Determinantal Hypertrees

Andrew Vander Werf *

August 19, 2022

Abstract

We deduce a structurally inductive description of Kalai’s famous tree generalization, the $Q$–acyclic complex, which extends to an analogous description of the determinantal probability measure associated with Kalai’s celebrated enumeration result. Along the way, we generalize the homological notion of a simplicial complex being $Q$–acyclic to a relative–homological notion of a pair of simplicial complexes being $Q$–acyclic relative to each other. We also apply these new results to random topology and the spectral analysis of random graphs.

1 Introduction

Forty years ago, Kalai [Kal83] introduced, to spectacular effect, a generalization of the graph–theoretic notion of a tree to higher–dimensional simplicial complexes, called $Q$–acyclic complexes for the triviality of their rational reduced homology groups in every dimension. For integers $0 \leq k < n$, let $T_{n,k}$ denote the set of $k$–dimensional $Q$–acyclic complexes on the vertex set $[n] := \{1, 2, \ldots, n\}$. Kalai noticed, among other things, that $\sum_{T \in T_{n,k}} |\tilde{H}_{k-1}(T)|^2 = n^{(n-2)}$, which is seen to be a generalization of Caley’s formula by recalling that $|\tilde{H}_k(T)| = 1$ for all $T \in T_{n,1}$, due to trees being connected. This suggests a natural probability measure $\nu_{n,k}$ on $T_{n,k}$ defined on atoms by $\nu_{n,k}(T) = n^{(n-2)}|\tilde{H}_{k-1}(T)|^2$.

Seemingly unrelated to this measure, consider the $k$–dimensional Linial–Meshulam complex, denoted $Y_k(n, p)$ and defined [LM06], [MW09] to be the random $k$–dimensional simplicial complex on $[n]$ with full $k-1$–skeleton wherein each $k$–face is included independently and with probability $p$. Let $\mu_{n,k}$ denote the probability density for $Y_k(n, (n+1)^{-1})$.

For the entirety of this paper, let $T_{n,k}$ denote a random complex distributed according to $\nu_{n,k}$, and let $Y_{n,k}$ denote a random complex distributed according to $\mu_{n,k}$. Our main result is the following structure theorem for random $Q$–acyclic complexes distributed according to $\nu_{n,k}$:

**Theorem 1.** Assume that $1 \leq k < n$. There exists a coupling of $T_{n,k}$, $T_{n-1,k}$, $T_{n-1,k-1}$, $Y_{n-1,k}$, $Y_{n-1,k-1}$ such that $T_{n-1,k}$ and $T_{n-1,k-1}$ are independent of $Y_{n-1,k}$ and $Y_{n-1,k-1}$ respectively. $T_{n-1,k}$ and $T_{n-1,k-1}$ are conditionally independent given $T_{n,k}$, and

$$T_{n,k} = \text{Cone}(n, T_{n-1,k-1} \cup Y_{n-1,k-1}) \cup T_{n-1,k} \setminus Y_{n-1,k}.$$  

We can easily identify this coned term as being the link of the vertex $n$ in $T_{n,k}$. So, with this coupling, we have $\text{Link}(n, T_{n,k}) = T_{n-1,k-1} \cup Y_{n-1,k-1}$. This can be taken to mean that the vertex link of $T_{n,k}$ can be

*The author gratefully acknowledges partial support from NSF-DMS # 1547357.
simulated by first sampling $\mathcal{T}_{n-1,k-1}$ and then adding each missing $k-1$–face independently with probability $1/n$. The remaining term in the displayed union gives a description of those $k$–faces in $\mathcal{T}_{n,k}$ which do not contain the vertex $n$. This set of faces can be simulated by sampling $\mathcal{T}_{n-1,k}$ and then deleting each of its $k$–faces independently with probability $1/n$. Note then that these two binomial processes, one of adding $k-1$–faces which are then coned with $n$ and one of deleting $k$–faces, must be correlated at least to the point of producing the same number of faces—this is because all trees of a given dimension and vertex count have the same number of top–dimensional faces.

1.1 Applications to random topology

This result has several applications to random topology. Indeed, Garland's method [Gar73] and its refinements (see [Z96], [Z03], [Opp18], [Opp20]), Žuk's criterion among them, have proven to be very effective tools for extracting global information about a pure $k$–dimensional simplicial complex using only information found in the $k - 2$–dimensional links of the complex.

**Theorem 2** (Garland’s method). Let $X$ be a pure $k$–dimensional simplicial complex. Suppose that, for all $(k-2)$–faces $\tau \in X$, we have $\lambda(0)(\operatorname{Link}(\tau, X)) \geq 1 - \varepsilon > 0$. Then $\lambda(k-1)(X) \geq 1 - k\varepsilon$.

**Theorem 3** (Žuk’s criterion). Let $X$ be a pure 2–dimensional simplicial complex. Suppose that, for all 0–faces $\tau \in X$, we have that $\lambda(0)(\operatorname{Link}(\tau, X)) > 1/2$ and $\operatorname{Link}(\tau, X)$ is connected. Then the fundamental group, $\pi_1(X)$, has Kazhdan’s property $(T)$.

The $\lambda(k-1)(X)$ mentioned in the above statement of Garland’s method refers to the smallest nonzero eigenvalue of the top–dimensional up–down Laplacian of $X$ under a particular weighted inner product (see [Lub18] or [GW14] for greater detail). This eigenvalue will be referred to as the spectral gap of $X$. In the special case where $k = 1$ and $X$ is a connected graph, $\lambda(0)(X)$ corresponds to the second smallest eigenvalue of the reduced Laplacian of $X$. Fortunately, the spectral gap of a random graph, in one form or another, is already quite well studied [FO05], [COL09], [Oli09], [TY19], [HKP19]. Combining our characterization of the law of $\operatorname{Link}(n, \mathcal{T}_{n,2})$ with the best known techniques for the kind of spectral gap estimation we would like to do, we find the following:

**Proposition 4.** Let $\delta > 0$ be an arbitrary fixed constant, and let $\mathcal{X}$ be the union of $[\delta \log n]$ jointly independent copies of $\mathcal{T}_{n,2}$. Then, for any fixed $s > 0$, we have, with probability at least $1 - o(n^{-s})$, that $\pi_1(\mathcal{X})$ has Kazhdan’s property $(T)$ and $\lambda^{(1)}(\mathcal{X}) \geq 1 - O(1/\sqrt{\log n})$.

**Proof.** This follows immediately from the previous two theorems in combination with Lemma 30 and a union bound on the probability that any vertex link of $\mathcal{X}$ fails to meet the criterion of Garland’s method. \hfill $\square$

The same results were proven for $\mathcal{X} = \mathcal{Y}_2(n, p)$ with $p > 2\log n/n$ in [HKP19]. Moreover, the authors show that this is a sharp threshold for $\mathcal{Y}_2(n, p)$ to have a large spectral gap. In contrast, the $\mathcal{X}$ in the above proposition is able to achieve these same results, and, moreover, it can do so using only an arbitrarily small fraction of the number of faces $\mathcal{Y}_2(n, p)$ requires.

1.2 Related work

Unfortunately, other work concerning $Q$–acyclic complexes (in dimension greater than 1, at least) is somewhat sparse. It began, of course, with Kalai [Kal83] as a primarily combinatorial idea which, quite satisfyingly, generalized the enumerative results of Caley and Prüfer. These results were notably extended by Adin [Adi92], but the subject otherwise remained largely stagnant until Duval, Klivans, and Martin [DKM08] extended Kalai’s original results to also generalize Kirchhoff’s famous Matrix–Tree theorem. Shortly thereafter, Lyons [Lyo09] revisited the idea of $Q$–acyclic complexes through the lens of determinantal probability measures on matroids and, in doing so, revealed several new fruitful avenues for exploring $Q$–acyclic complexes as well as the measures $\nu_{n,k}$. Armed with probabilistic techniques, Linial and Peled [LP19] sharpen Kalai’s original bounds on the number of $Q$–acyclic complexes, Kahle and Newman [KN20] explore the random–topological properties of 2–dimensional $Q$–acyclic complexes under $\nu_{n,2}$, and Mészáros [Mé21] determines the local weak limit of $\nu_{n,k}$ as $n \to \infty$ and $k$ remains fixed.
2 Homology, simplicial and relative

In this section, we will briefly cover the homological notions and notations that appear throughout this paper. Some familiarity with linear algebra, point–set topology, and (finitely–generated) abelian groups—particularly the structure theorem for finitely–generated abelian groups—is expected of the reader.

