Symmetry and Singularity Properties of Steen-Ermakov-Milne-Pinney Equations

K Krishnakumar
Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India.
krishapril09@gmail.com

A Durga Devi
Department of Physics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India.
raghanadurga@gmail.com

R Sinuvasan
Department of Mathematics, VIT-AP University, Amaravathi-522 237, Andhra Pradesh, India.
rsinuvasan@gmail.com

PGL Leach
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001
and
Institute for Systems Science,
Department of Mathematics, Durban University of Technology, POB 1334, Durban 4000, Republic of South Africa
leach@ucy.ac.cy

ABSTRACT
We examine the general element of the class of ordinary differential equations, $yy^{(n+1)} + \alpha y'y^{(n)} = 0$, for its Lie point symmetries.
We observe that the algebraic properties of this class of equations display an attractive set of patterns, the general member of the class can have three type of Algebra, $(n + 1)A_1 \oplus s \{A_1 \oplus sl(2,R)\}$, $A_1 \oplus sl(2,R)$ or $A_2 \oplus A_1$, for different values of $\alpha$. We look at the singularity properties of these equations for various values of $\alpha$.

**MSC Subject Classification:** 34A05; 34A34; 34C14; 22E60.

**Key Words and Phrases:** Symmetries; Singularities; Integrability.

## 1 Introduction

The Steen-Ermakov-Milne-Pinney Equation [19, 6, 9, 16]

\[ \rho''(t) + \omega^2(t) \rho(t) = \frac{1}{\rho(t)^3} \]

is well known for its frequent occurrence in various applications. It is probably less well known as an integral of a third-order equation of maximal symmetry [11] and this may well be the source of Pinney’s famous solution which was presented without proof. The simplest form of the equation is obtained by setting $\omega = 0$, that is,

\[ \rho''(t) = \frac{1}{\rho(t)^3}. \]

A natural generalisation, namely

\[ yy^{(n+1)} + \alpha y'y^{(n)} = 0, \quad (1) \]

was studied by Moyo et al. [12] with particular reference to its integrability.

We examine (1) in terms of its symmetry and singularity properties.

## 2 Symmetry Properties

We consider the ordinary differential equation

\[ yy^{(n+1)} + \alpha y'y^{(n)} = 0 \quad (2) \]
We examine Equation (2) for its symmetry properties. If \( n = 1 \), there are eight Lie point symmetries given by

\[
\begin{align*}
\Gamma_1 &= \partial_x \\
\Gamma_2 &= x\partial_x \\
\Gamma_3 &= y\partial_y \\
\Gamma_4 &= xy\partial_y \\
\Gamma_5 &= \log(y)\partial_x \\
\Gamma_6 &= y\log(y)\partial_y \\
\Gamma_7 &= x^2\partial_x + xy\log(y)\partial_y \\
\Gamma_8 &= x\log(y)\partial_x + (y\log(y))^2\partial_y
\end{align*}
\]

for \( \alpha = -1 \). Equally for \( \alpha \neq -1 \) there are eight Lie point symmetries. Now the value of \( \alpha \) intrudes into the expressions for some of the symmetries. The symmetries are

\[
\begin{align*}
\Gamma_1 &= \partial_x \\
\Gamma_2 &= x\partial_x \\
\Gamma_3 &= y\partial_y \\
\Gamma_4 &= y^{-\alpha}\partial_y \\
\Gamma_5 &= xy^{-\alpha}\partial_y \\
\Gamma_6 &= \frac{y^{1+\alpha}}{\alpha+1}\partial_x \\
\Gamma_7 &= (1+\alpha)x^2\partial_x + xy\partial_y \\
\Gamma_8 &= (1+\alpha)xy^{1+\alpha}\partial_x + y^{2+\alpha}\partial_y.
\end{align*}
\]

Because the maximal number of Lie point symmetries for a scalar second-order ordinary differential is eight [8][p 405], for \( n = 1 \) the algebra is \( sl(3, \mathbb{R}) \) irrespective of the value of \( \alpha \).

