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A PROXIMAL-LIKE ALGORITHM FOR VIBRO-IMPACT PROBLEMS WITH A NON-SMOOTH SET OF CONSTRAINTS

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Abstract. We consider a discrete mechanical system subjected to perfect uni-lateral constraints characterized by some geometrical inequalities \( f_\alpha(q(t)) \geq 0, \ \forall \alpha \in \{1, \ldots, \nu\}, \ \nu \geq 1 \). We assume that the transmission of the velocities at impacts is governed by a Newton’s impact law with a restitution coefficient \( e \in [0, 1] \), allowing for conservation of kinetic energy if \( e = 1 \), or loss of kinetic energy if \( e \in [0, 1) \), when the constraints are saturated. Starting from a formulation of the dynamics as a first order measure-differential inclusion for the unknown velocities, time-stepping schemes inspired by the proximal methods can be proposed. Convergence results in the single-constraint case (\( \nu = 1 \)) are recalled and extended to the multi-constraint case (\( \nu > 1 \)), leading to new existence results for this kind of problems.

Keywords Frictionless unilateral constraints, Newton’s impact law, multi-constraint case, time-stepping scheme.

1. Description of the dynamics. We consider a mechanical system with a finite number of degrees of freedom. We denote by \( q \in \mathbb{R}^d \) the representative point of the system in generalized coordinates and by \( M(q) \) the inertia operator. The unconstrained dynamics is described by a second order Ordinary Differential Equation

\[
M(q)\ddot{q} = g(t, q, \dot{q}).
\] (1)

We assume that the system is subjected to unilateral constraints described by some geometrical inequalities

\[
f_\alpha(q(t)) \geq 0, \ \forall t, \ \forall \alpha \in \{1, \ldots, \nu\}, \ \nu \geq 1
\] (2)

with smooth (at least of class \( C^1 \)) functions \( f_\alpha \). The set of admissible configurations is then defined by

\[
K = \{ q \in \mathbb{R}^d; f_\alpha(q) \geq 0 \ \forall \alpha \in \{1, \ldots, \nu\} \}
\]

and we denote by \( J(q) \) the set of active constraints at \( q \) given by

\[
J(q) = \{ \alpha \in \{1, \ldots, \nu\}; \ f_\alpha(q) \leq 0 \}, \ \forall q \in \mathbb{R}^d.
\]

When the constraints are saturated, i.e. when at least one of the inequalities (2) is an equality, a reaction force appears and should be added to the right hand side of equation (1):

\[
M(q)\ddot{q} = g(t, q, \dot{q}) + R, \ \text{Supp}(R) \subset \{ t; J(q(t)) \neq \emptyset \}.
\] (3)

We assume moreover that the constraints are perfect i.e.
• there is no adhesion
\((R,v) \geq 0 \ \forall v \in T_K(q)\)

• the contact is frictionless
\((R,v) = 0 \ \forall v \in T_K(q) \cap (-T_K(q))\)

where \((\cdot,\cdot)\) denotes the Euclidean inner product in \(\mathbb{R}^d\) and \(T_K(q)\) is the set of kinematically admissible right velocities at \(q\) given by
\[
T_K(q) = \left\{ v \in \mathbb{R}^d; (\nabla f_\alpha(q), v) \geq 0 \ \forall \alpha \in J(q) \right\}.
\]

Using Farkas’s lemma we infer that
\[
R = \sum_{\alpha \in J(q)} \lambda^\alpha \nabla f_\alpha(q), \quad \lambda^\alpha \geq 0. \quad (4)
\]

If \(J(q(t)) \neq \emptyset\) the inequalities (2) imply
\[
\dot{q}^+(t) \in T_K(q(t)), \quad \dot{q}^-(t) \in -T_K(q(t)). \quad (5)
\]

Thus the velocities may be discontinuous at impacts. It follows that \(R\) is a measure and (3)-(4) lead to a second order Measure Differential Inclusion
\[
M(q) \dot{q} = g(t,q,\dot{q}) \in -N_K(q) \quad (6)
\]

with
\[
N_K(q) = \left\{ w \in \mathbb{R}^d; w = \sum_{\alpha \in J(q)} \mu^\alpha \nabla f_\alpha(q), \ \mu^\alpha \leq 0 \right\} \text{ if } q \in K,
\]
\[
\emptyset \text{ if } q \notin K.
\]

We infer that the jumps of velocities at impacts satisfy
\[
M(q(t)) (\dot{q}^+(t) - \dot{q}^-(t)) \in -N_K(q(t)) \quad (7)
\]

but relations (5) and (7) do not define uniquely \(\dot{q}^+(t)\) and we have to complete the description of the dynamics by a constitutive impact law.

