Research Article

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Degrees of the approximations by some special matrix means of conjugate Fourier series

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Abstract: In this paper we will present the pointwise and normwise estimations of the deviations considered by W. Łenski, B. Szal, [Acta Comment. Univ. Tartu. Math., 2009, 13, 11-24] and S. Saini, U. Singh, [Boll. Unione Mat. Ital., 2016, 9, 495-504] under general assumptions on the class considered sequences defining the method of the summability. We show that the obtained estimations are the best possible for some subclasses of $L^p$ by constructing the suitable type of functions.

Keywords: degree of approximation, Fourier series, matrix means

MSC: 42A24

1 Introduction

Let $L^p$ ($1 \leq p < \infty$) be the class of all $2\pi$-periodic real-valued functions, integrable in the Lebesgue sense, with $p$-th power over $Q = [-\pi, \pi]$ with the norm

$$
\|f\| = \|f\|_{L^p} = \left( \int_Q |f(t)|^p \, dt \right)^{1/p}
$$

when $1 \leq p < \infty$. (1)

Consider the trigonometric Fourier series

$$
S_f(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} \left( a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x \right)
$$

and its conjugate

$$
\tilde{S}_f(x) := \sum_{\nu=1}^{\infty} \left( b_\nu(f) \cos \nu x - a_\nu(f) \sin \nu x \right)
$$

with the partial sums $\tilde{S}_k$. We know that if $f \in L^p$ then

$$
\tilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_k(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon \to 0} \tilde{f}(x, \epsilon),
$$

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where
\[ \tilde{f}(x, \epsilon) := -\frac{1}{\pi} \int_{0}^{\pi} \psi_\epsilon(t) \frac{1}{2} \cot \frac{t}{2} dt \]
with
\[ \psi_\epsilon(t) := f(x + t) - f(x - t), \]
which exists for almost all \( x \) [1, Theorem (3.1)IV].

Let \( A := (a_{n,k}) \) be an infinite lower triangular matrix of real numbers such that
\[ a_{n,k} \geq 0 \text{ when } k = 0, 1, 2, \ldots, n, \quad a_{n,k} = 0 \text{ when } k > n, \]
and let, for \( m = 0, 1, 2, \ldots, n, \)
\[ A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad \text{and} \quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}. \]

Let the \( A \)-transformation of \( \tilde{S}_kf \) be given by
\[ \overline{f}_{n,A}(x) := \sum_{k=0}^{n} a_{n,k} \tilde{S}_k f(x) \quad (n = 0, 1, 2, \ldots,). \]

Following Leindler [2] (see also [3]), we assume that for every \( n \) and \( 0 \leq m < n \)
\[ \sum_{k=m+1}^{n} \left| a_{n,r} - a_{n,r+1} \right| \leq K \frac{1}{m + 1} \sum_{r=0}^{m} a_{n,r} \]
or
\[ \sum_{r=0}^{n-m-1} \left| a_{n,r} - a_{n,r+1} \right| \leq K \frac{1}{m + 1} \sum_{r=m+1}^{n} a_{n,r} \]
hold if \( (a_{n,r})_{r=0}^{n} \) belongs to \( MRBVS \) (Mean Rest Bounded Variation Sequence) or \( MHBVS \) (Mean Head Bounded Variation Sequence), for \( n = 1, 2, \ldots, \) respectively, and let
\[ |A|_{n,m} = \begin{cases} A_{n,m}, & \text{when } (a_{n,r})_{r=0}^{n} \in MRBVS, \\ \overline{A}_{n,m-n}, & \text{when } (a_{n,r})_{r=0}^{n} \in MHBVS. \end{cases} \]

As a measure of approximation, we will use the generalized modulus of continuity of function \( f \) in the space \( L^p \) defined for \( \beta \geq 0 \) by the formula
\[ \overline{\omega}_{\beta f}(\delta)_{L^p} := \sup_{0 < |t| < \delta} \left\{ \left| \frac{1}{\pi} \int_{0}^{\pi} \left| \psi_\epsilon(t) \delta \right|^p dx \right|^\frac{1}{p} \right. \]
It is clear that for \( \beta > \alpha \geq 0 \)
\[ \overline{\omega}_{\beta f}(\delta)_{L^p} \leq \overline{\omega}_{\alpha f}(\delta)_{L^p}, \]
and it is easily seen that \( \overline{\omega}_{0 f}(\cdot)_{L^p} = \overline{\omega}_f(\cdot)_{L^p} \) is the classical modulus of continuity.

Let us consider a function \( \overline{\omega} \) of modulus of continuity type on the interval \([0, 2\pi] \), i.e. a nondecreasing continuous function having the following properties: \( \overline{\omega}(0) = 0, \quad \overline{\omega}(\delta_1 + \delta_2) \leq \overline{\omega}(\delta_1) + \overline{\omega}(\delta_2) \) for any \( 0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi. \) It is easy to conclude that the function \( \delta^{-1}\overline{\omega}(\delta) \) is a quasi nonincreasing function of \( \delta \).

Namely the subadditivity implies \( \overline{\omega}(\lambda \delta) \leq \lambda \overline{\omega}(\delta) \), whence \( \overline{\omega}(\lambda \delta) \leq (\lambda + 1) \overline{\omega}(\delta) \) and therefore \( \frac{\overline{\omega}(\delta)}{\delta_2} \leq 2 \frac{\overline{\omega}(\delta_1)}{\delta_1} \) since
\[ \overline{\omega}(\delta_2) = \left( \frac{\delta_2}{\delta_1} + 1 \right) \overline{\omega}(\delta_1) \leq 2 \frac{\delta_2}{\delta_1} \overline{\omega}(\delta_1), \]
where \( n \in \mathbb{N}_0, \lambda \geq 0 \) and \( 0 \leq \delta_1 \leq \delta_2 \).

