New Dirichlet Mean Identities

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Abstract: An important line of research is the investigation of the laws of random variables known as Dirichlet means as discussed in Cifarelli and Regazzini (7). However there is not much information on inter-relationships between different Dirichlet means. Here we introduce two distributional operations, which consist of multiplying a mean functional by an independent beta random variable and an operation involving an exponential change of measure. These operations identify relationships between different means and their densities. This allows one to use the often considerable analytic work to obtain results for one Dirichlet mean to obtain results for an entire family of otherwise seemingly unrelated Dirichlet means. Additionally, it allows one to obtain explicit densities for the related class of random variables that have generalized gamma convolution distributions, and the finite-dimensional distribution of their associated Lévy processes. This has implications in, for instance, the explicit description of Bayesian nonparametric prior and posterior models, and more generally in a variety of applications in probability and statistics involving Lévy processes. We demonstrate how the technique applies to several interesting examples.

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1. Introduction

In this work we present two distributional operations which identify relationships between seemingly different classes of random variables which are representable as linear functionals of a Dirichlet process, otherwise known as Dirichlet means. Specifically the first operation consists of multiplication of a Dirichlet mean by an independent beta random variable and the second operation involves an exponential change of measure to the density of a related infinitely divisible random variable having a generalized gamma convolution distribution (GGC). This latter operation is often referred to in the statistical literature as exponential tilting or in mathematical finance as an Esscher transform. We believe our results add a significant component to the foundational work of Cifarelli and Regazzini (6,7). In particular, our results allow one to use the often considerable analytic work to obtain results for one Dirichlet mean to obtain results for an entire family of otherwise seemingly unrelated mean functionals. It also allows one to obtain explicit densities for the related class of infinitely divisible random variables.

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variables which are generalized gamma convolutions, and the finite-dimensional distribution of their associated Lévy processes, (see Bertoin [1] for the formalities of general Lévy processes). The importance of this latter statement is that Lévy processes now commonly appear in variety of applications in probability and statistics. A detailed summary and outline of our results may be found in section 1.2. Some background information, and notation, on Dirichlet processes and Dirichlet means, their connection with GGC random variables, recent references and some motivation for our work is given in the next section.

1.1. Background and motivation

Let $X$ be a non-negative random variable with cumulative distribution function $F_X$. Note furthermore for a measurable set $C$, we use the notation $F_X(C)$ to mean the probability that $X$ is in $C$.

One may define a Dirichlet process random probability measure, see (16) and (14; 15), say $P_\theta$, on $[0, \infty)$ with total mass parameter $\theta$ and prior parameter $F_X$, via its finite dimensional distribution as follows; for any disjoint partition on $[0, \infty)$, say $(C_1, \ldots, C_k)$, the distribution of the random vector $(P_\theta(C_1), \ldots, P_\theta(C_k))$ is a $k$-variate Dirichlet distribution with parameters $(\theta F_X(C_1), \ldots, \theta F_X(C_k))$. Hence for each $C$,

$$P_\theta(C) = \int_0^\infty \mathbb{1}(x \in C) P_\theta(dx)$$

has a beta distribution with parameters $(\theta F_X(C), \theta(1 - F_X(C)))$. Equivalently setting $\theta F_X(C_i) = \theta_i$ for $i = 1, \ldots, k$,

$$(P_\theta(C_1), \ldots, P_\theta(C_k)) \overset{d}{=} \left( \frac{G_{\theta_i}}{G_{\theta}}; i = 1, \ldots, k \right)$$

where $(G_{\theta_i})$ are independent random variables with gamma$(\theta_i, 1)$ distributions and $G_{\theta} = G_{\theta_1} + \cdots + G_{\theta_k}$ has a gamma$(\theta, 1)$ distribution. This means that one can define the Dirichlet process via the normalization of an independent increment gamma process on $[0, \infty)$, say $\gamma_\theta$, as

$$P_\theta(\cdot) = \frac{\gamma_\theta(\cdot)}{\gamma_\theta([0, \infty])}$$

where $\gamma_\theta(C_i) \overset{d}{=} G_{\theta_i}$, and whose almost surely finite total random mass is $\gamma_\theta([0, \infty)) \overset{d}{=} G_{\theta}$. A very important aspect of this construction is the fact that $G_\theta$ is independent of $P_\theta$, and hence any functional of $P_\theta$. This is a natural generalization of Lukacs [33] characterization of beta and gamma random variables, whose work is fundamental to what is now referred to as the beta-gamma algebra, (for more on this, see Chaumont and Yor (5), section 4.2)). See also Emery and Yor [11] for some interesting relationships between gamma processes, Dirichlet processes and Brownian bridges.

These simple representations and other nice features of the Dirichlet process have, since the important work of Ferguson [14; 15], contributed greatly to the
relevance and practical utility of the field of Bayesian non and semi-parametric statistics. Naturally, owing to the ubiquity of the gamma and beta random variables, the Dirichlet process also arises in other areas. One of the more interesting, and we believe quite important, topics related to the Dirichlet process is the study of the laws of random variables called Dirichlet mean functionals, or simply Dirichlet means, which we denote as

\[ M_\theta(F_X) \overset{d}{=} \int_0^\infty xP_\theta(dx), \]

initiated in the works of Cifarelli and Regazzini \(^6\),\(^7\). In \(^7\) the authors obtained an important identity for the Cauchy-Stieltjes transform of order \(\theta\). This identity is often referred to as the Markov-Krein identity as can be seen in for example, Diaconis and Kemperman \(^{10}\), Kerov \(^{26}\) and Vershik, Yor and Tsilevich \(^{37}\), where these authors highlight its importance to, for instance, the study of the Markov moment problem, continued fraction theory and exponential representation of analytic functions. This identity is later called the Cifarelli-Regazzini identity in \(^{19}\). Cifarelli and Regazzini \(^7\), owing to their primary interest, used this identity to then obtain explicit density and cdf formulae for \(M_\theta(F_X)\). The density formulae may be seen as Abel type transforms and hence do not always have simple forms, although we stress that they are still useful for some analytic calculations. The general exception is the case of \(\theta = 1\) which has a nice form. Some examples of works that have proceeded along these lines are Cifarelli and Melilli \(^8\), Regazzini, Guglielmi and di Nunno \(^{35}\), Regazzini, Lijoi and Prünster \(^{30}\), Hjort and Ongaro \(^{18}\), Lijoi and Regazzini \(^{30}\), and Epifani, Guglielmi and Melilli \(^{12; 13}\)). Moreover, the recent work of Bertoin, Fujita, Roynette and Yor \(^2\) and James, Lijoi and Prünster \(^{23}\) (see also \(^{21}\) which is a preliminary version of this work) show that the study of mean functionals is relevant to the analysis of phenomena related to Bessel and Brownian processes. In fact the work of James, Lijoi and Prünster \(^{23}\) identifies many new explicit examples of Dirichlet means which have interesting interpretations.

Related to these last points, Lijoi and Regazzini \(^{30}\) have highlighted a close connection to the theory of generalized gamma convolutions (GGC) distributions may be represented as

\[ T_\theta \overset{d}{=} G_\theta M_\theta(F_X) \overset{d}{=} \int_0^\infty x\gamma_\theta(dx). \]  

We call these random variables \(\text{GGC}(\theta, F_X)\). In addition we see from \(^{1.1}\) that \(T_\theta\) is a random variable derived from a weighted gamma process, and hence the calculus discussed in Lo \(^{31}\) and Lo and Weng \(^{32}\) applies. In general GGC random variables are an important class of infinitely divisible random variables whose properties have been extensively studied by \(^3\) and others. We note further that although we have written a \(\text{GGC}(\theta, F_X)\) random variable as \(G_\theta M_\theta(F_X)\) this representation is not unique and in fact it is quite rare to see
$T_\theta$ represented in this way. We will show that one can in fact exploit this non-uniqueness to obtain explicit densities for $T_\theta$ even when it is not so easy to do so for $M_\theta(F_X)$. While the representation $G_\theta M_\theta(F_X)$ is not unique it helps one to understand the relationship between the Laplace transform of $T_\theta$ and the Cauchy-Stieltjes transform of order $\theta$ of $M_\theta(F_X)$, which indeed characterizes respectively the law of $T_\theta$ and $M_\theta(F_X)$.