We will assume that the tangential part of the left velocity is conserved while its normal part is reversed and multiplied by a coefficient \(e \in [0,1]\), i.e.
\[
\dot{q}^+(t) = \dot{q}^-_T(t) - e\dot{q}^-_N(t)
\]

where \(\dot{q}^-_T(t)\) and \(\dot{q}^-_N(t)\) are defined respectively as the projection of \(\dot{q}^-(t)\) on the convex cones \(T_K(q(t))\) and \(N_K^*(q(t)) = M^{-1}(g(t)) N_K(q(t))\) relatively to the kinetic metric at \(q(t)\). More precisely we define the kinetic metric at \(q\) by
\[
\|x\|_{M(q)} = \left( x^T M(q) x \right)^{1/2} \quad \forall (x,q) \in \mathbb{R}^d \times \mathbb{R}^d
\]

and we have
\[
\dot{q}^+(t) = \text{Proj}_{M(q(t))} (T_K(q(t)), \dot{q}^-(t)) - e \text{Proj}_{M(q(t))} (N_K^*(q(t)), \dot{q}^- (t)) \quad (8)
\]

which is a Newton’s law with a restitution coefficient \(e \in [0,1]\). When \(e = 0\) we simply get
\[
\dot{q}^+(t) = \text{Proj}_{M(q(t))} (T_K(q(t)), \dot{q}^- (t))
\]

and we recognize the description of standard inelastic shocks introduced by J.J. Moreau ([9]). When \(e \neq 0\), we can use the lemma of the two cones (see [8]) to rewrite (8) as
\[
\dot{q}^+(t) = -e\dot{q}^- (t) + (1 + e) \text{Proj}_{M(q(t))} (T_K(q(t)), \dot{q}^- (t))
\]
We can observe that this model is mechanically consistent. Indeed the kinetic energy $\mathcal{E} = \frac{1}{2} \| \dot{q} \|^2_{M(q)}$ satisfies

$$
\mathcal{E}^+(t) = \frac{1}{2} \| \dot{q}^+(t) \|^2_{M(q(t))}
= \frac{1}{2} \left( \| \text{Proj}_{M(q(t))} (T_K(q(t)), \dot{q}^-(t)) \|_{M(q(t))}^2
+ e^2 \| \text{Proj}_{M(q(t))} (N^*_K(q(t)), \dot{q}^-(t)) \|_{M(q(t))}^2 \right)
\leq \frac{1}{2} \left( \| \text{Proj}_{M(q(t))} (T_K(q(t)), \dot{q}^-(t)) \|_{M(q(t))}^2
+ \| \text{Proj}_{M(q(t))} (N^*_K(q(t)), \dot{q}^-(t)) \|_{M(q(t))}^2 \right)
= \mathcal{E}^-(t)
$$

and we have conservation of energy at impacts if $e = 1$ (elastic shocks).

Following J.J.Moreau’s approach (see [9] or [10] for instance) we can describe the dynamics at the velocity level by the following first order Measure Differential Inclusion

$$
g(t, q, \dot{q}) dt - M(q) \ddot{q} \in \partial \psi_{T_K(q)} \left( \frac{\dot{q}^+ + e \dot{q}^-}{1 + e} \right)
$$

where $\psi_{T_K(q)}$ denotes the indicator function of $T_K(q)$ and $\partial \psi_{T_K(q)}$ its subdifferential given by

$$
\partial \psi_{T_K(q)}(v) = \begin{cases} 
\{ z \in \mathbb{R}^d; (z, w - v) \leq 0 \ \forall w \in T_K(q) \} & \text{if } v \in T_K(q), \\
\emptyset & \text{otherwise.}
\end{cases}
$$

For a more detailed study of the equivalence of the system (6)-(8) with the MDI (9) the reader is referred to [13].

