Multiple Vortices for the Shallow Water Equation

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Abstract

In this paper, we construct stationary classical solutions of the shallow water equation with vanishing Froude number $Fr$ in the so-called lake model. To this end we need to study solutions to the following semilinear elliptic problem

\[
\begin{cases}
-\varepsilon^2 \text{div}(\nabla u) = b(u - q \log \frac{1}{\varepsilon})^p, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

for small $\varepsilon > 0$, where $p > 1$, $\text{div}(\nabla q) = 0$ and $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain.

We showed that if $\frac{q^2}{b}$ has $m$ strictly local minimum(maximum) points $\bar{z}_i$, $i = 1, \ldots, m$, then there is a stationary classical solution approximating stationary $m$ points vortex solution of shallow water equations with vorticity $\sum_{i=1}^m \frac{2\pi q(\bar{z}_i)}{b(\bar{z}_i)}$. Moreover, strictly local minimum points of $\frac{q^2}{b}$ on the boundary can also give vortex solutions for the shallow water equation. As a further study we construct vortex pair solutions as well.

Existence and asymptotic behavior of single point non-vanishing vortex solutions were studied by S. De Valeriola and J. Van Schaftingen in [9].

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1 Introduction and Main Results

We consider fluid contained in a basin by a uniform gravitational acceleration $g$ and fixed vertical lateral boundaries(i.e. no sloping beaches). Suppose that $(x, y)$ is horizontal spatial coordinate which is confined to a fixed bounded domain $\Omega$ with boundary $\partial \Omega$. The vertical coordinate is chosen so that the mean height of the fluid’s free upper surface is at $z = 0$. Let $z = -b(x, y)$ give the fixed bottom topography, so $b$ is a strict positive function over $\Omega$. Let $z = h(x, y)$ be the free upper surface. We assume that both $b$ and $\partial \Omega$ vary over distances $L$ which are large compared to the mean depth $B$, that is, the ratio $\delta = \frac{L}{B}$ is small.

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Let \( u \) and \( w \) denote the horizontal and vertical components respectively of the fluid velocity. We will consider only those motion for which \( u, w \) and \( h \) each vary in \((x, y)\) over distances \( L \), in other words, we will make the long-wave approximation. The "Froude number" is denoted as \( Fr = \frac{U}{\sqrt{gB}} \), where \( U \) is the characteristic magnitude of \( u \). We will consider the case of small "Froude number" \( Fr \) and \( h \) is small compared to \( B \). In such cases, from [1, 3, 4, 15], the leading-order evolution of \( u(x, y, t) \) and \( h(x, y, t) \) will be governed by equations that have the non-dimensional form

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v &= -\nabla h, \\
\text{div}(bv) &= 0,
\end{align*}
\]

where \( \nabla \) is the horizontal gradient. Since these equations apply to a domain which is shallow compared to its width and whose free surface exhibits negligible surface motion, they are called the 'lake' equations (see [4], for instance).

The first equation in (1.1) can be rewritten in terms of the vorticity \( \omega = \nabla \times v \) as

\[
\partial_t v + \omega \times v = -\nabla \left( \frac{|v|^2}{2} + h \right).
\]

This model is analogous to the two-dimensional Euler equation for an ideal incompressible fluid and has been recently studied by many authors. For instance, see [1, 3, 4, 15] and the references cited therein.

Recently, De Valeriola and Van Schaftingen [9] studied the desingularization of vortices for (1.1) with stream function method, which consists in observing that if \( \psi \) satisfies

\[
-\text{div} \left( \frac{\nabla \psi}{b} \right) = bf(\psi)
\]

for \( f \in C^1(\mathbb{R}) \), then \( v = \frac{\text{curl} \psi}{b} \) and \( h = -F(\psi) - \frac{|v|^2}{2} \) with \( F(s) = \int_0^s f(s) ds \) form a stationary solution to the shallow water equation. Moreover, the velocity \( v \) is irrotational on the set where \( f(\psi) = 0 \). It is easy to see that if \( \psi_0 \) satisfies \( \text{div} \left( \frac{\nabla \psi_0}{b} \right) = 0 \), then \( v_0 = \frac{\text{curl} \psi_0}{b} \) is an irrotational stationary solution of (1.1). In [9], they studied the asymptotics of solutions of

\[
\begin{align*}
-\varepsilon^2 \text{div} \left( \frac{\nabla \psi}{b} \right) &= b\psi_+^p, \quad \text{in } \Omega, \\
\psi &= \psi_0 \ln \frac{1}{\varepsilon}, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( p > 1 \), \( \Omega \subset \mathbb{R}^2 \) is smooth bounded domain.

To obtain their results, De Valeriola and Van Schaftingen investigated the following problem

\[
\begin{align*}
-\varepsilon^2 \text{div} \left( \frac{\nabla u}{b} \right) &= b(u - q_\varepsilon)_+^p, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( p > 1 \), \( q = -\psi_0, q_\varepsilon = q \ln \frac{1}{\varepsilon}, \Omega \subset \mathbb{R}^2 \) is smooth bounded domain.

More precisely, they first obtained the existence of solutions by using mountain pass lemma and studied the asymptotic behavior of solutions by giving exact estimates to the upper and lower energy bounds of the least energy solutions. As a consequence, they obtained
that the “vortex core” shrinks to a point $x_0$ which is the minimum point of $\frac{\psi_0^2}{b}$. However, it is hard to apply their method to construct multiple vortices for (1.1).

Motivated by [9], our goal in this paper is to construct multiple stationary vortices for shallow water equations. More specifically, we want to find some high energy solutions whose “vortex core” consists of multiple components which shrink to several distinct points in $\Omega$ as $\varepsilon \to 0$ under some additional assumptions on $\frac{\psi_0^2}{b}$.

Our main results in this paper can be stated as follows:

**Theorem 1.1.** Suppose that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. Let $b \in C^1(\bar{\Omega})$, $\psi_0 \in C^2(\Omega)$ be such that $\text{div}(\frac{\nabla \psi_0}{b}) = 0$ and let $\mathbf{v}_0 = \text{curl}\psi_0$. If $\inf_{\Omega} b > 0$ and $\sup_{\Omega} \psi_0 < 0$, then for any given strictly local minimum(maximum) points $\bar{z}_1, \cdots, \bar{z}_m$ of $\frac{\psi_0^2}{b}$, there exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$, there exists a family of solutions $\mathbf{v}_\varepsilon \in C^1(\Omega, \mathbb{R}^2)$ and $h_\varepsilon \in C^1(\Omega)$ of

$$\begin{cases}
\text{div}(b\mathbf{v}_\varepsilon) = 0, & \text{in } \Omega, \\
(\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon = -\nabla h_\varepsilon, & \text{in } \Omega, \\
\mathbf{v}_\varepsilon \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \ln \frac{1}{\varepsilon}, & \text{on } \partial \Omega,
\end{cases}$$

where $\mathbf{n}$ is the unit outward normal. Furthermore the corresponding vorticity $\omega_\varepsilon := \text{curl}\mathbf{v}_\varepsilon$ satisfying

$$\text{supp } \omega_\varepsilon \subset \bigcup_{i=1}^m B(z_{i,\varepsilon}, C\varepsilon) \text{ for } z_{i,\varepsilon} \in \Omega, \ i = 1, \cdots, m$$

and as $\varepsilon \to 0$,

$$\int_{\Omega} \omega_\varepsilon \to -\sum_{i=1}^m \frac{2\pi \psi_0(\bar{z}_i)}{b(\bar{z}_i)},$$

$$(z_{1,\varepsilon}, \cdots, z_{m,\varepsilon}) \to (\bar{z}_1, \cdots, \bar{z}_m).$$

The next result shows that strictly local minimum points of $\frac{\psi_0^2}{b}$ on the boundary $\partial \Omega$ can also give vortex solutions for (1.1).

**Theorem 1.2.** Suppose that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. Let $b \in C^1(\bar{\Omega})$, $\psi_0 \in C^2(\Omega)$ be such that $\text{div}(\frac{\nabla \psi_0}{b}) = 0$ and let $\mathbf{v}_0 = \text{curl}\psi_0$. If $\inf_{\Omega} b > 0$ and $\sup_{\Omega} \psi_0 < 0$, then, for any given strictly local minimum points $\hat{z}_1, \cdots, \hat{z}_n$ of $\frac{\psi_0^2}{b}$ on the boundary $\partial \Omega$, there exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$, there exists a family of solutions $\mathbf{v}_\varepsilon \in C^1(\Omega, \mathbb{R}^2)$ and $h_\varepsilon \in C^1(\Omega)$ of

$$\begin{cases}
\text{div}(b\mathbf{v}_\varepsilon) = 0, & \text{in } \Omega, \\
(\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon = -\nabla h_\varepsilon, & \text{in } \Omega, \\
\mathbf{v}_\varepsilon \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} \ln \frac{1}{\varepsilon}, & \text{on } \partial \Omega,
\end{cases}$$

where $\mathbf{n}$ is the unit outward normal. The corresponding vorticity $\omega_\varepsilon := \text{curl}\mathbf{v}_\varepsilon$ satisfying

$$\text{supp } \omega_\varepsilon \subset \bigcup_{i=1}^n B(z_{i,\varepsilon}, C\varepsilon) \text{ for } z_{i,\varepsilon} \in \Omega, \ i = 1, \cdots, n$$

and as $\varepsilon \to 0$,

$$\int_{\Omega} \omega_\varepsilon \to -\sum_{i=1}^n \frac{2\pi \psi_0(\hat{z}_i)}{b(\hat{z}_i)}.$$
Moreover,
\[ |z_{i,\varepsilon} - \hat{z}_i| \leq C \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|} \right), \quad \text{dist}(z_{i,\varepsilon}, \partial \Omega) \geq \frac{1}{|\ln \varepsilon|^\alpha}, \]
where \( \alpha \) is a positive constant.

**Remark 1.3.** If \( \frac{q^2}{b} \) has strictly local minimum points in \( \Omega \) and on the boundary \( \partial \Omega \), then there is, from Theorem 1.1 and 1.2, a stationary solution of the shallow water equation such that its vorticity set shrinks to corresponding strictly local minimum points.

It is worthwhile to pointing out that although the structure of shallow water equations is very analogous to that of two dimensional Euler equations for an ideal incompressible fluid, the position of vortex for (1.1) exhibits a striking difference with that of the Euler equations. The position of vortex for Euler equation is closely related to Kirchhoff-Routh function. The interested reader can refer to [5, 6, 16, 14] for more results on this problem.

Theorem 1.1 and Theorem 1.2 are proved via the following results concerning problem (1.3):

**Theorem 1.4.** Suppose that \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain. Let \( b \in C^1(\bar{\Omega}), q \in C^2(\bar{\Omega}), \inf_{\Omega} b > 0 \) and \( \inf_{\Omega} q > 0 \). Then, for any given strictly local minimum(maximum) points \( \hat{z}_1, \ldots, \hat{z}_m \) of \( \frac{q^2}{b} \), there exists \( \varepsilon_0 > 0 \), such that for each \( \varepsilon \in (0, \varepsilon_0) \), (1.3) has a solution \( u_{\varepsilon} \), such that the set \( \Omega_{\varepsilon} = \{ x : u_{\varepsilon} - q \ln \frac{1}{\varepsilon} > 0 \} \) has exactly \( m \) components \( \Omega_{\varepsilon,i}, i = 1, \ldots, m \) and as \( \varepsilon \to 0 \), each \( \Omega_{\varepsilon,i} \) shrinks to the point \( \hat{z}_i \).

