About the (Hahn) classical character of 2-orthogonal solutions of two families of differential equations of third order

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Abstract

Considering a differential operator of third order that does not increase the degree of polynomials, we analyse some properties of elements of the dual space of 2-orthogonal polynomial eigenfunctions. In two classes of such generic operator, we prove that a 2-orthogonal polynomial solution fulfils Hahn’s property.

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1 Basic definitions and notation

Let \( \mathcal{P} \) be the vector space of polynomials with coefficients in \( \mathbb{C} \) and let \( \mathcal{P}' \) be its topological dual space. We denote by \( \langle u, p \rangle \) the action of the form or linear functional \( u \in \mathcal{P}' \) on \( p \in \mathcal{P} \). In particular, \( \langle u, x^n \rangle := (u)_n, n \geq 0 \) represent the moments of \( u \). In the following, we will call polynomial sequence (PS) to any sequence \( \{P_n\}_{n \geq 0} \) such that deg \( P_n = n \), \( \forall n \geq 0 \). We will also call monic polynomial sequence (MPS) to a PS so that all polynomials have leading coefficient equal to one.

If \( \{P_n\}_{n \geq 0} \) is a MPS, there exists a unique sequence \( \{u_n\}_{n \geq 0}, u_n \in \mathcal{P}' \), called the dual sequence of \( \{P_n\}_{n \geq 0} \), such that,

\[
\langle u_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0. \tag{1.1}
\]

On the other hand, given a MPS \( \{P_n\}_{n \geq 0} \), the expansion of \( xP_{n+1}(x) \), defines sequences in \( \mathbb{C} \), \( \{\beta_n\}_{n \geq 0} \) and \( \{\chi_{n,\nu}\}_{0 \leq \nu \leq n, n \geq 0} \), such that

\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \tag{1.2}
\]

\[
xP_{n+1}(x) = P_{n+2}(x) + \beta_{n+1}P_{n+1}(x) + \sum_{\nu=0}^{n} \chi_{n,\nu} P_{\nu}(x). \tag{1.3}
\]
This relation is usually called the structure relation of \( \{P_n\}_{n \geq 0} \), and \( \{\beta_n\}_{n \geq 0} \) and \( \{\chi_{n,\nu}\}_{0 \leq \nu \leq n, n \geq 0} \) are called the structure coefficients (SCs) [15].

In this paper, we will deal with 2-orthogonal MPSs, hence it is important to mention in this section the definition of \( d \)-orthogonal MPSs (where \( d \) is a positive integer) along with basic definitions, in particular regarding the differential operator \( D \).

**Definition 1.1.** [9, 17, 24] Given \( \Gamma_1, \Gamma_2, \ldots, \Gamma_d \in \mathcal{P}' \), \( d \geq 1 \), the polynomial sequence \( \{P_n\}_{n \geq 0} \) is called \( d \)-orthogonal polynomial sequence (\( d \)-OPS) with respect to \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) if it fulfils

\[
\langle \Gamma^\alpha, P_m P_n \rangle = 0, \quad n \geq md + \alpha, \quad m \geq 0, \quad (1.4)
\]

\[
\langle \Gamma^\alpha, P_m P_{md+\alpha-1} \rangle \neq 0, \quad m \geq 0, \quad (1.5)
\]

for each integer \( \alpha = 1, \ldots, d \).

**Lemma 1.2.** [16] For each \( u \in \mathcal{P}' \) and each \( m \geq 1 \), the two following propositions are equivalent.

a) \( \langle u, P_{m-1} \rangle \neq 0, \quad \langle u, P_n \rangle = 0, \quad n \geq m \).

b) \( \exists \lambda_\nu \in \mathbb{C}, \quad 0 \leq \nu \leq m - 1, \quad \lambda_{m-1} \neq 0 \) such that \( u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu \).

The conditions (1.4) are called the \( d \)-orthogonality conditions and the conditions (1.5) are called the regularity conditions. In this case, the functional \( \Gamma \), of dimension \( d \), is said regular.

The \( d \)-dimensional functional \( \Gamma \) is not unique. Nevertheless, from Lemma [12] we have:

\[
\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha u_\nu, \quad \lambda_{\alpha-1}^\alpha \neq 0, \quad 1 \leq \alpha \leq d.
\]

Therefore, since \( U = (u_0, \ldots, u_{d-1}) \) is unique, we use to consider the canonical functional of dimension \( d \), \( U = (u_0, \ldots, u_{d-1}) \), saying that \( \{P_n\}_{n \geq 0} \) is \( d \)-OPS (\( d \geq 1 \)) with respect to \( U = (u_0, \ldots, u_{d-1}) \) if

\[
\langle u_\nu, P_m P_n \rangle = 0, \quad n \geq md + \nu + 1, \quad m \geq 0,
\]

\[
\langle u_\nu, P_m P_{md+\nu} \rangle \neq 0, \quad m \geq 0,
\]

for each integer \( \nu = 0, 1, \ldots, d-1 \). It is important to remark that when \( d = 1 \) we meet again the notion of regular orthogonality. Furthermore, the \( d \)-orthogonality corresponds to the generalisation of the well-known recurrence relation fulfilled by the orthogonal polynomials, as the next Theorem recalls.

**Theorem 1.3.** [17] Let \( \{P_n\}_{n \geq 0} \) be a MPS. The following assertions are equivalent:

a) \( \{P_n\}_{n \geq 0} \) is \( d \)-orthogonal with respect to \( U = (u_0, \ldots, u_{d-1}) \).
b) \( \{ P_n \}_{n \geq 0} \) satisfies a \((d + 1)\)-order recurrence relation \((d \geq 1)\):

\[
P_{m+d+1}(x) = (x - \beta_{m+d}) P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \quad m \geq 0,
\]

with initial conditions

\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0 \quad \text{and if} \quad d \geq 2:
\]

\[
P_n(x) = (x - \beta_{n-1}) P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x), \quad 2 \leq n \leq d,
\]

and regularity conditions: \( \gamma_{m+1}^0 \neq 0, \quad m \geq 0 \).

The classical orthogonal polynomials fulfil many important properties (e.g. [2]) and they are characterised by the Hahn’s property, that is to say, the monic polynomial sequence obtained through the standard differentiation \( D \),

\[
\{(n+1)^{-1} DP_{n+1}(x)\}_{n \geq 0},
\]

is also orthogonal. The \( d \)-orthogonal MPSs fulfilling this property are also called classical \( d \)-orthogonal MPSs in the Hahn’s sense.

**Definition 1.4.** [9] A PS \( \{ P_n \}_{n \geq 0} \) is \( d \)-symmetric if it fulfils

\[
P_n(\xi_k x) = \xi_k^n P_n(x), \quad n \geq 0, \quad k = 1, 2, \ldots, d,
\]

where \( \xi_k = \exp \left( \frac{2ik\pi}{d+1} \right) \), \( k = 1, \ldots, d \), \( \xi_d^{d+1} = 1 \).