Specifically, using the independence property of $G_\theta$ and $M_\theta(F_X)$, leads to, for $\lambda \geq 0$,

$$E[e^{-\lambda T_\theta}] = E[(1 + \lambda M_\theta(F_X))^{-\theta}] = e^{-\theta \psi_{F_X}(\lambda)}$$  \quad (1.2)

where

$$\psi_{F_X}(\lambda) = \int_0^\infty \log(1 + \lambda x)F_X(dx) = E[\log(1 + \lambda X)].$$  \quad (1.3)

is the Lévy exponent of $T_\theta$. We note that $T_\theta$ and $M_\theta(F_X)$ exist if and only if $\psi_{F_X}(\lambda) < \infty$ for $\lambda > 0$, (see for instance [3] and [4]). The expressions in (1.2) equates with the the identity obtained by Cifarelli and Regazzini [5], mentioned previously.

Despite these interesting results, there is very little work on the relationship between different mean functionals. Suppose, for instance, that for each fixed value of $\theta > 0$, $M_\theta(F_X)$ denotes a Dirichlet mean and $(M_\theta(F_{Z_c}); c > 0)$ denotes a collection of Dirichlet mean random variables indexed by a family of distributions $(F_{Z_c}; c > 0)$. Then one can ask the question, for what choices of $X$ and $Z_c$ are these mean functionals related, and in what sense? In particular, one may wish to know how their densities are related. The rationale here is that if such a relationship is established, then the effort that one puts forth to obtain results such as the explicit density of $M_\theta(F_X)$, can be applied to an entire family of Dirichlet means $(M_\theta(F_{Z_c}); c > 0)$. Furthermore since Dirichlet means are associated with GGC random variables this would establish relationships between a GGC($\theta, F_X$) random variable and a family of GGC($\theta, F_{Z_c}$) random variable. Simple examples are of course the choices $Z_c = X + c$ and $Z_c = cX$, which, due to the linearity properties of mean functionals, results easily in the identities in law

$$M_\theta(F_{X+c}) = c + M_\theta(F_X) \text{ and } M_\theta(F_{cX}) = cM_\theta(F_X)$$

Naturally, we are going to discuss more complex relationships, but with the same goal. That is, we will identify non-trivial relationships so that the often considerable efforts that one makes in the study of one mean functional $M_\theta(F_X)$ can be then used to obtain more easily results for other mean functionals, their corresponding GGC random variables and Lévy processes. In this paper we will describe two such operations which we elaborate on in the next subsection.

1.2. Outline and summary of results

Section 1.3 reviews some of the existing formulae for the density and cdf of Dirichlet means. In Section 2 we will describe the operation of multiplying a
mean functional $M_{\theta,\sigma}(F_X)$ by an independent beta random variable with parameters $(\theta \sigma, \theta(1 - \sigma))$, say, $\beta_{\theta \sigma, \theta(1 - \sigma)}$ where $0 < \sigma < 1$. We call this operation beta scaling. Theorem 2.1 shows that the resulting random variable

$$\beta_{\theta \sigma, \theta(1 - \sigma)} M_{\theta \sigma}(F_X)$$

is again a mean functional but now of order $\theta$. In addition, the GGC($\theta \sigma, F_X$) random variable $G_{\theta \sigma} M_{\theta \sigma}(F_X)$ is equivalently a GGC random variable of order $\theta$. Now keeping in mind that tractable densities of mean functionals of order $\theta = 1$ are the easiest to obtain, Theorem 2.1 shows that by setting $\theta = 1$, the densities of the uncountable collection of random variables $(\beta_{\sigma,1-M_{\sigma}}(F_X); 0 < \sigma \leq 1)$, are all mean functionals of order $\theta = 1$. Theorem 2.2 then shows that efforts used to calculate the explicit density of any one of these random variables, via the formulae of (7), lead to the explicit calculation of the densities of all of them. Additionally, Theorem 2.2 shows that the corresponding GGC random variables may all be expressed as GGC random variables of order $\theta = 1$, representable in distribution as $G_1 \beta_{\sigma,1-M_{\sigma}}(F_X)$. A key point here is that Theorem 2.2 gives a tractable density for $\beta_{\sigma,1-M_{\sigma}}(F_X)$ without requiring knowledge of the density of $M_{\sigma}(F_X)$, which is usually expressed in a complicated manner. These results also will yield some non-obvious integral identities. Furthermore, noting that a GGC($\theta, F_X$) random variable, $T_\theta$, is infinitely divisible, we associate it with an independent increment process $(\zeta_\theta(t); t \geq 0)$ known as a subordinator, (a non-decreasing non-negative Lévy process), where for each fixed $t$,

$$E[e^{-\lambda \zeta_\theta(t)}] = E[e^{-\lambda T_{\theta t}}] = e^{-t \theta \psi_{F_X}(\lambda)}.$$ 

That is, marginally $\zeta_\theta(1) \overset{d}{=} T_\theta$ and $\zeta_\theta(t) \overset{d}{=} \zeta_{\theta t}(1) \overset{d}{=} T_{\theta t}$. In addition, for $s < t$, $\zeta_\theta(t) - \zeta_\theta(s) \overset{d}{=} \zeta_{\theta t} - \zeta_{\theta s}$, is independent of $\zeta_\theta(s)$. We say that the process $(\zeta_\theta(t); t \geq 0)$ is a GGC($\theta, F_X$) subordinator. Proposition 2.1 shows how Theorems 2.1 and 2.2 can be used to address the usually difficult problem of describing explicitly the densities of the finite-dimensional distribution of a subordinator (see (21)). This has implications in, for instance, the explicit description of densities of Bayesian nonparametric prior and posterior models.

But clearly is of wider interest in terms of the distribution theory of infinitely divisible random variables and associated processes.

In Section 3, we describe how the operation of exponentially titling the density of a GGC($\theta, F_X$) random variable leads to a relationship between the densities of the mean functional $M_{\theta}(F_X)$ and yet another family of mean functionals. This is summarized in Theorem 3.1. Section 3.1 then discusses a combination of the two operations. Proposition 3.1 describes the density of beta scaled and tilted mean functionals of order 1. Using this, Proposition 3.2 describes a method to calculate a key quantity in the explicit description of the density and cdf of mean functionals. In Section 3, we demonstrate how our results can be applied to extend and explain results related to two well known cases of Dirichlet mean functionals. However, we emphasize that Proposition 3.1, 3.2 and 3.3 are genuinely new results to the literature. More complex applications, which may be viewed as extensions of section 4.2, may be found in an unpublished preliminary version of this work in (21). We discuss and develop these further in James (22).
5 presents a more involved result relative to those in section 4, but which does not require a great deal of background material. Here we show how the results in section 2 are used to derive the finite dimensional distribution and related quantities of a class of subordinators recently studied in (2).

1.3. Preliminaries

Suppose that \( X \) is a positive random variable with distribution \( F_X \), and define the function

\[
\Phi_{F_X}(t) = \int_0^\infty \log(|t-x|) \mathbb{I}(t \neq x) F_X(dx) = \mathbb{E}[\log(|t-X|) \mathbb{I}(t \neq X)].
\]

Furthermore, define

\[
\Delta_\theta(t|F_X) = \frac{1}{\pi} \sin(\pi \theta F_X(t)) e^{-\theta \Phi_{F_X}(t)},
\]

where using a Lebesque-Stieltjes integral, \( F_X(t) = \int_0^t F_X(dx) \). Cifarelli and Regazzini (7) (see also (8)), apply inversion formula to obtain the distributional formula for \( M_\theta(F_X) \) as follows. For all \( \theta > 0 \), the cdf can be expressed as

\[
\int_0^x (x-t)^{\theta-1} \Delta_\theta(t|F_X)dt
\]

provided that \( \theta F_X \) possesses no jumps of size greater than or equal to one. If we let \( \xi_{\theta F_X}(\cdot) \) denote the density of \( M_\theta(F_X) \), it takes its simplest form for \( \theta = 1 \), which is

\[
\xi_{F_X}(x) = \Delta_1(x|F_X) = \frac{1}{\pi} \sin(\pi F_X(x)) e^{-\Phi_X(x)}.
\]

Density formulae for \( \theta > 1 \) are described as

\[
\xi_{\theta F_X}(x) = (\theta - 1) \int_0^x (x-t)^{\theta-2} \Delta_\theta(t|F_X)dt.
\]

An expression for the density, which holds for all \( \theta > 0 \), was recently obtained by James, Lijoi and Prünster (23) as follows,

\[
\xi_{\theta F_X}(x) = \frac{1}{\pi} \int_0^x (x-t)^{\theta-1} d_\theta(t|F_X)dt
\]

where

\[
d_\theta(t|F_X) = \frac{d}{dt} \sin(\pi \theta F_X(t)) e^{-\theta \Phi_X(t)}.
\]

For additional formula, see (7; 35; 30).

