INTERIOR AND $\mathfrak{h}$ OPERATORS ON THE CATEGORY OF LOCALES

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Abstract. We present the concept of interior operator $I$ on the category $\text{Loc}$ of locales and then we construct a topological category $(\text{I-Loc}, U)$, where $U : \text{I-Loc} \to \text{Loc}$ is a forgetful functor; and we also introduce the notion of $\mathfrak{h}$ operator on the category $\text{Loc}$ and discuss some of their properties for constructing the topological category $(\mathfrak{h}\text{-Loc}, U)$ associated to the forgetful functor $U : \mathfrak{h}\text{-Loc} \to \text{Loc}$.

0. Introduction

Kuratowski operators (closure, interior, exterior, boundary and others) have been used intensively in General Topology ([2], [5], [6]). For a topological space it is well-known that, for example, the associated closure and interior operators provide equivalent descriptions of the topology; but this is not always true in other categories, consequently it makes sense to define and study separately these operators. In this context, we study an interior operator $I$ on the coframe $\mathcal{S}_c(L)$ of sublocales of every object $L$ in the category $\text{Loc}$.

On the other hand, a new topological operator $\mathfrak{h}$ was introduced by M. Suarez [12] in order to complete a Boolean algebra with all topological operators in General Topology. Following his ideas, we study an operator $\mathfrak{h}$ on the collection $\mathcal{S}_c(L)$ of all complemented sublocales of every object $L$ in the category $\text{Loc}$.

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The paper is organized as follows, we begin presenting, in section 1, the basic concepts of Heyting algebras, Frames, locales, sublocales, images and preimages of sublocales for the morphisms of Loc and the notions of closed and open sublocales; these notions can be found in Picado and Pultr [10] and A. L. Suarez [11]. In section 2, we present the concept of interior operator I on the category Loc and then we construct a topological category (I-Loc, U), where U : I-Loc → Loc is a forgetful functor. Finally in section 3 we present the notion of h operator on the category Loc and discuss some of their properties for constructing the topological category (h-Loc, U) associated to the forgetful functor U : h-Loc → Loc.

1. Preliminaries

For a comprehensive account on the the categories of frames and locales we refer to Picado and Pultr [10] and A. L. Suarez [11], from whom we take the following useful facts.

1.1. Heyting algebras. A bounded lattice L is called a Heyting algebra if there is a binary operation $x \rightarrow y$ (the Heyting operation) such that for all $a, b, c$ in L,

$$a \land b \leq c \text{ iff } a \leq b \rightarrow c.$$ 

Thus for every $b \in L$ the mapping $b \rightarrow (-) : L \rightarrow L$ is a right adjoint to $(-) \land b : L \rightarrow L$ and hence, if it exists, is uniquely determined.

In a complete Heyting algebra we have $(\bigvee A) \land b = \bigvee_{a \in A}(a \land b)$ for any $A \subseteq L$, $b \rightarrow (\bigvee A) = \bigvee_{a \in A}(b \rightarrow a)$, and $(\bigvee A) \rightarrow b = \bigvee a \in A(a \rightarrow b)$.

1.2. Frames. A frame is a complete lattice $L$ satisfying the distributive law

$$(\bigvee A) \land b = \bigvee_{2}\{a \land b \mid a \in A\}$$
for all $A \subseteq L$ and $b \in L$ (hence a complete Heyting algebra); a \textit{frame homomorphism} preserves all joins and all finite meets.

The lattice $\Omega(X)$ of all open subsets of a topological space $X$ is an example of a frame, and if $f : X \to Y$ is continuous we obtain a frame homomorphism $\Omega(f) : \Omega(Y) \to \Omega(X)$ by setting $\Omega(f)(U) = f^{-1}[U]$. Thus we have a contravariant functor $\Omega : \textbf{Top} \to \textbf{Frm}^{\text{op}}$ from the category of topological spaces into that of frames.

