One-Loop Renormalization of QCD with Lorentz Violation

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The explicit one-loop renormalizability of the gluon sector of QCD with Lorentz violation is demonstrated. The result is consistent with multiplicative renormalization as the required counterterms are consistent with a single re-scaling of the Lorentz-violation parameters. In addition, the resulting beta functions indicate that the CPT-even Lorentz-violating terms increase with energy scale in opposition to the asymptotically free gauge coupling and CPT-odd terms. The calculations are performed at lowest-order in the Lorentz-violating terms as they are assumed small.

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I. INTRODUCTION

As is well known, the Standard Model is defined by a Lagrangian which exhibits ultraviolet divergences arising from the structure of the theory at small distances. These divergences can be removed by a singular redefinition of the parameters defining the theory through the process of renormalization. The required calculations involve a number of remarkable cancellations which are most easily obtained by exploiting the various symmetries of the theory. The symmetries of the conventional Standard Model include the Lorentz group.

The investigation of the renormalizability properties of the SM in the presence of Lorentz violation began in [1, 2] where the authors studied one-loop radiative correction for QED with Lorentz violation. These calculations were carried out in the framework of an explicit theory called the Standard Model Extension (SME), which has been formulated to include possible Lorentz-violating background couplings to Standard Model fields [3, 4]. In this theory, the one-loop renormalizability of general Lorentz and CPT violating QED has been established [1]. The manuscript [1] includes an analysis of the explicit one-loop structure of Lorentz-violating QED and the resulting running of the couplings. The authors establish that conventional multiplicative renormalization succeeds and they find that the beta functions indicate a variety of running behaviors, all controlled by the running of the charge. Portions of this analysis have been extended to allow for a curved-space background [5], while other analysis involved finite, but undetermined radiative corrections due to CPT violation [6, 7, 8, 9, 10, 11, 12].

The Lorentz violating QED results of [1] were extended to non-abelian gauge theories in [13] where the authors established that the associated pure Yang-Mills theory is renormalizable at one-loop. More precisely, conventional multiplicative renormalization succeeds and the beta functions indicate that CPT-even Lorentz violating terms increase with energy scales in opposition to the asymptotically free gauge couplings and the CPT-odd couplings. The primary purpose of this paper is to extend the results of [13] to include fermions as well as to remove certain technical restrictions on the trace of the CPT-even terms. The results are directly applicable to one-loop renormalizability of the gluon sector of QCD in the presence of Lorentz violation. In addition, the running of the couplings is studied using the associated beta functions. New methods for carrying out renormalization calculations inside the SME using functional determinants are provided, giving a second, alternative derivation of the results presented in [13].

This work should be viewed as part of an extensive, systematic investigation of Lorentz violation and its possible implications for Planck-scale physics [14, 15, 16, 17, 18, 19, 20, 21, 22]. Extensive calculations using the SME have led to numerous experiments (see, for example [22]), which have in turn placed stringent bounds on parameters in the theory associated with electrons, photons, neutrinos, and hadrons. Recent work involving Lorentz violation and cosmic microwave background data [24] suggest that the SME might play a useful role in cosmology. In addition to the above, the SME formalism has been extended to include gravity [25, 26, 27], where it has been suggested that Lorentz violation provides an alternative means of generating General Relativity [28].

Some other related work includes a study of deformed instantons in the theory [29, 30], an analysis of the Casimir effect in the presence of Lorentz violation [31], an analysis of gauge invariance of Lorentz-violating QED at higher-orders [32], and possible effects due to nonpolynomial interactions [33]. Some investigations into possible Lorentz-violation induced from the ghost sector of scalar QED have also been performed [34]. More recently, functional determinants have been used to compute finite corrections to CPT-violating gauge terms arising from fermion violation [35].
II. NOTATION AND CONVENTIONS

