ABSTRACT

During inflation explicit perturbative computations of quantum field theories which contain massless, non-conformal fields exhibit secular effects that grow as powers of the logarithm of the inflationary scale factor. Starobinskii’s technique of stochastic inflation not only reproduces the leading infrared logarithms at each order in perturbation theory, it can sometimes be summed to reveal what happens when inflation has proceeded so long that the large logarithms overwhelm even very small coupling constants. It is thus a cosmological analogue of what the renormalization group does for the ultraviolet logarithms of quantum field theory, and generalizing this technique to quantum gravity is a problem of great importance. There are two significant differences between gravity and the scalar models for which stochastic formulations have so far been given: derivative interactions and the presence of constrained fields. We use explicit perturbative computations in two simple scalar models to infer a set of rules for stochastically formulating theories with these features.

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1 Introduction

During inflation, the energy-time uncertainty principle allows any massless virtual particle that emerged from the vacuum to persist forever. If classical conformal invariance is present, the rate of emergence redshifts so that very few such virtual particles are present. It is only for gravitons and minimally coupled scalars that classical conformal invariance is broken in such a way that inflation can give strong enhancements of quantum effects.

• de Sitter Inflation: A locally de Sitter geometry provides the simplest paradigm for inflation. To see why, consider a general homogeneous, isotropic and spatially flat geometry:

\[ ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} . \]

Derivatives of the scale factor \( a(t) \) give the Hubble parameter \( H(t) \) and the deceleration parameter \( q(t) \):

\[ H(t) \equiv \frac{\dot{a}}{a} , \quad q(t) \equiv -\frac{\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} . \]

The nonzero components of the Riemann tensor are:

\[ R_{ij00} = -qH^2 g_{ij} , \quad R_{ijkl} = H^2 (\delta_{ik} g_{jl} - \delta_{il} g_{jk}) . \]

Inflation is defined as positive expansion \( (H(t) > 0) \) with negative deceleration \( (q(t) < 0) \). On the other hand, stability – in the form of the weak energy condition – implies \( q(t) \geq -1 \). At the limit of \( q = -1 \) we see from (3) that the Riemann tensor assumes the locally de Sitter form:

\[ \lim_{q \to -1} R_{\sigma \mu \nu} = H^2 (\delta_{\mu} \delta_{\nu \sigma} g_{\sigma \nu} - \delta_{\nu} \delta_{\mu \sigma} g_{\sigma \mu}) . \]

It follows from (4) that the Hubble parameter is actually constant, and that the zero of time can be chosen to make the scale factor take the simple exponential form we shall henceforth assume:

\[ de \ Sitter \ Inflation \quad \Rightarrow \quad a(t) = e^{Ht} . \]

• Particle Production: The homogeneity of spacetime expansion in (1) does not change the fact that particles have constant wave vectors \( \vec{k} \), but it
does alter their physical meaning. In particular, the energy of a particle with mass \( m \) and wave number \( k \) becomes time dependent:

\[
E(t, k) = \sqrt{m^2 + \frac{k^2}{a^2(t)}} \quad , \quad k \equiv \|\vec{k}\| . \quad (6)
\]

This results in an interesting change in the energy-time uncertainty principle which restricts how long a virtual pair of such particles with wave vectors \( \pm \vec{k} \) can exist. If the pair was created at time \( t \), it can last a time \( \Delta t \) given by the integral:

\[
\int_t^{t+\Delta t} dt' \ E(t', k) \sim 1 . \quad (7)
\]

Just as in flat space, particles with the smallest masses persist longest. For the fully massless case, the integral is simple to evaluate,

\[
\int_t^{t+\Delta t} dt' \ E(t', k) \bigg|_{m=0} = \left[ 1 - e^{-H\Delta t} \right] \frac{k}{Ha(t)} . \quad (8)
\]

We, therefore, conclude that any massless virtual particle which happens to emerge from the vacuum can persist forever provided:

\[
\text{Unbounded Lifetime} \quad \Longrightarrow \quad k \leq Ha(t) . \quad (9)
\]

- **Conformal Invariance:** Most massless particles possess conformal invariance. A simple change of variables defines a conformal time \( \eta \) in terms of which the invariant element \((1)\) is just a conformal factor times that of flat space:

\[
ds^2 = -dt^2 + a^2(t) \ d\vec{x} \cdot d\vec{x} = a^2(\eta) \left( -d\eta^2 + d\vec{x} \cdot d\vec{x} \right) , \quad d\eta \equiv \frac{dt}{a(t)} . \quad (10)
\]

In the \((\eta, \vec{x})\) coordinates, conformally invariant theories are locally identical to their flat space counterparts. The rate at which virtual particles emerge from the vacuum per unit conformal time must be the same constant \( \Gamma \) – as in flat space.

\footnote{\textsuperscript{1}Of course “energy” is not a good quantum number in de Sitter background. What we mean by “\( E(t, k) \)” is the function whose integral times \((-i)\) determines the phase of plane wave mode functions. This statement is exact for \( m = 0 \) and true in the WKB limit for \( m \neq 0 \).}

\footnote{\textsuperscript{2}For particles which do not possess conformal invariance, the rates are generically different in de Sitter than in flat spacetime \([11, 12]\).}
Hence, the rate of emergence per unit physical time is:

\[
\frac{dN}{dt} = \frac{dN}{d\eta} \frac{d\eta}{dt} = \frac{\Gamma}{a(t)} \ . \tag{11}
\]

Consequently – although any sufficiently long wavelength, massless and conformally invariant particle emerging from the vacuum can persist forever during inflation – very few such particles will actually emerge.

- **Quantum Enhancement:** Gravitons and minimally coupled scalars are two kinds of massless particles which do not possess conformal invariance. To see that – unlike massless conformally invariant particles – the production of these two kinds of particles is not suppressed during inflation, note that each polarization and wave number behaves like a harmonic oscillator with time dependent mass and frequency:

\[
L = \frac{1}{2} m(t) \dot{q}^2(t) - \frac{1}{2} m(t) \omega^2(t) q^2(t) \ , \quad m(t) = a^3(t) \quad \& \quad \omega(t) = \frac{k}{a(t)} \ . \tag{12}
\]

The Heisenberg equation of motion can be exactly solved:

\[
\ddot{q} + 3H \dot{q} + \frac{k^2}{a^2} q = 0 \quad \Rightarrow \quad q(t) = u(t,k) \alpha + u^*(t,k) \alpha^\dagger \ , \tag{13}
\]

where the mode functions \( u \) and the commutation relations obeyed by the operators \( \alpha \) and \( \alpha^\dagger \) are given by:

\[
u(t,k) = \frac{H}{\sqrt{2k^3}} \left[ 1 - \frac{ik}{Ha(t)} \right] \exp \left( \frac{ik}{Ha(t)} \right) \ , \quad [\alpha, \alpha^\dagger] = 1 \ . \tag{14}
\]

The co-moving energy operator for this system is:

\[
E(t) = \frac{1}{2} m(t) \dot{q}^2(t) + \frac{1}{2} m(t) \omega^2(t) q^2(t) \ . \tag{15}
\]

Owing to the time dependent mass and frequency, there are no stationary states for this system. At any given time the minimum eigenstate of \( E(t) \) has energy \( \frac{1}{2} \omega(t) \), but which state this is changes for each value of time. The state \( |\Omega\rangle \) which is annihilated by \( \alpha \) has minimum energy in the distant past. The expectation value of the energy operator in its presence is,

\[
\langle \Omega | E(t) | \Omega \rangle = \frac{1}{2} a^3(t) |\dot{u}(t,k)|^2 + \frac{1}{2} a(t) k^2 |u(t,k)|^2 = \frac{k}{2a} + \frac{H^2 a}{4k} \ . \tag{16}
\]
If we think of each particle as having energy $k a^{-1}(t)$, it follows that the number of particles $N$ with any polarization and wave number $k$ grows as the square of the inflationary scale factor:

$$N(t, k) = \left[ \frac{Ha(t)}{2k} \right]^2 .$$  \hspace{1cm} (17)

Quantum field theoretic effects are driven by essentially classical physics operating in response to the source of virtual particles implied by quantization. On the basis of (17), one might expect inflation to dramatically enhance quantum effects from massless, minimally coupled scalars and gravitons. This has been confirmed explicitly and the oldest results are the cosmological perturbations induced by scalar inflatons [3] and by gravitons [4]. The more recent result which motivated the present analysis is that the gravitational back-reaction from the inflationary production of gravitons induces an ever greater slowing in the expansion rate [5, 6].

- **Quantum Cosmology:** The Lagrangian is the two-parameter effective gravitational theory:

$$\mathcal{L}_{GR} = \frac{1}{16\pi G} \left( -2\Lambda + R \right) \sqrt{-g} , \quad H^2 \equiv \frac{1}{3} \Lambda > 0 .$$ \hspace{1cm} (18)

What was actually computed [6] is the graviton one-point function, about a locally de Sitter background, in the presence of a state which is free Bunch-Davies vacuum at $t = 0$. However, if the resulting distortion of the background geometry was viewed in terms of an effective energy density and pressure the perturbative infrared results would be:

$$\rho(t) = \frac{\Lambda}{8\pi G} + \frac{(\kappa H)^2 H^4}{2^6\pi^4} \left\{ -\frac{1}{2} \ln^2 a + O(\ln a) \right\} + O(\kappa^4) ,$$ \hspace{1cm} (19)

$$p(t) = -\frac{\Lambda}{8\pi G} + \frac{(\kappa H)^2 H^4}{2^6\pi^4} \left\{ \frac{1}{2} \ln^2 a + O(\ln a) \right\} + O(\kappa^4) ,$$ \hspace{1cm} (20)

where $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. The one-loop effect from the kinematic energies of inflationary gravitons is a constant [7, 8] that must be subsumed into $\Lambda$. The next order effect is secular because the kinematic energies interact with the total graviton field strength which grows as more and more gravitons are produced. The induced energy density is negative because the gravitational interaction is attractive.
One would like to improve the aforementioned computation on the level of perturbation theory and beyond.

- **Invariant Regulation:** A perturbative improvement would be to use $D$-dimensional Feynman rules and dimensional regularization to obtain the fully renormalized answer and avoid complications that can arise from non-invariant counterterms. Such a procedure has been used in the simpler case of a massless, minimally coupled scalar field with a quartic self-interaction of strength $\lambda$ in a non-dynamical, locally de Sitter background:

$$\mathcal{L}_\varphi = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{\lambda}{4!} \sqrt{-g} \varphi^4 . \quad (21)$$

This theory can be renormalized so that, when released in free Bunch-Davies vacuum at $t = 0$, the energy density and pressure are [9, 10]:

$$\rho_{\text{ren}}(t) = \frac{\Lambda}{8\pi G} + \frac{\lambda H^4}{2^{6/4}} \left[ \frac{1}{2} \ln^2 a + \frac{2}{9} a^{-3} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{n + 2}{(n + 1)^2} a^{-n-1} \right] + O(\lambda^2) . \quad (22)$$

$$p_{\text{ren}}(t) = -\frac{\Lambda}{8\pi G} - \frac{\lambda H^4}{2^{6/4}} \left[ \frac{1}{2} \ln^2 a + \frac{1}{3} \ln a + \frac{1}{6} \sum_{n=1}^{\infty} \frac{n^2 - 4}{(n + 1)^2} a^{-n-1} \right] + O(\lambda^2) . \quad (23)$$

This is a completely renormalized result that – besides the leading infrared term – explicitly exhibits all the sub-leading infrared pieces. The one-loop effect from the kinematic energies of inflationary scalars is again a constant that must be subsumed into $\Lambda$. The next order effect is secular because the $\varphi^4$ self-interaction involves the total scalar field strength which grows as more and more scalars are produced. The induced energy density is positive, for $\lambda > 0$, because the $\varphi^4$ term adds to the energy density. In this model we might also guess that the effect is self-limiting because the classical restoring force tends to push the scalar back to towards zero energy density.

- **Infrared Logarithms:** Both gravitation and the scalar model, as [19, 20] and [22, 23] show, exhibit infrared logarithms – factors of $\ln(a) = Ht$. As inflation proceeds, these infrared logarithms grow without bound until they eventually overcome the small coupling constants – $(\kappa H)^2$ for gravity and $\lambda$ for the scalar model. We cannot conclude that there is actually a significant

\[\text{Secular terms are ubiquitous in quantum field theory; for example see [11, 12]}.\]
change in the background expansion rate because the higher order results remain unknown. The legitimate conclusions are rather that:

(i) The expansion rate decreases for gravitation and increases for the scalar model; and

(ii) Both effects eventually become non-perturbatively strong.

