A commutative noetherian local ring of embedding dimension 4 having a transcendental series of Betti numbers

Jan-Erik Roos
Department of Mathematics
Stockholm University
e-mail: jeroos@math.su.se

February 14, 2017

Abstract
We construct a ring with the properties of the title of the paper. We also construct some other local rings of embedding dimension 4 with exotic properties. Among the methods used are the Macaulay2-package DGAlgebras by Frank Moore, combined with and inspired by results by Anick, Avramov, Backelin, Katthän, Lemaire, Levin, Löfwall and others.

Mathematics Subject Classification (2010): Primary 13Dxx, 16Dxx, 68W30; Secondary 16S37, 55Txx

Keywords. local ring, Koszul complex, Yoneda Ext-algebra, Hilbert series, Macaulay2, Betti series, Avramov spectral sequence

0. Introduction

The aim of the present paper is to prove the following (if $V$ is a vector space over a field $k$ we write $|V|$ instead of $\dim_k V$):

THEOREM 1 – Let $k$ be a field and $k[x,y,z,u]$ the polynomial ring in four variables of degree one and let

$$R = \frac{k[x,y,z,u]}{(x^3, x^2 y, (x+y)(y^2+z^2) + yzu, zu^2, u^3)}$$

be the graded quotient ring. Let

$$P_R(Z) = \sum_{i \geq 0} |\text{Tor}^R_i(k,k)| Z^i$$

be the corresponding series of Betti numbers of $R$. Then $P_R(Z)$ is a transcendental function:

$$P_R(Z) = \frac{Z(1+Z)^2}{(1+Z)(1-Z-Z^2)^2 \prod_{n=1}^{\infty} (1-Z^{2n}) - 1 + 2Z + 2Z^2 - 4Z^3 - 4Z^4}$$
If one replaces \( k[x, y, z, u] \) with the formal power series ring \( k[[x, y, z, u]] \) one obtains a local ring with the same homological properties. The Theorem might be of interest in itself (previously known examples of transcendental series needed \( \geq 5 \) variables) but more interesting are probably the methods used to obtain the example (described in section 1 below) and the methods of proof described in section 2. There is now a possibility to obtain a new good insight in the theory of the homology of local rings of embedding dimension 4. In section 3 we will in particular indicate that the following ring which is a simpler variant of (1)

\[
R = \frac{k[x, y, z, u]}{(x^3, x^2y, (x + u)(y^2 + z^2), zu^2, u^3)}
\]

has a \( P_R(Z) = (1 - Z^2)(1 + Z)/(1 - 3Z - 3Z^3 - 2Z^5) \) i.e. a rational function but is such that the Yoneda Ext-algebra \( \text{Ext}^*_{R}(k, k) \) is not finitely generated as an algebra. This last phenomenon was also previously only known in the embedding dimension \( \geq 5 \) cases. We will also deduce some other transcendental results and also show that there are quotients of \( k[x, y, z, u] \) with an ideal with only six cubic generators which is non-Golod but has the multiplication in Koszul homology equal to zero (the first examples of this last phenomenon with more relations - monomials of higher degrees and more variables - were obtained by Lukas Katthän [KAT]).

### 1. How the example in Theorem 1 was found

In [A] (announced in 1980 [A-CR]), David Anick published the first example (example 7.1 page 29 of [A]) of an \( R \) in 5 variables and 7 quadratic relations for which \( P_R(Z) \) was transcendental. Here is a variant of his example with only 5 quadratic relations which has the same property and which will be useful for us:

\[
R = \frac{k[x_1, x_2, x_3, x_4, x_5]}{(x_1^2, x_1x_2, x_1x_3 + x_2x_4 + x_3x_5, x_4x_5, x_5^2)}
\]

In this case the Hilbert series of \( R \) is \( H(t) = (1 + 3t + t^2 - 2t^3 + t^4)/(1 - t)^2 \) and the Hilbert series of the Koszul dual \( R! \) of \( R \) is

\[
R!(t) = \frac{(1 + t)^2}{(1 - t - t^2)^2} \prod_{n=1}^{\infty} \frac{1 + t^{2n-1}}{1 - t^{2n}}
\]

and

\[
1/P_R(Z) = (1 + 1/Z)/R!(Z) - H(-Z)/Z
\]
Remark: For the case studied by Anick [A] we had that $H(t) = 1 + 5t + 7t^2$ and

$$R^1(t) = \frac{1}{(1 - 2t)^2} \prod_{n=1}^{\infty} \frac{1 + t^{2n-1}}{1 - t^{2n}}$$

and the formula (4) was also valid in his case.

