FREDHOLM DETERMINANTS, ANOSOV MAPS AND RUELLE RESONANCES

Abstract. I show that the dynamical determinant, associated to an Anosov diffeomorphism, is the Fredholm determinant of the corresponding Ruelle-Perron-Frobenius transfer operator acting on appropriate Banach spaces. As a consequence it follows, for example, that the zeroes of the dynamical determinant describe the eigenvalues of the transfer operator and the Ruelle resonances and that, for $C^\infty$ Anosov diffeomorphisms, the dynamical determinant is an entire function.

1. Introduction

In the last years there has been a considerable interest in the study of dynamical determinants and dynamical zeta functions (see [4, 13, 2, 6, 24, 17, 19], just to mention a few, [24, 20] for brief reviews of the field, [11] for a general introduction and [9] for a detailed discussion of physics related issues). Here, I will focus on Anosov diffeomorphisms $T$ and the associated Fredholm determinant $d^\flat$ (2.11) for the transfer operator $L$ (see (2.1) for a precise definition).

The most satisfactory results have been obtained for analytic systems [22, 10, 12, 25] and $C^{r+1}$ expanding maps [23, 11]. For axiom A analytic and $C^\infty$-expanding maps the above mentioned papers prove that the dynamical determinant is an entire function and its zeroes are exactly the inverse of the eigenvalues of the associated transfer operator; that is, it can be interpreted as a Fredholm determinant.

On the contrary for $C^{r+1}$ Axiom A maps or flows the situation is still unsatisfactory. The strongest result to date is [17] where it is showed that the dynamical determinant for a $C^{r+1}$ Anosov map, with expansion and contraction estimated by $\lambda$, is analytic in the disk $\lambda^2$. Nevertheless, in [17] the relation between the dynamical determinant and the transfer operator is only a formal one, in particular no information is available concerning the relation between the zeroes of such a function and the spectrum of the transfer operator. It was therefore a bit arbitrary to call such a function a Fredholm determinant.

In the present paper the missing relation is derived at the price of establishing the result in a smaller disk. Building on the results in [14] I will show that it is possible to make sense of the naïve idea of smoothing the singular kernel of the transfer operator. [11] page 103]. This yields a strategy greatly simplified with respect to previous approaches. In fact, it essentially boils down to a couple of pages computation. As a consequence one establishes the complete description of the correlation spectra (Ruelle resonances) in terms of periodic orbits. Finally, let
me remark that, most likely, the present approach can be extended to more general transfer operators (e.g., with smooth weights), systems (e.g., Axiom A) and to the study of dynamical zeta functions (since the latter can be expressed as ratios of dynamical determinants [22]).

The plan of the paper is as follows. Section 2 details and proves the main results of the paper. Given the existence of a scale of adapted Banach spaces (see Definition 1) and Lemma 2.9, the proofs are completely self-contained. Lemma 2.9 is proven in section 3 while Proposition 2.1, proven in section 4, states the existence of the adapted spaces. This last result relies on a scale of Banach spaces introduced in [14], yet it should be emphasized that other choices of adapted spaces are possible, e.g., V. Baladi has recently introduced a different choice that, in very special cases, enjoys some useful extra properties [8] and V. Baladi with M. Tsujii have announced a different possibility that could yield sharper bounds. Finally, an appendix contains an hardly surprising technical result that, for lack of references, needed to be proven somewhere.

Remark 1.1. In this paper $C$ stands for a generic constant depending only on the dynamical system $(X, T)$ under consideration. Its actual value can thus change from one occurrence to the next.

2. The results

In the following, I will discuss only the case $X = \mathbb{T}^d$ with the Euclidean metric. This simplifies the presentation and the notations since one can avoid the need to introduce local charts. The general case can be investigated in complete analogy by using partitions of unity and local charts along the lines exploited in [14]. Also, I will discuss only the transfer operator associated to the SRB measure although I do not see any real obstacle in treating more general, smooth, potentials.¹

Let $T \in \text{Diff}^{r+1}(X, X)$ and $\mathcal{D}'_r$ be the space of distributions of order $r$, the transfer operator $\mathcal{L} : \mathcal{D}'_r \to \mathcal{D}'_r$ is defined by

\[(2.1) \quad (\mathcal{L} h, \phi) := (h, \phi \circ T), \quad \forall \phi \in C^r(X, \mathbb{C}).\]

¹The main problem is than one needs the extension of [14] to such a setting. This is rather straightforward but to include it here would substantially increase the length of the paper without adding much to the presentation of the basic idea.

²Usually, the transfer operator is defined as acting on function but, in the present contest, it turns out to be essential that the operator can be defined also on distributions. In fact, by using the standard identification between functions and distributions (see also footnote 3), one can restrict the operator to $C^\omega$ obtaining, for each $h \in C^\omega(X, \mathbb{R})$, the usual formula $\mathcal{L} h(x) := f \circ T^{-1} |\det(D_T^{-1})|$ which describes the evolution, under the dynamics, of the density of the measures absolutely continuous with respect to the Lebesgue measure $m$. 
In addition, consider a convolution operator \( \hat{Q}_\varepsilon : C^\infty \to C^\infty \):
\[
\hat{Q}_\varepsilon f(x) := \int_{\mathbb{R}^d} q_\varepsilon(x-y) f(y) dy
\]
for each multi-index \( \alpha \) such that \( 0 < |\alpha| \leq r \), and where \( q_\varepsilon(x) = \varepsilon^{-d} \tilde{q}(\varepsilon^{-1}x) \), \( \tilde{q}(x) = \tilde{q}(-x) \), supp \( \tilde{q} \subset \{ x \in \mathbb{R}^d : |x| \leq 1 \} \), \( \tilde{q} \in C^\infty \).

By duality one can then define the operator \( Q_\varepsilon := \hat{Q}_\varepsilon^* : \mathcal{D}' \to \mathcal{D}' \) which can be easily seen to be an extension of \( \hat{Q}_\varepsilon \) to the space of distributions.