A chain complex is a sequence of abelian groups, \( \{C_j\}_{j \in \mathbb{Z}} \), called chain groups, linked by group homomorphisms, \( \partial_j : C_j \to C_{j-1} \), called boundary maps which, for all \( j \in \mathbb{Z} \), satisfy \( \partial_j \partial_{j+1} = 0 \) or, equivalently, \( \text{Ker} \partial_j \supseteq \text{Im} \partial_{j+1} \). The \( j \)th homology group of a chain complex such as this is defined to be the quotient group \( \text{Ker} \partial_j / \text{Im} \partial_{j+1} \). Since we will only be considering finitely–generated free \( \mathbb{Z} \)–modules for our chain groups, we can represent these boundary maps by integer matrices. By the structure theorem for finitely–generated abelian groups, the \( j \)th homology group can be expressed as the direct sum of a free abelian group, which is isomorphic to \( \mathbb{Z}^{\beta_j} \) and called its free part, and a finite abelian group, \( t_j \), called its torsion subgroup, which is a direct sum of finite cyclic groups. The rank, \( \beta_j \), of the free part is called the \( j \)th Betti number. We will require the following standard fact from homological algebra which gives two equivalent formulas for what is commonly called the Euler characteristic of a chain complex.

**Lemma 5.** Suppose \( \{C_\#, \partial_\#\} \) is a chain complex such that each chain group is freely and finitely generated and only finitely many of the chain groups are nontrivial. Let \( f_j \) denote the rank of \( C_j \) for each \( j \in \mathbb{Z} \). Then

\[
\sum_{j \in \mathbb{Z}} (-1)^j f_j = \sum_{j \in \mathbb{Z}} (-1)^j \beta_j.
\]

2.1 Simplicial

Fix an integer \( n \geq 1 \). The set \( [n] := \{1, 2, \ldots, n\} \) will be our vertex set. For \( -1 \leq j \leq n-1 \), a \( j \)–dimensional abstract simplex, or \( j \)–face, is a subset of \( [n] \) with cardinality \( j+1 \). We denote the set of all \( j \)–faces on \( [n] \) by \( \left( \begin{array}{c} [n] \\ j+1 \end{array} \right) \). All faces will be oriented according to the usual ordering on \( [n] \). As such, we will be denoting elements of \( \left( \begin{array}{c} [n] \\ j+1 \end{array} \right) \) by \( \{\tau_0, \tau_1, \ldots, \tau_j\} \), where it is to be implicitly understood that \( 1 \leq \tau_0 < \tau_1 < \cdots < \tau_j \leq n \).

Let \( \partial = \partial^{[n]}_k \) be the matrix that’s rows and columns are indexed respectively by \( \left( \begin{array}{c} [n] \\ k \end{array} \right) \) and \( \left( \begin{array}{c} [n] \\ k+1 \end{array} \right) \), and for which, given \( \sigma \in \left( \begin{array}{c} [n] \\ k \end{array} \right) \) and \( \tau \in \left( \begin{array}{c} [n] \\ k+1 \end{array} \right) \), we set

\[
\partial(\sigma, \tau) := \begin{cases} (-1)^j, & \sigma = \tau \setminus \{\sigma_j\} \\ 0, & \text{otherwise} \end{cases}
\]

(1)

to account for our choice of orientation for each face. In particular then, \( \partial^{[n]}_0 \) is the \( \left( \begin{array}{c} [n] \\ 0 \end{array} \right) \times \left( \begin{array}{c} [n] \\ 1 \end{array} \right) \) all–ones matrix.

An abstract simplicial complex on \( [n] \) is a nonempty subset \( X \subseteq \bigcup_{j \geq -1} \left( \begin{array}{c} [n] \\ j+1 \end{array} \right) \) which, for every pair of subsets \( \sigma \subseteq \tau \), satisfies \( \tau \in X \implies \sigma \in X \). Let \( \mathcal{A}_n \) denote the set of all abstract simplicial complexes on \( [n] \). It is easily verified with this definition that \( \mathcal{A}_n \) is closed under intersection and union. We write \( X_j := \left( \begin{array}{c} [n] \\ j+1 \end{array} \right) \cap X \) and define the dimension of \( X \) to be \( \sup\{k \geq -1 : X_k \neq \emptyset\} \). If \( X \) has dimension \( k \), \( X \) is said to be pure if, for every \( -1 \leq j < k \) and every \( j \)–face \( \sigma \in X \), there exists a \( \tau \in X_k \) such that \( \sigma \subseteq \tau \).

It should be addressed that our definition of an abstract simplicial complex on \( [n] \) differs in two ways from the usual definition of an abstract simplicial complex with vertex set \( [n] \). Namely, we do not require that any of the \( 0 \)–faces in \( \left( \begin{array}{c} [n] \\ 1 \end{array} \right) \) be present in \( X \) for it to be considered an abstract simplicial complex on \( [n] \). However, we do implicitly require that \( \left( \begin{array}{c} [n] \\ 0 \end{array} \right) = \{\emptyset\} \) be considered as contained in \( X \) by default. This first difference is really just a slightly broadened idea of what constitutes the vertex set of an abstract simplicial complex; we could return to the usual definition by just being more specific about what vertices from \( [n] \) are present in whatever complex we are defining. The entire effect of the second difference is that the homology theory we will end up with is equivalent to reduced simplicial homology.

Given a matrix \( M \) with entries indexed over the set \( S \times T \), for \( A \subseteq S \) and \( B \subseteq T \), we write \( M_{A,B} \) to denote the submatrix of \( M \) with rows indexed by \( A \) and columns indexed by \( B \). As a point of clarification
Lemma 8. For a combinatorial map $C$, the top–dimensional homology group of a complex is always a free group, this condition is equivalent to having $C_j(X) = \emptyset$ for all $j > k$ and all $j < -1$—because $|X_{-1}| = |\{\emptyset\}| = 1$. We denote the $j$th homology group of this chain complex by $H_j(X)$. As we mentioned, it is equivalent to the $j$th reduced simplicial homology group, hence the notation. Note that, with this chain sequence, it makes sense to consider $\tilde{H}_{-1} = \text{Ker} \partial_{-1}^{[n]} / \text{Im} \partial_{0}^{[n]}$ as well, although we can see that this group is always trivial since $\partial_{0}^{[n]}$ is surjective.

Lemma 6. $\partial_{k}^{[n]} \partial_{k}^{[n]} + \partial_{k-1}^{[n]} \partial_{k-1}^{[n]} = n \text{Id}$.

Proof. This follows by explicit computation of matrix entries via (1), combined with the chain sequence identity $\partial_{k}^{[n]} \partial_{k}^{[n]} = 0$, which can also be seen to hold by explicit computation of matrix entries via (1). A more detailed explanation along these lines for the cases $k \geq 2$ can be found in the proof of Lemma 3 in [Kal83], and the case $k = 0$ is straightforward. For the case $k = 1$, $\partial_{1}^{[n]}$ is the combinatorial Laplacian of the complete graph on $[n]$. Hence $\partial_{1}^{[n]} \partial_{1}^{[n]} = n \text{Id} - J$, where $J$ is the $n \times n$ all–ones matrix. Noticing that $\partial_{0}^{[n]} \partial_{0}^{[n]} = J$ then completes the proof. 

2.2 Relative

Suppose we have a pair $(X, Y)$ of abstract simplicial complexes on $[n]$ with $Y \subseteq X$. We can define another chain complex $C_j(X, Y) := \mathbb{Z}^{X_j \setminus Y_j}$ with boundary maps $\partial_{X_j \setminus Y_j \setminus X_{j-1}}^{Y_j \setminus Y_{j-1}}$. The homology groups of this chain complex are called relative homology groups, denoted $H_j(X, Y)$. We can recover ordinary reduced simplicial homology from this by taking $Y$ to be either $\{\emptyset\}$ or any complex with a single 0–face and no larger faces. Moreover, since homology is homotopy invariant, we also have that $H_j(X, Y) \cong H_j(X)$ as long as $Y$ is contractible. For our purposes, we will only require the following sufficient condition for an abstract simplicial complex to be contractible: For $X$ a pure $k$–dimensional abstract simplicial complex, $X$ is contractible if $\bigcap_{\tau \in X_k} \tau \neq \emptyset$.

Theorem 7 (Excision Theorem). For any $A, B \in \mathcal{A}_n$, we have $H_*(A, A \cap B) \cong H_*(A \cup B, B)$.