Let
\[
L^p(\omega)_\beta = \{ f \in L^p : \tilde{\omega}_\beta f(\delta)_{L^p} \leq \omega(\delta) \},
\]
where \( \tilde{\omega} \) is a function of modulus of continuity type. It is clear that for \( \beta > \alpha \geq 0 \)
\[
L^p(\omega)_\alpha \subset L^p(\omega)_\beta.
\]

The deviation \( \tilde{T}_{n,A}f - \tilde{f} \) was estimated by Qureshi [4] (with a special matrix \( A \)), the norm estimates we can find in the works of Lal and Nigam [5], Dhakal [6], Lal and Singh [7], Mishra, Khari et al. [8], Mishra and Mishra [9], Nigam and Sharma [10], Rhoades [11], Sonker and Singh [12] and Qureshi [13]. The next generalization was obtained by Łenski and Szal [14] in the following form:

**Theorem A.** Let \( f \in L^p(\omega)_\beta \) with \( \beta < 1 - \frac{1}{p} \), \( (a_{n,k})_{k=0}^n \in HBVS \) (Head Bounded Variation Sequence) or \( (a_{n,k})_{k=0}^n \in RBVS \) (Rest Bounded Variation Sequence), respectively, and let \( \tilde{\omega} \) be such that

\[
\left\{ \int_0^{\pi} \left( \frac{t|\psi_x(t)|}{\omega(t)} \right)^p \sin^{\frac{\beta p}{2}} \frac{t}{2} dt \right\}^{1/p} = O_x \left( (n+1)^{-1} \right),
\]
and

\[
\left\{ \int_{\pi}^{2\pi} \left( \frac{t^{-\gamma}|\psi_x(t)|}{\omega(t)} \right)^p \sin^{\frac{\beta p}{2}} \frac{t}{2} dt \right\}^{1/p} = O_x \left( (n+1)^{-1} \right).
\]

hold with \( 0 < \gamma < \beta + \frac{1}{p} \). Then

\[
\left| \tilde{T}_{n,A}f(x) - \tilde{f}(x, \frac{\pi}{n+1}) \right| = O_x \left( (n+1)^{\frac{\beta}{2}+\frac{1}{2} \frac{1}{n+1}} a_n(n+1)\tilde{\omega}\left(\frac{\pi}{n+1}\right) \right)
\]
for all \( x \), where

\[
a_n = \begin{cases} a_{n,0}, & \text{when } (a_{n,k})_{k=0}^n \in RBVS, \\ a_{n,n}, & \text{when } (a_{n,k})_{k=0}^n \in HBVS. \end{cases}
\]

**Theorem B.** Let \( f \in L^p(\omega)_\beta \) with \( \beta < 1 - \frac{1}{p} \), \( (a_{n,k})_{k=0}^n \in HBVS \) (or \( (a_{n,k})_{k=0}^n \in RBVS \)) and let \( \tilde{\omega} \) satisfy (3) with \( 0 < \gamma < \beta + \frac{1}{p} \),

\[
\left\{ \int_0^{\pi} \left( \frac{|\psi_x(t)|}{\omega(t)} \right)^p \sin^{\frac{\beta p}{2}} \frac{t}{2} dt \right\}^{1/p} = O_x \left( (n+1)^{-\frac{1}{2}} \right),
\]
and

\[
\left\{ \int_0^{\pi} \left( \frac{\tilde{\omega}(t)}{t \sin^{\frac{\beta p}{2}} \frac{t}{2}} \right)^q dt \right\}^{1/q} = O_x \left( (n+1)^{\frac{\beta}{2}+\frac{1}{2} \frac{1}{n+1}} \tilde{\omega}\left(\frac{\pi}{n+1}\right) \right)
\]
where \( q = p(p-1)^{-1} \). Then

\[
\left| \tilde{T}_{n,A}f(x) - \tilde{f}(x) \right| = O_x \left( (n+1)^{\frac{\beta}{2}+\frac{1}{2} \frac{1}{n+1}} a_n(n+1)\tilde{\omega}\left(\frac{\pi}{n+1}\right) \right)
\]
for all \( x \) such that \( \tilde{f}(x) \) exists, where

\[
a_n = \begin{cases} a_{n,0}, & \text{when } (a_{n,k})_{k=0}^n \in RBVS, \\ a_{n,n}, & \text{when } (a_{n,k})_{k=0}^n \in HBVS. \end{cases}
\]

Recently, Saini and Singh [15] have proved the following theorem:
Theorem C. Let \( f \) be a periodic function belonging to \( Lip(\partial \omega(t), p) \) class with \( p \geq 1 \) and let \( A = (a_{n,k}) \) be a lower triangular regular matrix with nonnegative and nondecreasing (with respect to \( 0 \leq k \leq n \)) entries and \( A_{n,0} = 1 \). Then the degree of approximation of \( \tilde{f} \), conjugate of \( f \), by matrix means of its conjugate Fourier series is given by

\[
\left\| \tilde{T}_n(f; x) - \tilde{f}(x) \right\|_p = O \left( \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} \tilde{\omega}(t) \frac{dt}{t^{1+1/p}} \right),
\]

provided \( \tilde{\omega}(t) \) is a positive increasing function satisfying the condition

\[
\int_0^v \frac{\tilde{\omega}(t)}{t^{1+1/p}} dt = O \left( \frac{\tilde{\omega}(v)}{v^{1/p}} \right),
\]

where \( 0 < v < \pi \).

We shall write \( J_1 \ll J_2 \), if there exists a positive constant \( C \), depending on some parameters, such that \( J_1 \leq CJ_2 \).

2 Statement of the results

In this paper, we will present the estimations of the deviations \( \tilde{T}_{n,A} f(\cdot) - \tilde{f}(\cdot) \) and \( \tilde{T}_n \omega f(\cdot) - \tilde{f}(\cdot, \frac{\pi}{n+1}) \) under general assumptions and we will show that the obtained degrees of approximations are the best for some subclasses of \( L^p \).

