Abstract. Modeling communication channels as thermal systems results in Hamiltonians which are an explicit function of the temperature. The first two authors have recently generalized the second thermodynamic law to encompass systems with temperature-dependent energy levels, $dQ = TdS + \langle dE/dT \rangle dT$, where $\langle \cdot \rangle$ denotes averaging over the Boltzmann distribution, recomputing the mutual information and other main properties of the popular Gaussian channel. Here the mutual information for the binary symmetric channel as well as for the discrete symmetric channel consisting of 4 input/output (I/O) symbols is explicitly calculated using the generalized second law of thermodynamics. For equiprobable I/O the mutual information of the examined channels has a very simple form, $-\gamma U(\gamma)\frac{\partial}{\partial \beta} \beta$, where $U$ denotes the internal energy of the channel. We prove that this simple form of the mutual information governs the class of discrete memoryless symmetric communication channels with equiprobable I/O symbols.
1. Introduction

The current scientific conception is that the theory of information is a creature of mathematics and has its own vitality independent of the physical laws of nature [1]. The first two authors have recently proved [2, 3] that the principal quantity in the theory of information, the mutual information, can be reformulated as a consequence of the fundamental laws of nature - the laws of thermodynamics. This corollary was originally exemplified for the Gaussian noisy channel.

The generic problem in information processing is the transmission of information over a noisy channel [4, 5, 6]. This central paradigm of information theory can be mathematically abstracted to having two random variables $X$ and $Y$ representing the desired information and its noisy replica, respectively. Noisy transmission can occur either via space from one geographical point to another, as happens in communications, or in time, for example, when sequentially writing and reading files from a hard disk in the computer.

Mutual information, $I(X;Y)$, quantifies the amount of information in common between two random variables and it is used to upper bound the attainable rate of information transferred across a channel. To put differently, mutual information measures the amount of information that can be obtained about one random variable (channel input $X$) by observing another (output $Y$). A basic property of the mutual information is that $I(X;Y) = H(X) - H(X|Y)$, where $H(\cdot)$ is the information (Shannon) entropy [1]. It measures the amount of uncertainty in a random variable, indicating how easily data can be losslessly compressed. Hence knowing $Y$, we can save an average of $I(X;Y)$ bits in encoding $X$ compared to not knowing $Y$.

As recently shown [2, 3], the modeling of the Gaussian channel as a thermal system requires the generalization of thermodynamics to include temperature-dependent Hamiltonians, as well as the redefinition of the notion of temperature. The generalized second thermodynamic law was proven to have the following form

$$dS = \frac{dQ}{T} - \frac{1}{T} \langle \frac{d\mathcal{E}(X)}{dT} \rangle dT$$

where $\langle \cdot \rangle$ denotes averaging over the Boltzmann distribution.

The generalized second law of thermodynamics (11) has a clear physical interpretation. For simplicity, let us assume that an examined system is characterized by a comb of discrete energy levels $\mathcal{E}_1, \mathcal{E}_2, \ldots$, see Figure 1a. The heat absorbed into the $T$-dependent system has the following dual effect: A first contribution of the heat, $dU - \langle d\mathcal{E}(X)/dT \rangle dT$, increases the temperature of the system, Figure 1b, while the second contribution, $\langle d\mathcal{E}(X)/dT \rangle dT$, goes for shifting the energy comb, Figure 1c. However, the shift of the energy comb does not affect the entropy, since the occupation of each energy level remains the same, and the entropy is independent of the energy values which stand behind the labels $\mathcal{E}_1, \mathcal{E}_2, \ldots$. The change in the entropy can be done only by moving part of the occupation of one tooth of the energy comb to the neighboring teeth, Figure 1b. Hence, the effective heat contributing to the entropy is
Figure 1. A system consisting of a discrete energy comb (levels), $\epsilon_m$, with the corresponding degeneracy $\Omega(\epsilon_m)$ which increases with the energy. For the simplicity of presentation we assume that the system is occupying only one tooth of the comb, depicted by a red tooth in Figure 1a. As heat is absorbed into the system, the system can either increase the temperature by jumping to the next tooth of the comb, Figure 1b, or shift the energy comb, Figure 1c. Note that only the jump to the next tooth, Figure 1b, changes the entropy of the system, where in Figure 1c the entropy remains the same as in Figure 1a.

