TWO DIMENSIONAL COMPLEX KLEINIAN GROUPS WITH FOUR COMPLEX LINES IN GENERAL POSITION IN ITS LIMIT SET

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Abstract. In this article we provide an algebraic characterization of those groups of $\text{PSL}(3, \mathbb{C})$ whose limit set in the Kulkarni sense has, exactly, four lines in general position. Also we show that, for this class of groups, the equicontinuity set of the group is the largest open set where the group acts discontinuously and agrees with the discontinuity set of the group.

Introduction

In a recent article, see [1], we have proven that for a complex Kleinian group without proper invariant subspaces and "enough" lines in the Kulkarni’s limit set, it holds that its discontinuity set agrees with the equicontinuity set of the group, is the largest open set where the group acts discontinuously and is a holomorphy domain. Such result enable us to understand the relationship, in the two dimensional case and for a "large class" of groups, between the different notions of limit sets which are usually studied as well as its geometry, see [3]. This article is a step to extent the results in [1] to the case when the groups has invariant subspaces and "enough" lines in the limit set. More precisely we prove:

**Theorem 0.1.** Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a discrete group. The limit set, in the Kulkarni sense, of $\Gamma$ has exactly four lines in general position if and only if $\Gamma$ has a hyperbolic toral group, see section 2, whose index is at most 8.

**Theorem 0.2.** Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a toral group. Thus the discontinuity set in the Kulkarni sense agrees with the equicontinuity region and is given by:

$$\Omega(\Gamma) = \bigcup_{\epsilon_1, \epsilon_2 = \pm 1} \mathbb{H}^{\epsilon_1} \times \mathbb{H}^{\epsilon_2},$$

where $\mathbb{H}^{+1}$ and $\mathbb{H}^{-1}$ are the upper half and lower half plane. Moreover $\Omega(\Gamma)$ is the largest open set on which $\Gamma$ acts properly discontinuously.

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This article is organized as follows: in section 1 we introduce some terms and notations which will be used along the text. In section 2 we construct some examples of groups with four lines in general position. Finally in section 3 we present the proof of theorem 0.1.

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1. Preliminaries and Notations

1.1. Projective Geometry. We recall that the complex projective plane \( \mathbb{P}^2_\mathbb{C} \) is

\[
\mathbb{P}^2_\mathbb{C} := (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*,
\]

where \( \mathbb{C}^* \) acts on \( \mathbb{C}^3 \setminus \{0\} \) by the usual scalar multiplication. This is a compact connected complex 2-dimensional manifold. Let \( [\cdot] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2_\mathbb{C} \) be the quotient map. If \( \beta = \{e_1, e_2, e_3\} \) is the standard basis of \( \mathbb{C}^3 \), we will write \([e_j] = e_j \) and if \( w = (w_1, w_2, w_3) \in \mathbb{C}^3 \setminus \{0\} \) then we will write \([w] = [w_1 : w_2 : w_3]\). Also, \( \ell \subset \mathbb{P}_\mathbb{C}^2 \) is said to be a complex line if \([\ell]^{-1} \cup \{0\}\) is a complex linear subspace of dimension 2. Given \( p, q \in \mathbb{P}^2_\mathbb{C} \) distinct points, there is a unique complex line passing through \( p \) and \( q \), such line will be denoted by \( \overrightarrow{p, q} \).

Consider the action of \( \mathbb{Z}_3 \) (viewed as the cubic roots of the unity) on \( SL(3, \mathbb{C}) \) given by the usual scalar multiplication, then

\[
PSL(3, \mathbb{C}) = SL(3, \mathbb{C})/\mathbb{Z}_3
\]

is a Lie group whose elements are called projective transformations. Let \( [\cdot] : SL(3, \mathbb{C}) \to PSL(3, \mathbb{C}) \) be the quotient map, \( \gamma \in PSL(3, \mathbb{C}) \) and \( \tilde{\gamma} \in GL(3, \mathbb{C}) \), we will say that \( \tilde{\gamma} \) is a lift of \( \gamma \) if there is a cubic root \( \tau \) of \( Det(\gamma) \) such that \([\tau \tilde{\gamma}] = \gamma \), also, we will use the notation \((\gamma_{ij})\) to denote elements in \( SL(3, \mathbb{C}) \). One can show that \( PSL(3, \mathbb{C}) \) is a Lie group that acts transitively, effectively and by biholomorphisms on \( \mathbb{P}_\mathbb{C}^2 \) by \([\gamma])([w]) = [\gamma(w)]\), where \( w \in \mathbb{C}^3 \setminus \{0\} \) and \( \gamma \in SL_3(\mathbb{C}) \).

1.2. Complex Kleinian Groups. Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a subgroup. We define (following Kulkarni, see [5]): the set \( L_0(\Gamma) \) as the closure of the points in \( \mathbb{P}_\mathbb{C}^2 \) with infinite isotropy group. The set \( L_1(\Gamma) \) as the closure of the set of cluster points of \( \Gamma z \) where \( z \) runs over \( \mathbb{P}_\mathbb{C}^2 \setminus L_0(\Gamma) \). Recall that \( q \) is a cluster point for \( \Gamma K \), where \( K \subset \mathbb{P}_\mathbb{C}^2 \) is a non-empty set, if there is a sequence \((k_m)_{m \in \mathbb{N}} \subset K \) and a sequence of distinct elements \((\gamma_m)_{m \in \mathbb{N}} \subset \Gamma \) such that \( \gamma_m(k_m) \xrightarrow{m \to \infty} q \). The set \( L_2(\Gamma) \) as the closure of cluster points of \( \Gamma K \) where
K runs over all the compact sets in \( \mathbb{P}^2_C \setminus (L_0(\Gamma) \cup L_1(\Gamma)) \). The Limit Set in the sense of Kulkarni for \( \Gamma \) is defined as:

\[
\Lambda(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma).
\]

The Discontinuity Region in the sense of Kulkarni of \( \Gamma \) is defined as:

\[
\Omega(\Gamma) = \mathbb{P}^2_C \setminus \Lambda(\Gamma).
\]

We will say that \( \Gamma \) is a Complex Kleinian Group if \( \Omega(\Gamma) \neq \emptyset \).