3 $k$–complexes

For integers $0 \leq k \leq n$, let $\mathcal{C}_{n,k} \subset \mathcal{A}_n$ denote the set of $k$–dimensional abstract simplicial complexes on $[n]$ which contain $\binom{[n]}{k}$. For convenience, we consider $\emptyset_{n,k} := \bigcup_{j=0}^{k} \binom{[n]}{j}$ to be both the minimal element of $\mathcal{C}_{n,k}$ as well as the maximal element of $\mathcal{C}_{n,k-1}$, i.e. $\emptyset_{n,k} \cap \emptyset_{n,k-1} = \emptyset_{n,k-1}$. We call these $(n, k)$–complexes or just $k$–complexes when the vertex count is irrelevant or can be inferred. Note that $\emptyset_{n,k} := \bigcup_{n \geq k} \mathcal{C}_{n,k}$ is closed under the operations of union and intersection. In particular, if $X \in \mathcal{C}_{n,k}$ and $Y \in \mathcal{C}_{m,k}$ with $m \leq n$, we have $X \cap Y \in \mathcal{C}_{n,m,k}$ and $X \cup Y \in \mathcal{C}_{n,k}$. We define the complement of an $(n, k)$–complex, $X$, by $X^c := \left(\binom{[n]}{k+1} \setminus X_k\right) \cup \emptyset_{n,k}$. A $k$–complex, $F$, is said to be a $k$–forest if $H_k(F) = 0$. Because the top–dimensional homology group of a complex is always a free group, this condition is equivalent to having $\beta_k(F) = 0$. Let $\mathcal{F}_{n,k}$ denote the set of $k$–forests on $[n]$. An $(n, k)$–forest, $T$, with $|T| = \binom{n-1}{k}$ is said to be a $k$–tree. Let $\mathcal{T}_{n,k}$ denote the set of $k$–trees on $[n]$. These are Kalai’s $k$–dimensional $\mathbb{Q}$–acyclic complexes mentioned in the introduction. In the setting of $\mathcal{C}_{n,k}$, we can consider the topological operation of coning as a combinatorial map $\mathcal{C}_{n,k} \to \mathcal{F}_{n+1,k+1}$ defined by $\text{Cone}(n + 1, X) := X \cup \{\sigma \cup \{n + 1\} : \sigma \in X\}$.

Lemma 8. For $X \in \mathcal{C}_{n,k}$ and $Y \in \mathcal{C}_{n,k-1}$, any two of the following conditions imply the remaining one:

- $|X_k| + |Y_{k-1}| = \binom{n}{k}$,
- $H_k(X, Y) = 0$,
- $|H_{k-1}(X, Y)| < \infty$. 

4
Proof. By Lemma 5, we have \( \sum_{j=1}^{k} (-1)^j (|X_j| - |Y_j|) = \sum_{j=1}^{k} (-1)^j \beta_j(X, Y) \), which simplifies to

\[
|X_k| - \binom{n}{k} + |Y_{k-1}| = \beta_k(X, Y) - \beta_{k-1}(X, Y).
\]

Indeed, since \( X_j = Y_j \) for all \( j < k - 1 \), we have \( \mathcal{C}_j(X, Y) = 0 \) for all \( j < k - 1 \). Since \( H_j(X, Y) \) is a quotient of \( \mathcal{C}_j(X, Y) \), the former vanishes with the latter. Since \( \beta_k(X, Y) = 0 \iff \beta_{k-1}(X, Y) = 0 \iff |H_{k-1}(X, Y)| < \infty \), the result follows.

Whenever two (and therefore all three) of the conditions of this lemma hold, we will call \((X, Y)\) a relative \((n, k)\)-pair, and we will denote the set of such pairs by \( \mathcal{S}_{n,k}^{rel} \). Note that, by taking \( Y = \text{Cone}(n, \emptyset, n-1, k-1) \), we have \( X \in \mathcal{T}_{n,k} \) if and only if \((X, Y) \in \mathcal{S}_{n,k}^{rel} \), since \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k-1} \) and, due to \( Y \) being contractible, \( \bar{H}_*(X) \cong H_*(X, Y) \). We therefore have the following corollary which gives the usual three equivalent necessary and sufficient pairs of conditions for a \( k \)-complex to be a \( k \)-tree:

**Corollary 9.** For \( X \in \mathcal{C}_{n,k} \), any two of the following conditions imply the remaining one and, in particular, are equivalent to the statement \( X \in \mathcal{T}_{n,k} \):

- \( |X_k| = \binom{n-1}{k} \),
- \( \bar{H}_k(X) = 0 \),
- \( |\bar{H}_{k-1}(X)| < \infty \).

**Lemma 10.** Suppose \( B \in \mathcal{C}_{n,k} \) and \( A \in \mathcal{C}_{n,k-1} \) satisfy \( |B_k| = |A_{k-1}| \), and set \( D = A^c \) so that \( |B_k| + |D_{k-1}| = \binom{n}{k} \). Then \( \det \partial_{A_{k-1}, B_k} =: \det \partial_{A, B} \neq 0 \iff \det \partial_{A, B} = |H_{k-1}(B, D)| \iff (B, D) \in \mathcal{S}_{n,k}^{rel} \).

**Proof.** The relative chain complex for the pair \((B, D)\) is \( 0 \longrightarrow \mathcal{C}_k(B) \xrightarrow{\partial_{A, B}} \mathcal{C}_{k-1}(A) \longrightarrow 0 \longrightarrow \cdots \) since \( D_k = \emptyset \), \( B_{k-1} = A_{k-1} \cup D_{k-1} \), and \( B_{k-2} = D_{k-2} \). We therefore have \( H_{k-1}(B, D) = \mathbb{Z}^A/\partial_{A, B} \mathbb{Z}^B \), and so, provided \( \det \partial_{A, B} \neq 0 \), we have \( |H_{k-1}(B, D)| = |\det \partial_{A, B}| \) —this is seen most easily by putting \( \partial_{A, B} \) in Smith normal form. Having \( |H_{k-1}(B, D)| = |\det \partial_{A, B}| \) clearly implies \( |H_{k-1}(B, D)| < \infty \), which, by the previous lemma, implies that \((B, D) \in \mathcal{S}_{n,k}^{rel} \) which, itself, implies that \( H_k(B, D) = 0 \). So, since \( H_k(B, D) = \ker \partial_{A, B} \), having \((B, D) \in \mathcal{S}_{n,k}^{rel} \) implies that \( \partial_{A, B} \) is injective, and thus that \( \det \partial_{A, B} \neq 0 \).

Define the submatrix \( \bar{\partial} \) of \( \partial \) by deleting all rows of \( \partial \) that correspond to elements of \( \binom{[n]}{k} \) which contain the vertex \( n \) (thus \( \bar{\partial}_0^n = \partial_0^n \)).

**Corollary 11.** Suppose \( T \in \mathcal{C}_{n,k} \) satisfies \( |T_k| = \binom{n-1}{k} \). Then \( \det \bar{\partial}_T \neq 0 \iff |\bar{\partial}_T| = |\bar{H}_{k-1}(T)| \iff T \in \mathcal{T}_{n,k} \).

**Proof.** Let \( D = \text{Cone}(n, \emptyset, n-1, k-1) \) so that \( |T_k| + |D_{k-1}| = \binom{n}{k} \), and \( \partial_{D^c, T} = \bar{\partial}_T \). This now follows from Corollary 9 and Lemma 10.

This last corollary was originally proven for the cases \( k \geq 1 \) ([Kalai83], Lemma 2) by Kalai, who combined this with the Cauchy–Binet formula and some deft linear algebra to show ([Kalai83], Theorem 1) that

\[
\binom{n}{k} = \det \bar{\partial}^2 = \sum_{T \in \mathcal{T}_{n,k}} \det \bar{\partial}_T^2 = \sum_{T \in \mathcal{T}_{n,k}} |\bar{H}_{k-1}(C)|^2.
\]

With our understanding that \( \bar{\partial}_0^n = \partial_0^n \) and \( \bar{H}_{-1}(T) = 0 \), the case \( k = 0 \) can also be seen to hold. This gives us a probability measure \( \nu = \nu_{n,k} \) on \( \mathcal{T}_{n,k} \). Namely,

\[
\nu_{n,k}(T) := \frac{\det \bar{\partial}_T^2}{\det \bar{\partial}^4} = \frac{|\bar{H}_{k-1}(T)|^2}{\binom{n}{k}}.
\]
Probability measures defined in this manner are said to be _determinantal_, as are the random variables distributed according to such measures. The following lemma is a special case of Theorem 5.1 from [Lyons03]:

**Lemma 12.** Let \( R, S \) be finite sets, and let \( M \) be an \( R \times S \) matrix of rank \(|R|\). Let \( \mu \) be the determinantal measure on \( S \) which is defined by \( \mu(T) = \frac{\det M^2_{T}}{\det(MM^t)} \) for all \( T \subseteq S \) of size \(|R|\). Let \( P := M'(MM^t)^{-1}M \) (this is the matrix of the projection onto the rowspace of \( M \)). Then, for any \( B \subseteq S \),

\[
\mu(T : T \supseteq B) = \det P_{B,B} \quad \text{and} \quad \mu(T : T \subseteq S \setminus B) = \det(\text{Id} - P)_{B,B}.
\]

Determinantal random variables enjoy the negative associations property (Theorem 6.5 of [Lyons03]), which can be stated for random variables on \( \mathcal{C}_{n,k} \) as follows: A function \( f : \mathcal{C}_{n,k} \to \mathbb{R} \) is called _increasing_ if \( f(X) \leq f(X \cup Y) \) for every \( X, Y \in \mathcal{C}_{n,k} \). A random variable \( X \in \mathcal{C}_{n,k} \) is said to have negative associations if, for every pair of increasing functions \( f_1, f_2 \) and every \( Y \in \mathcal{C}_{n,k} \), we have

\[
\mathbb{E}[f_1(X \cap Y)f_2(X \cap Y^c)] \leq \mathbb{E}[f_1(X \cap Y)]\mathbb{E}[f_2(X \cap Y^c)].
\]

We will make use of negative associations when we go to prove our applications to random topology.

