Quantum state of a free spin-$\frac{1}{2}$ particle and the inextricable dependence of spin and momentum under Lorentz transformations

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We revise the Dirac equation for a free particle and investigate Lorentz transformations on spinors. We study how the spin quantization axis changes under Lorentz transformations and evince the interplay between spin and momentum in this context.

I. INTRODUCTION

Authors like Alsing and Milburn [1], or Hacyan [2] concluded that entanglement is invariant under Lorentz transformations. Accordingly, Terashima and Ueda [3] showed that, under Lorentz transformations, the perfect anti-correlation of a singlet state is recovered if one appropriately corrects the detector’s orientation according to the Wigner rotation [4]. Caban and Remienlinski also did an interesting study of EPR correlations in the quantum field theory context[5]. On the other hand, Peres et al. [6, 7] considered a single free spin-$\frac{1}{2}$ particle, and literally stated that “the reduced density matrix for its spin is not covariant under Lorentz transformation. The spin entropy is not a relativistic scalar and has no invariant meaning”. This later conclusion by Peres et al. [6] may lead a careless reader to think that a mere change of reference frame could create entanglement. In Quantum Information, entanglement is a resource which allows for the realization of teleportation [8], quantum cryptography [9], quantum-enhanced global positioning [10], just to cite a few impressive possible applications. Though the Wigner rotation acts only on the spin degrees of freedom, it is not independent of the momentum, and a Lorentz transformation cannot be reduced solely to a Wigner rotation, without altering the momentum. In the relativistic context, spin and momentum are not independent degrees of freedom. It is not possible a change of inertial reference frame, which changes the momentum, i.e. a boost, without changing the quantization axis of the spin. Among others, this was realized by Gingrich and Adami [11], who concluded that “while spin and momentum entanglement separately are not Lorentz invariant, the joint entanglement of the wave function is.” As a matter of fact, it is simple to understand. If we insist in facing spin and momentum as independent degrees of freedom in the relativistic context, which they are not, a change of inertial reference frame is a global linear operation acting on spin and momentum. Even when one chooses a unitary representation for the inhomogeneous Lorentz group, and then the Wigner rotation comes into play, a Lorentz boost acts as a global unitary on spin and momentum, and not as local unitary solely on spin, or solely on momentum. A way out of this subtleties was proposed by Bartlett and Terno [12] in their “relativistic invariant quantum information” approach. Notwithstanding one can find in the literature calculations showing the variation of spin entanglement solely due to a Lorentz transformation [13].

The inextricable dependence of spin and momentum, in the relativistic quantum information context, was discussed by Czachor and Wilczewski [14], who considered the relativistic version of the Bennett-Brassard [16] cryptographic scheme. Even though, the subject is controversial, as exemplified by Czachor’s [14] comment on Peres et al. work [6]. Following ideas similar to Cza-
chor’s, Caban and Rembieliński [18] developed a covariant reduced spin density matrix, with the caveat that the momentum dependent Lorentz transformations could not be represented unitarily and should act on the Dirac spinors rather than on spin kets. Czachor recently wrote an extensive study of the covariance of quantum information [17], and also concluded that the entanglement does not change under Lorentz transformations.

As one can easily realize, the quantum information community is still trying to come to grips with the proper treatment of entanglement in the relativistic context. The present work is an attempt to guide the newcomer straightforwardly to the conclusion of the covariance of entanglement, using a very simple but yet rigorous formalism. The main difference of our approach and similar works in the literature is the way we treat the kinematic degrees of freedom. We explicitly identify the single particle Hilbert space as labeled by both spin and momentum, calling attention to the fact that they do not compose by means of a tensor product and, therefore, have no well defined partial trace, in the usual sense, but which does not hinder one to take averages on momentum, as done by Caban and Rembieliński [18]. The other crucial point is that we do not single out any particular spin operator, like Pauli-Lubanski, rather we obtain the correct basis for spin measurements directly from the form of the operators in the Lorentz algebra. This is illustrated by considering two observers in relative motion, sending particles with well defined momentum to each other and performing spin measurements, a scenario representative of quantum information exchange between an Earth station and a satellite [19].

