STRONG FACTORIZATIONS OF OPERATORS WITH APPLICATIONS TO FOURIER AND CESÁRO TRANSFORMS

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ABSTRACT. Consider two continuous linear operators $T: X_1(\mu) \to Y_1(\nu)$ and $S: X_2(\mu) \to Y_2(\nu)$ between Banach function spaces related to different $\sigma$-finite measures $\mu$ and $\nu$. We characterize by means of weighted norm inequalities when $T$ can be strongly factored through $S$, that is, when there exist functions $g$ and $h$ such that $T(f) = gS(hf)$ for all $f \in X_1(\mu)$. For the case of spaces with Schauder basis our characterization can be improved, as we show when $S$ is for instance the Fourier operator, or the Cesàro operator. Our aim is to study the case when the map $T$ is besides injective. Then we say that it is a representing operator—in the sense that it allows to represent each elements of the Banach function space $X(\mu)$ by a sequence of generalized Fourier coefficients—, providing a complete characterization of these maps in terms of weighted norm inequalities. Some examples and applications involving recent results on the Hausdorff-Young and the Hardy-Littlewood inequalities for operators on weighted Banach function spaces are also provided.

1. Introduction

Let $X_1(\mu), X_2(\mu), Y_1(\nu), Y_2(\nu)$ be Banach function spaces related to different $\sigma$-finite measures $\mu, \nu$ and consider two continuous linear operators $T: X_1(\mu) \to Y_1(\nu), S: X_2(\mu) \to Y_2(\nu)$. In this paper we provide a characterization in terms of weighted norm inequalities of when $T$ can be factored through $S$ via multiplication operators, that is, when there are functions $g$ and $h$ satisfying that $T(f) = gS(hf)$ for all $f \in X_1(\mu)$.

This problem was studied in [6] for the case when $\mu$ and $\nu$ are the same finite measure. However, the results developed there do not allow to face the problem we study here, in which different $\sigma$-finite measures $\mu$ and $\nu$ appear in order to consider the relevant case of the classical sequence spaces $\ell^p$. The reason is that we are interested in considering standard cases as the Fourier and the Cesàro operators, that will be in fact our main examples.

In this direction, we will show that in the case when the Köthe dual $Y_1(\nu)'$ of $Y(\nu)$ and $X_1(\mu)$ have Schauder basis, the norm inequality which characterizes the factorization of $T$ through $S$ can be weakened. After showing this, we will develop with some detail some the examples regarding Fourier operators, operators factoring though infinite matrices and the

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Let us explain briefly this notion. With the notation introduced above, assume that \( Y_1(\nu) \) and \( Y_2(\nu) \) have unconditional basis \( U_1 := \{ e_i : i \in \mathbb{N} \} \) and \( U_2 := \{ e_i : i \in \mathbb{N} \} \), respectively. Suppose that there exists a Schauder basis \( B : \{ f_i : i \in \mathcal{N} \} \) for the space \( X_2(\mu) \) and write \( \alpha_i(f) \) for the \( i \)-th basic coefficient of \( f \in X_2(\mu), \ i \in \mathbb{N} \). We will say that an operator \( T : X_1(\mu) \to Y_1(\nu) \) is a representing operator for \( X_1(\mu) \) on \( Y_1(\nu) \) (associated to the basis \( B \) of \( X_2(\mu) \)) if each element \( x \in X_1(\mu) \) can be represented univocally by a sequence of coefficients \( (\beta_i(x)) \) such that \( \sum_{i=1}^{\infty} \beta_i(x)w_i \in Y_1(\nu) \), where the coefficients \( \beta_i(x) \) can be computed by means of the associated values of \( \alpha_i \) by a simple transformation provided by multiplication operators.

Thus, the last part of the paper is devoted to find a characterization of such operators in terms of vector norm inequalities that they must satisfy. We provide also classical and recently published examples of such kind of maps, using for instance an improvement of the Hausdorff-Young inequality given in \([3]\), or the continuity of the Fourier operator \( H_p : L^p[-\pi,\pi] \to \ell^p(W) \) —where \( \ell^p(W) \) is a weighted \( \ell^p \)-space— that can be found in \([1]\).

2. Preliminaries

Let \((\Omega, \Sigma, \mu)\) be a \( \sigma \)-finite measure space and denote by \( L^0(\mu) \) the space of all measurable real functions defined on \( \Omega \), where functions which are equal \( \mu \)-a.e. are identified. By a Banach function space we mean a Banach space \( X(\mu) \subset L^0(\mu) \) with norm \( \| \cdot \|_X \) satisfying that if \( f \in L^0(\mu), \ g \in X(\mu) \) and \( |f| \leq |g| \) \( \mu \)-a.e. then \( f \in X(\mu) \) and \( \|f\|_X \leq \|g\|_X \). In particular \( X(\mu) \) is a Banach lattice for the \( \mu \)-a.e. pointwise order, in which the convergence in norm of a sequence implies the convergence \( \mu \)-a.e. for some subsequence. Note that every positive linear operator between Banach lattices is continuous, (see \([10\text{ p.}2])\. So, all inclusions between Banach function spaces are continuous. General information about Banach function spaces can be found for instance in \([17\text{ Ch.}15]) considering the function norm \( \rho \) defined there as \( \rho(f) = \|f\|_X \) if \( f \in X(\mu) \) and \( \rho(f) = \infty \) in other case.

A Banach function space \( X(\mu) \) is said to be saturated if there is no \( A \in \Sigma \) with \( \mu(A) > 0 \) such that \( f_{\chi_A} = 0 \) \( \mu \)-a.e. for all \( f \in X(\mu) \). This is equivalent to the existence of a function \( g \in X(\mu) \) such that \( g > 0 \) \( \mu \)-a.e.

Given two Banach function spaces \( X(\mu) \) and \( Y(\mu) \), the \( Y(\mu) \)-dual space of \( X(\mu) \) is defined by

\[
X^Y = \{ h \in L^0(\mu) : fh \in Y(\mu) \text{ for all } f \in X(\mu) \}.
\]

Every \( h \in X^Y \) defines a continuous multiplication operator \( M_h : X(\mu) \to Y(\mu) \) via \( M_h(f) = fh \) for all \( f \in X(\mu) \). The space \( X^Y \) is a Banach function space with norm

\[
\|h\|_{X^Y} = \sup_{f \in B_X} \|hf\|_Y, \quad h \in X^Y,
\]

if and only if \( X(\mu) \) is saturated. As usual \( B_X \) denotes the closed unit ball of \( X(\mu) \). Note that \( X^{L^1} \) is just the classical Köthe dual space \( X(\mu)' \) of \( X(\mu) \). If \( X(\mu) \) is saturated then \( X(\mu)' \) is also saturated. This does not hold in general for \( X^Y \). For issues related to generalized dual spaces see \([3]\) and the references therein.
A saturated Banach function space $X(\mu)$ is contained in its Köthe bidual $X(\mu)^{\prime\prime}$ with $\|f\|_{X^{\prime\prime}} \leq \|f\|_{X}$ for all $f \in X(\mu)$. It is known that $\|f\|_{X^{\prime\prime}} = \|f\|_{X}$ for all $f \in X(\mu)$ if and only if $X(\mu)$ is order semi-continuous, that is, if for every $f, f_n \in X(\mu)$ such that $0 \leq f_n \uparrow f$ $\mu$-a.e. it follows that $\|f_n\|_X \uparrow \|f\|_X$. Even more, $X(\mu) = X(\mu)^{\prime\prime}$ with equal norms if and only if $X(\mu)$ has the Fatou property, that is, if for every $f_n \in X(\mu)$ such that $0 \leq f_n \uparrow f$ $\mu$-a.e. and $\sup_n \|f_n\|_X < \infty$, we have that $f \in X(\mu)$ and $\|f_n\|_X \uparrow \|f\|_X$.

Denote by $X(\mu)^{\ast}$ the topological dual of a saturated Banach function space $X(\mu)$. Every function $h \in X(\mu)^{\prime}$ defines an element $\eta(h) \in X(\mu)^{\ast}$ via $\langle \eta(h), f \rangle = \int hf \, d\nu$ for all $f \in X(\mu)$. The map $\eta: X(\mu)^{\prime} \to X(\mu)^{\ast}$ is a continuous linear injection, since the norm of every $h \in X(\mu)^{\prime}$ can be computed as

$$\|h\|_{X^{\prime}} = \sup_{f \in B_X} \left| \int hf \, d\mu \right|$$

and so $\eta$ is an isometry. It is known that $\eta$ is surjective if and only if $X(\mu)$ is $\sigma$-order continuous, that is, if for every $(f_n) \subset X(\mu)$ with $f_n \downarrow 0$ $\mu$-a.e. it follows that $\|f_n\|_X \downarrow 0$. Note that $\sigma$-order continuity implies order semi-continuity.

The $\sigma$-order continuous part $X_\sigma(\mu)$ of a saturated Banach function space $X(\mu)$ is the largest $\sigma$-order continuous closed solid subspace of $X(\mu)$, which can be described as

$$X_\sigma(\mu) = \{ f \in X(\mu) : |f| \geq f_n \downarrow 0 \mu\text{-a.e. implies } \|f_n\|_X \downarrow 0 \}.$$ 

Also, a function $f \in X_\sigma(\mu)$ if and only if $f \in X(\mu)$ satisfies that $\|f_{X_\sigma(\mu)}\|_X \downarrow 0$ whenever $(A_n) \subset \Sigma$ is such that $A_n \downarrow$ with $\mu(\cap A_n) = 0$. Note that $X_\sigma(\mu)$ could be the trivial space as in the case of $X(\mu) = L^\infty(\mu)$ when $\mu$ is nonatomic. In the case when $X_\sigma(\mu)$ is saturated, $X_\sigma(\mu)$ is order dense in $L^0(\mu)$ and so by the Monotone Convergence Theorem, it follows easily that $X_\sigma(\mu)^{\prime} = X(\mu)^{\prime}$ with equal norms.

The $\pi$-product space $X\pi Y$ of two Banach function spaces $X(\mu)$ and $Y(\mu)$ is defined as the space of functions $h \in L^0(\mu)$ such that $|h| \leq \sum_n |f_n g_n| \mu$-a.e. for some sequences $(f_n) \subset X(\mu)$ and $(g_n) \subset Y(\mu)$ satisfying $\sum_n \|f_n\|_X \|g_n\|_Y < \infty$. For $h \in X\pi Y$, consider the norm

$$\pi(h) = \inf \left\{ \sum_n \|f_n\|_X \|g_n\|_Y \right\},$$

where the infimum is taken over all sequences $(f_n) \subset X(\mu)$ and $(g_n) \subset Y(\mu)$ such that $|h| \leq \sum_n |f_n g_n| \mu$-a.e. and $\sum_n \|f_n\|_X \|g_n\|_Y < \infty$. The space $X\pi Y$ is a saturated Banach function space with norm $\pi$ if and only if $X(\mu)$, $Y(\mu)$ and $X^{\prime\prime}$ are saturated and, in this case, $(X\pi Y)^{\prime} = Y^{\prime\prime}$ with equal norms (see [5 Proposition 2.2]). The calculus of product spaces is nowadays well-known (see [3, 4, 9, 16]); the reader can find all the information that is needed on this construction in these papers.

