On a Kinetic Equation Describing the Behavior of a Gas Interacting Mainly with Radiation

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Abstract
In this article we study a kinetic model which describes the interaction between a gas and radiation. Specifically, we consider a scaling limit in which the interaction between the gas and the photons takes place much faster than the collisions between the gas molecules themselves. We prove in the homogeneous case that the solutions of the limit problem solve a kinetic equation for which a well-posedness theory is considered. The proof of the convergence to a new kinetic equation is obtained analyzing the dynamics of the gas–photon system near the slow manifold of steady states.

Keywords Boltzmann equation · Radiative transfer equation · Nonelastic collision · Slow manifold

1 Introduction

In this paper we study a kinetic model that describes the interaction between the molecules of a gas and radiation. In the considered model it is assumed that the gas molecules can be in a ground and in an excited state. The transition between these two states takes place either due to nonelastic collisions between two molecules in the ground state, or by the absorption and emission of photons. The evolution of the radiation density is described also by means of a kinetic equation, specifically the radiative transfer equation in which the emission and absorption by the gas molecules are included. In spite of its simplicity the model provides valuable insights about the behavior of radiative gases.

We will denote as \( F^1 = F^1(t, v) \) the density in the phase space of gas molecules in the ground state, and as \( F^2 = F^2(t, v) \) the density of gas molecules in the excited state. We consider in this paper only homogeneous distributions in space (due to that \( F^1 \) and \( F^2 \) are independent of \( x \)).

The most peculiar feature of the model considered in this paper is that it is assumed that the fastest kinetic process is the interaction between the photons and the gas molecules. As
a consequence we have the following relation

\[ F^2 = \lambda F^1, \quad (1.1) \]

where \( \lambda = \lambda(t) \) is a function that depends on the intensity of radiation, which in fact depends on the initial amount of radiation present on the system. In the limit described above, which from now on will be referred to as the fast radiation limit, we will obtain that (1.1) holds approximately during most of the evolution. We will derive also equations describing the dynamic of \( \lambda(t) \) and \( F^1(t, v) \) which change in the time scale of the mean free time between the collisions of the gas.

### 1.1 The Model

The model considered in the paper, which has been already introduced in [15, 20, 24], is a simple model of gas–photon interaction. In this model it is assumed that the gas molecules can be in two states, namely the ground state \( A \) and the excited state \( \bar{A} \). The difference of energy between these states is given by \( \varepsilon_0 = h\nu_0 \), where \( \nu_0 \) is the frequency of the photons \( \gamma \). The radiative process in the system is assumed to be monochromatic. In particular all the effects yielding frequency dispersion like Doppler effect and line width are neglected. The considered system is also open with the exterior to the exchange of photons but not to the exchange of gas molecules. This is for example the situation of stellar atmospheres.

In our model we only deal with three possible types of interaction: elastic collisions, nonelastic collisions and radiative processes. In this paper we neglect all the possible processes which involve three or more particles.

The elastic collision is one of the following processes

\[ A + A \rightleftharpoons A + A, \]
\[ A + \bar{A} \rightleftharpoons A + \bar{A}, \]
\[ \bar{A} + \bar{A} \rightleftharpoons \bar{A} + \bar{A}. \quad (1.2) \]

The nonelastic collisions are only between ground state gas molecules, which colliding lose kinetic energy. This is absorbed by an electron yielding one of the molecule to be in the excited state. They can be schematically represented by

\[ A + A \rightleftharpoons A + \bar{A}. \quad (1.3) \]

Finally, the radiative processes are only of the following ones

\[ A + \gamma \rightleftharpoons \bar{A}. \quad (1.4) \]

In this case the absorption of a photon by a ground state molecule creates an excited state molecule. Also the emission (spontaneous or stimulated) of a photon due to the de-excitation of a molecule might happen. This is the so called bound-bound transition as explained in [16, 23, 25]. We neglect the momentum of the photons in all the interactions. Moreover, we assume that for every interaction the inverse process is also possible.

Elastic collisions as in (1.2) are characterized by the relation between pre-collisional velocities \( (v_1 \) and \( v_2) \) and post-collisional velocities \( (v_3 \) and \( v_4) \). This is given by the conservation of momentum and of kinetic energy

\[ v_1 + v_2 = v_3 + v_4 \quad \text{and} \quad |v_1|^2 + |v_2|^2 = |v_3|^2 + |v_4|^2. \quad (1.5) \]
Therefore we conclude also
\[ v_{3,4} = \frac{v_1 + v_2}{2} \pm \frac{|v_1 - v_2|}{2} \omega \quad \text{for} \quad \omega \in S^2. \tag{1.6} \]

Similarly, nonelastic collisions as in (1.3) are also characterized by the relation between pre-collisional velocities \((\overline{v}_1 \text{ and } \overline{v}_2)\) and post-collisional velocities \((\overline{v}_3 \text{ and } \overline{v}_4)\). From the conservation of momentum and total energy we compute
\[ \overline{v}_1 + \overline{v}_2 = \overline{v}_3 + \overline{v}_4 \quad \text{and} \quad |\overline{v}_1|^2 + |\overline{v}_2|^2 = |\overline{v}_3|^2 + |\overline{v}_4|^2 + 2\varepsilon_0. \tag{1.7} \]
and therefore the relation between pre- and post-collisional velocities is given by
\[ \overline{v}_{3,4} = \frac{\overline{v}_1 + \overline{v}_2}{2} \pm \omega \sqrt{\frac{|\overline{v}_1 - \overline{v}_2|^2}{4} - \varepsilon_0} \quad \text{for} \quad \omega \in S^2 \tag{1.8} \]
and
\[ \overline{v}_{1,2} = \frac{\overline{v}_3 + \overline{v}_4}{2} \pm \omega \sqrt{\frac{|\overline{v}_3 - \overline{v}_4|^2}{4} + \varepsilon_0} \quad \text{for} \quad \omega \in S^2. \tag{1.9} \]

We have already introduced the gas density functions. The behavior of the photons is usually described by the intensity of radiation \(I(t, x, n)\), where \(n \in S^2\). This is the energy flux due to the radiative process. However, for the description of this model we will use the photon number density \(Q(t, x, n)\), which is the number of photons per unit volume per unit frequency. This latter function is related to the former one by the identity \(I = ch\nu_0Q\) (cf. [23]), where \(c\) is the speed of light and \(h\) the Planck constant.

In this paper we will restrict ourselves to the homogeneous problem only, so \(F_i\) and \(Q\) would be independent of \(x\). We combine the kinetic equations for the elastic and nonelastic collisions, which involve several Boltzmann operators, with the radiative transfer equation. According to paper [15] in the homogeneous case the following system of kinetic equation is then obtained, which describes the behavior of the gas densities and the photons in the model under consideration.

\[
\frac{\partial F^1(v)}{\partial t} = \sum_{i=1}^{2} \int_{\mathbb{R}^3} d\nu_2 \int_{S^2} d\omega \ B_{el}(\nu, \nu_2) \left( F^1(\nu_2)F^i(\nu_4) - F^1(\nu)F^i(\nu_2) \right) \\
+ 2 \int_{\mathbb{R}^3} d\nu_2 \int_{S^2} d\omega \ B_{ne}^{12}(\nu, \nu_2) \left( F^2(\nu_3)F^1(\nu_4) - F^1(\nu)F^1(\nu_2) \right) \\
+ \int_{\mathbb{R}^3} d\nu_3 \int_{S^2} d\omega \ B_{ne}^{34}(\nu_3, \nu) \left( F^1(\nu_1)F^1(\nu_2) - F^2(\nu_3)F^1(\nu) \right) \\
+ \int_{S^2} dn \ A_0 F^2(v) + B_0 Q(n) \left( F^2(v) - F^1(v) \right) \tag{1.10}
\]

\[
\frac{\partial F^2(v)}{\partial t} = \sum_{i=1}^{2} \int_{\mathbb{R}^3} d\nu_2 \int_{S^2} d\omega \ B_{el}(\nu, \nu_2) \left( F^2(\nu_2)F^i(\nu_4) - F^2(\nu)F^i(\nu_2) \right) \\
+ \int_{\mathbb{R}^3} d\nu_4 \int_{S^2} d\omega \ B_{ne}^{34}(\nu, \nu_4) \left( F^1(\nu_1)F^1(\nu_2) - F^2(\nu_3)F^1(\nu) \right) \\
- \int_{S^2} dn \ A_0 F^2(v) + B_0 Q(n) \left( F^2(v) - F^1(v) \right)
\]

\[
\frac{\partial Q(n)}{\partial t} = \int_{\mathbb{R}^3} dv \ A_0 F^2(v) + B_0 Q(n) \left( F^2(v) - F^1(v) \right)
\]
We define $I := (F^1, F^2, Q)$ and we consider $F^i(t, v) \geq 0$, $Q(t, n) \geq 0$. Moreover, the collision kernels in the case of hard spheres are given by

\[ B_{el}(v_1, v_2) := |v_1 - v_2|, \]
\[ B_{ne}^{12}(\overline{v}_1, \overline{v}_2) := C_0 \frac{\sqrt{\overline{v}_1 - \overline{v}_2|^2 - 4\varepsilon_0}}{2|\overline{v}_1 - \overline{v}_2|}|\overline{v}_1 - \overline{v}_2|, \]
\[ B_{ne}^{34}(\overline{v}_3, \overline{v}_4) := C_0 \frac{\sqrt{\overline{v}_3 - \overline{v}_4|^2 + 4\varepsilon_0}}{2|\overline{v}_3 - \overline{v}_4|}|\overline{v}_3 - \overline{v}_4|. \]  
\(1.11\)

For their full derivation we refer to [15].

The constants in the radiative terms are defined as $A_0 = \frac{B_{12}2\varepsilon_0}{4\pi \nu^2} \geq 0$ and $B_0 = \frac{B_{12}c\nu_0}{4\pi} \geq 0$, where $B_{12}$ is the Einstein’s absorption coefficient. For a detailed derivation of the radiative transfer equation we refer to [16, 23, 25].

**Remark** It is well known that

\[ |v - w| \leq (1 + |v|^2)^{1/2}(1 + |w|^2)^{1/2}. \]  
\(1.12\)

This implies the following bounds of the kernels for the nonelastic collisions

\[ B_{ne}^{12}(\overline{v}_1, \overline{v}_2) \leq C_0 \frac{\sqrt{|\overline{v}_1 - \overline{v}_2|^2 + 4\varepsilon_0}}{2} \]
\[ \leq C_0 \frac{|\overline{v}_1 - \overline{v}_2| + 2\sqrt{\varepsilon_0}}{2} \]
\[ \leq \frac{C_0}{2}(1 + 2\sqrt{\varepsilon_0})(1 + |\overline{v}_1|^2)^{1/2}(1 + |\overline{v}_2|^2)^{1/2}, \]  
\(1.13\)

\[ B_{ne}^{34}(\overline{v}_3, \overline{v}_4) \leq \frac{C_0}{2}(1 + 2\sqrt{\varepsilon_0})(1 + |\overline{v}_3|^2)^{1/2}(1 + |\overline{v}_4|^2)^{1/2}. \]  
\(1.14\)

**Remark** In this paper we assume that the interaction of the gas molecules is described by means of hard spheres. We can obtain the same results also in the case of more general hard potentials. In that case instead of $|v_1 - v_2|$ we consider for the elastic kernel the function $b(\cos \theta)|v_1 - v_2|^\gamma$ for $\gamma \in (0, 2]$ and for the nonelastic kernels expressions of the form

\[ B_{ne}(\overline{v}, \overline{w}) := C_0 \frac{\sqrt{|\overline{v} - \overline{w}|^2 + 4\varepsilon_0}}{2|\overline{v} - \overline{w}|}b(\cos \theta)|\overline{v} - \overline{w}|^\gamma, \]

where $\theta$ is the angle between the difference of pre- and post-collisional velocities.

In order to simplify the reading we introduce the following notation

\[ K_1[I, F](v) = \sum_{i=1}^{2} \int \mathbb{R}^3 d\nu_2 \int S^2 d\omega B_{el}(v, \nu_2) \left( F^1(\nu_3)F^i(\nu_4) - F^1(v)F^i(\nu_2) \right) \]
\[ + 2 \int \mathbb{R}^3 d\nu_2 \int S^2 d\omega B_{ne}^{12}(\nu_1, \nu_2) \left( F^2(\nu_3)F^1(\nu_4) - F^1(v)F^1(\nu_2) \right) \]
\[ + \int \mathbb{R}^3 d\nu_3 \int S^2 d\omega B_{ne}^{34}(\nu_3, \nu) \left( F^1(\nu_1)F^1(\nu_2) - F^2(\nu_3)F^1(v) \right) \]  
\(1.15\)
and similarly
\[
\mathbb{K}_2 [\mathcal{F}, \mathcal{F}] (v) = \sum_{i=1}^{2} \int_{\mathbb{R}^3} d v_2 \int_{\mathbb{S}^2} d \omega \, B_{el}(v, v_2) \left( F^2(v_3) F^i(v_4) - F^2(v) F^i(v_2) \right) + \int_{\mathbb{R}^3} d \nu_4 \int_{\mathbb{S}^2} d \omega \, B_{nc}^{34}(v, \nu_4) \left( F^1(\nu_1) F^1(\nu_2) - F^2(v) F^1(\nu_4) \right).
\]

Moreover we define the following operators
\[
\mathbb{K}[\mathcal{F}, \mathcal{F}] := \begin{pmatrix} \mathbb{K}_1[\mathcal{F}, \mathcal{F}] \\ \mathbb{K}_2[\mathcal{F}, \mathcal{F}] \\ 0 \end{pmatrix}
\]
and
\[
\mathcal{R}[\mathcal{F}] = \begin{pmatrix} \int_{\mathbb{S}^2} d \nu \, A_0 F^2(v) + B_0 Q(n) (F^2(v) - F^1(v)) \\ - \int_{\mathbb{S}^2} d \nu \, A_0 F^2(v) + B_0 Q(n) (F^2(v) - F^1(v)) \\ \int_{\mathbb{R}^3} d v \, A_0 F^2(v) + B_0 Q(n) (F^2(v) - F^1(v)) \end{pmatrix}
\]
so that the kinetic system can be written as
\[
\partial_t \mathcal{F}(t, v) = \mathbb{K}[\mathcal{F}, \mathcal{F}](t, v) + \mathcal{R}[\mathcal{F}](t, v).
\]

In this paper we focus on the setting, when the gas is so much rarefied, that the collisions between gas molecules are much less frequent if compared to the radiative processes. We impose the following three conditions

1. \( m |v|^2 \approx h v_0 \),
2. \( Q \approx \frac{2 v^2}{c^3} \),
3. \( \alpha \int_{\mathbb{R}^3} (F^1 + F^2) \, d v \ll B_{12} \frac{2 h v_0^3}{c^2} \),

where \( \alpha \) is a parameter which measures the order of magnitude for the cross section for the collisions of the gas particles. With the relation \( \approx \) we mean that the quantities are of the same order.

Condition (1) in (1.19) means that the kinetic energy of the particles is comparable to the energy of the photons. Due to this assumption the Boltzmann ratio \( e^{-\frac{2 \nu_0}{K B T}} \) is of order one.

The physical meaning of (2) is that for the photon density in our model the number of occupied quantum states is of the same order of available quantum states. It indicates that we need to use the Bose-Einstein quantum statistic. In order to give a more mathematical meaning of condition (2), we recall the form of the radiative transfer equation in both the terms for the evolution of the density of gas molecules and of the photons. This is given by

\[
\partial_t F^i(t, v) = \mathbb{K}_i [\mathcal{F}, \mathcal{F}] \pm \int_{0}^{\infty} \int_{\mathbb{S}^2} \frac{B_{12}}{4 \pi} \delta(v - v_0) \left( \frac{2 h v^3}{c^2} F^2 \left( 1 + \frac{c^3}{2 v^2} Q \right) - \hbar c v Q F^1 \right) \, d n \, d v \quad \text{(1.20)}
\]

\[
\partial_t Q(t, n, v) = \int_{\mathbb{R}^3} \frac{B_{12}}{4 \pi} \left( \frac{2 h v^3}{c^2} F^2 \left( 1 + \frac{c^3}{2 v^2} Q \right) - \hbar c v Q F^1 \right) \, d v
\]

Imposing on \( Q \) the second condition yields the fact that both terms in \( \left( 1 + \frac{c^3}{2 v^2} Q \right) \) have the same order of magnitude. The term 1 is associated to the spontaneous emission, while \( \frac{c^3}{2 v^2} Q \) is related to the stimulated emission process. Assumption (2) means that both gas molecule densities \( F^1 \)
and $F^2$ contribute equally to the radiative process. Most likely this assumption is not needed in order to derive the model but it simplifies the third and most important assumption in (1.19). In a situation where $Q \ll \frac{2v_0^2}{c^3}$ the spontaneous radiation would be the dominant contribution, while whenever $Q \gg \frac{2v_0^2}{c^3}$ the stimulated radiation would be the most important term. Also in these limit situations we could obtain similar results as those we will obtain in this paper.

Finally, condition (3) implies exactly that the collision terms between gas molecules are much smaller than the terms describing the interaction between gas and radiation, i.e. $|K_i[F,F]| \ll \left| \int_0^\infty dv \delta(v - v_0) \int_{\mathbb{S}^2} dn \ A_0 F^2(v) + B_0 Q(n) (F^2(v) - F^1(v)) \right|$. This is true, since with (2) in (1.19) we have

$$\int_0^\infty dv \delta(v - v_0) \int_{\mathbb{S}^2} dn \ A_0 F^2(v) + B_0 Q(n) (F^2(v) - F^1(v)) \approx B_{12} \frac{2hv_0^3}{c^2} F^i(v).$$

(1.21)

Moreover, if $\alpha$ is the order of magnitude of the cross section, for the Boltzmann operators we see

$$K_i[F,F] \approx \alpha \left( \int_{\mathbb{R}^3} F^1 + F^2 \ dv \right) F^i(v).$$

(1.22)

This concludes the justification for condition (3) in (1.19), due to which we are assuming that the gas is very rarefied.

**Remark** We are not aware of a specific physical system, which can be described by these assumptions. Anyway, these conditions apply for example to systems of very rarefied hydrogen with an incoming radiation of frequency of visible light (more precisely $v_0 \approx 10^{15}$ s$^{-1}$ corresponding to the wavelength 122 nm) and with temperature below 7000 K. Already the sun photosphere, which is by far not very rarefied, with its density of order $10^{-4}$ kg m$^{-3}$ yields the ratio $\frac{a \int_{\mathbb{R}^3} (F^1 + F^2) \ dv}{B_{12} \frac{2hv_0^3}{c^2}}$ of order $10^{-28}$. We remark also that the temperature in the photosphere is lower than 6000 K and therefore the hydrogen is largely non-ionized.

It is convenient to introduce a scaling which defines in a more mathematically precise way the model under the assumptions in (1.19). Besides the fact that the energy of the photons and the kinetic energy of the particles are of the same order of magnitude and that the quantum and classical terms are comparable, the most relevant assumption in (1.19) is to assume that the collisions between photons and gas molecules take place much more frequently than the collision between the gas particles themselves. Hence, we rewrite the equation (1.10) under the scaling $K_i \mapsto \varepsilon K_i$. The dynamic of the pair $(\lambda(t), F^1(t))$, introduced in (1.1), will evolve with a time scale given by the relation between the collision and the radiative terms. Moreover, we scale the time like $t \mapsto \varepsilon t$ and, setting $A_0 = B_0 = 1$ for simplicity, we arrive to the equations

$$\partial_t F^1(v) = K_1[F,F](v) + \frac{1}{\varepsilon} \int_{\mathbb{S}^2} \left[ F^2(v) + Q(n)(F^2(v) - F^1(v)) \right] dn$$

$$\partial_t F^2(v) = K_2[F,F](v) - \frac{1}{\varepsilon} \int_{\mathbb{S}^2} \left[ F^2(v) + Q(n)(F^2(v) - F^1(v)) \right] dn$$

$$\partial_t Q(n) = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \left[ F^2(v) + Q(n)(F^2(v) - F^1(v)) \right] dv$$

(1.23)

Our goal is to find a new kinetic equation which defines the solution of this system as $\varepsilon \rightarrow 0$.

The mathematical properties of the homogeneous Boltzmann equation have been intensively studied (cf. [7]). The techniques for the homogeneous Boltzmann equation can be adapted for the case of nonelastic collision terms. The radiative transfer equation is linear in $Q$ and has good
properties. Therefore (1.10) has all usual conservation laws of mass and momentum of the gas and of the energy of the system. We recall again that we neglect the momentum of the photons. Moreover it is possible to define an entropy related to the equation (1.10), which satisfies an H-theorem.

In this paper we will not prove the well-posedness theory for (1.10), which has been developed in [11], but we focus on the analysis of the solution of (1.23) when \( \varepsilon \) tends to zero. We notice that formally if \( \varepsilon = 0 \) in equation (1.23) the possible solutions should satisfy the relation \( F^2 = \lambda F \) and \( Q = \frac{2}{1-\lambda} \) for some \( \lambda(t) \in [0, 1) \). We can call this set \( \mathcal{M} \) and it will be referred to as the manifold of steady states. We can expect that the solutions are close to \( \mathcal{M} \) for every time.

With a perturbative expansion we will be able to derive initially formally a new equation for the vector \((\lambda, F)\), from which we can recover the interaction of the gas molecules with the photons by the relation \( F = \left( \frac{F}{\lambda - \lambda} \right) \). The vector \((\lambda, F)\) takes value in \([0, 1) \times L^1_2(\mathbb{R}^3)\). The kinetic equation we will derive is defined as follows

\[
\frac{\partial \lambda}{\partial t} = -\frac{1}{4\pi} \frac{(1 - \lambda)^2(1 + \lambda)}{(1 + \lambda) + (1 - \lambda)^2} \int_{\mathbb{R}^3} K_1(F, \lambda) \, dv
\]

\[
\frac{\partial F}{\partial t} = \frac{1}{1 + \lambda} \left( K_1(F, \lambda) + K_2(F, \lambda) \right) + \frac{1}{4\pi} \frac{(1 - \lambda)^2 F}{(1 + \lambda) + (1 - \lambda)^2} \int_{\mathbb{R}^3} K_1(F, \lambda) \, dv
\]

(1.24)

Notice that this method can be thought as a generalized Chapman–Enskog expansion (cf. [9]). An important difference with the classical Chapman–Enskog expansion is that we do not derive an equation for the densities (mass, momentum and energy) but we construct a new equation for \( F \) and \( \lambda \). Moreover, the manifold of steady states for the homogeneous Boltzmann equation is only five dimensional, whereas here it is infinite dimensional. It is indeed characterized by the function \( F \) and \( \lambda \). Another way to understand the result of this paper is in terms of the theory of dynamical system. More precisely, we can understand the solution described in this paper by means of the dynamics of a system near a slow manifold. In the context of mathematical physics this approach was used to described the dynamic of a small charged particle in an electromagnetic field (cf. [26]).