Remark 1.1. Throughout for random variables \( R \) and \( X \), when we write the product \( RX \) we will assume unless otherwise mentioned that \( R \) and \( X \) are independent. This convention will also apply to the multiplication of the special random variables that are expressed as mean functionals. That is the product \( M_\theta(F_X)M_\theta(F_Z) \) is understood to be a product of independent Dirichlet means.
Remark 1.2. Throughout we will be using the fact that if $R$ is a gamma random variable, then the independent random variables $R, X, Z$ satisfying $RX \overset{d}{=} RZ$ imply that $X \overset{d}{=} Z$. This is true because gamma random variables are simplifiable. For precise meaning of this term and conditions, see Chaumont and Yor (5, sec. 1.12 and 1.13). This fact also applies to the case where $R$ is a positive stable random variable.

2. Beta Scaling

In this section we investigate the simple operation of multiplying a Dirichlet mean functional $M_\theta(F_X)$ by certain beta random variables. Note first that if $M$ denotes an arbitrary positive random variable with density $f_M$, then by elementary arguments it follows that the random variable $W \overset{d}{=} \beta_{a,b}M$, where $\beta_{a,b}$ is beta($a,b$) independent of $M$, has density expressible as

\[ f_W(w) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 f_M(w/u)u^{a-1}(-u)^{b-1}du. \]

However it is only in special cases where the density $f_W$ can be expressed in even simpler terms. That is to say, it is not obvious how to carry out the integration. In the next results we show how remarkable simplifications can be achieved when $M = M_\theta(F_X)$, in particular for the range $0 < \theta \leq 1$, and $\beta_{a,b}$ is a symmetric beta random variable. First we will need to introduce some additional notation. Let $Y_\sigma$ denote a Bernoulli random variable with success probability $0 < \sigma \leq 1$.

Then if $X$ is a random variable with distribution $F_X$, independent of $Y_\sigma$, it follows that the random variable $XY_\sigma$ has distribution denoted as

\[ F_{XY_\sigma}(dx) = \sigma F_X(dx) + (1-\sigma)\delta_0(dx), \]  

and cdf

\[ F_{XY_\sigma}(x) = \sigma F_X(x) + (1-\sigma)\mathbb{I}(x \geq 0). \]  

Hence, there exists the mean functional

\[ M_\theta(F_{XY_\sigma}) \overset{d}{=} \int_0^\infty y\tilde{P}_\theta(dy) \]

where $\tilde{P}_\theta(dy)$ denotes a Dirichlet process with parameters $(\theta, F_{XY_\sigma})$. In addition we have for $x > 0$,

\[ \Phi_{F_{XY_\sigma}}(x) = \mathbb{E}[\log(|x - XY_\sigma|)|XY_\sigma \neq x]| = \sigma \Phi_{F_X}(x) + (1-\sigma)\log(x). \]  

When $\sigma = 1$, $Y_\sigma = 1$ and hence $F_{XY_\sigma}(x) = F_X(x)$. Let $E_\sigma$ denote a set such that $\mathbb{E}[P_\theta(E_\sigma)] = \sigma$. Now notice that every beta random variable, $\beta_{a,b}$, where $a, b$ are arbitrary positive constants, can be represented as the simple mean functional,

\[ P_\theta(E_\sigma) \overset{d}{=} \beta_{\theta,\theta(1-\sigma)} \overset{d}{=} M_\theta(F_{Y_\sigma}). \]
by choosing
\[ \sigma = \frac{a}{a+b} \quad \text{and} \quad \theta = a + b. \]

We note however that there are other choices of \( F_X \) that will also yield beta random variables as mean functionals. Throughout we will use the convention that \( \beta_{0,0} := 1 \), that is the case when \( \sigma = 1 \). We now present our first result.

**Theorem 2.1.** For \( \theta > 0 \) and \( 0 < \sigma \leq 1 \), let \( \beta_{\theta\sigma,\theta(1-\sigma)} \) denote a beta random variable with parameters \((\theta\sigma,\theta(1-\sigma))\), independent of the mean functional \( M_{\theta\sigma}(F_X) \). Then

(i) \( \beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X) \overset{d}{=} M_{\theta}(F_{XY_{\alpha}}) \).

(ii) Equivalently, \( M_{\theta}(F_{Y_{\alpha}}) M_{\theta\sigma}(F_X) \overset{d}{=} M_{\theta}(F_{XY_{\alpha}}) \).

(iii) \( G_{\theta\sigma} M_{\theta\sigma}(F_X) \overset{d}{=} G_{\theta} M_{\theta}(F_{XY_{\alpha}}) \).

(iv) That is, \( GGC(\theta, \sigma) \overset{d}{=} GGC(\theta, F_{XY_{\alpha}}) \).

**Proof.** Since \( M_{\theta}(F_{Y_{\alpha}}) \overset{d}{=} \beta_{\theta\sigma,\theta(1-\sigma)} \) statements (i) and (ii) are equivalent. We proceed by first establishing (iii) and (iv). Note that using (1.2),

\[ \mathbb{E}[\log(1 + \lambda X_{\alpha})] = \mathbb{E}[\log(1 + \lambda X)] = \sigma \int_0^\infty \log(1 + \lambda x) F_X(dx). \]

Hence

\[ \mathbb{E}[e^{-\lambda G_{\theta\sigma} M_{\theta\sigma}(F_{XY_{\alpha}})}] = e^{-\theta \sigma \int_0^\infty \log(1 + \lambda x) F_X(dx)} = \mathbb{E}[e^{-\lambda G_{\theta\sigma} M_{\theta\sigma}(F_X)}], \]

which means that \( G_{\theta\sigma} M_{\theta\sigma}(F_{XY_{\alpha}}) \overset{d}{=} G_{\theta\sigma} M_{\theta\sigma}(F_X) \), establishing statements (iii) and (iv). Now writing \( G_{\theta\sigma} = G_{\theta} \beta_{\theta\sigma,\theta(1-\sigma)} \). It follows that

\[ G_{\theta} M_{\theta}(F_{XY_{\alpha}}) \overset{d}{=} G_{\theta} \beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X). \]

Hence \( \beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X) \overset{d}{=} M_{\theta}(F_{XY_{\alpha}}) \), by the fact that gamma random variables are simplifiable.

When \( \theta = 1 \), we obtain results for random variables \( \beta_{\sigma,1-\sigma} M_{\sigma}(F_X) \). The symmetric beta random variables \( \beta_{\sigma,1-\sigma} \) arise in a variety of important contexts, and are often referred to as generalized arcsine laws with density expressible as

\[ \frac{\sin(\pi \sigma)}{\pi} u^{\sigma-1}(1-u)^{-\sigma} \quad \text{for} \quad 0 < u < 1. \]

Now using (2.1) and (2.2), let \( C(F_X) = \{ x : F_X(x) > 0 \} \), then for \( x > 0 \),

\[ \sin(\pi F_{XY_{\alpha}}(x)) = \begin{cases} \sin(\pi \sigma(1 - F_X(x))) & \text{if} \ x \in C(F_X), \\ \sin(\pi(1 - \sigma)) & \text{if} \ x \notin C(F_X). \end{cases} \quad (2.4) \]

Note also that \( \sin(\pi[1 - F_X(x)]) = \sin(\pi F_X(x)) \). The next result yields another surprising property of these random variables.
Theorem 2.2. Consider the setting in the Theorem 2.1. Then when $\theta = 1$, it follows that for each fixed $0 < \sigma \leq 1$, the random variable $M_1(F_{XY_\sigma}) \overset{d}{=} \beta_{\sigma, 1-\sigma}M_\sigma(F_X)$ has density

$$\xi_{F_{XY_\sigma}}(x) = \frac{x^{\sigma-1}}{\pi} \sin(\pi F_{XY_\sigma}(x)) e^{-\sigma \Phi_{FX}(x)} \text{ for } x > 0,$$

specified by (2.4). Since $GGC(\sigma, F_X) = GGC(1, F_{XY_\sigma})$, this implies that the random variable $G_\sigma M_\sigma(F_X) \overset{d}{=} G_1 M_1(F_{XY_\sigma})$ has density

$$g_{\sigma, F_X}(x) = \frac{1}{\pi} \int_0^\infty e^{-\frac{y}{\sigma} y^{\sigma-2} \sin(\pi F_{XY_\sigma}(y)) e^{-\sigma \Phi_{FX}(y)}}dy \quad (2.6)$$

Proof. Since $M_1(F_{XY_\sigma}) \overset{d}{=} \beta_{\sigma, 1-\sigma}M_\sigma(F_X)$, the density is of the form (1.5), for each fixed $\sigma \in (0, 1]$. Furthermore we use the identity in (2.2). \hfill \Box

Remark 2.1. It is worthwhile to mention that transforming to the random variable $1/\beta_{\sigma, 1-\sigma}$, (2.5) is equivalent to the otherwise not obvious integral identity,

$$\frac{\sin(\pi \sigma)}{\pi} \int_1^\infty \frac{\xi_{\sigma,F_X}(xy)}{(y-1)^\sigma}dy = \frac{x^{\sigma-1}}{\pi} \sin(\pi F_{XY_\sigma}(x)) e^{-\sigma \Phi_{FX}(x)}.$$

This leads to interesting results when the density $\xi_{\sigma,F_X}(x)$ has a known form. On the other hand, we see that one does not need the explicit density of $M_\sigma(F_X)$ to obtain the density of $M_1(F_{XY_\sigma}) \overset{d}{=} \beta_{\sigma, 1-\sigma}M_\sigma(F_X)$. In fact, owing to our goal of yielding simple densities for many Dirichlet means from one mean, we see that the effort to calculate the density of $M_1(F_{XY_\sigma})$, for each $0 < \sigma \leq 1$, is no more than what is needed to calculate the density of $M_1(F_X)$.

