Pseudo-Hermiticity and Electromagnetic Wave Propagation in Dispersive Media

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Abstract

Pseudo-Hermitian operators appear in the solution of Maxwell’s equations for stationary non-dispersive media with arbitrary (space-dependent) permittivity and permeability tensors. We offer an extension of the results in this direction to certain stationary dispersive media. In particular, we use the WKB approximation to derive an explicit expression for the planar time-harmonic solutions of Maxwell’s equations in an inhomogeneous dispersive medium and study the combined affect of inhomogeneity and dispersion.

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1 Introduction

A linear operator $A$ that acts in a Hilbert space is called pseudo-Hermitian, if one can find an invertible Hermitian operator $\eta$ satisfying $A^\dagger = \eta H \eta^{-1}$, [1]. This notion of pseudo-Hermiticity arises naturally in the study of non-Hermitian Hamiltonian operators, such as $H = -\frac{d^2}{dx^2} + ix^3$, that admit a real spectrum [2]. It has also interesting applications in many different areas [3]. Perhaps one of the most remarkable of these is the appearance of pseudo-Hermitian operators in classical electrodynamics [4].

Consider the propagation of electromagnetic waves in a source-free medium with arbitrary (possibly space-dependent) permittivity and permeability tensors $\varepsilon$ and $\mu$. Then, the Maxwell equations read

\begin{align}
\vec{\nabla} \cdot \vec{D} &= 0, & \vec{\nabla} \cdot \vec{B} &= 0, \\
\dot{\vec{B}} + \mathcal{D}\vec{E} &= 0, & \dot{\vec{D}} - \mathcal{D}\vec{H} &= 0,
\end{align}

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, $\vec{D} = \varepsilon \vec{E}$, $\vec{H} = \mu^{-1} \vec{B}$, an over-dot means a time-derivative, and $\mathcal{D}$ is the curl operator, e.g., $\mathcal{D}\vec{E} := \vec{\nabla} \times \vec{E}$.

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For the cases that the medium is stationary, i.e., $\epsilon$ and $\mu$ do not depend on time, we can reduce Eqs. (2) to
\[ \vec{B}(\vec{r}, t) = \vec{B}_0(\vec{r}) - \int_0^t \mathcal{D} \vec{E}(\vec{x}, \tau) d\tau, \] (3)
\[ \ddot{\vec{E}} + \Omega^2 \vec{E} = 0, \] (4)
where $\vec{B}_0(\vec{r}) := \vec{B}(\vec{r}, 0)$ and
\[ \Omega^2 := \epsilon^{-1} \mathcal{D} \mu^{-1} \mathcal{D}. \] (5)
In view of (4), the initial-value problem for Maxwell’s equations admits the formal solution [4]:
\[ \vec{E}(\vec{r}, t) = \cos(\Omega t) \vec{E}_0(\vec{r}) + \Omega^{-1} \sin(\Omega t) \dot{\vec{E}}_0(\vec{r}), \] (6)
where $\vec{E}_0(\vec{r}) := \vec{E}(\vec{r}, 0)$, and $\dot{\vec{E}}_0(\vec{x}) := \dot{\vec{E}}(\vec{r}, 0) = \epsilon^{-1} \mathcal{D} \mu^{-1} \vec{B}_0(\vec{r})$.

