The Heston stochastic volatility model has a boundary trace at zero volatility

Bénédicte Alziary · Peter Takáč

Received: 13 August 2021 / Accepted: 1 December 2022 / Published online: 6 January 2023
© The Author(s) 2023

Abstract
We establish boundary regularity results in Hölder spaces for the degenerate parabolic problem obtained from the Heston stochastic volatility model in Mathematical Finance set up in the spatial domain (upper half-plane) \( \mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2 \). Starting with nonsmooth initial data \( u_0 \in H \), we take advantage of smoothing properties of the parabolic semigroup \( e^{-tA} : H \to H \), \( t \in \mathbb{R}_+ \), generated by the Heston model, to derive the smoothness of the solution \( u(t) = e^{-tA}u_0 \) for all \( t > 0 \). The existence and uniqueness of a weak solution is obtained in a Hilbert space \( H = L^2(\mathbb{H}; w) \) with very weak growth restrictions at infinity and on the boundary \( \partial \mathbb{H} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2 \) of the half-plane \( \mathbb{H} \). We investigate the influence of the boundary behavior of the initial data \( u_0 \in H \) on the boundary behavior of \( u(t) \) for \( t > 0 \).

Keywords Degenerate parabolic equation · Weighted Sobolev space · Holomorphic semigroup · Parabolic smoothing effect · Dynamic boundary conditions · Heston’s stochastic volatility model

Mathematics Subject Classification Primary 35B65 · 35K65 · Secondary 35K15 · 91G80

1 Introduction

The Heston stochastic volatility model for pricing the European call options on stocks (Heston [25]) has been of considerable interest to economists and mathematicians for almost three decades. Numerous articles have been written about mathematical treatment and solvability of this model in a number of settings. In our present work we focus on the degenerate
parabolic problem with prescribed initial and boundary conditions. The question of existence, uniqueness, and regularity of a weak solution to this problem is studied in Feehan and Pop [15], Chiarella et al. [7], Meyer [31], and Alziary and Takač [3, Sect. 4, pp. 16–17], to mention only a few. The analyticity of the solution in both, space and time variables, has been established in [3, Sect. 4, Theorem 4.2, pp. 16–17]. As a consequence, the completeness of the market (cf. Björk [5, Sect. 8, pp. 115–124] and Davis and Obloj [8]) described in Heston’s model is verified in [3, Sect. 5, Theorem 5.2, p. 19]. Thanks to the importance of Heston’s model in Mathematical Finance, there is a strong interest in efficient numerical methods applicable to computing the solution of this degenerate parabolic problem (cf. [7, 31]). A major obstacle to an efficient numerical method is the degeneracy of the diffusion coefficient at low volatility; see e.g. Düring and Fournié [12] and Ikonen and Toivanen [26]. This degeneracy causes serious problems in formulating and justifying the correct boundary conditions on the portion of the boundary with vanishing volatility, denoted by \( \partial \mathbb{H} \). A numerical scheme using a finite difference method in the domain \( \mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2 \) with the boundary \( \partial \mathbb{H} = \mathbb{R} \times \{0\} \) has to be designed with a mesh of points much too fine near the boundary \( \partial \mathbb{H} \), so that it finally becomes rather inefficient and unprecise there. This is one of the reasons why in this article we investigate the limiting boundary behavior of the solution of Heston’s model as the volatility approaches zero. We obtain a limiting partial differential equation of first order on the boundary \( \partial \mathbb{H} \), Eq. (4.4), thus specifying also the boundary conditions on \( \partial \mathbb{H} \).

It is worth of noticing that this equation on the space-time domain \( \partial \mathbb{H} \times (0, \infty) \) is coupled with the degenerate parabolic equation (3.9) inside the domain \( \mathbb{H} \times (0, \infty) \) solely through a linear term with the partial derivative with respect to the volatility (the volatility approaching zero) that appears in Eq. (4.5). This feature of Heston’s model is used in the recent work by Baustian et al. [4] with an orthogonal polynomial expansion in the spatial domain \( \mathbb{H} \). Orthogonal polynomial expansions have been used recently also in Ackerer and Filipović [1] for numerical approximations. Earlier, the authors [3, Sect. 11, pp. 48–51] have used orthogonal polynomial expansions with Hermite and Laguerre polynomials in Galérkin’s method to approximate functions in \( L^2(\mathbb{H}) \) by analytic functions.

Our derivation of Eq. (4.4) on \( \partial \mathbb{H} \times (0, \infty) \) is motivated by the limiting behavior of the diffusion part (second-order partial derivatives) in Eq. (3.9). The limit, equal to zero on \( \partial \mathbb{H} \times (0, \infty) \), has been obtained in Feehan and Pop [15], Lemma 3.1, Eq. (3.1), on p. 4409 (see also Daskalopoulos and Hamilton [11], Prop. I.12.1 on p. 940) for the corresponding (stationary) elliptic problem with the Heston operator \( \mathcal{A} \) given by Eq. (3.1). However, in order to fulfill the regularity hypothesis required in [15, Lemma 3.1], we need to establish a new regularity result for the weak solution \( u(\cdot, \cdot, t) : [0, \infty) \rightarrow H \) of the Heston model (see Proposition 4.1 with \( f \equiv 0 \)) which is given by the \( C^0 \)-semigroup of bounded linear operators \( e^{-t\mathcal{A}} : H \rightarrow H \), \( t \in \mathbb{R}_+ \), determined by the homogeneous initial value problem (3.9), that is to say, \( u(\cdot, \cdot, t) \equiv u(t) = e^{-t\mathcal{A}}u_0 \in H \), \( t \in \mathbb{R}_+ \), with an arbitrary initial value \( u_0 \in H \). The underlying Hilbert space \( H \) is a weighted \( L^2 \)-type Lebesgue space \( H = L^2(\mathbb{H}; w) \). Our regularity result is based on the smoothing property of the holomorphic semigroup \( e^{-t\mathcal{A}} \), \( t \in \mathbb{R}_+ \), acting on \( H \), see Theorem 4.2. This result contains a number of local and global partial regularity results which are new, as well. We stress the main difference between the classical Hölder-type regularity treated in Feehan and Pop [15, Theorem 1.1 on p. 4409] and the regularity obtained by parabolic smoothing: The Hölder-type regularity in [15] assumes the same spatial regularity already for the initial value \( u(0) = u_0 \) (in a suitable weighted Hölder space). As a consequence, analogous regularity for the solution \( u(t) \) is proved (by Schauder estimates) at all times \( t \in (0, T) \) in a bounded time interval. In contrast, we begin with nonsmooth initial data \( u_0 \in H \) at \( t = 0 \); then we apply the parabolic smoothing of the \( C^0 \)-semigroup \( e^{-t\mathcal{A}} \) for \( t \in (0, \infty) \), thus arriving at \( u(t) \in \mathcal{D}(\mathcal{A}^k) \subset H \) for all \( t \in (0, \infty) \).
and every \( k = 1, 2, 3, \ldots \). Since the domain \( D(A^k) \) of the \( k \)-th power of the Heston operator \( A \) is the image (range) of the \( k \)-th power of the bounded inverse \( (\lambda I + A)^{-1} : H \to H \) (the resolvent of \( -A \)), the solution \( u(t) = e^{-tA}u_0 \) has higher smoothness for all \( t > 0 \). This smoothing effect is essential for applications in Mathematical Finance where the initial data \( u_0 \in H \) are typically not continuously differentiable \( (u_0 \in W^{1,\infty}(\mathbb{H}) \setminus C^1(\mathbb{H})) \). Indeed, for our derivation of the limiting equation (4.4) on \( \partial \mathbb{H} \times (0, \infty) \) from equation (3.9) we need Hölder regularity of type \( C^{2+\alpha} \) over the closure of the open half-plane \( \mathbb{H} \) (cf. [15, Lemma 3.1]).

The proof of our main result, Theorem 4.2, makes use of the factorization \( (\lambda I + A)^{-j}(\lambda I + A)^k e^{-tA} \) of the bounded linear operator \( e^{-tA} \) for \( t > 0 \); with \( k = 1, 2, 3 \) and \( j = 0, 1, \ldots, k \). Thanks to the smoothing effect, the latter factor, \( (\lambda I + A)^k e^{-tA} \), is a bounded linear operator on \( H \) for each \( t > 0 \), whereas the former factor, \( (\lambda I + A)^{-j} \), is a bounded linear operator from \( H \) to the domain \( D(A^j) \) of the \( j \)-th power of the Heston operator \( A \). We use the resolvent \( (\lambda I + A)^{-j} : H \to H \) of \( -A \) in order to describe the function space \( D(A^j) \) (endowed with the graph norm) for \( 1 \leq j \leq k \leq 3 \). This factorization (in Sect. 6) is split into three consecutive steps in Paragraphs §6.1, §6.2, and §6.3, with the auxiliary functions \( f_{j,k}(\cdot, \cdot, \cdot, \cdot, t) \equiv f_{j,k}(t) \in H \) for \( 0 \leq j \leq k \leq 3 \) defined in eqs. (6.1) and (6.2) for a given \( u_0 \in H \) and \( t > 0 \). Clearly, for \( j = k; k = 1, 2, 3 \), and \( t > 0 \) we obtain \( f_{k,k}(t) = u(t) \) which yields the desired regularity of the solution \( u(t) \) for \( t > 0 \) as stated in Theorem 4.2.

Our second theorem (Theorem 4.4) is a weak maximum principle for the initial value Cauchy problem (3.9) in the unbounded space-time domain \( \mathbb{H} \times (0, T) \). As it is typical for parabolic problems posed in an unbounded spatial domain (the open half-plane \( \mathbb{H} \subset \mathbb{R}^2 \) in our case), the growth of the solution \( u(t) \equiv u(x, \xi, t) \) has to be limited with respect to the space variable \( (x, \xi) \in \mathbb{H} \) as \( x \to \pm \infty \) or \( \xi \to 0+ \) or \( \xi \to +\infty \), uniformly for all \( t \in (0, T) \). We find a positive “majorizing” function \( h_0 : \mathbb{H} \to (0, \infty) \) in Eq. (4.7) that provides the required limit on the solution \( u(x, \xi, t) \) in Theorem 4.4. This theorem has an important corollary applicable to a typical initial value problem in Mathematical Finance (see Corollary 4.5). The majorizing function \( U(x, \xi) \equiv K_1 e^{x+\sigma \xi} + K_0 \), for all \( (x, \xi) \in \mathbb{H} \), provides an important upper bound (independent from time) as the volatility \( \xi \in (0, \infty) \) approaches zero \( (\xi \to 0+) \). Here, \( K_0, K_1 \in \mathbb{R}_+ \) are arbitrary constants, and \( \sigma \in \mathbb{R} \) is another constant restricted by inequalities in (4.13). This choice of the majorizing function and the initial data covers the most typical alternatives for derivative contracts (which are determined by the choice of the initial data \( u_0 \)); see e.g. Fouque et al. [20, §1.2, pp. 8–12]. The case of \( u_0(x, \xi) \equiv u_0(x) \) being independent from the volatility \( \xi \in (0, \infty) \) is of special interest (e.g., European call and put options); we may set \( \sigma = 0 \). Derivative contracts do not seem to include the volatility level since volatility does not produce any direct returns such as dividends or interest. Volatility does not show long term upwards trends like equities, but typically shows periods of high volatility occurring within a short period of time (i.e. volatility “jumps”) and then shows a downward trend to return to the long run medium level.

This article is organized as follows. We begin with basic notations and function spaces of Hölder, Lebesgue, and Sobolev types, which involve weights. Most of these spaces were originally introduced in Daskalopoulos and Feehan [9] and [10, Sect. 2, p. 5048] and Feehan and Pop [17]. The mathematical problem resulting from Heston’s [25] model in Mathematical Finance (described in Appendix A in “economic” terms) is formulated in Sect. 3. The details of this formulation, especially a justification of the boundary conditions and restrictions imposed on some important constants (e.g., the volatility \( \sigma > 0 \) of the volatility, the rate of mean reversion \( \kappa > 0 \), and the long-term variance \( \theta > 0 \)), such as the well-known Feller
condition, can be found in our previous work [3, Sect. 2, pp. 6–13]. Our main results are collected in Sect. 4, in Theorems 4.2 and 4.4. In addition, also Proposition 4.1 (existence and uniqueness), Corollary 4.3 (boundary behavior), and Corollary 4.5 (maximum principle) are of importance. Our strategy of the proof of Theorem 4.2 (laid out above) is described in all details in Sect. 5. The first part of this strategy, obtaining Hölder regularity, is implemented in Sect. 6. The proof of Theorem 4.2 (and that of Corollary 4.3, as well) is completed in Sect. 7. The main part of this article ends up with the proofs of Theorem 4.4 and Corollary 4.5 in Sect. 8. We have postponed some rather technical results about weighted Sobolev spaces and boundary traces until Appendix C. Most of our regularity results gradually derived in Sect. 5 take advantage of difficult elliptic Schauder-type estimates for the degenerate Heston operator \( A \) in weighted Hölder spaces over the half-plane \( \mathbb{H} \) obtained in a series of articles by Feehan and Pop [16–18]. For reader’s convenience, we restate these results in Appendix C.

2 Basic notations, function spaces

We use the standard notation \( \mathbb{R} = (-\infty, +\infty) \), \( \mathbb{R}_+ = [0, \infty) \subset \mathbb{R} \) and \( \mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2 \) with the closure \( \overline{\mathbb{H}} = \mathbb{R} \times \mathbb{R}_+ \) for the open and closed upper half-planes, respectively. As usual, for \( x \in \mathbb{R} \) we abbreviate \( x^+ \triangleq \max\{x, 0\} \) and \( x^- \triangleq \max\{-x, 0\} \). The complex plane is denoted by \( \mathbb{C} = \mathbb{R} + i\mathbb{R} \). The complex conjugate of a number \( z \in \mathbb{C} \) is denoted by \( \bar{z} \), so that the absolute value of \( z \) is given by \( |z| = (\bar{z}z)^{1/2} \).

The basic function space, \( H \), in our treatment of the Heston model is defined as follows: We define the weight \( \omega : \mathbb{H} \to (0, \infty) \) by

\[
\omega(x, \xi) \overset{\text{def}}{=} \xi^{\beta-1} e^{-\gamma|\xi| - \mu \bar{x}} \quad \text{for} \quad (x, \xi) \in \mathbb{H},
\]

where \( \beta, \gamma, \mu \in (0, \infty) \) are suitable positive constants that will be specified later, in Sect. 3 (see also Appendix B). However, it is already clear that if we want that the weight \( \omega(x, \xi) \) tends to zero as \( \xi \to 0^+ \), we have to assume \( \beta > 1 \). Similarly, if we want that the function \( u_0(x, \xi) = K(e^x - 1)^+ \) of \( (x, \xi) \in \mathbb{H} \) (an initial condition in Heston’s model) belongs to \( H \), we must require \( \gamma > 2 \). Then \( H = L^2(\mathbb{H}; \omega) \) is the complex Hilbert space of all complex-valued Lebesgue-measurable functions \( f : \mathbb{H} \to \mathbb{C} \) with the finite norm

\[
\|f\|_H \overset{\text{def}}{=} \left( \int_{\mathbb{H}} |f(x, \xi)|^2 \omega(x, \xi) \, dx \, d\xi \right)^{1/2} < \infty.
\]

This norm is induced by the inner product

\[
(f, g)_H \equiv (f, g)_{L^2(\mathbb{H}; \omega)} \overset{\text{def}}{=} \int_{\mathbb{H}} f \bar{g} \cdot \omega(x, \xi) \, dx \, d\xi \quad \text{for} \quad f, g \in H.
\]

The domain, \( V \), of the sesquilinear form that defines the Heston operator is the weighted Sobolev space of all functions \( f \in H \), such that the first-order partial derivatives (in the sense of distributions), \( f_x \equiv \frac{\partial f}{\partial x} \), \( f_{\bar{x}} \equiv \frac{\partial f}{\partial \bar{x}} \), satisfy

\[
[f]_V^2 \overset{\text{def}}{=} \int_{\mathbb{H}} \left( |f_x|^2 + |f_{\bar{x}}|^2 \right) \cdot \xi \cdot \omega(x, \xi) \, dx \, d\xi < \infty.
\]

The Hilbert norm \( \| \cdot \|_V \) on \( V = H^1(\mathbb{H}; \omega) \), \( \|f\|_V^2 \overset{\text{def}}{=} \|f\|_H^2 + [f]_V^2 \) for \( f \in V \), is induced by the inner product

\[
(f, g)_V \equiv (f, g)_{H^1(\mathbb{H}; \omega)} \overset{\text{def}}{=} \int_{\mathbb{H}} \left( f_x \bar{g} + f_{\bar{x}} \bar{g}_{\xi} \right) \cdot \xi \cdot \omega(x, \xi) \, dx \, d\xi + \int_{\mathbb{H}} f \bar{g} \cdot \omega(x, \xi) \, dx \, d\xi.
\]
for \( f, g \in H^1(\mathbb{H}; \omega) \). In particular, the Sobolev imbedding \( V \hookrightarrow H \) is bounded (i.e., continuous).

We will see later that the domain of the Heston operator is contained in a local version of the following weighted Sobolev space, \( H^2(\mathbb{H}; \omega) \), of all functions \( f \in V \), such that also the second-order partial derivatives (in the sense of distributions), \( f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, f_{x\xi} \equiv \frac{\partial^2 f}{\partial x \partial \xi}, f_{\xi\xi} \equiv \frac{\partial^2 f}{\partial \xi^2} \), satisfy

\[
\int_{\mathbb{H}} \left( |f_{xx}|^2 + |f_{x\xi}|^2 + |f_{\xi\xi}|^2 \right) \cdot \xi^2 \cdot \omega(x, \xi) \, dx \, d\xi < \infty.
\]

In addition, we require that the Hilbert norm of \( f \) on \( H^2(\mathbb{H}; \omega) \), as defined below, is finite,

\[
\|f\|_{H^2(\mathbb{H}; \omega)}^2 \overset{\text{def}}{=} \int_{\mathbb{H}} \left( |f_{xx}|^2 + |f_{x\xi}|^2 + |f_{\xi\xi}|^2 \right) \cdot \xi^2 \cdot \omega(x, \xi) \, dx \, d\xi + \int_{\mathbb{H}} |f_{x}(x, \xi)|^2 \cdot (1 + \xi^2) \cdot \omega(x, \xi) \, dx \, d\xi + \int_{\mathbb{H}} |f(x, \xi)|^2 \cdot (1 + \xi) \cdot \omega(x, \xi) \, dx \, d\xi < \infty.
\]

It easy to see that the Sobolev imbeddings \( H^2(\mathbb{H}; \omega) \hookrightarrow V = H^1(\mathbb{H}; \omega) \hookrightarrow H = L^2(\mathbb{H}; \omega) \) are bounded (i.e., continuous).

We will make use of the following local version of the weighted Sobolev space \( H^2(\mathbb{H}; \omega) \):

Let \( B_R(x_0, \xi_0) \) denote the open disc in \( \mathbb{R}^2 \) with radius \( R > 0 \) centered at the point \((x_0, \xi_0) \in \mathbb{R}^2 \). If \( \xi_0 = 0 \), we define also the open upper half-disc

\[
B^+_R(x_0, 0) = \{ (x, \xi) \in \mathbb{R}^2 : (x - x_0)^2 + \xi^2 < R^2, \quad \xi > 0 \} \subset \mathbb{H}.
\]

Its closure in \( \mathbb{R}^2 \) (hence, also in \( \mathbb{H} \)) is denoted by

\[
\overline{B}^+_R(x_0, 0) = \{ (x, \xi) \in \mathbb{R}^2 : (x - x_0)^2 + \xi^2 \leq R^2, \quad \xi \geq 0 \} \subset \mathbb{H}.
\]

We denote by \( H^2(B^+_R(x_0, 0); \omega) \) the weighted Sobolev space of all functions \( f \in W^{2,2}_{\text{loc}}(B^+_R(x_0, 0)) \) whose norm defined below is finite,

\[
\left( \|f\|_{H^2(B^+_R(x_0, 0); \omega)}^2 \right)^2 \overset{\text{def}}{=} \int_{B^+_R(x_0, 0)} \left( |f_{xx}|^2 + |f_{x\xi}|^2 + |f_{\xi\xi}|^2 \right) \cdot \xi^2 \cdot \omega(x, \xi) \, dx \, d\xi + \int_{B^+_R(x_0, 0)} |f_{x}(x, \xi)|^2 \cdot (1 + \xi^2) \cdot \omega(x, \xi) \, dx \, d\xi + \int_{B^+_R(x_0, 0)} |f(x, \xi)|^2 \cdot (1 + \xi) \cdot \omega(x, \xi) \, dx \, d\xi < \infty.
\]

The half-disc \( B^+_R(x_0, 0) \) being bounded in \( \mathbb{H} \), this norm on \( H^2(B^+_R(x_0, 0); \omega) \) is equivalent with the following simpler norm defined by

\[
\|f\|_{H^2(B^+_R(x_0, 0); \omega)}^2 \overset{\text{def}}{=} \int_{B^+_R(x_0, 0)} \left( |f_{xx}|^2 + |f_{x\xi}|^2 + |f_{\xi\xi}|^2 \right) \cdot \xi^{\beta+1} \cdot dx \, d\xi + \int_{B^+_R(x_0, 0)} |f_{x}(x, \xi)|^2 \cdot \xi^{\beta-1} \cdot dx \, d\xi + \int_{B^+_R(x_0, 0)} |f(x, \xi)|^2 \cdot \xi^{\beta-1} \cdot dx \, d\xi < \infty.
\]
We will employ the weighted Sobolev space $H^2(B^+_R(x_0,0);w)$ in Sect. 6.