It is well known that the spectral properties of the transfer operator depend drastically on the space on which it acts. The space of distributions turns out to be too large of a space to be useful, yet it is well known that \( C^\infty \) is by far too small and the spectra of \( \mathcal{L} \) on such a space bears little relevance on the statistical properties of the system. Below we give an abstract characterization of some properties that good dynamical spaces should enjoy.

**Definition 1.** Given \( T \in \text{Diff}^{-1}(X, X) \), a scale of Banach spaces \( \{ \mathcal{B}^s \}_{s \in \mathbb{N}} \) is called adapted to \( T \) if there exists \( s_r \in \mathbb{N} \) such that, for \( s \in \{ 1, \ldots, s_r \} \), \( \mathcal{D}_r \supset \mathcal{B}^{s-1} \supset \mathcal{B}^s \supset \mathcal{C}^*, 4 \) More precisely, \( \| \cdot \|_{s-1} \leq \| \cdot \|_s \leq | \cdot |_{C^r} \), and \( C_s^\infty \mathcal{L}^s = \mathcal{B}^s \). In addition, there exist \( \theta \in (0, 1) \) and \( \varepsilon_1 > 0 \) such that, for all \( 0 < s \leq s_r \), \( \mathcal{L} \in \mathcal{L}(\mathcal{B}^s, \mathcal{B}^r) \) and, for each \( \varepsilon \in (0, \varepsilon_1) \), \( l < s \) and \( n \in \mathbb{N} \),

\[
\begin{align*}
(2.3) & \quad \left\| \int f \phi \right\|_{\mathcal{B}^s} \leq \| f \|_{\mathcal{B}^r} \| \mathcal{D}_r \phi \|_{C^\infty} \text{ for each } f, \phi \in C^\infty \\
(2.4) & \quad \mathcal{L} : \mathcal{B}^s \to \mathcal{B}^{s-1} \text{ is compact} \\
(2.5) & \quad \| \mathcal{L}^n h \|_{\mathcal{B}^s} \leq B \| h \|_{\mathcal{B}^r} \text{ for each } h \in \mathcal{B}^s \\
(2.6) & \quad \| \mathcal{L}^n h \|_{\mathcal{B}^s} \leq A \theta^n \| h \|_{\mathcal{B}^r} + B \| h \|_{\mathcal{B}^{s-1}} \text{ for each } h \in \mathcal{B}^s \\
(2.7) & \quad Q_\varepsilon \in \mathcal{L}(\mathcal{B}^s, \mathcal{C}^\infty) \text{ and } \| Q_\varepsilon - \text{Id} \|_{\mathcal{B}^s \to \mathcal{B}^{s-1}} \leq D \varepsilon^l \\
(2.8) & \quad \| h f \|_{\mathcal{B}^s} \leq D' \| h \|_{\mathcal{B}^r} \| f \|_{C^r} \text{ for each } h \in \mathcal{B}^s, \ f \in C^r
\end{align*}
\]

where \( A, B, D, D' \) do not depend on \( \varepsilon \) and \( n \). 6 If \( T \in \text{Diff}^\infty(X, X) \) and we have an adapted scale for each \( r \), with \( \lim_{r \to \infty} s_r = \infty \), then we say that we have a complete series of adapted Banach spaces.

From simple arguments (see, e.g., [14]) follows that on such spaces the spectrum of \( \mathcal{L} \) has a physical interpretation: it describes the rate of decay of correlations and it is stable with respect to a large family of perturbations. In addition, in section 4 I prove:

\footnote{For a general manifold \( X \), one must introduce coordinates charts \( \Psi_i \) and a subordinate partition of unity \( \{ \phi_i \} \), then, for \( \varepsilon \) small enough, one can define
\[
\hat{Q}_\varepsilon f(x) := \sum_i \int \mathcal{J} \phi_i(\Psi_i^{-1}(x) + \xi) q_\varepsilon \circ \Psi_i(\Psi_i^{-1}(x) + \xi) \phi_i \circ \Psi_i(\Psi_i^{-1}(x) + \xi) f \circ \Psi_i(\Psi_i^{-1}(x) + \xi)
\]
and the following holds essentially unchanged.}

\footnote{Of course, to make sense of such a scale it is necessary to slightly abuse notations and identify each functions \( f \in C^r(X, \mathbb{R}) \) with a linear functional (distribution) via the standard duality relation \( (f, \varphi) := \int_X f \varphi dm \).}

\footnote{In fact, Property 2.6, is needed only in the proof of Proposition 2.6.}
Proposition 2.1. If $T \in \text{Diff}^{r+1}(X, X)$ is Anosov,\(^7\) then there exists a scale of Banach spaces adapted to $T$ with $s_r = \left[ \frac{r}{2+1} \right].$\(^8\) If $T \in \text{Diff}^\infty(X, X)$ then the latter constitute a complete series of adapted spaces.

Remark 2.2. In section \(\S 4\) I will define the wanted spaces based on the Banach spaces introduced in \(\S 1\), yet the present results hold for any other choice of adapted Banach spaces satisfying \(\S 2, \S 3\).

Remark 2.3. The existence of an adapted scale of Banach spaces implies (see \(\S 1\)): For each $1 < s \leq s_r$ and $\sigma \in (\theta, 1)$, the operator $L$ is quasicompact on $B^s$, more precisely it can be decomposed as $L = P_{\sigma, s} + R_{\sigma, s}$ where $P_{\sigma, s}R_{\sigma, s} = R_{\sigma, s}P_{\sigma, s} = 0$, $P_{\sigma, s}$ is of finite rank and

\begin{equation}
\|P_{\sigma, s}\|_{B^s} \leq C\sigma^{-n}.
\end{equation}

For further use let us set

\begin{equation}
\Gamma_n := \sum_{x \in \text{Fix} T^n} |\det(\text{Id} - D_x T^n)|^{-1},
\end{equation}

The following estimate is more or less standard. The proof can be found at the end of the section and is enclosed only for completeness.

