INDEPENDENCE RATIO AND RANDOM EIGENVECTORS IN TRANSITIVE GRAPHS

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A theorem of Hoffman gives an upper bound on the independence ratio of regular graphs in terms of the minimum $\lambda_{\text{min}}$ of the spectrum of the adjacency matrix. To complement this result we use random eigenvectors to gain lower bounds in the vertex-transitive case. For example, we prove that the independence ratio of a 3-regular transitive graph is at least

\[ q = \frac{1}{2} - \frac{3}{4\pi} \arccos \left( \frac{1 - \lambda_{\text{min}}}{4} \right). \]

The same bound holds for infinite transitive graphs: we construct factor of i.i.d. independent sets for which the probability that any given vertex is in the set is at least $q - o(1)$.

We also show that the set of the distributions of factor of i.i.d. processes is not closed w.r.t. the weak topology provided that the spectrum of the graph is uncountable.

1. Introduction.

1.1. The independence ratio and the minimum eigenvalue. An independent set is a set of vertices in a graph, no two of which are adjacent. The independence ratio of a graph $G$ is the size of its largest independent set divided by the total number of vertices. If $G$ is regular, then the independence ratio is at most $1/2$, and it is equal to $1/2$ if and only if $G$ is bipartite.

The adjacency matrix of a $d$-regular graph has real eigenvalues between $-d$ and $d$. The least eigenvalue $\lambda_{\text{min}}$ is at least $-d$, and it is equal to $-d$ if and only if the graph is bipartite.

So the distance of the independence ratio from $1/2$ and the distance of $\lambda_{\text{min}}$ from $-d$ both measure how far a $d$-regular graph is from being bipartite. The following natural question arises: what kind of connection is there between these two graph parameters?

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A theorem of Hoffman [8] gives a partial answer to this question. It says that the independence ratio of a $d$-regular graph is at most

\[ \frac{-\lambda_{\min}}{d - \lambda_{\min}} = \frac{1}{2} - \frac{(1/2)(\lambda_{\min} + d)}{2d - (\lambda_{\min} + d)}; \]

for a simple proof, see [5], Theorem 11; also see [13], Section 4, for certain improvements.

Hoffman’s bound implies that $\lambda_{\min} \to -d$ as the independence ratio tends to $1/2$. The converse statement is not true in general: it is easy to construct $d$-regular graphs with $\lambda_{\min}$ arbitrarily close to $-d$ and the independence ratio separated from $1/2$. However, for transitive graphs the converse is also true. A graph $G$ is said to be vertex-transitive (or transitive in short) if its automorphism group $\text{Aut}(G)$ acts transitively on the vertex set $V(G)$.

**THEOREM 1.** Let $G$ be a finite, $d$-regular, vertex-transitive graph with least eigenvalue $\lambda_{\min}$. Then the independence ratio of $G$ is at least

\[ \frac{1}{2} - \frac{1}{3}\sqrt{d(\lambda_{\min} + d)}. \]

In particular, if $\lambda_{\min} \to -d$, then the independence ratio converges to $1/2$.

The idea behind the proof is to consider random eigenvectors with eigenvalue $\lambda_{\min}$. Let $\lambda$ be an arbitrary eigenvalue of the adjacency matrix of some transitive graph $G$, and let $E_{\lambda}$ denote the eigenspace corresponding to $\lambda$, that is, the space of eigenvectors with eigenvalue $\lambda$. (Note that $E_{\lambda}$ is typically more than one dimensional, since $G$ is transitive.) Furthermore, let $S_{\lambda}$ be the unit sphere in $E_{\lambda}$. Now we pick a uniform random vector from $S_{\lambda}$. Note that $S_{\lambda}$ is $\text{Aut}(G)$-invariant, therefore the distribution of this random vector is $\text{Aut}(G)$-invariant, too. Let us choose the vertices $v$ with the property that the value of the eigenvector at $v$ is larger than at each neighbor of $v$. (If $\lambda$ is negative, then we expect many of the vertices with positive value to have this property.) Clearly, these vertices form an independent set. Since our random vector is invariant, the probability $q$ that a given vertex is chosen is the same for all vertices. Therefore the expected size of this random independent set is $q|V(G)|$, and consequently, the independence ratio of $G$ is at least $q$. An estimate of $q$ yields Theorem 1 above. In many cases we obtain much sharper bounds.

When the graph has a lot of symmetry (e.g., when any pair of neighbors of a fixed vertex can be mapped to any other pair by a suitable graph automorphism), then the probability $q$ defined above is actually determined by $\lambda$. In this case it equals $q_d(\lambda)$, the relative volume of the $(d - 1)$-dimensional regular spherical simplex defined by normal vectors with pairwise scalar product $d^{-2} - \frac{\lambda}{2(d-1)}$, see Definition 2.9. There is a simple formula for $q_3(\lambda)$; see Theorem 3.

We conjecture that $q \geq q_d(\lambda)$ for arbitrary transitive graphs (provided that $\lambda$ is sufficiently small). In other words, the worst-case scenario is when the graph
has a lot of symmetry. Of course, this would yield a lower bound \( q_d(\lambda_{\min}) \) for the independence ratio. We managed to prove this conjecture for 3-regular transitive graphs and 4-regular arc-transitive graphs. We also showed that a natural geometric conjecture would imply the \( d \)-regular, arc-transitive case. [A graph is said to be arc-transitive or symmetric if for any two pairs of adjacent vertices \((u_1, v_1)\) and \((u_2, v_2)\), there is an automorphism of the graph mapping \( u_1 \) to \( u_2 \) and \( v_1 \) to \( v_2 \).]

The following theorems were obtained.

**Theorem 2.** Suppose that \( G \) is a finite, \( d \)-regular, arc-transitive graph with least eigenvalue \( \lambda_{\min} \). Then the independence ratio of \( G \) is at least

\[
\frac{1}{2} - \frac{1}{3}\sqrt{\lambda_{\min}} + d.
\]

In fact, a natural geometric conjecture (see Conjecture 2.13) would imply that the independence ratio is at least \( q_d(\lambda_{\min}) \). This has been proven in the case \( d = 4 \): the independence ratio of a finite, 4-regular, arc-transitive graph is at least

(2) \[ q_4(\lambda_{\min}) \geq \frac{1}{2} - \frac{1}{4}\sqrt{\lambda_{\min}} + 4. \]

**Theorem 3.** Suppose that \( G \) is a finite, 3-regular, vertex-transitive graph with minimum eigenvalue \( \lambda_{\min} \). Then the independence ratio of \( G \) is at least

\[
q_3(\lambda_{\min}) = \frac{1}{8} + \frac{3}{4\pi}\arcsin\left(\frac{1 - \lambda_{\min}}{4}\right) = \frac{1}{2} - \frac{3}{4\pi}\arccos\left(\frac{1 - \lambda_{\min}}{4}\right).
\]

In fact, the following stronger statement holds: \( G \) contains two disjoint independent sets \( I_1, I_2 \) with total size \(|I_1 \cup I_2| \geq 2q_3(\lambda_{\min})|V(G)|\). This means that the induced subgraph \( G[I_1 \cup I_2] \) is bipartite and has at least \( 2q_3(\lambda_{\min})|V(G)| \) vertices.

See Figure 1 to compare the lower bound given in Theorem 3 to Hoffman’s upper bound (1). Note that \(-3 \leq \lambda_{\min} \leq -2\) for any 3-regular transitive graph with

![FIG. 1. Hoffman’s upper bound (1) and the lower bound of Theorem 3 for \( \lambda_{\min} \in [-3, -1] \).](image-url)
the only exception of the complete graph $K_4$ for which $\lambda_{\min} = -1$; see Proposition A.3 in the Appendix.

1.2. Random wave functions on infinite transitive graphs. In order to generalize the above theorems, we define random wave functions on infinite transitive graphs $G$. A wave function with eigenvalue $\lambda$ on $G$ is a function $f : V(G) \to \mathbb{R}$ such that

$$\sum_{u \in N(v)} f(u) = \lambda f(v) \quad \text{for each vertex } v \in V(G),$$

where $N(v)$ denotes the set of neighbors of $v$ in $G$. So a wave function is basically an eigenvector of the adjacency operator of $G$, except that it does not need to be in $\ell^2(V(G))$.

These random wave functions will also let us answer an open question concerning factor of i.i.d. processes. Suppose that we have independent standard normal random variables $Z_u$ assigned to each vertex $u$ of an infinite transitive graph $G$. By a factor of i.i.d. process on $G$ we mean random variables $X_v, v \in V(G)$ that are all obtained as measurable functions of the random variables $Z_u, u \in V(G)$ and that are $\text{Aut}(G)$-equivariant [i.e., they commute with the natural action of $\text{Aut}(G)$]. It is easy to see that for any factor of i.i.d. process $X_v, v \in V(G)$ with $0 < \text{var}(X_v) < \infty$, the correlation of $X_v$ and $X_{v'}$ converges to 0 as the distance of $v$ and $v'$ goes to infinity; see Proposition A.4 in the Appendix. So a random process that is 0 everywhere with probability 1/2 and 1 everywhere with probability 1/2 cannot be a factor of i.i.d. However, it can be seen easily that this process can be approximated by factor of i.i.d. processes provided that $G$ is amenable. So the space of factor of i.i.d. processes is not closed; that is, the distributions of these processes do not form a closed set w.r.t. the weak topology. It has been an open question whether the same is true on nonamenable graphs, for example, on the $d$-regular tree; see [1], Section 4, Question 4. We will show that the space of factor of i.i.d. processes is not closed provided that the spectrum of $G$ is uncountable.

We say that a factor of i.i.d. process $X_v, v \in V(G)$ is a linear factor of i.i.d. if each $X_v$ is obtained as a (possibly infinite) linear combination of $Z_u, u \in V(G)$. Note that linear factors have the following properties.

**Definition 1.1.** We call a collection of random variables $X_v, v \in V(G)$ a Gaussian process on $G$ if they are jointly Gaussian, and each $X_v$ is centered (i.e., has mean 0). (Random variables are jointly Gaussian if any finite linear combination of them is Gaussian.) We say that a Gaussian process $X_v$ is $\text{Aut}(G)$-invariant (or simply invariant) if for any $\Phi \in \text{Aut}(G)$ the joint distribution of the Gaussian process $X_{\Phi(v)}$ is the same as that of the original process.

We will prove that the adjacency operator $A_G$ has approximate eigenvectors (satisfying a certain invariance property) for any $\lambda$ in the spectrum $\lambda \in \sigma(A_G)$. 
Then we will use these approximate eigenvectors as coefficients to define linear factor of i.i.d. processes converging in distribution to an invariant Gaussian process $X_v$ that satisfies the eigenvector equation at each vertex.

