Maximum Causal Tsallis Entropy Imitation Learning

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Abstract

In this paper, we propose a novel maximum causal Tsallis entropy (MCTE) framework for imitation learning which can efficiently learn a sparse multi-modal policy distribution from demonstrations. We provide the full mathematical analysis of the proposed framework. First, the optimal solution of an MCTE problem is shown to be a sparsemax distribution, whose supporting set can be adjusted. The proposed method has advantages over a softmax distribution in that it can exclude unnecessary actions by assigning zero probability. Second, we prove that an MCTE problem is equivalent to robust Bayes estimation in the sense of the Brier score. Third, we propose a maximum causal Tsallis entropy imitation learning (MCTEIL) algorithm with a sparse mixture density network (sparse MDN) by modeling mixture weights using a sparsemax distribution. In particular, we show that the causal Tsallis entropy of an MDN encourages exploration and efficient mixture utilization while Boltzmann Gibbs entropy is less effective. We validate the proposed method in two simulation studies and MCTEIL outperforms existing imitation learning methods in terms of average returns and learning multi-modal policies.

1 Introduction

In this paper, we focus on the problem of imitating demonstrations of an expert who behaves nondeterministically depending on the situation. In imitation learning, it is often assumed that the expert’s policy is deterministic. However, there are instances, especially for complex tasks, where multiple action sequences perform the same task equally well. We can model such nondeterministic behavior of an expert using a stochastic policy. For example, expert drivers normally show consistent behaviors such as keeping lane or keeping the distance from a frontal car, but sometimes they show different actions for the same situation, such as overtaking a car and turning left or right at an intersection, as suggested in [1]. Furthermore, learning multiple optimal action sequences to perform a task is desirable in terms of robustness since an agent can easily recover from failure due to unexpected events [2][3]. In addition, a stochastic policy promotes exploration and stability during learning [4][2][5]. Hence, modeling experts’ stochasticity can be a key factor in imitation learning.

To this end, we propose a novel maximum causal Tsallis entropy (MCTE) framework for imitation learning, which can learn from a uni-modal to multi-modal policy distribution by adjusting its supporting set. We first show that the optimal policy under the MCTE framework follows a sparsemax distribution [6], which has an adaptable supporting set in a discrete action space. Traditionally, the maximum causal entropy (MCE) framework [1][7] has been proposed to model stochastic behavior in demonstrations, where the optimal policy follows a softmax distribution. However, it often assigns non-negligible probability mass to non-expert actions when the number of actions increases [3][8].
On the contrary, as the optimal policy of the proposed method can adjust its supporting set, it can model various expert’s behavior from a uni-modal distribution to a multi-modal distribution.

To apply the MCTE framework to a complex and model-free problem, we propose a maximum causal Tsallis entropy imitation learning (MCTEIL) with a sparse mixture density network (sparse MDN) whose mixture weights are modeled as a sparsemax distribution. By modeling expert’s behavior using a sparse MDN, MCTEIL can learn varying stochasticity depending on the state in a continuous action space. Furthermore, we show that the MCTEIL algorithm can be obtained by extending the MCTE framework to the generative adversarial setting, similarly to generative adversarial imitation learning (GAIL) by Ho and Ermon [9], which is based on the MCE framework. The main benefit of the generative adversarial setting is that the resulting policy distribution is more robust than that of a supervised learning method since it can learn recovery behaviors from less demonstrated regions to demonstrated regions by exploring the state-action space during training. Interestingly, we also show that the Tsallis entropy of a sparse MDN has an analytic form and is proportional to the distance between mixture means. Hence, maximizing the Tsallis entropy of a sparse MDN encourages exploration by providing bonus rewards to wide-spread mixture means and penalizing collapsed mixture means, while the causal entropy of an MDN is less effective in terms of preventing the collapse of mixture means since there is no analytical form and its approximation is used in practice instead. Consequently, maximizing the Tsallis entropy of a sparse MDN has a clear benefit over the causal entropy in terms of exploration and mixture utilization.

To validate the effectiveness of the proposed method, we conduct two simulation studies. In the first simulation study, we verify that MCTEIL with a sparse MDN can successfully learn multi-modal behaviors from expert’s demonstrations. A sparse MDN efficiently learns a multi-modal policy without performance loss, while a single Gaussian and a softmax-based MDN suffer from performance loss. The second simulation study is conducted using four continuous control problems in MuJoCo [10]. MCTEIL outperforms existing methods in terms of the average cumulative return. In particular, MCTEIL shows the best performance for the reacher problem with a smaller number of demonstrations while GAIL often fails to learn the task.

2 Background

Markov Decision Processes  Markov decision processes (MDPs) are a well-known mathematical framework for a sequential decision making problem. A general MDP is defined as a tuple \( \{S, F, A, \phi, \Pi, d, T, \gamma, r\} \), where \( S \) is the state space, \( F \) is the corresponding feature space, \( A \) is the action space, \( \phi \) is a feature map from \( S \times A \) to \( F \), \( \Pi \) is a set of stochastic policies, \( d \) is the initial state distribution, \( T(s' | s, a) \) is the transition probability from \( s \in S \) to \( s' \in S \) by taking \( a \in A, \gamma \in (0, 1) \) is a discount factor, and \( r \) is the reward function from a state-action pair to a real value. In general, the goal of an MDP is to find the optimal policy distribution \( \pi^* \in \Pi \) which maximizes the expected discount sum of rewards, i.e., \( \mathbb{E}_\pi [r(s, a)] \triangleq \mathbb{E}[\sum_{t=0}^{\infty} r(s_t, a_t) | \pi, d] \). Note that, for any function \( f(s, a) \), \( \mathbb{E}[\sum_{t=0}^{\infty} f(s_t, a_t) | \pi, d] \) will be denoted as \( \mathbb{E}_\pi [f(s, a)] \).

Maximum Causal Entropy Inverse Reinforcement Learning  Zeibart et al. [11] proposed the maximum causal entropy framework, which is also known as maximum entropy inverse reinforcement learning (MaxEnt IRL). MaxEnt IRL maximizes the causal entropy of a policy distribution while the feature expectation of the optimized policy distribution is matched with that of expert’s policy. The maximum causal entropy framework is defined as follows:

\[
\begin{align*}
\text{maximize} & \quad \alpha H(\pi) \\
\text{subject to} & \quad \mathbb{E}_\pi [\phi(s, a)] = \mathbb{E}_{\pi_E} [\phi(s, a)],
\end{align*}
\]

