Abstract

In this second in a series of four articles we create a mathematical formalism sufficient to represent nontrivial hamiltonian quantum dynamics, including resonances. Some parts of this construction are also mathematically necessary. The specific construction is the transforming of a pair of quantized free oscillators into a resonant system of coupled oscillators by analytic continuation which is performed algebraically by the group of complex symplectic transformations, thereby creating dynamical representations of numerous semi-groups. The quantum free oscillators are the quantum analogue of classical action angle variable solutions for the coupled oscillators and quantum resonances, including Breit-Wigner resonances. Among the exponentially decaying Breit-Wigner resonances represented by Gamow vectors are hamiltonian systems in which energy transfers from one oscillator to the other. There are significant mathematical constraints in order that complex spectra be accommodated in a well defined formalism which represents dynamics, and these may be met by using the commutative real algebra \( \mathbb{C}(1,i) \) as the ring of scalars in place of the field of complex numbers. These mathematical constraints compel use of fundamental (spinor) representations rather than UIR’s. By including distributional solutions to the Schrödinger equation, placing us in a rigged Hilbert space, and by using the Hamiltonian as a generator of canonical transformation of the space of states, the Schrödinger equation is the equation for parallel transport of generalized energy eigenvectors, explicitly establishing the Hamiltonian as the generator of dynamical time translations in this formalism.

Key words: correlated quantum oscillators and quantum dynamics and weak symplectic structure

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1 Introduction

1.1 Motivation

The object of this second installment of a four part series is to illustrate our method for constructing the most general possible probabilistic description of dynamics which has the most well defined mathematical structures possible. We have indicated the general mathematical structures and basic notions of a probability amplitude description of correlated dynamics in installment one \[1\]. We will use the variable names $p$ and $q$ herein since they represent canonical variables. We will also understand the creation and destruction operators in a slightly different sense than is traditional: we will think of $A$ and $A^\dagger$ in the first instance as vectors generating displacements in coordinate directions in phase space (and its complexification) when viewed classically and as the corresponding operators in our function space representation of correlated dynamics by probability amplitudes. (They are clearly the generators of translations, but not infinitesimal generators. Hence, their identification as vectors. There will be much more justification for this in installment four of this series \[3\].) Since we will be working in the rigged Hilbert space, $A$ and $A^\dagger$ are not dual to each other as they were in the conventional Hilbert space approach: we will need another dual operation than the complex conjugation of the Hilbert space version of quantum theory. Therefore, $A$ and $A^\dagger$ are merely a pair of operators for us, both abstractly and in their function space representatives. We mention such things at the outset to give a taste of just how different things must be in order to bring a large number of mathematical structures into compatibility. We will see interesting physics emerge from the unfamiliar mathematics.

There are many lessons on many levels which emerge when we adopt the attitude that, if classical hamiltonian dynamics is grounded in the study of symplectic structures and symplectic transformations on phase space \[12\], then quantum dynamics should be based on the study of symplectic structures and symplectic transformations associated with the quantum mechanical space(s) of states. We will adopt such an attitude herein, and we will discover it takes very specific mathematical formalisms to insure our efforts are as well defined as it is possible to make them: be forewarned that unitary transformations are a subgroup of the symplectic transformations, and hermitean operators are a special (i.e., symmetric) case of the essentially self-adjoint operators, and we will adopt the more general types of transformations to our uses, thereby violating an orthodoxy entirely appropriate in another context. We will not be in von Neumann’s Hilbert space anymore! (Of course, the very act of analytic continuation is a non-unitary transformation, so the introduction of correlation notions by way of analytic continuation into any Hilbert space description takes you from the Hilbert space anyway \[15\]. The reader is reminded that the unit imaginary, $i$, is a type of correlation map.)
This approach also presents many threads to many notions currently under investigation in other contexts. Thus, if decoherence is the conversion of quantum probability into a classical probability, we showed in the first installment that our quantum construction can be thought of as the analytic continuation (complex symplectic transformations) of a type of classical probability theory. In this view, there should naturally be complex symplectic transformations which will have the effect of realification of our (spinorial) wave functions. We thus present the view that quantum theory is the result of the addition of coherence (e.g., correlation) to a particular form of classical probability theory, and the complimentary procedure of “decoherence” is also within the mathematical scope of the construction. In the third installment of this series, we will study many quantum analogues to classical dynamical systems, and their relation to classical statistical mechanics and thermodynamics.

From the combination of the seemingly simple problem of canonical transformations of two harmonic oscillators, and empowered by the mathematical tools of generalized functions (distributions) in a RHS, an astonishingly rich structure emerges exhibiting many of the features one associates with classical hard chaos and irreversible classical thermodynamics (in the $N = 2$ limit). The methods used here have transparent generalizations to larger numbers of oscillators.

The constructions of this paper make use of five major elements which distinguish it from the conventional Hilbert space treatment of quantum systems:

(1) It includes the weak, or distributional, solutions to the Schrödinger equation, mentioned above. These were not available to von Neumann.

(2) For our ring of scalars, rather than choosing the unit imaginary, $i$, as an element in the field of complex numbers, we will use $i$ as a 90 degree rotation in the complex plane, i.e., necessarily as an element of a real algebra. This means that for us $i$ is an element of the real algebra with two units, $\mathbb{C}(1,i)$. The result is a complex hyperbolic (Lobachevsky) structure in the tangent space at every point in our complex spaces of generalized states (e.g., tangent bundle). We are building spaces (modules) whose ring of scalars are real algebras rather than spaces (modules) over fields. This is the result of a uniqueness consideration, based on the old saw that complex conjugation is not a uniquely defined involution of the field of complex numbers regarded as an algebra–there is more than one possible conjugate structure to be obtained from the field $\mathbb{C}$. A linear space is an algebra plus a scalar product. If $\Phi \subset \Phi^\times$, then the dual (adjoint) operation works an involution of $\Phi$ when considered as a linear space in this sense. There is also the canonical inclusion $\mathfrak{g} \subset \mathfrak{g}^\times$, indicating that there is an algebra involution associated with the dual operation applied to an operator Lie algebra. This also distinguishes the present work from other recent work using the RHS formalism.

