ON THE SLOPE CONJECTURE OF BARJA AND STOPPINO
FOR FIBRED SURFACES

XIN LU AND KANG ZUO

Abstract. Let \( f : S \to B \) be a locally non-trivial relatively minimal fibration of genus \( g \geq 2 \) with relative irregularity \( q_f \). It was conjectured by Barja and Stoppino that the slope \( \lambda_f \geq \frac{4g-1}{b-q_f} \). We prove the conjecture when \( q_f \) is small with respect to \( g \); we also construct counterexamples when \( g \) is odd and \( q_f = (g+1)/2 \).

1. Introduction

A fibred surface, or simply a fibration, is a surjective proper morphism \( f : X \to B \) from a non-singular projective surface \( X \) onto a non-singular projective curve \( B \) with connected fibers. The general fiber of \( f \) is a smooth curve of genus \( g \), which will be assumed to be at least 2 in the paper. We always assume that \( f \) is relatively minimal, i.e., there is no \((-1)\)-curve contained in the fibers of \( f \). Here a curve \( C \) is called a \((-k)\)-curve if it is a smooth rational curve with self-intersection \( C^2 = -k \).

It is called smooth if all its fibers are smooth, isotrivial if all its smooth fibers are isomorphic to each other, locally trivial if it is both smooth and isotrivial, and semi-stable if all its singular fibers are reduced nodal curves.

Let \( \omega_X \) be the canonical sheaf of \( X \), and \( \omega_f = \omega_X \otimes f^*\omega_B^N \) the relative canonical sheaf of \( f \). The relative minimality of \( f \) implies that \( \omega_f \) is numerical effective (nef), i.e., \( \omega_f \cdot C \geq 0 \) for any curve \( C \subseteq X \). Let \( b = g(B) \), \( p_g = h^0(X, \omega_X) \), \( q = h^1(X, \omega_X) \), \( \chi(\mathcal{O}_X) = p_g - q + 1 \), and \( \chi_{\text{top}}(X) \) be the topological Euler characteristic of \( X \). Then we consider the following relative invariants of \( f \):

\[
\begin{align*}
\chi_f &= \deg f, \omega_f = \chi(\mathcal{O}_X) - (g-1)(b-1), \\
\omega_f^2 &= \omega_X^2 - 8(g-1)(b-1), \\
e_f &= \chi_{\text{top}}(X) - 4(g-1)(b-1).
\end{align*}
\]

They satisfy the following properties:

\[
\begin{align*}
12\chi_f &= \omega_f^2 + e_f, \\
e_f \geq 0; \text{ moreover, } e_f = 0 \text{ iff } f \text{ is smooth.} \\
\chi_f \geq 0; \text{ moreover, } \chi_f = 0 \text{ iff } f \text{ is locally trivial.}
\end{align*}
\]

2010 Mathematics Subject Classification. Primary 14D06, 14H10; Secondary 14D99, 14J29.

Key words and phrases. Fibrations, slope inequality, relative irregularity, double cover fibrations.

This work is supported by SFB/Transregio 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG (Deutsche Forschungsgemeinschaft), and partially supported by National Key Basic Research Program of China (Grant No. 2013CB834202).
If $f$ is not locally trivial, the slope of $f$ is defined to be
\[ \lambda_f = \frac{K_f^2}{\chi_f}. \]
It follows immediately that $0 < \lambda_f \leq 12$. The main known result is the slope inequality:

**Theorem 1.1** (Cornalba-Harris-Xiao, [7, 20]). If $f$ is not locally trivial, then
\[ \lambda_f \geq \frac{4(g-1)}{g}. \]

After that, it is natural to investigate the influence of some properties of the fibration on the behaviour of the slope. For instance, according to [13, 3], one knows that the Clifford index of the general fiber has some meaning to the lower bound of the slope. We would like to pay attention to the following conjecture of Barja and Stoppino (cf. [3, Conjecture 1.1]) about the influence of the relative irregularity $q_f := g - b$ on the lower bound of the slope.

**Conjecture 1.2** (Barja-Stoppino). If $f$ is not locally trivial and $q_f < g - 1$, then
\[ \lambda_f \geq \frac{4(g-1)}{g - q_f}. \] (1-3)

The first result in the direction is due to Xiao [20, Theorem 3], where he proved that if $q_f > 0$, then $\lambda_f \geq 4$ and the equality can hold only when $q_f = 1$. In [3, Theorem 1.3], Barja and Stoppino considered the influence of the Clifford index of the general fiber and the relative irregularity on the lower bound of the slope simultaneously, and obtained a lower bound which is close to the conjectured bound if the Clifford index is large. We proved the above conjecture for hyperelliptic fibrations in [14, Corollary 1.5]. In this paper, we show the following

**Theorem 1.3.** Let $f$ be a fibration of genus $g \geq 2$ which is not locally trivial.

(i) Conjecture 1.2 holds if $q_f \leq g/9$.

(ii) There exist fibrations with $q_f = (g + 1)/2$ violating (1-3) whenever $g$ is odd.

Pirola constructed in [19] the first example which does not satisfy (1-3), see also [3, Remark 4.6]. To our knowledge, the only known counterexamples to the bound (1-3) belong to the extremal case $q_f = g - 1$. According to [20, Corollary 4], the genus of fibrations with $q_f = g - 1$ is bounded from above ($g \leq 7$). In our construction, the genus has no upper bound.

Counterexamples will be given in section 5. The main idea of the proof of Theorem 1.3(i) is a combination of Xiao’s technique [20] and the second multiplication map (see subsection 2.2). It turns out that our conclusion follows directly from these two techniques if $f$ is not a double cover fibration.

**Definition 1.4.** The fibration $f$ is said to be a double (resp. triple) cover fibration of type $(g, \gamma)$ if there are morphisms $h': Y' \to B$ and $\pi: X \to Y'$ ($Y'$ may be singular) such that the general fiber of $h'$ is a genus-$\gamma$ curve, $\deg \pi = 2$ (resp. $\deg \pi = 3$) and $h' \circ \pi = f$. 

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y' \\
\downarrow f & & \downarrow h' \\
B & & \\
\end{array}
\]
Double cover fibrations were studied earlier by many authors, see [2, 4, 8] etc. We define certain local relative invariants for the double cover \( \pi \) and show that these relative invariants (1-1) of \( f \) can be expressed by these local relative invariants and relative invariants of the quotient fibration (cf. Theorem 4.3). Based on these formulas, we complete the proof of Theorem 1.3(i).

Our paper is organized as follows. In Section 2, we recall Xiao’s method in the study of the lower bound on the slope, and develop certain inequalities relying on the second multiplication map. In Section 3, we prove Theorem 1.3(i) based on a combination of these two techniques except for double cover fibrations. In Section 4, we treat the double cover fibrations. Meanwhile, we obtain various lower bounds on the slope of double cover fibrations in this section. Finally, in Section 5 we provide counterexamples to (1-3).

2. Preliminaries

2.1. Harder-Narasimhan filtration of the direct image sheaf. In the subsection, we briefly recall the Harder-Narasimhan filtration on the direct image sheaf \( f_* \omega_f \) and Xiao’s technique.

Let \( \mathcal{E} \) be a (non-zero) locally free sheaf over \( B \). The slope of \( \mathcal{E} \) is defined to be the rational number
\[
\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\text{rank} \mathcal{E}}.
\]
The sheaf \( \mathcal{E} \) is said to be stable (resp. semi-stable), if for any coherent subsheaf \( 0 \neq \mathcal{E}' \subseteq \mathcal{E} \) we have \( \mu(\mathcal{E}') < \mu(\mathcal{E}) \) (resp. \( \mu(\mathcal{E}') \leq \mu(\mathcal{E}) \)); it is said to be positive (resp. semi-positive), if for any quotient sheaf \( \mathcal{E} \rightarrow \mathcal{Q} \neq 0 \), one has \( \deg \mathcal{Q} > 0 \) (resp. \( \deg \mathcal{Q} \geq 0 \)).

It is well-known that the locally free sheaf \( \mathcal{E} := f_* \omega_f \) has a unique filtration, called the Harder-Narasimhan (H-N) filtration:
\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}, \tag{2-1}
\]
such that:
\begin{enumerate}
  \item the quotient \( \mathcal{E}_i/\mathcal{E}_{i-1} \) is a locally free semi-stable sheaf for each \( i \);
  \item the slopes are strictly decreasing \( \mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu_j := \mu(\mathcal{E}_j/\mathcal{E}_{j-1}) \) if \( i > j \).
\end{enumerate}

Note that we have \( \mu_n \geq 0 \) due to the semi-positivity of \( f_* \omega_f \), and
\[
\chi_f = \sum_{i=1}^{n} r_i (\mu_i - \mu_{i+1}), \quad \text{where } r_i := \text{rank} \mathcal{E}_i \text{ and } \mu_{n+1} := 0. \tag{2-2}
\]

**Definition 2.1 (20).** Let \( \mathcal{E}' \) be a locally free subsheaf of \( f_* \omega_f \). The fixed and moving parts of \( \mathcal{E}' \), denoted by \( Z(\mathcal{E}') \) and \( M(\mathcal{E}') \) respectively, are defined as follows. Let \( \mathcal{L} \) be a sufficiently ample line bundle on \( B \) such that the sheaf \( \mathcal{E}' \otimes \mathcal{L} \) is generated by its global sections, and \( \Lambda(\mathcal{E}') \subseteq |\omega_f \otimes f^* \mathcal{L}| \) be the linear subsystem corresponding to sections in \( H^0(B, \mathcal{E}' \otimes \mathcal{L}) \). Then we define \( Z(\mathcal{E}') \) to be the fixed part of \( \Lambda(\mathcal{E}') \), and \( M(\mathcal{E}') = \omega_f - Z(\mathcal{E}') \). Note that the definitions do not depend on the choice of \( \mathcal{L} \).

Let \( d_i = M(\mathcal{E}_i) \cdot F \), where \( \mathcal{E}_i \subseteq f_* \omega_f \) is any subsheaf in the H-N filtration of \( f_* \omega_f \) in (2.1), and \( F \) is a general fiber of \( f \). The next proposition, which is due to Xiao, is crucial to the study of the slope of fibrations (cf. [3, 4, 11, 20]).
Proposition 2.2 (20). For any sequence of indices \(1 \leq i_1 < \cdots < i_k \leq n\), one has
\[
\omega_f^2 \geq \sum_{j=1}^{k} (d_{i_j} + d_{i_{j+1}})(\mu_{i_j} - \mu_{i_{j+1}}), \quad \text{where } i_{k+1} = n + 1. \tag{2-3}
\]

2.2. Second multiplication map. In this subsection, we derive some inequalities based on the second multiplication map
\[\varrho : S^2(f_*\omega_f) \longrightarrow f_*(\omega_f^{\otimes 2}),\]
where \(S^2(f_*\omega_f)\) is the second symmetric power of \(f_*\omega_f\), and the map \(\varrho\) is induced by the canonical multiplication on the general fibers of \(f\). We assume in the subsection that \(f\) is non-hyperelliptic. Under this assumption, it is known that \(\varrho\) is generically surjective. Thus by studying the image of the map \(\varrho\), one may obtain a lower bound \(\deg f_*(\omega_f^{\otimes 2})\), hence also a lower bound of \(\omega_f^2\) due to the following simple fact:
\[
\deg f_*(\omega_f^{\otimes 2}) = \omega_f^2 + \chi_f. \tag{2-4}
\]

For any locally free sheaf \(E\) over \(B\), we define
\[\mu_f(E) = \max\{\deg F \mid E \otimes F' \text{ is semi-positive}\}.
\]

For those \(E_i\)'s occurring in the H-N filtration \([21]\) of \(f_*\omega_f\), it is easy to see that \(\mu_f(E_i) = \mu_i\). In particular, \(\mu_f(f_*\omega_f) = \mu_n\).

