Logarithmic penalty function method for invex multi-objective fractional programming problems

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ABSTRACT

In this paper, a new logarithmic penalty function method is used for solving nonlinear multi-objective fractional programming problems (MOFPP) involving invex objectives and constraints with respect to the same function \( f \). This approach is implemented by modifying fractional objective function to \( \alpha \)-invex function, no parameterizations to multi-objective fractional programming problem are required. Furthermore, the constrained multi-objective FPP has been converted to a sequence of unconstrained optimization problems by adding a new logarithmic penalty function to each objective function.

1. Introduction

Fractional programming problem (FPP) has received much interest from researchers in recent decades, particularly the multi-objective programming problem. Decision theory, game theory, economics, and many more are some of the practical problems required to be optimized in terms of the ratio of several linear and nonlinear functions.

The nonlinear multi-objective fractional programming problem to be studied in this article is as follows:

\[
\begin{align*}
\text{minimize} & \quad f(x) = \left( \frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_n(x)}{g_n(x)} \right) \\
\text{subject to} & \quad h_j(x) \leq 0, j = 1, \ldots, m, \text{ where } f_i : X \rightarrow \mathbb{R} \text{ and } g_i : X \rightarrow \mathbb{R}, i = 1, \ldots, n; h_j : X \rightarrow \mathbb{R}, j = 1, \ldots, m, \text{ are the differentiable functions on a nonempty open subset of the real number. Let assume that } f_i(x) \leq 0 \text{ and } g_i(x) > 0 \\
& \quad \text{for all } x \in D, \text{ where } D = \{ x \in X : h_j(x) \leq 0, j = 1, \ldots, m \} \text{ denote the set of all feasible solution for fractional programming problems (P).}
\end{align*}
\]

Generally, the multi-objective fractional programming problem is either parameterize or transformed to another suitable form, so that an equivalent multi-objective fractional programming can be obtained.

Bernard and Ferland [1] outline basic approaches and main types of available algorithms to deal with the fractional programming problems, and their convergence analysis was also reviewed.

Kuk, Lee, and Tanino [2] establishes generalized Kuhn-Tucker necessary and sufficient optimality conditions and derive the duality theorems regarding a class of nonsmooth multi-objective fractional programming problems involving \( V = \rho - \text{invex} \) functions.

Other researchers extend the idea to optimality and duality for nonsmooth multi-objective fractional programming problem (see for example [2–4]).

Santos, Osuna-Gómez and Rojas-Medar [5] use a notion of generalized convexity, called KT-invexity and study a class of nonconvex and nondifferentiable MOFP. Furthermore, a dual problem was defined and establish some duality results. The idea of invexity was first introduced by Hanson [6] and named by Craven [7].

There are several methods available for solving the considered optimization problem (P). In recent years, most of the powerful algorithms were designed explicitly for the unconstrained optimization problem, and these lead to inventing the penalty function approach that will enable the researchers to solve the constrained problem.

Penalty function method is one of the most critical approaches for solving an optimization problem, and the idea is implemented by incorporating constraints into an objective function by adding the penalty term, the penalty function ensures that the feasible solutions do not violate the constraints.

The concept of penalty function approach was first introduced by Mangwili [8], an algorithm was presented to handle the penalized optimization problem constructed based on nondifferentiable exact penalty function, the method appears to be more useful in the concave case.

The notion was extended by Eremin [9] via the exact penalty function method to solve nonlinear optimization with convex function. The assumption of

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convexity plays a vital role in most of the exact penalized optimization approaches in the literature. Antczak [10] established some characterization of the $l_1$ exact penalty method, and he used the technique to solve a new class of nonconvex optimization problems with inequality constraints. Some of the problems in operations research may be express in terms of the ratio of linear or nonlinear functions, Antczak [11] presents a new method for an invex optimization was discussed. Finally, in Section 5, some examples have been solved, which shows how the pareto optimal for both methods coincide.