where \( H(\pi) \triangleq \mathbb{E}_\pi [-\log(\pi(a | s))] \) is the causal entropy of policy \( \pi \), \( \alpha \) is a scale parameter, \( \pi_E \) is the policy distribution of the expert. Maximum causal entropy estimation finds the most uniformly distributed policy satisfying feature matching constraints. The feature expectation of the expert policy is used as a statistic to represent the behavior of an expert and is approximated from expert’s demonstrations \( D = \{\zeta_0, \cdots, \zeta_N\} \), where \( N \) is the number of demonstrations and \( \zeta_i \) is a sequence of state and action pairs whose length is \( T \), i.e., \( \zeta_i = \{(s_0, a_0), \cdots, (s_T, a_T)\} \). In [11], it is shown that the optimal solution of \( \mathbb{E}_\pi [\phi(s, a)] \) is a softmax distribution.
where $c$ is a cost function and $\psi$ is a convex regularization for cost $c$. As shown in [9], many existing IRL methods can be interpreted with this framework, such as MaxEnt IRL [1], apprenticeship learning [12], and multiplicative weights apprenticeship learning [13]. Existing IRL methods based on (2) often require to solve the inner minimization over $\pi$ for fixed $c$ in order to compute the gradient of $c$. In [11], Ziebart showed that the inner minimization is equivalent to a soft Markov decision process (soft MDP) under the reward $-c$ and proposed soft value iteration to solve the soft MDP. However, solving a soft MDP every iteration is often intractable for problems with large state and action spaces and also requires the transition probability which is not accessible in many cases. To address this issue, the generative adversarial imitation learning (GAIL) framework is proposed in [9] to avoid solving the soft MDP problem directly. The unified imitation learning problem (2) can be converted into the GAIL framework as follows:

$$\min_{\pi \in \Pi} \max_{D} E_{\pi} [\log(D(s,a))] + E_{\pi_E} [\log(1 - D(s,a))] - \alpha H(\pi),$$

where $D \in (0,1)^{|S||A|}$ indicates a discriminator, which returns the probability that a given demonstration is from a learner, i.e., 1 for learner’s demonstrations and 0 for expert’s demonstrations. Notice that we can interpret $\log(D)$ as cost $c$ (or reward of $-c$).

Since existing IRL methods, including GAIL, are often based on the maximum causal entropy, they model the expert’s policy using a softmax distribution, which can assign non-zero probability to non-expert actions in a discrete action space. Furthermore, in a continuous action space, expert’s behavior is often modeled using a uni-modal Gaussian distribution, which is not proper to model multi-modal behaviors. To handle these issues, we propose a sparsemax distribution as the policy of an expert and provide a natural extension to handle a continuous action space using a mixture density network with sparsemax weight selection.

**Sparse Markov Decision Processes** In [3], a sparse Markov decision process (sparse MDP) is proposed by utilizing the causal sparse Tsallis entropy $W(\pi) \triangleq \frac{1}{2} E_{\pi} [1 - \pi(a|s)]$ to the expected discounted rewards sum, i.e., $E_{\pi} [r(s,a)] + \alpha W(\pi)$. Note that $W(\pi)$ is an extension of a special case of the generalized Tsallis entropy, i.e., $S_{k,q}(p) = \frac{k}{q-1} (1 - \sum_i p_i^q)$, for $k = \frac{1}{2}, q = 2$, to sequential random variables. It is shown that the optimal policy of a sparse MDP is a sparse and multi-modal policy distribution [3]. Furthermore, sparse Bellman optimality conditions were derived as follows:

$$Q(s,a) \triangleq r(s,a) + \tau \sum_{s'} V(s')T(s'|s,a), \quad \pi(a|s) = \max \left( \frac{Q(s,a)}{\alpha} - \tau \left( \frac{Q(s,a)}{\alpha} \right)^2, 0 \right),$$

$$V(s) = \alpha \left[ \frac{1}{2} \sum_{a \in S(s)} \left( \frac{Q(s,a)}{\alpha} \right)^2 - \tau \left( \frac{Q(s,a)}{\alpha} \right)^2 \right] + \frac{1}{2},$$

where $\tau \left( \frac{Q(s,a)}{\alpha} \right) = \sum_{a \in S(s)} \frac{Q(s,a)}{K_s} - 1$. $S(s)$ is a set of actions satisfying $1 + \tau Q(s,a) > \sum_{i=1}^{K_s} \frac{Q(s,a)}{K_s}$, with $a(i)$ indicating the action with the $i$th largest state-action value $Q(s,a)$. $K_s$ is the cardinality of $S(s)$. In [3], a sparsemax policy shows better performance compared to a softmax policy since it assigns zero probability to non-optimal actions whose state-action value is below the threshold $\tau$. In this paper, we utilize this property in imitation learning by modeling expert’s behavior using a sparsemax distribution. In Section 3, we show that the optimal solution of an MCTE problem also has a sparsemax distribution and, hence, the optimality condition of sparse MDPs is closely related to that of MCTE problems.

### 3 Principle of Maximum Causal Tsallis Entropy

In this section, we formulate maximum causal Tsallis entropy imitation learning (MCTEIL) and show that MCTE induces a sparse and multi-modal distribution which has an adaptable supporting set. The
problem of maximizing the causal Tsallis entropy $W(\pi)$ can be formulated as follows:

$$\begin{align*}
\text{maximize} & \quad \alpha W(\pi) \\
\text{subject to} & \quad E_\pi [\phi(s, a)] = E_{\pi_E} [\phi(s, a)].
\end{align*}$$ (5)

In order to derive optimality conditions, we will first change the optimization variable from a policy distribution to a state-action visitation measure. The necessary and sufficient conditions for an optimal solution are derived from the Karush-Kuhn-Tucker (KKT) conditions using the strong duality and the optimal policy is shown to be a sparsemax distribution. Furthermore, we also provide an interesting interpretation of the MCTE framework as robust Bayes estimation in terms of the Brier score. Hence, the proposed method can be viewed as maximization of the worst case performance in the sense of the Brier score [14].

We can change the optimization variable from a policy distribution to a state-action visitation measure based on the following theorem.

**Theorem 1** (Theorem 2 of Syed et al. [15]). Let $M$ be a set of state-action visitation measures, i.e., $M \triangleq \{\rho|s, a, \rho(s, a) \geq 0, \sum_a \rho(s, a) = d(s) + \sum_{s', a'} T(s|s', a')\rho(s', a')\}$. If $\rho \in M$, then it is a state-action visitation measure for $\pi_\rho(a|s) \triangleq \frac{\rho(s, a)}{\sum_{s, a} \rho(s, a)}$, and $\pi_\rho$ is the unique policy whose state-action visitation measure is $\rho$.

**Proof.** The proof can be found in [15].

Theorem 1 guarantees the one-to-one correspondence between a policy distribution and state-action visitation measure. Then, the objective function $W(\pi)$ is converted into the function of $\rho$ as follows.

**Theorem 2.** Let $\bar{W}(\rho) = \frac{1}{2} \sum_{s, a} \rho(s, a) \left(1 - \frac{\rho(s, a)}{\sum_{s', a'} \rho(s', a')}\right)$. Then, for any stationary policy $\pi \in \Pi$ and any state-action visitation measure $\rho \in M$, $W(\pi) = W(\rho_\pi)$ and $\bar{W}(\rho) = W(\pi_\rho)$ hold.

