Linear or linearizable first-order delay ordinary differential equations and their Lie point symmetries

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Abstract
A recent article was devoted to an analysis of the symmetry properties of a class of first-order delay ordinary differential systems (DODSs). Here we concentrate on linear DODSs, which have infinite-dimensional Lie point symmetry groups due to the linear superposition principle. Their symmetry algebra always contains a two-dimensional subalgebra realized by linearly connected vector fields. We identify all classes of linear first-order DODSs that have additional symmetries, not due to linearity alone, and we present representatives of each class. These additional symmetries are then used to construct exact analytical particular solutions using symmetry reduction.

Keywords: Lie point symmetry, Lie group classification, delay ordinary differential equations

1. Introduction
The recent article [1] was devoted to the Lie group classification of first-order delay ordinary differential systems (DODSs). For motivation and a brief survey of the field we refer to [1] and the references therein.

The considered DODS consists of a pair of equations
\[ \dot{y} = f(x, y, y_-), \quad x_- = g(x, y, y_-), \quad x \in I, \quad \frac{\partial f}{\partial y_-}(x, y, y_-) \neq 0, \quad x_- < x, \quad g(x, y, y_-) \neq \text{const}, \]

where \( I \subset \mathbb{R} \) is some finite or semifinite interval. The functions \( f \) and \( g \) are locally smooth in some domain \( \Omega \subset \mathbb{R}^3 \). For the first equation in equation (1.1), we have to specify the delay point \( x_- \) where the delayed function value \( y_- = y(x_-) \) is taken, otherwise the problem is not fully determined.

In [1], use was made of the classification of the Lie algebras of vector fields in two real variables under the local diffeomorphisms presented in [2]. For each representative finite dimensional Lie algebra we calculated a basis for all group invariants in the space \( \{ x, y, x_-, y_, \dot{y} \} \). The invariants were then used to construct all DODSs of the form (1.1) with a symmetry algebra \( L \) satisfying \( \dim L \geq 1 \).

A similar approach to the construction of invariant differential delay equations was applied earlier in [3], with the restriction that the delay should be constant (and should satisfy \( x - x_- > 0 \)). The novelty of the approach used in the present article and our previous article [1] is that we allow the delay parameter \( x_- \) to be variable and transformable. Instead of constructing a differential delay equation out of group invariants we construct a differential delay system. The delay step is not assumed to be constant, nor is it a priori prescribed. It is obtained as part of the solution of the DODS. So far we have restricted ourselves to first-order ordinary DODSs, as in (1.1), but the generalization to higher order DODSs and to systems of equations with delay is immediate.

We are interested in DODSs which are invariant with respect to the point symmetry groups. Such groups consist of transformations generated by the vector fields of the same form as in the case of ordinary differential equations [4–8], namely

\[ X_\alpha = \xi_\alpha(x, y) \frac{\partial}{\partial x} + \eta_\alpha(x, y) \frac{\partial}{\partial y}, \quad \alpha = 1, ..., n. \]  

(1.2)

The prolongation of these vector fields acting on the system (1.1) has the form

\[ \text{pr}X_\alpha = \xi_\alpha \frac{\partial}{\partial x} + \eta_\alpha \frac{\partial}{\partial y} + \xi^-_\alpha \frac{\partial}{\partial x_-} + \eta^-_\alpha \frac{\partial}{\partial y_-} + \zeta_\alpha \frac{\partial}{\partial \dot{y}} \]  

(1.3)

with

\[ \xi_\alpha = \xi_\alpha(x, y), \quad \eta_\alpha = \eta_\alpha(x, y), \]

\[ \xi^-_\alpha = \xi_\alpha(x_-, y_-), \quad \eta^-_\alpha = \eta_\alpha(x_-, y_-), \]

\[ \zeta_\alpha(x, y, \dot{y}) = D(\eta_\alpha) - \dot{\gamma}D(\xi_\alpha), \]

where \( D \) is the total derivative operator. The operators (1.3) combine the prolongation for shifted discrete variables \( (x_-, y_-) \) [9–11] with standard prolongation for the derivative \( \dot{y} \) [4, 5].

In [1] the main emphasis was on nonlinear DODSs, on their Lie point symmetry algebras, and on the exact solutions obtained by symmetry reduction. In particular, it was shown that genuinely nonlinear DODSs can have symmetry algebras of dimension \( \dim L = n \) with \( n = 0, 1, 2, 3 \). By ‘genuinely nonlinear’ we mean nonlinear DODSs that cannot be linearized by a point transformation.
Two significant results concerning linear DODSs were also obtained in [1], namely the following. We consider the most general linear DODE of the form (1.1) with the solution independent delay point \(x_-\):

\[
\dot{y} = \alpha(x)y + \beta(x)y_- + \gamma(x), \quad x_- = g(x),
\]

where \(\alpha(x), \beta(x), \gamma(x)\) and \(g(x)\) are arbitrary real functions, satisfying

\[
\beta(x) \neq 0, \quad g(x) < x, \quad g(x) \neq \text{const},
\]

smooth in some interval \(x \in I\).

From now on we use the term \textit{linear DODS} for systems of the form (1.4). The symmetry algebra of (1.4) is infinite-dimensional for all functions \(\alpha(x), \beta(x), \gamma(x),\) and \(g(x)\). The reason for this result is quite simple. In order to solve the DODS (1.1) (in particular (1.4)) we must give some initial conditions (see [12, 13]). In contrast to the case of ordinary differential equations, the initial condition must be given by a function \(\varphi(x)\) on an initial interval \(I_0 \subset \mathbb{R}\), e.g.

\[
y(x) = \varphi(x), \quad x \in [x_-, x_0].
\]

Here it is assumed that \(x_- = g(x_0)\).

For linear DODSs the freedom in the choice of the initial function \(\varphi(x)\) is reflected in a linear superposition formula. This is formulated as a theorem of [1]: The linear DODS (1.4) admits an infinite-dimensional Lie algebra represented by the vector field:

\[
X(\rho) = \rho(x) \frac{\partial}{\partial y}, \quad Y(\sigma) = (y - \sigma(x)) \frac{\partial}{\partial y},
\]

where \(\rho(x)\) is the general solution of the homogeneous DODS obtained by putting \(\gamma(x) = 0\) in (1.4) and \(\sigma(x)\) is any chosen particular solution of the inhomogeneous DODS (1.4). The fact that (1.6) is a symmetry algebra of the DODS (1.4) is a consequence of linearity alone. We shall call this algebra \(\mathbb{A}_\infty\). As we shall see below, for some special cases of (1.4) the symmetry algebra can be larger.

The infinite-dimensional Lie algebra of (1.4) always contains a two-dimensional subalgebra realized by linearly connected vector fields. Two possibilities exist:

1. The solvable non-nilpotent algebra, a basis of which can be transformed into \(\mathbb{A}_{2,1}\):

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}.
\]

The invariant DODS is homogeneous and given by

\[
\dot{y} = f(x) \frac{\Delta y}{\Delta x}, \quad x_- = g(x), \quad g(x) < x, \quad g(x) \neq \text{const},
\]

where

\[
\Delta x = x - x_-, \quad \Delta y = y - y_-.
\]

We assume that \(f(x) \neq 0\).

