Supersymmetric Transformations in Coupled-Channel Systems

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A transformation of supersymmetric quantum mechanics for \(N\) coupled channels is presented, which allows the introduction of up to \(N\) degenerate bound states without altering the remaining spectrum of the Hamiltonian. Phase equivalence of the Hamiltonian can be restored by two successive supersymmetric transformations at the same energy. The method is successfully applied to the \(^3\)S\(_1\)-\(^3\)D\(_1\) coupled channels of the nucleon-nucleon system and a set of Moscow-type potentials is thus generated.

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\section*{I. INTRODUCTION}

The formalism of supersymmetric quantum mechanics (SQM), introduced by Witten \cite{1} in 1981, provides an elegant way to construct a hierarchy of Hamiltonians with well defined relations between their spectra \cite{1-5}. Specifically, it allows the elimination or introduction of bound states \cite{6} without altering the remaining part of the spectrum. Transformations of SQM, which we will refer to as supersymmetric (SUSY) transformations, are specific Darboux transformations \cite{7,8} of the Schrödinger equation and are related to the factorization method \cite{9}. Consequently, there is a close connection to inverse scattering theory \cite{4,10}.

The SUSY-transformations of the radial Schrödinger equation have been studied in detail \cite{5,6}. In particular emphasis was given to mathematical aspects as well as applications to specific physics phenomena. An important application is the relationship of equivalent effective interactions between composite particle systems. In such systems, because of the necessary suppression of the internal degrees of freedom, the resulting interactions are ambiguous \cite{11,12}. There is always a set of effective interactions leading to the same scattering phase shifts which sustain, however, a different number of additional unphysical bound states as a consequence of simulating differently the Pauli principle.

Baye \cite{5} pointed out that SQM is an elegant way to construct such phase equivalent potentials. Since then, more general transformations of the radial Schrödinger equation for uncoupled channels have been worked out \cite{5,6} which, apart from a simple removal and an addition of a ground state, allow arbitrary modifications of the bound state spectrum \cite{13}.

The concept of SUSY–transformations has been extended to coupled–channel systems by Amado \textit{et al.} \cite{14} but their transformations do not allow the construction of phase-equivalent potentials after the removal of a bound state \cite{15}. Recently, Sparenberg and Baye \cite{16} addressed this problem and presented SUSY–transformations for coupled–channel systems which in addition to the removal of bound states allow the construction of phase-equivalent potentials. However, the status of SUSY-transformations for coupled-channel systems is still incomplete because the process of introducing a bound state has not been considered so far. In this article we present a SUSY–transformation for the introduction of \(N\) degenerate bound states in a system of \(N\) coupled channels.

\section*{II. SUPERSYMMETRIC TRANSFORMATION}

We consider a system of \(N\) coupled channels which is described by the Schrödinger equation

\begin{equation}
H_0 \Psi_0(\epsilon, r) \equiv \left\{ -\frac{d^2}{dr^2} + U_0(r) \right\} \Psi_0(\epsilon, r) = \epsilon \Psi_0(\epsilon, r),
\end{equation}

where \(\epsilon = k^2 = 2mE/h^2\) and \(U_0(r) = 2mV_0(r)/h^2\). The potential \(V_0(r)\) is an \(N \times N\) matrix that may include the centrifugal barrier and thresholds which may be different in each channel. For simplicity, we assume that the mass \(m\) is equal for all channels (for unequal masses, see Ref. \cite{15}). The wave function \(\Psi_0(\epsilon, r) = (\psi_0^1(\epsilon, r), \cdots, \psi_0^N(\epsilon, r))\) is an \(N \times N\) matrix, where each column vector \(\psi_0^i(\epsilon, r)\) is a solution of Eq. \((1)\).

