A Borel-Weil Construction for Schur Modules

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Abstract

We present a generalization of the classical Schur modules of $GL(N)$ exhibiting the same interplay among algebra, geometry, and combinatorics. A generalized Young diagram $D$ is an arbitrary finite subset of $\mathbb{N} \times \mathbb{N}$. For each $D$, we define the Schur module $S_D$ of $GL(N)$. We introduce a projective variety $F_D$ and a line bundle $L_D$, and describe the Schur module in terms of sections of $L_D$.

For diagrams with the “northeast” property,

$$(i_1, j_1), (i_2, j_2) \in D \Rightarrow (\min(i_1, i_2), \max(j_1, j_2)) \in D,$$

which includes the skew diagrams, we resolve the singularities of $F_D$ and show analogs of Bott’s and Kempf’s vanishing theorems. Finally, we apply the Atiyah-Bott Fixed Point Theorem to establish a Weyl-type character formula of the form:

$$\text{char}_{S_D}(x) = \sum_t \frac{x^{\text{wt}(t)}}{\prod_{i,j} (1 - x_i x_j^{-1})^{d_{ij}(t)}},$$

where $t$ runs over certain standard tableaux of $D$.

Our results are valid over fields of arbitrary characteristic.

Introduction

The two main branches of the representation theory of the general linear groups $G = GL(N, F)$ began with the geometric Borel-Weil-Bott theory and the combinatorial analysis of Schur, Young, and Weyl. In the case when $F$ is of characteristic zero, the geometric theory realizes the irreducible representation of $G$ with highest weight $\lambda$ as the sections of a line bundle $L_\lambda$ over
the flag variety $\mathcal{F} = G/B$. By contrast, the Schur-Weyl construction obtains this representation as a subspace of $V^\otimes k$, where $V = F^N$, by a process of symmetrization and anti-symmetrization defined by $\lambda$ considered as a Young diagram. (See [3] for an accessible reference.)

Combinatorists have examined other symmetrization operations on $V^\otimes k$, such as those associated to skew diagrams, and recently more general diagrams $D$ of squares in the plane ([3], [11], [14], [21], [27], [28], [29], [30]). We call the resulting $G$-representations the Schur modules $S_D \subset V^\otimes k$. (In characteristic zero, $S_D$ is irreducible exactly when $D$ is a Young diagram.) Kraskiewicz and Pragacz [15] have shown that the characters of $S_D$, for $D$ running through the inversion diagrams of the symmetric group on $N$ letters, give an algebraic description of the Schubert calculus for the cohomology of the flag variety $\mathcal{F}$. (More precisely, the Schubert polynomials are characters of flagged Schur modules. We deal with this case in [16].)

In this paper, we attempt to combine the combinatorial and the geometric approaches. We give a geometric definition (valid for all characteristics) for the $G$-module $S_D$. That is, for any finite set $D \subset N \times N$, we produce $S_D$ as the space of sections of a line bundle over a projective variety $\mathcal{F}_D$, the configuration variety of $D$. (This is proved only for diagrams with a “direction” property, but a weaker statement is shown for general diagrams.) Our picture reduces to that of Borel-Weil when $D$ is a Young diagram. See also [4], [3], where similar varieties are introduced. We prove a conjecture of V. Reiner and M. Shimozono asserting the duality between the Schur modules of two diagrams whose disjoint union is a rectangular diagram.

We can carry out a more detailed analysis for diagrams satisfying a direction condition such as the northeast condition

$$(i_1, j_1), (i_2, j_2) \in D \Rightarrow (\min(i_1, i_2), \max(j_1, j_2)) \in D.$$ 

To accord with the literature, we will deal exclusively with northwest diagrams, but since the modules and varieties with which we are concerned do not change (up to isomorphism) if we switch one row of the diagram with another or one column with another, everything we will say applies with trivial modifications to skew, inversion, Rothe, and column-convex diagrams, and diagrams satisfying any direction condition (NE, NW, SE, SW).

In this case, we find an explicit resolution of singularities of $\mathcal{F}_D$, and we use Frobenius splitting arguments of Wilberd van der Kallen (based on work...
of Mathieu, Polo, Ramanathan, et al.) to show the vanishing of certain higher cohomology groups. In particular, the configuration varieties are projectively normal and have rational singularities. This allows us to apply the Atiyah-Bott Fixed Point Theorem to compute the character of the Schur modules.

For more general diagrams, the above program breaks down because we lack a suitable desingularization of $\mathcal{F}_D$. We can carry it through, however, for diagrams with at most 3 rows, for which the configuration variety is the space of triangles $[7], [17]$.

Those interested only in the algebraic and combinatorial side of our results can find the definitions and statements in Section 1, Theorem 5 of Section 2.3, and Section 5.2. Our discussion of geometry begins with Section 2.

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1 $GL(N)$ modules

1.1 Schur modules

Given a finite set $T$, we will also use the symbol $T$ to denote the order $|T|$ when appropriate. Thus $GL(T) \overset{\text{def}}{=} GL(|T|)$, etc. Let $\Sigma_T$ be the symmetric group permuting the elements of $T$. For any left $G$-space $X$, $\Sigma_T$ acts on the
right, and $G$ acts on the left, of the cartesian product $X^T$ by:

$$g(x_{t_1}, x_{t_2}, \ldots)\pi = (gx_{\pi t_1}, gx_{\pi t_2}, \ldots).$$

A diagram is a finite subset of $\mathbb{N} \times \mathbb{N}$. Its elements $(i, j) \in D$ are called squares. We shall often think of $D$ as a sequence $(C_1, C_2, \ldots, C_r)$ of columns $C_j \subset \mathbb{N}$. The Young diagram corresponding to $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)$ is the set $\{(i, j) \mid 1 \leq j \leq N, \ 1 \leq i \leq \lambda_j\}$. For any diagram $D$, we let

$$\text{Col}(D) = \{\pi \in \Sigma_D \mid \pi(i, j) = (i', j) \exists i'\}$$

be the group permuting the squares of $D$ within each column, and we define $\text{Row}(D)$ similarly for rows.

Let $F$ be a field. We shall always write $G = GL(N, F)$, $B = \text{the subgroup of upper triangular matrices}$, $H = \text{the subgroup of diagonal matrices}$, and $V = F^N$ the defining representation.

Now let $F$ have characteristic zero. Define the idempotents $\alpha_D, \beta_D$ in the group algebra $F[\Sigma_D]$ by

$$\alpha_D = \frac{1}{|\text{Row } D|} \sum_{\pi \in \text{Row } D} \pi, \quad \beta_D = \frac{1}{|\text{Col } D|} \sum_{\pi \in \text{Col } D} \text{sgn}(\pi)\pi,$$

where $\text{sgn}(\pi)$ is the sign of the permutation. Define the Schur module

$$S_D \overset{\text{def}}{=} V^{\otimes D} \alpha_D \beta_D \subset V^{\otimes D},$$

a representation of $G$.

Note that we get an isomorphic Schur module if we change the diagram by permuting the rows or the columns (i.e., for some permutation $\pi : \mathbb{N} \to \mathbb{N}$, changing $D = \{(i, j)\}$ to $D' = \{\pi(i), j \mid (i, j) \in D\}$, and similarly for columns).

### 1.2 Weyl modules

Let $W = V^*$, the dual of the defining representation of $G = GL(N, F)$, where $F$ is an infinite field. Given a diagram $D$, define the alternating product with respect to the columns

$$\bigwedge^D W = \{f : V^D \to F \mid f \text{ multilinear, and } f(v\pi) = \text{sgn}(\pi)f(v) \forall \pi \in \text{Col}(D)\},$$

where $\text{sgn}(\pi)$ is the sign of the permutation.
where multilinear means \( f(v_1, \ldots, v_d) \) is \( F \)-linear in each of the \( d = |D| \) variables. Consider the multidiagonal with respect to the rows
\[
\Delta^D V = \Delta^{R_1} V \times \Delta^{R_2} V \times \cdots \subset V^{R_1} \times V^{R_2} \times \cdots = V^D,
\]
where \( R_1, R_2, \ldots \) are the rows of \( D \). Now define the Weyl module
\[
W_D \overset{\text{def}}{=} \bigwedge^D W \upharpoonright_{\Delta^D V},
\]
where \( \Delta^D V \) denotes restriction of functions from \( V^D \) to \( \Delta^D V \). Since \( \Delta^D V \) is stable under the diagonal action of \( G \), \( W_D \) is naturally a \( G \)-module.

Remark. For \( F \) a finite field, we make the following modification. Consider \( W = W(F) \hookrightarrow W(\overline{F}) \), where \( \overline{F} \) is the algebraic closure. That is, identify
\[
W = \{ f : \overline{F}^N \to \overline{F} \mid f \text{ is } \overline{F} \text{-linear, and } f(F^N) \subset F \}.
\]
Then define
\[
W_D \overset{\text{def}}{=} \bigwedge^D W \upharpoonright_{\Delta^D \overline{V}(\overline{F})}.
\]
This keeps the restriction map from killing nonzero tensors which happen to vanish on the finite set \( \Delta^D V(F) \).

With this definition, \( W_D \) clearly has the base change property \( W_D(L) = W_D(F) \otimes_F L \) for any extension of fields \( F \subset L \).

Now consider \( W_D(Z) \). This is a free \( Z \)-module, since it is a submodule of the \( Z \)-valued functions on \( \Delta^D V \). Suppose \( D \) satisfies a direction condition. Then our vanishing results of Proposition 23 (a), along with the appropriate universal coefficient theorems, can be used to show that for any field \( F \),
\[
W_D(F) = W_D(Z) \otimes_Z F.
\]

**Proposition 1** If \( F \) has characteristic zero, then \( W_D \cong S^1_D \) as \( G \)-modules.

**Proof.** \( S_D \) is the image of the composite mapping
\[
V^{\otimes D} \alpha_D \hookrightarrow V^{\otimes D} \beta_D \Rightarrow V^{\otimes D} \beta_D.
\]
For \( W = V^* \), write
\[
W^{\otimes D} = \{ f : V^D \to F \mid f \text{ multilinear} \}.
\]
Sym$^D W = \{ f : V^D \rightarrow F \mid f \text{ multilinear, and } f(v \pi) = f(v) \forall \pi \in \text{Row}(D) \}$. 

Now, representations of $F[\Sigma_D]$ are completely reducible, so $S_D^*$ is the image of

$$W \otimes^D \beta_D \hookrightarrow W \otimes^D \alpha_D \twoheadrightarrow W \otimes^D \alpha_D,$$

and $W \otimes^D \beta_D \cong \Lambda^D W$, $W \otimes^D \alpha_D \cong \text{Sym}^D W$.

Now, let

$$\text{Poly}^D W = \{ f : V^l \rightarrow F \mid f \text{ homog poly of multidegree } (R_1, \ldots, R_l) \},$$

where $l$ is the number of rows of $D$. Then we have a $G$-equivariant map

$$\text{rest}_\Delta : \text{Sym}^D W \rightarrow \text{Poly}^D W$$

restricting functions from $V^D$ to the row-multidiagonal $\Delta^D V \cong V^l$. It is well known that $\text{rest}_\Delta$ is an isomorphism: the symmetric part of a tensor algebra is isomorphic to a polynomial algebra.

Thus we have the commutative diagram

$$
\begin{array}{c}
\bigwedge^D W \hookrightarrow W \otimes^D \alpha_D & \cong \text{Sym}^D W \\
\| & \| & \downarrow \text{rest}_\Delta \\
\bigwedge^D W \hookrightarrow W \otimes^D \alpha_D & \cong \text{Poly}^D W.
\end{array}
$$

Now, the image in the top row is $S_D^*$, the image in the bottom row is $W_D$, and all the vertical maps are isomorphisms, so we have $\text{rest}_\Delta : S_D^* \rightarrow W_D$ an isomorphism.