**Lemma 1.1.** (See [2]) Let \( \Gamma \subset PSL_3(\mathbb{C}) \) be a subgroup, \( p \in \mathbb{P}^2_C \) such that \( \Gamma p = p \) and \( \ell \) a complex line not containing \( p \). Define \( \Pi = \Pi_{p,\ell} : \Gamma \rightarrow Bihol(\ell) \) given by \( \Pi(g)(x) = \pi(g(x)) \) where \( \pi = \pi_{p,\ell} : \mathbb{P}^2_C \setminus \{p\} \rightarrow \ell \) is given by \( \pi(x) = \frac{x - p}{x \cap \ell} \), then:

(i) \( \pi \) is a holomorphic function.
(ii) \( \Pi \) is a group morphism.
(iii) If \( \text{Ker}(\Pi) \) is finite and \( \Pi(\Gamma) \) is discrete, then \( \Gamma \) acts discontinuously on \( \Omega = \bigcup_{z \in \Omega(\Pi(\Gamma))} \frac{x - p}{x \cap \ell} \setminus (\ell \cup \{p\}) \).
(iv) If \( \Gamma \) is discrete, \( \Pi(\Gamma) \) is non-discrete and \( \ell \) is invariant, then \( \Gamma \) acts discontinuously on \( \Omega = \bigcup_{z \in \text{Eq}(\Pi(\Gamma))} \frac{x - p}{x \cap \ell} \setminus (\ell \cup \{p\}) \).

**Lemma 1.2.** Let \( \Sigma \subset PSL(2,\mathbb{C}) \) be a non-discrete group, then:

(i) The set \( \mathbb{P}^1_C \setminus \text{Eq}(\Sigma) \) is either, empty, one points, two points, a circle or \( \mathbb{P}^1_C \).
(ii) If \( \mathcal{C} \) is an invariant closet set which contains at least 2 points, then \( \mathbb{P}^1_C \setminus \text{Eq}(\Sigma) \subset \mathcal{C} \).
(iii) The set \( \mathbb{P}^1_C \setminus \text{Eq}(\Sigma) \) is the closure of the loxodromic fixed points.

1.3. Counting Lines.

**Definition 1.3.** Let \( \Omega \subset \mathbb{P}^2_C \) be a non-empty open set. Let us define:

(i) The lines in general position outside \( \Omega \) as:

\[
LG(\Omega) = \{ \mathcal{L} \subset Gr_1(\mathbb{P}^2_C) \mid \text{The lines in } \mathcal{L} \text{ are in general position } \& \bigcup \mathcal{L} \subset \mathbb{P}^2_C \setminus \Omega \};
\]

(ii) The number of lines in general position outside \( \Omega \) as:

\[
LiG(\Omega) = \max(\{\text{card}(\mathcal{L}) : \mathcal{L} \in LG(\Omega)\}),
\]

where \( \text{card}(C) \) denotes the number of elements contained in \( C \).

(iii) Given \( \mathcal{L} \in LG(\Omega) \) and \( v \in \bigcup \mathcal{L} \), we will say that \( v \) is a vertex for \( \mathcal{L} \) if there are \( \ell_1, \ell_2 \in \mathcal{L} \) distinct lines and an infinite set \( \mathcal{C} \subset Gr_1(\mathbb{P}^2_C) \) such that \( \ell_1 \cap \ell_2 \cap (\bigcup \mathcal{C}) = \{v\} \) and \( \bigcup \mathcal{C} \subset \mathbb{P}^2_C \setminus \Omega \).

**Proposition 1.4.** Let \( \Gamma \subset PSL(3\mathbb{C}) \) be a complex Kleinian group. If \( LiG(\Omega(\Gamma)) = 4 \), then for each \( \mathcal{L} \in GL(\Omega) \) with \( \text{card}(\mathcal{L}) = 4 \), it falls out that:

(i) The array of lines \( \mathcal{L} \) contains exactly two vertexes;
(ii) For every vertex \( v \) of \( \mathcal{L} \), it follows that \( \text{Isot}(v,\Gamma) \) is a subgroup of \( \Gamma \) with finite index.
2. Toral Groups

**Definition 2.1.** Let $A \in SL(2, \mathbb{Z})$, then $A$ is said to be a Hyperbolic Toral Automorphism if none of the eigenvalues of $A$ lies on the unit circle.

**Theorem 2.2.** Let $A \in SL(2, \mathbb{Z})$ be an hyperbolic toral automorphism then:

1. The eigenvalues of $A$ are irrational numbers.
2. It holds
   \[
   \{x \in \mathbb{R}^2 : A^n(x) - x \in \mathbb{Z} \times \mathbb{Z} \text{ for some } n \in \mathbb{N}\} = \mathbb{Q} \times \mathbb{Q}.
   \]

**Definition 2.3.** Set $S : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N}$ which is given by:

\[S(x) = \min\{n \in \mathbb{N} : nx \in \mathbb{Z} \times \mathbb{Z}\}.\]

Also define $\text{Per} : SL(2, \mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q} \to \mathbb{N}$ by

\[\text{Per}(A, x) = \min\{n \in \mathbb{N} : B^n(x) - x \in \mathbb{Z} \times \mathbb{Z}\}.
\]

Finally define $\phi : SL(2, \mathbb{Z}) \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q} \times \mathbb{Q}$ by

\[\phi(B, x, l) = \begin{cases} 
\sum_{j=0}^{l-1} B^j(x) & \text{if } l > 0; \\
0 & \text{if } l = 0; \\
-\sum_{j=1}^{l} B^{-j}(x) & \text{if } l < 0.
\end{cases}
\]

The following straightforward lemmas will be useful

**Lemma 2.4.** Let $B \in SL(2, \mathbb{Z})$, $\nu \in \mathbb{Q}^2$ and $r, s, l \in \mathbb{N}$ with $0 < r, s < \text{Per}(B, \nu)$. If $l = K\text{Per}(B, \nu) + r$ and $K = \bar{n}S(\nu) + t$, where $K, r, \bar{n}, t \in \mathbb{N}$ are given by the division theorem,

\[
\begin{align*}
\delta_1 &= -B^r(\phi(B, B^{-K\text{Per}(B, \nu)}(\nu) - \nu), l); \\
\delta_2 &= -B^{rK\text{Per}(B, \nu)}(\phi(B, B^{-K\text{Per}(B, \nu)}(\nu) - \nu), l)); \\
\delta_3 &= S(\nu)B^{rK\text{Per}(B, \nu)}(\phi(B, B^{-K\text{Per}(B, \nu)}(\nu) - \nu, l)); \\
\delta_4 &= \sum_{j=0}^{K-1} \phi(B, B^{jK\text{Per}(B, \nu)}(\nu) - \nu, \text{Per}(B, \nu)); \\
\delta_5 &= \phi(B, B^{K\text{Per}(B, \nu)}(\nu) - \nu, r); \\
\delta_6 &= \bar{n}S(\nu)\phi(B, \nu, \text{Per}(B, \nu)); \\
\delta_7 &= \begin{cases} 
0 & \text{if } r + s < \text{Per}(B, \nu) \\
\phi(B, B^{r+s}\nu) - \nu, r + s - \text{Per}(B, \nu)) & \text{D.O.F.}
\end{cases} \\
\delta_8 &= \begin{cases} 
\sum_{j=r+s}^{r+s} B^j(\nu) & \text{if } r + s < \text{Per}(B, \nu) \\
\sum_{j=r}^{r+s} \text{Per}(B, \nu) - 1 B^j(\nu) + \phi(B, \nu, r + s - \text{Per}(B, \nu)) & \text{D.O.F.}
\end{cases}
\end{align*}
\]