To simplify notation we limit our investigation to the case of a single fermion. The associated Lagrangian with Lorentz violation is taken to be

\[ \mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi + \mathcal{L}_G, \]

where \( \mathcal{L}_A \) is the gauge field contribution, \( \mathcal{L}_\psi \) is the fermionic contribution, and \( \mathcal{L}_G \) is the ghost contribution. In computing the UV divergence, we treat each term in the Lagrangian separately. We begin with the primary Yang-Mills contribution \[3, 4]\n
\[ \mathcal{L}_A = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + (k_F)_{\mu\nu\sigma\beta} F^{\mu\nu} F^{\sigma\beta} + (k_{AF})^\kappa \epsilon_{\kappa\lambda\mu\nu} \langle A^\lambda F^{\mu\nu} - \frac{2i}{3} g A^\lambda A^\mu A^\nu \rangle + 2\lambda F[A]^2, \]

where \((k_F)_{\mu\nu\sigma\beta}\) and \((k_{AF})^\kappa\) are tensors governing the Lorentz violation in the Yang-Mills sector. For the Standard Model Extension the tensor \(k_F\) is CPT-even, satisfies a Jacobi identity and is constrained to have the symmetries of the Riemann tensor, while the tensor \(k_{AF}\) is CPT-odd. The parameter \(\lambda\) multiplies a gauge fixing term \(F\). The generators of the Lie Algebra defined by \(A^a = A^{a\mu} t^a\) are taken to satisfy

\[ [t^a, t^b] = i f^{abc} t^c, \]

and \(f^{abc}\) are totally anti-symmetric structure constants. The product of these generators is normalized to

\[ tr[t^a t^b] = C(r) \delta^{ab}, \]

where \(C(r)\) depends on the representation \(r\). In the adjoint representation used for the gauge fields, this is written \(C(G) = C_2(G)\) where \(C_2(r)\) is the quadratic Casimir operator

\[ t^a t^a = C_2(r) \cdot 1. \]

The field tensor is defined as

\[ F^{\mu\nu} = -\frac{i}{g} [D^{\mu}, D^{\nu}], \]

where the covariant derivative is \(D^\mu = \partial^\mu + ig A^\mu\).

The fermionic contribution to the Lagrangian is given by \[3, 4\]

\[ \mathcal{L}_\psi = \bar{\psi}(i\Gamma^\mu D_\mu - M)\psi, \]

where \(\Gamma^\nu = \gamma^\nu + \gamma^1\), \(M = m + M_1\), and \(\Gamma_1\) and \(M_1\) are of the form

\[ \Gamma_1 = \epsilon_{\nu\mu} \gamma^\mu + d_\nu \gamma^\mu \gamma^\nu + e_\nu + i f_\nu \gamma_5 + \frac{1}{2} g \sigma_\mu \sigma_\nu, \]

\[ M_1 = a_\nu \gamma^\mu + b_\nu \gamma_5 \gamma^\nu + \frac{1}{2} H_{\mu\nu} \sigma^\mu \sigma^\nu. \]

Here the \(\gamma^\mu\) are the standard gamma matrices, \(\sigma_\mu\) are the standard sigma matrices, and the remaining small parameters control Lorentz violation. The parameters \(e_{\nu\mu}\) and \(d_{\nu}\) are traceless, \(H_{\mu\nu}\) is antisymmetric, and \(q^{\lambda\mu\nu}\) is antisymmetric in the first two components. The parameters \(a_\nu\), \(b_\nu\), \(H_{\mu\nu}\), have the dimension of mass, while the remaining parameters are dimensionless.

Finally, the ghost Lagrangian is written in terms of the scalar, anticommuting field \(\phi\)

\[ \mathcal{L}_G = -\bar{\phi}(M - C_{\mu\nu} D^\mu D^\nu)\phi, \]

where \(M\) is the variation of the gauge fixing functional \(F\) with respect to the gauge transformation and the constants \(C_{\mu\nu}\) parameterize possible Lorentz violation in the ghost sector \[34\].

III. FUNCTIONAL DETERMINANTS AND BACKGROUND Fields

Recall, the one-loop effective action for the theory can be written as a functional integral over fields \(\Psi\):

\[ \exp i \Gamma[\Psi] = \int \mathcal{D}\Psi e^{i \int d^4 x \mathcal{L}[\Psi]}. \]

