REGULARITY OF FUNDAMENTAL SOLUTIONS FOR LÉVY-TYPE OPERATORS

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Abstract. For a class of non-symmetric non-local Lévy-type operators $L^\kappa$, which include those of the form

$$L^\kappa f(x) := \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - 1_{|z| < 1} \langle z, \nabla f(x) \rangle \right) \kappa(x, z) |z| dz,$$

we prove regularity of the fundamental solution $p^\kappa$ to the equation $\partial_t = L^\kappa$.

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1. Introduction

Lévy-type processes are stochastic models that can be used to approximate physical, biological or financial phenomena. A local expansion of such a process is described by its infinitesimal generator which is a Lévy-type operator $L$. The non-local part of that operator is responsible for and describes the jumps of the process. In the recent years a lot of effort has been put into understanding purely non-local Lévy-type operators. At first the case with constant coefficients attracted most of the attention, but the literature concerning non-constant coefficient is growing rapidly, including [12], [3], [10], [7], [15], [13], [1], [17], [9], [22], [23], [2], [11], [3]. The parametrix method [18], [5] used in those papers leads to a construction of the fundamental solution of the equation $\partial_t u = Lu$ or the heat kernel of the process that is a unique solution to the martingale problem for $L$.

The subject of the present paper is non-local Lévy-type operators with Hölder continuous coefficients considered in [7] and [22] (see Definition 1). A typical example here is the operator

$$L^\kappa f(x) = \int_{\mathbb{R}^d} \left( f(z + x) - f(x) - 1_{|z| < 1} \langle z, \nabla f(x) \rangle \right) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz,$$

where $\alpha \in (0, 2)$, $\kappa$ is bounded from below and above by positive constants, and $\beta$-Hölder continuous in the first variable with $\beta \in (0, 1)$. The usual result concerning the regularity of the fundamental solution of $\partial_t = L$ is $\gamma$-Hölder continuity with $\gamma < \alpha$. We improve that result in more general setting taking into account the $\beta$ regularity of the coefficient $\kappa$.

Of particular interest are the existence, estimates and regularity of the gradient of the fundamental solution for Lévy or Lévy-type operators [21],[16], [4], [8]; [14], [6]. In this context our assumptions will imply $\alpha > 1/2$. In a recent paper [19] this restriction was removed at the expense of additional constraints on the coefficient $\kappa$ requiring strong symmetry properties in $z$. As already mentioned, the purpose of the present paper is to cover a wide class of operators and coefficients discussed in [7] and [22]. In particular ones that are not symmetric in the $z$ variable, thus not considered in [19]. What is more, such non-symmetry may cause a (time

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dependent) non-zero internal drift with a coefficient that is unbounded as time tends to zero (see [22, Example 2]), which we cover in case (Q1) below.

Under certain conditions, which require \( \alpha > 1 \), we also prove existence, estimates and regularity of the second derivatives of the fundamental solution. To the best of the author’s knowledge, this result is new even for the operator \( \mathcal{L} \) above under any assumptions on the coefficient \( \kappa \) that is not constant in \( x \).

2. The setting and main results

Let \( d \in \mathbb{N} \) and \( \nu : [0, \infty) \to [0, \infty] \) be a non-increasing function satisfying

\[
\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(|x|) dx < \infty.
\]

For real numbers \( a \) and \( b \) we write as usual \( a \wedge b = \min(a, b) \) and \( a \vee b = \max(a, b) \). Consider \( J : \mathbb{R}^d \to [0, \infty] \) such that for some \( \gamma_0 \in [1, \infty) \) and all \( x \in \mathbb{R}^d \),

\[
\gamma_0^{-1} \nu(|x|) \leq J(x) \leq \gamma_0 \nu(|x|).
\]  

(1)

Further, suppose that \( \kappa(x, z) \) is a Borel function on \( \mathbb{R}^d \times \mathbb{R}^d \) such that

\[ 0 < \kappa_0 \leq \kappa(x, z) \leq \kappa_1, \]

and for some \( \beta \in (0, 1) \),

\[ |\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^\beta. \]  

(3)

For \( r > 0 \) we define

\[
h(r) := \int_0^\infty \left( 1 \wedge \frac{|x|^2}{r^2} \right) \nu(|x|) dx, \quad K(r) := r^{-2} \int_{|x| < r} |x|^2 \nu(|x|) dx.
\]

The above functions play a prominent role in the paper. Our main assumption is the weak scaling condition at the origin: there exist \( \alpha_h \in (0, 2] \) and \( C_h \in [1, \infty) \) such that

\[
h(r) \leq C_h \lambda^{\alpha_h} h(\lambda r), \quad \lambda \leq 1, r \leq 1,
\]  

(4)

and in a similar fashion, there exist \( \beta_h \in (0, 2] \) and \( c_h \in (0, 1] \) such that

\[
h(r) \geq c_h \lambda^{\beta_h} h(\lambda r), \quad \lambda \leq 1, r \leq 1.
\]  

(5)

Furthermore, suppose there are (finite) constants \( \kappa_3, \kappa_4 \geq 0 \) such that

\[
\sup_{x \in \mathbb{R}^d} \left| \int_{|z| < 1} z \kappa(x, z) J(z) dz \right| \leq \kappa_3 r h(r), \quad r \in (0, 1],
\]

(6)

\[
\left| \int_{|z| < 1} z \left[ \kappa(x, z) - \kappa(y, z) \right] J(z) dz \right| \leq \kappa_4 |x - y|^\beta r h(r), \quad r \in (0, 1].
\]  

(7)

Definition 1. We say that (PQ) holds if one of the following sets of assumptions is satisfied,

\begin{align*}
\text{(P1)} & \quad (1)-(4) \text{ hold and } 1 < \alpha_h \leq 2; \\
\text{(P2)} & \quad (1)-(5) \text{ hold and } 0 < \alpha_h \leq \beta_h < 1; \\
\text{(P3)} & \quad (1)-(4) \text{ hold, } J \text{ is symmetric and } \kappa(x, z) = \kappa(x, -z), \ x, z \in \mathbb{R}^d; \\
\text{(Q1)} & \quad (1)-(4) \text{ hold, } \alpha_h = 1; (6) \text{ and (7) hold}; \\
\text{(Q2)} & \quad (1)-(5) \text{ hold, } 0 < \alpha_h \leq \beta_h < 1 \text{ and } 1 - \alpha_h < \beta \land \alpha_h; (6) \text{ and (7) hold.}
\end{align*}

Our aim is to prove regularity of the heat kernel \( p^\kappa \) of a non-local non-symmetric Lévy-type operator \( \mathcal{L}^\kappa \), i.e., of a fundamental solution to the equation \( \partial_t u = \mathcal{L}^\kappa u \). For each of cases (P1), (Q1), (Q2) the operator under consideration is defined as

\[
\mathcal{L}^\kappa f(x) := \int_{\mathbb{R}^d} (f(x + z) - f(x) - 1_{|z| < 1} \langle z, \nabla f(x) \rangle) \kappa(x, z) J(z) dz.
\]  

(8)
If (P2) holds we consider
\[ \mathcal{L}^\kappa f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x))\kappa(x,z)J(z)\,dz. \] (9)

If (P3) holds we discuss
\[ \mathcal{L}^\kappa f(x) := \frac{1}{2} \int_{\mathbb{R}^d} (f(x+z) + f(x-z) - 2f(x))\kappa(x,z)J(z)\,dz. \] (10)

It was shown in [7, Theorem 1.1] and [22, Theorem 2.1] that under (PQ) the function \( p^\kappa \) exists and is unique within a certain class of functions. In fact, the heat kernel was constructed using the Levi’s parameterix method, i.e.,
\[ p^\kappa(t, x, y) = p^{\kappa^\ast}(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^{\kappa^\ast}(t-s, x, z)q(s, z, y)\,dz\,ds, \] (11)

where \( q(t, x, y) \) solves the equation
\[ q(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z)q(s, z, y)\,dz\,ds, \]

and \( q_0(t, x, y) = (\mathcal{L}_x^{\kappa^\ast} - \mathcal{L}_x^{\kappa^\ast})p^{\kappa^\ast}(t, x, y) \). The function \( p^{\kappa^\ast} \) is the heat kernel of the Lévy operator \( \mathcal{L}^{\kappa^\ast} \) obtained from the operator \( \mathcal{L}^\kappa \) by freezing its coefficient: \( \mathcal{R}_\kappa(z) = \kappa(w, z) \). For \( t > 0 \) and \( x \in \mathbb{R}^d \) we define the bound function
\[ \Upsilon_t(x) := \left( [h^{-1}(1/t)]^{-d} \land \frac{tK(|x|)}{|x|^d} \right). \] (12)

We refer the reader to [7, Theorems 1.2 and 1.4] and [22, Theorems 2.2 and 2.4] for a collection of properties of the function \( p^\kappa \), including estimates, Hölder continuity, differentiability and gradient estimates. For instance, under (PQ) for all \( T > 0 \) and \( \gamma \in [0, 1] \cap (0, \alpha_h) \), there is \( c > 0 \) such that for all \( t \in (0, T] \) and \( x, x', y \in \mathbb{R}^d \),
\[ |p^\kappa(t, x, y) - p^\kappa(t, x', y)| \leq c(|x - x'|^\gamma \land 1) \left[ h^{-1}(1/t) \right]^{-\gamma} \left( \Upsilon_t(y - x) + \Upsilon_t(y - x') \right). \]

Recall also that [7, Theorem 1.2 (6)] and [22, Theorem 2.2 (6)] provide an upper bound for \(|\nabla_x p^\kappa(t, x, y)|\) under conditions (PQ) and \( \alpha_h + \beta \land \alpha_h > 1 \).

