TWO-SIDED IDEALS IN THE RING OF DIFFERENTIAL OPERATORS ON A STANLEY-REISNER RING

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Abstract. Let $R$ be a Stanley-Reisner ring (that is, a reduced monomial ring) with coefficients in a domain $k$, and $K$ its associated simplicial complex. Also let $D_k(R)$ be the ring of $k$-linear differential operators on $R$. We give two different descriptions of the two-sided ideal structure of $D_k(R)$ as being in bijection with certain well-known subcomplexes of $K$; one based on explicit computation in the Weyl algebra, valid in any characteristic, and one valid in characteristic $p$ based on the Frobenius splitting of $R$. A result of Travers [Tra99] on the $D_k(R)$-module structure of $R$ is also given a new proof and different interpretation using these techniques.

1. Introduction

Rings of $k$-linear differential operators $D_k(R)$ on a $k$-algebra $R$ are generally difficult to study, even when the base ring $R$ is well-behaved. Some descriptions of $D_k(R)$ are given in e.g. [Mus94] for the case of toric varieties, [Bav10a] and [Bav10b] for general smooth affine varieties (in zero and prime characteristic respectively), and [Tra99], [Tri97] and [Eri98] for Stanley-Reisner rings. Some criteria for simplicity of $D_k(R)$ exist (see [SvdB97] and [Sai07] among others), and the study of their left and right ideals, through the theory of $D$-modules, is well developed.

When $D_k(R)$ is not simple, however, it is an interesting problem to give a description of its two-sided ideals; the purpose of this paper is to do this for the case of Stanley-Reisner rings. Every Stanley-Reisner ring is the face ring $R_K$ of a simplicial complex $K$, and we will give two different descriptions of the two-sided ideal structure of $R$ in terms of the combinatorial structure of $K$; namely the lattice of ideals is in a certain sense determined by the poset of subcomplexes of $K$ that are stars of some face of $K$. The first description is based on explicit computations with monomials in the Weyl algebra, and the second (valid only in prime characteristic) takes advantage of the Frobenius splitting of $R$.

2. Some preliminaries

Let us fix some notation. Throughout, $k$ is a commutative domain. $K$ will denote an abstract simplicial complex on vertices $x_1, \ldots, x_n$; we will not distinguish between $K$ as an abstract simplicial complex and its topological realization. In the corresponding face rings (see 2.1) the indeterminate corresponding to a vertex $x_i$ will also be named $x_i$ to avoid notational clutter. Elements of $K$ will be referred to as simplices or faces. For a face $\sigma \in K$, we let $x_\sigma := \prod_{x_i \in \sigma} x_i$. $R$ will always mean

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Lemma 2.3. Let $K$ be an abstract simplicial complex on vertices $x_1, \ldots, x_n$. The Stanley-Reisner ring, or face ring, of $K$ with coefficients in $k$ is the ring $R_K = k[x_1, \ldots, x_n]/I_K$, where $I_K = \langle x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\} \notin K \rangle$ is the ideal of square-free monomials corresponding to the non-faces of $K$, called the face ideal of $K$.

Geometrically, $R_K$ is the coordinate ring of the cone on $K$, so $\dim R_K = \dim K + 1$. Accordingly, when we talk about support of elements, we will refer to faces of $K$ when strictly speaking we mean the cones on these faces. If $K = \Delta_n$ is a simplex, $I_K$ is the zero ideal, and $R_K$ is the polynomial ring in $n$ variables. If $K = K' \ast K''$ is the simplicial join of complexes $K'$ and $K''$, then $R_K \simeq R_{K'} \otimes_k R_{K''}$. Face rings are exactly the reduced monomial rings, i.e. quotients of polynomial rings by square-free monomial ideals.

Given a simplicial complex $K$, we will have use for a well-known class of subsets of $K$:

Definition 2.2. Let $\sigma \in K$ be a face. The closed star of $\sigma$ in $K$ is the subcomplex

$$st(\sigma, K) := \{ \tau \in K \mid \tau \cup \sigma \in K \}.$$ 

The open star of $\sigma$ in $K$ is the set

$$st(\sigma, K)^\circ := \{ \tau \in K \mid \sigma \cup \tau \in K \land \sigma \cap \tau \neq \emptyset \};$$

$st(\sigma, K)^\circ$ is the interior of $st(\sigma, K)$ in $K$, and $st(\sigma, K)$ is the closure of $st(\sigma, K)^\circ$ in $K$. The open complement of $st(\sigma, K)$ is the set (not usually a subcomplex)

$$U_\sigma(K) = K \setminus st(\sigma, K) = \{ \tau \in K \mid \tau \cup \sigma \notin K \}.$$ 

Stars are important because the support of a principal monomial ideal of $R_K$, considered as an $R_K$-module, is exactly equal to the open star of some face, and the closed star is the smallest subcomplex containing it. For the remainder, we will take star to mean closed star. We will not have much need of comparing stars associated to different subcomplexes and so will often write simply $st(\sigma), U_\sigma$ if no confusion is likely to result. For completeness, we repeat a few simple facts:

Lemma 2.3. (i) If $\sigma \subset \tau$ are faces in $K$, $st(\sigma, K) \supset st(\tau, K)$;

(ii) If $L \subset K$ is a subcomplex containing $\sigma$, $st(\sigma, L) \subset st(\sigma, K)$;

(iii) For a face $\sigma = \tau \cup \{x\}$, $st(\sigma, K) = st(x, st(\tau, K))$.

(iv) $st(\tau) \subset st(\sigma)$ if and only if \{maximal simplices in $K$ that contain $\tau$\} \subset \{maximal simplices in $K$ that contain $\sigma$\}.

(v) $\sigma \in st(\tau) \iff \tau \in st(\sigma)$.

(vi) If $\sigma \cup \tau$ is a face of $K$, $st(\sigma)^\circ \cap st(\tau)^\circ = st(\sigma \cup \tau)^\circ$. 

Proof: (i), (ii) and (v) are obvious. (iv) follows from the fact that a complex is determined by its maximal cells. (iii) follows from unwrapping the definitions:

\[
\begin{align*}
(2.1) \quad & \text{st}(x, \text{st}(\tau, K)) = \{\alpha \in \text{st}(\tau, K) | \alpha \cup \{x\} \in \text{st}(\tau, K)\} \\
(2.2) \quad & = \{\alpha \in \text{st}(\tau, K) | \alpha \cup \{x\} \cup \tau \in K\} \\
(2.3) \quad & = \{\alpha \in \text{st}(\tau, K) | \alpha \cup \sigma \in K\} \\
(2.4) \quad & = \text{st}(\tau, K) \cap \text{st}(\sigma, K) \\
(2.5) \quad & = \text{st}(\sigma, K)
\end{align*}
\]

where the last equality follows from (i). To show (vi), note that for any $\sigma \in K$, $\text{st}(\sigma)^\circ$ is the interior of the union of maximal simplices containing $\sigma$. It follows that $\text{st}(\sigma \cup \tau)^\circ$ is the interior of the union of maximal simplices containing both $\sigma$ and $\tau$, in other words the maximal simplices in $\text{st}(\sigma) \cap \text{st}(\tau)$. □

We will need some properties of the face ideals $I_{\text{st}(\sigma)}$ and face rings $R_{\text{st}(\sigma)}$ of the subcomplexes $\text{st}(\sigma, K)$.