We can view $\mathcal{D}$, $\epsilon$, $\mu$, $\epsilon^{-1}$ and $\mu^{-1}$ as linear operators acting in the vector space of complex-valued vector fields $\vec{F} : \mathbb{R}^3 \to \mathbb{C}^3$. Endowing this space with the $L^2$-inner product, $\langle \vec{F}, \vec{G} \rangle := \int_{\mathbb{R}^3} \vec{F}(\vec{r})^* \cdot \vec{G}(\vec{r}) d^3x$, we find a Hilbert space $\mathcal{H}$ in which the curl operator $\mathcal{D}$ acts as a Hermitian operator, i.e., for all $\vec{F}, \vec{G} \in \mathcal{H}$ belonging to the domain of $\mathcal{D}$, we have $\langle \vec{F}, \mathcal{D} \vec{G} \rangle = \langle \mathcal{D} \vec{F}, \vec{G} \rangle$. For a lossless (and gainless) medium, the permittivity and permeability tensors (and their inverses) also define Hermitian operators acting in $\mathcal{H}$. This in turn implies $\Omega^2 \equiv \epsilon \Omega^2 \epsilon^{-1}$, i.e., as an operator acting in $\mathcal{H}$, $\Omega^2$ is a pseudo-Hermitian operator.

In fact, because for a lossless medium $\epsilon(\vec{r})$ is a positive-definite matrix for all $\vec{r}$, $\epsilon$ is a positive-definite operator, and $\Omega^2$ belongs to a special class of pseudo-Hermitian operators, called quasi-Hermitian [3]. A basic property of quasi-Hermitian operators is that they are related to Hermitian operators via similarity transformations [6]. For example, we can relate $\Omega^2$ to a Hermitian operator $h$ according to
\[ \Omega^2 \longrightarrow h := \epsilon^{1/2} \Omega^2 \epsilon^{-1/2} = \epsilon^{-1/2} \mathcal{D} \mu^{-1} \mathcal{D} \epsilon^{-1/2}. \] (7)
This observation has been used in [4] to devise a method of solving the initial-value problem for Maxwell’s equations. This method applies to arbitrary stationary inhomogeneous and anisotropic media, but it ignores the effects of dispersion. The purpose of the present article is to extend the approach of [4] to stationary dispersive media.

## 2 Time-Harmonic Solutions and Dispersion

In standard textbook discussions of the propagation of the electromagnetic waves in dispersive media, one usually begins by assuming a harmonic time-dependence for the electromagnetic waves,
\[ \vec{E}(\vec{r}, t) = e^{-i\omega t} \vec{E}(\vec{r}), \quad \vec{B}(\vec{r}, t) = e^{-i\omega t} \vec{B}(\vec{r}), \] (8)
and considers the possibility that the permittivity and permeability tensors $\epsilon$ and $\mu$ depend also on the frequency $\omega$. In Eqs. (8), $\vec{E}$ and $\vec{B}$ are time-independent complex-valued vector fields that
in view of (11) are subject to the constraints
\[
\vec{\nabla} \cdot (\hat{e}^\perp \vec{E}) = 0, \\
\vec{\nabla} \cdot \vec{B} = 0.
\] (9) (10)

Because according to (8) we can express the magnetic field in terms of the electric fields, we will confine our attention to the dynamics of the latter.

First, we consider the case of non-dispersive material.

Inserting the first of Eqs. (8) in (4) yields
\[
\Omega^2 \vec{E} = \omega^2 \vec{E}.
\] (11)

This reveals the interesting fact that the solutions of Maxwell’s equation having harmonic time-dependence are actually obtained from the eigenfunctions of the pseudo-Hermitian operator \(\Omega^2\).

In general, the eigenvalues of \(\Omega^2\) are highly degenerate, and the imposition of (9) does not lift this degeneracy completely. In the case that the wave propagates in vacuum, we have \(\hat{\varepsilon} = \varepsilon_0 \hat{1}\), \(\hat{\mu} = \mu_0 \hat{1}\), and \(\Omega^2 = c^2 \vec{D}^2\). Therefore, the eigenvalue equation (11) subject to the condition (9) coincides with that of the Laplacian subject to the same condition,
\[
-\nabla^2 \vec{\varphi} = k^2 \vec{\varphi}, \quad \vec{\nabla} \cdot \vec{\varphi} = 0.
\] (12)