The weighted Sobolev space $H^2(B^+_R(x_0,0);w)$ will be imbedded into the weighted $L^p$-Lebesgue space $L^p(B^+_R(x_0,0);w)$ ($1 \leq p < \infty$) of all complex-valued Lebesgue-measurable functions $f : B^+_R(x_0,0) \rightarrow \mathbb{C}$ with the finite norm

$$\|f\|_{L^p(B^+_R(x_0,0);w)} = \left( \int_{B^+_R(x_0,0)} |f(x,\xi)|^p \cdot \xi^{p-1} \cdot dx \, d\xi \right)^{1/p} < \infty.$$  \hspace{1cm} (2.3)

Finally, the local Schauder-type regularity results near the boundary $\partial \mathbb{H} = \mathbb{R} \times \{0\} = \mathbb{H} \setminus \mathbb{H}$ of the half-plane $\mathbb{H}$ established in Sect. 6 will be stated in the Hölder spaces $C^\alpha_s(B^+_R(x_0,0))$ and $C^{2+\alpha}_s(B^+_R(x_0,0))$ over any compact half-disc $B^+_R(x_0,0)$ with $x_0 \in \mathbb{R}$ and $R \in (0,\infty)$. The Hölder norm in these spaces corresponds to the so-called cycloidal Riemannian metric $s$ on $\mathbb{H}$ defined by $ds^2 = \xi^{-1}(dx^2 + d\xi^2)$. The associated cycloidal distance function on $\mathbb{H}$, denoted by $s_{\text{cyclo}}(P_1, P_2)$ for two different points $P_i = (x_i, \xi_i) \in \mathbb{H}$; $i = 1, 2$, is given by

$$s_{\text{cyclo}}(P_1, P_2) \overset{\text{def}}{=} \frac{|x_1 - x_2| + |\xi_1 - \xi_2|}{\sqrt{\xi_1^2 + \xi_2^2} + \sqrt{|(x_1, \xi_1) - (x_2, \xi_2)|^2}}.$$  \hspace{1cm} (2.4)

Of course, the expression $|P_1 - P_2| = |(x_1, \xi_1) - (x_2, \xi_2)|$ stands for the Euclidean distance on $\mathbb{R}^2$. We will use the following equivalent metric on $\mathbb{H}$ introduced in Koch [27, p. 11],

$$s(P_1, P_2) \overset{\text{def}}{=} \frac{|(x_1, \xi_1) - (x_2, \xi_2)|}{\sqrt{\xi_1^2 + \xi_2^2} + |(x_1, \xi_1) - (x_2, \xi_2)|}.$$  \hspace{1cm} (2.5)

As usual, $C(B^+_R(x_0,0))$ denotes the Banach space of all continuous functions $f : B^+_R(x_0,0) \rightarrow \mathbb{C}$ endowed with the maximum norm

$$\|f\|_{C(B^+_R(x_0,0))} \overset{\text{def}}{=} \max_{(x,\xi) \in B^+_R(x_0,0)} |f(x,\xi)| < \infty.$$  \hspace{1cm} (2.6)

Given $\alpha \in (0,1)$, we denote by $C^{\alpha}_s(B^+_R(x_0,0))$ the Hölder space of all functions $f \in C(B^+_R(x_0,0))$ that satisfy

$$[f]_{C^{\alpha}_s(B^+_R(x_0,0))} \overset{\text{def}}{=} \sup_{\begin{array}{c} P_1, P_2 \in B^+_R(x_0,0) \\ P_1 \neq P_2 \end{array}} \frac{|f(P_1) - f(P_2)|}{s(P_1, P_2)^\alpha} < \infty.$$  \hspace{1cm} (2.7)

Recall that $P_i = (x_i, \xi_i) \in \mathbb{H}$ for $i = 1, 2$. The norm on this vector space is defined by

$$\|f\|_{C^{\alpha}_s(B^+_R(x_0,0))} \overset{\text{def}}{=} \|f\|_{C(B^+_R(x_0,0))} + [f]_{C^{\alpha}_s(B^+_R(x_0,0))} < \infty.$$  \hspace{1cm} (2.8)

We denote by $C^{2+\alpha}_s(B^+_R(x_0,0))$ its vector subspace consisting of all functions $f \in C^{\alpha}_s(B^+_R(x_0,0))$ that are twice continuously differentiable in the open half-disc $B^+_R(x_0,0)$ and satisfy

$$\|f\|_{C^{2+\alpha}_s(B^+_R(x_0,0))} \overset{\text{def}}{=} \|f\|_{C^{\alpha}_s(B^+_R(x_0,0))} + \|f_x\|_{C^{\alpha}_s(B^+_R(x_0,0))} + \|f_{\xi}\|_{C^{\alpha}_s(B^+_R(x_0,0))}$$

$$\quad + \|\xi \cdot f_{xx}(x,\xi)\|_{C^{\alpha}_s(B^+_R(x_0,0))} + \|\xi \cdot f_{x\xi}(x,\xi)\|_{C^{\alpha}_s(B^+_R(x_0,0))}$$

$$\quad + \|\xi \cdot f_{\xi\xi}(x,\xi)\|_{C^{\alpha}_s(B^+_R(x_0,0))} < \infty.$$  \hspace{1cm} (2.9)
We endow $C^{2+\alpha}_{s}(\mathbb{B}^{+}_{R}(x_{0}, 0))$ with the norm $\| \cdot \|_{C^{2+\alpha}_{s}(\mathbb{B}^{+}_{R}(x_{0}, 0))}$ defined above. It is proved in Feehan and Pop [15], Lemma 3.1, Eq. (3.1), on p. 4409 (see also Daskalopoulos and Hamilton [11], Prop. I.12.1 on p. 940) that at every point $P^* = (x^*, 0) \in \partial \mathbb{H}$ with $x^* \in (x_{0} - R, x_{0} + R)$ we have the zero limit
\[
\lim_{P \to P^*} \xi \cdot D^2 f(x, \xi) = 0 \quad \text{for every } f \in C^{2+\alpha}_{s}(\mathbb{B}^{+}_{R}(x_{0}, 0)), \tag{2.7}
\]
where $P = (x, \xi) \in \mathbb{H}$ and $D^2 f = \begin{pmatrix} f_{xx} & f_{x\xi} \\ f_{\xi x} & f_{\xi\xi} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ stands for the Hessian matrix of $f$ in $B^{+}_{R}(x_{0}, 0)$ that consists of all second-order partial derivatives of $f$. This means that for any function $f \in C^{2}(B^{+}_{R}(x_{0}, 0))$ the weighted Hölder norm $\| f \|_{C^{2+\alpha}_{s}(\mathbb{B}^{+}_{R}(x_{0}, 0))} < \infty$ forces the zero limit (2.7) which thus may be regarded as an imposed homogeneous boundary condition.

3 Formulation of the mathematical problem

In this section we briefly describe Heston’s model [25, Sect. 1, pp. 328–332] and formulate the associated Cauchy problem as an evolutionary equation of (degenerate) parabolic type. A brief description of the “economic” model is provided in Appendix A. The reader is referred to our earlier work in Alziary and Takáč [3, Sect. 2, pp. 6–13] for a more detailed analytical treatment of Heston’s model.

3.1 Heston’s stochastic volatility model

We consider the Heston model given under a risk neutral measure via equations (1) – (4) in [25, pp. 328–329]. The model is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t})_{t \geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a risk neutral probability measure, and the filtration $(\mathcal{F}_{t})_{t \geq 0}$ satisfies the usual conditions. After a series of standard arguments based on Itô’s formula, a (terminal value) Cauchy problem for the price of a European call or put option is obtained (see [3, Eq. (2.4), p. 6]). This Cauchy problem is then transformed into an initial value problem in the parabolic domain $\mathbb{H} \times (0, T) \subset \mathbb{R}^{3}$ ( [3, Eq. (2.7), p. 8]) with the (autonomous linear elliptic) Heston operator, $A$, given by [3, Eq. (2.9), p. 8],
\[
(A u)(x, \xi) = -\frac{1}{2} \sigma \xi \cdot \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x}(x, \xi) + 2\rho \frac{\partial u}{\partial \xi}(x, \xi) \right) + \frac{\partial^{2} u}{\partial \xi^{2}}(x, \xi) \right] + (q_{r} + \frac{\sigma}{2} \sigma \xi) \cdot \frac{\partial u}{\partial x}(x, \xi) - \kappa (\theta_{\sigma} - \xi) \cdot \frac{\partial u}{\partial \xi}(x, \xi)
\equiv -\frac{1}{2} \sigma \xi \cdot \left[ (u_{x} + 2\rho u_{\xi})_{x} + u_{\xi \xi} \right] + (q_{r} + \frac{1}{2} \sigma \xi) \cdot u_{x} - \kappa (\theta_{\sigma} - \xi) \cdot u_{\xi} \quad \text{for } (x, \xi) \in \mathbb{H}, \tag{3.1}
\]
the boundary operator, $B$ ( [3, Eq. (2.10), p. 8]), on the boundary $\partial \mathbb{H} \times (0, T)$, and the boundary conditions as $x \to \pm \infty$ or $\xi \to +\infty$ ( [3, Eq. (2.11), p. 8]). Here, by $r - q \equiv -q_{r} \in \mathbb{R}$ we have abbreviated the instantaneous drift of the stock price returns with $-\infty < r < q < \infty$, and by $\theta_{\sigma} \equiv \theta / \sigma > 0$ the re-scaled long term (or long-run) variance with $\theta, \sigma \in (0, \infty)$. The correlation coefficient $\rho$ satisfies $\rho \in (-1, 1)$. Finally, $\kappa > 0$ denotes
the rate of mean reversion; see Eq. (A.1) (Appendix A) for motivation. We now give a rigorous mathematical formulation of this initial value Cauchy problem which follows [3, §2.2, pp. 9–11]. Earlier motivation for formulation in similar weighted Lebesgue and Sobolev spaces appears in Daskalopoulos and Feehan [9] and [10, Sect. 2, p. 5048] and Feeha and Pop [17].

We make use of the Gel’fand triple \( V \hookrightarrow H = H' \hookrightarrow V' \), i.e., we first identify the Hilbert space \( H \) with its dual space \( H' \), by the Riesz representation theorem, then use the imbedding \( V \hookrightarrow H \), which is dense and continuous, to construct its adjoint mapping \( H' \hookrightarrow V' \), a dense and continuous imbedding of \( H' \) into the dual space \( V' \) of \( V \) as well. The (complex) inner product on \( H \) induces a sesquilinear duality between \( V \) and \( V' \); we keep the notation \( (\cdot, \cdot)_H \) also for this duality. Now we define the linear operator \( A : V \rightarrow V' \) by the sesquilinear form (cf. [3, Eq. (2.21), p. 11]), for all \( u, w \in V \),

\[
(Au, w)_H = \frac{\sigma}{2} \int_H \left( u_x \cdot \bar{w}_x + 2\rho u_x \cdot \bar{w}_x + u_x \cdot \bar{w}_x \right) \cdot \bar{w}(x, \xi) \, dx \, d\xi \\
+ \frac{\sigma}{2} \int_H (1 - \gamma \text{ sign } x) \, u_x \cdot \bar{w} \cdot \bar{w}(x, \xi) \, dx \, d\xi \\
+ \int_H (\kappa - \gamma \rho \sigma \text{ sign } x - \frac{1}{2} \mu \sigma) \, u_x \cdot \bar{w} \cdot \bar{w}(x, \xi) \, dx \, d\xi \\
+ qr \int_H u_x \cdot \bar{w} \cdot \bar{w}(x, \xi) \, dx \, d\xi + \left( \frac{1}{2} \beta \sigma - \kappa \theta \sigma \right) \int_H u_x \cdot \bar{w} \cdot \bar{w}(x, \xi) \, dx \, d\xi. 
\]

(3.2)

All integrals on the right-hand side converge absolutely for any pair \( u, w \in V \) (by the proof of Prop. 6.1 in [3, pp. 21–23]).

In order to derive the right-hand side of Eq. (3.2) from the left-hand side which contains the formal expression (3.1) for \( A \) (see [3, Eq. (2.20), p. 10]), the following vanishing boundary conditions are employed ([3, Eqs. (2.18), p. 9, and (2.19), p. 10]):

\[
\begin{align*}
\xi^\beta \int_{-\infty}^{+\infty} u_x(x, \xi) \cdot \bar{w}(x, \xi) \cdot e^{-\gamma |x|} \, dx & \longrightarrow 0 \quad \text{as } \xi \to 0+; \\
\xi^\beta e^{-\mu \xi} \int_{-\infty}^{+\infty} u_x(x, \xi) \cdot \bar{w}(x, \xi) \cdot e^{-\gamma |x|} \, dx & \longrightarrow 0 \quad \text{as } \xi \to \infty, \\
\int_{0}^{+\infty} (u_x + 2\rho u_x) \, \bar{w}(x, \xi) \cdot \xi^\beta e^{-\mu \xi} \, d\xi & \longrightarrow 0 \quad \text{as } x \to \pm \infty.
\end{align*}
\]

(3.3)

(3.4)

for every function \( w \in V \). They are used in Alziary and Takáč [3, Eq. (2.20), p. 10] in order to perform integration by parts on all second-order partial derivatives of \( u \) that appear in the formal expression (3.1) for \( A \) inserted into the inner product \((Au, w)_H\) on the left-hand side of Eq. (3.2). The boundary conditions in (3.3) and (3.4) are guaranteed by the following (natural) zero boundary conditions valid for every function \( w \in V = H^1(\mathbb{H}; \bar{w}) \) (see [3,
Lemmas 10.2 and 10.3, pp. 44–45),

\[
\begin{align*}
&\xi^\beta \int_{-\infty}^{+\infty} |w(x, \xi)|^2 \cdot e^{-\gamma|\xi|} \, dx \rightarrow 0 \quad \text{as } \xi \rightarrow 0^+, \\
&\xi^\beta e^{-\mu \xi} \cdot \int_{-\infty}^{+\infty} |w(x, \xi)|^2 \cdot e^{-\gamma|\xi|} \, dx \rightarrow 0 \quad \text{as } \xi \rightarrow \infty,
\end{align*}
\]

(3.5)

and

\[
\begin{align*}
e^{-\gamma|x|} \cdot \int_{0}^{\infty} |w(x, \xi)|^2 \cdot \xi^\beta e^{-\mu \xi} \, d\xi \rightarrow 0 \quad \text{as } x \rightarrow \pm \infty,
\end{align*}
\]

(3.6)

which are combined with the following additional boundary conditions that we have to impose (cf. [3, Eqs. (2.23) and (2.24), p. 12]):

\[
\begin{align*}
&\xi^\beta \cdot \int_{-\infty}^{+\infty} |u_\xi(x, \xi)|^2 \cdot e^{-\gamma|\xi|} \, dx \leq \text{const} < \infty \quad \text{as } \xi \rightarrow 0^+; \\
&\xi^\beta e^{-\mu \xi} \cdot \int_{-\infty}^{+\infty} |u_\xi(x, \xi)|^2 \cdot e^{-\gamma|\xi|} \, dx \leq \text{const} < \infty \quad \text{as } \xi \rightarrow \infty^+, \\
&e^{-\gamma|x|} \cdot \int_{0}^{\infty} |u_x + 2\rho u_\xi|^2 \cdot \xi^\beta e^{-\mu \xi} \, d\xi \leq \text{const} < \infty \quad \text{as } x \rightarrow \pm \infty.
\end{align*}
\]

(3.7)

Indeed, we can apply the Cauchy-Schwarz inequality to the integrals in (3.3) and (3.4) to derive the zero limits from (3.5), (3.6), (3.7), and (3.8).

As we have just chosen a particular realization \( \mathcal{A} : V \rightarrow V' \) of the formal differential expression (3.1) defined by Eq. (3.2), we no longer need to impose the boundary conditions (3.7) and (3.8).

### 3.2 The Cauchy problem in the weighted \( L^2 \)-space \( H \)

The initial value Cauchy problem for the Heston model mentioned in the previous paragraph (§3.1) takes the following abstract form in the Hilbert space \( H = L^2(\mathbb{H}; \mathfrak{m}) \):

\[
\begin{align*}
&\frac{\partial u}{\partial t} + \mathcal{A} u = f(x, \xi, t) \quad \text{in } \mathbb{H} \times (0, T); \\
&u(x, \xi, 0) = u_0(x, \xi) \quad \text{for } (x, \xi) \in \mathbb{H},
\end{align*}
\]

(3.9)

with the function \( f(x, \xi, t) \equiv 0 \) on the right-hand side and the initial data \( u_0 \in H \) at \( t = 0 \). The letter \( T (0 < T \leq +\infty) \) stands for an arbitrary (finite or infinite) upper bound on time \( t \). The (autonomous linear) Heston operator \( \mathcal{A} : V \rightarrow V' \), defined by the sesquilinear form (3.2) is bounded, by the Lax-Milgram theorem. Namely, the boundedness and coercivity of this sesquilinear form are established in [3], Prop. 6.1 on p. 21 and Prop. 6.2 on p. 23, respectively, under certain restrictions on the constants which appear in the weight \( \mathfrak{m} \) and the operator \( \mathcal{A} \) (see Eqs. (2.1) and (3.1)). We will discuss these rather fundamental restrictions in Remark 3.2 at the end of this paragraph.

**Definition 3.1**  **Case** 0 < \( T < \infty \). Let \( f \in L^2((0, T) \rightarrow V') \) and \( u_0 \in H \). A function \( u : \mathbb{H} \times [0, T] \rightarrow \mathbb{R} \) is called a weak solution to the initial value problem (3.9) if it has the following properties:

(i) the mapping \( t \mapsto u(t) \equiv u(\cdot, \cdot, t) : [0, T] \rightarrow H \) is a continuous function, i.e., \( u \in C([0, T] \rightarrow H) \);

(ii) the initial value \( u(0) = u_0 \) in \( H \);

(iii) the mapping \( t \mapsto u(t) : (0, T) \rightarrow V \) is a Bôchner square-integrable function, i.e., \( u \in L^2((0, T) \rightarrow V) \); and
(iv) for every function 
\[ \phi \in L^2((0, T) \to V) \cap W^{1,2}((0, T) \to V') \hookrightarrow C([0, T] \to H), \]
the following equation holds,
\[ (u(T), \phi(T))_H - \int_0^T (u(t), \frac{\partial \phi}{\partial t}(t))_H \, dt + \int_0^T (Au(t), \phi(t))_H \, dt = (u_0, \phi(0))_H + \int_0^T (f(t), \phi(t))_H \, dt. \]

**Case** \( T = +\infty. \) Let \( f \in L^2_{\text{loc}}((0, \infty) \to V') \) (i.e., \( f \in L^2((0, T_0) \to V') \) for every \( 0 < T_0 < \infty \)) and let \( u_0 \in H. \) A function \( u : \mathbb{H} \times [0, \infty) \to \mathbb{R} \) is called a weak solution to the initial value problem (3.9) with \( T = +\infty, \) if it is a weak solution to the initial value problem (3.9) on every bounded time subinterval \( [0, T_0) \subset \mathbb{R}_+ \) with \( 0 < T_0 < T = +\infty, \) according to Case \( 0 < T < \infty \) above.

The following remarks are in order:

First, our definition of a weak solution is equivalent with that given in Evans [13, §7.1], p. 352. Here, for \( 0 < u = 0 \)

Case according to rate of mean reversion

We need to guarantee also

This means that we no longer need the boundary conditions in (3.7) and (3.8) imposed on \( u \in V. \)

**Remark 3.2** (Coercivity conditions). It is important to remark at this stage of our investigation of the Heston operator \( A \) that, in order to ensure the coercivity of \( A + c I \) on \( V, \) one has to assume the well-known **Feller condition** ([19, 23]),

\[ \frac{1}{2} \sigma^2 - \kappa \theta < 0. \]  

(3.10)

However, **Feller’s condition** (3.10) is not sufficient for obtaining the desired coercivity. We need to guarantee also

\[ c_{1, \text{max}} \equiv \frac{1}{2} \sigma \left[ \left( \frac{\kappa}{\sigma} - \gamma |\rho| \right)^2 - \gamma(1 + \gamma) \right] \geq 0; \]

cf. Ineq. (6.15) in Alziary and Takáč [3], proof of Prop. 6.2, pp. 23–27. That is, we need to assume the following **coercivity condition:**

\[ \kappa \geq \sigma \left( \gamma |\rho| + \sqrt{\gamma(1 + \gamma)} \right) \quad (> \sigma \gamma(|\rho| + 1)) . \]  

(3.11)

The last inequality is an additional condition to **Feller’s condition**, \( \frac{1}{2} \sigma^2 - \kappa \theta < 0, \) both of them requiring the rate of mean reversion \( \kappa > 0 \) of the stochastic volatility in Heston’s model to be sufficiently large. This additional condition is caused by the fact that Feller [19] considers only an analogous problem in one space dimension \( (\xi \in \mathbb{R}_+) \), so that the solution \( u = u(\xi) \) is independent from \( x \in \mathbb{R}. \) In particular, if the initial value \( u_0 = u(\cdot, \cdot, 0) \in H \) for
u(x, ξ, t) permits us to take γ > 0 arbitrarily small, then inequality (3.11) is easily satisfied, provided Feller’s condition $\frac{1}{2} \sigma^2 - \kappa \theta < 0$ is satisfied. This is the case for a European put option with the initial condition $u_0(x, \xi) = K (1 - e^x)^+ ( \leq K)$ for $(x, \xi) \in \mathbb{H}$. However, if we wish to accommodate also initial values of type $u_0(x, \xi) = K (e^x - 1)^+$ for $(x, \xi) \in \mathbb{H}$, attached to a European call option, then we are forced to take $\gamma > 2$ to ensure that $u_0 \in H$.

We refer the reader to the recent monograph by Meyer [31] for a discussion of the role of Feller’s condition in the boundary conditions in Heston’s model.

\[ \square \]

## 4 Main results

As our main results, Theorems 4.2 and 4.4, are only a priori results for existing weak and strong solutions, we state the following existence and uniqueness result taken from our previous work [3, Prop. 4.1, p. 16].

**Proposition 4.1** Let $\rho, \sigma, \theta, q_r$, and $\gamma$ be given constants in $\mathbb{R}$, $\rho \in (-1, 1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that $\kappa \in \mathbb{R}$ is sufficiently large, such that both inequalities, (3.10) (Feller’s condition) and (3.11) are satisfied. Set $\mu = \mu_{\text{max}}$ where

$$\mu_{\text{max}} \overset{\text{def}}{=} \frac{\kappa}{\sigma} - \gamma |\rho| \; (> 0) ; \quad \text{hence,} \quad c_{1,\text{max}} = \frac{1}{2} \sigma (\mu_{\text{max}}^2 - \gamma (1 + \gamma)) \geq 0. \quad (4.1)$$

Next, let us choose $\beta \in \mathbb{R}$ such that

$$1 < \beta \leq 2 \kappa \theta / \sigma^2. \quad (4.2)$$

Let $0 < T < \infty$, $f \in L^2((0, T) \to V')$, and $u_0 \in H$ be arbitrary. Then the initial value problem (3.9) (with $u_0 \in H$) possesses a unique weak solution

$$u \in C([0, T] \to H) \cap L^2((0, T) \to V)$$

in the sense of Definition 3.1. Moreover, this solution satisfies also $u \in W^{1,2}((0, T) \to V')$ and there exists a constant $C \equiv C(T) \in (0, \infty)$, independent from $f$ and $u_0$, such that

$$\sup_{t \in [0, T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 \, dt + \int_0^T \|\partial_t u(t)\|_{V'}^2 \, dt \leq C \left( \|u_0\|_H^2 + \int_0^T \|f(t)\|_{V'} \, dt \right).$$

If $T = +\infty$, $f \in L^2_{\text{loc}}((0, \infty) \to V')$, and $u_0 \in H$, the same existence and uniqueness result (in the sense of Definition 3.1) is valid with

$$u \in C([0, \infty) \to H) \cap L^2_{\text{loc}}((0, \infty) \to V).$$

Finally, if $u_0 : \mathbb{H} \to \mathbb{R}$ defined by $u_0(x, \xi) = K (e^x - 1)^+$, for $(x, \xi) \in \mathbb{H}$, should belong to $H$, one needs to take $\gamma > 2$.

The proof follows from the boundedness and coercivity of the sesquilinear form (3.2) in $V \times V$ which are assumed in Lions [29, Chapt. IV, §1], inequalities (1.1) (p. 43) and (1.9) (p. 46), respectively. For alternative proofs, see also e.g. Evans [13, Chapt. 7, §1.2(c)], Theorems 3 and 4, pp. 356–358, Lions [30, Chapt. III, §1.2], Theorem 1.2 (p. 102) and remarks thereafter (p. 103), or Friedman [21], Chapt. 10, Theorem 17, p. 316.

Our first theorem contains global and local regularity results for the weak solution $u : \mathbb{H} \times (0, T) \to \mathbb{R}$ obtained in Proposition 4.1 above for the special case $f \equiv 0$ in $\mathbb{H} \times (0, T)$. We formulate these regularity results using the $C^0$-semigroup representation of the (unique) weak solution $u(\cdot, \cdot, t) \equiv u(t) = e^{-tA}u_0 \in H$, $t \in \mathbb{R}_+$, to the homogeneous initial
value problem (3.9) (with \( f \equiv 0 \)), where we allow any \( 0 < T \leq +\infty \) and an arbitrary initial value \( u_0 \in H \). By the well-known properties of \( C^0 \)-semigroups, \( \lambda I + A : D(A) \subset H \to H \) is a closed linear operator in \( H \) with the domain \( D(A) \subset H \) which is invertible for all \( \lambda \in (\lambda_0, +\infty) \), with the bounded inverse \( (\lambda I + A)^{-1} : H \to H \). We denote by \( D((\lambda I + A)^k) \subset H \) the domain of the \( k \)-th power of \( \lambda I + A \); \( k = 1, 2, 3, \ldots \). Here, \( \lambda_0 \in (0, \infty) \) is a sufficiently large number (called the growth bound) determined by the well-known inequality (5.1) (in Sect. 5).