As we see from Lemma 12, for any \( B \in \mathcal{C}_{n,k} \) and \( A \in \mathcal{C}_{n,k-1} \), we have

\[
\nu_{n,k}(T : T \supseteq B) = \det(P_{n,k})_{B,B} \quad \text{and} \quad \nu_{n,k-1}(T : T \subseteq A^c) = \det(\text{Id} - P_{n,k-1})_{A,A}
\]

where \( P_{n,k} := \tilde{\partial}^t(\tilde{\partial}^t)^{-1} \tilde{\partial} \). Our use of \( \tilde{\partial} \) is actually a choice of basis for the rowspace of \( \partial \), and an arbitrary one at that. Let \( A \) be any \((n, k-1)\)-complex such that the rows of \( \partial_{A,\cdot} \) form a basis for the rowspace of \( \partial \)—this implies that \(|A_{k-1}| = \binom{n-1}{k-1} \) by Lemma 1 of [Kalai83]. Then, by applying a change of basis (see section 2 of [Lyons09] for more details), we also have the equivalent definition(s)

\[
\nu_{n,k}(T) := \frac{\det \partial_{A,T}^2}{\det(\partial \partial^t)_{A,A}}.
\]

all of which correspond to the same \( P_{n,k} \). Mészáros ([Meszaros21], Lemma 14) recently determined that

\[
P_{n,k} = \frac{1}{n} \tilde{\partial}_{k}^{[n]} \tilde{\partial}_{k}^{[n] t}.
\]

By Lemma 6, we also have \( \text{Id} - P_{n,k} = \frac{1}{n} \partial_{k}^{[n]} \partial_{k}^{[n] t} =: Q_{n,k} \).

**Corollary 13.** For any \( T \in \mathcal{T}_{n,k} \) and \( T' \in \mathcal{T}_{n,k-1} \), we have \(|H_{k-1}(T, T')| = |\hat{H}_{k-1}(T)||\hat{H}_{k-2}(T')|\). Moreover,

\[
\left\{(F, E) \in \mathcal{T}_{n,k}^{\text{rel}} : |F| = \binom{n-1}{k} \right\} = \mathcal{T}_{n,k} \times \mathcal{T}_{n,k-1}.
\]

**Proof.** Let \( A := T^c \) so that, as in expression (3), \(|A_{k-1}| = \binom{n-1}{k-1} \) and \(|A_{k-1}^c| = \binom{n-1}{k-1} \). Then, by Lemma 12, we have

\[
\nu_{n,k}(T) = \frac{\det \partial_{A,T}^2}{\det(\partial \partial^t)_{A,A}} = \frac{\det \partial_{A,T}^2}{n^{(n-1)}(n-1)_{\nu_{n,k-1}(\{S : S \subseteq A^c\})}} = \frac{\det \partial_{A,T}^2}{n^{(n-1)}(n-1)_{\nu_{n,k-1}(T')}}.
\]

Thus, by Lemma 10,

\[
\frac{|H_{k-1}(T, T')|^2}{n^{(n-1)}} = \frac{\det \partial_{A,T}^2}{n^{(n-1)}} = \nu_{n,k}(T)\nu_{n,k-1}(T') = \frac{|\hat{H}_{k-1}(T)|^2|\hat{H}_{k-2}(T')|^2}{n^{(n-1)+(n-2)}}.
\]

The set identity in the statement of the corollary now follows from Lemma 10, as the left hand side of this is finite if and only if the right hand side is finite, and \( T, T' \) are also the correct sizes for the statement to hold—i.e., \( \binom{n}{k} + \binom{n-1}{k-1} = \binom{n}{k} \).

\[\square\]
By Lemma 10 and Cauchy–Binet, we therefore have the following lemma:

**Lemma 14.** Suppose \( B \in \mathcal{C}_{n,k} \) and \( A \in \mathcal{C}_{n,k-1} \). Then

\[
\nu_{n,k-1}(T : T \subseteq A^c) = \det(Q_{n,k-1})_{A,A} = n^{-|A^c|} \sum_{B : (B,A^c) \in \mathcal{T}_{n,k}^{rel}} \det \partial^2_{A,B},
\]

and \( \nu_{n,k}(T : T \supseteq B) = \det(P_{n,k})_{B,B} = n^{-|B|} \sum_{A : (B,A^c) \in \mathcal{T}_{n,k}^{rel}} \det \partial^2_{A,B}. \)

For \( X \in \mathcal{C}_{n,k} \) we define

\[
\text{Proj}(n,X) := X \cap \bigcup_{j=0}^{k+1} \binom{n-1}{j} \in \mathcal{C}_{n-1,k}, \quad \text{and} \quad \nu(X) := \{ \sigma : \sigma \cup \{ n \} \in X \} \in \mathcal{C}_{n-1,k-1}.
\]

**Proposition 15.** Set \( T := F \cup \text{Cone}(n,E) \), where \( F \in \mathcal{C}_{n-1,k} \) and \( E \in \mathcal{C}_{n-1,k-1} \). Then \( T \in \mathcal{T}_{n,k} \) if and only if \( (F,E) \in \mathcal{T}_{n-1,k}^{rel} \). Either way, \( \tilde{H}_*(T) \cong H_*(F,E). \)

**Proof.** Since \( F \) has no \( k \)-faces containing \( n \) and \( E \subseteq F \), we have \( F \cap \text{Cone}(n,E) = E \). Thus, by excision, the pairs \( (T,\text{Cone}(n,E)) \) and \( (F,E) \) have the same relative homology in every dimension. But, since \( \text{Cone}(n,E) \) is contractible, \( \tilde{H}_*(T) \cong H_*(F,E) \). Therefore \( T \in \mathcal{T}_{n,k} \) if and only if \( (F,E) \in \mathcal{T}_{n-1,k}^{rel}. \)

This proposition implies a bijection between \( \mathcal{T}_{n-1,k}^{rel} \) and \( \mathcal{T}_{n,k} \). Indeed, we see the injection

\[
(F,E) \mapsto F \cup \text{Cone}(n,E)
\]

from the proposition, and we also have the inverse mapping

\[
T \mapsto (\text{Proj}(n,T), \text{Link}(n,T)).
\]

**Corollary 16.** Suppose \( T \in \mathcal{T}_{n,k} \), and let \( F = \text{Proj}(n,T) \) and \( E = \text{Link}(n,T) \). Then

\[
\nu_{n,k}(F \cup \text{Cone}(n,E)) = \frac{\det \partial^2_{E,F} n^{(n-k)^2}}{n^{(n-k)^2}}.
\]

**Proof.** This follows immediately from the previous proposition, the original definition \( \nu_{n,k}(T) = \frac{|\tilde{H}_{k-1}(T)|^2}{n^{(n-k)^2}} \), and Lemma 10.

**Corollary 17.** Suppose \( T \in \mathcal{T}_{n,k} \) is such that \( T' := \text{Proj}(n,T) \in \mathcal{T}_{n-1,k} \) and \( T'' := \text{Link}(n,T) \in \mathcal{T}_{n-1,k-1} \). Then \( \nu_{n,k}(T) = \nu_{n-1,k}(T') \nu_{n-1,k-1}(T'') (1-1/n)^{(n-k)^2} \).

**Proof.** By the previous corollary, Lemma 10, and Corollary 13,

\[
\nu_{n,k}(T) = \frac{|\tilde{H}_{k-1}(T',T'')|^2}{n^{(n-k)^2}} = \frac{|\tilde{H}_{k-1}(T')|^2 |\tilde{H}(T'')|^2}{(n-1)^{(n-k)^2}} \frac{|\tilde{H}(T')|^2}{(n-1)^{(n-k)^2}} \frac{(n-1)^{(n-k)^2} + (n-1)^{(n-k)^2}}{n^{(n-k)^2}}.
\]

**Lemma 18.** For all \( F \in \mathcal{C}_{n-1,k} \) and \( E \in \mathcal{C}_{n-1,k-1} \), we have

\[
\nu_{n,k}(T : \text{Proj}(n,T) = F) = \nu_{n-1,k}(T' : T' \supseteq F) (1-1/n)^{|F| n^{(n-k)^2}}
\]

and

\[
\nu_{n,k}(T : \text{Link}(n,T) = E) = \nu_{n-1,k-1}(T'' : T'' \subseteq E) (1-1/n)^{|E| n^{(n-k)^2}}.
\]
Proof. By Lemma 14, Corollary 16, and the Cauchy–Binet formula, we have
\[(n - 1)^{|E|} \nu_{n-1-k}(T : T \supseteq F) = \det(\partial^2 \partial_F F) = n^{(n-k)} \nu_{n,k}(T : \text{Proj}(n, T) = F)\]
and
\[(n - 1)^{|E|} \nu_{n-1,k-1}(T : T \subseteq E) = \det(\partial\partial^t_{E_c} E_c) = n^{(n-k)} \nu_{n,k}(T : \text{Link}(n, T) = E).\]