In Section 2 we explain why infrared logarithms occur, both from the mathematics of perturbation theory and on physical grounds. We also show that the field operator behaves like a stochastic random variable in the leading logarithm approximation.

- **Non-perturbative Extension:** Since time evolution makes the secular growth of the infrared logarithms unbounded, perturbation theory eventually breaks down. To reliably find out what happens after the breakdown, we must develop a non-perturbative technique. This seemingly impossible task may have a reasonable chance to produce a satisfactory method because inflationary evolution eventually makes the part of the quantum field responsible for particle production stochastic. Since it is precisely this part of the full field that drives the infrared effects of interest, we can try to isolate its contribution in the equations of motion and solve for its evolution non-perturbatively.

Starobinskiı has long argued that his technique of stochastic inflation should recover the leading infrared logarithms at each order in perturbation theory. A known example where the leading infrared late time evolution has been calculated beyond perturbation theory using stochastic techniques, is the scalar model; this was done by Starobinskiı and Yokoyama and its explication occupies Section 4. Their result agrees with the intuitive expectation that the growth of its field strength is eventually halted by the classical restoring force of the potential.

The gravitational system to which we should like to apply the same method, differs in many ways from the particular scalar model. Two basic differences are the presence of derivative interactions and constrained fields in the effective gravity theory. Both differences must be addressed and the stochastic approximation rules must be extended so that they can be applied to . This is done in Section 3 where we also show that the field operator behaves like a stochastic random variable in the leading logarithm approximation. In Sections 5 and 6, we consider and analyze suitable scalar

\footnote{For non-perturbative methods different from the one we shall describe here see .}
models with derivative interactions and a reasonable analogue of constrained fields. Our conclusions comprise Section 7.

The underlying idea behind the stochastic analysis of such theories is that the quantum field consists of two parts: a part which contains the ultraviolet effects that just redefine the parameters of the low energy interactions, and a part which contains the infrared effects from inflationary particle production and is responsible for the secular infrared logarithms. The ultraviolet sector decouples from the infrared except for:

(i) The constant renormalizations of the low energy parameters its interacting component furnishes; and

(ii) Its function as a reservoir of stochastic perturbations as the inflationary redshift pushes more and more modes into the infrared.

2 The Physics of Infrared Logarithms

The origin of the infrared logarithms in expressions (19-20) and (22-23) can be understood physically as well as from the mathematics of perturbation theory. Although the physical understanding is vastly more important in guiding the generalization we must make, we shall begin by explaining how perturbative computations are done invariantly in a locally de Sitter background. Besides revealing the sources of the infrared logarithms, these techniques provide an invariant separation between infrared and ultraviolet degrees of freedom. Moreover, these techniques shall be used in the computations of Sections 5 and 6 which relate exact results from perturbation theory to stochastic realizations.

• Invariant Regulation: We employ dimensional regularization in position space. The vertices are straightforward to obtain in any background, and the only difficulty comes in finding the propagators in arbitrary space-time dimension $D$. These propagators are expressed in terms of a de Sitter invariant length function we call $y(x; x')$:

$$y(x; x') \equiv a(t) a(t') \left[ H^2 \left\| \vec{x} - \vec{x}' \right\|^2 - \left( -\frac{1}{a} + \frac{1}{a'} \right)^2 \right]. \quad (24)$$

Its physical meaning in terms of the invariant length $\ell(x; x')$ between $x^\mu$ and $x'^\mu$ is:

$$y(x; x') = 4 \sin^2 \left[ \frac{1}{2} H \ell(x; x') \right]. \quad (25)$$
Conformal Scalar Propagator: The simplest propagator is that of a massless, conformally coupled scalar \([16]\):

\[
i\Delta_{\text{cf}}(x; x') = \frac{H^{D-2}}{(4\pi)^{D/2}} \Gamma\left(\frac{D}{2}-1\right) \left(\frac{4}{y}\right)^{D/2-1}.
\]

Because \(y(x; x')\) vanishes when \(x'^\mu = x^\mu\), and because one always interprets \(D\) so that zero is raised to a positive power in dimensional regularization, the coincidence limit of the conformally coupled propagator vanishes:

\[
i\Delta_{\text{cf}}(x; x) = 0.
\]

Scalar Propagator: Although conformal invariance suppresses interesting quantum effects, the conformal scalar propagator is quite useful for expressing the propagator of the minimally coupled scalar \([9, 10]\):

\[
i\Delta_{\text{cf}}(x; x') + \frac{H^{D-2}}{(4\pi)^{D/2}} \left(\frac{4}{y}\right)^{D/2-2} \Gamma\left(\frac{D}{2}\right) \Gamma\left(\frac{D}{2} - 1\right) \left[\pi \cot\left(\frac{\pi}{2} D\right) + \ln(aa')\right] + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} \Gamma\left(n + \frac{D}{2} - 1\right) \left(\frac{4}{y}\right)^n - \frac{1}{\Gamma(n + D/2)} \Gamma\left(n + \frac{D}{2} + 1\right) \left(\frac{4}{y}\right)^{n - \frac{D}{2} + 2} \right\}.
\]

This expression might seem daunting but it is actually simple to use in low order computations because the infinite sum on the final line vanishes in \(D = 4\), and each term in the series goes like a positive power of \(y(x; x')\). This means that the infinite sum can only contribute when multiplied by a divergence, and even then only a small number of terms can contribute.

Correlation Source: The explicit factor of \(\ln(aa')\) present in \((28)\) is one source of infrared logarithms. It gives the secular dependence of the coincidence limit \([17, 18, 19]\):

\[
i\Delta_{\text{cf}}(x; x) = \frac{H^{D-2}}{(4\pi)^{D/2}} \Gamma\left(\frac{D}{2}-1\right) \left\{ -\pi \cot\left(\frac{\pi}{2} D\right) + 2 \ln a \right\}.
\]

This factor of \(\ln(aa')\) is also interesting in that it breaks de Sitter invariance. Of course \(\varphi(x)\) transforms like a scalar; the breaking derives from the state in which the expectation value of the two free fields is taken. Allen and Folacci long ago proved that the massless, minimally coupled scalar fails to possess
normalizable de Sitter invariant states \[20\]. A final point is that the factor of \(\ln(aa')\) is actually an invariant, it just depends upon the initial value surface at which the state is released. In fact, it is the Hubble constant times the sum of the invariant times from \(x^\mu\) and \(x'^\mu\) to this initial value surface. The physical reason for the appearance of such a term is the increasing field amplitude due to inflationary particle production.

- **Graviton Propagator:** We define the graviton field \(\psi_{\mu\nu}(x)\) as follows:

\[
g_{\mu\nu}(x) \equiv a^2 \left( \eta_{\mu\nu} + \kappa \psi_{\mu\nu}(x) \right), \quad \kappa^2 \equiv 16\pi G . \tag{30}\]

Our gauge fixing Lagrangian takes the form:

\[
\mathcal{L}_{GF} = -\frac{1}{2} a^{D-2} \eta^{\mu\nu} F_\mu F_\nu , \tag{31}\]

where \(F_\mu\) is an analogue of the de Donder gauge fixing term of flat space \[21\]:

\[
F_\mu \equiv \eta^{\rho\sigma} \left[ \psi_{\mu\rho,\sigma} - \frac{1}{2} \psi_{\rho\sigma,\mu} + (D-2) H a \psi_{\mu\rho} \delta_\sigma^0 \right] . \tag{32}\]

Because space and time components are treated differently it is useful to have an expression for the purely spatial part of the Minkowski metric:

\[
\overline{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta_\mu^0 \delta_\nu^0 . \tag{33}\]

With these definitions the graviton propagator takes the form of a sum of three constant index factors times three scalar propagators:

\[
i \left[ \mu \nu \Delta_{\rho\sigma} \right](x; x') = \sum_{I=A,B,C} \left[ \mu \nu T^I_{\rho\sigma} \right] i \Delta_I(x; x') . \tag{34}\]

The explicit expressions for the index factors are \[22\]:

\[
\left[ \mu \nu T^A_{\rho\sigma} \right] = 2 \overline{\eta}_{\mu(\rho} \overline{\eta}_{\sigma)\nu} - \frac{2}{D-3} \overline{\eta}_{\mu\nu} \overline{\eta}_{\rho\sigma} , \tag{35}\]

\[
\left[ \mu \nu T^B_{\rho\sigma} \right] = -4 \delta_\mu^0 \overline{\eta}_{\nu(\rho} \delta_\sigma^0 , \tag{36}\]

\[
\left[ \mu \nu T^C_{\rho\sigma} \right] = \frac{2}{(D-2)(D-3)} \left[ (D-3) \delta_\mu^0 \delta_\nu^0 + \overline{\eta}_{\mu\nu} \right] \left[ (D-3) \delta_\rho^0 \delta_\sigma^0 + \overline{\eta}_{\rho\sigma} \right] . \tag{37}\]

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The $A$-type propagator is identical to (28), while the $B$-type and $C$-type are given by:

\[
i \Delta_B(x; x') = i \Delta_{C\text{F}}(x; x') - H_D^{-2} \left( \frac{4\pi}{D-2} \right) \sum_{n=0}^{\infty} \left\{ \frac{\Gamma(n+D-2)}{\Gamma(n+D/2)} \left( \frac{y}{4} \right)^n - \frac{\Gamma(n+D)}{\Gamma(n+2)} \left( \frac{y}{4} \right)^{n-D/2+2} \right\}, \tag{38}\n\]

\[
i \Delta_C(x; x') = i \Delta_{C\text{F}}(x; x') + H_D^{-2} \left( \frac{4\pi}{D-2} \right) \sum_{n=0}^{\infty} \left\{ (n+1) \frac{\Gamma(n+D-3)}{\Gamma(n+D/2)} \left( \frac{y}{4} \right)^n - (n-D/2+3) \frac{\Gamma(n+D-1)}{\Gamma(n+2)} \left( \frac{y}{4} \right)^{n-D/2+2} \right\}. \tag{39}\n\]

For completeness we also give the ghost propagator in this gauge:

\[
i \left[ \mu \Delta_\nu \right](x; x') = \pi_{\mu\nu} i \Delta_A(x; x') - \delta^0_\mu \delta^0_\nu i \Delta_B(x; x'). \tag{40}\n\]

Note that the infinite sums in (38) and (39) vanish for $D = 4$ so that, in this limit, the $B$-type and $C$-type propagators both agree with the conformal propagator. It is significant that only the $A$-type propagator contributes infrared logarithms. Since only physical graviton modes can experience the inflationary particle production responsible for infrared logarithms, we expect only the $A$-type propagator to contain them and this is indeed the case. A final point is that the breaking of de Sitter invariance apparent in these infrared logarithms is a real effect, not an artifact of having employed a de Sitter non-invariant gauge. One way to prove this is by defining a de Sitter transformation of the graviton field to include the compensating diffeomorphism needed to restore the gauge condition [23].

**Volume Source:** Although any infrared logarithms which appear in one-loop diagrams can only have come from the $A$-type propagator, there is another mechanism that can produce infrared logarithms in higher loop results such as (19-20) and (22-23). This other mechanism is the growth of the invariant volume of the past light-cone from the observation point back to the initial value surface:

\[
V_{\text{PLC}}(t) = \int_0^t dt' a^{D-1} \int d^{D-1}x' \theta[-y(x; x')] \equiv \frac{2\pi^{D-1}}{(D-1)\Gamma(D/2)} \ln a + O(1). \tag{41}\n\]
Factors of this quantity arise naturally whenever undifferentiated propagators connect an interaction vertex – at \( x'\mu \) – with the expectation value of some observable – at \( x\mu \).

To obtain true expectation values for cases – such as cosmology – in which the “in” and “out” vacua either do not agree or are not even well defined [24], the Schwinger-Keldysh formalism [25] must be employed. In this formalism the only net corrections come from interaction vertices which lie on or within the past light-cone of some observation point. The more familiar “in-out” matrix elements would harbor virulent infrared divergences from integrating over the exponentially large inflationary future volume [26]. The causality of the Schwinger-Keldysh formalism regulates these infrared divergences but simple correspondence implies that the regulated expressions must grow without bound at late times.