**Theorem 4.** Let $G$ be an infinite vertex-transitive graph with adjacency operator $A_G$. Then for each point $\lambda$ of the spectrum $\sigma(A_G)$ there exists a nontrivial invariant Gaussian process $X_v$, $v \in V(G)$ such that

\[
\sum_{u \in N(v)} X_u = \lambda X_v \quad \text{for each vertex } v \in V(G),
\]

where $N(v)$ denotes the set of neighbors of $v$ in $G$. Furthermore, the process $X_v$ can be approximated (in distribution) by linear factor of i.i.d. processes. Clearly, we can assume that these approximating linear factors have only finitely many nonzero coefficients.

An invariant Gaussian process satisfying (3) will be called a **Gaussian wave function** with eigenvalue $\lambda$. If the spectrum of $G$ is not countable, then we can conclude that some of these Gaussian wave functions cannot be obtained as factor of i.i.d. processes.

**Theorem 5.** Let $G$ be an infinite transitive graph such that the spectrum of the adjacency operator $A_G$ is not countable. Then there exist (linear) factor of i.i.d. processes on $G$ with the property that the weak limit of their distributions cannot be obtained as the distribution of a factor of i.i.d. process.

We can say more for Cayley graphs.

**Theorem 6.** Suppose that $G$ is the Cayley graph of a finitely generated infinite group. Then a Gaussian wave function with eigenvalue $\lambda_{\text{max}} \overset{\text{def}}{=} \sup \sigma(A_G)$ can never be obtained as the distribution of a factor of i.i.d. process.

In view of Theorems 4 and 6 there exists a Gaussian wave function with eigenvalue $\lambda_{\text{max}}$ that can be approximated by factor of i.i.d. processes but cannot be obtained as one. An independent and different proof of this result was given by Russell Lyons in the special case when $G$ is a regular tree [12], Corollary 3.3.

1.3. **Factor of i.i.d. independent sets.** Let $X_v$, $v \in V(G)$ be a random process on our infinite transitive graph $G$. As in the finite setting, $I_+ \overset{\text{def}}{=} \{v : X_v > X_u, \forall u \in N(v)\}$ is a random independent set. If our process is invariant, then the probability that $v \in I_+$ is the same for each vertex $v$, and thus this probability can be used to measure the size of $I_+$. If our process is a factor of some i.i.d. process $Z_v$, then the resulting independent set is also a factor of $Z_v$. 
In the infinite setting let $\lambda_{\min}$ denote the minimum of the spectrum $\sigma(A_G)$, and let $X_v$ be a linear factor of $Z_v$ approximating the Gaussian eigenvector with eigenvalue $\lambda_{\min}$; see Theorem 4. As the process $X_v$ converges in distribution to the Gaussian eigenvector, the probability $P(v \in I_\epsilon)$ approaches the corresponding probability for the Gaussian eigenvector process, which, as we will see, can be computed the exact same way as in the finite case.

**Theorem 7.** Theorems 1, 2 and 3 give lower bounds $q$ (in terms of $\lambda_{\min}$) for the independence ratio of finite transitive graphs with least eigenvalue $\lambda_{\min}$. These bounds remain true in the following framework. Let $\lambda_{\min}$ denote the minimum of the spectrum of an infinite transitive graph $G$. Then for any $\epsilon > 0$ there exists a factor of i.i.d. independent set on $G$ such that the probability that any given vertex is in the set is at least $q - \epsilon$.

A special case of this infinite setting was investigated in [4]. When $G$ is the $d$-regular tree $T_d$, then any factor of i.i.d. independent set on $G$ automatically gives a lower bound for the independence ratio of $d$-regular finite graphs with sufficiently large girth. In particular, for the 3-regular tree, $T_3$ one has $\lambda_{\min} = -2\sqrt{2}$. Therefore the infinite version of Theorem 3 tells us that there exists factor of i.i.d. independent set in $T_3$ with density

$$\frac{1}{2} - \frac{3}{4\pi} \arccos\left(\frac{1 + 2\sqrt{2}}{4}\right) \approx 0.4298.$$

In [4] the somewhat better bound 0.4361 was obtained. In fact, [4] was the starting point for the work in the present paper. For previous results on the independence ratio of large-girth graphs, see [3, 9, 11, 14–16].

2. Finite vertex-transitive graphs. Throughout this section $G$ will denote a vertex-transitive, finite graph with degree $d$ for some positive integer $d \geq 3$. The least eigenvalue of its adjacency matrix $A_G$ will be denoted by $\lambda_{\min}$. For now let $\lambda$ be an arbitrary eigenvalue of $A_G$. Eventually, we will choose $\lambda$ as the minimum eigenvalue. First we define what we mean by a random eigenvector.

**Definition 2.1.** Let $E_\lambda$ be the eigenspace corresponding to $\lambda$, that is,

$$E_\lambda = \{ x \in \ell_2(V(G)) : A_Gx = \lambda x \}.$$

We fix some orthonormal basis $e_1, \ldots, e_l$ in $E_\lambda$, and take independent standard normal random variables $\gamma_1, \ldots, \gamma_l$. We call $\sum_{i=1}^l \gamma_i e_i$ the random eigenvector with eigenvalue $\lambda$.

**Remark 2.2.** The (distribution of the) random eigenvector is clearly independent of the choice of the basis $e_1, \ldots, e_l$, so it is well defined. It also follows
that the distribution of the random eigenvector is $\text{Aut}(G)$-invariant. (Note that in the Introduction we defined the random eigenvector differently: a uniform random vector on the unit sphere of $E_\lambda$, which is just the normalized version of the random eigenvector of Definition 2.1.)

We will think of this random eigenvector as a collection of real-valued random variables $X_v, v \in V(G)$ with the property that they are jointly Gaussian and $\text{Aut}(G)$-invariant, each $X_v$ is centered, and

$$\sum_{u \in N(v)} X_u = \lambda X_v$$

for each vertex $v$, where $N(v)$ denotes the set of neighbors of $v$ in $G$. Since $G$ is transitive, each $X_v$ has the same variance. After multiplying these random variables with a suitable positive constant we might assume that $\text{var}(X_v) = 1$ for each vertex $v$. Next we define random independent sets by means of these random eigenvectors.

**Definition 2.3.** Let $X_v, v \in V(G)$ denote the random eigenvector corresponding to the eigenvalue $\lambda$ as explained above. The random sets $I_+$ and $I_-$ are defined as follows:

$$I_+ = I_+^\lambda \overset{\text{def}}{=} \{ v \in V(G) : X_v > X_u \text{ for each } u \in N(v) \}$$

and

$$I_- = I_-^\lambda \overset{\text{def}}{=} \{ v \in V(G) : X_v < X_u \text{ for each } u \in N(v) \}.$$  

Clearly, $I_+$ and $I_-$ are disjoint (random) independent sets in $G$.

The $\text{Aut}(G)$-invariance implies that the probability of the event $v \in I_+$ is the same for all vertices $v$. So from now on, we will focus on a fixed vertex and its neighbors. First we introduce the following notation.

**Notation 2.4.** Let $v$ be an arbitrary vertex of our vertex-transitive graph $G$. We will call $v$ the root. The neighbors of $v$ are denoted by $w_1, \ldots, w_d$. For $X_v$ and $X_{w_i}$ we will simply write $X$ and $Y_i$, respectively. We will assume that $\text{var}(X) = 1$, which implies that $\text{var}(Y_i) = 1$ for each $i$. Since $X_v, v \in V(G)$ is the random eigenvector with eigenvalue $\lambda$, we have

$$\sum_{i=1}^d Y_i = \lambda X. \quad (4)$$

The covariance $\text{cov}(Y_i, Y_j)$ will be denoted by $c_{i,j}$. Then it readily follows from (4) that

$$\lambda^2 = \text{cov}(\lambda X, \lambda X) = \sum_{i,j} c_{i,j} = d + 2 \sum_{i<j} c_{i,j}, \quad \text{thus } \sum_{i<j} c_{i,j} = \frac{\lambda^2 - d}{2}. \quad (5)$$
We also introduce the random variables \( U_i \overset{\text{def}}{=} X - Y_i \).

Since \( Y_1, \ldots, Y_d \) are centered and jointly Gaussian, they can be written as the linear combinations of independent standard normal variables: there exist independent standard Gaussians \( Z_1, \ldots, Z_d \) and (deterministic) vectors \( y_1, \ldots, y_d \in \mathbb{R}^d \) such that \( Y_i \) is the inner product of \( y_i \) and \( Z = (Z_1, \ldots, Z_d) \). Setting \( x = (y_1 + \cdots + y_d)/\lambda \) and \( u_i = x - y_i \), we have

\[
Y_i = y_i \cdot Z; \quad X = x \cdot Z; \quad U_i = u_i \cdot Z.
\]

It is easy to see that for any deterministic vectors \( a, b \in \mathbb{R}^d \) the covariance \( \text{cov}(a \cdot Z, b \cdot Z) \) is equal to the inner product \( a \cdot b \). In particular,

\[
x \cdot x = \text{var}(X) = 1; \quad y_i \cdot y_j = \text{cov}(Y_i, Y_j) = c_{i,j};
\]

(6)

\[
u_i \cdot u_j = \text{cov}(U_i, U_j).
\]

Finally, we introduce the following notation for the pairwise angles of the vectors \( u_i \):

\[
\varphi_{i,j} \overset{\text{def}}{=} \angle(u_i, u_j) = \arccos \left( \frac{u_i \cdot u_j}{\|u_i\| \|u_j\|} \right).
\]

(7)

Our goal is to give estimates for the probability that a certain vertex lies in our random independent set \( I_+ \). As we will see, this probability can be expressed as the volume of a certain spherical simplex.

**Definition 2.5.** Let \( S^{d-1} \) denote the unit sphere in \( \mathbb{R}^d \). A half-space is said to be **homogeneous** if the defining hyperplane (i.e., the boundary of the half-space) passes through the origin. A vector \( n \) orthogonal to the defining hyperplane and “pointing outward” is called an **outer normal vector**. Then the given (open) half-space consists of those \( x \in \mathbb{R}^d \) for which the inner product \( n \cdot x \) is negative.

A \((d - 1)\)-dimensional spherical simplex is the intersection of \( S^{d-1} \) and \( d \) homogeneous half-spaces in \( \mathbb{R}^d \). Up to congruence, a spherical simplex is determined by the \( \binom{d}{2} \) pairwise angles enclosed by the outer normal vectors of the \( d \) half-spaces. If these \( \binom{d}{2} \) angles are all equal, then we say that the spherical simplex is **regular**.

**Proposition 2.6.** The probability that any fixed vertex is in the random independent set \( I_+ \) is equal to the relative volume of the \((d - 1)\)-dimensional spherical simplex corresponding to the outer normal vectors \(-u_i\).