When we turn to the second member of the class, namely

\[
yy'''' + \alpha y'y'' = 0,
\]

we obtain three possible algebraic structures. These are

\[
\begin{align*}
\{\partial_x, x\partial_x, x^2\partial_x + 2xy\partial_y, y\partial_y, \partial_y, x\partial_y, x^2\partial_y \}, \\
\{\partial_x, x\partial_x, x^2\partial_x + xy\partial_y, y\partial_y, \frac{\partial_y}{y}, \frac{x\partial_y}{y}, \frac{x^2\partial_y}{y}\} \text{ and} \\
\{\partial_x, x\partial_x, y\partial_y \}
\end{align*}
\]

For the calculation of the symmetries we use the Mathematica add-on Sym [3, 4, 5, 2].
corresponding to $\alpha = 0$, $\alpha = \frac{2}{3}$ and general $\alpha$, respectively. For $\alpha = 0$ and $\alpha = \frac{3}{2}$ the algebra is $3A_1 \oplus s\{A_1 \oplus sl(2, R)\}$, for general $\alpha$ the algebra is $A_2 \oplus A_1$. (We make use of the Mubarakzyanov Classification Scheme [10, 13, 14, 15]).

The third member of the class is the prototype for subsequent equations and we find the symmetries

$$\{\partial_x, x\partial_x, x^2\partial_x + 3xy\partial_y, y\partial_y, \partial_y, x\partial_y, x^2\partial_y, x^3\partial_y\},$$

$$\{\partial_x, x\partial_x, x^2\partial_x + 2xy\partial_y, y\partial_y\}$$

and

$$\{\partial_x, x\partial_x, y\partial_y\}$$
corresponding to $\alpha = 0$, $\alpha = \frac{1}{2}$ and general $\alpha$. For $\alpha = 0$ the algebra is $4A_1 \oplus s\{A_1 \oplus sl(2, R)\}$, for $\alpha = \frac{1}{2}$ the algebra is $A_1 \oplus sl(2, R)$ and for general $\alpha$ the algebra is $A_2 \oplus A_1$.

These observations naturally lead to the following theorem for various values of $\alpha$.

**Theorem 1.** In general, symmetries of an $n^{th}$-order equation of the type $yy^{(n+1)} + \alpha y'y^{(n)} = 0$ are given by

| $\alpha$      | Symmetries                                      |
|---------------|-------------------------------------------------|
| $\alpha = 0$  | $\partial_y, x\partial_y, \cdots x^{(n)}\partial_y$ |
|               | $y\partial_y$                                  |
|               | $\partial_x, x\partial_x + \frac{n}{2}y\partial_y, x^2\partial_x + nxy\partial_y$ |
| $\alpha = \frac{n+1}{n-1}$ | $\partial_x, x\partial_x + \frac{n-1}{2}y\partial_y, x^2\partial_x + (n-1)xy\partial_y, y\partial_y$ |
| $\alpha = \text{else}$ | $\partial_x, x\partial_x, y\partial_y$. |

Equation $yy^{(n+1)} + \alpha y'y^{(n)} = 0$ has $n + 5$, 4 and 3 point symmetries corresponding to $\alpha = 0$, $\alpha = \frac{n+1}{n-1}$, and $\alpha = \text{else}$, respectively. The corresponding algebras are $(n+1)A_1 \oplus s\{A_1 \oplus sl(2, R)\}$, $A_1 \oplus sl(2, R)$ and $A_2 \oplus A_1$.

**Proof.** One can consider the general equation with left hand side in the form

$$\Omega : yy^{(n+1)} + \alpha y'y^{(n)}.$$  

The linearised symmetry condition is $X^{(n+1)}\Omega = 0$ when $\Omega = 0$, where

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

is the generator of the infinitesimal point transformation and

$$X^{(n)} = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \eta'\partial_y' + \cdots + \eta^{(n)}\partial_y^{(n)}$$

4
is its extension up to the $n^{th}$ derivative, that is,
\[- \alpha y' y^{(n)} \eta + \alpha y y^{(n)} \eta^{(1)} + \alpha y y' \eta^{(n)} + y^2 \eta^{(n+1)} = 0. \tag{7}\]

