Spike-and-Slab Group Lasso for Consistent Estimation and Variable Selection in Non-Gaussian Generalized Additive Models *

Ray Bai†

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Abstract

We study estimation and variable selection in non-Gaussian Bayesian generalized additive models (GAMs) under a spike-and-slab prior for grouped variables. Our framework subsumes GAMs for logistic regression, Poisson regression, negative binomial regression, and gamma regression, and encompasses both canonical and non-canonical link functions. Under mild conditions, we establish posterior contraction rates and model selection consistency when \( p \gg n \). For computation, we propose an EM algorithm for obtaining MAP estimates in our model, which is available in the R package sparseGAM. We illustrate our method on both synthetic and real data sets.

1. Introduction

1.1 Generalized Additive Models

GAMs are a useful extension of the linear model [6]. GAMs allow for more flexible modeling of a regression surface by permitting nonlinear covariate effects. However, the vast majority of methods and theoretical studies for Bayesian GAMs assume that the response variable is normally distributed [24, 22, 2, 3]. This assumption can be restrictive and is not appropriate for modeling non-Gaussian data such as binary data, count data, or skewed right positive data. In these cases, it is more tenable to assume that the

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†Department of Statistics, University of South Carolina, Columbia, SC. Email: RBAI@mailbox.sc.edu
response variables \( \{y_i\}_{i=1}^n \) belong to the more general exponential dispersion family. That is, we assume that the \( y_i \)'s are independently drawn from a density of the form,

\[
q_i(y_i) = \exp \left\{ \frac{y_i \theta_i - b(\theta_i)}{\tau} + c(y_i, \tau) \right\}, \quad i = 1, \ldots, n, \tag{1.1}
\]

and we relate the mean response \( \mathbb{E}(y_i) = b'(\theta_i) \) to a set of \( p \) covariates \( x_i = (x_{i1}, \ldots, x_{ip})' \in \mathbb{R}^p \) through a link function \( g \), i.e.

\[
g(b'(\theta_i)) = \mu + \sum_{j=1}^p f_j(x_{ij}), \quad i = 1, \ldots, n, \tag{1.2}
\]

where \( f_j \) is a possibly nonlinear function of the \( j \)th covariate and \( \mu \in \mathbb{R} \) is an overall mean. We assume throughout that \( \int f_j(x_j)dx_j = 0, j = 1, \ldots, p. \)

The exponential dispersion family (1.1) includes the Gaussian, binomial, Poisson, negative binomial, gamma, and beta distributions. However, in this paper, we shall be primarily concerned with GAMs for non-Gaussian data. Under (1.1), \( \theta_i \) is a natural parameter lying in \( \Theta \subset \mathbb{R} \), \( \tau \) is a dispersion factor which we assume to be known, and \( b \) and \( c \) are known functions. Following convention for generalized linear models (GLMs) [12], we assume that the function \( b \) in (1.1) is twice differentiable and strictly convex on \( \Theta \), with the second derivative \( b'' \) satisfying \( b''(u) > 0 \) for every \( u \in \Theta \). The link function \( g \) in (1.2) is chosen so that \( (g \circ b') : \Theta \mapsto \mathbb{R} \) is strictly increasing. These conditions are imposed so that we can map the range of the mean response \( \mathbb{E}(y_i) = b'(\theta_i) \) to the entire real number line. This in turn negates the need to place restrictions on \( \mu \in \mathbb{R} \) or the \( f_j(x_{ij}) \)'s on the right-hand side of (1.2). In practice, the canonical link function \( g = (b')^{-1} \) is often used. However, (1.2) also includes models with non-canonical link functions, such as binomial regression with a probit link and negative binomial regression with a log link.

When \( p \) is large, we often assume a low-dimensional structure such as sparsity. Under sparsity, we assume that only a small subset of covariates are significantly associated with the response. In this paper, we propose a general Bayesian modeling approach for jointly estimating and selecting the significant functions under (1.2) using the spike-and-slab group lasso of Bai et al. [2]. We call our approach SB-GAM for sparse Bayesian GAMs. We are only aware of one other paper by Scheipl et al. [18] that considers estimation and variable selection in non-Gaussian Bayesian GAMs. However, Scheipl
et al. [18] do not prove any theoretical results, and their implementation relies on Markov chain Monte Carlo (MCMC) rather than the optimization-based approach used in this work.

Theory for Bayesian estimation and variable selection in high-dimensional Gaussian nonparametric regression has been well-studied under basis expansions [24, 22, 2, 3], tree and forest priors [16, 11], and Gaussian process priors [26, 25, 8]. However, theory for high-dimensional Bayesian nonparametric regression when the response $y$ is non-Gaussian is much more deserted. In this paper, we move beyond the Gaussianity assumption and derive theoretical results for Bayesian nonparametric additive models (1.2) when $y$ belongs to the more general exponential dispersion family (1.1).

Outside of Gaussian regression, there are only a few theoretical results for high-dimensional Bayesian GLMs. Jiang [9] and Jeong and Ghosal [7] derived posterior contraction rates for Bayesian GLMs, but their results concern estimation rates only. Jiang [9] and Jeong and Ghosal [7] did not prove model selection consistency. Moreover, Jiang [9] and Jeong and Ghosal [7] focused solely on the case where the covariates have a perfectly linear relationship with $g(\mathbb{E}(y_i))$, that is, $\sum_{j=1}^{p} f_j(x_{ij}) = \sum_{j=1}^{p} \gamma_j x_{ij}$ in (1.2). In the context of high-dimensional binary regression with a logit link, Narisetty et al. [13] and Lee and Cao [10] recently established model selection consistency under spike-and-slab priors. However, Narisetty et al. [13] and Lee and Cao [10] focused only on logistic regression and did not study rates of estimation; their theory focused solely on model selection consistency.

The framework we introduce in this article goes beyond those of previous papers by: a) considering GAMs (1.2) with possibly nonlinear covariate effects, and b) proving both estimation rates and model selection consistency under Bayesian GAMs for any response $y$ belonging to the family (1.1). In particular, our main theorem on model selection consistency (Theorem 3 in Section 3) applies to Poisson regression, negative binomial regression, and gamma regression with either canonical or non-canonical links. To the best of our knowledge, our selection consistency result is the first of its kind in the Bayesian high-dimensional literature that applies to all of these families.

The rest of the paper is structured as follows. Section 2 introduces the SB-GAM framework for responses belonging to the exponential dispersion family (1.1). In Section 3, we establish the convergence rates and model selection consistency for SB-GAM. Section 4 presents an EM algorithm for implementing SB-GAM. In Section 5, we illustrate our method through simulation studies and analysis of a high-dimensional gene expression data set with dichotomous outcomes. Section 6 concludes the paper.
1.2 Notation and Preliminaries

We use the following notations for the rest of the paper. For two positive sequences \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n \asymp b_n \) to denote \( 0 < \liminf_{n \to \infty} a_n/b_n \leq \limsup_{n \to \infty} a_n/b_n < \infty \). If \( \lim_{n \to \infty} a_n/b_n = 0 \), we write \( a_n = o(b_n) \) or \( a_n \prec b_n \). We use \( a_n \preceq b_n \) or \( a_n = O(b_n) \) to denote that for sufficiently large \( n \), there exists a constant \( K > 0 \) independent of \( n \) such that \( a_n \leq Kb_n \). We write \( a_n \vee b_n \) to denote \( \max\{a_n, b_n\} \) and \( a_n \wedge b_n \) to denote \( \min\{a_n, b_n\} \). For a vector \( v = (v_1, \ldots, v_p)' \in \mathbb{R}^p \), we let \( \|v\|_2 = (\sum_{i=1}^p v_i^2)^{1/2} \) denote the \( \ell_2 \) norm.

For a symmetric matrix \( A \), we let \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote its minimum and maximum eigenvalue respectively. For a matrix \( C = (c_{jk}) \in \mathbb{R}^{m \times n} \), we denote \( \|C\|_{\max} = \max_{j,k} |c_{jk}| \) and \( \|C\|_2 = (\lambda_{\max}(C'C))^{1/2} \).

We assume that the response \( y \) depends on only a small subset \( S \subset [p] \) of the univariate functions and that the covariates \( x \) are fixed and have been rescaled to lie in \( [0,1]^p \). For a function \( f(x) \), we denote \( \|f\|_{\infty} = \sup_{x \in [0,1]^p} f(x) \). For \( \alpha > 0 \), we let \( C^\alpha[0,1] \) denote the Hölder class of \( \alpha \)-smooth univariate functions on \([0,1]\) that have continuously differentiable derivatives up to order \( \lfloor \alpha \rfloor \), with the \( \lfloor \alpha \rfloor \)th order derivative being Lipschitz continuous of order \( \alpha - \lfloor \alpha \rfloor \).

2 Sparse Bayesian GAMs (SB-GAMs)

Following [23], we assume that each \( f_j, j = 1, \ldots, p \), in (1.2) may be approximated by a linear combination of \( d \) basis functions \( B_j = \{h_{j1}, \ldots, h_{jd}\} \), that is,

\[
f_j(x_{ij}) \approx \sum_{k=1}^d h_{jk}(x_{ij}) \beta_{jk}, \tag{2.1}
\]

where \( \beta_j = (\beta_{j1}, \ldots, \beta_{jd})' \), and \( \beta = (\beta_1', \ldots, \beta_p')' \) is the vector of unknown basis coefficients. Let \( \bar{x}_{ij} = (h_{j1}(x_{ij}), \ldots, h_{jd}(x_{ij}))' \) be a \( d \)-dimensional vector of the \( d \) basis functions evaluated at \( x_{ij} \). With the approximation (2.1), we rewrite the GAM (1.2) as

\[
g(b'(\theta_{ij})) \approx \mu + \sum_{j=1}^p \bar{x}_{ij}' \beta_j, \quad i = 1, \ldots, n, \tag{2.2}
\]

To enforce the constraint that the \( f_j \)’s integrate to zero, we center the columns of the \( n \times d \) matrices, \( \bar{X}_j = [\bar{x}_{1j}, \ldots, \bar{x}_{nj}]', j = 1, \ldots, p \), to have mean zero. Under (2.2), the SB-GAM model places priors on \( (\mu, \beta) \). For the grand mean \( \mu \), we place the flat prior,

\[
\pi(\mu) \propto 1, \tag{2.3}
\]
so that \( \mu \) is not penalized. Next, we endow the basis coefficients \( \beta \) in (2.2) with the spike-and-slab group lasso (SSGL) prior [2],

\[
\pi(\beta | \kappa) = \prod_{j=1}^{p} \left[ (1 - \kappa)\Psi(\beta_j | \lambda_0) + \kappa\Psi(\beta_j | \lambda_1) \right],
\]

(2.4)

where \( \kappa \in (0, 1) \) is a mixing proportion and \( \Psi(\cdot | \lambda) \) denotes a multivariate density for a \( d \)-dimensional random vector indexed by hyperparameter \( \lambda \),

\[
\Psi(\beta_j | \lambda) = \frac{\lambda^d e^{-\lambda\|\beta_j\|^2}}{2^d \pi^{(d-1)/2} \Gamma((d+1)/2)}, \quad j = 1, \ldots, p.
\]

In (2.4), we typically set \( \lambda_0 \) to be large, so that the “spike” \( \Psi(\cdot | \lambda_0) \) is a heavily peaked density around the zero vector \( 0_d \). Meanwhile, we set \( \lambda_1 \) to be small, so that the “slab” \( \Psi(\cdot | \lambda_1) \) is very diffuse. To model the uncertainty in \( \kappa \) in (2.4), or the expected proportion of nonzero subvectors in \( \beta \), we endow \( \kappa \) with the beta prior,

\[
\kappa \sim B(a, b).
\]

(2.5)

The hierarchical prior (2.4)-(2.5), which we denote as \( \text{SSGL}(\lambda_0, \lambda_1, \kappa) \) going forward, is a two-group refinement of the group lasso [28] and a multivariate extension of the spike-and-slab lasso of Ročková and George [17]. Unlike the group lasso, however, \( \text{SSGL}(\lambda_0, \lambda_1, \kappa) \) ensures that the amount of shrinkage applied to each group is adaptive. Whereas the group lasso applies the same amount of shrinkage to every group, the slab component \( \Psi(\cdot | \lambda_1) \) in (2.4) prevents larger coefficients from being downward biased. The prior on \( \kappa \) (2.5) also renders the SSGL penalty on \( \beta \) non-separable; that is, the vectors \( \beta_1, \ldots, \beta_p \) are a priori dependent. This enables the \( \beta_j \)'s to share information across functions and self-adapt to ensemble information about sparsity.

