Properties of mathematical objects  
(Gödel on classes, properties and concepts)

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Abstract  
In terms of a sufficiently fine-grained theory we should distinguish between classes, properties and concepts. Since properties are best modeled as a kind of non-trivial intensions while mathematical objects are never non-trivial intensions we should not speak about properties of mathematical objects. When we do use the term property in mathematics (as Gödel did) we either mean classes, or the more fine-grained entities to be called concepts. In the latter case concepts have to be defined so that various distinct concepts could identify one and the same object. The notion of construction in transparent intensional logic makes it possible to construe concepts as abstract procedures. At the same time we have to distinguish between this notion and the notion of construction in constructivist systems: the former – unlike the latter – are objective and, therefore, acceptable for a realist.

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1. Gödel on classes, concepts and notions.
To introduce our problem I will adduce some quotations from [G]. Italics mine.
[t]he vicious circle principle in its first form applies only if one takes the constructivistic (or nominalistic) standpoint…toward the objects of logic and mathematics, in particular toward propositions, classes and notions, e.g., if one understands by a notion a symbol together with a rule for translating sentences containing the symbol into such sentences as do not contain it, so that a separate object denoted by the symbol appears as a mere fiction.

Classes and concepts may, however, also be conceived as real objects, namely classes as “pluralities of things” or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions. ...

I. [G], 128

I shall use the term “concept” in the sequel exclusively in this objective sense. One formal difference between the two conceptions of notions would be that any two different definitions of the form \( \alpha(x) = \phi(x) \) can be assumed to define two different notions \( \alpha \) in the constructivistic sense. … For concepts, on the contrary, this is by no means the case, since the same thing may be described in different
ways. … The difference may be illustrated by the following definition of the number two: “Two is the notion under which fall all pairs and nothing else.” There is certainly more than one notion in the constructivistic sense satisfying this condition, but there might be one common “form” or “nature” of all pairs.

II. [G], 128-129

As to classes in the sense of pluralities or totalities, it would seem that they are likewise not created but merely described by their definitions…

III. [G], 131

“no class” theory…considers the notions (or propositional functions) as something constructed out of propositions or sentences by leaving one or several constituents of them undetermined.

IV. [G], 137

If … one considers concepts as real objects, the theory of simple types is not very plausible since what one would expect to be a concept (such as, e.g., “transitivity” or the number two) would seem to be something behind all its various “realizations” on the different levels and therefore does not exist according to the theory of types. Nevertheless, there seems to be some truth behind this idea of realizations of the same concept on various levels…

V. Ibidem

[o]nly should take a more conservative course, such as would consist in trying to make the meaning of the terms “class” and “concept” clearer, and to set up a consistent theory of classes and concepts as objectively existing entities.

VI. [G], 140

Our problem (or its most general core) is: Speaking about properties in the area of extensional (mathematical) objects what do we mean? And a closely connected problem can be formulated as follows:

What is Gödel’s concept, how does it differ from properties?

2. Properties

2.1 Primitive or reducible?

As for property theories we can distinguish two groups of them:

a) For the theories of the first group properties are primitive entities, i.e., they are not defined.

b) The theories of the second group try to reduce properties to some more basic entities.

Ad a): Probably the most interesting representative of the first group is G. Bealer (see [Bealer 1], [Bealer 2]. For him properties are obviously “simple intensions”, i.e., “perfectly ‘natural’ properties and relations” (see [Bealer 2], 141), which “have no logical form”; they are “identical if they are necessarily equivalent” (ibidem).

What does mean the characteristics ‘perfectly ‘natural’ ‘? This can be understood if compared with some objections to the attempts to reduce properties in terms, e.g., of possible worlds (see below). We read, e.g., in [Swoyer], 14:

“I recognize the sound of an oboe or the taste of rhubarb; these are very direct and simple experiences that seem completely unrelated to functions from huge infinite sets of possible worlds”.

This kind of objections seems at first sight convincing: nobody needs to use the notion of possible worlds for identifying some property. But this very simplicity of such arguments is at least suspect. To show this let us use an analogy. Everybody can recognize an object as
being a tree or a cat. Yet from the vantage point of a scientific theory trees and cats are very complex objects, be it from the viewpoint of biology or physics or, say, chemistry. Does it mean that these scientific views are not necessary, that they only complicate our knowledge? Or take an example that is very simple and very close to the objections mentioned above: that Molière’s personage who wonders when hearing that he speaks prose.

Summing up: When, e.g., a property is reduced to a kind of intension as a function from possible worlds (and times) to classes of objects, then this reduction is part of what is called theoretical reconstruction, which can be construed as a sort of explanation and which can help in explaining other phenomena. The way the instantiations of the property are perceived is independent of the theoretical reconstruction in question.

Another objection can be found in [Swoyer], 14: “Properties are more useful in semantics than intensions because intensions are still too coarse-grained to explain many semantic phenomena involving intensional idioms.”

