THE BOULEAU-YOR IDENTITY FOR A BI-FRACTIONAL BROWNIAN MOTION∗

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ABSTRACT. Let $B$ be a bi-fractional Brownian motion with indices $H \in (0, 1)$, $K \in (0, 1]$, $2HK = 1$ and let $\mathcal{L}(x, t)$ be its local time process. We construct a Banach space $\mathcal{H}$ of measurable functions such that the quadratic covariation $[f(B), B]$ and the integral $\int_{\mathbb{R}} f(x) \mathcal{L}(dx, t)$ exist provided $f \in \mathcal{H}$. Moreover, the Bouleau-Yor identity

$$[f(B), B]_t = -2^{1-K} \int_{\mathbb{R}} f(x) \mathcal{L}(dx, t), \quad t \geq 0,$$

holds for all $f \in \mathcal{H}$.

1. INTRODUCTION

The bi-fractional Brownian motion (bi-fBm) with indices $H \in (0, 1)$ and $K \in (0, 1]$ is a zero mean Gaussian process $B = \{B_t, t \geq 0\}$ such that $B_0 = 0$ and

(1.1) $E[B_t B_s] = \frac{1}{2K} \left[(t^{2H} + s^{2H})^K - |t-s|^{2HK}\right]$ for all $s, t \geq 0$. Clearly, if $K = 1$, the process is a fractional Brownian motion with Hurst parameter $H$. Bi-fBm was first introduced by Houdré–Villa [11]. The process $B$ is HK-selfsimilar but it has no stationary increments. It has Hölder continuous paths of order $\delta < HK$ and its paths are not differentiable. An interesting property is that the bi-fBm has non-trivial quadratic variation equal with a constant times $t$ in the case $2HK = 1$, which is similar to this of the standard Brownian motion. That is

$$[B, B]_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t (B_{s+\varepsilon} - B_s)^2 ds = 2^{1-K} t, \quad t \geq 0$$

in $L^2(\Omega)$ (for this, see Russo–Tudor [19]). This motivates us to study the quadratic covariation and related to stochastic calculus of bi-fBm with $2HK = 1$. More works for bi-fBm can be found in Es-sebaiy–Tudor [7], Jiang-Wang [12], Kruk et al [13], Lei-Nualart [14], Russo-Tudor [19], Tudor-Xiao [24], Shen-Yan [23], Yan et al [26] and the references therein.

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Let now $2HK = 1$ and let $B = \{B_t, 0 \leq t \leq T\}$ be the bi-fBm on $\mathbb{R}$ with indices $H$ and $K$. In order to motivate our subject, let us first recall some known results concerning the quadratic variation and Itô's formula. Let $W$ be a standard Brownian motion and let $F$ be an absolutely continuous function with locally square integrable derivative $f$, that is,

$$F(x) = F(0) + \int_0^x f(y)dy$$

with $f$ being locally square integrable. Föllmer et al. [9] introduced the following Itô’s formula:

$$(1.2) \quad F(W_t) = F(0) + \int_0^t f(W_s)dW_s + \frac{1}{2} [f(W), W]_t.$$ 

Moreover, the result has been extended to some semimartingales and smooth nondegenerate martingales (see Russo–Vallois [20] and Moret–Nualart [15]). Thus, it is natural to ask whether the similar Itô formula for bi-fractional Brownian motion $B$ with $2HK = 1$, more general, for finite quadratic variation process $X$ holds or not. We will consider the question. Recall that a process $X$ is said to be of finite quadratic variation if quadratic variation $[X,X]$ is finite. For any continuous finite quadratic variation process $X$ and twice-differentiable function $f$, we have (see, for example, Russo-Vallois [21])

$$(1.3) \quad f(X_t) = f(0) + \int_0^t f'(X_s)dX_s + \frac{1}{2} [f'(X), X]_t,$$

where the integral $\int_0^t f(X_s)dX_s$ is the forward (pathwise) integral defined by

$$\int_0^t f(X_s)dX_s = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t f(X_s)(X_{s+\varepsilon} - X_s)ds$$

and the quadratic covariation $[f'(X), X]$ of $f'(X)$ and $X$ is defined as

$$(1.4) \quad [f'(X), X]_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \{f'(X_{s+\varepsilon}) - f'(X_s)\}(X_{s+\varepsilon} - X_s)ds,$$

provided the limit exists uniformly in probability. However, the formula (1.3) is only effective on twice-differentiable functions. It is impossible to list here all the contributors in previous topics. Some surveys and complete literatures could be found in Nualart [16], Russo-Vallois [22] and F. Russo-Tudor [19]. In this paper, our aim is to prove Itô’s formula (1.3) holds for $X = B$ with $2HK = 1$ whatever $f \in C^2(\mathbb{R})$, and obtain the relation between the forward (pathwise) integral and the Skorohod integral of bi-fractional Brownian motion with $2HK = 1$. Though our method is only effective on bi-fractional Brownian motion, the merit here has been to concentration fully on fBm in order to get a stronger statement by fully using bi-fractional Brownian motion’s regularity. In the present paper, we consider the case $2HK = 1$. Our start point is to consider the decomposition

$$(1.5) \quad \frac{1}{\varepsilon} \int_0^t \{f(B_{s+\varepsilon}) - f(B_s)\}(B_{s+\varepsilon} - B_s)ds$$

$$= \frac{1}{\varepsilon} \int_0^t f(B_{s+\varepsilon})(B_{s+\varepsilon} - B_s)ds - \frac{1}{\varepsilon} \int_0^t f(B_s)(B_{s+\varepsilon} - B_s)ds.$$
By estimating the two terms of the right hand side in the decomposition (1.5), respectively, we can construct a Banach space $H$ of measurable functions $f$ on $\mathbb{R}$ such that $\|f\|_H < \infty$, where

$$\|f\|_H^2 := \int_0^T \int_\mathbb{R} |f(x)|^2 \varphi_s(x) dx ds + \int_0^T \int_\mathbb{R} |f(x)x|^2 \varphi_s(x) \frac{dx ds}{s}$$

with $\varphi_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}$. We show that the quadratic covariation $[f(B), B]_t$ exists in $L^2$ for all $t \in [0, T]$ if $f \in H$. This allows us to write the following Itô’s formulas (Föllmer-Protter-Shiryayev’s formula):

$$F(B_t) = F(0) + \int_0^t f(B_s) dB_s + 2^{K-2} [f(B), B]_t,$$

where the integral $\int_0^t f(B_s) dB_s$ is the Skorohod integral, $f \in H$ is left continuous with right limit and $F$ is an absolutely continuous function with $\frac{d}{dx} F = f$. This extends the formula (1.3) for bi-fractional Brownian motion $B$ with $2HK = 1$. As an application we establish the following integral:

$$\int_\mathbb{R} f(x) \mathcal{L}(dx, t), \quad t \in [0, T],$$

and show that the Bouleau-Yor identity

$$[f(B), B]_t = -2^{1-K} \int_\mathbb{R} f(x) \mathcal{L}(dx, t)$$

holds provided $f \in H$, where

$$\mathcal{L}(x, t) = \int_0^t \delta(B_s - x) ds$$

is the local time of bi-fractional Brownian motion $B$.

For $K = 1$ and $H = \frac{1}{2}$, the process $B$ is classical Brownian motion $W$ and the above results first are studied by Bouleau–Yor [3] and Föllmer et al [9]. Moreover, these have also been extended to semimartingales by Bardina–Rovira [2], Eisenbaum [1, 5], Elworthy et al [6], Feng–Zhao [8], Peskir [17], Rogers–Walsh [18], Yan–Yang [28]. For $K = 1$ and $H \neq \frac{1}{2}$, the process $B$ is a standard fractional Brownian motion $B^H$ with Hurst index $H$. Yan et al [25, 27] studied the integration with respect to local time of fractional Brownian motion, and the weighted quadratic covariation $[f(B^H), B^H]^{(W)}$ of $f(B^H)$ and $B^H$. These deduce the fractional Itô formula for new classes of functions. For $2HK = 1$ and $K \neq 1$, this process is not fractional Brownian motion, and the question has not been studied. Recently, the long-range property has become an important aspect of stochastic models in various scientific area including hydrology, telecommunication, turbulence, image processing and finance. It is well-known that fractional Brownian motion is one of the best known and most widely used processes that exhibits the long-range property, self-similarity and stationary increments. It is a suitable generalization of classical Brownian motion. On the other hand, many authors have proposed to use more general self-similar Gaussian process and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. However, contrast to the extensive studies on fractional Brownian motion, there has been little
systematic investigation on other self-similar Gaussian processes. The main reason for 
this is the complexity of dependence structures for self-similar Gaussian processes which 
does not have stationary increments. The bi-fractional Brownian motion has properties 
analogous to those of fractional Brownian motion (self-similarity, long-range dependence, 
Hölder paths, the variation and the renormalized variation). However, in comparison with 
fractional Brownian motion, the bi-fractional Brownian motion has non-stationary incre-
ments and the increments over non-overlapping intervals are more weakly correlated and 
their covariance decays polynomially as a higher rate. The above mentioned properties 
make bi-fractional Brownian motion a possible candidate for models which involve long-
dependence, self-similarity and non-stationary. Therefore, it seems interesting to study 
the quadratic covariation and extension of Itô’s formula of bi-fractional Brownian motion 
with $2HK = 1$.

This paper is organized as follows. In Section 2 we present some preliminaries for bi-
fractional Brownian motion. In Section 3, we establish some technical estimates associated 
with bi-fBm with $2HK = 1$ and it seems interesting that these inequalities arising from 
the method. In Section 4 we will construct the Banach space $\mathcal{H}$ such that the quadratic 
covariation $\left[f(B), B\right]$ exists in $L^2$ for $f \in \mathcal{H}$. In section 5 our main object is to explain 
and prove the generalized Itô type formula (1.6). As an application we introduce the 
relationship between the forward (pathwise) integral and Skorohod integral 
\[
\int_0^t f(B_s)d^- B_s = \int_0^t f(B_s)dB_s + \frac{1}{2}(2K-1) \left[f(B), B\right]_t
\]
for all $f \in \mathcal{H}$. The result weakens the hypothesis of differentiability for $f$ (see Russo-
Tudor [19]). In Section 6 we study the integral (1.7) and show that the Bouleau-Yor 
identity (1.8) holds.

\section{Preliminaries for bi-fractional Brownian motion}

In this section, we briefly recall the definition and properties of stochastic integral with 
respect to bi-fBm. As a Gaussian process, it is possible to construct a stochastic calculus 
of variations with respect to $B$. We refer to Alós et al [11] and Nualart [16] for a complete 
description of stochastic calculus with respect to Gaussian processes. Here we recall only 
the basic elements of this theory (see Es-sebai–Tudor [7]). Throughout this paper we 
assume that $2HK = 1$. As we pointed out before, bi-fractional Brownian motion (bi-fBm 
in short) $B = \{B_t, 0 \leq t \leq T\}$, on the probability space $(\Omega, \mathcal{F}, P)$ with indices $H \in (0,1)$ 
and $K \in (0,1]$ is a rather special class of self-similar Gaussian processes such that $B_0 = 0$ and 
\begin{equation}
E[B_tB_s] = R(t,s) := \frac{1}{2K} \left[(t^{2H} + s^{2H})^K - |t-s|^{2HK}\right], \quad \forall s,t \geq 0.
\end{equation}

The process is $HK$-self similar and satisfies the following estimates (the quasi-helix prop-
erty)
\begin{equation}
2^{-K}|t-s|^{2HK} \leq E \left[(B_t - B_s)^2\right] \leq 2^{1-K}|t-s|^{2HK}.
\end{equation}
Thus, Kolmogorov’s continuity criterion implies that bi-fBm is Hölder continuous of order \( \delta \) for any \( \delta < HK \).

Let \( \mathcal{H} \) be the completion of the linear space \( \mathcal{E} \) generated by the indicator functions \( 1_{[0,t]}, t \in [0,T] \) with respect to the inner product \( \langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = R(s,t) \).