3 Infrared Dynamics and Their Rules

A crucially important consequence of our use of an invariant ultraviolet regularization is that there should be a constant dynamical impact from all modes whose physical wavelength ranges from zero – the far ultraviolet – to any fixed value. Recall that quanta are labeled by constant wave vectors \( \vec{k} \), and that the mode with wavenumber \( k = \| \vec{k} \| \) begins to experience significant inflationary particle production when the number of particles \( N(t, k) > 1 \) or, equivalently, when \( k < Ha(t) \). This suggests the following physical separation between “infrared” and “ultraviolet” modes:

\[
\text{Infrared} \quad \Rightarrow \quad H < k < Ha(t) , \quad (43)
\]
\[
\text{Ultraviolet} \quad \Rightarrow \quad k > Ha(t) . \quad (44)
\]

The range of the physical wavelengths \( \lambda_{ph} = 2\pi k^{-1} a(t) \) of the ultraviolet modes is from zero to the constant \( H^{-1} \). With any invariant regularization, the dynamical impact of such modes must be constant because they lie in an invariantly defined range.

Had we taken the lower limit in expression (43) down to \( k = 0 \), the infrared phase space would also have extended over a constant physical range. However, taking \( k \) down to zero has long been known to result in infrared divergences [27]. We regulate these by working on the manifold \( T^{D-1} \times \mathbb{R} \),

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with the range of the toroidal coordinates equal to a Hubble length \(26\):
\[-\frac{1}{2}H^{-1} < x^i \leq \frac{1}{2}H^{-1} . \] (45)

Other regulating techniques exist \[28\] but they all cause the effective infrared phase space to increase as the universe inflates. Indeed, this growth is the physical source of infrared logarithms. What happens is that the average field strength increases as it receives contributions from more and more infrared modes. If any interactions involve the undifferentiated field, then this growth can enter into physical quantities.

- **Self-interacting Scalar:** These assertions can be confirmed in the context of a self-interacting scalar theory in \(D = 3+1\) dimensions and in the presence of the inflationary background \(5\):

\[
\mathcal{L} = -\frac{1}{2}\sqrt{-g} \, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \sqrt{-g} \, V(\phi) ,
\] (46)

where, for stability reasons, the potential \(V(\phi)\) is to be bounded from below. The resulting field equation is:

\[
\ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V'(\phi) = 0 .
\] (47)

The full – ultraviolet plus infrared – perturbative initial value solution can be obtained by iterating a Yang-Feldman equation \[29\] for which the "in" time has been set to \(t = 0\):

\[
\phi(t, \vec{x}) = \phi_0(t, \vec{x}) - \int_0^t dt'\, a^3(t') \int d^3x' \, G_{\text{ret}}(t, \vec{x}; t', \vec{x}') \, V'(\phi)(t', \vec{x}') .
\] (48)

The retarded Green’s function equals:

\[
G_{\text{ret}}(t, \vec{x}; t', \vec{x}') \equiv \frac{H^2}{4\pi} \theta(t-t') \left\{ \frac{\delta(H\|\vec{x} - \vec{x}'\| + \frac{1}{a} - \frac{1}{a'})}{aa' \, H\|\vec{x} - \vec{x}'\|} + \theta(H\|\vec{x} - \vec{x}'\| + \frac{1}{a} - \frac{1}{a'}) \right\} .
\] (49)

The free field \(\phi_0(t, \vec{x})\) is expanded in terms of mode functions \(u(t, k)\) and operators \(\alpha(\vec{k})\) and \(\alpha^\dagger(\vec{k})\) obeying canonical commutation relations:

\[
\phi_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ e^{ik\cdot\vec{x}} \, u(t, k) \, \alpha(\vec{k}) + e^{-ik\cdot\vec{x}} \, u^*(t, k) \, \alpha^\dagger(\vec{k}) \right\} ,
\] (50)
where:

\[ u(t, k) = \frac{H}{\sqrt{2k^3}} \left[ 1 - \frac{ik}{Ha} \right] e^{i\frac{k}{Ha}} , \]  

(51)

\[ \left[ \alpha(\vec{k}), \alpha^+(\vec{k}') \right] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') . \]  

(52)

It should be noted that although \( \varphi_0(t, \vec{x}) \) is only the lowest order part of the solution, it and its first derivative agree exactly with the full field on the initial value surface.

- **Infrared Field:** To excise the ultraviolet modes (44), we iterate what is essentially the same equation:

\[ \Phi(t, \vec{x}) = \Phi_0(t, \vec{x}) - \int_0^t dt' \alpha^3(t') \int d^3x' \ G_{ret}(t, \vec{x}; t', \vec{x}') \ V'(\Phi)(t', \vec{x}') , \]  

(53)

but with the zeroth order solution restricted to only infrared modes:

\[ \Phi_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \theta(Ha(t) - k) \left\{ e^{i\vec{k} \cdot \vec{x}} u(t, k) \alpha_k + e^{-i\vec{k} \cdot \vec{x}} u^*(t, k) \alpha^+_k \right\} . \]  

(54)

This model is completely free of ultraviolet divergences, so we are justified in taking \( D = 3 + 1 \). To see that the model also reproduces the leading infrared logarithms at tree order it suffices to take the vacuum expectation value of \( \Phi_0^2 \): 

\[ \langle \Phi_0^2(t, \vec{x}) \rangle_{VEV} = \int \frac{d^3k}{(2\pi)^3} \theta(Ha - k) \left( \frac{H^2}{2k^3} \right) \left\{ 1 + \frac{k^2}{H^2a^2} \right\} , \]  

(55)

\[ = \frac{H^2}{4\pi^2} \int_H^{Ha} \frac{dk}{k} \left\{ 1 + \frac{k^2}{H^2a^2} \right\} , \]  

(56)

\[ = \left( \frac{H}{2\pi} \right)^2 \left\{ \ln a + \frac{1}{2} - \frac{1}{2a^2} \right\} . \]  

(57)

Comparison with (29) for \( D = 3 + 1 \) reveals exact agreement between the \( \ln(a) \) terms.

---

\(^5\)Vacuum expectation values are taken throughout in the presence of the free Bunch–Davies vacuum at \( t = 0 \).
Infrared Field Equation: Our purely infrared field operator \( \Phi(t, \vec{x}) \) does not quite obey the original field equation (47) because the kinetic operator fails to annihilate \( \Phi_0(t, \vec{x}) \). The reason is that, upon taking time derivatives of \( \Phi_0(t, \vec{x}) \), there are extra contributions due to the presence of the time dependent upper limit \( Ha(t) \) of the mode sum in (54). Thus, the action of the kinetic operator on \( \Phi_0(t, \vec{x}) \) produces momentum space surface terms:

\[
\ddot{\Phi}_0(t, \vec{x}) + 3H \dot{\Phi}_0(t, \vec{x}) - \frac{\nabla^2}{a^2(t)} \Phi_0(t, \vec{x}) = \dot{\mathcal{F}}(t, \vec{x}) + \mathcal{G}(t, \vec{x}) + 3H \mathcal{F}(t, \vec{x}) , \tag{58}
\]

involving the sources:

\[
\mathcal{F}(t, \vec{x}) \equiv H \int \frac{d^3k}{(2\pi)^3} k \delta(k-Ha(t)) \left\{ e^{i\vec{k} \cdot \vec{x}} u(t, k) \alpha_k + e^{-i\vec{k} \cdot \vec{x}} u^*(t, k) \alpha_k^\dagger \right\} \tag{59}
\]

\[
\mathcal{G}(t, \vec{x}) \equiv H \int \frac{d^3k}{(2\pi)^3} k \delta(k-Ha(t)) \left\{ e^{i\vec{k} \cdot \vec{x}} \dot{u}(t, k) \alpha_k + e^{-i\vec{k} \cdot \vec{x}} \dot{u}^*(t, k) \alpha_k^\dagger \right\} \tag{60}
\]

Note that the mode functions and their derivatives are simply constants at \( k = Ha(t) \):

\[
u(t, k) \bigg|_{k=Ha(t)} = \frac{H}{\sqrt{2k^3}} (1 - i) e^i \implies |u(t, k)|^2 \bigg|_{k=Ha(t)} = \frac{H^2}{k^3} , \tag{61}
\]

\[
\dot{u}(t, k) \bigg|_{k=Ha(t)} = -\frac{H^2}{\sqrt{2k^3}} e^i \implies |\dot{u}(t, k)|^2 \bigg|_{k=Ha(t)} = \frac{H^4}{2k^3} . \tag{62}
\]

In view of (58) the equation obeyed by the infrared field \( \Phi(t, \vec{x}) \) is not (47) but rather:

\[
(\ddot{\Phi} - \dot{\mathcal{F}} - \mathcal{G}) + 3H (\dot{\Phi} - \mathcal{F}) - \frac{\nabla^2}{a^2} \Phi + V'(\Phi) = 0 . \tag{63}
\]

Infrared Conservation: The infrared field equation (63) does not leave the original stress-energy tensor conserved because stress-energy is being continuously dumped into the truncated system by ultraviolet modes which redshift past the horizon. We can account for this by modifying what we call \( T_{\mu\nu} \). To motivate the modification it is useful to write down the form we expect for the divergence of the stress-energy. For the zero component we should get:

\[
T_{0\mu} = -\dot{T}_{00} - H \left[ 3T_{00} + g^{ij}T_{ij} + g^{ij}T_{0i, j} \right] , \tag{64}
\]

\[
= -\dot{\Phi} \left( \ddot{\Phi} - \dot{\mathcal{F}} - \mathcal{G} \right) + 3H (\dot{\Phi} - \mathcal{F}) - \frac{\nabla^2}{a^2} \Phi + V'(\Phi) , \tag{65}
\]

Infrared Field Equation: Our purely infrared field operator \( \Phi(t, \vec{x}) \) does not quite obey the original field equation (47) because the kinetic operator fails to annihilate \( \Phi_0(t, \vec{x}) \). The reason is that, upon taking time derivatives of \( \Phi_0(t, \vec{x}) \), there are extra contributions due to the presence of the time dependent upper limit \( Ha(t) \) of the mode sum in (54). Thus, the action of the kinetic operator on \( \Phi_0(t, \vec{x}) \) produces momentum space surface terms:

\[
\ddot{\Phi}_0(t, \vec{x}) + 3H \dot{\Phi}_0(t, \vec{x}) - \frac{\nabla^2}{a^2(t)} \Phi_0(t, \vec{x}) = \dot{\mathcal{F}}(t, \vec{x}) + \mathcal{G}(t, \vec{x}) + 3H \mathcal{F}(t, \vec{x}) , \tag{58}
\]

involving the sources:

\[
\mathcal{F}(t, \vec{x}) \equiv H \int \frac{d^3k}{(2\pi)^3} k \delta(k-Ha(t)) \left\{ e^{i\vec{k} \cdot \vec{x}} u(t, k) \alpha_k + e^{-i\vec{k} \cdot \vec{x}} u^*(t, k) \alpha_k^\dagger \right\} \tag{59}
\]

\[
\mathcal{G}(t, \vec{x}) \equiv H \int \frac{d^3k}{(2\pi)^3} k \delta(k-Ha(t)) \left\{ e^{i\vec{k} \cdot \vec{x}} \dot{u}(t, k) \alpha_k + e^{-i\vec{k} \cdot \vec{x}} \dot{u}^*(t, k) \alpha_k^\dagger \right\} \tag{60}
\]

Note that the mode functions and their derivatives are simply constants at \( k = Ha(t) \):

\[
u(t, k) \bigg|_{k=Ha(t)} = \frac{H}{\sqrt{2k^3}} (1 - i) e^i \implies |u(t, k)|^2 \bigg|_{k=Ha(t)} = \frac{H^2}{k^3} , \tag{61}
\]

\[
\dot{u}(t, k) \bigg|_{k=Ha(t)} = -\frac{H^2}{\sqrt{2k^3}} e^i \implies |\dot{u}(t, k)|^2 \bigg|_{k=Ha(t)} = \frac{H^4}{2k^3} . \tag{62}
\]

In view of (58) the equation obeyed by the infrared field \( \Phi(t, \vec{x}) \) is not (47) but rather:

\[
(\ddot{\Phi} - \dot{\mathcal{F}} - \mathcal{G}) + 3H (\dot{\Phi} - \mathcal{F}) - \frac{\nabla^2}{a^2} \Phi + V'(\Phi) = 0 . \tag{63}
\]

