Analytic toy-model for the ISCO shift

Shahar Hod

The Ruppin Academic Center, Emek Hefer 40250, Israel

and

The Hadassah Institute, Jerusalem 91010, Israel

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Abstract

A simple black-hole-ring system is proposed as a toy model for the two-body problem in general relativity. This toy-model yields the fractional shift $\Delta \Omega_{isco}/\Omega_{isco} = \frac{29}{81\sqrt{2}} \eta$ in the Schwarzschild ISCO (innermost stable circular orbit) frequency, where $\eta \equiv m/M_{ir} \ll 1$ is the dimensionless ratio between the mass of the particle and the irreducible mass of the black hole. Our model suggests that the second-order spin-orbit interaction between the black hole and the orbiting particle (the dragging of inertial frames) is the main element determining the observed value of the ISCO shift.
Introduction. The geodesic orbits of test particles in black-hole spacetimes have been studied extensively by many researches, see \cite{1-5} and references therein. The innermost stable circular orbit (ISCO) is especially interesting from both an astrophysical and theoretical points of view. This orbit is defined by the onset of a dynamical instability for circular geodesics – it separates stable circular orbits from orbits that plunge into the central black hole \cite{2}. This special orbit is therefore important in the context of inspiralling compact binaries since it represents the critical point where the character of the orbit (and thus also the character of the corresponding emitted gravitational radiation) sharply changes \cite{3}. In addition, this marginally stable orbit is usually regarded as the inner edge of accretion disks around central black holes \cite{2}.

An important physical quantity which characterizes the ISCO is the orbital angular frequency $\Omega_{\text{isco}}$ as measured by asymptotic observers. This gauge-invariant frequency is often regarded as the end point of the inspiral gravitational templates \cite{5}. For a test-particle in the Schwarzschild spacetime, this frequency is given by $M_{\text{ir}}\Omega_{\text{isco}} = 6^{-3/2}$ \cite{1-5}, where $M_{\text{ir}}$ is the irreducible mass \cite{6} of the central black hole.

What happens to the ISCO frequency if the mass $m$ of the particle can no longer be neglected as compared to the mass $M_{\text{ir}}$ of the black hole? In this case one should take into account the gravitational self-force (GSF) corrections to the orbit \cite{7-18}. The gravitational self-force has two distinct contributions: (1) It is responsible for dissipative effects that cause the orbiting particle to lose energy and angular momentum to gravitational waves. The location of the ISCO may become blurred due to these non-conservative effects \cite{5,7}. (2) The gravitational self-interaction is also responsible for conservative effects which preserve the characteristic constants of the orbital motion. This effect is responsible for a non-trivial shift in the orbital frequency of the ISCO.

Calculations of the GSF effects are mainly motivated by the need to analyze in detail extreme-mass-ratio inspirals in which a compact object interacts with a massive black hole \cite{8}. These two-body events with $m/M_{\text{bh}} \lesssim 10^{-3}$ are expected to be an important source for the Laser Interferometer Space Antenna (LISA) \cite{19}.

The computation of the GSF correction to the orbit is a highly non-trivial task. After a decade of intensive efforts by many groups of researches to evaluate the effects of the self-interaction terms on the orbital parameters, Barack and Sago \cite{17} have recently succeeded in calculating the shift in the ISCO frequency due to the conservative part of the GSF in
the Schwarzschild spacetime. Their numerical result for the ISCO frequency shift can be expressed in the form \[16, 17, 20\]:

\[M\Omega = 6^{-3/2}[1 + c \cdot \eta + O(\eta^2)] \quad \text{with} \quad c \simeq 1.251, \quad (1)\]

where \(M \equiv M_{\text{ir}} + m\) is the total mass and \(\eta \equiv m/M_{\text{ir}}\) is the dimensionless ratio between the masses. The result (1) is especially important because it provides gauge-invariant information about the strong-gravity effects in the vicinity of the black hole.

The main goal of the present study is to analyze a simple toy model which captures some of the essential features of the (physically more interesting) two-body problem in general relativity. In particular, we would like to provide a simple and intuitive explanation for the increase in the ISCO frequency in the extreme-mass-ratio limit [see Eq. (1)]. The proposed toy model is composed of a stationary axisymmetric ring of particles in orbit around a black hole. As shown by Will [21], this composed system is amenable to a perturbative analytic treatment.