**Theorem 1.5.** Suppose that \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain. Let \( b \in C^1(\bar{\Omega}), q \in C^2(\bar{\Omega}), \inf_{\Omega} b > 0 \) and \( \inf_{\Omega} q > 0 \). Then, for any given strictly local minimum points \( \hat{z}_1, \ldots, \hat{z}_n \) of \( \frac{q^2}{b} \) on the boundary \( \partial \Omega \), there exists \( \varepsilon_0 > 0 \), such that for each \( \varepsilon \in (0, \varepsilon_0) \), (1.3) has a solution \( u_{\varepsilon} \), such that the set \( \Omega_{\varepsilon} = \{ x : u_{\varepsilon} - q \ln \frac{1}{\varepsilon} > 0 \} \) has exactly \( m \) components \( \Omega_{\varepsilon,i}, i = 1, \ldots, m \) and as \( \varepsilon \to 0 \), each \( \Omega_{\varepsilon,i} \) shrinks to the point \( \hat{z}_i \).

Not as in [9] where (1.3) is investigated directly, we prove Theorem 1.4 and Theorem 1.5 by considering the following equivalent problem of (1.3) instead. Set \( \delta = \varepsilon (\ln \frac{1}{\varepsilon})^{1+p}, \)
\( w = \frac{u}{\ln \varepsilon} \), then (1.3) becomes

\[
\begin{aligned}
-\delta^2 \text{div}(\nabla \bar{w}) &= b(w - q)_{+}, & \text{in } \Omega, \\
 u &= 0, & \text{on } \partial \Omega.
\end{aligned}
\]

We will use a reduction argument to prove Theorem 1.4 and Theorem 1.5. To this end, we need to construct an approximate solution for (1.4). For the problem studied in this paper, the corresponding “limit” problem in \( \mathbb{R}^2 \) has no bounded nontrivial solution. So, we will follow the method in [7, 8] to construct an approximate solution. Since there are two parameters \( \delta, \varepsilon \) and \( b \) in problem (1.4), which causes some difficulty, we must take this influence into careful consideration and give delicate estimates in order to perform the reduction argument.

We will also apply the above idea and techniques to construct vortex pairs to shallow water equations in section 5, which has never been addressed before.
As a final remark, our results seem connected with the work of Wei, Ye and Zhou [17, 18, 19] on the anisotropic Emden-Fowler equation
\[
\begin{aligned}
\begin{cases}
\text{div}(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]

They constructed (boundary) bubbling solutions showing a striking difference with the isotropic case ($a \equiv \text{constant}$). Moreover, we point out that problem (1.4) can be considered as a free boundary problem. Similar problems have been studied extensively. The reader can refer to [5, 6, 7, 9, 8, 11, 13, 16] for more results on this kind of problems.

This paper is organized as follows. In Section 2, we construct the approximate solution for (1.4). We will carry out a reduction argument in Section 3 and prove the main results in Section 4. In Section 5, we give some further results on vortex pairs for the shallow water equations. Some basic estimates that used in sections 4 and 5 will be given in Section 6.

## 2 Approximate Solutions

In the section, we will construct approximate solutions for (1.4).

Let $R > 0$ be a large constant such that for any $x \in \Omega$, $\Omega \subset B_R(x)$. Consider the following Dirichlet problem:
\[
\begin{aligned}
\begin{cases}
\delta^2 \Delta w = (w - a)_+^p, & \text{in } B_R(0), \\
w = 0, & \text{on } \partial B_R(0),
\end{cases}
\end{aligned}
\]  
(2.1)

where $a > 0$ is a constant. Then, (2.1) has a unique solution $W_{\delta,a}$, which can be written as
\[
W_{\delta,a}(x) = \begin{cases}
a + \delta^{p-1} s_\delta^{-2} \phi(\frac{|x|}{s_\delta}), & |x| \leq s_\delta, \\
a \frac{|x|}{\ln s_\delta R}, & s_\delta \leq |x| \leq R,
\end{cases}
\]  
(2.2)

where $\phi(x) = \phi(|x|)$ is the unique solution of
\[
-\Delta \phi = \phi^p, \quad \phi > 0, \quad \phi \in H_0^1(B_1(0))
\]  
(2.3)

and $s_\delta \in (0, R)$ satisfies
\[
\delta^{p-1} s_\delta^{-2} \phi'(1) = \frac{a}{\ln s_\delta R},
\]
which implies
\[
\frac{s_\delta}{\delta |\ln \delta|^{p-2}} \to \left( \frac{|\phi'(1)|}{a} \right)^{p-1} > 0, \quad \text{as } \delta \to 0.
\]

Moreover, by Pohozaev identity, we can get that
\[
\int_{B_1(0)} \phi^{p+1} = \frac{\pi(p + 1)}{2} |\phi'(1)|^2 \quad \text{and} \quad \int_{B_1(0)} \phi^p = 2\pi |\phi'(1)|.
\]
For given \( \hat{b} > 0 \) and \( \hat{q} > 0 \), let \( V_{\delta, \hat{b}, \hat{q}}(x) \) be the solution of the following Dirichlet problem
\[
\begin{aligned}
-\delta^2 \Delta v &= \hat{b}^2 (v - \hat{q})^p_+, & \text{in } B_R(0), \\
v &= 0, & \text{on } \partial B_R(0).
\end{aligned}
\tag{2.4}
\]

By scaling, from (2.1) and (2.2), we obtain
\[
V_{\delta, \hat{b}, \hat{q}}(x) = \hat{b}^\frac{2}{p-1} W_{\delta, \hat{b}^\frac{2}{p-1} \hat{q}}(x) = \begin{cases} 
\hat{q} + \hat{b}^\frac{2}{p-1} \left( \frac{\delta}{s_\delta} \right)^\frac{2}{p-1} \phi\left( \frac{|x|}{s_\delta} \right), & |x| \leq s_\delta, \\
\ln \frac{|x|}{s_\delta}, & s_\delta \leq |x| \leq R.
\end{cases}
\tag{2.5}
\]

For any \( z \in \Omega \), define \( V_{\delta, \hat{b}, \hat{q}, z}(x) = V_{\delta, \hat{b}, \hat{q}}(x - z) \). Because \( V_{\delta, \hat{b}, \hat{q}} \) does not vanish on \( \partial \Omega \), we need to make a projection. Let \( PV_{\delta, \hat{b}, \hat{q}, z} \) be the solution of
\[
\begin{aligned}
-\delta^2 \Delta v &= \hat{b}^2 (V_{\delta, \hat{b}, \hat{q}, z} - \hat{q})^p_+, & \text{in } \Omega, \\
v &= 0, & \text{on } \partial \Omega,
\end{aligned}
\tag{2.6}
\]
and \( h(x, z) \) be the solution of
\[
\begin{aligned}
-\Delta h &= 0, & \text{in } \Omega, \\
h &= \frac{1}{2\pi} \ln \frac{1}{|x-z|}, & \text{on } \partial \Omega.
\end{aligned}
\]

Then
\[
P V_{\delta, \hat{b}, \hat{q}, z}(x) = V_{\delta, \hat{b}, \hat{q}, z}(x) - \frac{\hat{q}}{\ln \frac{R}{s_\delta}} g(x, z),
\tag{2.7}
\]
where \( g(x, z) = \ln R + 2\pi h(x, z) \).

We will construct solutions for \( \mathbf{1.4} \) of the following form
\[
\sum_{j=1}^m P V_{\delta, \hat{b}_j, \hat{q}_j, z_j} + \omega_\delta,
\]
where \( z_j \in \Omega \) for \( j = 1, \cdots, m \), \( \omega_\delta \) is a perturbation term. To obtain a good estimate for \( \omega_\delta \), we need to choose \( \hat{q}_{\delta, j} \) properly.

Denote \( Z = (z_1, \cdots, z_m) \in \mathbb{R}^{2m} \). In this paper, we always assume that \( z_j \in \Omega \) satisfies
\[
|z_i - z_j| \geq g^L, \quad \text{dist}(z_j, \partial \Omega) \geq g > 0, \quad \text{or}
\]
\[
|z_j - \hat{z}_j| < \eta, \quad \text{dist}(z_j, \partial \Omega) \geq \frac{1}{|\ln \varepsilon|^\alpha}, \quad i, j = 1, \cdots, m, \quad i \neq j
\tag{2.8}
\]
where \( g, \eta > 0 \) is a fixed small constant and \( L, \alpha > 0 \) is a fixed large constant.

Let \( \hat{b}_j = b(z_j) \) and \( \hat{q}_{\delta, j}(Z), j = 1, \cdots, m \) be the solution of the following problem:
\[
\hat{q}_i = q(z_i) + \frac{\hat{q}_i}{\ln \frac{R}{\varepsilon}} g(z_i, z_i) - \sum_{j \neq i} \frac{\hat{q}_j}{\ln \frac{R}{\varepsilon}} \hat{G}(z_i, z_j),
\tag{2.9}
\]
where \( \hat{G}(z_i, z_j) \) is the Green function of the Laplace operator in \( \Omega \).
where $\bar{G}(x,z_j) = \ln \frac{R}{|x-z_j|} - g(x,z_j)$. It is not difficult to see that since $\ln \frac{R}{\varepsilon} \to \infty$ as $\varepsilon \to 0$, (2.9) is a linear system with coefficient matrix, which is a small perturbation of a positively definite diagonal matrix for small $\varepsilon$. Thus we can obtain the solution $(\hat{q}_{\delta,1}(Z), \ldots, \hat{q}_{\delta,m}(Z))$ to (2.9). Moreover, we have

$$\hat{q}_{\delta,i}(Z) = \frac{q(z_i) - \sum_{j \neq i} \frac{\hat{q}_{i,j}(Z)}{\ln \frac{R}{\varepsilon}} \bar{G}(z_i, z_j)}{1 - \frac{q(z_i,z_i)}{\ln \frac{R}{\varepsilon}}}.$$  

For simplicity, for given $Z = (z_1, \ldots, z_m)$, in this paper, we will use $\hat{q}_{\delta,i}$ instead of $\hat{q}_{\delta,i}(Z)$. Define

$$V_{\delta,Z,i} = PV_{\delta,b_j,\hat{q}_{\delta,j},z_j}, V_{\delta,Z} = \sum_{j=1}^{m} V_{\delta,Z,j}. \quad (2.10)$$

Let $s_{\delta,i}$ be the solution of

$$\delta^2 \frac{2}{p-1}s \frac{2}{p-1} \phi'(1) = \frac{b_{\delta,i}^2}{\ln \frac{R}{s_{\delta,i}}},$$

then we have

$$\frac{1}{\ln \frac{R}{s_{\delta,i}}} = 1 \frac{1}{\ln \frac{R}{\varepsilon}} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).$$

