^2$ and $K = \nabla G$. Putting the representations for $\omega$ and for $u$ together we obtain

$$u(x, t) = \int \int K(x, y) \rho_u(0, \xi, t, x) \omega_0(\xi) d\xi dy$$

$$= \int E[K(x, X^\xi_t)] \omega_0(\xi) d\xi.$$

Therefore the SDE (1) can be closed in the following manner

$$dX^\xi_t = E[K(x, X^\eta_t)] \omega_0(\eta) d\eta |_{x=X^\xi_0} dt + \sqrt{2} \nu d\mathcal{B}_t,$$

SDE (6) can be used for designing numerical schemes for computing numerically the velocity field $u(x, t)$ accordingly.

While, like many other beautiful ideas in mathematics, when applying these ideas to concrete situations, one has to overcome several obstacles. This is not exceptional when one applies the ideas of random vortex methods to wall-bounded flows, and we need to reformulate the random vortex method which suits for the study of wall-bounded flows. One has to consider not only the boundary condition for the velocity, but also the possible constraint on the vorticity motion at the solid wall. The vorticity at the wall can not be determined before one is able to solve the Navier-Stokes equations, which therefore poses substantial difficulty in implementing the random vortex method. We make the following key observation: the Biot-Savart law holds well for a bounded domain as the velocity needs to satisfy the no slip condition, while we may recover the velocity $u(x, t)$ not only in terms of its vorticity $\omega$, but we may also obtain $u(x, t)$ by using some perturbations of $\omega$ which satisfy the right boundary condition but still
solve certain parabolic equations. Therefore we are able to formulate a family of random vortex dynamics for wall-bounded flows by using the reflection principle and one parameter family of perturbations of the vorticity.

We should mention that computational fluid dynamics (CFD) is a huge subject, and various numerical approaches have been developed such as Direct Numerical Simulations (DNS) (cf. [22], [29], [21], see also [6, 31, 33, 27] and the references therein for details), Large Eddy Simulations (LES) (cf. [28], [17], [2, 16] and the literature cited therein for further reading), Probability Density Function (PDF) (cf. [23] and [15] for example) and other technologies have been developed, and have become increasingly important. A few numerical experiments for fluid flows, including turbulent flows, within their boundary layers, have been carried out, which provide substantial information about wall-bounded fluid flows, for example [3], [10], [20], [7], [11], [25], [30], [34, 35, 36] and etc. Of course there is a large number of papers dealing with many different aspects of the wall-bounded flows, the authors must apologise for failing to mention many excellent contributions, and the interested reader may consult [12], [14], [26] and the papers cited therein. Stochastic simulation schemes have been studied too, mainly under the name of random vortex methods which were initiated in [4], cf. [5] and [19] for a comprehensive account. While to the best knowledge of the present authors, random vortex methods are mainly developed for fluid flows moving freely without boundary constraint. The goal of the present paper is therefore to formulate a Monte-Carlo method for numerically calculating solutions for simple wall bounded incompressible fluid flows. According to an important discovery by Prandtl [24], the viscosity however how small demonstrates its effect in a thin layer close to the solid wall, in order to ensure the fluid flow to obey the no slip boundary condition. The equations of motions in a thin boundary-layer can be simplified to the boundary layer equation which is much simpler than the Navier-Stokes equations. The integration of the boundary layer equation may be used to determine some very important aspects of wall bounded fluid flows such as the stress immediately applied to the solid wall, which provides with us the necessary information, together with the stochastic representations we are going to establish, to implement Monte-Carlo simulations for boundary turbulent layer flows. This line of research will be addressed however in a future work.

The rest of the paper is organised as follows. Section 2 presents several technical lemmas for later use. In Section 3, two dimensional flows are investigated, and a Monte-Carlo method is established in Section 4. The paper ends with several simulation experiments for the Monte-Carlo simulation.

2 The vorticity transport equation

In this paper we aim to implement the random vortex method and propose a Monte-Carlo scheme for numerically simulating the fluid motion near the thin boundary layer. We consider a fluid flow moving in the half space

$$D = \left\{ x = (x_1, \cdots, x_{d-1}, x_d) \in \mathbb{R}^d : x_d < 0 \right\}$$

(7)

where \( d = 2 \) or 3, whose velocity is denoted by \( u = (u^1, \ldots, u^d) \) is a time-dependent vector field in \( D \). The velocity \( u \) is determined by the Navier-Stokes equations:

$$\frac{\partial}{\partial t} u + (u \cdot \nabla) u - \nu \Delta u + \nabla P = F \quad \text{in} \ D$$

(8)

and

$$\nabla \cdot u = 0 \quad \text{in} \ D,$$

(9)

where \( P \) is the pressure and \( F \) is the external force applied to the fluid. The velocity \( u \) has to satisfy the no slip condition which says that \( u(x,t) \) vanishes for \( x \in \partial D \) and \( t > 0 \). By taking exterior derivative of both sides of the Navier-Stokes equation, we obtain

$$\frac{\partial}{\partial t} \omega + (u \cdot \nabla) \omega - \nu \Delta \omega - (\omega \cdot \nabla) u = G \quad \text{in} \ D$$
where $G = \nabla \wedge F$. The boundary value of $\omega$ along the solid wall $\partial D$ (it will be called the boundary vorticity in short) can not be specified, and must be computed via (8, 9).

Our first task is to identify the boundary vorticity in terms of measurable fluid dynamics variables. Observe that $\frac{\partial}{\partial x_d}$ is the unit normal pointing outward and $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{d-1}}$ form a basis of the tangent space along $\partial D$. It is perhaps convenient to introduce the following notations. If $T$ is a tensor field then $T^{\perp}$ and $T^{\parallel}$ denote the normal and tangential parts of $T$ respectively. Let $S = (S_{ij})$ be the symmetric tensor field, called the rate-of-strain. By definition

$$S_{ij} = \frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}$$

so along its boundary, the normal components of the stress tensor are $S_{13}, S_{23}$ and $S_{33}$. For an incompressible fluid, $\text{tr}(S_{ij}) = 0$, together with the no slip condition, it follows that $S_{11} = S_{22} = S_{33} = 0$ on $\partial D$. Therefore the normal part of the stress $S^{\perp} = (S_{13}, S_{23})$, which are the shear stress.

If $d = 3$, the vorticity $\omega = \nabla \wedge u$ has its components $\omega^i = \epsilon^{ijk} \frac{\partial u^j}{\partial x_k}$, so that the boundary vorticity

$$\omega^1|_{\partial D} = - \left. \frac{\partial u^2}{\partial x_3} \right|_{\partial D}, \quad \omega^2|_{\partial D} = \left. \frac{\partial u^1}{\partial x_3} \right|_{\partial D}, \quad \omega^3|_{\partial D} = 0.$$ (11)

Due to the no slip boundary condition,

$$\left. \frac{\partial u^3}{\partial x_3} \right|_{\partial D} = \left. \frac{\partial u^1}{\partial x_1} \right|_{\partial D} = 0$$

therefore

$$\omega^1|_{\partial D} = - S_{23}|_{\partial D}, \quad \omega^2|_{\partial D} = S_{13}|_{\partial D}, \quad \omega^3|_{\partial D} = 0.$$ (12)

Hence the tangent components $\omega^\parallel$ of the vorticity $\omega$ along the solid wall can be identified with the normal components $S^{\perp}$ of the stress at the wall.