The effective action is constructed by writing the underlying fields as the sum of a classical background and a fluctuating quantum field. The effective action is given by a classical term perturbed by terms quadratic in the fluctuation. The quadratic term gives rise to a Gaussian integral, which in turn can be described by a functional determinant \[36\]. Using \(\mathcal{L}_{ct} = \mathcal{L}_0 + \mathcal{L}_{c.t.}\) for the classical Lagrangian as a function of the background field where \(\mathcal{L}_{c.t.}\) is the counterterm Lagrangian, the expression becomes

\[ \exp i \Gamma[\Psi] = e^{i \int d^4 x \mathcal{L}_{c.t} \det(\Delta_\Delta)^{-\frac{1}{2}} \det(\Delta_\phi)^\frac{1}{2} \det(\Delta_\phi)^\frac{1}{2}}, \]

where the \(\Delta\) are operators which are given explicitly below. To compute the above determinants, dimensional regularization is used. Each determinant is treated separately, beginning with the pure Yang-Mills gauge field contribution. The calculation is performed to first order in Lorentz violating parameters. As this is the case, the computations of the various terms decouple and the CPT-even and CPT-odd cases can be treated independently.

We will write the gauge fields as the sum of a classical background field (denoted with an underline) and a fluctuating quantum field:

\[ A^\mu = \underline{A}^\mu + A^\mu. \]

With this convention the curvature can be expressed as

\[ F^{a\mu\nu} = F^{a\mu\nu} + (D^{a\mu\nu} A^\nu)^a - (\underline{D}^{a\mu} A^\mu)^a, \]

where the underline denotes background curvature and the covariant derivatives are taken with respect to the
background fields. The gauge fixing functional is chosen to be
\[ \mathcal{F}[A] = D^\mu A_\mu, \]
and \( \lambda \) is set equal to 1 incorporating Feynman gauge. Rescaling the vector potential to absorb \( g \), substituting \( P \) into \( \mathcal{L} \) and retaining terms which are quadratic in the perturbation of the background field we have
\[ L^{quad}_A = -\frac{1}{2 g^2} \text{tr} A^\mu \left[ -g_{\mu\nu} \partial^2 - 2i E_{\mu\nu} - i(k_F)_{\alpha\beta\rho\sigma} F^{\alpha\beta} - 2(k_F)_{\mu\alpha\rho\beta} D^\rho D^\beta + (k_{AF})_\alpha^\mu \epsilon_{\alpha\beta} D^\beta \right] A^\nu. \] (16)
The trace is performed over the Lie Algebra indices and all fields are written in the adjoint representation. The trace is extended to cover the Lorentz indices as well using the matrix notations
\[ (\tau_{\alpha\beta})_{\mu\nu} = i(g_{\mu\nu} g_{\alpha\beta} - g_{\alpha\nu} g_{\mu\beta}), \]
(17)
\[ (\epsilon_{\alpha\beta})_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu}, \]
(18)
\[ (k_{F\alpha\beta})_{\mu\nu} = (k_F)_{\alpha\mu\beta\nu}, \]
(19)
\[ (k_{F\alpha\beta})_{\mu\nu} = (k_F)_{\mu\alpha\nu\beta}, \]
(20)
the quadratic Lagrangian can be rewritten as
\[ L^{quad}_A = -\frac{1}{2 g^2} \text{tr} A \left[ -(g_{\alpha\beta} - 2k_{F\alpha\beta}) D^\alpha D^\beta - (\tau_{\alpha\beta} + i k_{F\alpha\beta}) \Gamma^{\alpha\beta} - (k_{AF})_\alpha^\mu \epsilon_{\alpha\beta} D^\beta \right] A \]
\[ = -\frac{1}{2 g^2} \text{tr} A \Delta_A A, \] (21)
where the trace is performed over both Lorentz and gauge spaces, \( \Delta_A = P_A + \Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)} \), \( \Delta_A^{(i)} \) is order \( i \) in the fields \( (i = 1, 2) \), and \( \Delta_A^{(F)} \) contains all curvature contributions:
\[ P_A = -(g_{\alpha\beta} + 2k_{F\alpha\beta}) \partial^\alpha \partial^\beta - k_{AF} \epsilon_{\alpha\beta} \partial^\beta \]
\[ \Delta_A^{(1)} = -i(g_{\alpha\beta} + 2k_{F\alpha\beta}) (\partial^\alpha \Delta^\beta - \Delta^\alpha \partial^\beta) - i k_{AF} \epsilon_{\alpha\beta} A^\beta \]
\[ \Delta_A^{(2)} = (g_{\alpha\beta} + 2k_{F\alpha\beta}) (\Delta^\alpha \Delta^\beta) \]
\[ \Delta_A^{(F)} = -[\tau_{\alpha\beta} + i k_{F\alpha\beta}] \Gamma^{\alpha\beta}. \] (22)
We compute an expansion for \( \logdet(P_A^{-1}) \) retaining terms which are linear in the small parameters \( k_F \) and \( k_{AF} \).
\[ \logdet(P_A^{-1}) = \logdet \left[ 1 + P_A^{-1}(\Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)}) \right]. \]
(23)
Note that \( P_A \) is actually independent of the background field and will cancel out of the functional determinant with proper normalization. The determinant is written in terms of the logarithm function using the relation \( \logdet S = \text{Tr} \log S \). The logarithm is then expanded as
\[ \text{Tr} \log(P_A^{-1}) = \text{Tr}(P_A^{-1}(\Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)})) - \frac{1}{2} \text{Tr}((P_A^{-1}(\Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)}))^2) + \text{h.o.} \] (24)
The analysis of the first term appearing in the expansion is straightforward: Since the Lie algebra elements trace to zero, the first term reduces to a quadratic divergence:
\[ \text{Tr}(P_A^{-1}(\Delta_A^{(1)} + \Delta_A^{(2)} + \Delta_A^{(F)})) = \text{Tr}(P_A^{-1}(\Delta_A^{(2)})) = tr \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \left( g^{\mu\nu} + 2(k_F)_{\mu\nu} \right) \partial^\mu(k_{AF}^\mu A^\nu(-k)). \] (25)
where, as before, the trace refers to both Lorentz and gauge space. To calculate the contribution which arises from the second order terms in the expansion, note that trace considerations immediately reduce the problem to studying the contribution arising from terms of the form \( P_A^{-1}(\Delta_A^{(1)}) P_A^{-1}(\Delta_A^{(1)}) \) and \( P_A^{-1}(\Delta_A^{(F)}) P_A^{-1}(\Delta_A^{(F)}) \). The first of the above terms produces a quadratic divergence that exactly cancels the quadratic divergence arising from the first order terms. A lengthy computation employing dimensional regularization gives the total Lorentz-violating divergent contribution as
\[ \logdet(P_A^{-1}) = \frac{i}{4} \text{Tr}(2 - \frac{d}{2}) \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \left( g^{\mu\nu} - 2k_{AF}^\mu A^\nu + k_{AF}^2 A^\mu A^\nu \right) - (12) \left( k_{F\mu\alpha\beta} (k_{AF}^\alpha k_{AF}^\beta A^\mu A^\nu) \right). \] (26)
Note that the trace over Lorentz indices has been performed in the above expression and only the gauge space trace remains. The contribution from the Lorentz violating CPT-odd terms is finite: there is no corresponding UV divergence. This calculation confirms the results obtained in [13].