**Proof.** The probability in question is

\[
P(v \in I_+) = P(X > Y_i, 1 \leq i \leq d) = P(U_i > 0, 1 \leq i \leq d).
\]

(8)
the intersection of the homogeneous half-spaces with outer normal vectors \(-u_i\) with respect to the standard multivariate Gaussian measure on \(\mathbb{R}^d\). This is simply the volume of the corresponding \((d - 1)\)-dimensional spherical simplex divided by the volume \(\text{vol}(S^{d-1})\) of the unit sphere \(S^{d-1}\). Note that this relative volume is determined by the pairwise angles \(\varphi_{i,j}\) [see (7) in Notation 2.4], which, in turn, can be expressed using the inner products \(y_i \cdot y_j = c_{i,j}\). □

The probability \(P(v \in I_+\) seems to be the smallest when \(G\) has a lot of symmetry. To make this more precise, we first define what we mean by a “lot of symmetry.”

**Definition 2.7.** We say that \(G\) is *cherry-transitive* if any cherry (path of length 2) in \(G\) can be mapped to any other cherry using a suitable graph automorphism of \(G\).

**Proposition 2.8.** If \(G\) is cherry-transitive, then

\[
c_{i,j} = \frac{\lambda^2 - d}{d(d-1)} \quad \text{for all } i \neq j,
\]

and, consequently, the pairwise angles \(\varphi_{i,j}\) are all equal to

\[
\arccos\left(\frac{d - 2 - \lambda}{2(d - 1)}\right).
\]

**Proof.** If \(G\) is cherry-transitive, then for any \(i_1 \neq j_1\) and \(i_2 \neq j_2\) there exists an automorphism \(\Phi \in \text{Aut}(G)\) such that \(\Phi\) fixes the root \(v\) and takes the unordered pair \((w_{i_1}, w_{j_1})\) to \((w_{i_2}, w_{j_2})\), that is,

\[
\Phi v = v, \quad \Phi w_{i_1} = \Phi w_{i_2}, \quad \Phi w_{j_1} = \Phi w_{j_2}
\]

or

\[
\Phi v = v, \quad \Phi w_{i_1} = \Phi w_{j_2}, \quad \Phi w_{j_1} = \Phi w_{i_2}.
\]

Together with the \(\text{Aut}(G)\)-invariance of the random eigenvector this implies that \(c_{i_1, j_1} = c_{i_2, j_2}\). Since this holds for any two pairs of indices, it follows that all \(c_{i,j}\), \(i \neq j\) are the same. Using (5) we conclude that for \(i \neq j\)

\[
c_{i,j} = \frac{\lambda^2 - d}{d(d-1)}.
\]

Then easy calculation shows (using notations introduced earlier) that

\[
\|u_i\|^2 = \|u_j\|^2 = \frac{2(d - \lambda)}{d} \quad \text{and} \quad u_i \cdot u_j = \frac{(d - \lambda)(d - 2 - \lambda)}{d(d - 1)}.
\]
Plugging this into (7) gives
\[ \varphi_{i,j} = \arccos\left( \frac{d - 2 - \lambda}{2(d - 1)} \right). \]
\[ \square \]

We are now in a position to define the functions \( q_d(\lambda) \).

**Definition 2.9.** For \(-d \leq \lambda \leq d\), let \( q_d(\lambda) \) denote the volume of the \((d - 1)\)-dimensional regular spherical simplex corresponding to the angle \( (9) \) divided by \( \text{vol}(S^{d-1}) \). Then \( P(v \in I_+) = q_d(\lambda) \) for any cherry-transitive \( G \). In particular, the independence ratio of any cherry-transitive graph \( G \) is at least \( q_d(\lambda_{\min}) \).

So \( P(v \in I_+) = q_d(\lambda) \) provided that \( G \) has enough symmetry. The following conjecture says that in the general (i.e., vertex-transitive) case the probability should be larger than that.

**Conjecture 2.10.** For any transitive graph \( G \) it holds that
\[ P(v \in I_+) \geq q_d(\lambda) \]
for any \( \lambda \), or at least for sufficiently small \( \lambda: \lambda \leq \lambda_0 \) for some \( \lambda_0 \).

This would, of course, imply that the independence ratio of \( G \) is at least \( q_d(\lambda_{\min}) \) provided that \( \lambda_{\min} \leq \lambda_0 \).

We will prove this conjecture for \( d = 3 \) and \( \lambda_0 = -2 \) in Section 2.1. The conjecture might be true for arbitrary \( \lambda \), but proving for \( \lambda \leq \lambda_0 = -2 \) will be sufficient for our purposes, because \( \lambda_{\min} \leq -2 \) for any 3-regular transitive graph except \( K_4 \).

**Remark 2.11.** In view of formula (8) the above conjecture would follow from the following statement. Let \((U_1, \ldots, U_d)\) be a multivariate Gaussian with each \( U_i \) centered and with pairwise covariances \( m_{i,j} \). We have the following constraints for the covariances:
\[ \sum_{1 \leq i, j \leq d} m_{i,j} = (d - \lambda)^2 \quad \text{and for each } i : m_{i,i} = \frac{2}{d - \lambda} \sum_{j=1}^{d} m_{i,j}. \]
[Or one might replace these by the weaker constraints \( \sum_{1 \leq i, j \leq d} m_{i,j} = (d - \lambda)^2 \) and \( \sum_{1 \leq i \leq d} m_{i,i} = 2(d - \lambda) \).] Then the orthant probability \( P(U_i > 0, 1 \leq i \leq d) \) is minimized when all \( m_{i,i}, 1 \leq i \leq d \) are equal and also all \( m_{i,j}, i \neq j \) are equal.

A few properties of the functions \( q_d(\lambda) \) are collected in the next proposition.

**Proposition 2.12.** For any \( d \geq 3 \), \( q_d \) is a monotone decreasing continuous function on \([-d, -1]\) with
\[ q_d(-d) = \frac{1}{2} \quad \text{and} \quad q_d(-1) = \frac{1}{d + 1}. \]
As for the behavior of $q_d$ around $-d$ we have

$$q_d(\lambda) \geq \frac{1}{2} - \frac{\pi \text{vol}(S^{d-2})}{4 \text{vol}(S^{d-1})} \sqrt{\frac{\lambda + d}{d}} \geq \frac{1}{2} - \frac{1}{3} \sqrt{\lambda + d}.$$  

**Proof.** The volume of a regular spherical simplex is clearly a continuous and monotone decreasing function of the corresponding angle. Since (9) is a continuous and monotone increasing function of $\lambda \in [-d, -1]$, monotonicity and continuity of $q_d$ follow.

For $\lambda = -d$ the angles $\varphi_{i,j}$ are $0$, so the corresponding (degenerate) spherical simplex is a hemisphere, thus $q_d(-d) = 1/2$ as claimed.

For $\lambda = -1$ the angles $\varphi_{i,j}$ are $\pi/3$. It is not hard to see that the vertices of our spherical simplex in that case will be the $d$ vertices of a face of a regular (Euclidean) simplex in $\mathbb{R}^d$. Then each of the $d + 1$ spherical simplices belonging to the $d + 1$ faces has volume $\text{vol}(S^{d-1})/(d + 1)$. [We could also argue that for $G = K_{d+1}$ and $\lambda = -1$ we have $P(v \in I_+) = 1/(d + 1)$, and since $K_{d+1}$ is cherry-transitive, $P(v \in I_+) = q_d(-1).$]

See Section 2.3 for a proof of the claimed behavior around $-d$. □

2.1. The 3-regular, vertex-transitive case. Now we turn to the proof of Theorem 3 that gives a lower bound for the independence ratio of 3-regular transitive graphs. We will basically show that Conjecture 2.10 is true when $d = 3$ and $\lambda_0 = -2$.

As we have seen, $P(v \in I_+)$ equals the relative volume of a certain spherical simplex. For $d = 3$ the surface of the unit sphere $S^{d-1} = S^2$ is $4\pi$ and the area of a spherical triangle with interior angles $\alpha, \beta, \gamma$ is equal to $\alpha + \beta + \gamma - \pi$. The spherical triangle in question is determined by the homogeneous half-spaces with outer normal vectors $-u_1, -u_2, -u_3$ (recall Proposition 2.6). We denoted the angle enclosed by the outer normal vectors $-u_i$ and $-u_j$ by $\varphi_{i,j}$. Then the interior angle at the intersection of the two corresponding planes is clearly $\pi - \varphi_{i,j}$. Therefore $P(v \in I_+)$ equals the relative surface area of a spherical triangle with angles $\pi - \varphi_{1,2}, \pi - \varphi_{1,3}, \pi - \varphi_{2,3},$

$$P(v \in I_+) = \frac{1}{4\pi} \left( \sum_{1 \leq i < j \leq 3} (\pi - \varphi_{i,j}) - \pi \right)$$  

$$= \frac{1}{4\pi} \left( \frac{\pi}{2} + \sum_{1 \leq i < j \leq 3} \left( \frac{\pi}{2} - \varphi_{i,j} \right) \right)$$  

(10)  

$$= \frac{1}{4\pi} \left( \frac{\pi}{2} + \sum_{1 \leq i < j \leq 3} \arcsin \left( \frac{u_i \cdot u_j}{\|u_i\| \|u_j\|} \right) \right).$$
By Proposition 2.8 we have \( c_{i,j} = (\lambda^2 - 3)/6 \) and \( \varphi_{i,j} = \arccos((1 - \lambda)/4) \) in the cherry-transitive case, thus
\[
q_3(\lambda) = \frac{1}{8} + \frac{3}{4\pi} \arcsin\left(\frac{1 - \lambda}{4}\right) = \frac{1}{2} - \frac{3}{4\pi} \arccos\left(\frac{1 - \lambda}{4}\right).
\]

**Proof of Theorem 3.** The statement of the theorem is true for the complete graph \( K_4 \) as the independence ratio is 1/4 and the minimum eigenvalue is \(-1\) in that case. For any other 3-regular transitive graph \( G \) we have \( \lambda_{\min} \leq -2 \); see Proposition A.3 in the Appendix. Therefore it suffices to prove that \( P(v \in I_+) \geq q_3(\lambda) \), whenever \( \lambda \leq -2 \).

Recall that \( Y_1, Y_2, Y_3 \) are standard Gaussians with pairwise covariances \( c_{i,j} \).

Therefore the matrix
\[
\begin{pmatrix}
1 & c_{1,2} & c_{1,3} \\
c_{1,2} & 1 & c_{2,3} \\
c_{1,3} & c_{2,3} & 1
\end{pmatrix}
\]
is positive semidefinite. In particular, its determinant is nonnegative:
\[
1 + 2c_{1,2}c_{1,3}c_{2,3} - c_{1,2}^2 - c_{1,3}^2 - c_{2,3}^2 \geq 0.
\]

Furthermore, according to (5) we have \( c_{1,2} + c_{1,3} + c_{2,3} = (\lambda^2 - 3)/2 \geq 1/2 \), because \( \lambda \leq -2 \). It follows that each \( c_{i,j} \) must be between \(-1/2\) and 1.