The prior (2.4)-(2.5) has been successfully employed in sparse semiparametric regression when the response \( y \) is normally distributed [1, 2]. However, its potential for non-Gaussian semiparametric regression has not been studied before.

3 Asymptotic Theory

3.1 Posterior Contraction Rates

We first derive posterior contraction rates for SB-GAM. We assume that there is a true data generating mechanism for \((x_i, y_i), i = 1, \ldots, n,\)

\[
g(b'(\theta_{0i})) = \mu_0 + \sum_{j=1}^{p} f_{0j}(x_{ij}), \quad i = 1, \ldots, n,
\]

(3.1)
where the $\theta_{0i}$'s are the true natural parameters in the densities of the $y_i$'s (1.1). Let $f_0 = \mu_0 + \sum_{j=1}^{p} f_{0j}$ denote the true additive function. Let $S_0 \subset [p]$ denote the indices of the true nonzero functions, with $f_j(x_j) = 0$ for all $j \in S_0^c$. We denote $s_0 = |S_0|$ as the cardinality of $S_0$. We assume that $f_0$ belongs to the class of functions,

$$
\mathcal{H}(s, \alpha) = \left\{ f : f(x) = \mu + \sum_{j \in S} f_j(x_j) : \mu \in \mathbb{R}, S \subset [p], 1 \leq |S| = s, \|f\|_\infty < \infty, \int_0^1 f_j(x)dx = 0, f_j \in C^\alpha[0,1], j \in [p], \alpha \in \mathbb{N} \right\}.
$$

Note that for the inactive covariates, the regression functions are zero functions, which trivially satisfy the smoothness assumption.

We further suppose that the true univariate functions $f_{0j}, j = 1, \ldots, p$, can be approximated by a linear combination of $d$ basis functions $\mathcal{B} = \{h_{j1}, \ldots, h_{jd}\}$, so that (3.1) can be written as

$$
g((b'(\theta_{0i})) = \mu_0 + \sum_{j=1}^{p} \tilde{x}_{ij} \beta_{0j} + \delta_{0i}, \ i = 1, \ldots, n, \quad (3.2)
$$

where $\tilde{x}_{ij} = (h_{j1}(x_{ij}), \ldots, h_{jd}(x_{ij}))'$ and $\delta_{0i}$ is the approximation error incurred by truncating the basis expansions to be of dimension $d$. As before, we let $\tilde{X} = [\tilde{x}_{1j}', \ldots, \tilde{x}_{nj}']'$ denote the $n \times d$ matrix corresponding to the $j$th basis expansion, and let $\tilde{X}_{n \times dp} = [\tilde{X}_1, \ldots, \tilde{X}_p]$. We let $\mathbb{E}_0$ and $\mathbb{P}_0$ denote the expectation and probability operators for (3.2) with the true parameters $(\mu_0, \beta_0)$, where $\beta_0 = (\beta_{01}', \ldots, \beta_{0p}')'$.

Define $\|\tilde{X}\|_* = \max_{1 \leq j \leq p}\|\tilde{X}_j\|_2$. We make the following assumptions about the basis functions.

**Condition 1.** $\|\tilde{X}\|_* \asymp n/d$.

**Condition 2.** The bias $\delta_0 = (\delta_{01}, \ldots, \delta_{0n})'$ satisfies $\|\delta_0\|_2 \lesssim (s_0nd^{-2\alpha})^{1/2}$.

An example of a set of basis functions $\mathcal{B}$ that satisfies Conditions 1-2 is the B-spline basis functions constructed with $d - \lfloor \alpha \rfloor$ knots and $\lfloor \alpha \rfloor - 1$ degree. See, for example, [27, 19, 22].

We also make the following mild assumption about the function $b$ in the family of distributions for the response (1.1). Note that this condition is satisfied by all practical applications of GAMs.

**Condition 3.** For any compact subset $\mathcal{K} \subset \Theta$, there exist constants $c_1, c_2 > 0$ such that $c_1 \leq \inf_{x \in \mathcal{K}} b''(x) \leq \sup_{x \in \mathcal{K}} b''(x) \leq c_2$. 

6
Let $q_i$ be the density of $y_i$ belonging to the family (1.1), where the natural parameter $\theta_i$ is related to the covariates through the additive function (1.2). Define $q_{0i}$ analogously with the true natural parameter $\theta_{0i}$. We denote the joint densities $q = \prod_{i=1}^{n} q_i$ and $q_0 = \prod_{i=1}^{n} q_{0i}$. We define the average squared Hellinger metric as $H_n^2(q, q_0) = n^{-1} \sum_{i=1}^{n} H^2(q_i, q_{0i})$, where $H(q_i, q_{0i}) = \left( f(\sqrt{q_i} - \sqrt{q_{0i}})^2 \right)^{1/2}$ is the Hellinger distance between $q_i$ and $q_{0i}$. Let $\mathcal{F}$ denote the set of all possible additive functions $f = \mu + \sum_{j=1}^{p} f_j$, where each univariate component $f_j$ can be represented by a $d$-dimensional basis expansion. Our first result establishes posterior contraction in the average squared Hellinger metric for the fitted density $q$ under SB-GAM.

**Theorem 1.** Suppose that Conditions 1-3 hold. Assume that $\log p = o(n)$, $s_0 = o((n \log p)^2/(2\alpha + 1))$, and $d \log n = O(\log p)$, and $d \asymp n^{1/(2\alpha + 1)}$. Under model (3.2), suppose that we endow $(\mu, \beta)$ with a Gaussian prior on $\mu$ and an SSGL($\lambda_0, \lambda_1, \kappa$) prior (2.4)-(2.5) on $\beta$, with hyperparameters $\lambda_0 = (1-\kappa)/\kappa$, $\lambda_1 \asymp 1/n$, $a = 1$, and $b = p^\epsilon$, where $c > 2$. Then for some $M_1 > 0$, we have

$$
\sup_{f_0 \in \mathcal{H}(s, \alpha)} \mathbb{E}_0 \Pi \left( f \in \mathcal{F} : H_n^2(q, q_0) > M_1 \epsilon_n^2 \mid y \right) \rightarrow 0,
$$

as $n, p \rightarrow \infty$, where $\epsilon_n^2 = s_0 \log p/n + s_0 n^{-2\alpha/(2\alpha + 1)}$.

**Proof.** Appendix C.

**Remark 1.** For Theorem 1, we needed to assume a proper prior on $\mu$ (with unbounded support) in order to utilize the standard theory on posterior concentration [5]. For implementation, we endowed $\mu$ with the improper prior (2.3), which poses no practical issues.

Theorem 1 states that the distance between the fitted density $q$ (indexed by $f$ under SB-GAM) and the true density $q_0$ indexed by the true $f_0$ becomes arbitrarily close as $n \rightarrow \infty$. However, Theorem 1 does not tell us anything about the estimation accuracy for the additive function $f$ itself. To quantify the estimation accuracy of $f$, we define the $L^2(\mathbb{P}_n)$ norm $\| \cdot \|_n$ on the function space $L^2([0, 1]^p)$ by $\| f - f_0 \|_n^2 = n^{-1} \sum_{i=1}^{n} [f(x_i) - f_0(x_i)]^2$. The next theorem gives the posterior contraction rate for $f$ under slightly stronger assumptions on the matrix of basis functions $\hat{X}$ and the size of the true model.

**Theorem 2.** Suppose that Conditions 1-3 hold. Assume that $\| \hat{X} \|_{\max} = O(1)$, $\log p = o(n^{2\alpha/(2\alpha + 1)})$, $s_0 = o((n/d \log p)^{1/2} \wedge n^{2\alpha/(2\alpha + 1)})$, $d \log n = O(\log p)$, and $d \asymp n^{1/(2\alpha + 1)}$. Under model (3.2), suppose that we endow $(\mu, \beta)$ with a Gaussian prior on $\mu$ and an SSGL($\lambda_0, \lambda_1, \kappa$) prior (2.4)-(2.5)
on $\beta$, with hyperparameters $\lambda_0 = (1 - \kappa)/\kappa$, $\lambda_1 \times 1/n$, $a = 1$, and $b = p^c$, where $c > 2$. Then for some $M_2 > 0$, we have
\[
\sup_{f_0 \in \mathcal{H}(s, \alpha)} \mathbb{E}_0 \Pi \left( f \in \mathcal{F} : \|f - f_0\|_n^2 > M_2 \epsilon_n^2 | y \right) \to 0,
\]
as $n, p \to \infty$, where $\epsilon_n^2 = s_0 \log p/n + s_0 n^{-2\alpha/(2\alpha+1)}$.

Proof. Appendix C.

Compared to Theorem 1, Theorem 2 requires the entries of the matrix $\tilde{X}$ to be uniformly bounded and the growth rate of the true model size to satisfy $s_0^2 = o(n/d \log p)$, rather than $s_0 = o(n/\log p)$. Since the covariates are rescaled to lie in $[0, 1]^p$, it is not difficult for most standard choices of basis functions to satisfy uniform boundedness. Theorem 2 shows that SB-GAM consistently estimates $f_0$ under (3.1) even when $p$ grows at nearly exponential rate with $n$. Our posterior contraction rate matches the minimax rate of estimation for Gaussian GAMs [26, 14]. However, our result is more general, allowing the response $y$ to belong to a very general family of distributions (1.1) and encompassing models with both canonical and non-canonical link functions.

3.2 Model Selection Consistency

We now establish that with an additional identifiability condition, SB-GAM also selects the correct set of active variables $S_0$ as $n \to \infty$. Under the basis expansion approximations for the functions $f_j, j = 1, \ldots, p$, in (2.2), selecting the nonzero functions $f_j$'s is equivalent to selecting the $d$-dimensional vectors $\beta_j$'s where $\beta_j \neq 0_d$.

While the posterior mode for $\beta$ under SB-GAM is exactly sparse, the priors (2.3)-(2.5) on $(\mu, \beta)$ are continuous, and therefore, the posterior puts zero mass at exactly sparse $\beta$'s. Given this, we designate the active covariates selected by SB-GAM as $S = \{j \in [p] : \|\beta_j\|_2 > \omega_d\}$, for some suitably small $\omega_d$. In other words, $S$ contains the indices of the functions whose corresponding vectors of basis coefficients have 2-norm greater than a small threshold close to zero. Like [1, 2], we use the threshold,
\[
\omega_d \equiv \omega_d(\lambda_0, \lambda_1, \kappa) = \frac{1}{\lambda_0 - \lambda_1} \log \left[ \frac{1 - \kappa \lambda_0^d}{\kappa \lambda_1^d} \right].
\]
(3.3)

As discussed in [1, 2], any vectors $\beta_j$ that satisfy $\|\beta_j\|_2 = \omega_d$ correspond to the intersection points between the spike and slab densities in the SSGL.
prior, or the turning point where the slab has dominated the spike. Thus, using this thresholding rule as a proxy for variable selection with the posterior mode is justified; when $\|\beta_j\|_2 > \omega_d$, $\beta_j$ typically belongs to the slab, i.e. $\beta_j$ has a non-sparse mode. When $\lambda_0 \gg \lambda_1$ (which will always be the case for large $p$ when $\lambda_0 = (1 - \kappa)/\kappa$ and $\kappa \sim \mathcal{B}(1, p^c), c \geq 1$), the threshold $\omega_d$ also rapidly approaches zero for large $n$.

To establish selection consistency, we need the following identifiability condition. Let $q_S = \prod_{i=1}^n q_i$ be the joint density in the exponential family (1.1) indexed by $f = \mu + \sum_{j \in S} f_j$, and recall that $H_n^2(p, q)$ is the average squared Hellinger distance between two $n$-variate densities $p, q$.