### 1.2 Definitions

Before stating the main results of this paper we shall give some important definitions and notation. First of all, as in the usual theory for the Boltzmann equation we will have to work with some weighted spaces.

**Definition 1.1** We define the Banach Space \( L^1_k(\mathbb{R}^3) \) as the space of all integrable functions \( f \) such that \( \| f \|_{L^1_k} := \int_{\mathbb{R}^3} dx |f(1 + |x|^2)^{k/2}| < \infty \).

This is the subspace of integrable functions used also in the well-posedness theory for the classical Boltzmann equation. We will also say that a function \( f \in L^1_k(\mathbb{R}^3) \) has a bounded k-th moment. The next lemma gives us a useful interpolation result for these spaces.

**Lemma 1.1** Let \( k \leq m \). If \( f \in L^1_m(\mathbb{R}^3) \) then also \( f \in L^1_k(\mathbb{R}^3) \) with \( \| f \|_{L^1_k} \leq \| f \|_{L^1_m} \). Moreover the following interpolation formula holds

\[
\| f \|_{L^1_1} \leq \| f \|_{L^1_k}^{1/2} \| f \|_{L^1_m}^{1/2}.
\]

(1.25)

**Proof** This is a consequence of the Hölder Inequality. See for example [4].
Another helpful notation that will be used in order to simplify the reading, is that given a vector \( F = (F^1, F^2, Q) \) we will denote \( F \) as the vector of the first two components: \( F = (F^1, F^2) \). So that we can also write \( F = (F, Q) \). Moreover we will also write for simplicity \( \|F\|_{L^1_2} = \|F^1\|_{L^1_2} + \|F^2\|_{L^1_2} \). With this notation we can define the space from where we consider the initial data.

**Definition 1.2** We define the Banach space \( \mathcal{X} = \left\{ F = \begin{pmatrix} F^1 \\ F^2 \\ Q \end{pmatrix} \in L^1_2(\mathbb{R}^3) \times L^1_2(\mathbb{R}^3) \times L^1(\mathbb{S}^2) \right\} \)

and we call \( \mathcal{X}_+ = \{ F \in \mathcal{X} : \| F \| \geq 0 \} \subset \mathcal{X} \).

We briefly give the definition of solutions for the kinetic equation (1.10).

**Definition 1.3** We call a non-negative vector \( F = (F^1, F^2, Q) \in C^1([0, \infty), \mathcal{X}) \) a strong solution if \( F(t) \in \mathcal{X}_+ \) for all \( t \) and it solves (1.10) almost everywhere.

Weak solutions are defined testing (1.10) against some proper test-functions.

**Definition 1.4** We say that a vector of non negative functions \( F \in C([0, \infty), \mathcal{X}) \) is a weak solution of the kinetic system (1.10), if for all \( \psi^i(v) \in C_c^\infty(\mathbb{R}^3), i = 1, 2 \) and \( \psi^3(n) \in C_c^\infty(\mathbb{S}^2) \) the following holds

\[
\begin{align*}
\partial_t &\left[ \int_{\mathbb{R}^3} dv \left( F^1 \psi^1(v) + F^2 \psi^2(v) \right) + \int_{\mathbb{S}^2} dn \ Q \psi^3(n) \right] \\
= &\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{S}^2} d\omega \ |\omega \cdot (v_1 - v_2)| F^i(v_1) F^j(v_2) \\
&\left( \psi^i(v_3) + \psi^i(v_4) - \psi^i(v_1) - \psi^i(v_2) \right) \\
+ &\int_{\mathbb{R}^{12}} d\overline{v}_1 d\overline{v}_2 d\overline{v}_3 d\overline{v}_4 \delta(\overline{v}_1 + \overline{v}_2 - \overline{v}_3 - \overline{v}_4) \delta(|\overline{v}_1|^2 + |\overline{v}_2|^2 - |\overline{v}_3|^2 - |\overline{v}_4|^2 - 2\epsilon_0) \\
W_{ne}(\overline{v}_1, \overline{v}_2; \overline{v}_3, \overline{v}_4) &\left[ F^1(\overline{v}_1) F^1(\overline{v}_2) \left( \psi^2(\overline{v}_3) + \psi^2(\overline{v}_4) - \psi^1(\overline{v}_1) - \psi^1(\overline{v}_2) \right) \\
&+ F^2(\overline{v}_3) F^1(\overline{v}_4) \left( \psi^1(\overline{v}_1) + \psi^1(\overline{v}_2) - \psi^2(\overline{v}_3) - \psi^2(\overline{v}_4) \right) \right] \\
+ &\int_{\mathbb{R}^3} dv \int_{\mathbb{S}^2} dn \ h_{rad} [F] \left[ \psi^1(v) - \psi^2(v) + \psi^3(n) \right],
\end{align*}
\]

(1.26)

where \( W_{ne}(\overline{v}_1, \overline{v}_2; \overline{v}_3, \overline{v}_4) = C_0 \frac{|\omega \cdot (\overline{v}_1 - \overline{v}_2)|}{|\overline{v}_1 - \overline{v}_2|} \).

We obtained this definition testing each side of the system (1.10) with the proper choice of \( \psi^i \) and then integrating both sides with respect to \( v \) and \( n \). As it can be read in the Appendix B of [15] evaluating the delta functions with respect to \( \overline{v}_1, \overline{v}_2 \) correspondingly \( \overline{v}_3, \overline{v}_4 \) we obtain the desired nonelastic kernels (1.11). We will refer to (1.26) often as to the weak formulation of (1.10).

**Remark** Using the inequalities (1.12), (1.13) and (1.14) we see that for \( F \in L^\infty([0, \infty), L^1_4(\mathbb{R}^3) \times L^1_4(\mathbb{R}^3) \times L^1(\mathbb{S}^2)) \) the right hand side of the equation (1.26) is well defined also for \( \psi^i(v) = C(1 + |v|^2) \) if \( i = 1, 2 \) and \( \psi^3(n) = C \) for all \( C > 0 \).
Moreover solutions with the control of at least the fourth moment satisfy also the conservation of mass, momentum and energy, which can be deduced by an approximating argument
\[
\partial_t \left[ \int_{\mathbb{R}^3} dV (F^{1} + F^{2}) \right] = 0 \quad \text{(mass)},
\]
for \( i = 1, 2, 3 \)
\[
\partial_t \left[ \int_{\mathbb{R}^3} dV (F^{1} + F^{2}) v_i \right] = 0 \quad \text{(momentum)},
\]
\[
\partial_t \left[ \int_{\mathbb{R}^3} (F^{1}|v|^2 + F^{2} (|v|^2 + 2\varepsilon_0)) dV + 2\varepsilon_0 \int_{\mathbb{S}^2} Q \ dn \right] = 0 \quad \text{(energy)}.
\]
(1.27)

Finally, we can also define an entropy.

**Definition 1.5** We define for \( \mathcal{F} \) a solution to the kinetic equation (1.10) the entropy at time \( t < \infty \) by
\[
\mathcal{H}(t) := \int_{\mathbb{R}^3} dV \left( F^{1} \log(F^{1}) + F^{2} \log(F^{2}) \right)
+ \int_{\mathbb{S}^2} dn \left( Q \log(Q) - \left( \frac{A_0}{B_0} + Q \right) \log \left( \frac{A_0}{B_0} + Q \right) \right).
\]
(1.28)

The entropy and its dissipation are important tools for the development of the well-posedness theory. We will be able to define them also for the new equation we will derive later.

### 1.3 Main Results of the Paper

We conclude the introduction with the following two theorems, which are the main results of this paper. The first one is about the well-posedness theory for the kinetic system (1.24).

**Theorem 1.1** Let \((\lambda_0, F_0) \in [0, 1) \times L^1_2(\mathbb{R}^3)\) with \(F_0 \log(F_0) \in L^1(\mathbb{R}^3)\) and \(F_0 \in L^4_1(\mathbb{R}^3)\). Let the initial energy \(E_0\) be finite. Assume \((\lambda_0, F_0)\) satisfies \(F_0(v) \geq Ce^{-\beta|v|^2}\). Let \(T > 0\) then there exists a weak solution \((\lambda, F) \in C([0, T], \mathbb{R} \times L^1_2(\mathbb{R}^3))\) with \(\lambda(t) \in [0, 1)\) of the kinetic system (1.24) for the initial value \((\lambda_0, F_0)\). This is the unique solution with \(F \in L^\infty([0, T], L^4_1(\mathbb{R}^3))\).

We will prove also rigorously that the kinetic system (1.24) really describes the behavior of the solutions to (1.23) as \(\varepsilon \to 0\).

**Theorem 1.2** Let \(\varepsilon > 0\) and \(T > 0\). Let \(\mathcal{F}_0\) defined by \((\lambda_0, F_0)\) and with \(\mathcal{F}_0 > 0\) and \(F_0(v) \geq Ce^{-\beta|v|^2}\). Assume it has mass \(\kappa_0 > 0\), energy \(E_0 \coloneqq E(\mathcal{F}_0) \leq \frac{1}{2} \) and \(F_0 \in L^4_1\).

Let \(\delta < \delta_0\) small enough for some \(\delta_0\) depending on \(\delta_0\) on the initial value \(\mathcal{F}_0\). We define \(\tilde{\mathcal{F}}_0 = \mathcal{F}_0 + W_0\) such that \(\tilde{\mathcal{F}}_0 > 0\) and \(\|W_0\| < \delta\).

Let \(\mathcal{F}^\varepsilon \in C^1([0, \infty), L^1_2(\mathbb{R}^3) \times L^1_2(\mathbb{S}^2) \times L^1(\mathbb{S}^2))\) be the solution to the equation (1.23) for the initial value \(\mathcal{F}^\varepsilon_0\).

Let \(\mathcal{F}\) be defined by \((\tilde{\lambda}, \mathcal{F}) \in C([0, \infty), \mathbb{R} \times L^1_2(\mathbb{R}^3))\), which are the solution of the derived equation (1.24) for the initial value \(\mathcal{F}_0\).

Then for all \(s > 0\) we have
\[
\lim_{\varepsilon \to 0} \sup_{0 < t \leq T} \|\mathcal{F}^\varepsilon - \mathcal{F}\|_X = 0.
\]
(1.29)

**Remark** We remark that there are two conditions on the initial value \(F_0\) which could be weakened using the available results for the general theory on the Boltzmann equation. First of all we assume that \(F_0 \in L^4_1(\mathbb{R}^3)\) has bounded fourth moment. This allows us to obtain immediately that the
energy is well-defined and constant in time and also that solutions to (1.24) and to (1.23) have a (uniformly) bounded fourth moment too. However, it is well-known as in [28] that the solution to the Boltzmann equation with initial finite energy, mass and momentum has also finite fourth moment because of the so-called moment production property. Concerning the lower estimate by means of the Maxwellian \( F_0(v) \geq Ce^{-\beta |v|^2} \), it allows us to show in an easy way that all terms in the dissipation of the entropy are well-defined. Most likely also the condition of the lower estimate for \( F_0 \) can be weakened and this result could be true also without this assumption. For the Boltzmann equation it is for example possible to show the \( H^2 \)-Theorem without the lower bound for the initial value using an approximating argument as in [1] and [2]. However, we have preferred to keep the assumptions of the bounded fourth moment \( F_0 \in L^4(\mathbb{R}^3) \) and of the lower bound \( F_0(v) \geq Ce^{-\beta |v|^2} \) in order to avoid too many technicalities, which are not the main focus of this paper.

1.4 A Summary of Mathematical Models for Photon-Gas Systems

The behavior of a gas interacting with photons is besides its mathematical interest also relevant in astrophysical applications. The radiative transfer equation is the main tool for the study of this problem. A detailed explanation of its derivation and its properties can be found in [8, 16, 23, 25].

The first studies of the interaction between gas and radiation can be found in [10, 17]. These models describe the diffusion of radiation through a gas which is trapping it. However, they do not consider collisions between gas molecules. The only interactions taken into account are the absorption and the successive emission of photons by the gas molecules.

Some papers which have studied the radiative transfer equation interacting with matter in a rigorous mathematical way are the following. In [3, 20] the interaction between a fluid system and radiation has been considered. Models close to the spirit of this paper, where the interaction is due to collisions and radiative processes, are developed and discussed in [5, 21, 22, 24]. In particular in [21, 22] the author studied also the well-posedness theory for a kinetic equation describing a system where the collisions between the gas molecules are only elastic and where in addition to the radiative process (1.4) the photon-gas interaction can be also of the form \( \overline{A} + \gamma \rightarrow A + 2\gamma \). The author of [5] develops a model which considers only two-levels gas molecules and monochromatic radiation, which are assumptions we also make in this paper. In the model in [24] all three interactions (1.2), (1.3) and (1.4) are included.

In [15] the authors studied the same model we will work with. They studied different scaling limits yielding either LTE (local thermal equilibrium) or Non-LTE situations and they developed via a hydrodynamic limit Euler equations coupled with the radiative transfer equation.

1.5 Outline of the Paper

In Sect. 2 we formally derive equation (1.24). Section 3 deals with the well-posedness theory for this equation, while in Sect. 4 the rigorous proof for the derivation of equation (1.24) will be shown. We postpone until Sect. 4 the proof of Theorem 1.2 since it requires the well-posedness theory developed in Sect. 3.

2 The Fast Radiation Limit

It is possible to develop a good well-posedness theory for the kinetic equation (1.10) when the collisions between gas molecules and the radiation processes are of the same order. In this paper
we aim to study the case when the collision terms are negligible compared to the terms describing the interaction between the gas and the radiation. In this section we will formally derive the new kinetic equation as it is stated in (1.24).

2.1 The Behavior of the Solutions

We recall the equation we stated in (1.23), which describes the behavior of the gas and the photons shortly after the initial time. After times of order $\varepsilon$, we can expect the distribution of gas molecules and radiation to solve

$$
0 = \int_{S^2} \left[ F^2(v) + Q(n)(F^2(v) - F^1(v)) \right] dn \quad \text{and}
0 = \int_{\mathbb{R}^3} \left[ F^2(v) + Q(n)(F^2(v) - F^1(v)) \right] dv
$$

(2.1)

Hence, $Q$ is independent of $n$ with $Q(n) = \frac{\int_{\mathbb{R}^3} F^2(v) dv}{\int_{\mathbb{R}^3} (F^2(v) - F^1(v)) dv}$. This is possible only assuming $\int_{\mathbb{R}^3} F^2(v) dv < \int_{\mathbb{R}^3} F^1(v) dv$, since $Q$ is non-negative. Moreover, substituting this photon number density into the first equation we see

$$
F^2(v) = \frac{\int_{S^2} Q(n) dn}{\int_{S^2} [1 + Q(n)] dn} F^1(v) = \frac{\int_{\mathbb{R}^3} F^2(v) dv}{\int_{\mathbb{R}^3} F^1(v) dv} F^1(v).
$$

Hence we have to look for solutions living in the manifold given by

$$
\begin{aligned}
\begin{cases}
F^2 &= \lambda F^1 \\
\overline{Q} &= \frac{\lambda}{1-\lambda}
\end{cases}
\end{aligned}
$$

(2.2)

where $\overline{F}^1 \geq 0$, $\overline{F}^1 \in L^1_1(\mathbb{R}^3)$ and $\lambda \in [0, 1)$ for all $t \geq 0$. In order to simplify the notation we define the manifold of steady states $\mathcal{M}$ as follows.

**Definition 2.1** We call $\mathcal{M} = \{ F \in \mathcal{X} : F = \left( \frac{F}{\lambda F} \right), \lambda < 1 \}$ and for the non-negative $F$ we denote

$\mathcal{M}_+ = \{ F \in \mathcal{M} : F \geq 0, \lambda \in [0, 1) \}$.

$\mathcal{M}_+$ plays the role of the so called critical manifold for the perturbative problem considered in [26], but we will call it the manifold of steady states.

We look for solutions of (1.23) that remain close to the manifold $\mathcal{M}_+$ for arbitrary times in powers of $\varepsilon$. Imposing suitable compatibility conditions we will obtain a kinetic equation which describes the evolution of functions in $\mathcal{M}_+$. The procedure is reminiscent to the classical Chapman–Enskog expansion which yields the hydrodynamic limits of the Boltzmann equation. We call the entire approach a generalized Chapman–Enskog expansion. We will hence study the new kinetic system, its entropy and we will develop a weak well-posedness theory for it. We will then prove Theorem 1.2.
2.2 The Generalized Chapman–Enskog Expansion

Setting $\overline{F}^i$, $\lambda$, and $Q$ as in (2.2), we look for a perturbative expansion for small variation of these functions. Therefore taking $H(t, n), G^i(t, v)$ we define

$$ F^1 = \overline{F}^1 (1 + \varepsilon G^1), $$

$$ F^2 = \overline{F}^2 (1 + \varepsilon G^2), $$

$$ Q = \frac{\lambda}{1 - \lambda} (1 + \varepsilon H). $$

Putting (2.3) into the kinetic system (1.23) and neglecting all terms of order $\varepsilon$ or small we obtain

$$ \partial_t \overline{F}^1 = \mathbb{K}_1 \left[ \overline{F}, \overline{F} \right] + \int_{S^2} \left[ \frac{\lambda}{1 - \lambda} \overline{F}^1 (G^2 - G^1) - \lambda H \overline{F}^1 \right] d\nu $$

$$ \partial_t \overline{F}^2 = \mathbb{K}_2 \left[ \overline{F}, \overline{F} \right] - \int_{S^2} \left[ \frac{\lambda}{1 - \lambda} \overline{F}^1 (G^2 - G^1) - \lambda H \overline{F}^1 \right] d\nu $$

$$ \partial_t \frac{\lambda}{1 - \lambda} = \int_{\mathbb{R}^3} \left[ \frac{\lambda}{1 - \lambda} \overline{F}^1 (G^2 - G^1) - \lambda \overline{F}^1 \right] dv $$

We can neglect such terms because the collision terms and all the other functions are of order 1. The collision terms $\mathbb{K}_i \left[ \overline{F}, \overline{F} \right]$ are as the one defined in (1.15) and (1.16) for the function $\overline{F} = (\overline{F}^1, \overline{F}^2, Q)$. When the vector $\overline{F}$ is an element of the steady manifold $\mathcal{M}_+$ we write $\mathbb{K}_i [F, \lambda]$ in order to stress out the fact that $F^1 = F$ and $F^2 = \lambda F$.

Let us fix $t \in [0, \infty)$. Let us consider the space $H := L^2 (\mathbb{R}^3) \times L^2 (\mathbb{R}^3) \times L^2 (S^2)$. Let us fix $\lambda \in [0, 1)$ and $\overline{F}^1$. For each $\lambda$, $\overline{F}^1$ we define an operator $L_{\lambda, \overline{F}^1} : H \rightarrow H$ by means of

$$ L_{\lambda, \overline{F}^1} \begin{pmatrix} G^1 \\ G^2 \\ H \end{pmatrix} = \begin{pmatrix} L_1 (G^1, G^2, H) \\ L_2 (G^1, G^2, H) \\ L_3 (G^1, G^2, H) \end{pmatrix} = \begin{pmatrix} \int_{S^2} \left[ \frac{\lambda}{1 - \lambda} \overline{F}^1 (G^2 - G^1) - \lambda H \overline{F}^1 \right] d\nu \\ - \int_{S^2} \left[ \frac{\lambda}{1 - \lambda} \overline{F}^1 (G^2 - G^1) - \lambda H \overline{F}^1 \right] d\nu \\ \int_{\mathbb{R}^3} \left[ \frac{\lambda}{1 - \lambda} \overline{F}^1 (G^2 - G^1) - \lambda \overline{F}^1 \right] dv \end{pmatrix} \begin{pmatrix} G^1 \\ G^2 \\ H \end{pmatrix} $$

On the Hilbert space with inner product $(H, \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \langle \cdot, \cdot \rangle_{S^2})$ the operator $L_{\lambda, \overline{F}^1}$ is self-adjoint. In order to find a new kinetic equation, we need to eliminate the operator $L_{\lambda, \overline{F}^1}$ in equation (2.4). Hence, we start looking at the tangent space of $\overline{F} := (\overline{F}^1, \overline{F}^2, Q)$ in the Hilbert space $H$. To do that we consider $F^i$ and $\tilde{\lambda}$ given by small perturbation of these $\overline{F}^1$ and $\lambda$ in the manifold as $F^1 = \overline{F}^1 (1 + \xi_1), F^2 = \overline{F}^2 (1 + \xi_2)$ and $Q = \frac{\tilde{\lambda}}{1 + \tilde{\lambda}} (1 + \eta)$. Defining $\tilde{\lambda} = \frac{Q}{1 + Q}$ and expanding with Taylor we see therefore that $\tilde{\lambda} = \lambda (1 + (1 - \lambda) \eta)$. Hence imposing $\tilde{F}^2 = \tilde{\lambda} F^1$ we obtain the condition for the tangent space: $\xi_2 = \xi_1 + (1 - \lambda) \eta$. Therefore we formally define the tangent space $T_{\overline{F}} \mathcal{M}$ at $\overline{F}$ by all vectors $(\xi, \xi, (1 - \lambda) \eta, \eta)^T$ for $\eta \in \mathbb{R}$ and $\xi \in L^2 (\mathbb{R}^3)$.

**Remark** Since the kernel of the operator $L_{\lambda, \overline{F}^1}$ are the vector $(\xi_1, \xi_2, \eta)^T$ satisfying the relation $\xi_2 = \xi_1 + (1 - \lambda) \eta$, we see that $L_{\lambda, \overline{F}^1} (\xi_1, \xi_1 + (1 - \lambda) \eta, \eta) = 0$. Therefore testing the equation with these vectors we can eliminate the operator in equation (2.4).

It is interesting also to look at the orthogonal space of $\overline{F} := (\overline{F}^1, \overline{F}^2, Q)$ in the Hilbert space $H$. It is not difficult to see that, by the definition of inner-product and the structure of the tangent space...
space, the orthogonal space to \(\mathcal{M}_+\) is formally given by all vectors
\[
\begin{pmatrix}
\alpha \\
-\alpha \\
\theta
\end{pmatrix}
\]
such that \(\int_{\mathbb{R}^3} \alpha \, dv = \int_{\mathbb{R}^2} \theta \, dn. \tag*{(2.6)}\)

The structure of the orthogonal space will play an important role for the proof of Theorem 1.2. We will indeed see that any vector \(\mathbb{F} > 0\) in a small neighborhood of the manifold \(\mathcal{M}\) can be decomposed as the sum of a vector in the manifold and a remainder, which has the same structure as the vectors in the orthogonal space.