We now see how this translates into the usually difficult problem of describing explicitly the density of the finite-dimensional distribution of a subordinator. In the next result we use the notation $\phi(C)$ to mean $\int_0^\infty I(s \in C) \phi(d\sigma)$.

Proposition 2.1. Let $(\xi_\phi(t); t \leq 1/\theta)$ denote a $GGC(\theta, F_X)$ subordinator on $[0, 1/\theta]$. Furthermore let $(C_1, \ldots, C_k)$ denote an arbitrary disjoint partition of the interval $[0, 1/\theta]$. Then the finite-dimensional distribution $(\xi_\phi(C_1), \ldots, \xi_\phi(C_k))$ has a joint density

$$\prod_{i=1}^k g_{\sigma_i, F_X}(x_i), \quad (2.7)$$

where each $\sigma_i = \theta |C_i| > 0$ and $\sum_{i=1}^k \sigma_i = 1$. The density $g_{\sigma_i, F_X}$ is given by (2.6). That is, $\xi_\phi(C_\ell) \overset{d}{=} G_1 M_1(F_{XY_{\sigma_\ell}})$ and are independent for $i = 1, \ldots, k$, where $M_1(F_{XY_{\sigma_\ell}}) \overset{d}{=} \beta_{\sigma_\ell, 1-\sigma_\ell}M_{\sigma_\ell}(F_X)$ has density

$$\frac{1}{\pi} x^{\sigma_i-1} \sin(\pi F_{XY_{\sigma_i}}(x)) e^{-\sigma_i \Phi_{FX}(x)}.$$
Proof. First, since \((C_1, \ldots, C_k)\) partitions the interval \([0, 1/\theta]\), it follows that their sizes satisfy \(0 < |C_i| \leq 1/\theta\) and \(\sum_{i=1}^{k} |C_i| = 1/\theta\). Since \(\zeta_\theta\) is a subordinator the independence of the \(\zeta_\theta(C_i)\) is a consequence of its independent increment property. In fact these are essentially equivalent statements. Hence, we can isolate each \(\zeta_\theta(C_i)\). It follows that for each \(i\) the Laplace transform is given by

\[ E[e^{-\lambda \zeta_\theta(C_i)}] = e^{-\theta |C_i| \psi_{F_X}(\lambda)} = e^{-\sigma_i \psi_{F_X}(\lambda)}, \]

which shows that each \(\zeta_\theta(C_i)\) is \(\text{GGC}(\sigma_i, F_X)\) for \(0 < \sigma_i \leq 1\). Hence the result follows from Theorem 2.2. □

3. Exponential Tilting/Esscher Transform

In this section we describe how the operation of exponential tilting of the density of a \(\text{GGC}(\theta, F_X)\) random variable leads to a non-trivial relationship between a mean functional determined by \(F_X\) and \(\theta\), and an entire family of mean functionals indexed by an arbitrary constant \(c > 0\). Additionally this will identify a non-obvious relationship between two classes of mean functionals. Exponential tilting is merely a catchy phrase for the operation of applying an exponential change of measure to a density or more general measure. In mathematical finance and other applications it is known as an Esscher Transform which is a key tool for option pricing. We mention that there is much known about exponential tilting of infinitely divisible random variables and in fact Bondesson \[3,\] example 3.2.5 discusses explicitly the case of GGC random variables, albeit not in the way we shall describe it. In addition, examining the gamma representation in \[11\] one can see a relationship to Lo and Weng \[32,\] Proposition 3.1) (see also Küchler and Sorensen \[28\] and James \[20,\] Proposition 2.1) for results on exponential tilting of Lévy processes). However, here our focus is on the properties of related mean functionals which leads to genuinely new insights.

Before we elaborate on this, we describe generically what we mean by exponential tilting. Suppose that \(T\) denotes an arbitrary positive random variable with density, say \(f_T\). It follows that for each positive \(c\), the random variable \(cT\) is well-defined and has density

\[ \frac{1}{c} f_T(t/c). \]

Exponential tilting refers to the exponential change of measure resulting in a random variable, say \(\hat{T}_c\), defined by the density

\[ f_{\hat{T}_c}(t) = \frac{e^{-t/(1/c)} f_T(t/c)}{E[e^{-cT}]} \]

Thus from the random variable \(T\) one gets a family of random variables \((\hat{T}_c; c > 0)\). Obviously the density for each \(\hat{T}_c\) does not differ much. However something interesting happens when \(T\) is a scale mixture of a gamma random variables, i.e., \(T = G_\theta M\), for some random positive random variable \(M\) independent of
In that case one can show, see [21], that $T_c = G_\theta \tilde{M}_c$ where \( \tilde{M}_c \) is sufficiently distinct for each value of \( c \). We demonstrate this for the case where \( M = M_\theta(F_X) \).

First note that obviously, \( cM_\theta(F_X) = M_\theta(F_{cX}) \), for each \( c > 0 \), which in itself is not a very interesting transformation. Now setting \( T_\theta = G_\theta M_\theta(F_X) \) with density denoted as \( g_{\theta,F_X} \), the corresponding random variable \( \tilde{T}_{\theta,c} \) resulting from exponential tilting has density

\[
e^{-t(1/c)}g_{\theta,F_X}(t/c)e^{\theta\psi_{F_X}(c)}
\]  

and Laplace transform

\[
E[e^{-c(1+\lambda)G_\theta M_\theta(F_X)}] = E[e^{-cG_\theta M_\theta(F_X)}] = e^{-\theta\psi_{F_X}(c(1+\lambda)) - \psi_{F_X}(c)}.
\]

Now for each \( c > 0 \), define the random variable

\[
A_c = \frac{cX}{cX + 1}.
\]

That is, the cdf of the random variable \( A_c \), can be expressed as,

\[
F_{A_c}(y) = F_X\left(\frac{y}{c(1-y)}\right) \text{ for } 0 < y < 1.
\]

In the next theorem we will show that \( M_\theta(F_X) \) relates to the family of mean functionals \( (M_\theta(F_{A_c}); c > 0) \), by the tilting operation described above. Moreover, we will describe the relationship between their densities.

**Theorem 3.1.** Suppose that \( X \) has distribution \( F_X \) and for each \( c > 0 \), \( A_c \) is a random variable with distribution \( F_{A_c} \). For each \( \theta > 0 \), let \( T_\theta = G_\theta M_\theta(F_X) \) denote a GGC \( (\theta,F_X) \) random variable having density \( g_{\theta,F_X} \). Let \( \tilde{T}_{\theta,c} \) denote a random variable with density and Laplace transform described by (3.1) and (3.2) respectively. Then \( \tilde{T}_{\theta,c} \) is a GGC \( (\theta,F_{A_c}) \) random variable and hence representable as \( G_\theta M_\theta(F_{A_c}) \). Furthermore, the following relationships exists between the densities of the mean functionals \( M_\theta(F_X) \) and \( M_\theta(F_{A_c}) \).

(i) Suppose that the density of \( M_\theta(F_X) \), say, \( \xi_{\theta,F_X} \) is known. Then the density of \( M_\theta(F_{A_c}) \) is expressible as

\[
\xi_{\theta,F_{A_c}}(y) = \frac{1}{c} e^{\theta\psi_{F_X}(c)}(1-y)^{\theta-2}\xi_{\theta,F_X}\left(\frac{y}{c(1-y)}\right),
\]

for \( 0 < y < 1 \).