thus $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7 \in \mathbb{Z}^2$ and

\[
\phi(B, \nu, -l) = \delta_1 + \delta_2 + \delta_3 + (S(\nu) - 1)B^{rK\text{Per}(B, \nu)}(\phi(B, \nu, l));
\]

\[
\phi(B, \nu, l) = \delta_4 + \delta_5 + \delta_6 + t\phi(B, \nu, \text{Per}(B, \nu)) + \phi(B, \nu, r);
\]

\[
B^r\phi(B, \nu, s + 1) = \delta_7 + \delta_8
\]

**Lemma 2.5.** Let $(a_m), (b_m) \subset \mathbb{C}$ be sequences, then:

1. If $(a_m)$ and $(b_m)$ diverges, then the accumulation points of
   \[\{[a_m : b_m : 1] : m \in \mathbb{N}\}
   \]

lies on $\mathbb{C}$.
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(ii) If \((a_m)\) converges and \((b_m)\) diverges, then \([a_m : b_m : 1] \xrightarrow{m \to \infty} [e_2];\)
(iii) If \((a_m)\) diverges and \((b_m)\) converges, then \([a_m : b_m : 1] \xrightarrow{m \to \infty} [e_1];\)
(iv) If \(k_m = [a_m : b_m : 1] \xrightarrow{m \to \infty} [z : 0 : 1],\) where \(z \neq 0,\) then there is a
subsequence of \((k_m),\) denoted \((\tilde{k}_m = [\tilde{a}_m : \tilde{b}_m : 1]),\) such that \((\tilde{a}_m)\)
and \((\tilde{b}_m)\) are convergent.

Definition 2.6. Let \(A, B \in SL(2, \mathbb{Z}), \nu \in \mathbb{Q} \times \mathbb{Q}, b \in M(1 \times 2, \mathbb{Z}), k, l \in \mathbb{Z},\)
thus we define:

\[
\langle k : l : b : \nu \rangle = \begin{pmatrix}
A^k & B^l & b + \phi(l, \nu) \\
0 & 1 & 1
\end{pmatrix}.
\]

From Lemma 2.4 it follows easily

Corollary 2.7. Let \(A, B \in SL(2, \mathbb{Z}), \nu \in \mathbb{Q} \times \mathbb{Q}, b \in M(1 \times 2, \mathbb{Z}), k, l \in \mathbb{Z},\)
then there are \(m_0, \ldots, m_{\text{Per}(B, \nu) - 1} \in \{0, \ldots, S(\nu) - 1\}\) and \(B \in \mathbb{Z}^2\) such that:

\[
\langle k : l : b : \nu \rangle = \begin{pmatrix}
A^k & B^l & b + \sum_{j=0}^{\text{Per}(B, \nu) - 1} m_j B^j(\nu) \\
0 & 1 & 1
\end{pmatrix}.
\]

Proposition 2.8. Let \(A, B \in SL(2, \mathbb{Z})\) be such that the group generated
by \(A, B\) is isomorphic to \(\mathbb{Z} \times \mathbb{Z}\) and each element in \(< A, B > \setminus \{Id\}\) is a
hyperbolic toral automorphism, also let \(\nu \in \mathbb{Q} \times \mathbb{Q}\) be such that \(A(\nu) - \nu \in \mathbb{Z} \times \mathbb{Z}.\) Then

\[
\Gamma_{A, B, \nu} = \{\langle k : l : b : \nu \rangle \mid k, l \in \mathbb{Z}, b \in M(1 \times 2, \mathbb{Z})\}
\]
is a complex Kleinian group. Moreover \(\Omega(\Gamma_{A, B, \nu})\) is projectively equivalent to:

\[
\bigcup_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \mathbb{H}^{41} \times \mathbb{H}^{42}.
\]

Proof. Let \(a = \langle k_1 : l_1 : b_1 : \nu \rangle, b = \langle k_2 : l_2 : b_2 : \nu \rangle \in \Gamma_{A, B, \nu}.\) Thus an easy
calculation shows:

\[
ab^{-1} = \langle k_1 - k_2 : l_1 - l_2 : b_1 + b_3 + b_4 \rangle
\]

where

\[
b_3 = -A^{k_1}B^{l_1}(A^{-k_2}B^{-l_2}(b_2) + B^{-l_2}\phi(A^{-k_2}(\nu) - \nu, l_2));
\]
\[
b_4 = B^{l_1}(\phi(A^{k_1}(\nu) - \nu, -l_2)).
\]

Since \(b_3, b_4 \in \mathbb{Z} \times \mathbb{Z},\) it follows that \(\Gamma_{A, B, \nu}\) is a group.

Now, since \(A, B \in SL(2, \mathbb{Z})\) are commuting hyperbolic toral automorphism, it follows that there is \(\hat{T} \in SL(2, \mathbb{R})\) such that \(T\hat{A}\hat{T}^{-1}, \hat{T}\hat{B}\hat{T}^{-1}\) are
diagonal matrices. Set

\[
T = \begin{pmatrix}
\hat{T} & 0 \\
0 & 1
\end{pmatrix}; T\hat{A}\hat{T}^{-1} = \begin{pmatrix}
\alpha & 0 \\
0 & \alpha^{-1}
\end{pmatrix}; T\hat{B}\hat{T}^{-1} = \begin{pmatrix}
\beta & 0 \\
0 & \beta^{-1}
\end{pmatrix};
\]
where $\alpha, \beta \in \mathbb{R} \setminus \{\pm 1\}$ and $\hat{T}(1,0), \hat{T}(0,1) \in \mathbb{R}^2$. Moreover, given $b \in M(1 \times 2, \mathbb{Z})$ and $k, l \in \mathbb{Z}$ and taking $\nu = (\nu_1, \nu_2)$, by Corollary 2.7 there are $m_0, \ldots, m_{\text{Per}(B, \nu)-1} \in \{0, \ldots, S(\nu) - 1\}$ and $b_1, b_2 \in \mathbb{Z}$ such that

$$
T(k \cdot l \cdot b : \nu) T^{-1} = \begin{pmatrix}
\alpha^k \beta^m & 0 \\
0 & \alpha^{-k} \beta^{-m}
\end{pmatrix} \begin{pmatrix}
\sum_{i=1}^{2} x_i (b + \nu \sum_{j=1}^{\text{Per}(\nu)-1} b_j \beta^j) \\
\sum_{i=1}^{2} y_i (b + \nu \sum_{j=1}^{\text{Per}(\nu)-1} b_j \beta^j)
\end{pmatrix} \quad (2.2)
$$