If \( d = 1 \), then \( \xi_1 = -1 \) and we meet the definition of a symmetric PS in which we have the following property \( P_n(-x) = (-1)^n P_n(x), \quad n \geq 0 \).

Finally, given \( \varpi \in \mathcal{P} \) and \( u \in \mathcal{P}' \), the form \( \varpi u \), called the left-multiplication of \( u \) by the polynomial \( \varpi \), is defined by

\[
\langle \varpi u, p \rangle = \langle u, \varpi p \rangle, \quad \forall p \in \mathcal{P}, \quad (1.6)
\]

and the transpose of the derivative operator on \( \mathcal{P} \) defined by \( p \rightarrow (Dp)(x) = p'(x) \), is the following (cf. [15]):

\[
u \rightarrow Du : \quad \langle Du, p \rangle = -\langle u, p' \rangle, \quad \forall p \in \mathcal{P}, \quad (1.7)
\]

so that we can retain the usual rule of the derivative of a product when applied to the left-multiplication of a form by a polynomial. Indeed, it is easily established that

\[
D(pu) = p'u + pD(u). \quad (1.8)
\]
2 Differential operators on $\mathcal{P}$ and technical identities

In this section, we summarise the operational point of view proposed in [22] that will be followed along the text. The notation here recalled help us in dealing with an operator $J$ that we may consider in the search of polynomial solutions of many different differential identities.

Given a sequence of polynomials $\{a_\nu(x)\}_{\nu \geq 0}$, let us consider the following linear mapping $J : \mathcal{P} \to \mathcal{P}$ (cf. [20], [23]).

$$J = \sum_{\nu \geq 0} \frac{a_\nu(x)}{\nu!} D^\nu, \quad \deg a_\nu \leq \nu, \quad \nu \geq 0. \quad (2.1)$$

Expanding $a_\nu(x)$ as follows:

$$a_\nu(x) = \sum_{i=0}^{\nu} a^i_\nu x^i,$$

and recalling that $D^\nu (\xi^n) (x) = \frac{n!}{(n-\nu)!} x^{n-\nu}$, we get the next identities about $J$:

$$J (\xi^n) (x) = \sum_{\nu=0}^{n} a_\nu(x) \binom{n}{\nu} x^{n-\nu} \quad (2.2)$$

$$J (\xi^n) (x) = \sum_{\tau=0}^{n} \left( \sum_{\nu=0}^{\tau} \binom{n}{n-\nu} a^{[n-\nu]}_{\tau-\nu} \right) x^\tau, \quad n \geq 0. \quad (2.3)$$

Most in particular, a linear mapping $J$ is an isomorphism if and only if

$$\deg (J (\xi^n) (x)) = n, \quad n \geq 0, \quad \text{and} \quad J (1) (x) \neq 0. \quad (2.4)$$

**Lemma 2.1.** [22] For any linear mapping $J$, not increasing the degree, there exists a unique sequence of polynomials $\{a_n\}_{n \geq 0}$, with $\deg a_n \leq n$, so that $J$ is read as in (2.1). Further, the linear mapping $J$ is an isomorphism of $\mathcal{P}$ if and only if

$$\sum_{\mu=0}^{n} \binom{n}{\mu} a^{|\mu|}_\mu \neq 0, \quad n \geq 0. \quad (2.5)$$

In brief, an operator that does not increase the degree can always be expressed in a $J$ format. Let us now recall some useful identities regarding any operator $J$, that are obtained by duality and taking into account (1.7).

$$\langle \iota J(u), f \rangle = \langle u, J(f) \rangle, \quad u \in \mathcal{P}', \quad f \in \mathcal{P},$$

$$= \sum_{n \geq 0} \langle u, \frac{a_n(x)}{n!} f^{(n)}(x) \rangle = \sum_{n \geq 0} \frac{(-1)^n}{n!} (D^n (a_n u), f);$$

thence

$$\iota J(u) = \sum_{n \geq 0} \frac{(-1)^n}{n!} D^n (a_n u), \quad u \in \mathcal{P}'. \quad (2.6)$$

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Henceforth, we will denote the transposed operator $t^*J(u)$ by $J(u)$, since most of the forthcoming calculations will be done on $P'$. The results presented in [22] about the $J$-image of the product of two polynomials $fg$ and of the form $fu$ require the definition of the following operator $J^{(m)}$, $m \geq 0$, on $P$

$$J^{(m)} = \sum_{n \geq 0} \frac{a_{n+m}(x)}{n!} D^n$$

whose transposed operator is given by

$$J^{(m)}(u) = \sum_{n \geq 0} \frac{(-1)^n}{n!} D^n (a_{n+m}u), \quad m \geq 0. \quad (2.7)$$

**Lemma 2.2.** [22] For any $f, g \in P$, $u \in P'$, we have:

$$J(fg)(x) = \sum_{n \geq 0} J^{(n)}(f)(x) \frac{g^{(n)}(x)}{n!} = \sum_{n \geq 0} J^{(n)}(g)(x) \frac{f^{(n)}(x)}{n!}, \quad (2.8)$$

$$J(fu) = \sum_{n \geq 0} \frac{(-1)^n}{n!} f^{(n)}(x)J^{(n)}(u). \quad (2.9)$$

### 2.1 Looking at $J$ as a lowering operator of order $k$

The generic operator $J$ defined in (2.1) does not increase the degree of polynomials and can be categorised in terms of its behaviour when compared with the standard derivative of order $k$, for $k = 0, 1, 2, \ldots$, as we show next.

Let us suppose that $J$ is an operator expressed as in (2.1), and acting as the derivative of order $k$, for some non-negative integer $k$, that is, it fulfils the following conditions.

$$J\left(\xi^k\right)(x) = a_0^k \neq 0 \quad \text{and} \quad \deg\left(J\left(\xi^{n+k}\right)(x)\right) = n, \quad n \geq 0; \quad (2.10)$$

$$J\left(\xi^i\right)(x) = 0, \quad 0 \leq i \leq k - 1, \quad \text{if} \quad k \geq 1. \quad (2.11)$$

**Lemma 2.3.** An operator $J$ fulfils (2.10)-(2.11) if and only if the next set of conditions hold.

a) $a_0(x) = \cdots = a_{k-1}(x) = 0$, if $k \geq 1$;

b) $\deg(a_\nu(x)) \leq \nu - k$, $\nu \geq k$;

c)

$$\lambda^{[k]}_{n+k} := \sum_{\nu=0}^n \binom{n+k}{n+k-\nu} a^{[n+k-\nu]}_{n+\nu} \neq 0, \quad n \geq 0. \quad (2.12)$$

**Remark 2.4.** Note that in (2.12) we find $\lambda^{[k]}_k = a^{[k]}_0$. If $k = 0$, then it is assumed that $\lambda^{[0]}_0 \neq 0$, $n \geq 0$, matching (2.5), so that $J$ is an isomorphism.