The new result in this theorem is a local Schauder-type regularity result near the boundary \( \partial\mathbb{H} \times (0, T) = \mathbb{R} \times \{0\} \times (0, T) \) of the parabolic domain \( \mathbb{H} \times (0, T) \subset \mathbb{R}^3 \) stated in the Hölder space \( C^{2+\alpha}_{s}((\overline{B}_R^+(x_0, 0))) \) for every time \( t \in (0, T) \).

**Theorem 4.2** (Local and global regularity). Let \( \rho, \sigma, \theta, q_r, \) and \( \gamma \) be given constants in \( \mathbb{R} \), \( \rho \in (-1, 1) \), \( \sigma > 0 \), \( \theta > 0 \), and \( \gamma > 0 \). Assume that \( \gamma, \kappa, \) and \( \mu \) are chosen as specified in Proposition 4.1 above and \( u_0 \in H \) is arbitrary. Finally, in addition to Ineq. (4.2), choose \( \beta \) such that also \( \beta(\beta - 1) < 4 \), i.e.,

\[
1 < \beta \leq 2\kappa\theta/\sigma^2 \quad \text{and} \quad \beta < \frac{1 + \sqrt{17}}{2} = 2.56 \ldots ,
\]

respectively. Then we have the following four statements for the weak solution \( u : (0, \infty) \to H \) obtained in Proposition 4.1:

(i) \( u(\cdot, \cdot, t) \equiv u(t) = e^{-tA}u_0 \in D_\infty = \bigcap_{k=1}^{\infty} D((\lambda I + A)^k) \subset H \) holds for every \( t \in (0, \infty) \).

(ii) \( u \in C^\infty(\mathbb{H} \times (0, \infty)) \), i.e., \( u \) is of class \( C^\infty \) in \( \mathbb{H} \times (0, \infty) \). Moreover, \( u \) is a (local) classical solution of the parabolic equation \( \frac{du}{dt} + Au = 0 \) in the strong sense (pointwise) in \( \mathbb{H} \times (0, \infty) \).

(iii) Given \( 0 < T \leq +\infty \) and any \( x_0 \in \mathbb{R} \), there are a radius \( R \in (0, \infty) \) and constants \( c_0, c'_0 \in (0, \infty) \) such that, for every \( t \in (0, T) \), we have \( u(t)|_{\overline{B}_R^+(x_0, 0)} \in C^{2+\alpha}_{s}(\overline{B}_R^+(x_0, 0)) \) and

\[
U_{\overline{B}_R^+(x_0, 0)}(t) : u_0 \longmapsto u(t)|_{\overline{B}_R^+(x_0, 0)} : H \to C^{2+\alpha}_{s}(\overline{B}_R^+(x_0, 0))
\]

is a bounded linear operator with the operator norm \( \|U_{\overline{B}_R^+(x_0, 0)}(t)\|_{\text{oper}} \leq (c'_0 t^{-3} + c_0)e^{c_0 t} \).

(iv) Moreover, in the situation of Part (iii) above, the mapping

\[
t \longmapsto u(t)|_{\overline{B}_R^+(x_0, 0)} = U_{\overline{B}_R^+(x_0, 0)}(t)u_0 : (0, T) \to C^{2+\alpha}_{s}(\overline{B}_R^+(x_0, 0))
\]

is continuous and differentiable, with

\[
\|u(t + \tau) - u(t)\|_{C^{2+\alpha}_{s}(\overline{B}_R^+(x_0, 0))} \leq (c'_1 t^{-4} + c_1 t^3)e^{c_0 t} \cdot \|u(\tau) - u_0\|_H
\]

and

\[
\left\| \frac{\partial u}{\partial t}(t) \right\|_{C^{2+\alpha}_{s}(\overline{B}_R^+(x_0, 0))} \leq (c'_1 t^{-4} + c_1 t^{-1})e^{c_0 t} \cdot \|u_0\|_H ,
\]

respectively, for all \( t \in (0, T) \) and for all \( \tau \in (0, \infty) \) such that \( t + \tau < T \). Here, \( c_1, c'_1 \in (0, \infty) \) are some other constants independent from \( t \) and \( u_0 \in H \).

Our proof of this theorem will be built up gradually in the next two sections (Sects. 5 and 6) and completed in Sect. 7.
We stress that the constants $c_0, c'_0 \in (0, \infty)$ in Part (iii) do not depend on the choice of $u_0 \in H$ or $t \in (0, T)$. However, the weighted norm on $H$ depends on the weight function $\varpi(x, \xi)$ which is not translation invariant with respect to $x \in \mathbb{R}$. This property of $\varpi$ means that the constants $c_0, c'_0 \in (0, \infty)$ may depend on $x_0 \in \mathbb{R}$. We will see in the course of the proof of Part (iii) (in Sect. 6) that these constants are rendered independent from the length of the time interval, $(0, T), 0 < T \leq +\infty$, thanks to the multiplicative exponential factor $e^{\lambda_0 t}$. The constant $x_0 \in (0, \infty)$ is determined solely by Ineq. (5.1) (in Sect. 5). In particular, we obtain $\|U_{\overline{B}_R(x_0,0)}(t)\|_{\text{oper}} \leq (c'_0^{-3} + c_0) e^{\lambda_0 t}$ for all $t \in (0, T)$ (even if $T = +\infty$).

Concerning the behavior of the weak solution $u(x, \xi, t)$ to the Cauchy problem (3.9) in $\mathbb{H} \times (0, T)$ near the boundary $\partial \mathbb{H} \times (0, T)$, Part (iii) of Theorem 4.2 has the following important consequence.

**Corollary 4.3** (Boundary behavior). Let $0 < T \leq +\infty$ and $t_0 \in (0, T)$. Under the hypotheses of Theorem 4.2, we have $u(\cdot, \cdot, t) = u(t) \in C^1(\mathbb{H})$ for every $t \in (0, T)$. Furthermore, the function $u(x, \xi, t)$ verifies the following initial value Cauchy problem on $\partial \mathbb{H} \times (t_0, T)$,

$$
\begin{align*}
\frac{\partial u}{\partial t}(x, 0, t) + q_r \cdot \frac{\partial u}{\partial x}(x, 0, t) - \kappa \sigma \frac{\partial u}{\partial \xi}(x, 0, t) &= 0 & \text{in } \mathbb{R} \times (t_0, T) ; \\
u(x, 0, t_0) &= u_0(x, 0) & \text{for } x \in \mathbb{R}.
\end{align*}
$$

(4.4)

Here, we have denoted $u_0(x, \xi) \overset{\text{def}}{=} u(x, \xi, t_0)$ for all $(x, \xi) \in \mathbb{H}$; hence, $u_0 \equiv u(\cdot, \cdot, t_0) \in C^1(\mathbb{H})$. This transport equation for the unknown function $u(x, 0, t)$ has a unique classical solution given by

$$
u(x, 0, t) = u(x - q_r(t - t_0), 0, t_0) + \kappa \sigma \int_{t_0}^{t} \frac{\partial u}{\partial \xi}(x - q_r(t - s), 0, s) \, ds$$

for $(x, t) \in \mathbb{R} \times (t_0, T)$. (4.5)

If, in addition, $u \in C^0(\mathbb{H} \times [0, T])$, then we may take $t_0 = 0$ above, in Eqs. (4.4) and (4.5).

This corollary will be proved in Sect. 7.

Our second theorem is a weak maximum principle which, in turn, implies a pointwise bound on the weak solution $u : \mathbb{H} \times (0, T) \rightarrow \mathbb{R}$ obtained in Proposition 4.1 above. We begin with some auxiliary notation:

First, whenever $0 < T \leq +\infty$, let us denote by $C^0(\mathbb{H} \times [0, T])$ the vector space of all continuous functions $u : \mathbb{H} \times [0, T) \rightarrow \mathbb{R}$ and by $C^{2,1}(\mathbb{H} \times (0, T))$ the vector space consisting of all continuous functions $u : \mathbb{H} \times (0, T) \rightarrow \mathbb{R}$ that are continuously differentiable in $\mathbb{H} \times (0, T)$ and also twice continuously differentiable with respect to the space variables $(x, \xi) \in \mathbb{H} = \mathbb{R} \times (0, \infty)$, i.e., all $u, u_t, u_x, u_{\xi}, u_{xx}, u_{x\xi}, u_{\xi\xi} \in C^0(\mathbb{H} \times (0, T))$.

Second, let $\gamma_0 \in (0, \infty)$ be an arbitrary constant, as large as needed. Assuming **Feller’s condition** (3.10), i.e., $\sigma^2 < 2\kappa \theta$, we allow any constants $\beta_0, \mu_0 \in (0, \infty)$ such that

$$
1 \leq \beta_0 < 2\kappa \theta / \sigma^2 \quad \text{and} \quad (0 \leq \beta_0 - 1 < \mu_0 < \infty).
$$

(4.6)

These two inequalities are motivated by conditions (8.4) and (8.6), respectively, in the proof of the theorem below. Notice that there is no upper bound on the constant $\mu_0$.

Third, define a “majorizing” function $h_0 : \mathbb{H} \rightarrow (0, \infty)$ by

$$
h_0(x, \xi) \overset{\text{def}}{=} \exp \left[ \gamma_0 (1 + x^2)^{1/2} + \mu_0 \xi - (\beta_0 - 1) \ln \xi \right]
\leq \xi^{-(\beta_0-1)} \exp \left[ \gamma_0 (1 + x^2)^{1/2} + \mu_0 \xi \right]
$$

for $(x, \xi) \in \mathbb{H}$.

(4.7)
A classical result on the weak maximum principle for a parabolic Cauchy problem in $\mathbb{R}^N \times (0, T)$ is valid under certain restrictions on the growth of a strong solution $u(x, t)$ as $|x| \to \infty$, $(x, t) \in \mathbb{R}^N \times (0, T)$; see e.g. Friedman [21, Chapt. 2, Sect. 4, Theorem 9, p. 43]. Such restrictions in our case are reflected in the function $h_0(x, \xi)$ introduced above.

Now we are ready to state our weak maximum principle. This is an a priori result for any strong subsolution $u$ to the parabolic Cauchy problem (4.8), (4.9), and (4.10) as described below. As a consequence, we do not need to assume hypothesis (3.11) or (4.2) (cf. Proposition 4.1).

**Theorem 4.4 (Weak maximum principle).** Let $0 < T \leq +\infty$. Assume that the constants $\sigma, \kappa, \theta \in (0, \infty)$ satisfy the Feller condition (3.10). Let $\gamma_0 \in (0, \infty)$ be arbitrary and assume that $\beta_0, \mu_0 \in (0, \infty)$ satisfy inequalities (4.6). Finally, assume that $u : \mathbb{H} \times [0, T) \to \mathbb{R}$ is a function that satisfies $u \in C^0(\mathbb{H} \times [0, T)) \cap C^{2,1}([\mathbb{H} \times (0, T))$ together with

$$
\begin{align*}
\frac{\partial u}{\partial t} + Au &\leq 0 \quad \text{in } \mathbb{H} \times (0, T), \\
u(x, \xi, t) &\leq C \cdot h_0(x, \xi) \quad \text{for } (x, \xi, t) \in \mathbb{H} \times (0, T); \\
u(x, \xi, 0) &\leq 0 \quad \text{for } (x, \xi) \in \mathbb{H},
\end{align*}
$$

where $C \in (0, \infty)$ is a positive constant independent from $(x, \xi, t) \in \mathbb{H} \times (0, T)$.

Then $u(x, \xi, t) \leq 0$ holds for all $(x, \xi, t) \in \mathbb{H} \times (0, T)$. In particular, the Cauchy problem (3.9) possesses at most one strong solution $u \in C^0(\mathbb{H} \times [0, T)) \cap C^{2,1}([\mathbb{H} \times (0, T))$ that satisfies the growth restriction

$$
|u(x, \xi, t)| \leq C \cdot h_0(x, \xi) \quad \text{for } (x, \xi, t) \in \mathbb{H} \times [0, T),
$$

where $C \in (0, \infty)$ is a positive constant.

An important feature of this theorem is that there are no upper bounds on the choice of the constants $\gamma_0, \mu_0 \in (0, \infty)$. Once they have been chosen, the constant $\beta_0 \in (0, \infty)$ must satisfy inequalities (4.6). Thus, any “fast” growth of the function $u(x, \xi, t)$, as $x \to \pm \infty$ and/or $\xi \to +\infty$, of type $\leq \text{const} \cdot e^{\gamma_0|x|+\mu_0\xi}$ is allowed in Ineq. (4.9). In contrast, as $x \to \pm \infty$ and $\xi \to 0+$, the growth of $u(x, \xi, t)$ is limited to $\leq \text{const} \cdot \xi^{-\beta_0} e^{\gamma_0|x|}$. A similar idea is offered by Corollary 4.5 to Theorem 4.4 below. As we will infer from our proof of Corollary 4.5 in Sect. 8, the case of $T < +\infty$ in Ineq. (4.9) is of special importance.

Our weak maximum principle in Theorem 4.4 differs from that in Feehan and Pop [15], Lemma 3.4 on p. 4416. Their conditions [15, Eq. (3.29)] imposed on the Heston operator $A$ are weaker than ours. We assume that the constants $\sigma, \kappa, \theta \in (0, \infty)$ satisfy the Feller condition (3.10). On the other hand, we do not need that the functions $u, u_t, u_x, u_\xi$, and $\xi u_{xx}, \xi u_{\xi x}, \xi u_{\xi \xi}$ be continuous up to the boundary $\partial \mathbb{H}$ of the half-plane $\mathbb{H} = \mathbb{R} \times (0, \infty)$; cf. [15, Eq. (3.30)]. Neither do we need the boundary condition in [15, Eq. (3.31)]. In fact, we will show that this boundary condition is satisfied also by our solutions to the Heston problem by combining our growth hypothesis (4.9) with Lemma 3.1 in [15, Eq. (3.1), p. 4409].

**Corollary 4.5** Let $0 < T \leq +\infty, \kappa \geq \sigma \rho$, and let $r_0 \in \mathbb{R}_+$ satisfy $r_0 + q_r = (r_0 - r) + q \geq 0$.

Assume that the constants $\sigma, \kappa, \theta \in (0, \infty)$ satisfy the Feller condition (3.10). Let $\gamma_0 \in [1, \infty)$ be arbitrary and assume that $\beta_0, \mu_0 \in (0, \infty)$ satisfy inequalities (4.6). Finally, assume that $u : \mathbb{H} \times [0, T) \to \mathbb{R}$ is a strong solution to the homogeneous Cauchy problem (3.9) with $f \equiv 0, u \in C^0(\mathbb{H} \times [0, T)) \cap C^{2,1}([\mathbb{H} \times (0, T))$, such that $u$ verifies the growth restriction (4.11) together with the following restriction at time $t = 0$,

$$
|u(x, \xi, 0)| \leq \text{U}(x, \xi, 0) \overset{\text{def}}{=} K_1 e^{x+\sigma \xi} + K_0 \quad \text{for all } (x, \xi) \in \mathbb{H}.
$$

(4.12)
Here, $K_0, K_1 \in \mathbb{R}_+$ are arbitrary constants, and $\sigma \in \mathbb{R}$ is another constant restricted by
\[
0 \leq \sigma < \mu_0 \quad \text{and} \quad (0 \leq) \; \sigma \leq \min \left\{ \frac{r_0 + q_r}{k \theta \sigma}, \frac{2(\kappa - \sigma \rho)}{\sigma} \right\}.
\]
Then $|u(x, \xi, t)| \leq U(x, \xi, 0) \overset{\text{def}}{=} e^{ot} U(x, \xi, 0)$ is valid in all of $\mathbb{H} \times [0, T)$, i.e., at all times $t \in [0, T)$.

This corollary will be proved in Sect. 8. We remark that the condition in (4.12) is satisfied for the initial value $u_0 : \mathbb{H} \to \mathbb{R}$ defined by $u_0(x, \xi) = K (e^t - 1)^+$, for $(x, \xi) \in \mathbb{H}$ (the European call option). One may set $\sigma = 0$ together with $K_0 = 0$ and $K_1 = 1$.

We recall from Theorem 4.2, Part (ii), that the (unique) weak solution $u(\cdot, \cdot, t) \equiv u(t) = e^{-t A} u_0 \in H$, $t \in \mathbb{R}_+$, to the homogeneous initial value problem (3.9) is of class $C^\infty$ in $\mathbb{H} \times (0, \infty)$, i.e., $u \in C^\infty(\mathbb{H} \times (0, \infty))$. Thus, we conclude that $u$ verifies the parabolic equation $\frac{du}{dt} + Au = 0$ in the strong sense (pointwise) in $\mathbb{H} \times (0, \infty)$, thanks to $u \in C^{2,1}(\mathbb{H} \times (0, \infty))$. However, in order that $u$ be a strong (classical) solution of problem (3.9) with $f \equiv 0$, the additional continuity hypothesis $u \in C^0(\mathbb{H} \times [0, T))$ has to be made.

## 5 Some smoothing properties of the Heston semigroup

This section is concerned with some standard properties of the $C^0$-semigroup $e^{-t A} (t \in \mathbb{R}_+)$ of bounded linear operators $e^{-t A} : H \to H$ on the complex Hilbert space $H$. This semigroup has been already mentioned in Sect. 4, Proposition 4.1, in connection with Theorem 4.2. It is shown in Alziary and Takah [3], Prop. 6.1 (p. 21) and Prop. 6.2 (p. 23), respectively, that under conditions (3.10) and (3.11) the sesquilinear form
\[
(u, w) \mapsto ( (\lambda I + A)u, w )_H = (Au, w)_H + \lambda (u, w)_H : V \times V \to \mathbb{R}
\]
defined in (3.2) (cf. [3, Eq. (2.21), p. 11]) is bounded and coercive on $V \hookrightarrow H$ (cf. also Lions [29, Chapt. IV, §1], inequalities (1.1) (p. 43) and (1.9) (p. 46), respectively), provided $\lambda \in (\lambda_0, \infty)$ where $\lambda_0 \in (0, \infty)$ is a sufficiently large constant, such that
\[
( (\lambda_0 I + A)u, u )_H = (Au, u)_H + \lambda_0 \|u\|_H^2 \geq 0 \quad \text{holds for all} \; u \in V.
\]

We recall from Sect. 3, §3.2, that the (autonomous linear) operator $A : V \to V'$, defined by this sesquilinear form, is bounded, by the Lax-Milgram theorem. The (unique) weak solution $u(x, \xi, t)$ to the Cauchy problem (3.9) with $f \equiv 0$ in $\mathbb{H} \times (0, T)$ and an arbitrary initial value $u_0 \in H$ defines the $C^0$-semigroup representation of the solution $u(\cdot, \cdot, t) \equiv u(t) = e^{-t A} u_0 \in H$, $t \in \mathbb{R}_+$, where $T \in (0, \infty)$ may be chosen arbitrarily large.

To be more precise, we denote by $-A : D(A) \subset H \to H$ the infinitesimal generator of this semigroup which is the restriction of the bounded linear operator $-A : V \to V'$ to the domain $D(A) = \{ w \in V : Aw \in H \}$. In what follows, we keep the notation $\pm A$ for this restriction. It is verified in [3, Sect. 7, §7.1, p. 29], that $e^{-t A} (t \in \mathbb{R}_+)$ is a holomorphic semigroup of bounded linear operators on $H$ with the operator norm
\[
\|e^{-t A}\|_{L(H \to H)} \leq M_0 e^{\lambda_0 t} \quad \text{for all} \; t \in \mathbb{R}_+.
\]
by [3, Ineq. (7.4), p. 29]. Here, $M_0 \geq 1$ and $\lambda_0 > 0$ are some constants. By a well-known smoothing property of a holomorphic semigroup (Pazy [33, Eqs. (6.5)–(6.7), p. 70]), we have

\[ Springer \]
\[ e^{-tA}u_0 \in D(A^k) \subset H \text{ for all } u_0 \in H \text{ and } t > 0; k = 1, 2, 3, \ldots, \]
together with the bound on the operator norm
\[
\| (\lambda I + A)^k e^{-tA} \|_{L^2(H \rightarrow H)} \leq M_k t^{-k} e^{\lambda_0 t} \quad \text{for all } t > 0, \quad (5.2)
\]
where \( M_k = (kM_1)^k > 0 \) is a constant independent from time \( t > 0 \). Indeed, employing the factorization \( (\lambda I + A)^k e^{-tA} = \left[ (\lambda I + A) e^{-(t/k)A} \right]^k \) for \( k = 1, 2, 3, \ldots, \) we deduce the value \( M_k = (kM_1)^k \) in Ineq. (5.2) for every \( k = 1, 2, 3, \ldots \) from the case \( k = 1 \).

Finally, the factorization
\[
e^{-tA} = (\lambda I + A)^{-k} \left[ (\lambda I + A)^k e^{-tA} \right] \quad \text{for } t > 0 \text{ and } k = 1, 2, 3, \ldots \quad (5.3)
\]
renders the smoothing property of the holomorphic \( C^0 \)-semigroup \( e^{-tA} (t \in \mathbb{R}_+) \) stated in the next lemma. As usual, we endow the domain \( D(A^k) = D((\lambda I + A)^k) \) of the \( k \)-th power of \( A \) with its graph norm for \( k = 1, 2, 3, \ldots \) (see Pazy [33, Def. 6.7 and Thm. 6.8, p. 72]). Hence, \( D(A^k) \) is a Banach space continuously imbedded into \( H \), thanks to the graph of each \( A^k \) being closed in \( H \times H \). Keeping the meaning of \( M_k = (kM_1)^k \) from above, we get

**Lemma 5.1 (Smoothing property).** Under the hypotheses of Theorem 4.2 (cf. Proposition 4.1), for any \( t > 0 \) and every \( k = 1, 2, 3, \ldots \), the bounded linear operator \( e^{-tA} : H \rightarrow H \) maps \( H \) into \( D(A^k) \) with the operator norm satisfying
\[
\| e^{-tA} \|_{L^2(H \rightarrow D(A^k))} \leq M_k t^{-k} e^{\lambda_0 t} \cdot \| (\lambda I + A)^{-k} \|_{L^2(H \rightarrow D(A^k))} \quad \text{for all } t > 0. \quad (5.4)
\]

**Proof** The estimate in Ineq. (5.4) is obtained by applying (5.2) to the right-hand side of (5.3). \( \square \)

## 6 Smoothing properties in Hölder spaces

We apply Lemma 5.1 step by step for \( k = 1, 2, 3 \). We define the auxiliary functions \( f_{j,k}(x, \xi, t) \) for \( 0 \leq j \leq k \leq 3 \) as follows: First, for any time \( t > 0 \) we set
\[
f_{0,k}(\cdot, \cdot, t) \equiv f_{0,k}(t) \overset{\text{def}}{=} (\lambda I + A)^k e^{-tA}u_0 \in H, \quad k = 1, 2, 3. \quad (6.1)
\]
Next, for \( t > 0 \) we introduce
\[
f_{j,k}(\cdot, \cdot, t) \equiv f_{j,k}(t) \overset{\text{def}}{=} (\lambda I + A)^{-j} f_{0,k}(t) \quad \text{for } 1 \leq j \leq k \leq 3. \quad (6.2)
\]
Clearly, for \( j = k; k = 1, 2, 3, \) and \( t > 0 \) we obtain \( f_{k,k}(t) = u(t) \).