Banach function spaces on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \lambda)$ with the counting measure $\lambda$ are called usually Banach sequence spaces. The classical Banach sequence spaces $\ell^p$ for $1 \leq p \leq \infty$ is saturated which is $\sigma$-order continuous if and only if $p < \infty$. As usual for each $n \in \mathbb{N}$, we denote by $(e^n)$ the standard unit vector basis in $c_0$. 
We recall the well known easily verified formula \((\ell^p)^{\ell^q} = \ell^{pq}\) with equal norms, where
\[
1 \leq s_{pq} = \begin{cases} \frac{pq}{p-q} & \text{if } 1 \leq q < p < \infty \\ q & \text{if } 1 \leq q < p = \infty \\ \infty & \text{if } 1 \leq p \leq q \leq \infty \end{cases}.
\] (2.1)
In particular, \((\ell^p)' = (\ell^p)^{\ell^q} = \ell^{p'}\) where \(p'\) denote the conjugate exponent of \(p\) \((\frac{1}{p} + \frac{1}{p'} = 1)\).
Note that \(\ell^p\) has the Fatou property as \((\ell^p)' = \ell^{p'}\). Also note that \(s_{pq} = 1\) if and only if \(q = 1\) and \(p = \infty\).

3. Strong factorization of operators on Banach function spaces

Let \((\Omega, \Sigma, \mu), (\Delta, \Gamma, \nu)\) be \(\sigma\)-finite measure spaces, \(X_1(\mu), X_2(\mu), Y_1(\nu), Y_2(\nu)\) saturated Banach function spaces and \(T : X_1(\mu) \to Y_1(\nu), S : X_2(\mu) \to Y_2(\nu)\) nontrivial continuous linear operators. For \(h \in X_1^{X_2}\), we will say that \(T\) factors strongly through \(S\) and \(M_h\) if there exists \(g \in Y_2^{Y_1''}\) such that the diagram

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

commutes. Here \(i\) denotes the inclusion map. Note that if \(Y_1(\nu)\) has the Fatou property the diagram above looks as

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

In the case when \(\mu\) and \(\nu\) are the same finite measure and under certain extra conditions, \cite{6} Theorem 4.1] characterizes when \(T\) factors strongly through \(S\) and \(M_h\). In this section we extend this theorem to our more general setting and improve it by relaxing the conditions. The extension will be obtained from the following broader factorization result.

**Theorem 3.1.** Assume that \(Y_2^{Y_1''}\) is saturated and consider a function \(h \in X_1^{X_2}\). The following statements are equivalent:

(a) There exists a constant \(C > 0\) such that the inequality

\[
\sum_{i=1}^{n} \int T(x_i)y'_i \, d\nu \leq C \left\| \sum_{i=1}^{n} S(hx_i)y'_i \right\|_{Y_2^{Y_1''}}, \quad n \in \mathbb{N},
\]

holds for every \(n \in \mathbb{N}, x_1, \ldots, x_n \in X_1(\mu)\) and \(y'_1, \ldots, y'_n \in Y_1(\nu)'\).

(b) There exists \(\xi^* \in (Y_2^{\pi Y_1^*})^*\) satisfying the following factorization between the operators \(T\) and \(S\):

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

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\begin{array}{ccc}
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\end{array}
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\end{array}
\]

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\end{array}
\]

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\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
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\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
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\begin{array}{ccc}
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\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

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\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

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\begin{array}{ccc}
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\begin{array}{ccc}
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\begin{array}{ccc}
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\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1(\mu) & \xrightarrow{T} & Y_1(\nu) \\
\downarrow{M_h} & & \downarrow{M_g}
\end{array}
\]

\[
\begin{array}{ccc}
X_2(\mu) & \xrightarrow{S} & Y_2(\nu) \\
\end{array}
\]
where \( \eta \) is the continuous linear injection of \( Y_1(\nu)^\prime \) into \( Y_1(\nu)^\prime \) and \( R_\xi^\prime \) is the continuous linear operator defined by \( \langle R_\xi^\prime(y_2), y_1' \rangle = \langle \xi^*, y_2 y_1' \rangle \) for \( y_2 \in Y_2(\nu) \) and \( y_1' \in Y_1(\nu)' \).

**Proof.** Note that the condition of \( Y_2 \to Y_1 \) being saturated assures that \( Y_2 \to Y_1 \) is a saturated Banach function space. Also note that the map \( R_\xi^\prime : Y_2(\nu) \to Y_1(\nu)' \) defined in (b) is a well defined continuous linear operator as

\[
\| \langle R_\xi^\prime(y_2), y_1' \rangle \| \leq \| \xi^* \|_{(Y_2 \to Y_1)'} \| y_2 \|_{Y_2} \| y_1' \|_{Y_1'}
\]

for all \( y_2 \in Y_2(\nu) \) and \( y_1' \in Y_1(\nu)' \).

(a) \( \Rightarrow \) (b) For every \( n \in \mathbb{N} \), \( x_1, ..., x_n \in X_1(\mu) \) and \( y_1', ..., y_n' \in Y_1(\nu)' \) we take the convex function \( \phi : B_{(Y_2 \to Y_1)'} \to \mathbb{R} \) given by

\[
\phi(\xi^*) = \sum_{i=1}^{n} T(x_i) y_i' d\nu - C \sum_{i=1}^{n} \langle \xi^*, S(h x_i) y_i' \rangle
\]

for all \( \xi^* \in B_{(Y_2 \to Y_1)'} \). Considering the weak* topology on \( (Y_2 \to Y_1) \) we have that \( \phi \) is a continuous map on a compact convex set. Moreover, from the Hahn-Banach Theorem there exists \( \xi^*_\phi \in B_{(Y_2 \to Y_1)'} \) such that

\[
\left\| \sum_{i=1}^{n} S(h x_i) y_i' \right\|_{Y_2 \to Y_1} = \langle \xi^*_\phi, \sum_{i=1}^{n} S(h x_i) y_i' \rangle
\]

and so, by (a), it follows that \( \phi(\xi^*_\phi) \leq 0 \).

Since the family \( \mathcal{F} \) of functions \( \phi \) defined in this way is concave, Ky Fan's lemma (see for instance [4, E.4]) guarantees the existence of an element \( \xi^*_\phi \in B_{(Y_2 \to Y_1)'} \) such that \( \phi(\xi^*_\phi) \leq 0 \) for all \( \phi \in \mathcal{F} \). In particular, for every \( x \in X_1(\mu) \) and \( y' \in Y_1(\nu)' \), we have that

\[
\int T(x) y' d\nu \leq C(\xi^*, S(h x) y').
\]

By taking \(-y'\) instead of \(y'\), we obtain that

\[
- \int T(x) y' d\nu \leq -C(\xi^*, S(h x) y')
\]

and so

\[
\langle \eta(T(x)), y' \rangle = \langle R_C \xi^*, (S(h x)) y' \rangle.
\]

Therefore, \( \eta(T(x)) = R_C \xi^*, (S(h x)) \) for all \( x \in X_1(\mu) \) and the factorization in (b) holds for \( C\xi^* \in (Y_2 \to Y_1)' \).

(b) \( \Rightarrow \) (a) For each \( n \in \mathbb{N} \) and every \( x_1, ..., x_n \in X_1(\mu) \), \( y_1', ..., y_n' \in Y_1(\nu)' \) we have that

\[
\sum_{i=1}^{n} \int T(x_i) y_i' d\nu = \sum_{i=1}^{n} \langle \eta(T(x_i)), y_i' \rangle = \sum_{i=1}^{n} \langle R_\xi^\prime(S(h x_i)), y_i' \rangle
\]

\[
= \sum_{i=1}^{n} \langle \xi^*, S(h x_i) y_i' \rangle = \langle \xi^*, \sum_{i=1}^{n} S(h x_i) y_i' \rangle
\]

\[
\leq \| \xi^* \|_{(Y_2 \to Y_1)'} \left\| \sum_{i=1}^{n} S(h x_i) y_i' \right\|_{Y_2 \to Y_1}.
\]

Note that \( \| \xi^* \|_{(Y_2 \to Y_1)'} > 0 \) as \( T \) is nontrivial. \( \square \)
Note that the condition of $Y_2^{Y''}$ being saturated is obtained for instance if $Y_2(\nu) \subset Y_1(\nu)''$ which is equivalent to $L^\infty(\nu) \subset Y_2^{Y''}$. Also note that the condition (a) of Theorem 3.1 is equivalent to

$$\left| \sum_{i=1}^{n} \int T(x_i) y'_i \, d\nu \right| \leq C \left\| \sum_{i=1}^{n} S(h x_i) y'_i \right\|_{Y_2 \pi Y_1'}, \quad n \in \mathbb{N}$$

for every $x_1, \ldots, x_n \in X_1(\mu)$ and $y'_1, \ldots, y'_n \in Y_1(\nu)'$. Indeed, we only have to take $-y'_1, \ldots, -y'_n$ instead of $y'_1, \ldots, y'_n$ in Theorem 3.1 (a).

As a consequence of Theorem 3.1 we obtain the following generalization and improvement of [6] Theorem 4.1.

**Corollary 3.2.** Assume that $Y_2^{Y''}$ is saturated and that $y_2 y'_1 \in (Y_2 \pi Y_1')_a$ for all $y_2 \in Y_2(\nu)$ and $y'_1 \in Y_1(\nu)'$. Given $h \in X_1^{X_2}$, the following statements are equivalent:

(a) $T$ factors strongly through $S$ and $M_h$.

(b) There exists a constant $C > 0$ such that the inequality

$$\sum_{i=1}^{n} \int T(x_i) y'_i \, d\nu \leq C \sum_{i=1}^{n} S(h x_i) y'_i \left\|_{Y_2 \pi Y_1'} \quad n \in \mathbb{N}$$

holds for all every $x_1, \ldots, x_n \in X_1(\mu)$ and $y'_1, \ldots, y'_n \in Y_1(\nu)'$.

