A MATRIX-VALUED BEREZIN-TOEPLITZ QUANTIZATION

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Abstract. We generalize some earlier results on a Berezin-Toeplitz type of quantization on Hilbert spaces built over certain matrix domains. In the present, wider setting, the theory could be applied to systems possessing several kinematic and internal degrees of freedom. Our analysis leads to an identification of those observables, in this general context, which admit a semiclassical limit and those for which no such limit exists. It turns out that the latter class of observables involve the internal degrees of freedom in an intrinsic way. Mathematically, the theory, being a generalization of the standard Berezin-Toeplitz quantization, points the way to applying such a quantization technique to possibly non-commutative spaces, to the extent that points in phase space are now replaced by $N \times N$ matrices.

1. Introduction

Let $\Omega$ be a symplectic manifold, with symplectic form $\omega$, and $\mathcal{H}$ a subspace of $L^2(\Omega, d\mu)$, for some measure $\mu$. For $\phi \in C^\infty(\Omega)$, the (generalized) Toeplitz operator $T_\phi$ with symbol $\phi$ is the operator on $\mathcal{H}$ defined by

$$T_\phi f = P(\phi f), \quad f \in \mathcal{H},$$

(1.1)

where $P : L^2(\Omega, d\mu) \to \mathcal{H}$ is the orthogonal projection. It is easily seen that $T_\phi$ is a bounded operator whenever $\phi$ is a bounded function, and $\|T_\phi\|_{\mathcal{H} \to \mathcal{H}} \leq \|\phi\|_\infty$, the supremum norm of $\phi$.

Suppose now that both the measure $\mu$ and the subspace $\mathcal{H}$ are made to depend on an additional parameter $h > 0$ (shortly to be interpreted as the Planck constant), in such a way that the associated Toeplitz operators $T^{(h)}_\phi$ on $\mathcal{H}$ satisfy, as $h \to 0$,

$$\|T^{(h)}_\phi\|_{\mathcal{H}_h \to \mathcal{H}_h} \to \|\phi\|_\infty, \quad \|T^{(h)}_\phi T^{(h)}_\psi - T^{(h)}_{\phi \psi}\|_{\mathcal{H}_h \to \mathcal{H}_h} \to 0,$$

(1.2)

and

$$\|2\pi i [T^{(h)}_\phi, T^{(h)}_\psi] - T^{(h)}_{\{\phi, \psi\}}\|_{\mathcal{H}_h \to \mathcal{H}_h} \to 0$$

(1.3)

(1.4)

(where $\{\cdot, \cdot\}$ is the Poisson bracket with respect to $\omega$), and, more generally,

$$T^{(h)}_\phi T^{(h)}_\psi \approx \sum_{j=0}^\infty h^j T^{(h)}_{C_j(\phi, \psi)} \quad \text{as } h \to 0,$$

(1.5)

for some bilinear differential operators $C_j : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega)$, with $C_0(\phi, \psi) = \phi \psi$ and $C_1(\phi, \psi) - C_1(\psi, \phi) = \frac{i}{2\pi}\{\phi, \psi\}$. Here the last asymptotic

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expansion means, more precisely, that
\begin{equation}
T^{(\hbar)}(\phi)T^{(\hbar)}(\psi) - \sum_{j=0}^{N} \hbar^j C_j(\phi, \psi) \|_{\mathcal{H}_\hbar \to \mathcal{H}_\hbar} = O(h^{N+1}) \quad \text{as } h \searrow 0, \quad \forall N = 0, 1, 2, \ldots
\end{equation}

One then speaks of the Berezin-Toeplitz quantization. Indeed, it is well known that the recipe
\[ \phi \ast \psi := \sum_{j=0}^{\infty} \hbar^j C_j(\phi, \psi) \]
then gives a star-product on \(\Omega\), and (1.3), (1.4) just amount to its correct semiclassical limit.

The simplest instance of the above situation is \(\Omega = \mathbb{R}^{2n} \simeq \mathbb{C}^n\), with the standard (Euclidean) symplectic structure, and
\begin{equation}
H_\hbar = L^2_{\text{hol}}(\Omega, d\mu_\hbar)
\end{equation}
the Segal-Bargmann space of all holomorphic functions square-integrable with respect to the Gaussian measure \(d\mu_\hbar(z) := e^{-|z|^2/h(\pi \hbar)^{-n}} dz\) (\(dz\) being the Lebesgue measure on \(\mathbb{C}^n\)). As shown by Coburn \cite{Cob}, (1.5) then holds with
\begin{equation}
C_j(\phi, \psi) = \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial^\alpha \phi \cdot \overline{\partial}^\alpha \psi.
\end{equation}
The resulting star-product coincides, essentially, with the familiar Moyal product.

Other examples of Berezin-Toeplitz quantization include the unit disc \(D\) with the Poincaré metric, bounded symmetric domains, strictly pseudo convex domains with metrics having reasonable boundary behaviour, or, provided one allows not only holomorphic functions but also sections of line bundles as elements of \(H_\hbar\), all compact Kähler manifolds whose Kähler form is integral. In all these cases, the choice of the spaces (1.7) which works are the weighted Bergman spaces \(H_\hbar = L^2_{\text{hol}}(\mathbb{D}, e^{-\Phi/\omega_n})\) (the subspaces of all holomorphic functions in \(L^2(\mathbb{D}, e^{-\Phi/\omega_n})\)), where \(n\) is the complex dimension of \(\Omega\) and \(\Phi\) is a Kähler potential for \(\omega\) (so, for instance, for the unit disc \(\mathcal{H}_\hbar = L^2_{\text{hol}}(D, \frac{1}{\pi \hbar}(1 - |z|^2)^{1/2} \, dz)\)). See \cite{KS}, \cite{BMS} or \cite{AE} for the details and further discussion.

Though this seems not to have been recorded explicitly in the literature, the whole formalism also extends seamlessly to spaces of vector-valued functions. In physical terms, this can be interpreted as accommodating the internal degrees of freedom of the quantized system. Namely, replacing the spaces \(\mathcal{H}_\hbar\) and \(L^2(\Omega, d\mu_\hbar)\) by the tensor products \(\mathcal{H}_\hbar \otimes \mathbb{C}^N\) and \(L^2(\Omega, d\mu_\hbar) \otimes \mathbb{C}^N\) (which can be viewed as spaces of \(\mathbb{C}^N\)-valued functions on \(\Omega\)), one can define in the same way the Toeplitz operators \(T^{(\hbar)}(\phi)\), where now the symbol \(\phi\) can even be allowed to be a \((N \times N)\)-matrix-valued function on \(\Omega\). It is a simple matter to check, however, that this Toeplitz operator is just the \(N \times N\) matrix \([T^{(\hbar)}(\phi_{jk})]_{j,k=1}^N\) of Toeplitz operators on \(\mathcal{H}_\hbar\), and it immediately follows that (1.5) remains in force in this vector-valued situation whenever it holds for the scalar-valued one. In particular, for scalar-valued functions \(\phi\) (i.e. \(\phi(z)_{jk} = \delta_{jk} \phi(z)\) for some \(\phi : \Omega \to \mathbb{C}\)), one recovers (1.5) completely, with the same cochains \(C_j\).