Indeed, let \( x, y, z \) be real numbers between \(-1\) and 1 with \( x + y + z \geq 1/2 \) and \( 1 + 2xyz - x^2 - y^2 - z^2 \geq 0 \). Assume that \( z < -1/2 \). Then
\[
0 \leq 1 + 2xyz - x^2 - y^2 - z^2 = 1 + 2(z + 1)xy - (x + y)^2 - z^2
\]
\[
\leq 1 + 2(z + 1)\left(\frac{x + y}{2}\right)^2 - (x + y)^2 - z^2 = 1 + z - 1 \left(\frac{x + y}{2}\right)^2 - z^2
\]
\[
\leq 1 + \frac{z - 1}{2} - \left(\frac{1}{2} - z\right)^2 - z^2 < 0,
\]
contradiction. Therefore \( z \geq -1/2 \). Similarly, \( x, y \leq -1/2 \), too.

Next we bound \( u_i \cdot u_j / (\|u_i\| \|u_j\|) \) from below. Using (6), \( x = (y_1 + y_2 + y_3)/\lambda \) and \( c_{1,2} + c_{1,3} + c_{2,3} = (\lambda^2 - 3)/2 \)
\[
x \cdot y_1 = \frac{1}{\lambda} \left(1 + c_{1,2} + c_{1,3}\right) = \frac{1}{\lambda} \left(1 + \frac{\lambda^2 - 3}{2} - c_{2,3}\right) = \frac{\lambda}{2} - \frac{1}{2\lambda} - \frac{1}{\lambda} c_{2,3},
\]
\[
\|u_1\|^2 = \|x - y_1\|^2 = 2 - 2x \cdot y_1 = 2 - \lambda + \frac{1}{\lambda} + \frac{2}{\lambda} c_{2,3}.
\]

Similar formulas hold for \( x \cdot y_i \) and \( \|u_i\|, i = 2, 3 \). By the inequality of arithmetic and geometric means it follows that
\[
\|u_1\|\|u_2\| \leq \frac{\|u_1\|^2 + \|u_2\|^2}{2} = 2 - \lambda + \frac{1}{\lambda} + \frac{1}{\lambda} (c_{1,3} + c_{2,3}) = -\frac{1}{\lambda} \left(\frac{1}{2} - 2\lambda + \frac{\lambda^2}{2} + c_{1,2}\right).
\]
Note that this holds with equality when all $c_{i,j}$ are equal. Furthermore,

$$u_1 \cdot u_2 = (x - y_1) \cdot (x - y_2) = 1 + c_{1,2} - x \cdot (y_1 + y_2)$$

$$= 1 + c_{1,2} + x \cdot (y_3 - \lambda x)$$

$$= 1 + c_{1,2} + \left(\frac{\lambda}{2} - \frac{1}{2\lambda} - \frac{1}{\lambda} c_{1,2}\right) - \lambda$$

$$= \frac{-1}{\lambda} \left(\frac{1}{2} - \lambda + \frac{\lambda^2}{2} + (1 - \lambda)c_{1,2}\right).$$

It follows that

$$\frac{u_1 \cdot u_2}{\|u_1\| \|u_2\|} \geq \frac{(1/2) - \lambda + (\lambda^2/2) + (1 - \lambda)c_{1,2}}{(1/2) - 2\lambda + (\lambda^2/2) + c_{1,2}},$$

because the numerator is positive (note that $-3 \leq \lambda \leq -2$ and $c_{1,2} \geq -1/2$). The analogous inequality holds for any other pair of indices $i, j$. Since arcsin is a monotone increasing function, (10) yields that

$$P(v \in I_+) \geq \frac{1}{8} + \frac{1}{4\pi} \sum_{1 \leq i < j \leq 3} \arcsin\left(\frac{(1/2) - \lambda + (\lambda^2/2) + (1 - \lambda)c_{i,j}}{(1/2) - 2\lambda + (\lambda^2/2) + c_{i,j}}\right).$$

Setting

$$f(t) \overset{\text{def}}{=} \arcsin\left(\frac{(1/2) - \lambda + (\lambda^2/2) + (1 - \lambda)t}{(1/2) - 2\lambda + (\lambda^2/2) + t}\right),$$

we have

$$P(v \in I_+) \geq \frac{1}{8} + \frac{1}{4\pi} \sum_{1 \leq i < j \leq 3} f(c_{i,j}).$$

(12)

On the other hand,

$$q_3(\lambda) = \frac{1}{8} + \frac{3}{4\pi} f\left(\frac{\lambda^2 - 3}{6}\right),$$

which follows from (11) and the definition of $f$. [It also follows from the fact that when each $c_{i,j}$ is equal to $(\lambda^2 - 3)/6$, then (12) should hold with equality.] In view of (12) and (13) we need to show that

$$\frac{1}{3} \sum_{1 \leq i < j \leq 3} f(c_{i,j}) \geq f\left(\frac{\lambda^2 - 3}{6}\right),$$

where each $c_{i,j}$ is between $-1/2$ and 1, and their average is $(\lambda^2 - 3)/6$. This, of course, would follow from the convexity of $f$. Unfortunately, $f$ is not convex on the entire interval $[-1/2, 1]$. We claim, however, that the tangent line to $f$ at $t_0 = \lambda$
\[(\lambda^2 - 3)/6\] is below \(f\) on the entire interval \([-1/2, 1]\), which still implies (14). The rather technical proof of this claim can be found in the Appendix (Lemma A.7).

Now let \(\lambda = \lambda_{\text{min}} \leq -2\), then \(P(v \in I_+) \geq q_3(\lambda_{\text{min}})\). So the expected size of the random independent set \(I_+\) is at least \(q_3(\lambda_{\text{min}})|V(G)|\); thus the independence ratio of \(G\) is at least \(q_3(\lambda_{\text{min}})\).

To prove the second part of the statement we notice that the random independent set \(I_-\) (see Definition 2.3) has the same expected size. Indeed, if we replace \(X_v, v \in V(G)\) with \(X'_v = -X_v\), then \(X'_v, v \in V(G)\) have the same joint distribution and the roles of \(I_+\) and \(I_-\) interchange. Since \(I_+\) and \(I_-\) are always disjoint, the expected size of their union \(I_+ \cup I_-\) is at least \(2q_3(\lambda_{\text{min}})|V(G)|\). Consequently, there must exist disjoint independent sets \(I_1, I_2\) in \(G\) with \(|I_1 \cup I_2| \geq 2q_3(\lambda_{\text{min}})|V(G)|\). \(\square\)

For graphs with very large odd-girth, Theorem A.1 of the Appendix gives a slightly better bound. The proof is based on the same random eigenvector, but uses a different method to find large independent sets.

### 2.2. The arc-transitive case

The following innocent-looking, and very plausible, conjecture is open in dimension \(n \geq 4\).

**Conjecture 2.13.** Let \(S\) be a sphere in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\). We have \(n+1\) spherical caps with the same given radius on \(S\). We want to find the configuration for which the volume of the union of the caps is maximal. It is conjectured that this optimal configuration is always the one where the \(n+1\) centers are the vertices of a regular simplex in \(\mathbb{R}^n\).

The statement of the conjecture is trivial for \(n = 2\), while the \(n = 3\) case follows from the so-called moment theorem of Fejes Tóth [6], Theorem 2; see also [7], Section 34. The general case would follow from the following conjecture: the volume of the intersection of a fixed spherical cap and a spherical simplex of fixed volume is maximal when the spherical simplex is regular, and its center coincides with the center of the spherical cap (Gábor Fejes Tóth, personal communication, 2012).

In what follows we will explain how the case \(n = d - 1\) of Conjecture 2.13 implies that \(P(v \in I_+) \geq q_d(\lambda)\) holds for every \(d\)-regular arc-transitive graph \(G\), and consequently the independence ratio of \(G\) is at least \(q_d(\lambda_{\text{min}})\). In particular, the \(d = 4\) case follows from the \(n = 3\) case of the conjecture, which is known to be true; see Theorem 2. Using our previous notation, \(P(v \in I_+)\) is the volume of the spherical simplex \(T\) determined by the half-spaces with outer normal vectors \(-u_i, i = 1, \ldots, d\), while \(q_d(\lambda)\) is the volume of the same simplex in the case when all the angles \(\phi_{i,j} = \angle(u_i, u_j), i \neq j\) are the same. In other words, we need to show that the volume of the spherical simplex \(T\) is minimal when the angles \(\angle(u_i, u_j)\) are the same.

If \(G\) is arc-transitive, then the covariances \(\text{cov}(X, Y_i) = x \cdot y_i\) are all equal. Since

\[x \cdot y_1 + \cdots + x \cdot y_d = x \cdot (y_1 + \cdots + y_d) = x \cdot (\lambda x) = \lambda,
\]
we get that \( x \cdot y_i = \lambda / d \) for each \( i \). It follows that the angle enclosed by \( x \) and \( u_i \)

\[
\angle(x, u_i) = \delta \overset{\text{def}}{=} \frac{\pi - \arccos(\lambda / d)}{2} = \frac{\arccos(-\lambda / d)}{2}
\]

for each \( i \).

Now let \( S_l \) be the set of points on \( S^{d-1} \) that has some fixed distance \( l \) from \( x \); thus \( S_l \) is a \((d - 2)\)-dimensional sphere for any \( l \). The intersection of \( S_l \) and the half-space with outer normal vector \( u_i \) is a spherical cap of radius depending only on \( l \) and \( \lambda \). So the intersection of \( S_l \) and our spherical simplex \( T \) can be obtained by removing \( d \) spherical caps of the same given radius from \( S_l \). If Conjecture 2.13 is true for \( n = d - 1 \), then the total volume of the removed area is maximal for the “regular configuration” when each \( \angle(u_i, u_j) \) is the same. Therefore the \((d - 2)\)-dimensional volume of \( T \cap S_l \) is minimal for the regular configuration for any \( l \). It follows that the \((d - 1)\)-dimensional volume of \( T \) is also minimal for the regular configuration, and this is what we wanted to prove.

2.3. Bounds near \(-d\). Even if Conjecture 2.13 is not assumed to be true, the above observations yield a lower bound for the independence ratio of \( d \)-regular arc-transitive graphs in the case when the least eigenvalue is close to \(-d\).

**Proof of Theorem 2.** We already explained in Section 2.2 why Conjecture 2.13 implies that the independence ratio is at least \( q_d(\lambda_{\min}) \).