(ii) Conversely, if the density of \( M_\theta(F_{A_c}) \), \( \xi_{\theta,F_{A_c}}(y) \), is known then the density of \( M_\theta(F_X) \) is given by

\[
\xi_{\theta,F_X}(x) = (1+x)^{\theta-2}\xi_{\theta,F_{A_c}}\left(\frac{x}{1+x}\right)e^{-\theta\psi_{F_X}(1)}.
\]
Proof. We proceed by first examining the Lévy exponent \( \psi_{F_X}(c(1 + \lambda)) - \psi_{F_X}(c) \) associated with \( \tilde{T}_{\theta,c} \) as described in (3.2). Notice that
\[
\psi_{F_X}(c(1 + \lambda)) = \int_0^{\infty} \log(1 + c(1 + \lambda)x)F_X(dx)
\]
and \( \psi_{F_X}(c) \) is of the same form with \( \lambda = 0 \). Hence isolating the logarithmic terms we can focus on the difference
\[
\log(1 + c(1 + \lambda)x) - \log(1 + cx).
\]
This is equivalent to
\[
\log \left(1 + \frac{cx}{1 + cx}\lambda\right) = \log \left(\frac{1}{1 + cx} + \frac{cx}{1 + cx}(1 + \lambda)\right),
\]
showing that \( \tilde{T}_{\theta,c} \) is GGC(\( \theta, F_{\mathcal{A}c} \)). This fact can also be deduced from Proposition 3.1 in Lo and Weng (32). The next step is to identify the density of \( M_{\theta}(F_{\mathcal{A}c}) \), in terms of the density of \( M_{\theta}(F_X) \). Using the fact that \( T_{\theta} = G_{\theta}M_{\theta}(F_X) \), one may write the density of \( T_{\theta} \) in terms of a gamma mixture as
\[
g_{\theta,F_X}(t) = \frac{t^{\theta-1}}{\Gamma(\theta)} \int_0^{\infty} e^{-t/m}m^{-\theta}\xi_{\theta,F_X}(m)dm.
\]
Hence, rearranging terms in (3.1), it follows that the density of \( \tilde{T}_{\theta,c} \) can be written as
\[
e^{\theta\psi_{F_X}(c)} \frac{t^{\theta-1}}{\Gamma(\theta)} \int_0^{\infty} e^{-\frac{cm+1}{cm}(cm)}\xi_{\theta,F_X}(m)dm.
\]
Now further algebraic manipulation makes this look like a mixture of a gamma(\( \theta, 1 \)) random variable, as follows,
\[
\frac{t^{\theta-1}}{\Gamma(\theta)} \int_0^{\infty} e^{-\frac{cm+1}{cm}} \left[\frac{cm+1}{cm}\right]^\theta e^{\theta\psi_{F_X}(c)}\xi_{\theta,F_X}(m) \frac{dm}{(1 + cm)^{\theta}}.
\]
Hence it is evident that \( M_{\theta}(F_{\mathcal{A}c}) \) has the same distribution as a random variable \( cm/(cM + 1) \) where \( M \) has density
\[
e^{\theta\psi_{F_X}(c)}(1 + cm)^{-\theta}\xi_{\theta,F_X}(m).
\]
Thus statements (i) and (ii) follow.

3.1. Tilting and Beta Scaling

This section describes what happens when one applies the exponentially tilting operation relative to a mean functional resulting from beta scaling. Recall that the tilting operation applied to \( G_{\theta}M_{\theta}(F_X) \) described in the previous section sets up a relationship between \( M_{\theta}(F_X) \) and \( M_{\theta}(F_{\mathcal{A}c}) \). Consider the random variable
β_{θ,σ(1−σ)}M_θ(σ(F_X)) \overset{d}{=} M_θ(F_{XY_σ}). Then tilting G_θ M_θ(F_{XY_σ}) as in the previous section leads to the random variable G_θ M_θ(F_{c,XY_σ/(c,XY_σ+1)}) and hence relates

$$\beta_{θ,σ(1−σ)}M_θ(σ(F_X)) \overset{d}{=} M_θ(F_{XY_σ})$$

to the Dirichlet mean of order θ,

$$M_θ(F_{c,XY_σ/(c,XY_σ+1)}).$$

Now letting F_{A,Y_σ} denote the distribution of A_σ Y_σ, one has

$$A_σ Y_σ \overset{d}{=} \frac{cXY_σ}{(cY_σ+1)}$$

and hence

$$M_θ(F_{c,XY_σ/(c,XY_σ+1)}) \overset{d}{=} M_θ(F_{A,Y_σ}) \overset{d}{=} \beta_{θ,σ(1−σ)}M_θ(σ(F_{A_σ})).$$

(3.3)

In a way this shows that the order of beta scaling and tilting can be interchanged. We now derive a result for the cases of

$$M_1(F_{XY_σ}) = \beta_{σ,1−σ}M_σ(σ(F_X))$$

and

$$M_1(F_{A,Y_σ}) = \beta_{σ,1−σ}M_σ(σ(F_{A_σ})).$$

Proposition 3.1. For each 0 < σ ≤ 1, the random variables M_1(F_{XY_σ}) = \beta_{σ,1−σ}M_σ(σ(F_X)) and M_1(F_{A,Y_σ}) = \beta_{σ,1−σ}M_σ(σ(F_{A_σ})) satisfy the following;

(i) The density of M_1(F_{A,Y_σ}) is expressible as

$$ξ_{F_{A,Y_σ}}(y) = \frac{e^{σφ_{FX}(c)}y^{σ-1}}{πc^{σ(1−y)}σ} \sin[πF_{XY_σ}\left(\frac{y}{c(1−y)}\right)] e^{−σφ_{FX}(c)} \left(\frac{y}{c(1−y)}\right)$$

for 0 < y < 1.

(ii) Conversely, the density of M_1(F_{XY_σ}) is given by

$$ξ_{F_{XY_σ}}(x) = \frac{e^{−σφ_{FX}(1)}x^{σ-1}}{π(1+x)} \sin[πF_{A,Y_σ}\left(\frac{x}{1+x}\right)] e^{−σφ_{FX}(1)} \left(\frac{x}{1+x}\right).$$

Proof. For clarity statement [(i)] is obtained by first using Theorem 3.1. Which gives,

$$ξ_{F_{A,Y_σ}}(y) = \frac{e^{φ_{FX}(c)}(1−y)^{−1}}{c} e^{−φ_{FX}(c)} \left(\frac{y}{c(1−y)}\right),$$

for 0 < y < 1. It then remains to substitute the form of the density (2.5) given in Theorem 2.2. Statement [(ii)] proceeds in the same way using 2.4.

Note that even if one can calculate Φ_{F_{A_σ}} for some fixed value of c, it may not be so obvious how to calculate it for another value. The previous results allow us to relate their calculation to that of Φ_{F_X} as described next.
Proposition 3.2. Set \( A_c = cX/(cX + 1) \) and define \( \Phi_{F_{A_c}}(y) = \mathbb{E}[\log(|y - A_c|)I(A_c \neq y)] \). Then for \( 0 < y < 1 \),

\[
\Phi_{F_{A_c}}(y) = \Phi_{FX} \left( \frac{y}{c(1 - y)} \right) - \psi_{FX}(c) + \log(c(1 - y)).
\]

Proof. The result can be deduced by using Proposition 3.1 in the case of \( \sigma = 1 \).

First notice that \( \sin(\pi F_{A_c}(y)) = \sin(\pi cX/(cX + 1)) \). Now equating the form of the density of \( M_1(F_{A_c}) \) given by (1.5) with the expression given in Proposition 3.1. It follows that

\[
e^{-\Phi_{F_{A_c}}(y)} = e^{\psi_{FX}(c)}e^{-\Phi_{FX} \left( \frac{y}{c(1 - y)} \right)},
\]

which yields the result. \(\square\)

Remark 3.1. We point out that if \( G_\kappa \) represents a gamma random variable for \( \kappa \neq \theta \), independent of \( M_\theta(F_X) \), it is not necessarily true that \( G_\kappa M_\theta(F_X) \) is a GGC random variable. For this to be true \( M_\theta(F_X) \) would need to be equivalent in distribution to some \( M_\kappa(F_R) \). In that case, our results above would be applied for a GGC\((\kappa,F_R)\) model. We will encounter a variation of such a situation in section 4.4.

4. First Examples

In this section we will demonstrate how our results in section 2 and 3 can be applied to extend and explain results related to two well known cases of Dirichlet mean functionals.

