Combinatorial Multi-Armed Bandit and Its Extension to Probabilistically Triggered Arms

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Abstract

We define a general framework for a large class of combinatorial multi-armed bandit (CMAB) problems, where subsets of base arms with unknown distributions form super arms. In each round, a super arm is played and the base arms contained in the super arm are played and their outcomes are observed. We further consider the extension in which more based arms could be probabilistically triggered based on the outcomes of already triggered arms. The reward of the super arm depends on the outcomes of all played arms, and it only needs to satisfy two mild assumptions, which allow a large class of nonlinear reward instances. We assume the availability of an offline \((\alpha, \beta)\)-approximation oracle that takes the means of the outcome distributions of arms and outputs a super arm that with probability \(\beta\) generates an \(\alpha\) fraction of the optimal expected reward. The objective of an online learning algorithm for CMAB is to minimize \((\alpha, \beta)\)-approximation regret, which is the difference in total expected reward between the \(\alpha\beta\) fraction of expected reward when always playing the optimal super arm, and the expected reward of playing super arms according to the algorithm. We provide CUCB algorithm that achieves \(O(\log n)\) distribution-dependent regret, where \(n\) is the number of rounds played, and we further provide distribution-independent bounds for a large class of reward functions. Our regret analysis is tight in that it matches the bound of UCB1 algorithm (up to a constant factor) for the classical MAB problem, and it significantly improves the regret bound in a recent paper on combinatorial bandits with linear rewards. We apply our CMAB framework to two new applications, probabilistic maximum coverage (PMC) for online advertising and social influence maximization for viral marketing, both having nonlinear reward structures. In particular, application to social influence maximization requires our extension on probabilistically triggered arms.

keywords: combinatorial multi-armed bandit, online learning, upper confidence bound, social influence maximization, online advertising.

1 Introduction

Multi-armed bandit (MAB) is a problem extensively studied in statistics and machine learning. The classical version of the problem is formulated as a system of \(m\) arms (or machines), each having an unknown distribution of the reward with an unknown mean. The task is to repeatedly play these arms in multiple rounds so that the total expected reward is as close to the reward of the optimal arm as possible. An MAB algorithm needs to decide which arm to play in the next round given the outcomes of the arms played in the previous rounds. The metric for measuring the effectiveness of an MAB algorithm is its regret, which

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is the difference in the total expected reward between always playing the optimal arm (the arm with the largest expected reward) and playing arms according to the algorithm. The MAB problem and its solutions reflect the fundamental tradeoff between exploration and exploitation: whether one should try some arms that have not been played much (exploration) or one should stick to the arms that provide good reward so far (exploitation). Existing results show that one can achieve a regret of $O(\log n)$ when playing arms in $n$ rounds, and this is asymptotically the best.

In many real-world applications, the setting is not the simple MAB one, but has a combinatorial nature among multiple arms and possibly non-linear reward functions. For example, consider the following online advertising scenario. A website contains a set of web pages and has a set of users visiting the website. An advertiser wants to place an advertisement on a set of selected web pages on the site, and due to his budget constraint, he can select at most $k$ web pages. Each user visits a certain set of pages, and on each visited page has a certain click-through probability of clicking the advertisement on the page, but the advertiser does not know these probabilities. The advertiser wants to repeatedly select sets of $k$ web pages, observe the click-through data collected to learn the click-through probabilities, and maximize the number of users clicking his advertisement over time.

There are several new challenges raised by the above example. First, page-user pairs can be viewed as arms, but they are not played independently. Instead, these arms form certain combinatorial structures, namely bipartite graphs, and in each round, a set of arms (called a super arm) are played together. Second, the reward structure is not a simple linear function of the outcomes of all played arms but takes a more complicated form. In the above example, for all page-user pairs with the same user, the collective reward of these arms is either 1 if the user clicks the advertisement on at least one of the pages and 0 if the user does not click the advertisement on any page. Third, even the offline optimization problem when the probabilities on all edges of the bipartite graph are known is still an NP-hard problem. Thus, the online learning algorithm needs to deal with combinatorial arm structures, nonlinear reward functions, and computational hardness of the offline optimization task.

Consider another example of viral marketing in online social networks. In an online social network such as Facebook, companies carry out viral marketing campaigns by engaging with a certain set of seed users (e.g. providing free sample products to seed users), and hoping that these seed users could generate a cascade in the network promoting their products. The cascades follow certain stochastic diffusion model such as the independent cascade model (Kempe et al., 2003), but the influence probabilities on edges are not known in advance and have to be learned over time. Thus, the online learning task is to repeatedly select seed nodes in a social network, observe the cascading behavior of the viral information to learn influence probabilities between individuals in the social network, with the goal of maximizing the overall effectiveness of all viral cascades. Similar to the online advertising example given above, we can treat each edge in the social network as a base arm, and all outgoing edges from a seed set as a super arm, which is the unit of play. Besides sharing the same challenges such as the combinatorial arm structures, nonlinear reward functions, and computational hardness of the offline maximization task, this viral marketing task faces another challenge: in each round after some seed set is selected, the cascade from the seed set may probabilistically trigger more edges (or arms) in the network, and the reward of the cascade depends on all probabilistically or deterministically triggered arms.

A naive way to tackle both examples above is to treat every super arm as an arm and simply apply the classical MAB framework to solve the above combinatorial problems. However, such naive treatment has two issues. First, the number of super arms may be exponential to the problem instance size due to the combinatorial explosion, and thus classical MAB algorithms may need exponential number of steps just to go through all the super arms. Second, after one super arm is played, in many cases, we can observe some information regarding the outcomes of the underlying arms, which may be shared by other super arms. However, this information is discarded in the classical MAB framework, making it less effective.

In this paper, we define a general framework for the combinatorial multi-armed bandit (CMAB) problem to address the above issues and cover a large class of combinatorial online learning problems in the stochastic setting, including the two examples given above. In the CMAB framework, we have a set of $m$ base arms, whose outcomes follow certain unknown distributions. A super arm $S$ is a subset of base arms. In each
round, one super arm is played and all base arms contained in the super arm are played. To accommodate applications such as viral marketing, we allow that the play of a super arm $S$ may further trigger more base arms probabilistically, and the triggering depends on the outcomes of the already played base arms in the current round. The reward of the round is determined by the outcomes of all triggered arms, which are observed as the feedback to the online learning algorithm. A CMAB algorithm needs to use these feedback information from the past rounds to decide the super arm to play in the next round.

The framework allows an arbitrary combination of arms into super arms. The reward function only needs to satisfy two mild assumptions, and thus covering a large class of nonlinear reward functions. We do not assume the direct knowledge on how super arms are formed from underlying arms or how the reward is computed. Instead, we assume the availability of an offline computation oracle that takes such knowledge as well as the expectations of outcomes of all arms as input and computes the optimal super arm with respect to the input.

Since many combinatorial problems are computationally hard, we further allow (randomized) approximation oracles with failure probabilities. In particular, we relax the oracle to be an $(\alpha, \beta)$-approximation oracle for some $\alpha, \beta \leq 1$, that is, with success probability $\beta$, the oracle could output a super arm whose expected reward is at least $\alpha$ fraction of the optimal expected reward. As a result, our regret metric is not comparing against the expected reward of playing the optimal super arm each time, but against the $\alpha \beta$ fraction of the optimal expected reward, since the offline oracle can only guarantee this fraction in expectation. We refer to this as the $(\alpha, \beta)$-approximation regret.

For the general framework, we provide the CUCB (combinatorial upper confidence bound) algorithm, an extension to the UCB1 algorithm for the classical MAB problem (Auer et al., 2002a). We provide a tight analysis on the distribution-dependent regret of CUCB and show that it is still bounded by $O(\log n)$. Our tight analysis further allows us to provide a distribution-independent regret bound that works for arbitrary distributions of underlying arms, for a large class of CMAB instances. For the extension accommodating probabilistically triggered arms, we also provide distribution-dependent and -independent bounds with triggering probabilities as parameters.

We then apply our framework and provide solutions to two new bandit applications, the probabilistic maximum coverage problem for advertisement placement and social influence maximization for viral marketing. The offline versions of both problems are NP-hard, with constant approximation algorithms available. Both problems have nonlinear reward structures that cannot be handled by any existing work.

We also apply our result to combinatorial bandits with linear rewards, recently studied by Gai et al. (2012). We show that we significantly improve their distribution-dependent regret bound, even though we are covering a much larger class of combinatorial bandit instances. We also provide new distribution-independent bound not available in Gai et al. (2012).

This paper is an extension to our ICML’13 paper (Chen et al., 2013), with explicit modeling of probabilistically triggered arms and their regret analysis for the CUCB algorithm. We correct an erroneous claim in Chen et al. (2013), which states that the original CMAB model and result without probabilistically triggered arms can be applied to the online learning task for social influence maximization. Our correction includes explicit modeling of probabilistically triggered arms in the CMAB framework, and significant reworking of the regret analysis to incorporate triggering probabilities in the analysis and the regret bounds.

In summary, our contributions include: (a) defining a general CMAB framework that encompasses a large class of nonlinear reward functions, (b) providing CUCB algorithm with a tight regret analysis as a general solution to this CMAB framework, (c) further generalizing our framework to accommodate probabilistically triggered base arms, and applying this framework to the social influence maximization problem, and (d) demonstrating that our general framework can be effectively applied to a number of practical combinatorial bandit problems, including ones with nonlinear rewards. Moreover, our framework provides a clean separation of the online learning task and the offline computation task: the oracle takes care of the offline computation task, which uses the domain knowledge of the problem instance, while our CMAB algorithm takes care of the online learning task, and is oblivious to the domain knowledge of the problem instance.
Related work. Multi-armed bandit problem has been well studied in the literature, in particular in statistics and reinforcement learning (cf. Berry & Fristedt, 1985; Sutton & Barto, 1998). Our work follows the line of research on stochastic MAB problems, which is initiated by Lai & Robbins (1985), who show that under certain conditions on reward distributions, one can achieve a tight asymptotic regret of $O(\log n)$, where $n$ is the number of rounds played. Later, Auer et al. (2002a) demonstrate that $O(\log n)$ regret can be achieved uniformly over time rather than only asymptotically. They propose several MAB algorithms, including the UCB1 algorithm, which has been widely followed and adapted in MAB research.

For combinatorial multi-armed bandits, a few specific instances of the problem has been studied in the literature. A number of studies consider simultaneous plays of $k$ arms among $m$ arms (e.g. Anantharam et al., 1987; Caro & Gallien, 2007; Liu et al., 2011). Other instances include the matching bandit (Gai et al., 2010) and the online shortest path problem (Liu & Zhao, 2012).

The work closest to ours is a recent work by Gai et al. (2012), which also considers a combinatorial bandit framework with an approximation oracle. However, our work differs from theirs in several important aspects. Most importantly, their work only considers linear rewards while our CMAB framework includes a much larger class of linear and nonlinear rewards. Secondly, our regret analysis is much tighter, and as the result we significantly improve their distribution-dependent regret bound when applying our result to the linear reward case, and we are able to derive a distribution-independent regret bound close to the theoretical lower bound while they do not provide distribution-independent bounds. Moreover, we allow the approximation oracle to have a failure probability (i.e., $\beta < 1$), while they do not consider such failure probabilities.

In terms of types of feedbacks in combinatorial bandits (Audibert et al., 2011), our work belongs to the semi-bandit type, in which the player observes only the outcomes of played arms in one round of play. Other types include (a) full information, in which the player observes the outcomes of all arms, and (b) bandit, in which the player only observes the final reward but no outcome of any individual arm. More complicated feedback dependences are also considered by Mannor & Shamir (2011).

A different line of research considers adversarial multi-armed bandit, initiated by Auer et al. (2002b), in which an adversary controls the arms and tries to defeat the learning process. In the context of adversarial bandits, several studies also consider combinatorial bandits (Cesa-Bianchi & Lugosi, 2009; Audibert et al., 2011; Bubeck et al., 2012). For linear rewards, Kakade et al. (2009) have shown how to convert an approximation oracle into an online algorithm with sublinear regret both in the full information setting and the bandit setting. For non-linear rewards, various online submodular optimization problems with bandit feedback are studied in the adversarial setting (Streeter & Golovin, 2008; Radlinski et al., 2008; Streeter et al., 2009; Hazan & Kale, 2009). Notice that our framework deals with stochastic instances and we can handle reward functions more general than the submodular ones.

This paper is the full version of our ICML’13 paper (Chen et al., 2013) with the extension to include probabilistically triggered arms in the model and analysis. We made a mistake in Chen et al. (2013) by claiming that the online learning task for social influence maximization is an instance of the original CMAB model proposed in Chen et al. (2013) without explicitly modeling probabilistically triggered arms. In this paper we correct this mistake by allowing probabilistically triggered arms in the CMAB model, and by significantly revising the analysis to include triggering probabilities in the analysis and the regret bounds.