Proposition 2.3. Let \(\bar{\mu}_1 > \cdots > \bar{\mu}_k \geq 0\) (resp. \(0 < \bar{\mu}_1 < \cdots < \bar{\mu}_k \leq 3g - 3\)) be any decreasing (resp. increasing) sequence of rational (resp. integer) numbers. Assume that there exists a subsheaf \(F_i \subset f_*(\omega_f^{\otimes 2})\) such that \(\mu_f(F_i) \geq \bar{\mu}_i\) and \(\text{rank } F_i \geq \bar{\mu}_i\) for each \(i\). Then
\[
\omega_f^2 + \chi_f \geq \sum_{i=1}^{k} \bar{\mu}_i(\bar{\mu}_i - \bar{\mu}_{i+1}), \quad \text{where } \bar{\mu}_{k+1} = 0. \tag{2-5}
\]

Proof. Consider the H-N filtration of \(E' = f_*(\omega_f^{\otimes 2})\):
\[0 = E'_0 \subset E'_1 \subset \cdots \subset E'_m = E'.\]

Let \(\mu'_i = \mu(E'_i/E'_{i-1})\) and \(r'_i = \text{rank } E'_i\) for \(1 \leq i \leq m\), and \(\mu'_m = 0\). Due to the semi-positivity of \(f_*(\omega_f^{\otimes 2})\), one has \(\mu'_i \geq 0\). Hence by \([24]\), one has
\[
\omega_f^2 + \chi_f = \deg f_*(\omega_f^{\otimes 2}) = \sum_{i=1}^{m} r'_i(\mu'_i - \mu'_{i+1}).
\]

If we view \((r'_i, \mu'_i)\)'s as points in the two-dimensional coordinate system, then it is easy to see that \(\deg f_*(\omega_f^{\otimes 2})\) is nothing but the area of the shadow in the following picture.
If we also view \((\tilde{r}_i, \tilde{\mu}_i)\)'s as points in the above coordinate system, the assumptions in our lemma imply that every point \((\tilde{r}_i, \tilde{\mu}_i)\) lies in the shadow part. Hence (2-5) follows immediately.

The next lemma provides subsheaves of \(\mathcal{F}_i \omega^2_f\) with 'large' slopes.

**Lemma 2.4.** Consider the H-N filtration of \(\mathcal{F}_i \omega^2_f\) in (2-1). Let \(\mu_i = \mu_i^i = \mu_i^i(\mathcal{E}_i, \mathcal{E}_i)\) the linear subsystem defined in [Definition 2.1] and \(g_0\) the geometric genus of the image of \(F\) under the map defined by \(\Lambda(E_i)\), where \(F\) is a general fiber of \(f\). Then there exists a subsheaf \(\mathcal{F}_i \subseteq \mathcal{F}_i \omega^2_f\) such that

\[
\mu_i(\mathcal{F}_i) \geq 2\mu_i, \quad \text{rank} \mathcal{F}_i \geq \min \{3(r_i - 1), 2r_i + g_0 - 1\} \geq 2r_i - 1.
\]

In particular, if \(\Lambda(E_i)\) is a birational map for a general fiber \(F\) of \(f\), then there exists a subsheaf \(\mathcal{F}_i \subseteq \mathcal{F}_i \omega^2_f\) such that

\[
\mu_i(\mathcal{F}_i) \geq 2\mu_i, \quad \text{rank} \mathcal{F}_i \geq 3(r_i - 1).
\]

**Proof.** Consider the composition map

\[
\varphi_i : \mathcal{E}_i \otimes \mathcal{E}_i \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_i \omega^2_f.
\]

It is clear that \(\mu_i(\text{Im}(\varphi_i)) \geq 2\mu_i(\mathcal{E}_i) = 2\mu_i\). Hence it suffices to show that

\[
\text{rank}(\text{Im}(\varphi_i)) \geq \min \{3(r_i - 1), 2r_i + g_0 - 1\}. \tag{2-6}
\]

Let \(Z_i\) be the normalization of the image of \(F\) under the map defined by \(\Lambda(E_i)\), \(D_i = M(E_i)|_F\), and \(V_i \subseteq H^0(F, D_i)\) the subspace corresponding to \(\Lambda(E_i)|_F\). Then by assumption, there exists a divisor \(D_i'\) on \(Z_i\), and a subspace \(V_i' \subseteq H^0(Z_i, D_i')\) such that \(D_i = \psi_i^* D_i'\) and \(V_i = \psi_i^* V_i'\), where \(\psi_i : F \rightarrow Z_i\) is the corresponding map. Moreover, \(V_i'\) is base-point-free and defines a birational map on \(Z_i\). Consider the natural maps

\[
\rho_i : V_i \otimes V_i \rightarrow H^0(F, 2D_i), \quad \rho_i' : V_i' \otimes V_i' \rightarrow H^0(Z_i, 2D_i').
\]

Then \(\dim V_i = \dim V_i' = r_i\) and

\[
\dim \text{Im}(\rho_i') = \dim \text{Im}(\rho_i) = \text{rank}(\text{Im}(\varphi_i)). \tag{2-7}
\]

Therefore (2-6) follows from the next lemma. The proof is complete.

**Lemma 2.5.** Let \(D \in \text{Pic}(Z)\) be an effective divisor of a smooth curve \(Z\) of genus \(g_0\), \(V \subseteq H^0(Z, D)\) be a subspace with \(\dim V = r\), and

\[
\rho : V \otimes V \rightarrow H^0(Z, 2D)
\]

be the natural multiplication maps. Assume that \(V\) is base-point-free and \(V\) induces a birational map \(\phi_V\) on \(Z\). Then

\[
\text{dim} \text{Im}(\rho) \geq \min \{3(r - 1), 2r + g_0 - 1\}. \tag{2-8}
\]

**Proof.** Since \(\phi_V\) is birational, according to the general position theorem (cf. [II §III.1]), there exist \(r\) points \(\{p_1, \ldots, p_r\} \subseteq F\) such that any \(r - 1\) of them give linearly independent conditions for the vector space \(V\). Hence there exist \(\{v_1, \ldots, v_r\} \subseteq V\) such that

\[
v_i(p_i) \neq 0, \quad \text{and} \quad v_i(p_j) = 0, \quad \forall 1 \leq j \leq r \text{ and } j \neq i.
\]
Since \( \{p_1, \cdots, p_r\} \subseteq F \) are in general position, one has
\[
\dim H^0(Z, p_3 + \cdots + p_r) = \begin{cases} 1, & \text{if } r \leq g_0 + 1; \\ r - 1 - g_0, & \text{if } r > g_0 + 1. \end{cases}
\]

Let \( V_{12} \subseteq V \) be generated by \( v_1 \) and \( v_2 \). Consider the subspace \( W \subseteq \text{Im}(\varphi) \) generated by \( v_2^3, \cdots, v_r^2 \), and the restriction map
\[
\varphi : V_{12} \otimes V \rightarrow H^0(Z, 2D).
\]

According to the base-point-free pencil trick (cf. [1, § III.3]), one has
\[
\dim \text{Im}(\varphi_i) = 2r - \dim H^0(Z, p_3 + \cdots + p_r) = \begin{cases} 2r - 1, & \text{if } r \leq g_0 + 1; \\ r + g_0 + 1, & \text{if } r > g_0 + 1. \end{cases}
\]

By valuating at the points \( \{p_3, \cdots, p_r\} \), we see that
\[
\dim W = r - 2, \quad W \cap \text{Im}(\varphi) = 0.
\]

Hence
\[
\dim (\text{Im}(\varphi)) \geq \dim W + \dim (\text{Im}(\varphi)) = \begin{cases} 3r - 3, & \text{if } r \leq g_0 + 1; \\ 2r + g_0 - 1, & \text{if } r > g_0 + 1. \end{cases}
\]

Therefore (2-8) follows.\( \square \)

3. Slope of fibrations

In this section, we prove certain lower bounds on the slope of fibrations, based on the combination of Xiao’s method and the second multiplication map. To illustrate this new idea, we first prove the following lower bound for non-triple and non-double cover fibrations.

**Theorem 3.1.** Let \( f : X \to B \) be a fibration of genus \( g \geq 2 \), which is not locally trivial.

(i). If \( f \) is neither a triple nor a double cover fibration, then
\[
\lambda_f > \frac{14(g - 1)}{3(g + 1)}.
\]

(ii). If \( f \) is not a double cover fibration, then
\[
\lambda_f > \frac{18(g - 1)}{4g + 3}.
\]

**Proof.** (i). Consider the H-N filtration \( f_*\omega_f \) of \( f_*\omega_f \). Let \( \Lambda(\mathcal{E}_i) \) and \( M(\mathcal{E}_i) \) be defined in [Definition 2.1]. By assumption, \( f \) is not a double cover fibration. In particular, it is non-hyperelliptic, which implies that \( \Lambda(\mathcal{E}_n)|_F \) defines a birational map for a general fiber \( F \). Define
\[
l = \min \{i \mid \Lambda(\mathcal{E}_i)|_F \text{ defines a birational map for a general fiber } F \text{ of } f\}.
\]

According to [Proposition 2.2] one gets
\[
\omega_f^2 \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}).
\]

By [Proposition 2.3] and [Lemma 2.4] with the decreasing sequence
\[
\{2\mu_1, \cdots, 2\mu_{t-1}, 2\mu_t, \cdots, 2\mu_n\},
\]
and the increasing sequence
\[ \{2r_1 - 1, \cdots, 2r_{l-1} - 1, 3(r_1 - 1), \cdots, 3(r_n - 1)\}, \]
we obtain
\[ \omega_f^2 \geq \sum_{i=1}^{l-1} (3r_i - 2)(\mu_i - \mu_{i+1}) + \sum_{i=1}^{n} (5r_i - 6)(\mu_i - \mu_{i+1}). \quad (3-5) \]

Note that we use (2-2) above. Combining (3-5) with (3-4), we obtain
\[ \omega_f^2 \geq \sum_{i=1}^{l-1} \left( 2 \left( 3r_i - 2 \right) + \frac{1}{3}(d_i + d_{i+1}) \right)(\mu_i - \mu_{i+1}) + \sum_{i=1}^{n} \left( \frac{2}{3}(5r_i - 6) + \frac{1}{3}(d_i + d_{i+1}) \right)(\mu_i - \mu_{i+1}). \quad (3-6) \]

As \( f \) is neither a triple nor a double cover fibration, by the Castelnuovo’s Bound (cf. [11 §III.2]), we have
\[
\begin{cases}
  d_i \geq 4(r_i - 1), & \forall i < l;
  d_i \geq \frac{g}{m_i} + \frac{(m_i + 1)s_i}{2} - m_i, & \forall i \geq l, \\
\end{cases} \quad (3-7)
\]
where \( s_i = h^0(F, M(E_i)|_F) \geq r_i \) and \( m_i = \left\lceil \frac{d_i-1}{s_i-2} \right\rceil \). Hence it is easy to check that
\[
\begin{align*}
  & \frac{2}{3}(3r_i - 2) + \frac{1}{3}(d_i + d_{i+1}) \geq \frac{14}{3}r_i - \frac{8}{3}, & \forall i < l; \\
  & \frac{2}{3}(5r_i - 6) + \frac{1}{3}(d_i + d_{i+1}) \geq \frac{14}{3}r_i - \frac{11}{3}, & \forall l \leq i < n & r_i \neq g - 1; \\
  & \frac{2}{3}(5r_{n-1} - 6) + \frac{1}{3}(d_{n-1} + d_n) = \frac{14}{3}r_{n-1} - \frac{13}{3}, & \text{if } r_{n-1} = g - 1; \\
  & \frac{2}{3}(5r_n - 6) + \frac{1}{3}(d_n + d_{n+1}) = \frac{14}{3}g - \frac{16}{3}. \\
\end{align*}
\]

Combining the above inequalities with (3-6), we obtain
\[
\omega_f^2 \geq \begin{cases}
  \frac{14}{3} \chi_f - \frac{11}{3} \mu_1 - \frac{2}{3} \mu_{n-1} - \mu_n, & \text{if } r_{n-1} = g - 1; \\
  \frac{14}{3} \chi_f - \frac{11}{3} \mu_1 - \frac{5}{3} \mu_n, & \text{if } r_{n-1} \neq g - 1. \\
\end{cases} \quad (3-8)
\]

We use the simple fact that \( \mu_1 \geq \mu_l \) above. On the other hand, by Proposition 2.2 one obtains
\[
\begin{align*}
\omega_f^2 & \geq (d_1 + d_n)(\mu_1 - \mu_n) + (d_n + d_{n+1})\mu_n \\
& \geq (2g - 2)(\mu_1 - \mu_n) + (4g - 4)\mu_n = (2g - 2)(\mu_1 + \mu_n); \quad (3-9) \\
\omega_f^2 & \geq (2g - 3)\mu_1 + (2g - 2)\mu_{n-1}, \quad \text{if } r_{n-1} = g - 1.
\end{align*}
\]

Hence (3-1) follows immediately from (3-8) and (3-9).
(ii). First of all, according to [11], we may assume that $g \geq 6$. From (3-4) and (3-5) it follows that

$$\omega_f^2 \geq \sum_{i=1}^{i-l-1} \left( \frac{1}{2} (3r_i - 2) + \frac{1}{2} (d_i + d_{i+1}) \right) (\mu_i - \mu_{i+1})$$
$$+ \sum_{i=l}^{n} \left( \frac{1}{2} (5r_i - 6) + \frac{1}{2} (d_i + d_{i+1}) \right) (\mu_i - \mu_{i+1}). \tag{3-10}$$

We claim that

$$\begin{cases}
\frac{1}{2} (3r_i - 2) + \frac{1}{2} (d_i + d_{i+1}) > \frac{9}{2} r_i - 2, & \forall i < l; \\
\frac{1}{2} (5r_i - 6) + \frac{1}{2} (d_i + d_{i+1}) \geq \frac{9}{2} r_i - \frac{7}{2}, & \forall l \leq i < n; \\
\frac{1}{2} (5r_n - 6) + \frac{1}{2} (d_n + d_{n+1}) = \frac{9}{2} g - 5.
\end{cases} \tag{3-11}$$

Assume that (3-11) is true. Then (3-2) follows easily from (3-11) together with (3-9) and (3-10). Hence it suffices to show (3-11).