2. An Abelian Lie algebra \(\mathbb{A}_{2,3}\), a basis of which can be transformed into

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}.
\]
The invariant DODE is inhomogeneous and the DODS can be presented as

\[ \dot{y} = \frac{\Delta y}{\Delta x} + f(x), \quad x_- = g(x), \quad g(x) < x, \quad g(x) \neq \text{const.} \quad (1.11) \]

We assume that \( f(x) \neq 0 \), as otherwise (1.11) is a special case of (1.8).

**Comment.** If we know at least one solution \( \sigma_0(x) \) of the inhomogeneous equation (1.11), then we can transform it into a homogeneous DODS by the transformation \( \bar{x} = x, \bar{y} = y - \sigma_0(x) \). However, it is not assumed that such a solution \( \sigma_0(x) \) is known.

**Comment.** To explain the notations we note that two more inequivalent two-dimensional algebras of the vector field exist, represented by

\[ A_{2,2} : \quad X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}; \]

\[ A_{2,4} : \quad X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}. \quad (1.12) \]

The vector fields \( X_1 \) and \( X_2 \) are linearly nonconnected, and therefore the invariant DODS is nonlinear and cannot be transformed into a linear one.

The purpose of this article is to provide a classification of linear DODSs of the form (1.4), and to obtain solutions which are invariant under some one-dimensional subalgebras. To do this we analyze the finite-dimensional subalgebras of the infinite-dimensional symmetry algebras of linear DODSs. It turns out that it is sufficient to consider dimensions \( 2 \leq n \leq 4 \) to fully characterize a class of equations. We already know that the finite-dimensional subalgebras always contain the algebra (1.7) or (1.10), or possibly both. The presence of further vector fields simply imposes restrictions on the functions \( f(x) \) and \( g(x) \) in (1.8) and (1.11).

The article is organized as follows. In section 2 we describe how invariant DODSs are constructed for given symmetry algebras. We also show how symmetries can be used to find particular (namely, invariant) solutions of DODSs. Section 3 is devoted to invariant linear DODSs and their invariant solutions. In section 4 we consider a special DODE, which appears often in the classification. Finally, the conclusions are presented in section 5 and the results are summed up in table A.2.

### 2. Construction of the invariant linear DODSs and their invariant solutions

In section 3 below, we shall run through the list of Lie algebras of the vector fields [1] in two variables, concentrating on the Lie algebras excluded from [1]. These are the Lie algebras containing algebras (1.7) and (1.10) as subalgebras. The invariant DODS thus obtained is linear as in (1.8) or (1.11).

The invariant DODSs are obtained in the form

\[ F_1(I_1, ..., I_k) = 0, \quad F_2(I_1, ..., I_k) = 0, \quad (2.1) \]

\[ \det \left( \frac{\partial(F_1, F_2)}{\partial(y, x_-)} \right) \neq 0, \quad (2.2) \]

where \( I_1, ..., I_k \) are invariants of the considered Lie group. These can be ‘strong invariants’
\[ prX_\alpha F_a(x, y, x-, y-, \dot{y}) = 0, \quad \alpha = 1, \ldots, n, \quad a = 1, 2, \quad (2.3) \]

or ‘weak invariants’ satisfying

\[ \left. prX_\alpha F_a(x, y, x-, y-, \dot{y}) \right|_{F_1 = F_2 = 0} = 0, \quad \alpha = 1, \ldots, n, \quad a = 1, 2. \quad (2.4) \]

Weak invariants are only invariant on the manifold \( F_1 = F_2 = 0 \), i.e. on the solutions of the linear DODS (2.1).

In view of the Jacobian condition (2.2) it is always possible to solve the system (2.1) for \( \dot{y} \) and \( x- \) and hence transform (2.1) into the form (1.1). Generally, the solution can be implicit. It was shown in [1] that the results are explicit for most nonlinear DODSs. Since this article is devoted to the linear case instead of the general (1.1), we actually obtain one of the two special cases (1.8) or (1.11). Once the equations are obtained we search for exact solutions—the so-called group invariant solutions—using symmetry reduction.

In the case of the DODS (1.1) it is sufficient to consider one-dimensional subalgebras. All of them have the form

\[ X = \sum_{\alpha=1}^{n} c_\alpha X_\alpha, \quad c_\alpha \in \mathbb{R}, \quad (2.5) \]

where \( c_\alpha \) are constants and \( X_\alpha \) are of the form (1.2) and are elements of the symmetry algebra \( L \) of the considered DODS.

The method consists of several steps.

1. Construct a representative list of one-dimensional subalgebras \( L_i \) of the symmetry algebra \( L \) of the DODS. The subalgebras are classified under the group of inner automorphisms \( G = \exp L \).

2. For each subalgebra in the list calculate the invariants of the subgroup \( G_i = \exp L_i \) in the four-dimensional space with local coordinates \( \{ x, y, x-, y- \} \). There are three functionally independent invariants. For the method to be applicable, two of the invariants must depend on two variables only, namely \( (x, y) \) and \( (x-, y-) \) respectively. We denote three invariants \( J_1(x, y), J_2(x-, y-) \) and \( J_3(x, y, x-, y-) \). We make two of them equal to the constants, namely:

\[ J_1(x, y) = A, \quad J_3(x, y, x-, y-) = B. \quad (2.6) \]

They must satisfy the Jacobian condition

\[ \det \left( \frac{\partial(J_1, J_3)}{\partial(y, x-)} \right) \neq 0 \quad (2.7) \]

(we have \( J_2(x-, y-) = J_1(x-, y-) = A \), i.e. \( J_2(x-, y-) \) is obtained by shifting \( J_1(x, y) \) to \( (x-, y-) \)). All elements of the Lie algebra have the form (2.5). A necessary condition for invariants of the form (2.6) to exist is that at least one of the vector fields in \( L_i \) satisfies

\[ \xi(x, y) \neq 0. \quad (2.8) \]

Only vector fields in the representative list satisfying (2.8) will provide invariant solutions.

3. Solve equation (2.6) to obtain the reduction formulas

\[ y = h(x, A), \quad x- = k(x, A, B) \quad (2.9) \]

(we also have \( y_- = h(x-, A) \)).
4. Substitute the reduction formulas (2.9) into the DODS (1.1) and require that the equations are satisfied identically. This provides relations which define the constants and therefore determine the functions $h$ and $k$. It may also impose constraints on the functions $f(x, y, y_\cdot)$ and $g(x, y, y_\cdot)$ in (1.1), and on the values of parameters that may figure in the subalgebras used in the reduction. Once the relations are satisfied, the invariant solution is given by (2.9).

5. An additional step, which is not implemented in the present paper. Apply the entire group $G_i = \exp L_i$ to the obtained invariant solutions. This can provide a more general solution depending on up to three more parameters (corresponding to the factor algebra $L/L_i$). Check whether the DODS imposes further constraints involving the new parameters.

3. Invariant DODSs and invariant solutions

In this section we rely on realizations of three- and four-dimensional Lie algebras by real vector fields of the form (1.2), which are used in [1] and introduced in [2]. They are given in table A.1. For the present paper we select only realizations which contain two-dimensional subalgebras of linearly connected vector fields. Complete tables of all algebras can be found in [1].

We run through the list of representative algebras and for each of them construct all group invariants in the space with the local coordinates \( \{x, y, x_\cdot, y_\cdot, \dot{y}\} \). We construct both strong invariants (invariant in the entire space) and weak invariants (invariant on some manifolds). The invariants will be used to write the invariant DODSs; for convenience we use the notations (1.9).

We also look for invariant solutions as described in the preceding section, and the representative optimal systems of one-dimensional subalgebras are taken from [15].