Theorem 1. Let \( (a_{n,k})_{k=0}^n \in MHBVS \cup MRBVS \) with the condition \( |A|_{n,\tau} = O \left( \frac{1}{n+1} \right) \), where \( \tau = [\pi/t] \), \( (\frac{\pi}{n+1} \leq t \leq \pi) \). Let \( f \in L^p \) and let \( \tilde{\omega} \) be such that

\[
\left\{ \frac{1}{n+1} \int_0^{\pi} \left( \frac{t |\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} t \frac{dt}{2} \right\}^{1/p} = O \left( (n+1)^{-\gamma} \right) \quad (6)
\]

and

\[
\left\{ \frac{1}{n+1} \int_{\pi}^{\pi} \left( \frac{t^{-\gamma} |\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} t \frac{dt}{2} \right\}^{1/p} = O \left( (n+1)^{-\gamma} \right) \quad (7)
\]

hold with \( 0 < \gamma < \beta + \frac{1}{p} \). Then

\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f} \left( x, \frac{\pi}{n+1} \right) \right| = O \left( (n+1)^{\beta} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right) \quad (8)
\]

holds for all \( x \).

Theorem 2. Let \( (a_{n,k})_{k=0}^n \in MHBVS \cup MRBVS \) with the condition \( |A|_{n,\tau} = O \left( \frac{1}{n+1} \right) \), where \( \tau = [\pi/t] \), \( (\frac{\pi}{n+1} \leq t \leq \pi) \). Let \( f \in L^p \) and let \( \tilde{\omega} \) be such that (7) and

\[
\left\{ \frac{1}{n+1} \int_0^{\pi} \left( \frac{|\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} t \frac{dt}{2} \right\}^{1/p} = O \left( (n+1)^{-\frac{1}{2}} \right), \quad (9)
\]

\[
\left\{ \frac{1}{n+1} \int_0^{\pi} \left( \frac{\tilde{\omega}(t)}{t^{\beta+1}} \right)^q dt \right\}^{1/q} = O \left( (n+1)^{\beta+1} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right), \quad (10)
\]
hold with $0 < \gamma < \beta + \frac{1}{p}$. Then
\[ |\tilde{T}_{n,Af}(x) - \tilde{f}(x)| = O_x \left( (n + 1)^\beta \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \right) \] (11)
holds for all $x$ such that $\tilde{f}(x)$ exists.

**Theorem 3.** Let $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ with the condition $|A|_{n,\tau} = O \left( \frac{\tau}{n^{1/2}} \right)$, where $\tau = \lceil \pi/t \rceil$, $(\frac{\pi}{n+1} \leq t \leq \pi)$ and let $\tilde{\omega}$ satisfy the conditions (6) and (7) with $0 < \beta < \beta + \frac{1}{p}$. If the function $t^{-\beta} \tilde{\omega}(t)$ is nonincreasing, then
\[ (n + 1)^\beta \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \leq \sup_{f \in L^p(\omega)_\beta} |\tilde{T}_{n,Af}(x) - \tilde{f}(x)| = O_x(1)(n + 1)^\beta \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \] (12)
with $0 < \beta < 1 - \frac{1}{p}$ for all $x$.

**Theorem 4.** Let $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ with the condition $|A|_{n,\tau} = O \left( \frac{\tau}{n^{1/2}} \right)$, where $\tau = \lceil \pi/t \rceil$, $(\frac{\pi}{n+1} \leq t \leq \pi)$ and let $\tilde{\omega}$ satisfy the conditions (7), (9) and (10) with $0 < \beta < \beta + \frac{1}{p}$.

If the function $t^{-\beta} \tilde{\omega}(t)$ is nondecreasing and concave then
\[ (n + 1)^\beta \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \leq \sup_{f \in L^p(\omega)_\beta} |\tilde{T}_{n,Af}(x) - \tilde{f}(x)| = O_x(1)(n + 1)^\beta \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \] (13)
with $\beta > 0$, for all $x$ such that $\tilde{f}(x)$ exists.

**Remark 1.** If we consider $\tilde{\omega}(t) = t^\alpha$ with $\beta < \alpha < 1 + \beta$, then $t^{-\beta} \tilde{\omega}(t)$ is a nondecreasing and concave function of $t$.

**Theorem 5.** Let $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ with the condition $|A|_{n,\tau} = O \left( \frac{\tau}{n^{1/2}} \right)$, where $\tau = \lceil \pi/t \rceil$, $(\frac{\pi}{n+1} \leq t \leq \pi)$. Let $f \in L^p(\omega)_\beta$, where $\beta < 1 - \frac{1}{p}$ and $0 < \gamma < \beta + \frac{1}{p}$. Then
\[ \left| \frac{\tilde{T}_{n,Af}(\cdot)}{\tilde{T}_{n,Af}(\cdot)} - \tilde{f}(\cdot) \right|_{L^p} = O \left( (n + 1)^{\beta} \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \right). \] (14)

**Theorem 6.** Let $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ with the condition $|A|_{n,\tau} = O \left( \frac{\tau}{n^{1/2}} \right)$, where $\tau = \lceil \pi/t \rceil$, $(\frac{\pi}{n+1} \leq t \leq \pi)$. Let $f \in L^p(\omega)_\beta$ and $\tilde{\omega}$ be such that (10) holds with $0 < \gamma < \beta + \frac{1}{p}$. Then
\[ \left| \frac{\tilde{T}_{n,Af}(\cdot)}{\tilde{T}_{n,Af}(\cdot)} - \tilde{f}(\cdot) \right|_{L^p} = O \left( (n + 1)^{\beta} \tilde{\omega} \left( \frac{\pi}{n + 1} \right) \right). \] (15)

**Remark 2.** If $f \in L^p(\omega)_\beta$, where $\tilde{\omega}(t) = t_\alpha$ with $\beta < \alpha \leq 1 + \beta$, $\beta > 0$, then the conditions of our theorems are satisfied. Putting $A_0 = (a_{n,k})$, where $a_{n,k} = \frac{1}{n^{1/2}}$, when $k = 0, 1, 2, \ldots, n$ and $a_{n,k} = 0$, when $k > n$, in our theorems, we obtain the following degree of approximation $\pi^\alpha (n + 1)^{\beta - \alpha}$.