More precisely, for any given admissible initial data $(q_0, u_0) \in K \times T_K(q_0)$ we will consider the Cauchy problem (P):

**Problem (P)** Find two functions $q, u : [0, \tau] \to \mathbb{R}^d$, with $\tau > 0$, such that

**P1** $u \in BV([0, \tau]; \mathbb{R}^d),$

**P2** $u(t) = \frac{u^+(t) + e u^-(t)}{1 + e}$ for all $t \in (0, \tau),$

**P3** $q(t) = q_0 + \int_0^t u(s) \, ds$ for all $t \in [0, \tau],$

**P4** there exists a non-negative measure $\mu$ such that the Stieltjes measure $du$ and the Lebesgue’s measure $dt$ admit densities relatively to $d\mu$, denoted respectively $u'_\mu$ and $t'_\mu$, and

$$
g(t, q(t), u(t)) t'_\mu(t) - M(q(t)) u'_\mu(t) \in \partial \psi_{T_K(q(t))}(u(t)) \quad d\mu \text{ a.e. on } (0, \tau),$$

**P5** $u^+(0) = u_0.$
2. Time-stepping scheme. Let $h > 0$ be a given time-step and $(q_0, u_0)$ be any given admissible initial data. Starting from (9) the discrete positions and velocities are defined by

$$q_{h,0} = q_0, \quad u_{h,0} = u_0$$

and for all $i \geq 0$

$$\begin{cases} q_{h,i+1} = q_{h,i} + hu_{h,i} \\ g_{h,i+1} - M(q_{h,i+1}) \left( \frac{u_{h,i+1} - u_{h,i}}{h} \right) \in \partial \psi_{T_K(q_{h,i+1})} \left( \frac{u_{h,i+1} + e u_{h,i}}{1 + e} \right) \end{cases} \quad (11)$$

with $g_{h,i+1} = g(t_{h,i+1}, q_{h,i+1}, u_{h,i})$. Interpreting $u_{h,i+1}$ and $u_{h,i}$ as the right and left velocities at $t_{h,i+1} = (i + 1)h$, this inclusion is a very natural discretization of the MDI (9). Moreover, using the definition of $\partial \psi_{T_K(q)}$, we can rewrite it as

$$u_{h,i+1} = -e u_{h,i} + (1 + e) \text{Proj}_{M(q_{h,i+1})} \left( T_K(q_{h,i+1}), u_{h,i} + \frac{h}{1 + e} M^{-1}(q_{h,i+1}) g_{h,i+1} \right)$$

which leads to an approximation of the impact law (8).

This scheme has been introduced by J.J.Moreau in the 80’s (see [10] for instance). It is inspired by sweeping process techniques and can be interpreted as a proximal-like method for the MDI (9). In the single constraint case (i.e. $\nu = 1$) its convergence has been established first in the case of a trivial mass matrix and standard inelastic shocks (i.e. $M(q) \equiv \text{Id}_{\mathbb{R}^d}$ and $e = 0$) by M.Monteiro Marques ([6], [7]). This result has been extended to the case of partially or totally elastic shocks (i.e. $e \in [0,1]$) but still a trivial mass matrix by M.Mabrouk ([4]).

For a non-trivial mass matrix and a vanishing restitution coefficient, the convergence of the scheme has been established by B.Maury for a constant inertia operator $M$ and by R.Dzonou and M.Monteiro Marques for a position-dependent inertia operator $M = M(q)$, both in 2006 ([5] and [2]). Finally, the general single constraint case (i.e. $M = M(q) \neq \text{Id}_{\mathbb{R}^d}$ and $e \in [0,1]$) has been considered in [3].

Motivated by applications to systems of rigid bodies (see for instance [11] or [12] for examples of implementation with granular materials), we can wonder if the previous convergence results can be extended to the multi-constraint case. Unfortunately we meet a new difficulty: continuity on data does not hold in general when $\nu > 1$.

Indeed, let us consider the model problem of a material point moving in an angular domain of $\mathbb{R}^2$: it can be seen easily that continuity on initial data does not hold if the edge angle is obtuse (see figure 1 and [14] for detailed computations).