1.3. \textbf{Locales.} The adjunction $\Omega : \textbf{Top} \to \textbf{Frm}^{\text{op}}$, $\text{pt} : \textbf{Frm}^{\text{op}} \to \textbf{Top}$ with $\Omega \dashv \text{pt}$, connects the categories of frames with that of topological spaces. The functor $\Omega$ assigns to each space its lattice of opens, and $\text{pt}$ assigns to a frame $L$ the collection of the frame maps $f : L \to 2$, topologized by setting the opens to be exactly the sets of the form $\{f : L \to 2 \mid f(a) = 1\}$ for some $a \in L$.

A frame $L$ is \textit{spatial} if for $a, b \in L$ whenever $a \nleq b$ there is some frame map $f : L \to 2$ such that $f(a) = 1 \neq f(b)$. Spatial frames are exactly those of the form $\Omega(X)$ for some space $X$.

A space is \textit{sober} if every irreducible closed set is the closure of a unique point. Sober spaces are exactly those of the form $\text{pt}(L)$ for some frame $L$.

The adjunction $\Omega \dashv \text{pt}$ restricts to a dual equivalence of categories between spatial frames and sober spaces.

The category of sober spaces is a full reflective subcategory of $\textbf{Top}$. For each space $X$ we have a sobrification map $N : X \to \text{pt}(\Omega(X))$ mapping each point $x \in X$ to the map $f_x : (X) \to 2$ defined as $f(U) = 1$ if and only if $x \in U$. 
The category of spatial frames is a full reflective subcategory of \textbf{Frm}. For each frame we have a spatialization map $\phi : L \to \Omega(\text{pt}(L))$ which sends each $a \in L$ to $\{f : L \to 2 \mid f(a) = 1\}$.

This justifies to view the dual category $\textbf{Loc} = \textbf{Frm}^{op}$ as an extended category of spaces; one speaks of the category of locales.

Maps in the category of locales have a concrete description: they can be characterized as the right adjoints of frame maps (since frame maps preserve all joins, they always have right adjoints).

1.3.1. Sublocales. A sublocale of a locale $L$ is a subset $S \subseteq L$ such that it is closed under arbitrary meets, and such that $s \in S$ implies $x \to s \in S$ for every $x \in L$. This is equivalent to $S \subseteq L$ being a locale in the inherited order, and the subset inclusion being a map in $\textbf{Loc}$.

Sublocales of $L$ are closed under arbitrary intersections, and so the collection $\mathcal{S}_L(L)$ of all sublocales of $L$, ordered under set inclusion, is a complete lattice. The join of sublocales is (of course) not the union, but we have a very simple formula $\bigvee_i S_i = \{\bigvee M \mid M \subseteq \bigcup_i S_i\}$.

In the coframe $\mathcal{S}_L(L)$ the bottom element is the sublocale $\{1\}$ and the top element is $L$.

1.3.2. Images and Preimages of sublocales. Let $f : L \to M$ be a localic map and if $S \subseteq L$ is a sublocale then the standard set-theoretical image $f[S]$ is also a sublocale. The set-theoretic preimage $f^{-1}[T]$ of a sublocale $T \subseteq M$ is not necessarily a sublocale of $L$. To obtain a concept of a preimage suitable for our purposes we will, first, make the following observation: “Let $A \subseteq L$ be a subset closed under meets. Then $\{1\} \subseteq A$ and if $S_i \subseteq A$ for $i \in J$ then $\bigwedge_{i \in J} S_i \subseteq A$”. Consequently there exists the largest sublocale contained in $A$. It will be denoted by $A_{sloc}$.
The set-theoretic preimage $f^{-1}[T]$ of a sublocale $T$ is closed under meets (indeed, $f(1) = 1$, and if $x_i \in f^{-1}[T]$ then $f(x_i) \in T$, and hence $f(\bigwedge_{i \in J} x_i) = \bigwedge_{i \in J} f(x_i)$ belongs to $T$ and $\bigwedge_{i \in J} x_i \in f^{-1}[T]$ ) and we have the sublocale $f^{-1}[T] := f^{-1}[T]_{sloc}$. It will be referred to as the preimage of $T$, and we shall sat that $f^{-1}[\cdot]$ is the preimage function of $f$.