Here are the main results of the present paper. For the meaning of \( \sigma_e \) see Definition 2.

**Theorem 2.1.** Assume (PQ). Let \( r_0 \in [0, 1] \cap [0, \alpha_h + \beta \land \alpha_h) \). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma_e, r_0) \) such that for all \( t \in (0, T] \), \( x, x', y \in \mathbb{R}^d \) and \( r \in [0, r_0] \),
\[ |p^\kappa(t, x, y) - p^\kappa(t, x', y)| \leq c \left( |x - x'|^{\gamma} \land 1 \right) \left[ h^{-1}(1/t) \right]^{-\gamma} \left( \Upsilon_t(y - x) + \Upsilon_t(y - x') \right). \]

**Theorem 2.2.** Assume (PQ) and suppose that \( \alpha_h + \beta \land \alpha_h > 1 \). Let \( r_0 \in [0, 1] \cap [0, \alpha_h + \beta \land \alpha_h - 1) \). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma_e, r_0) \) such that for all \( t \in (0, T] \), \( x, x', y \in \mathbb{R}^d \) and \( r \in [0, r_0] \),
\[ |\nabla_x p^\kappa(t, x, y) - \nabla_x p^\kappa(t, x', y)| \leq c \left( |x - x'|^{\gamma} \land 1 \right) \left[ h^{-1}(1/t) \right]^{1-\gamma} \left( \Upsilon_t(y - x) + \Upsilon_t(y - x') \right). \]

**Theorem 2.3.** Assume (PQ) and suppose that \( \alpha_h + \beta \land \alpha_h > 2 \). Let \( r_0 \in [0, 1] \cap [0, \alpha_h + \beta \land \alpha_h - 2) \). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma_e, r_0) \) such that for all \( t \in (0, T] \), \( x, x', y \in \mathbb{R}^d \) and \( r \in [0, r_0] \),
\[ |\nabla^2_x p^\kappa(t, x, y)| \leq c \left[ h^{-1}(1/t) \right]^{-2} \Upsilon_t(y - x), \]
\[ |\nabla^2_x p^\kappa(t, x, y) - \nabla^2_x p^\kappa(t, x', y)| \leq c \left( |x - x'|^{\gamma} \land 1 \right) \left[ h^{-1}(1/t) \right]^{-2-\gamma} \left( \Upsilon_t(y - x') + \Upsilon_t(y - x) \right). \]
The condition \( \alpha_h + \beta \land \alpha_h > 1 \) equivalently means that there is \( \beta_1 \in [0, \beta] \cap [0, \alpha_h) \) such that \( \alpha_h + \beta_1 > 1 \), and it may hold only if \( \alpha_h > 1/2 \). Similarly, \( \alpha_h + \beta \land \alpha_h > 2 \) is equivalent to the existence of \( \beta_1 \in [0, \beta] \cap [0, \alpha_h) \) such that \( \alpha_h + \beta_1 > 2 \), and it requires \( \alpha_h > 1 \). We note that even the existence of second derivatives of \( p^\# \) in Theorem 2.3 is a new result.

**Definition 2.** Following [7], in the case (P1), (P2), (P3) we respectively consider the set of parameters \( \sigma_1 = (\gamma_0, \kappa_0, \kappa_1, \alpha_h, C_h, h) \), \( \sigma_2 = (\gamma_0, \kappa_0, \kappa_1, \alpha_h, \beta_h, C_h, C_h, h) \), \( \sigma_3 = (\gamma_0, \kappa_0, \kappa_1, \alpha_h, C_h, h) \), which we abbreviate to \( \sigma \). Similarly, after [22] we put \( \sigma = (\gamma_0, \kappa_0, \kappa_1, \kappa_2, \alpha_h, C_h, h) \) under (Q1) or (Q2). We extend the set of parameters \( \sigma \) to \( \sigma_e \) by adding constant \( \kappa_2 \) in the cases (P1), (P2), (P3), and \( \kappa_2, \kappa_4 \) in the cases (Q1), (Q2). Abusing the notation we have \( \sigma_e = (\sigma, \kappa_2) \) under (P1), (P2), (P3), and \( \sigma_e = (\sigma, \kappa_2, \kappa_4) \) under (Q1), (Q2).

In the whole paper we assume that (PQ) holds.

### 3. Preliminaries

To prove the main results we will obviously use the representation (11) of \( p^\#(t, x, y) \). This formula consists of \( p^\#(
 x, y) \), which is known in the literature as the zero order approximation, and the integral part called the remainder. In the whole paper we follow and use consistently the notation of [7] and [22]. To study the remainder we let

\[
\phi_y(t, x, s) := \int_{\mathbb{R}^d} p^\#(t - s, x, z)q(s, z, y) \, dz, \quad x \in \mathbb{R}^d, \ 0 < s < t, \tag{13}
\]

and

\[
\phi_y(t, x) := \int_0^t \phi_y(t, x, s) \, ds = \int_0^t \int_{\mathbb{R}^d} p^\#(t - s, x, z)q(s, z, y) \, dz \, ds. \tag{14}
\]

See [7, Theorem 3.7] and [22, Theorem 6.2] for the definition of \( q(t, x, y) \) as a series. In our proofs we only use the properties of \( q(t, x, y) \) (mostly already known in [7] and [22]) rather than its concrete structure. We start by investigating the regularity of the zero order approximation and we do so in a slightly more general context that will be useful when dealing with the remainder. We introduce the following expression: for \( t > 0 \) and \( x, y, z \in \mathbb{R}^d \),

\[
\mathcal{F}_2(t, x, y; z) := \Upsilon_t(y - x - z)1_{|z| > h^{-1}(1/t)} + \left[ \frac{|z|}{h^{-1}(1/t)} \right] \land 1 \right] \Upsilon_t(y - x).
\]

In [7] and [22] also \( \mathcal{F}_1 \) was considered, but we will not need it in our analysis. In what follows functions, \( \mathcal{R} \) and \( p^\# \) are as in [7, Section 2] and [22, Section 4]. Namely, \( \mathcal{R}: \mathbb{R}^d \to [0, \infty) \) is such that

\[
0 < \kappa_0 \leq \mathcal{R}(z) \leq \kappa_1.
\]

Now, we consider an operator \( \mathcal{L}^\# \) defined by taking \( \kappa(x, z) = \mathcal{R}(z) \) in (8) for (P1), (Q1), (Q2); (9) for (P2); and (10) for (P3). The operator uniquely determines a Lévy process and its density \( p^\#(t, x, y) = p^\#(t, y - x) \), see [7, Section 6].

**Lemma 3.1.** For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma) \) such that for all \( t \in (0, T] \) and \( x, y, z \in \mathbb{R}^d \),

\[
|p^\#(t, x + z, y) - p^\#(t, x, y)| \leq c \mathcal{F}_2(t, x, y; z), \tag{15}
\]

\[
|\nabla_x p^\#(t, x + z, y) - \nabla_x p^\#(t, x, y)| \leq c [h^{-1}(1/t)]^{-1} \mathcal{F}_2(t, x, y; z), \tag{16}
\]

\[
|\nabla_x^2 p^\#(t, x + z, y) - \nabla_x^2 p^\#(t, x, y)| \leq c [h^{-1}(1/t)]^{-2} \mathcal{F}_2(t, x, y; z). \tag{17}
\]

**Proof.** The inequalities follow from [7, Proposition 2.1] and [22, Proposition 4.1] that provide upper bounds for derivatives of \( p^\# \) in spatial variable. If \( |z| \geq h^{-1}(1/t) \) we bound each term of the difference separately using these bounds. If \( |z| < h^{-1}(1/t) \) it suffices to write the increment
as an integral of the derivative and then use the bounds while removing small shifts in arguments using [7, Corollary 5.10]; cf. the proofs of [7, Lemma 2.3], [22, Lemma 4.3]. \hfill \Box

Inequalities (15), (16), (17) can be written equivalently as
\[
|p^\theta(t, x', y) - p^\theta(t, x, y)| \leq c \left( \frac{|x' - x|}{h^{-1}(1/t)} \wedge 1 \right) (\Upsilon_t(y - x') + \Upsilon_t(y - x)),
\]
\[
|\nabla_x p^\theta(t, x', y) - \nabla_x p^\theta(t, x, y)| \leq c \left( \frac{|x' - x|}{h^{-1}(1/t)} \wedge 1 \right) \left[ h^{-1}(1/t) \right]^{-1} (\Upsilon_t(y - x') + \Upsilon_t(y - x)),
\]
\[
|\nabla_x^2 p^\theta(t, x', y) - \nabla_x^2 p^\theta(t, x, y)| \leq c \left( \frac{|x' - x|}{h^{-1}(1/t)} \wedge 1 \right) \left[ h^{-1}(1/t) \right]^{-2} (\Upsilon_t(y - x') + \Upsilon_t(y - x)).
\]

