Friendship, Altruism, and Reward Sharing in Stable Matching and Contribution Games∗

Elliot Anshelevich† Onkar Bhardwaj‡ Martin Hoefer§

May 10, 2014

Abstract

We study stable matching problems in networks where players are embedded in a social context, and may incorporate friendship relations or altruism into their decisions. Each player is a node in a social network and strives to form a good match with a neighboring player. We consider the existence, computation, and inefficiency of stable matchings from which no pair of players wants to deviate. When the benefits from a match are the same for both players, we show that incorporating the well-being of other players into their matching decisions significantly decreases the price of stability, while the price of anarchy remains unaffected. Furthermore, a good stable matching achieving the price of stability bound always exists and can be reached in polynomial time. We extend these results to more general matching rewards, when players matched to each other may receive different utilities from the match. For this more general case, we show that incorporating social context (i.e., “caring about your friends”) can make an even larger difference, and greatly reduce the price of anarchy. We show a variety of existence results, and present upper and lower bounds on the prices of anarchy and stability for various matching utility structures. Finally, we extend most of our results to network contribution games, in which players can decide how much effort to contribute to each incident edge, instead of simply choosing a single node to match with.

1 Introduction

Stable matching problems form the basis of many important assignment and allocation tasks in economics and computer science. The central approach to analyzing such scenarios is two-sided matching, which has been studied intensively since the 1970s in both the algorithms and economics literature [19, 34]. An important variant of stable matching is matching with cardinal utilities, when each match can be given numerical values expressing the quality or reward that the match yields for each of the incident players [3]. Cardinal utilities specify the quality of each match instead of just a preference ordering, and they allow the comparison of different matchings using measures such as social welfare. A particularly appealing special case of cardinal utilities is known as correlated stable matching, where both players who are matched together obtain the same reward. Apart from the wide-spread applications of correlated stable matching in, e.g., market sharing [18], job markets [6], social networks [20], and distributed computer networks [18, 31], this model also has favorable theoretical properties such as the existence of a potential function. It guarantees

∗This work was supported in part by NSF grants CCF-0914782 and CCF-1101495.
†Dept. of Computer Science, Rensselaer Polytechnic Institute, Troy, NY
‡Dept. of Electrical, Computer, & Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY
§Dept. of Computer Science, RWTH Aachen University, Germany
existence of a stable matching even in the non-bipartite case, where every pair of players is allowed to match \[1, 31\].

When matching individuals in a social environment, it is often unreasonable to assume that each player cares only about their own match quality. Instead, players may incorporate the well-being of their friends/neighbors as well, or that of friends-of-friends. Players may even be altruistic to some degree, and consider the welfare of all players in the network. Caring about friends and altruistic behavior is commonly observed in practice and has been documented in laboratory experiments \[15, 30\]. However, results in algorithmic game theory about the impact of social context on stable outcomes are only recently starting to appear \[7, 11–13, 21–23\]. In this paper, we study how social context influences stability and efficiency in matching scenarios. We use a general approach incorporating the social context of a player into its decisions. Every player may consider the well-being of every other player to some degree, with the degree of this regardfulness possibly decaying with the hop distance in the network. Players who only care about their neighbors, as well as fully altruistic players, are special cases of this model. Our model of altruism is a strict generalization of recent approaches in algorithmic game theory in which the social welfare of the whole population is (part of) the utility of each player.

Moreover, for matching in social environments, the standard model of correlated stable matching may be too constraining compared to general cardinal utilities, because matched players receive exactly the same reward. Such an equal sharing property is intuitive and bears a simple beauty, but there are a variety of other reward sharing methods that can be more natural in different contexts. For instance, in theoretical computer science it is common practice to list authors alphabetically, but in other disciplines the author sequence is carefully designed to ensure a proper allocation of credit to the different participants of a joint paper. Here the credit is often supposed to be allocated in terms of input, i.e., the first author should be the one that has contributed most to the project. Such input-based or proportional sharing is then sometimes overruled with sharing based on intrinsic or acquired social status, e.g., when a distinguished expert in a field is easily recognized and subconsciously credited most with authorship of an article. In this paper, we are interested in how such unequal reward sharing rules affect stable matching scenarios. In particular, we consider a large class of local reward sharing rules and characterize the impact of unequal sharing on existence and inefficiency of stable matchings, both in cases when players are embedded in a social context and when they are not.

Recently, correlated matching problems have become the basis for analyzing more general contribution and participation games in networks. In such games, each player must decide how much effort to contribute to each relationship or project that it is involved in. Insights on such problems may advance the understanding of contribution incentives in networked societies and improve the design of user-based platforms. As we show in Sections 5 and 6, we are able to extend most of our results about stable matching in the presence of social context and general reward sharing to network contribution games which have recently been introduced in \[5\].

### 1.1 Stable Matching and Contribution Games

In this paper we consider two classes of games: stable matching with cardinal utilities, and convex contribution games. We consider both scenarios in the presence of social context, and unequal reward sharing.

**Stable Matching** Correlated stable matching is a prominent subclass of general ordinary stable matching. In this game, we are given a (non-bipartite) graph \( G = (V, E) \) with edge weights \( r_e \). In a matching \( M \), if node \( u \) is matched to node \( v \), the utility of node \( u \) is defined to be exactly \( r_e \).
This can be interpreted as both $u$ and $v$ getting an identical reward from being matched together. We will also consider unequal reward sharing, where node $u$ obtains some reward $r_{e}^u$ and node $v$ obtains reward $r_{e}^v$ with $r_{e}^u + r_{e}^v = r_e$. Therefore, the preference ordering of each node over its possible matches is implied by the rewards that this node obtains from different edges. A pair of nodes $(u, v)$ is called a blocking pair in matching $M$ if $u$ and $v$ are not matched to each other in $M$, but can both strictly increase their rewards by being matched to each other instead. A matching with no blocking pairs is called a stable matching.

While the matching model above has been well-studied, in this paper we are interested in stable matchings that arise in the presence of social context. Denote the reward obtained by a node $v$ in a matching $M$ as $R_v$. We now consider the case when node $u$ not only cares about its own reward, but also about the rewards of its friends. Specifically, the perceived or friendship utility of node $v$ in matching $M$ is defined as

$$U_v = R_v + \sum_{d=1}^{\text{diam}(G)} \alpha_d \sum_{u \in N_d(v)} R_u,$$

where $N_d(v)$ is the set of nodes with shortest distance exactly $d$ from $v$, and $1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ (we use $\alpha$ to denote the vector of $\alpha_i$ values). In other words, for a node $u$ that is distance $d$ away from $v$, the utility of $v$ increases by an $\alpha_d$ factor of the reward received by $u$. Thus, if $\alpha_d = 0$ for all $d \geq 2$, this means that nodes only care about their neighbors, while if all $\alpha_d > 0$, this means that nodes are altruistic and care about the rewards of everyone in the graph. The perceived utility is the quantity that the nodes are trying to maximize, and thus, in the presence of friendship, a blocking pair would be a pair of nodes such that each could increase its perceived utility by matching to each other.

**Contribution Games** While most of the results in this paper concern stable matching, we also study convex contribution games (CCG) (for detailed definition and discussion see [5]). In these games, we are given a graph $G = (V, E)$ with a reward function $f_e: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ for each edge $e$, which is assumed to be nondecreasing and convex in each of its arguments, and obeys the property that $f_e(0, y) = f_e(x, 0) = 0$ for all $x, y$. The nodes are players of this game: each node $v$ has a budget $B_v$, and its strategy consists of deciding how to allocate this budget among its incident edges. The reward to node $v$ from edge $e = (v, u)$ is equal to $f_e(s_v(e), s_u(e))$, where $s_v(e)$ and $s_u(e)$ are the amounts of budget allocated to edge $e$ by nodes $v$ and $u$ respectively. For the case where reward to endpoints of an edge is allowed to be different, we instead have two functions: $f_e^u$ and $f_e^v$ such that $v$ receives $f_e^v(s_v(e), s_u(e))$ reward and $u$ receives $f_e^u(s_v(e), s_u(e))$ reward. The total reward of a node $v$ (which we denote by $R_v$) is simply the total reward it collects from incident edges.

In this paper, just as in [5], we will mostly be concerned with pairwise equilibria of CCG. A pairwise equilibrium (a.k.a. a 2-strong equilibrium) is a solution where no pair of players can switch their strategies (budget allocations) simultaneously such that both players strictly increase their rewards. Also, a pairwise equilibrium must not possess any unilateral improving deviations by any individual player.

Just as with stable matching, we are interested in the properties of CCG with social context and friendship utilities. This version is defined analogously: the perceived utility of a node $v$ is $U_v = R_v + \sum_{d=1}^{\text{diam}(G)} \alpha_d \sum_{u \in N_d(v)} R_u$, and this is what node $v$ is attempting to maximize. Therefore, a pairwise equilibrium in the presence of friendship means that there is no pair of nodes that can simultaneously improve their perceived utility.
Centralized Optimum and the Price of Anarchy  We study the social welfare of equilibrium solutions and compare them to an optimal centralized solution. The social welfare is the sum of rewards, i.e., the optimal solution is the one that maximizes $\sum_v R_v$. Notice that, while this is equivalent to maximizing the sum of player utilities when $\vec{\alpha} = 0$, this is no longer true with social context (i.e., when $\vec{\alpha} \neq 0$). Nevertheless, as in e.g. [12, 32], we believe this is a well-motivated and important measure of solution quality, as it captures the overall performance of the system, while ignoring the perceived “good-will” effects of friendship and altruism. For example, when considering projects done in pairs, the reward of an edge can represent actual productivity, while the perceived utility may not.

To compare stable solutions with the centralized optimum, we will often consider the price of anarchy and the price of stability. When considering stable matchings, by the price of anarchy (resp. stability) we will mean the ratio between the optimum centralized solution and the worst (resp. best) stable matching. Similarly, when considering CCG, by the price of anarchy (resp. stability) we will mean the ratio between the optimum centralized solution and the worst (resp. best) pairwise equilibrium.

1.2 Our Results

For stable matching with cardinal utilities we show the following.

- For friendship utilities and equal reward sharing, a stable matching exists and the price of anarchy (ratio of the maximum-weight matching with the worst stable matching) is at most 2, the same as in the case without friendship. The price of stability, on the other hand, improves in the presence of friendship, as we can show a tight bound of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$. Moreover, we present a dynamic process that converges to a stable matching of at least this quality in polynomial time, if initiated from the maximum-weight matching.

- When two nodes matched together may receive different rewards, a stable matching may not exist. However, for several natural local reward sharing rules (e.g., when reward shares depend on inherent properties of the two incident nodes, see Section 3, we show that a stable matching exists. Moreover, for arbitrary oblivious reward sharing (i.e., when rewards for the incident players are arbitrary but independent of the matching decisions of other players), we show that prices of anarchy and stability depend on the level of inequality among reward shares. Specifically, if $R$ is the maximum ratio over all edges $(u, v) \in E$ of the reward shares of node $u$ and $v$, then the price of anarchy is at most $1 + R$ without friendship, and at most $\frac{1+R(1+\alpha_1)}{1+\alpha_1 R}$ with friendship utilities. We also show tight or almost-tight lower bounds on the price of anarchy, and give improved results for several particular reward sharing rules.

Our results imply that for socially aware players, the price of stability can greatly improve: e.g., if $\alpha_1 = \alpha_2 = \frac{1}{2}$, then the price of stability is at most $\frac{5}{7}$, and a solution of this quality can be obtained efficiently. Moreover, if reward sharing is extremely unfair ($R$ is unbounded), then friendship becomes even more important: changing $\alpha_1$ from 0 to $\frac{1}{2}$ reduces the price of anarchy from being unbounded to being at most 3.

We next consider convex contribution games. While friendship changes the properties of these games (e.g., there might be instances without a strong equilibrium), we show that all of the results mentioned above for stable matching also hold for convex contribution games, replacing “stable matching” with “pairwise equilibrium”. For the case where players do not have to spend all of their budget, this is not difficult to show, as there is a one-to-one correspondence between stable matchings and pairwise equilibria. For the case where players must spend their entire budget,
however, this becomes somewhat trickier, as the types of deviations available to players in convex
contribution games are significantly more numerous than in stable matching models. Nevertheless,
we show that the same results hold for the case of local friendship, i.e., where $\alpha_i = 0$ for all $i \geq 2$.
We also show new results for specific reward sharing rules in convex contribution games, such as
proportional sharing, where a node’s share of reward from an edge is proportional to the amount
of effort it contributes to the edge.

### 1.3 Related Work

Stable matching problems have been studied intensively over the last few decades. On the algorithmic
side, existence, efficient algorithms, and improvement dynamics for two-sided stable matchings
have been of interest (for references, see standard textbooks [19,34]). In this paper we address
the more general stable roommates problem, in which every player can be matched to every other
player. For general preference lists, there have been numerous works characterizing and algorithmically
deciding existence of stable matchings [14,24,35,37]. For the correlated stable roommates problem, existence is guaranteed by a potential function argument [1,31], and convergence time of random improvement dynamics is polynomial [2]. In [4], price of anarchy and stability bounds for approximate correlated stable matchings were provided. In contrast, we study friendship, altruism,
and unequal reward sharing in stable roommate problems with cardinal utilities.

Another line of research closely connected to some of our results involves game-theoretic models
for contribution. A prominent example is the general approach by Ballester et al [9], in which equilibria exhibit similarities with a commonly known centrality index in social networks. There
are numerous extensions and variants of this game. In all these games, however, players contribute
quite generally to the whole society, and not to particular links or relationships. See [17] for an
analysis of a broad framework that includes this game and several others (such as public goods games [10]). Instead, in [5] we consider a contribution game tied more closely to matching problems.
Here players have a budget of effort and contribute parts of this effort towards specific projects and
relationships. For more related work on the contribution game, see [5]. All previous results for this
model concern equal sharing and do not address the impact of the player’s social context.

Analytical aspects of reward sharing have been a central theme in game theory since its begin-
ing, especially in cooperative games [33]. Recently, there have been prominent algorithmic results
also for network bargaining [26,27] and credit allocation problems [28]. In addition, the work in [8]
considers various reward sharing schemes in coalition formation; their motivation resembles ours, although they mostly consider Nash equilibrium solutions in hypergraphs, while we consider
pairwise equilibria in the presence of social context. Work such as [38] could also be considered a
generalization of contribution games, but in the cooperative setting and without the players having
a social context.

Our notion of a player’s social context is based on numerical influence parameters that determine
the impact of player rewards on the (perceived) utilities of other players. A recently popular
model of altruism is inspired by Ledyard [29] and has generated much interest in algorithmic game
theory [12,13,22,23]. Our model smoothly interpolates between this global approach and the idea
of surplus collaboration among players in a given social context put forward in [7,32] and considered
recently in [11].