2. Preliminary definitions

Definition 1 [6]: Let $X$ be a nonempty open subset of $R^n$ and $f : X \to R$ be a differentiable function defined on $X$. Then $f$ is said to be (strictly) invex at $u \in X$ with respect to $\eta$ if there exist a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X (x \neq u)$.

$$f(x) - f(u) \geq \nabla f(u) \eta(x, u) \quad (>)$$

(1)

If (1) is satisfied for any $u \in X$. Then $f$ is an invex function on $X$ with respect to $\eta$.

Definition 2 [15]: Let $X$ be a nonempty open subset of $R^n$ and $f : X \to R$ be a differentiable function defined on $X$. Then $f$ is said to be (strictly) invex at $u \in X$ with respect to $\eta$ if there exist a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X (x \neq u)$.

$$f(x) - f(u) \leq \nabla f(u) \eta(x, u) \quad (<)$$

(2)

If (2) is satisfied for any $u \in X$. Then $f$ is an incave function on $X$ with respect to $\eta$.

Definition 3 [11]: Let $X$ be a nonempty open set $X \subset R^n$ and $\psi : X \to R$ be a differentiable function defined on $X$. Then $\psi$ is said to be (strictly) $\alpha_i$-invex at $u \in X$ on $X$ with respect to $\eta$ if there exists a vector-valued function $\eta : X \times X \to R^n$ and $\alpha_i : X \times X \to R_+ [0], i = 1, 2, \ldots, n$. such that, for all $x \in X (x \neq u)$.

$$\psi(x) - \psi(u) \geq \alpha_i(x, u) \nabla \psi(u) \eta(x, u) \quad (>$$

(3)

If (3) is satisfied for any $u \in X$. Then $f$ is an $\alpha_i$-invex function on $X$ with respect to $\eta$.

Definition 4 [11]: A point $\bar{x} \in D$ is said to be an efficient (pareto optimal) for fractional programming problem (P) if and only if $\bar{x} \in D$ such that for some $s \in \{1, 2, \ldots, n\}$

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} < \frac{f_i(x)}{g_i(x)} \quad \text{and} \quad \frac{f_i(\bar{x})}{g_i(\bar{x})} < \frac{f_i(x)}{g_i(x)} \quad \text{for all } i \in \{1, 2, \ldots, n\}, i \neq s.$$  

(4)

Definition 5 [11]: A point $\bar{x} \in D$ is said to be weak efficient (weak pareto optimal) for multi-objective fractional programming if and only if $\bar{x} \in D$ such that

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} < \frac{f_i(x)}{g_i(x)} \quad \text{for all } i \in \{1, 2, \ldots, n\}, i \neq s.$$  

(5)

Definition 6 [8]: A continuous function $p : R^n \to R$ satisfying the following conditions:

(a) $p(x) = 0$ if $x$ is feasible (in other word, if $h_j(x) \leq 0$)

(b) $p(x) > 0$ otherwise (in other word, if $h_j(x) > 0$)

is said to be a penalty function for constrained optimization problem.

Conventionally, a penalty function approach introduced by Zang will [8] for both equality and inequality constraints was popularly known as absolute value penalty function, it is of the following form:

$$p(x) = \sum_{j=1}^{m} |h_j^+(x)| + \sum_{k=1}^{l} |h_k(x)|$$

Note that: $h_j^+(x) = \max(0, h_j(x))$ and $h_k(x) = 0$ (equality constraints), $\forall k \in K = \{1, 2, \ldots, l\}$

Theorem 1: If $f_i : X \to R^n$, $-g_i : X \to R^n$ for $i = \{1, 2, \ldots, n\}$ are incave functions with respect to the same function $\eta : X \times X \to R$ at $u$ on $X$. then the fractional function
\( (f_i(x))/g_i(x) \) is \( \alpha_i \)-invex with respect to the same function \( \eta \) at \( u \) on \( X \) and with respect to the function given by
\[
\alpha_i(x, u) = \frac{g_i(u)}{g_i(x)}
\]  

**Proof:** For \( i = 1, 2, \ldots, n \)
\[
f_i(x) = f_i(u) - \frac{g_i(u)}{g_i(x)} \left( g_i(x)[f_i(x) - f_i(u)] - f_i(x)[g_i(x) - g_i(u)] \right) \frac{g_i(x)}{g_i(u)}
\]  

