ON LEGENDRIAN PRODUCTS AND TWIST SPUNS

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Abstract. The Legendrian product of two Legendrian knots, as defined by Lambert-Cole, is a Legendrian torus. We show that this Legendrian torus is a twist spun whenever one of the Legendrian knot components is sufficiently large. We then study examples of Legendrian products which are not Legendrian isotopic to twist spuns. In order to do this, we prove a few structural results on the bilinearised Legendrian contact homology and augmentation variety of a twist spun. In addition, we show that the threefold Bohr–Sommerfeld covers of the Clifford torus and Chekanov torus are not twist spuns.

1. Introduction

1.1. Basic notions. Here we mainly consider Legendrian submanifolds of the standard contact vector space \((\mathbb{R}^{2n+1}, \alpha)\) with coordinates \((x_1, y_1, \ldots, x_n, y_n, z)\), where \(\alpha\) is the standard contact form \(\alpha := dz - \sum y_i dx_i\), whose associated Reeb vector field thus is \(\partial_z\). We will also consider general contactisations, i.e. contact manifolds \((P \times \mathbb{R}, \alpha)\) for general \(2n\)-dimensional Liouville manifold \((P, d\theta)\) with the contact form \(\alpha = dz + \theta\). A Legendrian submanifold is an \(n\)-dimensional smooth submanifold that satisfies \(T_x \Lambda \subset \ker \alpha\) for all \(x \in \Lambda\), and a Legendrian isotopy is a smooth isotopy through Legendrian submanifolds.

Integral trajectories of Reeb vector field \(\partial_z\) which start and end on a Legendrian submanifold \(\Lambda\) are called Reeb chords of \(\Lambda\). The set of Reeb chords of \(\Lambda\) will be denoted by \(Q(\Lambda)\), which is a finite set in the case of a generic compact Legendrian. To each \(c \in Q(\Lambda)\) we associate its length \(\ell(c) := \int_c \alpha > 0\).

Definition 1.1. For Legendrian submanifolds \(\Lambda_1, \Lambda_2\), we say that:

- \(\Lambda_1\) is smaller than \(\Lambda_2\) and write \(\Lambda_1 < \Lambda_2\) if the length of the longest Reeb chord on \(\Lambda_1\) is strictly smaller than the length of the shortest Reeb chord on \(\Lambda_2\);
- \(\Lambda_1\) and \(\Lambda_2\) have distinct Reeb chord lengths if no Reeb chord of \(\Lambda_1\) has the same length as a Reeb chord of \(\Lambda_2\).

Recall that an \(n\)-dimensional immersion \(i: L \hookrightarrow (P, d\theta)\) is exact Lagrangian if \(\theta\) pulls back to an exact form \(i^* \theta = df\). Any exact Lagrangian gives rise to a Legendrian immersion by the lift \(\{z \circ i = -f\}\). Conversely, the Lagrangian projection is given as \(\Pi_P : P \times \mathbb{R} \to P\) and sends a Legendrian submanifold to an exact Lagrangian immersion, whose double points moreover correspond bijectively to the Reeb chords.
In the case of $\mathbb{R}^{2n+1}$ we also recall the definition of the front projection given by $\Pi_{P_1} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{n+1}$, $\Pi_{P_1}(x,y,z) = (x,z)$. A Legendrian submanifold $\Lambda$ can be recovered from its front projection.

We are here interested in comparing two well-known geometric constructions that can be used to produce Legendrian submanifolds of higher dimension from lower dimensional ones.

1.2. The Legendrian product.

Definition 1.2. Consider two Legendrian submanifolds

$$\iota_i : \Lambda_i \hookrightarrow (P_i \times \mathbb{R}, dz_i + \theta_i), \ i = 1, 2.$$ 

The Legendrian product $\Lambda_1 \boxtimes \Lambda_2 \hookrightarrow (P_1 \times P_2 \times \mathbb{R}, dz + \theta_1 + \theta_2)$ is the Legendrian immersion defined by

$$\iota_1 \boxtimes \iota_2(u_1, u_2) = (\Pi_{P_1}(\iota_1(u_1)), \Pi_{P_2}(\iota_1(u_1)), z_1(\iota_1(u_1)) + z_2(\iota_2(u_2)))$$ 

The Legendrian product is embedded if and only if the two Legendrians have distinct Reeb chord lengths. Legendrian products were introduced and studied in [25] by Lambert-Cole who, among other things, computed their classical invariants. Also see the work [26] by the same author for results about existence of generating families for products as well as computations of certain Morse flow-trees.

Remark 1.3. The Legendrian isotopy class of $\Lambda_1 \boxtimes \Lambda_2$ is only invariant under Legendrian isotopy of the two factors as long as the pair has distinct Reeb chord lengths for each time in the isotopy; in particular, the Legendrian product thus depends on the choice of the Legendrian embeddings $\Lambda_i \hookrightarrow (P_i \times \mathbb{R}, dz + \theta_i)$ as opposed to just their general Legendrian isotopy classes.

For instance, if $\Lambda_1 < \Lambda_2$, the contact isotopy

$$(P_2 \times \mathbb{R}, dz_2 + \theta_2) \rightarrow (P_2 \times \mathbb{R}, e^{-t}(dz_2 + \theta_2)),$$

$$(p, z) \mapsto (\varphi_{\theta_2}(p), e^t z),$$

induced by the lift of the Liouville flow has the effect of rescaling the length of each Reeb chord of $\Lambda_2$ by $e^t$. This contact isotopy will be used repeatedly in constructions below.

The Chekanov–Eliashberg algebra of a Legendrian product is expected to contain more information than merely what is contained in the DGAs of the knots themselves [25]. For instance, pseudoholomorphic discs that have boundary on each knot and more than one positive puncture naturally enter the computation of the differential. Due to the transversality issues that arise in such considerations, we still lack a structural understanding of the DGAs of Legendrian products. For the same reason we have not been able to answer the following natural question:

Question 1.4. Is there a Legendrian torus in $\mathbb{R}^5$ which is not obtained as a Legendrian product?

We expect the answer to be negative; more precisely, we believe that the Legendrian tori from [10] that we also consider in Section 6 below are not products.
1.3. The twist spun. Given a loop \( \{\Lambda_\theta\} \), \( \theta \in S^1 \), of Legendrian submanifolds of \((P \times \mathbb{R}, dz + \theta)\) the corresponding mapping torus has a natural Legendrian embedding

\[
\Sigma\{\Lambda_\theta\} \subset (\mathbb{R}^2 \times P, dz - ydx + \theta)
\]

that was first constructed and studied by Ekholm–Kálmán in [17]. There are also higher dimensional versions; c.f. the construction below. In the special case when the loop of Legendrians is constant, we recover the so-called \( S^k \)-spun of Legendrians which first appeared in [14] for \( k = 1 \) and later was generalised and studied in the case of all \( k \geq 1 \) by the second author in [24].