**Lemma 2.4.**

1. If $K_1, K_2$ are subcomplexes of $K$, $I_{K_1} + I_{K_2} = I_{K_1 \cap K_2}$ and $I_{K_1} \cap I_{K_2} = I_{K_1 \cup K_2}$.
2. $I_{\text{st}(\sigma)} = \langle x_\tau | \tau \in U_\sigma \rangle$.
3. The minimal primes of $I_K$ are the face ideals $I_{\text{st}(\tau)}$ for the maximal simplices $\tau$.

**Proof.** The first two items follow from the definition of $I_{\text{st}(\sigma)}$. For the last item, observe that $I_{\text{st}(\sigma)}$ is clearly prime when $\sigma$ is a maximal simplex, as $I_{\text{st}(\sigma)} = \langle x_\tau | \tau \in U_\sigma \rangle$ and monomial ideals are prime exactly when they are generated by a subset of the variables; observe also that all $I_{\text{st}(\sigma)}$ are radical. These observations together with item 1 give the result, as $I_K = \bigcap_{\sigma \subset K} I_{\text{st}(\sigma)}$. □

We intend to study the ring of differential operators on $R$, so let us define what that is:

**Definition 2.5.** The ring $D_k(R)$ of $k$-linear differential operators on a $k$-algebra $R$ is defined inductively by

\[
D_k(R) = \bigcup_{n \geq 0} D_k^n(R)
\]

where $D_k^0(R) = R$ and for $n > 0$, $D_k^n(R) := \{\phi \in \text{End}_k(R) | \forall r \in R : [\phi, r] \in D_k^{n-1}(R)\}$. Elements of $D_k^n(R) \setminus D_k^{n-1}(R)$ are said to have order $n$, and there is a natural filtration

\[
D_k^0(R) \subset D_k^1(R) \subset D_k^2(R) \subset \cdots
\]
on $D_k(R)$ called the order filtration.

**Definition 2.6.** The Weyl algebra in $n$ variables over $k$ is the ring of differential operators on the polynomial ring $k[x_1, \ldots, x_n]$. It is generated as an $R$-algebra by the divided power operators $\partial_i = \frac{x_i^m}{m! \partial_i^m}$, with the relations $[x_i, x_j] = [\partial_i^{(a)}, \partial_j^{(b)}] = 0$ for $i \neq j$, $\partial_i^{(a)} \partial_i^{(b)} = (\partial_i^{a+b})^{(a+b)}$ and $[\partial_i^{(b)}, x_i] = \partial_i^{(b-1)}$ (in particular $[\partial_i, x_i] = 1$).

**Remark 2.7.** We use the divided power operators rather than the usual vector fields $\frac{\partial}{\partial x_i}$, as the latter do not generate the whole ring of differential operators in the case of characteristic $p$; the divided power operators however always generate
everything regardless of the characteristic, as they define differential operators on \( \mathbb{Z} \) and so descend to any commutative ring. In characteristic zero, the derivations \( \partial_i \) suffice to generate everything; in characteristic \( p \) we need the full set of elements \( \partial_i^p \) for \( r \geq 0 \), which suffice due to the relation \( \partial_i^a \partial_i^b = (\frac{a+b}{a}) \partial_i^{a+b} \).

In the following, \( k \) will always be fixed, so we will omit it from the notation and write simply \( D(R) \). Elements of \( k \) will be referred to as constants. One easily verifies that an element \( x^a \partial_i^b \) in the Weyl algebra has order \( |b| \).

3. The two-sided ideals of \( D(R) \)

When \( R = R_K \) is a face ring, there exist several descriptions of \( D(R) \) in the literature, see [Tri97], [En98] and [Tr99]. We wish to give a description of the two-sided ideals of \( D(R) \) in terms of the combinatorics of \( K \); for our purposes, the following description due to Traves ([Tr99]) is the most convenient.

**Theorem 3.1.** Let \( k \) be a commutative domain, and \( R = k[X]/J \) a reduced monomial ring. An element \( x^a \partial_i^b = \prod x_i^a \partial_i^b \) of the Weyl algebra over \( k \) is in \( D(R) \) if and only if for each minimal prime \( p \) of \( R \), we have either \( x^a \in p \) or \( x^b \notin p \). \( D(R) \) is generated as a \( k \)-algebra by these elements, and they form a free basis of \( D(R) \) as a left \( k \)-module.

**Example 3.2.** Let \( R = k[x_1, x_2, x_3]/(x_1 x_2 x_3) \). The associated simplicial complex \( K \) is the boundary of a 2-simplex. Then by [3.1] \( D(R) = R(x_i^a \partial_i^b) \langle a_i, b_i \in \mathbb{N} \rangle \).

**Example 3.3.** Let \( R = k[x_1, x_2, x_3, x_4]/I \) where \( I = (x_1 x_3, x_1 x_4, x_2 x_4) \). The associated complex \( K \) is a chain of three 1-simplices, connected in order \( x_1, x_2, x_3, x_4 \). Theorem 3.1 gives \( D(R) = R(x_1^a \partial_1^b, x_2^a \partial_2^b, x_3^a \partial_3^b, x_4^a \partial_4^b, x_1^a \partial_2^b, x_2^a \partial_3^b) \) (for \( a, b > 0 \)).

Note that in both examples, generators of the form \( x_i^a \partial_i^b \) appear; it is not hard to see that such “toric” operators are always in \( D(R) \). In [3.3] we also have generators of the form \( x_i^a \partial_j^b \) (where \( i \neq j \)). To understand when this happens, we may give a somewhat more geometric formulation of [3.1].

**Proposition 3.4.** Let \( K \) be a simplicial complex and \( R = R_K \) its face ring. Also let \( x^a = \prod x_i^a, x^b = \prod x_j^b \) be such that \( \text{supp}(x^a) = \text{st}(\sigma) \) and \( \text{supp}(x^b) = \text{st}(\tau) \), for some \( \sigma, \tau \in K \). Then \( x^a \partial_i^b = \prod x_i^a \partial_i^b \) is in \( D(R) \) if and only if \( \text{st}(\sigma) \subset \text{st}(\tau) \).

**Proof.** Let \( P_{x^a} \) denote the set of minimal primes in \( R \) that contain \( x^a \), and \( P_{x^a} \) the set of minimal primes that does not contain \( x^a \). Clearly, \( P_{x^a} \cup P_{-x^a} \) is equal to the set of minimal primes in \( R \); denote this by \( P \). Recalling from [2.4] that the minimal primes of \( R \) are the face ideals \( I_{\text{st}(\alpha)} \) for maximal simplices \( \alpha \), we can reformulate these definitions: \( P_{x^a} \) is the set of ideals \( I_{\text{st}(\alpha)} \) such that \( \alpha \) is maximal and \( x^a \in I_{\text{st}(\alpha)} \), in other words those ideals \( I_{\text{st}(\alpha)} \) such that \( \alpha \) is maximal and \( \alpha \in U_{\sigma} \); and \( P_{-x^a} \) is the set of ideals \( I_{\text{st}(\alpha)} \) with \( \alpha \) maximal and contained in \( \text{st}(\sigma) \). Again using [2.4] the ideal \( I_{\text{st}(\sigma)} \) defining \( \text{st}(\sigma) \) is equal to the intersection of all ideals in \( P_{x^a} \). Unwrapping definitions, we get

\[
\text{st}(\sigma) \subset \text{st}(\tau) \Leftrightarrow I_{\text{st}(\sigma)} \supset I_{\text{st}(\tau)} \\
\Leftrightarrow P_{-x^a} \supset P_{-x^b} \\
\Leftrightarrow P_{x^a} \subset P_{x^b}.
\]
Putting this together with 3.1 we have
\[ x^n \partial^{(b)} \in D(R) \iff \forall p \in P : x^n \in p \lor x^b \not\in p \]
\[ \iff \forall p \in P : p \in P_{x^n} \lor p \in P_{-x^b} \]
\[ \iff P = P_{x^n} \cup P_{-x^b} \]
\[ \iff P_{x^n} \subset P_{x^b} \lor P_{-x^b} \subset P_{-x^n} \quad \text{(and these are equivalent)} \]
\[ \iff \text{st}(\sigma) \subset \text{st}(\tau). \]

