Super-Calogero-Moser-Sutherland systems and free super-oscillators : a mapping

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Abstract

We show that the supersymmetric rational Calogero-Moser-Sutherland (CMS) model of $A_{N+1}$-type is equivalent to a set of free super-oscillators, through a similarity transformation. We prescribe methods to construct the complete eigen-spectrum and the associated eigen-functions, both in supersymmetry-preserving as well as supersymmetry-breaking phases, from the free super-oscillator basis. Further we show that a wide class of super-Hamiltonians realizing dynamical $OSp(2|2)$ supersymmetry, which also includes all types of rational super-CMS as a small subset, are equivalent to free super-oscillators. We study $BC_{N+1}$-type super-CMS model in some detail to understand the subtleties involved in this method.

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I. INTRODUCTION

The rational CMS Hamiltonian is described by \( N \) particles interacting with each other through an inverse square interaction and all particles are subjected to a common confining harmonic force. This model is exactly solvable and the eigen-values, including the degeneracy at each level, are exactly identical to the spectrum of \( N \) free oscillators, except for a constant shift in the ground state energy \([1, 5]\). There were enough indications in the literature in different context that a very close connection between the CMS and the free oscillators model might exists. In fact, it has been shown recently that the rational CMS Hamiltonian is equivalent to that of free oscillators through a similarity transformation \([6, 7]\), confirming all previous speculations. This equivalence has enriched our understanding of the model and also became a very useful tool for studying different aspects of CMS, like eigen functions, integrability, and symmetry algebra, in a new way.

The supersymmetric version of the rational CMS system has also been studied in the literature in different context \([8–14]\). The zero fermion sector of the supersymmetric CMS describes the usual CMS model, while the \( N \) fermion sector describes the CMS model at a shifted value of the coupling constant \([10]\). Such relation between the zero and the \( N \) fermion sector of the model is due to ‘shape invariance’ of the Hamiltonian, which is a very popular and useful concept in studying quantum mechanics with one degree of freedom \([15]\). For other sectors with fermion numbers ranging from one to \( N - 1 \), however, no such trivial identifications with the usual CMS can be made. Hamiltonian in these sectors are in fact related to CMS with internal degrees of freedom \([4,12]\).

The supersymmetric rational CMS model (SRCMSM) of \( A_{N+1} \)-type is exactly solvable in both supersymmetry-preserving and supersymmetry-breaking phases \([8,14]\). The spectrum in the supersymmetry-preserving phase is again identical to that of the free
super-oscillators \[8\]. It might be recalled at this point that the supersymmetry is always preserved in the super-oscillator model, once the convention for choosing the ground state in either zero or \(N\) fermion sector has been made. Thus, the spectrum of SRCMSM in the supersymmetry-breaking phase, has no counter-part in the super-oscillator model. However, it has some similarity with the spectrum of the super-oscillator model modulo a constant shift in the ground state energy \[8\]. It is intriguing at this point to ask, whether or not the SRCMSM, at least in the supersymmetry-preserving phase, can be shown to be equivalent to free super-oscillators through a similarity transformation, much akin to its non-supersymmetric version.

The purpose of this paper is to show that the SRCMSM of \(A_{N+1}\)-type is indeed equivalent to free super-oscillators through a similarity transformation. This equivalence is valid only in supersymmetry-preserving phase. This explains the identicalness of the spectrum of SRCMSM and free super-oscillators. The eigen functions of these two models are of-course different from each other and we outline a method to construct eigen functions of the SRCMSM from permutationally invariant super-oscillator basis functions. In case, one chooses a basis function which is not symmetric under the combined exchange of bosonic and fermionic coordinates, the corresponding eigenfunction of the SRCMSM is not normalizable. This is due to highly correlated nature of the many-body inverse-square interaction and this has also been observed in the usual CMS model \[8\].

We also prescribe on constructing eigen-spectrum of the SRCMSM in the supersymmetry-breaking phase, from the known super-oscillator basis by making use of a duality property of the model \[8\]. In particular, we construct a new super-Hamiltonian, which differs from the SRCMSM by the fermionic number operator and a constant. This implies that any eigen-function of this dual model is also a valid eigen-function of
the SRCMSM. Of course, the corresponding energy eigen-values are different from each other. We show through a similarity transformation that this dual Hamiltonian is again equivalent to a free super-oscillator Hamiltonian. It turns out that the eigen-spectrum of the SRCMSM obtained from this super-oscillator model via the dual Hamiltonian indeed correctly describes the supersymmetry-breaking phase of the model.

The symmetry algebra of the super-oscillator model is well understood in terms of a set of bosonic and fermionic operators. We define a set of such operators for the SRCMSM, which are obtained from the corresponding operators in the super-oscillator model through the inverse similarity transformation. This enables us to study the symmetry algebra of SRCMSM in a simple way, leading to the construction of the complete eigen-spectrum algebraically.

As a generalization of these results, we show that a wide class of models whose bosonic many-body potential is a homogeneous function of degree $-2$ and all the particles are restricted to move on a line by a common confining harmonic force, are equivalent to the super-oscillator model through a similarity transformation. These Hamiltonians are characterized by a dynamical $OSp(2|2)$ supersymmetry. The SRCMSM associated with different root-structures of the Lie-algebra appear as a special small subset of this class. Though the equivalence is valid at the operator level for the general inverse-square potential, one must show that the complete set of eigen-functions as well as eigen-values of such models are indeed obtained from the super-oscillator model. The equivalence relation at the operator level acts as a necessary condition, while the construction of the complete eigen-spectrum and associated wave-functions from the super-oscillator basis is sufficient to claim such relation between these two models. We show that both the necessary and the sufficient conditions are certainly satisfied by the SRCMSM of $A_{N+1}$ and $BC_{N+1}$ types. However, it appears that all other cases have to be treated
We organize the paper in the following way. We first give an overview of the supersymmetric quantum mechanics with many degrees of freedom in the next section. We mostly review the known results in a way which will become useful for our subsequent discussions. In Sec. III, we consider the $A_{N+1}$-type SRCMSM and show its equivalence to free super oscillator model. We first show the equivalence for the supersymmetry-preserving phase in Sec. III.A and outline a method to construct the eigen-spectrum from the known super-oscillator basis. Similar study for the supersymmetry-breaking phase has been discussed in Sec. III.B, using a duality property of the model. We generalize these results to SRCMSM associated with other root-structures of the Lie algebra in Sec. IV. Finally, in Sec. V, we summarize and discuss the implications of these results. We show how the dynamical $OSp(2|2)$ supersymmetry is realized by these systems in Appendix A.

