On sums of Szemerédi–Trotter sets

Shkredov I.D.

Annotation.

We prove new general results on sumsets of sets having Szemerédi–Trotter type. This family includes convex sets, sets with small multiplicative doubling, images of sets under convex/concave maps and others.

1 Introduction

Let \( A = \{a_1, \ldots, a_n\} \), \( a_i < a_{i+1} \) be a set of real numbers. We say that \( A \) is convex if

\[
a_{i+1} - a_i > a_i - a_{i-1}
\]

for every \( i = 2, \ldots, n - 1 \). Lower bounds for sumsets/difference sets of convex sets were obtained in several papers, see [4], [1], [2], [3], [5], [16], [13], [8], [11] and others. For example, in [8] the following theorem was proved.

**Theorem 1** Let \( A \subseteq \mathbb{R} \) be a convex set. Then

\[
|A + A| \gg |A|^{\frac{14}{9}} \log^{-\frac{2}{9}} |A|.
\]

Moreover, our method gives a generalization of Theorem [1] for sumset of two different convex sets, see Theorem [14] below.

In [9] the authors prove a general statement on addition of a set and its image under a convex map (the first result in the direction was obtained in [3]).

\*This work was supported by grant Russian Scientific Foundation RSF 14–11–00433.
Theorem 3 Let \( f \) be any continuous, strictly convex or concave function on the reals, and \( A, C \subset \mathbb{R} \) be any finite sets such that \(|A| = |C|\). Then
\[
|f(A) + C|^{10}|A + A|^9 \gg |A|^{24} \log^{-2} |A|.
\]
In particular, choosing \( C = f(A) \), we get
\[
\max\{|f(A) + f(A)|, |A + A|\} \gg |A|^{24 \log^{-2} |A|}.
\]
Finally
\[
|AA|^{10}|A + A|^9 \gg |A|^{24} \log^{-2} |A|.
\]

We refine the result.

Theorem 4 Let \( f \) be any continuous, strictly convex or concave function on the reals, and \( A, C \subset \mathbb{R} \) be any finite sets such that \(|A| = |C|\). Then
\[
|f(A) + C|^{42}|A + A|^37 \gg |A|^{100} \log^{-20} |A|.
\]
In particular, choosing \( C = f(A) \), we get
\[
\max\{|f(A) + f(A)|, |A + A|\} \gg |A|^{100 \log^{-20} |A|}.
\]
Finally
\[
|AA|^{42}|A + A|^37 \gg |A|^{100} \log^{-20} |A|.
\]

Another applications can be found in the last section 4.

In the proof we use so–called the eigenvalues method, see e.g. [15] and some observations from [14].

2 Notation

Let \( G \) be an abelian group and \(+\) be the group operation. In the paper we use the same letter to denote a set \( S \subseteq G \) and its characteristic function \( S : G \to \{0, 1\} \). By \(|S|\) denote the cardinality of \( S \).

Let \( f, g : G \to \mathbb{C} \) be two functions with finite supports. Put
\[
(f * g)(x) := \sum_{y \in G} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in G} f(y)g(y + x) \tag{1}
\]

Let \( A_1, \ldots, A_k \subseteq G \) be any sets. Put
\[
E_k(A_1, \ldots, A_k) = \sum_{x \in G} (A_1 \circ A_1)(x) \ldots (A_k \circ A_k)(x) \tag{2}
\]
be the higher energy of \(A_1, \ldots, A_k\). If \(A_j = A, \ j = 1, \ldots, k\) we simply write \(E_k(A)\) instead of \(E_k(A, \ldots, A)\). In the same way one can define \(E_k(A)\) for non-integer \(k\). In particular case \(k = 2\) we put \(E(A, B) := E_2(A, B)\) and \(E(A) = E_2(A)\). The quantity \(E(A)\) is called the additive energy of a set, see e.g. [17]. Similarly, we define

\[
E_k(f_1, \ldots, f_k) = \sum_x (f_1 \circ f_1)(x) \cdots (f_k \circ f_k)(x) .
\]

Denote by \(C_{k+1}(f_1, \ldots, f_{k+1})(x_1, \ldots, x_k)\) the function

\[
C_{k+1}(f_1, \ldots, f_{k+1})(x_1, \ldots, x_k) = \sum_z f_1(z) f_2(z + x_1) \cdots f_{k+1}(z + x_k) .
\]