In such a case, even if we can prove a theoretical convergence result for the algorithm (10)-(11), round-up errors may lead to a kind of unpredictibility. So we have to introduce some geometrical assumptions on the active constraints to ensure continuity on data and avoid this difficulty. Keeping in mind the previous model problem, it has been proved that continuity on data holds if for all $q \in \partial K$

$$\begin{align} (\nabla f_\alpha(q), M(q)^{-1} \nabla f_\beta(q)) & \leq 0 \quad \text{if } e = 0 \\ (\nabla f_\alpha(q), M(q)^{-1} \nabla f_\beta(q)) & = 0 \quad \text{if } e \neq 0 \end{align} \quad (12)$$

for all $(\alpha, \beta) \in J(q)^2$ such that $\alpha \neq \beta$ (see [1] and [14]). These relations mean that the active constraints create right or acute angles with respect to the momentum metric defined by $M^{-1}(q)$ if $e = 0$ or right angles if $e \neq 0$ and, in this framework, we will establish the convergence of the approximate trajectories.
3. Convergence result. Let us introduce some regularity assumptions on the data:

**H1** $M$ is a mapping of class $C^1$ from $\mathbb{R}^d$ to the set of symmetric positive definite $d \times d$ matrices,

**H2** $g$ is a function of class $C^1$ from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ($T > 0$) to $\mathbb{R}^d$,

**H3** for all $\alpha \in \{1, \ldots, \nu\}$, the function $f_\alpha$ belongs to $C^1(\mathbb{R}^d; \mathbb{R})$, $\nabla f_\alpha$ is Lipschitz continuous and does not vanish in a neighbourhood of $\{q \in \mathbb{R}^d : f_\alpha(q) = 0\}$,

**H4** the active constraints are functionally independent i.e., for all $q \in K$ the vectors $(\nabla f_\alpha(q))_{\alpha \in J(q)}$ are linearly independent.

Without further assumptions on the mappings $M$ and $g$, we cannot expect a global existence result for problem (P) on $[0, T]$ since global solutions may not exist, even if the constraints are never saturated. Indeed, for any solution $(q, u)$ defined on $[0, \tau]$ (with $\tau \in (0, T]$), we have the following energy estimate

$$\mathcal{E}^+(t) \leq \mathcal{E}^+(0) + \int_0^t \left(g(s, q(s), \dot{q}(s)), \dot{q}(s)\right) ds + \frac{1}{2} \int_0^t \left(\ddot{q}(s), (dM(q(s))\dot{q}(s))\dot{q}(s)\right) ds \quad \forall t \in [0, \tau)$$

with an equality when the constraints are never saturated and finite time explosion may occur. Nevertheless, we can establish that
Proposition 3.1. ([15],[16]) Let $C > \|u_0\|_{M(q_0)}$. Then, there exists $\tau(C) \in (0, T]$ such that, for any solution $(q, u)$ of problem (P) defined on $[0, \tau]$, we have
\[
\begin{align*}
\|q(t) - q_0\| &\leq C \quad \forall t \in \left[0, \min(\tau(C), \tau)\right], \\
\|u(t)\|_{M(q(t))} &\leq C \quad \mathrm{a.e.} \text{ on } \left[0, \min(\tau(C), \tau)\right].
\end{align*}
\]

So we will prove the following theorem:

Theorem 3.2. Assume that $(H1)$-(H4) and the geometrical property (12) hold. Let $(q_0, u_0) \in K \times T_K(q_0)$ be given admissible initial data. Let $h > 0$ be a given time-step and let $(q_h, u_h)$ be defined by
\[
q_h(t) = q_{h,i} + (t - ih)u_{h,i}, \quad u_h(t) = u_{h,i}
\]
for all $t \in [ih, (i+1)h)$, for all $i \in \{0, \ldots, \lfloor T/h \rfloor\}$, where $(q_{h,i}, u_{h,i})_{0 \leq i \leq \lfloor T/h \rfloor}$ is given by (10)-(11).