II. SUPERSYMMETRIC QUANTUM MECHANICS WITH MANY DEGREES OF FREEDOM: BRIEF REVIEW

The supercharge $Q$ and its conjugate $Q^\dagger$ are defined as,

$$Q = \sum_{i=1}^{N} \psi_i^\dagger a_i, \quad Q^\dagger = \sum_{i=1}^{N} \psi_i a_i^\dagger,$$

where the fermionic variables $\psi_i$’s satisfy the Clifford algebra,

$$\{\psi_i, \psi_j\} = 0 = \{\psi_i^\dagger, \psi_j^\dagger\}, \quad \{\psi_i, \psi_j^\dagger\} = \delta_{ij}, \quad i, j = 1, 2, \ldots, N. \quad (2)$$

The operators $a_i (a_i^\dagger)$’s are analogous to bosonic annihilation (creation) operators. They are defined in terms of the momentum operators $p_i = -i \frac{\partial}{\partial x_i}$ and the superpotential $W(x_1, x_2, \ldots, x_N)$ as,
\[ a_i = p_i - iW_i, \quad a_i^\dagger = p_i + iW_i, \quad W_i = \frac{\partial W}{\partial x_i}, \quad (3) \]

and satisfy the following commutation relations among themselves,

\[ [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger], \quad [a_i, a_j^\dagger] = [a_j, a_i^\dagger] = 2W_{ij}, \quad W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}. \quad (4) \]

Note that, by construction, \( W_i \)'s satisfy the so called ‘zero-curvature condition’ \( \partial_i W_j = \partial_j W_i \). Also, for translationally invariant superpotential, these \( W_i \)'s satisfy the ‘sum to zero’ condition, \( \sum_i W_i = 0 \). These two properties are useful ingredients in studying the usual CMS model.

The supersymmetric Hamiltonian is defined in terms of the supercharges as,

\[
H = \frac{1}{2} \{ Q, Q^\dagger \} \\
= \frac{1}{4} \sum_i (a_i, a_i^\dagger) + \frac{1}{4} \sum_{i,j} [a_i, a_j^\dagger] [\psi_i^\dagger, \psi_j], \quad (5)
\]

The Hamiltonian commutes with both \( Q \) and \( Q^\dagger \). The ground state of \( H \) is annihilated by both \( Q \) and \( Q^\dagger \). Thus, the ground states are given by,

\[ \phi_0 = e^{-W}|0>, \quad \phi_N = e^W|\bar{0}>, \quad (6) \]

where the fermionic vacuum \( |0> \) and its conjugate \( |\bar{0}> \) in the \( 2^N \) dimensional fermionic Fock space are defined as,

\[ \psi_i|0> = 0, \quad \psi_i^\dagger|\bar{0}> = 0. \quad (7) \]

The first equation of (7) defines the zero-fermion sector, while the second one defines the \( N \) fermion sector. In case, either \( \phi_0 \) or \( \phi_N \) is normalizable, the supersymmetry is preserved with zero ground state energy. On the other hand, the supersymmetry is broken if neither \( \phi_0 \) nor \( \phi_N \) is normalizable. The ground state energy in this case is positive-definite.
III. EQUIVALENCE : RATIONAL CMS OF $A_{N+1}$-TYPE AND FREE OSCILLATORS

The superpotential for the $A_{N+1}$-type SRCMSM is given by,

$$W = -\lambda n \prod_{i<j} x_{ij} + \frac{1}{2} \sum_i x_i^2, \quad x_{ij} = x_i - x_j. \quad (8)$$

The first term produces the many-body inverse square interaction, while the second term generates the term responsible for harmonic confinement. The Hamiltonian (5), with the above choice of $W$, has the following form,

$$H = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda (\lambda - 1) \sum_{i \neq j} x_{ij}^2 + \frac{1}{2} \sum_i x_i^2 - \frac{1}{2} N (1 + \lambda (N - 1))$$

$$+ \sum_i \psi_i^\dagger \psi_i + \lambda \sum_{i \neq j} x_{ij}^{-2} (\psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j). \quad (9)$$

The Hamiltonian $H$ is permutationally invariant under the combined exchange of bosonic and fermionic coordinates. Observe that the zero-fermion sector of (9) describes the usual CMS, apart from a constant equal to its ground state energy. The ground state of SRCMSM has the well-known form,

$$\Phi = e^{-W}|0>$$

$$= \prod_{i<j} x_{ij}^\lambda e^{-\frac{1}{2} \sum_i x_i^2} |0>. \quad (10)$$

Note that $\Phi$ is normalizable for $\lambda > -\frac{1}{2}$. However, a stronger criteria that each momentum operator $p_i$ is self-adjoint for the wave-functions of the form $\Phi$ requires $\lambda > 0$. The supersymmetry is preserved for $\lambda > 0$, while it is broken for $\lambda < 0$ [8].

A. Supersymmetry-preserving phase

Now we would like to show that the Hamiltonian (8) is equivalent to the free superoscillators model through a similarity transformation. In order to do so, let us first
consider the following transformation,

\[
H_1 = e^W H e^{-W} = \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right) - S, \tag{11}
\]

\[
S = \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \lambda \sum_{i \neq j} x_{ij}^{-1} \frac{\partial}{\partial x_i} - \lambda \sum_{i \neq j} x_{ij}^{-2} \left( \psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j \right). \tag{12}
\]

The total fermion number operator \( N_f = \sum \psi_i^\dagger \psi_i \) commutes with the Hamiltonian \( H \).

The fermionic part of \( H_1 \) is identical to that of \( H \). Thus, \( N_f \) commutes with \( H_1 \) and hence, also with \( S \). Making use of the following identities,

\[
\left[ \sum_i x_i \frac{\partial}{\partial x_i}, S \right] = -2S, \quad \left[ \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right), S \right] = -2S, \tag{13}
\]

we find,

\[
\left[ H_1, e^{-S/2} \right] = S e^{-S/2} \tag{14}
\]

\[
H_2 = e^{S/2} H_1 e^{-S/2} = \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right). \tag{15}
\]

The transformed Hamiltonian \( H_2 \) is nothing but the supersymmetric generalization of the Euler operator. The connection of \( H \) with the free super-oscillators is apparent from the expression of \( H_2 \). In particular, we get the familiar supersymmetric \( N \) particle free oscillators model in the following way,

\[
H_{sho} = e^{-\frac{1}{2} \sum_i x_i^2} e^{-\frac{1}{4} \sum_i \frac{\partial^2}{\partial x_i^2}} H_2 e^{\frac{1}{4} \sum_i \frac{\partial^2}{\partial x_i^2}} e^{\frac{1}{2} \sum_i x_i^2}
\]

\[
= \frac{1}{2} \sum_i \left( -\frac{\partial^2}{\partial x_i^2} + x_i^2 \right) + \sum_i \psi_i^\dagger \psi_i - \frac{N}{2}. \tag{16}
\]