Since our work in Chen et al. (2013), several studies are also related to combinatorial multi-armed bandits or in general combinatorial online learning. Qin et al. (2014) extends CMAB to contextual bandits and apply it to diversified online recommendations. Lin et al. (2014) addresses combinatorial actions with limited feedbacks. Gopalan et al. (2014) uses Thompson sampling method to tackle combinatorial online learning problems. Comparing with our CMAB framework, they allow more feedback models than our semi-bandit feedback model, but they require finite number of actions and observations, their regret contains a large constant term, and it is unclear if their framework supports approximation oracles for hard combinatorial optimization problems. Kveton et al. (2014) studies linear matroid bandits, which is a subclass of the linear combinatorial bandits we discussed in Section 4.2, and they provide better regret bounds than our general bounds given in Section 4.2, because their analysis utilizes the matroid combinatorial structure.
Paper organization. In Section 2 we formally define the CMAB framework. Section 3 provides the CUCB algorithm, the main results on its regret bounds and the proofs. Section 4 shows how to apply the CMAB framework and CUCB algorithm to the online advertising and viral marketing applications, as well as the class of combinatorial bandits with linear reward functions. We conclude the paper in Section 5.

2 General CMAB Framework

A combinatorial multi-armed bandit (CMAB) problem consists of $m$ base arms associated with a set of random variables $X_{i,t}$ for $1 \leq i \leq m$ and $t \geq 1$, with bounded support on $[0, 1]$. Variable $X_{i,t}$ indicates the random outcome of the $i$-th base arm in its $t$-th trial. The set of random variables $\{X_{i,t} | t \geq 1\}$ associated with base arm $i$ are independent and identically distributed according to some unknown distribution with unknown expectation $\mu_i$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be the vector of expectations of all base arms. Random variables of different base arms may be dependent.

The unit of play in CMAB is a super arm, which is a set of base arms. Let $S$ denote the set of all possible super arms that can be played in a CMAB problem instance. For example, $S$ could be the set of all subsets of base arms containing at most $k$ base arms. In each round, one of the super arms $S \in S$ is selected and played, and every base arm $i \in S$ is triggered and played as a result. The outcomes of base arms in $S$ may trigger other base arms not in $S$ to be played, and the outcomes of these arms may further trigger more arms to be played, and so on. Therefore, when super $S$ is played in round $t$, a superset of $S$ is triggered and played, and the final reward of this round depends on the outcomes of all triggered base arms.

For each $i \in [m]$, let $p_{i,S}$ denote the probability that base arm $i$ is triggered when super arm $S$ is played. Once super arm $S$ is fixed, the event of triggering of base arm $i$ is independent of the history of previous plays of super arms. It is clear that for all $i \in S$, $p_{i,S} = 1$. Note that the triggering of base arms may depend on certain combinatorial structure of the problem instance, and triggering of different base arms may not be independent from one another. Let $\hat{S} = \{i \in [m] | p_{i,S} > 0\}$ denote the set of possibly triggered base arms by super arm $S$, also referred to as the triggering set of $S$. Let $p_i \triangleq \min_{S \in \hat{S}, i \in S} p_{i,S}$ denote the minimum nonzero triggering probability of base arm $i$ under all super arms. When $p_i = 1$ for all $i \in [m]$, each super arm $S$ deterministically triggers all base arms in $\hat{S}$, in which case we treat $S$ and $\hat{S}$ as the same set. In our paper, we assume that for every super arm $S \in S$ and base arm $i \in [m]$, the triggering probability $p_{i,S}$ is fully determined by $S$ and the means of all base arms. This is trivially true for a large class of CMAB problems where $\hat{S} = S$ for all $S \in S$, i.e., no base arm outside the super arm $S$ is triggered. In the more general case that some base arms not in $S$ may be triggered by arms in $S$, our assumption is true when the joint probability distribution of the outcomes of all base arms are determined by their means, for example, when all base arms are independent Bernoulli random variables. Henceforth, for simplicity we usually refer to base arms simply as arms.

For each arm $i \in [m]$, let $T_{i,t}$ denote the number of times arm $i$ has been successfully triggered after the first $t$ rounds in which $t$ super arms are played. If an arm $i \in S \setminus S$ is not triggered in round $t$ when super arm $S$ is played, then $T_{i,t} = T_{i,t-1}$. Let $R_t(S)$ be a non-negative random variable denoting the reward of round $t$ when super arm $S$ is played. The reward depends on the actual problem instance definition, the super arm $S$ played, and the outcomes of all triggered arms in round $t$. The reward $R_t(S)$ might be as simple as a summation of the outcomes of the triggered arms in $S$: $R_t(S) = \sum_{i \in \hat{S}, i \text{ is triggered}} X_{i,T_{i,t}}$, but our framework allows more sophisticated nonlinear rewards, as explained below.

In this paper, we consider CMAB problems in which the expected reward of playing any super arm $S$ in any round $t$, $E[R_t(S)]$, is a function of $S$ and the expectation vector $\mu$ of all arms. For the linear reward case as given above, this is true because linear addition is commutative with the expectation operator, and the triggering probabilities are also determined by the expectation vector $\mu$ of the underlying arms. For non-linear reward functions not commutative with the expectation operator, it is still true if we know the type of distributions and only the expectations of arm outcomes are unknown. For example, the distribution of $X_{i,t}$’s are known to be independent 0-1 Bernoulli random variables with unknown mean $\mu_i$. Henceforth, we denote the expected reward of playing $S$ as $r_{\mu}(S) \triangleq E[R_t(S)]$. 


Definition 1 (Assumptions on expected reward function). To carry out our analysis, we make the following two mild assumptions on the expected reward $r_\mu(S)$:

- **Monotonicity.** The expected reward of playing any super arm $S \in \mathcal{S}$ is monotonically nondecreasing with respect to the expectation vector, i.e., if for all $i \in [m]$, $\mu_i \leq \mu'_i$, we have $r_\mu(S) \leq r_{\mu'}(S)$ for all $S \in \mathcal{S}$.

- **Bounded smoothness.** There exists a continuous, strictly increasing (and thus invertible) function $f(\cdot)$ with $f(0) = 0$, called bounded smoothness function, such that for any two expectation vectors $\mu$ and $\mu'$ and for any $\Lambda > 0$, we have $|r_\mu(S) - r_{\mu'}(S)| \leq f(\Lambda)$ if $\max_{i \in S} |\mu_i - \mu'_i| \leq \Lambda$.

Both assumptions are natural. In particular, they hold true for all the applications we considered.

Definition 2 (CMAB algorithm). A CMAB algorithm $A$ is one that selects the super arm of round $t$ to play based on the outcomes of revealed arms of previous rounds, without knowing the expectation vector $\mu$. Let $S^A_t \in \mathcal{S}$ be the super arm selected by $A$ in round $t$. Note that $S^A_t$ is a random super arm that depends on the outcomes of arms in previous rounds and potential randomness in the algorithm $A$ itself. The objective of algorithm $A$ is to maximize the expected reward of all rounds up to a round $n$, that is, $\mathbb{E}_{S,R}[\sum_{t=1}^{n} R_t(S^A_t)] = \mathbb{E}_S[\sum_{t=1}^{n} r_\mu(S^A_t)]$, where $\mathbb{E}_{S,R}$ denotes taking expectation among all random events generating the super arms $S^A_t$’s and generating rewards $R_t(S^A_t)$’s, and $\mathbb{E}_S$ denotes taking expectation only among all random events generating the super arms $S^A_t$’s.

We do not assume that the learning algorithm has the direct knowledge about the problem instance, e.g. how super arms are formed from the base arms, how base arms outside of a super arm is triggered, and how reward is defined. Instead, the algorithm has access to a computation oracle that takes the expectation vector $\mu$ as the input, and together with the knowledge of the problem instance, computes the optimal or near-optimal super arm $S$. Let $\text{opt}_\mu = \max_{S \in \mathcal{S}} r_\mu(S)$ and $S^*_\mu = \text{argmax}_{S \in \mathcal{S}} r_\mu(S)$. We consider the case that exact computation of $S^*_\mu$ may be computationally hard, and the algorithm may be randomized with a small failure probability. Thus, we resolve to the following $(\alpha, \beta)$-approximation oracle.

Definition 3 ($(\alpha, \beta)$-Approximation oracle). For some $\alpha, \beta \leq 1$, $(\alpha, \beta)$-approximation oracle is an oracle that takes an expectation vector $\mu$ as input, and outputs a super arm $S \in \mathcal{S}$, such that $\Pr[r_\mu(S) \geq \alpha \cdot \text{opt}_\mu, r_{\mu'}(S) \geq \beta]$. Here $\beta$ is the success probability of the oracle.

Many computationally hard problems do admit efficient approximation oracles (Vazirani, 2004). With an $(\alpha, \beta)$-approximation oracle, it is no longer fair to compare the performance of a CMAB algorithm against the optimal reward $\text{opt}_\mu$ as the regret of the algorithm. Instead, we compare against the $\alpha \cdot \beta$ fraction of the optimal reward, because only a $\beta$ fraction of oracle computations are successful, and when successful the reward is only an $\alpha$-approximation of the optimal value.

Definition 4 ($(\alpha, \beta)$-approximation regret). $(\alpha, \beta)$-approximation regret of a CMAB algorithm $A$ after $n$ rounds of play using an $(\alpha, \beta)$-approximation oracle under the expectation vector $\mu$ is defined as

$$
\text{Reg}_{A,\mu,\alpha,\beta}(n) = n \cdot \alpha \cdot \beta \cdot \text{opt}_\mu - \mathbb{E}_S \left[ \sum_{t=1}^{n} r_\mu(S^A_t) \right].
$$

Note that the classical MAB problem is a special case of our general CMAB problem, in which (a) the constraint $\mathcal{S} = \{\{i\} | i \in [m]\}$ so that each super arm is just a base arm; (b) $S = \mathcal{S}$ for all super arm $S$, that is, playing of a base arm does not trigger any other arms; (c) the reward of a super arm $S = \{i\}$ in its $t$’s trial is its outcome $X_{i,t}$; (d) the monotonicity and bounded smoothness hold trivially with function $f(\cdot)$ being the identity function; and (e) the $(\alpha, \beta)$-approximation oracle is simply the argmax function among all expectation vectors, with $\alpha = \beta = 1$. 

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Then we simply play the super arm returned by the oracle false. We have special notations for the minimum and the maximum 
\( \Delta \). Furthermore, define

\[ \Delta_{\text{opt}} = \alpha \cdot \text{opt}_{\mu} - r_{\mu}(S). \]

For a given base arm \( i \in [m] \) with \( K_i > 0 \) and index \( j \in [K_i] \), we define

\[ \Delta_{i,j} = \Delta_{s_{i,j}^j}. \]

We have special notations for the minimum and the maximum \( \Delta_{i,j} \) for a fixed \( i \) with \( K_i > 0 \):

\[ \Delta_{\text{max}}^i = \Delta_{i,1}^i, \]

\[ \Delta_{\text{min}}^i = \Delta_{i,K_i}^i. \]

Furthermore, define \( \Delta_{\text{max}} = \max_{i \in [m], K_i > 0} \Delta_{\text{max}}^i, \) \( \Delta_{\text{min}} = \min_{i \in [m], K_i > 0} \Delta_{\text{min}}^i, \) \( K = \max_{i \in [m]} K_i. \)

Our main theorem below provides the distribution-dependent regret bound of the CUCB algorithm using the \( \Delta \) notations. We use \( \mathbb{I}\{\cdot\} \) to denote the indicator function, and \( \mathbb{I}\{E\} = 1 \) if event \( E \) is true, and 0 if \( E \) is false.

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1Throughout the paper, \( \ln \eta t = \ln(\eta t) \), and we omit the pair of parentheses for simplicity.
Theorem 1. The \((\alpha, \beta)\)-approximation regret of the CUCB algorithm in \(n\) rounds using an \((\alpha, \beta)\)-approximation oracle is at most

\[
\sum_{i \in [m], K_i > 0} \left( \ell_n(\Delta^i_{\min}, p_i) \Delta^i_{\min} + \int_{\Delta^i_{\min}}^{\Delta^i_{\max}} \ell_n(x, p_i) \, dx \right) + \left( \frac{\pi^2}{3\eta^4} + 1 \right) \cdot m \cdot \Delta_{\max} + \frac{\pi^4}{90\eta^4} \sum_{i \in [m]} K_i \cdot \mathbb{I}\{p_i < 1\} \cdot \Delta_{\max},
\]

where

\[
\ell_n(\Delta, p) = \begin{cases} 
\frac{12 \ln n}{(f^{-1}(\Delta))^{p}} + \frac{2 \ln n}{p^2}, & \text{if } 0 < p < 1, \\
\frac{\ln n}{p}, & \text{if } p = 1;
\end{cases}
\]

and \(f(\cdot)\) is the bounded smoothness function.