If $D = \lambda$ a Young diagram, then $W_D$ is isomorphic to Carter and Lusztig’s dual Weyl module for $G = GL(N, F)$. This will follow from Proposition \ref{prop:iso} in the following section.

\section{Configuration varieties}

\textbf{N.B.} Although our constructions remain valid over $\mathbb{Z}$, for simplicity we will assume for the remainder of this paper that $F$ is an algebraically closed field.
2.1 Definitions and examples

Given a finite set $C$ (a column), and $V = F^N$, consider $V^C \cong M_{N \times |C|}(F)$, the $N \times |C|$ matrices, with a right multiplication of $GL(C)$. Let

$$\text{St}(C) = \{ X \in V^C \mid \text{rank} \ X = |C| \},$$

the Stiefel manifold, and

$$\text{Gr}(C) = \text{St}(C)/GL(C),$$

the Grassmannian.

Also, let

$$L^C = \text{St}(C) \times \det^{-1} \to \text{Gr}(C)$$

be the Plucker determinant bundle, whose sections are regular functions $f : \text{St}(C) \to F$ with $f(XA) = \det(A)f(X) \forall A \in GL(C)$. In fact, such global sections can be extended to polynomial functions $f : V^C \to F$.

For a diagram $D$ with columns $C_1, C_2, \ldots$, we let

$$\text{St}(D) = \text{St}(C_1) \times \text{St}(C_2) \times \cdots, \quad \text{Gr}(D) = \text{Gr}(C_1) \times \text{Gr}(C_2) \times \cdots, \quad L_D = L_{C_1} \boxtimes L_{C_2} \boxtimes \cdots.$$

Recall that $\Delta^D V \subset V^D$ is the row multidiagonal (as opposed to the column constructions above). Let

$$\mathcal{F}_D \overset{\text{def}}{=} \text{Im} \left[ \Delta^D V \cap \text{St}(D) \to \text{Gr}(D) \right],$$

and define the configuration variety of $D$ by

$$\mathcal{F}_D = \overline{\mathcal{F}_D} \subset \text{Gr}(D),$$

the Zariski closure of $\mathcal{F}_D$ in $\text{Gr}(D)$. We denote the restriction of $L_D$ from $\text{Gr}(D)$ to $\mathcal{F}_D$ by the same symbol $L_D$.

Some properties follow immediately from the definitions. For instance, $\mathcal{F}_D$ is an irreducible variety. Just as for Schur modules and Weyl modules, changing the diagram by permuting the rows or the columns gives an isomorphic configuration variety and line bundle. If we add a column $C$ to $D$ which already appears in $D$, we get an isomorphic configuration variety, but the line bundle is twisted to have higher degree. Since $L_D$ gives the Plucker embedding on $\text{Gr}(D)$, it is very ample on $\mathcal{F}_D$. 

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Examples. Set $N = 4$. Consider the diagrams

\[
\begin{align*}
D_1 &= \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square & \square \end{array} \\
D_2 &= \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \\
D_3 &= \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \end{array}
\end{align*}
\]

Identifying $\text{Gr}(k, F^N)$ with $\text{Gr}(k-1, P_{F}^{N-1})$, we may consider the $\mathcal{F}_D$’s as varieties of configurations in $P^3$:

(1) $\mathcal{F}_{D_1}$ is the variety of pairs $(l, l')$, where $l, l'$ are intersecting lines in $P^3$. It is singular at the locus where the two lines coincide.

(2) $\mathcal{F}_{D_2}$ is the variety of triples $(l, p, l')$ of two lines and a point which lies on both of them. The variety is smooth: indeed, it is a fiber bundle over the partial flag variety of a line containing a point. There is an obvious map $\mathcal{F}_{D_2} \to \mathcal{F}_{D_1}$, which is birational, and in fact a small resolution of singularities. (C.f. Proposition 13.)

(3) $\mathcal{F}_{D_3}$ is the variety of planes with two marked points (which may coincide). $\mathcal{F}_{D_3}^0$ is the locus where the marked points are distinct. The variety is smooth as in the previous example.

\[
\begin{align*}
D_4 &= \begin{array}{cc} \square & \square \\ \square & \square \end{array} \\
D_5 &= \begin{array}{cccc} \square & \square & \square \end{array} \\
D_6 &= \begin{array}{cccc} \square & \square & \square \end{array}
\end{align*}
\]

(4) $\mathcal{F}_{D_4}$ is the variety of triples of coplanar lines.

(5) $\mathcal{F}_{D_5}$ is the variety of triples of lines with a common point. This is the projective dual of the previous variety, since the diagrams are complementary within a $4 \times 3$ rectangle (up to permutation of rows and columns). (See Theorem 3.) The variety of triples of lines which intersect pairwise cannot be described by a single diagram, but consists of $\mathcal{F}_{D_4} \cup \mathcal{F}_{D_5}$. (See Section 3.4.)

(6) $\mathcal{F}_{D_6} \cong (P^3)^4$ contains the $GL(N)$-invariant subvariety where all four points in $P^3$ are colinear. Since the cross-ratio is an invariant of four points on a line, this subvariety contains infinitely many $GL(N)$ orbits.

\[
\begin{align*}
D_7 &= \begin{array}{cccc} \square & \square & \square \end{array} \\
D_8 &= \begin{array}{cccc} \square & \square & \square & \square \end{array}
\end{align*}
\]

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F\, D_7 \cong G \times X_\lambda$, the $G$-orbit version of the Schubert variety $X_\lambda \subset \text{Gr}(2, 4)$ associated to the partition $\lambda = (1, 2)$. This is the smallest example of a singular Schubert variety.

(8) $\mathcal{F}_{D_8}$ is a smooth variety which maps birationally to $\mathcal{F}_{D_7}$ by forgetting the point associated to the last column. In fact, this is essentially the same resolution as (1) and (2) above. Such resolutions of singularities can be given for arbitrary Schubert varieties of $G = \text{GL}(N)$, and generalize Zelevinsky’s resolutions in [31]. C.f. Section 3.

Theorem 2 If $F$ is an algebraically closed field, then

$$W_D \cong \text{Im} \left[ \text{rest}_\Delta : H^0(\text{Gr}(D), \mathcal{L}_D) \rightarrow H^0(\mathcal{F}_D, \mathcal{L}_D) \right],$$

where $\text{rest}_\Delta$ is the restriction map.

Proof. Note that for $\text{GL}(D) = \text{GL}(C_1) \times \text{GL}(C_2) \times \cdots,$

$$H^0(\text{Gr}(D), \mathcal{L}_D) = \{ f : V^D \rightarrow F \mid f(XA) = \text{det}(A) f(X) \ \forall A \in \text{GL}(D) \},$$

and recall

$$\bigwedge^D W = \{ f : V^D \rightarrow F \mid f \text{ multilinear, and } f(v\pi) = \text{sgn}(\pi) f(v) \ \forall \pi \in \text{Col}(D) \}.$$ 

But in fact these sets are equal, because a multilinear, anti-symmetric function $g : V^C \rightarrow F$ always satisfies $g(XA) = \text{det}(A) g(X) \ \forall A \in \text{GL}(C)$. Now $W_D$ and $H^0(\mathcal{F}_D, \mathcal{L}_D)$ are gotten by restricting functions in these identical sets to $\Delta^D V$, so we are done.

2.2 Diagrams with at most $N$ rows

We say $D$ has $\leq N$ rows if $(i, j) \in D \Rightarrow 1 \leq i \leq N$.

Proposition 3 If $D$ has $\leq N$ rows, then $\mathcal{F}_D$ has an open dense $\text{GL}(N)$-orbit $\mathcal{F}_D^{\text{gen}}$.

Proof. Let $D$ have columns $C_1, C_2, \ldots$. Consider a sequence of vectors $X = (v_1, \ldots, v_n) \in V^N$. For $C = \{i_1, i_2, \ldots\} \subset \{1, \ldots, N\}$, define $X(C) \overset{\text{def}}{=} \text{Span}_F(v_{i_1}, v_{i_2}, \ldots) \in \text{Gr}(C)$ (for $X$ sufficiently general). Consider an element $g \in \text{GL}(N)$ as a sequence of column vectors $g = (v_1, \ldots, v_n)$. Then

$$g(C) = g \cdot \text{Span}_F(e_{i_1}, e_{i_2}, \ldots) = g \cdot (I(C_1), I(C_2), \ldots),$$
where \( e_i \) denotes the \( i \)-th coordinate vector and \( I \) the identity matrix.

Now define the map
\[
\Psi : V^N \to \Delta^D V \subset V^D
\]
\[
(v_1, \ldots, v_N) \mapsto (v_i)_{(i,j) \in D},
\]
where \((u_{ij})_{(i,j) \in D}\) denotes an element of \( V^D \). Then the composite
\[
V^N \xrightarrow{\Psi} \Delta^D V \to F_D^0
\]
is an onto map taking \( g \mapsto (g(C_1), g(C_2), \ldots) = g \cdot (I(C_1), I(C_2), \ldots) \). Since \( GL(N) \) is dense in \( V^N \), its image is dense in \( F_D^0 \), and hence the composite image \( F_D^{\text{gen}} \defeq G \cdot (I(C_1), I(C_2), \ldots) \) is a dense \( G \)-orbit in \( F_D \).

**Proposition 4** If \( D \) is the Young diagram associated to a dominant weight \( \lambda \) of \( GL(N) \), then:

(a) \( F_D \equiv G/P \), a quotient of the flag variety \( F = G/B \).

(b) The Borel-Weil line bundle \( L_\lambda \equiv G \times (\lambda^{-1}) \to F \) is the pullback of \( L_D \) under the projection \( F \to F_D \).

(c) \( \text{rest}_\Delta : H^0(\text{Gr}(D), L_D) \to H^0(F_D, L_D) \) is surjective, and \( W_D \equiv H^0(F_D, L_D) \).

**Proof.**

(a) Let \( \mu = (\mu_1 \geq \mu_2 \geq \cdots) = \lambda^t \), the transposed diagram, and let \( P = \{(x_{ij}) \in GL(N) \mid x_{ij} = 0 \text{ if } \exists k, i > \mu_k \geq j > \mu_{k+1} \} \), a parabolic subgroup of \( G \). Then \( G/P \) is the space of partial flags \( V = F^N \supset V_1 \supset V_2 \supset \cdots \) consisting of subspaces \( V_j \) with \( \text{dim}(V_j) = \mu_j \). Clearly \( G/P \equiv F_D \).

(b) Let \( \Psi : G \to \Delta^D V \cap \text{St}(D) \) be the map in the proof of the previous proposition. Then the map
\[
G^P \times F_{\lambda^{-1}} \to L_D = (\text{St}(D) \times GL(D) \times \det^{-1}_D) |_{F_D}
\]
\[
(g, \alpha) \mapsto (\Psi(g), \alpha)
\]
is a \( G \)-equivariant bundle isomorphism. Then (b) follows by standard arguments.