When we use the extension formula
\[\eta^{(n)} = D^n \eta - \sum_{j=1}^{n} \binom{n}{j} y^{(n+1-j)} D^j \xi, \tag{8}\]
where $D$ is the total derivative, we can rewrite (7) as
\[- \alpha y' y^{(n)} \eta + \alpha y y^{(n)} (\eta_x + y'(\eta_y - \xi_x) - y^2 \xi_y) + \alpha y y' D^n \eta

\[- \alpha y y' \sum_{j=1}^{n} \binom{n}{j} y^{(n+1-j)} D^j \xi + y^2 D^{(n+1)} \eta

\[-y^2 \sum_{j=1}^{n+1} \binom{n+1}{j} y^{(n+2-j)} D^j \xi = 0. \tag{9}\]

Comparing the coefficients of $y'' y^{(n)}$ on both sides in equation (9) we get $\xi_y = 0$, that is,
\[\xi = a(x). \tag{10}\]

On substitution of equation (10) into equation (9) we have
\[- \alpha y' y^{(n)} \eta + \alpha y y^{(n)} (\eta_x + y'(\eta_y - a')) + \alpha y y' D^n \eta

\[- \alpha y y' \sum_{j=1}^{n} \binom{n}{j} y^{(n+1-j)} a^{(j)} + y^2 D^{(n+1)} \eta

\[-y^2 \sum_{j=1}^{n+1} \binom{n+1}{j} y^{(n+2-j)} a^{(j)} = 0. \tag{11}\]

By comparison of the coefficients of $y'' y^{(n-1)}$ in equation (11) we see that $\eta_{yy} = 0$, that is,
\[\eta = b(x) + yc(x). \tag{12}\]

When we use Leibnitz’ rule for differentiating a product, we compute $D^n \eta$ as
\[D^n \eta = b^n(x) + \sum_{k=0}^{n} \binom{n}{k} y^{(k)} c^{(n-k)}. \tag{13}\]
On substitution of equation (13) into equation (11) we have

\[-\alpha (b + yc)y'y^{(n)} + \alpha (b' + yc' + y'(c - a'))yy^{(n)} + \alpha b^{(n)}yy' + \alpha \sum_{k=0}^{n} \binom{n}{k} y^{(k)}c^{(n-k)}yy' - \alpha \sum_{j=1}^{n} \binom{n}{j} y^{(n+1-j)}a^{(j)}yy' + b^{(n+1)}y^2 + \sum_{k=0}^{n+1} \binom{n+1}{k} y^{(k)}c^{(n+1-k)}y^2\]

\[= \alpha c'yy^{(n)} + \alpha \sum_{k=0}^{n-1} \binom{n}{k} y^{(k)}c^{(n-k)}yy' - \alpha \sum_{j=2}^{n} \binom{n}{j} y^{(n+1-j)}a^{(j)}yy' + \sum_{k=0}^{n} \binom{n+1}{k} y^{(k)}c^{(n+1-k)}y^2 - \sum_{j=2}^{n+1} \binom{n+1}{j} y^{(n+2-j)}a^{(j)}y^2 = 0.\]  

(14)

When we compare the coefficients of $y'y^{(n)}$ in equation (14), we obtain

\[b(x) = 0.\]  

(15)

Using the equation (15) we rewrite equation (14) as

\[\alpha c'yy^{(n)} + \alpha \sum_{k=0}^{n-1} \binom{n}{k} y^{(k)}c^{(n-k)}yy' - \alpha \sum_{j=2}^{n} \binom{n}{j} y^{(n+1-j)}a^{(j)}yy' + \sum_{k=0}^{n} \binom{n+1}{k} y^{(k)}c^{(n+1-k)}y^2 - \sum_{j=2}^{n+1} \binom{n+1}{j} y^{(n+2-j)}a^{(j)}y^2 = 0.\]  

(16)

By comparison of the coefficients of $y'y^{(n-1)}$ and $y^{(n)}$ in equation (16), we obtain

\[2c' - (n - 1)a'' = 0\]  

(17)

\[2(n + 1 + \alpha)c' - n(n + 1)a'' = 0.\]  

(18)

