Higher-spin fields and charges in the periodic spinor space

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Abstract

The $sp(2M)$-invariant unfolded system is considered in the periodic twistor-like spinor space. Complete set of non-trivial charges corresponding to the global symmetry compatible with the periodicity conditions is constructed. Residual infinite-dimensional symmetry is realized in terms of the star-product algebra. It is shown that charges associated with integrations over different cycles are related by particular higher-spin symmetry transformations.

Keywords: higher-spin theory, $sp(2M)$-invariant formulation, conserved charges, toric geometry, Riemann theta-function, black holes

1. Introduction

The $sp(2M)$-invariance of the higher-spin (HS) field multiplet was first proposed in [1]. The idea that HS theories should admit a description in a larger manifestly $sp(2M)$-invariant space-time is as natural as the idea to describe supersymmetric theories in superspace. Formulations of HS theories in $sp(2M)$-invariant (super)spaces has been widely elaborated (see [2–19] and references therein). In this setup free massless HS bosonic and fermionic fields are described [4, 5] by a scalar field $C(X)$ and a svector field $C_A(X)$, respectively, in the generalized space-time $M_M$ with local coordinates $X^{AB} = X^{BA}$, $A, B = 1 \ldots M$.

Conserved charges corresponding to conformal and higher symmetries were constructed in [6] (see also [15, 20–22]). Unfolded dynamics approach to the $sp(2M)$-invariant equations was first considered in [4] and later extended in [15–17] to conserved currents and charges. As usual in unfolded dynamics, to this end the generalized space-time $M_M$ is extended by auxiliary twistor-like spinor variables $Y^A$ to $M_M \times \mathbb{R}^M$. Variables $Y^A$ together with derivatives $\frac{\partial}{\partial Y^A}$ form Heisenberg algebra $H_M$:

$$\left[ \frac{\partial}{\partial Y^A}, Y^B \right] = \delta^B_A,$$

(1.1)
while their bilinears

\[ P_{AB} = \frac{\partial}{\partial Y^A} \frac{\partial}{\partial Y^B}, \quad K^{AB} = Y^A Y^B, \quad L^{AB}_a = \frac{1}{2} \left( Y^A \frac{\partial}{\partial Y^B} + \frac{\partial}{\partial Y^B} Y^A \right) \]  

form \( sp(2M) \). Here \( P_{AB} \) and \( K^{AB} \) represent generalized translations and special conformal transformations. The \( gl(M) \) subalgebra spanned by \( L^{AB}_a \) decomposes into generalized dilatation generator \( D = L^A_A \) and \( sl(M) \) representing generalized Lorentz transformations generated by \( L^{iB}_a = L^B_i - \frac{1}{M^2} \delta^B_i D \) [4, 6].

In this paper we construct a complete set of conserved charges in the case of periodic coordinates \( Y^A \). Thus, the full space where fields live is \( M \times T^M \). Analogous problem for the non-compact spinor space was considered in [17].

Complete set of non-trivial conserved charges is constructed together with the residual global symmetry they correspond to. Conserved charges are represented as integrals of closed forms independent of local variations of the integration cycle. Despite considerable similarity with the non-compact case, periodicity in the twistor-like variables causes a number of peculiarities. One is that, since toric geometry allows inequivalent cycles non-contractible to each other, corresponding integrations give different sets of conserved charges. An interesting output of this paper is that nevertheless the latter are related to each other by some HS transformation. The complete set of charges can be obtained starting from some elementary cycle in the spinor space. Charges associated with other cycles result from those in the spinor space by virtue of higher symmetries. In fact, this implies that HS symmetries can affect topology of cycles.

The global symmetry compatible with the periodicity in \( Y \) is represented by an infinite-dimensional Lie algebra generated by basis elements \( T^q_{(\xi,\zeta)} \) with \( \xi^A \in [0,2\pi) \), \( nB \in \mathbb{Z} \) and \( r = 0, 1 \) obeying the following commutation relations

\[ \left[ T^q_{(m,\xi)} , T^r_{(n,\zeta)} \right] = T^{(q+r)_{\xi+\zeta}}_{(m+n,\xi+\zeta)} e^{-i(q-r)(\mu_c \xi - n_c \zeta)} - T^{(q+r)_{\xi+n\zeta}}_{(m+n,\xi+\zeta)} e^{-i(q-r)(\mu_c \xi - n_c \zeta)}, \]  

where \( |q+r| := (q+r) \mod 2 \). Subalgebra of (1.3) with \( q = r = 0 \) obeys commutation relations

\[ \left[ T^0_{(m,\xi)} , T^0_{(n,\zeta)} \right] = 2i \sin \left( \mu_c \xi - n_c \zeta \right) T^0_{(m+n,\xi+\zeta)} \]  

and is somewhat analogous to the sine algebra introduced in [23] which is reproduced at \( \xi, \zeta \in \mathbb{Z}^M \). Analogously to [23], relations (1.3) admit oscillator representation

\[ T^r_{(\xi,\zeta)} (k; v) = K^r \ast e^{i k \cdot \xi + i n \cdot v}, \]  

\( k \in \mathbb{Z}^M \) and \( v \in [0,2\pi) \), with respect to the Moyal-like star product

\[ (f \ast g) (k; v) = \frac{1}{(2\pi)^M} \sum_{m,n \in \mathbb{Z}^M} \int_0^{2\pi} d^M u d^M w \]  

\[ \cdot f (k + m; v + u) g (k + n; v + w) \exp \left[ i \left( \mu_c w - n_c u \right) \right] , \]  

acting on functions \( f (k; v) = \sum_N f_N (k) e^{i N c \phi} \) with half of arguments discrete and another half periodic. Klein operator \( K \) (see e.g. [24]) is defined to fulfill the following properties

\[ K \ast K = 1, \quad K \ast f (k; v) = f (-k; -v) \ast K. \]  

(1.7)

Note that \( \left[ k_c, e^{i N a \phi} \right] \ast = -2N_c e^{i N a \phi} \), where \( [f,g]_a \equiv f \ast g - g \ast f. \)
Riemann theta-function [25]

\[
\Theta(Y|X) = \sum_{n \in \mathbb{Z}^M} \exp \left[ i \pi n_A X^{AB} n_B + 2 \pi i n_A Y^A \right]
\]  
(1.8)
can be interpreted as an evolution operator (or \(\mathcal{D}\)-function) for fields, propagating in the \(X, Y\) space with periodic \(Y\)-variables [15]. Connection with field theory may lead to an alternative interpretation of some of the important theta-function identities [25, 26] (see also [15]) and, other way around, to applications of the apparatus of toric geometry to field theory.

Consideration of the periodic twistor-like space can open a way toward a uniform description of black holes in various dimensions. Black-hole solutions like Schwarzschild, Kerr–Newman, Reisner–Nordström solutions in asymptotically flat or \(AdS\) spacetime or like BTZ black hole in 3d are exact solutions of theories of gravity. Moreover BTZ black hole is locally \(AdS_3\), resulting from quotienting of the \(AdS_3\) group \(O(2,2)\) over some discrete subgroup [27]. Hence it is described by an \(AdS_3\) flat connection obeying certain periodic boundary conditions. Such construction gives a hint that black holes in various dimensions may be constructed with the aid of BTZ-like solutions in \(\mathcal{M}_M \times T^M\) for properly chosen flat connections in unfolded equations. Periodicity of solutions in generalized space-time \(\mathcal{M}_M\) is induced by periodicity in spinor variables via unfolded dynamics and black holes themselves perhaps could be obtained as projection of aforementioned ‘flat’ solutions onto surfaces in \(\mathcal{M}_M\) representing usual space-time. This conjecture first expressed in [28] provides the main motivation for the problem addressed in this paper, opening a vast area for further research.

The rest of the paper is organized as follows. In section 2 the main ingredients such as fields and currents, their unfolded equations of motion and symmetry transformations are introduced following the non-compact case [17]. In section 3 the periodicity conditions on twistor-like variables are imposed. In section 4 construction of conserved charges is described and on-shell current cohomology along with the full set of non-zero conserved charges are presented. In section 5 it is shown that charges resulting from integration over non-homotopic cycles turn out to be related by the action of HS symmetry. In section 6 conserved charges are represented as symmetry generators acting on quantized fields and the full symmetry of dynamics in the periodic spinor space is formulated in terms of an infinite-dimensional Lie algebra. In conclusion some peculiar features of symmetries in the periodic spinor space are discussed.

2. Fields and currents

2.1. Fields

As shown in [5] (infinite towers of) conformal fields in various dimensions \((d \geq 4)\) (see also [12] and references therein) can be conveniently described in terms of generalized space-time \(\mathcal{M}_M\) with symmetric real matrix coordinates \(X^{AB} = X^{BA}\) \((A, B = 1...M)\). For the unfolded formulation \(\mathcal{M}_M\) is extended by auxiliary twistor-like variables \(Y^A\) spanning \(\mathbb{R}^M\) [4]. Conformal fields are described by scalar functions \(C^\pm (Y|X)\) obeying rank-one unfolded equations [17, 20]

\[
\left( \frac{\partial}{\partial X^{AB}} \pm i \frac{\partial^2}{\partial Y^A \partial Y^B} \right) C^\pm (Y|X) = 0.
\]  
(2.1)

Equation (2.1) expresses covariant constancy condition with the flat connection

\[
W^\pm (Y, \partial_Y|X) = \pm i dX^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B}.
\]  
(2.2)
Bosons are described by even functions \((C^\pm (−Y|X) = C^\pm (Y|X))\) while fermions are described by odd ones \((C^\pm (−Y|X) = −C^\pm (Y|X))\) [17].

