Supersymmetry on Lattice Using Ginsparg-Wilson Relation

Hiroto. So, Naoya Ukita

Department of Physics, Niigata University, Ikarashi 2-8050, Niigata, 950-2181, Japan

The Ginsparg-Wilson (G-W) relation for chiral symmetry is extended for a supersymmetrical (SUSY) case on a lattice. It is possible to define exact lattice supersymmetry which are divided into two different cases according to using difference operators. $U(1)_R$ symmetry on the lattice is also realized as one of exact symmetries. For an application, the extended G-W relation is given for a two-dimensional model with chiral multiplets.

1. Introduction

Although numerical QCD studies have succeeded in computer simulations by the lattice framework, the problem of chiral fermion[1] remained. Kaplan’s domain wall fermion[2] and the overlap formula[3] as the hamiltonian formalism is a nice idea if the chiral charge is sufficiently separate because the total theory is vector-like and consistent with the no-go theorem[1]. Recently, Neuberger[4] found that the Dirac operator by his overlap formula is satisfied with the Ginsparg-Wilson (G-W) relation[5]. Furthermore, although the Dirac operator is not chiral-invariant, the new symmetry found by Lüscher[6] is a modified version of chiral symmetry. It is noted that the usual path-integral measure is not invariant under the new symmetry and the chiral anomaly is generated.

Lattice formulations of supersymmetry (SUSY) were attempted by several authors[7] - [10]. But there are many problems such as definition of SUSY on lattice, non-locality and breakdown of Leibniz rule, because the SUSY transformation need not only chiral fermions but also derivative operators. In this article, we explain a SUSY-extended G-W relation[11]. A new symmetry for lattice SUSY can be defined without any ambiguities in replacing derivative operators with difference ones. As the compensation, it is necessary for the block-spin function constructing a lattice theory. As an example, a SUSY free action in two-dimension is constructed. In addition to fermionic and bosonic G-W relations used by Aoyama-Kikukawa[9], we can derive the further two relations among fermions and bosons for the lattice action.

2. Derivation of SUSY Ginsparg-Wilson relation

The original G-W relation was derived as an identity for the Gaussian type effective action. It should be recognized as a kind of Ward-Takahashi identity for the chiral transformation of the Gaussian type effective action and as the property of the lattice theory near the continuum limit with a chiral symmetry[12].

Due to the no-go theorem[1], we must prepare two chiral matter fields. A SUSY transformations for two chiral-multiplets, $\Phi_j = (\phi_j, \psi_j, F_j)^T, j = 1, 2$ in the continuum theory is defined as

$$\delta \Phi_j = Q(\epsilon, \bar{\epsilon}) \Phi_j$$  \hspace{1cm} (1)

$$\delta_\epsilon \bar{\Phi}_j = \bar{\Phi}_j Q(\epsilon, \bar{\epsilon})$$  \hspace{1cm} (2)

where an Euclidean space is considered because we construct the Euclidean lattice theory. The dependency on index $j$ is trivial and we suppress it from now. We begin with a continuum theory, and define its regularized theory on a cubic lattice by performing a block-spin transformation. A lattice point is expressed by an integer vector, $\{n_a\}$, where $a$ is lattice constant. We take for simplicity $a = 1$. The block-spin transformation from $\Phi(x)$ to $\Phi_n$ is given by

$$\Phi_n \sim \int dx f_n(x) \Phi(x) \equiv \langle f_n, \Phi \rangle$$  \hspace{1cm} (3)
\[ \Phi_n \sim \int dx f_n(x) \Phi(x) = \langle f_n, \Phi \rangle \] (4)

where \( f_n(x) \) is a block-spin function with finite support around \( x_\mu = n_\mu \), and co-moving with \( x_\mu - n_\mu : f_n(x) = f(x-n) \). (,) implies the usual inner product in a function space.

