ON CONJECTURES OF A. EREMENKO AND A. GABRIELOV

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Abstract. We study polynomials $p(x)$ satisfying a differential equation of the form $p'' - h'p' + Hp = 0$, where $h = x^3/3 + ax$. We prove a conjecture of A. Eremenko and A. Gabrielov.

1. Introduction

In this note we consider the equations of the form

$$p''(x) - (x^2 + a)p' + H(x)p = 0 \tag{1.1}$$

which have a polynomial solution. Such equations appear in the study of the elementary eigenfunctions of the Schrödinger equation with quartic potential, see [EG].

We consider the corresponding local system. The cohomology of the system is two-dimensional. Our main result is the proof of a conjecture of [EG], which describes the cohomology class of the polynomial $p^2(-x)$.

The problem of computing the cohomology classes is formulated in an algebraic setting, see Section 3 but we use complex-analytic tools to solve it. Our main insight comes from the consideration of the bispectral dual equation to (1.1), see equation (5.1).

To prove the wanted equality of two constants we interpret them as values at zero of two a priori different solutions of equation (5.1) and then show that the two solutions actually are the same comparing their asymptotics via steepest descent method.

The paper is written as follows. In Section 2 we discuss the local system associated to (1.1). We proceed to describe an explicit basis in the cohomology in Section 3. In Section 4 we exhibit polynomials which are homologous to a constant multiple of the first basis element proving in particular Conjecture 1 from [EG]. We discuss the bispectral dual equation in Section 5. Section 6 is devoted to the elementary computation with the characteristic equation of the linear operator corresponding to (1.1). Finally, we prove our main results Theorem 7.1 and Corollary 7.2 in Section 7.

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2. Elementary remarks

Fix \( a \in \mathbb{C} \) and let
\[
h(x) = \frac{x^3}{3} + ax \in \mathbb{C}[x].
\]

Denote by the prime the operator of differentiation with respect to variable \( x \) and define a linear map on rational functions of \( x \):
\[
D : \mathbb{C}(x) \to \mathbb{C}(x), \quad q(x) \mapsto q'(x) + h'(x)q(x).
\]
The map \( D \) is inherited from the derivative map \( \frac{d}{dx} : \mathbb{C}(x)e^{h(x)} \to \mathbb{C}(x)e^{h(x)} \).

Let \( C \subset \mathbb{C}(x) \) be the image of \( D \). Let
\[
R = \{ q(x) \in \mathbb{C}(x) \mid \text{Res } q(x)e^{h(x)} = 0 \}
\]
be the subspace of rational function which have no residues after multiplication by the exponential of \( h(x) \). We have \( C \subset R \).

For \( q_1(x), q_2(x) \in \mathbb{C}(x) \), we write \( q_1(x) \sim q_2(x) \) if and only if \( q_1(x) - q_2(x) \in C \).

Let \( \gamma_j(t), j = 0, 1, 2 \), be any smooth curves in complex plane such that
\[
\lim_{t \to -\infty} \arg(\gamma_j(t)) = \pi(1/3 + 2j/3), \quad \lim_{t \to \infty} \arg(\gamma_j(t)) = \pi(1 + 2j/3)
\]
and \( \lim_{t \to \pm \infty} |\gamma_j(t)| = \infty \), see Figure 1.

Define functionals \( l_j \in R^* \), \( j = 0, 1, 2 \), by the formula
\[
l_j(q(x)) = \int_{\gamma_j} q(x)e^{h(x)}dx.
\]
Here we chose the contour \( \gamma_j \) so that it does not go through possible poles of \( q(x) \).

Clearly, the functionals \( l_i \) are well-defined and we have \( l_1 + l_2 + l_3 = 0 \).
Proposition 2.1. We have $\dim R/C = 2$. Moreover, for $q(x) \in R$ we have $q(x) \sim 0$ if and only if $l_j(q(x)) = 0$, $j = 1, 2$.