Corollary 19. Using the notation from Theorem 1, there are couplings \(\pi_{n,k}\) and \(\lambda_{n,k}\) of \(T_{n,k}, T_{n-1,k}, Y_{n-1,k}\) and \(T_{n,k}, T_{n-1,k-1}, Y_{n-1,k-1}\) respectively such that, marginally, \(T_{n-1,k}, T_{n-1,k-1}\) are independent of \(Y_{n-1,k}, Y_{n-1,k-1}\) respectively, and
\[\pi_{n,k}((T, T', Y') : \text{Proj}(T) = T' \setminus Y') = 1 = \lambda_{n,k}((T, T'', Y'') : \text{Link}(T) = T'' \cup Y'').\]

Namely,
\[\pi_{n,k}(T, T', Y') := \mu_{n-1,k}(Y') \nu_{n-1,k}(T') \frac{\nu_{n,k}(T) 1\{\text{Proj}(n, T) = T' \setminus Y'\}}{\nu_{n,k}(S : \text{Proj}(n, S) = T' \setminus Y')}\]
and
\[\lambda_{n,k}(T, T'', Y'') := \mu_{n-1,k-1}(Y'') \nu_{n-1,k-1}(T'') \frac{\nu_{n,k}(T) 1\{\text{Link}(n, T) = T'' \cup Y''\}}{\nu_{n,k}(S : \text{Link}(n, S) = T'' \cup Y'')}\]

Proof. It suffices to show that \(\pi_{n,k}\) has the correct marginal densities, as the proof for \(\lambda_{n,k}\) is basically identical. Summing over all \(T\) clearly produces the independent coupling of \(T_{n-1,k}, Y_{n-1,k}\). For the remaining marginal, Corollary 18 gives us that
\[\pi_{n,k}(T, T', Y') = \nu_{n,k}(T) \nu_{n-1,k}(T') \frac{1\{T' \supseteq \text{Proj}(n, T)\} \mu_{n-1,k}(Y') 1\{\text{Proj}(n, T) = T' \setminus Y'\}}{\nu_{n-1,k}(S : S \supseteq \text{Proj}(n, T)) \mu_{n-1,k}(S : T' \setminus S = \text{Proj}(n, T))},\]
which, summed over \(Y'\) and then \(T'\), gives the desired marginal.

Proof of Theorem 1. It suffices to show that
\[(T, T', T'', Y', Y'') \mapsto \frac{\pi_{n,k}(T, T', Y') \lambda_{n,k}(T, T'', Y'')}{\nu_{n,k}(T)}\]
is a probability density with the claimed marginal densities. Considering expression (4), summing (5) over \(Y'\) and \(T'\) gives \(\lambda_{n,k}(T, T'', Y'')\), which we know to have the desired marginal densities. One can deduce by a symmetric argument that the marginal densities with respect to \(T'\) and \(Y'\) are also correct.

4 Spectral Estimates

For this section, we use asymptotic notation, \(o()\) and \(O()\), to describe the behavior of a function of \(n\) as \(n \to \infty\). Let \(A\) be the adjacency matrix of a random graph \(G = G(n, p, E, M)\) on \([n]\) satisfying the following:

1. \(P[e \in G] = p \in (0, 1)\) for all \(e \in \binom{[n]}{2}\).

2. There is a fixed constant \(E \geq 1\) and an \(M = M(n) > 0\) such that, for every \(t \in [0, M]\), we have
\[E \exp \left(\sum_{1 \leq i < j \leq n} t_{ij} A_{ij}\right) \leq E \prod_{1 \leq i < j \leq n} (1 - p + p e^{t_{ij}}).\]

The choice to make \(E\) a fixed constant is just for convenience. All of the results of this section can be easily adapted to work for \(E = E(n)\) an arbitrary fixed polynomial in the variable \(n\).
Proposition 20. Let $G$ be defined as above with $p = \frac{\delta \log n}{n}$ (where $\delta$ is an arbitrary constant), and $M \geq \frac{\theta}{n} (n/p)^{\frac{3}{4}}$. Then, for any fixed $s > 0$, we have with probability at least $1 - o(n^{-s})$ that

$$v^tAv = O(\sqrt{n p})$$

for all unit vectors $v \perp 1$.

Proof. This will follow from Lemmas 24, 25, 26, and 27.

Corollary 21. Let $L := \text{Id} - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ where $D$ is the degree matrix of $G = G(n, \frac{\delta \log n}{n}, O(1), \frac{3\log n}{\sqrt{\delta \log n}})$. Suppose there are positive integers $s = O(1)$ and $m \geq \sqrt{\log n}$ such that $P[\min_{i \in [n]} \text{deg}_G(i) \geq m] = 1 - o(n^{-s})$. Let $\lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L)$ denote the eigenvalues of $L$. Then, with probability $1 - o(n^{-s})$, we have $\lambda_1(L) = 0$, and

$$\lambda_2(L) = 1 - O\left(\frac{\sqrt{\log n}}{m}\right).$$

Proof. Since $G$ is connected with sufficiently high probability, we will treat $G$ as though it were connected almost surely. As such, we know that $L$ has minimal eigenvalue 0 with multiplicity one, and the corresponding eigenvector is $D^{\frac{1}{2}}1$. Thus we are interested in bounding the quantity

$$\sup \left\{ \frac{y^tD^{-\frac{1}{2}}AD^{-\frac{1}{2}}y}{y^t y} : 0 \neq y \perp D^{\frac{1}{2}}1 \right\}$$

from above by some $\lambda := O(\sqrt{\log n/m})$. Equivalently, we would like to show that

$$x^tAx \leq \lambda x^tDx$$

for all $x \perp D1$.

Without loss of generality, we can assume that $x$ is a unit vector. Let

$$x = \cos \theta u + \sin \theta v \quad \text{where} \quad u = \frac{1}{\sqrt{n}}1, \quad v \perp 1, \quad \text{and} \quad |v| = 1.$$

Noting that $u^tAx = u^tDx = 0$ and $\cos \frac{\theta \text{tr}D}{\sqrt{n}} = -\sin \theta v^tD1$ (both of these follow from the assumption that $x \perp D1$), we have

$$x^tAx = \sin^2 \theta v^tAv - \cos^2 \theta \frac{\text{tr}D}{n} \quad \text{and} \quad x^tDx = \sin^2 \theta v^tDv - \cos^2 \theta \frac{\text{tr}D}{n}.$$

So we have

$$x^t(\lambda D - A)x = v^t(\lambda D - A)v \sin^2 \theta + \frac{(1 - \lambda) \text{tr}D}{n} \cos^2 \theta$$

$$= v^t(\lambda D - A)v \frac{(\text{tr}D)^2}{(\text{tr}D)^2 + n(v^tD1)^2} + \frac{(1 - \lambda) \text{tr}D}{n} \frac{n(v^tD1)^2}{(\text{tr}D)^2 + n(v^tD1)^2},$$

which we would like to show is positive. Solving for $v^tAv$, it suffices to show that

$$v^tAv \leq (1 - \lambda) \frac{(v^tD1)^2}{\text{tr}D} + \lambda v^tDv$$

for all unit vectors $v \perp 1$.

By our minimum degree assumption, it would even suffice to show that $v^tAv \leq \lambda m$. The result now follows from Proposition 20.

In order to prove Lemma 25, we are going to need a version of Bernstein’s inequality that works on weighted sums of centered edge indicators of $G$. 

9
Theorem 22 (Bernstein’s inequality). Let \( G \) be as above. Suppose \(|c_{ij}| \leq c\) for some fixed \( c \) and all \( i < j \), and that \( \varepsilon \geq 0 \) is such that \( M \geq \frac{\varepsilon}{p \sum_{i < j} c_{ij}^2 + 2c\varepsilon/3} \). Then, for any \( H \subseteq \binom{[n]}{2} \), we have

\[
\mathbb{P} \left[ \sum_{(i,j) \in H} c_{ij}(A_{ij} - p) \geq \varepsilon \right] \leq E \exp \left( -\frac{\varepsilon^2}{2p \sum_{(i,j) \in H} c_{ij}^2 + 2c\varepsilon/3} \right).
\]

Proof. By a Chernoff bound, the numerical bounds \( 1 + x \leq e^x \leq 1 + x + \frac{3x^2}{6 - 2x} \) for \( x \leq 3 \), and our assumptions about \( G \), we have for \( t \leq \frac{3}{4c} \) that

\[
\mathbb{P} \left[ \sum_{(i,j) \in H} c_{ij}(A_{ij} - p) \geq \varepsilon \right] \leq E \inf_{t \in (0, M]} \exp \left( -\varepsilon t + \mathbb{E} \sum_{(i,j) \in H} \frac{3t^2 c_{ij}^2 (A_{ij} - p)^2}{2 - 2ct} \right)
\]

where \( \sigma^2 := p \sum_{(i,j) \in H} c_{ij}^2 \). Taking \( t = \frac{\varepsilon}{\sigma^2 + 2c\varepsilon/3} \), we have \( 2 - 2ct/3 = \frac{2\sigma^2 + 2c\varepsilon/3}{\sigma^2 + 2c\varepsilon/3} \), and thus

\[
\frac{(\sigma^2 + 2c\varepsilon/3)^2 - 2\varepsilon t}{2 - 2ct/3} = \left(\frac{(\sigma^2 + 2c\varepsilon/3)t}{2 - 2ct/3}\right) - \frac{\varepsilon}{\sigma^2 + 2c\varepsilon/3} \sigma^2 + 2c\varepsilon/3 = -\frac{\varepsilon^2}{2\sigma^2 + 2c\varepsilon/3}.
\]

Recalling that we were required to assume that \( t \leq \frac{3}{4c} \), what we have shown holds for our choice of \( t \) as long as \( \varepsilon \geq -3\sigma^2 c^{-1} \).

