PRIMES IN BEATTY SEQUENCE

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ABSTRACT. For a polynomial $g(x)$ of degree $k \geq 2$ with integer coefficient, we prove an upper bound for the least prime $p$ such that $g(p)$ is in an irrational non-homogeneous Beatty sequence $\{\lfloor n\alpha + \beta \rfloor : n = 1, 2, 3, \ldots \}$, where $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$ and we prove an asymptotic formula for the number of primes $p$ such that $g(p) = \lfloor \alpha n + \beta \rfloor$. Next we obtain an asymptotic formula for number of primes $p$ of the form $p = \lfloor \alpha n + \beta \rfloor$ which also satisfies $p \equiv f \pmod{d}$ where $\alpha, \beta$ are real numbers, $\alpha$ is irrational and $f, d$ are integers with $1 \leq f < d$ and $(f, d) = 1$.

1. INTRODUCTION

Given a real number $\alpha > 0$ and a non-negative real $\beta$, the Beatty sequence associated with $\alpha, \beta$ is defined by

$$B(\alpha, \beta) = \{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. If $\alpha$ is rational, then $B(\alpha, \beta)$ is union of residue classes, hence we always assume that $\alpha$ is irrational. An irrational number $\gamma$ is said to be of finite type $t \geq 1$ if

$$t = \sup \{\rho \in \mathbb{R} : \liminf_{n \to \infty} n^{\rho}\|n\gamma\| = 0\},$$

where $\|x\|$ is the distance of $x$ from nearest integer. In 2016, Jörn Steuding and Marc Technau [8] proved that, for every $\varepsilon > 0$ there exists a computable positive integer $l$ such that for every irrational $\alpha > 1$ the least prime $p$ in the Beatty sequence $B(\alpha, \beta)$ satisfies the inequality

$$p \leq L^{35\varepsilon - 16\varepsilon \alpha^{2(1-\varepsilon)}} B_{p_{m+l}}^{1+\varepsilon},$$

where $B = \max\{1, \beta\}$, $L = \log(2\alpha B)$, $p_m$ denotes the numerator of the $n^{th}$ convergent to the regular continued fraction expansion of $\alpha = [a_0, a_1, \ldots]$ and $m$ is the unique integer such that

$$p_m \leq L^{16\alpha^2} < p_{m+1}.$$ The first result in this paper is the following

Theorem 1. Let $g(x) = a_0 + a_1 x + \cdots + a_k x^k$, where $a_0, \ldots , a_k \in \mathbb{Z}$ with $a_k \geq 1$ and $k \geq 2$. Put $\gamma = 4^{1-k}$. Then for any positive integer $N \geq 3$, positive real number $\alpha$ with $\left|\frac{a_k}{\alpha} - \frac{1}{\gamma}\right| \leq \frac{1}{\gamma^2}, (a, q) = 1$ and any $\varepsilon > 0$ we have

$$\sum_{\substack{p \leq N \\text{and } \alpha_i \in \mathbb{R}}} \log p = \frac{1}{\alpha} \sum_{p \leq N} \log p + O\left(N\varepsilon (Nq^{-\gamma} + N^{1-\gamma/2} + q^{-\gamma} N^{1-(k+1)\gamma} + q^\gamma N^{1-k\gamma})\right).$$

In particular, if $a_k/\alpha$ is an irrational number of finite type $t > 0$ then we have

$$\sum_{\substack{p \leq N \\text{and } \alpha_i \in \mathbb{R}}} \log p = \frac{1}{\alpha} \sum_{p \leq N} \log p + O(N^{1-\frac{k\gamma}{1+\varepsilon}} + N^{1-\frac{k\gamma}{1+\varepsilon} + \varepsilon}).$$

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Let \( \alpha > 1 \) be irrational, \( \gamma = 4^{-k} \) and \( m \) be the unique integer such that \( \gamma \leq 4^{-m} \).

Theorem 2. Let \( g(x) = a_0 + a_1 x + \cdots + a_k x^k \) where \( a_0, \ldots, a_k \in \mathbb{Z} \) with \( a_k \geq 1 \) and \( k \geq 2 \). For every \( \varepsilon > 0 \), there exits a computable positive integer \( l \) such that for every irrational \( \alpha > 1 \) the least prime number \( p \) such that \( g(p) \) is contained in the Beatty sequence \( B(\alpha, \beta) \) satisfies the inequality,

\[
p \leq \alpha^\varepsilon \sum_{n \in \mathbb{N}} \Lambda(n) \, p_n^{1/2} + \varepsilon^{1/2} \ln n,
\]

where \( B = \max \{1, \beta\} \), \( p_n \) denotes the numerator of the \( n^{th} \) convergent to the regular continued fraction expansion of \( \frac{\alpha}{\alpha} \) and \( m \) is the unique integer such that

\[
p_m \leq \alpha^\varepsilon B^{1/2} p_{m+1}^{1/3} < p_{m+1}.
\]