For every localic map $f : L \to M$, the preimage function $f^{-1}[\cdot]$ is a right Galois adjoint of the image function $f[\cdot] : \mathcal{S}_\ell(L) \to \mathcal{S}_\ell(M)$.

1.3.3. **Closed and Open sublocales.** Embedded in $\mathcal{S}_\ell(L)$ we have the coframe of closed sublocales which is isomorphic to $L^{op}$. The closed sublocale $c(a) \subseteq L$ is defined to be $\uparrow a$ for $a \in L$.

Embedded in $\mathcal{S}_\ell(L)$ we also have the frame of open sublocales which is isomorphic to $L$. The open sublocale is defined to be $\{a \to x \mid x \in L\}$ for $a \in L$.

The sublocales $o(a)$ and $c(a)$ are complements of one another in the coframe $\mathcal{S}_\ell(L)$ for any element $a \in L$. Furthermore, open and closed sublocales generate the coframe $\mathcal{S}_\ell(L)$ in the sense that for each $S \in \mathcal{S}_\ell(L)$ we have $S = \bigcap \{ o(x) \cup c(y) \mid S \subseteq o(x) \cup c(y) \}$.

A pseudocomplement of an element $a$ in a meet-semilattice $L$ with 0 is the largest element $b$ such that $b \wedge a = 0$, if it exists. It is usually denoted by $\neg a$. Recall that in a Heyting algebra $H$ the pseudocomplement can be expressed as $\neg x = x \to 0$.

2. **Interior Operators**

We shall be concerned in this section with a version on locales of the interior operator studied in [7].
Before stating the next definition, we need to observe that since for localic maps $f : L \to M$ and $g : M \to N$:

- the preimage function $f^{-1}[-]$ is a right Galois adjoint of the image function $f[-] : \mathcal{S}(L) \to \mathcal{S}(M)$;
- $g[-] \circ f[-] = (g \circ f)[-]$.

Therefore $g^{-1}[-] \circ f^{-1}[-] = (g \circ f)^{-1}[-]$ because given two adjunctions the composite functors yield an adjunction.

**Definition 2.1.** An interior operator $I$ of the category $\textbf{Loc}$ is given by a family $I = (i_L)_L \in \textbf{Loc}$ of maps $i_L : \mathcal{S}(L) \to \mathcal{S}(L)$ such that

1. (Contraction) $i_L(S) \subseteq S$ for all $S \in \mathcal{S}(L)$;
2. (Monotonicity) If $S \subseteq T$ in $\mathcal{S}(L)$, then $i_L(S) \subseteq i_L(T)$
3. (Upper bound) $i_L(L) = L$.

**Definition 2.2.** An $I$-space is a pair $(L, i_L)$ where $L$ is an object of $\textbf{Loc}$ and $i_L$ is an interior operator on $L$.

**Definition 2.3.** A morphism $f : L \to M$ of $\textbf{Loc}$ is said to be $I$-continuous if

\[(1) \quad f^{-1} [i_M(T)] \subseteq i_L (f^{-1} [T]) \]

for all $T \in \mathcal{S}(M)$. Where $f^{-1}[-]$ is the preimage of $f[-]$.

**Proposition 2.4.** Let $f : L \to M$ and $g : M \to N$ be two $I$-continuous morphisms of $\textbf{Loc}$ then $g \circ f$ is an $I$-continuous morphism of $\textbf{Loc}$.