If $k = 1$, then $J$ imitates the usual derivative and is commonly called a lowering operator (e.g. [14, 21]).
Given a MPS \( \{P_n\}_{n \geq 0} \) and a non-negative integer \( k \), let us define its (normalised) \( J \)-image sequence of polynomials as follows, and notate its dual sequence by \( \{\tilde{u}_n\}_{n \geq 0} \). Of course, given that \( J \) satisfies (2.10)-(2.11), the obtained polynomial sequence \( \{\tilde{P}_n\}_{n \geq 0} \) is also a MPS.

\[
\tilde{P}_n(x) = \left( \lambda_{n+k}^{[k]} \right)^{-1} J(P_{n+k}(x)), \quad n \geq 0. \tag{2.13}
\]

Moreover, the dual sequences of \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \) are related as the next Lemma points out.

**Lemma 2.5.** [22] Let us consider a MPS \( \{P_n\}_{n \geq 0} \) and an operator \( J \) of the form (2.1) fulfilling (2.10)-(2.11). Thus,

\[
J(\tilde{u}_n) = \lambda_{n+k}^{[k]} u_{n+k}.
\]

**Lemma 2.6.** [22] Let us consider a MPS \( \{P_n\}_{n \geq 0} \) and an operator \( J \) of the form (2.1) such that (2.10)-(2.11) hold. Thus,

\[
\tilde{P}_n(x) = P_n(x), \quad n \geq 0, \quad \text{if and only if} \quad J(u_n) = \lambda_{n+k}^{[k]} u_{n+k}.
\]

In brief, given a non-negative integer \( k \), we can address the general problem of finding the MPSs \( P = (P_0, P_1, \ldots)^T \) such that

\[
P = \Lambda_J^{[k]} J(P), \tag{2.14}
\]

where the matrix (with infinite dimensions) \( \Lambda_J^{[k]} \) contains de normalisation constants \( \left( \lambda_{n+k}^{[k]} \right)^{-1}, \quad n \geq 0 \), by working with the dual sequence \( u = (u_0, u_1, \ldots)^T \).

## 3 An isomorphism applied to a 2-orthogonal sequence

Throughout this paper, we consider that \( J \) is an isomorphism and \( a_\nu(x) = 0, \quad \nu \geq 4 \), thus

\[
J = a_0(x)I + a_1(x)D + \frac{a_2(x)}{2} D^2 + \frac{a_3(x)}{3!} D^3, \quad \text{where}
\]

\[
a_0(x) = a_0^{[0]}, \quad a_1(x) = a_0^{[1]} + a_1^{[1]} x, \quad a_2(x) = a_0^{[2]} + a_1^{[2]} x + a_2^{[2]} x^2,
\]

\[
a_3(x) = a_0^{[3]} + a_1^{[3]} x + a_2^{[3]} x^2 + a_3^{[3]} x^3,
\]

and we suppose that the MPS \( \{P_n\}_{n \geq 0} \) fulfil

\[
J(P_n(x)) = \lambda_n^{[0]} P_n(x), \quad \text{with} \quad \lambda_n^{[0]} \neq 0, \quad n \geq 0.
\]

Focusing on the 2-orthogonality, where a polynomial sequence fulfil specific orthogonal conditions towards the two initial elements of the dual sequence [17], we begin by recalling that a 2-orthogonal MPS can be recursively computed by means of three sequences of constants \( \{\beta_n\}_{n \geq 0}, \{\gamma_n\}_{n \geq 1} \) and \( \{\gamma_n\}_{n \geq 1} \), with \( \gamma_{n+1} \neq 0, \quad n \geq 0 \) [5, 18]. For the sake of simplicity, it has been used (e.g. [5]), the
The following notation for these structure coefficients of the 2-orthogonal polynomial sequences: \( \{\beta_n\}_{n \geq 0}, \{\alpha_n\}_{n \geq 1} \) and \( \{\gamma_n\}_{n \geq 1} \), respectively, as indicated below in (3.2)-(3.3), with \( \gamma_{n+1} \neq 0, \ n \geq 0 \).

\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad P_2(x) = (x - \beta_1)P_1(x) - \alpha_1; \quad (3.2)
\]

\[
P_{n+3}(x) = (x - \beta_{n+2})P_{n+2}(x) - \alpha_{n+2}P_{n+1}(x) - \gamma_{n+1}P_n(x). \quad (3.3)
\]

In a similar manner to the orthogonal case, its dual sequence fulfills a recurrence relation based on those structure coefficients, as follows [18].

\[
xu_n = u_{n-1} + \beta_n u_n + \alpha_{n+1}u_{n+1} + \gamma_{n+1}u_{n+2}, \text{ with } n \geq 0, \ u_{-1} = 0. \quad (3.4)
\]

Moreover, all elements of the dual sequence can be written in terms of the regular functional vector \((u_0, u_1)\). In particular, we have [18] (p. 307)

\[
u_{2n} = E_n(x)u_0 + A_{n-1}(x)u_1 \quad (3.5)
\]

\[
u_{2n+1} = B_n(x)u_0 + F_n(x)u_1 \quad (3.6)
\]

where \( \text{deg} (E_n(x)) = \text{deg} (F_n(x)) = n, \ \text{deg} (A_n(x)) \leq n, \ \text{deg} (B_n(x)) \leq n, \ E_0(x) = 1, \ A_{-1}(x) = 0, \ B_0(x) = 0, \ F_0(x) = 1. \) These polynomial coefficients fulfill the following recurrence relations [18].

\[
E_1(x) = \frac{1}{\gamma_1} (x - \beta_0); \quad A_0(x) = -\frac{\alpha_1}{\gamma_1};
\]

\[
\alpha_{2n+2}E_{n+1}(x) + E_n(x) = (x - \beta_{2n+1})B_n(x) - \gamma_{2n+2}B_{n+1}(x);
\]

\[
\gamma_{2n+3}E_{n+2}(x) - (x - \beta_{2n+2})E_{n+1}(x) = -B_n(x) - \alpha_{2n+3}B_{n+1}(x);
\]

\[
\gamma_{2n+2}F_{n+1}(x) - (x - \beta_{2n+1})F_n(x) = -A_{n-1}(x) - \alpha_{2n+2}A_n(x);
\]

\[
\alpha_{2n+3}F_{n+1}(x) + F_n(x) = (x - \beta_{2n+2})A_n(x) - \gamma_{2n+3}A_{n+1}(x), \quad n \geq 0.
\]

Recalling the definition (24.12), we have

\[
\lambda_n^{(0)} := \sum_{\nu=0}^{n} \left( \begin{array}{c} n \\ n - \nu \end{array} \right) a_{n-\nu}^{[n-\nu]} \neq 0, \ n \geq 0;
\]

so that, \( \lambda_0^{[0]} = a_0(x) = a_{0}^{[0]}; \ \lambda_1^{[0]} = a_0^{[0]} + a_1^{[1]}; \ \lambda_2^{[0]} = a_0^{[0]} + 2a_1^{[1]} + a_2^{[2]}, \) just to mention a few.