## 3 Corollaries

Finally, we give some corollaries as an application of our results.

**Corollary 1.** Under the assumptions of Theorems 1 and 2 we can obtain better orders of approximations than these in Theorems A and B.

**Corollary 2.** From Theorems 5 and 6 the result of Saini and Singh follows with more general assumptions on the matrix $A$. 

4 Auxiliary results

We begin this section with some notation following Zygmund [1, Section 5 of Chapter II]. It is clear that

\[
\widetilde{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \tilde{D}_k(t) \, dt
\]

and

\[
\widetilde{T}_{n,A} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \sum_{k=0}^{n} a_{n,k} \tilde{D}_k(t) \, dt,
\]

where

\[
\tilde{D}_k(t) = \frac{1}{\pi} \int_{0}^{\pi} \psi_\chi(t) \sum_{k=0}^{n} a_{n,k} \tilde{D}_k(t) \, dt + \frac{1}{\pi} \int_{0}^{\pi} \psi_\chi(t) \sum_{k=0}^{n} a_{n,k} \tilde{D}_\circ_k(t) \, dt.
\]

Hence

\[
\widetilde{T}_{n,A} f(x) - \tilde{f}(x, \frac{\pi}{n + 1}) = -\frac{1}{\pi} \int_{0}^{\pi} \psi_\chi(t) \sum_{k=0}^{n} a_{n,k} \tilde{D}_k(t) \, dt + \frac{1}{\pi} \int_{0}^{\pi} \psi_\chi(t) \sum_{k=0}^{n} a_{n,k} \tilde{D}_\circ_k(t) \, dt.
\]

Now, we formulate some estimates of the considered kernel.

**Lemma 1.** (see [1]) If \(0 < |t| \leq \pi/2\), then

\[
|\tilde{D}_k(t)| \leq \frac{\pi}{2 |t|}
\]

and for any real \(t\), we have

\[
|\tilde{D}_k(t)| \leq \frac{1}{2} k(k + 1) |t| \quad \text{and} \quad |\tilde{D}_\circ_k(t)| \leq k + 1.
\]

**Lemma 2.** (see [16]) If \((a_{n,k})_{n,k=0}^n \in \text{MHBVS}\), then

\[
\left| \sum_{k=0}^{n} a_{n,k} \tilde{D}_\circ_k(t) \right| = O \left( t^{-1} A_{n,2} \right)
\]

and if \((a_{n,k})_{n,k=0}^n \in \text{MRBVS}\), then

\[
\left| \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_\circ_k(t) \right| = O \left( t^{-1} A_{n,\tau} \right),
\]

for \(\frac{2\pi}{\omega} \leq t \leq \pi (n = 2, 3, \ldots)\), where \(\tau = \lfloor \pi/|t| \rfloor\).

**Lemma 3.** If \(t^{-\beta} \omega(t)\) is a concave and nondecreasing function of \(t\), then the function

\[
f_0(x) = \sum_{k=2}^{\infty} \left[ k^{\beta} \omega \left( \frac{1}{k} \right) - (k-1)^{\beta} \omega \left( \frac{1}{k-1} \right) \right] \sin kx, \quad (x \in [0,\pi])
\]

belongs to \(L^p(\omega)_{\beta}\).
Proof. Let \( k = 2, 3, \ldots \) and

\[
a_k := k^{\beta} \omega \left( \frac{1}{k} \right) - (k-1)^{\beta} k \frac{k-1}{k} \omega \left( \frac{1}{k-1} \right).
\]

First, we show that \( a_k \geq a_{k+1} \), i.e.

\[
k^{\beta} \omega \left( \frac{1}{k} \right) - (k-1)^{\beta} k \frac{k-1}{k} \omega \left( \frac{1}{k-1} \right) \geq (k+1)^{\beta} \omega \left( \frac{1}{k+1} \right) - k^{\beta} k \frac{k}{k+1} \omega \left( \frac{1}{k} \right) \tag{16}
\]

and \( 0 \leq k a_k \leq k^{\beta} \omega \left( \frac{k}{k+1} \right) \), i.e.

\[
0 \leq k \left[ k^{\beta} \omega \left( \frac{1}{k} \right) - (k-1)^{\beta} k \frac{k-1}{k} \omega \left( \frac{1}{k-1} \right) \right] \leq k^{\beta} \omega \left( \frac{1}{k} \right). \tag{17}
\]

From the relations (see equation [17])

\[
\frac{1}{k} = \frac{k+1}{2k+1} \frac{1}{k+1} + \frac{(k+1)(k-1)}{k(2k+1)} \frac{1}{k-1},
\]

and using the concavity of the function \( t^{-\beta} \omega(t) \), we obtain

\[
k^{\beta} \omega \left( \frac{1}{k} \right) = \frac{k+1}{2k+1} (k+1)^{\beta} \omega \left( \frac{1}{k+1} \right) + \frac{(k+1)(k-1)}{k(2k+1)} (k-1)^{\beta} \omega \left( \frac{1}{k-1} \right), \tag{18}
\]

i.e.

\[
2k + 1 \frac{k}{k+1} k^{\beta} \omega \left( \frac{1}{k} \right) \geq (k+1)^{\beta} \omega \left( \frac{1}{k+1} \right) + k \frac{1}{k} (k-1)^{\beta} \omega \left( \frac{1}{k-1} \right).
\]

Thus, we get

\[
k^{\beta} \omega \left( \frac{1}{k} \right) + \frac{k}{k+1} k^{\beta} \omega \left( \frac{1}{k} \right) \geq (k+1)^{\beta} \omega \left( \frac{1}{k+1} \right) + k \frac{1}{k} (k-1)^{\beta} \omega \left( \frac{1}{k-1} \right).
\]

Hence, we finally obtain estimation (16).