The proof is provided in the supplementary material. **Theorem 2** tells us that if $\bar{W}(\rho)$ has the maximum at $\rho^*$, then $W(\pi)$ also has the maximum at $\pi_{\rho^*}$. Based on Theorem 1 and 2, we can freely convert the problem (5) into

$$\begin{align*}
\text{maximize} & \quad \alpha \bar{W}(\rho) \\
\text{subject to} & \quad \sum_{s, a} \rho(s, a) \phi(s, a) = \sum_{s, a} \rho_E(s, a) \phi(s, a),
\end{align*}$$ (6)

where $\rho_E$ is the state-action visitation measure corresponding to $\pi_E$.

### 3.1 Optimality Condition of Maximum Causal Tsallis Entropy

We show that the optimal policy of the problem (6) is a sparsemax distribution using the KKT conditions. In order to use the KKT conditions, we first show that the MCTE problem is concave.

**Theorem 3.** $\bar{W}(\rho)$ is strictly concave with respect to $\rho \in M$.

The proof of Theorem 3 is provided in the supplementary material. Since all constraints are linear and the objective function is concave, (6) is a concave problem and, hence, strong duality holds. The dual problem is defined as follows:

$$\begin{align*}
\max_{\theta, c, \lambda} \min_{\rho} & \quad L_W(\theta, c, \lambda, \rho) \\
\text{subject to} & \quad \forall s, a \lambda_{sa} \geq 0,
\end{align*}$$ (7)

where $L_W(\theta, c, \lambda, \rho) = -\alpha \bar{W}(\rho) - \sum_{s, a} \rho(s, a) \theta^T \phi(s, a) + \sum_{s, a} \rho_E(s, a) \theta^T \phi(s, a) - \sum_{s, a} \lambda_{sa} \rho(s, a) + \sum_s c_s \left(\sum_{a} \rho(s, a) - d(s) - \gamma \sum_{s', a'} T(s|s', a')\rho(s', a')\right)$ and $\theta$, $c$, and $\lambda$ are Lagrangian multipliers and the constraints come from $M$. Then, the optimal solution of primal and dual variables necessarily and sufficiently satisfy the KKT conditions.
where

which determines a supporting set. If expert’s policy is multi-modal at state which assigns zero probability to an action whose auxiliary variable

Theorem 4. The maximum causal Tsallis entropy distribution minimizes the worst case prediction Brier score.

\[
\min_{\pi} \max_{\pi} \mathbb{E}_{\tilde{\pi}} \left[ \sum_{a'} \frac{1}{2} \left( \mathbb{1}_{\{a'=a\}} - \pi(a|s) \right)^2 \right] \quad \text{subject to} \quad \mathbb{E}_{\pi} [\phi(s, a)] = \mathbb{E}_{\tilde{\pi}} [\phi(s, a)]
\]

where \( \sum_{a'} \frac{1}{2} \left( \mathbb{1}_{\{a'=a\}} - \pi(a|s) \right)^2 \) is the Brier score.

Note that minimizing the Brier score minimizes the misprediction ratio while we call it a score here. Theorem 5 is a straightforward extension of the robust Bayes results in [14] to sequential decision problems. This theorem tells us that the MCTE problem can be viewed as a minimax game between a sequential decision maker \( \pi \) and the nature \( \tilde{\pi} \) based on the Brier score. In this regards, the resulting estimator can be interpreted as the best decision maker against the worst that the nature can offer.

3.2 Interpretation as Robust Bayes

In this section, we provide an interesting interpretation about the MCTE framework. In general, maximum entropy estimation can be viewed as a minimax game between two players. One player is called a decision maker and the other player is called the nature, where the nature assigns a distribution to maximize the decision maker’s misprediction while the decision maker tries to minimize it [16]. The same interpretation can be applied to the MCTE framework. We show that the proposed MCTE problem is equivalent to a minimax game with the Brier score [14].

Theorem 5. The maximum causal Tsallis entropy distribution minimizes the worst case prediction Brier score.

\[
\min_{\theta} \max_{\pi} \mathbb{E}_{\pi} \left[ \sum_{s,a} \frac{1}{2} \left( q_{sa} \phi(s, a) + \gamma \sum_{s'} c_{s'} T(s'|s, a), c_{s'} = \alpha \left[ \frac{1}{2} \sum_{a \in S(s)} \left( \frac{q_{sa}}{\alpha} \right)^2 - \tau \left( \frac{q_{sa}}{\alpha} \right)^2 \right] + \frac{1}{2} \right], \text{ and} \quad \pi_p(a|s) = \max \frac{q_{sa}}{\alpha} - \tau \left( \frac{q_{sa}}{\alpha} \right), 0 \right),
\]

where \( \pi_p(a|s) = \frac{q_{sa}}{\sum_a q_{sa}} \), \( q_{sa} \) is an auxiliary variable, and \( q_s = [q_{s1}, \cdots q_{s|A|}]^\top \).

The optimality conditions of the problem (6) tell us that the MCTE problem can be viewed as a minimax game between two players. One player is called a decision maker and the other player is called the nature, where the nature assigns a distribution to maximize the decision maker’s misprediction while the decision maker tries to minimize it [16]. Since the optimality condition is equivalent to the sparse Bellman optimality equation [3], we can compute the optimal policy and state-action value for the reward function \( r(s, a) = \theta^\top \phi(s, a) \), where \( \theta^\top \) is the optimal dual variable. In addition, \( c_{s'} \) and \( q_{sa} \) can be viewed as a state value and state-action value for the reward \( \theta^\top \phi(s, a) \), respectively.

Furthermore, we also discover an interesting connection between the optimality condition of an MCTE problem and the sparse Bellman optimality condition [4]. Since the optimality condition is equivalent to the sparse Bellman optimality equation [3], we can compute the optimal policy and Lagrangian multiplier \( c_{s'} \) by solving a sparse MDP under the reward function \( r(s, a) = \theta^\top \phi(s, a) \), where \( \theta^\top \) is the optimal dual variable. In addition, \( c_{s'} \) and \( q_{sa} \) can be viewed as a state value and state-action value for the reward \( \theta^\top \phi(s, a) \), respectively.

