Entanglement Entropy from String Field Theory (and a Higher-Spin Example)

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Abstract

We study the new class of solutions in linearized open string field theory (OSFT) involving higher-spin modes. Unlike the elementary OSFT solutions (on-shell vertex operators) that, acting on a vacuum, define wavefunctions of pure states (e.g. a tachyon), the solutions that we describe correspond to the reduced density matrices which eigenvalues describe the entanglement between higher-spin modes with different spin values. We compute the entanglement entropy on these OSFT solutions, and the answer is expressed in terms of converging series in inverse weighted partition numbers. In the case of $D$-dimensional bosonic string theory, the entanglement entropy of spin 1 subsystem and the system of all the spin values is given by $D \log \lambda_0 + \frac{D}{\lambda_0} \sum_{N=3}^{\infty} \frac{\beta(N)}{\lambda(N)} \log \left( \frac{\lambda(N)}{\beta(N)} \right)$, where $\lambda(N)$ is the weighted number of partitions of $N$, $\beta(N) = \frac{(N-1)(3-\zeta(2))}{(N-1)^2}$ and $\lambda_0 = \sum_{N=1}^{\infty} \frac{\beta(N)}{\lambda(N)}$. The first term, $D \log \lambda_0$, represents the entanglement swapping between string vacuum and string excitations. We generalize this result to obtain the entanglement for a subsystem of a given spin $s$ in a given space-time dimension. We also discuss how open string field theory may be used to study the entanglement of systems other than higher spin excitations in string theory.

1 Introduction

The concept of entanglement has recently attracted a lot of interest due to its relevance to building an interface between information theory, quantum gravity and strongly coupled field theories, including some condensed matter systems (a very incomplete and subjective list

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of conceptual works on the subject, both classical and recent, includes, but is not limited to [1, 3, 14, 15, 2, 3, 4, 7, 16, 5, 9, 17, 8], as well as many other remarkable works on the subject. The entanglement also appears to be a crucial ingredient in our attempts to understand the microscopic structure of space and time, the emergence of space-time geometry and to understand gravity in the context of quantum mechanics (e.g. see [2, 3]). Entanglement entropy is a particularly important quantity characterizing quantum-mechanical or field-theoretic subsystems in mixed states that interact with some other systems and are described by reduced density matrices (rather than wavefunctions). In some cases, it can be measured experimentally for certain systems, such as ultra-cold atoms or entangled photons, leading to some fascinating observations, such as quantum non-locality in space and time [11, 12]. One particularly interesting example of a system where the entanglement occurs naturally is the one of higher-spin fields, which by themselves constitute an important ingredient of gauge-to gravity correspondence and have been an subject of a deep interest and investigation over recent years (some conceptual works on the subject include, but are not limited to [39, 35, 36, 37, 34, 10]).

It is well-known, from the structure of the higher-spin symmetries, that it is impossible to consistently truncate these theories at spin values greater than 2: for example, a commutator of two spin 3 currents would inevitably contain a contribution with spin 4, and so on.

From the point of view of quantum mechanics this means that a system of particles with a given spin \( s \geq 3 \) cannot be described by a wavefunction, but has to be a part of a density matrix which structure reflects the entanglement of this system with other higher-spin fields. In particular, this raises natural questions about higher-spin modes appearing in string theory: constructing on-shell vertex operators for massive higher-spin states is straightforward in string-theoretic formalism, but each of these on-shell operators acting on the vacuum defines certain pure quantum-mechanical state, so neither non-locality of interactions, nor entanglement are obvious in the on-shell approach. In our work, we particularly address this question by describing the off-shell solution in open string field theory, which particularly reflects the entanglement between the higher-spin modes. In fact, it turns out that even lower-spin system with \( s < 3 \) is entangled with the higher spins, although classically the higher-spin algebra can be truncated to the lower-spin currents, generating the underlying space-time isometries (e.g. \( \text{AdS} \) isometry algebra). This by itself makes higher-spin systems an instructive example to study entanglement.

In general, the entanglement entropy is hard to compute in quantum field theory since the computation involves complicated functional integrals. For example, to calculate the \( n \)'th Renyi entropy in 2d conformal field theory one has to evaluate the partition function on \( n \) glued copies of a Riemann surface which in general is highly non-trivial.

String field theory, on the other hand, can be regarded a natural framework to explore the entanglement - in particular, in the case of higher-spin modes in string theory. First of all, a string field in the second-quantized theory is by definition an expansion in operators, with the spin being a natural expansion parameter. Also, the underlying equations for higher-spin fields have a form similar to Vasiliev’s equations [19, 20, 23] making components of string fields the objects reminiscent of the differential forms in the higher-spin equations. At the
same time, string field theory is background independent: shifting a string field by an analytic solution of the equations of motion leaves the form of the equations invariant, with the new nilpotent BRST charge, which cohomology defines an on-shell theory in a new background. The space-time geometry is therefore emergent in string field theory, making it a natural interface to test the entanglement, in the context of the recent ideas relating the space-time origin to quantum information [2, 3].

Finding analytic solutions to the string field theory equations is generally hard and, despite some progress over recent years, very limited number of non-trivial solutions is known. There exists, however, a well-known class of elementary solutions of these equations \( Q\Psi + \Psi \Psi = 0 \). For example, any on-shell vertex operator in string theory having the form \( V \sim c P(\partial X, \partial^2 X, ...) e^{i k X} \phi(k) \) solves the linearized equation \( Q\Psi = 0 \) if \( V \) is a dimension 0 primary field - both the linearized (here \( P \) is polynomial in derivatives of the target space field \( X_m(z) \) in bosonic string theory). Each of these solutions, acting on a vacuum, defines a physical state in open string theory. From the quantum-mechanical point of view these are the pure states, with \( V \) defining the wavefunction. From the space-time point of view, each of these states belongs to some irreducible representation of the the Lorentz group and is labelled by eigenvalues of Casimir operators of the Poincare algebra in space-time. As we show in this work, apart from these elementary solutions there exists a class of SFT solutions defining mixed states on ensemble of wavefunctions; in fact, these solutions already appear at the linearized level which we discuss in our work. This new class of solutions describes the states that are not Casimir eigenvectors and have no definite mass or spin; In general, these solutions have the form of infinite formal series in higher-spin operators with different spins and masses, and with the expansion coefficients describing the entanglement between the sectors with different spins. So if we understand the string states with definite spins and masses as pure states, the BRST cohomology solutions that we present in our work describe the nontrivial ensembles consisting of the above states, i.e. the mixed states from the quantum-mechanical point of view.

The entanglement entropy can therefore be defined and computed on these solutions and, despite the complexity of the correlation functions involved, the final answer turns out to be finite and surprisingly simple: the entropies are expressed in terms of converging series involving the partition numbers for the restricted partitions, with the restriction details depending on values of the entangled spins. This entropy is “classical” in a sense that it is defined on the solution of the SFT equation of motion, which is classical from the second-quantized point of view. In case of the entanglement of spin one subsystem with the higher-spin system the answer is particularly simple and instructive; thus for large \( N \) the contribution of a spin \( N \) subsystem to the entanglement entropy of the spin 1 subsystem is, in the leading order, given by

\[
S_{1-N} \sim e^{-\alpha \sqrt{N}}
\]  

(1.1)

where \( \alpha \) is constant which can be evaluated asymptotically.

In case of the entanglement of a spin \( s \) subsystem with the rest-of-the-spins system the
structure of partitions involved becomes more complicated but still can be computed explicitly; in particular we expect that consistency conditions for the partial entanglements may lead to new non-trivial identities in number theory. In this paper, we limit this question to the discussion section, leaving the details for the future work. The rest of the paper is organized as follows.

In the Section 2 we discuss the simplest example of the mixed state type solution appearing in the linearized bosonic open string field theory, describing the entanglement of the lowest spin 1 subsystem to the system including all the tower of the higher spins. The density matrix, as well as the entanglement entropy are expressed in terms of convergent series in the inverse weighted partition numbers $\lambda^{-1}(N)$ of integers $N > 0$. To compute the OSFT correlators, relevant to the solution, we use the singularization transformation (described in the paper), which is the conformal transformation making it possible to express the reduced density matrix and the entropy in terms of generalized Schwarzians and ordered Bell numbers that in turn can be simplified and expressed in terms of simple combinations of the partitions.

In Section 3 we generalize the calculation to obtain solutions describing the entanglement of a given spin $s$ subsystem with the ensemble containing other higher-spin modes. It turns out that, for $s > 1$ it is more convenient to use the framework of RNS superstring theory \[23\] rather than that bosonic theory. Namely, we identify the analytic solutions at superconformal ghost number $s \geq -3$ in the cohomological gauge (used instead of the standard gauge $\eta_0 \Psi = 0$) with mixed states with the reduced density matrix describing the entanglement of the spin $s$ field with the ensemble. The result is again expressed in terms of relatively simple convergent series involving restricted partition numbers, with the character of the restrictions depending on $s$.

