Minimal crystallizations of 3-manifolds with boundary

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September 18, 2021

Abstract

Let $(\Gamma, \gamma)$ be a crystallization of connected compact 3-manifold $M$ with $h$ boundary components. Let $\mathcal{G}(M)$ and $k(M)$ be the regular genus and gem-complexity of $M$ respectively, and let $\mathcal{G}(\partial M)$ be the regular genus of $\partial M$. We prove that

$$k(M) \geq 3(\mathcal{G}(M) + h - 1) \geq 3(\mathcal{G}(\partial M) + h - 1).$$

These bounds for gem-complexity of $M$ are sharp for several 3-manifolds with boundary. Further, we show that if $\partial M$ is connected and $k(M) < 3(\mathcal{G}(\partial M) + 1)$ then $M$ is a handlebody. In particular, we prove that $k(M) = 3\mathcal{G}(\partial M)$ if $M$ is a handlebody and $k(M) \geq 3(\mathcal{G}(\partial M) + 1)$ if $M$ is not a handlebody. Further, we obtain several combinatorial properties for a crystallization of 3-manifolds with boundary.

MSC 2020: Primary 57Q15. Secondary 05C15; 57K30; 57K31; 57Q05.

Keywords: PL-manifolds, Crystallizations, Regular genus, Gem-complexity, Handlebody.

1 Introduction

A crystallization $(\Gamma, \gamma)$ of a connected compact PL $d$-manifold (possibly with boundary) is a certain type of edge colored graph which represents the manifold (for details and related notations we refer Subsection 2.1). The journey of crystallization theory has begun due to Pezzana who gives the existence of a crystallization for every closed connected PL $d$-manifold (see [28]). Later the existence of a crystallization has been proved for every connected compact PL $d$-manifold with boundary (see [18, 24]). A beautiful proof of the classification of closed surfaces using crystallization theory can be found in [3]. In [26], Gagliardi gave a combinatorial characterization of a 4-colored graph to be a crystallization of a closed connected 3-manifold. In [18] the authors extended the above result to a connected compact 3-manifold with connected boundary, and in [24] Gagliardi further extended the result to a connected compact 3-manifold with several boundary components. The lower bound for the number of vertices of a crystallization can be found in [7] for closed connected 3-manifolds, and in [6] for closed connected 4-manifolds. In this article, we gave a lower bound for the number of vertices of a crystallization for connected compact 3-manifolds with several boundary components.

Extending the notion of genus in dimension 2, the notion of regular genus $\mathcal{G}(M)$ for a closed connected PL $d$-manifold $M$, has been introduced in [25], which is strictly related

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to the existence of regular embeddings of crystallizations of the manifold into surfaces (cf. Subsection 2.2 for details). Later, in [23], the concept of regular genus has been extended for a connected compact PL $d$-manifold with boundary, for $d \geq 2$. The regular genus of a closed connected orientable (resp., a non-orientable) surface equals the genus (resp., half of the genus) of the surface. Several classification results according to the gem-complexity and regular genus can be found in [2, 3, 17, 20]. Let $M$ be a connected compact 3-manifold with boundary, and let $G(M)$ and $G(\partial M)$ be the regular genera of $M$ and $\partial M$ respectively. Then from [10, 19], we know that $G(M) \geq G(\partial M)$. For $3 \leq d \leq 5$, a classification result for a $d$-dimensional manifold with connected boundary can be found in [11, 12, 13] when the regular genus of the manifold is same as the regular genus of its boundary.

The gem-complexity is another interesting and useful combinatorial invariant for classifying connected compact PL $d$-manifolds $M$, and is defined as the non-negative integer $k(M) = p - 1$, where $2p$ is the minimum number of vertices of a crystallization of $M$. A catalogue of closed connected 3-manifolds up to gem-complexity 14 can be found in [5, 16]. A catalogue of PL 4-manifolds by gem-complexity can be found in [14]. Such results for 3-manifolds with boundary are not very well known. The estimations of Matveev’s complexity for 3-manifolds with boundary can be found in [11, 12, 13]. In this article, we prove that if $M$ is a connected compact 3-manifold with $h$ boundary components then $k(M) \geq 3(G(M) + h - 1)$ (cf. Theorem 12). This bound is sharp for several 3-manifolds with boundary (cf. Remark 20). Since $G(M) \geq G(\partial M)$, we also have $k(M) \geq 3(G(\partial M) + h - 1)$. Further, we have shown that if $M$ is a 3-manifold with connected boundary and $k(M) < 3(G(\partial M) + 1)$ then $M$ is a handlebody (cf. Theorem 14). In particular, we prove that if $M$ is a handlebody then $k(M) = 3G(\partial M)$ and if $M$ is not a handlebody then $k(M) \geq 3(G(\partial M) + 1)$. We have shown the sharpness of this bound for several 3-manifolds which are not handlebodies. (cf. Remark 20).

2 Preliminaries

2.1 Crystallization

Crystallization theory is a combinatorial representation tool for piecewise-linear (PL) manifolds of arbitrary dimension. A multigraph is a graph where multiple edges are allowed but loops are forbidden. For a multigraph $\Gamma = (V(\Gamma), E(\Gamma))$, a surjective map $\gamma : E(\Gamma) \to \Delta_d := \{0, 1, \ldots, d\}$ is called a proper edge-coloring if $\gamma(e) \neq \gamma(f)$ for any pair $e, f$ of adjacent edges. The elements of the set $\Delta_d$ are called the colors of $\Gamma$. A graph $(\Gamma, \gamma)$ is called $(d+1)$-regular if degree of each vertex is $d+1$ and is said to be $(d+1)$-regular with respect to a color $c$ if after removing all the edges of color $c$ from $\Gamma$, the resulting graph is $d$-regular. We refer to [9] for standard terminology on graphs.