Let us now return to the formal derivation of the kinetic equation for \(\lambda\) and \(\mathbb{F}^1\). Let \(\xi_1 \in C_c^\infty (\mathbb{R}^3)\) and \(\eta\) be a constant. Using the property that the self-adjoint operator \(L_{\lambda, \mathbb{F}^1}\) is zero on the tangent space \(T_{\mathbb{F}^1} \mathcal{M}\) we obtain that for any \((G^1, G^2, H) \in \mathbb{H}\)
\[
\left\langle \left(\frac{\xi_1}{\eta} + (1 - \lambda) \eta\right), \left(\frac{\partial_t \mathbb{F}^1}{\partial_t \mathbb{F}^2}, \frac{\partial_t \mathbb{F}^1}{\partial_t \mathbb{F}^2} \right) \right\rangle_\mathbb{H} = 0. \tag*{(2.7)}
\]

Putting that into the initial equation (2.4) we obtain the following result
\[
\left\langle \left(\frac{\xi_1}{\eta} + (1 - \lambda) \eta\right), \left(\frac{\partial_t \mathbb{F}^1}{\partial_t \mathbb{F}^2}, \frac{\partial_t \mathbb{F}^1}{\partial_t \mathbb{F}^2} \right) \right\rangle_\mathbb{H} = \left\langle \left(\frac{\xi_1}{\eta} + (1 - \lambda) \eta\right), \left(\frac{\mathbb{K}_1 [\mathbb{F}^1, \lambda]}{\mathbb{K}^1_0}, \frac{\mathbb{K}_2 [\mathbb{F}^1, \lambda]}{\mathbb{K}^1_0} \right) \right\rangle_\mathbb{H}. \tag*{(2.8)}
\]

From now on we define \(F := \mathbb{F}^1\). Suppose first \(\eta = 0\). Since \(\xi_1 \in C_c^\infty (\mathbb{R}^3)\) is arbitrary we have
\[
\partial_t (1 + \lambda) F = \mathbb{K}_1 [F, \lambda] + \mathbb{K}_2 [F, \lambda]. \tag*{(2.9)}
\]

Similarly we now consider \(\xi = 0\) and \(\eta = \frac{1}{1 - \lambda}\) in (2.8). The weak formulation in Sect. 1 implies \(\int_{\mathbb{R}^3} dv \, \mathbb{K}_2 [F, \lambda] = - \int_{\mathbb{R}^3} dv \, \mathbb{K}_1 [F, \lambda]\). Hence, we obtain
\[
4 \pi \partial_t \frac{\lambda}{1 - \lambda} = \partial_t \int_{\mathbb{R}^3} dv \, F - \int_{\mathbb{R}^3} dv \, \mathbb{K}_1 [F, \lambda]. \tag*{(2.10)}
\]

We now rewrite (2.10) with the goal to find an equation for the derivative of \(\lambda\) only. Surely, since
\[
\partial_t \frac{\lambda}{1 - \lambda} = \frac{\partial_t \lambda}{(1 - \lambda)^2},
\]
we see that
\[
\partial_t \lambda = \frac{(1 - \lambda)^2}{4 \pi} \int_{\mathbb{R}^3} (\partial_t F - \mathbb{K}_1 [F, \lambda]) \, dv. \tag*{(2.11)}
\]

Leibniz product rule and an integration over \(\mathbb{R}^3\) imply \(\int_{\mathbb{R}^3} dv \, \partial_t F = - \frac{\partial_t \lambda}{(1 + \lambda)^2} \int_{\mathbb{R}^3} dv \, F\). Hence, (2.11) reads
\[
\partial_t \lambda = - \frac{1}{4 \pi} \frac{(1 - \lambda)^2 (1 + \lambda)}{(1 + \lambda) + (1 - \lambda)^2 \int_{\mathbb{R}^3} dv \, F} \int_{\mathbb{R}^3} \mathbb{K}_1 [F, \lambda] dv. \tag*{(2.12)}
\]

Substituting \(\partial_t \lambda\) in (2.9) with (2.12) we conclude the derivation of the equation for \(F\) and we summarize the new kinetic system we already defined in Sect. 1.3 and which we will study in the next sections as follows
\[
\begin{align*}
\partial_t \lambda &= - \frac{1}{4 \pi} \frac{(1 - \lambda)^2 (1 + \lambda)}{(1 + \lambda) + (1 - \lambda)^2 \int_{\mathbb{R}^3} dv \, F} \int_{\mathbb{R}^3} \mathbb{K}_1 [F, \lambda] dv \\
\partial_t F &= \frac{1}{1 + \lambda} (\mathbb{K}_1 [F, \lambda] + \mathbb{K}_2 [F, \lambda]) + \frac{1}{4 \pi} \frac{(1 - \lambda)^2 F}{(1 + \lambda) + (1 - \lambda)^2 \int_{\mathbb{R}^3} dv \, F} \int_{\mathbb{R}^3} \mathbb{K}_1 [F, \lambda] dv 
\end{align*}
\tag*{(2.13)}
\]
2.3 Stability Condition for the Generalized Chapman–Enskog Expansion

We want now to briefly look at the stability condition for the generalized Chapman–Enskog expansion. To this end we start with the usual kinetic system as in (1.10). We expect $F^1$ to change in time $t$ of order 1, hence it shall change very slowly on the time scale $\varepsilon t$ for small $\varepsilon > 0$. Due to this we assume $F^1(\nu, t)$ to be constant in time. Moreover, according to the model we assume $K_2[F^1, F^2]$ to be very small and $\int_{\mathbb{R}^3} dv F^1 \geq \int_{\mathbb{R}^3} dv F^2$. Multiplying (1.23) by $\varepsilon$ and defining $\tau := t/\varepsilon$ the system becomes

$$
\partial_\tau F^2 = - \int_{\mathbb{S}^2} dn \left[ F^2 + Q(F^2 - F^1) \right]
$$

$$
\partial_\tau Q = \int_{\mathbb{R}^3} dv \left[ F^2 + Q(F^2 - F^1) \right]
$$

(2.14)

We would like to show the following proposition, which ensures the stability of the expansion we made in Sect. 2.2.

**Proposition 2.1** Let $F^1$ be constant in time and $\int_{\mathbb{R}^3} dv F^1 \geq \int_{\mathbb{R}^3} dv F^2$. Let $F^2$ and $Q$ satisfy equation (2.14). Then as $\tau \to \infty$, the density $Q$ approaches to a constant and the particle density $F^2$ satisfies $F^2(\nu) = \frac{Q}{\varepsilon + Q} F_1$.

**Proof** The proof is based on ODE theory. Integrating in $\mathbb{R}^3$ or in $\mathbb{S}^2$, (2.14) becomes

$$
\partial_\tau \rho_2 = - \rho_2 (I + 1) + I a
$$

$$
\partial_\tau I = I(\rho_2 - a) + \rho_2
$$

(2.15)

where we used the notation $a := \int_{\mathbb{R}^3} dv F^1(\nu) = \text{constant}$, $\rho_2(\tau) := \int_{\mathbb{R}^3} dv F^2(\nu)$ and $I(\tau) := \int_{\mathbb{S}^2} dn Q(\nu)$. First of all we see from the equation (2.15) that $c_0 := \rho_2 + I$ is a constant with $c_0 < I + a$. Hence we rewrite the equation for $I$ as

$$
\partial_\tau I = I(c_0 - I - a) + c_0 - I = -I^2 - (1 - c_0 + a)I + c_0.
$$

(2.16)

The solution $I$ to this ODE satisfies $I \to -\frac{(1-c_0+a)-\sqrt{(1-c_0+a)^2+4c_0}}{2} := I^\infty > 0$ as $\tau \to \infty$ and therefore also $\rho_2 = c_0 - I \to \rho_2^\infty$. Solving now the equation $\partial_\tau Q = Q(\rho_2 - a) + \rho_2$ and letting $\tau \to \infty$ we conclude $Q \to I^\infty$. Similarly, the solution to $\partial_\tau F^2 = -(1 + I) F^2 + I F^1$ satisfies $F^2 \to \frac{I^\infty}{I^\infty + 1} F^1$ as $\tau \to \infty$.

\[\square\]

3 Well-Posedness Theory for the Fast Radiation Limit

We anticipated that one can prove a well-posedness theory for equation (1.10). Indeed, following the standard proof as in [12, 18, 19] one can prove existence and uniqueness of weak solutions for the kinetic system. On the other hand, by mean of the ODE’s theory in general Banach spaces one can prove as for example in [4, 13] a strong well-posedness theory. Finally, the long time convergence to the unique equilibrium is achieved with the standard method as in [1, 2]. One can read these results in the first part of [11]. In Sect. 3 we will prove similar results for the kinetic equation (2.13) we formally derived in Sect. 2. We start the analysis by looking at its properties and giving some definitions. Afterwards we will develop a well-posedness theory, which we recall to be important for the proof of Theorem 1.2 in Sect. 4.
3.1 Properties of the Kinetic System

The solutions for the equation (2.13) satisfy some conservation laws. Indeed, let \( F(t, v) \geq 0 \) and \( \lambda(t) \) be solution with \( F \in L^1_2(\mathbb{R}^3) \). Then putting \( \xi_1 = 1 \) and \( \eta = 0 \) in (2.8) we obtain the conservation of mass for \( F \)

\[
\partial_t \int_{\mathbb{R}^3} dv \ (1 + \lambda) F = 0. \tag{3.1}
\]

Here we also used the properties of the collision terms when integrated. Similarly taking \( \xi_1 = \nu_i \) for \( i = 1, 2, 3 \) we deduce the conservation of momentum

\[
\partial_t \int_{\mathbb{R}^3} dv \ (1 + \lambda) F \nu_i = 0. \tag{3.2}
\]

Finally, we also have the conservation of energy in the following sense. Let us consider \( \xi_1 = |v|^2 \) and \( \eta = \frac{2\nu_0}{1-\lambda} \), then we have

\[
\partial_t \left( \int_{\mathbb{R}^3} dv \left[ (|v|^2(1 + \lambda) F) + 2\nu_0 \lambda F \right] \right) + 2\nu_0 4\pi \partial_t \frac{\lambda}{1-\lambda} = 0. \tag{3.3}
\]

Clearly we shall work on a different space than \( \mathcal{X} \).

**Definition 3.1** We define the Banach space \( X := \{(\lambda, F) : \lambda \in \mathbb{R}, \ F \in L^1_2(\mathbb{R}^3)\} \) and the subset \( X_+ := \{(\lambda, F) \in X : \lambda \in [0, 1), \ F \geq 0\} \).

We will also consider the subspace \( \tilde{X} := \{(\lambda, F) \in X : F \in L^1_3(\mathbb{R}^3)\} \).

With these spaces we can define the following operator, that will simplify the notation.

**Definition 3.2** We define the operator \( T : \tilde{X} \to X \) by

\[
T \begin{pmatrix} \lambda \\ F \end{pmatrix} = \begin{pmatrix} -\frac{1}{4\pi} \frac{(1-\lambda)^2(1+\lambda)}{(1+\lambda)(1-\lambda)^2} \int_{\mathbb{R}^3} \mathbb{K}_1[F, \lambda] dv \\ \frac{1}{1+\lambda} (\mathbb{K}_1[F, \lambda] + \mathbb{K}_2[F, \lambda]) + \frac{1}{4\pi} \frac{(1-\lambda)^2 F}{(1+\lambda)(1-\lambda)^2} \int_{\mathbb{R}^3} \mathbb{K}_1[F, \lambda] dv \end{pmatrix}.
\]

**Remark** With this operator we can rewrite the kinetic system (2.13) as

\[
\begin{align*}
\partial_t \lambda &= T_1(\lambda, F) \\
\partial_t F &= T_2(\lambda, F)
\end{align*}
\]

Next we define the notion of weak solutions, energy and entropy.

**Definition 3.3** We call a pair \( (\lambda, F) \in C(\{0, \infty\}, X) \) a weak solution to (2.13) with initial values \( (\lambda_0, F_0) \in X_+ \) if \( (\lambda, F) \in X_+ \) for all \( t \geq 0 \) and if

\[
\lambda(t) = \int_0^t T_1(\lambda, F) \ d\tau + \lambda_0
\]

and if for all \( \varphi \in C^\infty_c(\mathbb{R}^3) \)

\[
\partial_t \int_{\mathbb{R}^3} dv \ F(t, v) \varphi(v) = \int_{\mathbb{R}^3} T_2(\lambda, F) \varphi(v) \ dv.
\]

**Definition 3.4** Given a solution to (2.13) \( (\lambda, F) \) in the sense of Definition 3.3 we define the energy \( E(t) \) by

\[
E(t) = \left( \int_{\mathbb{R}^3} dv \left[ (1 + |v|^2(1 + \lambda) F) + 2\nu_0 \lambda F \right] \right) + 2\nu_0 4\pi \frac{\lambda}{1-\lambda}. \tag{3.4}
\]
We already know that $E(t) = E(0) = \text{constant}$ for strong solutions. As in the kinetic system (1.10), this holds for weak solutions with bounded fourth moment too. We also define the entropy of the new system (2.13) by

$$\mathcal{H}(t) = \left( \int_{\mathbb{R}^3} dv \left[ (1 + \lambda) F \log(F) + F \lambda \log(\lambda) \right] \right) + 4\pi \left( \frac{\lambda}{1 - \lambda} \log(\lambda) + \log(1 - \lambda) \right). \quad (3.5)$$

One of the ways in which we will use the conservation of energy is in the proof of the boundedness of $\lambda$. Indeed it provides $0 \leq \lambda(t) \leq \lambda_0 (E(0)) < 1$. The entropy can be derived from the Definition 1.5 under the assumption $\frac{A_0}{\lambda_0} = 1$. Moreover, it decreases, as we will show in Proposition 3.1.

We will assume throughout this section that the initial data satisfies

$$F_0(v) \geq C e^{-\beta |v|^2} \quad (3.6)$$

for some constant $C > 0$ and $\beta > 0$. This is a strong assumption, but it allows us to have a global well-posedness without using too many technical arguments. In this paper we will not focus on the most general well-posedness theory, but on the derivation of the effective kinetic equation (2.13) for the fast radiation limit and on the rigorous proof of Theorem 1.2 in Sect. 4, for which a well-posedness theory is required.

With assumption (3.6) we can estimate for any solution to (2.13) which satisfies $\|F\|_{L^1_x} \leq E(0)$ that

$$\partial_t F(v) \geq - (1 + \lambda) \left( \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{S}^2} d\omega \, B_{\delta}(v, v_2) F(v_2) \right) F(v) - \frac{2}{1 + \lambda} \left( \int_{\mathbb{R}^3} d\bar{v}_2 \int_{\mathbb{S}^2} d\omega \, B_{ne}^{12}(v, \bar{v}_2) F(\bar{v}_2) \right) F(v) - \frac{2\lambda}{1 + \lambda} \left( \int_{\mathbb{R}^3} d\bar{v}_3 \int_{\mathbb{S}^2} d\omega \, B_{ne}^{34}(\bar{v}_3, v) F(\bar{v}_3) \right) F(v)$$

$$\geq - C(E(0), \varepsilon_0) F(v) \geq - C(E(0), \varepsilon_0) (1 + |v|) F(v), \quad (3.7)$$

where we used the properties $0 \leq \frac{(1 - \lambda)^2}{(1 + \lambda) + (1 - \lambda)^2} \int_{\mathbb{R}^3} dF \leq 1$ and

$$\int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{S}^2} d\omega \, (1 + \lambda) B_{\delta}(v_1, v_2) F(v_2) F(v_1) + 2 \int_{\mathbb{R}^3} d\bar{v}_1 \int_{\mathbb{R}^3} d\bar{v}_2 \int_{\mathbb{S}^2} d\omega \, B_{ne}^{12}(\bar{v}_1, \bar{v}_2) F(\bar{v}_2) F(\bar{v}_2) + \lambda \int_{\mathbb{R}^3} d\bar{v}_4 \int_{\mathbb{R}^3} d\bar{v}_3 \int_{\mathbb{S}^2} d\omega \, B_{ne}^{34}(\bar{v}_3, \bar{v}_4) F(\bar{v}_3) F(\bar{v}_4) \leq C(E(0), \varepsilon_0) \quad (3.8)$$

coming from the estimates (1.12), (1.13), (1.14) for the collisional kernels.

The estimate (3.8) implies that we have the following estimate for the solution $F$

$$F(t, v) \geq C e^{-\beta |v|^2 - C(E(0), \varepsilon_0)(1 + |v| t)} \quad (3.9)$$

and so we conclude that for every $t \geq 0$ there exists some constant $\tilde{\beta} > 0$ and $\tilde{C}$ such that $F(t, v) \geq \tilde{C} e^{-\tilde{\beta} |v|^2}$. 

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The assumption (3.6) will be used in order to justify a formal calculation, which yields the formula for the time derivative of the entropy. Notice in addition that it implies $F > 0$ for all $t \geq 0$.

**Proposition 3.1** Assume that the initial value $(\lambda_0, F_0)$ satisfies the assumption (3.6) and that its energy is bounded, i.e. $E(0) < \infty$. If $F$ has bounded fourth moment, the solution $(\lambda, F)$ satisfies $\lambda(t) \in [0, 1]$ for all $t \geq 0$. Moreover if for a strong solution $\|F\|_{L^1} \leq \kappa$ for some $0 < \kappa < \infty$ and for all $t \geq 0$, then $\partial_t \mathcal{H}(t) \leq 0$.

**Proof** We start with the first claim: the well-definedness of $\lambda$. Using the definition of energy and its conservation we see

$$\frac{\lambda}{1-\lambda} \leq \frac{1}{4\pi} \frac{E(t)}{2\varepsilon_0} = \frac{1}{4\pi} \frac{E(0)}{2\varepsilon_0} \leq \frac{E(0)}{2\varepsilon_0} \tag{2.13}$$

This implies that $\lambda(t) < 1$ for all $t$, since $\lambda \leq \frac{E(0)}{2\varepsilon_0 + E(0)} < 1$.

Assuming now $\lambda(t_0) = 0$ for some $t_0 \in [0, 1)$, the equation (2.13) implies

$$\partial_t \lambda(t_0) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^3} \mu_1 \{ F, 0 \} \, dv$$

$$= -\frac{1}{4\pi^2} \int_{\mathbb{R}^3} d\bar{v}_1 d\bar{v}_2 d\bar{v}_3 d\bar{v}_4 \delta(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4)$$

$$\delta(|\bar{v}_1|^2 + |\bar{v}_2|^2 - |\bar{v}_3|^2 - |\bar{v}_4|^2 - 2\varepsilon_0) W_{ne}(\bar{v}_1; \bar{v}_2, \bar{v}_3, \bar{v}_4) (-F(\bar{v}_2) F(\bar{v}_1))$$

$$\geq 0.$$ 

This conclude the first claim, indeed we have just seen that $\lambda(t) \in \left[0, \frac{E(0)}{2\varepsilon_0 + E(0)}\right]$ for all $t$.

For the second claim, we shall simply differentiate the entropy and using the kinetic system (2.13) we can conclude that it is non-increasing.

$$\partial_t \mathcal{H}(t) = \int_{\mathbb{R}^3} dv \left[ (1 + \lambda) \log(F) \partial_t F + F \log(F) \partial_t \lambda + (1 + \lambda) \partial_t F \right]$$

$$+ \lambda \log(\lambda) \partial_t F + F \log(\lambda) \partial_t \lambda + F \partial_t \lambda$$

$$+ 4\pi \left( \frac{\partial_t \lambda}{(1-\lambda)^2} \log(\lambda) + \frac{\partial_t \lambda}{1-\lambda} - \frac{\partial_t \lambda}{1-\lambda} \right)$$

$$= \int_{\mathbb{R}^3} dv \left[ \partial_t \left((1 + \lambda) F\right) + \partial_t \left((1 + \lambda) F\right) \log(F) + \partial_t (\lambda F) \log(\lambda) \right]$$

$$+ 4\pi \log(\lambda) \partial_t \frac{\lambda}{1-\lambda}$$

$$= \int_{\mathbb{R}^3} dv \left[ \mu_1 \{ F, \lambda \} + \mu_2 \{ F, \lambda \} \right] \log(F) + \int_{\mathbb{R}^3} dv \mu_2 \{ F, \lambda \} \log(F)$$

$$= \frac{1}{4} \int_{\mathbb{R}^3} d\nu_1 \int_{\mathbb{R}^3} d\nu_2 \int_{\mathbb{R}^3} \omega \, B_{el}(\nu_1, \nu_2) (1 + \lambda)^2 \left( F(\nu_3) F(\nu_4) - F(\nu_1) F(\nu_2) \right)$$

$$+ \int_{\mathbb{R}^3} d\bar{v}_1 \int_{\mathbb{R}^3} d\bar{v}_2 \int_{\mathbb{R}^3} \omega \, B_{ne}^2(\bar{v}_1, \bar{v}_2) \left( \lambda F(\bar{v}_3) F(\bar{v}_4) - F(\bar{v}_1) F(\bar{v}_2) \right)$$

$$\leq 0,$$ 

(3.10)
where we used that the integral of the sum of the collision terms is zero and equation (2.10), the weak formulation of the kinetic equation and the symmetry for the elastic collision. Now the inequality

\[(z - y) \log \left( \frac{y}{z} \right) \leq 0 \quad \forall z, y > 0\]

implies that each integrand on the right hand side of (3.10) is negative, which implies that the entropy is decreasing.

In order to justify more rigorously this calculation we remark that \[\|F\|_{L^1_{\infty}} \leq \kappa\] for every \(t \geq 0\) and that for every collisional kernel \(B \in \{B_{el}, B_{ne}^f\}\) there exists some constant \(C > 0\) such that \(0 \leq B \leq C (1 + |v| + |w|)\). Hence we know that for every \(i = 1, 2\) we have that \(\mathbb{K}_i [F, \lambda] \in L^1_{\infty}(\mathbb{R}^3)\), which implies that \(T_1 (\lambda, F) \in L^\infty\) and \(T_2 (\lambda, F) \in L^1(\mathbb{R}^3)\). Given the assumption (3.6) we know that \(|\log (F)| \leq C (|v|^2 + 1)\) for some \(C\) and for all \(0 \leq s \leq t\). Therefore using the Leibniz rule we can compute for all \(\varepsilon > 0\)

\[
\partial_t (F \log (F + \varepsilon)) = T_2 (\lambda, F) \log (F + \varepsilon) + \frac{F}{F + \varepsilon} T_2 (\lambda, F) \quad (3.11)
\]

and also

\[
\partial_t (\lambda F \log (\lambda F + \varepsilon)) = [FT_1 (\lambda, F) + \lambda T_2 (\lambda, F)] \log (\lambda F + \varepsilon) + \frac{\lambda F}{\lambda F + \varepsilon} [FT_1 (\lambda, F) + \lambda T_2 (\lambda, F)] \quad (3.12)
\]

These imply using Fubini that for any fixed \(t\)

\[
\int_{\mathbb{R}^3} dv \ F(t, v) \log (F + \varepsilon) (t, v) = \int_{\mathbb{R}^3} dv \ F_0(v) \log (F_0 + \varepsilon) (v)
+ \int_0^t ds \int_{\mathbb{R}^3} dv \ T_2 (\lambda, F) (s, v) \log (F + \varepsilon) (s, v) + \frac{F(s, v)}{F(s, v) + \varepsilon} T_2 (\lambda, F) (s, v)
\]

\[
(3.13)
\]

and similarly

\[
\int_{\mathbb{R}^3} dv \ \lambda F(t, v) \log (\lambda F + \varepsilon) (t, v) = \int_{\mathbb{R}^3} dv \ \lambda_0 F_0(v) \log (\lambda_0 F_0 + \varepsilon) (v)
+ \int_0^t ds \int_{\mathbb{R}^3} dv \ [FT_1 (\lambda, F) + \lambda T_2 (\lambda, F)] (s, v) \log (\lambda F + \varepsilon) (s, v)
\]

\[
+ \frac{\lambda F(s, v)}{\lambda F(s, v) + \varepsilon} [FT_1 (\lambda, F) + \lambda T_2 (\lambda, F)] (s, v) \quad (3.14)
\]

Here everything is uniformly integrable, and therefore letting then \(\varepsilon \to 0\) we conclude the formula in (3.10), since moreover the function \(f(x) = \frac{1}{x - 1} \log(x) + \log(1 - x)\) is already well-defined and differentiable for \(x \in [0, 1)\). \(\square\)

We proceed now with the well-posedness theory. We will adapt here the standard method for the classical Boltzmann equation. We will use a weakly convergent sequence of approximating solutions. This is the method used in the seminal paper [1].
3.2 The Cut-Off Kinetic System

We define some auxiliary cut-off kernels. We define for \( n \in \mathbb{N} \)

\[
B_{cl,n}(v, w) := \min (|v - w|, n),
\]

\[
B_{he,n}^1(\bar{v}_1, \bar{v}_2) := B_{he}^1(\bar{v}_1, \bar{v}_2) \chi_{|\bar{v}_1 - \bar{v}_2| \leq n} (\bar{v}_1, \bar{v}_2),
\]

\[
B_{he,n}^3(\bar{v}_3, \bar{v}_4) := B_{he}^3(\bar{v}_3, \bar{v}_4) \chi_{|\bar{v}_3 - \bar{v}_4| \leq \sqrt{n^2 - 4\varepsilon_0}} (\bar{v}_3, \bar{v}_4).
\]

It is not difficult to see that the following is true.