According to Ginsparg and Wilson, we may define a Gaussian effective action \( A \) by using a SUSY action \( A_c \) in the continuum theory\[1\]

\[
\exp(-A_{\text{eff}}[\Psi, \bar{\Psi}]) = \int D\Phi(x) D\bar{\Phi}(x) \exp(-A_c[\Phi, \bar{\Phi}]) - \sum_{n,m} (\Psi_n - \bar{\Psi}_n) \alpha_{n,m}(\Psi_m - \bar{\Psi}_m)).
\] (5)

Here \( \alpha_{n,m} \) is a matrix acting on the multiplet, \( \Psi_n \).

\[
\alpha_{n,m} = \alpha \delta_{n,m} \begin{pmatrix} 0 & 0 & 1 \\ 0 & V & 0 \\ 1 & 0 & 0 \end{pmatrix}
\] (6)

where \( V \) is some anti-symmetric matrix determined by a mass-term Lagrangian. This \( \alpha_{n,m} \) term is a SUSY invariant mass term with a large mass \( O(a^{-1}) \). In the naive sense, our effective action, \( A_{\text{eff}} \) is SUSY invariant but the transformation is modified as seen later.

We may define naive SUSY on the lattice:

\[
\delta^L \Phi_n = \int f_n(x) \delta \Phi(x) dx
\] (7)

\[
\Phi_n \equiv \Phi_n(x) dx = Q_L(\epsilon; \bar{\epsilon}; \nabla) \Phi_n
\]

\[
\delta^N \bar{\Phi}_n = \int f_n(x) \delta \bar{\Phi}(x) dx
\] (8)

\[
\bar{\Phi}_n \equiv \bar{\Phi}_n(x) dx = \bar{\Phi}_n Q_L(\epsilon; \bar{\epsilon}; \nabla)
\]

This is just an interpretation from continuum SUSY to lattice SUSY, where a derivative operator in the continuum SUSY is replaced by a difference operator in the lattice SUSY using the relation \( \delta f_n(x) = -\partial^\mu f_n(x) \). Although the explicit form of the difference operator depends on the block-spin function, it is possible to choose the reasonable function in the realization of lattice SUSY. Under this naive lattice SUSY transformation, our effective action changes by

\[
\exp(-A_{\text{eff}}[\Psi', \bar{\Psi}'])
\]

\[
= \int D\Phi(x) D\bar{\Phi}(x) \exp(-A_c[\Phi, \bar{\Phi}]) - (\bar{\Psi}' - \bar{\Phi}) \alpha (\Psi' - \Phi)
\]

\[
= \int D\Phi(x) D\bar{\Phi}(x) \exp(-A_c[\Psi', \bar{\Psi}'])
\]

\[
- (\bar{\Psi} - \bar{\Phi}) e^{Q_L \alpha e Q_L (\Psi - \Phi)}
\]

where the lattice site and the spinor indices are both omitted. It is assumed that \( A_c \) is invariant under SUSY transformation in the continuum theory,

\[
A_c[\Phi, \bar{\Phi}] = A_c[\Psi', \bar{\Psi}'].
\] (9)

Although the path-integral measure is naively unchanged

\[
D\Phi' D\bar{\Phi}' = D\Phi D\bar{\Phi},
\] (10)

we include the Jacobian factor in the following calculation and shall find the relation with the anomaly\[13\].

Let us derive a SUSY extension of the G-W relation for a free theory described by

\[
A_{\text{eff}}[\Psi, \bar{\Psi}] = \sum_{n,m} \bar{\Psi}_n S_{(n,m)} \Psi_m.
\] (11)

Under the naive SUSY, it transforms as

\[
\exp(-A_{\text{eff}}[\Psi, \bar{\Psi}]) \{ 1 - \bar{\Psi}(SQ_L + \bar{Q}_L)S\Psi \}
\]

\[
= \{ 1 + \delta J + \text{str}(Q_L + \bar{Q}_L) \}
\]

\[
- \text{str}(Q_L \alpha^{-1} S + S \alpha^{-1} \bar{Q}_L) - \bar{\Psi} S(Q_L \alpha^{-1} + \alpha^{-1} \bar{Q}_L) S\Psi \}
\]

\[
\times \exp(-A_{\text{eff}}[\Psi, \bar{\Psi}]),
\] (12)

*Since we have two chiral multiplets, it is possible to construct a Dirac mass term.
where $\delta J$ comes from a Jacobian factor. So, we can get following two relations;

$$
\delta J = \text{str} \left( Q_L \alpha^{-1} S + S \alpha^{-1} Q_L \right)
- \text{str} \left( Q_L + \bar{Q}_L \right),
$$

\hspace{1cm} (13)

and

$$
\bar{\Psi} (SQ_L + \bar{Q}_L S) \Psi
= \Psi (Q_L \alpha^{-1} + \alpha^{-1} Q_L) S \Psi.
$$