4 Maximum Causal Tsallis Entropy Imitation Learning

In this section, we propose a maximum causal Tsallis entropy imitation learning (MCTEIL) algorithm to solve a model-free IL problem in a continuous action space. In many real-world problems, state and action spaces are often continuous and transition probability of a world cannot be accessed. To apply the MCTE framework for a continuous space and model-free case, we follow the extension of GAIL [9], which trains a policy and reward alternatively, instead of solving RL at every iteration. We extend the MCTE framework to a more general case with reward regularization and it is formulated by replacing the causal entropy \( H(\pi) \) in the problem (2) with the causal Tsallis entropy \( W(\pi) \) as follows:

\[
\max_{\theta} \min_{\pi} \mathbb{E}_{\pi} \left[ -\alpha W(\pi) - \mathbb{E}_{\pi} [\theta^\top \phi(s, a)] + \mathbb{E}_{\pi} [\theta^\top \phi(s, a)] - \psi(\theta) \right].
\]

Similarly to [9], we convert the problem (2) into the generative adversarial setting as follows.
p \text{model mixture weights using a sparsemax distribution, the number of mixtures used to model } 
\sum \text{where } \pi \text{ is called a sparse MDN. In many imitation learning algorithms, a Gaussian network is often } 
\text{implemented using a softmax distribution. Modeling the expert's policy using an MDN with } 
\text{sparsemax weight selection, which can model sparse and multi-modal behavior of an expert, } 
\text{mixtures can be interpreted as separating continuous action space into } K \text{ mixtures of Gaussians: mixture weights } \{ w_i \text{, means } \{ \mu_i \}, \text{and covariance matrices } \{ \Sigma_i \}. \text{ A sparse MDN policy is defined as } 
\pi(a|s) = \sum_{i=1}^{K} w_i(s) \mathcal{N}(a; \mu_i(s), \Sigma_i(s)), \text{ where } \mathcal{N}(a; \mu, \Sigma) \text{ indicates a multivariate Gaussian density at point } a \text{ with mean } \mu \text{ and covariance } \Sigma. \text{ In our implementation, } w(s) \text{ is computed as a sparsemax distribution, while most existing MDN implementations utilize a softmax distribution. Modeling the expert's policy using an MDN with } K \text{ mixtures can be interpreted as separating continuous action space into } K \text{ representative actions. Since we model mixture weights using a sparsemax distribution, the number of mixtures used to model the expert's policy can vary depending on the state. In this regards, the sparsemax weight selection has an advantage over the soft weight selection since the former utilizes mixture components more efficiently as unnecessary components will be assigned with zero weights.}

\textbf{Tsallis Entropy of Mixture Density Network} \quad \text{An interesting fact is that the causal Tsallis entropy of an MDN has an analytic form while the Gibbs-Shannon entropy of an MDN is intractable.}
Theorem 7. Let $\pi(a|s) = \sum_i^K w_i(s) \mathcal{N}(\mu_i(s), \Sigma_i(s))$. Then,

$$W(\pi) = \frac{1}{2} \sum_x \rho_x(s) \left( 1 - \sum_i^K \sum_j^K w_i(s)w_j(s) \mathcal{N}(\mu_i(s); \mu_j(s), \Sigma_i(s) + \Sigma_j(s)) \right).$$ (11)

The proof is included in the causal Tsallis trial. The analytic form of the Tsallis entropy shows that the Tsallis entropy is proportional to the distance between mixture means. Hence, maximizing the Tsallis entropy of a sparse MDN encourages exploration of diverse directions during the policy optimization step of MCTEIL. In imitation learning, the main benefit of the generative adversarial setting is that the resulting policy is more robust than that of supervised learning since it can learn how to recover from a less demonstrated region to a demonstrated region by exploring the state-action space during training. Maximum Tsallis entropy of a sparse MDN encourages efficient exploration by giving bonus rewards when mixture means are spread out. (11) also has an effect of utilizing mixtures more efficiently by penalizing for modeling a single mode using several mixtures. Consequently, the Tsallis entropy $W(\pi)$ has clear benefits in terms of both exploration and mixture utilization.

5 Experiments

To verify the effectiveness of the proposed method, we compare MCTEIL with several other imitation learning methods. First, we use behavior cloning (BC) as a baseline. Second, generative adversarial imitation learning (GAIL) with a single Gaussian distribution is compared. While several variants of GAIL exist \cite{18,19}, they are all based on the maximum causal entropy framework and utilize a single Gaussian distribution as a policy function. Hence, we choose GAIL as the representative method. We also compare a straightforward extension of GAIL for a multi-modal policy by using a softmax weighted mixture density network (soft MDN) in order to validate the efficiency of the proposed sparsemax weighted MDN. In soft GAIL, due to the intractability of the causal entropy of a mixture of Gaussians, we approximate the entropy term by adding $-\alpha \log(\pi(a_t|s_t))$ to $-\log(D(s_t, a_t))$ since $\mathbb{E}_\pi [-\alpha \log(D(s, a))] + \alpha H(\pi) = \mathbb{E}_\pi [-\log(D(s, a)) - \alpha \log(\pi(a|s))]$. The other related imitation learning methods for multi-modal task learning, such as \cite{20,21}, are excluded from the comparison since they focus on the task level multi-modality, where the multi-modality of demonstrations comes from multiple different tasks. In comparison, the proposed method captures the multi-modality of the optimal policy for a single task. We would like to note that our method can be extended to multi-modal task learning as well.

5.1 Multi-Goal Environment

To validate that the proposed method can learn multi-modal behavior of an expert, we design a simple multi-goal environment with four attractors and four repulsors, where an agent tries to reach one of the attractors while avoiding all repulsors as shown in Figure 1(a). The agent follows the point-mass dynamics and get a positive reward (resp., a negative reward) when getting closer to an attractor (resp., repulsor). Intuitively, this problem has multi-modal optimal actions at the center. We first train the optimal policy using \cite{3} and generate 300 demonstrations from the expert’s policy. For both soft GAIL and MCTEIL, 500 episodes are sampled at each iteration. In every iterations, we measure the average return using the underlying rewards and the reachability which is measured by counting how many goals are reached. If the algorithm captures the multi-modality of expert’s demonstrations, then, the resulting policy will show high reachability.

The results are shown in Figure 1(b) and 1(c). Since the rewards are multi-modal, it is easy to get a high return if the algorithm learns only uni-modal behavior. Hence, the average returns of soft GAIL and MCTEIL increases similarly. However, when it comes to the reachability, MCTEIL outperforms soft GAIL when they use the same number of mixtures. In particular, MCTEIL can learn all modes in demonstrations at the end of learning while soft GAIL suffers from collapsing mixture means. This advantage clearly comes from the maximum Tsallis entropy of a sparse MDN since the analytic form of the Tsallis entropy directly penalizes collapsed mixture means while $-\log(\pi(a|s))$ indirectly prevents modes collapsing in soft GAIL. Consequently, MCTEIL efficiently utilizes each mixture for wide-spread exploration.
5.2 Continuous Control Environment

We test MCTEIL with a sparse MDN on MuJoCo [10], which is a physics-based simulator, using Halfcheetah, Walker2d, Reacher, and Ant. We train the expert policy distribution using trust region policy optimization (TRPO) [22] under the true reward function and generate 50 demonstrations from the expert policy. We run algorithms with varying numbers of demonstrations, 4, 11, 18, and 25, and all experiments have been repeated three times with different random seeds. To evaluate the performance of each algorithm, we sample 50 episodes from the trained policy and measure the average return value using the underlying rewards. For methods using an MDN, we use the best number of mixtures using a brute force search.