One very useful property of this construction is that the Chekanov–Eliashberg algebra of the resulting Legendrian can be explicitly determined in terms of the Chekanov–Eliashberg algebra of the original one by a type of Künneth formula; c.f. the result from [17] as well as the partial results in high dimension by the authors in [9]. (In fact, in the case of \( S^1 \)-spuns, the DGA is the circle analogue of the Baues–Lemaire cylinder object in the category of DG-algebras from [3].)

In the case of a twist spun of a loop of Legendrians \( \Lambda_\theta \subset P = \mathbb{R}^{2n} \) we obtain a new Legendrian inside \( \{ (\tilde{x}, \tilde{y}, \tilde{z}) \} = \mathbb{R}^{2(n+1)+1} \) that can be explicitly expressed as follows. Given the parametrisation \((x(\theta, q), y(\theta, q), z(\theta, q)) \in \mathbb{R}^{2n+1}\) in locally defined coordinates on the mapping torus, we can write

\[
\begin{align*}
\tilde{x} &= (x_1(\theta, u) \cos \theta, x_1(\theta, u) \sin \theta, x_2(\theta, u), \ldots, x_n(\theta, u)), \\
\tilde{y} &= (y_1(\theta, u) \cos \theta - \partial_{\theta} z(\theta, u) \sin \theta, y_1(\theta, u) \sin \theta + \partial_{\theta} z(\theta, u) \cos \theta, y_2(\theta, u), \ldots, y_n(\theta, u)), \\
\tilde{z} &= z(\theta, u),
\end{align*}
\]

We now proceed to give a more general presentation of the twist spin, which also has the advantage that it exhibits the relation to Legendrian products more clearly. Denote by \( W^k \subset \mathbb{R}^{2k+1} \) the \( k \)-dimensional standard Legendrian sphere with a unique Reeb chord, that can be constructed as e.g. the Legendrian lift of the Whitney immersion in \( \mathbb{R}^{2n} \). When \( n = 1 \), this is simply a self-transverse figure-8 curve that bounds a total of zero area; in general, its front projection is given by the rotationally symmetric “flying saucer”, see Figure 1.

![Figure 1](image-url)
The subset
\[ P \times W^k \subset (P \times \mathbb{R}^{2k} \times \mathbb{R}, dz + \theta - ydx) \]
has a standard neighbourhood which can be identified with
\[ (P \times D_{\varepsilon}T^*S^k \times [-\varepsilon, \varepsilon], dz + \theta - \theta_{S^k}) \cong (P \times \mathbb{R}^{2k} \times \mathbb{R}, dz + \theta - ydx) \]
by a contact-form preserving contactomorphism that takes \( P \times 0_{S^k} \) to \( P \times W^k \). Here \( \theta_{S^k} \) is the tautological one-form on some radius-\( \varepsilon \) codisc bundle \( D_{\varepsilon}T^*S^k \). See e.g. [23] for more details about this standard contact neighbourhood theorem.

**Remark 1.5.** In the case when \( P = \mathbb{R}^{2n} \), the fact that \( \mathbb{R}^{2n} \) is subcritical makes it possible to do better: we can find a contact form preserving embedding of the entire neighbourhood \( (T^*\mathbb{R}^n \times S^k) \times \mathbb{R}, dz - y_idx_i - \theta_{S^k} \), which e.g. can be taken to be induced by the canonical inclusion
\[ \mathbb{R}^{n-1} \times \mathbb{R} \times S^k \subset \mathbb{R}^{n-1} \times \mathbb{R}^{k+1}. \]

Inside the above neighbourhood we then consider the suspension of \( \Lambda_{\theta} \subset P \times \mathbb{R}, \theta \in S^k \), which is the unique Legendrian
\[ \Sigma(\Lambda_{\theta}) \subset (P \times T^*S^k \times \mathbb{R}, dz + \theta - ydx) \]
whose canonical projection to \( P \times S^k \times \mathbb{R} \) is parametrised by
\[ (u, \theta) \mapsto (i_\theta(u), \theta, z(t_\theta(u))), \ u \in \Lambda_{\theta}, \theta \in S^k. \]

In the case \( P = \mathbb{R}^{2n} \) and \( k = 1 \) we recover the above formula for the twist spun. For general \( P \) it is necessary for the construction to first apply the contactomorphism \((\varphi_{\theta}^t, e^{-t}. \cdot)\) induced by the backwards Liouville flow to the loop \( \Lambda_{\theta} \) in order to make its \( z \)-coordinate sufficiently \( C^1 \)-small for all \( \theta \in S^1 \).

Of course, this construction generalises to arbitrary embeddings of products of cotangent bundles and Liouville domains, and can be seen as a version of the Lagrangian bundle construction due to Audin–Lalonde–Polterovich [1].

### 1.4. Results.
It was shown by Lambert-Cole in [25, Corollary 1.5] that the Legendrian product \( \Lambda \boxtimes W^k \) is Legendrian isotopic to the front \( S^k \)-spun of \( \Lambda \) whenever \( \Lambda < W^k \) is satisfied. The result of Lambert-Cole has the following generalisation:

**Theorem 1.6.** Assume that \( \Lambda_i \subset (\mathbb{R}^{2n_i+1}, \alpha) \) are Legendrian submanifolds \( i = 1, 2 \). In the case when \( \Lambda_1 < \Lambda_2 \), it follows that \( \Lambda_1 \boxtimes \Lambda_2 \) is a twist spun. Furthermore, if \( n_2 = 1 \) then \( \Lambda_1 \boxtimes \Lambda_2 \) is the twist spun of \( \Lambda_1 \) for the Legendrian isotopy that covers a rotation by \( 2\pi \cdot \mathrm{rot}(\Lambda_1) \) of the first \( \mathbb{R}^2 \) factor of \( \mathbb{R}^{2n_1} \) under the canonical \( U(1) \)-action.