Unfolded formulation is useful in many respects. In particular, equation (2.1) reconstructs \(X\)-dependence from a given function \(C^\pm (Y|0)\),

\[
C^\pm (Y|X) = \exp \left[ ±iX^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B} \right] C^\pm (Y|0) .
\]  

(2.3)

Fourier decomposition of \(C^\pm (Y|0)\) gives the following representation for general solution of (2.1)

\[
C^\pm (Y|X) = \int d^d\xi \ c^\pm (\xi) \exp \left[ ±i \left( \xi_\alpha X^{\alpha B} + \xi_\beta Y^\beta \right) \right]
\]

(2.4)

with the elementary solutions

\[
C^\pm (Y|0) = \exp \left[ ±i \left( \xi_\alpha X^{\alpha B} + \xi_\beta Y^\beta \right) \right].
\]

(2.5)

As explained in [15, 17] (see also [5]), the superscript \(\pm\) in (2.4) distinguishes between positive- and negative-frequency modes corresponding to particles and antiparticles upon quantization. The two modes are complex conjugated

\[
C^+ (Y|X) = C^- (Y|X) \iff c^+ (\xi) = c^- (\xi).
\]

(2.6)

Another useful feature of unfolded formulation is that it allows one to describe symmetries of a system in a regular way. Namely, consider a transformation

\[
\eta (Y, \partial_Y |X) \rightarrow C^\pm (Y|X).
\]

(2.7)

To be a symmetry, \(\eta (Y, \partial_Y |X)\) should commute with the differential operator on the lhs of (2.1),

\[
\left[ \frac{\partial}{\partial X^{AB}} ± i \frac{\partial^2}{\partial Y^A \partial Y^B}, \eta (Y, \partial_Y |X) \right] = 0 .
\]

(2.8)

Condition (2.8) is formally consistent since connection in (2.1) is flat. The first-order differential operators

\[
A^C_\pm (Y|X) = Y^C ± 2iX^{CB} \frac{\partial}{\partial Y^B}, \quad B^\pm c (Y|X) = \frac{\partial c}{\partial Y^c}
\]

(2.9)

verify (2.8). Each pair \(A^+, B^+\) and \(A^-, B^-\) obeys Heisenberg algebra \(H_M\) [16]

\[
[B^\pm a, A^\pm b] = \delta^a_b, \quad [A^\pm b, A^\pm c] = 0, \quad [B^\pm b, B^\pm c] = 0 .
\]

(2.10)

Since operators (2.9) are covariantly constant, they will be referred to as covariant oscillators\(^1\). Any function of covariant oscillators \(\eta (A^\pm; B^\pm)\) is a solution of (2.8) and hence is a symmetry of (2.1).

Covariant oscillators act on (2.5) as follows

\[
B^\pm c \theta^\pm_\xi = ±i\xi c \theta^\pm_\xi, \quad A^C_\pm \theta^\pm_\xi = ±i \frac{\partial}{\sigma c} \theta^\pm_\xi.
\]

(2.11)

Exponentiation of (2.11) gives

\[
\exp \left[ ±i\xi c A^C_\pm \right] \theta^\pm_\xi = \theta^\pm_{\xi + \xi}.
\]

(2.12)

\(^1\) Here and after notations of covariant oscillators correspond to [17].
allowing to generate the whole basis (2.5) from a single vacuum vector \( \theta_0 := \theta_{01} \partial_1 \xi \) \( Y[X] = 1 \).

\[
B^\pm \theta_0 = 0, \quad \exp \left[ \pm i \xi C A^\pm \right] \theta_0 = \theta_{0}^\pm . \tag{2.13}
\]

Any solution to (2.1) can thus be written as

\[
C^\pm (Y[X]) = \int d^4 \xi \, c^\pm (\xi) \, \exp \left[ \pm i \xi C A^\pm \right] \theta_0 . \tag{2.14}
\]

The result of the action of a symmetry transformation can be represented as

\[
\eta \left( A^\pm ; B^\pm \right) C^\pm (Y[X]) = \int d^4 \xi \, c^\pm (\xi) \eta \left( \mp i \partial_\xi \pm \pm i \xi \right) \theta^\pm (Y[X]) . \tag{2.15}
\]

Evolution of a particular field configuration in \( X \)-variables from \( C^\pm (Y[X]) \) to \( C^\pm (Y[\hat{X}]) \) is given by a \( D \)-function via the following transformation [17]

\[
C^\pm (Y[X]) = \int d^4 Y' \, D^\pm (Y - Y'[X - \hat{X}]) \, C^\pm (Y'[X]) . \tag{2.16}
\]

The \( D \)-function is a solution to (2.1) with the \( \delta \)-functional initial data

\[
\left( \frac{\partial}{\partial X^{AB}} \pm i \frac{\partial^2}{\partial Y^A_1 \partial Y^B_1} \right) D^\pm (Y - Y'[X - \hat{X}]) = 0, \quad \left[ D^\pm (Y - Y'[0]) = \delta (Y - Y') \right] . \tag{2.17}
\]

Hence,

\[
D^\pm (Y[X]) = \frac{1}{(2\pi)^M} \int d^4 \xi \, \theta^\pm (Y[X]) . \tag{2.18}
\]

### 2.2. Currents

Doubling of spinor variables leads to a rank-two unfolded equation [20]

\[
\left( \frac{\partial}{\partial X^{AB}} + i \frac{\partial^2}{\partial Y^A_1 \partial Y^B_1} - i \frac{\partial^2}{\partial Y^A_2 \partial Y^B_2} \right) J(Y_1, Y_2|X) = 0 . \tag{2.19}
\]

Its solutions \( J(Y_1, Y_2|X) \) are called current fields or simply currents. They describe conserved currents in the unfolded formulation. The flat connection

\[
W^{(2)} (Y_{1,2} | \partial Y_{1,2} | X) = i d^4 \xi \left( \frac{\partial^2}{\partial Y^A_1 \partial Y^B_1} - \frac{\partial^2}{\partial Y^A_2 \partial Y^B_2} \right) = W^+ (Y_1, \partial Y_1 | X) + W^- (Y_2, \partial Y_2 | X) \tag{2.20}
\]

is the sum of flat connections for positive- and negative-frequency modes of (2.1). Hence bilinears of rank-one fields

\[
J(Y_{1,2} | X) = C^+ (Y_1 | X) \, C^- (Y_2 | X) , \tag{2.21}
\]

called bilinear currents, verify (2.19). The straightforward generalization of currents (2.21) via extension of the set of rank-one fields (2.4) by a color index \( i = 1...N \)

\[
J(Y_{1,2} | X) = \sum_{i=1}^{N} C^+_i (Y_1 | X) \, C^-_i (Y_2 | X) , \tag{2.22}
\]

together with the flat connection (2.20) can be easily reinserted at any moment we will not consider it in the sequel.
Bilinear current \((2.21)\) is a particular case of a more general current field

\[
J_\eta(Y_{1,2}|X) = \eta(Y_{1,2}, \partial Y_{1,2}|X) C^+(Y_1|X) C^-(Y_2|X),
\]

where \(\eta(Y_{1,2}, \partial Y_{1,2}|X)\) is a symmetry of \((2.19)\). Analogously to the rank-one case \(\eta\) commutes with covariant differential of \((2.19)\)

\[
\left[ \frac{\partial}{\partial X^{\alpha \beta}} + i \frac{\partial^2}{\partial Y_1^{\alpha \beta} Y_2^{\alpha \beta}} - i \frac{\partial^2}{\partial Y_2^{\alpha \beta} Y_1^{\alpha \beta}} \right] \eta(Y_{1,2}, \partial Y_{1,2}|X) = 0.
\]

Covariant oscillators \((2.9)\)

\(\mathcal{A}_+^C(Y_1|X), B_+^C(Y_1|X)\) and \(\mathcal{A}_-^C(Y_2|X), B_-^C(Y_2|X)\)

verify \((2.24)\), hence any function \(\eta(\mathcal{A}_+, \mathcal{A}_-; B_+, B_-)\) is a symmetry of \((2.19)\) and the most general form of a bilinear current is [17]

\[
J_\eta(Y_{1,2}|X) = \eta(\mathcal{A}_+, \mathcal{A}_-; B_+, B_-) C^+(Y_1|X) C^-(Y_2|X).
\]