In this paper, we work out a formalism, based upon certain spaces of matrix-valued functions, which could be looked upon, in appropriate cases, as a possible different approach to the quantization of the internal degrees of freedom of systems...
whose kinematics is defined on complex phase spaces $\Omega$. Moreover, the more general
setting adopted here, in that points in phase space are replaced by $N \times N$ matrices,
could potentially be used to describe systems defined over non-commutative spaces.

In more concrete terms, our spaces $\mathcal{H}_h$ will be suitable subspaces (actually, rather
small ones, in terms of codimension) of the spaces $L^2(\Omega, d\mu_h) \otimes \mathbb{C}^N$ of $\mathbb{C}^N$-valued
functions on certain domains $\Omega$ in $\mathbb{C}^{n \times N \times N}$ associated to $\Omega$ in a natural way.
(Here, as before, $n$ is the complex dimension of $\Omega$ and $N$ is related to the number
of internal degrees of freedom.) The Toeplitz operators are again defined by the
formula (1.1), only with $P$ replaced by the orthogonal projection onto $\mathcal{H}_h$, and
the symbol $\phi$ can now be allowed to be a $\mathbb{C}^{N \times N}$-valued function on $\Omega$. Finally,
there exists a canonical unitary isomorphism $\iota : \mathcal{H}_h \rightarrow \mathcal{H}_h \otimes \mathbb{C}^N$ (with $\mathcal{H}_h$ as in
(1.7)). This means that the quantum system defined on $\mathcal{H}_h$ can be thought of as
one possessing $N$ internal degrees of freedom and moving on the phase space $\Omega$.

The following facts then emerge from our analysis.

(a) To any function $\phi$ on $\Omega$ one can associate, in a canonical way, a function $\tilde{\phi}$
on $\Omega$. (For reasons which will become apparent later, functions $\phi$ that arise
in this way will be called spectral functions.) For any two functions $\phi, \psi$
of this form, the corresponding Toeplitz operators turn out to be unitarily
equivalent via $\iota$ to $T^{(\phi)}_\psi \otimes I$ and $T^{(\psi)}_\phi \otimes I$, respectively, acting on $\mathcal{H}_h \otimes \mathbb{C}^N$.
Consequently, (1.5) must hold (with the same cochains $C_j$, and, in this
sense, our quantization contains the original scalar-valued Berezin-Toeplitz
quantization, as well as its vector-valued analogue obtained by tensoring
with $\mathbb{C}^N$ (and using only scalar-valued symbols $\phi$), mentioned above.

(b) Let the unitary group $U(N)$ of order $N$ act on $\mathbb{C}^{n \times N \times N}$ by

$$Z^U = (U^* Z_1 U, U^* Z_2 U, \ldots, U^* Z_n U) \quad \text{with} \quad Z = (Z_1, \ldots, Z_n) \in \mathbb{C}^{n \times N \times N}.$$ 

The domain $\Omega$ is invariant under this action, and functions $\phi$ satisfying
$\phi(Z^U) = U^* \phi(Z) U \ \forall U \in U(N)$ will be called $U$-invariant. All spectral
functions are $U$-invariant, but not vice versa. It is then the case that for any
$U$-invariant function $\phi$, the Toeplitz operator $T^{(\phi)}_\psi$ is unitarily equivalent via
$\iota$ to the operator $T^{(\phi)}_{\pi_h} \otimes I$ on $\mathcal{H}_h \otimes \mathbb{C}^N$, where $\pi_h \phi \in C^\infty(\Omega)$ is a certain
“average” of $\phi$ over the internal variables (reminiscent of “spin averaging”
in quantum mechanical scattering theory). Not surprisingly, the operator $\pi_h$
behaves nicely as $h \rightarrow 0$; owing to this, for any two $U$-invariant functions $\phi, \psi$
one obtains a semiclassical expansion of the product $T^{(\phi)}_\psi T^{(h)}_\phi$ of the form

$$T^{(\phi)}_\psi T^{(h)}_\phi \approx \sum_{r=0}^\infty h^r T^{(h)}_{\psi_\phi},$$

for some uniquely determined spectral functions $\psi_\phi$. Thus, in this sense,
the internal degrees of freedom disappear in the semiclassical limit, as they
should. Using the isomorphism $\iota$ and the facts mentioned in (a), this can
also be recast into the language of the traditional vector-valued quantization
discussed before; note, however, that now we are able to quantize not only
the scalar-valued functions (which we have seen in (a) to correspond to the
spectral functions on $\Omega$), but a much wider class of observables corresponding
to $U$-invariant functions.
Finally, for completely general functions $\phi, \psi \in C^\infty(\Omega)$, the semiclassical expansion of the product $T^{(h)}_\phi T^{(h)}_\psi$ in the usual sense (i.e., in the sense of [1.5]) does not exist. (There may be one, but the cochains $C_j$ are then no longer uniquely determined unless one requires that their values always be spectral functions, and then they are no longer local (i.e., differential) operators, but rather involve some kind of averaging over a sort of $U(N)$-orbit of $Z$.) Consequently, such functions lead to quantum observables that have no classical counterparts. The following situation is thus seen to emerge: while the Toeplitz operator corresponding a general function $\phi$ could be a legitimate quantum observable, only those functions which are $U$-invariant, and consequently involve the internal degrees of freedom only in a “controlled” way, admit a semi-classical limit. In other words, only observables kinematically related to the phase space have semi-classical limits. (Note that even in the case where internal degrees of freedom are absent, i.e., $N = 1$, the model of quantum mechanics being used here is one where the wave functions are defined on phase space and not on configuration space.)

The whole approach is applicable to any phase space $\Omega \subset \mathbb{C}^n$ admitting the ordinary (i.e. scalar-valued) Berezin-Toeplitz quantization. At the moment, we do not know how to extend it from domains in $\mathbb{C}^n$ to manifolds.

For the simplest case of $\Omega = \mathbb{C}$, corresponding to a free particle on the real line, the results above have been obtained in [AE2]. For the reader’s convenience, we review, in Section 2 below, the necessary material from that paper (without proofs), as well as from its precursor [AEG], where spaces of matrix-valued functions of this type were first introduced. The quantization procedure is spelled out in Section 3. Hidden under surface in all these developments are also certain vector- and matrix-valued analogues of some reproducing kernels and coherent states; these in fact make sense in several more general situations as well (even though the quantization procedure may not lead to physically meaningful theories). We describe these in the last Section 4.