For irrational \( \alpha \) of finite type \( \tau = \tau(\alpha) \), Banks and Y. proved in \([2] \), Theorem 2) that for any fixed \( \varepsilon > 0 \), for all integers \( 1 \leq c < d < N^\varepsilon \) with \( \gcd(c, d) = 1 \), we have

\[
\sum_{p \leq N \atop p \equiv f \pmod{d}} \log p = \frac{1}{\alpha} \sum_{p \leq N \atop p \equiv f \pmod{d}} \log p + O(N^{-1/4 + \varepsilon} d^{1/4} + N^{-1/2 + \varepsilon} d^{1/2} + \frac{N^{1/3}}{d^{1/3}}).
\]

Furthermore if \( \frac{1}{n} \) is an irrational number of finite type \( t > 0 \) then for all integers \( 1 \leq f < d \leq \min\{N^\varepsilon, N^{1/6}\} \) with \( (f, d) = 1 \) and for any \( 0 < \varepsilon < \frac{1}{4(t+1)} \), we have

\[
\sum_{p \leq N \atop p \equiv f \pmod{d}} \log p = \frac{1}{\alpha} \sum_{p \leq N \atop p \equiv f \pmod{d}} \log p + O(N^{1/2 + \varepsilon} d^{1/2} + N^{1/3 + \varepsilon} d^{1/3}).
\]

The proof of Theorem 3 depends on estimation of exponential sum of the type

\[
S(\vartheta) = \sum_{|l| \leq L} \left| \sum_{n \leq N \atop n \equiv f \pmod{d}} \Lambda(n) e(l n \vartheta) \right|,
\]

where \( \vartheta \) is irrational, \( L \geq 1 \) and \( f < d, (f, d) = 1 \). We obtain an upper bound for \( S(\vartheta) \) in Proposition 2 which is of independent interest.

Remark 1. Let \( f, d \) be natural numbers such that \( 1 \leq f < d \leq 500 \) and \( (f, d) = 1 \). For every \( \varepsilon > 0 \) there exists a computable positive integer \( l \) such that for every irrational \( \alpha > 1 \); the least prime number \( p \in B(\alpha, \beta) \) such that \( p \equiv f \pmod{d} \) satisfies the inequality,

\[
p \leq \alpha^{3 - \varepsilon} B^{1/2} d^{3 - 10 \varepsilon} p_{m+1}^{1/3} + \varepsilon^{1/3} \ln n,
\]

where \( B = \max \{1, \beta\} \) and \( p_n \) denotes the numerator of the \( n^{th} \) convergent to the regular continued fraction expansion of \( \alpha \) and \( m \) is the unique integer such that

\[
p_m \leq \alpha^{7/3} B^{1/2} d^{10/3} < p_{m+1}.
\]

This fact can be proved in a similar way as Theorem 2 using Corollary 1.6 of [3].
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2. Notation

Throughout this paper, the implied constants in the symbols $O$ and $\ll$ may depend on $\alpha$ and $\varepsilon$ otherwise are absolute. We recall that the notation $f = O(g)$ and $f \ll g$ are equivalent to the assertion that the inequality $|f| \leq cg$ holds for some constant $c > 0$. The notation $f \sim g$ means that $f \ll g$ and $f \gg g$. It is important to note that our bounds are uniform with respect to all of the involved parameters other than $\alpha, \varepsilon$ and degree of the polynomial $k$; in particular, our bounds are uniform with respect to $\beta$.

The letters $a, d, f, q$ always denote non-negative integers and $m, n, l, u, v$ and $t$ denotes integers. We use $\lfloor x \rfloor$ and $\{ x \}$ to denote the greatest integer less than or equal to $x$ and the fractional part of $x$ respectively. Finally, recall that the discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, a_2, \ldots, a_M \in [0, 1)$ is defined by

$$D(M) = \sup_{I \subset [0, 1)} \left| \frac{V(I, M)}{M} - |I| \right|,$$

where the supremum is taken over all sub-intervals $I$ of $[0, 1)$, $V(I, M)$ is the number of positive integers $m \leq M$ such that $a_m \in I$ and $|I|$ is the length of $I$.

3. Preliminaries

3.1. Case of polynomial values of prime. Note that an integer $m \in B(\alpha, \beta)$ if and only if

$$\left\| \frac{m}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right\| < \frac{1}{2\alpha}.$$

Hence

$$\# \{ m \leq N : m \in B(\alpha, \beta) \} = \sum_{m \leq N} \chi_{\delta} \left( \frac{m}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right),$$

where $\chi_\delta$ for $\delta > 0$ is defined by

$$\chi_\delta(\theta) = \begin{cases} 1 & \text{if } ||\theta|| < \delta, \\ 0 & \text{otherwise} \end{cases}$$

for $\theta \in \mathbb{R}$. Let $g(x) = a_0 + a_1x + \cdots + a_kx^k$, where $a_0, \ldots, a_k \in \mathbb{Z}, a_k \geq 1$. Therefore

$$\# \{ p \leq N : g(p) \in B(\alpha, \beta) \} = \sum_{p \leq N} \chi_{\delta} \left( \frac{g(p)}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right).$$