### 6.1 Smoothing with the factor \((\lambda I + A)^{-1}\)

For \( j = 1 \) and \( k = 1, 2, 3 \) we get \((\lambda I + A)f_{1,k}(t) = f_{0,k}(t) \in H, \ t > 0 \). We apply an interior (local) \( H^2 \)-type regularity result due to Feehan and Pop [17], Theorem 3.16, Eq. (3.12), on p. 385 (stated in Lemma C.1, Appendix C), to conclude that \( f_{1,k}(t) \in H^2 \left( B_{R_1}^+(x_0, 0); w \right) \) holds with any radius \( R_1 \in (0, \infty) \). More precisely, there is a constant \( C_1 > 0 \) depending only on the center point \( x_0 \in \mathbb{R} \) and the radii \( 0 < R_1 < R_0 < \infty \), but independent from \( u_0 \in H \) and \( t > 0 \), such that
\[
\| f_{1,k}(t) \|_{H^2 \left( B_{R_1}^+(x_0, 0); w \right)} \leq C_1 \left( \| f_{0,k}(t) \|_{L^2 \left( B_{R_0}^+(x_0, 0); w \right)} + \| f_{1,k}(t) \|_{L^2 \left( B_{R_0}^-(x_0, 0); w \right)} \right). \quad (6.3)
\]
(The weighted Sobolev norm on the left-hand side has been introduced in Eq. (2.2).

For \( k = 1 \) we take advantage of the well-known fact that the operator norms of the family of bounded linear operators \( t(\lambda I + A) e^{-tA} : H \rightarrow H \) are bounded above by \( M_1 e^{\lambda_0 t} \) for all \( t > 0 \), by Ineq. (5.2). Consequently, we get the estimate

\[
\| f_{0,1}(t) \|_{L^2(B_{R_0}^+(x_0);w)} \leq \| f_{0,1}(t) \|_H \leq M_1 t^{-1} e^{\lambda_0 t} \| u_0 \|_H \quad \text{for } t > 0. \quad (6.4)
\]

Recalling \( u(t) = f_{1,1}(t) = e^{-tA}u_0 \) with the operator norms \( \| e^{-tA} \|_{\mathcal{L}(H \rightarrow H)} \leq M_0 e^{\lambda_0 t} \) for \( t > 0 \), by Ineq. (5.1), and applying (6.4) to (6.3) to deduce

\[
\| u(t) \|_{H^2(B_{R_1}^+(x_0);w)} \leq C_1 (M_1 t^{-1} e^{\lambda_0 t} \| u_0 \|_H + \| u(t) \|_H) \\
\leq (C_1 t^{-1} + C_1,0) e^{\lambda_0 t} \| u_0 \|_H \quad \text{for all } t \in (0, \infty).
\]

The constants \( C_{1,1}, C_{1,0} > 0 \) are given by \( C_{1,1} = C_1 M_1 \) and \( C_{1,0} = C_1 M_0 \).

We conclude that, for every \( t > 0, u_0 \mapsto u(t)|_{B_{R_1}^+(x_0,0)} : H \mapsto H^2\left(B_{R_1}^+(x_0,0);w\right) \) is a bounded linear operator with the operator norm bounded above by \( (C_1 t^{-1} + C_{1,0}) e^{\lambda_0 t} \).

### 6.2 Smoothing with the factor \((\lambda I + A)^{-2}\)

Now we take \( j = 2 \) and \( k = 2, 3 \). Hence, we get \( (\lambda I + A) f_{2,k}(t) = f_{1,k}(t) \in H^2\left(B_{R_1}^+(x_0,0);w\right), t > 0 \). Thanks to our hypotheses \( \beta > 1 \) and \( \beta(\beta - 1) < 4 \) in Ineq. (4.3), there is a number \( p > 4 \) such that \( 2 + \beta < p < 2 + \frac{4}{\beta-1} \). In particular, Ineq. (B.20) is valid. For instance, if \( 1 < \beta < 2 \), we may choose \( p = 6 \). By Lemma B.5, Ineq. (B.22) (Appendix B), the restricted imbedding

\[
\{ x \in B_{R_1}^+(x_0,0) \mapsto u|_{B_{R_1}^+(x_0,0)} : H^2\left(B_{R_1}^+(x_0,0);w\right) \hookrightarrow L^p\left(B_{R_1}^+(x_0,0);w\right) \quad (6.5)
\]

is continuous, whenever \( R_1' = R_1/2 \) and \( 0 < R_1 < R_0 < \infty \). (We refer to Adams and Fournier [2, Chapt. 6, §6.1, p. 167] for the definition of a restricted imbedding concerning Sobolev and Lebesgue function spaces. Typically, a restricted imbedding is not injective.) Consequently, we have also \( (\lambda I + A) f_{2,k}(t) = f_{1,k}(t) \in L^p\left(B_{R_1}^+(x_0,0);w\right), t > 0 \). This is an elliptic equation for the unknown function \( f_{2,k}(t) \in V \hookrightarrow H \). This observation allows us to apply a local Hölder regularity result from Feehan and Pop [18], Theorem 1.11, Eq. (1.31), on p. 1083 (stated in Lemma C.2, Appendix C; see also [17], Theorem 2.5, Eq. (2.12), pp. 375–376) in order to derive \( f_{2,k}(t) \in C_5^\alpha\left(\overline{B}_{R_2}^+(x_0,0)\right) \) for \( t \in (0, T) \), together with the estimate

\[
\| f_{2,k}(t) \|_{C_5^\alpha\left(\overline{B}_{R_2}^+(x_0,0)\right)} \leq C_2\left(\| f_{1,k}(t) \|_{L^p\left(B_{R_1}^+(x_0,0);w\right)} + \| f_{2,k}(t) \|_{L^2\left(B_{R_1}^+(x_0,0);w\right)}\right) \quad (6.6)
\]

with some \( 0 < R_2 < R_1' \). In this local Hölder regularity result (Lemma C.2), only the condition \( p > \max\{4, 2 + \beta\} \) is needed. (The Hölder norm on the left-hand side has been introduced in Eq. (2.5).) All constants \( R_2 \equiv R_2(R_1') (0 < R_2 < R_1'), \alpha \in (0, 1), \) and \( C_2 > 0 \) depend on the center point \( x_0 \in \mathbb{R} \) and the radius \( R_1' \) with \( R_1' = R_1/2 \) (\( < R_1 < R_0 < \infty \)),
but are independent from $u_0\in H$ and $t > 0$. We now employ Lemma B.5, Ineq. (B.22) (Appendix B), again to estimate the norm of the restricted Sobolev imbedding in (6.5),

$$
\|f_{1,k}(t)\|_{L^p\left(B_{R'_1}^+(x_0,0);w\right)} \leq C'(R_1) \|f_{1,k}(t)\|_{H^2\left(B_{R_1}^+(x_0,0);w\right)},
$$

where $0 < C'(R_1) < \infty$ is a constant depending only on the center point $x_0 \in \mathbb{R}$ and the radius $R_1 > 0$, but neither on $u_0 \in H$ nor on $t > 0$. We combine the last estimate with (6.3) in order to estimate the right-hand side of Ineq. (6.6) by

$$
\|f_{2,k}(t)\|_{L^2\left(B_{R_2}^+(x_0,0);w\right)} \leq C_2 \cdot C'(R_1) \|f_{1,k}(t)\|_{H^2\left(B_{R'_1}^+(x_0,0);w\right)} + C_2 \|f_{2,k}(t)\|_{L^2\left(B_{R_1}^+(x_0,0);w\right)}
$$

$$
\leq C_1 C_2 \cdot C'(R_1) \left(\|f_{0,k}(t)\|_{L^2\left(B_{R_0}^+(x_0,0);w\right)} + \|f_{1,k}(t)\|_{L^2\left(B_{R_0}^+(x_0,0);w\right)}\right) + C_2 \|f_{2,k}(t)\|_{L^2\left(B_{R_1}^+(x_0,0);w\right)}
$$

(6.7)

with some $0 < R_2 < R'_1 = R_1/2$ and $0 < R_1 < R_0 < \infty$.

For $k = 2$ we now employ the fact that the operator norms of both families of bounded linear operators, $t(\lambda I + A) e^{-tA} : H \to H$ and $t^2(\lambda I + A)^2 e^{-tA} : H \to H$, are bounded above by $M_1 e^\lambda t$ and $(2M_1)^2 e^\lambda t$, respectively, by Ineq. (5.2), i.e., by const $\cdot e^\lambda t$ for all $t > 0$. We thus estimate

$$
\|f_{0,2}(t)\|_{L^2\left(B_{R_0}^+(x_0,0);w\right)} \leq \|f_{0,2}(t)\|_{H} \leq (2M_1)^2 t^{-2} e^\lambda t \|u_0\|_{H} \quad \text{and} \quad (6.8)
$$

$$
\|f_{1,2}(t)\|_{L^2\left(B_{R_0}^+(x_0,0);w\right)} = \|f_{0,1}(t)\|_{L^2\left(B_{R_0}^+(x_0,0);w\right)} \leq \|f_{0,1}(t)\|_{H} \leq M_1 t^{-1} e^\lambda t \|u_0\|_{H} \quad \text{for } t \in (0, \infty). \quad (6.9)
$$

The latter estimate follows directly from $f_{1,2}(t) = f_{0,1}(t)$ and Ineq. (6.4). Consequently, recalling $u(t) = f_{2,2}(t) = e^{-tA}u_0$ with the operator norms $\|e^{-tA}\|_{L^2(H \to H)} \leq M_0 e^\lambda t$ for $t > 0$, we apply the estimates in (6.8) and (6.9) to (6.7), thus arriving at

$$
\|u(t)\|_{C^j_t\left(B_{R_2}^+(x_0,0)\right)} \leq C_1 C_2 \cdot C'(R_1) \left(2M_1\right)^2 t^{-2} + M_1 t^{-1} e^\lambda t \|u_0\|_{H} + C_2 \|u(t)\|_{H}
$$

$$
\leq C_1 C_2 \cdot C'(R_1) \left(2M_1\right)^2 t^{-2} + M_1 t^{-1} e^\lambda t \|u_0\|_{H} + C_2 M_0 e^\lambda t \|u_0\|_{H}
$$

$$
= \left(C_2 M_0 e^\lambda t + C_2 M_0 e^\lambda t \right) e^\lambda t \|u_0\|_{H} \quad \text{for all } t \in (0, \infty). \quad (6.10)
$$

The constants $C_{2,j} > 0$; $j = 0, 1, 2$, are given by $C_{2,2} = C_1 C_2 \cdot C'(R_1)(2M_1)^2$, $C_{2,1} = C_1 C_2 \cdot C'(R_1) M_1$, and $C_{2,0} = C_2 M_0$.

We have shown that, for every $t > 0$, $u_0 \mapsto u(t)|_{B_{R_2}^+(x_0,0)} : H \mapsto C^j_t\left(B_{R_2}^+(x_0,0)\right)$ is a bounded linear operator with the operator norm bounded above by

$$
\left(2M_1\right)^2 t^{-2} + C_2 M_0 e^\lambda t\right) e^\lambda t.
$$

### 6.3 Smoothing with the factor $(\lambda I + A)^{-3}$

Here, we take $j = k = 3$, that is, we factorize $u(t) = f_{3,3}(t) = (\lambda I + A)^{-1} f_{2,3}(t)$ with $f_{2,3}(t) = f_{0,1}(t) = (\lambda I + A) e^{-tA} \in H$. In Paragraph §6.2 above we have obtained the
local Hölder regularity $f_{2,3}(t) \in C_3^\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)$ for $t \in (0, \infty)$, together with the estimate (6.7) ($k = 3$). Applying Ineq. (5.2) to (6.7) with $k = 3$, where $f_{1,3} = f_{0,2}$ and $f_{2,3} = f_{0,1}$, we obtain further

$$
\| f_{2,3}(t) \|_{C^\alpha_s \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)}
\leq C_1 C_2 \cdot C'(R_1) \left( (3M_1)^3 t^{-3} + (2M_1)^2 t^{-2} \right) e^\lambda t^f \| u_0 \|_H + C_2 M_1 t^{-1} e^\lambda t^f \| u_0 \|_H
= (c_3, t^{-3} + c_3, t^{-2} + c_3, t^{-1}) e^\lambda t^f \| u_0 \|_H \quad \text{for all} \ t \in (0, \infty), \quad (6.11)
$$

with some constant $R_2 \in \mathbb{R}$ satisfying $0 < R_2 < R_1' = R_1/2$ and $0 < R_1 < R_0 < \infty$. The constants $c_{3,j} > 0$; $j = 1, 2, 3$, are given by $c_{3,3} = C_1 C_2 \cdot C'(R_1)(3M_1)^3$, $c_{3,2} = C_1 C_2 \cdot C'(R_1)(2M_1)^2 = C_{2,2}$, and $c_{3,1} = C_{2,1}$. The function $u(t) = f_{3,3}(t) \in V$ verifies the elliptic equation $(\lambda I + A) f_{3,3}(t) = f_{2,3}(t) \in C_3^\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)$, $t > 0$. In Paragraph §6.2 we have shown also $u(t) = f_{2,2}(t) \in C_3^\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)$, $t > 0$, together with the norm estimate (6.10). We apply another local Hölder regularity result from Feehan and Pop [16], Theorem 8.1, Eq. (8.4), pp. 937–938 (stated in Lemma C.3, Appendix C; see also [14], Theorem 1.1, Part 2, on pp. 2487–2488) in order to derive $u(t) = f_{3,3}(t) \in C_3^2+\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)$, $t > 0$, together with the estimate

$$
\| u(t) \|_{C_3^2+\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)} = \| f_{3,3}(t) \|_{C_3^2+\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)}
\leq C_3 \left( \| f_{2,3}(t) \|_{C_3^1 \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)} + \| f_{3,3}(t) \|_{C \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)} \right),
$$

provided $0 < R_2' < R_2 < R_1' = R_1/2$ and $0 < R_1 < R_0 < \infty$. We estimate the right-hand side by a combination of inequalities (6.10) and (6.11), thus arriving at

$$
\| u(t) \|_{C_3^2+\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)} = \| f_{3,3}(t) \|_{C_3^2+\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)}
\leq C_3 \left[ (c_3, t^{-3} + c_3, t^{-2} + c_3, t^{-1}) + \left( (C_2, t^{-2} + C_2, t^{-1} + C_2, 0) \right) e^\lambda t^f \| u_0 \|_H
= (C_3, t^{-3} + C_3, t^{-2} + C_3, t^{-1} + C_3, 0) e^\lambda t^f \| u_0 \|_H \quad \text{for all} \ t \in (0, \infty). \quad (6.12)
$$

We have abbreviated the constants $c_{3,j} > 0$; $j = 0, 1, 2, 3$, given by

$$
C_{3,3} = C_3 c_{3,3} = C_1 C_2 C_3 \cdot C'(R_1)(3M_1)^3, \quad C_{3,2} = C_3 (c_{3,2} + c_{2,2}) = 2C_1 C_2 C_3 \cdot C'(R_1)(2M_1)^2, \quad C_{3,1} = C_3 (c_{3,1} + c_{2,1}) = (1 + C_1 \cdot C'(R_1))C_2 C_3 M_1, \quad C_{3,0} = C_3 c_{3,0} = C_2 C_3 M_0.
$$

In particular, we have shown that

$$
U_{\mathcal{B}_{R_2}^+(x_0)}(t) : u_0 \rightarrow u(t)|_{\mathcal{B}_{R_2}^+(x_0, 0)} = (e^{-tA}u_0)|_{\mathcal{B}_{R_2}^+(x_0, 0)} : H \rightarrow C_3^2+\alpha \left( \mathcal{B}_{R_2}^+(x_0, 0) \right)
$$

is a bounded linear operator with the operator norm

$$
\| U_{\mathcal{B}_{R_2}^+(x_0)}(t) \|_{\text{oper}} \leq (C_3, t^{-3} + C_3, t^{-2} + C_3, t^{-1} + C_3, 0) e^\lambda t^f \quad \text{for all} \ t \in (0, \infty).
$$
7 Completion of the proof of the main regularity result

In this section we finish the proof of our main regularity result, Theorem 4.2, started in the two previous sections, Sects. 5 and 6, and prove also its Corollary 4.3.

Proof of Theorem 4.2. The regularity statement in Part (i) follows directly from the results in Sect. 5, Ineq. (5.2). The $C^\infty$-regularity in Part (ii) is a (local) interior regularity result for (local) weak solutions to a locally strictly parabolic equation established (in a more general setting) in Friedman [21, Chapt. 10, Sect. 4], Theorem 11 (p. 302) and its Corollary (p. 303). The complete proof of Part (iii) has been given in Sect. 6. The radius $R \in (0, \infty)$ stands for the radius $R_2 \in (0, \infty)$ that appears in Eq. (6.12).

Finally, we derive Part (iv) from Part (iii) as follows. The continuity and differentiability of the mapping $t \mapsto u(t)|_{\bar{B}_R^{+}(x_0,0)}$ from $(0, T)$ to the Hölder space $C^{2+\alpha}_{s}(\bar{B}_R^{+}(x_0,0))$ follow from the respective formulas

$$\left(\frac{\partial u}{\partial t}(t+\tau) - u(t)\right)|_{\bar{B}_R^{+}(x_0,0)} = U_{\bar{B}_R^{+}(x_0,0)}(t)(u(\tau) - u_0) \quad \text{and}$$

$$\left(\frac{\partial u}{\partial t}(t+\tau)\right)|_{\bar{B}_R^{+}(x_0,0)} = (-A)(u(\tau))|_{\bar{B}_R^{+}(x_0,0)} = U_{\bar{B}_R^{+}(x_0,0)}(t)\left(\frac{\partial u}{\partial t}(\tau)\right)$$

for all $t \in (0, T)$ and for all $\tau \in (0, \infty)$ such that $t + \tau < T$, combined with the locally uniform upper bound on the operator norm of the bounded linear operator $U_{\bar{B}_R^{+}(x_0,0)}(t)$.

Whereas the norm in the Hölder space $C^{2+\alpha}_{s}(\bar{B}_R^{+}(x_0,0))$ of the expression in Eq. (7.1) above is estimated easily by the operator norm of $U_{\bar{B}_R^{+}(x_0,0)}(t)$ from Part (iii), estimating the expression in Eq. (7.2) requires also the following estimate which follows from inequalities (5.1) and (5.2) ($k = 1$),

$$\|( - A ) e^{-\tau A}\|_{L(H\to H)} \leq \| (\lambda I + A ) e^{-\tau A}\|_{L(H\to H)} + |\lambda| \cdot \| e^{-\tau A}\|_{L(H\to H)}$$

$$\leq (M_1 \tau^{-1} + \lambda_0 M_0) e^{\lambda_0 \tau} \quad \text{for all } \tau > 0.$$
and at any time \( t \in [t_0, T_0] \). The Hölder norms of \( u(t) \) and \( \frac{\partial u}{\partial t}(t) \) satisfy

\[
\|u(t)\|_{C^{2+\alpha}_t(B^+_{R_0}(x_0,0))} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{C^{2+\alpha}_t(B^+_{R_0}(x_0,0))} \leq \Gamma \quad \text{for every } t \in [t_0, T_0].
\]  

(7.3)

The constant \( \Gamma \equiv \Gamma(R_0, t_0, T_0) \in (0, \infty) \) does not depend on the choice of \( t \in [t_0, T_0] \). Moreover, \( u \) is a (local) classical solution of the parabolic equation \( \frac{\partial u}{\partial t} + Au = 0 \) in the strong sense (pointwise) in \( \mathbb{H} \times (0, T) \). Our next step is to take the limit (as \( \xi \to 0^+ \)) of the function \( u(x, \xi, t) \), its first-order partial derivatives, and the expressions \( \xi \cdot u_{xx}(x, \xi, t), \xi \cdot u_{x\xi}(x, \xi, t) \), \( \xi \cdot u_{\xi\xi}(x, \xi, t) \), for an arbitrary, but fixed pair \((x, \xi) \in (-R_0, R_0) \times [t_0, T_0] \). More generally, we fix any pair \((x^*, \xi) \in (-R_0, R_0) \times [t_0, T_0] \) which means that \( P^* = (x^*, 0) \in \partial \mathbb{H} \cap B^+_{R_0}(x_0, 0) \). We will take any point \( P = (x, \xi) \in B^+_{R_0}(x_0, 0) \) and calculate the limit (as \( P \to P^* \)) of the functions \( u(x, \xi, t), u_t \), etc. (as indicated above).

To this end, let us abbreviate the function

\[
g(x, \xi, t) \overset{\text{def}}{=} \frac{1}{2} \sigma^2 \cdot \left( \frac{\partial^2 u}{\partial x^2}(x, \xi, t) + 2\rho \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial \xi^2}(x, \xi, t) \right) - \xi \cdot \left( \frac{1}{2} \sigma \cdot \frac{\partial u}{\partial x}(x, \xi, t) + \kappa \cdot \frac{\partial u}{\partial \xi}(x, \xi, t) \right)
\]

of \((x, \xi, t) \in \mathbb{H} \times (0, T)\) and the boundary operator, \( \mathcal{B} \) (cf. [3, Eq. (2.10), p. 8]), near the boundary \( \partial \mathbb{H} \times (0, T) \),

\[
(\mathcal{B} u)(x, \xi, t) \overset{\text{def}}{=} q_r \cdot \frac{\partial u}{\partial x}(x, \xi, t) - \kappa \theta \sigma \cdot \frac{\partial u}{\partial \xi}(x, \xi, t)
\]

for \((x, \xi, t) \in \mathbb{H} \times (0, T) \). Notice that \( Au = \mathcal{B} u - g \) holds in \( \mathbb{H} \times (0, T) \). Hence, the parabolic equation \( \frac{\partial u}{\partial t} + Au = 0 \) for a (local) classical solution \( u \in C^{2+1}_{t}(\mathbb{H} \times (0, T)) \) is equivalent with

\[
\frac{\partial u}{\partial t} + (\mathcal{B} u)(x, \xi, t) = g(x, \xi, t) \equiv (\mathcal{B} - A) u \quad \text{for } (x, \xi, t) \in \mathbb{H} \times (0, T).
\]

(7.4)

Fixing any \( \xi \in (0, \infty) \) (arbitrarily small for our purpose), we can easily solve Eq. (7.4) as a first-order transport equation for the unknown function \((x, t) \mapsto u^{(\xi)}(x, t) \overset{\text{def}}{=} u(x, \xi, t) : \mathbb{R} \times (0, T) \to \mathbb{R} \), thus obtaining the following formula, valid for any \((x, t) \in \mathbb{R} \times [t_0, T] \):

\[
u^{(\xi)}(x, t) = u(x, \xi, t) = u(x - q_r(t - t_0), \xi, t_0)
+ \kappa \theta \sigma \int_{t_0}^{t} \frac{\partial u}{\partial \xi}(x - q_r(t - s), \xi, s) \, ds + \int_{t_0}^{t} g(x - q_r(t - s), \xi, s) \, ds.
\]

(7.5)

To complete our proof, let us recall the (local) boundary regularity results obtained above, in addition to \( u \in C^{\infty}(\mathbb{H} \times (0, \infty)) \), namely, \( u(t)|_{B^+_{R_0}(x_0,0)}, \frac{\partial u}{\partial t}(t)|_{B^+_{R_0}(x_0,0)} \in C^{2+\alpha}_{t}(\mathbb{R}^+_R(x_0,0)) \) with any finite radius \( R_0 \in (0, \infty) \) and at any time \( t \in [t_0, T_0] \). Moreover, Ineq. (7.3) holds for every \( t \in [t_0, T_0] \), with a constant \( \Gamma \equiv \Gamma(R_0, t_0, T_0) \in (0, \infty) \). Let \( x^* \in \mathbb{R} \) be given. We choose \( x_0 \in \mathbb{R} \) arbitrary and \( R_0 \in (0, \infty) \) large enough, such that \( x_0 - R_0 < x^* - q_rT_0 < x^* < x_0 + R_0 \). All these inequalities are guaranteed by choosing \( R_0 > |x_0 - x^*| + q_rT_0 \). We apply the Hölder regularity from (7.3) to all expressions in Eq. (7.4) in order to conclude that all these expressions belong to the Hölder space \( C^{\alpha}_{t}(\mathbb{R}^+_R(x_0,0)) \), at any fixed time \( t \in [t_0, T_0] \). In particular, we may take the limit (as
ξ → 0+) of all these expressions in order to conclude that \( g(x, \xi, t) \to g(x^*, 0, t) = 0 \) owing to \( P = (x, \xi) \to P^* = (x^*, 0) \in \partial \mathbb{H} \). Here, the limits of both first-order partial derivatives \( u_x \) and \( u_{\xi} \) as \( \xi \to 0+ \) exist and are bounded by (7.3) and the definition of the Hölder space \( C^{2+\alpha}_s(B^+_0(x,0)) \), cf. Eq. (2.6), whereas the limits of all expressions containing the second-order partial derivatives, \( \xi \cdot u_{xx}, \xi \cdot u_{x\xi}, \) and \( \xi \cdot u_{\xi\xi} \), vanish as \( \xi \to 0+ \), by Feehan and Pop [15], Lemma 3.1, Eq. (3.1), on p. 4409 (see also Daskalopoulos and Hamilton [11], Prop. I.12.1 on p. 940). We complete our proof by applying these limits to Eqs. (7.4) and (7.5), thus arriving at Eqs. (4.4) and (4.5), as desired. □

8 A maximum principle and growth at low and high volatilities

According to a classical result on the weak maximum principle for a uniformly parabolic Cauchy problem in \( \mathbb{R}^N \times (0, T) \), see e.g. Friedman [21, Chapt. 2, Sect. 4, Theorem 9, p. 43], the weak maximum principle is valid under “very weak” restrictions on the growth of a strong solution \( u(x, t) \) as \( |x| \to \infty, (x, t) \in \mathbb{R}^N \times (0, T) \). Consequently, one may speak of practically no boundary conditions being imposed on the strong solution \( u(x, t) \) as \( |x| \to \infty \), at least in contrast with classical boundary conditions of Dirichlet, Neumann, or oblique derivative (Robin) types. Nevertheless, thanks to the weak maximum principle, the uniqueness of any strong solution to the Cauchy problem with prescribed initial data is still guaranteed.