Let $C > \|q_0\|_q$ and $\tau(C) \in (0, T]$ given by proposition 3.1. There exist $(q, u) \in C^{\alpha}([0, \tau(C)]; \mathbb{R}^d) \times BV(0, \tau(C); \mathbb{R}^d)$ and a subsequence of $(q_h, u_h)_{h>0}$, denoted $(q_{h_n}, u_{h_n})_{n \in \mathbb{N}}$, such that
\[
\begin{align*}
q_{h_n}(t) &\to q(t) \text{ strongly in } C^{\alpha}([0, \tau(C)]; \mathbb{R}^d), \\
u_{h_n}(t) &\to u(t) \text{ except perhaps on a countable subset of } [0, \tau(C)],
\end{align*}
\]
and $(q, u)$ is a solution of problem (P).

Let us emphasize that this convergence result provides an existence result for problem (P) under weaker regularity assumptions on the data than in [1]. Since the Cauchy problem may have several solutions (see [17] or [1] for counter-examples to uniqueness) we can not expect the convergence of the whole sequence of approximate solutions $(q_h, u_h)_{h>0}$.

Sketch of the proof The proof is divided in four steps.

Step 1 We begin with an estimate of the discrete velocities.

With the lemma of the two cones we can check easily that
\[
\|\text{Proj}_{M(q)}(T_K(q), x) - e\text{Proj}_{M(q)}(N^*_K(q), x)\|_{M(q)} \leq \|x\|_{M(q)} \quad \forall (q, x) \in \mathbb{R}^d \times \mathbb{R}^d.
\]
By definition of the algorithm
\[
\begin{align*}
u_{h,i+1} &= \text{Proj}_{M(q)}(T_K(q), x_{h,i}) - e\text{Proj}_{M(q)}(M^{-1}(q)N_K(q), x_{h,i}) \\
&\quad + \frac{eh}{1+\epsilon} M^{-1}(q_{h,i+1})g(t_{h,i+1}, q_{h,i+1}, u_{h,i})
\end{align*}
\]
with $x_{h,i} = u_{h,i} + \frac{h}{1+\epsilon} M^{-1}(q_{h,i+1})g(t_{h,i+1}, q_{h,i+1}, u_{h,i})$ for all $i \geq 0$.

Thus,
\[
\begin{align*}
\|u_{h,i+1}\|_{M(q_h,i+1)} &\leq \|u_{h,i}\|_{M(q_h,i)} + \frac{h}{1+\epsilon} M^{-1/2}(q_{h,i+1}) \|g(t_{h,i+1}, q_{h,i+1}, u_{h,i})\| \\
&\leq \|u_{h,i}\|_{M(q_h,i)} + \frac{h}{1+\epsilon} M^{-1/2}(q_{h,i+1}) \|g(t_{h,i+1}, q_{h,i+1}, u_{h,i})\|
\end{align*}
\]
It follows that there exists $\tau \in (0, T]$ such that the sequence $(u_h)_{h>0}$ is uniformly bounded in $L^\infty(0, \tau; \mathbb{R}^d)$ and $(q_h)_{h>0}$ is uniformly Lipschitz continuous on $[0, \tau]$. Using Ascoli’s theorem, and possibly extracting a subsequence denoted $(q_{h_n}, u_{h_n})_{n \in \mathbb{N}}$, we get
\[
\begin{align*}
q_{h_n} &\to q \quad \text{strongly in } C^{\alpha}([0, \tau]; \mathbb{R}^d), \\
u_{h_n} &\rightharpoonup v \quad \text{weakly* in } L^\infty(0, \tau; \mathbb{R}^d).
\end{align*}
\]
Furthermore, observing that the constraints are satisfied at each time step at the velocity level by the average discrete velocity \( \frac{u_{h,i+1} + c u_{h,i}}{1 + e} \) (which belongs to \( T_K(q_{h,i+1}) \)), we can prove that

\[ q(t) \in K \quad \forall t \in [0, \tau]. \]

Thus there exists a compact set \( B \) such that \( q(t) \in K \cap B \) for all \( t \in [0, \tau] \). We infer that there exists a compact neighbourhood \( K \) of \( K \cap B \) such that (H3) holds on \( K \) and \( q_{h,n}(t) \in K \) for all \( t \in [0, \tau] \) and for all \( n \) large enough. As a consequence we can prove

