Tunneling Analysis of Kerr-Newman Black Hole-Like Solution in Rastall Theory

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Hamilton-Jacobi ansatz is used to analyze boson charged particles tunneling in Rastall gravity Kerr-Newman black hole (BH) surrounded by perfect fluid matter through the horizon. The geometrical BH parameters are observed, how these are affecting the radiation utilizing the Lagrangian gravity equation. In this article, the corrected Hawking temperature is computed by considering the effects of quantum gravity. In our analysis, the Rastall gravity BH solution surround by perfect fluid matter effects on Hawking radiation is analyzed graphically. Moreover, the stability and instability of Rastall gravity BH are investigated. The influence of quantum gravity and rotation parameters on BH radiation are also observed by graphical representation.

Keywords: Rastall Gravity Black Hole; Lagrangian Gravity Equation; Hawking Radiation

I. INTRODUCTION

Hawking [1] observed that a black hole (BH) radiates all kinds of particles and the radiated spectrum of particles are absolutely thermal and the spectrum of emission probability for different types of particles has been studied. The phenomenon of tunneling is established on the physical action of particles theory which gives BHs radiation. The law of energy conservation and momentum conservation has been adapted to this phenomenon. The particular temperature value at which vector particles are emitted from the BH horizon is called boson radiation. The Hawking radiation for scalar and fermion particles from different spacetimes have been investigated [2, 3]. The tunneling probability of charged scalars and fermions outgoing from the event horizon from these BHs have been studied. The temperature and entropy in the cosmological horizon of Schwarzschild de-Sitter spacetime have been investigated in [4]. The charged particles radiated through tunneling from 4-dimensional as well as 5-dimensional BHs and Vaidya BH have been investigated [5, 6], however, the Hawking temperatures and tunneling rates which depend on the properties of space-time have been observed.

The noncommutative acoustic BHs of quantum-corrected finite entropy have been discussed [7] and the corrections in terms of electric charge are conserved. The energy and mass of outgoing fermions and Kerr and Kerr-Newman BHs have been investigated [8–10] and the tunneling particles would be stopped at some particular range and it is also concluded that the tunneling rate of radiated particles and Hawking temperature for both BHs which is depends on different parameters. The Hawking radiation spectrum under the influence of quantum gravity has been analyzed from different types of BH in [32, 33]. The energy spectrum and the Hawking temperature contribute holomorphic and anti-holomorphic functions, GUP and conformal gravity effects on tunneling radiation from various BHs have been observed [11–14], which are consistent with original results. The BHs tunneling radiation with the assumption of GUP [15], unitary theory [16], loop quantum theory[17] and coordinate system [18] have been examined and found that which depends on GUP, unitary theory, string and loop quantum theory but independent on coordinate system.

The tunneling particles from the BH, BTZ like BH and black ring have been examined [19–21] and it contains the hypothesis; BHs and black rings have both unstable and stable properties. Many authors computed [22–31, 34–47] the tunneling radiation of boson and fermion particles to get the Hawking temperature for different wormholes and
BHs. The researcher computed [48, 49] the quantum gravity effect on tunneling radiation and it is a phenomenon for different types of BHs.

The Newmananis algorithmic rule to analyze the Hayward BH solution with rotation parameter has been computed in [50]. The temperature for regular Hayward BH metric with rotation parameter has been derived with the help of semi-classical phenomenon. An extension for temperature of 3 and 4 dimensional BH metric as well as higher dimensional BHs metric under the influences of quantum gravity have been analyzed in [51, 52]. The Rastall rotating BH with surrounding quintessence and dust parameter has been discussed in [53] and showed that the Rastall theory metric solution is different from the standard metric solution.

The charged bosons particles tunneling with spin equals 1 that is $Z$ and $W_{\pm}$ play a significant role for the standard model of electro-weak interaction so that the radiation of charged boson particles should be importance in the observation of tunneling radiation. Many different approaches have been analyzed for the investigation of the tunneling model of electro-weak interaction so that the radiation of charged bosons particles should be importance in the observation of tunneling radiation. Many different approaches have been analyzed for the investigation of the tunneling model of electro-weak interaction so that the radiation of charged bosons particles should be importance in the observation of tunneling radiation.