**Proposition 3.2** For the velocities of the nonelastic collisions the following is equivalent

\[
|\bar{v}_1 - \bar{v}_2| \leq n \iff |\bar{v}_3 - \bar{v}_4| \leq \sqrt{n^2 - 4\varepsilon_0}
\]

**Proof** We recall the definition of pre- and post-collisional nonelastic velocities as in (1.8) which implies that \(|\bar{v}_3 - \bar{v}_4| = \sqrt{|\bar{v}_1 - \bar{v}_2|^2 - 4\varepsilon_0}\). Moreover, we also have the relation (1.9) which again implies \(|\bar{v}_1 - \bar{v}_2| = \sqrt{|\bar{v}_3 - \bar{v}_4|^2 + 4\varepsilon_0}\). Hence considering \(|\bar{v}_1 - \bar{v}_2| \leq n\) respectively \(|\bar{v}_3 - \bar{v}_4| \leq \sqrt{n^2 - 4\varepsilon_0}\) in the relations (1.8) and (1.9) we conclude the proof.

An important implication of Proposition 3.2 is that the following sets are the same

\[
\left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^{12} : |v_1 - v_2| \leq n, \ v_3, v_4 = \frac{v_1 + v_2}{2} \pm \omega \sqrt{\frac{|v_1 - v_2|^2}{4} - \varepsilon_0}, \ \omega \in \mathbb{S}^2 \right\}
\]

\[
= \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^{12} : |v_3 - v_4| \leq \sqrt{n^2 - 4\varepsilon_0}, \ v_{1,2} = \frac{v_3 + v_4}{2} \pm \omega' \sqrt{\frac{|v_3 - v_4|^2}{4} + \varepsilon_0}, \ \omega' \in \mathbb{S}^2 \right\}
\]

\[
:= \Omega_n,
\]

We can now define the cut-off collisions term \(K_{\varepsilon_i,n}\) for \(i = 1, 2\) replacing of the usual hard sphere’s kernels with the cut-off kernel as in (3.15), (3.16) and (3.17). We also define a new operator \(T_n : X \rightarrow X\) by

\[
T_n \left( \lambda, F_n \right) = \left( \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(1-\lambda)^2(1+\lambda)^2}{(1+\lambda)(1-\lambda)^2} F \int_{\mathbb{R}^3} \mathbb{K}_{1,n} [F, \lambda] \, dv 
+ \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathbb{K}_{2,n} [F, \lambda] \, F \int_{\mathbb{R}^3} \mathbb{K}_{1,n} [F, \lambda] \, dv \right).
\]

**Lemma 3.1** A solution \((\lambda_n, F_n) \in C^1 ([0, 1), X)\) of the equation \(\partial_t \left( \frac{\lambda_n}{F_n} \right) = T_n \left( \frac{\lambda_n}{F_n} \right)\) still satisfies the conservation of mass, momentum and energy as written in (3.1), (3.2) and (3.3).

**Proof** We first find the weak formulation for the collisional operator of the cut-off system. This is the analogous of the weak formulation for the original kinetic system (1.26). We use on the one hand the symmetry of the elastic kernels with respect to pre- and post-collisional velocities.
and on the other hand Proposition 3.2. Hence,
\[
\int_{\mathbb{R}^3} dv \left[ \mathcal{K}_{1,n} \{ F_n, \lambda_n \} \varphi^1(v) + \mathcal{K}_{2,n} \{ F_n, \lambda_n \} \varphi^2(v) \right]
\]
\[
= \frac{(1 + \lambda_n)^2}{2} \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{S}^2} d\omega \min(|v_1 - v_2|, n) F_n(v) F_n(v_2)
\]
\[
(\varphi^1(v_1) + \varphi^1(v_4) - \varphi^1(v_1) - \varphi^1(v_2))
\]
\[
+ \int_{\Omega_0} dv_1 dv_2 dv_3 v_3 \delta(\bar{v}_1 + v_2 - \bar{v}_3 - v_4) \delta(|\bar{v}_1|^2 + |\bar{v}_2|^2 - |\bar{v}_3|^2 - |v_4|^2 - 2\varepsilon_0) W_{ne}
\]
\[
(F_n(\bar{v}_1) F_n(\bar{v}_2) - \lambda_n F_n(\bar{v}_3) F_n(\bar{v}_4)) (\varphi^2(v_3) + \varphi^1(v_4) - \varphi^1(v_1) - \varphi^1(v_2)).
\]
(3.20)

Hence, testing with \(\varphi^1(v) = 1, v, |v|^2 + \delta i_2 \varepsilon_0\) we can immediately conclude that
\[
\int_{\mathbb{R}^3} dv \left[ \mathcal{K}_{1,n} \{ F_n, \lambda_n \} \varphi^1(v) + \mathcal{K}_{2,n} \{ F_n, \lambda_n \} \varphi^2(v) \right] = 0.
\]
This implies as for the solution to the kinetic system (2.13) the conservation of mass, momentum and energy. Moreover, we remark that testing with the functions \(\varphi^1\) is always well-defined since the solution \((\lambda_n, F_n) \in C^1([0, \infty), X)\) and since the collisional kernels are bounded. Thus, \(\mathcal{K}_{1,n} \{ F_n, \lambda_n \} \in L^1_+(\mathbb{R}^3)\).

With this Lemma and in particular with the conservation of energy, we can again that for a solution \((\lambda_n, F_n) \in C^1([0, 1), X)\), the first term satisfies \(\lambda_n \in \mathbb{R}^+ \) and \(E_n(0) + \int_0^t \lambda_n F_n(v) dv = E_n(t)\) for all \(t \geq 0\). Next we look at what happens to the entropy for the cut-off system. We define therefore the entropy \(H_n(t) := \mathcal{H}_n[\lambda_n, F_n](t)\) for a solution \((\lambda_n, F_n)\) according to Definition 3.4. The following two lemmas summarize the properties of the entropy in this case.

**Lemma 3.2** Let \((\lambda_n, F_n) \in C^1([0, 1), X)\) be a solution of the equation \(\partial_t \left( \frac{\lambda_n}{F_n} \right) = T_n \left( \frac{\lambda_n}{F_n} \right)\) for the initial value \((\lambda_0, F_0) \in X_+ \cap \tilde{X}\) which satisfies (3.6). Assume \(H_n(t)\) is well-defined and assume \(\| F_n \|_{L^1} \leq \kappa\) for some \(0 < \kappa < \infty\) and for all \(t \geq 0\). Then
\[
\partial_t H_n(t) = \int_{\mathbb{R}^3} dv \left[ \mathcal{K}_{1,n} \{ F_n, \lambda_n \} \log(F_n) + \mathcal{K}_{2,n} \{ F_n, \lambda_n \} \log(\lambda_n F_n) \right] \leq 0.
\]

**Proof** Using the results of Lemma 3.1 we proceed as in Proposition 3.1.

**Lemma 3.3** Let \((\lambda_n, F_n) \in C^1([0, 1), X)\) be a solution of the equation \(\partial_t \left( \frac{\lambda_n}{F_n} \right) = T_n \left( \frac{\lambda_n}{F_n} \right)\) for the initial value \((\lambda_0, F_0) \in X_+ \cap \tilde{X}\) which satisfies (3.6). Assume \(\| F_n \|_{L^1} \leq \kappa\) for some \(0 < \kappa < \infty\) and for all \(t \geq 0\), \(F_0 \log(F_0) \in L^1(\mathbb{R}^3)\) and \(E_n(0) =: E < \infty\). Then there exists a constant \(K\) depending on \(E\) such that \(H_n(t) \geq K\) for all \(t\).

**Proof** As we already have seen the assumptions imply \(\lambda_n \in \left[ 0, \frac{E^2}{2\varepsilon_0 + E} \right]\). We now estimate each term of the entropy. So we start with \((1 + \lambda_n) \leq 2\), also using the well-known inequality \(x \log(x) \geq -y + x \log(y)\) with \(x = F\) and \(y = e^{-|v|^2}\) we estimate
\[
\int_{\mathbb{R}^3} dv \ F_n \log(F_n) \geq \int_{\mathbb{R}^3} dv \ F_n \log^{-}(F_n) \geq -\int_{\mathbb{R}^3} dv \left[ F_n |v|^2 + e^{-|v|^2} \right]
\]
\[
\geq -E - \int_{\mathbb{R}^3} dv \ e^{-|v|^2} := -C_E
\]
(3.21)

It is well-known that \(\log(x) \geq -\frac{1}{x}\) and also that \(\log(1-x) \geq \log \left( 1 - \frac{E^2}{2\varepsilon_0 + E} \right) = \log \left( \frac{2\varepsilon_0}{2\varepsilon_0 + E} \right)\). Finally, \(\frac{1}{x} \log(x) \geq 1\) for all \(0 \leq x < 1\). All these estimates imply now the lower bound for
the entropy
\[ \mathcal{H}_n(t) \geq -2 C_E - \frac{E}{e} - 4 \pi - 4 \pi \log \left( \frac{2 \varepsilon_0}{2 \varepsilon_0 + E} \right) := K. \] (3.22)

Remark Lemmas 3.2 and 3.3 imply together an important result for any strong solution \((\lambda_n, F_n) \in C^1 \{(0, 1), X \}\) of \(\partial_t \left( \begin{array}{c} \lambda_n \\ F_n \end{array} \right) = T_n \left( \begin{array}{c} \lambda_n \\ F_n \end{array} \right)\) for the initial value \((\lambda_0, F_0) \in X_+ \cap \tilde{X}\) which satisfies (3.6). If \(\|F_n\|_{L^1_k} \leq \kappa\) for some \(0 < \kappa < \infty\) and for all \(t \geq 0\), \(F_0 \log(F_0) \in L^1(\mathbb{R}^3)\) and \(E_n(0) := E < \infty\), then there exists a \(K \in \mathbb{R}\) such that \(K \leq \mathcal{H}_n(t) \leq \mathcal{H}(0)\). This implies, as we will see in the next Lemma, that \(F_n \log(F_n)\) has uniformly bounded integral for any \(n\) and \(t\).

Lemma 3.4 Let \((\lambda_n, F_n)\) and \((\lambda_0, F_0)\) satisfy the same assumption as in Lemma 3.3. Then the sequence \((\lambda_n, F_n)\) is bounded in \(C \{(0, \infty), X\}\) and there exists some \(C \in \mathbb{R}\) independent of \(n\) and \(t\) such that \(\int_{\mathbb{R}^3} dv \ F_n \log(F_n) \leq C\) for all \(n\) and \(t\).

Proof The conservation of energy implies the first claim
\[ \sup_{t \geq 0} |\lambda_n(t)| \leq \frac{E}{2 \varepsilon_0 + E} \text{ and } \sup_{t \geq 0} \|F_n(t)\|_{L^1_2} \leq E. \]

The second claim follows from Lemma 3.3. Indeed
\[ \int_{\mathbb{R}^3} dv \ F_n \log(F_n) \leq \int_{\mathbb{R}^3} dv \ (1 + \lambda_n) F_n \log(F_n) \]
\[ = \mathcal{H}(t) - \left[ \int_{\mathbb{R}^3} dv \lambda_n \log(\lambda_n) + 4 \pi \log(1 - \lambda_n) \right] \]
\[ \leq \mathcal{H}(0) + \frac{E}{e} + 4 \pi + 4 \pi \log \left( \frac{1}{2 \varepsilon_0 + E} \right) := C. \]

Moreover, we already know that \(\int_{\mathbb{R}^3} dv \ F_n \log^-(F_n) \geq -C_E\), this together with the last estimate implies
\[ \int_{\mathbb{R}^3} dv \ F_n \log^+(F_n) = \int_{\mathbb{R}^3} dv \ F_n \log(F_n) - \int_{\mathbb{R}^3} dv \ F_n \log^-(F_n) \leq C + C_E. \]

Since all these estimates are uniformly in time and \(n\), we conclude \(F_n \log(F_n) \in L^1(\mathbb{R}^3)\). □

The next and last Lemma of this section is very important for the existence of strong solution of the cut-off kinetic system.

Lemma 3.5 Let \(T_n \left( \begin{array}{c} \lambda \\ F \end{array} \right) = T_n \left( \begin{array}{c} \mu \\ G \end{array} \right) \in X_+\) with bounded energies \(E(\lambda, F) \leq E_0\) and \(E(\mu, G) \leq E_0\), then there exists some \(C_n\) (which can depend on \(n\)) such that
\[ \left\| T_n \left( \begin{array}{c} \lambda \\ F \end{array} \right) - T_n \left( \begin{array}{c} \mu \\ G \end{array} \right) \right\|_X \leq C_n \left\| \left( \begin{array}{c} \lambda \\ F \end{array} \right) - \left( \begin{array}{c} \mu \\ G \end{array} \right) \right\|_X. \]

Proof The proof is not difficult, but it requires several estimates. We start with some preliminary calculations. They will be useful also at the end of this paper. For most of the next estimates we use that for given functions \(F, G, H, K\) and velocities \(v, w, u, v\) the following holds true
\[ (H(v) F(u) - F(v) F(w) - K(v) G(u) + G(v) G(w)) \]
\[ = (H(v) - K(v)) F(u) + (K(v) F(u) - G(u)) - (F(v) - G(v)) F(w) - G(v) (F(w) - G(w)). \]

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We can proceed with all preliminary estimates. First of all, using (3.23) together with the triangle inequality and the well-known inequality \( \min(|v_1 - v_2|, n) \leq |v_1 - v_2| \leq (1 + |v_1|^2)^{1/2}(1 + |v_2|^2)^{1/2} \) we compute with the help of the monotonicity of the \( L^1_k \)-norms in the exponent \( k \) the following

\[
\int_{\mathbb{R}^3} dv \left| \mathbb{K}_{1,n}^{\text{elastic}} [F, \lambda F] - \mathbb{K}_{1,n}^{\text{elastic}} [G, \lambda G] \right| \\
\leq 4 \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \left( 1 + |v_1|^2 \right) \left( 1 + |v_2|^2 \right) \left( |F(v_1)| F_2 + |F(v_2)| G_1 \right) \\
\leq 8E_0 \| F - G \|_{L^1_2}.
\]

(3.24)

We obtain a similar result (the constant multiplying \( \| F - G \|_{L^1_2} \) will be slightly different) for the first nonelastic collision terms. Here the only properties we need are again that \( B_{ne,n}^{ij} \leq B_{ne}^{ij} \), the estimate in (1.14) and (3.23) together with the triangle inequality. We have therefore

\[
\int_{\mathbb{R}^3} dv \left| \mathbb{K}_{1,n}^{\text{nonelastic}} [F, \lambda F] - \mathbb{K}_{1,n}^{\text{nonelastic}} [G, \lambda G] \right| \leq 6C_0 (1 + 2\sqrt{\varepsilon_0}) E_0 \| F - G \|_{L^1_2}.
\]

(3.25)

This means that there exists a constant \( \overline{C} \) depending on \( E_0 \) and on \( \varepsilon_0 \) such that

\[
\int_{\mathbb{R}^3} dv \left| \mathbb{K}_{1,n} [F, \lambda F] - \mathbb{K}_{1,n} [G, \lambda G] \right| \leq \overline{C} E_0.
\]

(3.26)

This immediately implies the useful inequality \( \int_{\mathbb{R}^3} dv \left| \mathbb{K}_{1,n} [F, \lambda F] \right| \leq \overline{C} E_0 \).

We now estimate

\[
\int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left[ \left| \mathbb{K}_{1,n} [F, \lambda F] - \mathbb{K}_{1,n} [G, \lambda G] \right| + \left| \mathbb{K}_{2,n} [F, \lambda F] - \mathbb{K}_{2,n} [G, \lambda G] \right| \right].
\]

We proceed similarly as before using (3.23) and estimating the cut-off kernel by \( n \). In the change of variable we use this time the relation between pre- and post-collisional velocities as in (1.5) and we calculate

\[
\int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left[ \left| \mathbb{K}_{1,n}^{\text{elastic}} [F, \lambda F] - \mathbb{K}_{1,n}^{\text{elastic}} [G, \lambda G] \right| \\
+ \left| \mathbb{K}_{2,n}^{\text{elastic}} [F, \lambda F] - \mathbb{K}_{2,n}^{\text{elastic}} [G, \lambda G] \right| \right] \\
\leq 4n \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \left( 1 + |v_1|^2 \right) \left[ |F(v_1)| F_2 + |F(v_2)| G_1 \right] \\
+ 4n \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \left( 1 + |v_3|^2 + 1 + |v_4|^2 \right) \left[ |F(v_3)| F_4 + |F(v_4)| G_3 \right] \\
\leq 24n E_0 \| F - G \|_{L^1_2}.
\]

(3.27)

For the nonelastic collisions we proceed in a similar way, using the inequality \( B_{ne,n}^{ij} \leq n \), and obtain a similar result as the estimate in (3.27)

\[
\int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left[ \left| \mathbb{K}_{1,n}^{\text{nonelastic}} [F, \lambda F] - \mathbb{K}_{1,n}^{\text{nonelastic}} [G, \lambda G] \right| \\
+ \left| \mathbb{K}_{2,n}^{\text{nonelastic}} [F, \lambda F] - \mathbb{K}_{2,n}^{\text{nonelastic}} [G, \lambda G] \right| \right] \\
\leq 16nC_0 (1 + 2\sqrt{\varepsilon_0})(1 + \varepsilon_0) E_0 \| F - G \|_{L^1_2}.
\]

(3.28)
In all these calculations we estimated \( \lambda \) by 1. All these results together imply the existence of a constant \( \overline{C}_n > 0 \) which depends on \( n \) and for which

\[
\int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left[ \| \mathbb{K}_{1,n} [F, \lambda F] - \mathbb{K}_{1,n} [G, \lambda G] \| \right]
+ \| \mathbb{K}_{2,n} [F, \lambda F] - \mathbb{K}_{2,n} [G, \lambda G] \| \leq \overline{C}_n \| F - G \|_{L^1} \tag{3.29}
\]
holds.

Other additional estimates involve the difference of the operator \( T_n \) applied this time to the pair \( (G, \lambda G) \) and \( (G, \mu G) \). With a similar computation as for equation (3.26) (now many terms of the form \(|G(v) - G(v)\| \) cancel out) we conclude the existence of a constant \( \overline{C} > 0 \) independent of \( n \) but which depends on \( E_0 \) such that

\[
\int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left[ \| \mathbb{K}_{1,n} [G, \lambda G] - \mathbb{K}_{1,n} [G, \mu G] \| + \| \mathbb{K}_{2,n} [G, \lambda G] - \mathbb{K}_{2,n} [G, \mu G] \| \right]
\leq \overline{C} E_0 |\lambda - \mu|.	ag{3.30}
\]

This can be also explicitly computed by means of the definition of the collision operator, the relation (3.23), the triangle inequality and finally estimating the kernels as for equations (3.24) and (3.26).

In an analogous way as we did for equation (3.29) we can find a constant \( \widetilde{C}_n \) dependent on \( n \) and on \( E_0 \) such that

\[
\int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left[ \| \mathbb{K}_{1,n} [G, \lambda G] - \mathbb{K}_{1,n} [G, \mu G] \| + \| \mathbb{K}_{2,n} [G, \lambda G] - \mathbb{K}_{2,n} [G, \mu G] \| \right]
\leq \widetilde{C}_n E_0 |\lambda - \mu|.	ag{3.31}
\]

Again this can be seen easily doing similar computation as for (3.27) for the elastic kernels, while for the nonelastic part all the terms depending on \( \overline{v}_1 \) and \( \overline{v}_2 \) eliminate, due to the fact that they are not multiplied by \( \lambda \) or \( \mu \).

Equipped with estimates (3.26), (3.29), (3.30) and (3.31), the proof of Lemma 3.5 can be concluded. We can bound the norm of the difference of the maps \( T_n \) by the sum of four terms

\[
\left\| T_n \left( \frac{\lambda}{F} \right) - T_n \left( \frac{\mu}{G} \right) \right\|_X \leq \left| T_{1,n} \left( \frac{\lambda}{F} \right) - T_{1,n} \left( \frac{\lambda}{G} \right) \right| + \left| T_{1,n} \left( \frac{\lambda}{G} \right) - T_{1,n} \left( \frac{\mu}{G} \right) \right|
+ \left| T_{2,n} \left( \frac{\lambda}{F} \right) - T_{2,n} \left( \frac{\lambda}{G} \right) \right| + \left| T_{2,n} \left( \frac{\lambda}{G} \right) - T_{2,n} \left( \frac{\mu}{G} \right) \right|
= I + II + III + IV. \tag{3.32}
\]

We estimate here explicitly only the first term since all remaining terms can be handled in a similar way adding and subtracting suitable integrals. For the first term we need (3.26). Then we see that there exists some \( C_1 > 0 \) depending on \( E_0 \) but not on \( n \) such that

\[
I \leq \frac{(1 - \lambda)^2 (1 + \lambda)}{(1 + \lambda)(1 - \lambda)^2} \int_{\mathbb{R}^3} dv \left[ \| \mathbb{K}_{1,n} [F, \lambda F] - \mathbb{K}_{1,n} [G, \lambda G] \| \right]
\times \int_{\mathbb{R}^3} dv \left[ \mathbb{K}_{1,n} [G, \lambda G] \right]
\leq 2 \int_{\mathbb{R}^3} dv \left[ \| \mathbb{K}_{1,n} [F, \lambda F] - \mathbb{K}_{1,n} [G, \lambda G] \| \right]
+ 2 \int_{\mathbb{R}^3} dv \left[ \| F - G \|_{L^1} \right] \leq C_1 \| F - G \|_{L^1}. \tag{3.33}
\]
For the second term we use the result in (3.26) and in (3.30). We find in this way a constant \( C_2 > 0 \) independent of \( n \) which bind the second term as follows
\[
II \leq C_2 |\lambda - \mu|.
\]
Now we consider the estimates for the \( L_2^1 \)-norms. By means of equations (3.26) and (3.30) we can compute for the third term
\[
III \leq C_3 \| F - G \|_{L_2^1},
\]
for some \( C_3 > 0 \) depending on \( n \) and \( E_0 \).

