A REMARK ON MEASURES OF SECTIONS OF $L_p$-BALLS

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Abstract. We prove that there exists an absolute constant $C$ so that

$$\mu(K) \leq C\sqrt{p} \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) |K|^{1/n}$$

for any $p > 2$, any $n \in \mathbb{N}$, any convex body $K$ that is the unit ball of an $n$-dimensional subspace of $L_p$, and any measure $\mu$ with non-negative even continuous density in $\mathbb{R}^n$. Here $\xi^\perp$ is the central hyperplane perpendicular to a unit vector $\xi \in S^{n-1}$, and $|K|$ stands for volume.

1. Introduction

The slicing problem [Bo1, Bo2, Ba1, MP], a major open question in convex geometry, asks whether there exists a constant $C$ so that for any $n \in \mathbb{N}$ and any origin-symmetric convex body $K$ in $\mathbb{R}^n$,

$$|K|^{n-1} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,$$

where $|K|$ stands for volume of proper dimension, and $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to a unit vector $\xi$. The best-to-date result $C \leq O(n^{1/4})$ is due to Klartag [Kl], who improved an earlier estimate of Bourgain [Bo3]. The answer is affirmative for unconditional convex bodies (as initially observed by Bourgain; see also [MP, J2, BN]), intersection bodies [G, Theorem 9.4.11], zonoids, duals of bodies with bounded volume ratio [MP], the Schatten classes [KMP], $k$-intersection bodies [KPY, K6]; see [BGVV] for more details.

The case of unit balls of finite dimensional subspaces of $L_p$ is of particular interest in this note. It was shown by Ball [Ba2] that the slicing problem has an affirmative answer for the unit balls of finite dimensional subspaces of $L_p$, $1 \leq p \leq 2$. Junge [J1] extended this result to every $p \in (1, \infty)$, with the constant $C$ depending on $p$ and going to infinity when $p \to \infty$. Milman [M1] gave a different proof for

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subspaces of $L_p$, $2 < p < \infty$, with the constant $C \leq O(\sqrt{p})$. Another proof of this estimate can be found in [KPY].

A generalization of the slicing problem to arbitrary measures was considered in [K3, K4, K5, K6]. Does there exist a constant $C$ so that for every $n \in \mathbb{N}$, every origin-symmetric convex body $K$ in $\mathbb{R}^n$, and every measure $\mu$ with non-negative even continuous density $f$ in $\mathbb{R}^n$,

$$\mu(K) \leq C \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) |K|^{1/n},$$

where $\mu(K) = \int_K f$, and $\mu(K \cap \xi^\perp) = \int_{K \cap \xi^\perp} f$?

Inequality (1) was proved with an absolute constant $C$ for intersection bodies [K3] (by [K2], this includes the unit balls of subspaces of $L_p$ with $0 < p \leq 2$), unconditional bodies and duals of bodies with bounded volume ratio in [K6], for $k$-intersection bodies in [K5]. For arbitrary origin-symmetric convex bodies, (1) was proved in [K4] with $C \leq O(\sqrt{n})$. A different proof of the latter estimate was recently given in [CGL], where the symmetry condition was removed.

For the unit balls of subspaces of $L_p$, $p > 2$, (1) was proved in [K5] with $C \leq O(n^{1/2-1/p})$. In this note we improve the estimate to $C \leq O(\sqrt{p})$, extending Milman’s result [M1] to arbitrary measures in place of volume. In fact, we prove a more general inequality

$$\mu(K) \leq (C \sqrt{p})^k \max_{H \in Gr_{n-k}} \mu(K \cap H) |K|^{k/n},$$

where $1 \leq k < n$, $Gr_{n-k}$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, $K$ is the unit ball of any $n$-dimensional subspace of $L_p$, $p > 2$, $\mu$ is a measure on $\mathbb{R}^n$ with even continuous density, and $C$ is a constant independent of $p, n, k, K, \mu$.

The proof is a combination of two known results. Firstly, we use the reduction of the slicing problem for measures to computing the outer volume ratio distance from a body to the class of intersection bodies established in [K6]; see Proposition 1. Note that outer volume ratio estimates have been applied to different cases of the original slicing problem by Ball [Ba2], Junge [J1], and E.Milman [M1]. Secondly, we use an estimate for the outer volume ratio distance from the unit ball of a subspace of $L_p$, $p > 2$, to the class of origin-symmetric ellipsoids proved by E.Milman in [M1]. This estimate also follows from results of Davis, V.Milman and Tomczak-Jaegermann [DMT]. We include a concentrated version of the proof in Proposition 2.