In the concluding section, we discuss physical implications of our results for the interplays between string dynamics and entanglement, as well as their generalizations for systems beyond higher spins.

## 2 Bosonic SFT and lower-higher spin entanglement

Consider open bosonic string field theory equation of motion:

$$Q \Psi + \Psi \star \Psi = 0 \quad (2.1)$$

and its linearization

$$Q \Psi = 0 \quad (2.2)$$

where $\star$ is conformal transformation putting worldsheets of interacting strings on wedges of a disc,

$$Q = \oint \frac{dz}{2i\pi} \left[ -\frac{1}{2} c \partial X^m \partial X^m (z) + bc \partial c (z) \right] \quad (2.3)$$

is the BRST charge in bosonic string theory, skipping the Liouville terms (that ensure the overall nilpotence of $Q$ in non-critical space-time dimensions, but that shall play no role
in our calculations). $X^m; m = 1,...D$ are the target space coordinates, $b, c$ are fermionic reparametrization ghosts. The equation (2.2) has a class of elementary solutions having the form

$$\Psi_0 = \sum_i cV_i(X(z))\phi_i(p)$$

(2.4)

where $V_i$ are primary fields of dimension 1 and ghost number 0 (so that $\Psi_0$ is a primary of ghost number 1 and conformal dimension 0). In this case, $\Psi_0$ simply defines the open string spectrum in the unperturbed theory, modulo gauge transformations. Each term in the sum (2.4) then defines a physical operator in open string theory which, acting on a vacuum, defines a wavefunction of such a string mode in space-time. From the quantum-mechanical point of view all such states are the pure states, with their wavefunctions satisfying the low-energy effective action’s equations of motion (e.g. a Klein-Gordon equation for a tachyon, solving the linearized equation (2.2)). Apart from this class of elementary solutions, the known examples of nontrivial solutions to the full cubic OSFT equation (2.1) are few since in general the conformal transformations induced by the star product act on $\Psi$ in a highly nontrivial way. One remarkable example of such a solution is the one found by Schnabl [27], describing the background with nonperturbative configuration of a tachyon potential (in some sense, the Schnabl’s solution can be thought of as a “nonperturbative tachyon vertex operator at zero momentum”). One may wonder, however, if the linearized equation (2.2) admits any nontrivial solutions too, other than the elementary class (2.4). It turns out that the nontrivial solutions do exist at the linearized level, and they describe the mixed states related to the entanglement of different spin modes in open string theory. Below we shall describe these solutions and compute the higher-spin entanglement entropy for such solutions. The answer for the density matrix and for the entropy turns out to be remarkably simple, despite the seeming complexity of the correlators involved. For simplicity, let us start from the $D = 1$ case, which will be straightforward to generalize to higher space-time dimensions. Consider a general ghost number one string field, with the Siegel gauge constraint

$$b_0\Psi = 0$$

(2.5)

More specifically, consider the string field in the Siegel gauge with gost number 1 and with the following expansion in infinite formal series in derivatives of $X$:

$$\Psi_0 = c \sum_{N=1}^\infty \sum_{p=1}^N \sum_{N|n_1...n_p} \alpha_{n_1...n_p} \frac{\partial^{n_1} X}{n_1!} ... \frac{\partial^{n_p} X}{n_p!}$$

(2.6)

where $\sum_{N|n_1...n_p}$ stands for the summation over ordered length $p$ partitions of $N$:

$$N = n_1 + ... + n_p$$

$$n_1 \geq n_2 ... \geq n_p > 0$$

(2.7)

and $\alpha_{n_1...m_p}$ are some coefficients. The numbers $N$ and $p$ are thus useful parameters of such an expansion; although not directly related to higher-spin currents in space-time in $D = 1$, ...
in higher space-time dimensions conformal dimension (worldsheet spin) \( N \) of a string field component actually can be related to the space-time spin \( N \) of the component, with the contributions from different \( p \) looking like “Stueckelberg-like” terms. It is therefore convenient to cast \( \Psi_0 \) as

\[
\Psi_0 \equiv c \sum_{N=1}^{\infty} \sum_{p=1}^{N} \Psi_0^{(N,p)}
\]  

(2.8)

Our initial goal will be to find the choice of the coefficients \( \alpha \) for which \( \Psi_0 \) is the analytic solution of the linearized equation (2.2). In practice, it is convenient to use the following definition: we shall define \( \Psi_0 \) as the solution of the equation

\[
\langle \langle Q \Psi_0, \Psi \rangle \rangle \equiv \langle Q \Psi_0(0) I \circ \Psi(0) \rangle = 0
\]

(2.9)

for any string field \( \Psi \) (not necessarily in the Siegel’s gauge). Since \( \Psi \) is arbitrary, this identity, once true for any two-point correlator, will also be true for the insertion of \( Q \Psi_0 \) into any other SFT correlator, due to the closedness of the full operator algebra in CFT, which is equivalent to the statement that \( Q \Psi_0 \) vanishes identically. Here the double brackets stand for the standard OSFT correlator and the conformal transformation \( I(z) = -\frac{1}{z} \) maps \( \Psi \) to infinity.

Let us start with evaluating \( Q \Psi_0 \). Simple calculation gives:

\[
Q(c \sum_{N,p} \Psi^{N,p}_0) = \sum_{N,p} (N-1) \partial_{cc} \Psi^{N,p}_0
\]

\[
+ \sum_{N,p} \sum_{N|n_1...n_p} \sum_{j=1}^{n_j} \sum_{k=2}^{\infty} \frac{\partial^k \alpha_{n_1...n_p} \partial^{n_1} X \partial^{n_{j-1}} X \partial^{n_j-k} X \partial^{n_{j+1}} X... \partial n_p X}{n_1!...n_{j-1}!(n_j-k)!n_{j+1}!...n_p!}
\]

\[
\times \alpha_{n_1...n_p} \partial^{n_1} X... \partial^{n_{i-1}} X \partial^{n_{i+1}} X... \partial^{n_{j-1}} X \partial^{n_{j+1}} X... \partial^{n_p} X
\]

(2.10)

It can be shown, however that, with \( \Psi_0 \) having the ghost structure (2.6), (2.8) only the first term in \( Q \Psi_0 \) contributes to the correlator \( \langle \langle Q \Psi_0 \Psi \rangle \rangle \). To prove this, note that \( Q \Psi_0 \) has ghost number 2, so the only \( \Psi \) components contributing to the correlator are those having ghost number 1. The operators having ghost number 1 in general have the ghost part proportional to

\[
\sim \partial^{m_1} b \partial^{m_r} \partial^{n_{r+1}} c \sim G(\partial \sigma, \partial^2 \sigma,...)e^\sigma
\]

where \( m_j, n_j \) are non-negative integers and \( G \) is some polynomial in derivatives of \( \sigma \), with the standard bosonization relations: \( b = e^{-\sigma}; c = e^\sigma \). First of all, it is clear that only the terms with \( r = 0 \) or \( 1 \) can contribute (otherwise there would be \( b \)-fields left with no contractions). Let us first check our claim for \( r = 0, n_1 = 0 \) and then generalize it to the arbitrary case.
In the case of \( r = 0, n_1 = 0 \) (\( \Psi \)-field proportional to the \( c \)-ghost) (ghost number zero or contain powers of the \( b \)-ghost) the ghost part of the correlator \( \langle \partial^k c c(0) I \circ c(0) \rangle \) has the form \( \langle \partial^k c c(0) I \circ c(0) \rangle \). It is then easy to check that the only nonzero correlator is the one at \( k = 1 \). Indeed, write \( I \circ c = (\frac{d}{dz})^{-1}|_{z=0} c(\infty) = \lim_{w \to \infty} w^{-2} c(w) \). Then, at \( k = 1 \),

\[
\langle \partial c c(0) I \circ c(0) \rangle = \lim_{w \to \infty} w^{-2} \langle \partial c c(0) c(w) \rangle = \lim_{w \to \infty} w^{-2} w^2 = 1 \quad (2.11)
\]

At \( k = 2 \),

\[
\langle \partial^2 c c(0) I \circ c(0) \rangle = \lim_{w \to \infty} w^{-2} \langle \partial^2 c c(0) c(w) \rangle = \lim_{w \to \infty} w^{-2} (-2w) = 0 \quad (2.12)
\]