**Condition 4.** For any $S \not\supset S_0$, $H_n(q_0, q_S) > \epsilon_{n,s_0}$, where $\epsilon_{n,s_0}^2$ is the contraction rate in Theorem 2.

Condition 4 ensures that all of the active functions $f_j, j \in S_0$, are sufficiently distinguishable from the zero functions. In the special cases of Gaussian and binomial regression, this condition is equivalent to the “beta-min” condition, and we can replace Condition 4 with the requirement that $\min_{j \in S_0} \|\beta_0_j\|_2 > \epsilon_{n,s_0}$. However, in order to cover all densities of the form (1.1) (not just Gaussian and binomial), we need to impose the slightly more abstract Condition 4, which guarantees the existence of sufficiently powerful tests for testing $H_0 : q = q_0$ vs. $H_1 : q = q_S$, where $H_n(q_0, q_S) > \epsilon_{n,s_0}$.

The next theorem shows that SB-GAM consistently selects the correct set of nonzero functions as $n \to \infty$.

**Theorem 3.** Suppose that Conditions 1-4 hold. Assume that $\log p = o(n)$, $0 < s_0 = O(1)$, $d \log n = O(\log p)$, and $d \asymp n^{1/(2\alpha+1)}$. Under model (3.2), suppose that we endow $(\mu, \beta)$ with a Gaussian prior on $\mu$ and an $\text{SSGL}(\lambda_0, \lambda_1, \kappa)$ prior (2.4)-(2.5) on $\beta$, with hyperparameters $\lambda_0 = (1-\kappa)/\kappa$, $\lambda_1 \asymp 1/n$, $a = 1$, and $b = p^c$, where $c > 2$. Let $S = \{j \in [p] : \|\beta_j\|_2 > \omega_d\}$, where $\omega_d$ is as in (3.3). Then as $n, p \to \infty$,

$$\sup_{f_0 \in \mathcal{H}(s,\alpha)} \mathbb{E}_{0}\|f \in \mathcal{F} : S = S_0 \mid y\| \to 1.$$

**Proof.** Appendix C.

Unlike previous results on nonparametric Bayesian model selection consistency which have only considered Gaussian responses [24, 22, 11, 25, 8], Theorem 3 allows $y$ to belong to any distribution in the exponential dispersion family (1.1). Theorem 3 also allows for both canonical and noncanonical link functions.
4 Computational Strategy

4.1 Expectation-Maximization (EM) Algorithm

In order to implement SB-GAM, we adopt the penalized likelihood perspective and perform MAP estimation. Namely, we extend the EM variable selection (EMVS) approach of Ročková and George [15] to the non-Gaussian setting with grouped variables. The complete derivations for our algorithm are provided in Appendix A.

The EMVS approach of Ročková and George [15] introduces latent variables \( \nu = (\nu_1, \ldots, \nu_p)' \), \( \nu_j \in \{0, 1\} \), where \( \nu_j = 1 \) indicates that the \( j \)th group of basis coefficients \( \beta_j \) (or equivalently, the \( j \)th function \( f_j \)) should be included in the model. These indicator variables are treated as missing data in the E-step of our algorithm. To be precise, we reparameterize the SSGL\((\lambda_0, \lambda_1, \kappa)\) prior (2.4) as a beta-Bernoulli prior. We rewrite (2.4) as

\[
\pi(\beta \mid \nu) = \prod_{j=1}^{p} \left[ (1 - \nu_j) \Psi(\beta_j \mid \lambda_0) + \nu_j \Psi(\beta_j \mid \lambda_1) \right], \\
\pi(\nu \mid \kappa) = \prod_{j=1}^{p} \kappa^{\nu_j} (1 - \kappa)^{1 - \nu_j},
\]

where \( \nu \) is a binary vector. Let \( \Xi \) denote the collection \( \{\mu, \beta, \kappa\} \). As shown in Appendix A, \( \mathbb{E}[\nu_j \mid y, \Xi] = p^*_j(\beta_j, \kappa) \), where

\[
p^*_j(\beta_j, \kappa) = \frac{\kappa \Psi(\beta_j \mid \lambda_1)}{\kappa \Psi(\beta_j \mid \lambda_1) + (1 - \kappa) \Psi(\beta_j \mid \lambda_0)}
\]

is the conditional posterior probability that \( \beta_j \) is drawn from the slab distribution rather than from the spike.

In the E-step, we compute \( p^*_j = p^*(\beta_j^{(t-1)}, \kappa^{(t-1)}) = \mathbb{E}[\nu_j \mid y, \Xi^{(t-1)}], j = 1, \ldots, p \), given the previous estimate \( \Xi^{(t-1)} \). In the M-step, we update \( \kappa \) as

\[
\kappa^{(t)} = \frac{a - 1 + \sum_{j=1}^{p} p^*_j}{a + b + p - 2},
\]

and then we update \( (\mu, \beta) \) as

\[
(\mu^{(t)}, \beta^{(t)}) = \arg \max_{\mu, \beta} \left\{ \ell(\mu, \beta) - \sum_{j=1}^{p} \lambda_j^* \|\beta_j\|_2 \right\}, \tag{4.4}
\]

where \( \ell(\mu, \beta) \) is the log-likelihood, and each \( \lambda_j^* = \lambda_1 p^*_j + \lambda_0 (1 - p^*_j) \) is an adaptive weight ensuring that insignificant groups of basis coefficients are
shrunk aggressively to zero, while significant groups incur minimal shrinkage. The objective (4.4) is simply a group lasso optimization with known weights $\lambda^* = (\lambda_1^*, \ldots, \lambda_p^*)$. In Appendix A, we describe how to efficiently solve (4.4).

Once we have our MAP estimator of $\hat{\beta}$, we can estimate the $p$ functions $f_j(x_j)$ in (1.2) straightforwardly as $\hat{f}_j(x_j) = \bar{X}_j \hat{\beta}_j, j = 1, \ldots, p$. Since $\hat{\beta}_j = 0_d$ for some indices $j$, our EM algorithm automatically performs variable selection from the $p$ functions.

### 4.2 Choice of Hyperparameters

The performance of SB-GAM is mainly governed by the three parameters in the $SSGL(\lambda_0, \lambda_1, \kappa)$ prior (2.4) on $\beta$. We recommend fixing $\lambda_1 = 1$, so that the $\beta_j$’s with large entries receive minimal shrinkage. To induce sparsity, the mixing proportion $\kappa$ should also be small with high probability. To this end, we recommend setting $a = 1, b = p$ for the $B(a, b)$ prior (2.5) on $\kappa$. This ensures that most of the $\beta_j$’s belong to the spike.

The spike hyperparameter $\lambda_0$ in (2.4) controls how sparse our final model is, with larger values of $\lambda_0$ leading to more sparsity. Since we are mainly interested in predictive accuracy for GAMs, we recommend tuning $\lambda_0$ using $K$-fold cross-validation, with $K = 5$ or $K = 10$. We then choose the $\lambda_0$ which minimizes the average prediction error $\sum_i (y_i - b'(\hat{\theta}_i))^2$ across all $K$ validation sets. In practice, we have observed that choosing $\lambda_0$ from an equispaced grid between 1 and 100 works well.

### 5 Simulations and Data Analysis

In this section, we illustrate SB-GAM and compare its performance to other regularized GAMs. All GAMs were implemented using B-spline basis functions with $d = 6$ basis functions and interior knots placed at the 25th, 50th, and 75th percentiles of the data for each covariate. We compared SB-GAM to GAMs with other group regularization penalties: group lasso ($gLASSO$-GAM), group smoothly clipped absolute deviation ($gSCAD$-GAM), and group minimax concave penalty ($gMCP$-GAM) [28, 4]. All of these methods are implemented in the R package sparseGAM, which is available on the Comprehensive R Archive Network (CRAN).
5.1 Simulations

We illustrate SB-GAM on logistic regression and Poisson regression. For logistic regression, \( b(u) = \log(1 + e^u) \) and \( \tau = 1 \) in (1.1), and \( g(u) = \log\{u/(1 - u)\} \) in (1.2). For Poisson regression, we have \( b(u) = e^u \), \( \tau = 1 \), and \( g(u) = \log(u) \). In Appendix B, we report simulation results for SB-GAM in negative binomial regression with a log link, which is an example of SB-GAM with a non-canonical link function.

For our logistic regression simulation study, we set \( n = 100 \) and \( p = 500 \). We generated data \((x_i, y_i), i = 1, \ldots, n\), where the \( x_i \)'s came from a standard uniform distribution. The \( y_i \)'s were generated from Bernoulli(\( \theta_i \)), where the true regression function was

\[
\log\left( \frac{\theta_i}{1 - \theta_i} \right) = 5 \sin(2\pi x_{i1}) - 4 \cos(\pi x_{i2}) + 1.5 e^{x_{i3} - 1} - 2x_{i4};
\]

that is, only \( x_1, x_2, x_3, \) and \( x_4 \) were active covariates, while \( f_j(x_j) = 0 \) for \( j = 5, \ldots, 500 \). To assess out-of-sample estimation accuracy of the true additive function \( f \), we generated 100 new data points \((x_{\text{new}}, y_{\text{new}})\) and computed the mean squared error (MSE), where \( \text{MSE} = n^{-1} \sum_{i=1}^{n} (f(x_{i,\text{new}}) - f(x_{i,\text{new}}))^2 \). To examine accuracy of variable selection, we used Matthews correlation coefficient (MCC),

\[
\text{MCC} = \frac{TP \times TN - FP \times FN}{\sqrt{(TP + FP)(TP + FN)(TN + FP)(TN + FN)}}.
\]
where TP, TN, FP, and FN are true positives, true negatives, false positives, and false negatives, respectively. Finally, to assess prediction performance, we computed the area under the curve (AUC) of the receiver operating characteristic (ROC) curve for $y_{new}$. A higher AUC indicates more accurate predictions of $y$.

Figure 1 shows our results for 100 replications of logistic GAMs. We see that SB-GAM had the lowest median MSE, indicating that SB-GAM predicted $f$ the best. In terms of selection of the univariate functions, SB-GAM performed the second best, only slightly worse than gMCP-GAM. Finally, SB-GAM had the highest median AUC, indicating that it gave the best predictions of the binary outcomes $y$ ("0" or "1").

For Poisson GAMs, we set $n = 100$, $p = 500$. We generated $(x_i, y_i), i = 1, \ldots, n$, where the $x_i$'s came from a standard uniform distribution. The $y_i$'s were generated from Poisson($\theta_i$), where the true regression function was

$$\log(\theta_i) = 1.5 \sin(2\pi x_{i1}) - \cos(\pi x_{i2}) + e^{x_{i3}} - x_{i4}^2.$$ 

For Poisson GAMs, we also computed out-of-sample MSE and MCC to gauge estimation accuracy and variable selection accuracy. To assess predictive performance, we computed the mean squared prediction error (MSPE), where $MSPE = n^{-1} \sum_{i=1}^{n} (y_{i,new} - \exp(\theta_{i,new}))^2$. For Poisson regression, a lower MSPE indicates better predictions of $y$. Figure 2 shows our results for 100 replications of Poisson GAMs. SB-GAM had the best function recovery (i.e. the lowest median MSE) and the best variable selection performance.
(i.e. the highest median MCC), SB-GAM’s median MSPE was also the lowest, indicating the best predictive accuracy of the count responses $y$.

### 5.2 Real Data Analysis

We now use the SB-GAM method to predict incidence of prostate cancer. The data we analyzed came from a microarray experiment consisting of gene expression levels of $p = 6033$ genes for $n = 102$ subjects [20]. The response $y$ was either “1” if the subject had prostate cancer or “0” if they did not. Of the 102 subjects, 52 were patients who were diagnosed with prostate cancer, and the remaining 50 were normal control subjects. For each of the subjects, we first rescaled their 6033 gene expression values to lie in $[0, 1]^p$. We then fit a logistic regression model with SB-GAM.