(3) We shall adopt an algebraic approach, emphasizing the involution of real algebras, so that, e.g., the dual operation will be an involution of some sort. We
will wish ultimately to identify the algebra of observables with the Lie algebra of dynamical transformations, which is the Lie algebra of the symplectic group of our phase space. This dual as involution then effects the canonical inclusion of the Lie algebra into its dual Lie algebra.

(4) We will use the weak symplectic structure available on $\Phi \times \Phi^\times$ (e.g., associated with the scalar product on $\Phi$) to reflect (represent) the canonical weak symplectic structure on $g \times g^\times$ for Lie algebra $g$ (which is represented on $\Phi$). For the complex Hilbert space of von Neumann $\mathcal{H}$, there is a strong symplectic structure on $\mathcal{H} \times \mathcal{H}^\times$, e.g., associated the scalar product on $\mathcal{H}$ [45]. You cannot have a non-trivial dynamics in an Euclidean geometry.

(5) In order to proceed uniquely, we are required to work with the real form (symplectic form) of various complex Lie algebras (of complex simple Lie groups), and representations of those complex Lie algebras. This will operate as a major determinant of structure for our spaces of states (which are, due to this mathematical necessity, spin spaces, and our representations will be fundamental representations and not unitary irreducible representations).

The basic program underlying the present work is conceptually quite austere, although there are a myriad of details to the actual implementation as we seek to implement compatibility of many disparate mathematical structures. We regard $i$ as, e.g., an element of the commutative real algebra $\mathbb{C}(1,i) \equiv \mathbb{R} \oplus i \mathbb{R}$ [46]. We will regard the Hamiltonian of our oscillator system as a realization of one of the generators of infinitesimal canonical transformations of the associated phase space. This makes it a generator in the Lie algebra of the appropriate symplectic (semi-)group representation, e.g., for two pairs of canonical coordinates, $H$ belongs to $\text{sp}(4,R)$. Since $Sp(4,\mathbb{C})$ is simple, its representation provides a connected covering structure in which to exponentiate our continuous infinitesimal symplectic transvections. In fact, the Schrödinger equation would emerge as the analogue of Hamilton’s equation if we should chose a variational (i.e., Hamilton-Jacobi) treatment mathematically alternative to the approach adopted here.

There is a significant incompleteness to our work at the present stage, as convergence of the exponential map will not be addressed until the fourth part of our series of papers [3]. There we will see the the application of an extraordinary result for spinors: conjugation by a group element, $gAg^{-1}$ can be uniquely identified with a transformation from the left only from the relevant spin group: $gAg^{-1} = h_{spinA}$. The conjugation is well defined only in a finite local neighborhood (coordinate patch) for our semigroups, but the spin transformation which is thus fixed has global import. This admits an extension to the semigroups of interest to us, and the multicomponent nature of the vectors which is necessary for other reasons in this part of our work will become crucial in another context when we address the exponential map, the spin nature of our constructions, and related topological, group, semigroup and other issues in part four.

Our Hamiltonian (and its Lie algebraic dual) can also be defined to have an in-
finitesimal action on the spaces of states in such a way that its action on the space(s)
of states is symplectic as well. This will ultimately result in it (the representation of
the Hamiltonian) being associated with generating flows of hamiltonian (integrable)
vector fields on our generalized spaces of states (which are, in turn, representations
of vector fields on phase space.

We are creating a forum in which we may speak of a quantum dynamics evolving
through dynamical (=symplectic) transformations in a manner which follows much
of the spirit of the classical treatment of dynamical systems. (The “quantum chaos”
associated with the flow of topologically transitive hyperbolic affine transforma-
tions on the spaces of quantum states, which also have a symplectic action on the
spaces of states, will be discussed in the third installment of this series [2].) These
mathematical structures are not compatible with the conventional formulation of
quantum mechanics using von Neumann’s Hilbert space, and so we are pursuing
a description of a “quantum dynamical system” which is unique to our variant of
the RHS formulation of quantum mechanics. In the course of these investigation of
formulation of a quantum dynamics, many insights emerge illuminating mathematical
structures that naturally arise when one formulates an hamiltonian quantum
field theory based on harmonic oscillators dynamically evolving in our version of
the RHS formalism. There is a lot of mathematical physics we will show some
consistent connection to which is above and beyond the machinery we actually in-
vok for our immediate purposes. (Roger Newton dealt with enlarging the physical
Hilbert space in order to obtain quantum action-angle variables using “spin like”
multi-valued quantum numbers [6]. We are using spinors explicitly. See installment
four [3].)

Every complex semi-simple Lie algebra has some complex Lie group, and on a
semisimple Lie group with a complex Lie algebra, exp is holomorphic [7]. In con-
sequence, the complex simple Lie (semi-)groups relevant to our problem are con-
ected and locally path connected. Since exp is holomorphic for \( \mathfrak{sp}(4, \mathbb{R})^\mathbb{C} \),
the exponential map of \( H \) can locally be identified with the transformation of parallel
transport along geodesics in \( Sp(4, \mathbb{R})^\mathbb{C} \). Formally, we work with the real (or sym-
plectic) form of the Lie algebra, e.g., \( \mathfrak{sp}(4, \mathbb{R})^\mathbb{C} \) rather than \( \mathfrak{sp}(4, \mathbb{C}) \) itself, in order
that adjoints be well defined for both the algebra and the group simultaneously.
(See Section 2.) We will construct a representation of this Lie group/Lie algebra
structure, and, with the only mathematical limitation being the assumption that the
potentials in the Hamiltonian are analytic, we may make use of the creation and de-
struction operator formalism. (See [42].) There are however some wrinkles, since
our distributional solutions are associated with semigroups rather than full groups,
and our geodesics may have only local uniqueness due to topological considera-
tions which will become apparent in installment four [3] - this is a local quantum
theory in many senses. (We will eventually elaborate in installment four [3] how
the exponential map may be holomorphic and also how the semigroups of unitary
transformations—viewed as a sub-semigroup of the symplectic semigroup—differ at
most on a set of measure zero from the related unitary group.)
We construct our representation spaces as modules over $\mathbb{C}(1,i)$, and corresponding to the geodesics in $Sp(4,\mathbb{R})\mathbb{C}$ generated by $exp(sp(4,\mathbb{R})\mathbb{C})$ there will be geodesics of evolution generated in our representation space by the representation of the Lie group and associated Lie algebra: among the equations of parallel transport along these geodesics, it is possible to find an equation equivalent to the Schrödinger equation. We will (necessarily) allow weak solutions to these equations of geodesic evolution on our representation space, so that our chosen representation spaces include spaces of generalized functions, and thereby our work naturally falls into the RHS formalism. As part of our construction, we endeavor to respect all canonical constructions, including canonical inclusions (Section 2) and canonical symplectic forms (Section 3 and Section 2). The investigation of the mathematical and physical structures resulting from this basic program is the subject of the remainder of this series of papers. We shall begin, however, under the guise of attacking the concrete problem of one harmonic oscillator dissipating (transferring) energy to another oscillator, and our objective is to describe and constructively illustrate the structures and methods which figure in providing mathematically respectable solutions to that problem.