In an equivalent manner, we learn from Lemma 2.6 that \( J(u_n) = \lambda_n^{(0)}u_n \), which means

\[
D \left( -a_1(x)u_n + \frac{1}{2!}D(a_2(x)u_n) - \frac{1}{3!}D^2(a_3(x)u_n) \right) = \left( \lambda_n^{(0)} - a_0(x) \right) u_n, \ n \geq 0. \quad (3.7)
\]

Taking into account the definition of \( J \) established in (3.1), identity (2.9) allow us to expand the image of the form \( fu \), with \( f \in \mathcal{P} \) and \( u \in \mathcal{P}' \), as follows:

\[
J(fu) = f(x)J(u) - f'(x)J^{(1)}(u) + \frac{1}{2!}f''(x)J^{(2)}(u) - \frac{1}{3!}f'''(x)J^{(3)}(u). \quad (3.8)
\]
Furthermore, the forms $J^{(1)}(u)$, $J^{(2)}(u)$ and $J^{(3)}(u)$ have the following definitions that can be deduced from identity (2.7), for an operator $J$ of third order (3.1):

\[
J^{(1)}(u) = a_1(x)u - D(a_2(x)u) + \frac{1}{2!}D^2(a_3(x)u) \quad (3.9)
\]

\[
J^{(2)}(u) = a_2(x)u - D(a_3(x)u) \quad (3.10)
\]

\[
J^{(3)}(u) = a_3(x)u \quad (3.11)
\]

\[
J^{(m)}(u) = 0, \ m \geq 4. \quad (3.12)
\]

When we apply $J$ on (3.5) and (3.6) taking into account (3.8), we get for $n \geq 0$, respectively:

\[
\lambda_{2n}^{[0]}u_{2n} = E_n(x)J(u_0) - E'_n(x)J^{(1)}(u_0) + \frac{1}{2!}E''_n(x)J^{(2)}(u_0)
\]

\[
- \frac{1}{3!}F''_n(x)J^{(3)}(u_0)
\]

\[
+ A_n(x)J(u_0) - A'_n(x)J^{(1)}(u_0) + \frac{1}{2!}A''_n(x)J^{(2)}(u_0)
\]

\[
- \frac{1}{3!}A'^{(3)}(x)J^{(3)}(u_0);
\]

\[
\lambda_{2n+1}^{[0]}u_{2n+1} = B_n(x)J(u_0) - B'_n(x)J^{(1)}(u_0) + \frac{1}{2!}B''_n(x)J^{(2)}(u_0)
\]

\[
- \frac{1}{3!}B'^{(3)}(x)J^{(3)}(u_0)
\]

\[
+ F_n(x)J(u_1) - F'_n(x)J^{(1)}(u_1) + \frac{1}{2!}F''_n(x)J^{(2)}(u_1)
\]

\[
- \frac{1}{3!}F'^{(3)}(x)J^{(3)}(u_1).
\]

Considering these two latest functional identities with $n = 1$ and $n = 2$ we put in evidence the following set of relations, while taking $n = 0$ only yields trivial identities.

\[
\lambda_2^{[0]}u_2 = E_1(x)\lambda_0^{[0]}u_0 - \frac{1}{\gamma_1}J^{(1)}(u_0) + A_0(x)\lambda_1^{[0]}u_1; \quad (3.15)
\]

\[
\lambda_4^{[0]}u_4 = E_2(x)\lambda_0^{[0]}u_0 - E'_2(x)J^{(1)}(u_0) + \frac{1}{2}E''_2(x)J^{(2)}(u_0)
\]

\[
+ A_1(x)\lambda_1^{[0]}u_1 - A'_1(x)J^{(1)}(u_1);
\]

\[
\lambda_3^{[0]}u_3 = B_1(x)\lambda_0^{[0]}u_0 - B'_1(x)J^{(1)}(u_0) + F_1(x)\lambda_1^{[0]}u_1 - F'_1(x)J^{(1)}(u_1); \quad (3.17)
\]

\[
\lambda_5^{[0]}u_5 = B_2(x)\lambda_0^{[0]}u_0 - B'_2(x)J^{(1)}(u_0) + \frac{1}{2}B''_2(x)J^{(2)}(u_0)
\]