SB-GAM selected 21 genes out of 6033 as significantly associated with prostate cancer. Figure 3 plots the functions $f_j(x_j)$ for these 21 genes. In addition, we assessed the predictive power of SB-GAM on 50 data sets. For each data set, we randomly assigned 82 (or roughly 80%) of the observations to the training set, fit the SB-GAM logistic model on the training data, and then used our fitted model to predict the incidence of cancer on the remaining 20 observations. We obtained an average AUC of 0.89, indicating that SB-GAM was quite good at predicting the presence or absence of prostate cancer.
cancer in out-of-sample patients. For this particular data set, SB-GAM both selected a parsimonious model and had high predictive power.

6 Conclusion

In this paper, we introduced SB-GAM, a general framework for Bayesian estimation and variable selection in high-dimensional GAMs. Namely, we extended the SSGL prior of [2] to the non-Gaussian setting where the response variable belongs to the exponential dispersion family (1.1). We derived both posterior contraction rates and model selection consistency for Bayesian GAMs in the exponential dispersion family (1.1). Our results are fairly broad. For example, we are not aware of any other Bayesian model selection consistency results for Poisson regression, negative binomial regression, or gamma regression. Our main result on model selection consistency (Theorem 3) covers all of these families and encompasses both canonical and non-canonical link functions. For practical implementation of SB-GAM, we proposed using an EM algorithm to rapidly select and estimate the univariate functions.

Although we have addressed Bayesian variable selection and estimation in GAMs, we have not addressed the question of uncertainty quantification. While Theorem 2 implies that the full SB-GAM posterior potentially provides valid inference, computationally efficient posterior sampling in high dimensions is a very challenging task. We will address fast and scalable inference with the full SB-GAM posterior in future work.

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A Additional Computational Details

A.1 Derivation of the EM Algorithm

Define the function $\xi = (g \circ b')^{-1}$, where $b$ is the cumulant function in (1.1) and $g$ is the link function in (1.2). Under (1.1)-(1.2) and the approximation of the $f_j$’s with basis expansions in (2.2), the log-likelihood of the data (up to an additive constant) is

$$
\ell(\mu, \beta) = \frac{1}{\tau} \sum_{i=1}^{n} [y_i \theta_i - b(\theta_i)] \\
= \frac{1}{\tau} \sum_{i=1}^{n} [y_i \xi(\mu + \sum_{j=1}^{p} \bar{x}_{ij} \beta_j) - b(\xi(\mu + \sum_{j=1}^{p} \bar{x}_{ij} \beta_j))]. 
$$

(A.1)

Recall that $\Xi$ denotes the collection $\{\mu, \beta, \kappa\}$. With the reparameterization of the spike-and-slab group lasso (SSGL) prior $SSGL(\lambda_0, \lambda_1, \kappa)$ in (4.1), we can write the log-posterior for SB-GAM under priors (2.3)-(2.5) as

$$
\log \pi(\mu, \beta, \kappa | y) = \ell(\mu, \beta) + \sum_{j=1}^{p} \log \left( (1 - \nu_j) \lambda_0^d e^{-\lambda_0 \|\beta_j\|^2} + \nu_j \lambda_1^d e^{-\lambda_1 \|\beta_j\|^2} \right) \\
+ \left( a - 1 + \sum_{j=1}^{p} \nu_j \right) \log \kappa + \left( b - 1 + p - \sum_{j=1}^{p} \nu_j \right) \log(1 - \kappa). 
$$

(A.2)

From (A.2), it is straightforward to verify that $E[\nu_j | y, \Xi] = p_j^*(\beta_j, \kappa)$, where $p_j^*(\beta_j, \kappa)$ is as in (4.2).

In the E-step our EM algorithm, we treat $\nu$ as missing data and take expectation with respect to $\nu_j, j = 1, \ldots, p$, holding the parameters $\Xi$ fixed.
at their previous values. That is, we compute
\[ p^*_j := p^*(\beta_j^{(t-1)}), \]
where \[ \nu_j \mid y, \Xi^{(t-1)}, j = 1, \ldots, p, \]
given the previous estimate \[ \Xi^{(t-1)} \].

For the M-step, we then maximize the following function:
\[ \mathbb{E}[\log \pi(\Xi \mid y) \mid \Xi^{(t-1)}] = \ell(\mu, \beta) - \sum_{j=1}^p \lambda^*_j \| \beta_j \|_2 \]
\[ + \left( a - 1 + \sum_{j=1}^p p^*_j \right) \log \kappa + \left( b - 1 + p - \sum_{j=1}^p p^*_j \right) \log(1 - \kappa), \] (A.3)

where \[ \lambda^*_j = \lambda_1 p^*_j + \lambda_0 (1 - p^*_j) \]. From (A.3), it is easy to derive the update for \[ \kappa \] as in (4.3) by taking the derivative of (A.3) with respect to \[ \kappa \]. The update for \( (\mu, \beta) \) in (4.4) is obtained by isolating the terms on the right-hand side of (A.3) that only depend on \( (\mu, \beta) \). We discuss the M-step for \( (\mu, \beta) \) in detail in the next section.

### A.2 M-step for Updating the Grand Mean and Regression Coefficients

In the M-step of our EM algorithm, we need to solve the optimization problem (4.4), or equivalently,
\[ (\mu^{(t)}, \beta^{(t)}) = \arg \min_{\mu, \beta} \left\{ -\ell(\mu, \beta) + \sum_{j=1}^p \lambda^*_j \| \beta_j \|_2 \right\}, \] (A.4)
where \(-\ell(\mu, \beta)\) is the negative of the log-likelihood in (A.1). While (A.4) may appear to be intractable, we can apply the standard iterated reweighted least squares (IRLS) algorithm [12] for generalized linear models (GLMs) to efficiently solve (A.4).

The IRLS algorithm in GLMs is based on a quadratic approximation of the negative log-likelihood. Let \[ U = [\mu_1, \ldots, \mu_n, \bar{X}] \] and \[ \gamma = (\mu, \beta)' \]. Denote the mean response as \( \mu_i = b'(\theta_i) \), and let \( V(\mu_i) = b''((b')^{-1}(\mu_i)) \) be the variance function, where \( b \) is the cumulant function in (1.1). Letting \( g \) be the link function in (1.2), we define the “working response” vector \( z \) at the \( k \)th iteration of the optimization problem (A.4) as \( z = U\gamma^{(k)} + \zeta^{(k)} \), where \( \zeta^{(k)} = g'(\mu_i^{(k)})(y_i - \mu_i^{(k)}), i = 1, \ldots, n \). Similarly, we define the weights matrix \( W = \text{diag}(w_1^{(k)}, \ldots, w_n^{(k)}) \), where the weights are \( w_i^{(k)} = [V(\mu_i^{(k)})(g'(\mu_i^{(k)}))^2]^{-1} \). As shown in [12], the negative log-likelihood for any GLM in the exponential dispersion family (1.1) can then be approximated as
\[ -\ell(\gamma) \approx \frac{1}{2} (z - U\gamma)' W (z - U\gamma). \] (A.5)
Thus, substituting the approximation (A.5) for $-\ell(\mu, \beta)$ in (A.4), we see that the M-step (A.4) for $\gamma = (\mu, \beta)$ can alternatively be written as

$$
\gamma^{(t)} = \arg\min_{\gamma} \left\{ \frac{1}{2}(z - U\gamma)'W(z - U\gamma) + \sum_{j=1}^{p} \lambda_j^{*}\|\beta_j\|_2 \right\}.
$$

(A.6)

With the quadratic approximation to the negative log-likelihood, (A.6) is now a standard group penalized linear regression model with weights $\lambda_j^{*}$ for each group of basis coefficients $\beta_j, j = 1, \ldots, p$. Meanwhile, the intercept $\mu$ is not penalized. Solving (A.6) can be done with any standard block coordinate descent algorithm for the regular group lasso [28] in linear regression. In particular, if the canonical link function $g = (b')^{-1}$ is used, then the Fisher information matrix and the Hessian matrix are equal, and we use the majorization-minimization (MM) algorithm for group lasso in [4] to solve (A.6). On the other hand, if a non-canonical link function is used, then we use the least squares approximation (LSA) approach of [21] with a group lasso penalty.

**B Negative Binomial Regression with a Non-Canonical Link**

Since the mean and variance are equal for the Poisson distribution, Poisson regression may be inappropriate for modeling count data with many zeros. In this case, it may be more appropriate to use negative binomial regression; that is, we assume that $y_i \sim \text{NegativeBinomial}(\alpha, \mu_i)$, where the probability density function (pdf) for each $y_i, i = 1, \ldots, n$, is

$$
f(y_i | \alpha, \mu_i) = \frac{\Gamma(y_i + \alpha)}{y_i!\Gamma(\alpha)} \left( \frac{\mu_i}{\mu_i + \alpha} \right)^{y_i} \left( \frac{\alpha}{\mu_i + \alpha} \right)^{\alpha}, \quad y_i = 0, 1, 2, \ldots
$$

and we assume the size parameter $\alpha > 0$ is known. Since $\text{Var}(y_i) = \mu_i + \mu_i^2/\alpha$, the negative binomial distribution is better equipped to handle zero-inflated count data than the Poisson distribution. For negative binomial responses, the natural parameter is $\theta_i = \log(\mu_i/(\mu_i + \alpha))$. With the log link function, we have $b(u) = -\alpha \log(1 - e^u)$ and $\tau = 1$ in (1.1) and $g(u) = \log(u)$ in (1.2). It is readily seen that $E(y_i) = b'(\theta_i) = \mu_i$. Defining the function $\xi = (g \circ b')^{-1}$, we have $\xi(u) = -\log(\alpha e^{-u} + 1)$. Since $\xi(u) \neq u$, negative binomial regression with the log link is an example of a regression model with a non-canonical link function.
We now present simulation results in negative binomial regression for the SB-GAM model, as well as GAMs with the group lasso penalty (gLASSO-GAM), GAMs with the group smoothly clipped absolute deviation penalty (gSCAD-GAM), and GAMs with the minimax concave penalty (gMCP-GAM). We fixed the size parameter $\alpha = 1$. We set $n = 500$, $p = 50$, and used $d = 6$ B-spline basis functions to approximate the $f_j$'s. We generated the $p$ independent covariates from a standard uniform distribution. We then generated the responses as $y_i \sim \text{NegativeBinomial}(1, \mu_i)$, where

$$\log(\mu_i) = 1.5 \sin(2\pi x_{i1}) - \cos(\pi x_{i2}) + e^{x_{i3}} - 2x_{i4}^2.$$  \hspace{1cm} (B.1)

Similarly as in our experiments for logistic GAMs and Poisson GAMs in Section 5.1 of the main manuscript, we computed the out-of-sample mean squared error (MSE) on a new data set and the Matthews correlation coefficient (MCC) to gauge estimation and variable selection performance. To assess predictive accuracy, we computed the mean squared prediction error (MSPE) as $\text{MSPE} = n^{-1} \sum_{i=1}^{n} [y_{i,\text{new}} - \exp(\hat{\mu}_{i,\text{new}})]^2$. Models with lower MSE, higher MCC, and lower MSPE were preferred.

Figure B.1 shows our simulation results for 100 replications of negative binomial GAMs with the log link. We see that SB-GAM had the lowest median MSE and MSPE, indicating the best estimation of the additive function $f(x) = \mu + \sum_{j=1}^{p} f_j(x_j)$ and best prediction of the response $y$. For variable selection, SB-GAM also outperformed its competitors, with a median MCC of one.
C Proofs of Main Results

We use the following notation. For two densities \( p \) and \( q \), we let \( K(p, q) = \int \log(p/q) \) and \( V(p, q) = \int f|\log(p/q) - K(p, q)|^2 \) denote the Kullback-Leibler (KL) divergence and variation respectively. We also define the \( \varepsilon \)-covering number for a set \( \Omega \) with semimetric \( d \) as the minimum number of \( d \)-balls of radius \( \varepsilon \) needed to cover \( \Omega \) and denote the \( \varepsilon \)-covering number as \( N(\varepsilon, \Omega, d) \) and the metric entropy as \( \log N(\varepsilon, \Omega, d) \).