The action of covariant oscillators on the rank-two basis vectors \(\theta_\xi^C(Y_1|X) \theta_\zeta^C(Y_2|X)\)

\[
B_+^C \theta_\xi^C \theta_\zeta^C = i \xi C \theta_\xi^C \theta_\zeta^C, \quad B_-^C \theta_\xi^C \theta_\zeta^C = -i \xi C \theta_\xi^C \theta_\zeta^C, \quad A_+^C \theta_\xi^C \theta_\zeta^C = -i \frac{\partial}{\partial \xi} \theta_\xi^C \theta_\zeta^C, \quad A_-^C \theta_\xi^C \theta_\zeta^C = i \frac{\partial}{\partial \xi} \theta_\xi^C \theta_\zeta^C
\]

generates the complete basis from a single vacuum vector \(\theta_0^{(2)} := \theta_0^+ \theta_0^- = 1\),

\[
B_+^C \theta_0^{(2)} = 0, \quad \theta_\xi^C \theta_\zeta^C = \exp \left[ i \xi C \mathcal{A}_+^C - i \xi C \mathcal{A}_-^C \right] \theta_0^{(2)}.
\]

The action of a symmetry parameter in \((2.26)\) is

\[
J_\eta(Y_{1,2}|X) = \int d^4 \xi d^4 \zeta \, c^+(\xi) c^-(\zeta) \, \eta\left(-i \partial \xi, i \partial \zeta; i \xi, -i \zeta\right) \theta_\xi^C(Y_1|X) \theta_\zeta^C(Y_2|X).
\]

It is convenient to introduce the following linear combinations of the covariant oscillators \(\mathcal{A}, \mathcal{B}\)

\[
\mathcal{B}_C = B_-^C - B_+^C, \quad \tilde{\mathcal{B}}_C = B_-^C + B_+^C, \\
\mathcal{B}^C = \frac{1}{2} (\mathcal{A}_-^C - \mathcal{A}_+^C), \quad \mathcal{B}_C = \frac{1}{2} (\mathcal{A}_-^C + \mathcal{A}_+^C)
\]

with the non-zero commutation relations

\[
[\mathcal{B}_A, \tilde{\mathcal{B}}_B] = \delta_A^B, \quad [\tilde{\mathcal{B}}_A, \mathcal{B}_B] = \delta_A^B.
\]

These oscillators are most conveniently represented as differential operators

\[
\mathcal{B}_C = \frac{\partial}{\partial \Sigma^C}, \quad \tilde{\mathcal{B}}_C = \frac{\partial}{\partial \Sigma^C}, \quad \mathcal{B}_C = U^C + i X^{CB} \frac{\partial}{\partial \Sigma^C}, \quad \mathcal{B}_C = V^C + i X^{CB} \frac{\partial}{\partial \Sigma^C}
\]

in terms of the variables [17]

\[
Y_1 = V - U, \quad Y_2 = V + U.
\]

Let us introduce an involutive antiautomorphism \(\rho\) of the algebra of covariant oscillators that acts as follows

\[
\rho \left( \mathcal{A}_\pm \right) = \mathcal{A}_\mp, \quad \rho \left( B_\pm^C \right) = - B_\mp^C.
\]

The oscillators \(\mathcal{B}, \tilde{\mathcal{B}}\) are \(\rho\)-even and \(\rho\)-odd, respectively,
\[ \rho(\mathcal{B}) = \mathcal{B}, \quad \rho(\mathcal{B}) = -\mathcal{B}. \]  \tag{2.35}

For practical computations it is convenient to chose a specific ordering prescription for functions of covariant oscillators. We will use the totally symmetric Weyl ordering described by the Weyl star product. In these terms, any two symbols (i.e. functions of commuting variables) \( f(A_+, A_-; B^+, B^-) \) and \( g(A_+, A_-; B^+, B^-) \) are star-multiplied as follows

\[ (f \ast g)(A; B) = f(A; B) \exp \frac{1}{2} \sum_{\alpha = +, -} \left( \frac{\partial}{\partial B_{\alpha}} \frac{\partial}{\partial A_{\alpha}} - \frac{\partial}{\partial A_{\alpha}} \frac{\partial}{\partial B_{\alpha}} \right) g(A; B). \]  \tag{2.36}

In terms of the star product (2.36) the vacuum vector \( \theta_0^{(2)} \) obeying \( B^\pm c \ast \theta_0^{(2)} = 0 \) is realized as

\[ \theta_0^{(2)} = \exp \left[ -2 \sum_{a = +, -} A_a c B^\alpha c \right]. \]  \tag{2.37}

Symbols of the basis star-product elements \( \theta^+_\xi \theta^-_\zeta \) can be generated from the vacuum \( \theta_0^{(2)} \) by the left star-multiplication (2.36) via (2.28)

\[ \theta^+_\xi \theta^-_\zeta = \exp \left[ i \xi C A_+^c, i \zeta C A_-^c \right] \ast \theta_0^{(2)} = \exp \left[ 2i \xi C A_+^c - 2i \zeta C A_-^c \right] \ast \theta_0^{(2)}. \]  \tag{2.38}

In terms of the star product, symmetry parameters are represented by their symbols \( \eta(A; B) \). Since Weyl ordering is totally symmetric, antiautomorphism \( \rho \) (2.34) acts on a symbol \( f(A_+, A_-; B^+, B^-) \) simply as

\[ \rho(f(A_+, A_-; B^+, B^-)) = f(A_-, A_+; -B^+, -B^-). \]  \tag{2.39}

Indeed, it is straightforward to check that \( \rho \) is an antiautomorphism of the star-product algebra (2.36), i.e. \( \rho(f \ast g) = \rho(g) \ast \rho(f) \).

3. Periodic spinor space

To construct periodic solutions it suffices to put (2.4) on a lattice by setting \( \xi_A = \frac{2\pi}{\ell(1) \cdots \ell(M)} n_A \) with \( n_A \in \mathbb{Z} \) (or in condensed notation \( \xi = \frac{2\pi}{\ell} n \) for \( n \in \mathbb{Z}^M \))

\[ C^\pm(Y|X) = \frac{(2\pi)^M}{\ell(1) \cdots \ell(M)} \sum_n c \left( \frac{2\pi}{\ell} n \right) \theta^\pm_{2\pi n/\ell}(Y|X), \]  \tag{3.1}

\[ \theta^\pm_{2\pi n/\ell}(Y|X) = \exp \left[ \frac{\pm i}{4\pi^2 n_A n_B} \frac{Y^{AB}}{\ell(A) \ell(B)} + \frac{2\pi n_A Y^A}{\ell(1) \ell(1)} \right]. \]  \tag{3.2}

Such solutions are \( \ell(A) \)-periodic in \( Y^A \)-variables. The non-compact limit corresponds to \( \ell(A) \rightarrow \infty \).

It is convenient to use rescaled variables and change notations as follows

\[ Y^A : = \frac{2\pi}{\ell(A)} Y^A, \quad X^{AB} : = \frac{4\pi^2}{\ell(A) \ell(B)} X^{AB}. \]  \tag{3.3}

Since this is equivalent to setting

\[ \ell(A) = 2\pi, \]  \tag{3.4}
in the sequel we will not distinguish between primed and unprimed variables. The dependence on $\ell(A)$ can be easily reconstructed in the very end if necessary.

In terms of rescaled variables basis vectors (3.2) are

$$\theta_n^\pm (Y|X) := \exp \left[ \pm i \left( n_C A^+ \pm n_B A^B \right) \right],$$

while any periodic solution (3.1) can be written as follows

$$C^\pm (Y|X) = \sum_{n \in \mathbb{Z}^M} c_n^\pm \theta_n^\pm (Y|X).$$

Basis functions (3.5) (and hence functions (3.6)) are $2\pi$-periodic in $Y$-variables, $2\pi$-periodic in $X^{AA}$ and $\pi$-periodic in $X^{AB}$ with $A \neq B$. Hence, unfolded dynamics induces periodicity in $\mathcal{M}_M$ from that in the spinor variables. It also implies that, reintroducing arbitrary radii, periods of the $X$-variables factorize into products of periods of $Y$-variables. Namely, periods of $Y$- and $X$-variables are $\ell(A)$ for $Y^A \ell(B)$ for $X^{AA}$ and $\frac{\ell(A) \ell(B)}{4\pi}$ for $X^{AB}$ ($A \neq B$). So, possible periods of the $\mathcal{M}_M$-dimensional space $\mathcal{M}_M$, that can be respected by solutions of the rank-one equation (2.1), are parametrized by $M$ numbers.