We will also want to be able to apply a near–optimal Chernoff bound to uniformly weighted sums of edge indicators.

Lemma 23. For \( G \) as above with \( M \geq \log \frac{1-p\varepsilon}{1-p} \) where \( \varepsilon \geq 3 \), we have

\[
\mathbb{P} \left[ \sum_{(i,j) \in H} A_{ij} \geq \varepsilon p|H| \right] \leq E \exp \left( -\frac{\varepsilon \log \varepsilon}{3} p|H| \right).
\]

Proof. As with the previous proof,

\[
\mathbb{P} \left[ \sum_{(i,j) \in H} A_{ij} \geq \varepsilon p|H| \right] \leq E \inf_{t \in (0, M]} \exp \left( -\varepsilon p|H| \frac{1 - p}{1 - p\varepsilon} \right) \left( \frac{1 - p}{1 - p\varepsilon} \right)^{|H|}
\]

\[
= E \exp \left( -\varepsilon p|H| \frac{1}{1 - p\varepsilon} \int_{(1-p)(1-p\varepsilon)}^{|H|} \left( \frac{1 - p}{1 - p\varepsilon} \right)^{|H|} \right)
\]

\[
= E \exp \left( -\varepsilon p|H| \frac{1}{1 - p\varepsilon} \int_{(1-p)(1-p\varepsilon)}^{|H|} \left( \frac{1}{1 - p\varepsilon} \right)^{|H|} \right)
\]

For \( \varepsilon \geq 3 \), this is bounded above by \( E \exp \left( -\frac{\varepsilon \log \varepsilon}{3} p|H| \right) \).
To prove Proposition 20, we will adapt the Kahn–Szemerédi argument. Let
\[ S := \{ v \in \mathbb{R}^n : |v| = 1 \text{ and } v \perp 1 \} \quad \text{and} \quad T := \left\{ x \in \frac{1}{\sqrt{4n}} \mathbb{Z}^n : |x| \leq 1 \text{ and } x \perp 1 \right\}. \]
The following lemma is a special case of Claim 2.4 in [FO05].

Lemma 24. Suppose $|x^tAy| \leq c$ for all $x, y \in T$. Then $|x^tAv| \leq 4c$ for all $v \in S$.

We now write
\[ \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| = \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| + \sum_{(i,j) \in \mathcal{H}} |x_iA_{ij}y_j| \]
where $\mathcal{L} := \{ (i, j) \in [n]^2 : (x_iy_j)^2 \leq \frac{2}{n} \}$ and $\mathcal{H} := [n]^2 \setminus \mathcal{L}$.

4.1 Light Couples

Lemma 25. Suppose $M \geq \frac{3}{8}(n/p)^{\frac{1}{2}}$ in the definition of $G$. For any constant $s > 0$, we have
\[ P \left[ \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| \geq 7\sqrt{np} \text{ for some } x, y \in T \right] \leq E(18e^{-3}n) = o(n^{-s}). \]

Proof. We can bound the contribution from the light couples as follows: It is known ([FO05], Claim 2.9) that $|T| \leq 18^n$. So, by applying a union bound, we have
\[ P \left[ \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| \geq 7\sqrt{np} \text{ for some } x, y \in T \right] \leq 18^n \sup_{x, y \in T} P \left[ \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| \geq 7\sqrt{np} \right]. \]
Towards applying Bernstein’s inequality, define centered random variables
\[ B_{ij} := (|x_iy_j|1\{(i, j) \in \mathcal{L}\} + |x_jy_i|1\{(j, i) \in \mathcal{L}\}) (A_{ij} - p) \]
so that
\[ \sum_{\{i,j\} : (i,j) \in \mathcal{L} \text{ or } (j,i) \in \mathcal{L}} B_{ij} = \sum_{(i, j) \in \mathcal{L}} |x_iA_{ij}y_j| - E \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j|. \]
Note that each $B_{ij}^2 \leq 4\epsilon^2$ a.s., and $\sum_{i,j} E B_{ij}^2 \leq p \sum_{i,j} (x_iy_j)^2 + (x_jy_i)^2 \leq 2p$ by taking advantage of the fact that $|x_i|, |y_j| \leq 1$. Thus, by Bernstein’s inequality,
\[ P \left[ \sum_{\{i,j\} : (i,j) \in \mathcal{L} \text{ or } (j,i) \in \mathcal{L}} B_{ij} \geq 6\epsilon\sqrt{np} \right] \leq E \exp \left( -\frac{(6\epsilon^2 np}{4p + 6\epsilon \sqrt{p} \sqrt{np} \epsilon} \right) = E \exp \left( -\frac{9\epsilon^2}{1 + 2\epsilon^2} n \right). \]
Taking $\epsilon = 1$, this shows that $\sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| \leq 6\sqrt{np} + E \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j|$ with probability at least $1 - Ec^{-3n}$. By Lemma 2.6 of [FO05], we also have $E \sum_{(i,j) \in \mathcal{L}} |x_iA_{ij}y_j| \leq \sqrt{np}$, thus giving us the desired bound. Our use of Bernstein’s inequality requires us to have $M \geq \frac{6\epsilon^2}{2p + \epsilon \sqrt{np}} = \frac{3}{8}(n/p)^{\frac{1}{2}}$. \(\square\)

4.2 Heavy Couples

For $B, C \subseteq [n]$, let $c(B, C) := |\{(i, j) \in G : i \in B, j \in C\}|$ and $\mu(B, C) := p|B||C|$. The following is a weakened form of Lemma 9.1 in [HKP19].

Lemma 26. Suppose we have constants $c_0, c_1, c_2 > 1$ and a graph $G$ on $[n]$ with $\max_{i \in [n]} \deg_G(i) \leq c_0np$, and, for all $B, C \subseteq [n]$, one or more of the following hold:
• $e(B, C) \leq c_1 \mu(B, C)$

• $e(B, C) \log \frac{e(B, C)}{\mu(B, C)} \leq c_2 (|B| \lor |C|) \log \frac{n}{|B| \lor |C|}$

Then $\sum_{(i, j) \in \mathcal{I}} |x_i A_{ij} y_j| = O(\sqrt{n \delta})$.

**Lemma 27.** Suppose that $p = \frac{\delta \log n}{n}$ for some fixed but arbitrary $\delta > 0$, and $M \geq \frac{3}{2} (n/p)^{\frac{1}{2}}$. For any $s > 0$, there are fixed constants $c_0, c_1, c_2 > 1$ so that the conditions of Lemma 26 hold for $G$ with probability at least $1 - o(n^{-s})$.

In terms of probabilistic bounds, the proof of this will only rely on Lemma 23. We note then that the condition $M \geq \frac{3}{2} (n/p)^{\frac{1}{2}}$ is overkill since Lemma 23 only ever requires that we have $M$ be greater than a fixed constant.

**Proof.** Assume without loss of generality that $c_0 \geq 3$. First note that, due to the monotonicity of $x \log x$ for $x \geq 1$, the second bulleted condition is equivalent to the statement $e(B, C) \leq r_1 \mu(B, C)$ where $r_1 \geq 1$ solves $r_1 \log r_1 = \frac{c_2 (|B| \lor |C|) \log n}{\mu(B, C)}$. So we can rewrite the two bulleted conditions as the single condition $e(B, C) \leq r_1 \mu(B, C)$ where $r := r_1 \lor c_1$. By Lemma 23 and a union bound, we have

$$\mathbb{P}[\max_{i \in [n]} \deg_{\mathcal{G}}(i) > c_0 n p] \leq n \mathbb{P}[\deg_{\mathcal{G}}(n) \geq c_0 n p] \leq Ex \exp \left(-\frac{\delta c_0 \log c_0}{3} \log n \right)$$

and

$$\mathbb{P}[e(B, C) > r_1 \mu(B, C)] \leq \mathbb{P}[e(B, C) > r_1 \mu(B, C)] \leq E \exp \left(-\frac{c_2 (|B| \lor |C|) \log n}{3} \mu(B, C) \right).$$