Finally for the last term we proceed in a similar way, using again equations (3.26), (3.30) and (3.31). We therefore find a constant \( C_4 \) depending on \( n \) and \( E_0 \) such that the fourth term is bounded from above by
\[
IV \leq C_4 |\lambda - \mu|.
\]

We are ready now to conclude the proof of this Lemma. Indeed the estimates (3.33), (3.34), (3.35) and (3.36) imply the existence of a constant \( C_n \) depending on \( n \) such that
\[
\| T_n(\lambda F) - T_n(\mu G) \|_{X} \leq (C_2 + C_4) |\lambda - \mu| + (C_1 + C_3) \| F - G \|_{L_2^1}
\]
\[
\leq C_n \| (\lambda F) - (\mu G) \|_{X}.
\]
\[
(3.37)
\]

This Lemma ends Sect. 3.2 and we can now prove the existence of weak solutions for the system (2.13).

### 3.3 Existence of Weak Solutions

We are now ready to prove the existence of weak solutions. Because of the uniform bound of the entropy, the cut-off system of Sect. 3.2 gives us a sequence of functions which converges to the desired weak solutions. However, we still need one result before the actual theorem. We namely need, that for all \( n \geq M \) the \( L_4^1 \)-norm is uniformly bounded. This is proved in the next Lemma, which can be proved adapting the theory of the classical Boltzmann equation.

**Lemma 3.6** Let \((\lambda_n, F_n) \in C^1([0, \infty), X)\) be the unique solution of the approximating kinetic system
\[
\partial_t \left( \frac{\lambda_n}{F_n} \right) = T_n \left( \frac{\lambda_n}{F_n} \right) \text{ with initial value } (\lambda_0, F_0) \in X_+ \text{ which satisfies (3.6). Assume also that } \| F_0 \|_{L_4^1} < \infty. \text{ Then there exist constants } M \text{ and } C \text{ such that for every } n \geq M \text{ it holds}
\]
\[
\sup_{t \geq 0} \| F_n \|_{L_4^1} \leq C \quad \text{ and } \quad \int_0^t \| F_n(\cdot, \tau) \|_{X} \| F_n(\cdot, \tau) \|_{L_4^1} \leq C(1 + t).
\]

**Proof** We adapt to our system the proof of Lemma 4.2 in [19]. We will proceed in two steps. We first prove that there exist \( M \) and \( C \) such that
\[
\sup_{t \geq 0} \| F_n \|_{L_4^1} \leq C \quad \text{ and } \quad \int_0^t \| F_n(\cdot, \tau) \|_{X} \| F_n(\cdot, \tau) \|_{L_4^1} \leq C(1 + t)
\]
and then with the help of this result, we will prove the Lemma.

Since there exist constants \( c_3 \) and \( c_4 \) such that \((1 + |v|^2)^{3/2} \leq c_3 (1 + |v|^3) \) and \((1 + |v|^2)^2 \leq c_4 (1 + |v|^4)\), we will show the result for the weighted norm defined by

\[
\| T_n(\lambda F) - T_n(\mu G) \|_{X} \leq (C_2 + C_4) |\lambda - \mu| + (C_1 + C_3) \| F - G \|_{L_2^1}
\]
\[
\leq C_n \| (\lambda F) - (\mu G) \|_{X}
\]
\[
(3.37)
\]
\[ \|F\|_s := \int_{\mathbb{R}^3} dv \left( F^1(v) + F^2(v) \right) |v|^s \quad \text{for } s = 3, 4. \] Indeed, since the solutions have the \(L_2\)-norm bounded, the initial mass is conserved, and so we would conclude the lemma.

We need now some estimates for the elastic and nonelastic kernels. First of all for the elastic kernels we obtain the following relation

\[ \frac{1}{4} \min ((|v_1|, n) - |v_2|) \leq \min ((|v_1| - |v_2|), n) \leq 4 \left( |v_2| + \min (|v_1|, n) \right). \quad (3.40) \]

We recall the well-known Povzner inequality for the elastic collisions, as given e.g. in [27] and [28]

\[ (|v_3|^s + |v_4|^s - |v_1|^s - |v_2|^s) \leq -K(\theta) |v_1|^s + C \left( |v_1|^{s-1} |v_2| + |v_1| |v_2|^{s-1} \right), \]

where \( s > 2, \theta \) is the angle between \( v_1 - v_2 \) and \( v_3 - v_4 \) and \( K(\theta) \geq 0 \) and strictly positive for \( 0 < \theta \leq \frac{\pi}{2} \). Therefore, using (3.41) we have the following inequality

\[ \int_{\mathbb{R}^3} d\omega \left( |v_3|^3 + |v_4|^3 - |v_1|^3 - |v_2|^3 \right) \leq C_1 \left( |v_1|^2 |v_2| + |v_1| |v_2|^2 \right) - c_1 |v_1|^3. \quad (3.42) \]

Hence, putting (3.40) and (3.42) together and using that \( 0 \leq \lambda_n < 1 \) we obtain

\[ \frac{(1 + \lambda_n)_n^2}{2} \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{S}^2} d\omega \ B_{el,n}(v_1, v_2) F_n(v_1) F_n(v_2) \left( |v_3|^3 + |v_4|^3 - |v_1|^3 - |v_2|^3 \right) \]

\[ \leq C_2 \left( \|F_n\|^2_{L^1} + \|F_n\|_{L^1} \right) \int_{\mathbb{R}^3} dv \left( F_n(v) |v|^2 \min (|v|, n) \right) + C_3 \|F_n\|^3_{L^1} \quad (3.43) \]

\[ - C_4 \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv \left( F_n(v) \right) |v|^3 \min (|v|, n), \]

for some constants \( C_2, C_3, C_4 > 0 \) depending on \( C_1 \) and \( c_1 \).

We proceed now with some estimates for the terms of the nonelastic collisions. We notice first of all that for the kernels we have following relation due to (1.13)

\[ B_{ne,n}^{12}(\overline{v}_1, \overline{v}_2) \leq \frac{C_0}{2} \left| \overline{v}_1 - \overline{v}_2 \right| \frac{\chi}{|\overline{v}_1 - \overline{v}_2|} (\overline{v}_1, \overline{v}_2) \]

\[ \leq \frac{C_0}{2} \min (|\overline{v}_1 - \overline{v}_2|, n) \leq 2C_0 \left( |\overline{v}_2| + \min (|\overline{v}_1|, n) \right) \]

\[ (3.44) \]

and also since \( \sqrt{n^2 - 4\varepsilon_0} \leq n \) we estimate

\[ B_{ne,n}^{34}(\overline{v}_3, \overline{v}_4) \leq \frac{C_0}{2} \left( |\overline{v}_3 - \overline{v}_4| + 2\sqrt{\varepsilon_0} \right) \frac{\chi}{|\overline{v}_3 - \overline{v}_4|} (\overline{v}_3, \overline{v}_4) \]

\[ \leq 2C_0 \left( |\overline{v}_4| + \min (|\overline{v}_3|, n) + 2\varepsilon_0 \right). \]

Moreover we have also some estimates for the nonelastic velocities. We can use the well-known relation \( x^3 + y^3 \leq \left( x^2 + y^2 \right)^{\frac{3}{2}} \) for \( x, y \geq 0 \) and also that there exists some \( C > 0 \) such that \( \left( x + y \right)^{3/2} - x^{3/2} - y^{3/2} \leq C \left( x^{1/2}y + xy^{1/2} \right) \) for \( x, y \geq 0 \).

By the relation for the velocities of the non elastic collisions (1.7) we see on one hand that

\[ |\overline{v}_3|^3 + |\overline{v}_4|^3 - |\overline{v}_1|^3 - |\overline{v}_2|^3 \leq C \left( |\overline{v}_1| |\overline{v}_2|^2 + |\overline{v}_1|^2 |\overline{v}_2| \right). \]

\[ (3.46) \]

And on the other hand there exists also a constant \( C_{\varepsilon_0} \) depending on \( \varepsilon_0 \) such that

\[ |\overline{v}_1|^3 + |\overline{v}_2|^3 - |\overline{v}_3|^3 - |\overline{v}_4|^3 \leq C_{\varepsilon_0} \left( 1 + |\overline{v}_3| |\overline{v}_4|^2 + |\overline{v}_3|^2 |\overline{v}_4| + |\overline{v}_3| + |\overline{v}_4| + |\overline{v}_3|^2 + |\overline{v}_4|^2 \right). \]

\[ (3.47) \]
Therefore using (3.44) and (3.46), we compute
\[
B_{\delta, n}^{12} (\overline{v}_1, \overline{v}_2) \left( |\overline{v}_3|^3 + |\overline{v}_4|^3 - |\overline{v}_1|^3 - |\overline{v}_2|^3 \right)
\leq C_1 \left( |\overline{v}_1|^3 + |\overline{v}_2|^3 + |\overline{v}_1|^2 - |\overline{v}_2|^2 \min ((|\overline{v}_1|, n) |\overline{v}_2|^2 + |\overline{v}_1|^2 min ((|\overline{v}_1|, n) |\overline{v}_2|) \right). \tag{3.48}
\]

Equations (3.45) and (3.47) imply a similar estimate for \(B_{\delta, n}^{44} (\overline{v}_3, \overline{v}_4) \left( |\overline{v}_1|^3 + |\overline{v}_2|^3 - |\overline{v}_3|^3 - |\overline{v}_4|^3 \right)\), which together with (3.48) and the weak formulation (3.20) yields evaluating the delta functions with respect to \((\overline{v}_1, \overline{v}_2)\) or \((\overline{v}_3, \overline{v}_4)\) the following
\[
\begin{align*}
\int_{\Omega_n} d\overline{v}_1 d\overline{v}_2 d\overline{v}_3 d\overline{v}_4 \delta (\overline{v}_1 - \overline{v}_2 - \overline{v}_3 - \overline{v}_4) & \delta (|\overline{v}_1|^2 + |\overline{v}_2|^2 - |\overline{v}_3|^2 - |\overline{v}_4|^2 - 2\epsilon_0) W_{\delta, n} \\
& \leq (F_n(\overline{v}_1) F_n(\overline{v}_2) + \lambda_n F_n^2(\overline{v}_3) F_n(\overline{v}_4)) \left( |\overline{v}_3|^3 + |\overline{v}_4|^3 - |\overline{v}_1|^3 - |\overline{v}_2|^3 \right) \\
& \leq \tilde{C} \left( \|F_n\|^2_{L^2} + \|F_n\|_{L^1}^2 \|F_n\|_3 + \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv (F_n(v)|v|^2 \min (|v|, n)) \right). \tag{3.49}
\end{align*}
\]

Here we also used the monotonicity of the \(L^1\)-norms estimating all moments less than \(2\) by the \(L^2\)-norm.

Therefore putting together the estimates (3.43) and (3.49) we conclude the existence of constants \(\tilde{C}_2, \tilde{C}_3\) and \(C_4\) such that the following holds
\[
\begin{align*}
\frac{1}{1 + \lambda_n} \int_{\mathbb{R}^3} dv |v|^3 \left( \|F_n\|_{L^1} + \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv \|F_n(v)|v|^2 \min (|v|, n) \right) \\
& \leq \tilde{C}_2 \left( \|F_n\|^2_{L^2} + \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv \|F_n(v)|v|^2 \min (|v|, n) \right) + \tilde{C}_3 \|F_n\|_{L^1} \|F_n\|_3 \\
& - C_4 \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv \|F_n(v)|v|^3 \min (|v|, n). \tag{3.50}
\end{align*}
\]

Using the sharper estimate for the collisional kernels in both elastic and nonelastic terms
\(|v_1 - v_2| \leq (1 + |v_2|^2)(1 + |v_1|^2)^{1/2} \leq (1 + |v_2|^2)(1 + |v_1|)\) and arguing as in (3.24) and (3.25) we find a constant \(\overline{C}\) which depends on the initial energy, which we know to be conserved, such that \(\int_{\mathbb{R}^3} dv \|\mathbb{K}_1[F_n, \lambda_n]\| \leq \overline{C} \|F_n\|_{L^1}\). Thus we easily see that
\[
\begin{align*}
(1 + \lambda_n)^{2} (1 + (1 - \lambda_n)^2)^{1/2} \int_{\mathbb{R}^3} F \int_{\mathbb{R}^3} dv \mathbb{K}_1[F_n, \lambda_n] \int_{\mathbb{R}^3} dv \|F_n(v)|v|^3 \leq \overline{C} \|F_n\|_{L^1} \|F_n\|_3.
\end{align*}
\]

Therefore there exist constants \(\tilde{C}_2, \tilde{C}_3\) and \(C_4\) such that for the time derivative of the third moment the following holds
\[
\begin{align*}
\partial_t \|F_n\|_3 \leq & 2 \tilde{C}_2 \left( \|F_n\|^2_{L^2} \|F_n\|_{L^2} + \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv \|F_n(v)|v|^2 \min (|v|, n) \right) \\
& + \tilde{C}_3 \|F_n\|_{L^1} \|F_n\|_3 - C_4 \|F_n\|_{L^1} \int_{\mathbb{R}^3} dv \|F_n(v)|v|^3 \min (|v|, n). \tag{3.51}
\end{align*}
\]

Arguing exactly as in [19] using Young’s inequality we can prove the existence of a constant \(C_{\delta, M} > 0\) for \(\delta > 0\) and \(M > 0\) such that for all \(n \geq M\) the following estimate holds true
\[
\begin{align*}
\partial_t \|F_n\|_3 + \left( \frac{C_4}{2} \|F_n\|_{L^1} - C_5 \left( \|F_n\|_{L^1} + \|F_n\|_{L^2} \right) \right) \left( \delta + \frac{1}{M} \right) \\
& \leq \|F_n\|_3 + \int_{\mathbb{R}^3} dv (F_n(v)) |v|^3 \min (|v|, n) \tag{3.52}
\end{align*}
\]

Taking now \(\delta\) small enough and \(M\) large enough Grönwall’s Lemma implies the estimate (3.39).

In order to conclude the proof for the \(L^1\)-norm notice that it is enough to follow the same procedure as we have just done for the \(L^3\)-norm, since now the third moment is uniformly
bounded. We need again the Povzner inequality (3.41), the well-known estimate \( x^4 + y^4 \leq (x^2 + y^2)^2 \) and Young’s inequality.

Finally we are ready for the proof of the existence of weak solutions of the new kinetic system (2.13).

**Theorem 3.1** Let \((\lambda_0, F_0) \in X_+ \cap \tilde{X}\) with \(F_0 \log(F_0) \in L^1(\mathbb{R}^3)\) and let \(E(\lambda_0, F_0) := E_0 < \infty\). Assume \((\lambda_0, F_0) \in \tilde{X}\) satisfies (3.6). Let \(T > 0\) then there exist a weak solution \((\lambda, F) \in C([0, T], X)\) of the kinetic system (1.24) for the initial value \((\lambda_0, F_0)\).

**Remark** Since \(T\) is arbitrary, this theorem implies the existence of a weak solution \((\lambda, F) \in C([0, \infty), X)\).

**Proof** The proof follow the usual strategy for this kind of kinetic equations. Such proof structure is the one we can see for example in [1, 2]. First of all we start constructing a sequence of functions in \(C^1([0, \infty), X)\) which converge in some sense to a solution of the desired equation.

Let \(n \in \mathbb{N}\). We consider the cut-off equation \(\partial_t \begin{pmatrix} \lambda_n \\ F_n \end{pmatrix} = T_n \begin{pmatrix} \lambda_n \\ F_n \end{pmatrix}\) with initial value \((\lambda_0, F_0)\) as in the assumption. Lemma 3.5 together with the ODE theory implies the existence of unique global solutions \((\lambda_n, F_n) \in C^1([0, \infty), X)\) with values in \(X_+\).

From Lemma 3.4 we know that the sequence \((\lambda_n, F_n)\) is uniformly bounded in \(C([0, T], X)\). Moreover, \((\lambda_n(t))\) is uniformly equicontinuous. Indeed the computation in equation (3.26) implies that \(|T_{1,n}(\lambda_n, F_n)| \leq 2 \int_{\mathbb{R}^3} dv \|\mathbb{K}_{1,n}[F_n, \lambda_n]\| \leq 2C E_0\) for all \(n\). Therefore the definition of weak solution yields the claimed uniformly equicontinuity since \(|\lambda_n(t_1) - \lambda_n(t_2)| \leq 2C E_0|t_1 - t_2|\). Applying Arzela-Ascoli’s Theorem we find a subsequence \(\lambda_{nk}\) and a function \(\lambda \in C([0, T], \mathbb{R})\) with \(\lambda \in [0, 1]\) such that \(\lim_{k \to \infty} \lambda_{nk} \to \lambda\) in \(C([0, T], \|\cdot\|_{sup})\).

There exist also a subsequence \(F_{nk}\) and a function \(F \in C([0, T], L^1_2(\mathbb{R}^3))\), \(F \geq 0\) such that \(F_{nk} \to F\) for all \(t \in [0, T]\) in \(L^1(\mathbb{R}^3)\). By Lemma 3.6 there exists some constant \(C > 0\) such that \(\|F_n\|_{L^1_4} \leq C\) for all \(t \in [0, \infty)\) and for all \(n \geq M\). Also Lemma 3.4 implies the existence of a constant \(C\) independent of \(n\) such that \(\int_{\mathbb{R}^3} dv \ F_n \log(F_n) \leq C\) for all \(n \geq M\) and hence \(F_0 \log(F_0) \in L^1_1(\mathbb{R}^3)\). The uniform boundedness of the \(L^1_4\)-norm implies also, as we remarked on page 18, that \(\mathbb{K}_{1,n}[F_n, \lambda_n] \in L^1_3(\mathbb{R}^3)\) uniformly for all \(n \geq M\). Therefore, the following estimate holds true

\[
\sup_{t \leq T} \|\partial_t F\|_{L^1_2} \leq \sup_{t \leq T} \left( \|\mathbb{K}_{1,n}[F_n, \lambda_n]\| + \|\mathbb{K}_{2,n}[F_n, \lambda_n]\|_{L^1_2} + \|F_n \int_{\mathbb{R}^3} dv \mathbb{K}_{1,n}[F_n, \lambda_n]\|_{L^1_2} \right) \leq C
\]

(3.53)

for some constant \(C > 0\) depending on the uniform bound of the \(L^1_4\)-norm and on the initial mass and energy. Thus, Dunford-Pettis Theorem and the uniformly equicontinuity of \(F_n \in C([0, T], L^1_2(\mathbb{R}^3))\) imply the weak-L^1 compactness of this sequence. Indeed the sequence \(F_n\) is uniformly bounded in \(L^1_2(\mathbb{R}^3)\) by the conservation of initial energy. Moreover, the sequence is tight by the uniformly boundedness of the \(L^1_4\)-norm. Finally, the result of Lemma (3.4) implies the uniform integrability of the sequence. Thus, we conclude the weak \(L^1\)-convergence for a subsequence \(F_{nk}\) (which can be chosen common to the one for \(\lambda_{nk}\)).

Moreover, since the \(L^1_4\)-norm is uniformly bounded, we obtain

\[
\int_{|v| \geq R} dv F_n(v) \left(1 + |v|^l\right) \to 0 \quad \text{for any} \ 0 \leq l \leq 2.
\]
This implies
\[
\int_{\mathbb{R}^3} dv \, F_{nk}(v) \varphi(v) \xrightarrow{k \to \infty} \int_{\mathbb{R}^3} dv \, F(v) \varphi(v) 
\] (3.54)
for all \( \varphi \in L^\infty(\mathbb{R}^3) \). Hence, \( F \) satisfies the conservation of mass and momentum and also \( F \in L^1(\mathbb{R}^3) \). Moreover by the uniform convergence of \( \lambda_{nk} \) we see that \((\lambda, F)\) still satisfies the conservation of energy. Now it only remains to pass to the limit in the integrals. Equation (3.54) implies the weak \( L^1 \)-convergence of
\[
BF_{nk}(v) F_{nk}(w) \rightharpoonup BF(v) F(w) 
\] (3.55)
for a kernel \( B \in \{B_{el}, B_{ne}, B_{we}\} \).

Since \( 0 \leq (B - B_{nk}) (1 + |v|)^2 \leq (B - B_{nk}) (1 + |v|)^2 \chi_{|v-w| > R} \leq (B - B_{nk}) (1 + |v|)^2 \chi_{|v-w| > R} \)
we conclude using again (3.54) that \( \|B - B_{nk}\|_{L^1} \to 0 \) as \( k \to \infty \). This result, together with the uniform convergence of \( \lambda_{nk} \) and the weak \( L^1 \)-convergence of \( F_{nk} \) implies that \((\lambda, F) \in C([0, T], X) \) is the desired weak solution. Moreover, since the \( L^4 \)-norm of the sequence \( F_{nk} \) is uniformly bounded, also the solution \( F \) has this property. Indeed for all \( R > 0 \)
\[
\int_{|v| \leq R} dv \, F(v) (1 + |v|^2)^2 \leq \lim_{k \to \infty} \int_{|v| \leq R} dv \, F_{nk}(v) (1 + |v|^2)^2 \leq \tilde{C}. 
\] (3.56)

\[
\square
\]

### 3.4 Uniqueness of Weak Solutions

Now that we have shown the existence of weak solutions, we can show that they are unique.

**Theorem 3.2** Under the same assumption as in Theorem 3.1 there is at most one weak solution \((\lambda, F)\) for the same initial value problem with \( F \in L^\infty([0, T], L^4(\mathbb{R}^3)) \).