On the space $\Phi$ of the Gel’fand triplet $\Phi \subset \mathcal{H} \subset \Phi^\times$, essentially self adjoint operators need not be also be symmetric, i.e., may be non-hermitean. When working in these RHS’s the usual notions of hermiticity or anti-hermiticity are not controlling, and we will need a radically different notion of what is the proper form of adjoint involution in order to provide dynamical (including unitary) representations [47] of the connected Lie groups which our complex Lie algebras—and associated representations—integrate into through using the exponential map.

Item 1 in the list above gives us a more general set of solutions to work with. Item 2, we will see, gives us a geometric context in which it is mathematically proper to speak of the Hamiltonian as the generator of infinitesimal time translations (Item 5). This interpretation of the Hamiltonian does not automatically follow just because the Hamiltonian has this meaning when restricted to another space (and another topology). This is also significant to establishing the existence of Lie algebra valued connections, leading to a gauge theoretic interpretation of some associated structures which we will explore in the fourth installments of this series [3]. The third item is significant as indicating a possible source of obstructions which must be avoided by proceeding carefully, and our treatment demonstrates the inability of the standard Hilbert space formalism to address issues in the same mathematical generality we have available in our RHS construction.

1.2 Symplectic transformations of oscillators

In the non-relativistic RHS formalism, one works in a subspace of the Schwartz space (of functions of rapid decrease) for the function space realization of the
abstract space $\Phi$ of the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$. The Lie algebra and Lie group of symplectic transformations used in this present work have been associated with problems in quantum optics. One therefore infers the methodology of this paper is likely to find concrete application with electromagnetic fields. The group of symplectic transformations is in fact the group of squeezing transformations of quantum optics, and provide the source of the transfer matrices of classical optics in the paraxial approximation. Hence, the present work can be considered the study of the squeezing transformations of generalized Gaussian wave packets [48]. Physically, we are dealing with families of generalized Gaussian minimum uncertainty states. (If a field theoretic interpretation is adopted for the oscillators, a natural candidate for the representation of a “particle” is a stable member of these families of minimum uncertainty states.) Two free oscillators are “squeezed” into correlation (or coherence) by the symplectic (=dynamical) transformations, and further “squeezing” results in resonant decay of the coupled oscillator system.

1.3 Order of proceeding

The appendix contain some important calculations. Appendix A recapitulates the Feshbach-Tikochinsky $SU(1, 1)$ dissipative oscillator system calculation [11] in slightly changed notation (and with changed physical content, in order to put things in a form which is not otherwise remarkable).

We begin with a bit of naive algebra which requires substantial justification, which we also begin in this section. This is a variant of the Feshbach-Tikochinsky calculation in which we take the transformations to be part of a realization of the canonical (symplectic) transformations for the two oscillator phase space and also define the action of the appropriate Lie group (representation) to have a symplectic action on our spaces of states. The simple computation in Section 2 has implications at many levels. It is shown that the dissipative oscillator system can be obtained from a realization of the (semi-)group of symplectic transformations applied to the system of two free quantum oscillators. Further (complex) symplectic transformations yield vectors which decay exponentially (without regeneration). It is also demonstrated that similar constructions can generate a representation of $SU(2)$ in terms of creation and destruction operators, and further complex symplectic extensions will yield a complex spectrum for it as well, indicating a quite general process is involved.

The present results ultimately must be compared to the analytic continuation used by Bohm to obtain his Gamow vectors [13,14]. The two methods both describe complex symplectic transformations, so the necessary and sufficient “very well behaved” starting point for obtaining the Breit-Wigner resonance poles to associate to Gamow vectors [15] for the dissipative coherence of two oscillators problem is not the F-T system of coupled oscillators, but should probably be thought of as the
Section 3 provides a summary descriptions of aspects of well-known structures which will play important roles for us. These establish a basic general mathematical context in which we operate. In Section 3, the subject is the multitude of symplectic and Poisson structures associated with representation of both real and complex semi-simple Lie algebras in a RHS format. The structure of the space $\Phi^\times$ of the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$ is not completely known, but in the case of a Lie algebra representation, the structure of $\Phi^\times_{\mathfrak{g}_{\pm}}$ follows the structure of $\mathfrak{g}^\times$ in many important regards, and there is some elaboration on this.

The fact that the momentum map for a semi-simple Lie algebra $\mathfrak{g}$ is an element of $\mathfrak{g}^\times$ provides us with the first indication of a mandatory topology change to a weak-$\times$ topology: viewing a formal “adjoint” complex symplectic transformation on $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R})$, taking “$Ad_G\mathfrak{g}$”, as transitive and also having a symplectic action on a space requires us to view it as giving rise to a co-adjoint representation of $\mathfrak{g}^C$ in $(\mathfrak{g}^C)^\times$. This co-adjoint orbit structure within the individual symplectic sheaves making up the Poisson manifold $(\mathfrak{g}^C)^\times$ is faithfully reflected by the associated orbits of transformations on the representation spaces $\Phi_{\mathfrak{g}_{\pm}}$ and $\Phi^\times_{\mathfrak{g}_{\pm}}$. By appropriate identifications, the symplectic (and therefore Poisson) mapping of inclusion (of the representation of $\mathfrak{g}$) into these co-adjoint orbit structures (lying inside the representation of $(\mathfrak{g}^C)^\times$) can be made to provide essentially self adjoint extensions of generators and a representation of the associated transformation (semi-)groups whereby the exponential mapping for $\mathfrak{g}^C$ can be identified properly with the exponential mapping of $\mathfrak{g}^C \subset (\mathfrak{g}^C)^\times$.