**Proof.** First note that $(Y_2 \pi Y_1')_a$ is saturated. Indeed, by taking $0 < y_2 \in Y_2(\nu)$ and $0 < y'_1 \in Y_1(\nu)'$ we have that $0 < y_2 y'_1 \in (Y_2 \pi Y_1')_a$. Then,

$$(Y_2 \pi Y_1')' = (Y_2 \pi Y_1')' = Y_2^{Y''},$$

(b) $\Rightarrow$ (a) From Theorem 3.1 there exists $\xi^* \in (Y_2 \pi Y_1')^*$ such that

$$\langle \eta(T(x)), y' \rangle = \langle R_{\xi^*}(S(hx)), y' \rangle = \langle \xi^*, S(hx)y' \rangle$$

for all $x \in X_1(\mu)$ and $y' \in Y_1(\nu)'$. Denote by $\tilde{\xi}^*$ the restriction of $\xi^*$ to $(Y_2 \pi Y_1')_a$. Since $(Y_2 \pi Y_1')_a$ is $\sigma$-order continuous and $\tilde{\xi}^* \in (Y_2 \pi Y_1')^*_a$, we can identify $\tilde{\xi}^*$ with a function $g \in (Y_2 \pi Y_1')_a = Y_2^{Y''}$, that is, $\langle \tilde{\xi}^*, z \rangle = \int g \, d\nu$ for all $z \in (Y_2 \pi Y_1')_a$. Then, for every $x \in X_1(\mu)$ and $y' \in Y_1(\nu)'$, we have that

$$\langle \eta(T(x)), y' \rangle = \langle \tilde{\xi}^*, S(hx)y' \rangle = \int g S(hx) y' \, d\nu = \langle \eta(g S(hx)), y' \rangle$$

and so $T(x) = g S(hx)$.

(a) $\Rightarrow$ (b) Let $g \in Y_2^{Y''} = (Y_2 \pi Y_1')'$ be such that $T(x) = g S(hx)$ for all $x \in X_1(\mu)$. Consider the continuous linear injection $\tilde{\eta}: (Y_2 \pi Y_1')' \rightarrow (Y_2 \pi Y_1')^*$. Then $\tilde{\eta}(g) \in (Y_2 \pi Y_1')^*$ satisfies

$$\langle R_{\tilde{\eta}(g)}(S(hx)), y' \rangle = \langle \tilde{\eta}(g), S(hx)y' \rangle = \int g S(hx) y' \, d\nu = \int T(x) y' \, d\mu = \langle \eta(T(x)), y' \rangle$$

for all $x \in X_1(\mu)$ and $y' \in Y_1(\nu)'$ and so Theorem 3.1 (b) holds for $\xi^* = \tilde{\eta}(g)$. \qed

**Remark 3.3.** Of course, the condition $y_2 y'_1 \in (Y_2 \pi Y_1')_a$ for all $y_2 \in Y_2(\nu)$ and $y'_1 \in Y_1(\nu)'$ holds when $Y_2 \pi Y_1'$ is $\sigma$-order continuous. But also this condition is obtained for instance if any of $Y_2(\nu)$ or $Y_1(\nu)'$ is $\sigma$-order continuous. Indeed, suppose that $Y_2(\nu)$ is $\sigma$-order continuous...
and take $y_2 \in Y_2(\nu) = (Y_2)_a(\nu)$ and $y'_1 \in Y_1(\nu)'$. For every $(A_n) \subset \Sigma$ such that $A_n \downarrow$ with $\nu(\cap A_n) = 0$, we have that

$$\|y_2y'_1\chi_{A_n}\|_{Y_2\pi Y_1'} \leq \|y_2\chi_{A_n}\|_{Y_2} \cdot \|y'_1\|_{Y_1'} \to 0$$

and so $y_2y'_1 \in (Y_2\pi Y_1')_a$. We get the case when $Y_1(\nu)'$ is $\sigma$-order continuous in a similar way.

**Remark 3.4.** Note that if the $\sigma$-order continuous part $X_a(\mu)$ of a saturated Banach function space $X(\mu)$ is also saturated then $\|x\|_X = \|x\|_{X''}$ for all $x \in X_a(\mu)$. Indeed, for every $x \in X_a(\mu)$ we have that $\|x\|_X = \|x\|_{X''}$ since $X_a(\mu)$ is order semi-continuous and $\|x\|_{X''} = \|x\|_{X''}$ since $X_a(\mu)' = X(\mu)'$ with equal norms. Then, the norm in Corollary 3.2(b) can be computed as

$$\left\| \sum_{i=1}^n S(hx_i)y'_i \right\|_{Y_2\pi Y_1'} = \left\| \sum_{i=1}^n S(hx_i)y'_i \right\|_{Y_2\pi Y_1''} = \left\| \sum_{i=1}^n S(hx_i)y'_i \right\|_{(Y_2\pi Y_1'')} \sup_{f \in B_{Y_2\pi Y_1''}} \int_{\Omega} f \left( \sum_{i=1}^n S(hx_i)y'_i \right) d\nu.$$  

4. **Strong factorization involving Schauder basis**

Let $(\Omega, \Sigma, \mu)$, $(\Delta, \Gamma, \nu)$ be $\sigma$-finite measure spaces, $X_1(\mu)$, $X_2(\mu)$, $Y_1(\nu)$, $Y_2(\nu)$ saturated Banach function spaces and $T: X_1(\mu) \to Y_1(\nu)$, $S: X_2(\mu) \to Y_2(\nu)$ nontrivial continuous linear operators. In this section we assume the existence of a Schauder basis $(\gamma_n)$ for $Y_1(\nu)'$ and denote by $(\gamma_n^a)$ the sequence of coefficient functionals with respect to this basis.

**Theorem 4.1.** Assume that $Y_2^{Y_1''}$ is saturated and that any of $Y_2(\nu)$ or $Y_1(\nu)'$ is $\sigma$-order continuous. Given $h \in X_1^{X_2}$, the following statements are equivalent:

(a) $T$ factors strongly through $S$ and $M_h$.

(b) There exists a constant $C > 0$ such that the inequality

$$\sum_{i=1}^n \int T(x_i)\gamma_i d\nu \leq C \left\| \sum_{i=1}^n S(hx_i)\gamma_i \right\|_{Y_2\pi Y_1'}, \quad n \in \mathbb{N}$$

holds for every $x_1, \ldots, x_n \in X_1(\mu)$.

Moreover, if $Y_2(\nu) \subset Y_1(\nu)''$ and the functions $(\gamma_n)$ have pairwise disjoint support, then the condition

(c) There exists a constant $C > 0$ such that the inequality

$$\int T(x)\gamma_n d\nu \leq C \int |S(hx)\gamma_n| d\nu, \quad n \in \mathbb{N}$$

holds for every $x \in X_1(\mu)$ and $n \geq 1$.

implies (a)-(b). In the case when $Y_2^{Y_1''} = L^\infty(\nu)$, we have that (c) is equivalent to (a)-(b).

**Proof.** (a) $\iff$ (b) From Remark 3.3 we only have to prove that the condition (b) of the present theorem implies the condition (b) of Corollary 3.2. The converse implication follows.
by taking \( y'_i = \gamma_i \). Let \( x_1, \ldots, x_n \in X_1(\mu) \) and \( y'_1, \ldots, y'_n \in Y_1(\nu)' \). Fix \( m \in \mathbb{N} \) and denote \((y'_i)^m = \sum_{k=1}^{m} \langle \gamma'_k, y'_i \rangle \gamma_k\). It follows that

\[
\sum_{i=1}^{n} \int T(x_i)(y'_i)^m \, d\nu = \sum_{i=1}^{n} \sum_{k=1}^{m} \langle \gamma'_k, y'_i \rangle \int T(x_i) \gamma_k \, d\nu
\]

\[
= \sum_{k=1}^{m} \int \left( \sum_{i=1}^{n} \langle \gamma'_k, y'_i \rangle T(x_i) \right) \gamma_k \, d\nu
\]

\[
= \sum_{k=1}^{m} \int T \left( \sum_{i=1}^{n} \langle \gamma'_k, y'_i \rangle x_i \right) \gamma_k \, d\nu
\]

\[
\leq C \left\| \sum_{k=1}^{m} S \left( \sum_{i=1}^{n} \langle \gamma'_k, y'_i \rangle x_i \right) \gamma_k \right\|_{Y_2^\pi Y'_1}
\]

\[
= C \left\| \sum_{i=1}^{n} S (h x_i) (y'_i)^m \right\|_{Y_2^\pi Y'_1}. \tag{4.1}
\]

Since \((y'_i)^m \to y'_i\) in \( Y_1(\nu)' \) as \( m \to \infty \) and

\[
\left| \int z y'_i \, d\nu - \int z (y'_i)^m \, d\nu \right| = \left| \int z (y'_i - (y'_i)^m) \, d\nu \right| \leq \|z\|_{Y_1} \|y'_i - (y'_i)^m\|_{Y'_1}
\]

for every \( z \in Y_1(\nu) \), we have that \( \sum_{i=1}^{n} \int T(x_i)(y'_i)^m \, d\nu \to \sum_{i=1}^{n} \int T(x_i)y'_i \, d\nu \) as \( m \to \infty \). On other hand, since

\[
\|z y'_i - z (y'_i)^m\|_{Y_2^\pi Y'_1} = \|z (y'_i - (y'_i)^m)\|_{Y_2^\pi Y'_1} \leq \|z\|_{Y_2} \|y'_i - (y'_i)^m\|_{Y'_1}
\]

for every \( z \in Y_2(\nu) \), we have that \( \sum_{i=1}^{n} S (h x_i) (y'_i)^m \to \sum_{i=1}^{n} S (h x_i) y'_i \) in \( Y_2^\pi Y'_1 \) as \( m \to \infty \). Then, taking limit as \( m \to \infty \) in (4.1), we obtain

\[
\sum_{i=1}^{n} \int T(x_i) y'_i \, d\nu \leq C \left\| \sum_{i=1}^{n} S (h x_i) y'_i \right\|_{Y_2^\pi Y'_1}.
\]

Assume that \( Y_2(\nu) \subset Y_1(\nu)'' \) and that the functions \((\gamma_n)\) have pairwise disjoint support. Let us see that (c) implies (b). The condition \( Y_2(\nu) \subset Y_1(\nu)'' \) is equivalent to \( L^\infty(\nu) \subset Y_2^{Y_1''} = (Y_2^\pi Y_1)' \) and so \( Y_2^{Y_1''} \subset (Y_2^\pi Y_1)' \subset L^\infty(\nu) = L^1(\nu) \). Denote by \( K \) the continuity constant of the inclusion \( Y_2^\pi Y'_1 \subset L^1(\nu) \). For every \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in X_1(\mu) \), noting that \( \sum_{i=1}^{n} |S(h x_i) \gamma_i| = \left| \sum_{i=1}^{n} S(h x_i) \gamma_i \right| \) pointwise (as \((\gamma_k)\) have disjoint support), we have that

\[
\sum_{i=1}^{n} \int T(x_i) \gamma_i \, d\nu \leq C \sum_{i=1}^{n} \int |S(h x_i) \gamma_i| \, d\nu = C \int \left| \sum_{i=1}^{n} S(h x_i) \gamma_i \right| \, d\nu
\]

\[
\leq C K \left\| \sum_{i=1}^{n} S(h x_i) \gamma_i \right\|_{Y_2^\pi Y'_1}.
\]

If moreover \( L^\infty(\nu) = Y_2^{Y_1''} \) then (a) implies (c), as if \( g \in Y_2^{Y_1''} \) is such that \( T(x) = g S(h x) \) for all \( x \in X_1(\mu) \), it follows that

\[
\int T(x) \gamma_n \, d\nu = \int g S(h x) \gamma_n \, d\nu \leq \int |g S(h x) \gamma_n| \, d\nu \leq \|g\|_{\infty} \int |S(h x) \gamma_n| \, d\nu.
\]
Now suppose that there is also a Schauder basis \((\beta_n)\) for \(X_1(\mu)\) and denote by \((\beta^*_n)\) the sequence of its coefficient functionals. Then, the equivalent inequalities for the strong factorization can be relaxed.