Solving the equation (17) for $c$ we obtain

\[c = c_1 + \frac{n - 1}{2}a',\]  

(19)

where $c_1$ is the constant of integration. On substitution of equation (19) into equation (18) we have

\[(-\alpha + \alpha n - n - 1)a'' = 0.\]  

(20)
2.1 Case 1: $\alpha = \frac{n+1}{n-1}$

The equation (21) is satisfied for $\alpha = \frac{n+1}{n-1}$. Comparing the coefficients of $y^{(n-1)}$ we get

$$3c'' - (n - 1)a''' = 0. \quad (21)$$

We substitute (19) into equation (21) to obtain $a''' = 0$, that is,

$$a = a_1 + a_2 x + a_3 x^2. \quad (22)$$

The coefficient functions of the symmetries of the case $\alpha = \frac{n+1}{n-1}$ are

$$\xi = a_1 + a_2 x + a_3 x^2, \quad (23)$$
$$\eta = c_1 y + \frac{n - 1}{2}a_2 y + (n - 1)a_3 y. \quad (24)$$

2.2 Case 2: $\alpha = \text{else}$

If $\alpha = \text{else}$, then $-\alpha + \alpha n - n - 1 \neq 0$. From (21) we obtain $a''' = 0$, that is,

$$a = a_1 + a_2 x. \quad (25)$$

The coefficient functions of the symmetries of the case $\alpha = \text{else}$ are

$$\xi = a_1 + a_2 x, \quad (26)$$
$$\eta = c_1 y + \frac{n - 1}{2}a_2 y. \quad (27)$$

2.3 Case 3: $\alpha = 0$

The equation is

$$y^{(n+1)} + \sum_{k=0}^{n+1} \binom{n+1}{k} y^{(k)} c^{(n+1-k)} - \sum_{j=1}^{n+1} \binom{n+1}{j} y^{(n+2-j)} a^{(j)} = 0. \quad (28)$$

We collect the constant and $y$ coefficients in equation (28) and find that

$$y^{(n+1)}(x) = 0, \ c^{(n+1)}(x) = 0. \quad (29)$$

We rewrite equation (28) as

$$\sum_{k=1}^{n} \binom{n+1}{k} y^{(k)} c^{(n+1-k)} - \sum_{j=2}^{n+1} \binom{n+1}{j} y^{(n+2-j)} a^{(j)} = 0 \quad (30)$$
and collect the coefficients of $y^{(n)}$ and $y^{(n-1)}$ in equation (2) to obtain

\begin{align*}
2c^{(1)} - na^{(2)} &= 0 \quad (31) \\
3c^{(2)} - (n-1)a^{(3)} &= 0. \quad (32)
\end{align*}

From equation (31) $c = c_1 + \frac{n}{2} a^{(1)}$ and we substitute this into equation (32) to obtain $a^{(3)} = 0$, that is,

\begin{equation}
a = a_1 + a_2 x + a_3 x^2. \quad (33)
\end{equation}

The coefficient functions of the symmetries of the case $\alpha = 0$ are

\begin{align*}
\xi &= a_1 + a_2 x + a_3 x^2 \quad \text{and} \\
\eta &= b_1 + b_2 x + \cdots + b_{n+1} x^n + c_1 y + \frac{n}{2} a_2 y + a_3 n x y.
\end{align*}

3 Singularity Analysis

We examine the specific class of equations for the value of $\alpha = -(n + 1)$ introduced above in terms of singularity analysis. We examine the sequence of equations introduced above in terms of singularity analysis. We follow the general method as outlined in [20, 21] with the modification for negative nongeneric resonances introduced by Andriopoulos et al [1]. We illustrate the method on the fifth-order equation,

\begin{equation}
yy^{(5)} - 5y' y^{(4)} = 0. \quad (34)
\end{equation}

To determine the leading-order behaviour we set $y = \alpha \chi^p$, where $\chi = x - x_0$ and $x_0$ is the location of the putative singularity. We obtain

\begin{equation*}
a^2(p-4)(p-3)(p-2)(p-1)p\chi^{2p-5} - 5a^2(p-3)(p-2)(p-1)p^2 \chi^{2p-5}
\end{equation*}

which is zero if $(p-4) = 5p$, ie, $p = -1^2$. Note that the coefficient of the leading-order term is arbitrary.