The paper is organized as follows. We start by reviewing the Dirac equation for a free particle. We then investigate how the free spin-\(\frac{1}{2}\) particle state transforms under an inertial reference frame change. In order to do that, we use the invariance of the Hilbert space internal product [4] and derive a non-unitary representation of the Lorentz group. Our motivation to do this is that the choice of representation does not change the physics, and thus we can highlight the interdependence of spin and momentum in the Lorentz transformation. In the sequence, we verify that the purity (or mixedness) of the state is Lorentz invariant, and show how the spin quantization axis changes with the momentum. Finally we do some illustrative calculations on how to perform the momentum dependent spin measurements on moving frames, and conclude.

II. DIRAC EQUATION FOR A FREE PARTICLE

In this section we recall the arguments that lead to the fundamental equation of the relativistic quantum mechanics, the Dirac equation [20–22], whose solution is a particle with 4-momentum \(p = (E, \vec{p})\) and spin 1/2 (or an antiparticle with 4-momentum \(p = (-E, \vec{p})\)). We first write a Schrödinger like equation (natural units are assumed, \(\hbar = c = 1\)):

\[
i \frac{\partial \Psi(x)}{\partial t} = H \Psi(x),
\]

where the variable \(x = (t, \vec{x})\) is the 4-position in the Minkowski space. The Hamiltonian \(H\) is the relativistic energy \(H = \sqrt{\vec{p}^2 + m^2}\), where \(\vec{p}\) is the momentum vector and \(m\) is the mass in rest frame. The Dirac equation is obtained by imposing linear derivatives on both space and time in Eq.1, which guarantees Lorentz covariance (Lorentz invariance of form), i.e. the physical content of the equation is the same in all relativistic inertial frames. Therefore, the Dirac Hamiltonian has the form:

\[
H = \vec{\alpha} \cdot \vec{p} + \beta m,
\]

and the Dirac equation for a free particle reads:

\[
i \frac{\partial \Psi(x)}{\partial t} = (\vec{\alpha} \cdot \vec{p} + \beta m) \Psi(x).
\]
As the Dirac Hamiltonian should be consistent with the relativistic energy, i.e. \( H = (\vec{\alpha} \cdot \vec{p} + \beta m)^2 = |\vec{p}|^2 + m^2 \), the constants \( \alpha_i \) and \( \beta \) must satisfy the following relations:

\[
\begin{align*}
\alpha_i \alpha_j + \alpha_j \alpha_i &= 2 \delta_{i,j}, \\
\alpha_i \beta + \beta \alpha_i &= 0, \\
\beta^2 &= 1.
\end{align*}
\]

The simplest solution to the above relations is \( 4 \times 4 \) traceless unitary Hermitian matrices [21, 22]. Thus the Dirac equation is a \( 4 \times 4 \) matrix equation. The solutions of the Dirac equation, known as spinors, have four components, like the 4-vectors in Minkowski space, but they do not transform like vectors under Lorentz transformations. The spinors call for a new representation of the Lorentz transformations.

A possible choice for the \( \alpha_i \) and \( \beta \) matrices, known as Weyl or Chiral representation (see for example Appendix A.3 in [21]) is:

\[
\begin{align*}
\alpha_i &= \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},
\end{align*}
\]

where \( \sigma_i \) are the Pauli matrices and \( I \) is the \( 2 \times 2 \) identity. We now define the gamma matrices, \( \gamma^i = \alpha^i \beta \) and \( \gamma^0 = \beta \), such that we have a 4-vector of matrices \( \gamma^\mu = (\gamma^0, \vec{\gamma}) \), i.e.:

\[
\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{bmatrix}.
\]

It is convenient to define the Pauli matrix 4-vectors \( \sigma^\mu = (I, \vec{\sigma}) \) and \( \sigma^\mu = (I, -\vec{\sigma}) \). Finally, the the Dirac equation in covariant form reads:

\[
(\gamma^\mu p_\mu - m) \Psi(x) = 0.
\]

For a free particle with 4-momentum \( p = (E, \vec{p}) \), in a given reference frame, the Dirac equation solution is given in the form:

\[
\Psi(x) = \exp (-ip \cdot x) u(p).
\]