**Corollary 3.2.** For every $T > 0$ there exists a constant $c = c(d, T, \sigma)$ such that for all $t \in (0, T]$, $x, x', y \in \mathbb{R}^d$ and $\gamma \in [0, 1]$,
\[
|p^\theta(t, x', y) - p^\theta(t, x, y)| \leq c(|x' - x|^{\gamma} \wedge 1) \left[ h^{-1}(1/t) \right]^{-\gamma} (\Upsilon_t(y - x') + \Upsilon_t(y - x)),
\]
\[
|\nabla_x p^\theta(t, x', y) - \nabla_x p^\theta(t, x, y)| \leq c(|x' - x|^{\gamma} \wedge 1) \left[ h^{-1}(1/t) \right]^{1-\gamma} (\Upsilon_t(y - x') + \Upsilon_t(y - x)),
\]
\[
|\nabla_x^2 p^\theta(t, x', y) - \nabla_x^2 p^\theta(t, x, y)| \leq c(|x' - x|^{\gamma} \wedge 1) \left[ h^{-1}(1/t) \right]^{2-\gamma} (\Upsilon_t(y - x') + \Upsilon_t(y - x)).
\]

Corollary 3.2 already covers the targeted regularity for the zero order approximation, because it can be applied to $\mathcal{R}_u(z) = \kappa(w, z)$ to get inequalities which are uniform in $w \in \mathbb{R}^d$, hence in particular uniform for the family of functions $\{p^\theta(t, x, y) : t \in (0, T], x, y \in \mathbb{R}^d\}$.

Now, we focus on the remainder (14). We will clearly have to use estimates for $q(t, x, y)$ and again inequalities (15)–(17). However, if we simply apply these two properties, we immediately run into the problem of integrability of $[h^{-1}(1/(t - s))]^{-\gamma}$ with respect to $s \in [t/2, t)$, imposing constraints on the value of $\tilde{\tau}$, which does not lead to our main results. Therefore, we have to find and exploit cancellations that take place in the integrals defining (14). We write
\[
\phi_y(t, x) - \phi_y(t, x') = \int_{0}^{t/2} (\phi_y(t, x, s) - \phi_y(t, x', s)) \, ds + \int_{t/2}^{t} (\phi_y(t, x, s) - \phi_y(t, x', s)) \, ds
\]
and we treat the second term as follows:
\[
\int_{t/2}^{t} (\phi_y(t, x, s) - \phi_y(t, x', s)) \, ds = \int_{t/2}^{t} \int_{\mathbb{R}^d} \left( p^\theta(t - s, x, z) - p^\theta(t - s, x', z) \right) q(s, z, y) \, dz \, ds
\]
\[
= \int_{t/2}^{t} \int_{\mathbb{R}^d} \left( p^\theta(t - s, x, z) - p^\theta(t - s, x', z) \right) (q(s, z, y) - q(s, x, y)) \, dz \, ds
\]
\[
+ \int_{t/2}^{t} \int_{\mathbb{R}^d} \left( p^\theta(t - s, x, z) - p^\theta(t - s, x', z) \right) q(s, x, y) \, dz \, ds.
\]
Similar, though slightly different, decompositions are used for the first and second order derivatives under appropriate assumptions. Roughly speaking, the goal is achieved by making use of the regularity of the coefficient $\kappa(x, z)$ in $x$, and the so called convolution inequalities that involve space or space-time integrals of the following functions for certain $\gamma, \beta \in \mathbb{R}$:
\[
\rho_{\beta}^\gamma(t, x) := \left[ h^{-1}(1/t) \right]^\gamma (|x|^{\beta} \wedge 1) t^{-1} \Upsilon_t(x).
\]
The inequalities are collected in [7, Lemma 5.17], and we will simply refer to that result whenever using them. The very initial step to detect cancellations uses the inequalities in [7, Theorem 2.11] and [22, Proposition 5.3]. We also note that for the second derivatives we have to prove more, because this case was less studied in [7] or [22].
**Remark 3.3.** We often use the monotonicity of the function $h$ and $h^{-1}$, see [7, Lemma 5.1], in particular $[h^{-1}(1/(t - s))]^\gamma \leq [h^{-1}(1/t)]^\gamma \leq h^{-1}(1/T) \vee 1$ holds for all $0 < s < t \leq T$, $\gamma \in [0, 1]$.

**Remark 3.4.** Certain technical results of [7], e.g., [7, Lemma 5.3] with $u = 1/t$, in view of (4) provide inequalities that hold for $t < 1/h(1)$. Using [7, Remark 5.2] we may extend those inequalities to hold for $t \in (0, T]$ by increasing the constant $C_h$ to $C_h[h^{-1}(1/T) \vee 1]^2$.

We will also need a slight improvement of [7, (38)] and [22, (54)] concerning the dependence of the constant on the parameter $\gamma > 0$.

**Lemma 3.5.** Let $\beta_1 \in (0, \beta] \cap (0, \alpha_{\bar{h}})$ and $0 < \gamma_1 \leq \beta_1$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \beta_1, \gamma_1)$ such that for all $t \in (0, T]$, $x, x', y \in \mathbb{R}^d$ and $\gamma \in [\gamma_1, \beta_1],$

\[
|q(t, x, y) - q(t, x', y)|
\leq c \left( |x - x'|^{\beta_1 - \gamma} \wedge 1 \right) \left\{ (\rho^0_{\gamma} + \rho^{\beta_1}_{\gamma - \beta_1})(t, x - y) + (\rho^0_{\gamma} + \rho^{\beta_1}_{\gamma - \beta_1})(t, x' - y) \right\} .
\]

**Proof.** This formulation of the Hölder continuity of $q$ has the same proof as [7, (38)]. One only needs to pay attention to explicit constants when applying [7, Lemma 5.17(c)]. In particular, the monotonicity of the Beta function is used. See also Remark 3.3. \qed

### 4. Regularity - Part I

We start with several technical lemmas before we prove the key Proposition 4.4.

**Lemma 4.1.** For every $T > 0$ there exists a constant $c = c(d, T, \sigma_e)$ such that for all $t \in (0, T]$ and $x, x', y, w, w' \in \mathbb{R}^d$ satisfying $|x - x'| \leq h^{-1}(1/t),$

\[
|p^{\bar{h}w}(t, x, y) - p^{\bar{h}w}(t, x', y) - (p^{\bar{h}w}(t, x, y) - p^{\bar{h}w}(t, x', y))|
\leq c \left( \frac{|x - x'|}{h^{-1}(1/t)} \right) \left( |w - w'|^\beta \wedge 1 \right) \Upsilon_t(y - x).
\]

**Proof.** Let $w_0 \in \mathbb{R}^d$ be fixed. Define $\mathcal{R}_0(z) = (\kappa_0/(2\kappa_1))\kappa(w_0, z)$ and $\bar{h}w(z) = \mathcal{R}_w(z) - \mathcal{R}_0(z)$. By the construction of the Lévy process we have

\[
p^{\bar{h}w}(t, x, y) = \int_{\mathbb{R}^d} p^{\bar{h}w}(t, x, \xi)p^{\bar{h}w}(t, \xi, y) d\xi .
\]

Thus,

\[
p^{\bar{h}w}(t, x, y) - p^{\bar{h}w}(t, x', y) - (p^{\bar{h}w}(t, x, y) - p^{\bar{h}w}(t, x', y))
= \int_{\mathbb{R}^d} \left( p^{\bar{h}w}(t, x, \xi) - p^{\bar{h}w}(t, x', \xi) \right) \left( p^{\bar{h}w}(t, \xi, y) - p^{\bar{h}w}(t, \xi, y) \right) d\xi .
\]

By (15) and [7, Theorem 2.11], [22, Proposition 5.3], for $|x - x'| \leq h^{-1}(1/t)$ we get

\[
|p^{\bar{h}w}(t, x, \xi) - p^{\bar{h}w}(t, x', \xi)| \leq c \left( \frac{|x - x'|}{h^{-1}(1/t)} \right) \Upsilon_t(\xi - x),
\]

\[
|p^{\bar{h}w}(t, \xi, y) - p^{\bar{h}w}(t, \xi, y)| \leq c(|w - w'|^\beta \wedge 1) \Upsilon_t(y - \xi) .
\]

Therefore,

\[
|p^{\bar{h}w}(t, x, y) - p^{\bar{h}w}(t, x', y) - (p^{\bar{h}w}(t, x, y) - p^{\bar{h}w}(t, x', y))|
\leq c \int_{\mathbb{R}^d} \left( \frac{|x - x'|}{h^{-1}(1/t)} \Upsilon_t(\xi - x)(|w - w'|^\beta \wedge 1) \Upsilon_t(y - \xi) d\xi .
\]
Now, by [7, Corollary 5.14 and Lemma 5.6] we get
\[ \int_{\mathbb{R}^d} \Upsilon_t(y - x) \Upsilon_t(x - x) \, dx \leq c \int_{\mathbb{R}^d} \Upsilon_{2t}(y - x) \left( \Upsilon_t(y - \xi) + \Upsilon_t(\xi - x) \right) \, d\xi \leq c \Upsilon_{2t}(y - x). \]

Finally, as in Remark 3.3, we get by the monotonicity that \([h^{-1}(1/(2t))]^{-d} \leq [h^{-1}(1/t)]^{-d},\)
which further gives \(\Upsilon_{2t}(y - x) \leq 2 \Upsilon_t(y - x). \) This completes the proof. \(\square\)