Lemma 1. ([5, Lemma 2.1]). For any $L \in \mathbb{N}$ there are coefficients $C_i^\pm$ such that

$$2\delta - \frac{1}{L + 1} + \sum_{1 \leq i \leq L} C_i^- e(i\theta) \leq \chi_\delta(\theta) \leq 2\delta + \frac{1}{L + 1} + \sum_{1 \leq i \leq L} C_i^+ e(i\theta),$$

with $|C_i^\pm| \leq \min \left( 2\delta + \frac{1}{L + 1}, \frac{3}{2\pi} \right)$. 
Then we have

\[ \sum_{n \leq N} \Lambda(n) \chi_{\frac{a}{\alpha}} \left( \frac{g(n)}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right) = \frac{1}{\alpha} \sum_{n \leq N} \Lambda(n) + O \left( \frac{N}{L + 1} \right) \]

\[ + O \left( \sum_{1 \leq |\delta| \leq L} |C_\delta| \sum_{n \leq N} \Lambda(n) e \left( \frac{g(n)}{\alpha} \right) \right), \] \hspace{1cm} (3)

where \( |C_\delta| \leq \min \left( \frac{1}{\alpha} + \frac{1}{L + 1}, \frac{3}{2\alpha} \right) \). To estimate the exponential sum we use the following Proposition

**Proposition 1.** (Equation (22), [4]) Suppose \( \varepsilon > 0 \) is given. Let \( f(x) \) be a real valued polynomial in \( x \) of degree \( k \geq 2 \). Put \( \gamma = 4^{k-1} \). Suppose \( \alpha \) is the leading coefficient of \( f \) and there are integers \( a, q \) with \( (a, q) = 1 \) such that \( |q\alpha - a| < q^{-1} \).

Then we have

\[ \sum_{l \leq L} \left| \sum_{n \leq N} \Lambda(n) e(\ell f(n)) \right| \ll (NL)^{1+\varepsilon} (q^{-1} + N^{-1/2} + qN^{-k}L^{-1})^\gamma. \]

3.2. **Case of primes in arithmetic progression.** Now we are interested in prime numbers \( p \) of the form \( p \equiv f \mod d \), which is in \( B(\alpha, \beta) \), where \( (f, d) = 1 \) and \( f < d \). As we discussed above, in order to find a prime number \( p \in B(\alpha, \beta) \) and we need to show that

\[ \left| \frac{p}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right| < \frac{1}{2\alpha}. \]

By Lemma 1 we have

\[ \sum_{\substack{n \leq N \\text{mod} \ f(d) \\text{\ and} \ \frac{a}{\alpha} \equiv f(d) \mod d}} \Lambda(n) \chi_{\frac{a}{\alpha}} \left( \frac{n}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right) = \frac{1}{\alpha} \sum_{\substack{n \leq N \\text{mod} \ f(d) \\text{\ and} \ \frac{a}{\alpha} \equiv f(d) \mod d}} \Lambda(n) + O \left( \frac{N}{L\varphi(d)} \right) \]

\[ + O \left( \sum_{1 \leq |\delta| \leq L} |C_\delta| \sum_{\substack{n \leq N \\text{mod} \ f(d) \\text{\ and} \ \frac{a}{\alpha} \equiv f(d) \mod d}} \Lambda(n) e(\ell n/\alpha) \right), \] \hspace{1cm} (4)

where \( |C_\delta| \leq \min \left( \frac{1}{\alpha} + \frac{1}{L + 1}, \frac{3}{2\alpha} \right) \). Now we want to estimate the exponential sum of the form (2).

To estimate the exponential sum we use the following Proposition

**Proposition 2.** Let \( S(\vartheta) \) is defined by [2] with \( \vartheta - \frac{a}{\alpha} \leq q^{-2} \), where \( a \) and \( q \) are positive integers satisfying \( (a, q) = 1 \). Then for any real number \( \varepsilon > 0 \); we have

\[ S(\vartheta) \ll \varepsilon (NL)^{\varepsilon} \left( \frac{NL}{q^{1/2}} + L^{1/2}N^{1/2}q^{1/2} + LN^{3/4}d^{1/2} + \frac{LN^{3/5}}{d^{3/5}} \right). \] \hspace{1cm} (5)

We will give the proof of Proposition 2 in Section 5.

4. **Proof of Theorem 1 and Theorem 2**

In the previous section we stated essential results to prove Theorem 1 and Theorem 2. In this section we will give proof of these theorems.