We can analyze the contribution arising due to \( L_\psi \) in a similar manner. The mass-term \( M \) does not contribute any divergences to lowest order, so it is omitted from the calculation. The contribution from the kinetic piece is squared to facilitate computation
\[ - (\Gamma^\mu D_\mu)^2 = -P_\psi(1 - P_\psi^{-1}(\Delta_\psi^{(1)} + \Delta_\psi^{(2)} + \Delta_\psi^{(F)})) \] (27)
where
\[ P_\psi = (g^{\alpha\beta} + \{\gamma^\alpha, \Gamma_\beta^\gamma\}) \partial^\alpha \partial^\beta \]
\[ \Delta_\psi^{(1)} = -i(g^{\alpha\beta} + \{\gamma^\alpha, \Gamma_\beta^\gamma\}) (\partial^\alpha \Delta^\beta - \Delta^\alpha \partial^\beta) \]
\[ \Delta_\psi^{(2)} = (g^{\alpha\beta} + \{\gamma^\alpha, \Gamma_\beta^\gamma\}) (\Delta^\alpha \Delta^\beta) \]
\[ \Delta_\psi^{(F)} = - (S_{\alpha\beta}^\gamma + \frac{1}{2} [\gamma^\alpha, \Gamma_\gamma^\beta]) E_{\alpha\beta}. \] (31)
where \( \gamma \) are the standard gamma-matrices, \( \Gamma_1 \) is as defined in \( S_{\alpha\beta} \), and \( S_{\alpha\beta} \) are the generators for the Lorentz transformations in the spinor representation. With these adjustments, the expansion \([21]\) holds for the case of \( L_\psi \). Symmetry considerations imply that, to lowest order in the parameters, terms involving parameters other than \( \epsilon_{\mu\nu} \) and \( b_{\mu} \) do not contribute to the UV divergences in the pure Yang-Mills sector. Similarly, explicit calculation confirms that terms involving \( b_{\mu} \) do not contribute to
UV divergences. Thus, for the purpose of computing UV divergences arising from the Lorentz violating fermionic Lagrangian given by \([33]\) and \([39]\), only the terms involving \(c_{\mu\nu}\) are relevant, and we can proceed with the computation assuming that \(\Gamma^\mu_i\) is replaced by \(c^{\mu\nu}g_{\nu\mu}\) and \(M_i\) is replaced by 0 in \([33]\) and \([39]\), respectively, and that the associated simplifications are made in \([28\text{–}31]\). With these simplifications, we proceed as we did in the case of pure Yang-Mills. The analysis of the first order contribution leads to a quadratic divergence. More precisely, since \(\Delta_{\phi}^{(1)}\) is linear in the fields and the \(\Delta_{\phi}^{(F)}\) term will trace to zero, we have the Lorentz-violating contribution

