Equivalent Versions of Total Flow Analysis

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Abstract—Total Flow Analysis (TFA) is a method for conducting worst-case analysis of time-sensitive networks that are without cyclic dependencies. In networks with cyclic dependencies, Fixed-Point TFA introduces artificial cuts, analyses the resulting cycle-free network with TFA, and iterates on it. If it converges, it does provide valid performance bounds. We show that the choice of the specific cuts used by Fixed-Point TFA does not affect its convergence or the obtained performance bounds, and that it can be replaced by an alternative algorithm that does not use any cuts at all, while still applying to cyclic dependencies.

Index Terms—Delay bound, service curve, network calculus, FIFO system, deterministic networks, Total Flow Analysis (TFA)

1 INTRODUCTION

In time-sensitive networks, obtaining deterministic bounds is required on worst case delay, delay-jitter, and backlog. Total Flow Analysis (TFA) is a method for conducting worst-case analysis in such settings for per-class networks [1]. It is implemented in industrial software (WoPANets [2]) and can consider several important features such as the effect of packetization, regulators, and line shaping [3]. In a per-class network, the traffic of a given class is isolated from other classes by using mechanisms such as Deficit Round-Robin [4] or the Credit-Based Shaper [5]; inside a class, packets of all flows are handled in a FIFO manner. One of the main issues is that the burstiness of a flow increases as the flow travels through the network. If there is no cyclic dependency, i.e., if the graph induced by flow paths has no cycle, TFA first analyses edge-nodes by using network calculus [6, Section 1.4], then computes hop-by-hop the propagated burstiness. Propagated burstiness is derived from bounds on delay-jitters between a source and a point inside the network; such a bound is the sum of the delay-jitter bounds at the nodes on the path of a flow.

Computing performance bounds when there are cyclic dependencies is more challenging, as it is generally not known under which conditions deterministic bounds exist [7]. For such cases, the most recent version of TFA, Fixed-Point TFA (FPTFA [8]) introduces artificial network cuts; the resulting cycle-free network is analysed, and the output is fed back into the analysis. If the scheme converges, which is assumed to occur when the utilization is not too large, then [8] shows that the resulting fixpoint provides valid performance bounds. [8] shows that finding the minimal number of cuts to realize on a cyclic network is the most time consuming part of their algorithm, which raises the following question: Do the results of the worst-case analysis performed by FPTFA depend on the specific cuts chosen by the algorithm? Are some cuts better than others? Is there benefit in computing minimal cuts?

We show that the answer to all of these questions is negative: cuts do not matter. We obtain this by investigating alternative formulations of TFA that do not make cuts, while still applying to networks with cyclic dependencies. All formulations presented in this article are based on the same principle of TFA, but they differ by the instant and the nodes at which the burstiness and the delay-jitter bounds are updated. We distinguish two types of algorithms:

• synchronous: at every iteration, delay-jitter and burstiness bounds are updated at all nodes, based on the values of delay-jitter and burstiness bounds at the previous iteration;
• asynchronous: at every iteration, a set of nodes is visited, delay-jitter and output burstiness bounds are updated based on the burstiness bounds computed in previous iterations; nodes are visited according to some arbitrary scheme.

The synchronous algorithm is of theoretical use, it serves to show the equivalence of all schemes. The asynchronous algorithm is practical, as it combines a flexible choice of the sets of nodes visited at every iteration with the use of the most recent information. The alternating TFA algorithm, which is of particular interest in symmetric networks, is a special case of the asynchronous algorithm.

The contributions of this paper are as follows:

• We introduce versions of TFA that do not make cuts and apply to networks with or without cyclic dependencies. We show that all of these versions are equivalent; furthermore they are equivalent to TFA in networks without cyclic dependencies and to FPTFA in networks with cyclic dependencies. All algorithms converge or they all diverge; and if they all converge, they all give the same performance bounds.
• It follows that the behaviour and the result obtained by FPTFA do not depend on the chosen cut.
• We prove that all algorithms are correct, i.e., if they converge, they do provide valid delay-jitter bounds. The proof does not require any assumption on propagation times, unlike the original proof of FPTFA.
• To obtain these results, we provide a variant of the Knaster-Tarski fixpoint theorem and a new fixpoint theorem for concave functions.
In Section 2, we introduce the system model and, in Section 3, we describe TFA and FPTFA in a compact form suitable for analysis. In Section 4, we describe Synchronous TFA, and establish its validity (Theorem 1) and its equivalence with TFA/FPTFA (Theorem 3). In Section 5, we do the same for Asynchronous TFA (Theorem 4) and introduce Alternating TFA as a special case. In Section 6, we introduce a variant of the Knaster-Tarski fixpoint theorem (Theorem 6) and a new fixpoint theorem for concave functions (Theorem 7), both are at the heart of the theoretical analysis performed in the previous sections. In Section 7, we give a numerical illustration to a ring network.

2 System Model

2.1 Time-Sensitive Packet-Switched Network

We consider a packet-switched network. Each device in the network is composed of input ports, output ports, and a switch fabric, for example in Figure 1. Each input port contains a packetizer that releases the data when an entering packet has been entirely received. The packet is then transmitted through the switch fabric to the scheduler of one specific output port (as indicated by the routing table); then, it is serialized on the output line at the transmission rate of the line. The output port schedulers that separate traffic classes and are first-in-first-out (FIFO) inside a class. Each scheduler has a minimum service curve that is assumed to be a rate-latency service curve [6]. This models the service isolation provided to a traffic class; for example, rate-latency service curves are given in [4] for Deficit Round-Robin and [5] for the Credit-Based Shaper. A rate-latency service curve has two parameters: a rate that captures the service rate guaranteed to the class; and a latency that captures an additional delay that has to be considered when computing delay bounds.

In the remainder of the paper, we focus on one of these traffic classes that is assumed to receive a deterministic service [9]. This means that every flow in the class (which can be unicast or multicast) has a fixed path and is constrained at the source. Specifically, we assume that flow $f$ is constrained at its source by an arrival curve $\alpha_f(t)$ of the form:

$$\alpha_f(t) = r_f t + b_f,$$

i.e., a “leaky bucket” arrival curve with rate $r_f > 0$ and burstiness $b_f > 0$. The arrival curve constraint means that over any time interval of any duration $t$, flow $f$ should not generate more than $\alpha_f(t)$ bits. Let $L_{max} > 0$ and $L_{min}$ be, respectively, the maximum and minimum packet size over all flows.

The deterministic service means that bounds on delay, delay-jitter (defined as the difference between worst-case and best-case delays) and backlog are computed at every node, using the source arrival curves and the rate-latency service curves. The challenge with such computations is that the burstiness of a flow increases after crossing a node and the resulting “propagated burstiness” needs to be estimated. We assume that local stability holds at every node $i$, i.e., the sum of the rates $r_f$ for all flows that cross node $i$ is less than or equal to the rate of the rate-latency service curve of node $i$. This is a necessary condition for the existence of finite performance bounds (but not sufficient [7]).
dition mentioned in the previous section ensures that the network is stable, and deterministic performance bounds can be computed at every node. In contrast, cyclic dependencies can make the network unstable in the sense that no finite bound exists for worst-case delay-jitter [7] even when local stability holds. Algorithms, such as FPTFA, and the algorithms in this paper compute delay-jitter bounds when they converge; when they diverge, it could be that the network is truly unstable or not. In practice, this latter case occurs when the network utilization is close to 100%.

2.3 Notation List

We use the following notation, illustrated on the example of Figure 1.

