Towards NNLO corrections to muon-electron scattering

RADCOR-LoopFest 2021

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based on:  *JHEP* 11 (2020) 028

May 17-21, 2021

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Outline

- Motivation
- MUonE
- NNLO Photonic corrections
- NNLO Leptonic corrections
- MIs for Leptonic corrections on Vertices
- Outlook
Motivation
$a_\mu^{\text{EXP}} = (116592089 \pm 63) \times 10^{-11} [0.54\text{ppm}]$ BNL E821

$a_\mu^{\text{EXP}} = (116592040 \pm 54) \times 10^{-11} [0.46\text{ppm}]$ FNAL E989 Run 1

$a_\mu^{\text{EXP}} = (116592061 \pm 41) \times 10^{-11} [0.35\text{ppm}]$ WA

- **FNAL** aims at $16 \times 10^{-11}$. First 3 runs completed, 4th in progress.
- **Muon g-2 proposal at J-PARC**: Phase-1 with ~ BNL precision.

WP20 = White Paper of the Muon g-2 Theory Initiative: arXiv:2006.04822

See M. Passera’s talk
\[ a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HVP, LO}} + a_\mu^{\text{HVP, NLO}} + a_\mu^{\text{HVP, NNLO}} + a_\mu^{\text{HLbL}} + a_\mu^{\text{HLbL, NLO}} \]
\[ = 116591810(43) \times 10^{-11}. \]

[WP20]

Types of corrections that goes inside SM prediction
$a_\mu^{\text{QED}}(\alpha(\text{Cs})) = 116\ 584\ 718.931(104) \times 10^{-11}$.

$a_\mu^{\text{EW}} = 153.6(1.0) \times 10^{-11}$

$a_\mu^{\text{HVP, LO}} = 6931(40) \times 10^{-11}$

$a_\mu^{\text{HVP, NLO}} = -98.3(7) \times 10^{-11}$

$a_\mu^{\text{HVP, NNLO}} = 12.4(1) \times 10^{-11}$

$a_\mu^{\text{HLbL}(\text{phenomenology + lattice QCD})} = 90(17) \times 10^{-11}$

$a_\mu^{\text{HLbL, NLO}} = 2(1) \times 10^{-11}$

[WP20]

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$a_{\mu}^{\text{HVP, NNLO}} = 12.4(1) \times 10^{-11}$

$a_{\mu}^{\text{HLLbL (phenomenology + lattice QCD)}} = 90(17) \times 10^{-11}$

$a_{\mu}^{\text{HLLbL, NLO}} = 2(1) \times 10^{-11}$

Types of corrections that goes inside SM prediction
[Borsányi et. al. (BMWC), Nature 2021]
In the following, focus on $a_{\mu}^{\text{HLO}}$, which contributes (with $a_{\mu}^{\text{HLbL}}$) to the SM uncertainty.

\[ a_{\mu}^{\text{HLO}} = \frac{1}{4\pi^3} \int_0^\infty ds K(s) \frac{\sigma_0}{s^2} = \frac{\alpha m_\mu}{3\pi} \int_0^\infty ds K(s) \frac{\sigma_0}{s^2} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_0^\infty ds K(s) \frac{\sigma_0}{s^2}. \]

Using dispersion relations and the Optical Theorem,

\[ a_{\mu}^{\text{HLO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left[ \int_0^{E_{\text{cut}}} ds \frac{K(s)R_{\text{data}}(s)}{s^2} + \int_0^\infty ds \frac{K(s)R_{\text{QCD}}(s)}{s^2} \right] \]

\[ K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)\frac{m_\mu^2}{s}} \sim \frac{1}{s}, \quad R_{\text{QCD}}(s) = \frac{\sigma_0}{4\pi^2} \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \frac{1}{s}. \]

Alternatively (exchanging $s$ and $x$ integrations in $a_{\mu}^{\text{HLO}}$)

\[ a_{\mu}^{\text{HLO}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \Delta \alpha_{\text{had}}[t(x)] \]

\[ t(x) = \frac{x^2 m_\mu^2}{x - 1}, \quad \Delta \alpha_{\text{had}}[t(x)] = 0. \]

Essentially the same formula used in lattice QCD calculation of $a_{\mu}^{\text{HLO}}$.