The application \( \varphi \in \mathcal{E} \to B(\varphi) \) is an isometry from \( \mathcal{E} \) to the Gaussian space generated by \( B \) and it can be extended to \( \mathcal{H} \). For \( 2HK = 1 \) we can characterize \( \mathcal{H} \) as

\[
\mathcal{H} = \{ f : [0,T] \to \mathbb{R} | \| f \|_{\mathcal{H}} < \infty \},
\]

where

\[
\| f \|_{\mathcal{H}}^2 = \int_0^T \int_0^T f(t)f(s)\phi(s,t)dsdt
\]

with \( \phi(s,t) = (s^{2H} + t^{2H})^{K-2} t^{2H-1}s^{2H-1} \). Let us denote by \( \mathcal{S} \) the set of smooth functionals of the form

\[
F = f(B(\varphi_1), B(\varphi_2), \ldots, B(\varphi_n)),
\]

where \( f \in C_0^\infty(\mathbb{R}^n) \) and \( \varphi_i \in \mathcal{H} \). The Malliavin derivative \( D^{H,K} \) of a functional \( F \) as above is given by

\[
D^{H,K} F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\varphi_1), B(\varphi_2), \ldots, B(\varphi_n))\varphi_j.
\]

The derivative operator \( D^{H,K} \) is then a closable operator from \( L^2(\Omega) \) into \( L^2(\Omega; \mathcal{H}) \). We denote by \( \mathbb{D}^{1,2} \) the closure of \( \mathcal{S} \) with respect to the norm

\[
\| F \|_{1,2} := \sqrt{E|F|^2 + E\| D^{H,K} F \|_{\mathcal{H}}^2}.
\]

The divergence integral \( \delta^{H,K} \) is the adjoint of derivative operator \( D^{H,K} \). That is, we say that a random variable \( u \) in \( L^2(\Omega; \mathcal{H}) \) belongs to the domain of the divergence operator \( \delta^{H,K} \), denoted by \( \text{Dom}(\delta^{H,K}) \), if

\[
E|\langle D^{H,K} F, u \rangle_{\mathcal{H}}| \leq c\| F \|_{L^2(\Omega)}
\]

for every \( F \in \mathbb{D}^{1,2} \), where \( c \) is a constant depending only on \( u \). In this case \( \delta^{H,K}(u) \) is defined by the duality relationship

\[
E \left[ F \delta^{H,K}(u) \right] = E \langle D^{H,K} F, u \rangle_{\mathcal{H}}
\]

for any \( F \in \mathbb{D}^{1,2} \). We have \( \mathbb{D}^{1,2} \subset \text{Dom}(\delta^{H,K}) \) and for any \( u \in \mathbb{D}^{1,2} \)

\[
E \left[ \delta^{H,K}(u)^2 \right] = E\| u \|_{\mathcal{H}}^2 + E\langle D^{H,K} u, (D^{H,K} u)^* \rangle_{\mathcal{H} \otimes \mathcal{H}}
\]

\[
= E\| u \|_{\mathcal{H}}^2 + E \int_{[0,T]^2} D^{H,K}_\xi u_r D^{H,K}_\eta u_s \phi(\eta, r)\phi(\xi, s)dsdrd\xi d\eta,
\]

where \( (D^{H,K} u)^* \) is the adjoint of \( D^{H,K} u \) in the Hilbert space \( \mathcal{H} \otimes \mathcal{H} \). We will denote

\[
\delta^{H,K}(u) = \int_0^T u_s dB_s
\]

for an adapted process \( u \), and it is called Skorohod integral.
**Theorem 2.1** (Itô’s formula [7]). Let \( f \in C^2(\mathbb{R}) \) such that
\[
\max \{ |f(x)|, |f'(x)|, |f''(x)| \} \leq \kappa e^{\beta x^2},
\]
where \( \kappa \) and \( \beta \) are positive constants with \( \beta < (4T)^{-1} \). Suppose that \( 2HK = 1 \), then we have
\[
f(B_t) = f(0) + \int_0^t \frac{d}{dx}f(B_s)dB_s + \frac{1}{2} \int_0^t \frac{d^2}{dx^2}f(B_s)ds.
\]

Recall that bi-fBm \( B \) has a local time \( L(x,t) \) continuous in \( (x,t) \in \mathbb{R} \times [0,\infty) \) which satisfies the occupation formula (see Geman-Horowitz [10])
\[
\int_0^t \psi(B_s,s)ds = \int_\mathbb{R} dx \int_0^t \psi(x,s)L(x,ds)
\]
for every continuous and bounded function \( \psi(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) and any \( t \geq 0 \), and such that
\[
\mathcal{L}(x,t) = \int_0^t \delta(B_s - x)ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda(s \in [0,t], |B_s - x| < \epsilon),
\]
where \( \lambda \) denotes Lebesgue measure and \( \delta \) is the Dirac delta function. Moreover \( \mathcal{L} \) has a compact support in \( x \) for all \( t \geq 0 \) and the following Tanaka formula holds:
\[
|B_t - x| = |x| + \int_0^t \text{sign}(B_s - x)dB_s + \mathcal{L}(x,t).
\]
For these see Es-sebaiy–Tudor [7] and Tudor–Xiao [24].

3. **Some estimates associated with bi-fBm with \( 2HK = 1 \)**

In this section we will establish some technical estimates associated with bi-fBm. For simplicity throughout this paper we let \( C \) stand for a positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to \( c \).

**Lemma 3.1.** Let \( 2HK = 1 \), and for all \( s, r \in [0,T] \), \( s \geq r \) we denote
\[
\rho_{s,r}^2 = sr - \mu^2
\]
where \( \mu_{s,r} = E(B_sB_r) \). Then we have
\[
r(s - r) \leq \rho_{s,r}^2 \leq (1 + 2^{1-2K})r(s - r).
\]

By the local nondeterminacy of bi-fBm we can prove the lemma. Moreover, one can also obtain the estimates by considering the asymptotic property of some functions. Here, we shall prove these estimates (3.1) by an elementary method, and it seems interesting that these inequalities arising from the method. We shall use the following inequalities:
\[
(1 + x)^\alpha \leq 1 + (2^\alpha - 1)x^\alpha
\]
\[
(1 + x)^\beta \geq 1 + (2^\beta - 1)x^\beta
\]
for all $0 \leq x \leq 1$, $0 \leq \alpha \leq 1$ and $\beta \geq 1$. The inequalities above are two calculus exercises, and they are stronger than the well known inequalities

$$(1 + x)^\alpha \leq 1 + \alpha x^\alpha \leq 1 + x^\alpha$$
$$(1 + x)^\beta \geq 1 + x^\beta$$

because of $2^\alpha - 1 \leq \alpha$ and $2^\beta - 1 \geq 1$ for all $0 \leq \alpha \leq 1$ and $\beta \geq 1$. Furthermore, by applying the inequality (3.2) one can improve the left estimate in (2.2) as (see Yan et al [26])

$$|t - s|^{2HK} \leq E [(B_t - B_s)^2],$$
for all $H \in (0, 1)$ and $K \in (0, 1]$.

**Proof of Lemma 3.1.** Clearly, by (3.2) we have

$$\rho_{s,r} = sr - \mu_{s,r}^2 = sr - \frac{1}{4K} [(s^{2H} + r^{2H})^K - (s - r)]^2 \geq sr - r^2 = r(s - r).$$

In order to show that the right estimate in (3.1), we have $\frac{1}{2} \leq K \leq 1$ and

$$(s^{2H} + r^{2H})^{2K} \geq s^{4HK} + (2^{2K} - 1)r^{4HK} = s^2 + (2^{2K} - 1)r^2,$$
which deduces

$$\rho_{s,r} = sr - \mu_{s,r}^2 = sr - \frac{1}{4K} [(s^{2H} + r^{2H})^K - (s - r)]^2$$
$$= \frac{1}{4K} \left( 4K sr - (s^{2H} + r^{2H})^{2K} + 2(s^{2H} + r^{2H})^K (s - r) - (s - r)^2 \right)$$
$$\leq \frac{1}{4K} \left( 4K sr - (s^2 + (2^{2K} - 1)r^2) + 2(s + r)(s - r) - (s - r)^2 \right)$$
$$= \frac{1}{4K} \left( (4K + 2)sr - (4K + 2)r^2 \right)$$
$$= (1 + 2^{1-2K})r(s - r)$$

by (3.3) and the inequality

$$(s^{2H} + r^{2H})^K \leq s + r$$
for all $s, r \geq 0$. This completes the proof. □

**Lemma 3.2.** Let $2HK = 1$. Then we have

$$c_{H,K} \frac{r}{s} (s - r) \leq r - \mu_{s,r} \leq C_{H,K} \frac{r}{s} (s - r)$$

and

$$s - r \leq s - \mu_{s,r} \leq C_{H,K} (s - r)$$

for all $s > r \geq 0$. 
Proof. In order to show that the estimates (3.5), we have

\[ r - \mu_{s,r} = r - \frac{1}{2K} \left[ (s^{2H} + r^{2H})^K - (s - r) \right] \]

\[ = \frac{1}{2K} s \left\{ 2^K x - (1 + x^{2H})^K + 1 - x \right\} \]

\[ = \frac{1}{2K} s \left\{ 1 + (2^K - 1)x - (1 + x^{2H})^K \right\} > 0 \]

for all \( x = \frac{s}{r} \in (0, 1) \) by the inequality (3.2). Elementary calculus can show that

\[ \lim_{x \to 0} \frac{1 + (2^K - 1)x - (1 + x^{2H})^K}{x(1 - x)} = 2^K - 1 \]

(3.7)

\[ \lim_{x \to 1} \frac{1 + (2^K - 1)x - (1 + x^{2H})^K}{x(1 - x)} = 1 - 2^{K-1} \]

(3.8)

which deduce the estimates (3.5) by continuity.

On the other hand, by the inequality (3.2) we have

\[ s - \mu_{s,r} = s - \frac{1}{2K} \left[ (s^{2H} + r^{2H})^K - (s - r) \right] \]

\[ \geq s - 2^K \left( s + (2^K - 1)r \right) + 2^{-K} (s - r) \]

\[ = s - r \]

for all \( s \geq r \), and moreover we have

\[ s - \mu_{s,r} = s - r + (r - \mu_{s,r}) \leq C_{H,K}(s - r) \]

by the right estimates in (3.5). □

Lemma 3.3. Let \( 2HK = 1 \). Then we have

\[ |E(B_t - B_s)(B_{t'} - B_{s'})| \leq (2H - 1)2^{-K} (t - s)(t' - s') \]

(3.9)

holds for all \( T \geq t > s \geq t' > s' > 0 \).

Proof. For \( y > 0 \) we define the function \( x \mapsto G_y(x) \) on \([0, T]\) by

\[ G_y(x) = (y^{2H} + x^{2H})^{K-1}. \]

Thanks to mean value theorem, we see that there is an \( \xi_t \in (s', t') \) such that

\[ G_t(t') - G_t(s') = 2H(K - 1)\xi_t^{2H-1}(t' - s') \left( t'^{2H} + \xi_t^{2H} \right)^{K-2}. \]
It follows from the duality relationship that

\[ -E(B_t - B_s)(B_{t'} - B_{s'}) = - \int_s^t \int_{s'}^{t'} \frac{\partial^2}{\partial r \partial l} R(r, l) dr dl \]

\[ = 2H(1 - K)2^-K \int_s^t \int_{s'}^{t'} (r^{2H} + l^{2H})^{K-2} r^{2H-1} l^{2H-1} dr dl \]

\[ = 2H(1 - K)2^-K \int_s^t l^{2H-1} dr \int_{s'}^{t'} (r^{2H} + l^{2H})^{K-2} r^{2H-1} dr \]

\[ = -2^-K \int_s^t l^{2H-1} \{ G_l(t') - G_l(s') \} dl \]

\[ = -2H(K - 1)2^-K (t' - s') \int_s^t l^{2H-1} (l^{2H} + \xi_l^{2H})^{K-2} \xi_l^{2H-1} dl. \]

Notice that

\[ \frac{1}{(l^{2H} + \xi_l^{2H})^{2-K}} \leq \frac{1}{l^{2H\alpha(2-K)} \xi_l^{2H(1-\alpha)(2-K)}} \]

\[ = \frac{1}{l^{\alpha(4H-1)} \xi_l^{(1-\alpha)(4H-1)}} = \frac{1}{l^{2H} \xi_l^{2H-1}} \]

with \( 1 - \alpha = \frac{2H-1}{4H-1} \) by Young’s inequality. We get

\[ \int_s^t l^{2H-1} (l^{2H} + \xi_l^{2H})^{K-2} \xi_l^{2H-1} dl \leq \int_s^t l^{-1} dl \leq \frac{t - s}{s}, \]

which deduces

\[ |E(B_t - B_s)(B_{t'} - B_{s'})| \leq \frac{(t - s)(t' - s')}{s}. \]

This completes the proof. \( \square \)

From the proof of the above lemma we also have

(3.10) \[ 0 \leq -E(B_t - B_s)(B_{t'} - B_{s'}) \leq \frac{(t - s)(t' - s')^{1-H}}{(1-H)s^{1-H}} \]

holds for all \( t > s \geq t' > s' > 0 \). In fact, under the notations of proof of the above lemma we have

\[ 0 \leq -[G_r(t) - G_r(s)] = 2H(1 - K)(t - s) \frac{\xi_l^{2H-1}}{(r^{2H} + \xi_l^{2H})^{2-K}} \]

\[ = 2H(1 - K)(t - s) \frac{\xi_l^{2H-1}}{(r^{2H} + \xi_l^{2H})^{1-K} (r^{2H} + \xi_l^{2H})} \]

\[ \leq (t - s) \frac{\xi_l^{2H-1}}{r^{2H(1-K)}r^H \xi_l H} \leq \frac{t - s}{r^{3H-1} s^{1-H}} \]
by Cauchy’s inequality. It follows that

\[ |E(B_t - B_s)(B_{t'} - B_{s'})| = 2H(1 - K) \int_{s'}^{t'} \int_s^t (r^{2H} + l^{2H})^{K-2} r^{2H-1} l^{2H-1} drdl \]

\[ \leq \int_{s'}^{t'} r^{2H-1} |G_r(t) - G_r(s)| dr \leq \int_{s'}^{t'} \frac{t - s}{r H s^{1-H}} dr \]

\[ = \frac{1}{(1 - H)s^{1-H}} (t - s) (t'^{1-H} - s^{1-H}) \]

\[ \leq \frac{(t - s)(t' - s)^{1-H}}{(1 - H)s^{1-H}}, \]

which deduces the estimate (3.10).