3.1. Dimension 3

\( \text{A}_{3,1} \): The nilpotent Lie algebra \( \mathfrak{n}_{3,1} \) can be realized as

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x}.
\]

(3.1)

A basis of the invariants is given by

\[
I_1 = \Delta x, \quad I_2 = \dot{y} - \frac{\Delta y}{\Delta x}.
\]

This provides us with an invariant DODS

\[
\dot{y} = \frac{\Delta y}{\Delta x} + C_1, \quad \Delta x = C_2, \quad C_2 > 0.
\]

(3.2)

We now show how to find invariant solutions following the procedure given in section 2:

Step 1. The representative algebra of one-dimensional subalgebras (see [15]) is

\[
\{X_1\}, \quad \{\cos(\varphi)X_2 + \sin(\varphi)X_3\}.
\]

Here and below \( 0 \leq \varphi < \pi \). Due to the condition \( \xi(x, y) \neq 0 \), invariant solutions can only exist for the second element with \( \sin(\varphi) \neq 0 \). In this case, we can rewrite this operator as \( aX_2 + X_3 \). Here and in the following cases \( -\infty < a < \infty \), unless another interval is given.
Step 2. The invariants for $aX_2 + X_3$ in the space $\{x, y, x-, y-\}$ are

$$J_1 = y - \frac{a}{2}x^2 = A, \quad J_3 = x - x_- = B.$$ 

Step 3. The reduction formulas take the form

$$y = \frac{a}{2}x^2 + A, \quad x_- = x - B.$$ \hspace{1cm} (3.3)

Step 4. The substitution in (3.2) gives restrictions

$$\frac{a}{2}B = C_1, \quad B = C_2 \neq 0.$$ 

Thus, for $a = 2\frac{\Delta x}{\Delta y}$ there is an invariant solution (3.3) with an arbitrary $A$ and $B = C_2$. The invariant DODS (3.2) depends on two constants: $C_1$ and $C_2$. The solution (3.3) depends on three constants: one of them ($A$) is free, the other two—$a$ and $B$—are expressed in terms of $C_1$ and $C_2$.

For the following cases we employ the same steps while providing fewer details.

A3.3**: The algebra $a_{3,1}$ has two realizations [1]. One of them, namely

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = (1 - a)x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad 0 < |a| \leqslant 1,$$ \hspace{1cm} (3.4)

contains linearly connected vector fields (the nilradical $\{X_1, X_2\}$).

We consider two subcases:

• $a \neq 1$

A basis of the invariants is

$$I_1 = \frac{x_-}{x}, \quad I_2 = |x|^{\frac{\Delta y}{\Delta x}} \left( \frac{\Delta y}{\Delta x} - \frac{\Delta x}{\Delta y} \right)$$

and the general invariant DODS can be written as

$$\dot{y} = \frac{\Delta y}{\Delta x} + C_1 |x|^{\frac{\Delta y}{\Delta x}}, \quad x_- = C_2x, \quad (1 - C_2)x > 0.$$ \hspace{1cm} (3.5)

The representative list of one-dimensional subalgebras has four elements:

$$\{X_1\}, \quad \{X_2\}, \quad \{X_1 \pm X_2\}, \quad \{X_3\}.$$ 

Invariant solutions exist only for the last element. For $X_3$ we find invariant solutions in the form

$$y = A|x|^{\frac{\Delta y}{\Delta x}}, \quad x_- = Bx,$$ \hspace{1cm} (3.6)

with constants $A$ and $B$ given by the system

$$A \left( \frac{1}{1 - a} - \frac{1 - |C_2|^{\frac{\Delta y}{\Delta x}}}{1 - C_2} \right) = C_1 \text{sgn} \,(x), \quad B = C_2,$$

where

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\[
\text{sgn}(x) = \begin{cases} 
1, & x > 0; \\
0, & x = 0; \\
-1, & x < 0.
\end{cases}
\]

• \(a = 1\)

There are two invariants

\[ I_1 = x, \quad I_2 = x_- \]

and an invariant manifold

\[ \dot{y} - \frac{\Delta y}{\Delta x} = 0. \]

We obtain the invariant DODS

\[ \dot{y} = \frac{\Delta y}{\Delta x}, \quad x_- = g(x), \quad g(x) < x, \quad g(x) \neq \text{const.} \quad (3.7) \]

In this case, there are no invariant solutions because there are no symmetries satisfying \(\xi(x,y) \neq 0\).

\[ A_{3,5}: \] The solvable algebra \(s_{3,2}\) has two realizations, and one of them, namely

\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{x}{\partial y}, \quad X_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (3.8) \]

contains the subalgebra \(A_{2,3}\). A basis of the invariants is

\[ I_1 = \Delta x, \quad I_2 = e^{-x} \left( \dot{y} - \frac{\Delta y}{\Delta x} \right). \]

We find the general invariant DODS

\[ \dot{y} = \frac{\Delta y}{\Delta x} + C_1 e^x, \quad \Delta x = C_2, \quad C_2 > 0. \quad (3.9) \]

The optimal system of one-dimensional subalgebras is

\[ \{X_1\}, \quad \{X_2\}, \quad \{X_3\} \]

and invariant solutions exist only for \(\{X_3\}\). We obtain the solution

\[ y = A e^x, \quad x_- = x - B, \quad (3.10) \]

for constants \(A\) and \(B\) satisfying the system

\[ A = \frac{C_1 C_2}{C_2 - 1 + e^{-C_1}}, \quad B = C_2. \]

\[ A_{3,7}^b: \] There are two realizations of the algebra \(s_{3,3}\), and one of them contains the subalgebra \(A_{2,3}\). This realization is

\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{x}{\partial y}, \quad X_3 = (1 + x^2) \frac{\partial}{\partial x} + (x + b) y \frac{\partial}{\partial y}, \quad b \geq 0. \quad (3.11) \]

A basis of the invariants
\[ I_1 = \frac{x-x_-}{1+xx_-}, \quad I_2 = \sqrt{1+x^2}e^{-b \arctan(x)} \left( \frac{\Delta y}{\Delta x} \right) \]

provides us with the most general invariant DODS
\[ \dot{y} = \frac{\Delta y}{\Delta x} + C_1 \frac{e^{b \arctan(x)}}{\sqrt{1+x^2}}, \quad x_- = \frac{x-C_2}{1+C_2^2}, \quad \frac{C_2}{1+C_2^2} > 0. \]  

(3.12)

The representative list of subalgebras is \{X_2\}, \{X_3\}; invariant solutions exist only for \(X_3\). We obtain solutions of the form
\[ y = A \sqrt{1+x^2}e^{b \arctan x}, \quad x_- = \frac{x-B}{1+Bx} \]  

(3.13)

with constants \(A\) and \(B\) defined by the system
\[ A \left( b - \frac{1}{C_2} + \text{sgn} (1+C_2^2) \frac{\sqrt{1+C_2^2}}{C_2} e^{-b \arctan C_2} \right) = C_1, \quad B = C_2. \]

\(A_{3,11}\): There are four realizations of the algebra \(sl(2, \mathbb{R})\). Only one of them contains linearly connected vector fields. It can be presented as
\[ X_1 = \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial y}, \quad X_3 = y^2 \frac{\partial}{\partial y}. \]  

(3.14)

Note that all operators are linearly connected. There are two invariants
\[ I_1 = \Delta x, \quad I_2 = x_- \]

and two invariant manifolds
\[ y - y_- = 0 \quad \text{and} \quad \dot{y} = 0. \]

In this case there is no invariant DODS.