For a two dimensional flow, where $d = 2$, the vorticity $\omega$ can be identified with the scalar function $\omega = \frac{\partial u^2}{\partial x_1} - \frac{\partial u^1}{\partial x_2}$. As we have indicated already, the vorticity transport equation has a simpler form:

$$\frac{\partial}{\partial t} \omega + (u \cdot \nabla) \omega - \nu \Delta \omega = G,$$ in $D$. (13)

that is, the vorticity equation is a scalar “linear” parabolic equation (if $u$ is considered as a known dynamic variable) without non-linear stretching term. It is however in contrast with the case of isotropic turbulence, turbulence may be built up for two dimensional flows (i.e. flows with certain symmetries) due to boundary layer phenomena near the solid wall. For two dimensional flows, the boundary vorticity

$$\omega|_{\partial D} = - S_{12}|_{\partial D}$$ (14)

where $S_{12}$ is the (normal) rate-of-strain at the solid wall.

### 3 Random vortex dynamics

In this section the mathematical framework is set up for implementing Monte-Carlo simulations for wall-bounded flows. The method is based on the random vortex dynamics, and the key step is to derive a functional integral representations for a family of modified vorticity dynamical variables.

Some of the techniques work for

$$D = \left\{ x = (x_1, \ldots, x_{d-1}, x_d) \in \mathbb{R}^d : x_d < 0 \right\}.$$ (15)

for any dimension $d \geq 2$, while the reflection in $\mathbb{R}^d$ about the hyperspace: $x_d = 0$ is the mapping which sends $x = (x_1, \ldots, x_{d-1}, x_d)$ to $\bar{x} = (x_1, \ldots, x_{d-1}, -x_d)$.
3.1 The Biot-Savart law

The first ingredient needed for deriving the random vortex dynamics is a version of the Biot-Savart law for $D$. By definition $\omega = \nabla \times u$ and $\nabla \cdot u = 0$, hence $\Delta u = -\nabla \times \omega$ in $D$. Since $u$ vanishes on $\partial D$, according to Green formula

$$u(x,t) = -\int_D \Gamma_D(x,y)\nabla \cdot \omega(y,t)\,dy$$

for $x \in D$, where $\Gamma_D$ denotes the Green kernel of $D$. Suppose $u(x,t)$ and $\omega(x,t)$ decay to zero sufficiently fast, then we may perform integration by parts, using the no slip condition for $u$, it follows that

$$u(x,t) = \int_D K(x,y) \cdot \omega(y,t)\,dy$$

where $K(x,y) = \nabla \cdot \Gamma_D(x,y)$, which is called the Biot-Savart law for $D$. In dimension two one may prefer to formulate it in slightly different form as $\omega$ is scalar. That is, in dimension two

$$u^i(x,t) = \int_D K^i(x,y) \omega(y,t)\,dy, \quad \text{for } i = 1, 2,$$

where

$$K^1(x,y) = \frac{\partial}{\partial y_2} \Gamma_D(x,y), \quad K^2(x,y) = -\frac{\partial}{\partial y_1} \Gamma_D(x,y).$$

It will reduce the computational cost if an explicit formula for the singular integral kernel $K(x,y)$ is available. To this end we recall that the Green function in $\mathbb{R}^d$ is given by

$$\Gamma(x,y) = \frac{1}{2\pi} \log |x-y| \quad \text{for } d = 2$$

and

$$\Gamma(x,y) = \frac{1}{\kappa_d(2-d)} |x-y|^{2-d} \quad \text{if } d \geq 3$$

where $\kappa_d$ is the area of the unit sphere in $\mathbb{R}^d$. By the reflection principle, the Green function for $D$ (subject to the Dirichlet boundary condition) has an explicit formula

$$\Gamma_D(x,y) = \Gamma(x,y) - \Gamma(x,\bar{y})$$

for $x, y \in D$.

For example in dimension two we have the following facts which can be established for $D$ (which is unbounded) with slight modifications of the arguments in any standard textbooks.

**Lemma 1.** Suppose $u \in C^2(D) \cap C^1(\overline{D})$ solves the Poisson equation

$$\Delta u = f \quad \text{in } D, \quad u|_{\partial D} = 0$$

where $f \in L^1(D)$, and $u \to 0$ and $\nabla u \to 0$ at infinity. Then

$$u(x) = \int_D \Gamma_D(x,y)f(y)\,dy \quad \text{for } x \in D.$$  

**Proof.** The only fact we need is the asymptotic of $\Gamma_D(x,y)$ as $y \to \infty$. If $x = (x_1, x_2) \in D$ and $y = (y_1, y_2) \in D$, then

$$\Gamma_D(x,y) = \frac{1}{4\pi} \log \left(1 - \frac{4x_2y_2}{|y-x|^2}\right) \quad \text{for } x, y \in D,$$

which implies that

$$\Gamma_D(x,y) \sim -\frac{1}{\pi} \frac{x_2y_2}{|y-x|^2} \quad \text{as } |y| \to \infty.$$  

The Green formula then follows from the standard argument. □
Lemma 2. Suppose $u = (u^1, u^2) \in C^2(D) \cap C^1(\overline{D})$, $u = 0$ on $\partial D$, such that $u \to 0$ and $|\nabla u| \to 0$ at the infinity and $\nabla \cdot u = 0$ in $D$. Let $\omega = \nabla \wedge u$ and assume that both $\omega, \nabla \omega \in L^1(D)$. Then

$$u^i(x) = \int_D K^i(x, y) \omega(y) dy \quad \text{for all } x \in D$$

(27)

for $i = 1, 2$, where

$$K^1(x, y) = \frac{1}{2\pi} \left( \frac{y_2 - x_2}{|y - x|^2} - \frac{y_2 + x_2}{|y - x|^2} \right)$$

and

$$K^2(x, y) = \frac{1}{2\pi} \left( \frac{y_1 - x_1}{|y - x|^2} - \frac{y_1 + x_1}{|y - x|^2} \right).$$

(28)

(29)

Proof. This follows from the Green formula and integration by parts. □

The following lemma provides an elementary fact which will be used later on.

Lemma 3. Let $a < b$ be two numbers and $x = (x_1, x_2) \in D$ fixed. Then $K^1(x, y)$ and $K^2(x, y)$ are integrable on $[a, b] \times (-\infty, 0)$. Let

$$H^i(x) = \int_{(a, b) \times (-\infty, 0)} K^i(x, y) dy$$

where $i = 1, 2$. Then

$$H^1(x) = -\frac{1}{2} x_2 (\text{sgn}(b - x_1) - \text{sgn}(a - x_1))$$

(30)

and

$$H^2(x) = \frac{x_2}{2\pi} \ln \left( \frac{(b - x_1)^2 + x_2^2}{(a - x_1)^2 + x_2^2} \right) + \frac{1}{\pi} \left( (b - x_1) \arctan \frac{x_2}{b - x_1} - (a - x_1) \arctan \frac{x_2}{a - x_1} \right).$$

(31)

In particular

$$\int_{-\infty}^0 \int_{x_1 - A}^{x_1 + A} K^2(x, y) dy_1 dy_2 = 0$$

(32)

for every $A > 0$ and $x = (x_1, x_2)$ with $x_2 < 0$.

3.2 Taylor’s diffusion

Suppose $b(x, t) = (b^1(x_1, t), \ldots, b^d(x_1, t))$ is a time dependent vector field on $\overline{D}$, which is differential up to the boundary $\partial D$, bounded, Borel measurable, and vanishes along the boundary $\partial D$. The vector field $b(x, t)$ is extended to the whole space $\mathbb{R}^d$ through reflection. That is, if $x_d > 0$, then

$$b^i(x, t) = b^i(\overline{x}, t) \quad \text{for } i = 1, \ldots, d - 1 \text{ and } b^d(x, t) = -b^d(\overline{x}, t).$$

(33)

Since $b(x, t) = 0$ along $\partial D$, $b(x, t) = \overline{b(t, \overline{x})}$ for all $x \in \mathbb{R}^d$.