The representations of the associated simple complex semi-groups are defined below in such a way that the transformations have a symplectic action on the representation spaces themselves. (See Section 2).

The overall procedures closely follow the prescription in [17]. There is a topology change mandated by continuous transformations of analytic continuation, and this point is very subtle and easy to overlook, but after the continuous analytic continuation one’s solutions have been extended to the distributions and only a weak-dual topology is appropriate [15]. (Once again, we’re not in Hilbert space anymore!)

2 “Scattering” of Simple Oscillators

In this section, we study the algebraic structure of the “dissipative oscillator” system, but this section may also be interpreted as a brief exercise undertaken to indicate the possibility of an algebraic theory of scattering, which is rigorous. Such systems for the description of scattering have been considered before, e.g., [19] uses $SU(1,1)$ as the continuation of $SU(2)$, and uses $Sp(4,\mathbb{R})$, to construct a par-
tial theory of this sort. Herein, a pair of free oscillators is subjected to canonical transformations to become an interacting (correlated or coupled) system in which one oscillator is ready to transfer energy to the other. Further canonical transformations lead to a dissipative oscillator system in which the flow of energy from one oscillator (with higher energy) in the coupled system to the other oscillator (with lower energy) is described by Gamow vectors which are energy eigenvectors with complex energy (and which therefore exhibit exponential growth or decay in their time evolution). This is the result of the identification of the interaction Hamiltonian of the coupled system with a non-compact generator of the algebra \( \mathfrak{su}(1,1) \subset \mathfrak{sp}(4,\mathbb{R}) \), which is extended (analytically continued) from that real algebra to a complex algebra with the same generators and commutation relations defined (i.e., extended to a complex covering algebra). This use of symplectic transformations is a generalization of the F-T results recapitulated in Appendix A. Any required additional justifications of the naive algebraic manipulations of the present section will be given in later sections.

The commutation relations useful to us for a unitary representation of \( \mathfrak{sp}(4,\mathbb{R}) \) on Hilbert space, \( \mathcal{H} \), are \([20]\) (\( i, j, k = 1, 2, 3 \), sums implied):

\[
\begin{align*}
[J_i, J_j] &= \epsilon_{ijk} J_k & (1) \\
[J_i, J_0] &= 0 & (2) \\
[K_i, K_j] &= -\epsilon_{ijk} J_k & (3) \\
[K_i, J_j] &= \epsilon_{ijk} K_k & (4) \\
[Q_i, Q_j] &= -\epsilon_{ijk} J_k & (5) \\
[Q_i, J_j] &= \epsilon_{ijk} Q_k & (6) \\
[K_i, Q_j] &= \delta_{ij} J_0 & (7) \\
[K_i, J_0] &= Q_i & (8) \\
[Q_i, J_0] &= -K_i & (9)
\end{align*}
\]

An appropriate realization of this algebra in terms of two mode creation and annihilation operators is \([20][21]\);
\[ iJ_1 = \frac{1}{2} \left( A^\dagger B + B^\dagger A \right) \]  
\[ iJ_2 = -\frac{i}{2} \left( A^\dagger B - B^\dagger A \right) \]  
\[ iJ_3 = \frac{1}{2} \left( A^\dagger A - B^\dagger B \right) \]  
\[ iJ_0 = \frac{1}{2} \left( A^\dagger A + BB^\dagger \right) \]  
\[ iK_1 = -\frac{1}{4} \left( A^\dagger A^\dagger + AA - B^\dagger B^\dagger - BB \right) \]  
\[ iK_2 = \frac{i}{4} \left( A^\dagger A^\dagger - AA + B^\dagger B^\dagger - BB \right) \]  
\[ iK_3 = \frac{1}{2} \left( A^\dagger B^\dagger + AB \right) \]  
\[ iQ_1 = \frac{i}{4} \left( A^\dagger A^\dagger - AA - B^\dagger B^\dagger + BB \right) \]  
\[ iQ_2 = -\frac{1}{4} \left( A^\dagger A^\dagger + AA + B^\dagger B^\dagger + BB \right) \]  
\[ iQ_3 = \frac{i}{2} \left( A^\dagger B^\dagger - AB \right) \]

Identifying \( X = iJ_1, Y = iJ_2, \) and \( Z = J_3, \) we obtain the realization of the \( SU(1,1) \) Lie algebra generators used in Appendix A.

The Baker-Campbell-Hausdorff relation

\[ e^B \, A \, e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} [B, [B, \ldots [B, A] \ldots]] \]

containing \( n \) factors of \( B \) in each term, can be applied to semi-simple Lie groups and algebras, e.g., to \( B = \mu X \in su(1,1) \subset sp(4, \mathbb{R}), \ A = Y \in su(1,1) \subset sp(4, \mathbb{R}), \ \mu \in \mathbb{R}, \) to yield:

\[ e^{i\mu X} (iY) e^{-i\mu X} = (iY) \cos \mu - [X,Y] \sin \mu. \]

For \( su(1,1) \) and for pure imaginary \( \mu, \) the cosine becomes hyperbolic cosine (cosh) and sine becomes hyperbolic sine (sinh). For the semi-simple group \( SU(1,1) \) and its semi-simple algebra \( su(1,1), \) it follows that:

\[ e^{i\mu X} (iY) e^{-i\mu X} = (iY) \cos \mu - Z \sin \mu \]

so that

\[ e^{i(\pi/2)X} (iY) e^{-i(\pi/2)X} = -Z \]

or

\[ Y = i \, e^{-i(\pi/2)X} \, Z \, e^{i(\pi/2)X}, \quad Z = -i \, e^{i(\pi/2)X} \, Y \, e^{-i(\pi/2)X} \]
Note that we are not in Hilbert space, so that “hermiticity” is not a relevant concept for us—it is not the case that conjugation of an hermitean operator by a seemingly “unitary” transformation has resulted in a non-hermitean operator.