In addition, our work is more generally related to the area of strategic network creation games,
in which selfish players build networks and optimize different trade-offs between creation cost and
benefits from network structure. For an introduction to this literature see recent expositions [25,36].
In this literature, there also originated a notion of pairwise equilibrium, which allows fewer player
deviations than what we term pairwise equilibrium here. In our case, it corresponds exactly to
Figure 1: biswivel deviation

Figure 2: swivel deviation

2-strong equilibrium; for a discussion see [5].

2 Stable Matching with Friendship Utilities

We begin by considering correlated stable matching in the presence of friendship utilities. In this section, the reward received by both nodes of an edge in a matching is the same, i.e., we use equal reward sharing, where every edge $e$ has an inherent value $r_e$ and both endpoints receive this value if edge $e$ is in the matching. We consider more general reward sharing schemes in Sections 3 and 4.

Recall that the friendship utility of a node $v$ increase by $\alpha_d R_u$ for every node $u$, where $d$ is the shortest distance between $v$ and $u$. We abuse notation slightly, and let $\alpha_{uv}$ denote $\alpha_d$, so if $u$ and $v$ are neighbors, then $\alpha_{uv} = \alpha_1$.

Given a matching $M$, we begin by classifying the following types of improving deviations that a blocking pair can undergo.

Definition 1. We call an improving deviation a biswivel whenever two neighbors $u$ and $v$ switch to match to each other, such that both $u$ and $v$ were matched to some other nodes before the deviation in $M$.

See Figure 1 for explanation. For such a biswivel to exist in a matching, the following necessary and sufficient conditions must hold.

\[
(1 + \alpha_1) r_{uv} > (1 + \alpha_1) r_{uw} + (\alpha_1 + \alpha_{uz}) r_{vz} \quad (1)
\]
\[
(1 + \alpha_1) r_{uv} > (1 + \alpha_1) r_{vz} + (\alpha_1 + \alpha_{vw}) r_{uw} \quad (2)
\]

Inequality (1) can be explained as follows: The left side quantifies the utility gained by $u$ because of getting matched to $v$ and the right side quantifies the utility lost by $u$ because of $u$ and $v$ breaking their present matchings with $w$ and $z$ respectively. Hence Inequality (1) implies that $u$ gains more utility by getting matched with $v$ than it loses because of $u$ and $v$ breaking their matchings with $v$ and $z$. Inequality (2) can similarly be explained in the context of node $v$.

Definition 2. We call an improving deviation a swivel whenever two neighbors get matched such that at least one node among the two neighbors was not matched before the deviation.
See Figure 2 for explanation. For such a swivel to occur, the following set of conditions must hold.

\[(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{uw}\]  
\[(1 + \alpha_1)r_{uv} > (\alpha_1 + \alpha_{vw})r_{uw}\]

Inequality (3) says that \(u\) gains more utility by getting matched with \(v\) than it loses by breaking its matching with \(w\). Inequality (4) says that \(v\) gains more utility by getting matched with \(u\) than the utility it loses because of \(u\) breaking its matching with \(w\). As \(\alpha_1 + \alpha_{vw} \leq 1 + \alpha_1\), Inequality (4) is implied by Inequality (3). This means that if \(v\) is unmatched, the only condition for \((u, v)\) to be a blocking pair is that \(u\) should have net increase in utility by getting matched with \(v\). This is true even if \(v\) and \(w\) are neighbors. Canceling the factor of \(1 + \alpha_1\), we can thus summarize this (necessary and sufficient) condition for swivel to be an improving deviation as:

\[r_{uv} > r_{uw}\]  

All improving deviations by a blocking pair can be classified as either a biswivel or a swivel, depending only on whether both nodes are matched or not. Now we make the following observation which will be useful later:

**Lemma 1.** Suppose a node \(u\) is matched to \(w\) in matching \(M\). If \((u, v)\) form a blocking pair, then \(r_{uv} > r_{uw}\).

**Proof.** It is straightforward to see it from inequalities (1) and (2) for a biswivel and inequality (5) in case of a swivel. \(\square\)

### 2.1 Existence and Price of Anarchy of Stable Matching with Friendship Utilities

**Theorem 1.** A stable matching exists in stable matching games with friendship utilities. Moreover, the set of stable matchings without friendship (i.e., when \(\vec{\alpha} = 0\)) is a subset of the set of stable matchings with friendship utilities on the same graph.

**Proof.** For \(\vec{\alpha} = 0\), our model is a subcase of correlated stable matching, so a stable matching \(M\) exists. All we need to prove now is that the same \(M\) is stable when we have friendship utilities.

Suppose it is not the case, i.e., \(M\) is unstable for some value of \(\vec{\alpha}\). This is possible only if we have a blocking pair \((u, v)\). But this cannot happen because:

- If both \(u\) and \(v\) were unmatched in \(M\) then \(M\) could not have been stable for \(\vec{\alpha} = 0\).
- If exactly one of \(u\) and \(v\) is unmatched in \(M\), say \(u\) is matched to \(w\) and \(v\) is unmatched, then for \((u, v)\) to be a blocking pair, \(r_{uv} > r_{uw}\) by Lemma 1. But in such a case, \(M\) could not have been stable for \(\vec{\alpha} = 0\).
- Suppose both \(u\) and \(v\) are matched in \(M\), say \(u\) is matched to \(w\) and \(v\) is matched to \(z\). In such a case if \((u, v)\) forms a blocking pair corresponding to a biswivel, then by Lemma 1, we have \(r_{uv} > r_{uw}\) and \(r_{uw} > r_{uz}\) and thus \(M\) could not have been stable for \(\vec{\alpha} = 0\).

Hence we have shown that no blocking pair exists in \(M\) with friendship utilities, thus proving the theorem. \(\square\)

**Theorem 2.** The price of anarchy in stable matching games with friendship utilities is at most 2, and this bound is tight.
Proof. This theorem is simply a special case of our much more general Theorem 10, which proves a price of anarchy bound of $1 + \frac{R + \alpha_1}{1 + \alpha_1 R}$, with $R$ being a measure of how unequally players can share rewards on an edge. When players share edge rewards equally, the price of anarchy bound in Theorem 10 reduces to $1 + \frac{1 + \alpha_1}{1 + \alpha_1} = 2$, as desired. To show that this bound is tight, simply consider a 3-edge path with all edge rewards being 1, for any value of $\bar{\alpha}$.

2.2 Price of Stability and Convergence

The main result in this section bounds the price of stability in stable matching games with friend-ship utilities to $\frac{2 + 2\alpha_1}{1 + 2\alpha_1 + \alpha_2}$, and this bound is tight (see Theorem 4 below). This bound has some interesting implications. It is decreasing in each $\alpha_1$ and $\alpha_2$, hence having friendship utilities always yields a lower price of stability than without friendship utilities. Also, note that values of $\alpha_3, \alpha_4, ..., \alpha_{\text{diam}(G)}$ have no influence. Thus, caring about players more than distance 2 away does not improve the price of stability in any way. Also, if $\alpha_1 = \alpha_2 = 1$, then PoS = 1, i.e., there will exist a stable matching which will also be a social optimum. Thus loving thy neighbor and thy neighbor’s neighbor but nobody beyond is sufficient to guarantee that there exists at least one socially optimal stable matching. In fact, due to the shape of the curve, even small values of friendship quickly decrease the price of stability; e.g., setting $\alpha_1 = \alpha_2 = 0.1$ already decreases the price of stability from 2 to $\sim 1.7$.

We will establish the price of stability bound by defining an algorithm that creates a good stable matching in polynomial time. One possible idea to create a stable matching that is close to optimum is to use a BEST-BLOCKING-PAIR algorithm: start with the best possible matching, i.e. a social optimum, which may or may not be stable. Now choose the “best” blocking pair $(u, v)$: the one with maximum edge reward $r_{uv}$. Allow this blocking pair to get matched to each other instead of their current partners. Check if the resulting matching is stable. If it is not stable then allow the best blocking pair for this matching to get matched. Repeat the procedure until there are no more blocking pairs, thereby obtaining a stable matching.

This algorithm gives the desired price of stability and running time bounds for the case of “altruism” when all $\alpha_i$ are the same, see Corollary 1 below. To provide the desired bound with general friendship utilities, we must alter this algorithm slightly using the concept of relaxed blocking pair.

Definition 3. Given a matching $M$, we call a pair of nodes $(u, v)$ a relaxed blocking pair if either $(u, v)$ form an improving swivel, or $u$ and $v$ are matched to $w$ and $z$ respectively, with the following inequalities being true:

\[
(1 + \alpha_1) r_{uv} > (1 + \alpha_1) r_{uw} + (\alpha_1 + \alpha_2) r_{vz} \quad (6)
\]

\[
(1 + \alpha_1) r_{uw} > (1 + \alpha_1) r_{vz} + (\alpha_1 + \alpha_2) r_{uw} \quad (7)
\]

In other words, a relaxed blocking pair ignores the possible edges between nodes $u$ and $z$, and has $\alpha_2$ in the place of $\alpha_{uz}$ (similarly, $\alpha_2$ in the place of $\alpha_{uw}$). It is clear from this definition that a blocking pair is also a relaxed blocking pair, since the conditions above are less constraining than Inequalities (1) and (2). Thus a matching with no relaxed blocking pairs is also a stable matching. Moreover, it is easy to see that Lemma 1 still holds for relaxed blocking pairs. We will call a relaxed blocking pair satisfying Inequalities (6) and (7) a relaxed biswivel, which may or may not correspond to an improving deviation, since a relaxed blocking pair is not necessarily a blocking pair.

Now we present the algorithm to compute a stable matching that is close to optimal.
2.2.1 Best-Relaxed-Blocking-Pair Algorithm

1. Initialize $M = M^*$ where $M^*$ is a socially optimum matching.

2. If there is no relaxed blocking pair, terminate. Otherwise make the relaxed blocking pair $(u, v)$ with maximum edge reward $r_{uv}$ be matched to each other. In other words, remove the edges of $M$ containing $u$ and $v$, and add the edge $(u, v)$ to $M$.

3. Repeat step 2.

2.2.2 Dynamics of Best-Relaxed-Blocking-Pair

To establish the efficient running time of Best-Relaxed-Blocking-Pair and the price of stability bound of the resulting stable matching, we first analyze the dynamics of this algorithm and prove some helpful lemmas. We can interpret the algorithm as a sequence of swivel and relaxed biswivel deviations, each inserting one edge into $M$, and removing up to two edges. Note that it is not guaranteed that the inserted edge will stay forever in $M$, as a subsequent deviation can remove this edge from $M$. Let $O_1, O_2, O_3, \ldots$ denote this sequence of deviations, and $e(i)$ denote the edge which got inserted into $M$ because of $O_i$. Now let us analyze the dynamics of the algorithm by using the following two lemmas.

Lemma 2. The first deviation $O_1$ during the execution of Best-Relaxed-Blocking-Pair is a relaxed biswivel.

Proof. Having $O_1$ as a swivel will strictly improve the value of matching by Lemma 1. Hence if we begin the algorithm with $M = M^*$, having $O_1$ as a swivel will produce a matching with value strictly greater than $M^*$, which is a contradiction.

Lemma 3. Let $O_j$ be a relaxed biswivel that takes place during the execution of the best relaxed blocking pair algorithm. Suppose a deviation $O_k$ takes place before $O_j$. Then we have $r_{e(k)} \geq r_{e(j)}$. Furthermore, if $O_k$ is a relaxed biswivel then $e(k) \neq e(j)$ (thus at most $|E(G)|$ relaxed biswivels can take place during the execution of the algorithm).

It is important to note that this lemma does not say that $r_{e(i)} \geq r_{e(j)}$ for $i < j$. We are only guaranteed that $r_{e(i)} \geq r_{e(j)}$ for $i < j$ if $O_j$ is a relaxed biswivel. Between two successive relaxed biswivels $O_k$ and $O_j$, the sequence of $r_{e(i)}$ for consecutive swivels can and does increase as well as decrease, and the same edge may be added to the matching multiple times. All that is guaranteed is that $r_{e(j)}$ for a biswivel $O_j$ will have a lower value than all the preceding $r_{e(j)}$'s. Thus, this lemma suggests a nice representation of Best-Relaxed-Blocking-Pair in terms of phases, where we define a phase as a subsequence of deviations that begins with a relaxed biswivel and continues until the next relaxed biswivel. Lemma 3 guarantees that at the start of each phase, the $r_{e(j)}$ value is smaller than the values in all previous phases, and that there is only a polynomial number of phases. Now we proceed to prove Lemma 3.

Proof. Let $e(j) = (vz)$ get inserted in $M$ because of a relaxed biswivel $O_j$. We first give a brief outline of the proof. Suppose that the claim $r_{e(k)} \geq r_{e(j)}$ for $k < j$ is false and we have an $O_k$ with $k < j$ such that $r_{e(k)} < r_{e(j)}$. Clearly $(v, z)$ could not have been a relaxed blocking pair just before $O_k$, otherwise the algorithm would have chosen $(v, z)$ as the best relaxed blocking pair instead of $O_k$. We will show that this leads to a conclusion that $(v, z)$ cannot be a relaxed blocking pair even for $O_j$. This is a contradiction, hence our assumption of $r_{e(k)} < r_{e(j)}$ could not have been correct.
Thus for all $O_k$ such that $k < j$ we will have $r_{e(k)} \geq r_{e(j)}$. Later we will use similar reasoning to prove that if $O_t$ with $i < j$ is a relaxed biswivel that takes place before a relaxed biswivel $O_j$ then $e(i) \neq e(j)$. Now let us proceed to the proof.

Suppose to the contrary that we have $O_k$ with $k < j$ such that $r_{e(k)} < r_{e(j)}$ with $O_j$ being a relaxed biswivel. As discussed in the outline of the proof, this implies that $(v, z)$ was not a relaxed blocking pair at the time $O_k$ was selected. Let $S$ be the set of nodes with whom $v$ and $z$ are matched at the time that $O_k$ is selected. As long as $S$ does not change, $v$ and $z$ will not be a relaxed blocking pair, since the change in utility experienced by $v$ and $z$ from matching to each other depends only on their partners in the current matching, i.e., the set $S$. Thus for the relaxed biswivel $O_j$ to occur, $S$ must change between $O_k$ and $O_j$. We will show that this leads to a contradiction: that $(v, z)$ cannot be a relaxed blocking pair for the time $O_j$ is selected.