The toy-model. It is well-known [21] that local inertial frames are dragged by an orbiting particle. In fact, because of the dragging of inertial frames by the orbiting particle, one can have a Schwarzschild-like black hole with zero angular momentum but with a non-zero angular velocity, see Eq. (6) below.

We propose to model the conservative behavior of the black-hole-particle system using the analytically solvable model of the composed black-hole-ring system. We expect this toy model to capture the essential features of the original black-hole-particle system, at least qualitatively. In particular, like the orbiting particle, the rotating ring can drag the generators of the black-hole horizon [21]. We therefore expect the (analytically-known) spin-orbit interaction between the black hole and the ring to mimic, at least qualitatively, the corresponding spin-orbit interaction in the original black-hole-particle system.

To analyze the influence of the frame-dragging effect (caused by the azimuthal motion of the ring) on the ISCO frequency, we must focus on the second-order spin-orbit interaction [22] between the ring and the black hole. This interaction introduces terms of order \(O((mj)^2/M_{\text{ir}}^3)\) into the energy budget of the system [see Eqs. (2) and (6) below] and an analogous term of order \(O(mj/M_{\text{ir}}^3)\) to the expression of the orbital frequency [see Eq. (9) below], where \(mj\) is the orbital angular momentum of the ring.

The black-hole-ring system, which is composed of a stationary axisymmetric ring of par-
articles in orbit around a slowly rotating black hole, was analyzed by Will [21]. This system is characterized by five physical parameters [21]: The irreducible mass $M_{ir}$ of the black hole, the angular velocity $\omega_H$ of the horizon, the rest mass $m$ of the ring, the proper circumferential radius $R$ of the ring, and the half-thickness $r \ll R$ of the ring.

To second order in the mass $m$ of the ring and to second order in the angular velocity $\omega_H$ of the black-hole horizon, a sequence of black-hole-ring equilibrium configurations was obtained in [21]. The total gravitational mass of the composed black-hole-ring configuration is given by [21]

$$E(x; M_{ir}, \omega_H, j) = M_{ir} + 2M_{ir}^3\omega_H^2 + m - m\Phi(x) - \omega_Hmj\Psi(x) - \frac{m^2x}{2\pi M_{ir}} \ln \left(\frac{8M_{ir}}{xr}\right),$$  \hspace{1cm} (2)

where

$$x \equiv \frac{M_{ir}}{R}; \quad \Phi(x) \equiv 1 - \frac{1 - 2x}{(1 - 3x)^{1/2}}; \quad \Psi(x) \equiv 12\frac{x^3 - 2x^4}{1 - 3x},$$  \hspace{1cm} (3)

and $j$ is the angular momentum per unit mass of the ring which is given (to the necessary order) by [21]

$$j = \frac{M_{ir}}{x(1 - 3x)^{1/2}}.$$  \hspace{1cm} (4)

Each term on the r.h.s. of Eq. (2) has a clear physical interpretation [21]:

- The first two terms on the r.h.s. of (2) represent the contribution of the black hole to the total mass of the system. Remembering that a slowly rotating vacuum Kerr black hole is characterized by the relation $M_{Kerr} = M_{ir} + 2M_{ir}^3\omega_H^2 + O(\omega_H^4)$, one can identify the second term of (2) as the rotational-energy of the spinning black hole. Note, however, that for the black-hole-ring system $\omega_H$ contains a term linear in $mj$ [see Eq. (3) below], and thus the rotational energy of the black hole contains within it a second-order self-interaction term of order $O((mj)^2)$.

- The third term on the r.h.s. of (2) is simply the “bare” (rest) mass of the ring.

- The fourth term on the r.h.s. of (2) represents the first-order mutual interaction between the ring and the black hole. In the large-$R$ (small-$x$) limit it becomes $-M_{ir}m/2R$, which can be identified as the negative Newtonian potential energy of the black-hole-ring system plus the Newtonian rotational energy of the ring.

- The fifth term on the r.h.s. of Eq. (2) represents a spin-orbit interaction between the spinning black hole (which is characterized by the horizon angular velocity $\omega_H$)
and the rotating ring (which is characterized by the angular momentum $m_j$). We expect this term to capture, at least qualitatively, the essential features of a similar spin-orbit interaction in the original black-hole-particle system. Since $\omega_H$ contains a term linear in $m_j$ [see Eq. (6) below], this spin-orbit interaction term contains within it a second-order self-interaction term of order $O((m_j)^2)$.