**Theorem 4.2.** Assume that \(Y_2^{X''} \) is saturated and that any of \(Y_2(\nu)\) or \(Y_1(\nu)'\) is \(\sigma\)-order continuous. Given \(h \in X_1^{X'}\), the following statements are equivalent:

(a) \(T\) factors strongly through \(S\) and \(M_h\).

(b) There exists \(g \in Y_2^{X''}\) such that \(T(\beta_n) = gS(h\beta_n)\) for each \(n \in \mathbb{N}\).

(c) There exists a constant \(C > 0\) such that the inequality

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \int T(\beta_j) \gamma_i \, d\nu \leq C \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} S(h\beta_j) \gamma_i \right\|_{Y_2^{X} Y_1'}, \quad n, m \in \mathbb{N}
\]

holds for every \((r_{ij}) \subset B_{\ell^\infty}\).

Moreover, if \(Y_2(\nu) \subset Y_1(\nu)'\) and the functions \((\gamma_n)\) have pairwise disjoint support, then the condition

(d) There exists a constant \(C > 0\) such that the inequality

\[
\sum_{j=1}^{m} r_j \int T(\beta_j) \gamma_n \, d\nu \leq C \left\| \sum_{j=1}^{m} r_j S(h\beta_j) \gamma_n \right\| d\nu
\]

holds for every \(n, m \in \mathbb{N}\) and \((r_j) \subset B_{\ell^\infty}\).

implies (a)-(c). In the case when \(L^\infty(\nu) = Y_2^{X''}\), we have that (d) is equivalent to (a)-(c).

**Proof.** (a) \(\Rightarrow\) (b) Let \(g \in Y_2^{X''}\) be such that \(T(x) = gS(hx)\) for all \(x \in X_1(\mu)\). In particular, for \(x = \beta_n\) we obtain (b).

(b) \(\Rightarrow\) (c) Since \(g \in Y_2^{X''} = (Y_2^{X} Y_1)'), \) for every \(n, m \in \mathbb{N}\) and \((r_{ij}) \subset B_{\ell^\infty}\), it follows that

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \int T(\beta_j) \gamma_i \, d\nu = \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \int gS(h\beta_j) \gamma_i \, d\nu
\]

\[
= \int g \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} S(h\beta_j) \gamma_i \, d\nu
\]

\[
\leq \left\| g \right\|_{(Y_2^{X} Y_1')} \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} S(h\beta_j) \gamma_i \right\|_{Y_2^{X} Y_1'}
\]

(c) \(\Rightarrow\) (a) Let us show that the condition (b) of Theorem 4.1 holds. Let \(x_1, \ldots, x_n \in X_1(\mu)\) which can be assumed to be non-null. Fix \(m \in \mathbb{N}\) large enough such that \((x_i)^m = \sum_{j=1}^{m} \langle \beta^*_j, x_i \rangle \beta_j \neq 0\) and denote \(\alpha = \max_{i=1, \ldots, n} |\langle \beta^*_j, x_i \rangle|\). By taking \(r_{ij} = \frac{\langle \beta^*_j, x_i \rangle}{\alpha}\) it follows
that
\[
\sum_{i=1}^{n} \int T((x_i)^m) \gamma_i \, d\nu = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \beta_j^*, x_i \rangle \int T(\beta_j) \gamma_i \, d\nu
\]
\[
= \alpha \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \int T(\beta_j) \gamma_i \, d\nu
\]
\[
\leq \alpha C \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} S(h \beta_j) \gamma_i \right\|_{Y_2 \pi Y_1^*}
\]
\[
= C \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \beta_j^*, x_i \rangle S(h \beta_j) \gamma_i \right\|_{Y_2 \pi Y_1^*}
\]
\[
= C \left\| \sum_{i=1}^{n} S(h(x_i)^m) \gamma_i \right\|_{Y_2 \pi Y_1^*}
\] (4.2)

Denoting by \(\|T\|\) the operator norm of \(T\), since \((x_i)^m \rightarrow x_i\) in \(X_1(\mu)\) as \(m \rightarrow \infty\) and
\[
\left| \int T(x_i) z \, d\nu - \int T((x_i)^m) z \, d\nu \right| = \left| \int T(x_i - (x_i)^m) z \, d\nu \right|
\]
\[
\leq \|z\|_{Y_1'} \|T(x_i - (x_i)^m)\|_{Y_1}
\]
\[
\leq \|z\|_{Y_1'} \|T\| \|x_i - (x_i)^m\|_{X_1}
\]
for every \(z \in Y_1(\nu)'\), we have that \(\sum_{i=1}^{n} \int T((x_i)^m) \gamma_i \, d\nu \rightarrow \sum_{i=1}^{n} \int T(x_i) \gamma_i \, d\nu\) as \(m \rightarrow \infty\).

On other hand, denoting by \(\|S\|\) the operator norm of \(S\), since
\[
\|S(h x_i) z - S(h(x_i)^m) z\|_{Y_2 \pi Y_1^*} = \|S(h(x_i - (x_i)^m)) z\|_{Y_2 \pi Y_1^*}
\]
\[
\leq \|z\|_{Y_1'} \|S(h(x_i - (x_i)^m))\|_{Y_2}
\]
\[
\leq \|z\|_{Y_1'} \|S\| \|h(x_i - (x_i)^m)\|_{X_2}
\]
\[
\leq \|z\|_{Y_1'} \|S\| \|h\|_{X_1 x_2} \|x_i - (x_i)^m\|_{X_1}
\]
for every \(z \in Y_1(\nu)'\), we have that \(\sum_{i=1}^{n} S(h(x_i)^m) \gamma_i \rightarrow \sum_{i=1}^{n} S(h x_i) \gamma_i\) in \(Y_2 \pi Y_1^*\) as \(m \rightarrow \infty\).

Then, taking limit as \(m \rightarrow \infty\) in (4.2), we obtain
\[
\sum_{i=1}^{n} \int T(x_i) \gamma_i \, d\nu \leq C \left\| \sum_{i=1}^{n} S(h x_i) \gamma_i \right\|_{Y_2 \pi Y_1^*}.
\]

Assume that \(Y_2(\nu) \subset Y_1(\nu)''\) and that the functions \((\gamma_n)\) have pairwise disjoint support. We have already noted that in this case \(Y_2 \pi Y_1' \subset L^1(\nu)\) (denote by \(K\) its continuity constant) and \(\sum_{i=1}^{n} |f_i \gamma_i| = \sum_{i=1}^{n} f_i \gamma_i\) pointwise for every \(n \in \mathbb{N}\) and \((f_i) \subset L^0(\nu)\). Let us see that (d) implies (c). For every \(n, m \in \mathbb{N}\) and \((r_{ij}) \subset B_{\ell^\infty}\) we have that
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \int T(\beta_j) \gamma_i \, d\nu \leq C \sum_{i=1}^{n} \int \left| \sum_{j=1}^{m} r_{ij} S(h \beta_j) \gamma_i \right| \, d\nu
\]
\[
= C \int \left| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} S(h \beta_j) \gamma_i \right| \, d\nu
\]
\[
\leq C K \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} S(h \beta_j) \gamma_i \right\|_{Y_2 \pi Y_1^*}.
\]
If moreover \( L_\infty(\nu) = Y_2^Y \) then (a) implies (d), as if \( g \in Y_2^Y \) is such that \( T(x) = gS(hx) \) for all \( x \in X_1(\mu) \), it follows that

\[
\sum_{j=1}^m r_j \int T(\beta_j) \gamma_n d\nu = \sum_{j=1}^m r_j \int gS(h\beta_j) \gamma_n d\nu = \int g \sum_{j=1}^m r_j S(h\beta_j) \gamma_n d\nu \\
\leq \int \left| g \sum_{j=1}^m r_j S(h\beta_j) \gamma_n \right| d\nu \leq \|g\|_\infty \int \left| \sum_{j=1}^m r_j S(h\beta_j) \gamma_n \right| d\nu.
\]

\( \square \)

5. **Examples: the Fourier and Cesàro operators**

In this section we show how the results obtained in the previous one can be applied in concrete contexts. In particular, we will deal with the Fourier operator acting in different weighted \( L^p \)-spaces, we will show factorization through infinite matrices and, as a special case, we will analyze the case provided by the Cesàro operator.

5.1. **Strong factorization through the Fourier operator.** Consider the measure space given by the interval \( \mathbb{T} = [-\pi, \pi] \), its Borel \( \sigma \)-algebra and the Lebesgue measure \( m \) and denote by \((\phi_n)\) the real trigonometric system on \( \mathbb{T} \), that is,

\[
\phi_n(x) = \begin{cases} 
\frac{1}{\sqrt{2\pi}} & \text{if } n = 1 \\
\frac{\cos(kx)}{\sqrt{\pi}} & \text{if } n = 2k \\
\frac{\sin(kx)}{\sqrt{\pi}} & \text{if } n = 2k + 1
\end{cases}
\]

Note that \( \int_{-\pi}^{\pi} \phi_i(x)\phi_j(x) \, dx = 0 \) if \( i \neq j \) and \( \int_{-\pi}^{\pi} \phi_i(x)\phi_i(x) \, dx = 1 \). Each function \( f \in L^1(m) \) is associated to its Fourier series \( S(f) = \sum_{n \geq 1} a_n \phi_n \) where \( a_n = \int_{\mathbb{T}} f \phi_n \, dm \). If \( f \in L^r(m) \) for \( 1 < r < \infty \) then \( S(f) \) converges to \( f \) in \( L^r(m) \) and so \((\phi_n)\) is a Schauder basis on \( L^r(m) \).