- $\Delta \alpha_{\text{had}}(t)$ (and $a_{\mu}^{\text{HLO}}$) can be directly measured in a (single) experiment involving a space-like scattering process.

- Still a data-driven evaluation of $a_{\mu}^{\text{HLO}}$, but with space-like data.

From Carlo’s talk, INFN Roma Tre, 2019.
Abbiendi, Carloni Calame, Marconi, Matteuzzi, Montagna, Nicrosini, MP, Piccinini, Tenchini, Trentadue, Venanzoni
EPJC 2017 - arXiv:1609.08987
\[ \Delta \alpha_{had}(t) \] can be measured from elastic $\mu e \rightarrow \mu e$ scattering.

- $150$ GeV muon beam on a fixed electron target.
- Each module consists of a low-Z target (Berillium) and two silicon tracking stations located at a distance of one meter.

- Systematic effects must be known at $\leq 10$ ppm
- Test run approved for 2021.
- Hopefully full run from 2022-24.
The ratio of the SM cross section in the signal and the normalization region must be known at $\leq 10$ ppm

M. Passera’s talk
NNLO Photonic Corrections

Published in --
*JHEP* 11 (2020) 028
Sample topologies for NNLO QED corrections on electron/muon line

[Carlo Calame et. Al. ‘20]
Sample topologies for NNLO photonic corrections on electron/muon line

Nf=1 subset is removed

Exact calculation

Two loop Formfactors are taken from Mastrolia et. al. arXiv:hep-ph/0302162

[Carlo Calame et. Al. ‘20]
Sample topologies for one loop boxes

All relevant one loop boxes and pentagons are calculated exactly.
Sample topologies for NNLO photonic corrections to box like structure

Not known exactly with full mass effect. Recent progress in calculation of MIs keeping the muon mass and neglecting electron mass

[Massification]

Yennie-Frautschi-Suura (YFS) approximation used including full mass dependence
NNLO photonic corrections

\[ \mathcal{M}^{\alpha^0} = \mathcal{T} \]

\[ \widetilde{\mathcal{M}}^{\alpha^2} = \mathcal{M}_e^{\alpha^2} + \mathcal{M}_\mu^{\alpha^2} + \mathcal{M}_{e\mu}^{\alpha^2} + \frac{1}{2} Y_{e\mu}^2 \mathcal{T} + Y_{e\mu} (Y_e + Y_\mu) \mathcal{T} + (Y_e + Y_\mu) \mathcal{M}_{e\mu}^{\alpha^1,R} + Y_{e\mu} \mathcal{M}_{e\mu}^{\alpha^1,R}. \]

\[ Y = \sum_{i,j=1,4}^{j \geq i} Y_{ij} = Y_e + Y_\mu + Y_{e\mu} \]

\[ Y_{ij} = \begin{cases} \frac{1}{8\pi} Q_i^2 \left[ B_0 \left( 0, m_i^2, \frac{m_i^2}{2} \right) - 4m_i^2 C_0 \left( m_i^2, 0, \frac{m_i^2}{2}, \lambda^2, m_i^2, \frac{m_i^2}{2} \right) \right] & \text{for } i = j \\ \frac{\alpha}{\pi} Q_i Q_j \vartheta_i \vartheta_j \left[ p_i \cdot p_j C_0 \left( m_i^2, (\vartheta_i p_i + \vartheta_j p_j)^2, m_j^2, \lambda^2, m_i^2, m_j^2 \right) + \frac{1}{4} B_0 \left( (\vartheta_i p_i + \vartheta_j p_j)^2, m_i^2, m_j^2 \right) \right] & \text{for } i \neq j \end{cases} \]

\[ Y_e = Y_{24} + Y_{22} + Y_{44} \]

\[ Y_\mu = Y_{13} + Y_{11} + Y_{33} \]

\[ Y_{e\mu} = Y_{12} + Y_{14} + Y_{23} + Y_{34} \]

[Carlo Calame et. Al. ’20]
Phenomenological results are obtained by using fully differential MC code, MESMER. Structure of the code is completely general. YFS can be replaced by exact calculation. We adopt the typical running condition of the MUonE experiment. Energy of the incoming muon beam is taken to be 150 GeV.