Due to the second relation in (2.11) which does not respect periodicity, polynomials of covariant oscillators $A_{\pm c}$ do not act properly on (3.5). The generators respecting periodicity are

$$B^\pm c \theta_n^\pm = \pm i n_C \theta_n^\pm, \quad \exp \left[ \pm i n_C A_{\pm c} \right] \theta_n^\pm = \theta_{n+m}^\pm$$

for any $m,n \in \mathbb{Z}^M$. As in the non-compact case, basis vectors are generated from a single vacuum vector $\theta_0$

$$B^\pm c \theta_0 = 0, \quad \exp \left[ \pm i n_C A_{\pm c} \right] \theta_0 = \theta_n^\pm.$$

Periodicity demands any symmetry transformation to be $2\pi$-periodic in the oscillators $A$

$$\eta = \eta \left( e^{i A_{\pm C}}; B^\pm \right).$$

The parameter $\eta$ can be viewed as a polynomial of $B^\pm$ and a Laurent polynomial of $e^{i A_{\pm c}}$. The action of a symmetry transformation can be written as

$$\eta \left( e^{i A_{\pm c}}; B^\pm \right) C^\pm (Y|X) = \sum_n c_n^\pm \eta \left( e^{\pm \frac{\theta}{2\pi}}; i n \right) \theta_n^\pm (Y|X).$$

For rank-two equation (2.19) periodic Ansatz is introduced in the same manner. Bilinear currents (2.21) are built from positive- and negative-frequency rank-one fields (3.6)

$$J(Y_1, Y_2|X) = \sum_{m,n} c_m^+ c_n^- \theta_m^+ (Y_1|X) \theta_n^- (Y_2|X).$$

Basis elements $\theta_m^+ (Y_1|X) \theta_n^- (Y_2|X)$ are constructed from the vacuum vector $\theta_0^{(2)}$ analogously to (2.28)

$$B^\pm c \theta_0^{(2)} = 0, \quad \exp \left[ i n_C A_{+, c} - i n_C A_{-, c} \right] \theta_0^{(2)} = \theta_m^+ \theta_n^-.$$

In terms of $Y_1, Y_2$ and $U, V$ (2.33) they have the form

$$\theta_m^+ (Y_1|X) \theta_n^- (Y_2|X) = \exp \left[ i \left( (m+n)_C X^{BC} + m_C Y^C_1 - n_C Y^C_2 \right) \right]$$

$$= \exp \left[ i \left( (m+n)_C (m-n)_C X^{BC} - (m+n)_C U^C + (m-n)_C V^C \right) \right].$$

(3.13)
Periodicity properties of the vector (3.13) in $X^{AB}$ are the same as of (3.5).

The global symmetry transformation respects periodicity in $Y_1$ and $Y_2$ variables iff they are generated by $e_{iA}^+$ and $B^C_c$,

$$\eta = \eta (e^{+iA_c}, B^c), \quad a, b = +, -.$$  \hfill (3.14)

Within the Weyl ordering the periodic star-product symbols of parameters (3.14) admit Fourier decomposition,

$$\eta (A; B) = \sum_{k,l \in \mathbb{Z}^M} \eta_{kl} (B^+, B^-) e^{ia_k A_+} e^{ib_k A_-}.$$  \hfill (3.15)

This gives the following explicit formula for the symmetry action

$$J_q (Y|X) := \eta (A; B) \ast J(Y|X) = \sum_{m,n,k,l} e_m^+ e_n^- \eta_{kl} \left( im + \frac{ik}{2}, -in + \frac{il}{2} \right) \theta_{m+k}^+ \theta_{n-l}^-.$$  \hfill (3.16)

In terms of oscillators (2.30) decomposition (3.15) is

$$\eta (\mathfrak{B}, \mathfrak{B}) = \sum_{k,l \in \mathbb{Z}^M} \eta_{kl} (\mathfrak{B}_C, \mathfrak{B}_D) e^{i(k+l) \mathfrak{B}^A} e^{-i(k-l) \mathfrak{B}^a}.$$  \hfill (3.17)

Let, for $N\in \mathbb{Z}$, $|N|_2 = N \mod 2$, and for $N \in \mathbb{Z}^M$, $|N|_2 \in \mathbb{Z}_2^M$ is understood component-wise. Then $|k + l|_2 = |k - l|_2$ and hence decomposition (3.17) can be rewritten as follows

$$\eta (\mathfrak{B}, \mathfrak{B}) = \sum_{|N|_2 = |\tilde{N}|_2} \eta_{N, \tilde{N}} (\mathfrak{B}_C, \mathfrak{B}_D) e^{iN\mathfrak{B}^a} e^{i\tilde{N}\mathfrak{B}^a}.$$  \hfill (3.18)

$D$-functions for periodic solutions of (2.1) can be introduced analogously to the non-compact case [15]. In the positive-frequency sector, the $D$-function

$$\theta (Y|X) = \frac{1}{(2\pi)^M} \sum_{n} \exp \left[ i \left( nA_B X^{AB} + na Y^A \right) \right]$$  \hfill (3.19)

is a solution to (2.1) with the $\delta$-functional initial data on a torus,

$$\theta (Y|0) = \delta (Y), \quad Y^A \in [-\pi, \pi) \text{ for } A = 1...M.$$  \hfill (3.20)

Up to simple redefinitions of arguments expression (3.19) represents Riemann theta-function [25]

$$\Theta (Y|X) = \sum_{n \in \mathbb{Z}^M} \exp \left[ i\pi nA_B X^{AB} + 2\pi i na Y^A \right].$$  \hfill (3.21)

Action of covariant oscillators $e^{ib} B e^{\mu A} \theta (Y|X)$ gives rise to theta-functions with rational characteristics $a_C \in \mathbb{Q}$ and $b^C \in \mathbb{Q} (C = 1...M)$ [25, see also [26]]

$$\Theta_{a,b} (Y|X) := \sum_{n} \exp \left[ i\pi \left( n + a \right)_B \left( n + a \right)_C X^{BC} + 2\pi i \left( n + a \right)_B \left( Y + b \right)^\theta \right].$$  \hfill (3.22)

Along with $Y$-periodicity (and aforementioned $X$-periodicity) (3.19) is quasi-periodic with $X$ being the matrix of quasi-periods [25]

$$S_a \theta (Y|X) := \theta \left( Y^C + 2m_B X^{BC} |X\right) = e^{-i\left( m_{ac} X^{BC} + nu Y^a \right)} \theta (Y|X).$$  \hfill (3.23)
Quasi-translations $S_m$, $m \in \mathbb{Z}^M$, defined as semi-linear operators acting on the rank-one fields (3.5)

$$S_m \left( \lambda \theta^\pm (Y|X) : = \bar{X} \theta^{\pm}_{m} (Y | X) \theta^{\pm}_{m+n} (Y | X) \right),$$

are not symmetries of (2.1). However, for $m = -n$ composition of quasi-translation and complex conjugation gives an involutive linear symmetry operator $K$,

$$K \theta^\pm (Y|X) := S_{-n} \theta^\pm (Y | X) = \theta^{-\pm}_{n} (Y | X), \quad K^2 = \text{Id.}$$

(3.25)

It is straightforward to verify that $K B^\pm C = -B^\pm C K$ and $K e^{i A \pm C} = e^{-i A \pm C} K$, therefore $K$ is Klein operator (see e.g. [24]). It doubles the symmetries of (2.1)

$$\varepsilon \left( e^{i A \pm C} B^\pm \right) = \eta \left( e^{i A \pm C} B^\pm \right) + K \bar{\eta} \left( e^{i A \pm C} B^\pm \right).$$

(3.27)

4. Charges

4.1. Charge components and integration surfaces

Conserved charges can be represented as integrals of on-shell-closed current differential forms. Current forms are constructed from an arbitrary rank-two field [15, 17, 22], and, in particular, from the bilinear currents (2.26). In the non-compact case the closed on-shell current $M$-form is [17] (see also [15])

$$\Omega (J_\eta) = W^1 \wedge ... \wedge W^M \left. J_\eta (Y|X) \right|_{U = 0},$$

(4.1)

with $U,V (2.33)$. $W^A$ is the operator-valued 1-form

$$W^A = d V^A + i d X^{AB} \frac{\partial}{\partial U^B}.$$ 

(4.2)

Conserved charges result from integration over an $M$-dimensional surface $\Sigma \subset \mathcal{M} \times \mathbb{R}^M$ which is spacelike in $X$-variables [17] ($\mathbb{R}^M$ is parametrized by spinor variables $V^A$),

$$Q_\eta = \int_\Sigma \Omega (J_\eta).$$

(4.3)

Charge (4.3) is independent of local variations of $\Sigma$ since $d \Omega (J_\eta) = 0$ by virtue of the current equation. Non-trivial charges correspond to the on-shell de Rham cohomology of the set of forms (4.1). As presented in [17], in the non-compact case non-zero charges are completely represented by the $\mathfrak{B}$-independent symmetry parameters $\eta (\mathfrak{B})$. In other words, given $\eta (\mathfrak{B}, \bar{\mathfrak{B}})$ there exists such $\eta^\prime (\mathfrak{B})$ that $\Omega (J_{\eta^\prime}) - \Omega (J_\eta) = d \omega$.

Another set of dual closed current forms $\tilde{\Omega} (J_{\eta})$ is constructed via exchange $U \leftrightarrow V$ [17]. Nontrivial conserved charges for such current forms are represented by $\tilde{\eta} (\bar{\mathfrak{B}})$. Hence the complete set of charges is doubled giving rise to the $\mathcal{N} = 2$ supersymmetric HS algebra [17]. For definiteness in this paper we mostly focus on current forms (4.1). The dual set of charges can be considered analogously.