A word of clarification is, perhaps, in order at this juncture. We are not suggesting here that the current formalism be used to replace the traditional quantum mechanical setup for describing systems with internal degrees of freedom. As far as traditional quantum mechanics is concerned, the present formalism, with wave functions described over matrix domains is an interesting alternative to it. Besides being well-adapted to studying the semi-classical limit, the present formalism can also be easily employed to build “quantum systems” which show no limiting semi-classical behaviour at all! One might venture a guess that such quantum systems (which even admit a proper probability interpretation on “phase space”) could point to some underlying non-commutative geometry.

2. The case of the complex plane

For the reader’s convenience, we briefly review here the salient facts from [AE2] and [AEG], which correspond to the simplest case of the quantization on $\Omega = \mathbb{C}$.

Consider the domain $\Omega = \{Z \in \mathbb{C}^{N \times N} : Z^*Z = ZZ^*\}$ of all normal matrices in $\mathbb{C}^{N \times N}$. By the spectral theorem, any $Z \in \Omega$ can be written in the form

\begin{equation}
Z = U^*DU,
\end{equation}

where $U$ is a unitary matrix and $D$ is a diagonal matrix with non-negative entries.
with \( U \in U(N) \) unitary and \( D \) diagonal; \( D \) is determined by \( Z \) uniquely up to permutation of the diagonal elements, and if the latter are all distinct and their order has been fixed in some way, then \( U \) is unique up to left multiplication by a diagonal matrix with unimodular elements. Consequently, there exists a unique measure \( d\mu_h(Z) \) on \( \Omega \) such that

\[
\int_\Omega f(Z) d\mu_h(Z) = (\pi h)^{-N} \int_{U(N)} \int_{\mathbb{C}^N} f(U^*DU) e^{-\|D\|^2/h} dU \, dD \quad \forall f,
\]

where \( dU \) is the normalized Haar measure on \( U(N) \), \( dD \) is the Lebesgue measure on \( \mathbb{C}^N \), where we are identifying the diagonal matrix \( D = \text{diag}(d_1, \ldots, d_N) \) with the vector \( d = (d_1, \ldots, d_N) \in \mathbb{C}^N \), and \( \|D\|^2 = \|d\|^2 := |d_1|^2 + \cdots + |d_N|^2 \). It can be shown [AEG] that

\[
\int_\Omega Z^j Z^k d\mu_h(Z) = \delta_{jk} k! h^k I,
\]

so that the elements

\[
\frac{Z^j \chi_j}{\sqrt{k! h^k}} \quad j = 1, \ldots, N, \ k = 0, 1, 2, \ldots,
\]

where \( \chi_1, \ldots, \chi_N \) is the standard basis of \( \mathbb{C}^N \), are orthonormal in \( L^2(\Omega, d\mu_h) \otimes \mathbb{C}^N \).

Let \( \mathcal{H}_h \) be the subspace spanned by these functions.

In analogy with the scalar-valued situation, we next define for any \( \phi \in C^\infty(\Omega) \otimes \mathbb{C}^{N \times N} \) the Toeplitz operator \( T_\phi \) on \( \mathcal{H}_h \) by the recipe

\[
T_\phi^{(h)} f = P_h(\phi f),
\]

where \( P_h : L^2(\Omega, d\mu_h) \to \mathcal{H}_h \) is the orthogonal projection. Note that the last formula implies that \( \|T_\phi^{(h)}\|_{\mathcal{H}_h \to \mathcal{H}_h} \leq \|\phi\|_\infty := \sup_{X \in \Omega} |\phi(X)|_{\mathbb{C}^N \to \mathbb{C}^N} \).

A \( \mathbb{C}^{N \times N} \)-valued function \( \phi(Z) \) of \( Z \in \Omega \) will be called spectral if it is a function of \( Z \) in the sense of the Spectral Theorem for matrices: that is, if there exists a function \( \phi : \mathbb{C} \to \mathbb{C} \) such that \( \phi = \phi^\# \), where

\[
\phi^\#(Z) := U \cdot \text{diag}_j(\phi(d_j)) \cdot U^* \quad \text{if} \quad Z = U \cdot \text{diag}_j(d_j) \cdot U^*.
\]

Further, as was already mentioned in the Introduction, the function \( \phi \) will be called \( U \)-invariant if

\[
\phi(U^* Z U) = U^* \phi(Z) U \quad \forall U \in U(N) \ \forall Z \in \Omega.
\]

Clearly, a spectral function is \( U \)-invariant, but not vice versa: an example is the function \( \phi(Z) = |\det Z|^2 I \).

The following results have been established in [AEG].

**Proposition.** ([AEG], Proposition 12) A function \( \phi \) is \( U \)-invariant if and only if there exists a function \( \phi(d_1; d_2, \ldots, d_N) \) from \( \mathbb{C} \times \mathbb{C}^{N-1} \) into \( \mathbb{C} \), symmetric in the last \( N-1 \) variables \( d_2, \ldots, d_N \), such that \( \phi = \phi^\# \), where

\[
\phi^\#(U^* DU) := U^* \cdot \text{diag}_j(\phi(d_j; d_1, \ldots, \hat{d}_j, \ldots, d_N)) \cdot U.
\]

The function \( \phi \) is uniquely determined by \( \phi^\# \).

Further, \( \phi \) is spectral if and only if \( \phi \) depends only on the first variable, i.e. if and only if \( \phi(d_1; d_2, \ldots, d_N) = \phi(d_1; 0, \ldots, 0) \).

(Here we are using the notation \( \phi^\# \) both in the sense of (2.7) and (2.5), but there is no danger of confusion.)
Theorem. ([AE2], Theorem 10) If $\phi = \phi^#$ and $\psi = \psi^#$ are two smooth spectral functions on $\Omega$, then there exist unique spectral functions $\nu_r$, $r = 0, 1, 2, \ldots$, such that

$$T_{\phi}^{(h)} T_{\psi}^{(h)} \approx \sum_{r=0}^{\infty} h^r T_{\nu_r}^{(h)}$$

as $h \to 0$ in the sense of operator norms (i.e. as in (1.6)). In fact, $\nu_r = C_r(f, g)^#$, where

$$(2.8) \quad C_r(f, g) = \frac{1}{r!} \partial^r f \cdot \overline{g}$$

are the operators (1.8) for $n = 1$.