Note that when \(p_i = 1\) for all \(i \in [m]\), each super arms \(S\) deterministically triggers base arms in \(S\) and no probabilistic triggering of other arms. In this case, if we set the CUCB parameter \(\eta = 1\), the above theorem falls back to Theorem 1 of Chen et al. (2013). When \(p_i < 1\) for some \(i \in [m]\), the regret bound has a bit more complicated term, with \(p_i\) appearing in the denominators of the leading \(\ln n\) term, and the constant term includes \(K_i\) as a factor, where \(K_i\) is the number of bad super arms that may trigger \(i\). Since \(K_i\) could be exponential to the problem instance size, if we still set \(\eta = 1\), we may have a constant term in the regret bound that is exponential to the problem instance size. In this case, if we want to avoid the exponentially large constant term in the regret, we can consider setting a large \(\eta\). For example we can set \(\eta = |S|^{1/4}\), such that \(K_i \leq \eta^i\), and thus the exponential \(K_i\) is absorbed. The cost for setting such a large \(\eta\) is that we will have an additional constant regret term related to \(O(\ln |S|)\), which is reasonably small. Essentially, by setting a large \(\eta\), the CUCB algorithm will do more explorations to compensate the inaccuracy of estimation due to probabilistically triggered arms.

Based on the same analytical framework, we have a slightly different bound which might be helpful in some circumstances.

Theorem 2. The \((\alpha, \beta)\)-approximation regret of the CUCB algorithm in \(n\) rounds using an \((\alpha, \beta)\)-approximation oracle is at most

\[
\sum_{i \in [m], K_i > 0} \ell_n(\Delta^i_{\min}, p_i) \cdot \Delta_{\max} + \left( \frac{\pi^2}{3\eta^4} + 1 \right) \cdot m \cdot \Delta_{\max} + \frac{\pi^4}{90\eta^4} \sum_{i \in [m]} \mathbb{I}\{p_i < 1\} \cdot \Delta_{\max}.
\]

Comparing Theorem 2 with Theorem 1 when using the same parameter \(\eta\), we see that (a) when \(p_i = 1\) for all \(i \in [m]\), Theorem 1 provides a tighter bound than Theorem 2; (b) when \(p_i < 1\) for some \(i \in [m]\), the regret bounds provided by the two theorems are not comparable — Theorem 1 provides a better bound for the leading \(\ln n\) term, but Theorem 2 provides a better bound for the constant term (removing the dependency on \(K_i\)’s, the number of super arms that could trigger a base arm \(i\)). Theorem 2 shows that, even if we use \(\eta = 1\), the distribution-dependent regret bound of CUCB can avoid the exponential constant term, at the expense of a larger bound for the leading \(\ln n\) term.

In Theorem 1 or 2, when \(\Delta^i_{\min}\) is extremely small, the regret would be approaching infinity. Below we prove a distribution-independent regret for arbitrary distributions with support in \([0, 1]\) on all arms, for a large class of problem instances with a polynomial bounded smoothness function \(f(x) = \gamma x^\omega\) for \(\gamma > 0\) and \(0 < \omega \leq 1\). The rough idea of the proof is, if \(\Delta^i_{\min} \leq 1/\sqrt{n}\), it can only contribute \(\sqrt{n}\) regret at time horizon \(n\). The proof of the following theorem relies on the tight regret bound of Theorem 1 on the leading \(\ln n\) term.
Theorem 3. Consider a CMAB problem with an \((\alpha, \beta)\)-approximation oracle. Let \(p^* = \min_{i \in[m]} p_i\). If the bounded smoothness function \(f(x) = \gamma \cdot x^\omega\) for some \(\gamma > 0\) and \(\omega \in (0, 1]\), the regret of CUCB is at most:

\[
\begin{align*}
\text{for } p^* = 1: & \quad \frac{2n}{\omega} \cdot \left(6m \ln m \right)^{\omega/2} \cdot n^{1-\omega/2} + \left(\frac{2}{m^2} + 1\right) \cdot m \cdot \Delta_{\max}, \\
\text{for } 0 < p^* < 1: & \quad \frac{2n}{\omega} \cdot \left(12m \ln m \right)^{\omega/2} \cdot n^{1-\omega/2} + \left(\frac{2}{m^2} + 1\right) \cdot m \cdot \Delta_{\max} + \sum_{i\in[m]} \left(\frac{2m \ln m}{p_i^2} + \frac{\omega^2 K_i \ln m}{p_i^4}\right) \Delta_{\max},
\end{align*}
\]

Similar to the remark after Theorem 1, for \(p^* = 1\), we can set \(\eta = 1\) while for \(p^* < 1\), we can set \(\eta = |S|^{1/4}\) to absorb the exponential term \(K_i\). Note that for all applications discussed in Section 4, we have \(\omega = 1\). For the classical MAB setting with \(\eta = 1\), \(\omega = 1\) and \(p^* = 1\), we obtain a distribution-independent bound of \(O(\sqrt{mn \ln n})\), which matches (up to logarithmic factors) the original UCB1 algorithm (Auer et al., 2009). In the linear combinatorial bandit setting, i.e., semi-bandit with \(L_\infty\) assumption in Audibert et al. (2011), our regret is \(O(\sqrt{m^3 n \log n})\), which is a factor \(\sqrt{m}\) off the optimal bound in the adversarial setting.

### 3.1 Proof of the Theorems

#### 3.1.1 Proof of Theorem 1

Before getting to the proof of our theorem, we need more definitions and lemmas. First, we have a convenient notation for the case when the oracle outputs non-\(\alpha\)-approximation answers.

**Definition 7** (Non-\(\alpha\)-approximation output). In the \(t\)-th round, let \(F_i\) be the event that the oracle fails to produce an \(\alpha\)-approximate answer with respect to \(\hat{\bar{\mu}} = (\hat{\bar{\mu}}_1, \hat{\bar{\mu}}_2, \ldots, \hat{\bar{\mu}}_m)\). We have \(\Pr[\{F_i\}] = \mathbb{E}[\|F_i\|] \leq 1 - \beta\).

Since the value of many variables are changing in different rounds, we also define notations for their value in round \(t\). All of them are random variables.

**Definition 8** (Variables in round \(t\)). For variable \(T_i\), let \(T_{i,t}\) be the value of \(T_i\) at the end of round \(t\), that is, \(T_{i,t}\) is the number of times arm \(i\) is played in the first \(t\) rounds. For variable \(\mu_i\), let \(\mu_{i,s}\) be the value of \(\mu_i\) after arm \(i\) is played \(s\) times, that is, \(\mu_{i,s} = \frac{\sum_{j=1}^s X_{i,j}}{s}\). Then, the value of variable \(\mu_i\) at the end of round \(t\) is \(\mu_{i,t}\). For variable \(\bar{\mu_i}\), let \(\bar{\mu}_{i,t}\) be the value of \(\bar{\mu_i}\) at the end of round \(t\).

Next, we introduce an important parameter in our proof called sampling threshold.

**Definition 9** (Sampling threshold). For a probability value \(p \in (0, 1]\) and reward difference value \(\Delta \in \mathbb{R}^+\), the value \(\ell_n(\Delta, p)\) defined below is called the sampling threshold for round \(n\):

\[
\ell_n(\Delta, p) = \begin{cases} 
\frac{12 \ln m}{(f^{-1}(\Delta))^2} p + \frac{2 \ln m}{p^2}, & \text{if } 0 < p < 1, \\
\frac{6 \ln m}{f^{-1}(\Delta)^2}, & \text{if } p = 1.
\end{cases}
\]

Informally, base arm \(i \in [m]\) at round \(n\) is considered as sufficiently sampled if the number of times \(i\) has been played by round \(n\), \(T_{i,n}\), is above its sampling threshold \(\ell_n(\Delta_{\min}, p_i)\). When all base arms are sufficiently sampled, with high probability we would obtain accurate estimates of their sample means and would be able to distinguish the \(\alpha\)-approximate super arms from bad super arms.

We rely on the following well known Chernoff-Hoeffding bound in our analysis, as summarized in Auer et al. (2002a).

**Fact 1** (Chernoff-Hoeffding bound). Let \(X_1, \ldots, X_n\) be random variables with common support \([0, 1]\), and \(\mathbb{E}[X_t | X_1, \ldots, X_{t-1}] = \mu\) for every \(t \leq n\). Let \(Y = X_1 + \cdots + X_n\), then for all \(t \geq 0\),

\[
\Pr\{Y \geq n\mu + t\} \leq e^{-2t^2/n} \quad \text{and} \quad \Pr\{Y \leq n\mu - t\} \leq e^{-2t^2/n}.
\]

Based on the above fact, it is immediate to get an extended corollary.
**Corollary 1.** Let \( X_1, \ldots, X_n \) be random variables with common support \([0, 1]\), and for every \( t \leq n \) we have \( \mathbb{E}[X_i | X_1, \ldots, X_{i-1}] \geq \mu \). Let \( Y = X_1 + \cdots + X_n \), then for all \( t \geq 0 \),

\[
\Pr\{Y \leq n\mu - t\} \leq e^{-2t^2/n}.
\]

Using Chernoff-Hoeffding bound, we can prove that with high probability, the empirical mean of a set of independently sampled variables is close to the actual mean. Below we give a definition on the standard difference between the empirical mean and the actual expectation.

**Definition 10** (Standard difference). For the random variable \( T_{i,t-1} \), standard difference is defined as a random variable \( \Lambda_{i,t} = \sqrt{\frac{3 \ln n t}{2 T_{i,t-1}}} \). The maximum standard difference is defined as a random variable \( \Lambda_t = \max\{\Lambda_{i,t} \mid i \in S_1\} \). The universal difference bound is defined as \( \Lambda^{i,l} = \sqrt{\frac{3 \ln n t}{2 (\Delta^{i,l})^2}} = \frac{t^{-1} (\Delta^{i,l})^2}{2} \), which is not a random variable.

If in the round \( t \), the difference between the empirical mean and the actual expectation is below the standard difference, we call the process a “nice process”. See the formal definition below.

**Definition 11** (Nice run). The run of Algorithm 1 is nice at time \( t \) (denoted as the indicator \( N_{i,t} \)) if:

\[
\forall i \in [m], \ |\hat{\mu}_{i,T_{i,t-1}} - \mu_i| < \Lambda_{i,t}. \tag{4}
\]

**Lemma 1.** The probability that the run of Algorithm 1 is nice at time \( t \) is at least \( 1 - \frac{2m}{n^3 t^2} \).

**Proof.** By the Chernoff-Hoeffding bound in Fact 1, for any \( i \in [m] \),

\[
\Pr\{|\hat{\mu}_{i,T_{i,t-1}} - \mu_i| \geq \Lambda_{i,t}\} = \sum_{s=1}^{t-1} \Pr\{|\hat{\mu}_{i,s} - \mu_i| \geq \Lambda_{i,t}, T_{i,t-1} = s\} \leq \sum_{s=1}^{t-1} \Pr\{|\hat{\mu}_{i,s} - \mu_i| \geq \sqrt{\frac{3 \ln n t}{2s}}\} \leq t \cdot 2e^{-3 \ln n t} = \frac{2}{n^3 t^2}. \tag{5}
\]

The lemma follows by taking union bound on \( i \). \[ \square \]

Lemma 1 tells us that if at time \( t \), \( T_{i,t-1} \) is large, then we can get a good estimation of \( \mu_i \). Intuitively, if we estimate all \( \mu_i \)’s pretty well, it is unlikely that we will choose a bad super arm using the approximation oracle. On the other hand, in the case that for some \( i \) \( T_{i,t-1} \) is small, although we may not have a good estimate of \( \mu_i \), it indicates that arm \( i \) has not been played for many times, which gives us an upper bound on the number of times that the algorithm plays a bad super arm containing arm \( i \). Based on this idea, it is crucial to find a sampling threshold, which separates these two cases.

Now we need to define the way that we count the sampling times of each arm \( i \). Please notice the difference between \( N_{i,t} \) (defined below) and \( T_{i,t} \), especially when \( p \neq 1 \).

**Definition 12** (Counter for arm \( i \)). After the \( m \) initialization rounds, we maintain a counter \( N_{i} \) for each arm \( i \). Let \( N_{i,t} \) be the value of \( N_{i} \) at the end of round \( t \) and \( N_{i,m} = 1 \). Note that \( \sum_{i \in [m]} N_{i,m} = m \). \{\( N_{i} \)\} is updated in the following way.

For a round \( t > m \), let \( S_t \) be the super arm selected in round \( t \) by the oracle (line 7 of Algorithm 1). Round \( t \) is bad if the oracle selects a bad super arm \( S_t \in S_B \). If round \( t \) is bad, let \( i = \arg\min_{j \in S_t} N_{j,t-1} \). If the above \( i \) is not unique, we pick an arbitrary one. Then we increment the counter \( N_{i} \), i.e., \( N_{i,t} = N_{i,t-1} + 1 \) while not changing other counters \( N_j \) with \( j \neq i \). If round \( t \) is not bad, i.e., \( S_t \notin S_B \), no counter \( N_{i} \) is incremented.
Note that the counter $N_t$ is for the purpose of analysis, and its maintenance is not part of the algorithm. Intuitively, for each round $t$ where a bad super arm $S_t$ is played, we increment exactly one counter $N_i$, where $i$ is selected among all possibly triggered base arms $S_t$ such that the current value of $N_i$ is the lowest. In the special case when $p_i = 1$ for some $i \in [m]$, we know that $i \notin \hat{S} \setminus S$ for any super arm $S$. Therefore, whenever arm $i$ is selected to increment its counter $N_i$ in a round $t$, $i$ must have been played in round $t$, and thus we have $T_{i,t} \geq N_{i,t}$ for any $i \in [m]$ with $p_i = 1$ and all time $t$. However, this may not holds for $i \in [m]$ with $p_i < 1$.