- \( \mathcal{I} \) is the set of nodes in the graph, they correspond to output ports; \( n = |\mathcal{I}| \).
- Nodes are called terminal when they lead to flow sinks (denoted with \( s \) in Figure 1).
- Nodes are called non-terminal when they lead to other nodes (denoted with \( O \) in Figure 1).
- Each flow \( f \) has a path \( \text{Path}(f) \) that is a sequence of connected nodes where the last element is terminal. We assume that a path has no loop, i.e., that a node appears at most one time in the path of \( f \). For example, for flow \( f_{br} \):

\[
\text{Path}(f_{br}) = (O5, O4, O11, O8, O9, os4) .
\]

- A flow \( f \) is called fresh at a node if this node is the first element of \( \text{Path}(f) \).
- \( \mathcal{L} \) is the set of transit edges in the graph, i.e., the set of edges that lead to non-terminal nodes (transit links in Figure 1 top). In the physical network, it corresponds to the transit links, i.e., the set of links that carry at least one transit flow (i.e., a flow that is neither fresh nor terminal). For example,

\[
(O8, O9) \in \mathcal{L} \text{ but } (O9, os1) \notin \mathcal{L} .
\]

- \( \forall i \in \mathcal{I} \), \( \text{In}(i) \subset \mathcal{L} \) is the set of transit edges that are incident to node \( i \). For example,

\[
\text{In}(O6) = \{(O1, O6)\}.
\]

- \( \forall i \in \mathcal{I} \), \( \text{Out}(i) \subset \mathcal{L} \) is the set of transit edges that leave node \( i \). For example,

\[
\text{Out}(O7) = \{(O7, O8), (O7, O12)\} .
\]

- \( \forall \ell \in \mathcal{L} \), \( \text{Pred}(\ell) \) denotes the set of nodes that are crossed by at least one flow that is present in \( \ell \), and that are upstream of \( \ell \) for such flows. In other words, node \( i \in \text{Pred}(\ell) \) if and only if there exists one flow \( f \) such that node \( i \) and edge \( \ell \) are on the path of flow \( f \), in this order. For example, for \( \ell = (O3, O2) \),

\[
\text{Pred}(\ell) = \{O3, O4, O5, O6, O7, O8, O9, O10, O12\} .
\]

We also introduce \( \text{Pred}_{f}(\ell) \) as the set of nodes that are crossed by flow \( f \) and that are upstream of \( \ell \). For \( \ell = (O3, O2) \), and flow \( f_{j} \):

\[
\text{Pred}_{f}(\ell) = \{O3, O4, O5\} .
\]

Recall that flow \( f \) is constrained at the source by a leaky bucket with rate \( r_{f} \) and burstiness \( b_{f} \). The algorithms in this paper estimate bounds on delay-jitter and propagated burstiness. The notation for these is as follows.

- \( \forall \ell \in \mathcal{L} \), \( z_{\ell} \) is a vector that has one component for every transit flow that is carried by \( \ell \). Every component is an upper bound on the propagated burstiness of the flow on the transit link \( \ell \) (denoted by \( b_{f}^{\ell} \)). With \( \ell = (O6, O7) \):

\[
z_{\ell} = \left( b_{f}^{O6, O7}, b_{f}^{O6, O7}, b_{f}^{O6, O7} \right) .
\]

- \( \forall M \subset \mathcal{L} \), \( z_{M} \) is the set of \( z_{\ell} \) for all \( \ell \in M \). In particular, \( z_{\mathcal{L}} \) denotes the collection of all bounds on propagated burstiness.

- \( \forall i \in \mathcal{I} \), \( d_{i} \) is a vector that is meant to contain an upper bound on the delay-jitters at node \( i \) for every flow \( f \) (transit or not) that uses node \( i \). The delay-jitter at a node is defined as the worst-case delay minus the best-case delay; it is computed per bit, from entrance to the packetizer to exit out of the output port. The delay includes the packetization delay; if all flows have maximum and minimum packet length and all links have same rate, then all flows have the same delay-jitter bound and \( d_{i} \) is a single number; else the delay will depend on the input port and the packet sizes, hence depends on the flow, not just the output port \( i \) [10].

- \( \forall J \subset \mathcal{I} \), \( d_{j} \) denotes the collection of all \( d_{i} \) for \( i \in J \). In particular, \( d_{\mathcal{L}} \) is the collection of all delay-jitter bounds.

The following operators are introduced in subsequent sections and are given here for completeness.

\[
\begin{align*}
\mathcal{D} & \quad d = D(z) \text{ is a vector of delay-jitter bounds derived from propagated burstinesses (Section 3.1 used by all algorithms).} \\
\mathcal{G} & \quad G(z, d) \overset{\text{def}}{=} (Y(d), D(z)) \text{ (Section 4.1).} \\
\mathcal{Y} & \quad y = Y(d) \text{ is a vector of propagated burstinesses derived from delay-jitter bounds (Section 4.1 used by SyncTFA, AsyncTFA and AltTFA).} \\
\mathcal{Z} & \quad z' = Z(d, z) \text{ is a vector of output burstinesses derived from delay-jitter bounds and from input burstinesses (Section 3.1 used by TFA and FPTFA).}
\end{align*}
\]

Beyond delay-jitter, the deterministic performance bounds of delay and backlog are derived from \( z_{\mathcal{L}} \) and from fixed parameters of the network, hence we do not need to consider them further.

3 State-of-the-Art Algorithms: TFA [11] and FPTFA [8]

In this section, we present two state-of-the-art algorithms that compute worst case bounds: The former, Total Flow Analysis (TFA) [11], is used for networks without cyclic dependencies on the graph induced by flow path; the latter, Fixed-Point Total Flow Analysis (FPTFA) [8], applies to cases with cyclic dependencies. Our presentation is more compact than in the original references and is used to support the theoretical analysis in this paper.
3.1 Total Flow Analysis (TFA) [11]

The Total Flow analysis (TFA) algorithm [11] performs worst-case analysis on graphs without cyclic dependencies (see an example of feed-forward network on Figure 2). TFA uses network calculus to analyse each node after the other, starting with nodes that have only fresh input flows (such nodes must exist when there is no cyclic dependency). For each node, TFA computes the delay-jitter bound and the output burstiness bound. The output burstiness is then used as input for the following nodes, in the sense of the flows. Iteratively TFA computes all delay-jitter bounds and burstiness bounds, for all nodes and links of the network.

TFA [11] bounds are then improved by [3] with TFA++ by taking into account the effect of line-shaping constraints at the input and the output of each node. Then [3] obtains a tighter delay-jitter bound within nodes by considering the effect of packetizer.

The first step of the algorithm is to create a topological order [12] of the acyclic directed graph by labelling the nodes of the network \( i_1, \ldots, i_n \). It means that the input of a node \( i_k \) belongs to the output of the previous nodes \( \{i_1, \ldots, i_{k-1}\} \). Specifically, \( \text{In}(i_1) = \emptyset \) (all flow that enters in node \( i_1 \) are fresh) and

\[
\forall k \in \{2, \ldots, n\}, \text{In}(i_k) \subset \text{Out}(i_1) \cup \cdots \cup \text{Out}(i_{k-1}).
\]

Let us take the example of the feed-forward network in Figure 2. This network is composed of three flows \( f_r, f_{bl}, f_{gr} \), that first cross node \( O6 \). The only possible labelling is \( \{O6, O7, O8, O9, O10\} \).

Then the second step is the Algorithm 1 that performs an analysis of each node in the order of the labelling by computing delay-jitter and output bursts.

Algorithm 1 Total Flow Analysis (TFA)

1. \( z_l \leftarrow 0, \forall l \in L \)
2. \( d_i \leftarrow 0, \forall i \in I \)
3. for \( k \leftarrow 1 \) to \( n \) do
4. \( d_{i_k} \leftarrow D_{i_k}(z_{\text{In}(i_k)}) \);
5. \( \forall l \in \text{Out}(i_k), z_l = Z_{i_k}(d_{i_k}, z_{\text{In}(i_k)}) \);
6. end for
7. print\((z, d)\);

To perform this analysis, TFA uses network calculus methods to derive delay-jitter and burstiness bounds at every node, abstracted as follows:

- \( \forall i \in I, D_i \) is the function that gives a delay-jitter bound \( d_i \) for a node \( i \). It is defined as follows: \( d_i = D_i(z_{\text{In}(i)}) \), where \( z_{\text{In}(i)} \) is the burstiness bounds of every transit flow that enters into \( i \). A formulation of the \( D_i \) function is given in [8, Eq.(3)]. Note that \( D_i \) accounts for the burstiness of all flows, transit or fresh, but only the burstiness of transit flows is captured by the argument, as the burstiness at the source is assumed to be fixed. As shown in [8, Eq.(3)], the delay-jitter at each node is the minimum of affine functions in the input burstinesses.