Yes, of course. Therefore Bealer (e.g. in [Bealer 1]) has distinguished two kinds of intensions (see [Bealer 1], 185):

i) qualities, connections and conditions (“that pertain to the world”) and

ii) thoughts and concepts (“that pertain to thinking”)

Independently of Bealer’s way of defining and handling these two kinds of intensions we can state that the intensions sub i) correspond to intensions in the PWS semantics: they are functions from worlds (and times) – not for Bealer, who leaves them undefined, primitive. The ‘intensions’ sub ii) correspond to what Cresswell baptized as ‘hyperintensional’ objects and Tichý defined as constructions (see below), which is in harmony with [Bealer 2], 141: “[l]ogically complex intensions (i.e., the ii)-kind, P.M.) are identical if they have the same complete analysis trees”.

In other words, the i) kind intensions are not structured, unlike the ii) kind intensions.

For Bealer, the fact that the PWS intensions cannot be used in every analysis serves as an argument for a type-free algebraic approach (as confronted with Church’s typed approach connected with the possible-worlds semantics. See [Bealer 2], 145: “[t]here us good reason to have a unified semantic method in which the full spectrum of candidate conceptions can be represented and compared with one another. The algebraic semantic method does this; the possible-worlds method does not.”

Swoyer (p.15) seems content with some objections to intensional approach and concludes: “[i]t is much better to view properties (including relations, and perhaps propositions) as primitive entities.”

Now I would like to show that a unified semantic method distinct from Bealer’s algebraic approach can be defined so that properties remain to be PWS intensions and the cases of ‘hyperintensionality’ are handled in full harmony with the overall approach. I mean transparent intensional logic (TIL) founded by Tichý (see, e.g., [Tichý 1], [Tichý 2], [Materna 2004] ). TIL makes it possible to construe properties as being a kind of intensions and to handle the ‘hyperintensional cases’ in terms of constructions of intensions.

2.2 Fundamentals of TIL

A very good general characteristics of TIL can be found in B. Jespersen’s article The Foundations of Tichý’s Logic, which appeared as a sort of introduction to [Tichý 2], 9-23. We will begin with quoting from [Tichý 1], 194-195:
"It is to Frege that we owe the insight that the mathematical notion of function is a universal medium of explication not just in mathematics but in general. To explicate a system of intuitive, pre-theoretical notions is to assign to them, as surrogates, members of the functional hierarchy over a definite objectual base. Relations between the intuitive notions are then represented by the mathematically rigorous relationships between the functional surrogates."

This ‘explication of explication’ is what we need when being confronted with the objections mentioned above under a). When we claim, for example, that properties are best explicated as functions from worlds to chronologies of classes of objects (of the given type) then we do not claim thereby that, e.g., recognizing the taste of rhubarb is a mental process that as if consisted in finding the respective function defined on the huge set of possible worlds and returning another huge function from time moments to classes. Maybe those naïve fantastic images are responsible for the frequent manifestations of the aversion to the idea of possible worlds. So let us get rid of any psychological interpretations of the following explications. We should rather notice that they could prove their worth in the logical analysis of natural language, i.e., in the analysis that makes it possible to detect logical consequences of the claims articulated in a natural language.

The components of the pre-theoretical idea are:

i) **Objectual base.** The indefinable notions are truth-values, individuals + (unlike Frege) possible worlds and time moments (“times”). The respective surrogates are
- o  (the set \{T, F\}, interpreted as **TRUE, FALSE**, respectively)
- i  (the set called *Universe*, the members **individuals**)
- τ  (the set of **time moments**, “times”, also: **real numbers**)
- ω  (the set of **possible worlds**, see below)

ii) **Functions over the objectual base** (Simple hierarchy of types)
1. The members of the objectual base are types of order 1.
2. Let \( \alpha, \beta_1, \ldots, \beta_m \) be *types of order 1*. Then the set of partial functions with domain \( \beta_1 \times \ldots \times \beta_m \), \( m \geq 1 \), and values in \( \alpha \), denoted by \((\alpha \beta_1 \ldots \beta_m)\), is a *type of order 1*.
3. Only what falls under 1. and 2. is a *type of order 1*.

**Remark:** The notion of possible worlds is frequently suspected as being a controversial notion. In [Tichý 1], 199-200, we can find the best explication thereof.

iii) **Constructions.**
The most fundamental category of constructions can be informally characterized as follows: Constructions are abstract procedures like algorithms. As procedures they are complex (structured), i.e., consist of at least one ‘step’, ‘instruction’, being themselves again an ‘instruction’ (so not being a set of its subprocedures). As abstract they are not spatio-temporally localizable, they are, e.g., not time consuming. (Notice the analogy with computer algorithm, which is the ‘sense’ of a computer program, see [Moschovakis].) It follows that constructions are extra-linguistic entities.