Suppose $v$ is matched to $x$ and $z$ is matched to $y$ just before biswivel $O_j$. Since $(v, z)$ is a relaxed blocking pair at this point, we thus have

\[
(1 + \alpha_1)r_{vx} > (1 + \alpha_1)r_{vy} + (\alpha_1 + \alpha_2)r_{zy} \tag{8}
\]

\[
(1 + \alpha_1)r_{vx} > (1 + \alpha_1)r_{zy} + (\alpha_1 + \alpha_2)r_{xz}. \tag{9}
\]

Recall that $(v, z)$ was not a relaxed blocking pair just before $O_k$, and to make it a relaxed blocking pair for $O_j$, $S$ must change between $O_k$ and $O_j$. Let $O_t$ be the last deviation which changed $S$ to $\{x, z\}$. Without loss of generality, we can assume that $O_t$ adds the edge $(v, x)$. Now we have two cases:

- $(v, z)$ was a relaxed blocking pair at the time $O_t$ is selected: in this case $(v, x)$ could not have been the best relaxed blocking pair for $O_t$ because inequality $(8)$ tells us $r_{vx} > r_{ex}$.

- $(v, z)$ was not a relaxed blocking pair at the time $O_t$ is selected: Suppose $v$ was matched with $w$ before $O_t$. As $(v, z)$ was not a relaxed blocking pair just before $O_t$ we have

\[
\text{Either } (1 + \alpha_1)r_{vx} \leq (1 + \alpha_1)r_{vw} + (\alpha_1 + \alpha_2)r_{zy} \tag{10}
\]

\[
\text{OR } (1 + \alpha_1)r_{vx} \leq (1 + \alpha_1)r_{zy} + (\alpha_1 + \alpha_2)r_{wv}. \tag{11}
\]

(If $v$ was not matched just before $O_t$ then substitute $r_{vw} = 0$ to obtain appropriate condition.) Assume that it is inequality $(10)$ that holds. Then, because $O_t$ removes edge $(v, w)$ and adds edge $(v, x)$, we have $r_{vx} > r_{wv}$ as Lemma 1 holds for relaxed blocking pairs. Thus, the following must be true:

\[
(1 + \alpha_1)r_{vx} \leq (1 + \alpha_1)r_{vx} + (\alpha_1 + \alpha_2)r_{zy} \tag{12}
\]

This contradicts inequality $(8)$, and thus $(v, z)$ cannot be a relaxed blocking pair at the time $O_j$ is selected. The same conclusion can be reached if we assume inequality $(11)$ holds true.

Either way we arrive at a contradiction, thus showing that if $O_j$ is a relaxed biswivel then for all $O_k$ with $k < j$, we have $r_{e(k)} < r_{e(j)}$.

Now the only remaining piece is to prove $e(k) \neq e(j)$ if $O_k$ is a relaxed biswivel. All we need to notice that if $e(k) = e(j) = (v, z)$ then $S$ has to change between $O_k$ and $O_j$. Now we use exactly the reasoning from the previous paragraph to arrive at a contradiction, thus proving that $e(k) \neq e(j)$.

\[ \square \]
2.2.3 Convergence of Best-Relaxed-Blocking-Pair

For the case where $\alpha_1 = \alpha_2$, the conditions for a blocking pair are identical to the conditions for a relaxed blocking pair. Hence, our algorithm corresponds to letting the best blocking pair deviate at each step. As a special case, for $\alpha = 0$ and correlated stable matching, this algorithm is known to provide a stable matching in polynomial time [2]. For friendship utilities, however, (quick) convergence was previously unknown. Here we will show that even with the addition of friendship, Best-Relaxed-Blocking-Pair (and thus Best-Blocking-Pair for the case when $\alpha_1 = \alpha_2$) terminates and produces a stable matching. Moreover, it does this in polynomial time.

Note that if instead of the best we pick some arbitrary blocking pair, then there exists an instance in which, starting from the empty matching, a sequence of blocking pairs of length $\Omega(n)$ exists until reaching a stable matching, even without friendship. This is directly implied by recent results in correlated stable matching [20].

A trivial adjustment of the gadget in [20] allows us to construct the exponential sequence even when starting from the social optimum. We scale the reward of each (original) edge $i \in \{1, \ldots, m\}$ in the gadget from $i$ to $1 + i \cdot \epsilon$, for some tiny $\epsilon > 0$. This preserves all incentives and the structure of all blocking pairs. Then, we add an auxiliary neighbor for each (original) player and connect it via an auxiliary edge of reward 1. The social optimum is obviously given by matching each original player with his auxiliary neighbor. However, the exponential sequence of blocking pairs still exists, as auxiliary edges are not rewarding enough to influence blocking pairs among original players. Due to the fact that such exponential-length sequences exist, it is perhaps surprising that our algorithm indeed finds a stable matching and it terminates in polynomial time.

**Theorem 3.** Best-Relaxed-Blocking-Pair outputs a stable matching after $O(m^2)$ iterations, where $m$ is the number of edges in the graph.

**Proof.** Consider the three possible changes that can occur to the matching $M$ during each iteration: a swivel could add a new edge, or it could delete an edge and add an edge with strictly higher $r_e$ value. A relaxed biswivel deletes two edges, and adds an edge with higher $r_e$ value than either. Thus, without any biswivels taking place, the total number of consecutive swivels is at most $m^2$, since no edges are deleted by swivels. Each relaxed biswivel can allow at most $m$ extra swivels to occur, since it deletes one edge. As there are at most $m$ relaxed biswivel deviations possible by Lemma 3, the algorithm terminates after at most $m^2 + m^2$ deviations. Since there are no more relaxed blocking pairs for the algorithm to continue, and since a blocking pair is also a relaxed blocking pair, then the final matching produced by the algorithm is a stable matching. \(\square\)

As we can have only a polynomial number of consecutive swivel deviations between each relaxed biswivel, we know that every phase (defined as a maximal subsequence of consecutive swivels) lasts only a polynomial amount of time, and there are only $O(m)$ phases by Lemma 3. Moreover, in each phase, the value of the matching only increases, since swivels only remove an edge if they add a better one. Below, we use the fact that only relaxed biswivel operations reduce the cost of the matching to bound the cost of the stable matching this algorithm produces.

2.2.4 Upper Bound on Price of Stability

Before proceeding to prove the bound, we will introduce some notation and prove some useful lemmas.

We define a sequence of mappings from $M^*$ to $E(G)$. Define $h_0 : M^* \rightarrow E(G)$ as $h_0(e) = e$. Depending on $O_i$, we will define $h_i$ as follows: Suppose $O_i$ is a deviation that removes edge $h_{i-1}(e_j)$ from $M$. If $O_i$ inserts edge $e_l$ in $M$ then set $h_i(e_j) = e_l$. For all other $e_k \in M^*$, keep $h_i(e_k)$ same
as $h_{i-1}(e_k)$. Let us note that a deviation $O_i$ may not remove any edges from $\{h_{i-1}(e_j) : e_j \in M^*\}$. This can happen because during the course of the algorithm, two unmatched nodes can get matched, say to insert $e_p$ into $M$. No edges in $M^*$ get mapped to $e_p$. If this edge is removed from $M$ by a later deviation, the mapping may not change, since no edge is mapped to $e_p$. To summarize, $h_i$ may be the same as $h_{i-1}$, or may differ from $h_{i-1}$ in one location (in case of a swivel), or in two locations (in case of a relaxed biswivel). Denote the resulting mapping when our algorithm terminates by $h_M$.

Coupling Lemma 4 with the definition of mappings $h_i$, we immediately have the following result:

**Lemma 4.** $\{r_{h_i(e)}\}_{i \geq 0}$ is a nondecreasing sequence and $r_{h_{i+1}(e)} > r_{h_i(e)}$ whenever $h_{i+1}(e) \neq h_i(e)$.

The following lemma will be instrumental in proving the price of stability bound.

**Lemma 5.** If $h_M(e_i) = h_M(e_j)$ with $e_i \neq e_j$ then

1. There must exist a relaxed biswivel $O_k$ such that $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ but $O_k$ makes $h_k(e_i) = h_k(e_j)$. Furthermore, for all $p \geq k$ we have $h_p(e_i) = h_p(e_j)$.

2. There does not exist another $e_l \in M^*$ such that $h_M(e_l) = h_M(e_i) = h_M(e_j)$.

3. $r_{e_l} + r_{e_j} < \frac{2 + 2\alpha_1}{1 + 2\alpha_1 + \alpha_2} \times r_{h_M(e_i)}$

**Proof.** To prove the first part, say $O_l$ was the first deviation such that $h_{l-1}(e_i) \neq h_{l-1}(e_j)$ and $h_l(e_i) = h_l(e_j)$. It cannot happen because of a swivel deviation because a swivel can make $h_l(e) \neq h_{l-1}(e)$ for at most for one $e \in M^*$. Thus $O_l$ must be a relaxed biswivel. Set $k = l$ and it is easy to see that for $p \geq k$ we have $h_p(e_i) = h_p(e_j)$. Hence the first part is proven.

To prove the second part, suppose there exists an $e_l$ with $e_l \neq e_i \neq e_j$ such that $h_M(e_l) = h_M(e_i) = h_M(e_j)$. From the first part, there must exist a relaxed biswivel $O_k$ s.t. $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ but $h_k(e_i) = h_k(e_j)$. Similarly there must exist a relaxed biswivel $O_p$ s.t. $h_{p-1}(e_i) \neq h_{p-1}(e_j)$ but $h_p(e_i) = h_p(e_j)$. Without loss of generality say $p > k$. Using Lemma 4 we get $r_{e(k)} \geq r_{e(p)}$. But from Lemma 4 we have $r_{e(k)} < r_{e(p)}$, since $e(p) = h_p(e_i) > h_k(e_i) = e(k)$. Hence we have a contradiction here, thus proving that there does not exist another $e_l \in M^*$, with $h_M(e_l) = h_M(e_i) = h_M(e_j)$.

To prove the third part, consider a relaxed biswivel $O_k$ such that $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ and $h_k(e_i) = h_k(e_j)$. Substitute $r_{uw} = r_{h_k(e_i)}$, $r_{uu} = r_{h_{k-1}(e_i)}$ and $r_{uv} = r_{h_{k-1}(e_j)}$ in inequalities (1) and (2). Adding these inequalities and simplifying, we get

\[
r_{h_{k-1}(e_i)} + r_{h_{k-1}(e_j)} < \frac{2 + 2\alpha_1}{1 + 2\alpha_1 + \alpha_2} \times r_{h_k(e_i)}
\]

From Lemma 4 we have that $\{r_{h_i(e)}\}_{i \geq 0}$ as a nondecreasing sequence. Using this in (13) we get

\[
r_{e_i} + r_{e_j} < \frac{2 + 2\alpha_1}{1 + 2\alpha_1 + \alpha_2} \times r_{h_M(e_i)}
\]

Using Lemma 4, we can partition edges of $M^*$ into two sets as follows: Let $B$ denote the set of edges $e_i \in M^*$ such that $h_M(e_i) = h_M(e_j)$ for some $e_j \in M^*$ and let $A$ denote the remaining edges in $M^*$. We can further partition set $B$ into two sets $P$ and $Q$ as follows: choose a pair $e_i$ and $e_j$ in $B$ such that $h_M(e_i) = h_M(e_j)$. Denote $e_j$ by $\mu(e_i)$. Put $e_i$ in $P$ and $\mu(e_i)$ in $Q$. Notice that value of the matching $M$ that BEST-RELAXED-BLOCKING-PAIR gives as output is at least $\sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)}$. (The possible additional edges in $M$ are produced because of swivels which match two previously unmatched nodes with each other.)

We are now in position to prove the main theorem of this section:
Theorem 4. The price of stability in stable matching games with friendship utilities is at most $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$, and this bound is tight.

Proof. The value of $M^*$ is given by

$$w(M^*) = \sum_{e \in A} r_e + \sum_{e \in P} r_e + \sum_{e \in Q} r_e = \sum_{e \in A} r_e + \sum_{e \in P} (r_e + r_{\mu(e)})$$

Using Lemma 5 for $e \in P$ we have $r_e + r_{\mu(e)} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \times r_{h_M(e)}$. Using Lemma 11 for $e \in A$ we have $r_e \leq r_{h_M(e)}$. Thus we get

$$w(M^*) \leq \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \times r_{h_M(e)}$$

$$\leq \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \left( \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)} \right)$$

Note that this inequality may not be strict since $A$ may be empty. This could happen if each edge in $M^*$ gets removed because of a relaxed bi-wivel as the algorithm proceeds (though it may be possible that it is inserted later). We also have $w(M) \geq \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)}$ for the final matching $M$ that the algorithm gives. Using this,

$$w(M^*) \leq \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} w(M),$$

which proves the bound on the price of stability, since $M$ is a stable matching.

To prove the tightness of the bound, let us make $\alpha_2 = 0$ and assign $r_{uw} = \frac{1+2\alpha_1+\epsilon}{1+\alpha_1}$, $r_{uw} = r_{vz} = 1$ in Fig 1. Then we have $\{(uw)\}$ as the only stable matching but the social optimum is $\{(uw), (vz)\}$. Thus we get $\text{PoS} = \frac{2+2\alpha_1}{1+2\alpha_1+\epsilon}$ which can be taken arbitrarily close to $\frac{2+2\alpha_1}{1+2\alpha_1}$. This gives us a tight bound given that we are using $\alpha_2 = 0$. □

From Theorems 3 and 4 we immediately get the following corollary about the behavior of best blocking pair dynamics. This corollary applies in particular to the model of altruism when $\alpha_i = \alpha$ for all $i = 1, \ldots, \text{diam}(G)$.

Corollary 1. If $\alpha_1 = \alpha_2$ and we start from the centrally optimum matching, Best-Blocking-Pair converges in $O(m^2)$ time to a stable matching that is at most a factor of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$ worse than the optimum.

Proof. Immediate since when $\alpha_1 = \alpha_2$, the conditions for a blocking pair are identical to the conditions for a relaxed blocking pair. Hence, Best-Relaxed-Blocking-Pair is Best-Blocking-Pair. □

3 General Reward Sharing without Friendship

In the previous section we considered the case where if $(uv) \in M$ then $u$ and $v$ get the same reward from edge $(uv)$, namely $r_{uv}$. Now we will look into the case where $u$ and $v$ may possibly share the edge reward $r_{uv}$ if $(uv) \in M$. Let us define $r_{xy}$ as the reward node $x$ gets from edge $(xy)$ if $(xy) \in M$. We assume $r_{xy} = r_{xy} + r_{xy}$ as $x$ and $y$ share the edge reward $r_{xy}$. Note that while
in Section 2 we had both nodes $u$ and $v$ getting reward of $r_{uv}$ from edge $(uv)$, this is actually equivalent to $u$ and $v$ sharing the edge reward equally. To see this we can redefine the reward $u$ and $v$ get from edge $(uv)$ as $r_{uv}^2$ if $(uv) \in M$. This scaling of edge reward does not affect any of the results.