Theorem. ([AE2], Theorem 16) For a function $f$ on $\mathbb{C}^N$ and $h > 0$, let $\pi_h f$ be the function on $\mathbb{C}$ defined by

$$\pi_h f(z_1) := \int_{\mathbb{C}^{N-1}} f(z_1, z_2, \ldots, z_N) e^{-\left(|z_1|^2 + \cdots + |z_N|^2\right)/h} \frac{dz_2 \cdots dz_N}{(\pi h)^{N-1}}.$$

Let $\phi = \phi^#$, $\psi = \psi^#$ be smooth $U$-invariant functions on $\Omega$ such that the partial derivatives of $\phi$ and $\psi$ of all orders are bounded, and let $C_r$ be the bidifferential operators (2.8). Then

$$(2.9) \quad T_{\phi}^{(h)} T_{\psi}^{(h)} \approx \sum_{r=0}^{\infty} h^r T_{\nu_r}^{(h)}$$

in the sense of operator norms, where

$$\nu_r = \sum_{j,k,l: j \geq 0, j+k+l = r} \frac{1}{j!k!l!} \partial^j (\Delta' \phi)^l \cdot \overline{\Delta'(\Delta' \psi)^k},$$

where $\Delta'$ denotes the Laplacian with respect to the last $N-1$ variables $z_2, \ldots, z_N$, and $f^l(z) := f(z; 0, \ldots, 0)$.

Finally, for functions which are not $U$-invariant, things seem to go wrong regarding quantization: namely, there is evidence that in general the semiclassical expansion of the form (1.5) either does not exist, or if it exists then the cochains $C_j$ have rather pathological properties (for instance, are not local operators — the value of $C_j(\phi, \psi)$ at a point $Z$ need not depend only on the jets of $\phi$ and $\psi$ at $Z$). In more detail: first of all, there exist functions $\phi$ (even very nice and $U$-invariant ones — for instance, $\phi(Z) = |\det(Z)|^2 e^{-\text{Tr}(Z^* Z)/2}$) for which $\|T_{\phi}^{(h)}\| \to 0$ as $h \searrow 0$; as a result, the cochains $C_j$ in (1.5) have no chance of being uniquely determined, unless they are subjected to some additional condition. The only such condition which gives the right answer for spectral functions seems to be that $C_j$ take values in spectral functions; let us therefore assume that this is the case. Second, there exist families of elements $k_{Z, \chi}^{(h)} \in \mathfrak{H}_h$, labelled by $Z \in \Omega$ and $\chi \in \mathbb{C}^N$ (interpretable...
as normalized reproducing kernels, or vector coherent states — see Section 3 below for more information), such that as $\hbar \to 0$, there are asymptotic expansions

$$
\langle T^{(h)}_{\phi} T^{(h)}_{\psi} \rangle_{Z,\chi, k_{Z,n}} \approx \sum_{r=0}^{\infty} \hbar^r \eta^* l_r[\phi](Z) \chi,
$$

$$
\langle T^{(h)}_{\phi} \rangle_{Z,\chi, k_{Z,n}} \approx \sum_{r=0}^{\infty} \hbar^r \eta^* m_r[\phi, \psi](Z) \chi,
$$

for some $\mathbb{C}^{N \times N}$-valued functions $l_r[\phi]$ and $m_r[\phi, \psi]$ on $\Omega$. If (1.5) holds, then we must therefore have $m_0[\phi, \psi] = l_0[C_0(\phi, \psi)]$. For spectral functions, $l_0$ turns out to be just the identity operator; since we have agreed that $C_j$ takes values in spectral functions, it follows that $C_0(\phi, \psi) = m_0[\phi, \psi]$. Now computations show that $m_0[\phi, \psi](Z)$ is given by a rather complicated expression involving integration over the whole orbit $\{ U^* (d_j^j (\chi_j) \chi_j) U \}$ of the spectral projections $d_j^j (\chi_j) \chi_j$, $j = 1, \ldots, N$, of $Z$ under the unitary group $U(N)$. The reader is referred to Sections 4–6 of [AL2] for the full story.

The appearance of $\phi^\phi$ and $\psi^x$, and not $\phi$ and $\psi$, in (2.9) means that the $\mathbb{C}^{N-1}$ part of $\phi$ disappears in the semiclassical limit $\hbar \to 0$, and only the projection $\phi^\phi$, which lives on $\mathbb{C}$, survives; that is, only the “spectral component” of the corresponding $U$-invariant function $\phi$ on $\Omega$. As mentioned before, all this means that we are dealing here with a quantum system which has $N$ internal degrees of freedom, and that the full set of quantum observables of this system includes those which do not have classical counterparts, while even for those having the classical counterparts, the internal degrees of freedom — being purely quantum in this case — do not survive in the semi-classical limit.

### 3. General domains

We proceed to describe how the spaces from the preceding section can be adapted from the complex plane to any phase-space $\Omega \subset \mathbb{C}^n$ admitting the ordinary (scalar-valued) Berezin-Toeplitz quantization.

The appropriate matrix domain is

$$
\Omega = \{ Z = (Z_1, Z_2, \ldots, Z_n) \in \mathbb{C}^{n \times N \times N} : \\
Z_j Z_k^* = Z_k^* Z_j \forall j,k = 1, \ldots, n, \text{ and } \sigma(Z) \subset \Omega \},
$$

i.e. the set of all commuting $n$-tuples of normal $N \times N$ matrices whose joint spectrum $\sigma(Z)$ is contained in $\Omega$. In other words, this means that in the decomposition (2.1) for the entries $Z_j$, the unitary parts will be the same for all $j$:

$$
Z \in \Omega \iff Z = (U^* D^{(1)} U, \ldots, U^* D^{(n)} U),
$$

with $U \in U(N)$ and $D^{(1)}, \ldots, D^{(n)}$ diagonal, and, if we denote the diagonal entries of the matrices $D^{(j)}$ by $d_{jk}^{(j)}$ ($j = 1, \ldots, n$, $k = 1, \ldots, N$),

$$
(d_{k1}^{(1)}, \ldots, d_{kn}^{(n)}) \in \Omega, \quad \forall k = 1, \ldots, N.
$$

With this notation, we define the measure on $\Omega$ by

$$
d\mu_{h}(Z) = \frac{1}{\mu_h(\Omega) N!} dU \prod_{k=1}^{N} d\mu_h(d_{k}),
$$

where $\mu_h(\Omega)$ is the volume of $\Omega$ with respect to the standard inner product on $\mathbb{C}^n$. Finally, we define

$$
\{ Z_{l} \}_{l=0}^{\infty} = \{ Z_{l} \}_{l=0}^{\infty}
$$
where \( d_k := (d_k^{(1)}, \ldots, d_k^{(n)}) \). We will also sometimes use the shorthand
\[
diag(\mathbf{d}_1, \ldots, \mathbf{d}_n)
\]
to denote the \( n \)-tuple of diagonal matrices \((D^{(1)}, \ldots, D^{(n)}) \in \Omega\).