For higher \( k > 2 \) the ghost correlators vanish identically; that is, using the bozonized expression \( c = e^\sigma \) we write \( \partial^k c c = B^{(k)}_\sigma e^\sigma \) where \( B^{(k)}_\sigma \) is the degree \( k \) Bell polynomial in derivatives of \( \sigma \); its OPE with \( e^\sigma \) has the form:

\[
B^{(k)}_\sigma(z)e^\sigma(w) = (z-w)^{-k}B^{(k-1)}_\sigma(z)e^\sigma(w) + O(z-w)^0, \quad (2.13)
\]

so

\[
\partial^k c c := k : B^{(k-1)}_\sigma e^{2\sigma} : \quad (2.14)
\]

As it is clear from the OPE (2.13), for \( k > 2 \) the polynomial \( : B^{(k-1)}_\sigma : (0) \) cannot fully contract with the \( c \)-ghost at infinity and all such correlators vanish identically. This constitutes the proof that only the terms proportional to \( N \partial c c \Psi^{(N,p)}_0 \) \((k = 1)\) in \( Q \Psi_0 \) contribute to the correlator \( \langle \partial^k c c(0) I \circ c(0) \rangle \) with the components of \( \Psi \) satisfying \( r = 0, n_1 = 0 \). Now let us show that, once this is true for \( r = 0, n_1 = 0 \), this is also true for arbitrary components of \( \Psi \). For the reasons pointed out above, it is sufficient to show that this is the case for \( r = 1 \), i.e. for the components of \( \Psi \) with the ghost structure \( \sim \partial^{n_1} b \partial^{n_2} c \partial^{n_3} c \). First of all, note that, since the correlator \( \langle \partial^k c c(0) I \circ c(0) \rangle = 0 \) on the half-plane for \( k = 0 \), it also vanishes under any conformal transformation: \( z \to f(z) \) of the half-plane. Now let us consider the half-plane correlator \( \langle \partial^k c c(0) (I \circ (\partial^{n_1} b \partial^{n_3} c \partial^{n_2} c)(w \to \infty) \rangle \) (for the certainty, on the upper half-plane) and apply the conformal transformation \( z \to f(z) = e^{iz} \). This transformation is well-defined everywhere on the upper half-plane (including the real axis) and vanishes exponentially fast at infinity. Under this transformation, the \( : \partial^{n_1} b \partial^{n_3} c \partial^{n_2} c : (z) \) operators transform as

\[
: \partial^{n_1} b \partial^{n_3} c \partial^{n_2} c : (w \to \infty) \equiv : H(\partial \sigma, \partial^2 \sigma, \ldots) e^\sigma : (w \to \infty) \\
\to \lim_{w \to \infty} \{ S(m_1 | n_1, n_2) (e^{iw}; w)c(w) + O(e^{iw}) \}
\]

where we skipped the terms of orders of \( e^{iw} \) and higher (suppressed exponentially when \( w \) is taken to infinity) and \( S(m_1 | n_1, n_2) (e^{iw}; w) \) is the generalized Schwarzian of the conformal transformation \( z \to e^{iz} \) of the upper half-plane, appearing as a result of the regularization of the internal singularities in operator products between the derivatives between of the \( b \) and \( c \)-ghosts. For the exponential conformal transformation of the half-plane
\( S(m_1|n_1,n_2)(e^{i\omega};w) \) are constant numbers that do not depend on \( w \) (see below for the discussion of some essential properties of the generalized Schwarzians). For this reason, the correlator \( \lim_{w \to \infty} \langle \partial^{k_1} \xi \partial^{m_1} \eta \partial^{m_2} \zeta(w) \rangle \), computed on the Riemann surface as the result of the conformal transformation of the upper half-plane, is proportional to the correlator \( \lim_{w \to \infty} \langle \partial^{k_1} \xi \partial^{m_1} \zeta(w) \rangle \) on the same Riemann surface (with the coefficient given by constant generalized Schwarzian factor) and therefore vanishes for \( k > 1 \). This constitutes the proof that only the terms proportional to \( N \partial^{cc} \Psi_0^{(N,p)} \) need to be considered in \( Q \Psi_0 \). We are now prepared to analyze the correlator \( \langle Q \xi \partial^{cc} \Psi(0) I \circ \Psi(0) \rangle \) for \( \Psi_0 \) of the form (2.6) and an arbitrary string field \( \Psi \).

The string fields of this correlator are located on the halfplane’s boundary; the crucial next step to compute the correlator is the conformal transformation of the half-plane:

\[ z \to f(z) = e^{iz} \quad (2.15) \]

taking the upper half-plane to compact Riemann surface, with \( Q \Psi_0 \) taken from zero to 1 and \( \Psi \) from infinity to zero. This conformal transformation (which we will also refer to as the “singularization transformation”) maps the upper half-plane to a compact Riemann surface which we shall call the “singularoid”.

Consider the behavior of the \( \langle \langle Q \Psi_0, \Psi \rangle \rangle \) correlator under such a conformal map. For that, one crucial relation that we shall need is the transformation law of the \( : \partial^{n_1} X \partial^{n_2} X : (z) \)-operator under \( z \to f(z) \), given by

\[
\frac{1}{n_1!n_2!} : \partial^{n_1} X \partial^{n_2} X : (z) \to \\
\frac{1}{n_1!n_2!} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} B_{n_1|k_1}(f(z);z) B_{n_2|k_2}(f(z);z) : \partial^{k_1} X \partial^{k_2} X : (f(z)) \\
+ S_{n_1|n_2}(f(z);z) \quad (2.16)
\]

where \( B_{n|k} \) are the incomplete Bell polynomials in the \( z \)-derivatives of \( f \). The general definition of \( B_{n|k} \) is:

\[
B_{n|k}(g_1, \ldots g_{n-k+1}) = n! \sum_{p_1! \cdots p_{n-k+1}!} \frac{1}{1!} (g_1)^{p_1} \cdots \frac{g_{n-k+1}}{(n-k+1)!} p_{n-k+1} \quad (2.17)
\]

with the sum taken over all the non-negative \( p_1, \ldots p_{n-k+1} \) satisfying

\[
p_1 + \cdots + p_{n-k+1} = k \\
p_1 + 2p_2 + \cdots + (n-k+1)p_{n-k+1} = n
\]

In particular, the incomplete Bell polynomials \( B_{n|k}(f; z) \) in the derivatives (or the expansion coefficients) of \( f(z) \), are given by \( g_k = \partial^k f(z) \equiv \frac{d^k f}{dz^k} \) (although the partial derivative sign is not necessary, we keep it to shorten our notations) or equivalently.
For the conformal transformation under study, the second kind need, the summation, so for the conformal transformation that we exemplified to give:

\[
B_{n|k}(f(z); z) = n! \sum_{n_1 \ldots n_k} \frac{\partial^{n_1} f(z) \ldots \partial^{n_k} f(z)}{n_1! \ldots n_k! q(n_1)! \ldots q(n_k)!}
\]

with the sum \(n|n_1 \ldots n_k\) taken over all ordered \(n_1 \geq n_2 \ldots \geq n_k > 0\) length \(k\) partitions of \(n\) and with \(q(n_j)\) denoting the multiplicity of \(n_j\) element of the partition (e.g. for the partition \(7 = 2 + 2 + 3\) we have \(q(2) = 2, q(3) = 1\), so the appropriate term would read \(\sim \frac{\partial^2 f \partial^2 f \partial^3 f}{2!2!3! \times 2!1!}\).