**Proof** Let \( T, (\lambda_0, F_0) \) be as in the assumption. Let us assume that \((\lambda, F)\) and \((\mu, G)\) are two weak solutions for the same initial value \((\lambda_0, F_0)\). Then using the computation in Lemma 3.5 we see
\[
|\lambda(t) - \mu(t)| \leq \left| \int_0^t d\tau \, (T_1 [\lambda, F] - T_1 [\mu, G]) \right|
\leq \max(C_1, C_2) \int_0^t \left( |\lambda - \mu| + \|F - G\|_{L^1} \right).
\]

We claim that there exists a constant \( C_T > 0 \) depending on \( T \) and on \( \sup_{t \leq T} \left( \|F\|_{L^1}, \|G\|_{L^1} \right) \)
such that
\[
\int_{\mathbb{R}^3} dv \,(1 + |v|^2) \text{sign}(F - G) (\mathcal{K}_1 [F, \lambda] + \mathcal{K}_2 [F, \lambda] - \mathcal{K}_1 [G, \lambda] - \mathcal{K}_2 [G, \lambda]) \leq C_T \|F - G\|_{L^1}.
\] (3.57)
Indeed, defining $\phi(v) = \text{sign}(F - G)(v)(1 + |v|^2)$ we can do the following computation, first for the elastic terms and then for the nonelastic one

$$
\int_{\mathbb{R}^3} dv (1 + |v|^2) \text{sign}(F - G) \left( K_{el}^1 [F, \lambda] + K_{el}^2 [F, \lambda] - K_{el}^1 [G, \lambda] - K_{el}^2 [G, \lambda] \right)
$$

$$
\leq 4 \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \int_{S^2} d\omega \ (1 + |v_1|^2)^{1/2} (1 + |v_2|^2)^{1/2} |F(v_1) - G(v_1)| |G(v_2)(1 + |v_2|^2) + 4 \int_{\mathbb{R}^3} dv_1 \int_{\mathbb{R}^3} dv_2 \int_{S^2} d\omega \ (1 + |v_1|^2)^{1/2} (1 + |v_2|^2)^{1/2} |F(v_1)| |F(v_2) - G(v_2)(1 + |v_2|^2)
$$

$$
\leq 8 \sup_{t \leq T} \left( \|F\|_{L^1_t}, \|G\|_{L^1_t} \right) \|F - G\|_{L^1_t},
$$

(3.58)

where we used in addition to the estimate for the elastic kernel (1.12) the relation (3.23) we already saw in Lemma 3.5. We applied also the triangle inequality, the definition of sign function and the relation between the elastic velocities. In a very similar way we can bound the nonelastic terms applying this time the estimate for the nonelastic kernels and the relation between the nonelastic velocities. Using the weak formulation we see again

$$
\int_{\mathbb{R}^3} dv (1 + |v|^2) \text{sign}(F - G) \left( K_{ne}^1 [F, \lambda] + K_{ne}^2 [F, \lambda] - K_{ne}^1 [G, \lambda] - K_{ne}^2 [G, \lambda] \right)
$$

$$
\leq 4C_0 (1 + 2\sqrt{\epsilon_0})(1 + \epsilon_0) \|F - G\|_{L^1_t} \sup_{t \leq T} \left( \|F\|_{L^1_t}, \|G\|_{L^1_t} \right).
$$

We also see that, since $\sup_{t \leq T} \left( \|F\|_{L^1_t}, \|G\|_{L^1_t} \right) < \infty$ in equation (3.26) we can actually find a constant $\tilde{C}_T$ which depends on the initial energy $E(0)$ and on $\sup_{t \leq T} \left( \|F\|_{L^1_t}, \|G\|_{L^1_t} \right)$ such that $IV \leq \tilde{C}_T |\lambda - \mu|.$

This is true because for any collision kernel $B \in \{B_{el}, B_{ne}^{12}, B_{ne}^{34}\}$ we have the estimate $B(v, w) \leq K (1 + |v|^2)^{1/2} (1 + |w|^2)^{1/2}$ so that for the key estimate (3.31) we can use

$$
\int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dw \int_{S^2} d\omega \ B(v, w) G(v) G(w) (1 + |v|^2) \leq \|G\|_{L^1_t} \|G\|_{L^1_t}
$$

$$
\leq E(0) \sup_{t \leq T} \left( \|F\|_{L^1_t}, \|G\|_{L^1_t} \right).
$$

(3.60)

Thus, estimating the third integral term $III$ in equation (3.35) with the help of the above relation (3.57) and using also the estimate for the term $IV$ together with an approximating argument for $\phi$ we see that

$$
\partial_t \int_{\mathbb{R}^3} dv (1 + |v|^2) \text{sign}(F - G)(F - G) \leq \bar{C}_T \left( |\lambda - \mu| + \|F - G\|_{L^1_t} \right),
$$

(3.61)

where $\bar{C}_T$ depends on the initial energy $E(0)$ and on $\sup_{t \leq T} \left( \|F\|_{L^1_t}, \|G\|_{L^1_t} \right).$ These results imply the existence of a constant $C = \max(C_1, C_2, \bar{C}_T)$ such that

$$
\left\| \left( \frac{\lambda}{F} \right) - \left( \frac{\mu}{G} \right) \right\|_X \leq \int_0^t d\tau \ C \left\| \left( \frac{\lambda}{F} \right) - \left( \frac{\mu}{G} \right) \right\|_X.
$$

(3.62)
The integral form of Grönwall’s Lemma implies the claim.

4 The Rigorous Proof of the Derivation of the Kinetic Equation in the Fast Radiation Limit

In Sect. 2 we formally developed the equation (2.13), which describes the behavior of the system immediately after the initial time. In Sect. 3 a well-posedness theory was proven. In this new section we are ready to prove Theorem 1.2. We want to prove that the solutions $(\tilde{x}, \tilde{F})$ of the formal derived system (2.13) are the limit of the solutions of (1.23) when $\varepsilon$ goes to zero. The theorem will be stated and proven in Sect. 4.3.

For the classical Chapman–Enskog expansion of the Boltzmann equation around the equilibrium, i.e. the Maxwellian, a good convergence result was already shown. In this case the Chapman–Enskog expansion leads in the hydrodynamic limit to the Euler equation (or to the Navier-Stokes equation). For example in [6] it is proven that the smooth solution to the $\varepsilon$-Boltzmann equation converges for small times to the Maxwellian constructed with the smooth solutions to the corresponding Euler equation. Similar is also the result in [14], which deals with the more general notion of renormalized solutions.

In our case we are dealing with a slightly different situation. The manifold we considered for the generalized Chapman–Enskog expansion is infinitely dimensional. Also the derived kinetic system (2.13) is different from the Euler Equation. We will proceed therefore in a similar approach as in the literature but with different tools. We will consider an initial data $F_0$ in a small neighborhood of $M^+$ and the solution $F^\varepsilon$ of (1.23) to that initial value. The most important step is that in the neighborhood of the manifold of the steady states $M^+$ we can decompose each vector as the sum of an element in $M^+$ and a remainder. So we will prove that $F^\varepsilon$ converges to $F$, for the latter being given by the solution of the derived system (2.13) with the initial value being the projection of $F_0$ into $M^+$.

4.1 The Decomposition Lemma

We start with the decomposition lemma, to this end we define a new space

$$ \mathcal{Y} = \{ (F, \alpha, \theta, \lambda) \in L^1_2(\mathbb{R}^3) \times L^1_2(\mathbb{R}^3) \times L^1(S^2) \times \mathbb{R} \} . $$

**Lemma 4.1** Let $\tilde{F} \in X$ with $\tilde{F} > 0$. Let $\delta > 0$. Assume there exists some $F_0 \in M^+$ such that $\|F - F_0\|_{\mathcal{X}} < \varepsilon$ for some $\varepsilon > 0$ small enough. Then there exist $F \in \mathcal{M}$ and $W = \left( \begin{array}{c} \alpha \\ -\alpha \\ \theta \\ \lambda \end{array} \right) \in \mathcal{X}$ with $\int_{\mathbb{R}^3} \alpha = \int_{S^2} \theta$ such that $\tilde{F} = F + W$, $\|W\|_{\mathcal{X}} < \delta$ and $\lambda \in (0, 1)$. Moreover, $\tilde{F} \in M^+$.

**Proof** This is an application of the Implicit Function Theorem for Banach Spaces. Let $A = L^1_2(\mathbb{R}^3) \times L^1_2(\mathbb{R}^3) \times L^1(S^2) \times (-\infty, 1) \subset \mathcal{Y}$, which is an open subset of $\mathcal{Y}$. We denote by $H : \mathcal{X} \times A \to \mathcal{Y}$ the function given by

$$ H \left( \begin{array}{c} \tilde{F} \\ F \\ \alpha \\ \theta \\ \lambda \end{array} \right) = \left( \begin{array}{c} \int_{\mathbb{R}^3} F - F - \alpha \\ \int_{\mathbb{R}^3} F^2 - \lambda F + \alpha \\ \int_{S^2} Q - \frac{\lambda}{1 - \lambda} \theta - \theta \\ \int_{\mathbb{R}^3} \alpha - \int_{S^2} \theta \end{array} \right) . \tag{4.1} $$

$H$ is well-defined in $\mathcal{Y}$. Moreover clearly $H(F_0, 0, 0, \lambda_0) = 0$ by the definition of the manifold $M^+$. By a simple computation using the Taylor expansion we calculate for
\[ h = (h_t, h_4, h_5, h_6, h_7) \in \mathcal{X} \times \mathcal{Y} \]

\[
H(\mathcal{F}_0 + \overline{h}, F_0 + h_4, h_5, h_6, \lambda + h_7) - H(\mathcal{F}_0, F_0, 0, 0, \lambda_0)
= \begin{pmatrix}
\overline{h}_1 - h_4 - h_5 \\
\overline{h}_2 - \lambda_0 h_4 + h_5 - F_0 h_7 \\
\overline{h}_3 - \frac{1}{(1 - \lambda_0)^2} \\
\int_{\mathbb{R}^3} h_5 - \int_{\mathbb{S}^2} h_6
\end{pmatrix}
\begin{pmatrix}
0 \\
-h_7 h_4 \\
\mathcal{O}(h_7^2) \\
0
\end{pmatrix}.
\]

(4.2)

This implies that \( H \) is Fréchet differentiable in \((\mathcal{F}_0, F_0, 0, 0, \lambda_0)\). Denoting by \( D_\mathcal{Y} H(\mathcal{F}_0, F_0, 0, 0, \lambda_0) \) the Fréchet derivative of \( H \) when the first entrance is constant \( \mathcal{F}_0 \) we see that it is a bounded linear operator from \( \mathcal{Y} \) to \( \mathcal{Y} \) defined by

\[
D_\mathcal{Y} H(\mathcal{F}_0, F_0, 0, 0, \lambda_0)((f, g, \omega, x)) = \begin{pmatrix}
-f - g \\
-\lambda_0 g + g - F_0 x \\
-\frac{1}{(1 - \lambda_0)^2} x - \omega \\
\int_{\mathbb{R}^3} g - \int_{\mathbb{S}^2} \omega
\end{pmatrix}.
\]

(4.3)

This is also an invertible operator. The injectivity is easy to see: we assume that for some element in \( \mathcal{Y} \) we have \( \|D_\mathcal{Y} H(\mathcal{F}_0, F_0, 0, 0, \lambda_0)((f, g, \omega, x))\|_\mathcal{Y} = 0 \). This implies that each row of the vector defined in (4.3) must be equal 0 in \( \mathcal{Y} \). Therefore we have first \( f = -g \) almost everywhere, which implies also \( f = \frac{F_0}{1 + \lambda_0} x \) almost everywhere. From the relation \( \frac{1}{(1 - \lambda_0)^2} x = -\omega \) for almost every \( n \in \mathbb{S}^2 \) we conclude, since \( x \in \mathbb{R} \), that also \( \omega \) has to be constant almost everywhere.

Hence, putting the new definitions for \( f, g, \omega, x \) makes \( \lambda \) equal to \( \frac{F_0}{1 + \lambda_0} \) almost everywhere. By the assumption on \( F \) we see that \( f = -g \) almost everywhere, and the one for \( \omega \) into \( \int_{\mathbb{R}^3} g - \int_{\mathbb{S}^2} \omega = 0 \) we conclude that \( x = 0 \). This is because by assumption \( \frac{\int_{\mathbb{R}^3} F_0}{1 + \lambda_0} > 0 \) and \( \frac{1}{(1 - \lambda_0)^2} > 0 \) and because of the computation

\[
0 = \int_{\mathbb{R}^3} g - \int_{\mathbb{S}^2} \omega = -\int_{\mathbb{R}^3} \frac{F_0}{1 + \lambda_0} x - \frac{1}{(1 - \lambda_0)^2} x.
\]

(4.4)

Since we have proved that \( x = 0 \), this implies \( \omega = 0 \) almost everywhere and also \( g = F = 0 \) almost everywhere. Thus, the claimed injectivity holds true.

For the surjectivity we have to do a similar computation. Let \((F, \alpha, \theta, \lambda) \in \mathcal{Y} \). We want to find a vector \((f, g, \omega, x) \in \mathcal{Y} \) such that

\[
\begin{pmatrix}
-f - g \\
-\lambda_0 g + g - F_0 x \\
-\frac{1}{(1 - \lambda_0)^2} x - \omega \\
\int_{\mathbb{R}^3} g - \int_{\mathbb{S}^2} \omega
\end{pmatrix} = \begin{pmatrix}
F \\
\alpha \\
\theta \\
\lambda
\end{pmatrix}.
\]

From this equation we have first

\[
-g = F + f \quad \text{a.e}
\]

and therefore also

\[
f = \frac{-\alpha - F_0 x - F}{1 + \lambda_0} \quad \text{a.e.}
\]

(4.6)

From the equation \(-\frac{1}{(1 - \lambda_0)^2} x - \omega = \theta \) for almost every \( n \in \mathbb{S}^2 \) we see that \( \omega + \theta = -\frac{1}{(1 - \lambda_0)^2} x \) almost everywhere, which implies that \( \omega + \theta \) is constant. So we can conclude with the following computation

\[
\lambda = \int_{\mathbb{R}^3} g - \int_{\mathbb{S}^2} \omega = \int_{\mathbb{R}^3} \frac{\alpha - \lambda_0 F}{1 + \lambda_0} - \int_{\mathbb{S}^2} \theta + x \left( \int_{\mathbb{R}^3} \frac{F_0}{1 + \lambda_0} + \frac{1}{(1 - \lambda_0)^2} \right).
\]

(4.7)

By the assumption on \( \mathcal{F}_0 \) the coefficient of \( x \) is positive, which implies the resolvibility of this equation. Having found the value of \( x \) we can find the desired vector \((f, g, \omega, x) \in \mathcal{Y} \).
We have just proved the assumption for the implicit function theorem. Therefore there exist open neighborhoods \( V \subset X \) of \( F_0 \) and \( U \subset A \subset Y \) of \((F_0, 0, \lambda_0)\) and a unique continuous function \( \Phi : V \to U \) such that \( \Phi(F_0) = (F_0, 0, \lambda_0) \) and such that for every \((F, F, \alpha, \theta, \lambda) \in V \times U\) we have \( H(F, F, \alpha, \theta, \lambda) = 0 \) if and only if \( \Phi(F) = (F, \alpha, \theta, \lambda) \). This implies that every \( F \in V \) can be uniquely written as \( F = \overline{F} + W \) for \( \overline{F} \in M \) and \( W = (\alpha, -\alpha, \theta) \in X \) with \( \int_{\mathbb{R}^3} \alpha = \int_{S^2} \theta \). Now let us consider 
\[
\tilde{U} = U \cap \left( L^2_1(\mathbb{R}^3) \times B^L_{3/2}(0) \times B^L_{3/2}(0) \times (0, 1) \right),
\]
which is still an open neighborhood of \((F_0, 0, 0, \lambda_0)\). Here we denote by \( B^L_r(z_0) \) the ball of radius \( r \) and center \( z_0 \in Z \) with respect to the norm of the Banach space \( Z \). Then taking \( \varepsilon > 0 \) small enough such that \( B^L_{\varepsilon}(z_0) \subset \Phi^{-1} \left( \tilde{U} \right) \subset V \) we can conclude the lemma. By the continuity of \( \Phi \) the preimage of an open set is still an open set, in this case also a neighborhood of \( F_0 \).

Now that we have the decomposition, it is not difficult to see that \( \overline{F} > 0 \). The assumption that \( \overline{F} > 0 \) implies that \( \alpha > 0 \) on the set where \( F \leq 0 \). We already have seen that \( \lambda \in (0, 1) \). Hence, on the set where \( F \leq 0 \) we obtain \( \lambda\overline{F} - \alpha < 0 \). But this is possible only on a set of measure zero. Therefore we conclude the claim. \( \square \)

This Lemma can be generalized also for continuous functions with value in the Banach Space \( X \).

**Corollary 4.1** Let \( F \in C ([0, T], X) \) with \( \overline{F} > 0 \). Let \( \delta > 0 \). Assume there exists some \( F_0 \in M \) such that \( \|F - F_0\|_X < \varepsilon \) for all \( t \in [0, T] \) for some \( \varepsilon > 0 \) small enough. Then there exist \( \overline{F} \in C ([0, T], X) \) with \( \overline{F}(t) \in M \) and \( W = \left( \begin{array}{c} \alpha \\ -\alpha \\ \theta \end{array} \right) \in C ([0, T], X) \) with \( \int_{\mathbb{R}^3} \alpha = \int_{S^2} \theta \) such that \( F = \overline{F} + W \), \( \sup_{t \leq T} \|W\|_X < \delta \) and \( \lambda(t) \in (0, 1) \) for all \( t \in [0, T] \). Moreover \( \overline{F} > 0 \).

**Proof** The proof is analogous as the one of Lemma 4.1. Note that here \( F_0 \) is constant in time. In this case we consider the open set \( A = C ([0, T], L^2_1(\mathbb{R}^3) \times L^1_1(\mathbb{R}^3) \times L^1_1(S^2) \times (-\infty, 1)) \subset C ([0, T], Y) \) and the function \( H : C ([0, T], X) \times A \to C ([0, T], Y) \) is defined as before in (4.1). It is well-defined and Fréchet differentiable at \((\overline{F}, F_0, 0, \lambda_0)\) and as in the Lemma 4.1 its derivative is a bounded invertible operator. By the Implicit Function Theoreum we have again the existence of open neighborhoods \( V \subset C ([0, T], X) \) of \( F_0 \) and \( U \subset A \subset C ([0, T], Y) \) of \((F_0, 0, 0, \lambda_0)\) and of a unique continuous function \( \Phi : V \to U \) with the same property as before. Defining now 
\[
\tilde{U} = U \cap \left( C ([0, T], L^2_1(\mathbb{R}^3)) \times B^C_{3/2}(0, T), L^2_1(0) \right),
\]
we conclude as in Lemma 4.1. \( \square \)

Lemma 4.1 and Corollary 4.1 are fundamental for the proof of Theorem 1.2, which is the goal of this section. Before moving to that we need some useful notation and technical results.

### 4.2 Some Technical Results

**Definition 4.1** We denote by \( \kappa_0 = \int_{\mathbb{R}^3} \left( F_0^1 + F_0^2 \right) \) the initial mass of the particles.

**Remark** We recall that both the solutions to the kinetic equation (1.10) as to the derived kinetic equation (2.13) conserve the initial mass. Moreover they conserve also the initial energy \( E_0 \).
We also recall, that solutions to the derived kinetic equation (2.13) for initial values with bounded $L^4$-norm have also uniformly bounded $L^4$-norm. Moreover it can be proved, that strong solutions to the equation (1.10) for initial values bounded in $L^4$ have also the fourth moment bounded, by a constant which only depends on $E_0$ and $\kappa_0$. This can be proven using the ODE’s theory in general Banach spaces. We will not prove the whole well-posedness theory here. However, the most important step for the uniform boundedness of the $L^4$-norm is given by Lemma 4.3, which is proved for the homogeneous Boltzmann Equation in Lemma 6.5 of [4]. Before taking examine that proof carefully, we state an helpful result.

Lemma 4.2 ([4], pp. 17–19)

For each $s > 2$ and any $\lambda \in \left(\frac{4}{s+2}, 1\right)$, one can find a constant $\alpha$ large enough so that the following holds. If $|\xi| \geq \alpha (1 + |\xi_s|)$, then

$$
\frac{2}{\pi |\xi - \xi_s|^2} \int_{S_{\xi,\xi_s}} (1 + |\xi'|^2)^{s/2} d\sigma \leq \lambda \left[ \left(1 + |\xi|^2\right)^{s/2} + \left(1 + |\xi_s|^2\right)^{s/2} \right],
$$

where $d\sigma$ denotes the surface area on $S_{\xi,\xi_s}$, which is the sphere with diameter the segment, joining $\xi$ with $\xi_s$.

Now we prove the desired result for the uniform $L^4$-boundedness of the sequence $F^\varepsilon$ of strong solution to the equation (1.23).

Lemma 4.3 Let $F$ be a solution to the kinetic system (1.10). Assume $\|F\|_{L^4_1} \leq \kappa_2$, $\|F\|_{L^1_1} = \kappa_0$ and that $\|F\|_{L^4_1}$ is bounded. Then there exists a constant $\overline{\kappa}$ depending on $\kappa_2$ and on $\kappa_0$ such that

$$
\frac{d}{dt} \|F\|_{L^4_1} \leq 0 \quad \text{whenever} \quad \|F\|_{L^4_1} \geq \overline{\kappa}.
$$

Proof We follow almost one-to-one the proof of Lemma 6.5 of [4]. The only difference is that we are working with a system and also with some nonelastic interactions. Using the inequality $x^4 + y^4 \leq \left(x^2 + y^2\right)^2$ together with the relation of the nonelastic velocities as in (1.7) and estimating the kernel $B_{ne}^{ij}(\overline{v}, \overline{w}) \leq C_{\varepsilon_0} (|\overline{v}| + |\overline{w}|)$ we see that there exists some constant $\tilde{C}$ such that defining $\varphi'(v) = (1 + |v|^2)^2$ the following is true that for every such $F$

$$
\int_{\mathbb{R}^2} d\overline{v}_1 d\overline{v}_2 d\overline{v}_3 d\overline{v}_4 \delta(\overline{v}_1 + \overline{v}_2 - \overline{v}_3 - \overline{v}_4) \delta(|\overline{v}_1|^2 + |\overline{v}_2|^2 - |\overline{v}_3|^2 - |\overline{v}_4|^2 - 2\varepsilon_0)
\leq \tilde{C} \|F\|_{L^1_2} \|F\|_{L^1_1}.
$$

Let us take $\lambda$ and $\alpha$ according to Lemma 6.4 of [4] as written in Lemma 4.2. Without loss of generality we assume $\alpha > 1$. Let $\rho = \sqrt{\frac{2\kappa_2}{\kappa_0}}$. By the conservation of mass we obtain that $\int_{|v| \leq \rho} dv \left[ F_1(v) + F_2(v) \right] > \frac{\kappa_0}{2}$. Moreover let us define $\Gamma'$ by $\Gamma := \{(v_1, v_2) : |v_1| \leq \rho, \ |v_2| \geq \alpha(1 + \rho)\} \subset \mathbb{R}^3 \times \mathbb{R}^3$. We denote also $\Gamma := \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Gamma'$. Following the same computation as in [4], estimating the nonelastic term by (4.10) and applying Lemma 4.2 we obtain

$$
\frac{d}{dt} \left( \frac{F}{\kappa_0} \right)_{L^4_1} \leq \frac{C_1}{\kappa_0} \|F\|_{L^2_1} \|F\|_{L^1_1} - \frac{(1 - \lambda)}{4} \int_{|v| \geq \alpha(1 + \rho)} dv_1 (1 + |v_1|^2)^{5/2} \left( F_1^1 + F_1^2 \right).
$$

(4.11)
To conclude the Lemma we apply Jensen’s inequality for the convex function $h(x) = x^{5/4}$ and the probability measure $\frac{(F_1(v) + F_2(v))}{\kappa_0} dv$. Hence we see

$$\kappa_0^{-1/4} \left( \int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right) \left( F_1(v) + F_2(v) \right) \right)^{5/4} \leq \int_{\mathbb{R}^3} dv \left( 1 + |v|^2 \right)^{5/2} \left( F_1(v) + F_2(v) \right) \leq \int_{|v|\geq\alpha(1+\rho)} dv \left( 1 + |v|^2 \right)^{5/2} \left( F_1(v) + F_2(v) \right) + \kappa_0 \left( 1 + \alpha^2 (1 + \rho)^2 \right)^{5/2}.$$  \hfill (4.12)

Therefore putting all estimates together we obtain

$$\frac{d}{dt} \|F\|_{L^4_1} \leq C_1 \|F\|_{L^4_1} \|F\|_{L^4_1} - \frac{(1 - \lambda) \kappa_0^{3/4}}{4} \|F\|_{L^4_1}^{5/4} + \frac{(1 - \lambda) \kappa_0^2}{4} (1 + \alpha^2 (1 + \rho)^2)^{5/2}. \hfill (4.13)$$

The right hand side of this estimate is non-positive provided $\|F\|_{L^4_1}$ is large enough. This conclude the Lemma. \hfill \Box

With Lemma 4.3 we can conclude that for a constant $\kappa_4 > \kappa$ we have that for any $t \geq 0$ the $L^1_4$-norm of the solution is uniformly bounded $\|F\|_{L^4_1} \leq \kappa_4$. Note that this computation never used the radiative term of the equation (1.18). This means that for an initial value with fourth moment bounded by $\kappa_4$ for every $\varepsilon > 0$ any strong solution $F^\varepsilon$ has the same bound $\|F^\varepsilon\|_{L^4_1} \leq \kappa_4$. It is very important to notice that this constant does not depend on $\varepsilon$.