This shows the equivalence between SRCMSM and the free super-oscillators.
1. Construction of eigen-functions

The eigen-spectrum of (9) can be constructed either from (15) or (16). We prefer to work with Eq. (15). If $P_{n,k}$ is an eigen-function of (15) with the eigen-value $E_{n,k}$, then, $H$ has the same eigen-value $E_{n,k}$ with the eigen-function given by,

$$
\chi = e^{-W} e^{-\frac{S}{2}} P_{n,k} |0 > .
$$

(17)

We have to choose $P_{n,k}$ to be a permutationally symmetric polynomial of $x_i$ and $\psi_i$, under the combined exchange of the bosonic and fermionic coordinates. Otherwise, the action of $S$ on $P_{n,k}$ produces non-vanishing singular terms, thereby, making $\chi$ non-normalizable.

It is worth recalling at this point that similar constraint on $P_{n,k}$ has been noticed also for the usual CMS case, reflecting the highly correlated nature of these systems. The highly correlated nature of this model is also present in the supersymmetric version.

There are many choices for the polynomial $P_{n,k}$. Let us choose the following form of $P_{n,k}$,

$$
P_{n,k} = r^{2n} \sum i x_i^{k-1} \psi_i^\dagger, \quad r^2 = \sum i x_i^2,
$$

(18)
as the $N_f = 1$ solution of $H_2$ with $E_{n,k} = 2n + k$. The quantum numbers $n$ and $k - 1$ are nonnegative integers. It can be checked easily that the action of $S^m$ on $P_{n,k}$ does not produce any singularity for positive $m$. Let us first consider the action of $S$ on $P_{n,k}$,

$$
SP_{n,k} = b_1 r^{2(n-1)} \sum i x_i^{k-1} \psi_i^\dagger + b_2 r^{2n} \sum i x_i^{k-3} \psi_i^\dagger + \lambda r^{2n} \sum \sum (k - l - 2)x_i^{k-3} x_j^{l-3} \psi_i^\dagger, 
$$

$$
b_1 = n [N + 2\lambda N(N - 1) + 2(n + k - 2)], \quad b_2 = \frac{1}{2}(k - 1)(k - 2).
$$

(19)
The first two terms on the right hand side of the first equation in (19) has the same form as that of $P_{n,k}$, except for powers of $r$ and $x_i$. Thus, the contribution of these two
terms to $S^2 P_{n,k}$ can not contain a singular term. The third term has a different form than $P_{n,k}$. A term like this, which has a general form,

$$\eta = \sum_{i_1 \neq i_2 \neq \ldots \neq i_N} x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_N}^{k_N} \psi_{i_1} \ldots \psi_{i_N}, \tag{20}$$

keeps on appearing on each successive operation of $S$ on the left hand side of the first equation of (19). The integers $k_i$’s are determined in terms of $k$. However, we keep them as arbitrary nonnegative integers in (20). The first term of $S$, a generalized Laplacian operator $\nabla = \sum_i \frac{\partial^2}{\partial x_i^2}$, acting on $\eta$ can not produce any singularity. Further, we have the following identity,

$$S' \eta = \left( S - \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} \right) \eta$$

$$= \lambda \sum_{i_1 \neq i_2 \neq \ldots \neq i_N} \sum_{j(\neq i_p)} \sum_{p=1}^{N} \sum_{l=0}^{k_p-2} \beta_{k_{p,l}} x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_p}^{k_p-2} x_{i_1}^{l} \ldots x_{i_N}^{k_N} \psi_{i_1} \ldots \psi_{i_N}, \tag{21}$$

$$\beta_{k_1,l} = k_1 - l - 1, \quad \beta_{k_p,l} = \frac{k_p}{2} \text{ for } p \geq 2.$$

Note that $S' \eta$ has the same form as that of $\eta$, once the summation over the indices $j, p$ and $l$ has been performed. This proves that $S^m P_{n,k}$ can not contain a singular term, instead terminates as a finite degree polynomial. Thus, the well-behaved eigenfunctions $\chi$ of $H$ can be constructed from the super-oscillator basis $P_{n,k}$. It may be worth mentioning here that the exact solution for $N_f = 1$ and certain small values of $k$, obtained in [8], can be reproduced in a systematic way from Eqs. (17) and (18).

Similar results for other values of $N_f$ can also be obtained. For example, one may choose $P_{n,k}$ for an arbitrary $N_f$ as,

$$P_{n,k} = \frac{1}{N_f!} \frac{1}{2^{2n}} \sum_{i_{1},i_{2},\ldots,i_{N_f}} f_{i_{1}i_{2}\ldots i_{N_f}}(x_1, x_2, \ldots, x_N) \psi_{i_1} \psi_{i_2} \ldots \psi_{i_{N_f}}, \tag{22}$$

where $f_{i_{1}i_{2}\ldots i_{N_f}}$ is anti-symmetric under the exchange of any two indices and is a homogeneous function of degree $k - N_f$. The anti-symmetric nature of $f$ ensures that
$P_{n,k}$ is permutationally invariant under the combined exchange of bosonic and fermionic coordinates. Though we do not present here results concerning normalizability of eigenfunctions $\chi$ constructed from (22) for arbitrary $N_f$, it is expected that the certain specific choices of $f$ would indeed produce well-behaved and physically accepted $\chi$. This is because of the result [9] that the eigen-value equation of $H_1$ has permutationally symmetric polynomials in $x_i$ and $\psi_i$ as the solution. This implies that the solution for the eigen equation of $S$ are also permutationally symmetric polynomials. Thus, the action of $S^m$ on these permutationally symmetric polynomials for any positive $m$ are not expected to produce singular terms. We outline a method in the next section to construct the eigenstates in an algebraic way.