2. Slicing inequalities

We need several definitions and facts. A closed bounded set $K$ in $\mathbb{R}^n$ is called a star body if every straight line passing through the origin
crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the Minkowski functional of $K$ defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The radial function of a star body $K$ is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n, \; x \neq 0.$$ 

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of $K$ in the direction of $x$.

We use the polar formula for volume of a star body

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \quad (3)$$

The class of intersection bodies was introduced by Lutwak [L]. Let $K, L$ be origin-symmetric star bodies in $\mathbb{R}^n$. We say that $K$ is the intersection body of $L$ if the radius of $K$ in every direction is equal to the $(n-1)$-dimensional volume of the section of $L$ by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^{|\cdot\|^{-1}}|$$

$$= \frac{1}{n-1} \int_{S^{n-1} \cap \xi^{|\cdot\|^{-1}}} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R(\|\cdot\|_L^{-n+1})(\xi),$$

where $R : C(S^{n-1}) \to C(S^{n-1})$ is the spherical Radon transform

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^{|\cdot\|^{-1}}} f(x) dx, \quad \forall f \in C(S^{n-1}).$$

All bodies $K$ that appear as intersection bodies of different star bodies form the class of intersection bodies of star bodies. A more general class of intersection bodies is defined as follows. If $\mu$ is a finite Borel measure on $S^{n-1}$, then the spherical Radon transform $R\mu$ of $\mu$ is defined as a functional on $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

A star body $K$ in $\mathbb{R}^n$ is called an intersection body if $\|\cdot\|_K^{-1} = R\mu$ for some measure $\mu$, as functionals on $C(S^{n-1})$, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

Intersection bodies played a crucial role in the solution of the Busemann-Petty problem and its generalizations; see [K1, Chapter 5].
A generalization of the concept of an intersection body was introduced by Zhang [Z] in connection with the lower dimensional Busemann-Petty problem. For $1 \leq k \leq n - 1$, the $(n-k)$-dimensional spherical Radon transform $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1} \cap H} g(x) \, dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

We say that an origin symmetric star body $K$ in $\mathbb{R}^n$ is a generalized $k$-intersection body, and write $K \in BP^n_k$, if there exists a finite Borel non-negative measure $\mu$ on $Gr_{n-k}$ so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|^k_K g(x) \, dx = \int_{Gr_{n-k}} R_{n-k}g(H) \, d\mu(H). \quad (4)$$

When $k = 1$ we get the class of intersection bodies. It was proved by Goodey and Weil [GW] for $k = 1$ and by Grinberg and Zhang [GZ, Lemma 6.1] for arbitrary $k$ (see also [M2] for a different proof) that the class $BP^n_k$ is the closure in the radial metric of $k$-radial sums of origin-symmetric ellipsoids. In particular, the classes $BP^n_k$ contain all origin-symmetric ellipsoids in $\mathbb{R}^n$ and are invariant with respect to linear transformations. Recall that the $k$-radial sum $K +_k L$ of star bodies $K$ and $L$ is defined by

$$\rho^k_{K+_k L} = \rho^k_K + \rho^k_L.$$

For a convex body $K$ in $\mathbb{R}^n$ and $1 \leq k < n$, denote by

$$o.v.r.(K, BP^n_k) = \inf \left\{ \left( \frac{|C|}{|K|} \right)^{1/n} : K \subset C, \ C \in BP^n_k \right\}$$

the outer volume ratio distance from a body $K$ to the class $BP^n_k$.

Let $B^n_2$ be the unit Euclidean ball in $\mathbb{R}^n$, let $| \cdot |_2$ be the Euclidean norm in $\mathbb{R}^n$, and let $\sigma$ be the uniform probability measure on the sphere $S^{n-1}$ in $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$, let $x_1$ be the first coordinate of $x$. We use the fact that for every $p > -1$

$$\int_{S^{n-1}} |x_1|^p d\sigma(x) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)}; \quad (5)$$

see for example [K1, Lemma 3.12], where one has to divide by $|S^{n-1}| = 2\pi^{(n-1)/2}/\Gamma\left(\frac{n}{2}\right)$, because the measure $\sigma$ on the sphere is normalized.

In [K6], the slicing problem for arbitrary measures was reduced to estimating the outer volume ratio distance from a convex body to the classes $BP^n_k$, as follows.
Proposition 1. For any \( n \in \mathbb{N} \), \( 1 \leq k < n \), any origin-symmetric star body \( K \) in \( \mathbb{R}^n \), and any measure \( \mu \) with even continuous density on \( K \),
\[
\mu(K) \leq (\text{o.v.r.}(K, \mathcal{B}_k^n))^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(K \cap H) |K|^{k/n},
\]
where \( c_{n,k} = |B_2^n|^{(n-k)/n}/|B_2^{n-k}| \in (e^{-k/2}, 1) \).