The electron is assumed to be in rest inside a bulk target and thus $\sqrt{s} \approx 0.405541$ GeV

1. $\theta_e, \theta_\mu < 100$ mrad and $E_e > 1$ GeV (i.e. $t_{ee} \lesssim -1.02 \cdot 10^{-3}$ GeV$^2$). The angular cuts model the typical acceptance conditions of the experiment and the electron energy threshold is imposed to guarantee the presence of two charged tracks in the detector (Setup 1);

2. the same criteria as above, with the additional acoplanarity cut $|\pi - |\phi_e - \phi_\mu|| \leq 3.5$ mrad. We remind the reader that this event selection is considered in order to mimic an experimental cut which allows to stay close to the elasticity curve given by the tree-level relation between the electron and muon scattering angles (Setup 2)

where $t_{ee} = (p_2 - p_4)^2$, $(\theta_e, \phi_e, E_e)$ and $(\theta_\mu, \phi_\mu, E_\mu)$ are the scattering and azimuthal angles and the energy, in the laboratory frame, of the outgoing electron and muon, respectively.
NNLO photonic corrections

\[ \alpha = \frac{1}{137.03599907430637} \quad m_e = 0.510998928 \text{ MeV} \quad m_\mu = 105.6583715 \text{ MeV} \]

| \( \sigma (\mu b) \) | Setup 1 | | Setup 2 | |
|---------------------|--------|---|--------|---|
| \( \mu^-e^- \rightarrow \mu^-e^- \) | \( \mu^+e^- \rightarrow \mu^+e^- \) | \( \mu^-e^- \rightarrow \mu^-e^- \) | \( \mu^+e^- \rightarrow \mu^+e^- \) |
| \( \sigma_{LO} \) | | 245.038910(1) |
| \( \sigma_{NLO}^e \) | 255.5500(7) | | 223.4387(6) |
| \( \sigma_{NLO}^\mu \) | 244.9707(1) | | 244.4136(1) |
| \( \sigma_{NLO}^f \) | 255.1176(5) | 255.8437(5) | 222.8545(3) | 222.7714(3) |
| \( \sigma_{NNLO}^e \) | 255.5725(5) | | 224.4796(4) |
| \( \sigma_{NNLO}^\mu \) | 244.9706(1) | | 244.4154(1) |
| \( \sigma_{NNLO}^f \) | 255.205(1) | 256.092(1) | 224.041(1) | 224.088(1) |

Cross sections (in \( \mu b \)) and relative corrections for the processes \( \mu^-e^- \rightarrow \mu^-e^- \) and \( \mu^+e^- \rightarrow \mu^+e^- \), in the two different setups described in the text. The symbols \( \sigma_{(N)(N)LO}^{e/\mu/f} \) stand for the cross sections with corrections along the electron line only, along the muon line only and the full approximate contributions, respectively, with the perturbative accuracy given by the subscripts. The digits in parenthesis correspond to 1\( \sigma \) MC error. Italicized numbers in the last row indicate that in this cross-section the full two-loop amplitude is approximated.
NNLO photonic corrections

\[ \Delta_{NLO}^i = 100 \times \frac{d\sigma_{NLO}^i - d\sigma_{LO}}{d\sigma_{LO}} \]

\[ \Delta_{NNLO}^i = 100 \times \frac{d\sigma_{NNLO}^i - d\sigma_{NLO}^i}{d\sigma_{LO}} \]

[Carlo Calame et. Al. ‘20]
NNLO photonic corrections

[Carlo Calame et. Al. ‘20]
NNLO Leptonic Corrections

(Work in progress)
NNLO Leptonic Corrections

\[ d\sigma_{N_f}^{\alpha^2} = d\sigma_{\text{virt}}^{\alpha^2} + d\sigma_{\gamma}^{\alpha^2} + d\sigma_{\text{real}}^{\alpha^2} \]