Infrared Conservation: The infrared field equation (63) does not leave the original stress-energy tensor conserved because stress-energy is being continually dumped into the truncated system by ultraviolet modes which redshift past the horizon. We can account for this by modifying what we call \( T_{\mu\nu} \). To motivate the modification it is useful to write down the form we expect for the divergence of the stress-energy. For the zero component we should get:

\[
T_{0\mu} = -\dot{T}_{00} - H \left[ 3T_{00} + g^{ij}T_{ij} + g^{ij}T_{0i, j} \right] , \tag{64}
\]

\[
= -\dot{\Phi} \left( \ddot{\Phi} - \dot{\mathcal{F}} - \mathcal{G} \right) + 3H (\dot{\Phi} - \mathcal{F}) - \frac{\nabla^2}{a^2} \Phi + V'(\Phi) . \tag{65}
\]
while for the spatial components:

\[ T_{\mu i}^{(\nu)} = -T_{0i} - 3HT_{0i} + \frac{1}{a^2} T_{ij,j}, \quad (66) \]

\[ = -\partial_i \Phi \left[ (\ddot{\Phi} - \ddot{\mathcal{F}} - \mathcal{G}) + 3H(\dot{\Phi} - \dot{\mathcal{F}}) - \frac{\nabla^2}{a^2} \Phi + V'(\Phi) \right]. \quad (67) \]

We can enforce (65) and (67) with a stress-energy of the form:

\[ T_{00} = \frac{1}{2} (\dot{\Phi} - \mathcal{F})^2 + \frac{1}{2a^2} \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi + V(\Phi), \quad (68) \]

\[ T_{0i} = (\dot{\Phi} - \mathcal{F}) \partial_i \Phi, \quad (69) \]

\[ T_{ij} = \partial_i \Phi \partial_j \Phi - g_{ij} \left[ -\frac{1}{2} (\dot{\Phi} - \mathcal{F})^2 + \frac{1}{2a^2} \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi + V(\Phi) \right] \]

\[ + \partial_i S_j + \partial_j S_i - \frac{1}{2} (\delta_{ij} - 3 \partial_i \partial_j) S^L + \frac{1}{2} (\delta_{ij} - \partial_i \partial_j) S. \quad (70) \]

Once this form is assumed, the non-local source terms of the purely spatial components are determined by conservation:

\[ S_i = \frac{a^2}{\nabla^2} \left[ \mathcal{G} \partial_i \Phi + (\dot{\Phi} - \mathcal{F}) \partial_i \mathcal{F} \right] - \frac{a^4}{\nabla^4} \partial_i \partial_k \left[ \mathcal{G} \partial_k \Phi + (\dot{\Phi} - \mathcal{F}) \partial_k \mathcal{F} \right], \quad (71) \]

\[ S^L = \frac{a^2}{\nabla^2} \partial_k \left[ \mathcal{G} \partial_k \Phi + (\dot{\Phi} - \mathcal{F}) \partial_k \mathcal{F} \right], \quad (72) \]

\[ S = \frac{a^2}{H} \mathcal{F} \left[ \ddot{\Phi} - \ddot{\mathcal{F}} + 3H(\dot{\Phi} - \dot{\mathcal{F}}) - \frac{\nabla^2}{a^2} \Phi \right] - \frac{a^2}{H} (\dot{\Phi} - \mathcal{F}) \mathcal{G} - H \vec{\nabla} \mathcal{F} \cdot \vec{\nabla} \Phi \quad (73) \]

\[ = -\frac{a^2}{H} (\dot{\Phi} - \mathcal{F}) \mathcal{G} - \frac{1}{H} \vec{\nabla} \mathcal{F} \cdot \vec{\nabla} \Phi - \frac{a^2}{H} \mathcal{F} V'(\Phi). \quad (74) \]

We have, therefore, achieved a completely consistent model of just the infrared modes that reproduces the leading infrared logarithms. Note that this establishes the decoupling of the ultraviolet sector.

- **Leading Infrared Field:** Although the zeroth order field \( \Phi_0(t, \vec{x}) \) contains only infrared modes, it is still quantum mechanical in that the field and its first time derivative do not commute:

\[ \left[ \Phi_0(t, \vec{x}), \dot{\Phi}_0(t, \vec{x}') \right] = \int \frac{d^3k}{(2\pi)^3} \theta \left( Ha(t) - k \right) \left[ u \dot{u}^* - u^* \dot{u} \right] e^{i \vec{k} \cdot \Delta \vec{x}}, \quad (75) \]
\[
\frac{i}{4\pi^2 a^3} \int_H \, dk \, k^2 \frac{\sin(k\Delta x)}{k\Delta x}, \tag{76}
\]
\[
= \frac{i}{4\pi^2(a\Delta x)^3} \left\{ \sin(aH\Delta x) - aH\Delta x \cos(aH\Delta x) \right\} \tag{77}
\]

where \(\Delta \vec{x} \equiv \vec{x} - \vec{x}'\) and \(\Delta x \equiv \|\Delta \vec{x}\|\). However, the leading infrared logarithm in (57) derives entirely from the constant first term of the long wavelength expansion of the mode function:

\[
u(t,k) = \frac{H}{\sqrt{2k^3}} \left[ 1 - \frac{ik}{Ha} \right] e^{\frac{ik}{Ha}} , \tag{78}
\]
\[
= \frac{H}{\sqrt{2k^3}} \left\{ 1 + \frac{1}{2} \left( \frac{k}{Ha} \right)^2 + \frac{i}{3} \left( \frac{k}{Ha} \right)^3 + O(k^4) \right\} . \tag{79}
\]

By constructing a free field, which we shall call \(\phi_0(t,\vec{x})\), with only this first term as its mode function:

\[
\text{Leading Infrared} \implies u_{\text{IR}}(t,k) \sim \frac{H}{\sqrt{2k^3}} , \tag{80}
\]

we would get precisely the same leading infrared logarithm:

\[
\phi_0(t,\vec{x}) \equiv \int \frac{d^3k}{(2\pi)^3} \theta(Ha(t) - k) \frac{H}{\sqrt{2k^3}} \left\{ e^{ik\vec{x}} \alpha_{\vec{k}} + e^{-ik\vec{x}} \alpha_{\vec{k}}^\dagger \right\} . \tag{81}
\]

Because – unlike a usual quantum field – the creation and annihilation parts of \(\phi_0(t,\vec{x})\) are both multiplied by a phase factor with identical time dependence, the leading infrared field commutes with its time derivative:

\[
[\phi_0(t,\vec{x}) , \phi_0(t',\vec{x}')] = 0 . \tag{82}
\]

Consequently, \(\phi_0\) behaves like a classical variable except for the operators \(\alpha\) and \(\alpha^\dagger\) which can take random values. Such a random but commuting variable might be termed “stochastic”.

- **Leading Infrared Field Equation:** Replacing \(\Phi_0\) with \(\phi_0\) in the basic equation (53) does capture the leading infrared logarithms, but the resulting field theory is still more complicated than necessary. The leading infrared logarithms are preserved if we retain only the first term in the long wavelength expansion of the mode function:

\[
u(t,k) = \frac{H}{\sqrt{2k^3}} \left[ 1 - \frac{ik}{Ha} \right] e^{\frac{ik}{Ha}} , \tag{78}
\]
\[
= \frac{H}{\sqrt{2k^3}} \left\{ 1 + \frac{1}{2} \left( \frac{k}{Ha} \right)^2 + \frac{i}{3} \left( \frac{k}{Ha} \right)^3 + O(k^4) \right\} . \tag{79}
\]
expansion of the retarded Green’s function. From the expression:
\[
G_{\text{ret}}(t, \vec{x} ; t', \vec{x}') = i\theta(t - t') \times \int \frac{d^3k}{(2\pi)^3} \left[ u(t, k) u^*(t', k) - u^*(t, k) u(t', k) \right] e^{i\vec{k} \cdot \Delta \vec{x}} ,
\]
we can determine the long wavelength expansion of the bracketed term using (83):
\[
\left[ u(t, k) u^*(t', k) - u^*(t, k) u(t', k) \right] = \frac{i}{3H} \left( \frac{1}{a'^3} - \frac{1}{a^3} \right) + O(k^2) .
\]
Retaining only the leading term from (83) in (84) we conclude:
\[
G_{\text{ret}}(t, \vec{x} ; t', \vec{x}') \longrightarrow \frac{1}{3H} \theta(t - t') \delta^3(\vec{x} - \vec{x}') \left( \frac{1}{a'^3} - \frac{1}{a^3} \right) .
\]
Recall that the Green’s function is multiplied by a factor of \( a'^3 \) from the measure of integration in the Yang-Feldman equation. Whereas the term \((a'/a)^3 = 1\) contributes coherently over the full range of integration, the term \(-(a'/a)^3\) is only significant for \( t' \sim t \). Hence, the second term in (85) is irrelevant in the leading logarithm approximation:
\[
\text{Leading Infrared} \quad \Rightarrow \quad a'^3 G_{\text{ret}}^{\text{IR}}(t, \vec{x} ; t', \vec{x}') \sim \frac{1}{3H} \theta(t - t') \delta^3(\vec{x} - \vec{x}') .
\]
We are at length lead to the following simple iterative equation for recovering the leading infrared logarithms:
\[
\phi(t, \vec{x}) = \phi_0(t, \vec{x}) - \frac{1}{3H} \int_0^t dt' \left( V(t', \vec{x}) \phi(t', \vec{x}) \right) .\]
To infer the local differential equation which the full stochastic field \( \phi \) obeys, we note that the time derivative of the stochastic free field \( \phi_0 \) is a momentum space surface term:
\[
\dot{\phi}_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \delta(k-Ha(t)) \frac{H^2}{\sqrt{2k}} \left\{ e^{i\vec{k} \cdot \vec{x}} \alpha_{\vec{k}} + e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger_{\vec{k}} \right\} ,
\]
\[
\equiv f_\phi(t, \vec{x}) .
\]
Taking the time derivative of (87) gives the Langevin equation for the stochastic field $\phi(t, \vec{x})$ first obtained by Starobinski˘ı [14, 15]:

$$\dot{\phi}(t, \vec{x}) = f_\phi(t, \vec{x}) - \frac{1}{3H} V'(\phi(t, \vec{x})) . \tag{90}$$

Physically, $f_\phi$ is a stochastic source caused by the ultraviolet modes that are instantaneously redshifting across the horizon at time $t$. Because each increment is uncorrelated, this source represents Gaussian white noise:

$$\langle f_\phi(t, \vec{x}) f_\phi(t', \vec{x}) \rangle_{\text{VEV}} = \frac{H^3}{4\pi^2} \delta(t - t') . \tag{91}$$

The sort of system comprised by (89-90) has been much studied [30]. Expectation values of functionals of the stochastic field can be computed in terms of a probability density $\rho(t, \phi)$ as follows:

$$\langle F[\phi(t, \vec{x})] \rangle_{\text{VEV}} = \int_{-\infty}^{+\infty} d\omega \ \rho(t, \omega) F(\omega) . \tag{92}$$

The probability density satisfies a Fokker-Planck equation whose first term is given by the interaction on the right hand side of equation (90) and whose second term is fixed by the normalization of the white noise (91):

$$\dot{\rho}(t, \phi) = \frac{1}{3H} \frac{\partial}{\partial \phi} \left[ V'(\phi) \rho(t, \phi) \right] + \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} \left[ \rho(t, \phi) \right] . \tag{93}$$

- **Leading Infrared Rules**: Relation (91) derives from the original field equation (47) by applying the following rules:

(I) *At each order in the field* $\varphi^1, \varphi^2, \text{and so forth} - \text{retain only the term with the smallest number of derivatives.} \quad \tag{94}$

(II) *For the linear terms in the field, each time derivative has a stochastic source subtracted.* \quad \tag{95}

For the kinetic part of (47), by applying rule I (94) we select the Hubble friction term and, thereafter, by applying rule II (95) a stochastic source must be subtracted off because we are at linear order: \(^6\)

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2}{a^2} \varphi \rightarrow 3H\dot{\varphi} \rightarrow 3H\left( \dot{\varphi} - f_\phi \right) . \tag{96}$$

\(^6\)Since the scale factor varies much faster than the field during inflation, it is preferable to have derivatives act on the scale factor instead of the field. Thus, the single time derivative of the Hubble friction term dominates over the second time and space derivatives terms.
There are no derivatives or linear terms in the interaction part of (47) so we merely replace the full field by its stochastic counterpart:

$$V'(\varphi) \rightarrow V'(\phi). \quad (97)$$

Hence the full equation (47) reduces as follows,

$$\dddot{\varphi} + 3H \dot{\varphi} - \frac{\nabla^2}{a^2} \varphi + V'(\varphi) \rightarrow 3H \left( \dot{\phi} - f_{\phi} \right) + V'(\phi), \quad (98)$$

which is indeed equivalent to (90).