(c) The surjectivity is a special case of Proposition 23 in Section 4. (See also [9].) The other statement then follows by Prop 4. •
2.3 Complementary diagrams

**Theorem 5** Suppose the rectangular diagram $\text{Rect} = \{1, \ldots, N\} \times \{1, \ldots, r\}$ is the disjoint union of two diagrams $D, D'$. Let $W_D, W_{D'}$ be the corresponding Weyl modules for $G = \text{GL}(N, F)$. Then:

(a) there is an $F$-linear bijection $\tau : W_D \rightarrow W_{D'}$ such that $\tau(gw) = \det_{N \times N}(g') g' \tau(w)$, where $g'$ is the inverse transpose in $\text{GL}(N)$ of the matrix $g$;

(b) the characters obey the relation $\text{char} W_{D'}(h) = \det_{N \times N}(h) \text{char} W_D(h^{-1})$, for diagonal matrices $h \in G$ ;

(c) if $F$ has characteristic zero, then as $G$-modules

\[ W_{D'} \cong \det^{-r} \otimes W_D^* \quad \text{and} \quad S_{D'} \cong \det^r \otimes S_D^*. \]

**Proof.** (a) Given $C \subset \{1, \ldots, N\}$ (a column set), we considered above the Plucker line bundle

\[ \text{St}(C) \xrightarrow{\text{GL}(C)} \det^{-1} \rightarrow \text{Gr}(D). \]

We may equally well write this as

\[ \text{GL}(N) \xrightarrow{P_C} \det^{-1}_C \rightarrow \text{Gr}(D), \]

where $P_C \equiv \{ (x_{ij}) \in \text{GL}(N) \mid x_{ij} = 0 \text{ if } i \notin C, j \in C \}$ is a maximal parabolic subgroup of $\text{GL}(N)$ (not necessarily containing $B$), and $\det_C : P_C \rightarrow F$ is the multiplicative character $\det_C(x_{ij})_{N \times N} \equiv \det_C(x_{ij})_{i,j \in C}$.

Hence, if $C_1, C_2, \ldots, C_r$ are the columns of $D$, we may write

\[ \text{Gr}(D) \cong G^r / P_D, \]

and the bundle

\[ \mathcal{L}_D \cong G^r \circledast \det^{-1}_D, \]

where $P_D \equiv P_{C_1} \times \cdots \times P_{C_r}$ and $\det_D(X_1, \ldots, X_r) \equiv \det_{C_1}(X_1) \times \cdots \times \det_{C_r}(X_r)$. Under this identification,

\[ \mathcal{F}_D \cong \text{closure} \text{ Im} [\Delta G \hookrightarrow G^r \rightarrow \text{Gr}(D)] \]

(c.f. Proposition 3).
Now let $\tau : G^r \to G^r$, $\tau(g_1, \ldots, g_r) = (g'_1, \ldots, g'_r)$, where $g' = g^{-1}$, the inverse transpose of a matrix $g \in G$. Then $\tau(P_D) = P_{D'}$, and $\tau$ induces a map

$$\tau : \text{Gr}(D) \to \text{Gr}(D'),$$

as well as a map of line bundles

$$\tau : \mathcal{L}_D \to \mathcal{L}_{D'},$$

with

$$\begin{align*}
\left( g_1, \ldots, g_r, \alpha \right) &\mapsto \left( g'_1, \ldots, g'_r, \det(g'_1, \ldots, g'_r) \alpha \right).
\end{align*}$$

This map is not $G$-equivariant. Rather, if we have a section of $\mathcal{L}_D$, $f : G^r \to F$ (with $f(gp) = \det_D(p)f(g)$ for $p \in P_D$), then for $g_0 \in G$, we have $\tau(g_0f) = g'_0 \det(g'_0)^r \tau(f)$ (a section of $\mathcal{L}_{D'}$).

Since $W_D$ is the restriction of such functions $f$ to $\Delta G \subset G^r$, and $\tau(\Delta G) \subset \Delta G$, we have an induced map

$$\tau : W_D \to W_{D'}$$

(an isomorphism of $F$ vector spaces), satisfying $\tau(g_0w) = g'_0 \det(g'_0)^r \tau(w)$ for $g_0 \in G$, $w \in W_D$. This is the map required in (a), and now (b), (c) follow trivially. •

### 3 Resolution of singularities

In this section, we define the class of northwest direction diagrams, which includes (up to a permutation of rows and columns) the skew, inversion, Rothe, and column-convex diagrams. We construct an explicit resolution of singularities of the associated configuration varieties by means of “blowup diagrams”. We also find defining equations for these varieties. One should note that the resolutions constructed are not necessarily geometric blowups, and can sometimes be small resolutions, as in Example 8 above.

We shall, as usual, think of a diagram $D$ either as a subset of $\mathbb{N} \times \mathbb{N}$, or as a list $(C_1, C_2, \ldots, C_r)$ of columns $C_j \subset \mathbb{N}$. We shall examine only configuration varieties, as opposed to line bundles on them, so we shall assume that the columns are without multiplicity: $C_j \neq C_{j'}$ for $j \neq j'$. 12
3.1 Northwest and lexicographic diagrams

A diagram $D$ is northwest if it possesses the following property:

$$(i_1, j_1), (i_2, j_2) \in D \Rightarrow (\min(i_1, i_2), \min(j_1, j_2)) \in D.$$ 

Given two subsets $C = \{i_1 < i_2 < \ldots < i_l\}$, $C' = \{i'_1 < i'_2 < \ldots < i'_l\} \subset \mathbb{N}$, we say $C$ is lexicographically less than $C'$ ($C \preceq C'$) if

$l < l'$ and $i_1 = i'_1$, \ldots, $i_l = i'_l$, 

or $\exists m : i_1 = i'_1$, \ldots, $i_{m-1} = i'_{m-1}, i_m < i'_m.$

In the first case, we say $C$ is an initial subset of $C'$ ($C \subset C'$).

A diagram $D = (C_1, C_2, \ldots)$ is lexicographic if $C_1 \preceq C_2 \preceq \cdots$. Note that any diagram can be made lexicographic by rearranging the order of columns.

**Lemma 6** If $D$ is northwest, then the lexicographic rearrangement of $D$ is also northwest.

**Proof.** (a) I claim that if $j < j'$, then either $C_j \preceq C_{j'}$, or $C_j \supseteq C_{j'}$. Let $C_j = \{i_1 < i_2 < \ldots\}$, $C'_{j'} = \{i'_1 < i'_2 < \ldots\}$. We have assumed $C_j \not\preceq C'_{j'}$.

Thus $C_j \preceq C'_{j'}$ or $C_j \supseteq C'_{j'}$. In the second case, $C_j \supseteq C'_{j'}$ or there is an $r$ such that $i_1 = i'_1, \ldots, i_{r-1} = i'_{r-1}, i_r < i'_r$. By the northwest property, this last case would mean $i'_r \in C_j$, with $i_{r-1} = i'_{r-1} < i'_r < i_r$. But this contradicts the definition of $C_j$. Thus the only possibilities are those of the claim.

(b) It follows immediately from (a) that if $C_1 \preceq C_2 \preceq \cdots \preceq C_{s-1} > C_s$, then there is a $t < s$ with $C_t \preceq C_s$, $C_s \supseteq C_t$, $C_t \subset C_{t+1}, \ldots, C_s \subset C_{s-1}$.

(c) From (b), we see that to rearrange the columns lexicographically requires only the following operation: we start with $C_1, C_2, \ldots$, and when we encounter the first column $C_s$ which violates lexicographic order, we move it as far left as possible, passing over those columns $C_i$ with $C_i \supseteq C_s$. This operation does not destroy the northwest property, as we can easily check on boxes from each pair of columns in the new diagram. By repeating this operation, we get the lexicographic rearrangement, which is thus northwest.
3.2 Blowup diagrams

The combinatorial lemmas of this section will be used to establish geometric properties of configuration varieties.

Given a northwest diagram $D$ and two of its columns $C, C' \subset N$, the intersection blowup diagram $\hat{D}_{C,C'}$ is the diagram with the same columns as $D$ except that the new column $C \cap C'$ is inserted in the proper lexicographic position (provided $C \cap C' \neq C, C'$).

**Lemma 7** Suppose $D$ is lexicographic and northwest, and $C \leq_{\text{lex}} C'$ are two of its columns. Then: (a) $C \cap C' \leq_{\text{init}} C'$, and (b) if $C \subset C'$, then $C \leq_{\text{init}} C'$.

**Proof.** (a) If $i \in C_j \cap C'_{j'}$ and $i > i' \in C'_{j'}$, then $i' \in C_j$ by the northwest property. Similarly for (b). •

**Lemma 8** If $D$ is lexicographic and northwest, then $\hat{D}_{C,C'}$ is also lexicographic and northwest.

**Proof.** If $C = C_j, C' = C'_{j'}$ with $j < j'$, and we insert the column $C \cap C' \leq_{\text{init}} C'$ immediately before $C'$, then we easily check that the resulting diagram is again northwest. Hence $\hat{D}_{C,C'}$, which is the lexicographic rearrangement of this, is also northwest by a previous lemma. •

Consider the columns $C_1, C_2, \ldots \subset N$ of a northwest diagram $D$, and take the smallest collection $\{\hat{C}_1 \leq_{\text{lex}} \hat{C}_2 \leq_{\text{lex}} \cdots\}$ of subsets of $N$ which contains the $C_i$ and is closed under taking intersections. Then we define a new diagram $\hat{D} = (\hat{C}_1, \hat{C}_2, \ldots)$ which we call the maximal intersection blowup diagram of $D$. Clearly $\hat{D} = \hat{D}$. Repeated application of the above lemma shows that if $D$ is lexicographic and northwest, then so is $\hat{D}$.

**Examples.** For one of the (non-northwest) diagrams considered previously, we have:

\[
\begin{align*}
D_4 = & \quad \square \quad \square \quad \square \quad \square \quad \square \\
\hat{D}_4 = & \quad \square \quad \square \quad \square \quad \square \quad \square \\
& \quad \square \quad \square \quad \square \quad \square \quad \square 
\end{align*}
\]

For the diagrams $D_7$ and $D_8$ in the previous examples, $D_8 = \hat{D}_7$. •

Consider the columns $C \subset N$ of a diagram $D$ as a partially ordered set under $\subset$, ordinary inclusion. Given two distinct columns $C, C''$, we say $C''$
minimally covers $C$ (or simply $C'$ covers $C$) if $C \subset C'$ and there is no column of $D$ strictly included between $C$ and $C'$.

**Lemma 9** Let $D$ be a lexicographic northwest diagram, and $C_L$ be the last column of $D$. Then:
(a) there is a column $C_l \neq C_L$ such that
\[ ( \bigcup_{C \neq C_L} C) \cap C_L = C_l \cap C_L; \]
(b) if $\tilde{D} = D$, then $C_L$ covers at most one other column $C_l$ and is covered by at most one other column $C_u$.

**Proof.** (a) Now, by Lemma 7, $C \cap C_L \subset C_L$ for any column $C$. Hence the sets $C \cap C_L$ for $C \neq C_L$ are linearly ordered under inclusion, and there is a largest one $C_l \cap C_L$. Thus
\[ ( \bigcup_{C \neq C_L} C) \cap C_L = \bigcup_{C \neq C_L} (C \cap C_L) = C_l \cap C_L. \]

(b) By Lemma 7, the columns with $C \subset C_L$ satisfy $C^{\text{init}} \subset C_L$ and are linearly ordered, so there is at most one maximal $C_u$.