\(A_{3,12}\): There exist two realizations of the decomposable algebra \(\mathfrak{n}_{1,1} \oplus \mathfrak{g}_{2,1}\) and both of them contain linearly connected vector fields. The first realization can be taken as
\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial y}. \]  

(3.15)

We find the invariants
\[ I_1 = \Delta x, \quad I_2 = \frac{\Delta x}{\Delta y} \dot{y} \]

and obtain the invariant DODS
\[ \dot{y} = C_1 \frac{\Delta y}{\Delta x}, \quad \Delta x = C_2, \quad C_2 > 0. \]  

(3.16)

The optimal system
\[ \{X_2\}, \quad \{X_1 \pm X_2\}, \quad \{\sin(\varphi)X_1 + \cos(\varphi)X_3\} \]

contains two elements for which we can find invariant solutions:
For \( X_1 \pm X_2 \) we find the invariant solutions

\[ y = \pm x + A, \quad x_- = x - B, \]

(3.17)

only if \( C_1 = 1 \). In this case \( A \) is arbitrary and \( B = C_2 \).

For \( X_1 + aX_3 \) (which is \( \sin(\varphi)X_1 + \cos(\varphi)X_3 \)) with \( \sin(\varphi) \neq 0 \) we find the solution

\[ y = Ae^{ax}, \quad x_- = x - B, \]

(3.18)

with constants \( A \) and \( B \) satisfying

\[ Aa = C_1A \frac{1 - e^{-ac_2}}{C_2}, \quad B = C_2. \]

If \( a \) satisfies

\[ a = C_1 \frac{1 - e^{-ac_2}}{C_2}, \]

we can take arbitrary \( A \) and \( B = C_2 \). If not, we get only the trivial solution \( y = 0 \), \( x_- = x - C_2 \).

\( A_{3,14} \): The second realization of the algebra \( \mathfrak{n}_{1,1} \oplus \mathfrak{g}_{2,1} \) is

\[ X_1 = x \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \]

(3.19)

We obtain the invariants

\[ I_1 = \frac{x_-}{x}, \quad I_2 = \frac{y}{\Delta x} \]

and invariant DODS

\[ \dot{y} = \frac{\Delta y}{\Delta x} + C_1, \quad x_- = C_2x, \quad (1 - C_2)x > 0. \]

(3.20)

The representative list of one-dimensional subalgebras is

\{ \( X_2 \), \( X_1 \pm X_2 \), \{ \( \sin(\varphi)X_1 + \cos(\varphi)X_3 \) \} \}.

There can only be invariant solutions for \( \sin(\varphi)X_1 + \cos(\varphi)X_3 \) with \( \cos(\varphi) \neq 0 \). We rewrite these operators as \( aX_1 + X_3 \). The invariant solutions have the form

\[ y = ax \ln |x| + Ax, \quad x_- = Bx, \]

(3.21)

with

\[ a \left( 1 + \frac{C_2 \ln |C_2|}{1 - C_2} \right) = C_1, \quad B = C_2. \]

(3.22)

Thus, invariant solutions only exist for \( a \) satisfying (3.22). In this case, \( A \) is arbitrary and \( B = C_2 \).

\( A_{3,15} \): For the Abelian algebra \( 3\mathfrak{n}_{1,1} \) we get the following realization:

\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \chi(x) \frac{\partial}{\partial y}, \quad \ddot{\chi}(x) \neq 0. \]

(3.23)

We remark that all operators are linearly connected.
There are two invariants
\[ I_1 = x, \quad I_2 = x_- \]
and an invariant manifold
\[ \dot{y} - \frac{\Delta y}{\Delta x} = 0 \]
provided that
\[ \dot{\chi}(x) = \frac{\chi(x) - \chi(x_-)}{x - x_-}. \]

Thus, we can present the most general invariant DODSs as
\[ \dot{y} = \frac{\Delta y}{\Delta x} + f(x), \quad \dot{\chi}(x) = \frac{\chi(x) - \chi(x_-)}{x - x_-}, \tag{3.24} \]
where \( f(x) \) is an arbitrary function. Note that the second equation is the delay relation which defines \( x_- \) as an implicit function of \( x \). Only solutions satisfying \( x_- < x \) will provide a DODS.

The symmetries (3.23) provide no invariant solutions.

To sum up, for three-dimensional Lie algebras containing \( A_{2,1} \) or \( A_{2,3} \) as subalgebras, we have presented all the corresponding DODSs and each class is represented by a linear DODS. For \( A_{3,1}, A_{3,3}^a \) with \( a \neq 1 \), \( A_{3,5}, A_{3,7}, A_{3,13}, A_{3,14} \), their DODSs depend on two or three parameters. \( A_{3,13}^a \) with \( a = 1 \) provides the DODS (3.7) which depends on one function \( g(x) \). For \( A_{3,15} \) the DODS involves a function \( \chi(x) \), which defines the delay relation (3.24). All obtained invariant solutions are quite explicit and correspond to very specific invariant conditions. The constants in the solutions are expressed in terms of those in the DODS, or may take arbitrary values.

### 3.2. Dimension 4

In this section, we go through the realizations of four-dimensional Lie algebras given in table A.1. For most of these realizations there are no invariant DODSs. We present only the four realizations which provide us with invariant DODSs.

\( A_{4,5} \): One of the realizations of the solvable algebra \( s_{4,3} \) is given by the operators
\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \chi(x) \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial y}, \quad \dot{\chi}(x) \neq 0. \tag{3.25}
\]
All four vector fields are linearly connected. There are two invariants
\[ I_1 = x, \quad I_2 = x_- \]
and an invariant manifold
\[ \dot{y} - \frac{\Delta y}{\Delta x} = 0 \]
provided that
\[ \dot{\chi}(x) = \frac{\chi(x) - \chi(x_-)}{x - x_-}. \]
Thus we get the invariant DODS
\[ \dot{y} = \frac{\Delta y}{\Delta x}, \quad \dot{\chi}(x) = \frac{\chi(x) - \chi(x_-)}{x - x_-}, \quad (3.26) \]

where the last equation is the delay relation, which defines \( x_- \) as a function of \( x \). Only solutions satisfying \( x_- < x \) will provide a DODS.

There are no invariant solutions for symmetries (3.25) since they do not satisfy the condition (2.8).

**A\(_{4,12}\):** One of two realizations of algebra \( a_{4,11} \) is given by the operators

\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = y \frac{\partial}{\partial y}. \quad (3.27) \]

There is one invariant

\[ I = \Delta x \]

and one invariant manifold

\[ \dot{y} - \frac{\Delta y}{\Delta x} = 0. \]

We obtain the most general invariant DODS

\[ \dot{y} = \frac{\Delta y}{\Delta x}, \quad \Delta x = C, \quad C > 0. \quad (3.28) \]

The representative list of one-dimensional subalgebras consists of five algebras

\{\( X_1 \), \( X_2 \), \( X_3 \), \( X_2 \pm X_3 \), \( aX_3 + X_4 \)\}.