It is important to note that we would get the same leading infrared logarithms by solving (63) for $\Phi$. The enormous advantage of solving the stochastic equation (90) instead, is that the field $\phi$ can be regarded as classical. Whereas there is little hope of being able to exactly solve the Heisenberg field equations for an interacting, four dimensional quantum field, classical field equations can sometimes be solved.

### 4 Scalar Field with Quartic Self-Interaction

For the quartic self-interaction (21) the full field equation is:

$$\dddot{\varphi} + 3H \dot{\varphi} - \frac{\nabla^2}{a^2} \varphi + \frac{\lambda}{6} \varphi^3 = 0. \quad (99)$$

Applying the two stochastic reduction rules gives:

$$3H \left( \dot{\phi} - f_{\phi} \right) + \frac{\lambda}{6} \phi^3 = 0. \quad (100)$$

- **Non-perturbative Solution:** To compute expectation values we need the probability density $\rho(t, \phi)$ which obeys the Fokker-Planck equation:

$$\frac{\partial}{\partial t} \rho(t, \phi) = \frac{\lambda}{18H} \frac{\partial}{\partial \phi} \left( \phi^3 \rho(t, \phi) \right) + \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2} \rho(t, \phi). \quad (101)$$

To recover the non-perturbative late time solution of Starobinski\'i and Yokoyama [15] we make the ansatz:

$$\lim_{t \rightarrow \infty} \rho(t, \phi) = \varrho_{\infty}(\phi), \quad (102)$$
because the scalar force should eventually balance the tendency of inflationary particle production to force the scalar up its potential. This ansatz results in a first order equation:

$$\frac{d\varrho_\infty(\phi)}{\varrho_\infty(\phi)} = -\frac{4\pi^2 \lambda}{9H^4} \phi^3 d\phi \ . \quad (103)$$

The solution is straightforward:

$$\varrho_\infty(\phi) = \frac{2}{\Gamma(\frac{1}{4})} \left( \frac{\pi^2 \lambda}{9H^4} \right)^{\frac{1}{4}} \exp \left[ -\frac{\pi^2}{9} \lambda \left( \frac{\phi}{H} \right)^4 \right] \ . \quad (104)$$

and its non-perturbative nature is clear. It follows that the stochastic expectation value of the 2n-th power of the field has the following late time limit:

$$\lim_{t\to\infty} \left\langle \phi^{2n}(t, \vec{x}) \right\rangle_{\text{VEV}} = \int_{-\infty}^{+\infty} d\omega \varrho_\infty(\omega) \omega^{2n} \ , \quad (105)$$

$$= \frac{\Gamma(\frac{n}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left( \frac{9H^4}{\pi^2 \lambda} \right)^{\frac{n}{2}} \ . \quad (106)$$

- **Perturbative Agreement**: To make contact between stochastic expectation values and perturbation theory, we first differentiate the vacuum expectation value of $\phi^{2n}$ and then use the Fokker-Planck equation [31, 32]:

$$\frac{\partial}{\partial t} \left\langle \phi^{2n}(t, \vec{x}) \right\rangle_{\text{VEV}} = \int_{-\infty}^{+\infty} d\omega \dot{\varrho}(t, \omega) \omega^{2n} \ , \quad (107)$$

$$= \frac{n(2n-1)H^3}{4\pi^2} \left\langle \phi^{2n-2}(t, \vec{x}) \right\rangle_{\text{VEV}} - \frac{n\lambda}{9H} \left\langle \phi^{2n+2}(t, \vec{x}) \right\rangle_{\text{VEV}} \ . \quad (108)$$

This relation can be more revealingly expressed in terms of a new time variable $\alpha$ and a rescaled coupling constant $\bar{\lambda}$:

$$\alpha \equiv \frac{1}{4\pi^2} \ln a \quad , \quad \bar{\lambda} \equiv \frac{4\pi^2}{9} \lambda \ . \quad (109)$$

A differential recursion relation emerges:

$$\frac{\partial}{\partial \alpha} \left\langle \left( \frac{\phi}{H} \right)^{2n} \right\rangle_{\text{VEV}} = n(2n-1) \left\langle \left( \frac{\phi}{H} \right)^{2n-2} \right\rangle_{\text{VEV}} - n\bar{\lambda} \left\langle \left( \frac{\phi}{H} \right)^{2n+2} \right\rangle_{\text{VEV}} \ , \quad (110)$$
whose solution has the form:
\[
\left\langle \left( \frac{\phi}{H} \right)^{2n} \right\rangle_{\text{VEV}} = (2n - 1)!! \alpha^n F(\lambda \alpha^2)
\]
(111)
where the function \( F_n(z) \) obeys:
\[
2zF'_n(z) + nF_n(z) = nF_{n-1}(z) - n(2n+1)zF_{n+1}(z).
\]
(112)
It is straightforward to obtain the first few terms of the series expansion:
\[
F_n(z) = 1 - \frac{n}{2}(n+1)z + \frac{n}{280}(35n^3 + 170n^2 + 225n + 74)z^2 + O(z^3).
\]
(113)
Hence, the stochastic expectation value of \( \phi^{2n} \) has the following time dependence in perturbation theory:
\[
\left\langle \phi^{2n}(t, \vec{x}) \right\rangle_{\text{VEV}} = (2n - 1)!! \left( \frac{H^2}{4\pi^2} \ln a \right)^n \left\{ 1 - \frac{n}{2} (n+1) \frac{\lambda}{36\pi^2} \ln^2 a \right.
\]
\[
\left. + \frac{n}{280}(35n^3 + 170n^2 + 225n + 74)\left[ \frac{\lambda}{36\pi^2} \ln^2 a \right]^2 + \ldots \right\}
\]
(114)
Part of this result is in precise agreement with detailed and highly non-trivial calculations of the same quantities using quantum field theory \[33\].

5 Derivative Interactions

A basic difference between the scalar model of Section 4 and gravitation, is the presence of derivative interactions. Since our main purpose is to derive the proper stochastic equation for quantum gravity, we first study a simple scalar model with derivative interactions defined by the following Lagrangian:
\[
\mathcal{L}_D = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu A \partial_\nu A - \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu B \partial_\nu B
\]
\[
- \frac{\lambda}{4} \sqrt{-g} g^{\mu\nu} A^2 \partial_\mu B \partial_\nu B.
\]
(115)
Whereas \( A \) and \( \dot{A} \) have the same equal-time commutation relation as a free field, \( B \) and \( \dot{B} \) do not:
\[
\left[ A(t, \vec{x}) , \dot{A}(t, \vec{x}') \right] = \frac{i\delta^3(\vec{x} - \vec{x}')}{{a^3(t)}} ,
\]
\[
\left[ B(t, \vec{x}) , \dot{B}(t, \vec{x}') \right] = \frac{i\delta^3(\vec{x} - \vec{x}')}{{a^3(t)} \left[ 1 + \frac{\lambda}{2} A^2(t, \vec{x}) \right]} .
\]
(116)
This means we will need to include a homogeneous term in writing the Yang-Feldman equation for $B$.

The full equations of motion are:

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A} = A^{\mu}_{\mu} - \frac{\lambda}{2} g^{\mu\nu} A B_{\mu} B_{\nu} = 0 \ , \quad (118)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B} = B^{\mu}_{\mu} + \lambda g^{\mu\nu} A A_{\mu} B_{\nu} + \frac{\lambda}{2} A^2 B^{\mu}_{\mu} = 0 \ . \quad (119)$$

Each equation of motion can be re-written to give the d’Alembertian of the field in terms of lower derivatives:

$$A^{\mu}_{\mu} = \frac{\lambda}{2} A B_{\mu} B^{\mu} \ , \quad (120)$$

$$B^{\mu}_{\mu} = -\frac{\lambda A A_{\mu} B^{\mu}}{1 + \frac{\lambda}{2} A^2} \ . \quad (121)$$

The full perturbative initial value solution comes from iterating the Yang-Feldman equations:\footnote{Field arguments are sometimes compressed to 4-vector notation: $(t, \vec{x}) \equiv x$.}

$$A(t, \vec{x}) = A_0(t, \vec{x}) - \frac{\lambda}{2} \int_0^t dt' \ a^3(t') \int d^3 x' \ G_{\text{ret}}(t, \vec{x} ; t', \vec{x}') \times \ A(x') B_{\mu}(x') B^{\mu}(x') \ , \quad (122)$$

$$B(t, \vec{x}) = B_0(t, \vec{x}) - \frac{\lambda}{2} \int d^3 x' \ G_{\text{ret}}(t, \vec{x} ; 0, \vec{x}') A^2(0, \vec{x}') B(0, \vec{x}') \ + \lambda \int_0^t dt' \ a^3(t') \int d^3 x' \ G_{\text{ret}}(t, \vec{x} ; t', \vec{x}') \left\{ \frac{A(x') A_{\mu}(x') B^{\mu}(x')}{1 + \frac{\lambda}{2} A^2(x')} \right\} \ . \quad (123)$$

Because both $A$ and $B$ are massless, minimally coupled scalar fields, the retarded Green’s functions in (122,123) are identical to (49). The free field expansions are similarly identical to (50,52), except that each field has an independent canonically normalized set of creation and annihilation operators:

$$A_0(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \left\{ e^{i \vec{k} \cdot \vec{x}} u(t, k) \alpha_k + e^{-i \vec{k} \cdot \vec{x}} u^*(t, k) \alpha_k^\dagger \right\} \ , \quad (124)$$

$$B_0(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \left\{ e^{i \vec{k} \cdot \vec{x}} u(t, k) \beta_k + e^{-i \vec{k} \cdot \vec{x}} u^*(t, k) \beta_k^\dagger \right\} \ . \quad (125)$$
The second homogeneous term in (123) accounts for the non-canonical commutation relation (117). This second term can be expressed as the integral of a total derivative:

\[- \frac{\lambda}{2} \int d^3x' \ G_{\text{ret}}(t, \vec{x}'; 0, \vec{x}') A^2(0, \vec{x}') \dot{B}(0, \vec{x}') \]

\[= - \frac{\lambda}{2} \int_0^t dt' \int d^3x' \ \partial'_\mu \left\{ \sqrt{-g(x')} \ g^{\mu\nu}(x') \ G_{\text{ret}}(x; x') A^2(x') \partial'_\nu B(x') \right\} , \]

\[= - \frac{\lambda}{2} \int_0^t dt' \ a'^3 \int d^3x' \partial'_\mu G_{\text{ret}}(x; x') g^{\mu\nu}(x') A^2(x') \partial'_\nu B(x') \]

\[\quad - \lambda \int_0^t dt' \ a'^3 \int d^3x' \ G_{\text{ret}}(x; x') \left\{ \frac{A(x') \ A_{\mu}(x') \ B_{\mu}(x')}{{1 + \frac{\lambda}{2} A^2(x')}} \right\} . \quad (126)\]

In reaching this final expression we have made use of the equation of motion (121). Combining terms gives our final form for the Yang-Feldman equation of the field \(B\),

\[B(t, \vec{x}) = B_0(t, \vec{x}) - \frac{\lambda}{2} \int_0^t dt' \ a'^3 \int d^3x' \ G_{\text{ret}}(t, \vec{x}'; t, \vec{x}') \ g^{\mu\nu}(x') A^2(x') \partial'_\nu B(x') . \quad (127)\]

**Stochastic Realization:** We need to make the transition from the full fields \(A\) and \(B\) to the stochastic fields \(A_{\text{IR}}\) and \(B_{\text{IR}}\). Most of this proceeds as the reduction of Section 3, with slight modifications to accommodate derivative interactions. The free fields suffer the same truncation of their ultraviolet modes but we must now include the next order term in the long wavelength expansion of the mode functions because the first term has zero time derivative:

\[A_0(t, \vec{x}) \rightarrow A_{\text{IR}0}(t, \vec{x}) \equiv \]

\[\int \frac{d^3k}{(2\pi)^3} \ \theta(Ha(t) - k) \ \frac{H}{\sqrt{2k^3}} \left[ 1 + \frac{1}{2} \left( \frac{k}{Ha(t)} \right)^2 \right] \left\{ e^{i\vec{k} \cdot \vec{x}} \alpha_k + e^{-i\vec{k} \cdot \vec{x}} \alpha_k^\dagger \right\} , \quad (128)\]

\[B_0(t, \vec{x}) \rightarrow B_{\text{IR}0}(t, \vec{x}) \equiv \]

\[\int \frac{d^3k}{(2\pi)^3} \ \theta(Ha(t) - k) \ \frac{H}{\sqrt{2k^3}} \left[ 1 + \frac{1}{2} \left( \frac{k}{Ha(t)} \right)^2 \right] \left\{ e^{i\vec{k} \cdot \vec{x}} \beta_k + e^{-i\vec{k} \cdot \vec{x}} \beta_k^\dagger \right\} . \quad (129)\]