Thus, \(C_2(f_1, f_2)(x) = (f_1 \circ f_2)(x)\). If \(f_1 = \cdots = f_k+1 = f\) then write \(C_{k+1}(f)(x_1, \ldots, x_k)\) for \(C_{k+1}(f_1, \ldots, f_{k+1})(x_1, \ldots, x_k)\). Note that

\[
\sum_{x_1, \ldots, x_k} C_{k+1}^2(f_1, \ldots, f_{k+1})(x_1, \ldots, x_k) = E_{k+1}(f_1, \ldots, f_{k+1}) .
\]

Let \(g : \mathbb{G} \to \mathbb{C}\) be a function, and \(A \subseteq \mathbb{G}\) be a finite set. By \(T_A^g\) denote the matrix with indices in the set \(A\)

\[
T_A^g(x, y) = g(x - y) A(x) A(y) .
\]

It is easy to see that \(T_A^g\) is hermitian iff \(g(-x) = g(x)\). The corresponding action of \(T_A^g\) is

\[
\langle T_A^g a, b \rangle = \sum_z g(z) (\overline{b} \circ a)(z) .
\]

for any functions \(a, b : A \to \mathbb{C}\). In the case \(g(-x) = g(x)\) by \(\text{Spec}(T_A^g)\) we denote the spectrum of the operator \(T_A^g\)

\[
\text{Spec}(T_A^g) = \{ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|A|} \} .
\]

Write \(\{f\}_\alpha, \alpha \in [|A|]\) for the corresponding eigenfunctions. We call \(\mu_1\) as the main eigenvalue and \(f_1\) as the main function.

In the asymmetric case let \(g : \mathbb{G} \to \mathbb{C}\) be a function, and \(A, B \subseteq \mathbb{G}\) be two finite sets. Suppose that \(|B| \leq |A|\). By \(T_{A,B}^g\) denote the rectangular matrix

\[
T_{A,B}^g(x, y) = g(x - y) A(x) B(y) ,
\]

and by \(\tilde{T}_{A,B}^g(x, y)\) denote the another rectangular matrix

\[
\tilde{T}_{A,B}^g(x, y) = g(x + y) A(x) B(y) .
\]

As in (6), we arrange the singular values in order of magnitude

\[
\lambda_1(T_{A,B}^g) \geq \lambda_2(T_{A,B}^g) \geq \cdots \geq \lambda_{|B|}(T_{A,B}^g) \geq 0 ,
\]

\[
T_{A,B}^g(x, y) = \sum_{j=1}^{|B|} \lambda_j u_j(x) v_j(y)
\]

and similar for \(\tilde{T}_{A,B}^g\). Here \(u_j(x), v_j(y)\) are singular functions of the operators. General theory of such operators was developed in [15].

All logarithms are base 2. Signs \(\ll\) and \(\gg\) are the usual Vinogradov’s symbols.
3 The main definition

We begin with a rather general definition of families of sets which are usually obtained by Szemerédi–Trotter’s theorem, see [17].

Definition 5 A set \( A \subset \mathbb{R} \) has SzT–type (in other words \( A \) is called Szemerédi–Trotter set) with parameter \( \alpha \geq 1 \) if for any set \( B \subset \mathbb{R} \) and an arbitrary \( \tau \geq 1 \) one has

\[
|\{x \in A + B : (A * B)(x) \geq \tau\}| \ll c(A)|B|^\alpha \cdot \tau^{-3},
\]

where \( c(A) > 0 \) is a constant depends on the set \( A \) only. We define the quantity \( c(A)|B|^\alpha \) as \( c(A,B) \).

From the definition one can see that if \( A \) has SzT–type then \((-A)\) has the same SzT–type with the same parameters \( \alpha \) and \( c(A) \).

Remark 6 We put parameter \( \alpha \geq 1 \) because otherwise there is no any SzT–type set. Indeed, take \( B = C - A \), where set \( C \) will be chosen later. One has \((A * B)(x) \geq |A|\) for any \( x \in C \). Then by (7), we obtain

\[
|A|^2|C| \ll c(A)|A + C|^\alpha \leq c(A)|A|^\alpha |C|^\alpha.
\]

Taking \(|C|\) sufficiently large and having the set \( A \) is fixed, we see that \( \alpha \geq 1 \).