A graph $(\Gamma, \gamma)$ is called $(d+1)$-regular colored graph if $\Gamma$ is a $(d+1)$-regular and $\gamma$ is a proper edge-coloring. A $(d+1)$-colored graph with boundary is a pair $(\Gamma, \gamma)$, where $\Gamma$ is a $(d+1)$-regular graph with respect to a color $c \in \Delta_d$ but not $(d+1)$-regular and $\gamma$ is a proper edge-coloring. If $(\Gamma, \gamma)$ is a $(d+1)$-regular colored graph or a $(d+1)$-colored graph with boundary then we simply call $(\Gamma, \gamma)$ as a $(d+1)$-colored graph. For each $B \subseteq \Delta_d$ with $h$ elements, the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$ is an $h$-colored graph with edge-coloring $\gamma|_{\gamma^{-1}(B)}$. For a color set $\{i_1, i_2, \ldots, i_k\} \subset \Delta_d$, $\Gamma_{\{i_1, i_2, \ldots, i_k\}}$ denotes the subgraph restricted to the color set $\{i_1, i_2, \ldots, i_k\}$ and $g_{i_1, i_2, \ldots, i_k}$ denotes the number of connected components of the graph $\Gamma_{\{i_1, i_2, \ldots, i_k\}}$. A graph $(\Gamma, \gamma)$ is called contracted if subgraph $\Gamma_{\hat{c}} := \Gamma_{\Delta_d \setminus \{c\}}$ is
Let \( G_d \) denote the set of graphs \((\Gamma, \gamma)\) which are \((d+1)\)-regular with respect to the fixed color \( d \). Thus \( G_d \) contains all the \((d+1)\)-regular colored graphs as well as all \((d+1)\)-colored graphs with boundary. If \((\Gamma, \gamma) \in G_d\) then the vertices with degree \( d + 1 \) are called the internal vertices and the vertices with degree \( d \) are called the boundary vertices. Let \( C_{ij} \) denote the number of \( \{i, j\} \)-colored cycles in \( \Gamma \). Then \( C_{ij} = g_{ij} \) for \( i, j \in \Delta_d \setminus \{d\} \). For each graph \((\Gamma, \gamma) \in G_d\), we define its boundary graph \((\partial \Gamma, \partial \gamma)\) as follows:

- there is a bijection between \( V(\partial \Gamma) \) and the set of boundary vertices of \( \Gamma \);
- \( u_1, u_2 \in V(\partial \Gamma) \) are joined in \( \partial \Gamma \) by an edge of color \( j \) if and only if \( u_1 \) and \( u_2 \) are joined in \( \Gamma \) by a path formed by \( j \) and \( d \) colored edges alternatively.

Note that, if \((\Gamma, \gamma)\) is \((d+1)\)-regular then \((\Gamma, \gamma) \in G_d \) and \( \partial \Gamma = \emptyset \). For each \((\Gamma, \gamma) \in G_d\), a corresponding \( d \)-dimensional simplicial cell-complex \( K(\Gamma) \) is determined as follows:

- for each vertex \( u \in V(\Gamma) \), take a \( d \)-simplex \( \sigma(u) \) and label its vertices by \( \Delta_d \);
- corresponding to each edge of color \( j \) between \( u, v \in V(\Gamma) \), identify the \( (d-1) \)-faces of \( \sigma(u) \) and \( \sigma(v) \) opposite to \( j \)-labeled vertices such that the vertices with same label coincide.

We refer to [8] for CW-complexes and related notions. We say \((\Gamma, \gamma)\) represents connected compact PL \( d \)-manifold \( M \) (possibly with boundary) if the geometrical carrier \( |K(\Gamma)| \) is PL homeomorphic to \( M \). It is not hard to see that \( |K(\Gamma)| \) is orientable if and only if \( \Gamma \) is a bipartite graph. If \((\Gamma, \gamma) \in G_d\) represents a connected compact PL \( d \)-manifold with boundary then we can define its boundary graph \((\partial \Gamma, \partial \gamma)\), and each component of the boundary-graph \((\partial \Gamma, \partial \gamma)\) represents a component of \( \partial M \). From the construction it is easy to see that, for \( B \subset \Delta_d \) of cardinality \( k + 1 \), \( K(\Gamma) \) has as many \( k \)-simplices with vertices labeled by \( B \) as many connected components of \( \Gamma_{\Delta_d \setminus B} \) are (cf. [22]).

Let \((\Gamma, \gamma) \in G_d\) represent a connected compact PL \( d \)-manifold with \( h \) boundary components, then \( K(\Gamma) \) has at least \( d \cdot \max\{1, h\} + 1 \) vertices, for \( h \geq 0 \). For \( h \geq 1 \), it is easy to see that \( \Gamma_{\partial} \) is connected and each component of \( \partial \Gamma \) is contracted if and only if \( K(\Gamma) \) has exactly \( dh + 1 \) vertices.