**Proof of Theorem 2.** It follows from Proposition 1 and partial summation formula

\[ \sum_{1 \leq |\delta| \leq L} C_\delta \sum_{n \leq N} \Lambda(n) e \left( \ell \left( \frac{g(n)}{\alpha} + \frac{1 - 2\beta}{2\alpha} \right) \right) \ll \varepsilon N^{1+\varepsilon} L^\varepsilon \left( q^\gamma + N^{-\gamma/2} + q^\gamma N^{-k\gamma}L^{-\gamma} \right) \]

\[ + N^{1+\varepsilon} q^\gamma N^{-k\gamma}. \] \hspace{1cm} (6)
Proof of Theorem 2:

This completes the proof of Theorem 1.

where

Then by (10) and (11) there exists a convergent to the simple continued fraction expansion of $Q$ theorem with

By (3) and (6) we have

Choosing $L = q^{-\gamma} N^{1-\epsilon}$ we have

This leads to

where

The number of prime powers $p^\nu \leq N$ with $\nu \geq 2$ is $O(\pi(N^{1/2}))$, thus we have

Suppose we assume $\frac{a_k}{\alpha}$ is an irrational number of finite type $t$. Using Dirichlet’s approximation theorem with $Q = N^{1/t}$, we obtain a rational $p/q$ with $1 \leq q \leq N^{1/t}$ such that

By definition of finite type of irrational, for any positive $\epsilon$, there is positive constant $c$ such that

Then by (10) and (11) there exists a convergent to the simple continued fraction expansion of $\frac{a_k}{\alpha}$ whose denominator satisfies

Therefore by (9) and (12) we obtain

This completes the proof of the Theorem 1.

Proof of Theorem 2 By (7) and (8) we have

where

\[
\xi(N, q) \leq \epsilon N^\epsilon (Nq^{-\gamma} + N^{1-\gamma/2} + q^\gamma N^{-k}\gamma + q^{2\gamma} N^{1-(k+1)\gamma})
\]
By Lemma 2, the second sum on the left hand side of (13) is less than \(1.04(\alpha + \beta - 1)\).

We will use inequality \(\log p \geq 1/\alpha \sum_{p \leq N} \log p + \xi(N, q) - 1.04(\alpha + \beta - 1) + \left(\frac{1}{\alpha} - 1\right) \sum_{p' \leq N} \log p\).

Notice that the last term is negative, it is obviously bounded by:

\[
\left(1 - \frac{1}{\alpha}\right) \sum_{p \leq N} \log p < \left(1 - \frac{1}{\alpha}\right) \pi(N^{1/2}) \log N.
\]

We will use inequality (2.18) Rosser and Schoenfeld for \(\pi(x)\), we have

\[
\left(1 - \frac{1}{\alpha}\right) \sum_{p' \leq N} \log p < \left(1 + \frac{3}{\log N}\right) N^{1/2}
\]

we also use inequality (3.16) of Rosser and Schoenfeld which is,

\[
\sum_{p \leq N} \log p > N - \frac{N}{\log N}
\]

for \(N \geq 41\). Therefore we obtain

\[
\sum_{p \leq N} \log p \geq \frac{N}{\alpha} \left(1 - \frac{1}{\log N}\right) + \xi(N, q) - 1.04(\alpha + \beta - 1) - \left(1 + \frac{3}{\log N}\right) N^{1/2}.
\]

We thus find a prime \(p \leq N\) and \(p^2 \in B(\alpha, \beta)\) if we show that the following inequality

\[
\frac{N}{\alpha} \left(1 - \frac{1}{\log N}\right) > \xi(N, q) + 1.04(\alpha + \beta - 1) - \left(1 + \frac{3}{\log N}\right) N^{1/2},
\]

which we may also replace by

\[
0.73 \frac{N}{\alpha} > 1.04(\alpha + \beta - 1) + 1.81 N^{1/2} + \xi(N, q).
\]

By (14) we have,

\[
0.73 > 1.04 \frac{\alpha}{N}(\alpha + \beta - 1) + 1.81 \frac{\alpha}{N^{1/2}} + C(\varepsilon) N^{\varepsilon}(q^{\gamma} + N^{-\gamma/2} + q^{-\kappa \gamma} + q^{-\gamma}\gamma^{-\gamma} N^{-\gamma/2})
\]

and appropriate absolute constant \(C(\varepsilon)\) depending only on \(\varepsilon\) but not \(\alpha\).

Obviously \(N\) need to be larger than \(\text{Max}\{\alpha^{2/\gamma}, B\}\) and \(q\) larger than \(\alpha^{1/\gamma}\). We shall take both \(N\) and \(q\) somewhat larger so that above inequality holds, now choose

\[
N = \alpha^{2k+1} B^{\frac{1}{k+\gamma}} q^{\frac{1}{\gamma}} \eta, \quad q = \alpha^{\frac{2k+1}{\gamma}} B\eta
\]

with some large parameter \(\eta\) to be specified later and \(B=\text{Max}\{1, \beta\}\). Then the latter inequality can be rewritten as

\[
0.73 > 1.04(\alpha + \beta - 1)(\alpha^{\frac{2k+1}{\gamma}} B^{-1} \eta^{-\frac{1}{k+\gamma}}) + 1.81(\alpha^{\frac{2k+1}{\gamma} - 2k} B^{-1/2} \eta^{-\frac{k-2}{k+\gamma}})
\]