Next we prove the first part of Theorem 2. As we have seen in (15), \( \angle(x, u_i) = \delta \) for each \( i \), which means that each point of \( S^{d-1} \) at (spherical) distance less than \( \pi / 2 - \delta \) from \( x \) is contained in our spherical simplex \( T \). These points form a spherical cap with center \( x \) and radius \( \pi / 2 - \delta \). (In fact, this spherical cap is the “inscribed ball” of \( T \).) Using (15) and that \( \arccos(t) \leq \pi / 2 \sqrt{1 - t} \) for any \( t \in [0, 1] \), we get

\[
\delta = \frac{\arccos(-\lambda / d)}{2} \leq \frac{\pi}{4} \sqrt{1 + \lambda / d}
\]

provided that \( \lambda \leq 0 \).

This spherical cap can be obtained by taking the hemisphere (around \( x \)), and removing a strip of “width” \( \delta \) (in spherical distance). The volume of this strip is clearly at most \( \delta \text{vol}(S^{d-2}) \). Therefore the volume of the spherical cap is at least \( \text{vol}(S^{d-1}) / 2 - \delta \text{vol}(S^{d-2}) \), whence

\[
P(v \in I_+) \geq \frac{\text{vol}(S^{d-1}) / 2 - \delta \text{vol}(S^{d-2})}{\text{vol}(S^{d-1})} = \frac{1}{2} - \frac{\pi \text{vol}(S^{d-2})}{4 \text{vol}(S^{d-1})} \sqrt{\frac{\lambda + d}{d}}.
\]

For \( d = 4 \) we have \( \text{vol}(S^2) / \text{vol}(S^3) = (4\pi) / (2\pi^2) = 2 / \pi \), so the bound is

\[
\frac{1}{2} - \frac{1}{4} \sqrt{\lambda + 4}.
\]
For general $d$, we use the estimate $\frac{1}{2} - \frac{\sqrt{\pi}}{4\sqrt{2}} \sqrt{\lambda + d} > \frac{1}{2} - \frac{\sqrt{\pi}}{3} \sqrt{\lambda + d}$.

These are lower bounds for the probability $P(v \in I_+)$, in particular, for $q_d(\lambda)$. Thus the first part of Theorem 2 follows, as well as the estimate (2) for $q_4(\lambda)$ and the last statement of Proposition 2.12. □

PROOF OF THEOREM 1. In the general (vertex-transitive) case, we use that

$$x \cdot y_1 + \cdots + x \cdot y_d = \lambda \quad \text{and} \quad x \cdot y_j \geq -1,$$

Therefore the angle $\angle (x, y_i)$ is at least $\arccos(\lambda + d - 1)$. Using that $\arccos(t) \leq \frac{\pi}{2} \sqrt{1 - t}$ for any $t \in [0, 1]$, it follows that

$$\angle (x, u_i) \leq \delta' \overset{\text{def}}{=} \frac{\pi - \arccos(\lambda + d - 1)}{2} = \frac{\arccos(1 - \lambda - d)}{2} \leq \frac{\pi}{4} \sqrt{\lambda + d}$$

provided that $\lambda \leq -d + 1$. This means that our spherical simplex $T$ contains the spherical cap with center $x$ and radius $\pi/2 - \delta'$. Therefore

$$P(v \in I_+) \geq \frac{\text{vol}(S^{d-1})/2 - \delta' \text{vol}(S^{d-2})}{\text{vol}(S^{d-1})} = \frac{1}{2} - \frac{\pi \text{ vol}(S^{d-2})}{4 \text{ vol}(S^{d-1})} \sqrt{\lambda + d} \geq \frac{1}{2} - \frac{\sqrt{\pi}}{4\sqrt{2}} \sqrt{d(\lambda + d)}.$$

Since $\sqrt{\pi}/(4\sqrt{2}) < 1/3$, Theorem 1 follows. □

3. Infinite transitive graphs.

3.1. Random wave functions. Our goal now is to generalize the random eigenvectors we introduced in Section 2 for infinite transitive graphs $G$. For an infinite graph $G$ the adjacency operator $A_G : \ell_2(V(G)) \to \ell_2(V(G))$ might not have any eigenvectors (i.e., the point spectrum might be empty). So the approach we used in the finite setting will not work here. Instead, we will define random wave functions as the limit of linear factor of i.i.d. processes. The coefficients of these linear factors will be approximate eigenvectors of $A_G$ that are invariant under automorphisms fixing some root $x \in V(G)$. We start with proving that such approximate eigenvectors exist for any $\lambda$ in the spectrum $\sigma(A_G)$. Let $\text{Stab}_x(G)$ denote the stabilizer subgroup, that is, the group of automorphisms fixing $x$.

**Theorem 3.1.** Let $G$ be an infinite vertex-transitive graph with adjacency operator $A_G$ and with some fixed root $x$. Then for any $\varepsilon > 0$ and any $\lambda$ in the spectrum $\sigma(A_G)$, there exists a $\text{Stab}_x(G)$-invariant vector $\alpha \in \ell_2(V(G))$ such that

$$\|\alpha\| = 1 \quad \text{and} \quad \|A_G\alpha - \lambda\alpha\| \leq \varepsilon.$$
PROOF. Consider the projection-valued measure $P_\lambda$ corresponding to the self-adjoint operator $A_G$. This “measure” assigns an orthogonal projection $P_S$ to each Borel set $S \subseteq \mathbb{R}$. According to spectral theory, one can integrate with respect to this measure. For instance, the following formula holds:

$$A_G = \int_{\mathbb{R}} \lambda \, dP_\lambda.$$ 

Furthermore, the projections $P_S$ have the property that if an operator $T$ commutes with $A_G$, then it also commutes with each projection $P_S$. There is a unitary operator $U_\Phi$ corresponding to each $\Phi \in \text{Aut}(G)$ [the one that permutes the coordinates of $\ell_2(V(G))$ according to $\Phi$]. Since $U_\Phi$ commutes with $A_G$, it also commutes with the projections $P_S$.

Now let $\lambda_0$ be an arbitrary element of the spectrum $\sigma(A_G)$, and set $S = [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. We define $\alpha$ as the image of the indicator function $1_x$ under the projection $P_S$,

$$\alpha \overset{\text{def}}{=} P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_x.$$ 

Note that $1_x$ is a fixed point of $U_\Phi$ for any $\Phi \in \text{Stab}_x(G)$. Therefore

$$U_\Phi \alpha = U_\Phi P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_x = P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} U_\Phi 1_x = P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_x = \alpha,$$

and thus $\alpha$ is $\text{Stab}_x(G)$-invariant. On the other hand, since $P_S P_{\mathbb{R} \setminus S} = 0$, we have

$$A_G \alpha - \lambda_0 \alpha = \left( \int_{\mathbb{R}} (\lambda - \lambda_0) \, dP_\lambda \right) \alpha = \left( \int_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} (\lambda - \lambda_0) \, dP_\lambda \right) \alpha,$$

which clearly implies that

$$\|A_G \alpha - \lambda_0 \alpha\| \leq \varepsilon \|\alpha\|.$$ 

It remains to show that $\alpha = P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_x \neq 0$. Assume that $P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_x = 0$. It follows that $P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_v = 0$ for every vertex $v \in V(G)$. Indeed, let $\Phi \in \text{Aut}(G)$ such that $\Phi x = v$. Then $U_\Phi 1_x = 1_v$ and

$$P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_v = P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} U_\Phi 1_x = U_\Phi P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} 1_x = 0.$$ 

This holds for each vertex $v$, which clearly implies that $P_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} = 0$. Then the operator

$$B = \int_{[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]} \frac{1}{\lambda - \lambda_0} \, dP_\lambda$$

would be the inverse of $A_G - \lambda_0 I$ contradicting our assumption that $\lambda_0 \in \sigma(A_G)$.

REMARK 3.2. There is a general theorem for Hilbert spaces saying that every point of the spectrum of a self-adjoint operator is an approximate eigenvalue [17],
Corollary 4.1.3. So the real content of the above theorem is that one can find approximate eigenvectors that are Stab$_x(G)$-invariant. This invariance will be crucial for us later on, when we will use these approximate eigenvectors as coefficients to define linear factor of i.i.d. processes.

Suppose now that we have an i.i.d. process on $G$: independent standard normal random variables $Z_u$ assigned to each vertex $u$. We will consider processes $X_v, v \in V(G)$, where each $X_v$ is a (possibly infinite) linear combination of $Z_u, u \in V(G)$. We collected some obvious properties of such processes in the next proposition.

**Proposition 3.3.** Let $\beta_{v,u}$, $v, u \in V(G)$ be real numbers, and let

$$X_v = \sum_{u \in V(G)} \beta_{v,u} Z_u. \quad (16)$$

The infinite sum in (16) converges almost surely if and only if

$$\sum_{u \in V(G)} \beta_{v,u}^2 < \infty. \quad (17)$$

If (17) is satisfied, then $X_v$ is a centered Gaussian with variance $\text{var}(X_v) = \sum_{u \in V(G)} \beta_{v,u}^2$.

The process $X_v, v \in V(G)$ is Aut($G$)-invariant if and only if

$$\beta_{v,u} = \beta_{v,\Phi u} \quad \text{for all } \Phi \in \text{Aut}(G). \quad (18)$$

Now we are in a position to formally define linear factor of i.i.d. processes.

**Definition 3.4.** We say that a process $X_v, v \in V(G)$ is a linear factor of the i.i.d. process $Z_u$ if it can be written as in (16) for some real numbers $\beta_{v,u}$, $v, u \in V(G)$ satisfying (17) and (18).

**Remark 3.5.** Let us fix a root $x \in V(G)$. For a linear factor the coefficients $\alpha_u \overset{\text{def}}{=} \beta_{x,u}$ clearly determine each $\beta_{v,u}$. Here $\alpha = (\alpha_u)_{u \in V(G)}$ can be any Stab$_x(G)$-invariant vector in $\ell^2(V(G))$. So there is a one-to-one correspondence between linear factor of i.i.d. processes on $G$ and Stab$_x(G)$-invariant vectors $\alpha \in \ell^2(V(G))$. Also, by Proposition 3.3 we have $\text{var}(X_v) = \|\alpha\|^2$.

Recall Definition 1.1 of invariant Gaussian processes.

**Definition 3.6.** We call an invariant Gaussian process $X_v, v \in V(G)$ a Gaussian wave function with eigenvalue $\lambda$ if

$$\sum_{u \in N(v)} X_u = \lambda X_v \quad \text{for each vertex } v \in V(G),$$

where $N(v)$ denotes the set of neighbors of $v$ in $G$. 
EXAMPLE 3.7. It was shown in [4] that for the $d$-regular tree $T_d$ there exists an essentially unique Gaussian wave function for each $\lambda \in [-d,d]$. Furthermore, this Gaussian wave function can be approximated by factor of i.i.d. processes provided that $\lambda$ is in the spectrum $\sigma(T_d) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$.

In general, it is not clear for which $\lambda$ such Gaussian wave functions exist and whether they are unique.