Thus, we find that for $x \in B_{Ls_{\delta,i}}(z_i)$, where $L > 0$ is any fixed constant,

$$V_{\delta,Z,i}(x) - q(x) = V_{\delta,b_i,\hat{q}_{\delta,i},z_i}(x) - \frac{\hat{q}_{\delta,i}}{\ln \frac{R}{s_{\delta,i}}} g(x,z_i) - q(x)$$

$$= V_{\delta,b_i,\hat{q}_{\delta,i},z_i}(x) - q(z_i) - \frac{\hat{q}_{\delta,i}}{\ln \frac{R}{s_{\delta,i}}} g(z_i,z_i) + O(s_{\delta,i}) + O \left( \frac{s_{\delta,i} |Dg(z_i,z_i)|}{\ln \frac{R}{s_{\delta,i}}} \right)$$

$$= V_{\delta,b_i,\hat{q}_{\delta,i},z_i}(x) - q(z_i) - \frac{\hat{q}_{\delta,i}}{\ln \frac{R}{\varepsilon}} g(z_i,z_i) + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right) g(z_i,z_i)$$

and for $j \neq i$ and $x \in B_{Ls_{\delta,i}}(z_i)$,

$$V_{\delta,Z,j}(x) = V_{\delta,b_j,\hat{q}_{\delta,j},z_j}(x) - \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} g(x,z_j) = \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} G(x,z_j)$$

$$= \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} \bar{G}(z_i,z_j) + \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} \left( \bar{G}(x,z_j) - \bar{G}(z_i,z_j) \right)$$

$$= \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{\varepsilon}} \bar{G}(z_i,z_j) + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).$$

So, by using (2.9), we obtain
\[ V_{\delta,Z}(x) - q(x) = V_{\delta,Z}(x) - \hat{q}_{\delta,i} + O \left( \frac{\ln \ln \varepsilon}{\ln \varepsilon^2} g(z_i, z_i) \right), \quad x \in B_{Ls_{\delta,i}}(z_i). \]  

We end this section by giving the following formula which can be obtained by direct computation and will be used in the next two sections.

\[
\begin{align*}
\frac{\partial V_{\delta,b_i,\hat{q}_{\delta,i},z_i}(x)}{\partial z_{i,h}} &= \\
&= \begin{cases} \\
\hat{b}_{i}^{-\frac{2}{p-1}} \frac{1}{s_{\delta,i}} \left( \frac{\delta}{s_{\delta,i}} \right)^{\frac{2}{p-1}} \phi' \left( \frac{|x - z_i|}{s_{\delta,i}} \right) \frac{z_{i,h} - x_h}{|x - z_i|} + O(1), & x \in B_{s_{\delta,i}}(z_i), \\
- \frac{\hat{q}_{\delta,i}}{\ln s_{\delta,i}} \frac{z_{i,h} - x_h}{|x - z_i|^2} + O(1), & x \in \Omega \setminus B_{s_{\delta,i}}(z_i). 
\end{cases}
\end{align*}
\]  

3 The Reduction

Let \( V_{\delta,Z} \) be given as in (2.10), we are to find solutions of the form \( V_{\delta,Z} + \omega_{\delta,Z} \), where \( \omega_{\delta,Z} \) is a small perturbation (obtained in Proposition 3.3). We will show that for any given \( Z \), there exists \( \omega_{\delta,Z} \) such that \( w_{\delta,Z} = V_{\delta,Z} + \omega_{\delta,Z} \) satisfies

\[
\int_{\Omega} \left[ \frac{\delta^2}{b} \nabla w_{\delta,Z} \nabla v - b(w_{\delta,Z} - q)^{p_+} v \right] = 0, \text{ for any } v \in H^1_0(\Omega) \cap W^{2,p}(\Omega) \setminus H^s, \tag{3.1}
\]

where \( H^s \) is a finite dimensional subspace of \( H^1_0(\Omega) \cap W^{2,p}(\Omega) \). In the next section, we will choose \( Z \) properly so that \( V_{\delta,Z} + \omega_{\delta,Z} \) is a solution of (1.4).

To show (3.1), we need to study the kernel of \( \mathcal{L}w := -\delta^2 \text{div}(\nabla w) - pb(V_{\delta,Z} - q)^{p-1} w \). To do this first we need to understand the kernel of the linearized equation of

\[-\Delta w = w^p_+, \quad \text{in } \mathbb{R}^2. \tag{3.2}\]

Let

\[ w(x) = \begin{cases} \\
\phi(|x|), & |x| \leq 1, \\
\phi'(1) \ln |x|, & |x| > 1,
\end{cases} \]

where \( \phi \) is the solution of (2.3), then \( w \in C^1(\mathbb{R}^2) \) is the unique solution of (3.2). Since \( \phi'(1) < 0 \) and \( \ln |x| \) is harmonic for \( |x| > 1 \). Moreover, since \( w_+ \) is Lip-continuous, by the Schauder estimate, \( w \in C^{2,\alpha} \) for any \( \alpha \in (0, 1) \).

The linearized equation of (3.2) at \( w \) is as follows

\[-\Delta v - pw_+^{p-1} v = 0, \quad v \in L^\infty(\mathbb{R}^2). \tag{3.3}\]

It is easy to see that \( \frac{\partial w}{\partial x_i}, i = 1, 2, \) is a solution of (3.3). Moreover, from Dancer and Yan [8], we know that \( w \) is also non-degenerate, in the sense that the kernel of the operator \( Lw := -\Delta v - pw_+^{p-1}v, \) \( v \in D^{1,2}(\mathbb{R}^2) \) is spanned by \( \{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2} \} \).
Let $V_{\delta,Z,j}$ be the function defined in (2.10). Set

$$F_{\delta,Z} = \left\{ u : u \in L^p(\Omega), \int_{\Omega} \frac{\partial V_{\delta,Z,j}}{\partial z,j,h} u = 0, j = 1, \ldots, m, h = 1,2 \right\},$$

and

$$E_{\delta,Z} = \left\{ u : u \in W^{2,p}(\Omega) \cap H^1_0(\Omega), \int_{\Omega} \Delta \left( \frac{\partial V_{\delta,Z,j}}{\partial z,j,h} \right) u = 0, j = 1, \ldots, m, h = 1,2 \right\}.$$

Define, for any $u \in L^p(\Omega)$, $Q_\delta u$ as follows:

$$Q_\delta u := u - \sum_{j=1}^m \sum_{h=1}^2 c_{j,h} \left( -\delta^2 \Delta \left( \frac{\partial V_{\delta,Z,j}}{\partial z,j,h} \right) \right),$$

(3.4)

where the constants $c_{j,h}(j = 1, \ldots, m, h = 1,2)$ are chosen to satisfy

$$\sum_{j=1}^m \sum_{h=1}^2 c_{j,h} \left( -\delta^2 \int_{\Omega} \Delta \left( \frac{\partial V_{\delta,Z,j}}{\partial z,j,h} \right) \frac{\partial V_{\delta,Z,i}}{\partial z,i,h} \right) = \int_{\Omega} u \frac{\partial V_{\delta,Z,i}}{\partial z,i,h}.$$  

(3.5)

Since $\int_{\Omega} \frac{\partial V_{\delta,Z,j}}{\partial z,j,h} Q_\delta u = 0$, the operator $Q_\delta$ can be regarded as a projection from $L^p(\Omega)$ to $F_{\delta,Z}$. In order to show the existence of $c_{j,h}$ satisfying (3.5), we just need the following estimate (by (2.12)):

$$-\delta^2 \int_{\Omega} \Delta \left( \frac{\partial V_{\delta,Z,j}}{\partial z,j,h} \right) \frac{\partial V_{\delta,Z,i}}{\partial z,i,h}$$

$$= p \delta_j^2 \int_{\Omega} \left( V_{\delta,b_j,q_{\delta,b_j}} \right)^{p-1} \left( \frac{\partial V_{\delta,b_j,q_{\delta,b_j},z_j}}{\partial z,j,h} \frac{\partial V_{\delta,b_j,q_{\delta,b_j},z_j}}{\partial z,j,h} \right) + O \left( \frac{\varepsilon}{|\ln \varepsilon|^{p+1}} \right),$$

(3.6)

where $c' > 0$ is a constant, $\delta_{ij,h} = 1$, if $i = j$ and $h = \bar{h}$; otherwise, $\delta_{ij,h} = 0$.

Define

$$L_\delta u = -\delta^2 \text{div} \left( \frac{\nabla u}{b} \right) - pb (V_{\delta,Z} - q)^{p-1} u.$$

For the operator $Q_\delta L_\delta$ we have the following lemma.

**Lemma 3.1.** There are constants $\rho_0 > 0$ and $\delta_0 > 0$, such that for any $\delta \in (0, \delta_0]$, $Z$ satisfying (2.8), $u \in E_{\delta,Z}$ with $Q_\delta L_\delta u = o(1)$ in $L^p(\Omega \setminus \cup_{j=1}^m B_{L\delta,\bar{h}}(z_j))$ for some $L > 0$ large, then

$$\|Q_\delta L_\delta u\|_{L^p(\Omega)} \geq \frac{\rho_0 \varepsilon^{\frac{2}{p}}}{|\ln \varepsilon|^{p-1}} \|u\|_{L^\infty(\Omega)}.$$
Proof. We will use $\| \cdot \|_p, \| \cdot \|_\infty$ to denote $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_{L^\infty(\Omega)}$ respectively. We argue by contradiction. Suppose that there are $\delta_n \to 0, Z_n = (z_{1,n}, \cdots, z_{m,n})$ satisfying (2.8) and $u_n \in E_{\delta_n, Z_n}$ with $Q_{\delta_n} L_{\delta_n} u_n = o(1)$ in $L^p(\Omega \cup \cup_{j=1}^m B_{L_{\delta_n,j}}(z_{j,n}))$, such that

$$\|Q_{\delta_n} L_{\delta_n} u_n\|_p \leq \frac{1}{n} \frac{\varepsilon_n^2}{\ln \varepsilon_n |p-1|},$$

and $\|u_n\|_\infty = 1$, where and in the sequel we set $s_{n,j} = s_{\delta_n,j}$ to simplify notation.