\[
\text{Tr}(P^{-1}(\Delta_{\phi}^{(1)} + \Delta_{\phi}^{(2)} + \Delta_{\phi}^{(F)})) = \text{tr} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} (2c_{\mu\nu}) A^\mu(k) A^\nu(-k). \tag{32}
\]

Using dimensional regularization and following the same calculation as was carried out for pure Yang-Mills, the quadratic piece of the divergence is exactly cancelled by a term arising from a second order contribution. The total contribution associated to the second term in the expansion \([24]\) is

\[
\log\det(P^{-1}\Delta_{\phi}) = -\frac{2}{3} \frac{C(r)}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) c_{\mu\nu} Q^{\mu\nu}. \tag{33}
\]

where \(C(r)\) is given by the relation \(\text{tr}(\gamma^\mu \gamma^\nu) = C(r)\delta^{\mu\nu}\) and \(r\) refers to the fermion representation. The other new factor is defined as

\[
Q^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} (k^\mu k^\nu A^2 - 2k^\mu A^\nu k \cdot A + k^2 A^\mu A^\nu). \tag{34}
\]

To treat the ghost we write the quadratic contribution

\[
\mathcal{L}_G = -\bar{\phi}(\partial^\mu D^\mu - C_{\mu\nu} D^\mu D^\nu) \phi, \tag{35}
\]

and express the functional determinant using

\[
P_\phi = (-i^2 - C_{\mu\nu}\partial^\mu \partial^\nu) \quad \Delta_{\phi}^{(1)} = -i(g_{\mu\nu} + C_{\mu\nu}) (A^\mu \partial^\nu + \partial^\mu A^\nu) \quad \Delta_{\phi}^{(2)} = (g_{\mu\nu} + C_{\mu\nu}) (A^\mu A^\nu). \tag{37-38}
\]

Computing as above and using the notation introduced in \([34]\), the ghost contribution is given by

\[
\log\det(P^{-1}\Delta_{\phi}) = -\frac{i}{6} \frac{C_2(G)}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) C_{\mu\nu} Q^{\mu\nu}. \tag{39}
\]

**IV. RENORMALIZATION FACTORS**

In this section we complete the explicit one-loop renormalizability calculation. We begin by taking logarithms of the expression \([12]\) and substituting for the resulting divergences using \([26]\), \([29]\), and \([30]\). To treat the divergences arising from the pure Yang-Mills term, we proceed by noting that \(k_F\) can be treated as the sum of selfdual and anti-selfdual parts \([37]\). Noting that the selfdual contribution is trace-free, we match the structure of the Lagrangian to the structure of the corresponding singularity in the expression of the divergence \([20]\). The sum of corresponding terms is given by

\[
\mathcal{L}_0 + \delta \mathcal{L} = -\frac{1}{g^2} \left(1 - \frac{6g^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \right) (k_F)_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}. \tag{40}
\]

Rescaling the bare parameters \(g\) and \(k_F\) via

\[
g_f = Z_g g, \quad (k_F)_f = Z_{k_F} (k_F)_r, \tag{41-42}
\]

leads to

\[
\frac{(k_F)_f}{g_f^2} = \frac{Z_{k_F}^2}{Z_g^2} \frac{(k_F)_r}{g_r^2}. \tag{43}
\]