DEFINITION 3.8. For a transitive graph $G$ we call the closed set
\[ \tilde{\sigma}(G) \overset{\text{def}}{=} \{ \lambda : \text{there exists a Gaussian wave function on } G \text{ with eigenvalue } \lambda \} \]
the Gaussian spectrum of $G$.

Theorem 4 claims that for any $\lambda \in \sigma(A_G)$ there exists a Gaussian wave function on $G$, which can be approximated by linear factor of i.i.d. processes. Therefore $\tilde{\sigma}(G) \supseteq \sigma(A_G)$.

PROOF OF THEOREM 4. We use the $\text{Stab}_x(G)$-invariant approximate eigenvectors of Theorem 3.1 to define linear factor of i.i.d. processes. So let $\varepsilon > 0$ be arbitrary and $\alpha^\varepsilon$ a $\text{Stab}_x(G)$-invariant vector with $\|\alpha^\varepsilon\| = 1$ and $\|A_G\alpha^\varepsilon - \lambda \alpha^\varepsilon\| \leq \varepsilon$. By Remark 3.5 for each $\alpha^\varepsilon$ there is a corresponding linear factor $X^\varepsilon_v$, $v \in V(G)$. Note that the process
\[ Y^\varepsilon_v \overset{\text{def}}{=} \sum_{u \in N(v)} X^\varepsilon_u - \lambda X^\varepsilon_v \]
is also a linear factor, and the corresponding coefficient vector is $\delta^\varepsilon \overset{\text{def}}{=} A_G\alpha^\varepsilon - \lambda \alpha^\varepsilon$. Therefore $X^\varepsilon_v$ is an invariant Gaussian process with $\text{var}(X^\varepsilon_v) = \|\alpha^\varepsilon\|^2 = 1$ and
\[ \text{var}\left(\sum_{u \in N(v)} X^\varepsilon_u - \lambda X^\varepsilon_v\right) = \text{var}(Y^\varepsilon_v) = \|\delta^\varepsilon\|^2 = \|A_G\alpha^\varepsilon - \lambda \alpha^\varepsilon\|^2 \leq \varepsilon^2. \]
Since the space of invariant Gaussian processes with variance 1 is compact, it follows that there exists a sequence $\varepsilon_n$ converging to 0 such that the processes $X^\varepsilon_n_v$ converge in distribution. The limit process will be a nontrivial invariant Gaussian process $X_v$ that satisfies the eigenvector equation (3) at each vertex. \( \square \)

3.2. Fact of i.i.d. processes. For a graph $G$ we defined an i.i.d. process on $G$ as independent standard normal random variables $Z_v$, $v \in V(G)$. In other words, $Z = (Z_v)_{v \in V(G)}$ is a random point in the measure space $(\Omega, \mu)$, where $\Omega$ is $\mathbb{R}^{V(G)}$ with the product topology, and $\mu$ is the product of standard Gaussian measures.
INDEPENDENCE RATIO AND RANDOM EIGENVECTORS

The natural action of Aut$\left( G \right)$ on $V( G )$ gives rise to an action of Aut$\left( G \right)$ on $\Omega$. For $\Phi \in$ Aut$\left( G \right)$ and $\omega = (\omega_v)_{v \in V( G )} \in \Omega$ let 

$$ (\Phi \cdot \omega)_v \stackrel{\text{def}}{=} \omega_{\Phi^{-1} v}. $$

Let $G$ be an infinite transitive graph, and suppose that $F$ is a measurable $\Omega \rightarrow \Omega$ function that is Aut$\left( G \right)$-equivariant [i.e., commutes with the Aut$\left( G \right)$-action]. Then $X = F(Z)$ is an invariant process on $G$. Such a process $X = (X_v)_{v \in V(G)}$ is called a factor of the i.i.d. process $Z$.

An Aut$\left( G \right)$-equivariant $F: \Omega \rightarrow \Omega$ function is determined by 

$$ f = \pi_x \circ F, $$

where $\pi_x: \Omega \rightarrow \mathbb{R}$ is the projection corresponding to the coordinate of some fixed root $x$. Here $f$ can be any Stab$\left( x\left( G \right) \right)$-invariant $\Omega \rightarrow \mathbb{R}$ function. So factor of i.i.d. processes can be identified with measurable, Stab$\left( x\left( G \right) \right)$-invariant functions $f: \Omega \rightarrow \mathbb{R}$.

Next we will prove Theorems 5 and 6 by showing that certain Gaussian wave functions $X_v$, $v \in V(G)$ cannot be obtained as factor of i.i.d. processes. Since $X_v$ has finite variance in that case, we can restrict ourselves to functions $f \in L_2(\Omega, \mu)$. Let $H_{\text{inv}} \subset L_2(\Omega, \mu)$ be the subspace containing those $f \in L_2(\Omega, \mu)$ that are Stab$\left( x\left( G \right) \right)$-invariant. There is a natural way to define an adjacency operator $A$ on the Hilbert space $H_{\text{inv}}$. Let

$$ (Af)(\omega) \stackrel{\text{def}}{=} \sum_{y \in N(x)} f(\Phi_{y \rightarrow x} \cdot \omega), $$

where $\Phi_{y \rightarrow x}$ is an (arbitrary) automorphism of $G$ taking $y$ to $x$. Since $f$ is Stab$\left( x\left( G \right) \right)$-invariant, $A$ is well defined.

Suppose now that we have a Gaussian wave function with eigenvalue $\lambda$ that can be obtained as a factor of i.i.d. process. Then the corresponding $f$ satisfies the eigenvector equation $Af = \lambda f$. In particular, $\lambda$ needs to be in the point spectrum of $A$. [Note that an eigenvector $f$ of $A$ does not necessarily give us a Gaussian wave function: although the corresponding factor of i.i.d. process will satisfy the eigenvector equation at each vertex, $f(Z)$ might not have a Gaussian distribution.]

**PROOF OF THEOREM 5.** Since $L_2(\Omega, \mu)$ is a separable Hilbert space, so is $H_{\text{inv}}$, and consequently the point spectrum of $A: H_{\text{inv}} \rightarrow H_{\text{inv}}$ is countable.

Therefore only for countably many $\lambda$’s can we have a Gaussian wave function on $G$ that can be obtained as a factor of i.i.d. process. However, if $\sigma(A_G)$ is uncountable, then by Theorem 4, $G$ has Gaussian wave functions for uncountably many different eigenvalues $\lambda$; moreover, they can all be approximated by linear factor of i.i.d. processes. □

**PROOF OF THEOREM 6.** We will use two basic facts about the point spectra of the adjacency operators $A_G$ and $A$. First, $\lambda_{\text{max}}$ is never in the point spectrum $\sigma_p(A_G)$ (we will give a short proof for this in the Appendix; see Lemma A.5). Second, $\sigma_p(A) \subseteq \sigma_p(A_G) \cup \{d\}$ for Cayley graphs (this will be explained after the
proof). Therefore $\lambda_{\text{max}}$ is not in the point spectrum of $A$ provided that $\lambda_{\text{max}} < d$, and consequently, a Gaussian wave function with eigenvalue $\lambda_{\text{max}}$ cannot be obtained as a factor of i.i.d. process.

In the case $\lambda_{\text{max}} = d$ the Gaussian wave function has to be constant; that is, $X_u = X_v$ for any two vertices $u,v$. However, for a factor of i.i.d. process the correlation between $X_u$ and $X_v$ should tend to 0 as the distance of $u$ and $v$ goes to infinity; see Proposition A.4 in the Appendix. □

Next we will explain the relation between the adjacency operators $A_G$ and $A$. This can be found in [10], Section 3, in a more general setting; see also [13], Theorem 2.1 and Corollary 2.2. Let $\nu$ denote the standard Gaussian measure. Since $L^2(\mathbb{R},\nu)$ is a separable Hilbert space, it has a countable orthonormal basis: $g_0, g_1, g_2, \ldots$, where $g_0$ will be assumed to be the constant 1 function. Let $\mathcal{I}$ denote the set of finitely supported $V(G) \to \{0, 1, 2, \ldots\}$ functions. For each $q \in \mathcal{I}$ we define an $\Omega \to \mathbb{R}$ function:

$$W_q(\omega) \overset{\text{def}}{=} \prod_{v \in V(G)} g_{q(v)}(\omega_v).$$

Note that this is actually a finite product, since all but finitely many terms are equal to $g_0 \equiv 1$. According to [10], Lemma 3.1, the functions $W_q, q \in \mathcal{I}$ form an orthonormal basis of $L^2(\Omega, \mu)$. It follows that $L^2(\Omega, \mu)$ is separable, which fact was used in the proof of Theorem 5.

We defined the operator $A$ on the space $H_{\text{inv}} \subset L^2(\Omega, \mu)$ containing $\text{Stab}_x(G)$-invariant functions. When $G$ is a Cayley graph, there is a natural way to extend $A$ to an adjacency operator over the whole space $L^2(\Omega, \mu)$. Suppose that $\Gamma$ is a finitely generated infinite group. Let $S$ be a finite, symmetric set of generators, and let $G$ be the corresponding Cayley graph; that is, $V(G) = \Gamma$ and the vertex $v \in \Gamma$ is adjacent to the vertices $\gamma v$, $\gamma \in S$. The natural action of $\Gamma$ on itself gives rise to the following $\Gamma$-action on $\Omega$:

$$(\gamma \cdot \omega)_v \overset{\text{def}}{=} (\omega)_{\gamma^{-1}v}.$$ 

(This is often called the generalized Bernoulli shift.) Then for $f \in L^2(\Omega, \mu)$, let

$$(Af)(\omega) \overset{\text{def}}{=} \sum_{\gamma \in S} f(\gamma \cdot \omega).$$

This clearly extends our earlier definition of $A$.

There is a natural $\Gamma$-action on $\mathcal{I}$ as well: for $q \in \mathcal{I}$

$$(\gamma \cdot q)(v) \overset{\text{def}}{=} q(\gamma^{-1}v).$$

It is compatible with the $\Gamma$-action on $\Omega$ in the following sense:

$$W_{\gamma \cdot q}(\omega) = W_q(\gamma^{-1} \cdot \omega).$$
It means that
\[ A W_q = \sum_{\gamma \in S} W_{\gamma \cdot q}. \]

We now consider the orbit \( \{ \gamma \cdot p : \gamma \in \Gamma \} \) of a given element \( p \in I \) and the closure of the space spanned by the corresponding functions \( W_{\gamma \cdot p} \),

\[ H_p \overset{\text{def}}{=} \text{cl}(\text{span}\{W_{\gamma \cdot p} : \gamma \in \Gamma\}) \subset L_2(\Omega, \mu). \]

It is clear from (19) that \( H_p \) is \( A \)-invariant. If \( p \equiv 0 \), then \( H_p \) consists of the constant functions on \( \Omega \) and both the point spectrum and the spectrum of \( A|_{H_p} \) is \( \{d\} \).