It appears that for the unit balls of subspaces of \( L_p \), \( p > 2 \) the outer volume ratio distance to the classes of intersection bodies does not depend on the dimension. As mentioned in the introduction, the following estimate was proved in [M1] and also follows from results of [DMT]. We present a short version of the proof.

Proposition 2. Let \( p > 2 \), \( n \in \mathbb{N} \), \( 1 \leq k < n \), and let \( K \) be the unit ball of an \( n \)-dimensional subspace of \( L_p \). Then
\[
\text{o.v.r.}(K, \mathcal{B}_k^n) \leq C \sqrt{p},
\]
where \( C \) is an absolute constant.

Proof: Since the classes \( \mathcal{B}_k^n \) are invariant under linear transformations, we can assume that \( K \) is in the Lewis position. By a result of Lewis in the form of [LYZ, Theorem 8.2], this means that there exists a measure \( \nu \) on the sphere so that for every \( x \in \mathbb{R}^n \)
\[
\|x\|^p_K = \int_{S^{n-1}} |(x, u)|^p d\nu(u),
\]
and
\[
|x|^2_2 = \int_{S^{n-1}} |(x, u)|^2 d\nu(u).
\]
Also, by the same result of Lewis [Le], \( K \subset n^{1/2-1/p} B_2^n \).

Let us estimate volume of \( K \) from below. By the Fubini theorem, formula (5) and Stirling’s formula, we get
\[
\int_{S^{n-1}} \|x\|^p_K d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} |(x, u)|^p d\sigma(x) d\nu(u)
\]
\[
= \int_{S^{n-1}} |x_1|^p d\sigma(x) \int_{S^{n-1}} d\nu(u) \leq \left( \frac{Cp}{n+p} \right)^{p/2} \int_{S^{n-1}} d\nu(u).
\]
Now
\[
\frac{Cp}{n+p} \left( \int_{S^{n-1}} d\nu(u) \right)^{2/p} \geq \left( \int_{S^{n-1}} \|x\|^p_K d\sigma(x) \right)^{2/p}
\]
\[
\geq \left( \int_{S^{n-1}} \|x\|^{-n} d\sigma(x) \right)^{-2/n} = \left( \frac{|K|}{|B_2^n|} \right)^{-2/n} \sim \frac{1}{n} |K|^{-2/n},
\]
because $|B^n_2| \sim n^{-1/2}$. On the other hand,
\[
1 = \int_{S^{n-1}} |x|^2 d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} (x, u)^2 d\nu(u) d\sigma(x)
\]
\[
= \int_{S^{n-1}} \int_{S^{n-1}} |x_1|^2 d\sigma(x) d\nu(u) = \frac{1}{n} \int_{S^{n-1}} d\nu(u),
\]
so
\[
\frac{Cp}{n + p} n^{2/p} \geq \frac{1}{n} |K|^{-2/n},
\]
and
\[
|K|^{1/n} \geq cn^{-1/p} \sqrt{\frac{n + p}{np}} \geq \frac{cn^{1/2 - 1/p}}{\sqrt{p}} |B^n_2|^{1/n}.
\]
Finally, since $K \subset n^{1/2 - 1/p} B^n_2$, and $B^n_2 \in \mathcal{B}^n_k$ for every $k$, we have
\[
\text{o.v.r.}(K, \mathcal{B}^n_k) \leq \left( \frac{|n^{1/2 - 1/p} B^n_2|}{|K|} \right)^{1/n} \leq C \sqrt{p},
\]
where $C$ is an absolute constant.

We now formulate the main result of this note.

**Corollary 1.** There exists a constant $C$ so that for any $p > 2$, $n \in \mathbb{N}$, $1 \leq k < n$, any convex body $K$ that is the unit ball of an $n$-dimensional subspace of $L_p$, and any measure $\mu$ with non-negative even continuous density in $\mathbb{R}^n$,
\[
\mu(K) \leq (C \sqrt{p})^k \max_{H \in \text{Gr}_{n-k}} \mu(K \cap H) |K|^{k/n}.
\]

**Proof:** Combine Proposition 1 with Proposition 2. Note that $\frac{n}{n-k} \in (1, e^k)$, and $c_{n,k} \in (e^{-k/2}, 1)$, so these constants can be incorporated in the constant $C$.

\[
\square
\]

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