Claim 1. Let $z \neq 0$, if $[z : 0 : 1]$ lies on $L_1(T\Gamma_{A,B,\nu}T^{-1})$, then $z \in \mathbb{R}$. Let us assume that $\text{Im}(z) \neq 0$. Thus there are $w = [a : b : 1]$ and $(\gamma_m) \subset \Gamma_{A,B,\nu}$ a sequence of distinct elements in $\Gamma_{A,B,\nu}^{-1}$ such that $T\gamma_m T^{-1}(w) \xrightarrow{m \to \infty} x$. From equation (2.2), it follows that for each $m \in \mathbb{N}$ there are $n_m, k_m, b_1 m, b_2 m \in \mathbb{Z}$ and $(l_j)_{j=0}^{\text{Per}(\nu)-1} \in \{0, \ldots, S(\nu) - 1\}$ such that $\gamma_m(w) = [a_m : b_m : 1]$ where

$$a_m = \alpha^{k_m} \beta^{n_m} a + \sum_{i=1}^{2} x_i (b_{l_i} + \nu \sum_{j=0}^{\text{Per}(\nu)-1} l_j \beta^j);$$

$$b_m = \alpha^{-k_m} \beta^{-n_m} b + \sum_{i=1}^{2} y_i (b_{l_i} + \nu \sum_{j=0}^{\text{Per}(\nu)-1} l_j \beta^j).$$

By Lemma 2.5 we can assume that $a_m \xrightarrow{m \to \infty} z$ and $b_m \xrightarrow{m \to \infty} 0$. Since $\text{Im}(a_m) = \alpha^{k_m} \beta^{n_m} \text{Im}(a) \to \text{Im}(z) \neq 0$. We conclude that $(k_m)$ and $(n_m)$ are eventually constant. In consequence we conclude that $(p_{1m})$ and $(p_{2m})$ are eventually constant. Thus $(\gamma_m)$ is eventually constant. Which is contradiction.

Observe that by a similar argument, we can show the claim in the case $x = [0 : z : 1] \in L_1(T\Gamma_{A,B,\nu}T^{-1})$.

Now let $\gamma \in T\Gamma_{A,B,\nu}T^{-1}$ induced by the linear map:

$$
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

then is straightforward to check that $\overline{e_1, e_3} \cup \overline{e_2, e_3} \subset \Lambda(T\Gamma_{A,B,\nu}T^{-1})$.

On the other hand, from equation (2.2), we conclude that $\ell_1 = \{e_2, e_3\}$, $\ell_2 = \{e_1, e_3\}$, $e_1$ and $e_2$, are $T\Gamma_{A,B,\nu}T^{-1}$-invariant. Thus, taking $\pi_i = \pi_{e_i, \ell_i}$ we can define $\Pi_i : \Gamma_0 \to \text{Bihol}(l_i)$. Thus, from Equation 2.2 we conclude that $\Pi_j(T\Gamma_{A,B,\nu}T^{-1})$ leaves $e_k, k \in \{1, 2\} \setminus \{i\}$, and $\left\{[r \alpha e_k + se_3, s \in \mathbb{R}] \setminus \{0\}\right\}$ invariant, moreover, it contains loxodromic and parabolic elements. Thus Lemma 1.2 yields

$$\ell_j \setminus \text{Eq}(\Pi_j(T\Gamma_{A,B,\nu}T^{-1})) = \left\{[r \alpha e_k + se_3, s \in \mathbb{R}] \setminus \{0\}\right\}.$$
Thus a straightforward calculation shows
\[
T \Gamma_{A,B,ν}^{-1}(ℓ_1 \cup ℓ_2) = \mathbb{P}_C^2 \setminus \bigcup_{j \in \{1,2\}} \bigcup_{p \in \mathcal{R}(ℓ_j)} \overrightarrow{e_j, p} \subset \Lambda(T \Gamma_{A,B,ν}^{-1}).
\]

Thus \( Ω = \bigcup_{ǫ_1, ǫ_2 \in \{±1\}} H_{ǫ_1} \times H_{ǫ_2} \) is an open \( T \Gamma_{A,B,ν}^{-1} \)-invariant set, with \( \text{Li}_G(Ω) = 4 \). In consequence Theorem 3.5 in [1], yields \( Ω \subset \text{Eq}(T \Gamma_{A,B,ν}^{-1}) \subset Ω(Γ) \). Which clearly concludes the proof.

By means of similar arguments the following proposition can be showed.

**Proposition 2.9.** Let \( A \in \text{SL}(2, \mathbb{Z}) \) be an hyperbolic toral automorphism, then the following set is a discrete group of \( \text{PSL}(3, \mathbb{C}) \)
\[
Γ_A = \left\{ \left( \begin{array}{ccc} A^k & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \mid b \in M(1 \times 2, \mathbb{Z}), k \in \mathbb{Z} \right\}
\]
Moreover \( Ω(Γ_{A,B,ν}) \) is projectively equivalent to \( \bigcup_{e_1, e_2 \in \{±1\}} \mathbb{H}^{e_1} \times \mathbb{H}^{e_2} \).

**Definition 2.10.** A subgroup \( Γ \subset \text{PSL}(3, \mathbb{C}) \) is said to be a *Hyperbolic Toral Group* if \( Γ \) is conjugated to the group described either in proposition 2.9 or the one in proposition 2.8.

### 3. Four lines Groups

Through this section \( Γ \subset \text{PSL}(3, \mathbb{C}) \) is a complex Kleinian group with \( \text{Li}_G(Ω(Γ)) = 4 \), \( L \in \text{LG}(Ω(Γ)) \) with \( \text{card}(L) = 4 \), the vertex of \( L \) are \( e_1, e_2, ℓ_1 = \overrightarrow{e_2, e_3}, ℓ_2 = \overrightarrow{e_1, e_3}, Γ_0 = \text{Stab}(e_1, Γ) \cap \text{Stab}(e_2, Γ) \), \( π_i = π_{e_i, ℓ_i} \) and \( Π_i = Π_{e_i, ℓ_i} \).