Note the “brazen” use of the exponential map in the preceding. Full justification for this will have to wait until installment four [3], but we will make some preliminary comments here and further comment later on. The exponential map is to be understood geometrically here, translating a tangent vector back and forth along a geodesic (in the tangent space to the relevant group). For now, we will merely comment that this does not mean that we are defining a general inverse for any group element, but are working locally for limited displacements along single geodesics only. In general, inverses may not be unique. However, the maximal sub-semigroup of the dynamical transformations - the semigroup of unitary transformations - differs from the unitary group by no more than a set of measure zero, so at least some of the dynamical transformations can be viewed as fully invertible. In fact, we will see in part four of this series [3] that conjugation can be identified with a spin transformation acting from the left only, but preliminary to that we must establish that the multicomponent vectors we define in this part are in fact truly spinors, deal with the non-trivial issue of convergence of the exponential map. Physically, this approach is equivalent to insisting upon relative physical isolation: as time evolves, there are no additional dynamical interactions which further change our Hamiltonian and redefine the tangent to the geodesic of dynamical time evolution. The period of time of relative dynamical isolation may be billions of years for some photons to an astronomer or the minutest fraction of a second in a high energy experiment. As a practical matter, some sort of relative causal isolation is required for any precision experiment, so, although the mathematical meaning here may be a little vague (awaiting fuller development in part four [3]), the physical meaning is untroubling.

Note that for $\mathfrak{su}(2)$, where $[J_k, J_l] = \epsilon_{klm} J_m$, for real $\mu$

$$e^{i\mu J_k} (iJ_l) e^{-i\mu J_k} = (iJ_l) \cosh \mu - [J_k, J_l] \sinh \mu .$$

(25)

For pure imaginary $\mu$, the corresponding expression is $(iJ_l) \cos \mu + i [J_k, J_l] \sin \mu$. This is consistent with the $\mathfrak{su}(1, 1)$ results, because there is a (“dangerous to use”) mapping $\mathfrak{su}(2) \rightarrow \mathfrak{su}(1, 1)$ given by $J_1 \mapsto X = iJ_1$, $J_2 \mapsto Y = iJ_2$, $J_3 \mapsto Z$.

The Baker-Campbell-Hausdorf relation applies to both compact and non-compact groups, and to pure real and pure imaginary coefficients.

This transformation procedure (which is based on the Baker-Campbell-Hausdorf relations) results in “complex spectra for $SU(2)$” just as a similar transformation resulted in “complex spectra for $SU(1, 1)$”. This illustrates that a general process is going on applicable to all locally compact semisimple Lie subgroups of a complex semisimple Lie covering group, and, in particular, applicable to both non-compact and compact sub(semi)groups alike.
Consider a system of two independent simple harmonic oscillators. The Hamiltonian for this system is:

\[ H = \frac{1}{2} \left( p_x^2 + q_x^2 + p_y^2 + q_y^2 \right). \]  

(26)

This is the same operator as \( iJ_0 \), equation (13): \( H = iJ_0 \). If we subject this system to as “preparation procedure” the symplectic (=dynamical) transformation:

\[
\alpha e^{i(\pi/2)J_1} e^{i(\pi/2)K_2} (iJ_0) e^{-i(\pi/2)K_2} \alpha e^{-i(\pi/2)J_1} + \beta e^{i(\pi/2)Q_1} e^{i(\pi/2)K_2} (iJ_0) e^{-i(\pi/2)K_2} \beta e^{-i(\pi/2)Q_1} = \\
= -\alpha e^{i(\pi/2)J} [K_2, J_0] \alpha e^{-i(\pi/2)J_1} - \beta e^{i(\pi/2)Q_1} [K_2, J_0] \beta e^{-i(\pi/2)Q_1} \\
= +i\alpha e^{i(\pi/2)J_1} (iQ_2) \alpha e^{-i(\pi/2)J_1} + i\beta e^{i(\pi/2)Q_1} (iQ_2) \beta e^{-i(\pi/2)Q_1} \\
= i\alpha^2 [J_1, Q_2] + i\beta^2 [Q_1, Q_2] \\
= i\alpha^2 (-\epsilon_{123} Q_3) + i\beta^2 (-\epsilon_{123} J_3) \\
= -\alpha^2 (iQ_3) - \beta^2 (iJ_3) \]  

(27)

Choosing

\[
\alpha^2 = -\frac{\Gamma}{2} \quad \beta^2 = -\Omega 
\]  

(28)

we have recovered the Hamiltonian of the \( SU(1, 1) \) dissipative oscillator system, equations (A.6), (A.7), (A.8) (See Appendix A). Our Hamiltonian eigenstates have changed from \( \{|n_A\} \oplus \{|n_B\} \} \in \Phi \cap \mathcal{H} \) for the two free oscillators (energy/number representation) into two-component state vectors representing mixed states which provide the foundations for the representations of the two semi-groups \( SU(1, 1) \subset Sp(4, \mathbb{R}) \). The function space realization of these extensions runs the from the space spanned by the direct sum of the “very well behaved” free oscillator energy eigenstates representing the creation and destruction operator algebra, to the relevant representations of \( SU(1, 1) \subset Sp(4, \mathbb{R}) \).

Physically, the free oscillator system has been transformed by the introduction of correlation into a system now composed of two correlated oscillators—and the correlated system representation is split into a pair of semigroup representations split along time domains of definition. The states and related spaces are now mixed. That correlated system, in turn, is able to represent further internal transformation representing the redistribution the energy between the two oscillators. See Appendix A. Note that the correlated system acts as a (formally) closed system in which both source and sink for the “dissipation” are dealt with. We have a system which can not only represent the boundary conditions of an irreversible quantum dynamical process (without regeneration), but we can represent such a process in a dissipative context such as a thermodynamic reservoir (because the annihilation operator is a bounded continuous operator in the appropriate topology, this reservoir can be infinitely deep).
In [22], use is made of rigged Hilbert spaces for representation of \(SU(1, 1)\), and considerable care is taken to choose a topology which insures any anomalous complex eigenvalues for this real group are properly avoided. Herein, the anomalous complex eigenvalues are not excluded, meaning that at a minimum we must be working in some complex covering group, such as \(SL(2, \mathbb{C})\) or \(Sp(4, \mathbb{R})^C\), for which complex scalars are defined—and we note that analytic continuation is a complex symplectic transformation, which narrows our choices. Now making an additional extension of the full Hamiltonian of the dissipative oscillator system to \(\Phi^\times_{sp(4, \mathbb{R})_\pm}\) using the \(e^{i\mu K_3}\) map used in Appendix [A] (as an exemplar of the extension process—there are nine other possible generators for the liftings and other possible scalar coefficients), for the value \(\mu = \pi/2\) we find:

\[
e^{i(\pi/2)K_3} H e^{-i(\pi/2)K_3} = \alpha^2 e^{i(\pi/2)K_3} (iQ_3) e^{-i(\pi/2)K_3} + \beta^2 e^{i(\pi/2)K_3} (iJ_3) e^{-i(\pi/2)K_3}
\]

\[
= -\alpha^2 [K_3, Q_3] - \beta^2 [K_3, J_3]
\]

\[
= -\alpha^2 J_0 - \beta^2 (0)
\]

\[
= i\alpha^2 (iJ_0)
\]