Here \(k := \omega/c\) and the superscript \(\varnothing\) refers to the vacuum solution. As is well-known, Eqs. (12) admit the following complete set of plane-wave solutions
\[
\vec{E}_{\varnothing \hat{k}, \hat{e}^\pm_k}(\vec{r}, t) := \frac{e^{i \vec{k} \cdot \vec{r}}}{(2\pi)^{3/2}} \hat{e}^\pm_k(t),
\] (13)

where \(\hat{k}\) and \(\hat{e}^\pm_k\) are any pair of orthogonal unit vectors, \(\hat{e}^-_k := \hat{k} \times \hat{e}^+_k\), and \(\hat{k} := k \hat{k}\). The unit vectors \(\hat{k}\) and \(\hat{e}^+_k\) play the role of the degeneracy labels for the eigenvectors. Substituting (13) for \(\vec{E}(\vec{r})\) in (8), we find the following solutions of Maxwell’s equations having harmonic time-dependence
\[
\vec{E}_{\varnothing \hat{k}, \hat{e}^\pm_k}(\vec{r}, t) := \frac{e^{i(k \cdot \vec{r} - \omega t)}}{(2\pi)^{3/2}} \hat{e}^\pm_k(t).
\] (14)

The general solution of Maxwell’s equations in vacuum is a superposition of these plane wave solutions,
\[
\vec{E}(\vec{x}, t) = \int_{\mathbb{R}^3} d^3\vec{k} \int_0^{2\pi} d\varphi_k \left[ c_+(\vec{k}, \varphi_k) \vec{E}_{\varnothing \hat{k}, \hat{e}^+_k(\varphi_k)}(\vec{x}, t) + c_-(\vec{k}, \varphi_k) \vec{E}_{\varnothing \hat{k}, \hat{e}^-_k(\varphi_k)}(\vec{x}, t) \right].
\] (15)

Here for each \(\vec{k} \neq \vec{0}\) we fix a polar coordinate system \((\rho, \varphi_k)\) in the plane normal to \(\vec{k}\), \(\hat{e}^+_k(\varphi)\) is the unit vector specified by the polar angle \(\varphi_k\) in this plane, \(\hat{e}^-_k(\varphi_k) = \hat{k} \times \hat{e}^+_k(\varphi_k)\), and \(c_\pm\) are complex-valued mode functions.

Now, consider the more general case that the medium is locally inhomogeneous and/or anisotropic. In this case, we will suppose that the degeneracy structure of the eigenvectors of \(\Omega^2\) is the same as in
the case of vacuum. Therefore, we denote these eigenvectors by $\delta_{k,\pm}^\omega$, and express the corresponding solutions of Maxwell’s equations with harmonic time-dependence \([8]\) as
\[
\tilde{E}_{k,\pm}(\vec{r}, t) = e^{-i\omega t} \delta_{k,\pm}^\omega(\vec{r}).
\] (16)

It is usually customary to take the general solution of Maxwell’s equations to be a superposition of these time-harmonic solutions.

For the case that the medium is lossless, $\Omega^2$ is a quasi-Hermitian operator with a complete set of eigenvectors $\delta_{k,\pm}^\omega$. This ensures the existence of an eigenfunction expansion for $\Omega^2$ and provides the mathematical justification for the claim that the most general solution $\tilde{E}$ of Maxwell’s equations is a superposition of $\tilde{E}_{k,\pm}$:
\[
\tilde{E}(\vec{r}, t) = \int_{\mathbb{R}^3} d^3k \int_0^{2\pi} d\varphi_k \left[ c_+(k, \varphi_k) \tilde{E}_{k,\pm}^+(\varphi_k)(\vec{x}, t) + c_-(k, \varphi_k) \tilde{E}_{k,\pm}^-(\varphi_k)(\vec{x}, t) \right].
\] (17)