The choice of the procedures to be called constructions in TIL has been inspired by the typed \( \lambda \)-calculus, where the procedures are essentially reducible to two of them: constructing a function and constructing the value of the function on the given arguments. The artificial expressions used in TIL to handle constructions cannot be construed as being expressions of a formal language: the expressions of the latter have to be interpreted and in general we can expect more interpretations. The expressions of the ‘constructional language’ (if you like) in TIL are only a kind of shorthand.
determining unambiguously the procedure (= construction) in question. Both the choice of the objectual base and of the kinds of construction are in a sense flexible, i.e., dependent on the kind of problem to be solved by a TIL analysis. Thus analyzing mathematical languages we do not need possible worlds, for arithmetic of natural numbers the type $\tau$ has to be replaced by the type $\nu$ (natural numbers), in some cases a type for ordered tuples is needed; as for constructions, we will confine ourselves to defining four most important ones (Tichý himself defines six of them). Informally:

1. **Variables** are ‘atomic incomplete constructions’: they are not letters, i.e., expressions (letters $x$, $y$, ..., $p$, $q$, ..., $f$, $g$, ...etc. are names of variables); denumerably infinitely many variables are at our disposal for any type; total functions (‘valuations’) associate every variable with just one object of the given type. So we say that variables $\nu$-construct, where $\nu$ is the parameter of valuations. The same holds for any construction which contains some (free) variable.

2. **Trivializations** are most simple complete constructions: they construct the presented object without any change. That TIL can represent not only using but also mentioning an object is enabled by trivialization. Where $X$ is an object or construction, trivialization $0X$ constructs just $X$.

3. **Composition** is a construction that determines the value of a function on the given arguments. Where constructions $X$, $X_1$, ..., $X_m$ $\nu$-construct objects of the type $(\alpha_1 \beta_1 ... \beta_m)$, $\beta_1$, ..., $\beta_m$, respectively, the composition $[XX_1...X_m] \nu$-constructs the value of the function $\nu$-constructed by $X$ on the arguments $\nu$-constructed by $X_1$, ..., $X_m$, respectively, or $\nu$-constructs nothing (partiality!) (is ‘$\nu$-improper’).

4. **Closure** $\nu$-constructs a function (‘$\lambda$-abstracting’). Let $x_1$, ..., $x_m$ be pairwise distinct variables ranging over $\beta_1$, ..., $\beta_m$, respectively, and let $X$ be a construction $\nu$-constructing objects of the type $\alpha$. Then $[\lambda x_1...x_mX] \nu$-constructs a function (type $(\alpha_1 \beta_1 ... \beta_m)$).

The exact definitions can be found, e.g., in [Tichý 1].

**iv) Higher-order types.** (Ramified hierarchy of types.)

See again [Tichý 1]. The principle is: constructions $\nu$-constructing members of a type $\alpha$, where $\alpha$ is a type of order $n$, are constructions of order $n$ (this is a little simplifying formulation, the exact definition is inductive). Now $*_n$ denotes the collection of all constructions of order $n$. Higher types are defined as follows:

1. $*_n$ and types of order $n$ are types of order $n + 1$.
2. If $\alpha$, $\beta_1$, ..., $\beta_m$ are types of order $n + 1$, then $(\alpha_1 \beta_1 ... \beta_m)$ is a type of order $n + 1$.
3. (Nothing other...) 

So every construction is a member of some $*_n$.

**2.3 Examples**

Each logically reachable object gets as its surrogate a function. The members of the objectual base are sets of nullary functions, classes and relations are the respective characteristic functions, so where $\alpha$ is a type, $(\alpha \alpha)$ is the type of classes of $\alpha$-objects etc. A special kind of objects are intensions: for a type $\alpha$ they are objects of the type $((\alpha \alpha) \alpha)$, abbreviated $\alpha_{\tau \alpha}$. So they are functions mapping possible worlds to chronologies of $\alpha$-objects.
Intensions are independent of possible worlds and times but their values do depend on the state of the world at the given time moment.

Some frequently handled Intensions:

Propositions. Propositions are conceived of in the sense of PWS: they (their surrogates, that is) are functions that associate every possible world with a chronology of truth-values, thus their type is $\omega_\alpha$ (remember: $((\omega_\alpha))$). No need of Russellian structured propositions arises: one and the same proposition can be constructed in infinitely many ways, but it is constructions what is structured. To adduce an example: the proposition associated with the sentences

*Mount Everest is higher than MtBlanc*

is one and the same, whereas the constructions are distinct: Let ME and MB be Mount Everest, MtBlanc, respectively (so ME, MB/ $\iota$) and H, L be the relations *higher than, lower than*, respectively. The constructions that play the role of Frege’s and Bealer’s ‘thoughts’ and construct the respective proposition are $(\lambda w \lambda t [H_{wt} ME MB], \lambda w \lambda t [L_{wt} MB ME])$, where $X_{wt}$ stands instead of $[[Xw][t]]$.

Another example should bring us closer to our main problems.

We have already mentioned that (for any type $\alpha$) $(\omega_\alpha)$ is the type of the classes of the objects of the type $\alpha$. So the expression *prime number* denotes the class of prime numbers and this class is of the type (i.e., belongs to the type) $(\omega_\alpha)$ (supposing that primes are a subclass of real numbers). Now could we find some natural example when $\alpha$ is $\iota$?