For example, in Figure 2, the delay-jitter at node \( i = O6 \) is

\[
d_{\text{O6}} = d_{\text{O6}}^r + d_{\text{O6}}^{bl} + d_{\text{O6}}^{gr} = D(z_{\text{InO6}})
\]

with \( z_{\text{InO6}} \) the input vector of burstinesses at node \( O6 \) defined in Eq. (3).

\[
z_{\text{InO6}} = (b_{\text{InO6}}^r, b_{\text{InO6}}^{bl}, b_{\text{InO6}}^{gr}).
\]

- \( \forall i \in I, Z_i \) is the mapping that provides a collection of burstiness bounds for all transit flows at the output of node \( i \). It is defined as follows: \( z_{\text{Out}(i)} = Z_i(d_i, z_{\text{In}(i)}) \) where \( d_i \) is the delay-jitter bound of node \( i \) and \( z_{\text{In}(i)} \) is the input-burstiness bounds at this node. Every coordinate of \( Z_i \) is affine in its arguments, as propagated burstiness is simply equal to the input burstiness plus the rate of the flow times the delay-jitter for the flow at this node.

For example in Figure 2, the output burstiness vector at node \( O6 \) is:

\[
z_{\text{Out}(O6)} = Z_i(d_{\text{O6}}, z_{\text{InO6}}) = (b_{\text{InO6}}^r, b_{\text{InO6}}^{bl}, b_{\text{InO6}}^{gr}).
\]

Note that in this example \( \text{In}(O6) = \emptyset \), \( z_{\text{Out}(O6)} \) depends on the burstinesses at source:

\[
\begin{align*}
b_{\text{InO6}}^r &= b_{\text{InO6}}^r + r_f d_{\text{O6}} \quad (4) \\
b_{\text{InO6}}^{bl} &= b_{\text{InO6}}^{bl} + r_{bl} d_{\text{O6}} \quad (5) \\
b_{\text{InO6}}^{gr} &= b_{\text{InO6}}^{gr} + r_{gr} d_{\text{O6}} \quad (6)
\end{align*}
\]

At the output of node \( O6 \), with function \( D \) and \( Z \), delay-jitter and output burstiness bounds have been computed (Lines 4-5 of Algorithm 1). Then the same analysis is performed for all the other nodes in sequence \( \{O7, O8, O9, O10\} \) to finish the worst-case analysis.

Let \( Z \) and \( D \) be compact notations for these mappings, i.e.,

\[
(z' = Z(d, z) \iff (z_{\text{Out}(i)} = Z_i(d_i, z_{\text{In}(i)}), \forall i \in I) \quad (7)
\]

\[
(d = D(z)) \iff (d_i = D_i(z_{\text{In}(i)}), \forall i \in I) \quad (8)
\]

The local stability conditions ensures that \( D \) is well defined, i.e., returns finite values for all finite values of its arguments.

3.2 Fixed-Point Total Flow Analysis (FPTFA) [8]

In the case where we have cyclic dependencies, such as in Figure 3, an example of feed-forward network on Figure 2, TFA is applied.

Fig. 2. A Feed-forward network with three flows \( f_r, f_{bl}, f_{gr} \) on which TFA is applied.
In the example of the bottom figure in Figure 3, a possible node labelling is 
\[(O_6, O_7, O_5, O_4, O_{11}, O_{12}, O_8, O_3, O_9, O_2, O_{10}, O_1).\]

Note that, in this paper, In() and Out() always refer to the original, uncut, graph.

FPTFA is described in Algorithm 2. In one iteration, FPTFA computes a new value \(z'_f\) of the vector of burstinesses at the cut and then prints the value of the delay-jitter and burstiness bounds \((z, d)\) valid for the cut network. We call \(F^L\) the mapping that transforms the vector of burstinesses \(z_L\) into \(z'_f\), specifically, as described in lines 4-8 of Algorithm 2. This is the same as the function called \(FF\) in [8], where we highlight the dependency on the cut \(L\). It follows that FPTFA computes the successive iterated of \(F^L\), starting with initial value \(z_L = 0\).

Algorithm 2 Fixed-Point TFA (FPTFA)

1: \(z_L \leftarrow 0, \forall \ell \in L\);
2: \(d_i \leftarrow 0, \forall i \in I\);
3: while true do
4:     for \(k \leftarrow 1\) to \(n\) do
5:         \(d_{ik} \leftarrow D_{ik} \big( z_{in(i_k)} \big) \);
6:         \(\forall \ell \in \text{Out}(i_k) \setminus L, z'_\ell = Z_{ik} \big( d_{ik}, z_{in(i_k)} \big) \);
7:         \(\forall \ell \in \text{Out}(i_k) \cap L, z'_\ell = Z_{ik} \big( d_{ik}, z_{in(i_k)} \big) \);
8:     end for
9:     \(z_L \leftarrow z'_f, \forall \ell \in L\)
10:     print \((z, d)\);
11: end while

Theorem 2 in [8] proves that if the successive iterated of the burstinesses at cut \(z_L\) converge, and if the network is empty at time 0, then values of \(d\) and \(z\) computed by FPTFA are valid bounds for the original, non cut, network.

We notice that when the network is without cyclic dependencies and the cut of FPTFA is \(L = \emptyset\), then FPTFA and TFA compute the same bounds. Indeed, indefinitely FPTFA prints at each iteration the same burstinesses and delays bounds as TFA outputs.

Back to a network with cyclic dependency, a question that arises is whether the choice of a cut influences the convergence and the value of the bounds computed by FPTFA. We answer this question in the following section by introducing a new algorithm SyncTFA. We prove that FPTFA with cut \(L\) and SyncTFA compute the same bounds, therefore that the choice of cut has no influence on the end-result.

4 **Synchronous TFA**

We now introduce SyncTFA, a new algorithm that simultaneously updates delay-jitter bounds at every node and propagated burstinesses at all transit links. It applies to networks without or with cyclic dependencies. In the former case, it stops after a number of iterations and is equivalent to TFA. In the latter case, it iterates until it either finds a fixpoint or it diverges. The particularity of this new algorithm is that, contrary to FPTFA, it does not require cutting the network.
4.1 Definition of SyncTFA

SyncTFA updates the delay-jitter bounds by using the same method as TFA and FPTFA but for, propagated burstiness, uses a method that slightly differs, and that now describe. We need to introduce \( \forall \ell \in \cal{L}, \cal{Y}_\ell \) a mapping such that \( z_\ell = \cal{Y}_\ell (d_{\text{Pre}(\ell)}) \) is a collection of valid burstiness bounds for all transit flows present in edge \( \ell \), when the delay-jitters at all upstream nodes are given by \( d_{\text{Pre}(\ell)} \). Every coordinate of \( \cal{Y}_\ell \) is a propagated burstiness; it is equal to the burstiness at the source (considered to be a constant) plus the rate of the flow times the accumulated delay-jitter bound on the path of the flow. For example, for \( \ell = (O3, O2) \), the first coordinate of \( \cal{Y}_\ell \) returns

\[
z_\ell^1 = b_{(O3,O2)}^f + \sum_{i \in \{O3,O12,O7,O6\}} d_i^f + b_{S2}^f.
\]

In contrast, \( Z \), used in TFA and FPTFA, estimates the output burstiness at a node from the delay-jitter bounds at this node and the input burstiness.

Note that \( \cal{Y}_\ell \) is affine.

Also introduce \( \cal{G} \), such that \( \cal{G}(z, d) \) is a propagated burstiness; it is equal to the burstiness at a node from the delay-jitter bounds at this node and the input burstiness.

The iteration \( k \) of SyncTFA prints \((z^k, d^k) = \cal{G}^k(0, 0)\).

4.2 Correctness of SyncTFA

In this Section we show the correctness of SyncTFA, which means that if the successive iterates of SyncTFA converge, they provide valid bounds for propagated burstiness and delay-jitter. This is a similar result as Theorem 2 in [8] (which is for FPTFA), but our method of proof is simpler and does not require any assumptions on propagation delays or initial network state.