We now address the problem of constructing a 4-variable version of the ring (3). We start with the ring $S_2 = k[x, y, z, u]/(x^2, xy, zu, u^2)$ where $x, y, z, u$ correspond to $x_1, x_2, x_4, x_5$ and try to add an extra relation corresponding to $x_1x_3 + x_2x_4 + x_3x_5$. How to do this is not at all evident and instead we try to add a new relation which is a linear combination with coefficients 0 or 1 of the 6 nonzero quadratic monomials $yu, xu, z^2, yz, xz, y^2$ in $S_2$. There are 64 such linear combinations but each of them leads to a rational $P_R(Z)$ (they all occur in the Appendix to [R4]). Instead we turn to the study of cubic relations, i.e. we start with a new ring

$$S = \frac{k[x, y, z, u]}{(x^3, x^2y, zu^2, u^3)}$$

and try to add a linear combination (coefficients 0 or 1) of the 16 non-zero cubic monomials

$$yu^2, xu^2, z^2u, yzu, xzu, y^2u, xyu, x^2u, z^3, yz^2, xz^2, y^2z, xyz, x^2z, y^3, xy^2$$

in $S$. There are $2^{16} = 65536$ cases to study and this can be done automatically using the following input programme to Macaulay 2 which was written at my request several years ago by Mike Stillman (it is an elegant version of a programme I wrote for Macaulay 1 a long time ago (cf. [R3, pp. 294-296]))

```plaintext
binaries = (n) -> (  
  if n === 0 then {}
  else if n === 1 then {{0},{1}}
  else (r := binaries(n-1));
  join(apply(r, i->prepend(0,i)),
    apply(r, i->prepend(1,i))))

doit = (i) -> (  
  h := hypers(0,i);
  J1 := J + ideal(h);
  << newline << flush;
  << '--- n = ' << i << ' ideal = ' << hypers(0,i) << '---' << flush;
  << newline << flush;
```
E := res J1;
<< newline << flush;
<< ' ' ' ' << betti E << flush;
<< newline << flush;
A := (ring J1)/J1;
C := res(coker vars A, LengthLimit=>6);
<< newline << flush;
<< ' ' ' ' << betti C << flush;
<< newline << flush;
<< ' ' ' ' << hilbertSeries(A,Order=>12) << flush;
<< newline << flush;
)
makeHyperplanes = (J) -> ( 
I := matrix basis(3,coker gens J);
c := numgens source I;
m := transpose matrix binaries c;
I * m)
R = QQ[x,y,z,u]
J = ideal(x^3,x^2*y,z*u^2,u^3)
time hypers = makeHyperplanes J;
time scan(numgens source hypers, i -> doit i);

We obtain 20 possibilities for the $P_R(z)$ up to degree 6 presented here in increasing order (of course there can be variations inside each case $P_i(z)$ due e.g. to different Koszul homology):

$P_1(z)=1+4z+11z^2+27z^3+62z^4+137z^5+295z^6+...$  
$P_2(z)=1+4z+11z^2+27z^3+64z^4+152z^5+364z^6+...$

$P_3(z)=1+4z+11z^2+28z^3+69z^4+168z^5+407z^6+...$  
$P_4(z)=1+4z+11z^2+28z^3+69z^4+169z^5+414z^6+...$

$P_5(z)=1+4z+11z^2+28z^3+69z^4+172z^5+431z^6+...$  
$P_6(z)=1+4z+11z^2+28z^3+70z^4+173z^5+427z^6+...$

$P_7(z)=1+4z+11z^2+28z^3+70z^4+177z^5+451z^6+...$  
$P_8(z)=1+4z+11z^2+29z^3+74z^4+188z^5+476z^6+...$

$P_9(z)=1+4z+11z^2+29z^3+75z^4+193z^5+496z^6+...$  
$P_{10}(z)=1+4z+11z^2+29z^3+75z^4+193z^5+498z^6+...$

$P_{11}(z)=1+4z+11z^2+29z^3+75z^4+194z^5+503z^6+...$  
$P_{12}(z)=1+4z+11z^2+30z^3+80z^4+213z^5+567z^6+...$

$P_{13}(z)=1+4z+11z^2+30z^3+81z^4+218z^5+587z^6+...$  
$P_{14}(z)=1+4z+11z^2+30z^3+82z^4+224z^5+612z^6+...$
\[ P_{15}(z) = 1 + 4z + 11z^2 + 31z^3 + 86z^4 + 238z^5 + 660z^6 + \ldots \quad P_{16}(z) = 1 + 4z + 11z^2 + 31z^3 + 87z^4 + 244z^5 + 685z^6 + \ldots \]

\[ P_{17}(z) = 1 + 4z + 11z^2 + 31z^3 + 88z^4 + 249z^5 + 705z^6 + \ldots \quad P_{18}(z) = 1 + 4z + 11z^2 + 32z^3 + 92z^4 + 263z^5 + 755z^6 + \ldots \]

\[ P_{19}(z) = 1 + 4z + 11z^2 + 32z^3 + 93z^4 + 269z^5 + 780z^6 + \ldots \quad P_{20}(z) = 1 + 4z + 11z^2 + 33z^3 + 99z^4 + 294z^5 + 877z^6 + \ldots \]

Which one of these cases could give a transcendental \( P_R(Z) \)?