Firstly, we estimate $c_{j,h,n}$ corresponding to $u_n$ in (3.4). By definition $c_{j,h,n}$ satisfies:

$$Q_{\delta_n} L_{\delta_n} u_n = L_{\delta_n} u_n - \sum_{j=1}^m \sum_{h=1}^{2} c_{j,h,n} \left( -\delta_n^2 \Delta \frac{\partial V_{\delta_n, Z_{n,j}}}{\partial z_{j,h}} \right).$$

For each fixed $i$, multiplying (3.7) by $\frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}}$, noting that

$$\int_\Omega (Q_{\delta_n} L_{\delta_n} u_n) \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} = 0,$$

we obtain

$$\int_\Omega u_n L_{\delta_n} \left( \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} \right) = \int_\Omega (L_{\delta_n} u_n) \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}}
= \sum_{j=1}^m \sum_{h=1}^{2} c_{j,h,n} \int_\Omega \left( -\delta_n^2 \Delta \frac{\partial V_{\delta_n, Z_{n,j}}}{\partial z_{j,h}} \right) \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}}.$$

Using (2.11) and Lemma 6.1 we obtain

$$\int_\Omega u_n L_{\delta_n} \left( \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} \right)
= \int_\Omega u_n \left[ -\delta_n^2 \text{div} \left( \frac{1}{b_i} \nabla \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} \right) - p \hat{b}_i (V_{\delta_n, Z_n} - q)^{p-1} \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} \right]
+ \int_\Omega u_n \left[ \delta_n^2 \text{div} \left( \left( \frac{1}{b_i} - \frac{1}{b} \right) \nabla \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} \right) \right]
+ \int_\Omega u_n \left[ (\hat{b}_i - b) (V_{\delta_n, Z_n} - q)^{p-1} \frac{\partial V_{\delta_n, Z_{n,i}}}{\partial z_{i,h}} \right]
= O \left( \frac{\varepsilon_n \ln^2 |\ln \varepsilon_n|}{|\ln \varepsilon_n|^{p+1}} \right) + O \left( \frac{\varepsilon_n^2}{|\ln \varepsilon_n|^{p-1}} \right) + O \left( \frac{\varepsilon_n^2}{|\ln \varepsilon_n|^p} \right)
= O \left( \frac{\varepsilon_n \ln^2 |\ln \varepsilon_n|}{|\ln \varepsilon_n|^{p+1}} \right).$$

Using (3.6), we find that

$$c_{i,h,n} = O \left( \varepsilon_n \ln^2 |\ln \varepsilon_n| \right).$$
Therefore,

\[
\begin{align*}
\sum_{j=1}^{m} \sum_{h=1}^{2} c_{j,h,n} \left( -\delta_n^2 \frac{\partial V_{\delta_n, z_{n,j}}}{\partial z_{j,h}} \right) \\
= \sum_{j=1}^{m} \sum_{h=1}^{2} p \delta_n^2 c_{j,h,n} \left( V_{\delta_n, \tilde{b}_j, n, \tilde{a}_j, n, z_{j,n}} - \tilde{q}_{j,n} \right)^p + \left( \frac{\partial V_{\delta_n, \tilde{b}_j, n, \tilde{a}_j, n, z_{j,n}}}{\partial z_{j,h}} - \frac{\partial \tilde{q}_{j,n}}{\partial z_{j,h}} \right) \\
+ \sum_{j=1}^{m} \sum_{h=1}^{2} 2 \delta_n \frac{\partial b_{j,n}}{\partial z_{j,h}} \left( V_{\delta_n, b_{j,n}, \tilde{a}_j, n, z_{j,n}} - \tilde{q}_{j,n} \right)^p \\
= O \left( \sum_{j=1}^{m} \sum_{h=1}^{2} \frac{2}{p} |c_{j,h,n}| \frac{1}{| \ln \varepsilon_n |^p} \right) \\
= O \left( \frac{2 \ln | \ln \varepsilon_n |}{| \ln \varepsilon_n |^p} \right) \text{ in } L^p(\Omega).
\end{align*}
\]

Thus, we obtain

\[
L_{\delta_n} u_n = Q_{\delta_n} L_{\delta_n} u_n + O \left( \frac{2 \ln | \ln \varepsilon_n |}{| \ln \varepsilon_n |^p} \right) = O \left( \frac{1}{n} \frac{2 \ln \varepsilon_n}{| \ln \varepsilon_n |^{p-1}} \right).
\]

For any fixed \( i \), define

\[
\tilde{u}_{i,n}(y) = u_n(s_{n,i} y + z_{i,n}).
\]

Let

\[
\tilde{L}_n u = -\text{div} \left( \frac{\nabla u}{\sigma_{s_{n,i} y + z_{i,n}}} \right) - pb(s_{n,i} y + z_{i,n})^2 s_{n,i}^2 (V_{s_{n,i} y + z_{i,n}}) \left( V_{s_{n,i} y + z_{i,n}} - q(s_{n,i} y + z_{i,n}) \right)^p u.
\]

Then

\[
\frac{2}{s_{n,i}^p} \frac{\delta_n^2}{s_{n,i}^2} \| \tilde{L}_n \tilde{u}_{i,n} \|_p = \| L_{\delta_n} u_n \|_p.
\]

Noting that

\[
\left( \frac{\delta_n}{s_{n,i}} \right)^2 = O \left( \frac{1}{| \ln \varepsilon_n |^{p-1}} \right),
\]

we find that

\[
L_{\delta_n} u_n = o \left( \frac{2 \varepsilon_n^2}{| \ln \varepsilon_n |^{p-1}} \right).
\]

As a result,

\[
\tilde{L}_n \tilde{u}_{i,n} = o(1), \text{ in } L^p(\Omega_n),
\]

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where $\Omega_n = \{ y : s_n,i y + z_{i,n} \in \Omega \}$.

Since $\| \tilde{u}_{i,n} \|_\infty = 1$, by the regularity theory of elliptic equations, we may assume that

$$
\tilde{u}_{i,n} \to u_i, \quad \text{in } C^1_{loc}(\mathbb{R}^2).
$$

It is easy to see that

$$
\frac{s_{n,i}^2}{\delta_n^2} (V_{\delta_n} z_n (s_n,i y + z_{i,n}) - q(s_n,i y + z_{i,n}))_{+}^{p-1} = \frac{s_{n,i}^2}{\delta_n^2} \left( V_{\delta_n} h_{n,i} z_{i,n} + \hat{q}_{i,n} + O \left( \frac{\ln^2 |\ln \varepsilon_n|}{|\ln \varepsilon_n|^2} \right) \right)_{+}^{p-1} = \frac{1}{\hat{b}_{i,n}^2} \left( \phi(y) + O \left( \frac{\ln^2 |\ln \varepsilon_n|}{|\ln \varepsilon_n|^2} \right) \right)_{+}^{p-1}.
$$

Then, by Lemma 6.1, we find that $u_i$ satisfies

$$
-\Delta u_i - p w_{+}^{p-1} u_i = 0.
$$

Now from the Proposition 3.1 in [8], we have

$$
u_i = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2}.
$$

(3.8)

Since

$$
\int_\Omega \Delta \left( \frac{\partial V_{\delta_n} z_{n,i}}{\partial z_{i,h}} \right) u_n = 0,
$$

we find that

$$
\int_{\mathbb{R}^2} \phi_{+}^{p-1} \frac{\partial \phi}{\partial z_h} u_i = 0,
$$

which, together with (3.8), gives $u_i \equiv 0$. Thus,

$$
\tilde{u}_{i,n} \to 0, \quad \text{in } C^1(B_L(0)),
$$

for any $L > 0$, which implies that $u_n = o(1)$ on $\partial B_{L\varepsilon_n,i}(z_{i,n})$.

By assumption,

$$
Q_{\delta_n} L_{\delta_n} u_n = o(1), \quad \text{in } L^p(\Omega \setminus \bigcup_{i=1}^k B_{L\delta_{n,i}}(z_{i,n})).
$$

On the other hand, by Lemma 6.1, we have

$$(V_{\delta_n} z_n - q)_{+} = 0, \quad x \in \Omega \setminus \bigcup_{i=1}^k B_{L\delta_{n,i}}(z_{i,n}).$$

Thus, we find that

$$
-\text{div} \left( \frac{\nabla u_n}{b} \right) = o(1), \quad \text{in } \Omega \setminus \bigcup_{i=1}^m B_{L\delta_{n,i}}(z_{i,n}).
$$
However, $u_n = 0$ on $\partial \Omega$ and $u_n = o(1)$ on $\partial B_{Ls_n,i}(z_{i,n})$, $i = 1, \cdots, m$. So we have

$$u_n = o(1).$$

This is a contradiction.

\[ \square \]

**Proposition 3.2.** $Q_\delta L_\delta u$ is one to one and onto from $E_\delta, Z$ to $F_\delta, Z$.

**Proof.** Suppose that $Q_\delta L_\delta u = 0$. Then, by Lemma 3.1, $u \equiv 0$. Thus, $Q_\delta L_\delta$ is one to one.

Next, we prove that $Q_\delta L_\delta$ is an onto map from $E_\delta, Z$ to $F_\delta, Z$.

Denote

$$\tilde{E} = \{ u : u \in H^1_0(\Omega), \int_\Omega \nabla \partial V_{\delta, Z,j} \nabla u = 0, j = 1, \cdots, m, h = 1, 2 \}.$$ 

Note that $E_\delta, Z = \tilde{E} \cap W^{2,p}(\Omega)$.

For any $\tilde{h} \in F_\delta, Z$, by the Riesz representation theorem, there is a unique $u \in H^1_0(\Omega)$, such that

$$\delta^2 \int_\Omega \nabla u \nabla \varphi = \int_\Omega \tilde{h} \varphi, \quad \forall \varphi \in H^1_0(\Omega). \quad (3.9)$$

On the other hand, from $\tilde{h} \in F_\delta, Z$, we find that $u \in \tilde{E}$. Moreover, by the $L^p$-estimate, we deduce that $u \in W^{2,p}(\Omega)$. As a result, $u \in E_\delta, Z$. Thus, we see that $Q_\delta(\delta^2 \Delta) = -\delta^2 \Delta$ is an one to one and onto map from $E_\delta, Z$ to $F_\delta, Z$. On the other hand, $Q_\delta L_\delta u = \tilde{h}$ is equivalent to

$$u = \delta^{-2}(-Q_\delta \Delta)^{-1} [Tu + bh], \quad u \in E_\delta, Z \quad (3.10)$$

where

$$Tu = b \delta^2 \nabla \frac{1}{b} \nabla u + pb^2 (V_{\delta, Z} - q)^{p-1} u + \sum_{j=1}^m \sum_{h=1}^2 \hat{b}_{j,h} \left( -\delta^2 \Delta \frac{\partial V_{\delta, Z,j}}{\partial z_{j,h}} \right).$$

It is easy to check that $\delta^{-2}(-Q_\delta \Delta)^{-1} Tu$ is a compact operator in $E_\delta, Z$. By the Fredholm alternative, \((3.10)\) is solvable if and only if

$$u = \delta^{-2}(-Q_\delta \Delta)^{-1} Tu$$

has only trivial solution, which is true since $Q_\delta L_\delta$ is a one to one map.

Now consider the equation

$$Q_\delta L_\delta \omega = Q_\delta l_\delta + Q_\delta R_\delta(\omega), \quad (3.11)$$

where

$$l_\delta = b (V_{\delta, Z} - q)_+ - \sum_{j=1}^m \frac{\hat{b}^2}{b} \left( V_{\delta, j, \hat{b}_{\delta, j, z_j} - \hat{q}_{\delta, j} } \right)_+ + \delta^2 \left( \nabla \frac{1}{b} \nabla V_{\delta, Z} \right). \quad (3.12)$$
and

\[ R_\delta(\omega) = b (V_{\delta,Z} + \omega - q)_+^p - b (V_{\delta,Z} - q)_+^p - p b (V_{\delta,Z} - q)_+^{p-1} \omega. \]  

Using Proposition 3.2, we can rewrite (3.11) as

\[ \omega = G_\delta \omega = (Q_\delta L_\delta)^{-1} Q_\delta (l_\delta + R_\delta(\omega)). \]  

(3.14)

The next proposition enables us to reduce the problem of finding a solution for (3.14) to a finite dimensional one.