### 4.3 The Convergence Theorem

We are ready now for the rigorous prove of the derivation of the kinetic equation (2.13).

**Theorem 4.1** Let $\varepsilon > 0$. Let $F_0 \in \mathcal{M}_+$, $F_0 > 0$ satisfying assumption (3.6) with mass $\kappa_0 > 0$, energy $E_0 := E(F_0^\varepsilon) \leq \frac{1}{4}$ and $F_0 \in L^1_4$.

Let $\delta_0 := \frac{2\varepsilon \kappa_0}{64(2\varepsilon + E_0)} > 0$ and let $\delta < \delta_0$ small enough. We define $\bar{F}_0 = F_0 + W_0 \in X$ such that $F_0^\varepsilon > 0$, $W_0 = \begin{pmatrix} \alpha_0 \\ -\alpha_0/\theta_0 \end{pmatrix} \in \mathcal{X}$ with $\int_{\mathbb{R}^3} \alpha_0 = \int_{\mathbb{R}^3} \theta_0$ and $\|W_0\|_{\mathcal{X}} < \delta$.

Let $\bar{F}^\varepsilon \in C^1 ((0, \infty), \mathcal{X})$ be the solution to the equation (1.23) for the initial value $\bar{F}_0$.

Let $\bar{F} = \begin{pmatrix} \bar{\chi} \bar{F} \\ \bar{\chi} \bar{F} \end{pmatrix} \in \mathcal{M}_+$ with $(\bar{\chi}, \bar{F}) \in C ([0, \infty), [0, 1) \times L^1_2 (\mathbb{R}^3))$ being the solution of the kinetic equation (2.13) for the initial value $\bar{F}_0$.

Then there exists some $t_0 > 0$ such that for all $s > 0$ we have

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq t_0} \|F^\varepsilon - \bar{F}\|_{\mathcal{X}} = 0. \hfill (4.14)$$

**Proof** We proceed in several steps. We first decompose $F^\varepsilon$ according to Lemma 4.1. Then using the equations for the functions in the manifold and for the rest term we estimate the norm of $F^\varepsilon - \bar{F}$.

**STEP 1: Decomposition**

By the continuity of $\|F^\varepsilon - F_0\|_{\mathcal{X}}$ and by the assumption there exists some $t_1 > 0$ such that

$$\|F^\varepsilon - F_0\|_{\mathcal{X}} < \delta_1 < \delta_0 \quad \text{for all } t \leq t_1 \hfill (4.15)$$
for some $\delta_1 > \delta$ small enough. $\delta_1$ is chosen so that by Lemma 4.1 and its Corollary 4.1 there exist unique vectors $\vec{F}^e, \bar{W}_\varepsilon \in C([0, t_1], \mathcal{X})$ such that $\vec{F}^e = \bar{F}^e + W_\varepsilon$ and $\bar{F}^e \in \mathcal{M}$ and $W_\varepsilon = (\alpha_\varepsilon, -\alpha_\varepsilon, \theta_\varepsilon)$ with $\int_{\mathbb{R}^3} \alpha_\varepsilon = \int_{\mathbb{S}^2} \theta_\varepsilon$. Moreover they satisfy

$$\|W_\varepsilon\|_\mathcal{X} \leq \delta_0 \quad \text{and} \quad \lambda_\varepsilon \in (0, 1).$$  \hfill (4.16)

We can also take $\delta_1$ small enough so that by the uniqueness of the decomposition we have that $\bar{F}^e(0) = \bar{F}(0) = F_0$. Therefore there exists by continuity also some $0 < t_0 \leq t_1$ such that

$$\|\vec{F}^e - \bar{F}\|_\mathcal{X} < \frac{1}{16} \frac{2\varepsilon_0}{2\varepsilon_0 + E_0}.$$  \hfill (4.17)

**Some Important Remarks**

We recall that since both initial data $\bar{F}_0$ and $\bar{F}_0$ are strictly positive, then also $\vec{F}^e$ and $\bar{F}$ are strictly positive for every $t \geq 0$ and every $\varepsilon$. This implies that the mass of $\vec{F}^e$ is actually the mass of $\bar{F}^e$

$$\kappa_0 = \int_{\mathbb{R}^3} F^1_\varepsilon(v) + F^2_\varepsilon(v) \, dv$$

$$= \int_{\mathbb{R}^3} F^e_\varepsilon(v) + \alpha_\varepsilon(v) + \bar{\alpha_\varepsilon} F^e_\varepsilon(v) - \alpha_\varepsilon(v) \, dv = (1 + \bar{\alpha_\varepsilon}) \int_{\mathbb{R}^3} F^e_\varepsilon(v) \, dv,$$  \hfill (4.18)

which by the positivity of $\bar{\alpha_\varepsilon}$ implies $\int_{\mathbb{R}^3} F^e_\varepsilon(v) \, dv \in \left( \frac{\kappa_0}{2}, \kappa_0 \right)$. In the same way we can see that the energy of $\vec{F}^e$ is the same as the energy of $\bar{F}^e$ which is constant $E_0$. Moreover the $L^1_\mathcal{X}$-norm of both $\bar{F}^e$ and $W_\varepsilon$ is bounded

$$\int_{\mathbb{R}^3} d\varepsilon |\bar{F}^e| (1 + |v|^2) \leq (1 + \bar{\alpha_\varepsilon}) \int_{\mathbb{R}^3} d\varepsilon |\vec{F}^e| (1 + |v|^2)$$

$$= \int_{\mathbb{R}^3} d\varepsilon |\bar{F}^e + \alpha_\varepsilon + \bar{\alpha_\varepsilon} F^e - \alpha_\varepsilon| (1 + |v|^2)$$

$$\leq \int_{\mathbb{R}^3} d\varepsilon |\bar{F}^e + \alpha_\varepsilon| + |\bar{\alpha_\varepsilon} F^e - \alpha_\varepsilon| (1 + |v|^2) = \|F^e\|_{L^1_\mathcal{X}} \leq E_0.$$  \hfill (4.19)

Similarly we compute $\int_{\mathbb{R}^3} d\varepsilon |\alpha_\varepsilon| (1 + |v|^2) \leq \int_{\mathbb{R}^3} d\varepsilon |\bar{F}^e + \alpha_\varepsilon| + |\bar{F}^e| (1 + |v|^2) \leq 2E_0$. For simplicity we will call $\kappa_4$ the maximum between the uniform bound of the $L^1_\mathcal{X}$-norm of $\bar{F}^e$ and $\bar{F}$. We know already that this is a constant independent of $\varepsilon$. With the very same calculation as in the derivation of equation (4.19) we see that $\int_{\mathbb{R}^3} d\varepsilon |\bar{F}^e| (1 + |v|^2)^2 \leq \kappa_4$ and therefore also $\int_{\mathbb{R}^3} d\varepsilon |\alpha_\varepsilon| (1 + |v|^2)^2 \leq 2\kappa_4$.

**STEP 2: The equation for the decomposition**

In order to estimate the norm of $\vec{F}^e - \bar{F}$ we start looking for the equation that $\vec{F}^e$ and $W_\varepsilon$ satisfy. We define two operators. Let $\bar{F} \in \mathcal{M}$, then we define for $W \in \mathcal{X}$

$$L_\bar{F}(W) = \begin{pmatrix} \int_{\mathbb{S}^2} dn \left[ W^2(v) + \frac{\lambda}{1-\lambda} \left( W^2(v) - W^1(v) \right) + W^3(n)(\lambda - 1)F(v) \right] \\ \int_{\mathbb{R}^3} d\varepsilon \left[ W^2(v) + \frac{\lambda}{1-\lambda} \left( W^2(v) - W^1(v) \right) + W^3(n)(\lambda - 1)F(v) \right] \end{pmatrix}$$

$$\int_{\mathbb{R}^3} d\varepsilon \left[ W^2(v) + \frac{\lambda}{1-\lambda} \left( W^2(v) - W^1(v) \right) + W^3(n)(\lambda - 1)F(v) \right]$$  \hfill (4.20)

and

$$\mathcal{L}(W) = \begin{pmatrix} \int_{\mathbb{S}^2} dn \left[ W^3(n) (W^2(v) - W^1(v)) \right] \\ -\int_{\mathbb{S}^2} dn \left[ W^3(n) (W^2(v) - W^1(v)) \right] \\ \int_{\mathbb{R}^3} d\varepsilon \left[ W^3(n) (W^2(v) - W^1(v)) \right] \end{pmatrix}. \quad (4.21)$$
Now, since \( \mathcal{R}[\overline{W}^\epsilon] = 0 \) using the definition of the decomposition and the just new defined operators we see that for the radiation part we have \( \mathcal{R}[\overline{W}^\epsilon] = L_{\overline{W}^\epsilon}(W_\epsilon) + \mathcal{L}(W_\epsilon) \) for all \( t \in [0, t_0] \).

We therefore work with the following equation

\[
\partial_t \overline{W}^\epsilon + \partial_x W_\epsilon = \frac{1}{\epsilon} (L_{\overline{W}^\epsilon}(W_\epsilon) + \mathcal{L}(W_\epsilon)) + \mathcal{K} \left[ \overline{W}^\epsilon + W_\epsilon, \overline{W}^\epsilon + W_\epsilon \right].
\]

(4.22)

Our aim is now to find the equation which \( \overline{W}^\epsilon \) satisfies, and then subtracting it from equation (4.22) we obtain the equation satisfied by \( W_\epsilon \). Let therefore \( \varphi \in C^1([0, t_0], C^\infty(\mathbb{R}^3)) \) and let \( \eta \in C ([0, t_0], \mathbb{R}) \) be both arbitrary. We can define a linear bounded operator on \( \mathcal{X} \) by

\[
A_{\varphi, \eta}(f, g, h) = \int_{\mathbb{R}^3} \varphi f \, dv + \int_{\mathbb{R}^3} \varphi g + (1 - \overline{\alpha}_x)^2 \eta g \, dv + \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \eta h \, dn.
\]

(4.23)

For every arbitrary pair \( \varphi, \eta \) we have \( A_{\varphi, \eta} \left[ L_{\overline{W}^\epsilon}(W) \right] = 0 \) and \( A_{\varphi, \eta} \left[ \mathcal{L}(W) \right] = 0 \). Indeed we can compute

\[
A_{\varphi, \eta} \left[ L_{\overline{W}^\epsilon}(W) \right] = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \varphi \left[ W^2 + \frac{\overline{\alpha}_x}{1 - \overline{\alpha}_x} (W^2 - W^1) + W^3(\overline{\alpha}_x - 1) \overline{W}^\epsilon \right] \, dn \, dv
\]

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \varphi \left[ W^2 + \frac{\overline{\alpha}_x}{1 - \overline{\alpha}_x} (W^2 - W^1) + W^3(\overline{\alpha}_x - 1) \overline{W}^\epsilon \right] \, dn \, dv
\]

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \eta \left[ W^2 + \frac{\overline{\alpha}_x}{1 - \overline{\alpha}_x} (W^2 - W^1) + W^3(\overline{\alpha}_x - 1) \overline{W}^\epsilon \right] \, dn \, dv
\]

\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \eta \left[ W^2 + \frac{\overline{\alpha}_x}{1 - \overline{\alpha}_x} (W^2 - W^1) + W^3(\overline{\alpha}_x - 1) \overline{W}^\epsilon \right] \, dn \, dv.
\]

(4.24)

and similarly also

\[
A_{\varphi, \eta} \left[ \mathcal{L}(W) \right] = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \varphi \left[ W^3(\alpha) (W^2(v - W^1(v))) \right] \, dn \, dv
\]

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \varphi \left[ W^3(\alpha) (W^2(v - W^1(v))) \right] \, dn \, dv
\]

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \eta \left[ W^3(\alpha) (W^2(v - W^1(v))) \right] \, dn \, dv
\]

\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \eta \left[ W^3(\alpha) (W^2(v - W^1(v))) \right] \, dn \, dv.
\]

(4.25)

Now, using the definition of \( W_\epsilon \) we also see easily that

\[
A_{\varphi, \eta} \left[ W_\epsilon \right] = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \varphi \alpha_x \, dn \, dv - \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \varphi \alpha_x \, dn \, dv
\]

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \alpha_x \, dn \, dv + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (1 - \overline{\alpha}_x)^2 \eta \theta_x \, dn \, dv
\]

\[
= (1 - \overline{\alpha}_x)^2 \eta \left( - \int_{\mathbb{R}^3} \alpha_x \, dv + \int_{\mathbb{S}^2} \theta_x \, dn \right) = 0.
\]

(4.26)

Applying this operator \( A_{\varphi, \eta} \) on both sides of equation (4.22) and using partial integration we see that we can simplify that relation as

\[
A_{\varphi, \eta} \left[ \partial_t \overline{W}^\epsilon \right] + A_{\varphi, \eta} \left[ \partial_x W_\epsilon \right] = A_{\varphi, \eta} \left[ \mathcal{K} \left[ \overline{W}^\epsilon + W_\epsilon, \overline{W}^\epsilon + W_\epsilon \right] \right].
\]

(4.27)
Since this result holds for every arbitrary \( \varphi, \eta \) we can conclude choosing first \( \eta = 0 \) and \( \varphi \) arbitrary and then taking \( \varphi = 0 \) and \( \eta = \frac{1}{(1-\lambda)^2} \) that the following must be satisfied

\[
\partial_t (1 + \lambda_\varepsilon) \bar{F}^\varepsilon = K_1 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] + K_2 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] \quad (4.28)
\]

and

\[
\partial_t \left( \int \limits_{\mathbb{R}^3} dv \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} \bar{F}^\varepsilon \right) + 4\pi \partial_t \left( \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} \right) = \int \limits_{\mathbb{R}^3} dv \ K_2 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right]. \quad (4.29)
\]

We can see that this is true by the following computations

\[
A_{\varphi,0} [\partial_t W_\varepsilon] = \int \limits_{\mathbb{R}^3} dv \ \varphi \left( \partial_t \alpha_\varepsilon - \partial_t \alpha_\varepsilon \right) = 0 \quad (4.30)
\]

and using the structure of \( W_\varepsilon \)

\[
A_{0,(1-\lambda)^{-2}} [\partial_t W_\varepsilon] = - \int \limits_{\mathbb{R}^3} dv \ \partial_t \alpha_\varepsilon + \int \limits_{\mathbb{R}^3} \partial_t dn \theta_\varepsilon = \partial_t \left( \int \limits_{\mathbb{R}^3} \partial_t \theta_\varepsilon - \int \limits_{\mathbb{R}^3} dv \ \alpha_\varepsilon \right) = 0. \quad (4.31)
\]

Equations (4.28) and (4.29) are very similar to the one we had in Sect. 2.2 in equations (2.9) and (2.10). Therefore in the same way we conclude that \( \bar{F}^\varepsilon \) must satisfy the following equation

\[
\partial_t \bar{F}^\varepsilon = \frac{1}{1 + \lambda_\varepsilon} \left( K_1 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] \right) + K_2 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] \]

\[
+ \frac{1}{4\pi} \left( 1 - \lambda_\varepsilon \right)^2 \bar{F}^\varepsilon \int \limits_{\mathbb{R}^3} dv \ K_1 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] d v
\]

\[
\partial_t \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} \bar{F}^\varepsilon = \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} \left( K_1 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] \right) + K_2 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] \quad (4.32)
\]

\[
- \frac{1}{4\pi} \left( 1 + \lambda_\varepsilon \right) + \left( 1 - \lambda_\varepsilon \right)^2 \int \limits_{\mathbb{R}^3} dv \ K_1 \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] d v
\]

In order to simplify the notation we will say \( \partial_t \bar{F}^\varepsilon = \mathbb{P}_{\mathcal{T}_{\mathcal{M}}} \left( \mathbb{K} \left[ \bar{F}^\varepsilon + W_\varepsilon, \bar{F}^\varepsilon + W_\varepsilon \right] \right) \). With a similar notation, since we already know that \( \bar{F} \) satisfies

\[
\partial_t \bar{F} = \frac{1}{1 + \lambda_\varepsilon} \left( K_1 \left[ \bar{F}, \bar{F} \right] \right) + K_2 \left[ \bar{F}, \bar{F} \right] \]

\[
+ \frac{1}{4\pi} \left( 1 + \lambda_\varepsilon \right) + \left( 1 - \lambda_\varepsilon \right)^2 \int \limits_{\mathbb{R}^3} dv \ K_1 \left[ \bar{F}, \bar{F} \right] d v
\]

\[
\partial_t \frac{\lambda_\varepsilon}{1 + \lambda_\varepsilon} \bar{F} = \frac{\lambda_\varepsilon}{1 + \lambda_\varepsilon} \left( K_1 \left[ \bar{F}, \bar{F} \right] \right) + K_2 \left[ \bar{F}, \bar{F} \right] \quad (4.33)
\]

\[
- \frac{1}{4\pi} \left( 1 + \lambda_\varepsilon \right) + \left( 1 - \lambda_\varepsilon \right)^2 \int \limits_{\mathbb{R}^3} dv \ K_1 \left[ \bar{F}, \bar{F} \right] d v
\]

\[
\partial_t \left( \frac{\lambda_\varepsilon}{1 - \lambda_\varepsilon} \right) = - \frac{1}{4\pi} \left( 1 + \lambda_\varepsilon \right) + \left( 1 - \lambda_\varepsilon \right)^2 \int \limits_{\mathbb{R}^3} dv \ K_1 \left[ \bar{F}, \bar{F} \right] d v
\]
We will say \( \partial_t \mathcal{F} = \mathbb{P}_{\mathcal{T} \mathcal{M}} \left( \mathbb{K} \left[ \mathcal{F}, \mathcal{F} \right] \right) \). We can now conclude this STEP 2 deriving the following equation for the remainder \( W_\varepsilon \)

\[
\partial_t W_\varepsilon = \frac{1}{\varepsilon} \left( L_{\mathcal{F}^\varepsilon} (W_\varepsilon) + \mathcal{L}(W_\varepsilon) \right) + \mathbb{K} \left[ \mathcal{F}^\varepsilon + W_\varepsilon, \mathcal{F}^\varepsilon + W_\varepsilon \right] - \mathbb{P}_{\mathcal{T} \mathcal{M}} \left( \mathbb{K} \left[ \mathcal{F}^\varepsilon + W_\varepsilon, \mathcal{F}^\varepsilon + W_\varepsilon \right] \right).
\]

(4.34)

STEP 3: The estimate for \( W_\varepsilon \)

In this step we want to estimate the norm of the remainder \( W_\varepsilon \) and see that it will converge to zero for every positive time \( t \in (0, t_0) \). We aim to use Grönwall’s Lemma, therefore we proceed testing the right hand side of equation (4.34) with the vector \((\phi(v), -\phi(v), \varphi(n))\), where \( \phi(v) = \text{sign}(\alpha_\varepsilon) (1 + |v|^2) \) and \( \varphi(n) = \text{sign}(\theta_\varepsilon) \). Using the estimates for the elastic and nonelastic kernels (1.12) and (1.14) and the assumption on \( \lambda_\varepsilon \) it is not difficult to see that

\[
\int_{\mathbb{R}^3} d v \left| K_1[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] + K_2[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] \right| \left( 1 + |v|^2 \right) \leq C(C_0, \varepsilon_0) E_0 \kappa_4
\]

and similarly

\[
\int_{\mathbb{R}^3} d v \left| K_1[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] + K_2[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] \right| \leq C(C_0, \varepsilon_0) E_0^2.
\]

(4.35)

(4.36)

For this reason there exists a constant \( C_1 > 0 \) which depends on the elastic and nonelastic kernels, on the initial energy \( E_0 \) and on \( \kappa_4 \) but independent of \( \varepsilon \) such that for every \( t \in [0, t_0] \) the following holds true

\[
\int_{\mathbb{R}^3} d v \phi(v) \left( K_1[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] + K_2[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] - \mathbb{P}_{\mathcal{T} \mathcal{M}} \left( \mathbb{K}[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] \right) \right) - \int_{\mathbb{S}^2} d n \varphi(n) \mathbb{P}_{\mathcal{T} \mathcal{M}} \left( \mathbb{K}[\mathcal{F}^\varepsilon, \mathcal{F}^\varepsilon] \right)
\]

\[
\leq C_1.
\]

(4.37)

It is also easy to compute the following estimate for \( \mathcal{L} \)

\[
\int_{\mathbb{R}^3} d v \phi(v) \mathcal{L}^1(W_\varepsilon) - \int_{\mathbb{R}^3} d v \phi(v) \mathcal{L}^2(W_\varepsilon) + \int_{\mathbb{S}^2} d n \varphi(n) \mathcal{L}^3(W_\varepsilon)
\]

\[
= 2 \int_{\mathbb{R}^3} d v \phi(v) \mathcal{L}^1(W_\varepsilon) + \int_{\mathbb{S}^2} d n \varphi(n) \mathcal{L}^3(W_\varepsilon)
\]

\[
= -4 \int_{\mathbb{R}^3} d v |\alpha_\varepsilon| (1 + |v|^2) \left( \int_{\mathbb{S}^2} d n \theta_\varepsilon \right) - 2 \int_{\mathbb{R}^3} d v \alpha_\varepsilon \left( \int_{\mathbb{S}^2} d n |\theta_\varepsilon| \right)
\]

\[
\leq 2 \|W_\varepsilon\|_{\mathcal{X}}.
\]