One can guess a possibility by studying how the transcendental \( P_R(z) \) comes up in five variables when we add linear combinations of quadratic monomials to the ring in 5 variables: \( \mathbb{Q}[x, y, z, u, v]/(x^2, xy, uv, v^2) \) i.e. modifying the previous input programme by replacing \( R \) by \( \mathbb{Q}[x, y, z, u, v] \), \( J \) by \( (x^2, x \ast y, u \ast v, v^2) \) and the line

\[ I := \text{matrix basis}(3, \text{coker gens } J) \]

by the line

\[ I := \text{matrix basis}(2, \text{coker gens } J) \]

There are now “only” \( 2^{11} = 2048 \) cases to study and now we obtain only 10 possibilities for the \( P_R(z) \) up to degree 6 presented here in increasing order:

\[ P_1(z) = 1 + 5z + 15z^2 + 37z^3 + 82z^4 + 170z^5 + 337z^6 + \ldots \quad P_2(z) = 1 + 5z + 15z^2 + 38z^3 + 88z^4 + 192z^5 + 406z^6 + \ldots \]

\[ P_3(z) = 1 + 5z + 15z^2 + 38z^3 + 89z^4 + 199z^5 + 432z^6 + \ldots \quad P_4(z) = 1 + 5z + 15z^2 + 38z^3 + 91z^4 + 216z^5 + 516z^6 + \ldots \]

\[ P_5(z) = 1 + 5z + 15z^2 + 39z^3 + 95z^4 + 222z^5 + 506z^6 + \ldots \quad P_6(z) = 1 + 5z + 15z^2 + 39z^3 + 96z^4 + 231z^5 + 553z^6 + \ldots \]

\[ P_7(z) = 1 + 5z + 15z^2 + 39z^3 + 97z^4 + 237z^5 + 575z^6 + \ldots \quad P_8(z) = 1 + 5z + 15z^2 + 39z^3 + 99z^4 + 254z^5 + 659z^6 + \ldots \]

\[ P_9(z) = 1 + 5z + 15z^2 + 40z^3 + 104z^4 + 268z^5 + 689z^6 + \ldots \quad P_{10}(z) = 1 + 5z + 15z^2 + 41z^3 + 112z^4 + 306z^5 + 836z^6 + \ldots \]

There are e.g. 1024 cases of an extra quadratic relation corresponding to \( P_1(z) \) and 576 cases corresponding to \( P_4(z) \), but only 8 for \( P_2(z) \) including the case of the extra relation \( xz + yu + zv \) leading to the transcendental series mentioned in the introduction (the other 7 cases of \( P_2(z) \) lead to the same total series). Furthermore, there are also 8 cases for \( P_6(z) \) and they all lead to cases where the Yoneda Ext-algebra \( \text{Ext}_R^*(k, k) \) is not finitely generated but the series of Betti numbers is rational (this phenomenon was first found by me in [R1] (inspired by Lemaire [LEM]).

We now try to use this information to try to guess what happens for the 65536 embedding dimension 4 cases above. In this situation there are only 80 cases of \( P_{10}(z) \) and only two of them correspond to adding a cubic relation with is a sum of 5 cubic monomials (the other cases need more monomials): case 9257 which corresponds to \( xy^2 + xz^2 + y^2u + yzu + z^2u \) and case 13345 which corresponds to \( xy^2 + xyz + xz^2 + y^2u + z^2u \). Here one of
these cases can be obtained from the other one by permuting x and u. We can therefore restrict ourselves to the homological study of the ring

\[
\frac{k[x, y, z, u]}{(x^3, x^2y, (x + u)(y^2 + z^2) + yzu, zu^2, u^3)}
\]

mentioned in the introduction and which will be the subject of the next section.

2. The treatment of the example in Theorem 1

We will now use the programme DGA\text{Algebra} of [MOO] on the ring (1).