(\(iJ_0\)) is the same as the realization of the operator \(iZ\) previously identified in the \(SU(1, 1)\) algebra, and this recovers the Feshbach-Tikochinsky result, equation (A.18) or equation (24). There are other liftings, however, for which the commutator which results from the lifting of \(H_0\) does not vanish. There are other decay constants (and decay processes) besides the single one identified in the F-T construction, although not all of these relate simply to the \(su(1, 1)\) algebra which the F-T procedure purports to be associated with. (Reiterating, it is \(sp(4, \mathbb{R})^C \cong sp(4, \mathbb{C}) \supset sl(2, \mathbb{C}) \supset su(1, 1)\) which sets the overall structure. This is only a substructure of that larger \(sp(4, \mathbb{R})^C\) structure.) [49]

The essentially self adjoint extensions of the \textit{generators} of the realization of the Lie algebra can be understood as, for instance:

\[
(iY)^\times = iY = e^{-i(\pi/2)K_3} \left[ e^{+i(\pi/2)K_3} (iY) e^{-i(\pi/2)K_3} \right] = e^{+i(\pi/2)K_3} \left[ e^{-i(\pi/2)K_3} \right] Z e^{+i(\pi/2)K_3}
\]

(30)

This amounts to identifying the simple inclusion of an operator belonging to \(g_\pm\) into \((g^C_\pm)^\times\) with a certain point on the co-adjoint orbit of another operator also within \((g^C_\pm)^\times\), which can be explained as the failure of the weak-\(\times\) topology to separate them.

Beyond those justifications for such an interpretation already given, we might add that this is of the form \(gAg^{-1}, g \in Sp(4, \mathbb{R})^C, A \in sp(4, \mathbb{R})\), and that on the semigroup representation space for which \(g\) is continuous, the operator \(g^{-1}\) cannot be a
continuous operator except for limited (finite) translations of the relevant generators along the relevant geodesic, according to the exponential map. (The conjugacy of $iY$ and $Z$ will be addressed at length in the fourth installment [3] of these articles, when we discuss the nature of the covering structure which makes all the exponential maps well defined, and which in turn makes the structure spanned by the related Gamow vectors with their associated Breit-Wigner resonances also well defined.) From the perspective of Lie algebra representations on $\Phi \subset \Phi \times$, esa operators need not be symmetric, contrary to the situation in $\mathcal{H}$. As to $\Phi$ and $\Phi \times$, there is no violation of any sense of hermiticity in the above. (Again, we’re not in Hilbert space anymore, so insisting on strictly applying notions of hermiticity or anti-hermiticity is unduly restrictive.)

Many readers may be asking why we do not just go ahead and speak of anti-self adjoint (anti-self dual, or asd) extensions. The generators of the Lie algebra $g$ are also canonically generators of the dual Lie algebra $g^\times$ due to the canonical $\pm$-inclusion $g \subset g^\times$. The generators of the algebra are taken to be identified with generators of infinitesimal translations (directional derivatives), and these have invariant geometrical meaning as a certain tangent vector. If we wish to identify representations of these generators in the Lie algebra with connections associated to covariant derivatives, i.e., identify $U(1)$ subgroups as geodesic subgroups, then we must understand that the semi-simple Lie algebras we are working with here may be associated with a group or with either of a pair of semi-groups, and the related duals: we are speaking of the same generator in all cases. (Because generators are first order derivatives, on analytic spaces the generator is locally independent of the direction of travel along the path of $\exp$.) Hence, because $g \subset g^\times$, the generators of $g$ are canonically (essentially) self adjoint (or self dual). To define a dynamical inner product transformation structure, on the other hand, requires an anti-self adjoint (e.g., anti-self dual or asd) extension of the Lie algebra [50].

It is necessary of symplectic transformations that their group obey the dynamical law. I.e., to say that a transformation $t$ belongs to the group of symplectic transformations of a complex symplectic space necessarily implies that $t \cdot t^\times$ must be the identity transformation [23]. This is a necessary and sufficient requirement. Therefore, there is really no other alternative for the form of adjoint involution other than the alternative chosen here, assuming we persist in the requirement that our Hamiltonian generate infinitesimal dynamical (=symplectic) transformations. We continue to mean that the ”inverse” here is to be understood only locally, for a period of relative dynamical isolation.

For unitary transformations on Hilbert space, one conventionally thinks of the dual transformation in terms of hermitean (i.e., complex) conjugation, e.g., for esa Hamiltonian

$$
(e^{-iHt})^\dagger = e^{(-i)^*H^\dagger t} = e^{(iH^\dagger)^t} = e^{iH^\dagger t} = “(e^{-iHt})^{-1}”.
$$

(31)

The unitary transformations on $\mathcal{H}$ are thus dynamical in their action on $\mathcal{H}$, and in-
deed the symplectic group contains a unitary group as maximal compact subgroup. This means the unitary transformations form only a proper subgroup of the group of all possible dynamical (symplectic) transformations on $\mathcal{H}$. On $\Phi$ and $\Phi^\times$, there one must think in terms of, e.g.,

$$
(e^{-iHt})^\times = (e^{-(iH)t})^\times = e^{-(iH)^\times(-t)} = e^{+(iH)t} = e^{+iHt} = "(e^{-iHt})^{-1},"
$$

(32)

because it must then also be the case that

$$
(e^{Ht})^\times = e^{H^\times(-t)} = e^{-Ht} = "(e^{Ht})^{-1},"
$$

(33)

in order that the group scalar product be properly defined [51,52]. (The quotations on the right hand sides above are cautions that care must be exercised interpreting these relations, because, e.g., defining the rotation group by $RR^T = I$ or by $RR^{-1} = I$ leads to different spin groups covering the rotation group.) This should look more natural when we look at the spinorial nature of these constructions in the fourth installment of this series of papers [3]. There are also topological obstructions to general inverses in the present setting, as mentioned earlier. The back and forth along a geodesic must be understood as mathematically local and relatively isolated physically. In installment four [3], we will see that our semigroup is made up of a series of local patches - one solves hyperbolic differential equation problems in such local patches (and worries about the permissible size of the patches a great deal) - and the understanding we assert here amounts to working only within a single of those local coordinate patches when we are describing our dynamics in general, but that there is the possibility of a broader description with regard to the areas of compact dynamics, e.g., resonances are locally restricted but you may work “globally” when within the islands of stability providing you use the appropriate topology.