\[
+ \alpha^{-\gamma + \frac{2k+1}{\gamma} + \frac{1}{k+\gamma}} B^{-\frac{1}{k+\gamma}} \eta^{\frac{1}{\gamma}} \gamma^{-\gamma} + \alpha^{-\frac{2k+1}{\gamma} + \frac{2k+1}{\gamma}} B^{-\frac{k+1}{k+\gamma}} \eta^{-\frac{k+1}{k+\gamma}} (2k+1)^{\frac{1}{k+\gamma}}
\]

\[
+ \alpha^{2k+1} \left(\frac{2k+1}{\gamma} + \frac{1}{\gamma}\right) B^{(1-k)\gamma + \frac{1}{\gamma}} \eta^{\frac{1}{\gamma}} + \alpha^{-\frac{2k+1}{\gamma} + \gamma} B^{\frac{1}{k+\gamma}} \eta^{-\frac{1}{k+\gamma}} + \alpha^{\frac{1}{k+\gamma} + \frac{1}{\gamma} + \frac{1}{\gamma}} B^{\frac{1}{k+\gamma}}
\]

Since \(k \geq 2\) and \(\gamma = 4^{1-k}\) assuming \(\varepsilon < \frac{\gamma^2}{2(2k+1)}\) as we may, all exponents of \(\alpha, B, \text{and} \eta\) are negative. Therefore the above inequality is satisfied for all sufficiently large \(\eta\), say \(\eta \geq \eta_0\). Since \(\eta\) is interwined with \(q\) a little care needs to be taken. In order to find a suitable \(\eta\) recall \(\alpha\) is irrational. Hence, by Dirichlet’s approximation theorem, there are infinitely many solution \(\frac{a}{\eta}\) to inequality
Thus we have $|\alpha - \frac{m}{q}| < \frac{1}{q^2}$; in view of $a = \frac{1}{\alpha}$ we may take the reciprocals of the convergents $\frac{m}{q_n}$ to the continued fraction expansion of $\alpha$. We shall choose $l$ such that $\eta_0 \leq \frac{m}{q_{m+1}}$, where $m$ is defined by (1), for then the choice $q = q_{m+l}$ will yield an $\eta \geq \eta_0$. The choice of $\eta$ follows from (12) of [6]. Therefore the choice of $l$ as it depends on $\eta$. This completes the proof of the Theorem [2].

5. PROOF OF THEOREM 3

The present section is devoted to a proof of Theorem 3.

Proof of Theorem 3. It follows from Proposition 2 and partial summation formula

$$\sum_{1 \leq |l| \leq L} C_l \sum_{n \leq N, n \equiv f(d)} A(n) \chi \left( \frac{n}{\alpha} + 1 - \frac{2\beta}{2\alpha} \right) = \left( \frac{1}{\alpha} \right) \sum_{n \leq N, n \equiv f(d)} A(n) + \left( \frac{1}{L} \right) \frac{N}{\varphi(d)}$$

By (4) and (5), we obtain

$$+ O \left( (NL)^\epsilon \left( \frac{N}{q^{1/2}} + N^{1/2} q^{1/2} + \frac{N^{1/2} q^{1/2}}{L^{1/2}} + N^{3/4} d^{1/2} + \frac{N^{4/5}}{d^{1/5}} \right) \right).$$

Choose

$$L = \frac{N}{qd^2}.$$

Therefore we obtain an estimate

$$\sum_{n \leq N, n \equiv f(d)} A(n) \chi \left( \frac{n}{\alpha} + 1 - \frac{2\beta}{2\alpha} \right) = \left( \frac{1}{\alpha} \right) \sum_{n \leq N, n \equiv f(d)} A(n)$$

$$+ O \left( N^\epsilon \left( \frac{N}{q^{1/2}} + N^{1/2} q^{1/2} + qd + N^{3/4} d^{1/2} + \frac{N^{4/5}}{d^{1/5}} \right) \right).$$

We rewrite above equality as

$$\sum_{p^\nu \leq N, p^\nu \equiv f(d)} \log p + \sum_{p^\nu \leq N, p^\nu \equiv f(d)} \log p = \frac{1}{\alpha} \sum_{p^\nu \leq N, p^\nu \equiv f(d)} \log p$$

$$+ O \left( N^\epsilon \left( \frac{N}{q^{1/2}} + N^{1/2} q^{1/2} + qd + N^{3/4} d^{1/2} + \frac{N^{4/5}}{d^{1/5}} \right) \right).$$

By (4.3.3) of [6] we have

$$\frac{1}{\alpha} \sum_{p^\nu \leq N, p^\nu \equiv f(d), \nu \geq 2} \log p \leq \frac{1.0012}{\alpha} (\sqrt{N} + N^{1/3}).$$