Recall that for additive functions \( f \) and \( f_0 \),
\[
 f(x_i) = \mu + \sum_{j=1}^{p} f_j(x_{ij}) \quad \text{and} \quad f_0(x_i) = \mu_0 + \sum_{j=1}^{p} f_{0j}(x_{ij}).
\]
\( \mathcal{F} \) is the set of all functions \( f \) where each additive component \( f_j \) can be written as a \( d \)-dimensional basis expansion. For notational convenience, we denote \( \eta_i = f(x_i) \) and \( \eta_{0i} = f_0(x_i) \).

C.1 Proofs for Theorems 1 and 2

Our proofs for Theorems 1 and 2 make use of the foundational theory of Ghosal and van der Vaart [5]. We synthesize recent theoretical results for Gaussian GAMs by Bai et al. [2] and high-dimensional generalized linear models by Jeong and Ghosal [7] to derive the posterior contraction rate for GAMs where the response is not necessarily Gaussian. The proofs can be sketched as follows. We first establish that the posterior concentrates on sparse sets whose sizes do not exceed a constant multiple of the true model size \( s_0 \). Then we prove posterior contraction with respect to the root-average squared Hellinger metric, which we then relate to posterior contraction in the empirical \( \ell^2 \) norm.

Lemma C.1 (evidence lower bound). Suppose that the conditions of Theorem 1 hold. Then \( \sup_{f_0 \in \mathcal{H}(s_0, \alpha)} P_0(\mathcal{E}_n^c) \to 0 \), where the set \( \mathcal{E}_n \) is
\[
 \mathcal{E}_n \equiv \left\{ \int \prod_{i=1}^{n} \frac{q_0(y_i)}{q_0(y_i)} d\Pi(q) \geq e^{-C_1 n \epsilon_n^2} \right\}, \tag{C.1}
\]
for some constant \( C_1 > 0 \) and \( \epsilon_n^2 = s_0 \log p/n + s_0 n^{-2\alpha/(2\alpha+1)} \).

Proof of Lemma C.1. By Lemma 10 of [5], this statement will be proven if we can show that
\[
 \Pi \left( \frac{1}{n} \sum_{i=1}^{n} K(q_0, q_i) \leq \epsilon_n^2, \frac{1}{n} \sum_{i=1}^{n} V(q_0, q_i) \leq \epsilon_n^2 \right) \geq e^{-C_1 n \epsilon_n^2}. \tag{C.2}
\]
Let \( B_n \) denote the event, \( B_n = \{ K(q_0, q) \leq n \epsilon_n^2, V(q_0, q) \leq n \epsilon_n^2 \} \). As established in Lemma 1 of [7], the KL divergence and variation for the exponential
dispersion family of distributions are given by

\[ K(q_0, q_i) = E_0 \log \left( \frac{q_0}{q_i} \right) = \frac{1}{\tau} \left[ (\theta_{0i} - \theta_i) b'(\theta_{0i}) - b(\theta_{0i}) + b(\theta_i) \right], \]

\[ V(q_0, q_i) = E_0 \left[ \left( \log \left( \frac{q_0}{q_i} \right) - K(q_0, q_i) \right)^2 \right] = \frac{b''(\theta_{0i})}{\tau^2} (\theta_i - \theta_{0i})^2. \]

Note that \( \theta_i = \xi(\eta_i) \) and \( \theta_{0i} = \xi(\eta_{0i}) \), and by Taylor expansion in \( \eta_i \) at \( \eta_{0i} \), we have

\[ \max\{K(q_0, q_i), V(q_0, q_i)\} \leq \frac{b''(\theta_{0i})(\xi'(\eta_{0i}))^2}{\tau^2} (\eta_i - \eta_{0i})^2 + o((\eta_i - \eta_{0i})^2). \]

Since \( f_0 \in \mathcal{H}(s_0, \alpha) \), we have that \( \|f_0\|_\infty < B \) for some constant \( B \). Thus, \( \theta_{0i} \) is uniformly bounded for all \( n \). This follows from the fact that \( |\theta_{0i}| = |\xi(\eta_{0i})| \leq \sup_{x \in [0,1]^p} |\xi(\mu + f(x))| \leq \sup_{x \in [0,1]^p} |\xi(\mu) + B| < \infty \), since \( \xi \) is continuous and strictly increasing, and \( |\mu| + B < \infty \). Coupled with Condition 3, we have that \( b''(\theta_{0i}) \) is uniformly bounded above by a constant. Similarly, one can also show that \( (\xi'(\eta_{0i}))^2 \) is bounded above by a constant for \( f_0 \in \mathcal{H}(s_0, \alpha) \). Therefore, both \( K(q_0, q_i) \) and \( V(q_0, q_i) \) can be bounded above by a constant multiple of \( (\eta_i - \eta_{0i})^2 \). Thus, we have for some constant \( b_1 > 0 \) and sufficiently large \( n \),

\[ \Pi(B_n) \geq \Pi \left( \sum_{i=1}^n (\eta_i - \eta_{0i})^2 \leq b_1^2 n \epsilon_n^2 \right) \]

\[ = \Pi_{\mu, \beta} \left( \| \mu_1 + \bar{X} \beta \| - (\mu_0 1_n + \bar{X} \beta_0 + \delta_0) \| \leq b_1^2 n \epsilon_n^2 \right) \]

\[ \geq \Pi_{\mu, \beta} \left( n |\mu - \mu_0|^2 + \| \bar{X} (\beta - \beta_0) - \delta_0 \|^2 \leq \frac{b_1^2 n \epsilon_n^2}{2} \right) \]

\[ \geq \Pi_\mu \left( |\mu - \mu_0| \leq \frac{b_1 \epsilon_n}{2} \right) \Pi_\beta \left( \| \bar{X} (\beta - \beta_0) - \delta_0 \|^2 \leq \frac{b_1^2 n \epsilon_n^2}{4} \right) \]

\[ \geq \exp \left( -\frac{C_1 n \epsilon_n^2}{2} \right) \Pi_\beta \left( \| \bar{X} (\beta - \beta_0) - \delta_0 \|^2 \leq \frac{b_1^2 n \epsilon_n^2}{4} \right). \quad (C.3) \]

for some \( C_1 > 0 \) is the appropriately chosen constant in the statement of the lemma. In the second line of the display, \( 1_n \) denotes the \( n \times 1 \) vector of all ones. The last line of the display follows from the fact that the prior on \( \mu \) is a Gaussian distribution, so we can easily show that the first probability in the second to last line is bounded below by \( \exp(-C_1 n \epsilon_n^2/2) \). We now focus on bounding the second probability in the final line of (C.3) below by
\[ \exp(-C_1 n r_n^2 / 2) \] as well. Let \( r_n^2 = \epsilon_n^2 - s_0 n^{-2\alpha/(2\alpha+1)} = s_0 \log p/n. \) For some \( b_2 > 0, \) we have

\[
\Pi_\beta \left( \| \bar{X}(\beta - \beta_0) - \delta_0 \|^2_2 \leq \frac{b_1 n \epsilon_n^2}{4} \right)
\]

\[
\geq \Pi_\beta \left( \| \bar{X}(\beta - \beta_0) \|^2_2 + \| \delta_0 \|^2_2 \leq \frac{b_1 n \epsilon_n^2}{8} \right)
\]

\[
\geq \Pi_\beta \left( \| \tilde{X} \|^2_2 \left( \sum_{j=1}^p \| \beta_j - \beta_{0j} \|^2 \right) \leq b_2^2 n r_n^2 \right)
\]

\[
\geq \Pi_\beta \left( \sum_{j=1}^p \| \beta_j - \beta_{0j} \|^2 \leq b_2^2 n^{1/(2\alpha+1)} r_n^2 \right)
\]

\[
\geq \Pi_\beta \left( \sum_{j=1}^p \| \beta_j - \beta_{0j} \|^2 \leq b_2 r_n \right)
\]

\[
\geq \int_0^1 \Pi_{S_0^c} \left( \sum_{j \in S_0} \| \beta_j - \beta_{0j} \|^2 \leq \frac{b_2 r_n}{2} \right) \Pi_{S_0^c} \left( \sum_{j \in S_0} \| \beta_j \|^2 \leq \frac{b_2 r_n}{2} \right) d\Pi(\kappa)
\]

\[
\geq \int_0^1 \left\{ \Pi_{S_0} \left( \| \beta_{S_0} - \beta_{0S_0} \|^2 \leq \frac{b_2^2 r_n^2}{4s_0} \right) \right\} \left\{ \Pi_{S_0^c} \left( \| \beta_{S_0^c} \|^2 \leq \frac{b_2^2 r_n^2}{4p s_0} \right) \right\} d\Pi(\kappa)
\]

\[
\geq \exp \left( -C_1 n \epsilon_n^2 / 2 \right). \tag{C.4}
\]

The third line of the display follows from Condition 2 and our assumption that \( d \asymp n^{1/(2\alpha+1)}, \) and the fourth line follows from Condition 1. The second to last line of the display follows from an application of the Cauchy-Schwarz inequality, and the final inequality can be obtained by suitably modifying the arguments used to prove (D.24) in the proof of Theorem 2 of Bai et al. [2] (note that since \( \| f_0 \| < \infty, \) it must be that \( \| \beta_0 \|_\infty < \infty). \) Combining (C.3)-(C.4) gives the lower bound (C.2), and so the lemma is proven.

Next, we prove that the SB-GAM posterior concentrates on sparse sets. Let \( S = \{ j \in [p] : \| \beta_j \|^2 > \omega_d \}, \) where \( \omega_d \) is the small threshold in (3.3). Let \( s = |S|. \) The next lemma will allow us to focus our attention only on models that do not overshoot the true dimension of \( f_0 \) as \( n, p \to \infty. \)

\textbf{Lemma C.2} (effective dimension). Suppose that the conditions of Theorem
2 hold. Then for sufficiently large $C_2 > 1$,

$$\sup_{f_0 \in \mathcal{H}(s_0, \alpha)} \mathbb{E}_0 \Pi(\beta : s > C_2 s_0 \mid y) \to 0,$$

as $n, p \to \infty$.

**Proof of Lemma C.2.** Let $\mathcal{E}_n$ as $\mathcal{E}_n = \{ \int \int \prod_{i=1}^{n} \frac{q_i(y_i)}{q_0(y_i)} d\Pi(\beta) d\Pi(\mu) \geq e^{-C_1 n e_n^2} \}$, where $e_n^2$ is the rate in Lemma C.1. In what follows, we work on the function space $\mathcal{H}(s_0, \alpha)$. Define $A_n = \{ \beta : s \leq C_2 s_0 \}$, where $C_2 > C_1 \vee 1$. Then we have

$$\mathbb{E}_0 \Pi(A_n^c \mid y) \leq \mathbb{E}_0 \Pi(A_n^c \mid y) 1_{\mathcal{E}_n} + \mathbb{P}_0(\mathcal{E}_n^c).$$

By Lemma C.1, $\mathbb{P}_0(\mathcal{E}_n^c) \to 0$ as $n \to \infty$, so to prove that $\mathbb{E}_0 \Pi(A_n^c \mid y) \to 0$, it suffices to show that $\mathbb{E}_0 \Pi(A_n^c \mid y) 1_{\mathcal{E}_n} \to 0$. Now,

$$\Pi(A_n^c \mid y) = \frac{\int \int \prod_{i=1}^{n} \frac{q_i(y_i)}{q_0(y_i)} d\Pi(\beta) d\Pi(\mu)}{\int \int \prod_{i=1}^{n} \frac{q_i(y_i)}{q_0(y_i)} d\Pi(\beta) d\Pi(\mu)}. \quad (C.5)$$

On the event $\mathcal{E}_n$, the denominator in (C.5) is bounded below by $e^{-C_1 n e_n^2}$. On the other hand, an upper bound for the expected value of the numerator is

$$\mathbb{E}_0 \left( \int \int \prod_{i=1}^{n} \frac{q_i(y_i)}{q_0(y_i)} d\Pi(\beta) d\Pi(\mu) \right) \leq \int \int d\Pi(\beta) = \Pi(\beta : s > C_2 s_0). \quad (C.6)$$