Let \( \mathcal{F} \) be the Fourier operator defined by

\[
\mathcal{F}(f) = \left( \int_{\mathbb{T}} f \phi_n \, dm \right), \quad f \in L^1(m).
\]

The Hausdorff-Young inequality (see for instance [7 (8.5.7)]) guarantees that

\( \mathcal{F} : L^r(m) \to \ell^r \)

is a well defined continuous operator for every \( 1 < r \leq 2 \).

Fix \( 1 < r \leq 2, \, r \leq p < \infty, \, 1 < q \leq \infty \) and let \( T : L^p(m) \to \ell^q \) be a non-trivial continuous linear operator. We have that \((\phi_n)\) is a Schauder basis for \( L^p(m) \) (as \( 1 < p < \infty \)) and \((e_n)\) is a Schauder basis for \((\ell^q)'\) (as \( q > 1 \)). Also, \( L^p(m) \subset L^r(m) \) (as \( r \leq p \)) and so \( \chi_{\mathbb{T}} \in (L^p)_{L^r} \).

**Proposition 5.1.** The following statements are equivalent:
(a) \( T \) factors strongly through \( F \), that is, there exists \( g \in \ell^{s_{r,q}} \) such that

\[
\begin{array}{ccc}
L^p(m) & T & \ell^q \\
\downarrow & & \downarrow \\
L^r(m) & F & \ell^{r'}
\end{array}
\]

(see (2.1) in the preliminaries for the definition of \( s_{r,q} \)).

(b) \( T(\phi_n)_i = 0 \) for all \( i \neq n \) and \( T(\phi_i)_i \in \ell^{s_{r,q}} \).

(c) There exists a constant \( C > 0 \) such that the inequality

\[
\sum_{i=1}^n \sum_{j=1}^m r_{ij} T(\phi_j)_i \leq C \left( \sum_{i=1}^{\min\{n,m\}} |r_{ii}|^{s_{r,q}} \right)^{s_{r,q}}, \quad n, m \in \mathbb{N}
\]

holds for every \( (r_{ij}) \subset B_{\ell^\infty} \).

Moreover, in the case when \( r' \leq q \), the conditions (a)-(c) are equivalent to

(d) There exists a constant \( C > 0 \) such that the inequality

\[
\sum_{j=1}^m r_j T(\phi_j)_n \leq C \begin{cases} |r_n| & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}
\]

holds for each \( n, m \in \mathbb{N} \) and all \( (r_j) \subset B_{\ell^\infty} \).

**Proof.** Note that both \( \ell^{r'} \) and \( (\ell^q)' \) are \( \sigma \)-order continuous (as \( r, q > 1 \)) and that \( (\ell^{r'})'' = (\ell^q)' = \ell^{s_{r,q}} \) where \( s_{r,q} \) is defined as in (2.1). For the equivalence among (a), (b) and (c), let us see that conditions (b) and (c) are just respectively conditions (b) and (c) of Theorem 4.2 rewritten for \( X_1(\mu) = L^p(m) \), \( X_2(\mu) = L^r(m) \), \( Y_1(\nu) = \ell^q \) (\( \nu \) being the counting measure \( \lambda \) on \( \mathbb{N} \)), \( Y_2(\nu) = \ell^{r'} \), \( S = F \), \( h = \chi_T \), \( (\beta_n) = (\phi_n) \) and \( (\gamma_n) = (e^n) \).

(b) \( \Rightarrow \) Theorem 4.2(b). Take \( g = (T(\phi_i)_i) \in \ell^{s_{r,q}} \). Then, for every \( n, i \in \mathbb{N} \) we have that \( T(\phi_n)_i = T(\phi_i)_i, F(\phi_n)_i = g_i F(\phi_n)_i \) and so \( T(\phi_n) = gF(\phi_n) \).

Theorem 4.2(b) \( \Rightarrow \) (b). Let \( g \in \ell^{s_{r,q}} \) be such that \( T(\phi_n) = gF(\phi_n) \) for all \( n \in \mathbb{N} \). Then

\[
T(\phi_n)_i = g_i F(\phi_n)_i = \begin{cases} g_i & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}
\]

and so \( (T(\phi_i)_i) = g \in \ell^{s_{r,q}} \).

(c) \( \Leftrightarrow \) Theorem 4.2(c). From Remark 3.4 and noting that \( (\ell^{s_{r,q}})' = (\ell^{s_{r,q}})'' = (\ell^{s_{r,q}})' = \ell^{s_{r,q}} \) with equals norms and \( s_{r,q}' < \infty \) (as \( s_{r,q} > 1 \)), for each \( n, m \in \mathbb{N} \) and all...
\[(r_{ij}) \subset B_{\ell^\infty}\) it follows that
\[
\left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \mathcal{F}(\phi_j) e^i \right\|_{\ell^p} = \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \mathcal{F}(\phi_j) e^i \right\|_{(\ell^p)^\prime} = \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \mathcal{F}(\phi_j) e^i \right\|_{\ell^q} = \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \mathcal{F}(\phi_j) e^i \right\|_{(\ell^q)^\prime} = \left( \sum_{i=1}^{\min\{n,m\}} |r_{ii}| \right)^{\frac{1}{q}}
\]
and
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \int T(\phi_j) e^i \, d\lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} T(\phi_j)_i.
\]

In the case when \(r' \leq q\) we have that \(s_{r,q} = \infty\) and so \((\ell^p)'' = \ell^\infty\). Then (d) is equivalent to (a)-(c) as (d) is to rewrite condition (d) of Theorem 4.2. Indeed,
\[
\sum_{j=1}^{m} r_j \int T(\phi_j) e^n \, d\lambda = \sum_{j=1}^{m} r_j T(\phi_j)_n
\]
and
\[
\int \left| \sum_{j=1}^{m} r_j \mathcal{F}(\phi_j) e^n \right| \, d\lambda = \sum_{j=1}^{m} r_j \mathcal{F}(\phi_j)_n = \begin{cases} |r_n| & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}.
\]

5.2. Strong factorization for infinite matrices and the Cesàro operator. Consider the measure space \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \lambda)\) with \(\lambda\) being the counting measure on \(\mathbb{N}\). Let \(X_1(\lambda), X_2(\lambda), Y_1(\lambda), Y_2(\lambda)\) be saturated Banach function spaces in which \((e^n)\) is a Schauder basis and \(T: X_1(\lambda) \to Y_1(\lambda), S: X_2(\lambda) \to Y_2(\lambda)\) be nontrivial continuous linear operators. Then, the operators \(T\) and \(S\) can be described by infinite matrices \((a_{ij})\) and \((b_{ij})\) respectively, namely \(a_{ij} = T(e^i)_i\) and \(b_{ij} = S(e^j)_i\). We also require that \((e^n)\) is a Schauder basis for \(Y_1(\lambda)'\).

**Proposition 5.2.** Assume that \(Y_2^{Y_1'}\) is saturated and that any of \(Y_2(\lambda)\) or \(Y_1(\lambda)'\) is \(\sigma\)-order continuous. Given \(h \in X_1^{X_2}\), the following statements are equivalent:

(a) \(T\) factors strongly through \(S\) and \(M_h\).

(b) There exists \(g \in Y_2^{Y_1'}\) such that \(\frac{a_{ij}}{b_{ij}} = g h_j\) whenever \(b_{ij} \neq 0\) and \(a_{ij} = 0\) whenever \(b_{ij} = 0\).

(c) There exists a constant \(C > 0\) such that the inequality
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} a_{ij} \leq C \left\| \sum_{i=1}^{n} \left( \sum_{j=1}^{m} h_j r_{ij} b_{ij} \right) e^i \right\|_{Y_2 Y_1'}, \quad n, m \in \mathbb{N}
\]
holds for every \((r_{ij}) \subset B_{\ell^\infty}\).
Moreover, if $Y_2(\lambda) \subset Y_1(\lambda)^\prime$, then the condition

(d) There exists a constant $C > 0$ such that the inequality

$$\sum_{j=1}^{m} r_j a_{nj} \leq C \left| \sum_{j=1}^{m} h_j r_j b_{nj} \right|, \quad n, m \in \mathbb{N}$$

holds for every $(r_j) \subset B_{\ell^\infty}$, implies (a)-(c). In the case when $\ell^\infty = Y_2^\prime$, we have that (d) is equivalent to (a)-(c).

Proof. We only have to see that conditions (b), (c) and (d) are just respectively conditions (b), (c) and (d) of Theorem 4.2 rewritten for $\mu = \nu$ being the counting measure $\lambda$ and $(\beta_n) = (\gamma_n) = (e^n)$. Note that for every $i, j \in \mathbb{N}$ we have that $a_{ij} = T(e^i)_i$ and $g_i h_j b_{ij} = g_i h_j S(e^j)_i = g_i S(h_j e^j)_i = g_i S(e^j)_i$. So (b) $\iff$ Theorem 4.2(b). Since $\int T(e^j)_i d\lambda = T(e^j)_i = a_{ij}$ and $S(h_j e^j)_i = S(h_j e^j)_i e^i = h_j S(e^j)_i e^i = h_j b_{ij} e^i$ we have that (c) $\iff$ Theorem 4.2(c).

Moreover as

$$\int \left| \sum_{j=1}^{m} r_j S(h e^j)_i e^n \right| d\lambda = \int \left| \sum_{j=1}^{m} r_j h_j b_{nj} e^n \right| d\lambda = \left| \sum_{j=1}^{m} r_j h_j b_{nj} \right|,$$

it follows that (d) $\iff$ Theorem 4.2(d).

□

Let $C$ be the Cesàro operator which maps a real sequence $x = (x_n)$ into the sequence of its Cesàro means $C(x) = (\frac{1}{n} \sum_{i=1}^{n} x_i)$. It is well known that $C: \ell^r \to \ell^r$ continuously for every $1 < r < \infty$ (see [7, Theorem 326]) and it can be described by the infinite matrix $(b_{ij})$ where $b_{ij} = \frac{1}{i}$ if $j \leq i$ and $b_{ij} = 0$ if $j > i$, that is,

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}.$$

Fix $1 \leq p < \infty$, $1 < q, r < \infty$ and let $T: \ell^p \to \ell^q$ be a nontrivial continuous operator described by the infinite matrix $(a_{ij})$ with $a_{ij} = T(e^i)_i$. Note that $(e^n)$ is a Schauder Basis on $\ell^p$, $\ell^q$, $\ell^r$ and $(\ell^r)'$.