2. Algebraic structure

The algebraic structure of the super-oscillators can be exploited to construct the eigenstates of $H$ in an algebraic way. Consider the following set of operators,

$$
b^-_i = ip_i = \frac{\partial}{\partial x_i}, \quad b^+_i = 2x_i,
$$

$$
B^-_n = \sum_{i=1}^N T^{-1} b^{-n}_i T, \quad B^+_n = \sum_{i=1}^N T^{-1} b^{+n}_i T, \quad T = e^{2T} e^W
$$

$$
F^-_n = T^{-1} \left( \sum_i \psi_i b^{-n-1}_i \right) T, \quad F^+_n = T^{-1} \left( \sum_i \psi_i^* b^{+n-1}_i \right) T,
$$

$$
a^-_n = T^{-1} \left( \sum_i \psi_i^* b^{-n}_i \right) T, \quad a^+_n = T^{-1} \left( \sum_i \psi_i b^{+n}_i \right) T.
$$

(23)

Note that we are using a particular form of $b^-_i$ and $b^+_i$, such that $[b^-_i, b^+_j] = 2\delta_{ij}$. This choice has been made to make one to one correspondence between the usual annihilation (creation) operator of the harmonic oscillator and the $b^-_i (b^+_i)$. In particular, it can be checked easily,

$$
-ib^-_i = t^{-1} a^-_i t, \quad ib^+_i = t^{-1} a^+_i t, \quad a^\pm_i = p_i \pm ix_i, \quad t = e^{-\frac{1}{2} \sum_i x^2_i} e^{-\frac{1}{4} \sum_i \frac{\partial^2}{\partial x^2_i}}.
$$

(24)
The operators in (23) satisfy the following algebra among themselves.

\[
\{ F_m^+, F_n^+ \} = 0, \quad [B_m^+, F_n^+] = 0, \quad [B_m^+, B_n^+] = 0, \\
\{ q_1^+, F_n^+ \} = 0, \quad \{ q_1^+, F_n^+ \} = B_n^+, \quad [H, F_n^+] = nF_n^+, \\
[q_1^-, B_n^+] = 2nF_n^+, \quad [q_1^+, B_n^+] = 0, \quad [H, B_n^+] = nB_n^+. 
\] (25)

This is also the algebra of the corresponding operators of super-oscillators. Thus, the eigen-functions can be created in a similar way by acting different powers of \( B_n^+ \) and \( F_n^+ \) on the ground state. In particular [8],

\[
\chi_{n_1\ldots n_N \nu_1 \ldots \nu_N} = \prod_{k=1}^{N} B_k^{n_k} F_k^{\nu_k} \Phi,
\] (26)

is the eigenfunction with the eigen-value \( E = \sum_{k=1}^{N} k(n_k + \nu_k) \). The bosonic quantum numbers \( n_k \)'s are nonnegative integers, while the fermionic quantum numbers \( \nu_k \)'s are either 0 or 1. Note that a set of \( N \) independent super-oscillators with the frequencies \( 1, 2, \ldots, N \) have the same energy \( E \). Thus, the spectrum of SRCMSM is identical to that of \( N \) independent super-oscillators with the frequencies \( 1, 2, \ldots, N \).

A particular realization of the operators \( B_2^+, B_3^+, F_2^+ \) and \( F_3^+ \) was obtained in [8]. One can easily check that the explicit forms of these operators found in [8], are indeed identical to those obtained from (23). This equivalence is valid modulo an overall normalization factor. Thus, we have given a systematic way to determine \( B_n^+ \) and \( F_n^+ \) for arbitrary \( n \). It might be noted here that the particular basis we choose for the definitions of these operators is over-complete. However, one may always choose a basis similar to one given in [8] to avoid the over-completeness.

B. Supersymmetry-breaking phase

Consider the following supercharges,
\[ \tilde{Q} = \sum_{i} \psi_i (p_i - i \tilde{W}_i), \quad \tilde{Q}^\dagger = \sum_{i} \psi_i^\dagger (p_i + i \tilde{W}_i), \quad \tilde{W} = \lambda \ln \prod_{i<j} x_{ij} + \frac{1}{2} \sum_i x_i^2. \]  

These supercharges can be obtained from Eqs. (1) and (8) by making \( \lambda \rightarrow -\lambda \) and \( \psi_i \leftrightarrow \psi_i^\dagger \). The dual Hamiltonian \( H_d = \frac{1}{2} \{ \tilde{Q}, \tilde{Q}^\dagger \} \) differs from \( H \) by the fermionic number operator \( N_f \) and a constant. In particular,

\[ H_d = \frac{-1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda (\lambda - 1) \sum_{i \neq j} x_{ij}^{-2} + \frac{1}{2} \sum_i x_i^2 + \frac{1}{2} N (1 + \lambda (N - 1)) \]

\[ - \sum_i \psi_i^\dagger \psi_i + \lambda \sum_{i \neq j} x_{ij}^{-2} \left( \psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j \right), \]

\[ H = H_d + 2N_f - N (1 + \lambda (N - 1)). \]  

The ground state of \( H_d \) is in the \( N \) fermion sector,

\[ \Phi = e^{-\tilde{W}} | \tilde{0} > = \prod_{i<j} x_{ij}^{-\lambda} e^{-\frac{1}{2} \sum_i x_i^2} | \tilde{0} >, \]

which is normalizable for \( \lambda < \frac{1}{2} \). A stronger criteria that each momentum operator \( p_i \) is self-adjoint for wave-functions of the form \( \Phi \) determines \( \lambda < 0 \). The supersymmetric phase of \( H_d \) is described by \( \lambda < 0 \). The wave-function \( \Phi \) is also an eigen-state of \( H \) with positive energy. This is, in fact, the ground state of \( H \) in the supersymmetry-breaking phase \[8\]. The complete spectrum of \( H \) in this phase can be obtained from \( H_d \) by making use of the second equation of (28).

We get the super-oscillator Hamiltonian under the following transformations,

\[ \tilde{H}_2 = e^{\frac{\tilde{W}}{2}} e^{\tilde{W}} H_d e^{-\tilde{W}} e^{-\frac{\tilde{W}}{2}} \]

\[ = \sum_i \left( x_i \frac{\partial}{\partial x_i} - \psi_i^\dagger \psi_i \right) + N \]

\[ \tilde{H}_{sho} = e^{-\frac{1}{4} \sum_i x_i^2} e^{-\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{4} \sum_i \frac{\partial^2}{\partial x_i^2}} e^{\frac{1}{4} \sum_i \frac{\partial^2}{\partial x_i^2}} e^{\frac{1}{2} \sum_i x_i^2} \]

\[ = \frac{1}{2} \sum_i \left( - \frac{\partial^2}{\partial x_i^2} + x_i^2 \right) - \sum_i \psi_i^\dagger \psi_i + \frac{N}{2}. \]  

Note the difference between \( H_{sho} \) and \( \tilde{H}_{sho} \). The ground state is in the \( N_f = 0 \) sector for the former case, while it is in the \( N_f = N \) sector for the latter one. This is expected
also, since the original many-body Hamiltonians $H$ and $H_d$ have ground states in the $N_f = 0$ and $N_f = N$, respectively.