**Lemma 3.4.** For $2HK = 1$ we have

(3.11) \[ |E[B_s(B_t - B_s)]| \leq C_{H,K} \frac{s}{t} (t - s), \]

(3.12) \[ |E[B_r(B_t - B_s)]| \leq \frac{r}{s} (t - s), \]

(3.13) \[ |E[B_s(B_t - B_r)]| \leq 4(t - r), \]

(3.14) \[ |E[B_t(B_s - B_r)]| \leq 2(s - r) \]

for all $t > s > r > 0$.

**Proof.** Keeping the notation in the proof of Lemma 3.3. For the estimate (3.11) we have

\[ |E[B_s(B_t - B_s)]| = \frac{1}{2K} |(t^{2H} + s^{2H})^K - (t - s) - 2K s| \]

\[ = \frac{t}{2K} \left( 1 + (2K - 1)x - (1 + x^{2H})^K \right) \]

\[ \leq C_{H,K} x(1 - x) = C_{H,K} \frac{s}{t} (t - s) \]

with $x = \frac{s}{t}$ by the identities (3.7) and (3.8).

In order to prove the other estimates we define the function $g_r : \mathbb{R}+ \to \mathbb{R}$ for $r > 0$ by

\[ x \mapsto g_r(x) = (r^{2H} + x^{2H})^K. \]

We then have by mean value theorem,

\[ g_r(t) - g_r(s) = (t - s)\xi_r^{2H-1}(r^{2H} + \xi_r^{2H})^{K-1} = (t - s)\xi_r^{2H-1} G_{\xi_r}(r) \]

for some $\xi_r \in (s,t)$, and

\[ |\xi_r^{2H-1} G_{\xi_r}(r) - 1| = 1 - \frac{\xi_r^{2H-1}}{(r^{2H} + \xi_r^{2H})^{1-K}} = 1 - \left( \frac{\xi_r^{2H}}{r^{2H} + \xi_r^{2H}} \right)^{1-K} \]

\[ \leq 1 - \frac{\xi_r^{2H}}{r^{2H} + \xi_r^{2H}} \leq \frac{r^{2H}}{r^{2H} + \xi_r^{2H}} \leq \frac{r^{2H}}{s^{2H}}, \]

which deduces

\[ |E[B_r(B_t - B_s)]| = \frac{1}{2K} |g_r(t) - g_r(s) - (t - s)| \]

\[ \leq (t - s)|\xi_r^{2H-1} G_{\xi_r}(r) - 1| \leq \frac{r^{2H}}{s^{2H}} (t - s) \leq \frac{r}{s} (t - s). \]
This gives the estimate (3.12).

For (3.13), by mean value theorem we have

\[
|E[B_s(B_t - B_r)]| = \frac{1}{2K} |g_s(t) - g_s(r) - (t - r) + 2(s - r)|
\]

\[
= \frac{1}{2K} |(t - r)\xi_s^{2H-1}(s^{2H} + \xi_s^{2H})K^{-1} - (t - r) + 2(s - r)|
\]

\[
\leq \frac{1}{2K} ((t - r)\xi_s^{2H-1}(s^{2H} + \xi_s^{2H})K^{-1} + (t - r) + 2(s - r))
\]

\[
\leq \frac{1}{2K}(t - r) (\xi_s^{2H-1}(s^{2H} + \xi_s^{2H})K^{-1} + 3)
\]

\[
\leq \frac{4}{2K}(t - r)
\]

for some \( \xi_s \in (r, t) \). Similarly, we also have

\[
|E[B_t(B_s - B_r)]| = \frac{1}{2K} |g_t(s) - g_t(r) + (s - r)| \leq \frac{2}{2K}(s - r),
\]

which obtains (3.14). Thus, we complete the proof. \( \square \)

Let \( \varphi(x, y) \) be the density function of \( (B_s, B_r) \) \((s > r > 0)\). That is

\[
\varphi(x, y) = \frac{1}{2\pi\rho} \exp \left\{ - \frac{1}{2\rho^2} \left( rx^2 - 2\mu_{xy} + sy^2 \right) \right\},
\]

where \( \mu_{s,r} = E(B_sB_r) \) and \( \rho^2_{s,r} = rs - \mu^2 \).

**Lemma 3.5.** Let \( f \in C^1(\mathbb{R}) \) admit compact support. Then we have

\[
|E[f''(B_s)f(B_r)]| \leq C_{H,K} \frac{1}{s^{1/4}} \int_{\mathbb{R}} f^2(x) \varphi_s(x)dx
\]

and

\[
|E[f'(B_s)f'(B_r)] + E[f''(B_s)f(B_r)]| \leq C_{H,K} \left( \frac{1}{s^{3/4}r^{1/4}} + \frac{1}{r^{3/4}s^{1/4}} \right) \int_{\mathbb{R}} f^2(x)(s - r + x^2)\varphi_s(x)dx
\]

for all \( s > r > 0 \) and \( 2HK = 1 \), where \( \varphi_s(x) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{x^2}{2s}} \).

**Proof.** Elementary calculus can show that

\[
E[f''(B_s)f(B_r)] = \int_{\mathbb{R}^2} f(x)f(y) \frac{\partial^2}{\partial x^2} \varphi(x,y)dxdy
\]

\[
= \int_{\mathbb{R}^2} f(x)f(y) \left\{ \frac{1}{\rho^4} (rx - \mu_{s,r}y)^2 - \frac{r}{\rho^2_{s,r}} \right\} \varphi(x,y)dxdy
\]
and

\[
\int_{\mathbb{R}^2} \left| f(y) \right|^2 \left| \frac{r}{\rho_{s,r}^2} (x - \frac{\mu_{s,r}}{r} y)^2 - 1 \right|^2 \varphi(x,y) dxdy
= \int_{\mathbb{R}} \left| f(y) \right|^2 \varphi_r(y) dy \int_{\mathbb{R}} \left| \frac{r}{\rho_{s,r}^2} (x - \frac{\mu_{s,r}}{r} y)^2 - 1 \right|^2 \varphi_{\rho_{s,r}^2} (x - \frac{\mu_{s,r}}{r} y) dx
= \int_{\mathbb{R}} \left| f(y) \right|^2 \varphi_r(y) dy \int_{\mathbb{R}} (u^2 - 1)^2 \varphi_1(u) du
= 2 \int_{\mathbb{R}} \left| f(y) \right|^2 \varphi_r(y) dy.
\]

We have

\[
|E[f''(B_s)f(B_r)]| \leq \frac{r}{\rho_{s,r}^2} \int_{\mathbb{R}^2} \left| f(x) f(y) \left\{ \frac{r}{\rho_{s,r}^2} (x - \frac{\mu_{s,r}}{r} y)^2 - 1 \right\} \varphi(x,y) dxdy
\leq \frac{r^2}{\rho_{s,r}^2} \left( \int_{\mathbb{R}} \left| f(x) \right|^2 \varphi_s(x) dx \int_{\mathbb{R}} \left| f(y) \right|^2 \varphi_r(y) dy \right)^{1/2}
= C_{H,K} \frac{r}{\rho_{s,r}^2} \left( \int_{\mathbb{R}} \left| f(x) \right|^2 \varphi_s(x) dx \int_{\mathbb{R}} \left| f(y) \right|^2 \varphi_r(y) dy \right)^{1/2}
\leq C_{H,K} \frac{r}{\rho_{s,r}^2} \sqrt{s/r} \int_{\mathbb{R}} \left| f(x) \right|^2 \varphi_s(x) dx \leq C_{H,K} \frac{s^{1/4}}{r^{1/4}(s-r)} \int_{\mathbb{R}} \left| f(x) \right|^2 \varphi_s(x) dx
\]

by the inequalities (3.1) and the fact

\[
(3.17) \quad E \left[ f^2(B_r) \right] = \int_{\mathbb{R}} \left| f(x) \right|^2 \varphi_r(x) dx
\leq \sqrt{\frac{s}{r}} \int_{\mathbb{R}} \left| f(x) \right|^2 \varphi_s(x) dx = \sqrt{\frac{s}{r}} E \left[ f^2(B_s) \right]
\]

with \( s \geq r > 0 \).

On the other hand, we have

\[
E[f'(B_s)f'(B_r)] = \int_{\mathbb{R}^2} f(x)f(y) \frac{\partial^2}{\partial x \partial y} \varphi(x,y) dxdy
= \int_{\mathbb{R}^2} f(x)f(y) \left\{ \frac{1}{\rho_{s,r}^4} (sy - \mu_{s,r} x) (rx - \mu_{s,r} y) + \frac{\mu_{s,r}}{\rho_{s,r}^2} \right\} \varphi(x,y) dxdy,
\]

which deduce, by the following identity

\[
(3.18) \quad (sy - \mu x)(rx - \mu_{s,r} y) = \rho_{s,r}^2 y (x - \frac{\mu_{s,r}}{r} y) - \mu_{s,r} r (x - \frac{\mu_{s,r}}{r} y)^2,
\]
\[
E[f'(B_s)f''(B_r)] + E[f''(B_s)f(B_r)]
= \frac{1}{\rho_{s,r}^4} \int_{\mathbb{R}^2} f(x)f(y) \left[ (sy - \mu_{s,r}x)(rx - \mu_{s,r}y) + (rx - \mu_{s,r}y)^2 \right] \varphi(x,y)\,dx\,dy
\]
\[
+ \frac{\mu_{s,r} - r}{\rho_{s,r}^2} \int_{\mathbb{R}^2} f(x)f(y)\varphi(x,y)\,dx\,dy
\]
\[
= \frac{1}{\rho_{s,r}^2} \int_{\mathbb{R}^2} f(x)f(y)(y - \frac{\mu_{s,r}}{r})\varphi(x,y)\,dx\,dy
\]
\[
+ \frac{(r - \mu_{s,r})}{\rho_{s,r}^2} \int_{\mathbb{R}^2} f(x)f(y)(y - \frac{\mu_{s,r}}{r})^2\varphi(x,y)\,dx\,dy
\]
\[
+ \frac{r - \mu_{s,r}}{\rho_{s,r}^2} \int_{\mathbb{R}^2} f(x)f(y)\varphi(x,y)\,dx\,dy \equiv \Lambda_1 + \Lambda_2 + \Lambda_3.
\]
Notice that
\[
\int_{\mathbb{R}^2} f^2(y)|x - \frac{\mu_{s,r}}{r} y|^{2m} \varphi(x,y)\,dx\,dy
= \int_{\mathbb{R}^2} f^2(y)\varphi_r(y)\,dy \int_{\mathbb{R}} |x - \frac{\mu_{s,r}}{r} y|^{2m} \frac{\sqrt{r}}{2\pi\rho_{s,r}} e^{-\frac{s}{2\rho_{s,r}^2} (y - \frac{\mu_{s,r}}{s} x)^2} \,dy
\]
\[
= C_m \left( \frac{\rho_{s,r}^2}{r} \right)^m \int_{\mathbb{R}} f^2(y)\varphi_r(y)\,dy,
\]
for all \( m \geq 0 \), and by the inequalities (3.1)
\[
\int_{\mathbb{R}^2} |f(x)y|^2 \varphi(x,y)\,dx\,dy = \int_{\mathbb{R}} f^2(x)\varphi_s(x)\,dx \int_{\mathbb{R}} y^2 \frac{\sqrt{s}}{2\pi\rho_{s,r}} e^{-\frac{s}{2\rho_{s,r}^2} (y - \frac{\mu_{s,r}}{s} x)^2} \,dy
\]
\[
= \int_{\mathbb{R}} f^2(x)\varphi_s(x)\,dx \int_{\mathbb{R}} \left( (y - \frac{\mu_{s,r}}{s} x)^2 + \frac{\mu_{s,r}}{s} x \right)^2 \frac{\sqrt{s}}{2\pi\rho_{s,r}} e^{-\frac{s}{2\rho_{s,r}^2} (y - \frac{\mu_{s,r}}{s} x)^2} \,dy
\]
\[
= \int_{\mathbb{R}} f^2(x) \left( \frac{\rho_{s,r}^2}{s} + \frac{\mu_{s,r}^2}{s^2} x^2 \right) \varphi_s(x)\,dx
\]
\[
\leq C_{H,K} \frac{r}{s} \int_{\mathbb{R}} f^2(x) (s - r + x^2) \varphi_s(x)\,dx
\]
for all \( 0 < r < s \). We see that, by the fact (3.17)
\[
|\Lambda_1| \leq \frac{1}{\rho_{s,r}^2} \left( \int_{\mathbb{R}^2} f^2(y)(x - \frac{\mu_{s,r}}{r} y)^2 \varphi(x,y)\,dx\,dy \right)^{1/2}
\]
\[
\leq \frac{C_{H,K}}{\rho_{s,r}^2} \left( \frac{\rho_{s,r}^2}{r} \int_{\mathbb{R}} f^2(y)\varphi_r(y)\,dy \cdot \frac{r}{s} \int_{\mathbb{R}} f^2(x)(s - r + x^2)\varphi_s(x)\,dx \right)^{1/2}
\]
\[
\leq \frac{C_{H,K}}{\sqrt{s} \rho_{s,r}} \left( \frac{s}{r} \int_{\mathbb{R}} f^2(y)\varphi_r(y)\,dy \int_{\mathbb{R}} f^2(x)(s - r + x^2)\varphi_s(x)\,dx \right)^{1/2}
\]
\[
\leq \frac{C_{H,K}}{(rs)^{1/4} \rho_{s,r}} \int_{\mathbb{R}} f^2(x)(s - r + x^2)\varphi_s(x)\,dx,
\]
\[ |\Lambda_2| \leq \frac{(r - \mu_{s,r})r}{\rho_{s,r}^4} \left( \int_{\mathbb{R}^2} f^2(y)(x - \frac{\mu_{s,r}}{r} y)^4 \varphi(x, y) dy \int_{\mathbb{R}^2} f^2(x) \varphi(x, y) dx \right)^{1/2} \]

\[ = \frac{\sqrt{3}(r - \mu_{s,r})}{\rho_{s,r}^2} \sqrt{\frac{8}{r} \int f^2(x) \varphi_s(x) dx} \]

and

\[ |\Lambda_3| \leq \frac{r - \mu_{s,r}}{\rho_{s,r}^2} \left( \int_{\mathbb{R}^2} f^2(y) \varphi(x, y) dy \int_{\mathbb{R}^2} f^2(x) \varphi(x, y) dx \right)^{1/2} \]

\[ = \frac{r - \mu_{s,r}}{\rho_{s,r}^2} \sqrt{\frac{8}{r} \int f^2(x) \varphi_s(x) dx}. \]