**Lemma 3.3.** For all \( i \in \{0, \ldots, \lfloor \tau/h_n \rfloor \} \), there exist non-positive real numbers \( (\mu_{h,n,i+1}^\alpha)_{\alpha \in J(q_{h,n,i+1})} \) such that

\[ M(q_{h,n,i+1})(u_{h,n,i} - u_{h,n,i+1}) + h_n g_{h,n,i+1} = \sum_{\alpha \in J(q_{h,n,i+1})} \mu_{h,n,i+1}^\alpha \nabla f_\alpha(q_{h,n,i+1}) \]

and there exists a constant \( C_1 \) (independent of \( n \) and \( i \)) such that \( |\mu_{h,n,i+1}^\alpha| \leq C_1 \).

**Step 2** Next we prove an a priori estimate for the discrete accelerations. More precisely

**Lemma 3.4.** There exists \( C_2 > 0 \) (independent of \( n \)) such that

\[ \sum_{i=1}^{\lfloor \tau/h_n \rfloor} ||u_{h,n,i} - u_{h,n,i-1}|| \leq C_2 \quad \forall n \in \mathbb{N}. \]

Indeed, with lemma 3.3, we only need to estimate \( \sum_{i=1}^{\lfloor \tau/h_n \rfloor} \sum_{\alpha \in J(q_{h,n,i})} |\mu_{h,n,i}^\alpha| \). But, for all \( i \in \{1, \ldots, \lfloor \tau/h_n \rfloor \} \), the family \( (\nabla f_\alpha(q_{h,n,i+1}))_{\alpha \in J(q_{h,n,i+1})} \) is linearly independent, so we can complete it as a basis of \( \mathbb{R}^d \) denoted by \( (v_j(q_{h,n,i+1}))_{1 \leq j \leq d} \) and we define by \( (w_j(q_{h,n,i+1}))_{1 \leq j \leq d} \) its dual basis. Then we introduce \( \mu_{h,n,i}^\beta = 0 \) if \( \beta \notin J(q_{h,n,i}) \) and we have

\[ |\mu_{h,n,i}^\alpha| = \sum_{\beta=1}^d -\mu_{h,n,i}^\beta (v_\beta(q_{h,n,i}), w_\alpha(q_{h,n,i})). \]

We rewrite it in order to get a telescopic sum

\[ |\mu_{h,n,i}^\alpha| = \sum_{\beta=1}^d -\mu_{h,n,i}^\beta (v_\beta(q_{h,n,i}), w_\alpha(q_{h,n,i})) = (M(q_{h,n,i})(u_{h,n,i} - u_{h,n,i-1}) - h_n g_{h,n,i}, w_\alpha(q_{h,n,i})) = (M(q_{h,n,i})u_{h,n,i}, w_\alpha(q_{h,n,i})) - (M(q_{h,n,i-1})u_{h,n,i-1}, w_\alpha(q_{h,n,i-1})) + \mathcal{O}(h_n) \]

which allows us to conclude.

We infer that the sequence \((u_{h,n})_{n \in \mathbb{N}}\) is uniformly bounded in \( BV(0, \tau; \mathbb{R}^d) \). Thus we can pass to the limit with Helly’s theorem: possibly extracting another subsequence, still denoted \((q_{h,n}, u_{h,n})_{n \in \mathbb{N}}\) and possibly modifying \( v \) on a negligible subset of \([0, \tau]\) we obtain

\[ u_{h,n}(t) \to v(t) \quad \forall t \in [0, \tau] \]
and \( v \in BV(0, \tau; \mathbb{R}^d) \). We define \( u \in BV(0, \tau; \mathbb{R}^d) \) by
\[
  u(0) = \frac{v^+(0) + ev(0)}{1 + e}, \quad u(\tau) = \frac{v(\tau) + ev^-(\tau)}{1 + e}, \quad u(t) = \frac{v^+(t) + ev^-(t)}{1 + e} \quad \forall t \in (0, \tau).
\]
Clearly \( u \) satisfies properties (P1)-(P2) and we have
\[
  q(t) = q_0 + \int_0^t u(s) \, ds \quad \forall t \in [0, \tau].
\]
Furthermore, using the same techniques as in [3] we prove that (P4) holds at the continuity points of \( u \) and that \( u^+(0) = u_0 \).