For general (unequal) reward sharing, we will give results about existence of a stable matching, as well as bounds on prices of anarchy and stability. In addition to that, we will also focus on the following specific reward sharing rules:

- **Matthew Effect sharing:** In sociology, “Matthew Effect” is a term coined by Robert Merton to describe the phenomenon which says that, when doing similar work, the more famous person tends to get more credit than other less-known collaborators. We model such phenomena for our network by associating brand values $\lambda_u$ with each node $u$, and defining the reward that node $u$ gets by getting matched with node $v$ as $r_{uw} = \frac{\lambda_u}{\lambda_u + \lambda_v} \cdot r_{uv}$. Thus nodes $u$ and $v$ split the edge reward in the ratio of $\lambda_u : \lambda_v$, and a node with high $\lambda_u$ value gets a disproportionate amount of reward.

- **Parasite sharing:** This effect is opposite to the Matthew effect in the sense that by collaborating with a renowned person, a less-known person becomes famous, whereas the reputation of the already renowned person does not change significantly from such a collaboration. We model this situation by defining the reward that node $u$ gets by getting matched with node $v$ as $r_{uw} = \frac{\lambda_v}{\lambda_u + \lambda_v} \cdot r_{uv}$. Thus nodes $u$ and $v$ split the edge reward in the ratio of $\lambda_v : \lambda_u$, in the exactly opposite way to the Matthew Effect sharing.

- **Trust sharing:** Often people collaborate based on not only the quality of a project but also how much they trust each other. We model such a situation by associating a value $\beta_u$ with each node $u$, which represents the trust value of player $u$, or how pleasant they are to work with. Each edge $(u, v)$ also has an inherent quality $h_{uv}$. Then, the reward obtained by node $u$ from partnering with node $v$ is $r_{uw} = h_{uv} + \beta_v$.

For the sake of analysis, Matthew Effect sharing and Parasite sharing are the same if we change $\lambda_u$ of Parasite sharing to $1/\lambda_u$ of Matthew Effect sharing. We will refer to both the models as Matthew Effect sharing from now on. In the next few sections, we will give results about the existence of stable matchings, and give upper bounds on prices of anarchy and stability for Matthew Effect sharing and Trust sharing, as well as for general reward sharing. Note that this analysis is for the case when friendship is absent; we consider the more general case of unequal sharing with friendship utilities in Section 4.

### 3.1 Existence of a Stable Matching

Without friendship utilities, our stable matching game reduces to the stable roommate problem, since reward shares can be arbitrary and thus induce arbitrary preference lists for each node. It is well known that a stable matching may not exist in a stable roommate problem [16]. However, we will prove in this section that for Matthew Effect sharing and Trust sharing, a stable matching can always be found.

Let us define a preference cycle as a cycle $(u_1, u_2, \ldots, u_k)$ in the graph $G$ such that $r_{u_i,u_{i+1}} \geq r_{u_{i-1},u_i}$ with at least one inequality being strict. Chung [14] defines odd rings and proves that if a graph does not contain odd rings, then a stable matching exists. It is straightforward to see that absence of preference cycles implies absence of odd rings. Hence, if a graph has no preference cycles, then a stable matching must exist. Below we prove the stronger statement that such a matching can also be found efficiently.
Theorem 5. A stable matching always exists in stable matching games with unequal sharing and no preference cycles. Furthermore, a stable matching can be found in $O(|V||E|)$ time.

Proof. In brief, we show below that whenever there exist no preference cycles in a graph, we can always find two nodes which prefer getting matched to each other over other nodes. We allow them to get matched to each other and eliminate such matched nodes from the graph. Neither of these two nodes will ever deviate from this matching. Applying the same greedy scheme on the reduced graph will give us a stable matching. Then we will prove that this algorithm produces a stable matching in $O(|V||E|)$ time. Let us now proceed to the details.

Let $T_u$ denote the sets of “best” neighbors of $u$ as follows:

$$T_u = \{ v \in N_1(u) : r_{uw}^u \geq r_{uv}^u \forall (uw) \in G \} \quad (15)$$

Now we construct a directed graph $G_D$ as follows: for all nodes $u$, choose a node $v \in T_u$ and draw an edge from $u$ directed to $v$. Every node in this graph has one outgoing edge hence this graph contains a (directed) cycle. If we find a cycle of length 2 then we have found two nodes which prefer each other the most. If a (directed) cycle $(u_1, u_2, \ldots, u_k)$ has length $k > 2$, then we have $r_{u_iu_{i+1}}^u \geq r_{u_iu_{i-1}}^u$. Now we cannot have $r_{u_2u_3}^u > r_{u_1u_2}^u$, otherwise in the original graph $G$, $(u_1, u_2, \ldots, u_k)$ would have constituted a preference cycle. Hence we have $r_{u_1u_2}^u = r_{u_2u_3}^u$. Thus $u_1$ and $u_3$ both are $u_2$’s most preferred nodes. But we also have $u_1$ prefer $u_2$ the most as $G_D$ has an edge from $u_1$ to $u_2$. Hence $u_1$ and $u_2$ is the pair of nodes that prefer each other the most. Therefore we will always be able to find two nodes in $G$ which prefer each other the most in their preference lists. Match them to each other and they will never have incentive to deviate from this matching. Remove these two nodes and repeat the procedure until no more nodes can be matched. Because no nodes matched in this process will ever deviate, we have a stable matching.

It takes $O(|E|)$ time to find each matched pair because for each edge we check if two nodes prefer each other the most. Since total number of nodes to be matched are $O(|V|)$, we find a stable matching in $O(|V||E|)$ time. \qed

Now we can prove the following theorem:

Theorem 6. No preference cycles exist with Matthew Effect sharing and Trust sharing. Hence, a stable matching exists with Matthew Effect sharing and Trust sharing and can be found efficiently.

Proof. Suppose a preference cycle exists in Matthew Effect sharing. Then there exists a cycle $(u_1, u_2, \ldots, u_k)$ such that

$$\frac{\lambda_{u_i}}{\lambda_{u_i} + \lambda_{u_{i+1}}} r_{u_iu_{i+1}} \geq \frac{\lambda_{u_i}}{\lambda_{u_i} + \lambda_{u_{i-1}}} r_{u_iu_{i-1}} \quad (16)$$

with at least one inequality being strict. Multiplying all these inequalities and canceling common factors, we reach a contradiction that $1 > 1$. Thus a preference cycle cannot exist in Matthew Effect sharing.

Suppose a preference exists in Trust sharing. Then there exists a cycle $(u_1, u_2, \ldots, u_k)$ such that

$$h_{u_iu_{i+1}} + \beta_{u_{i+1}} \geq h_{u_iu_{i-1}} + \beta_{u_{i-1}} \quad (17)$$

with at least one inequality being strict. Adding all these inequalities and canceling common factors, we reach a contradiction that $0 > 0$. Thus a preference cycle cannot exists in Trust sharing.

Since preference cycles cannot exist, we only need to apply Theorem 5 to obtain the desired result. \qed

\[15\]
3.2 Prices of Anarchy and Stability with General Reward Sharing

In this section, we will investigate prices of anarchy and stability with general reward sharing. First we will prove that for general reward sharing, the price of anarchy is upper bounded by $1 + \max_{(uv) \in G} \left( \frac{r_{uv}}{r_{uw}} \right)$. This implies a bound of $1 + \max_{(uv) \in G} \frac{\lambda_u}{\lambda_v}$ for Matthew Effect sharing. We will further prove that for the special case of Trust sharing, the upper bound on the price of anarchy is 3.

Let us define $R$ as

$$R = \max_{(uv) \in G} \frac{r_{uv}}{r_{uw}}$$  \hspace{1cm} (18)

Note that we will always have $R \geq 1$. We have the following theorem:

**Theorem 7.** If a stable matching exists, both prices of anarchy and stability in stable matching games with unequal reward sharing (without friendship utilities) are at most $R + 1$ and the bound is tight.

The tightness of this bound implies that as sharing becomes more unfair, i.e., as $R \to \infty$, we can find instances where both prices are unbounded. Thus unequal sharing can make things much worse for the stable matching game. In Section 4, however, we will see that this bound will significantly improve if we introduce friendship utilities. Thus, caring about others when reward sharing is unfair makes a significant difference to the price of anarchy, much more so than in equal sharing.

Now let us proceed to the proof of Theorem 7.

**Proof.** This theorem is simply a special case of our much more general Theorem 10 which proves a price of anarchy bound of $1 + \frac{R + \alpha_1}{1 + \alpha_1} R$. Without friendship utilities, the price of anarchy bound in Theorem 10 reduces to $1 + \frac{R}{1 + \alpha_1} = 1 + R$, as desired. To show that this bound is tight, we will use an instance of Matthew Effect sharing. We assign the following values in Fig. 1: $r_{uv} = 2$, $r_{uw} = R + 1$, $r_{vz} = R + 1$. Let $\lambda_u = 1$, $\lambda_v = 1$, $\lambda_w = R$, $\lambda_z = R$. Thus we have $r_{uv} = r_{uw} = r_{vz} = 1$. Now the matching $\{(uv)\}$ is stable and hence we get $\text{PoA} = R + 1$. For tightness of PoS bound, change $r_{uv}$ to $2 + 2\epsilon$. Now we have $r_{uv} = r_{uw} = 1 + \epsilon$ but $r_{uw} = r_{vz} = 1$, hence the matching $\{(uv)\}$ is the only stable matching. Thus we get $\text{PoS} = \frac{R + 1}{1 + \epsilon}$ which can be taken arbitrarily close to $R + 1$. $\square$

**Theorem 8.** The price of anarchy in stable matching games with Trust sharing is at most 3.

**Proof.** A stable matching always exists for Trust sharing by Theorem 6. Now we will prove that the price of anarchy can be at most 3.

Let $M^*$ denote a socially optimum matching and let $M$ denote a stable matching. Let $w_u^*$ denote the reward a node $u$ gets in $M^*$ and $w_u$ denote the reward a node $u$ gets in $M$. Consider an edge $(uv) \in M^* \setminus M$. As $(u,v)$ is not a blocking pair in $M$, without loss of generality, we can assume that the utility of $u$ does not increase by getting matched with $v$ in $M$. Now $u$ must be matched to some other node, say $z$. Call $u$ a witness node for $(uv) \in M^* \setminus M$. Since $u$ does not want to switch to $(uv)$ from $M$, then

$$h_{uv} + \beta_v \leq h_{uz} + \beta_z$$  \hspace{1cm} (19)

Adding $\beta_u$ to both sides

$$h_{uv} + \beta_v + \beta_u \leq h_{uz} + \beta_z + \beta_u$$  \hspace{1cm} (20)

$$\square$$
From Inequality (19), we get $h_{uv} \leq h_{uz} + \beta_z$. Adding this to Inequality (20), we obtain

$$2h_{uv} + \beta_v + \beta_u \leq 2h_{uz} + 2\beta_z + \beta_u$$

(21)

Suppose we form such inequalities for all $(uv) \in M^* \setminus M$ and add them. Let us investigate the coefficients of terms appearing on right hand side after such addition. If a term $h_{uz}$ appears on right hand side, then its coefficient can be at most 4: counting one inequality for $u$ acting as witness, and possibly one more inequality for $z$ acting as witness. However, the coefficient for a term $\beta_z$ appearing on right hand side can be at most 3, because $2\beta_z$ comes in when $u$ acts as witness and $\beta_z$ comes in when $z$ acts as witness. Hence adding these inequalities will give us

$$\sum_{(uv) \in M^* \setminus M} 2h_{uv} + \beta_v + \beta_u \leq \sum_{(uv) \in M \setminus M^*} 4h_{uv} + 3\beta_v + 3\beta_u$$

$$\Rightarrow \sum_{(uv) \in M^* \setminus M} 2h_{uv} + \beta_v + \beta_u \leq 3 \sum_{(uv) \in M \setminus M^*} 2h_{uv} + \beta_v + \beta_u$$

(22)

But $r_{xy} = r_{xy}^x + r_{xy}^y = 2h_{xy} + \beta_x + \beta_y$ for Trust sharing. Substituting this in inequality (22), we get

$$\sum_{(uv) \in M^* \setminus M} r_{uv} \leq 3 \sum_{(uv) \in M \setminus M^*} r_{uv}$$

$$\Rightarrow \frac{w(M^*)}{w(M)} = \frac{\sum_{(uv) \in M^*} r_{uv}}{\sum_{(uv) \in M} r_{uv}} \leq 3$$

As this is valid for any stable matching $M$, we have proved that for stable matching games with trust sharing, PoA $\leq 3$. \[\square\]

4 Stable Matching with Friendship and General Reward Sharing

In this section, we consider general stable matching games where players may have both friendship utilities and unequal reward sharing. We show general bounds on price of anarchy, and establish that friendship can make a much larger difference in the context of unequal sharing than in the case of fair sharing. First, just as at the start of Section 2, we write explicit conditions for nodes to form a blocking pair in this context, and define some helpful notation.

The necessary and sufficient conditions for nodes $(u, v)$ to form a biswivel from nodes $w$ and $z$ (See Fig. 1) in reward sharing with friendship are:

$$r_{uv}^u + \alpha_1 r_{uv}^v > r_{uw}^u + \alpha_1 (r_{uv}^w + r_{vz}^v) + \alpha_{uz} r_{vz}^z$$

$$r_{uv}^v + \alpha_1 r_{uv}^u > r_{vz}^v + \alpha_1 (r_{vz}^u + r_{uw}^u) + \alpha_{vw} r_{uw}^w.$$ 

Let us define $q_{xy}^x = r_{xy}^x + \alpha_1 r_{xy}^y$. Then the conditions for biswivel such as shown in Fig. 1 are:

$$q_{uv}^u > q_{uw}^u + \alpha_1 q_{uw}^v + \alpha_{uz} r_{vz}^z$$

(23)

$$q_{uv}^v > q_{vz}^v + \alpha_1 q_{uw}^u + \alpha_{vw} r_{uw}^w.$$ 

(24)

Similarly, the necessary and sufficient conditions for swivel (See Fig. 2) are

$$r_{uv}^u + \alpha_1 r_{uw}^v > r_{uw}^u + \alpha_1 r_{uw}^w$$

$$r_{uw}^v + \alpha_1 r_{uw}^u > \alpha_1 r_{uw}^v + \alpha_{vw} r_{uw}^w.$$ 

(25)

(26)
Using the definition of $q_{xy}(\cdot,\cdot)$, the conditions for swivel become:

\begin{align}
q_{uw}^u & > q_{uv}^u \quad (25) \\
q_{uv}^v & > \alpha_1 r_{uv}^u + \alpha_v r_{uw}^w \quad (26)
\end{align}

We also have

\[
\frac{q_{xy}^u}{q_{xy}^v} = \frac{r_{xy}^u + \alpha_1 r_{xy}^y}{r_{xy}^y + \alpha_1 r_{xy}^x}
\]

Using the fact that $\frac{p + \alpha_1}{1 + \alpha_1 p}$ is an increasing function of $p$ and using the definition of $R$, we thus obtain

\[
\frac{q_{xy}^u}{q_{xy}^v} \leq \frac{R + \alpha_1}{1 + \alpha_1 R} \quad (27)
\]

Let us define $q_{xy} = q_{xy}^u + q_{xy}^v$. Thus we obtain $q_{xy} = (1 + \alpha_1)r_{xy}$.