Lemma 2.4. If $T \in \text{Diff}^{r+1}(X, X)$ is Anosov, for all $n \in \mathbb{N}, \Gamma_n \leq C$.

The main result of the paper is the following.

Theorem 1. If $T \in \text{Diff}^{r+1}(X, X)$ is Anosov, define, for $|z| < 1$,

\begin{equation}
d^b(z) := \text{Exp} \left( -\sum_{n=1}^\infty \frac{z^n}{n} \sum_{x \in \text{Fix} T^n} |\det(\text{Id} - D_x T^n)|^{-1} \right),
\end{equation}

and consider its analytic extension. Then, if $\bar{s}_r := \left[ \frac{2r}{4+1} \right] > 0$, $d^b(z) \det(\text{Id} - zP_{\bar{s}_r})^{-1}$ is holomorphic and never zero in the disk $|z| < \theta^{-2\bar{s}_r}$. Thus in such a disk $d^b$ is holomorphic and its zeroes are in one one correspondence with the eigenvalues of the operator $L$. In addition, the algebraic multiplicity of the zeroes equal the dimension of the associated eigenspaces. In particular, if $T \in C^\infty$, then $d^b$ is an entire function.

Remark 2.5. Using the spaces in \(\S 1\), see section \(\S 4\) the above Theorem gives the analyticity domain $\lambda^{\frac{n(1-a/2s_r)}{2+1}}$. This is certainly far from optimal and can be easily improved.\(^9\) Yet, since I do not see how to obtain the expected Kitaev-like bound $\lambda^{-\frac{1}{2}}$, I will not strive for superficial improvements at the expenses of clarity, simplicity and brevity.

In the present language the SRB measures are the eigenvectors associated to the eigenvalue one. Probably the most interesting physical consequence of Theorem \(\S 1\) is the following.

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\(^7\)Anosov means that there exists a continuous splitting $E^n \otimes E^s$, $\dim(E^s) = d_s$ and $\dim(E^n) = d_n$, of the tangent bundle and a constant $\lambda > 1$ such that $\|D_x T^n v\| \leq \lambda^{-n} \|v\|$ for all $v \in E^n(x), x \in X$ and $\|D_x T^{-n} v\| \leq \lambda^{-n} \|v\|$ for all $v \in E^n(x), x \in X$.

\(^8\)Given $a \in \mathbb{R}, \lfloor a \rfloor$ stands for the largest integer $n \leq a$.

\(^9\)For example, one should be able to prove $Q_{\varepsilon} \in L(B^s, B^b)$, $d$ can be replaced by $d_n$, etc.
**Proposition 2.6.** Given a transitive (hence mixing) Anosov map $T \in \text{Diff}^\infty(X, X)$, let $\mu_{\text{SRB}}$ be the SRB measure and let $f, g \in C^\infty(X, \mathbb{R})$, $\mu_{\text{SRB}}(f) = \mu_{\text{SRB}}(g) = 0$, then the function (the correlation spectra)

$$C_{f,g}(e^{i\omega}) = \sum_{n \in \mathbb{Z}} e^{i\omega n} \mu_{\text{SRB}}(fg \circ T^n); \quad \omega \in \mathbb{R}$$

extends to a meromorphic function on $\mathbb{C} \setminus \{0\}$ and its poles (often called Ruelle resonances) are exactly described by the zeroes of $d^\phi$

**Proof.** Fix any $r \in \mathbb{N}$ and consider an associated scale of adapted spaces. Calling $m$ the Lebesgue measure\(^{10}\) the SRB measure can be defined as

$$\mu_{\text{SRB}}(\phi) = \lim_{n \to \infty} m(\phi \circ T^n) = \lim_{n \to \infty} \mathcal{L}^n1(\phi).$$

Note that, since the map is mixing, then one must be a simple eigenvalue and no other eigenvalues can be present on the unit circle.\(^{11}\) Consequently, remembering Remark 2.3, $\mathcal{L}$ has a spectral gap, hence there exists $\rho > 1$ such that, for each $h \in \mathcal{B}^s$

$$\|\mathcal{L}^n h - (h,1)\mu_{\text{SRB}}\|_s \leq C\|h\|_s \rho^{-n}.$$ 

Since, $\mu_{\text{SRB}}(fg \circ T^n) = \lim_{p \to \infty} (\mathcal{L}^p1, fg \circ T^n)$ it is natural to define the measures $m_{p,f}(h) := (\mathcal{L}^p1, fh)$. In fact, by (2.8), we have $m_{p,f} \in \mathcal{B}^s$, $s \leq s_r$ and

$$m_{p,f}(g \circ T^n) = \mathcal{L}^n m_{p,f}(g) = \mu_{\text{SRB}}(g)m_{p,f}(1) + O(\rho^{-n}) = O(\rho^{-n}).$$

The above means that, for each $|z| < \rho$,

$$\sum_{n \in \mathbb{N}} z^n \mu_{\text{SRB}}(fg \circ T^n) = \lim_{p \to \infty} \sum_{n \in \mathbb{N}} z^n \mathcal{L}^n m_{p,f}(g)$$

$$= \lim_{p \to \infty} \left[ (\text{Id} - z\mathcal{L})^{-1} m_{p,f}(g) \right]$$

$$= \left[ (\text{Id} - z\mathcal{L})^{-1} \lim_{p \to \infty} m_{p,f}(g) \right]$$

$$= \left[ (\text{Id} - z\mathcal{L})^{-1} \mu_f(g) \right] =: G_{f,g}(z),$$

where $\mu_f(h) := \mu_{\text{SRB}}(fh)$. Since $\mu_f \in \mathcal{B}^s$, for each $s \leq s_r$, by Remark 2.3 follows that the function $G_{f,g}$ can be extended to a meromorphic function on $\{z \in \mathbb{C} : |z| < \rho^{-s_r}\}$. On the other hand, if $|z| > 1$,

$$\sum_{n \in \mathbb{N}} z^{-n} \mu_{\text{SRB}}(fg \circ T^{-n}) = \sum_{n \in \mathbb{N}} z^{-n} \mu_{\text{SRB}}(gf \circ T^n) = G_{g,f}(z^{-1}).$$

Hence the formula

$$C_{f,g}(e^{i\omega}) = G_{f,g}(e^{i\omega}) + G_{g,f}(e^{-i\omega}) - \mu_{\text{SRB}}(fg)$$

together with (2.12), shows that $C_{f,g}$ is meromorphic in the annulus $\{z \in \mathbb{C} : \theta^{s_r} < |z| < \theta^{-s_r}\}$. By Theorem 1 its poles are the inverse of the zeroes of $d^\phi$ in the

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\(^{10}\)Note that, since $(1,\phi) = \int \varphi dm$ (see footnote 4) then $m$ can also be seen as the element 1 of $\mathcal{B}^s$ or $\mathcal{D}^f$.