Examples. Let us give some examples of SzT–type sets with parameter \( \alpha = 2 \).
1) If \( A \subset \mathbb{R} \) is a convex set then \( A \) has SzT–type with \( c(A) = |A| \), see [13].
2) Let \( f \) be a strictly convex/concave function. Then \( f(A) \) has SzT–type with \( c(A) = q(A) \), where

\[
q(A) := \min_C \frac{|A + C|^2}{|C|},
\]

and \( A \) has SzT–type with \( c(A) = q(f(A)) \), see [12], [9].
3) Let \(|AA| \leq M|A|\). Then \( A \) has SzT–type with \( c(A) = M^2|A| \). This is a particular case of the family from (2). Indeed, take \( f(x) = \log x \), and apply (10) with \( C = \log(A \cap \mathbb{R}^+) \) or \( C = \log |(A \cap \mathbb{R}^-)| \).
4) Let \( A \subset \mathbb{R}^+ \), and \( a \in \mathbb{R} \setminus \{0\} \). Then \( \log A \) has SzT–type with \( c(A) = q'(A) \), where

\[
q'(A) := \min_C \frac{|(A + a)C|^2}{|C|},
\]

see [6], [10].

There are another families of SzT–type sets, for example see a family of complex sets in [7].

Using definition 5 and easy calculations, one can obtain upper bounds for some simple characteristics of SzT–type sets see, e.g. papers [12], [13], [8], [9]. It is more convenient do not use parameter \( \alpha \) in the statements.
Lemma 7 Let $A$ be a SzT–type set. Then

$$E_3(A) \ll c(A, A) \log |A|, \quad E_3^2(A) \ll E_{3/2}^2(A) c(A, A),$$

and for any $B$ one has

$$E(A, B) \ll (c(A, B) |A| |B|)^{1/2}.$$

We need in one more technical lemma.

Lemma 8 Let $A, A_*$ be a SzT–type set with the same parameter $\alpha > 1$. Then

$$E^{2\alpha-1}(A_*, A) \ll_{(\alpha-1)^{-1}} \left( \sum_x (A_* \circ A_*)^{1/2}(x)(A \circ A)(x) \right)^{2\alpha-2} \cdot c^{1/3}(A) c^{\alpha/3}(A_*) |A|^{2/3} |A_*|^{\alpha/3}. $$

Proof. Put $c_* = c(A_*)$, $c = c(A)$, $a = |A|$, $a_* = |A_*|$. Splitting the sum, we get with some inaccuracy

$$E(A_*, A) \ll \tau^{1/2} \left( \sum_x (A_* \circ A_*)^{1/2}(x)(A \circ A)(x) \right) + \tau \sum_x S_\tau(x)(A \circ A)(x) = \tau^{1/2} \omega_1 + \tau \omega_2, \quad (11)$$

where $S_\tau = \{ x : (A_* \circ A_*)(x) \geq \tau \}$. Because of $A_*$ is Szemerédi–Trotter set, we have $|S_\tau| \ll c_*(a_*)^{\alpha-3} \tau^{-3}$. On the other hand, $A$ is also Szemerédi–Trotter set, so

$$\sum_x S_\tau(x)(A \circ A)(x) = \sum_{x \in A} (S_\tau \ast A)(x) \ll c^{1/3} |S_\tau|^{\alpha/3} a_*^{2/3} \ll c^{1/3} a^{2/3} c_*/c_*(a_*)^{\alpha/3} \tau^{-\alpha}. \quad (12)$$

Combining (11) and (12), we obtain

$$E(A_*, A) \ll_{(\alpha-1)^{-1}} \tau^{1/2} \omega_1 + \tau^{-\alpha} c^{1/3} a^{2/3} c_*/c_*(a_*)^{\alpha/3}. $$

An optimal choice of parameter $\tau$ is $\tau^{1/2} = \omega_1^{1/(2\alpha-1)} c_*/c_*(2\alpha-1) c_*/c_*(2\alpha-1) a_*/a_*(2\alpha-1)^{\alpha/2}$. Thus

$$E^{2\alpha-1}(A_*, A) \ll_{(\alpha-1)^{-1}} \left( \sum_x (A_* \circ A_*)^{1/2}(x)(A \circ A)(x) \right)^{2\alpha-2} \cdot c^{1/3} a^{2/3} a_*/a_*(2\alpha-1)^{\alpha/2}$$

as required. □
4 The proof of the main result

We begin with a lemma from [15].