**Definition 1** ([24]). Let \((\Gamma, \gamma) \in G_d\) be a connected graph such that \( \partial \Gamma \) has \( h \) components, for \( h \geq 1 \). Then \((\Gamma, \gamma)\) is called \( \partial \)-contracted if (a) \( \Gamma_{\partial} \) is connected, and (b) for every \( 0 \leq c \leq d - 1 \), \( \Gamma_{c} \) has \( h \) components.

A connected graph \((\Gamma, \gamma) \in G_d\) is said to be a crystallization of a connected compact PL \( d \)-manifold \( M \) with \( h \) boundary components if \( K(\Gamma) \) has exactly \( d \cdot \max\{1, h\} + 1 \) vertices, for \( h \geq 0 \). In other words, a connected graph \((\Gamma, \gamma) \in G_d\) is a crystallization of a manifold \( M \) with (non-empty) boundary if \((\Gamma, \gamma)\) is \( \partial \)-contracted. Note that, if \( \partial M \) is connected (resp., empty) then \((\Gamma, \gamma)\) is contracted.

The starting point of the whole crystallization theory is the following Pezzana’s Existence Theorem (cf. [23]).

**Proposition 2.** Every closed connected PL \( d \)-manifold admits a crystallization.

Pezzana’s existence theorem has been extended to the boundary case (cf. [18, 24]).
Proposition 3 ([18] [24]). Let $M$ be a connected compact PL $d$-manifold with (possibly non-connected) boundary. For every crystallization $(\Gamma', \gamma')$ of $\partial M$, there exists a crystallization $(\Gamma, \gamma)$ of $M$, whose boundary graph $(\partial \Gamma, \partial \gamma)$ is isomorphic with $(\Gamma', \gamma')$.

It is known that a connected compact PL $d$-manifold with boundary can always be represented by a $(d + 1)$-colored graph $(\Gamma, \gamma)$ which is regular with respect to a fixed color $k$, for some $k \in \Delta_d$. Without loss of generality, we can assume that $k = d$, i.e., $(\Gamma, \gamma) \in \mathbb{G}_d$.

In [26], Gagliardi gave a combinatorial characterization of a contracted 4-colored graph to be a crystallization of a closed connected 3-manifold. In [24], Gagliardi extended the above theorem to connected compact 3-manifold with several boundary components. For graph $(\Gamma, \gamma)$ with boundary, let $2p$ and $2\bar{p}$ denote the number of vertices and boundary vertices of $(\Gamma, \gamma)$ respectively.

Proposition 4 ([24]). A 4-colored graph with boundary $(\Gamma, \gamma)$ is a crystallization of a connected compact 3-manifold $M$ with $h$ boundary components ($h \geq 1$) if and only if the following conditions hold.

(i) $(\Gamma, \gamma)$ is connected, $\partial$-contracted element of $\mathbb{G}_3$, and $\partial \Gamma$ has $h$ components.

(ii) $g_{03} - g_{12} = g_{13} - g_{02} = g_{23} - g_{01} = \frac{\bar{p}}{2} + \frac{h}{2} - 1$.

(iii) $g_{01} + g_{02} + g_{12} = 2 + p$.

A new invariant ‘gem-complexity’ has been defined. Given a connected compact PL $d$-manifold $M$, its gem-complexity is the non-negative integer $k(M) = p - 1$, where $2p$ is the minimum number of vertices of a crystallization of $M$.

Let $(\Gamma_1, \gamma_1)$ and $(\Gamma_2, \gamma_2)$ be two disjoint $(d + 1)$-colored graphs (possibly with boundary) with the same color set $\Delta_d$, and let $v_i \in V(\Gamma_i)$ $(1 \leq i \leq 2)$. The connected sum $(\Gamma_1 \#_{v_1, v_2} \Gamma_2, \gamma_1 \#_{v_1, v_2} \gamma_2)$ of $\Gamma_1, \Gamma_2$ with respect to vertices $v_1, v_2$ is the graph obtained from $(\Gamma_1 \setminus \{v_1\}) \cup (\Gamma_2 \setminus \{v_2\})$ by adding $d + 1$ new edges $e_0, \ldots, e_d$ with colors $0, \ldots, d$ respectively, such that the end points of $e_i$ are $u_{j,i}$ and $u_{j,2}$, where $v_i$ and $u_{j,i}$ are joined in $(\Gamma_1, \gamma_1)$ with an edge of color $j$ for $0 \leq j \leq d$, $1 \leq i \leq 2$. From the construction it is clear that $K(\Gamma_1 \#_{v_1, v_2} \Gamma_2)$ is obtained from $K(\Gamma_1)$ and $K(\Gamma_2)$ by removing the $d$-simplices $\sigma(v_1)$ and $\sigma(v_2)$ and pasting together all $(\Delta_d \setminus \{j\})$-colored $(d - 1)$-simplices of $\sigma(u_{j,1})$ and $\sigma(u_{j,2})$ for $0 \leq j \leq d$. For $0 \leq j \leq d$, if the $j$-colored vertex of $\sigma(v_1)$ or $\sigma(v_2)$ is not a boundary vertex of $K(\Gamma_1)$ or $K(\Gamma_2)$ respectively then $(\Gamma_1 \#_{v_1, v_2} \Gamma_2, \gamma_1 \#_{v_1, v_2} \gamma_2)$ represents $|K(\Gamma_1)|/|K(\Gamma_2)|$.