**Lemma 4.2.** Let \(\beta_1 \in [0, \beta] \cap [0, \alpha_b)\). For every \(T > 0\) there exists a constant \(c = c(d, T, \sigma_e, \beta_1)\) such that for all \(t \in (0, T], \ x, x' \in \mathbb{R}^d,\)
\[ \left| \int_{\mathbb{R}^d} \left( p^{R_h}(t, x, y) - p^{R_h}(t, x', y) \right) \, dy \right| \leq c \left[ h^{-1}(1/t) \right]^{\beta_1} \left( \frac{|x - x'|}{h^{-1}(1/t)} \right)^{\beta_1} \cdot \]

**Proof.** Let I be the left hand side of the inequality. Since \(\int_{\mathbb{R}^d} p^{R_h}(t, x, y) \, dy = 1, \) by [7, Theorem 2.11], [22, Proposition 5.3] and [7, Lemma 5.17(a)] we have
\[ \left| \int_{\mathbb{R}^d} p^{R_h}(t, x, y) \, dy - 1 \right| = \left| \int_{\mathbb{R}^d} \left( p^{R_h}(t, x, y) - p^{R_h}(t, x, y) \right) \, dy \right| \]
\[ \leq c \int_{\mathbb{R}^d} (|y - x|^{\beta_1} \amalg 1) \Upsilon_t(y - x) \, dy \leq c \left[ h^{-1}(1/t) \right]^{\beta_1} \cdot \]

Now, for \(|x - x'| \geq h^{-1}(1/t)\) we add and subtract 1 as follows:
\[ I = \left( \int_{\mathbb{R}^d} p^{R_h}(t, x, y) \, dy - 1 \right) - \left( \int_{\mathbb{R}^d} p^{R_h}(t, x, y) \, dy - 1 \right) \leq c \left[ h^{-1}(1/t) \right]^{\beta_1} \cdot \]

For \(|x - x'| \leq h^{-1}(1/t)\) we subtract zero, and use Lemma 4.1 and [7, Lemma 5.17(a)] to get
\[ I = \left| \int_{\mathbb{R}^d} \left( p^{R_h}(t, x, y) - p^{R_h}(t, x', y) - \left( p^{R_h}(t, x, y) - p^{R_h}(t, x', y) \right) \right) \, dy \right| \]
\[ \leq c \int_{\mathbb{R}^d} \left( \frac{|x - x'|}{h^{-1}(1/t)} \right) (|y - x|^{\beta_1} \amalg 1) \Upsilon_t(y - x) \, dy \leq c \left[ h^{-1}(1/t) \right]^{\beta_1} \left( \frac{|x - x'|}{h^{-1}(1/t)} \right)^{\beta_1} \cdot \]

**Lemma 4.3.** Let \(r_0 \in [0, 1] \cap [0, \alpha_b + \beta \amalg \alpha_b).\) For every \(T > 0\) there exists a constant \(c = c(d, T, \sigma_e, r_0)\) such that for all \(t \in (0, T], \ x, x', y \in \mathbb{R}^d \) and \(r \in [0, r_0],\)
\[ \int_{t/2}^t \int_{\mathbb{R}^d} \left( p^{R_h}(t, x, z) - p^{R_h}(t, x', z) \right) q(s, z, y) \, dz \, ds \leq c \left( |x - x'|^{\beta_1} \amalg 1 \right) \left[ h^{-1}(1/t) \right]^{-1 - \tau} \left( \Upsilon_t(y - x) + \Upsilon_t(y - x') \right). \]

**Proof.** We denote
\[ I := \int_{\mathbb{R}^d} \left( p^{R_h}(t, x, z) - p^{R_h}(t, x', z) \right) q(s, z, y) \, dz \]
\[ \leq \int_{\mathbb{R}^d} \left| p^{R_h}(t, x, z) - p^{R_h}(t, x', z) \right| \, |q(s, z, y) - q(s, x, y)| \, dz \]
\[ + \int_{\mathbb{R}^d} \left( p^{R_h}(t, x, z) - p^{R_h}(t, x', z) \right) \, |q(s, x, y)| \, dz \cdot \]

We start by investigating
\[ I_0 := \int_{\mathbb{R}^d} \left| p^{R_h}(t, x, z) - p^{R_h}(t, x', z) \right| \, |q(s, z, y) - q(s, x, y)| \, dz. \]
By (15) and $|q(s, z, y) - q(s, x, y)| \leq |q(s, z, y) - q(s, x', y)| + |q(s, x', y) - q(s, x, y)|$ we get

\[
I_0 \leq c \int_{\mathbb{R}^d} \left( \frac{|x - x'|}{h^{-1}(1/(t - s))} \right)^{\wedge} 1) |\gamma_{t-s}(x - z)| q(s, z, y) - q(s, x, y)|dz
\]

\[+ c \int_{\mathbb{R}^d} \left( \frac{|x - x'|}{h^{-1}(1/(t - s))} \right)^{\wedge} 1) |\gamma_{t-s}(x' - z)| q(s, z, y) - q(s, x', y)|dz
\]

\[+ c \int_{\mathbb{R}^d} \left( \frac{|x - x'|}{h^{-1}(1/(t - s))} \right)^{\wedge} 1) |\gamma_{t-s}(x' - z)| q(s, x, y) - q(s, x', y)|dz.
\]

Define

\[
I_1 = \int_{\mathbb{R}^d} (t - s)\rho_0^{\beta_1 - \gamma} (t - s, x - z) \left\{ (\rho_0^0 + \rho_0^{\beta_1}) (s, z - y) + (\rho_0^0 + \rho_0^{\beta_1}) (s, x - y) \right\} dz,
\]

\[
I_2 = \int_{\mathbb{R}^d} (t - s)\rho_0^{\beta_1 - \gamma} (t - s, x' - z) \left\{ (\rho_0^0 + \rho_0^{\beta_1}) (s, z - y) + (\rho_0^0 + \rho_0^{\beta_1}) (s, x' - y) \right\} dz,
\]

\[
I_3 = \left( |x - x'|^{\beta_1 - \gamma} \wedge 1 \right) \int_{\mathbb{R}^d} (t - s)\rho_0^0 (t - s, x' - z) \times \left\{ (\rho_0^0 + \rho_0^{\beta_1}) (s, x - y) + (\rho_0^0 + \rho_0^{\beta_1}) (s, x' - y) \right\} dz.
\]

Now, let $\beta_1 \in (0, \beta] \cap (0, \alpha_h)$ be such that $\alpha_h + \beta_1 - r_0 > 0$ and fix $0 < \gamma_1 \leq (\alpha_h + \beta_1 - r_0) \wedge \beta_1$. By Lemma 3.5 there is a constant $c = c(d, T, \sigma_{\varepsilon}, \beta_1, \gamma_1)$ such that for all $\gamma \in [\gamma_1, \beta_1]$,

\[
I_0 \leq c \left( \frac{|x - x'|}{h^{-1}(1/(t - s))} \right)^{\wedge} 1) \left( I_1 + I_2 + I_3 \right).
\]

In what follows we frequently replace $s \in (t/2, t)$ with $t$ due to the comparability of $h^{-1}(1/s)$ and $h^{-1}(1/t)$. More precisely, using monotonicity as in Remark 3.3, and additionally by (4), [7, Lemma 5.3] and Remark 3.4 for the lower bound,

\[
\frac{h^{-1}(1/t)}{(2C_1[h^{-1}(1/T) \wedge 1]^2)^{1/\alpha_h}} \leq h^{-1}(2/t) \leq h^{-1}(1/s) \leq h^{-1}(1/t).
\]

Next, by [7, Lemma 5.17(b)] with $\beta_0 = \beta_1$, $m_1 = n_1 = n_2 = \beta_1 - \gamma$, $m_2 = 0$, we have

\[
\int_{\mathbb{R}^d} (t - s)\rho_0^{\beta_1 - \gamma} (t - s, x - z)\rho_0^0 (s, z - y) dz \\
\leq c(t - s) \left[ h^{-1}(1/s) \right]^{\gamma} \left[ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma} + s^{-1} \left[ h^{-1}(1/s) \right]^{\beta_1 - \gamma} \right] \rho_0^0 (t, x - y) \\
\text{+} (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma} \rho_0^0 (t, x - y) + s^{-1} \rho_0^{\beta_1 - \gamma} (t, x - y) \\
\leq c \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma} \rho_0^0 (t, x - y) + c(t - s)^{-1} (\rho_0^0 + \rho_0^{\beta_1 - \gamma}) (t, x - y).
\]