**Step 3** We study now the jumps of the velocities. First we observe that

**Lemma 3.5.** For all \( t \in (0, \tau) \) we have
\[
  M(q(t))(v^-(t) - v^+(t)) \in N_K(q(t)).
\]

Next we consider \( t \in (0, \tau) \) such that \( u \) is discontinuous at \( t \), i.e. \( u^-(t) = v^-(t) \neq v^+(t) = u^+(t) \). We have to prove that
\[
  v^+(t) = -ev^-(t) + (1 + e)\text{Proj}_{M(q(t))}(T_K(q(t)), v^-(t)). \tag{13}
\]
With the previous results, we already know that
\[
  \dot{q}^+(t) = v^+(t) \in T_K(q(t)), \quad \dot{q}^-(t) = v^-(t) \in -T_K(q(t)),
\]
\[
  M(q(t))(v^-(t) - v^+(t)) \in N_K(q(t)).
\]

It follows that \( J(q(t)) \neq \emptyset \) and there exist non-positive real numbers \((\mu^\alpha)_{\alpha \in J(q(t))}\) such that
\[
  M(q(t))(v^+(t) - v^-(t)) = -\sum_{\alpha \in J(q(t))} \mu^\alpha \nabla f_\alpha(q(t)).
\]
Hence (13) is satisfied if and only if
\[
  \left\{ \begin{array}{l}
    v^+(t) + ev^-(t) \in T_K(q(t)) \\
    M(q(t))(v^+(t) - v^-(t)), v^+(t) + ev^-(t) = 0
  \end{array} \right.
\]
i.e.
\[
  (\nabla f_\alpha(q(t)), v^+(t) + ev^-(t)) \geq 0 \tag{14}
\]
and
\[
  \mu^\alpha (\nabla f_\alpha(q(t)), v^+(t) + ev^-(t)) = 0, \tag{15}
\]
for all \( \alpha \in J(q(t)) \).

Recalling lemma 3.3, we have
\[
  M(qh_{n,i+1})(uh_{n,i} - uh_{n,i+1}) + h_n g_{n,i+1} = \sum_{\alpha \in J(qh_{n,i+1})} \mu^\alpha_{h_n,i+1} \nabla f_\alpha(qh_{n,i+1}),
\]
with \( \mu^\alpha_{h_n,i+1} \leq 0 \) for all \( i \in \{0, \ldots, \lfloor \tau/h_n \rfloor - 1\} \), for all \( n \in \mathbb{N} \). So, if \( \mu^\alpha \neq 0 \), we can prove with a contradiction argument that, in any neighbourhood \( \mathcal{V} \) of the impact instant \( t \), there exists at least one discrete impact \( t_{h_{n,i+1}} \) such that \( \alpha \in J(qh_{n,i+1}) \) and \( \mu^\alpha_{h_n,i+1} < 0 \) for all \( n \) large enough. With the definition of the scheme we get also
\[
  \frac{uh_{n,i+1} + evh_{n,i}}{1 + e} = \text{Proj}_{M(qh_{n,i+1})}(T_K(qh_{n,i+1}), uh_{n,i} + \frac{h_n}{1 + e} g_{n,i+1}).
\]
We infer that \( \frac{u_{h_{n,i}} + e u_{h_{n,i}}}{1 + e} \in T_K(q_{n,i+1}) \) and since \( T_K(q_{n,i+1}) \) is a cone
\[
(M(q_{n,i+1})(u_{h_{n,i}} - u_{h_{n,i+1}}) + h_n g_{h_{n,i+1}} u_{h_{n,i+1}}) = 0
\]
if \( \mu_{h_{n,i+1}}^\alpha < 0 \). Thus
\[
(\nabla f_\alpha(q_{n,i+1}), u_{h_{n,i+1}} + e u_{h_{n,i}}) = 0.
\]

We distinguish now two cases.