(4.38)

We can come to the conclusion for the estimate of the remainder looking at the behavior of the operator \( L_{\mathcal{F}^\varepsilon}(W_\varepsilon) \). Here we really want to bound that expression by a negative constant multiplied to the norm of \( W_\varepsilon \). This would guarantee us the good decay for the remainder. Again we shall
use also the properties of $W_\varepsilon$. We conclude

$$
\int_{\mathbb{R}^3} dv \phi(v)L_{\varepsilon}^1(W_\varepsilon) - \int_{\mathbb{R}^3} dv \phi(v)L_{\varepsilon}^2(W_\varepsilon) + \int_{\mathbb{S}^2} dn \varphi(n)L_{\varepsilon}^3(W_\varepsilon)
= 2 \int_{\mathbb{R}^3} dv \phi(v)L_{\varepsilon}^1(W_\varepsilon) + \int_{\mathbb{S}^2} dn \varphi(n)L_{\varepsilon}^3(W_\varepsilon)
= -\frac{1 + \lambda_\varepsilon}{1 - \lambda_\varepsilon} \int_{\mathbb{R}^3} dv |\alpha_\varepsilon| (1 + |v|^2) - \left( \int_{\mathbb{S}^2} dn \text{sign}(\theta_\varepsilon) \right) \frac{1 + \lambda_\varepsilon}{1 - \lambda_\varepsilon} \int_{\mathbb{R}^3} dv \alpha_\varepsilon
- 2(1 - \lambda_\varepsilon) \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \text{sign}(\alpha_\varepsilon) (1 + |v|^2) \right) \left( \int_{\mathbb{S}^2} dn \theta_\varepsilon \right)
- (1 - \lambda_\varepsilon) \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \right) \left( \int_{\mathbb{S}^2} dn |\theta_\varepsilon| \right)
= I + II + IIII.
$$

(4.39)

Now we should estimate these terms. The first one is a consequence of taking the absolute value

$$
I \leq -\frac{1 + \lambda_\varepsilon}{1 - \lambda_\varepsilon} \int_{\mathbb{R}^3} dv |\alpha_\varepsilon| (1 + |v|^2) \leq - \int_{\mathbb{R}^3} dv |\alpha_\varepsilon| (1 + |v|^2).
$$

(4.40)

For the second and third terms we use that the initial energy $E_0$ is bounded by $\frac{1}{4}$ and that as we saw after the first step $\int_{\mathbb{R}^3} dv |\bar{F}_\varepsilon|(1 + |v|^2) \leq E_0$. Since also $\lambda_\varepsilon \in [0, \frac{2E_0}{2\varepsilon_0 + E_0}]$ and $\int_{\mathbb{R}^3} \bar{F}_\varepsilon(v)dv \in (\frac{\varepsilon_0}{2}, \kappa_0)$ we compute

$$
II + IIII \leq -2(1 - \lambda_\varepsilon) \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \right) \left(1 - \frac{\kappa_0}{4}\right) \int_{\mathbb{R}^3} dv \alpha_\varepsilon + \frac{\kappa_0}{4} \int_{\mathbb{S}^2} dn \theta_\varepsilon
- (1 - \lambda_\varepsilon) \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \right) \left( \int_{\mathbb{S}^2} dn |\theta_\varepsilon| \right)
\leq \left(1 - \frac{\kappa_0}{4}\right) \int_{\mathbb{R}^3} dv |\alpha_\varepsilon| - 2(1 - \lambda_\varepsilon) \frac{\kappa_0}{4} \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \right) \left(1 + |v|^2\right) \int_{\mathbb{S}^2} dn \theta_\varepsilon
- (1 - \lambda_\varepsilon) \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \right) \left( \int_{\mathbb{S}^2} dn |\theta_\varepsilon| \right)
- (\lambda_\varepsilon - \lambda_\varepsilon) \frac{\kappa_0}{4} \left( \int_{\mathbb{R}^3} dv \bar{F}_\varepsilon \right) \left( \int_{\mathbb{S}^2} dn |\theta_\varepsilon| \right)
$$

(4.41)

Combining these estimates and equation (4.17) in STEP 1 we see that for every $t \in [0, t_0]$ we have

$$
I + II + IIII \leq -\frac{\kappa_0}{8} \frac{2\varepsilon_0}{2\varepsilon_0 + E_0} \|W_\varepsilon\|X + \kappa_0 \|\bar{F}_\varepsilon\|X - \|\bar{F}_\varepsilon\|X \|W_\varepsilon\|X \leq -\frac{\kappa_0}{16} \frac{2\varepsilon_0}{2\varepsilon_0 + E_0} \|W_\varepsilon\|X.
$$

(4.42)
Putting together equations (4.37), (4.38) and (4.42) and using the result of STEP 1 that
\[ \| W_\varepsilon \|_X \leq \delta_0 := \frac{2 \varepsilon_0 \kappa_0}{6(2 \varepsilon_0 + E_0)} \] , we conclude
\[ \frac{\partial}{\partial t} \| W_\varepsilon \|_X \leq - \frac{C_2}{\varepsilon} \| W_\varepsilon \|_X + C_1 \quad \text{for all } t \in [0, t_0], \] (4.43)
where \( C_1 = \frac{\kappa_0}{32} \frac{2 \varepsilon_0}{2 \varepsilon_0 + E_0} \) is independent of \( \varepsilon \) and \( t \).

Using that \( \| W_0 \|_X < \delta < \delta_0 \) Gronwall’s Lemma implies the good decay of the remainder as \( \varepsilon \to 0 \)
\[ \| W_\varepsilon \| \leq \| W_0 \|_X e^{\frac{C_2}{\varepsilon} t} + \frac{C_1}{C_2} \leq \delta e^{\frac{C_2}{\varepsilon} t} + \frac{C_1}{C_2}. \] (4.44)
Therefore we conclude also for all \( s > 0 \)
\[ \lim_{\varepsilon \to 0} \sup_{s \leq t \leq t_0} \| W_\varepsilon \| = \lim_{\varepsilon \to 0} \delta e^{\frac{C_2}{\varepsilon} s} + \frac{C_1}{C_2} = 0. \] (4.45)

**STEP 4: The estimate for** \( \overline{F}_\varepsilon - \overline{F} \)
We proceed estimating the norm of \( \overline{F}_\varepsilon - \overline{F} \) using also in this case the Gronwall’s inequality.
This step will then imply the theorem, as we will see in STEP 5.

We start recalling that as we have seen in STEP 1, namely \( \overline{\lambda}_\varepsilon \in (0, 1) \). Now, recalling that \( \overline{F} > 0 \) and \( \overline{F}_\varepsilon > 0 \) almost everywhere, since the initial data is positive. It is important to notice, that by Lemma 4.1 also \( \overline{F}_\varepsilon > 0 \). We want first to estimate the time derivative of \( \| \overline{F}_\varepsilon - \overline{F} \|_X \). We start with some preliminary estimates. We define two functions \( \phi(v) = \text{sign}(\overline{F}_\varepsilon - \overline{F}) |v|^2 \) and \( \psi(v) = \text{sign}(\overline{\lambda}_\varepsilon \overline{F}_\varepsilon - \overline{\lambda} \overline{F}) |v|^2 \). Then since both functions \( \overline{F}_\varepsilon \) and \( \overline{F} \) are non-negative and their \( L^1 \)-norm is bounded uniformly by \( \kappa_4 \), estimates (3.58) and (3.59) imply
\[ \int_{\mathbb{R}^3} dv \phi(v) \left[ K_1[\overline{F}_\varepsilon, \overline{\lambda}_\varepsilon] + K_2[\overline{F}_\varepsilon, \overline{\lambda}_\varepsilon] - K_1[\overline{F}, \overline{\lambda}_\varepsilon] - K_2[\overline{F}, \overline{\lambda}_\varepsilon] \right] \leq C(\varepsilon_0, C_0) \kappa_4 \| \overline{F}_\varepsilon - \overline{F} \|_{L^1_2}. \] (4.46)

where the constant is independent of the time and of \( \varepsilon \) but depends only on the kernels. Using now that
\[ \overline{\lambda}_\varepsilon \overline{F}_\varepsilon (v) \overline{F}_\varepsilon (w) - \overline{\lambda} \overline{F}(v) \overline{F}(w) \]
\[ = (\overline{\lambda}_\varepsilon \overline{F}_\varepsilon (v) - \overline{\lambda} \overline{F}(v)) \overline{F}_\varepsilon (w) + \overline{F}(v) (\overline{\lambda}_\varepsilon \overline{F}_\varepsilon (w) - \overline{\lambda} \overline{F}(w)) + (\overline{\lambda}_\varepsilon - \overline{\lambda}) \overline{F}(v) \overline{F}_\varepsilon (w), \] (4.47)

we can compute completely analogously as in (3.58) and (3.59) that there exists some constant \( C = C(\varepsilon_0, C_0, \kappa_4) \) independent of time and \( \varepsilon \) such that
\[ \int_{\mathbb{R}^3} dv \psi(v) \left[ \overline{\lambda}_\varepsilon K_1[\overline{F}_\varepsilon, \overline{\lambda}_\varepsilon] + \overline{\lambda}_\varepsilon K_2[\overline{F}_\varepsilon, \overline{\lambda}_\varepsilon] - \overline{\lambda} K_1[\overline{F}, \overline{\lambda}_\varepsilon] - \overline{\lambda} K_2[\overline{F}, \overline{\lambda}_\varepsilon] \right] \]
\[ \leq C \left( \| \overline{F}_\varepsilon - \overline{F} \|_{L^1_2} + \| \overline{\lambda}_\varepsilon \overline{F}_\varepsilon - \overline{\lambda} \overline{F} \|_{L^1_2} + \left| \frac{\overline{\lambda}_\varepsilon}{1 - \overline{\lambda}_\varepsilon} - \frac{\overline{\lambda}}{1 - \overline{\lambda}} \right| \right). \] (4.48)
Now we are ready for looking at the last estimates. First of all we can write down the equation for \( \partial_t \left( \mathcal{F} - \mathcal{F} \right) \) as follows

\[
\partial_t \left( \mathcal{F} - \mathcal{F} \right) = \frac{1}{1 + \lambda} \left( \mathcal{K}_1 + \mathcal{K}_2 \right) \left[ \mathcal{F}, \mathcal{X}_c \right] - \frac{1}{1 + \lambda} \left( \mathcal{K}_1 + \mathcal{K}_2 \right) \left[ \mathcal{F}, \mathcal{X} \right]
\]

\[
+ \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}^e, \mathcal{X}_c \right] d\mathbf{v}
\]

\[
- \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}, \mathcal{X} \right] d\mathbf{v}
\]

\[
+ \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}^e, \mathcal{X}_c \right] d\mathbf{v}
\]

\[
+ \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}, \mathcal{X} \right] d\mathbf{v}
\]

\[
(4.49)
\]

We can thus estimate using (4.46) and \(| \mathcal{X}_c - \mathcal{X} | \leq \frac{\mathcal{X}_c}{1 - \lambda} - \frac{\mathcal{X}}{1 - \lambda} \) and testing (4.49) against \( \phi \)

\[
\partial_t \| \mathcal{F}^e - \mathcal{F} \|_{L^1_x} \leq C(\varepsilon_0, C_0, \kappa_4, \kappa_2) \left( \| \mathcal{F}^e - \mathcal{F} \|_{L^1_x} + \| W_\varepsilon \|_{L^{1/2}_x} \right).
\]

(4.50)

for some \( C(\varepsilon_0, C_0, \kappa_4, \kappa_2) > 0 \). In order to prove this relation we need the estimates of Theorem 3.2 and of Lemma 3.5. For the last two terms in (4.49) we use the estimates on the kernels as in (1.12) and (1.13) to see that

\[
\int_{\mathbb{R}^3} \left[ \phi(\mathbf{v}) \left( \mathcal{K}_1 + \mathcal{K}_2 \right) \left[ \mathcal{F}^e, \mathcal{X}_c \right] + \left[ W_\varepsilon, \mathcal{F}^e \right] + \left[ W_\varepsilon, W_\varepsilon \right] \right] d\mathbf{v}
\]

\[
\leq C(\varepsilon_0, C_0) \left( \| \mathcal{F}^e \|_{L^1_x} \| W_\varepsilon \|_{L^1_x} + \| W_\varepsilon \|_{L^{1/2}_x} + \| W_\varepsilon \|_{L^{1/2}_x} \right)
\]

(4.51)

where in the end we used the interpolation formula for the \( L^1_x \)-norm.

In a completely analogous way we can estimate also \( \partial_t \| \mathcal{X}_c \mathcal{F} - \mathcal{X} \mathcal{F} \|_{L^1_x} \) and \( \partial_t \left| \frac{\mathcal{X}_c}{1 - \lambda} - \frac{\mathcal{X}}{1 - \lambda} \right| \).

In the first case, we notice

\[
\partial_t \left( \mathcal{X}_c \mathcal{F} - \mathcal{X} \mathcal{F} \right) = \frac{\mathcal{X}_c}{1 + \lambda} \left( \mathcal{K}_1 + \mathcal{K}_2 \right) \left[ \mathcal{F}, \mathcal{X}_c \right] - \frac{\mathcal{X}}{1 + \lambda} \left( \mathcal{K}_1 + \mathcal{K}_2 \right) \left[ \mathcal{F}, \mathcal{X} \right]
\]

\[
+ \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}^e, \mathcal{X}_c \right] d\mathbf{v}
\]

\[
- \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}, \mathcal{X} \right] d\mathbf{v}
\]

\[
+ \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}^e, \mathcal{X}_c \right] d\mathbf{v}
\]

\[
+ \frac{1}{4} \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\mathbb{R}^3} \mathcal{K}_1 \left[ \mathcal{F}, \mathcal{X} \right] d\mathbf{v}
\]

(4.52)
We can proceed as we did in equation (4.50). We test equation (4.52) against \( \psi \) and we recall results (4.47) and (4.48). Hence, performing the same calculations we did in (4.50) we estimate
\[
\partial_t \| \lambda e F^\varepsilon - \lambda \bar{F} \|_{L^1_\varepsilon} \leq C(\varepsilon_0, C_0, \kappa_4, \kappa_2) \left( \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} + \| \mathcal{W} \|_{X^\varepsilon}^{1/2} \right). \tag{4.53}
\]

We conclude with the estimate for the photon number density. Here it is even easier since we estimate everything by means of the absolute value following the estimates for the second, third and fifth terms on the right hand side of equation (4.49). We conclude therefore that there exists some constant \( C = C(\varepsilon_0, C_0, \kappa_4) \) independent of \( t \) and \( \varepsilon_0 \) such that for all \( \varepsilon > 0 \) and \( t \in [0, t_0] \) the following holds true
\[
\| F^\varepsilon - \bar{F} \|_{X^\varepsilon} \leq \int_0^t C \left( \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} + \| \mathcal{W} \|_{X^\varepsilon}^{1/2} \right) dt. \tag{4.54}
\]

**STEP 5: Conclusion**

Now we are ready to conclude the claim of the theorem. Given equation (4.44) we see that equation (4.54) can be estimated by
\[
\| F^\varepsilon - \bar{F} \|_{X^\varepsilon} \leq \int_0^t C \left( \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} + \sqrt{\delta e - \frac{C_2}{2}} t + \sqrt{\frac{C_1}{C_2} \varepsilon t} \right) dt \tag{4.55}
\]

Hence, the Grönwall’s Inequality implies
\[
\| F^\varepsilon - \bar{F} \|_{X^\varepsilon} \leq C \left( \frac{C \sqrt{2 \delta e}}{C_2} + \sqrt{\frac{C_1}{C_2} \varepsilon t} \right) + \int_0^t C \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} dt.
\]

which converges to zero uniformly in \([0, t_0]\) as \( \varepsilon \to 0 \). Since \( \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} \leq \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} + \| \mathcal{W} \|_{X^\varepsilon} \) equations (4.45) and (4.56) imply the theorem. \( \Box \)

We can extend the result of the theorem for larger time. We can argue using the fact that for some positive time \( t_0 \) the solution \( F^\varepsilon \) is in a small neighborhood of \( \mathcal{M}_+ \) and so we can again apply Theorem 4.1. We repeat then the same procedure.

**Corollary 4.2** Let \( T > 0 \). Under the assumption of Theorem 4.1 the following holds for all \( s > 0 \)
\[
\lim_{\varepsilon \to 0} \sup_{s \leq t \leq T} \| F^\varepsilon - \bar{F} \|_{X^\varepsilon} = 0.
\]

**Proof** We only need to check Step 1 in Theorem 4.1 at time \( t_0 \). There exists some \( \varepsilon_1 > 0 \) such that for all \( \varepsilon \leq \varepsilon_1 \) we have on one hand that
\[
\| F^\varepsilon(t_0) - \bar{F}^\varepsilon(t_0) \|_{X^\varepsilon} = \| \mathcal{W}^\varepsilon(t_0) \|_{X^\varepsilon} < \delta < \delta_0
\]
and on the other hand also
\[
\| F^\varepsilon(t_0) - \bar{F}(t_0) \|_{X^\varepsilon} < \tilde{\delta}.
\]
This holds for some \( \tilde{\delta} > 0 \) small enough such that there exists some \( t_2 = t_0 + \nu > t_0 \) so that for every \( t \in [t_0 - \nu, t_2] \) we have
\[
\| F^\varepsilon(t) - \bar{F}^\varepsilon(t_0) \|_{X^\varepsilon} < \delta_1 < \delta_0
\]
for some $\delta_1$ small enough. $\delta_1$ is chosen so that we have again a decomposition in $F^\varepsilon = \overline{F} + W^\varepsilon$ as in the Theorem 4.1. The time $t_2$ is also chosen so that for all $t \in [0, t_2]$ the following holds

$$\|F^\varepsilon(t) - \overline{F}(t)\|_{\mathcal{X}} < \frac{1}{16} \frac{2\varepsilon_0}{2\varepsilon_0 + E_0}.$$  (4.61)

Then the corollary follows from all the steps in Theorem 4.1 and from the uniform continuity of the $\mathcal{X}$-norm in $[0, T]$, i.e. we can repeat a finite number of times the above argument. \hfill \Box

### 5 Summary and Final Comments

In this paper we considered a simplified model describing the interaction of a gas with monochromatic radiation. We focused on the fast radiation limit, i.e. on the situation when the radiative processes are much more frequent than the collisions between gas molecules. Starting from a scaled version of the initial kinetic system (cf. Eq. (1.23)), we formally derived an effective kinetic equation which describes the behavior of the gas–photon system immediately after the initial time. To this end we analyzed the dynamic of the system near the slow manifold of steady states $\mathcal{M}_+$. We performed a perturbative expansion, which we called generalized Chapman–Enskog expansion due to its reminiscence to the classical Chapman–Enskog expansion of the Boltzmann equation, and we derived equation (2.13). After having developed a well-posedness theory for the resulting equation (cf. Theorem 3.1 and Theorem 3.2) we proved rigorously that the solutions $F^\varepsilon$ of (1.23) converge to the solutions of (2.13) as $\varepsilon \to \infty$. The main technical idea for this is that every vector in a small neighborhood of the manifold of steady states can be written as the sum of a vector in $\mathcal{M}_+$ (the projection) and an orthogonal vector (cf. Lemma 4.1).

Decomposing in this way the sequence $F^\varepsilon$ we concluded the proof of Theorem 4.1 observing that the sequence of orthogonal vectors converges to zero uniformly for every positive time, while the sequence of projections converges to the solution of the formal derived equation (2.13).

An interesting question we did not consider in this paper is the long-time behavior of the solutions of (2.13). In the case, when the collision terms and the radiative terms are of the same order, it is possible to show that strong solutions of (1.10) converge to the equilibrium whenever $t \to \infty$. We refer to [11]. The crucial result one needs for the proof of this claim, is that the entropy of strong solutions is bounded from above and from below and it is non-increasing, the so called H-Theorem. Also for the kinetic system (2.13) we can define an equilibrium. As in the classical Boltzmann equation, at the equilibrium the gas densities is given by a Maxwellian.

**Proposition 5.1** The equilibrium solution to the system (2.13) is given by the pair $\left( e^{-2k\varepsilon_0} A e^{-k|v-u|^2}, \right)$, where $A, k > 0$ and $u \in \mathbb{R}^3$.

Moreover, in the fast radiation limit it is still possible to show that the lower semi-continuity of the entropy yields the upper and lower uniform bound of the entropy for the solution to (2.13) as in Definition 3.3.

**Proposition 5.2** Under the assumptions of Theorem 3.1 the entropy $\mathcal{H}(t) = \mathcal{H}[\lambda, F](t)$ of the weak solution $(\lambda, F)^\top \in C ([0, \infty), \mathcal{X})$ satisfies

$$K \leq \mathcal{H}(t) \leq \mathcal{H}(0)$$

for all $t \geq 0$ and for some $K \in \mathbb{R}$.

However, in order to have also for the weak solutions a version of the H-theorem as in Lemma 3.1, we shall pass to the limit in the equation for the dissipation of the cut-off approximating solutions. Unfortunately, the weak $L^1$-convergence is not strong enough for that.
Nonetheless, for strong solutions to special bounded kernels the long-time convergence to the equilibrium can be proved. Indeed, we can use the theory developed for the cut-off system in Sect. 3. For given positive constants $\kappa_0, \kappa_2$ and $\kappa_4$ we can consider the set

$$\Omega := \left\{ (\lambda, F) \in X : \lambda \in [0, 1), \ F \geq 0, \ \| F \|_{L^1} \leq \kappa_0, \ E(\lambda, F) \leq \kappa_2 \text{ and } \| F \|_{L^4} \leq \kappa_4 \right\}.$$ 

We define $M$ to be the $M \in \mathbb{N}$ as in Lemma 3.6 such that the solution $F_M$ of the $M$-cut-off kinetic system for the initial data $(\lambda_0, F_0) \in \Omega$ satisfies $\| F_M \|_{L^1} \leq C$ for all $t \geq 0$ ($C$ depends only on initial energy, mass and $L^4$-norm). We can consider the kinetic equation (2.13) with the bounded cut-off kernels as in (3.15), (3.16) and (3.17) for $n = M$. As a consequence of Lemma 3.5 and Theorem 3.1 and adapting the proof of Theorem 3.6 in [2] we can show that the solution $(\lambda, F)$ converge to the unique equilibrium as $t \to \infty$.

**Theorem 5.1** Let $(\lambda_0, F_0) \in \Omega$, $F_0 \log(F_0) \in L^1(\mathbb{R}^3)$. Assume $F_0$ satisfies assumption (3.6). Let $(\lambda, F) \in C^1([0, \infty), X)$ be the unique strong solution of this kinetic equation with bounded kernels. Then $(\lambda, F)$ converges to the unique equilibrium solution $\left( e^{-2k \varepsilon_0} A e^{-k |v-u|^2} \right)$ which conserves the initial mass, momentum and energy. The convergence is pointwise for $\lambda$ and weak in $L^1(\mathbb{R}^3)$ for $F$.

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