Example 3.5. Let \( R = k[x_1, x_2, x_3, x_4, x_5]/(x_1 x_3, x_1 x_4, x_2 x_4), \) the associated \( K \) is three 2-simplices \( \{x_1, x_2, x_5\}, \{x_2, x_3, x_5\}, \{x_3, x_4, x_5\} \) glued along the edges \( \{x_2, x_5\} \) and \( \{x_3, x_5\} ; x_5 \) is a common vertex to all faces. Note that this makes \( K \) a simplicial join of \( \{x_5\} \) with the complex from Example 3.3. Looking at the closed stars of the faces, we see that
\[ \text{st}(x_1) \subset \text{st}(x_2) \subset \text{st}(x_3) \subset \text{st}(x_4) \subset \text{st}(x_5). \]

As \( \text{st}(x_1) = \text{st}(\{x_1, x_2\}) \), \( \text{st}(x_4) = \text{st}(\{x_4, x_3\}) \) and for any face \( \sigma \), \( \text{st}(\sigma) = \text{st}(\sigma \cup x_5) \) this accounts for all the stars. From this we should by 3.3 have the “toric” generators \( x_i^a \partial_{i}^{(b)} \), and also \( x_i^a \partial_{i}^{(b)} \), \( x_i^a \partial_{j}^{(b)} \partial_{i}^{(c)} \), \( x_i^a \partial_{j}^{(b)} \) and the same with \( x_1 \) and \( x_2 \) replaced by \( x_4 \) and \( x_3 \) respectively (by symmetry). In fact, \( \text{st}(x_5) = \text{st}(\emptyset) = K \), so we should also have \( \partial_5^{(a)} = 1 \cdot \partial_5^{(a)} \) and the description is somewhat redundant.

From 3.3 we deduce the following very useful criterion.

Corollary 3.6. \( \langle x_\tau \rangle \subset \langle x_\sigma \rangle \) if and only if \( \text{st}(\tau) \subset \text{st}(\sigma) \).

Proof. If \( \text{st}(\tau) \subset \text{st}(\sigma) \), it follows from 3.3 that \( x_\tau \partial_\sigma = x_\tau \prod_{i \in \sigma} x_i \partial_i \) is in \( D(R) \). Now observe that \( \prod_{i \in \sigma} x_i | \cdots | x_i | \cdots | x_i \) is equal to \( x_\sigma = \prod_{i \in \sigma} x_i \), where \( x_\sigma = \prod_{i \in \sigma} x_i \). So we have \( x_\tau \in \prod_{i \in \sigma} x_i = \langle x_\sigma \rangle \).

To show the converse, note that by definition of \( I_{\text{st}(\tau)} \), we have \( I_{\text{st}(\sigma)} \cap \langle x_\sigma \rangle = \emptyset \). If now \( \text{st}(\tau) \subset \text{st}(\sigma) \), it follows that \( \tau \subseteq U_\sigma \), so \( x_\tau \in I_{\text{st}(\sigma)} \), which finally implies \( x_\tau \not\in \langle x_\sigma \rangle \).

The following very useful result is surprising.

Theorem 3.7. Any proper two-sided ideal in \( D(R) \) is generated by reduced monomials in the “ordinary” variables \( x_1, \ldots, x_n \).

Proof. The proof is in three parts:

1. The ideal \( \langle \sum_{(a,b) \in S} c_{ab} x^a \partial^{(b)} \rangle \) for some index set \( S \subset \mathbb{N}^{2n} \) is equal to the ideal \( \langle x^n \partial^{(b)} | (a, b) \in S \rangle \);
2. the ideal \( \langle x^n \rangle \) is equal to the ideal \( \langle \prod_{i \neq 0} x_i \rangle \);
3. the ideal \( \langle x^n \partial^{(b)} \rangle \) is equal to the ideal \( \langle \prod_{i \neq 0} x_i \rangle \).

We will make heavy use of the fact that for any two-sided ideal \( I \) and any element \( \phi \in D(R) \), the set of commutators \( [\phi, I] \) is contained in \( I \).

For the first part, recall that we have two natural concepts of grading on the Weyl algebra, that descend to \( D(R) \). First, the natural \( \mathbb{Z}^n \)-grading on the Weyl algebra given by the degree
\[ \deg(x^n \partial^{(b)}) = (a_1 - b_1, \ldots, a_n - b_n), \]
which induces a grading on $D(R)$; second we have the $\mathbb{N}^n$-grading given by the order 

$$\text{ord}(x^a \partial^{(b)}) = (b_1, \ldots, b_n).$$

Note that 

$$[x_i \partial_i, x^a \partial^{(b)}] = x_i \partial_i x^a \partial^{(b)} - x^a \partial^{(b)} x_i \partial_i$$

$$= x_i (x^a \partial_i + a_i x^{a-1}) \partial^{(b)} - x^a (x_i \partial^{(b)} + \partial^{(b-1,i)}) \partial_i$$

$$= x_i x^a \partial_i \partial^{(b)} + a_i x_i x^{a-1} \partial^{(b)} - x^a x_i \partial^{(b)} \partial_i - x^a \partial^{(b-1,i)} \partial_i$$

$$= a_i x^a \partial^{(b)} - \begin{pmatrix} b_i - 1 + 1 \\ 1 \end{pmatrix} x^a \partial^{(b)}$$

$$(a_i - b_i) x^a \partial^{(b)}$$

(in the remainder we omit the proof of such identities to avoid tedium), and in the case of characteristic $p$, if $a_i - b_i = cp^r$, we have $[x_i^p \partial_i^{(p^r)}, x^a \partial^{(b)}] = c x^a \partial^{(b)}$. In other words, the operators $[x_i \partial_i, -]$ and $[x_i^p \partial_i^{(p^r)}, -]$ give different weight to each degree-graded component. Note also that 

$$x^a \partial^{(b)}, x_i \partial_i = x^a \partial^{(b-1,i)} \partial_i = b_i x^a \partial^{(b)},$$

and if $b_i = cp^r$, we have $[x^a \partial^{(b)}, x_i^p \partial_i^{(p^r)}] = c x^a \partial^{(b)}$. In other words the operators $[-, x_i] \partial_i$ (and $[-, x_i^p \partial_i^{(p^r)}]$) give different weight to each order-graded component. Putting these together, we can isolate any term $x^a \partial^{(b)}$ by applying a suitable polynomial in the operators $[x_i \partial_i, -], [x_i^p \partial_i^{(p^r)}, -], [-, x_i \partial_i]$ and $[-, x_i^p \partial_i^{(p^r)}]$. 