Thus, the estimate (3.16) follows from Lemma 3.1 and Lemma 3.2.

From the above proof of Lemma 3.5 we also have

\[ |E[f(B_r)f''(B_s)]| \leq C_{H,K} r^{5/4} \frac{r}{s^{5/4}(r - s)} \int f^2(x) \varphi_r(x) dx \]

(3.19)

and

\[ |E[f'(B_s)f'(B_r)] + E[f(B_r)f''(B_s)]| \]

\[ \leq C_{H,K} \left( \frac{r^{1/4}}{s^{5/4}(r - s)^{1/4}} + \frac{r^{1/4}}{s^{5/4}} \right) \int f^2(x) \left( r - s + x^2 \right) \varphi_r(x) dx \]

(3.20)

for all 0 < s < r and 2HK = 1.

4. Existence of quadratic covariation

In this section, we study the quadratic covariation \([f(B), B]\). Denote

\[ J_\varepsilon(f, t) := \frac{1}{\varepsilon} \int_0^t \{ f(B_{s+\varepsilon}) - f(B_s) \} (B_{s+\varepsilon} - B_s) ds \]

for \( \varepsilon > 0 \) and 0 \( \leq t \leq T \). Recall that the quadratic covariation, the forward integral and the backward integrals are defined as

\[ [f(B), B]: = \lim_{\varepsilon \to 0} J_\varepsilon(f, t), \]

(4.1)

\[ \int_0^t f(B_s)d^- B_s := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t f(B_s)(B_{s+\varepsilon} - B_s) ds, \]

(4.2)

\[ \int_0^t f(B_s)d^+ B_s := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t f(B_{s+\varepsilon})(B_{s+\varepsilon} - B_s) ds, \]

(4.3)

provided the corresponding limits exist in \( L^1 \), and we have

\[ [f(B), B]_t = \int_0^t f'(B_s)d[B, B]_s = 2^{1-K} \int_0^t f'(B_s) ds \]

(4.4)

for all 0 \( \leq t \leq T \) and \( f \in C^1(\mathbb{R}) \) (see Russo-Tudor [19] and Russo-Vallois [21, 22]).
Now, we study the existence in $L^2$ of the forward integral, backward integral and quadratic covariation. Consider the set $\mathcal{H}$ of measurable functions $f$ on $\mathbb{R}$ such that 

$$
\|f\|_{\mathcal{H}} := \int_0^T \int_\mathbb{R} |f(x)|^2 \varphi_s(x) dx ds + \int_0^T \int_\mathbb{R} |f(x)x| \varphi_s(x) \frac{dx ds}{s}
$$

with $\varphi_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}$. Clearly, $\mathcal{H}$ is a Banach space and the set $\mathcal{E}$ of elementary functions

$$ f_\Delta(x) = \sum_j f_j \mathbb{1}_{(a_j, a_{j+1})}(x), \quad f_j \in \mathbb{R}; -\infty < a_0 < a_1 < \cdots < a_N < \infty $n

is dense in $\mathcal{H}$, and moreover every $f \in \mathcal{H}$ is locally square integrable and the space of measurable functions

$$ \mathcal{H}_K = \left\{ f \mid \int_0^T \int_\mathbb{R} |f(x)|^{2/K} \varphi_s(x) dx ds < \infty \right\} $$

is a simple subspace of $\mathcal{H}$.

**Lemma 4.1.** Let $2HK = 1$. If $f \in \mathcal{H}$, then we have

$$
E \left| \frac{1}{\varepsilon} \int_0^t f(B_s)(B_{s+\varepsilon} - B_s) ds \right|^2 \leq C_{H,K} \|f\|_{\mathcal{H}}^2, \tag{4.5}
$$

and

$$
E \left| \frac{1}{\varepsilon} \int_0^t f(B_{s+\varepsilon})(B_{s+\varepsilon} - B_s) ds \right|^2 \leq C_{H,K} \|f\|_{\mathcal{H}}^2 \tag{4.6}
$$

for all $0 < \varepsilon < T$ and $0 \leq t \leq T$.

**Proof.** Without loss of generality one may assume that $T = 1$. We prove only the estimate (4.5) and similarly one can prove (4.6). Let $0 < \varepsilon < T$ and $0 < s, r < T$. By approximating we may assume that $f$ is an infinitely differentiable function with compact
In order to establish (4.5) we first show that
\[ E[f(B_s)f(B_r)(B_{s+\varepsilon} - B_s)(B_{r+\varepsilon} - B_r)] \]
\[ = E \left[ f(B_s)f(B_r)(B_{s+\varepsilon} - B_s) \int_r^{r+\varepsilon} dB_i \right] \]
\[ = E \left( \int_0^T \int_0^T 1_{[r,r+\varepsilon]}(u)D^H_v \left[ f(B_s)f(B_r)(B_{s+\varepsilon} - B_s) \right] \phi(u,v) dudv \right) \]
\[ + \left( \int_0^T \int_0^T 1_{[r,\varepsilon]}(u)1_{[0,r]}(u) \phi(u,v) dudv \right) E \left[ f(B_s)f'(B_r)(B_{s+\varepsilon} - B_s) \right] \]
\[ + \left( \int_0^T \int_0^T 1_{[r,\varepsilon]}(u)1_{[s,s+\varepsilon]}(u) \phi(u,v) dudv \right) E \left[ f(B_s)f(B_r) \right] \]
\[ = E \left[ B_s(B_{r+\varepsilon} - B_r) \right] E \left[ f'(B_s)f(B_r)(B_{s+\varepsilon} - B_s) \right] \]
\[ + E \left[ B_r(B_{r+\varepsilon} - B_r) \right] E \left[ f(B_s)f'(B_r)(B_{s+\varepsilon} - B_s) \right] \]
\[ + E \left[ [B_{r+\varepsilon} - B_r](B_{s+\varepsilon} - B_s) \right] E \left[ f(B_s)f(B_r) \right] \]
\[ \equiv \Psi_\varepsilon(s,r,1) + \Psi_\varepsilon(s,r,2) + \Psi_\varepsilon(s,r,3). \]

In order to establish (4.5) we first show that
\[ \frac{1}{\varepsilon^2} \left| \int_0^1 \int_0^1 \Psi_\varepsilon(s,r,3) dsdr \right| \leq C_{H,K} \|f\|^2_{\mathcal{F}} \]
for all \( \varepsilon > 0 \) small enough. we have
\[ \frac{1}{\varepsilon^2} \left| \int_0^1 \int_0^1 \Psi_\varepsilon(s,r,3) dsdr \right| \]
\[ \leq \frac{2}{\varepsilon^2} \int_0^1 \int_0^s |E \left[ (B_{r+\varepsilon} - B_r)(B_{s+\varepsilon} - B_s) \right] |E \left[ f(B_s)f(B_r) \right] |d\varepsilon \]
\[ = \frac{2}{\varepsilon^2} \int_0^1 ds \int_0^{s-\varepsilon} |E \left[ (B_{r+\varepsilon} - B_r)(B_{s+\varepsilon} - B_s) \right] |E \left[ f(B_s)f(B_r) \right] |dr \]
\[ + \frac{2}{\varepsilon^2} \int_0^1 \int_0^s |E \left[ (B_{r+\varepsilon} - B_r)(B_{s+\varepsilon} - B_s) \right] |E \left[ f(B_s)f(B_r) \right] |d\varepsilon \]
\[ + \frac{2}{\varepsilon^2} \int_0^1 \int_{s-\varepsilon}^s |E \left[ (B_{r+\varepsilon} - B_r)(B_{s+\varepsilon} - B_s) \right] |E \left[ f(B_s)f(B_r) \right] |d\varepsilon \]
\[ \equiv \Lambda_{31} + \Lambda_{32} + \Lambda_{33} \]
for all \( 0 < \varepsilon \leq 1 \). Clearly, Lemma (3.3) and the fact (3.17) imply that
\[ \Lambda_{31} \leq 2 \int_0^1 ds \int_0^{s-\varepsilon} \frac{1}{s} \sqrt{E(|f(B_s)|^2)E(|f(B_r)|^2)} dr \]
\[ \leq 2 \int_0^1 ds \int_0^s \frac{1}{s^{3/4}} E(|f(B_s)|^2) dr \]
\[ = 2 \int_0^1 ds E(|f(B_s)|^2) \leq 2 \|f\|^2_{\mathcal{F}}. \]
Notice that
\[
\int_0^\varepsilon ds \int_0^s |Ef(B_s)f(B_r)| dr \leq \int_0^\varepsilon ds \int_0^s \sqrt{E(|f(B_s)|^2)E(|f(B_r)|^2)} dr \\
\leq \int_0^\varepsilon ds \int_0^s \frac{s^{1/4}}{r^{1/4}} E(|f(B_s)|^2) dr \\
\leq \frac{4}{3} \varepsilon \int_0^1 E(|f(B_s)|^2) ds \leq \varepsilon \|f\|_{\mathcal{H}}^2
\]
and
\[
\int_{\varepsilon}^1 ds \int_{s-\varepsilon}^s |Ef(B_s)f(B_r)| dr \leq \int_{\varepsilon}^1 ds \int_{s-\varepsilon}^s \sqrt{E(|f(B_s)|^2)E(|f(B_r)|^2)} dr \\
\leq \int_{\varepsilon}^1 ds \int_{s-\varepsilon}^s \frac{s^{1/4}}{r^{1/4}} E(|f(B_s)|^2) dr \\
= \frac{4}{3} \int_{\varepsilon}^1 E(|f(B_s)|^2)s^{1/4} \left(s^{3/4} - (s - \varepsilon)^{3/4}\right) ds \\
\leq \frac{4}{3} \varepsilon \int_{\varepsilon}^1 E(|f(B_s)|^2) ds \leq \frac{4}{3} \varepsilon \|f\|_{\mathcal{H}}^2
\]
for all $0 < \varepsilon \leq 1$. We get
\[
\Lambda_{32} + \Lambda_{33} \leq \frac{2}{\varepsilon^2} \int_0^\varepsilon \int_0^s \sqrt{E[(B_{r+\varepsilon} - B_r)^2]E[(B_{s+\varepsilon} - B_s)^2]} E |f(B_s)f(B_r)| dsdr \\
+ \frac{2}{\varepsilon^2} \int_\varepsilon^1 \int_{s-\varepsilon}^s \sqrt{E[(B_{r+\varepsilon} - B_r)^2]E[(B_{s+\varepsilon} - B_s)^2]} E |f(B_s)f(B_r)| dsdr \\
\leq C_{H,K} \frac{1}{\varepsilon} \int_0^\varepsilon \int_0^s E |f(B_s)f(B_r)| dsdr + C_{H,K} \frac{1}{\varepsilon} \int_{\varepsilon}^1 \int_{s-\varepsilon}^s E |f(B_s)f(B_r)| dsdr \\
\leq C_{H,K} \|f\|_{\mathcal{H}}^2.
\]
It follows that
\[
\frac{1}{\varepsilon^2} \left| \int_0^1 \int_0^1 \Psi_\varepsilon(s, r, 3) dsdr \right| \leq \Lambda_{31} + \Lambda_{32} + \Lambda_{33} \leq C_{H,K} \|f\|_{\mathcal{H}}^2
\]
for all $0 < \varepsilon \leq 1$.