The calculation of \(Z_g\) produces the same result as the standard calculation for renormalizability of standard Yang-Mills

\[
Z_g = 1 - \frac{g^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \left(\frac{1}{16} C_2(G) - \frac{1}{3} n_f C(r)\right), \tag{44}
\]

where \(n_f\) is the number of fermion species assumed to be all in representation \(r\). When \(k_F\) is selfdual (that is, when \((k_F)_{\mu\nu\alpha\beta} = 1_4 \eta_{\mu\nu}(k_F)_{\lambda\kappa\sigma} \epsilon_{\rho\sigma\alpha\beta}\) see \([37]\)). This leads immediately to the scaling for \(k_F:\)

\[
Z_{k_F} = 1 + \frac{g^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \left(\frac{1}{16} C_2(G) + \frac{1}{3} n_f C(r)\right), \tag{45}
\]

which coincides with the trace-free result given in \([13]\) in the absence of fermions.

For the anti-selfdual contribution, we note that \((k_F)_{\mu\nu\alpha\beta}\) can be written in terms of \(\Lambda^{\mu\nu} = \frac{1}{2} k_F^{-1} \epsilon^{\mu\nu}\phi\),

\[
\Lambda^{\mu\nu} = \Lambda^{[\mu [\nu]} \delta^{\rho]} \beta]. \tag{46}
\]

In the absence of fermions and ghosts, term matching leads to the expression

\[
\mathcal{L}_0 + \delta \mathcal{L} = -\frac{1}{g^2} \left(1 - \frac{6g^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \right) \Lambda^{\mu\nu} Q^{\mu\nu}, \tag{47}
\]

where \(Q^{\mu\nu}\) is as given in \([31]\). Given the rescaling of \(g\) in Eq. \([44]\), we see that in this case \(\Lambda\) (and hence \(k_F\)) is unaffected by renormalization due to the pure Yang-Mills sector. Only the fermions and ghosts contribute.

Adding a fermion and the ghosts we use \([35]\) and \([39]\) and match terms to obtain

\[
\mathcal{L}_0 + \delta \mathcal{L} = -\frac{1}{g^2} \left(1 - \frac{6g^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \right) n_f C(r) \quad \mathcal{L}_0 + \mathcal{L}_F + \delta \mathcal{L} = \Lambda^{\mu\nu} Q^{\mu\nu}, \tag{48}
\]

with \(n_f = 1\) for one fermion. Defining the renormalized \(\Lambda\) parameter using

\[
\mathcal{L}_0 + \delta \mathcal{L} = -\frac{1}{g^2} \Lambda^{\mu\nu} Q^{\mu\nu} \tag{49}
\]
and defining
\[ \Lambda_b^{\mu\nu} = (Z_\Lambda)_{\alpha\beta}^{\mu\nu} \Lambda_\alpha^\beta \]
yields the relationship
\[ (Z_\Lambda)_{\alpha\beta}^{\mu\nu} \left[ \Lambda_\alpha^\beta + \frac{g_0^2}{6(4\pi)^2} \Gamma(2 - \frac{d}{2}) \left( C(r)C^{\alpha\beta} + C_2(G)C^{\alpha\beta} \right) \right] = Z_S \Lambda_b^{\mu\nu}, \]  
with
\[ Z_S = \left( 1 + \frac{4}{3} \frac{g_0^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) n_f C(r) \right). \]  
Finally, incorporating terms for arbitrary fermions is straightforward as the contributions simply add. This is accomplished in the above formula by letting \( n_f \) be arbitrary and making the replacement \( c \rightarrow \sum_f c_f \) as a sum over fermion species.

The CPT-odd terms \( k_{AF} \) contain no divergent contributions, however, they still need to be renormalized as the combination \( k_{AF}/g^2 \) appears in the classical action. This means that \( Z_{k_{AF}} = Z_g^2 \) at one-loop, in agreement with the result found in [13].