Proof. Let $q(x) \in R$. Write $q(x)$ as a sum of simple fractions. If we have a term $1/(x-z)^k$, for some $z \in \mathbb{C}$, then we subtract $D(1/(x-z)^{k-1})$ and decrease the order of the pole modulo $C$. Note, that since $q(x) \in R$, $k = 1$ is impossible. Therefore there exists a polynomial $f(x)$, such that $q(x) \sim f(x)$. Now if $\deg f(x) = n > 1$ we subtract $D(x^{n-2})$ and reduce the degree of $f(x)$ modulo $R$. It follows that $q(x) \sim ax+b$ for some choice $a,b \in \mathbb{C}$.

Since $|e^{h(t)}| \to 0$ as $t \to \pm\infty$, we have $l_j(C) = 0$. Therefore to finish the proof of the proposition, it is sufficient to show that $l_1,l_2 \in R^*$ are linearly independent functionals. Thus, it is sufficient to show that $\det(\int_{\gamma_j} x^{k-1} e^{h(x)} dz)_{j,k=1,2} = 0$. But this determinant is non-zero because it equals the Wronski determinant $W(\phi_1(u),\phi_2(u))$ where $\phi_j(u) = \int_{\gamma_j} e^{h(x)+ux} dx$ are fundamental solutions of the Airy equation $f''(u) + (u+a)f(u) = 0$.

We remark that one can replace the cubic odd polynomial $h(x)$ with an arbitrary polynomial of degree $k$ and similarly define $k$ functionals and prove a generalization of Proposition 2.1 with $\dim R/C = k - 1$.

Let $p(x) \in \mathbb{C}[x]$ be a polynomial of degree $n$ with simple roots only. Let

$$R(p(x)) = R \cap \frac{\mathbb{C}[x]}{p^2(x)}, \quad C(p(x)) = C \cap \left(\frac{\mathbb{C}[x]}{p(x)}\right).$$

Then clearly $C(p(x)) \subset R(p(x))$. If $q_1(x), q_2(x) \in R(p(x)) \subset R$ then clearly $q_1(x) \sim q_2(x)$ if and only if $q_1(x) - q_2(x) \in C(p(x))$.

Lemma 2.2. We have $\dim R(p(x))/C(p(x)) = 2$. Moreover, for $q(x) \in R(p(x))$ we have $q(x) \in R(p(x))$ if and only if $l_i(q(x)) = 0$, $i = 1, 2$.

Proof. The proof is similar to the proof of Proposition 2.1. □

3. When $p(x)$ is a wave polynomial

We start with the following lemma.

Lemma 3.1. Let $p(x) \in \mathbb{C}[x]$ be a polynomial with simple roots only. We have $1/p^2(x) \in R$ if and only if there exists $b \in \mathbb{C}$ such that $p(x)$ is a solution of the equation

$$y''(x) - h'(x)y'(x) + (nx + b)y(x) = 0.$$ \quad (3.1)

Proof. Let $z \in \mathbb{C}$ be such that $p(z) = 0$. We compute

$$\text{Res}_{x=z} \frac{e^{h(x)}}{p^2(x)} = \lim_{x \to z} \frac{e^{h(z)}(x-z)^2}{p^2(z)}' = e^{h(z)} \frac{h'(z)p'(z) - p''(z)}{(p'(z))^3}$$

by applying the L’Hopital rule. The lemma follows. □
We call a polynomial $p(x)$ satisfying (3.1) the wave polynomial of degree $n$. It is known that for each $n \in \mathbb{Z}_{\geq 1}$, $a \in \mathbb{C}$, there exists at least one wave polynomial of degree $n$, see also Section 6 below.

Note that all roots of all non-zero wave polynomials are simple.

**Lemma 3.2.** Let $p(x)$ be a wave polynomial. Let $f(x)$ be the polynomial such that $f'(x) = p(x)$, $f(0) = 0$. Then $f(x)/p^2(x) \in R(p(x))$ and for any $q(x) \in R(p(x))$, there exist unique $c, d \in \mathbb{C}$ such that $q(x) \sim (c + df(x))/p^2(x)$.