The stochastic limit of the retarded Green’s function is unchanged:

\[a'^3 \ G_{\text{ret}}(t, \vec{x}'; t', \vec{x}') \rightarrow \frac{1}{3H} \theta(t - t') \delta^3(\vec{x} - \vec{x}') . \quad (130)\]
The new feature of these equations is that each interaction contains two derivatives. From the analysis of Section 3 it is apparent that differentiating a propagator precludes it from contributing an infrared logarithm. Said another way, it is always preferable to have a derivative act on the rapidly varying scale factor rather than the slowly varying field. We, therefore, expect that the Hubble friction term dominates the scalar d’Alembertian:

\[ B^\mu_\mu = -\ddot{B} - 3H\dot{B} + \frac{\nabla^2}{a^2} B \rightarrow -3H\dot{B}_{ir} \]  \quad (131)

There are no d’Alembertians in our Yang-Feldman equations but the time derivative of the retarded Green’s function in (127) contributes a similar term. The space derivatives always give two derivatives acting upon fields, and no term is comparably important to (131) when the two derivatives act on different fields:

\[ B_\mu B^\mu = -\dot{B}^2 + \frac{1}{a^2} \nabla B \cdot \nabla B \rightarrow 0 \] \quad (132)

\[ A_\mu B^\mu = -\dot{A}\dot{B} + \frac{1}{a^2} \nabla A \cdot \nabla B \rightarrow 0 \] \quad (133)

By dropping all double derivatives of the fields we arrive at the equations which recover the stochastic fields:

\[ A_{ir}(t, \vec{x}) = A_{ir0}(t, \vec{x}) \]  \quad (134)

\[ B_{ir}(t, \vec{x}) = B_{ir0}(t, \vec{x}) - \frac{\lambda}{2} \int_0^t dt' A_{ir}^2(t', \vec{x})\dot{B}_{ir}(t', \vec{x}) \]  \quad (135)

As in Section 3, the local stochastic field equations are obtained by differentiating the stochastic Yang-Feldman equations (134-135) with respect to time. Also, as in Section 3, derivatives of free fields produce stochastic source terms:

\[ \dot{A}_{ir0}(t, \vec{x}) \equiv f_A(t, \vec{x}) \]  \quad (136)

\[ = \int \frac{d^3k}{(2\pi)^3} \delta(k-Ha(t)) \frac{H^2}{\sqrt{2k}} \left\{ e^{ik\cdot\vec{x}} \alpha_\vec{k} + e^{-ik\cdot\vec{x}} \alpha_\vec{k}^\dagger \right\} \]

\[ - \frac{1}{a^2(t)} \int \frac{d^3k}{(2\pi)^3} \theta(Ha(t)-k) \sqrt{\frac{k}{2}} \left\{ e^{ik\cdot\vec{x}} \alpha_\vec{k} + e^{-ik\cdot\vec{x}} \alpha_\vec{k}^\dagger \right\} , \]

\[ \dot{B}_{ir0}(t, \vec{x}) \equiv f_B(t, \vec{x}) \]  \quad (137)

24
\[ \int \frac{d^3 k}{(2\pi)^3} \delta(k-Ha(t)) \frac{H^2}{\sqrt{2k}} \left\{ e^{ik \cdot \vec{x}} \beta_k^1 + e^{-ik \cdot \vec{x}} \beta_k^1 \right\} \]
\[ - \frac{1}{a^2(t)} \int \frac{d^3 k}{(2\pi)^3} \theta(Ha(t)-k) \sqrt{\frac{k}{2}} \left\{ e^{ik \cdot \vec{x}} \beta_k^1 + e^{-ik \cdot \vec{x}} \beta_k^1 \right\} \cdot \]

Because no derivatives of \( A_{IR} \) appear in the Yang-Feldman stochastic field equations we can neglect the final term in (136) but it must be retained in (137) on account of the \( \dot{B}_{IR} \) which appears in (135). The resulting local stochastic equations are:

\[ \dot{A}_{IR}(t, \vec{x}) = f_A(t, \vec{x}) \quad (138) \]
\[ \dot{B}_{IR}(t, \vec{x}) = f_B(t, \vec{x}) - \frac{\lambda}{2} A_{IR}^2(t, \vec{x}) \dot{B}_{IR}(t, \vec{x}) \quad (139) \]

These are in good agreement with the equations which are obtained by directly applying the reduction rules I-II (94-95) of Section 3 to the full equations of motion (118-119):

\[ -3H \left( \dot{A}_{IR} - f_A \right) = 0 \quad (140) \]
\[ -3H \left( \dot{B}_{IR} - f_B \right) - \frac{3H \lambda}{2} A_{IR}^2 \dot{B}_{IR} = 0 \quad (141) \]

Note the curious fact that interactions containing derivatives are free of stochastic source terms.

It is amusing to note that we can obtain explicit operator solutions to the stochastic field equations (138-139). The exact solution for \( A_{IR} \) follows trivially from (138):

\[ A_{IR}(t, \vec{x}) = \int_0^t dt' f_A(t', \vec{x}) = A_{IR0}(t, \vec{x}) \quad (142) \]

and shows that \( A_{IR} \) receives no corrections to its free field value \( A_{IR0} \). A few simple re-arrangements in (139) give the closed form operator solution for \( B_{IR} \):

\[ B_{IR}(t, \vec{x}) = \int_0^t dt' \frac{f_B(t', \vec{x})}{1 + \frac{1}{2} \lambda A_{IR0}^2(t', \vec{x})} \quad (143) \]

From (143) we conclude that \( B_{IR} \) exhibits a kind of spacetime-dependent field strength renormalization whereby each set of \( \beta \)-modes which experiences horizon crossing is attenuated by the factor \( 1 + \frac{\lambda}{2} A_{IR0}^2 \). In other words,
\[
\begin{align*}
x & \quad x' = i \Delta_A(x, x') \\
\cdots \cdots \cdots & \quad = i \Delta_A(x, x') \\
\begin{figure}[h]
\begin{align*}
\langle A_{(x)}^2 \rangle & = x + x + \ldots \\
\langle B_{(x)}^2 \rangle & = x + x + \ldots
\end{align*}
\end{figure}
\end{align*}
\]

\textbf{Figure 1:} The Feynman rules and the diagrammatic expansion to } O(\lambda) \text{ of the vacuum expectation value of the squares of the fields in the scalar model with derivative interactions. Solid lines correspond to the scalar field } A \text{ and segmented lines to the scalar field } B.
the stochastic field $B_{ir}$ acquires corrections from its free field form which diminish with time and it reaches some constant value asymptotically.

- **Perturbative Agreement:** It is possible to calculate the one and two loop contributions to the expectation values of the squares of the full fields $A$ and $B$. Because they are both massless and minimally coupled fields, they have the same propagator given by (28). The basic vertex is:

$$V_{14,23}(x) = -i\lambda \sqrt{-g} g^{\mu\nu} \partial_\mu \partial_\nu ,$$

where we have used the notation of Figure 1 which also depicts all relevant graphs. In the Appendix we use these Feynman rules to compute:

$$\langle A^2(t, \vec{x}) \rangle_{ir} = \left(\frac{H}{2\pi}\right)^2 \ln a + O(\lambda^2) ,$$

$$\langle B^2(t, \vec{x}) \rangle_{ir} = \left(\frac{H}{2\pi}\right)^2 \ln a + \frac{\lambda H^4}{2^{n-4}} [-\ln^2 a + O(\ln a)] + O(\lambda^2) .$$

It is apparent that (145) agrees with the stochastic result (142). To see that (146) agrees as well with (143), we perturbatively expand the stochastic operator solution:

$$B_{ir}(t, \vec{x}) = B_{ir0}(t, \vec{x}) - \lambda \int_0^t dt' A^2_{ir}(t', \vec{x}) f_b(t', \vec{x}) + O(\lambda^2) ,$$

and square it:

$$B_{ir}^2(t, \vec{x}) = B_{ir0}^2(t, \vec{x}) - \lambda B_{ir}(t, \vec{x}) \int_0^t dt' A^2_{ir}(t', \vec{x}) f_b(t', \vec{x}) + O(\lambda^2) .$$

Taking the vacuum expectation value of (148) gives:

$$\langle B_{ir}^2(t, \vec{x}) \rangle_{\text{VEV}} = \langle B_{ir0}^2(t, \vec{x}) \rangle_{\text{VEV}}$$

$$- \lambda \int_0^t dt' \langle A^2_{ir0}(t', \vec{x}) \rangle_{\text{VEV}} \langle B_{ir0}(t, \vec{x}) f_b(t', \vec{x}) \rangle_{\text{VEV}} + O(\lambda^2) .$$

For the case in which $t > t'$ we have:

$$\langle B_{ir0}(t, \vec{x}) f_b(t', \vec{x}) \rangle_{\text{VEV}} = \frac{H^2}{4\pi^2} \int \frac{dk}{k} \left\{ H^2 a' \delta(Ha' - k) - \frac{k^2}{Ha'^2} \theta(Ha' - k) \right\} = \frac{H^3}{8\pi^2} .$$

27
Note that we discard the time dependent corrections to the wave function in $B_{\text{IR}}$ which would in any case give sub-dominant terms like $a^{-2}a'^2$. Hence we obtain:

$$\langle B_{\text{IR}}^2(t, \vec{x}) \rangle_{\text{VEV}} = \frac{H^2}{4\pi^2} \ln a - \lambda \int_0^t dt' \frac{H^2}{4\pi^2} \ln a' \frac{H^3}{8\pi^2} + O(\lambda^2), \quad (151)$$

$$= \frac{H^2}{4\pi^2} \ln a - \frac{\lambda H^4}{2^6 \pi^4} \ln^2 a + O(\lambda^2), \quad (152)$$

which establishes the non-trivial agreement between (143) and (146).

### 6 Constrained Fields

Besides dynamical degrees of freedom, gravitation contains constrained fields. Because these fields possess no dynamical degrees of freedom there should be no independent stochastic sources for them. However, a constrained variable can depend non-linearly and non-locally upon the dynamical fields and it is by no means clear how to include the extra stochastic jitter the constrained field acquires due to each increment of dynamical modes which experience horizon crossing.

No purely scalar model can mimic the gravitational system we really need to understand. It would be interesting – and it seems feasible – to gain insight into constrained degrees of freedom by attempting a stochastic formulation of scalar quantum electrodynamics. This model certainly possesses a gauge constraint, and the one loop vacuum polarization has been shown to contain an infrared logarithm [34]. However, as a first step we shall attempt to gain insight from purely scalar models by exploiting the existence of a covariant gauge for gravity in which all components of the perturbed metric are either minimally coupled or else conformally coupled [35].