\hspace{1cm} (14)

These are SUSY extended GW relations. Note that the right hand side of Eq.(14) vanishes if

the difference operator $(\nabla)$ of $Q_L(\nabla)$, $\bar{Q}_L(\nabla)$ is symmetric. Therefore eq.(14) is divided into

two cases according to the difference operator $(\nabla)$. For a symmetric difference operator $\nabla_s$, the above relations suggest us to define

$$
q_s \equiv Q_L(\nabla_s)
$$

\hspace{1cm} (15)

$$
\bar{q}_s \equiv \bar{Q}_L(\nabla_s)
$$

\hspace{1cm} (16)

under which we can show the invariance of our effective action:

$$
\delta A_{\text{eff}} = \bar{\Psi} (S q_s + \bar{q}_s S) \Psi = 0.
$$

\hspace{1cm} (17)

For a non-symmetric difference operator $\nabla$,

$$
q \equiv Q_L(\nabla) - Q_L(\nabla) \alpha^{-1} S
$$

\hspace{1cm} (18)

$$
\bar{q} \equiv \bar{Q}_L(\nabla) - \bar{Q}_L(\nabla) \alpha^{-1} S
$$

\hspace{1cm} (19)

and

$$
\delta A_{\text{eff}} = \bar{\Psi} (S q + \bar{q} S) \Psi = 0.
$$

\hspace{1cm} (20)

This is a SUSY extension of Lüscher’s symmetry[@].

For $U(1)_R$ charge $Q_R$, we can find the G-W relations similar to the SUSY case:

$$
\delta J_R = \text{str} \left( Q_R \alpha^{-1} S + S \alpha^{-1} Q_R \right)
- \text{str} \left( Q_R + \bar{Q}_R \right),
$$

\hspace{1cm} (21)

and

$$
\bar{\Psi} (SQ_R + \bar{Q}_R S) \Psi
= \bar{\Psi} S (Q_R \alpha^{-1} + \alpha^{-1} Q_R) S \Psi.
$$

\hspace{1cm} (22)

The conserved $'U(1)_R'$ charge $q_R$,$

$$
q_R \equiv Q_R (1 - \alpha^{-1} S),
$$

\hspace{1cm} (23)

can be also found as one of exact symmetries on the lattice.

3. An example: 2-Dimensional chiral multiplets

We consider two chiral-multiplets $\Phi_j$, $j = 1, 2$, consisted of real scalars $\phi_j$, auxiliary fields $F_j$ and complex Weyl spinors $\chi_j$. There are arranged in a complex multiplet $\Phi = (\phi_1 + i\phi_2, F_1 + iF_2; \chi_1 + i\chi_2, \chi_1^* + i\chi_2^*) \equiv (\phi, F; \chi, \bar{\chi})$ and its conjugate $\bar{\Phi} = (\phi_2 - i\phi_2, F_2 - iF_2; \chi_2 - i\chi_2, \chi_2^* - i\chi_2^*) \equiv (\phi^*, F^*; \bar{\chi}^1, \chi^1)$. We define N=1 SUSY transformation;

$$
\begin{align*}
\delta_F \phi &= i(\epsilon^* \chi + \epsilon \bar{\chi}) \\
\delta_F F &= -2\epsilon \partial_z \chi + 2\epsilon^* \partial_z \bar{\chi} \\
\delta_F \chi &= -2\epsilon^* \partial_z \phi + i\epsilon F \\
\delta_F \bar{\chi} &= -2\epsilon \partial_z \phi - i\epsilon^* F.
\end{align*}
\hspace{1cm} (24)

$$
\text{We consider a SUSY-invariant massless Lagrangian:}
\begin{equation}
\mathcal{L} = 2\partial_z \phi \partial_z \phi - \frac{1}{2} F^* F + i(\chi^1 \partial_z \bar{\chi} + \bar{\chi}^1 \partial_z \chi).
\end{equation}
\hspace{1cm} (25)

\text{Now, we obtain the matrix } V \text{ in Eq.(3),}
\begin{equation}
V = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\end{equation}
\hspace{1cm} (26)

\text{from a mass-term in a SUSY-invariant Lagrangian,}
\begin{equation}
\mathcal{L}_m = -\frac{m}{2} (F^* \phi + F \phi^* + \chi^1 \chi - \bar{\chi}^1 \bar{\chi}).
\end{equation}
\hspace{1cm} (27)

The naïve lattice SUSY takes of the matrix form

$$
Q_L = \begin{pmatrix}
0 & 0 & i\epsilon^* & i\epsilon \\
0 & 0 & -2\epsilon \nabla_z & +2\epsilon^* \nabla_z \\
-2\epsilon \nabla_z & i\epsilon & 0 & 0 \\
-2\epsilon^* \nabla_z & -i\epsilon^* & 0 & 0
\end{pmatrix},
$$

\hspace{1cm} (28)

and
We denote the lattice effective action one.