Well, we can assume that particular mountains are individuals (as we have assumed in the preceding example); then we can say that the class with the members Mount Everest, MtBlanc, Popocatepetl is an object of the type $(\omega_\iota)$. But the expression *mountain* does not denote a class: if it did, it would have to denote distinct classes when some object ceased being a mountain (cf. [Tichý 2], 314-315), and being a mountain would be a necessarily ascribed attribute, since membership in a class is independent of any empirical event. We could, of course, say that the expression *mountain* denotes the class of ‘actual mountains’ but denotation should be a semantic relation, i.e., a relation that holds a priori. We can therefore abolish the nonsensical identification of denotation and reference admitting that the class of ‘actual mountains’ is the reference of the expression *mountain* in the actual world and time, and say that this expression denotes a property of individuals.

We will first generalize and then return to properties.

The fundamental intuition connected with the approach of TIL to analyzing language expressions is that any expression that can be rightly called ‘empirical’ denotes a non-trivial intension, i.e., an intension whose values are distinct in at least two worlds/times. The reference of such an expression in the given world W at the given time T is the value of its denotation in W at T. Thus the denotation is connected with the (disambiguated) expression necessarily, unlike its reference, whose connection with the expression is dependent on the state of the world at the given time and is therefore contingent. (As for a more detailed argumentation see also [Materna 1, 2]).

Summarizing: Classes are extensions, properties are intensions. And just as there can be classes of objects of any type we can have properties of any types: their type schema is $(\omega_\alpha)_\tau$ for any type $\alpha$. 
Question: Can we speak about properties in the area of mathematical objects, as Gödel did?

3. Concepts

First of all it is necessary to stress that there is a sense in which mathematical objects can be said to possess properties, and that Gödel was surely not interested in the properties of this kind. I mean properties like: to be an equation solved by Gauss, to be a prime known before 1950, to be less than the number of the stars in our Galaxy etc. etc.

So which properties were of interest for Gödel?

Inspecting II.[G] and V.[G] we can see that we can consider a) the property of all pairs and of nothing other called the number two and b) the property of binary relations called transitivity.

Ad a) The Fregean conception of natural numbers (see [Frege 1]) enables us to define them as ‘properties of concepts’, where Fregean concepts are functions with values in (as we have it) the set \( \omega \). The key definition (§ 68 of [Frege 1]) does not use the term ‘property’, instead we read

“The number that belongs to the concept \( F \) is the extension of the concept ‘equinumerous with the concept \( F \)’.”

Now we can see that the property of all pairs (and of nothing other) is – as well as the property transitivity, where Gödel explicitly states the problem – type-theoretically polymorph: let \( \alpha \) be again any type. \( F \) is then of the type (\( \omega \alpha \)). The number belonging to \( F \) is the class of all (\( \omega \alpha \))-objects that are equinumerous with \( F \). Let however \( \alpha \) be any type, the Fregean number is of the type (\( \omega (\omega \alpha) \)), i.e., it is a class (of classes), an extension, not a property. In particular, the number two is a class (of classes with just two members, if we ignore the sophisticated way of avoiding vicious circle in definition).

Ad b) For Gödel, transitivity is a concept, and since he says (I. [G]) that concepts are properties … transitivity is a property as well. But again, let \( \alpha \) be any type, transitivity is a class of binary relations (it can be constructed – omitting trivializations, using infix notation and other abbreviations – by

\[
\lambda r \ ( \forall xyz ((rxy \land ryz) \supset rzx) ),
\]

where \( x, y, z \) range over \( \alpha \) and \( r \) over (\( \omega \alpha \)).

Again, no dependence on extra-linguistic factors: transitivity (or: transitivities) is no property.

A very simple solution of the problem could be, of course, that Gödel simply used the term “property” as denoting properties as well as classes; after all, he was interested primarily in mathematics, and the distinction between intensions and extensions (in the PWS sense) finds no application in mathematics. Also, we must realize that this ‘generalized’ use of the term ‘property’ is very frequent.

Using this generalized term ‘property’ and talking about properties of mathematical objects we can simply identify any property of a mathematical object \( M \) with one of the following cases:

1) Any set \( S \) such that \( M \in S \); let \( E, P \) be respectively the set of even numbers, of prime numbers. Then \( E, P \) are properties of the number 2. Or: Let \( I \) be the set of the sets whose cardinality is \( \aleph_0 \). Then \( I \) is a property of \( E \) and of \( P \).

2) Let \( M \) be a set. Then any superset of \( M \) is a property of any member of \( M \). Thus where \( O \) is the set of odd numbers, the set \( O \cup \{2\} \) may be said to be a property of \( P \).
Of course, 2) is a consequence of 1). I adduce this case because it seems that it corresponds to our way of talking: we can say “a property of the primes is that they are odd or equal to 2”.

All these considerations are rather banal; to get some more interesting results we will first return to our type-theoretical definitions that help us in classifying the objects we speak about.