Substituting this function in the Dirac equation, we obtain for the spinors \( u(p) \):

\[
(\gamma^\mu p_\mu - m) u(p) = 0.
\]

For a particle in its rest frame, with 4-momentum \( p_0 = (m, 0, 0, 0) \), Eq.11 reduces to:

\[
m \begin{bmatrix} -I & I \\ I & -I \end{bmatrix} u(p_0) = 0.
\]

The solutions of Eq.12 are of the form:

\[
u(p_0) = N \begin{bmatrix} \xi \\ \overline{\xi} \end{bmatrix},
\]

where \( N \) is a normalization factor. Since the Dirac equation is a \( 4 \times 4 \) matrix equation, the \( \xi \) must be 2-component vectors and form an orthonormal basis in the Hilbert space. Therefore we write:

\[
\begin{bmatrix} \xi^0 \\ \xi^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \xi^0 \\ \xi^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
These vectors label a new degree of freedom, that we had not taken into account, the particle’s spin. Therefore, \( u(p_0) \) needs one more label, the spin orientation (\( \alpha \)):

\[
u(p_0, \alpha) = N \begin{bmatrix} \xi^\alpha \\ \xi_{\bar{\alpha}} \end{bmatrix}.
\]

(15)

As we are in the particle’s rest frame, the chosen basis for the vectors \( \xi^\alpha \) are usually the eigenstates of the \( \sigma_z \) Pauli matrix, so the spin quantization axis is in \( z \) direction and we have the spin orientations \(+1/2\) when \( \alpha = 0 \) and \(-1/2\) when \( \alpha = 1 \).

If we now consider a free antiparticle with negative energy,

\[
\Psi(x) = \exp(i p \cdot x) \nu(p, \alpha),
\]

(16)

the Dirac equation for the spinors \( \nu(p, \alpha) \) reads

\[
(\gamma^\mu p_\mu + m) \nu(p, \alpha) = 0,
\]

(17)

and the solution in the rest frame is:

\[
\nu(p_0, \alpha) = M \begin{bmatrix} \eta^\alpha \\ -\eta_{\bar{\alpha}} \end{bmatrix},
\]

(18)

with \( \eta^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and \( \eta^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

III. LORENTZ TRANSFORMATIONS ON DIRAC SPINORS

In this section, we obtain the representation of the Lorentz transformations that act on the spinors. Let \( S^{\mu\nu} \) and \( S^{\rho\sigma} \) be elements of the Lorentz algebra, satisfying the following commutation relation [21, 22]:

\[
[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho}).
\]

(19)

The metric of Minkowski space is \( g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \), which can be obtained from the anti-commutation relation of the gamma matrices \( g^{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \). On the other hand, the \( S^{\mu\nu} \) can be obtained from the commutation relation of the gamma matrices:

\[
S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].
\]

(20)

This algebra has six elements, namely three for the boosts (translations), and other three for spin rotations. \( S^{0k} \) is the Lorentz boost generator in the direction \( k = 1, 2, 3 \), while \( S^{ij} \) is the angular momentum operator of the spin, responsible for its rotation in the plane \( i, j = 1, 2, 3 \). Note that as consequence of the tensorial character of the spin operator \( S^{ij} [21] \), spin observables intrinsically have tensorial character in the context of relativistic quantum mechanics.

Now writing explicitly the Lorentz operators:

\[
S^{0k} = \frac{-i}{2} \begin{bmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{bmatrix} \text{ and } S^{ij} = \frac{1}{2} \epsilon_{ijk} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix},
\]

(21)

we obtain the Lorentz group as an exponential function of the Lorentz algebra. If \( D(\omega) \) is an element of the group, and \( \omega \) is the transformation parameter, we have:

\[
D(\omega) = \exp \left( \frac{-i}{2} \omega_{\mu\nu} S^{\mu\nu} \right).
\]