Now suppose $C_u, C'_u \leq C_L$ are columns of $D$ both covering $C_L$. Then again by Lemma 7, we have $C_u \cap C'_u \leq C_u$ or $\leq C'_u$, so that $C_u \cap C' \neq C_L$. But $C_u \cap C'_u$ is between $C_L$ and $C_u$, and between $C_L$ and $C'_u$. Hence $C_u = C_u \cap C'_u = C'_u$. •

### 3.3 Blowup varieties

Let $D = (C_1, C_2, \ldots)$ be a lexicographic northwest diagram, and $\tilde{D} = (\tilde{C}_1, \tilde{C}_2, \ldots)$ be its maximal intersection blowup. Recall that $\tilde{D}$ is obtained by adding certain columns to $D$, so there is a natural projection map
\[ \text{pr} : \text{Gr}(\tilde{D}) \to \text{Gr}(D) \]
\[ (V_{\tilde{C}})_{\tilde{C} \in \tilde{D}} \mapsto (V_C)_{C \in D}, \]

obtained by forgetting some of the linear subspaces $V_{\tilde{C}} \in \text{Gr}(\tilde{C})$. 15
Proposition 10  If \( D \) has \( \leq N \) rows, then

\[
pr : \text{Gr}(\hat{D}) \to \text{Gr}(D)
\]

induces a birational map of algebraic varieties

\[
pr : \mathcal{F}_{\hat{D}} \to \mathcal{F}_D.
\]

Proof. Consider the dense open sets \( \mathcal{F}^{\text{gen}}_{\hat{D}} \subset \mathcal{F}_{\hat{D}} \) and \( \mathcal{F}^{\text{gen}}_D \subset \mathcal{F}_D \) of Proposition 3, consisting of subspaces in general position. If we consider an element \( g \in GL(N) \) as a sequence of column vectors \( g = (v_1, \ldots, v_n) \), and \( C = \{i_1, i_2, \ldots\} \subset \{1, \ldots, N\} \), recall that we define \( g(C) = \text{Span}_F(v_{i_1}, v_{i_2}, \ldots) \in \text{Gr}(C) \). By definition, any element of \( \mathcal{F}^{\text{gen}}_{\hat{D}} \) can be written as \( (g(\hat{C}_1), g(\hat{C}_2), \ldots) \in \text{Gr}(D) \) for some \( g \in GL(N) \).

Now, any column of \( \hat{D} \) can be written as an intersection of columns of \( D \): \( \hat{C} = C_{j_1} \cap C_{j_2} \cap \cdots \). Then we have \( g(\hat{C}) = g(C_{j_1}) \cap g(C_{j_2}) \cap \cdots \), so the projection map

\[
pr : \mathcal{F}^{\text{gen}}_{\hat{D}} \to \mathcal{F}^{\text{gen}}_D
\]

\[
(g(\hat{C}))(\hat{C} \in \hat{D}) \mapsto (g(C))(C \in D)
\]

can be inverted:

\[
pr^{-1} : \mathcal{F}^{\text{gen}}_D \to \mathcal{F}^{\text{gen}}_{\hat{D}}
\]

\[
(g(C))(C \in D) \mapsto (g(\hat{C})) = g(C_{j_1}) \cap g(C_{j_2}) \cap \cdots \hat{C} \in \hat{D}.
\]

Hence the map is birational on the configuration varieties as claimed. •

3.4 Intersection varieties

Now, given a diagram \( D \), define the intersection variety \( \mathcal{I}_D \) of \( D \) by:

\[
\mathcal{I}_D = \{(V_C)_{C \in D} \in \text{Gr}(D) \mid \forall C, C', \ldots \in D, \dim(V_C \cap V_C' \cap \cdots) \geq |C \cap C' \cap \cdots|\}.
\]

Clearly \( \mathcal{I}_D \) is a projective subvariety of \( \text{Gr}(D) \), and \( \mathcal{F}_D \subset \mathcal{I}_D \).

If \( \hat{D} = D \) (up to rearrangement of column order), then the intersection conditions reduce to inclusions:

\[
\mathcal{I}_D = \{(V_C)_{C \in D} \in \text{Gr}(D) \mid C \subset C' \Rightarrow V_C \subset V_{C'}\}.
\]
**Example.** For the diagram $D_4$ of Section 2.1, and $N = 4$, $\mathcal{I}_{D_4}$ has two irreducible components, $\mathcal{F}_{D_4}$ and $\mathcal{F}_{D_5}$. That is, as before, if we have three lines in $\mathbb{P}^3$ with non-empty pairwise intersections, then either they are coplanar, or they all intersect in a point.

**Lemma 11** Let $D$ be a northwest diagram, and $\mathcal{I}_D$ its intersection variety. Then any configuration $(V_C)_{C \in D} \in \mathcal{I}_D$ satisfies

$$\dim(V_C + V_{C'} + \cdots) \leq |C \cup C' \cup \cdots|$$

for any columns $C, C', \ldots$ of $D$.

**Proof.** Without loss of generality, assume $D$ is lexicographic. We use induction on the number of columns in $D$. Now any list $C, C', \ldots$ of columns of $D$ also constitutes a lexicographic northwest diagram, so to carry through the induction we need only prove the statement for all the columns $C_1, C_2, \ldots, C_L$ of $D$. Now, by Lemma 9, there is a column $C_l \neq C_L$ such that $$(\bigcup_{C \neq C_L} C) \cap C_L = C_l \cap C_L.$$ Then we have

$$\dim( (\sum_{C \neq C_L} V_C) \cap V_{C_L} ) \geq \dim( \sum_{C \neq C_L} (V_C \cap V_{C_L}) )$$

$$\geq \dim( V_{C_l} \cap V_{C_L} )$$

$$\geq |C_l \cap C_L|$$

since $(V_C) \in \mathcal{I}_D$

$$= |( \bigcup_{C \neq C_L} C ) \cap C_L |.$$ 

Thus we may write

$$\dim( \sum_{C \in D} V_C ) = \dim( \sum_{C \neq C_L} V_C ) + \dim(V_{C_L}) - \dim( (\sum_{C \neq C_L} V_C) \cap V_{C_L} )$$

$$\leq |\bigcup_{C \neq C_L} C| + |C_L| - |( \bigcup_{C \neq C_L} C ) \cap C_L | \quad \text{by induction}$$

$$= |\bigcup_{C \in D} C | \quad \bullet$$

**Lemma 12** If $D$ is a northwest diagram with $\leq N$ rows and $\widehat{D} = D$ (up to rearrangement of column order), then $\mathcal{F}_D$ is an irreducible component of $\mathcal{I}_D$. 17
**Proof.** Recall that $F_D$ is always irreducible. Thus it suffices to show that $F_D^{gen}$ is an open subset of $I_D$.

Consider the set $I_D^{gen}$ of configurations $(V_C)_{C \in D}$ satisfying, for every list $C, C', \ldots$ of columns in $D$,

$$\dim(V_C + V_{C'} + \cdots) = |C \cup C' \cup \cdots|$$

and

$$\dim(V_C \cap V_{C'} \cup \cdots) = |C \cap C' \cap \cdots|.$$  

This is an open subset of $I_D$ by the previous lemma.

I claim that $F_D^{gen} = I_D^{gen}$. To see this equality, let $(V_C)_{C \in D} \in I_D$ satisfy the above rank conditions, and we will find a basis $g = (v_1, \ldots, v_N)$ of $V = F^N$ such that $V_C = g(C)$ for all $C$. (C.f. the proof of Proposition 3.)

As before, we consider the columns as a poset under ordinary inclusion. We begin by choosing mutually independent bases for those $V_C$ where $C$ is a minimal element of the poset. This is possible because $\dim \text{Span}(V_C \mid C \text{ minimal}) = \sum_{C \text{ minimal}} |C|$.

Now we consider the $V_C$ where $C$ covers a minimal column. We start with the basis vectors already chosen, and add enough vectors, all mutually independent, to span each space. Again, the dimension conditions ensure there will be no conflict in choosing independent vectors, since the $V_C$ can have no intersections with each other except those due to the intersections of columns. The condition $\hat{D} = D$ ensures that all these intersections are (previously considered) columns.

We continue in this way for the higher layers of the poset. We will not run out of independent basis vectors because all the columns of $D$ are contained in $\{1, \ldots, N\}$. ●

### 3.5 Smoothness and equations defining varieties

**Proposition 13** Let $D$ be a northwest diagram with $\leq N$ rows and $\hat{D} = D$ (up to rearrangement of column order). Then $F_D = I_D$, and $F_D$ is a smooth variety.

**Proof.** (a) Let $C_L$ be the last column of $D$, and let $D'$ be $D$ without the last column. By lemma 9, $C_L$ is covered by at most one other column $C_u$, and covers at most one other column $C_l$. If these columns do not exist, take
(b) Now I claim that there is a fiber bundle

\[
\begin{array}{ccc}
\text{Gr}(C_l, C_L, C_u) & \rightarrow & Z \\
& \downarrow & \\
& \text{Gr}(D') & \\
\end{array}
\]

where \( \text{Gr}(C_l, C_L, C_u) \) denotes the Grassmannian of \(|C_L|\)-dimensional linear spaces which contain a fixed \(|C_l|\)-dimensional space and are contained in a fixed \(|C_u|\)-dimensional space; and

\[
Z = \{((V_{C'})_{C'} , V_L) \in \text{Gr}(D') \times \text{Gr}(C_L) \mid V_{C_l} \subset V_L \subset V_{C_u}\}.
\]

This is clear. See also [4].

(c) Note that \( \mathcal{I}_D = (\mathcal{I}_D \times \text{Gr}(C_L)) \cap Z \). This is because of the uniqueness of \( C_l \) and \( C_u \). Thus the above fiber bundle restricts to

\[
\begin{array}{ccc}
\text{Gr}(C_l, C_L, C_u) & \rightarrow & \mathcal{I}_D \\
& \downarrow & \\
& \mathcal{I}_{D'} & \\
\end{array}
\]

which is thus also a fiber bundle.

(d) Now apply the above construction repeatedly, dropping columns of \( D \) from the end. Finally we obtain \( \mathcal{I}_D \) as an iterated fiber bundle whose fibers at each step are smooth and connected (in fact they are Grassmannians). In particular, \( \mathcal{I}_D \) is smooth and connected.

(e) Since \( \mathcal{I}_D \) is a smooth, connected, projective algebraic variety, it must be irreducible. But by a previous lemma, \( \mathcal{F}_D \) is an irreducible component of \( \mathcal{I}_D \). Therefore \( \mathcal{F}_D = \mathcal{I}_D \), a smooth variety.

\[\text{Proposition 14} \]

Let \( D \) be a northwest diagram with \( \leq N \) rows. Then \( \mathcal{F}_D = \mathcal{I}_D \), and the birational projection map \( \mathcal{F}_D \rightarrow \mathcal{F}_D \) has connected fibers.

\[\text{Proof.} \quad (a) \text{ I claim the following: if } \hat{C} \text{ is a column of } \hat{D} \text{ such that for all } C \in \hat{D} \text{ with } C \notin \hat{C} \text{ we have } C \in D, \text{ then the projection map } \mathcal{I}_{D,\hat{C}} \rightarrow \mathcal{I}_D \text{ is onto, with connected fibers.} \]
Suppose \((V_C)_{C \in \mathcal{D}}\) is a configuration in \(\mathcal{I}_D\). Let
\[
V_u = \bigcap_{C \in \mathcal{D}, C \supseteq \hat{C}} V_C \quad \text{and} \quad V_l = \sum_{C \in \mathcal{D}, C \subseteq \hat{C}} V_C.
\]
Then \(\dim(V_u) \geq |\hat{C}|\) since \((V_C) \in \mathcal{I}_D\), and \(\dim(V_l) \leq |\hat{C}|\) by Lemma \[\Box\].

Clearly \(V_l \subseteq V_u\). Now choose an arbitrary \(V_{\hat{C}}\) between \(V_l\) and \(V_u\) with \(\dim(V_{\hat{C}}) = |\hat{C}|\). Then for any list of columns \(C, C', \ldots \in \mathcal{D}\), we have either:
(i) \(\hat{C} \cap C \cap C' \cdots = \hat{C}\), and
\[
V_{\hat{C} \cap C \cap C' \cdots} = V_{\hat{C}} = V_{\hat{C}} \cap V_u \subseteq V_{\hat{C}} \cap V_C \cap V_{C'} \cap \cdots;
\]
or (ii) \(\hat{C} \cap C \cap C' \cdots \not\subset \hat{C}\), so that \(\hat{C} \cap C \cap C' \cdots \in \mathcal{D}\) by hypothesis, and
\[
V_{\hat{C} \cap C \cap C' \cdots} \subseteq V_l \cap V_C \cap V_{C'} \cap \cdots \subseteq V_{\hat{C}} \cap V_C \cap V_{C'} \cap \cdots \subseteq V_{\hat{C}} \cap V_{C'} \cap \cdots.
\]

In either case \((V_C)_{C \in \mathcal{D} \cup \hat{C}} \in \mathcal{I}_{\mathcal{D} \cup \hat{C}}\). Thus \(\mathcal{I}_{\mathcal{D} \cup \hat{C}} \rightarrow \mathcal{I}_D\) is onto, and the fibers are the Grassmannians \(\text{Gr}(V_l, |C|, V_u)\).