2.2 Regular Genus of PL $d$-manifolds (possibly with boundary)

Let $(\Gamma, \gamma) \in \mathbb{G}_d$ be a $(d + 1)$-colored graph which represents a connected compact $d$-manifold $M$ (possibly with boundary $\partial M$). For each boundary vertex $u$ (possibly empty), a new vertex $u'$ and a new $d$-colored edge is added between $u$ and $u'$. In this way a new graph $(\Gamma', \gamma')$ is obtained. If $\Gamma$ has no boundary vertex then $\Gamma'$ is same as $\Gamma$. Let $2p$ and $2\bar{p}$ denote the number of vertices and boundary vertices of $(\Gamma, \gamma)$ respectively. Then the number of interior vertices is $2\bar{p} := 2p - 2\bar{p}$.

Now, given any cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d = d)$ of the color set $\Delta_d$, a regular imbedding of $\Gamma'$ into a surface $F$ is simply an imbedding $i : |\Gamma'| \to F$ such that the vertices of $i(\Gamma') \cap \partial F$ are the images of the new added vertices and the regions of the imbedding are bounded by either a cycle (internal region) or by a walk (boundary region) of $\Gamma'$ with $\varepsilon_i, \varepsilon_{i+1}(i \mod d + 1)$ colored edges alternatively.
Using Gross ‘voltage theory’ (see [27]), in the bipartite case, and Stahl ‘embedding schemes’ (see [29]), in the non-bipartite case, one can prove that for every cyclic permutation \( \varepsilon \) of \( \Delta_d \), a regular embedding \( i_\varepsilon : \Gamma' \hookrightarrow F_\varepsilon \) exists, where orientable (resp., non-orientable) surface \( F_\varepsilon \) is of Euler characteristic

\[
\chi_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}^{d+1}} C_{\varepsilon i, i+1} + (1 - d)\hat{p} + (2 - d)\bar{p}
\]

and \( \lambda_\varepsilon = \partial g_{\varepsilon_0 \varepsilon_{d-1}} \) holes where \( \partial g_{ij} \) denotes the number of \( \{i, j\} \)-colored cycles of \( \partial \Gamma \). For more details we refer [1, 23].

In the orientable (resp., non-orientable) case, the integer

\[
\rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma)/2 - \lambda_\varepsilon/2
\]

is equal to the genus (resp., half of the genus) of the surface \( F_\varepsilon \). Then, the regular genus of \((\Gamma, \gamma)\) denoted by \( \rho(\Gamma) \) and the regular genus of \( M \) denoted by \( \mathcal{G}(M) \) are defined as follows (cf. [25]):

\[
\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d = d) \text{ is a cyclic permutation of } \Delta_d\};
\]

\[
\mathcal{G}(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ represents } M\}.
\]

In dimension two, it is easy to see that if \((\Gamma, \gamma)\) represents a surface \( F \), then the corresponding \((\Gamma', \gamma')\) regularly imbeds into \( F \) itself. Hence, for each surface \( F \),

\[
\mathcal{G}(F) = \begin{cases} 
\text{genus}(F) & \text{if } F \text{ is orientable}, \\
\frac{1}{2} \times \text{genus}(F) & \text{if } F \text{ is non-orientable}.
\end{cases}
\]

If \( M \) is a connected compact 3-manifold with \( h \) boundary components say \( \partial^1 M, \ldots, \partial^h M \), then \( \mathcal{G}(\partial M) \) is defined as \( \sum_{i=1}^h \mathcal{G}(\partial^i M) \). Further, from [11, 12, 13], we have the following result.

**Proposition 5.** For \( 3 \leq d \leq 5 \), let \( M \) be a \( d \)-dimensional (orientable or non-orientable) manifold with connected boundary \( \partial M \). Then the regular genus of \( M \) is equal to the regular genus of the boundary \( \partial M \) (say, \( g \)) if and only if \( M \) is a \( d \)-dimensional genus \( g \) handlebody.

### 2.3 3-dimensional handlebodies

Let \( M \) be a connected compact 3-manifold with boundary. Then each component of the boundary \( \partial M \) of \( M \) is a closed connected surface. A handlebody can be defined as the simplest 3-manifold with connected boundary - in the sense that it contains pairwise disjoint, properly embedded 2-discs such that the manifold resulting from cutting along the discs is a 3-ball. Up to homeomorphism, there are exactly two handlebodies (one is orientable and another is non-orientable) of any positive integer genus.
3 Main results

Let \((\Gamma, \gamma)\) be a 4-colored graph with the color set \(\Delta_3 = \{0, 1, 2, 3\}\) regular with respect to the color 3, i.e., \((\Gamma, \gamma) \in G_3\). Let \(2p\) and \(2\overline{p}\) denote the number of vertices and boundary vertices of \((\Gamma, \gamma)\) respectively. Thus the number of interior vertices is \(2p - 2\overline{p}\) and number of 3-colored edges is \(p - \overline{p}\). Let \(g_{ij}, C_{ij}\) denote the number of components and cycles in \(\Gamma_{ij}\) for \(0 \leq i, j \leq 3\). Then it is easy to see that \(g_{i3} = \overline{p} + C_{i3}\), for \(0 \leq i \leq 2\).