\footnote{The positive integral values of $p$ are not acceptable for singularity analysis.}
To establish the terms at which the remaining constants of integration occur in the Laurent Expansion we make the substitution

\[ y = \alpha \chi^{-1} + m \chi^{-1+s}, \]

where the various values at which \( s \) may take are determined by the coefficient of terms linear in \( m \) being zero and so \( m \) is arbitrary. The coefficient of \( m \) is a fifth-order polynomial in \( s \), the roots of which are

\[ s = -1, 0, 6, \frac{1}{2} \left( 5 - i \sqrt{39} \right), \frac{1}{2} \left( 5 + i \sqrt{39} \right). \]

The resonances are discordant. However, this problem can be overcome by the substitution \( y(x) \to \frac{1}{w(x)} \). From Table1 the value of the leading-order, \(-1\), is always present. For a system to possess the Painlevé Property the resonance must be an integer. If \( \alpha < n \), the Laurent expansion is known as a Right Painlevé Series because the exponents commence at \(-1\) and increase to a presumed infinity. If \( \alpha > n \), the Laurent expansion is known as a Full Painlevé Series.

In the case that the resonance is a rational number the expansion can be made in terms of fractional powers – the same be true if the exponent of the leading-order behaviour be rational. In this case the solution cannot be analytic. Rather, it has branch point singularities. Provided the denominator of the fractional power is not great, the expansion is acceptable. If the dominator is large, the complex plane is divided by so many branch cuts as to be effectively useless for the almost inevitable numerical computations used in the solution. When fractional powers are included in the expansion, the system is said to possess the weak Painlevé Property.

Consistency is automatically satisfied as all terms in the equation are dominant. We deduce the following theorem.

**Theorem 2.** The exponent of the leading-order term and the resonances of the \( n \)th member of the class of equations,

\[ yy^{(n+1)} + \alpha y^{(n)} = 0, \quad n > 1, \alpha \text{ rational}, \]

under the transformation \( y(x) \to \frac{1}{w(x)} \) are \( p = -n + 1, -n + 2, \cdots, -1, 0, -\frac{n}{1+\alpha} \) and \( s = -1, 0, 1, \cdots, n - 2, n - 1 - \alpha \).
Table 1: Leading order and resonances of Equation \(2\) under the transformation \(y(x) \rightarrow \frac{1}{w(x)}\)

| \(n\) | Leading-order | Resonances          |
|-----|--------------|---------------------|
| 2   | \(-1, 0, -\frac{2}{1+\alpha}\) | \(-1, 0, 1-\alpha\) |
| 3   | \(-2, -1, 0, -\frac{3}{1+\alpha}\) | \(-1, 0, 1-2-\alpha\) |
| 4   | \(-3, -2, -1, 0, -\frac{4}{1+\alpha}\) | \(-1, 0, 1, 2-3-\alpha\) |
| \(n\) | \(-n+1, -n+2, \ldots, -1, 0, -\frac{n}{1+\alpha}\) | \(-1, 0, 1, \ldots, n-2, n-1-\alpha\) |

**Proof.** Substituting \(y = w^{-1}\) in equation (2) then we have

\[ w^{-1}(w^{-1})^{(n+1)} + \alpha(w^{-1})'(w^{-1})^n = 0 \]  

(35)

To find the leading order of equation (35), let us take \(w = \beta x^p\), Obviously

\[ w^{-1} = \frac{1}{\beta} x^{-p} \]  

(36)

Then we may obtained the corresponding first and second order derivative is

\[ (w^{-1})' = \frac{1}{\beta} (-1) p x^{-p-1} \]  

(37)

\[ (w^{-1})'' = \frac{1}{\beta} (-1)^2 p(p+1) x^{-p-2} \]  

(38)

we can rewriting the general form as

\[ (w^{-1})^{(n)} = \frac{1}{\beta} (-1)^n p(p+1)(p+2)\ldots(p+(n-1)) x^{-p-n} \]  