We use the time-stopping method [6], which can be cast as follows. Consider that the network starts at time 0 in some arbitrary state and fix some time \( \tau \geq 0 \). Modify the sources such that they stop sending after time \( \tau \) and call \( \cal{N}^\tau \) the resulting modified network. Since all sources are constrained at the sources and \( \tau \) is finite, the number of bits that ever exist in \( \cal{N}^\tau \) is finite. Therefore, the worst-case delays and propagated burstinesses, \( d^\tau \) and \( z^\tau \) are finite. By network calculus, \( \cal{Y}(d) \) provides valid burstiness bounds at every transit link \( \ell \) and \( \cal{D}(z) \) provides valid burstiness bounds at every node \( i \) whenever \( d \) and \( z \) are also valid bounds. Therefore, in \( \cal{N}^\tau \), \( \cal{Y}(d^\tau) \) and \( \cal{D}(z^\tau) \) are valid bounds. Since \( d_i^\tau \) and \( z_i^\tau \) are minimal bounds, it follows that

\[
z_i^\tau \leq \cal{Y}_i(d^\tau) \text{ and } d_i^\tau \leq \cal{D}_i(z^\tau) \text{ for every node } i \text{ and transit link } \ell.\]

In compact form:

\[
(z^\tau, d^\tau) \leq \cal{G}(z^\tau, d^\tau)
\]

Let us introduce the following set:

\[
\text{Low}(\cal{G}) = \{(z, d) \in \cal{G}(z, d) \}
\]

It follows that, for every \( \tau \geq 0 \)

\[
(z^\tau, d^\tau) \in \text{Low}(\cal{G})
\]

Back to the original network, let \((z(t), d(t))\) be the worst-case burstinesses and delays observed in the interval \([0, t]\), where \( t \geq 0 \). By causality, in any network, the worst-case delays and burstinesses that can be observed in the time interval \([0, t]\) depend only on the history of the network up to time \( t \). It follows that

\[
(d(t), z(t)) \leq (d^\tau, z^\tau) \text{ if } t \leq \tau
\]

Combining \([13]\) and \([14]\) we see that, if \( \text{Low}(\cal{G}) \) is bounded, an upper-bound on \( \text{Low}(\cal{G}) \) provides valid bounds for the original network. This explains why sets such as \( \text{Low}(\cal{G}) \) play an important role in the analysis of the algorithms in this paper.

The following theorem establishes a proof of correctness of SyncTFA.

**Theorem 1.** If \( \text{Low}(\cal{G}) \) is bounded, then

1) \( \cal{G} \) has a unique fixpoint \((\bar{z}, \bar{d})\).
2) \( \bar{d} \) is a set of valid delay-jitter bounds at all nodes and \( \bar{z} \) gives valid burstiness bounds at all transit links.
3) The SyncTFA sequence \((z^k, d^k)\) converges to \((\bar{z}, \bar{d})\).

If \( \text{Low}(\cal{G}) \) is unbounded, then \( \cal{G} \) has no fixpoint and the SyncTFA sequence does not converge.

It follows from Theorem [1] that if the SyncTFA sequence \((z^k, d^k)\) converges, the limit gives a set of valid delay-jitter bounds at all nodes.

**Proof.** \( \cal{G} \) is isotonic, i.e., \( \cal{G}(z, d) \leq \cal{G}(z', d') \) whenever \( z \leq z' \) and \( d \leq d' \) (comparison is coordinatewise). Now \((0, 0) = (z^0, d^0) \leq (z^1, d^1) \), hence \((z^k, d^k) \leq (z^{k+1}, d^{k+1}) \) for all \( k \geq 0 \). Furthermore, \((z^k, d^k) \in \text{Low}(\cal{G}) \) because \((z^{k+1}, d^{k+1}) = \cal{G}(z^k, d^k) \). Also, \( \cal{G} \) is concave because \( \cal{Y} \) is affine and \( D \) is also affine by the definition of \( D \) in [8] Eq.(3). Last, \( \cal{G}(0, 0) > 0 \), because

- There is a packetizer at each input port and \( \ell_{\text{max}} > 0 \), thus \( D(0) > 0 \).
- By Theorem 1 in [8] the burstiness increase after a packetizer is \( \frac{c}{\ell_{\text{max}}} \) (with \( r \) the rate of the flow aggregate and \( c \) the transmission rate of the line at the input port), thus \( \cal{Y}(0) > 0 \).

Now assume that \( \text{Low}(\cal{G}) \) is bounded. By Corollary [1], \( \cal{G} \) has a unique fixpoint, say \((\bar{z}, \bar{d})\), which proves 1). Also \((z(t), d(t)) \leq (z^\tau, d^\tau) \in \text{Low}(\cal{G}) \) by the time-stopping method and, by Corollary [1], \((\bar{z}, \bar{d}) \) is the largest element of \( \text{Low}(\cal{G}) \); this proves 2). Also, \((z^k, d^k) \leq (\bar{z}, \bar{d}) \), hence the sequence converges to a finite limit. Since \( \cal{G} \) is continuous, the limit is a fixpoint of \( \cal{G} \); by uniqueness of the fixpoint, the limit is \((\bar{z}, \bar{d})\), which proves 3).

If \( \text{Low}(\cal{G}) \) is unbounded, then by Corollary [2], \( \cal{G} \) has no fixpoint and diverges.
SyncTFA, like all algorithms in this paper, is built for networks with cyclic dependencies, but also applies to networks without cyclic dependencies. When the network has no cyclic dependencies, Theorem 1 establishes that SyncTFA is stationary after a finite number of iteration and that it provides valid delay-jitter bounds.

To define Theorem 1, we use the notion of level number, defined as follows. The level number of a node i, denoted with Level(i), is the length of the longest directed path to this node on the graph induced by flows. Here, note that a path is a concatenation of adjacent edges in the graph induced by flows. It need not be a path followed by a flow of the original network.

The following properties hold for the level numbers:
1) If a node i ∈ I has a level number of k = 1, then there are only fresh flows at i. Thus for this node i, we have In(i) = ∅.
2) If node i ∈ I has level number k, then:
∀f s.t. i ∈ Path(f), ∀j ∈ Pred(f)((·, i)), Level(j) ≤ k − 1 (15)

The number of levels of a network is the maximum level number of all its nodes.

For example, in the bottom of Figure 3, the graph induced by flows has 6 levels: Nodes with level number 1 are O6 and O5. O1 and O0 have level number 6. They have a respective longest directed path (O6, O7, O12, O3, O2, O1) and (O5, O4, O11, O8, O9, O10).

Theorem 2. In a network without cyclic dependencies and with k levels, SyncTFA is stationary in at most 2k steps.

It follows from Theorems 1 and 2 that, in a network without cyclic dependencies, Low(Ḡ) is bounded and SyncTFA always computes valid delay-jitter and burstiness bounds.

Proof. We note z(i,j) the value of z after iterations of the while loop of FPTFA (and, respectively, d(i,j) for d). Let us prove by induction on the level u ∈ {1, · · · , k} the following property H(u):

\[ H(u) = \begin{cases} \forall i \in I \text{ s.t. Level}(i) = u, d_i^{(2u)} = d_i^{(2u-1)} \\ \forall i \in I \text{ s.t. Level}(i) = u, z_i^{(2u)} = z_i^{(2u-1)} \end{cases} (16a) \]

\[ \forall i \in I \text{ s.t. Level}(i) = u, z_i^{(2u)} = z_i^{(2u-1)} \] (16b)

• Base step, u = 1: Let i ∈ I such that Level(i) = 1.
B-1) By applying Line 4 of Algorithm 3 at the first loop iteration, we have d_i^{(1)} = D_i(z_{In(i)}^{(0)}). The same line for the second loop iteration gives d_i^{(2)} = D_i(z_{In(i)}^{(1)}).
By definition of the level number, as Level(i) = 1, there are only fresh flows at input of node i. Hence, In(i) = ∅, and thus d_i^{(1)} and d_i^{(2)} are independent of z. Therefore, d_i^{(2)} = d_i^{(1)} and Eq. (16a) is true.
B-2) As there are only fresh flows, for each fresh flow f at node i, the estimated propagated burstinesses at the output links are

\[ [z_{(i,j)}^{(3)}]_f = [Y_{(i,j)}(d_{Pred((i,j))}^{(2)})]_f = r_fd_i^{(2)} + b_f \]

With B-1), d_i^{(2)} = d_i^{(1)}, thus z_{(i,j)}^{(3)} = z_{(i,j)}^{(2)} by definition of Y. As a result, Eq. (16b) is true, and H(1) is true.