By [7, Lemma 5.17(b)] with $\beta_0 = \beta_1$, $m_1 = \beta_1 - \gamma$, $m_2 = \beta_1$, $n_1 = n_2 = \beta_1$, we have

\[
\int_{\mathbb{R}^d} (t - s)\rho_0^{\beta_1 - \gamma} (t - s, x - z)\rho_0^{\beta_1 - \beta_1} (s, z - y) dz \\
\leq c(t - s) \left[ h^{-1}(1/s) \right]^{\gamma} \left[ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1} + s^{-1} \left[ h^{-1}(1/s) \right]^{\beta_1} \right] \rho_0^0 (t, x - y) \\
\text{+} (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1} \rho_0^0 (t, x - y) + s^{-1} \rho_0^{\beta_1 - \gamma} (t, x - y) \\
\leq c \left[ h^{-1}(1/(t - s)) \right]^{\beta_1} \rho_0^0 (t, x - y) + c(t - s)^{-1} (\rho_0^0 + \rho_0^{\beta_1 - \gamma}) (t, x - y) \\
\text{+} c \left[ h^{-1}(1/(t - s)) \right]^{\beta_1} \rho_0^{\beta_1 - \beta_1} (t, x - y).
This gives uniformly in $\gamma \in [\gamma_1, \beta_1]$,
\[
I_1 \leq c \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-\gamma} (\rho^{\alpha}_{\hat{\gamma}} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1})(t, x-y) \\
+ c (t-s)^{-1} (\rho^{\alpha}_{\beta_1} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1} + \rho^{\gamma}_{\hat{\gamma}})(t, x-y) \\
+ c \left[ h^{-1}(1/(t-s)) \right]^{\beta_1} \rho^{\beta_1}_{\hat{\gamma}-\beta_1}(t, x-y).
\]
We treat $I_2$ alike. Further, by [7, Lemma 5.6] we have
\[
\int_{\mathbb{R}^d} (t-s) \rho^{\alpha}_0(t-s, x'-z) (\rho^{\alpha}_{\hat{\gamma}} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1})(s, x-y) \, dz \leq c (\rho^{\alpha}_{\hat{\gamma}} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1})(t, x-y)
\]
and so uniformly in $\gamma \in [\gamma_1, \beta_1]$,
\[
I_3 \leq c (|x-x'|^{\beta_1-\gamma} \land 1) \left\{ (\rho^{\alpha}_{\hat{\gamma}} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1})(t, x-y) + (\rho^{\alpha}_{\hat{\gamma}} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1})(t, x'-y) \right\}.
\]
For each $r \in [0, r_0]$ we take $\gamma = \gamma_1 \lor (\beta_1 - r)$. Using the inequalities in Remark 3.3 we have
\[
\left( \frac{|x-x'|}{h^{-1}(1/(t-s))} \land 1 \right) \leq (|x-x'|^r \land [h^{-1}(1/T)]^r) \left[ h^{-1}(1/(t-s)) \right]^{-r} \\
\leq \left[ h^{-1}(1/T) \lor 1 \right]^{r_0} (|x-x'|^r \land 1) \left[ h^{-1}(1/(t-s)) \right]^{-r}.
\]
Note that $\beta_1 - \gamma - r \leq 0$ and, by considering $r \leq \beta_1 - \gamma_1$ and $r > \beta_1 - \gamma_1$, we obtain
\[
\frac{\beta_1 - \gamma - r}{\alpha_k} + 1 \geq \min \{ 1, \frac{\beta_1 - \gamma_1 - r_0}{\alpha_k} + 1 \} > 0.
\]
We also have $(-r/\alpha_k) + 2 \geq (-r_0/\alpha_k) + 2 > 0$ and $((\beta_1 - r)/2) \land ((\beta_1 - r)/\alpha_k) + 1 \geq ((\beta_1 - r_0)/2) \land ((\beta_1 - r_0)/\alpha_k) + 1 > 0$. Therefore, [7, Lemma 5.15] and the monotonicity of the Beta function provide, uniformly for all $r \in [0, r_0]$,
\[
\int_{t/2}^t \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-\gamma-r} ds \leq ct \left[ h^{-1}(1/t) \right]^{\beta_1-\gamma-r},
\]
\[
\int_{t/2}^t (t-s) \left[ h^{-1}(1/(t-s)) \right]^{-r} ds \leq ct^2 \left[ h^{-1}(1/t) \right]^{-r},
\]
\[
\int_{t/2}^t \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-r} ds \leq ct \left[ h^{-1}(1/t) \right]^{\beta_1-r}.
\]
Thus
\[
\int_{t/2}^t \left( \frac{|x-x'|}{h^{-1}(1/(t-s))} \land 1 \right) I_1 \, ds \leq c (|x-x'|^r \land 1) t \left[ h^{-1}(1/t) \right]^{-r} (\rho^{\alpha}_{\beta_1} + \rho^{\beta_1}_{\hat{\gamma}-\beta_1} + \rho^{\gamma}_{\hat{\gamma}})(t, x-y).
\]
We deal with the part containing $I_2$ in the same way. Similarly,
\[
\left( \frac{|x-x'|}{h^{-1}(1/(t-s))} \land 1 \right) \leq [h^{-1}(1/T) \lor 1]^{r_0} (|x-x'|^{r-(\beta_1-\gamma)} \land 1) \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-\gamma-r}
\]
and
\[
\int_{t/2}^t \left( \frac{|x-x'|}{h^{-1}(1/(t-s))} \land 1 \right) I_3 \, ds \leq c \int_{t/2}^t (|x-x'|^{r-(\beta_1-\gamma)} \land 1) \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-\gamma-r} I_3 \, ds \\
\leq c (|x-x'|^r \land 1) t \left[ h^{-1}(1/t) \right]^{-r} \left\{ (\rho^{\alpha}_{\beta_1} + \rho^{\beta_1}_{\hat{\gamma}})(t, x-y) + (\rho^{\alpha}_{\beta_1} + \rho^{\beta_1}_{\hat{\gamma}})(t, x'-y) \right\}.
\]
To sum up, we have, uniformly for all $r \in [0, r_0]$,
\[
\int_{t/2}^t I_0 \, ds \leq c (|x-x'|^r \land 1) \left[ h^{-1}(1/t) \right]^{-r} (\Upsilon_t(y-x) + \Upsilon_t(y-x')).
\]
Now, since \(|q(s, x, y)| \leq c(\rho_0^{\beta_1} + \rho_0^0)(s, x - y) \leq c(\rho_0^{\beta_1} + \rho_0^0)(t, x - y)\) (see \([7, (37)], [22, (53)]\)), together with Lemma 4.2 we get, uniformly for all \(r \in [0, r_0]\),

\[
\int_{t/2}^t \int_{\mathbb{R}^d} \left( p^{\xi_+}(t - s, x, z) - p^{\xi_+}(t, x, z) \right) dz \left| q(s, x, y) \right| ds \\
\leq c (|x - x'|^r \wedge 1) \int_{t/2}^t \left[ h^{-1}(1/(t - s)) \right]^\beta_1^{-r} ds (\rho_0^{\beta_1} + \rho_0^0)(t, x - y) \\
\leq c (|x - x'|^r \wedge 1) \left[ h^{-1}(1/t) \right]^{-r} (\Upsilon_t(y - x) + \Upsilon_t(y - x')).
\]

Finally, since

\[
\int_{t/2}^t I ds \leq \int_{t/2}^t I_0 ds + \int_{t/2}^t \int_{\mathbb{R}^d} \left( p^{\xi_+}(t - s, x, z) - p^{\xi_+}(t, x, z) \right) dz \left| q(s, x, y) \right| ds,
\]

the proof is complete. □

**Proposition 4.4.** Let \(r_0 \in [0, 1] \cap [0, \alpha_h + \beta \wedge \alpha_h]\). For every \(T > 0\) there exists a constant \(c = c(d, T, \sigma, r_0)\) such that for all \(t \in (0, T], x, x', y \in \mathbb{R}^d\) and \(r \in [0, r_0]\),

\[
|\phi_y(t, x) - \phi_y(t, x')| \leq c (|x - x'|^r \wedge 1) \left[ h^{-1}(1/t) \right]^{-r} (\Upsilon_t(y - x) + \Upsilon_t(y - x')).
\]

**Proof.** For \(s \in (0, t/2]\) we use Corollary 3.2 and \([7, (37)], [22, (53)]\) to get, for all \(r \in [0, 1]\),

\[
|\phi_y(t, x, s) - \phi_y(t, x', s)| \leq c (|x - x'|^r \wedge 1) \left[ h^{-1}(1/(t - s)) \right]^{-r} \int_{\mathbb{R}^d} (t - s) (\rho_0^{0}(t - s, x, z) + \rho_0^0(t - s, x', z) (s, z - y) dz.
\]

Here \(\beta_1 \in (0, \beta) \cap (0, \alpha_h)\) is fixed. Since \([7, Lemma 5.17(b)]\) and Remark 3.3 give

\[
\int_{\mathbb{R}^d} (t - s) \rho_0^0(t - s, x, z) (\rho_0^{\beta_1} + \rho_0^0)(s, z - y) dz \\
\leq c \left[ h^{-1}(1/(t - s)) \right]^{\beta_1} + h^{-1}(1/s) \left[ h^{-1}(1/s) \right]^{\beta_1} \rho_0^0(t, x - y) + \rho_0^{\beta_1}(t, x - y) \\
\leq c \left[ h^{-1}(1/t) \right]^{\beta_1} + ts^{-1} \left[ h^{-1}(1/s) \right]^{\beta_1} \rho_0^0(t, x - y) + \rho_0^{\beta_1}(t, x - y),
\]

and \([7, Lemma 5.3]\) gives \([h^{-1}(1/(t - s))]^{-r} \leq c [h^{-1}(1/t)]^{-r}\), by \([7, Lemma 5.15]\) we get

\[
\int_0^{t/2} \left| \phi_y(t, x, s) - \phi_y(t, x', s) \right| ds \\
\leq c (|x - x'|^r \wedge 1) \left[ h^{-1}(1/t) \right]^{-r} t \left( (\rho_0^{\beta_1} + \rho_0^0)(t, x - y) + (\rho_0^{\beta_1} + \rho_0^0)(t, x' - y) \right).
\]

For the remaining part of the integral with integration in \(s\) over \((t/2, t)\) we apply Lemma 4.3. □

**Proof of Theorem 2.1.** The result follows from \((11), Corollary 3.2\) and Proposition 4.4. □
5. REGULARITY - part II

In this section we assume that $\alpha_h + \beta \wedge \alpha_h > 1$. This condition necessitates $\alpha_h > 1/2$. The proofs here differ from those in Section 4; see Lemma 5.3.