In every bad round, exactly one counter in $\{N_i\}$ is incremented, so the total number of bad rounds in the first $n$ rounds is at most $\sum N_{i,n}$ (and equal if all the initialization rounds are bad).

Below we give the definition of refined counters.

**Definition 13 (Refined counters).** Each time $N_i$ gets updated, one of the bad super arm containing $i$ is played. We further separate $N_i$ into a set of counters as follows:

$$\forall l \in [K], N^l_{i,n} = \sum_{t=m+1}^{n} I\{S_t = S^l_{i,B}, N_{i,t} > N_{i,t-1}\}.$$

That is, each time $N_i$ is updated, we also record which bad super arm is played.

With these counters in hands, we shall define the two stages “sufficiently sampled” and “under-sampled” using the sampling threshold, which further split the counter $N^l_{i,n}$ into two counters.

**Definition 14 (Sufficiently sampled and under-sampled).** Consider time horizon $n$ and current time $t \leq n$. When counter $N_{i,t}$ is incremented at time $t$, we inspect the counter $N_{i,t-1}$. If $N_{i,t-1} > \ell_n(\Delta^i_{\min, p_i})$, we say the bad super arm selected in round $t$ is sufficiently sampled; otherwise, it is under-sampled. We split the counter $N_{i,n}$ into two parts, which are sufficiently sampled part and under-sampled part, as follows:

$$N^{\text{suf}}_{i,n} = \sum_{t=m+1}^{n} I\{S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} > \ell_n(\Delta^i_{\min, p_i})\},$$

$$N^{\text{und}}_{i,n} = \sum_{t=m+1}^{n} I\{S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n(\Delta^i_{\min, p_i})\}.$$

For the refined counter $N^l_{i,n}$’s, we also separate them into sufficiently sampled part and under-sampled part, as defined below. When counter $N^l_{i,t}$ is incremented at time $t$, i.e., $S_t = S^l_{i,B}$, we inspect the counter $N^l_{i,t-1}$. If $N^l_{i,t-1} > \ell_n(\Delta^{i,l}_{\min, p_i})$, we say that the bad super arm $S^l_{i,B}$ is sufficiently sampled; otherwise, it is under-sampled. Thus counter $N^l_{i,n}$ is separated into the following sufficiently sampled part and under-sampled part:

$$N^{l,\text{suf}}_{i,n} = \sum_{t=m+1}^{n} I\{S_t = S^l_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} > \ell_n(\Delta^{i,l}_{\min, p_i})\},$$

$$N^{l,\text{und}}_{i,n} = \sum_{t=m+1}^{n} I\{S_t = S^l_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n(\Delta^{i,l}_{\min, p_i})\}.$$

Note that in the above definition, the counter $N^{\text{suf}}_{i,n}$ uses the more stringent criteria for sufficient sampling (namely $N_{i,t-1} > \ell_n(\Delta^i_{\min, p_i})$) than the counter $N^{l,\text{suf}}_{i,n}$ for each individual index $l$ (namely $N_{i,t-1} > \ell_n(\Delta^{i,l}_{\min, p_i})$). Therefore, we have $N^{\text{suf}}_{i,n} \leq \sum_{l \in [K]} N^{l,\text{suf}}_{i,n}$. We will use these two different counting methods to prove two different regret bounds as shown in Theorems 1 and 2.

Following the definition, we have $N^{l,\text{und}}_{i,n} \leq \ell_n(\Delta^{i,l}_{\min, p_i})$, and $N_{i,n} = 1 + \sum_{l \in [K]} (N^{l,\text{suf}}_{i,n} + N^{l,\text{und}}_{i,n})$. Using this notation, the total reward at time horizon $n$ is at least

$$n \cdot \alpha \cdot \text{opt} = \sum_{i \in [m], K_i > 0} \left( \Delta^i_{\max} + \sum_{l \in [K]} (N^{l,\text{suf}}_{i,n} + N^{l,\text{und}}_{i,n}) \cdot \Delta^{i,l}_{\min} \right), \quad (6)$$
where $\Delta^i_{\max}$ is for the initialization.

To get an upper bound on the regret, we want to upper bound $N^{l,suf}_{i,n}$ and $N^{l,und}_{i,n}$ separately. Before doing that, we prove an important connection as follows.

**Lemma 2** (Connection between $N_{i,t-1}$ and $T_{i,t-1}$). Let $n$ be the time horizon. For every round $t$ with $m < t \leq n$, every base arm $i \in [m]$, and every $\Delta > 0$, we have,

$$
\Pr \left\{ N_{i,t-1} > \ell_n(\Delta, p_i) , T_{i,t-1} \leq \frac{6 \cdot \ln \eta t}{\eta^{-1}(\Delta)^2} \right\} \leq \frac{1}{\eta^4 t^4}.
$$

(7)

Specifically, if $p_i = 1$, we have

$$
\Pr \left\{ N_{i,t-1} > \ell_n(\Delta, p_i) , T_{i,t-1} \leq \frac{6 \cdot \ln \eta t}{\eta^{-1}(\Delta)^2} \right\} = 0.
$$

**Proof.** Fix a base arm $i$. The case of $p_i = 1$ is trivial since in this case $T_{i,t-1} \geq N_{i,t-1}$ and $n \geq t$. Now we only consider the case of $0 < p_i < 1$.

In a run of CUCB algorithm (Algorithm 1), let $t^{(j)}$ be the $j$-th round that $N_{i,t}$ gets updated. Suppose that in round $t^{(j)}$, super arm $S^{(j)}$ is played. Note that both $t^{(j)}$ and $S^{(j)}$ are random, depending on the randomness of the environment sampling the outcomes of base arms and the triggering of base arms from super arms in all historical rounds. For each $j = 1, 2, \ldots N_{i,t-1}$, we fix the random round $t^{(j)}$ to be some fixed value $r_j$, fix the random super arm $S^{(j)}$ to be some fixed super arm $A_j$, and fix $N_{i,t-1}$ to be any value $\ell > \ell_n(\Delta, p_i)$. Let event $E$ be the event of $\{N_{i,t-1} = \ell, t^{(j)} = r_j, S^{(j)} = A_j, \forall j = 1, 2, \ldots \ell \}$. Our plan is to prove

$$
\Pr \left\{ N_{i,t-1} > \ell_n(\Delta, p_i), T_{i,t-1} \leq \frac{6 \cdot \ln \eta t}{\eta^{-1}(\Delta)^2} \bigg| E \right\} \leq \frac{1}{\eta^4 t^4}.
$$

(8)

Then, since this inequality holds for any conditional event $E$ with any fixed value $\ell > \ell_n(\Delta, p_i)$, and fixed rounds $r_1, r_2, \ldots, r_\ell$, and any fixed super arms $A_1, A_2, \ldots, A_\ell$, then the original Inequality (7) holds unconditionally.

Let $X^{(j)}$ be the Bernoulli random variable indicating whether arm $i$ is triggered by the play of super arm $S^{(j)}$ in round $t^{(j)}$. Since $T_{i,t-1}$ is the number of times $i$ is triggered by the end of round $t - 1$, we have $T_{i,t-1} \geq \sum_{j=1}^{N_{i,t-1}} X^{(j)}$. When conditioned on event $E$, by our CMAB model we know that once the super arm played in round $t^{(j)} = r_j$, namely $S^{(j)}$, is fixed to be $A_j$, the random event of triggering base arms in round $t^{(j)} = r_j$ is independent of historical random events before round $r_j$. Therefore, conditioned on event $E$, $X^{(j)}$’s are mutually independent. Moreover, conditioned on event $E$, the probability that base arm $i$ is triggered in round $t^{(j)} = r_j$ is $p_i^{A_j} \geq p_i$, i.e., $\Pr(X^{(j)} = 1 \mid E) = \mathbb{E}[X^{(j)} \mid E] = p_i^{A_j} \geq p_i$. With these
preparations, we can now apply Chernoff-Hoeffding bound, in particular Corollary 1, to obtain the following:

\[
\Pr \left\{ N_{i,t-1} > \ell_n(\Delta, p_i), T_{i,t-1} \leq \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta)^2} \mid E \right\} \\
\leq \Pr \left\{ N_{i,t-1} > \ell_n(\Delta, p_i), \sum_{j=1}^{N_{i,t-1}} X^{(j)} \leq \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta)^2} \mid E \right\} \\
\leq \Pr \left\{ \sum_{j=1}^{\ell_n(\Delta, p_i)} X^{(j)} \leq \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta)^2} \mid E \right\} \\
\leq \Pr \left\{ \sum_{j=1}^{\ell_t(\Delta, p_i)} X^{(j)} \leq \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta)^2} \mid E \right\} \quad \{n \geq t \Rightarrow \ell_n(\Delta, p_i) \geq \ell_t(\Delta, p_i)\} \\
= \Pr \left\{ \sum_{j=1}^{\ell_t(\Delta, p_i)} X^{(j)} \leq \ell_t(\Delta, p_i) \cdot p_i - \frac{6 \ln \eta t}{f^{-1}(\Delta)^2} - \frac{2 \ln \eta t}{p_i} \mid E \right\} \quad \{\text{definition of } \ell_t(\Delta, p_i)\} \\
= \Pr \left\{ \sum_{j=1}^{\ell_t(\Delta, p_i)} X^{(j)} \leq [\ell_t(\Delta, p_i)] \cdot p_i - \delta \cdot p_i - \frac{6 \ln \eta t}{f^{-1}(\Delta)^2} - \frac{2 \ln \eta t}{p_i} \mid E \right\} \quad \{\text{Let } \delta = \ell_t(\Delta, p_i) - \ell_t(\Delta, p_i)\} \\
\leq \exp \left\{ -2 \cdot \frac{6 \ln \eta t}{f^{-1}(\Delta)^2} + \frac{2 \ln \eta t}{p_i} + \delta \cdot p_i \right\} \quad \{\text{Corollary 1 and } \mathbb{E}[X^{(j)} \mid E] \geq p_i\} \\
\leq \exp \left\{ -2 \cdot \frac{24 \ln^2 \eta t}{f^{-1}(\Delta)^2} + \frac{4 \ln^2 \eta t}{p_i^2} + 4 \delta \frac{\ln \eta t}{p_i^2} \right\} \\
\leq \exp \left\{ -2 \cdot \frac{12 \ln \eta t}{f^{-1}(\Delta)^2} + \frac{2 \ln \eta t}{p_i^2} + \delta \right\} \\
\leq \exp \left\{ -4 \ln \eta t \right\} = \frac{1}{\eta^4 t^2}.
\]

Therefore, Inequality (8) holds, and thus the unconditional Inequality (7) also holds. \(\square\)

Recall that a nice run at time \(t\) (Definition 11, denoted as \(N_t\)) means that the difference between the empirical mean and the actual mean is bounded by the standard difference \(\Lambda_{i,t}\) for every arm \(i \in [m]\). By Lemma 1, we know that with probability \(1 - \frac{2m}{\eta^2 t^2}\), \(N_t\) holds. According to line 6 of Algorithm 1, we have \(\bar{\mu}_{i,t} - \mu_i > 0\), \(\bar{\mu}_{i,t} - \mu_i < 2\Lambda_t\). Thus:

\[
N_t \Rightarrow \forall i \in [m], \bar{\mu}_{i,t} - \mu_i > 0, \quad \text{(9)}
\]

\[
N_t \Rightarrow \forall i \in \hat{S}_t, \bar{\mu}_{i,t} - \mu_i < 2\Lambda_t, \quad \text{(10)}
\]

Meanwhile, by Definition 10, we know that for any \(i \in [m], l \in [K_i]\) and any time \(t:\)

\[
\left\{ S_t = S_{i,B}, \forall s \in \hat{S}_t, T_{s,t-1} > \frac{6 \ln \eta t}{f^{-1}(\Delta)^2} \right\} \Rightarrow \Lambda_{i,l} > \Lambda_t. \quad \text{(11)}
\]

With the previous observations, we have the following lemma. Informally, it says that in a nice run in round \(t\), it is impossible that the algorithm would select a bad super arm \(S_t\) using the oracle, which outputs a correct \(\alpha\)-approximation answer, while every arm in \(S_t\) has been tested for enough times.