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This paper aims to study the quantum gravity effects on BH thermodynamics. This paper is organized as follows. In the Sec. (II) deals the introduction of Rastall gravity Kerr-Newman BH surrounded by perfect fluid matter. We analyze the tunneling radiation of BH by assuming the Lagrangian equation in the scenario of GUP and calculated the temperature. The graphical representation of temperature in terms of rotation and quantum gravity are discussed in Sec. (III). The last section deals with our results.

## II. KERR-NEWMAN BLACK HOLE IN RASTALL GRAVITY THEORY

By applying Newman-Janis algorithmic rule to study the rotating Rastall BH space-time, which analyzes by mass ($M$), rotation parameter ($a$), surrounding fluid structure parameter ($\sigma$), state parameter of surrounding fluid $u$ and Rastall coupling parameter $\alpha$. The Rastall BH metric in the Boyer-Lindquist coordinates from [55, 56] can be expressed as

$$ds^2 = -Adt^2 + Bd\tau^2 + Cd\theta^2 + Dd\phi^2 + 2Fdt\phi,$$

where $A, B, C, D$ and $E$ are given by the following equations:

$$A = 1 - \frac{2Mr + \sigma r^\gamma}{\Sigma}, \quad B = \frac{\Sigma}{\lambda_r}, \quad C = \Sigma,$$

$$D = \sin^2\theta \left( r^2 + a^2 + \frac{(2Mr + \sigma r^\gamma)a^2\sin^2\theta}{\Sigma} \right),$$

$$F = -\sin^2\theta \left( 2Mr + \sigma r^\gamma \right).$$

Here, these $\lambda_r, \Sigma$ and $\gamma$ parameters are defined as

$$\lambda_r = r^2 - 2Mr + a^2 - \sigma r^\gamma,$$

$$\Sigma = a^2\cos^2\theta + a^2,$$

$$\gamma = \frac{1 - 3\alpha}{1 - 3\alpha(1 + u)}$$

It is important to mention that, if $a = 0$ and $-1 < u < -\frac{1}{3}$, the metric (1) constitutes the Kerr BH surrounded by quintessence [57] and if $\sigma$ and $a$ both are zero, we get Schwarzschild BH solution in Ref. [56].

In order to study the tunneling radiation of charged particles via the BH horizon, we shall analyze electromagnetic effects of the Lagrangian gravity equation. The electromagnetic-charged fields, which describe the particle’s motion in the Lagrangian charged equation [58]

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} \Psi^{\mu} \right) + \frac{i}{\hbar} e A_{\mu} \Psi^{\mu} + \frac{m^2}{\hbar^2} \Psi^{\mu} + \hbar^2 \beta \partial_{\mu} \partial_{\nu} (g^{\mu\nu} \Psi^{\mu} - \hbar^2 \beta \partial_{\nu} \partial_{\mu} \Psi^{\mu}) = 0,$$

where $g$, $\Psi^{\mu\nu}$ and $m$ are the matrix determinant of coefficients, particle mass and anti-symmetric tensor, respectively.

$$\Psi_{\nu\mu} = (1 - \hbar^2 \beta \partial_{\nu}^2) \partial_{\nu} \Psi_{\mu} - (1 - \hbar^2 \beta \partial_{\mu}^2) \partial_{\mu} \Psi_{\nu} + (1 - \hbar^2 \beta \partial_{\nu}^2) \frac{i}{\hbar} e A_{\nu} \Psi_{\mu} - (1 - \hbar^2 \beta \partial_{\mu}^2) \frac{i}{\hbar} e A_{\mu} \Psi_{\nu}.$$
Here, $\nabla_\mu$, $e$ and $A_\mu$ are the covariant derivative, particle charged and BH potential, respectively. So the tunneling of positive and negative particles is similar but opposite sign ($\bar{W}_+ = - \bar{W}_-$). For explanation, here we shall study the positive particles tunneling and this tunneling can be changed to negative particles tunneling. The values of $\Psi^\mu$ and $\Psi^{\nu\mu}$ are given by