**Lemma 5.1.** For every $T > 0$ there exists a constant $c = c(d, T, \sigma_e)$ such that for all $t \in (0, T]$, $x, x', y, w, w' \in \mathbb{R}^d$ satisfying $|x - x'| \leq h^{-1}(1/t)$,

$$\left| \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) - \left( \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) \right) \right|$$

$$\leq c \left( \frac{|x - x'|}{h^{-1}(1/t)} \right)^{\beta} \left[ h^{-1}(1/t) \right]^{-1} \mathcal{Y}_t(y - x).$$

**Proof.** Let $w_0 \in \mathbb{R}^d$ be fixed. Define $\mathcal{R}_0(z) = (\kappa_0/(2\kappa_1))\kappa(z, w_0)$ and $\mathcal{R}_w(z) = \mathcal{R}_w(z) - \mathcal{R}_0(z)$. By (18), (16) and [7, Theorem 2.11], [22, Proposition 5.3], for $|x - x'| \leq h^{-1}(1/t)$ we get

$$\left| \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) - \left( \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) \right) \right|$$

$$= \left| \int_{\mathbb{R}^d} \left( \nabla_x p^{\alpha}(t, x, \xi) - \nabla_x p^{\alpha}(t, x', \xi) \right) \left( p^{\alpha}(t, \xi, y) - p^{\alpha}(t, \xi, y) \right) d\xi \right|$$

$$\leq c \int_{\mathbb{R}^d} \left( \frac{|x - x'|}{h^{-1}(1/t)} \right)^{\beta} \mathcal{Y}_t(\xi - x) \left( |y - w'|^{\beta} \wedge 1 \right) \mathcal{Y}_t(y - \xi) d\xi$$

$$\leq c \left[ h^{-1}(1/t) \right]^{-1} \left( \frac{|x - x'|}{h^{-1}(1/t)} \right)^{\beta} \left( |y - w'|^{\beta} \wedge 1 \right) \mathcal{Y}_t(\xi - x).$$

[7, Corollary 5.14 and Lemma 5.6] have also been used in the last inequality. It remains to apply $\mathcal{Y}_t(y - x) \leq 2\mathcal{Y}_t(y - x)$; see Remark 3.3.

**Lemma 5.2.** Let $\beta_1 \in [0, \beta] \cap [0, \alpha_h]$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma_e, \beta_1)$ such that for all $t \in (0, T]$, $x, x' \in \mathbb{R}^d$,

$$\left| \int_{\mathbb{R}^d} \left( \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) \right) dy \right| \leq c \left[ h^{-1}(1/t) \right]^{-1+\beta_1} \left( \frac{|x - x'|}{h^{-1}(1/t)} \wedge 1 \right).$$

**Proof.** Let $I$ be the left hand side of the inequality. For $|x - x'| \geq h^{-1}(1/t)$ we conclude from [7, (29)] and [22, Lemma 5.5] that

$$I \leq c \left[ h^{-1}(1/t) \right]^{-1+\beta_1}.$$

For $|x - x'| \leq h^{-1}(1/t)$ we subtract zero, and use Lemma 5.1 and [7, Lemma 5.17(a)] to get

$$\left| \int_{\mathbb{R}^d} \left( \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) - \nabla_x p^{\alpha}(t, x, y) - \nabla_x p^{\alpha}(t, x', y) \right) dy \right|$$

$$\leq c \int_{\mathbb{R}^d} \left[ h^{-1}(1/t) \right]^{-1} \left( \frac{|x - x'|}{h^{-1}(1/t)} \wedge 1 \right) \mathcal{Y}_t(y - x) dy$$

$$\leq c \left[ h^{-1}(1/t) \right]^{-1+\beta_1} \left( \frac{|x - x'|}{h^{-1}(1/t)} \wedge 1 \right).$$

**Lemma 5.3.** Let $r_0 \in [0, 1] \cap [0, \alpha_h + \beta \wedge \alpha_h - 1]$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma_e, r_0)$ such that for all $t \in (0, T]$, $x, x' \in \mathbb{R}^d$ and $r \in [0, r_0]$,

$$\int_{t/2}^T \int_{\mathbb{R}^d} \left( \nabla_x p^{\alpha}(t - s, x, z) - \nabla_x p^{\alpha}(t - s, x', z) \right) q(s, z, y) dz ds \leq c \left( |x - x'|^r \wedge 1 \right) \left[ h^{-1}(1/t) \right]^{-1-r} \left( \mathcal{Y}_t(y - x) + \mathcal{Y}_t(y - x') \right).$$
Proof. Denote
\[ I := \left| \int_{\mathbb{R}^d} (\nabla x p^{R_s}(t-s,x,z) - \nabla x' p^{R_s}(t-s,x',z)) q(s,z,y) \, dz \right|. \]
Let \( \beta_1 \in (0,\beta] \cap (0,\alpha_k) \) be such that \( \alpha_k + \beta_1 - 1 - r_0 > 0 \). Fix \( 0 < \gamma < (\alpha_k + \beta_1 - 1 - r_0) \wedge \beta_1 \). We fist show that
\[ I \leq c \left( \frac{|x-x'|}{h^{-1}(1/(t-s))} \wedge 1 \right) \left( V(t,x-y;s) + V(t,x'-y;s) \right). \tag{19} \]
where
\[ V(t,x-y;s) := [h^{-1}(1/(t-s))]^{-1} \left\{ [h^{-1}(1/(t-s))]^{-\beta_1} \left( \rho_0^0 + \rho_{\gamma-\beta_1}^0 \right)(t,x-y) \right. \]
\[ + (t-s)t^{-1} \left( \rho_0^0 + \rho_{\gamma-\beta_1}^0 \right)(t,x-y) \left. \right\} \]
In what follows we frequently replace \( h^{-1}(1/s) \) with \( h^{-1}(1/t) \); see [7, Lemmas 5.1 and 5.3] and Remark 3.4. For \( |x-x'| \geq h^{-1}(1/(t-s)) \) we have, by [7, Proposition 2.1, (38), (29) and (37)] and [22, Proposition 4.1, (54), Lemma 5.5 and (53)],
\[ \left| \int_{\mathbb{R}^d} \nabla_x p^{R_s}(t-s,x,z) q(s,z,y) \, dz \right| \]
\[ \leq \int_{\mathbb{R}^d} \left| \nabla_x p^{R_s}(t-s,x,z) \right| q(s,z,y) - q(s,x,y) \, dz + \int_{\mathbb{R}^d} \nabla_x p^{R_s}(t-s,x,z) \, dz \right| q(s,x,y) \]
\[ \leq c \left[ h^{-1}(1/(t-s)) \right]^{-1} \left\{ \int_{\mathbb{R}^d} (t-s) \rho_0^0 \right. \gamma(t-s,x-z) (\rho_0^0 + \rho_{\gamma-\beta_1}^0)(s,z-y) \, dz \]
\[ + \left. (t-s) \rho_0^0 \gamma(t-s,x-z) \, dz (\rho_0^0 + \rho_{\gamma-\beta_1}^0)(t,x-y) \right. \]
\[ + \left. [h^{-1}(1/(t-s))]^{-\beta_1} (\rho_0^0 + \rho_{\gamma-\beta_1}^0)(t,x-y) \right\} =: R. \]
Now, (19) follows in this case from [7, Lemma 5.18(a) and (b)] (once with \( n_1 = n_2 = \beta_1 \)). For \( |x-x'| \leq h^{-1}(1/(t-s)) \), by (16), Lemma 5.2 and [7, (38), (37)], [22, (54), (53)] we have
\[ I \leq \int_{\mathbb{R}^d} \left| \nabla_x p^{R_s}(t-s,x,z) - \nabla_x p^{R_x}(t-s,x',z) \right| q(s,z,y) - q(s,x,y) \, dz \]
\[ + \int_{\mathbb{R}^d} \left| (\nabla_x p^{R_s}(t-s,x,z) - \nabla_x p^{R_s}(t-s,x',z)) \right| q(s,x,y) \, dz \leq c \left( \frac{|x-x'|}{h^{-1}(1/(t-s))} \right) R. \]
Here again (19) follows from [7, Lemma 5.18(a) and (b)]. Finally, since by our assumptions \((\beta_1-\gamma-1-r)/\alpha_k+1 \geq (\beta_1-\gamma-1-r_0)/\alpha_k+1 > 0 \) and \((1-r)/\alpha_k+2 \geq (1-r_0)/\alpha_k+2 > 0 \), inequality (19) and [7, Lemma 5.15] with the monotonicity of Beta function give, uniformly for all \( r \in [0,r_0] \),
\[ \int_{t/2}^t I \, ds \leq c (|x-x'| \wedge 1) \int_{t/2}^t \left[ h^{-1}(1/(t-s)) \right]^{-r} \left( V(t,x-y;s) + V(t,x'-y;s) \right) \, ds \]
\[ \leq c (|x-x'| \wedge 1) t \left[ h^{-1}(1/t) \right]^{-1-r} \left\{ (\rho_0^0 + \rho_{\gamma-\beta_1}^0)(t,x-y) + (\rho_0^0 + \rho_{\gamma-\beta_1}^0)(t,x'-y) \right\}. \]
This ends the proof (see Remark 3.3). \( \square \)
Proposition 5.4. Let $r_0 \in [0, 1] \cap [0, \alpha_h + \beta \wedge \alpha_h - 1]$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma_r, r_0)$ such that for all $t \in (0, T]$, $x, x', y \in \mathbb{R}^d$ and $r \in [0, r_0]$,
$$
|\nabla_x \phi_y(t, x) - \nabla_{x'} \phi_y(t, x')| \leq c (|x - x'|^r \wedge 1) \left[ h^{-1}(1/t) \right]^{-1-r} (\Upsilon_t(y - x) + \Upsilon_t(y - x')).
$$
Proof. Applying [7, (43) and (45)] and [22, (59) and (56)] we have
$$
\nabla_x \phi_y(t, x) - \nabla_{x'} \phi_y(t, x') = \int_0^t \left( \nabla_x \phi_y(t, x, s) - \nabla_{x'} \phi_y(t, x', s) \right) ds
$$
$$
= \int_0^t \int_{\mathbb{R}^d} \left( \nabla_x p^R(t - s, x, z) - \nabla_{x'} p^R(t - s, x', z) \right) q(s, z, y) dz ds.
$$
For $s \in (0, t/2]$ we find by Corollary 3.2 and [7, (37)], [22, (53)] that for all $r \in [0, 1]$,
$$
|\nabla_x \phi_y(t, x, s) - \nabla_{x'} \phi_y(t, x', s)| \leq c (|x - x'|^r \wedge 1) \left[ h^{-1}(1/(t - s)) \right]^{-1-r}
$$
$$
\times \int_{\mathbb{R}^d} (t - s) (\rho_0^0(t - s, x - z) + \rho_0^0(t - s, x' - z)) (\rho_0^{\beta_1} + \rho_0^0)(s, z - y) dz,
$$
where $\beta_1 \in (0, [\beta] \cap (0, \alpha_h)$ is fixed. The rest of this part of the proof is the same as that of Proposition 4.4, and relies on [7, Lemmas 5.17(b), 5.3 and 5.15], integration in $s \in (0, t/2]$ and Remark 3.3. For integration in $s$ over $(t/2, t)$ we apply Lemma 5.3.