The Koszul homology of the ring has the Betti numbers given by Macaulay:

| total: | 1 | 5 | 12 | 12 | 4 |
|--------|---|---|----|----|---|
| 0:     | 1 | . | . | . | . |
| 1:     | . | . | . | . | . |
| 2:     | . | 5 | 2 | . | . |
| 3:     | . | . | 1 | . | . |
| 4:     | . | . | 8 | 10 | 3 |
| 5:     | . | . | . | . | . |
| 6:     | . | . | 1 | 2 | 1 |

From this we can guess that the multiplication in \( HKR = \text{Tor}^{k[x,y,z,u]}_{*}(k[x, y, z, u]/J, k) \), where \( J \) is the ideal in (1) is rather complicated. We determine details about that multiplication as a preparation for using the Avramov spectral sequence ([AV0],[AV1]) where \( \tilde{R} = k[x, y, z, u] \) and \( R = \tilde{R}/J \):

\[
E^2_{p,q} = \text{Tor}^{HKR}_{p,q}(k, k) \Rightarrow \text{Tor}_{*}^{\tilde{R}}(k, k)/\text{Tor}_{*}^{R}(k, k)
\]

by using the following infile for Macaulay:

```macaulay
loadPackage('DGAlgebras')
R=QQ[x,y,z,u]/ideal(x^3,x^2*y,z*u^2,u^3,x*y^2+x*z^2+y^2*u+y*z*u+z^2*u)
res(ideal R); betti oo
res(coker vars R,LengthLimit => 6); betti oo
HKR=HH koszulComplexDGA(R)
generators HKR
for n from 1 to length(generators HKR) list degree X_n
ideal HKR
```
\[ C = \frac{k[X_1, X_2, X_3, X_4, X_5, X_6, X_7]}{(X_3 X_5, X_1 X_3 + X_2 X_4 - X_4 X_5, X_1 X_6, X_2 X_6, X_3 X_7, X_5 X_7, X_6 X_6, X_7 X_7)} \]

Then the Koszul dual \( HKR^l \) of \( HKR \) can be presented as a coproduct

\[ C^l \sqcup k < x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19} > \]

where the second algebra in the coproduct is the free algebra on the dual generators \( x_8, \ldots x_{19} \) of \( X_8, \ldots X_{19} \).
Furthermore, it is well-known (cf. e.g. [LEM]) that for the coproduct $A \sqcup B$ of two graded connected algebras, we have the following formula for the respective Hilbert series

$$\frac{1}{(A \sqcup B)(z)} + 1 = \frac{1}{A(z)} + \frac{1}{B(z)}$$

(this works also for the bigraded case) and it follows that

$$\frac{1}{HKR'(z, z)} + 1 = \frac{1}{C'(z, z)} + 1 - 3z^3 - 6z^4 - 3z^5$$

and we have therefore reduced our problem to calculate $C'(z, z)$ where $C$ only involves the first 7 variables $X_1, X_2, X_3, X_4, X_5, X_6, X_7$.

We start with an algebra $D$ that involves only the first 5 variables $X_1, X_2, X_3, X_4, X_5$:

$$D = \frac{k[X_1, X_2, X_3, X_4, X_5]}{(X_3X_5, X_1X_3 + X_2X_4 - X_4X_5, X_1X_2)}$$

The variables skewcommute. The Koszul dual $D^!$ is the quotient of the free algebra in the dual variables $x_i$ of the $X_i$:

$$D^! = \frac{k < x_1, x_2, x_3, x_4, x_5 >}{([x_1, x_4], [x_1, x_5], [x_2, x_3], [x_2, x_5], [x_3, x_4], [x_1, x_3] + [x_4, x_5], [x_2, x_4] + [x_4, x_5]}$$

where the $[x_i, x_j]$ are Lie commutators. From this one sees exactly as in Löfwall’s and mine study of the Anick example [LR1] that the underlying Lie algebra $g_D$ of $D^!$ sits in the middle of an extension

$$0 \rightarrow \oplus_{i=1}^\infty a_i \rightarrow g_D \rightarrow f_1 \times f_2 \rightarrow 0$$

i.e. is the extension of the product of two free Lie algebras $f_1$ generated by $x_1$ and $x_2$ and $f_2$ generated by $x_3$ and $x_5$ with the infinite abelian Lie algebra

$$a = \oplus_{i=1}^\infty a_i$$

where the $a_i$ are onedimensional generated by $a_1 = x_4$, $a_2 = [x_5, x_4]$, $a_3 = [x_5, [x_5, x_4]]$ etc. which commute and by the 2-cocycle $\gamma : f_1 \times f_2 \rightarrow a$ defined by $\gamma(x_1, x_3) = -\gamma(x_3, x_1) = a_2$ and $\gamma(x_2, x_5) = -\gamma(x_5, x_2) = a_2$. It follows that

$$D^!(z) = \frac{1}{(1 - 2z)^2} \prod_{n=1}^\infty \frac{1}{1 - z^n}$$
But since we need $D^l(z, z)$ we have to replace the $z$ by $z^2$ in the formula (9) to get $D^l(z, z)$. We next want to incorporate the variables $X_6, X_7$ (which commute with the $X_1, X_2, X_3, X_4, X_5$) into the picture. It turns out that we have to study the underlying Lie algebra of the Koszul dual $C^l$ of the $C$ in (6), i.e. the quotient of