$$
\Psi^\nu = \Psi_{\mu\nu} \delta^{\mu\nu},
$$

$$
\Psi_0 = \frac{D\Psi_{0} - F\Psi_{3}}{AD + F^2}, \quad \Psi_1 = \frac{\Psi_1}{B}, \quad \Psi_2 = \frac{\Psi_2}{C}, \quad \Psi_3 = \frac{A\Psi_{3} - F\Psi_{0}}{AD + F^2},
$$

$$
\Psi^{\nu\mu} = \Psi_{\mu\nu} \delta^{\nu\mu} S^\nu.
$$

Applying the WKB approximation,

$$
\psi_v = k_v \exp \left[ \frac{i}{\hbar} N_0(t, r, \theta, \phi) + \Sigma h^i N_i(t, r, \theta, \phi) \right],
$$

(3)

The term $\hbar$ is assumed only for the lowest (1st) order and the higher (higher order $> 1$) dictate contributions are neglected in the Lagrangian gravity Eq. (2), we get set of field equations. The variables separation technique in [58], we can take

$$
N_0 = -E_0 t + W(r) + V(\theta) + J\phi,
$$

(4)

where $E_0 = E - \omega \Omega$, $\omega$ and $E$ represent particle’s angular momentum and energy respectively. The set of field equations, we can get a matrix field equation

$$
K(k_0, k_1, k_2, k_3)^T = 0,
$$

which implies the "$K$" is label as $4 \times 4$ matrix, whose elements are given as follows:

$$
K_{00} = \frac{D}{B} \bar{W}^2 W_1 - \frac{D}{C} \bar{J}^2 J_1 - V^2 V_1 = eA_3 [2V + 2\beta \bar{V}^3 + eA_3 + \beta eA_3 \bar{V}^2] - m^2 D,
$$

$$
K_{01} = \frac{D}{B} \bar{W} E_0 [1 + \beta E_0^2 - eA_0 E_0 - \beta eA_0 E_0] - \frac{F}{B} \bar{W} [V + \beta \bar{V}^3 + eA_3 + \beta eA_3 \bar{V}^2],
$$

$$
K_{02} = -\frac{D}{C} \bar{J} [E_0 - \bar{E}_0^3 + eA_0 + \beta eA_0 E_0] - \frac{F}{C} \bar{J} [V + \beta \bar{V}^3 + eA_3 + \beta \bar{V}^2],
$$

$$
K_{03} = \frac{F}{C} \bar{W}^2 W_1 + \frac{F}{C} \bar{J}^2 J_1 - E_0 E_1 V + eA_0 V E_1 - eA_3 E_0 E_1 + e^2 A_0 A_3 E_1 + m^2 F,
$$

$$
K_{10} = -D \bar{W} E_0 W_1 - F \bar{V} \bar{W} W_1 + \bar{D} A_0 \bar{W} W_1 - \bar{F} A_3 \bar{W} W_1,
$$

$$
K_{11} = -D E_0^2 E_1 - D A_3 E_0 E_1 - F \bar{V} E_0 V_1 - \bar{F} A_3 E_0 V_1 - \frac{F}{C} \bar{J}^2 J_1 - \bar{V}^2 V_1 - eA_3 \bar{V} V_1 - \bar{F} \bar{E} \bar{V} E_1 + eA_3 \bar{V} E_1 + \bar{F} \bar{A} \bar{A} \bar{A} E_1 + eA_3 \bar{A} \bar{A} \bar{A} E_1 - m^2 F_0
$$