If $\Lambda(E_i) |_F : F \rightarrow \Gamma_i$ is a finite map of degree at least 4 for any $i < l$, then we have (3-7), from which (3-11) follows immediately. Let

$$\bar{l} = \min \left\{ i \mid \Lambda(E_i) |_F \text{ is a finite map of degree 3 for a general fiber } F \right\}. \tag{3-12}$$

By assumption, such an $\bar{l}$ exists. Let $g_{\bar{l}}$ be the genus of the image $F$ under the map defined by $\Lambda(E_i) |_F$. Then we may assume that $g_{\bar{l}} \geq 1$: otherwise, $f$ is a trigonal fibration, in which case (3-2) follows from the result of [12]. Hence by the Castelnuovo’s Bound (cf. [1, §III.2]), we have

$$\begin{cases}
d_i \geq 6(r_i - 1), & \forall i < \bar{l}; \\
d_i \geq 3r_i, & \forall \bar{l} \leq i < l; \\
d_i \geq \frac{g}{m_i} + \frac{(m_i + 1)s_i}{2} - m_i, & \forall i \geq \bar{l},
\end{cases}$$

where $s_i = h^0 (F, M(E_i) |_F) \geq r_i$ and $m_i = \left\lceil \frac{d - 1}{n_i - 2} \right\rceil$. Note that the first inequality above follows from the fact that the map defined by $\Lambda(E_i) |_F$ factors through that by $\Lambda(E_j) |_F$ for $i < j$. By computation, we obtain (3-11) from the above inequalities. The proof is complete. \qed

**Remark 3.2.** Since the double covers are well-studied in the surface theory (cf. [5]), the above theorem turns to be useful if one can construct a fibration with $\lambda_f < \frac{18(g - 1)}{4g + 3}$. For instance, in [10] we apply the above theorem to study the geography of irregular surfaces of general type and maximal Albanese dimension. For fibrations with double cover fibration structure, we also have the following conditional result.

**Proposition 3.3.** Let $f : X \rightarrow B$ be a fibration of genus $g \geq 2$, which is not locally trivial. If $\gamma \geq g/3$ for any double cover fibration structure of type $(g, \gamma)$ on $f$, then

$$\lambda_f > \frac{18(g - 1)}{4g + 3}, \tag{3-13}$$
Proof. Let
\[ l' = \min \{ i \mid \Lambda(E_i)|_F \text{ is a finite map of degree 2 for a general fiber } F \} . \] (3-14)
According to the proof of Proposition 2.3 and Lemma 2.4, we may assume that such an \( l' \) exists. Note that the map defined by \( \Lambda(E_i)|_F \) factors through that by \( \Lambda(E_j)|_F \) for \( i < j \). Hence
\[ \deg (\Lambda(E_i)|_F) = 2, \quad \forall \ l' < i < l; \quad \deg (\Lambda(E_i)|_F) \geq 4, \quad \forall \ i < l', \] (3-15)
where \( l \) is defined in (3-3). Let \( \gamma \) be the geometric genus of the image of \( F \) under \( \Lambda(E_i)|_F \), and
\[ \theta_i = \min \{ 3r_i - 3, 2r_i + \gamma - 1 \}, \quad \forall \ l' \leq i \leq l - 1. \]
By Proposition 2.3 and Lemma 2.4, we have the decreasing sequence
\[ \{ 2\mu_1, \ldots, 2\mu_n \}, \]
and the increasing sequence
\[ \{ 2r_1 - 1, \ldots, 2r_{l'-1} - 1, \theta_{l'}, \ldots, \theta_{l-1}, 3(r_l - 1), \ldots, 3(r_n - 1) \}, \]
we obtain
\[ \omega_f^2 \geq \sum_{i=1}^{l'-1} (3r_i - 2)(\mu_i - \mu_{i+1}) + \sum_{i=l'}^{l-1} (2\theta_i - r_i)(\mu_i - \mu_{i+1}) \]
\[ + \sum_{i=l}^{n} (5r_i - 6)(\mu_i - \mu_{i+1}). \] (3-16)
Combining (3-16) with (3-4), we obtain
\[ \omega_f^2 \geq \sum_{i=1}^{l'-1} \left( \frac{1}{2}(3r_i - 2) + \frac{1}{2}(d_i + d_{i+1}) \right) (\mu_i - \mu_{i+1}) \]
\[ + \sum_{i=l'}^{l-1} \left( \frac{1}{2}(2\theta_i - r_i) + \frac{1}{2}(d_i + d_{i+1}) \right) (\mu_i - \mu_{i+1}) \]
\[ + \sum_{i=l}^{n} \left( \frac{1}{2}(5r_i - 6) + \frac{1}{2}(d_i + d_{i+1}) \right) (\mu_i - \mu_{i+1}). \] (3-17)
By (3-17), one has
\[ d_i \geq 4(r_i - 1), \quad \forall \ i < l'; \quad d_i \geq 2 \min \{ 2(r_i - 1), r_i + \gamma - 1 \}, \quad \forall \ l' \leq i < l. \]
Combining this with the assumption \( \gamma \geq g/3 \) and the Castelnuovo’s Bound (cf. [1], §III.2), we have
\[
\begin{cases}
\frac{1}{2}(3r_i - 2) + \frac{1}{2}(d_i + d_{i+1}) \geq \frac{9}{2} r_i - 2, & \forall \ i < l' ; \\
\frac{1}{2}(2\theta_i - r_i) + \frac{1}{2}(d_i + d_{i+1}) \geq \frac{9}{2} r_i - 2, & \forall \ l' \leq i < l ; \\
\frac{1}{2}(5r_i - 6) + \frac{1}{2}(d_i + d_{i+1}) \geq \frac{9}{2} r_i - \frac{7}{2}, & \forall \ l \leq i < n ; \\
\frac{1}{2}(5r_n - 6) + \frac{1}{2}(d_n + d_{n+1}) = \frac{9}{2} g - 5 .
\end{cases}
\] (3-18)
Hence (3-13) follows from the above inequalities together with (3-17) and (3-9). \( \Box \)
Proof of Theorem 1.3(i). According to Theorem 1.1 and [20] Theorem 3, one may assume that $q_f \geq 2$, which implies that $g \geq 9q_f \geq 18$ by assumption.

Consider first the case when $f$ is a double cover fibration of type $(g, \gamma)$ with $g \geq 3\gamma + 1$:

- If $g \geq 4\gamma + 1$, then according to Theorem 4.7 we may assume that $g > 9\gamma$ and $q_f > \gamma$. Hence $f$ is an irregular double cover (cf. (4-5)), and $g \geq 6\gamma + 7$ since $g \geq \min\{18, 9\gamma + 1\}$. Therefore (3-13) follows from (4-20).

- If $4\gamma + 1 > g \geq 3\gamma + 1$, then (1-3) follows from (4-22), since in this case
\[
\frac{4(g-1)(3g+1-4\gamma)}{(g+1-2\gamma)^2+4\gamma(2g+1-3\gamma)} > \frac{9(g-1)}{2g} \geq \frac{4(g-1)}{g-q_f}.
\]

Hence we may assume that $\gamma \geq g/3$ for any double cover fibration structure of type $(g, \gamma)$ on $f$. Consider the H-N filtration of $f_\omega f$ as in (2-1). According to [9, Theorem 3.1], one sees that $\mu_n = 0$ and $r_{n-1} \leq g-q_f \leq g-2$. Let $l$ be defined as in (3-3). Then according to the Castelnuovo’s Bound (cf. [11 §III.2]), we have
\[
\frac{1}{2}(5r_i - 6) + \frac{1}{2}(d_i + d_{i+1}) \geq \min\left\{\frac{9}{2}r_i - 2, \frac{8r_i + g - 8}{2}\right\}, \quad \forall \; l \leq i < n - 1,
\]
\[
\frac{1}{2}(5r_{n-1} - 6) + \frac{1}{2}(d_{n-1} + d_n) \geq \min\left\{\frac{9}{2}r_i - 2, \frac{13r_i + 5g - 20}{4}\right\},
\]
Hence if $r_{n-1} \leq g - 3$, then
\[
\frac{1}{2}(5r_i - 6) + \frac{1}{2}(d_i + d_{i+1}) \geq \frac{9}{2}r_i - 2, \quad \forall \; l \leq i \leq n - 1. \tag{3-19}
\]

Therefore, under the assumption that $r_{n-1} \leq g - 3$, we obtain from (3-10), (3-11), (3-17) and (3-18) that
\[
\omega_f^2 \geq \frac{9}{2} \lambda_f - 2\mu_1.
\]
Combining this with (3-9), we obtain that
\[
\lambda_f \geq \frac{9(g-1)}{2g} \geq \frac{4(g-1)}{g-q_f}. \tag{3-20}
\]

Finally, we consider the case when $r_{n-1} = g - 2$. By assumption, $f$ is not hyperelliptic, hence $d_{n-1} \geq 2g - 4 = 2r_{n-1}$. Thus one checks that we still have (3-19). Therefore, similar as above, (3-20) holds. The proof is complete. \(\square\)

Remarks 3.4. (i) One sees from the above proof that the condition $q_f \leq g/9$ might be relaxed a little. But the proof requires a much more complicated computation.

(ii) We only deal with the case when $q_f$ is small with respect to $g$. If $q_f$ is big, we refer to [4, Theorem 3.2] for the case when $f$ is not a double cover fibration.

4. Double cover fibrations

In this section, we treat the double cover fibrations. So we always assume in the section that $f : X \to B$ is a locally non-trivial double cover fibration of type $(g, \gamma)$ as in [Definition 1.4]. Since the case where $\gamma = 0$ has been studied in [22, 14] (see also [7, 15] for the semi-stable case), $\gamma$ is assumed to be positive in the section unless other explicit statements.
4.1. **Invariants of double cover fibrations.** In this subsection, we first define the local invariants of the induced double cover, and then show in [Theorem 4.3]{#11} that the relative invariants of $f$ can be expressed by these local invariants and relative invariants of the quotient fibration.