Given a Lagrangian cobordism \( L \) from \( \Lambda_1^- \) to \( \Lambda_1^+ \subset P_1 \times \mathbb{R} \) (see Section 2.2 below) we can again form a product
\[ L \boxtimes \Lambda_2 \cong (\mathbb{R} \times P_1 \times P_2 \times \mathbb{R}, dz + \theta_1 + \theta_2), \]
which similarly to the Legendrian product we define to be the Lagrangian immersed cobordism
\[ \iota_1 \boxtimes \iota_2(u_1, u_2) = (\Pi_{\mathbb{R} \times P_1}(\iota_1(u_1)), \Pi_{P_2}(\iota_1(u_1)), z_1(\iota_1(u_1)) + z_2(\iota_2(u_2))) \]
from $\Lambda_1^- \boxtimes \Lambda_2$ to $\Lambda_1^+ \boxtimes \Lambda_2$. Note that one still can speak about Reeb chords on $L$, by which we mean an integral curve of the Reeb vector field $\partial_z$ that is contained inside some slice \{t\} $\times P_1 \times \mathbb{R}$ and which has endpoints on $L$. In other words, the condition that $L$ and $\Lambda_2$ have distinct Reeb chord lengths still makes sense.

**Theorem 1.7.**

(i) The cobordism $L \boxtimes \Lambda_2$ is an immersed exact Lagrangian cobordism from $\Lambda_1^- \boxtimes \Lambda_2$ to $\Lambda_1^+ \boxtimes \Lambda_2$ which is embedded when the Reeb chord lengths of $L$ are distinct from those of $\Lambda_2$. This can be arranged e.g. in the case when $\Lambda_1^+ < \Lambda_2$, since we then can rescale $\Lambda_2$ by an arbitrary amount by applying the contact isotopy $(\varphi_t^\theta, e^t \cdot)$ induced by the positive Liouville flow; and

(ii) If $\Lambda_1 < \Lambda_2$ and $\Lambda_1$ is stabilised or loose, then $\Lambda_1 \boxtimes \Lambda_2$ is loose as well.

The main aim of this note is to give an example of a Legendrian product which is not Legendrian isotopic to a twist spun:

**Theorem 1.8.** There are Legendrian tori $\Lambda = \Lambda_1 \boxtimes \Lambda_2 \subset \mathbb{R}^5$ which are not Legendrian isotopic to any twist spun.

For instance $\Lambda_1$ can be taken to be the standard unknot shown to the left in Figure 4 while $\Lambda_2$ is the stabilised unknot shown in the same figure to the right.

In addition, in Section 6 we prove that the threefold Bohr–Sommerfeld covers of the Clifford torus and Chekanov torus discussed by the authors in [10] are not twist spuns. Finally, we conjecture that no threefold Bohr–Sommerfeld cover of one of Vianna’s monotone Lagrangian tori in $\mathbb{C}P^2$ is a twist spun (this infinite family was constructed in [33]).

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**2. Background**

Here we consider some basic notions from the theory of Legendrian submanifolds. In particular, we introduce two important and distinct classes of Legendrian submanifolds: loose Legendrian submanifolds and fillable Legendrian submanifolds. These two classes have very different properties. Loose Legendrian submanifolds belong to “flexible contact topology.” More precisely, they satisfy an h-principle due to Murphy [30]. Fillable Legendrian submanifolds of contactisations belong to “rigid contact topology”; the Chekanov–Eliashberg algebra has been shown to be a rich invariant that is able to detect many obstructions in this case. We end the section by recalling the necessary background on this invariant.
2.1. **Loose Legendrians.** The class of *loose Legendrian* submanifolds were introduced by Murphy in [30], where they were shown to satisfy an \( h \)-principle. In particular, they are classified up to Legendrian isotopy by their topological properties. We now recall the definition.

![Figure 2. The front projection of \( \gamma \).](image)

We say that a Legendrian submanifold \( \Lambda \subset (P^{2n} \times \mathbb{R}, \alpha), \ n \geq 2 \), is loose if there exists a pair of neighbourhoods \( (U, \Lambda_0) \subset (P \times \mathbb{R}, \Lambda) \) that admits a contactomorphism to standard loose chart \( (R_{abc}, \Lambda_0) \) with \( a < bc \). Here \( R_{abc} \subset (\mathbb{R}^{2n+1}, dz - y_i dx_i) \) is a standard Darboux neighbourhood defined by

\[
R_{abc} = \{(x, y, x_1, \ldots, y_{n-1}, z); \ |x|, |y| \leq 1, \|(x_1, \ldots, x_{n-1})\| \leq b, \|(y_1, \ldots, y_{n-1})\| \leq c, |z| \leq a\} \subset (\mathbb{R}^{2n+1}, \alpha)
\]

and \( \Lambda_0 \) is the Legendrian solid cylinder, which is the product of

\[
D_b = \{(x_1, y_1, \ldots, x_{n-1}, y_{n-1}); \ y_1 = \cdots = y_{n-1} = 0, \|(x_1, \ldots, x_{n-1})\| \leq b\}
\]

and a Legendrian curve \( \gamma \subset \mathbb{R}^3 \) with coordinates \((x, y, z)\) and whose front projection is described in Figure 2. The Legendrian arc \( \gamma \) is contained in the box

\[
Q_a = \{|x| \leq 1, |y| \leq 1, |z| \leq a\}
\]

with \( \partial \gamma \subset \partial Q_a \).

2.2. **Lagrangian cobordisms and fillings.** An *exact Lagrangian cobordism from \( \Lambda^- \) to \( \Lambda^+ \subset P \times \mathbb{R} \) is a properly embedded \((n + 1)\)-dimensional submanifold

\[
L \subset (\mathbb{R} \times P^{2n} \times \mathbb{R}, d(e^t(dz + \theta)))
\]

which

- coincides with a cylinder over \( \Lambda^+ \) inside \([T, +\infty) \times P \times \mathbb{R}, \)
- coincides with a cylinder over \( \Lambda^- \) inside \((-\infty, -T] \times P \times \mathbb{R}, \) and
- is exact Lagrangian in the sense that \( e^t(dz + \theta)|_{TL} \) is exact with a globally constant primitive on \((-\infty, -T] \times \Lambda^- \subset L \).

We also allow the case when \( \Lambda^- = \emptyset \); if this holds then we call \( L \) an *exact Lagrangian filling of \( \Lambda^+ \), and we say that \( \Lambda^+ \) is (exact) fillable.
2.3. The Chekanov–Eliashberg algebra. The Chekanov–Eliashberg algebra is a Legendrian invariant introduced by Chekanov [6] and Eliashberg [19], and is a part of the Symplectic Field Theory [20] by Eliashberg–Hofer–Givental. The version that we use for Legendrians of contactisations of Liouville domains is due to Ekholm–Etnyre–Sullivan [15]. We proceed to sketch the definition, and refer to the latter article for more details.