$$
+ F eA_3 \bar{A} \bar{A} \bar{A} E_1 - A_3 \bar{V} \bar{V} V_1 - \bar{F} A_3 \bar{V} E_1 + eA_3 \bar{V} E_1 + \bar{F} \bar{A} \bar{A} \bar{A} E_1 - m^2 F_0
$$
Here, \( W_+ \) and \( W_- \) represent the solution of absorbing and radiating charged boson particles action respectively and \( X_1 = \frac{m^2A + AC^{-1}J^2}{2} \) and \( X_2 = E_0^2 - eA_0E_0^2 + \frac{A^2}{2}W^2 + \frac{A}{C}J^2 - eA_0E_0^2 + e^2A_3^2V_1 \). In the series of Taylor’s, we are expanding the functions \( A(r) \) and \( B(r) \) near the horizon, we get

\[
A(r_+) \approx A(r - r_+) \quad B(r_+) \approx B(r)(r - r_+)
\]

(6)

Since applying Eqs. (5) and (6) and integrate the around the pole, we get

\[
imW_+ = \pi\left(\frac{E_0 - eA_0}{2\kappa(r_+)}\right)^2(1 + \Xi\beta),
\]

(7)

where \( \Xi \) is arbitrary parameter and \( \kappa(r_+) \) is a surface gravity of BH. The surface gravity of BH defines as;

\[
\kappa(r_+) = \frac{2M(M - r_+)}{\left(a^2 + a^2\cos^2\theta \right)^2}
\]

(8)

Here, the tunneling probability \( \Gamma \) forbidden trajectories of the classically of the s-waves coming from inside to outside from the BH horizon. Applying the WKB approximation, by the terms of classical action \( S_0 \) of charged boson particles tunneling across the BH horizon as trajectories up to leading order is,

\[
\Gamma = \frac{\Gamma_{\text{emission}}}{\Gamma_{\text{absorption}}} = \frac{\exp(-2imW_+ - 2imV)}{\exp(-2imW_- - 2imV)} = e^{-4imW_+}
\]

\[
= \exp \left[ -2\pi\frac{E - \omega\Omega - eA_0}{\kappa(r_+)}(1 + \Xi\beta) \right].
\]

(9)

Thus, for computing the Hawking temperature and we expand the action in terms of particles energy \( E \). Hence, the Hawking temperature is calculated at linear order given by

\[
T \approx \frac{M(r_+ - M)}{\pi(a^2 + a^2\cos^2\theta)^2} [1 - \Xi\beta].
\]

(10)
The charged boson massive particles case is similar as for massless case for Rastall gravity Kerr-Newman BH surrounded by perfect fluid matter from the above Hawking temperature. Moreover, the temperature for spin-up particles are similar \((-W_+ = W_-)\) as for spin-down case with the change \(r_+\) into \(r_-\). We observe that both spin-down and spin-up charged boson particles are radiate with like rate the temperature is assumed for this case.

### III. GRAPHICAL TEMPERATURE ANALYSIS

In this section, we analyze graphical behavior of thermodynamical quantities in Rastall gravity Kerr-Newman BH surrounded by perfect fluid matter. For this purpose, we take a parameter \((\Xi = 1)\) and observe the effects of the thermodynamical quantities (quantum gravity and rotation parameter) on BH radiation. The BH remains unstable if temperature increases or decreases. From Fig. 1, the Hawking temperature is decreasing due to the more quantum gravity. The stability of the BH depends on quantum gravity and also shows that BH remains stable unless the quantum gravity parameter is zero \((\beta = 0)\). Also, temperature decreases in the approximation range \((0 < r_+ < 5)\). If the quantum gravity is assumed to be non-zero, then Hawking temperature attains the similar value in figure 2. The Hawking temperature is decrease to rotation parameter in the values 0.1-0.3. The gravity parameter is fixed and varying the rotation parameter then we concluded that rotation parameter is inversely proportional to Hawking temperature and stability of BH. The stability of BH remains constant in very small range, i.e., approximation \(0 < r_+ < 0.5\) and then temperature decreases with a wide variation. The rotation parameter is inversely proportional to the temperature of BH by keeping the quantum gravity constant in Fig. 2 for 2D. The temperature is behaving constant when \(r_+\) is very small, but at a certain value of rotation parameter, it attain a constantly decrease value in Fig. 2 for 2D. Hence our BH is stable when \(r_+\) is very small range otherwise is unstable.