We will assume that $\nabla \cdot b(x, t) = 0$ for $x_d > 0$. Then the extension via the reflection is divergence-free, that is, $\nabla \cdot b(\cdot, t) = 0$ in $\mathbb{R}^d$ in distribution sense. For such a time dependent vector field we consider the differential operator of second order on $\mathbb{R}^d$:

$$L_{b(x, t)} = \nabla \Delta + b(x, t) \cdot \nabla$$

(34)

where the differential operators $\Delta$ and $\nabla$ apply only to the space variable $x$. The variables $(x, t)$ in the sub-script will be omitted if no confusion may arise. Since $\nabla \cdot b = 0$ in distribution, the formal adjoint operator $L^*_b = L_{-b}$, which is again a diffusion operator of the same type.

Suppose $\Omega \subset \mathbb{R}^d$, then $\Gamma^*_{\Omega, b}(\bar{x}, t; \bar{\xi}, \tau)$; where $0 \leq \tau < t$ and $\bar{\xi}, x \in \Omega$; denotes the Green function to the (forward) parabolic equation

$$\left( \frac{\partial}{\partial t} - L_{b(x, t)} \right) f(x, t) = 0 \quad \text{in } (0, \infty) \times \Omega,$$  

(35)
Suppose that $f(\cdot,t)|_{\partial\Omega} = 0$, in the sense that (i) for every $\xi \in \Omega$, $\tau \geq 0$, as a function $(x,t)$, $f(x,t) = \Gamma_{\Omega,b}(x,t;\xi,\tau)$ solves the previous boundary problem of (35) for $t > \tau$; and for every bounded and continuous function $\phi$ on $\Omega$

$$\lim_{t \uparrow \tau} \int_{\Omega} \phi(\xi) \Gamma_{\Omega,b}(x,t;\xi,\tau) d\xi = \phi(x) \quad (36)$$

for all $x \in \Omega$.

If $\Omega = \mathbb{R}^d$, then we will use $\Gamma_b$ to denote $\Gamma_{\mathbb{R}^d,b}$ for simplicity.

Similarly, $\Gamma^*_b(x,t;\xi,\tau)$ (defined for $0 \leq t < \tau, x, \xi \in \mathbb{R}^d$) denotes a Green function to the backward parabolic equation

$$\left( \frac{\partial}{\partial t} + L^*_b(x,t) \right) f(x,t) = 0 \quad \text{in } \Omega. \quad (37)$$

Since $L^*_b = L_{-b}$, so that the backward equation can be written as

$$\left( \frac{\partial}{\partial t} + L_{-b(x,t)} \right) f(x,t) = 0 \quad \text{in } \Omega.$$

**Lemma 4.** Suppose that $\nabla \cdot b = 0$ in distribution and $b(x,t) = \overline{b(x,t)}$ for all $x \in \mathbb{R}^d, t > 0$. Then

$$\Gamma_{D,b}(x,t;\xi,\tau) = \frac{1}{2} \left( \Gamma_b(x,t;\xi,\tau) - \Gamma_b(x,s;\xi,\tau) - \Gamma_b(x,t;\xi,\tau) + \Gamma_b(x,t;\xi,\tau) \right) \quad (38)$$

and

$$\Gamma^*_b(x,t;\xi,\tau) = \frac{1}{2} \left( \Gamma^*_b(x,t;\xi,\tau) - \Gamma^*_b(x,s;\xi,\tau) - \Gamma^*_b(x,t;\xi,\tau) + \Gamma^*_b(x,t;\xi,\tau) \right) \quad (39)$$

for any $x, \xi \in D$ and $\tau < t$.

**Proof.** Since $b(x,t) = \overline{b(x,t)}$, so by definition, it is easy to see that $\Gamma_b(x,t;\xi,\tau), \Gamma_b(x,t;\xi,\tau), \Gamma_b(x,t;\xi,\tau)$ and $\Gamma_b(x,t;\xi,\tau)$ are solutions to the parabolic equation

$$\left( \frac{\partial}{\partial t} - L_{b(x,t)} \right) f(x,t) = 0$$

in $\mathbb{R}^d$ for $t > \tau$. It follows that

$$\Gamma(x,t;\xi,\tau) = \frac{1}{2} \left( \Gamma_b(x,t;\xi,\tau) - \Gamma_b(x,t;\xi,\tau) - \Gamma_b(x,t;\xi,\tau) + \Gamma_b(x,t;\xi,\tau) \right)$$

solves the boundary problem and $f(\cdot,t)|_{x=x} = 0$, and

$$\lim_{t \uparrow \tau} \int_{D} \phi(\xi) \Gamma(x,t;\xi,\tau) d\xi = \phi(x).$$

Therefore $\Gamma = \Gamma_{D,b}$. \qed

It is known that $\Gamma_b(x,t;\xi,\tau) = \Gamma_b^*(\xi,\tau;x,t)$ where $t > \tau$ and $\tau, \xi \in \mathbb{R}^d$, so that as a consequence, we have $\Gamma_{D,b}(x,t;\xi,\tau) = \Gamma_{D,b}^*(\xi,\tau;x,t)$.

Let $p_b(x,t;\xi,y)$ denote the transition probability density function of the diffusion process $X$ with its infinitesimal generator $L_b$. $X$ may be constructed as a (weak) solution to the stochastic differential equation:

$$dX = b(X,t)dt + \sqrt{2}dB, \quad X_\tau = \xi$$

where $B$ is a Brownian motion on some probability space. Then

$$p_b(\tau,\xi,y) dx = P[X_\tau = dx|X_\tau = \xi]$$

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for $\tau < t$. Since $L^*_b = L_{-b}$ as $b$ is divergence-free, so that
\[ p_b(\tau, \xi, t, x) = \Gamma_{-b}(\xi, \tau; x, t) = \Gamma_{-b}(x, t; \xi, \tau), \tag{40} \]
by Lemma 4,
\[ \Gamma_{D,b}(x, t; \xi, \tau) = p_{D,-b}(\tau, \xi, t, x) \tag{41} \]
for $t \geq \tau$ and $x, \xi \in D$, where $p_{D,-b}(\tau, \xi, t, x)$ is the transition probability density function of the $L_{-b}$-diffusion killed on leaving $D$. On the other hand, since $b(x, t) = 0$ for $x \in \partial D$ and $b(\bar{x}, \tau) = b(x, t)$, so we must have
\[ p_b(\tau, \xi, t, x) = p_b(\tau, \bar{\xi}, t, \bar{x}) \quad \text{for any } \xi, x \in \mathbb{R}^d \text{ and } t > \tau. \tag{42} \]
By combining (39), (40) and (42) together, we deduce the following lemma.

**Lemma 5.** Under the same assumptions on $b(x, t)$ as in Lemma 4. It holds that
\[ \Gamma_{D,b}(x, t; \xi, \tau) = p_{D,-b}(\tau, \xi, t, x) = p_{-b}(\tau, \xi, t, x) - p_{-b}(\tau, \bar{\xi}, t, \bar{x}) \tag{43} \]
for $\tau < t$ and $\xi, x \in D$.

As a consequence we have the following lemma which provides another ingredient needed in formulating the random vortex system.