**Proof.** It follows that for any $c, d \in \mathbb{C}$, we have $(c + df(x))/(p(x))^2 \in R$. If $\deg p(x) = n$ then $\deg f(x) = n + 1$. If $g(x)/p^2(x) \in C$ then $\deg g(x) \geq n + 2$. The lemma follows from Lemma 2.2.

In what follows we study the constants $c, d$ for $q(x)$ of the form $p(-x)r(x)$, where $r(x)$ is a polynomial of degree at most $n$. In particular, we prove Conjecture 1 and formula (18) from [EG] describing the constants $c, d$ for $q(x) = p^2(-x)$.

4. When the constant $d$ is zero

Let $p(x)$ be a wave polynomial of degree $n$.

**Theorem 4.1.** For any polynomial $r(x)$ with $\deg r(x) \leq n$, there exists $c \in \mathbb{C}$ such that $r(x)p(-x) \sim c/p^2(x)$.

**Proof.** For $j = 0, 1, 2$, consider

\[(4.1) \quad y_j(x) = p(x) \int_{\gamma_{j,x}} \frac{e^{h(z)}}{p^2(z)} \, dz,\]

where the integration is taken over the contour $\gamma_{j,x}(t)$, $t \in (0, 1]$, such that $\gamma_{j,x}(0) = x$ and $\lim_{t \to \infty} \arg \gamma_{j,x}(t) = \pi(1/3 + 2/3j)$, $\lim_{t \to \infty} |\gamma_{j,x}(t)| = \infty$. Clearly $y_j(x)$ are holomorphic functions satisfying (3.1). Let $H_j \subset \mathbb{C}$ be the half-planes given by

\[ H_j = \{ z \in \mathbb{C} \mid \pi(-1/6 + 2j/3) < \arg z < \pi(5/6 + 2j/3) \}, \]

see Figure 1. We have the following asymptotics:

\[ y_j(x) = \frac{e^{h(x)}}{p^{n+2}(1 + o(1))}, \quad x \to \infty, \quad x \in H_j. \]

It implies the following connection formulas

\[ y_{j+1}(x) = y_j(x) - J_j p(x), \quad j = 0, 1, 2, \]

where

\[ J_j = l_j \left( \frac{1}{p^2(x)} \right) = \int_{\gamma_j} \frac{e^{h(x)}}{p^2(x)} \, dx \in \mathbb{C}, \quad j = 0, 1, 2, \]

and $y_3(x) = y_0(x), J_3 = J_0$. 


By Proposition 2.1 and Lemma 2.2 it is sufficient to prove
\[ \left| \int_{\gamma_1} x^k p(-x)e^{h(x)} \, dx - \int_{\gamma_2} x^k p(-x)e^{h(x)} \, dx \right| = 0, \quad k = 0, 1, \ldots, n. \]

Make the change of variables \( x \to -x \) in the integrals and using the connection formulas we obtain that the determinant up to a sign is equal to
\[ J_2 \int_{\gamma_1} x^k p(x)e^{-h(x)} \, dx - J_1 \int_{\gamma_2} x^k p(x)e^{-h(x)} \, dx \]
\[ = \int_{\gamma_1} x^k (y_0 - y_2)e^{-h(x)} \, dx - \int_{\gamma_2} x^k (y_2 - y_1)e^{-h(x)} \, dx \]
\[ = \int_{\gamma_1} x^k y_0 e^{-h(x)} \, dx + \int_{\gamma_2} x^k y_1 e^{-h(x)} \, dx + \int_{\gamma_0} x^k y_2 e^{-h(x)} \, dx. \]

Here \( \tilde{\gamma}_j \) are the contours given by \( \tilde{\gamma}_j(t) = -\gamma_j(t) \) for all \( t \in \mathbb{R} \). Note that \( \tilde{\gamma}_{j+1} \subset H_j \), \( j = 0, 1, 2 \), and therefore the contour of integration in each of the last three integrals can be sent to infinity inside of \( H_j \). It follows that each of the three integrals is zero due to the asymptotics of \( y_j(x) \).

5. Bispectral dual equation

Motivated by [MTV] we consider the bispectral dual equation to (3.1):
\[ u\ddot{g}(u) - ng(u) - (u^2 - au + b)g(u) = 0, \]
where the dot denotes the derivative with respect to the variable \( u \).