(22)
The boost representation will be \( D(\omega_k) = \exp \left( \frac{-i}{\hbar} \omega_k S^{0k} \right) \), where \( \omega_{0k} = 2\eta_k \), being \( \eta_k \) the rapidity of the particle in direction \( k \). The rapidity can be defined as a function of the relative velocity, \( \tanh \eta = \beta \). Therefore, we can write the boosts having the rapidity as a parameter of the transformation: \( D(\eta) = \exp \left( -i S^{0k} \eta_k \right) \). In the case that the particle has a 4-momentum \( p = (E, \vec{p}) \), the rapidity is \( \cosh \eta = E/m \) and \( \sinh \eta = p/m \), this results in the Lorentz boost:

\[
D(\eta) = \frac{1}{\sqrt{2m(E + m)}} \begin{bmatrix}
 m + p \cdot \sigma \\
 0 \\
 m + p \cdot \bar{\sigma}
\end{bmatrix}.
\] (23)

Note that the boost generator \( S^{0k} \) is not Hermitian, therefore the implementation of the homogeneous Lorentz group is not unitary [23].

Now we can calculate the spinor with arbitrary 4-momentum \( p = (E, \vec{p}) \), from the spinor in the rest frame, i.e. \( u(p, \alpha) = D(\eta)u(p_0, \alpha) \):

\[
u(p, \alpha) = \frac{N}{\sqrt{2m(E + m)}} \begin{bmatrix} (m + p \cdot \sigma) \xi^\alpha \\
 (m + p \cdot \bar{\sigma}) \xi^\alpha
\end{bmatrix}.
\] (24)

Alternatively, we can also write:

\[
u(p, \alpha) = \frac{N}{\sqrt{m}} \begin{bmatrix}\sqrt{p \cdot \sigma} \xi^\alpha \\
-\sqrt{p \cdot \bar{\sigma}} \eta^\alpha
\end{bmatrix}.
\] (25)

Analogously for the spinors \( v(p_0, \alpha) \) (corresponding to the antiparticle), we have:

\[
u(p, \alpha) = \frac{M}{\sqrt{m}} \begin{bmatrix}\sqrt{p \cdot \sigma} \eta^\alpha \\
-\sqrt{p \cdot \bar{\sigma}} \xi^\alpha
\end{bmatrix}.
\] (26)

To calculate the normalization constants we need to define the dual of the Dirac spinors. A good choice is

\[
\overline{\nu}(p, \alpha) = u^\dagger(p, \alpha) \gamma^0, \quad \overline{v}(p, \alpha) = v^\dagger(p, \alpha) \gamma^0.
\] (27)

The justification of such a choice is that it guarantees the Lorentz invariance of the inner product. With \( \overline{\nu}(p, \alpha)u(p, \alpha) = 1 \) and \( \overline{v}(p, \alpha)v(p, \alpha) = 1 \), we obtain for the normalization constants: \( N = 1/\sqrt{2} \) and \( M = 1/\sqrt{-2} \). Finally, the Dirac spinors in an arbitrary frame can be written as:

\[
u(p, \alpha) = \frac{1}{\sqrt{2m}} \begin{bmatrix}\sqrt{p \cdot \sigma} \xi^\alpha \\
-\sqrt{p \cdot \bar{\sigma}} \eta^\alpha
\end{bmatrix},
\] (28)

\[
u(p, \alpha) = \frac{1}{\sqrt{-2m}} \begin{bmatrix}\sqrt{p \cdot \sigma} \eta^\alpha \\
-\sqrt{p \cdot \bar{\sigma}} \xi^\alpha
\end{bmatrix}.
\] (29)

With these expressions we have a complete and orthonormal basis for the Hilbert space of the spinors. This space is denoted by \( \mathcal{H}(m, 1/2, \pm) \), which is a direct sum of the Hilbert space of the particle (rest mass \( m \), spin \( 1/2 \) and positive energy) and the Hilbert space of its antiparticle (rest mass \( m \), spin \( 1/2 \), but with negative energy) [23]. The Hilbert space labels come from the Casimir operators of the Lorentz and Poincaré group, which are \( p^\mu p_\mu = m^2 \) and \( S^2 \). Therefore the mass and spin are the kinematic labels for the quantum systems [4]. The attentive reader certainly noted that the spinors (Eqs[28] and [29]) are not tensor products of kets of momentum and spin (e.g. \( u(p, \alpha) \neq |p\rangle |\alpha\rangle \)), and it is not possible to transform spin independently of momentum (Eqs[22] [23] and [24]), let alone a transformation on a reduced spin density matrix, which is ill defined in the relativistic context [15].
IV. THE SPINOR DENSITY MATRIX

In this section we start discussing general superpositions of spinors, then we introduce mixed states, and investigate covariance properties of expectation values, and finally unify the representation of pure and mixed states in a Bloch sphere. The motivation is to show that a properly defined mixed state should be covariant, which has the important consequence of the invariance of pure states’ entanglement under Lorentz transformations. This conclusion, though obvious, seems to be still misunderstood in the literature \[13\].