**Lemma 9** Let $A \subseteq G$ be a set and $g$ be a nonnegative function on $G$. Suppose that $f_1$ is the main eigenfunction of $T_A^g$ or $T_A$, and $\mu_1$ is the correspondent eigenvalue. Then

$$\langle T_A^g f_1, f_1 \rangle \geq \frac{\mu_1^3}{\|g\|_2 \cdot \|g\|_\infty}.$$ 

A particular case $\alpha = 2$ of the next lemma is contained inside Theorem 8 of paper [14]. We give the proof for the sake of completeness.

**Lemma 10** Let $A$ be a SzT–type set and let $\Delta \geq 1$ be a real number. Suppose that

$$B \subseteq \{ x : (A \circ A)(x) \geq \Delta \} \quad \text{or} \quad B \subseteq \{ x : (A * A)(x) \geq \Delta \}.$$ 

Then

$$\mathcal{E}_3(A, A, B) \ll \Delta^{-\frac{4}{3(\alpha - 1)}} c(A) \frac{5a + 1}{2(3a - 1)} |A|^{\frac{2a^2 + 5a - 1}{6a - 2}} |B|^{\frac{3(a^2 - 1)}{6a - 2}} \log |A|. \quad (13)$$

**Proof.** Put $a = |A|$, $L = \log a$ and $c = c(A)$. By the pigeonhole principle there is a set $Q$ such that

$$\sigma := \mathcal{E}_3(A, A, B) \ll L \sum_{x \in Q} (A \circ A)(x)(B \circ B)(x),$$

and the values of the convolution $(A \circ A)(x)$ differ at most twice on $Q$. Denote by $q$ the maximum of $(A \circ A)(x)$ on $Q$. Because of $A$ is SzT–type set, we have $|Q| \ll ca^3 q^{-3}$. Using Lemma 7, we obtain

$$\sigma \ll Lq \mathcal{E}(A, B) \ll Lq(c(A)a|B|^{\alpha + 1})^{1/2}. \quad (14)$$

On the other hand, by the definition of the set $B$ and the Cauchy–Schwarz inequality one has

$$\sigma \ll Lq^2 \Delta^{-1} \mathcal{E}(Q, B, A) \ll Lq^2 \Delta^{-1} \mathcal{E}^{1/2}(Q, A) \mathcal{E}^{1/2}(B, A). \quad (15)$$

Combining the last two bounds and the upper bound for size of $Q$, we have

$$\sigma \ll Lq \mathcal{E}^{1/2}(A, B)(\mathcal{E}^{1/2}(A, B) + q\Delta^{-1}c^{1/4}a^{1/4}|Q|^{(\alpha + 1)/4} \ll$$

$$\ll Lq \mathcal{E}^{1/2}(A, B)(\mathcal{E}^{1/2}(A, B) + q^{-(3\alpha - 1)/4} \Delta^{-1}c^{(\alpha + 2)/4}a^{(\alpha^2 + \alpha + 1)/4}).$$

The optimal choice of $q$ is $q = E^{2/(3\alpha - 1)}(A, B)\Delta^{-4/(3\alpha - 1)}c^{(\alpha + 2)/(3\alpha - 1)}d(a^2 + \alpha + 1)/(3\alpha - 1)$. Here we have used the fact that $\alpha > 1/3$. Substituting $q$ into the last formula and using Lemma 7 again, we get

$$\sigma \ll Lq \mathcal{E}(A, B) \ll L(E^{3(\alpha - 1)/(3\alpha - 1)}(A, B) \cdot \Delta^{-4/(3\alpha - 1)}c^{(\alpha + 2)/(3\alpha - 1)}d(a^2 + \alpha + 1)/(3\alpha - 1) \right) =$$

$$= L\Delta^{-\frac{4}{3(\alpha - 1)}} |B|^{\frac{3(a^2 - 1)}{6a - 2}} c^{\frac{5a + 1}{6a - 2}} a^{\frac{2a^2 + 5a - 1}{6a - 2}}$$

as required. \[\square\]