\[
dQ - < dE(X)/dT > dT, \]
and this is the physical explanation to the generalized second law [11]. A schematic picture of communication-heat-engine is depicted in Figure 2, where the heat is devoted to the change of the Hamiltonian (without altering the thermodynamic entropy) is denoted by the term ‘working channel’. Note that for T-independent Hamiltonians the traditional picture of heat engine is recovered as well as the traditional second thermodynamic law [7, 8, 9].

Using the generalized second law of thermodynamics, (1), one can show [2, 3] that the generalized expression for the mutual information is given by

\[
I = -\gamma U(\gamma)|_0^\beta + \left<\int_0^\beta \left(U(\gamma, y) + \gamma < \frac{dE}{d\gamma} >_x|y\right) d\gamma\right>_y.
\]

Note that this thermodynamic expression for the mutual information holds for any channel which can be described by a thermal system exhibiting quasi-static heat transfer. For the Gaussian channel with a standard Gaussian input and signal-to-noise ratio, $\text{snr}$,
one can show \[2, 3\] that the celebrated formula for the Shannon capacity \[1\], is obtained from (2), \(I(X; Y) = \frac{1}{2} \log (1 + \beta)\), where \(\beta = \text{snr}\).

2. Binary Symmetric Channel

We turn now to derive the mutual information for the archetypal discrete memoryless channel, the binary symmetric channel (BSC), with input \(x\) and output \(y\). The input’s prior distribution obeys \(P(x = 1) = P(x = -1) = 1/2\) and the probability for a symbol to flip during the transmission is denoted by \(\delta\), \(P(y = \pm | x = \mp) = \delta\), see Figure 3. Hence, the conditional probability of the output given the input is

\[
P(y|x) = \delta \frac{x-y}{2} (1-\delta)^{\frac{x+y}{2}} = \exp \left[ \frac{xy}{2} \ln \left( \frac{1-\delta}{\delta} \right) + \frac{1}{2} \ln(\delta(1-\delta)) \right].
\]

Equation (3)

A comparison of the channel’s a-posteriori probability distribution, given by Bayes’ law

\[
P(X = x|Y = y) \propto P(Y = y|X = x)
\]

Equation (4)

with the Boltzmann distribution law yields the following mapping of the energy and the inverse temperature, \(\beta\), of the equivalent thermal system

\[
E = -\frac{xy}{2} - \frac{1}{2\beta} \ln(\delta(1-\delta)),
\]

\[
\beta = \ln \frac{1-\delta}{\delta}.
\]

Equation (5)

Note that since the inverse temperature, \(\beta(\delta)\), is positive, then \(\delta = 1/(1 + \exp(\beta)) < 1/2\) \[10, 11\]. This definition of \(\delta\) coincides with two limiting cases. For \(\delta = 1/2\),
\[ \beta = 0 \ (T \to \infty) \], where for \( \delta = 0, \ \beta \to \infty \ (T = 0) \). Using \( \langle x|y \rangle = y(1 - 2\delta) \), one can find that the internal energy is given by

\[ U(\beta) = -\frac{1 - 2\delta}{2} - \frac{1}{2} + \ln(1 + \exp(\beta)). \tag{6} \]

Note that for the BSC specifically the internal energy is independent of \( y \) due to the binary nature of the I/O symbols. Similarly to the internal energy one can find

\[ \langle \beta \frac{dE}{d\beta} \rangle_{x|y} = -\frac{\ln(1 + \exp(\beta))}{\beta} + \frac{\exp(\beta)}{1 + \exp(\beta)}. \tag{7} \]