It is important to stress that we are treating a free Dirac particle, whose Hilbert space is a direct sum of the particle’s and antiparticle’s sub-spaces, \( \mathcal{H}(m, 1/2, \pm) = \mathcal{H}(m, 1/2, +) \oplus \mathcal{H}(m, 1/2, -) \). Only CPT transformations (i.e., inversion of both charge and parity, and time reversal) can connect these sub-spaces \[21\], transforming a particle in an antiparticle, and vice-versa. As CPT transformations are not considered in the present context, we can write a pure spin state for a particle with momentum \( p = (E, \vec{p}) \) as:

\[
\psi(p) = \sum_{\alpha=0}^{1} a(\alpha) u(p, \alpha),
\]

with \( \sum_{\alpha} |a(\alpha)|^2 = 1 \). If Eq 30 represents a quantum state, the squared coefficients \( |a(\alpha)|^2 \) must correspond to some probability distribution. We can check if this is so by means of the conservation of the 4-current:

\[
\partial_{\mu} j^{\mu} = 0.
\]

\( j^0 \) is the probability distribution, and \( \vec{j} \) is the probability current associated with the spinor \( \Psi(x) \), such that \( j^{\mu} = (j^0, \vec{j}) = (\Psi(x)^\dagger \Psi(x), \Psi(x)^\dagger \vec{\alpha} \Psi(x)) \). \( \Psi(x) \) is a solution of the Dirac equation,

\[
\Psi(x) = \sum_{\alpha} a(\alpha) u(p, \alpha) \exp (-ip \cdot x).
\]

Now the probability distribution \( j^0 \) can be written in the basis \( \{u(p, \alpha)\} \) as:

\[
j^0 = \sum_{\alpha, \beta} a(\alpha)^* a(\beta) u(p, \alpha)^\dagger u(p, \beta),
\]

and using the inner product relation \( u(p, \alpha)^\dagger u(p, \beta) = \frac{E}{m} \delta_{\alpha, \beta} \), we finally arrive at:

\[
j^0 = \frac{E}{m} \sum_{\alpha} |a(\alpha)|^2.
\]

Therefore, as \( E/m \) is a positive constant, the coefficients \( |a(\alpha)|^2 \) are the \( j^0 \) probability distribution coefficients.

Now we check that though the Lorentz boost is not unitary, the normalization of the spinor is invariant under Lorentz transformations \[22\]. Let the dual of the spinor \( \psi(p) \) in the frame \( S \) be:

\[
\bar{\psi}(p) = \psi(p)^\dagger \gamma^0.
\]

In other frame \( S' \), the spinor \( \psi'(p') \) and its dual \( \bar{\psi}'(p') \) are related to frame \( S \) through the Lorentz transformation \( D(\omega) = \exp (-i\omega_{\mu\nu} S^{\mu\nu}/2) \):

\[
\psi'(p') = D(\omega) \psi(p) \quad \text{and} \quad \bar{\psi}'(p') = \bar{\psi}(p) D^{-1}(\omega).
\]
The transformation for the spinors is trivial, but it is not so obvious in the case of the duals, for we have to use the inverse Lorentz transformation given by

\[ D^{-1}(\omega) = \gamma^0 D^\dagger(\omega)\gamma^0. \]  

(37)

The explicit calculation is as follows:

\[ \bar{\psi}'(p') = \psi'^\dagger(p')\gamma^0 \]  

(38)

\[ = \psi^\dagger(p) D(\omega)^\dagger \gamma^0 \]  

(39)

\[ = \psi^\dagger(p)\gamma^0 D(\omega)^{-1} \]  

(40)

\[ = \bar{\psi}(p) D(\omega)^{-1}. \]  

(41)

Finally we have:

\[ \bar{\psi}'(p')\psi'(p') = \bar{\psi}(p)\psi(p), \]  

(42)

which implies that the normalization is invariant under Lorentz transformations, as expected.