Equation (5.1) is obtained from (3.1) by formal replacing the operator of multiplication by \( x \) with the operator \( d/du \) and the operator \( d/dx \) with operator of multiplication by \( u \) and placing all derivatives \( d/du \) to the right of the operators of multiplication by \( u \).

The solution sets of bispectral dual operators are often related by suitable transforms. We describe such transforms for the case of bispectral dual operators (3.1) and (5.1).

**Lemma 5.1.** Let \( p(x) \) be a polynomial solution of (3.1). Then for \( j = 0, 1, 2 \), the integral
\[ g_j^{[1]}(u) = \int_{\gamma_j} p(-x)e^{h(x)-ux} \, dx \]
is well-defined and \( g_j^{[1]}(u) \) is a holomorphic solution of (5.1).

**Proof.** The integral is well-defined since \( e^{h(x)} \) is decaying along \( \gamma_j \). We twice use the integration by parts to compute
\[ u\ddot{g}_j^{[1]} - n\dot{g}_j^{[1]} - (u^2 - au + b)g_j^{[1]} = \int_{\gamma_j} u(e^{h-ux})'p(-x) - (nx - b)p(-x)dx \]
\[ = \int_{\gamma_j} -(p(-x))'e^{h}(e^{-ux})' - (nx - b)p(-x)dx = \int_{\gamma_j} (p(-x)e^{h})' - (nx - b)p(-x)dx = 0. \]
Lemma 5.2. Let $y_j(x)$ be a solution of (3.1) given by (4.1). Then the integral

$$g_j^{[2]}(u) = u^{n+1} \int_{\gamma_j} y_j(x) e^{-ux} dx$$

is convergent for $u \in \mathbb{C}$ such that $\text{Re}(ue^{i\pi(1+2j/3)}) < 0$ and $g(u)$ is a solution of (5.1).

Proof. The integral converges as $t \to -\infty$ on $\gamma_j(t)$ since $y_j(x)$ is decaying and for $t \to \infty$ since $e^{-ux}$ is decaying.

Similarly to the proof of Lemma 5.1, we use twice the integration by parts and obtain

$$\frac{1}{u^{n+1}}(u g_j^{[2]} - y_j^{[2]} - (u^2 - au + \lambda) y_j^{[2]}) = \int_{\gamma_j} (uh' - (n+2)x - u^2 - \lambda)y_j e^{-ux} dx$$

$$= \int_{\gamma_j} (-y_j' + h'y_j) (e^{-ux})' - ((n+2)x + \lambda)y_j dx = 0.$$

Using integration by parts the function $g_j^{[2]}(u)$ can be rewritten as follows:

$$g_j(u) = u^{n+1} \int_{\gamma_j} p(x)e^{-ux} \left( \int_0^x \frac{e^{h(z)}}{p^2(z)} dz \right) dx = \int_{\gamma_j} \sum_{r=0}^n u^{n-r} p^{(r)}(x) \frac{e^{h(x)-ux}}{p^2(x)} dx,$$

where $p^{(r)}(x)$ denotes the $r$-th derivative of $p(x)$. In particular, the integral on the right hand side of (5.2) converges for all values of $u \in \mathbb{C}$ and the function $g_j^{[2]}(u)$ is holomorphic in $\mathbb{C}$.

Proposition 5.3. For $j = 0, 1, 2$, we have $g_j^{[1]}(u) = (-1)^n g_j^{[2]}(u)$.

Proof. We compute the asymptotics of $g_j^{(i)}$ using the steepest descend method similarly to the computation of asymptotics of the Airy functions, see [S]. We obtain for $j = 0, 1, 2$,

$$g_j^{[1]}(u) = i(-1)^{n+j'} \sqrt{\pi u^{n/2-1/4}} e^{-\frac{4}{3} u^{3/2} + au^{1/2}} (1 + o(1))$$

as $|u| \to \infty$, with arg $u$ fixed such that

$$\frac{\pi}{3} < \text{arg } u < \frac{7\pi}{3} \quad (j = 0),$$

$$-\frac{7\pi}{3} < \text{arg } u < -\frac{\pi}{3} \quad (j = 1),$$

$$-\pi < \text{arg } u < \pi \quad (j = 2).$$

Here $j' = 0$ for $j = 2$ and $j' = 1$ for $j = 0, 1$.