**Lemma 3** (Impossible case). Let \(F_t\) be the indicator defined in Definition 7. For any \(i \in [m], l \in [K_i]\) and any time \(t\), the event \(\left\{ N_t, \neg F_t, S_t = S_{i,B}, \forall s \in \hat{S}_t, T_{s,t-1} > \frac{6 \ln \eta t}{f^{-1}(\Delta)^2} \right\} \) is empty.
Proof. Indeed, if all the conditions hold, we have:

\[ r_\mu(S_t) + f(2\Lambda_{i,l}) > r_\mu(S_t) + f(2\Lambda_i) \] \{ strict monotonicity of \( f(\cdot) \) and Eq.(11) \}

\[ \geq r_\mu(S_t) \] \{ bounded smoothness property and Eq.(10) \}

\[ \geq \alpha \cdot \text{opt}_\mu, \] \{ \( -F_t \Rightarrow S_t \) is an \( \alpha \) approximation w.r.t. \( \mu \) \}

\[ \geq \alpha \cdot r_\mu(S_*^\mu) = \alpha \cdot \text{opt}_\mu. \] \{ definition of \text{opt}_\mu \}

\[ \geq \alpha \cdot r_\mu(S_*^\mu) = \alpha \cdot \text{opt}_\mu. \] \{ monotonicity of \( r_\mu(S) \) and Eq.(9) \}

So we have

\[ r_\mu(S_*^\mu) + f(2\Lambda_{i,l}) > \alpha \cdot \text{opt}_\mu. \] (12)

However, by Definition 10, \( f(2\Lambda_{i,l}) = f(f^{-1}(\Delta_{i,l})) = \Delta_{i,l} \). Thus, Inequality (12) contradicts the definition of \( \Delta_{i,l} \) in Definition 6.

Now we are ready to prove the bound on sufficiently sampled part.

Lemma 4. [Bound on sufficiently sampled part] For any time horizon \( n > m \),

\[ \mathbb{E} \left[ \sum_{i \in [m], K_i > 0} \sum_{t \in [K_i]} N_{i,n}^{t,suf} \right] \leq (1 - \beta)n + \frac{m\pi^2}{3\eta^3} + \frac{\pi^4}{90\eta^4} \sum_{i \in [m]} K_i \cdot \mathbb{I}\{p_i < 1\}. \] (13)

Proof. Based on Definition 14 on \( N_{i,n}^{t,suf} \), it is sufficient to show that for any \( m < t \leq n \),

\[ \mathbb{E} \left[ \sum_{i \in [m], K_i > 0} \sum_{t \in [K_i]} \mathbb{I}\{S_t = S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} > \ell_n(\Delta_{i,l}, p_i)\} \right] \leq \sum_{i \in [m], K_i > 0} \sum_{t \in [K_i]} \mathbb{P}\{S_t = S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, N_{s,t-1} > \ell_n(\Delta_{i,l}, p_i)\} \] (14)

\[ \leq (1 - \beta) + \frac{2m}{\eta^4\mu^2} + \frac{1}{\eta^4\mu^4} \sum_{i \in [m]} K_i \cdot \mathbb{I}\{p_i < 1\}. \] (15)

Note that Inequality (14) is due to our way of updating counter \( N_i \) by Definition 12: When \( N_i \) is incremented in round \( t \) such that \( N_{i,t} > N_{i,t-1} \), we know that \( N_{i,t-1} \) has the lowest counter value among all \( N_{s,t-1} \) for \( s \in \tilde{S}_t \). The reason that Inequality (15) is sufficient because we may then take the union bound on all \( t \)'s, and get a bound of

\[ \sum_{t=m+1}^{n} \left( (1 - \beta) + 2mt^{-2} + \sum_{i \in [m]} K_i \cdot t^{-4} \cdot \mathbb{I}\{p_i < 1\} \right) \leq (1 - \beta)n + \frac{m\pi^2}{3\eta^3} + \frac{\pi^4}{90\eta^4} \sum_{i \in [m]} K_i \cdot \mathbb{I}\{p_i < 1\}. \]

Thus, in order to prove our claim, it suffices to prove Inequality (15).

Based on Lemma 2, we may split the term in Eq.(14) into two parts:
\[ \sum_{i \in [m], K_i > 0} \sum_{l \in [K_i]} \Pr \left\{ S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in S_{i,l}^t, N_{s,t-1} > \ell_n \left( \Delta_{i,t}^l, p_i \right) \right\} \]

\[ = \sum_{i \in [m], K_i > 0} \sum_{l \in [K_i]} \left( \Pr \left\{ S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in S_{i,l}^t, N_{s,t-1} > \ell_n \left( \Delta_{i,t}^l, p_i \right), T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} \right) \]

\[ + \Pr \left\{ S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in S_{i,l}^t, T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} \leq \sum_{i \in [m], K_i > 0} \sum_{l \in [K_i]} \left( \Pr \left\{ S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in S_{i,l}^t, T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} \right) \]

\[ + \frac{1}{\eta^4 t^4} \sum_{i \in [m]} K_i \cdot \mathbb{I}\{p_i < 1\}. \]  

(16)

By Lemma 3, we have:

\[ \forall i \in [m], \forall l \in [K_i], \Pr \left\{ N_i, \neg F_l, S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} = 0 \Rightarrow \]

\[ \Pr \left\{ N_i, \neg F_l, \exists i \in [m], \exists l \in [K_i], S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} = 0 \Rightarrow \]

\[ \Pr \left\{ \exists i \in [m], \exists l \in [K_i], S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} \leq \]

\[ \Pr[F_l \vee \neg N_i] \leq (1 - \beta) + \frac{2m}{\eta^4 t^2} \]  

(18)

\[ \Rightarrow \sum_{i \in [m], K_i > 0} \sum_{l \in [K_i]} \Pr \left\{ S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta t}{f^{-1}(\Delta_{i,t}^l)^2} \right\} \leq (1 - \beta) + \frac{2m}{\eta^4 t^2}. \]  

(19)

The inequality in Eq. (18) uses the definition of \( F_l \) (Definition 7) and Lemma 1. The left side of Inequality (17) is the same as the left side of Inequality (19) because events \( \{ S_t = S_{i,l}^t, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln t}{f^{-1}(\Delta_{i,t}^l)^2} \} \) for all \( i \in [m] \) and \( l \in [K_i] \) are mutually exclusive. This is because by the definition of counter \( N_i \) (Definition 12) in each round \( t \) only one counter \( N_i \) for some \( i \in [m] \) is incremented and thus only this \( i \) has \( N_{i,t} > N_{i,t-1} \). For this \( i \), only one \( l \) has \( S_t = S_{i,l}^t \). Eq. (19) together with Eq. (16) leads to Eq. (15).

Now we consider the bound on under-sampled part, i.e., the number of times that the played bad super arms are under-sampled. For a particular arm \( i \), its counter \( N_i \) will increase from 1 to \( \ell_n(\Delta_{i,K_i}^i, p_i) \). Assume \( N_{i,t-1} \in \left( \ell_n(\Delta_{i,t-1}^i, p_i), \ell_n(\Delta_{i,t}^i, p_i) \right) \) when \( N_i \) is incremented at time \( t \) with an under-sampled super arm \( S_{i,l}^t \). We can conclude that \( \Delta_{i,t}^l \leq \Delta_{i,j}^1 \), which will be used as an upper bound for the regret. Otherwise, we must have \( \Delta_{i,l}^l \geq \Delta_{i,j}^1 \) and \( S_{i,l}^t \) is already sufficiently sampled.

To simplify the notation, set \( \ell_n(\Delta_{i,0}^0, p_i) = 0 \). (Notice that \( N_{i,m} = 1 \) for all \( i \).) Before we go into the detail, we discuss the essential idea first. We break the range of the counter \( N_{i,t} \) into discrete segments, i.e., \( \ell_n(\Delta_{i,t-1}^i, p_i), \ell_n(\Delta_{i,t}^i, p_i) \) for \( j \in [K_i] \). Let us assume that the round \( t \) is bad and \( N_{i,t} \) is incremented. Assume \( N_{i,t-1} \in \left( \ell_n(\Delta_{i,t-1}^i, p_i), \ell_n(\Delta_{i,t}^i, p_i) \right) \) for some \( j \). Notice that we are only interested in the case that \( S_t \) is under-sampled. In particular, if this is indeed the case, we have \( \ell_n(\Delta_{i,t}^i, p_i) \) for some \( l \geq j \). (Otherwise, \( S_t \) is sufficiently sampled based on the counter \( N_{i,t} \). Therefore, we will suffer a regret of \( \Delta_{i,l}^l \leq \Delta_{i,j}^1 \) (See Definition 6). Consequently, for counter \( N_{i,t} \) in range \( \ell_n(\Delta_{i,j}^1, p_i), \ell_n(\Delta_{i,t}^i, p_i) \), we will suffer a total regret for those under-sampled arms at most \( \ell_n(\Delta_{i,j}^1, p_i) - \ell_n(\Delta_{i,t-1}^i, p_i) \cdot \Delta_{i,j}^1 \) in rounds that \( N_{i,t} \) is incremented.

**Lemma 5** (Bound on under-sampled part). For any time horizon \( n > m \), we have,

\[ \sum_{i \in [m], K_i > 0} \sum_{l \in [K_i]} N_{i,t} \cdot \Delta_{i,t}^l \leq \sum_{i \in [m], K_i > 0} \left( \ell_n(\Delta_{i,n}^0, p_i) \Delta_{i,n}^l + \int_{\Delta_{i,n}^0}^{\Delta_{i,n}^l} \ell_n(x, p_i) dx \right). \]  

(20)
Proof. It suffices to show that for any arm \( i \in [m] \) with \( K_i > 0 \),
\[
\sum_{l \in [K_i]} N^{l, \text{und}}_{i,n} \cdot \Delta^{i,l} = \ell_n (\Delta_{\text{min}}^{i,1}, p_i) \Delta^{i,1} + \frac{\Delta_{\text{min}}^{i,l}}{\Delta_{\text{min}}^{i}} \int_{\Delta_{\text{min}}^{i}} x \leq n (\Delta_{\text{min}}^{i,1}, p_i) \Delta^{i,1} + \int_{\Delta_{\text{min}}^{i}} x \leq n (x, p_i) dx.
\]

Now, by definition and discussion on the interval that \( N_{i,t-1} \) lies in, we have
\[
\sum_{l \in [K_i]} N^{l, \text{und}}_{i,n} \cdot \Delta^{i,l}
= \sum_{t=m+1}^{n} \sum_{l \in [K_i]} \I \{ S_t = S_{i,B}^l, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n (\Delta^{i,l}, p_i) \} \cdot \Delta^{i,l}
= \sum_{t=m+1}^{n} \sum_{l \in [K_i]} \I \{ S_t = S_{i,B}^l, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n (\Delta^{i,l-1}, p_i), \ell_n (\Delta^{i,l}, p_i) \} \cdot \Delta^{i,l}
\leq \sum_{t=m+1}^{n} \sum_{l \in [K_i]} \I \{ S_t = S_{i,B}^l, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n (\Delta^{i,l-1}, p_i), \ell_n (\Delta^{i,l}, p_i) \} \cdot \Delta^{i,l}.
\]

The last inequality holds since \( \Delta^{i,j} \geq \Delta^{i,l} \) for \( j \leq l \). Now it is safe to switch the summations, and we have,
\[
\leq \sum_{t=m+1}^{n} \sum_{l \in [K_i]} \sum_{l \in [K_i]} \I \{ S_t = S_{i,B}^l, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n (\Delta^{i,l-1}, p_i), \ell_n (\Delta^{i,l}, p_i) \} \cdot \Delta^{i,j}
= \sum_{t=m+1}^{n} \sum_{l \in [K_i]} \I \{ S_t = S_{i,B}^l, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n (\Delta^{i,l-1}, p_i), \ell_n (\Delta^{i,l}, p_i) \} \cdot \Delta^{i,j}.
\]

The last equality comes from merging all \( S_{i,B}^l \) into \( S_{i,B} \), so we may now switch the summations again, and get
\[
\leq \sum_{j \in [K_i]} \sum_{t=m+1}^{n} \I \{ S_t = S_{i,B}^l, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n (\Delta^{i,l-1}, p_i), \ell_n (\Delta^{i,l}, p_i) \} \cdot \Delta^{i,j}
\leq \sum_{j \in [K_i]} (\ell_n (\Delta^{i,j}, p_i) - \ell_n (\Delta^{i,j-1}, p_i)) \cdot \Delta^{i,j}.
\]

The last inequality uses a relaxation on the indicators. Now we simply expand the summation, and some terms will be cancelled. Then, we upper bound the new summation using an integral:
\[
= \ell_n (\Delta^{i,K_i}, p_i) \Delta^{i,K_i} + \sum_{j \in [K_i-1]} \ell_n (\Delta^{i,j}, p_i) \cdot (\Delta^{i,j} - \Delta^{i,j+1})
\leq \ell_n (\Delta^{i,K_i}, p_i) \Delta^{i,K_i} + \int_{\Delta^{i,K_i}} \ell_n (x, p_i) dx
\]

Inequality (25) comes from the fact that \( \ell_n (x, p_i) \) is decreasing in \( x \).

Finally we are ready to prove our main theorem. We just need to combine the upper bounds from the sufficiently sampled part and the under-sampled part together.
Proof of Theorem 1. Using the counters defined in Definition 13, we may get the expectation of the regret by computing the expectation of the value of the counters after the $n$-th round. More specifically, according to Definition 4, the expected regret is the difference between $n \cdot \alpha \cdot \beta \cdot \text{opt}_\mu$ and the expected reward, which is at least $\alpha \cdot n \cdot \text{opt}_\mu$, minus the expected loses from playing bad super arms.