**Proof.** Since $g : M \to N$ is $I$-continuous, we have

\[g^{-1} [i_N(S)] \subseteq i_M (g^{-1} [S])\]
for all $S \in \mathcal{S}_\ell(N)$, it follows that

$$f^{-1}\left[ g^{-1}(i_N(S)) \right] \subseteq f^{-1}\left[ i_M(g^{-1}[S]) \right];$$

now, by the $I$-continuity of $f$,

$$f^{-1}\left[ i_M(g^{-1}[S]) \right] \subseteq i_L\left( f^{-1}[g^{-1}[S]] \right),$$

therefore

$$f^{-1}\left[ g^{-1}(i_N(S)) \right] \subseteq i_L\left( f^{-1}[g^{-1}[S]] \right),$$

that is to say

$$(g \cdot f)^{-1}[i_N(S)] \subseteq i_L((g \cdot f)^{-1}[S])$$

As a consequence we obtain

**Definition 2.5.** The category $I$-Loc of $I$-spaces comprises the following data:

1. **Objects:** Pairs $(L, i_L)$ where $L$ is an object of Loc and $i_L$ is an interior operator on $L$.

2. **Morphisms:** Morphisms of Loc which are $I$-continuous.

**2.1. The lattice structure of all interior operators.**

For the category Loc we consider the collection

$$\text{Int}(\text{Loc})$$

of all interior operators on Loc. It is ordered by

$$I \leq J \iff i_L(S) \subseteq j_L(S), \quad \text{for all } S \in \mathcal{S}_\ell \text{ and all } L \text{ object of Loc}.$$
Proposition 2.6. Every family $(I_\lambda)_{\lambda \in \Lambda}$ in $\text{Int}(\mathbf{Loc})$ has a join $\bigvee_{\lambda \in \Lambda} I_\lambda$ and a meet $\bigwedge_{\lambda \in \Lambda} I_\lambda$ in $\text{Int}(\mathbf{Loc})$. The discrete interior operator

$$I_D = (i_{DL})_{L \in \mathbf{Loc}} \text{ with } i_{DL}(S) = S \text{ for all } S \in \mathcal{S}_\mathbf{L}$$

is the largest element in $\text{Int}(\mathbf{Loc})$, and the trivial interior operator

$$I_T = (i_{TL})_{L \in \mathbf{Loc}} \text{ with } i_{TL}(S) = \begin{cases} 1 & \text{for all } S \in \mathcal{S}_\mathbf{L}, \ S \neq L \\ L & \text{if } S = L \end{cases}$$

is the least one.

Proof. For $\Lambda \neq \emptyset$, let $\hat{I} = \bigvee_{\lambda \in \Lambda} I_\lambda$, then

$$\hat{i}_L = \bigvee_{\lambda \in \Lambda} i_{\lambda L}$$

for all $L$ object of $\mathbf{Loc}$, satisfies

- $\hat{i}_L(S) \subseteq S$, because $i_{\lambda L}(S) \subseteq S$ for all $S \in \mathcal{S}_\mathbf{L}$ and for all $\lambda \in \Lambda$.
- If $S \leq T$ in $\mathcal{S}_\mathbf{L}$ then $i_{\lambda L}(S) \subseteq i_{\lambda L}(T)$ for all $S \in \mathcal{S}_\mathbf{L}$ and for all $\lambda \in \Lambda$, therefore $\hat{i}_S(S) \subseteq \hat{i}_L(T)$.
- Since $i_{\lambda L}(L) = L$ for all $\lambda \in \Lambda$, we have that $\hat{i}_L(L) = L$.

Similarly $\bigwedge_{\lambda \in \Lambda} I_\lambda$, $I_D$ and $I_T$ are interior operators. \hfill \blacksquare

Corollary 2.7. For every object $L$ of $\mathbf{Loc}$

$$\text{Int}(L) = \{i_L | i_L \text{ is an interior operator on } L\}$$

is a complete lattice.