The degree-two morphism $\pi$ in [Definition 1.3]{#11} induces an involution $\sigma$ on $X$. Let $\vartheta : \tilde{X} \to X$ be the composition of all the blowing-ups of the isolated fixed points of $\sigma$, and $\bar{\sigma}$ the induced involution on $\tilde{X}$. Then the quotient $\tilde{Y} := \tilde{X}/(\bar{\sigma})$ is a smooth surface with a natural fibration $h : \tilde{Y} \to B$ of genus $\gamma$, which may not be relatively minimal. Let $h : Y \to B$ be its relatively minimal model.

\[
\begin{array}{c}
X \xrightarrow{\vartheta} \tilde{X} \xrightarrow{\tilde{\pi}} \tilde{Y} \xrightarrow{\psi} Y \\
\downarrow f \quad \downarrow \tilde{f} \quad \downarrow \tilde{h} \quad \downarrow h \\
\uparrow B \quad \uparrow Y
\end{array}
\]

The double cover $\tilde{\pi}$ induces a double cover $\pi_0 : X_0 \to Y_0 := Y$, which is determined by the relation $\mathcal{O}_Y(R) = L^{\otimes 2}$ with $R = \psi(\tilde{R})$ and $\tilde{R}$ being the branch locus of $\tilde{\pi}$. According to Hurwitz formula, one has

\[
R \cdot \Gamma = 2g + 2 - 4\gamma \geq 0, \text{ for any fiber } \Gamma \text{ of } h. \tag{4-1}
\]

The surface $X_0$ is normal but not necessarily smooth. Moreover, $\tilde{\pi}$ is in fact the canonical resolution of $\pi_0$ (cf. [5, §III.7]):

\[
\begin{array}{c}
\tilde{X} \\
\Phi = \pi_i \\
\tilde{Y}
\end{array}
\xrightarrow{\phi_t} \begin{array}{c}X_i \xrightarrow{\phi_{i-1}} \cdots \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1} X_0 \\
Y_i \xrightarrow{\psi_{i-1}} \cdots \xrightarrow{\psi_2} Y_1 \xrightarrow{\psi_1} Y_0
\end{array}
\]

**Figure 1.** Canonical resolution.

Here $\psi_i$'s are successive blowing-ups resolving the singularities of $R$, and $\pi_i : X_i \to Y_i$ is the double cover determined by $\mathcal{O}_{Y_i}(R_i) = L_i^{\otimes 2}$ with

\[
R_i = \psi_i^*(R_{i-1}) - 2[m_{i-1}/2]E_i, \quad L_i = \psi_i^*(L_{i-1}) \otimes \mathcal{O}_{Y_i}\left(E_i^{-[m_{i-1}/2]}\right),
\]

where $E_i$ is the exceptional divisor of $\psi_i$, $m_{i-1}$ is the multiplicity of the singular point $y_{i-1}$ in $R_{i-1}$ (also called the multiplicity of the blowing-up $\psi_i$), $[\ ]$ stands for the integral part, $R_0 = R$ and $L_0 = L$. A singularity $y_j \in R_j \subseteq Y_j$ is said to be infinitely closed to $y_i \in R_i \subseteq Y_i$ ($j > i$), if $\psi_{i+1} \circ \cdots \circ \psi_j(y_j) = y_i$.

We remark that the order of these blowing-ups contained in $\psi$ is not unique. If $y_{i-1}$ is a singular point of $R_{i-1}$ of odd multiplicity $2k + 1$ ($k \geq 1$) and there is a unique singular point $y$ of $R_i$ on the exceptional curve $E_i$ of multiplicity $2k + 2$, then we always assume that $\psi_{i+1} : Y_{i+1} \to Y_i$ is a blowing-up at $y_i = y$. We call such a pair $(y_{i-1}, y_i)$ a **singularity of $R$ of type** $(2k + 1 \to 2k + 1)$, and $y_{i-1}$ (resp. $y_i$) the first (resp. second) component.

**Definition 4.1.** For any singular fiber $F$ of $f$ and $j \geq 2$, we define
• if \( j \) is odd, \( s_j(F) \) equals the number of \((j \to j)\) type singularities of \( R \) over the image \( f(F) \);
• if \( j \) is even, \( s_j(F) \) equals the number of singularities of multiplicity \( j \) or \( j + 1 \) of \( R \) over the image \( f(F) \), neither belonging to the second component of type \((j - 1 \to j)\) singularities nor to the first component of type \((j + 1 \to j + 1)\) singularities.

Let \( \omega_R = \omega_Y \otimes \hat{h}^*\omega_{\tilde{R}}^{-1} \) and \( \tilde{R}' = \tilde{R} \setminus \tilde{V} \), where \( \tilde{V} \) is the union of vertical isolated \((-2)\)-curves in \( \tilde{R} \). Here a curve \( C \subseteq \tilde{R} \) is called to be isolated in \( \tilde{R} \), if there is no other curve \( C' \subseteq \tilde{R} \) such that \( C \cap C' \neq \emptyset \). We define

\[
\begin{align*}
s_2 &:= (\omega_R + \tilde{R}') \cdot \tilde{R}' + 2 \sum_{F \text{ is singular}} s_2(F), \\
s_j &:= \sum_{F \text{ is singular}} s_j(F), \quad \forall \ j \geq 3.
\end{align*}
\]

Note that the contraction \( \psi \) is unique since \( \gamma > 0 \) (although the order of these blowing-ups contained in \( \psi \) is not unique). Hence the invariants \( s_j \)’s are well-defined. By definition, \( s_j \) is non-negative for \( j \geq 3 \), but it is not clear whether \( s_2 \) is non-negative or not.

**Lemma 4.2.** Let \( F \) be a singular fiber of the fibration \( f \), and \( \tilde{F} \) (resp. \( \tilde{\Gamma} \), resp. \( \Gamma \)) the corresponding fiber in \( \tilde{X} \) (resp. \( \tilde{Y} \), resp. \( Y \)). Then the \((-1)\)-curves in \( \tilde{F} \) are in one-to-one correspondence to the isolated \((-2)\)-curves of \( \tilde{R} \), which are also contained in \( \tilde{\Gamma} \). And the number of these \((-1)\)-curves is equal to

\[
n_2(F) + \sum_{k \geq 1} s_{2k+1}(F),
\]

where \( n_2(F) \) is the number of isolated \((-2)\)-curves of \( R \), which are also contained in \( \Gamma \).

**Proof.** Note that the \((-1)\)-curves in \( \tilde{F} \) are exactly the inverse image of the isolated fixed points of \( \sigma \) on \( F \), hence fixed by \( \tilde{\sigma} \). It follows that these \((-1)\)-curves in \( \tilde{F} \) are in one-to-one correspondence to the isolated \((-2)\)-curves of \( \tilde{R} \), which are also contained in \( \tilde{\Gamma} \).

Let \( E \) be such a \((-2)\)-curve of \( \tilde{R} \). Then it is the strict inverse image of either an exceptional curve \( \mathcal{E}_i \) or an irreducible curve \( C \) on \( \Gamma \). In the first case, it is easy to see that \( y_{i-1} = \psi_i(\mathcal{E}_i) \) is a singularity of \( R_{i-1} \) with odd multiplicity \( 2k + 1 \), and that \( R_i \) has a unique singularity on \( \mathcal{E}_i \) with multiplicity \( 2k + 2 \). Equivalently, it corresponds to a singularity of \( R \) of type \((2k + 1 \to 2k + 1)\). In the later case, let

\[
E = \psi^*(C) - \sum a_j \mathcal{E}_j, \quad \text{with } a_j \geq 0.
\]

Then

\[
-2 = E^2 = C^2 - \sum a_j^2, \quad 0 = \omega_Y \cdot E = \omega_Y \cdot C + \sum a_j.
\]

On the other hand, one has \( C^2 \leq 0 \) and \( C^2 = 0 \) if and only if \( \Gamma = nC \) for some \( n \), since \( C \subseteq \Gamma \). Hence it follows that \( C^2 \neq 0 \) since \( \gamma > 0 \), and that \( C^2 \neq -1 \); otherwise by construction \( C \) must be smooth and hence is \((-1)\)-curve, which is impossible due to the relative minimality of \( h \). Therefore, \( C \) must be an isolated \((-2)\)-curve of \( R \), which is also contained in \( \Gamma \).
Conversely, it is clear that each singularity of $R$ of type $(2k+1 \to 2k+1)$ creates an isolated $(-2)$-curve contained in $\tilde{R}$, and that the inverse image of each isolated $(-2)$-curve in $R$ is still an isolated $(-2)$-curve in $\tilde{R}$. The proof is complete. □

**Theorem 4.3.** Let $f$ be a double cover fibration of type $(g, \gamma)$, and $s_i$’s the singularity indices as above. Then

\[
(2g + 1 - 3\gamma)\omega_f^2 = x \cdot \frac{\omega_h^2}{\gamma - 1} + yT + zs_2 + \sum_{k \geq 1} a_k s_{2k+1} + \sum_{k \geq 2} b_k s_{2k},
\]

\[
(2g + 1 - 3\gamma)\chi_f = \tilde{x} \cdot \frac{\omega_h^2}{\gamma - 1} + 2(2g + 1 - 3\gamma)\chi_h + \tilde{y}T
\]

\[+ \tilde{z}s_2 - \frac{2g + 1 - 3\gamma}{4} \cdot n_2 + \sum_{k \geq 1} \bar{a}_k s_{2k+1} + \sum_{k \geq 2} \bar{b}_k s_{2k},
\]

\[e_f = 2e_h + s_2 - 3n_2 + \sum_{k \geq 1} s_{2k+1} + \sum_{k \geq 2} 2s_{2k},
\]

where we set $\frac{\omega_h^2}{\gamma - 1} = 0$ if $\gamma = 1$, $n_2 = \sum_{F \text{ is singular}} n_2(F)$, and

\[x = \frac{(3g + 1 - 4\gamma)(g - 1)}{2}, \quad y = \frac{3}{2}, \quad z = g - 1;\]

\[\tilde{x} = \frac{(g + 1 - 2\gamma)^2}{8}, \quad \tilde{y} = \frac{1}{8}, \quad \tilde{z} = g - \gamma;\]

\[a_k = 12\bar{a}_k - (2g + 1 - 3\gamma), \quad b_k = 12\bar{b}_k - 2(2g + 1 - 3\gamma),\]

\[\bar{a}_k = k(g - 1 + (k - 1)(\gamma - 1)), \quad \bar{b}_k = k(g - 1 + (k - 2)(\gamma - 1))\]

\[T = -\left(\frac{(g + 1 - 2\gamma)\omega_h - (\gamma - 1)R}{\gamma - 1}\right)^2 - 2(\gamma - 1)n_2 \geq 0.\]

**Proof.** Recall the canonical resolution $\psi$ exhibited in Figure 1. By Lemma 4.2 one has

\[
\left(\omega_h + \tilde{R}'\right) \cdot \tilde{R}' - 2\left(n_2 + \sum_{k \geq 1} s_{2k+1}\right)
\]

\[= \left(\omega_h + \tilde{R}\right) \cdot \tilde{R} = (\omega_h + R) \cdot R - \sum_{i=1}^{l} \left(\left[m_i \right] - 1\right) \cdot \left[m_i \right]
\]

\[= (\omega_h + R) \cdot R - \sum_{k \geq 1} (8k^2 + 4k + 2)s_{2k+1} - \sum_{k \geq 2} (4k^2 - 2k)s_{2k} - 2 \sum_{F \text{ is singular}} s_2(F).
\]

Combining this with the definition of $s_2$, we get

\[
(\omega_h + R) \cdot R = s_2 - 2n_2 + \sum_{k \geq 1} 4k(2k + 1)s_{2k+1} + \sum_{k \geq 2} 2k(2k - 1)s_{2k}.
\]
Thus by the formulas for double covers (cf. [5, § V.22]), one obtains:

\[
\omega^2_f = 2 \left( \omega^2_h + \omega_h \cdot R + \frac{R^2}{4} \right) - 2 \sum_{k \geq 1} (2k^2 - 2k + 1)s_{2k+1} + \sum_{k \geq 2} (k-1)^2s_{2k}
\]

\[
\omega^2_f = x' \cdot \frac{\omega^2_h}{\gamma - 1} + y' (T + 2(\gamma - 1)n_2) + z' (\omega_h + R) \cdot R
\]

\[
-2 \left( \sum_{k \geq 1} (2k^2 - 2k + 1)s_{2k+1} + \sum_{k \geq 2} (k-1)^2s_{2k} \right),
\]

\[
\chi_f = 2\chi_h + \frac{1}{2} \left( \frac{\omega_h \cdot R}{2} + \frac{R^2}{4} \right) - \left( \sum_{k \geq 1} k^2s_{2k+1} + \sum_{k \geq 2} \frac{k(k-1)}{2}s_{2k} \right)
\]

\[
- \left( \sum_{k \geq 1} k^2s_{2k+1} + \sum_{k \geq 2} \frac{k(k-1)}{2}s_{2k} \right),
\]

where \(s' = \frac{s}{2r+1-3\gamma}\) for \(* = x, y, z, \bar{x}, \bar{y}\) or \(\bar{z}\). Note that \(\omega^2_f = \omega^2_f + n_2 + \sum_{k \geq 1} s_{2k+1}\) and \(\chi_f = \chi_f\) by Lemma 4.2. Therefore, the formulas in our theorem follow from the above equalities together with (1-2) and (1-2).

Note that \(T = 2(g - 1)\omega_h \cdot R \geq 0\) if \(\gamma = 1\). It remains to show that \(T \geq 0\) if \(\gamma > 1\). For this purpose, let \(V \subseteq R\) be these isolated \((-2)\)-curves contracted by \(h\), and \(R' = R \setminus V\). By Lemma 4.2, the number of components contained in \(V\) is \(n_2\).