\[
+ F_2(x)\lambda_1^{[0]}u_1 - F'_2(x)J^{(1)}(u_1) + \frac{1}{2}F''_2(x)J^{(2)}(u_1).
\]
In view of (3.23) and (3.24), we highlight the next functional identities
\[
\begin{align*}
  u_2 &= E_1(x)u_0 + A_0(x)u_1 \\
  u_3 &= B_1(x)u_0 + F_1(x)u_1 \\
  u_4 &= E_2(x)u_0 + A_1(x)u_1 \\
  u_5 &= B_2(x)u_0 + F_2(x)u_1
\end{align*}
\]  
where
\[
\begin{align*}
  E_1(x) &= \frac{1}{\gamma_1} (x - \beta_0); \quad A_0(x) = -\frac{\alpha_1}{\gamma_1}; \quad B_1(x) = -\frac{\alpha_2}{\gamma_1\gamma_2} (x - \beta_0) - \frac{1}{\gamma_2}; \\
  F_1(x) &= \frac{1}{\gamma_2} \left( x - \beta_1 + \frac{\alpha_2 \alpha_1}{\alpha_1} \right); \quad E_2(x) = \frac{1}{\gamma_3} \left( (x - \beta) E_1(x) - \alpha_3 B_1(x) \right); \\
  A_1(x) &= -\frac{1}{\gamma_3} \left( \alpha_3 F_1(x) + 1 + \frac{\alpha_1}{\gamma_1} (x - \beta_2) \right); \\
  B_2(x) &= \frac{1}{\gamma_4} \left( (x - \beta_3) B_1(x) - \alpha_4 E_2(x) - E_1(x) \right); \\
  F_2(x) &= \frac{1}{\gamma_4} \left( (x - \beta_3) F_1(x) - \alpha_4 A_1(x) - A_0(x) \right).
\end{align*}
\]  
From (3.15) and (3.17), together with (3.19)-(3.20), we get respectively
\[
\begin{align*}
  J^{(1)} (u_0) &= p_0(x)u_0 + p_1(x)u_1, \quad \text{with} \\
  p_0(x) &= \gamma_1 E_1(x) \left( \lambda_0^{[0]} - \lambda_2^{[0]} \right), \quad p_1(x) = \gamma_1 A_0(x) \left( \lambda_1^{[0]} - \lambda_2^{[0]} \right); \\
  J^{(1)} (u_1) &= f_0(x)u_0 + f_1(x)u_1 \quad \text{with} \\
  f_0(x) &= \gamma_2 \left( \lambda_0^{[0]} - \lambda_3^{[0]} \right) B_1(x) + \alpha_2 E_1(x) \left( \lambda_0^{[0]} - \lambda_2^{[0]} \right), \\
  f_1(x) &= \gamma_2 \left( \lambda_1^{[0]} - \lambda_3^{[0]} \right) F_1(x) + \alpha_2 A_0(x) \left( \lambda_1^{[0]} - \lambda_2^{[0]} \right) u_1.
\end{align*}
\]  
Let us now focus on (3.16) and (3.21), having learned the above expansions (3.23) and (3.24). The polynomial $E_2(x)$ has degree two and $E_2^{(2)}(x) = \frac{2}{\gamma_1\gamma_3} \neq 0$, thus we may isolate in (3.10) the term $J^{(2)}(u_0)$, obtaining:
\[
\begin{align*}
  J^{(2)}(u_0) &= \mathfrak{p}_0(x)u_0 + \mathfrak{p}_1(x)u_1, \quad \text{with} \\
  \mathfrak{p}_0(x) &= \gamma_1 \gamma_3 \left( \lambda_4^{[0]} - \lambda_0^{[0]} \right) E_2(x) + E_2^{(2)}(x)p_0(x) + A_1(x)f_0(x), \\
  \mathfrak{p}_1(x) &= \gamma_1 \gamma_3 \left( \lambda_1^{[0]} - \lambda_4^{[0]} \right) A_1(x) + E_2^{(2)}(x)p_1(x) + A_1(x)f_1(x).
\end{align*}
\]  
We remark that $\deg (\mathfrak{p}_0(x)) \leq 2$ and $\deg (\mathfrak{p}_1(x)) \leq 1$.

Analogously, identity (3.18) allows us to write $J^{(2)}(u_1)$ in terms of the pair $(u_0, u_1)$, with the help of the information settled up to this point, in particular (3.22). More precisely, the polynomial $F_2(x)$ has degree two and $F_2^{(2)}(x) =
\[ \frac{2}{\gamma_2 \gamma_4} \neq 0, \text{ and we may affirm that} \]
\[
J^{(2)}(u_1) = \overline{f}_0(x)u_0 + \overline{f}_1(x)u_1, \quad \text{with} 
\]
\[
\overline{f}_0(x) = \gamma_2 \gamma_4 \left( \left( \lambda^0_0 - \lambda^0_1 \right) B_2(x) + B_2'(x) p_0(x) + F_2'(x) f_0(x) - \frac{1}{2} B_2^{(2)}(x) \overline{p}_0(x) \right),
\]
\[
\overline{f}_1(x) = \gamma_2 \gamma_4 \left( \left( \lambda^0_0 - \lambda^0_1 \right) F_2(x) + B_2'(x) p_1(x) + F_2'(x) f_1(x) - \frac{1}{2} B_2^{(2)}(x) \overline{p}_1(x) \right).
\]

We remark that \( \deg(\overline{f}_0(x)) \leq 2 \) and \( \deg(\overline{f}_1(x)) \leq 2 \).

Next, we list a small set of functional identities that are fulfilled by the fundamental pair of functionals \((u_0, u_1)\) when the identity \( J(P_n(x)) = \lambda^0_n P_n(x) \) holds.

**Lemma 3.1.** Considering an isomorphism \( J \) defined by (3.1) and a 2-orthogonal MPS \( \{P_n(x)\}_{n \geq 0} \) such that \( J(P_n(x)) = \lambda^0_n P_n(x), \quad n \geq 0, \) the initial elements of the corresponding dual sequence \( \{u_n\}_{n \geq 0} \) fulfill the following three identities:

\[
D \left( a_2(x)u_0 \right) = (2p_0(x) + 4a_1(x)) u_0 + 2p_1(x) u_1, 
\]
\[
\frac{1}{2} D^2 \left( a_2(x)u_1 \right) - 3a_1^1 u_1 = D \left( f_0(x)u_0 + (2a_1(x) + f_1(x)) u_1 \right), \quad \text{(3.27)}
\]
\[
D \left( \overline{p}_0(x)u_0 + \overline{p}_1(x)u_1 \right) + (2a_1(x) + 4p_0(x)) u_0 + 4p_1(x) u_1 = 0, 
\]
\[
\text{(3.28)}
\]
\[
\text{(3.29)}
\]

where polynomials \( p_0(x) \) and \( p_1(x) \) are defined in (3.23), the polynomials \( f_0(x) \) and \( f_1(x) \) are defined in (3.24), and the polynomials \( \overline{p}_0(x) \) and \( \overline{p}_1(x) \) are defined in (3.25).

**Proof.** Taking \( n = 0 \) in (3.27), we get

\[
D \left( -a_1(x) u_0 + \frac{1}{2!} D (a_2(x) u_0) - \frac{1}{3!} D^2 (a_3(x) u_0) \right) = 0
\]

which implies \( -a_1(x) u_0 + \frac{1}{2!} D (a_2(x) u_0) - \frac{1}{3!} D^2 (a_3(x) u_0) = 0 \), that is,

\[
D^2 (a_3(x) u_0) = -6a_1(x) u_0 + 3D (a_2(x) u_0) \quad \text{(3.30)}
\]

On the other hand, identity (3.9) asserts

\[
J^{(1)}(u_0) = a_1(x) u_0 - D (a_2(x) u_0) + \frac{1}{2!} D^2 (a_3(x) u_0)
\]

by which we get the following identity replacing \( D^2 (a_3(x) u_0) \) by the above expression (3.30) and \( J^{(1)}(u_0) \) by \( p_0(x) u_0 + p_1(x) u_1 \):

\[
D (a_2(x) u_0) = (2p_0(x) + 4a_1(x)) u_0 + 2p_1(x) u_1.
\]

The second identity is deduced in a similar way. We begin by considering \( n = 1 \) in (3.27) with \( D^2 (a_3(x) u_1) \) replaced by \( 2J^{(1)}(u_1) - 2a_1(x) u_1 + 2D (a_2(x) u_1) \):

\[
D \left( -\frac{4}{6} a_1(x) u_1 + \frac{1}{6} D (a_2(x) u_1) - \frac{1}{3} J^{(1)}(u_1) \right) = \left( \lambda^0_1 - a_0(x) \right) u_1.
\]