(a) \hspace{2cm} (b) 

(c) \hspace{2cm} (d) 

(e) \hspace{2cm} (f) 

Ettore Budassi, Carlo M. Carloni Calame, Mauro Chiesa, Syed Mehedi Hasan, Guido Montagna, Oreste Nicrosini, Fulvio Picinini. (Work in progress)
MIs for Leptonic corrections on Vertices
Vertex with two different mass:
-> Amplitude given by Feynman diagrams

\[ A = \sum_i a_i I_i \]
-> Amplitude given by Feynman diagrams

\[ A = \sum_{i} a_i I_i \]

-> Project onto basis using Integration by Parts identities

\[ A = \sum_{i} c_i f_i \]

Implemented in public codes

LiteRed (Lee)
REDUZE (Studerus, von Manteuffel)
Fire (Smirnov)
Air (Anastasiou, Lazopolus)
Kira (Maierhoefer, Usovitsch, Uwer)
-> Amplitude given by Feynman diagrams

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Implemented in public codes

- LiteRed (Lee)
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-> Calculate basis elements via differential equations
Differential Equation

-> Kinematic derivative in space spanned by MIs

$$\partial_x \bar{f} = A_x \bar{f}$$
Differential Equation

-> Kinematic derivative in space spanned by MIs

\[ \partial_x \bar{f} = A_x \bar{f} \]

-> Conjecture: There is a basis such that:

\[ \partial_x \bar{g} = \epsilon \tilde{A}_x \bar{g} \]

(Henn)

-> There are many strategies to get the epsilon factorized form

- Magnus Theorem
  (Ageri, Di Vita, Mastrolia, Mirabella, Schlenk, Tancredi, US)
- Unit leading Singularity
  (Henn)
- Reduction to fuchsian form and Eigenvalue normalization
  (Lee, Smirnov)
- Factorization of Picard-Fuchs operator
  (Adams, Chaubey, Weinzierl)
Solving Canonical Differential Equation

Canonical form

\[ \partial_x \bar{g}(x, \epsilon) = \epsilon \tilde{A}_x(x) \bar{g}(x, \epsilon) \]

\[ d\bar{g}(x, \epsilon) = \epsilon \sum_i M_i d \log(\eta_i) \bar{g}(x, \epsilon) \]

- Kinematic dependence encoded in \( \eta \)
- \( \eta \)'s form the alphabet

Solution given by

\[ \bar{g}(x, \epsilon) = \left[ 1 + \sum_{i=1}^{\infty} \int_\gamma dA \ldots dA \right] \bar{g}(x_0, \epsilon) \]

**Algebraic \( \eta \)s : Chen Iterated Integrals**

\[ C(\eta^n; x) = \int_\gamma d \log(\eta_1) \ldots d \log(\eta_n) \]

**Rational \( \eta \)s : Generalized Polylogarithms**

\[ G(\bar{\eta}_n; x) = \frac{1}{n!} \log^n(x) \]

\[ G(\bar{\omega}_n; x) = \int_0^x \frac{dt}{t - \omega_1} G(\bar{\omega}_{n-1}; t) \]
Boundary Conditions

Solution given by

\[ \tilde{g}(x, \epsilon) = \left[ 1 + \sum_{i=1}^{\infty} \int_{\gamma} dA \ldots dA \right] g(x_0, \epsilon) \]

Two general ways to fix the boundary

| Known Limit | Pseudo-thresholds |
|-------------|-------------------|
| • Taking the limit \( x \) to \( x_0 \) | • Solution has unphysical divergences |
| • Fix boundary constant by matching the solution to known function | • Demanding absence of unphysical divergences gives relations between boundary constant |
| ![Diagram](image) | • Leftover constants must be provided |
Boundary Constants are fixed using PSLQ

Numerically
Checked against
SecDec 3
(Borowka et. al
arXiv:1502.06595)
Outlook

- MUonE is on track
- NNLO Photonic corrections to muon-electron scattering are presented
- NNLO Leptonic corrections are in progress
- New multi scale Master integrals are calculated
● Thank you for the attention