The model we shall study consists of a massless, minimally coupled scalar $A$, a massless, conformally coupled scalar $C$ and an elementary interaction between the two:

$$\mathcal{L}_c = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu A \partial_\nu A - \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu C \partial_\nu C$$

$$- \frac{D-2}{8(D-1)} \sqrt{-g} R C^2 - \frac{1}{2} \kappa H^2 \sqrt{-g} A^2 C. \quad (153)$$
The equations of motion are:

\[
\frac{1}{\sqrt{-g}} \delta S = A^\mu_{\mu} - \kappa H^2 AC = 0 , \quad (154)
\]

\[
\frac{1}{\sqrt{-g}} \delta S = C^\mu_{\mu} - \frac{1}{6} RC - \frac{1}{2} \kappa H^2 A^2 \\
= C^\mu_{\mu} - 2H^2 C - \frac{1}{2} \kappa H^2 A^2 = 0 . \quad (155)
\]

As usual, the perturbative initial value solution comes from iterating the Yang-Feldman equations:

\[
A(t, \vec{x}) = A_0(t, \vec{x}) - \kappa H^2 \int_0^t dt' a^3(t') \int d^3x' G_{\text{ret}}(t, \vec{x} ; t', \vec{x}') A(x') C(x') \quad (156)
\]

\[
C(t, \vec{x}) = C_0(t, \vec{x}) - \frac{1}{2} \kappa H^2 \int_0^t dt' a^3(t') \int d^3x' G_{\text{ret}}^C(t, \vec{x} ; t', \vec{x}') A^2(x') . \quad (157)
\]

The minimally coupled free field \(A_0\) is identical to (124). Its conformally coupled cousin is:

\[
C_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ e^{i\vec{k} \cdot \vec{x}} v(t, k) \gamma^\dagger_{\vec{k}} + e^{-i\vec{k} \cdot \vec{x}} v^*(t, k) \gamma_{\vec{k}} \right\} , \quad (158)
\]

where \(\gamma^\dagger_{\vec{k}}\) and \(\gamma_{\vec{k}}\) are canonically normalized creation and annihilation operators and \(v(t, k)\) is the conformal mode function:

\[
v(t, k) = \frac{-i}{\sqrt{2k}} \exp \left[ \frac{ik}{a(t)} \right] \quad (159)
\]

The minimally coupled retarded Green’s function \(G_{\text{ret}}(t, \vec{x} ; t', \vec{x}')\) is the same one \((129)\) we have been using since Section 3. The conformally coupled retarded Green’s function is:

\[
G_{\text{ret}}^C(t, \vec{x} ; t', \vec{x}') = \frac{H^2}{4\pi} \theta(t - t') \left\{ \frac{\delta(H\|\vec{x} - \vec{x}'\| + \frac{1}{a} - \frac{1}{a'})}{a a' H\|\vec{x} - \vec{x}'\|} \right\} . \quad (160)
\]

**Stochastic Realization:** We seek a simplification of the Yang-Feldman equations for stochastic fields \(A_{\text{ir}}(t, \vec{x})\) and \(C_{\text{ir}}(t, \vec{x})\) which preserves the leading infrared logarithms. The A equation \((156)\) can be reduced as in the previous examples:

\[
A_{\text{ir}}(t, \vec{x}) = A_{\text{ir}0}(t, \vec{x}) - \frac{\kappa H}{3} \int_0^t dt' A_{\text{ir}}(t', \vec{x}) C_{\text{ir}}(t', \vec{x}) . \quad (161)
\]

\(^8\text{Field arguments are sometimes compressed to 4-vector notation: } (t, \vec{x}) \equiv x.\)
The free stochastic mode sum \( A_{\text{ir}0}(t, \vec{x}) \) – as in the previous examples – is given by (124). We obtain the local stochastic field equation by differentiation:

\[
\text{Leading Infrared} \implies \dot{A}_{\text{ir}}(t, \vec{x}) = f_A(t, \vec{x}) - \frac{\kappa H}{3} A_{\text{ir}}(t, \vec{x}) C_{\text{ir}}(t, \vec{x})
\]  

(162)

The stochastic source \( f_A(t, \vec{x}) \) is given by (136).

To obtain the analogous reduction for the \( C \) equation (157) note first that a free field does not produce infrared logarithms unless its mode function goes like \( k^{-\frac{5}{2}} \) near \( k = 0 \). The conformally coupled mode function (159) goes like \( k^{-\frac{1}{2}} \) so its associated free field truncates to zero:

\[
\text{Leading Infrared} \implies C_0(t, \vec{x}) \rightarrow C_{\text{ir}0}(t, \vec{x}) = 0 .
\]  

(163)

Then, we determine the infrared limit of the conformally coupled Green’s function by beginning with the generic expression:

\[
G_{\text{ret}}^C(t, \vec{x} ; t', \vec{x}') = i\theta(t - t') \times 
\int \frac{d^3k}{(2\pi)^3} \left[ v(t, k) v^*(t', k) - v^*(t, k) v(t', k) \right] e^{i\vec{k} \cdot \Delta \vec{x}} .
\]  

(164)

Using (159) the long wavelength expansion of the bracketed term is:

\[
\left[ v(t, k) v^*(t', k) - v^*(t, k) v(t', k) \right] = \frac{i}{\theta(t - t')} \left( \frac{1}{aa'} - \frac{1}{a^2 a'} \right) + O(k^2) .
\]  

(165)

Consequently, the infrared limit of the conformally coupled Green’s function is:

\[
a^3 G_{\text{ret}}^{C_{\text{ir}}}(t, \vec{x} ; t', \vec{x}') \rightarrow \frac{1}{H} \theta(t - t') \delta^3(\vec{x} - \vec{x}') \left( \frac{a'}{a} - \frac{a'^2}{a^2} \right) .
\]  

(166)

The stochastic Yang-Feldman equation for \( C_{\text{ir}} \) could be written as:

\[
C_{\text{ir}}(t, \vec{x}) = -\frac{\kappa H}{2} \int_0^t dt' \left( \frac{a'}{a} - \frac{a'^2}{a^2} \right) A_{\text{ir}}^2(t', \vec{x}) .
\]  

(167)

However, a further simplification is possible. By neglecting the slow variation in \( A_{\text{ir}}^2(t', \vec{x}) \) relative to the rapidly growing scale factors we can perform the integration over \( t' \):

\[
\text{Leading Infrared} \implies C_{\text{ir}}(t, \vec{x}) = -\frac{\kappa}{4} A_{\text{ir}}^2(t, \vec{x}) .
\]  

(168)
Substituting (168) for $C_{IR}$ in (162) reveals the amusing fact that $A_{IR}$ obeys the same stochastic field equation as the self-interacting scalar $\phi$ of Section 3 with the identification $\lambda = -\frac{3}{2}\kappa^2 H^2$. Because the effective 4-point coupling is negative this system ought to exhibit a runaway instability.

Equations (162) and (168) result from applying the reduction rules I-II (94-95) of Section 3 to the full equations of motion (154-155). First apply rules I-II (94-95) to the kinetic term of the minimally coupled scalar in the usual way:

$$A^{\mu} \mu = -\ddot{A} - 3H \dot{A} + \frac{\nabla^2}{a^2} A \rightarrow -3H \left( \dot{A}_{IR} - f_A \right). \quad (169)$$

By rule I (94), the leading contribution to the conformally coupled kinetic term is undifferentiated, so by rule II (95) there is no stochastic source:

$$C^{\mu} \mu - 2H^2 C = -\ddot{C} - 3H \dot{C} + \frac{\nabla^2}{a^2} C - 2H^2 C \rightarrow -2H^2 C_{IR}. \quad (170)$$

The complete stochastic field equations are therefore:

$$-3H \left( \dot{A}_{IR} - f_A \right) - \kappa H^2 A_{IR} C_{IR} = 0, \quad (171)$$

$$-2H^2 C_{IR} - \frac{1}{2} \kappa H^2 A_{IR} C_{IR} = 0, \quad (172)$$

and the rules I-II (94-95) are again shown to give correct results.

**Perturbative Agreement:** It is simple to compute the expectation values of squares of the full quantum fields $A$ and $C$ to a few orders in perturbation theory. The Feynman rules and relevant diagrams are shown in Figure 2; the explicit form of the basic vertex is:

$$V_{12,3}(x) = -i\kappa H^2. \quad (173)$$

The lowest order contributions are just the coincidence limits of the two propagators:

$$\langle A^2(t, \bar{x}) \rangle_{\text{VEV}} = i\Delta_A(x; x) + O(\kappa^2) = \frac{H^2}{4\pi^2} \{ \text{UV} + \ln a \} + O(\kappa^2), \quad (174)$$

$$\langle C^2(t, \bar{x}) \rangle_{\text{VEV}} = i\Delta_{C\Phi}(x; x) + O(\kappa^2) = 0 + O(\kappa^2), \quad (175)$$
\[ \begin{align*}
\overline{x} & \quad \underline{x'} = i \Delta_{\Delta}(x, x') \\
\overline{x} & \quad \overline{x'} = i \Delta_{\Delta}(x, x')
\end{align*} \]
\[ \overline{x} \quad \overline{x'} \quad \overline{x} \quad \overline{x'} = V_{1,2,3}(x) \]

\[ < A_{(x)} > = \quad + \quad + \quad ... \]

\[ < C_{(x)} > = \quad + \quad + \quad ... \]

**Figure 2:** The Feynman rules and the diagrammatic expansion to \( O(\kappa^2) \) of the vacuum expectation value of the squares of the fields in the scalar model with constraints. Solid lines correspond to the scalar field \( A \) and segmented lines to the scalar field \( C \).
where “UV” stands for a constant, ultraviolet divergence. These results are completely consistent with the perturbative solution of the stochastic equations (162) and (168):

\[
A_{ir}(t, \vec{x}) = A_{ir0}(t, \vec{x}) + \frac{\kappa^2 H}{12} \int_0^t dt' A_{ir0}^3(t', \vec{x}) + O(\kappa^4),
\]

(176)

\[
C_{ir}(t, \vec{x}) = -\frac{\kappa}{4} A_{ir0}^2(t, \vec{x}) + O(\kappa^3).
\]

(177)

Based on the above expansions (176-177) we make the following predictions for the next order – that is, two loop – perturbative results:

\[
\langle A_{ir}^2(t, \vec{x}) \rangle_{\text{VEV}} = \frac{H^2}{4\pi^2} \ln a + \frac{\kappa^2 H^4}{2^5 3\pi^4} \ln^3 a + O(\kappa^4),
\]

(178)

\[
\langle C_{ir}^2(t, \vec{x}) \rangle_{\text{VEV}} = \frac{3\kappa^2 H^4}{2^8 \pi^4} \ln^2 a + O(\kappa^4).
\]

(179)

It will be interesting to check these predictions.

7 Epilogue

The subject of this paper has been infrared logarithms. These are powers of the number of inflationary e-foldings from when the system was released in a prepared state. They can arise in the expectation values of operators in quantum field theories which contain massless, minimally coupled scalars or gravitons. We demonstrated that infrared logarithms derive entirely from long wavelength quanta. In Section 3 we were able to obtain a completely finite system, with a conserved stress tensor, by truncating the ultraviolet modes from the free field of the Yang-Feldman equation.

Infrared logarithms are very exciting because they may signal important quantum effects. However, their continued growth implies the breakdown of perturbation theory. A natural approach for obtaining non-perturbative information is the leading logarithm approximation in which one attempts to sum the series comprised of just the leading infrared logarithms at each order. Starobinskii has long argued that his formalism of stochastic inflation [14, 15] recovers the leading infrared logarithms for scalar fields with non-derivative interactions. We have confirmed this using the infrared-truncated Yang-Feldman equation of Section 3. The leading infrared logarithms are
not changed when the free field mode functions are simplified to retain only the first nonzero term in the long wavelength expansion and a similar simplification is done to the retarded Green’s function. The fields of the resulting system are commuting, even though they depend upon creation and annihilation operators, and they obey precisely Starobinskii’s Langevin equation.

Whereas there is little hope of obtaining non-perturbative solutions for interacting quantum field theories in $3 + 1$ dimensions, classical field equations can sometimes be solved exactly. And simple classical equations are especially likely to possess exact solutions. In Section 4 we reviewed the exact solution obtained by Starobinskii and Yokoyama [15] for the late time limit of a massless, minimally coupled scalar with a quartic self-interaction. We also reviewed Starobinskii’s technique [31, 32] for predicting the infrared logarithms of explicit perturbative computations, and we worked out the general result at orders $\lambda^0$, $\lambda^1$ and $\lambda^2$.

The remainder of the paper concerned extending Starobinskii’s formalism to more complicated theories with derivative interactions and constraints. We have written down a set of simple rules (94-95) for accomplishing this. By going through the same procedure of first infrared-truncating the Yang-Feldman equations and then making the long wavelength approximation on their mode functions and retarded Green’s function, we showed that rules I-II give the correct results for a scalar model with derivative interactions (Section 5) and for a reasonable scalar analogue of a model with gauge constraints (Section 6). We also checked our rules against explicit perturbative computations.

We emphasize that just because the rules I-II (94-95) can be stated generally does not mean they apply generally. For any particular model the stochastic Langevin equations implied by our rules should be derived by infrared-truncating the Yang-Feldman equations and then imposing the long wavelength approximation. Specific perturbative results should also be checked.