**Proposition 5.3.** Let $h \in \ell^{s_{pr}}$ (see (2.1) for the definition of $s_{pr}$). The following statements are equivalent:

(a) $T$ factors strongly through $C$ and $M_h$, that is, there exists $g \in \ell^{s_{rq}}$ such that

$$\begin{array}{ccc}
\ell^p & \xrightarrow{T} & \ell^q \\
M_h & \downarrow & M_g \\
\ell^r & \xrightarrow{C} & \ell^r
\end{array}$$

(b) There exists $g \in \ell^{s_{rq}}$ such that

$$a_{ij} = \begin{cases}
\frac{g_i h_j}{i} & \text{if } j \leq i \\
0 & \text{if } j > i
\end{cases}$$
(c) There exists a constant $C > 0$ such that the inequality
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij}a_{ij} \leq C \left( \sum_{i=1}^{n} \frac{1}{s'_{r_q}} \sum_{j=1}^{m} h_{j}r_{ij} \right) \frac{1}{s'_{r_q}}, \quad n, m \in \mathbb{N}
\]
holds for every $(r_{ij}) \in B_{l^\infty}$.
Moreover, in the case when $r \leq q$, the conditions (a)-(c) are equivalent to

(d) There exists a constant $C > 0$ such that the inequality
\[
\sum_{j=1}^{m} r_{j} a_{nj} \leq C \frac{1}{n} \sum_{j=1}^{m} h_{j} r_{j}, \quad n, m \in \mathbb{N}
\]
holds for every $(r_{j}) \subset B_{l^\infty}$.

Proof. Note that both $\ell^r$ and $(\ell^q)'$ are $\sigma$-order continuous (as $r < \infty$ and $q > 1$), $(\ell^r)^{(\ell^q)''} = (\ell^r)^{\ell_q}$ and $(\ell^q)^{\ell_r} = \ell^{s_{pq}}$. Also note that if $r \leq q$ then $s_{r_q} = \infty$ and so $(\ell^r)^{(\ell^q)''} = \ell^\infty$.

Then, we only have to see that (b), (c), (d) is just to rewrite respectively conditions (b), (c), (d) of Proposition 5.2. for $X_1(\lambda) = \ell^r$, $X_2(\lambda) = \ell^q$, $Y_1(\lambda) = \ell^q$, $Y_2(\lambda) = \ell^r$ and $S = C$.

As noted above, the elements of the matrix of $C$ are $b_{ij} = \frac{1}{r}$ if $j \leq i$ and $b_{ij} = 0$ if $j > i$, so (b) $\Leftrightarrow$ Proposition 5.2 (b).

By Remark 3.3 and noting that $((\ell^r \pi (\ell^q))')'' = ((\ell^r)^{(\ell^q)''})' = (\ell^s_{pq})' = \ell^{s_{pq}}$ with equals norms and $s_{pq} < \infty$ (as $s^q > 1$), for every $n, m \in \mathbb{N}$ and $(r_{ij}) \subset B_{l^\infty}$ it follows that
\[
\left\| \sum_{i=1}^{n} \left( \sum_{j=1}^{m} h_{j}r_{ij}b_{ij} \right) e^i \right\|_{(\ell^r \pi (\ell^q))'} = \left\| \sum_{i=1}^{n} \left( \sum_{j=1}^{m} h_{j}r_{ij}b_{ij} \right) e^i \right\|_{(\ell^r \pi (\ell^q))''}
\]
\[
= \left\| \sum_{i=1}^{n} \left( \sum_{j=1}^{m} h_{j}r_{ij}b_{ij} \right) e^i \right\|_{\ell^r}
\]
\[
= \left( \sum_{i=1}^{n} \sum_{j=1}^{m} h_{j}r_{ij}b_{ij} \right) \frac{1}{s'_{r_q}}
\]
\[
= \left( \sum_{i=1}^{n} \frac{1}{s'_{r_q}} \sum_{j=1}^{m} h_{j}r_{ij} \right) \frac{1}{s'_{r_q}}.
\]

Hence, (c) $\Leftrightarrow$ Proposition 5.2 (c).

(d) $\Leftrightarrow$ Proposition 5.2 (d) holds as
\[
\sum_{j=1}^{m} h_{j}r_{j}b_{hj} = \frac{1}{n} \sum_{j=1}^{m} h_{j}r_{j}.
\]

Finally we show how the matrix of $T$ must looks for $T$ can be strongly factored through the Cesàro operator.

**Proposition 5.4.** Let $h \in \ell^s_{pr}$ and suppose that $h_1 \neq 0$. The following statements are equivalent:

(a) $T$ factors strongly through $\mathcal{C}$ and $M_h$.
Then, from Proposition 5.3, (a) holds. □

(b) \( a_{ij} = 0 \) for \( j > i \), \( a_{ij} = \frac{h_{ja_1}}{h_1} \) for \( j \leq i \) and \( (ia_1) \in \ell^{r,q} \).

(c) The matrix of \( T \) looks as

\[
\begin{pmatrix}
    h_1\alpha_1 & 0 & 0 & 0 & 0 & \cdots \\
    h_1\alpha_2 & h_2\alpha_2 & 0 & 0 & 0 & \cdots \\
    h_1\alpha_3 & h_2\alpha_3 & h_3\alpha_3 & 0 & 0 & \cdots \\
    h_1\alpha_4 & h_2\alpha_4 & h_3\alpha_4 & h_4\alpha_4 & 0 & \cdots \\
    h_1\alpha_5 & h_2\alpha_5 & h_3\alpha_5 & h_4\alpha_5 & h_5\alpha_5 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

where \( (\alpha_n) \in \mathbb{R}^N \) is such that \((na_n) \in \ell^{r,q}\).

Proof. (a) \( \Rightarrow \) (b) From Proposition 5.3 there exists \( g \in \ell^{r,q} \) such that

\[
a_{ij} = \begin{cases} \frac{g_{ih_j}}{i} & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}
\]

Then \( a_{ii} = \frac{a_{ih_i}}{i} \) for all \( i \) and so \( a_{ij} = \frac{h_{ja_1}}{h_1} \) for every \( j \leq i \). Also note that \((ia_1) = h_1g \in \ell^{r,q}\).

(b) \( \Rightarrow \) (c) Taking \((\alpha_n) = \left( \frac{a_{ih}}{h} \right)\) we have that \( h_j\alpha_i = \frac{h_{ja_1}}{h_1} = a_{ij} \) for every \( j \leq i \) and \((na_n) = \frac{1}{h_1}(na_{n1}) \in \ell^{r,q}\).

(c) \( \Rightarrow \) (a) Taking \( g = (i\alpha_i) \in \ell^{r,q} \) it follows that

\[
a_{ij} = \begin{cases} h_j\alpha_i = \frac{g_{ih_j}}{i} & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}
\]

Then, from Proposition 5.3 (a) holds. □

If \( T \) factors strongly through \( \mathcal{C} \) and \( M_h \) then there exists \( h_j \neq 0 \) as \( T \) is non trivial. So, given \( 0 \neq h \in \ell^{r,p} \) and denoting \( j_0 = \min \{ j \in \mathbb{N} : h_j \neq 0 \} \), similarly to Proposition 5.4 we have that \( T \) factors strongly through \( \mathcal{C} \) and \( M_h \) if and only if its matrix looks as

\[
\begin{pmatrix}
    0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & \cdots & 0 & h_{j_0}\alpha_1 & 0 & 0 & \cdots \\
    0 & \cdots & 0 & h_{j_0}\alpha_2 & h_{j_0+1}\alpha_2 & 0 & \cdots \\
    0 & \cdots & 0 & h_{j_0}\alpha_3 & h_{j_0+1}\alpha_3 & h_{j_0+2}\alpha_3 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

for some \((\alpha_n) \in \mathbb{R}^N \) such that \((na_n) \in \ell^{r,q} \) (note that the element \( h_{j_0}\alpha_1 \) is positioned at the \( j_0 \)-th row and the \( j_0 \)-th column of the matrix).

6. Domination by basis operators and representing operators

As a result of the active research in several branches of the Harmonic Analysis, a lot of information is known about weighted norm inequalities for classical operators on weighted Banach function spaces, mainly regarding weighted \( L^p \) and Lorentz spaces. The bibliography on the subject is extremely broad; we refer the reader to [7] for the classical inequalities, and
Definition 6.1. Let $X(\mu)$ be a Banach function space over $\mu$ and $\ell$ a sequence space over the counting measure $c$ on $\mathbb{N}$. Let $B$ be a Schauder basis of a Banach function space $L(\mu)$ and suppose that the basic coefficients of the functions of $L(\mu)$ are in a sequence space $\Lambda$ defined as the Banach lattice given generated by an unconditional basis of a Banach space. Consider an operator $T : X(\mu) \to \ell$. We will say that $T$ is a representing operator for $X(\mu)$ (with respect to $\mathcal{F}$) if it is an injective two-sides-diagonal transformation of the basis operator $\mathcal{F}$.

Thus, technically a representing operator is an injective map such that there are a sequence $g = (g_j) \in (\Lambda)^\ell$ with $g_j \neq 0$ for all $j \in \mathbb{N}$ and a function $h \in X^L$, $h \neq 0$ $\mu$-a.e., such that for every $x \in X(\mu_1)$, the sequence $T(x) = (\beta(x)_j) \in \ell$ can be written as

$$\beta(x)_j = P_j \circ T(x) = g_j \mathcal{F}(hx) = g_j \alpha_j(hx).$$

That is, for the elements $y \in h \cdot X(\mu) \subseteq L(\mu)$ we have that

$$\alpha_j(y) = \mathcal{F}(y) = g_j^{-1} \beta(h^{-1}y)_j.$$

Equivalently, for each $x \in X(\mu)$, there is a sequence $(\beta_j) \in \ell$ such that

$$x = T^{-1}((\beta_j)) = h^{-1} \mathcal{F}^{-1}((g_j^{-1} \beta_j)).$$

Example 6.2.

(i) An easy example of the above introduced notion is the so called generalized Fourier series. Consider $p = 2$, an interval $I$ of the real line, the space $L^2(I)$ endowed with Lebesgue measure $dx$ and a weight function $w : I \to \mathbb{R}^+$, $w > 0$. Note that the multiplication operator $M_{w^{1/2}} : L^2(wdx) \to L^2(I)$ defines an isometry. Take a sequence of functions $(\phi_n)_n$ belonging to $L^2(wdx)$ and such that the associated sequence $(b_n)_n$, where $b_n = w^{1/2} \phi_n$ for all $n$ defines an orthonormal basis $B$ in $L^2(I)$, that is, it is orthogonal, norm one and complete. Note that this is equivalent to say that it defines an orthonormal basis in the weighted space $L^2(wdx)$. Consider the Fourier operator $\mathcal{F}_B$ associated to the basis $B$ of $L^2(I)$. Then the operator $T : L^2(wdx) \to \ell^2$ given by $T := id_{\ell^2} \circ \mathcal{F}_B \circ M_{w^{1/2}}$ is a representing operator for $L^2(wdx)$.