Recapitulating, in order to accomplish this properly dynamical construction, we see from the above that must:

- Deal with the complex Lie algebra as a \textit{real} algebra. Note that there is a unique decomposition of complex operators (e.g., a Lie algebra as an operator algebra) into their real or symplectic form, e.g., the complex Lie algebra $\mathfrak{g}_C$ can be identified with $\mathfrak{g}_C \rightarrow \mathfrak{g}_C \cong \mathfrak{g} \oplus i \mathfrak{g}$ [25,26,27]. Thus, we distinguish $H$ and $(iH)$ as distinct generators of a real operator algebra containing $H$ itself as a generator, because there is a complex plane structure present.
- Accompanying the dual extension of this real operator algebra must be an involution, transforming the real algebra used as semi-group ring scalars in such a way that the resulting semi-group transformations satisfy the tests for symplectic action.
- These structures must be represented in representation spaces.

This form of extension which is self-dual (esa) as to the generators of the realization of the semi-algebra but anti-self-dual as to the realization of elements of the Lie
algebra and associated semi-group sets up the connections due to the “$U(1)_{\pm}^C$” sub-semi-groups which figure in the identification of a generalized Yang-Mills gauge structure associated to the group representations in generalized spaces used in this present body of work (see installment four [3] of this series), and also figures in setting up the quasi-invariant measures which are part of establishing the ergodicity of the Gamow vectors. (See installment three of this series [2].)

This esa extension makes possible the geometric identification of the roles of the Lie algebra generators in their role as generators of infinitesimal translations in the related semi-group. Their being esa also enables us to apply the nuclear spectral theorem to the generators of the Lie algebra, meaning when we proceed in the manner chosen for the adjoint involution, the generators can be associated with physical observables. Thereby, we are simultaneously creating an identification between dynamical quantities and physical observables by means of careful mathematical construction.

The existence of a complex structure on $\Phi_{sp(4,\mathbb{R})_{\pm}^C}$ (induced by the representation of complex Lie algebras $sp(4,\mathbb{R})_{\pm}^C$) means, for instance, that $Y$ and $iY$ define different connections associated to different directional derivatives along transverse geodesics on $\Phi_{sp(4,\mathbb{R})_{\pm}^C}$, and each must be given an esa extension individually. It is obviously not possible to accommodate this structure in the traditional Hilbert space formalism.

There are multiple one parameter families of symplectic esa extensions of the generators from $\Phi$ to $\Phi^{\times}$, since there are other generators of symplectic transformations and the lifting parameters for all are continuous. This is a radically different structure from the structure one is more familiar with in Hilbert space. These group extensions satisfy the often encountered and standard criteria for “unitary representations”, e.g., if $\theta : G \rightarrow GL(V)$, then as a standard criterion for a unitary representation of $G$ on $V$ we require $\langle \theta \circ g v , \theta \circ g w \rangle = \langle v , w \rangle$, $\forall g \in G, \forall v , w \in V$. See, e.g., [18]. However, in the present context we see them as dynamical (=symplectic) in their action rather than as unitary, since the theorem of Porteous [23] applies to all the symplectic transformations, of which the unitary transformations are a subpart.

For symmetric esa extensions (e.g., of hermitean operators on Hilbert space), the eigenvalues in the unitary representation of the group are necessarily unimodular, and the categorization of the present extensions as quasi-invariant rather than invariant is intended to clearly signal that the eigenvalues of the group representation need not be unimodular. The scalar product is only “quasi-invariant” (see [37], page 311), even though a test similar to the familiar test for unitarity is satisfied. Following the complexification–dual-extension, the physical expectation values may be different from those previously obtained, since in general the extended operators are no longer symmetric. Eigenvalues for the quasi-invariant semi-group representation are not necessarily unimodular any longer, so that the semi-group operator

16
\[ e^{-iH^\times t} \] responsible for time evolution during \( t \geq 0 \) need no longer contribute a simple unimodular phase oscillation for energy eigenvectors (Recall the Gadella diagrams in installment one \([1]\)–it is \( H^\times \) and not \( H \) which governs the time evolution during \( t \geq 0 \)).

Many details will not be covered exhaustively herein, such as the existence of different left and right quasi-invariant measures, which will be touched on very lightly in the third and fourth installment of this series \([2,3]\). The “inverses” associated with the quasi-invariant measures, e.g., the scalar product in our RHS representation, must again be understood locally rather than globally (if global, we would call them invariant). We are thus restricted in how far our time parameter can run, for instance.

2.1 Physical interpretation.

What is physically important are the matrix elements emerging from this structure. For a connected complex semi-simple Lie group \( G \), such as \( Sp(4, \mathbb{C}) \), integration of forms on the group is equivalent to integration on forms on the algebra. From bi-duality, one associates the scalar product \( \langle g, h \rangle \) on \( G_\pm \) and \( G^\times_\pm \) with a scalar product on the representation space \( V \), \( \langle \cdot, \cdot \rangle_V \), \( \theta: G_\pm \rightarrow Aut(V) \), \( \theta: g_\pm \rightarrow aut(V) \), by \([35]\):

\[
\langle \theta(\tau)v, \theta(\tau)w \rangle_V = \langle v, w \rangle_V, \forall w \in V^\times, \forall v \in V, \forall \tau \in G_\pm.
\] (34)

For \( G \) complex, there is a canonical scalar product on \( V \) and \( V^\times \) which is “unitary” (quasi-isometric):

\[
\langle v, w \rangle_V = \int_{G_\pm} \langle \theta(\tau)v, \theta(\tau)w \rangle_V d\tau
\] (35)