• Induction step: Assume H(1), · · · , H(u − 1) and let us show H(u). Let i ∈ I such that Level(i) = u.
I-1) By applying Line 4 of Algorithm 3 at iteration 2u, d_i^{(2u)} = D_i(z_{In(i)}^{(2u-1)}).
By definition, In(i) is the set of transit edges that are incident to node i, thus

∀j ∈ In(i), ∃f such that i ∈ Path(f)

With Eq. (15), Level(j) ≤ u − 1. And by induction assumption, ∀j ∈ In(i), z_j^{(2(u-1)+1)} = z_j^{(2(u-1))}, thus z_j^{(2u-1)} = z_j^{(2u-2)}. Therefore d_i^{(2u)} = D_i(z_{In(i)}^{(2u-1)}) = D_i(z_{In(i)}^{(2u-2)}) = d_j^{(2u-1)}. Hence, Eq. (16a) is true.
I-2) Let f such that i ∈ Path(f), the output burstiness at node i for iteration 2u + 1 for flow f is

\[ [z_{(i,j)}^{(2u+1)}]_f = Y_{(i,j)}(d_{Pred((i,j))}^{(2u)}) \]

∀j ∈ Pred((i,j)), Level(j) ≤ u, hence with I-1).

Therefore,

\[ [z_{(i,j)}^{(2u+1)}]_f = Y_{(i,j)}(d_{Pred((i,j))}^{(2u-1)}) = \ldots = z_{(i,j)}^{(2u)} \]

where the second equality comes from the definition of Y. Therefore, z_{(i,j)}^{(2u+1)} = z_{(i,j)}^{(2u)}, Hence, Eq. (16b) is true.

\[ \square \]

4.3 Equivalence of SyncTFA and FPTFA / TFA

We now return to the question of Section 3.2: Does the choice of a cut influence the convergence and the value of the bounds computed by FPTFA? To prove that this is not the case, we prove that SyncTFA and FPTFA with cut L compute the same thing. Specifically, Theorem 3 shows that Algorithm 3 and 2 both diverge or both converge; and if they both converge, they obtain the same delay-jitter and burstiness bounds. This holds for any valid cut, i.e., any cut that leaves the network free of cyclic dependency.

Theorem 3. 1) Low(Ḡ) bounded ⇔ Low(F L) bounded.
2) If Low(Ḡ) and Low(F L) are bounded then
   a) G has a unique fixpoint (, ) and F L has a unique fixpoint z L;
   b) Let d∗ be the collection of delay-jitter bounds computed by FPTFA; then z L = z _L and d = d∗.

Proof. Part A. In this part, we show that if (, ) is a (finite) fixpoint of G, then F L (z L) = z L.

To prove this implication, we assume that (, ) is a (finite) fixpoint of G and we construct z f, z L and d by applying the same construction as in the inner loop of FPTFA, which enables us to compute F L (z L). We execute specifically the following algorithm with input value z L, i.e., we compute FPTFAITER(z L).

1: function FPTFAITER(z L)
2:   ∀ℓ ∈ L, z ℓ = z ℓ
3: for k ← 1 to n do
4:   d ik ← D ik (z Ik)
5:   ∀ℓ ∈ Out(ik) \ L, z ℓ = Z ik (d ik, z Ik)
6:   ∀ℓ ∈ Out(ik) \ L, z ℓ = Z ik (d ik, z Ik)
7: end for
8. Output($z'_L, z_{\text{Out}}(L), d$)
9. \[ z'_L \text{ is equal to } \mathcal{F}^L(z'_L) \]
10. end function

We note $z^k_\ell$ the value of $z_\ell$ after the $k$-th iteration (respectively $z_\ell^{k+1}$ for $z'_\ell$). FPTFAiter($z_L$) has the following properties:

\[ \forall \ell \in L, \forall k \in \{1, ..., n\}, z_\ell^k = z_\ell \quad (18) \]

If \( \ell \in \text{Out}(i_{k'}) \setminus L, \forall k \geq k', z_\ell^k = z_\ell^{k'} \) \quad (19)

If \( \ell \in \text{Out}(i_{k'}) \cap L, \forall k \geq k', z_\ell^k = z_\ell^{k'} \) \quad (20)

Eq. (18) holds as in line 4 of \( \forall \ell \in L, z_{\ell} \) is not assigned. Eq. (19) - (20) hold as \( \forall \ell, \exists k, \text{s.t. } \ell \in \text{Out}(i_k) \). By induction on \( k \in \{1, ..., n\} \), we now prove the following property $P_1(k)$:

\[
P_1(k) = \begin{cases} 
\forall \ell \in \text{Out}(i_{k}) \setminus L, \forall f \text{ s.t. } i_k \in \text{Pred}_f(\ell), \\
z_{\ell,f}^k = z_{\ell,f} \\
\forall \ell' \in \text{Out}(i_{k}) \cap L, \forall f \text{ s.t. } i_k \in \text{Pred}_f(\ell'), \\
z_{\ell,f}^k = z_{\ell,f} 
\end{cases}
\]

\[
\text{ Base step, } k=1: 
\]

B1) By line 4, $d_{i_1} = D_{i_1}(z_{\text{In}(i_1)})$. By Eq. (11a), $z_{\text{In}(i_1)} \subseteq L$. By fixpoint assumption of $G$, $d_{i_1} = D_{i_1}(z_{\text{In}(i_1)})$.

Thus, $d_{i_1} = d_{i_1}$.

B2) Let $\ell \in \text{Out}(i_{k}) \cap L$, let $f$ a flow such that $i_k \in \text{Pred}_f(\ell)$.

a) Either $i_k$ is the first hop on the path of $f$ (fresh flow at $i_k$) and by line 5 and definition of $Z$, $z_{\ell,f}^k = b_f + r_f d_{i_k}$.

As $d_{i_k} = d_{i_k}$, $z_{\ell,f}^k = b_f + r_f d_{i_k}$.

As a result, $z_{\ell,f}^k = z_{\ell,f}$.

b) Or $i_k$ is not the first hop on the path of $f$ and \exists $\ell'' \in \text{In}(i_k)$ such that $\ell'' \in L$, and by line 2 and definition of $Z$, $z_{\ell,f}^k = z_{\ell,f} + r_f d_{i_k}$.

By fixpoint of $G$ and definition of $\mathcal{Y}$, $z_{\ell,f} = b_f + r_f d_{i_k}$. By concatenation of paths, since $\ell'' \in \{i_1, \ldots, i_{k-1}\}$, and $\ell = (i_{k-1}, \ldots, i_1)$, $\text{Pred}_f(\ell) = \{i_1\} \cup \text{Pred}_f(\ell'')$.

Therefore, we have

\[
z_{\ell,f}^k = b_f + r_f \sum_{u \in \text{Pred}_f(\ell'')} d_u.
\]

By fixpoint of $G$ and definition of $\mathcal{Y}$,

\[
z_{\ell,f} = b_f + r_f \sum_{u \in \text{Pred}_f(\ell)} d_u.
\]

As a result, $z_{\ell,f}^k = z_{\ell,f}$ and Eq. (21a) is true for $k = 1$.

B3) Let $\ell \in \text{Out}(i_{k}) \cap L$, let $f$ a flow such that $i_k \in \text{Pred}_f(\ell)$. Either $i_k$ is the first hop on the path of $f$ (fresh flow), or it is not and \exists $\ell'' \in \text{In}(i_k)$ such that $\ell'' \in L$. Then with then $\ell \in \text{In}(i_k)$, $z_{\text{Out}(i_k),i_k} = z_{\text{Out}(i_k),i_k}$, and so $d_{i_k} = d_{i_k}$.