We know (see inequality [17]) that from the concavity of the function \( t^{-\beta} \omega(t) \), we have

\[
kk^{\beta} \omega \left( \frac{1}{k} \right) \geq (k-1)(k-1)^{\beta} \omega \left( \frac{1}{k-1} \right),
\]

which implies immediately the left side of inequality (17). Using the monotonicity of the function \( t^{-\beta} \omega(t) \) we get

\[
k^{\beta} \omega \left( \frac{1}{k} \right) - (k-1)^{\beta} k \frac{k-1}{k} \omega \left( \frac{1}{k-1} \right) \leq k^{\beta} \omega \left( \frac{1}{k} \right) - k^{\beta} k \frac{k}{k+1} \omega \left( \frac{1}{k} \right) = \frac{1}{k} k^{\beta} \omega \left( \frac{1}{k} \right),
\]

which gives the right side of inequality (17).

Let us denote

\[
\psi_x^0(t) = f_0(x+t) - f_0(x) - f_0(x-t) = f_0(x) = S_x(t) - S_x(-t).
\]

Hence, we get

\[
\bar{\omega}_{\beta} f_0(\delta)_{L^p} := \sup_{0 \leq \delta \leq r} \left\{ \left| \frac{t}{2} \right| \int_0^\pi |\psi_x^0(t)|^p \, dx \right\}^{\frac{1}{p}} \leq \sup_{0 \leq \delta \leq r} \left[ \left\{ \frac{t}{2} \right\}^\beta \int_0^\pi |f_0(x+t) - f_0(x)|^p \, dx \right]^{\frac{1}{p}} + \left\{ \frac{t}{2} \right\}^\beta \int_0^\pi |f_0(x-t) - f_0(x)|^p \, dx \right\}^{\frac{1}{p}} \leq 2 \sup_{0 \leq \delta \leq r} \left\{ \left\{ \frac{t}{2} \right\}^\beta \int_0^\pi |S_x(t)|^p \, dx \right\}^{\frac{1}{p}}.
\]
Let \( \frac{\pi}{m} < t < \frac{\pi}{m-1}, \ t \leq x \leq \pi - t \). We have

\[
|S_x(t)| = \left| \sum_{k=2}^{\infty} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| (\sin k(x+t) - \sin kx) \\
\leq \sum_{k=2}^{m} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| (\sin k(x+t) - \sin kx) \\
+ \sum_{k=m+1}^{\infty} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| \sin kx =: |s_1| + |s_2| + |s_3|.
\]

Using the mean value theorem and the left side of inequality (17), we get for \( x < z < x + t \)

\[
|s_1| \leq |t| \sum_{k=2}^{m} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| |\cos kz| \\
\leq |t| \sum_{k=2}^{m} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| .
\]

Thus by summation, we obtain the estimate

\[
|s_1| \leq |t| m \sum_{k=2}^{m} k^{\beta} \bar{\omega} \left( \frac{1}{m} \right) \leq |t|^{-\beta} \bar{\omega} (|t|).
\]

For the terms \( |s_2| \) and \( |s_3| \), with \( x \neq 0 \), we get using the left side of inequality (17) and following Totik (see estimation [17])

\[
|s_2| \leq \sum_{k=m+1}^{\infty} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| |\sin k(x+t)| \\
\leq \frac{4}{x+t} \left[ (m+1)^{\beta} \bar{\omega} \left( \frac{1}{m+1} \right) - m^{\beta} \frac{m}{m+1} \bar{\omega} \left( \frac{1}{m} \right) \right],
\]

\[
|s_3| \leq \sum_{k=m+1}^{\infty} k^{\beta} \bar{\omega} \left( \frac{1}{k} \right) - (k-1)^{\beta} \frac{k-1}{k} \bar{\omega} \left( \frac{1}{k-1} \right) \right| |\sin kx| \\
\leq \frac{4}{x} \left[ (m+1)^{\beta} \bar{\omega} \left( \frac{1}{m+1} \right) - m^{\beta} \frac{m}{m+1} \bar{\omega} \left( \frac{1}{m} \right) \right].
\]

Thus by inequalities (16) and (17), we obtain

\[
|s_2| \leq 4m \left[ (m+1)^{\beta} \bar{\omega} \left( \frac{1}{m+1} \right) - m^{\beta} \frac{m}{m+1} \bar{\omega} \left( \frac{1}{m} \right) \right] \\
\leq 4m \left[ m^{\beta} \bar{\omega} \left( \frac{1}{m} \right) - (m-1)^{\beta} \frac{m-1}{m} \bar{\omega} \left( \frac{1}{m-1} \right) \right] \\
\leq 4m^{\beta} \bar{\omega} \left( \frac{1}{m} \right) \leq 4|t|^{-\beta} \bar{\omega} (|t|)
\]

and analogously

\[
|s_3| \leq 4|t|^{-\beta} \bar{\omega} (|t|).
\]

If \( x = 0 \), then we can prove that \( |s_1| \ll t^{-\beta} \bar{\omega} (|t|) \), \( |s_2| \ll t^{-\beta} \bar{\omega} (|t|) \) and \( |s_3| = 0 \). Collecting these estimates, we get \( |S_x(t)| \ll |t|^{-\beta} \bar{\omega} (|t|) \). Hence

\[
\bar{\omega}_\beta f_0(x) \in L^p (\bar{\omega})_\beta.
\]

Thus we have proved that \( f_0 \) belongs to \( L^p (\bar{\omega})_\beta \).
Lemma 4. If $t^\beta \omega(t)$ is a nondecreasing function of $t$, then the function

$$f_1(x) = (n + 1)^\beta \omega\left(\frac{\pi}{n + 1}\right) \sin x, \quad (x \in [0, \pi])$$

belongs to $L^p(\omega)_\beta$.