Thus we have

$$\sum_{p^\nu \leq N, p^\nu \equiv f(d)} \log p = \frac{1}{\alpha} \sum_{p^\nu \leq N, p^\nu \equiv f(d)} \log p + O \left( \frac{N^{1/2}}{\alpha} \right)$$

$$+ O \left( N^\epsilon \left( \frac{N}{q^{1/2}} + N^{1/2} q^{1/2} + qd + N^{3/4} d^{1/2} + \frac{N^{4/5}}{d^{1/5}} \right) \right). \quad (15)$$
Suppose we assume \( \frac{1}{\alpha} \) is an irrational number of finite type \( t \). By using Dirichlet’s approximation theorem with \( Q = N^{1+t} \), we obtain a rational \( p/q \) with \( 1 \leq q \leq N^{1+t} \) such that

\[
\left| \frac{1}{\alpha} - \frac{p}{q} \right| \leq \frac{1}{qN^{1+t}}.
\]  

(16)

And by definition of finite type of irrational, for any positive \( \varepsilon \), there is positive constant \( c \) such that

\[
\left| \frac{1}{\alpha} - \frac{p}{q} \right| \geq \frac{c}{q^{t+1} + \varepsilon}.
\]  

(17)

Then by (16) and (17) there exists a convergent to the simple continued fraction expansion of \( \frac{1}{\alpha} \) whose denominator satisfies

\[
N^{1+t} \ll q \leq N^{1+t}.
\]  

(18)

Therefore by (15) and (18) we obtain

\[
\sum_{\substack{p \leq N \cr p \equiv f(d)}} \log p = \frac{1}{\alpha} \sum_{\substack{p \leq N \cr p \equiv f(d)}} \log p + O(N^{1/2}(N^{1-t} + N^{3/4}d^{1/2} + N^{4/5}d^{-1/5})).
\]

This completes the proof of the Theorem 3.

6. PROOF OF PROPOSITION 2

Proof of Proposition 2 is based on work of Balog and Perelli [1].

6.1. Some Lemmas. Here we list several lemmas required for the proof. The following lemma gives explicit bound for average of von Mangoldt function

**Lemma 2.** [7] For any \( N \in \mathbb{N} \)

\[
\sum_{n \leq N} \Lambda(n) \leq c_0 N,
\]

for some constant \( c_0 \), where one may take \( c_0 = 1.04 \).

**Lemma 3.**

\[
\sum_{x < m \leq x'} e(m\theta) \ll \min \left( \frac{x'}{d} + 1, \frac{\|\theta d\|^{-1}}{d} \right).
\]

**Lemma 4.** [10] Suppose that \( X, Y \geq 1 \) are positive integers, Also suppose that \( |\alpha - a/q| < q^{-2} \), where \( \alpha \) is a real number, \( a \) and \( q \) integers satisfying \((a, q) = 1\). Then

\[
\sum_{x \leq X} \min(Y, \|\alpha x\|^{-1}) \ll \frac{XY}{q} + (X + q) \log 2q,
\]

\[
\sum_{x \leq X} \min \left( \frac{XY}{x}, \|\alpha x\|^{-1} \right) \ll \frac{XY}{q} + (X + q) \log(2XYq).
\]

**Lemma 5.** [9] For any real number \( \vartheta \) and natural numbers \( N, l \) and \( 1 \leq f < d \) such that \((f, d) = 1\), we have

\[
\sum_{\substack{n \leq N \cr n \equiv f(d)}} \Lambda(n)e(ln\vartheta) = O(N^{1/2}) + S_1 - S_2 - S_3,
\]
Proof. By using Cauchy Schwarz inequality we obtain

\[ S_1 = \sum_{m \leq U} \sum_{n \leq N/m, mn \equiv f(d)} \mu(m)(\log n)e(lmn\vartheta), \]

\[ S_2 = \sum_{m \leq U^2} \sum_{n \leq N/m, mn \equiv f(d)} \phi_1(m)e(lmn\vartheta), \]

\[ S_3 = \sum_{U < m \leq N/U} \sum_{U < n \leq N/m, mn \equiv f(d)} \phi_2(m)\Lambda(n)e(lmn\vartheta), \]

and

\[ \phi_1(m) \ll \log m, \quad \phi_2(m) \ll d_2(m). \]

Here \( U \) is an arbitrary parameters to be chosen later satisfying \( 1 \leq U \leq N^{1/2} \).