We have said that empirical expressions denote non-trivial intensions, so non-empirical expressions denote either extensions or trivial intensions. The traditional interpretation of Frege’s schema has it that if an expression denotes an object, then intensions in the sense of PWS occupy the place of the Fregean sense (we use today the term meaning). This interpretation of Frege’s schema would be perhaps approved by Frege but is connected with one próton pseudos (Tichý: ‘Fregean Thesis’), viz. the claim that an (empirical) expression denotes not what is given by linguistic convention but what happens to be determined by it due to the contingent state of the world. According to this claim the morning star does not denote the condition fulfilled by an individual if it plays the role of the morning star: it denotes Venus, which only happens to play this role; an empirical sentence does not denote the truth conditions, it denotes the truth-value contingently corresponding to these truth conditions; the expression a dog does not denote a property, i.e., the conditions satisfied by a class if it is the class of dogs, it denotes the class of ‘actual dogs’, which only happens to fulfill the condition in the given world/time, etc. etc.

Now when the non-semantic character of the Fregean denotation (which is properly speaking a reference in the given world-time) is shown a follower of Frege can raise a query: If intensions are denotations rather than meanings, what should be the meaning of an expression?

We can raise however a similar query: If intensions could not play the role of a meaning in the area of mathematical objects (since no intensions are needed in this area), what should be the meaning of a mathematical expression?

We will show that the answer to the preceding, more general question will be at the same time the answer to the second, specific question.

Let us begin with a most simple example.

Consider the sentence

(1) \[3 \text{ is a prime number.}\]

We will surely agree (as Frege would) that this sentence denotes a truth-value (in this case the value \(T\)). The idea of meaning (sense) has been introduced to explain the way from an expression to its denotation. (Frege has argued for this explanatory role of meaning when he stated that there can be more meanings that result in the same denotation.) So the meaning of the sentence (1) must be some way, some procedure, not however a concrete process (called often ‘procedure’) but an abstract procedure. As a procedure it must be structured, but the idea of structured meaning – suggested in a rudimentary form by Carnap in 1947 – has been first exactly formulated not in 1975 (Cresswell) but in 1968 and 1969 (see [Tichý 2], 77-109).

We have seen that abstract procedures are modeled in TIL as constructions. To find the meaning of an expression E – be it an empirical or a mathematical expression – can be therefore identified with finding a construction that underlies E. An attempt to describe a systematic method of finding “the best construction” can be found in Philosophia 32:1-4 (May 2005) 157-182 (Duží, M. , Materna, P.: Parmenides Principle.)
We will offer a construction that underlies the sentence (1) if the following type-theoretical analysis is applied:

3 is a name of the number 3. If the objectual base from 2.2 is accepted 3 gets the type \( \tau \). (Since the context is about primes a better typing would be \( \nu \), i.e., the type of natural numbers. Just now this distinction is not important.) What about a prime number? Obviously this is a predicate that denotes the class \( P \) of primes, so the type of \( P \) is \( (\sigma \tau) \) (see however the note in the parentheses above). The procedure that will lead to the truth-value (here \( T \)) is represented by the construction

\[
(1') \quad [^0P \, ^03].
\]

Our definitions say that \( (1') \) is a composition: \( ^0P \), as a trivialization, constructs the set of primes, and \( ^03 \) the number 3; the composition applies the former to the latter, and since 3 really is a member of \( P \), the outcome is \( T \).

**Remark:** As for constructing (by trivialization) infinite classes, this should not be interpreted as constructing the actually infinite class. This problem is dealt with in [Duží, Materna].

Thus the question *what is the meaning of (1)* is answered: it is an abstract procedure that consists in applying the class of primes – or more precisely the characteristic function of this class - (as determined by the simple procedure of trivialization) to the number 3 (as determined by the simple procedure of trivialization).

The core of this conception of meaning is well illustrated by this example. No doubt, this example is extremely simple and, besides, some problems connected with trivializations were not mentioned here (see however [Materna 2] ). But we can say that a simple example demonstrated the main point.

To exploit this conception to define concepts let us define bound variables in TIL. The exact inductive definition can be found in [Materna 1, 2], here we will use a ‘prose’:

There are two ways the variables can be bound in a construction.

a) Let \( X \) be a construction. Any variable occurring in \( ^0X \) is \( ^0 \text{bound in} \ ^0X \).

b) Let \( X \) be a construction of the form \([\lambda x_1 \ldots x_m Y]\). Any occurrence of \( x_i \) for \( 1 \leq i \leq m \) is \( \lambda \text{-bound in} \ ^X \) unless it is \( ^0 \text{bound in} \ Y \).

Any occurrence of a variable \( x \) in a construction \( X \) which is not \( ^0 \text{bound or} \ \lambda x \text{-bound is free in} \ ^X \). Any variable having at least one free occurrence in \( X \) is free in \( X \).

A construction with no free occurrences of a variable is called a closed construction.

There are two interesting (from our present viewpoint) kinds of equivalence of closed constructions (CC):

i) A CC \( X \) is \( \alpha \)-equivalent to a CC \( Y \) iff \( X \) differs from \( Y \) just by replacing \( n \) \( (n \geq 0) \) \( \lambda \)-bound variables via a correct substitution.

ii) A CC \( X \) is \( \eta \)-equivalent to a CC \( Y \) iff \( X \) arises as an \( \eta \)-reduction of \( Y \) or vice versa.