By differential calculus, we have the following
\[
\nabla \left( f_i \over g_i \right) (u) = \frac{g_i(u)[\nabla f_i(u)] - f_i(u)[\nabla g_i(u)]}{[g_i(u)]^2}
\]  

(8) can be written in this form
\[
\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \geq \frac{g_i(u)[\nabla f_i(u)] - f_i(u)[\nabla g_i(u)]}{[g_i(u)]^2} \eta(x, u)
\]  

(10)

\[
\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \geq \frac{g_i(u)[\nabla f_i(u)] - f_i(u)[\nabla g_i(u)]}{[g_i(u)]^2} \nabla \left( f_i \over g_i \right)(u) \eta(x, u)
\]  

(11)

For simplicity, we can represent \( \left( \frac{f_i}{g_i} \right) = \psi_i \)

By (6), (11) can be re-written in the following form
\[
\psi_i(x) - \psi_i(u) \geq \alpha_i(x, u) \nabla \psi_i(u) \eta(x, u)
\]  

(12)

Therefore, by Definition 3 \( f_i/g_i \) is \( \alpha_i \)-invex with respect to the function \( \eta \) at \( u \) on \( X \) and with respect to the function \( \alpha_i(x, u) \).

For simplicity, since \( \alpha_i(x, u) > 0 \) for all \( i \) we consider \( \psi_i \) to be an invex function, such that
\[
\psi_i(x) - \psi_i(u) \geq \nabla \psi_i(u) \eta(x, u)
\]  

(13)

Now that we have a modified multi-objective fractional programming problem (P)
- Minimize \( \psi_i(x) \)
- Subject to \( h_j(x) \leq 0, j = 1, 2, \ldots, m \)
\[
x \in X
\]

where \( \psi_i : X \to \mathbb{R}, \ i \in I \) and \( h_j : X \to \mathbb{R}, \ j \in J, \) are nonempty differentiable functions on an open set \( X \subset \mathbb{R}^n \).

### 3. Kuhn–Tucker multiplier for logarithmic penalty function

In any nonlinear optimization problem, the first order necessary conditions for a nonlinear optimization problem to be optimal is Karush–Kuhn–Tucker (KKT) conditions, considering that some constraints qualifications are satisfied. However, Courant–Beltrami penalty function may not be differentiable at a point \( h_j(x) = 0 \) for some \( i \). But for the constrained optimization problem both objective function and constraints may be partially differentiable on \( \mathbb{R}^n \) while at the same time the penalized problem is not, being differentiability is not among the properties of max(0, \( h_j(x) \)). Therefore, some additional hypothesis may be imposed on the constraint function \( h_j(x) \), i.e. if the constraint \( h_j(x) \) has continuous first-order partial derivatives on \( \mathbb{R}^n \), for this reason \( (h_j^+(x))^2 \) admit the same. Therefore,
\[
\frac{\partial}{\partial x_r} [h_j^+(x)]^2 = 2[h_j^+(x)] \frac{\partial}{\partial x_r} h_j(x)
\]  

(14)

where \( r \) is the multi-variable indexes.

Considering Equation (14), if \( p(x) : \mathbb{R}^n \to \mathbb{R} \) is a logarithmic penalty function and the constraints \( h_j(x) \) has continuous first-order partial derivative on \( \mathbb{R}^n \), then
\[
\nabla p(x) = \sum_{j=1}^m \nabla [\ln((h_j^+(x))^2 + 1)]
\]  

(15)

From (15), we can define Kuhn–Tucker multiplier as follows:
\[
\mu_j = \frac{2h_j^+(x)}{([h_j^+(x)]^2 + 1)}
\]  