**Lemma 3.1.** Either \( Π_1(Γ_0) \) or \( Π_2(Γ_0) \) contains loxodromic elements.

**Proof.** On the contrary, let us assume that \( Π_j(Γ_0), j \in \{1,2\} \), does not contains loxodromic elements. Thus each element \( γ \in Γ \) has a lift \( \tilde{γ} \in GL(3, \mathbb{C}) \) which is given by:
\[
\tilde{γ} = \left( \begin{array}{ccc} γ_{11} & 0 & γ_{13} \\ 0 & γ_{22} & γ_{23} \\ 0 & 0 & 1 \end{array} \right)
\]
where \( |γ_{11}| = |γ_{22}| = 1 \). A straightforward calculation shows that \( \text{Eq}(Γ_0) = \mathbb{P}_C^2 \setminus \overrightarrow{e_1, e_2} \). Thus \( \text{Eq}(Γ) = \text{Eq}(Γ_0) \). Which is a contradiction. \( \square \)

**Lemma 3.2.** If \( Π_i(Γ_0) \) contains a loxodromic element then \( \bigcap_{τ \in Π_i(Γ_0)} \text{Fix}(τ) \) contains a single point.

**Proof.** Without loss of generality we may assume that \( i_0 = 2 \). Now, if \( F_2 = \bigcap_{τ \in Π_2(Γ_0)} \text{Fix}(τ) \) contains more than one point, we deduce that \( F_2 = \{e_1, z\} \) for some \( z \in ℓ_2 \setminus \{e_1\} \). By conjugating by a projective transformation, if it
is necessary, we may assume that $z = e_3$. Thus each element $\gamma \in \Gamma_0$ has a
lift $\tilde{\gamma} \in SL(3, \mathbb{C})$ which is given by:

$$
\tilde{\gamma} = \begin{pmatrix}
\gamma_{11} & 0 & 0 \\
0 & \gamma_{22} & \gamma_{23} \\
0 & 0 & 1
\end{pmatrix}
$$

where $abc = 1$. In consequence $\ell_1$ and $e_1$ are invariant under the action of
$\Gamma_0$. By Lemma 1.1 $W = \mathbb{P}^2_C \setminus (\ell_1 \cup \{e_1\})$ is a discontinuity region for $\Gamma_0$
which is contained in $Eq(\Gamma_0)$. In consequence $Lin(\Gamma W) < \infty$, which is a
contradiction, since $\Gamma W \subset \Gamma(Eq(\Gamma_0) = Eq(\Gamma)$. \hfill \Box

**Lemma 3.3.** The groups $\Pi_1(\Gamma_0)$ and $\Pi_2(\Gamma_0)$ contains loxodromic elements.

**Proof.** By Lemma 3.1, either $\Pi_1(\Gamma_0)$ or $\Pi_2(\Gamma_0)$ contains loxodromic elements.
Without loss of generality let us assume that $\Pi_1(\Gamma_0)$ contains a
loxodromic element. If $\Pi_2(\Gamma_0)$ does not contains loxodromic elements,
every element $\gamma \in \Gamma_0$ has a lift $\gamma \in GL(3, \mathbb{C})$ which is given by:

$$
\tilde{\gamma} = \begin{pmatrix}
\gamma_{11} & 0 & \gamma_{13} \\
0 & \gamma_{22} & \gamma_{23} \\
0 & 0 & 1
\end{pmatrix}
$$

where $|\gamma| = 1$. In consequence there are $\gamma, \tau \in \Gamma_0$ such that $\Pi_1(\gamma)$ and
$\Pi_1(\tau)$ are loxodromic elements with $Fix(\Pi_1(\gamma)) \neq Fix(\Pi_1(\tau))$, $\Pi_2(\gamma)$ is
either parabolic or the identity and $\Pi_2(\tau)$ is either parabolic or the identity.
An easy calculation shows $\Pi_1(\tau \gamma^{-1} \gamma^{-1})$ is parabolic and $\Pi_1(\tau \gamma^{-1} \gamma^{-1})$ is
the identity. In consequence $\kappa = \tau \gamma^{-1} \gamma^{-1}$ has a lift $\tilde{\kappa} \in SL(3, \mathbb{C})$ given by:

$$
\tilde{\kappa} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \kappa_{23} \\
0 & 0 & 1
\end{pmatrix}
$$

Finally, let $\gamma_0 \in \Gamma_0$ be such that $\Pi_1(\gamma_0)$ is a loxodromic element. By conjugating with a projective transformation, if it is necessary, we may assume that $Fix(\Pi_1(\gamma_0)) = \{e_2, e_3\}$. Also, by taking the Inverse of $\gamma$, if it is necessary, we may assume that $e_2$ is an attracting point for $\Pi_1(\gamma_0)$. In consequence, if $\tilde{\gamma}_0 = (\gamma_{ij} \in SL(3, \mathbb{C}))$ is a lift of $\gamma$, we conclude that:

$$
\tilde{\gamma}^m \tilde{\kappa} \tilde{\gamma}^{-m} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \gamma_{23}^m \kappa_{23} \\
0 & 0 & 1
\end{pmatrix} \xrightarrow{m \to \infty} Id,
$$

which is a contradiction, since $\Gamma$ is discrete. \hfill \Box

**Lemma 3.4.** There is an element $\gamma_0 \in \Gamma_0$ such that $\Pi_1(\gamma_0)$ and $\Pi_2(\gamma_0)$ are
loxodromic.

**Proof.** If this is not the case, there are $\gamma_1, \gamma_2 \in \Gamma_0$ such that $\Pi_j(\gamma_k)$ is
loxodromic if $j = k$ and either the identity or parabolic in other case. It is
straightforward to check that $\Pi_j(\gamma_1 \gamma_2), j \in \{1, 2\}$, is loxodromic. Which is
a contradiction.
From now on $\gamma_L$ will denote a fixed element in $\Gamma_0$ such that $\Pi_j(\gamma_L)$, $j \in \{1, 2\}$, is loxodromic. Also, by conjugating with a projective transformation, if it is necessary, we may assume that $\gamma_L$ has a lift $\tilde{\gamma}_L = (\gamma_{Lij})$ which is a diagonal matrix.

**Lemma 3.5.** There is an element $\gamma \in \Gamma_0$ such that $\Pi_1(\gamma_0)$ and $\Pi_2(\gamma_0)$ are parabolic elements.