It is easy now to verify that the second term of the generalized mutual information, (2), vanishes

\[ U(\beta) + \langle \beta \frac{dE}{d\beta} \rangle_{x|y} = -\frac{\ln(1 + \exp(\beta))}{\beta} + \frac{\exp(\beta)}{1 + \exp(\beta)} = 0, \tag{8} \]

and the mutual information for the equiprobable input-output (I/O) BSC (which is also the Shannon capacity of the BSC in general) has a very simple form given explicitly by

\[ I = -\gamma U(\gamma)|_{\beta}^0 = \beta(1 - \delta) - \ln(1 + \exp(\beta)) + 1 = (1 - \delta) \ln(1 - \delta) + \delta \ln(\delta) + 1 = 1 - H(\delta). \tag{9} \]

Hence, the mutual information for the BSC is recovered using the generalized second thermodynamic law. Note that in contrary to the Gaussian channel, in the BSC case the second term of the mutual information, (2), vanishes, and the mutual information is proportional to the internal energy

\[ I = -\gamma U(\gamma)|_{\beta}^0. \tag{10} \]

### 3. 4-I/O Symbols

Is this simple thermodynamic form, (10), a coincidence of the BSC only or it occurs for the general class of discrete memoryless channels? To find the answer to this interesting question, we first turn to discuss in detail the case of 4-I/O symbols. The input is represented by two binary units \( x_1 \) and \( x_2 \), and similarly the output units are \( y_1 \) and \( y_2 \). We assume that the 4-input symbols are equiprobable. The conditional probability of \( P(y_1, y_2|x_1, x_2) \) obeys the following symmetry. The probability that both output units are equal to the input units, respectively, is \( 1 - \delta \), the probability that only one unit is equal is \( \epsilon \delta \ (\epsilon \leq 1) \), and the probability that the two output units differ from the input is \( \delta - 2\epsilon \delta \), Figure 4. Hence,

\[ P(y|x) = (1 - \delta)^{1 + \frac{x_1y_1}{2}} \frac{1 - \frac{x_1y_1}{2}}{2} (\epsilon \delta)^{1 + \frac{x_2y_2}{2}} \frac{1 - \frac{x_2y_2}{2}}{2} (\delta - 2\epsilon \delta)^{1 + \frac{x_1y_1}{2}} \frac{1 - \frac{x_1y_1}{2}}{2}, \tag{11} \]

and it is easy now to verify that

\[ P(X = x|Y = y) \propto \exp \left\{ \frac{x_1y_1 + x_2y_2}{4} \ln\left( \frac{1 - \delta}{\delta(1 - 2\epsilon)} \right) + \frac{1}{4} \ln((1 - \delta)\delta^3 \epsilon^2 (1 - \epsilon)) + \frac{x_1x_2y_1y_2}{4} \ln\left[ \frac{1 - \delta (1 - 2\epsilon)}{\delta \epsilon^2} \right] \right\}. \]

Similarly to (5) one can find in this case
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$E = \frac{1}{4} (x_1 y_1 + x_2 y_2 + x_1 x_2 y_1 y_2) - \frac{1}{4} \ln((1 - \delta)\delta^3) + \frac{1}{4\beta} f(x_1 y_1, x_2 y_2, \epsilon), \quad (12)$

where

$f \triangleq (x_1 y_1 + x_2 y_2) \ln(1 - 2\epsilon) - x_1 x_2 y_1 y_2 \ln(\frac{1 - 2\epsilon}{\epsilon^2}) - \ln(\epsilon^2 (1 - \epsilon)), \quad (13)$

\[\beta = \ln(\frac{1 - \delta}{\delta}), \quad (14)\]

and the two limiting cases are $\delta = 3/4$ and $\epsilon = 1/3$ ($\beta = 0$) and $\delta = 0$ ($\beta \to \infty$). Using the following conditional expectations