We concluded that BH radiation depends on BH mass, rotation parameter and gravity parameter but keeping constant \(\cos^2 \theta = 1\).

### IV. CONCLUSIONS

In this paper, we have observed radiation spectrum by considering Hawking temperature for Rastall gravity Kerr-Newman BH is surrounded by perfect fluid matter. For this purpose, we have utilized the WKB approximation to the Hamilton-Jacobi ansatz for massive bosons charged spin-1 particles. The calculation yields, the Hawking temperature reliable with BH universality. In our investigation, we have changed the Lagrangian field equation in curved space time by effect of incorporate GUP quantum gravity. We have computed the tunneling probability at event horizon as well as the equating Hawking temperature. We have observed temperature in detail the effect of quantum gravity and rotation parameters in graphically.

The paper have studied graphical investigation of the result and the BH will emit away all types of particles as the black-body radiation and the BH lose completely its information. In adjustment to break this problem, the after-effect rotation and gravity to be modified. We assume into account the law of conservation of momentum and energy. In our observation, the back reaction of the radiated boson particles of the self-gravitational interaction and BH geometry are reasonably ignored, the calculated Hawking temperatures depend on quantum gravity and the leading terms of metric. The modified form of temperature depends on BH mass, rotation parameters and quantum gravity parameter. When the gravity parameter \((\beta = 0)\) influences are ignored, we have obtained the temperature of Rastall gravity Kerr-Newman BH. So, the quantum gravity effect minimum then stability of BH are increased.

The quantum gravity and rotation are present in the BH, which shows that the instability of BH. The BH is completely stable when rotation and quantum gravity do not appear.

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FIG. 1: $T_H$ versus $r_+$ for $M > 0, \ a = 10, \ \cos^2 \theta = 1$ and $\Xi = 1$.

FIG. 2: $T_H$ versus $r_+$ for $M > 0, \ \beta > 0, \ \cos^2 \theta = 1$ and $\Xi = 1$. 
Appendix

In the WKB approximation the term $\hbar$ is considered just for the 1st order and the higher dictate contributions are neglected in the Lagrangian Eq. (2), we get following field equations;

$$
\begin{align*}
\frac{D}{B}[k_1 (0 \partial N_0) (\partial_1 N_0) + \beta k_1 (0 \partial N_0)^3 (\partial_1 N_0) + k_0 (\partial_1 N_0)^2 - \beta k_0 (\partial_1 N_0)^4 + eA_0 k_1 (0 \partial N_0)^2 (\partial_1 N_0) + eA_0 k_1 (0 \partial N_0)^2 (\partial_1 N_0)]
- \frac{F}{B}[k_1 (\partial_1 N_0) (\partial_3 N_0) + \beta k_1 (\partial_1 N_0) (\partial_3 N_0)^3 - k_3 (\partial_1 N_0)^2 - \beta k_3 (\partial_1 N_0)^4 + eA_3 k_1 (\partial_1 N_0) + eA_3 k_1 (\partial_1 N_0) (\partial_3 N_0)^2]
+ \frac{D}{C}[k_2 (0 \partial N_0) (\partial_2 N_0) + \beta k_2 (0 \partial N_0)^3 (\partial_2 N_0) - k_0 (\partial_2 N_0)^2 - \beta k_0 (\partial_2 N_0)^4 + eA_0 k_2 (\partial_2 N_0) + eA_0 k_2 (\partial_2 N_0)^2 (\partial_2 N_0)]
- \frac{F}{C}[k_2 (\partial_2 N_0) (\partial_3 N_0) + \beta k_2 (\partial_2 N_0) (\partial_3 N_0)^3 - k_3 (\partial_2 N_0)^2 - \beta k_3 (\partial_2 N_0)^4 + eA_3 k_2 (\partial_2 N_0) + eA_3 k_2 (\partial_2 N_0)^2 (\partial_3 N_0)^2]
\end{align*}
$$