The Chekanov–Eliashberg algebra of a closed Legendrian \( \Lambda \subset P \times \mathbb{R} \) is a noncommutative semifree DGA \((A(\Lambda), \partial)\) generated by the Reeb chords on \( \Lambda \) over the group ring \( \mathbb{F}[H_1(\Lambda)] \).

In this article we may restrict attention to \( \mathbb{F} = \mathbb{Z}_2 \), but when \( \Lambda \) is spin we can also take e.g. \( \mathbb{F} = \mathbb{Q} \) or \( \mathbb{C} \). Since our main interest are Legendrians diffeomorphic to \((S^1)^2\) we get an identification of \( \mathbb{F}[H_1(\Lambda)] \) with a Laurent polynomial ring \( \mathbb{F}[\mu^{\pm 1}, \lambda^{\pm 1}] \).

The degree of a Reeb chord generator \( c \) is determined by the so-called Conley–Zehnder index via \( |c| = CZ(c) - 1 \). The differential \( \partial \) satisfies the Leibniz rule

\[
\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)
\]

and is defined on the generators by a count of pseudoholomorphic polygons in \( P \) with boundary on \( \Pi_P(\Lambda) \), a single positive puncture at the input chord, and negative punctures at the output chords.

The Legendrian invariance comes from the fact that the stable-tame isomorphism type of the Chekanov–Eliashberg algebra is independent of the choice of almost complex structure and invariant under Legendrian isotopy. Here we will only concern a slightly weaker notion, which is that of DG-homotopy; see e.g. [16] for the definition.

An augmentation is a unital DGA-morphism \( \varepsilon : A(\Lambda) \to \mathbb{F} \), where the latter field is considered as a unital DGA with an empty set of generators. An augmentation is said to be graded if all generators in degrees different from zero are sent to zero under \( \varepsilon \).

Not all Legendrian have augmentations; for instance in the Chekanov–Eliashberg algebra of a loose Legendrian the unit 1 is a boundary, so it admits no augmentations. On the contrary, in accordance with the principles of symplectic field theory;

**Theorem 2.1** ([20, 12, 13]). An exact Lagrangian filling \( L \) of \( \Lambda \subset P \times \mathbb{R} \) induces an augmentation \( \varepsilon_L : A(\Lambda) \to \mathbb{Z}_2 \) of its Chekanov–Eliashberg algebra with coefficients in \( \mathbb{Z}_2 \). If the filling is spin, then one obtains an augmentation with arbitrary coefficient. If the Maslov class of \( L \) vanishes, then the augmentation is moreover graded.

Given an augmentation one can perform Chekanov’s linearisation procedure [6] to obtain a complex \( LCC_{\varepsilon}^*(\Lambda) \) which is an \( \mathbb{F} \)-vector spaces spanned by the Reeb chords. This was generalised in [4] by Bourgeois–Chantraine to the bilinearised complex \( LCC_{\varepsilon_1,\varepsilon_2}^*(\Lambda) \) induced by two augmentations (in fact, they showed that the augmentations form an \( A_\infty \)-category). Observe that graded augmentations must be used if we want have a well-defined \( \mathbb{Z} \)-grading of the bilinearised complex.

The invariance of the bilinearised complex under Legendrian isotopy is more complicated to state than the invariance of the DGA, since it depends on the augmentations. The main result that we need here is that
Theorem 2.2 (Corollary 5.12 & Proposition 5.17 [31]). The quasi-isomorphism class of the homology

$$LCH^{ε_1,ε_2}(Λ) := H(LCC^{ε_1,ε_2}_*(Λ))$$

of the bilinearised Legendrian complex for a Legendrian knot $Λ \subset \mathbb{R}^3$ does not depend on the DG-homotopy classes of the augmentations $ε_i$ involved.

3. Proofs of Theorems [1.6] and [1.7]

3.1. Proof of Theorem [1.6] Here we follow the argument of [11, Section 6.3] which uses the h-principle for subcritical isotropic curves to prove a flexibility for Lagrangians constructed using the circle-bundle construction of [1]. We argue in the case $k = 1$, i.e. a loop of Legendrians. The argument is completely analogous for higher-dimensional families. Recall that we here restrict attention to the case $P_1 = \mathbb{R}^{2n}$ and $P_2 = \mathbb{R}^2$.

Consider the Lagrangian projection $Π_{P_1 \times P_1}(Λ_1 \boxtimes Λ_2)$ of the product. After a suitable deformation, and further rescaling the first factor by the contactomorphism $((φ_1^{-t}, e^{-t}·))$ induced by the negative Liouville flow, we obtain an exact Lagrangian immersion which can be perturbed through exact Lagrangian immersions with embedded Legendrian lifts to one that is obtained from the circle bundle construction of [1] in the following sense:

Start with the exact subcritical isotropic immersion $\{0\} \times Π_{P_2}(Λ_2) \subset P_1 \times P_2$ which has non-generic intersection points corresponding to the Reeb chords. A generic perturbation through exact immersions produces an exact subcritical isotropic embedding. Its symplectic normal neighbourhood is of the form

$$(D_2^\epsilon \times D_2T^*S^1, ydx + dθ_{S^1})$$

and $Λ_1 \boxtimes Λ_2$ can be readily seen to be Legendrian isotopic to a Legendrian lift of $δ \cdot Π_{P_1}(Λ_1) \times 0_{S^1}$ when $δ > 0$ is sufficiently small; see e.g. [28].

Now recall the following h-principle due to Gromov: two exact embedded isotropic curves inside a symplectic manifold are Hamiltonian isotopic [21] whenever they are homotopic. This can be proven by using e.g. the h-principle for open exact Lagrangians, after fattening the isotropic curves to open Lagrangian ribbons. We can use this h-principle to construct an Hamiltonian isotopy that moves the above standard neighbourhood to the symplectic normal neighbourhood of an exact embedded perturbation of the subcritical exact Whitney immersion

$$Π_{\mathbb{R}^2}(W^k) \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^2.$$ 

By passing to the Legendrian lifts, after choosing $δ > 0$ sufficiently small, we also obtain a Legendrian isotopy from the Legendrian product under consideration to a Legendrian obtained as the “bundle construction” in the latter symplectic normal neighbourhood.