$< x_m | y_m > = y_m (1 - 2\delta + 2\epsilon\delta) = y_m (1 - 2\delta (1 - \epsilon)),$

$< x_1 x_2 | y_1 y_2 > = y_1 y_2 (1 - 4\epsilon\delta), \quad (15)$

one can find that the internal energy

$U(\beta) = -\frac{3}{4} + \delta - \frac{\ln((1 - \delta)\beta)}{4\beta} + \frac{\delta}{\beta} [-2\epsilon \ln(\epsilon) + \ln(1 - 2\epsilon)] \quad (16)$

and again verify that

$U(\beta) + \beta < \frac{dE}{d\beta} >_{x|y} = 0. \quad (17)$

Hence the mutual information has again the simple form, (10), and is given explicitly by

$I = 2 + (1 - \delta) \ln(1 - \delta) + 2\epsilon\delta \ln(\epsilon\delta) + \delta (1 - 2\epsilon) \ln((1 - 2\epsilon)\delta), \quad (18)$

which can be verified (by direct computation of the mutual information out of its definition) to have the correct form. The mutual information as a function of $(\delta, \epsilon)$ is depicted in Figure 5, where the mutual information $I = 0$ for $\delta = 3/4$ and $\epsilon = 1/3$. 

![Figure 4](image_url)

Figure 4. The input/output transition probabilities for a channel of 4-I/O symbols.
4. Equiprobable Discrete Memoryless Symmetric Channels

For the general case of discrete memoryless symmetric channels with $2^n$ equiprobable input symbols, the conditional probabilities has the following form

$$P(y_1, ..., y_n|x_1, ..., x_n) = \delta \epsilon(x_1 y_1, ..., x_n y_n), \quad \text{(19)}$$

and one can verify that similarly to (5) and (12) the energy is given now by

$$E = -\frac{1}{2^n} [-1 + \Pi_{k=1}^{n}(1 + x_k y_k)] - \frac{1}{2^n} \ln((1 - \delta)\delta^{2n-1})$$

$$+ \frac{1}{\beta} f_n(\{x_i y_i\}, \{\epsilon(x_1 y_1, ..., x_n y_n)\}), \quad \text{(20)}$$

where $f_n$ is a function of the number of symbols only. Now it is clear that

$$2^n(U(\beta) + < \beta \frac{dE}{d\beta} >_{x|y}) = - < -1 + \Pi_{k=1}^{n}(1 + x_k y_k) > - \frac{d\ln((1 - \delta)\delta^{2n-1})}{d\delta} \frac{d\delta}{d\beta},$$

where as in (5) and (12) $\beta = \ln((1 - \delta)/\delta)$ and $d\delta/d\beta = -\delta(1 - \delta)$. Hence

$$2^n(U(\beta) + < \beta \frac{dE}{d\beta} >_{x|y}) = - < -1 + \Pi_{k=1}^{n}(1 + x_k y_k) > + 2^n - 1 - 2^n \delta =$$

$$- < \Pi_{k=1}^{n}(1 + x_k y_k) > + 2^n (1 - \delta) = 0. \quad \text{(21)}$$

The identity to 0 is a result of $< \Pi_{k=1}^{n}(1 + x_k y_k) >$ which is equal to the trace of the conditional probability, (19). Hence we proved that for a general memoryless symmetric channel consisting of $2^n$ equiprobable I/O symbols

$$I = -\gamma U(\gamma)|_{\beta}^{\beta},$$

$$U(\beta) + < \beta \frac{dE}{d\beta} >_{x|y} = 0. \quad \text{(22)}$$

Note that the identity $I = -\gamma U(\gamma)|_{\beta}^{\beta}$ is in contrast to the case of a Gaussian channel with Bernoulli-1/2 or Gaussian inputs [2, 3].

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**Figure 5.** The mutual information as a function of $\delta$ and $\epsilon$ for the discussed case of 4-I/O symbols.
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