- The sixth term on the r.h.s. of Eq. (2) represents the second-order gravitational self-energy of the ring [23] (not considered in [21]). This term reflects the inner interactions between the many particles that compose the ring. Since our main interest here is the original two-body system with a single orbiting particle (we only use the analytically-known properties of the rotating ring to model the rotation of the particle), we shall not consider this many-particle term [and an analogous term in Eq. (9) below] here. This will allow us to focus on the influence of the frame-dragging effect alone on the ISCO frequency. In this respect, the ring considered in [21] should be regarded as a quasi test ring.

The total angular momentum of the composed black-hole-ring system is given by [21]

$$J(x; M_\text{ir}, \omega_H, j) = 4M_\text{ir}^3\omega_H - 8mjx^3 + mj.$$  \hspace{1cm} (5)

The first two terms on the r.h.s. of (5) represent the angular momentum $J_H$ of the black hole while the third term is the angular momentum of the ring [21]. As emphasized in [21], the simple relation $\omega_H = J_{Kerr}/4M_{bh}M_\text{ir}^2$ between the angular momentum of the black hole and the angular velocity of the horizon, which was valid for vacuum Kerr black holes [and, in particular, the simple relation $\omega_H(J_{Kerr} = 0) = 0$] no longer holds when matter (the ring) is present in the black-hole exterior. In particular, due to the dragging of inertial frames [21] by the rotating ring, a zero angular momentum black hole ($J_H = 0$) is characterized by a non-zero angular velocity [see Eq. (5)]:

$$\omega_H = \frac{2x^3}{M_\text{ir}^3} \cdot mj.$$ \hspace{1cm} (6)

Substituting Eqs. (4) and (6) into Eq. (2), one finds

$$E(x; M_\text{ir}, \omega_H, j) = M_\text{ir} + \frac{1 - 2x}{(1 - 3x)^{1/2}} \cdot m + \frac{8x^5(-2 + 3x)}{(1 - 3x)^2} \cdot \frac{m^2}{M_\text{ir}}.$$ \hspace{1cm} (7)

for the total energy of the composed black-hole-ring system.
The ISCO. A standard way to identify the location of the ISCO is by finding the minimum of the total energy \([5, 18, 24, 25]\). A simple derivation of (7) with respect to \(x\) reveals that the perturbed ISCO is characterized by

\[
x_{\text{isco}} = \frac{1}{6} + \frac{5\sqrt{2}}{243} \eta + O(\eta^2)
\]  

(8)

Substituting (8) into the expression \([21]\)

\[
M_{\text{ir}} \Omega = x^{3/2} - 4M_{\text{ir}}\omega_\text{H}x^3 + O(\omega_\text{H}^2)
\]

(9)

for the angular velocity of the rotating ring, one finds

\[
M_{\Omega_{\text{isco}}} = 6^{-3/2} \left[1 + \left(1 + \frac{29}{81\sqrt{2}}\right)\eta + O(\eta^2)\right]
\]

(10)

for the perturbed ISCO frequency.

Summary. A black-hole-ring system was proposed as a simple toy model for the (physically more interesting) two-body problem in general relativity. In particular, we have used the (analytically-known) second-order spin-orbit coupling between the black hole and the orbiting ring to describe the essential features of a similar spin-orbit interaction in the original black-hole-particle system. Our original motivation was to provide a simple qualitative explanation for the increase in the ISCO frequency in the extreme-mass-ratio limit. Admittedly, we had no a priori reason to believe that this simple toy model would be able to provide a good quantitative description of the ISCO shift phenomena of the original black-hole-particle system. However, somewhat surprisingly, our analytical expression \(c = 1 + \frac{29}{81\sqrt{2}} \simeq 1.253\) for the value of the ISCO shift-parameter is astonishingly close to the numerically computed \([17, 26]\) value \(c \simeq 1.251\). This fact suggests that the second-order spin-orbit interaction between the black hole and the orbiting object (the dragging of inertial frames) is the main element determining the observed value of the ISCO shift.

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