Now we are ready to prove our Theorem 4.4.

Proof of Theorem 4.4. Let us recall that \( \gamma_0 \in (0, \infty) \) is an arbitrary constant, as large as needed, the constants \( \beta_0, \mu_0 \in (0, \infty) \) satisfy inequalities (4.6), and the function \( h_0 \) is defined in (4.7).

We will compare the function \( u : \mathbb{H} \times (0, T) \to \mathbb{R} \) to the smooth function \( h \) defined as follows:

\[
h(x, \xi, t) \overset{\text{def}}{=} \exp \left( \frac{\gamma_1 (1 + x^2)^{1/2} + \mu_1 \xi - (\beta_1 - 1) \ln \xi}{1 - \omega t} + vt \right) \quad (8.1)
\]

for \( (x, \xi, t) \in \mathbb{H} \times (0, T) \), where \( \beta_1 \geq 1, \gamma_1 > 0, \mu_1 > 0, \nu \geq 0, \) and \( \omega > 0 \) are suitable positive constants to be specified later in the proof. Clearly, \( h(x, \xi, t)^{-1} \) replaces the weight function \( w : \mathbb{H} \to (0, \infty) \) defined in Eq. (2.1).

We calculate the partial derivatives of \( h(x, \xi, t) \) at \( (x, \xi) \in \mathbb{H} \) and \( 0 < t < T \):

\[
h^{-1} \frac{\partial h}{\partial t} = \frac{\omega}{(1 - \omega t)^2} \left[ \gamma_1 (1 + x^2)^{1/2} + \mu_1 \xi - (\beta_1 - 1) \ln \xi \right] + \nu,
\]
\[
h^{-1} \frac{\partial h}{\partial x} = \frac{\gamma_1}{1 - \omega t} \frac{x}{(1 + x^2)^{1/2}}, \quad h^{-1} \frac{\partial h}{\partial \xi} = \frac{1}{1 - \omega t} \left( \mu_1 - \frac{\beta_1 - 1}{\xi} \right).
\]

Similarly, we calculate the second-order partial derivatives:

\[
h^{-1} \frac{\partial^2 h}{\partial x^2} = \left( \frac{\gamma_1}{1 - \omega t} \right)^2 \frac{x^2}{1 + x^2} + \frac{\gamma_1}{1 - \omega t} \left[ \frac{1}{(1 + x^2)^{1/2}} - \frac{x^2}{(1 + x^2)^{3/2}} \right]
\]
\[
= \frac{\gamma_1^2}{(1 - \omega t)^2} \left( 1 - \frac{1}{1 + x^2} \right) + \frac{\gamma_1}{1 - \omega t} \frac{1}{(1 + x^2)^{3/2}}.
\]
\[
- h^{-1} \left( \frac{\partial h}{\partial t} + Ah \right)
= \frac{1}{2} \sigma \xi \left[ \frac{\gamma_1^2}{(1 - \omega t)^2} \left( 1 - \frac{1}{1 + x^2} \right) + \frac{\gamma_1}{1 - \omega t} \left( \frac{1}{1 + x^2} \right)^{3/2} + \frac{2 \rho \gamma_1}{1 - \omega t} \right]
\times \left( \mu_1 - \frac{\beta_1 - 1}{\xi} \right) \frac{x}{(1 + x^2)^{1/2}} + \frac{\gamma_1}{1 - \omega t} \left( \frac{1}{1 + x^2} \right)^{1/2} \left( \mu_1 - \frac{\beta_1 - 1}{\xi} \right) \frac{x}{(1 + x^2)^{1/2}} + \frac{\gamma_1}{1 - \omega t} \left( \frac{1}{1 + x^2} \right)^{1/2} \left( \mu_1 - \frac{\beta_1 - 1}{\xi} \right) \frac{x}{(1 + x^2)^{1/2}}
\]
\[\]
We begin with estimating the last expression, $J_{-1}$. We fix any $\gamma_1 \in (0, \infty)$ such that $\gamma_1 > \gamma_0$. Recalling Feller’s condition (3.10) and the first inequality in (4.6), let us choose $\beta_1 \in [1, \infty)$ such that
\[
(1 \leq \beta_0 < \beta_1 \leq 1 + (1 - \tau) \left( \frac{2\kappa\theta}{\sigma^2} - 1 \right)) < \frac{2\kappa\theta}{\sigma^2}.
\] (8.4)
This choice guarantees the following inequality, whenever $0 < t \leq T_\omega$,
\[
J_{-1} \leq \frac{\beta_1 - 1}{1 - \omega t} \left( \frac{\sigma}{2} (\beta_1 - 1) + \frac{1}{2} \sigma - \kappa \theta \sigma \right)
= \frac{\sigma (\beta_1 - 1)}{2(1 - \omega t)(1 - \tau)} \left[ \beta_1 - 1 - (1 - \tau) \left( \frac{2\kappa\theta}{\sigma^2} - 1 \right) \right] \leq 0.
\] (8.5)
We fix a suitable constant $\mu_1 > 0$ in the first expression, $J_1$, as follows:
\[
(0 \leq \beta_0 - 1 < \max\{\beta_1 - 1, \mu_0\} < \mu_1 < \infty).
\] (8.6)
This choice, combined with the standard inequality $\ln \xi \leq \xi - 1$ for all $\xi > 0$, guarantees the following estimate for the expression in the parentheses of the last summand in $J_1$, Eq. (8.3),
\[
- \frac{\omega}{(1 - \omega t)^2} \left( \mu_1 - (\beta_1 - 1) \frac{\ln \xi}{\xi} \right) \leq - \frac{\omega}{(1 - \omega t)^2} \left( \mu_1 - (\beta_1 - 1) \frac{\xi - 1}{\xi} \right)
\]
\[
= - \frac{\omega}{(1 - \omega t)^2} \left( \mu_1 - (\beta_1 - 1) + \frac{\beta_1 - 1}{\xi} \right) \leq - \frac{\omega}{(1 - \omega t)^2} \left[ \mu_1 - (\beta_1 - 1) \right].
\]
We apply this inequality and the trivial relation $|x| \leq (1 + x^2)^{1/2}$ for all $x \in \mathbb{R}$ to estimate $J_1$, whenever $0 < t \leq T_\omega$:
\[
J_1 \leq \frac{1}{2} \frac{\sigma}{1 - \omega t} \left( \frac{\gamma_1^2}{1 - \tau} + \gamma_1 + \frac{2 |\rho| \gamma_1 \mu_1}{1 - \tau} + \frac{\mu_1^2}{1 - \tau} + \gamma_1 \right) - \kappa \mu_1 - \omega [\mu_1 - (\beta_1 - 1)]
\]
\[
\leq \frac{\sigma}{1 - \tau} \left( \frac{\gamma_1^2 + 2 |\rho| \gamma_1 \mu_1 + \mu_1^2}{2(1 - \tau)} + \gamma_1 \right) - \kappa \mu_1 - \omega [\mu_1 - (\beta_1 - 1)].
\]
Recall that the correlation coefficient $\rho$ satisfies $\rho \in (-1, 1)$. All constants $\beta_1 \geq 1, \gamma_1 > 0$, and $\mu_1 > 0$ having been fixed, such that all inequalities (3.10), (8.4), and (8.6) are valid, we now choose $\omega \in (0, \infty)$ large enough to guarantee $\omega \geq \tau/T$ and also
\[
J_1 \leq \frac{\sigma}{1 - \tau} \left( \frac{\gamma_1 + \mu_1^2}{2(1 - \tau)} + \gamma_1 \right) - \kappa \mu_1 - \omega [\mu_1 - (\beta_1 - 1)] \leq 0
\] (8.7)
whenever $0 < t \leq \tau/\omega \,(= T_\omega \leq T)$.

The constant $\nu$ appears in the expression $J_0$ only; we take $\nu \in \mathbb{R}_+ = [0, \infty)$ arbitrary. Since $|x| \leq (1 + x^2)^{1/2}$ holds for every $x \in \mathbb{R}$, we can choose $\omega \in (0, \infty)$ even greater than above to obtain also
\[
J_0 + \nu \leq \frac{\sigma \rho \gamma_1 (\beta_1 - 1)}{(1 - \tau)^2} - \sigma \mu_1 (\beta_1 - 1) + \frac{q_\nu \gamma_1}{1 - \tau} + \frac{\kappa [\theta \sigma \mu_1 + (\beta_1 - 1)]}{1 - \tau} - \omega \gamma_1
\]
\[
= \sigma (\beta_1 - 1) \left( \frac{\rho \gamma_1}{(1 - \tau)^2} - \mu_1 \right) + \frac{q_\nu \gamma_1 + \kappa [\theta \sigma \mu_1 + (\beta_1 - 1)]}{1 - \tau} - \omega \gamma_1 \leq 0
\] (8.8)
whenever $0 < t \leq T_\omega = \tau/\omega \ (\leq T)$. In other words, the constant $\omega \in (0, \infty)$ must be large enough in order to obey all three inequalities, $\omega \geq \tau/T$, (8.7), and (8.8).

We remark that, in the works by Daskalopoulos and Feehan [9] and [10, Sect. 2, p. 5048], the constants $\beta$ and $\mu$ are chosen to be $\beta = 2\kappa/\sigma^2 > 1$ and $\mu = 2\kappa/\sigma^2 = \beta/\theta$.

Finally, we apply inequalities (8.5), (8.7), and (8.8) to Eq. (8.2) to infer that

$$-h^{-1} \left( \frac{\partial h}{\partial t} + Ah \right) = J_1 \xi + J_0 + J_{-1} \xi^{-1} \leq -v \leq 0$$

is valid for all $(x, \xi) \in \H$ and for all $0 < t \leq T_\omega$. (8.9)

In order to obtain a weak maximum principle for a strong solution $u : \H \times (0, T) \to \R$ of the initial value problem (3.9), such that $u(x, \xi, t) \leq \text{const} \cdot h(x, \xi, t)$ for all $(x, \xi) \in \H$ and $t \in (0, T)$, from the parabolic equation in the Cauchy problem (3.9) we derive an analogous parabolic equation for the ratio $w(x, \xi, t) \overset{\text{def}}{=} u(x, \xi, t)/h(x, \xi, t) \leq \text{const} < \infty$. Using $u = wh$ we have

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} h + w \frac{\partial h}{\partial t}, \quad \frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} h + w \frac{\partial h}{\partial x}, \quad \frac{\partial u}{\partial \xi} = \frac{\partial w}{\partial \xi} h + w \frac{\partial h}{\partial \xi},$$

and similarly

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} h + 2 \frac{\partial w}{\partial x} \frac{\partial h}{\partial x} + w \frac{\partial^2 h}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x \partial \xi} = \frac{\partial^2 w}{\partial x \partial \xi} h + \frac{\partial w}{\partial x} \frac{\partial h}{\partial \xi} + \frac{\partial w}{\partial \xi} \frac{\partial h}{\partial x} + w \frac{\partial^2 h}{\partial x \partial \xi}.$$ We plug these partial derivatives of $u$ into formula (3.1) to calculate

$$-h^{-1} \left( \frac{\partial u}{\partial t} + Au \right) = -\frac{\partial w}{\partial t} - Aw - h^{-1} \left( \frac{\partial h}{\partial t} + Ah \right) \cdot w(x, \xi, t) + \sigma \xi \left[ \frac{\partial w}{\partial x} \frac{\partial h}{\partial x} + \rho \left( \frac{\partial w}{\partial x} \frac{\partial h}{\partial \xi} + \frac{\partial w}{\partial \xi} \frac{\partial h}{\partial x} \right) + \frac{\partial w}{\partial \xi} \frac{\partial h}{\partial \xi} \right],$$

or equivalently, for all $(x, \xi, t) \in \H \times (0, T)$,

$$-h^{-1} \left( \frac{\partial u}{\partial t} + Au \right) = -\frac{\partial w}{\partial t} - Aw - h^{-1} \left( \frac{\partial h}{\partial t} + Ah \right) \cdot w(x, \xi, t) + \sigma \xi \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial \xi} \right) \cdot \frac{\partial w}{\partial x} + \sigma \xi \left( \frac{\partial h}{\partial \xi} + \frac{\partial h}{\partial \xi} \right) \cdot \frac{\partial w}{\partial \xi}. \quad (8.10)$$

We recall that the multiplicative coefficient at $w(x, \xi, t)$ is $\leq -v \leq 0$, by Ineq. (8.9).

Recalling formula (8.1) for $h$, the ratio

$$\tilde{h}(x, \xi, t) \overset{\text{def}}{=} \frac{h_0(x, \xi)}{h(x, \xi, t)} = \exp \left( - (\gamma_1 - \gamma_0)(1 + x^2)^{1/2} - (\mu_1 - \mu_0)\xi + (\beta_1 - \beta_0) \ln \xi \right) \times \exp \left( - \frac{\omega t}{1 - \omega t} \left[ \gamma_1 (1 + x^2)^{1/2} + \mu_1 \xi - (\beta_1 - 1) \ln \xi \right] - vt \right) \leq \exp \left( - (\gamma_1 - \gamma_0)(1 + x^2)^{1/2} - (\mu_1 - \mu_0)\xi + (\beta_1 - \beta_0) \ln \xi \right)$$
has the following asymptotic behavior, for \((x, \xi) \in \mathbb{H}\) and \(0 < t < T_\omega = \tau/\omega (\leq T)\):

\[
\lim_{|x| \to \infty} \sup_{(\xi, t) \in (0, \infty) \times (0, T_\omega)} \tilde{h}(x, \xi, t) = 0,
\]

\[
\lim_{\xi \to 0^+} \sup_{(x, t) \in \mathbb{H} \times (0, T_\omega)} \tilde{h}(x, \xi, t) = 0, \quad \lim_{\xi \to +\infty} \sup_{(x, t) \in \mathbb{H} \times (0, T_\omega)} \tilde{h}(x, \xi, t) = 0.
\]

These limits follow from inequalities (8.4) and (8.6) combined with \(\omega \geq \tau/T\). From the limits above we derive analogous results for the ratio

\[
w(x, \xi, t) \equiv \frac{u(x, \xi, t)}{h(x, \xi, t)} = \frac{h_0(x, \xi)}{h(0, x, \xi)} \leq C \cdot \tilde{h}(x, \xi, t)
\]

for \((x, \xi) \in \mathbb{H}\) and \(0 < t < T_\omega (\leq T)\), namely,

\[
\lim_{|x| \to \infty} \sup_{(\xi, t) \in (0, \infty) \times (0, T_\omega)} w(x, \xi, t) \leq 0, \quad \text{(8.11)}
\]

\[
\lim_{\xi \to 0^+} \sup_{(x, t) \in \mathbb{H} \times (0, T_\omega)} w(x, \xi, t) \leq 0, \quad \lim_{\xi \to +\infty} \sup_{(x, t) \in \mathbb{H} \times (0, T_\omega)} w(x, \xi, t) \leq 0. \quad \text{(8.12)}
\]

In order to complete our proof, we recall the parabolic equation (8.10) for \(w\) with the right-hand side \(\geq 0\), by Ineq. (4.8), and the multiplicative coefficient \(\leq -v \leq 0\) at \(w(x, \xi, t)\), by Ineq. (8.9), or equivalently, for all \((x, \xi, t) \in \mathbb{H} \times (0, T)\),

\[
\frac{\partial w}{\partial t} + Aw - \sigma \xi \left( \frac{\partial h}{\partial x} + \rho \frac{\partial h}{\partial \xi} \right) \cdot \frac{\partial w}{\partial x} - \sigma \xi \left( \frac{\partial h}{\partial \xi} + \rho \frac{\partial h}{\partial x} \right) \cdot \frac{\partial w}{\partial \xi} = h^{-1} \left( \frac{\partial u}{\partial t} + Au \right) - h^{-1} \left( \frac{\partial h}{\partial t} + Ah \right) \cdot w(x, \xi, t) \leq 0.
\]

Taking advantage of the initial condition (4.10) at time \(t = 0\), which is equivalent with \(w(x, \xi, 0) \leq 0\) for all \((x, \xi) \in \mathbb{H}\), in addition to the boundary behavior (8.11) and (8.12), we may apply the weak maximum principle from A. Friedman [21, Chapt. 2, Sect. 4, Lemma 5, p. 43] to conclude that \(w(x, \xi, t) \leq 0\) holds for all \((x, \xi) \in \mathbb{H}\) at all times \(t \in [0, T_\omega)\). We may apply this result in any subinterval \([t_0, t_0 + T_\omega) \subset [0, T)\) of length \(0 < T_\omega = \tau/\omega \leq T\) to extend the weak maximum principle in \(\mathbb{H} \times (0, T_\omega)\) to the entire domain \(\mathbb{H} \times (0, T)\). The corresponding result for \(u = wh\) now follows exactly as in [21, Chapt. 2, Sect. 4, Theorem 9, p. 43].

The uniqueness for the Cauchy problem (3.9) follows from the weak maximum principle exactly as in [21, Chapt. 2, Sect. 4, Theorem 10, p. 44].

Theorem 4.4 is proved.

It remains to give

Proof of Corollary 4.5 Our strategy of the proof is to verify that the weak maximum principle in Theorem 4.4 can be applied to both functions \(W_\pm(x, \xi, t) = -U(x, \xi, t) \pm u(x, \xi, t)\) for \((x, \xi, t) \in \mathbb{H} \times [0, T)\). In Theorem 4.4, we take \(T \in (0, \infty)\) arbitrarily large, but finite.

We begin with the growth restriction (4.11). The strong solution to the homogeneous Cauchy problem (3.9) with \(f \equiv 0\), \(u \in C^0(\mathbb{H} \times [0, T)) \cap C^{2,1}(\mathbb{H} \times (0, T))\), obeys this restriction by hypothesis. Hence, it remains to verify that so does the function \(U : \mathbb{H} \times (0, \infty) \to \mathbb{R}\), that is to say,

\[
(0 \leq t) \quad U(x, \xi, t) = e^{r_0 \tau} \left( K_1 e^{r_0 \xi} + K_0 \right) \leq C e^{r_0 T} \cdot h_0(x, \xi),
\]

\[
= C e^{r_0 T} \cdot \xi^{-(\beta_0 - 1)} \exp \left[ y_0(1 + x^2)^{1/2} + \mu_0 \xi \right].
\]
holds for all \((x, \xi, t) \in \mathbb{H} \times (0, \infty)\), with some constant \(C \in (0, \infty)\); see Eqs. (4.7) and (4.12). We recall from the hypotheses in Theorem 4.4 (the weak maximum principle) that the constant \(\gamma_0 \in [1, \infty)\) is arbitrary and \(\beta_0, \mu_0 \in (0, \infty)\) satisfy inequalities (4.6). Consequently, we have to choose \(\gamma_0 \geq 1\), as we have already done in the hypotheses, and \(0 \leq \sigma < \mu_0\) assumed in Ineq. (4.13), as well. We conclude that the functions \(W_{\pm} : \mathbb{H} \times [0, T) \to \mathbb{R}\) obey the growth restriction (4.11).

Furthermore, the restriction at the initial time \(t = 0\) in Ineq. (4.12) guarantees \(W_{\pm}(x, \xi, 0) \leq 0\) for all \((x, \xi) \in \mathbb{H}\).

Thus, conditions (4.9) and (4.10) having been verified above, only Ineq. (4.8) for \(W_{\pm}\) in place of \(u\) remains to be proved. Notice that \(\frac{\partial u}{\partial t} = r_0 U\) and \(\frac{u}{\partial t} + Au = 0\) in the strong sense (pointwise) in \(\mathbb{H} \times (0, T)\), thanks to \(u \in C^{2,1}(\mathbb{H} \times (0, T))\).

The first and second partial derivatives of \(U\) are
\[
U_x = K_1 e^{\rho t} \cdot e^{x+\sigma \xi}, \quad U_{\xi} = \sigma K_1 e^{\rho t} \cdot e^{x+\sigma \xi}, \\
U_{xx} = K_1 e^{\rho t} \cdot e^{x+\sigma \xi}, \quad U_{\xi\xi} = \sigma K_1 e^{\rho t} \cdot e^{x+\sigma \xi}, \quad U_{\xi\xi} = \sigma^2 K_1 e^{\rho t} \cdot e^{x+\sigma \xi}.
\]

We insert them into the Heston operator (3.1), \(A\),
\[
e^{-\rho t} \left( \frac{\partial U}{\partial t} + (AU)(x, \xi) \right) = r_0 K_0 + K_1 \cdot e^{x+\sigma \xi} \left[ 1 - 2 \rho \sigma + \sigma^2 \right] + (qr + \frac{1}{2} \sigma \xi) \cdot K_1 \cdot e^{x+\sigma \xi} - \kappa (\theta \sigma - \xi) \cdot \sigma K_1 \cdot e^{x+\sigma \xi}
\]
\[
= r_0 K_0 + K_1 \cdot e^{x+\sigma \xi} \left[ \xi \left[ 1 + 2 \rho \sigma + \sigma^2 \right] + \frac{1}{2} \sigma + \kappa \sigma \right] + r_0 + qr - \kappa \theta \sigma \sigma \geq 0.
\]
The last inequality follows from \(r_0, K_0 \in \mathbb{R}_+\) combined with our conditions on \(\sigma\) in (4.13). Finally, we combine this inequality with \(\frac{\partial u}{\partial t} + Au = 0\) in \(\mathbb{H} \times (0, T)\) to derive the desired inequality (4.8) for \(W_{\pm} = -U \pm u\) in place of \(u\).