Similarly, using (5.2), we conclude that the function $g_j^{[2]}(u)$ has the asymptotics different from that of $g_j^{[1]}(u)$ only by a factor of $(-1)^n$ and since there is a unique solution of (5.1) with such asymptotics, the proposition follows.
Lemma 5.4. Let \( g(u) \) be a solution of (5.1) holomorphic at \( u = 0 \). Then
\[
p(x) = \text{Res}_{u=0} \frac{g(u)e^{ux}}{u^{n+1}}
\]
is a polynomial solution of (3.1).

Proof. We again use twice the integration by parts:
\[
2\pi i \left( p'' - h'p' + (nx+b)p \right) = \int_{|u|=\epsilon} \frac{(u^2 - au - ux^2u + nx + b)g e^{ux}}{u^{n+1}}
\]
\[
= \int_{|u|=\epsilon} \frac{((u^2 - au + b)g + n\dot{g} - u\ddot{g})e^{ux}}{u^{n+1}}
\]
\[= 0.\]

\[\square\]

6. Some linear algebra

Let \( V = \mathbb{C}^{n+1} \) be the vector space with a scalar product. We fix an orthonormal basis \( \{e_0, \ldots, e_n\} \) in \( V \). For \( v \in V \) we denote \( v_i = v_i \cdot e_i \) the coordinates of \( v \). For an operator \( A \in \text{End}(V) \) we denote \( A_{ij} = e_i \cdot Ae_j \) the matrix coefficients of \( A \). We also denote \( \hat{A} \) the adjoint operator of \( A \). We have \( \hat{A}A = AA = (\det A)I \).

Let \( A : V \to V \) be a linear operator with eigenvalue \(-b\). Let \( v \) and \( v^* \) be the corresponding eigenvectors of \( A \) and \( A^* \). We have \((A+b)v = 0\), and \((A^*+b)v^* = 0\).

Lemma 6.1. Assume that \( v_jv_k^* \neq 0 \). Then
\[
\left( \frac{d}{d\lambda} \det(A + \lambda) \right) |_{\lambda=b} = \frac{v \cdot v^*}{v_jv_k^*} (\hat{A} + b)_{jk}.
\]

Proof. Since \(-b\) is an eigenvalue, there exists \( \alpha \in \mathbb{C} \) such that \((\hat{A} + b)_{sl} = \alpha v_s v_l^* \) for all \( s, l = 0, 1, \ldots, n \). Thus
\[
\left( \frac{d}{d\lambda} \det(A + \lambda) \right) |_{\lambda=b} = \text{tr} \hat{A} + b = \alpha \sum_{s=0}^n v_s v_s^* = \frac{(\hat{A} + b)_{jk}}{v_jv_k^*} v \cdot v^*.
\]

\[\square\]

We apply Lemma 6.1 to the case \( V = \mathbb{C}[x] \) the space of polynomials of degree at most \( n \) and
\[
A = (d/dx)^2 - h'(x)(d/dx) + nx.
\]
Clearly \( A \) is a linear operator which preserves \( V \). We choose the basis of \( V \) as follows: let
\[
e_k(x) = x^k/k!, \quad k = 0, \ldots, n.
\]
We set \( e_{-1}(x) = e_{-2}(x) = 0 \). Then
\[
Ae_k = e_{k-2} - ae_{k-1} + (n-k)(k+1)e_{k+1}, \quad k = 0, \ldots, n.
\]
Clearly, there exists a wave polynomial \( p(x) \) of operator (3.1) if and only if \(-b\) is an eigenvalue of \( A \). Moreover, in such a case the rank of \( D + b \) is \( n \), \( p(x) \) is unique up to a multiplicative constant and the degree of \( p(x) \) is exactly \( n \).
Let \( p(x) = \sum_{s=0}^{n} p_s e_s(x) \), \( p_s \in \mathbb{C} \), be a wave polynomial: \((A + c)p(x) = 0\). Then, clearly, \( p^*(x) = \sum_{s=0}^{n} p_{n-s} e_s(x) \) satisfies \((A^* + c)p^*(x) = 0\).