Since \(\Gamma \cdot \left((g + 1 - 2\gamma)\omega_h - (\gamma - 1)R'\right) = 0\), one gets by Hodge index theorem that

\[
0 \geq (g + 1 - 2\gamma)\omega_h - (\gamma - 1)R' \geq (g + 1 - 2\gamma)\omega_h - (\gamma - 1)R' \geq 2(\gamma - 1)^2n_2.
\]

Hence \(T \geq 0\) as required.

4.2. Irregular double cover fibrations. In this subsection, we would like to prove the following restrictions on the invariants of irregular double cover fibrations. Here, the double cover fibration \(f\) of type \((g, \gamma)\) is called irregular if

\[
q_\tau := q(\bar{X}) - q(\bar{Y}) > 0,
\]

where \(\bar{X}\) and \(\bar{Y}\) are the same as in the last subsection.

**Proposition 4.4.** Let \(f : X \to B\) be a double cover fibration of type \((g, \gamma)\) with the double cover \(\pi\) as in [Definition 1.4]

(i) If the double cover \(\pi\) is irregular, i.e., \(q_\tau > 0\), then

\[
2(g + 1 - 2\gamma)s_2 
\]

\[
\leq (g + 1 - 2\gamma)^2 \cdot \frac{\omega^2_h}{\gamma - 1} + T + \sum_{k \geq 1} 2(4\alpha_k + 2g + 1 - 3\gamma)s_{2k+1} + \sum_{k \geq 2} 8\bar{b}_ks_{2k}.
\]
(ii) If the image \( J_0(\tilde{X}) \subseteq \text{Alb}_0(\tilde{X}) \) is a curve of geometric genus \( g' > 0 \), then

\[
2(g + 1 - 2\gamma) \left( s_2 + \sum_{k \geq 1}^{g'-1} 4(2k + 1)ks_{2k+1} + \sum_{k \geq 2}^{g'} 2(2k - 1)ks_{2k} \right) \tag{4-7}
\]

\[
\leq (g + 1 - 2\gamma)^2 \cdot \frac{\omega_2^2}{\gamma - 1} + T + \sum_{k \geq g'} 2(4a_k + 2g + 1 - 3\gamma)s_{2k+1} + \sum_{k \geq g'+1} 8b_k s_{2k};
\]

where \( a_k \)'s, \( b_k \)'s are defined in Theorem 4.3, and \( J_0 \) will be defined in (4-8).

The main tool to prove the above proposition is the usage of Albanese varieties. We first review the Albanese varieties and show that the ramified divisor is contracted by \( J_0 \). Then the proposition follows from the semi-negativity of the divisors contracted by some non-trivial map.

Let \( \tilde{R} = \tilde{\pi}^{-1}(\tilde{R}) \subseteq \tilde{X} \) the ramified divisor. Let \( \text{Alb}(\tilde{X}) \) (resp. \( \text{Alb}(\tilde{Y}) \)) be the Albanese variety of \( \tilde{X} \) (resp. \( \tilde{Y} \)), and \( \tau \) the generator of the Galois group \( \text{Gal}(\tilde{X}/\tilde{Y}) \cong \mathbb{Z}/2\mathbb{Z} \). Then we have a natural map \( \text{Alb}(\tilde{\pi}) : \text{Alb}(\tilde{X}) \to \text{Alb}(\tilde{Y}) \) and \( \tau \) has a natural action on \( \text{Alb}(\tilde{X}) \). Let \( \text{Alb}_0(\tilde{X}) \) be the kernel of the action of \( \tau \) on \( \text{Alb}(\tilde{X}) \). Then it is clear that \( \text{Alb}(\tilde{X}) \) is isogenous to \( \text{Alb}_0(\tilde{X}) \oplus \text{Alb}(\tilde{\pi})^{-1}(\text{Alb}(\tilde{Y})) \) and \( \dim \text{Alb}_0(\tilde{X}) = q_\pi \). Denote by

\[
J_0 : \tilde{X} \to \text{Alb}_0(\tilde{X}) \tag{4-8}
\]

the induced map.

**Lemma 4.5.** The ramified divisor \( \tilde{R} \) is contracted by the map \( J_0 \).

**Proof.** Let \( C \subseteq \tilde{R} \) be any irreducible component, \( \tilde{C} \) its normalization, \( j : \tilde{C} \to \tilde{X} \) the induced map and \( \varphi = J_0 \circ j : \tilde{C} \to \text{Alb}_0(\tilde{X}) \) the composition. We have to prove that \( \varphi(\tilde{C}) \) is a point.

We argue by contradiction. Assume that \( \varphi(\tilde{C}) \) is not a point. Then the induced map

\[
\varphi^* : H^0 \left( \text{Alb}_0(\tilde{X}), \Omega_{\text{Alb}_0(\tilde{X})}^1 \right) \longrightarrow H^0 \left( \tilde{C}, \Omega_{\tilde{C}}^1 \right)
\]

is non-zero. On the other hand, it is clear that \( \varphi^* \) factors through

\[
H^0 \left( \text{Alb}_0(\tilde{X}), \Omega_{\text{Alb}_0(\tilde{X})}^1 \right) \xrightarrow{J_0^*} H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right) \xrightarrow{j^*} H^0 \left( \tilde{C}, \Omega_{\tilde{C}}^1 \right).
\]

Note that the generator \( \tau \) of the Galois group \( \text{Gal}(\tilde{X}/\tilde{Y}) \) acts on \( H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right) \). Let

\[
H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right) \cong H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right)_{-1} \oplus H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right)_{1}
\]

be the eigenspace decomposition. Then by construction, the image of \( J_0^* \) is contained in \( H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right)_{-1} \). To deduce a contradiction, it suffices to prove that the restricted map

\[
j^* \big|_{H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right)_{-1}} : H^0 \left( \tilde{X}, \Omega_{\tilde{X}}^1 \right)_{-1} \longrightarrow H^0 \left( \tilde{C}, \Omega_{\tilde{C}}^1 \right)
\]

is zero.
Lemma 4.6. Hence if \( j \) in the canonical resolution in Figure 16, one has

\[ \omega = \alpha(x, y)dx + \beta(x, y)dy \in H^0 \left( \bar{X}, \Omega^1_X \right), \]

one has

\[ \omega \in H^0 \left( \bar{X}, \Omega^1_X \right) \iff \alpha(x, y) = y\tilde{\alpha}(x, y^2), \beta(x, y) = \tilde{\beta}(x, y^2). \]

Hence if \( \omega \in H^0 \left( \bar{X}, \Omega^1_X \right) \), one gets that \( j^*\omega|_{j^{-1}(p)} = 0 \), from which it follows that \( j^*\omega = 0 \) since \( p \) is arbitrary. The proof is complete.

Lemma 4.6. Let \( y_j \in R_j \subseteq Y_j \) be a singularity infinitely closed to \( y_i \in R_i \subseteq Y_i \) as in the canonical resolution in Figure 1. Then

\[ m_j \leq m_i, \quad \text{if } m_i \text{ is even}; \quad m_j \leq m_i + 1, \quad \text{if } m_i \text{ is odd.} \]

Proof. It suffices to consider the case where \( j = i + 1 \) and \( \psi_{i+1}(y_{i+1}) = y_i \). But this is clear because if \( m_i \) is even, then \( E_{i+1} \subseteq R_{i+1} \); and if \( m_i \) is odd, then \( E_{i+1} \subseteq R_{i+1} \).

Proof of Proposition 4.4. Recall that those blowing-ups \( \psi_i \)'s are contained in the canonical resolution \( \psi \). For convenience, we view \( \psi_i \circ \psi_{i+1} : Y_{i+1} \to Y_{i-1} \) as a single blowing-up (but with two exceptional curves) if

\[ Y_{i+1} \xrightarrow{\psi_{i+1}} Y_i \xrightarrow{\psi_i} Y_{i-1} \]

are blowing-ups of a type-\((2k + 1 \to 2k + 1)\) singularity. For a blowing-up \( \psi' \) contained in \( \psi \), the order of \( \psi' \) is defined to be \( k + 1 \) if \( \psi' \) is a blowing-up of a type-\((2k + 1 \to 2k + 1)\) singularity, and to be \([m'/2]\) if \( \psi' \) is a blowing-up of a singularity of the branch divisor with multiplicity \( m' \). Now we introduce a partial order on these blowing-ups contained in \( \psi \): we say \( \psi' \geq \psi'' \) if \( k' \geq k'' \), where \( k' \) (resp. \( k'' \)) is the order of \( \psi' \) (resp. \( \psi'' \)). According to Lemma 4.6 we can reorder these blowing-ups contained in \( \psi \) such that \( \psi_i \geq \psi_j \) if \( i < j \). Let \( M \) be the maximal order of these blowing-ups contained in \( \psi \). Then \( \psi \) can be decomposed as

\[ \hat{Y} = \cdots \xrightarrow{\psi} \hat{Y}_2 \xrightarrow{\psi_1} \hat{Y}_1 \xrightarrow{\psi} \hat{Y}_0 = Y \]

such that the order of each blowing-up contained in \( \psi_i \) is \( M + 1 - i \).

Consider any blowing-up \( \psi' \) contained in \( \psi_1 \). If it is a blowing-up of a type-\((2(M - i) + 1 \to 2(M - i) + 1)\) singularity, let \( E_1 \) and \( E_2 \) be the two exceptional curves. By construction, one of them, saying \( E_1 \) is contained in the branch divisor, hence its strict inverse image on \( \bar{X} \) is a rational curve; another one, saying \( E_2 \), is not contained in the branch divisor and intersects the branch divisor at most \( 2(M - i) + 2 \) points, hence the geometric genus of its strict inverse image on \( \bar{X} \) is at most \( M - i \) by Hurwitz formula (cf. 14 §IV.2). If \( \psi' \) is an ordinary blowing-up with one exceptional curve \( E \), then one can prove similarly that the geometric genus of its strict inverse image on \( \bar{X} \) is also at most \( M - i \). In any case, we obtain that the strict inverse image of any exceptional curve of \( \psi_i \) has geometric genus at most \( M - i \).
Consider first the case when $J_0(\tilde{X})$ is a curve of geometric genus $g' > 0$. In this case, any curve of geometric genus less than $g'$ is contracted by $J_0$. Hence combining this with the above arguments and Lemma 4.5, we conclude that the total inverse image of $\hat{R}_{M-g'}$ in $\tilde{X}$ is contracted by $J_0$, where $\hat{R}_{M-g'} \subseteq \hat{Y}_{M-g'}$ is the image of $\hat{R}$. In particular, the total inverse image of $\hat{R}_{M-g'}$ is semi-negative definite, which implies that $\hat{R}_{M-g'}$ is also semi-negative definite. By construction, each blowing-up contained in

$$\hat{\psi}_{M-g'+1} \circ \cdots \circ \hat{\psi}_M : \hat{Y} = \hat{Y}_M \to \hat{Y}_{M-g'}$$

has order less than or equal to $g'$. Thus there exist $n_2 + \sum_{k \geq g'} s_{2k+1}$ vertical isolated $(-2)$-curves contained in $\hat{R}_{M-g'}$ by Lemma 4.2, since the image of any isolated $(-2)$-curve contained in $\tilde{R}$ is still an isolated $(-2)$-curve contained in $\hat{R}_{M-g'}$. Therefore

$$\hat{R}_{M-g'}^2 \leq -2 \left( n_2 + \sum_{k \geq g'} s_{2k+1} \right). \quad (4-9)$$

By construction, we have

$$\hat{R}_{M-g'}^2 = R^2 - \left( \sum_{k \geq g'} 4(2k^2 + 2k + 1)s_{2k+1} + \sum_{k \geq g'+1} 4k^2 s_{2k} \right)$$

$$= \hat{x} \cdot \frac{\omega_h^2}{\gamma - 1} + \hat{y}(T + 2(\gamma - 1)n_2) + \hat{z}(\omega_h + R) \cdot R$$

$$- \left( \sum_{k \geq g'} 4(2k^2 + 2k + 1)s_{2k+1} + \sum_{k \geq g'+1} 4k^2 s_{2k} \right),$$

where

$$\hat{x} = \frac{(g+1-2\gamma)^2}{(2g+1-3\gamma)}, \quad \hat{y} = \frac{-1}{(2g+1-3\gamma)}, \quad \hat{z} = \frac{2g + 2 - 4\gamma}{2g + 1 - 3\gamma}.$$

Hence (4-9) follows from the above equation together with (4-2) and (4-9).