Now that we have well defined pure states, we introduce the mixed states by means of the convex sum \[24\]:

\[ \rho(p) = \sum_k q_k \psi_k(p)\bar{\psi}_k(p), \]  

(43)

where \( \psi_k(p) \) are pure states given by Eq.30 and \( q_k \) represents the probability to obtain the \( k \)th state, such that \( \sum_k q_k = 1 \).

Now we shall calculate how these mixed states behave under Lorentz transformations. A state \( \rho'(p') \) in the frame \( S' \) is written as:

\[ \rho'(p') = \sum_k q_k \psi_k'(p')\bar{\psi}_k'(p'). \]  

(44)

Invoking the Lorentz transformation Eq.36 it is evident that:

\[ \rho'(p') = \sum_k q_k D(\omega)\psi_k(p)\bar{\psi}_k(p) D(\omega)^{-1}, \]  

(45)

and therefore

\[ \rho'(p') = D(\omega)\rho(p) D(\omega)^{-1}. \]  

(46)

Now we check that the expectation value of a Hermitian operator \( (A(p)) \) is the same in all frames:

\[ Tr[A'(p')\rho'(p')] = Tr[D(\omega)A(p)D(\omega)^{-1}D(\omega)\rho(p)D(\omega)^{-1}] \]

\[ Tr[A'(p')\rho'(p')] = Tr[A(p)\rho(p)]. \]

It follows that the trace and eigenvalues of the density matrix \( (\rho) \) are covariant. Of course, to maintain invariance, we must transform both state and observable.

Another important consequence of the covariance of expectation values is the invariance of the purity (or mixedness) of the density matrix, i.e. \( Tr[\rho'(p')^2] = Tr[\rho(p)^2] \). A corollary of this straightforward result is the invariance of entanglement of pure states under Lorentz transformations, for in this case the purity of the marginals characterize the entanglement.
Let us write a Bloch like decomposition for the density matrix \[24\]. First we define the Pauli matrices decomposed in the Dirac spinors as:

\[
\Sigma_x(p) = u(p, 0) \bar{\pi}(p, 1) + u(p, 1) \bar{\pi}(p, 0),
\]

\[
\Sigma_y(p) = i[u(p, 1) \bar{\pi}(p, 0) - u(p, 0) \bar{\pi}(p, 1)],
\]

\[
\Sigma_z(p) = u(p, 0) \bar{\pi}(p, 0) - u(p, 1) \bar{\pi}(p, 1).
\]

The matrix \(I(p)/2\) is the maximally mixed (or depolarized) state, which can be decomposed in the Dirac spinors as:

\[
I(p)/2 = \frac{1}{2} \sum_\alpha u(p, \alpha) \bar{\pi}(p, \alpha).
\]

\(I(p)\) also satisfies the relation \[21\]:

\[
I(p) = \sum_\alpha u(p, \alpha) \bar{\pi}(p, \alpha) = (\gamma^\mu p_\mu + m)/2m.
\]

Finally we have a continuous manifold, for \(p\) is continuous, that has a Bloch like sphere in each definite momentum \(p = (E, \vec{p})\):

\[
\rho(p) = \frac{1}{2} I(p) + \frac{1}{2} r_l \Sigma_l(p).
\]

As in the nonrelativistic case, we have pure states for \(|\vec{r}| = 1\), and mixed states for \(|\vec{r}| < 1\). Note that to each momentum value there is associated a Bloch sphere on spins, such that the spin quantization axis, or the antipodal points on a canonical basis orientation (like \(\sigma_z\) eigenstates), depends explicitly on the momentum of the reference frame.

Note that we have defined a probability distribution Eq.34 on the spin degrees of freedom, because we assumed a particle with a known definite momentum. In a quantum information context this is justified because two parties interested in performing some task, a protocol, must share a common reference frame, in order to have a well defined quantization axis. Remember, for instance, that in the quantum teleport protocol \[8\], Alice must inform Bob the directions she measured spin, in order to Bob perform his measurements to recover the quantum information.

