**NON-REGULARITY OF \([\alpha + \log_k n]\)**

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**Abstract.** This paper presents a new proof that if \(k^\alpha\) is irrational then the sequence \(\{\lfloor \alpha + \log_k n \rfloor\}_{n \geq 1}\) is not \(k\)-regular. Unlike previous proofs, the methods used do not rely on automata or language theoretic concepts. The paper also proves the stronger statement that if \(k^\alpha\) is irrational then the generating function in \(k\) non-commuting variables associated with \(\{\lfloor \alpha + \log_k n \rfloor\}_{n \geq 1}\) is not algebraic.

Fix an integer \(k \geq 2\). A sequence \(\{a(n)\}_{n \geq 0}\) is \(k\)-regular if the \(\mathbb{Z}\)-module generated by the subsequences \(\{a(k^n + i)\}_{n \geq 0}\) for \(e \geq 0\) and \(0 \leq i < k^e\) is finitely generated. Regular sequences were introduced by Allouche and Shallit [1] and have several nice characterizations, including the following characterization as rational power series in non-commuting variables \(x_0, x_1, \ldots, x_{k-1}\). If \(n = n_1 \cdots n_1 n_0\) is the standard base-\(k\) representation of \(n\), then let \(\tau(n) = x_{n_0} x_{n_1} \cdots x_{n_l}\). The sequence \(\{a(n)\}_{n \geq 0}\) is \(k\)-regular if and only if the power series \(\sum_{n \geq 0} a(n)\tau(n)\) is rational. In this sense, regular sequences are analogous to constant-recursive sequences (sequences that satisfy linear recurrence equations with constant coefficients), the set of which coincides with the set of sequences whose generating functions in a single variable are rational.

The sequence \(\{\lfloor \log_2(n+1) \rfloor\}_{n \geq 0}\) is an example of a 2-regular sequence, and the associated power series in non-commuting variables \(x_0, x_1\) is

\[
f(x_0, x_1) = \sum_{n \geq 0} \lfloor \log_2(n+1) \rfloor \tau(n)
\]

\[
= x_1 + x_0 x_1 + 2x_1 x_1 + 2x_0 x_0 x_1 + 2x_1 x_0 x_1 + 2x_0 x_1 x_1 + 3x_1 x_1 x_1 + \cdots.
\]

The rational expression for this series is somewhat large; however its commutative projection is quite manageable:

\[
x_1 \frac{1-x_0-x_1+x_0^2+x_0 x_1}{(1-x_1)(1-x_0-x_1)^2}
\]

Allouche and Shallit [2] open problem 16.10] asked whether the sequence \(\{\lfloor \frac{1}{2} + \log_2(n+1) \rfloor\}_{n \geq 0}\) is 2-regular. Bell [3] and later Moshe [5] Theorem 4] gave proofs that this sequence is not 2-regular. Moreover, they proved the following.

**Theorem.** Let \(k \geq 2\) be an integer and \(\alpha\) be a real number. The sequence \(\{\lfloor \alpha + \log_k(n+1) \rfloor\}_{n \geq 0}\) is \(k\)-regular if and only if \(k^\alpha\) is rational.

In this paper we prove the following theorem, which is a slightly weaker statement than the previous theorem but still establishes that if \(k^\alpha\) is irrational then \(\{\lfloor \alpha + \log_k(n+1) \rfloor\}_{n \geq 0}\) is not \(k\)-regular. Let \(|\tau(n)|\) be the length of the word \(\tau(n)\), i.e., \(|\tau(0)| = 0\) and \(|\tau(n)| = |\log_k n| + 1\) for \(n \geq 1\).

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Theorem. Let $k \geq 2$ be an integer and $\alpha$ be a real number. The series $f(x) = \sum_{n \geq 0} (\alpha + \log_k(n + 1)|x|^{\tau(n)})$ is rational if and only if $k^\alpha$ is rational.

The proof given here is similar to Moshe’s but does not require the notion of a regular language. Note that, given the associated power series

$$f(x_0, x_1, \ldots, x_{k-1}) = \sum_{n \geq 0} (\alpha + \log_k(n + 1)|x|^{\tau(n)}),$$

the series in the theorem is the power series $f(x) = f(x, x, \ldots, x)$ in one variable obtained by setting $x_0 = x_1 = \cdots = x_{k-1} = x$. Therefore non-rationality of $f(x)$ implies non-regularity of $\{ (\alpha + \log_k(n + 1)) \}_{n \geq 0}$.