**Lemma 6.** Under the same assumptions on $b(x, t)$ as in Lemma 4. If $w(x, t)$ is the solution to the initial value problem
\[ \left( \frac{\partial}{\partial t} - L_{b(x,t)} \right) w(x, t) = g(x, t) \text{ in } D \times [\tau, \infty) \tag{44} \]
for $t > \tau$, with initial data that $w(x, \tau) = \varphi(x)$, and satisfies the Dirichlet boundary condition that $w(x, t)|_{x \in \partial D} = 0$ for $t > 0$. Then
\[ w(x, t) = \int_D (p_{-b}(\tau, \xi, t, x) - p_{-b}(\tau, \bar{\xi}, t, x)) \varphi(\xi) d\xi + \int_{\tau}^t \int_D (p_{-b}(s, \xi, t, x) - p_{-b}(s, \bar{\xi}, t, x)) g(\xi, s) d\xi ds. \tag{45} \]

**Proof.** By definition we have
\[ w(x, t) = \int_D \Gamma_{D,b}(x, t; \xi, \tau) \varphi(\xi) d\xi + \int_{\tau}^t \int_D \Gamma_{D,b}(x, t; \xi, s) g(\xi, s) ds d\xi. \tag{46} \]
Thanks to the duality (41) this equality may be rewritten as
\[ w(x, t) = \int_D p_{D,-b}(\tau, \xi, t, x) \varphi(\xi) d\xi + \int_{\tau}^t \int_D p_{D,-b}(s, \xi, t, x) g(\xi, s) ds d\xi + \int_{\tau}^t \int_D p_{D,-b}(s, \bar{\xi}, t, \bar{x}) g(\xi, s) ds d\xi \tag{47} \]
for $t > \tau$ for $x \in D$, and the representation follows from (42) and (43) immediately. 

### 3.3 Two dimensional wall-bounded flows

From now on we only consider the two dimensional case, so that $D = \{(x_1, x_2) : x_2 < 0\}$ and the vorticity $\omega = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2}$ evolves according to the vorticity transport equation (13) in $D$, which is a solution to the linear parabolic equation (13) if $u$ is considered given. While unlike the whole space case, unfortunately, one is unable to apply the representation (45) to $\omega$ as we have pointed already, $\omega$ satisfies a non-homogeneous boundary condition containing unknown data. Therefore in order to apply Lemma 6, it remains to deal with the boundary vorticity, which is given in (14).

To simplify our arguments, we make two technical assumptions:
We assume that the initial velocity \( u_0(x) = u(x, 0) \) is smooth up to the boundary, and of course we assume that \( u_0(x_1, 0) = 0 \) for all \( x_1 \). We may assume that \( u_0(x_1, x_2) = 0 \) if \( x_1 \not\in (a, b) \) where \( a < b \) are two numbers.

The external force \( F = (F^1, F^2) \) is smooth, has a compact support in \( \bar{D} \), and vanishes at the boundary \( \partial D \).

These assumptions can be weaken greatly, and they are imposed for simplifying our derivation of the random vortex dynamics below.

The velocity \( u(x, t) \) satisfies the no slip condition, so that it can be extended to the whole space \( \mathbb{R}^2 \), still denoted by \( u(x, t) \), such that \( u(x, t) = \bar{u}(\bar{x}, \bar{t}) \) for every \( x \in \mathbb{R}^2 \) and \( t \geq 0 \). \( u(x, t) \) is divergence-free in \( \mathbb{R}^2 \) in distribution sense. Let \( p_u(\tau, \xi, t, x) \) be the probability transition function of Taylor’s diffusion, i.e. a diffusion with its infinitesimal generator \( \nu \Delta + u \cdot \nabla \).

For simplicity denote the boundary vorticity \( \omega_{|\partial D} = -\frac{\partial u^1}{\partial x_2} \) by \( \theta \). Then \( \theta \) is a function on \( \partial D : x_2 = 0 \), so \( \theta \) depends only on \( x_1 \). Under our assumptions

\[
\theta(x_1, t) = -\frac{\partial u^1}{\partial x_2}(x_1, 0, t) \quad \text{for } x_1 \in \mathbb{R} \text{ and } t \geq 0.
\]

We introduce a family of perturbations of the vorticity \( \omega \) modified near the boundary by using a cut-off function, so that the modified vorticity vanishes along the boundary \( \partial D \). More precisely \( \theta \) is extended to the interior of \( D \) as the following: for any given \( \varepsilon > 0 \) set

\[
\sigma_\varepsilon(x_1, x_2, t) = \theta(x_1, t)\phi(-x_2/\varepsilon),
\]

where \( \phi : [0, \infty) \rightarrow [0, 1] \) is a proper cut-off function to be chosen later, and \( \phi \) is smooth, such that \( \phi(r) = 1 \) for \( r \in [0, 1/3] \) and \( \phi(r) = 0 \) for \( r \geq 2/3 \). Let \( W^\varepsilon = \omega - \sigma_\varepsilon \). Then it is easy to verify that

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla - \nu \Delta \right) W^\varepsilon = g_\varepsilon \quad \text{in } D, \quad \text{and } W^\varepsilon_{|\partial D} = 0,
\]

where

\[
g_\varepsilon(x, t) = G(x, t) + \frac{\nu}{\varepsilon^2} \phi''(-x_2/\varepsilon)\theta(x_1, t) + \frac{1}{\varepsilon} \phi'(x_2/\varepsilon)u^2(x, t)\theta(x_1, t)
\]

\[
+ \phi(-x_2/\varepsilon) \left( \nu \frac{\partial^2 \theta}{\partial x_1^2}(x_1, t) - \frac{\partial \theta}{\partial t}(x_1, t) \right) - \phi(-x_2/\varepsilon)u^1(x, t)\frac{\partial \theta}{\partial x_1}(x_1, t)
\]

for any \( x = (x_1, x_2), x_2 \geq 0 \). The initial data for \( W^\varepsilon \) is identified with the following

\[
W^\varepsilon_0(x) = \omega_0(x_1, x_2) - \omega_0(x_1, 0)\phi(-x_2/\varepsilon) \quad \text{for } x \in D.
\]

Under our technical assumptions, \( K(x, y)\sigma_\varepsilon(y, t) \) for every \( x \) and \( t \), as a function of \( y \), is integrable on \( D \). Therefore, according to the integral representation (cf. Lemma 6) applying to (49) we then obtain that

\[
\omega(y, t) = \int_D \left( p_u(0, \xi, t, y) - p_u(0, \bar{\xi}, t, y) \right) W^\varepsilon_0(\xi) d\xi + \sigma_\varepsilon(y, t)
\]

\[
+ \int_0^t \int_D \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) g_\varepsilon(\xi, s) d\xi ds
\]

for \( y \in D \) and \( t > 0 \). On the other hand \( u(x, t) \) can be recovered via the Biot-Savart law (cf. Lemma 2) to obtain

\[
u(x, t) = \int_D \int_D K(x, y) \left( p_u(0, \xi, t, y) - p_u(0, \bar{\xi}, t, y) \right) W^\varepsilon_0(\xi) d\xi dy + \int_D K(x, y)\sigma_\varepsilon(y, t) dy
\]
and a cut-off function \( \phi \) is given. Then
\[
K(x, y) (p_u(s, \xi, t, y) - p_u(s, \xi', t, y)) \geq C
\]
for \( i = 1, 2, x \in D \) and \( t > 0 \), and \( K = (K^1, K^2) \) is given by (28, 29).