Now, let us prove
\[
\frac{1}{\varepsilon^2} \int_0^t \int_0^t |\Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2)| dr ds \leq C_{H,K} \|f\|_{\mathcal{H}}^2
\]
for all $\varepsilon > 0$. We have
\[
\Psi_\varepsilon(s, r, 1) = E \left[ B_s(B_{r+\varepsilon} - B_r) \right] E \left[ f'(B_s)f(B_r)(B_{s+\varepsilon} - B_s) \right] \\
= E \left[ B_s(B_{r+\varepsilon} - B_r) \right] E \left[ B_s(B_{s+\varepsilon} - B_s) \right] E \left[ f''(B_s)f(B_r) \right] \\
+ E \left[ B_s(B_{r+\varepsilon} - B_r) \right] E \left[ B_r(B_{s+\varepsilon} - B_s) \right] E \left[ f'(B_s)f'(B_r) \right],
\]
\[
\Psi_\varepsilon(s, r, 2) = E \left[ B_r(B_{r+\varepsilon} - B_r) \right] E \left[ B_s(B_{s+\varepsilon} - B_s) \right] E \left[ f'(B_s)f'(B_r) \right] \\
+ E \left[ B_r(B_{r+\varepsilon} - B_r) \right] E \left[ B_r(B_{s+\varepsilon} - B_s) \right] E \left[ f'(B_s)f'(B_r) \right].
\]
For $s > r > 0$ we decompose $\Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2)$ as follows

$$
\Psi_\varepsilon(s, r, 1) = E\left[B_s(B_{r+s} - B_r)\right] E\left[B_r(B_{s+t} - B_s)\right]
\cdot \left(E\left[f''(B_s)f(B_r)\right] + E\left[f'(B_s)f'(B_r)\right]\right)
+ \left\{ E\left[B_s(B_{r+s} - B_r)\right] E\left[B_r(B_{s+t} - B_s)\right]
- E\left[B_s(B_{r+s} - B_r)\right] E\left[B_r(B_{s+t} - B_s)\right]\right\} E\left[f'(B_s)f'(B_r)\right]
\equiv \Psi_\varepsilon(s > r, 1) + \Psi_\varepsilon(s > r, 2)
$$

and

$$
\Psi_\varepsilon(s, r, 2) = E\left[B_r(B_{r+s} - B_r)\right] E\left[B_s(B_{s+t} - B_s)\right]
\cdot \left\{ E\left[f'(B_s)f'(B_r)\right] + E\left[f(B_s)f''(B_r)\right]\right\}
+ \left\{ E\left[B_r(B_{r+s} - B_r)\right] E\left[B_s(B_{s+t} - B_s)\right]
- E\left[B_r(B_{r+s} - B_r)\right] E\left[B_s(B_{s+t} - B_s)\right]\right\} E\left[f'(B_s)f'(B_r)\right]
\equiv \Psi_\varepsilon(s > r, 3) + \Psi_\varepsilon(s > r, 4) .
$$

Notice that

$$
|\Psi_\varepsilon(s > r, 1)| \leq C_{H,K} \frac{r^2}{s^2} |E[f''(B_s)f(B_r)] + E[f'(B_s)f'(B_r)]|,
$$
$$
|\Psi_\varepsilon(s > r, 2)| = |E[B_s(B_{r+s} - B_r)] E[(B_s - B_r)(B_{s+t} - B_s)] E[f''(B_s)f(B_r)]|,
$$
$$
\leq C_{H,K} \varepsilon \frac{2s - r}{s} |E[f''(B_s)f(B_r)]|,
$$
$$
|\Psi_\varepsilon(s > r, 3)| \leq \varepsilon^2 |E[f'(B_s)f'(B_r)] + E[f(B_s)f''(B_r)]|,
$$
$$
|\Psi_\varepsilon(s > r, 4)| = |E[B_r(B_{r+s} - B_r)] E[(B_s - B_r)(B_{s+t} - B_s)] E[f'(B_s)f''(B_r)]|,
$$
$$
\leq C_{H,K} \varepsilon \frac{2s - r}{s} |E[f'(B_s)f''(B_r)]| .
$$

by Lemma 3.3 and Lemma 3.4. We get

$$
\frac{1}{\varepsilon^2} \int_0^t ds \int_0^s |\Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2)| dr
\leq \frac{1}{\varepsilon^2} \sum_{i=1}^4 \int_0^t ds \int_0^s |\Psi_\varepsilon(s > r, i)| dr
\leq C_{H,K} \|f\|_{\mathcal{L}^2}
$$

by Lemma 3.5. Similarly, for $r > s > 0$ in order to decompose $\Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2)$, we have

$$
\Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2) = E\left[B_s(B_{r+s} - B_r)\right] E\left[B_r(B_{s+t} - B_s)\right]
\cdot \left\{ E\left[f'(B_s)f'(B_r)\right] + E\left[f(B_s)f''(B_r)\right]\right\}
+ \left\{ E\left[B_r(B_{r+s} - B_r)\right] E\left[B_s(B_{s+t} - B_s)\right]
- E\left[B_s(B_{r+s} - B_r)\right] E\left[B_r(B_{s+t} - B_s)\right]\right\} E\left[f'(B_s)f'(B_r)\right]
\equiv \Psi_\varepsilon(s < r, 1) + \Psi_\varepsilon(s < r, 2)
$$
From Lemma 4.1, it is enough to show that

\[ E \left[ f'(B_s) f'(B_r) \right] + E \left[ f''(B_s) f(B_r) \right] \]

which gives

\[ \Psi_\varepsilon(s < r, 3) + \Psi_\varepsilon(s < r, 4) \]

Clearly, Lemma 3.3 and Lemma 3.4 implies that

\[ |\Psi_\varepsilon(s < r, 1)| \leq C_{H,K} \varepsilon^2 S \left[ E \left[ f'(B_s) f'(B_r) \right] + E \left[ f''(B_s) f(B_r) \right] \right], \]

\[ |\Psi_\varepsilon(s < r, 2)| \leq E \left[ B_s(B_{s+\varepsilon} - B_s) \right] E \left[ (B_r - B_s)(B_{s+\varepsilon} - B_r) \right] E \left[ f''(B_s) f(B_r) \right] \]

\[ |\Psi_\varepsilon(s < r, 3)| \leq C_{H,K} \varepsilon^2 \left[ E \left[ f'(B_s) f'(B_r) \right] + E \left[ f''(B_s) f(B_r) \right] \right] \]

\[ \Psi_\varepsilon(s < r, 4) \leq E \left[ B_s(B_{s+\varepsilon} - B_s) \right] E \left[ (B_r - B_s)(B_{s+\varepsilon} - B_r) \right] E \left[ f''(B_s) f(B_r) \right] \]

\[ \leq \varepsilon^2 \frac{t - s}{s} \left[ E \left[ f''(B_s) f(B_r) \right] \right] \]

for all \( r > s > 0 \). It follows from (3.19) and (3.20) that

\[ \frac{1}{\varepsilon^2} \int_0^t dr \int_0^r |\Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2)| ds \]

\[ \leq \frac{1}{\varepsilon^2} \sum_{i=1}^4 \int_0^t ds \int_0^s |\Psi_\varepsilon(s < r, i)| dr \leq C_{H,K} \|f\|_{\mathcal{H}}^2. \]

Thus, we have given the desired estimate \((4.9)\), and the lemma follows. \( \square \)

In this section our main result is the following theorem which shows that \( J_\varepsilon(f, t) \) converges in \( L^2 \) as \( \varepsilon \) tends to 0.

**Theorem 4.1.** Let \( 2HK = 1 \). If \( f \in \mathcal{H} \), then the forward, backward integrals \( \int_0^t f(B_s) dB_s \) and the quadratic covariation \( [f(B), B] \) exist in \( L^2 \), and

\[ E \left[ \left| \int_0^t f(B_s) dB_s \right|^2 \right] \leq C_{H,K} \|f\|_{\mathcal{H}}^2 \]

for all \( 0 \leq t \leq T \).

**Proof.** From Lemma 4.1, it is enough to show that

\[ E \left| J_{\varepsilon_1}^- - J_{\varepsilon_2}^- \right|^2 \rightarrow 0, \]

and

\[ E \left| J_{\varepsilon_1}^+ - J_{\varepsilon_2}^+ \right|^2 \rightarrow 0 \]
as \( \varepsilon_1, \varepsilon_2 \downarrow 0 \), where

\[
J^-_\varepsilon = \frac{1}{\varepsilon} \int_0^t f(B_s)(B_{s+\varepsilon} - B_s)ds \quad \text{and} \quad J^+_\varepsilon = \frac{1}{\varepsilon} \int_0^t f(B_{s+\varepsilon})(B_{s+\varepsilon} - B_s)ds.
\]

Without loss of generality we assume that \( \varepsilon_1 > \varepsilon_2 \). We prove only the convergence (4.15) and similarly one can prove (4.16). It follows that

\[
E|J^-_{\varepsilon_1} - J^-_{\varepsilon_2}|^2 = \frac{1}{\varepsilon_1^2} \int_0^t \int_0^t E\{f(B_s)f(B_r)(B_{s+\varepsilon_1} - B_s)(B_{r+\varepsilon_1} - B_r)dsdr
\]

\[
- \frac{2}{\varepsilon_1^2} \int_0^t \int_0^t E\{f(B_s)f(B_r)(B_{s+\varepsilon_1} - B_s)(B_{r+\varepsilon_2} - B_r)dsdr
\]

\[
+ \frac{1}{\varepsilon_2^2} \int_0^t \int_0^t E\{f(B_s)f(B_r)(B_{s+\varepsilon_2} - B_s)(B_{r+\varepsilon_2} - B_r)dsdr
\]

\[
\equiv \frac{1}{\varepsilon_1^2} \int_0^t \int_0^t \{\varepsilon_2 \Phi_{s,r}(1, \varepsilon_1) - \varepsilon_1 \Phi_{s,r}(2, \varepsilon_1, \varepsilon_2) \} dsdr
\]

\[
+ \frac{1}{\varepsilon_2^2} \int_0^t \int_0^t \{\varepsilon_1 \Phi_{s,r}(1, \varepsilon_2) - \varepsilon_2 \Phi_{s,r}(2, \varepsilon_1, \varepsilon_2) \} dsdr,
\]

where

\[
\Phi_{s,r}(1, \varepsilon_1) = E(f(B_s)f(B_r)(B_{s+\varepsilon} - B_s)(B_{r+\varepsilon} - B_r)),
\]

and

\[
\Phi_{s,r}(2, \varepsilon_1, \varepsilon_2) = E(f(B_s)f(B_r)(B_{s+\varepsilon_1} - B_s)(B_{r+\varepsilon_2} - B_r)).
\]

We have by (4.7)

\[
\Phi_{s,r}(1, \varepsilon_1) = \Psi_\varepsilon(s, r, 1) + \Psi_\varepsilon(s, r, 2) + \Psi_\varepsilon(s, r, 3)
\]

\[
= E[B_s(B_{r+\varepsilon} - B_r)]E[B_s(B_{s+\varepsilon} - B_s)]E[f''(B_s)f(B_r)]
\]

\[
+ E[B_s(B_{r+\varepsilon} - B_r)]E[B_r(B_{s+\varepsilon} - B_s)]E[f'(B_s)f'(B_r)]
\]

\[
+ E[B_r(B_{r+\varepsilon} - B_r)]E[B_s(B_{s+\varepsilon} - B_s)]E[f'(B_s)f'(B_r)]
\]

\[
+ E[B_r(B_{r+\varepsilon} - B_r)]E[B_r(B_{s+\varepsilon} - B_s)]E[f(B_s)f''(B_r)]
\]

\[
+ E[(B_{r+\varepsilon} - B_r)(B_{s+\varepsilon} - B_s)]E[f(B_s)f(B_r)]
\]

and

\[
\Phi_{s,r}(2, \varepsilon_1, \varepsilon_2) = E[B_s(B_{r+\varepsilon_2} - B_r)]E[f'(B_s)f(B_r)(B_{s+\varepsilon_1} - B_s)]
\]

\[
+ E[B_r(B_{r+\varepsilon_2} - B_r)]E[f(B_s)f'(B_r)(B_{s+\varepsilon_1} - B_s)]
\]

\[
+ E[(B_{s+\varepsilon_1} - B_s)(B_{r+\varepsilon_2} - B_r)]E[f(B_s)f(B_r)]
\]

\[
+ E[B_r(B_{r+\varepsilon_2} - B_r)]E[B_s(B_{s+\varepsilon_1} - B_s)]E[f'(B_s)f'(B_r)]
\]

\[
+ E[B_r(B_{r+\varepsilon_2} - B_r)]E[B_r(B_{s+\varepsilon_1} - B_s)]E[f(B_s)f''(B_r)]
\]