4.1 Existence of a Stable Matching with Friendship and General Reward Sharing

In Section 2.1 we showed that for the case of equal sharing with friendship utilities, a stable matching always exists. We showed this by proving that for equal sharing, a stable matching without friendship utilities (i.e. $\bar{\alpha} = \mathbf{0}$) is also a stable matching when we have friendship utilities.

However, for unequal reward sharing with friendship, the set of stable matchings for $\bar{\alpha} = \mathbf{0}$ is no longer a subset of the set of stable matchings when we have friendship utilities. Moreover, existence of a stable matching for $\bar{\alpha} = \mathbf{0}$ no more guarantees the existence of a stable matching with friendship utilities. We will give examples below to justify both claims. Finally we will conclude this section by giving a sufficient condition for the existence of a stable matching for stable matching games with unequal reward sharing and friendship utilities.

The following is an example which has non-overlapping sets of stable matchings with and without friendship: Assign $r_{uw}^u = r_{uw}^w = 0, r_{uw}^u = 10/11, r_{uv}^u = 100/11$ with $\alpha_1 = 1/2$ and $\alpha_2 = \alpha_3 = \cdots = 0$ in Fig. 2. Without friendship utilities, $\{uw\}$ is the only stable matching as node $u$ will always want to get matched to node $w$. However, with friendship utilities we have $q_{uw}^u = \frac{60}{11}, q_{uw}^u = \frac{3}{2}, q_{uw}^v = \frac{105}{11}, q_{uw}^v = \frac{3}{2}$. Thus using inequalities (25) and (26) we see that with friendship utilities, the only stable matching is $\{(uv)\}$ as the node $u$ will always want to get matched with node $v$. Thus for unequal reward sharing with friendship utilities, the set of stable matchings can be completely nonoverlapping with the set of stable matchings for unequal reward sharing but without friendship utilities.

Now we give an example where we have a stable matching with $\bar{\alpha} = \mathbf{0}$ but no stable matching with friendship utilities. Consider the Matthew Effect sharing example as shown in Fig. 3. The values on edges are edge rewards of those edges. The values in the brackets beside a node label is the brand value ($\lambda$ value) of that node. By Theorem 6 for $\bar{\alpha} = \mathbf{0}$ a stable matching always exists for Matthew Effect sharing. However, let us investigate the example for values shown in Fig. 3 with $\alpha_1 = 4/5, \alpha_2 = \alpha_3 = \cdots = 0$. Here we have

\[
\begin{align*}
q_{qx}^q &= 90 \quad > \quad q_{pq}^q = 89.1667 \\
q_{xy}^x &= 91.7493 \quad > \quad q_{xx}^x = 90 \\
q_{yz}^y &= 92.1545 \quad > \quad q_{xy}^y = 91.8507 \\
q_{zp}^z &= 112 \quad > \quad q_{zp}^z = 111.2455 \\
q_{zp}^p &= 103.3433 \quad > \quad q_{zp}^p = 102.2
\end{align*}
\]
Figure 3: Existence of a stable matching without friendship does not guarantee existence of a stable matching with friendship

Suppose there exists a stable matching for this example. In such a matching exactly one node would stay unmatched. Say the matching \{\((qx), (zp)\)\} is a candidate for stable matching. Now the node \(y\) is unmatched. In such a situation, \((x, y)\) will form blocking pair because we have \(q^x_{xy} > q^x_{qx}\) and \(q^y_{xy} > \alpha_1 r^x_{qx}\) (See inequalities (25) and (26) substituting \(\alpha_2 = 0\) as we use it in this example). Hence \{\((qx), (zp)\)\} is not a stable matching. Similarly every other matching can be shown to be not stable. Hence here we do not have a stable matching with friendship utilities, even though with \(\bar{\alpha} = 0\) a stable matching exists.

Now we give a sufficient condition for the existence of a stable matching in unequal reward sharing with friendship utilities. Let us denote by \(SRP_q\) the instance of stable roommate problem where we have exactly the same edges in the graph as our network but in \(SRP_q\) the nodes will prepare their preference lists based on \(q^x_{xy}\), i.e. a node \(u\) will prefer node \(v\) as roommate over \(w\) iff \(q^u_{uv} > q^u_{uw}\), breaking ties arbitrarily. Note that in an instance of stable roommate problem like \(SRP_q\) friendship utilities plays no role.

**Theorem 9.** A stable matching for \(SRP_q\) is a stable matching for matching games with unequal reward sharing and friendship utilities. Hence, existence of a stable matching for \(SRP_q\) implies the existence of a stable matching for general reward sharing with friendship utilities.

**Proof.** Suppose a stable matching \(M\) for \(SRP_q\) is not a stable matching for unequal reward sharing with friendship utilities. Then there exists a blocking pair \((u, v)\) with one of the following two possibilities:

- In \(M\), both \(u\) and \(v\) are matched: Let \(u\) and \(v\) be matched with \(w\) and \(z\) respectively. In such case, for \((u, v)\) to be a blocking pair the inequalities (23) and (24) must hold true. These inequalities imply \(q^u_{uw} > q^u_{wu}\) and \(q^v_{uv} > q^v_{uz}\). But then \((u, v)\) would be a blocking pair in \(SRP_q\). Hence \(M\) could not have been stable in \(SRP_q\).

- In \(M\), only one of the nodes \(u\) and \(v\) is matched: Say \(u\) is matched with \(w\) but \(v\) is unmatched. (Both cannot be unmatched otherwise \(M\) would not be stable in \(SRP_q\).) Then for \((u, v)\) to be a blocking pair inequalities (25) and (26) must hold true. But these inequalities imply \(q^u_{uv} > q^u_{uw}\) and thus \((u, v)\) would be a blocking pair in \(SRP_q\). Hence \(M\) could not have been stable in \(SRP_q\).

Either way we reach a contradiction. Hence \(M\) must be stable with unequal reward sharing and friendship utilities. Moreover, the set of stable matchings in \(SRP_q\) is a subset of the set of stable matchings in unequal reward sharing with friendship utilities.

\(\square \)  \(\square \)
4.2 Price of Anarchy with Friendship and General Reward Sharing

This section is about proving the following theorem:

**Theorem 10.** If a stable matching exists for general reward sharing with friendship utilities, then price of anarchy is at most $1 + Q$, where $Q = \max_{(uv) \in E(G)} \frac{q_{uv}}{q_{uw}} = \frac{R + \alpha_1}{1 + \alpha_1 R}$, and this bound is tight.

**Proof.** Let $M^*$ be a socially optimum matching, i.e., a matching with maximum $\sum_{(uv) \in M^*} r_{uv}$. Let $M$ be any stable matching. We will use $w_u^*$ (or $w_u$) to denote $q_{uv}$ if $u$ is matched to $v$ in $M^*$ (or in $M$). Because $q_{xy} = (1 + \alpha_1)r_{xy}$, we have

$$\text{PoA} = \max_{M \text{ is stable}} \frac{\sum_{(uv) \in M^*} q_{uv}}{\sum_{(uv) \in M} q_{uv}}$$

(28)

Using the definitions of $w_u^*$ and $w_u$ and letting $w_u^* = 0$ (or $w_u = 0$) in case $u$ is unmatched in $M^*$ (or $M$), we get

$$\text{PoA} = \max_{M \text{ is stable}} \frac{\sum_{u \in G} w_u^*}{\sum_{u \in G} w_u}$$

(29)

If edge $(uv) \in M^* \setminus M$, then the utility of at least one node among $u$ and $v$ does not increase if they were to deviate in $M$ to get matched with each other. Say the utility of node $u$ does not increase. Now we have three cases:

- Both $u$ and $v$ are matched in $M$: Say $u$ and $v$ are matched to $w$ and $z$ respectively. In such a case if getting matched to $v$ does not increase the utility of $u$ then we have

  $$q_{uv} \leq q_{uw} + \alpha_1 r_{vz} + \alpha_{uz} r_{vz}$$

  (30)

- $u$ is matched but $v$ is unmatched in $M$: Say $u$ is matched to $w$. In such a case if getting matched to $v$ does not increase the utility of $u$ then we have

  $$q_{uv} \leq q_{uw}$$

  (31)

- $u$ is unmatched but $v$ is matched in $M$: Say $v$ is matched to $z$. In such a case if getting matched to $v$ does not increase the utility of $u$ then we have

  $$q_{uv} \leq \alpha_1 r_{vz} + \alpha_{uz} r_{vz}$$

  (32)

Noticing that $\alpha_1 r_{vz} + \alpha_{uz} r_{vz} \leq r_{vz} + \alpha_1 r_{vz} = q_{vz}$, each of the inequalities (30), (31), and (32) imply that:

$$q_{uv} \leq q_{uw}^* + q_{vz}^*$$

A little algebraic manipulation gives us:

$$q_{uw}^* + q_{vz}^* \geq q_{uv} = \left(\frac{1}{1 + \alpha_1} \frac{q_{uv}}{q_{uw}}\right) \cdot q_{uv}$$

$$w_u + w_v \geq \frac{1}{1 + Q} \cdot (w_u^* + w_v^*)$$

(33)
Adding such inequalities for all \((uv) \in M^* \setminus M\), we obtain
\[
\sum_{(uv) \in M^* \setminus M} (w_u + w_v) \geq \frac{1}{1+Q} \cdot \sum_{(uv) \in M^* \setminus M} (w_u^* + w_v^*)
\] (34)

Notice that if a node \(u\) appears in the above inequality then \(u\) is matched to different nodes in \(M^*\) and \(M\). Denote the set of all such nodes by \(B\). Hence inequality (34) becomes
\[
\sum_{u \in B} w_u \geq \frac{1}{1+Q} \cdot \sum_{u \in B} w_u^*,
\] (35)

and so the price of anarchy is at most \(1 + Q\), as desired.

**Tightness of the bound:** Consider the 3-length path as shown in Fig. 1. Make \(\alpha_2 = \alpha_3 = \cdots = 0\).

Substitute the following values:
\[
\begin{align*}
 r_{uv}^u &= \frac{1}{1 + \alpha_1} \\
r_{uw}^u &= \frac{1}{1 + \alpha_1 R} \\
r_{vz}^v &= \frac{1}{1 + \alpha_1 R}
\end{align*}
\]
\[
\begin{align*}
 r_{uv}^v &= \frac{1}{1 + \alpha_1} \\
r_{uw}^w &= \frac{R}{1 + \alpha_1 R} \\
r_{vz}^z &= \frac{R}{1 + \alpha_1 R}
\end{align*}
\]

Note that as desired, \(\max_{(xy) \in E(G)} \frac{r_{xy}^x}{r_{xy}^y} = R\). Using \(q_{xy}^x = r_{xy}^x + \alpha r_{xy}^y\), we obtain
\[
\begin{align*}
 q_{uv}^u &= 1 \\
 q_{uw}^u &= 1 \\
 q_{vz}^v &= 1 \\
 q_{uw}^w &= Q \\
 q_{vz}^z &= Q
\end{align*}
\]

Note that as desired, \(\max_{(xy) \in E(G)} \frac{q_{xy}^x}{q_{xy}^y} = \frac{R + \alpha_1}{1 + \alpha_1 R} = Q\). We have \(\{(uv)\}\) as a stable matching because given this matching, \((u, w)\) is not a blocking pair as we have \(q_{uw}^u \leq q_{uw}^w\). Similarly \((v, z)\) too is not a blocking pair in matching \(\{(uv)\}\). Another stable matching is \(\{(uw), (vz)\}\) because given this matching, \((u, v)\) will not be a blocking pair as we have \(q_{uw}^u < q_{uw}^u + \alpha_1 r_{vz}^v\), hence the condition in inequality (23) is violated. Since there are no other stable matchings for this graph, the price of anarchy will be determined by the value of the worst stable matching which is \(\{(uv)\}\). It is given by
\[
\text{PoA} = \frac{r_{uw} + r_{vz}}{r_{uv}} = \frac{q_{uw} + q_{vz}}{q_{uv}} = 1 + Q
\]

Hence the bound is tight.

**Discussion** We have \(\text{PoA} \leq 1 + Q\) where \(Q = \frac{R + \alpha_1}{1 + \alpha_1 R}\). Let us consider the implications of this bound. If \(\alpha_1 = 0\), we have \(\text{PoA} \leq 1 + R\) which agrees with Theorem 2. If \(R = 1\), we have \(\text{PoA} = 2\). This result implies Theorem 2 since when we have \(R = 1\), then both \(u\) and \(v\), if they are matched to each other, get the same reward from \((uv)\).

Notice also that \(\frac{R + \alpha_1}{1 + \alpha_1 R}\) is a decreasing function of \(\alpha_1\). As \(\alpha_1\) goes from 0 to 1, the bound goes from 1 + \(R\) to 2. Without friendship utilities, we have a tight bound \(\text{PoA} \leq 1 + R\). Thus for \(\alpha = 0\), it can be extremely bad if \(R\) is large. As \(\alpha_1\) gets close to 1, however, no matter how large \(R\) is, \(\text{PoA}\) comes down to 2 from \(R + 1\). For example, if \(\alpha_1 = 1/2\), then it is only 3. Thus, social context can drastically improve the outcome for the society, especially in the case of unfair and unequal reward sharing.
4.3 Price of Stability with Friendship and General Reward Sharing

In this section, we give a simple lower bound $Q'$ on the price of stability for stable matching games with friendship and reward sharing. Furthermore, we show that this bound is within an additive factor of 1 of optimum, i.e., $Q < Q' \leq \text{PoS} \leq 1 + Q$.