\(^{11}\)Indeed, if $e^{i\theta} \in \sigma(\mathcal{L})$, then $\Pi_\theta := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\theta k} L^k$ is well defined and is exactly the projector on the associated eigenspace. Moreover, from (2.8) and (2.10) follows that $\text{Range}(\Pi_\theta) \subset \mathcal{B}_0$. Hence, by (2.8),

$$|\Pi_\theta h, \varphi| = |(\Pi_\theta^2 h, \varphi)| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |(\Pi_\theta h, \varphi \circ T^k)| \leq \|\Pi_\theta h\|_s \varphi_\infty.$$ 

That is the eigenspace would consist of measures, whereby violating the mixing assumption.
annulus \{ z \in \mathbb{C} : \theta^{\bar{s}_r} < |z| < \theta^{-\bar{s}_r} \}$. The Lemma easily follows since we have a complete series of spaces and \( r \) can be chosen arbitrarily. \( \square \)

**Remark 2.7.** For \( C^{r+1} \) maps the above argument shows that the correlation function is meromorphic in the annulus \( \{ \lambda^{-\frac{|n|+|r|}{n+1}} < |z| < \lambda^{-\frac{|n|+|r|}{n+1}} \} \), but the relations between the poles and the zeroes of the dynamical determinant can be established only in the smaller annulus \( \{ \lambda^{-\frac{|n|+|r|}{n+1}} < |z| < \lambda^{-\frac{|n|+|r|}{n+1}} \} \).

**Remark 2.8.** Since \( C_{f,g}(e^{i\omega}) \) is the Fourier transform of the correlation function, it is a physically accessible function. Its poles on the complex plane (the Ruelle resonances) can be computed, e.g. via Pade approximants, hence they are physically observable as well.

The proof of Theorem 1 rests on the next basic estimate proven in section 3.

**Lemma 2.9.** For each \( n \in \mathbb{N}, \sigma \in (\theta, 1) \) and \( \bar{s}_r = \left\lceil \frac{\bar{n}}{2} \right\rceil \), holds true\(^{12} \)

\[
|\Gamma_n - \text{Tr} P^n_{\sigma, \bar{s}_r}| \leq C \sigma^{-\frac{\bar{s}_r}{\bar{n}+1}}. 
\]

**Proof of Theorem 1.** For \( |z| < 1 \), let

\[
g(z) := \det(\text{Id} - z P_{\sigma,s})^{-1} d^n(z) = \exp \left( -\sum_{n=0}^{\infty} \frac{z^n}{n} (\Gamma_n - \text{Tr} P^n_{\sigma,s}) \right).
\]

By the estimate of Lemma 2.9, \( g \) is analytic and different from zero, in the disk \( |z| < \sigma^{-\frac{\bar{s}_r}{\bar{n}+1}} \). Since

\[
d^n(z) = \det(\text{Id} - z P_{\sigma,s}) g(z)
\]

the theorem trivially follows from the arbitrariness of \( \sigma \). \( \square \)

**Proof of Lemma 2.9.** Clearly we must worry only about large \( n \). Consider \( x \in \text{Fix} T^n \), choose a coordinate system \((\xi, \eta)\) in a neighborhood of \( x \) such that \( W^u(x) = \{(\xi, 0)\} \) and \( W^s(x) = \{(0, \eta)\} \). In such coordinates

\[
D_x T^n = \begin{pmatrix} A_n(x) & 0 \\ 0 & B_n(x) \end{pmatrix}
\]

where \( \|A_n(x)^{-1}\| \leq C \lambda^{-n} \) and \( \|B_n(x)\| \leq C \lambda^{-n} \). Accordingly

\[
|\det(\text{Id} - D_x T^n)| = |\det A_n(x) \det(\text{Id} - A_n(x)^{-1}) \det(\text{Id} - B_n(x))| 
\geq C^{-1} |\det B_n(x)| = C^{-1} |\det(D_x T^n)_{E_u}|.
\]

Let us now consider a small fixed \( \rho > 0 \) and let \( W_{\rho, u}^u(z) \) be the unstable and stable manifolds of \( z \) of size \( \rho \), respectively. By (2.13) and standard distortion arguments

\[
|\det(\text{Id} - D_x T^n)|^{-1} \leq C \rho^{-d_u} \int_{W_{\rho, u}^u(x)} |\det(D T^n)_{E_u}| d\xi = C \rho^{-d_u} \int_{T^{-n} W_{\rho, u}^u(x)} d\xi.
\]

Next, let us consider the sets \( Z_{\rho}(x) := \bigcup_{y \in T^{-n} W_{\rho, u}^u(y)} W_{\rho, s}^s(y) \). Indeed, let \( z \in \text{Fix} T^n \cap Z_{2\rho}(x) \), then, if \( \rho \) has been chosen small enough, \( W_{2\rho}^s(z) \cap W_{2\rho}^u(x) \) contain only one point, let it be \( y \). But, by construction \( y \in W_{2\rho}^s(z) \cap T^{-n} W_{\rho, u}^u(x) \), hence \( T^n y \in W_{2\rho}^s(z) \cap W_{2\rho}^u(x) \), that is \( T^n y = y \). But \( y = \lim_{n \to \infty} T^{-n} y = x \) and \( y = \lim_{n \to \infty} T^n y = z \), hence \( x = y = z \).