**Proposition 3.3.** There is a \( \delta_0 > 0 \), such that for any \( \delta \in (0, \delta_0) \) and \( Z \) satisfying (2.8), (3.11) has a unique solution \( \omega_\delta \in E_{\delta,Z} \), with

\[ \| \omega_\delta \|_\infty = O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right). \]

**Proof.** It follows from Lemma 6.1 that if \( L \) is large enough, \( \delta \) is small then

\[ (V_{\delta,Z} - q)_+ = 0, \quad x \in \Omega \setminus \bigcup_{j=1}^m B_{L_{s_\delta,j}}(z_j). \]

Let

\[ M = E_{\delta,Z} \cap \left\{ \| \omega \|_\infty \leq \frac{1}{|\ln \varepsilon|^{2-\theta}} \right\}, \]

where \( \theta > 0 \) is a small constant.

Then \( M \) is complete under \( L^\infty \) norm and \( G_\delta \) is a map from \( E_{\delta,Z} \) to \( E_{\delta,Z} \). We will show that \( G_\delta \) is a contraction map from \( M \) to \( M \) by two steps in the following.

Step 1. \( G_\delta \) is a map from \( M \) to \( M \).

For any \( \omega \in M \), similar to Lemma 6.1, it is easy to prove that for large \( L > 0 \), \( \delta \) small

\[ (V_{\delta,Z} + \omega - q)_+ = 0, \quad x \in \Omega \setminus \bigcup_{j=1}^m B_{L_{s_\delta,j}}(z_j). \]  

(3.15)

Note also that for any \( u \in L^\infty(\Omega) \),

\[ Q_\delta u = u \quad \text{in} \quad \Omega \setminus \bigcup_{j=1}^m B_{L_{s_\delta,j}}(z_j). \]

Direct computations yield that

\[ \left\| \delta^2 Q_\delta \left( \nabla \frac{1}{b} \nabla V_{\delta,Z} \right) \right\|_p = \begin{cases} \frac{\varepsilon^2}{|\ln \varepsilon|^p}, & 1 < p < 2, \\ \frac{\varepsilon^2}{p+1}, & p \geq 2. \end{cases} \]

Therefore, using Lemma 6.1 (3.12) and (3.13), we find that for any \( \omega \in M \),

\[ Q_\delta l_\delta + Q_\delta R_\delta(\omega) = o(1), \quad \text{in} \quad L^p(\Omega \setminus \bigcup_{j=1}^m B_{L_{s_\delta,j}}(z_j)). \]
So, we can apply Lemma 3.1 to obtain

\[ \left\| (Q_\delta L_\delta)^{-1} (Q_\delta l_\delta + Q_\delta R_\delta(\omega)) \right\|_\infty \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} \| Q_\delta l_\delta + Q_\delta R_\delta(\omega) \|_p. \]

Thus, for any \( \omega \in M \), we have

\[ \| G_\delta(\omega) \|_\infty = \left\| (Q_\delta L_\delta)^{-1} (Q_\delta l_\delta + Q_\delta R_\delta(\omega)) \right\|_\infty \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} \| Q_\delta l_\delta + Q_\delta R_\delta(\omega) \|_p. \tag{3.16} \]

It follows from (3.5)–(3.6) that the constant \( c_{j,h} \), corresponding to \( u \in L^\infty(\Omega) \), satisfies

\[ |c_{j,h}| \leq C |\ln \varepsilon|^{p+1} \sum_{i,h} \left| \frac{\partial V_{\delta,Z,i}}{\partial z_{i,h}} \right| |u|. \]

Hence, we find that the constant \( c_{j,h} \), corresponding to \( l_\delta + R_\delta(\omega) \) satisfies

\[ |c_{j,h}| \leq C |\ln \varepsilon|^{p+1} \sum_{i,h} \left| \frac{\partial V_{\delta,Z,i}}{\partial z_{i,h}} \right| |l_\delta + R_\delta(\omega)| \]

\[ \leq C |\ln \varepsilon|^{p+1} \sum_{i,h} \left| \int_{B_{L^\infty(\delta Z,i)}} \left| \frac{\partial V_{\delta,Z,i}}{\partial z_{i,h}} \right| |l_\delta + R_\delta(\omega)| + C \varepsilon^2 |\ln \varepsilon| \right| \]

\[ \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p} \|l_\delta + R_\delta(\omega)\|_p + C \varepsilon^2 |\ln \varepsilon|. \]

where

\[ \tilde{\delta} = b(V_\delta,Z - q)^+_\delta - \sum_{j=1}^m \frac{\hat{\delta}_j^2}{b} (V_{\delta,j},\hat{q}_\delta,j,\hat{z}_j - \hat{q}_\delta,j)^+_\delta. \]

As a result,

\[ \| Q_\delta(l_\delta + R_\delta(\omega)) \|_p \]

\[ \leq \| l_\delta + R_\delta(\omega) \|_p + C \sum_{j,h} |c_{j,h}| \left\| -\delta^2 \Delta \left( \frac{\partial V_{\delta,Z,j}}{\partial z_{j,h}} \right) \right\|_p \]

\[ = C \| \tilde{l}_\delta \|_p + C \| R_\delta(\omega) \|_p + \left\| \frac{\delta^2 \nabla^1 b}{b} \nabla V_{\delta,Z} \right\|_p + C \varepsilon^{1+\frac{2}{p}} |\ln \varepsilon|^{p-1} \]

\[ = C \| \tilde{l}_\delta \|_p + C \| R_\delta(\omega) \|_p + R(p, \varepsilon) \]

where

\[ R(p, \varepsilon) = \begin{cases} 
\frac{C \varepsilon^2}{|\ln \varepsilon|^p}, & 1 < p < 2, \\
\frac{C \varepsilon^{1+\frac{2}{p}}}{|\ln \varepsilon|^{p-1}}, & p \geq 2.
\end{cases} \]

On the other hand, from Lemma 6.1 and (2.11), we can deduce
\[ \| \tilde{l}_\delta \|_p \leq \left\| b(V_\delta, Z - q)_+^p - \sum_{j=1}^{m} \frac{\hat{b}}{b}(V_\delta, \hat{b}_j, \hat{d}_j, z_j - \hat{q}_\delta j)_+^p \right\|_p \]
\[ \leq \sum_{j=1}^{m} \frac{C \ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \left\| (V_\delta, \hat{b}_j, \hat{d}_j, z_j - \hat{q}_\delta j)_+^{p-1} \right\|_p \]
\[ \leq C \frac{\varepsilon^{\frac{2}{p}} |\ln \varepsilon|}{|\ln \varepsilon|^{p+1}}. \]

For the estimate of \( \| R_\delta(\omega) \|_p \), we have

\[ \| R_\delta(\omega) \|_p = \| b(V_\delta, Z + \omega - q)_+^p - b(V_\delta, Z - q)_+^p - pb(V_\delta, Z - q)_+^{p-1} \omega \|_p \]
\[ \leq C \| \omega \|_2^2 \left\| (V_\delta, Z - q)_+^{p-2} \right\|_p \]
\[ \leq C \frac{\varepsilon^{\frac{2}{p}} \| \omega \|_2^2}{|\ln \varepsilon|^{p-2}}. \]

Thus, we obtain

\[ \| G_\delta(\omega) \|_\infty \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} \left( \| \tilde{l}_\delta \|_p + \| R_\delta(\omega) \|_p + R(p, \varepsilon) \right) \]
\[ \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} \left( \frac{\varepsilon^{\frac{2}{p}} |\ln \varepsilon|}{|\ln \varepsilon|^{p+1}} + \frac{\varepsilon^{\frac{2}{p}} \| \omega \|_2^2}{|\ln \varepsilon|^{p-2}} \right) \]
\[ \leq \frac{1}{|\ln \varepsilon|^{2-\theta}}. \]

Thus, \( G_\delta \) is a map from \( M \) to \( M \).

Step 2. \( G_\delta \) is a contraction map.

In fact, for any \( \omega_i \in M, i = 1, 2 \), we have

\[ G_\delta \omega_1 - G_\delta \omega_2 = (Q_\delta L_\delta)^{-1} Q_\delta (R_\delta(\omega_1) - R_\delta(\omega_2)). \]

Noting that

\[ R_\delta(\omega_1) = R_\delta(\omega_2) = 0, \quad \text{in } \Omega \setminus \bigcup_{j=1}^{m} B_{L_\delta j}(z_j), \]

we can deduce as in Step 1 that

\[ \| G_\delta \omega_1 - G_\delta \omega_2 \|_\infty \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} \| R_\delta(\omega_1) - R_\delta(\omega_2) \|_p \]
\[ \leq C |\ln \varepsilon|^{p-1} \left( \frac{\| \omega_1 \|_\infty}{|\ln \varepsilon|^{p-2}} + \frac{\| \omega_2 \|_\infty}{|\ln \varepsilon|^{p-2}} \right) \| \omega_1 - \omega_2 \|_\infty \]
\[ \leq \frac{C}{|\ln \varepsilon|^{1-\theta}} |\omega_1 - \omega_2 \|_\infty \leq \frac{1}{2} |\omega_1 - \omega_2 \|_\infty. \]
Combining Step 1 and Step 2, we have proved that $G_δ$ is a contraction map from $M$ to $M$. As a consequence, there is a unique $ω_δ ∈ M$ such that $ω_δ = G_δω_δ$. Moreover, it follows from (3.18) that
\[ \|ω_δ\|_∞ \leq C\frac{\ln |\ln ε|}{|\ln ε|^2}. \]

4 Proof of Main Results

In this section, we will choose $Z$ properly so that $V_{δ,Z} + ω_δ$, where $ω_δ$ is the map obtained in Proposition 3.3, is a solution of (1.4).

Define
\[ I(u) = \frac{δ^2}{2} \int_Ω \frac{|∇u|^2}{b} - \frac{1}{p+1} \int_Ω b (u - q(x))_+^{p+1} \]
and
\[ K(Z) = I(V_{δ,Z} + ω_δ). \]  \hspace{1cm} (4.1)

It is well known that if $Z$ is a critical point of $K(Z)$, then $V_{δ,Z} + ω_δ$ is a solution of (1.4).

In the following, we will prove that $K(Z)$ has a critical point. To do this let us first show that $I(V_{δ,Z})$ is the leading term in $K(Z)$.

**Lemma 4.1.** We have
\[ K(Z) = I(V_{δ,Z}) + O\left( \frac{ε^2 \ln |\ln ε|}{|\ln ε|^{p+2}} \right). \]

**Proof.** Recall that
\[ V_{δ,Z} = \sum_{j=1}^m V_{δ,Z,j}. \]

By the definition of $K(Z)$
\[ K(Z) = I(V_{δ,Z}) + δ^2 \int_Ω \frac{1}{b} ∇V_{δ,Z} ∇ω_δ + \frac{δ^2}{2} \int_Ω \frac{|∇ω_δ|^2}{b} \]
\[ - \frac{1}{p+1} \int_Ω b \left[ (V_{δ,Z} + ω_δ - q)_+^{p+1} - (V_{δ,Z} - q)_+^{p+1} \right]. \]

Using Proposition 3.3 and (3.15), we have
\[ \int_Ω b \left[ (V_{δ,Z} + ω_δ - q)_+^{p+1} - (V_{δ,Z} - q)_+^{p+1} \right] \]
\[ = \sum_{j=1}^m (p+1) \int_{B_{r(ε)}(z_j)} b(V_{δ,Z} - q)_+^p ω_δ + O\left( \frac{ε^2 \ln^2 |\ln ε|}{|\ln ε|^{p+3}} \right) \]
\[ = O\left( \frac{S_δ^2 ∥ω∥_∞}{|\ln ε|^p} \right) = O\left( \frac{ε^2 \ln |\ln ε|}{|\ln ε|^{p+2}} \right). \]
On the other hand,
\[ \delta^2 \int_\Omega \frac{1}{b} \nabla V_{\delta, Z} \nabla \omega = \sum_{j=1}^{m} \int_\Omega \frac{\hat{b}_j^2}{b} \left( V_{\delta, b_j, \hat{b}_{\delta, j}, z_j} - \hat{q}_{\delta, j} \right)_+ \omega - \delta^2 \int_\Omega \frac{1}{b} \nabla V_{\delta, Z} \omega \]
\[ = O \left( \frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+2}} \right). \]

Finally, we estimate \( \delta^2 \int_\Omega \frac{1}{b} |\nabla \omega|^2 \).