Thus we can choose a constant $c_0$ large enough to make $\mathbb{P}[\max_{i \in [n]} \deg_{\mathcal{G}}(i) > c_0 n p] = o(n^{-s})$. Using this fact and the resulting (high probability) inequality

$$\frac{e(B, C)}{\mu(B, C)} \leq \frac{c_0 (|B| \lor |C|) n p}{\mu(B, C)} = \frac{c_0 n}{|B| \lor |C|},$$

we have, in the case $|B| \lor |C| \geq n/e$, that

$$\mathbb{P}[\exists B, C \text{ s.t. } e(B, C) > c_0 n p(B, C) \text{ and } |B| \lor |C| \geq n/e] \leq \mathbb{P}[\max_{i \in [n]} \deg_{\mathcal{G}}(i) > c_0 n p] = o(n^{-s}).$$

So suppose that $|B| \lor |C| < n/e$. By a union bound over all possible pairs $\{i, j\} \subset [\lfloor n/e \rfloor]$ ($i \leq j$, without loss of generality) of sizes for the sets $B, C$, it suffices to show that

$$\binom{n}{i} \binom{n}{j} \exp \left(-\frac{c_2 j \log \frac{n}{j}}{3} \right) \leq n^{-s-3}.$$

Recalling that $\binom{n}{j} \leq \left(\frac{en}{j^2} \right)^j$ for all $j \in [n]$, this can be done by showing that

$$(s + 3) \log n + j \left(1 + \log \frac{n}{j} \right) + i \left(1 + \log \frac{n}{i} \right) \leq \frac{c_2 j \log \frac{n}{j}}{3}$$

for all $1 \leq i \leq j < n/e$. Indeed, since $j \log \frac{n}{j}$ is monotone increasing for $1 \leq j < n/e$ (Lemma 2.12 of [FO05]), we have

$$(s + 3) \log n + j \left(1 + \log \frac{n}{j} \right) + i \left(1 + \log \frac{n}{i} \right) \leq (s + 3) \log n + 4j \log \frac{n}{j} \leq (s + 7) j \log \frac{n}{j}.$$

Taking $c_2 \geq 3s + 21$ therefore gives the desired bound. \qed
4.3 Link unions

Let $T_1, T_2, \ldots, T_m$ be jointly independent copies of $T_{n,k-1}$. Then, if $T'_1, T'_2, \ldots, T'_m$ are jointly independent copies of $T_{n+1,k}$, we can use Theorem 1 to couple these random trees so that

$$L_{m-1}^k(n,p) := \text{Link} \left( n+1, \bigcup_{i \in [m]} T'_i \right) = \bigcup_{i \in [m]} \text{Link} \left( n+1, T'_i \right) = Y_{k-1}(n,p) \cup \bigcup_{i \in [m]} T_i,$$

where $p := 1 - \left( 1 - \frac{1}{n+1} \right)^m \sim \frac{m}{n}$. Since each $T_i$ is determinantal, the set of $k$-faces in each $T_i$ are negatively associated (to be abbreviated NA). Moreover, the set of $k$-faces in $T := \bigcup_{i \in [m]} T_i$ are also NA, as

$$1 \{ f \in T \} = 1 - \prod_{i \in [m]} (1 - 1 \{ f \in T_i \})$$

is increasing in $T$ as a function $\mathcal{C}_{n,k} \to \mathbb{R}$, and any set of increasing functions defined on disjoint subsets of an NA set is itself an NA set [JDP83]. By the same reasoning and the fact that sets of jointly independent random variables are NA sets, we also have that the set of $k$-faces of $L_m^k(n,p)$ is NA. We now narrow our focus to the case $k = 1$, in which

$$G := L_1^1(n,p)$$

is a graph with negatively associated edges each appearing with probability $q := 1 - (1 - p)(1 - \frac{2}{m})^m \sim \frac{2m}{n}$. This implies that $G$ is of type $\mathcal{G}(n,q,1,\infty)$. Toward applying Corollary 21 to this $G$, we will assume that $m = \lfloor \delta \log n \rfloor$ for some arbitrary constant $\delta > 0$. In order to get a sufficiently strong lower bound on the minimum degree of $G$, we need to have a very strong bound on the moment generating function of $\text{deg}_G(n)$ for negative inputs.

**Lemma 28.** Let $p_2 := 1 - \left( \frac{n-1}{n} \right)^m \sim \frac{2m}{n}$. Then, for all $t < 0$,

$$\mathbb{E} \left[ \exp \left( t \text{deg}_G(n) \right) \right] \leq e^{nt} \left( 1 + p_2(e^t - 1) \right)^{n-1-m} \exp \left( \frac{m^2 e^{1-t}}{2n} \right).$$

**Proof.** Let $\ell := \text{Link}(n, \bigcup_{i \in [m]} T_i)$, and let $B_i := 1 \{ (i, n) \in \mathcal{G}(n,p) \}$, where $\mathcal{G}(n,p)$ is as above and, in particular, independent of $\ell$. Then

$$\mathbb{E} \left[ \exp \left( t \text{deg}_G(n) \right) | \ell \right] = \prod_{i \in [n-1]} \mathbb{E} \left[ \exp \left( t \left( 1 \{ i \in \ell \} + 1 \{ i \notin \ell \} B_i \right) \right) \right] | \ell \right]$$

$$= \prod_{i \in [n-1]} \left( (1 - p) \exp \left( t 1 \{ i \in \ell \} \right) + pe^t \right).$$

Note that $\ell \not\sim \text{Binom} \left( [n-1], p' \right) \cup R$ where $p' := 1 - (1 - \frac{1}{m})^m = p + O(m/n^2)$, and $R$ is a random subset of $[n-1]$ formed by independently and uniformly choosing (with replacement) $m$ vertices from $[n-1]$—this follows from the $k = 1$ case of Theorem 1. Thus

$$\mathbb{E} \left[ \exp \left( t \text{deg}_G(n) \right) | R \right] = \prod_{i \in [n-1]} \mathbb{E} \left[ (1 - p) \exp \left( t 1 \{ i \in \ell \} \right) + pe^t | R \right].$$

Now note that

$$\mathbb{E} \left[ \exp \left( t 1 \{ i \in \ell \} \right) | R \right] = \mathbb{E} \left[ \exp \left( t (1 \{ i \in R \} + 1 \{ i \notin R \} \text{Bernoulli}(p')) \right) \right] | R \right]$$

$$= \exp \left( t 1 \{ i \in R \} \right) \left( 1 - p' + p' \exp \left( t 1 \{ i \notin R \} \right) \right)$$

$$= (1 - p') \exp \left( t 1 \{ i \in R \} \right) + p' e^t.$$
Thus
\[
\mathbb{E} \left[ \exp(t \deg_G(n)) \mid R \right] = \prod_{i \in [n-1]} \left(1 - p \right) \left((1 - p') \exp(t \{i \in R\}) + p'e^t\right) + pe^t
\]
\[
= \prod_{i \in [n-1]} \left((1 - p_2) \exp(t \{i \in R\}) + p_2e^t\right). 
\]
We therefore have
\[
\mathbb{E} \left[ \exp(t \deg_G(n)) \right] = \sum_{j \in [m]} e^{jt} \left(1 - p_2 + p_2e^t\right)^{n-1-j} \mathbb{P} \left[ |R| = j \right]
\]
\[
= e^{mt} \left(1 - p_2 + p_2e^t\right)^{\frac{n-1-m}{2}} \sum_{j \in [m]} e^{(j-m)t} \left(1 - p_2 + p_2e^t\right)^{m-j} \mathbb{P} \left[ |R| = j \right].
\]
Since \( t < 0 \), we have \( 1 - p_2 + p_2e^t \leq 1 \), and thus
\[
\mathbb{E} \left[ \exp(t \deg_G(n)) \right] \leq e^{mt} \left(1 - p_2 + p_2e^t\right)^{\frac{n-1-m}{2}} \sum_{j \in [m]} e^{-t(m-j)} \mathbb{P} \left[ |R| = j \right].
\]
Toward controlling the size of \( R \), let \( R = R_m \), and let \( R_j \) be defined as \( R \) after only the first \( j \) independent samplings of \([n-1]\). Then, letting \( v_1, v_2, \ldots, v_m \) be the random vertex selections, we have
\[
|R_{j+1}| = |R_j| + \mathbf{1}\{v_{j+1} \notin R_j\}.
\]
Thus, for \( s > 0 \),
\[
\mathbb{E} \left[ \exp(-s|R_{j+1}|) \mid R_j \right] = \exp(-s|R_j|) \left( \frac{|R_j|}{n-1} + \frac{n-1-|R_j|}{n-1}e^{-s} \right)
\]
\[
\leq \exp(-s|R_j|) \left( \frac{j}{n-1} + \frac{n-1-j}{n-1}e^{-s} \right) = \exp(-s|R_j| - s) \left(1 + \frac{j(e^s - 1)}{n-1} \right).
\]
Taking expectations and iterating then gives
\[
\mathbb{E} \exp(-s|R_m|) \leq e^{-ms} \prod_{j=1}^{m-1} \left(1 + \frac{j(e^s - 1)}{n-1} \right).
\]
Thus, by Chernoff’s inequality, we have
\[
\mathbb{P} \left[ |R_m| \leq m - k \right] \leq e^{-ks} \prod_{j=1}^{m-1} \left(1 + \frac{j(e^s - 1)}{n-1} \right) \leq \exp \left(-ks + \frac{e^s - 1}{n-1} \left(\frac{m}{2} \right) \right)
\]
for any \( s > 0 \). Taking \( s = \log \frac{2k(n-1)}{m(m-1)} \), gives
\[
\mathbb{P} \left[ |R_m| \leq m - k \right] \leq \exp \left(k - \frac{m(m-1)}{2(n-1)} \right) \left(\frac{m(m-1)}{2k(n-1)} \right)^{k} \leq \left(\frac{em^2}{2kn} \right)^{k}.
\]
Thus \( |R| \geq m - k + 1 \) with probability at least \( 1 - \left(\frac{em^2}{2kn} \right)^{k} \). Recall the summation by parts formula:
\[
\sum_{j \in [m]} f_jg_j = f_m \sum_{j \in [m]} g_j - \sum_{j \in [m]} (f_j - f_{j-1}) \sum_{i \in [j-1]} g_i.
\]
Setting $f_j = e^{-t(m-j)}$ and $g_j = \mathbb{P}[|R| = j]$, we have