The next step is to express the integrals involving the transition probability density \( p_u(0, \xi, t, y) \) in terms of the distribution of the diffusion with infinitesimal generator \( \nu \Delta + u \cdot \nabla \). Recall that, if \( (X_t^{\xi, x}) \) is the diffusion started from \( \xi \in \mathbb{R}^2 \) at instance \( s \geq 0 \), that is, a solution to the stochastic differential equation
\[
dX_t^{\xi, x} = u(X_t^{\xi, x}, t) dt + \sqrt{2} \nu dB_t, \quad X_0^{\xi, x} = \xi
\]
for \( t \geq s \) (of course we may define \( X_t^{\xi, x} = \xi \) for all \( t \leq s \)), where \( B \) is a two dimensional Brownian motion, then
\[
\int_D f(y) p_u(s, \xi, t, y) dy = \mathbb{E} \left[ 1_D(X_s^{\xi, x}) f(X_s^{\xi, x}) \right].
\]
By using this elementary fact and Fubini’s theorem we may rewrite (52) and obtain the following general representation theorem.

**Theorem 7.** Let \( u(x, t) \) be a solution to Navier-Stokes equations (8, 9) in \( D = \{ x : x_2 > 0 \} \). Let \( \epsilon > 0 \) and a cut-off function \( \phi \) be given. Then
\[
u \Delta + u \cdot \nabla
\]
and \( \partial \nabla \) form the closed random vortex dynamics:
\[
\omega_0(x) = \nabla \wedge u_0(x), \quad W_0(x) = \omega_0(x_1, x_2) + \frac{\partial u_0}{\partial x_2}(x_1, 0)
\]
$u_0(x) = u(x, 0)$ are the initial data, and

$$g(x, t) = G(x, t) + \nu \frac{\partial^2 \theta}{\partial x_1^2}(x_1, t) - \frac{\partial \theta}{\partial t}(x_1, t) - u_1(x, t) \frac{\partial \theta}{\partial x_1}(x_1, t).$$  \hspace{1cm} (59)

**Proof.** The proof of this theorem is based on the representation (52) and (63) by sending $\varepsilon \uparrow \infty$. Let $\varepsilon \uparrow \infty$ to obtain (with $x = (x_1, x_2)$)

$$\lim_{\varepsilon \uparrow \infty} W_0^\varepsilon(x) = W_0(x) \equiv \omega_0(x) + \frac{\partial u_0^1}{\partial x_2}(x_1, 0),$$  \hspace{1cm} (60)

$$\lim_{\varepsilon \uparrow \infty} \sigma_\varepsilon(x_1, x_2, t) = \theta(x_1, t),$$  \hspace{1cm} (61)

and

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(x, t) = g(x, t).$$  \hspace{1cm} (62)

Hence the conclusion follows immediately.

The following stochastic representation provides another approach which avoids the discussion of the dynamics of the boundary vorticity $\theta$ and therefore it has some advantage when the viscosity $\nu > 0$ is small.

**Theorem 9.** Let $u(x, t)$ be a solution to Navier-Stokes equations (8, 9) in $D = \{x : x_2 < 0\}$, and $G = \nabla \wedge F$. Assume that both $u(x, t)$ and $G(x, t)$ have twice continuous derivatives on $\bar{D}$, and assume that $u$ and $G$ and their derivatives are integrable on $D$. Then $u(x, t)$ and the Taylor diffusion with its infinitesimal generator $\nu \Delta + u \cdot \nabla$ form the closed random vortex dynamics

$$\begin{align*}
\left\{ \begin{array}{l}
u \Delta u(x, t) = \int_D \mathbb{E} \left[ 1_D(X_t^\varepsilon, 0)K(x, X_t^\varepsilon, 0) - 1_D(X_t^\varepsilon, 0)K(x, X_t^\varepsilon, 0) \right] \omega_0(\xi)d\xi \\
+ \int_0^t \int_D \mathbb{E} \left[ 1_D(X_s^\varepsilon, 0)K(x, X_s^\varepsilon, 0) - 1_D(X_s^\varepsilon, 0)K(x, X_s^\varepsilon, 0) \right] G(\xi, s) d\xi ds \\
- 2\nu \int_0^t \int_D \frac{\partial}{\partial \xi_1} \mathbb{E} \left[ 1_D(X_s^\varepsilon, 0)K(x, X_s^\varepsilon, 0) \right] \theta(\xi_1, s) d\xi_1 ds, \quad & \text{for } x_2 < 0, \\
u u(x, t) = u(\bar{x}, t), \quad & \text{for } x_2 > 0, \\
dX_t^\varepsilon = u(X_t^\varepsilon, s) dt + \sqrt{2\nu} dB_t, \quad & \text{for } \xi \in \mathbb{R}^2,
\end{array} \right. \\
(63) u(x, t) = u(\bar{x}, t),
\right. \\
(63)
\end{align*}$$

where $\omega_0(\xi) = \nabla \wedge u_0(x)$ and $u_0(x) = u(x, 0)$ are the initial data.

**Proof.** The proof of this theorem is based on the representation (52) as well, and the stochastic representation (63) by sending $\varepsilon \downarrow 0$. Let us take the following explicit cut-off function defined by

$$\phi(r) = \begin{cases} 1 & \text{for } r \in [0, 1/3), \\ 1/2 + 54 \left( r - \frac{1}{2} \right)^3 - \frac{9}{2} \left( r - \frac{1}{2} \right) & \text{for } r \in [1/3, 2/3], \\ 0 & \text{for } r \geq 2/3 \end{cases} \hspace{1cm} (64)$$

Then $-54 \leq \phi'' \leq 54$, $-\frac{9}{2} \leq \phi' \leq 0$ on $[1/3, 2/3]$ and $\phi' = 0$ for $r \leq 1/3$ or $r \geq 2/3$. In fact

$$\phi'(r) = \begin{cases} 162 \left( r - \frac{1}{2} \right)^2 - \frac{9}{2} & \text{for } r \in [1/3, 2/3], \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (65)$$

and

$$\phi''(r) = \begin{cases} 324 \left( r - \frac{1}{2} \right) & \text{for } r \in [1/3, 2/3], \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (66)$$

The key is to show that the last term on the right-hand side of (52), that is,

$$E_\varepsilon(x, t) \equiv \int_D \int_0^t \int_D K(x, y) \left( p_u(s, \xi, t, y) - p_u(s, \xi, t, y) \right) g_\varepsilon(\xi, s) d\xi ds dy$$

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has a limit as $\varepsilon \downarrow 0$, where $g_\varepsilon$ is given by (50). To this end we consider the integral

$$Q^\varepsilon(s) = \int_D \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) g_\varepsilon(\xi, s)d\xi.$$ 

Since $\phi(-x_2/\varepsilon) \to 1_{\{0\}}(x_2)$ as $\varepsilon \downarrow 0$, so that there are no contributions towards $Q^\varepsilon(s)$ from the last two terms on the right-hand side (50). In fact the sum of the two terms

$$\phi(-x_2/\varepsilon) \left( v \frac{\partial^2 \theta}{\partial x_1^2}(x_1, t) - \frac{\partial \theta}{\partial t}(x_1, t) \right) - \phi(-x_2/\varepsilon) u^1(x, t) \frac{\partial \theta}{\partial x_1}(x_1, t)$$

tends to, as $u^1(x, t) = 0$ on $x_2 = 0$,

$$1_{\{0\}}(x_2) \left( v \frac{\partial^2 \theta}{\partial x_1^2}(x_1, t) - \frac{\partial \theta}{\partial t}(x_1, t) \right)$$

as $\varepsilon \downarrow 0$, which equals to zero almost surely on $D$. Let us handle the singular terms involving $1/\varepsilon$ in $Q^\varepsilon(s)$. There are two integrals we need to consider:

$$Q^\varepsilon_1(s) = \int_D \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) \frac{1}{\varepsilon} \theta(-x_2/\varepsilon) u^2(\xi, s) \theta(\xi_1, s)d\xi$$

and

$$Q^\varepsilon_2(s) = \int_D \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) \frac{v}{\varepsilon^2} \theta''(-x_2/\varepsilon) \theta(\xi_1, s)d\xi.$$ 