Note that
\[ -\delta^2 \text{div} \left( \frac{1}{b} \nabla \omega \right) = b(V_{\delta, Z} + \omega - q)_+ - \sum_{j=1}^{m} \frac{\hat{b}_j^2}{b} \left( V_{\delta, b_j, \hat{b}_{\delta, j}, z_j} - \hat{q}_{\delta, j} \right)_+ \]
\[ + \delta^2 \nabla \frac{1}{b} \nabla V_{\delta, Z} + \sum_{j=1}^{m} \left( -\delta^2 \Delta \frac{\partial V_{\delta, Z, j}}{\partial z_{j, h}} \right), \]

hence
\[ \delta^2 \int_\Omega \frac{1}{b} |\nabla \omega|^2 = \int_\Omega b(V_{\delta, Z} + \omega - q)_+ \omega - \sum_{j=1}^{m} \frac{\hat{b}_j^2}{b} \left( V_{\delta, b_j, \hat{b}_{\delta, j}, z_j} - \hat{q}_{\delta, j} \right)_+ \omega \]
\[ + \delta^2 \int_\Omega \omega \nabla \frac{1}{b} \nabla V_{\delta, Z} + \sum_{j=1}^{m} \sum_{h=1}^{2} c_{j, h} \int_\Omega \left( -\delta^2 \Delta \frac{\partial V_{\delta, Z, j}}{\partial z_{j, h}} \right) \omega \]
\[ = O \left( \frac{\varepsilon^2 \ln^3 |\ln \varepsilon|}{|\ln \varepsilon|^{p+3}} \right) + O \left( \frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+2}} \right) \]
\[ = O \left( \frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+2}} \right). \]

\[ \square \]

**Proof of Theorem 1.4.** By Proposition 6.2, we have
\[ K(Z) = \sum_{j=1}^{n} \pi \delta^2 \frac{q^2(z_j)}{b(z_j)} + O \left( \frac{\delta^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right). \]

Since \( \tilde{z}_1, \ldots, \tilde{z}_m \) is strictly local minimum(maximum) points of \( \frac{q^2}{b} \), for \( \delta > 0 \) small enough, there exists a neighborhood \( O_{\varepsilon, i} \) of \( \tilde{z}_i, i = 1, \ldots, m \), such that the reduced function \( K(Z) \) admits at least one critical point in \( O_{\varepsilon, i} \). Hence, we get a solution \( w_\delta \) for (1.4). Let \( u_\varepsilon = |\ln \varepsilon| w_\delta \) and \( \delta = \varepsilon |\ln \varepsilon|^{\frac{1}{p+2}} \), it is not difficult to check that \( u_\varepsilon \) has all the properties listed in Theorem 1.4 and thus the proof of Theorem 1.4 is complete. \( \square \)

**Proof of Theorem 1.5.** Define
\[ \mathcal{M} = \left\{ Z = (z_1, \ldots, z_n) \in \Omega^n : |z_j - \hat{z}_j| \leq \eta, \text{dist}(z_j, \partial \Omega) \in \left( \frac{1}{|\ln \varepsilon|^\tau_1}, \frac{1}{|\ln \varepsilon|^\tau_2} \right) \right\} \]
where \( \tau_1 \) and \( \tau_2 \) will be determined later.

Consider the following minimizing problem

\[
\min_{Z \in \bar{\mathcal{M}}} K(Z).
\]

There exists a minimizer \( Z_\varepsilon \) for \( K(Z) \) in \( \bar{\mathcal{M}} \). Now, as in Theorem 1.4, we just need to verify that \( Z_\varepsilon \) is an interior point of \( \bar{\mathcal{M}} \) and hence is a critical point of \( K(Z) \).

By Proposition 6.2, we have

\[
K(Z) = \sum_{j=1}^{n} \frac{\pi \delta^2 q^2(z_j)}{\ln \frac{R}{\varepsilon}} \left( 1 + \frac{g(z_j, z_j)}{\ln \frac{R}{\varepsilon}} \right) + O \left( \frac{\delta^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).
\]

Let \( \tilde{Z}_\varepsilon = (\tilde{z}_{\varepsilon,1}, \cdots, \tilde{z}_{\varepsilon,n}) \in \bar{\mathcal{M}} \) be such that

\[
|\tilde{z}_{\varepsilon,j} - \tilde{z}_j| = \text{dist}(\tilde{z}_{\varepsilon,j}, \partial \Omega) = \frac{1}{|\ln \varepsilon|^2},
\]

then

\[
\frac{q^2(\tilde{z}_{\varepsilon,j})}{b(\tilde{z}_{\varepsilon,j})} = \frac{q^2(\tilde{z}_j)}{b(\tilde{z}_j)} + O \left( \frac{1}{|\ln \varepsilon|^2} \right), \quad g(\tilde{z}_{\varepsilon,j}, \tilde{z}_{\varepsilon,j}) = O(|\ln \varepsilon|), \quad j = 1, \cdots, n.
\]

As a result,

\[
K(\tilde{Z}_\varepsilon) = \sum_{j=1}^{n} \frac{\pi \delta^2 q^2(\tilde{z}_j)}{\ln \frac{R}{\varepsilon}} + O \left( \frac{\delta^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).
\]

Note that

\[
K(Z_\varepsilon) \leq K(\tilde{Z}_\varepsilon),
\]

we find

\[
\frac{q^2(z_{\varepsilon,j})}{b(z_{\varepsilon,j})} - \frac{q^2(\tilde{z}_j)}{b(\tilde{z}_j)} \leq C \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|} \right)
\]

and

\[
g(z_{\varepsilon,j}, z_{\varepsilon,j}) \leq C \ln |\ln \varepsilon|, \quad j = 1, \cdots, n,
\]

where \( C \) is independent of \( \tau_1 \) and \( \tau_2 \).

Hence, for \( j = 1, \cdots, n \), we have

\[
|z_{\varepsilon,j} - \tilde{z}_j| \leq C \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|} \right), \quad \text{dist}(z_{\varepsilon,j}, \partial \Omega) \geq \frac{1}{|\ln \varepsilon|^C}.
\]

Thus, \( Z_\varepsilon \) is an interior point of \( \bar{\mathcal{M}} \) if we choose \( \tau_1 \) to be sufficiently large and \( \tau_2 \) sufficiently small in the definition of \( \bar{\mathcal{M}} \).

Proof of Theorem 1.1 and 1.2. By Theorem 1.4 and 1.5, we obtain that \( u_\varepsilon \) is a solution of (1.3).

Define for \( x \in \Omega \),

\[
\begin{cases}
  v_\varepsilon = \frac{\text{curl}(u_\varepsilon - q_\varepsilon)}{b}, \\
  h = -\frac{b(u_\varepsilon - q_\varepsilon)^{p+1}}{(p+1)\varepsilon^2} - \frac{|v_\varepsilon|^2}{2}.
\end{cases}
\]
Then, $v^\varepsilon$ is a stationary solution of (1.1) with \( \text{curl} v^\varepsilon = \frac{b}{\varepsilon^2} (u^\varepsilon - q^\varepsilon)^p_+ \).

What remains to do is just to verify, as $\varepsilon \to 0$, that
\[
\int_\Omega \text{curl} v^\varepsilon \to \sum_{j=1}^m 2\pi q(\bar{z}_i) b(\bar{z}_i).
\]

By direct calculations, we can obtain for $\varepsilon$ small that
\[
\int_\Omega \text{curl} v^\varepsilon = \int_\Omega \frac{b}{\varepsilon^2} (u^\varepsilon - q^\varepsilon)^p_+ \\
= \frac{|\ln \varepsilon|^p}{\varepsilon^2} \int_{\bigcup_{i=1}^m B_{L,\delta_i}(z_i)} b(V_{\delta_i} + \omega_\delta - q)^p_+ \\
= \sum_{i=1}^m \frac{|\ln \varepsilon|^p}{\varepsilon^2} \int_{B_{L,\delta_i}(z_i)} \hat{b}_i (V_{\delta_i} + \omega_\delta - q)^p_+ + \sum_{i=1}^m \frac{|\ln \varepsilon|^p}{\varepsilon^2} \int_{B_{L,\delta_i}(z_i)} (b - \hat{b}_i) (V_{\delta_i} + \omega_\delta - q)^p_+ \\
= \sum_{i=1}^m \frac{|\ln \varepsilon|^p}{\varepsilon^2} \int_{B_{L,\delta_i}(z_i)} \hat{b}_i (V_{\delta_i,\hat{b}_i,\hat{\delta}_i} - \hat{q}_{\delta,i})^p_+ + O \left( \frac{\ln^2 |\ln \varepsilon|}{|\ln \varepsilon|} \right) \\
= \sum_{i=1}^m \frac{2\pi \hat{q}_{\delta,i}}{b_i} \ln \frac{R}{\delta_i} + O \left( \frac{\ln^2 |\ln \varepsilon|}{|\ln \varepsilon|} \right) \to \sum_{i=1}^m \frac{2\pi q(\bar{z}_i)}{b(\bar{z}_i)}.
\]

Therefore, the result follows.

\[\square\]

5 Further Results

In this section, we will use the idea and techniques in the previous sections to construct vortex pairs for the shallow water equations. For this purpose, instead of (1.3), similar to [10], we now consider the following boundary value problem:
\[
\left\{ \begin{array}{ll}
-\varepsilon^2 \text{div}(\frac{\sum u_b}{b}) = b(u - q)^p_+ - b(-u - q)^p_+, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{array} \right.
\]

where $p > 1$, $q = -u_0$, $q_\varepsilon = q \ln \frac{1}{\varepsilon}$, $\Omega \subset \mathbb{R}^2$ is smooth bounded domain.

Set $\delta = \varepsilon \left( \ln \frac{1}{\varepsilon} \right)^{1-p}$, $w = \frac{u}{|\ln \varepsilon|}$, then (5.1) becomes
\[
\left\{ \begin{array}{ll}
-\delta^2 \text{div}(\frac{\sum w}{b}) = b(w - q)^p_+ - b(-w - q)^p_+, & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega.
\end{array} \right.
\]

Let $\hat{b}^\pm_i = b(z^\pm_i)$ and $\hat{q}^\pm_{\delta,i}$ be the solutions of the following system:
\[
\left\{ \begin{array}{l}
\hat{q}^+_i = q(z^+_i) + \frac{\hat{q}^+_i}{\ln \frac{R}{\varepsilon}} g(z^+_i, z^+_i) - \sum_{k \neq i} \frac{\hat{q}^+_i}{\ln \frac{R}{\varepsilon}} G(z^+_i, z^+_k) + \sum_{l=1}^m \frac{\hat{q}^+_i}{\ln \frac{R}{\varepsilon}} G(z^+_i, z^+_l), \\
\hat{q}^-_j = q(z^-_j) + \frac{\hat{q}^-_j}{\ln \frac{R}{\varepsilon}} g(z^-_j, z^-_j) - \sum_{k \neq j} \frac{\hat{q}^-_j}{\ln \frac{R}{\varepsilon}} G(z^-_j, z^-_k) + \sum_{l=1}^m \frac{\hat{q}^-_j}{\ln \frac{R}{\varepsilon}} G(z^-_j, z^-_l),
\end{array} \right.
\]

(5.3)
where \( i = 1, \ldots, m, j = 1, \ldots, n \).