It remains to define the spaces \( \mathcal{S}_h \). Observe that since the \( Z_j \) commute and \( \sigma(Z) \in \Omega \), the spectral theorem implies that, for any function \( f : \Omega \to \mathbb{C} \), we can form the matrix 
\[
f(Z, \ldots, Z_n) := f^\#(Z) \in \mathbb{C}^{N \times N}:
\]
specifically,
\[
f^\#(Z) = U^* \diag_k(f(d_k^{(1)}), \ldots, d_k^{(n)}))U = U^* \diag_k(f(d_k))U.
\]
Functions on \( \Omega \) of this form will be called \textit{spectral functions}. We now define spaces \( \mathcal{S}_h \subset L^2(\Omega, d\mu_h) \otimes \mathbb{C}^N \) as
\[
\mathcal{S}_h = \text{span}\{ f^\#(Z) \chi : f \in L^2_{\text{hol}}(\Omega, d\mu_h), \chi \in \mathbb{C}^N \}.
\]

Finally, recall also from the Introduction that a function \( \phi : \Omega \to \mathbb{C}^{N \times N} \) is called \textit{U-invariant} if \( \phi(ZU) = U^*\phi(Z)U \) for all \( Z \in \Omega \) and \( U \in U(N) \), where \( ZU = (U^*Z_1U, U^*Z_2U, \ldots, U^*Z_nU) \). (Clearly, this reduces to the definition from Section 2 if \( n = 1 \).)

Our main result is the following.

\textbf{Theorem 1.} (i) The mapping
\[
i : f(z) \otimes \chi \mapsto f^\#(Z)\chi
\]
is a unitary isomorphism of \( L^2_{\text{hol}}(\Omega, \mu_h) \otimes \mathbb{C}^N \) onto \( \mathcal{S}_h \).

(ii) Under this isomorphism, the Toeplitz operator \( T_\phi^{(h)} \), for a spectral function \( \phi = \phi^\# \), corresponds to the tensor product \( T_\phi^{(h)} \otimes I \) of the scalar Toeplitz operator \( T_\phi \) on \( L^2_{\text{hol}}(\Omega, d\mu_h) \) with the identity operator on \( \mathbb{C}^N \). (In other words — to the Toeplitz operator \( T_\phi^{(h)} \) on \( L^2_{\text{hol}}(\Omega, d\mu_h) \otimes \mathbb{C}^N \) with scalar matrix-valued symbol \( \phi I \) discussed in the second paragraph after (1.3) in the Introduction.)

(iii) Consequently, if \( \phi = \phi^\# \) and \( \psi = \psi^\# \) are two smooth spectral functions on \( \Omega \), then there exist unique spectral functions \( v_r \), \( r = 0, 1, 2, \ldots \), such that
\[
T_\phi^{(h)}T_\psi^{(h)} \approx \sum_{r=0}^\infty h^rT_{v_r}^{(h)} \quad \text{as } h \to 0
\]
in the sense of operator norms (i.e. as in (1.0)). In fact,
\[
v_r = C_r(f, g)^#, \quad \text{where}
\]
where \( C_r \) are the cochains \( \{E\} \) from the ordinary (i.e. scalar-valued) Berezin-Toeplitz quantization on \( \Omega \).

(iv) A function \( \phi : \Omega \to \mathbb{C}^{N \times N} \) is \textit{U-invariant} if and only if there exists a function \( \phi(\mathbf{d}_1; \mathbf{d}_2, \ldots, \mathbf{d}_N) \) from \( \Omega \times \Omega^{N-1} \) into \( \mathbb{C} \), symmetric in the last \( N - 1 \) variables \( \mathbf{d}_2, \ldots, \mathbf{d}_N \), such that \( \phi = \phi^\# \), where
\[
\phi^\#(Z) \equiv U^*(\diag_k(\phi(\mathbf{d}_k; \mathbf{d}_1, \ldots, \mathbf{d}_k, \ldots, \mathbf{d}_N)))U.
\]
The function \( \phi \) is uniquely determined by \( \phi \), and \( \phi \) is spectral if and only if \( \phi \) depends only on the first variable, i.e. if and only if \( \phi(\mathbf{d}_1; \mathbf{d}_2, \ldots, \mathbf{d}_N) = \phi(\mathbf{d}_1) \).
(v) For a $U$-invariant function $\phi = \phi^\#$, the Toeplitz operator $T^{(h)}_\phi$ corresponds, under the isomorphism \((3.2)\), to the tensor product $T^{(h)}_{\pi h \phi} \otimes I$, where

$$\pi_h f(z) := \frac{1}{\mu_h(\Omega)^{N-1}} \int_{\Omega^{N-1}} f(z_1, z_2, \ldots, z_N) \prod_{j=2}^N d\mu_h(z_j).$$

(vi) Consequently, for any two smooth $U$-invariant functions $\phi = \phi^\#$, $\psi = \psi^\#$ on $\Omega$ such that the ordinary (scalar-valued) Berezin-Toeplitz quantization on $\Omega$ is applicable to $\phi$ and $\psi$,

$$T^{(h)}_\phi T^{(h)}_\psi \approx \sum_{r=0}^\infty h^r T^{(h)}_{C_r(\pi h \phi, \pi h \psi)^\#}$$

in the sense of operator norms, where $C_r$ are the cochains \((1.5)\) from the ordinary Berezin-Toeplitz quantization on $\Omega$.

(vii) Finally, if, in addition, $\Omega$ is one of the domains mentioned in the paragraph after \((1.8)\) in the Introduction (examples of domains on which the scalar-valued Berezin-Toeplitz quantization is currently known to work), and $\phi$ and $\psi$ have compact support, then \((3.4)\) can be converted into an asymptotic expansion in powers of $h$, i.e. there exist uniquely determined spectral functions $\nu_r$, $r = 0, 1, \ldots$, such that

$$T^{(h)}_\phi T^{(h)}_\psi \approx \sum_{r=0}^\infty h^r T^{(h)}_{\nu_r^\#}$$

in the sense of operator norms.

The hypotheses in the part (vii) are made only for technical reasons, and can probably be weakened or dropped altogether.