Next, consider the case of a stationary dispersive medium. Then, one usually accounts for the $\omega$-dependence of $\tilde{\varepsilon}$ and $\tilde{\mu}$ in determining the time-Harmonic solutions $\tilde{E}_{k,\pm}^\omega$ and considers the superposition of these solutions to describe waves having more general initial conditions, e.g., localized wave packets. This requires solving \([11]\). Note, however, that the operator $\Omega^2$ depends on $\omega$. Therefore, this equation can no longer be viewed as a standard eigenvalue equation. Yet one can still perform the transformation \([7]\) to determine an $\omega$-dependent operator $h$ and express the solutions of Eq. \([11]\) in terms of those of
\[
h \tilde{\psi} = \omega^2 \tilde{\psi},
\] (18)
according to
\[
\tilde{\vec{E}} = \tilde{\varepsilon}^{-\frac{1}{2}} \tilde{\vec{E}}.
\] (19)

In the absence of dispersion, $h$ is a genuine Hermitian operator with a complete orthonormal set of eigenvectors. Ref. \([4]\) uses this observation to evaluate the right-hand side of the formal solution of Maxwell’s equation given by \([11]\). Here we explore the consequences of allowing $\tilde{\varepsilon}, \tilde{\mu}$, and $h$ to depend on $\omega$. Because a direct extension of the results of \([4]\) to general dispersive media is beyond our knowledge, we will confine our attention to separable cases where
\[
\tilde{\varepsilon}(\vec{r}, \omega) = \varepsilon_1(\omega)\tilde{\varepsilon}_2(\vec{r}), \quad \tilde{\mu}(\vec{r}, \omega) = \mu_1(\omega)\tilde{\mu}_2(\vec{r}),
\] (20)
$\varepsilon_1, \mu_1$ are positive real-valued functions of $\omega$, and $\tilde{\varepsilon}_2, \tilde{\mu}_2$ are positive $3 \times 3$ matrix-valued functions of $\vec{r}$. The separability condition \([20]\) is not quite realistic, but as we will see in the following, it provides us with a concrete analytically treatable class of toy models. Furthermore, we can use it to establish the completeness of the eigenfunctions of $h$.

1In terms of the transformed permeability and permittivity tensors, $\tilde{\varepsilon}(t, t', \vec{r}, \vec{r}')$ and $\tilde{\mu}(t, t', \vec{r}, \vec{r}')$, that appear in the constitutive relations \([11]\): $\tilde{B}(\vec{r}, t) = \int_{-\infty}^{t} dt' \int_{\mathbb{R}^3} d^3\vec{r}' \tilde{\varepsilon}(t, t', \vec{r}, \vec{r}') \tilde{E}(\vec{r}', t')$, $\tilde{B}(\vec{r}, t) = \int_{-\infty}^{t} dt' \int_{\mathbb{R}^3} d^3\vec{r}' \tilde{\mu}(t, t', \vec{r}, \vec{r}') \tilde{H}(\vec{r}', t')$. The separability condition \([20]\) takes the form $\tilde{\varepsilon}(t, t', \vec{r}, \vec{r}') = \varepsilon_1(t - t')\tilde{\varepsilon}_1(\vec{r})\delta(\vec{r}' - \vec{r}), \tilde{\mu}(t, t', \vec{r}, \vec{r}') = \mu_1(t - t')\tilde{\mu}_1(\vec{r})\delta(\vec{r}' - \vec{r})$. 

4
For the cases for which (20) holds, the \( \omega \)-dependence of \( \Omega^2 \) and \( h \) factors, and we have

\[
\Omega^2 = n_1(\omega)^2 \Omega^2_2, \quad h = n_1(\omega)^2 h_2, \tag{21}
\]

where

\[
n_1(\omega) := \sqrt{\varepsilon_1(\omega) \mu_1(\omega)}, \quad \Omega^2_2 := \varepsilon_2^{-1} \mathcal{D} \mu_2^{-1} \mathcal{D}, \quad h_2 := \varepsilon_2^{-1} \mathcal{D} \mu^{-1}_2 \mathcal{D} \varepsilon_2^{-\frac{1}{2}}. \tag{22}
\]

As seen from (22), \( h_2 \) is a Hermitian (\( \omega \)-independent) operator acting in \( \mathcal{H} \), and the eigenfunctions \( \psi \) of \( h \) with eigenvalue \( \omega^2 \) coincide with the eigenfunctions \( \tilde{\psi}_2 \) of \( h_2 \) with eigenvalue \( \omega^2 n_1(\omega)^2 \).