4.1. An example connected to Diaconis and Kemperman

Set \( X = U \), denoting a uniform\([0,1]\) random variable with \( F_U(u) = u \) for \( 0 \leq u \leq 1 \). It is known from Diaconis and Kemperman (10) that the density of \( M_1(F_U) \) is

\[
e^{-Xy}y^{-y}(1 - y)^{-(1 - y)} \text{ for } 0 < y < 1.
\]

(4.1)

Note furthermore that \( \tilde{T}_1 \overset{d}{=} G_1M_1(F_U) \) is GGC\((1,F_U)\) and has a rather strange Laplace transform,

\[
\mathbb{E}[e^{-\lambda G_1M_1(F_U)}] = e^{-\psi_{F_U}(\lambda)} = e(1 + \lambda)^{-\lambda + 1}.
\]

We can use this fact combined with the previous results to obtain a new explicit expression for the density of what we believe should be an important mean functional and corresponding infinitely divisible random variable.
Proposition 4.1. Let \( W = G_1/G_1' \) be the ratio of two independent exponential (1) random variables having density \( F_W(dx)/dx = (1 + x)^{-2} \) for \( x > 0 \). Now let \( T_1 \) denote a GGC\((1,F_W)\) random variable, with \( \psi_{F_W}(\lambda) = \lambda^{-1} \log(\lambda) \). Then \( T_1 \overset{d}{=} G_1 M_1(F_W) \), where \( M_1(F_W) \) has density,

\[
\xi_{F_W}(x) = \frac{1}{\pi} \sin \left( \frac{\pi x}{1 + x} \right) x^{-\frac{1}{1+x}} \text{ for } x > 0.
\]

**Proof.** First note that it is straightforward to show that

\[
\mathbb{E}[e^{-T_1}] = \mathbb{E}[(1 + M_1(F_W))^{-1}] = e^{-1}.
\]

This fact also establishes the existence of \( T_1 \). Now we see that

\[
W = \frac{G_1}{G_1 + G_1'} \overset{d}{=} U.
\]

Hence Theorem 3.1 shows that \( M_1(F_U) \) arises from tilting the density of \( G_1 M_1(F_W) \). The density is obtained by applying statement (ii) of Theorem 3.1 to (4.1). Or by statement (ii) of Proposition 3.1. We now apply Theorem 2.2 and Proposition 2.1 to give a description of the finite-dimensional distribution of the subordinators associated with the two random variables above. Here \( U \) and \( W = G_1/G_1' \) are as described previously.

Proposition 4.2. Let \( U \) denote a uniform\([0,1]\) random variable and let \( W = G_1/G_1' \) denote a ratio of independent exponential(1) random variables.

(i) Suppose that \( (\tilde{\zeta}_i(t); 0 < t < 1) \) is a GGC\((1,F_U)\) subordinator, then for \( (C_1, \ldots, C_k) \) a disjoint partition of \((0,1)\), the finite-dimensional distribution has a joint density as in (2.7), with

\[
g_{\sigma_i,F_U}(x_i) = \int_0^1 e^{-\sigma_i y} \frac{e^{\sigma_i y}}{\pi} \sin(\pi \sigma_i (1 - y)) y^{\sigma_i (1 - y) - 3} (1 - y)^{-\sigma_i (1 - y)} dy,
\]

for \( i = 1, \ldots, k \).

(ii) That is \( \tilde{\zeta}_i(C_i) \overset{d}{=} G_1 M_1(F_U Y_{\sigma_i}) \) and are independent for \( i = 1, \ldots, k \). Furthermore, the density of \( M_1(F_U Y_{\sigma_i}) \) is

\[
e^{\sigma_i y} \frac{\sin(\pi \sigma_i (1 - y))}{\pi} y^{\sigma_i (1 - y) - 1} (1 - y)^{-\sigma_i (1 - y)}
\]

for \( 0 < y < 1 \).

(iii) If \( (\zeta_1(t); 0 < t < 1) \) is a GGC\((1,F_W)\) subordinator then the finite-dimensional distribution \( (\zeta_1(C_1), \ldots, \zeta_1(C_k)) \) is described now with

\[
g_{\sigma_i,F_W}(x_i) = \int_0^{\infty} e^{- \frac{x_i}{w}} \frac{1}{\pi} \sin \left( \frac{\pi \sigma_i}{1 + w} \right) w^{\sigma_i} w^{-3} dw.
\]
(iv) That is $\zeta_1(C_i) \overset{d}{=} G_1 M_1(F_{W\sigma_i})$ and are independent for $i = 1, \ldots, k$.
Furthermore, the density of $M_1(F_{W\sigma_i}) \overset{d}{=} \beta_{\sigma_i,1-\sigma_i} M_i(F_W)$ is
\[
\frac{1}{\pi} \sin \left( \frac{\pi \sigma_i}{1 + x} \right) x^{\frac{\sigma_i}{1-\sigma_i} - 1}
\]
for $x > 0$.

Proof. This now follows from Theorem 2.2, Proposition 2.1 and (4.1). Specifically, note that $\mathcal{C}(F_U) = (0, \infty)$, hence for any $0 < \sigma < 1$,
\[
\sin(\pi F_{UY\sigma}(u)) = \sin(\pi \sigma(1-u))
\]
for $0 < u < 1$ and 0 otherwise. Furthermore from (4.1), or by direct argument, it is easy to see that,
\[
\Phi_{F_U}(y) = -\log \left( y^{-\nu}(1-y)^{-(1-\nu)} \right) - 1.
\]
This fact also is evident from Diaconis and Kemperman [10]. It follows that $M_1(F_{UY\sigma})$ has density
\[
\frac{e^\sigma}{\pi} \sin(\pi \sigma(1-y)) y^{\sigma(1-\nu)}(1-y)^{-\sigma(1-y)} \text{ for } 0 < y < 1.
\]
The density for $M_1(F_{W\sigma})$ is obtained in a similar fashion by Proposition 3.1.

Remark 4.1. Setting
\[
A_c \overset{d}{=} \frac{c G_1}{c G_1 + G'_1},
\]
one can easily obtain the density of the random variable $M_1(F_{A_c})$ for each $c > 0$ by using statement (ii) of Theorem 3.1. Note also that one can deduce from the density of $M_1(F_W)$ that $\Phi_{F_W}(x) = [x/(1 + x)] \log(x)$. Hence in this case an application of Proposition 3.2 shows that,
\[
\Phi_{F_{A_c}}(y) = \frac{y}{c(1-y) + y} \log \left[ \frac{y}{c(1-y)} \right] - \frac{c \log(c)}{c-1} + \log[c(1-y)].
\]
We note that otherwise it is not easy to calculate $\Phi_{A_c}$, in this case, by direct arguments.

4.2. Reconciling some results of Cifarelli and Mellili

To further illustrate our point we show how to reconcile two apparently unrelated results given in Cifarelli and Mellili [8]. Let $\beta_{1/2,1/2}$ denote a beta$(1/2,1/2)$ random variable with cdf
\[
F_{\beta_{1/2,1/2}}(y) = \frac{2}{\pi} \tan^{-1}\left( \sqrt{\frac{y}{1-y}} \right) = 1 - \frac{2}{\pi} \cot^{-1}\left( \sqrt{\frac{y}{1-y}} \right)
\]
often referred to as the arcsine law. Cifarelli and Melilli [8, p.1394-1395] show that for all \( \theta > 0 \), \( M_\theta(F_{\beta_{1/2,1/2}}) \xrightarrow{d} \beta_{\theta+1/2,\theta+1/2} \). Now define the probability density

\[
\vartheta_{1/2}(dx)/dx = \frac{1}{\pi}x^{-1/2}(1+x)^{-1}.
\]

Cifarelli and Mellili [8, p.1394-1395] also show that for \( \theta \geq 1 \), \( M_\theta(\vartheta_{1/2}) \) has the density,

\[
\frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1/2)\Gamma(\theta + 1/2)}x^{\theta-1/2}(1+x)^{-(\theta+1)}.
\]  (4.2)

Hjort and Ongaro [18] recently extend this result for all \( \theta > 0 \).