To prove the lower bound, consider the 3-length path as shown in Fig. 1. Make $\alpha_2 = \alpha_3 = \cdots = 0$ and and use the following values:

\[
\begin{align*}
    r_{uv}^u &= \frac{1}{1 + \alpha_1} \left( \frac{1 + \alpha_1 (R + 1)}{1 + \alpha_1 R} + \epsilon \right) \quad r_{uv}^v = \frac{1}{1 + \alpha_1} \left( \frac{1 + \alpha_1 (R + 1)}{1 + \alpha_1 R} + \epsilon \right) \\
    r_{uw}^u &= \frac{1}{1 + \alpha_1 R} \quad r_{uw}^w = \frac{R}{1 + \alpha_1 R} \\
    r_{vz}^v &= \frac{1}{1 + \alpha_1 R} \quad r_{vz}^z = \frac{1}{1 + \alpha_1 R}
\end{align*}
\]

As desired we have $\max_{(xy) \in E(G)} \frac{r_{xy}^u}{r_{xy}^v} = R$. Using $q_{xy}^u = r_{xy}^u + \alpha_1 r_{xy}^v$, we obtain

\[
\begin{align*}
    q_{uw}^u &= \frac{1 + \alpha_1 (R + 1)}{1 + \alpha_1 R} \\
    q_{uw}^u &= 1 \\
    q_{vz}^v &= 1
\end{align*}
\]

As desired, we have $\max_{(xy) \in E(G)} \frac{q_{xy}^u}{q_{xy}^v} = \frac{R + \alpha_1}{1 + \alpha_1 R} = Q$. We have $\{(uv)\}$ as a stable matching because $(u, w)$ is not a blocking pair as $q_{uw}^u \leq q_{uw}^v$. Similarly $(v, z)$ will not be a blocking pair. But the matching $\{(uw), (vz)\}$ is no longer stable because $(u, v)$ is a blocking pair as inequalities (23) and (24) are satisfied. However $\{(uw), (vz)\}$ is still the socially optimal matching. Hence the price of stability for this graph will be given by

\[
\text{PoS} = \frac{r_{uw}^u + r_{vz}^v}{r_{uw}^u} = \frac{q_{uw}^u + q_{vz}^v}{q_{uw}^u} = \frac{(1 + \alpha_1)(1 + R)}{1 + \alpha_1 (R + 1)}
\]

Let us define $Q' = \frac{(1 + \alpha_1)(1 + R)}{1 + \alpha_1 (R + 1)}$. Because in the above instance we have $\text{PoS} = Q'$, the lower bound on the price of stability can be expressed as $\text{PoS} \geq Q'$, where $Q \leq Q' \leq Q + 1$. Since $Q + 1$ is an upper bound on the price of stability, this means that the lower bound of $Q'$ is within an additive term of 1 of optimum.

**Theorem 11.** The worst-case price of stability of stable matching games with friendship and general reward sharing is in $[Q', Q + 1]$, with $Q < Q' \leq Q + 1$.

**Proof.** The only part that is yet to be proven is $Q \leq Q'$ and $Q' \leq 1 + Q$. We have

\[
Q' - Q = \frac{(1 - \alpha_1 + \alpha_1 R)(1 + \alpha_1)}{(1 + \alpha_1 + \alpha_1 R)(1 + \alpha_1 R)}
\]

As $(1 - \alpha_1 + \alpha_1 R) \leq (1 + \alpha_1 + \alpha_1 R)$ and $1 + \alpha_1 \leq 1 + \alpha_1 R$, we have that $Q' - Q \leq 1$. As $R \geq 1$, the numerator is always positive. Hence $0 < Q' - Q \leq 1$. Using this with $Q' \leq \text{PoS}$, we have that $Q < Q' \leq \text{PoS} \leq 1 + Q$. \qed
5 Convex Contribution Games (CCGs)

In this section we consider convex contribution games (CCGs), as defined in Section 1.1. In this version of CCG, players do not have to spend all their budget: the total contribution of a player to its incident edges must be at most $B_i$. This corresponds to the fact that players may decide to keep some budget for themselves, instead of spending it all on friendships/projects that the links represent. We consider the case when players must spend their entire budget in Section 6.

For each CCG we define a corresponding stable matching game denoted $SM(G, \bar{\alpha})$ as follows. The edge rewards in stable matching are rewards when both players invest their full budget on an edge in the CCG. For equal sharing this means $r_{uv} = f_{uv}(B_u, B_v)$, for unequal sharing $r_{uv}^u = f_{uv}^u(B_u, B_v)$. For games with friendship we assume the same values for $0 \leq \alpha_1 \leq \ldots \leq \alpha_{\text{diam}(G)} \leq 1$ in both games. For simplicity, we use the following notation: $g_{uv}^u(x, y) = f_{uv}^u(x, y) + \alpha_1 f_{uv}^u(x, y)$ and $g_{uv}(x, y) = f_{uv}^u(x, y) + f_{uv}^v(x, y)$ for all $x, y \geq 0$.

In general, we will show that properties like existence and total reward of pairwise equilibria in CCGs can be derived from the properties of stable matchings in the corresponding games.

5.1 Existence of a Pairwise Equilibrium

We start by showing a general reduction for existence of a pairwise equilibrium for arbitrary $\bar{\alpha}$. Recall that all reward functions of CCG are assumed to be convex in both its parameters, and satisfy the property that $f(x, 0) = f(y, 0) = 0$ for all $x, y$. We call the class of such functions $C_0$.

Theorem 12. If all reward functions $f_{uv}^u(\cdot, \cdot) \in C_0$, then for every stable matching of the corresponding $SM(G, \bar{\alpha})$ there is an equivalent pairwise equilibrium in the CCG. The pairwise equilibrium has the same assignment structure and total reward.

Proof. Let $M$ be a stable matching in $SM(G, \bar{\alpha})$ and consider the following strategy profile for the CCG: if node $u$ is matched to node $v$ in $M$, set $s_u(uv) = B_u$. If $u$ is not matched in $M$, then set $s_u(uv) = 0$ for all incident edges $(uv) \in E$. We will show that $s$ is a pairwise equilibrium. Obviously, $s$ has the same structure as $M$ and, in particular, yields the same total reward.

First, note that $f_{uv}^u(x, y)$ is increasing and convex in both arguments, which implies the same for $g_{uv}^u(x, y)$. Second, note that in $s$ for each edge we have both players contributing the full budget or nothing. Thus, players can deviate unilaterally or bilaterally only by reallocating budget onto edges $(uv) \notin M$.

Suppose two players $u$ and $v$ deviate and do not move any additional effort to their common edge $(uv)$ (because, e.g., $(uv) \notin E$, or $(uv) \in M$ and both already spend all budget there). They cannot increase reward on incident edges in $M$ (if any), because they are spending their full budget. For every other incident edge $e \notin M, e \neq (uv)$ they cannot increase the reward beyond 0, because the other player keeps putting 0 effort. Hence, the only possibility to strictly improve their reward is when both players move some non-zero effort to $(uv)$. This, in particular, shows that there are no improving unilateral deviations.

Hence, let us focus on bilateral deviations of players $u$ and $v$ by moving some effort to a common edge. If both players are unmatched in $M$ and have such a improving deviation, this contradicts that $M$ is a stable matching. Hence, the following two cases remain.

- Suppose there exists a improving bilateral deviation onto $(uv) \notin M$ and exactly one player, say $v$, is unmatched in $M$. Let $u$ be matched to $w$ in $M$. We assume that in the CCG $u$ and $v$ can improve by moving $\epsilon_1$ and $\epsilon_2$ of budget to $(uv)$, respectively. This implies

$$g_{uw}^u(B_u, B_w) < g_{uw}^u(\epsilon_1, \epsilon_2) + g_{uw}^u(B_u - \epsilon_1, B_w)$$
As both $g^u_{uw}$ and $g^u_{uw}$ are convex in both arguments, this means that
\[ g^u_{uw}(B_u, B_w) < g^u_{uw}(B_u, \epsilon_2) < g^u_{uw}(B_u, B_v). \]
For $SM(G, \alpha)$ this shows $g^u_{uv} > g^u_{uw}$, but then $v$ cannot be unmatched, because this would contradict that $M$ is a stable matching.

\[ \bullet \text{ Suppose there exists a profitable bilateral deviation onto an edge } (uv) \notin M, \text{ where } u \text{ is matched to } w \text{ and } v \text{ to } z \text{ in } M. \text{ If } u \text{ and } v \text{ transfer } \epsilon_1 \text{ and } \epsilon_2 \text{ to } (uv), \text{ respectively, then for node } u \text{ we have} \]
\[ g^u_{uv}(\epsilon_1, \epsilon_2) > g^u_{uw}(B_u, B_w) - g^u_{uw}(B_u - \epsilon_1, B_w) + \alpha_1(f^u_{uv}(B_u, B_z) - f^u_{uz}(B_v - \epsilon_2, B_z)) + \alpha_{uz}(f^z_{uz}(B_v, B_z) - f^z_{uv}(B_v - \epsilon_2, B_z)) \]
\[ = g^u_{uv}(B_u, B_w) - g^u_{uw}(B_u - \epsilon_1, B_w) + \alpha_1(f^u_{uv}(B_u, B_w) - f^u_{uw}(B_u - \epsilon_1, B_w)) + \alpha_{uv}(f^w_{uw}(B_u, B_w) - f^w_{uw}(B_u - \epsilon_1, B_w)) \] (36)
which formally states that there is a net increase in the utility of $u$ because of the transfer. Similarly for node $v$ we have
\[ g^v_{uv}(\epsilon_1, \epsilon_2) > g^v_{uv}(B_v, B_v) - g^v_{uv}(B_v - \epsilon_2, B_z) + \alpha_1(f^v_{uv}(B_u, B_w) - f^v_{uw}(B_u - \epsilon_1, B_w)) + \alpha_{uv}(f^w_{uw}(B_u, B_v) - f^w_{uw}(B_u - \epsilon_1, B_v)) \] (37)
As all functions $f$ and $g$ are convex and increasing in both arguments, we get
\[ g^u_{uw}(B_u, B_v) > g^u_{uw}(B_u, B_u) + \alpha_1f^u_{uv}(B_u, B_z) + \alpha_{uz}f^z_{uv}(B_v, B_z) \]
\[ g^v_{uv}(B_u, B_v) > g^v_{uv}(B_v, B_v) + \alpha_1f^v_{uw}(B_u, B_w) + \alpha_{uv}f^w_{uw}(B_u, B_w) \]
but then in $SM(G, \alpha)$ the following must hold true
\[ g^u_{uv} > g^u_{uw} + \alpha_1f^u_{uv} + \alpha_{uv}f^w_{uw} \]
\[ g^v_{uv} > g^v_{uv} + \alpha_1f^v_{uw} + \alpha_{uv}f^w_{uw}. \]
This means that in $SM(G, \alpha)$, nodes $u$ and $v$ would prefer getting matched to each other (see inequalities (23) and (24)), i.e., $M$ is not a stable matching in $SM(G, \alpha)$ which contradicts our assumption.

\[ \square \]

The conditions for existence of a pairwise equilibrium can be weakened for CCGs without friendship. In this case, convexity of reward share in the other player’s contribution is not necessary.

**Corollary 2.** If $f^u_{uv}(s_u(uv), s_v(uv))$ are increasing in $s_u(uv)$ and $s_v(uv)$ and convex in $s_v(uv)$, then for every stable matching of the corresponding game SM($G, \alpha = 0$), there is an equivalent pairwise equilibrium in the CCG without friendship. The pairwise equilibrium has the same assignment structure and total reward.

**Proof.** We just need to observe that $f^u_{uv}(s_u(uv), s_v(uv))$ does not need to be convex in $s_v(uv)$. In the proof of the previous theorem, convexity in $s_v(uv)$ is only required in inequality (36) and (37) with the $\alpha_2$ coefficient.

We proceed to specify more detailed results for particular reward sharing rules.
Equal Sharing We first consider equal sharing with (or without) friendship. In this case, $SM(G, \bar{\alpha})$ always has a stable matching from Theorem 1. Also, because $f_{uv}(s_u(uv), s_v(uv)) = f_{uv}^u(s_u(uv), s_v(uv)) = f_{uv}^v(s_u(uv), s_v(uv))$ and $f_{uv}^v(\cdot, \cdot) \in C_0$, we also have $f_{uv}^u(\cdot, \cdot) \in C_0$ and $f_{uv}(\cdot, \cdot) \in C_0$. Hence all the conditions for existence of a pairwise equilibrium are satisfied. The following corollary extends a main result from [5] to CCGs with arbitrary friendship.

Corollary 3. A pairwise equilibrium always exists in a CCG with equal sharing.

Matthew Effect Sharing Next, let us consider Matthew Effect CCGs defined as follows: Each node $u \in G$ has an associated brand value $\lambda_u$. If nodes $u$ and $v$ invest $s_u(uv)$ and $s_v(uv)$ respectively on edge $(uv)$, then $u$ obtains a reward of 

$$f_{uv}^u(s_u(uv), s_v(uv)) = \frac{\lambda_u}{\lambda_u + \lambda_v} f_{uv}(s_u(uv), s_v(uv))$$

from edge $(uv)$. Consequently in Matthew Effect CCG, we have 

$$g_{uv}^u(s_u(uv), s_v(uv)) = \frac{\lambda_u + \alpha \lambda_v}{\lambda_u + \lambda_v} f_{uv}(s_u(uv), s_v(uv)) .$$

It can be easily seen that in Matthew Effect CCG, $f_{uv}^u(\cdot, \cdot)$ are increasing and convex in the investment of $u$ and $v$. Hence we have the following corollaries from Theorem 12 and Lemma 3.

Corollary 4. For every stable matching of the corresponding game $SM(G, \bar{\alpha})$ with Matthew Effect Sharing, there is an equivalent pairwise equilibrium in the Matthew Effect CCG. The pairwise equilibrium has the same assignment structure and total reward.

As a special case, we have guaranteed existence for Matthew Effect CCGs without friendship.

Corollary 5. A pairwise equilibrium always exists in Matthew Effect CCGs without friendship.