Otherwise the stabilizer \( \Gamma_p \) of \( p \) is a finite subgroup of \( \Gamma \), and \( A|_{H_p} \) is closely related to the original adjacency operator \( A_G \). Indeed, let \( T_p : H_p \rightarrow \ell_2(V(G)) \cong \ell_2(\Gamma) \) be the operator defined by

\[ T_p : W_q \mapsto 1_{\{\gamma \in \Gamma : \gamma \cdot p = q\}}, \]

where \( q \) is in the orbit of \( p \). It is easy to see that \( T_p \) is a bounded operator for which \( T_p A|_{H_p} = A_G T_p \). Since \( T_p \) is also bounded below, it follows that

\[ \sigma(A|_{H_p}) \subseteq \sigma(A_G) \quad \text{and} \quad \sigma_p(A|_{H_p}) \subseteq \sigma_p(A_G) \]

with equality when the stabilizer \( \Gamma_p \) is trivial.

Therefore for Cayley graphs the operators \( A_G : \ell_2(V(G)) \rightarrow \ell_2(V(G)) \) and \( A : L_2(\Omega, \mu) \rightarrow L_2(\Omega, \mu) \) have the same spectra and point spectra with the possible exception of the point \( d \),

\[ \sigma(A) = \sigma(A_G) \cup \{d\} \quad \text{and} \quad \sigma_p(A) = \sigma_p(A_G) \cup \{d\}. \]

Consequently,

\[ \sigma_p(A|_{H_{\text{inv}}}) \subseteq \sigma_p(A) = \sigma_p(A_G) \cup \{d\}, \]

which we used in the proof of Theorem 6.

3.3. Independent sets. Let \( G \) be an infinite transitive graph and \( \lambda_{\text{min}} \) be the minimum of its spectrum \( \sigma(A_G) \). Consider linear factor of i.i.d. processes \( X^n \) converging in distribution to a Gaussian wave function \( X_v \) with eigenvalue \( \lambda_{\text{min}} \) as \( n \rightarrow \infty \) as in Theorem 4. We define the following independent sets on \( G \):

\[ I_+ \overset{\text{def}}{=} \{ v : X_v > X_u, \forall u \in N(v) \} \quad \text{and} \quad I^n_+ \overset{\text{def}}{=} \{ v : X^n_v > X^n_u, \forall u \in N(v) \}. \]

Then for each \( n \) the independent set \( I^n_+ \) is a factor of the i.i.d. process \( Z_v \); that is, it is obtained as a measurable function of \( Z_v, v \in V(G) \) that commutes with the natural action of \( \text{Aut}(G) \). Furthermore, since the event \( v \in I_+ \) corresponds to an open set, we have

\[ \liminf_{n \rightarrow \infty} P(v \in I^n_+) \geq P(v \in I_+). \]
Therefore whenever we have a lower bound $q$ for $P(v \in I_+)$, it yields that for any $\varepsilon > 0$ there exists a factor of i.i.d. independent set with “size” greater than $q - \varepsilon$.

Bounding $P(v \in I_+)$, however, leads us to the same optimization problem as in the finite case. We need to estimate the volume of the same spherical simplex with the exact same constraints. [Of course, there might be a difference between the finite and infinite setting in terms of what covariances $c_{i,j}$ can actually come up, but our proofs used only the trivial constraints that they form a positive semidefinite matrix and their sum is $(\lambda_{\min}^2 - d)/2$, which are true in the infinite case, too.] Thus we obtain the exact same bounds, and Theorem 7 follows.

Actually, in Theorem 3 we proved the bound only for graphs with $\lambda_{\min} \leq -2$ and argued that the only finite, 3-regular, transitive graph for which this does not hold is the complete graph $K_4$. For infinite transitive graphs $\lambda_{\min} \leq -2$ holds with no exception. This follows from the fact that they contain arbitrarily long paths as induced subgraphs.

**APPENDIX**

**THEOREM A.1.** Suppose that $G$ is a finite, 3-regular, vertex-transitive graph with minimum eigenvalue $\lambda_{\min}$ and odd-girth $g$. Then the independence ratio of $G$ is at least

$$\frac{5g - 3}{16g} + \frac{g + 1}{2g} \frac{3}{4\pi} \arcsin\left(\frac{\lambda_{\min}^2 - 3}{6}\right) \geq \frac{5}{16} + \frac{3}{8\pi} \arcsin\left(\frac{\lambda_{\min}^2 - 3}{6}\right) - \frac{3}{16g}.$$ 

In fact, there exist two disjoint independent sets in $G$ such that their average size divided by $|V(G)|$ is not less than the above bound.

**PROOF.** It is easy to check the statement for $K_4$. According to Proposition A.3 $\lambda_{\min} \leq -2$ holds for any other finite, 3-regular, transitive graph $G$. Let $X_v, v \in V(G)$ be the random eigenvector corresponding to $\lambda_{\min}$. Let $V_+$ denote the set of “positive vertices,” that is,

$$V_+ \overset{\text{def}}{=} \{v \in V(G) : X_v > 0\}.$$

The expected size of $V_+$ is $|V(G)|/2$.

Since $\lambda_{\min}$ is negative, a vertex and its three neighbors cannot all be positive. Therefore each vertex has degree at most two in the induced subgraph $G[V_+]$. Thus each connected component of this subgraph is a path or a cycle. We want to choose an independent set from each component. We can choose at least half the vertices from paths and even cycles. From an odd cycle of length $l \geq g$ we can choose $(l - 1)/2$ vertices, which is at least a $(g - 1)/(2g)$ proportion of all vertices in that component. (Recall that $g$ denotes the odd-girth of $G$, i.e., the length of the shortest odd cycle in $G$.)

We need one more observation, namely, that many of the components actually contain only one vertex. Using our earlier notation, let $v$ be an arbitrary vertex with
neighbors $w_1, w_2, w_3$, the corresponding random variables are $X$ and $Y_1, Y_2, Y_3$. Note that $Y_1 < 0$, $Y_2 < 0$ and $Y_3 < 0$ imply that $X > 0$. Therefore the probability $p$ that $v$ is an isolated vertex in $G[V_+]$ is

$$p \overset{\text{def}}{=} P(X > 0; Y_1 < 0; Y_2 < 0; Y_3 < 0) = P(Y_1 < 0; Y_2 < 0; Y_3 < 0)$$

$$= P(y_i \cdot Z < 0; i = 1, 2, 3) = \frac{1}{2} - \frac{1}{4\pi} \sum_{1 \leq i, j \leq 3} \arccos(c_{i,j})$$

$$= \frac{1}{8} + \frac{1}{4\pi} \sum_{1 \leq i, j \leq 3} \arcsin(c_{i,j}).$$

Note that $\arcsin$ it is a monotone increasing odd function on $[-1, 1]$, which is convex on $[0, 1]$. Furthermore, the average of $c_{i,j}$ is $(\lambda_{\text{min}}^2 - 3)/6 \geq (2^2 - 3)/6 > 0$. It is easy to see that these imply that the right-hand side of (20) decreases (not increases) if we replace each $c_{i,j}$ with their average $(\lambda_{\text{min}}^2 - 3)/6$. Thus

$$p \geq \frac{1}{8} + \frac{3}{4\pi} \arcsin\left(\frac{\lambda_{\text{min}}^2 - 3}{6}\right).$$

Our independent set will contain all isolated vertices and at least a $(g - 1)/(2g)$ proportion of all the other vertices in $V_+$. This yields the following lower bound for the independence ratio of $G$:

$$p + \frac{g - 1}{2g} \left(\frac{1}{2} - p\right) = \frac{g - 1}{4g} + \frac{g + 1}{2g} p.$$

Combining this with (21) yields the desired bound.

We can choose an independent set with the same expected size from the “negative vertices”

$$V_- \overset{\text{def}}{=} \{v \in V(G) : X_v < 0\}.$$

This implies the second part of the theorem.

We mention that the proof also works in the infinite setting, so there is an analogous theorem for infinite transitive graphs (as in Theorem 7).

**Remark A.2.** Any nontrivial lower bound for the density of components of size 3, 5, ... in $G[V_+]$ would immediately yield an improvement in the above theorem. In [4] such nontrivial bounds were obtained for the 3-regular tree $T_3$.

**Proposition A.3.** Suppose that $G$ is a finite, connected, 3-regular, vertex-transitive graph. Then either $G$ is isomorphic to the complete graph $K_4$, or the least eigenvalue $\lambda_{\text{min}}$ of its adjacency matrix is at most $-2$.

The proof below is due to Péter Csikvári.
**Proof of Proposition A.3.** Let $G$ be a connected, 3-regular, vertex-transitive graph with $\lambda_{\min}(G) > -2$. We need to show that $G$ must be the complete graph $K_4$.

Cauchy’s interlacing theorem implies that $\lambda_{\min}(G) \leq \lambda_{\min}(H)$ whenever $H$ is an induced subgraph of $G$. Therefore $\lambda_{\min}(H) > -2$ must hold for any induced subgraph. Let $T$ denote the tree shown in Figure 2. It is easy to see that the smallest eigenvalue of $T$ is $-2$. We also have $\lambda_{\min}(C_{2k}) = -2$ for the cycle of length $2k$ for any $k \geq 2$. Therefore $G$ can contain neither $T$, nor $C_{2k}$ as an induced subgraph.

We will distinguish three cases.

---

**Case 1.** $G$ does not contain triangles.

Let $u, v$ be two neighboring vertices, and let $u_1, u_2$ and $v_1, v_2$ denote the remaining two neighbors of $u$ and $v$, respectively. Since $G$ contains no triangles, $u_1, u_2, v_1, v_2$ are pairwise distinct vertices. The induced subgraph on the set \{u, $u_1, u_2, v, v_1, v_2$\} must be isomorphic to $T$ (the graph shown in Figure 2), otherwise $G$ would contain a triangle or an induced $C_4$. Since $G$ cannot contain $T$ as an induced subgraph, this is a contradiction.

**Case 2.** $G$ contains triangles, but no two share a common edge.

Since $G$ is vertex-transitive, there must be at least one triangle through every vertex. We claim that any two triangles must be disjoint. If they had two common vertices, then they would share an edge, and if they had exactly one common vertex, then that vertex would have degree at least 4.

So we have disjoint triangles in $G$, exactly one through every vertex. We claim that there can be at most one edge between two triangles (with one endpoint in one triangle and one in the other). Indeed, otherwise we would either have an induced $C_4$ or a vertex with degree at least 4.