Here, however, we note that if \( X \) has distribution \( \vartheta_{1/2} \) then \( X \xrightarrow{d} G_{1/2}/G_{1/2}' \) where \( G_{1/2} \) and \( G_{1/2}' \) are independent and identically distributed gamma random variables. That is \( M_\theta(\vartheta_{1/2}) = M_\theta(G_{1/2}/G_{1/2}') \). Now using the known fact that

\[
\beta_{1/2,1/2} \xrightarrow{d} \frac{G_{1/2}/G_{1/2}'}{G_{1/2}/G_{1/2}'+1},
\]

we see that \( \beta_{1/2,1/2} \) is a special case of \( A_1 \) in Proposition 3.2. That is, \( M_\theta(F_{\beta_{1/2,1/2}}) \) results from exponentially tilting the density of \( G_{1/2}M_\theta(F_{G_{1/2}/G_{1/2}'}) \), with \( c = 1 \). Hence one could have obtained the density of \( M_\theta(F_{G_{1/2}/G_{1/2}'}) \) by using the result of Cifarelli and Melili that \( M_\theta(F_{\beta_{1/2,1/2}}) \xrightarrow{d} \beta_{\theta+1/2,\theta+1/2} \), and simply applying statement (ii) of Theorem 3.1

**Remark 4.2.** Note that the following identity for the normalizing constant,

\[
\frac{\Gamma(2\theta+1)}{\Gamma(\theta+1/2)\Gamma(\theta+1/2)}4^{-\theta} = \frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1/2)\Gamma(\theta + 1/2)},
\]

is obtained from the gamma duplication formula, where in particular

\[
\Gamma(2\theta + 1) = \frac{4^{\theta}}{\Gamma(1/2)^2\Gamma(\theta + 1/2)}\Gamma(\theta + 1)\Gamma(\theta + 1/2).
\]

Hereafter we shall use the notation \( X_{1/2} \xrightarrow{d} G_{1/2}/G_{1/2}' \). Moreover, from the density in (4.2), it follows that, using the notation \( X_{1/2,\theta} \),

\[
X_{1/2,\theta} \xrightarrow{d} M_\theta(F_{X_{1/2}}) \xrightarrow{d} G_{\theta+1/2}/G_{1/2}.
\]

In particular \( X_{1/2,0} = X_{1/2} \). Furthermore,

\[
G_{\theta}X_{1/2,\theta} \xrightarrow{d} G_{\theta}M_\theta(F_{X_{1/2}}) \xrightarrow{d} G_{\theta}G_{\theta+1/2}/G_{1/2}
\]

is \( \text{GGC}(\theta,F_{X_{1/2}}) \), with Laplace transform

\[
\mathbb{E}[e^{-\lambda G_{\theta}M_\theta(F_{X_{1/2}})}] = (1 + \sqrt{\lambda})^{-2\theta}.
\]  (4.3)
Now, for $0 < p < 1$, setting $c = p^2/(1 - p)^2$ one can extend the result of Cifarelli and Melilli for mean functionals defined by the random variable,

$$B_{1/2,p} = \frac{p^2 G_{1/2}}{p^2 G_{1/2} + q^2 G'_{1/2}} = \frac{cX_{1/2}}{cX_{1/2} + 1}$$

having density

$$\frac{pq}{\pi} \frac{y^{-1/2}(1 - y)^{-1/2}}{q^2 y + p^2 (1 - y)}$$

for $0 < y < 1$, as follows, first the density of the random variable

$$cM_\theta(F_{cX_{1/2}}) = M_\theta(F_{X_{1/2}})$$

is simply,

$$\xi_{\theta F_{cX_{1/2}}}(x) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1/2)\Gamma(1/2)} c^{1/2} x^{\theta - 1/2} (c + x)^{-(\theta + 1)}.$$

The density of $M_\theta(F_{B_{1/2,p}})$ is then obtained by applying statement (ii) of Theorem 3.1 to the density above, to get,

$$\xi_{\theta F_{B_{1/2,p}}}(y) = pq \frac{\Gamma(\theta + 1)}{\Gamma(\theta + 1/2)\Gamma(1/2)} \frac{y^{\theta - 1/2} (1 - y)^{\theta - 1/2}}{p^2 (1 - y) + q^2 y}. \quad (4.4)$$

However this density equates with the density of a two parameter $(1/2, \theta)$ Poisson-Dirichlet random probability measure evaluated at a sect $C$ say, $\tilde{P}_{1/2,\theta}(C)$, such that $\mathbb{E}[\tilde{P}_{1/2,\theta}(C)] = p$, given in Carlton (4, Remark 3.1) (see also (23) and (34)). Hence we see that $M_\theta(F_{B_{1/2,p}}) = \tilde{P}_{1/2,\theta}(C)$ arises from exponentially tilting the density of $G_\theta M_\theta(F_{X_{1/2}})$.

4.2.1. Results for $\theta > -1/2$

Notice that the random variable $X_{1/2,\theta}$ is well defined for all $\theta > -1/2$, and still has density (4.2) for the negative range of $\theta$. However obviously in the range $-1/2 < \theta < 0$ it is not a Dirichlet mean functional of order $\theta$, and as such is not covered by the result in (8). The next result shows that for all $\theta > -1/2$ the random variable $X_{1/2,\theta}$ is a mean functional of order $1 + \theta$.

**Proposition 4.3.** For each $\theta > -1/2$, the random variable $X_{1/2,\theta}$ satisfies the distributional equality.

$$X_{1/2,\theta} \overset{d}{=} \beta_{\theta+1/2,1/2} X_{1/2,\theta+1/2} \overset{d}{=} M_{1+\theta}(F_{X_{1/2} Y_{2+1/2}})$$
As a consequence the random variable, $G_{1+\theta}X_{1/2,\theta} \overset{d}{=} G_{\theta+1/2}X_{1/2,\theta+1/2}$ is $GGC(\theta+1/2, F_{X_{1/2},\theta+1/2}) = GGC(\theta + 1/2, F_X)$. 

**Proof.** To see this note the simple identity

$$G_{\theta+1/2} \overset{d}{=} \beta_{\theta+1/2}$$

The first equality in the Proposition is then obtained by dividing both sides by an independent $G_{1/2}$ random variable. The second equality is obtained by using Theorem 2.1. That is using the fact that $\beta_{\theta+1/2} \overset{d}{=} \beta_{\theta+1/2} M_{\theta+1/2}(F_{X_{1/2}})$.

Thus we see that for $\theta > 0$,

$$X_{1/2,\theta} \overset{d}{=} M_{\theta}(F_{X_{1/2}}) \overset{d}{=} M_{1+\theta}(F_{X_{1/2},\theta+1/2})$$

**Remark 4.3.** The random variable $X_{1/2} = G_{1/2}/G'_{1/2}$ is a special case of an important random variable studied by Lamperti [29]. The general case is defined for $0 < \alpha < 1$, $X_\alpha = S_\alpha/S'_\alpha$, where $S_\alpha$ and $S'_\alpha$ are iid positive stable random variables of index $\alpha$. The $GGC(\theta, F_{X_{1/2}})$ random defined by its Laplace transform (4.3) is an example of the generalized positive Linnik random variable which can be represented as $(G_{\theta/\alpha})^{1/\alpha} S_\alpha$. Where in this case $\alpha = 1/2$. A thorough investigation of this class, and its various implications, initiated in [21], will be discussed in a forthcoming manuscript [22] (see also [25]).

5. Obtaining the finite dimensional distribution of a subordinator of BFRY

The previous examples revisited some cases that have existed in the literature for some time. Our final example shows how one can apply the results in section 2 to obtain new results for a subordinator recently studied by Bertoin, Fujita, Roynette and Yor [2]. We first supply some background. For $0 < \alpha < 1$, let $S_\alpha$ denote a positive $\alpha$-stable random variable specified by its Laplace transform

$$E[e^{-\lambda S_\alpha}] = e^{-\lambda^\alpha}.$$ 

In addition define

$$Z_\alpha = \left(\frac{S_\alpha}{S'_\alpha}\right)^\alpha$$

where $S'_\alpha$ is independent of $S_\alpha$ and has the same distribution. The density of this random variable was obtained by Lamperti [29] (see also Chaumont and Yor [3], exercise 4.2.1) and has the remarkably simple form,

$$f_{Z_\alpha}(y) = \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{1}{y^2 + 2y \cos(\pi\alpha) + 1} \text{ for } y > 0.$$
In order to avoid confusion we will now denote relevant random variables appearing originally in (2) as $\Delta_\alpha$ and $G_\alpha$, as $\Sigma_\alpha$ and $G_\alpha$ respectively. From (2), let $(\Sigma_\alpha(t); t > 0)$ denote a subordinator such that

$$
\mathbb{E}[e^{-\lambda \Sigma_\alpha(t)}] = ((\lambda + 1)\alpha - \lambda \alpha)^t = \exp(-t(1-\alpha)\mathbb{E}[\log(1 + 1/G_\alpha)])
$$

where from ((2), Theorems 1.1 and 1.3), $G_\alpha$ denotes a random variable such that

$$
G_\alpha \overset{d}{=} Z_1^{1/\alpha} - Z_1^{1/\alpha - 1} + Z_1^{1/\alpha - 1}
$$

and has density on $(0, 1)$ given by

$$
f_{G_\alpha}(u) = \frac{\alpha \sin(\pi \alpha)}{(1-\alpha)\pi} \frac{u^{\alpha - 1}(1-u)^{\alpha - 1}}{u^{2\alpha} - 2(1-u)\cos(\pi \alpha) + (1-u)^2}
$$