\(^{12}\text{Since } P_{\sigma,s} \text{ is a finite rank operator, the usual trace } \text{Tr} \text{ and the determinant are well defined.}
The above discussion implies that if \( x_1, x_2 \in \text{Fix } T^n \), \( x_1 \neq x_2 \), then \( Z_\rho(x_1) \cap Z_\rho(x_2) = \emptyset \). Hence
\[
\sum_{x \in \text{Fix } T^n} |\det(\text{Id} - D_x T^n)|^{-1} \leq C \rho^{-d} \sum_{x \in \text{Fix } T^n} m(Z_\rho(x)) \leq C \rho^{-d} m(X).
\]

3. Proof of Lemma 2.9

The first step in the proof of Lemma 2.9 is to define, given an integral operator \( Kh(x) := \int \kappa(x, y) h(y) dy, \kappa \in C^0(X^2) \),
\[
\text{(3.1) } \text{Tr } K := \int_X \kappa(x, x) dx.
\]
The first key ingredient is a representation of such an integral trace for small \( \varepsilon \).

**Lemma 3.1.** For each \( \varepsilon \) small enough, holds true
\[
\text{Tr } Q_\varepsilon \mathcal{L}^n = \Gamma_n + O(\varepsilon^{r+1}).
\]

**Proof.** Since \( Q_\varepsilon \mathcal{L}^n h(x) = \int_X q_\varepsilon(x - T^n y) h(y) dy =: \int_X \kappa_{\varepsilon, n}(x, y) h(y) dy \), we have
\[
\text{(3.2) } \text{Tr } Q_\varepsilon \mathcal{L}^n = \int_X \kappa_{\varepsilon, n}(x, x) dx = \int_X q_\varepsilon(x - T^n x) dx.
\]
Next, we consider the change of variable \( z = \Phi_n(x) := x - T^n x \), clearly
\[
\text{(3.3) } \det D_x \Phi_n = \det(\text{Id} - D_x T^n)
\]
Let \( B(0, \varepsilon) \subset \mathbb{R}^d \) be the ball of radius \( \varepsilon \) and center zero. If \( z \in B(0, \varepsilon) \), then it turns to be useful to define the map \( F_0 : \mathbb{T}^d \to \mathbb{T}^d \)
\[
\text{(3.4) } F_0(x) := T^n(x) + z \mod 1.
\]
For \( \varepsilon \) small enough, \( F_0 \) is still hyperbolic hence for each \( x \in \text{Fix } (T^n) \) we can consider the \( F_0 \varepsilon \)-orbit \( \{x, x, \ldots\} \). By shadowing the exists a unique point \( x_z \), in a neighborhood of \( x \), such that \( x_z = F_0(x_z) = T^n(x_z) + z \). The latter fact means that \( \Phi_n(x_z) = z \), that is \( B(0, \varepsilon) \subset \text{Range } \Phi_n \). On the other hand, If \( x \in \Phi_n^{-1}(B(0, \varepsilon)) \), then, by shadowing, it is associated to a unique periodic point of period \( n \). Indeed, given \( z \in B(0, \varepsilon) \) and \( x \) such that \( x - T^n x = z \), then there exists a unique periodic orbit of period \( n \) in a neighborhood of the periodic \( \varepsilon \)-pseudo-orbit \( \{x, T x, \ldots, T^{n-1} x\} \). We can then define the function \( \Psi : \Phi_n^{-1}(B(0, \varepsilon)) \to \text{Fix } T^n \). For each \( x \in \text{Fix } T^n \) we can then define the set \( \Delta_x := \Psi^{-1}(x) \). Due to hyperbolicity of \( T \) it is easy to verify that \( \Phi : \Delta_x \to B(0, \varepsilon) \) is one-to-one beside being onto. We can then label the inverse branches of \( \Phi_n \) by the elements of \( \text{Fix } T^n \).

**SubLemma 3.2.** There exists a constant \( M \) such that, for each inverse branch,
\[
\|\Phi_n^{-1}\|_{C^{r+1}} \leq M;
\]
\[
\|\det(D\Phi_n^{-1})\|_{C^r} \leq M \|\det(D_x \Phi_n^{-1})\| \forall s \in \{0, \ldots, r\},
\]
where \( x_s \in \text{Fix } T^n \) labels the inverse branch.

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\(^{13}\)Notice that the definition below may not coincide necessarily with the usual trace even when the latter is well defined, e.g. it does not necessarily correspond to the sum of the eigenvalues.
The above estimate, whose technical but straightforward proof is postponed to the appendix, together with Lemma 2.4 and 3.2 yields

\[ \text{Tr} Q_n^\varepsilon L^n = \sum_{z \in \text{Fix } T^n} \int d q(z) \left| \det \left( D_{\Phi_n^{-1}} \Phi_n(z) \right) \right| dz \]

\[ = \sum_{z \in \text{Fix } T^n} \left| \det (\text{Id} - D_z T^n) \right|^{-1} + O(\varepsilon^r). \]

Hence, \[ \text{Tr} Q_n^\varepsilon L^n = \text{Tr} L^n Q_\varepsilon = \lim_{\varepsilon \to 0} \text{Tr} Q_\varepsilon L^n Q_\varepsilon \]

\[ = \lim_{\varepsilon \to 0} \text{Tr} (Q_{\varepsilon_\delta} P_{\sigma,s}^n Q_\varepsilon) + \lim_{\varepsilon \to 0} \text{Tr} (Q_{\varepsilon_\delta} R_{\sigma,s}^n Q_\varepsilon). \]

Setting \( \varphi_{\varepsilon,y}(x) := q_\varepsilon(x - y) \in C^\infty \), for each \( A \in L(B^s,B^s) \) and \( f \in C^s \) holds\[ (3.6) \]

\[ A Q_\varepsilon f = \int A \varphi_{\varepsilon,y}(y) dy. \]

Using (3.6) and remembering (2.3), (2.7), (2.9) yields

\[ \text{Proof.}\] Since \( \delta \to Y \), \( \sum_{i} \| \varphi_{\varepsilon_\delta} \|_{B^r} \leq C \varepsilon^{-d-s}. \]

The last step is given by the following perturbation result.