Then, \(S_{n_1|n_2}(f(z); z)\) are the generalized Schwarzians of the conformal transformation, given by

\[
S_{n_1|n_2}(f; z) = \frac{1}{n_1! n_2!} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{m_1 \geq 0} \sum_{m_2 \geq 0} \sum_{p \geq 0} \sum_{q=1}^{p} (-1)^{k_1+m_2+q} 2^{-m_1-m_2}(k_1 + k_2 - 1)! \cdot \frac{\partial^{m_1} B_{n_1|k_1}(f(z); z) \partial^{m_2} B_{n_2|k_2}(f(z); z) B_{p|q}(g_1, \ldots, g_{p-q+1})}{m_1! m_2! p! (f'(z))^{k_1+k_2}}
\]

\[
g_s = 2^{-s-1}(1 + (-1)^s) \frac{d^{s+1} f}{(s + 1) f'(z)}; s = 1, \ldots, p - q + 1 \quad (2.18)
\]

with the sum over the non-negative numbers \(m_1, m_2\) and \(p\) taken over all the combinations satisfying

\(m_1 + m_2 + p = k_1 + k_2\)

For \(n_1 = n_2 = 1\) \(S_{1|1}\) becomes the usual Schwarzian derivative (up to the conventional normalization factor of \(\frac{1}{6}\)). Note that the exponential factors proportional to powers of \(\sim e^{iz}\) cancel out in all the terms of the summation, so for the conformal transformation that we need, \(f(z) = e^{iz}\), the generalized Schwarzians \(S_{n_1|n_2}\) do not depend on \(z\) and are constant. For the conformal transformation under study, \(f(z) = e^{iz}\), the value of the Bell polynomials \(B_{n|p}(f(z); z)\) and their derivatives at can be expressed in terms of the Stirling numbers of the second kind \(S(n; k)\):

\[
B_{n|k}(e^{iz}; z) = i^n S(n; k) e^{ikz}
\]

\[
\partial^p z B_{n|k}(f(z); z) = i^{n+p} k^p S(n; k) e^{ikz} \quad (2.19)
\]

and accordingly, for \(f(z) = e^{iz}\) the explicit form of the generalized Schwarzians can be simplified to give:

\[
S_{n_1|n_2}(f; z) = \frac{1}{n_1! n_2!} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{m_1 \geq 0} \sum_{m_2 \geq 0} \sum_{p \geq 0} \sum_{q=1}^{p} (-1)^{k_1+m_2+q} 2^{-k_1-k_2}(k_1 + k_2 - 1)! \cdot \frac{i^{-p} S(n_1, k_1) S(n_2, k_2) k_1^{m_1} k_2^{m_2} B_{p|q}(g_1, \ldots, g_{p-q+1})}{m_1! m_2! p!}
\]

\[
g_s = \frac{2^{-s} \cos\left(\frac{s\pi}{2}\right)}{s + 1} \quad (2.20)
\]
with the summations subject to the same constraints (2.18). The transformation law (2.16)

is straightforward to generalize for any monomial in the derivatives of \( X \). Namely, under 

\[ z \rightarrow f(z) \]

we have

\[
\sum_{q=1}^{[\frac{p}{2}]} \sum_{\{i_1...i_p\} \rightarrow \{i_1...i_{2q};i_{1+2q}...i_{p-2q}\}} \sum_{k_1=1}^{n_{i_1}} ... \sum_{k_{p-2q}+1}^{n_{p-2q}} S_{n_1|n_{i_2}}(f(z);z)...S_{n_{p-2q-1}|n_{i_{2q}}}(f;z) B_{n_{i_1}|k_1}(f(z);z)...B_{n_{p-2q}|k_{p-2q}}(f(z);z) \partial^{k_1} X...\partial^{k_{p-2q}} X : (f(z)) \quad (2.21)
\]

where \( \sum_{\{i_1...i_p\} \rightarrow \{i_1...i_{2q};i_{1+2q}...i_{p-2q}\}} \) stands for the summation over the permutations \( \{i_1...i_p\} \rightarrow \{i_1...i_{2q};i_{1+2q}...i_{p-2q}\} \) such that \( i_1 \neq i_2 ... \neq i_{2q} \neq j_1 ... \neq j_{p-2q}; \ 1 \leq i_k \leq p; 1 \leq j_k \leq p \) and 

\( i_{2k-1} \leq i_{2k} \) (the last constraint is imposed in order to ensure that the redundant combinations of 

Schwarzians \( S_{n|n} \) do not appear in the permutations).

In what follows, we will be particularly interested in the terms with \( p = 2q \) in the sum 

(2.21) that contain no operators but are just the numbers only depending on \( f(z) \). We shall 

call these terms pure Schwarzian contributions, and they will be play an important role 

in the calculations below. To simplify the notations, it is convenient to write

\[
S_{n_1...n_p}(f(z);z) = \sum_{\{i_1...i_p\} \rightarrow \{i_1...i_p\}} S_{n_1|n_{i_2}}(f(z);z)...S_{n_{p-1}|n_{i_p}}(f(z);z) 
\]

with the summation over permutations of \( 1...p \) defined as above. We are now prepared 

to return to the conformal transformation (2.15) of \( \langle \langle Q\Psi_0,\Psi \rangle \rangle \). First, consider the 

transformation of \( I \circ \Psi \) located at infinity. Note that, in general \( \Psi \) has the form similar to 

(2.6), except that, generally speaking, \( \alpha_{n_1...n_p} \)-coefficients may depend on \( X \). According to the 

transformation formula (2.21), each term in \( \Psi \) gets multiplied by \( e^{ihz}|z\rightarrow\infty \) with \( h \geq p - 2q \). 

Therefore all the contributions, except for the one with \( p = 2q \) (that is, the pure Schwarzian 

contribution \( S_{n_1...n_p} \)) are exponentially dumped and vanish identically at infinity. So for any 

positive \( N = n_1 + ... + n_p \) the only surviving part in any component of \( \Psi \) upon the conformal 

transformation (2.15) is the pure Schwarzian (which is constant, given by sum of combinations 

of the products involving Stirling numbers according to (2.20)). The only possible exception 

to it is the component with \( N = 0 \) which, in principle, also may be present in \( \Psi \). This 

component is just a function of \( X \) with no derivatives having the form : \( f(X) \). But such a 

component a priori does not contribute to the contractions with \( \Psi_0 \) in the correlator (note 

that \( \Psi_0 \) by construction contains no \( N = 0 \) terms). To see this, it is convenient to apply 

the conformal transformation \( I(z) \) to the correlator \( \langle \langle Q\Psi_0(0)I_0(\cdot;X^n:(0)) \rangle \rangle \) for any 

\( n \), taking : \( X^n : \) from infinity to zero and \( \Psi_0 \) at 0 to \( Q\Psi_0 \) at infinity, with \( Q\Psi_0 \) having the 

same form (2.6) as \( Q\Psi_0 \), but with some new coefficients \( \tilde{\alpha}_{n_1...n_p} \), straightforward to determine 

from the conformal transformation. Note that \( X^n \) doesn’t change as the resulting conformal 

transformation applied to it, \( I \circ I \) , is an identity. Then, using the transalational invariance, 

take \( f(X) \) to \( z = -\pi \), and apply another transformation \( f(z) = e^{iz} \) to the correlator

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\[
<X^n\left(-\frac{\pi}{2}\right)(I \circ Q\Psi_0)(\infty)>. 
\]

Similarly to what we explained before, only the pure Schwarzian terms remain out of \(\tilde{\psi}_0\) upon the transformation, implying that the entire correlator is proportional to the pure Schwarzian factor of \(X^n\) which does not contract. But this factor is proportional to

\[
(S_{0|0}(f(z); z))^{\tilde{\psi}}|_{f(z)=e^{iz}; z=-\frac{\pi}{2}}
\]

where \(S_{0|0} = \log(f'(z))\), i.e. vanishes at \(z = -\frac{\pi}{2}\). This shows that the only possible string field component of \(I \circ \Psi\), that does not vanish under \(f(z) = e^{iz}\), except for the pure Schwarzian part, does not contribute to the correlator \(<Q\Psi_0 I \circ \Psi>>\). But then, since only the pure Schwarzian (non-contracting) terms of \(\Psi\) contribute to the correlator, the same is true for \(Q\Psi_0\); therefore we conclude that the correlator \(<Q\Psi_0(0)I \circ (\Psi(0))>\) evaluated on the singularoid has the form:

\[
<<Q\Psi_0\Psi>> = G_{\psi} \sum_{N=1}^{\infty} \sum_{p=1}^{N} \sum_{N|n_1...n_p} \alpha_{n_1...n_p} S_{n_1...n_p} \tag{2.23}
\]

where \(G_{\psi}\) is some constant which only depends on particulars of \(\Psi\) and independent on \(\Psi_0\). The coefficients \(\alpha_{n_1...n_p}\) are now to be chosen so that the correlator involving the summation over \(N\) vanishes. At the first glance, this doesn’t seem to be a simple problem because of the complexity of \(S_{n_1...n_p}\)-factors involving cumbersome summations over products of generalized Schwarzians. There is, however, a simplification trick making it possible to deduce \(S_{n_1...n_p}\) (as previously, we consider \(p\) even). Consider the correlator of \(Q\Psi_0\) with \(\frac{1}{p!} : I \circ (\partial X)^p :\) (multiplied by the c-ghost, as usual) in OSFT for some \(p\). The relevant terms in the part of \(<<Q\Psi_0,(\partial X)^p>>\) for a given \(N\) are

\[
\lim_{w \to \infty} \frac{1}{p!} \sum_{N|n_1...n_p} \frac{\partial^{n_1} X...\partial^{n_p} X(0)}{n_1!...n_p!} \frac{I(\circ(\partial X)^p(0))}{U_0(w)} = (U_0(w)) \tag{2.24}
\]

where \(U_0(w)\) is the overlap factor accounting for for the correlator change as a result of the integration of conformal Ward identities (note that the correlator (2.24), computed naively without this factor would have been proportional to to \(w^{p-N}\), i.e. would have vanished, as the \(w^{2p}\)-factor due to the conformal transformation of \((\partial X)^p\) by \(I\) would have been multiplied by \(w^{-N+p}\) as a result of the contractions).