Because \( h_2 \) is Hermitian, the latter form a complete orthonormal basis of \( \mathcal{H} \). This in turn implies that the eigenfunctions of \( h \) will also form a complete orthonormal basis of \( \mathcal{H} \) provided that the function

\[
f(\omega) := \omega n_1(\omega) \tag{23}
\]

is a monotonically increasing function of \( \omega \) and we normalize the eigenfunctions properly. This together with (19) constitute a mathematical basis for expanding the general solution of Maxwell’s equations in terms of time-Harmonic solutions for dispersive media satisfying (20).

3 Planar Waves in an Inhomogeneous Dispersive Medium

Consider an isotropic but inhomogeneous dispersive medium defined by

\[
\varepsilon = \varepsilon(z, \omega) \hat{i}, \quad \mu = \mu(z, \omega) \hat{j}, \tag{24}
\]

where \( \varepsilon \) and \( \mu \) are positive real-valued functions. Suppose that a linearly polarized planar wave propagates along the \( z \)-axis in this medium in such a way that both the electric and magnetic fields are independent of the \( x \)- and \( y \)-coordinates. The initial data have the form

\[
\bar{E}_0(\vec{r}) = E_0(z) \hat{i}, \quad \bar{B}_0(\vec{r}) = B_0(z) \hat{j}, \tag{25}
\]

where \( \hat{i} \) and \( \hat{j} \) are respectively the unit vectors along the \( x \)- and \( y \)-axes.

Introducing \( p := -i \frac{d}{dz} \), we can express the operators \( \Omega^2 \) and \( h \) as

\[
\Omega^2 = \varepsilon(z, \omega)^{-1} p \mu(z, \omega)^{-1} p, \quad h = \varepsilon(z, \omega)^{-1/2} p \mu(z, \omega)^{-1/2} \varepsilon(z, \omega)^{-1/2}. \tag{26}
\]

Therefore, similarly to the non-dispersive case considered in [4], Eq. (18) reads as

\[
- \frac{1}{\sqrt{\varepsilon(z, \omega)}} \frac{d}{dz} \left[ \frac{1}{\mu(z, \omega)} \frac{d}{dz} \left( \frac{\psi(z)}{\sqrt{\varepsilon(z, \omega)}} \right) \right] = \omega^2 \psi(z). \tag{27}
\]

Here we have dropped the vector sign from \( \tilde{\psi} \), because it is always parallel to the \( x \)-axis; \( \tilde{\psi}(z) = \psi(z) \hat{i} \).

Eq. (27) is the time-independent Schrödinger equation for a point particle moving along the \( z \)-axis and having a position- and energy-dependent-mass. Following [4], we will employ WKB approximation to solve this equation. This yields solutions of the form

\[
\tilde{\psi}_\omega(z) := \frac{\nu(\omega)}{\sqrt{v(\omega)}} e^{i\omega u(z, \omega)}, \tag{28}
\]

5
where \( \nu(\omega) \) are nonzero complex normalization constants, and

\[
\begin{align*}
    u(z, \omega) &:= \int_0^z \frac{d\delta}{v(3, \omega)}, \\
    v(z, \omega) &:= \frac{1}{\sqrt{\varepsilon(z, \omega)\mu(z, \omega)}}.
\end{align*}
\]  
(29)