V. SPIN QUANTIZATION AXIS UNDER LORENTZ TRANSFORMATIONS

In this section we discuss how a spin measurement depends on the particle’s momentum under Lorentz transformations. The scenario is a source, in the rest frame \(S\) on Earth, emitting particles with velocity \(v\hat{z}\) and spin up in the \(z\) direction, such that

\[
S_z u(m, 0) = \frac{1}{2} u(m, 0).
\]

An observer at rest in a frame \(S’\) on a satellite, which moves with velocity \(-\beta\hat{z}\) in relation to \(S\), measures the spin of the particles. Our problem is to preview the measurement outcomes in \(S’\) as a function of the detector’s orientation.

We start by writing the momentum of the particle in the frame \(S\),

\[
p = m(cosh \eta, 0, 0, -\sinh \eta).
\]
This momentum is obtained performing a Lorentz transformation (boost in $z$ direction) on a particle at rest ($p_0 = (m, 0, 0, 0)$),

$$ p = L(\eta)p_0, \tag{55} $$

where

$$ L(\eta) = \begin{bmatrix} \cosh \eta & 0 & 0 & -\sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \eta & 0 & 0 & \cosh \eta \end{bmatrix}, \tag{56} $$

being $\eta$ the particle’s rapidity ($\tanh \eta = v$).

The particle’s momentum in the satellite ($p'$) is obtained by means of a boost in the $-x$ direction on $p$:

$$ p' = L(\omega)p, \tag{57} $$

where

$$ L(\omega) = \begin{bmatrix} \cosh \omega & -\sinh \omega & 0 & 0 \\ -\sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{58} $$

being $\omega$ the rapidity of the satellite ($\tanh \omega = \beta$). Finally we arrive at $p' = (p'_0, \vec{p}')$, where

$$ p'_0 = m \cosh \omega \cosh \eta, \tag{59} $$
$$ \vec{p}' = (m \sinh \omega \cosh \eta, 0, -m \sinh \eta). \tag{60} $$

Substituting the momentum $p'$ in Eq.24 we obtain the particle’s spinor in the satellite:

$$ u(p', \alpha) = \frac{1}{\sqrt{4m(p'_0 + m)}} \begin{bmatrix} (m + p'_0 - \vec{p}' \cdot \vec{\sigma})\xi^\alpha \\ (m + p'_0 + \vec{p}' \cdot \vec{\sigma})\xi^\alpha \end{bmatrix}, \tag{61} $$

Performing the matrix products in the above equation results in:

$$ u(p', \alpha) = \frac{1}{\sqrt{4m(p'_0 + m)}} \begin{bmatrix} (m + p'_0 - (-1)^\alpha p'_z)\xi^\alpha - p'_x\xi^{\alpha+1} \\ (m + p'_0 + (-1)^\alpha p'_z)\xi^\alpha + p'_x\xi^{\alpha+1} \end{bmatrix}. \tag{62} $$

In order to define the spin measurement in the satellite, we introduce the following projectors:

$$ P_\pm = |\theta, \phi, \pm\rangle \langle \theta, \phi, \pm|, \tag{63} $$

where

$$ |\theta, \phi, +\rangle = \cos \theta/2\xi^0 + \exp(i\phi) \sin \theta/2\xi^1, \tag{64} $$
$$ |\theta, \phi, -\rangle = \sin \theta/2\xi^0 - \exp(i\phi) \cos \theta/2\xi^1. \tag{65} $$

The spin measurement in the satellite reveals that, in relation to Earth, the quantization axis changes by $\theta$ and the spinor gains a relative phase $\phi$. The spin measurement operator has to
FIG. 1: (First panel) Inclination ($\theta$) of the spin quantization axis on the satellite in relation to Earth, as a function of the rapidities of the particle ($\eta$) and of the satellite ($\omega$). (Second panel) The effect of the speed of the particle is just to attenuate how the quantization axis ($z$ in the rest frame) goes to the $x$ direction with increasing rapidities.

belong to both the Lorentz algebra and the angular momentum Lie algebra. A pair of operators satisfying these restrictions and defining a complete measurement is:

$$M_{\pm} = \begin{bmatrix} \langle \theta, \phi, \pm | \theta, \phi, \pm \rangle & 0 \\ 0 & \langle \theta, \phi, \pm | \theta, \phi, \pm \rangle \end{bmatrix}. \tag{66}$$