We finish our proof by applying the weak maximum principle (Theorem 4.4) to the functions \(W_{\pm} : \mathbb{H} \times [0, T) \to \mathbb{R}\) which guarantees \(W_{\pm} \leq 0\) throughout \(\mathbb{H} \times [0, T)\).\]

\[\square\]

9 Discussion of the boundary conditions

It is not difficult to see, as we will show below, that at any time \(t \in (0, T)\) the Cauchy problem for the Heston model (§3.1 and Appendix A) imposes on the solution \(u(\cdot, t) : \mathbb{R}^N \to \mathbb{R}\) the “boundary” behavior at infinity (as \(|x| \to \infty\)) exhibited precisely by the initial value \(u(\cdot, 0) = u_0 : \mathbb{R}^N \to \mathbb{R}\). More specifically, this is the case for the European call and put options, \(u_0(x, \xi) = K(1 - e^x)^+\) and \(u_0(x, \xi) = K(1 - e^x)^+\), respectively, for \((x, \xi) \in \mathbb{H}\); cf. Eq. (A.10) (for the European call option) and Fouque et al. [20, Fig. 1.2 (p. 17) and Fig. 1.3 (p. 18)] (for both, European call and put options, respectively). This means that, at least in the case of European call and put options, the boundary conditions for \(u(x, t)\) as \(|x| \to \infty\) are determined by the asymptotic behavior of the initial data \(u_0(x)\) as \(|x| \to \infty\). Hence, if any boundary conditions at infinity (independent from time \(t \in (0, T)\)) are to be imposed on the strong solution to the Cauchy problem, they should be obeyed also by the initial data (at time \(t = 0\)). An apparent open question is if those boundary conditions (i.e., boundary behavior) at infinity obeyed by the initial data \(u_0(x)\) as \(|x| \to \infty\) are inherited by the (unique) solution for all times \(t \in (0, T)\) and in what sense.

\[\text{\copyright} \ Springer\]
To illustrate this question, one may consider the well-known Black-Scholes model as treated in [20, §1.3, pp. 12–18] with the closed-form solution provided in [20, Eq. (1.37), p. 16]. Of economic importance is the Delta hedging ratio, \( e^{-x \cdot \frac{\partial}{\partial t}} \), defined in [20, Eq. (1.32), p. 14] and calculated in [20, §1.3.3, p. 15]. The limit of this ratio as \( |x| \to \infty \) is obeyed by the closed-form solution to the Black-Scholes model for the European call and put options. In the analogous form it is imposed also in the Heston model for the European call option; cf. Eq. (A.10) (equivalent to Eq. (A.9)) in the next section (Appendix A). It is well-known (see, e.g., [20, §1.3.2, p. 26]) that the Black-Scholes partial differential equation [20, Eq. (1.35), p. 14] can be easily transformed (by a few elementary substitutions of variables) into the standard diffusion (i.e., heat) equation over the space-time domain \( \mathbb{R} \times (0, \infty) \). The solution of this standard evolutionary equation is given by the classical formula

\[
 u(x, t) = \int_{-\infty}^{+\infty} G(|x - y|; t) u_0(y) \, dy \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty),
\]

where \( G(|x - y|; t) \triangleq \frac{1}{\sqrt{4\pi t}} \cdot \exp \left( -\frac{|x-y|^2}{4t} \right) \). (9.1)

In order to obtain a classical solution \( u : \mathbb{R} \times (0, \tau) \to \mathbb{R} \) on a sufficiently short time interval \( (0, \tau) \subset (0, \infty) \), any Lebesgue-measurable initial data \( u_0 : \mathbb{R} \to \mathbb{R} \) satisfying the growth restriction \( |u_0(x)| \leq M e^{cx^2} \) for a.e. \( x \in \mathbb{R} \) will do. Here, \( M, c \in (0, \infty) \) are some positive constants. Applying this procedure in any time interval \( (t_0, t_0 + \tau) \subset (0, \infty) \) of length \( \tau > 0 \), one obtains a classical solution \( u : \mathbb{R} \times (0, \infty) \to \mathbb{R} \), global in time. This solution is unique among all classical solutions satisfying the growth restriction \( |u(x, t)| \leq M e^{cx^2} \) for all \( (x, t) \in \mathbb{R} \times (0, \infty) \). Let us rewrite Eq. (9.1) as

\[
 u(x, t) - u_0(x) = \int_{-\infty}^{+\infty} G(|x - y|; t) [u_0(y) - u_0(x)] \, dy \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty). \tag{9.2}
\]

Next, given any \( \delta \in (0, 1) \), we fix a number \( A_\delta \in (0, \infty) \) large enough, such that

\[
 \int_{A_\delta \sqrt{t}}^{+\infty} G(s; t) \, ds = \int_{A_\delta}^{+\infty} G(s'; 1) \, ds' < \delta/2 \quad \text{for} \quad t \in (0, \infty). \tag{9.3}
\]

If we wish to impose (possibly inhomogeneous) Dirichlet boundary conditions on the initial data \( u_0(x) \) as \( |x| \to \infty \), for the sake of simplicity, let us assume that \( u_0 : \mathbb{R} \to \mathbb{R} \) is a bounded continuous function, \( |u_0(x)| \leq M \equiv \text{const} < \infty \) for all \( x \in \mathbb{R} \), with the limits \( \lim_{x \to -\infty} u_0(x) = u_0(-\infty) \) and \( \lim_{x \to +\infty} u_0(x) = u_0(+\infty) \). Making use of Ineq. (9.3), we now estimate the difference in Eq. (9.2) as \( |x| \to \infty \):

\[
 |u(x, t) - u_0(x)| \leq \int_{-A_\delta \sqrt{t}}^{+A_\delta \sqrt{t}} G(|z|; t) \, |u_0(x + z) - u_0(x)| \, dz \]
\[
 + \int_{-\infty}^{-A_\delta \sqrt{t}} G(|z|; t) \, |u_0(x + z) - u_0(x)| \, dz + \int_{+A_\delta \sqrt{t}}^{+\infty} G(|z|; t) \, |u_0(x + z) - u_0(x)| \, dz \]
\[
 \leq \left( \int_{-\infty}^{+\infty} G(|z|; t) \, dz \right) \cdot \sup_{x \in \mathbb{R}} \sup_{|z| \leq A_\delta \sqrt{t}} |u_0(x + z) - u_0(x)| + 2M \int_{|z| \geq A_\delta} G(|z'|; 1) \, dz' \]
\[
 \leq \sup_{x \in \mathbb{R}} \sup_{|z| \leq A_\delta \sqrt{t}} |u_0(x + z) - u_0(x)| + 2M \delta \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty). \]

Letting \( x \to \pm \infty \) we arrive at

\[
 \limsup_{x \to \pm \infty} |u(x, t) - u_0(x)| \leq 2M \delta \quad \text{for every} \quad t \in (0, \infty). \]

\[\square\] Springer
The number $\delta \in (0, 1)$ being arbitrary, we conclude that $\lim_{x \to -\infty} u(x, t) = u_0(-\infty)$ and $\lim_{x \to +\infty} u(x, t) = u_0(+\infty)$ as desired.

Neumann boundary conditions can be treated in a similar manner using the following formula derived from Eq. (9.2) by simple differentiation:

$$\frac{\partial u}{\partial x}(x, t) - \frac{\partial u_0}{\partial x}(x) = \int_{-\infty}^{+\infty} G(|x - y|; t) \left[ \frac{\partial u_0}{\partial x}(y) - \frac{\partial u_0}{\partial x}(x) \right] dy$$

for $(x, t) \in \mathbb{R} \times (0, \infty)$.

However, caution must be paid to the “weighted” Neuman boundary conditions suggested by Eq. (A.10) (Appendix A); The European call option prescribes the limits

$$L_{-\infty} = \lim_{x \to -\infty} \left( e^{-x} \cdot \frac{\partial u}{\partial x}(x, t) \right) = 0 \quad \text{and} \quad L_{+\infty} = \lim_{x \to +\infty} \left( e^{-x} \cdot \frac{\partial u}{\partial x}(x, t) \right) = 1$$

for the Delta hedging ratio (cf. [20, Eq. (1.37), p. 16]). In the case of the Black-Scholes model, these limits can be verified in a manner similar to the Dirichlet boundary conditions above. We leave the details to an interested reader.

The boundary condition as $|x| \to \infty$, given by Eq. (3.8) with $\gamma > 2$, that we have used in the definition of the Heston operator by Eq. (3.1) (cf. [3, Eq. (2.24), p. 12]), is in fact weaker than the corresponding boundary condition in Eq. (A.10). Nevertheless, our condition (3.8) is still sufficient for obtaining a unique solution to the Heston model. We recall that the choice of $\gamma > 2$ is necessary to ensure $u_0 \in H$ for the European call option with the initial condition $u_0(x, \xi) = K (e^x - 1)^+$ for $(x, \xi) \in \mathbb{H}$.

Acknowledgements A part of this research was performed while the second author (P.T.) was a visiting professor at Toulouse School of Economics, I.M.T., Université de Toulouse – Capitole, Toulouse, France.

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

A Appendix: The Heston model in finance

A number of stochastic volatility models for derivative pricing (e.g., of call or put options on stocks) are known in the literature; see Fouque et al. [20, Table 2.1, p. 42]. We focus our attention on Heston’s model [25] which has attracted significant attention of a broad community of researchers from Finance and Mathematics. We consider this model under a risk neutral measure via equations (1) – (4) in [25, pp. 328 – 329]. The model is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a risk neutral probability measure, and the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. Denoting by $S_t$ the stock price and by $V_t$ the (stochastic) variance of the stock market at (the real) time $t \geq 0$, the Heston model requires that the unknown pair $(S_t, V_t)_{t \geq 0}$ satisfies the following system of stochastic
differential equations,
\[ \frac{dS_t}{S_t} = -q_r \, dt + \sqrt{V_t} \, dW_t, \quad dV_t = \kappa (\theta - V_t) \, dt + \sigma \sqrt{V_t} \, dZ_t. \] (A.1)

Here, \((W_t)_{t \geq 0}\) and \((Z_t)_{t \geq 0}\) are two Brownian motions with the correlation coefficient \(\rho \in (-1, 1)\), a constant given by \(d(W, Z)_t = \rho \, dt\). Furthermore, \(q_r = q - r \in \mathbb{R}\) and \(\sigma, \kappa, \theta \in (0, \infty)\) are some given constants whose economic meaning is explained, e.g., in Alziary and Takáč [3, Sect. 1, pp. 3–4] or Chiarella et al. [7, Chapt. 2, pp. 3–5].

If \(X_t = \ln(S_t/K)\) denotes the (natural) logarithm of the scaled stock price \(S_t/K\) at time \(t \geq 0\) relative to the strike price \(K > 0\) at maturity \(T > 0\), then the pair \((X_t, V_t)_{t \geq 0}\) satisfies the following system of stochastic differential equations,
\[ dX_t = -\left(q_r + \frac{1}{2}V_t\right) \, dt + \sqrt{V_t} \, dW_t, \quad dV_t = \kappa (\theta - V_t) \, dt + \sigma \sqrt{V_t} \, dZ_t. \] (A.2)

Following [8, Sect. 4], let us consider a European call option written in this market with payoff \(h(S_T, V_T) = h(S_T) \geq 0\) at maturity \(T\), where \(h(s) = (s - K)^+\) for all \(s > 0\). Recalling Heston’s notation in [25, Eq. (11), p. 330], we denote by \(x = X_t(\omega) \in \mathbb{R}\) the logarithm of the spot price of stock and by \(v = V_t(\omega) \in \mathbb{R}\) the variance of stock market at time \(t\). We set \(h(x, v) = h(x) = K (e^x - 1)^+\) for all \(x = \ln(s/K) \in \mathbb{R}\), so that \(h(x) = h(s) = h(K e^x)\) for \(x \in \mathbb{R}\). Hence, if the instant values \((X_t(\omega), V_t(\omega)) = (x, v)\) are known at time \(t \in (0, T)\), where \(\mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2\), the arbitrage-free price \(P^h_t\) of the European call option at this time is given by the following expectation formula (with respect to the risk neutral probability measure \(\mathbb{P}\)) which is justified in [8] and [34]:
\[ P^h_t = p(X_t, V_t, t) \] where
\[ p(x, v, t) = e^{-r(T-t)} \mathbb{E}_\mathbb{P} \left[ h(S_T) \mid \mathcal{F}_t \right] = e^{-r(T-t)} \mathbb{E}_\mathbb{P} \left[ h(X_T) \mid \mathcal{F}_t \right] \] (A.3)

Furthermore, \(p\) solves the (terminal value) Cauchy problem
\[ \begin{cases} \frac{\partial p}{\partial t} + \mathcal{G}_t \, p - r \, p = 0, & (x, v, t) \in \mathbb{H} \times (0, T); \\ p(x, v, T) = h(x), & (x, v) \in \mathbb{H}, \end{cases} \] (A.4)

with \(\mathcal{G}_t\) being the (time-independent) infinitesimal generator of the time-homogeneous Markov process \((X_t, V_t); cf. Friedman [22, Chapt. 6] or Øksendal [32, Chapt. 8]. Indeed, to justify Eq. (A.4), we take advantage of Itô’s formula to derive the following equation from eqs. (A.2) and (A.3):
\[ \left( \frac{\partial}{\partial t} + \mathbf{A} \right) U(s, v, t) = 0, \] (A.5)

where we use the instant values \((s, v) = (S_t(\omega), V_t(\omega)) \in (0, \infty)^2\) and substitute \(U(s, v, t) = p(x, v, t)\) with \(s = K e^x, ds/dx = s\), and
\[ (\mathbf{A}U)(s, v, t) \overset{\text{def}}{=} \frac{1}{2} v \cdot \left( s^2 \frac{\partial^2 U}{\partial s^2} (s, v, t) + 2 \rho \sigma s \frac{\partial^2 U}{\partial s \partial v} (s, v, t) + \sigma^2 \frac{\partial^2 U}{\partial v^2} (s, v, t) \right) + (r - q) s \frac{\partial U}{\partial s} (s, v, t) + [\kappa (\theta - v) - \lambda \cdot v] \frac{\partial U}{\partial v} (s, v, t) - r \, U (s, v, t) \quad \text{for } s > 0, v > 0, \text{ and } 0 \leq t \leq T, \] (A.6)
denotes the (usual) **Black-Scholes-\(\text{-}\)Itô operator**. Eq. (A.5) entails the desired diffusion equation (A.4) using

\[
\frac{\partial p}{\partial x}(x, v, t) = s \frac{\partial U}{\partial s}(s, v, t), \\
\frac{\partial^2 p}{\partial x^2}(x, v, t) = s \frac{\partial U}{\partial s}(s, v, t) + s^2 \frac{\partial^2 U}{\partial s^2}(s, v, t) = \frac{\partial p}{\partial x}(x, v, t) + s^2 \frac{\partial^2 U}{\partial x^2}(s, v, t).
\]

Hence, the function \(\bar{p} : (x, v, t) \mapsto p(x, v, T - t)\) verifies a linear (initial value) Cauchy problem derived from (A.4). The functional-analytic formulation of this problem is given in Eq. (3.9) with the initial data corresponding to \(\bar{p}(x, v, 0) = p(x, v, T) = h(x)\) at \(t = 0\).

We have replaced the meaning of the temporal variable \(t\) as real time \((t \leq T)\) by the time to maturity \(\tau = T - t\), so that the real time (time to maturity) has become \(\tau = T - t\). According to Heston [25, Eq. (6), p. 329], the unspecified term \(\lambda(x, v, T - t)\) in the second drift term on the right-hand side of Eq. (A.6) (with \(\partial U/\partial v\)) represents the **price of volatility risk** and is specifically chosen to be \(\lambda(x, v, T - t) \equiv \lambda v\) with a constant \(\lambda \geq 0\). As we have already pointed out in the Introduction (Sect. 1), we can treat much more general initial conditions \(\bar{p}(x, v, 0) = h(x, v)\) than just those given by the terminal condition for the European call option, \(p(x, v, T) = h(x) = K (e^t - 1)^+\) for \((x, v) \in \mathbb{H}\), which does not depend on the instant value \(v = V_T(\omega)\) of the variance at maturity \(T\); see Sect. 4.

Next, we eliminate the constants \(r \in \mathbb{R}\) and \(\lambda \geq 0\), respectively, from Eq. (A.4) by substituting

\[
p^\ast(x, v, t) \overset{\text{def}}{=} e^{\tau r} \bar{p}(x, v, t) = e^{\tau r} p(x, v, T - t) = e^{\tau r} U(x, v, T - t)
\]

for \(\bar{p}(x, v, t)\) and replacing \(\kappa\) by \(\kappa^\ast = \kappa + \lambda > 0\) and \(\theta\) by \(\theta^\ast = \frac{\kappa \theta}{\kappa + \lambda} > 0\). Hence, we may set \(r = \lambda = 0\). Finally, we introduce also the re-scaled variance \(\xi = v/\sigma > 0\) for \(v \in (0, \infty)\) and abbreviate \(\theta_\sigma \overset{\text{def}}{=} \theta/\sigma \in \mathbb{R}\). These substitutions have a simplifying effect on our calculations. Eq. (A.4) then yields the initial value problem (3.9) for the unknown function \(u(x, \xi, t) = p^\ast(x, \sigma \xi, t)\), with the initial data \(u_0(x, \xi) = p^\ast(x, \sigma \xi, 0) \equiv h(x)\) at \(t = 0\), where the (autonomous linear) **Heston operator** \(\mathcal{A}\), derived from Eq. (A.4), takes the standard elliptic form [3, Eq. (2.8), p. 8]; we prefer to use the asymmetric “divergence” form of \(\mathcal{A}\) given by Eq. (3.1) (cf. [3, Eq. (2.9), p. 8]).

The original work by Heston [25, Eq. (9), p. 330] imposes the following **boundary conditions**: The **boundary operator** as \(v \to 0^+\), defined by

\[
(BU)(s, 0, t) \overset{\text{def}}{=} (r - q) s \frac{\partial U}{\partial s}(s, 0, t) + \kappa \theta \frac{\partial U}{\partial v}(s, 0, t) - r U(s, 0, t) \\
\text{for } s > 0, v = 0, \text{ and } 0 \leq t \leq T,
\]

transforms the left-hand side of the boundary condition as \(v \to 0^+\),

\[
\left(\frac{\partial}{\partial t} + B\right) U(s, 0, t) = 0,
\]

into the following (logarithmic) form on the boundary \(\partial \mathbb{H} = \mathbb{R} \times \{0\}\) of \(\mathbb{H}\):

\[
e^{-\tau r} \left(\frac{\partial}{\partial \tau} + B\right) U(s, 0, \tau) \bigg|_{\tau = T - t} = -\left(\frac{\partial}{\partial t} + B\right) u(x, 0, t) \\
= -\frac{\partial u}{\partial t}(x, 0, t) - qr \frac{\partial u}{\partial x}(x, 0, t) + \kappa \theta_\sigma \frac{\partial u}{\partial \xi}(x, 0, t) \\
\text{for } x \in \mathbb{R} \text{ and } 0 < t < \infty.
\]
The remaining boundary conditions (in addition to \((A.7)\)),
\[
\begin{align*}
U(0, v, t) &= 0; \\
\lim_{s \to -\infty} \frac{\partial}{\partial s}(U(s, v, t) - s) &= 0; \\
\lim_{t \to \infty} (U(s, v, t) - s) &= 0,
\end{align*}
\]
for \(s > 0, v > 0,\) and \(0 \leq t \leq T,\) become (in addition to \((A.8)\)),
\[
\begin{align*}
u(-\infty, \xi, t) &\overset{\text{def}}{=} \lim_{x \to -\infty} \left(u(x, \xi, t) - Ke^{x+rt}\right) = 0 \quad \text{for } \xi > 0; \\
\lim_{x \to +\infty} \left[e^{-x} \cdot \frac{\partial}{\partial x} \left(u(x, \xi, t) - Ke^{x+rt}\right)\right] &= 0 \quad \text{for } \xi > 0; \\
\lim_{\xi \to \infty} \left(u(x, \xi, t) - Ke^{x+rt}\right) &= 0 \quad \text{for } x \in \mathbb{R},
\end{align*}
\]
at all times \(t \in (0, \infty).\) We remark that the first equation in \((A.10)\) is a consequence of the initial conditions (for a European call option) \(0 \leq u_0(x, \xi) = K (e^x - 1)^+\) for \((x, \xi) \in \mathbb{H},\) the lower bound \(u(x, \xi, t) \geq 0\) for all \((x, \xi, t) \in \mathbb{H} \times (0, \infty)\) obtained from the weak maximum principle in Theorem 4.4, and the upper bound \(u(x, \xi, t) \leq Ke^{x+rt}\) for all \((x, \xi, t) \in \mathbb{H} \times (0, \infty)\) obtained from the weak maximum principle in Corollary 4.5 with \(K_1 = K,\) \(K_0 = 0,\) and \(\sigma = 0.\)

B Appendix: Weighted Sobolev spaces and boundary traces

We denote by \(H^1(B_R^+(x_0, 0); \mathfrak{w})\) the weighted Sobolev space of all functions \(f \in W_{\text{loc}}^{1,2}(B_R^+(x_0, 0))\) whose norm defined below is finite,
\[
\|f\|_{H^1(B_R^+(x_0, 0); \mathfrak{w})}^2 \overset{\text{def}}{=} \int_{B_R^+(x_0, 0)} \left(|f_x|^2 + |f_\xi|^2\right) \cdot \xi^\beta \cdot \text{d}x \text{d}\xi \\
+ \int_{B_R^+(x_0, 0)} |f(x, \xi)|^2 \cdot \xi^{\beta-1} \cdot \text{d}x \text{d}\xi < \infty.
\]
Let us recall that the weighted Sobolev space \(H^2(B_R^+(x_0, 0); \mathfrak{w})\) has been defined by its norm in Eq. (2.2). As we will see later, in Lemma B.5, functions from the weighted Sobolev spaces \(H^j(B_R^+(x_0, 0); \mathfrak{w}); j = 1, 2,\) must satisfy certain homogeneous boundary conditions as \(\xi \to 0+,\) i.e., near the boundary \(\partial \mathbb{H} \cap B_R^+(x_0, 0) = (x_0 - R, x_0 + R) \times \{0\}.\) We will see in the proof of this result (Lemma B.5), as well, that these boundary conditions, if satisfied by a function, imply that this function belongs to a particular weighted Sobolev space.