Finally, let’s consider the case when $q_\pi > 0$. In this case, $J_0(\tilde{X})$ is of positive dimension since $J_0(\tilde{X})$ generates $\text{Alb}_0(\tilde{X})$ by construction, and any rational curve in $\tilde{X}$ is contracted by $J_0$. Hence similarly as above, one sees that $\hat{R}_{M-1}$ is semi-negative definite and

$$\hat{R}_{M-1}^2 \leq -2 \left( n_2 + \sum_{k \geq 1} s_{2k+1} \right). \quad (4-10)$$

Therefore, (4-6) follows from a similar argument as above. \hfill \Box

4.3. **Slope of double cover fibrations.** In this subsection, we would like to consider the question on the lower bound of the slope for double cover fibrations. The main techniques are [Theorem 4.3] and [Proposition 4.4].

Based on [Theorem 4.3], we can reprove the following lower bound of the slope for a double cover fibration, which was proved earlier by Barja, Zucconi, Cornalba and Stoppino.
Theorem 4.7 ([1] Cor. 2.6 & [2] Thm. 2.1 & [8] Thm. 3.1). Let $f$ be a double cover fibration of type $(g, \gamma)$. If $h$ is locally trivial or $g \geq 4\gamma + 1$, then

$$\lambda_f \geq \frac{4(g-1)}{g-\gamma}.$$  \hfill (4-11)

Proof. By Theorem 4.3, for any $\lambda$, one has

$$(2g+1-3\gamma)(\omega_f^2 - \lambda \cdot \chi_f) = \left(\frac{(3g+1-4\gamma)(g-1)}{2} - \frac{(g+1-2\gamma)^2\lambda}{8}\right) \cdot \frac{\omega_h^2}{\gamma-1} - 2(2g+1-3\gamma)\lambda \cdot \chi_h + \frac{12-\lambda}{8} T + \frac{4(g-1) - (g-\gamma)\lambda}{4} \cdot s_2 + \frac{(2g+1-3\gamma)\lambda}{4} \cdot n_2$$

$$+ \sum_{k \geq 1} \left((12-\lambda)k((g-1)+(k-1)(\gamma-1)) - (2g+1-3\gamma)\right) \cdot s_{2k+1}$$

$$+ \sum_{k \geq 2} \left((12-\lambda)k((g-1)+(k-2)(\gamma-1)) - 2(2g+1-3\gamma)\right) \cdot s_{2k}.$$

Taking $\lambda = \frac{4(g-1)}{g-\gamma}$ in (4-12), it is easy to see that the coefficients of $n_2$ and $s_j$'s for $j \geq 3$ are all non-negative due to (4-11). Since $T$, $n_2$ and $s_j$'s for $j \geq 3$ are also all non-negative by definition, it follows from (4-12) that

$$\omega_f^2 - \frac{4(g-1)}{g-\gamma} \cdot \chi_f \geq \frac{1}{2(g-\gamma)} \left((g-1)^2 \cdot \frac{\omega_h^2}{\gamma-1} + T - 16(g-1) \cdot \chi_h\right).$$  \hfill (4-13)

If $h$ is locally trivial, then $\omega_f^2 \geq \chi_h = 0$ and $T \geq 0$, from which together with (4-11) the inequality (4-11) follows immediately.

If $g \geq 4\gamma + 1$ and $\gamma = 1$, then by [5] § V-Theorem 12.1, one has

$$\omega_h \sim_{(\text{numerically equivalent})} \left(\chi_h + \sum_{i=1}^{n} \frac{l_i-1}{l_i}\right) \Gamma,$$  \hfill (4-14)

where $\Gamma$ is a general fiber of $h$ and $\{\Gamma_i\}_{i=1, \ldots, n}$ are the union of multiple fibers of $h$ with multiplicities $\{l_i\}_{i=1, \ldots, n}$. Hence $T = 2(g-1)\omega_h \cdot R \geq 4(g-1)^2\chi_h$. Therefore, it follows from (4-13) that $\omega_f^2 - 4\gamma \chi_f \geq 2(g-5)\chi_h \geq 0$.

If $g \geq 4\gamma + 1$ and $\gamma > 1$, then one has $\omega_f^2 \geq \frac{4(g-1)}{\gamma} \cdot \chi_h \geq 0$ and $T \geq 0$. Hence by (4-13), we get

$$\omega_f^2 - \frac{4(g-1)}{g-\gamma} \cdot \chi_f \geq \frac{4(g-1)(g-4\gamma-1)}{2(g-\gamma)\gamma} \cdot \chi_h \geq 0 \quad \text{as required. \quad \Box}$$

When $f$ is an irregular double cover, we have the following better bounds, which is a generalization of [14] Theorem 1.4].

Theorem 4.8. Let $f$ be an irregular double cover fibration of type $(g, \gamma)$, and

$$F(g, \gamma, q_\gamma) = (g-1)^2 - 4(g-1)(\gamma q_\gamma + \gamma + q_\gamma) - 4q_\gamma^2(\gamma^2 - 1).$$  \hfill (4-15)

(i) If $h$ is locally trivial or $F(g, \gamma, 1) \geq 0$, then

$$\lambda_f \geq 6 + \frac{4(g-1)}{g-1}. \quad \hfill (4-16)$$
(ii) Assume moreover that \( J_0(\overline{X}) \) is a curve, where \( J_0 \) is defined in (4-8). If \( h \) is locally trivial or \( F(g, \gamma, q_\pi) \geq 0 \), then
\[
\lambda_f \geq \lambda_{g, \gamma, q_\pi} := 8 - \frac{4(g + 1 - 2\gamma)}{(q_\pi + 1)((g - 1) + (q_\pi - 1)(\gamma - 1))}. \tag{4-17}
\]

**Proof.** We only prove (ii) here, for the proof of (i) is completely the same except replacing the usage of (4-7) by (4-6) in the following.

Note that \( J_0(\overline{X}) \) generates \( \text{Alb}_0(\overline{X}) \) by construction. Hence the geometric genus of \( J_0(\overline{X}) \) is at least \( q_\pi = \dim \text{Alb}_0(\overline{X}) \). Note also that \( \lambda_{g, \gamma, q_\pi} \geq \frac{4(g - 1)}{\gamma - 1} \), since \( g + 1 - 2\gamma \geq 0 \) by (4-11). Hence by (4-7) and (4-12) with \( \lambda = \lambda_{g, \gamma, q_\pi} \), we obtain
\[
\omega_f^2 - \lambda_{g, \gamma, q_\pi} \cdot \chi_f \geq \frac{8(g - 1) - (g + 1 - 2\gamma)\lambda_{g, \gamma, q_\pi}}{8} \cdot \frac{\omega_h^2}{\gamma - 1} - 2\lambda_{g, \gamma, q_\pi} \cdot \chi_h + \frac{8 - \lambda_{g, \gamma, q_\pi}}{8(g + 1 - 2\gamma)} \cdot T
\]
\[
+ \frac{\lambda_{g, \gamma, q_\pi}}{4} \cdot n_2 + \sum_{k=1}^{q_\pi - 1} \xi_k \cdot s_{2k} + \sum_{k=2}^{q_\pi} \eta_k \cdot s_{2k} + \sum_{k \geq q_\pi} \mu_k \cdot s_{2k} + \sum_{k \geq q_\pi + 1} \nu_k \cdot s_{2k},
\]
where
\[
\xi_k = k^2\lambda_{g, \gamma, q_\pi} - (2k - 1)^2,
\eta_k = \frac{(k - 1)(k\lambda_{g, \gamma, q_\pi} - 4(k - 1))}{2k},
\mu_k = \frac{(4k(g - 1) + (2k - 2)(\gamma - 1))(8 - \lambda_{g, \gamma, q_\pi}) - (g + 1 - 2\gamma)\lambda_{g, \gamma, q_\pi}}{4(g + 1 - 2\gamma)},
\nu_k = \frac{k((g - 1) + (k - 1)(\gamma - 1))(8 - \lambda_{g, \gamma, q_\pi}) - 4(g + 1 - 2\gamma)}{2(g + 1 - 2\gamma)}.
\]

It is easy to see that \( \xi_k \geq 0 \) for any \( 1 \leq k \leq q_\pi - 1 \), \( \eta_k \geq 0 \) for any \( 2 \leq k \leq q_\pi \), and
\[
\mu_k \geq \mu_{q_\pi} = \frac{2(q_\pi - 1)}{q_\pi + 1} + \frac{g - \gamma}{(q_\pi + 1)((g - 1) + (q_\pi - 1)(\gamma - 1))} \geq 0, \quad \forall k \geq q_\pi,
\]
\[
\nu_k \geq \nu_{q_\pi + 1} = 0, \quad \forall k \geq q_\pi + 1.
\]
Hence by (4-18), one has
\[
\omega_f^2 - \lambda_{g, \gamma, q_\pi} \cdot \chi_f \geq \frac{8(g - 1) - (g + 1 - 2\gamma)\lambda_{g, \gamma, q_\pi}}{8} \cdot \frac{\omega_h^2}{\gamma - 1} - 2\lambda_{g, \gamma, q_\pi} \cdot \chi_h + \frac{8 - \lambda_{g, \gamma, q_\pi}}{8(g + 1 - 2\gamma)} \cdot T
\]
\[
\geq \frac{8(g - 1) - (g + 1 - 2\gamma)\lambda_{g, \gamma, q_\pi}}{8} \cdot \frac{\omega_h^2}{\gamma - 1} - 2\lambda_{g, \gamma, q_\pi} \cdot \chi_h + \frac{8 - \lambda_{g, \gamma, q_\pi}}{8(g + 1 - 2\gamma)} \cdot T
\]
\[
\geq 2\lambda_{g, \gamma, q_\pi} \cdot \chi_f \geq \frac{2(g - 8q_\pi - 5)}{q_\pi + 1} \cdot \chi_h.
\]

Note that the assumption \( F(g, \gamma, q_\pi) \geq 0 \) implies that \( g \geq 8q_\pi + 5 \) when \( \gamma = 1 \). Thus the above inequality implies that (4-17) holds if \( \gamma = 1 \).
Finally, we consider the case when \( F(g, \gamma, q_\pi) \geq 0 \) and \( \gamma > 1 \). In this case one has \( \omega_f^2 \geq \frac{4(\gamma - 1)}{\gamma} \cdot \chi_f \geq 0 \) and \( T \geq 0 \). Hence by (4.19), we get
\[
\omega_f^2 - \lambda_f \gamma \chi_f \geq \frac{2F(g, \gamma, q_\pi)}{\gamma(q_\pi + 1)((g - 1) + (q_\pi - 1)(\gamma - 1))} \cdot \chi_f \geq 0.
\]

**Remark 4.9.** Let \( f \) be an irregular double cover fibration of type \((g, \gamma)\). Similar to the above proof, one can show that
\[\lambda_f \geq 6, \quad \text{if } g \geq 6\gamma + 7.\] (4.20)

In the following, we would like to consider Conjecture 1.2 for double cover fibrations, i.e., consider the influence of \( q_f \) on the lower bound of \( \lambda_f \). In order to use Proposition 4.4 and Theorem 4.8, we first have to know when \( J_0(X) \) is a curve, where \( J_0 \) is defined in \((4.8)\).

**Lemma 4.10** ([6]). If \( q_\pi > \gamma + 1 \), then the image \( J_0(X) \subset \text{Alb} \) is a curve.

**Proof.** Let \( \tilde{F} \) be a general fibre of \( f \), and \( \tilde{F} = \tilde{\pi}(\tilde{F}) \subset \tilde{Y} \). Consider the linear map
\[\zeta : \wedge^2 H^{1,0}(\text{Alb}(X)) \cong H^{2,0}(\text{Alb}(X)) \rightarrow H^{1,0}(\tilde{F})\]
on obtained by composing the linear map
\[H^{2,0}(\text{Alb}(X)) \rightarrow H^{2,0}(X)\]
with the restriction map
\[H^{2,0}(X) \cong H^0(\tilde{S}, \omega_{\tilde{S}}) \rightarrow H^0(\tilde{F}, \omega_{\tilde{F}}) \cong H^{1,0}(\tilde{F}),\]
where \( \omega_{\tilde{X}} \) (resp. \( \omega_{\tilde{F}} \)) is the canonical sheaf of \( \tilde{X} \) (resp. \( \tilde{F} \)). Note that the generator \( \tau \) of the Galois group \( \text{Gal}(\tilde{X}/\tilde{Y}) \) acts on \( H^{1,0}(\text{Alb}(X)) \) by multiplying \(-1\), from which it follows that the image \( \text{Im}(\zeta) \) is contained in the invariant subspace \( H^{1,0}(\tilde{F}, \omega_{\tilde{F}})^- \cong H^0(\tilde{C}, \omega_{\tilde{C}}) \). In particular, one has
\[\dim \text{Im}(\zeta) \leq \dim H^0(\tilde{C}, \omega_{\tilde{C}}) = \gamma.\]

On the other hand, if \( J_0(X) \) is a surface, then it is proved by Xiao (cf. [21, Theorem 2], see also [18, Lemma 1] by Pirola) that
\[\dim \text{Im}(\zeta) \geq q_\pi - 1.\]
From the two above inequalities it follows that \( J_0(X) \) is a curve if \( q_\pi > \gamma + 1 \). □

**Theorem 4.11.** Let \( f \) be a double cover fibration of type \((g, \gamma)\). Then Conjecture 1.2 holds for \( f \) if one of the following is satisfied:

(i) \( h \) is locally trivial, \( g \geq 2(\gamma + 1) + \sqrt{8\gamma^2 + 1} \), and \( q_f < (g + 1)/2 \);
(ii) \( h \) is locally non-trivial, \( g \geq 2\gamma + 2q_f + 1 > 6\gamma + 3 \).

