Abstract

We study the local geometry of the zero pattern of a weighing matrix $W(23, 16)$. The geometry consists of 23 lines and 23 points where each line contains 7 points. The incidence rules are that every two lines intersect in an odd number of points, and the dual statement holds as well. We show that more than 50% of the pairs of lines must intersect at a single point, and construct a regular weighted graph out of this geometry. This might indicate that a weighing matrix $W(23, 16)$ does not exist.
On the finite geometry of $W(23, 16)$

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November 2011

A weighing matrix of size $n$ and weight $k$, generally denoted $W(n, k)$, is an orthogonal $n \times n \{0, 1, -1\}$-matrix with rows of length $\sqrt{k}$. Weighing matrices have applications in Chemistry, Spectroscopy, Quantum Computing and Code Theory. The main mathematical interest is to find or prove inexistence of a $W(n, k)$. For references about weighing matrices, see for example, [1] or [2].

To date, the smallest weighing matrix whose existence is unknown (see [3]) is $W(23, 16)$. In this note we study the underlying finite geometry of a $W(23, 16)$ and conjecture that it does not exist.

Suppose that $W$ is a $W(23, 16)$ matrix. We define a graph structure on the rows of $W$ by assigning an edge connecting the rows $W_i$ and $W_j$ if the zeros of $W_i$ overlap those of $W_j$ at only one position. We will show that there is at least one such edge, and in fact more than 50% of the pairs $W_i$ and $W_j$ are connected.

We begin by looking at a set $S$ of cardinality 23, and a collection of subsets (called lines) $\mathcal{G}(W) := \{L_1, L_2, \ldots, L_{23}\}$, such that $L_i$ is the characteristic set of the zeros of $W_i$. Then (i) each $L_i$ has cardinality 7, and by the orthogonality relations (ii) $|L_i \cap L_j| = 1, 3, 5, 7$ for all $i, j$. The fact that $W^T$ is also of type $W(23, 16)$ implies the duality statement: By interchanging between lines and points, any statement that we can prove in general from (i) and (ii) for $\mathcal{G}(W)$, is also true in $\mathcal{G}(W^T)$. For example, for any two points in $S$ there is an odd number of lines (i.e. 1, 3, 5 or 7) passing through them. We construct a graph $\Gamma(W)$ on the index set $\{1, 2, \ldots, 23\}$ by connecting $i$ and $j$ if $|L_i \cap L_j| = 1$. We claim

**Proposition 1.** $\Gamma(W)$ must contain at least one edge.

**Proof.** We consider the correspondence relation between unordered pairs of points $\{i, j\}$ $i \neq j$ and lines $L_k$ containing them. More precisely, the correspondence is the set

$$\mathcal{C} := \{\{i, j\}, L_k\} \mid i \neq j \text{ and } i, j \in L_k\}$$

We shall count the elements of $\mathcal{C}$ in two ways. First we sum over pairs $\{i, j\}$ where $\sigma(i, j) \in \{1, 3, 5, 7\}$ is the number of lines $L_{\sigma(i, j)}$ containing this pair. Second, we can sum over lines $L_k$, where for each $L_k$ we count the $\pi_k$ number of pairs $\{i, j\}$ contained in it. Thus we have the identity

$$\sum_{k=1}^{23} \pi_k = \sum_{1 \leq i < j \leq 23} \sigma(i, j).$$
Now, \( \pi_k = \binom{7}{2} \) so the left hand side equals \( 23 \cdot 7 \cdot 3. \) Suppose now that \( \sigma(i, j) > 1 \) for all \( \{i, j\}. \) Then in fact \( \sigma(i, j) \geq 3 \) and we have that

\[
23 \cdot 7 \cdot 3 = \sum_{k=1}^{23} \pi_k = \sum_{1 \leq i < j \leq 23} \sigma(i, j) \geq 3 \cdot \binom{23}{2} = 23 \cdot 11 \cdot 3.
\]

This is a contradiction, leading us to the conclusion that \( \sigma(i, j) = 1 \) for at least one pair. By duality, there are two lines \( L_i, L_j \) with \( |L_i \cap L_j| = 1. \) This is an edge of the graph, and we are done.