Let us formulate the main result of the paper.
Theorem 11 Suppose that \( A \subset \mathbb{R} \) has \( SzT \)-type with parameter \( \alpha \). Then

\[
|A + A| \gg c(A) \frac{1-11\alpha}{3\alpha^2 + 12\alpha + 1} |A| A^{\frac{8\alpha^2 + 5\alpha - 3}{3\alpha^2 + 12\alpha + 1}} \cdot (\log |A|)^{-\frac{4(3\alpha - 1)}{5\alpha^2 + 12\alpha + 1}}.
\]  

(16)

In particular, for \( \alpha = 2 \) one has

\[
|A + A| \gg c(A) \frac{21}{37} |A| \frac{39}{59} \cdot (\log |A|)^{-\frac{20}{37}}.
\]

(17)

Proof. Let \( S = A + A, \ |S| = d, \ a = |A| \). Let also \( L = \log a, \ c = c(A) \). We have

\[
|A|^2 = \sum_{x,y} A(x)A(y)S(x + y) \leq 2 \sum_{z \in S_1} (A + A)(z),
\]

where \( S_1 = \{z \in S : (A + A)(z) \geq 2^{-1}a^2d^{-1}\} \). Denote by \( f_j, \mu_j \) the eigenfunctions and eigenvalues of hermitian operator \( T^A \). From (18) it follows that \( \mu_1 \geq a/2 \). Applying Lemma 9, we see that

\[
\langle T^A f_1, f_1 \rangle \geq \mu_1^2(T^A) \geq 2^{-3}a^3d^{-1}.
\]

(19)

Further, by nonnegativity of the operator \( T^A \) as well as inequality (19) and the lower bound for \( \mu_1 \), we get

\[
\sigma := \sum_{x,y,z} T^A f_1, f_1 \geq \mu_1^2(T^A f_1, f_1) \geq 2^{-5}a^5d^{-1}.
\]

(20)

On the other hand

\[
\sigma = \sum_{x,y,z} (A \circ A)(x + y)S_1(x + z)S_1(y + z) = \sum_{\alpha, \beta} S_1(\alpha)S_1(\beta)(A \circ A)(\alpha - \beta)C_3(-A, A, A)(\alpha, \beta).
\]

(21)

Combining (20), (21) and using (4), we obtain by the Cauchy–Schwarz inequality that

\[
a^{10}d^{-2} \ll E_3(A)E_3(A, A, S_1).
\]

(22)

Applying the first formula of Lemma 7 to estimate the quantity \( E_3(A) \) and Lemma 10 to estimate \( E_3(A, A, S_1) \), we have

\[
a^{10}d^{-2} \ll L^2 a^\alpha c \cdot \Delta^{-\frac{4}{5\alpha+1}} e^{\frac{5\alpha+1}{2(3\alpha-1)} a} a^{\frac{2\alpha^2 + 5\alpha - 3}{2(3\alpha-1)}} \cdot d^{\frac{3(3\alpha - 1)}{10}},
\]

where \( \Delta = 2^{-1}a^2d^{-1} \). After some calculations, we get

\[
d \gg L \frac{4(3\alpha - 1)}{3\alpha^2 + 12\alpha + 1} c^\frac{11-11\alpha}{3\alpha^2 + 12\alpha + 1} a^{-\frac{8\alpha^2 + 5\alpha - 3}{3\alpha^2 + 12\alpha + 1}}
\]

as required.

\[
\square
\]

Proof of Theorems 2, 4. To obtain Theorem 2 just recall that \( \alpha = 2, \ c(A) = |A| \) for convex sets. Remembering the definition of \( q(f(A)) \) from (11), we have \( c(A) = q(f(A)) \leq |f(A) + C|^2|C|^{-1} \). After that applying the main Theorem 11 we get Theorem 4.