**Case 1:** \( e = 0 \). Then (14)-(15) reduces to the following complementarity conditions
\[
\mu^\alpha(\nabla f_\alpha(q(t)), v^+(t)) = 0 \quad \text{for all } \alpha \in J(q(t)).
\]
So, if \( \mu^\alpha \neq 0 \), for any neighbourhood \( V \) of the impact instant \( t \), we may consider
the last discrete impact in \( V \): we have
\[
(\nabla f_\alpha(q_{n,i+1}), u_{h_{n,i+1}}) = 0.
\]
Using assumption (12) we obtain
\[
(\nabla f_\alpha(q_{n,i+1}), u_{h_{n,i+1}}) \leq \mathcal{O}(\|q - q_{h_n}\|_{C^0([0,\tau];\mathbb{R}^d)} + |V|)
\]
where \( i_+ = \max\{i \in \mathbb{N} ; \ i h_n \in V \} \). So we can pass to the limit as \( n \) tends to \( +\infty \)
first, then as \( |V| \) tends to zero, and we finally obtain \( (\nabla f_\alpha(q(t)), v^+(t)) \leq 0 \). But
\( v^+(t) \in T_K(q(t)) \), thus \( (\nabla f_\alpha(q(t)), v^+(t)) \geq 0 \) and we may conclude.

**Case 2:** \( e \neq 0 \). The geometrical assumption (12) is now an orthogonality property
for the active constraints which implies that
\[
(\nabla f_\alpha(q(t)), v^+(t)) = (\nabla f_\alpha(q(t)), v^-(t)) - \mu^\alpha \|M^{-1/2}(q(t))\nabla f_\alpha(q(t))\|^2
\]
for all \( \alpha \in J(q(t)) \). Since \( v^+(t) \in T_K(q(t)) \) and \( v^-(t) \in -T_K(q(t)) \), we also have
\[
(\nabla f_\alpha(q(t)), v^+(t)) \geq 0, \quad (\nabla f_\alpha(q(t)), v^-(t)) \leq 0.
\]
Hence
\[
(\nabla f_\alpha(q(t)), v^+(t)) = (\nabla f_\alpha(q(t)), v^-(t)) = 0
\]
if \( \mu^\alpha = 0 \) and (14)-(15) hold. Otherwise we have to prove that
\[
(\nabla f_\alpha(q(t)), v^+(t) + e v^-(t)) = 0.
\]
Once again, for any given neighbourhood \( V \) of \( t \), we know that there exists
at least one discrete impact \( t_{h_{n,i+1}} \in V \) for all \( n \) large enough. It follows that
\[
(\nabla f_\alpha(q_{n,i+1}), u_{h_{n,i+1}} + e u_{h_{n,i}}) = 0.
\]
Using (12) we infer that
\[
(u_{h_{n,i+1}}, \nabla f_\alpha(q_{n,i+1})) = -e (u_{h_{n,i-1}}, \nabla f_\alpha(q_{n,i-1}))
+ \mathcal{O}(\|q - q_{h_n}\|_{C^0([0,\tau];\mathbb{R}^d)} + |V| + h_n)
\]
where \( i_+ = \max\{i \in \mathbb{N} ; \ i h_n \in V \} \) and \( i_- = \min\{i \in \mathbb{N} ; \ i h_n \in V \} \) which allows us
to conclude.

**Step 4** Let \( C > \|u_0\|_{M(q_0)} \). With the previous steps of the proof we already know
that the convergence holds on a non-trivial time interval \([0,\tau]\), with \( \tau \in (0,T) \) and
it remains to extend it to \([0,\tau(C)]\). With proposition 3.1 we get
\[
\lim_{n \to +\infty} \|u_{h_{n}}(t)\|_{M(q_{h_{n}}(t))} = \|u(t)\|_{M(q(t))} \leq C \ dt \text{ a.e. on } [0, \min(\tau(C),\tau)].
\]
But we can establish the following stronger property

**Lemma 3.6.** We have

\[
\limsup_{n \to +\infty} \{ \|u_{n,i}\|_{M(q_{n,i})}; t_{n,i} \in [0, \min(\tau(C), \tau)] \} \\
\leq \text{essup} \{ \|u(t)\|_{M(q(t))}; t \in [0, \min(\tau(C), \tau)] \} \leq C.
\]

Then we use a contradiction argument to conclude.

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