To get a sense of computing $f(x)$ in the proof of the theorem, first we examine the case where $k = 2$ and $\alpha = \frac{1}{2}$. The power series in this case is

$$f(x_0, x_1) = \sum_{n \geq 0} \left\lfloor \frac{1}{2} + \log_2(n + 1) \right\rfloor \tau(n)$$

$$= x_1 + 2x_0x_1 + 2x_1x_1 + 2x_0x_0x_1 + 3x_1x_0x_1 + 3x_0x_1x_1 + 3x_1x_1x_1 + \cdots,$$

and

$$f(x) = \sum_{n \geq 0} \left\lfloor \frac{1}{2} + \log_2(n + 1) \right\rfloor x^{\tau(n)}$$

$$= x + 2x^2 + 2x^2 + 2x^3 + 3x^3 + 3x^3 + 3x^4 + 3x^4 + 3x^4 + 4x^4 + \cdots$$

$$= x + 4x^2 + 11x^3 + 29x^4 + 74x^5 + 179x^6 + 422x^7 + 971x^8 + 2198x^9 + \cdots$$

$$= \sum_{m \geq 0} b(m)x^m.$$

To write $b(m)$ in closed form, we observe how the first few terms of $\{ \left\lfloor \frac{1}{2} + \log_2(n + 1) \right\rfloor \}_{n \geq 0}$ gather by exponent:

$$012233333444444444445555555555555555555566666666666666666666\cdots$$

Since the length of $n$ in binary is $|\tau(n)| = 1 + \lfloor \log_2 n \rfloor$ for $n \geq 1$, the difference $|\tau(n)| - \lfloor \frac{1}{2} + \log_2(n + 1) \rfloor$ between exponent and coefficient in each term of the first sum above is either 1 or 0. In other words, the only terms that contribute to $b(m)x^m$ are of the form $(m-1)x^m$ and $mx^m$, so for some sequence $\{ c(m) \}_{m \geq 1}$ we have

$$b(m) = (m-1)(c(m) - 2^{m-1}) + m(2^m - c(m))$$

for $m \geq 1$. In fact $c(m)$ is the smallest value of $n$ for which $\frac{1}{2} + \log_2(n + 1) \geq m$, so $c(m) = \lceil 2^{m-\frac{1}{2}} \rceil$ and $b(m) = (m+1)2^{m-1} - \lceil 2^{m-\frac{1}{2}} \rceil$ for $m \geq 1$. Therefore

$$f(x) = \frac{1}{2(1-2x)^2} - \frac{1}{2} - \sum_{m \geq 0} \left\lfloor 2^{m-\frac{1}{2}} \right\rfloor x^m,$$

where the term $-1/2$ is needed because $b(0) = 0$.

We carry out the preceding computation more generally to prove the theorem.
Proof. Let $\frac{\alpha}{k^n} = \alpha - \lfloor \alpha \rfloor$ denote the fractional part of $\alpha$. Then
\[
f(x) = \sum_{n \geq 0} [\alpha + \log_k(n + 1)] x^{[\tau(n)]}
= [\alpha + \log_k 1] + \sum_{m \geq 1} \sum_{i=k^{m-1}}^{k^m-1} [\alpha + \log_k(i + 1)] x^m
= [\alpha] + \sum_{m \geq 1} \left( \sum_{i=k^{m-1}}^{k^m-1} [\alpha + \log_k(i + 1)] + \frac{k^m - 1}{k - 1} \right) x^m.
\]
Since
\[
[\alpha + \log_k(i + 1)] = \begin{cases} 
[\alpha] + m - 1 & \text{if } k^{m-1} + 1 \leq i + 1 \leq k^m - 1 \\
[\alpha] + m & \text{if } k^m - 1 \leq i + 1 \leq k^m,
\end{cases}
\]
we have
\[
f(x) = [\alpha] + \sum_{m \geq 1} \left( \frac{k^{m-1}((k-1)(m+\lfloor \alpha \rfloor)+1)+1-k^m}{k-1} \right) x^m
= \frac{1-x}{1-kx} \frac{[\alpha] (1-kx)}{1-x} + \frac{x}{1-x} + \sum_{m \geq 1} \left\lfloor k^{-m} \right\rfloor x^m.
\]
The series $f(x)$ is therefore rational if and only if
\[
g(x) = -\left\lfloor k^{-m} \right\rfloor + \left\lfloor \frac{1}{x} - k \right\rfloor \sum_{m \geq 1} \left\lfloor k^{-m} \right\rfloor x^m
= \sum_{m \geq 1} \left( \left\lfloor k^{-m-1} \right\rfloor - k \left\lfloor k^{-m} \right\rfloor \right) x^m
\]
is rational. The expression $\lfloor k^{-m} \rfloor - k \lfloor k^{-m-1} \rfloor$ is the $(m)$th base-$k$ digit of $x$, so the coefficients of $g(x)$ are the base-$k$ digits of $\frac{\alpha}{k^n}$, which is rational precisely when $k^n$ is rational.

If $k^n$ is rational, then the coefficients of $g(x)$ are eventually periodic, so $g(x)$ and hence $f(x)$ is rational. If $k^n$ is irrational, then $g(x)$ is not rational, since in particular $g\left(\frac{1}{x}\right) = \frac{\alpha}{k^n}$ is irrational; therefore $f(x)$ is not rational. \qed

In fact we may show something stronger: Not only does $f(x_0, x_1, \ldots, x_{k-1})$ fail to be rational when $k^n$ is irrational, but it fails to be algebraic. Bell, Gerhold, Klazar, and Luca [3] Proposition 13) prove that if a polynomial-recursive sequence (a sequence satisfying a linear recurrence equation with polynomial coefficients) has only finitely many distinct values, then it is eventually periodic. It follows that the coefficient sequence of $g(x)$ is not polynomial-recursive, hence $g(x)$ is not algebraic, and $f(x, x, \ldots, x)$ is not algebraic.

References

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