Hence it follows that the random variable $1/G_\alpha$ takes its values on $(1, \infty)$ with probability one and has cdf satisfying,

$$
1 - F_{1/G_\alpha}(x) = F_{G_\alpha}(1/x) = F_{Z_{1-\alpha}}((x-1)^{-\alpha}),
$$

where using properties of the cdf $Z_{1-\alpha}$, (see (17) and ((22), Proposition 2.1)), it follows that for $x > 1$

$$
F_{Z_{1-\alpha}}\left(\frac{1}{x-1}\right) = 1 - \frac{1}{\pi(1-\alpha)\cot^{-1}\left(\frac{\cos(\pi(1-\alpha)) + (x-1)^{-\alpha}}{\sin(\pi(1-\alpha))}\right)}.
$$

It follows as noted by (2) that $(\Sigma_\alpha(t); t > 0)$ is a GGC$(1-\alpha, F_1/G_\alpha)$ subordinator. Where from ((2), Theorem 1.1) the GGC$(1-\alpha, F_1/G_\alpha)$ random variable $\Sigma_\alpha \overset{d}{=} \Sigma_\alpha(1)$ satisfies,

$$
\Sigma_\alpha \overset{d}{=} U_1^{-\beta_{\alpha,1}} \overset{d}{=} U_1^{-1/\alpha}
$$

where $U$ denotes a uniform$[0, 1]$ random variable and for clarity $G_{1-\alpha}$ is a gamma$(1-\alpha, 1)$ random variable. It is evident, as investigated in (17), that

$$
M_{1-\alpha}(F_1/G_\alpha) \overset{d}{=} \frac{1}{\beta_{\alpha,1}} \overset{d}{=} U^{-1/\alpha},
$$

So this constitutes a rare known example where a Dirichlet mean of order $0 < \theta < 1$ has a simple recognizable form. Note also that $M_{1-\alpha}(F_\alpha) \overset{d}{=} U$ for all $0 < \alpha \leq 1$. The above points may also be found in the survey paper of (24).

Note that when $\alpha = 1/2$, $G_{1/2} \overset{d}{=} \beta_{1/2,1/2}$, which then relates to the results in section 4.2. In addition, it is known that for each fixed $t$,

$$
\Sigma_{1/2}(t) \overset{d}{=} \frac{G_{1/2}}{\beta_{1/2,1/2+t/2}}.
$$
This result may be found in [25] where the case of $\alpha = 1/2$ also belongs to another class of subordinators indexed by $\alpha$.

Now using the fact discussed in this section we will show how to use the results in section 2 to explicitly describe the finite dimensional distribution of the subordinator $\Sigma_\alpha(t)$ over the range $0 < t \leq 1/(1-\alpha)$, hence by infinite divisibility for all $t$. Additionally the analysis will also yield expressions for mean functionals based on $F_{1/G_\alpha}$. We first show how to calculate

$$R_\alpha(x) = \Phi_{F_{1/G_\alpha}}(x) = \mathbb{E}[\log(|t - 1/G_\alpha|)I(t \neq 1/G_\alpha)].$$

(5.1)

Using simple beta gamma algebra one has

$$\Sigma_\alpha \overset{d}{=} G_{1-\alpha}/\beta_{\alpha,1} \overset{d}{=} G_1 \beta_{1-\alpha,\alpha}/U^{1/\alpha}$$

Hence applying Theorem 2.1, with $\theta = 1$, and $\sigma = 1-\alpha$, it follows that $\Sigma_\alpha$ is also GGC$(1,F_{Y_{1-\alpha}/G_\alpha})$ and

$$\beta_{1-\alpha,\alpha} \overset{d}{=} \beta_{1-\alpha,\alpha}/U^{1/\alpha} \overset{d}{=} M_1(F_{Y_{1-\alpha}/G_\alpha})$$

(5.2)

Furthermore, using (2.4), one has

$$\sin(\pi F_{Y_{1-\alpha}/G_\alpha}(x)) = \begin{cases} \sin(\pi F_{Y_{1-\alpha}/G_\alpha}(x)), & \text{if } x > 1, \\ \sin(\pi(1-\alpha)), & \text{if } 0 < x \leq 1. \end{cases}$$

(5.3)

where again using the properties of $F_{Z_{1-\alpha}}$, as deduced from (22), Proposition 2.1, [(iii)],

$$\sin(\pi(1-\alpha)F_{G_\alpha}(1/x)) = \frac{\sin(\pi(1-\alpha))}{[(x-1)^{2\alpha} - 2(x-1)^\alpha \cos(\pi \alpha) + 1]^{1/2}}$$

(5.4)

for $x > 1$. These points lead to the following description of (5.1).

**Proposition 5.1.** For $0 < \alpha < 1$, consider $R_\alpha(x)$ as defined in (5.1). Then,

$$R_\alpha(x) = \begin{cases} \frac{1}{2(1-\alpha)}[\log((x-1)^{-\alpha} - 2(x-1)^{-\alpha} \cos(\pi \alpha) + 1)], & \text{if } x > 1, \\ \frac{1}{1-\alpha} \log(x/[1 - (1-x)^{\alpha}]), & \text{if } 0 < x \leq 1. \end{cases}$$

(5.5)

**Proof.** By standard calculations the density of $B_\alpha = \beta_{1-\alpha,\alpha}/\beta_{\alpha,1}$ is given by

$$f_{B_\alpha}(x) = \frac{\sin(\pi(1-\alpha))}{\pi} x^{-\alpha-1}(1 - (1-x)^{\alpha})|I(x \leq 1)|$$

However we see from (5.2) that $B_\alpha \overset{d}{=} M_1(F_{Y_{1-\alpha}/G_\alpha})$. Hence Theorem 2.2 applies and the density of $B_\alpha$ can be written as

$$f_{B_\alpha}(x) = \frac{x^{-\alpha}}{\pi} \sin(\pi F_{Y_{1-\alpha}/G_\alpha}(x))e^{-(1-\alpha)R_\alpha(x)}$$

Now equating the two forms of the density of $B_\alpha$ and using (5.3) and (5.4), one then obtains the expression for $R_\alpha$.\[\square\]
This leads to our next result,

**Proposition 5.2.** For $0 < \sigma \leq 1$ the density of $M_1(F_{Y_\sigma/G_\alpha})$ is given by,

$$f_{\alpha,\sigma}(x) = \frac{x^{\frac{\sigma}{1-\alpha}-1}}{\pi} \sin(\pi \sigma) [1 - (1 - x)^\alpha]^\frac{\sigma}{1-\alpha},$$

for $0 < x \leq 1$, and for $x > 1$

$$f_{\alpha,\sigma}(x) = \frac{x^{\sigma-1}}{\pi} \sin(\pi \sigma F_{G_\alpha}(1/x))[(x-1)^{2\sigma} - 2(x-1)^\sigma \cos(\pi \alpha) + 1]^\frac{1}{2(1-\alpha)}.$$

**Proof.** From Theorem 2.2 we have the general form of the density of $M_1(F_{Y_\sigma/G_\alpha})$ is given by

$$\int_0^\infty e^{-z/w} f_{\alpha,\sigma}(w) dw$$

The result is then concluded by applying Proposition 5.1 and (5.3) and (5.4). □

We now can close with the finite dimensional distribution of the subordinator, which follows from Proposition 2.1 and Proposition 5.2.

**Theorem 5.1.** Consider the GGC$(1-\alpha, F_1/G_\alpha)$ subordinator $(\Sigma_\alpha(t), t \leq 1/(1-\alpha))$. Then for a disjoint partition of the interval $(0, 1/(1-\alpha)]$, $(C_1, \ldots, C_k)$, the finite dimensional distribution of $(\Sigma_\alpha(C_1), \ldots, \Sigma_\alpha(C_k))$ is such that each $\Sigma_\alpha(C_i)$ is independent with density

$$\int_0^\infty e^{-z/w} w^{-1} f_{\alpha,\sigma_i}(w) dw$$

where $\sigma_i = (1-\alpha)|C_i|$ and $f_{\alpha,\sigma_i}(w)$ represents the density of $M_1(F_{Y_\sigma_i/G_\alpha})$ given in Proposition 5.2. That is,

$$\Sigma_\alpha(C_i) \overset{d}{=} G_1 M_1(F_{Y_\sigma_i/G_\alpha}).$$

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