2.2. Initial interior operators. Let $\mathbf{I-Loc}$ be the category of $I$-spaces. Let $(M, i_M)$ be an object of $\mathbf{I-Loc}$ and let $L$ be an object of $\mathbf{Loc}$. For each morphism $f : L \to M$ in $\mathbf{Loc}$ we define on $L$ the operator

$$(2) \quad i_{L,f} := f_1 \cdot i_M \cdot f_*.$$

Proposition 2.8. The operator $(2)$ is an interior operator on $L$ for which the morphism $f$ is $I$-continuous.
Proof.

\((I_1)\) (Contraction) \(i_{L_f}(S) = f^{-1} \cdot i_M \cdot f_*[S] \subseteq f^{-1} \cdot f_*[S] \subseteq S\) for all \(S \in \mathcal{S}_L\);

\((I_2)\) (Monotonicity) \(S \subseteq T\) in \(\mathcal{S}_L\) implies \(f_*[S] \subseteq f_*[T]\), then \(i_M \cdot f_*[S] \subseteq i_M \cdot f_*[T]\), consequently \(f^{-1} \cdot i_M \cdot f_*[S] \subseteq f^{-1} \cdot i_M \cdot f_*[T]\);

\((I_3)\) (Upper bound) \(i_{L_f}(L) = f^{-1} \cdot i_M \cdot f_*[L] = L\).

Finally,

\[
f^{-1}(i_M(T)) \subseteq f^{-1}(i_M \cdot f_* \cdot f^{-1}(T)) = (f^{-1} \cdot i_M \cdot f_*)(f^{-1}(T)) = i_{L_f}(f^{-1}(T)),
\]

for all \(T \in \mathcal{S}_L\). \(\blacksquare\)

It is clear that \(i_{L_f}\) is the coarsest interior operator on \(L\) for which the morphism \(f\) is \(I\)-continuous; more precisely

**Proposition 2.9.** Let \((L, i_L)\) and \((M, i_M)\) be objects of \(I\)-Loc, and let \(N\) be an object of \(\text{Loc}\). For each morphism \(g : N \to L\) in \(\text{Loc}\) and for \(f : (L, i_L) \to (M, i_M)\) an \(I\)-continuous morphism, \(g\) is \(I\)-continuous if and only if \(f \cdot g\) is \(I\)-continuous.

**Proof.** Suppose that \(g \cdot f\) is \(I\)-continuous, i. e.

\[(f \cdot g)^{-1}(i_M(T)) \subseteq i_N((f \cdot g)^{-1}(T))\]

for all \(T \in \mathcal{S}(N)\). Then, for all \(S \in \mathcal{S}_L\), we have

\[
g^{-1}(i_{L_f}(S)) = g^{-1}(f^{-1} \cdot i_M \cdot f_*(S)) = (f \cdot g)^{-1}(i_M(f_*(S))) \subseteq i_N((f \cdot g)^{-1}(f_*(S))) = i_N(g^{-1} \cdot f^{-1} \cdot f_*(S)) \subseteq i_N(g^{-1}(S))
\]

i.e. \(g\) is \(I\)-continuous. \(\blacksquare\)
As a consequence of corollary (2.7), proposition (2.8) and proposition (2.9) (cf. [1] or [?]), we obtain

**Theorem 2.10.** The forgetful functor $U : \mathbf{I-Loc} \to \mathbf{Loc}$ is topological, i.e. the concrete category $(\mathbf{I-Loc}, U)$ is topological.

### 2.3. Open subobjects.

We introduce a notion of open subobjects different from the one alluded in 1.3.3.

**Definition 2.11.** An sublocale $S$ of a locale $L$ is called $I$-open (in $L$) if it is isomorphic to its $I$-interior, that is: if $i_L(S) = S$.

The $I$-continuity condition (1) implies that $I$-openness is preserve by inverse images:

**Proposition 2.12.** Let $f : L \to M$ be a morphism in $\mathbf{Loc}$. If $T$ is $I$-open in $M$, then $f^{-1}(T)$ is $I$-open in $L$.