Proof of Theorem 2.2. The result follows from (11), Corollary 3.2 and Proposition 5.4. □

6. Regularity - part III

In this section we assume that $\alpha_h + \beta \wedge \alpha_h > 2$. Note that this may only hold if $\alpha_h > 1$, which in turn puts us into case (P1) or (P3). We first prove that the second order derivatives of $p^\alpha(t, x, y)$ in $x$ actually exist. Such a result is missing in [7], therefore we first need to prepare several technical lemmas.

Lemma 6.1. For every $T > 0$ there exists a constant $c = c(d, T, \sigma_r)$ such that for all $t \in (0, T]$ and $x, x', y, w, w' \in \mathbb{R}^d$,
$$
|\nabla_x^2 p^R(t, x, y) - \nabla_{x'}^2 p^R(t, x, y)| \leq c (|w - w'|^{\beta} \wedge 1) \left[ h^{-1}(1/t) \right]^{-2} \Upsilon_t(y - x),
$$
and if $|x - x'| \leq h^{-1}(1/t)$, then
$$
|\nabla_x^2 p^R(t, x, y) - \nabla_{x'}^2 p^R(t, x', y) - (\nabla_x^2 p^R(t, x, y) - \nabla_{x'}^2 p^R(t, x', y))|
$$
$$
\leq c \left( \frac{|x - x'|}{h^{-1}(1/t)} \right) (|w - w'|^{\beta} \wedge 1) \left[ h^{-1}(1/t) \right]^{-2} \Upsilon_t(y - x).
$$
Proof. By (18) ((15) and (16) allow differentiating under the integral) we have
$$
\nabla_x^2 p^R(t, x, y) - \nabla_{x'}^2 p^R(t, x, y) = \int_{\mathbb{R}^d} \nabla_x^2 p^R(t, x, \xi) \left( p^\alpha(t, \xi, y) - p^\alpha(t, \xi, y) \right) d\xi,
$$
and
$$
|\nabla_x^2 p^R(t, x, y) - \nabla_{x'}^2 p^R(t, x', y) - (\nabla_x^2 p^R(t, x, y) - \nabla_{x'}^2 p^R(t, x', y))|
$$
$$
= \left| \int_{\mathbb{R}^d} \left( \nabla_x^2 p^R(t, x, \xi) - \nabla_{x'}^2 p^R(t, x', \xi) \right) \left( p^\alpha(t, \xi, y) - p^\alpha(t, \xi, y) \right) d\xi \right|.
$$
The desired inequalities follow from [7, Proposition 2.1, Theorem 2.11, Corollary 5.14 and Lemma 5.6] and (17); cf. proof of Lemma 5.1. □
Lemma 6.2. Let $\beta_1 \in [0, \beta] \cap [0, \alpha_b)$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \beta_1)$ such that for all $t \in (0, T)$ and $x, x' \in \mathbb{R}^d$,

$$\left| \int_{\mathbb{R}^d} \nabla_x^2 p_{R^3}(t, x, y) \, dy \right| \leq c \left[ h^{-1}(1/t) \right]^{-2+\beta_1}, \quad (22)$$

$$\left| \int_{\mathbb{R}^d} \left( \nabla_x^2 p_{R^3}(t, x, y) - \nabla_{x'}^2 p_{R^3}(t, x', y) \right) \, dy \right| \leq c \left[ h^{-1}(1/t) \right]^{-2+\beta_1} \left( \frac{|x-x'|}{h^{-1}(1/t)} \wedge 1 \right). \quad (23)$$

**Proof.** The proof of (22) is like that of [7, (29)], but it requires the use of (20). For the proof of (23) we employ (22) if $|x-x'| \geq h^{-1}(1/t)$, and we use (21) if $|x-x'| \leq h^{-1}(1/t)$; cf. proof of Lemma 5.2.

Lemma 6.3. For all $0 < s < t$ and $x, y \in \mathbb{R}^d$,

$$\nabla_x^2 \phi_y(t, x, s) = \int_{\mathbb{R}^d} \nabla_x^2 p_{R^1}(t-s, x, z) q(s, z, y) \, dz.$$  \hspace{1cm} (24)

**Proof.** By [7, (45)] we have $\nabla_x \phi_y(t, x, s) = \int_{\mathbb{R}^d} \nabla_x p_{R^1}(t-s, x, z) q(s, z, y) \, dz$. We obtain the result by applying this equality to the difference quotient $(\nabla_x \phi_y(t, x+\varepsilon e_i, s) - \nabla_x \phi_y(t, x, s))/\varepsilon$ and using the dominated convergence theorem justified by (16) and [7, (37), Lemma 5.17(b)].

Proposition 6.4. For every $T > 0$ there exists a constant $c = c(d, T, \alpha_e)$ such that for all $t \in (0, T)$ and $x, y \in \mathbb{R}^d$,

$$\nabla_x^2 \phi_y(t, x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x^2 p_{R^2}(t-s, x, z) q(s, z, y) \, dz \, ds,$$  \hspace{1cm} (25)

$$|\nabla_x^2 \phi_y(t, x)| \leq c \left[ h^{-1}(1/t) \right]^{-2} \Upsilon_t(x-y).$$

**Proof.** We choose $\beta_1 \in (0, \beta] \cap (0, \alpha_b)$ such that $\alpha_b + \beta_1 - 2 > 0$. Let $0 < |\varepsilon| \leq h^{-1}(3/t)$ and $x = x + \varepsilon \theta e_i$. Based on [20, Theorem 7.21] and Lemma 6.3 we have

$$I_0 := \left| \frac{1}{\varepsilon} \left( \frac{\partial}{\partial x_j} \phi_y(t, x+\varepsilon e_i, s) - \frac{\partial}{\partial x_j} \phi_y(t, x, s) \right) \right| = \left| \int_0^t \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_i \partial x_j} p_{R^1}(t-s, x, z) q(s, z, y) \, dz \, d\theta \right|.$$  \hspace{1cm} (26)

For $s \in (0, t/2)$ by [7, Proposition 2.1, (37), Lemmas 5.17(b) and 5.3, and Proposition 5.8] and Remark 3.3,

$$I_0 \leq c \int_0^t \int_{\mathbb{R}^d} \rho_{\alpha_b}^0(t-s, x-z) \left( \rho_{\beta_1}^0 + \rho_{\beta_1}^0 \right)(s, z-y) \, dz \, d\theta$$

$$\leq c \left[ h^{-1}(1/t-s) \right]^{\beta_1} \left( 1 + [h^{-1}(1/s)]^{\beta_1} + (t-s)s^{-1} [h^{-1}(1/s)]^{\beta_1} \right)$$

$$\leq c \left[ h^{-1}(1/t) \right]^{\beta_1} \rho_{\alpha_b}^0(t, x-y) \left( 1 + t s^{-1} [h^{-1}(1/s)]^{\beta_1} \right).$$