For the $Y$-periodic case the situation is somewhat different. Now functions (3.6) live on a torus $\mathcal{T}_L \times T^M := (\mathcal{M} \times \mathbb{R}^M) / L$ where $L \subset \mathcal{M} \times \mathbb{R}^M$ is the lattice corresponding to the periods of rescaled coordinates (3.3)

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\[ Y O Goncharov and M A Vasiliev \]
\[
L = \left\{ \begin{array}{l}
X^A = 2\pi p, \quad X^{AB} = 2\pi q, \quad X^{AB}|_{A\neq B} = \pi r \text{ for } p, q, r \in \mathbb{Z}
\end{array} \right\}. \tag{4.4}
\]

Integration surfaces \( \Sigma \), being \( M \)-dimensional cycles in \( \mathcal{T}_M \times \mathcal{T}^M \), may belong to different homotopy classes. These are anticipated to generate different charges.

In the periodic case, the question whether it is possible to eliminate the \( \mathfrak{B} \)-dependence from the symmetry parameters (3.17) has to be reconsidered. The goal is to find the essential part of \( \mathfrak{B} \)- and \( \mathfrak{B}^\text{c} \)-dependence of the symmetry parameters, associated with the current cohomology in the periodic case.

For symmetry parameters (3.17) and periodic solutions (3.6) the current forms (4.1) are

\[
\Omega(J_i) = \sum_{m,n,k,l} \left( dV^A + (m + n + k - l)_C dX^{CA} \right)^{J_M}
\]

\[
c^m_n c^l_k \eta_{ij} \left(-i \left( m + n + \frac{k - l}{2} \right), i \left( m - n + \frac{k + l}{2} \right)\right) \cdot \exp \left[ \lim_{(m - n + k + l)_B (V^B + (m + n + k - l)_D X^{DB})} \right], \tag{4.5}
\]

where \( (W^A)^{J_M} := W^1 \wedge \ldots \wedge W^M \). Current form (4.5) is defined on \( \mathcal{T}_M \times \mathcal{T}^M \) where the twistor-like sector \( \mathcal{T}^M \) is parametrized by variables \( V^A \in [0, 2\pi] \). Integration of (4.5) over a compact surface \( \Sigma \subset \mathcal{T}_M \times \mathcal{T}^M \) gives

\[
\int_{\Sigma} \Omega(J_i) = \sum_{m,n,k,l} c^m_n c^l_k \eta_{ij} \left(-i \left( m + n + \frac{k - l}{2} \right), i \left( m - n + \frac{k + l}{2} \right)\right) \cdot q^{(m-n+k+l}_{(m+n+k-l)} \nu_{\Sigma} \cdot w^A := V^A + \tilde{\nu}_C X^{CA}, \tag{4.7}
\]

will be referred to as charge components and

\[
\nu_C = (m - n + k + l)_C, \quad \tilde{\nu}_C = (m + n + k - l)_C. \tag{4.8}
\]

Charge components are independent of local variations of \( \Sigma \) since the differential form in (4.7) is closed.

Integration in (4.3) and (4.7) should be performed over space-like \( M \)-cycles. Space-like directions in \( \mathcal{M}_M \) are associated with the traceless parts of \( X^{AB} \) [5]

\[
\sum_{A=1}^{M} X^{AA} = 0. \tag{4.9}
\]

Consider the following parametrization of \( \mathcal{M}_M \) by variables \( t, y_{ij+1}, z_i \) (\( i \leq j \) and \( i, j = 1 \ldots M - 1 \)):

\[
X^{11} = z_1, \quad X^{22} = -z_1 + z_2, \ldots, \quad X^{M-1,M-1} = -z_{M-2} + z_{M-1}, \quad X^{MM} = t = z_{M-1}, \quad y_{ij+1} = X^{i+1,j} = X^{ij+1}. \tag{4.10}
\]

Here \( t = \sum_{A=1}^{M} X^{AA} \) parametrizes time while \( y \) and \( z \), parametrizing the traceless part of \( X^{AB} \), are space coordinates. Note that transformation (4.10) is from \( \text{SL} \left( \frac{M(M+1)}{2} \right) \). Therefore it preserves the lattice (4.4) acting properly on the torus \( \mathcal{T}_M \subset \mathcal{M}_M \).

The freedom in the choice of parametrization of space-like directions in (4.9), i.e. of traceless part of \( X^{AB} \) is not essential. Different parametrizations resulting from \( \text{SL} \left( \frac{M(M+1)}{2} \right) \)
transformations of $X^{AB}$ preserve the lattice and give equivalent sets of conserved charges. Indeed, in this case fundamental cycles corresponding to one parametrization are expressed as integer combinations of those for the other. The same is true for conserved charges being integrals over $M$-dimensional space-like cycles in $T_M \times T^M$. More generally, parametrizations of $T_M \times T^M$ resulting from $\text{SL}(M(M+1)/2 + M|\mathbb{Z})$ coordinate transformations of $X^{AB}, Y^A$ give equivalent charges.

For instance, consider parametrization (4.10). Consider fundamental space-like $M$-cycles $\{\sigma_a\}$ of $T_M \times T^M$ with a single winding parametrized by all sets of $M$ pairwise different $y, z$ and $V$. A single-winding cycle $\sigma_0$ in the spinor space parametrized by the variables $V$ will be referred to as the lower cycle. Other space-like cycles will be called higher. Any space-like cycle $\Sigma$ is homotopic to a sum of fundamental cycles $\Sigma = \sum_a b_a \sigma_a$ (4.11)

with the coefficients $b_a \in \mathbb{Z}$ representing the number of windings over the respective fundamental cycle.

Charge components (4.7) being linear functions on the space of cycles are determined by their values $q_{\sigma_a}^{(\nu, \bar{\nu})}$ on the fundamental cycles. Using (4.10) for (4.7) and setting $t = 0$ one arrives at the sum of monomials of the $M$th power in $dV, dy, dz$ which correspond to integration over space-like fundamental cycles. One can see that for any $\sigma_a$

$q_{\sigma_a}^{(\nu, \bar{\nu})} \propto p_{\sigma_a}(\bar{\nu}) \delta_{\nu,0} \delta_{\bar{\nu},0}$. (4.12)

where $p_{\sigma_a}(\bar{\nu})$ is a homogeneous polynomial of $\bar{\nu}_A$. Indeed, formula (4.12) results from the change of variables $w^A$ in (4.7) to those among $V, y, z$ (4.10) that parametrize $\sigma_a$, with $p_{\sigma_a}(\bar{\nu})$ being the Jacobian. As a result, for any cycle $\sigma_a$ there is a linear transformation $F_a[\bar{\nu}]$ of variables $\nu$ leading to (4.12) in the form

$q_{\sigma_a}^{(\nu, \bar{\nu})} \propto \det F_a[\bar{\nu}] \delta_{F_a[\bar{\nu]},0}$. (4.13)

Generally, different cycles may give the same polynomials. Because charge components (4.7) are linear functions of cycles, for any cycle $\Sigma (4.11)$

$q_{\Sigma}^{(\nu, \bar{\nu})} \propto p_{\Sigma}(\bar{\nu}) \delta_{\nu,0} \delta_{\bar{\nu},0}$. $p_{\Sigma} = \sum_a b_a p_{\sigma_a}$. (4.14)

A useful consequence of (4.14) is the expression for charge components of any cycle in terms of those for the lower fundamental one

$q_{\Sigma}^{(\nu, \bar{\nu})} \propto p_{\Sigma}(\bar{\nu}) q_{\sigma_0}^{(\nu, \bar{\nu})}$. (4.15)

Note that $p_{\sigma_0}(\bar{\nu}) \propto 1$.

4.2. $M = 2$ example

As an example, consider the $M = 2$ case in some detail. General parametrization (4.10) for $X^{AB}$ is

$X = t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (4.16)

The volume form in (4.7) for $t = 0$ is
\[ dw^1 \land dw^2 = (d V^1 + \tilde{\nu}_2 dy + \tilde{\nu}_1 dz) \land (d V^2 + \tilde{\nu}_1 dy - \tilde{\nu}_2 dz) \]
\[ = d V^1 \land d V^2 + \tilde{\nu}_1 d V^1 \land d y - \tilde{\nu}_2 d V^2 \land d y \]
\[- \tilde{\nu}_2 d V^1 \land d z - \tilde{\nu}_1 d V^2 \land d z - (\tilde{\nu}_1^2 + \tilde{\nu}_2^2) dy \land dz. \]

(4.17)

With the particular parametrization (4.16) fundamental 2-cycles of \( T_M \times T^M \) are associated with the following pairs of variables \( V^1 V^2 \) (the lower cycle), \( V^1 y \), \( V^2 z \), \( V^1 z \) and \( y z \).