**Proof.** (i) Note that \( q_f = q_\pi + q_h \leq q_\pi + \gamma \). If \( q_\pi = 0 \), then \( q_f \leq \gamma \), and hence Conjecture 1.2 follows from Theorem 4.7. Now assume that \( q_\pi > 0 \), i.e., \( f \) is an irregular double cover. In this case, if \( q_\pi \leq \gamma + 1 \), then \( q_f \leq 2\gamma + 1 \). Hence according to Theorem 4.8(i), one gets
\[\lambda_f \geq 6 + \frac{4(\gamma - 1)}{g - 1} \geq \frac{4(g - 1)}{g - 2\gamma - 1} \geq \frac{4(g - 1)}{g - q_f}.\]
If \( q_\pi > \gamma + 1 \), then the image \( J_0(\tilde{X}) \subseteq \text{Alb}_0(\tilde{X}) \) is a curve by Lemma 4.10. Hence according to Theorem 4.8(ii), one obtains

\[
\lambda_f \geq 8 - \frac{4(g + 1 - 2\gamma)}{(q_\pi + 1)((g - 1) + (q_\pi - 1)(\gamma - 1))}
\]

\[
\geq \begin{cases} 
\frac{4(g-1)}{g-q_\pi-\gamma}, & \text{if } q_\pi + \gamma < \frac{g+1}{2}, \\
8 - \frac{8}{g}, & \text{if } q_\pi + \gamma \geq \frac{g+1}{2}, \\
\frac{4(g-1)}{g-q_f}, & \text{since } q_f < \frac{g+1}{2}.
\end{cases}
\]

(ii) The assumption implies that the image \( J_0(\tilde{X}) \subseteq \text{Alb}_0(\tilde{X}) \) is a curve by Lemma 4.10, where \( J_0 \) is defined in (4.8). Let \( \lambda_0 = \frac{4(g-1)}{g-q_f} = \frac{4(g-1)}{g-q_\pi-q_n} \). Then \( \lambda_0 \geq \frac{4(g-1)}{g-\gamma} \) by assumption. Hence similar to the proof of Theorem 4.8(ii), one obtains

\[
\omega_f^2 - \lambda_0 \cdot \chi_f \geq \frac{8(g-1)-(g+1-2\gamma)\lambda_0}{8} \cdot \frac{\omega_h^2}{\gamma-1} - 2\lambda_0 \cdot \chi_h + \frac{8 - \lambda_0}{8(g+1-2\gamma)} \cdot T.
\]

If \( \gamma = 1 \), then by (4.14) one has \( T = 2(g-1)\omega_h \cdot R \geq 4(g-1)^2\chi_h \). Hence it follows from (4.21) that

\[
\omega_f^2 - \lambda_0 \cdot \chi_f \geq \frac{8(g-1)-(g+3)\lambda_0}{2} \cdot \chi_h \geq 0.
\]

If \( \gamma > 1 \), then by (4.21) with \( \omega_h^2 \geq \frac{4(\gamma-1)}{\gamma} \cdot \chi_h \geq 0 \) and \( T \geq 0 \) we get

\[
\omega_f^2 - \lambda_0 \cdot \chi_f \geq \frac{8(g-1)-(g+1+2\gamma)\lambda_0}{2\gamma} \cdot \chi_h \geq 0. \]

\[\square\]

Remark 4.12. There do exist double cover fibrations of type \((g, \gamma)\) with \( q_f = (g+1)/2 \) but violating Conjecture 1.2; see Example 5.2.

We end this section with the following lower bound on the slope of double cover fibrations, which was used in section 3 for the proof of Theorem 1.3(i). It can be viewed as a supplement to Theorem 4.7.

Theorem 4.13. Let \( f \) be a double cover fibration of type \((g, \gamma)\). If \( g \leq 4\gamma + 1 \) and \((g + 1 - 2\gamma)^2 \geq 2(2g + 1 - 3\gamma)\), then

\[
\lambda_f \geq \frac{4(g-1)(3g+1-4\gamma)}{(g+1-2\gamma)^2 + 4\gamma(2g+1-3\gamma)}.
\]

Proof. Let \( \lambda_0 := \frac{4(g-1)(3g+1-4\gamma)}{(g+1-2\gamma)^2 + 4\gamma(2g+1-3\gamma)} \). Then \( 4 \leq \lambda_0 \leq \frac{4(g-1)}{g-\gamma} \) by assumptions.

If \( \gamma = 1 \), then the assumptions imply that \( \lambda_0 = 4 \) and \( g = 5 \). Hence (4.22) follows from (4.11). If \( \gamma > 1 \), taking \( \lambda = \lambda_0 \) in (4.12) and using Lemma 4.14 below.
to eliminate $s_2$, one obtains

$$
\omega_j^2 - \lambda_0 \cdot \chi_f
\geq \left( \frac{(3g + 1 - 4\gamma)(g - 1)}{2(2g + 1 - 3\gamma)} - \frac{(g + 1 - 2\gamma)^2 \lambda_0}{8(2g + 1 - 3\gamma)} \right) \cdot \frac{\omega_h^2}{\gamma - 1} - 2\lambda_0 \cdot \chi_h + \frac{(\lambda_0 - 4)}{8(\gamma - 1)} \cdot T
\geq \frac{\lambda_0}{4} \cdot n_2 + \sum_{k \geq 1} \left( (k^2 \lambda_0 - (2k - 1)^2) \cdot s_{2k + 1} + \sum_{k \geq 2} \left( \frac{k(k - 1)}{2} \lambda_0 - 2(k - 1)^2 \right) \cdot s_{2k} \right)
\geq \left( \frac{(3g + 1 - 4\gamma)(g - 1)}{2(2g + 1 - 3\gamma)} - \frac{(g + 1 - 2\gamma)^2 \lambda_0}{8(2g + 1 - 3\gamma)} \right) \cdot \frac{\omega_h^2}{\gamma - 1} - 2\lambda_0 \cdot \chi_h
\geq \left( \frac{(3g + 1 - 4\gamma)(g - 1)}{2(2g + 1 - 3\gamma)} - \frac{(g + 1 - 2\gamma)^2 \lambda_0}{8(2g + 1 - 3\gamma)} \right) \cdot \frac{4}{\gamma} \cdot (2\lambda_0) \cdot \chi_h = 0,
$$

where the second inequality follows from the non-negativity of $T$, $n_2$ and $s_j$'s for $j \geq 3$; and the third inequality comes from the slope inequality $\omega_h^2 \geq \frac{4(\gamma - 1)}{\gamma} \chi_h$ of the fibration $h$. The proof is complete.

**Lemma 4.14.**

$$
T + (\gamma - 1) \left( s_2 + \sum_{k \geq 1} 4k(2k + 1)s_{2k + 1} + \sum_{k \geq 2} 2(2k - 1)s_{2k} \right) \geq 0. \quad (4-23)
$$

**Proof.** We may assume that $\gamma > 1$. By $(4-24)$, the inequality $(4-23)$ is equivalent to

$$
T + (\gamma - 1)((\omega_h + R) \cdot R + 2n_2) \geq 0. \quad (4-24)
$$

Let $R = \sum_{i=1}^m D_i$ be the decomposition into connected components, such that

$$
D_i \cdot \Gamma > 0, \quad \forall 1 \leq i \leq l; \quad D_i \cdot \Gamma = 0, \quad \forall l + 1 \leq i \leq m,
$$

where $\Gamma$ is a general fiber of $h$. We claim that

$$(\omega_h + D_i) \cdot D_i \geq 0, \quad \forall 1 \leq i \leq l; \quad (\omega_h + D_i) \cdot D_i \geq -2, \quad \forall l + 1 \leq i \leq m. \quad (4-25)
$$

Indeed, let $\tilde{D}_i = \sum_{j=1}^{k_i} \tilde{D}_{ij} \to D_i$ be the normalization, and $\sum_{j=1}^{l_i} \tilde{D}_{ij}$ be the irreducible components which are mapped surjectively onto $B$. Then

$$
(\omega_h + D_i) \cdot D_i = (2g(B) - 2) \Gamma \cdot D_i + (\omega_Y + D_i) \cdot D_i
\geq (2g(B) - 2) \Gamma \cdot D_i + \sum_{j=1}^{k_i} (2g(\tilde{D}_{ij}) - 2) + 2(k_i - 1)
\geq \sum_{j=l_i + 1}^{k_i} (2g(\tilde{D}_{ij}) - 2) + 2(k_i - 1) \geq 2(k_i - l_i - 1).
$$

Hence $(4-25)$ follows. Let $D = D_i$ and $D' = \sum_{i=l + 1}^m D_i$. Then $(\omega_h + D) \cdot D \geq 0$ by $(4-25)$. Since $\Gamma \cdot ((g + 1 - 2\gamma)\omega_h - (\gamma - 1)D) = 0$, one gets by Hodge index theorem
admits a double cover to $E$ of factors through $\subseteq$ Let $\Delta$ that $g$ where $'$ is the addition on the elliptic curve $E$ of odd genus $g$ and $X$ is smooth. According to [17 §2.4-Example (b)], we know that $X$ is minimal of general type with $q(X) = g_0$ and

$$\chi(O_X) = \frac{(g_0 - 1)^2 - (g_0 - 1)}{2}, \quad \omega_X^2 = 4(g_0 - 1)^2 - 5(g_0 - 1).$$

To obtain a fibration on $X$, we consider first the fibration on $C \times C$ defined by

$h : C \times C \rightarrow E, \quad (x_1, x_2) \mapsto \eta(x_1) + \eta(x_2),$

where $\cdot$ is the addition on the elliptic curve $E$. It is easy to see that the morphism $h$ factors through $X$ and so induces a fibration $f : X \rightarrow E$:

\[
\begin{array}{ccc}
C \times C & \xrightarrow{\pi} & X \\
\downarrow h & & \downarrow f \\
E & \xrightarrow{\eta} & f
\end{array}
\]

It is clear that $f$ is relatively minimal since $X$ is minimal, and $q_f = q(Y) - q(E) = g_0 - 1$. To compute the genus $g$ of a general fiber of $f$, let $H$ be a general fiber of $h$, $F = \pi(H) \subseteq X$, $p = h(H) \in E$, and $pr_1$ (resp. $pr_2$) be the projection of $C \times C$ to the first (resp. the second) factor $C$. Then for any $(x_1, x_2) \in H$, one has $\eta(x_1) + \eta(x_2) = p$, i.e., $\eta(x_1) = -\eta(x_2) + p$. In other word, one has the following commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{pr_1|H} & C \\
\downarrow pr_2|H & & \downarrow \eta \\
C & \xrightarrow{-\eta+p} & E
\end{array}
\]
The maps in the above diagram are all double covers, and the branch divisor of $pr_2|_H$ is
\[ T = \{ x \in C \mid y := -\eta(x) + p \text{ is a branch point of } \eta : C \to E \}, \]
which is of degree $4g_0 - 4$. Hence one obtains that $g(H) = 4g_0 - 3$. Note that $H \cdot \Delta = 8$. Thus by Hurwitz formula, we get that
\[ 2g(H) - 2 = 2(2g(F) - 2) + 8. \]
Hence $g = g(F) = 2g_0 - 3$. Therefore $q_f = g_0 - 1 = \frac{g+1}{2}$, and
\[ \lambda_f = \frac{\omega_f^2}{\chi_f} = \frac{\omega_0^2}{\chi_0} = \frac{8g_0 - 18}{g_0 - 2} = 8 - \frac{4}{g-1} < 8 = \frac{4(g-1)}{g-q_f}, \text{ as required.} \]