(b) We now see that \(\mathcal{I}_{\hat{D}} \rightarrow \mathcal{I}_D\) is onto (with connected fibers) by repeated application of (a), starting with \(\hat{C}\) minimal in the poset of columns of \(\hat{D}\) and proceeding upward.

(c) By the previous proposition, the projection map takes \(\mathcal{I}_{\hat{D}} = \mathcal{F}_{\hat{D}} \rightarrow \mathcal{F}_D\). But \(\mathcal{I}_{\hat{D}} \rightarrow \mathcal{I}_D\) is onto, so \(\mathcal{F}_D = \mathcal{I}_D\), and we are done. \(\bullet\)

The above proposition shows that for northwest diagrams, \(\mathcal{F}_D\) is defined by the rank conditions of \(\mathcal{I}_D\). In general, we state the

**Conjecture 15** For an arbitrary diagram \(D\), \(\mathcal{F}_D\) is the set of configurations satisfying
\[
\dim(V_C + V_{C'} + \cdots) \leq |C' \cup C'' \cup \cdots|
\]
\[
\dim(V_C \cap V_{C'} \cap \cdots) \geq |C \cap C' \cap \cdots|
\]
for every list \(C, C', \ldots\) of columns of \(D\). Equivalently(?), the variety defined by these equations is irreducible.
4 Cohomology of line bundles

Using the technique of Frobenius splitting, we show certain surjectivity and vanishing theorems for line bundles on configuration varieties. In particular, we show that for any northwest diagram $D$, $\mathcal{F}_D$ is normal, and projectively normal with respect to $\mathcal{L}_D$ (so that global sections of $L_D$ on $\mathcal{F}_D$ extend to $\text{Gr}(D)$); and $\mathcal{F}_D$ has rational singularities. The material of section 4.1 was shown to me by Wilberd van der Kallen.

4.1 Frobenius splittings of flag varieties

The technique of Frobenius splitting, introduced by V.B. Mehta, S. Ramanan, and A. Ramanathan [20], [24], [25], [26], is a method for proving certain surjectivity and vanishing results.

Given two algebraic varieties $Y \subset X$ defined over an algebraically closed field $F$ of characteristic $p > 0$, with $Y$ a closed subvariety of $X$, we say that the pair $Y \subset X$ is compatibly Frobenius split if:

(i) the $p^{th}$ power map $F : \mathcal{O}_X \to F^*\mathcal{O}_X$ has a splitting, i.e. an $\mathcal{O}_X$-module morphism $\phi : F^*\mathcal{O}_X \to \mathcal{O}_X$ such that $\phi F$ is the identity; and

(ii) we have $\phi(F^*I) = I$, where $I$ is the ideal sheaf of $Y$.

Mehta and Ramanathan prove the following

**Theorem 16** Let $X$ be a projective variety, $Y$ a closed subvariety, and $L$ an ample line bundle on $X$. If $Y \subset X$ is compatibly split, then $H^i(Y, L) = 0$ for all $i > 0$, and the restriction map $H^0(X, L) \to H^0(Y, L)$ is surjective.

Furthermore, if $Y$ and $X$ are defined and projective over $\mathbb{Z}$ (and hence over any field), and they are compatibly split over any field of positive characteristic, then the above vanishing and surjectivity statements also hold for all fields of characteristic zero.

Our aim is to show that, for $D$ a northwest diagram, $\mathcal{F}_D \subset \text{Gr}(D)$ is compatibly split. The above theorem and Theorem 2 will then imply that $S^*_D \cong H^0(\mathcal{F}_D, \mathcal{L}_D) = \sum_i (-1)^i H^i(\mathcal{F}_D, \mathcal{L}_D)$, the Euler characteristic of $\mathcal{L}_D$.

We will also need the following result of Mehta and V. Srinivas [19]:

**Proposition 17** Let $Y$ be a projective variety which is Frobenius split, and suppose there exists a smooth irreducible projective variety $Z$ which is mapped onto $Y$ by an algebraic map with connected fibers. Then $Y$ is normal.
Furthermore, if $Y$ is defined over $\mathbf{Z}$, and is normal over any field of positive characteristic, then $Y$ is also normal over all fields of characteristic zero.

**Proposition 18** Let $f : Z \to X$ be a separable morphism with connected fibers, where $X$ and $Z$ are projective varieties and $X$ smooth. If $Y \subset Z$ is compatibly split, then so is $f(Y) \subset X$.

We will show our varieties are split by using the above proposition to push forward a known splitting due to Ramanathan [26] and O. Mathieu [?].

For an integer $n$ and $n$ permutations $w, w', \ldots$, define

$$X_n = \left( G \times G \times \cdots \times G \right) / B,$$

and the twisted multiple Schubert variety

$$Y_{w, w', \ldots} = BwB \times Bw'B \times \cdots \subset X_n.$$

Note that we have an isomorphism

$$X_n \to (G/B)^n : (g, g', g'', \ldots) \mapsto (g, gg', gg'g'', \ldots).$$

**Proposition 19** (Ramanathan-Mathieu) Let $G$ be a reductive algebraic group over a field of positive characteristic with Weyl group $W$ and Borel subgroup $B$, and let $w_0, w_1, \ldots, w_r \in W$. Then $Y_{w_0, w_1, \ldots} \subset X_{r+1}$ is compatibly split.

Now, for Weyl group elements $u_1, \ldots, u_r$, define a variety $\mathcal{F}_{u_1, \ldots, u_r} \subset (G/B)^r$ by

$$\mathcal{F}_{u_1, \ldots, u_r} = G \cdot (u_1 B, \ldots, u_r B).$$

**Proposition 20** (van der Kallen) Let $w_1, \ldots, w_r$ be Weyl group elements, and define $u_1 = w_1, u_2 = w_1 w_2, \ldots, u_r = w_1 \cdots w_r$. Suppose $w_1, \ldots, w_r$ satisfy $\ell(w_1 w_2 \cdots w_r) = \ell(w_1) + \ell(w_2) + \cdots + \ell(w_r)$, or equivalently $\ell(u_j) = \ell(u_{j-1}) + \ell(u_{j-1}^{-1} u_j)$ for all $j$. Then the pair $\mathcal{F}_{u_1, \ldots, u_r} \subset (G/B)^r$ is compatibly split.
Proof. Define

\[ f : X_{r+1} \to (G/B)^r \]
\[ (g_0, g_1, \ldots, g_r) \mapsto (g_0g_1, g_0g_1g_2, \ldots, g_0g_1\cdots g_r). \]

We will examine the image under this map of

\[ Y \overset{\text{def}}{=} Y_{w_0, w_1, \ldots w_r} = G^B \times Bw_1B \times \cdots \times Bw_rB \subset X_{r+1}, \]

where \( w_0 \) is the longest element of the Weyl group.

It is well known that, under the given hypotheses, we have \((Bw_1B)\cdots(Bw_rB) = Bw_1\cdots w_rB\) and that the multiplication map

\[ Bw_1B \times \cdots \times Bw_rB \to Bw_1\cdots w_rB \]

is bijective. Thus any element \((g, b_1w_1b_1', \ldots, b_rw_rb_r')\) (for \( b_i, b_i' \in B \)) can be written as \((g, bw_1, w_2, \ldots, w_rB)\) for some \( b \in B \), and

\[ f(G \times Bw_1B \times \cdots \times Bw_rB) = f(G \times w_1B \times \cdots \times w_rB) = G(u_1B, \ldots, u_rB). \]

Hence \( f(Y) = \mathcal{F}_{u_1, \ldots, u_r} \), since our varieties are projective.

Now, \( f \) is a separable map with connected fibers between smooth projective varieties, so the compatible splitting of the previous proposition pushes forward by Proposition 18.

We will need the following lemmas to show that our configuration varieties have rational singularities.

Lemma 21 (Kempf [13]) Suppose \( f : Z \to X \) is a separable morphism with generically connected fibers between projective algebraic varieties \( Z \) and \( X \), with \( X \) normal. Let \( L \) be an ample line bundle on \( X \), and suppose that \( H^i(Z, f^*L^{\otimes n}) = 0 \) for all \( i > 0 \) and all \( n \gg 0 \).

Then \( R^if_*\mathcal{O}_Z = 0 \) for all \( i > 0 \). 

Resuming the notation of Prop 19, let \( w_1, \ldots w_n \) be arbitrary Weyl group elements, and let \( \lambda_1, \ldots \lambda_n \) be arbitrary weights of \( G \). Let \( X_n \) be as before, and define the line bundle \( \mathcal{L}_{\lambda_1, \ldots, \lambda_n} \) on \( X_n \) and on \( Y_{w_1, \ldots, w_n} \subset X_n \) as the quotient of \( G^n \times F \) by the \( B^n \)-action

\[ (b_1, \ldots, b_n) \cdot (g_1, g_2, \ldots, g_n, a) \overset{\text{def}}{=} (g_1b_1, b_1^{-1}g_2b_2, \ldots, b_1\cdots b_{n-1}^{-1}g_nb_n, \lambda_1(b_1)\cdots \lambda_n(b_n)a). \]

Note that under the identification \( X_n \cong (G/B)^n \), \( \mathcal{L}_{\lambda_1, \ldots, \lambda_n} \) is isomorphic to the Borel-Weil line bundle \( G^n B^n \times (\lambda_1^{-1}, \ldots \lambda_n^{-1}) \).
Lemma 22 Assume $\lambda_1, \ldots, \lambda_n$ are dominant weights (possibly on the wall of the Weyl chamber). Then $H^i(Y_{w_1, \ldots, w_n}, L_{\lambda_1, \ldots, \lambda_n}) = 0$ for all $i > 0$.

Proof (van der Kallen). Note that $L_{\lambda_1, \ldots, \lambda_n}$ is effective, but not necessarily ample, so we cannot deduce the conclusion directly from Theorem 16.

Recall the following facts from $B$-module theory [23], [12]:
(a) An excellent filtration of a $B$-module is one whose quotients are isomorphic to Demazure modules $H^0(B w B, L_{\lambda_1, \ldots, \lambda_n})$, for Weyl group elements $w$ and dominant weights $\lambda$.
(b) If $M$ has an excellent filtration, and $E(M) \overset{\text{def}}{=} G \times M$ is the corresponding vector bundle on $G/B$, then $H^i(G/B, E(M)) = 0$ for all $i > 0$, and $H^0(G/B, E(M))$ has an excellent filtration.
(c) Polo’s Theorem: If $M$ has an excellent filtration, then so does $(\lambda - 1) \otimes M$ for any dominant weight $\lambda$.