\[
+ E[(B_{s+\varepsilon_1} - B_s)(B_{r+\varepsilon_2} - B_r)]E[f(B_s)f(B_r)].
\]
Denote
\[
A_1(s, r, \varepsilon, j) := \varepsilon_j E [B_s(B_{r+\varepsilon} - B_r)] E [B_s(B_{s+\varepsilon} - B_s)] \\
- \varepsilon E [B_s(B_{r+\varepsilon} - B_r)] E [B_s(B_{s+\varepsilon} - B_s)]
\]
\[
A_{21}(s, r, \varepsilon, j) := \varepsilon_j E [B_s(B_{r+\varepsilon} - B_r)] E [B_r(B_{s+\varepsilon} - B_s)] \\
- \varepsilon E [B_s(B_{r+\varepsilon} - B_r)] E [B_r(B_{s+\varepsilon} - B_s)]
\]
\[
A_{22}(s, r, \varepsilon, j) := \varepsilon_j E [B_r(B_{r+\varepsilon} - B_r)] E [B_s(B_{s+\varepsilon} - B_s)] \\
- \varepsilon E [B_r(B_{r+\varepsilon} - B_r)] E [B_s(B_{s+\varepsilon} - B_s)]
\]
\[
A_{3}(s, r, \varepsilon, j) := \varepsilon_j E [B_r(B_{r+\varepsilon} - B_r)] E [B_r(B_{s+\varepsilon} - B_s)] \\
- \varepsilon E [B_r(B_{r+\varepsilon} - B_r)] E [B_r(B_{s+\varepsilon} - B_s)]
\]
\[
A_4(s, r, \varepsilon, j) := \varepsilon_j E [(B_{r+\varepsilon} - B_r)(B_{s+\varepsilon} - B_s)] - \varepsilon E [(B_{s+\varepsilon} - B_s)(B_{r+\varepsilon} - B_r)]
\]
with \( j = 1, 2 \). It follows that
\[
\varepsilon_j \Phi_{s,r}(1, \varepsilon_i) - \varepsilon_i \Phi_{s,r}(2, \varepsilon_1, \varepsilon_2)
\]
\[
= \left( A_1(s, r, \varepsilon_i, j) E [f''(B_s)f(B_r)] + (A_{21}(s, r, \varepsilon_i, j) + A_{22}(s, r, \varepsilon_i, j)) E [f'(B_s)f'(B_r)] \\
+ A_3(s, r, \varepsilon_i, j) E [f(B_s)f''(B_r)] \right) + A_4(s, r, \varepsilon_i, j) E [f(B_s)f(B_r)]
\]
\[
\equiv \Upsilon(s, r, \varepsilon_i, j) + A_4(s, r, \varepsilon_i, j) E [f(B_s)f(B_r)]
\]
with \( i, j = 1, 2 \) and \( i \neq j \). In order to end the proof we claim that the following convergence hold:
\[
(4.17) \quad \frac{1}{\varepsilon_i^2 \varepsilon_j} \int_0^t \int_0^t \left\{ \varepsilon_j \Phi_{s,r}(1, \varepsilon_i) - \varepsilon_i \Phi_{s,r}(2, \varepsilon_1, \varepsilon_2) \right\} dsdr \rightarrow 0 \quad (i, j = 1, 2, i \neq j),
\]
as \( \varepsilon_1, \varepsilon_2 \to 0 \). This will be done in three parts. Keeping the notations in the proof of Lemma 3.4.

**Part A.** The following convergence hold:
\[
(4.18) \quad \frac{1}{\varepsilon_i^2 \varepsilon_j} \int_0^t \int_0^t \Upsilon(s, r, \varepsilon_i, j) dsdr \rightarrow 0 \quad (i, j = 1, 2, i \neq j)
\]
as \( \varepsilon_1, \varepsilon_2 \to 0 \). For \( s > r > 0 \) we decompose \( \Upsilon(s, r, \varepsilon_i, j) \) as follows
\[
\Upsilon(s, r, \varepsilon_i, j) = A_1(s, r, \varepsilon_i, j) E [f''(B_s)f(B_r)] + A_3(s, r, \varepsilon_i, j) E [f(B_s)f''(B_r)] \\
+ (A_{21}(s, r, \varepsilon_i, j) + A_{22}(s, r, \varepsilon_i, j)) E [f'(B_s)f'(B_r)]
\]
\[
= A_{21}(s, r, \varepsilon_i, j) \left\{ E [f'(B_s)f'(B_r)] + E [f''(B_s)f(B_r)] \right\} \\
+ A_{22}(s, r, \varepsilon_i, j) \left\{ E [f'(B_s)f'(B_r)] + E [f(B_s)f''(B_r)] \right\} \\
+ \{ A_1(s, r, \varepsilon_i, j) - A_{21}(s, r, \varepsilon_i, j) \} E [f''(B_s)f(B_r)] \\
+ \{ A_3(s, r, \varepsilon_i, j) - A_{22}(s, r, \varepsilon_i, j) \} E [f(B_s)f''(B_r)]
\]
with \( i, j = 1, 2 \) and \( i \neq j \). By symmetry, we only need to show that this holds for \( i = 1, j = 2 \). We will establish the convergence (4.18) with \( i = 1, j = 2 \) in two steps.
Step A-1. The following convergence hold:

\[(4.19) \quad \frac{1}{\varepsilon_1^2} \int_0^t ds \int_0^s \epsilon\{ E [f'(B_s) f'(B_r)] + E [f''(B_s) f(B_r)] \} dr \to 0,\]

\[(4.20) \quad \frac{1}{\varepsilon_1^2} \int_0^t ds \int_0^s \epsilon\{ E [f'(B_s) f'(B_r)] + E [f(B_s) f''(B_r)] \} dr \to 0,\]

as \(\varepsilon_1, \varepsilon_2 \to 0\). In order to prove the convergence (4.19) we need to estimate

\[A_2(s, r, \varepsilon_1, 2).\]

Notice that, by Lemma 3.4

\[\frac{1}{\varepsilon_1^2} |A_2(s, r, \varepsilon_1, 2)| \leq \frac{1}{\varepsilon_1^2} |E[B_r(B_{s+\varepsilon_1} - B_s)]|\]

\[(4.21) \quad \cdot (|\varepsilon_2 E[B_s(B_{r+\varepsilon_1} - B_r)]| + |\varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)]|)\]

\[\leq C_{H,K} \frac{r}{s}\]

for \(s > r > 0\). We get

\[\frac{1}{\varepsilon_1^2} \int_0^t ds \int_0^{s-\varepsilon_1} |A_2(s, r, \varepsilon_1, 2)| \left| E \left[ f'(B_s) f'(B_r) \right] + E \left[ f''(B_s) f(B_r) \right] \right| dr ds\]

\[\leq C_{H,K} \int_0^t ds \int_0^{s-\varepsilon_1} \frac{r}{s} \left( \frac{1}{s^{3/4}r^{1/4}} + \frac{1}{r^{3/4}s^{1/4}} \right) dr ds \left( f(x) \right) \left( s-r+x^2 \right) \eta_s(x) dx\]

\[\leq C_{H,K} \| f \|_B\]

by Lemma 3.5. Moreover, for \(\varepsilon_1 < s < t, 0 < r < s - \varepsilon_1\) we have

\[\varepsilon_2 E[B_s(B_{r+\varepsilon_1} - B_r)] - \varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)]\]

\[= 2^{-K} \xi_{\varepsilon_1} \{ g_s(r + \varepsilon_1) - g_s(r) - (s-r-\varepsilon_1) + (s-r) \} - 2^{-K} \xi_{\varepsilon_1} \{ g_s(r + \varepsilon_2) - g_s(r) - (s-r-\varepsilon_2) + (s-r) \}\]

\[(4.22) \quad = 2^{-K} \{ [g_s(r + \varepsilon_1) - g_s(r)] \varepsilon_2 - [g_s(r + \varepsilon_2) - g_s(r)] \varepsilon_1 \}

\[= 2^{-K} \{ [g_s(\xi) - g_s(\eta)] \} \varepsilon_1 \varepsilon_2\]

\[= 2^{-K} \left\{ \frac{\xi^{2H-1}}{(s^{2H} + \xi^{2H})^{1-K}} - \frac{\eta^{2H-1}}{(s^{2H} + \eta^{2H})^{1-K}} \right\} \varepsilon_1 \varepsilon_2\]

for some \(\xi \in (r, r+\varepsilon_1)\) and \(\eta \in (r, r+\varepsilon_2)\) by Mean Value Theorem, which implies that

\[\frac{1}{\varepsilon_1^2} |A_2(s, r, \varepsilon_1, 2)| = |E[B_r(B_{s+\varepsilon_1} - B_s)]|\]

\[\cdot |\varepsilon_2 E[B_s(B_{r+\varepsilon_1} - B_r)] - \varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)]|\]

\[(4.23) \quad \leq C_{H,K} \frac{r}{s} \left| \frac{\xi^{2H-1}}{(s^{2H} + \xi^{2H})^{1-K}} - \frac{\eta^{2H-1}}{(s^{2H} + \eta^{2H})^{1-K}} \right| \varepsilon_1 \varepsilon_2\]

\[\to 0\]

for all \(s > r > 0\), as \(\varepsilon_1, \varepsilon_2 \to 0\). This proves

\[\frac{1}{\varepsilon_1^2} \int_0^t ds \int_0^{s-\varepsilon_1} |A_2(s, r, \varepsilon_1, 2)| \left| E \left[ f'(B_s) f'(B_r) \right] + E \left[ f''(B_s) f(B_r) \right] \right| dr ds \to 0\]
by Lebesgue’s dominated convergence theorem. On the other hand, Lemma 4.5 and (4.21) imply that
\[
\frac{1}{\varepsilon_1^2} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} ds \int_0^{\varepsilon_1} \frac{r}{s} |A_{21}(s, r, \varepsilon_1, 2)| |E[f'(B_s) f'(B_r)] + E[f''(B_s) f(B_r)]| dr \\
\leq C_{H,K} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} \frac{r}{s} \left( \frac{1}{s^{3/4} r^{1/4}} \right) \int_R |f(x)|^2 (s + x^2) \varphi_s(x) dx \\
= C_{H,K} \left( \int_0^{\varepsilon_1} (s + \sqrt{s}) ds \int_R |f(x)|^2 \varphi(x) dx + \int_0^{\varepsilon_1} (1 + \frac{1}{\sqrt{s}}) ds \int_R |f(x)|^2 \varphi(x) dx \right) \\
\leq C_{H,K}(\varepsilon_1 + \sqrt{\varepsilon_1}) \|f\|^2_{\mathcal{M}} \rightarrow 0 \quad (\varepsilon_1, \varepsilon_2 \rightarrow 0)
\]
and
\[
\frac{1}{\varepsilon_1^2} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} ds \int_{s-\varepsilon_1}^{s} |A_{21}(s, r, \varepsilon_1, 2)| |E[f'(B_s) f'(B_r)] + E[f''(B_s) f(B_r)]| dr ds r \\
\leq C_{H,K} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} \frac{r}{s} \left( \frac{1}{s^{3/4} r^{1/4}} \right) \int_R |f(x)|^2 (s + x^2) \varphi_s(x) dx \\
= C_{H,K} \left( \int_0^{\varepsilon_1} (s-3/4(s^{3/4} + (s-\varepsilon_1)^{3/4}) + \sqrt{\varepsilon_1}) ds \int_R |f(x)|^2 \varphi(x) dx \\
+ C_{H,K} \left( \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} \frac{1}{s} \left( s^{-3/4} (s^{3/4} + (s-\varepsilon_1)^{3/4}) + \sqrt{\varepsilon_1} \right) ds \int_R |f(x)|^2 \varphi(x) dx \right) \right) \\
\leq C_{H,K}(\varepsilon_1 + \sqrt{\varepsilon_1}) \|f\|^2_{\mathcal{M}} \rightarrow 0 \quad (\varepsilon_1, \varepsilon_2 \rightarrow 0).
\]
Thus, we have shown that
\[
\frac{1}{\varepsilon_1^2} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} ds \int_0^{\varepsilon_1} \{A_{21}(s, r, \varepsilon_1, 2) - A_{22}(s, r, \varepsilon_1, 2)\} E[f''(B_s) f'(B_r)] dr ds \rightarrow 0
\]
as $\varepsilon_1, \varepsilon_2 \rightarrow 0$, which obtains the convergence (4.19). In a same way one can prove the convergence (4.20).