Proof. If $T = i_M(T)$ then $f^{-1}(T) = f^{-1}(i_M(T)) \subseteq i_L(f^{-1}(T))$, so $i_L(f^{-1}(T)) = f^{-1}(T)$. ■

### 3. $\mathfrak{h}$ Operators

In this section we shall be concerned with a weak categorical version of a topological function studied by M, Suarez M. in [12]. For that purpose we will use the collection $\mathcal{S}_f(L)$ of all complemented sublocales of a locale $L$ (See P, T. Johnston [3], for example).

**Definition 3.1.** An $\mathfrak{h}$ operator of the category $\mathbf{Loc}$ is given by a family $\mathfrak{h} = (h_L)_{L \in \mathbf{Loc}}$ of maps $h_L : \mathcal{S}_f(L) \to \mathcal{S}_f(L)$ such that

1. $S \cap h_L(S) \subseteq S$, for all $S \in \mathcal{S}_f(L)$;
2. If $S \subseteq T$ then $S \cap h_L(S) \subseteq T \cap h_L(T)$, for all $S,T \in \mathcal{S}_f(L)$;
(h_3) \ h_L(L) = L.

**Definition 3.2.** An \( h \)-space is a pair \((L, h_L)\) where \( L \) is an object of \( \text{Loc} \) and \( h_L \) is an \( h \) operator on \( L \).

**Definition 3.3.** A morphism \( f : L \to M \) of \( \text{Loc} \) is said to be \( h \)-continuous if

\[
(3) \quad f^{-1} [T \cap h_M(T)] \subseteq f^{-1} [T] \cap h_L (f^{-1} [T])
\]

for all \( T \in \mathcal{S}^c(M) \). Where \( f^{-1}[\cdot] \) is the inverse image of \( f[\cdot] \).

**Proposition 3.4.** Let \( f : L \to M \) and \( g : M \to N \) be two \( h \)-continuous morphisms of \( \text{Loc} \) then \( g \circ f \) is an \( h \)-continuous morphism of \( \text{Loc} \).

**Proof.** Since \( g : M \to N \) is \( I \)-continuous, we have

\[
g^{-1} [V \cap h_N(V)] \subseteq g^{-1} [V] \cap h_M (g^{-1} [V])
\]

for all \( V \in \mathcal{S}^c(N) \), it follows that

\[
f^{-1} [g^{-1} [V \cap h_N(V)]] \subseteq f^{-1} [g^{-1} [V] \cap h_M (g^{-1} [V])]
\]

now, by the \( h \)-continuity of \( f \),

\[
f^{-1} [g^{-1} [V] \cap h_M (g^{-1} [V])] \subseteq f^{-1} [g^{-1} [V]] \cap h_L (f^{-1} [g^{-1} [V]])
\]

therefore

\[
(g \circ f)^{-1} [V \cap h_N(V)] \subseteq (g \circ f)^{-1} \cap h_L ((g \circ f)^{-1} [V]).
\]

This complete the proof. \( \blacksquare \)

As a consequence we obtain

**Definition 3.5.** The category \( h \cdot \text{Loc} \) of \( h \)-spaces comprises the following data:
(1) **Objects**: Pairs \((L, h_L)\) where \(L\) is an object of \(\text{Loc}\) and \(h_L\) is an \(h\)-operator on \(L\).

(2) **Morphisms**: Morphisms of \(\text{Loc}\) which are \(h\)-continuous.

### 3.1. The lattice structure of all \(h\) operators.

For the category \(\text{Loc}\) we consider the collection

\[ h(\text{Loc}) \]

of all \(h\) operators on \(\text{Loc}\). It is ordered by

\[ h \leq h' \iff h_L(S) \subseteq h'_L(S), \text{ for all } S \in S_L(L) \text{ and all } L \text{ object of } \text{Loc}. \]

This way \(h(\text{Loc})\) inherits a lattice structure from \(S_L\).