Next, for $s \in (t/2, t)$ we take $0 < \gamma < (\alpha_b + \beta_1 - 2) \wedge \beta_1$ and by [7, Proposition 2.1, (38), (37)] and (22) we obtain

$$I_0 \leq \int_0^t \int_{\mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_j} p_{R^1}(t-s, x, z) \right| q(s, z, y) - q(s, x, y) \, dz \, d\theta$$

$$+ \int_0^1 \left| \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_i \partial x_j} p_{R^1}(t-s, x, z) \, dz \right| q(s, x, y) \, d\theta.$$




\[ I_1 \leq c \left( [h^{-1}(1/(t-s))]^{-2+\beta_1-\gamma} \rho^0_{\gamma-\beta_1}(t, x-y) + (t-s) [h^{-1}(1/(t-s))]^{-2} t^{-1} \rho^0_h(t, x-y) \right), \]
\[ I_2 \leq c \left( [h^{-1}(1/(t-s))]^{-2+\beta_1-\gamma} \rho^0_{\gamma-\beta_1}(t, x-y), \quad I_3 \leq c \left( [h^{-1}(1/(t-s))]^{-2+\beta_1} \rho^0_h(t, x-y) \right). \]

Thus \( I_0 \) is bounded by a function independent of \( \varepsilon \), which is integrable in \( s \) over \( (0, t) \) due to \([7, \text{Lemma 5.15}]\) and our assumptions that guarantee \((\beta_1-\gamma-2)/\alpha_h+1 > 0\) and \((-2)/\alpha_h+2 > 0\).

Now, by \([7, (43)\) and \((45)\) we have
\[ \frac{1}{\varepsilon} \left( \frac{\partial}{\partial x_j} \phi_y(t, x+\varepsilon e_i) - \frac{\partial}{\partial x_j} \phi_y(t, x) \right) = \int_0^{1/t} \frac{1}{\varepsilon} \left( \frac{\partial}{\partial x_j} \phi_y(t, x+\varepsilon e_i, s) - \frac{\partial}{\partial x_j} \phi_y(t, x, s) \right) ds, \]
and we use the dominated convergence theorem and Lemma 6.3 to reach (24). The estimate (25) follows from integrating the upper bound of \( I_0 \) and applying \([7, \text{Lemma 5.15}]\). \(\square\)

We now concentrate on the regularity of \( \nabla^2_x \phi_y(t, x) \).

**Lemma 6.5.** Let \( r_0 \in [0, 1] \cap [0, \alpha_h + \beta \land \alpha_h - 2) \). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma_e, r_0) \) such that for all \( t \in (0, T] \), \( x, x' \in \mathbb{R}^d \) and \( r \in [0, r_0] \),
\[ \left| \int_{t/2}^t \left| \int_{\mathbb{R}^d} (\nabla^2_x p^h(t-s, x, z) - \nabla^2_x p^h(t-s, x', z)) q(s, z, y) \, dz \right| ds \right| \leq c (|x-x'|^r \land 1) \left[ h^{-1}(1/t) \right]^{-2-r} \left( \Upsilon_t(y-x) + \Upsilon_t(y-x') \right). \]

**Proof.** The proof goes by the same lines as the proof of Lemma 5.3 with 1 replaced by 2 in the choice of \( \beta_1 \) and \( \gamma \), \([h^{-1}(1/u)]^{-1} \) replaced by \([h^{-1}(1/u)]^{-2} \), \([16] \) by \([17], [7, (29)]\) by \((22)\), and Lemma 5.2 by \((23)\). Note also that by our assumptions, \((\beta_1-\gamma-2-r)/\alpha_h+1 \geq (\beta_1-\gamma-2-r_0)/\alpha_h+1 > 0\) and \((-2-r)/\alpha_h+2 \geq (-2-r_0)/\alpha_h+2 > 0\). \(\square\)

**Proposition 6.6.** Let \( r_0 \in [0, 1] \cap [0, \alpha_h + \beta \land \alpha_h - 2) \). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma_e, r_0) \) such that for all \( t \in (0, T] \), \( x, x', y \in \mathbb{R}^d \) and \( r \in [0, r_0] \),
\[ \left| \nabla^2_x \phi_y(t, x) - \nabla^2_x \phi_y(t, x') \right| \leq c (|x-x'|^r \land 1) \left[ h^{-1}(1/t) \right]^{-2-r} \left( \Upsilon_t(y-x) + \Upsilon_t(y-x') \right). \]

**Proof.** The result follows from (24), Lemma 6.3, Corollary 3.2, \([7, (37)\), Lemmas 5.17(b), 5.3 and 5.15], integration in \( s \in (0, t/2) \), Remark 3.3 and Lemma 6.5; cf. proof of Proposition 5.4. \(\square\)

**Proof of Theorem 2.3.** From (11) and (24) we get the second order differentiability of \( p^h(t, x, y) \) in \( x \). By \([7, \text{Proposition 2.1}]\) and (25) we obtain the upper bound. Finally, Corollary 3.2 and Proposition 6.6 give the regularity. \(\square\)
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References

[1] K. Bogdan, P. Sztonyk, and V. Knopova. Heat kernel of anisotropic nonlocal operators. Doc. Math., 25:1–54, 2020.
[2] B. Böttcher. A parametrix construction for the fundamental solution of the evolution equation associated with a pseudo-differential operator generating a Markov process. Math. Nachr., 278(11):1235–1241, 2005.
[3] Z.-Q. Chen and X. Zhang. Heat kernels and analyticity of non-symmetric jump diffusion semigroups. Probab. Theory Related Fields, 165(1-2):267–312, 2016.
[4] K. Du and X. Zhang. Optimal gradient estimates of heat kernels of stable-like operators. Proc. Amer. Math. Soc., 147(8):3559–3565, 2019.
[5] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei. Analytic methods in the theory of differential and pseudo-differential equations of parabolic type, volume 152 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2004.
[6] C. L. Epstein and C. A. Pop. Regularity for the supercritical fractional Laplacian with drift. J. Geom. Anal., 26(2):1231–1268, 2016.
[7] T. Grzywny and K. Szczypkowski. Heat kernels of non-symmetric Lévy-type operators. J. Differential Equations, 267(10):6004–6064, 2019.
[8] T. Grzywny and K. Szczypkowski. Estimates of heat kernels of non-symmetric Lévy processes. Forum Math., 33(5):1207–1236, 2021.
[9] P. Jin. Heat kernel estimates for non-symmetric stable-like processes. Preprint, arXiv:1709.02836.
[10] P. Kim, R. Song, and Z. Vondraček. Heat Kernels of Non-symmetric Jump Processes: Beyond the Stable Case. Potential Anal., 49(1):37–90, 2018.
[11] V. Knopova and A. Kulik. Parametrix construction for certain Lévy-type processes. Random Oper. Stoch. Equ., 23(2):111–136, 2015.
[12] V. Knopova and A. Kulik. Intrinsic compound kernel estimates for the transition probability density of Lévy-type processes and their applications. Probab. Math. Statist., 37(1):53–100, 2017.
[13] V. Knopova and A. Kulik. Parametrix construction of the transition probability density of the solution to an SDE driven by α-stable noise. Ann. Inst. Henri Poincaré Probab. Stat., 54(1):100–140, 2018.
[14] A. Kohatsu-Higa and L. Li. Regularity of the density of a stable-like driven SDE with Hölder continuous coefficients. Stoch. Anal. Appl., 34(6):979–1024, 2016.
[15] F. Kühn. Transition probabilities of Lévy-type processes: parametrix construction. Math. Nachr., 292(2):358–376, 2019.
[16] T. Kulczycki and M. Ryznar. Gradient estimates of harmonic functions and transition densities for Lévy processes. Trans. Amer. Math. Soc., 368(1):281–318, 2016.
[17] T. Kulczycki and M. Ryznar. Transition density estimates for diagonal systems of SDEs driven by cylindrical α-stable processes. ALEA Lat. Am. J. Probab. Math. Stat., 15(2):1335–1375, 2018.
[18] E. E. Levi. Sulle equazioni lineari totalmente ellittiche alle derivate parziali. Rend. Circ. Mat. Palermo, 24:275–317, 1907.
[19] W. Liu, R. Song, and L. Xie. Gradient estimates for the fundamental solution of Lévy type operator. Adv. Nonlinear Anal., 9(1):1453–1462, 2020.
[20] W. Rudin. Real and complex analysis. McGraw-Hill Book Co., New York, third edition, 1987.
[21] R. L. Schilling, P. Sztonyk, and J. Wang. Coupling property and gradient estimates of Lévy processes via the symbol. Bernoulli, 18(4):1128–1149, 2012.
[22] K. Szczypkowski. Fundamental solution for super-critical non-symmetric Lévy-type operators. Accepted in Adv. Differential Equations, arXiv:1807.04257.
[23] L. Xie and X. Zhang. Heat kernel estimates for critical fractional diffusion operators. Studia Math., 224(3):221–263, 2014.

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