(39)

\[ (w^{-1})^{(n+1)} = \frac{1}{\beta} (-1)^{n+1} p(p+1)(p+2)\ldots(p+(n-1))(p+n) x^{-p-n-1} \]  

(40)

By substituting the equation (39) and (40) to the equation (35)

\[ \frac{1}{\beta^2} (-1)^{n+1} p(p+1)(p+2)\ldots(p+n-1)(p+n) x^{-2p-n-1} \]

\[ + \frac{\alpha}{\beta^2} (-1)^{n+1} p^2(p+1)(p+2)\ldots(p+n-1) x^{-2p-n-1} = 0 \]  

(41)
Collecting the coefficients of $x^{-2p-n-1}$ and equating it into zero

$$p(p + 1)(p + 2)...(p + n - 1)(p + n + \alpha p) = 0 \quad (42)$$

We get the following values for $p$, $p = 0, -1, -2, \ldots -n + 1, \frac{-n}{1+\alpha}$

To find the resonances let us take $w = \beta x^{-1} + mx^{-1+s}$ therefore

$$w^{-1} = \frac{x}{\beta^2}\left(1 + \frac{m}{\beta}x^s\right)^{-1} \quad (43)$$

$$w^{-1} = \frac{1}{\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{m}{\beta}\right)^k x^{ks+1} \quad (44)$$

$$(w^{-1})' = \frac{1}{\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{m}{\beta}\right)^k (ks + 1)x^{ks} \quad (45)$$

$$(w^{-1})'' = \frac{1}{\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{m}{\beta}\right)^k (ks + 1)(ks)x^{ks-1} \quad (46)$$

$$(w^{-1})^{(n)} = \frac{1}{\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{m}{\beta}\right)^k (ks+1)(ks)(ks-1)\ldots(ks-(n-2))x^{ks-(n-1)} \quad (47)$$

$$(w^{-1})^{(n+1)} = \frac{1}{\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{m}{\beta}\right)^k (ks+1)(ks)(ks-1)\ldots(ks-(n-2))(ks-(n-1))x^{ks-n} \quad (48)$$

Substituting equations $(44, 45, 47)$ and $(49)$ in equation $(35)$ then the resultant equation is given by

$$\frac{1}{\beta^2}x^{-n+1} \left( \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} (-1)^j \left(\frac{m}{\beta}\right)^j (js + 1)(js)(js - 1)\ldots(js - (n - 2))(js - (n - 1)) \right\} \right)$$

$$-\alpha \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} (-1)^j \left(\frac{m}{\beta}\right)^j (js + 1)(js)(js - 1)\ldots(js - (n - 2))((k - j)s + 1) \right\} x^{ks}$$

To collect the coefficient of $m$, we have to take $k = 1$ and $j = 1$ therefore

$$(s + 1)(s - 1)\ldots(s - (n - 2))(s - n + 1 + \alpha) = 0 \quad (49)$$

Hence the resonances are $s = -1, 0, 1, \ldots, n - 2, n - 1 - \alpha$
4 Discussion

We have examined the equation,

\[ yy^{(n+1)} + \alpha y^{(n)} = 0, \]

in terms of the algebraic properties of its Lie point symmetries and its integrability in terms of analytic functions. We found that the Lie point symmetries for general \( n \) are

\[
\alpha = 0 : \partial_y, x\partial_y, \cdots x^{(n)}\partial_y, y\partial_y, \partial_x, x\partial_x + \frac{n}{2}y\partial_y, x^2\partial_x + nxy\partial_y
\]

\[
\alpha = \frac{n+1}{n-1} : \partial_x, x\partial_x + \frac{n-1}{2}y\partial_y, x^2\partial_x + (n-1)xy\partial_y, y\partial_y
\]

\[
\alpha = \text{else} : \partial_x, x\partial_x, y\partial_y
\]

The algebras are \((n + 1)A_1 \oplus_s \{A_1 \oplus sl(2, R)\}, A_1 \oplus sl(2, R)\) and \(A_2 \oplus A_1\), respectively. In terms of singularity analysis, under the transformation \( y(x) \to \frac{1}{w(x)} \), the solution for \( w(x) \) is either analytic over the complex plane or on a portion of it defined by branch cuts. It follows that \( y(x) \) is also analytic.