\[
\square
\]
Remark 12 Of course, one can replace the condition $|A| = |C|$ in Theorems 3, 4 to $c_1|A| \leq |C| \leq c_2|A|$, where $c_1, c_2 > 0$ are any absolute constants. Certainly, signs $\ll, \gg$ should be changed by $\ll_{c_1, c_2}, \gg_{c_1, c_2}$ in the case. Even more, it is possible to prove the results for sets $A$ and $C$, having incomparable sizes. We do not make such calculations here (and also below), note only that because in Theorem 4, we have $c(A) \leq |f(A) + C|^{-1}$ this implies
$$|f(A) + C|^2 |A + A|^{37} \gg |A|^{79} |C|^{21} \log^{-20} |A|.$$  

We conclude the section proving a result which generalize, in particular, Theorem 1.3 from [9] as well as the results on sumsets/difference sets of convex sets from [13]. The arguments are in the spirit of Theorem 11. We need in a lemma from [15].

Lemma 13 Let $A, B \subseteq G$ be finite sets, $|B| \leq |A|$, $D, S \subseteq G$ be two sets such that $A - B \subseteq D$, $A + B \subseteq S$. Then the main eigenvalues and singular functions of the operators $T_{A,B}^D, \tilde{T}_{A,B}^S$ equal $\lambda_1 = (|A||B|)^{1/2}$, and
$$v_1(y) = B(y)/|B|^{1/2}, \quad \text{and} \quad u_1(x) = A(x)/|A|^{1/2}.$$  

All other singular values equal zero.

Using lemma above, we prove our second main result, although one can use a more elementary approach as in [13].

Theorem 14 Suppose that $A, A_\ast \subseteq \mathbb{R}$ have SzT-type with the same parameter $\alpha$. Then

$$|A \pm A_\ast| \gg$$
$$\min\{c(A_\ast)^{-2} c(A)^{-\frac{13}{3(\alpha^2 + 4\alpha - 5)}} |A|^{\frac{2(24 - \alpha)}{3(\alpha^2 + 4\alpha - 5)}} c(A)^{-\frac{33 - 10\alpha}{3(\alpha^2 + 4\alpha - 5)}} |A_\ast| \}$$
$$\times (\log(|A||A_\ast|))^{-\frac{2}{\alpha + 1}},$$

and for $\alpha > 1$

$$|A \pm A_\ast| \gg_{(\alpha-1)^{-1}} c(A)^{-\frac{4\alpha - 2}{3(\alpha^2 + 4\alpha - 5)}} c(A_\ast)^{-\frac{7\alpha - 5}{3(\alpha^2 + 4\alpha - 5)}} |A|^{-\frac{28\alpha - 4\alpha^2 - 16}{3(\alpha^2 + 4\alpha - 5)}} |A_\ast|^{-\frac{25\alpha - 4\alpha^2 - 21}{3(\alpha^2 + 4\alpha - 5)}}$$
$$\times (\log(|A||A_\ast|))^{-\frac{2(\alpha-1)}{\alpha^2 + 4\alpha - 5}}.$$  

In particular, for $\alpha = 2$ one has

$$|A \pm A_\ast| \gg \max\{c(A_\ast)^{-\frac{1}{2}} c(A)^{-\frac{11}{12}} |A|^{\frac{1}{12}} c(A)^{-\frac{11}{12}} |A_\ast|^{\frac{11}{12}} |A_\ast|^{\frac{1}{12}} c(A)^{-\frac{11}{12}} c(A_\ast)^{-\frac{11}{12}} A|^{\frac{11}{12}} |A_\ast|^{\frac{11}{12}} \}$$
$$\times (\log(|A||A_\ast|))^{-\frac{1}{2}}.$$  

Finally

$$|A \pm A_\ast|^{\frac{1 + \alpha}{2}} |A - A| \gg |A|^{\frac{33 - 4\alpha}{3}} |A_\ast|^{\frac{6 - \alpha}{3}} c^{-7/6}(A)c^{-1/3}(A_\ast) \log^{-1}(|A||A_\ast|).$$
After some calculations, we have

\[ aa_s = \sum_{x,y} A(x)A_s(y)S(x+y) \leq d^{1/2}E^{1/4}(A)E^{1/4}(A_s). \]  (27)

Let us begin with (23). One can assume that \( E(A) \geq d^{-1}a^2a_s^2 \), the opposite case is similar. By Lemma 7, we get

\[ \frac{E_{3/2}(A)}{a} \geq d^{-3/2}c^{-1/2}a^2a_s^2. \]  (28)