Proof. We have

$$\tilde{\omega}_\beta f_1(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ \left| \sin \frac{t}{2} \right|^\beta \cdot \left( \int_0^1 |\psi_1(t)|^p \, dt \right)^{\frac{1}{p}} \right\}.$$

Let $\frac{n}{n+1} < t < \frac{n}{n+1}, \ t \leq x \leq \pi - t$. We have

$$\int_0^1 |\psi_1(t)|^p \, dt = \int_0^1 |f_1(x + t) - f_1(x)|^p \, dx = \int_0^1 \left[ (n + 1)^\beta \omega\left(\frac{\pi}{n + 1}\right) \right]^p |2 \sin t \cos|^p \, dx$$

$$\ll \left[ (n + 1)^\beta \omega\left(\frac{\pi}{n + 1}\right) \right]^p \ll |t|^{-\beta} \omega(|t|)^p.$$

Hence, we get

$$\tilde{\omega}_\beta f_1(\delta)_{L^p} \ll \sup_{0 \leq |t| \leq \delta} \left\{ |t|^{\beta p} \cdot |t|^{-\beta p} \omega(|t|)^p \right\}^{\frac{1}{p}} \ll \tilde{\omega}(\delta).$$

Thus, we have proved that $f_1$ belongs to $L^p(\omega)_\beta$.

5 Proofs of the results

5.1 Proof of Theorem 1

Let us start with the obvious relations

$$\tilde{T}_{n,a} f(x) - \tilde{f}(x, \frac{\pi}{n + 1}) = -\frac{1}{n} \int_0^\pi \psi(t) \sum_{k=0}^{n-1} a_k \delta_k(t) \, dt + \frac{1}{n} \int_0^\pi \psi(t) \sum_{k=0}^{n-1} a_k \delta_k(t) \, dt =: \tilde{I}_1 + \tilde{I}_2$$

and

$$|\tilde{T}_{n,a} f(x) - \tilde{f}(x, \frac{\pi}{n + 1})| \leq |\tilde{I}_1| + |\tilde{I}_2|.$$

By the Hölder inequality $\left( \frac{1}{p} + \frac{1}{q} = 1 \right)$, Lemma 1 and equation (6)

$$|\tilde{I}_1| \leq (n + 1)^2 \int_0^\pi \frac{t |\psi(t)|}{\omega(t)} \, dt$$

$$\leq (n + 1)^2 \left\{ \int_0^\pi \left[ \frac{t |\psi(t)|}{\omega(t)} \right]^p \sin^\beta \frac{t}{2} \, dt \right\}^{\frac{1}{p}} \left\{ \int_0^\pi \left[ \sin^\beta \frac{t}{2} \right]^q \, dt \right\}^{\frac{1}{q}} \ll (n + 1)^{1-\frac{1}{p}} \left\{ \int_0^\pi \left[ \omega(t) \right]^q \, dt \right\}^{\frac{1}{q}} \ll (n + 1)^{\frac{1}{p}} \omega\left(\frac{\pi}{n + 1}\right),$$

for $\beta < 1 - \frac{1}{p}$.
By the Hölder inequality \( \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \), Lemma 2, monotonicity of the function \( t^{-1} \tilde{\omega}(t) \) and equation (7)

\[
\left| \tilde{I}_2 \right| \leq \frac{1}{\pi} \int_0^{\pi} \frac{\left| \psi_x(t) \right| |A_{n,r}| dt}{t^2} \leq (n+1)^{-1} \int_0^{\pi} \left| \frac{\psi_x(t)}{t} \right| dt \left( n+1 \right)^{1-\frac{\gamma}{p}} \leq \left( n+1 \right)^{1-\frac{\gamma}{p}} \left( \frac{\pi}{n+1} \right) \tilde{\omega}
\]

\[
\ll (n+1)^{-\frac{\gamma}{p}} \left( \frac{\pi}{n+1} \right) \tilde{\omega} \leq (n+1)^{\beta} \left( \frac{\pi}{n+1} \right) \tilde{\omega}
\]

for \( 0 < \gamma < \beta + \frac{1}{p} \).

Collecting these estimates, we obtain the desired result. \( \Box \)

### 5.2 Proof of Theorem 2

As usual, let us start with the obvious relations

\[
\tilde{T}_{n,Af}(x) - \tilde{f}(x) = \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \hat{f}_k(t) dt + \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \hat{f}_k(t) dt =: \tilde{I}_1 + \tilde{I}_2
\]

and

\[
\left| \tilde{T}_{n,Af}(x) - \tilde{f}(x) \right| \leq \left| \tilde{I}_1 \right| + \left| \tilde{I}_2 \right|.
\]

By the Hölder inequality \( \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \), Lemma 1, equations (9) and (10),

\[
\left| \tilde{I}_1 \right| \leq \frac{1}{\pi} \int_0^{\pi} \frac{\left| \psi_x(t) \right| dt}{t} = \frac{1}{\pi} \left\{ \int_0^{\pi} \left| \frac{\psi_x(t)}{\omega(t)} \right| \frac{\sin^{\beta} \frac{t}{2}}{t} \right\}^{\frac{1}{p}} \left\{ \int_0^{\pi} \left( \frac{\omega(t)}{t \sin^{\beta} \frac{t}{2}} \right)^q dt \right\}^{\frac{1}{q}} \ll (n+1)^{\frac{\gamma}{p}} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \tilde{\omega} \ll (n+1)^{\beta} \left( \frac{\pi}{n+1} \right) \tilde{\omega}
\]

By the previous proof

\[
\left| \tilde{I}_2 \right| \ll (n+1)^{\beta} \tilde{\omega} \left( \frac{\pi}{n+1} \right)
\]

for \( 0 < \gamma < \beta + \frac{1}{p} \).