**Step A-2.** The following convergence holds:
\[
(4.24) \quad \frac{1}{\varepsilon_1^2} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} ds \int_0^{\varepsilon_1} \{A_1(s, r, \varepsilon_1, 2) - A_{21}(s, r, \varepsilon_1, 2)\} E[f''(B_s) f'(B_r)] dr \rightarrow 0,
\]
\[
(4.25) \quad \frac{1}{\varepsilon_1^2} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} ds \int_0^{\varepsilon_1} \{A_3(s, r, \varepsilon_1, 2) - A_{22}(s, r, \varepsilon_1, 2)\} E[f'(B_s) f''(B_r)] dr \rightarrow 0,
\]
as $\varepsilon_1, \varepsilon_2 \rightarrow 0$. We have
\[
A_1(s, r, \varepsilon_1, 2) - A_{21}(s, r, \varepsilon_1, 2) \\
= \varepsilon_2 E[B_s(B_{r+\varepsilon_1} - B_r)] E[B_s(B_{s+\varepsilon_1} - B_s)] - \varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)] E[B_s(B_{s+\varepsilon_1} - B_s)] \\
- \varepsilon_2 E[B_s(B_{r+\varepsilon_1} - B_r)] E[B_s(B_{r+\varepsilon_2} - B_r)] + \varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)] E[B_s(B_{s+\varepsilon_1} - B_s)] \\
= \{\varepsilon_2 E[B_s(B_{r+\varepsilon_1} - B_r)] - \varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)]\} E[(B_s - B_r)(B_{s+\varepsilon_1} - B_s)],
\]
which deduces
\[
|A_1(s, r, \varepsilon_1, 2) - A_21(s, r, \varepsilon_1, 2)| \leq \varepsilon_2 [E[B_s(B_{r+\varepsilon_1} - B_r)]E[(B_s - B_r)(B_{s+\varepsilon_1} - B_s)] + \varepsilon_1 E[B_s(B_{r+\varepsilon_2} - B_r)]E[(B_s - B_r)(B_{s+\varepsilon_1} - B_s)]
\]
\[
\leq C_{H,K}\varepsilon_2^2 \frac{s-r}{s}
\]
by Lemma 3.3, Lemma 3.4 and the estimate (3.10). It follows from Lemma 3.5 that
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} \int_0^t ds \int_0^s |A_1(s, r, \varepsilon_1, 2) - A_21(s, r, \varepsilon_1, 2)|E[f''(B_s)f(B_r)]dr
\]
\[
\leq \int_0^t ds \int_0^s \frac{s-r}{s} E[f''(B_s)f(B_r)]dr
\]
\[
\leq C_{H,K} \int_0^t ds \int_0^s \frac{1}{r^{1/4}s^{3/4}} \int_{\mathbb{R}} f^2(x) \varphi_s(x)dx
\]
\[
\leq C_{H,K} \|f\|_2^2.
\]
On the other hand, by (4.26), (4.22), (4.23) and Lemma 3.4 we have
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} |A_1(s, r, \varepsilon_1, 2) - A_21(s, r, \varepsilon_1, 2)| \rightarrow 0,
\]
as \(\varepsilon_1, \varepsilon_2 \rightarrow 0\), for all \(s > r > 0\), which implies that the convergence (4.24) holds by Lebesgue’s dominated convergence theorem. Similarly, one can prove (4.25).

**Part B.** The following convergence hold:

\[
(4.27) \quad \frac{1}{\varepsilon_1^2 \varepsilon_j} \int_0^t \int_0^r Y(s, r, \varepsilon_i, j)dsdr \rightarrow 0 \quad (i, j = 1, 2, i \neq j)
\]
as \(\varepsilon_1, \varepsilon_2 \rightarrow 0\). For \(r > s > 0\) we can decompose \(Y(s, r, \varepsilon_i, j)\) as follows

\[
Y(s, r, \varepsilon_i, j) = A_1(s, r, \varepsilon_i, j)E[f''(B_s)f(B_r)] + A_3(s, r, \varepsilon_i, j)E[f(B_s)f''(B_r)]
\]
\[
+ (A_21(s, r, \varepsilon_i, j) + A_22(s, r, \varepsilon_i, j)) E[f'(B_s)f'(B_r)]
\]
\[
= A_21(s, r, \varepsilon_i, j) \{E[f'(B_s)f'(B_r)] + E[f(B_s)f''(B_r)]\}
\]
\[
+ A_22(s, r, \varepsilon_i, j) \{E[f'(B_s)f'(B_r)] + E[f(B_s)f''(B_r)]\}
\]
\[
+ \{A_3(s, r, \varepsilon_i, j) - A_21(s, r, \varepsilon_i, j)\} E[f''(B_s)f(B_r)]
\]
\[
+ \{A_1(s, r, \varepsilon_i, j) - A_22(s, r, \varepsilon_i, j)\} E[f(B_s)f''(B_r)]
\]
with \(i, j = 1, 2\) and \(i \neq j\). By the same method proving (4.19) we can show that the following convergence hold

\[
\frac{1}{\varepsilon_1^2 \varepsilon_j} \int_0^t dr \int_0^r A_1(s, r, \varepsilon_i, j) \{E[f'(B_s)f'(B_r)] + E[f(B_s)f''(B_r)]\} ds \rightarrow 0,
\]
\[
\frac{1}{\varepsilon_1^2 \varepsilon_j} \int_0^t dr \int_0^r A_22(s, r, \varepsilon_i, j) \{E[f'(B_s)f'(B_r)] + E[f''(B_s)f(B_r)]\} ds \rightarrow 0,
\]
as $\varepsilon_1, \varepsilon_2 \to 0$. On the other hand, clearly, we have

$$A_3(s, r, \varepsilon, j) - A_{21}(s, r, \varepsilon, j)$$

$$= \varepsilon_j E [B_r(B_{s+\varepsilon} - B_s)] E [(B_r - B_s)(B_{r+\varepsilon} - B_r)]$$

$$- \varepsilon E [B_r(B_{s+\varepsilon_1} - B_s)] E [(B_r - B_s)(B_{r+\varepsilon_2} - B_r)]$$

$$A_1(s, r, \varepsilon, j) - A_{22}(s, r, \varepsilon, j)$$

$$= -\varepsilon_j E [B_s(B_{s+\varepsilon} - B_s)] E [(B_r - B_s)(B_{r+\varepsilon} - B_r)]$$

$$+ \varepsilon E [B_s(B_{s+\varepsilon_1} - B_s)] E [(B_r - B_s)(B_{r+\varepsilon_2} - B_r)]$$

for all $r > s > 0$. Thus, in the same way as proof of (4.24) and (4.25) one can prove the convergence

$$\frac{1}{\varepsilon_i^2 \varepsilon_j} \int_0^t \int_0^r \{A_3(s, r, \varepsilon_i, j) - A_{21}(s, r, \varepsilon_i, j)\} E [f''(B_s)f(B_r)] ds \to 0$$

$$\frac{1}{\varepsilon_i^2 \varepsilon_j} \int_0^t \int_0^r \{A_1(s, r, \varepsilon_i, j) - A_{22}(s, r, \varepsilon_i, j)\} E [f(B_s)f''(B_r)] ds \to 0$$

as $\varepsilon_1, \varepsilon_2 \to 0$, and the convergence (4.27) follows.

**Part C.** The following convergence holds:

(4.28) \[ \frac{1}{\varepsilon_i^2 \varepsilon_j} \int_0^t ds \int_0^r A_4(s, r, \varepsilon_i, j)E [f(B_s)f(B_r)] dr \to 0 \quad (i, j = 1, 2, i \neq j) \]

as $\varepsilon_i, \varepsilon_j \to 0$. We have

$$A_4(s, r, \varepsilon_1, 2) = \varepsilon_2 E [(B_r - B_s)(B_{s+\varepsilon_1} - B_{s+\varepsilon_2})] - \varepsilon_1 E [(B_{s+\varepsilon_1} - B_s)(B_{r+\varepsilon_2} - B_r)]$$

$$= 2^{-K} \left( [g_{s+\varepsilon_1}(r + \varepsilon_1) - g_s(r + \varepsilon_1) - [g_{s+\varepsilon_1}(r) - g_s(r)]\varepsilon_2$$

$$- [g_{s+\varepsilon_1}(r + \varepsilon_2) - g_s(r + \varepsilon_2) - [g_{s+\varepsilon_1}(r) - g_s(r)]\varepsilon_1$$

$$+ 2^{-K} \left( \varepsilon_2 [-|s-r| + |s+\varepsilon_1 - r| + |s-r-\varepsilon_1| - |s-r|$$

$$- \varepsilon_1 [-|s+\varepsilon_1 - r - \varepsilon_2| + |s+\varepsilon_1 - r| + |s-r-\varepsilon_2| - |s-r|]$$

$$\right) \right)$$

$$= 2^{-K} A_{41}(s, r, \varepsilon_1, 2) + 2^{-K} A_{42}(s, r, \varepsilon_1, 2)$$

for $s, r > 0$. By Mean Value Theorem we have

(4.29) \[ A_{41}(s, r, \varepsilon_1, 2) = \varepsilon_1 \varepsilon_2 \left( [g'_{s+\varepsilon_1}(\xi) - g'(\xi)] - [g'_{s+\varepsilon_1}(\eta) - g'(\eta)] \right) \]

for some $\xi \in (r, r + \varepsilon_1)$ and $\eta \in (r, r + \varepsilon_2)$. Now, the convergence (4.28) will be varied in three cases.

For $0 < r, s < \varepsilon_1$. It is easy to verify that

(4.30) \[ |A_{42}(s, r, \varepsilon_1, 2)| \leq 2\varepsilon_1 \varepsilon_2. \]

Combining this with

(4.31) \[ |g_y'(x)| = \frac{x^{2H-1}}{(y^{2H} + x^{2H})^{1-K}} = \left( \frac{x^{2H}}{y^{2H} + x^{2H}} \right)^{1-K} \leq 1, \quad x, y \geq 0, \]
we get
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} \int_0^{\varepsilon_1} ds \int_0^{\varepsilon_1} |A_4(s, r, \varepsilon_1, 2) E[f(B_s) f(B_r)]| dr \\
\leq \frac{2}{\varepsilon_1} \int_0^{\varepsilon_1} \int_0^{\varepsilon_1} \{E[f^2(B_s)] + E[f^2(B_r)]\} dr ds \\
= \int_0^{\varepsilon_1} E[f^2(B_s)] ds \longrightarrow 0,
\]
as \varepsilon_1, \varepsilon_2 \to 0, by Lebesgue’s dominated convergence theorem. Similarly, we can show that
the following convergence holds:
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} \int_0^{\varepsilon_1} ds \int_0^{\varepsilon_1} |A_4(s, r, \varepsilon_1, 2) E[f(B_s) f(B_r)]| dr \longrightarrow 0,
\]
as \varepsilon_1, \varepsilon_2 \to 0. For \( s > r + \varepsilon_1 \), by using Mean Value Theorem to the function
\[
x \mapsto g_{s+\varepsilon_1}'(x) - g_s'(x), \quad x \geq 0,
\]
again, we get
\[
A_{41}(s, r, \varepsilon_1, 2) = \varepsilon_1 \varepsilon_2 (\xi - \eta) (g_{s+\varepsilon_1}''(\theta) - g_s''(\theta))
\]
for a \( \theta \in (\xi \land \eta, \xi \lor \eta) \), which gives
\[
|A_{41}(s, r, \varepsilon_1, 2)| \leq \varepsilon_1^2 \varepsilon_2 |g_{s+\varepsilon_1}''(\theta) - g_s''(\theta)| \longrightarrow 0
\]
for all \( s > r > 0 \), as \( \varepsilon_1, \varepsilon_2 \to 0 \). It follows that
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} \int_0^{\varepsilon_1} ds \int_0^{s-\varepsilon_1} |A_4(s, r, \varepsilon_1, 2) E[f(B_s) f(B_r)]| dr \longrightarrow 0,
\]
as \( \varepsilon_1, \varepsilon_2 \to 0 \) because
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} |A_4(s, r, \varepsilon_1, 2)| \leq \frac{2}{s}
\]
for \( s > r + \varepsilon_1 \). Finally, by symmetry we have that
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} \int_0^t dr \int_0^r A_4(s, r, \varepsilon_1, 2) E[f(B_s) f(B_r)] dr \longrightarrow 0,
\]
as \( \varepsilon_1, \varepsilon_2 \to 0 \), and moreover, in the same way we can establish the convergence
\[
\frac{1}{\varepsilon_1^2 \varepsilon_2} \int_0^t ds \int_0^t A_4(s, r, \varepsilon_2, 1) E[f(B_s) f(B_r)] dr \longrightarrow 0,
\]
as \( \varepsilon_1, \varepsilon_2 \to 0 \). Thus, we have established the convergence \( 4.17 \), and the theorem follows.

\( \square \)

**Corollary 4.1.** Let \( 2HK = 1 \). If \( f \) is uniformly bounded, then the quadratic covariation \( [f(B), B] \) exists in \( L^2 \) and
\[
E |[f(B), B]|^2 \leq \left( C_{H,K} \max_x |f(x)| \right) t^2
\]
for all \( 0 \leq t \leq T \).
5. An Itô Formula

Our main object of this section is to explain and prove the following theorem which gives a generalized Itô formula.

**Theorem 5.1.** Let $2HK = 1$ and let $f \in \mathcal{H}$ be left continuous with right limits. If $F$ is an absolutely continuous function with the derivative $F' = f$, then the following Itô type formula holds:

\[
F(B) = F(0) + \int_0^t f(B_s)dB_s + 2^{K-2} [f(B), B]_t.
\]

Clearly, the formula (5.1) is an analogue of Föllmer-Prottet-Shiryayev’s formula (see Eisenbaum [4], Föllmer et al [9], Moret–Nualart [15], Russo–Vallois [20], and the references therein). It is an improvement in terms of the hypothesis on $f$ and it is also quite interesting itself. As an application we get the relationship between the forward (pathwise) integral and Skorohod integral

\[
\int_0^t f(B_s)d^{-}B_s = \int_0^t f(B_s)dB_s + \frac{1}{2}(2^{K-1} - 1) [f(B), B]_t
\]

for all $f \in \mathcal{H}$ left continuous with right limits. The result weakens the hypothesis of differentiability for $f$ (see Russo-Tudor [19]).