Using Lemma 6.1 with \( j = n, k = 0 \), we have

\[
\left( \frac{d}{d\lambda} \det(A + \lambda) \right) \bigg|_{\lambda = c} = (-1)^n \frac{(n!)^2}{p^2_n} \sum_{s=0}^{n} p_s p_{n-s}.
\]

7. The constant \( c \)

We are now ready to compute the constants \( c \).

**Theorem 7.1.** Let \( p(x) = \sum_{s=0}^{n} p_s e_s(x) \) be a monic wave polynomial of degree \( n \). Then for \( k = 0, \ldots, n \) we have

\[
e_k(-x)p(-x) \sim \frac{(-1)^n p_{n-k}}{p^2(x)}.
\]

**Proof.** By Lemma 2.2, there exists a constant \( c_k \) such that \( e_k(-x)p(-x) \sim c_k/p^2(x) \) and for \( j = 0, 1, 2 \),

\[
J_j c_k = \int_{\gamma_j} e_k(-x)p(-x)dx.
\]

Choose any \( j \in \{0, 1, 2\} \) and set \( g_j(u) = g^{[1]}_j(u) = (-1)^n g^{[2]}_j(u) \), see Proposition 5.3.

Using the presentation \( g_j(u) = g^{[1]}_j(u) \), we obtain

\[
J_j c_k = \int_{\gamma_j} e_k(-x)p(-x) e^k dx = \frac{g^{(k)}_j(0)}{k!}.
\]

where \( g^{(k)}_j(u) \) denotes the \( k \)-th derivative of \( g_j(u) \).

Expanding \( g_j(u) \) in the Taylor series at \( u = 0 \) and using Lemma 5.4, we compute

\[
\alpha p(x) = \text{Res}_{u=0} \sum_{s=0}^{\infty} g^{(s)}_j(0) \frac{u^{s-n-1}}{s!} e^{ux} = \sum_{s=0}^{\infty} g^{(s)}_j(0) \frac{1}{s!} e_{n-s}(x),
\]

Since \( p(x) \) is monic, we have \( p_n = n! \) and the constant \( \alpha \) is given by \( g_j(0)/n! \). It follows that

\[
c_k = \frac{g^{(k)}_j(0)}{k! J_j} = \frac{g_j(0)}{n! J_j} p_{n-k}.
\]

Finally, using that \( g_j(u) = (-1)^n g^{[2]}_j(u) \) and equation (5.2), we obtain

\[
g_j(0) = (-1)^n n! J_j.
\]

The theorem follows. \( \square \)

**Corollary 7.2.** Let \( p(x) \) be a monic wave polynomial of degree \( n \). Then

\[
p^2(-x) \sim \frac{c}{p^2(x)}, \quad c = \left( \frac{d}{d\lambda} \det(A + \lambda) \right) \bigg|_{\lambda = b},
\]

where \( A \) is given by (6.1) or (6.2).
Proof. By Theorem 7.1 we have

\[ p^2(-x) = \left( \sum_{s=0}^{n} p_s e_s(-x) \right) p(-x) \sim (-1)^n \frac{\sum_{s=0}^{n} p_s p_{n-s}}{p^2(x)}. \]

The corollary now follows from (6.3). \qed

Corollary 7.3. Conjecture 1 and formula (18) in [EG] is true. \qed

Proof. Conjecture 1 and formula (18) in [EG] follow from Theorem 4.1 and Corollary 7.2 respectively after the change of variables:

\[ x = \beta z_{EG}, \quad p(x) = p_{EG}(z) \beta^n, \quad \beta a = 2b_{EG}, \quad \beta^2 b = 2a_{EG}, \]

where \( \beta^3 = -2 \) and we denote the objects from [EG] by the same letters as there but with the index \( EG \) to distinguish from the notation used in this note. \qed

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