In our case, this factor is not difficult to compute explicitly. Infinitezimally, it is given by
the integral

\[
\int dz \frac{1}{2i\pi} \epsilon(z) \partial X \partial X(z); \sum_{N|n_1...n_p} \frac{\partial^{n_1} X...\partial^{n_p} X(0)}{n_1!...n_p!} (I \circ ((\partial X)^p(0))) > |_{\text{overlap} \xi = 0, w \to \infty}
\]

\[
= \sum_{N|n_1...n_p} \sum_{j=1}^{p} \frac{1}{(p-1)!n_1!...n_{j-1}!n_{j+1}!...n_p!} \times \frac{\partial^{n_1} X...\partial^{n_{j-1}} X\partial X(w)(\partial X(w))^{p-1}}{n_j!} |_{\xi = 0, w \to \infty}
\]

\[
	imes \int dz \frac{\epsilon(z)}{2i\pi (z - \xi)^{n_j+1}(z - w)^2}
\]

with one of the $\partial X$’s in the stress tensor $T(z)$ acting on the operator at $\xi$ and another on the operator at $w$ (i.e. the infinitesimal overlap transformation gives the change of the entire correlator under the conformal transformation excluding the contributions due to infinitesimal conformal transformations of the vertex operators themselves). The integral over $z$ is straightforward to evaluate, however, since the conformal transformation by $I(z)$ only acts on the second operator in $\langle \langle Q\Psi_0; \Psi \rangle \rangle = \langle Q\Psi_0(0) I \circ \Psi(\infty) \rangle$ (in our case, $(\partial X)^p$), only the pole at $w$ contributes to the overlap function, so the $z$-integral’s contribution to the infinitesimal overlap transformation is

\[
\sum_{j=1}^{p} \partial_w \left[ \frac{\epsilon(w)}{(w - \xi)^{n_j+1}} \right] = \sum_{j=1}^{p} \frac{\partial \epsilon(w)}{w - \xi} - (n_j + 1) \frac{\epsilon(w)}{(w - \xi)^{n_j+2}}
\]

This is easily integrated to give the finite transformation, i.e. the overlap function for $I(z)$:

\[
U_0(w) = \prod_{j=1}^{p} \frac{dI}{dz} \bigg|_{z = w} (I(w) - I(\xi))^{n_j+1} |_{\xi = 0, w \to \infty} = w^{N-p}
\]

Multiplying by the overlap function thus precisely cancels the vanishing $w^{p-N}$-factor discussed above, keeping the correlator finite and relating the correlators before and after the conformal transformations. The correlator is then easy to compute, as each given combination $n_1...n_p$, divided by $p!$, contributes exactly 1 to the correlator. Therefore the overall correlator simply equals the number of such combinations, i.e. the number of partitions $\lambda(N|p)$ of number $N$ with the length $p$:

\[
\sum_{N|n_1...n_p} \frac{\partial^{n_1} X...\partial^{n_p} X(0)}{n_1!...n_p!} (I \circ (\partial X)^p)(\infty)) = \lambda(N|p)
\]
Next, apply the conformal transformation $f(z) = e^{iz}$ to the correlator (2.28). Similarly to the explained above, the correlator computed on singularoid is contributed by the pure Schwarzian terms only with the overlap function computed to be

$$U_0(w) = \frac{p! w^{-2p}}{((p-1)!!)^2(S_{1|1}(e^{iz}; z))^p} + O(e^{iw})$$

(2.29)

where the Schwarzian of the exponential transformation $S_{1|1}(e^{iz}; z)$ is simply $\frac{1}{12}$. Therefore the correlator (2.28) computed on the singularoid, is given by

$$\sum_{N|n_1...n_p} S_{n_1...n_p} \left( (p-1)!!(S_{1|1}(e^{iz}; z))^\frac{p}{2} \right)$$

(2.30)

This identity particularly expresses the number of partitions of the length $p$ in terms of summation (2.19), (2.20), (2.30) over Stirling numbers of the second kind. Given (2.31), it is now straightforward to get the OSFT analytic solution of the form (2.6) for $\Psi_0$. First of all, it is necessary to pick $\alpha_{n_1...n_p} = 0$ for any $p$ odd, since for the odd $p$ values the factorization (2.23) of the OSFT correlator $<< Q\Psi_0; \Psi >>$ doesn’t appear to exist. For even $p$, writing $p = 2k$, the solution for $\Psi_0$ is

$$\Psi_0 = c \sum_{N=2}^{\infty} \frac{\beta(N)}{\lambda(N)} \sum_{k=1}^{|N/2|} \sum_{N|n_1...n_{2k}} \frac{\prod_{j=1}^{2k} \partial^{(n_j)} X}{n_j! \sqrt{12}} \lambda(N) = \sum_{k=1}^{|N/2|} (2k - 1)!! \lambda(N|2k)

\beta(N) = \frac{(N-1)\zeta(3) - \zeta(2)}{(N-1)^4}$$

(2.32)

where $\zeta$ is the Riemann’s zeta-function and the $\lambda(N)$ coefficients are sums over the partitions of $N$ with even lengths $2k$, weighted with $(2k-1)!!$. Note that $\zeta(2) = \frac{\pi^2}{6} \approx 1.64$ and $\zeta(3) \approx 1.2$ is the Apery’s constant. Indeed, it is now easy to check that, with the string field given by (2.32) one has

$$<< Q\Psi_0, \Psi >> = G_0 \sum_{N=2}^{\infty} \left( \frac{\zeta(3)}{(N-1)^2} - \frac{\zeta(2)}{(N-1)^3} \right) = G_0 (\zeta(3)\zeta(2) - \zeta(2)\zeta(3)) = 0$$

(2.33)
Note that all the $\beta(N)$ coefficients are positive, except for the $N = 2$; in particular, we will discuss below the implications of that for the entanglement of the bosonic string states. This OSFT solution is straightforward to generalize to $D$ space-time dimensions; one just has to take the product of $D$ copies of $\Psi_0$:

$$\Psi_0^{(D)} = c \prod_{m=1}^{D} \Psi_0^{(m)}$$

$$\Psi_0^{(m)} = \Psi_0(X \to X_m) \quad (2.34)$$

($X$ is replaced with $X_m$ with the $c$-ghost factor removed)

The solution (2.32) has a structure quite different from the class of the elementary solutions (2.4). The summation over $N$ is essentially the summation over space-time spin values coinciding with conformal dimensions of the string field’s components; the components with different $k$ with $N$ fixed could then be understood as Stueckelberg terms for a given spin $N$. Clearly, unlike the elementary solutions (2.4) defining wavefunctions of pure states, with given spins and masses, the solution (2.32) - (2.34) sums over the ensemble of the states with different spins and masses, with the coefficients defining the reduced density matrix of a certain subsystem. As our solution carries he $b - c$ ghost number 1 (and in fact can be extended to superstring theory with no coupling to the $\beta - \gamma$ ghost system), it belongs to the same ghost sector as the generators of Poincare isometries in space-time. It is therefore natural to identify the solution (2.32) - (2.34) with the reduced density matrix of the subsystem of the lower spin 1 entangled with tower of higher spins in open string theory, with the terms at a given $N$ corresponding to contribution from the spin $N$ subsystem to the entanglement. In the next section we will give a more systematic explanation for such an identification; it appears that the formalism of RNS superstring theory is more convenient for that; in particular it makes it far easier technically (in comparison with bosonic theory) to analyze the entanglement of higher spin subsystems with systems including all the spin ensembles. Given the BRST cohomology solution described above, we can now compute the entanglement entropy associated with the SFT solution (2.32)-(2.34). There is one subtlety though, that has to be pointed out. We aim to express $\Psi_0$ as a sum over the ensemble of the pure states (each of them characterized by a certain mass and a spin), with the summation coefficients defining the eigenvalues of the reduced density matrix. All of these coefficients must be positive (since they correspond to classical probabilities). The coefficients that we computed are, on the other hand, proportional to $\sim \frac{\beta(N)}{\lambda(N)}$ and indeed are all positive, with the exception of first term with $N = 2$. Since each $N$ represents the entanglement of spin 1 excitations with those of higher spin $N$, to keep the density matrix Hermitian, it is sufficient to invert the sigh of the graviton’s wavefunction (while keeping all the higher-spin wavefunctions invariant). Writing

$$\tilde{\Psi}_0 = c \sum_{N=3}^{\infty} \frac{\beta(N)}{\lambda(N)} \varphi_N \quad (2.35)$$
where by definition

\[ \varphi_N = \sum_{k=1}^{[N/2]} \sum_{N_1 \ldots n_{2k}} \prod_{j=1}^{2k} \frac{\partial^{(n_j)} X}{n_j! \sqrt{12}} \]

we see that in \( D \) space-time dimensions the SFT solution \( \Psi_0^{(D)} \) can be written as

\[ \Psi_0^{(D)} = c \sum_{N_1 \ldots N_D} \rho_{N_1 \ldots N_D} \varphi_{N_1} \ldots \varphi_{N_D} \]