Therefore, combining with Eq.(13) and Eq.(20), the overall regret of our algorithm is

$$\text{Reg}_\mu^{A,\alpha,\beta}(n) \leq \mathbb{E} \left[ n \cdot \alpha \cdot \beta \cdot \text{opt}_\mu - \left( \alpha \cdot n \cdot \text{opt}_\mu - \sum_{i \in [m], K_i > 0} \left( \Delta^\mu_{\text{max}} + \sum_{i \in [K_i]} \left( N_{i,n}^{\text{suf}} + N_{i,n}^{\text{und}} \right) \cdot \Delta^i_{\text{opt}} \right) \right] \right] \quad (27)$$

$$\leq \Delta^\mu_{\text{max}} \cdot \mathbb{E} \left[ \sum_{i \in [m], K_i > 0} \sum_{i \in [K_i]} N_{i,n}^{\text{suf}} \right] + m \Delta^\mu_{\text{max}}$$

$$+ \sum_{i \in [m], K_i > 0} \left( \ell_n (\Delta^\mu_{\text{min}}, p_i) \Delta^i_{\text{min}} + \int_{\Delta^\mu_{\text{min}}}^{\Delta^i_{\text{max}}} \ell_n (x, p_i) \, dx \right) - (1 - \beta) \cdot n \cdot \alpha \cdot \text{opt}_\mu$$

$$\leq \sum_{i \in [m], K_i > 0} \left( \ell_n (\Delta^\mu_{\text{min}}, p_i) \Delta^i_{\text{min}} + \int_{\Delta^\mu_{\text{min}}}^{\Delta^i_{\text{max}}} \ell_n (x, p_i) \, dx \right) + \left( \frac{\pi^2}{3n^2} + 1 \right) \cdot m \cdot \Delta^\mu_{\text{max}}$$

$$+ \frac{\pi^4}{90n^4} \sum_{i \in [m]} K_i \cdot \mathbb{1}\{p_i < 1\} \cdot \Delta^\mu_{\text{max}} + (1 - \beta) n \cdot \Delta^\mu_{\text{max}} - (1 - \beta) \cdot n \cdot \alpha \cdot \text{opt}_\mu \quad (28)$$

$$\leq \sum_{i \in [m], K_i > 0} \left( \ell_n (\Delta^\mu_{\text{min}}, p_i) \Delta^i_{\text{min}} + \int_{\Delta^\mu_{\text{min}}}^{\Delta^i_{\text{max}}} \ell_n (x, p_i) \, dx \right) + \left( \frac{\pi^2}{3n^2} + 1 \right) \cdot m \cdot \Delta^\mu_{\text{max}}$$

$$+ \frac{\pi^4}{90n^4} \sum_{i \in [m]} K_i \cdot \mathbb{1}\{p_i < 1\} \cdot \Delta^\mu_{\text{max}}. \quad (29)$$

The last step of derivation from Eq.(28) to Eq.(29) uses the fact that all rewards are nonnegative and thus $\Delta^\mu_{\text{max}} \leq \alpha \cdot \text{opt}_\mu$ by Definition 6.

3.1.2 Proof of Theorem 2

The proof of Theorem 2 follows the same analytical framework as the proof of Theorem 1, but it uses a different set of counters.

Proof of Theorem 2. Instead of using refined counters $N_{i,t}^l$’s, we use the original counters $N_{i,t}$’s for each arm $i$, and then the sampling threshold for each arm $i$ is simply $\ell_n (\Delta^\mu_{\text{min}}, p_i)$. Then, we follow the same framework as the previous proof, which split the counters into two parts, $N_{i,t}^{\text{suf}}$ and $N_{i,t}^{\text{und}}$ (Definition 14).

For the under-sampled part, by Definition 14 $N_{i,t} \leq \ell_n (\Delta^\mu_{\text{min}}, p_i)$, thus for each $i \in [m]$ the under-sampled part contributes at most $\ell_n (\Delta^\mu_{\text{min}}, p_i) \cdot \Delta^\mu_{\text{max}}$ to the regret, which is a simple union bound on each time a bad super arm with at most $\Delta^\mu_{\text{max}}$ difference to an $\alpha$-approximation reward is selected.

For the sufficiently sampled part, we need a different version Lemma 4.

Lemma 6 (Another bound on sufficiently sampled part). For any time horizon $n > m$,

$$\mathbb{E} \left[ \sum_{i \in [m], K_i > 0} N_{i,n}^{\text{suf}} \right] \leq (1 - \beta) n + \frac{m \pi^2}{3n^2} + \frac{\pi^4}{90n^4} \sum_{i \in [m]} \mathbb{1}\{p_i < 1\}.$$
Proof. The proof bear similarities to the proof of Lemma 4, and thus we omit some of the detailed explanations here. By the definition of $N_{i,n}^{suf}$ (Definition 14), it suffices to show that for any $t > m$,

$$ E \left[ \sum_{i \in [m], K_i > 0} I \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} > \ell_n \left( \Delta_{\min}^i, p_i \right) \right\} \right] $$

$$ \leq \sum_{i \in [m], K_i > 0} \Pr \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, N_{s,t-1} > \ell_n \left( \Delta_{\min}^i, p_i \right) \right\} $n \Delta_{\min}^i \geq \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} + \frac{1}{\eta^2 t^2} \sum_{i \in [m]} I \left\{ p_i < 1 \right\}. \quad (30) $$

Based on Lemma 2, we may split the term in Eq. (30) into two parts:

$$ \sum_{i \in [m], K_i > 0} \Pr \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, N_{s,t-1} > \ell_n \left( \Delta_{\min}^i, p_i \right) \right\} $$

$$ = \sum_{i \in [m], K_i > 0} \left( \Pr \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, N_{s,t-1} > \ell_n \left( \Delta_{\min}^i, p_i \right), T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} + \Pr \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} \right\} $$

$$ \leq \sum_{i \in [m], K_i > 0} \left( \Pr \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} + \frac{1}{\eta^2 t^2} \sum_{i \in [m]} I \left\{ p_i < 1 \right\}. \quad (31) $$

By Lemma 3, we have:

$$ \forall i \in [m], \Pr \left\{ N_i, \neg F_i, S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} = 0 \Rightarrow $$

$$ \Pr \left\{ N_i, \neg F_i, \exists i \in [m], S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} = 0 \Rightarrow $$

$$ \Pr \left\{ \exists i \in [m], S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} \leq \Pr \left\{ F_i \cup \neg N_i \right\} \leq \left( 1 - \beta \right) + 2mt^{-2} \Rightarrow $$

$$ \sum_{i \in [m], K_i > 0} \Pr \left\{ S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, \forall s \in \tilde{S}_t, T_{s,t-1} > \frac{6 \cdot \ln \eta_t}{f^{-1} \left( \Delta_{\min}^i \right)^2} \right\} \leq \left( 1 - \beta \right) + \frac{2m}{\eta^2 t^2}. $$

Together with (31), we prove the lemma.

To prove the theorem, we combine the two parts together, similar as we have done in the derivation from
\[ \text{Eq.}(27) \text{ to Eq.}(29). \]

\[ \text{Reg}_t^{A, \alpha, \beta}(n) \leq \mathbb{E} \left[ n \cdot \alpha \cdot \beta \cdot \text{opt}_\mu - \left( \alpha \cdot n \cdot \text{opt}_\mu - \sum_{i \in [m], K_i > 0} (\Delta_{\text{max}} + (N_{i,n}^{\text{uf}} + N_{i,n}^{\text{und}}) \cdot \Delta_{\text{max}}) \right) \right] \]

\[ \leq \Delta_{\text{max}} \cdot \mathbb{E} \left[ \sum_{i \in [m], K_i > 0} N_{i,n}^{\text{uf}} \right] + m \cdot \Delta_{\text{max}} + \sum_{i \in [m], K_i > 0} \ell_n(\Delta_{i}^{i}, p_i) \cdot \Delta_{\text{max}} \]

\[ \leq \sum_{i \in [m], K_i > 0} \ell_n(\Delta_{i}^{i}, p_i) \cdot \Delta_{\text{max}} + \left( \frac{\pi^2}{3\eta^2} + 1 \right) \cdot m \cdot \Delta_{\text{max}} + \frac{\pi^4}{90 \eta^4} \sum_{i \in [m]} \mathbb{I}\{p_i < 1\} \cdot \Delta_{\text{max}} \]

\[ + (1 - \beta)n \cdot \alpha \cdot \text{opt}_\mu \]

\[ \leq \sum_{i \in [m], K_i > 0} \ell_n(\Delta_{i}^{i}, p_i) \cdot \Delta_{\text{max}} + \left( \frac{\pi^2}{3\eta^2} + 1 \right) \cdot m \cdot \Delta_{\text{max}} + \frac{\pi^4}{90 \eta^4} \sum_{i \in [m]} \mathbb{I}\{p_i < 1\} \cdot \Delta_{\text{max}}. \]

\[ \square \]

3.1.3 Proof of Theorem 3

The proof of Theorem 3 relies on the tight regret bound for the leading \( \ln n \) term giving by Theorem 1.

**Proof of Theorem 3.** We first prove the case of \( p^* = 1 \). Following the proof of Theorem 1, we only need to consider the base arms that are played when they are under-sampled. Following the intuition, we need to quantify when \( \Delta \) is too small. In particular, we measure the threshold for \( \Delta_{\text{min}} \) based on \( N_{i,n} \), i.e., the counter of arm \( i \) at time horizon \( n \). Let \( \{n_j \mid j \in [m]\} \) be a set of possible counter values at time horizon \( n \). Our analysis will then be conditioned on event \( \mathcal{E} = \{\forall j \in [m], N_{j,n} = n_j\} \).

For an arm \( i \in [m] \) with \( K_i > 0 \), we have

\[ \sum_{l \in [K_i]} N_{i,n}^{l,\text{und}} \cdot \Delta_{i,l} \mid \mathcal{E} \]

\[ = \sum_{i=m+1}^{n} \sum_{l \in [K_i]} \mathbb{I}\{S_i = S_{i,l}, N_{i,t} > N_{i,t-1}, N_{i,t-1} \leq \ell_n(\Delta_{i,l}, 1) \mid \mathcal{E}\} \cdot \Delta_{i,l}. \]

With \( f(x) = \gamma x^\omega \), we have \( f^{-1}(x) = \left( \frac{x}{\overline{x}} \right)^{1/\omega} \). Define \( \Delta^*(n_i) = \left( \frac{\omega^2 \gamma \ln \eta n_i}{\alpha} \right)^{\omega/2} \), i.e., \( \ell_n(\Delta^*(n_i), 1) = n_i \).

Now we consider two cases.

Case (1): \( \Delta_{\text{min}} > \Delta^*(n_i) \). Following the same derivation as in the proof of Lemma 5, in particular the derivation leading to Inequality (26), we have

\[ \sum_{l \in [K_i]} N_{i,n}^{l,\text{und}} \cdot \Delta_{i,l} \leq \ell_n(\Delta_{\text{min}}^{l-1,1}) \Delta_{\text{min}}^{l} + \int_{\Delta_{\text{min}}^{l}}^{\Delta_{\text{max}}^{l}} \ell_n(x, 1) dx \]

\[ = \frac{6 \gamma \frac{2}{\omega - \omega} \ln \eta n_i}{(\Delta_{\text{min}}^{l})^{\omega/2}} + \frac{\omega}{2 - \omega} 6 \gamma \frac{2}{\omega - \omega} \ln \eta n_i \left( (\Delta_{\text{min}}^{l})^{1-\frac{2}{\omega}} - (\Delta_{\text{max}}^{l})^{1-\frac{2}{\omega}} \right) \]

\[ \leq \frac{2}{2 - \omega} \cdot \frac{6 \gamma \frac{2}{\omega - \omega} \ln \eta n_i}{(\Delta_{\text{min}}^{l})^{\omega/2}} \leq \frac{2}{2 - \omega} \cdot (6 \ln \eta n_i)^{\omega/2} n_i^{1-\omega/2}. \]