For the second part, we may reduce to a single variable. We separate the cases by characteristic. If $\text{char}(k) = p$, we have $[x_i \partial_i^{(p^r)}, x_i^p] = x_i$, so $x_i$ is in the ideal generated by $x_i^p$; choosing a power of $p$ larger than $a_i$ we have $x_i^p = x_i^{a_i} \cdot x_i^{p^r - a_i}$ and so $x_i \in \langle x_i^{a_i} \rangle$. If $\text{char}(k) = 0$, on the other hand, we have 

$$[x_i \partial_i^{(2)}, x_i^{a_i}] = a_i x_i^{a_i} \partial_i + \begin{pmatrix} a_i \end{pmatrix} x_i^{a_i - 1}$$

and 

$$[x_i^2 \partial_i^{(3)}, x_i^{a_i}] = a_i x_i^{a_i+1} \partial_i^{(2)} + \begin{pmatrix} a_i \end{pmatrix} x_i^{a_i} \partial_i + \begin{pmatrix} a_i \end{pmatrix} x_i^{a_i - 1}.$$

If $a_i = 0, 1$ there is nothing to prove, and if $a_i > 1$, we can invert $\frac{a_i - 1}{2} \left( a_i^2 - 1 \right) = \frac{1}{a_i} a_i (a_i^2 - 1)$ and get 

$$x_i^{a_i - 1} = \frac{12}{a_i (a_i^2 - 1)} \left( a_i - 1 \right) \left( x_i \partial_i^{(2)}, x_i^{a_i} \right) - \frac{1}{a_i} a_i x_i^{a_i} \cdot x_i \partial_i^{(2)}.$$ 

This gives $\langle x_i^{a_i - 1} \rangle \subset \langle x_i^{a_i} \rangle$ and by iterating this procedure, $\langle x_i \rangle = \langle x_i^{a_i} \rangle$. 

For the third part, observe that $[x^a \partial^{(m)}, x_j] = x^a \partial^{(m-1,j)}$ (for $j$ such that $m_j \neq 0$) is a valid identity for all $n, m > 0$. Iterating this beginning with $n = a, m = b$ gives $\langle x^a \rangle \subset \langle x^a \partial^{(b)} \rangle$. By applying part 2 this becomes $\langle \prod_{a_i \neq 0} x_i \rangle \subset \langle x^a \partial^{(b)} \rangle$. 

To show the reverse implication $\langle x^a \partial^{(b)} \rangle \subset \langle \prod_{a_i \neq 0} x_i \rangle$ we show $\langle x^a \partial^{(b)} \rangle \subset \langle x_i \rangle$ for the two cases $a_i, b_i \neq 0$ and $a_i \neq 0, b_i = 0$. For the first case, $x_i^{a_i} \partial_i^{(b_i)}$ is a factor of $x^a \partial^{(b)}$; and applying the above argument we have that $x_i^{a_i} \partial_i^{(b_i)} \in \langle x_i \rangle$; it follows that $x^a \partial^{(b)} \in \langle \prod_{i,a_i \neq 0, b_i \neq 0} x_i \rangle$. 


For the second case, \( a_i \neq 0, b_i = 0 \), we may assume \( a_i = 1 \), for if \( a_i > 1 \), then clearly \( x^a \partial(b) = x_1 x^{a-1} \partial(b) \in \langle x_1 \rangle \). By the previous case, \( x_i \partial_i^{(2)} \) is in \( \langle x_i \rangle \), and so is \( x^{a+1} \partial(b) = x_i x^a \partial(b) \); then of course their commutator

\[
[x^{a+1} \partial(b), x_i \partial_i^{(2)}] = - (a_i + 1) x^{a+1} \partial_i \partial(b) - a_i x^a \partial(b)
\]
is also in \( \langle x_i \rangle \). Rewriting this (with \( a_i = 1 \) as we have assumed) we get

\[
x^a \partial(b) = [x^{a+1} \partial(b), x_i \partial_i^{(2)}] - 2 x^{a+1} \partial_i \partial(b)
\]
and so \( x^a \partial(b) \in \langle x_i \rangle \); it follows that \( x^a \partial(b) \in \langle \prod_{i:a_i \neq 0, b_i = 0} x_i \rangle \). Taking both cases together we have shown that \( x^a \partial(b) \in \langle \prod_{i:a_i \neq 0, b_i = 0} x_i \rangle \).

We have shown that all ideals in \( D(R) \) are generated by reduced monomials \( \prod x_i \) in the variables of \( R \); the next question is of course which ones? Recall that we will not distinguish between the vertices of the simplicial complex \( K \) and the variables of the associated face ring \( R \), but refer to either by the same name, e.g. \( x_i \). We also remind of the notation \( x_\sigma = \prod_{i \in \sigma} x_i \).

**Theorem 3.8.** Any proper ideal in \( D(R) \) is generated by monomials \( x_\sigma \) with \( \sigma \in K \) such that \( st(\sigma) \neq K \).

**Proof.** From 3.7 it follows that any ideal in \( D(R) \) is generated by reduced monomials in the variables \( x_i \), and clearly the monomials corresponding to non-faces cannot occur as they are in \( I_K \), so what remains are the monomials \( x_\sigma \) for \( \sigma \in K \). Only those \( x_\sigma \) such that \( st(\sigma) \neq K \) generate proper ideals, as otherwise we have \( st(\sigma) = K \) and by 4.4 the elements \( 1 \cdot \partial_i \) where \( x_i \in \sigma \) are in \( D(R) \), as both \( 1 \) and \( \partial_i \) are monomials with support contained in \( st(\sigma) = K \); if we write \( \sigma = \{ x_{i_1}, \ldots, x_{i_r} \} \), we have \( [\partial_{i_1}, [\partial_{i_2}, [\cdots, [\partial_{i_r}, x_\sigma]\cdots]]] = 1 \) and so \( \langle x_\sigma \rangle = \langle 1 \rangle = R \).

This now gives us all the ideals in \( D(R) \), as by sums of principal ideals \( \langle x_\sigma \rangle \) we can make everything. We may however also take a different approach: Any two-sided ideal in \( D(R) \) is the kernel of some ring homomorphism; the combinatorial structure of the associated simplicial complex \( K \) gives rise to several such maps. An obvious choice for candidate homomorphisms is the localization at an element \( x_\sigma \); we will see that the kernels of such maps is another generating set for the lattice of two-sided ideals in \( D(R) \). We introduce the notation \( \overline{J} \) for the extension to \( D(R) \) of an ideal \( J \subset R \).

**Theorem 3.9.** The kernel of the localization map \( D(R) \to D(R) [ \frac{1}{x_\sigma} ] \) is the extension \( \overline{I_{st(\sigma)}} \) of the ideal \( I_{st(\sigma,K)} \subset R \) to \( D(R) \).