Proportional Sharing Finally, let us consider a natural model of sharing that is specific to CCGs (this model was not considered in Section 3). In Proportional Sharing CCG, the reward a node gets is proportional to the effort it contributes to an edge. In other words, if nodes $u$ and $v$ invest $s_u(uv)$ and $s_v(uv)$ respectively on edge $(uv)$, then $u$ obtains a reward of 

$$f_{uv}^u(s_u(uv), s_v(uv)) = \frac{s_u(uv)}{s_u(uv) + s_v(uv)} f_{uv}(s_u(uv), s_v(uv))$$

from edge $(uv)$. Consequently, in Proportional Sharing CCG, we have 

$$g_{uv}^u(s_u(uv), s_v(uv)) = \frac{s_u(uv) + \alpha s_v(uv)}{s_u(uv) + s_v(uv)} f_{uv}(s_u(uv), s_v(uv)) .$$

For a proportional sharing CCGs, it can be verified that $f_{uv}^u(s_u(uv), s_v(uv))$ are increasing in $s_u(uv)$ and $s_v(uv)$ and convex in $s_u(uv)$. It is easy to observe that the corresponding stable matching game is, in fact, an instance of the Matthew Effect model with $\lambda_u = B_u$ for all nodes $u \in G$. Hence, without friendship a stable matching always exists in the stable matching game and provides the following corollary based on Corollary 2.

Corollary 6. For every stable matching of the corresponding game $SM(G, \bar{\alpha} = 0)$ with Matthew Effect Sharing, there is an equivalent pairwise equilibrium in the Proportional Sharing CCG without friendship. The pairwise equilibrium has the same assignment structure and total reward. There always exists at least one such stable matching with corresponding pairwise equilibrium.
5.2 Prices of Anarchy and Stability

In the previous section, we have seen that stable matchings can easily be translated into pairwise equilibria for CCGs. However, there could potentially be other pairwise equilibria that are, in particular, much worse in terms of total reward. In this section, we show that this is not the case and translate the bounds for prices of anarchy and stability from stable matching to CCGs. Thus, these bounds apply for all reward functions \( f^u_{uv}(\cdot, \cdot) \in C_0 \).

Let us define a tight edge as an edge on which both of the nodes invest their full budget. The social optimum with maximum total reward does not depend on the reward sharing scheme at hand or on the values of \( \overrightarrow{\alpha} \). Thus, Claim 2.10 in [5] shows that there always exists a tight social optimum, i.e., a social optimum \( s^* \) such that players invest only in tight edges. In particular, as the CCG allows more flexibility than the corresponding stable matching game \( SM(G, \overrightarrow{\alpha}) \), a tight social optimum in the CCG is in 1-to-1 correspondence to a social optimum in \( SM(G, \overrightarrow{\alpha}) \). Whenever stable matchings in \( SM(G, \overrightarrow{\alpha}) \) correspond to pairwise equilibria in the CCG and a tight social optimum in the CCG corresponds to a social optimum in the corresponding \( SM(G, \overrightarrow{\alpha}) \), we can directly translate our upper bounds on the price of stability to CCGs. This implies the following corollary.

**Corollary 7.** The price of stability in CCGs with equal sharing and friendship is at most \( \frac{2 + 2\alpha_1}{1 + 2\alpha_1 + \alpha_2} \). The Best-Relaxed-Blocking-Pair starting from a tight social optimum converges in polynomial time to a pairwise equilibrium that achieves this bound.

For the price of anarchy, we could possibly have worse equilibria in the CCG that do not correspond to matchings in \( SM(G, \overrightarrow{\alpha}) \). However, the same bound as in Theorem 10 can be proved. We use parameter \( Q \) (as detailed in Section 4) for the corresponding stable matching game \( SM(G, \overrightarrow{\alpha}) \).

**Theorem 13.** The price of anarchy in CCGs is bounded by \( \text{PoA} \leq Q + 1 \).

**Proof.** It suffices to compare against a tight social optimum which we denote by \( s^* \). Let \( s \) denote a pairwise equilibrium. Let \( s_u(uv) \) denote the investment of node \( u \) on edge \( (uv) \) in the pairwise equilibrium \( s \). Let us define \( w_u \) as:

\[
  w_u = \sum_{z \in N_1(u)} g^u_{uz}(s_u(uz), s_z(uz))
\]  

(38)

Using the definition of \( w_u \), it can be verified that the total reward \( w(s) \) of \( s \) is given by

\[
  w(s) = \frac{1}{1 + \alpha_1} \sum_{u \in G} w_u
\]  

(39)

and the total reward \( w(s^*) \) of \( s^* \) is

\[
  w(s^*) = \frac{1}{1 + \alpha_1} \sum_{(uv) \in s^*} g_{uv}(B_u, B_v)
\]  

(40)

where by \((uv) \in s^* \) we mean the tight edges in \( s^* \). Hence the price of anarchy can also be expressed as

\[
  \text{PoA} = \max_s \frac{w(s^*)}{w(s)} = \max_s \frac{\sum_{(uv) \in s^*} g_{uv}(B_u, B_v)}{\sum_{u \in G} w_u}
\]  

(41)

We will use this alternative expression for the price of anarchy for proving the claim.
Let us construct a set of witness nodes in one-to-one correspondence with tight edges in \( s^* \) as follows: For each tight edge \((uv)\) of \( s \) we make either \( u \) or \( v \) a witness for \((uv)\). On other edges, we have either \( s_u(uv) < B_u \) or \( s_v(uv) < B_v \). Now as \( s \) is a pairwise equilibrium, if \( u \) and \( v \) both transfer their full budget to \((uv)\), utility of at least one – we w.l.o.g. assume node \( u \) – will not increase, and we make \( u \) witness for \((uv)\).

As the deviation towards \((uv)\) is not improving, we can examine the utility \( u \) and bound

\[
g^u_{uv}(B_u, B_v) \leq \sum_{y \in N_1(u)} g^y_{uy}(s_u(uy), s_y(uy)) + \alpha \sum_{z \in N_1(v) - u} f^v_{uv}(s_v(v), s_z(v))
\]

\[
+ \sum_{z \in N_1(v) - u} \alpha_{uz} f^z_{uv}(s_v(v), s_z(v))
\]

As a consequence of \( \alpha\sum f^v_{uv}(s_v(v), s_z(v)) + \alpha_{uz} f^z_{uv}(s_v(v), s_z(v)) \leq \alpha \sum f^v_{uv}(s_v(v), s_z(v)) + \alpha \sum f^z_{uv}(s_v(v), s_z(v)) \leq g^v_{uv}(s_v(v), s_z(v)) \), we obtain

\[
g^u_{uv}(B_u, B_v) \leq \sum_{y \in N_1(u)} g^y_{uy}(s_u(uy), s_y(uy)) + \sum_{z \in N_1(v) - u} g^v_{uv}(s_v(v), s_z(v)) \tag{42}
\]

\[
\Rightarrow \frac{1}{1 + Q} \cdot g_{uv}(B_u, B_v) \leq \sum_{y \in N_1(u)} g^y_{uy}(s_u(uy), s_y(uy)) + \sum_{z \in N_1(v) - u} g^v_{uv}(s_v(v), s_z(v)) \tag{43}
\]

\[
\Rightarrow \frac{1}{1 + Q} \cdot g_{uv}(B_u, B_v) \leq w_u + w_v \tag{44}
\]

Note that the last inequality \( \text{(44)} \) also holds for tight edges \((uv)\). Thus we have one inequality due to witnessing each edge. Adding these inequalities,

\[
\frac{1}{1 + Q} \sum_{(uv) \in s^*} g_{uv}(B_u, B_v) \leq \sum_{(uv) \in s^*} (w_u + w_v) \leq \sum_{u \in G} w_u
\]

Hence we get

\[
\frac{\sum_{(uv) \in s^*} g_{uv}(B_u, B_v)}{\sum_{u \in G} w_u} \leq 1 + Q
\]

As this is valid for any pairwise equilibrium \( s \), using Eqn. \( \text{(4)} \), we can complete the proof by finding

\[
\text{PoA} \leq 1 + Q
\]

\[ \square \]

## 6 Contribution Games With Tight Budget Constraints

In this section we consider the version of contribution games where all player budget must be spent on adjacent edges, i.e., the sum of each player \( v \)'s contributions to incident edges \( \sum_{(v, u)} s_v(vu) \) exactly equals \( B_v \). At first glance, this version of the game does not seem very different than the case when \( \sum_{(v, u)} s_v(vu) \leq B_v \). And in fact, when node utilities simply consist of \( R_u(s) \) (i.e., there is no “friendship” component), then it can be easily shown that all the results from \( \text{[5]} \) and from Section \( \text{[5]} \) still hold.
Theorem 14. Consider a CCG with equal reward sharing and local friendship in which all players corresponding to a CCG must contribute their entire budget to incident edges. Let a matching be a pairwise equilibrium. If, however, the two endpoint nodes must contribute their entire budget on the edge between them, and the two endpoints do not contribute anything, is a solution where the two middle nodes put all of their budget, or less than their budget, then the solution where the two middle nodes put all of their budget on the edge between them, and the two endpoints do not contribute anything, is a pairwise equilibrium. If, however, the two endpoint nodes must contribute their budget to their incident edges, then this is no longer a pairwise equilibrium, as the two middle nodes are able to simultaneously move their budgets to the outer edges, obtaining \((1 + 2\alpha_1)(1 - \varepsilon) > (1 + \alpha_1)\) utility each. In general, the argument for existence of pairwise equilibrium from Section 5 no longer works, as stable matchings may no longer correspond to pairwise equilibrium as they did in Theorem 12. Moreover, the existence argument from 5 is based on forming a maximal greedy matching, which, as the above example shows, is not necessarily a pairwise equilibrium.

Fortunately, we are able to extend many of our results to the version where players must spend their entire budgets as well. Specifically, we show that all our results still hold for the case of equal sharing, with \(\alpha_i = 0\) for all \(i \geq 2\). We call this type of perceived utility local friendship, since nodes only care about their neighbors, but not their neighbors-of-neighbors. In the rest of this section, we let \(\alpha = \alpha_1\), since it is the only non-zero \(\alpha_i\).

As the example above shows, even for the case of local friendship, contribution games with tight budget constraints can behave peculiarly. Essentially, the complication here arises from the fact that a stable matching is stable with respect to swivel and biswivel deviations only. On the other hand, a pairwise equilibrium has to be stable with respect to all bilateral deviations, including two non-adjacent nodes switching contributions, or two adjacent nodes moving their contributions away from their shared edge. If all unmatched nodes do not contribute anything to incident edges, then all these deviations cannot be improving (see Theorem 12), but in the model where all budget must be spent, these deviations can and do occur.

The key to our results in this section is the following theorem. To state the theorem, we first need the concept of forbidden edges, defined below. As in Section 5, \(SM(G, \vec{\alpha})\) is the stable matching game corresponding to a CCG, and \(r_{uv} = f_{uv}(B_u, B_v)\) as before.

**Definition 4.** We call an edge \(e = (u, v)\) in a contribution game forbidden if both of the following hold:

1. There exist edges \((u, x)\) and \((v, y)\) with \(x \neq v, y \neq u,\) and both \(x, y\) having degree 1.
2. \(u\) and \(v\) would be willing to deviate by putting all their budget on edges \((u, x)\) and \((v, y)\) even if both \(u\) and \(v\) are putting all their budget on edge \(e\). In other words, \(e\) is such that \(r_{uv} + \alpha r_{uv} < r_{ux} + \alpha r_{ux} + \alpha r_{vy}\) and \(r_{uv} + \alpha r_{uv} < r_{vy} + \alpha r_{vy} + \alpha r_{ux}\).

**Theorem 14.** Consider a CCG with equal reward sharing and local friendship in which all players must contribute their entire budget to incident edges. Let a matching \(M\) be a stable matching in the corresponding \(SM(G, \vec{\alpha})\). If \(M\) does not contain any forbidden edges, then there exists an equivalent pairwise equilibrium, with the same assignment structure and total reward, obtained by having all unmatched nodes split their contributions equally among all incident edges.

**Proof.** Set the strategy of a node \(u\) with an edge \((u, v) \in M\) to put all its budget onto edge \((u, v)\). If a node is unmatched in \(M\), then we set its strategy to spread its budget equally among all its incident edges. Call this solution \(s\); our goal is to show that \(s\) is a pairwise equilibrium. Since all functions \(f_e\) are nondecreasing and convex in both parameters, we can restrict our attention, wlog, to deviations where nodes move all their budget to a single edge.
It is clear that no unilateral improving deviations exist in s. This is because if a node v could gain utility by unilaterally moving effort to an edge e = (u, v), then u must be unmatched (recall that \( f_e(B_v, 0) = 0 \) by definition of CCG), and so v would gain by performing a swivel to u in M, contradicting the stability of M. Similarly, swivels and biswivels (i.e., deviations where two nodes u and v move all their effort to the edge (u, v)) cannot be improving deviations, since otherwise M would not be a stable matching. The above argument also implies that no unmatched node would participate in an improving deviation from s, since it can only obtain positive reward by putting effort on an edge to another unmatched node, which contradicts the stability of M, since all stable matchings are maximal.

Now we must consider all other types of deviations. Specifically, we must now show that for every pair of nodes u and v that are matched in M (not necessarily to each other), there is no improving bilateral deviation of u and v. Let \( e_1 = (u, w) \) and \( e_2 = (v, z) \) be the edges of M incident to u and v respectively. Note that \( e_1 \) may equal \( e_2 \).

Suppose to the contrary that u and v have an improving bilateral deviation in s, and let \( e_3 = (u, x) \) be the edge that u moves its budget to, and \( e_4 = (v, y) \) be the edge that v moves its budget to. We know that \( e_3 \neq e_4 \), since otherwise this deviation corresponds to an improving biswivel in M, which is not possible since M is a stable matching. This means that wlog, in a bilateral deviation u and v will move all their budget to an edge incident to an unmatched node: this is because by moving its budget to an edge incident to a matched node that is neither \( e_1 \) or \( e_2 \), u will end up with 0 reward. Thus, if an improving bilateral deviation of u and v exists, then in this deviation u and v move their budgets from \( e_1 \) and \( e_2 \) (which may be the same edge \( e_1 = (u, v) = e_2 \)) to edges \( e_3 = (u, x) \) and \( e_4 = (v, y) \), which are not the same edges. It is still possible, however, that x may equal y.

Denote by \( \gamma_x \) the reward on edge \( e_3 \) obtained if node u puts all its budget onto this edge, i.e., \( \gamma_x = f_{ux}(B_u, s_x(ux)) \). Note that if x has only a single incident edge, then \( \gamma_x = r_{ux} \), since \( s_x(ux) = B_x \) in this case. If instead x has degree at least 2, then \( \gamma_x \leq r_{ux}/2 \), since x is splitting its budget evenly among all incident edges, and since \( f_{ux} \) is convex in the contribution of node x. Similarly, define \( \gamma_y \) as \( f_{vy}(B_v, s_y(vy)) \). Finally, we will use notation \([a]_P\) to denote a if property P holds, and 0 otherwise.