(\( \varphi_{N}^{(m)} \) is obtained from \( \varphi_{N} \) by replacing \( X \to X^m; \ m = 1, \ldots, D \))

with the products of \( \varphi_{N_j} \)'s defining the ensemble of states for the reduced density matrix of the lower-spin subsystem, with \( \rho_{N_1 \ldots N_D} \) defining the entanglement probabilities of this subsystem with the higher spins. Note that the factors of \( n_j! \sqrt{S_{[1]}^{12}} = n_j! \sqrt{12} \) appearing in \( \varphi_N \) can be absorbed by rescaling \( \partial^{(n_j)} X \)'s in the products. Such a rescaling only affects an overall normalization constant for the density matrix (call it \( \lambda_0 \)), which in any case can be fixed from the condition:

\[ \text{Tr} \rho = \sum_{N_1 \ldots N_D} \rho_{N_1 \ldots N_D} = 1 \] (2.38)

In particular, in \( D = 1 \) the normalization condition reads

\[ \lambda_0^{-1}(\zeta(2) - \zeta(3)) + \sum_{N=3}^{\infty} \beta(N)(\lambda(N))^{-1} = 1 \] (2.39)

so SFT solution (2.32) must be divided by

\[ \lambda_0 = \zeta(2) - \zeta(3) + \sum_{N=3}^{\infty} \beta(N)\lambda^{-1}(N) \equiv \sum_{N=2}^{\infty} |\beta(N)|\lambda^{-1}(N) \] (2.40)

to give the normalized density matrix (note that the summation over \( N \) converges fast since the partition numbers \( \lambda(N) \) grow exponentially with \( N \)). Accordingly, in \( D \) dimensions the solution \( \Psi_0^{(D)} \) (2.37) is to be divided by \( \lambda_0^D \) to ensure the correct normalization. That said, the entanglement entropy for the solution (2.37) is

\[ S_{ent}^{spin_{1 \ldots allspins}} = D \log \lambda_0 + \frac{D}{\lambda_0} \left( \sum_{N=3}^{\infty} \frac{\beta(N)}{\lambda(N)} \log \left( \frac{\lambda(N)}{\beta(N)} \right) - (\zeta(2) - \zeta(3)) \log(\zeta(2) - \zeta(3)) \right) \] (2.41)

The series in \( N \) again converges fast as \( \lambda(N) \) grows exponentially. It is tempting to assume that each term in the summation represents contribution of spin \( N \) to the entanglement. This concludes the computation of the entanglement entropy of the lower spins as a subsystem of the higher-spin system.

In the next section we shall generalize this computation to obtain the entanglement entropy of a given spin \( s \) subsystem, as a part of the entire higher-spin system. This will also provide an additional explanation for the interpretation of the entropy (2.41) as the one for the entanglement of the lower-spin subsystem, discussed above.
3 Entanglement of spin $s$ subsystems: general case

In this section we will generalize the main result of the previous one and compute the entanglement entropy of any spin $s$ subsystem. For reasons that will become clear below, it appears that the framework of RNS superstring theory is more convenient for this purpose, compared to bosonic string theory. The action for the RNS superstring theory in superconformal gauge is

$$S \sim \int d^2z \left[ -\frac{1}{2} \partial X^m \bar{\partial} X_m - \frac{1}{2} \bar{\partial} \psi^m \psi_m - \frac{1}{2} \partial \bar{\psi}^m \bar{\psi}_m ight. $$

$$+ b \bar{\partial} c + \bar{b} \partial \bar{c} + \beta \bar{\partial} \gamma + \bar{\beta} \bar{\partial} \bar{\gamma} \left. \right] + S_{\text{Liouville}}, \quad (3.1)$$

the BRST charge (ignoring the Liouville terms) is now

$$Q = \oint \frac{dz}{2i\pi} \left[ cT - b c \partial c - \frac{1}{2} \gamma \psi^m \partial X_m - \frac{1}{4} b \gamma^2 \right] \quad (3.2)$$

where $T$ is the full matter+ghost stress-energy tensor and, as before, we are searching for the solutions of the linearized SFT equation $Q \Psi_0 = 0$, equivalent to finding $\Psi_0$ such that $< \langle Q \Psi_0, \Psi \rangle > = 0$ for any $\Psi$ (subject to the gauge constraint $b_0 \Psi = 0$). The bosonization relations for the ghost fields are, as usual

$$c = e^\sigma, b = e^{-\sigma}$$

$$\gamma = e^{\phi - \chi}, \beta = e^{\chi - \phi} \partial \chi \quad (3.3)$$

First of all, it is straightforward to extend the solution found in the previous section to superstring theory. For simplicity, consider the number $D$ of space-time dimensions even. Then, bosonize the RNS fermions according to

$$\psi_{2j-1} \pm \psi_{2j} = \sqrt{2} e^{\pm i \varphi_j} (j = 1, ..., \frac{D}{2}) \quad (3.4)$$

implying

$$: \psi_{2j-1} \psi_{2j} : = - \partial \varphi_j \quad (3.5)$$

Since the stress tensor for $\psi_m$:

$$T_{\psi} = -\frac{1}{2} \partial \psi^m \psi_m = -\frac{1}{2} \partial \varphi_j \partial \varphi^j$$

doesn’t have a background charge, conformal transformations of products of $\varphi$ derivatives involve the generalized Schwarzians identical to those appearing in (2.18) for the $X$-fields.
Therefore the corresponding solution for $\Psi_0$ in superstring theory is simply

$$\Psi_0^{(D)} = c \prod_{m=1}^{\frac{3D}{2}} \Psi_0^{(m)}$$

$$\Psi_0^{(m)} = \Psi_0(X \rightarrow X_m); m = 1, \ldots, D$$

$$\Psi_0^{(m)} = \Psi_0(X \rightarrow \varphi_{m-D}); m = D + 1, \ldots, \frac{3D}{2}$$  \hspace{1cm} (3.6)

($X$ is replaced with $X_m$ or $\varphi_{m-D}$ with the $c$-ghost factor removed) and the entanglement entropy is

$$S_{spin|allspins} = \frac{3D}{2} \log \lambda_0 + \frac{3D}{2\lambda_0} \sum_{N=2}^{\infty} \frac{|\beta(N)|}{\lambda(N)} \log \left( \frac{\lambda(N)}{|\beta(N)|} \right)$$  \hspace{1cm} (3.7)

where $\Psi_0$ has the same form as in (2.32) with $X$ replaced with $X_m$ for $m = 1, \ldots, D$ in $\Psi_0^{(m)}$ and with $\varphi_1, \ldots, \varphi_{\frac{3D}{2}}$ in the remaining $\frac{3D}{2}$ factors. The entropy is then obtained from the one in the bosonic theory simply by replacing $D \rightarrow \frac{3D}{2}$. Now let us consider the entanglement of a given spin $s \geq 3$ system, regarded as the subsystem of string excitations with all the spins. As previously, the first step is to determine the appropriate solution in linearized string field theory. To identify the structure of the solution we are looking for, it is useful to recall the general relation between the SFT solutions and the physical vertex operators and currents in string theory. For example, consider the Schnabl’s solution \cite{27,28,29} for nonperturbative tachyonic vacuum that was used to prove Sen’s conjecture \cite{30,31,32}. This is the pure ghost solution, with the ghost number +1. Since at zero momentum the tachyon vertex operator is just a $c$-ghost, the solution found by Schnabl was identified, based on its ghost-matter structure, to the nonperturbative tachyonic vacuum, defined by acting with this solution on the initial string vacuum state. In the similar spirit, we have identified the string field theory solutions (2.32) (carrying $b-c$ ghost number 1 and $\beta-\gamma$ ghost number zero, just as Poincare generators at unintegrated $b-c$ picture) with the reduced density matrix of the spin 1 system, considered as a subsystem of string modes with all the spins. The above arguments make it quite clear what type of the solutions we should be looking for. Namely, to describe the reduced density matrix of the subsystem with a given spin $s$, we have to search for the SFT solutions in the superconformal ghost sector containing the currents - the primaries of dimension one integrated over the worldsheet’s boundary (or multiplied by the $c$-ghost at unintergrated picture). However, not all such operators generate authentic space-time symmetries, with some of them being BRST exact and some being the picture-changing transformations of operators with lower values of the ghost numbers. In fact, the spin $s$ operators we need are the superconformal ghost number $-s$ dimension zero primaries satisfying the constraints \cite{21,22}. 