**Lemma 6.** Suppose that \( \epsilon > 0 \) and that \( \phi(u) \) and \( \psi(v) \) are real valued functions such that \( |\phi(u)| \ll T, |\psi(v)| \ll F. \) Suppose that \(|\vartheta - a/q| < q^{-2} \), where \( \vartheta \) is a real number, \( a \) and \( q \) integers satisfying \( (a, q) = 1. \) For positive integers \( N, W, X, \) and \( L \) write

\[ S = \sum_{|l| \leq L} \sum_{X < v < 2X} \sum_{u \leq W} \phi(u)\psi(v)e(luv\vartheta). \quad (19) \]

Then

\[ S \ll TF \left( \frac{LWX^{1/2}}{d^{1/2}} + (LWd^{1/2} + LXW^{1/2}d^{1/2} + L^{1/2}q^{1/2}X^{1/2}W^{1/2}) \right). \]

Proof. For the moment we shall ignore the condition \( uv \leq N \) in \( (19) \). Consider

\[ S = \sum_{|l| \leq L} \sum_{X < v < 2X} \sum_{u \leq W} \phi(u)\psi(v)e(luv\vartheta). \quad (20) \]

We observe that

\[ S = \sum_{f_1, f_2 \equiv f(d)} \sum_{(f_1, d) = (f_2, d) = 1} \left| R_{f_1, f_2} \right|, \quad (21) \]

where

\[ R_{f_1, f_2} = \sum_{|l| \leq L} \sum_{X < v < 2X} \sum_{u \leq W} \phi(u)\psi(v)e(luv\vartheta). \]

By using Cauchy Schwarz inequality we obtain

\[ |R_{f_1, f_2}|^2 \leq L \sum_{|l| \leq L} \left( \sum_{u \leq W} \sum_{X < v < 2X} \phi(u)\psi(v)e(luv\vartheta) \right)^2 \]

\[ \leq \sum_{|l| \leq L} \left( \sum_{u \leq W} |\phi(u)|^2 \right) \left( \sum_{X < v < 2X} \left| \sum_{u \equiv f_1(d)} \psi(v)e(luv\vartheta) \right|^2 \right) \]

\[ \leq T^2LW / d^{1/2} + R_1, \quad (22) \]

where

\[ R_1 = \sum_{|l| \leq L} \sum_{u \leq W} \sum_{X < v_1, v_2 \leq 2X} \psi(v_1)\psi(v_2)e(l(u_1 - v_2)\vartheta). \]
Lemma 7. Suppose we have the hypotheses and notations of Lemma 6 with either \( \phi(X) \equiv 1 \) or \( \phi(x) = \log x \) for all \( x \). Then
\[
S \ll F(LX) \frac{q^{-1} + LXd + q}{d}.
\]

Proof. The \( \log x \) factor may easily be removed by partial summation formula so we presume that \( \phi(x) \equiv 1 \). Again we may ignore the condition \( uw \leq N \). Therefore we need to estimate
\[
S = \sum_{|l| \leq L} \left| \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \psi(v_1) \psi(v_2) e(luv\theta) \right|
\leq F \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\leq F \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\]

where \( \bar{u} \) is defined by \( u\bar{u} \equiv 1 \pmod{d} \). Then by using Lemma 5 we have
\[
S \leq F \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\leq \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\]

We apply Lemma 3 for innermost sum of (23), we get
\[
|R_1| \leq \frac{F^2 X}{d} \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\]

Let \( r = lkd \) so that \( 1 \leq |r| \leq 2LXd \) and \( r \) will run through all the integers in the interval above, also number of representations of \( r \) is not more than \( d_2(r) \). Therefore we have
\[
|R_1| \leq \frac{F^2 X}{d} (LXd)^s \sum_{|r| \leq 2LXd} \min \left( \frac{W}{d} + 1, ||r\bar{r}||^{-1} \right).
\]

Then by using Lemma 4 we obtain
\[
R_1 \ll F^2 (LXd)^s \left( \frac{LX^2 W}{qd} + LX^2 + qX^2 \right).
\]

By (22) and (24) we have
\[
R_{f_1, f_2} \ll TF \left( \frac{LW X^{1/2}}{d^{3/2}} + (LXd)^s \left( \frac{LXW^{1/2}}{q^{1/2} d} + \frac{LXW^{1/2}}{d^{1/2}} + \frac{L^{1/2} q^{1/2} X^{1/2} W^{1/2}}{d} \right) \right). \tag{25}
\]

Thus Lemma follows from (21) and (25). \( \square \)

Lemma 7. Suppose we have the hypotheses and notations of Lemma 6 with either \( \phi(X) \equiv 1 \) or \( \phi(x) = \log x \) for all \( x \). Then
\[
S \ll F(LX) \frac{q^{-1} + LXd + q}{d}.
\]

Proof. The \( \log x \) factor may easily be removed by partial summation formula so we presume that \( \phi(x) \equiv 1 \). Again we may ignore the condition \( uw \leq N \). Therefore we need to estimate
\[
S = \sum_{|l| \leq L} \left| \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \psi(v_1) \psi(v_2) e(luv\theta) \right|
\leq F \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\]

where \( \bar{u} \) is defined by \( u\bar{u} \equiv 1 \pmod{d} \). Then by using Lemma 5 we have
\[
S \leq F \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\leq \sum_{|l| \leq L} \sum_{X < v_1, v_2 \leq 2X} \sum_{u \leq W} \epsilon(luv\theta) |u\bar{u}| \equiv f(d), v \equiv f\bar{u}(d)\}
\]