**Proof.** (i) For any $f, g \in L^2_{hol}(\Omega, d\mu_h)$ and $\chi, \eta \in \mathbb{C}^N$, we have

$$\langle f^\#(\mathbf{Z}) \chi, g^\#(\mathbf{Z}) \eta \rangle = \int_{\Omega} \eta^* g^\#(\mathbf{Z})^* f^\#(\mathbf{Z}) \chi d\mu_h(\mathbf{Z})$$

$$= \mu_h(\Omega)^{-N} \int_{U(N)} \int_{\Omega^N} \eta^* U^* \text{diag}(g(d_k)) \text{diag}(f(d_k)) U \chi dU \prod_j d\mu_h(d_j).$$

Since, for any matrix $X$,

$$\int_{U(N)} U^* X U \ dU = \frac{\text{Tr}(X)}{N} I,$$

we can continue the computation by

$$= \frac{1}{N} \mu_h(\Omega)^{-N} \eta^* \chi \int_{\Omega^N} \sum_k \text{diag}(g(d_k)) f(d_k) \prod_j d\mu_h(d_j)$$

$$= \frac{1}{N} \mu_h(\Omega)^{-N} \eta^* \chi \sum_k \left( \int_{\Omega} d\mu_h \right)^{N-1} \int_{\Omega} g(d_k) f(d_k) d\mu_h(d_k)$$

$$= \frac{1}{N} \eta^* \chi N \langle f, g \rangle$$

$$= \langle \chi, \eta \rangle \langle f, g \rangle,$$

and the claim follows.
(i) For $f, g \in L^2_{\text{hol}}(\Omega, d\mu_h)$, $\chi, \eta \in \mathbb{C}^N$ and any function $\phi$ on $\Omega$, we have by a similar computation as in (i),
\[
\langle T^{(h)}_{\phi^*} f^\#, \chi \rangle = \int_{\Omega} \eta^* g^\#(Z)^* \phi^*(Z) f^\#(Z) \chi \, d\mu_h(Z) = \mu_h(\Omega)^{-1} \int_{U(N)} \int_{\Omega^N} \eta^* U^* \text{diag}(g(d_k)) \text{diag}(\phi(d_k)) \, d\mu_h(d_k)
\]
\[
= \mu_h(\Omega)^{-1} \int_{U(N)} \int_{\Omega^N} \eta^* \sum_k g(d_k) \phi(d_k) f(d_k) \, d\mu_h(d_k)
\]
\[
= \mu_h(\Omega)^{-1} \eta^* \chi \int_{\Omega^N} \sum_k g(d_k) \phi(d_k) f(d_k) \, d\mu_h(d_k)
\]
\[
= \mu_h(\Omega)^{-1} \eta^* \chi \int_{\Omega^N} \sum_k g(d_k) \phi(d_k) f(d_k) \, d\mu_h(d_k)
\]
\[
= \mu_h(\Omega)^{-1} \eta^* \chi \int_{\Omega^N} \sum_k g(d_k) \phi(d_k) f(d_k) \, d\mu_h(d_k)
\]
\[
= \mu_h(\Omega)^{-1} \eta^* \chi \int_{\Omega^N} \sum_k g(d_k) \phi(d_k) f(d_k) \, d\mu_h(d_k)
\]
\[
= \langle \chi, \eta \rangle \langle T^{(h)}_{\phi^*} f, g \rangle.
\]

(iii) follows immediately from (ii) and the ordinary Berezin-Toeplitz quantization on $\Omega$, upon tensoring with $\mathbb{C}^N$.

(iv) Let $\phi : \Omega \to \mathbb{C}^{N \times N}$ be a $U$-invariant function, and let $D = (D^{(1)}, \ldots, D^{(n)})$ be an element of $\Omega$ whose entries are diagonal matrices. For any complex numbers $\epsilon_1, \ldots, \epsilon_N$ of modulus one, consider the matrix $\epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_N)$. Then $\epsilon \in U(N)$ and $\epsilon D \epsilon^* = D$ for any diagonal matrix $D$, whence $D^\epsilon = D$; thus by the $U$-invariance condition,
\[
\phi(D) = \epsilon^* \phi(D) \epsilon \quad \forall \epsilon_1, \ldots, \epsilon_N \in \mathbb{T}.
\]
Consequently, $\phi(D)$ is also a diagonal matrix. Define the functions $f_1, \ldots, f_N : \Omega^N \to \mathbb{C}$ by
\[
f_j(d_1, d_2, \ldots, d_N) := \phi(D))_{jj} \quad \text{where } D = \text{diag}(d_1, \ldots, d_N).
\]
For any permutation $\sigma$ of the set $\{1, 2, \ldots, N\}$, let $F_\sigma$ denote the permutation matrix $[F_\sigma]_{jk} = \delta_{\sigma(j), k}$. Then $F_\sigma \in U(N)$ and $F_\sigma D F_\sigma^* = \text{diag}(d_{\sigma(1)}, \ldots, d_{\sigma(N)})$ if $D = \text{diag}(d_1, \ldots, d_N)$. Thus by the $U$-invariance condition again
\[
f_{\sigma(j)}(d_1, d_2, \ldots, d_N) = f_j(d_{\sigma(1)}, d_{\sigma(2)}, \ldots, d_{\sigma(N)}).
\]
It follows that $f_j$ is symmetric with respect to the last $N - 1$ variables $d_1, \ldots, d_{j-1}, \hat{d}_j, \ldots, d_N$ and $\phi = f^\#$ for $f = f_1$.

Conversely, it is easily seen that any function of the form \ref{spectral} is $U$-invariant, and $f^\# = g^\# \iff f = g$.

Finally, the assertion concerning spectral functions is immediate upon comparing \ref{spectral} and \ref{1}.

(v) Using \ref{spectral}, the assertion (v) now follows by a similar computation as in the proofs of (i) and (ii): for $f, g \in L^2_{\text{hol}}(\Omega, d\mu_h)$, $\chi, \eta \in \mathbb{C}^N$ and any $\phi : \Omega^N \to \mathbb{C}$ as in (iv), we have
\[
\langle T^{(h)}_{\phi^*} f^\#, \chi \rangle = \int_{\Omega} \eta^* g^\#(Z)^* \phi^*(Z) f^\#(Z) \chi \, d\mu_h(Z)
\]
\[ = \mu_h(\Omega)^{1-N} \int_{\Omega} \int_{\Omega}^N \eta^* U^* \operatorname{diag}_k(g(d_k)) \]
\[ \quad \text{(diag}_k(\phi(d_k; d_1, \ldots, d_N))) \operatorname{diag}_k(f(d_k))U \chi dU \prod_j d\mu_h(d_j) \]
\[ = \frac{1}{N} \mu_h(\Omega)^{1-N} \eta^* \chi \int_{\Omega}^N \sum_k g(d_k) \phi(d_k; d_1, \ldots, d_N) f(d_k) \prod_j d\mu_h(d_j) \]
\[ = \mu_h(\Omega)^{1-N} \eta^* \chi \int_{\Omega} g(d_1) \phi(d_1; d_2, \ldots, d_N) f(d_1) \prod_{j=2}^N d\mu_h(d_j) \]
\[ = \mu_h(\Omega)^{1-N} \eta^* \chi \int_{\Omega} \left( \int_{J=\Omega} \phi(d_1; d_2, \ldots, d_N) \prod_{j=2}^N d\mu_h(d_j) \right) f(d_1) d\mu_h(d_1) \]
\[ = \eta^* \chi \int_{\Omega} \frac{g(d_1)}{\pi_h} \phi(d_1) f(d_1) d\mu_h(d_1) \]
\[ = \eta^* \chi \langle \pi_h \phi, f, g \rangle \]
\[ = \langle \chi, \eta \rangle \langle T_{\pi_h \phi}, f, g \rangle. \]