II) Let $\ell \in \text{Out}(i_{k}) \cap L$, let $f$ a flow such that $i_k \in \text{Pred}_f(\ell)$.

a) Either $i_k$ is the first hop on the path of $f$ (fresh flow) and with same arguments as B1 and B2, $z_{\ell,f}^k = z_{\ell,f}$

b) Or $i_k$ is not the first hop on the path of $f$ and $\exists \ell'' \in \text{In}(i_k)$ such that $\ell'' \in L$. By line 5, $z_{\ell,f} = z_{\ell,f} + r_f d_{i_k}$.

As previously, the analysis is done for cases $\ell'' \in L$, and $\ell'' \notin L$. In all cases, the arguments in B2 lead to $z_{\ell,f}^k = z_{\ell,f}$.

Eq. (21b) is true for $k$, hence $P_1(k)$ is true.

Part B. Conversely, in this way we show that if $z^*_\ell$ is a (finite) fixpoint of $F^L$, then we can extend $z^*_\ell$ for all $\ell \in L$ and give values to $d^*_i$ for all nodes $i$ such that $(z^*, d^*)$ is a fixpoint of $G$.

We thus assume that $z^*_\ell$ is a (finite) fixpoint of $F^L$; by applying FPTFAiter($z^*_L$) = $[\mathcal{F}^L(z^*_L)]$, we extend $z^*_\ell$ for all $\ell \in L$ and give values to $d^*_i$ for all nodes $i$.

\[
\forall \ell \in L, z'_\ell = [\mathcal{F}^L(z^*_L)]_\ell \\
\forall \ell \in L, z'_\ell = z^*_\ell
\]

Let us prove $P^*_1(h)$ by induction on $h \in [1, n_h]$, with $n_h$ the number of hops of flow $f$ and $i^{h,f}$ its $h$-th hop:

\[
P^*_1(h) = \begin{cases} 
\forall \ell \in \text{Out}(i^{h,f}) \setminus L, \text{s.t. } i^{h,f} \in \text{Pred}_f(\ell), \\
z_{\ell,f}^k = b_f + r_f \sum_{u \in \text{Pred}_f(\ell)} d^*_u \\
\forall \ell' \in \text{Out}(i^{h,f}) \cap L, \text{s.t. } i^{h,f} \in \text{Pred}_f(\ell'), \\
z_{\ell',f}^k = b_f + r_f \sum_{u \in \text{Pred}_f(\ell')} d^*_u
\end{cases}
\]
PB1) Let $\ell \in \text{Out}(\ell_0) \setminus L$, by line 5 and definition of $Z$, $z_{\ell,j}^* = b_j + r_j d_{\ell,j}^*$. As $f$ is a fresh flow, $\text{Pred}_f(\ell) = \{\ell_0, j\}$, hence Eq. (24a) is true.

PB2) Let $\forall \ell' \in \text{Out}(\ell_j) \cap L$, Eq. (24b) is satisfied with same arguments as in PB1. Therefore, $P_2^e(h)$ is true.

• Induction step: Assume $P_2^e(h - 1)$, and show $P_2^e(h)$.

PII) Let $\ell \in \text{Out}(\ell_j) \setminus L$. As $h > 1$, $\ell_j$ is not the first hop of flow $f$, thus $\exists \ell'' \in \text{In}(\ell_j)$ such that flow $f$ crosses the link $\ell''$.

a) Either $\ell'' \in L$, by line 5 and definition of $Z$, $z_{\ell,j}^* = z_{\ell''} + r_j d_{\ell,j}^*$. Since $\ell'' \in L$, by Eq. (23), $z_{\ell''}^* = z_{\ell''}^*$, and $\ell'' \in \text{Out}(\ell_j)$, so by induction assumption, $z_{\ell''}^* = b_j + r_j \sum_{u \in \text{Pred}_j(v)} d_u^*$. As $\text{Pred}_f(\ell) = \{\ell_j\}$, Eq. (24a) is satisfied.

b) Or, $\ell'' \not\in L$. By line 5 and definition of $Z$, $z_{\ell,j}^* = z_{\ell''} + r_j d_{\ell,j}^*$. Since $\ell \in \text{Out}(\ell_{j-1})$, then by induction assumption, $z_{\ell''}^* = b_j + r_j \sum_{u \in \text{Pred}_j(v)} d_u^*$. As $\text{Pred}_f(\ell) = \{\ell_j\}$, Eq. (24a) is satisfied.

PIII) Let $\ell \in \text{Out}(\ell_j) \cap L$, and analyse line 6. As $h > 1$, $\ell_j$ is not the first hop of flow $f$, thus $\exists \ell'' \in \text{In}(\ell_j)$ such that flow $f$ crosses the link $\ell''$. As in PII, the analysis is done for cases $\ell'' \in L$, or $\ell'' \not\in L$. All these cases lead to $z_{\ell,j}^* = b_j + r_j \sum_{u \in \text{Pred}_j(v)} d_u^*$ by using the same arguments as in PIIa and PIIb. Eq. (24b) is true, hence $P_1(k)$ is true.

Therefore, we have

$$\forall f, \forall \ell \in L, z_{\ell,j}^* = b_j + r_j \sum_{u \in \text{Pred}_j(v)} d_u^* = \gamma_\ell(d_{\ell,j}^*).$$

Also, by definition of $D$:

$$\forall i, d_i^* = D(z_{\text{In}(i)}^*).$$

Thus $(\gamma_\ell, d^*)$ is a fixedpoint of $G$.

Part C. The mapping $F_L$ is isotonic and concave by properties of the delay-jitter and burstiness functions. Indeed, by the definition in [3] Eq.(3)], at each node, the delay-jitter at a node is the minimum of affine functions in input bursts at this node. Thus, the delay-jitter is concave in the input bursts at this node.

By strong recurrence, for all nodes belonging to the feedforward network (linked with the function $F^L$), the delay-jitter and burstiness functions are concave in the burstiness at cuts $z_L$ and so $F^L$ is concave in $z_L$. Furthermore, $F^L(0) > 0$ as every component in $z_L$ is lower bounded by the burstiness at the source of the corresponding flow.

Therefore, we can apply Corollary 1 to $F^L$:

- Either, Low($F^L$) is unbounded,

- Or, $F^L$ has a unique fixpoint, that is also the largest element of Low($F^L$).

In the previous section, we show that the same holds for $G$. It follows from parts A and B that $G$ has a fixpoint $\Rightarrow F^L$ has a fixpoint, that shows item 1).

Now, assume that Low($G$) and Low($F^L$) are bounded. Item 2) similarly follows from parts A and B and from the uniqueness of the fixpoints. □

Remark 1. We can also apply FPTFA with $L = \emptyset$ on a network without cyclic dependencies. In such case, as noticed in Section 2, FPTFA and TFA compute the same bounds. Theorem 3 ensures that SyncTFA and FPTFA also compute the same bounds. Therefore bounds computed by SyncTFA and TFA are equal.

5 ASYNCHRONOUS TFA

We introduce the asynchronous algorithm, a new algorithm that, contrary to SyncTFA algorithm seen in Section 4 updates together the delay and burst. Here, we perform asynchronously the TFA updates, at one or several nodes at a time and in some arbitrary order.

The advantage of AsynTFA is that several nodes can be analysed at the same time, hence the update of the delays and the burstiness can be realised for several nodes in parallel.

Specifically, at every round $k = 1, 2, ..., $, we pick a set of nodes $I_k \subseteq I$; then, the TFA update of delay and burstiness are done simultaneously for every node $i \in I_k$.

We assume

(H1) Every node $i$ is visited infinitely often, i.e., $\forall i$ there is an infinite number of rounds $k$ such that $i \in I_k$.