Charge components for \( \sigma_0 = V^1 V^2 \) are
\[ q^{(\nu, \delta)}_{\sigma_0} = \int \int_0^{2\pi} dV^1 dV^2 e^{i \nu_1 V^1 + i \nu_2 V^2} = 4\pi^2 \delta_{\nu_1,0} \delta_{\nu_2,0} \propto \delta_{\nu,0}. \]

(4.18)

Analogous computation for \( V^1 y \) gives
\[ q^{(\nu, \delta)}_{\sigma_1} = \int_0^{2\pi} dy \int_0^{2\pi} dV^1 e^{i \nu_1 V^1 + i(\nu_1 \tilde{\nu}_2 + \tilde{\nu}_1 \nu_2)} y = 2\pi^2 \tilde{\nu}_1 \delta_{\nu_1,0} \delta_{\nu_1 \tilde{\nu}_2 + \nu_2 \tilde{\nu}_1,0}. \]

(4.19)

In agreement with (4.12) this is equivalent to
\[ q^{(\nu, \delta)}_{\sigma_1} \propto \tilde{\nu}_1 \delta_{\nu,0}. \]

(4.20)

Analogous computation of the full set of charge components for fundamental cycles gives
\[ q^{(\nu, \delta)}_{\sigma_{12}} \propto \delta_{\nu,0}, \]
\[ q^{(\nu, \delta)}_{\sigma_{1v}} \propto \tilde{\nu}_1 \delta_{\nu,0}, \]
\[ q^{(\nu, \delta)}_{\sigma_{1z}} \propto \tilde{\nu}_2 \delta_{\nu,0}, \]
\[ q^{(\nu, \delta)}_{\sigma_{2v}} \propto \tilde{\nu}_1 \delta_{\nu,0}, \]
\[ q^{(\nu, \delta)}_{\sigma_{2z}} \propto \tilde{\nu}_2 \delta_{\nu,0}. \]

(4.21)

The respective polynomials in (4.14) are integer combinations of \( 1, \tilde{\nu}_1, \tilde{\nu}_2 \) and \( \tilde{\nu}_1^2 + \tilde{\nu}_2^2 \).

4.3. On-shell current cohomology and non-zero charges

4.3.1. \( \mathcal{B}_C \)-dependence. Here we show that, analogously to the non-compact case [17] the dependence on \( \mathcal{B}_C \) can be eliminated from the parametrization of non-zero conserved charges. Namely, for a given symmetry parameter \( \eta_{\nu} (\mathcal{B}_A, \mathcal{B}_B) \) we introduce another parameter
\[ \eta'_{\nu} (\mathcal{B}_A) = \eta_{\nu} (\mathcal{B}_A, -i \frac{k + l}{2}) \]

(4.22)

depending solely on \( \mathcal{B}_A \) such that current forms (4.5) with \( \eta_{\nu} (\mathcal{B}_A, \mathcal{B}_B) \) and \( \eta'_{\nu} (\mathcal{B}_A) \) differ by an exact form.

Indeed, using (4.5), (4.7), (4.8) and (4.22),
\[ \Omega (J_{\nu}) - \Omega (J_{\nu}') = \sum_{m,n,k,l} (\text{d} \omega^A)^{\lambda M} e^A_m e_n^A \exp [i \nu_B n^B] \]
\[ \Delta \eta_{\nu} \left( -i \left( m + n + \frac{k + l}{2} \right), i \left( m - n + \frac{k + l}{2} \right) \right), \]

(4.23)
\[ \Delta \eta_{kl} = -i \left( m + n + \frac{k - l}{2} \right) i \left( m - n + \frac{k + l}{2} \right) \]
\[ = \eta_{kl} \left( -i \left( m + n + \frac{k - l}{2} \right) i \left( m - n + \frac{k + l}{2} \right) \right) - \eta_{kl}' \left( -i \left( m + n + \frac{k - l}{2} \right) \right) - i k + l + i \nu \]
\[ = \int_0^1 dt \frac{d}{d t} \eta_{kl} \left( -i \left( m + n + \frac{k - l}{2} \right), -i \frac{k + l}{2} + i \nu \right) . \tag{4.24} \]

Hence
\[ \Omega (J_q) - \Omega (J_q') = d \beta, \tag{4.25} \]
where
\[ \beta \propto \sum_{m,n,k,l} c_{m}^{+} c_{n}^{-} \epsilon_{A_1 A_2} \ldots \epsilon_{A_{N-1}} \epsilon_{A_{N}} e^{i \nu \cdot w} . \]
\[ \int_0^1 dt \frac{d}{d t} \eta_{kl} \left( -i \left( m + n + \frac{k - l}{2} \right), -i \frac{k + l}{2} + i \nu \right) . \tag{4.26} \]

### 4.3.2. $\mathcal{B}^C$ and $\overline{\mathcal{B}}^C$ dependence.
Symmetry parameters (3.18) with reduced $\overline{\mathcal{B}}_{\lambda}$-dependence are
\[ \eta (\mathcal{B}_C; \mathcal{B}_A, \overline{\mathcal{B}}^B) = \sum_{|N|_2 = |\overline{N}|_2} \eta_{N\overline{N}} (\mathcal{B}_C) e^{i N_k \overline{N}_{\overline{k}}} e^{i N_2 \overline{N}_2} . \tag{4.27} \]

The non-compact case \[17\] suggests that oscillators $\mathcal{B}_A$ play the central role in parametrization of non-trivial charges. On the other hand, since for a symmetry parameter (3.18) periodicity implies that $|N|_2 = |\overline{N}|_2$, the $\mathcal{B}_A$-dependence cannot be fully eliminated in the periodic case. Indeed, parameters (4.27) depending solely on parameters $\mathcal{B}$,
\[ \eta (\mathcal{B}) = \sum_N \eta_N (\mathcal{B}_C) e^{2 N_k \overline{N}_{\overline{k}}} , \tag{4.28} \]
give rise to charges for the lower cycle $\sigma_0$ (see (4.6), (3.17)–(3.18) and (4.15))
\[ Q_{\eta} \propto \sum_{m,n,N} c_{m}^{+} c_{n}^{-} \eta_N (-i (m + n)) \delta_{m-n+2N,0} . \tag{4.29} \]

where $c_{m}^{+}$ and $c_{m}^{-}$ enter $Q_{\eta}$ with $|m|_2 = |n|_2$. The latter condition implies that parameters of the form (4.28) do not represent the full set of conserved charges. Since the form of charge components (4.14) gives $\delta_{m-n+2N,0}$ in (4.29) for any cycle $\Sigma$ this is true for the general case. In the sequel we focus on integration over the lower cycle $\sigma_0$ showing in the next section that this allows us to obtain the full set of non-trivial conserved charges.

The $\mathcal{B}_A$-dependence can be however minimized as follows. For a parameter (4.27) the conserved charge for $\sigma_0$ is
\[ Q_{\eta} \propto \sum_{m,n,|N|_2 = |\overline{N}|_2} c_{m}^{+} c_{n}^{-} \eta_{N\overline{N}} \left( -i \left( m + n - \overline{N} \right) \right) \delta_{m-n+2N,0} . \tag{4.30} \]
Let \( \eta_{N,\tilde{N}} \) have definite grading \( \Gamma = |N|_2 \in \mathbb{Z}_2^M \). Since the charge depends only on the following combinations of parameters

\[
\eta_N (-ik) = \sum_{\tilde{N}:|\tilde{N}|_2 = \Gamma} \eta_{N,\tilde{N}} \left( -ik + i \frac{\tilde{N}}{2} \right), \tag{4.31}
\]

it can be represented by any term with \( |\tilde{N}|_2 = \Gamma \). The simplest options are

\[
\eta_{(\pm \Gamma)} = \sum_{N:|N|_2 = \Gamma} \eta_N (\mathcal{B}_C) e^{\mathcal{B}_A} e^{\pm i \Gamma} \tilde{\mathcal{B}_A}. \tag{4.32}
\]

The antiautomorphism \( \rho \) acts on (4.32) as follows

\[
\rho \left( \eta_{(\pm \Gamma)} \right) = \eta_{(\mp \Gamma)}. \tag{4.33}
\]

Since \( \rho \) is involutive, parameters \( \eta \) can be decomposed into \( \rho \)-even and \( \rho \)-odd parts

\[
\eta^\pm = \frac{1 \pm \rho}{2} \eta. \tag{4.34}
\]

For the case of \( \Gamma = 0, \eta^- = 0 \).

Note that star product of two parameters (4.32) is not necessarily of the form (4.32) because, generally, \( \Gamma + \Gamma' \in \mathbb{Z}_2^M \). However, parameters of the form (4.32) are not demanded to form an algebra and will only be used for calculation of charges which are

\[
Q_\eta = \sum_{m,|m-n|_2 = \Gamma} c_n^+ c_m^- \eta_{m-n} \left(-1 (m + n) \right) \quad \eta_{(\pm \Gamma)} (k) = \eta_N \left( k \pm \frac{i \Gamma}{2} \right). \tag{4.35}
\]

The grading \( \Gamma \in \mathbb{Z}_2^M \) can be interpreted as distinguishing between bosonic and fermionic degrees of freedom. This suggests an extension of the initial periodic spinor Ansatz by allowing anti-periodic (Neveu-Schwarz) conditions. Detailed consideration of this issue is, however, beyond the scope of this paper.