Let us consider the following graph $G^*$. To each triangle in $G$ corresponds a vertex in $G^*$, and we join two such vertices with an edge if there is an edge between the corresponding triangles. It is easy to see that $G^*$ will be 3-regular as well. Take a cycle in $G^*$ with minimum length $g \geq 3$. There is a corresponding cycle of length $2g$ in the original graph $G$. It is easy to see that this must be an induced cycle, contradiction.

**Case 3.** $G$ contains two triangles sharing an edge.

Let $xy$ be an edge shared by triangles $xyu$ and $xyv$; see Figure 3.
Then $x$ and $y$ already have degree 3, while $u$ and $v$ still need an edge. We claim that $uv$ must be an edge. Otherwise $v$ would have a neighbor $z$ different from $x$, $y$, $u$. Since $z$ cannot be adjacent to $x$ and $y$, there is only one triangle through $v$, while there are two triangles through $x$, contradicting the transitivity of $G$. So $uv$ is an edge, and therefore each of $x$, $y$, $u$, $v$ has degree 3. Since $G$ is connected, $G$ cannot have any other vertices and thus isomorphic to $K_4$. □

For the sake of completeness we include a simple proof for the following well-known result.

**Proposition A.4.** Let $G$ be a vertex-transitive graph, and let $X_v$, $v \in V(G)$ be a factor of the i.i.d. process $Z_u$, $v \in V(G)$ with $0 < \text{var}(X_v) < \infty$. Then $\text{corr}(X_v, X_v') \to 0$ as the distance of $v$ and $v'$ goes to infinity.

See [2] for an explicit (and sharp) bound on the correlation decay on the $d$-regular tree.

**Proof of Proposition A.4.** For any vertex $v \in V(G)$, one can define the following Lévy martingales:

$$X_v^{(n)} = E(X_v | Z_u, \text{dist}(u, v) \leq n).$$

According to martingale convergence theorems, $X_v^{(n)}$ converges to $X_v$ almost surely and in $L^2$ as well. Since $E(X_v^{(n)}) = E(X_v)$, the latter means that $\text{var}(X_v - X_v^{(n)}) \to 0$ as $n \to \infty$.

Moreover, $X_v^{(n)}$, $v \in V(G)$ is a so-called block factor of $Z_u$, $u \in V(G)$, that is, $X_v^{(n)}$ depends only on those $Z_u$’s for which $u$ is in some finite neighborhood of $v$.

Now let $\varepsilon > 0$ be arbitrary and let us pick $n$ such that $\text{var}(X_v - X_v^{(n)}) < \varepsilon$. If the distance of $v$ and $v'$ is more than $2n$, then $X_v^{(n)}$ and $X_v^{(n)}$ are independent (because they depend on disjoint sets of $Z_u$’s). Therefore $\text{cov}(X_v^{(n)}, X_v^{(n)}) = 0$ and hence

$$\text{cov}(X_v, X_v') = \text{cov}(X_v - X_v^{(n)}, X_v' - X_v^{(n)}) + \text{cov}(X_v^{(n)}, X_v' - X_v^{(n)})$$

$$+ \text{cov}(X_v - X_v^{(n)}, X_v^{(n)}),$$
which can be bounded by \( \varepsilon + 2\sqrt{\varepsilon \text{var}(X_v)} \), and the statement of the proposition follows. \( \square \)

The following lemma is probably known, but we did not find an explicit reference, so we give a short proof.

**Lemma A.5.** If \( G \) is an infinite transitive graph, then the maximum \( \lambda_{\text{max}} \) of the spectrum of \( A_G \) is never in the point spectrum of \( A_G \).

**Proof.** In the case \( \lambda_{\text{max}} = d \), the equation \( A_G f = df \) means that the vector \( f \) is harmonic. However, the maximum principle implies that there are no \( \ell^2 \) harmonic functions. Thus there is no eigenvector for \( \lambda_{\text{max}} \), which is equivalent to saying that \( \lambda_{\text{max}} \) is not in the point spectrum of \( A_G \).

For the nonamenable case (i.e., \( \lambda_{\text{max}} < d \)), Theorem II.7.8 in [18] implies that for any vertex \( v \),

\[
\sum_{n=0}^{\infty} \lambda_{\text{max}}^{-2n} \langle 1_v, A_G^{2n} 1_v \rangle < \infty,
\]

where the left-hand side can be written in terms of the spectral measure \( \mu_G \) as

\[
\sum_{n=0}^{\infty} \lambda_{\text{max}}^{-2n} \int x^{2n} d\mu_G(x) \geq \sum_{n=0}^{\infty} \lambda_{\text{max}}^{-2n} \lambda_{\text{max}}^{2n} \mu_G(\{\lambda_{\text{max}}\}).
\]

This forces \( \mu_G(\{\lambda_{\text{max}}\}) = 0 \), which means that \( \lambda_{\text{max}} \) is not in the point spectrum of \( A_G \). \( \square \)

**Lemma A.6.**

\[
\frac{\text{vol}(S^{d-2})}{\text{vol}(S^{d-1})} < \frac{\sqrt{d}}{\sqrt{2\pi}}.
\]

**Proof.** Using the formula

\[
\text{vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)},
\]

we need to show that

\[
\frac{\Gamma((d - 1)/2)}{\Gamma((d - 2)/2)} < \sqrt{\frac{d}{2}}.
\]

Since \( \Gamma \) is log-convex, the increments of its logarithm over intervals of length, say, 1/2 are increasing. Thus

\[
\frac{\Gamma((d - 1)/2)}{\Gamma((d - 2)/2)} \leq \frac{\Gamma(d/2)}{\Gamma((d - 1)/2)}
\]
and multiplying both sides by the left-hand side, we get

\[
\left( \frac{\Gamma((d - 1)/2)}{\Gamma((d - 2)/2)} \right)^2 \leq \frac{\Gamma(d/2)}{\Gamma((d - 2)/2)} = \frac{d - 2}{2} < \frac{d}{2}
\]
as required. □

**Lemma A.7.** Let \( \lambda \in [-3, -2] \) and

\[
f(t) \overset{\text{def}}{=} \arcsin \left( \frac{(1/2) - \lambda + (\lambda^2/2) + (1 - \lambda)t}{(1/2) - 2\lambda + (\lambda^2/2) + t} \right).
\]

Then the tangent line to \( f \) at \( t_0 = (\lambda^2 - 3)/6 \) is below \( f \) on the entire interval \([-0.5, 1]\); see Figure 4 for the case \( \lambda_{\text{min}} = -2 \).

**Proof.** We need to prove that

\[ f(t) - f'(t_0)t \]
takes its minimum value at \( t_0 \) on the interval \([-0.5, 1]\). This will follow from the fact that \( f'(t) < f'(t_0) \) for \(-0.5 \leq t < t_0 \) and \( f'(t) > f'(t_0) \) for \( t_0 < t < 1 \).

In order to make calculations easier, we will use the following notation:

\[
a = \frac{1}{2} - \lambda + \frac{\lambda^2}{2} \geq 4.5; \quad b = 1 - \lambda \geq 3; \quad c = a + b - 1 \geq 6.5;
\]
then

\[
f(t) = \arcsin \left( \frac{a + bt}{c + t} \right).
\]
It is easy to see that $0 < a + bt < c + t$ for $t \in [-0.5, 1)$. Therefore we have

$$f'(t) = \left(1 - \left(\frac{a + bt}{c + t}\right)^2\right)^{-1/2} b(c + t) - (a + bt) \frac{b(c + t)}{(c + t)^2}$$

$$= \frac{bc - a}{(c + t)^2 - (a + bt)^2}.$$

Since $bc - a > 0$ it follows that $f'$ is positive on $[-0.5, 1)$, and thus $f$ is monotone increasing. Next we study the intervals of monotonicity of $f'$. First we note that

$$(c + t)^2 - (a + bt)^2 = (c + a + (1 + b)t)(c - a + (1 - b)t).$$

Using $c - a = b - 1$ we get that

$$(c + t)^2 - (a + bt)^2 = (b^2 - 1)(t + d)(1 - t),$$

where

$$d = \frac{c + a}{b + 1} = \frac{1 - 3\lambda + \lambda^2}{2 - \lambda} = 1 - \lambda - \frac{1}{2 - \lambda} \geq \frac{11}{4}.$$ 

It follows that

$$\frac{1}{(f'(t))^2} = \frac{b^2 - 1}{(bc - a)^2} (t + c)^2 (t + d)(1 - t).$$

If we restrict ourselves to the interval $[-0.5, 1)$ (where $f'$ is positive), then it suffices to examine the function

$$g(t) = (t + c)^2 (t + d)(1 - t).$$

Wherever $g$ is monotone increasing, $f'$ is monotone decreasing, and vice versa.

So we have a fourth-degree polynomial $g$ with leading coefficient $-1$, whose roots are $-c$ (with multiplicity 2), $-d$ and 1. Consequently, the derivative $g'$ is a third-degree polynomial with negative leading coefficient and with roots $-c, u, v$, where $-c < u < -d < v < 1$. We distinguish the following two cases.

**Case 1.** $v \leq -0.5$. Then $g$ is monotone decreasing on $[-0.5, \infty)$, and therefore $f'$ is monotone increasing on $[-0.5, 1)$, and thus $f$ is convex on the whole interval, which clearly implies the statement of the lemma.

**Case 2.** $v > -0.5$. Since the other two roots of $g'$ are less than $-d < -0.5$, we know that $g$ is monotone increasing on $[-0.5, v]$ and monotone decreasing on $[v, 1)$. We claim that

$$g\left(-\frac{1}{2}\right) > g\left(\frac{1}{6}\right).$$

This would yield that $v < 1/6$. Since $1/6 \leq t_0 = (\lambda^2 - 3)/6$, we have $g\left(-1/2\right) > g\left(1/6\right) > g(t_0)$. This means that $g(t) > g(t_0)$ for $-0.5 \leq t < t_0$ and $g(t) < g(t_0)$
for $t_0 < t < 1$. As for $f'$, $f'(t) < f'(t_0)$ for $-0.5 < t < t_0$ and $f'(t) > f'(t_0)$ for $t_0 < t < 1$, and the statement of the lemma clearly follows.

It remains to show (22). Let $-1/2 = t_2 < t_1 = 1/6$. Then $t_1 - t_2 = 2/3$; $t_2 + c ≥ 6$ and $t_2 + d ≥ 9/4$, and consequently,

$$\frac{g(t_1)}{g(t_2)} = \frac{1 - t_1}{1 - t_2} \left(1 + \frac{t_1 - t_2}{t_2 + c}\right)^2 \left(1 + \frac{t_1 - t_2}{t_2 + d}\right)$$

$$≤ \frac{5/6}{3/2} \left(1 + \frac{2/3}{6}\right)^2 \left(1 + \frac{2/3}{9/4}\right) = \frac{5}{9} \left(\frac{10}{9}\right)^2 = \frac{17,500}{19,683} < 1.$$

\[\square\]

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