We want to find their limits as $\varepsilon \downarrow 0$. Suppose $\beta$ is a smooth function on $\mathbb{R}^2$ with a compact support, then

$$\int_{\mathbb{R}^2} \beta(x) \theta(-x_2/\varepsilon)dx = \varepsilon \int_{-\infty}^{2/3} \beta(x_1, -\varepsilon x_2) \theta(x_2)dx_2dx_1$$

$$= \varepsilon^2 \int_{\mathbb{R}}^{2/3} \frac{\partial \beta}{\partial x_2}(x_1, -\varepsilon x_2) \phi(x_2)dx_2dx_1$$

$$+ \varepsilon \int_{\mathbb{R}} \left[ -\beta(x_1, -\varepsilon/3) \right] dx_1$$

so that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \beta(x) \theta(-x_2/\varepsilon)dx = -\int_{\mathbb{R}} \beta(x_1, 0)dx_1.$$ 

In other words,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \phi'(-x_2/\varepsilon)dx_1dx_2 = -dx_1 \delta_0(dx_2) \quad \text{as } \varepsilon \to 0.$$ 

Hence

$$\lim_{\varepsilon \downarrow 0} Q^\varepsilon_1(s) = \lim_{\varepsilon \downarrow 0} \int_D \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) \frac{1}{\varepsilon} \phi''(-x_2/\varepsilon) u^2(\xi, t) \theta(\xi_1, t)d\xi$$

$$= -\int_{\mathbb{R}} \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) u^2(\xi, t) \theta(\xi_1, t)d\xi = 0.$$ 

Similarly, since

$$\int_{\mathbb{R}^2} \beta(x) \phi''(-x_2/\varepsilon)dx = \varepsilon \int_{\mathbb{R}}^{2/3} \beta(x_1, -\varepsilon x_2) \phi''(x_2)dx_2dx_1$$

$$= \varepsilon^2 \int_{\mathbb{R}}^{2/3} \frac{\partial \beta}{\partial x_2}(x_1, -\varepsilon x_2) \phi'(x_2)dx_2dx_1$$

$$+ \varepsilon \int_{\mathbb{R}} \left[ -\beta(x_1, -\varepsilon/3) \right] dx_1$$
\[ = \varepsilon^3 \int_{\mathbb{R}} \int_{1/3}^{2/3} \frac{\partial^2 \beta}{\partial x^2} (x_1, -\varepsilon x_2) \phi(x_2) dx_2 dx_1 \]

so that

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \beta(x) \phi''(-x_2/\varepsilon) dx = -\int_{\mathbb{R}} \frac{\partial \beta}{\partial x_2}(x_1, 0) dx_1. \tag{70}
\]

Therefore

\[
\lim_{\varepsilon \downarrow 0} Q^\varepsilon(s) = \lim_{\varepsilon \downarrow 0} \int_D \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) \frac{\nu}{\varepsilon^2} \phi''(-\xi_2/\varepsilon) \theta(\xi_1, s) d\xi
\]

\[
- \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\partial}{\partial \xi_2} \left| \left. p_u(s, (\xi_1, \xi_2), t, y) \theta(\xi_1, s) \right|_{\xi_2 = 0} d\xi_1. \tag{71}
\]

Limits (69) and (71) together imply that

\[
\lim_{\varepsilon \downarrow 0} Q^\varepsilon(s) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) g_\varepsilon(\xi, s) d\xi
\]

\[
- \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\partial}{\partial \xi_2} \left| \left. p_u(s, (\xi_1, \xi_2), t, y) \theta(\xi_1, s) \right|_{\xi_2 = 0} d\xi_1,
\]

and it in turn yields that

\[
\lim_{\varepsilon \downarrow 0} E_\varepsilon(x, t) = \int_{\mathbb{R}} \int_{0}^{t} \int_{D} K(x, y) \left( p_u(s, \xi, t, y) - p_u(s, \bar{\xi}, t, y) \right) G(\xi, s) d\xi ds dy
\]

\[
- \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\partial}{\partial \xi_2} \left| \left. p_u(s, (\xi_1, \xi_2), t, y) \theta(\xi_1, s) \right|_{\xi_2 = 0} d\xi_1 ds dy
\]

\[
- \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\partial}{\partial \xi_2} \left| \left. E_\varepsilon(x, t) \right|_{\xi_2 = 0} d\xi_1 ds dy
\]

We then deal with other two terms in (52). Since

\[
W_0^\varepsilon(x) = \omega_0(x) + \phi(-x_2/\varepsilon) \frac{\partial u_0}{\partial x_2}(x_1, 0) \tag{72}
\]

for \( x = (x_1, x_2) \) with \( x_2 \leq 0 \), it follows easily that

\[
\lim_{\varepsilon \downarrow 0} W_0^\varepsilon(x) = \omega_0(x_1, x_2) + 1_{(0)}(x_2) \frac{\partial u_0}{\partial x_2}(x_1, 0) \equiv W_0(x), \tag{73}
\]

which allows to replace \( W_0^\varepsilon \) by \( \omega_0 \) in the first term on the right-hand side of (52).

In the next step we show the limit of the second term on the right-hand side (52) vanishes as \( \varepsilon \downarrow 0 \). Hence we need to handle the integrals

\[
I_i^\varepsilon(x) = \int_D K^\varepsilon(x, y) \sigma_\varepsilon(y, t) dy
\]

where \( i = 1, 2 \). By the definition of the cut-off function \( \phi \),

\[
\int_{D} K(x, y) \sigma_\varepsilon(y, t) dy = \varepsilon \int_{-\infty}^{\infty} \left( \theta(y_1, t) \int_{0}^{2/3} K(x, (y_1, -\varepsilon y_2) \phi(y_2) dy_2 \right) dy_1.
\]
Since the integral against $y_2$ can be calculated as the following:

$$J_{\epsilon}^1(x, y_1) = \int_0^{2/3} K^1(x, (y_1, -\epsilon y_2)) \phi(y_2) dy_2$$

$$= \frac{1}{2\pi} \int_0^{2/3} \left( \frac{-\epsilon y_2 - x_2}{(y_1 - x_1)^2 + (x_2 + \epsilon y_2)^2} - \frac{-\epsilon y_2 + x_2}{(y_1 - x_1)^2 + (x_2 - \epsilon y_2)^2} \right) \phi(y_2) dy_2$$

$$\to \frac{1}{\pi} \frac{x_2}{(y_1 - x_1)^2 + (x_2)^2} \int_0^{2/3} \phi(y_2) dy_2$$

as $\epsilon \downarrow 0$, and similarly

$$J_{\epsilon}^2(x, y_1) = \int_0^{2/3} K^2(x, (y_1, -\epsilon y_2)) \phi(y_2) dy_2$$

$$= \frac{1}{2\pi} \int_0^{2/3} \left( \frac{1}{(y_1 - x_1)^2 + (x_2 - \epsilon y_2)^2} - \frac{1}{(y_1 - x_1)^2 + (x_2 + \epsilon y_2)^2} \right) \phi(y_2) dy_2$$

$$\to 0$$

as $\epsilon \downarrow 0$. Therefore

$$\lim_{\epsilon \downarrow 0} \int_D K(x, y) \sigma_\epsilon(y, t) dy = 0$$

which allows us to drop the second term in (52) as $\epsilon \downarrow 0$. The conclusion of the theorem now follows immediately. 

\[ \square \]

### 4 Monte-Carlo simulations

We retain the assumptions in Section 3.3 and we assume that the technical conditions in Theorem 9 are satisfied. To implement Monte-Carlo simulations via the stochastic formulation, Theorem 7, we appeal to the ideas from random vortex method. That is, either dropping the mathematical expectations by running independent Brownian motions, or appealing to the strong law of large numbers so that the expectations are replaced by running a number of independent copies of the Taylor diffusion. Let us describe the methods in more details.