We use the first equation of (30) to construct eigen-spectrum of $H$. The eigenfunction is given by,

$$\hat{\Phi} = e^{-\tilde{W}} e^{-\frac{\delta}{2} \tilde{P}_{n,k}} |\bar{0}>,$$

where $\tilde{P}_{n,k}$ is a permutationally invariant polynomial under the combined exchange of $x_i$ and $\psi_i$. We may choose $\tilde{P}_{n,k}$ to have the same form as $P_{n,k}$, except for the replacement $\psi_i^\dagger \rightarrow \psi_i$. Following the discussions on the supersymmetry-preserving phase in Sec. III.A.1, it can be checked easily that this choice of $\tilde{P}_{n,k}$ results in well-behaved, normalizable eigenfunction for $H$.

The complete eigenstates can also be constructed with the help of bosonic creation operator $\hat{B}_n^+$ and the fermionic creation operator $\hat{F}_n^+$. We define,

$$\hat{B}_n^+ = \sum_i \hat{\mathcal{T}}^{-1} b_i^{+n} \hat{\mathcal{T}}, \quad \hat{F}_n^+ = \hat{\mathcal{T}}^{-1} \left( \sum_i \psi_i b_i^{+n-1} \right) \hat{\mathcal{T}}, \quad \hat{q}_n^+ = \hat{\mathcal{T}}^{-1} \left( \sum_i \psi_i^\dagger b_i^{+n} \right) \hat{\mathcal{T}},$$

with $\hat{\mathcal{T}} = e^{\frac{\delta}{2} \hat{W}}$. The eigenstates are,

$$\hat{\Phi}_{n_1,\ldots,n_N,\nu_1,\ldots,\nu_N} = \prod_{k=1}^{N} \hat{B}_k^{+n_k} \hat{F}_k^{+\nu_k} \hat{\Phi},$$

with the eigen-values, $E = N(1 - \lambda(N - 1)) + \sum_{k=1}^{N}(kn_k + (k - 2)n_k)$. The bosonic quantum numbers $n_k$’s are non-negative integers, while the fermionic quantum numbers $\nu_k$’s are either 0 or 1.

**IV. GENERALIZATION**

We have constructed a similarity transformation which shows the equivalence between the SRCMSM and free super-oscillators. The particular SRCMSM we considered
is associated with the $A_{N+1}$ type root-structure of the Lie algebra. SRCMSM associated with other root structures also can be shown to be equivalent to free super-oscillators. Instead of considering each model separately, we prove below a general result, which is applicable to all types of SRCMSM and also to a new class of rational models considered in [16] having nearest-neighbor and next-nearest-neighbor interactions. In particular, we consider a super-Hamiltonian $\mathcal{H}$ whose bosonic many-body potential is a homogeneous function of degree $-2$ and all the particles are confined on the line by a common harmonic oscillator potential. It is worth recalling at this point that all types of SRCMSM and models considered in [16], indeed satisfy this criteria. We construct a similarity transformation which shows the equivalence between $\mathcal{H}$ and free super-oscillators $H_{sho}$.

Let us decompose the superpotential $W$ in terms of superpotentials for the many-body interaction and the harmonic term as,

$$ W = -w + \frac{1}{2} \sum_i x_i^2, \quad w = \ln G(x_1, x_2, \ldots, x_N), $$

(34)

where $G$ is a homogeneous function of any arbitrary positive degree $d$,

$$ \sum_i x_i \frac{\partial G}{\partial x_i} = dG. $$

(35)

This property of $G$ ensures that each $w_i$ is a homogeneous function of degree $-1$ and hence, the bosonic potential is always homogeneous function of degree $-2$, apart from the harmonic term. The Hamiltonian is given by,

$$ \mathcal{H} = \frac{1}{2} \sum_i \left[ -\frac{\partial^2}{\partial x_i^2} + w_i^2 + w_{ii} + x_i^2 \right] - (d + \frac{N}{2}) + \sum_i \psi_i^\dagger \psi_i - \sum_{i,j} w_{ij} \psi_i^\dagger \psi_j. $$

(36)

This Hamiltonian has a dynamical $OSp(2|2)$ supersymmetry. The full $OSp(2|2)$ algebra and the operators realizing this algebra are given in Appendix-A.

The bosonic sub-algebra $O(2, 1) \times U(1)$ of $OSp(2|2)$ is present for a wide class of Hamiltonians $\mathcal{H}$, due to the constraint (35) on the superpotential. This class of Hamil-
tonians having $O(2,1) \times U(1)$ symmetry can even be made larger by adding a term $T$ having the following properties,

$$[N_f, T] = 0, \quad \left[ \sum_i x_i \frac{\partial}{\partial x_i}, T \right] = -2T,$$

(37)

to the Hamiltonian $H$. However, the new Hamiltonian $H' = H + T$ will not be supersymmetric anymore for general $T$. It is worth mentioning at this point that the presence of the symmetry algebra $O(2,1) \times U(1)$ is enough to show the equivalence between $H'$ and free super-oscillators. The supersymmetry of the Hamiltonian does not play any role.

In other words, our results are valid even if the $OSp(2|2)$ symmetry of $H'$ is lost, but, has only $O(2,1) \times U(1)$ symmetry. However, we restrict our discussions in this paper to $OSp(2|2)$ supersymmetric Hamiltonian $H$ only.

Observe that $H$ can be transformed to a new Hamiltonian $H_1$ under the following similarity transformation,

$$H_1 = e^W H e^{-W}$$

$$= \sum_i \left( x_i \frac{\partial}{\partial x_i} + \psi_i^\dagger \psi_i \right) - \hat{S}$$

(38)

$$\hat{S} = \sum_i \left( \frac{1}{2} \frac{\partial^2}{\partial x_i^2} + w_i \frac{\partial}{\partial x_i} \right) + \sum_i w_{ii} \psi_i^\dagger \psi_i + \sum_{i \neq j} w_{ij} \psi_i^\dagger \psi_j.$$  

(39)

The total fermion number $N_f = \sum_i \psi_i^\dagger \psi_i$ commutes with $\hat{S}$, $[N_f, \hat{S}] = 0$. The commutation relation between the Euler operator $E = \sum_i x_i \frac{\partial}{\partial x_i}$ and $\hat{S}$ is given by,

$$[E, \hat{S}] = -2\hat{S}, \quad [H_2, \hat{S}] = [E + N_f, \hat{S}] = -2\hat{S}.$$  

(40)

The homogeneity property (35) of $G$ has been used in deriving the above equations.

Now it is easy to show that $H_1$ is transformed to $H_2$ under the following transformation,

$$H_2 = e^{\frac{1}{2} \hat{S}} H_1 e^{-\frac{1}{2} \hat{S}}.$$  

(41)

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The super-oscillator Hamiltonian can be obtained from (41) by using the same transformation as used in equation (16).