Each of these equivalences induces an equivalence class of constructions, say, \( \alpha \)-class and \( \eta \)-class, respectively. A normalization process (see [Horák] ) selects from the \( \alpha \)-class that construction which contains the first unused variable, and from the \( \eta \)-class that construction which is no more \( \eta \)-reducible. Horák has proved that an unambiguously selected construction arises if \( \alpha \)-normalization is applied to \( \eta \)-normalized construction: \( \text{N}(X) = \text{N}^\alpha(\text{N}^\eta(X)) \).

So let \( X \) be a CC. The concept determined by \( X \) is \( \text{N}(X) \). The other, not normalized constructions equivalent to \( X \) are said to point to the concept \( \text{N}(X) \).

This operation makes it possible to say that any concept is a closed construction and to use any CC as a representative of a concept.
Thus the shift made in the Fregean schema by TIL consists in placing constructions/concepts where Frege’s Sinn took place, and using the PWS intensions as denotations. (Frege’s Bedeutung was the reference rather than denotation.)

No essential problem is connected with conceiving meanings of mathematical expressions as being concepts (found by logical analysis). Moreover, the place of a Fregean sense/meaning in the case of empirical expressions is solved as well. To illustrate this claim let us adduce again a simple example (admitting that some degree of simplification is present): consider the expression

(2) \( \text{being a mountain higher than MtBlanc} \).

As an empirical predicate it obviously denotes (not a class but) a property of individuals, so the type is \((\omega_1)_{\tau_0}\). Which other types can we associate with the subexpressions of the predicate? We know already that Mountain is of the type \((\omega_1)_{\tau_0}\) as well. (Notation: M/\((\omega_1)_{\tau_0}\).) Also, Mt(Blanc) is obviously an individual, so Mt/\(\iota\). Higher than is a binary (empirical) relation of individuals, so H/(\((\omega_1)\)\(\omega\)). Our task is now to find such a construction that would construct a property of individuals and were derivable of constructions of M, Mt, H.

Omitting obvious brackets we can begin with stating that in general, the following schema of constructions is a schema of constructing properties of individuals:

\[
\lambda w \lambda t \lambda x X,
\]

where \(w, t, x\) range over \(\omega, \tau, \iota\), respectively and \(X\) \(v\)-constructs a truth-value. (According to our definitions – using \(\to\) as “\(v\)"-constructs objects of the type” – we have:

\[
X \to \omega, \lambda x X \to (\omega_1\iota), \lambda w \lambda t \lambda x X \to (((\omega_1\iota)\omega) (= (\omega_1)_{\omega_0}).
\]

Thus \(X\) must \(v\)-construct a truth-value and contain at most \(w, t, x\) as free variables. The ‘truth-conditions’ can be verbally formulated as follows: to be a mountain and to be higher than MtBlanc. So we need one construction more, namely a construction of conjunction, \(\land\) \((\omega_0)_{\omega_0})\).

The resulting construction is:

\[
(2') \lambda w \lambda t \lambda x [0 \land [0^6 M_{\omega_1} X^6 H_{\omega_1} X^6 Mt]].
\]

Thus while (2) is an expression, (2') is the concept (do not forget: a procedure!) that is the meaning of (2), and the denotation of (2) = what is constructed by (2'), i.e., a property of individuals. (The reference of (2) = the value of the denotation of (2) in the actual world and time, imagine the newest (still valid) atlas of mountains with marked mountains higher than MtBlanc.)

Notice that the main distinction between meaning and denotation in this conception consists in the fact that the meaning (concept) is structured unlike the denotation. The denotation is always a set-theoretical object (at least in the most frequent cases where an expression does not denote a construction – these latter cases can be handled within the ramified hierarchy, where it holds that the meaning is more structured than the denotation), we must see that intensions – although something new for some philosophers – are mappings, so set-theoretical objects.

At this opportunity let me comment the frequently occurring example which is used to show that properties are not extensions: I mean the heart – lungs example, where the argumentation looks as follows:

It is obvious, we are told, that the class of those animals who possess lungs is the same class as that of the animals possessing heart. All the same, our intuition says us that having heart is not the same property as having lung. Thus two distinct properties can share their extension, i.e., a class.
The use of this argument is a little sloppy. The class of those animals etc. is not well-defined. Two interpretations are possible: a) the sameness holds in some worlds/times only (among which the actual world/time is), b) the sameness is necessary, i.e., holds in all worlds/times. It is only the case a), where the argument is valid. The case b) is more interesting: we accept this case if we feel that having lungs without heart or heart without lungs is not something what can happen (so that the co-possession of both organs is not a contingent, empirical phenomenon) but something what is logically impossible (like having the father without having the mother). But if we accept b) then we must see that to have lungs and to have heart are two distinct concepts of one and the same property, since two properties must differ in their population in at least two worlds/times.

This comment is characteristic of the general situation. It was Bolzano (see [Bolzano]), who – to my knowledge first – sharply distinguished between concepts (Begriffe) and, say, classes. If, for example, a class is defined as the class of equilateral triangles, and then as the class of equiangular triangles, then the definitions determine two distinct concepts but the class is the same. The respective concepts are two distinct procedures which define one and the same class.