What is left is to investigate how the Hamiltonian isotopy acts on the framing of the symplectic normal bundle of the isotropic embeddings. To that end it suffices to compare the Maslov class of $W^k$ (which vanishes) with that of $Λ_2$ which is $2 \cdot \text{rot}(Λ_2)$. The Hamiltonian isotopy that takes the subcritical isotropic submanifolds to each other thus compensates the difference in Maslov classes by an additional twisting of the trivialisation of the symplectic normal bundle (or, equivalently, after twisting the Lagrangian fattening appropriately). □
3.2. Proof of Theorem 1.7. Part (i): To check the exact Lagrangian condition we compute the pull-back
\[(\iota_1 \boxtimes \iota_2)^* e^t (dz + \theta_1 + \theta_2) = \iota_1^* (e^t (dz_1 + \theta_1)) + e^{t \iota_2} \iota_2^* (dz_2 + \theta_2) = \iota_1^* (e^t (dz_1 + \theta_1))\]
which clearly is exact with a globally constant primitive for \(t \ll 0\).

Part (ii): Let \(R_{abc} \subset (P_1^{2n_1} \times \mathbb{R}, dz + \theta)\) be a stabilised (in the case \(n = 1\)) or loose (in the case \(n \geq 2\)) neighbourhood of \(\Lambda_1\). After a small Legendrian isotopy of \(\Lambda_2\), we may find a Darboux neighbourhood \(U \subset (P_2^{2n_2}, d\theta)\) symplectomorphic to some \([[-\epsilon, \epsilon][2n_1, dx_i \wedge dy_i])\), under which \(\Pi_{\varphi_2}(\Lambda_2)\) is identified with \(\{y_i = 0\}\), and such that the \(z_2\)-coordinate of the Legendrian lift is constant inside the same neighbourhood.

After rescaling the second factor by the positive Liouville flow \(\varphi_{\theta_2}^t, e^t \cdot \cdot\), we can in addition make the assumption that \(U\) is symplectomorphic to the Darboux neighbourhood
\[([-e^{t/2}\epsilon, e^{t/2}\epsilon][2n_1, dx_i \wedge dy_i])\]
for an arbitrarily large \(t \gg 0\). (Here we use the assumption that \(\Lambda_1 < \Lambda_2\) in order to infer that the rescaling is a Legendrian isotopy.) After such a rescaling the product
\[(R_{abc} \times U, (R_{abc} \cap \Lambda_1) \times (\Pi_{\varphi_2}^{-1}(U) \cap \Lambda_2)) \subset (P_1 \times P_2 \times \mathbb{R}, \Lambda_1 \boxtimes \Lambda_2),\]
is readily seen to be contactomorphic to a loose neighbourhood of \(\Lambda_1 \boxtimes \Lambda_2\).

Remark 3.1. Theorem 1.7 in particular implies that
- if \(\Lambda_1 < \Lambda_2\), \(\Lambda_1\) is loose and \(\Lambda_2\) is fillable, then \(\Lambda_1 \boxtimes \Lambda_2\) is loose;
- if \(\Lambda_1 < \Lambda_2\), \(\Lambda_1\) is fillable and \(\Lambda_2\) is loose, then \(\Lambda_1 \boxtimes \Lambda_2\) is fillable.

In other words, without any extra assumptions on the sizes of Reeb chords, Legendrian product construction neither preserve looseness, nor fillability of the components.

4. Structural results of DGAs of twist spuns

In this section we restrict ourselves to the case of Legendrian knots that live in \(\mathbb{R}^3\). For simplicity we will be mainly interested in the case when the twist spun is diffeomorphic to \(\mathbb{T}^2\).

The following general form for the Chekanov–Eliashberg algebra of a twist \(S^1\)-spuns of Legendrian knots follows from the results of Ekholm–Kálmán [17, Theorem 1.1] that we now recall. We want to use the results to compute the bilinearised Legendrian contact homology (see [4]) of a spin.

Denote by \((A(\Lambda_0), \partial)\) the Chekanov–Eliashberg algebra of the knot where we use coefficients \(\mathbb{F}\) (i.e. without the Novikov coefficients), and let
\[\Phi: (A(\Lambda_0), \partial) \rightarrow (A(\Lambda_0), \partial)\]
be the unital DGA endomorphism induced by the loop. The Legendrian torus twist spun \(\Sigma\{\Lambda_\theta\} \subset \mathbb{R}^5\) has a Chekanov–Eliashberg algebra \((A(\Sigma\{\Lambda_\theta\}), D)\) which after a suitable perturbation is of the following form.

Generators: For each Reeb chord generator \(x \in A(\Lambda_0)\) there are two generators \(x\) and \(\hat{x}\) of \((A(\Sigma\{\Lambda_\theta\}), D)\) where the degree of \(x\) agrees in both algebras, while \(|\hat{x}| = |x| + 1\).
Differential: For any $x \in A(\Lambda_0)$ we have $D(x) = \partial x$, while for $\hat{x}$ we have

$$D(\hat{x}) = \Phi(x) - x + \sum_{bcd} \langle \partial(x), bcd \rangle \Phi(b) \hat{c}d.$$ 

Note that, even when $\Lambda_0$ has rotation number zero, it could be the case that the isotopy $\Lambda_\theta$ induces a shift of Maslov potentials; in this case the corresponding torus does not have a vanishing Maslov class.

The above structure of the Chekanov–Eliashberg algebra of the twist spun immediately implies that the DGA of the knot sits include inside of it. It is not difficult to show that this also is the case when Novikov coefficients are used (see Part (1) in the theorem below). The following is the structural result that we need for the DGA of a twist spun:

**Theorem 4.1.** Let $\Lambda_\theta \subset \mathbb{R}^3$, $\theta \in S^1$, be a loop of Legendrian knots. Assume that the twist spun $\Sigma\{\Lambda_\theta\}$ is a torus which has an augmentation $\tilde{\varepsilon}$.