The SUSY transformations are based on the factorization of the Hamiltonian

\begin{equation}
H_0 = A^+ A^- + \epsilon,
\end{equation}

where \(A^+ A^- = \left(\begin{array}{ccc} 0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \end{array}\right)\) is a \(N \times N\) matrix with all \(0\)s except for \(a_{ij} = 1\) if \(i = j\) and \(0\) otherwise. The supersymmetric partners of the Hamiltonian are

\begin{align*}
H_0^{(1)} &= A^- A^+ + \epsilon, \\
H_0^{(2)} &= A^+ A^- + \epsilon.
\end{align*}
where the energy $\bar{\epsilon}$ is smaller than or equal to the ground state $\epsilon_0$ of $H_0$ and
\begin{equation}
A^\pm = \pm \frac{d}{dr} + W(r).
\end{equation}

The superpotential $W(r)$ is an $N \times N$ matrix satisfying the differential equation
\begin{equation}
\frac{dW}{dr} + W^2 = U_0 - \bar{\epsilon}.
\end{equation}

The supersymmetric partner Hamiltonian $H_1$ is given by
\begin{equation}
H_1 = A^- A^+ + \bar{\epsilon} \quad \text{with} \quad U_1(r) = U_0(r) - 2\frac{d}{dr} W(r).
\end{equation}

The solutions of the corresponding Schrödinger equation, at any energy $\epsilon$, are directly given in terms of $\Psi_0(\epsilon, r)$ by $\Psi_1(\epsilon, r) = A^- \Psi_0(\epsilon, r)$. The above equations are formally equivalent to those for uncoupled channels for which the SUSY transformations are known in closed form [3–5]. In the present work we consider a SUSY transformation for coupled–channel systems generated via the ansatz
\begin{equation}
W(r) = \Psi_0^U(\bar{\epsilon}, r) \Psi_0^{-1}(\bar{\epsilon}, r) + \left( \Psi_0^1(\bar{\epsilon}, r) \right)^{-1} \Lambda \\
x \left[ 1 + \int_a^r dr \Psi_0^{-1}(\bar{\epsilon}, t) \left( \Psi_0^1(\bar{\epsilon}, t) \right)^{-1} \Lambda \right]^{-1} \Psi_0^1(\bar{\epsilon}, r)
\end{equation}
where we have used the hermiticity of the first term of Eq. (6). The constant $N \times N$ matrix $\Lambda$ must be symmetric and should be chosen in such a way that $W(r)$ has no singularities for $r > 0$. The first term of Eq. (6) is a straightforward extension of the simplest expression for the superpotential in uncoupled systems.

The superpotential (6) can also be put in the simplest form $\Psi_0^U \Psi_0^{-1}$, where $\Psi_0$ is now the most general solution of the Schrödinger equation [17]. The associated transformation depends on the choice of $\Psi_0(\bar{\epsilon}, r)$, the integration bound $a$, and the constant matrix $\Lambda$. In the present work we are concerned with choices that lead to the addition of a bound state.

For an energy $\epsilon \leq \epsilon_0$, it turns out that the regular solution of $H_0$ together with the bound $a = \infty$ is the appropriate choice. Then the transformed wave function $\Psi_1(\epsilon, r)$ at the energy $\epsilon = \bar{\epsilon}$, is simply given by
\begin{equation}
\Psi_1(\bar{\epsilon}, r) = A_0^- \Psi_0(\bar{\epsilon}, r) \\
= \left( \Psi_0^1(\bar{\epsilon}, r) \right)^{-1} \Lambda \\
x \left[ 1 - \int_r^\infty dt \Psi_0^{-1}(\bar{\epsilon}, t) \left( \Psi_0^1(\bar{\epsilon}, t) \right)^{-1} \Lambda \right]^{-1}.
\end{equation}

The regular solutions $\Psi_0$, at energies below the lowest ground state energy of $H_0$, exhibit a diverging behavior for $r \to \infty$. Hence, all columns of $\Psi_1(\bar{\epsilon}, r)$ are exponentially vanishing solutions of $H_1$ at asymptotic distances. Furthermore $\Psi_1(\bar{\epsilon}, r)$ is bounded (for $0 < r < \infty$) if
\begin{equation}
\det \left( 1 - \int_r^\infty dt \Psi_0^{-1}(\bar{\epsilon}, t) \left( \Psi_0^1(\bar{\epsilon}, t) \right)^{-1} \Lambda \right) \neq 0
\end{equation}
which is the case when the matrix $\Lambda$ is chosen negative semidefinite. This guarantees also that $U_1(r)$ is bounded except for $r = 0$.