Concrete examples of this situation are given by classical orthogonal basis of polynomials in weighted $L^2$-spaces. For example, for the trivial case of the weight equal to
1 and the space $L^2[-1, 1]$, we can define the functions $\phi_n$ to be the Legendre polynomials, that are solutions to the Sturm-Liouville problem and define the corresponding Fourier-Legendre series. Other non trivial cases also for $I = (-1, 1)$ are given by the weight functions $w(x) = (1 - x^2)^{-1/2}(x)$ and $w(x) = (1 - x^2)^{1/2}(x)$ and the Chebyshev polynomial of the first and second kinds, respectively. Laguerre polynomials give other example for $I = (0, \infty)$ and weight function $w(x) = e^{-x}$.

(ii) Take a function $0 < h \in X(\mu)^{L^2(\mu)}$ and consider a sequence $0 < \lambda = (\lambda_i) \in \ell^2$. Let us write $c$ for the counting measure in $\mathbb{N}$. Consider the space $\ell^1(\lambda c) = \{ (\tau_i) : (\lambda_i \tau_i) \in \ell^1 \}$ with the corresponding norm $\| (\tau_i) \|_{\ell^1(\lambda c)} = \sum |\lambda_i \tau_i|$. Then we have that $\ell^2 \hookrightarrow \ell^1(\lambda c)$, and so the space of multiplication operators $(\ell^2)^{\ell^1(\lambda c)}$ is not trivial. A direct computation shows also that $(\ell^2)^{\ell^1(\lambda c)} = \ell^2(\lambda^2 c)$. Then, for every $\tau = (\tau_i) \in \ell^2(\lambda^2 c)$ with $\tau_i \neq 0$ for all $i \in \mathbb{N}$ we have that the operator $T : X(\mu) \rightarrow \ell^1(\lambda c)$ given by $T(\cdot) = \tau F(h \cdot)$ is a representing operator for the space $X(\mu)$.

Let $J$ be a finite subset of $\mathbb{N}$, and write $P_J : \ell \rightarrow \ell$ for the standard projection on the subspace generated by the elements of $B$ with subindexes in $J$. If $T : X(\mu) \rightarrow \ell$ is an operator, consider the net $\{ P_J \circ T : T_J : \mathbb{N} \supset J \text{ finite} \}$, where the order is given by the inclusion of the set of subindexes, that is $P_J \circ T \leq P_J' \circ T$ if and only if $J \subseteq J'$. By definition, 

$$T = \lim_B P_B \circ T$$

as a pointwise limit. In what follow we will characterize representing operators in terms of inequalities using this approximation procedure and a compactness argument. Thus, considering the basic (biorthogonal) functionals $b_i' \in L(\mu)'$, $i \in J$, associated to the basis of $L(\mu)$ that defines the Fourier operator that we are considering, we have

$$P_J(x) := \sum_{j \in J} \langle f, b_j' \rangle e_j, \quad f \in L^p(\mu).$$

Fix a function $h \in (X(\mu))^X(\mu)$ and suppose that $\Lambda^\ell$ is non-trivial. Assume that the conditions are given in order to obtain that $(\Lambda)_{\ell^\ell} = (\ell^\ell)^{\Lambda^\ell} = (\Lambda^\ell)^*$. The domination inequality that must be considered in this case is given by the following expression.

$$\sum_{i=1}^n \int P_J \circ T(x_i)y_i' \, dc \leq \left\| \left( \sum_{i=1}^n \sum_{j \in J} \langle h x_i, b_i' \rangle (e_j, y_i') \right) \right\|_{\Lambda^\ell}$$

$$= \sup_{g \in B_{\Lambda^\ell}} \left( \sum_{i=1}^n \sum_{j \in J} \langle h x_i, b_i' \rangle (e_j, y_i') \right),$$

that is, we are considering the sequence $\left( \sum_{i=1}^n \langle h x_i, b_i' \rangle (y_i') \right)_{j \in J} \in \Lambda^\ell$ as the functional of the dual of $(\Lambda)_{\ell^\ell}$ given by

$$\left( \sum_{i=1}^n \sum_{j \in J} \langle h x_i, b_i' \rangle (e_j, y_i') \right) : \Lambda^\ell \rightarrow \mathbb{R}.$$
After taking into account the particular descriptions of the elements of the spaces involved, we get the equivalent expression for the inequality
\[
\sum_{i=1}^{n} \sum_{j \in J} \langle T(x_i), e_j \rangle \langle e_j, ((y'_i)_j) \rangle
\leq \sup_{g \in B_{\Lambda'}^\ell} \left( \sum_{i=1}^{n} \sum_{j \in J} \langle hx_i, b'_j \rangle \langle e_j, (g_j(y'_i)_j) \rangle \right) = \sup_{g \in B_{\Lambda'}^\ell} \left( \sum_{i=1}^{n} \sum_{j \in J} \alpha_j(hx_i) g_j(y'_i)_j \right),
\]
and so the initial inequality is equivalent to the following one,
\[
\sum_{i=1}^{n} \sum_{j \in J} T(x_i)_j (y'_i)_j \leq \sup_{\ell \in B_{\Lambda'}^\ell} \left( \sum_{i=1}^{n} \sum_{j \in J} \alpha_j(hx_i) g_j(y'_i)_j \right),
\]
where \(\alpha_j(hx_i) = \int hx_i b'_j d\mu, j \in J\), are the \(j\)-th Fourier coefficients of the function \(hx_i\) associated to the basis \(B\).

Thus, the assumptions on the properties of \((X(\mu))^L(\mu)\) and \(\Lambda^\ell\) provides the following

**Theorem 6.3.** Suppose that \(\Lambda'\) and \(\ell\) satisfies that \(\Lambda'\pi^\ell\) is saturated and \((\Lambda'\pi^\ell)^* = (\Lambda')^\ell\), and let \(h\) be a measurable function such that \(0 < |h| \in (X(\mu))^L(\mu)\). The following statements are equivalent for an operator \(T : X(\mu) \to \ell\).

(i) For every finite set \(J \subseteq \mathbb{N}\) the inequality
\[
\sum_{i=1}^{n} \sum_{j \in J} T(x_i)_j (y'_i)_j \leq C \sup_{\ell \in B_{\Lambda'}^\ell} \left( \sum_{i=1}^{n} \sum_{j \in J} \alpha_j(hx_i) g_j(y'_i)_j \right),
\]
holds for every \(x_1, ..., x_n \in X_1\) and \(y'_1, ..., y'_n \in \ell'\).

(ii) \(T\) is a representing operator with respect to \(\mathcal{F}\), that is, there is a sequence \(g \in \Lambda^\ell\) such that \(T(x))_j = g_j \cdot \alpha_j(hx)\) for all \(x \in X(\mu)\) and \(j \in \mathbb{N}\). In other words, \(T\) factors through \(\mathcal{F}\) as
\[
\begin{align*}
X(\mu) & \xrightarrow{T} \ell \\
L(\mu) \xrightarrow{M_g} \Lambda.
\end{align*}
\]

**Proof.** Let us see that (i) implies (ii). We can assume without loss of generality that \(C = 1\). Note that as a consequence of Remark 3.2, the requirements on \(\Lambda\) and \(\ell\) provides the conditions on these spaces for applying Corollary 3.2. By the computations above, we obtain that for each finite set \(J\) we have a norm one sequence \(g_J \in \Lambda^\ell\) satisfying that
\[
P_J \circ T(x) = g_J \cdot P_J \circ \mathcal{F}(hx).
\]
Consider the net \(\mathcal{N} := \{g_J : J \subseteq \mathbb{N} finite\}\), where the order is given by the inclusion of the finite sets used for the subindexes. We can assume without loss of generality that the support of each function \(g_J\) is in \(J\), that is, the coefficients \((g_J)_k\) of the sequence \(g_J\) are 0 for \(k \notin J\).
Since all the functions of the net are in the unit ball and due to the product compatibility of the pair defined by \(\Lambda'\) and \(\ell\), we have that the net is included in the weak* compact set \(B_{\Lambda'}\).
Therefore, it has a convergent subnet \(\mathcal{N}_0\), that is, there is a sequence \(g_0 \in B_{\Lambda'}\) such that
\[
\lim_{\eta \in \mathcal{N}_0} g_{\eta} = g_0
\]
in the weak* topology given by the dual pair \( \langle \Lambda', \pi \ell, \Lambda \rangle \).

Note now that for a fixed \( x \in X(\mu) \), due to the fact that we are assuming that \( \ell \) has an unconditional basis with associated projections \( P_J \), we have
\[
\lim_{J \in \mathcal{N}_{\text{finite}}} P_J \circ T(x) = T(x).
\]

Then,
\[
T(x) = \lim_{J \in \mathcal{N}_{\text{finite}}} P_J \circ T(x) = \lim_{\eta \in \mathcal{N}_\eta} P_\eta \circ T(x) = \lim_{\eta \in \mathcal{N}_\eta} g_\eta \cdot \mathcal{F}(hx) = g_0 \cdot \mathcal{F}(hx).
\]

This gives (ii) and finishes the proof, since the converse holds by a direct computation. \( \square \)

Let us provide an example. Consider again Example 6.2(ii), and recall that \((\ell^2)_{\ell^1(\lambda c)} = \ell^2(\lambda^2 c)\). Theorem 6.3 gives that an injective operator \( T : X(\mu) \to \ell^1(\lambda c) \) is a representing operator by means of the Fourier operator if and only if for every finite set \( J \subseteq \mathbb{N} \) the inequality
\[
\sum_{i=1}^n \sum_{j \in J} T(x_i)_j (y'_i)_j \leq C \sup_{g \in B_{\ell^2(\lambda^2 c)}} \left( \sum_{i=1}^n \sum_{j \in J} \alpha_j(hx_i)_j g_j(y'_i)_j \right),
\]
holds for every \( x_1, \ldots, x_n \in X(\mu) \) and \( y'_1, \ldots, y'_n \in \ell^1(\lambda c)' \).