As expectation values are Lorentz invariant, and our particle was prepared on Earth in a state with spin up in the $z$ direction, if the detectors in the satellite are properly oriented, we should obtain:

$$\bar{u}(p', 0) M_+ u(p', 0) = 1 \tag{67}$$

and

$$\bar{u}(p', 0) M_- u(p', 0) = 0. \tag{68}$$
These expectation values lead to the following non-linear system of equations for $\theta$ and $\phi$:

$$\bar{u}(p', 0) M_+ u(p', 0) = \frac{1}{2m(m + p'_0)} \left\{ [(m + p'_0)^2 - p'_z^2] \cos^2 \theta/2 - p'_x^2 \sin^2 \theta/2 - \right.$$ \left. \begin{align*}
-2p'_x p'_z \cos \theta/2 \sin \theta/2 \cos \phi \\
\right\} = 1, 
$$

$$\bar{u}(p', 0) M_- u(p', 0) = \frac{1}{2m(m + p'_0)} \left\{ [(m + p'_0)^2 - p'_z^2] \sin^2 \theta/2 - p'_x^2 \cos^2 \theta/2 - \right.$$ \left. \begin{align*}
-2p'_x p'_z \cos \theta/2 \sin \theta/2 \cos \phi \\
\right\} = 0.
$$

The solution of the non-linear system results in null relative phase ($\phi = 0$), and the angle $\theta$ depends on both the rapidities of the particle ($\eta$) and of the satellite ($\omega$), as expected. In Fig.1, we see that the quantization axis tends to $\hat{z}$ ($\cos \theta/2 = 1$), as the particle’s velocity tends to zero, as expected. On the other hand, for a particle moving near to the speed of light, the spin quantization axis tends to $\hat{x}$. Of course a massive particle never reaches the speed of light, and the plot in Fig.1 never touches the $x$ axis. This result is nice, for it is well known that a massless particle always has its spin, or rather its helicity, parallel to the momentum.

It is interesting to note that the Lorentz transformation, though not unitary, acts on the spin degree of freedom like a rotation in the quantization axis. This rotation is like a little group representation of the Poincaré (inhomogeneous) group, and belongs to $SU(2)$ [25]. Therefore, as entanglement is invariant under local unitaries, we conclude that the entanglement of a system under Lorentz transformation cannot change, what changes is just the spin quantization axis as a function of momentum, as we see in Fig.1.

We could analyze a simpler situation where the particle is at rest, with spin up, and it is to be measured by a moving observer with rapidity $\omega$. This observer sees the spin quantization axis rotate according to (Fig.2):

$$\cos^2 \theta/2 = \frac{2(1 + \cosh \omega) + \sinh^2 \omega}{(1 + \cosh \omega)^2 + \sinh^2 \omega}.$$ 

This result is just an evidence that what really matters for the relativity principle is the relative movement.
VI. CONCLUSION

In order to discuss quantum information in the relativistic context, one has first to properly describe one-particle states under Lorentz transformations, and that was what we did. After revising the Dirac equation for a free particle, we obtained the covariance of expectation values, which implies the covariance of the eigenvalues of the spinor density matrix. The covariance of the eigenvalues also implies the covariance of the purity of the density matrix. It follows then that entanglement of a bipartite pure state is covariant, for it can be characterized by the purity of the marginal density matrix. We saw that what changes in a spin-$\frac{1}{2}$ particle under Lorentz transformations is the spin quantization axis as a function of the momentum. The Lorentz transformation acts on the spin degree of freedom as a local rotation, and this is another way to understand why the entanglement does not change. As a matter of fact, from the point of view of Lorentz transformations, spin and momentum are not independent degrees of freedom, but just labels of the Hilbert space. Finally, in Fig.1 and Fig.2, we illustrated how the detectors should be aligned, depending on the momentum, for a proper spin measurement. Note that if the observer in the moving frame ignores his momentum in relation to Earth, and therefore cannot calculate the proper alignment of his detectors according to the Lorentz transformation, this is just classical ignorance and cannot induce any quantum effect.

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