We may write \( R_1 \) in the form
\[
= \sum_{|l| \leq L} \sum_{u \leq W} \sum_{k \equiv 0(d)} \sum_{v_1, v_2 \equiv f_1(d)} \psi(v_1) \psi(v_2) e(luv\theta)
\leq \sum_{|l| \leq L} \sum_{k \equiv 0(d)} \sum_{\substack{X < v_1, v_2 \leq 2X \vspace{1pt} \ \ v_1 \neq v_2 \ v_1 - v_2 = k}} \psi(v_1) \psi(v_2) e(luv\theta), \tag{23}
\]

where
\[
\zeta_1(k) = \sum_{X < v_1, v_2 \leq 2X} \sum_{\substack{v_1 \neq v_2 \ v_1 - v_2 = k}} \psi(v_1) \psi(v_2) \ll \frac{F^2 X}{d}.
\]
Let \( r = lvd \) so that \( 1 \leq |r| \leq 2LXd \) and \( r \) will run through all the integers in the interval above, also number of representations of \( r \) is not more than \( d_2(r) \). Therefore we have

\[
S \leq F(LXd) \sum_{|r| \leq 2LXd} \min \left( \frac{W}{d} + 1, ||r\|^{-1} \right). \tag{27}
\]

Thus (26) follows from (27) and Lemma 4. \( \square \)

**Proof of the proposition 2.** We may assume that

\[
N \geq \max(qd^2L^{-1}, d^2), \quad q \geq d^2
\]

otherwise (5) is a consequence of the trivial bound,

\[
S(\vartheta) \leq \frac{LN}{d}.
\]

Using Lemma 5 we have the following sums to estimate

\[
S_1' = \sum_{0 \leq l \leq L} \left| \sum_{m \leq U} \sum_{n \leq N/m} \mu(m)(\log n)e(\text{lmmn}) \right|
\]

\[
S_2' = \sum_{0 \leq l \leq L} \left| \sum_{m \leq U^2} \sum_{n \leq N/m} \phi_1(m)e(\text{lmmn}) \right|
\]

\[
S_3' = \sum_{0 \leq l \leq L} \left| \sum_{U < m \leq N/U} \sum_{U < n \leq N/m} \phi_2(m)\Lambda(n)e(\text{lmmn}) \right|
\]

By dyadic division we write

\[
S_1 = \sum_{t=0}^{[\log U/d]} S_{1t},
\]

where

\[
S_{1t} = \sum_{l \leq L} \left| \sum_{2^t \leq m \leq 2^{t+1}} \sum_{n \leq N/m} \mu(m)(\log n)e(\text{lmmn}) \right|
\]

Then using Lemma 7 we get

\[
S_1' \ll (NL)^t(LN q^{-1} + LU d + q).
\]

\( S_2' \) can be estimated similarly as \( S_1' \) by partitioning into dyadic subsums say \( S_{2t} \). We estimate \( S_{2t} \) using Lemma 7 and we get

\[
S_2' \ll (NL)^{2t}(LN q^{-1} + LU^2 d + q).
\]

We write \( S_3' = S_{31}' + S_{32}' \), where

\[
S_{31}' = \sum_{l \leq L} \sum_{U < m \leq N^{1/2}} \sum_{U < n \leq N/m} \phi_2(m)\Lambda(n)e(\text{lmmn}) \]

\[
S_{32}' = \sum_{l \leq L} \sum_{U < n \leq N^{1/2}} \sum_{N^{1/2} < m \leq N/n} \phi_2(m)\Lambda(n)e(\text{lmmn}) \]

By dividing \( S_{31}' \) dyadically we obtain

\[
S_{31}' = \sum_{t=0}^{R} S_{31t},
\]
where
\[
S_{31t} = \sum_{l \leq L} \sum_{U^2 < m \leq U^2 + 1} \sum_{U < n \leq N/m, \text{gcd}(m,n) = f(d)} \mu(m)(\log n)e(lmn\theta) \quad \text{and} \quad R = \left[ \frac{\log \left( \frac{U^{1/2}}{d} \right)}{\log 2} \right].
\]

Then using Lemma 6
\[
S'_{31} \ll (NL)^{3\epsilon} \left( \frac{LN}{q^{1/2}} + LN^{1/2}q^{1/2} + LN^{3/4}d^{1/2} + \frac{LN}{U^{1/2}d^{1/2}} \right).
\]

Similarly we can show that \(S'_{32}\) has the same upper bound. Therefore
\[
S(\theta) \ll_{\epsilon} (NL)^{\epsilon} \left( \frac{NL}{q^{1/2}} + L^{1/2}N^{1/2}q^{1/2} + LN^{3/4}d^{1/2} + LU^{2}q + \frac{LN}{U^{1/2}d^{1/2}} \right). \tag{29}
\]

Then (5) follows from (29) with the choice of
\[
U = \frac{N^{2/5}}{d^{3/5}}
\]
and the observation \(q \leq L^{1/2}N^{1/2}q^{1/2}\).

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