Substituting (28) for \( \vec{\psi} \) in (19) and using (24), we find a set of solutions \( \vec{E}_\omega \) of Eq. (11). Using these in (8) yields the time-harmonic solutions for the system. In order to address the completeness of these solutions, we should address the completeness of (28). We are able to do this for the separable cases where

\[
\varepsilon(z, \omega) = \varepsilon_1(\omega)\varepsilon_2(z), \quad \mu(z, \omega) = \mu_1(\omega)\mu_2(z).
\]  
(30)

The result is the following WKB-approximate eigenfunctions of \( h \).

\[
\psi_\omega(z) := \sqrt{\frac{f'(\omega)}{2\pi v_2(z)}} e^{i f(\omega) u_2(z)},
\]  
(31)

where \( f \) is the function given in (23), a prime stands for a derivative, and

\[
\begin{align*}
    v_2(z) &:= \frac{1}{\sqrt{\varepsilon_2(z)\mu_2(z)}}, \\
    u_2(z) &:= \int_0^z \frac{d\delta}{v_2(\delta)}.
\end{align*}
\]  
(32)

Under the assumption that \( f \) is a monotonically increasing function, we can easily establish the following orthonormality and completeness relations for \( \psi_\omega \).

\[
\int_{-\infty}^{\infty} \psi_\omega(z) \psi_\omega'(z) \, dz = \delta(\omega - \omega'), \quad \int_{-\infty}^{\infty} \psi_\omega(z) \psi_\omega(z')^* \, d\omega = \delta(z - z').
\]  
(33)

Having obtained a complete set of eigenfunctions of \( h \), we can construct time-Harmonic WKB-approximate solutions of Maxwell’s equations. In view of (8) and (19), these have the form

\[
\vec{E}_\omega(z, t) = \frac{e^{-i\omega t} \psi_\omega(z) \hat{i}}{\sqrt{\varepsilon_1(\omega)\varepsilon_2(\omega)}} \left( \frac{\mu_1(\omega)\mu_2(z)}{\varepsilon_1(\omega)\varepsilon_2(\omega)} \right)^{1/4} \sqrt{\frac{1}{2\pi}} \left[ 1 + \frac{\omega n_1'(\omega)}{n_1(\omega)} \right] \exp \{ i\omega [n_1(\omega)u_2(z) - t] \} \hat{i}.
\]  
(34)

Note that the condition of the validity of the WKB approximation is the same as in the non-dispersive case (\( \varepsilon_1 = 1 \)) discussed in [4], namely

\[
\frac{v_2^2}{2} \left| \frac{2v_2 v_2' - v_2^2}{2v_2^2} + \frac{2\mu_2\mu_2' - 3\mu_2^2}{2\mu_2^2} \right| \ll \omega^2,
\]  
(35)

where we have suppressed the \( z \)-dependence of \( v_2 \) and \( \mu_2 \) for simplicity.

In light of (35), the general WKB solution of Maxwell’s equations with initial conditions of the form (25) reads

\[
\vec{E}(z, t) = \int_{-\infty}^{\infty} c(\omega) \vec{E}_\omega(z, t) \, d\omega,
\]  
(36)

where the modulus of the mode function \( c(\omega) \) is supposed to be negligibly small for all \( \omega \) violating (35). We can use (33), (34), and (36) to express the mode function \( c(\omega) \) in terms of the initial electric field. This yields

\[
c(\omega) = \sqrt{\varepsilon_1(\omega)} \int_{-\infty}^{\infty} \sqrt{\varepsilon_2(z)} \psi_\omega(z)^* E_0(z) \, dz,
\]  
(37)
where $\mathcal{E}_0(z) := \vec{E}(z, 0) \cdot \hat{i}$. Therefore, the condition of the validity of the WKB approximation restricts the choice of the initial condition.