**Lemma 3.3.** There exists \( \varepsilon_1 > 0 \) such that, for each \( \varepsilon \in (0, \varepsilon_1) \) and \( n \in \mathbb{N} \),

\[ \left| \text{Tr} P_{\sigma,s}^n - \lim_{\varepsilon \to 0} \text{Tr} Q_{\varepsilon_\delta} P_{\sigma,s}^n Q_\varepsilon \right| \leq C \varepsilon^{s_r-s}. \]

**Proof.** Since \( P_{\sigma,s} \) is finite dimensional the usual trace of \( P_{\sigma,s}^n \), \( \text{Tr} P_{\sigma,s}^n \), is well defined. More precisely, \( P_{\sigma,s} h = \sum_i w_i^1, w_i^1 \in B^{s_r}, \ell_i \in (B^r)' \), and \( \text{Tr} P_{\sigma,s} = \sum_i \ell_i(w_i) \).

Hence, for each \( h \in C^\infty \), by (3.6) and (2.7),

\[ (Q_{\varepsilon_\delta} P_{\sigma,s}^n Q_\varepsilon h)(x) = \int (Q_{\varepsilon_\delta} P_{\sigma,s}^n \varphi_{\varepsilon,y} ) h(y) dy \]

\[ = \sum_i (Q_{\varepsilon_\delta} w_i^1(x) \ell_i(P_{\sigma,s}^{-1} \varphi_{\varepsilon,y}) h(y) dy. \]

Thus, by (3.6) and (2.7),

\[ \lim_{\varepsilon \to 0} \text{Tr} Q_{\varepsilon_\delta} P_{\sigma,s}^n Q_\varepsilon = \lim_{\varepsilon \to 0} \sum_i \ell_i(Q_{\varepsilon_\delta} P_{\sigma,s}^{-1} Q_\varepsilon w_i) = \sum_i \ell_i(P_{\sigma,s}^{-1} Q_\varepsilon w_i) \]

\[ = \text{Tr} P_{\sigma,s}^n + \sum_i O(\| P_{\sigma,s}^{-1} (Q_\varepsilon - \text{Id}) w_i \|_s) = \text{Tr} P_{\sigma,s}^n + O(\varepsilon^{s_r-s}). \]

---

14The following equalities can be easily verified by direct computation.

15This follows immediately from the fact that, if \( f_n \to f \) in \( L^1 \), then \( Q_\varepsilon f_n \to Q_\varepsilon f \) in \( C^s \), hence \( \lim_{n \to \infty} A Q_\varepsilon f_n = A Q_\varepsilon f. \) One can then approximate \( f \) by piecewise constant function and compute the corresponding Riemann sums. Taking the limit and since \( y \to \varphi_{\varepsilon,y} \in B^s \) is continuous one recovers the integral on the right which is meant in Bochner sense.
Collecting (3.5), (3.7) and Lemma 3.3 yields
\[
\text{Tr} \left( Q_x L^n \right) = \text{Tr} \left( P^\alpha_{\sigma, \omega} \right) + O \left( \varepsilon^{s-\delta} + \varepsilon^{s-\delta} \right).
\]

Proof of Lemma 2.5. Lemma 3.1 and (3.8) imply
\[
|\Gamma_n - \text{Tr} \left( P^\alpha_{\sigma, \omega} \right)| \leq C \left( \varepsilon^r + \varepsilon^{s+\delta} + \varepsilon^{s+\delta} \right).
\]

Finally, choose \( \varepsilon = \sigma^{r+\delta} \), \( s = \bar{s}_r = \left[ \frac{2}{2} \right] \), and hence the lemma. \( \square \)

4. PROOF OF PROPOSITION 2.1

Let us start by recalling the scale of Banach spaces introduced in [14].

It is well known that being Anosov is equivalent to the existence of a continuous strictly invariant vector field \( C \). Let \( C' \) be another continuous cone field contained in \( \text{Int}(C) \). Consider \( \delta > 0 \) and a set of \( d_x \)-dimensional manifolds (with boundary) \( \Omega \) such that, if \( W \in \Omega \), then there exists \( x_W \in W \) and a \( d_x \)-dimensional hyperplane \( E_W \) contained in \( C' \) such that, making an isometric change of coordinates such that \( E_W = \{ (\xi, 0) : \xi \in \mathbb{R}^d \} \), \( W = \{ x_W + (\xi, \gamma_W(\xi)) : \xi \in \mathbb{R}^d, ||\xi|| \leq \delta \} \), \( \Gamma_W \subset C(x) \), for each \( x \in W \), and \( |\gamma_W|_{C^{r+1}(\mathbb{R}^d, \mathbb{R}^d)} \leq M \) for some fixed \( M \) large enough.

Given \( W \in \Omega \), we will denote by \( C^q_0(W, \mathbb{R}) \) the set of functions from \( W \) to \( \mathbb{R} \) which are \( [q] \) times continuously differentiable and such that the \( [q] \) derivative is \( q - [q] \) Hölder continuous on \( W \) and vanish on a neighborhood of the boundary of \( W \). For each \( h \in C^\infty(X, \mathbb{R}) \) and \( q \in \mathbb{R}_+, p \in \mathbb{N}^* \), let
\[
||h||_{p,q} := \sup_{|\alpha| \leq p} \sup_{W \in \Omega} \sup_{\varphi \in C^q_0(\mathbb{R}^d, \mathbb{R})} \int_W \partial^\alpha h \cdot \varphi,
\]
and define the Banach spaces \( B^{p,q} := C^\infty \| \cdot \|_{p,q} \).