Beside on the localization argument and smooth approximation one can prove Theorem 5.1. The so-called the localization argument is that one can localize the domain $\text{Dom}(\delta^{H,K})$ of the operator $\delta^{H,K}$ (see Nualart [16]). Suppose that $\{(\Omega_n, u_n), n \geq 1\} \subset \mathcal{F} \times \text{Dom}(\delta^{H,K})$ is a localizing sequence for $u$, i.e., the sequence $\{(\Omega_n, u_n), n \geq 1\}$ satisfies

(i) $\Omega_n \uparrow \Omega$, a.s.;
(ii) $u = u_n$ a.s. on $\Omega_n$.

If $\delta(u^{(n)}) = \delta(u^{(m)})$ a.s. on $\Omega_n$ for all $m \geq n$, then, the divergence $\delta^{H,K}$ is the random variable determined by the conditions

$\delta^{H,K}(u)|_{\Omega_n} = \delta^{H,K}(u^{(n)})|_{\Omega_n}$ for all $n \geq 1$,

but it may depend on the localizing sequence. Under the localization argument one may assume that the function $f \in \mathcal{H}$ is uniformly bounded. In fact, for any $k \geq 0$ we may consider the set

$\Omega_k = \left\{ \sup_{0 \leq t \leq T} |B_t| < k \right\}$

and let $f^{[k]}$ be a measurable function such that $f^{[k]} = f$ on $[-k, k]$ and vanishes outside. Then $f^{[k]}$ is uniformly bounded and $f^{[k]} \in \mathcal{H}$ for every $k \geq 0$. Set $\frac{d}{dx}F^{[k]} = f^{[k]}$ and $F^{[k]} = F$ on $(-k, k)$. If the formula (5.1) is true for all uniformly bounded functions, then we get the desired formula

$F^{[k]}(B_t) = F^{[k]}(0) + \int_0^t f^{[k]}(B_s)dB_s + 2^{K-2} \left[f^{[k]}(B), B\right]_t$

on the set $\Omega_k$. Letting $k$ tend to infinity we deduce the Itô formula (5.1) for all $f \in \mathcal{H}$ being left continuous with right limits. Thus, we may assume that $f \in \mathcal{H}$ is uniformly bounded in the next discussion.
Lemma 5.1 (Nualart [10], Es-sebaiy and Tudor [7]). Let \( \{u^{(n)}\} \) be a sequence such that \( u_n \to u \) in \( L^2 \), as \( n \to \infty \) and let

\[
\delta^{H,K}(u^{(n)}) = \int_0^T u^{(n)}_s dB_s, \quad n \geq 1
\]

exist in \( L^2 \). If \( \delta^{H,K}(u^{(n)}) \to G \) in \( L^2 \), then \( \delta^{H,K}(u) = \int_0^T u_s dB_s \) exists in \( L^2 \) and equals to \( G \).

Lemma 5.2. Let \( f, f_1, f_2, \ldots \in \mathcal{H} \). If \( f_n \to f \) in \( \mathcal{H} \) as \( n \) tends to infinity, then we have

\[
(5.2) \quad \int_0^t f_n(B_s)d^\pm B_s \to \int_0^t f(B_s)d^\pm B_s
\]

and

\[
(5.3) \quad [f_n(B), B]_t \to [f(B), B]_t
\]

in \( L^2 \) as \( n \to \infty \).

Proof. The lemma follows from

\[
E \left| \int_0^t f_n(B_s)d^\pm B_s - \int_0^t f(B_s)d^\pm B_s \right|^2 \leq C_{H,K}\|f_n - f\|^2_{\mathcal{H}} \to 0,
\]

as \( n \) tends to infinity. \( \square \)

Proof of Theorem 5.1. If \( F \in C^2(\mathbb{R}) \), this is Itô’s formula since

\[
[f(B), B]_t = \int_0^t f'(B_s)d[B, B]_s = 2^{1-K}\int_0^t f'(B_s)ds.
\]

If \( F \notin C^2(\mathbb{R}) \), we let \( F' = f \in \mathcal{H} \) be uniformly bounded and left continuous. Consider the function \( \zeta \) on \( \mathbb{R} \) by

\[
(5.4) \quad \zeta(x) := \begin{cases} \frac{c e^{\frac{1}{(x-1)^2-1}}}{x-1}, & x \in (0, 2), \\ 0, & \text{otherwise}, \end{cases}
\]

where \( c \) is a normalizing constant such that \( \int_\mathbb{R} \zeta(x)dx = 1 \). Define the so-called mollifiers

\[
(5.5) \quad \zeta_n(x) := n\zeta(nx), \quad n = 1, 2, \ldots
\]

and the sequence of smooth functions

\[
(5.6) \quad F_n(x) := \int_\mathbb{R} F(x-y)\zeta_n(y)dy = \int_0^2 F(x - \frac{y}{n})\zeta(y)dy, \quad n = 1, 2, \ldots
\]

for all \( x \in \mathbb{R} \). Denote \( f_n = F'_n \) for \( n = 1, 2, \ldots \). Then \( F_n \in C^\infty(\mathbb{R}) \), \( f_n \in C^\infty(\mathbb{R}) \cap \mathcal{H} \) and

\[
f_n(x) = \int_\mathbb{R} f(x-y)\zeta_n(y)dy
\]

for all \( n \geq 1 \). It is easy to check that \( F'_n, f_n, f'_n (n \geq 1) \) satisfy the condition (2.4) in Theorem 2.1. Hence, Skorohod integral \( \int_0^t f_n(B_s)dB_s \) exists and Itô’s formula

\[
(5.7) \quad F_n(B_t) = F_n(0) + \int_0^t f_n(B_s)dB_s + \frac{1}{2} \int_0^t f'_n(B_s)ds
\]

holds for all \( n \geq 1 \).
On the other hand, using Lebesgue’s dominated convergence theorem, one can prove that as \( n \) tends to infinity, \( f_n \rightarrow f \) in \( \mathcal{H} \) and
\[
F_n(B_t) \rightarrow F(B_t), \quad f_n(B_t) \rightarrow f(B_t),
\]
in \( L^2 \), for all \( t \in [0, T] \). Thus, we get
\[
2^{1-K} \int_0^t f'_n(B_s)ds = [f_n(B), B]_t \rightarrow [f(B), B]_t
\]
in \( L^2 \) by Lemma 5.2, as \( n \) tends to infinity. It follows that
\[
\int_0^t f_n(B_s)dB_s = F_n(B_t) - F_n(0) - 2^{K-2}[f_n(B), B]_t
\]
\[
\rightarrow F(B_t) - F(0) - 2^{K-2}[f(B), B]_t
\]
in \( L^2 \), as \( n \) tends to infinity. This completes the proof by Lemma 5.1.

\[\square\]

6. The Bouleau-Yor identity

In this section we study one parameter integral of local time
\[
\int_{\mathbb{R}} f(x) \mathcal{L}(dx, t),
\]
and establish the Bouleau-Yor identity between the integral above and the quadratic co-
variation \([f(B), B]_t\), where \( f \) is a deterministic function and
\[
\mathcal{L}(x, t) = \int_0^t \delta(B_s - x)ds
\]
is the local time of bi-fBm \( B \). Recall that the quadratic covariation \([f(W), W]_t\) of Brownian
motion \( W \) can be characterized as
\[
[f(W), W]_t = -\int_{\mathbb{R}} f(x) \mathcal{L}^W(dx, t),
\]
where \( f \) is locally square integrable and \( \mathcal{L}^W(x, t) \) is the local time of Brownian motion
\( W \). This is called the Bouleau-Yor identity. More works for this can be found in Bouleau-
Yor [3], Eisenbaum [4], Föllmer et al [9], Feng–Zhao [8], Peskir [17], Rogers–Walsh [18],
Yang–Yan [27], and the references therein. Moreover, this has been extended to fractional
Brownian motion \( B^H \) by Yan et al [25, 27].

**Lemma 6.1.** Let \( f_\triangle \in \mathcal{E} \). If
\[
f_\triangle = \sum_{j=1}^{N_1} x_j 1_{(a_{j-1}, a_j)} = \sum_{i=1}^{N_2} y_i 1_{(b_{i-1}, b_i)},
\]
we then have
\[
\sum_j x_j [\mathcal{L}(a_j, t) - \mathcal{L}(a_{j-1}, t)] = \sum_i y_i [\mathcal{L}(b_i, t) - \mathcal{L}(b_{i-1}, t)]
\]
\[
= -2^{K-1}[f_\triangle(B), B]_t.
\]
Proof. Take \( F(x) = (x-a)^+ - (x-b)^+ \). Then \( F \) is absolutely continuous with the derivative \( F' = 1_{(a,b]} \in \mathcal{H} \) being left continuous and bounded, and the Itô formula (5.1) yields

\[
2^{K-2} \left[ 1_{(a,b]}(B), B \right]_t = F(B_t) - F(0) - \int_0^t 1_{(a,b]}(B_s) dB_s
\]

for all \( t \geq 0 \). On the other hand, the Tanaka formula (2.6) follows

\[
\mathcal{L}(a,t) - \mathcal{L}(b,t) = 2F(B_t) - 2F(0) - 2 \int_0^t 1_{(a,b]}(B_s) dB_s
\]

for all \( t \geq 0 \), which deduces

\[
\mathcal{L}(a,t) - \mathcal{L}(b,t) = 2^{K-1} \left[ 1_{(a,b]}(B), B \right]_t
\]

for all \( t \geq 0 \). Thus, the linearity property of the quadratic covariation implies that the lemma holds.

As a direct consequence of Lemma 6.1 we can define the integral

\[
\int f(x) \mathcal{L}(dx,t) := \sum_j x_j \left[ \mathcal{L}(a_j,t) - \mathcal{L}(a_{j-1},t) \right]
\]

for every \( f \in \mathcal{E} \). Together this and Lemma 5.2 lead to

\[
\lim_{n \to \infty} \int f_{\Delta,n}(x) \mathcal{L}(dx,t) = \lim_{n \to \infty} \int \tilde{f}_{\Delta,n}(x) \mathcal{L}(dx,t) \quad \text{in } L^2,
\]

if \( f_{\Delta,n} \to f \) and \( \tilde{f}_{\Delta,n} \to f \) in \( \mathcal{H} \), as \( n \) tends to infinity, where \( \{f_{\Delta,n}\}, \{\tilde{f}_{\Delta,n}\} \subset \mathcal{E} \). Thus, thanks to the density of \( \mathcal{E} \) in \( \mathcal{H} \), we can define integral of \( f \in \mathcal{H} \) with respect to \( x \to \mathcal{L}(x,t) \) in the following manner:

\[
\int f(x) \mathcal{L}(dx,t) := \lim_{n \to \infty} \int f_{\Delta,n}(x) \mathcal{L}(dx,t) \quad \text{in } L^2,
\]

provided \( f_{\Delta,n} \to f \) in \( \mathcal{H} \), as \( n \) tends to infinity, where \( \{f_{\Delta,n}\} \subset \mathcal{E} \).

**Corollary 6.1.** Let \( 2HK = 1 \) and let \( f \in \mathcal{H} \). Then the integral \( \int f(x) \mathcal{L}(dx,t) \) exists in \( L^2 \), and the Bouleau-Yor identity

\[
[f(B), B]_t = -2^{1-K} \int f(x) \mathcal{L}(dx,t)
\]

holds for all \( t \in [0,T] \).

**Corollary 6.2.** Let \( 2HK = 1 \) and let \( f, f_1, f_2, \ldots \in \mathcal{H} \). If \( f_n \to f \) in \( \mathcal{H} \), as \( n \) tends to infinity, we then have

\[
\int f_n(x) \mathcal{L}(dx,t) \to \int f(x) \mathcal{L}(dx,t)
\]

in \( L^2 \), as \( n \) tends to infinity.

According to Theorem 5.1, we get an analogue of Bouleau-Yor’s formula.
Corollary 6.3. Let $2HK = 1$ and let $f \in \mathcal{H}$ be left continuous with right limits. If $F$ is an absolutely continuous function with the derivative $F' = f$, then the following Itô type formula holds:

$$F(B_t) = F(0) + \int_{0}^{t} f(B_s)dB_s - \frac{1}{2} \int_{\mathbb{R}} f(x)L(dx,t).$$

Recall that if $F$ is the difference of two convex functions, then $F$ is an absolutely continuous function with derivative of bounded variation. Thus, the Itô-Tanaka formula (see Es-sebaiy and Tudor [7])

$$F(B_t) = F(0) + \int_{0}^{t} F'(B_s)dB_s + \frac{1}{2} \int_{\mathbb{R}} L(x,t)F''(dx)$$

$$\equiv F(0) + \int_{0}^{t} F'(B_s)dB_s - \frac{1}{2} \int_{\mathbb{R}} F'(x)L(dx,t)$$

holds.

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