A simple motivation for such a result is the classical Sobolev space \(W^{1,2}(0, 1):\) This space is continuously imbedded into the Hölder space \(C^{1/2}[0, 1];\) hence, the limit \(\lim_{\xi \to 0+} f(\xi) = f(0) \in \mathbb{R}\) is valid for every function \(f \in W^{1,2}(0, 1).\) By Hardy’s inequality (proved in Hardy et al. [24, Theorem 330, pp. 245–246]; see also Kufner [28, Section 5]), we have
\[
\int_0^1 |f(\xi) - f(0)|^2 \xi^{-2} \text{d}\xi \leq \text{const} \cdot \int_0^1 |f'(\xi)|^2 \xi^{-2} \text{d}\xi
\]
with a positive constant independent from \(f \in W^{1,2}(0, 1),\) whenever \(f\) satisfies \(f(1) = f(0).\) Clearly, the homogeneous boundary condition \(f(0) = 0\) is valid if and only if \(\int_0^1 |f(\xi)|^2 \xi^{-2} \text{d}\xi < \infty.\) If this is the case, then even \(\lim_{\xi \to 0+} (f(\xi)/\xi^{1/2}) = 0\) holds.
A much less trivial example appears in our earlier work [3, Sect. 10 (Appendix)]. We now show this example only for the bounded half-disc $B_R^+(x_0, 0) \subset \mathbb{H}$ near the boundary $\partial \mathbb{H} \cap B_R^+(x_0, 0)$. Let us fix any $r \in (0, R)$ and set $\varnothing \equiv \varnothing(r) = \sqrt{R^2 - r^2}$. Given any function $f \in W_{loc}^{1,2}(B_R^+(x_0, 0))$, we begin with the identity
\[
\frac{\partial}{\partial \xi} (\xi^{\beta-1} f(x, \xi)^2) = 2 \int x \varnothing \cdot \xi^{\beta-1} + (\beta - 1) f(x, \xi)^2 \cdot \xi^{\beta-2},
\]
for $(x, \xi) \in B_R^+(x_0, 0)$ satisfying $x \in (x_0 - r, x_0 + r)$ and $0 < \xi < \varnothing$. We apply Cauchy’s inequality
\[
2 |f \varnothing| \leq \frac{\beta - 1}{2} \xi^{\beta-2} f^2 + \frac{2}{\beta - 1} \xi f^2 \xi \varnothing^2
\]
to the equation above to estimate the partial derivative
\[
\frac{\beta - 1}{2} \xi^{\beta-2} f^2 - \frac{2}{\beta - 1} \xi f^2 \xi \varnothing \leq \frac{\partial}{\partial \xi} (\xi^{\beta-1} f(x, \xi)^2)
\]
\[
\leq \frac{3}{2} (\beta - 1) \xi^{\beta-2} f^2 + \frac{2}{\beta - 1} \xi f^2 \xi \varnothing^2. \quad (B.2)
\]
Assuming the integrability
\[
\int_{B_R^+(x_0, 0)} (f^2 \varnothing + f^2 \cdot \xi^{\beta-1}) \, dx \, d\xi < \infty,
\]
we deduce from the inequalities in (B.2) that the function
\[
\xi \mapsto F_{\beta, r}(\xi) \overset{\text{def}}{=} \xi^{\beta-1} \int_{x_0-r}^{x_0+r} f(x, \xi)^2 \, dx : (0, \varnothing) \rightarrow \mathbb{R}
\]
is absolutely continuous over the compact interval $[0, \varnothing]$ with finite boundary limits
\[
F_{\beta, r}(0) \overset{\text{def}}{=} \lim_{\xi \rightarrow 0^+} F_{\beta, r}(\xi) \quad \text{and} \quad F_{\beta, r}(\varnothing) = \varnothing \beta^{-1} \int_{x_0-r}^{x_0+r} f(x, \varnothing)^2 \, dx \quad \text{(as } \xi \rightarrow \varnothing^-)\]
if and only if
\[
\int_0^\varnothing F_{\beta, r}(\xi) \cdot \xi^{-1} \, d\xi = \int_0^\varnothing \xi^{\beta-2} \int_{x_0-r}^{x_0+r} f^2 \, dx \, d\xi < \infty.
\]
However, the last integral is finite if and only if $F_{\beta, r}(0) = 0$. If this is the case, then also the limit $\lim_{\xi \rightarrow 0^+} F_{\beta, r}(\xi) = 0$. We conclude that the homogeneous boundary condition given by $F_{\beta, r}(0) = 0$ is equivalent with the convergence of the last integral. Greater details can be found in [3, Sect. 10, pp. 43–48], Lemmas 10.1 through 10.5.

We will follow a similar procedure in treating the case $f \in H^2(B_R^+(x_0, 0); \varnothing)$. More precisely, we wish to show that if we take a weaker norm, $\| \cdot \|_{H^2(B_R^+(x_0, 0); \varnothing)}$ on $H^2(B_R^+(x_0, 0); \varnothing)$ defined in Eq. (B.4) below, the restriction mapping
\[
\tilde{H}^2(B_R^+(x_0, 0); \varnothing) \rightarrow H^2(B_R^+(x_0, 0); \varnothing)
\]
is still continuous from $\tilde{H}^2(B_R^+(x_0, 0); \varnothing)$ to $H^2(B_R^+(x_0, 0); \varnothing)$, where $\tilde{H}^2(B_R^+(x_0, 0); \varnothing)$ stands for the completion of the Sobolev space $H^2(B_R^+(x_0, 0); \varnothing)$ under the new norm
\[\| f \|_{H^2(R^+(x_0, 0); w)}^2 \triangleq \int_{B_R^+(x_0, 0)} (|f_{xx}|^2 + |f_{x\xi}|^2 + |f_{\xi\xi}|^2) \cdot \xi^{\beta+1} \cdot \, dx \, d\xi + \int_{B_R^+(x_0, 0)} (|f_x|^2 + |f_{\xi}|^2) \cdot \xi^{\beta} \cdot \, dx \, d\xi + \int_{B_R^+(x_0, 0)} |f(x, \xi)|^2 \cdot \xi^{\beta-1} \cdot \, dx \, d\xi = [f]_{H^2(B_R^+(x_0, 0); w)}^2 + \| f \|_{H^1(B_R^+(x_0, 0); w)}^2 < \infty, \quad (B.4)\]

where \([ \cdot ]_{H^2(B_R^+(x_0, 0); w)}\) is a seminorm on \(H^2(B_R^+(x_0, 0); w)\) defined by

\[\| f \|_{H^2(B_R^+(x_0, 0); w)} \triangleq \left( \int_{B_R^+(x_0, 0)} (|f_{xx}|^2 + |f_{x\xi}|^2 + |f_{\xi\xi}|^2) \cdot \xi^{\beta+1} \cdot \, dx \, d\xi \right)^{1/2}. \quad (B.5)\]

It is easy to see that \(H^2(B_R^+(x_0, 0); w)\) consists of all functions \(f \in W_{\text{loc}}^{2,2}(B_R^+(x_0, 0))\) that satisfy \(\| f \|_{H^2(B_R^+(x_0, 0); w)} < \infty\). The continuous restriction mapping in (B.3) is termed a restricted imbedding, by Adams and Fournier [2, Chapt. 6, §6.1, p. 167]. In the course of the proof of this restriction imbedding, we will obtain also certain boundary conditions (i.e., trace results as \(\xi \to 0^+\)) on the boundary \(\partial\mathbb{H} \cap \overline{B_R^+(x_0, 0)}\).

Keeping in mind that some of the constants in our estimates below may depend on the choice of \(x_0 \in \mathbb{R}\), we suppress the dependence on \(x_0\) in the notation for the half-disc \(B_R^+(x_0, 0) \subset \mathbb{H}\) and, thus, write only \(B_R^+ \equiv B_R^+(x_0, 0)\). We further denote by \(Q_r^+ \equiv Q_r^+(x_0, 0)\) the open rectangle

\[Q_r^+(x_0, 0) \triangleq (x_0 - r, x_0 + r) \times (0, r) \subset \mathbb{H}\]

(a “half-square”) with side lengths \(2r\) and \(r \in (0, \infty)\). Its closure in \(\mathbb{R}^2\) is denoted by \(\overline{Q_r^+}\).

Our first lemma is an essential estimate for obtaining the boundary trace as \(\xi \to 0^+\). Given a function \(u \in W_{\text{loc}}^{1,1}(B^+)\), we abbreviate the gradient \(\nabla u = (u_x, u_\xi)\).

**Lemma B.1 (\(\xi\)-derivative inequalities).** Let \(\beta > 0\) and \(R > 0\), and set \(r = R/\sqrt{2}\). Assume that \(u \in H^2(B_R^+; w)\). Then \(Q_r^+ \subset B_R^+\) and the following inequalities hold at almost every point \((x, \xi) \in Q_r^+\),

\[
\frac{\beta}{2} \xi^{\beta-1} \cdot |u_x(x, \xi)|^2 + \frac{2}{\beta} \xi^{\beta+1} \cdot |\partial_\xi u| \leq \frac{\partial}{\partial \xi} (\xi^\beta \cdot |\nabla u(x, \xi)|^2)
\]

\[
\leq \left( \frac{3\beta}{2} \xi^{\beta-1} \cdot |\nabla u(x, \xi)|^2 + \frac{2}{\beta} \xi^{\beta+1} \cdot |\partial_\xi u| \right)^{1/2}. \quad \text{(B.6)}
\]

**Proof** The following partial derivatives exist almost everywhere in \(B_R^+\); we first calculate

\[
\frac{\partial}{\partial \xi} (\xi^\beta \cdot |\nabla u(x, \xi)|^2) = \beta \xi^{\beta-1} \cdot |\nabla u(x, \xi)|^2 + 2\xi^\beta \cdot (\nabla u \cdot \partial_\xi \nabla u). \quad \text{(B.7)}
\]
with the scalar product $\nabla u \cdot \partial_\xi \nabla u = u_x u_{x\xi} + u_{x\xi} u_{\xi\xi}$ in $\mathbb{R}^2 \subset \mathbb{C}^2$, then estimate the scalar product on the right-hand side by Cauchy’s inequality,

$$\pm 2(\nabla u \cdot \partial_\xi \nabla u) \leq 2|\nabla u| \cdot |\partial_\xi \nabla u| \leq \frac{\beta}{2} \xi^{-1} \cdot |\nabla u|^2 + \frac{2}{\beta} \xi \cdot |\partial_\xi \nabla u|^2 \quad (B.8)$$

for a.e. $(x, \xi) \in B^+_R$. We apply Ineq. (B.8) to estimate the right-hand side of Eq. (B.7), thus arriving at (B.6) as desired.

**Lemma B.2 (Pointwise trace inequalities).** Let $\beta > 0$ and $R > 0$, and set $r = R/\sqrt{2}$. Assume that $u \in \tilde{H}^2(B^+_R; \mathbb{w})$. Then the following inequalities hold at almost every point $x \in (-r, r)$, for every $\xi \in (0, r)$:

$$\frac{\beta}{2} \int_0^\xi |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta-1} d\xi' - \frac{2}{\beta} \int_0^\xi |\partial_\xi \nabla u(x, \xi')|^2 \cdot (\xi')^{\beta+1} d\xi' \leq \xi^\beta \cdot |\nabla u(x, \xi)|^2 - \lim_{\xi' \to 0^+} [((\xi')^\beta \cdot |\nabla u(x, \xi')|^2]$$

$$\leq \frac{3\beta}{2} \int_0^\xi |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta-1} |\partial_\xi \nabla u(x, \xi')|^2 \cdot (\xi')^{\beta+1} d\xi'. \quad (B.9)$$

At almost every point $x \in (-r, r)$, all (Lebesgue) integrals above are finite and the limit (viewed as a boundary condition)

$$L_0(x) \overset{\text{def}}{=} \lim_{\xi' \to 0^+} [((\xi')^\beta \cdot |\nabla u(x, \xi')|^2] = 0 \quad \text{exists.} \quad (B.10)$$

**Proof** Let us set

$$\ell_0(x) \overset{\text{def}}{=} \liminf_{\xi' \to 0^+} [((\xi')^\beta \cdot |\nabla u(x, \xi')|^2] \quad \text{for every } x \in (-r, r);$$

hence, $0 \leq \ell_0(x) \leq \infty$. Clearly, the function $\ell_0 : x \mapsto \ell_0(x) : (-r, r) \rightarrow [0, +\infty]$ is Lebesgue-measurable. Fatou’s lemma yields

$$\int_{-r}^r \ell_0(x) \, dx \leq \ell_0 \overset{\text{def}}{=} \liminf_{\xi' \to 0^+} \left[ (\xi')^\beta \int_{-r}^r |\nabla u(x, \xi')|^2 \, dx \right] \leq \infty.$$

Furthermore, from the hypothesis $u \in \tilde{H}^2(B^+_R; \mathbb{w})$ combined with $Q^+_r \subset B^+_R$, we deduce that

$$\int_0^r |\partial_\xi \nabla u(x, \xi')|^2 \cdot (\xi')^{\beta+1} d\xi' + \int_0^r |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta} d\xi' < \infty \quad (B.11)$$

holds for every $x \in (-r, r) \setminus M_0$, where $M_0 \subset (-r, r)$ is a set of Lebesgue measure zero. As an easy consequence, we observe that, due to the change of weight $(\xi')^\beta \leftrightarrow (\xi')^{\beta-1}$, for every $x \in (-r, r) \setminus M_0$ we have

$$\int_0^r |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta-1} d\xi' = \infty \quad \text{if and only if}$$

$$\int_0^\xi |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta-1} d\xi' = \infty \quad \text{holds for all } \xi \in (0, r). \quad (B.12)$$
Integrating the first inequality in (B.6) (in Lemma B.1 above), for every \( x \in (-r, r) \setminus M_0 \) and every \( \xi \in (0, r) \) we obtain
\[
\frac{\beta}{2} \int_0^\xi |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta-1} \, d\xi' + \ell_0(x) \\
\leq \xi^\beta \cdot |\nabla u(x, \xi)|^2 + \frac{2}{\beta} \int_0^\xi |\partial_\xi \nabla u(x, \xi')|^2 \cdot (\xi')^{\beta+1} \, d\xi'
\]
with the limit \( 0 \leq \ell_0(x) \leq \infty \). Thus, if the integral over \((0, r)\) in (B.12) were infinite, so would be the integral over \((0, \xi)\) for every \( 0 < \xi \leq r \). As the same integral over \((0, \xi)\) appears in Ineq. (B.13) as well, thanks to (B.11) this would force \( \xi^\beta \cdot |\nabla u(x, \xi)|^2 = \infty \) for every \( 0 < \xi \leq r \), thus contradicting (B.11). We conclude that, for every \( x \in (-r, r) \setminus M_0 \), all integrals in (B.12) must be finite, whenever \( 0 < \xi \leq r \). Moreover, also \( \ell_0(x) < \infty \) must hold. However, if \( \ell_0(x) > 0 \) then all integrals in (B.12) would have to be infinite, another contradiction. It follows that \( \ell_0(x) = 0 \).

Similarly, integrating both inequalities in (B.6), combined with \( \ell_0(x) = 0 \), for every \( x \in (-r, r) \setminus M_0 \) and every pair \( \xi_1, \xi_2 \in \mathbb{R} \), \( 0 < \xi_1 < \xi_2 \leq r \), we get
\[
\left| \xi_2^\beta \cdot |\nabla u(x, \xi_2)|^2 - \xi_1^\beta \cdot |\nabla u(x, \xi_1)|^2 \right| \\
\leq \frac{3\beta}{2} \int_0^\xi |\nabla u(x, \xi')|^2 \cdot (\xi')^{\beta-1} \, d\xi' + \frac{2}{\beta} \int_0^\xi |\partial_\xi \nabla u(x, \xi')|^2 \cdot (\xi')^{\beta+1} \, d\xi'.
\]

Consequently, for every \( x \in (-r, r) \setminus M_0 \), the function
\[
\xi \mapsto \xi^\beta \cdot |\nabla u(x, \xi)|^2 : (0, r] \to \mathbb{R}_+
\]
is absolutely continuous with the vanishing limit \( L_0(x) = \ell_0(x) = 0 \) in Eq. (B.10) (as \( \xi \to 0^+ \)). In particular, the inequalities in (B.9) are valid for every \( x \in (-r, r) \setminus M_0 \) and almost every \( \xi \in (0, r) \), with the function \( \xi \mapsto \xi^\beta \cdot |\nabla u(x, \xi)|^2 : (0, r] \to \mathbb{R}_+ \) being absolutely continuous on \([0, r]\) with the limit \( L_0(x) = 0 \).

Finally, we integrate all equations and inequalities (B.11) – (B.14) with respect to \( x \in (-r, r) \) to derive the following corollary of Lemma B.2.

**Corollary B.3 (Global trace inequalities).** Let \( \beta > 0 \) and \( R > 0 \), and set \( r = R/\sqrt{2} \). Assume that \( u \in \widetilde{H}^2(B_R^+; \omega) \). Then the following inequalities hold for every \( \xi \in (0, r) \):
\[
\frac{\beta}{2} \int_0^\xi (\xi')^{\beta-1} \int_{-r}^r |\nabla u(x, \xi')|^2 \cdot \, dx \, d\xi' \\
- \frac{2}{\beta} \int_0^\xi (\xi')^{\beta+1} \int_{-r}^r |\partial_\xi \nabla u(x, \xi')|^2 \cdot \, dx \, d\xi' \\
\leq \xi^\beta \int_{-r}^r |\nabla u(x, \xi)^2 | \, dx - \lim_{\xi' \to 0^+} \left[ (\xi')^\beta \int_{-r}^r |\nabla u(x, \xi')|^2 \, dx \right] \\
\leq \frac{3\beta}{2} \int_0^\xi (\xi')^{\beta-1} \int_{-r}^r |\nabla u(x, \xi')|^2 \cdot \, dx \, d\xi' \\
+ \frac{2}{\beta} \int_0^\xi (\xi')^{\beta+1} \int_{-r}^r |\partial_\xi \nabla u(x, \xi')|^2 \cdot \, dx \, d\xi'.
\]
All (Lebesgue) integrals above are finite and the limit (viewed as a boundary condition)
\[
\hat{L}_0 \overset{\text{def}}{=} \lim_{\xi^+ \rightarrow 0^+} \left[ (\xi^+)^\beta \int_{-r}^r |\nabla u(x, \xi^+)|^2 \, dx \right] = 0 \quad \text{exists.} \tag{B.16}
\]
In addition, the restriction mapping (B.3) from $\tilde{H}^2(B_R^+; \mathfrak{w})$ to $\tilde{H}^2(B_{R/\sqrt{2}}^+; \mathfrak{w})$ is continuous.

**Proof** The inequalities in (B.15) follow directly from those in (B.9). Of course, Ineq. (B.11) has to be replaced by
\[
\int_{Q_r^+} \left| \frac{\partial \xi}{\partial v} \nabla u(x, \xi) \cdot (\xi')^{\beta + 1} \right| \, dx \, d\xi' + \int_{Q_r^+} |\nabla u(x, \xi)|^2 \cdot (\xi')^\beta \, dx \, d\xi' < \infty.
\]
The vanishing limit $\hat{L}_0 = 0$ in Eq. (B.16) (as $\xi^+ \rightarrow 0^+$) is derived from (B.15) in an analogous way as is $L_0(x) = 0$ from Eq. (B.10) in our proof of Lemma B.2 above. Finally, we employ the inequalities in (B.15) to compare the norms on $H^2(B_{R/\sqrt{2}}^+; \mathfrak{w})$ and $H^2(B_{R}^+; \mathfrak{w})$, defined by Eqs. (2.2) and (B.4), respectively. Recall that $B_{R/\sqrt{2}}^+ \subset Q_{R/\sqrt{2}}^+ \subset B_R^+$. The continuity of the restriction mapping (B.3) follows. \qed

Our results in Corollary B.3 above will lead us to a restricted imbedding lemma, Lemma B.5, needed in Sect. 6, §6.2. This lemma will be derived from the following Hardy-Sobolev-type inequality proved in Castro [6, Theorem 4, p. 594].

**Lemma B4** (A Hardy-Sobolev-type inequality). Let $2 \leq p < \infty$, $R > 0$, and set $r = R/\sqrt{2}$. Assume that $a$, $a'$, $b \in \mathbb{R}$ satisfy the following inequalities,

(i) $0 < a - b < 1$, $0 < a' < 1$, and

(ii) $1 - \frac{2}{p} < (a + a') - b \leq 1$.

Define $p^* \in (0, \infty)$ by
\[
\frac{1}{p^*} + \frac{b + 1}{2} = \frac{1}{p} + \frac{a + a'}{2}, \quad \text{whence} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{p} + \frac{1}{p} - \frac{1}{p} \leq \frac{1}{p}. \tag{B.17}
\]
Then there exists a constant $C \equiv C(R; p) \in (0, \infty)$ such that
\[
\|u\|_{L^{p^*}(B_{R/\sqrt{2}}^+; \xi^{b \cdot p^*})} \overset{\text{def}}{=} \left( \int_{B_{R/\sqrt{2}}^+} |u(x, \xi)|^{p^*} \cdot \xi^{b \cdot p^*} \, dx \, d\xi \right)^{1/p^*} \leq C \cdot \|u\|_{W^{1,p}(B_{R}^+; \xi^{a \cdot p})} \tag{B.18}
\]
holds for all $u \in W^{1,p}(B_{R/\sqrt{2}}^+; \xi^{a \cdot p})$, i.e., for all $u \in W^{1,p}_{\text{loc}}(B_{R}^+)$ with the norm
\[
\|u\|_{W^{1,p}(B_{R}^+; \xi^{a \cdot p})} \overset{\text{def}}{=} \left( \int_{B_{R}^+} \left( |\nabla u(x, \xi)|^p + |u(x, \xi)|^p \cdot \xi^{a \cdot p} \right) \, dx \, d\xi \right)^{1/p} < \infty.
\]
In particular, for $p = 2$ and $p^* = 2^* = \frac{2}{(a + a') - b} \geq 2$ the following analogue of the restricted imbedding (B.3) is continuous, this time considered as a linear mapping

\[
u \rightarrow u|_{B_{R/\sqrt{2}}^+} : W^{1,2}(B_{R/\sqrt{2}}^+; \xi^{2a}) \longrightarrow L^{2^*}(B_{R}^+; \xi^{2^*a}).
\]

**Proof** This lemma follows directly from Theorem 4 in Castro [6, p. 594]. Since Theorem 4 in [6] is formulated for a $C^1$ function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ with compact support, we have to apply it to the function $\phi u$, where $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a $C^\infty$ function with the following
properties: \( \phi(x, \xi) = 1 \) in \( B_r(x_0, 0) \), \( \phi(x, \xi) = 0 \) in \( \mathbb{R}^2 \setminus B_R(x_0, 0) \), and \( 0 \leq \phi(x, \xi) \leq 1 \) in \( B_R(x_0, 0) \) \( \setminus B_r(x_0, 0) \). Recall that \( B_R \equiv B_R(x_0, \xi_0) \) denotes the open disc in \( \mathbb{R}^2 \) with radius \( R > 0 \) centered at the point \( (x_0, \xi_0) \in \mathbb{R}^2 \). We have abbreviated the upper half-disc by \( B_R^+ \equiv B_R^+(x_0, \xi_0) \). Applying [6, Theorem 4, p. 594] with the compactly support product function \( \phi u \in W^{1,p}(B_R^+; |x|^{a'\xi^{-ap}}) \), we obtain the following Hardy-Sobolev-type inequality,

\[
\| (\phi u) \|_{L^{p^*}(B_R^+; |x|^{b p^*})} = \left( \int_{B_R^+} |\phi u|^{p^*} \cdot |x|^{b p^*} \cdot dx \, d\xi \right)^{1/p^*} \\
\leq C' \cdot \| (\phi u) \|_{W^{1,p}(B_R^+; |x|^{a'\xi^{-ap}})} \\
= C' \cdot \left( \int_{B_R^+} |\nabla (\phi u)|^p \cdot |x|^{a'\xi^{-ap}} \cdot dx \, d\xi \right)^{1/p}
\]

(B.19)

for all \( u \in W^{1,p}_0(B_R^+) \) with \( \| (\phi u) \|_{W^{1,p}(B_R^+; |x|^{a'\xi^{-ap}})} < \infty \). Here, \( C' \equiv C'(R; p) \in (0, \infty) \) is a constant independent from the product function \( \phi u \). Thanks to \( \nabla (\phi u) = \phi \nabla u + u (\nabla \phi) \) with both \( \phi, |\nabla \phi| \in L^\infty(\mathbb{R}^2) \) and \( |x|^{a'} \leq R^d < \infty \) for \( x \in (-R, R) \), as well, we can apply the triangle inequality in \( L^p(B_R^+) \) to the right-hand side of Ineq. (B.19) in order to derive the desired inequality (B.18).