Using very similar arguments as those used to prove (D.34) in the proof of Theorem 2 in Bai et al. [2] (and noting that for $f_0 \in \mathcal{H}(s_0, \alpha)$, $\|\beta_0\| < \infty$), we have

$$\Pi(\beta : s > C_2 s_0) \prec e^{-C_2 n e_n^2}. \quad (C.7)$$

Combining (C.6)-(C.7), we have that $\mathbb{E}_0 \Pi(A_n^c \mid y) 1_{\mathcal{E}_n} \prec e^{-C_2 - C_1 n e_n^2} \to 0$ since $C_2 > C_1$. This completes the proof. \qed

We now have the ingredients to prove Theorem 1, i.e. posterior contraction with respect to the average squared Hellinger metric $H_n^2(q, q_0)$ for the joint densities $q = \prod_{i=1}^{n} q_i$ and $q_0 = \prod_{i=1}^{n} q_0$. \[fo5]

**Proof of Theorem 1.** Let $\mathcal{M}_n = \{ s \leq C_2 s_0, \lambda_0 \geq p^2 \}$, where $C_2$ is the constant in Lemma C.2. Below, we work on the function space $\mathcal{H}(s_0, \alpha)$. Then for every $\epsilon > 0$,

$$\mathbb{E}_0 \Pi(H_n(q, q_0) > \epsilon \mid y)$$

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\[
\leq \mathbb{E}_0 \Pi(\beta \in \mathcal{M}_n : H_n(q, q_0) > \epsilon \mid y) \mathbb{1}_{\mathcal{E}_n} + \mathbb{E}_0 \Pi(\mathcal{M}_n^* \mid y) + \mathbb{P}_0 \mathcal{E}_n^c.
\]

(C.8)

where \( \mathcal{E}_n \) is the event in (C.1). By Lemma C.1, the final term on the right-hand side of (C.8) tends to zero uniformly over \( \mathcal{H}(s_0, \alpha) \). We now focus on the second term on the right-hand side in (C.8). Note that

\[
\mathbb{E}_0 \Pi(\mathcal{M}_n^* \mid y) \leq \mathbb{E}_0 \Pi(s > M_1 s_0) + \Pi(\lambda_0 < p^2).
\]

By Lemma C.2, the first term on the right-hand side of the display goes to zero. To bound \( \Pi(\lambda_0 < p^2) \) from above, first note that \( n\epsilon_n^2/p^2 \leq 1/(p^2 + 1) \), by the fact that \( c > 2 \) and \( n\epsilon_n^2 \leq p^{-2} \). Then using the fact that \( \kappa \sim \mathcal{B}(1, p^c) \), we have for sufficiently large \( n \),

\[
\Pi \left( \kappa > \frac{1}{p^2 + 1} \right) \leq \Pi \left( \kappa > \frac{n\epsilon_n^2}{p^c} \right) = \left( 1 - \frac{n\epsilon_n^2}{p^c} \right)^{p^c} \leq e^{-n\epsilon_n^2},
\]

Noting that \( \lambda_0 = (1 - \kappa)/\kappa = (1/\kappa) - 1 \), this gives \( \Pi(\lambda_0 < p^2) \leq e^{-n\epsilon_n^2} \). Therefore, \( \Pi(\lambda_0 < p^2) \) also goes to zero, and \( \mathbb{E}_0 \Pi(\mathcal{M}_n^* \mid y) \rightarrow 0 \). Hence, it suffices to show that the first term in the right-hand side of (C.8) goes to zero for \( \epsilon = C_4 \epsilon_n \) uniformly over \( \mathcal{H}(s_0, \alpha) \), for some sufficiently large \( C_4 \).

Define the event, \( \mathcal{M}_n^* = \{ (\mu, \beta) : |\mu| \leq \sqrt{n}, \beta \in \mathcal{M}_n, \max_{1 \leq j \leq p} \| \beta_j \|_2 \leq nd/\lambda_1 \} \), where \( \lambda_1 \) is the slab hyperparameter in the \( \text{SSGL}(\lambda_0, \lambda_1, \kappa) \) prior. By Lemma 2 of Ghosal and van der Vaart [5], there exists a test \( \varphi_n \) such that for any \( q_1 \) with \( H_n(q_0, q_1) < \epsilon \),

\[
\mathbb{E}_0 \varphi_n \leq \exp(-n\epsilon_n^2/2), \quad \sup_{q : H_n(q, q_1) \leq \epsilon/18} \mathbb{E}_q (1 - \varphi_n) \leq \exp(-n\epsilon_n^2/2).
\]

We now show that \( \log N(\epsilon_n/36, \mathcal{M}_n^*, H_n) \approx n\epsilon_n^2 \), so that we can use Lemma 9 of Ghosal and van der Vaart [5] to finish the proof. Suppose that \( q_{1i} \) and \( q_{2i} \) are two densities indexed by functions, \( f_1 \) and \( f_2 \) in \( \mathcal{H}(\alpha, s_0) \). Using the fact that the squared Hellinger distance is bounded from above by the KL divergence and the fact that the KL divergence for any univariate densities \( q_{1i} \) and \( q_{2i} \) can be bounded above by a constant multiple of \( (\eta_{1i} - \eta_{2i}) \) (from the proof of Lemma C.1), we then bound the average squared Hellinger metric for joint densities, \( q_1 = \prod_{i=1}^n q_{1i} \) and \( q_2 = \prod_{i=1}^n q_{2i} \), on \( \mathcal{M}_n^* \) as follows:

\[
H_n^2(q_1, q_2) \lesssim \frac{1}{n} \sum_{i=1}^n (\eta_{1i} - \eta_{2i})^2 = \frac{1}{n} \| (\mu_11_n + \bar{X}\beta_1) - (\mu_21_n + \bar{X}\beta_2) \|^2_2.
\]
\[
\leq 2|\mu_1 - \mu_2|^2 + \frac{2}{n} \|\overline{X}(\beta_1 - \beta_2)\|^2
\]
\[
\lesssim |\mu_1 - \mu_2|^2 + \frac{\|\overline{X}\|^2}{n} \left( \sum_{j=1}^{p} \|\beta_{1j} - \beta_{2j}\|_2 \right)^2
\]
\[
\lesssim |\mu_1 - \mu_2|^2 + \frac{d^2 s^2_{\beta_1 - \beta_2}}{d^{2\alpha}} \|\beta_1 - \beta_2\|_\infty^2
\]
\[
\lesssim |\mu_1 - \mu_2|^2 + C_2 s^2_0 \|\beta_1 - \beta_2\|_\infty^2,
\] (C.9)

where we used Condition 1, an application of the Cauchy-Schwarz inequality, and the fact that for \(v \in \mathbb{R}^q, \|v\|_2 \leq q^{1/2}\|v\|_\infty\). In the fifth line of the display, we also used the fact that \(\lambda_0 \geq p^2\) on \(\mathcal{M}_n\), so for any \(j \notin S_{\beta_1 - \beta_2}\), \(\|\beta_j\|_2 \leq \omega_d \ll 1/p\). Thus, the overall contribution of the \(\beta_j\)'s for \(j \notin S_{\beta_1 - \beta_2}\) satisfies \(\|\beta_{S_{\beta_1 - \beta_2}} - \beta_{S_{\beta_1 - \beta_2}}\|_2 \leq 2(p - s_{\beta_1 - \beta_2})\omega_d = o(1)\). In the final line of the display, we used the facts that \(\alpha \in \mathbb{N}, d > 1\), and \(s \leq C_2 s_0\) on \(\mathcal{M}_n\).

Using the upper bound in (C.9), we have for some \(b_3 > 0\) that an upper bound for \(\log N(\epsilon_n/36, \mathcal{M}_n^*, H_n)\) is
\[
\log N \left( b_3 \epsilon_n, \left\{ \mu : |\mu| < \sqrt{n}, |\cdot| \right\} \right) + \log N \left( \frac{b_3 \epsilon_n}{36 C_2 s_0}, \left\{ \beta \in \mathcal{M}_n : \|\beta - \beta_0\|_\infty \leq \frac{nd}{\lambda_1} \right\}, \|\cdot\|_\infty \right).
\] (C.10)

Clearly, the first term in (C.10) is bounded above by a constant multiple of \(n \epsilon_n^2\). To bound the second term in (C.10) from above, first note that
\[
N \left( \frac{b_3 \epsilon_n}{36 C_2 s_0}, \left\{ \beta \in \mathcal{M}_n : \|\beta - \beta_0\|_\infty \leq \frac{nd}{\lambda_1} \right\}, \|\cdot\|_\infty \right)
\leq \sum_{S, |S| \leq C_2 s_0} N \left( \frac{b_3 \epsilon_n}{36 C_2 s_0}, \left\{ \beta \in \mathcal{M}_n : |S| = s, \|\beta - \beta_0\|_\infty \leq \frac{nd}{\lambda_1} \right\}, \|\cdot\|_\infty \right)
\leq \sum_{s=0}^{[C_2 s_0]} \binom{p}{s} \left( \frac{108 C_2 s_0 n d}{b_3 \epsilon_n \lambda_1} \right)^s
\leq \left( \frac{108 C_2 s_0 n d}{b_3 \epsilon_n \lambda_1} \right)^{C_2 s_0} \sum_{s=0}^{[C_2 s_0]} \binom{p}{s}
\leq (C_2 s_0 + 1) \left( \frac{p}{[C_2 s_0]} \right) \left( \frac{108 C_2 s_0 n d}{b_3 \epsilon_n \lambda_1} \right)^{C_2 s_0}.
\] (C.11)

In the third line of the display, we used the fact that \(s \leq C_2 s_0\) and \(\lambda_0 \geq p^2\) on \(\mathcal{M}_n\), so for \(j \notin S\), \(\|\beta_j\|_2 \leq \omega_d \ll 1/p\); thus, the contribution to \(\|\beta - \beta_0\|_\infty\) from \(\beta_{S_{\beta_1 - \beta_2}}\) tends to zero as \(n, p \to \infty\). In the final line of the display, we used
the fact that $\sum_{i=k}^{n} i^n \leq (m - k + 1)n^m$ if $n > m$. Following from (C.11),
the second term in (C.10) can be bounded above as
\[
\log N \left( \frac{b_3\epsilon_n}{36C_2s_0}, \left\{ \beta \in M_n : \| \beta - \beta_0 \| \leq \frac{nd}{\lambda_1} \right\}, \| \cdot \| \right) 
\lesssim \log (C_2s_0 + 1) + C_2s_0 \log p + C_2ds_0 \log \left( \frac{108C_2s_0nd}{b_3\epsilon_n\lambda_1} \right) 
\lesssim \log (C_2s_0 + 1) + C_2s_0 \log p + C_2ds_0 \log \left( \frac{108C_2s_0^{1/2}n^{5/2}d}{b_3(\log p)^{1/2}} \right) 
\lesssim \log (C_2s_0 + 1) + C_2s_0 \log p + C_2ds_0 \log n 
\lesssim n\epsilon_n^2. \tag{C.12}
\]
for some $C_2 > 0$. In the second line of the display, we used the fact that \( \binom{p}{C_2s_0} \leq p^{C_2s_0} \). In the third line, we used our assumption that $\lambda_1 > 1/n$ and the fact that $\epsilon_n > \sqrt{s_0 \log p}/n$. In the fourth line, we used the fact that the final term the third line of the display is of the same order as $ds_0 \log n$. In the final line, we used the fact that $d \log n = O(\log p)$ by assumption and the fact that the first term in the fourth line is dominated by the other two terms. From (C.10)-(C.12), $\log N(\epsilon_n/36, M_n^*, H_n) \lesssim n\epsilon_n^2$. Now, we can use Lemma 9 of Ghosal and van der Vaart [5], which implies that for every $\epsilon > \epsilon_n$, there exists a test $\overline{\nu}_n$ and constant $b_4$ such that
\[
\mathbb{E}_0\overline{\nu}_n \leq \frac{1}{2} \exp(b_4n\epsilon_n^2 - n\epsilon^2/2),
\sup_{(\mu, \beta) \in M_n^*: \|H_n(q, q_0)\| > \epsilon} \mathbb{E}_0(1 - \overline{\nu}_n) \leq \exp(-n\epsilon^2/2).
\]
Thus, for some constant $b_5 > 0$, we have
\[
\mathbb{E}_0\Pi(\beta \in M_n : H_n(q, q_0) > \epsilon | y) 1\epsilon_n 
\leq \mathbb{E}_0\Pi(\beta \in M_n : H_n(q, q_0) > \epsilon | y) 1\epsilon_n (1 - \overline{\nu}_n) + \mathbb{E}_0\overline{\nu}_n 
\leq \left\{ \sup_{(\mu, \beta) \in M_n^*: \|H_n(q, q_0)\| > \epsilon} \mathbb{E}_0(1 - \overline{\nu}_n) + \Pi(M_n \setminus M_n^*) \right\} e^{b_5n\epsilon_n^2} + \mathbb{E}_0\overline{\nu}_n.
\]
The terms in the display other than $\Pi(M_n \setminus M_n^*)e^{b_5n\epsilon_n^2}$ all go to zero uniformly over $H(s_0, \alpha)$ by choosing $\epsilon = b_6\epsilon_n$ for a sufficiently large $b_6$. To complete the proof, note that
\[
\Pi(M_n \setminus M_n^*) \leq \Pi_{\mu} (|\mu| > \sqrt{n}) + \sum_{S: s \leq C_2s_0} \sum_{j \in S} \Pi(\| \beta_j \|_2 > nd/\lambda_1) 
\leq e^{-n/2} + \sum_{S: s \leq C_2s_0} \sum_{j \in S} \Pi(\| \beta_j \|_2 > nd/\lambda_1)
\]
\[ < e^{-b_5 n e_n^2}, \]  