Denote by \( f_j, \mu_j \) the eigenfunctions and the eigenvalues of hermitian nonnegative operator

\[ (\tilde{T}_{A,A_s}^{S} (\tilde{T}_{A,A_s}^{S})^*)(x,y) = C_3(A_s, S, S)(x,y)A(x)A(y). \]

By Lemma 13 we know that \( f_1(x) = A(x)/a^{1/2} \) and \( \mu_1 = aa_s \). Using the lemma again as well as bound (28), we obtain

\[ \sigma := \sum_{x,y \in A} \tau_A^{(A\circ A)}(x,y)C_3(A_s, S, S)(x,y) = \sum_{j=1}^a \mu_j\langle \tau_A^{(A\circ A)} \rangle f_j, f_j \]

\[ = a^{-1}\mu_1\langle \tau_A^{(A\circ A)} \rangle A_s \geq d^{-3/2}c^{-1/2}a^3a_s^3. \]  (29)

On the other hand, we have as in (21), (22) that

\[ \sigma^2 \leq E_3(A_s, A, A)E(A, S) \leq E_3(A_s, A, A)(cad^{a+1})^{1/2}. \]  (30)

Using calculations similar to Lemma 7, one can show that

\[ E_3(A_s, A, A) \ll (c_a c^2)^{1/3}(a_s a^2)^{\alpha/3}L. \]  (31)

Substituting (31) into (30) and combining the result with (29), we obtain

\[ d^{-3}c^{-1}a^6a^8 \leq (cad^{a+1})^{1/2}(c_a c^2)^{1/3}(a_s a^2)^{\alpha/3}L. \]

After some calculations, we have

\[ d \gg L^{-\frac{2}{\tau+\alpha}c_a^2 - \frac{3}{3(\tau+\alpha)c_a} - \frac{24(\alpha+1)}{3(\tau+\alpha)c_a^2} - \frac{10\alpha}{3(\tau+\alpha)c_a^2} - \frac{3}{3(\tau+\alpha)c_a^2} - \frac{10\alpha}{3(\tau+\alpha)c_a^2}} \]

as required.

To prove (24), returning to (27) and applying Lemma 8, we obtain

\[ (a^2 d_s^2 d^{-1})^{2\alpha-1} \leq E^{2\alpha-1}(A_s, A) \ll (\alpha-1)^{-1} \left( \sum_{x} (A_s \circ A_s)^{1/2}(x)(A \circ A)(x) \right)^{2\alpha-2} \cdot c_1^{1/3} c_a^{1/3} a_s^{2/3} a_s^{2/3}. \]

Thus

\[ a^{-1}\langle \tau_A^{(A\circ A)} \rangle A_s \gg (\alpha-1)^{-1} a^{\frac{3\alpha-1}{6\alpha-1}} c_{\alpha-1}^{\frac{12\alpha-6\alpha^2}{6\alpha-1}} d^{\frac{1-2\alpha}{2\alpha-1}} c_{\alpha-1}^{\frac{1}{6\alpha-1}}. \]  (32)
After that use previous arguments replacing $T_A^{(A \circ A)^{1/2}}$ onto $T_A^{(A \circ A)^{1/2}}$. By (32), we have

$$a^{\frac{12\alpha - 8}{3(\alpha - 1)}} \frac{18\alpha - 12 - \alpha^2}{d^{-1} c - \frac{1}{3(\alpha - 1)} c} \leq \left[ \mu_1 a^{-1} \langle T_A^{(A \circ A)^{1/2}} A, A \rangle \right]^2 \leq E_3(A, A, A) E(A, S) \leq (c_s a_s a^{\alpha + 1})^{1/2} (c_s c^2)^{1/3} (a_s a^2)^{\alpha/3} L.$$

After some calculations, we get the required bound.

Finally, to get (26) just apply the arguments above to get

$$a_1^2 E_{3/2}^2 \leq (c a d^{\alpha + 1})^{1/2} (c_s c^2)^{1/3} (a_s a^2)^{\alpha/3} L$$

and use the lower bound $E_{3/2}(A) \geq a^6 / |A - A|$. Of course, one can replace $A$ to $A$ in (26) and vice versa. This completes the proof.

Theorem above gives us a consequence to sumsets/difference sets for convex sets.