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When we substitute \(J^{(1)}(u_1)\) by the expression provided by (3.24) and calculate \(λ_1^{[0]} - a_0(x) = a_1^{[1]}\) we get (3.28). With respect to the third relation, taking into account (3.23) and (3.25), and in view of

\[
J^{(1)}(u) = a_1(x)u - D (a_2(x)u) + \frac{1}{2!} D^2 (a_3(x)u)
\]

we may write

\[
a_1(x)u_0 - D (a_2(x)u_0) + \frac{1}{2!} D^2 (a_3(x)u_0) = p_0(x)u_0 + p_1(x)u_1,
\]

\[
a_2(x)u_0 - D (a_3(x)u_0) = \overline{p}_0(x)u_0 + \overline{p}_1(x)u_1.
\]

Let us apply the operator \(D\) on the second identity in order to eliminate \(D^2 (a_3(x)u_0)\) in the first one, yielding

\[
D (a_2(x)u_0) + D \left( \overline{p}_0(x)u_0 + \overline{p}_1(x)u_1 \right) + (2p_0(x) - 2a_1(x)) u_0 + 2p_1(x)u_1 = 0.
\]

Inserting the information of (3.27), we end the proof.

4 First choice in the polynomial coefficients of operator \(J\)

We will now proceed by taking two assumptions regarding polynomial coefficients \(a_1(x)\) and \(a_2(x)\) of operator \(J\). These hypotheses were suggested by different calculations, performed during this research, that aimed to achieve functional identities corresponding to the Hahn classical set up, as described in p.182-183 of [8].

In this section, we require the use of Lemma 1.2, page 297 of [19] that has the following content.

**Lemma 4.1.** [19] Let \(M\) and \(N\) be two polynomials such that \(Mu_0 = Nu_1\). If the vector functional \(U = (u_0, u_1)^T\) is regular, then necessarily \(M = 0\) and \(N = 0\).

Let us analyse the identities (3.27), (3.28) and (3.29) assuming that

\[
a_2(x) = 0 \text{ and } a_1(x) = -\frac{1}{3} E_1(x) = -\frac{1}{3\gamma_1}(x - \beta_0).
\]

Under (4.1), equation (3.27) yields \((2p_0(x) + 4a_1(x))u_0 + 2p_1(x)u_1 = 0\). Taking into account Lemma 4.1, we get

\[
p_0(x) = -2a_1(x)
\]

\[
p_1(x) = 0, \text{ thus, } a_1 = 0.
\]
Looking at (3.24) with these new informations about \( p_0(x) \) and \( p_1(x) \) we get the following:

\[
D \left( \bar{p}_0(x)u_0 + \bar{p}_1(x)u_1 \right) - 6a_1(x)u_0 = 0
\]

or \( D \left( \bar{p}_0(x)u_0 + \bar{p}_1(x)u_1 \right) + 2E_1(x)u_0 + 2A_0(x)u_1 = 0 \quad (4.4) \)

with \( A_0(x) = 0 \).

Finally, from (3.28) we obtain:

\[
D \left( - \gamma_1 f_0(x)u_0 - \gamma_1 (2a_1(x) + f_1(x)) u_1 \right) + u_1 = 0. \quad (4.5)
\]

In brief, the two identities (4.5) and (4.4) allow us to affirm that the regular vector functional \( U = (u_0, u_1)^T \) fulfill \( D (\Phi(x)U) + \Psi(x)U = 0 \) with

\[
\Psi(x) = \begin{bmatrix} 0 & 1 \\ 2E_1(x) & 2A_0(x) \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} \phi_{1,1}(x) & \phi_{1,2}(x) \\ \phi_{2,1}(x) & \phi_{2,2}(x) \end{bmatrix},
\]

as it is established for the 2-orthogonal polynomial sequences fulfilling Hahn’s property in p.182-183 of [8] (see also p.320 of [19]).

In particular, we know that \( \deg (\phi_{1,1}(x)) \leq 1, \deg (\phi_{1,2}(x)) \leq 1, \deg (\phi_{2,1}(x)) \leq 2 \) and \( \deg (\phi_{2,2}(x)) \leq 1 \) as required in [8], as explained next with the detailed description of the functional relations and the polynomials involved.

\[
D (\phi_{1,1}(x)u_0 + \phi_{1,2}(x)u_1) + u_1 = 0,
\]

\[
D (\phi_{2,1}(x)u_0 + \phi_{2,2}(x)u_1) + 2E_1(x)u_0 + 2A_0(x)u_1 = 0, \quad \text{with } A_0(x) = 0,
\]

\[
\phi_{1,1}(x) = -\gamma_1 f_0(x), \text{ that is, } \quad (4.6)
\]

\[
\phi_{1,1}(x) = \alpha_2 \left( \frac{1}{3\gamma_1} - a_3 \right) x + a_3 \left( \alpha_2 \beta_0 - \gamma_1 \right) - \frac{\alpha_2 \beta_0}{3\gamma_1} + 1
\]

\[
\phi_{1,2}(x) = -\gamma_1 (2a_1(x) + f_1(x)), \text{ that is, } \quad (4.7)
\]

\[
\phi_{1,2}(x) = a_3 \gamma_1 x + \frac{1}{3} \left( -2\beta_0 + \beta_1 (2 - 3a_3 \gamma_1) \right)
\]

\[
\phi_{2,1}(x) = \bar{p}_0(x), \text{ that is, } \quad (4.8)
\]

\[
\phi_{2,1}(x) = 4a_3^3 x^2 + \frac{1}{3\gamma_1 \gamma_2} \left( \alpha_2 \alpha_3 \left( -1 + 9a_3^3 \gamma_1 \right) - 2 \left( \beta_2 (-1 + 6a_3^3 \gamma_1) + \beta_0 (1 + 6a_3^3 \gamma_1) \right) \gamma_2 \right) x
\]

\[
+ \frac{1}{3\gamma_1 \gamma_2} \left( \alpha_3 (\alpha_2 \beta_0 - \gamma_1) \left( 1 - 9a_3^3 \gamma_1 \right) + 2 \beta_0 \left( \beta_0 + \beta_2 \left( -1 + 6a_3^3 \gamma_1 \right) \right) \gamma_2 \right)
\]

\[
\phi_{2,2}(x) = \bar{p}_1(x), \text{ that is, } \quad (4.9)
\]

\[
\phi_{2,2}(x) = \frac{1}{3\gamma_2} a_3 \left( 1 - 9a_3^3 \gamma_1 \right) x + \frac{1}{3\gamma_2} \left( \alpha_3 \beta_1 \left( -1 + 9a_3^3 \gamma_1 \right) + 3 \left( 1 - 4a_3^3 \gamma_1 \right) \gamma_2 \right).
\]

Moreover, the characterisation of the classical \( d \)-orthogonal polynomial sequences (in Hahn’s sense) provided by [8] imposes that the coefficient of \( x^2 \) in
\( \phi_{2,1}(x) \) is different from \( \frac{2}{\gamma_1(m+1)} \), \( m \geq 0 \), (that is, different from \( c \times \frac{1}{m+1} \), being \( c \) the leading coefficient of \( \psi(x) = 2E_1(x) \)) and the coefficient of \( x \) in \( \phi_{1,2}(x) \) is different from \( \frac{1}{m+1} \), \( m \geq 0 \) (cf. p.183 of [8] or p.320 of [19]). This conducts us to the addition of the following restriction about the leading coefficient of polynomial \( a_3(x) \) in the definition of the operator \( J \):

\[
a^{[3]}_3 \neq \frac{1}{\gamma_1(m+1)}, \quad m \geq 0.
\]

We now summarise the conclusions of the above argumentation in the next result.