**Proposition 3.6.** Every family \(\langle h_\lambda \rangle_{\lambda \in \Lambda}\) in \(h(\text{Loc})\) has a join \(\bigvee_{\lambda \in \Lambda} h_\lambda\) and a meet \(\bigwedge_{\lambda \in \Lambda} h_\lambda\) in \(\text{Int}(\text{Loc})\). The discrete \(h\) operator

\[ h_D = (h_{DL})_{L \in \text{Loc}} \text{ with } h_{DL}(S) = S \text{ for all } S \in S_L(L) \]

is the largest element in \(h(\text{Loc})\), and the trivial \(h\) operator

\[ h_T = (h_{TL})_{L \in \text{Loc}} \text{ with } h_{TL}(S) = \begin{cases} \{1\} & \text{for all } S \in S_L(L), S \neq L \\ L & \text{if } S = L \end{cases} \]

is the least one.

**Proof.** For \(\Lambda \neq \emptyset\), let \(\widehat{h} = \bigvee_{\lambda \in \Lambda} h_\lambda\), then

\[ \widehat{h}_L = \bigvee_{\lambda \in \Lambda} h_{\lambda L}, \]

for all \(L\) object of \(\text{Loc}\), satisfies

- \(S \cap \widehat{h}_L(S) \subseteq S\), because \(S \cap h_{\lambda S}(L) \subseteq S\), for all \(S \in S_L(L)\) and for all \(\lambda \in \Lambda\).
- If \(S \subseteq T\) then \(S \cap \widehat{h}_L(S) \subseteq T \cap \widehat{h}_L(T)\), since \(S \cup h_{\lambda L}(S) \subseteq T \cup h_{\lambda L}(T)\), for all \(S, T \in S_L(L)\) and for all \(\lambda \in \Lambda\).
- \(L \cap \widehat{h}_L(L) = L\), because \(L \cap h_{\lambda L}(L) = L\) for all \(\lambda \in \Lambda\).
Similarly \( \bigwedge_{\lambda \in \Lambda} \mathfrak{h}_\lambda \), \( \mathfrak{h}_D \) and \( \mathfrak{h}_T \) are \( \mathfrak{h} \) operators.

\[ \mathfrak{h}(L) = \{ h_L \mid h_L \text{ is an } \mathfrak{h} \text{ operator on } L \} \]

is a complete lattice.

3.2. Initial \( \mathfrak{h} \) operators. Let \( \mathfrak{h}-\text{Loc} \) be the category of \( \mathfrak{h} \)-spaces. Let \( (M, h_M) \) be an object of \( \mathfrak{h}-\text{Loc} \) and let \( L \) be an object of \( \text{Loc} \). For each morphism \( f : L \to M \) in \( \text{Loc} \) we define on \( L \) the operator

\[ h_{L_f} := f^{-1} \circ h_M \circ f_* \]

Proposition 3.8. The operator (4) is an \( \mathfrak{h} \) operator on \( L \) for which the morphism \( f \) is \( \mathfrak{h} \)-continuous.

Proof.

\( (h_1) \quad S \cap h_{L_f}(S) = f^{-1}(f_*[S] \cap h_M(f_*[S])) \subseteq f^{-1}(f_*[S]) \subseteq S, \) for all \( S \in S^c_f(L) \).

\( (h_2) \quad S \subseteq T \text{ in } S^c_f(L), \text{ implies } f_*[S] \subseteq f_*[T], \text{ then } f_*[S] \cap h_M(f_*[S]) \subseteq f_*[T] \cap h_M(f_*[T]), \text{ therefore } f^{-1}(f_*[S] \cap h_M(f_*[S])) \subseteq f^{-1}(f_*[T] \cap h_M(f_*[T])), \text{ consequently } S \cap h_{L_f}(S) \subseteq T \cap h_{L_f}(T), \text{ for all } S, T \in S^c_f(L); \)