Acknowledgements

PGLL thanks the University of KwaZulu-Natal and the National Research Foundation of South-Africa for financial support.

References

[1] Andriopoulos K & Leach PGL (2006) An interpretation of the presence of both positive and negative nongeneric resonances in the singularity analysis Physics Letters A 359 199-203

[2] Andriopoulos K, Dimas S, Leach PGL & Tsoubelis D, On the systematic approach to the classification of differential equations by group theoretical methods Journal of Computational and Applied Mathematics, 230 (2009) 224 – 232 (DOI: 10.1016/j.cam.2008.11.002).
[3] Dimas S & Tsoubelis D (2005) SYM: A new symmetry-finding package for Mathematica Group Analysis of Differential Equations Ibragimov NH, Sophocleous C & Damianou PA edd (University of Cyprus, Nicosia) 64 – 70

[4] Dimas S & Tsoubelis D (2006) A new Mathematica-based program for solving overdetermined systems of PDEs 8th International Mathematica Symposium (Avignon, France)

[5] Dimas S (2008) Partial Differential Equations, Algebraic Computing and Nonlinear Systems (Thesis: University of Patras, Patras, Greece)

[6] Ermakov V P (1880) Differenzialniya uravneniya vtorago poryadka. Univ. Izv., Kiev, Series III (9) 1-25

[7] Feix MR, Geronimi C, Leach PGL, Lemmer RL & Bouquet S (1997) On the singularity analysis of ordinary differential equations invariant under time translation and rescaling Journal of Physics A: Mathematical and General 30 (21) 7437-7461

[8] Lie Sophus (1967) Differentialgleichungen (Chelsea, New York)

[9] Milne W E (1930) The numerical determination of characteristic numbers. Physical Review, 35(7), 863

[10] Morozov, V. V., Classification of six-dimensional nilpotent Lie algebras. Izvestia Vysshikh Uchebn Zavendenii Matematika, 5, 1958, p. 161-171.

[11] Morris R M & Leach P G L (2017) The Effects of Symmetry-Breaking Functions on the Ermakov-Pinney Equation. Applicable Analysis and Discrete Mathematics , 11(1), 62-73

[12] Moyo S & Leach P G L (2006) Reduction properties of ordinary differential equations of maximal symmetry, J. Calmet/WM Seiler/RW Tucker (Eds.), 253

[13] Mubarakzyanov, G. M.: On solvable Lie algebras. Izvestia Vysshikh Uchebn Zavendenii Matematika, 32, p. 114-123.
[14] Mubarakzyanov, G. M.: Classification of real structures of five-dimensional Lie algebras. Izvestia Vysshikh Uchebn Zavedenii Matematika, 34, 1963, p. 99-106.

[15] Mubarakzyanov, G. M.: Classification of solvable six-dimensional Lie algebras with one nilpotent base element. Izvestia Vysshikh Uchebn Zavedenii Matematika, 35, 1963, p. 104-116.

[16] Pinney E (1950) The nonlinear differential equation $y'' + p(x)y + cy^3 = 0$. Proceedings of the American Mathematical Society, 1(5), 681

[17] Ramani, A., Grammaticos, B., Bountis, T.: The Painlevé property and singularity analysis of integrable and nonintegrable systems. Physics Reports, 180, 1989, p. 159-245. (doi.org/10.1016/0370-1573(89)90024-0)

[18] Sinuvasan R, Tamizhmani K M & Leach P G L (2016) Algebraic and singularity properties of a class of generalisations of the Kummer-Schwarz equation. Differential Equations and Dynamical Systems 1-10.

[19] Steen A (1874) Om Formen for Integralet af den lineaere Differentialligning af an den Orden. Overs over d K Danske Vidensk Selsk Forh 1

[20] Ramani A, Grammaticos B & Bountis T The Painlevé property and singularity analysis of integrable and nonintegrable systems Phys Rep 180 (1989) 159-245

[21] Tabor M (1989) Chaos and Integrability in Nonlinear Dynamics (John Wiley, New York)