$$
\begin{align*}
\frac{D}{[k_0 (\partial_0 N_0) (\partial_1 N_0) + \beta k_0 (\partial_0 N_0)^3 (\partial_1 N_0) - k_1 (\partial_0 N_0)^2 - \beta k_1 (\partial_0 N_0)^4 - eA_0 k_0 (\partial_0 N_0) - eA_0 k_0 (\partial_0 N_0)^3] - F[k_3 (\partial_0 N_0) (\partial_1 N_0) + \beta k_3 (\partial_0 N_0) (\partial_3 N_0)^3 - k_3 (\partial_0 N_0)^2 - \beta k_3 (\partial_0 N_0)^4] + eA_3 k_0 (\partial_1 N_0) + eA_3 k_0 (\partial_0 N_0)^3 (\partial_1 N_0) + eA_3 k_0 (\partial_0 N_0)^3 (\partial_3 N_0) - \beta k_3 (\partial_3 N_0)^3 - eA_3 k_1 (\partial_0 N_0) - eA_3 k_1 (\partial_0 N_0)^3 (\partial_3 N_0) + eA_3 k_1 (\partial_1 N_0) - \beta k_3 (\partial_1 N_0)^4 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_0 N_0) + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2] - m^2 k_1
\end{align*}
$$

$$
\begin{align*}
D[k_0 (\partial_0 N_0) (\partial_2 N_0) + \beta k_0 (\partial_0 N_0)^3 (\partial_2 N_0)^3 - k_2 (\partial_0 N_0)^2 - \beta k_2 (\partial_0 N_0)^4 - eA_0 k_2 (\partial_0 N_0) + eA_0 k_2 (\partial_0 N_0)^3]^3 - F[k_3 (\partial_0 N_0) (\partial_2 N_0) + \beta k_3 (\partial_0 N_0)^3 (\partial_2 N_0)^3 - k_2 (\partial_0 N_0)^2 - \beta k_2 (\partial_0 N_0)^4] + eA_3 k_0 (\partial_0 N_0) + eA_3 k_0 (\partial_0 N_0)^3 (\partial_2 N_0) + eA_3 k_0 (\partial_0 N_0)^3 (\partial_3 N_0) - \beta k_3 (\partial_3 N_0)^3 - eA_3 k_1 (\partial_0 N_0) - eA_3 k_1 (\partial_0 N_0)^3 (\partial_3 N_0) + eA_3 k_1 (\partial_1 N_0) - \beta k_3 (\partial_1 N_0)^4 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_0 N_0) + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2] - m^2 k_2
\end{align*}
$$

$$
\begin{align*}
[k_0 (\partial_0 N_0) (\partial_3 N_0) + \beta k_0 (\partial_0 N_0)^3 (\partial_3 N_0)^3 + k_3 (\partial_0 N_0)^2 - \beta k_3 (\partial_0 N_0)^4 + eA_3 k_0 (\partial_0 N_0) + eA_3 k_0 (\partial_0 N_0)^3 (\partial_3 N_0)^2 - eA_0 k_0 (\partial_0 N_0) - eA_0 k_0 (\partial_0 N_0)^3 [\partial_3 N_0) - \beta k_3 (\partial_3 N_0)^3 + eA_3 k_1 (\partial_0 N_0) - eA_3 k_1 (\partial_0 N_0)^3 (\partial_3 N_0) + eA_3 k_1 (\partial_1 N_0) - \beta k_3 (\partial_1 N_0)^4 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_0 N_0) + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2 + eA_3 k_1 (\partial_1 N_0)^2 (\partial_3 N_0)^2] - m^2 k_3
\end{align*}
$$

(11)

(12)

(13)
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