1. There exists an inclusion $A_{\mathbb{F}[\mu^\pm 1]}(\Lambda_0) \subset A_{\mathbb{F}[\mu^\pm 1, \lambda^\pm 1]}(\Sigma\{\Lambda_\theta\})$ of unital DGAs, extending the natural inclusion $\mathbb{F}[\mu^\pm 1] \subset \mathbb{F}[\mu^\pm 1, \lambda^\pm 1]$ on the level of coefficients (for suitable identifications); in particular, the augmentation $\tilde{\varepsilon}$ pulls back to an augmentation $\varepsilon$ of $A_{\mathbb{F}[\mu^\pm 1]}(\Lambda_0)$ that induces a canonical inclusion

$$LCH^\varepsilon_*(\Lambda_0) \subset LCC^\tilde{\varepsilon}_*(\Sigma\{\Lambda_\theta\})$$

of complexes;

2. The unital DGA-morphism $\Phi: A(\Lambda_0) \to A(\Lambda_0)$ of Chekanov–Eliashberg algebras induced by the loop $\Lambda_\theta$ of Legendrians extends to a unital DGA-morphism $A(\Sigma\{\Lambda_\theta\}) \to A(\Sigma\{\Lambda_\theta\})$ under the inclusion in Part (1); and

3. The linearised homology satisfies

$$LCH^\varepsilon_*(\Sigma\{\Lambda_\theta\}) \cong H(Cone(\psi))$$

for some (graded) chain-map

$$\psi: LCC^{\varepsilon \circ \Phi, \varepsilon}_*(\Lambda_0) \to LCC^\varepsilon_*(\Lambda_0).$$

Under the additional assumption that $A(\Sigma\{\Lambda_\theta\})$ has a unique (graded) augmentation up to DG-homotopy, it moreover follows that $LCH^{\varepsilon \circ \Phi, \varepsilon}_*(\Lambda_0) = LCH^\varepsilon_*(\Lambda_0)$.

**Remark 4.2.** The map $\psi$ vanishes when $\Lambda_0$ is the constant family, which gives back the Künneth formula for the spin.

Proof of Theorem 4.1. (1): When coefficients are taken in $\mathbb{F}$ as opposed to the Novikov ring the corresponding result is immediate from the above form of the algebra. The refined result with Novikov coefficients is similarly seen to follow by the analysis in [17]. To that end we briefly recall that there is a geometrically induced bijective correspondence between the discs...
that contribute to $D|_{A(\Lambda_0)}$ and the discs that contribute to $\partial$. The bijection is a feature of the geometric perturbation of the twist spun that is used to obtain the aforementioned form of the DGA.

Roughly speaking, one uses a Morse function on $S^1$ with precisely two critical points to perturb the Legendrian. All generators corresponding to the generators of $A(\Lambda_0)$ sit above the minimum $\theta_m \in S^1$. More precisely, there exists a symplectic hypersurface $\{\theta_m\} \times \mathbb{R}^2 \subset T^*S^1 \times \mathbb{R}^2$ which intersects the Lagrangian projection $\Pi_{\mathbb{R}^2}(\Sigma\{\Lambda_\theta\})$ of the twist spun in precisely $\{\theta_m\} \times \Pi_{\mathbb{R}^2}(\Lambda_0)$. Furthermore, the discs that contribute to $D(x)$ are identified with the discs corresponding to $\partial(x)$ that live inside this hypersurface. It is now straightforward to do the identification of first homology classes that contribute to the Novikov coefficients.

(2): The twist spun itself sits inside a loop of Legendrians induced by the $S^1$-family of loops $\{\Lambda_{\tau+\theta}\}_\theta$ that is parametrised by $\tau \in S^1$. The result now readily follows by the analysis from [17]. For a suitable perturbation of the loop of twist spuns, the bifurcations of the Chekanov–Eliashberg algebras parametrised by $\tau \in S^1$ satisfies the property that it restricts to bifurcations of the sub algebra $A(\Lambda_\tau)$ that live inside the symplectic hypersurface above the minimum that was considered in Part (1).

(3): The quotient $LCC^\varepsilon(\Sigma\{\Lambda_\theta\})/LCC^\varepsilon(\Lambda_0)$ induced by the inclusion from Part (1) is isomorphic to the bilinearised complex $LCC^\varepsilon_{\Phi,\varepsilon}(\Lambda_0)$. This can be explicitly checked by using the above expression for the DGA of a twist spun; see [4] for the definition of bilinearised Legendrian contact homology. The cone structure is evident from this.

Under the additional assumptions, Parts (1) and (2) combined implies that $\varepsilon \circ \Phi$ is DG-homotopic to $\varepsilon$. (The pull-back of a DG homotopy is again a DG-homotopy.) It now follows from the result Theorem 2.2 from [31] that the bilinearised complexes have homologies that are isomorphic, i.e.

$$LCH^\varepsilon_{\Phi,\varepsilon}(\Lambda_0) \cong LCH^\varepsilon_{\varepsilon}(\Lambda_0)$$

as sought.

Leverson [27] has shown that any graded augmentation of a knot must send the Novikov coefficient $\mu \mapsto -1$ (for a suitable choice of spin structure). Recall that the augmentation variety of a Legendrian torus is the algebraic closure of those points in $(\mathbb{C}^*)^2$, thought of as $\mathbb{C}$-algebra maps $\mathbb{C}[\mu^{\pm1},\lambda^{\pm1}] \to \mathbb{C}$, which extend to graded augmentations of its Chekanov–Eliashberg algebra. From Part (1) of Theorem 4.1 above we thus conclude that:

**Corollary 4.3.** The augmentation variety of a twist-span is contained inside the line

$$\{\mu = -1\} \subset (\mathbb{C}^*)^2$$

for a suitable choice of identification of $H_1(T^2) = \mathbb{Z}\mu \oplus \mathbb{Z}\lambda$ and spin structure on the torus.

5. **Proof of Theorem 1.8**

In this section we will analyse the simplest possible examples of Legendrian products $\Lambda_1 \boxtimes \Lambda_2$ of two Legendrian knots $\Lambda_1, \Lambda_2 \subset \mathbb{R}^3$ for which neither $\Lambda_1 < \Lambda_2$ nor $\Lambda_1 > \Lambda_2$ is satisfied. In particular, we will consider it from the point of view of fillability and also perform some partial computations of their Chekanov–Eliashberg algebras. Since Theorem
Figure 3. Left: the front projection of $\Lambda_1$ with $\ell(a) = 1$ and $|a| = 1$. Right: the front projection of the two Legendrian unknots $\Lambda_2^-$ together with the surgery disc $D$.

1.6 does not apply to products of this kind, it is a priori not clear whether such a product is a twist spun or not. Indeed, we will exhibit examples which are not Legendrian isotopic to any twist spun of a knot.