where the second line follows from the fact that \( \mu \) has a Gaussian prior. The final line follows directly from the arguments used to prove (E.14) in the proof of Theorem 2 of Bai et al. [1] (and thus, the second term in the second line is less than \( e^{-b_5 n e_n^2} \) for large \( n \), as well as the fact that \( n e_n^2 = o(n) \), so \( e^{-n/2} < e^{-b_5 n e_n^2} \) for large \( n \). Therefore, \( \Pi(M_n \setminus M^*_n) e^{b_5 n e_n^2} \) also goes to zero uniformly over \( \mathcal{H}(s_0, \alpha) \), and this completes the proof. \( \square \)

Having established posterior contraction in the average squared Hellinger metric, we proceed to prove Theorem 2.

**Proof of Theorem 2.** Note that since \( s_0 = o((n/d \log p)^{1/2}) \) by assumption, \( s_0 = o(n/\log p) \). By Lemma C.2, the posterior is asymptotically supported on the event \( \mathcal{A}_n = \{ \beta : s \leq C_2 s_0 \} \), so we can confine our attention to the set \( \mathcal{A}_n \). Let \( N_{n \times (d p + 1)} = [1_n, \tilde{X}] \), \( \gamma = (\mu, \beta_1', \ldots, \beta_p') \), and \( \gamma_0 = (\mu_0, \beta_0', \ldots, \beta_{0p}') \). Let \( f_{0j}(x_j) \) denote the \( n \times 1 \) vector \( (f_{0j}(x_{1j}), \ldots, f_{0j}(x_{nj}))' \), for all \( j = 1, \ldots, p \). Note that in order to prove Theorem 2, we can equivalently show that for \( f_0 \in \mathcal{H}(s_0, \alpha) \),

\[
\mathbb{E}_0 \Pi \left( (\mu, \beta) : \frac{1}{n} \| (\mu 1_n + \tilde{X} \beta) - (\mu_0 1_n + \sum_{j=1}^{p} f_{0j}(x_j)) \|_2^2 > M_2 e_n^2 | y \right) \mathbb{I}_{\mathcal{A}_n} \\
= \mathbb{E}_0 \Pi \left( \gamma : \| W(\gamma - \gamma_0) - \delta_0 \|_2^2 > M_2 n e_n^2 | y \right) \mathbb{I}_{\mathcal{A}_n} \rightarrow 0,
\]

as \( n, p \rightarrow \infty \). The rest of the proof follows from a suitable modification of the proof of Theorem 3 in [7]. Let us first introduce some notation. We define the uniform compatibility number as

\[
\phi_1(s) = \inf_{\gamma : \|S\| \leq s} \frac{\|W\gamma\|_2^2 (d|S|)^{1/2}}{n^{1/2} \|\gamma\|_1}.
\]

Also, let \( \zeta_i(\eta_i) = H^2(q_i, q_{0i}) \), and define the set \( I_n = \{ 1 \leq i \leq n : \delta_1 (\eta_i - \eta_{0i})^2 \geq \delta_2 \} \), where \( \delta_1, \delta_2 > 0 \) are constants that satisfy \( \zeta_i(\eta_i) \geq \zeta''_i(\eta_{0i}) \min\{\delta_1 (\eta_i - \eta_{0i})^2, \delta_2 \} \).

By Theorem 1, the posterior is asymptotically supported on the event \( \{ H_n(q, q_0) \leq \epsilon_n^2 \} \) for joint densities \( q = \prod_{i=1}^{n} q_i \) and \( q_0 = \prod_{i=1}^{n} q_{0i} \), where the \( q_i \)'s and \( q_{0i} \)'s belong to the exponential dispersion family (1.1). From (A2) in the proof of Theorem 3 in [7], we have

\[
\epsilon_n^2 \geq H_n(q, q_0) \geq \frac{\delta_1}{n} \sum_{i \in I_n} \zeta''_i(\eta_{0i}) + \frac{\delta_2}{n} \sum_{i \notin I_n} \zeta''_i(\eta_{0i})(\eta_i - \eta_{0i})^2
\]

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\[
\geq \frac{1}{n} \sum_{i \in I_n, \delta} \zeta''(\eta_{0i})(\eta_i - \eta_{0i})^2
\geq \frac{1}{n} \sum_{i=1}^n \zeta''(\eta_{0i})(\eta_i - \eta_{0i})^2 - \frac{1}{n} \sum_{i \in I_n} \zeta''(\eta_{0i}) \max_{1 \leq i \leq n} (\eta_i - \eta_{0i})^2.
\]

(C.15)

But \( \zeta''(\eta_{0i}) = b''(\theta_{0i})(\xi'(\eta_{0i}))/4 \), and \( |\theta_{0i}| \) and \( |\eta_{0i}| \) are bounded by Condition 3, so we have \( |\zeta''(\eta_{0i})| \leq 1 \), for all \( i = 1, \ldots, n \). Thus, from (C.15), we have that for some constant \( C_3 > 0 \),

\[
\epsilon_n^2 \geq \frac{1}{n} \sum_{i=1}^n (\eta_i - \eta_{0i})^2 - \frac{1}{n} \sum_{i \in I_n} \max_{1 \leq i \leq n} (\eta_i - \eta_{0i})^2
\geq \frac{1}{n} \| W(\gamma - \gamma_0) - \delta_0 \|_2^2 - C_3 \epsilon_n^2 \left( \| W \|_{\text{max}}^2 \| \gamma - \gamma_0 \|_1^2 + \| \delta_0 \|_\infty^2 \right)
\geq \frac{1}{n} \| W(\gamma - \gamma_0) \|_2^2 - \frac{1}{n} \| \delta_0 \|_2^2 - C_3 \epsilon_n^2 \left( \| W \|_{\text{max}}^2 \| \gamma - \gamma_0 \|_1^2 + \| \delta_0 \|_\infty^2 \right).
\]

(C.16)

By Condition 2 and the fact that \( d \asymp n^{1/(2\alpha + 1)} \), \( \epsilon_n^2 + n^{-1} \| \delta_0 \|_2^2 \) is of the same order as \( \epsilon_n^2 \). Also, \( \| \delta_0 \|_\infty^2 \leq s_0 d^{-2\alpha} < \epsilon_n^2 \), and by assumption, \( \| W \|_{\text{max}} \leq 1 \). Thus, from (C.16) and the definition of the compatibility number, we have that on set \( A_n \),

\[
\epsilon_n^2 \geq \frac{\phi_n^2 (2C_2 s_0)}{8C_2 d s_0} \| \gamma - \gamma_0 \|_1^2 - C_4 \epsilon_n^2 \| \gamma - \gamma_0 \|_1^2 - C_5 \epsilon_n^4
\geq \frac{1}{8C_2 d s_0} \| \gamma - \gamma_0 \|_1^2 - C_4 \epsilon_n^2 \| \gamma - \gamma_0 \|_1^2 - C_5 \epsilon_n^4,
\]

(C.17)

for some \( C_4, C_5 > 0 \). In the second line of the display, we used the fact that \( \| W \gamma_0 \|_\infty \lesssim 1 \) since \( f \in H(\alpha, s_0) \), and therefore, on \( A_n \), \( \phi_1 (2C_2 s_0) \) is bounded away from zero. By assumption, \( s_0^3 = o(n/d \log p) \), and so (C.17) can be further bounded below by a constant multiple of \( \| \gamma - \gamma_0 \|_1^2/d s_0 \). This implies that for some \( M_3 > 0 \),

\[
\sup_{f \in H(s_0, \alpha)} E_0 \Pi \left( \gamma : \| \gamma - \gamma_0 \|_1^2 \leq M_3 d s_0 \epsilon_n^2 \mid \eta \right) 1_{A_n} \rightarrow 1,
\]

(C.18)

as \( n, p \rightarrow \infty \). Combining results from (C.16)-(C.18), we also have that on \( A_n \) and some \( C_6 > 0 \),

\[
\| W(\gamma - \gamma_0) - \delta_0 \|_2^2 \lesssim n \epsilon_n^2 + C_4 n \epsilon_n^2 \| \gamma - \gamma_0 \|_1^2 + C_5 n \epsilon_n^4
\]
≤ nε₂ + C₀ds₀nε₄ + C₅nε₄
≤ nε₂
\hspace{1cm} (C.19)

Since the expected posterior probability of the event \{W(γ − γ₀) − δ₀∥₂ \lessapprox nε₂\} tends to one as \(n, p \to \infty\) on \(A_n\), this proves (C.14). \(\square\)

### C.2 Proofs for Theorem 3

To prove Theorem 3, we utilize the framework of Yang and Pati [25]. Yang and Pati [25] provided general conditions for proving Bayesian model selection consistency when the marginal likelihood is intractable. We let

\[ S = \{ j \in [p] : \|β_j\|_2 > ω_d \} \]

Meanwhile, \(S₀ \subset [p]\) denotes the true model. For a generic set \(S\) with cardinality \(s = |S|\), associate with it the rate \(ε_{n,s} = s \log p/n + sn^{-2α/(2α+1)}\). Thus, \(ε_{n,s₀}\) is the same as the posterior contraction rate derived in Theorems 1 and 2.

**Proof of Theorem 3.** It suffices to verify conditions (B1)-(B4) in Theorem 4 of Yang and Pati [25]. First, note that Condition 4 in the present manuscript corresponds to condition (B3) in [25] for identifiability. Therefore, the condition (B3) in [25] is already satisfied by assumption. We now verify their other conditions (B1), (B2), and (B4) in Parts I, II, and III respectively.