Now, when the place of Frege’s Sinn has been occupied by concepts, we can be more fine-grained than many current attempts at analysis of natural language expressions. Thus we can not only show that two distinct intensions can share their value in some worlds/times but also that two distinct concepts can construct one and the same intension. This latter idea can be viewed as a formulation of ‘hyperintensionality’ (see [Cresswell1], [Cresswell2]; Cresswell’s approach can be however criticized – see [Tichý 2], 835-841).

To say that two expressions are equivalent means that they denote the same object.

To say that two equivalent expressions are synonymous means that they, moreover, express the same concept.

The expressions that denote distinct intensions whose value in the actual world + time happens to be the same can be called coincident (see [Materna 1, 2]).

This classification is more fine-grained than most of such classifications. Some examples can illustrate this claim:

I. Mathematical claims – if true – denote the truth-value T. So all true mathematical claims are mutually equivalent. Does it mean that they express one and the same (Davidson’s) ‘Big Fact’?

First of all, the true mathematical claims are not synonymous. They express various distinct constructions/concepts of T. All the same, one could object that what counts in the case of ‘Big Fact’ is denotation rather than meaning. Then two solutions are viable:

i) In his [Tichý 1], 224, Tichý decided that semantic relations can be reduced to denotation but then what is denoted are constructions. That there cannot be any Big Fact in mathematics either is then obvious. See also Tichý “Constructions as the Subject-Matter of Mathematics”, [Tichý 2], 873-885.

ii) The original notion of denotation can be retained but due to the specific character of mathematics we can say either that what semantically counts in mathematics is expressing rather than denoting, or that mathematical expressions denote constructions (see [Materna 3]). In both cases there is no Big Fact.

II. What about Frege’s morning star vs. evening star example? Frege would probably say that these expressions are equivalent: according to him they both denote one and the same object, viz. Venus. We have seen, however, that they denote distinct objects: as empirical expressions they denote intensions, here individual roles (or offices, as
Tichý used to say, i.e., objects of the type \( \tau_{\omega} \): functions, that is, that associate with every world-time at most one individual. Venus is from this viewpoint the reference of both expressions, whereas the roles are distinct: the condition an individual must fulfill to be the morning star differs from such a condition to be fulfilled by the candidate for being the evening star. (The meanings are then the respective concepts, i.e., procedures constructing the respective roles. These concepts can be made ‘visible’ when we replace morning star and evening star with the descriptions like the first celestial body visible in the evening or suchlike.)

Thus we can see that morning star and evening star are only coincident expressions. Their agreement is the weakest one: they happen to share their reference.

The morning star – evening star example can be generalized: according to Frege any two true sentences would be equivalent since they are both true. From our viewpoint two empirical true sentences are equivalent only if one of them is the result of a logical transformation of the other one, i.e., if they denote the same proposition. Otherwise they are only coincident. The first case can be illustrated by the sentences Jupiter is the biggest planet, It is not the case that Jupiter is not the biggest planet, or even Mt Blanc is lower than Mount Everest, Mount Everest is taller than Mt Blanc.

The second case: Jupiter is the biggest planet, Warsaw is the capital city of Poland.

III. The following example is important for our analysis of the problem formulated in the title of the present paper. I apologize for some simplification connected with a little uncritical use of trivializations (see [Materna 1, 2] for this problem).

Let \( \nu \) be the type of natural numbers, Div(isible) is of type \((\nu \nu)\), Card(inality) of the type \((\nu \nu)\), type of 1, 2 is \( \nu \), of \( > \) and of \( = \) is \((\nu \nu)\). \( x, y \) range over \( \nu \), \( \land, \lor \) are respectively conjunction, disjunction, so type \((\oo)\). Compare two concepts:

\[
C_1 \quad \lambda x \left[ (x > 0) \land (x > 0) \right] \land \lambda y \left[ (\text{Div } x y) \land (y = y \land 0) \right]
\]

\[
C_2 \quad \lambda x \left[ (x = (\text{Card } y) \land (\text{Div } x y)) \land (y > 0) \right]
\]

C1 constructs the class of natural numbers greater than 1 that are divisible just by themselves or by 1. C2 constructs the class of natural numbers \( x \) such that the class of the factors of \( x \) contains just two members. Clearly, the class constructed by C1 and C2 is one and the same (and is called prime numbers). Now reading II.[G] we must state the following claim:

C1 and C2 represent two distinct definitions (“of the form \( \alpha(x) = \phi(x) \)” so that in the “constructivist sense” we have two notions. For Gödel there would be one concept only – “one common ‘form’ or ‘nature’ “. Comparing with I.[G], where concepts are “properties and relations of things…” we have to admit that in the “constructivist” sense and also in the sense of our definitions (and for Bealer as well) C1 and C2 are distinct concepts whereas the Gödelian ‘concept’ would reduce to the class of prime numbers.

Defining

D1 **Prime numbers** are (natural) numbers that are divisible just by themselves and by 1 and are greater than 1 (see C1)

D2 **Prime numbers** are (natural) numbers that possess just two factors (see C2) we either define one and the same concept (Gödel’s view) or introduce two equivalent concepts (Bealer, TIL, also Bolzano). The problem with the former case is that the Gödelian ‘concept’ must be either simply a class or otherwise is very indefinite.