**Proof.** Let $j \in \{1, 2\}$, then there is an element $\gamma_j$ such that $\Pi_j(\gamma_j)$ is loxodromic and $Fix(\Pi_j(\gamma_j)) \neq Fix(\Pi_j(\gamma_L))$. Set $\gamma_j = \gamma_L \gamma_j \gamma_L^{-1} \gamma_j^{-1}$, then $\Pi_i(\gamma_j)$ is parabolic if $i = j$ and is either the identity or parabolic in other case. Thus the only interesting case is $\Pi_1(\gamma_1) = Id$ and $\Pi_2(\gamma_2) = Id$. But in such case a simple calculation shows that $\Pi_j(\gamma_i \gamma_j)$, $j \in \{1, 2\}$, is parabolic. □

From now on $\gamma_P$ will denote a fixed element in $\Gamma_0$ with a lift $(\gamma_{Pij})$, such that $\Pi_j(\gamma_0)$, $j \in \{1, 2\}$, is parabolic.

**Lemma 3.6.** If $|\gamma_{L11}| < |\gamma_{L33}|$, then $|\gamma_{L22}| > |\gamma_{L33}|$.

**Proof.** On the contrary, let us assume that $|\gamma_{L22}| < |\gamma_{L33}|$. Then a straightforward calculation shows that:

$$
\kappa_m = \begin{pmatrix}
1 & 0 & \gamma_{L11}^m \gamma_{P13} \gamma_{L33}^{-m} \\
0 & 1 & \gamma_{L22}^m \gamma_{P23} \gamma_{L33}^{-m} \\
0 & 0 & 1
\end{pmatrix},
$$

is a lift of $\gamma_{L}^{-m} \gamma_{P} \gamma_{L}^m$. And clearly $[[\kappa_m]] \xrightarrow{m \to \infty} Id$, which is a contradiction, since $\Gamma_0$ is discrete. □

**Lemma 3.7.** The sets $\mathbb{P}^2_\mathbb{C} \setminus Eq(\Pi_1(\Gamma_0))$ and $\mathbb{P}^2_\mathbb{C} \setminus Eq(\Pi_2(\Gamma_0))$ are circles.

**Proof.** Since the vertex of $\mathcal{L}$ are $e_1, e_2$ it follows that

$$
\mathcal{C}_j = \Pi_j \left( \bigcup \{ \ell \in \text{Gr}_1(\mathbb{P}^2_\mathbb{C}) | e_j \in \ell, \ell \subset \Lambda(\Gamma) \} \right),
$$

is a closet, infinite and $\Pi_j(\Gamma_0)$-invariant set. Thus Lemma 1.2 yields the result. □

**Lemma 3.8.** Up to conjugacy $\Gamma_0$ leaves $\mathbb{P}^3_\mathbb{R}$ invariant.

**Proof.** By Lemma 3.5 there is an element $\gamma_0 \in \Gamma_0$ such that $\Pi_1(\gamma_0)$ and $\Pi_2(\gamma_0)$ are loxodromic elements. Thus after conjugating with a projective transformation, if it is necessary, we may assume that $Fix(\Pi_1(\gamma_0)) = \{[e_2], [e_3]\}$ and that $Fix(\Pi_2(\gamma_0)) = \{[e_1], [e_3]\}$. In consequence $\gamma_0$ has a lift $\tilde{\gamma}_0 \in SL(3, \mathbb{C})$ given by:

$$
\tilde{\gamma}_0 = \begin{pmatrix}
\gamma_{11} & 0 & 0 \\
0 & \gamma_{22} & 0 \\
0 & 0 & \gamma_{22}
\end{pmatrix},
$$

where $\gamma_{11} \gamma_{22} \gamma_{33} = 1$. Thus there are $\alpha_1, \alpha_2 \in \mathbb{C}^*$ such that:

$$
\ell_1 \setminus Eq(\Pi_1(\Gamma_0)) = \{r \alpha_1 e_2 + se_3 | r, s \in \mathbb{R} \} \setminus \{0\};
$$

$$
\ell_2 \setminus Eq(\Pi_2(\Gamma_0)) = \{r \alpha_2 e_3 + se_3 | r, s \in \mathbb{R} \} \setminus \{0\};
$$

$$
\ell_3 \setminus Eq(\Pi_3(\Gamma_0)) = \{r \alpha_2 e_3 + se_3 | r, s \in \mathbb{R} \} \setminus \{0\};
$$

and

$$
\ell_4 \setminus Eq(\Pi_4(\Gamma_0)) = \{r \alpha_1 e_2 + se_3 | r, s \in \mathbb{R} \} \setminus \{0\};
$$

where $\alpha_1, \alpha_2 \in \mathbb{C}^*$ and $\ell_1, \ell_2, \ell_3, \ell_4$ are circles.
\[ \ell_2 \setminus Eq(\Pi_2(\Gamma_0)) = \{ \{ r \alpha_2 e_1 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \} \}. \]

Let \( \eta \in PSL(3, \mathbb{C}) \) be the element induced by the linear map:

\[ \tilde{\eta} = \begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Thus a straightforward calculation shows that

\[ \Pi_1(\eta^{-1} \Gamma_0 \eta)[\{ re_2 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \}] = \{ \{ re_2 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \} \}; \]

\[ \Pi_2(\eta^{-1} \Gamma_0 \eta)[\{ re_1 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \}] = \{ \{ re_1 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \} \].

In consequence \( \mathbb{P}_2^R \) is \( \eta^{-1} \Gamma_0 \eta \)-invariant. \( \square \)

From now on we will assume that \( \mathbb{P}_2^R \) is \( \Gamma_0 \)-invariant. Also let us define:

\[ \text{Par}(\Gamma_0) = \{ \gamma \in \Gamma_0 \mid \Pi_j(\gamma), j \in \{ 1, 2 \}, \text{ is either parabolic or the identity} \}. \]