**Theorem 4.2.** Let \( \{P_n(x)\}_{n \geq 0} \) be a 2-orthogonal MPS such that \( J(P_n(x)) = \lambda_n^0 P_n(x), \quad n \geq 0 \), with \( J \) defined by (3.1). If the polynomial coefficients of operator \( J \) fulfil:

\[
a_2(x) = 0 \land a_1(x) = -\frac{1}{3\gamma_1}(x - \beta_0) \land a_3^{[3]} \neq \frac{1}{\gamma_1(m+1)}, \quad m \geq 0,
\]

then \( \{P_n(x)\}_{n \geq 0} \) is (Hahn) classical and the regular vector functional \( U = (u_0, u_1)^T \) fulfils

\[
D(\Phi(x)U) + \Psi(x)U = 0,
\]

\[
\Psi(x) = \begin{bmatrix} 0 & 1 \\ 2E_1(x) & 0 \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} \phi_{1,1}(x) & \phi_{1,2}(x) \\ \phi_{2,1}(x) & \phi_{2,2}(x) \end{bmatrix},
\]

with entries \( \phi_{i,j}(x) \) defined by (4.6)-(4.9).

5 Second choice in the polynomial coefficients of operator \( J \)

In this section, we consider the following hypotheses regarding the polynomial coefficients of operator \( J \), defined in (3.1):

\[
a_3(x) = \tau a_2(x), \text{ for a certain nonzero constant } \tau , \text{ and } \deg(a_1(x)) = 1.
\]

From the definition of \( J^{(2)}(u) \) and the content of (3.26), we may write

\[
a_2(x)u_1 - D(a_3(x)u_1) = T_0(x)u_0 + T_1(x)u_1
\]

which in view of \( a_3(x) = \tau a_2(x) \) becomes

\[
\tau D(a_2(x)u_1) = -T_0(x)u_0 + (a_2(x) - T_1(x)) u_1
\]

and thus

\[
D^2(a_2(x)u_1) = \frac{1}{\tau} D(-T_0(x)u_0 + (a_2(x) - T_1(x)) u_1).
\]
We can now eliminate the term on \( D^2 (a_2(x) u_1) \) of the general identity \((3.28)\) of Lemma 3.1 in other words, reading \((3.28)\) with the information of \((5.1)\) yields
\[
D\left( \varpi_{1,1}(x) u_0 + \varpi_{1,2}(x) u_1 \right) + u_1 = 0 , \quad \text{with (5.2)}
\]
\[
\varpi_{1,1}(x) = \frac{1}{3a_1^{[3]}} \left( \frac{1}{2\tau} f_0(x) + f_0(x) \right) , \quad \text{with (5.3)}
\]
\[
\varpi_{1,2}(x) = \frac{1}{3a_1^{[3]}} \left( 2a_1(x) + f_1(x) - \frac{1}{2\tau} (a_2(x) - \bar{f}_1(x)) \right) . \quad \text{with (5.4)}
\]
Looking at the definition of the polynomials \( \bar{f}_0(x) \) and \( \bar{f}_1(x) \) involved in \((3.26)\), we know that \( \deg(\varpi_{1,1}(x)) \leq 2 \) and \( \deg(\varpi_{1,2}(x)) \leq 2 \).

The detailed computation of these latest polynomials (confirmed by the use of a computer algebra software) allows us to conclude that \( \deg(\varpi_{1,1}(x)) \leq 1 \) and \( \deg(\varpi_{1,2}(x)) \leq 1 \) if and only if \( a_2^{[2]} = 0 \) and \( \alpha_4 = \frac{\alpha_2 \gamma_3}{\gamma_2} \), respectively.

Subsequently, we add to the set of hypotheses of this discussion the conditions:
\[ a_2^{[2]} = 0 \] \text{and} \[ \alpha_4 = \frac{\alpha_2 \gamma_3}{\gamma_2} . \]

Let us now look at \((3.29)\) of Lemma 3.1
\[
D\left( \overline{p}_0(x) u_0 + \overline{p}_1(x) u_1 \right) + (2a_1(x) + 4p_0(x)) u_0 + 4p_1(x) u_1 = 0 \]
taking into account the information considered up to this moment, namely, \( a_3(x) = \tau a_2(x) \) (thus \( a_3^{[3]} = 0 \) , \( a_1^{[3]} \neq 0 \), \( a_2^{[2]} = 0 \) and \( \alpha_4 = \frac{\alpha_2 \gamma_3}{\gamma_2} \).

Looking carefully at the term \((2a_1(x) + 4p_0(x)) u_0\) we conclude that
\[
2a_1(x) + 4p_0(x) = 2E_1(x) \iff a_1(x) = -\frac{1}{3\gamma_1}(x - \beta_0) .
\]

Hence, considering this choice of coefficient \( a_1(x) \), identity \((3.29)\) has the form
\[
D\left( \overline{p}_0(x) u_0 + \overline{p}_1(x) u_1 \right) + 2E_1(x) u_0 + \frac{4}{3} A_0(x) u_1 = 0
\]
or
\[
D\left( \overline{p}_0(x) u_0 + \overline{p}_1(x) u_1 \right) + 2E_1(x) u_0 + 2A_0(x) u_1 - \frac{2}{3} A_0(x) u_1 = 0 .
\]

Replacing the final term \( -\frac{2}{3} A_0(x) u_1 \) by the information provided by \((5.2)\), specifically \( \frac{2}{3} A_0(x) D\left( \varpi_{1,1}(x) u_0 + \varpi_{1,2}(x) u_1 \right) \), we get the following:
\[
D\left( \varpi_{2,1}(x) u_0 + \varpi_{2,2}(x) u_1 \right) + 2E_1(x) u_0 + 2A_0(x) u_1 = 0 \quad \text{with (5.5)}
\]
\[
\varpi_{2,1}(x) = \overline{p}_0(x) + \frac{2}{3} A_0(x) \varpi_{1,1}(x) \quad \text{with (5.6)}
\]
\[
\varpi_{2,2}(x) = \overline{p}_1(x) + \frac{2}{3} A_0(x) \varpi_{1,2}(x) \quad \text{with (5.7)}
\]