**Part I: Prior concentration.** As per condition (B1) in [25], we need to verify that for large \(n\),

\[ Π(S₀) \geq e^{-nε_{n,s₀}^2}, \hspace{1cm} (C.20) \]

and

\[ Π_S₀(KL(q₀, q) \leq nε_{n,s}^2, V(q₀, q) \leq nε_{n,s₀}^2) \geq e^{-Dnε_{n,s₀}^2}, \hspace{1cm} (C.21) \]

for some constant \(D\). The conditions (C.20)-(C.21) ensure that the prior puts enough mass near the true model \(S₀\) and enough prior concentration in the parameter space under the true model. By the proof of Theorem 1, \(P₀(Mₙ \mid y) \to 0\), where \(Mₙ = \{s \leq C₂s₀, λ₀ \geq p²\}\) and \(C₂ > 1\). Therefore, we can proceed conditioning only on models of size \(s \leq C₂s₀\) and spike parameter \(λ₀ \geq p²\). For a \(d\)-dimensional vector \(β_j\), denote \(\bar{p} = Pr(β_j : \|β_j\|_2 > ω_d)\). The implied prior for any given model \(S\) of size \(s\) is then

\[ Π(S) \propto \bar{p}^{s}(1 − \bar{p})^{p−s}1(s \leq C₂s₀). \hspace{1cm} (C.22) \]

By assumption, \(s₀ = O(1)\), so \(C₂s₀\) is bounded above by a constant for all \(n\). Therefore, we may bound \(Π(S₀)\) from below, for some constants \(K, K’ > 0\),

\[ Π(S₀) ≥ K\bar{p}^{s₀}(1 − \bar{p})^{p−s₀} \geq K’\bar{p}^{s₀}. \hspace{1cm} (C.23) \]
It thus suffices to prove that $\bar{p}^s_0$ is bounded below by $\exp(-n\epsilon^2_{n,s_0})$. Let $C_d = 2^{-d}\pi^{-(d-1)/2}[\Gamma((d+1)/2)]^{-1}$ be the normalizing constant in the multivariate $d$-dimensional density $\Phi(\beta_j | \lambda)$. Since the prior on each $\beta_j$ is the SSGL prior (2.4)-(2.5), we have that

$$\bar{p}^s_0 = \left[ \Pr(\beta_j : ||\beta_j||^2 > \omega_d) \right]^{s_0}$$

$$\geq \left[ C_d \lambda_1^d \int_{0}^{1} \kappa \left( \int_{\{\beta_j : ||\beta_j||^2 > \omega_d\}} e^{-\lambda_1 ||\beta_j||^2} d\beta_j \right) d\Pi(\kappa) \right]^{s_0}$$

$$> \left[ C_d \lambda_1^d \int_{0}^{1} \kappa \left( \int_{\{\beta_j : \omega_d < ||\beta_j||^2 < 1\}} e^{-\lambda_1 ||\beta_j||^2} d\beta_j \right) d\Pi(\kappa) \right]^{s_0}$$

$$\geq e^{-\lambda_1 s_0 C_d \lambda_1^d s_0} \left[ \int_{0}^{1} \kappa d\Pi(\kappa) \right]^{s_0}$$

$$\geq e^{-\lambda_1 s_0 C_d \lambda_1^d s_0} \left( 1 + p^c \right)^{-s_0}$$

$$\geq e^{-b_6 s_0 \log p}$$

$$> e^{-n\epsilon^2_{n,s_0}}, \quad (C.24)$$

for some constant $b_6 > 0$. In the second line, we used the fact that with the SSGL prior on $\beta_j$, $\pi(\beta_j | \kappa) > \kappa \Psi(\beta_j | \lambda_1)$. In the third line, we used the fact that $\omega_d \ll 1$ for large $n$, and in the fifth line, we used the fact that $\kappa \sim B(1,p^c)$ by assumption, and thus, the expectation of $\kappa$ is $1/(1 + p^c)$. In the sixth line, we used the fact that $\lambda_1 \sim 1/n$ by assumption, and so one can easily show that the first three terms in the product in the fifth line can be lower bounded by $\exp(-Ks_0 \log p)$, for some $K > 0$. Meanwhile $(1 + p^c)^{-s_0} \sim \exp(-c s_0 \log p)$. Thus, combining (C.23)-(C.24) verifies the condition (C.20).

As argued in Lemma C.1, both KL divergence and KL variation can be upper bounded by a constant multiple of $(\eta_i - \eta_{0i})^2$. Therefore, for some constants $b_7 > 0$ and $D > 0$, and sufficiently large $n$, a lower bound for the left-hand side of (C.21) is

$$\Pi_{S_0} \left( \sum_{i=1}^{n} (\eta_i - \eta_{0i})^2 \leq b_7^2 \frac{n\epsilon^2_{n,s_0}}{2} \right)$$

$$\geq \Pi_{S_0} \left( \mu : |\mu - \mu_0| \leq b_7 \epsilon_{n,s_0} \right) \Pi_{S_0} \left( \beta : ||\bar{X}(\beta - \beta_0) - \delta_0||^2 \leq b_7^2 \frac{n\epsilon^2_{n,s_0}}{4} \right)$$

$$\geq \left( e^{-Dn\epsilon^2_{n,s_0}/2} \right) \left( e^{-Dn\epsilon^2_{n,s_0}/2} \right) = e^{-Dn\epsilon^2_{n,s_0}},$$

where the last two lines can be easily verified using very similar reasoning as that used to prove Lemma C.1. Therefore, (C.21) also holds.
Part II: Prior anti-concentration. Condition (B2) in [25] introduces the anti-concentration condition for overfitting models $S \supset S_0$:

$$\Pi_S(H_n(q, q_0) \leq M\epsilon_{n,s}) \leq e^{-Hn\epsilon_{n,s}^2},$$

where $\epsilon_{n,s} \geq \epsilon_{n,s_0}$ and $M, H > 0$ are sufficiently large constants. In other words, the posterior probability of overly large models that contain the truth should tend to zero as $n \to \infty$. As pointed out in Liu et al. [11] and conditioning on $M_n$, it is sufficient to prove

$$\sum_{S \supset S_0: s \leq C_s} \Pi(S) \Pi_S(H_n(q, q_0) \leq M\epsilon_{n,s}) \leq e^{-Hn\epsilon_{n,s}^2}. \tag{C.25}$$

We first find an upper bound for $\Pi(S)$. Note that when $\|\beta_j\|_2 > \omega_d$, $\pi(\beta_j | \kappa) < 2\kappa C_d \lambda_1^d \exp(-\lambda_1 \|\beta_j\|_2) < \kappa \lambda_1^d$. Thus, from (C.22), we have

$$\Pi(S) \lesssim \bar{p}^s = \left[ \int_0^1 \left( \int_{\{\beta_j: \|\beta_j\|_2 > \omega_d\}} \pi(\beta_j | \kappa) d\beta_j \right) d\Pi(\kappa) \right]^s$$

$$< \left[ \lambda_1^d \int_0^1 \kappa d\Pi(\kappa) \right]^s$$

$$= \lambda_1^d s (1 + p^c)^{-s}$$

$$\leq \exp(-b_8 s (\log p + n))$$

$$< e^{-bn\epsilon_{n,s}^2}, \quad \tag{C.26}$$

for some constant $b_8 > 2$. In the third line of the display, we used the fact that $\kappa \sim B(1, p^c)$, so $\mathbb{E}(\kappa) = 1/(1 + p^c)$, and in the fourth line, we used the fact that $\lambda_1 \asymp 1/n$ by assumption, so $\lambda_1^d \asymp \exp(-ds \log n)$. Therefore, from (C.26), we can upper bound the left-hand side of (C.25) by

$$\sum_{S \supset S_0: s \leq C_s} \Pi(S) \leq \sum_{S \supset S_0: s \leq C_s} e^{-bn\epsilon_{n,s}^2}$$

$$\leq e^{-bn\epsilon_{n,s_0}^2} \left( \frac{2ep}{C_s} \right)^{C_s + 1}$$

$$\leq e^{-Hn\epsilon_{n,s_0}^2},$$

by choosing $H < b_8 - 1$. Therefore, (C.25) holds.

Part III: Testing and entropy conditions. The final condition we need to verify is condition (B4) in [25]. We first need to show that for any models $S \neq S_0$, there exists an exponentially powerful test for testing $H_0 : q = q_0$ vs. $H_1 : q = q_S$, where $H_n(q_S, q_0) > \epsilon_{n,s_0}$. By [5], such a test always exists for the average Hellinger distance, so we are done.
We also need to verify certain entropy conditions. Specifically, we need to show that the complexity of overfitting models \( S \supset S_0 \) and underfitting models \( S \nsubseteq S_0 \) is not too large inside a sieve \( T_n \), where there is exponentially small probability outside of \( T_n \). As before, our analysis below is conditioned on the event \( \mathcal{M}_n = \{ s \leq C_2 s_0, \lambda_0 \geq p^2 \} \). For both overfitting and underfitting models \( S \) such that \( |S| \leq C_2 s_0 \), we construct the sieve \( T_n = \{ \beta : \max_{1 \leq j \leq p} \| \beta_j \|_2 \leq nd/\lambda_1 \} \). Then we have \( \mathbb{P}_0(T_n^c)1_{\mathcal{M}_n} \leq \sum_{S : s \leq C_2 s_0} \sum_{j \in S} \Pi(\| \beta_j \|_2 > nd/\lambda_1) < e^{-b_9 n\epsilon^2_{n,s}} \), for some \( b_9 > 0 \), where the final inequality follows directly from the arguments used to prove (E.14) in the proof of Theorem 2 of [1]. So there is exponentially small probability outside of \( T_n \cap \mathcal{M}_n \).

We first consider overfitting models \( S \supset S_0 \). Let \( \beta_S \) be the corresponding vector of basis coefficients indexed by any arbitrary overfitting model \( S \) of size \( s > s_0 \). Arguing similarly as in (C.9)-(C.11), we have that on \( T_n \setminus \mathcal{M}_n \) and for some \( b_9 > 0 \),

\[
\log N(\epsilon_{n,s}, T_n \cap \mathcal{M}_n, H_n) \\
\leq \log N(b_9\epsilon_{n,s}, \{ \mu : |\mu| < \sqrt{n}, |\cdot| \}) \\
+ \log N \left( \frac{b_9\epsilon_{n,s}}{s}, \left\{ \beta_S \in \mathcal{M}_n : \| \beta_S - \beta_0 \|_\infty \leq \frac{nd}{\lambda_1} \right\}, \| \cdot \|_\infty \right) \\
\lesssim n\epsilon^2_{n,s} + \log \left( \frac{3snd}{b_9\epsilon_{n,s}\lambda_1} \right)^{ds} \\
\lesssim n\epsilon^2_{n,s} + ds(\log d + \log n) \\
\lesssim n\epsilon^2_{n,s}.
\]

For underfitting models \( S \nsubseteq S_0 \), we also need to verify an entropy condition for some sequence \( \delta_n \) satisfying \( \delta_n > \epsilon_{n,s_0}, \delta_n \to 0, \) and \( n\delta_n^2 \to \infty \). This follows from the same arguments as above (replacing \( \epsilon_{n,s} \) with \( \delta_n \)). Thus, we obtain for any arbitrary underfitting model \( S \),

\[
\log N(\delta_n, T_n \cap \mathcal{M}_n, H_n) \lesssim n\delta_n^2.
\]

Thus, we have controlled the metric entropy for both overfitting models and underfitting models. The final requirement in Assumption (B4) of [25] is to verify that

\[
\sum_{S \nsubseteq S_0 : |S| \leq C_2 s_0} e^{-C_7 n\delta_n^2} + \sum_{S \supset S_0 : |S| \leq C_2 s_0} e^{-C_7 n\epsilon^2_{n,s}} \leq 1, \tag{C.27}
\]

for a large constant \( C_7 > 0 \). We must have \( n\delta_n^2 > n\epsilon^2_{n,s_0} \) based on our choice of \( \delta_n \). Meanwhile, for any overfitting model \( S \supset S_0, s > s_0 \) so \( n\epsilon^2_{n,s} > n\epsilon^2_{n,s_0} \)

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as well. Therefore, we can bound the left-hand side of (C.27) from above by

\[
\sum_{k=0}^{C_2s_0} \sum_{S:s=k} e^{-C_7n\epsilon_{n,s_0}^2} \leq e^{-C_7n\epsilon_{n,s_0}^2} \sum_{k=0}^{C_2s_0} \binom{p}{k} \\
\leq e^{-C_7n\epsilon_{n,s_0}^2} \left( \frac{2ep}{C_2s_0} \right)^{2C_2s_0+1} \\
= \left( e^{-C_7n\epsilon_{n,s_0}^2} \right) \left( e^{(2C_2s_0+1)(\log p + \log(2e/C_2s_0))} \right).
\]

In the final line of the display, the dominating term in the exponent of the second term is \(2C_2s_0 \log p\). Meanwhile, \(n\epsilon_{n,s_0}^2 > s_0 \log p\) (since \(\epsilon_{n,s_0}^2 = s_0 \log p/n + s_0n^{-2\alpha/(2\alpha+1)}\)), so the final line of the display can be made to be strictly less than one by choosing \(C_7\) to be large enough.

Having verified all four conditions of Theorem 4 of Yang and Pati [25], the model selection consistency result holds. \(\square\)