The results are shown in Figure 2. For three problems, except Walker2d, MCTEIL outperforms the other methods with respect to the average return as the number of demonstrations increases. For Walker2d, MCTEIL and soft GAIL show similar performance. Especially, in the reacher problem, we obtain the similar results reported in [9], where BC works better than GAIL. However, our method shows the best performance for all demonstration counts. It is observed that the MDN policy tends to show high performance consistently since MCTEIL and soft GAIL are consistently ranked within the top two high performing algorithms. From these results, we can conclude that an MDN policy explores better than a single Gaussian policy since an MDN can keep searching multiple directions during training. In particular, since the maximum Tsallis entropy makes each mixture mean explore in different directions and a sparsemax distribution assigns zero weight to unnecessary mixture components, MCTEIL efficiently explores and shows better performance compared to soft GAIL with a soft MDN. Consequently, we can conclude that MCTEIL outperforms other imitation learning methods and the causal Tsallis entropy has benefits over the causal Gibbs-Shannon entropy as it encourages exploration more efficiently.

6 Conclusion

In this paper, we have proposed a novel maximum causal Tsallis entropy (MCTE) framework, which induces a sparsemax distribution as the optimal solution. We have also provided the full mathematical analysis of the proposed framework, including the concavity of the problem, the optimality condition, and the interpretation as robust Bayes. We have also developed the maximum causal Tsallis entropy
imitation learning (MCTEIL) algorithm, which can efficiently solve a MCTE problem in a continuous action space since the Tsallis entropy of a mixture of Gaussians encourages exploration and efficient mixture utilization. In experiments, we have verified that the proposed method has advantages over existing methods for learning the multi-modal behavior of an expert since a sparse MDN can search in diverse directions efficiently. Furthermore, the proposed method has outperformed BC, GAIL, and GAIL with a soft MDN on the standard IL problems in the MuJoCo environment. From the analysis and experiments, we have shown that the proposed MCTEIL method is an efficient and principled way to learn the multi-modal behavior of an expert.

A Analysis

Proof of Theorem 2 The proof is simply done by checking two equalities. First,

\[
W(\pi) = \frac{1}{2} \mathbb{E}_\pi [1 - \pi(a|s)] = \frac{1}{2} \sum_{s,a} \rho_\pi(s,a) (1 - \pi(a|s)) \\
= \frac{1}{2} \sum_{s,a} \rho_\pi(s,a) \left( 1 - \frac{\rho(s,a)}{\sum_{a'} \rho(s,a')} \right)
\]

and, second,

\[
\bar{W}(\rho) = \frac{1}{2} \sum_{s,a} \rho(s,a) \left( 1 - \frac{\rho(s,a)}{\sum_{a'} \rho(s,a')} \right) = \frac{1}{2} \sum_{s,a} \rho_\pi(s,a) \left( 1 - \pi_\rho(a|s) \right) \\
= W(\pi_\rho).
\]

A.1 Concavity of Maximum causal Tsallis Entropy

Proof of Theorem 2 Proof of concavity of \( \bar{W}(\rho) \) is equivalent to show that following inequality is satisfied for all state \( s \) and action \( a \) pairs:

\[
(\lambda_1 \rho_1(s,a) + \lambda_2 \rho_2(s,a)) \left( 1 - \frac{\lambda_1 \rho_1(s,a) + \lambda_2 \rho_2(s,a)}{\lambda_1 \sum_{a'} \rho_1(s,a') + \lambda_2 \sum_{a'} \rho_2(s,a')} \right) \\
\geq \lambda_1 \rho_1(s,a) \left( 1 - \frac{\rho_1(s,a)}{\sum_{a'} \rho_1(s,a')} \right) + \lambda_2 \rho_2(s,a) \left( 1 - \frac{\rho_2(s,a)}{\sum_{a'} \rho_2(s,a')} \right)
\]

where \( \lambda_1 \geq 0, \lambda_2 \geq 0, \) and \( \lambda_1 + \lambda_2 = 1. \) For notational simplicity, \( \rho_i(s,a) \) and \( \sum_{a'} \rho_i(s,a') \) are replaced with \( a_i \) and \( b_i, \) respectively. Then, the right-hand side is

\[
\sum_{i=1,2} \lambda_i a_i \left( 1 - \frac{a_i}{b_i} \right) = \sum_{i=1,2} \lambda_i a_i \left( 1 - \frac{a_i}{\lambda_i b_i} \right) \\
= \left( \sum_{j=1,2} \lambda_j b_j \right) \sum_{i=1,2} \left[ \frac{\lambda_i b_i}{\sum_{j=1,2} \lambda_j b_j} \right] \left( 1 - \frac{a_i}{\lambda_i b_i} \right).
\]

Let \( F(x) = x(1-x), \) which is a concave function. Then the above equation can be expressed as follows,

\[
\sum_{i=1,2} \lambda_i a_i \left( 1 - \frac{a_i}{b_i} \right) = \left( \sum_{i=1,2} \lambda_i b_j \right) \sum_{i=1,2} \left[ \frac{\lambda_i b_i}{\sum_{j=1,2} \lambda_j b_j} \right] F \left( \frac{\lambda_i a_i}{\lambda_i b_i} \right).
\]
By using the property of concave function $F(x)$ we obtain the following inequality:

$$
\left( \sum_{j=1,2} \lambda_j b_j \right) \sum_{i=1,2} \left[ \frac{\lambda_i b_i}{\sum_{j=1,2} \lambda_j b_j} F\left( \frac{\lambda_i a_i}{\lambda_i b_i} \right) \right] \\
\leq \left( \sum_{j=1,2} \lambda_j b_j \right) F\left( \sum_{i=1,2} \left[ \frac{\lambda_i b_i}{\sum_{j=1,2} \lambda_j b_j} \lambda_i a_i \right] \right) = \left( \sum_{i=1,2} \lambda_i b_j \right) F\left( \sum_{i=1,2} \lambda_i a_i \right) \\
= \left( \sum_{j=1,2} \lambda_j b_j \right) \frac{\sum_{i=1,2} \lambda_i a_i}{\sum_{j=1,2} \lambda_j b_j} \left( 1 - \frac{\sum_{i=1,2} \lambda_i a_i}{\sum_{j=1,2} \lambda_j b_j} \right) = \sum_{i=1,2} \lambda_i a_i \left( 1 - \frac{\sum_{i=1,2} \lambda_i a_i}{\sum_{j=1,2} \lambda_j b_j} \right).
$$

Finally, we have the following inequality for every state and action pair,

$$(\lambda_1 \rho_1(s, a) + \lambda_2 \rho_2(s, a)) \left( 1 - \frac{\lambda_1 \rho_1(s, a) + \lambda_2 \rho_2(s, a)}{\lambda_1 \sum_{a'} \rho_1(s, a') + \lambda_2 \sum_{a'} \rho_2(s, a')} \right) \\
\geq \lambda_1 \rho_1(s, a) \left( 1 - \frac{\rho_1(s, a)}{\sum_{a'} \rho_1(s, a')} \right) + \lambda_2 \rho_2(s, a) \left( 1 - \frac{\rho_2(s, a)}{\sum_{a'} \rho_2(s, a')} \right),$$

and, by summing up with respect to $s, a$, we get

$$W(\lambda_1 \rho_1 + \lambda_2 \rho_2) \geq \lambda_1 W(\rho_1) + \lambda_2 W(\rho_2).$$

Therefore, $W(\rho)$ is a concave function.