It seems that Parsons in [Parsons] is also somewhat uncertain. He says:
“By ‘concepts’ Gödel evidently means objects signified in some way (italics mine. P.M.) by predicates”. Parsons tries to offer some interpretation via some hints (relevance of simple theory of types, impredicative theories of properties) but then he has to state
“Gödel’s remarks about realistic theories of concepts…have an inconclusive character; no available theory satisfies him.” (110).

Gödel himself is however a little more definite when he writes
“the meaning of the term ‘concept’ seems to imply that every propositional function defines a concept” (139),
which, of course, can be conceived of in the “constructivistic” as well as in the set-theoretical sense, dependently on whether two equivalent propositional functions do or do not define two different concepts. Commenting, however, Russell’s no-class theory from his realistic viewpoint he recommends
“to make the meaning of the terms ‘class’ and ‘concept’ clearer and to set up a consistent theory of classes and concepts as objectively existing entities.” (140)

4. Summing up: Properties of mathematical objects

A. For TIL, properties are not primitives, they are definable in terms of possible worlds and times. From our preceding definitions we know that, in general, properties of the objects of a type $\alpha$ are objects of a type $(\omega\alpha)_{\tau\omega}$. Empirical properties are non-trivial intensions (see 2.3).

B. Mathematical objects – be it numbers, functions, sets, proofs etc. – are never non-trivial intensions.

C. Trivial intensions can be handled as extensions. A proposition whose truth-value is $T$ in all worlds-times can be replaced simply by $T$, an intension that takes as its value the number 0 in all worlds-times behaves as the number 0. Therefore, mathematical objects can be handled as extensions without any loss of information.

D. Mathematical objects possess indefinitely many empirical properties (see 3.). No such property is ever of interest for a mathematician qua mathematician.

E. Let a mathematical object $X$ be of a type $\alpha$. From C. and D. it follows that the only properties of $X$ that are of interest for a mathematician are objects of a type $(\omega\alpha)_{\tau\omega}$. (If $X$ is an $n$-tuple the type of such a property is $(\omega\beta_1…\beta_n)$, etc. etc., of course.) Thus the mathematically relevant properties of mathematical objects are classes.

F. Let us accept the following assumption (see 3.,I,i): The mathematical objects themselves are constructions. The properties of such objects would be again classes (of ‘higher orders’), e.g., (primitive) recursiveness, finiteness, Turing solvability etc. In many cases such properties-classes will be type-theoretically polymorph, as Gödel seems to suggest in V.[G].

Let us consider for a while the point F.: The class recursiveness is the same class as Turing solvability. It is well-known however that the proof of this identity belongs to important results of mathematical logic. So this identity is not of the kind

\[ 2 = 2 \]

but rather of the kind

\[ 2 = \text{the least prime} \]
(Frege’s $a = b$ as against $a = a$. Notice that Frege’s first place in [Frege 2] where the term sense is introduced is the mathematical example with medians.)

Thus being recursive is the same class as being Turing solvable but both differ (not by being distinct intensions, distinct properties but) by being distinct concepts (in our sense, i.e., kind of construction). This is – on a higher level – the same phenomenon as the difference between the two concepts of primes (see D1, D2).

So let us consider the problem of properties of mathematical objects from the viewpoint of the ‘hyperintensional’ perspective.

In both examples just mentioned we have two options:

Example 1:

D1 Prime numbers are (natural) numbers that are divisible just by themselves and by 1 and are greater than 1 (see C1)

D2 Prime numbers are (natural) numbers that possess just two factors (see C2)

Option 1: Prime numbers are a class definable in various distinct ways.

Option 2 (a preliminary formulation): Any prime number has the property being greater than 1 and being divisible just by itself or by 1, and also the property possessing just two factors.

Applying our points E. and F. we can accept the option 1: According to these points there are no distinct properties mentioned in Option 2. But: It does not seem that what Option 2 “wants to say” is the same banality as stating that primes make up a class (be it defined any way). Option 2 speaks (in a not just correct way) about the procedures that result in determining primes: it speaks (in our terms) about concepts. So let us reformulate Option 2:

Option 2’: Any prime number falls under the concept being greater than 1 and being divisible just by itself or by 1, and also under the concept possessing just two factors.

Since these two concepts are really distinct the new formulation is correct.

An analogous consideration can be connected with the second example.

Now we can ask: Why Gödel did not accept the ‘constructivist’ view according to which two distinct definitions determine two distinct concepts? Why did he instead choose to talk in a rather vague way about a common ‘form’ or ‘nature’ (cf. II. [G])?

In my opinion, the reason can be seen in Gödel’s realism: he connected the idea of constructing (via definitions) with the subjective factor, whereas the objects constructed were clear representatives of objective extra-linguistic abstract objects. Constructions in the sense of TIL are, however, objective abstract procedures, they could be acceptable for Gödel.

**Conclusion:**

In terms of the well-defined notions of TIL we should not speak about properties of mathematical objects: what we mean in particular contexts is either classes or concepts, the latter in the structured, procedural way.

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