One might wonder at this point that any super-Hamiltonian with the superpotential \( W \) described by (34) and (35) is exactly solvable, due to its equivalence to free super oscillators through similarity transformations. We would like to point out that this may not be true always, because, merely showing the equivalence of different models is not sufficient for such conclusions. We have to make sure that the similarity transformation, which is responsible for such equivalence, keeps the original Hamiltonian in its own Hilbert space. Thus, as a check, one should show that the complete spectrum and the corresponding well-behaved, normalizable eigen-functions of \( \mathcal{H} \) can be constructed from \( H_{sho} \) or \( H_2 \) through inverse similarity transformation. The equation (41) act as a necessary condition, while the construction of the complete spectrum and associated well-behaved eigen-functions of the original Hamiltonian from the super-oscillator model is sufficient to claim the equivalence between these two Hamiltonians. We discuss these points below with the example of \( BC_{N+1} \)-type SRCMSM.

A. \( BC_{N+1} \)-type SRCMSM and super-half-oscillator

The superpotential for the \( BC_{N+1} \)-type SRCMSM is described by,

\[
G(\lambda, \lambda_1, \lambda_2) = \prod_{i<j} \left( x_i^2 - x_j^2 \right)^{\lambda/2} \prod_k x_k^{\lambda_1} \prod_l (2x_l)^{\lambda_2},
\]

where \( \lambda \), \( \lambda_1 \) and \( \lambda_2 \) are arbitrary parameters. The \( D_{N+1} \)-type model is described by \( \lambda_1 = \lambda_2 = 0 \), while \( \lambda_1 = 0(\lambda_2 = 0) \) describes \( C_{N+1}(B_{N+1}) \)-type Hamiltonian. Without loss of any generality, we restrict our discussions to the \( B_{N+1} \)-type Hamiltonian only. The Hamiltonian is given by,

\[
H_{B_{N+1}} = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \lambda(\lambda - 1) \sum_{i \neq j} \left[ x_{ij}^{-2} + (x_i + x_j)^{-2} \right] + \frac{1}{2} \lambda_1(\lambda_1 - 1) \sum_i x_i^{-2}
\]
\[ + \frac{1}{2} \sum_i x_i^2 - \frac{1}{2} N [1 + 2\lambda(N - 1) + \lambda_1] + \sum_i \psi_i^\dagger \psi_i + \lambda_1 \sum_i \psi_i^\dagger \psi_i x_i^{-2} \]
\[ + \lambda \sum_{i \neq j} \left[ x_{ij}^{-2} \left( \psi_i^\dagger \psi_i - \psi_j^\dagger \psi_j \right) + (x_i + x_j)^{-2} \left( \psi_i^\dagger \psi_i + \psi_j^\dagger \psi_j \right) \right]. \quad (43) \]

The many-body potential is not translationally invariant like \( A_{N+1} \)-type SRCMSM. Each particle interacts with the images of all other particles and also with itself. This kind of Hamiltonians are suitable for describing systems with boundaries. We choose to work in the \( 0 < x_1 < x_2 < \ldots < x_N \) sector of the phase space. Solutions in other sectors can be obtained by using the fact that the Hamiltonian is permutationally invariant under the combined exchange of \( x_i \) and \( \psi_i \). The Hamiltonian also has a very interesting discrete symmetry. It is invariant under any pair \( (x_i, \psi_i) \rightarrow (-x_i, -\psi_i) \). This reflection symmetry has a consequence on the spectrum.

The ground-state of (43) in the supersymmetric phase is given by,
\[ \Phi = \prod_{i < j} \left( x_i^2 - x_j^2 \right)^\lambda \prod_k x_k^{\lambda_1} e^{-\frac{1}{2} \sum_i x_i^2}, \quad (44) \]
with \( \lambda, \lambda_1 > 0 \). We would like to emphasize here that \( \Phi \) is normalizable for \( \lambda, \lambda_1 > -\frac{1}{2} \). However, a stronger criteria that each momentum operator \( p_i \) is self-adjoint for the wave-function of the form \( \Phi \) has been imposed. This requires \( \lambda \) and \( \lambda_1 \) to be positive definite.

The supersymmetry-breaking phase of the \( BC_{N+1} \)-type model has a richer structure than the \( A_{N+1} \)-type model. In the parameter space of \( \lambda \) and \( \lambda_1 \), there are three regions for which the supersymmetry is broken. They are, (i) \( \lambda < 0, \lambda_1 < 0 \), (ii) \( \lambda < 0, \lambda_1 > 0 \) and (iii) \( \lambda > 0, \lambda_1 < 0 \). We first discuss the spectrum in the supersymmetric phase in the next section. The spectrum in the supersymmetry-breaking phase will be discussed subsequently.
1. Supersymmetric phase

The complete spectrum of $H_{B_{N+1}}$ is described by a subset of the spectrum of super-oscillators,

$$E_{B_{N+1}} = 2(n + k + N_f), \quad E_{sho} = 2n + k + N_f.$$  \quad (45)

At a first thought, this observation might lead to a wrong conclusion regarding the validity of the similarity transformation for $B_{N+1}$-type SRCMSM. This apparent contradiction is removed once the discrete reflection symmetry of the $H_{B_{N+1}}$ is imposed on the eigen-functions of $H_2$. In particular, we have to choose,

$$
\hat{P}_{n,k} = \frac{1}{N_f!} r^{2n} \sum_{i_1, i_2, \ldots, i_{N_f}} f_{i_1 i_2 \ldots i_{N_f}} (x_1, x_2, \ldots, x_N) (x_{i_1} \psi^\dagger_{i_1})(x_{i_2} \psi^\dagger_{i_2}) \ldots (x_{i_{N_f}} \psi^\dagger_{i_{N_f}}),
$$  \quad (46)