In fact we can conclude a much stronger statement, namely

**Proposition 2.** There are at least 138 edges in \( \Gamma(W). \)

**Proof.** This follows from the same identity, that \( 23 \cdot 7 \cdot 3 = \sum_{1 \leq i < j \leq 23} \sigma(i, j). \) If \( e \) is the number of (dual) edges, then \( 23 \cdot 7 \cdot 3 \geq e + 3(\binom{23}{2} - e), \) which implies our statement.

Notice that there can be at most \( \binom{23}{2} = 253. \) This means that there is more than \( 1/2 \) probability that two edges will be connected!

In fact we can prove more.

**Proposition 3.** Each vertex of the graph has at least 12 neighbors.

**Proof.** Consider a vertex, say \( L_1. \) We consider the correspondence

\[
C(L_1) := \{(i, L_j) \mid i \in L_1 \cap L_j \text{ and } j > 1\}.
\]

For each \( i \) let \( f_i \) count the number of set containing \( i \) besides \( L_1. \) For each \( j \) let \( \psi_j = |L_1 \cap L_j|. \) Then

\[
\sum_i f_i = \sum_j \psi_j.
\]

Let \( e \) be the number of \( j \) such that \( \psi_j = 1. \) Since \( f_i = 6 \) for all \( i, \) then \( 6 \cdot 7 \geq e + 3(23 - e) \) which implies that \( e \geq 12. \) Notice that \( e \) is the number of neighbors of \( L_1. \)

**Corollary 4.** The graph \( \Gamma(W) \) contains a triangle at any vertex of the graph.

**Proof.** Take a vertex \( v \) and twelve neighbors \( v_1, \ldots, v_{12}. \) The vertex \( v_1 \) has 12 neighbors (at least), so some of them come from \( v_2, \ldots, v_{12}. \) Hence we get a triangle.

**Corollary 5.** The graph \( \Gamma(W) \) has diameter 2 (at most).

**Proof.** Taking the neighbors of any two non-neighboring vertices, there must be a common neighbor.

We now introduce a weighted graph \( \overline{\Gamma}(W) \) which in a sense may serve as the complement graph of \( \Gamma(W). \)
Definition 6. The graph $\tilde{\Gamma}(W)$ is a weighted graph whose vertex set is the set $\{1, 2, \ldots, 23\}$ and we connect $i$ and $j$ with an edge of weight $1, 2, 3$ if $|\ell_i \cap \ell_j| = 3, 5, 7$ respectively.

Proposition 7. $\tilde{\Gamma}(W)$ is a regular weighted graph of degree $10$.

Proof. The proof follows from examining more carefully the proof of Proposition 3. Again we take a vertex, say $1$, and study the correspondence

$$C(L_1) := \{(i, L_j) \mid i \in L_1 \cap L_j, j > 1\}.$$

For each $i$ let $f_i$ count the number of lines containing $i$ except for $L_1$. For each $j$ let $\psi_j = |L_1 \cap L_j|$. Then we have

$$\sum_i f_i = \sum_j \psi_j. \quad (1)$$

Let $n_k$ be the number of lines $L_j$ that intersect $L_1$ with cardinality $k$. Then the right hand side of (1) is rewritten as $\sum_k kn_k = n_1 + 3n_3 + 5n_5 + 7n_7$. The left hand side equals $6 \cdot 7 = 42$ since $f_i = 6$ for all $i \in L_1$. It follows that

$$n_1 + 3n_3 + 5n_5 + 7n_7 = 42.$$

However, $n_1 + n_3 + n_5 + n_7 = 22$ as there are 23 lines. Combining these two facts together yields

$$2n_3 + 4n_5 + 6n_7 = 20 \implies n_3 + 2n_5 + 3n_7 = 10.$$

The proof is finished. □

Conjecture 8. A weighing matrix $W(23, 16)$ does not exist.

We base our conjecture on the fact that (1) The underlying set of the geometry has only 23 points, (2) Most pairs of lines have a single intersection point, and (3) there are many triples with pairwise intersection of a single point. Each triple covers at least 18 points. It seems unlikely that all this can be packed in a small set of size 23.

References

[1] I. S. Kotsireas, C. Koukouvinos, J. Seberry, New weighing matrices constructed from two circulant submatrices, *Optimizations Letters*, January 2012, Volume 6, Issue 1, pp 211-217

[2] C. Koukouvinos and J. Seberry, Weighing matrices and their applications, *JSPI*, 62 (1997) 91-101.

[3] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs, Second Edition*, Taylor and Francis 2006, ISBN-13: 978-1584885061