In [14] it is proven that, setting \( q_p = \frac{q}{p-1} \) and \( B^*: = B^{q, q_p} \), \( L \in L(B^*, B^*) \) and it satisfies \( \text{(2.5)}, \text{(2.6)}, \text{(2.8)} \) and \( \text{(2.9)} \), provided \( s \leq \left[ \frac{q}{p-1} \right] =: s_r \).

Lemma 4.1. For each \( l < s \leq s_r \), holds \( Q_\varepsilon \in L(B^*, C^\infty) \) and
\[
||Q_\varepsilon - 1||_{B^*, B^*} \leq C \varepsilon^{s-l}.
\]

Proof. Consider \( W \in \Omega \). Let \( W_\rho := \{ x_W + (\xi, (1 - \rho)\gamma(\xi)) : \xi \in \mathbb{R}^d, ||\xi|| \leq \delta \} \). Clearly, provided \( \delta \) has been chosen small enough, for each \( \rho \in [0, 1) \), \( W_\rho \in \Omega \) with \( x_{W_\rho} = x_W \), \( E_{W_\rho} = E_W \). Then, for each multi-index \( \alpha \), \( |\alpha| \leq l \),
\[
\int_W \partial^\alpha h = \int_{||\xi|| \leq \delta} \partial^\alpha h(\xi, \gamma(\xi)) \varphi(\xi, \gamma(\xi)) J_W(\xi) d\xi
\]
\[
= \sum_{i=0}^{s-l-1} \frac{1}{i!} \int_{||\xi|| \leq \delta} \frac{d^i}{d\rho^i} h(\xi, (1 - \rho)\gamma(\xi)) \big|_{\rho = \varepsilon} \varphi(\xi, \gamma(\xi)) J_W(\xi) d\xi
\]
\[
= \sum_{i=0}^{s-l-1} \frac{(1 - \varepsilon)^i}{i!} \int_{W_\varepsilon} \gamma^\beta \partial^\alpha + \beta \varphi + \sum_{|\beta| = s-l} \frac{(-1)^{s-l} \int_{W_\varepsilon} d\rho_1 \cdots d\rho_{s-l}}{l!}
\]
\[
\times \int_{W_{\rho, s-l}} \gamma^\beta \partial^\alpha + \beta \varphi d\mu_{\rho, s-l}
\]
Thus, in order to estimate the norms, it suffices to consider \( \Omega_z := \{ W_\varepsilon : W \in \Omega \} \), that is manifolds uniformly strictly inside the cone field. Let \( W \in \Omega_\varepsilon \), then

\[
\int_W \partial^\alpha Q \phi - \int_W \partial^\alpha \phi = \int_d q(z) \int_W dq^\alpha(y) - \partial^\alpha h(y) \phi(y).
\]

If \( W = \{ \xi, \gamma(\xi) \} \), then \( W_z = \{ (\xi, \gamma(\xi)) + z \} \in \Omega \), provided \( z \) is small enough. Thus, for \( |\alpha| = l \), remembering (2.2),

\[
\int_W \partial^\alpha Q \phi = \sum_{|\beta| = l} \int_d q(z) \int_0^1 dt_1 \cdots \int_0^{t_l-1} dt_{s-l} dy \partial^\beta h(y) \phi(y) \leq \| h \|_{s,q} \varepsilon^{s-l}.
\]

Finally, (2.8) follows easily from (4.1). Clearly if \( T \in \text{Diff}^\infty(X, X) \) we have a complete series of adapted spaces.

**Appendix A. Proof of Sub-Lemma 3.2**

Let us choose a periodic point \( x_0 \in \text{Fix}(T^n) \) and limit our considerations to the associated inverse branch, that, by a slight abuse of notation, I will designate simply by \( \Phi_n^{-1} \). Again we will use the map \( F_\varepsilon \) introduced in (3.1). Clearly \( D_z \Phi_n^{-1} = (I - D_x T^n)^{-1} = (I - D_x F_\varepsilon)^{-1} \), where \( \Phi_n(x) = z \).

To study the regularities properties at a given point \( z_0 \), small enough, it is convenient to perform an affine change of coordinates \( \Lambda(z_0) \) such that \( \Lambda(z_0)(x_{z_0}) = 0 \), \( |(D_z \Lambda)|_{C^0} + |(D_z \Lambda)^{-1}|_{C^0} \leq C \), and \( 16 \)

(A.1)

\[
D_{\Lambda(z_0)}(x_{z_0}) = \begin{pmatrix}
A(z, z_0) & B(z, z_0) \\
C(z, z_0) & D(z, z_0)
\end{pmatrix}
\]

where \( \Lambda(z_0) \circ F_\varepsilon \circ \Lambda(z_0)^{-1} \), \( |A(z_0, z_0)| , |D(z_0, z_0)^{-1}| \leq \lambda^{-n} \) and \( B(z_0, z_0) = C(z_0, z_0) = 0 \). In other words, in the new coordinates, \( \{ (\xi, 0) \} \) corresponds to the stable manifold at \( z_0 \) and \( \{ (0, \eta) \} \) to the unstable one. In such coordinates, \( 17 \)

(A.2)

\[
D_{z_0} \Phi_n^{-1} \big|_{z = z_0} = \begin{pmatrix}
(Id - A)^{-1} & 0 \\
-(Id - D^{-1})^{-1} & (Id - D^{-1})^{-1}
\end{pmatrix}.
\]

Given the simpler structure of \( F_{z_0, z_0} \) it would be much easier to study its regularity rather than the one of \( F_{z_0} \). Yet, the two are equivalent only if the change of coordinates \( \Lambda(z) \) is uniformly \( C^r \). To prove the latter is our first task.