**A.2 Optimality Condition from Karush–Kuhn–Tucker (KKT) conditions**

The following proof explains the optimality condition of the maximum causal Tsallis entropy problem and also tells us that the optimal policy distribution has a sparse and multi-modal distribution.

**Proof of Theorem** These conditions are derived from the stationary condition of KKT, where the derivative of $L_W$ is equal to zero,

$$\frac{\partial L_W}{\partial \rho(s, a)} = 0.$$

We first compute the derivative of $W$ as follows:

$$\frac{\partial W}{\partial \rho(s, a)} = \frac{1}{2} - \sum_{a'} \rho(s, a') + \frac{1}{2} \sum_{a'} \left( \sum_{a'} \rho(s, a') \right)^2.$$

We also check the derivative of Bellman flow constraints as follows:

$$\frac{\partial}{\partial \rho(s', a'')} \sum_s c_s \left( \sum_{a'} \rho(s, a') - d(s) - \gamma \sum_{s', a'} T(s|s', a') \rho(s', a') \right) = c_{s''} - \gamma \sum_{s} c_s T(s|s', a'').$$

Hence, the stationary condition can be obtained as

$$\frac{\partial L_W}{\partial \rho(s, a)} = \alpha \left[ -\frac{1}{2} + \sum_{a'} \rho(s, a') - \frac{1}{2} \sum_{a'} \left( \sum_{a'} \rho(s, a') \right)^2 \right] - \theta T \phi(s, a) \\
+ c_s - \gamma \sum_{s'} c_s T(s'|s, a) - \lambda_{sa} = 0. \tag{12}$$

First, let us consider a positive $a \in S(s) = \{a|\rho(s, a) > 0\}$. From the complementary slackness, the corresponding $\lambda_{sa}$ is zero. By replacing $\sum_{a'} \rho(s, a')$ with $\pi(a|s)$ and using the definition of $q_{sa}$, the following equation is obtained from the stationary condition [12],

$$\pi(a|s) - \frac{q_{sa}}{\alpha} = \frac{1}{2} + \frac{1}{2} \sum_{a'} \left( \pi(a'|s) \right)^2 - \frac{c_s}{\alpha}. \tag{13}$$

$^4 \sum_i \mu_i F(x_i) \leq F(\sum_i \mu_i x_i)$, for some $(x_1, \ldots, x_n)$ and $(\mu_1, \ldots, \mu_n)$ such that $\mu_i \geq 0$ and $\sum_i \mu_i = 1$. 

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It can be observed that the right hand side of the equation only depends on the state \( s \) and is constant for the action \( a \). In this regards, by summing up with respect to the action with positive \( \rho(s, a) > 0 \), \( c_s \) is obtained as follows:

\[
1 - \frac{\sum_{a \in S(s)} q_{sa}}{\alpha} = K \left( \frac{1}{2} + \frac{1}{2} \sum_{a'} (\pi(a'|s))^2 - \frac{c_s}{\alpha} \right)
\]

\[
\frac{c_s}{\alpha} = \frac{1}{2} + \frac{1}{2} \sum_{a' \in S(s)} (\pi(a'|s))^2 + \frac{\sum_{a \in S(s)} q_{sa}}{K} - \frac{1}{\alpha},
\]

where \( K \) is the number of actions with positive \( \rho(s, a) > 0 \). By plug in \( \frac{c_s}{\alpha} \) into (13), we obtain a policy as follows:

\[
\pi(a|s) = \frac{q_{sa}}{\alpha} - \left( \frac{\sum_{a \in S(s)} q_{sa}}{K} - \frac{1}{\alpha} \right)
\]

Now, we define \( \tau(\frac{q_{sa}}{\alpha}) \triangleq \frac{\sum_{a \in S(s)} q_{sa}}{K} - \frac{1}{\alpha} \), and, interestingly, \( \tau \) is the same as the threshold of a sparsemax distribution (6). Then, we can obtain the optimality condition for the policy distribution \( \pi(a|s) \) as follows:

\[
\forall s, a \quad \pi(a|s) = \max \left( \frac{q_{sa}}{\alpha} - \tau(s), 0 \right).
\]

where \( \tau(s) \) indicates \( \tau(\frac{q_{sa}}{\alpha}) \).

The Lagrangian multiplier \( c_s \) can be found from \( \pi \) as follows:

\[
\frac{c_s}{\alpha} = \frac{1}{2} + \frac{1}{2} \sum_{a' \in S(s)} (\pi(a'|s))^2 + \tau(s)
\]

\[
= \frac{1}{2} + \frac{1}{2} \sum_{a' \in S(s)} \left( \frac{q_{sa'}}{\alpha} - \tau(s) \right)^2 + \tau(s)
\]

\[
= \frac{1}{2} + \frac{1}{2} \sum_{a' \in S(s)} \left( \frac{q_{sa'}}{\alpha} \right)^2 - \sum_{a' \in S(s)} \frac{q_{sa'}}{\alpha} \tau(s) + \frac{K}{2} \tau(s)^2 + \tau(s)
\]

\[
= \frac{1}{2} + \frac{1}{2} \sum_{a' \in S(s)} \left( \frac{q_{sa'}}{\alpha} \right)^2 - K \sum_{a' \in S(s)} \frac{q_{sa'}}{\alpha} - \frac{1}{\alpha} \tau(s) + \frac{K}{2} \tau(s)^2
\]

\[
c_s = \alpha \left[ \frac{1}{2} \sum_{a \in S(s)} \left( \frac{q_{sa}}{\alpha} - \tau \left( \frac{q_s}{\alpha} \right) \right)^2 + \frac{1}{2} \right]
\]

To summarize, we obtain the optimality condition of (6) as follows:

\[
q_{sa} \triangleq \theta^T \phi(s, a) + \gamma \sum_{s'} c_{s'} T(s'|s, a),
\]

\[
c_s = \alpha \left[ \frac{1}{2} \sum_{a \in S(s)} \left( \frac{q_{sa}}{\alpha} - \tau \left( \frac{q_s}{\alpha} \right) \right)^2 + \frac{1}{2} \right],
\]

\[
\pi(a|s) = \max \left( \frac{q_{sa}}{\alpha} - \tau \left( \frac{q_s}{\alpha} \right), 0 \right).
\]

\[\square\]