Recall that we have established the following representation

$$u(x, t) = \int_D \mathbb{E} \left[ 1_D(X_0^\xi)K(x, X_0^\xi) - 1_D(X_0^\xi)K(x, X_0^\xi) \right] \omega_0(\xi) d\xi$$

$$+ \int_D \int_0^t \mathbb{E} \left[ 1_D(X_s^\xi)K(x, X_s^\xi) - 1_D(X_s^\xi)K(x, X_s^\xi) \right] G(\xi, s) d\xi_1 ds$$

$$-2\nu \int_0^t \int_D \frac{\partial}{\partial \xi} \mathbb{E} \left[ 1_D(X_s^\xi)K(x, X_s^\xi) \right] \theta(\xi_1, s) d\xi_1 ds,$$

for $x_2 < 0$,

$$u(x, t) = u(\bar{x}, t),$$

for $x_2 > 0$,

$$X_0^\xi = \xi + \int_0^t u(X_0^\xi, r) dr + \sqrt{2\nu}(B_t - B_0),$$

for $\xi \in \mathbb{R}^2$ and for $s \geq 0$, for $\xi \in \mathbb{R}^2$,

where $\omega_0(\xi) = \nu \wedge u_0(x)$ and $B$ is a two dimensional Brownian motion. In numerical schemes described below the boundary vorticity $\theta$ has to be updated over the time during iterations, rather than through modelling or boundary layer equations. However, for small viscosity $\nu$, the term involving the boundary vorticity may be dropped, and for the case where there is no external force, then the previous system can be approximated by the following simpler random vortex dynamics

$$\tilde{u}(x, t) = \int_D \mathbb{E} \left[ 1_D(X_t^\xi)K(x, X_t^\xi) - 1_D(X_t^\xi)K(x, X_t^\xi) \right] \omega_0(\xi) d\xi,$$

for $x_2 < 0$,

$$\tilde{u}(x, t) = \tilde{u}(\bar{x}, t),$$

for $x_2 > 0$,

$$X_t^\xi = \xi + \int_0^t u(X_0^\xi, r) dr + \sqrt{2\nu}B_t,$$

for $\xi \in \mathbb{R}^2$, for $\xi \in \mathbb{R}^2$. 

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The use of this approximation will reduce the computing cost, and therefore more computing hours may be saved for performing simulations of boundary turbulent flows where the viscosity is small while demands for finer scales.

Let us describe the numerical schemes we are going to perform the numerical experiments. We divide the schemes into two steps. During the first step we set up the discretization procedure for dealing with (finite dimensional) integrals in time and in the space variables. The discretization of this type appears in any numerical methods, but a bit care is needed due to the appearance of boundary layer phenomena for wall-bounded flows. According to Prandtl [24], there is a thin layer near the boundary within which the main stream velocity decreases to zero sharply, hence there is substantial stress at the wall, which in turn generates significant boundary vorticity. Turbulence may be generated near the solid wall if the Reynolds number is large. The boundary layer thickness \( \delta \) for the \( x_2 \)-coordinate has to be far smaller than the boundary layer thickness \( \delta \sim \sqrt{1/Re} \), which leads to the first constraint:

\[
h_2 \ll L \sqrt{\frac{1}{Re}}. \tag{76}
\]

The mesh size \( h_1 \) for \( x_1 \)-coordinate should be comparable to \( h_2 \) but not need to be larger than \( h_2 \). For the region outside the boundary layer, we can use reasonable size of the mesh \( h_0 \), and in general we choose \( h_0 \geq h_1 \geq h_2 \). We may choose integers \( N_1, N_2 \) and \( N_0 \) according to the following constraints:

\[
N_1 h_1 \sim L, \quad N_2 h_2 \geq \delta \quad \text{and} \quad N_0 h_0 \sim L. \tag{77}
\]

We calculate the values at the lattice points

\[
x^{i_1,i_2} = \begin{cases} (i_1 h_1, i_2 h_2) & \text{if } |i_1| \leq N_1 \text{ and } |i_2| \leq N_2; \\ (i_1 h_0, i_2 h_0) & \text{if } |i_1| \leq N_0 \text{ and } N_2 < |i_2| \leq N_0 + N_2. \end{cases} \tag{78}
\]

That is, \( x^{i_1,i_2} = (x_1^{i_1,i_2}, x_2^{i_1,i_2}) \) where

\[
\begin{aligned}
x_1^{i_1,i_2} &= i_1 h_1 \quad \text{and} \quad x_2^{i_1,i_2} = i_2 h_2 \quad \text{if } |i_1| \leq N_1 \text{ and } |i_2| \leq N_2; \\
x_1^{i_1,i_2} &= i_1 h_0 \quad \text{and} \quad x_2^{i_1,i_2} = i_2 h_0 \quad \text{if } |i_1| \leq N_0 \text{ and } N_2 < |i_2| \leq N_0 + N_2. \quad \tag{79}
\end{aligned}
\]

The total number of the lattice points we require is given by

\[
n = 2(N_1 + 1)(N_2 + 1) + 2(N_0 + 1)^2
\]

which largely determine the computational cost. Thus far we have described the discretization of space variables.

For time \( t \), we can use a unified scheme, at each step we use time duration \( h > 0 \), to be chosen properly.

The Taylor diffusion (53) has to be issued from the lattice points in the numerical scheme at starting time \( t_0 = 0, t_1 = h \), and so on on \( t_k = kh \) for \( k = 0, 1, 2, \ldots \). A simple scheme, but it is not claimed to be the best one, for the discretization of the SDE can be set up as the following:

\[
X_{t_{l+1}}^{i_1,i_2} = x^{i_1,i_2} + h \bar{u}(x^{i_1,i_2}, 0) + \sqrt{2V}(B_{t_{l+1}} - B_{t_l})
\]

(for \( l = 0, 1, 2, \ldots \)), then

\[
X_{t_{k+1}}^{i_1,i_2} = X_{t_k}^{i_1,i_2} + h \bar{u}(X_{t_k}^{i_1,i_2}, t_k) + \sqrt{2V}(B_{t_{k+1}} - B_{t_k})
\]
for $k = l, l + 1, \cdots$, where

$$\tilde{u}(x, t_0) = u_0(x),$$

$$\tilde{u}(x, t_{k+1}) = \sum_{i_1, i_2} A_{i_1, i_2} \omega_{i_1, i_2} \mathbb{E} \left[ K_D(x, X_{k}^{i_1, i_2, 0}) - K_D(x, X_{k}^{i_1, -i_2, 0}) \right]$$

$$+ \sum_{i_1, i_2} A_{i_1, i_2} \sum_{l=0}^{k} h G_{i_1, i_2, l} \mathbb{E} \left[ K_D(x, X_{k+l}^{i_1, i_2 l}) - K_D(x, X_{k}^{i_1, -i_2, l}) \right]$$

$$+ 2v \sum_{l=1}^{k} \frac{h}{h_2} \sum_{i_1, i_2} h \left( \mathbb{E} \left[ K_D(x, X_{k+l}^{i_1, i_2 l}) - \mathbb{E} \left[ K_D(x, X_{k}^{i_1, 0, l}) \right] \right] \right) \theta_{i_1, l},$$