(vi) With (v) in hands, we obtain from the ordinary Berezin-Toeplitz quantization on \( \Omega \), for any \( \phi, \psi : \Omega^N \rightarrow \mathbb{C} \) as in (iv),
\[ T_{\phi}^{(h)} T_{\psi}^{(h)} \approx T_{\pi_h \phi}^{(h)} T_{\pi_h \psi}^{(h)} \otimes I \]
\[ \approx \sum_{r=0}^{\infty} h^r T_{C_r(\pi_h \phi, \pi_h \psi)}^{(h)} \otimes I \]
\[ \approx \sum_{r=0}^{\infty} h^r T_{C_r(\pi_h \phi, \pi_h \psi)^{\#}}. \]
in the sense of operator norms (the last isomorphism being the one for spectral functions from part (i)), which proves (vi).

(vii) Finally, to convert the last expansion into one of the form \( (1.5) \) (i.e. in powers of \( h \)), we only need to exhibit a uniform asymptotic expansion for \( \pi_h \phi \), i.e. show that

\[(3.7) \quad \pi_h \phi \approx \sum_{r=0}^{\infty} h^r L_r \phi, \]
in the sense of norms in \( L^\infty(\Omega) \), for some linear operators \( L_r \) acting from functions on \( \Omega^N \) into functions on \( \Omega \). Indeed, since \( C_r \) are bidifferential operators with smooth coefficients and \( \phi, \psi \) are assumed to have compact support, it will then follow that
\[ C_r(\pi_h \phi, \pi_h \psi) \approx \sum_{j,k \geq 0} h^{j+k} C_r(L_k \phi, L_m \psi) \]
in the sense of \( L^\infty \) norms, and in view of the inequality \( \| T_0^{(h)} \| \leq \| \psi \|_\infty \), we can "apply \( T^{(h)}n \) to both sides.

In order to prove (3.7), it suffices in turn to show that there is an expansion of that form for
\[ (3.8) \quad h^k \mu_h(\Omega)^{N-1} \pi_h \phi(z) = h^k \int_{\Omega^N-1} \phi(z; z_2, \ldots, z_N) d\mu_h(z_2) \ldots d\mu_h(z_n), \]
\[ \text{for all $z$ in $\Omega$ and all $k$.} \]
for some $k \geq 0$, with leading coefficient which is positive on $\Omega$ when $\phi$ is identically 1. Indeed, specializing this to $\phi$ the constant one and dividing the two expansions gives (3.7). (The leading coefficient is needed to make sure that we are not dividing by zero.)

Finally, for the situations where the ordinary Berezin-Toeplitz quantization is nowadays known to work (as summarized in the paragraph following (1.8) in the Introduction), the measures $\mu_h$ are taken to be $e^{-\Phi/h} d\mu$, where $\Phi$ is a real-valued potential for the Kähler form $\omega$ and $d\mu(z) = \omega(z)^n = \det[\partial\partial\Phi(z)] dz = g(z) dz$ is the Liouville measure. However, in that case the right-hand side of (3.8) reduces to

$$\int_{\Omega^{N-1}} \phi(z; z_2, \ldots, z_N) e^{-\Phi(z_2)+\cdots+\Phi(z_N)}/h g(z) \ldots g(z_N) dz_2 \ldots dz_N,$$

which has an asymptotic expansion of the desired form by the usual stationary phase method (or, rather, Laplace’s method), with $k = (N-1)(n - \text{(the dimension of the variety on which } -\Phi \text{ attains its global minimum)})$, and the leading coefficient being essentially the integral of $\phi(z; z_2, \ldots, z_N) g(z_1) \ldots g(z_N)$ over that variety with respect to the corresponding Hausdorff measure; see e.g. [Fed], §4 of Chapter II, [Hrm], Section 7.7, or [Me], Chapter 7. This completes the proof. □

From a physical point of view, the sort of domains and Hilbert spaces envisaged in the above theorem could be used to describe system having $n$ kinematic and $N$ internal degrees of freedom.

4. SOME RELATED REPRODUCING KERNELS

The original motivation that led the authors to the spaces like $\mathcal{S}_h$ above did not actually come from quantization, but rather from an attempt to generalize to various vector- and matrix-valued setups the multifarious existing notions of coherent states from quantum optics (see e.g. [AAG]). In our case here, these are given essentially by the “normalized” reproducing kernels of the respective spaces, see [AEG]; for instance, as shown [AE2], for the spaces $\mathcal{S}_h$ of Section 2 they are just the family $k^{(h)}_{Z,\chi}$ of elements of $\mathcal{S}_h$, indexed by $Z \in \Omega$ (= the set of all normal $N \times N$ matrices) and vectors $\chi \in \mathbb{C}^N$, given by

$$k^{(h)}_{Z,\chi}(X) = K^{(h)}(X, Z) K^{(h)}(Z, Z)^{-1/2} \chi,$$

where

$$K^{(h)}(X, Y) = \sum_{k=0}^{\infty} \frac{X^k Y^* k!}{h^k}$$

is the reproducing kernel of $\mathcal{S}_h$.

Note that, despite the isomorphism from part (i) of Theorem 1 and the fact that the reproducing kernel for the corresponding space $\mathcal{H}_h$ is well known to be simply $e^{i <x, y>/h}$, the reproducing kernel (4.1) cannot be evaluated in a closed form, since the matrices $X$ and $Y^*$ do not commute. For the same reason, it is impossible to evaluate in closed form any of the kernels from Theorem 1 even if the corresponding kernels for $\mathcal{H}_h$ are known. All one can do is to write them again in the form (4.1), only the monomials need to be replaced by some general orthonormal basis of the space: namely, if $\{\psi_k\}$ is an arbitrary orthonormal basis of $\mathcal{H}_h = L^2_{hol}(\Omega, d\mu_h)$,
then by the well-known formula of Bergman [Be] the reproducing kernel of $\mathcal{H}_h$ is given by
\[
\sum_{k=0}^{\infty} \psi_k(x)\overline{\psi_k(y)}
\]
(the sum of the series does not depend on the choice of the orthonormal basis).