The invariant solution can be found for the following three:

- The invariant solution for \( X_3 \) is

\[ y = A, \quad x_- = x - B \quad (3.29) \]

with arbitrary \( A \) and \( B = C \).
- For \( X_2 \pm X_3 \) the invariant solutions have the form

\[ y = \pm \frac{x^2}{2} + A, \quad x_- = x - B. \quad (3.30) \]

In this case, invariant solutions do not exist because the obtained conditions \( B = 0 \) and \( B = C \) are not compatible with \( C > 0 \).
- For \( aX_3 + X_4 \) (only for \( a \neq 0 \)) the invariant solutions have the form

\[ y = Ae^{x/a}, \quad x_- = x - B \quad (3.31) \]

with the restrictions

\[ A \left( \frac{1}{a} - \frac{1 - e^{-C/a}}{C} \right) = 0, \quad B = C. \]

Since there are no real numbers \( a \neq 0 \) and \( C > 0 \) satisfying

\[ \frac{1}{a} = \frac{1 - e^{-C/a}}{C} \]
we get only the trivial solution
\[ y = 0, \quad x_\sigma = x - C. \]

\[ \textbf{A}_{4,14}: \] Algebra \(a_{4,12}\) has a realization
\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial y}, \quad X_4 = (1 + x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \]  (3.32)

We find one invariant
\[ I = \frac{x - x_\sigma}{1 + xx_\sigma}, \]
and an invariant manifold
\[ \dot{y} - \frac{\Delta y}{\Delta x} = 0. \]

We obtain the most general invariant DODS
\[ \dot{y} = \frac{\Delta y}{\Delta x}, \quad x_\sigma = \frac{x - C}{1 + Cx}, \quad \frac{C}{1 + Cx} > 0. \]  (3.33)

The optimal system of subalgebras contains three elements
\[ \{X_1\}, \quad \{X_3\}, \quad \{aX_3 + X_4\} \]
and invariant solutions exist only for the element \(aX_3 + X_4\). They have the form
\[ y = A \sqrt{1 + x^2} e^{a \arctan x}, \quad x_\sigma = \frac{x - B}{1 + Bx} \]  (3.34)
providing that \(a, A\) and \(B\) satisfy the restrictions
\[ A \left( a - \frac{1}{C} + \text{sgn} (1 + Cx) \frac{\sqrt{1 + C^2}}{C} e^{-a \arctan C} \right) = 0, \quad B = C. \]

If \(a\) and \(C\) satisfy
\[ a - \frac{1}{C} + \text{sgn} (1 + Cx) \frac{\sqrt{1 + C^2}}{C} e^{-a \arctan C} = 0, \]
we obtain the invariant solution (3.34) with arbitrary \(A\) and \(B = C\). If not, we only get the trivial invariant solution \(A = 0\) (i.e. \(y = 0\)) and \(B = C\).

\[ \textbf{A}_{4,21}: \] For the decomposable algebra \(a_{2,1}\) there is a realization
\[ X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial y}, \quad X_4 = x \frac{\partial}{\partial x}. \]  (3.35)

We find one invariant
\[ I = \frac{x_\sigma}{x} \]
and the invariant manifold
\[ \dot{y} - \frac{\Delta y}{\Delta x} = 0. \]

These provide us with the most general invariant DODS
\[ \dot{y} = \frac{\Delta y}{\Delta x}, \quad x_+ = Cx, \quad (1 - C)x > 0. \]  

(3.36)

For convenience, we chose a simpler basis for the realization (3.35) as

\[ Y_1 = x \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y}, \quad Y_3 = x \frac{\partial}{\partial y}, \quad Y_4 = y \frac{\partial}{\partial y}. \]  

(3.37)

Then we obtain the following optimal system of one-dimensional subalgebras

\{Y_1\}, \quad \{Y_2\}, \quad \{Y_1 \pm Y_2\}; \quad \{Y_3\}, \quad \{Y_2 \pm Y_3\}, \quad \{aY_1 + Y_4\}.

We find the following invariant solutions:

- Invariant solutions for \( Y_1 \) have the form
  \[ y = A, \quad x_+ = Bx, \]  
  with arbitrary \( A \) and \( B = C \).
- For \( Y_1 \pm Y_2 \) we look for solutions
  \[ y = \pm \ln |x| + A, \quad x_+ = Bx \]  
  and get conditions
  \[ \frac{\ln |C|}{C - 1} = 1, \quad B = C. \]
  The invariant solutions exist only if \( \ln |C| = C - 1 \). In this case \( A \) can be arbitrary.
- For \( aY_1 + Y_4 \) (only for \( a \neq 0 \)) we look for invariant solutions in the form
  \[ y = A|x|^{1/a}, \quad x_+ = Bx \]  
  and get the restrictions
  \[ A \left( \frac{1}{a} - 1 - \frac{|C|^{1/a}}{1 - C} \right) = 0, \quad B = C. \]
  If \( a \) and \( C \) satisfy the relation
  \[ \frac{1}{a} - 1 - \frac{|C|^{1/a}}{1 - C} = 0, \]  
  we obtain invariant solutions (3.40) with arbitrary \( A \) and \( B = C \). If not, we only get the trivial invariant solution \( y = 0, \quad x_+ = Cx \).

3.3. Higher dimensional cases

Direct computation shows that for realizations of Lie algebras of dimension \( n \geq 5 \) by vector fields in a plane, we do not get new cases of DODSs. However, some cases that are determined by Lie algebras of dimension \( 2 \leq n \leq 4 \) can also be invariant under higher dimensional finite Lie algebras. This happens because linear DODEs with solution-independent delay relations admit infinite-dimensional symmetry groups that may contain finite-dimensional subalgebras of a higher dimension.
For example, the DODS (3.28) admits infinitely many symmetries of the form
\[ X = e^{\lambda x} \frac{\partial}{\partial y}, \]
where \( \lambda \) is a solution of the characteristic equation
\[ \lambda = \frac{1 - e^{-\lambda C}}{C}. \]
This equation can be rewritten as
\[ 1 + z = e^z \]
for \( z = -\lambda C \). Except for the operator \( \frac{\partial}{\partial y} (\lambda = 0) \), the symmetries (3.41) are complex, but we can obtain real symmetries taking real and imaginary parts of these complex symmetries. Some subsets of such symmetries together with symmetries (3.27) are higher dimensional realizations of finite-dimensional Lie algebras. Let us consider a particular example.

**Example 3.1.** Equations (3.28) are specified by the four-dimensional symmetry algebra (3.27). The same equations also allow the six-dimensional symmetry algebra
\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial y}, \\
X_5 = e^{a x} \cos(b x) \frac{\partial}{\partial y}, \quad X_6 = e^{a x} \sin(b x) \frac{\partial}{\partial y},
\]
where \( \lambda = a + ib \neq 0 \) is a solution of (3.42).

Since we do not get new DODSs for Lie algebras of dimension \( n \geq 5 \) we conclude that we have listed all linear DODSs which can be specified by finite-dimensional Lie symmetry algebras. The larger symmetry algebras may, however, provide new invariant solutions. For instance, in example 3.1 one can find particular solutions which are invariant for the subalgebras \( X_1 + \gamma X_5 \).