**Case 1:** \( e_1 = e_2 = (u, v) \) In this case, it cannot be that both x and y have degree 1, since this would imply that \((u, v)\) is a forbidden edge, and thus could not be in M. Therefore, we can assume that, wlog, the degree of x is at least 2. The only rewards that change during the deviation are the rewards on \((u, v)\), \( e_3 \), and \( e_4 \). The reward u receives from edges \( e_3 \) and \( e_4 \) after the deviation is at most \( \gamma_x + \alpha \gamma_x + \alpha \gamma_y + \alpha \gamma_y \); the last term is only present if node y is adjacent to u. Since x has degree at least 2, then in order for this to be a profitable deviation for u, it must be that

\[
\frac{r_{ux} + \alpha r_{ux}}{2} + \alpha r_{vy} > r_{uv} + \alpha r_{uv}.
\] (45)

Recall that nodes x and y are unmatched in M. Due to stability of M, it must be that \( r_{uv} \geq r_{ux} \) and \( r_{uv} \geq r_{vy} \), since otherwise swiveling from \((u, v)\) to \((u, x)\) or from \((u, v)\) to \((v, y)\) would be an improving swivel deviation in the stable matching. Thus, Inequality (45) implies that \((1 + \alpha)r_{uv} < (\frac{1}{2} + \frac{\alpha}{2})r_{uv}\), which is a contradiction since \( \alpha \leq 1 \).

**Case 2:** \( e_1 \neq e_2 \) We now have the final case to consider, in which \( e_1 \neq e_2 \) (recall also that \( e_3 \neq e_4 \)). The total contribution of rewards of \( e_1 = (u, w) \) and \( e_2 = (v, z) \) to the utilities of u and v before the deviation was at least

\[
(1 + \alpha + \lceil a \rceil_{(u,v) \in E})(r_{uw} + r_{vz}).
\] (46)

29
Note that the contribution can be even larger if, for example, $u$ is adjacent to $z$, but it is at least as large as (46). The total contribution of rewards of $e_3$ and $e_4$ to $u$ and $v$ after the deviation is at most

$$ (1 + \alpha + [\alpha]_{(u,v) \in E})(\gamma_x + \gamma_y) + [\alpha \gamma_x]_{\deg(x) > 1} + [\alpha \gamma_y]_{\deg(y) > 1}, \quad (47) $$

where $\deg(x)$ is the degree of $x$. Thus for this deviation to be strictly improving, it must be that (47) > (46).

If $\deg(x) = 1$, then $$(1 + \alpha + [\alpha]_{(u,v) \in E})(\gamma_x) + [\alpha \gamma_x]_{\deg(x) > 1} = (1 + \alpha + [\alpha]_{(u,v) \in E})r_{ux},$$ since in this case $\gamma_x = r_{ux}$. If $\deg(x) > 1$, then $$(1 + \alpha + [\alpha]_{(u,v) \in E})(\gamma_x) + [\alpha \gamma_x]_{\deg(x) > 1} \leq (1 + 2\alpha + [\alpha]_{(u,v) \in E})(r_{ux}/2),$$ since in this case $\gamma_x \leq r_{ux}/2$, as argued above. Thus in either case, $(1 + \alpha + [\alpha]_{(u,v) \in E})(r_{ux} + r_{vy})$, and so quantity (47) is at most

$$ (1 + \alpha + [\alpha]_{(u,v) \in E})(r_{ux} + r_{vy}) \quad (48) $$

As argued above, since both $x$ and $y$ are unmatched in $M$, then we know that $r_{uw} \geq r_{ux}$ and $r_{uv} \geq r_{vy}$, since otherwise $M$ would have an improving swivel. Therefore, quantity (48) is at least (48), and so (47) < (46). Therefore, this cannot be an improving deviation.

Using the above theorem, we can proceed similarly to Section 5 and show existence of pairwise equilibrium, convergence results, and the same bounds for price of stability as before.

**Theorem 15.** Consider a CCG with equal reward sharing and local friendship utilities in which all players must contribute their entire budget to incident edges. Then, the price of stability is at most \(\frac{2+2\alpha}{1+2\alpha}\), and a pairwise equilibrium that achieves this bound exists and can be found in polynomial time.

**Proof.** Recall that, by the discussion at the start of Section 5.2, the optimal solution in this game simply corresponds to the maximum-weight matching $M^*$. First, notice that $M^*$ does not contain forbidden edges. This is easy to see, since if a forbidden edge $(u,v) \in M^*$, then the nodes $x$ and $y$ of degree 1 adjacent to $u$ and $v$ must be unmatched. Removing $(u,v)$ from $M^*$, and adding $(u,x)$ and $(v,y)$, increases the weight of the matching. This is because, by definition of forbidden edge,

$$ r_{uw} < \frac{1 + 2\alpha}{2 + 2\alpha}(r_{ux} + r_{vy}), $$

and thus $r_{uw} < r_{ux} + r_{vy}$. Since $M^*$ is the maximum weight matching, this is a contradiction, and thus $M^*$ cannot contain forbidden edges. Moreover, every node adjacent to a forbidden edge must be matched in $M^*$, otherwise we could add the edge between this node and a node of degree 1 to increase the weight of $M^*$.

Now consider the same game, but with all forbidden edges removed from the graph. The maximum-weight matching does not change, and thus the optimum solution does not change. Let a matching $M$ be the matching found by BEST-RELAXED-BLOCKING-PAIR in the corresponding matching game $SM(G, \tilde{\alpha})$ (this is the game with all forbidden edges removed). Define a solution $s$ to be such that all nodes matched in $M$ put all their effort on edges of $M$, and all unmatched nodes split their effort equally among all incident edges. By Theorem 4, the weight of $M$ is at least \(\frac{1+2\alpha}{2+2\alpha}\) of the weight of $M^*$, and thus $s$ meets the desired price of stability bound. $s$ can clearly be found in poly-time, since $M$ can be found in poly-time.

To show that $s$ is a pairwise equilibrium, we will use Theorem 14 $M$ clearly does not contain forbidden edges, since all of these edges were removed before forming $M$. It is also stable with respect to deviations to non-forbidden edges, since BEST-RELAXED-BLOCKING-PAIR forms a stable
matching. Thus, all we need to show is that $M$ is stable with respect to deviations to forbidden edges.

To show this, consider a forbidden edge $(u, v)$, with nodes $x$ and $y$ defined as in the definition of forbidden edge. In matching $M$, $u$ must be matched to some node $w \neq v$: this is because it cannot be matched to $v$ (we removed edge $(u, v)$ when running our algorithm to form the matching), and it cannot be unmatched since nodes $u$ and $x$ would then form a blocking pair in $M$. Moreover, $r_{uw} \geq r_{ux}$, since otherwise $u$ would have an improving swivel to node $x$ in matching $M$. The same holds for node $v$: it must be matched to some node $z$ such that $r_{vz} \geq r_{vy}$. By definition of forbidden edge, this implies that $(u, v)$ does not form a blocking pair, even with edge $(u, v)$ present.

We have now shown that $M$ is a stable matching that does not contain forbidden edges, and thus $s$ is a pairwise equilibrium, as desired.

Finally, since a pairwise equilibrium when nodes spend their entire budget is also a pairwise equilibrium for the CCG without tight budget constraints, then we obtain the following corollary of Theorem 13.

**Corollary 8.** The price of anarchy in CCGs in which all players must contribute their entire budget to incident edges is bounded by $\text{PoA} \leq 1 + Q$.

References

[1] David Abraham, Ariel Levavi, David Manlove, and Gregg O’Malley.: The stable roommates problem with globally ranked pairs. *Internet Math.*, 5(4):493–515, 2008.

[2] Heiner Ackermann, Paul Goldberg, Vahab Mirrokni, Heiko Röglin, and Berthold Vöcking.: Uncoordinated two-sided matching markets. *SIAM J. Comput.*, 40(1):92–106, 2011.

[3] Elliot Anshelevich and Sanmay Das.: Matching, cardinal utility, and social welfare. *SIGecom Exchanges*, 9(1):4, 2010.

[4] Elliot Anshelevich, Sanmay Das, and Yonatan Naamad.: Anarchy, stability, and utopia: Creating better matchings. In *Proc. 2nd Intl. Symp. Algorithmic Game Theory (SAGT)*, pages 159–170, 2009.

[5] Elliot Anshelevich and Martin Hoefer.: Contribution games in networks. *Algorithmica*, Volume 63, 1-2 (2012), pp. 51–90. Conference version appeared in *Proc. 18th European Symposium on Algorithms (ESA)*, volume 1, pages 158–169, 2010.

[6] Esteban Arcaute and Sergei Vassilvitskii.: Social networks and stable matchings in the job market. In *Proc. 5th Intl. Workshop Internet & Network Economics (WINE)*, pages 220–231, 2009.

[7] Itai Ashlagi, Piotr Krysta, and Moshe Tennenholtz.: Social context games. In *Proc. 4th Intl. Workshop Internet & Network Economics (WINE)*, pages 675–683, 2008.

[8] John Augustine, Ning Chen, Edith Elkind, Angelo Fanelli, Nick Gravin, and Dmitry Shiryaev. Dynamics of Profit-Sharing Games. In *Proc. of IJCAI 2011*.

[9] Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou.: Who’s who in networks. Wanted: The key player. *Econometrica*, 74(5):1403–1417, 2006.
10. Yann Bramoullé and Rachel Kranton.: Public goods in networks. *J. Econ. Theory*, 135(1):478–494, 2007.

11. Russell Buehler, Zachary Goldman, David Liben-Nowell, Yuechao Pei, Jamie Quadri, Alexa Sharp, Sam Taggart, Tom Wexler, and Kevin Woods.: The price of civil society. In *Proc. 7th Intl. Workshop Internet & Network Economics (WINE)*, 2011.

12. Po-An Chen, Bart De Keijzer, David Kempe, and Guido Schaefer.: On the robust price of anarchy of altruistic games. In *Proc. 7th Intl. Workshop Internet & Network Economics (WINE)*, 2011.

13. Po-An Chen and David Kempe. Altruism, selfishness, and spite in traffic routing.: In *Proc. 9th Conf. Electronic Commerce (EC)*, pages 140–149, 2008.

14. Kim-Sau Chung. On the existence of stable roommate matchings.: *Games Econom. Behav.*, 33(2):206–230, 2000.

15. Ilan Eshel, Larry Samuelson, and Avner Shaked.: Altruists, egoists and hooligans in a local interaction model. *Amer. Econ. Rev.*, 88(1):157–179, 1998.

16. David Gale and Lloyd Shapley.: College admissions and the stability of marriage. *Amer. Math. Monthly*, 69(1):9–15, 1962.

17. Andrea Galeotti, Sanjeev Goyal, Matthew Jackson, Fernando Vega-Redondo, and Leeat Yariv.: Network games. *Rev. Econom. Studies*, 77(1):218–244, 2010.

18. Michel Goemans, Li Li, Vahab Mirrokni, and Marina Thottan.: Market sharing games applied to content distribution in ad-hoc networks. *IEEE J. Sel. Area Comm.*, 24(5):1020–1033, 2006.

19. Dan Gusfield and Robert Irving.: *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, 1989.

20. Martin Hoefer.: Local matching dynamics in social networks. In *Proc. 38th Intl. Coll. Automata, Languages and Programming (ICALP)*, volume 2, pages 113–124, 2011.

21. Martin Hoefer, Michal Penn, Maria Polukarov, Alexander Skopalik, and Berthold Vöcking.: Considerate equilibrium. In *Proc. 22nd Intl. Joint Conf. Artif. Intell. (IJCAI)*, pages 234–239, 2011.

22. Martin Hoefer and Alexander Skopalik.: Altruism in atomic congestion games. In *Proc. 17th European Symposium on Algorithms (ESA)*, pages 179–189, 2009.

23. Martin Hoefer and Alexander Skopalik.: Stability and convergence in selfish scheduling with altruistic agents. In *Proc. 5th Intl. Workshop Internet & Network Economics (WINE)*, pages 616–622, 2009.

24. Robert Irving.: An efficient algorithm for the "stable roommates" problem. *J. Algorithms*, 6(4):577–595, 1985.

25. Matthew Jackson.: *Social and Economic Networks*. Princeton University Press, 2008.

26. Yashodhan Kanoria, Mohsen Bayati, Christian Borgs, Jennifer Chayes, and Andrea Montanari.: Fast convergence of natural bargaining dynamics in exchange networks. In *Proc. 22nd Symp. Discrete Algorithms (SODA)*, pages 1518–1537, 2011.
[27] Jon Kleinberg and Éva Tardos.: Balanced outcomes in social exchange networks. In Proc. 40th Symp. Theory of Computing (STOC), pages 295–304, 2008.

[28] Jon M. Kleinberg and Sigal Oren.: Mechanisms for (mis)allocating scientific credit. In Proc. 43rd Symp. Theory of Computing (STOC), pages 529–538, 2011.

[29] John Ledyard.: Public goods: A survey of experimental research. In John Kagel and Alvin Roth, editors, Handbook of Experimental Economics, pages 111–194. Princeton University Press, 1997.

[30] David Levine.: Modeling altruism and spitefulness in experiments. Rev. Econom. Dynamics, 1:593–622, 1998.

[31] Fabien Mathieu.: Self-stabilization in preference-based systems. Peer-to-Peer Netw. Appl., 1(2):104–121, 2008.

[32] Dominic Meier, Yvonne Anne Oswald, Stefan Schmid, and Roger Wattenhofer.: On the windfall of friendship: Inoculation strategies on social networks. In Proc. 9th Conf. Electronic Commerce (EC), pages 294–301, 2008.

[33] Bezalel Peleg and Peter Sudhölter.: Introduction to the Theory of Cooperative Games. Kluwer Academic Publishers, 2003.

[34] Alvin Roth and Marilda Oliveira Sotomayor.: Two-sided Matching: A study in game-theoretic modeling and analysis. Cambridge University Press, 1990.

[35] Alvin Roth and John H. Vande Vate.: Random paths to stability in two-sided matching. Econometrica, 58(6):1475–1480, 1990.

[36] Éva Tardos and Tom Wexler.: Network formation games. In Noam Nisan, Éva Tardos, Tim Roughgarden, and Vijay Vazirani, editors, Algorithmic Game Theory, chapter 19. Cambridge University Press, 2007.

[37] Chung-Piaw Teo and Jay Sethuraman.: The geometry of fractional stable matchings and its applications. Math. Oper. Res., 23(3):874–891, 1998.

[38] Yair Zick, Georgios Chalkiadakis, and Edith Elkind. Overlapping Coalition Formation Games: Charting the Tractability Frontier. AAMAS 2012.