Let \(M\) be a connected compact 3-manifold with connected boundary \(\partial M\). If \(\partial M\) is a non-orientable surface then \(\partial M\) must be an \(n\)-connected sum of Klein bottles, for some \(n \geq 1\). Thus, the regular genus of the boundary surface \(\partial M\) is a non-negative integer.

Lemma 6. For \(n \in \mathbb{N} \cup \{0\}\), let \((\Gamma, \gamma)\) be a crystallization of a connected compact 3-manifold \(M\) with connected boundary surface of regular genus \(n\). Then \(|V(\partial \Gamma)| = 2 + 4n\).

Proof. Since \((\Gamma, \gamma)\) is a crystallization of a connected compact 3-manifold \(M\) with connected boundary, \((\partial \Gamma, \partial \gamma)\) is a crystallization of \(\partial M\). Because \(\partial M\) has regular genus \(n\), we conclude that \(\chi(K(\partial \Gamma)) = 2 - 2n\). Let \(V, E, F\) be the number of the vertices, edges and triangles of the corresponding simplicial cell complex \(K(\partial \Gamma)\) respectively. Then \(V = 3\) and \(2E = 3F\). Thus,

\[
V - E + F = \chi(\partial M) = 2 - 2n.
\]

This implies, \(F = 2 + 4n\), i.e., \(|V(\partial \Gamma)| = 2 + 4n\). \(\Box\)

Lemma 7. Let \((\Gamma, \gamma) \in G_3\) be a crystallization of a connected compact 3-manifold with connected boundary. Let \(|V(\partial \Gamma)| = 2 + 4n\), for some \(n \in \mathbb{N} \cup \{0\}\). Then for \(\{i, j, k\} = \{0, 1, 2\}\),

\[ (i) \quad g_{i3} = 1 + 2n + C_{i3}, \]
\[ (ii) \quad g_{i3} = n + g_{jk}, \]
\[ (iii) \quad g_{jk} = 1 + n + C_{i3}. \]

Proof. Since \(2\overline{p} = 2 + 4n\) and \(g_{i3} = \overline{p} + C_{i3}\), it follows that \(g_{i3} = 1 + 2n + C_{i3}\).

Further, it follows from Proposition 4 that

\[
g_{i3} - g_{jk} = \frac{\overline{p}}{2} - \frac{1}{2} = n.
\]

These prove Parts (i) and (ii), and Part (iii) follows from Parts (i) and (ii). \(\Box\)

From Proposition 4 we have \(g_{01} + g_{02} + g_{12} = 2 + p\). Therefore, it is easy to prove the following.

Corollary 8. Let \((\Gamma, \gamma) \in G_3\) be a crystallization of a connected compact 3-manifold with connected boundary surface of regular genus \(n\). Then \(|V(\Gamma)| \geq 2 + 6n\) and

\[
g_{03}, g_{13}, g_{23} \geq 1 + 2n, \]
\[
g_{01}, g_{02}, g_{12} \geq 1 + n.
\]
Moreover, if $|V(\Gamma)| = 2 + 6n$ then
\[
g_{03}, g_{13}, g_{23} = 1 + 2n, \\
g_{01}, g_{02}, g_{12} = 1 + n.
\]

**Lemma 9.** For $n \in \mathbb{N} \cup \{0\}$, let $(\Gamma, \gamma) \in G_3$ be a crystallization of a connected compact 3-manifold with connected boundary of regular genus $n$. Then $\Gamma$ represents a handlebody if and only if at least one of $g_{01}, g_{02}$ and $g_{12}$ attains the minimum (i.e., equals to $1 + n$).

**Proof.** Let $2\overline{p}$ and $2\hat{p}$ denote the number of boundary vertices and internal vertices respectively, and $2p = 2\overline{p} + 2\hat{p}$. Since $(\Gamma, \gamma)$ is a crystallization of a connected compact 3-manifold $M$ with connected boundary surface of regular genus $n$, we have $2\overline{p} = 2 + 4n$ by Lemma 6. Moreover, Corollary 8 implies, $2p \geq 2 + 6n$. Let $2p = 2r + 6n$, for $r \geq 1$. Because $2\overline{p} = 2 + 4n$, we have $2\hat{p} = 2r + 2n - 2$. Further, it follows from Lemma 7 that $g_{12} - C_{03} = g_{02} - C_{13} = g_{01} - C_{23} = n + 1$. Let $C_{23} = a, C_{13} = b$ and $C_{03} = c$. Then
\[
g_{01}, g_{02}, g_{12} (1 + n + a) (1 + n + b) (1 + n + c) c b a
\]

Since $g_{01} + g_{02} + g_{12} = p + 2 = r + 3n + 2$, it follows that $r = a + b + c + 1$. Thus, for $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 = 3)$, we have
\[
\chi_\epsilon(\Gamma) = g_{\epsilon_0\epsilon_1} + g_{\epsilon_1\epsilon_2} + C_{\epsilon_2\epsilon_3} + C_{\epsilon_0\epsilon_3} - (2r + 2n - 2) - (1 + 2n) = g_{\epsilon_0\epsilon_1} + g_{\epsilon_1\epsilon_2} + C_{\epsilon_2\epsilon_3} + C_{\epsilon_0\epsilon_3} - 1 - 4n - 2(a + b + c).
\]