Non-trivial dual charges resulting from the substitution \( V \leftrightarrow U \) in (4.1) are parametrized by

\[
\tilde{\eta}^{(\Gamma)} = \sum_{\tilde{N}:|\tilde{N}|_2 = \Gamma} \tilde{\eta}_{\tilde{N}} (\tilde{\mathcal{B}_C}) e^{\tilde{\mathcal{B}_A}} e^{i \Gamma} \tilde{\mathcal{B}_A} \tag{4.36}
\]

having the form

\[
\tilde{\eta} = \sum_{m,|m-n|_2 = \Gamma} c_n^+ c_m^- \eta_{m-n} \left(1 (m - n) \right) \quad \tilde{\eta}^{(\Gamma)} (k) = \tilde{\eta}_N \left( k \pm \frac{i \Gamma}{2} \right). \tag{4.37}
\]

Note that at \( \tilde{\Gamma} = 0 \) parameters (4.36) are \( \mathcal{B} \)-independent giving \( \tilde{\eta}^\pm = 0 \) whenever \( \tilde{\eta} (\tilde{\mathcal{B}}) = \mp \tilde{\eta} (\tilde{\mathcal{B}}) \).

4.4. Non-compact limit

The \( \mathbb{Z}_2^M \)-grading \( \Gamma \) accounting for even and odd components \( N_A \) in (4.32) degenerates in the non-compact limit \( \ell \to \infty \). Indeed, in terms of oscillators (2.30), the rescaled oscillator \( \tilde{\mathcal{B}_C} \) is \( \frac{1}{\sqrt{\ell}} \tilde{\mathcal{B}_C} \). Hence
This reproduces the result of [17] for the non-compact case that non-trivial charges are parametrized solely by the oscillators \( \mathcal{B} \). Fourier components \( c_n \) in (3.6) reproduce their non-compact analogs \( c(\xi) \) in (2.4) with \( \xi = \frac{2\pi}{\ell} n \)

\[
(2\pi)^M \sum_n \to \int d^M \xi...
\]

(4.39)

Independence of non-compact charges of the integration surface [17] is also reproduced in the limit \( \ell \to \infty \). Indeed, charge components (4.7) for fundamental cycles where shown to be of the form (4.13). They are different for different integration cycles because charge for the lower cycle corresponding to the higher symmetry parameter. In more detail, let

\[
\Omega(J_\eta) = \sum_{m,n,k,l} (dV^A + (m+n+k-l)_{C} dX^{CA})^{\wedge M}_{C} c^+_m c^-_n \eta_{kl} \left( -i \left( m+n + \frac{k-l}{2} \right) \right)
\]

\[
\quad \cdot \exp \left[ i (m-n+k+l) \bar{B} (V^B + (m+n+k-l)_{D} X^{DB}) \right].
\]

(5.1)

The charge resulting from (5.1) by integration over any cycle \( \Sigma \) equals to a charge associated with the lower fundamental cycle \( \sigma_0 = V^1 \cdots V^M \) with appropriately modified symmetry parameter. In more detail, let

\[
\eta'_{kl} \left( -i \left( m+n + \frac{k-l}{2} \right) \right) = p_{\Sigma} (m+n+k-l) \eta_{kl} \left( -i \left( m+n + \frac{k-l}{2} \right) \right).
\]

(5.2)

Integration of (4.1) with \( \eta' \) over \( \sigma_0 \) gives the same charge as for \( \eta_{kl} \left( -i \left( m+n + \frac{k-l}{2} \right) \right) \) integrated over \( \Sigma \). Indeed, using (4.15), integration over \( \Sigma \) with parameter \( \eta \) gives a factor of \( p_{\Sigma} (m+n+k-l) \), while on the rhs of (5.2) it is included into \( \eta' \).

In terms of star product (2.36) relation (5.2) takes the form

\[
\eta' \left( \mathcal{B}; \mathcal{B}^A; \mathcal{B}^B \right) = p_{\Sigma} (i \mathcal{B} \eta) \left( \mathcal{B}; \mathcal{B}^A; \mathcal{B}^B \right).
\]

(5.3)

This implies that a charge for a higher cycle \( \Sigma \) corresponding to some symmetry \( \eta \) equals to a charge for the lower cycle corresponding to the higher symmetry \( p_{\Sigma} \eta \).

Let conserved charges be considered as pairings between cycles and symmetry parameters

\[
\langle \sigma_{\Sigma}, \eta \rangle = \int_{\sigma_{\Sigma}} \Omega(J_\eta).
\]

(5.4)
Consider a transformation $\Xi_a$ mapping the lower cycle $\sigma_0$ to a higher one $\sigma_a = \Xi_a(\sigma_0)$. By (5.3)

$$\langle \Xi_a(\sigma_0), \eta \rangle = \langle \sigma_0, p_{\sigma_a} \ast \eta \rangle.$$  \hspace{1cm} (5.5)

As a result, transformation of symmetry parameters (5.3) is conjugate to a transition from the lower cycle to $\Sigma$ represented by an integer combination $\Sigma = \sum a b a a(\sigma_0)$ (see (4.11)). Note that only specific polynomials $p_{\sigma}$ described in (4.12) and their integer combinations generate transformation (5.3) conjugate to mappings of the lower cycles to higher ones. An important outcome is that any conserved charge can be obtained by integration over the spinor space, i.e. over the lower cycle $\sigma_0$, for some symmetry parameter $\eta$. This is somewhat analogous to the situation in the non-compact case, where, for a given parameter $\eta$, conserved charge is independent of the integration cycle. In the periodic case transition to the integration over the spinor space is always possible with the symmetry parameters transformed according to (5.3) and hence involving HS algebra into play.

Transformation (5.5) relates different geometric structures to algebraic properties of the symmetry transformations, i.e. higher integration cycles correspond to higher symmetries which are naturally included into the whole framework. On the other hand, in the customary lower-symmetry framework there is no room for algebraic relations between different integration cycles.

An interesting remaining question is to describe inverse transformation, i.e. conditions on the symmetry parameters $\eta$ allowing to obtain the same charge from a higher cycle $\Sigma$ with some symmetry parameter $\eta'$ such that $\langle \sigma_0, \eta \rangle = \langle \Sigma, \eta' \rangle$. According to (5.5) this is possible provided that $\eta = p_{\Sigma} \ast \eta'$. Analysis of this issue is less trivial, demanding a definition of a proper class of (may be nonpolynomial) functions $\eta'$. Its detailed consideration is beyond the scope of this paper.

6. Algebra of charges and symmetries

6.1. Charges as symmetry operators

Conserved charges correspond to symmetries of the rank-one system (2.1) via Noether’s theorem. Constructed in terms of rank-two fields they can be realized as symmetry generators acting on rank-one fields. Via quantization of rank-one fields Fourier amplitudes $c^\pm_n$ become operators $\hat{c}^\pm_n$ with non-trivial commutation relations (see [5] for details of the quantization procedure and [17] for algebra of charges). Analogously to [15], the non-zero commutation relations in the periodic case are

$$[\hat{c}^-_m, \hat{c}^+_n] = \delta_{mn}. \hspace{1cm} (6.1)$$

With the symmetry parameters $\eta^{(\Gamma)}$ (4.32) quantized conserved charges (4.35) become operators

$$\hat{Q}_\eta = \sum_{m,n} \eta^{(\Gamma)}_{m-n} (m+n) \hat{c}^+_m \hat{c}^-_n. \hspace{1cm} (6.2)$$

As in the non-compact case [17] they form closed algebra with respect to commutators

$$[\hat{Q}_\eta', \hat{Q}_\eta] = \hat{Q}_{[\eta', \eta]}, \hspace{1cm} (6.3)$$

where $\eta(k; \nu) = \sum N \eta_N(k)e^{i\nu_k}$ ($k_B \in \mathbb{Z}, \nu_C \in [0, 2\pi)$) are Weyl symbols for the Moyal-like star product

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\[ (f \star g) (k; v) = \sum_{m,n \in 2\mathbb{Z}} \int_0^{2\pi} \frac{d\mu d\nu}{(2\pi)^3} f (k + m; v + u) g (k + n; v + w) \exp \left[ i \left( mcw - ncu \frac{v}{e} \right) \right], \]

and the corresponding star commutator is \( [f, g] \star = f \star g - g \star f, \left[ k_C, e^{i\hat{Q}_v} \right] \star = -2N_C e^{i\hat{Q}_v} \).

One-to-one correspondence between symbols of the form
\[ \hat{\eta} (k; v) = \sum_N \eta_N (-ik) e^{i\hat{Q}_v} \]

and charges (6.2) results from the substitution \( k \mapsto m + n \) and \( N \mapsto n - m \) for Fourier components \( \eta_N (-ik) \). The charge \( \hat{Q}_{[v', \eta]} \) in (6.3) is associated with the symbol \( [\hat{\eta}', \hat{\eta}] \star (k; v) \). The dual set of charges for parameters \( \hat{\eta} \) is constructed from the symbols \( K \star \hat{\eta} \), where \( K \) is Klein operator obeying
\[ K \star K = 1, \quad K \star f (k; v) = f (-k; -v) \star K. \]

In terms of star product (6.4) it is represented by the delta-function
\[ K = (2\pi)^M \delta_{0,0} \delta (v). \]

The whole algebra of symmetries is thus \( \mathbb{Z}_2 \)-graded by \( K \) and parametrized by symbols of the form \( \varepsilon = \hat{\eta} + K \star \hat{\eta}' \) with symmetry operators obeying commutation relations
\[ \left[ \hat{Q}_{[\varepsilon]}, \hat{Q}_{[\varepsilon']} \right] = \hat{Q}_{[\varepsilon', \varepsilon]}. \]

This is the straightforward generalization of the non-compact construction of [17].