5.1. The family of examples. Consider the standard Legendrian unknot $\Lambda_1 \subset \mathbb{R}^3$ with a single Reeb chord of length one. Then we consider a Legendrian knot $\Lambda_2^r$, $r \geq 0$, that satisfies the following properties:

- $\text{rot}(\Lambda_2^r) = 0$,
- there is a single transversely cut out Reeb chord $b$ on $\Lambda_2^r$ in degree $2r \in \mathbb{Z}$ which is of length $< 1$, while all other Reeb chords are of length $> 1$.

Here we construct a particular family of examples $\Lambda_2^r$ by taking the cusp connect sum, as defined in [22] by Etnyre–Honda, between the union $\Lambda_2^r$ of two Legendrian unknots. More precisely, denote by $\Lambda_2^r$ the a Legendrian unknot of $\text{rot} = r$ and its image of under the rotation $(x, y, z) \mapsto (-x, -y, z)$ (or, equivalently, after a reflection of the front) and then separated by a horizontal translation. We then perform a cusp connect sum between two cusp-edges that face each other with the same $z$-coordinate; the small Reeb chord produced by the connect sum is then seen to be of degree precisely $2r$ as sought, and can be assumed to be arbitrarily small compared to the other remaining chords.

In the case $r = 0$ we simply take two copies of the standard Legendrian unknot of $\text{rot} = 0$ and $tb = -1$. In the case $r = 2$ we have depicted $\Lambda_2^2$ in Figure 3 while the resulting $\Lambda_2^2$ is depicted on the right in Figure 4.

5.2. Preliminary results. We do not compute the full DGA of the products under considerations. However, since they are exact fillable as shown below the quasi-isomorphism class of the completed DGA can be determined by the topology of the filling by the work [18] of Ekholm–Lekili.

Proposition 5.1. The Legendrian $\Lambda_1 \boxtimes \Lambda_2^{2r}$ is Legendrian isotopic to a representative whose Reeb chords consists of precisely the following:
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\[ \Lambda_a \quad \text{with } \ell(a) = 1 \quad \text{and} \quad |a| = 1. \]

\[ \Lambda_b \quad \text{with } \ell(b) < 1 \quad \text{and} \quad |b| = 2. \]

All other Reeb chords have lengths greater than one.

- \( a \) and \( A \) in degree \( |a| = |A| - 1 = 1 \) and of length \( \ell(a) = \ell(A) - \epsilon = 1 \);
- \( b \) and \( B \) in degrees \( |b| = |B| - 1 = 2r \) and of length \( \ell(b) = \ell(B) - \epsilon < 1 - 2\epsilon \);
- \( c_{a+b} \) in degree \( |c_{a+b}| = |a| + |b| + 1 = 2 + 2r \) and of length \( \ell(c_{a+b}) = \ell(a) + \ell(b) \); and
- \( c_{a-b} \) in degree \( |c_{a-b}| = |a| - |b| = 1 - 2r \) and of length \( \ell(c_{a-b}) = \ell(a) - \ell(b) \),

all of which are transverse.

Proof. The product has many chords involving the long chords on \( \Lambda_2^{2r} \). We need to argue that these can be removed after an application of a Legendrian isotopy.

The argument is the same as in the proof of Theorem 1.6. Consider the Legendrian unlink \( \Lambda_2 \cup D \) together with the Legendrian arc \( D \) that connects the rightmost cusp edge on the left knot to the leftmost cusp edge on the right knot. The cusp connect sum can be performed in an arbitrarily small neighbourhood of \( D \); c.f. \[8\]. Away from a neighbourhood of the disc \( D \) we can perform a Legendrian isotopy of the product to a (Legendrian lift of a) “bundle construction” over a subcritical isotropic embedding of two arcs. In other words, we isotope the Legendrian to the lift of a normal bundle construction over a perturbation of \( \{0\} \times \Pi_{f^2}(\Lambda_2) \).

While performing the above deformation we want to simultaneously fix the Legendrian in a neighbourhood of \( D \) so that, in the same neighbourhood, the Legendrian still coincides with the Legendrian product of \( \Lambda_1 \) and the two arcs involved in the standard Legendrian surgery. However, since we need to shrink \( \Lambda_1 \) in order to isotope the product Legendrian to the version that arises from the bundle construction in the previous paragraph, it is necessarily to simultaneously shrink the Legendrian \( \Lambda_2^{2r} \) in the the neighbourhood of \( D \) where the surgery takes place. (If we were to shrink only the factor \( \Lambda_1 \), we will produce a self-intersection at some moment due to the existence of the small chord \( b \) on \( \Lambda_2^{2r} \).

In this manner we are left with the Reeb chords:

- the \( S^1 \)-family of Reeb chords \( a \otimes \Lambda_2^{2r} \);
- the \( S^1 \)-family of Reeb chords \( \Lambda_1 \otimes b \);
• the transversely cut out Reeb chord $c_{a+b}$ that starts at the pair $(s_a, s_b)$ of starting points and ends at the pair $(e_a, e_b)$ of endpoints of the two Reeb chords $a$ and $b$ (this chord is of length $\ell(a) + \ell(b)$); and

• the transversely cut out Reeb chord $c_{a-b}$ that starts at the pair $(s_a, e_b)$ and ends at the pair $(e_a, s_b)$ (this chord is of length $\ell(a) - \ell(b)$).

After a Morse perturbation of the first two families of Reeb chords we obtain the sought situation. We leave the degree computations to the reader; c.f. [25]. □

Let $L$ denote the compact three-dimensional manifold with boundary $\partial L \cong \mathbb{T}^2$ obtained from $S^2 \times [0,1]$ by two oriented one-handle attachments.

**Proposition 5.2.** All Legendrians $\Lambda_1 \boxtimes \Lambda_2^{2r}$ are exact Lagrangian fillable by an exact Lagrangian filling diffeomorphic to $L$. (Whose Maslov class vanishes if and only if $2r = 0$.)

**Proof.** By Theorem 1.6 the product $\Lambda_1 \boxtimes \Lambda_2^{2r}$ consists of two unlinked twist spuns of the standard unknot. Hence they admit fillings by exact Lagrangian solid tori (whose Maslov class vanishes if and only if $r = 0$). Since the handle-attachment cobordism from $\Lambda_2^{2r}$ to $\Lambda_2^{-2r}$ as constructed in e.g. [8] can be taken to be contained in an arbitrarily small neighbourhood, Part (i) of Theorem 1.7 shows that taking the product gives rise to an embedded exact Lagrangian cobordism from $\Lambda_1 \boxtimes \Lambda_2^{2r}$ to $\Lambda_1 \boxtimes \Lambda_2^{-2r}$. □

**Corollary 5.3.** When $r \geq 1$, the Chekanov–Eliashberg algebra of the Legendrian $\Lambda_1 \boxtimes \Lambda_2^{2r}$ has a unique zero-graded augmentation up to DG-homotopy. (This augmentation is necessarily the trivial one for the above representative of the Legendrian.)