$$ k < x_1, x_2, x_3, x_4, x_5, x_6, x_7 > $$

with the ideal generated by

$$ [x_1, x_4], [x_1, x_5], [x_2, x_3], [x_2, x_5], [x_3, x_4], [x_1, x_3] + [x_4, x_5], [x_2, x_4] + [x_4, x_5], $$

as before and the extra ideal generators:

$$ [x_1, x_7], [x_2, x_7], [x_4, x_7], [x_6, x_7], [x_3, x_6], [x_4, x_6], [x_6, x_7] $$

where $x_6$ and $x_7$ are dual to $X_6$ and $X_7$. From the preceding presentation one sees that the Lie algebra $g_C$ of $C^l$ sits in the middle of an extension:

$$ 0 \rightarrow \oplus_{i=1}^\infty a_i \rightarrow g_C \rightarrow F_1 \times F_2 \rightarrow 0 $$

where $F_1$ is the free Lie algebra generated by $x_1, x_2, x_6$ of degrees 1, 1, 2 and $F_2$ is the free Lie algebra generated by $x_3, x_5, x_7$ of degrees 1, 1, 2 and the $a_i$’s are as before, and the $x_6$ and $x_7$ operate in the trivial way on the $a_i$. Furthermore we have a new cocycle

$$ \Gamma : F_1 \times F_2 \rightarrow \oplus_{i=1}^\infty a_i $$

where as before $\Gamma$ is defined by its value on the generators by $-\Gamma(x_3, x_1) = \Gamma(x_1, x_3) = a_2$ and $-\Gamma(x_5, x_2) = \Gamma(x_2, x_5) = a_2$ and being zero for all other pairs of generators. By calculating $H^2(F_1 \times F_2, a)$ for (8) one sees that the relations are those of $C^l$ and therefore the Hilbert series

$$ C^l(z) = \frac{1}{(1 - 2z - z^2)^2} \prod_{n=1}^{\infty} \frac{1}{1 - z^n} $$

if the variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are given the degrees 1, 1, 1, 1, 1, 2, 2. But for the calculation of the series of $HKR^l(z, z)$ these variables should be given the degrees 2, 2, 2, 2, 2, 3, 3 so that we get the series

$$ \frac{1}{(1 - 2z^2 - z^3)^2} \prod_{n=1}^{\infty} \frac{1}{1 - z^{2n}} $$
Using the formulae (8) and (12) we obtain

$$\frac{1}{HKR^i(z, z)} = (1 - 2z^2 - z^3)^2 \prod_{n=1}^{\infty} (1 - z^{2n}) - 3z^3 - 6z^4 - 3z^5$$

But $I^3 = 0$ and

$$I^2 = (X_4X_5, X_2X_5, X_1X_5, X_3X_4, X_2X_4, X_1X_4, X_2X_3, X_4X_7, X_2X_7, X_1X_7, X_5X_6, X_4X_6, X_3X_6, X_6X_7)$$

and this gives that $HKR(x, y) = 1 + 5xy + 5xy^2 + 6xy^3 + 3xy^4 + 7x^2y^2 + 6x^2y^3 + x^2y^4$,

so that

$$HKR(-z, z) = 1 - 5z^2 - 5z^3 + z^4 + 3z^5 + z^6$$

and finally the formula

$$\frac{1}{P_{HKR}(z, z)} = (1 + 1/z)/HKR^i(z, z) - HKR(-z, z)/z$$

gives that

$$\frac{1}{P_{HKR}(z, z)} = (1 + 1/z)[(1 - 2z^2 - z^3)^2 \prod_{n=1}^{\infty} (1 - z^{2n}) - 3z^3 - 6z^4 - 3z^5]$$

$$- (1 - 5z^2 - 5z^3 + z^4 + 3z^5 + z^6)/z$$

and if the Avramov spectral sequence degenerated this series should be the same as

$(1 + z)^4/P_R(z)$ so the final result should be

$$P_R(z) = \frac{z(1 + z)^2}{(1 + z)(1 - z - z^2)^2 \prod_{n=1}^{\infty} (1 - z^{2n}) - 1 + 2z + 2z^2 - 4z^3 - 4z^4}$$

$$= 1 + 4z + 11z^2 + 29z^3 + 75z^4 + 193z^5 + 498z^6 + 1289z^7 + 3341z^8 + 8663z^9 + 22466z^{10} + \ldots$$

as asserted in our Theorem 1. The following calculation in Macaulay2

```
res(coker vars R, LengthLimit => 11); betti oo
```

gives the diagram of Betti numbers (16) below whose first row (the total Betti numbers) gives support to this assertion. But the Theorem 5.9 of [AV0] gives first that the differentials $d^r$ of the Avramov spectral sequence are 0 for $r > 2$ but furthermore since the dimensions of the two sides of the Avramov spectral sequence are the same in degrees $\leq 10$
it follows from the proof of Theorem 5.9 in [AV0] and the structure of the matrix Massey products that there are no non-zero differentials, and Theorem 1 is proved. We can also get a bigraded more precise version of all this: Indeed, we also have a series