Algorithm 4 Asynchronous TFA (AsynTFA)

```
1: $z_\ell \leftarrow 0, \forall \ell \in L;$
2: $d_i \leftarrow 0, \forall i \in I;$
3: $k \leftarrow 0;$
4: while true do
5:     $k \leftarrow k + 1;$
6:     parfor $i \in I_k$ do // Parallel for loop
7:         parallelSections do
8:             section1
9:                 $d_i \leftarrow D_i(z_{\text{In}(i)});$
10:             end section1
11:         section2
12:             $\forall \ell \in \text{Out}(i), z_{\ell}^* = \gamma_\ell(d_{\text{Pred}(\ell)});$
13:             end section2
14:     end parallelSections
15:     end parfor
16:     $\forall i \in I_k, \forall \ell \in \text{Out}(i), z_\ell = z_{\ell}^*;$
17:     print($z, d$);
18: end while
```

Algorithm 4 is a parallelized and non deterministic algorithm. The parfor loop from lines 6 to 15 performs a TFA update for all nodes in the subset $I_k$. These updates are done in parallel for each element of $I_k$. In addition, there are parallel sections inside the parfor loop. Indeed, for each TFA update, each update of $d$ and $z$ are also done in parallel (lines 9 and 12). $d$ and $z$ are shared variables, so the most recent values of $d$ and $z$ are used to compute lines 9 and 12.

At round $k$, burst update at line 12 can be done with a new value of $d$ updated in round $k$ or with a former value of $d$ from round $k - 1$.

Theorem 4. The sequence of $(z,d)$ at each loop of AsynTFA is (monotone) increasing. If Low($G$) is bounded then the AsynTFA sequence converges to the largest element $(\bar{z},d)$ of Low($G$). Else it does not converge.
Proof. As we start at 0 and all operators are isotonic, by induction, the sequence of \((z, d)\) in Algorithm 4 is (widesense) increasing.

Assume that \(\text{Low}(G)\) is bounded, by Theorem 6, it has a larger element \((\bar{z}, d)\) which is also a fixpoint of \(G\). Let \((z^k, d^k)\) be the values printed at the end of iteration \(k\). We have \((z^k, d^k) = (0, 0) \leq (\bar{z}, d)\), hence, by monotonicity of \(D_i\) and \(\mathcal{Y}_i\) in lines 7 and 8, \((z^k, d^k) \leq G(z, d) = (\bar{z}, d)\). By induction it comes that \((z^k, d^k) \leq (\bar{z}, d)\) for all \(k \geq 0\). As a result, the sequence is bounded and converges. By \((H1)\), every node is visited infinitely often, hence Line 7 and 12 are executed infinitely often. By continuity of \(D\) and \(\mathcal{Y}\), the limit is a fixpoint of \(G\). We have shown in the proof of Theorem 1 that \(G\) satisfies the hypotheses of Theorem 7; thus \(G\) has a unique fixpoint. This shows the second statement.

Conversely, if the sequence \((z^k, d^k)\) of AsyncTFA converges, then by continuity of \(D\) and \(\mathcal{Y}\) and by definition of \(G\), the limit is a fixpoint of \(G\). By contraposition, \(\text{Low}(G)\) unbounded implies the non-convergence of the AsyncTFA sequence.

It follows that AsyncTFA produces the same end-results as SyncTFA, FPTFA/TFA. It should converge more quickly than SyncTFA, as it always uses the most recently available values.

AltTFA is a special case of AsyncTFA where all nodes are visited simultaneously at every round, i.e., \(I_k = I\) for every \(k\), and for all \(i \in I_k\), some workers execute line 9 in parallel before some workers execute line 12 in parallel. It corresponds to the case where each parallel section \(1\) and \(2\) are done sequentially one after the other for all nodes at round \(k\). As a result, AltTFA computes delay-jitter bounds at every node, assuming propagated burstinesses are known, then it updates the propagated burstinesses, until it finds a fixpoint or diverges. \((H1)\) holds, hence all the statements in Theorem 5 hold for AltTFA. In particular, AltTFA produces the same end-results as SyncTFA, FPTFA/TFA.

Algorithm 5 Alternating TFA (AltTFA)

1: \(z_t \leftarrow 0, \forall \ell \in \mathcal{L};\)
2: \(d_t \leftarrow 0, \forall i \in I;\)
3: while true do
4: \(d \leftarrow D(z);\)
5: \(z \leftarrow \mathcal{Y}(d);\)
6: print \((z, d);\)
7: end while

AsyncTFA algorithms are also valid for networks without cyclic dependencies and becomes stationary in a finite number of iteration. Theorem 5 specifies the maximal number of iteration steps before stability for AltTFA.

Theorem 5. In a network without cyclic dependencies, AsyncTFA converges in a finite number of iterations. The number of iterations depends on the organisation of the subsets of nodes on Line 6 in Algorithm 4. In the specific case of AltTFA with a network of \(k\) levels, AltTFA is stationary in at most \(k\) steps.

Proof. AsyncTFA converges in a finite number of iterations, due to Assumption \((H1)\). The proof of stationary of AltTFA in at most \(k\) steps is similar to the proof of Theorem 5.

6 Background: Fixpoints and Maximal Elements

We recall that for SyncTFA and AsyncTFA algorithms, worst-case delay and burstiness bounds in a network (respectively, \(d(l)\) and \(z(t)\)) belong to the specific set \(\text{Low}(G) = \{(z, d), \text{such that} (z, d) \leq G(z, d)\}\), where \(G\) corresponds to the function that computes burstinesses and delay at each loop of the algorithms. Bounds belong to a similar set for FPTFA algorithm with the \(F^L\) function. The problem here is to find an upper bound on \(\text{Low}(G)\) (respectively, \(\text{Low}(F^L)\) for FPTFA). In this section, we introduce two theorems:

- **Theorem 6** is introduced to prove that our algorithms are valid in the sense that when they converge they find valid worst-case bounds.
- **Theorem 7** proves that the solution is unique. This is used to show the equivalence of all algorithms.

For every \(b \in \mathbb{R}^n\), define the set \(S(b)\) by

\[S(b) \overset{\text{def}}{=} \{z \in \mathbb{R}^n, z \geq b\},\]

where comparison is coordinatewise. For a function \(F : S(b) \rightarrow S(b)\), we say that \(z\) is a fixpoint of \(F\) if \(z \in S(b)\) and \(F(z) = z\), i.e., we consider only finite fixpoints.

We say that \(F\) is isotonic if and only if

\[\forall z, z' \in S(b), z \leq z' \Rightarrow F(z) \leq F(z').\]

**Theorem 6.** Let \(F : S(b) \rightarrow S(b)\) be isotonic. If the set \(\text{Low}(F) \overset{\text{def}}{=} \{z \in S(b), z \leq F(z)\}\) is bounded, it has a largest element \(\bar{z}\) and \(F\) has at least one fixpoint. \(\bar{z}\) is also the largest fixpoint of \(F\).

**Theorem 6** is a variant of the Knaster-Tarski theorem [14]. In Remark 1, we present an example that highlights the differences between Theorem 6 and Knaster-Tarski theorem.

**Proof.** This proof is based on the proof of the Knaster-Tarski theorem [14].

Let us assume that the set \(\text{Low}(F) \overset{\text{def}}{=} \{z \in S(b), z \leq F(z)\}\) is bounded. We introduce \(\bar{z}\) by:

\[\bar{z} \overset{\text{def}}{=} \sup(\text{Low}(F)).\]  

(25)

Since \(\text{Low}(F)\) is bounded, \(\bar{z}\) is finite. \(\forall z \in \text{Low}(F), z \leq \bar{z}\).

Since \(F\) is isotonic, \(F(z) \leq F(\bar{z})\) and since \(z \leq F(z)\), it follows that \(z \leq F(z)\). Hence, \(\bar{z} \leq \sup(\text{Low}(F)) = \bar{z}\). This shows that \(\bar{z} \in \text{Low}(F)\), i.e., \(\text{Low}(F)\) has a largest element, \(\bar{z}\).

Second, \(F(F(z)) \geq F(z)\), thus \(F(\bar{z}) \in \text{Low}(F)\). With Eq. (25) and previous inequality, we have \(F(\bar{z}) \leq \bar{z}\). Therefore, we conclude that \(F(\bar{z}) = \bar{z}\). This shows that \(\bar{z}\) has a fixpoint \(\bar{z}\). Any other fixpoint is in \(\text{Low}(F)\) hence is lower than \(\bar{z}\).