Set

\[
V_{\delta,Z,j}^\pm = PV_{\delta,b_j,\delta^\pm_{i,j},z_j^\pm}, \quad V_{\delta,Z}^+ = \sum_{i=1}^m V_{\delta,Z,i}^+, \quad V_{\delta,Z}^- = \sum_{j=1}^n V_{\delta,Z,j}^-,
\]

and

\[
J(u) = \frac{\delta^2}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega b[(u - q)^{p+1} + (-u - q)^{p+1}].
\]

Let \( s_{\delta,i}^\pm \) be the solution of

\[
\delta^2 s^\pm \cdot s^\pm \phi'(1) = \frac{(\hat{b}_{\delta}^\pm)^2 \hat{q}_{\delta,i}^\pm}{\ln \frac{s_{\delta,i}^\pm}{R}}.
\]

Then, we have

\[
\frac{1}{\ln \frac{R}{s_{\delta,i}^\pm}} = \frac{1}{\ln \frac{R}{\varepsilon}} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).
\]

Thus, as in \((2.11)\), we find, for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \) that

\[
V_{\delta,Z}^+ - V_{\delta,Z}^- - q(x) = V_{\delta,b_i,\delta_{i,j}^+,z_j^+} - \hat{q}_{\delta,i}^+ + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} g(z_i^+, z_j^+) \right), \quad x \in B_{Ls_i^+}(z_i^+),
\]

and

\[
V_{\delta,Z}^- - V_{\delta,Z}^+ - q(x) = V_{\delta,b_j,\delta_{j,i}^-,-z_i^-} - \hat{q}_{\delta,j}^- + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} g(z_j^-, z_j^-) \right), \quad x \in B_{Ls_j^-}(z_j^-).
\]

Similar to Proposition \(6.2\), we have the following energy expansion:

\[
J(V_{\delta,Z}^+ - V_{\delta,Z}^-) = \sum_{i=1}^m \frac{\pi \delta^2 g^2(z_i^+)}{\ln \frac{R}{\varepsilon} b(z_i^+)} \left( 1 + \frac{g(z_i^+, z_i^+)}{\ln \frac{R}{\varepsilon}} \right) + \sum_{j=1}^n \frac{\pi \delta^2 g^2(z_j^-)}{\ln \frac{R}{\varepsilon} b(z_j^-)} \left( 1 + \frac{g(z_j^-, z_j^-)}{\ln \frac{R}{\varepsilon}} \right) + O \left( \frac{\delta^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right) . \tag{5.4}
\]

From \((5.4)\), we can deduce the following result:

**Theorem 5.1.** Suppose that \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain. Let \( b \in C^1(\bar{\Omega}), \psi_0 \in C^2(\bar{\Omega}) \) be such that \( \text{div}(\nabla \psi_0) = 0 \) and let \( \psi_0 = \text{curl}\psi_0 \). If \( \inf_{\Omega} b > 0 \) and \( \sup_{\Omega} \psi_0 < 0 \), then for any given strictly local minimum(maximum) points \( \hat{z}_1^+, \ldots, \hat{z}_m^+, \hat{z}_1^-, \ldots, \hat{z}_n^- \) of \( \frac{\psi_0^2}{b} \), there exists \( \varepsilon_0 > 0 \), such that for each \( \varepsilon \in (0, \varepsilon_0) \), we can find a family solutions \( \psi_\varepsilon \in C^1(\Omega, \mathbb{R}^2) \) and \( h_\varepsilon \in C^1(\Omega) \) of

\[
\begin{align*}
\text{div}(b\psi_\varepsilon) &= 0, & \text{in } \Omega, \\
(\psi_\varepsilon \cdot \nabla)\psi_\varepsilon &= -\nabla h_\varepsilon, & \text{in } \Omega, \\
\psi_\varepsilon \cdot n &= \psi_0 \cdot n \ln \frac{1}{\varepsilon}, & \text{on } \partial \Omega,
\end{align*}
\]

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where \( \mathbf{n} \) is the outward normal direction. Furthermore, as \( \varepsilon \to 0 \), the corresponding vorticity \( \omega_\varepsilon := \text{curl} \mathbf{v}_\varepsilon \) satisfying

\[
\text{supp}(\omega_\varepsilon^+) \subset \bigcup_{i=1}^m B(z_{i,\varepsilon}^+, C\varepsilon) \text{ for } z_{i,\varepsilon}^+ \in \Omega, \quad i = 1, \cdots, m,
\]

\[
\text{supp}(\omega_\varepsilon^-) \subset \bigcup_{j=1}^n B(z_{j,\varepsilon}^-, C\varepsilon) \text{ for } z_{j,\varepsilon}^- \in \Omega, \quad j = 1, \cdots, n,
\]

\[
\int_{\Omega} \omega_\varepsilon \to -\sum_{i=1}^m \frac{2\pi \psi_0(z_{i,\varepsilon}^+)}{b(z_{i,\varepsilon}^+)} + \sum_{j=1}^n \frac{2\pi \psi_0(z_{j,\varepsilon}^-)}{b(z_{j,\varepsilon}^-)},
\]

\[
(z_{1,\varepsilon}^+, \cdots, z_{m,\varepsilon}^+, z_{1,\varepsilon}^-, \cdots, z_{n,\varepsilon}^-) \to (z_1^+, \cdots, z_m^+, z_1^-, \cdots, z_n^-).
\]

**Proof.** Since the arguments are similar to those used in section 3 and section 4 we will not give detail here. Let \( \omega_\delta \) be the map obtained in the reduction procedure. Define

\[
\tilde{K}(Z) = J(V_{\delta,Z}^+ - V_{\delta,Z}^- + \omega_\delta).
\]

Then, as in Lemma 4.1 we can prove

\[
\tilde{K}(Z) = J(V_{\delta,Z}^+ - V_{\delta,Z}^-) + O \left( \frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+2}} \right).
\]

Similar to Theorem 1.4 we can obtain a solution \( u_\varepsilon \) for (5.1).

Define for \( x \in \Omega \),

\[
\begin{align*}
\mathbf{v}_\varepsilon &= \frac{\text{curl}(u_\varepsilon - q_\varepsilon)}{b}, \\
h &= -\frac{b(u_\varepsilon - q_\varepsilon)_+^{p+1}}{(p+1)\varepsilon^2} + \frac{b(-u_\varepsilon - q_\varepsilon)_+^{p+1}}{(p+1)\varepsilon^2} - \frac{|\mathbf{v}_\varepsilon|^2}{2}.
\end{align*}
\]

Thus, \( \mathbf{v}_\varepsilon \) is a stationary solution of (1.1) with

\[
\text{curl} \mathbf{v}_\varepsilon = \frac{b}{\varepsilon^2}(u_\varepsilon - q_\varepsilon)_+^p - \frac{b}{\varepsilon^2}(-u_\varepsilon - q_\varepsilon)_+^p.
\]
Now, as in Theorem 1.1, we have

\[
\int_{\Omega} \text{curl } v_\xi = \int_{\Omega} \frac{b}{\varepsilon^2} (u_\xi - q_\xi)^p \text{ }_+ - \int_{\Omega} \frac{b}{\varepsilon^2} (-u_\xi - q_\xi)^p \\
= \left[ \frac{\ln \varepsilon}{\varepsilon^2} \right]^p \left( \int_{\Omega} ^m b \left( V_{\delta}^+ - V_{\delta}^- - (\omega_{\delta} - q)^p \right) \right) + \\
- \int_{\Omega} ^n b \left( V_{\delta}^+ - V_{\delta}^- - (\omega_{\delta} - q)^p \right)
\]

\[
= \sum_{i=1}^m \frac{|\ln \varepsilon|}{\varepsilon^2} \int_{B_{\delta}^{z_i} (z_i^+)} \hat{b}_i^+ \left( V_{\delta}^+ - q_{\delta,i}^+ - \hat{q}_{\delta,i}^+ \right) + O \left( \frac{\ln \varepsilon}{|\ln \varepsilon|^2} \right)
\]

\[
= \sum_{j=1}^n \frac{|\ln \varepsilon|}{\varepsilon^2} \int_{B_{\delta}^{z_j} (z_j^-)} \hat{b}_j^- \left( V_{\delta}^- - q_{\delta,j}^- - \hat{q}_{\delta,j}^- \right) + O \left( \frac{\ln \varepsilon}{|\ln \varepsilon|^2} \right)
\]

\[
= \sum_{i=1}^m \frac{2\pi \hat{q}_{\delta,i}^+}{b(\hat{z}_i^+)} \ln \frac{R}{s_{\delta,i}} - \sum_{j=1}^n \frac{2\pi \hat{q}_{\delta,j}^-}{b(\hat{z}_j^-)} \ln \frac{R}{s_{\delta,j}} + O \left( \frac{\ln \varepsilon}{|\ln \varepsilon|} \right)
\]

\[
\rightarrow \sum_{i=1}^m \frac{2\pi \hat{q}(\hat{z}_i^+)}{b(\hat{z}_i^+)} - \sum_{j=1}^n \frac{2\pi \hat{q}(\hat{z}_j^-)}{b(\hat{z}_j^-)}, \text{ as } \varepsilon \rightarrow 0.
\]

So, the result follows. \(\square\)

**Remark 5.2.** For any given strictly local minimum points \(\hat{z}_1^+, \ldots, \hat{z}_m^+, \hat{z}_1^-, \ldots, \hat{z}_n^-\) of \(\frac{\partial^2}{b}\) on the boundary \(\partial \Omega\), we can also obtain the corresponding results as in Theorem 1.2.

### 6 Technical Estimates

In this section we will give precise expansions of \(I(V_{\delta,z})\), which has been used in section 4. Let

\[
I(u) = \frac{\delta^2}{2} \int_{\Omega} \frac{1}{b} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} b(u - q)^{p+1}.
\]

Recall that

\[
V_{\delta,z} = PV_{\delta,z} = \sum_{j=1}^m V_{\delta,z,j}.
\]

**Lemma 6.1.** There is a large constant \(L > 0\) such that

\[
V_{\delta,z}(x) - q(x) < 0, \ x \in \Omega \setminus \cup_{j=1}^m B_{Ls}(z_j).
\]
Proof. The proof is similar to Lemma A.1 in [8]. For reader’s convenience, we give a sketch here.