The last inequality above is by replacing \( \Delta_{\text{min}}^{l} \) with \( \Delta^*(n_i) \). Note that the above inequality holds unconditionally, so it also holds when conditioned on the event \( \mathcal{E} = \{\forall j \in [m], N_{j,n} = n_j\} \).
Case (2): $\Delta_{\min} \leq \Delta^*(n_i)$. Let $l^* = \min\{l \in [K_i] \mid \Delta_{i,l} \leq \Delta^*(n_i)\}$. Notice that $\Delta_{i,l}^* \leq \left(\frac{2\gamma}{2-\omega} \cdot 6 \ln \eta n\right)^{\omega/2}$. In the following derivation, we follow the derivation of Eq. (23), and then we critically use the fact that the counter $N_i$ cannot go beyond $n_i$ (in the first term in Inequality (34)):

\[
\sum_{l \in [K_i]} N_{i,l} \cdot \Delta_{i,l} | \mathcal{E}
\]

\[
\leq \sum_{l \in [K_i]} \sum_{t=m+1}^{n} \mathbb{I}\{S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} \in (\ell_n(\Delta_{i,j}^{-1}, 1), \ell_n(\Delta_{i,j}, 1)] \mid \mathcal{E}\} \cdot \Delta_{i,j}
\]

\[
\leq \sum_{j \geq l^*} \sum_{t=m+1}^{n} \mathbb{I}\{S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} \in (\ell_n(\Delta_{i,j}^{-1}, 1), \ell_n(\Delta_{i,j}, 1)] \mid \mathcal{E}\} \cdot \Delta^*(n_i)
\]

\[
+ \sum_{j \in [k-1]} \sum_{t=m+1}^{n} \mathbb{I}\{S_t \in S_{i,B}, N_{i,t} > N_{i,t-1}, N_{i,t-1} \in (\ell_n(\Delta_{i,j}^{-1}, 1), \ell_n(\Delta_{i,j}, 1)] \mid \mathcal{E}\} \cdot \Delta_{i,j}
\]

\[
\leq (n_i - \ell_n(\Delta_{i,j}^{-1}, 1)) \cdot \Delta^*(n_i) + \sum_{j \geq l^*} (\ell_n(\Delta_{i,j}^{-1}, 1) - \ell_n(\Delta_{i,j}, 1)) \cdot \Delta_{i,j}
\]

\[
\leq n_i \cdot \Delta^*(n_i) + \int \Delta_{i,1} \cdot \ell_n(x, 1) dx \leq \frac{2\gamma}{2-\omega} \cdot (6 \ln \eta n)^{\omega/2} n_i^{1-\omega/2}. \tag{35}
\]

Therefore, Eq.(35) holds in both cases. We then have

\[
\sum_{l \in [m], K_i > 0} \sum_{\ell \in [K_i]} N_{l,n} \cdot \Delta_{l,n} \mid \mathcal{E} \leq \frac{2\gamma}{2-\omega} \cdot (6 \ln \eta n)^{\omega/2} \cdot \sum_{l \in [m], K_i > 0} n_i^{1-\omega/2}
\]

\[
\leq \frac{2\gamma}{2-\omega} \cdot (6 m \ln \eta n)^{\omega/2} \cdot n_i^{1-\omega/2}. \tag{36}
\]

The last inequality comes from Jesen’s inequality and $\sum_i n_i \leq n$. Since the final inequality does not depend on $n_i$, we can drop the condition $\mathcal{E}$ above. With the bound on the under-sampled part given in Inequality (36), we combine it with the result on sufficiently sampled part given in Lemma 4, then we can following the similar derivation as shown from Eq.(27) to Eq.(29) to derive the distribution-independent regret bound given in Theorem 3 for the case of $p^s = 1$.

We now prove the case of $p^s < 1$. The proof is essentially the same, but with a different definition of $\ell_n(\Delta, p)$. In this case, we define $\Delta^*_i(n_i) = \left(\frac{12 \ln \eta n}{p_i n_i}\right)^{\omega/2}$.

For Case (1): $\Delta_{\min} > \Delta^*_i(n_i)$, following the same derivation, we have

\[
\sum_{l \in [K_i]} N_{l,n} \cdot \Delta_{l,n} \leq \frac{2\gamma}{2-\omega} \cdot \left(\frac{12 \ln \eta n}{p_i}\right)^{\omega/2} n_i^{1-\omega/2} + \frac{2 \ln \eta n}{p_i^2} \cdot \Delta_{\max}^i.
\]

For Case (2): $\Delta_{\min} \leq \Delta^*_i(n_i)$, again following the same derivation, we have

\[
\sum_{l \in [K_i]} N_{l,n} \cdot \Delta_{l,n} | \mathcal{E} \leq \frac{2\gamma}{2-\omega} \cdot \left(\frac{12 \ln \eta n}{p_i}\right)^{\omega/2} n_i^{1-\omega/2} + \frac{2 \ln \eta n}{p_i^2} \cdot \Delta_{\max}^i.
\]
Together, we have
\[
\sum_{i \in [m], K_i > 0} \sum_{t \in [K_i]} N_{i, t}^{\text{und}} \cdot \Delta_{i,t}^{\text{ld}} | \mathcal{E} \\
\leq \frac{2 \gamma}{2 - \omega} \left( \frac{12 \ln \eta n}{\ln n} \right)^{\omega/2} \sum_{i \in [m], K_i > 0} n_i^{1 - \omega/2} + \sum_{i \in [m], K_i > 0} \frac{2 \ln \eta n}{\ln n} \cdot \Delta_{i, \text{max}}
\]
\[
\leq \frac{2 \gamma}{2 - \omega} \left( \frac{12 m \ln \eta n}{\ln n} \right)^{\omega/2} n^{1 - \omega/2} + \sum_{i \in [m]} \frac{2 \ln \eta n}{\ln n} \cdot \Delta_{i, \text{max}}.
\]

Finally, combining Lemma 4 and the derivation for the regret bound as shown from Eq. (27) to Eq. (29), we obtain the regret bound for the case of \( p^* < 1 \).

### 3.2 Discussions

We may further improve the bound in Theorem 1 as follows, when all the triggering probabilities are 1.

**Improving the coefficient of the leading term when \( \forall i, p_i = 1 \).** In general, we can set \( \tilde{\mu}_i = \mu_i + \sqrt{y/(2T_i)} \) for some \( y \) in line 6 in the CUCB algorithm. The corresponding regret bound obtained is

\[
\sum_{i \in [m], K_i > 0} \left( \frac{2 \cdot y}{(f^{-1}(\Delta_{i, \text{min}}))^2} \cdot \Delta_{i, \text{min}} + \frac{2 \cdot y}{(f^{-1}(\Delta_{i, \text{max}}))^2} \cdot \Delta_{i, \text{max}} \right) + \left( 1 + \sum_{t=m+1}^{n} \frac{2 t}{\ln n} \right) \cdot m \cdot \Delta_{\text{max}}.
\]

What we need is to make sure the term \( \sum_{t=m+1}^{n} \frac{2 t}{\ln n} \) in the above regret bound converges. We can thus set \( y \) appropriately to guarantee convergence while improving the constant in the leading term. One way is setting \( y = (1 + c) \ln t \) with a constant \( c > 1 \), or equivalently setting \( \mu_i = \hat{\mu}_i + \sqrt{(1 + c) \ln t/(2T_i)} \), so that \( \sum_{t=m+1}^{n} \frac{2 t}{\ln n} = 2 \sum_{t=m+1}^{n} t^{-c} \leq 2 \zeta(c) \), where \( \zeta(c) = \sum_{t=1}^{\infty} \frac{1}{t^c} \) is the Riemann's zeta function, and has a finite value when \( c > 1 \). Then the regret bound is

\[
\sum_{i \in [m], K_i > 0} \left( \frac{2 \cdot (1 + c) \cdot \ln n}{(f^{-1}(\Delta_{i, \text{min}}))^2} \cdot \Delta_{i, \text{min}} + \frac{2 \cdot (1 + c) \cdot \ln n}{(f^{-1}(\Delta_{i, \text{max}}))^2} \cdot \Delta_{i, \text{max}} \right) + (2 \cdot \zeta(c) + 1) \cdot m \cdot \Delta_{\text{max}}.
\]

We can also further improve the constant factor from \( 2(1 + c) \) to 4 by setting \( \tilde{\mu}_i = \hat{\mu}_i + \sqrt{2 \ln \ln n} \) at the cost of a second order \( \ln \ln n \) term as in Garivier & Cappé (2011), with regret at most

\[
\sum_{i \in [m], K_i > 0} \left( \frac{2 \cdot (2 \ln n + \ln \ln n)}{(f^{-1}(\Delta_{i, \text{min}}))^2} \cdot \Delta_{i, \text{min}} + \frac{2 \cdot (2 \ln n + \ln \ln n)}{(f^{-1}(\Delta_{i, \text{max}}))^2} \cdot \Delta_{i, \text{max}} \right) + (1 + 2 \ln \ln n) \cdot m \cdot \Delta_{\text{max}}.
\]

This is because \( \sum_{t=m+1}^{n} \frac{1}{t \ln t} \leq \int_{m+1}^{n} \frac{1}{t \ln t} \, dt \leq \ln \ln n \) when \( m > e \).

**Comparing to classical MAB.** As we discussed earlier, the classical MAB is a special instance of our CMAB framework in which each super arm is a simple arm, \( p_i = p \) for all \( i \in [m] \), function \( f(\cdot) \) is the identity function, and \( \alpha = \beta = 1 \). For this case, we set \( \eta = 1 \). Notice that \( \Delta_{\text{max}}^i = \Delta_{\text{min}}^i \). Thus, by Theorem 1, the regret bound of the classical MAB is

\[
\sum_{i \in [m], \Delta_i > 0} \frac{6 \ln n}{\Delta_i} + \left( \frac{\pi^2}{3} + 1 \right) \cdot m \cdot \Delta_{\text{max}}.
\]

where \( \Delta_i = \max_{j \in [m]} \mu_j - \mu_i \). Comparing with the regret bound in Theorem 1 of Auer et al. (2002a), we see that we even have a better coefficient \( \sum_{i \in [m], \Delta_i > 0} 6/\Delta_i \) in the leading \( \ln n \) term than the one
\[ \sum_{i \in [m], \Delta_i > 0} 8/\Delta_i \] in the original analysis of UCB1.\(^2\) The improvement is due to a tighter analysis, and is the reason that we obtained improved regret over Gai et al. (2012). Therefore, our CUCB algorithm does not lose accuracy comparing to the UCB1 for the classical MAB problem.

4 Applications

In this section, we describe two applications with non-linear reward functions as well as the class of linear reward applications that fit our CMAB framework. Notice that, the probabilistic maximum coverage bandit and social influence maximization bandit are also instances of the online submodular maximization problem, which can be addressed in the adversarial setting by Streeter & Golovin (2008), but we are not aware of their counterpart in the stochastic setting.

4.1 Probabilistic maximum coverage bandit

The online advertisement placement application discussed in the introduction can be modeled by the bandit version of the probabilistic maximum coverage (PMC) problem. PMC has as input a weighted bipartite graph \( G = (L, R, E) \) where each edge \((u, v)\) has a probability \( p(u, v) \), and it needs to find a set \( S \subseteq L \) of size \( k \) that maximizes the expected number of activated nodes in \( R \), where a node \( v \in R \) can be activated by a node \( u \in S \) with an independent probability of \( p(u, v) \). In the advertisement placement scenario, \( L \) is the set of web pages, \( R \) is the set of users, and \( p(u, v) \) is the probability that user \( v \) clicks the advertisement on page \( u \). PMC problem is NP-hard, since when all edge probabilities are 1, it becomes the NP-hard Maximum Coverage problem.

Using submodular set function maximization technique (Nemhauser et al., 1978), it can be easily shown that there exists a deterministic \((1 - 1/e)\) approximation algorithm for the PMC problem, which means that we have a \((1 - 1/e, 1)\)-approximation oracle for PMC.

The PMC bandit problem is that edge probabilities are unknown, and one repeatedly selects \( k \) targets in \( L \) in multiple rounds, observes all edge activations and adjusts target selection accordingly in order to maximize the total number of activated nodes over all rounds.

We can formulate this problem as an instance in the CMAB framework. Each edge \((u, v) \in E\) represents an arm, and each play of the arm is a 0-1 Bernoulli random variable with parameter \( p_{u,v} \). A super arm is the set of edges \( E_S \) incident to a set \( S \subseteq L \) of size \( k \). The reward of \( E_S \) is the number of activated nodes in \( R \), which is the number of nodes in \( R \) that are incident to at least one edge in \( E_S \) with outcome 1. Since all arms are independent Bernoulli random variables, we know that the expected reward only depends on the probabilities on all edges. In particular we have that the expected reward function is not linear in \( \mu = \{p(u, v)\}_{(u, v) \in E} \). For all arm \( i \in E \), we have \( p_i = 1 \), that is, we do not have probabilistically triggered arms. The monotonicity property is straightforward. The bounded smoothness function is \( f(x) = |E| \cdot x \), i.e., increasing all probabilities of all arms in a super arm by \( x \) can increase the expected number of activated nodes in \( V \) by at most \(|E| \cdot x \). Since \( f(\cdot) \) is a linear function, the integral in Eq. (2) has a closed form. In particular, by Theorem 1, we know that the distribution-dependent \((1 - 1/e, 1)\)-approximation regret bound of our CUCB algorithm (with parameter \( \eta = 1 \)) on PMC bandit is

\[
\sum_{i \in E, K_i > 0} \frac{12 \cdot |E|^2 \cdot \ln n}{\Delta_i^{\min}} + \left( \frac{\pi^2}{3} + 1 \right) \cdot |E| \cdot \Delta_{\max}.
\]

Notice that all edges incident to a node \( u \in L \) are always played together. In other words, these edges can share one counter. We call these arms (edges) as clustered arms. It is possible to exploit this property to improve the coefficient of the \( \ln n \) term, so that the summation is not among all edges but only nodes in \( L \). (See Section 4.1 and the supplementary material of Chen et al. (2013) for the regret bound and analysis for the case of clustered arms).