Now consider the fiber bundle
$$Y_{w_2, \ldots, w_n} \to Y_{w_1, w_2, \ldots, w_n} \to B w_1 B$$
which leads to the spectral sequence
$$H^i(B w_1 B, E((\lambda - 1) \otimes H^j(Y_{w_2, \ldots, w_n}, L_{\lambda_2, \ldots, \lambda_n}))) \Rightarrow H^{i+j}(Y_{w_1, w_2, \ldots, w_n}, L_{\lambda_1, \lambda_2, \ldots, \lambda_n}).$$

By induction, assume that $H^j(Y_{w_2, \ldots, w_n}, L_{\lambda_2, \ldots, \lambda_n}) = 0$ for $j > 0$, and that $H^0(Y_{w_2, \ldots, w_n}, L_{\lambda_2, \ldots, \lambda_n})$ has an excellent filtration. Then applying (b) and (c), we find
$$H^i(Y_{w_1, w_2, \ldots, w_n}, L_{\lambda_1, \lambda_2, \ldots, \lambda_n}) = H^i(B w_1 B, E((\lambda - 1) \otimes H^0(Y_{w_2, \ldots, w_n}, L_{\lambda_2, \ldots, \lambda_n}))) = 0$$
for $i > 0$, and that $H^0(Y_{w_1, w_2, \ldots, w_n}, L_{\lambda_1, \lambda_2, \ldots, \lambda_n})$ has an excellent filtration. •

4.2 Frobenius splitting of Grassmannians

We would now like to push forward the Frobenius splittings found above for flag varieties to get splittings of configuration varieties. For this we need a combinatorial prerequisite.

Given a diagram $D = (C_1, C_2, \ldots, C_r)$ with $\leq N$ rows, consider a sequence of permutations (Weyl group elements) $u_1, u_2, \ldots \in \Sigma_N$ such that,
for all \( j \):

(\( \alpha \)) \( \ell(u_j) = \ell(u_{j-1}) + \ell(u_{j-1}^{-1}u_j) \), and

(\( \beta \)) \( u_j(\{1, 2, \ldots, |C_j|\}) = C_j \). The first condition says that the sequence is increasing in the weak order on the Weyl group. In the next section, we will give an algorithm which produces such a sequence for any northwest diagram, so that the following theorem will apply:

**Proposition 23** If \( D \) a diagram which admits a sequence of permutations \( u_1, u_2, \ldots \) satisfying (\( \alpha \)) and (\( \beta \)) above, then the pair \( \mathcal{F}_D \subset \text{Gr}(D) \) is compatibly split for any field \( F \) of positive characteristic.

Hence over an algebraically closed field \( F \) of arbitrary characteristic,

(a) the cohomology groups \( H^i(\mathcal{F}_D, \mathcal{L}_D) = 0 \) for \( i > 0 \);

(b) the restriction map \( \text{rest}_\Delta : H^0(\text{Gr}(D), \mathcal{L}_D) \rightarrow H^0(\mathcal{F}_D, \mathcal{L}_D) \) is surjective;

(c) \( \mathcal{F}_D \) is a normal variety.

**Proof.** By (\( \beta \)), the maximal parabolic subgroups \( P_C = \{(x_{ij}) \in GL(N) | x_{ij} = 0 \text{ if } i \notin C, j \in C\} \) satisfy \( u_i B u_i^{-1} \subset P_C \). Write

\[ \text{Gr}(D) = \text{Gr}(C_1) \times \cdots \times \text{Gr}(C_r) \cong G/P_{C_1} \times \cdots \times G/P_{C_r}, \]

and consider the \( G \)-equivariant projection

\[ \phi : (G/B)^r \rightarrow \text{Gr}(D) \]

\[ (g_1 B, \ldots, g_r B) \mapsto (g_1 u_1^{-1} P_{C_1}, \ldots, g_r u_r^{-1} P_{C_r}) \]

Then we have \( \phi(u_1 B, \ldots, u_r B) = (I P_{C_1}, \ldots, I P_{C_r}) \) and \( \phi(\mathcal{F}_{u_1, \ldots, u_r}) = \mathcal{F}_D \).

Since \( \phi \) is a map with connected fibers between smooth projective varieties, we can push forward the compatible splitting for \( \mathcal{F}_{u_1, \ldots, u_r} \subset (G/B)^r \) found in the previous section. Applying Theorem 16 and Propositions 17 and 14, we have the assertions of the theorem. \( \bullet \)

Note that (b) and (c) of the Proposition are equivalent to the projective normality of \( \mathcal{F}_D \) with respect to \( \mathcal{L}_D \).

**Conjecture 24** For any diagram \( D \), and any Weyl group elements \( u_1, \ldots u_r \), the pairs \( \mathcal{F}_D \subset \text{Gr}(D) \) and \( \mathcal{F}_{u_1, \ldots, u_r} \subset (G/B)^r \) are compatibly split.

In order to prove the character formula in the last section of this paper, we will need stronger relations between the singular configuration varieties and their desingularizations. In particular, we will show that our varieties have rational singularities.
Lemma 25 Let $X$, $Y$ be algebraic varieties with an action of an algebraic group $G$, and $f : X \to Y$ an equivariant morphism. Assume that $X$ has an open dense $G$-orbit $G \cdot x_0$, and take $y_0 = f(x_0)$, $G_0 = \text{Stab}_G y_0$

Then $f^{-1}(y_0) = G_0 \cdot x_0$. In particular, if $G_0$ is connected, then $f^{-1}(y_0)$ is connected and irreducible.

Proof. For $F = \mathbb{C}$, this is trivial. Take $x_1 \in f^{-1}(y_0)$, and consider a path $x(t) \in X$ such that $x(0) = x_1$ and $x(t) \in G \cdot x_0$ for small $t > 0$. Then the path $f(x(t))$ lies in $G \cdot y_0$ for small $t \geq 0$, and we can lift it to a path $g(t) \in G$ such that $g(0) = \text{id}$ and $f(x(t)) = g(t) \cdot y_0$ for small $t \geq 0$. Then $\tilde{x}(t) = g(t)^{-1} \cdot x(t)$ satisfies $\tilde{x}(0) = x_1$, $\tilde{x}(t) \in G_0 \cdot x_0$ for small $t > 0$.

For general $F$, T. Springer has given the following clever argument. Assume without loss of generality that $X$ is irreducible and $G \cdot y_0$ is open dense in $Y$. Since an algebraic map is generically flat, and $G \cdot y_0$ is open, all the irreducible components $C$ of $f^{-1}(y_0)$ have the same dimension $\dim C = \dim X - \dim Y$. Let $Z = G \cdot C$ be the closure of one of these components. Now, the restriction $f : Z \to Y$ also satisfies our hypotheses, with $C \subset Z$ again a component of the fiber of the restricted $f$, so we again have $\dim C = \dim Z - \dim Y$, and $\dim Z = \dim X$. Thus $G \cdot C$ is an open subset of $X$, since $X$ is irreducible.

Now consider the open set $G \cdot C \cap G \cdot x_0 \subset X$. Choose a point $z$ in this set which does not lie in any other component $C'$ of our original $f^{-1}(y_0)$. For any other component $C'$, choose a similar point $z'$. But we have $g \cdot z \in C$, $g' \cdot z' = g_0 g \cdot z \in C'$ for some $g, g', g_0 \in G$, and in fact $g_0 \in G_0$. Thus $C' = g_0 \cdot C$, and $G_0$ permutes the components transitively. Hence, $G_0 \cdot x_0$ has at least as many irreducible components as the whole $f^{-1}(y_0)$, and the lemma follows. \hfill ⊗

Proposition 26 (Inamdar-van der Kallen) Suppose $D_1$, $D_2$ are diagrams admitting sequences of permutations with (α) and (β) as above, such that $D_2$ is obtained by removing some of the columns of $D_1$. Denote $\mathcal{F}_1 = \mathcal{F}_{D_1}$, $\mathcal{F}_2 = \mathcal{F}_{D_2}$, $\mathcal{L}_2 = \mathcal{L}_{D_2}$, and consider the projection $\text{pr} : \mathcal{F}_1 \to \mathcal{F}_2$.

Then:
(a) $H^0(\mathcal{F}_1, \text{pr}^* \mathcal{L}_2) = H^0(\mathcal{F}_2, \mathcal{L}_2)$, and this $G$-module has a good filtration (one whose quotients are isomorphic to $H^0(G/B, \mathcal{L}_\lambda)$ for dominant weights $\lambda$).
(b) $H^i(\mathcal{F}_1, \text{pr}^* \mathcal{L}_2) = H^i(\mathcal{F}_2, \mathcal{L}_2) = 0$ for all $i > 0$. 

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(c) $R^i \Pr_* O_{\mathcal{F}_2} = 0$ for all $i > 0$.
(d) If $F$ has characteristic zero, then $\mathcal{F}_D$ has regular singularities for any northwest diagram $D$.

Proof. (i) Consider a sequence of permutations $w_1, w_2, \ldots, w_r$ (where $r$ is the number of columns in $D_1$) such that $u_1 = w_1, u_2 = w_1 w_2, \ldots$ satisfies $(\alpha)$ and $(\beta)$, and let $Y = Y_{w_0, w_1, \ldots, w_r}$, (where $w_0$ is the longest permutation). Then we have a commutative diagram of surjective morphisms

$$
\begin{array}{ccc}
Y & \xrightarrow{\Phi_1} & \mathcal{F}_1 \\
\downarrow \Phi_2 & & \downarrow \Pr \\
\mathcal{F}_2 & & 
\end{array}
$$

where $\Phi_j = \phi \circ f$, where $\phi$ and $f$ are the maps defined in the proofs of Propositions 20 and 23 in the cases $D = D_j$. All of these spaces have dense $G$-orbits. Furthermore, the stabilizer of a general point in $\mathcal{F}_D$ is an intersection of parabolic subgroups and is connected. Thus, by the above lemma, the fibers of $\Phi_1$ are generically connected.

(ii) Now (i) and Lemma 22 insure that the hypotheses of Kempf’s lemma (Proposition 21) are satisfied. Thus $R^i(\Phi_1)_* O_Y = 0$ for $i > 0$, and by the Leray spectral sequence we have, for all $i \geq 0$,

$$
H^i(Y, \Phi_1^* \Pr^* \mathcal{L}_2) = H^i(\mathcal{F}_1, (\Phi_1)_*(\Phi_1)^* \Pr^* \mathcal{L}_2).
$$

(iii) Furthermore, $\mathcal{F}_1$ is normal by the previous Proposition, and $\Phi_1$ is separable with connected fibers, so

$$
(\Phi_1)_*(\Phi_1)^* \Pr^* \mathcal{L}_2 \cong [(\Phi_1)_*(\Phi_1)^* O_{\mathcal{F}_1}] \otimes \Pr^* \mathcal{L}_2 \\
\cong \Pr^* \mathcal{L}_2.
$$

Thus $H^i(Y, \Phi_1^* \Pr^* \mathcal{L}_2) = H^i(\mathcal{F}_1, \Pr^* \mathcal{L}_2)$ for all $i \geq 0$.

(iv) An exactly similar argument shows that $H^i(Y, \Phi_2^* \mathcal{L}_2) = H^i(\mathcal{F}_2, \mathcal{L}_2)$ for all $i \geq 0$. But $\Phi_2^* = \Phi_1^* \Pr^*$, so for all $i$,

$$
H^i(\mathcal{F}_1, \Pr^* \mathcal{L}_2) = H^i(Y, \Phi_2^* \mathcal{L}_2) = H^i(\mathcal{F}_2, \mathcal{L}_2).
$$

But we saw in Lemma 22 that $H^i(Y, \Phi_2^* \mathcal{L}_2)$ vanishes for $i > 0$, so (b) of the present Proposition follows.
(iv) We also saw in the proof of Lemma 22 that \( H^0(Y, \Phi_2^*L_2) \) has an excellent filtration as a \( B \)-module. But this is equivalent to it having a good filtration as a \( G \)-module, so (a) follows.

(v) Now consider the spectral sequence
\[
R^i \text{pr}_* R^j (\Phi_1)_* \mathcal{O}_Y \Rightarrow R^{i+j} (\Phi_2)_* \mathcal{O}_Y.
\]
For \( i > 0 \), we have \( R^i (\Phi_1)_* \mathcal{O}_Y = 0 \) and \( R^i (\Phi_2)_* \mathcal{O}_Y = 0 \) by (ii) above. Because of this and the normality of \( F_1 \), we have for all \( i > 0 \),
\[
0 = R^i (\Phi_2)_* \mathcal{O}_Y = R^i \text{pr}_* (\Phi_1)_* \mathcal{O}_Y = R^i \text{pr}_* \mathcal{O}_{F_1}.
\]
this shows (c).