\[ P_R(x, y) = \sum_{p,q} |\text{Tor}^R_{p,q}(k, k)| x^p y^q \]

which takes into account the extra grading of \( R \) (we have that \( P_R(z, 1) \) is the old \( P_R(z) \) given as

\begin{equation}
(15)
P_R(x, y) = \frac{(1 + xy)^4}{(1 + 1/x)A - H/x}
\end{equation}

where

\[
1/A = (1 - 2x^2 y^3 - x^3 y^4)^2 \prod_{n=1}^{\infty} (1 - x^{-2n} y^{3n}) - x^3 y^5 - x^3 y^6 - x^3 y^8 - 4x^4 y^7 - 2x^4 y^9 - 2x^5 y^8 - x^5 y^{10}
\]

and \( H \) is equal to

\[
1 - 5x^2 y^3 - 2x^3 y^4 - x^3 y^5 - x^3 y^6 - x^3 y^8 - 4x^4 y^7 - 2x^4 y^9 - 2x^5 y^8 - x^5 y^{10} + 7x^4 y^6 + 6x^5 y^7 + x^6 y^8
\]

and the expansion of \( P_R(x, y) \) of (15) up to degree 11 gives the same diagram of graded Betti numbers that was earlier found by the Macaulay2 calculation:

|      | 0: | 1: | 2: | 3: | 4: | 5: | 6: | 7: | 8: | 9: | 10: | 11: | 12: | 13: | 14: | 15: | 16: |
|------|----|----|----|----|----|----|----|----|----|----|-----|-----|----|----|----|----|----|
| total| 1  | 4  | 11 | 29 | 75 | 193| 498| 1289| 3341| 8663| 22466| 58264|    |    |    |    |
| 0:   | 1  | 4  | 6  | 4  | 1  | .  | .  | .  | .  | .  | .   | .   | .  | .  | .  | .  |
| 1:   | .  | .  | 5  | 22 | 38 | 32 | 13 | 2  | .  | .  | .   | .   | .  | .  | .  | .  |
| 2:   | .  | .  | .  | 1  | 22 | 92 | 171| 169| 92  | 26 | 3   | .   | .  | .  | .  | .  |
| 3:   | .  | .  | .  | 1  | 8  | 34 | 135| 389| 689 | 744 | 493 | 196 |    |    |    |    |
| 4:   | .  | .  | .  | 10 | 87 | 346| 923| 1870| 2835| 3066|    |    |    |    |    |    |
| 5:   | .  | .  | 1  | 6  | 15 | 22 | 96 | 645 | 2426| 5739| 9705|    |    |    |    |    |
| 6:   | .  | .  | .  | 1  | 10 | 65 | 186| 348 | 964 | 4302| 14387|    |    |    |    |    |
| 7:   | .  | .  | .  | 2  | 73 | 459| 1446| 3296| 8595|    |    |    |    |    |    |    |
| 8:   | .  | .  | .  | 2  | 20 | 112| 669 | 3025| 9538|    |    |    |    |    |    |    |
| 9:   | .  | .  | .  | .  | 3  | 320| 1668| 6469|    |    |    |    |    |    |    |    |
| 10:  | .  | .  | .  | 1  | 8  | 28 | 62  | 396 | 3282|    |    |    |    |    |    |    |
| 11:  | .  | .  | .  | .  | 15 | 129| 510 | 1377|    |    |    |    |    |    |    |    |
| 12:  | .  | .  | .  | .  | .  | 3  | 153 | 1278|    |    |    |    |    |    |    |    |
| 13:  | .  | .  | .  | .  | .  | .  | 3  | 36  | 246 |    |    |    |    |    |    |    |
| 14:  | .  | .  | .  | .  | .  | .  | .  | .   | 60  |    |    |    |    |    |    |    |
| 15:  | .  | .  | .  | .  | .  | .  | 1 | 10 | 45  |    |    |    |    |    |    |    |
| 16:  | .  | .  | .  | .  | .  | .  | .  | .   | .   | 20 |    |    |    |    |    |    |
3. Other embedding dimension 4 cases

The example we have just studied is certainly not unique. Let us illustrate this with two examples: If one adds the relation $z^3$ to the example in Theorem 1 we obtain the following

$$Ra = \frac{k[x, y, z, u]}{(x^3, x^2y, (x+u)(y^2+z^2) + yzu, z^3, zu^2, u^3)}$$

which has Hilbert series

$$\frac{1 + 3T + 6T^2 + 4T^3 - T^4 - 7T^5}{1 - T}$$

Furthermore the homology of the Koszul complex of $R_a$ is

| total | 1 | 6 | 18 | 20 | 7 |
|-------|---|---|----|----|---|
| 0     | 1 | . | .  | .  | . |
| 1     | . | . | .  | .  | . |
| 2     | . | 6 | 2  | .  | . |
| 3     | . | . | 2  | .  | . |
| 4     | . | . | 14 | 20 | 7 |