**Theorem 2 [16]:** Let \( \bar{x} \) be the optimal solution in the problem (P) and assume that any suitable constraint qualification in [17] be satisfied at \( \bar{x} \). Then there exists a Lagrange multiplier \( \tilde{\mu} \in \mathbb{R}^n \) such that
\[
\nabla \psi(\bar{x}) + \sum_{j=1}^m \tilde{\mu}_j \nabla h_j(\bar{x}) = 0, \quad (i)
\]
\[
\tilde{\mu}_j h_j(\bar{x}) = 0, \quad j \in J, \quad (ii)
\]
\[
\tilde{\mu} \geq 0. \quad (iii)
\]

### 4. The logarithmic penalty method for an invex optimization problems

Transforming a constrained optimization to a single unconstrained problem for a single objective mathematical programming, or a sequence of an unconstrained problem for multi-objective optimization problem can be actualized employing the penalty function. If we consider the new logarithmic penalty function
introduced by Hassan and Baharum [14] for equality constraints, we modified a Courant–Beltrami penalty function of the form

\[ p(x) = \sum_{j=1}^{m} [h_j^+(x))^2 \]

for inequality constraints, the modified Courant–Beltrami penalty should be constructed as follows:

\[ p(x) = \sum_{j=1}^{m} \ln[(h_j^+(x))^2 + 1]\]

This leads to the following logarithmic penalized optimization problem for multi-objective fractional programming (P);

\[ \text{minimize } P_c(x) = \varphi(x) + c \sum_{j=1}^{m} \ln[(h_j^+(x))^2 + 1]. \quad (16) \]

We can now completely characterize solutions for the minimization for the problem (P) in terms of the minimizers of the logarithmic penalty parameter that exceeds some suitable threshold. For a sufficiently large value of \( c \) under imposed suitable invexity assumption on the functions in the problem (P). The KKT point minimizes the auxiliary function \( P_c(x) \) if and only if it minimizes optimization problem (P).

We are required to show that a KKT point in the optimization problem yields the minimizer of logarithmic penalty function in the associated penalized optimization problem.

**Theorem 3 [15]:** Let \( \bar{x} \) be a feasible solution in the mathematical programming problem (P), and the KKT necessary optimality conditions hold at \( \bar{x} \) with the Lagrange multipliers \( \bar{\lambda}_j, j \in J \). Furthermore, assume that the objective function \( \varphi \) is invex at \( \bar{x} \) on \( X \) with respect to \( \eta \) and the function \( \sum_{j=1}^{m} h_j(x) \) is an invex at \( \bar{x} \) on \( X \) with respect to the same function \( \eta \). If \( c \) is assumed to be sufficiently large (it is sufficient to set \( c > \max \{\bar{\lambda}_j, i \in I\} \), where \( \bar{\lambda}_j, i \in I \), are Lagrange multipliers associated with the constraints \( h_j \), respectively), then \( \bar{x} \) is also a minimizer in the associated penalized optimization problem (\( P_c(x) \)) with the \( I \) exact penalty function.

**Proof:** To prove that \( \bar{x} \) is optimal to the associated penalized optimization problem \( P_c(x) \), we proceed by contradiction. In contrary to the assumption, suppose that \( \bar{x} \) is not an optimal solution of the associated penalized optimization problem \( P_c(x) \) with logarithmic penalty function. Therefore, there exists \( \tilde{x} \in X \) such that

\[ P_c(x) < P_c(\tilde{x}). \]

By (16), we have

\[ \varphi(\tilde{x}) + c \sum_{j=1}^{m} \ln[(h_j^+(\tilde{x}))^2 + 1] < \varphi(\bar{x}) + c \sum_{j=1}^{m} \ln[(h_j^+(\bar{x}))^2 + 1] \quad (17) \]

Since \( \bar{x} \) is a feasible solution in the mathematical programming problem (P),

\[ \sum_{j=1}^{m} \ln[(h_j^+(\bar{x}))^2 + 1] = 0. \quad (18) \]

Moreover, by Equation (18), Equation (17) becomes

\[ \varphi(\bar{x}) + c \sum_{j=1}^{m} \ln[(h_j^+(\bar{x}))^2 + 1] < \varphi(\tilde{x}) \quad (19) \]