The behavior of $\Psi_1(\bar{\epsilon}, r)$ at $r \sim 0$ requires more attention as it depends on the coupled–channel system considered. Specifically, if the singularity of the potential at the origin is of the form $\nu(\nu + 1)/r^2$ with the same $\nu$ for all channels, one obtains the behavior $\lim_{r \to 0} \Psi_1(\epsilon, r) r^{-\nu} = \text{const} \neq 0$. Hence, for $\nu \geq 1$ the function $\Psi_1(\epsilon, r)$ vanishes at the origin and with an appropriate choice of $\Lambda$, satisfying (8), each column vector of $\Psi_1(\epsilon, r)$ corresponds to a bound state of $H_1$.

Thus the transformation associated with Eq. (6) enables us to add $N$ degenerate bound states to the spectrum of $H_0$ at the energy $\bar{\epsilon}$. More precisely, analytical examples show that the degeneracy at $\epsilon = \bar{\epsilon}$ seems to depend on the rank of the matrix $\Lambda$ [17]. For $\nu = 0$ the function $\Psi_1(\epsilon, r)$ does not correspond to bound states. The situation becomes even more intriguing, when the potential exhibits different singularities in the coupled channels, i.e. $\nu_i$, $i = 1, \ldots, N$. It can be shown that if the coupling vanishes near the origin, the $i$th component of each column vector of $\Psi_1(\epsilon, r)$
behaves as \( r^{\nu_i} \) there and leads, with an appropriate choice of \( \Lambda \), to the addition of up to \( N \) degenerate bound states to the spectrum of \( H_0 \).

For a nonvanishing coupling near the origin and different \( \nu_i \) the behavior of \( \Psi_1(\bar{\epsilon}, r) \) near the origin cannot be given in closed form. Therefore, definite conclusions about the number of bound states that can be added are difficult to draw. However, the numerical examples discussed below demonstrate that the transformation \( \Psi^1 \) can also lead, in the general case, to the addition of up to \( N \) degenerate bound states.

### III. Phase-Equivalence

The SUSY transformation generated by \( W(r) \) of Eq. (1) modifies, similarly to the single channel case, the S-matrix [14]. To compensate this, another transformation is required which is associated with a superpotential that has the opposite sign at asymptotic distances [14]. Therefore we first perform a transformation generated by the simple superpotential (first term of Eq. (1)) using asymptotically vanishing solutions \( \eta_0(\bar{\epsilon}, r) \) of the Schrödinger equation with \( H_0 \) \([\lim_{r \to \infty} \eta_0(\bar{\epsilon}, r) \exp(\sqrt{\bar{\epsilon}} r) = A, \) where \( A \) is an \( N \times N \) matrix\]. The existence of the transformed potential requires a non-vanishing determinant of \( \eta_0(\bar{\epsilon}, r) \) for all finite \( r \)-values. This can always be satisfied because \( \bar{\epsilon} \) is assumed to be smaller than the deepest bound state of \( H_0 \). The first SUSY transformation modifies the S-matrix but does not change the number of bound states. In a second step we transform \( H_1 \) via the transformation mediated by the superpotential of Eq. (1). A direct verification shows that \( (\eta_0^{-1}(\bar{\epsilon}, r))^I \) is a regular solution of \( H_1 \). We use this solution to construct the superpotential of the second transformation, following the principle of the preceding section. The potential \( U_2 \) resulting from the two successive transformations, as well as the corresponding solution \( \Psi_2 \), can be expressed in terms of the solutions of the initial equation only. They read

\[
U_2(r) = U_0(r) - 2 \frac{d}{dr} \left\{ \chi_2(\bar{\epsilon}, r) \eta_0(\bar{\epsilon}, r) \right\}
\]

and

\[
\Psi_2(\epsilon, r) = -(\epsilon - \bar{\epsilon}) \Psi_0(\epsilon, r) - \chi_2(\bar{\epsilon}, r) W[\eta_0(\bar{\epsilon}, r), \Psi_0(\epsilon, r)],
\]

where

\[
\chi_2(\bar{\epsilon}, r) = \eta_0(\bar{\epsilon}, r) \Lambda \left[ 1 - \int^n_{r} dt \eta_0^I(\bar{\epsilon}, t) \eta_0(\bar{\epsilon}, t) \Lambda \right]^{-1}
\]

corresponds to the added bound state(s) of \( H_2 \) at \( \epsilon = \bar{\epsilon} \). The quantity \( \Lambda \) is a constant \( N \times N \) matrix.