**Proof.** By 3.8 it is enough to examine what happens in the localization to monomials \( x_\alpha \) for \( \alpha \in K \). Assume first that \( x_\sigma = x_i \) (in other words, \( \sigma \) is a vertex). Inverting \( x_i \) has the effect that for any non-face \( \beta = \cup x_j \) containing \( x_i \), the monomial \( \overline{x_\sigma} = \prod_{x_{i' \in \beta, j \neq i}} x_j \) is zero in the localization. It is clear that no other monomials are killed, so what remains after localization are those monomials supported on a face \( \tau \) such that \( \tau \cup x_i \) is not a non-face, or clearing negations, that \( \tau \cup x_i \) is a face in \( K \); in other words the remaining monomials are those supported on a face of \( st(x_i) \). For the general case, note that inverting \( x_\sigma = \prod x_i \) is the same as inverting each \( x_i \) successively, and observing that we have from 2.3 (iii) that \( st(\sigma,K) = st(x_1, st(\sigma \setminus x_1, K)) \), we are done by recursion.
Theorem 3.10. The lattice of two-sided ideals in $D(R)$ is generated by the ideals $\mathcal{I}_{st(\sigma)} \subset D(R)$.

Proof. After applying \[3.8\] the question is whether we can generate any proper ideal $\langle x_\tau \rangle$ by sums and intersections of the ideals $\mathcal{I}_{st(\sigma)}$. Considering that $\mathcal{I}_{st(\sigma)} = \langle x_\alpha | \alpha \in U_\sigma \rangle$, we can look at the intersection of all such ideals that contain $x_\tau$:

$$\bigcap_{\sigma \tau \in U_\sigma} \mathcal{I}_{st(\sigma)} = \langle x_\alpha | \alpha \in \bigcap_{\sigma \tau \in U_\sigma} U_\sigma \rangle$$

(3.1)

$$= \langle x_\alpha | \forall \sigma \in K : \tau \in U_\sigma \Rightarrow \alpha \in U_\sigma \rangle$$

(3.2)

$$= \langle x_\alpha | \forall \sigma \in K : \alpha \not\in U_\sigma \Rightarrow \tau \not\in U_\sigma \rangle$$

(3.3)

$$= \langle x_\alpha | \forall \sigma \in K : \alpha \cup \sigma \in K \Rightarrow \tau \cup \sigma \in K \rangle$$

(3.4)

$$= \langle x_\alpha | \forall \sigma \in K : \sigma \in st(\alpha) \Rightarrow \sigma \in st(\tau) \rangle$$

(3.5)

$$= \langle x_\alpha | st(\alpha) \subset st(\tau) \rangle$$

(3.6)

$$= \langle x_\tau \rangle$$

(3.7)

where the last step is applying Corollary \[3.9\] \qed

Example 3.11. Consider again the ring from \[3.3\] $R = k[x_1, x_2, x_3, x_4]/I$ where $I = (x_1, x_2, x_1x_4, x_2x_4)$; the associated complex $K$ is a chain of three 1-simplices. Inverting $x_1$ gives us that $x_3$ and $x_4$ go to zero in the localization as $x_3 = \frac{1}{x_1} x_1 x_3 \in I$, etc; it follows that the generators $x_4^a \partial_3^{(b)}$ are also killed; the kernel of the localization $D(R) \rightarrow D(R)[\frac{1}{x_1}]$ is then (using \[3.4\] and \[3.3\]) the ideal $\langle x_3, x_4 \rangle$, which is the face ideal of $st(x_1, K)$. Localizing at $x_2$ gives $x_3 = \frac{1}{x_2} x_2 x_4 = 0$, and the kernel of the localization is indeed equal to the ideal $\langle x_4 \rangle$, the face ideal of $st(x_2, K)$. Proceeding in the same manner for the remaining faces $x_3, x_2, \{x_1, x_2\}, \{x_2, x_3\}$, and $\{x_3, x_4\}$, we get as possible kernels the ideals $\langle x_1 \rangle, \langle x_4 \rangle, \langle x_1, x_2 \rangle, \langle x_3, x_4 \rangle$ and $\langle x_2, x_3 \rangle$. By \[3.3\] we have $\langle x_1, x_2 \rangle = \langle x_2 \rangle$ and $\langle x_3, x_4 \rangle = \langle x_3 \rangle$; in other words our possible kernels of localization are the ideals $\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle$ and $\langle x_4 \rangle$; in light of \[3.7\] these obviously generate all the ideals by sums and intersections.

Let us round off this section with some applications. In \[1999\], Traves examines the $D(R)$-module structure of $R$ when $k$ is a field, and determines what the (left) $D(R)$-submodules of $R$ are. These are the ideals $I \subset R$ such that $D(R) \bullet I = I$, so we follow Traves’ terminology and call such a submodule a $D(R)$-stable ideal. The reason for restricting $k$ to be a field is that elements of $D_k(R)$ are $k$-linear endomorphisms of $R$, so any ideal of $k$ extends to a $D_k(R)$-submodule of $R$.

Theorem 3.12 (Traves). When $k$ is a field, the $D_k(R)$-submodules of the reduced monomial ring $R$ are exactly the ideals given by intersections of sums of minimal primes of $R$.

Based on our results about the ideal structure of $D(R)$, we can give a new proof of this result. We denote the module action of $D(R)$ by $\bullet$ (e.g. $D(R) \bullet I$) and the product in $D(R)$ by $\cdot$ (e.g. $D(R) \cdot I$). We prove the result by means of a general fact which to our knowledge is previously unknown.

Proposition 3.13. Let $k$ be a field and $R$ be a $k$-algebra. An ideal $J \subset R$ is $D(R)$-stable if and only if $J = \overline{J} \cap R$, where $\overline{J}$ denotes the extension of $J$ to $D(R)$.

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Proof. Observe first that $R$ is isomorphic as a $D(R)$-module to $D(R)/D^>(0)(R)$, the quotient by the left ideal of positive order elements; we can see this by writing $D(R) = D^0(R) + D^>(0)(R) = R + D^>(0)(R)$, as $R = D^0(R)$. In other words, if $S \subset R$ is a subset, then under this isomorphism $D(R) \cdot S = D(R) \cdot S + D^>(0)(R)$. Further, if $J \in D(R)$ is some subset, then

$$J \cdot D(R) + D^>(0)(R) = J \cdot (D(R)^0 + D^>(0)(R)) + D^>(0)(R)$$

$$= J \cdot R + D^>(0)(R).$$

Now, if $I \subset R$ is an ideal, the extension of $I$ to $D(R)$ is $\overline{I} = D(R) \cdot I \cdot D(R)$, so we have

$$\overline{I} + D^>(0)(R) = D(R) \cdot I \cdot D(R) + D^>(0)(R)$$

$$= D(R) \cdot I + D^>(0)(R)$$

$$= D(R) \cdot I.$$

A $D$-stable ideal is an ideal $I \subset R$ such that $D(R) \cdot I = I$, so it follows that the $D$-stable ideals are exactly those such that $\overline{I} + D^>(0)(R) = I$.

It remains to show that for an ideal $J \subset D(R)$, $J + D^>(0)(R) = J \cap R$. Let $f \in J$ be some element, and write it as the sum $f = f_0 + f_1 + \cdots + f_{\text{ord}(f)}$ where $f_i$ are the terms of order $i$; it then follows from $\overline{I}$ that also each $f_i \in J$. Reducing modulo $D^>(0)(R)$ we get $J + D^>(0)(R) = \{ f_0 | f \in J \}$, and restricting to the homogenous elements of order zero we have $J \cap R = J \cap D^0(R) = \{ f \in J | f = f_0 \}$; these sets clearly are equal. \hfill $\square$

Theorem 3.14. The $D(R)$-stable ideals of $R$ are those generated by sums and intersections of the ideals $I_{st(\sigma)}$ for $\sigma \in K$.