**Remark 3.1.** It is interesting to note that equation (3.43) can be represented with the help of Bernoulli numbers (see [16, 17])
\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1.
\]

4. Comments on the special case of a linear DODE

In this section we show how the integration procedure is applied to a particular DODS. We consider the delayed differential equation
\[
\dot{y} = \frac{\Delta y}{\Delta x}
\]
for the delay relation
\[
x_- = qx - \tau.
\]
where \( q > 0 \) and \( \tau \geq 0 \) are constants such that \( \tau^2 + (q - 1)^2 \neq 0 \). This delay relation can be considered as a generalization of the constant delay \( \Delta x = \tau \) (for \( q = 1 \)) and \( q \)-delay relation \( x_\tau = qx \) (for \( \tau = 0 \)). Note that relation (4.2) is equivalent to delay relation
\[
\Delta x = (1 - q)x + \tau. \tag{4.3}
\]

### 4.1. Symmetries

The DODS admits the symmetries
\[
X_1 = ((q - 1)x - \tau) \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y}, \quad X_\alpha = \alpha(x) \frac{\partial}{\partial y}, \tag{4.4}
\]
where \( \alpha(x) \) is an arbitrary solution of the DODE (4.1) with the delay relation (4.2). Generally, solutions \( \alpha(x) \) are piecewise-smooth.

**Remark 4.1.** This symmetry algebra includes a subalgebra with smooth coefficients given by the operators
\[
X_1 = ((q - 1)x - \tau) \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = x \frac{\partial}{\partial y}. \tag{4.5}
\]

If we require invariance with respect to these symmetries, we obtain the DODS (4.1) and (4.2). Operators (4.5) are equivalent to operators (3.27) representing algebra \( A_{4,12} \) if \( q = 1 \) and \( \tau \neq 0 \) and to operators (3.35) representing algebra \( A_{4,21} \) if \( q \neq 1 \) (in the first case we can simply divide \( X_1 \) by \( \tau \), in the second we can divide \( X_1 \) by \( q - 1 \) and then remove \( \tau \) by the translation \( x \rightarrow x - \tau/(q - 1) \)). Therefore, one can transform the invariant solutions of cases \( A_{4,12} \) and \( A_{4,21} \) into the invariant solutions of DODS (4.1) and (4.2).

### 4.2. Sequences of intervals

Consider several cases of the delay relation (4.2) in detail. Solving equation (4.2) for the initial interval \([x_{-1}, x_0]\), such that \( x_{-1} = qx_0 - \tau < x_0 \), we can obtain the sequence of intervals explicitly.

1. **If** \( q = 1 \), we consider the delay relation
\[
\Delta x = \tau, \tag{4.6}
\]
which is invariant with respect to the symmetry
\[
X_1 = \frac{\partial}{\partial x}.
\]

We obtain the points
\[
x_n = x_0 + n\tau, \quad \tau = x_0 - x_{-1}, \quad n = -1, 0, 1, 2, \ldots \tag{4.7}
\]

2. **If** \( q \neq 1 \), we obtain the points
\[
x_n = \frac{x_0}{1 - q} \left( \frac{1}{q^n} - q \right) + \frac{x_{-1}}{q - 1} \left( \frac{1}{q^n} - 1 \right) = \frac{x_0}{q^n} + \frac{\tau}{1 - q} \left( \frac{1}{q^n} - 1 \right), \quad n = -1, 0, 1, 2, \ldots \tag{4.8}
\]
Note that
\[ x_n \rightarrow \begin{cases} 
\infty, & 0 < q < 1 \\
\tau^{\frac{1}{q-1}}, & q > 1 
\end{cases} \] as \( n \rightarrow \infty \).

In the particular case \([x_{-1}, x_0] = [\tau, 0]\) we get
\[ x_n = \frac{\tau}{1-q} \left( 1 - q^n \right), \quad n = -1, 0, 1, 2, ... \]

3. For \( 0 < q < 1, \tau = 0 \) we obtain a particular subcase of case two. We get the \( q \)-delay relation
\[ x_n = q x, \quad (4.9) \]
which is invariant with respect to the symmetry operator
\[ X_1 = x \frac{\partial}{\partial x}. \]

For \( 0 < x_{-1} < x_0 \) this relation can be solved as
\[ x_n = \frac{x_0}{q^n}, \quad q = \frac{x_{-1}}{x_0}, \quad n = -1, 0, 1, 2, ... \quad (4.10) \]

4.3. Integration

Let us show how the DODE \((4.1)\) can be explicitly solved by the method of steps \([12, 13]\) starting from the initial condition
\[ y(x) = \varphi(x), \quad x \in [x_{-1}, x_0]. \]

We need to consider different cases of the delay relation \((4.2)\).

1. \( q = 1 \).

The delay relation provides points \((4.7)\). The recursive relation
\[ y(x) = y(x_n) e^{\frac{x-x_n}{\tau}} - \frac{1}{\tau} e^{\frac{x}{\tau}} \int_{x_n}^{x} e^{-\frac{u}{\tau}} y(u - \tau) du \]
allows us to find the solution on the interval \([x_n, x_{n+1}]\) using the solution from the previous interval \([x_{n-1}, x_n]\).

2. \( q \neq 1 \).

The delay relation gives points \((4.8)\). We find the recursive relation
\[ y(x) = y(x_n) \left( \frac{(1-q)x_n + \tau}{(1-q)x + \tau} \right)^{\frac{1}{q-1}} \]
\[ = \left( \frac{1}{(1-q)x + \tau} \right)^{\frac{1}{q-1}} \int_{x_n}^{x} ((1-q)u + \tau)^{\frac{2-q}{q-1}} y(qu - \tau) du, \]
where \( \tau = q x_0 - x_{-1} \).

3. \( 0 < q < 1 \) and \( \tau = 0 \) (subcase of case two).

The delay relation gives points (4.10). We find the recursive relation

\[
y(x) = y(x_n) \left( \frac{x_n}{x} \right)^{1/\alpha} - \frac{1}{(1 - q) x^{1/\alpha}} \int_{x_n}^x u^{1/\alpha} y(u) du,
\]

where \( q = \frac{x_{-1}}{x_0} \).

### 4.4. Symmetries with piecewise-smooth coefficients

The simplest linear DODE (4.1) with a constant delay parameter

\[
\dot{y} = \frac{y - y_{-}}{x - x_{-}}, \quad x - x_{-} = \text{const}, \tag{4.11}
\]

is close to the second-order ODE \( \ddot{y} = 0 \) in the sense of approximation order. Indeed, substituting the Taylor series into (4.11), we obtain

\[
\frac{\tau}{2} \left( \dot{y} - \frac{\tau}{3} \ddot{y} + \ldots \right) = 0.
\]

We suppose that \( x - x_{-} = \tau = \text{const} \ll 1 \).

Equation \( \ddot{y} = 0 \) has a general solution \( y = Ax + B \), which is a special solution of (4.11). Meanwhile (4.11) has infinitely many nonsmooth solutions. In particular, one can verify that it has the following solution in a point \( x_0 \)

\[
y(x) = (x - x_0 + \tau)^2, \quad x_0 - \tau \leq x \leq x_0
\]

\[
y(x) = -\tau^2 e^{(x - x_0)/\tau} + (x - x_0 + \tau)^2 + \tau^2, \quad x_0 \leq x \leq x_0 + \tau.
\]

Equation (4.11) possesses, in particular, the following symmetry

\[
X = \alpha(x) \frac{\partial}{\partial y}, \quad \dot{\alpha}(x) = \frac{\alpha(x) - \alpha(x_{-})}{x - x_{-}}. \tag{4.12}
\]

Substituting the solution given above into the Lie equations, we can present the following transformation group acting in the neighborhood of point \( x_0 \):

\[
y^*(x) = y + (x - x_0 + \tau)^2 a, \quad x^* = x, \quad x_0 - \tau \leq x \leq x_0
\]

\[
y^*(x) = y + \left( -\tau^2 e^{(x - x_0)/\tau} + (x - x_0 + \tau)^2 + \tau^2 \right) a, \quad x^* = x, \quad x_0 \leq x \leq x_0 + \tau.
\]

where \( a \) is a group parameter. Thus, the transformation exists and it is defined by the nonsmooth function, which has a brake in a first derivative at the point \( x_0 \).