If \( \sigma > 0 \) is small and \(|x - z_j| \geq s_{\delta,j}^\sigma\), \( j = 1, \ldots, m\),

\[
V_{\delta,Z} - q(x) = \sum_{j=1}^{m} \left( V_{\delta,\hat{b}_j,\hat{q}_{\delta,j},z_j} - \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} g(x, z_j) \right) - q(x)
\]

\[
\leq \sum_{j=1}^{m} \hat{q}_{\delta,j} \ln \frac{R}{s_{\delta,j}} - \hat{c}
\]

\[
\leq \sum_{j=1}^{m} \hat{q}_{\delta,j} \sigma (1 + o(1)) - \hat{c}
\]

\[
< 0.
\]

If \( Ls_{\delta,j} \leq |x - z_j| \leq s_{\delta,j}^\sigma\), then it follows from (2.11) that

\[
V_{\delta,Z} - q(x) = V_{\delta,\hat{b},\hat{q}_{\delta,i},z_i} - \hat{q}_{\delta,i} + O \left( \frac{\ln \ln \epsilon}{\ln \epsilon^2} g(z_i, z_i) \right)
\]

\[
\leq V_{\delta,\hat{b},\hat{q}_{\delta,i}} \left( Ls_{\delta,i} \right) - \hat{q}_{\delta,i} + O \left( \frac{\ln^2 \ln \epsilon}{\ln \epsilon^2} \right)
\]

\[
= - \frac{\hat{q}_{\delta,i} \ln L}{\ln \frac{R}{s_{\delta,i}}} + O \left( \frac{\ln^2 \ln \epsilon}{\ln \epsilon^2} \right)
\]

\[
< 0.
\]

Proposition 6.2. We have

\[
I(V_{\delta,Z}) = \sum_{j=1}^{n} \frac{\pi \delta^2}{\ln R} \frac{q^2(z_j)}{b(z_j)} \left( 1 + \frac{g(z_j, z_j)}{\ln \frac{R}{\epsilon}} \right) + O \left( \frac{\delta^2 \ln \ln \epsilon}{\ln \epsilon^2} \right).
\]

Proof. Taking advantage of (2.6), we find that

\[
\delta^2 \int_{\Omega} \frac{1}{b} |\nabla V_{\delta,Z}|^2
\]

\[
= \sum_{j=1}^{m} \int_{\Omega} \hat{b}_j V_{\delta,\hat{b}_j,\hat{q}_{\delta,j},z_j} - \hat{q}_{\delta,j} \p V_{\delta,Z,j} + \sum_{j \neq i} \int_{\Omega} \hat{b}_j V_{\delta,\hat{b}_j,\hat{q}_{\delta,j},z_j} - \hat{q}_{\delta,j} \p V_{\delta,Z,i}
\]

\[
+ \sum_{j=1}^{m} \delta^2 \int_{\Omega} \left( \frac{1}{b} - \frac{1}{\hat{b}_j} \right) |\nabla V_{\delta,Z,j}|^2 + \sum_{j \neq i} \delta^2 \int_{\Omega} \left( \frac{1}{b} - \frac{1}{\hat{b}_j} \right) \nabla V_{\delta,Z,i} \nabla V_{\delta,Z,j}.
\]

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First, we estimate 
\[
\hat{b}_j \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^p + V_{\delta, z,j} = \hat{b}_j \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^p + \hat{b}_j \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^{p+1} 
\]
\[
- \frac{\hat{b}_j \hat{q}_{\delta,j}}{\ln R_{\delta,j}} \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^p 
\]
\[
= \hat{b}_j \hat{q}_{\delta,j} \frac{2p}{p-1} \int_{B_1(0)} \phi(x) + \hat{b}_j \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^{p+1} 
\]
\[
- \frac{2\pi^2 \hat{q}_{\delta,j}^2 \hat{b}_j}{\ln R_{\delta,j}} \int_{B_1(0)} \phi(x) g(z_j + s_{\delta,j} x, z_j) 
\]
\[
= \frac{2p^2 \hat{q}_{\delta,j}^2}{b_j} \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^p + \frac{2p^2 \delta^2 g(z_j, z_j)}{b_j} + O \left( \frac{|\nabla g(z_j, z_j)| s_{\delta,j}^3}{|\ln \varepsilon|^{p+1}} \right). 
\]

Next, for \( j \neq i \),
\[
\hat{b}_j \int_{\Omega} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^p + V_{\delta, z,i} 
\]
\[
= \frac{\hat{b}_j \hat{q}_{\delta,i}}{\ln R_{\delta,i}} \int_{B_{\delta,i}(z_i)} (V_{\hat{b}_j, \hat{q}_{\delta,j}, z_j} - \hat{q}_{\delta,j})^p G(x, z_i) 
\]
\[
= \frac{\hat{b}_j \hat{q}_{\delta,i}^2 \hat{b}_j}{\ln R_{\delta,i}} \int_{B_1(0)} \phi(x) G(z_j + s_{\delta,i} x, z_i) 
\]
\[
= \frac{2p^2 \hat{q}_{\delta,i}^2}{\hat{b}_j \ln R_{\delta,i} \ln R_{\delta,j}} + O \left( \frac{|\nabla G(z_i, z_j)| s_{\delta,i}^3}{|\ln \varepsilon|^{p+1}} \right). 
\]

Note that (2.12) and \( |\frac{1}{\hat{b}_i} - \frac{1}{\hat{b}_j}| = O(|x - z_j|) \), we can obtain that 
\[
\delta^2 \int_{\Omega} \left( \frac{1}{\hat{b}_i} - \frac{1}{\hat{b}_j} \right) |\nabla V_{\delta, z,j}|^2 
\]
\[
= \delta^2 \int_{B_{\delta,j}(z_j)} \left( \frac{1}{\hat{b}_i} - \frac{1}{\hat{b}_j} \right) |\nabla V_{\delta, z,j}|^2 + \delta^2 \int_{\Omega \setminus B_{\delta,j}(z_j)} \left( \frac{1}{\hat{b}_i} - \frac{1}{\hat{b}_j} \right) |\nabla V_{\delta, z,j}|^2 
\]
\[
= O \left( \frac{\delta^2}{|\ln \varepsilon|^2} \right). 
\]

Similarly,
\[
\delta^2 \int_{\Omega} \left( \frac{1}{\hat{b}_i} - \frac{1}{\hat{b}_j} \right) \nabla V_{\delta, z,i} \nabla V_{\delta, z,j} = O \left( \frac{\delta^2}{|\ln \varepsilon|^2} \right). 
\]
By Lemma 6.1, we have
\[
\int_\Omega b(V_{\delta,Z} - q)^{p+1} = \sum_{j=1}^m \int_{B_{Ls\delta_j}(z_j)} b \left( V_{\delta_i,\hat{q}_{\delta,i},z_i}(x) - \hat{q}_{\delta,i} + O \left( \frac{\ln^2 |\ln \varepsilon|}{|\ln \varepsilon|^2} \right) \right) \right)^{p+1}.
\]
\[
= O \left( \frac{\delta^2}{|\ln \varepsilon|^2} \right).
\]

Thus, we find
\[
I(V_{\delta,Z}) = \sum_{j=1}^m \pi \delta^2 \frac{\hat{q}_{\delta,j}^2}{b_j} - \sum_{j=1}^m \pi \delta^2 g(z_j, z_j) \frac{\hat{q}_{\delta,j}^2}{b_j} + \sum_{j \neq i} \pi \delta^2 \hat{q}_{\delta,i} \hat{q}_{\delta,j} b_j \ln \frac{R}{s_{\delta,i}} \ln \frac{R}{s_{\delta,j}} G(z_i, z_j) + O \left( \frac{\delta^2}{|\ln \varepsilon|^2} \right).
\]

The result follows from \( \hat{b}_j = b(z_j) \) and the fact that
\[
\hat{q}_{\delta,i} = q(z_i) \left( 1 + \frac{g(z_i, z_i)}{\ln \frac{R}{\varepsilon}} + O \left( \frac{1}{|\ln \varepsilon|} \right) \right), i = 1, \ldots, m.
\]

References

[1] G.K. Batchelor, An introduction to Fluid Dynamics, Cambridge University Press, (1967).

[2] M.S. Berger and L.E. Fraenkel, Nonlinear desingularization in certain free-boundary problems, Comm. Math. Phys., 77(1980), 149–172.

[3] R. Camassa, D.D. Holm and C.D. Levermore, Long-time effects of bottom topography in shallow water, Phys. D, 98(1996), 258–286.

[4] R. Camassa, D.D. Holm and C.D. Levermore, Long-time shallow-water equations with a varying bottom, J. Fluid Mech., 349(1997), 173–189.

[5] D. Cao, Z. Liu and J. Wei, Regularization of point vortices for the Euler equation in dimension two. arXiv:1208.3002.

[6] D. Cao, Z. Liu and J. Wei, Regularization of point vortices for the Euler equation in dimension two, part II. arXiv:1208.5540.

[7] D. Cao, S. Peng and S. Yan, Multiplicity of solutions for the plasma problem in two dimensions, Adv. Math., 225(2010), 2741–2785.

[8] E.N. Dancer and S. Yan, The Lazer-McKenna conjecture and a free boundary problem in two dimensions, J. London Math. Soc., 78(2008), 639–662.

[9] S. De Valeriola and J. Van Schaftingen, Desingularization of vortex rings and shallow water vortices by semilinear elliptic problem. arXiv:1209.3988.
[10] L.E. Fraenkel and M.S. Berger, A global theory of steady vortex rings in an ideal fluid, *Acta Math.*, 132(1974), 13–51.

[11] F. Flucher and J. Wei, Asymptotic shape and location of small cores in elliptic free-boundary problems, *Math. Z.*, 228(1998), 638–703.

[12] G. Li, S. Yan and J. Yang, An elliptic problem related to planar vortex pairs, *SIAM J. Math. Anal.*, 36(2005), 1444–1460.

[13] Y. Li and S. Peng, Multiple solutions for an elliptic problem related to vortex pairs, *J. Diff. Equat.*, 250(2011), 3448–3472.

[14] C.C. Lin, On the motion of vortices in two dimension – I. Existence of the Kirchhoff-Routh function, *Proc. Natl. Acad. Sci. USA*, 27(1941), 570–575.

[15] G. Richardson, Vortex motion in shallow water with varying bottom topography and zero Froude number, *J. Fluid Mech.*, 411(2000), 351–374.

[16] D. Smets and J. Van Schaftingen, Desingulariation of vortices for the Euler equation, *Arch. Rational Mech. Anal.*, 198(2010), 869–925.

[17] J. Wei, D. Ye and F. Zhou, Bubbling solutions for an anisotropic Emden-Fowler equation, *C. R. Acad. Sci. Pairs, Sect. I*, 343(2006), 253–258.

[18] J. Wei, D. Ye and F. Zhou, Bubbling solutions for an anisotropic Emden-Fowler equation, *Calc. Var. Partial Differential Equations*, 28(2007), 217–247.

[19] J. Wei, D. Ye and F. Zhou, Analysis of boundary bubbling solutions for an anisotropic Emden-Fowler equation, *Ann. Inst. Poincaré Anal. Non Linéaire*, 25(2008), 425–447.

[20] J. Yang, Existence and asymptotic behavior in planar vortex theory, *Math. Models Methods Appl. Sci.*, 1(1991), 461–475.

[21] J. Yang, Global vortex rings and asymptotic behaviour, *Nonlinear Anal.*, 25(1995), 531–546.