$$
\sum_{j \in [m]} e^{-t(m-j)} \mathbb{P}[|R| = j] = 1 + (e^{-t} - 1) \sum_{j \in [m]} \mathbb{P}[|R| < j] e^{-t(m-j)}
$$

$$
\leq 1 + (e^{-t} - 1) \sum_{j \in [m]} \left( \frac{e^{m^2}}{2(m-j+1)n} \right)^{m-j+1} e^{-t(m-j)}
$$

$$
= 1 + (1 - e^t) \sum_{j \in [m]} \left( \frac{e^{1-t}m^2}{2jn} \right)^j
$$

$$
\leq 1 + (1 - e^t) \left( \exp \left( \frac{m^2e^{1-t}}{2n} \right) - 1 \right),
$$

where the last inequality uses the bound $j^j \geq j!$. Thus, for $t < 0$, we have

$$
\mathbb{E} [\exp (t \deg_G(n))] \leq e^{mt} \left( 1 - p_2 + p_2 e^t \right)^{n-1-m} \sum_{j \in [m]} e^{-t(m-j)} \mathbb{P}[|R| = j]
$$

$$
\leq e^{mt} \left( 1 + p_2(e^t - 1) \right)^{n-1-m} \left( 1 + (1 - e^t) \left( \exp \left( \frac{m^2e^{1-t}}{2n} \right) - 1 \right) \right)
$$

$$
\leq e^{mt} \left( 1 + p_2(e^t - 1) \right)^{n-1-m} \exp \left( \frac{m^2e^{1-t}}{2n} \right).
$$

\[\square\]

**Lemma 29.** For $0 \leq j \leq m$, we have $\mathbb{P}[\min_{i \in [n]} \deg_G(i) \leq m - j] \leq (1 + o(1))n^{1-j}e^{-2m}$.

**Proof.** For any $t > 0$, we have by a union bound that

$$
\mathbb{P}[\min_{i \in [n]} \deg_G(i) \leq m - j] \leq n \mathbb{P}[\exp (-t \deg_G(n)) \geq e^{(j-m)t}]
$$

$$
\leq ne^{-jt} \left( 1 + p_2(e^{-t} - 1) \right)^{n-1-m} \exp \left( \frac{m^2e^{1+t}}{2n} \right)
$$

$$
\leq \exp \left( \log n - jt + (n-1-m)p_2(e^{-t} - 1) + \frac{m^2e^{1+t}}{2n} \right).
$$

Letting $t = r \log n$ for any $r \in (0, 1)$ gives

$$
\mathbb{P}[\min_{i \in [n]} \deg_G(i) \leq m - j] \leq \exp \left( (1 - jr) \log n - (n-1-m)p_2(1-n^{-r}) + \frac{em^2}{2n^{1-r}} \right)
$$

$$
\leq (1 + o(1)) \exp \left( (1 - jr) \log n - 2m \right).
$$

Taking $r \to 1$ from the left gives $(1 + o(1))n^{1-j}e^{-2m}$.

\[\square\]

**Lemma 30.** Let $L$ be the reduced Laplacian of $G$ as defined above with $m = [\delta \log n]$ and $\delta > 0$ an arbitrary constant (so that $\mathbb{P}[\min_{i \in [n]} \deg_G(i) \geq m - j + 2] = 1 - o(n^{-2})$ for any fixed $j$). Then, for any fixed $s$, we have $\lambda_1(L) = 0$ and $\lambda_2(L) = 1 - O\left(\frac{1}{\sqrt{\log n}}\right)$ with probability $1 - o(n^{-s})$.

**Proof.** This follows from the previous lemma and Corollary 21.

\[\square\]
References

[Adi92] Ron M. Adin. “Counting colorful multi-dimensional trees”. In: Combinatorica 12 (1992), pp. 247–260.

[COL09] Amin Coja-Oghlan and André Lanka. “The spectral gap of random graphs with given expected degrees”. In: the electronic journal of combinatorics (2009), R138–R138.

[DKM08] Art Duval, Caroline Klivans, and Jeremy Martin. “Simplicial Matrix-Tree Theorems”. In: Transactions of the American Mathematical Society 361 (Feb. 2008). doi: 10.1090/S0002-9947-09-04898-3.

[FO05] Uriel Feige and Eran Ofek. “Spectral techniques applied to sparse random graphs”. In: Random Structures & Algorithms 27.2 (2005), pp. 251–275. doi: https://doi.org/10.1002/rsa.20089.

[Gar73] Howard Garland. “p-Adic Curvature and the Cohomology of Discrete Subgroups of p-Adic Groups”. In: Annals of Mathematics 97 (1973), p. 375.

[GW14] Anna Gundert and Uli Wagner. “On Eigenvalues of Random Complexes”. In: Israel Journal of Mathematics 216 (Nov. 2014). doi: 10.1007/s11856-016-1419-1.

[HKP19] Christopher Hoffman, Matthew Kahle, and Elliot Paquette. “Spectral Gaps of Random Graphs and Applications”. In: International Mathematics Research Notices 2021.11 (May 2019), pp. 8353–8404. DOI: 10.1093/imrn/rnz077.

[JDP83] Kumar Joag-Dev and Frank Proschan. “Negative Association of Random Variables with Applications”. In: The Annals of Statistics 11.1 (1983), pp. 286–295. doi: 10.1214/aos/1176346079.

[Kal83] Gil Kalai. “Enumeration of Q-acyclic simplicial complexes”. In: Israel Journal of Mathematics 45 (1983), pp. 337–351.

[KN20] Matthew Kahle and Andrew Newman. Topology and geometry of random 2-dimensional hypertrees. 2020.

[LM06] Nathan Linial and Roy Meshulam. “Homological Connectivity Of Random 2-Complexes”. In: Combinatorica 26 (2006), pp. 475–487.

[LP19] Nati Linial and Yuval Peled. “Enumeration and randomized constructions of hypertrees”. In: Random Structures & Algorithms 55.3 (2019), pp. 677–695. DOI: https://doi.org/10.1002/rsa.20841.

[Lub18] Alexander Lubotzky. “High dimensional expanders”. In: Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018. World Scientific. 2018, pp. 705–730.

[Lyo03] Russell Lyons. “Determinantal probability measures”. eng. In: Publications Mathématiques de l'IHÉS 98 (2003), pp. 167–212.

[Lyo09] Russell Lyons. “Random complexes and ℓ²-Betti numbers”. In: Journal of Topology and Analysis 01.02 (2009), pp. 153–175. DOI: 10.1142/S1793525309000072.

[MW09] Roy Meshulam and N. Wallach. “Homological connectivity of random k-dimensional complexes”. In: Random Structures & Algorithms 34 (2009).

[Mé21] András Mészáros. The local weak limit of k-dimensional hypertrees. Jan. 2021. DOI: 10.1090/tran/8711.

[Oli09] Roberto Imbuzeiro Oliveira. “Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges”. In: arXiv preprint arXiv:0911.0600 (2009).

[Opp18] Izhar Oppenheim. “Local spectral expansion approach to high dimensional expanders part I: Descent of spectral gaps”. In: Discrete & Computational Geometry 59.2 (2018), pp. 293–330.

[Opp20] Izhar Oppenheim. “Local spectral expansion approach to high dimensional expanders part II: Mixing and geometrical overlapping”. In: Discrete & Computational Geometry 64.3 (2020), pp. 1023–1066.

[TY19] Konstantin Tikhomirov and Pierre Youssef. “The spectral gap of dense random regular graphs”. In: The Annals of Probability 47.1 (2019), pp. 362–419. DOI: 10.1214/18-AOP1263.
[Ż03] Andrzej Źuk. “Property (T) and Kazhdan constants for discrete groups”. In: Geometric And Functional Analysis 13 (June 2003), pp. 643–670. DOI: 10.1007/s00039-003-0425-8.

[Ż96] Andrzej Źuk. “La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres”. In: Comptes rendus de l’Académie des sciences. Série 1, Mathématique 323.5 (1996), pp. 453–458.