Thus, for different permutations $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 = 3)$, the possible choices for $\chi_\epsilon(\Gamma)$ are $1 - 2n - 2a, 1 - 2n - 2b$ or $1 - 2n - 2c$. Since $(\partial\Gamma, \partial\gamma)$ is a crystallization of $\partial M$, $\partial g_{ij} = 1$, i.e., $\lambda_\epsilon = 1$. Thus, the possible choices for $\rho_\epsilon(\Gamma)$ are $n + a, n + b$ or $n + c$. Therefore, $\rho(\Gamma) = \min\{n + a, n + b, n + c\}$. From Proposition 5 we know that $\Gamma$ represents a handlebody if and only if $\rho(\Gamma) = n$. Thus, $\Gamma$ represents a handlebody if and only if at least one of $g_{01}, g_{02}$ and $g_{12}$ equals to $(1 + n)$. \qed

**Lemma 10.** Let $M$ be a connected compact 3-manifold with connected boundary. Then, $k(M) \geq 3G(M)$.

**Proof.** Let $(\Gamma, \gamma) \in G_3$ be a crystallization for $M$ with $2p$ vertices. Let $G(\partial M) = n$. Then, from the proof of Lemma 9, we have the regular genus of $(\Gamma, \gamma)$, $\rho(\Gamma) = \min\{n + a, n + b, n + c\}$, where $C_{23} = a, C_{13} = b$ and $C_{03} = c$. It follows from Lemma 7 that $g_{jk} = 1 + n + C_{ij}$ for $i, j, k \in \{0, 1, 2\}$. Since $p = (g_{01} + g_{02} + g_{12}) - 2$, $p = 1 + 3n + a + b + c$. Thus, $p - 1 = (n + a) + (n + b) + (n + c) \geq 3\rho(\Gamma) \geq 3G(M)$. Since $(\Gamma, \gamma) \in G_3$ is an arbitrary crystallization for $M$, we have $k(M) \geq 3G(M)$. \qed

**Example 11** (Gem-complexity of a handlebody $M$ is $3G(\partial M)$). Let $M$ be a 3-dimensional handlebody. It is known that $\partial M$ is either $S^2$, $\#_n(S^1 \times S^1)$ or $\#_n(S^1 \times S^1)$ for some $n \geq 1$. For $n \in \mathbb{N} \cup \{0\}$, we have given a crystallization of a connected compact 3-manifold $M$ with boundary surface $\#_n(S^1 \times S^1)$ (cf. Part (a) of Figure 1) and $\#_n(S^1 \times S^1)$ (cf. Part (b) of Figure 1) with exactly $2 + 6n$ vertices, where $n = G(\partial M)$. Since $\rho(\Gamma) = G(\partial M)$, $M$ is a handlebody. Therefore, by Lemma 10 we have the gem-complexity of a handlebody $M$ is $3G(\partial M)$.\[7]
(a) A crystallization of the orientable handlebody.

(b) A crystallization of the non-orientable handlebody.

Figure 1: Crystallizations of the orientable and non-orientable handlebodies with $6n + 2$ vertices.

**Theorem 12.** Let $M$ be a connected compact 3-manifold with $h$ boundary components. Then, $k(M) \geq 3(G(M) + h - 1)$.

**Proof.** Let $(\Gamma, \gamma) \in \mathbb{G}_3$ be a crystallization for $M$ with $2p$ vertices. Let $2\overline{p}$ and $2\hat{p}$ be the number of boundary vertices and interior vertices of $(\Gamma, \gamma)$ respectively. Let $(\Gamma', \gamma')$ be a crystallization for a 3-manifold with connected boundary and $|V(\Gamma')| = |V(\Gamma)| = 2p$. Let $2p', 2\overline{p'}$ be the number of interior and boundary vertices of $(\Gamma', \gamma')$ respectively. For $i, j \in \{0, 1, 2, 3\}$, let $g'_{ij}$ and $C_{ij}'$ denote the number of components and cycles in $\Gamma'_ij$ respectively. Let $\partial g'_{ij}$ be the number of $\{i, j\}$-colored cycles in $\partial \Gamma'$. Then $2p' = 2\overline{p} + 2(h-1)$, $2\overline{p'} = 2\overline{p} - 2(h - 1)$, $C_{ij}' = C_{ij}$, $g'_{ij} = g_{ij}$ and $\partial g'_{ij} = \partial g_{ij} - (h - 1)$, for $0 \leq i, j \leq 2$. Therefore, for any permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_3 = 3)$ of $\Delta_3$, $\chi_\varepsilon(\Gamma) = \chi_\varepsilon(\Gamma') + (h - 1)$. Then, $\rho_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma') - (h - 1)$. Since $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_3 = 3)$ is an arbitrary permutation of $\Delta_3$, $\rho(\Gamma') = \rho(\Gamma) + h - 1$. Now, by the proof of Lemma 10, we have $p - 1 \geq 3\rho(\Gamma') = 3(\rho(\Gamma) + h - 1) \geq 3(G(M) + h - 1)$. Since $(\Gamma, \gamma) \in \mathbb{G}_3$ is an arbitrary crystallization for $M$, $k(M) \geq 3(G(M) + h - 1)$. \qed
Let \( M \) be a connected compact 3-manifold with boundary. Then from \([10, 19]\), we know that \( G(M) \geq G(\partial M) \). Thus we have the following result.

**Corollary 13.** Let \( M \) be a connected compact 3-manifold with \( h \) boundary components. Then, \( k(M) \geq 3(G(M) + h - 1) \geq 3(G(\partial M) + h - 1) \).