where $f$ is anti-symmetric under the exchange of any two indices and a homogeneous function of degree 2$k$. Note that $\hat{P}_{n,k}$ is invariant under, (a) $(x_i, \psi^\dagger_i) \leftrightarrow (x_j, \psi^\dagger_j)$ and (b) $(x_i, \psi^\dagger_i) \rightarrow (-x_i, -\psi^\dagger_i)$. With this choice of $\hat{P}_{n,k}$, the eigen-value $E'_{sho}$ of $H_2$ is identical with $E_{B_{N+1}}$, $E_{B_{N+1}} = E'_{sho} = 2(n + k + N_f)$. Also, the action of $\hat{S}^m$ on $\hat{P}_{n,k}$ does not produce any singularity for positive $m$. Thus, $H_{B_{N+1}}$ is equivalent to a set of free super-half-oscillators. An explanation on the use of the term ‘super-half-oscillator’ is in order. Note that both $E_{B_{N+1}}$ and $E'_{sho}$ are always even for any integer $n$, $k$ and $N_f$. On the other hand, there is no such restriction on $E_{sho}$. It can be both even and odd. Thus, $E'_{sho}$ or $E_{BN+1}$ describes only half of the spectrum described by $E_{sho}$. This is because the eigen-value $E_{sho}$ is for $N$ super-oscillators defined on the full-line. On the contrary, the super-Hamiltonian $H_{B_{N+1}}$ is defined only on the positive half-line and, hence, the $E_{B_{N+1}}$ or $E'_{sho}$ corresponds to the eigen-value of a set of free super-oscillators on the half-line. Thus, in analogy with the similar problem for a single particle oscillator Hamiltonian, we use the term ‘super-half-oscillator’.
The eigen-spectrum also can be constructed in an algebraic way. We define the creation and annihilation operators as,

\[ B_n^+ = T^{-1} \sum_i b_i^{2^n} \mathcal{T}, \quad F_n^+ = T^{-1} \sum_i \psi_i \psi_i^{\dagger} b_i^{2^n-1} \mathcal{T}, \quad \mathcal{T} = e^{\hat{S} e^W}. \tag{47} \]

Note that these operators are invariant under (a) and (b). Thus, the eigen-functions obtained by operating these operators on the ground-state also are invariant under (a) and (b). The eigen-states are obtained as,

\[ \chi_{n_1 \ldots n_N \nu_1 \ldots \nu_N} = \prod_{k=1}^{N} B_k^{n_k} F_k^{\nu_k} \Phi, \tag{48} \]

with the energy \( E = \sum_{k=1}^{N} 2k(n_k + \nu_k) \). The bosonic quantum numbers are non-negative integers, while the fermionic quantum numbers are 0 or 1. Note that the energy \( E \) can be interpreted as that of \( N \) independent super-half-oscillators with the frequencies 1, 2, \ldots, \( N \).

2. Supersymmetry-breaking phase

The eigen-spectrum of the Hamiltonian in the region (i) can be obtained in a similar way as described in Sec. III.B. In particular, we construct a dual Hamiltonian \( H_{B_{N+1}}^d \) from \( H_{B_{N+1}} \) by the transformations, \( \psi_i \leftrightarrow \psi_i^{\dagger}, \lambda \rightarrow -\lambda \) and \( \lambda_1 \rightarrow -\lambda_1 \). The relation between these two Hamiltonians is given by,

\[ H_{B_{N+1}} = H_{B_{N+1}}^d + 2N_f - N \left[ 1 + 2\lambda(N - 1) + \lambda_1 \right]. \tag{49} \]

Using this relation, the complete eigen-spectrum of \( H_{B_{N+1}} \) in the supersymmetry-breaking phase can be obtained. In particular, the bosonic and fermionic creation operators can be obtained from (47) by replacing \( \lambda \rightarrow -\lambda, \lambda_1 \rightarrow -\lambda_1 \) and \( \psi_i \leftrightarrow \psi_i^{\dagger} \). These operators acting on the ground state of \( H_{B_{N+1}}^d \) produces the eigenstates of \( H_{B_{N+1}} \) with
the eigen-value, \( E = N(1 - 2\lambda(N - 1) - \lambda_1) + \sum_{k=1}^{N} 2(kn_k + (k - 1)\nu_k) \). This method does not work for the regions (ii) and (iii) in a straightforward way.

The method for constructing eigen-spectrum in the regions described by (ii) and (iii) are similar. We first study the Hamiltonian in the region (ii). The ground state wave-function in this region is given by,

\[
\psi(\lambda, \lambda_1) = e^{-\theta}|\bar{0}\rangle = \prod_{i<j} \left(x_i^2 - x_j^2\right)^{-\lambda} \prod_k x_k^{1+\lambda_1} e^{-\frac{1}{2} \sum_i x_i^2}|\bar{0}\rangle, \tag{50}
\]

with the ground-state energy \( E = \frac{3N^2}{2} - 2\lambda N(N - 1) \). It may be noted here that \( \psi(1-\lambda, \lambda_1-1) \) is also an exact eigenstate in the \( N_f = 0 \) sector. However, the associated energy eigen-value is greater than \( E \) for \( N \geq 3 \). We would also like to point out here that the particular form of \( \psi \) is due to the shape invariance of the model, relating \( N_f = 0 \) sector to \( N_f = N \) sector \cite{[10],[15]}. Now we introduce a new Hamiltonian \( H_3 \), which is related to \( H_{BN+1} \) by the following relation,

\[
H_{BN+1} = H_3 + 2N_f - \frac{N}{2} - 2\lambda N(N - 1). \tag{51}
\]

The above relation is similar to \((49)\). However, unlike \( H_{BN+1}^d \) or \( H_d \) for the \( A_{N+1} \)-type model, \( H_3 \) is not supersymmetric. Thus, we can not use the methods of supersymmetric theory to determine the ground-state energy of \( H_3 \). Instead, we find by inspection that \( \psi(\lambda, \lambda_1) \) is the zero-energy eigenstate of \( H_3 \). Now one can check easily,

\[
\tilde{H}_2 = e^{\hat{S}} e^{\theta} H_3 e^{-\theta} e^{\frac{-\hat{S}}{2}}, \tag{52}
\]

where \( \hat{S} \) can be calculated from \((39)\) for the choice of \( w \) as \( w = \ln G(-\lambda, \lambda_1, \lambda_2 = 0) \). The bosonic and fermionic creation operators can be obtained from \((17)\) by replacing \( \lambda \rightarrow -\lambda, \ \lambda_1 \rightarrow \lambda_1 \) and \( \psi_i \leftrightarrow \psi_i^\dagger \). These operators acting on \( \psi(\lambda, \lambda_1) \) produces the eigenstates of \( H_{BN+1} \) in region (ii) with the eigen-value, \( E = N\left(\frac{3}{2} - 2\lambda(N - 1)\right) + \sum_{k=1}^{N} 2(kn_k + (k - 1)\nu_k) \). Finally, we mention that the ground-state wave-function in the
region (iii) is $\psi(-1+\lambda, 1-\lambda_1)$ with the ground-state energy $E = \frac{2N}{2} - 2\lambda_1N(N-1)$. Rest of the analysis in this region can be done in a straightforward way.