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\[ \{Q, V^{(-s)}\} = 0 \]
\[ V^{(-s)} \neq [Q, ...] \]
\[ : \Gamma V^{(-s)} := 0 \]  
(3.8)

where

\[ \Gamma = -\frac{1}{2} e^{\phi} \psi_m \partial X^m - \frac{1}{4} be^{2\phi - \chi}(\partial \chi + \partial \sigma) + ce^\chi \partial \chi \]  
(3.9)

is the picture-changing operator for \( \beta - \gamma \) pictures or, in the dual positive \( s - 2 \)-picture:

\[ \{Q, V^{(s-2)}\} = 0 \]
\[ V^{(s-2)} \neq [Q, ...] \]
\[ : \Gamma^{-1} V^{(s-2)} := 0 \]  
(3.10)

Sets of operators with such properties define the dual negative and positive ghost homologies \( H_{-s} \sim H_{s-2} \) \[21\].

In the manifest form such operators can be constructed as follows:

Take a massless spin 3 operator, the element of \( H_{-3} \), given by

\[ V^{(-3)} = \Omega^{n_1 n_2 n_3}(p) \oint dze^{-3\phi} \psi_{n_1} \partial X_{n_2} \partial X_{n_3} e^{ipX} \]  
(3.11)

with the symmetric space-time spin 3 field \( \Omega \) satisfying Fronsdal’s on-shell constraints. Consider the operator product of 2 \( V^{(-3)} \)’s which is straightforward to calculate. It is straightforward to check that this product will have the form

\[ V^{(-3)} V^{(-3)} \sim \partial c \sum_W C_{w|v}^{(-3)} W^{(-3)} + C_{w|v}^{(-4)} W^{(-4)} \]  
(3.12)

where \( C_{w|v} \) are the OPE structure constants with \( W^{(-3)} \) and \( W^{(-4)} \) being the vertex operators from \( H_{-3} \) and \( H_{-4} \) respectively (with no operators from \( H_{-5} \) and \( H_{-6} \), despite that the right-hand side of the product has ghost number \(-6\)). In particular, the \( W^{(-4)} \) terms contain an operator which, after double picture-changing transformation from picture \(-6\) to picture \(-4\), takes the form \( \oint dze^{-4\phi} \partial \psi_{(m_1} \psi_{m_2)} \partial X^{n_1} \partial X^{n_2} \partial X^{n_3} \) (times the structure constants multiplied by \( \Omega^2 \)). This operator is the vertex operator for the two-row field \( \Omega_{3|1} \) which, in Vasiliev’s description, corresponds to the symmetric frame-like field of spin 4 and the structure constants define the quadratic contribution of spin 3 field to the \( \beta \)-function of the spin 4 field. This, in turn, produces a cubic \( 3 - 3 - 4 \) vertex in the lower-energy effective action and the appropriate term in the higher-spin algebra (that is, the spin 4 term in the commutator of two spin 3 currents). For general \( s \), the operator algebra has the form:
\[ V(-s_1)V(-s_2) \sim \partial c \sum_{s_3} \sum_{W} C^{s_1,s_2|s_3} W(-s_3) \]  

(3.13)

(the OPE coefficients \( C^{s_1,s_2|s_3} \) vanish for \( s_3 = 0, 1, 2 \)). The general fusion rule for the cohomologies:

\[ H_{-s_1} \otimes H_{-s_2} \sim \sum_{s_3} H_{-s_3} \]  

(3.14)

reproduces the structure of the higher-spin symmetry algebra with the structure constants generating the cubic couplings for the higher-spin frame-like fields in space-time. All the above arguments altogether instruct us about the form of the SFT solution to search, in order to describe the spin \( s \) entanglement. While still retaining the gauge condition \( b_0 \Psi = 0 \), it is appropriate to replace the constraint \( \xi_0 \Psi = 0 \) with the cohomological gauge constraint:

\[ : \Gamma \Psi := 0 \]  

(3.15)

for each negative ghost number sector and

\[ : \Gamma^{-1} \Psi := 0 \]  

(3.16)

for each positive ghost number sector. We will refer to the gauge choice (3.15), (3.16) as cohomological gauge. This gauge choice is natural for our purposes since, with such a choice, the SFT solutions at host number \(-s\) can be clearly related to the reduced density matrices of the spin \( s \) system; with other gauge choices, the ghost number \(-s\) solutions would mix contributions from different spins, with no obvious way to identify the entanglement.

First of all, the cohomological gauge imposes stringent limits on the possible number of bosonized RNS fermions (\( \varphi \)'s) in the solution. That is, unlike the lower-spin \( s < 3 \) SFT solution (2.32) with the number of of \( \varphi \)'s unrestricted, the cohomological gauge restricts this number to \( s - 1 \) at most, since the OPE of \( \partial \varphi \) with \( \psi \) in \( \Gamma \) has the structure \( \partial \varphi(z)\psi(w) \sim (z-w)^{-1}\psi(w) \) implying that any string field containing product of more than \( s - 1 \) derivatives of \( \varphi \)'s would violate cohomological condition. This precisely corresponds to the number of the extra fields for a symmetric frame-like field of spin \( s \) in Vasiliev’s formalism. This is again useful to compare with the structure of the higher-spin operators in the on-shell limit. Generally, the operators for the two-row \( \Omega^{t|t} \) extra field with \( t \) derivatives \( 0 \leq t \leq s - 1 \) contain \( t \) \( \psi \)-fields, with the \( t = 0 \) field the only one being dynamical. In other words, the \( \psi \)-fields do not carry information about real physical degrees of freedom in the cohomological gauge and should be excluded from the structure of the solution we are looking for. Similarly, we shall ignore the components containing the derivatives of the \( \varphi \)-ghost: in the on-shell limit inclusion of the ghost derivatives in the vertex operator effectively reduces the spin of the
matter part in space-time; for this reason the operators containing the ghost derivatives are related to the Stueckelberg-type terms (to ensure the overall BRST invariance of the operator) and do not contribute to actual physical degrees of freedom.

That said, we shall search for the SFT solution in the cohomological gauge having the form:

$$\Psi_s = c e^{\chi - s \phi} \sum_{N_1=1}^{\infty} \cdots \sum_{N_D=1}^{\infty} \alpha(N_1, \ldots, N_D)$$

$$\times \sum_{k_1=1}^{[N_1]} \cdots \sum_{k_D=1}^{[N_D]} \prod_{j=1}^{(s-2)} \sum_{N_j[n_1 \ldots n_{2k_j}]}^{(s-2)} \partial^{n_j} X_j$$

$$= \sum_{N=n_1+\ldots+n_k}^{\infty} \sum_{n_1 \geq n_2 \geq \ldots \geq n_k > 0} \lambda^{(m)}(N_j) \prod_{j=1}^{D} \Psi_s(X_j; N_j)$$

(3.17)

where $\sum_{n_1 \ldots n_k}^{(m)}$ stands for the summation over the length k ordered partitions of $N = n_1 + \ldots + n_k$ such that $n_1 \geq n_2 \geq \ldots \geq n_k > 0$ with values of the partition elements not bigger than m (so $n_1 \leq m$). Next, the BRST charge acting on $\Psi_s$ gives:

$$Q \Psi_s = \partial c e^{\chi - s \phi} \sum_{N_1=1}^{\infty} \cdots \sum_{N_D=1}^{\infty} \alpha(N_1, \ldots, N_D)(N_1 + \ldots N_D - \frac{1}{2}s^2 + s - 1)$$

$$\times \prod_{j=1}^{D} \Psi_s(X_j; N_j)$$

(3.18)

where we skipped the irrelevant terms, as was explained in the previous section. From this we deduce

$$\alpha(N_1, \ldots, N_D) = (N_1 + \ldots N_D - \frac{1}{2}s^2 + s - 1)^{-1} \prod_{j=1}^{D} \beta(N_j)$$

(3.19)

where

$$\lambda^{(m)}(N_j) \equiv \sum_{k=\left[\frac{N_j}{2}\right]}^{\left[\frac{N_j}{m}\right]} (2k-1)!! \lambda^{(m)}(N_j|2k)$$