We get same Hilbert series and the same diagram of Koszul homology if we replace $z^3$ with $y^3 + z^3$ to get the ring

$$R_b = \frac{k[x, y, z, u]}{(x^3, x^2y, (x+u)(y^2+z^2) + yzu, y^3 + z^3, zu^2, u^3)}$$

Both these ring have transcendental series of Betti numbers. These two series can be determined and they are different. For the case of $R_a$ we have using DGAAlgebras that $HKR_a$ is generated by 6 skew-commuting variables $X_1, X_2, X_3, X_4, X_5, X_6$ of degree (1,3) (and 27 variables of higher degrees). The first 6 variables satisfy the following relations $X_5X_6, X_3X_6, X_3X_5, X_1X_3 + X_2X_4 - X_4X_6, X_1X_2$ and the Hilbert series of the corresponding Koszul dual is

$$\frac{1}{(1 - 2z)(1 - 3z) \prod_{n=1}^{\infty}(1 - z^n)}$$

On the other hand, using DGAAlgebras for $R_b$ we find that $HKR_b$ is still generated by 6 variables $X_1, X_2, X_3, X_4, X_5, X_6$ of degree (1,3) (but only 24 variables of higher degrees). In this case the first 6 variables satisfy the following relations $X_3X_6, X_1X_3 + X_2X_4 - X_4X_6, X_1X_2$ and in this case the Hilbert series of the corresponding Koszul dual is

$$\frac{1}{(1 - 2z)^2(1 - z) \prod_{n=1}^{\infty}(1 - z^n)}$$
Now we can continue our analysis as in section 2 and we obtain the following formulae:

\[ P_{R_a}(Z) = \frac{Z(1 + Z)^2}{(1 - Z - Z^2)(1 - 3Z^2 - Z^3) \prod_{n=1}^{\infty}(1 - Z^{2n}) - 1 + 2Z + 3Z^2 - 6Z^3 - 7Z^4} \]

and

\[ P_{R_b}(Z) = \frac{Z(1 + Z)^2}{(1 - Z)(1 - 2Z^2 - Z^3)^2 \prod_{n=1}^{\infty}(1 - Z^{2n}) - 1 + 2Z + 3Z^2 - 6Z^3 - 7Z^4} \]

Note also that we could have started with many more examples in embedding dimension 4 than just our variation of the Anick case that we took in section 1.

Now let us also briefly analyze the case when

\[ S = k[x, y, z, u] / (x^3, x^2y, (x + u)(y^2 + z^2), zu^2, u^3) \]

Using DGAlgebras we obtain as before that

\[ P_S(Z) = \frac{(1 - Z)(1 + z)^2}{1 - 3Z + 3Z^3 - 2Z^5} \]

but we can also prove that neither \( \text{Ext}_{S}(k, k) \) nor \( \text{Ext}_{HKS}(k, k) \) are finitely generated as algebras equipped with the Yoneda product. The case \( S \) is the 4-variable of the 5-variable \( k[x, y, z, u, v] / (x^2, xy, xz + zv, uv, v^2) \), a variant of which we treated in [R1], inspired by Lemaire [LEM]. An alternative way to treat \( S \) is to observe that the quotient map

\[ \frac{k[x, y, z, u]}{(x^3, x^2y, zu^2, u^3)} \rightarrow S \]

is a Golod map [LEV] so that \( k[x, y, z, u] \rightarrow S \) is a composite of three Golod maps, and use [R2].

All this leads support to the surmise that the embedding dimension 4 case could be equally complicated as the general case, and one could even pose the

**QUESTION 1:** Let \( \mathcal{E}^4 \) be the set of series \( \sum_{n \geq 0} |\text{Tor}_n^R(k, k)| Z^n \) for \((R, m)\) a local commutative noetherian ring of embedding dimension \( \leq 4 \) where \( k = R/m \). Can \( \mathcal{E}^4 \) be added to the list of 17 series of [A-Gu] that are all rationally related?