(vi) Now take \( D_2 = D \) an arbitrary northwest diagram, and \( D_1 = \hat{D} \) its maximal blowup. Then pr is a resolution of singularities by Proposition 13. Assume, as we will show in the next section, that \( D_1 \) admits a sequence of permutations as required. Then (c) holds, and this is precisely the definition of rational singularities in characteristic zero, so we have (d).

4.3 Monotone sequences of permutations

Let \( D = (C_1, C_2, \ldots, C_r) \) be a northwest diagram with \( \leq N \) rows. In this section, we will construct by a recursive algorithm a sequence of permutations
\[
u_1, \nu_2, \ldots \in \Sigma_N\]
satisfying the conditions of the previous section: for all \( j \),
(a) \( \ell(u_j) = \ell(u_{j-1}) + \ell(u_{j-1}^{-1} u_j) \), and
(b) \( u_j(\{1, 2, \ldots, |C_j|\}) = C_j \).

For each column \( C \) of \( D \), define the integer
\[
gap_N(C) = \begin{cases} \max \{i \mid i \notin C, \exists i' \in C : i < i'\}, & \text{if this set is } \neq \emptyset \\ N, & \text{if the above set is empty} \end{cases}
\]
Since \( D \) is northwest, there is an integer \( J_N \geq 1 \) such that
\[
N = \gap_N(C_1) = \cdots = \gap_N(C_{J_N-1}) > \gap_N(C_{J_N}) = \cdots = \gap_N(C_r).
\]
Now define the derived diagram \( D' \) of \( D \) as follows. Given a column \( C \) of \( D \), there is a corresponding column \( C' \) of \( D' \):
\[
C' = \{i \mid i \in C, i < \gap_N(C)\} \cup \{i - 1 \mid i \in C, i > \gap_N(C)\}.
\]
That is, we take $C$ and push all squares below the $\text{gap}_N(C)$-th row upward by one place.

**Lemma 27** If $D$ is northwest with $\leq N$ rows, then $D'$ is northwest with $\leq N - 1$ rows.

**Proof.** The only doubtful case in checking the northwest property is that of two squares $(i_1, j_1)$ and $(i_2, j_2)$ in $D'$ with $j_1 < J_N \leq j_2$ and $i_1 > i_2$. Since $j_1 < J_N$, we have $C_{j_1} = \{1, 2, \ldots, i_1, i_1 - 1, \ldots\}$, so that $i_2 \in C_{j_1}$ and $i_2 \in C'_{j_1}$. Hence $(i_2, j_1) \in D'$ as required. □

Now, consider the following elements of $\Sigma_N$:

$$\kappa_n^{(N)}(i) = \begin{cases} i & \text{if } i < n \\ i + 1 & \text{if } n \leq i < N \\ n & \text{if } i = N \end{cases}$$

Then $\kappa_1^{(N)}, \ldots, \kappa_N^{(N)}$ are minimum length coset representatives of the quotient $\Sigma_N/\Sigma_{N-1}$, and for any permutation $\pi \in \Sigma_{N-1}$, we have $\ell(\kappa_n \pi) = \ell(\kappa_n) + \ell(\pi)$.

Now, starting with $D$, a northwest diagram with $\leq N$ rows, we can define a sequence of derived diagrams $D = D^{(N)}, D^{(N-1)}, \ldots, D^{(1)}$, where $D^{(i)} = (D^{(i+1)})'$ is a northwest diagram with $\leq i$ rows. Let the columns of $D^{(i)}$ be $C_1^{(i)}, \ldots, C_r^{(i)}$, and define $\text{gap}(i, j) = \text{gap}(i, C_j^{(i)})$. For each $i$, we have

$$i = \text{gap}(i, 1) = \cdots = \text{gap}(i, J_i - 1) > \text{gap}(i, J_i) = \cdots = \text{gap}(i, r).$$

Then either $\kappa_{\text{gap}(i, j)}^{(i)}(\{1, 2, \ldots, i - 1\}) \supset C_j^{(i)}$, or $C_j^{(i)} = \{1, 2, \ldots, i\}$. Notice that $J_N \leq J_{N-1} \leq \cdots$.

Finally, for each column $j = 1, \ldots, r$, define

$$u_j = \kappa_{\text{gap}(N, j)}^{(N)} \kappa_{\text{gap}(N-1, j)}^{(N-1)} \cdots \kappa_{\text{gap}(1, j)}^{(1)}.$$

This is a reduced decomposition of $u_j$, in the sense that $\ell(u_j)$ is the sum of the lengths of the factors. Since $\kappa_i^{(i)} = \text{id}$, and $J_N \leq J_{N-1} \leq \cdots$, each $u_j$ is an initial string of $u_{j+1}$. Thus the $u_j$ have the desired monotonicity property $(\alpha)$.

It only remains to show property $(\beta)$:

**Lemma 28** For each column $C_j$ of $D$, $u_j(\{1, 2, \ldots, |C_j|\}) = C_j$. 29
Proof. For each $i$, we have a $u_j^{(i)}$ associated to $D^{(i)}$, with $u_j^{(i+1)} = \kappa_{\text{gap}(i+1,j)}^{(i+1)} u_j^{(i)}$.

For a given $i < N$, assume that $u_j^{(i)}(\{1, 2, \ldots, |C_j^{(i)}|\}) \subset C_j^{(i)}$. Then I claim the same is true for $i + 1$.

This is clear, because $C_j^{(i+1)}$ is $C_j^{(i)}$ with some of its squares pushed down, and $\kappa_{\text{gap}(i+1,j)}^{(i+1)}$ pushes down these squares to the proper positions. In the case that $C_j^{(i+1)} = \{1, 2, \ldots, l\}$, for some $l$, we have $u_j^{(i+1)} = u_j^{(i)} = \text{id}$, and the claim is again true.

The lemma now follows by induction on $i$. 

5. A Weyl character formula

The results of the last two sections allow us to apply the Atiyah-Bott Fixed Point Theorem to compute the characters of the Schur modules for northwest diagrams. To apply this theorem, we must examine the points of $\mathcal{F}_D$ fixed under the action of $H$, the group of diagonal matrices. We must also understand the action of $H$ on the tangent spaces at the fixed points.

5.1 Fixed points and tangent spaces

The following formula is due to Atiyah and Bott \cite{Atiyah68} in the complex analytic case, and was extended to the algebraic case by Nielsen \cite{Nielsen93}, \cite{Atiyah68}.

Theorem 29 Let $F$ be an algebraically closed field, and suppose the torus $H = (F^{\times})^N$ acts on a smooth projective variety $X$ with isolated fixed points, and acts equivariantly on a line bundle $L \to X$. Then the character of $H$ acting on the cohomology groups of $L$ is given by:

$$\sum_i (-1)^i \text{tr}(h \mid H^i(X, L)) = \sum_{p \text{ fixed}} \frac{\text{tr}(h \mid L|_p)}{\det(\text{id} - h \mid T^*_pX)},$$

where $p$ runs over the fixed points of $H$, $L|_p$ denotes the fiber of $L$ above $p$, and $T^*_pX$ is the cotangent space. 

We will apply the formula for $X = \mathcal{F}_D$ a smooth configuration variety, where $D = (C_1, C_2, \ldots)$ is a lexicographic northwest diagram with $\leq N$ rows and $\hat{D} = D$. 

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**Fixed points.** Assume for now that the columns are all distinct. Let $H = \{ h = \text{diag}(x_1, \ldots, x_N) \in GL(N) \}$ act on $\text{Gr}(D)$ and $F_D$ by the restriction of the $GL(N)$ action. Then by Proposition 13, we have $I_D = \{ (V_C)_{C \in D} \in \text{Gr}(D) \} | C \subset C' \Rightarrow V_C \subset V_{C'}$, a smooth variety.

A point in $F_D \subset \text{Gr}(D) = \text{Gr}(C_1) \times \text{Gr}(C_2) \times \cdots$ is fixed by $H$ if and only if each component is fixed. Now, the fixed points of $H$ in $\text{Gr}(l, F^N)$ are the coordinate planes $E_{k_1 \ldots, k_l} = \text{Span}(e_{k_1}, \ldots, e_{k_l})$, where the $e_k$ are coordinate vectors in $F^N$ (c.f. 5). For instance, the fixed points in $P^{N-1}$ are the $N$ coordinate lines $F e_k$. We may naturally identify the tangent space $E$ of $\text{Gr}(C)$ if each component is fixed. Now, the fixed points of $\text{Gr}_{C_0}(l) = \bigcup_{C \subset C'} \text{Gr}(C)$ are the points in $\text{Gr}(C)$.

Hence the fixed points in $F_D$ are as follows: Take a function $t$ which assigns to any column $C$ a set $t(C) \subset \{ 1, \ldots, N \}$ with $|t(C)| = |C|$, and $C \subset C' \Rightarrow t(C) \subset t(C')$. We will call such a $t$ a standard column tableau for $D$. Then the fixed point corresponding to $t$ is $E_t = (E_{t(C)})_{C \in D}$.

**Tangent spaces at fixed points.** We may naturally identify the tangent space $T_{V_0} \text{Gr}(l, F^N) = \text{Hom}_F(V_0, F^N/V_0)$. If $V_0$ is a fixed point (that is, a space stable under $H$), then $h \in H$ acts on a tangent vector $\phi \in \text{Hom}_F(V_0, F^N/V_0)$ by $(h \cdot \phi)(v) = h(\phi(h^{-1}v))$. For $(V_C)_{C \in D} \in \text{Gr}(D)$, we have $T_{(V_C)} \text{Gr}(D) = \bigoplus_{C \subset C'} \text{Hom}(V_C, F^N/V_C)$. Furthermore, if $(V_C)_{C \in D} \in F_D$, then

$$T_{(V_C)} F_D = \{ \phi = (\phi_C)_{C \in D} \in \bigoplus_{C \subset C'} \text{Hom}(V_C, F^N/V_C) \mid C \subset C' \Rightarrow \phi_{C'} |_{V_C} \equiv \phi_C \mod V_{C'} \}$$

(that is, the values of $\phi_C$ and $\phi_{C'}$ on $V_C$ agree up to translation by elements of $V_{C'}$). See 5.

For a fixed point $E_t = (E_{t(C)})$, we will find a basis for $T_{E_t}$ consisting of eigenvectors of $H$. Now, the eigenvectors in $T_{E_t} \text{Gr}(D) = \bigoplus_{C \subset C'} \text{Hom}(V_C, F^N/V_C)$ are precisely $\phi^{ij}_{C_0} = (\phi^{ij}_{C_0})_{C \in D}$, where $i, j \leq N$, $C_0$ is a fixed column of $D$, and $\phi^{ij}_{C_0}(e_i) \overset{\text{def}}{=} \delta_{C_0,C} \delta_{i,j} (\delta$ being the Kronecker delta). The eigenvalue is

$$h \cdot \phi^{ij}_{C_0} = \text{diag}(x_1, \ldots, x_N) \cdot \phi^{ij}_{C_0} = x_i^{-1} x_j \phi^{ij}_{C_0}.$$

To obtain eigenvectors of $T_{E_t} F_D$, we must impose the compatibility conditions. An eigenvector $\phi$ with eigenvalue $x_i^{-1} x_j$ must be a linear combination $\phi = \sum_{C \in D} a_C \phi^{ij}_{C}$ with $a_C \in F$. By the compatibility, we have that

$$C \subset C', i \in t(C), j \notin t(C') \Rightarrow a_C = a_{C'}.$$
We wish to find the number $d_{ij}$ of linearly independent solutions of this condition for $a_C$.