It is straightforward to see that for parameters \( \eta^{(\Lambda)} \) and \( \eta^{(\Gamma)} \), their product \( \eta^{(\Lambda)} \star \eta^{(\Gamma)} \) has grading \( \Lambda + \Gamma \mod 2 \).

Though Klein operators (3.25) and (6.6) are different: the former acts on the rank-one fields (3.5) as defined in (3.25) while the latter is introduced in the star-product algebra (6.4) of symmetry parameters (6.5) of quantized charges (6.2), there is manifest correspondence between the operators (3.25) and (6.6) allowing us to use the same notation for both of them. Indeed, charges (6.2) act on quantized rank-one fields \( \hat{\mathcal{C}}^\pm (Y|X) = \sum_n c_n^\pm \theta_n^\pm (Y|X) \) by commutator.

For instance, for the symmetry parameters \( \mathfrak{B}_C \) and \( \eta_N = e^{i\hat{Q}_v} \mathfrak{B}_N e^{i[Q_v; \mathfrak{B}]} \), the charges
\[ \hat{Q}_{2\mathfrak{B}_C} = -2i \sum_n nC c_n^+ c_n^-, \quad \hat{Q}_{\eta_N} = \sum_n c_n^+ \theta_n^+, \]

act on \( \hat{\mathcal{C}}^+ (Y|X) \) as follows
\[ \left[ \hat{Q}_{2\mathfrak{B}_C}, \hat{\mathcal{C}}^+ \right] = -2 \sum_n nC c_n^+ \theta_n^+, \quad \left[ \hat{Q}_{\eta_N}, \hat{\mathcal{C}}^+ \right] = \sum_n c_n^+ \theta_n^+. \]

This is equivalent to the action (3.7) of operators \( -2B_C^- \) and \( e^{i\hat{Q}_v} \mathfrak{B}_N e^{i[Q_v; \mathfrak{B}]} \) on \( \mathcal{C}^+ (Y|X) \). Computation for \( \hat{\mathcal{C}}^- (Y|X) \) and the operators \( -2B_C^- \) and \( -e^{i\hat{Q}_v} \mathfrak{B}_N e^{i[Q_v; \mathfrak{B}]} \) is analogous.

Dual set of charges (4.37) constructed via the star-multiplication of parameters (6.5) by Klein operator (6.6) is related to the operators (3.7) via operator \( K (3.25) \). Namely, for the symmetry parameters \( \mathfrak{B}_C \) and \( \hat{\eta}_N = e^{i\hat{Q}_v} \mathfrak{B}_N e^{i[Q_v; \mathfrak{B}]} \), the charges are
\[ \hat{\mathfrak{B}}_C = 2i \sum_n nC c_n^+ c_n^- \quad \text{and} \quad \hat{\mathfrak{B}}_N = \sum_n c_n^+ \theta_n^- \]

and parametrized by symbols of the form
Their action on $\tilde{\mathcal{C}}^+(Y|X)$ is the same as of operators $-2B^cK$ and $e^{i\eta^CN}A^c_+K$ on $C^+(Y|X)$, while for $\mathcal{C}^-(Y|X)$ charges (6.11) correspond to operators $-2K\tilde{B}^c$ and $-K\tilde{e}^{i\eta^CN}A^c_-$ acting on $C^-(Y|X)$ with $K$ defined in (3.25). This makes the correspondence between the conserved charges and symmetries of rank-one system (2.1) manifest.

6.2. Symmetry algebra

Periodicity of $Y$-variables changes symmetries of the rank-one system compared to the non-compact case. Residual symmetry algebra that respects periodicity is presented by conserved charge operators (6.8) as functionals of symmetry parameters (6.5), acting on quantized rank-one fields via commutator. Symmetries of the rank-one system are thus generated by polynomials in $s$ can be obtained via differentiation over $s$. The generators with $r=0$ form a subalgebra obeying

$$[T_{(m,\xi)}^q, T_{(n,\xi)}^r] = 2i \sin (mc^c - nc^C) \ T_{(m+n,\xi+\zeta)}^0,$$

which is analogous to the sine algebra introduced in [23], where its oscillator representation was also presented. The difference is that a half of indices in (6.12) are continuous, while for the sine algebra all of them are discrete. Relations (6.13) obey Jacobi identity and hence elements (6.12) form a Lie algebra with respect to star commutator (6.13). This infinite-dimensional Lie algebra represents the symmetry of rank-one system (2.1) with periodic variables $Y$.

7. Conclusion

Analysis of conserved charges of the HS equations with periodic twistor-like coordinates $Y^A$ performed in this paper exhibits several interesting features. The charges are represented as integrals of closed current forms in the extended $X^{AB}$, $Y^A$ space. Since periodicity in the $Y$-variables implies periodicity in $X$ variables, one can consider charges associated with different cycles in the $X^{AB}$, $Y^A$ space.

Closed current forms may depend on symmetry parameters $\eta$ parametrizing different charges like momentum, electric charge, conformal weight as well as their HS generalizations. In the non-compact case the most general symmetry parameters $\eta$ depend on the four types of oscillators $A_{\pm}$ and $B_{\pm}$ (2.9). In the periodic case, the symmetry parameters depend on $B_{\pm}$ and $e^{iA_{\pm}}$. The set of symmetries is doubled via Klein operator $K$ (3.25) (see (3.26) and (3.27)). In the framework of toric geometry it admits geometric interpretation as quasi-translation (3.25), the fundamental operation underlying quasi-periodicity of Riemann theta-function [25].

Nontrivial charges are represented by the current cohomology, i.e. those closed current forms that are not exact. In the non-compact case the current cohomology was shown in [17] to be represented by the symmetry parameters that depend solely on the oscillators $B$ or $B$.
(2.30). In the periodic case the situation is slightly different with the current cohomology represented by various $\mathbb{Z}_2^M$-graded parameters of the form

$$\eta^\Gamma(\mathfrak{B}) = \sum_{N: |N| = \Gamma} \eta^N(\mathfrak{B}) e^{iN \cdot \mathfrak{B}^s} e^{i\mathfrak{B}^s \tilde{\mathfrak{B}}^s}, \quad \Gamma \in \mathbb{Z}_2^M$$

(7.1)

and

$$\tilde{\eta}^\Gamma(\tilde{\mathfrak{B}}) = \sum_{\tilde{N}: |\tilde{N}| = \tilde{\Gamma}} \tilde{\eta}^N(\tilde{\mathfrak{B}}) e^{i\tilde{N} \cdot \tilde{\mathfrak{B}}^s} e^{i\tilde{\mathfrak{B}}^s \mathfrak{B}^s}, \quad \tilde{\Gamma} \in \mathbb{Z}_2^M$$

(7.2)

for the dual set of charges. Doubling of charges reflects the doubling of symmetries of (2.1) by Klein operator $K$ (3.25).

Another peculiarity of the periodic case is that naive expectation that charges evaluated as integrals over non-equivalent cycles are different is not quite true. Namely, the complete set of charges can be obtained by integration over the lower fundamental cycle $\sigma_0 = V^1 \ldots V^M$ constituted solely by spinor variables. Other cycles $\Sigma$ for a given symmetry parameter $\eta$ give charges which can be also obtained from the lower cycle with appropriate higher symmetry $\eta' \eta' = p\Sigma (i \mathfrak{B}) \ast \eta$.

This means that HS symmetries act on different non-contractible to each other cycles and hence connect them algebraically. Let us stress that there is no room for such connection unless higher symmetries are around. On the other hand, from this perspective (some of) HS symmetries acquire a nontrivial geometric meaning as relating nonequivalent cycles. An interesting remaining question is whether it is possible for a given parameter $\eta$ for the charge $\langle \sigma_0, \eta \rangle$ to find $\eta'$ such that $\langle \sigma_0, \eta \rangle = \langle \Sigma, \eta' \rangle$ for a higher cycle $\Sigma$. Expression (5.5) gives only sufficient condition for this to be true.

In accordance with the Noether’s theorem, quantized conserved charges resulting from the lower fundamental cycle generate symmetry transformations (3.7) of quantized rank-one fields via the commutator. Charges are parametrized by elements of the star-product algebra (6.4) which are conveniently packed into generating functions (6.12) closed under the star-product commutator as in (6.13) and which subalgebra (6.14) resembles sine algebra introduced in [23]. The Lie algebra (6.13) represents the full residual global symmetry of the unfolded system (2.1) after imposing periodic conditions on the spinor variables $Y^A$. Composed to the non-compact case, periodicity implies that half of indices in (6.12) are discrete.

The results of this paper may have several applications mentioned in Introduction. One related to black hole solutions in the HS theory seems to be the most interesting. We hope to consider this issue in the future.

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