Proof. As we have shown $\overline{I}$ and $3.10$ that any ideal of $D(R)$ is an extension of an ideal of $R$, we only have to restrict these to $R$ to recover the $D(R)$-stable ideals. Theorem $3.10$ tells us that the lattice of ideals in $D(R)$ is generated by sums and intersections of ideals $I_{st(\sigma)}$, and it is easy to see that $\overline{I_{st(\sigma)}} \cap R = I_{st(\sigma)}$. Indeed, the only possible problem is that in $D(R)$, $\langle x_{\alpha} \rangle \subset \langle x_{\beta} \rangle$ if and only if $st(\alpha) \subset st(\beta)$, and this may cause additional monomials not in $I$ to appear in $\overline{I} \cap R$. For $I_{st(\sigma)}$ however, this does not happen. Consider that $I_{st(\sigma)} = \langle x_{\tau} | \tau \in U_{\sigma} \rangle$ and $\overline{I_{st(\sigma)}} \cap R = \langle x_{\tau} | \tau \in U_{\sigma} \rangle = \langle x_{\alpha} | \exists \tau \in U_{\sigma} : st(\alpha) \subset st(\tau) \rangle$. In other words, we need to check if there are faces $\tau \in U_{\sigma}$ and $\alpha \in st(\sigma)$ such that $st(\alpha) \subset st(\tau)$, as then $x_{\alpha}$ would be in $I_{st(\sigma)} \cap R$, but not in $I_{st(\sigma)}$. This is impossible, however: by $2.3$ $(v)$, $\alpha \in st(\sigma)$ if and only if $\sigma \in st(\alpha)$, and if $st(\alpha) \subset st(\tau)$, we have $\sigma \in st(\tau)$, which again by $2.3$ $(v)$ gives $\tau \in st(\sigma)$, which contradicts the assumption $\tau \in U_{\sigma}$. \hfill $\square$

To recover $3.12$ recall that by $2.4$ the minimal primes are exactly the face ideals of the maximal faces of $K$, and any $I_{st(\sigma)}$ is the intersection of the face ideals of the maximal faces of $st(\sigma)$.

Remark 3.15. Recall that the partially ordered set of two-sided ideals of $D(R)$ (or bijectively, the $D(R)$-stable ideals of $R$) is in order-reversing bijection with the partially ordered set of closed stars of $K$. This partially ordered set can be completed to a simplicial complex $(\bar{K}, \bar{L})$, say, homotopic to the nerve of the cover of $K$ by open stars. The results about two-sided ideals of $D(R)$ and $D(R)$-stable ideals of $R$ imply that subcomplexes $L$ of $K$ such that $I_L$ is $D(R)$-stable or $\overline{I_L}$ is a
two-sided ideal of $D(R)$ are exactly those that are unions of intersections of closed stars; in other words the complex $K$ classifies such subcomplexes. This interesting connection is perhaps worthy of further study.

4. Characteristic $p$

The constructions in the previous section are independent of the characteristic of $k$, and so solve the problem of finding the two-sided ideal structure of $D(R)$. In characteristic $p$ however, there is a qualitatively different construction of $D(R)$, which perhaps offers more interesting possibilities for generalization. From here on, we assume $k$ is a field of characteristic $p$.

The major tool when working in characteristic $p$ is the Frobenius automorphism of $k$, given by $x \mapsto x^p$. This induces an endomorphism $F : R \to R$ given by $F(f) = f^p$, and the image $F(R)$ is the subring $R^p \subset R$ of $p$’th powers; as $R$ is reduced $F$ is also an isomorphism onto its image. Any $R$-module $M$ gets a new $R$-module structure through the pullback by the Frobenius map, namely $F_*M$, and it is equal to $M$ as an abelian group, but has $R$-module structure given by $f \cdot m = f^p m$. This is equivalent to considering $M$ as an $R^p$-module, as the maps $F : R \to R$ and $R^p \to R$ both are injections with image $R^p$. We will have need for considering also iterates of $F$, so if we let $q = p^r$ we write $F^r : R \to R$ or $R^q = R^{p^r} \subset R$. For our purposes in examining $D(R)$, it will be most convenient to use the description in terms of the subrings $R^q$, as we will see.

Considering the behaviour $R$ itself as an $R^p$-module gives rise to several classifying properties of the ring $R$. We will simply recall the definitions of the particular properties that are relevant for us, other such properties and further details may be found in [SVdB97]. If $R$ is finitely generated as an $R^p$-module, we say that $R$ is $F$-finite; if $R$ is $F$-finite and the map $R^p \to R$ splits as a map of $R^p$-modules we say $R$ is $F$-split; if $F_*^* R \simeq M_1^q \oplus \cdots \oplus M_{m(r)}^q$ as an $R$-module and the set of isomorphism classes $\{[M_i^q] | r \in \mathbb{N}, 1 \leq i \leq m(r)\}$ of modules appearing in such a decomposition for some $r$ is finite, we say that $R$ has finite $F$-representation type, or FFRT.

For our purposes, the key property of face rings $R_K$ in this respect is that they are $F$-split and have FFRT. Even better, we can give a concrete decomposition of $R$ as an $R^q$-module:

**Lemma 4.1.** As an $R^q$-module, $R$ is isomorphic to $\bigoplus_{st(\sigma) \subset K} (R_{st(\sigma)}^q)^{m_{st(\sigma)}(q)}$, where $m_{st(\sigma)}(q) = \sum_{\alpha : st(\alpha) = st(\sigma)}(q - 1)^{(\dim(\alpha)+1)}$.

Note that the direct sum runs over those subcomplexes of $K$ that is the star of some simplex.

**Proof.** As we have $R \simeq R^p \oplus R^p x_1 \oplus \cdots \oplus R^p x_n^{p-1} \cdots x_n^{p-1}$ (where only the appropriate monomials appear), this expresses $R$ as an $R^p$-module. We can rewrite this using $R^p \cdot x^\alpha \simeq R^p / \text{Ann}_{R^p}(x^\alpha)$, and observing that as the monomials $x^\alpha$ that appear in the decomposition are those supported on a face $\text{supp}(\alpha) =: \sigma$, and that the annihilator of $x^\alpha$ is the face ideal of the complex $st(\sigma, K)$, we get the decomposition $R = \bigoplus_{\sigma \in K} (R_{st(\sigma)}^p)^{m_{st(\sigma)}},$ where $R_{st(\sigma)}^p$ is the $(p$’th power) face ring of $st(\sigma)$ and by simply counting monomials we have $m_{st(\sigma)} = \sum_{\alpha : st(\alpha) = st(\sigma)}(p - 1)^{(\dim(\alpha)+1)}$ (using the convention that $\dim(\emptyset) = -1$). Iterating the same construction, we get $R = \bigoplus_{\sigma \in K} (R_{st(\sigma)}^q)^{m_{st(\sigma)}(q)}$, where $m_{st(\sigma)}(q) = (q - 1)^{(\dim(\sigma)+1)}$. \qed
Let us make use of this to compute some invariants of $R$ that only make sense in characteristic $p$, namely the Hilbert-Kunz function and the Hilbert-Kunz multiplicity. This invariant was introduced by Kunz \cite{Kun69} for local rings, and extended to graded rings by Conca \cite{Con96}; see also \cite{Hun13} and \cite{Mon83}.