**Lemma 3.9.** The set \( \text{Par}(\Gamma_0) \) is a group isomorphic to \( \mathbb{Z} \times \mathbb{Z} \).

**Proof.** Clearly \( \text{Par}(\Gamma_0) \) is a group. Moreover \( \text{Par}(\Gamma_0) \) can be lifted to a group \( \tilde{\text{Par}}(\Gamma_0) \subset SL(3, \mathbb{C}) \) where each element has the form:

\[ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}. \]

where \( a, b \in \mathbb{R} \). Also observe that the group morphism \( Lat : \tilde{\text{Par}}(\Gamma_0) \to \mathbb{R}^2 \)

given by \( Lat((\gamma_{ij})) = (\gamma_{13}, \gamma_{23}) \) enable us to show that \( \text{Par}(\Gamma_0) \) is isomorphic to a lattice in \( \mathbb{R}^2 \). Thus to get the claim, will be enough to show that there are two elements in \( Lat(\tilde{\text{Par}}(\Gamma_0)) \) which are \( \mathbb{R} \)-linearly independent. Clearly

\[ \kappa_1 = \begin{pmatrix} 1 & 0 & \gamma_{L11} \gamma_{P13} \gamma_{L33}^{-1} \\ 0 & 1 & \gamma_{L22} \gamma_{P23} \gamma_{L33}^{-1} \\ 0 & 0 & 1 \end{pmatrix}. \]

is a lift in \( \tilde{\text{Par}}(\Gamma_0) \) of \( \gamma_L^{-1} \gamma_P \gamma_L. \) To conclude observe that the system of linear equations

\[ rLa(\kappa_1) + sLa(\gamma_P) = 0 \]

has determinant \( \gamma_{P23} \gamma_{P13} \gamma_{L33}^{-1} (\gamma_{L11} - \gamma_{L22}) \neq 0. \) Which conclude the proof. \( \square \)

Set

\[ \mathbb{R}(\ell_1) = \{ \{ r \alpha_1 e_2 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \} \}; \]
\[ \mathbb{R}(\ell_2) = \{ \{ r \alpha_1 e_1 + s e_3 \mid r, s \in \mathbb{R} \} \setminus \{ 0 \} \}; \]

**Proposition 3.10.** The equicontinuity set of \( \Gamma \) is given by:

\[ Eq(\Gamma) = \bigcup_{\epsilon_1, \epsilon_2 \in \{ \pm 1 \}} \mathbb{H}^{\epsilon_1} \times \mathbb{H}^{\epsilon_2}. \]
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\[ \Omega = \mathbb{P}^2 \backslash \bigcup_{j \in \{1, 2\}} \bigcup_{p \in \mathbb{R}(l_j)} \hat{e}_j \hat{p}. \]

Clearly \( \Omega = \bigcup_{\varepsilon_1, \varepsilon_2 \in \{\pm 1\}} H^{e_1} \times H^{e_2} \) and is a \( \Gamma_0 \)-invariant with \( \text{LiG}(\Omega) = 4 \). Thus Theorem 3.5 in \( \text{[1]} \), yields \( \Omega \subset \text{Eq}(\Gamma_0) \). On the other hand, let \( j \in \{1, 2\} \) and \( l_j \in Gr_1(\mathbb{P}^2) \) be such that \( e_j \in l_j \subset \Lambda(\Gamma) \). Then

\[ \bigcup_{p \in \mathbb{R}(l_j)} \hat{e}_j \subset \bigcup_{\gamma \in \Gamma_0} \Pi_j(\gamma)(\pi_j(l_j \backslash \{e_j\})) = \Pi_j. \]

In consequence \( \text{Eq}(\Gamma) \subset \Omega \). Finally, since \( \Gamma_0 \) is a subgroup of finite index of \( \Gamma \), we conclude that \( \text{Eq}(\Gamma_0) = \text{Eq}(\Gamma) \), which concludes the proof. \( \square \)

In the sequel let \( \tilde{\Gamma} = \bigcap_{\varepsilon_1, \varepsilon_2 \in \{\pm 1\}} \text{Stab}(H^{e_1} \times H^{e_2}, \Gamma) \). As an immediate consequence one has:

**Lemma 3.11.** The group \( \tilde{\Gamma} \) is a subgroup of \( \Gamma \) of index at most 4, which contains \( \text{Par}(\Gamma_0) \). Moreover \( \tilde{\Gamma} \) can be lifted to a subgroup \( \tilde{\Gamma} \) of \( GL(3, \mathbb{R}) \) where each element has the form:

\[
\begin{pmatrix}
a & 0 & c \\
0 & b & d \\
0 & 0 & 1
\end{pmatrix}
\]

where \( a, b > 0 \) and \( c, d \in \mathbb{R} \).

**Definition 3.12.** From now on \( \widetilde{\text{Par}}(\Gamma_0) \) will denote the lift of \( \text{Par}(\Gamma_0) \) in \( \tilde{\Gamma} \) and \( \text{Lat} : \widetilde{\text{Par}}(\Gamma_0) \to \mathbb{R}^2 \) will denote the group morphism given by \( \text{Lat}(\gamma_{ij}) = (\gamma_{13}, \gamma_{23}) \).

**Corollary 3.13.** For each \( j \in \{1, 2\} \), it follows that \( \Pi_j(\tilde{\Gamma}) \) does not contains elliptic elements.

**Lemma 3.14.** Let \( j \in \{1, 2\} \) and \( \gamma \in \tilde{\Gamma} \). If \( \Pi_j(\gamma) \) is parabolic, then \( \Pi_i(\gamma) \neq \text{Id} \), where \( j \) is the unique element in \( \{1, 2\} \setminus \{j\} \).

**Proof.** On the contrary let us assume that there \( j \in \{1, 2\} \) and \( \gamma \in \tilde{\Gamma} \) such that \( \Pi_j(\gamma) \) is parabolic and \( \Pi_i(\gamma) = \text{Id} \). Now by taking the inverse of \( \gamma_L \) if it is necessary, we may assume that \( |\gamma_{L13}| < 1 \). Thus an easy calculation shows:

\[
\gamma_{L}^{-m} = \begin{pmatrix}
1 & 0 & \gamma_{L13}^{-m} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \to Id.
\]

Which is a contradiction. \( \square \)

**Lemma 3.15.** Let \( j \in \{1, 2\} \) and \( \gamma \in \tilde{\Gamma} \). If \( \Pi_j(\gamma) \) is loxodromic, then \( \Pi_i(\gamma) \) is loxodromic, where \( j \) is the unique element in \( \{1, 2\} \setminus \{j\} \).
Proof. On the contrary let us assume that there $j \in \{1, 2\}$ and $\gamma \in \hat{\Gamma}$ such that $\Pi_j(\gamma)$ is loxodromic and $\Pi_i(\gamma)$ is not loxodromic. Thus $\pi_j(\gamma)$ is either $Id$ or a parabolic element. Set $\tau = \gamma \gamma_1$, then $\pi_j(\tau)$ is loxodromic with $\text{Fix}(\Pi_j(\tau)) \neq \text{Fix}(\Pi_j(\gamma))$. Thus an easy calculation shows that $\Pi_j(\gamma \gamma^{-1} \gamma^{-1})$ is parabolic and $\Pi_i(\gamma \gamma^{-1} \gamma^{-1})$ is the identity. Which contradicts Lemma 3.14. $\square$

**Definition 3.16.** Let $e\text{Lat} : \hat{\Gamma} \to GL(2, \mathbb{R}^+)$ given by

$$e\text{Lat}(\gamma_{ij}) = \begin{pmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{22} \end{pmatrix}$$

**Proposition 3.17.** $e\text{Lat}$ is a group morphism whose kernel is $\text{Par}(\Gamma_0)$.

**Proposition 3.18.** $\text{Lat}(\text{Par}(\Gamma_0))$ is invariant, as set of $\mathbb{R}^2$, under the action of the group $e\text{Lat}(\hat{\Gamma})$.