Collecting these estimates, we obtain the desired result. \( \Box \)

### 5.3 Proof of Theorem 3

Let us fix a point \( x \) and let us consider the class \( L^p(\tilde{\omega}), \beta \), with \( \beta > 0 \), of all functions \( f \in L^p \) such that

\[
\tilde{\omega}_\beta f(\delta)_{L^p} \leq \tilde{\omega}(\delta), \quad (0 \leq \delta \leq 2\pi).
\]
Then Theorem 1 implies the following estimate
\[
\sup_{f \in L^p(\Omega)_{\beta}} \left| \tilde{T}_{n,Af} (x) - \tilde{f} \left( x, \frac{\pi}{n+1} \right) \right| \ll (n+1)^{\beta} \left( \frac{\pi}{n+1} \right)
\]
for $\beta < 1 - \frac{1}{p}$.

On the other hand, the function
\[
f_1(x) = (n+1)^{\beta} \left( \frac{\pi}{n+1} \right) \sin x
\]
by Lemma 4 belongs to class $L^p(\Omega)_{\beta}$, if $t^{1-\beta} (\tilde{f})$ is a nondecreasing function of $t$, and $f_1$ satisfies the conditions (6) and (7) of Theorem 1. Indeed, we have
\[
\left\{ \int_0^{\pi} \left( \frac{t | \psi^1_x(t) |}{\omega(t)} \right)^p \sin^{\alpha p} t dt \right\}^{1/p} \ll \left\{ \int_0^{\pi} \left( \frac{t^{1-\beta} \omega(t)}{\omega(t)} \right)^p \sin^{\alpha p} t dt \right\}^{1/p} = O(n+1)^{-\frac{1-\beta}{p}}.
\]
Moreover, there exists $\gamma$ such that $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$ and
\[
\left\{ \int_0^{\frac{\pi}{n\pi}} \left( \frac{t^{-\gamma} | \psi^1_x(t) |}{\omega(t)} \right)^p \sin^{\alpha p} t dt \right\}^{1/p} \ll \left\{ \int_0^{\frac{\pi}{n\pi}} \left( \frac{t^{-\gamma} \omega(t)}{\omega(t)} \right)^p \sin^{\alpha p} t dt \right\}^{1/p} = O(n+1)^{-\gamma \frac{1}{p}}.
\]

Let $\frac{\pi}{n} < t < \frac{\pi}{n+1}$. We have
\[
\tilde{f}_1 \left( x, \frac{\pi}{n+1} \right) = -\frac{1}{\pi} \int_0^{\pi} \left[ f_1(x+t) - f_1(x-t) \right] \frac{\cos \frac{t}{2}}{2 \sin \frac{t}{2}} dt
\]
\[
= -\frac{1}{\pi} (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right) \int_0^{\pi} 2 \sin t \cos x \frac{\cos \frac{t}{2}}{2 \sin \frac{t}{2}} dt
\]
\[
= -\frac{2}{\pi} (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right) \cos x \int_0^{\pi} \frac{\cos^2 \frac{t}{2}}{2} dt
\]
\[
= -\frac{2}{\pi} (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right) \cos x \left[ \frac{\pi}{2} - \frac{\pi}{2} \frac{1}{2n+1} - \frac{1}{2} \sin \frac{\pi}{n+1} \right].
\]

We get
\[
\sup_{f \in L^p(\Omega)_{\beta}} \left| \tilde{T}_{n,Af} (x) - \tilde{f} \left( x, \frac{\pi}{n+1} \right) \right| \geq \left| \tilde{T}_{n,Af_1} (x) - \tilde{f}_1 \left( x, \frac{\pi}{n+1} \right) \right|
\]
\[
= \left[ \sum_{k=2}^{n} a_{n,k} (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right) \cos x + n(n+1)^{\beta-1} \left( \frac{\pi}{n+1} \right) \cos x \right.
\]
\[
- \frac{1}{\pi} (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right) \cos x \sin \frac{\pi}{n+1} \left[ \right].
\]
Thus in a special case, for $x = 0$, we get
\[
\left| (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right) \left[ 1 + \frac{\pi}{2} - \frac{\pi}{2n+1} - \frac{1}{2} \sin \frac{\pi}{n+1} \right] \right| \gg (1 + n)^{\beta} \left( \frac{\pi}{n+1} \right).
\]

Hence, we finally obtain equation (12).

When $x = x_0$, we can consider the function $f_{x_0}(\cdot) = f_1(\cdot - x_0)$ instead of $f_1(\cdot)$. Thus our proof is complete.

\[\square\]
5.4 Proof of Theorem 4

Let us fix a point $x$ and let us consider the class $L^p(\tilde{\omega})_\beta$, with $\beta < 1 - \frac{1}{p}$, of all functions $f \in L^p$ such that

$$
\tilde{\omega}_f(\delta)_L \leq \tilde{\omega}(\delta), \quad 0 \leq \delta \leq 2\pi.
$$

The Theorem 2 implies the estimate

$$
\sup_{f \in L^p(\tilde{\omega})_{\beta}} \left| \tilde{T}_nAf(x) - \tilde{f}(x) \right| \ll (n + 1)^{\beta} \tilde{\omega} \left( \frac{\pi}{n + 1} \right).
$$

On the other hand, the function

$$
f_0(x) = \sum_{k=1}^{\infty} \left[ k^{\beta} \tilde{\omega} \left( \frac{1}{k} \right) - (k - 1)^{\beta} \frac{k - 1}{k} \tilde{\omega} \left( \frac{1}{k - 1} \right) \right] \sin kx,
$$

by Lemma 3 belongs to class $L^p(\tilde{\omega})_\beta$, if $t^{\beta} \tilde{\omega}(t)$ is a concave and nondecreasing function of $t$.

We can see that the function $f_0$ satisfies the condition (7) with $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$ and the condition (9).