Given a poset with a relation $\subset$, define its **connected components** as the equivalence classes generated by the elementary relations $x \sim y$ for $x \subset y$. Now for a given $i, j$ consider the poset whose elements are those columns $C$ of $D$ such that $i \in t(C)$, $j \not\in t(C)$, with the relation of ordinary inclusion. Then $d_{ij}$ is the number of components of this poset.

Note that the eigenvectors for all the eigenvalues span the tangent space. Thus

$$\det(\id - h | T_{E_t}^*) = \prod_{i \neq j} (1 - x_i x_j^{-1})^{d_{ij}(t)}.$$

**Bundle fibers above fixed points.** Finally, let us examine the line bundles $L$ on $\mathcal{F}_D$ obtained by giving each column $C$ a multiplicity $m(C) \geq 0$. If $m(C) > 0$ for all columns $C$ of $D$, then $L \cong \mathcal{L}_{D'}$ for the diagram $D'$ with the same columns as $D$, each repeated $m(C)$ times. If some of the $m(C) = 0$, then $L$ is the pullback of $\mathcal{L}_{D'}$ for the diagram $D'$ with the same columns as $D$, each taken $m(C)$ times, where 0 times means deleting the column. In the second case, $L$ is effective, but not ample.

It follows easily from the definition that

$$\text{tr}(h | L|_{E_t}) = x_1^{-\text{wt}_1(t)} \cdots x_N^{-\text{wt}_N(t)},$$

where

$$\text{wt}_i(t) = \sum_{C \in t(C)} m(C).$$

Hence we obtain:

$$\sum_i (-1)^i \text{tr}(h | H^i(\mathcal{F}_D, \mathcal{L}_D)) = \sum_t \frac{\prod_i x_i^{-\text{wt}_i(t)}}{\prod_{i \neq j} (1 - x_i x_j^{-1})^{d_{ij}(t)}},$$

where $t$ runs over the standard column tabloids of $D$.

### 5.2 The character formula

We summarize in combinatorial language the implications of the previous section.
We think of a diagram $D$ as a list of columns $C_1, C_2, \ldots \subset \mathbb{N}$, possibly with repeated columns. Given a diagram $D$, the blowup diagram $\widehat{D}$ is the diagram whose columns consist of all the columns of $D$ and all possible intersections of these columns. We will call the columns which we add to $D$ to get $\widehat{D}$ the phantom columns.

We may define a standard column tabloid for the diagram $\widehat{D}$ with respect to $GL(N)$, to be a filling (i.e. labeling) of the squares of $\widehat{D}$ by integers in $\{1, \ldots, N\}$, such that:

(i) the integers in each column are strictly increasing, and
(ii) if there is an inclusion $C \subset C'$ between two columns, then all the numbers in the filling of $C$ also appear in the filling of $C'$.

Given a tabloid $t$ for $\widehat{D}$, define integers $w_i(t)$ to be the number of times $i$ appears in the filling, but not counting $i$'s which appear in the phantom columns. Also define integers $d_{ij}(t)$ to be the number of connected components of the following graph: the vertices are columns $C$ of $\widehat{D}$ such that $i$ appears in the filling of $C$, but $j$ does not; the edges are $(C, C')$ such that $C \subset C'$ or $C' \subset C$. (An empty graph has zero components.)

Recall that a diagram $D$ is northwest if $i \in C_j$, $i' \in C_{j'} \Rightarrow \min(i, i') \in C_{\min(j, j')}$. The following theorem applies without change to northeast diagrams and any other diagrams obtainable from northwest ones by rearranging the order of the rows and the order of the columns. Also, we can combine it with Theorem 5 to compute the character for the complement of a northwest diagram in an $N \times r$ rectangle.

Denote a diagonal matrix by $h = \text{diag}(x_1, \ldots, x_N)$.

**Theorem 30** Suppose $D$ is a northwest diagram with $\leq N$ rows, and $F$ an algebraically closed field. Then:

(a) The character of the Weyl module $W_D$ (for $GL(N, F)$) is given by

$$\text{char}_{W_D}(h) = \sum_t \prod_i x_i^{-w_i(t)} \prod_{i \neq j} (1 - x_i x_j^{-1})^{d_{ij}(t)}.$$ 

where $t$ runs over the standard tabloids for $\widehat{D}$.

(b) For $F$ of characteristic zero, the character of the Schur module $S_D$ (for $GL(N, F)$) is given by

$$\text{char}_{S_D}(h) = \sum_t \prod_i x_i^{w_i(t)} \prod_{i \neq j} (1 - x_i^{-1} x_j) d_{ij}(t).$$
where $t$ runs over the standard tabloids for $\hat{D}$.

Example. Consider the following diagram and some of its standard tabloids for $N = 3$:

\[
\begin{array}{cc}
D &=& \begin{array}{c}
\square \ \square \\
\square \ \square
\end{array} & t_1 = \begin{array}{cc}
1 & 1 \\
2 & 2
\end{array} & t_2 = \begin{array}{cc}
1 & 1 \\
2 & 1
\end{array} & t_3 = \begin{array}{cc}
3 & 2 \\
3 & 3
\end{array}
\end{array}
\]

The tabloid $t_1$ has $d_{12} = d_{13} = d_{21} = d_{23} = 1$, $d_{31} = d_{32} = 0$, and $t_2$ has $d_{12} = 2$, $d_{13} = d_{23} = 1$, $d_{21} = d_{31} = d_{32} = 0$. The other standard tabloids can be obtained from $t_1$ and $t_2$ by applying a permutation of $\{1,2,3\}$ to the entries, and rearranging the entries in the middle column to make them increasing. For instance, $t_3 = \pi t_2$, where $\pi$ is the transposition $(13)$. Note that the standard tabloids are standard tableaux in the usual sense: they are fillings with the columns strictly increasing, and the rows non-increasing. This is true in general when $\hat{D}$ is a skew diagram with no repeated columns, though not all the standard tableaux are obtained in this way.

Applying our formula we find that $\text{char } S_D = s_{(3,1,0)} + s_{(2,2,0)}$, where $s_{(\lambda_1,\lambda_2,\lambda_3)}$ is a classical Schur function, the character of an irreducible Schur module. Since $D$ is a skew diagram, we could have obtained this result using the Littlewood-Richardson Rule. It should be possible to prove this rule using the present methods. 

**Proof of the Theorem.** (i) Consider the map $pr : \mathcal{F}_{\hat{D}} \to \mathcal{F}_D$, and the pullback line bundle $pr^*\mathcal{L}_D$. This is the bundle on $\mathcal{F}_{\hat{D}}$ corresponding to giving the phantom columns $C$ of $\hat{D}$ multiplicity $m_C = 0$. Let RHS denote the right hand side of our formula in (a). Then by the analysis of Section 5.1, RHS is equal to the right hand side of the Atiyah-Bott formula (Theorem 29) for $X = \mathcal{F}_{\hat{D}}$, $L = pr^*\mathcal{L}_D$. Thus

$$\text{RHS} = \text{char } \sum (-1)^i H^i(\mathcal{F}_{\hat{D}}, pr^*\mathcal{L}_D).$$

(ii) By Proposition 27, we have $H^i(\mathcal{F}_{\hat{D}}, pr^*\mathcal{L}_D) = 0$ for $i > 0$, and $H^0(\mathcal{F}_{\hat{D}}, pr^*\mathcal{L}_D) = H^0(\mathcal{F}_D, \mathcal{L}_D)$. Thus RHS $= \text{char } H^0(\mathcal{F}_D, \mathcal{L}_D)$.

(iii) By Proposition 23, the restriction of global sections of $\mathcal{L}_D$ from $\text{Gr}(D)$ to $\mathcal{F}_D$ is surjective, and we have

$$\text{RHS} = \text{char } H^0(\mathcal{F}_D, \mathcal{L}_D)$$

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\[ \text{char Im} \left( \text{rest} : H^0(\text{Gr}(D), \mathcal{L}_D) \to H^0(\mathcal{F}_D, \mathcal{L}_D) \right) \]
\[ = \text{char} W_D. \]

The last equality holds by Proposition 2, and we have proved (a). Then (b) follows because \( S_D = (W_D)^* \).

### 5.3 Betti numbers

In this section, we compute the betti numbers of the smooth configuration varieties of Section 5.1.

**Proposition 31** (Bialynicki-Birula [3]) Let \( X \) be a smooth projective variety over an algebraically closed field \( F \), acted on by the one-dimensional torus \( F^\times \) with isolated fixed points. Then there is a decomposition

\[ X = \bigcup_{p \text{ fixed}} X_p, \]

where the \( X_p \) are disjoint, locally closed, \( H \)-invariant subvarieties, each isomorphic to an affine space \( X_p \cong \mathbb{A}^{d^+(p)} \).

The dimensions \( d^+(p) \) are given as follows. Let the tangent space \( T_{p}X \cong \bigoplus_{n \in \mathbb{Z}} a_n(p)F_n \), where \( a_n(p) \in \mathbb{N} \) and \( F_n \) is the one-dimensional representation of \( F^\times \) for which the group element \( t \in F^\times \) acts as the scalar \( t^n \). Then

\[ d^+(p) = \sum_{n > 0} a_n(p). \]

Over \( \mathbb{C} \), the above proposition does not quite give a CW decomposition for \( X \), since the boundaries of the cells need not lie in cells of lower dimension. Nevertheless, \( \dim_{\mathbb{R}} \partial X_p \leq \dim_{\mathbb{R}} X_p - 2 \), and this is enough to fix the betti numbers \( \beta_i = \dim_{\mathbb{R}} H^i(X, \mathbb{R}) \): \( \beta_{2i} = \# \{ p \mid d^+(p) = i \} \), and \( \beta_{2i+1} = 0 \).

Now, in our case consider the spaces \( X = \mathcal{F}_D \) of Section 5.1 acted on by the \( N \)-dimensional torus \( H \). Consider the embedding

\[ \hat{\rho} : F^\times \to H \]
\[ t \mapsto \text{diag}(t^{N-1}, t^{N-2}, \ldots, t, 1), \]
corresponding to the coweight \( \check{\rho} = \) half sum of the positive coroots. Then \( \check{\rho}(F^\times) \) has the same (isolated) fixed points as \( H \), since none of the eigenvectors of \( H \) on \( T_\rho X \) is fixed by \( \check{\rho}(F^\times) \). (I.e., \( (\alpha, \check{\rho}) \neq 0 \) for any root \( \alpha \).)

Also, a given eigenvector of weight \( x_i x_j^{-1} \) is of positive \( \check{\rho}(F^\times) \) weight exactly when \( i < j \). Thus, for a fixed point (standard tabloid) \( t \) of \( D \), define

\[
d^+(t) = \sum_{i<j} d_{ij}(t).
\]

We then have the

**Proposition 32** Suppose \( F = \mathbb{C} \), and \( D \) is a northwest diagram with \( \leq N \) rows and \( \check{D} = D \). Then the betti numbers

\[
\beta_{2i} = \# \{ t \mid d^+(t) = i \}, \quad \beta_{2i+1} = 0,
\]

and the Poincare polynomial

\[
P(x, \mathcal{F}_D) \overset{\text{def}}{=} \sum_i \beta_i x^i = \sum_t x^{2d^+(t)},
\]

where \( t \) runs over the standard tabloids of \( D \). ●

In fact, our proof shows the above proposition for a broader class of spaces. Suppose \( D = (C_1, C_2, \ldots) \) is an arbitrary diagram such that the variety

\[
\text{Inc}_D \overset{\text{def}}{=} \{(V_C)_{C \in D} \in \text{Gr}(D) \mid C \subset C' \Rightarrow V_C \subset V_{C'}\}
\]

is smooth. Then the proposition holds with \( \mathcal{F}_D \) replaced by \( \text{Inc}_D \).

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