The above result suggests that a crystallization \((\Gamma, \gamma)\) of a connected compact 3-manifold with connected boundary has at least \( 6G(\partial M) + 2 \) vertices. Further, it is easy to see that if \( G(\partial M) = 0 \) then \( |V(\Gamma)| < 8 \) implies \( M \) is the 3-ball \( D^3 \), which we consider as a trivial handlebody. It is also easy to construct an 8-vertex crystallization of a connected compact 3-manifold \( M \) with spherical boundary such that \( M \) is not \( D^3 \). Now, we state and prove a similar result when \( M \) is a connected compact 3-manifold with non-spherical boundary.

**Theorem 14.** If \( M \) is a 3-manifold with connected boundary and \( k(M) < 3(G(\partial M) + 1) \) then \( M \) is a handlebody.

**Proof.** Let \((\Gamma, \gamma)\) be a crystallization of the 3-manifold \( M \) such that \( 2p = |V(\Gamma)| < 8 + 6G(\partial M) \). From Corollary 13, we know that \( 2p \geq 6G(\partial M) + 2 \) and we have a crystallization of a ‘handlebody’ \( M \) with boundary surface \( \#_n(S^1 \times S^1) \) or \( \#_m(S^1 \times S^1) \) with exactly \( 2 + 6G(\partial M) \) vertices. Now, we claim that if \( 2p = 2 + 6G(\partial M), 4 + 6G(\partial M) \) or \( 6 + 6G(\partial M) \), then \( M \) is a handlebody.

Let \( n \) be the regular genus of the boundary surface \( \partial M \). Then Lemma 6 implies, \( 2p = 2 + 4n \). It follows from Lemma 7 that \( g_{jk} = 1 + n + C_{i3} \) for \( i, j, k \in \{0, 1, 2\} \). If \( C_{i3} \geq 1 \) for all \( i \in \{0, 1, 2\} \) then \( 2p = 2g_{01} + 2g_{02} + 2g_{12} - 4 \geq 8 + 6n = 8 + 6G(\partial M) \), which is a contradiction. Thus, at least one of \( g_{01}, g_{02} \) and \( g_{12} \) equals to \( 1 + n \). It follows from Lemma 9 that \( M \) is a handlebody.

**Corollary 15.** Let \( M \) be a connected compact 3-manifold with connected boundary. Then

\( (a) \) If \( M \) is a handlebody then \( k(M) = 3G(\partial M) \).

\( (b) \) If \( M \) is not a handlebody then \( k(M) \geq 3(G(\partial M) + 1) \).

**Remark 16.** Let \((\Gamma_1, \gamma_1)\) be a crystallization of the orientable or non-orientable handlebody \( M_1 \) with \( 2 + 6G(\partial M) \) vertices as constructed in Figure 1. Let \((\Gamma_2, \gamma_2)\) be the unique 8-vertex crystallization of \( S^2 \times S^1 \) or \( S^2 \times S^1 \) or \( \mathbb{RP}^3 \) (cf. [7, Figure 2]). Let \( v_1 \in V(\Gamma_1) \) be an interior vertex and \( v_2 \in V(\Gamma_2) \). Then the graph connected sum \((\Gamma, \gamma) = (\Gamma_1 \# v_1v_2 \Gamma_2, \gamma_1 \# v_1v_2 \gamma_2)\) is a crystallization of a connected compact 3-manifold \( M \) with boundary such that \( M \) is not a handlebody and \( \partial M = \partial M_1 \). Here \( |V(\Gamma)| = 6G(\partial M) + 2 + 8 - 2 = 6G(\partial M) + 8 \). Thus, \( k(M) = 3(G(\partial M) + 1) \).

A 1-dipole of color \( j \in \Delta_d \) of a \((d+1)\)-colored graph (possibly with boundary) \((\Gamma, \gamma) \in \mathbb{G}_d \) is a subgraph \( \theta \) of \( \Gamma \) consisting of two vertices \( x, y \) joined by color \( j \) such that \( \Gamma(j)(x) \neq \Gamma(j)(y) \), where \( \Gamma_j(u) \) denotes the component of \( \Gamma_j \) containing \( u \). The cancellation of 1-dipole from \( \Gamma \) consists of two steps: first deleting \( \theta \) from \( \Gamma \) and second welding the same colored hanging edges (see [21] for more details).

**Corollary 17.** Let \( M \) be a 3-manifold with \( h \) boundary components such that \( k(M) < 3(G(\partial M) + h) \). Let \((\Gamma, \gamma) \in \mathbb{G}_3 \) be a crystallization of \( M \) such that \( k(M) \leq |V(\Gamma)|/2 - 1 < 3(G(\partial M) + h) \). Then the new graph \((\Gamma', \gamma') \in \mathbb{G}_3 \) after cancelling all possible 1-dipoles from \((\Gamma, \gamma)\), represents a handlebody.
Proof. Let \((\Gamma, \gamma) \in \mathbb{G}_3\) be a crystallization of \(M\) such that \(|V(\Gamma)| = 2p < 6G(\partial M) + 6h + 2\). It follows from Corollary 13 that \(2p \geq 6G(\partial M) + 6h - 4\). Let \((\Gamma', \gamma') \in \mathbb{G}_3\) be the new crystallization after cancelling \((h - 1)\) number of 1-dipoles of color \(j\) from \((\Gamma, \gamma)\), for \(0 \leq j \leq 2\). Then, \(|V(\Gamma')| = 2p - 6(h-1)\). Let \(M'\) denote the manifold with connected boundary with crystallization \((\Gamma', \gamma')\). Then \(G(\partial M) = G(\partial M')\). Further,

\[
6G(\partial M) + 6h - 4 \leq 2p < 6G(\partial M) + 6h + 2
\]

\[
\Rightarrow 6G(\partial(M')) + 2 \leq |V(\Gamma')| < 6G(\partial M) + 8.
\]