In brief, we can assert the next result.
Theorem 5.1. Let \( \{P_n(x)\}_{n \geq 0} \) be a 2-orthogonal MPS fulfilling \( \alpha_4 = \frac{\alpha_2 \gamma_3}{\gamma_2} \) and \( J(P_n(x)) = \lambda_n^{[0]} P_n(x) \), \( n \geq 0 \), with \( J \) defined by (3.1). If the polynomial coefficients of operator \( J \) fulfill:

\[
\begin{align*}
a_3(x) &= \tau a_2(x) \text{, for some } \tau \neq 0 \, , \quad a_1(x) = -\frac{1}{3\gamma_1}(x - \beta_0) \, , \quad \deg(a_2(x)) \leq 1, \\
a_1^{[2]} &\neq \frac{2\tau}{\gamma_1(m + 1)} - \frac{2}{3\gamma_1}(\beta_1 - \beta_3) \, , \quad m \geq 0,
\end{align*}
\]

then \( \{P_n(x)\}_{n \geq 0} \) is (Hahn) classical and the regular vector functional \( U = (u_0, u_1)^T \) fulfills

\[
D(\Phi(x)U) + \Psi(x)U = 0,
\]

\[
\Psi(x) = \begin{bmatrix} 0 & 1 \\ 2E_1(x) & 0 \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} \varpi_{1,1}(x) & \varpi_{1,2}(x) \\ \varpi_{2,1}(x) & \varpi_{2,2}(x) \end{bmatrix}, \quad \text{with entries } \varpi_{i,j}(x)
\]
defined by (5.3), (5.4), (5.6) and (5.7), that under the conditions described have the expressions indicated in Table 1.

Proof. The arguments presented in this section demonstrate this result, except for the need of the condition

\[
a_1^{[2]} \neq \frac{2\tau}{\gamma_1(m + 1)} - \frac{2}{3\gamma_1}(\beta_1 - \beta_3) \, , \quad m \geq 0.
\]

As mention in the previous section, the characterisation of the classical \( d \)-orthogonal polynomial sequences (in Hahn’s sense) provided by [8] imposes that the coefficient of \( x^2 \) in \( \varpi_{2,1}(x) \) is different from \( \frac{2}{\gamma_1(m+1)} \), \( m \geq 0 \), and the coefficient of \( x \) in \( \varpi_{1,2}(x) \) is different from \( \frac{1}{m+1} \), \( m \geq 0 \) (cf. p.183 of [8]). The first requirement is naturally assured since \( \deg(\varpi_{2,1}(x)) \leq 1 \) and from the second we get the above inequality, since the coefficient of \( x \) in \( \varpi_{1,2}(x) \) is given by \( \frac{2(\beta_1 - \beta_3) + 3\gamma_1 a_1^{[2]}}{6\tau} \). \( \square \)
\begin{align*}
\varpi_{1,1}(x) & = \frac{3\beta_0 \gamma_2 + \alpha_2 \beta_0 (\beta_1 + \beta_2 - 2 (\beta_3 + \tau)) + \gamma_1 (-2 \beta_0 - 3 \beta_1 + 2 \beta_3 + 6 \tau) \gamma_1 \tau}{6 \gamma_1 \tau} \\
& \quad + \frac{x (3 \gamma_1 - \gamma_2 - \alpha_2 (\beta_1 + \beta_2 - 2 (\beta_3 + \tau))) \gamma_1 \tau}{6 \gamma_1 \tau} \\
\varpi_{1,2}(x) & = \frac{\gamma_1 \left(3 \gamma_1 a_0^{[2]} - 2 (\alpha_2 + 2 \beta_0 \tau + \beta_1 (\beta_1 - \beta_3 - 2 \tau))\right) + \alpha_1 (-\gamma_1 + 3 \gamma_2 + \alpha_2 (\beta_1 + \beta_2 - 2 (\beta_3 + \tau))) \gamma_1 \tau}{6 \gamma_1 \tau} \\
& \quad + \frac{x \left(3 \gamma_1 a_1^{[2]} + 2 \beta_1 - 2 \beta_3 \right) \gamma_1 \tau}{6 \gamma_1 \tau} \\
\varpi_{2,1}(x) & = \frac{-3 \alpha_1 \beta_0 \gamma_2^2 + \gamma_2 \gamma_1 (\alpha_1 (2 \beta_0 - 2 \beta_3 + 3 (\beta_1 + \tau)) + 6 \beta_0 (\beta_0 - \beta_2) \tau) + \alpha_2 \beta_0 (3 \alpha_3 \gamma_1 \tau - \alpha_1 \gamma_2 (\beta_3 + \beta_2 - 2 \beta_3 + \tau)) - 3 \alpha_3 \gamma_2^2 \tau \gamma_1 \gamma_2 \tau}{9 \gamma_1 \gamma_2 \tau} \\
& \quad + \frac{x (\alpha_2 (\alpha_1 \gamma_2 (\beta_1 + \beta_2 - 2 \beta_3 + \tau) - 3 \alpha_3 \gamma_1 \tau) + \gamma_2 (\alpha_1 (\gamma_2 - \gamma_1) + 2 (\beta_3 - \beta_0) \gamma_1 \tau)) \gamma_1 \tau}{9 \gamma_1 \gamma_2 \tau} \\
\varpi_{2,2}(x) & = \frac{\alpha_1 \gamma_1 \left(\gamma_2 \left(-3 \gamma_1 a_0^{[2]} + 7 \beta_0 \tau + 2 (\beta_1 + \beta_1 (\tau - \beta_3 - 3 \beta_2 \tau))\right) + \alpha_2 (2 \gamma_2 + 3 \alpha_3 \tau)\right) + \alpha_1^2 (-\gamma_2) (-\gamma_1 + 3 \gamma_2 + \alpha_2 (\beta_1 + \beta_2 - 2 \beta_3 + \tau)) - 3 \gamma_2^2 \tau (\alpha_3 \beta_1 - \gamma_2) \gamma_1 \gamma_2 \tau}{9 \gamma_1 \gamma_2 \tau} \\
& \quad + \frac{x \left(3 \alpha_3 \gamma_1 \tau - \alpha_1 \gamma_2 \left(3 \gamma_1 a_1^{[2]} + 2 \beta_1 - 2 \beta_3 + 3 \tau\right)\right) \gamma_1 \gamma_2 \tau}{9 \gamma_1 \gamma_2 \tau}
\end{align*}

Table 1: Polynomial entries of matrix \( \Phi(x) \) of Theorem 5.1.
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