**Remark 6.4.** Let us give some sufficient conditions for the product sequence space appearing in Theorem 6.3 to satisfy what is needed. The product compatibility of the pair \( \ell^\prime \) and \( \ell \) means that
\[
(\ell^\prime)_{\ell} = (\ell^\prime)_{\ell^\prime} = (\ell^\prime \pi \ell^\prime)^*.
\]
For example, if \( \ell^\prime \) is \( p \)-convex we have that \( \ell^\prime \pi \ell^\prime \) is saturated and so, a Banach function space (see Proposition 2.2 in [15]). Moreover, the quoted result provides also the equality (under the assumption of saturation of the product)
\[
(\ell^\prime \pi \ell^\prime)' = (\ell^\prime)'^\ell.
\]

Consequently, if the product is order continuous, we get the desired result. Conditions under which this space is order continuous are given in Proposition 5.3 in [3]: for example, if the norm of the product is equivalent to
\[
\| \lambda \|_{\pi} \sim \inf \left\{ \| \eta \|_{\ell^\prime}, \| \gamma \|_{\ell^\prime} : |\lambda| = \eta \cdot \gamma, \eta \in \ell^\prime, \gamma \in \ell^\prime \right\},
\]
the space is order continuous if \( \ell^\prime \) is assumed to be order continuous (recall that \( p > 1 \) and so \( \ell^\prime \) is order continuous too). The formula above for the product space works for example if \( \ell \) is \( p^\prime \)-concave, since this implies that \( \ell^\prime \) is \( p \)-convex that together with the \( p^\prime \)-concavity of \( \ell^\prime \) provides the result. Concrete examples for \( \ell^p \) spaces has been given in Example 6.2.

7. Operators associated to trigonometric series

Relevant historical examples are the ones associated to the Fourier series and the corresponding Fourier coefficients. We finish the paper by explicitly writing the results presented previously in this setting. We will write \( \hat{x}(\cdot) \) for the \( i \)-th Fourier (real) coefficients of the function \( x \) with indexes in the set \( \mathbb{Z} \), writing the coefficients \( a_n \) associated to \( \text{cos} \) functions as \( \hat{x}(i) \) with positive \( i \) and the coefficients \( b_n \) for the functions \( \text{sin} \) as \( \hat{x}(i) \) with negative \( i \).
• Due to the Hausdorff-Young inequality, we know that for $1 < p \leq 2$, the Fourier transform $\mathcal{F}_p$: sending $L^p[-\pi, \pi] \to \ell^p$, that assigns to each function the sequence of its Fourier coefficients is well-defined and continuous. The Fourier transform is defined as $\mathcal{F}_2 : L^2 \to \ell^2$. Suppose that we want to check if a particular operator $\mathcal{G}_2 : L^2[-\pi, \pi] \to \ell^2$ can be extended to $L^p[-\pi, \pi]$ through $\mathcal{F}_p$. That is, is there a factorization for $\mathcal{G}_2$ as

$$L^2[-\pi, \pi] \xrightarrow{\mathcal{G}_2} \ell^2$$

for the operator $\mathcal{G}_2$ for some multiplication operator given by a sequence $\lambda$.

We have shown that this is equivalent to the following inequalities to hold for the operator $\mathcal{G}_2$. For each $x_1, \ldots, x_n \in L^2[-\pi, \pi]$ and $\lambda_i \in \ell^2$,

$${\sum_{k=1}^{\infty} \sum_{i=1}^{n} (\mathcal{G}_2(x_i))(\lambda_i)_k} \leq C\| (\sum_{i=1}^{n} |\hat{x}_i(k)(\lambda_i)_k|^{q})^{1/r} \|_{\ell^r} = C\left(\sum_{k=1}^{\infty} \| \hat{x}_i(k)(\lambda_i)_k \|_{\ell^r} \right)^{1/r}.$$

• For $1 < p \leq 2$ again, Kellogg proved an improvement of the Hausdorff-Young inequality, that assures that the corresponding Fourier coefficients of the functions in $L^p$ can be found in the smaller mixed norm space $L^{p,q}_{\ell} \subseteq \ell^p$. Fix $1 \leq p, q \leq \infty$. The mixed norm sequence space $L^{p,q}$ was defined in $\mathbb{R}$ as the space of sequences $\lambda = (\lambda_k)_{k=1}^{\infty}$ such that

$$\| \lambda \| = \left( \sum_{m=\infty}^{\infty} \left( \sum_{k \in I(m)} |\lambda_k|^p \right)^{q/p} \right)^{1/q} < \infty,$$

where $I(m) = \{ k \in \mathbb{Z} : 2^{m-1} \leq k \leq 2^m \}$ if $m > 0$, $I(0) = \{ 0 \}$ and $I(m) = \{ k \in \mathbb{Z} : -2^{-m} \leq k \leq -2^{-m-1} \}$ if $m < 0$. It is easy to see that $L^{p,2}_{\ell} \subseteq \ell^p$, and so we have a factorization for the Fourier map as

$$L^p[-\pi, \pi] \xrightarrow{\mathcal{F}_p} \ell^p$$

In Theorem 1 of $[5]$, it is proved that the space of multiplication operators (multipliers) from $L^{p,2}_{\ell}$ to $\ell^p$ can in fact be identified with $\ell^\infty$. Consequently, our results imply that for every finite set $J \subseteq \mathbb{Z}$ the inequality

$$\sum_{i=1}^{n} \sum_{j \in J} (\mathcal{F}_p x_i)_j (\lambda_i')_j \leq C \sup_{g \in B_{L^\infty}} \left( \sum_{i=1}^{n} \sum_{j \in J} |\hat{x}_i(j) g_j(\lambda_i')_j| \right),$$

holds for every $x_1, \ldots, x_n \in X_1$ and $\lambda_1', \ldots, \lambda_n' \in \ell^p$, what is obvious. However, note that this is essentially a characterization, since any other operator $\mathcal{G}_p$ from $L^p$ and having
values in a sequence space \( \ell \) such that \( (L^p', 2)^\ell = \ell^\infty \) satisfying these inequalities has to be of the form \( g \cdot K_p \) for a certain sequence \( g \in \ell^\infty \).

- The Hardy-Littlewood inequality, also for \( 1 < p \leq 2 \), provides an example of an operator \( \mathcal{H}_p \) sending the Fourier coefficients of the functions in \( L^p \) to a weighted \( \ell^p \) space. For \( 1 < p < 2 \), consider the weighted sequence space \( \ell^p(W) \), where the weight \( W \) is given by \( W = (W_n) = (1/(n+1)^{2-p}) \). The Hardy-Littlewood inequality can be understood as the fact that the Fourier operator can be defined as \( \mathcal{H}_p : L^p[-\pi, \pi] \to \ell^p(W) \) (see [1] S.2, in particular Theorem B). Note that the multiplication operator \( M_\gamma : \ell^p(W) \to \ell^p \) given by the sequence \( \gamma = \left( (1/(n+1)^{\frac{2-p}{p}}) \right) \) defines an isometry. Therefore, the factorization scheme

\[
L^p[-\pi, \pi] \xrightarrow{\gamma \cdot \mathcal{H}_p} \ell^p \xrightarrow{M_\gamma} \ell^p(W)
\]

provides other example of the situation we are describing. Indeed, for every multiplication operator \( \tau \) for \( \tau \in (\ell^p(W))^{\ell^p} \) we can give an operator \( \tau \cdot \mathcal{H}_p \) satisfying this factorization. Our results implies the class of all these operators is characterized in the following way: if \( T : L^p[-\pi, \pi] \to \ell^p \) satisfies the inequalities

\[
\sum_{i=1}^{n} \sum_{j \in J} (T(x_i))_j (\lambda'_i)_j \leq C \sup_{g \in B(\ell^p(W))^{\ell^p}} \left( \sum_{i=1}^{n} \sum_{j \in J} x_i(j) g_j (\lambda'_i)_j \right),
\]

for each finite subset \( J \subset \mathbb{Z} \), for every \( x_1, \ldots, x_n \in L^p[-2\pi, 2\pi] \) and \( \lambda'_1, \ldots, \lambda'_n \in \ell^{p'} \), then it has a factorization as the one above for a certain \( \tau \in (\ell^p(W))^{\ell^p} \).

- Let us recall Example 6.21). A representing operator \( T : L^2(wdx) \to \ell^2 \) associated to a weight function \( w \) and an orthogonal basis \( \mathcal{B} \) with respect to the corresponding weight function was considered. It allowed a factorization as

\[
L^2(wdx) \xrightarrow{T} \ell^2 \xrightarrow{id} \ell^2
\]

for a given constant \( C > 0 \) and for each finite subset \( J \subset \mathbb{N} \), for every \( x_1, \ldots, x_n \in L^2(wdx) \) and \( \lambda'_1, \ldots, \lambda'_n \in \ell^2 \).
References

[1] J. M. Ash, S. Tikhonov and J. Tung, *Wiener’s positive Fourier coefficients theorem in variants of Lp spaces*, Michigan Math. J. 59 (2010), no. 1, 143–151.

[2] J. J. Benedetto and H. P. Heinig, *Weighted Fourier inequalities: new proofs and generalizations*, J. Fourier Anal. Appl. 9 (2003), no. 1, 1–37.

[3] J. M. Calabuig, O. Delgado and E. A. Sánchez Pérez *Generalized perfect spaces*, Indag. Math. 19 (2008), 359–378.

[4] D. Cruz-Uribe, J. M. Martell and C. Pérez, *Sharp weighted estimates for classical operators*, Adv. Math. 229 (2012), no. 1, 408-441.

[5] O. Delgado and E. A. Sánchez Pérez, *Summability properties for multiplication operators on Banach function spaces*, Integr. Equ. Oper. Theory 66 (2010), 197–214.

[6] O. Delgado and E. A. Sánchez Pérez, *Strong factorizations between couples of operators on Banach function spaces*, J. Convex Anal. 20 (2013), 599–616.

[7] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

[8] C. N. Kellogg, *An extension of the Hausdorff-Young theorem*, Michigan Math. J. 18 (1971), 121–127.

[9] P. Kolwicz, K. Leśnik and L. Maligranda, *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. 266 (2014), no. 2, 616–659.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, vol. II*, Springer-Verlag, Berlin, 1979.

[11] L. Maligranda and L. E. Persson, *Generalized duality of some Banach function spaces*, Indag. Math. 51 (1989), 323–338.

[12] H. Mhaskar and S. Tikhonov, *Wiener type theorems for Jacobi series with nonnegative coefficients*, Proc. Amer. Math. Soc. 140 (2012), no. 3, 977–986.

[13] C. Pérez, *Sharp Lp-weighted Sobolev inequalities*, Ann. Inst. Fourier 45 (1995), no. 3, 809-824.

[14] A. Pietsch, *Operator Ideals*, North Holland, Amsterdam, 1980.

[15] E. A. Sánchez Pérez, *Factorization theorems for multiplication operators on Banach function spaces*, Integr. Equ. Oper. Theory 80.1 117-135 (2014).

[16] A. R. Schep, *Products and factors of Banach function spaces*, Positivity 14 (2010), no. 2, 301-319.

[17] A. C. Zaanen, *Integration*, 2nd rev. ed. North Holland, Amsterdam, 1967.

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