This procedure transforms a regular solution \( \Psi_0 \) into a regular solution \( \Psi_2 \) as long as the singularity at the origin is sufficiently strong (\( \nu \geq 2 \)). In this case the new potential \( U_2 \) is phase equivalent to \( U_0 \) [Eq. (10) shows that both \( \Psi_0 \) and \( \Psi_2 \) have the same asymptotic behaviour, up to a normalization factor, which confirms phase equivalence], and sustains (besides the bound states of \( H_0 \)) up to \( N \) additional bound states at \( \bar{\epsilon} \). Similarly to the single SUSY transformation, discussed in the previous section, the degeneracy of the introduced bound states at \( \bar{\epsilon} \) can be controlled by the number of non-vanishing eigenvalues of the constant \( \Lambda \) [17].

### IV. Examples

As a demonstration we consider an \( \ell = 2 \) two-channel system without threshold and a potential of Gauß form, \( V_{ij} = V_{ij}^{(0)} \exp(-r^2/R^2) + \delta_{ij} \hbar^2/2m r^2 \), with \( R = 2 \) fm, \( 2mc^2 = 938.9185 \) MeV and the depths \( V_{11}^{(0)} = -100 \) MeV, \( V_{22}^{(0)} = -60 \) MeV, and \( V_{12}^{(0)} = V_{21}^{(0)} = -56.56 \) MeV. This potential does not sustain a bound state. A pair of degenerate bound states at \( E = -200 \) MeV has been introduced using the twice iterated SUSY transformation. The new potentials are shown in Fig. 1 together with the corresponding wave functions of the two added bound states. It is seen that there is a clear spatial separation between the two degenerate states which is also reflected in the potential matrix. Choosing \( \Lambda \) of rank one results in the introduction of only one bound state at \( \bar{\epsilon} \). This leads to the potential and bound state wave functions displayed in Fig. 2. It must be emphasized that the potentials of Fig. 1 and Fig. 2 are phase equivalent.

The introduction of bound states via the twice iterated SUSY transformation (1) is also possible for systems where channels with different quantum numbers of orbital angular momentum are coupled. Of particular interest is the
coupling of an s-channel ($\ell_1 = 0$) and a d-channel ($\ell_2 = 2$) when the associated potential has only in the d-channel an $1/r^2$-singularity at the origin. In such a system the strength of the singularity suffices for the introduction of one additional bound state via the SUSY transformation (11). The transformed potential has no $r^{-2}$ singularity at the origin which, taking into account the centrifugal term of the d-channel explicitly, implies an attractive $1/r^2$-contribution to the residual potential. We have verified this special case numerically assuming the previous two-channel system with slight modifications, i.e. $\ell_1 = 0, \ell_2 = 2$ and $V_{ij} = V^{(0)}_{ij} \exp(-r^2/R^2) + \delta_{ij} 6\hbar^2/(2mr^2)$. This system has a bound state at $E = -29.45$ MeV. The introduction of an additional bound state at $E = 200$ MeV leads to the potential and bound state wave function displayed in Fig. 3. The transformed potential confirms the previous discussion and its phase equivalence has been verified numerically. On the contrary, for a $\Lambda$ matrix of rank 2, we have verified that the final potential is not phase equivalent to $U_0$.

As a realistic example we consider the $^3S_1-^3D_1$ partial waves of the nucleon–nucleon (NN) system which are coupled by a tensor potential that sustains a single physical bound state at $E_d = -2.22$ MeV (the deuteron). Within the quark model the NN force is an effective interaction of a composite particle system and may give rise to Pauli–forbidden states which are simulated by deep unphysical bound states in local potential models [11]. Neudatchin et al. [18] were the first to use the concept of forbidden states for the NN potential. A more detailed study led to the well known Moscow potential which reproduces the scattering data up to 400 MeV [19]. Elimination of the additional unphysical bound state via the twice iterated SUSY transformation of Sparenberg and Baye [16] leads to a phase equivalent potential having a close similarity to the standard NN interactions derived in the meson-exchange picture. Qualitatively the same result has been obtained by Leeb et al. [20] by applying a numerical SUSY transformation to an early version of the Moscow potential.