Then Theorem 14 implies \(M'\) is a handlebody. \(\square\)

Remark 18. Let \(M\) be a 3-manifold with \(h\) boundary components as in Corollary 17. Then \(M\) need not be the connected sum of handlebodies. For an example, we have a crystallization for 3-manifold \(\mathbb{R}^2 \times [0, 1]\) in Figure 2 which satisfies the hypothesis of Corollary 17 but is not a connected sum of handlebodies.

Corollary 19. Let \(S\) be a closed connected surface. Then

\[
6G(S) + 3 \leq k(S \times [0, 1]) \leq 8G(S) + 3.
\]

Proof. Let \((\bar{\Gamma}, \bar{\gamma})\) be a crystallization of \(M = S \times [0, 1]\). Here \(\partial M\) has exactly two components. Therefore, by Theorem 12 we have \(k(S \times [0, 1]) \geq 3(\mathcal{G}(M) + 1) \geq 6G(S) + 3\). On the other hand, it is easy to construct a crystallization of \(M = S \times [0, 1]\) with \(16G(S) + 8\) vertices by the following procedure:

Let \((\Gamma, \gamma)\) be a crystallization of \(S\) with color set \(\Delta_2 = \{0, 1, 2\}\). Let \(2p\) be the number of vertices of \(\Gamma\). Then \(2p = 4G(S) + 2\). Let \(v_1, v_2, \ldots, v_{2p}\) be the vertices of \(\Gamma\). Now, choose a fixed order of the colors say, \((0, 1, 2)\). For \(1 \leq m \leq 4\), let \(G_m(i, j, k)\) be the graph obtained from \((\Gamma, \gamma)\) by replacing vertices \(v_l\) by \(v_l^{(m)}\) and by replacing the triplet of colors \((0, 1, 2)\) by \((i, j, k)\), where \(1 \leq l \leq 2p\) and \(0 \leq i \neq j \neq k \leq 3\). Now consider the four 2-colored graphs \(G_1(0, 1, -)\), \(G_2(-, 1, 3)\), \(G_3(2, -3)\) and \(G_4(2, 0, -)\), where by ‘−’, we mean the corresponding color is missing. Let \((\bar{\Gamma}, \bar{\gamma})\) be a graph obtained by (i) adding \(2p\) edges of color 2 between \(v_l^{(1)}\) of \(G_1(0, 1, -)\) and \(v_{l'}^{(2)}\) of \(G_2(-, 1, 3)\), for \(1 \leq l \leq 2p\), (ii) adding \(2p\) edges of color 0 between \(v_l^{(2)}\) of \(G_2(0, 1, -)\) and \(v_{l'}^{(3)}\) of \(G_3(-, 1, 3)\), for \(1 \leq l \leq 2p\), and (iii) adding \(2p\) edges of color 1 between \(v_l^{(3)}\) of \(G_3(0, 1, -)\) and \(v_{l'}^{(4)}\) of \(G_4(-, 1, 3)\), for \(1 \leq l \leq 2p\).

Then \((\bar{\Gamma}, \bar{\gamma})\) is a crystallization (cf. Figure 2 for \(S = \mathbb{R}^2\)) of \(S \times [0, 1]\) with \(8p = 16G(S) + 8\) vertices. Therefore, \(k(S \times [0, 1]) \leq 8G(S) + 3\). \(\square\)

Figure 2: A crystallization of \(\mathbb{R}^2 \times [0, 1]\).
Remark 20. Let $D^3$ denote the 3-ball, and $H_3^g, \overline{H}_3^g$ denote the orientable and non-orientable handelbody of genus $g$. Then our bounds $k(M) \geq 3(\mathcal{G}(M) + h - 1)$ and $k(M) \geq 3(\mathcal{G}(\partial M) + h - 1)$ both are sharp for the compact 3-manifolds $(#_{h_1} H_3^g) \# (#_{h_2} \overline{H}_3^g)$, where $h_1, h_2 \geq 0$, $h_1 + h_2 = h$, $g_i, \bar{g}_j \geq 0$. Further, our bounds $k(M) \geq 3(\mathcal{G}(M) + h - 1)$ is sharp for the compact 3-manifolds $M \# (#_{h_1} D^3)$, $M \# (#_{h_1} H_3^g), M \# (#_{h_1} \overline{H}_3^g)$ where $M = \mathbb{R}P^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1$. In Corollary 15 we prove that if $M$ has connected boundary and is not a handelbody then $k(M) \geq 3(\mathcal{G}(\partial M) + 1)$. This bound is also sharp for $M \# D^3$, $M \# H_g^3, M \# H_g^3$ where $M = \mathbb{R}P^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1$.

Acknowledgement: The first author is supported by DST INSPIRE Faculty Research Grant (DST/INSPIRE/04/2017/002471).

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