We start by computing the derivatives of \( x_{z_0}^i := T^i x_z \) with respect to \( z \):

\[
\frac{\partial x_{z}^i}{\partial z} = (D_x T^i (Id - D_x T^n)^{-1})
\]

---

16Here, and in the following, given \( I \subset \mathbb{R}^p \) and a function \( f \) from \( I \) to some Banach algebra \( \mathfrak{B} \), by \( | \cdot |_C^p \) we mean the norm \( \sup_{x \in I} \sum_{0 \leq |\alpha| \leq p} \| \partial^\alpha f(x) \|_\mathfrak{B} \) so that \( C^p(I, \mathfrak{B}) \) is itself a Banach algebra.

17Here and in the following I suppress the dependence on \( z, z_0 \) and \( \Lambda \) when no confusion arises.
It is convenient to use in the tangent space at \( x \) the coordinates pushed forward from the tangent space of \( x \). In such coordinates holds

\[
(A.3) \quad D_{x^i}T = \begin{pmatrix} A_i(z, z_0) & B_i(z, z_0) \\ C_i(z, z_0) & D_i(z, z_0) \end{pmatrix}
\]

where \( \|A_i\|, \|D_i^{-1}\| \leq \lambda^{-1} \) and \( B_i(z, z_0) = C_i(z, z_0) = 0 \). Hence,

\[
(A.4) \quad \partial_{x^i} T \bigg|_{z=0} = \left( \prod_{j=1}^n A_j \right) \begin{pmatrix} 0 & 0 \\ \prod_{j=i+1}^n D_j^{-1} & 0 \end{pmatrix} \begin{pmatrix} (\text{Id} - A)^{-1} & 0 \\ 0 & -(\text{Id} - D^{-1})^{-1} \end{pmatrix},
\]

which readily implies \( |\partial_{x^i} T|_{\lambda^0} \leq \lambda^{-C} \), for the stable coordinates, and \( |\partial_{x^i} T|_{\lambda^0} \leq \lambda^{-n}C \) for the unstable ones.

An hyperplane \( E \) in the stable direction is uniquely determined by a linear operator \( U : \mathbb{R}^{d_x} \to \mathbb{R}^{d_x} : E = \{ (\xi, U\xi) : \xi \in \mathbb{R}^{d_x} \} \). A simple computation shows that, by defining

\[
H(z, z_0, U) := (C(z, z_0) + D(z, z_0)U)(A(z, z_0) + B(z, z_0)U)^{-1},
\]

the stable hyperplane for \( \bar{F}_z \) at the point \( \Lambda(z_0)(x) \) is determined by the fixed point of \( H(z_0, z, U(z)) = U(z) \).

18Applying the implicit function theorem, since by construction \( U(z_0) = 0 \), yields

\[
(A.5) \quad \partial_{x^i} U \bigg|_{z=0} = -(\text{Id} - G)^{-1}(D^{-1}\partial_x C) \bigg|_{z=0},
\]

where \( G : GL(d_x, d_u) \to GL(d_x, d_u) \) is defined by \( G(V) := D^{-1}VA \). In addition,

\[
(A.6) \quad \partial_{x^i} C \bigg|_{z=0} = \sum_{p=1}^n \sum_{k=1}^d \left[ \prod_{j=1}^n D_j \right] c_{2,p}(x_{k_0}^{-1}) \left[ \prod_{j<k} A_j \right] \frac{\partial(x^{-1})_p}{\partial x_i} \bigg|_{z=0}
\]

This, by equations \( A.1 \) and \( A.3 \) implies \( |U|_{\lambda^1} \leq C \). Since the exact same argument can be carried out for the unstable space, remembering \( A.3 \) we have \( |\Lambda|_{\lambda^1} + |D_x \Lambda|_{\lambda^1} + |D_x \Lambda^{-1}|_{\lambda^1} \leq C \). We can thus conclude the argument by induction: let \( l \leq r \) and suppose that \( |x^i|_{\lambda^1} \leq C \) and \( |D_x \Lambda|_{\lambda^1} + |D_x \Lambda^{-1}|_{\lambda^1} \leq C \). Then by \( A.3 \) follows \( A_i(z, z), B_i(z, z), C_i(z, z), D_i(z, z) \), seen as functions of \( z \) have \( C_i \) norms equibounded. In turn, by \( A.3 \), it follows \( |x^i_{c_{l+1}}| \leq C \). Equations, \( A.5 \) and \( A.6 \), imply then \( |D_x \Lambda|_{\lambda^{l+1}} + |D_x \Lambda^{-1}|_{\lambda^{l+1}} \leq C \), provided \( l+1 \leq r \). This proves that the change of coordinates \( \Lambda(z_0) \) is uniformly \( C^r \).

It is now easy to verify that \( \bar{F}_z \) is \( C^r \) and, remembering \( A.2 \), the first inequality of the sub-lemma can be readily proven. To conclude note that

\[
\det(D\Phi_n^{-1}) = -\det(\text{Id} - A)^{-1} \det(D - D^{-1}) \det(D^{-1}).
\]

Next, since given any smooth function \( \Delta(z) \) with values in the invertible matrices, \( \partial_{z_i} \det \Delta(z) = \lim_{h \to 0} \frac{\det(\text{Id} + [\Delta(z) + h\epsilon_i] - \Delta(z)]\det(\Delta(z)^{-1}) - 1}{h} \det(\Delta(z)) \)

\[
= \lim_{h \to 0} \frac{e^{\text{Tr}\ln(\text{Id} + [\partial_{z_i} \Delta(z)]\det(\Delta(z)^{-1})h)} - 1}{h} \det(\Delta(z)) = \text{Tr}([\partial_{z_i} \Delta(z)]\det(\Delta(z)^{-1}) \cdot \det(\Delta(z)),
\]

18In fact, it is known that \( U(z) \) is \( C^r \), e.g. see [21 Propositions 1, 2], yet here we need explicit estimates. This forces us to redo the argument.
also in view of (A.4), holds true
\[ \| \det(D\Phi_n^{-1}) \|_{C^s} \leq C |\det(D^{-1})|. \]
Finally, since \(|\ln \det D| \leq C_1\), it follows \(\| \det(D^{-1}) \|_{C^0} \leq |\det(D^{-1}(x_\star))|\) and the lemma.

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