\( (h_3) \quad L \cap h_{L_f}(L) = f^{-1}(f_*[L] \cap h_M(f_*[L])) = L. \)

It is clear that \( h_{L_f}(L) \) is the coarsest \( \mathfrak{h} \) operator on \( L \) for which the morphism \( f \) is \( \mathfrak{h} \)-continuous; more precisely
**Proposition 3.9.** Let \((L, h_L)\) and \((M, h_M)\) be objects of \(\mathfrak{h}\)-\text{Loc}, and let \(N\) be an object of \text{Loc}. For each morphism \(g : N \to L\) in \text{Loc} and for \(f : (L, h_L) \to (M, h_M)\) an \(\mathfrak{h}\)-continuous morphism, \(g\) is \(\mathfrak{h}\)-continuous if and only if \(f \circ g\) is \(\mathfrak{h}\)-continuous.

**Proof.** Suppose that \(g \circ f\) is \(I\)-continuous, i.e.
\[
(f \circ g)_{-1}(T \cap h_M(T)) \subseteq T \cap h_N((f \circ g)_{-1}(T))
\]
for all \(T \in T \in S_{C}(N)\). Then, for all \(S \in T \in S_{C}(L)\), we have
\[
g_{-1}(S \cap (h_{L}(S))) = g_{-1}(f_{-1}(f_{*}(S) \cap h_{M} \cdot f_{*}(S))) = (f \circ g)_{-1}(f_{*}(S) \cap h_{M}(f_{*}(S)))
\]
\[
\subseteq f \cdot g_{-1}(f_{*}(S)) \cap h_{N}(f_{-1}(f_{*}(S)))
\]
\[
= (f \circ g)_{-1}(f_{*}(S)) \cap h_{N}(f_{-1} \cdot f_{*}(S))
\]
\[
\subseteq g_{-1}(S) \cap h_{N}(g_{-1}(S)),
\]
i.e. \(g\) is \(I\)-continuous. \(\blacksquare\)

As a consequence of corollary[3.7], proposition[3.8] and proposition[3.9] (cf. [1] or [2]), we obtain

**Theorem 3.10.** The forgetful functor \(U : \mathfrak{h}\)-\text{Loc} \to \text{Loc} is topological, i.e. the concrete category \((\mathfrak{h}\)-\text{Loc}, \(U\)) is topological.

**References**

[1] JIRI ADAMEK, HORST HERRLICH, GEORGE STRECKER, *Abstract and Concrete Categories*, John Wiley & Sons, New York, 1990.
[2] J. DUGUNDJII, *Topology*, Allyn and Bacon, Inc., Boston / London / Sydney / Toronto, 1966.
[3] P. T. JOHNSTONE, *Complemented sublocales and open maps*, Annals of Pure and Applied Logic 137, 2006.
[4] P. T. JOHNSTONE, *Stone spaces*, Cambridge University Press, Cambridge, 1982.
[5] K. KURATOWSKI, *Topology*, Vol. 1, Academic Press, New York and London, 1966.
[6] K. KURATOWSKI, *Topology*, Vol. 2, Academic Press, New York and London, 1968.
[7] J. LUNA-TORRES, C. OCHOA, *Interior operators and topological categories*, Adv. Appl. Math. Sci., 10, 2011.
[8] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York / Heidelberg / Berlin, 1971.

[9] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic, A first introduction to Topos theory*, Springer-Verlag, New York / Heidelberg / Berlin, 1992.

[10] J. Picado, A. Pultr, *Frames and Locales: Topology Without Points*, Frontiers in Mathematics, vol. 28, Springer, Basel, 2012.

[11] A. L. Suarez, *Revisiting the relation between subspaces and sublocales*, arXiv:2010.05284v1 [math.FA] 11 Oct 2020.

[12] M. Suarez M., *La función topológica h*, Universidad Pedagógica y Tecnológica de Colombia, Tunja, 1988.

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