If the medium happens to be homogeneous, i.e., $\varepsilon_2$ and $\mu_2$ are constants, the WKB approximation is exact, $v_2$ is constant, $u_2$ takes the form $u_2(z) = z/v_2$, and the time-harmonic solution \((34)\) coincides with a plane wave, $A e^{i(kz-\omega t)}$, where

$$A := \left(\frac{\mu(\omega)}{\varepsilon(\omega)}\right)^{1/4} \sqrt{\frac{1}{2\pi} \left[ 1 + \frac{\omega n_1'(\omega)}{n_1(\omega)} \right]},$$

$$k := \frac{\omega n_1(\omega)}{v_2} = \omega \sqrt{\varepsilon(\omega)} \mu(\omega).$$

Therefore, if we consider the time-evolution of an initial plane wave:

$$\mathcal{E}_0(z) = A e^{ikz},$$

we will not be able to observe the effect of dispersion. The situation is the opposite in an inhomogeneous dispersive medium. Because of the $z$-dependence of $v_2$ and $\varepsilon_2$, inserting \((40)\) in \((37)\) gives a mode function $c(\omega)$ that is not proportional to a Dirac delta function. Consequently, the shape of the field changes in time; in an inhomogeneous medium an initial plane wave feels the effect of dispersion.

For example, consider a nonmagnetic material ($\mu_1(\omega) = 1$, $\mu_2(z) = \mu_0$) with a Lorentzian inhomogeneity:

$$\varepsilon_2(z) = \varepsilon_0 \left( 1 + \frac{a}{1 + z^2/\gamma^2} \right), \quad a, \gamma \in \mathbb{R}^+, \quad (41)$$

and an arbitrary dispersion relation given by $\varepsilon_1(\omega)$. Then to the first order in the strength of the inhomogeneity $a$, the mode function \((37)\) corresponding to the plane wave \((40)\) takes the form

$$c(\omega) = \mathfrak{A}(\omega) \left\{ \delta(k - \mathfrak{R}(\omega)) + \left[ \frac{e^{-\gamma|k-\mathfrak{R}(\omega)|}}{8(k - \mathfrak{R}(\omega))} \right] \gamma a \right\} + \mathcal{O}(a^2),$$

where

$$\mathfrak{A}(\omega) := A \sqrt{\frac{2\pi \varepsilon_0 \varepsilon_1(\omega)^{3/2}}{c}} \left( 1 + \frac{\omega \varepsilon_1'(\omega)}{2\varepsilon_1(\omega)} \right), \quad \mathfrak{R}(\omega) := \frac{\omega \sqrt{\varepsilon_1(\omega)}}{c}. \quad (43)$$

The term in the square bracket in \((42)\) reflects the dispersion of the plane wave \((40)\) due to the inhomogeneity. It is interesting to see that this term is exponentially suppressed for the wave numbers violating the dispersion relation $k = \mathfrak{R}(\omega)$.

### 4 Conclusion

In this paper we have examined the role of pseudo-Hermitian operators in the description of electromagnetic waves propagating in a stationary, possibly inhomogeneous or anisotropic dispersive medium.

In the absence of dispersion the properties of the pseudo-Hermitian operator $\Omega^2$ associated with the Maxwell equations lead to a powerful spectral method for solving the initial-value problem for these equations \([4]\). In particular, for an effectively one-dimensional model, it yields an
explicit expression for the propagating wave provided that one can employ the WKB approximation. When the medium is dispersive the same approach can not be pursued. Nevertheless, the pseudo-Hermitian operator $\Omega^2$ still generates the time-harmonic solutions as its eigenfunctions. This raises the question of the completeness of these eigenfunctions. In this article we addressed this question for the cases that the frequency- and the space-dependence of the permittivity and permeability tensors are separable. Although this condition was rather unrealistic, it allowed for a concrete implementation of our general method. In particular, for an effectively one-dimensional (planar) system with this separability property, we were able to construct a complete set of WKB-approximate eigenfunctions of $\Omega^2$ and study the combined effect of dispersion and inhomogeneity showing that unlike for a homogenous medium in an inhomogeneous medium a plane wave also undergoes dispersion.

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