It should be emphasized that these symmetries are not classical because they are defined by piecewise-smooth functions.

### 5. Conclusions

In this paper we considered first-order delay ordinary differential equations which admit two linearly connected symmetries. Consideration of a DODE requires an additional equation which specifies the delayed argument \( x_{-} \). This equation is called the delay relation. A
DODE and a delay relation form a delay ordinary differential system. The obtained DODEs are all linear with a delay equation of the form $x_→ = g(x)$.

We provided a classification of all first-order DODSs specified by two-, three- and four-dimensional Lie algebras, which contain two linearly connected symmetry operators. These results are presented in table A.2. It was noted that higher dimensional Lie algebras do not provide new DODSs. However, the infinite-dimensional symmetry algebra of a DODS specified by a finite-dimensional subalgebra of dimension $2 \leq n \leq 4$ may contain finite-dimensional subalgebras of arbitrary higher dimension.

As an application of the symmetry properties, we have constructed exact analytic solutions of the obtained DODSs with the help of symmetry reduction. Such solutions, which are called invariant solutions, were found using the representative lists of one-dimensional subalgebras.

One point to emphasize is the role of the infinite-dimensional symmetry algebra $A_∞$ with the basis (1.6). Similarly to the case of PDEs [7], the existence of such an infinite-dimensional symmetry algebra for a nonlinear DODS indicates that this DODE is linearizable by a point transformation, which is an important result.

On the other hand, the symmetry algebra (1.6) alone does not provide any invariant solutions, simply because it does not contain any elements of the form (1.2) with $ξ(x, y) \not\equiv 0$. In some cases the symmetry algebra $A_∞$ can be extended to a larger algebra $\tilde{A}_∞$ with $A_∞$ as an ideal (at least locally, in intervals where the derivatives of the functions $ρ(x)$ and $σ(x)$ in (1.6) exist). It was shown in [1], theorem 5.4, that at most one element with $ξ \not\equiv 0$ can be added to the basis of $A_∞$. However, this can only be done if the functions $α(x)$, $β(x)$ and $g(x)$ in (1.4) satisfy a compatibility condition given in [1]. The additional element has the form

$$Z = ξ(x) \frac{∂}{∂x} + (A(x)y + B(x)) \frac{∂}{∂y}, \quad ξ(x) \not\equiv 0. \quad (5.1)$$

Invariant solutions are obtained using elements of the form

$$Z + pX, \quad X \text{ contained in } A_∞, \quad p = \text{const.} \quad (5.2)$$

All such cases up to equivalence are listed in section 3. For instance, we have $Z = ∂/∂x$ in $A_3,1$, $A_3,13$ and $A_4,12$, $Z = (1 - a)x∂/∂x + y∂/∂y$ for $A_4,13$, $Z = ∂/∂x + y∂/∂y$ for $A_3,5$, etc.

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Appendix

Table A.1. Lie symmetry algebras and their realizations containing two linearly connected vector fields.

In column 1 we give the isomorphism class using the notations of [14]. Thus $n_{i,k}$ denotes the $k$th nilpotent Lie algebra of dimension $i$ in the list. The only nilpotent algebras in table A.1 are $n_{1,1}$, $n_{3,1}$ and $n_{4,1}$. Similarly, $s_{i,k}$ is the $k$th solvable Lie algebra of dimension $i$ in the list. The simple Lie algebra $sl(2, R)$ is identified by its usual name. In column 2 $A_{i,k}$ runs through all algebras in the list of subalgebras of $\text{diff}(2, R)$ and $i$ is again the dimension of the algebra. The numbers in brackets correspond to the notations used in the list of [2]. In column 3 we give vector fields spanning each representative algebra.
Indecomposable Lie algebras precede the decomposable ones (like $2n_{1,1}$ or $n_{1,1} \oplus s_{2,1}$) in the list. Isomorphic Lie algebras can be realized in more than one manner by vector fields. For nilpotent Lie algebras, elements of the derived algebra precede a semicolon, e.g. $X_1, X_2$ in $n_{4,1}$. For solvable Lie algebras, the nilradical precedes a semicolon, e.g. $X_1, X_2, X_3$ in $s_{4,6}$.

### Dimension 2

| Lie algebra | Case   | Operators                  |
|-------------|--------|----------------------------|
| $s_{2,1}$   | $A_{2,1}(10)$ | $X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}$ |
| $2n_{1,1}$ | $A_{2,3}(20)$ | $\left\{ X_1 = \frac{\partial}{\partial y} \right\}, \quad \left\{ X_2 = x \frac{\partial}{\partial y} \right\}$ |

### Dimension 3

| Lie algebra | Case   | Operators                  |
|-------------|--------|----------------------------|
| $n_{3,1}$   | $A_{3,1}(22)$ | $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x}$ |
| $s_{3,1}$   | $A_{3,5}^{1,2,2}(21,22)$ | $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = (1 - a)x \frac{\partial}{\partial y} + y \frac{\partial}{\partial y}, \quad 0 < |a| \leq 1$ |
| $s_{3,2}$   | $A_{3,5}(22)$ | $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial y} + y \frac{\partial}{\partial y}$ |
| $s_{3,3}$   | $A_{3,5}^{3,2}(22)$ | $X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = (1 + x^2) \frac{\partial}{\partial y} + (x + b)y \frac{\partial}{\partial y}, \quad b \geq 0$ |
| $sl(2,\mathbb{R})$ | $A_{3,11}(11)$ | $X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}, \quad X_3 = y^2 \frac{\partial}{\partial y}$ |
| $n_{1,1} \oplus s_{2,1}$ | $A_{3,13}(23)$ | $\left\{ X_1 = \frac{\partial}{\partial y} \right\}, \quad \left\{ X_2 = \frac{\partial}{\partial y} \right\}, \quad X_3 = y \frac{\partial}{\partial y}$ |
| $A_{3,14}(22)$ | $\left\{ X_1 = x \frac{\partial}{\partial y} \right\}, \quad \left\{ X_2 = \frac{\partial}{\partial y} \right\}, \quad X_3 = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial y}$ |
| $3n_{1,1}$ | $A_{3,15}(20)$ | $\left\{ X_1 = \frac{\partial}{\partial y} \right\}, \quad \left\{ X_2 = x \frac{\partial}{\partial y} \right\}, \quad \left\{ X_3 = \chi(x) \frac{\partial}{\partial y} \right\}, \quad \ddot{\chi}(x) \neq 0$ |
## Dimension 4