**Remark 2.** The usual form of the Knaster-Tarski theorem claims that, if \(L\) is a complete lattice and \(\phi : L \rightarrow L\) is isotone, then the least upper bound of \(\{x \in L \mid x \leq \phi(x)\}\) is a fixpoint of \(\phi\). It is tempting to apply the Knaster-Tarski theorem by allowing \(+\infty\) in the domain of definition of \(F\) and in its values, in order to obtain a substitute for Theorem 6; however, as we show next, this does not work.

Indeed applying the Knaster-Tarski theorem to an isotone function \(F : [b, +\infty]^n \rightarrow [b, +\infty]^n\) gives that \(F\) has a fixpoint.
the set \( \{ z \in [b; +\infty)^n, z \leq \bar{F}(z) \} \) has a largest element, and this largest element is also the largest fixpoint of \( \bar{F} \). However, as the following example shows, this does not provide the largest finite solution of \( z \leq \bar{F}(z) \).

- Consider \( n = 1 \) and \( F \) defined by:
  \[
  F : [b; +\infty) \to [b; +\infty] \quad (26)
  \]
  where the last equality is because \( z^* \) is a fixpoint. Now \( F_r \) is concave, hence
  \[
  F_r(z') \geq (1 - \frac{1}{\gamma}) F_r(b) + \frac{1}{\gamma} F_r(z) \geq (1 - \frac{1}{\gamma}) F_r(b) + \frac{1}{\gamma} z_r
  \]
  where the latter inequality is because \( F(z) \geq z \).

- Also, consider \( F' \) defined by:
  \[
  F' : [b; +\infty) \to [b; +\infty] \quad (27)
  \]
  \[
  F' \text{ can also be extended to } \bar{F}' : [b; +\infty] \to [b; +\infty] \text{ by setting } \bar{F}'(\infty) = +\infty.
  \]
  The set \( \{ z \in [b; +\infty)^n, z \leq \bar{F}'(z) \} \) is equal to \( \{ 0; 1 \} \cup \{ +\infty \} \). Here \( F' \) has one fixpoint \( z = 1 \) and \( \bar{F}' \) has two fixpoints: 1 and \(+\infty\).

- Consider \( n = 1 \) and \( F \) defined by:
  \[
  F : [b; +\infty) \to [b; +\infty] \quad (28)
  \]
  where the last equality is because \( z^* \) is a fixpoint. Now \( F_r \) is concave, hence
  \[
  F_r(z') \geq (1 - \frac{1}{\gamma}) F_r(b) + \frac{1}{\gamma} F_r(z) \geq (1 - \frac{1}{\gamma}) F_r(b) + \frac{1}{\gamma} z_r
  \]
  where the latter inequality is because \( F(z) \geq z \).

- Also, consider \( F' \) defined by:
  \[
  F' : [b; +\infty) \to [b; +\infty] \quad (29)
  \]
  \[
  F' \text{ can also be extended to } \bar{F}' : [b; +\infty] \to [b; +\infty] \text{ by setting } \bar{F}'(\infty) = +\infty.
  \]
  The set \( \{ z \in [b; +\infty)^n, z \leq \bar{F}'(z) \} \) is equal to \( \{ 0; 1 \} \cup \{ +\infty \} \). Here \( F' \) has no fixpoint and \( \bar{F}' \) has one fixpoint: \(+\infty\).

In both cases, the Knaster-Tarski theorem obtains the same result, i.e., that \(+\infty\) is the largest fixpoint of \( \bar{F} \) and the largest solution of \( z \leq \bar{F}(z) \), which is not helpful. In contrast, our results obtain that either \( \{ z \in [b; +\infty)^n, z \leq \bar{F}(z) \} \) is unbounded and \( F \) has no fixpoint (as in the latter case), or \( F \) has a unique fixpoint (as in the former case) that is also the largest (finite) solution of \( z \leq \bar{F}(z) \).

Theorem 7 enables us to show the unicity of the fixpoint, by using additional properties of \( G \) and \( F^L \).

**Theorem 7.** Consider an isotonic function \( F : S(b) \to S(b) \) and assume in addition that \( F \) is concave and \( F(b) > b \). If \( F \) has a fixpoint \( z^* \), then

1) \( \text{Low}(F) \) is bounded.

2) The largest element of \( \text{Low}(F) \) is \( z^* \).

**Proof.** The proof is inspired by the proof of Lemma 6 in [15 Section 6.2].

Assume that \( F \) has a fixpoint \( z^* \in S(b) \). We have \( z^* \geq b \) hence \( z_j^* \leq F_j(z^*) \geq F_j(b) > b_j \) for all \( j \).

Now fix some arbitrary \( z \in \text{Low}(F) \). We prove by contradiction that \( z \leq z^* \). Assume this does not hold, i.e., there exists some \( j \) such that \( z_j^* < z_j - b_j \) and thus

\[
0 < z_j^* - b_j < z_j - b_j
\]

and

\[
\frac{z_j - b_j}{z_j^* - b_j} > 1
\]

Let \( r \) be such that \( \frac{z_j - b_j}{z_j^* - b_j} \) is maximum and \( \gamma = \frac{z_j - b_j}{z_j^* - b_j} > 1 \).

It follows that \( z_i - b_i \leq \gamma(z_i^* - b_i) \) for all \( i \). Now, let \( z' = \frac{1}{\gamma}(z - b) + b \). Thus \( z'_j \leq z^* \) and \( z'_j = z_j^* \). By the former inequality and the isotonicity of \( F_r \)

\[
F_r(z') \leq F_r(z^*) = z^*_r
\]

This contradicts Equation (30). Therefore, we have \( z \leq z^* \). This shows that \( \text{Low}(F) \) is bounded and since \( z^* \in \text{Low}(F) \), this shows item 2).

We can notice that the assumptions in the theorem are satisfied by the functions \( G \) and \( F^L \) (with \( b = 0 \)).

**Corollary 1.** Under the conditions of Theorem 7 one of the following mutually exclusive conditions must hold:

1) \( F \) has a unique fixpoint, \( \text{Low}(F) \) is bounded and the fixpoint of \( F \) is the largest element of \( \text{Low}(F) \);

2) \( F \) has no fixpoint and \( \text{Low}(F) \) is unbounded.

In particular for a function \( F \) that satisfies the conditions of Theorem 7 whenever a sequence converges to a fixpoint of \( F \), this proves that \( \text{Low}(F) \) is bounded and the limit point must be the largest element of \( \text{Low}(F) \). This is a key argument that allowed us to prove the equivalence of all algorithms in this paper.

7 **NUMERICAL ILLUSTRATION**

Figure 4 represents the execution times needed to compute the end-to-end delay-jitter for FPTFA and the AltTFA algorithms presented in this article for a flow that crosses \( n - 1 \) different servers on a ring network of \( n \) servers. Every server has a rate-latency service curve with rate \( R = 10^7 \) bits/s and latency \( T = 0.001 \) s. There are \( n - 1 \) other flows, each of them starts from a different node and crosses \( n - 1 \) nodes. They all have the same rate and same input burstinesses \( (r = 0.7R/n \text{ bits/s}) \). The execution time in Figure 4 is the mean on 10000 simulations for each ring size. 99% confidence intervals are less than 0.3%.

![Fig. 4. Execution time on the ring network, for FPTFA and AltTFA algorithms and their respective confidence intervals at 95% (CI).](image-url)
As analysed in Section 5, both algorithms compute the same end-to-end delay-jitter bounds, and the asynchronous algorithms converges faster (see Figure 4).

8 CONCLUSION

In this paper, we have presented new formulations of the TFA and FPTFA algorithms. They do not use any cut and work whether the network has cyclic dependencies or not.

These new formulations show us that TFA methods from the literature all produce the same results, in terms of convergence, and compute the same delay and burstiness bounds. Furthermore, the asynchronous version, presented in Section 5 and its simulation show a gain in computing time.

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