Indeed, using the estimation (18) obtained in the proof of Lemma 3, we have

$$
|\tilde{T}_nA\phi(t)| = |f_0(x + t) - f_0(x - t)| \ll |t|^{\beta} \tilde{\omega}(|t|).
$$

Thus, we easily get

$$
\left\{ \frac{\pi}{p} \left( \frac{t^{-\gamma} \psi_0(t)}{\tilde{\omega}(|t|)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \ll \left\{ \frac{\pi}{p} \left( \frac{t^{-\gamma} \tilde{\omega}(|t|)}{\tilde{\omega}(|t|)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p}
$$

and

$$
\left\{ \frac{\pi}{p} \left( \frac{\psi_0(t)}{\tilde{\omega}(|t|)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \ll \left\{ \frac{\pi}{p} \left( \frac{t^{-\beta} \tilde{\omega}(|t|)}{\tilde{\omega}(|t|)} \right)^p t^{\beta p} dt \right\}^{1/p} = O(n + 1)^{\gamma - \frac{1}{2}}
$$

Hence, by Theorem 2 the estimation (11) holds for the function $f_0$.

On the other hand using the fact

$$
\tilde{f}_0(x) = \sum_{k=2}^{\infty} \left[ k^{\beta} \tilde{\omega} \left( \frac{1}{k} \right) - (k - 1)^{\beta} \frac{k - 1}{k} \tilde{\omega} \left( \frac{1}{k - 1} \right) \right] \cos kx
$$

we get

$$
\sup_{f \in L^p(\tilde{\omega})_{\beta}} \left| \tilde{T}_nAf(x) - \tilde{f}(x) \right| \geq \left| \tilde{T}_nAf_0(x) - \tilde{f}_0(x) \right| = \left| \sum_{k=2}^{\infty} \sum_{l=k+1}^{\infty} \left[ l^{\beta} \tilde{\omega} \left( \frac{1}{l} \right) - (l - 1)^{\beta} \frac{l - 1}{l} \tilde{\omega} \left( \frac{1}{l - 1} \right) \right] \cos lx \right|.
$$

Thus in a special case, for $x = 0$, we get

$$
\left| \sum_{k=2}^{n} \sum_{l=k+1}^{\infty} \left[ l^{\beta} \tilde{\omega} \left( \frac{1}{l} \right) - (l - 1)^{\beta} \frac{l - 1}{l} \tilde{\omega} \left( \frac{1}{l - 1} \right) \right] \cos lx \right|.
$$

Using the inequality (18), we have

$$
k^{\beta} \tilde{\omega} \left( \frac{1}{k} \right) \geq \frac{k + 1}{2k + 1} (k + 1)^{\beta} \tilde{\omega} \left( \frac{1}{k + 1} \right) + \frac{k + 1}{k(2k + 1)} (k - 1)^{\beta} \tilde{\omega} \left( \frac{1}{k - 1} \right)
$$

and

$$
\frac{1}{2} (k + 1)^{\beta} \tilde{\omega} \left( \frac{1}{k + 1} \right).
$$
which implies
\[
\sup_{f \in L^p(\omega)} \left| \tilde{T}_{n,A} f(x) - \tilde{f}(x) \right| \geq \frac{1}{q} \sum_{k=2}^{n} a_{n,k} (k+1)^\beta \tilde{\omega} \left( \frac{1}{k+1} \right) \gg (n+1)^\beta \tilde{\omega} \left( \frac{1}{n+1} \right).
\]
Hence, we finally obtain (13).

When \( x = x_0 \), we can consider the function \( f_{x_0}(\cdot) = f_0(\cdot - x_0) \) instead of \( f_0(\cdot). \) Thus our proof is complete.

\[\square\]

### 5.5 Proof of Theorem 5

Note that if \( f \in L^p(\omega)_\beta \), then Theorem 1 implies
\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f}(x, \frac{\pi}{n+1}) \right| \ll (n+1)^\beta \tilde{\omega} \left( \frac{\pi}{n+1} \right).
\]

Thus, we easily get
\[
\left\| \tilde{T}_{n,A} f(\cdot) - \tilde{f}(\cdot, \frac{\pi}{n+1}) \right\|_{L^p} = \left\{ \int_0^\pi \left| \tilde{T}_{n,A} f(x) - \tilde{f}(x, \frac{\pi}{n+1}) \right|^p dx \right\}^{1/p} \ll (n+1)^\beta \tilde{\omega} \left( \frac{\pi}{n+1} \right) = O \left( (n+1)^\beta \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]

The conditions (6) and (7) from Theorem 1 are satisfied in the following form
\[
\left\| \left\{ \int_0^\pi \left( \frac{t}{\omega(t)} \right) \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \leq \left\{ \int_0^\pi \left( \frac{t}{\omega(t)} \right) \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O \left( (n+1)^{-\frac{1}{p}} \right)
\]
\[
\left\| \left\{ \int_0^{\pi/n} \left( \frac{t}{\omega(t)} \right) \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \leq \left\{ \int_0^{\pi/n} \left( \frac{t}{\omega(t)} \right) \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O \left( (n+1)^{-\frac{1}{p}} \right).\]

Hence our proof is complete.

\[\square\]

### 5.6 Proof of Theorem 6

Similarly to the previous proof, note that if \( f \in L^p(\omega)_\beta \), then Theorem 2 implies
\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f}(x) \right| \ll (n+1)^\beta \tilde{\omega} \left( \frac{\pi}{n+1} \right).
\]

Thus, we easily get
\[
\left\| \tilde{T}_{n,A} f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} = \left\{ \int_0^\pi \left| \tilde{T}_{n,A} f(x) - \tilde{f}(x) \right|^p dx \right\}^{1/p} \ll (n+1)^\beta \tilde{\omega} \left( \frac{\pi}{n+1} \right) = O \left( (n+1)^\beta \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]

We know by the previous proof that the condition (7) is satisfied and the condition (9), from Theorem 2, is satisfied in the following form
\[
\left\| \left\{ \int_0^\pi \left( \frac{t}{\omega(t)} \right) \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \leq \left\{ \int_0^\pi \left( \frac{t}{\omega(t)} \right) \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O \left( (n+1)^{-\frac{1}{p}} \right).
\]

Hence, our proof is complete.

\[\square\]
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