(3.20)

where $\lambda^{(m)}(N_j|2k)$ is the number of the length 2k partitions of $N_j = n_1 + \ldots + n_{2k}; n_1 \geq n_2 \geq \ldots \geq n_{2k} > 0$ with all the elements of the partition being not greater than m or, in other words,

$$n_1 \leq m$$

(3.21)

The remaining steps are identical to those described in the previous section. With the conformal transformation $z \to f(z) = e^{iz}$ we reduce the SFT correlator

$$< Q \Psi_s; \eta_0 \Psi >/ = < Q \Psi_s(0)(I \circ (\eta_0 \Psi))(\infty) >$$

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to pure Schwarzian contributions from $\eta_0\Psi$ at infinity and, consequently, $\Psi_s$. The structure of the result is then identical to (2.32), given by series in combinations of generalized Schwarzians for $\Psi_s$ (which orders are now restricted by the cohomological gauge constraints), multiplied by the factor depending on $\eta_0\Psi$ only (times the constant given by the ghost part of the correlator). Then we compare it to the test correlator $\langle < Q\Psi_s; \eta_0\Phi \rangle \rangle$ with the matter part of $\Phi$ given by $\Phi_{\text{matter}} = \sum_{k_1, \ldots, k_D=1}^{\infty} \prod_{j=1}^{D} \frac{1}{k_j!} : (\partial X_j)^{k_j} :$. Using the identity $\langle < Q\Psi_s(0)I \circ (\Phi_{\text{matter}})(\infty) \rangle \rangle = \langle < Q\Psi_s(0)\eta_0\Phi_{\text{matter}}(1) \rangle \rangle$ we express $\langle < Q\Psi_s; \eta_0\Phi \rangle \rangle$ in terms of the weighted numbers of restricted partitions $\lambda^{(s)}$ (3.20). Finally, applying the transformation $f(z) = e^{iz}$ to the test correlator, we relate the sum over the combinations of the generalized Schwarzians to the restricted partition numbers. It is then straightforward to check that, with the choice (3.19) of $\alpha$ the string field $\Psi_s$ satisfies

$$\langle < Q\Psi_s; \eta_0\Psi \rangle \rangle = \tilde{G}_\Psi \times (\zeta(2)\zeta(3) - \zeta(3)\zeta(2))^{D} = 0$$

(3.22)

for any $\Psi$ where, as before, $\tilde{G}_\Psi$ is the factor that only depends on the string field $\Psi$ (but not on $\Psi_s$). Thus the string field $\Psi_s$ (3.17), (3.19) defines the ghost number $-s$ linearized OSFT solution in the cohomological gauge, which defines the reduced density matrix for a spacetime spin $s$ subsystem in superstring theory. With this density matrix, it is straightforward to obtain the entanglement entropy for the spin $s$ subsystem, with the result given by:

$$S_{\text{ent}}(s) = \log \tilde{\lambda}^{(s)} - \frac{1}{\lambda^{(s)}} \sum_{\tilde{N}=D;\tilde{N}\neq \frac{s^2}{2}-s}^{\infty} \rho_s(\tilde{N}) \log \rho_s(\tilde{N})$$

(3.23)

where

$$\rho_s(\tilde{N}) = \frac{1}{|\tilde{N} + s - \frac{s^2}{2}|} \sum_{\{N_1, \ldots, N_D; N_1 + \ldots + N_D = \tilde{N}\}} |\beta(N_1)\ldots\beta(N_D)| \left/ \lambda^{(s)}(N_1)\ldots\lambda^{(s)}(N_D) \right|$$

(3.24)

with the sum is taken over all positive values of $N_1, \ldots, N_D$ satisfying

$$N_1 + \ldots + N_D = \tilde{N}$$

(3.25)

and

$$\tilde{\lambda}^{(s)} = \sum_{\tilde{N}=D;\tilde{N}\neq \frac{s^2}{2}-s}^{\infty} \rho_s(\tilde{N})$$

(3.26)

This concludes the calculation of the entanglement for the spin $s$ subsystems. It should be noted that the SFT solutions (3.17), (3.19) are GSO-odd and GSO-even for the odd and even spin values respectively. To preserve the algebraic structures of SFT, such as cyclicity of the correlators, one can assign the internal Chan-Paton factors to the operators, e.g. by multiplying the GSO-even operators by $2 \times 2$ identity matrix, GSO-odd operators by $\sigma_1$ Pauli matrix, while multiplying $Q$ and $\eta_0$ by $\sigma_3$. Then, upon computing the SFT correlators, one has to take the trace over the resulting $2 \times 2$ matrix.

In the following concluding section, we shall discuss some properties of our SFT solutions and the results for the entanglement.
4 Conclusions and Discussion

In this work we have calculated the entanglement entropies for the subsystems of spin \( s \) excitations in string theory, using the solutions in linearized open string and superstring field theories. Unlike the elementary solutions of the linearized OSFT that typically define the pure states (on-shell vertex operators acting on the vacuum), the solutions that we find and analyze in this work define reduced density matrices for various spin \( s \) excitations and the related entanglements with other spins. Despite the overall complexity of the operators involved, the conformal transformations described in this paper allow to express them in terms of series over generalized Schwarzians and, subsequently, to relate these series to weighted partition numbers. The final answer for the entropies is remarkably simple - they all are expressed in terms of convergent series in the inverse partition numbers, with no the restrictions on the partition elements for the lower-spin (spin 1) entanglement and with the values of the partitions restricted by the spin value \( s \) for the higher-spin entanglement.

The restrictions on values of the partition elements for the higher-spin reduced density matrices and entanglements is the direct consequence of the cohomological gauge condition, necessary to single out authentic higher-spin currents amidst higher ghost number string fields. This restriction clearly reduces the number of relevant partitions \( \lambda(N_j|2k) \) and hence the number of weighted partitions \( \lambda^{(s)}(N_j) \) entering the solution. Since the density matrix elements are divided by the normalization factors \( \tilde{\lambda}^{(s)} \) involving summations over inverse \( \lambda^{(s)}(N_j) \), converging faster with \( s \), this clearly implies that the entanglement entropy generally grows with \( s \) for \( s \geq 3 \). Next, our results for the entanglement entropies imply that the entanglements for any spin contain universal contributions which are purely logarithmic and have the form \( \sim \log \tilde{\lambda}^{(s)} \) (where \( \tilde{\lambda}^{(s)} \) is given by the series in terms of inverse weighted partition numbers with the partition elements restricted by \( s-2 \) for \( s \geq 3 \) and with no restrictions for the lower spins. These purely logarithmic contributions are collective in a sense that they can’t be viewed as sums of individual contributions from different spins to the entanglement (unlike the terms linear in inverse \( \tilde{\lambda}^{(s)} \) in (3.23). The structure of these contributions hints at their possible interpretation: these terms represent the entanglement swappings between spin \( s \) subsystems and the string vacuum, representing nonlocality of time in string theory, reminiscent of the entanglement between non-coexisting photons that has been observed experimentally \cite{12}. In string theory context, this swapping is the entanglement between the spin \( s \) excitations of a string and the vacuum state in the past.

In this paper we have calculated the lower (spin 1) entanglement in both bosonic OSFT and in superstring field theory, while the calculation of the higher-spin entanglement \( (s \geq 3) \) was limited to superstring field theory only. Calculating the higher-spin entanglement in bosonic string field theory seems to be much harder to do because the analogue of the cohomological gauge, used to identify the higher-spin density matrices in the set of higher ghost number SFT string fields, is far more complicated in the bosonic theory. This is because in bosonic theory the cohomological gauge has to be defined with respect to the \( b-c \) picture changing operator \( Z =: b\delta(T) : \) which is a highly nonlocal object (unlike \( \Gamma \)) due the delta-function of the stress tensor (an object with conformal dimension \(-2\)). One needs to have a better understanding...
of the OPE structure of the $Z$-operators in order to extend our results to the bosonic theory.

In this work we limited ourselves to calculating the entanglement on the solutions of the linearized theory. It would be obviously extremely interesting and important to extend our results to the full interacting SFT, by the identifying the reduced density matrix type solutions. Although finding analytic solutions in interacting SFT isn’t simple in general, we hope that the singularization method that we used in this work, can be extended to the interacting theory with some modifications, in order to obtain new classes of solutions. Given the background independence of string field theory, it can be holographically related to very different quantum field theories and systems, such as holographic fluids and condensed matter systems. With the interplays between quantum entanglement and concepts of string field theory, mentioned in the beginning our work, our hope is that SFT will prove to be a new powerful framework for computing the entanglement entropies in various systems and for our understanding of quantum entanglement in general (including its relevance to the origin of space and time). We hope to address these questions in our future works.

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