### V. BETA FUNCTIONS

Tacitly assuming for the moment that our renormalization prescription can be extended to all orders, the renormalization constants \( Z_{k_F} \) and \( Z_{k_{AF}} \) can be used to deduce the one-loop beta functions for these parameters. Following the developments presented in [1], use is made of
\[ \beta_{x_j} = \lim_{\epsilon \to 0} \left[ -\rho_{x_j} a_1^j + \sum_{k=1}^{N} \rho_{x_k} x_k \frac{\partial a_1^j}{\partial x_k} \right], \]  
where \( x_j \) represents an arbitrary running coupling in the theory, the parameters \( \rho_{x_j} \) are determined by comparing the mass dimension of the renormalized parameters to the bare parameters in \( d = 4 - 2\epsilon \) dimensions, and the \( a_1^j \) represent the first order divergent contribution to the rescaling factor associated to the variable \( x_j \). In more detail, writing
\[ x_{jb} = \mu^{\rho_{x_j} + \epsilon} Z_{x_j} x_j, \]
gives the values
\[ \rho_g = 1, \quad \rho_{k_F} = \rho_{k_{AF}} = 0, \]  
and the \( a_1^j \) are defined by the expansion
\[ Z_{x_j} x_j = x_j + \sum_{n=1}^{\infty} \frac{a_1^j}{\epsilon^n}. \]  
As in the QED case [1], the coupling \( g \) completely controls the running of the Lorentz-violating parameters. The resulting beta function for \( g \) is given by
\[ \beta_g = -\frac{g_0^3}{(4\pi)^2} \left( \frac{14}{3} C_2(G) - \frac{8}{3} n_f C(r) \right), \]  
the same as the conventional case. The beta function corresponding to \( k_{AF} \) is
\[ \beta_{k_{AF}} = -\frac{g_0^2}{(4\pi)^2} \left( \frac{14}{3} C_2(G) - \frac{8}{3} n_f C(r) \right) k_{AF}. \]  
where the Lorentz indices have been suppressed for simplicity. The selfdual part of \( k_F \) has the beta function
\[ \beta_{k_F} = \frac{g_0^2}{(4\pi)^2} \left( \frac{14}{3} C_2(G) + \frac{8}{3} n_f C(r) \right) k_F. \]
The anti-selfdual contributions coupled to the fermions and ghosts give
\[ \beta_{\Lambda^{\mu\nu}} = -\frac{g_0^2}{3(4\pi)^2} \left( C(r) \left[ \sum_j \epsilon_{j}^{\mu\nu} - 8 n_f \Lambda^{\mu\nu} \right] + C_2(G) C^{\mu\nu} \right). \]  
Note that special values \( c \) and \( \Lambda \) can lead to cancelation in the beta function.

### VI. BRST SYMMETRY

Due to the preservation of gauge invariance in the perturbed theory, the Lorentz-violating action satisfies a standard Becchi-Rouet-Stora-Tyutin (BRST) symmetry provided that there is no explicit ghost violation introduced. The ghost violation terms are not invariant under a standard BRST transformation, but a specific form for the gauge fixing term can be chosen to maintain invariance. However, this introduces additional violation into the photon propagator which can be absorbed by a better choice of gauge. This means that explicit Lorentz violation in the ghost sector alone can violate the gauge symmetry as well.

This symmetry should ensure that the multiplicative renormalization will be consistent to all orders by fixing the ratios of the relevant counter-terms in the renormalized lagrangian. This implies that all coupling constants \( g, k_F, \) and \( k_{AF} \) are universal as \( g \) is in the conventional case. An explicit proof of this fact to all orders is beyond the scope of the present paper. This fact agrees with the explicit one loop calculations performed in this paper.

### VII. SUMMARY

The functional determinant technique is particularly well suited to Lorentz violation loop calculations as the traces conveniently preserve both the observer Lorentz invariance and the gauge invariance throughout the calculation. Sums over special subclasses of diagrams are required to maintain a similar invariance using the diagrammatic approach [13]. In addition to the ease of organization of the calculation, the present approach is well suited to exploring renormalization in more complicated versions of the Lorentz-violating standard model.
New results of this paper include the contribution of the trace components of $k_f$, terms that were neglected in the previous paper [13]. These terms in fact renormalize differently than the trace-free $k_f$ components indicating their fundamentally different properties. Additional new results include the incorporation of fermions and explicit ghost violation. Both effects are accommodated by a renormalization of the trace components of $k_f$, while the trace-free components are unaffected. These additional results give the entire one-loop gluon sector effective action in Lorentz-violating QCD and demonstrate renormalizability at this order.

In addition, explicit BRST symmetry is present in the full Lorentz violating theory (without explicit ghost violation) and should be crucial in eventually establishing full renormalizability of the theory.

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