In the present work, we apply the procedure described by Eqs. (11) in the opposite direction. Starting from the Reid soft core potential (RSC) [21] for the $^3S_1-^3D_1$ channel, we would like to generate a series of NN potentials of Moscow-type where an additional bound state has been introduced at different energies $\bar{E}$. In Fig. 4 we show a series of phase equivalent NN–potentials obtained by this procedure. Variation of the matrix $\Lambda$ results in modifications of the $r$-dependence while the energy of the added bound state is reflected in the depth of the potentials. The repulsive core of the effective potential $V_{22}(r)$ is a consequence of the repulsive core of the RSC potential. It must be emphasized that the obtained potentials are not exactly of the same nature as the Moscow potential since the central part of their D-channel component has an $r^{-2}$ attractive core.

V. CONCLUSIONS

In summary, in this work we have presented a closed form expression for a SUSY transformation which enables us to introduce single or degenerate bound states in a coupled channel system. The novel transformation works for a general hermitean N-channel system with and without thresholds. The method has been successfully tested for the case of two coupled partial waves in the NN system where a whole set of phase equivalent NN potentials of Moscow-type has been generated. The method has also been successfully applied to two–channel systems with potentials of Gauß form.

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FIGURE CAPTIONS

Figure 1
Introduction of two degenerate bound states at \( E = -200 \text{ MeV} \) into a two-channel system with Gaussian potential (see text) at \( \ell = 2 \) using \( \Lambda_{11} = -5 \cdot 10^4 \), \( \Lambda_{22} = -5 \cdot 10^6 \), \( \Lambda_{12} = \Lambda_{21} = 0 \) and \( \eta_0(\bar{\epsilon}, r) \) associated with \( A_{11} = A_{12} = 1 \) and \( A_{21} = -A_{22} = 0.5 \). The upper graph shows the matrix elements of the potential; \( V_{11}(r) \) (solid line), \( V_{22}(r) \) (dashed line) and \( V_{12}(r) = V_{21}(r) \) (dotted line). The two lower graphs show the components of the wave functions (solid line for the first channel, dashed line for the second channel) of the two added bound states.

Figure 2
The same system as shown in Fig. 1 but entering only one bound state at \( E = -200 \text{ MeV} \) using \( \Lambda_{11} = -5 \cdot 10^4 \), \( \Lambda_{12} = \Lambda_{21} = \Lambda_{22} = 0 \) and the same \( \eta_0 \). For the notation see Fig. 1.

Figure 3
Introduction of a bound state at \( E = -200 \text{ MeV} \) into a two-channel system of coupled s- and d-partial waves with Gaussian potential (see text) using \( \Lambda_{11} = -5 \cdot 10^4 \), \( \Lambda_{12} = \Lambda_{21} = \Lambda_{22} = 0 \) and \( \eta_0(\bar{\epsilon}, r) \) associated with \( A_{11} = A_{12} = 1 \), \( A_{21} = -A_{22} = 0.5 \). The upper graph shows the matrix elements of the potential; \( V_{11}(r) \) (solid line), \( V_{22}(r) \) (dashed line) and \( V_{12}(r) = V_{21}(r) \) (dotted line). The graph at bottom shows the components of the wave function (solid line for the first channel, dashed line for the second channel) of the added bound state.

Figure 4
NN potentials for the \(^3\text{S}_1-^3\text{D}_1\)-channel, phase equivalent to the RSC generated by the twice iterated SUSY transformation \( \text{RSC} \). The unphysical bound states are at \( E = -100 \text{ MeV} \) (long dashed line), \( E = -200 \text{ MeV} \) (dashed line), \( E = -400 \text{ MeV} \) (dotted line) and \( E = -800 \text{ MeV} \) (solid line). The constant matrix \( \Lambda \) was fixed to \( \Lambda_{22} = -5 \cdot 10^6 \), \( \Lambda_{11} = \Lambda_{12} = \Lambda_{21} = 0 \) and \( \eta_0(\bar{\epsilon}, r) \) associated with \( A_{11} = A_{12} = 1 \) and \( A_{21} = -A_{22} = 0.5 \).
Figure 1
Figure 2
Figure 3
Figure 4