| Lie algebra | Case | Operators |
|-------------|------|-----------|
| $\mathfrak{n}_{4,1}$ | $A_{4,1}(22)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = x^2 \frac{\partial}{\partial y}$, $X_4 = \frac{\partial}{\partial x}$ |
| $\mathfrak{g}_{4,1}$ | $A_{4,2}(22)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = e \frac{\partial}{\partial y}$, $X_4 = \frac{\partial}{\partial x}$ |
| $\mathfrak{g}_{4,2}$ | $A_{4,3}(22)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = x^2 \frac{\partial}{\partial y}$, $X_4 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ |
| $\mathfrak{g}_{4,3}$ | $A_{4,4}(22)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = x^2 \frac{\partial}{\partial y}$, $X_4 = (1 - a)x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, $a \in [-1, 0) \cup (0, 1)$, $\alpha \neq \{0, 1\}$ |
| | | (see [14] for additional restrictions on $a$ and $\alpha$) |
| | $A_{4,5}(21)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = \chi(x) \frac{\partial}{\partial y}$, $X_4 = y \frac{\partial}{\partial y}$, $\chi(x) \neq 0$ |
| $\mathfrak{g}_{4,4}$ | $A_{4,6}^{\alpha}(22)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = e^{\alpha x} \frac{\partial}{\partial y}$, $X_4 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, $\alpha \neq 0, 1$ |
| $\mathfrak{g}_{4,5}$ | $A_{4,7}^{\alpha, \beta}(22)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = e^{\alpha x} \cos(\beta x) \frac{\partial}{\partial y}$, $X_3 = e^{\alpha x} \sin(\beta x) \frac{\partial}{\partial y}$, $X_4 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, $\beta \neq 0$ |
| $\mathfrak{g}_{4,6}$ | $A_{4,8}(24)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = x \frac{\partial}{\partial y}$, $X_4 = x \frac{\partial}{\partial x}$ |
| $\mathfrak{g}_{4,8}$ | $A_{4,9}^{\alpha}(24)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = x \frac{\partial}{\partial y}$, $X_4 = x \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}$, $\alpha \neq 0, 1$ |
| $\mathfrak{g}_{4,10}$ | $A_{4,10}(25)$ | $X_1 = \frac{\partial}{\partial y}$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = x \frac{\partial}{\partial y}$, $X_4 = x \frac{\partial}{\partial x} + (2y + x^2) \frac{\partial}{\partial y}$ |
| Lie algebra | Case | Operators |
|------------|------|-----------|
| $\mathfrak{s}_{4,11}$ | $\mathbf{A}_{4,11}(24)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y}$, $X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ |
| | $\mathbf{A}_{4,12}(23)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = x \frac{\partial}{\partial x}$, $X_3 = y \frac{\partial}{\partial y}$, $X_4 = y \frac{\partial}{\partial y}$ |
| $\mathfrak{s}_{4,12}$ | $\mathbf{A}_{4,14}(23)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = x \frac{\partial}{\partial x}$, $X_3 = y \frac{\partial}{\partial y}$, $X_4 = (1 + x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$ |
| $\mathfrak{n}_{1,1} \oplus \mathfrak{s}_{3,1}$ | $\mathbf{A}_{4,15}(22)$ | $X_1 = x |\frac{\partial}{\partial x}|$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = x \frac{\partial}{\partial x}$, $X_4 = (1 - a)x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, $a \in [-1, 0) \cup (0, 1)$ |
| $\mathfrak{n}_{1,1} \oplus \mathfrak{s}_{3,2}$ | $\mathbf{A}_{4,16}(22)$ | $X_1 = e^a \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = x \frac{\partial}{\partial x}$, $X_4 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ |
| $\mathfrak{n}_{1,1} \oplus \mathfrak{s}_{3,3}$ | $\mathbf{A}_{4,17}(22)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = e^{\alpha x} \cos x \frac{\partial}{\partial y}$, $X_4 = e^{\alpha x} \sin x \frac{\partial}{\partial y}$, $\alpha \geq 0$ |
| $\mathfrak{n}_{1,1} \oplus \mathfrak{s}(2, \mathbb{R})$ | $\mathbf{A}_{4,18}(14)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y}$, $X_3 = y \frac{\partial}{\partial x}$, $X_4 = y^2 \frac{\partial}{\partial y}$ |
| | $\mathbf{A}_{4,19}(19)$ | $X_1 = x \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y}$, $X_3 = x \frac{\partial}{\partial y}$, $y \frac{\partial}{\partial y}$, $X_4 = 2xy \frac{\partial}{\partial y} + y^2 \frac{\partial}{\partial y}$ |
| $2\mathfrak{s}_{2,1}$ | $\mathbf{A}_{4,20}(13)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = x \frac{\partial}{\partial x}$, $X_3 = \frac{\partial}{\partial y}$, $X_4 = y \frac{\partial}{\partial y}$ |
| | $\mathbf{A}_{4,21}(23)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, $X_3 = x \frac{\partial}{\partial x}$, $X_4 = x \frac{\partial}{\partial x}$ |
| $4\mathfrak{n}_{1,1}$ | $\mathbf{A}_{4,22}(20)$ | $X_1 = \frac{\partial}{\partial x}$, $X_2 = x \frac{\partial}{\partial x}$, $X_3 = x_1(x) \frac{\partial}{\partial x}$, $X_4 = x_2(x) \frac{\partial}{\partial x}$, $1$, $x$, $x_1(x)$ and $x_2(x)$ are linearly independent |

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Table A.2. Invariant linear DODSs: linear DODEs and delay relations which do not depend on the solutions.

**Dimension 2**

| Case  | DODE                      | Delay relation |
|-------|----------------------------|----------------|
| \( A_{2,1} \) | \( \dot{y} = f(x) \frac{\Delta y}{\Delta x} \) | \( x_- = g(x) \) |
| \( A_{2,3} \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + f(x) \) | \( x_- = g(x) \) |

**Dimension 3**

| Case  | DODE                      | Delay relation |
|-------|----------------------------|----------------|
| \( A_{3,1} \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + C_1 \) | \( \Delta x = C_2 \) |
| \( A_{3,3}^a \) \( a \neq 1 \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + C_1 |x|^a \) | \( x_- = C_2 x \) |
| \( A_{3,3}^a \) \( a = 1 \) | \( \dot{y} = \frac{\Delta y}{\Delta x} \) | \( x_- = g(x) \) |
| \( A_{3,5} \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + C_1 e^t \) | \( \Delta x = C_2 \) |
| \( A_{3,7} \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + C_1 \frac{\sin(\Delta x)}{\sqrt{1+\Delta x^2}} \) | \( x_- = \frac{x-C_2}{1+\Delta x^2} \) |
| \( A_{3,11} \) | No DODE                   |                |
| \( A_{3,13} \) | \( \dot{y} = C_1 \frac{\Delta y}{\Delta x} \) | \( \Delta x = C_2 \) |
| \( A_{3,14} \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + C_1 \) | \( x_- = C_2 x \) |
| \( A_{3,15} \) | \( \dot{y} = \frac{\Delta y}{\Delta x} + f(x) \) | \( \dot{x}(x) = \frac{\chi(x)-\chi(x_-)}{x-x_-} \) |
**Dimension 4**

| Case | DODE | Delay relation |
|------|------|----------------|
| $A_{4,5}$ | $\dot{y} = \frac{\Delta y}{\Delta t}$ | $\dot{\chi}(x) = \frac{\chi(x) - \chi(x_-)}{x-x_-}$ |
| $A_{4,12}$ | $\dot{y} = \frac{\Delta y}{\Delta t}$ | $\Delta x = C$ |
| $A_{4,14}$ | $\dot{y} = \frac{\Delta y}{\Delta t}$ | $x_- = \frac{x-C}{1+Cx}$ |
| $A_{4,21}$ | $\dot{y} = \frac{\Delta y}{\Delta t}$ | $x_- = Cx$ |

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