### A.3 Interpretation as Robust Bayes

In this section, the connection between MCTE estimation and a minimax game between a decision maker and the nature is explained. We prove that the proposed MCTE problem is equivalent to a minimax game with the Brier score.
Proof of Theorem 3. The objective function can be reformulated as
\[
E_{\tilde{\pi}} \left[ \sum_{a'} \frac{1}{2} \left( \mathbb{1}_{(a'=a)} - \tilde{\pi}(a'|s) \right)^2 \right] = E_{\tilde{\pi}} \left[ B(s,a) \right] = \sum_{s,a} \rho_{\tilde{\pi}}(s,a)B(s,a) \\
= \frac{1}{2} \sum_{s,a} \rho_{\tilde{\pi}}(s,a) \left( 1 - 2\tilde{\pi}(a|s) + \sum_{a'} \pi(a'|s)^2 \right),
\]
Hence, the objective function is quadratic with respect to \( \pi(a|s) \) and is linear with respect to \( \rho_{\tilde{\pi}}(s,a) \). By using the one-to-one correspondence between \( \tilde{\pi} \) and \( \rho_{\tilde{\pi}} \), we change the variable of inner maximization into the state action visitation. After changing the optimization variable, by using the minmax theorem \[17\], the minimization and maximization of the problem (8) are interchangeable as follows:
\[
\min_{\pi \in \Pi} \max_{\rho_{\tilde{\pi}} \in M} E_{\tilde{\pi}} \left[ \sum_{a'} \frac{1}{2} \left( \mathbb{1}_{(a'=a)} - \tilde{\pi}(a|s) \right)^2 \right] \\
= \max_{\rho_{\tilde{\pi}} \in M} \min_{\pi \in \Pi} E_{\tilde{\pi}} \left[ \sum_{a'} \frac{1}{2} \left( \mathbb{1}_{(a'=a)} - \tilde{\pi}(a|s) \right)^2 \right]
\]
where sum-to-one, positivity, and Bellman flow constraints are omitted here. After converting the problem, the optimal solution of inner minimization with respect to \( \pi \) is easily computed as \( \pi = \tilde{\pi} \) using \( \nabla_{\pi(a'|s)} E_{\tilde{\pi}}[B(s,a)] = 0 \). After applying \( \pi = \tilde{\pi} \) and recovering the variables from \( \rho_{\tilde{\pi}} \) to \( \tilde{\pi} \), the problem (8) is converted into
\[
\max_{\tilde{\pi} \in \Pi} \frac{1}{2} \sum_s \rho_{\tilde{\pi}}(s) \left( 1 - \sum_a \tilde{\pi}(a|s)^2 \right) = \max_{\tilde{\pi} \in \Pi} W(\tilde{\pi}),
\]
which equals to the causal Tsallis entropy. Hence, the problem (8) is equivalent to the maximum causal Tsallis entropy problem.

A.4 Generative Adversarial Setting with Maximum Causal Tsallis Entropy

Proof of Theorem 4. We first change the variable from \( \pi \) to \( \rho \) as follows:
\[
\max_{\theta} \min_{\rho} - \alpha W(\rho) - \theta^T \sum_{s,a} \rho(s,a)\phi(s,a) - \theta^T \sum_{s,a} \rho_E(s,a)\phi(s,a) - \psi(\theta) \\
\text{subject to } \forall s,a, \sum_{s,a} \rho(s,a)\phi(s,a) = \sum_{s,a} \rho_E(s,a)\phi(s,a), \quad (14)
\]
where \( \rho_E \) is \( \rho_{\pi_E} \). Let
\[
\bar{L}(\rho,\theta) \triangleq - \alpha W(\rho) - \psi(\theta) - \theta^T \sum_{s,a} \rho(s,a)\phi(s,a) + \theta^T \sum_{s,a} \rho_E(s,a)\phi(s,a). \quad (15)
\]
From Theorem 3, \( W(\rho) \) is a concave function with respect to \( \rho \) for a fixed \( \theta \). Hence, \( \bar{L}(\rho,\theta) \) is also a concave function with respect to \( \rho \) for a fixed \( \theta \). From the convexity of \( \psi \), \( \bar{L}(\rho,\theta) \) is a convex function with respect to \( \theta \) for a fixed \( \rho \). Furthermore, the domain of \( \rho \) is compact and convex and the domain of \( \theta \) is convex. Based on this property of \( \bar{L}(\rho,\theta) \), we can use minimax duality \[17\]:
\[
\max_{\theta} \min_{\rho} \bar{L}(\rho,\theta) = \min_{\rho} \max_{\theta} \bar{L}(\rho,\theta).
\]
Hence, the maximization and minimization are interchangable. By using this fact, we have:

$$\max_{\theta} \min_{\rho} \tilde{L}(\rho, \theta) = \min_{\rho} \max_{\theta} \tilde{L}(\rho, \theta)$$

$$= \min_{\rho} -\alpha \tilde{W}(\rho) + \max_{\theta} \left( -\psi(\theta) + \theta^T \sum_{s,a} (\rho(s,a) - \rho_E(s,a)) \phi(s,a) \right)$$

$$= \min_{\rho} -\alpha \tilde{W}(\rho) + \psi^* \left( \sum_{s,a} (\rho(s,a) - \rho_E(s,a)) \phi(s,a) \right)$$

$$= \min_{\pi} \psi^* \left( \mathbb{E}_{\pi} [\phi(s,a)] - \mathbb{E}_{\pi E} [\phi(s,a)] \right) - \alpha \tilde{W}(\pi)$$

\[\Box\]

A.5 Tsallis Entropy of a Mixture of Gaussians

Proof of Theorem In our implementation of maximum causal Tsallis entropy imitation learning (MCTEIL), we approximate $W(\pi)$ using sampled trajectories as follows:

$$W(\pi) = \frac{1}{2} \sum_s \rho_\pi(s) \left( 1 - \int_A \pi(a|s)^2 \, da \right)$$

$$= \frac{1}{2} \sum_s \rho_\pi(s) \left( 1 - \int_A \left( \sum_i^K w_i(s) \mathcal{N}(a; \mu_i(s), \Sigma_i(s)) \right)^2 \, da \right)$$

$$= \frac{1}{2} \sum_s \rho_\pi(s) \left( 1 - \sum_i^K \sum_j^K w_i(s) w_j(s) \int_A \mathcal{N}(a; \mu_i(s), \Sigma_i(s)) \mathcal{N}(a; \mu_j(s), \Sigma_j(s)) \, da \right)$$

$$= \frac{1}{2} \sum_s \rho_\pi(s) \left( 1 - \sum_i^K \sum_j^K w_i(s) w_j(s) \mathcal{N}(\mu_i(s); \mu_j(s), \Sigma_i(s) + \Sigma_j(s)) \right)$$

\[\Box\]