\(^2\)We remark that the constant of UCB1 has been tightened to the optimum (Garivier & Cappé, 2011).
From Theorem 3 (with $\eta = 1$), we also have the distribution-independent regret bound of

$$\sqrt{24|E|^3n \ln n} + \left(\frac{\pi^2}{3} + 1\right) \cdot |E| \cdot \Delta_{\max}.$$  

Note that for the PMC bandit, $\Delta_{\max}$ is at most the number of vertices covered in $R$, and thus $\Delta_{\max} \leq |R|$.

4.2 Combinatorial bandits with linear rewards

Gai et al. (2012) studied the Learning with Linear Reward policy (LLR). Their formulation is close to ours except that their reward function must be linear. In their setting, there are $m$ underlying arms. Each super arm consists of a set of underlying arms $S$ together with a set of coefficients $\{w_{i,S} \mid i \in S\}$. The reward of playing super arm $S$ is $\sum_{i \in S} w_{i,S} \cdot X_i$, where $X_i$ is the random outcome of arm $i$. The formulation can model a lot of bandit problems appeared in the literature, e.g., multiple plays, shortest path, minimum spanning tree and maximum weighted matching.

Our framework contains such linear reward problems as special cases. In particular, let $L = \max_S |S|$ and $a_{\max} = \max_{i,S} w_{i,S}$, and we have the bounded smoothness function $f(x) = a_{\max} \cdot L \cdot x$. In this setting we have $p_i = 1$ for all $i \in [m]$. By applying Theorem 1 (with $\eta = 1$), the regret bound is

$$\left(\sum_{i \in [m]; K_i > 0} \frac{12 \cdot a_{\max}^2 \cdot L^2 \cdot \ln n}{\Delta_i^{\min}}\right) + \left(\frac{\pi^2}{3} + 1\right) \cdot m \cdot \Delta_{\max}.$$  

Our result significantly improves the coefficient of the leading $\ln n$ term comparing to Theorem 2 of Gai et al. (2012) in two aspects: (a) we remove a factor of $L + 1$; and (b) the coefficient $\sum_{i \in [m]; \Delta_i^{\min} > 0} 1/\Delta_i^{\min}$ is likely to be much smaller than $m \cdot \Delta_{\max}/(\Delta_{\min})^2$ in Gai et al. (2012). This demonstrates that while our framework covers a much larger class of problems, we are still able to provide much tighter analysis than the one for linear reward bandits. Moreover, applying Theorem 3 (with $\eta = 1$) we can obtain distribution-independent bound for combinatorial bandits with linear rewards, which is not provided in Gai et al. (2012):

$$a_{\max} L \sqrt{24mn \ln n} + \left(\frac{\pi^2}{3} + 1\right) \cdot m \cdot \Delta_{\max}.$$  

Note that, for the class of linear bandits, the reward is at most $a_{\max} \cdot L$, and thus $\Delta_{\max} \leq a_{\max} \cdot L$.

4.3 Application to social influence maximization

In social influence maximization with the independent cascade model (Kempe et al., 2003), we are given a directed graph $G = (V, E)$, where every edge $(u, v)$ is associated with an unknown influence probability $p_{u,v}$. Initially, a seed set $S \subseteq V$ is selected and activated. In each iteration of the diffusion process, each node $u$ activated in the previous iteration has one chance of activating its inactive outgoing neighbor $v$ independently with probability $p_{u,v}$. The reward of $S$ after the diffusion process is the total number of activated nodes in the end. Influence maximization is to find a seed set $S$ of at most $k$ nodes that maximize the expected reward, also referred to as the influence spread of seed set $S$. Kempe et al. (2003) show that the problem is NP-hard and provide an algorithm with approximation ratio $1 - 1/e - \varepsilon$ with success probability $(1 - 1/|E|)$ for any fixed $\varepsilon > 0$. This means that we have a $(1 - 1/e - \varepsilon, 1 - 1/|E|)$-approximation oracle.

In the CMAB framework, we do not know the activation probabilities of edges and want to learn them during repeated seed selections while maximizing overall reward. Each edge in $E$ is considered as a base arm, and a super arm in this setting is the set $E_S$ of edges incident to the seed set $S$. Note that these edges will be deterministically triggered, but other edges not in $E_S$ may also be triggered, and the reward is related

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3To include the linear reward case, we allow two super arms with the same set of underlying arms to have different sets of coefficients. This is fine as long as the oracle could output super arms with appropriate parameters.
to all the triggered arms. Therefore, this is an instance where arms may be probabilistically triggered, and thus $p_i < 1$ for some $i \in E$.

It is straightforward to see that the expected reward function is still a function of probabilities on all edges, and the monotonicity holds. However, bounded smoothness property is nontrivial to argue, as we will show in the following lemma.

**Lemma 7.** The social influence maximization instance satisfies the bounded smoothness property with bounded smoothness function $f(x) = |E||V|x$.

**Proof.** For the social influence maximization bandit, the expectation vector $\mu$ is the vector of all probabilities on all edges. For a seed set $S \subseteq V$, the corresponding super arm is the set $E_S$ of edges incident to vertices in $S$. Without loss of generality, we assume that for any edge $i \in E$, its probability $\mu_i > 0$. Then for super arm $E_S$, the set of base arms that can be triggered by $E_S$, denoted as $\tilde{E}_S$, is exactly the set of edges reachable from seed set $S$ (an edge $(u,v)$ reachable from a set $S$ means its starting vertex $u$ is reachable from $S$). By Definition 1, to show any two expectation vectors $\mu$ and $\mu'$ and for any $\Lambda > 0$, we have $|r_\mu(E_S) - r_{\mu'}(E_S)| \leq f(\Lambda)$ if $\max_{i \in \tilde{E}_S} |\mu_i - \mu'_i| \leq \Lambda$.

Since we know that monotonicty holds, it is sufficient to assume that for all $i \in \tilde{E}_S$, $\mu_i = \mu'_i + \Lambda$. This is because without loss of generality, we can assume $r_\mu(E_S) \geq r_{\mu'}(E_S)$, and if $\mu_i < \mu'_i + \Lambda$ we can increase $\mu_i$ and decrease $\mu'_i$ such that $\mu_i = \mu'_i + \Lambda$, and this only increase the gap between $r_\mu(E_S)$ and $r_{\mu'}(E_S)$. Thus, henceforth let us assume that $i \in \tilde{E}_S$, $\mu_i = \mu'_i + \Lambda$.

Starting from $\mu'$, we take one edge $i_1$ in $\tilde{E}_S$, and increase $\mu'_i$ to $\mu'_i + \Lambda = \mu_i$ to get a new vector $\mu^{(1)}$. Suppose the edge $i_1$ is $(u_1,v_1)$. Comparing $\mu'$ with $\mu^{(1)}$, the only difference is that the probability on edge $(u_1,v_1)$ increases by $\Lambda$. For the influence spread of seed set $S$, the above change increases the activation probability of $v_1$ and only node reachable from $v_1$ by at most $\Lambda$. Thus the total increase of influence spread is at most $|V|\Lambda$. Then we select the second edge $i_2$ in $\tilde{E}_S$ and increases its probability by $\Lambda$. By the same argument, the influence spread increases at most $|V|\Lambda$. Repeating the above process, after selecting all edges in $\tilde{E}_S$, we obtain probability vector $\mu^{(s)}$ where $s = |\tilde{E}_S|$, and the increase in influence spread is at most $s|V|\Lambda$. Comparing vector $\mu^{(s)}$ with $\mu$, they are the same on all edges in $\tilde{E}_S$, and may only differ in the rest of edges. However, since the rest of edges cannot be reachable from $S$, their difference will not affect the influence spread of $S$. Therefore, we know that the difference between influence spread $r_\mu(E_S)$ and $r_{\mu'}(E_S)$ is at most $s|V|\Lambda \leq |E||V|\Lambda$. This concludes that if we use function $f(x) = |E||V|x$, the bounded smoothness property holds.

**Remark.** In Sectin 4.2 of Chen et al. (2013), we made a claim that social influence maximization bandit satisfies the bounded smoothness property (with function $f(x) = |E||V|x$) that does not consider probabilistically triggered arms, that is, it satisfies the property that for any two expectation vectors $\mu$ and $\mu'$ and for any $\Lambda > 0$, $|r_\mu(E_S) - r_{\mu'}(E_S)| \leq f(\Lambda)$ if $\max_{i \in E_S} |\mu_i - \mu'_i| \leq \Lambda$. This claim is incorrect. For example, all edges in $E_S$ could have the same probability (and thus we could have $\Lambda$ to be arbitrarily small), but other edges reachable from $E_S$ have different probabilities, and thus the gap between $r_\mu(E_S)$ and $r_{\mu'}(E_S)$ will not be arbitrarily small and cannot be bounded by $f(\Lambda)$ for any continuous $f$ tending to zero when $\Lambda$ tends to zero.

Applying Lemma 7 and Theorem 1 together and setting $\eta = 1$, we know that the distribution-dependent $(1 - 1/e - \varepsilon, 1 - 1/|E|)$-approximation regret bound of the CUCB algorithm on influence maximization is:

$$
\sum_{i \in E, K_i > 0} \left( \frac{2 \cdot |V|^2 |E|^2}{\Delta_i^\min \cdot p_i} \cdot \ln n \right) + \frac{2 \ln n \cdot \Delta_i^\max}{p_i^2} + \left( \frac{\pi^2}{3} + 1 \right) \cdot |E| \cdot \Delta_i^\max + \frac{\pi^4}{90} \sum_{i \in E} K_i \cdot I(p_i < 1) \cdot \Delta_i^\max.
$$

Alternatively, if we want to avoid $K_i$ and setting $\eta = |S|^{1/4}$ where $|S| = \binom{|V|}{k}$ in this case, we have

$$
\sum_{i \in E, K_i > 0} \left( \frac{2 \cdot |V|^2 |E|^2}{\Delta_i^\min \cdot p_i} + \frac{2 \cdot \Delta_i^\max}{p_i^2} \right) \left( \frac{\ln |S|}{4} + \ln n \right) + \left( \frac{\pi^2}{3|S|^{1/4} + 1} \right) \cdot |E| \cdot \Delta_i^\max + \frac{\pi^4}{90} |E| \cdot \Delta_i^\max.
$$

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We may also apply Lemma 7 and Theorem 2 with $\eta = 1$, and obtain the following distribution-dependent bound:

$$
\sum_{i \in E, K_i > 0} \left( \frac{12 \cdot |V|^2 |E|^2 \cdot \ln n}{(\Delta_{\min}^n)^2 \cdot p_i} + \frac{2 \ln n}{p_i^2} \right) \cdot \Delta_{\max} + \left( \frac{\pi^2}{3} + 1 \right) \cdot |E| \cdot \Delta_{\max} + \frac{\pi^4}{90} |E| \cdot \Delta_{\max}.
$$

With Theorem 3 and $\eta = 1$, we obtain the distribution-independent bound:

$$
|V| \sqrt{\frac{48 |E|^3 n \ln n}{p^*}} + \left( \frac{\pi^2}{3} + 1 \right) \cdot m \cdot \Delta_{\max} + \sum_{i \in E} \left( \frac{2 \ln n}{p_i^2} + \frac{\pi^4 \cdot K_i \cdot \mathbb{I}[p_i < 1]}{90} \right) \cdot \Delta_{\max}.
$$

Finally, if we apply Theorem 3 and $\eta = |S|^{1/4}$, we obtain the following alternative distribution-independent bound:

$$
|V| \sqrt{\frac{48 |E|^3 n (\ln n + \ln |S|)}{p^*}} + \left( \frac{\pi^2}{3 \cdot |S|^{3/4}} + 1 \right) \cdot m \cdot \Delta_{\max} + \left( \sum_{i \in E} \frac{4 \ln n + \ln |S|}{2p_i^2} + \frac{\pi^4 |E|}{90} \right) \cdot \Delta_{\max}.
$$

Note that for the social influence maximization bandit, $\Delta_{\max}$ is at most the number of nodes in the network, that is, $\Delta_{\max} \leq |V|$.

## 5 Conclusion

In this paper, we propose the first general stochastic CMAB framework that accommodates a large class of nonlinear reward functions among combinatorial and stochastic arms, and it even accommodates probabilistically triggered arms such as occurred in the viral marketing application. We provide CUCB algorithm with tight analysis on its distribution-dependent and distribution-independent regret bounds and applications to new practical combinatorial bandit problems.

There are many possible future directions from this work. One may study the CMAB problems with Markovian outcome distributions on arms, or the restless version of CMAB, in which the states of arms continue to evolve even if they are not played. Another direction is to study further the case of probabilistically triggered arms such as the viral marketing application to see if one can design better learning algorithms that provided better regret bounds. One may also see if any technique of this work can be applied to the study of adversarial combinatorial bandits with nonlinear rewards, or combinatorial bandits with bandit feedback or other feedback forms.

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