The primary motivation for developing these rules is their application to gravitation where some interesting perturbative results already exist [5, 6]. We close with an amusing and thought-provoking argument which is completely independent of these results and assumes only that a stochastic formulation of inflationary quantum gravity exists. If so, then it is straightforward to prove that back-reaction must become important. Suppose the null hypothesis is correct: there is no significant back-reaction at any order. In that case the stochastic metric field is just the inflationary background plus the
free field mode sum of long wavelength gravitons in the transverse-traceless field $h_{ij}^{TT}(t, \vec{x})$:

$$\text{Null Hypothesis} \implies g_{ij}(t, \vec{x}) = a^2(t) \left[ \delta_{ij} + h_{ij}^{TT}(t, \vec{x}) \right].$$  \hspace{1cm} (181)

Hence, the volume element operator is:

$$\sqrt{-g(t, \vec{x})} = a^3(t) \left[ 1 - \frac{1}{4} h_{ij}^{TT}(t, \vec{x}) h_{ij}^{TT}(t, \vec{x}) + O(h^3) \right].$$  \hspace{1cm} (182)

Its stochastic vacuum expectation value is simple to compute:

$$\langle \sqrt{-g(t, \vec{x})} \rangle_{\text{VEV}} = a^3(t) \left[ 1 - \frac{4}{\pi} GH^2 \ln a + O(G^2) \right].$$  \hspace{1cm} (183)

For large observation times this eventually shrinks to zero, at which point back-reaction must have become significant. Hence it cannot be consistent to ignore back-reaction!

In addition to illustrating the power of the stochastic formalism, the argument we have just given establishes the need to include quantum gravitational back-reaction. However, this does not necessarily mean there will be significant modifications to the de Sitter background. It may be instead that non-linear corrections to the graviton field, and to the various constrained fields, add up to give only a small shift that approaches a constant at late times. Precisely this happens to the scalar model considered in Section 4. However, it was obvious in this model that the classical force pushing the scalar down its potential would eventually compensate the tendency of inflationary particle production to push it ever higher. No such mechanism is apparent in gravity. Indeed, the physical picture seems rather to be that the self-gravitation from continual production of inflationary gravitons must eventually bring the universe to the verge of gravitational collapse. It would be as premature to ignore the possibility of a significant effect from quantum gravity on the basis of the scalar model as it would have been to discount the prospects for asymptotic freedom – in a somewhat different leading logarithm approximation – based on the positive beta functions of QED and $\phi^4$ theory.
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References

[1] D. Boyanovsky and H. J. de Vega, Phys. Rev. D70 (2004) 063508, arXiv:astro-ph/0406287

[2] D. Boyanovsky, H. J. de Vega and N. G. Sanchez, Phys. Rev. D71 (2005) 023509, arXiv:astro-ph/0409406

[3] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33 (1981) 532.

[4] A. A. Starobinskiĭ, JETP Lett. 30 (1979) 682.

[5] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B474 (1996) 235, arXiv:hep-ph/9602315

[6] N. C. Tsamis and R. P. Woodard, Ann. Phys. 253 (1997) 1, arXiv:hep-ph/9602317

[7] L. H. Ford, Phys. Rev. D31 (1985) 710.

[8] F. Finelli, G. Marozzi, G. P. Vacca and G. Venturi, “Adiabatic Regularization of the Graviton Stress-Energy Tensor in de Sitter Space-Time,” arXiv:gr-qc/0407101

[9] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. 19 (2002) 4607, arXiv:gr-qc/0204065

[10] V. K. Onemli and R. P. Woodard, Phys. Rev. D70 (2004) 107301, arXiv:gr-qc/0406098

[11] L.-Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. Lett. 73 (1994) 1311,

[12] D. Boyanovsky and H. J. de Vega, Ann. Phys. 307 (2003) 335, arXiv:hep-ph/0302055

[13] D. Boyanovsky and H. J. de Vega, “Out of Equilibrium Fields in Self-consistent Inflationary Dynamics: Density Fluctuations,” in Current Topics in Astrophysical Physics: The Cosmic Microwave Background, ed. N. G. Sanchez (Kluwer, Dordrecht, The Netherlands, 2001), arXiv:astro-ph/0006446
[14] A. A. Starobinski˘ı, “Stochastic de Sitter (inflationary) stage in the early universe,” in *Field Theory, Quantum Gravity and Strings*, ed. H. J. de Vega and N. Sanchez (Springer-Verlag, Berlin, 1986) pp. 107-126.

[15] A. A. Starobinski˘ı and J. Yokoyama, Phys. Rev. D50 (1994) 6357, arXiv:astro-ph/9407016.

[16] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[17] A. Vilenkin and L. H. Ford, Phys. Rev. D26 (1982) 1231.

[18] A. D. Linde, Phys. Lett. 116B (1982) 335.

[19] A. A. Starobinski˘ı, Phys. Lett. 117B (1982) 175.

[20] B. Allen and A. Folacci, Phys. Rev. D35 (1987) 3771.

[21] N. C. Tsamis and R. P. Woodard, Commun. Math. Phys. 162 (1994) 217.

[22] R. P. Woodard, “de Sitter Breaking in Field Theory,” arXiv:gr-qc/0408002.

[23] G. Kleppe, Phys. Lett. B317 (1993) 305.

[24] R. D. Jordan, Phys. Rev. D33 (1986) 444.

[25] J. Schwinger, J. Math. Phys. 2 (1961) 407.

[26] N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. 11 (1994) 2969.

[27] L. H. Ford and L. Parker, Phys. Rev. D16 (1977) 245.

[28] S. A. Fulling, M. Sweeny and R. M. Wald, Commun. Math. Phys. 63 (1978) 257.

[29] C. N. Yang and D. Feldman, Phys. Rev. 79 (1950) 972.

[30] L. Accardi, Y. G. Lu and I. Volovich, *Quantum Theory and Its Stochastic Limit* (Springer-Verlag, Berlin, 2002).

[31] A. A. Starobinski˘ı, correspondence of July 17, 2002.
[32] F. Finelli, G. Marozzi, A. A. Starobinskiï, G. P. Vacca and G. Venturi, “Generation of fluctuations during inflation: comparison of stochastic and field-theoretic approaches,” preprint in preparation.

[33] N. C. Tsamis and R. P. Woodard, Phys. Lett. 426B (1998) 21, arXiv:hep-ph/9710466.

[34] T. Prokopec, O. Tornkvist and R. P. Woodard, Ann. Phys. 303 (2003) 251, arXiv:gr-qc/0205130.

[35] L. R. Abramo and R. P. Woodard, Phys. Rev. D65 (2002) 063515, arXiv:astro-ph/0109272.
8 Appendix: Perturbative Computations

We shall derive expression (136) from Section 5. The one loop contribution derives from the coincidence limit (29) of the $A$-type propagator. At two loop order the contribution is:

$$\langle B^2(t, \vec{x}) \rangle_{\text{VEV}}^{\text{2-loop}} = \int d^D x' \sqrt{-g(x')} g^{\mu\nu}(x') \left[ -\frac{i\lambda}{2} i\Delta_A(x'; x') - i\delta Z \right] \times$$

$$\left\{ \left[ \partial'_\mu i\Delta_{A++}(x; x') \right] \partial'_\nu i\Delta_{A++}(x; x') \right.$$  

$$- \left[ \partial'_\mu i\Delta_{A+-}(x; x') \right] \partial'_\nu i\Delta_{A+-}(x; x') \right\},$$  

(184)

where we have used the Feynman rules of Figure 1 and equations (28), (144). In (184), the subscript pairs ++ and +− refer to the two possible variations of the interaction vertex at $x'$ as required when computing true expectation values in quantum field theory [25, 24, 6]. The symbol $\delta Z$ represents the field strength renormalization necessary to absorb the ultraviolet divergence coming from the coincidence limit (29) of the propagator at the interaction point $x'$.

We first partially integrate both the ++ and the +− sectors of (184) using the identity:

$$\sqrt{-g(x')} g^{\mu\nu}(x') \left[ \partial'_\mu i\Delta_A(x; x') \right] \partial'_\nu i\Delta_A(x; x') =$$

$$\partial'_\mu \left\{ \sqrt{-g(x')} g^{\mu\nu}(x') i\Delta_A(x; x') \partial'_\nu i\Delta_A(x; x') \right\}$$

$$- i\Delta_A(x; x') \partial'_\mu \left\{ \sqrt{-g(x')} g^{\mu\nu}(x') \partial'_\nu i\Delta_A(x; x') \right\}.$$  

(185)

Because the propagators obey:

$$\partial'_\mu \left\{ \sqrt{-g(x')} g^{\mu\nu}(x') \partial'_\nu i\Delta_{A++}(x; x') \right\} = i\delta^D(x - x'),$$  

(186)

$$\partial'_\mu \left\{ \sqrt{-g(x')} g^{\mu\nu}(x') \partial'_\nu i\Delta_{A+-}(x; x') \right\} = 0,$$  

(187)

and we can trivially write:

$$i\Delta_A(x; x') \partial'_\mu i\Delta_A(x; x') = \frac{1}{2} \partial'_\mu i\Delta_A^2(x; x'),$$  

(188)
the partial integration gives:

\[
\langle B^2(t, \vec{x}) \rangle_{\text{VEV}}^{2\text{loop}} = i \Delta_A(x; x) \left[ -\frac{\lambda}{2} i \Delta_A(x; x) - \delta Z \right] + \tag{189}
\]

\[
\frac{i \lambda}{4} \int d^D x' \sqrt{-g(x')} g^\mu \nu(x') \partial'_\nu i \Delta_A(x'; x') \left\{ \partial'_\mu i \Delta_{A+ +}^2(x; x') - \partial'_\mu i \Delta_{A+ -}^2(x; x') \right\}
\]

Note that the only surface term is at \( t' = 0 \) because the ++ and +- propagators cancel whenever \( x'^\mu \) lies outside the past light-cone of \( x^\mu \). That surface term vanishes if \( \delta Z \) is chosen – as it must be – to cancel the one loop contribution to the self-mass squared at \( t = 0 \):

\[
\delta Z = \frac{\lambda}{2} \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(D/2)} \ln a' , \tag{190}
\]

Then we have:

\[
\left[ -\frac{\lambda}{2} i \Delta_A(x'; x') - \delta Z \right] = -\frac{\lambda H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(D/2)} \ln a' , \tag{191}
\]

which vanishes at \( t' = 0 \).

The first term in (189) contributes a double infrared logarithm:

\[
i \Delta_A(x; x) \left[ -\frac{\lambda}{2} i \Delta_A(x; x) - \delta Z \right] = -\frac{\lambda H^4}{2^5 \pi^3} \ln^2 a + O(\ln a) . \tag{192}
\]

To see that the second term in (189) also contributes a double infrared logarithm first note that the only the temporal derivative survives:

\[
\partial'_\nu i \Delta_A(x'; x') = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(D/2)} H a' \delta'_\nu . \tag{193}
\]

Hence we can partially integrate on \( \eta' \):

\[
-\frac{i \lambda}{4} \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(D/2)} H \int d^D x' a'^{D-1} \frac{\partial}{\partial \eta'} \left\{ i \Delta_{A+ +}^2(x; x') - i \Delta_{A+ -}^2(x; x') \right\}
\]

\[
= \left( \text{surface term at } \eta' = -\frac{1}{H} \right) + \tag{194}
\]

\[
-\frac{i \lambda}{4} \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D)}{\Gamma(D/2)} H^2 \int d^D x' a'^D \left\{ i \Delta_{A+ +}^2(x; x') - i \Delta_{A+ -}^2(x; x') \right\} .
\]

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The surface term contributes no infrared logarithms. In another context we have previously evaluated the leading infrared logarithm contribution from an integral of any power of the propagator $[33]$. For the case involving the squares one finds:

\[ \frac{i\lambda H^{D-2}}{4(4\pi)^\frac{D}{2}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2})} H^2 \int d^D x' a'^D \left\{ i\Delta^2_{A++}(x; x') - i\Delta^2_{A+-}(x; x') \right\} \]

\[ = \frac{\lambda H^4}{2^6\pi^4} \ln^2 a + O(\ln a) \quad . \tag{195} \]

Consequently, the leading infrared behavior of $[189]$ in $D = 4$ is:

\[ \left\langle B^2(t, \vec{x}) \right\rangle_{\text{2-loop}}^{\text{VEV}} = \frac{\lambda H^4}{2^6\pi^4} \left[ - \ln^2 a + O(\ln a) \right] , \tag{196} \]

which is precisely the term appearing in expression $[148]$. 

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