One would certainly get a smaller set than \( \mathcal{E}^4 \) if one restricted oneself to rings of the form \( k[x, y, z, u] / (f_1, \ldots, f_s) \) where the relations \( f_i \) were of degree \( \leq 3 \) or \( \leq 4 \).
So far no embedding dimension 4 variant of the rings in [LR2] has been found. But even in this restricted case of degree 3 relations one can get surprises: The ring

\[ S_I = Q[x, y, z, u]/(u^3, xy^2, (x+y)z^2, x^2u + zu^2, y^2u + xzu, y^2z + yz^2) \]

classified in [R5] (inspired by Lukas Katthän [KAT]) which has only six relations of degree 3, has as homology of the Koszul complex:

\[
\begin{array}{cccccc}
\text{Total:} & 1 & 6 & 12 & 9 & 2 \\
0: & 1 & . & . & . & . \\
1: & . & . & . & . & . \\
2: & . & 6 & . & . & . \\
3: & . & . & 12 & 5 & 1 \\
4: & . & . & . & 4 & . \\
5: & . & . & . & . & 1 \\
\end{array}
\]

Furthermore the multiplication of elements of positive degree in this homology is zero (use DGA1gebra), and

\[
\frac{(1 + z)^4}{(1 - 6z^2 - 12z^3 - 9z^4 - 2z^5)} = 1 + 4z + 12z^2 + 40z^3 + 130z^4 + 422z^5 + 1376z^6 + 4476z^7 + \ldots
\]

whereas the series of the total Betti numbers of \( S_I \) is given by Macaulay2 as:

\[
1 + 4z + 12z^2 + 40z^3 + 130z^4 + 421z^5 + 1371z^6 + 4454z^7 + \ldots
\]

so that \( S_I \) is not a Golod ring. In [R5] we show that if one accepts relations of degree 3 and 4, there are hundreds of such exotic non-Golod rings.

QUESTION 2: For local rings \( R \) of embedding dimension 4 with only cubic relations, are my \( R \) in Theorem 1 and the \( R_a \) and \( R_b \) essentially the only examples where we have a transcendental \( P_R(Z) \)?

4. References

[A] D.J. Anick, A counterexample to a conjecture of Serre, Ann. of Math. 115, 1982, pp. 1-33; Correction, Ann. of Math. 116, p. 661.

[A-CR] D.J. Anick, Construction d’espaces de lacets et d’anneaux locaux à séries de Poincaré-Betti non rationnelles, C.R. Acad. Sc. Paris 290, (1980), pp. A733-A736.
[A-Gu] D.J. Anick and Tor H. Gulliksen, *Rational dependence among Hilbert and Poincare series*, Journal of Pure and Applied Algebra 38, 1985, pp. 135-157.

[AV0] Avramov, L. L., *The Hopf algebra of a local ring*. (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 38, (1974), pp. 253-277. English translation [Math. USSR-Izv.8 (1974), 259-284].

[AV1] Avramov, L. L., *Obstructions to the existence of multiplicative structures on minimal free resolutions*. Amer. J. Math., 103 (1981), pp. 1-31.

[KAT] L. Katthän, *A non-Golod ring with a trivial product on its Koszul homology*, http://arxiv.org/pdf/1511.04883.pdf

[LEM] Lemaire, Jean-Michel, *Algèbres connexes et homologie des espaces de lacets*, Lecture Notes in Mathematics, 422, Springer-Verlag, Berlin-New York, 1974.

[LEV] Levin, Gerson, *Local rings and Golod homomorphisms*, Journal of Algebra 37, 1975, pp 266289.

[L1] Löfwall, C., *On the subalgebra generated by one-dimensional elements in the Yoneda Ext–algebra*, in Algebra, algebraic topology, and their interactions, (J.–E. Roos, ed), Lecture Notes in Math., vol.1183, Springer-Verlag, Berlin–New York, 1986, pp. 291-338.

[LR1] C. Löfwall and J.-E. Roos, *Cohomologie des algèbres de Lie graduées et séries de Poincaré-Betti non rationnelles*, C.R. Acad. Sc. Paris 290, 1980, pp. A733-A736.

[LR2] C. Löfwall and J.-E. Roos, *A nonnilpotent 1-2-presented graded Hopf algebra whose Hilbert series converges in the unit circle*, Adv. Math.130, 1997, pp. 161-200.

[MOO] Moore, Frank, DGAlgebras. A package for Macaulay2 http://www.math.uiuc.edu/Macaulay2/Packages/

[R1] J.-E. Roos, *Relations between the Poincaré-Betti series of Loop Spaces and of Local rings*, Springer Lecture Notes in Math.740,1979, 285-322.

[R2] J.-E. Roos, *On the use of graded Lie algebras in the theory of local rings*, Commutative algebra: Durham 1981 (R. Y. Sharp, ed.) London Math. Soc. Lecture Notes Ser. vol. 72, Cambridge Univ. Press, Cambridge, 1982, pp. 204230.

[R3] J.-E. Roos, *A computer-aided study of the graded Lie-algebra of a local commutative noetherian ring* (with an Appendix by Clas Löfwall), Journal of Pure and Applied Algebra 91, 1994, pp. 255-315.

[R4] J.-E. Roos, *Homological properties of the homology algebra of the Koszul complex of a local ring: Examples and questions*, Journal of Algebra 465, 2016, pp. 399-436.

[R5] J.-E. Roos, *On some unexpected rings that are close to Golod rings*, in preparation.