Again, by hypothesis \( c > \max \{\bar{\lambda}_j, i \in I\} \), where \( \bar{\lambda}_j, i \in I \), are Lagrange multipliers associated with the constraints \( h_j \), respectively. Then for each \( \bar{x} \in X \), Equation (19) can be transformed to the following form;

\[ \varphi(\bar{x}) + \sum_{j=1}^{m} \bar{\lambda}_j h_j^+(\bar{x}) < \varphi(\tilde{x}) \]

Since the KKT necessary optimality conditions are fulfilled. Being that \( \bar{x} \) is a feasible point in the mathematical programming (P), it follows that

\[ \varphi(\bar{x}) + \sum_{j=1}^{m} \bar{\lambda}_j h_j^+(\bar{x}) < \varphi(\tilde{x}) + \sum_{j=1}^{m} \bar{\lambda}_j h_j^+(\tilde{x}) \]

Thus,

\[ \varphi(\bar{x}) - \varphi(\tilde{x}) < - \sum_{j=1}^{m} \bar{\lambda}_j h_j^+(\bar{x}) \]

By assumption, the objective function \( \varphi \) is an invex at \( \bar{x} \) on \( X \) with respect to \( \eta \). Therefore, by Definitions (1) and (2), respectively, we rewrite the above inequality as follows:

\[ \nabla \varphi_0(\bar{x}) \eta(\bar{x}, \tilde{x}) < - \sum_{j=1}^{m} \bar{\lambda}_j h_j^+(\bar{x}) \eta(\bar{x}, \tilde{x}) \]

Hence,

\[ [\nabla \varphi_0(\bar{x}) + \sum_{j=1}^{m} \bar{\lambda}_j h_j^+(\bar{x})] \eta(\bar{x}, \tilde{x}) < 0 \]

The above strict inequality contradicts the KKT necessary optimality (i). Therefore, the conclusion of the theorem is established.
5. Numerical examples

Example 1 [11]: Let us now consider the following nonlinear multi-objective fractional programming problem with single constraint:

\[
\frac{f(x)}{g(x)} = \left[ \frac{\ln(x + 1) - 2}{x^2 + x^2 + x + 1} \right],
\]

subject to \(x \geq 0\), \(R\), \(i = 1, 2\), and \(\psi_1, \psi_2\) are defined as

\[
\psi_1(x) = (4x_1^2 + 2x_2 + 5)/(x_1^2 + x_2 + 2.5), \quad \psi_2(x) = (x_1^2 + 2x_2 + 6)/(x_1^2 + 2).
\]

Now, we construct the unconstrained multi-objective fractional programming based on logarithmic penalized optimization problem as in Equation (14),

\[
\min P_c(x) = \left( \psi_1(x) + c \sum_{j=1}^{m} \ln(\psi_j^+(x)) \right) - 11, \psi_2(x) + c \sum_{j=1}^{m} \ln(\psi_j^+(x)) + 1
\]

Table 1. Feasible solutions based on the set of feasible points.

| \(x_1\) | \(x_2\) | \(\psi_1\) | \(\psi_2\) |
|---|---|---|---|
| 1 | -1 | -1 | 2.8000E00 |
| 2 | -1 | 0 | 2.5714E00 |
| 3 | -1 | 1 | 2.4444E00 |
| 4 | 0 | -1 | 2.0000E00 |
| 5 | 0 | 0 | 2.0000E00 |
| 6 | 0 | 1 | 2.0000E00 |
| 7 | 1 | -1 | 2.8000E00 |
| 8 | 1 | 0 | 2.5714E00 |
| 9 | 1 | 1 | 2.4444E00 |

6. Conclusion

In this paper, a new logarithmic penalty function method has been used and implemented on a modified multi-objective fractional programming problem, and if the functions constituting the original mathematical programming are invex/incave and differentiable with respect to the same function \(\eta\), then the modified objective functions are also invex and differentiable. The notion of transforming constrained into an
unconstrained optimization problem via penalty function yield the same pareto optimal between the original optimization problem and it associated penalized optimization problem. The result obtained shows how crucial is the logarithmic penalty function method is.

Future works will be mainly on solving practical applications via metaheuristic algorithms.

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No potential conflict of interest was reported by the authors.

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