**Proof.** Since $a$ is the only generator of degree 1 it is sufficient to show that $\partial(a) = 0$. Indeed, since there are no generators in degree zero in this case, it then follows that the trivial augmentation is the unique graded augmentation.

To see that $\partial(a) = 0$ we argue as follows. Since $r \geq 1$, there are no words of action less than $\ell(a) = 1$ and of degree equal to zero. Hence $\partial(a) \in \mathbb{F}$ by the action preserving properties of the differential. The exact Lagrangian filling provided by Proposition 5.2 implies the existence of a (possibly ungraded) augmentation; see Theorem 2.1 proven in [12, 13]. This implies that $\partial(a) = 0$ holds as sought. □

### 5.3. The product is not a twist spun (Proof of Theorem 1.8)

We show that the Legendrian $\Lambda_1 \boxtimes \Lambda_2^{2r}$ are not Legendrian isotopic to a twist spun whenever $r \geq 1$. We believe that the statement is true also when $r = 0$, but in that a more involved computation is necessary.

It follows from Proposition 5.1 together with Corollary 5.3 that the linearised Legendrian contact homology of $\Lambda_1 \boxtimes \Lambda_2^{2r}$ satisfies

\[
LCH^c_\ast(\Lambda_1 \boxtimes \Lambda_2^{2r}) = \begin{cases} 
\mathbb{F}, & 1 - 2r = * < 0, \\
0, & 1 - 2r \neq * \leq 0.
\end{cases}
\]

in negative degrees. For this we do not need to compute any differentials, we can simply argue by degree reasons (but we of course need to use the nontrivial property that an augmentation exists in the first place).
Argue by contradiction and assume that $\Lambda_1 \boxtimes \Lambda_2^{2r}$ is Legendrian isotopic to a twist spun. Since Part (3) of Theorem 4.1 applies in view of Corollary 5.3, we deduce that

$$LCH^e_{\ast} \cong H(\text{Cone}(\psi)),$$

where $\psi$ moreover is chain map between two chain complexes with isomorphic homology $LCH_{\ast}$. The induced long exact sequence in homology by (5.1) takes the form

$$0 \to LCH^e_{1-2r}(\Lambda) \xrightarrow{\delta} LCH^e_{1-2r}(\Lambda) \to H_{1-2r}(\text{Cone}(\psi)) \to LCH^e_{2r}(\Lambda) \xrightarrow{\delta} LCH^e_{2r}(\Lambda) \to 0$$

and implies that the connecting homomorphisms $\delta_{1-2r}$ as well as $\delta_{-2r}$ are injective and surjective, respectively. By finite dimensionality they are hence both isomorphisms. Since, again by (5.1), we have

$$H_{1-2r}(\text{Cone}(\psi)) \cong \mathbb{F} \neq 0,$$

exactness now leads to the sought contradiction. \qed

6. OTHER EXAMPLES OF LEGENDRIANS THAT ARE NOT TWIST SPUNS

The two Legendrian tori constructed in [10], corresponding to suitable threefold covers of the Clifford and Chekanov torus, are here shown to not be Legendrian isotopic to twist spuns. This is done by mere considerations of their augmentation varieties, while considering the structural result for the DGA of a twist spun from Part (1) of Theorem 4.1.

![Figure 5. Front projection of $\Lambda_{Cl}$.

We consider a conical special Lagrangian inside $\mathbb{R}^6$, whose intersection with the standard contact sphere $S^5$ is a Legendrian torus which projects to $\mathbb{C}P^2$ as a threefold cover of the monotone Clifford torus (the Legendrian link of the Harvey–Lawson cone). After a Legendrian isotopy into a small contact Darboux ball, we get the Legendrian $\Lambda_{Cl} \subset J^1(\mathbb{R})$ with the front projection in Figure 5. The computation of it appeared in [7] and [10]:

**Proposition 6.1.** For the Lie group spin structure and suitable choices of capping paths and basis $\{\mu, \lambda\}$ of $H_1(\Lambda_{Cl})$, the augmentation variety of $\Lambda_{Cl}$ is equal to the one-dimensional complex pair of pants

$$\text{Sp}(\mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}]/(1 + \lambda(1 + \mu))).$$
We consider the Legendrian lift $\Lambda_{\text{Ch}}$ of the threefold Bohr–Sommerfeld cover of the Chekanov torus placed inside a Darboux ball, the front projection of it is described in Figure 6. The augmentation variety of $\Lambda_{\text{Ch}}$ has been computed by the authors in [10]:

**Proposition 6.2.** For the Lie group spin structure and suitable choices of capping paths and basis $\langle \mu, \lambda \rangle$ of $H_1(\Lambda_{\text{Ch}})$, the augmentation variety of $\Lambda_{\text{Ch}}$ is equal to the one-dimensional complex pair of pants

$$\text{Sp}(\mathbb{C}[\mu^{\pm 1}, \lambda^{\pm 1}]/(1 + \lambda(1 + \mu)^2)).$$

Since, in particular, the augmentation varieties of $\Lambda_{\text{Cl}}$ and $\Lambda_{\text{Ch}}$ do not contain a component which is a two-punctured sphere, we can use Corollary 4.3 to deduce that

**Corollary 6.3.** Neither $\Lambda_{\text{Cl}}$ nor $\Lambda_{\text{Ch}}$ is the twist spun of a Legendrian knot.

**Remark 6.4.** By the result of Vianna [33], there exists an infinite family of different monotone Lagrangian tori inside $\mathbb{C}P^2$. Since all these tori have superpotentials whose Newton polytopes are nondegenerate triangles by the same author (which means that their zero loci have at least three punctures, see for example [29, Section 1.5]), and since their superpotentials cannot have critical value equal to zero by [2, Theorem 1.6], we expect that their threefold Bohr–Sommerfeld covers as constructed in [10] give an infinite family of Legendrians which are not twist spuns. This expectation is based on [10, Conjecture 8.1] formulated by the authors, by which the augmentation variety is a certain cover of the zero set of the superpotential, together with Corollary 5.3.

**Remark 6.5.** In addition, we expect that none of the Legendrian tori discussed in this section is a Legendrian product.

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