**Example 5.2.** We construct a relatively minimal double cover fibration $f : X \to \mathbb{P}^1$ of type $(g, \gamma)$ with $0 < \gamma < (g + 1)/2$, $q_f = (g + 1)/2$, and
\[ \lambda_f = 8 - \frac{4}{(g + 1 - 2\gamma)\gamma} < 8 = \frac{4(g-1)}{g-q_f}. \]

Consider the ruled surface $\eta_0 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $\Lambda_0$ be a pencil on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $H_0$ is a section of $\eta_0$ and $H^2 = 2$ for a general member $H_0 \in \Lambda_0$. Assume that $\Lambda_0$ has two distinct base-points, which are mapped to $\{p, p'\} \subseteq \mathbb{P}^1$. By $\eta_0$. Let $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be a double cover branched exactly over $\{p, p'\}$, and consider the Cartesian product
\[ \mathbb{P}^1 \times \mathbb{P}^1 \quad \xrightarrow{\eta} \quad \mathbb{P}^1 \times \mathbb{P}^1 \quad \xrightarrow{\psi} \quad \mathbb{P}^1 \]

Let $\Lambda$ be the pulling-back of $\Lambda_0$. Then $\Lambda$ also has two distinct base-points ($H$ and $H'$ are tangent to each other at each of these two base-points for any two general $H, H' \in \Lambda$). Let $\xi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be another fibration, and $\{D_1, D_2, \cdots, D_{2\gamma+2}\}$ be $2\gamma+2$ fibers of $\xi$ such that these two base-points of $\Lambda$ are contained in $D_1$ and $D_2$ respectively. Let $\Gamma \to \mathbb{P}^1$ be the double cover branched over $\{\xi(D_1), \xi(D_2), \cdots, \xi(D_{2\gamma+2})\}$, and
\[ Y = (\mathbb{P}^1 \times \mathbb{P}^1) \times_{\mathbb{P}^1} \Gamma = \mathbb{P}^1 \times \Gamma \]
the fiber-product. Let $\Lambda_Y$ be the inverse of $\Lambda$ on $Y$. Then $\Lambda_Y$ has also exactly two base-points (each of the base-points is of multiplicity two). Blowing up the base-points of the pencil $\Lambda_Y$, we obtain a fibration
\[ \varphi : \tilde{Y} \to \mathbb{P}^1. \]

By construction, the strict inverse images of $D_1$ and $D_2$ in $\tilde{Y}$ are contracted by $\varphi$. Let $\tilde{p}, \tilde{p}'$ be the images, and $\Gamma' \to \mathbb{P}^1$ the double cover branched over $\{\tilde{p}, \tilde{p}', x_1, \cdots, x_{2\gamma'}\}$, where $\gamma' = (g + 1)/2 - \gamma$, and $x_1, \cdots, x_{2\gamma'}$ are distinct general points on $\mathbb{P}^1$. Let $X$ be the normalization of the fiber-product $\tilde{Y} \times_{\mathbb{P}^1} \Gamma'$ and...
Let $\tilde{C}_i = \varphi^*(x_i)$ be the fibers of $\varphi$ for $1 \leq i \leq 2\gamma'$. Then it is clear that

$$\omega_{\tilde{Y}}^2 = -8(\gamma - 1) - 2, \quad \chi(O_{\tilde{Y}}) = -1(\gamma - 1), \quad \omega_{\tilde{Y}} \cdot \tilde{C} = 4\gamma - 4.$$

Note that the fibers of $\varphi$ over $\tilde{p}$ and $\tilde{p}'$ are of multiplicity two. Hence $\pi$ is a double cover branched exactly over $\tilde{R} = \{\tilde{C}_1, \cdots, \tilde{C}_{2\gamma'}\}$. Therefore, $f$ is a relatively minimal fibration of genus $g$, and

$$\omega_f^2 = 2 \left( \omega_{\tilde{Y}} + \frac{1}{2} \bar{R} \right)^2 + 8(g - 1) = 8(g + 1 - 2\gamma)\gamma - 4,$$

$$\chi_f = 2\chi(O_{\tilde{Y}}) + \frac{1}{2} \left( \omega_{\tilde{Y}} + \frac{1}{2} \bar{R} \right) \cdot \frac{\bar{R}}{2} + (g - 1) = (g + 1 - 2\gamma)\gamma.$$

Hence $f$ has the required slope. Note that $q(\tilde{Y}) = \gamma$ and $q_{\pi} = \gamma'$ since $\pi$ is the normalization of the fiber-product $\tilde{Y} \times_{P^1} \Gamma'$. Therefore $q_f = \gamma + \gamma' = (g + 1)/2$ as required.

**Example 5.3.** We construct a relatively minimal double cover fibration $f : X \to P^1$ of type $(g, \gamma)$ with $q_{\pi} = \frac{g + 1 - 2\gamma - 1}{d} (d \geq 2)$ and

$$\lambda_f = \lambda_{g, \gamma, q_{\pi}} := 8 - \frac{4(g + 1 - 2\gamma)}{(q_{\pi} + 1)((g - 1) + (q_{\pi} - 1)(\gamma - 1))}.$$

Let $C$ be a smooth genus-$\gamma$ curve admitting a morphism $\zeta : C \to P^1$ of degree $d \geq 2$. Consider the following diagram

\[
\begin{array}{ccc}
Y \cong C \times P^1 & \xrightarrow{(\zeta, id)} & P^1 \times P^1 \\
\downarrow h & & \downarrow \eta \\
P^1 & \xrightarrow{\pi} & P^1
\end{array}
\]

Let $H_0 \subseteq P^1 \times P^1$ be any section of $\eta$ with $H_0^2 = 2b_0$ for some $b_0 > 0$. It is well-known that $H_0$ is very ample (cf. [10 §V.2]). Take two general members $D, D' \in |H_0|$, and let $\Lambda$ be the pencil on $Y$ generated by $\pi_{\lambda}^0(D)$ and $\pi_{\lambda}^0(D')$. Then $\Lambda$ defines a rational map $\varphi_{\lambda} : Y \to P^1$. By blowing up the base points of $\Lambda$, we get a fibration $h' : \tilde{Y} \to P^1$. Let $\Delta \subseteq P^1$ be a set of $2(q_{\pi} + 1)$ general points,
\( \tilde{R} = (\tilde{h}')^*(\Delta) \) the corresponding fibers of \( \tilde{h}', \pi' : B' \to \mathbb{P}^1 \) the double cover ramified over \( \Delta \), and \( \tilde{X} \) the normalization of the fiber-product \( \tilde{Y} \times_{\mathbb{P}^1} B' \): 

\[ 
\begin{array}{c}
\overline{\mathbb{P}^1} \\
\cap_{\varphi_{\Delta}} \\
\mathbb{P}^1
\end{array}
\]

\[ 
\begin{array}{c}
\tilde{X} \leftarrow \tilde{Y} \\
\downarrow \tilde{h} \leftarrow \downarrow h
\end{array}
\]

\[ 
\begin{array}{c}
\pi' \leftarrow \pi \\
\downarrow \leftarrow \\
B' \leftarrow \mathbb{P}^1
\end{array}
\]

Since \( \Delta \) is general on \( \mathbb{P}^1 \), \( \tilde{R} \) is both reduced and smooth. Hence \( \tilde{X} \) is also smooth. Let \( \tilde{\Gamma}' \) be a general fiber of \( \tilde{h}' \). By construction, one has \( q(\tilde{X}) - q(\tilde{Y}) = g(B') = q_{\pi} \), and 

\[
\omega_f^2 = -2db_0, \quad \omega_f \cdot \tilde{\Gamma}' = 2(\gamma - 1 + d)b_0, \quad (\tilde{\Gamma}')^2 = 0.
\]

Hence by the standard formulas for double covers (cf. \[5, \S V.22\]), we obtain 

\[
\omega_f^2 = 2 \left( \omega_{\tilde{h}} + \frac{\tilde{R}}{2} \right)^2 = 4(2q_{\pi} + 1)(\gamma - 1 + d - d)b_0,
\]

\[
\chi_f = 2\chi_{\tilde{h}} + \frac{1}{2} \left( \omega_{\tilde{\gamma}} + \frac{\tilde{R}}{2} \right) \cdot \frac{\tilde{R}}{2} = (q_{\pi} + 1)(\gamma - 1 + d)b_0.
\]

Actually, \( f \) is relatively minimal. To see this, let \( R \) be the image of \( \tilde{R} \) in \( Y \). Then the singular points of \( R \) are all of multiplicity \( 2q_{\pi} + 1 \). Hence \( s_{2k+1} = 0 \) for all \( k \geq 1 \). Combining this with the triviality of \( h \), we obtain that \( \tilde{X} \) is relatively minimal by **Lemma 4.2**. Hence \( f \) is a relatively minimal double cover fibration \( f : X \to \mathbb{P}^1 \) of type \((g, \gamma)\) with \( q_{\pi} = \frac{g+1-2\gamma}{d} - 1 \) and

\[
\lambda_f = \frac{\omega_f^2}{\chi_f} = \frac{4(2q_{\pi} + 1)(\gamma - 1 + d - d)}{(q_{\pi} + 1)(\gamma - 1 + d)} = \lambda_{g, \gamma, q_{\pi}}, \quad \text{as required}.
\]

**References**

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[2] M. A. Barja. *On the slope of bielliptic fibrations*. Proc. Amer. Math. Soc., 129(7): 1899–1906 (electronic), 2001.

[3] M. Á. Barja and L. Stoppino. *Linear stability of projected canonical curves with applications to the slope of fibred surfaces*. J. Math. Soc. Japan, 60(1): 171–192, 2008.

[4] M. Á. Barja and F. Zucconi. *On the slope of fibred surfaces*. Nagoya Math. J., 164: 103–131, 2001.

[5] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, second edition, 2004.

[6] J.-X. Cai. *Irregularity of certain algebraic fiber spaces*. Manuscripta Math., 95(3): 273–287, 1998.

[7] M. Cornalba and J. Harris. *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*. Ann. Sci. École Norm. Sup. (4), 21(3): 455–475, 1988.

[8] M. Cornalba and L. Stoppino. *A sharp bound for the slope of double cover fibrations*. Michigan Math. J., 56(3): 551–561, 2008.
[9] T. Fujita. On Kähler fiber spaces over curves. J. Math. Soc. Japan, 30(4): 779–794, 1978.
[10] R. Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
[11] K. Konno. Nonhyperelliptic fibrations of small genus and certain irregular canonical surfaces. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 20(4): 575–595, 1993.
[12] K. Konno. A lower bound of the slope of trigonal fibrations. Internat. J. Math., 7(1): 19–27, 1996.
[13] K. Konno. Clifford index and the slope of fibered surfaces. J. Algebraic Geom., 8(2): 207–220, 1999.
[14] X. Lu and K. Zuo. On the slope of hyperelliptic fibrations with positive relative irregularity. arXiv:1311.7271v4, to appear in Trans. Amer. Math. Soc., 2013.
[15] X. Lu and K. Zuo. The Oort conjecture on Shimura curves in the Torelli locus of curves. arXiv:1405.4751, 2014.
[16] X. Lu and K. Zuo. On the Severi type inequalities for irregular surfaces. preprint, 2015.
[17] M. Mendes Lopes and R. Pardini. The geography of irregular surfaces. In Current developments in algebraic geometry, volume 59 of Math. Sci. Res. Inst. Publ., pages 349–378. Cambridge Univ. Press, Cambridge, 2012.
[18] G. P. Pirola. Curves on generic Kummer varieties. Duke Math. J., 59(3): 701–708, 1989.
[19] G. P. Pirola. On a conjecture of Xiao. J. Reine Angew. Math., 431: 75–89, 1992.
[20] G. Xiao. Fibered algebraic surfaces with low slope. Math. Ann., 276(3): 449–466, 1987.
[21] G. Xiao. Irregularity of surfaces with a linear pencil. Duke Math. J., 55(3): 597–602, 1987.
[22] G. Xiao. π1 of elliptic and hyperelliptic surfaces. Internat. J. Math., 2(5): 599–615, 1991.

Department of Mathematics, East China Normal University, Shanghai, China, 200241
Current address: Institut für Mathematik, Universität Mainz, Mainz, Germany, 55099
E-mail address: lvxinwillv@gmail.com

Institut für Mathematik, Universität Mainz, Mainz, Germany, 55099
E-mail address: zuok@uni-mainz.de