(80)

$$\theta_{i_1, l} = \frac{\tilde{u}^1(i_1 h_1, -h_2, t_l)}{h_2}, \quad \text{for } l = 0, \ldots, k,$$

for $x_2 < 0$, and

$$(\tilde{u}^1(x, t), \tilde{u}^2(x, t)) = (\tilde{u}^1(x, t), -\tilde{u}^2(x, t)) \quad \text{for } x_2 > 0.$$ 

which is solved for $i_1, i_2$ within the regions assigned, where

$$A_{i_1, i_2} = \begin{cases} h_1 h_2 & \text{if } |i_1| \leq N_1 \text{ and } |i_2| \leq N_2; \\ h_0 h_0 & \text{if } |i_1| \leq N_1 \text{ and } N_2 < |i_2| \leq N_0 + N_2, \end{cases}$$

(81)

and

$$\omega_{i_1, i_2} = \omega_0(x^{i_1, i_2}), \quad G_{i_1, i_2, l} = G(x^{i_1, i_2}, t_l)$$

and $\tilde{u}(x, 0) = u_0(x)$. Note that $\tilde{u} = (\tilde{u}^1, \tilde{u}^2)$. Also here for simplicity, we have introduce the notation that

$$K_D(x, y) = \begin{cases} K(x, y) & \text{if } y \in D, \\ 0 & \text{if } y \notin D. \end{cases}$$

(83)

In the second step, one has to handle the mathematical expectations, which is the core of the Monte-Carlo schemes. Slightly different approaches lead to the following schemes.

### 4.1 Numerical scheme 1

We are now in a position to formulate our first Monte-Carlo scheme based on the simple random vortex dynamic (75). In this scheme, we drop the expectation by running independent (two dimensional) Brownian motions. More precisely we run the following stochastic differential equations:

$$X_{l+1}^{i_1, -i_2 l} = X_{l}^{i_1, i_2 l} + h \tilde{u}(X_{l}^{i_1, i_2 l}, 0) + \sqrt{2v} (B_{l+1}^{i_1, i_2 l} - B_{l}^{i_1, i_2 l})$$

(for $l = 0, 1, 2, \cdots$), then

$$X_{k+1}^{i_1, i_2 l} = X_{k}^{i_1, i_2 l} + h \tilde{u}(X_{k}^{i_1, i_2 l}, t_k) + \sqrt{2v} (B_{k+1}^{i_1, i_2 l} - B_{k}^{i_1, i_2 l})$$

for $k = l, l + 1, \cdots$, and

$$\tilde{u}(x, 0) = u_0(x),$$

$$\tilde{u}(x, t_{k+1}) = \sum_{i_1, i_2} A_{i_1, i_2} \omega_{i_1, i_2} \left[ K_D(x, X_{k}^{i_1, i_2 l}) - K_D(x, X_{k}^{i_1, -i_2, l}) \right]$$

$$+ \sum_{i_1, i_2} A_{i_1, i_2} \sum_{l=0}^{k} h G_{i_1, i_2, l} \left[ K_D(x, X_{k+l}^{i_1, i_2 l}) - K_D(x, X_{k}^{i_1, -i_2, l}) \right]$$

$$+ 2v \sum_{l=1}^{k} \frac{h}{h_2} \sum_{i_1, i_2} h \left( \mathbb{E} \left[ K_D(x, X_{k+l}^{i_1, i_2 l}) - \mathbb{E} \left[ K_D(x, X_{k}^{i_1, 0, l}) \right] \right] \right) \theta_{i_1, l},$$

(84)
where $A_{i_1,i_2}, \omega_{i_1,i_2}$ are given (81) and (82), and $B^{m,i}$ (where $(i_1,i_2)$ runs through the assigned indices) is a family of independent two dimensional Brownian motions.

The convergence of this scheme of course cannot be taken as granted which itself is an important mathematical problem, for the case without surface boundary, the convergence has been proved in [1], [18] and [9]. We will investigate this problem for surface boundary case in a separate work.

4.2 Numerical scheme 2

In this scheme we appeal to the strong law of large numbers, that is we replace the expectation by its average. Therefore we solve the following stochastic differential equations:

$$X_{t_{k+1}}^{i_1,i_2;i_1;i_2} = X_{t_k}^{i_1,i_2;i_1;i_2} + h\tilde{u}(x_{t_k}^{i_1,i_2;i_1;i_2}, 0) + \sqrt{2}V(B_{t_{k+1}}^m - B_{t_k}^m)$$

(for $l = 0, 1, 2, \cdots$), then

$$X_{t_{k+1}}^{m;i_1;i_2} = X_{t_k}^{m;i_1;i_2} + h\tilde{u}(x_{t_k}^{m;i_1;i_2}, t_k) + \sqrt{2}V(B_{t_{k+1}}^m - B_{t_k}^m)$$

for $k = l, l+1, \cdots, m = 1, \ldots, N$, and

$$\tilde{u}(x, 0) = u_0(x),$$

$$\begin{align*}
\tilde{u}(x, t_{k+1}) &= \sum_{i_1,i_2} A_{i_1,i_2} \omega_{i_1,i_2} \sum_{l=0}^{N-1} \frac{1}{N} \sum_{m=1}^{N} \left[ K_D(x, X_{t_k}^{m;i_1;i_2,0}) - K_D(x, X_{t_k}^{m;i_1, -i_2,0}) \right] \\
&+ \sum_{i_1,i_2} A_{i_1,i_2} \sum_{l=0}^{k} hG_{i_1,i_2} \sum_{l=0}^{N-1} \frac{1}{N} \sum_{m=1}^{N} \left[ K_D(x, X_{t_k}^{m;i_1;i_2,l}) - K_D(x, X_{t_k}^{m;i_1, -i_2,l}) \right] \\
&+ 2\sqrt{2}h_1 \sum_{l=0}^{k} \frac{1}{h_2} \sum_{l=0}^{N-1} \frac{1}{N} \sum_{m=1}^{N} \left[ K_D(x, X_{t_k}^{m;i_1, -h_2;l}) - K_D(x, X_{t_k}^{m;i_1, 0;l}) \right] \theta_{l;i},
\end{align*}$$

(85)

The data $A_{i_1,i_2}, \omega_{i_1,i_2}$ and so on are the same as in the previous scheme.

4.3 Numerical experiment

Since the aim of the paper is to report a theoretical framework for performing Monte-Carlo simulations for incompressible fluid flows passing over a solid wall, to demonstrate the usefulness of the method, we report a simple numerical experiment for the fluid flow with small viscosity $v > 0$, for which the updating of the boundary stress $\theta$ seems not necessary, and therefore less demanding in computing power. We hope to improve the numerical experiment results in future work.

Recall the Reynolds number is given by $Re = \frac{U_0 L}{v}$, where $U_0$ the main stream velocity and $L$ is a typical size, which is determined by the initial velocity in our experiments. For simplicity, in our experiment, we set

$$v = 0.15.$$ 

(86)

The typical length scale

$$L = 2\pi.$$ 

(87)

This gives the initial main stream velocity

$$U_0 = \frac{V}{L}Re = 0.02388535 \times Re.$$ 

(88)

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The initial velocity in our experiments is chosen as the following:

\[ u_0(x_1, x_2) = (-U_0 \sin x_2, 0) \]

so that the initial vorticity

\[ \omega_0(x_1, x_2) = U_0 \cos x_2. \]  

(89)

In the numerical experiment, we choose \( \text{Re} = 2500 \). Then \( U_0 = \alpha = 59.7 \). Choose

\[ h_2 \ll \delta = L \sqrt{\frac{1}{\text{Re}}} = 0.1256. \]

Let \( h_1 = 0.3 \), and \( h_0 = 0.4 \). The numerical result is demonstrated in the figure.

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