B Causal Entropy Approximation

In our implementation of maximum causal Tsallis entropy imitation learning (MCTEIL), we approximate $W(\pi)$ using sampled trajectories as follows:

$$W(\pi) = \mathbb{E}_\pi \left[ \frac{1}{2} (1 - \pi(a|s)) \right] \approx \frac{1}{N} \sum_{i=0}^N \sum_{t=0}^{T_i} \gamma^t \left( 1 - \int_A \pi(a|s_i,t)^2 \, da \right),$$

(17)

where $\{(s_{i,t}, a_{i,t})_{i=0}^N\}$ are $N$ trajectories and $T_i$ is the length of the $i$th trajectory. Since the integral part of (17) is analytically computed by Theorem there is no additional computational cost. We have also tested the following approximation:

$$W(\pi) = \mathbb{E}_\pi \left[ \frac{1}{2} (1 - \pi(a|s)) \right] \approx \frac{1}{N} \sum_{i=0}^N \sum_{t=0}^{T_i} \gamma^t \left( 1 - \pi(a_{i,t}|s_{i,t}) \right).$$

However, this approximation has performed poorly compared to (17).

For soft GAIL, $H(\pi)$ is approximated as the sum of discounted likelihoods

$$H(\pi) = \mathbb{E}_\pi \left[ -\log \left( \pi(a|s) \right) \right] \approx \frac{1}{N} \sum_{i=0}^N \sum_{t=0}^{T_i} -\gamma^t \log \left( \pi(a_{i,t}|s_{i,t}) \right).$$

Note that the same approximation (17) of $W(\pi)$ is not available for $H(\pi)$ since $-\int_A \pi(a|s) \log \left( \pi(a|s) \right) \, da$ is intractable when we model $\pi(a|s)$ as a mixture of Gaussians.
C Additional Experimental Results

In the multi-goal environment, the experimental results with other hyperparameters are shown in Figure 3.

Figure 3: (a) and (b) show the average return and reachability of MCTEIL, respectively. (c) and (d) show the average return and reachability of soft GAIL, respectively. $k$ indicates the number of mixtures and $\alpha$ indicates an entropy regularization coefficient.

References

[1] B. D. Ziebart, A. L. Maas, J. A. Bagnell, and A. K. Dey, “Maximum entropy inverse reinforcement learning,” in Proceedings of the 23rd AAAI Conference on Artificial Intelligence, July 2008, pp. 1433–1438.

[2] T. Haarnoja, H. Tang, P. Abbeel, and S. Levine, “Reinforcement learning with deep energy-based policies,” in Proceedings of the 34th International Conference on Machine Learning, August 2017, pp. 1352–1361.

[3] K. Lee, S. Choi, and S. Oh, “Sparse Markov decision processes with causal sparse Tsallis entropy regularization for reinforcement learning,” IEEE Robotics and Automation Letters, vol. 3, no. 3, pp. 1466–1473, 2018.

[4] N. Heess, D. Silver, and Y. W. Teh, “Actor-critic reinforcement learning with energy-based policies,” in Proceedings of the 10th European Workshop on Reinforcement Learning, June 2012, pp. 43–58.

[5] P. Vamplew, R. Dazeley, and C. Foale, “Softmax exploration strategies for multiobjective reinforcement learning,” Neurocomputing, vol. 263, pp. 74–86, Jun 2017.

[6] A. F. T. Martins and R. F. Astudillo, “From softmax to sparsemax: A sparse model of attention and multi-label classification,” in Proceedings of the 33nd International Conference on Machine Learning, June 2016, pp. 1614–1623.
[7] M. Bloem and N. Bambos, “Infinite time horizon maximum causal entropy inverse reinforcement learning,” in *Proceedings of the 53rd International Conference on Decision and Control*, December 2014, pp. 4911–4916.

[8] O. Nachum, Y. Chow, and M. Ghavamzadeh, “Path consistency learning in tsallis entropy regularized mdps,” 2018. [Online]. Available: [https://arxiv.org/abs/1802.03501](https://arxiv.org/abs/1802.03501)

[9] J. Ho and S. Ermon, “Generative adversarial imitation learning,” in *Advances in Neural Information Processing Systems*, December 2016, pp. 4565–4573.

[10] E. Todorov, T. Erez, and Y. Tassa, “MuJoCo: A physics engine for model-based control,” in *Proceedings of the International Conference on Intelligent Robots and Systems*, October 2012, pp. 5026–5033.

[11] B. D. Ziebart, “Modeling purposeful adaptive behavior with the principle of maximum causal entropy,” Ph.D. dissertation, Carnegie Mellon University, Pittsburgh, PA, USA, 2010, aAI3438449.

[12] P. Abbeel and A. Y. Ng, “Apprenticeship learning via inverse reinforcement learning,” in *Proceedings of the 21st International Conference of Machine Learning*, July 2004.

[13] U. Syed and R. E. Schapire, “A game-theoretic approach to apprenticeship learning,” in *Advances in neural information processing systems*, December 2007, pp. 1449–1456.

[14] G. W. Brier, “Verification of forecasts expressed in terms of probability,” *Monthly Weather Review*, vol. 78, no. 1, pp. 1–3, 1950.

[15] U. Syed, M. Bowling, and R. E. Schapire, “Apprenticeship learning using linear programming,” in *Proceedings of the 25th international conference on Machine learning*. ACM, 2008, pp. 1032–1039.

[16] P. D. Grünwald and A. P. Dawid, “Game theory, maximum entropy, minimum discrepancy and robust Bayesian decision theory,” *Annals of Statistics*, pp. 1367–1433, 2004.

[17] P. W. Millar, “The minimax principle in asymptotic statistical theory,” in *Ecole d’Eté de Probabilités de Saint-Flour XI—1981*. Springer, 1983, pp. 75–265.

[18] N. Baram, O. Anschel, I. Caspi, and S. Mannor, “End-to-end differentiable adversarial imitation learning,” in *Proceedings of the 34th International Conference on Machine Learning*, August 2017, pp. 390–399.

[19] Y. Li, J. Song, and S. Ermon, “Infogail: Interpretable imitation learning from visual demonstrations,” in *Advances in Neural Information Processing Systems*, December 2017, pp. 3815–3825.

[20] K. Hausman, Y. Chebotar, S. Schaal, G. S. Sukhatme, and J. J. Lim, “Multi-modal imitation learning from unstructured demonstrations using generative adversarial nets,” in *Advances in Neural Information Processing Systems*, December 2017, pp. 1235–1245.

[21] Z. Wang, J. S. Merel, S. E. Reed, N. de Freitas, G. Wayne, and N. Heess, “Robust imitation of diverse behaviors,” in *Advances in Neural Information Processing Systems*, December 2017, pp. 5326–5335.

[22] J. Schulman, S. Levine, P. Abbeel, M. I. Jordan, and P. Moritz, “Trust region policy optimization,” in *Proceedings of the 32nd International Conference on Machine Learning*, July 2015, pp. 1889–1897.