**Definition 4.2.** Let $R$ be a local ring with maximal ideal $\mathfrak{m}$, or a graded ring with homogenous maximal ideal $\mathfrak{m}$, over a field $k$ of characteristic $p$, and let $q = p^r$. The Hilbert-Kunz function of a ring $R$ is the function

$$HK_R(q) = l(R/\mathfrak{m}^{[q]})$$

where $I^{[q]}$ is the ideal generated by $q$'th powers of elements in the ideal $I$. The Hilbert-Kunz multiplicity is the number

$$e_{HK}(R) = \lim_{q \to \infty} \frac{HK_R(q)}{q^{\dim R}},$$

in other words the leading coefficient of $HK_R(q)$.

The Hilbert-Kunz function gives a measure of singularity of $R$, roughly speaking higher multiplicities correspond to worse singularities. It is a theorem of Kunz that $HK_R(q) = q^\dim R$ if and only if $R$ is regular (see \cite{Kun69}), so if $R$ is regular, $e_{HK}(R) = 1$. The converse holds for unmixed rings, but not in general, and in particular not for face rings. The following is equivalent to Remark 2.2 in \cite{Con96}, though we prove it in a different way.

**Proposition 4.3.** Let $R_K$ be a face ring, then $HK_R(q) = \sum_{i=1}^{\dim R-1} f_i(q-1)^{i+1}$, where $f_i$ is the number of $i$-simplices in $K$, so $(f_1, \ldots, f_{\dim R-1})$ is the $f$-vector of $K$ (we recall the usual convention $\dim(\emptyset) = -1$, so $f_{-1} = 1$). In particular, $e_{HK}(R_K) = f_{\dim K}$, the number of top-dimensional faces of $K$.

**Proof.** The number of indecomposable summands of $R$ as an $R^n$-module is $\sum_{\alpha \in K} (q-1)^\dim(\alpha)+1$ by \cite{HK02}. By simply rearranging the sum, this is equal to $\sum_{i=1}^{\dim R-1} f_i(q-1)^{i+1}$. The claim now follows from the fact that none of the generators of these summands are in $\mathfrak{m}^{[q]} = \langle x_1^q, \ldots, x_n^q \rangle$, so the number of summands in the splitting of $R$ is the same as the length of $R/\mathfrak{m}^{[q]}$. \hfill $\square$

The promised different construction of $D(R)$ is due to Yekutieli \cite{Yek92}. We omit the proof here, but mention that in addition to \cite{Yek92}, the reader can find an excellent exposition in \cite{SVdB97}.

**Proposition 4.4.** $D_k(R) \simeq \bigcup_q End_{R^q}(R)$, where $q = p^r$, $r \in \mathbb{N}$ and $R^q$ is the subring of $q$-th powers.

Let us now give the summands appearing in \cite{HK02} a more convenient notation, and define $M_{st(\sigma)}^q := (R_{st(\sigma)}^q)^{\mathcal{M}_{st(\sigma)}(q)}$. It follows from \cite{HK02} that

$$End_{R^q}(R) \simeq \bigoplus_{st(\sigma), st(\tau) \subseteq K} Hom_R(M_{st(\sigma)}^q, M_{st(\tau)}^q).$$

As each $M_{st(\sigma)}^q$ is generated as an $R^q$-module by monomials of degree in each variable up to $q-1$, we can see that as an $R^m$-module it is contained in $\bigoplus_{st(\alpha) \subseteq st(\sigma)} M_{st(\alpha)}^{pq}$, because the elements of $M_{st(\sigma)}^q$ contain monomials of degree larger than $q-1$, which have support on smaller stars (recall that as $q = p^r$, $pq = p^{r+1}$). In particular this implies the following:
Lemma 4.5. \( \text{Hom}_{R^s}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \subset \bigoplus_{\sigma \subset st(\tau)} \text{Hom}_{R^s}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \). 

This lets us think of elements \( \phi \in \text{End}_{R^s}(R) \) as block matrices with each block having entries in some \( R^s/I_{st(\tau)} \); it is vital to remember that this means that the entries have degree equal to a multiple of \( q \).

Definition 4.6. Let \( J_q(st(\alpha), st(\beta)) \) denote the ideal in \( D(R) \) generated by the elements of \( \text{Hom}_{R^s}(M^q_{st(\alpha)}, M^q_{st(\beta)}) \), and let \( J(st(\alpha), st(\beta)) := \sum_q J_q(st(\alpha), st(\beta)) \). For convenience we denote \( J(st(\sigma), st(\tau)) \) by simply \( J(st(\sigma)) \).

The following result is essentially the same as \( \square \) in a different guise.

Proposition 4.7. Assume \( st(\sigma) \supseteq st(\tau) \), and let \( \phi \in \text{Hom}_{R^s}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \) be a nonzero element. Then \( \langle \phi \rangle \), the ideal in \( D(R) \) generated by \( \phi \), is equal to the ideal \( J(st(\tau)) \). Furthermore, we have that \( J(st(\tau)) \subseteq J(st(\sigma)) \).

Proof. Clearly, \( J(st(\tau)) \) is generated by the identity maps \( id^q_{st(\tau)} : M^q_{st(\tau)} \to M^q_{st(\tau)} \) (for each \( q \)), so it suffices to show that these are in \( \langle \phi \rangle \).

Recall that any element of \( \text{End}_{R^s}(R) \) has entries with degree a multiple of \( q \).

We claim that for \( s > q \) a sufficiently large power of \( p \), \( \phi \) considered as an element of \( \text{End}_{R^s}(R) \) will have at least some constant entries in each block \( \text{Hom}_{R^s}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \).

To see this, suppose \( \phi \) (as an element of \( \text{End}_{R^s}(R) \) has an entry \( x_{ij}^q \) in a block \( \text{Hom}_{R^s}(R^{a_q} \cdot x^a, R^{b_q} \cdot x^b) \) (with all \( 0 \leq a_q, b_q < q \), in other words \( \phi(x^{a+q}) = x^{a+(c+1)q+b} \). It follows from \( \square \) that this block has image in \( \text{End}_{R^s}(R) \) contained in \( \bigoplus_{0 \leq a_q < p} \text{Hom}_{R^s}(R^{pq} \cdot x^{a_q+1}c, R^{pq} \cdot x^{b_q+1}d) \), and as \( \phi(x^{a+q}) = x^{a+(c+1)q+b} \) this yields the entry 1 in the blocks \( \text{Hom}_{R^s}(R^{pq} \cdot x^{a+q}, R^{pq} \cdot x^{b+1}q) \).

In similar fashion an entry with degree \( nq \) will yield constant entries somewhere when considered as an \( R^s \)-linear map for \( s > q \) a sufficiently large power of \( p \).

Now let \( s \) be such a sufficiently large power of \( p \), and consider \( \phi \) as an element of \( \text{End}_{R^s}(R) \); by \( \square \) \( \text{Hom}_{R^s}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \) is contained in \( \bigoplus_{\sigma \subset st(\tau)} \text{Hom}_{R^s}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \).

We can see that \( \phi \), considered as a matrix \( (\phi_{ij}) \) in \( \text{End}_{R^s}(R) \), will have (among others) some constant entries in each block \( \text{End}_{R^s}(M^q_{st(\beta)}) \) such that \( st(\beta) \subset st(\tau) \).