V. SUMMARY AND DISCUSSIONS

We have constructed a similarity transformation which maps the SRCMSM Hamiltonian of $A_{N+1}$-type to that of a supersymmetric free harmonic oscillators. This equivalence is valid only in the supersymmetry-preserving phase of SRCMSM. We have outlined methods for the construction of eigen-functions of SRCMSM from the eigen-functions of super-oscillators. Even though there is no equivalence between SRCMSM in supersymmetry-breaking phase and super-oscillators, we are able to construct eigen-spectrum in this phase by using a duality property of the model. We observed that only those eigen functions of the free super-oscillators, which are symmetric under the combined exchange of both bosonic and fermionic coordinates, produce normalizable wave-function for the SRCMSM. This has also been observed in the pure bosonic case. Thus, this brings out the highly correlated nature of these systems.

We have generalized these results to a wide class of super-Hamiltonians whose bosonic many-body interaction is a homogeneous function of degree $-2$ and all the particles are subjected to a common harmonic confinement. These Hamiltonians are characterized by a dynamical $OSp(2|2)$ supersymmetry. Though this equivalence is certainly valid at the operator level, it turns out that the individual super-Hamiltonians should be analyzed carefully to see if the similarity transformation is keeping the Hamiltonian in its original Hilbert space or not. As a check to ascertain this, one should be able to construct the complete set of eigen-values and associated eigen functions of the original Hamiltonian from the super-oscillator model. We discussed the $BC_{N+1}$-type SRCMSM as an example and showed its equivalence to half of the spectrum of super-oscillators. To the best of
our knowledge, this is the first instance in the literature where the complete spectrum and the eigenstates of the $BC_{N+1}$-type SRCMSM has been obtained.

The SRCMSM associated with the root structures other than $A_{N+1}$ and $BC_{N+1}$ type have not been touched upon in this paper. We believe that permutationally symmetric super-oscillator basis with additional symmetry requirements coming from the specific nature of the root structure, as in the case of $BC_{N+1}$-type model, would produce the complete spectrum and associated well-behaved eigen-functions of these models. An universal formulation of the method described here, valid for SRCMSM associated with all the root structures along the line of investigations carried out in [11,17], is desirable.

The equivalence relation (41) is valid for a wide class of super-models realizing $OSp(2|2)$ supersymmetry. The CMS systems form only a small subset. We have seen that Eq. (41) act as a necessary condition for the equivalence between the original Hamiltonian and super-oscillators. The sufficient condition for the equivalence is to construct the complete spectrum and associated wave-functions of the original Hamiltonian from the super-oscillator basis. Thus, it would be interesting to construct new exactly solvable super-models using the method described here.

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APPENDIX A: \( OSP(2|2) \) SUPER-ALGEBRA

We show in this appendix that the Hamiltonian \( \mathcal{H} \) in (36) has dynamical \( OSp(2|2) \) supersymmetry. We first define the following four supercharges,

\[
q = \frac{1}{2} \sum_{i=1}^{N} \psi_i^\dagger (p_i + iw_i - ix_i), \quad q^\dagger = \frac{1}{2} \sum_{i=1}^{N} \psi_i (p_i - iw_i + ix_i),
\]

\[
\bar{q} = \frac{1}{2} \sum_{i=1}^{N} \psi_i (p_i - iw_i - ix_i), \quad \bar{q}^\dagger = \frac{1}{2} \sum_{i=1}^{N} \psi_i^\dagger (p_i + iw_i + ix_i) .
\]

(A1)

The Hamiltonian \( \mathcal{H} \) is given in terms of \( q \) and \( q^\dagger \) as,

\[
\mathcal{H} = 2\{ q, q^\dagger \},
\]

The dual Hamiltonian \( \mathcal{H}^d \) can be constructed in terms of \( \bar{q} \) and \( \bar{q}^\dagger \), \( \mathcal{H}^d = 2\{ \bar{q}, \bar{q}^\dagger \} \). We define the following operators,

\[
h = \frac{1}{2} \left( \mathcal{H} + \mathcal{H}^d \right), \quad U = \frac{1}{2} \left( \mathcal{H} - \mathcal{H}^d \right),
\]

\[
\mathcal{B}^-_2 = \mathcal{B}_0 - \frac{1}{4} \sum_i x_i^2 - \frac{1}{4} (N + 2E), ~ \mathcal{B}^+_2 = \mathcal{B}_0 - \frac{1}{4} \sum_i x_i^2 + \frac{1}{4} (N + 2E),
\]

\[
\mathcal{B}_0 = \frac{1}{4} \sum_i \left( p_i^2 + w_i^2 + 2 \sum_j w_{ij} \psi_i^\dagger \psi_j \right).
\]

(A2)

The bosonic operators \( \mathcal{B}^\pm_2 \) and \( h \) satisfies the following relations,

\[
[h, \mathcal{B}^\pm_2] = \pm 2\mathcal{B}^\pm_2, \quad [\mathcal{B}^-_2, \mathcal{B}^+_2] = h.
\]

(A3)

The commutator relation \( [E, \mathcal{B}_0] = -2\mathcal{B}_0 \) has been used in deriving the above equations.

The \( U(1) \) generator \( U \) commutes with \( \mathcal{B}^\pm_2 \) and \( h \).

The non-vanishing anticommutators among \( q, q^\dagger, \bar{q} \) and \( \bar{q}^\dagger \) are,

\[
\{ q, q^\dagger \} = \frac{1}{2} (h + U), \quad \{ \bar{q}, \bar{q}^\dagger \} = \frac{1}{2} (h - U), \quad \{ q^\dagger, q^\dagger \} = \mathcal{B}^+_2, \quad \{ q, \bar{q} \} = \mathcal{B}^-_2 .
\]

(A4)

Observe that the relation,

\[
\mathcal{H} = \mathcal{H}^d + 2U,
\]

(A5)
which is useful in determining the spectrum in the supersymmetry-breaking phase, follows easily from the first two equations of (A4).

The other non-vanishing commutators are,

\[
    [\mathcal{B}_2^+, q] = -\tilde{q}^\dagger, \quad [\mathcal{B}_2^+, \tilde{q}] = -q^\dagger, \quad [\mathcal{B}_2^-, \tilde{q}^\dagger] = q, \quad [\mathcal{B}_2^-, q^\dagger] = \tilde{q},
\]

\[
    [h, q^\dagger] = q^\dagger, \quad [h, q] = -q, \quad [h, \tilde{q}] = -\tilde{q}, \quad [h, \tilde{q}^\dagger] = \tilde{q}^\dagger,
\]

\[
    [U, \tilde{q}] = -\tilde{q}, \quad [U, \tilde{q}^\dagger] = q^\dagger, \quad [U, q^\dagger] = -q^\dagger, \quad [U, q] = q.
\] (A6)
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