**Corollary 15** Let $A, A_* \subset \mathbb{R}$ be two convex sets. Then

$$|A \pm A_*| \gg \max\{|A|^\frac{5}{2} |A|^{\frac{3}{2}}, |A_*|^{\frac{5}{2}} |A|^\frac{3}{2}\} \cdot \log^{-\frac{2}{3}} (|A||A_*|).$$

In another corollary we obtain Theorem 1.3 from [9] as well as Corollary 1.4 from the paper. These results can be considered as theorems on lower bounds for sums of SzT–type sets of special form.

**Corollary 16** We have

$$|A + f(A)| \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}} \quad (33)$$

for any strictly convex or concave function $f$. Further

$$|A A|^{1/6} |A - A|^5 \gg \frac{|A|^{14}}{\log^2 |A|}. \quad (34)$$

In particular

$$\max\{|A A|, |A - A|\} \gg |A|^{14/11} \log^{-2/11} |A|.$$

**Proof.** Indeed, to obtain (33) just apply (25) with $A = A, A_* = f(A)$ and $c(A) = c_s(A) = |f(A) + A|^2 |A|^{-1}$. To get (34), we use (26) with $A = A_*$ having

$$|A - A|^5/2 \gg |A|^{11/2} c^{-3/2} (A) \log^{-1} |A|. \quad (35)$$

After that recall $c(A) = M^2 |A|$ with $M = |AA|/|A|$. This concludes the proof. □
References

[1] M. Z. Garaev, *On lower bounds for $L_1$–norm of exponential sums*, Mathematical Notes 68 (2000), 713–720.

[2] M. Z. Garaev, K-L. Kueh, *On cardinality of sumsets*, J. Aust. Math. Soc. 78 (2005), 221–224.

[3] G. Elekes, M. Nathanson, I. Ruzsa, *Convexity and sumsets*, Journal of Number Theory, 83:194–201, 1999.

[4] N. Hegyvári, *On consecutive sums in sequences*, Acta Math. Acad. Sci. Hungar. 48 (1986), 193–200.

[5] A. Iosevich, V. S. Konyagin, M. Rudnev, V. Ten, *On combinatorial complexity of convex sequences*, Discrete Comput. Geom. 35 (2006), 143–158.

[6] T. G. F. Jones, O. Roche–Newton, *Improved bounds on the set $A(A+1)$*, J. Combin. Theory Ser. A 120 (2013), no. 3, 515–526.

[7] S. V. Konyagin, M. Rudnev, *On new sum-product type estimates*, SIAM J. Discrete Math. 27 (2013), no. 2, 973–990.

[8] L. Li, *On a theorem of Schoen and Shkredov on sumsets of convex sets*, [arXiv:1108.4382v1 [math.CO]].

[9] L. Li, O. Roche–Newton, *Convexity and a sum–product type estimate*, Acta Arith. 156 (2012), no. 3, 247–255.

[10] B. Murphy, O. Roche–Newton, I. D. Shkredov, *Variations on the sum–product problem*, [arXiv:1312.6438v2 [math.CO]] 8 Jan 2014.

[11] T. Schoen, *On convolutions of convex sets and related problems*, preprint.

[12] T. Schoen, I. D. Shkredov, *Higher moments of convolutions*, J. Number Theory 133 (2013), no. 5, 1693–1737.

[13] T. Schoen, I. D. Shkredov, *On sumsets of convex sets*, Comb. Probab. Comput. 20 (2011), 793–798.

[14] I. D. Shkredov, *Some new inequalities in additive combinatorics*, Moscow J. Combin. Number Theory 3 (2013), 237–288.

[15] I. D. Shkredov, *Some new results on higher energies*, Transactions of MMS, 74:1 (2013), 35–73.

[16] J. Solymosi, *Sumas contra productos*, Gaceta de la Real Sociedad Matematica Espanola, ISSN 1138–8927, 12 (2009).

[17] T. Tao, V. Vu, *Additive Combinatorics*, Cambridge University Press (2006).
I.D. Shkredov
Steklov Mathematical Institute,
ul. Gubkina, 8, Moscow, Russia, 119991
and
IITP RAS,
Bolshoy Karetny per. 19, Moscow, Russia, 127994
ilya.shkredov@gmail.com