Unfortunately, earlier results of this kind (Feehan and Pop [18], Lemma 2.2, Eq. (2.2), on p. 1091, and Koch [27, Lemma 4.2.4, p. 62]) seem to be useless in our case due to the hypothesis \( u \in H^1(B_R^+; m) \) that is weaker than \( u \in W^{1,2}(B_R^+; \xi^{-\beta-1}) \) owing to the seminorm

\[
\| \nabla u \|_{L^2(B_R^+; \xi^{-\beta})} \overset{\text{def}}{=} \left( \int_{B_R^+} \left( |u_x|^2 + |u_\xi|^2 \right) \cdot \xi^{-\beta} \cdot dx \, d\xi \right)^{1/2}
\]

in Eqs. (B.1) and (B.4) being weaker than

\[
\| \nabla u \|_{L^2(B_R^+; \xi^{\beta-1})} \overset{\text{def}}{=} \left( \int_{B_R^+} \left( |u_x|^2 + |u_\xi|^2 \right) \cdot \xi^{\beta-1} \cdot dx \, d\xi \right)^{1/2}
\]

which appears in Eq. (2.2), thanks to \( \xi^{-\beta}/\xi^{\beta-1} = \xi^{-1} \).

In contrast to these results, the next lemma enables us to establish the restricted Sobolev imbedding (6.5) (see §6.2) by replacing the pair \( (p, p^*) \) by \( (2, p) = (2, 2^*) \). In this pair we allow for \( p = 2^* \in (2, \infty) \) arbitrary to which we associate suitable constants \( a, a', b \in (0, \infty) \) that verify conditions (i), (ii), and Eq. (B.17) of Lemma B4.

**Lemma B.5 (Two Sobolev-type imbeddings).** Let \( 2 < p < \infty \) and \( R > 0 \) be arbitrary, and set \( r = R/\sqrt{2} \). Let \( \beta \in \mathbb{R} \) satisfy

\[
0 < \beta - 1 < \frac{4}{p - 2}.
\]

(B.20)

Then there exists a constant \( C_\beta \equiv C_\beta(R; p) \in (0, \infty) \) such that

\[
\| u \|_{L^p(B_r^+; \xi^{\beta-1})} = \left( \int_{B_r^+} |u(x, \xi)|^p \cdot \xi^{\beta-1} \cdot dx \, d\xi \right)^{1/p} \\
\leq C_\beta \cdot \| u \|_{W^{1,2}(B_r^+; \xi^{\beta-1})} \\
= C_\beta \cdot \left( \int_{B_r^+} (|\nabla u|^2 + |u|^2) \cdot \xi^{\beta-1} \cdot dx \, d\xi \right)^{1/2}
\]

(B.21)
holds for all $u \in W^{1,2}(B^+_{r}; \xi^{\beta-1})$.

Furthermore, there exists another constant $C'_\beta \equiv C'_\beta(R; p) \in (0, \infty)$ such that

$$
\|u\|_{L^p(B^+_{r/2}; \xi^{\beta-1})} \leq C'_\beta \cdot \|u\|_{H^2(B^+_{r}; \omega)} \quad \text{for all } u \in H^2(B^+_{r}; \omega).
$$

(B.22)

In particular, the restricted Sobolev imbedding (cf. Eq. (6.5))

$$
u|_{B^+_{r/2}} \mapsto u|_{B^+_{r/2}} : H^2(B^+_{r}; \omega) \hookrightarrow L^p\left(B^+_{r/2}; \omega\right)
$$

(B.23)
is continuous.

**Proof** We wish to apply Lemma B4 stated above with the weight $\xi^{\beta-1}$ as indicated in (6.5). We replace the pair $(p, p^*)$ by $(2, 2^*)$ and forget the former one entirely; thus, from now on, we may write $p = 2^*$ with $2 < p < \infty$. We need to fix the constant $\beta \in (1, \infty)$ in such a way that Lemma B4 is applicable with suitable constants $a, a', b \in \mathbb{R}$. Consequently, we choose the constants

$$a = \frac{\beta - 1}{2} \quad \text{and} \quad b = \frac{\beta - 1}{2^*} = \frac{\beta - 1}{p}.$$

Clearly, we have $a > 0$, $b > 0$, and $a - b > 0$, by $p > 2$. In order to fulfill also the condition $a - b < 1$, we have to choose $\beta \in (1, \infty)$ such that $a - b = (\beta - 1)\left(\frac{1}{2} - \frac{1}{p}\right) < 1$ or, equivalently, $1 < \beta < 1 + \frac{2p}{p-1}$. These inequalities follow from our choice of $p > 2$ and $\beta$ obeying the conditions in (B.20). We have no other restriction on $p \in (2, \infty)$. The remaining constant, $a' \in (0, 1)$, must be chosen in such a way that Eq. (B.17) holds with the pair $(2, 2^*)$ in place of $(p, p^*)$, i.e., $a' = 2^*/2 - (a - b)$, together with the inequalities $a' > 0$ and $(a + a') - b \leq 1$. Since $2^* = p > 2$, we get $a' = \frac{2^*}{2^*} - (a - b) < 1 - (a - b)$, i.e., $a + a' < b + 1$. It remains to verify $a' > 0$ which is equivalent with (from now on we write $2^* = p > 2$)

$$
\frac{2}{p} > (\beta - 1)\left(\frac{1}{2} - \frac{1}{p}\right).
$$

This inequality is equivalent with the condition in (B.20).

The desired inequality in (B.21) now follows directly from Lemma B4. Finally, we apply Corollary B.3, Eq. (B.15) with $R$ and $r = R/\sqrt{2}$, to the right-hand side of Ineq. (B.21) with $r$ and $R/2 = r/\sqrt{2}$ (in place of $R$ and $r = R/\sqrt{2}$, respectively) to derive (B.22). \hfill $\square$

**C Appendix: Some known elliptic regularity results**

In this appendix we collect a few known results on the local regularity of a weak solution $u \in V$ to the degenerate elliptic problem $(\lambda I + A)u = f \in H$, i.e., for $u = (\lambda I + A)^{-1} f \in V$. Recall that $\lambda > \lambda_0$ with the constant $\lambda_0 > 0$ determined by Ineq. (5.1).

The first regularity result is due to Feehan and Pop [17], Theorem 3.16, Eq. (3.12), on p. 385.

**Lemma C.1** ($H^2$-smoothing property). Let $\rho, \sigma, \theta, q_r$, and $\gamma$ be given constants in $\mathbb{R}$, $\rho \in (-1, 1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that $\beta, \gamma, \kappa$, and $\mu$ are chosen as specified in Proposition 4.1 and $\lambda > \lambda_0$. Then, given any $x_0 \in \mathbb{R}$ and $R_0$, $R_1 \in \mathbb{R}$ with $0 < R_1 < R_0$, and any function $f \in H$, the restriction $u|_{B^+_{R_1}(x_0, 0)}$ of the function $u = (\lambda I + A)^{-1} f \in V$
to the open half-disc \( B^+_{R_1}(x_0, 0) \) satisfies \( u|_{B^+_{R_1}(x_0, 0)} \in H^2 \left( B^+_{R_1}(x_0, 0); \mathfrak{w} \right) \). Furthermore, there is a constant \( C_1 \in (0, \infty) \) independent from \( f \) and \( u \), such that

\[
\|u\|_{H^2_B(x_0, 0); \mathfrak{w}} \leq C_1 \left( \|u\|_{L^2(B^+_{R_0}(x_0, 0); \mathfrak{w})} + \|f\|_{L^2(B^+_{R_0}(x_0, 0); \mathfrak{w})} \right).
\]  

(C.1)

(The weighted Sobolev norm on the left-hand side has been introduced in Eq. (2.2).)

This lemma gets us from \( H = L^2(\mathbb{R}; \mathfrak{w}) \) into the (local) interior regularity of weighted \( H^2 \)-type over an open half-disc \( B^+_{R_1}(x_0, 0) \). Our restricted Hardy-Sobolev-type imbedding (Lemma B.5) brings an \( H^2 \)-type function into a weighted \( L^p \)-space over a smaller open half-disc \( B^+_{R_1'}(x_0, 0) \) with the radius \( R_1' = R_1/2 \) \((0 < R_1' < R_1 < R_0)\). (This step will require an additional upper bound on \( \beta > 1 \), in addition to Ineq. (4.2), in order to allow for \( p > 2 \) large enough in Lemma C.2 below and still fulfill Ineq. (B.20) in Lemma B.5.)

Now we continue with another local regularity result for a weak solution \( u = (\lambda I + A)^{-1} f \in V \), this time for \( u \in H \) with \( f \in V \) satisfying also \( f|_{B^+_{R_1}(x_0, 0)} \in L^p \left( B^+_{R_1}(x_0, 0); \mathfrak{w} \right) \).

This weighted \( L^p \)-space has been introduced in Eq. (2.3).)

Lemma C.2 (\( C^\alpha_s \)-smoothing property). Let \( \rho, \sigma, \theta, q_r \), and \( \gamma \) be given constants in \( \mathbb{R} \), \( \rho \in (-1, 1), \sigma > 0, \theta > 0, \) and \( \gamma > 0 \). Assume that \( \beta, \gamma, \kappa, \) and \( \mu \) are chosen as specified in Proposition 4.1 and \( \lambda > \lambda_0 \). Finally, let \( p \) satisfy \( \max\{4, 2 + \beta\} < p < \infty \). Then, given any \( x_0 \in \mathbb{R} \) and \( R_1' \in (0, R_1) \), there are constants \( R_2 \equiv R_2(R_1') \), which depends on \( R_1', \alpha \in (0, 1) \), and \( C_2 \in (0, \infty) \) with the following properties:

(a) \( 0 < R_2 < R_1' \),

(b) given any function \( f \in H \) with the restriction \( f|_{B^+_{R_1'}(x_0, 0)} \in L^p \left( B^+_{R_1'}(x_0, 0); \mathfrak{w} \right) \), the restriction \( u|_{B^+_{R_2}(x_0, 0)} \) of the function \( u = (\lambda I + A)^{-1} f \in V \) to the closed half-disc \( B^+_{R_2}(x_0, 0) \) satisfies \( u|_{B^+_{R_2}(x_0, 0)} \in C^\alpha_s \left( B^+_{R_2}(x_0, 0); \mathfrak{w} \right) \), and

(c) for all pairs \( f \) and \( u \) from Part (b), the following inequality holds,

\[
\|u\|_{C^\alpha_s \left( B^+_{R_2}(x_0, 0) \right)} \leq C_2 \left( \|u\|_{L^2 \left( B^+_{R_1'}(x_0, 0); \mathfrak{w} \right)} + \|f\|_{L^p \left( B^+_{R_1'}(x_0, 0); \mathfrak{w} \right)} \right).
\]

(C.2)

(The weighted Hölder norm on the left-hand side has been introduced in Eq. (2.5).)

This lemma improves the (local) interior regularity of \( u \in H \) from \( L^p \left( B^+_{R_1'}(x_0, 0); \mathfrak{w} \right) \) to the weighted Hölder space \( C^\alpha_s \left( B^+_{R_2}(x_0, 0) \right) \), \( 0 < R_2 < R_1' \) \((< R_1 < R_0)\). The proof of this lemma is given in Feehan and Pop [18], Theorem 1.11, Eq. (1.31), on p. 1083; see also [17], Theorem 2.5, Eq. (2.12), pp. 375–376. We stress that the constant \( R_2 \equiv R_2(R_1') \) depends on \( R_1' \), while \( R_1' \in (0, \infty) \) is arbitrary.

The last (local) interior regularity results for \( u \in H \) brings \( u \) from \( C^\alpha_s \left( B^+_{R_2}(x_0, 0) \right) \), to another weighted Hölder space \( C^{2+\alpha}_s \left( B^+_{R_2'}(x_0, 0) \right) \), \( 0 < R_2' < R_2 \) \((< R_1' < R_1 < R_0)\). (The weighted Hölder space above has been introduced in Eq. (2.6).) Here, the constants \( R_2 \) and \( R_2' \) are arbitrary with \( 0 < R_2' < R_2 < \infty \).

Lemma C.3 (\( C^{2+\alpha}_s \)-smoothing property). Let \( \rho, \sigma, \theta, q_r \), and \( \gamma \) be given constants in \( \mathbb{R} \), \( \rho \in (-1, 1), \sigma > 0, \theta > 0, \) and \( \gamma > 0 \). Assume that \( \beta, \gamma, \kappa, \) and \( \mu \) are chosen as specified.
in Proposition 4.1 and λ > λ_0. Finally, let α ∈ (0, 1) be arbitrary. Then, given any x_0 ∈ ℝ and R_2, R'_2 ∈ ℝ with 0 < R'_2 < R_2, and any function f ∈ H with the restriction f|_{B^*_R(x_0, 0)} ∈ C^α_s\left(\overline{B^+_R(x_0, 0)}\right), the restriction u|_{B^*_R(x_0, 0)} of the function u = (λI + A)^{-1} f ∈ V to the closed half-disc \overline{B^+_R(x_0, 0)} satisfies u|_{B^*_R(x_0, 0)} ∈ C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right). Furthermore, there is a constant C_3 ∈ (0, ∞) independent from f and u, such that
\[
\|u\|_{C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right)} ≤ C_3\left(\|f\|_{C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right)} + \|u\|_{C\left(\overline{B^+_R(x_0, 0)}\right)}\right). \tag{C.3}
\]

This lemma is proved in Feehan and Pop [16], Theorem 8.1, Eq. (8.4), pp. 937–938 (see also Feehan [14], Theorem 1.1, Part 2, on pp. 2487–2488).

Lemma C.3 has the following important consequence for the boundary limits (as \(ξ \to 0^+\)) of the functions \(ξ \cdot f_{xx}, ξ \cdot f_{xξ}, \) and \(ξ \cdot f_{ξξ}, \) provided \(f ∈ C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right).\)

**Corollary C.4 (Boundary limits in \(C^{2+α}_s\)).** Let \(α ∈ (0, 1)\) be arbitrary, \(x_0 ∈ ℝ\) and \(R ∈ (0, ∞).\) Then every function \(f ∈ C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right)\) has the following behavior near the boundary ∂\(H = ℝ \times \{0\}\) of the half-plane \(H = ℝ \times (0, ∞) \subset ℜ^2:\)
\[
\lim_{ξ → 0^+} \left[ξ \cdot \left(|f_{xx}(x, ξ)| + |f_{xξ}(x, ξ)| + |f_{ξξ}(x, ξ)|\right)\right] = 0 \tag{C.4}
\]
for every \(x ∈ (x_0 − R, x_0 + R).\) In addition, there exists a constant \(c_α ∈ (0, ∞)\) such that
\[
|f_{xx}(x, ξ)| + |f_{xξ}(x, ξ)| + |f_{ξξ}(x, ξ)| ≤ c_α \|f\|_{C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right)} \cdot ξ^{1-(α/2)} \tag{C.5}
\]
for all \((x, ξ) ∈ B^+_R(x_0, 0).\)

**Proof** In a somewhat stronger version stated in Eq. (2.7), the limit (C.4) in this corollary is proved in Feehan and Pop [15], Lemma 3.1, Eq. (3.1), on p. 4409 (see also Daskalopoulos and Hamilton [11], Prop. I.12.1 on p. 940). The estimate (C.5) is derived from this limit combined with \(f ∈ C^{2+α}_s\left(\overline{B^+_R(x_0, 0)}\right);\) cf. Eq. (2.6) with all \(ξ \cdot f_{xx}, ξ \cdot f_{xξ}, ξ \cdot f_{ξξ} ∈ C^α_s\left(\overline{B^+_R(x_0, 0)}\right).\)

As far as the Hölder norm in \(C^α_s\left(\overline{B^+_R(x_0, 0)}\right),\) given by Eq. (2.5), is concerned, notice that for every pair of points \(P^* = (x, 0) ∈ ∂\(H\) and \(P = (x, ξ) ∈ H,\) with \(x ∈ (x_0 − R, x_0 + R)\) and \((x, ξ) ∈ B^+_R(x_0, 0),\) we have the s-distance \(s(P, P^*) = \sqrt{ξ^2/2}\) (see Eq. (2.4)). Hence, the inequality in (C.5) follows.

\[\square\]

**References**

1. Ackerer, D., Filipović, D.: Option pricing with orthogonal polynomial expansions. Math. Finance 30(1), 47–84 (2020). https://doi.org/10.1111/mafi.12226. (Online)
2. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, 2nd edn. Academic Press, New York Oxford (2003)
3. B. Alziary and P. Takač. Analytic Solutions and Complete Markets for the Heston Model with Stochastic Volatility, Electronic J. Diff. Equations, 2018(168) (2018), 1–54. ISSN: 1072-6691. Online: http://ejde.math.txstate.edu. Preprint arXiv:1711.04536v1 [math.AR], 13th November 2017
4. F. Baustian, K. Filipová, and J. Pospíšil.: Solution of option pricing equations using orthogonal polynomial expansion, Appl. Math. 66(4), 553–582 (2021). https://doi.org/10.21136/AM.2021.0361-19
5. Björk, T.: Arbitrage Theory in Continuous Time, 3rd Ed., Oxford Univ. Press, Oxford, (2011)
6. Castro, Hernán: Hardy–Sobolev-type inequalities with monomial weights. Annali Mat. Pura Appl (4) 196(2), 579–598 (2017). https://doi.org/10.1007/s10231-016-0587-2. (Online)
7. Chiarella, C., Kang, B., Meyer, G.H.: The Numerical Solution of the American Option Pricing Problem, Finite Difference and Transform Approaches. World Scientific Publ. Co., New JerseyLondonSingapore (2015)
8. Davis, Mark H. A., Oblój, Jan.: Market Completion Using Options, Banach Center Publ., Vol. 83 (2008), pp. 49–60. Polish Acad. Sci., Warsaw, 2008. https://doi.org/10.4064/bc83-0-4
9. Daskalopoulos, P., Feehan, P. M. N.: Existence, uniqueness, and global regularity for variational inequalities and obstacle problems for degenerate elliptic partial differential operators in mathematical finance, Preprint at arXiv:1109.1075v1
10. Daskalopoulos, P.,Feehan, P.M.N.: $C^{1,1}$-regularity for degenerate elliptic obstacle problems. J. Differ. Equ. 260, 5043–5074 (2016)
11. Daskalopoulos, P., Hamilton, R.: Regularity of the free boundary for the porous medium equation. J. Am. Math. Soc. 11(4), 899–965 (1998)
12. Düring, B., Fourniè, M.: High-order compact finite difference scheme for option pricing in stochastic volatility models. J. Comput. Appl. Math. 236, 4462–4473 (2012). https://doi.org/10.1016/j.cam.2012.04.017. (Online)
13. Evans, L.C.: Partial Differential Equations, in Graduate Studies in Mathematics, vol. 19. Amer. Math. Society, Providence, R.I. (1998)
14. Feehan, P.M.N.: A classical Perron method for existence of smooth solutions to boundary value and obstacle problems for degenerate-elliptic operators via holomorphic maps. J. Differ. Equ. 263, 2481–2553 (2017). https://doi.org/10.1016/j.jde.2017.04.003. (Online)
15. Feehan, P.M.N., Pop, C.A.: A Schauder approach to degenerate-parabolic partial differential equations with unbounded coefficients. J. Differ. Equ. 254, 4401–4445 (2013). https://doi.org/10.1016/j.jde.2013.03.006. (Online)
16. Feehan, P.M.N., Pop, C.A.: Schauder a priori estimates and regularity of solutions to boundary-degenerate elliptic linear second-order partial differential equations. J. Differ. Equ. 256, 895–956 (2014). https://doi.org/10.1016/j.jde.2013.08.012. (Online)
17. Feehan, P.M.N., Pop, C.A.: Degenerateelliptic operators in mathematical finance and highorder regularity for solutions to variational equations. Adv. Differ. Equ. 20(3/4), 361–432 (2015)
18. Feehan, P.M.N., Pop, C.A.: Boundary-degenerate elliptic operators and Hölder continuity for solutions to variational equations and inequalities, Annales Inst. Henri Poincaré C - Analyse non linéaire 34(5), 1075–1129 (2017). https://doi.org/10.1016/j.anihpc.2016.07.005. (Online)
19. Feller, W.: Two singular diffusion problems. Annals of Math., 2-nd Series, 54(1) (1951), 173–182
20. Fouque, J.-P., Papanicolaou, G., Sircar, K.R.: Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press, Cambridge, U.K. (2000)
21. Friedman, A.: Partial Differential Equations of Parabolic Type. PrenticeHall, Englewood Cliffs, N.J. (1964)
22. Friedman, A.: Stochastic Differential Equations and Applications, Vol. 1, in Probability and Mathematical Statistics, Vol. 28. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975
23. Guo, S., Grzelak, L.A., Oosterlee, C.W.: Analysis of an affine version of the Heston-Hull-White option pricing partial differential equation. Appl. Numer. Math. 72, 143–159 (2013). https://doi.org/10.1016/j.apnum.2013.06.004. (Online)
24. G. H. Hardy, J. E. Littlewood, and G. Pólya, “Inequalities”, 2nd ed., Cambridge Univ. Press, Cambridge, U.K., (1952)
25. Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6(2), 327–343 (1993). (http://www.jstor.org/stable/2962057, Online: Stable)
26. Ikonen, S., Toivanen, J.: Operator splitting methods for pricing American options under stochastic volatility. Numer. Math. 113, 299–324 (2009). https://doi.org/10.1007/s00211-009-0227-5. (Online)
27. Koch, H.: Non-Euclidean Singular Integrals and the Porous Medium Equation, Habilitation Thesis, University of Heidelberg, Germany, (1999)
28. Kufner, A.: Weighted Sobolev Spaces, in Teubner Texts in Mathematics, vol. 31. A WileyInterscience Publication. John Wiley & Sons Inc, New York (1985)
29. Lions, J.-L.: Équations différentielles opérationnelles et problèmes aux limites, in Grundlagen der mathematischen Wissenschaften, vol. 111. SpringerVerlag, BerlinGöttingenHeidelberg (1961)
30. J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, in Grundlagen der mathematischen Wissenschaften, Vol. 170. Springer-Verlag, Berlin-Heidelberg-New York, 1971. (Translated from the French by S. K. Mitter.)
31. Meyer, G.H.: The TimeDiscrete Method of Lines for Options and Bonds, A PDE Approach. World Scientific Publ. Co., New JerseyLondonSingapore (2015)
32. Øksendal, B.: Stochastic Differential Equations: An Introduction with Applications, 6th Ed., Springer-Verlag, Berlin-Heidelberg-New York, (2003)
33. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, in Applied Mathematical Sciences, vol. 44. SpringerVerlag, New YorkBerlinHeidelberg (1983)
34. Takáč, P.: Space-time analyticity of weak solutions to linear parabolic systems with variable coefficients. J. Funct. Anal. 263(1), 50–88 (2012). https://doi.org/10.1016/j.jfa.2012.04.008. (Online)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.