This result  is mentioned in Eq.(14). It follows that

\[ \alpha Q_L + \bar{Q}_L \alpha = \alpha \begin{pmatrix} 0 & 0 & -2\epsilon \frac{\nabla}{\nabla z} & -2\epsilon \frac{\nabla}{\nabla z} \\ 0 & 0 & -i\epsilon & -i\epsilon^* \\ -i\epsilon^* & 2\epsilon \frac{\nabla}{\nabla z} & 0 & 0 \\ -i\epsilon & -2\epsilon^* \frac{\nabla}{\nabla z} & 0 & 0 \end{pmatrix}, \] (29)

where \((TD)_z\) means a total derivative \(\nabla_z + \frac{\nabla}{\nabla z}\).

Therefore, if one uses the symmetric difference operator, lattice SUSY is exactly same as naive one.

Let us derive SUSY extended GW relations. We denote the lattice effective action \(A_{\text{eff}}\):

\[ A_{\text{eff}}[\Psi, \bar{\Psi}] = \sum_{n,m} \bar{\Psi}_n S_{(n,m)} \Psi_m, \] (31)

\[ S = \begin{pmatrix} D_B & 0 \\ 0 & D_F \end{pmatrix}, \] (32)

where \(D_B\) is a bosonic kinetic part, and \(D_F\) is a fermionic one. From Eq.(14), we obtain in our approach not only the original GW relation but also the relation among fermions and bosons:

\[ \sigma_3 D_F + D_F \sigma_3 = \alpha^{-1} D_F \sigma_3 D_F, \] (33)

\[ \Sigma_3 D_B + D_B \Sigma_3 = \alpha^{-1} D_B \Sigma_3 D_B, \] (34)

\[ i D_B^{\phi} \nabla^\phi - 2 \nabla^z D_F^z \chi = \alpha^{-1} D_B^{\phi} (TD)_z D_F^z \chi, \] (36)

where \(\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), \(\Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), and the superscript \(\chi^\dagger \chi\) of \(D_F^z \chi\) denotes the \((\chi^\dagger, \chi)\) component of \(D_F\). Eq.(33) for fermions is just the original GW relation for chiral symmetry, and due to SUSY Eq.(34) for bosons was obtained as the corresponding relation. Therefore SUSY can live with \(U(1)_R\) symmetry on the lattice. Also Eqs.(33),(34) represent the supersymmetric relation among fermions and bosons. Once we find the fermion kinetic term \(D_F^{z} \chi\), it is easy to express the total action explicitly from these relations.

REFERENCES

1. H. B. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 20; ERRATUM B195 (1982) 541; B193 (1981) 173.
2. D. Kaplan, Phys. Lett. B288 (1992) 342.
3. R. Narayanan and H. Neuberger, Phys. Rev. Lett. 71 (1993) 325; Phys. Lett. B302 (1993) 62; Nucl. Phys. B443 (1995) 305.
4. H. Neuberger, Phys. Lett. B417 (1998) 141; B427 (1998) 353.
5. P. Ginsparg and K. Wilson, Phys. Rev. D25 (1982) 2649.
6. M. Lüscher, Phys. Lett. B428 (1998) 342.
7. N. Sakai and M. Sakamoto, Nucl. Phys. B229 (1983) 173.
8. S. Nojiri, Prog. Theor. Phys. 74 (1985) 819.
9. T. Aoyama and Y. Kikutaka, Phys. Rev. D59 (1999) 054507.
10. W. Bietenholz, Mod. Phys. Lett. A14 (1999) 51.
11. H. So and N. Ukita, Phys. Lett. B457 (1999) 314.
12. M. Koseki, H. So and N. Ukita, hep-lat/0011009.
13. K. Fujikawa, Phys. Rev. D21 (1980) 2848; ERRATUM D22 (1980) 1499.