Proof. Let $(a, b) \in \text{Lat}(\text{Par}(\Gamma_0))$ and $(\gamma_{ij}) \in e\text{Lat}(\hat{\Gamma})$. Thus there is $\gamma \in \hat{\Gamma}$ and $\tau \in \text{Par}(\Gamma_0)$ such that $e\text{Lat}(\gamma) = (\gamma_{ij})$ and $\text{Lat}(\tau) = (a, b)$. Thus $\kappa = \gamma \tau \gamma^{-1} \in \text{Par}(\Gamma_0)$ and $\text{Lat}(\kappa) = (\gamma_{11} a, \gamma_{22} b)$. $\square$

**Lemma 3.19.** $e\text{Lat}(\hat{\Gamma})$ is a commutative discrete group with at most 2 generators.

Proof. Since the map $\rho : e\text{Lat}(\hat{\Gamma}) \to \mathbb{R}^2$ given by

$$\rho \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) = (\text{Log}(a), \text{Log}(b))$$

is an isomorphism of groups and from the Bierberbach Theorem (see [7]), it will be enough to show $e\text{Lat}(\hat{\Gamma})$ is discrete. If this is not the case, then there are sequences of distinct elements $(\alpha_m), (\beta_m) \in \mathbb{R}^+$ such that $\left( \begin{array}{cc} \alpha_m & 0 \\ 0 & \beta_m \end{array} \right) \in e\text{Lat}(\hat{\Gamma})$ and $\alpha_m, \beta_m \xrightarrow{m \to \infty} 1$. Let $\gamma_m = (\gamma_{ij}^{(m)}) \in \hat{\Gamma}$ such that $e\text{Lat}(\gamma_m) = (\alpha_m, \beta_m)$. Since $\text{Lat}(\text{Par}(\Gamma_0))$ is a lattice of rank 2, it follows that there is a sequence $(\tau_m) \in \text{Lat}(\text{Par}(\Gamma_0))$ such that $(\text{Lat}(\tau_m) + (\gamma_{13}, \gamma_{23}))$ is a bounded sequence. Thus we can assume that there are $c, d \in \mathbb{R}$ such that $(\text{Lat}(\tau_m) + (\gamma_{13}, \gamma_{23})) \xrightarrow{m \to \infty} (c, d)$. Now a straightforward calculation shows:

$$\gamma_m \tau_m = \begin{pmatrix} e\text{Lat}(\gamma_m) & \text{Lat}(\tau_m) + (\gamma_{13}^{(m)}, \gamma_{23}^{(m)}) \\ 0 & 1 \end{pmatrix} \xrightarrow{m \to \infty} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix},$$

which is a contradiction. $\square$

In the sequel $\sigma$ is a subset of $\hat{\Gamma}$ such that $\text{Card}(\sigma) = \text{rank}(e\text{Lat}(\hat{\Gamma}))$ and its lift $\tilde{\sigma}$ in $\hat{\Gamma}$ satisfies $< \tilde{\sigma} > = e\text{Lat}(\hat{\Gamma})$. Also, take $\tau_0 \in \sigma$ be a a fixed element, thus, by conjugating with a projective transformation if it is necessary, we
may assume that $\tau_0$ has a lift $\tilde{\tau} \in \tilde{\Gamma}$ which is a diagonal matrix. Finally, let $\gamma_1 \gamma_2 \in \text{Par}(\Gamma_0)$ be such that $\langle \gamma_1, \gamma_2 \rangle = \text{Par}(\Gamma_0)$.

**Lemma 3.20.** It holds that $\langle \sigma, \gamma_1, \gamma_2 \rangle = \tilde{\Gamma}$. 

**Proof.** Let $\gamma \in \tilde{\Gamma}$ and $\hat{\gamma} \in \hat{\Gamma}$ be a lift. Thus there is $\tau \in \langle \sigma \rangle$ with a lift $\hat{\tau} \in \tilde{\Gamma}$ such that $e\text{Lat}(\hat{\gamma}) = e\text{Lat}(\hat{\tau})$. In consequence $\gamma \tau^{-1} \in \text{Par}(\Gamma_0)$, which concludes the proof. □

**Lemma 3.21.** It is verified that $\tilde{\Gamma}$ is a hyperbolic toral group.

**Proof.** Let $(v_{11}, v_{21}), (v_{12}, v_{22}) \in \mathbb{R}^2$ be linearly independent vectors such that $\text{Lat}(\text{Par}(\Gamma_0)) = \langle v_1, v_2 \rangle$. Also set

$$\hat{T} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} ; \quad T = \begin{pmatrix} \hat{T} & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Thus an easy calculation shows:

$$T^{-1} \gamma_j T = \begin{pmatrix} I & e_j \\ 0 & 1 \end{pmatrix} \text{ for each } j \in \{1, 2\}$$

$$T^{-1} h T = \begin{pmatrix} \hat{T}^{-1} h \hat{T} & \nu_h \\ 0 & 1 \end{pmatrix} \text{ for each } h \in \tilde{\sigma}$$

where $\nu_h = 0$ if $h = e\text{Lat}(\tilde{\tau}_0)$. Clearly Proposition 3.18 yields that $G = \langle \{\hat{T}^{-1} h : h \in \tilde{\sigma}\} \rangle$ is a commutative group, where each element different from the identity is a hyperbolic toral automorphism and whose rank is either 1 or 2. To conclude the proof, let us show the following claim.

Claim 1.- If there is $h \in \tilde{\sigma} \setminus \{\tau\}$, then $e\text{Lat}(\tilde{\tau}_0) - \nu_{e\text{Lat}(h)} \in \mathbb{Z}^2$, where $\tilde{h} \in \tilde{\sigma}$ is the lift of $h$. An easy calculation shows:

$$\tilde{h} \tilde{\tau}^{-1} \tilde{h}^{-1} = \begin{pmatrix} I & e\text{Lat}(\tilde{\tau}_0) - \nu_{e\text{Lat}(h)} \\ 0 & 1 \end{pmatrix}.$$ 

Which concludes the proof. □

Now theorem 0.1 follows easily.

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