In view of the isomorphism (3.2), it therefore transpires that
\[
K^{(h)}(X, Y) := \sum_{k=0}^{\infty} \psi_k^*(X)\overline{\psi_k(Y)}^*
\]
is the reproducing kernel of the space $\mathcal{H}_h$, in the sense that it has the reproducing property
\[
f(Z) = \int_{\Omega} K^{(h)}(Z, X)f(X) \, d\mu_h(X) \quad \forall Z \in \Omega, \ f \in \mathcal{H}_h.
\]

In this short section we want to call attention to some situations when the quantization procedure from Theorem 1 does not apply, but there still exists a formula for the reproducing kernels like (4.2). They all arise as Cartesian products of the spaces from Theorem 1 for the complex plane $\mathbb{C}$, the unit disc $\mathbb{D}$, and, more generally, any one-dimensional domains $\Omega \subset \mathbb{C}$ for which the ordinary Berezin-Toeplitz quantization works; for simplicity of ideas, we describe the space corresponding to $\mathbb{C}^n$ (the construction for the Cartesian product of any other $n$ spaces of the above-mentioned type contains no additional new ideas).

The domain in this case will consist of all (not just commuting) $n$-tuples of normal matrices:
\[
\Omega = \{Z = (Z_1, \ldots, Z_n) \in \mathbb{C}^{n \times N \times N} : Z_j^* Z_j = Z_j Z_j^* \forall j\}.
\]

For the measure we take
\[
d\mu_h(Z) = d\mu_h(Z_1) \ldots d\mu_h(Z_n),
\]
where, abusing the notation a little, the $d\mu_h$ on the right-hand side stand for the measure (2.2) on the set of all $N \times N$ normal matrices from Section 2.

Finally, we define the space $\mathcal{H}_h$ to be the span in $L^2(\Omega, d\mu_h) \otimes \mathbb{C}^N$ of the functions
\[
Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n} \chi_j, \quad k_1, \ldots, k_n \geq 0, \ j = 1, \ldots, N.
\]

**Theorem 2.** The mapping
\[
Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n} \chi \mapsto z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} \otimes \chi
\]
is a unitary isomorphism of $\mathcal{H}_h$ onto $L^2(\mathbb{C}^n, e^{-\|z\|^2/(\pi h)^{-n}} \, dz_1 \ldots dz_n) \otimes \mathbb{C}^N$. Consequently, the reproducing kernel of $\mathcal{H}_h$ is given by
\[
K^{(h)}(X, Y) = \sum_{k_1, \ldots, k_n=0}^{\infty} \frac{X_1^{k_1} \ldots X_n^{k_n} Y_1^{*k_1} \ldots Y_n^{*k_n}}{k_1! \ldots k_n! h_1^{k_1} \ldots h_n^{k_n}}
\]

**Proof.** Let $Z_j = U_j^* D_j U_j$ be the spectral decomposition (2.1) of $Z_j$. Then by the definition of $d\mu_h$,
\[
\langle Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n} \chi, Z_1^{j_1} Z_2^{j_2} \ldots Z_n^{j_n} \eta \rangle = \int_{\Omega} \eta^* Z_1^{j_1} \ldots Z_1^{j_n} Z_1^{k_1} \ldots Z_n^{k_n} \chi \, d\mu_h(Z)
\]
\[
= (\pi h)^{-nN} \int_{\mathbb{C}^N} \ldots \int_{\mathbb{C}^N} \int_{U(N)} \ldots \int_{U(N)} \eta^* U_1^{*j_1} U_2^{*j_2} \ldots U_n^{*j_n} U_1^{k_1} \ldots U_n^{k_n} \, d\mu_h(Z)
\]
Applying successively (3.6) to \( U_1 \), \( U_2, \ldots, U_n \), we obtain

\[
\frac{(\pi h)^{-nN}}{N} \int_{U(N)} \cdots \int_{U(N)} \eta^* U_n^* D_n^{*j_m} U_n C \epsilon U_n e^{-\text{Tr}(D_1^* D_1 + \cdots + D_n^* D_n)/h} \, dD_1 \cdots dD_n
\]

\[
\frac{(\pi h)^{-nN}}{N^2} \int_{U(N)} \cdots \int_{U(N)} \eta^* U_n^* D_n^{*j_m} U_n C \epsilon U_n e^{-\text{Tr}(D_1^* D_1 + \cdots + D_n^* D_n)/h} \, dD_1 \cdots dD_n
\]

\[
\frac{(\pi h)^{-nN}}{N^n} \int_{U(N)} \cdots \int_{U(N)} \eta^* \text{Tr}(D_1^* D_1) \cdots \text{Tr}(D_n^* D_n) C \epsilon U_n e^{-\text{Tr}(D_1^* D_1 + \cdots + D_n^* D_n)/h} \, dD_1 \cdots dD_n
\]

This settles the first claim. Besides, it shows that the function \( K^{(h)}(X, Y) \) satisfies

\[
\int_{\Omega} K^{(h)}(X, Z) Z_1^{k_1} \cdots Z_n^{k_n} \epsilon \, d\mu_h(Z)
\]

\[
= \sum_{j_1, \ldots, j_n=0}^{\infty} \frac{X_1^{j_1} \cdots X_n^{j_n}}{k_1! \cdots k_n! h^{k_1 + \cdots + k_n}} \int_{\Omega} Z_1^{*j_1} \cdots Z_n^{*j_n} \epsilon \, d\mu_h(Z)
\]

\[
= \sum_{j_1, \ldots, j_n=0}^{\infty} \frac{X_1^{j_1} \cdots X_n^{j_n}}{k_1! \cdots k_n! h^{k_1 + \cdots + k_n}} \delta_{j_1 k_1} \cdots \delta_{j_n k_n} \epsilon \, d\mu_h(Z)
\]

\[
= X_1^{k_1} \cdots X_n^{k_n},
\]

i.e., has the reproducing property (4.3) for functions of the form \( Z_1^{k_1} \cdots Z_n^{k_n} \); since the latter span all of \( \mathcal{H}_h \) by definition, the second part of the theorem also follows.

\[\Box\]
Although the reproducing kernels and the isomorphism \(^{(3.2)}\) work out fine, what breaks down is that part (ii) of Theorem \([1]\) the Toeplitz operators on \(\mathcal{S}_h\) do not correspond, under the isomorphism above, to the Toeplitz operators on the Segal-Bargmann space \(L^2_{\text{hol}}(\mathbb{C}^n, e^{-\|z\|^2/(\pi h)} dz_1 \ldots dz_n)\). The reason is the noncommutativity of \(Z_1, \ldots, Z_n\) — the reader can try to go through the beginning of the proof of part (ii) of Theorem \([1]\) to see what is happening. For the very same reason, it also not possible to define spectral functions (and much less to describe the \(U\)-invariant ones), and it is totally unclear at the moment how to achieve anything similar to the quantization from the previous section.

We take this occasion to remark that one lands in even greater difficulties if one tries to deal with domains of arbitrary (rather than just normal) matrices. See Section 4 of \([AE2]\) for details.

On the other hand, it is clearly possible to use any other ordering of the entries of \(Z\) in Theorem \([2]\) than \(Z_1 \ldots Z_n\) (for instance, \(Z_n \ldots Z_1\)); we omit the details.

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