Each of these entries can be “picked out” in the following manner: Let \( 1_{ii} \) be the matrix in \( \text{End}_{R^s}(R) \) with the appropriate identity map in position \( (i,i) \) and zeroes otherwise. It is clear that \( 1_{ii} \cdot \phi \cdot 1_{jj} \) is the matrix with entry \( \phi_{ij} \) in position \( (i,j) \) and zeroes otherwise; we may assume \( \phi_{ij} = 1 \) as it is constant. Applying permutations of \( \text{End}_{R^s}(M^q_{st(\beta)}) \) (on both sides), we can now place this entry 1 wherever we want within the matrix block corresponding to \( \text{End}_{R^s}(M^q_{st(\beta)}) \); taking sums of these we can produce any matrix with constant entries. In particular, we can make \( id^q_{st(\beta)} \).

Thus, we have that each \( id^q_{st(\beta)} \) such that \( st(\beta) \subset st(\tau) \) is in \( \langle \phi \rangle \), and in the same way any such \( id^q_{st(\beta)} \) for \( t > s \) any larger power of \( p \). To recreate \( id^q_{st(\tau)} \) for smaller powers \( t < s \) we observe that those maps, considered as elements of \( \text{End}_{R^s}(R) \), are in \( \bigoplus_{st(\beta) \subset st(\tau)} \text{End}_{R^s}(M^q_{st(\beta)}) \) and as such are contained in the ideal generated by the identity maps \( id^q_{st(\beta)} \), in other words contained in \( \langle \phi \rangle \). We have shown \( J(st(\tau)) \subset \langle \phi \rangle \); the opposite inclusion follows from the observation that \( \phi = id^q_{st(\tau)} \circ \phi \), and so \( \phi \in J(st(\tau)) \).

The final claim is similar: \( \phi = \phi \circ id^q_{st(\sigma)} \), and so \( \phi \in J(st(\sigma)) \). \( \square \)

Proposition 4.8. The ideal \( J(st(\sigma), st(\tau)) \) is equal to \( J(st(\sigma \cup \tau)) \), if \( \sigma \cup \tau \) is a face of \( K \), and the zero ideal otherwise.
Theorem 4.9. The ideals \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \) has support \( st(\sigma) \cap st(\tau) \). From [2.3] it follows that this is \( st(\sigma \cup \tau) \), if \( \sigma \cup \tau \in K \).

If \( \sigma \cup \tau \) is a non-face, \( st(\sigma) \cap st(\tau) \) does not contain any maximal simplices, and so the cone on \( st(\sigma) \cap st(\tau) \) is not a union of irreducible components of \( \text{Spec}(\mathcal{R}) \), and so is not the closure of the support of any element in \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \), so this must be the zero module. It follows that \( J(st(\sigma), st(\tau)) \) is the zero ideal.

Proof. By 4.7 and 4.8 this is equal to \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \).

In particular, there will be entries in the block \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma \cup \tau)}, M^q_{st(\sigma \cup \tau)}) \), so by 4.7 we have that \( J(st(\sigma \cup \tau)) \subset J(st(\sigma), st(\tau)) \).

For the converse, note that as an \( R^s \)-module, \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \approx (I^q_{st(\tau)} : I^q_{st(\sigma)}/I^q_{st(\tau)})^{m_{st(\sigma)}(q) \times m_{st(\tau)}(q)} \) (where \( I^q \) is the restriction of \( I \subset R \) to \( R^q \)). Any element of \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \) has, as a matrix, entries with degree (in each variable) a multiple of \( q \), with constant (nonzero) entries only when \( st(\beta) \subset st(\alpha) \), as then \( I^q_{st(\beta)}/I^q_{st(\alpha)} \) is the unit ideal in \( R^q \) (otherwise it is generated by elements of degree \( \geq q \)). It follows that elements of the image of \( \text{Hom}_{\mathcal{R}}(M^q_{st(\sigma)}, M^q_{st(\tau)}) \) in \( \text{End}_{\mathcal{R}^q}(R) \) for \( s > q \) (considered as matrices) have entries with degree some multiple of \( s \), with constant (nonzero) entries only in those blocks \( \text{Hom}_{\mathcal{R}^q}(M^s_{st(\alpha)}, M^s_{st(\beta)}) \) with \( st(\beta) \subset st(\alpha) \). In the direct limit, these elements become infinite matrices with entries in \( k \), in other words there can only be nonzero entries in those blocks corresponding to \( st(\beta) \subset st(\alpha) \) (any nonzero entry in a different block must have infinite degree, which is impossible). This implies that \( J(st(\sigma), st(\tau)) \) is contained in \( \sum_{st(\alpha) \supset st(\beta) \subset st(\tau)} J(st(\alpha), st(\beta)) \), which by 4.7 is equal to \( \sum_{st(\alpha) \supset st(\beta) \subset st(\tau)} J(st(\beta)) \). We are done.

Theorem 4.9. The ideals \( J(st(\sigma)) \) generate the lattice of ideals in \( D(R) \) by sums and intersections.

Proof. Let \( I \) be an ideal in \( D(R) \); it is of course true in general that \( I = \sum_{\phi \in I} \langle \phi \rangle \). By 4.7 and 4.8 this is equal to \( \sum J(st(\sigma)) \), where the sum goes over all \( \sigma \in K \) such that \( I \) contains elements from some \( \text{Hom}_{\mathcal{R}}(M^q_{st(\alpha)}, M^q_{st(\sigma)}) \).

Finally, the intersection \( J(st(\sigma)) \cap J(st(\tau)) \) contains elements in those \( \text{End}_{\mathcal{R}}(M^q_{st(\alpha)}) \) with \( st(\alpha) \subset st(\sigma) \cap st(\tau) \); the maximal such star is \( st(\sigma \cup \tau) \) if \( \sigma \cup \tau \) is a face of \( K \), and if \( \sigma \cup \tau \) is not a face, there are no such \( \alpha \); in other words \( J(st(\sigma)) \cap J(st(\tau)) = J(st(\sigma \cup \tau)) \).

We have now given two essentially different descriptions of the ideals of \( D(R) \), and we may wonder how to translate between the two languages. This is not too hard, as the obvious suggestion turns out to be true.

Theorem 4.10. The ideal \( J(st(\sigma)) \) is equal to the ideal \( \langle x_\sigma \rangle \).

Proof. It follows from 4.7 and 4.8 that \( J(st(\sigma)) = \bigoplus_{\beta > 0, st(\beta) \subset st(\sigma)} \text{Hom}_{\mathcal{R}^q}(M^q_{st(\alpha)}, M^q_{st(\beta)}) \), in other words all the endomorphisms with support contained in \( st(\sigma) \). We can think of \( x_\sigma \) as an endomorphism of \( R \), given by \( f \mapsto fx_\sigma \), and considering that whatever element \( f \) we choose, \( fx_\sigma \) has support contained in \( st(\sigma) \). This means that the endomorphism \( x_\sigma \) is in \( J(st(\sigma)) \) and not in any larger ideal, and as \( x_\sigma(1) = x_\sigma \) has
support equal to \( st(\sigma)^0 \), it is not in any smaller ideal \( J(st(\tau)) \) with \( st(\tau) \subset st(\sigma) \). From [1.7] it follows that \( x_\sigma \) generates all of \( J(st(\sigma)) \) and the two ideals are equal. □

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