This is a trivial extension of \([35]\), p 151-153, to more general function spaces and to semi-groups. Note that for the typical \( e^{\pm iH^\times t} \) time evolution semi-groups, this scalar product would be phrased

\[
\int_0^\infty \langle e^{-iH^\times t}v, e^{-iH^\times t}w \rangle d\mu_t = \int_0^\infty \left[ \langle e^{-iH^\times t}v, e^{-iH^\times t}w \rangle \right] d\mu_t
\]

\[
= \langle v, w \rangle_V \int_0^\infty d\mu_t = \langle v, w \rangle_V
\] (36)

where \( d\mu_t \) is the left invariant 1-form on the one parameter time evolution sub-semi-group, properly oriented and normalized. For a real semi-group generated by the essentially self adjoint operator \( A \), one must make the association \( (e^{\alpha A}v)^\times = v^\times e^{-\alpha A}, \alpha \in \mathbb{R}_+ \), in order for this to also result in a dynamical transformation of the inner product, i.e., physically one should think of the above integral as

\[
\int_{G_+} \langle e^{-iH^\times t}v, e^{-iH^\times t}w \rangle d\omega_t = \int_{G_+} v^\times e^{-iH^\times (-t)} \circ e^{-iH^\times t} d\mu_t
\] (37)
or as the “time reversed scalar product” of F-T in some sense.

This mathematical structure (which is forced upon us in order to provide both uniquely defined adjoints and a proper dynamical representation of the connected complex covering group) has a sensible physical interpretation, as on our generalized space of states it effectively measures the overlap of a state (=probability amplitude representing a dynamical system) prepared during \( t \leq 0 \) with the results of a measurement (=probability amplitude representation of a second dynamical system used as a measurement apparatus) during \( t \geq 0 \). Working in a rigged Hilbert space suggests it may be reasonable to adopt an untraditional but physically meaningful physical interpretation of evolution operators appearing in the integrals of evolving state vectors. This is an unexpected ancillary result of these quasi-invariant measures.

See also [29,4] for extensive discussions of these semi-groups of time evolution and their physical interpretation. This present work finally establishes they truly deserve to be called semi-groups of dynamical time evolution, and that they are not merely generalizations gotten from the \( U(1) \) groups of time evolution obtained with the Schrödinger equation in the HS formulation, and for which the same physical meaning is conjectured. Note that the form for the semi-groups of time evolution and their duals obtained by formal analysis in [29,4] is here obtained by an alternative means based on careful algebra and geometry.

In an algebraic theory of scattering based on the RHS and this procedure, the Møller operators \( \Omega^{\pm} \) would be represented by sums and products of canonical transformations representing the actual dynamical transformation processes, such as in [27]. (This represents a generalization of the unitary Møller operators.) The procedures dealt with here certainly do not constitute a complete theory of scattering, but do suggest that the formulation of quantum mechanics may be enlarged from a Hilbert space to a RHS to provide the vehicle from which a rigorous algebraic theory of scattering may be developed.

The present algebraic approach to scattering is distinguished by:

1. Representation of the preparation and registration processes as continuous quantum dynamical transformations (which preserve some internal symmetry of the system). There is no classical apparatus anywhere.
2. Dynamics can be separated into internal and external components, and both are based on symmetries. The external evolutionary impetus takes the form of symplectic transformations which preserve some internal symmetry. Internally, the identity of the oscillators remains fixed by the internal symmetry (creation and destruction operator algebra) which remains unbroken. The interpretation of this seems consistent with “noisy” external perturbations of the internal system and deliberate action upon the internal system both being represented by canonical transformations.
(3) There are no infinite reservoirs in the present construction, which involves a (formally) closed system of two oscillators dynamically evolving, and there is no privileged role for any apparatus or observer.

(4) The preparation or registration process is treated as a true dynamical process, driven in a self consistent way by the interaction of the system, and evolving continuously according to the appropriate $U(1)$ sub-semigroup of dynamical evolution. Thus, for the preparation procedure represented by $e^{i\alpha A}$, the preparation advances with the transformation parameter $\alpha$.

(5) Only essentially self adjoint operators are used as infinitesimal generators, and although the self adjoint Hamiltonian is transformed by the preparation procedure (and so may depend parametrically on the preparation process), it has no explicit time dependence.

There are substantial open questions concerning the formalism, particularly relating to interpretation of the physical meaning of the mathematics, and, conversely, how to phrase a given physical situation mathematically. For instance, systems seem to evolve independently of any preparation or registration apparatuses, so one must distinguish the decay of a resonance from its observation, yet both seem to be representable by the same dynamical evolution parameter and transformation. Apparently, one must encounter and properly interpret both active and passive canonical transformations, e.g., the decay process itself might be considered passive and the registration of the decay event by an experimental apparatus might be considered active. A fuller account of stability against small perturbations also needs to be provided for these quantum states which decay, and this will begin in the third installment of this series [2].

3 Poisson and Symplectic Structures

The choice of spaces upon which to realize an operator representation of the algebras and semi-groups is very significant. By a theorem of Marsden, Darboux’s theorem does not hold for weak symplectic structures [28]. Thus, during the course of representing a Lie algebra structure (which participates in defining a weak symplectic form on $g \times g^*$) on a subspace of the Schwartz space (and its dual), Darboux’s theorem is of no use to construct a local euclidean grid which is complete, e.g., there exists no way to completely diagonalize the matrix elements of that representation space and its dual (which is based on Darboux’s theorem). This has the physical implication that there is a generic possibility of mixing occurring during transformations on a space with a weak symplectic structure. (Mixing is a condition precedent for entropy increase, discussed in installment three [2].)

Semi-simple operators locally may be diagonalizable individually (since they are “linear”), but in the present situation there is no possibility of a complete set of commuting operators being obtained using Darboux’s theorem. (A complete set of
commuting operators may still be obtained at the even higher level of the universal enveloping algebra in the case of semi-simple groups.) This weak symplectic structure also permits the faithful representation of some sub-semi-groups which may contain elements not describable by the exponential mappings of single generators \[16\], meaning we may add certain sets (e.g., countable sets which are therefore of measure zero) to our spaces without adverse effect on our mathematics.

Marsden’s Theorem means that there are non-trivial local invariants (such as curvature or torsion) for the structure on \( \Phi \times \Phi \). Thus, the antilinear functionals \( F(\phi) = \langle \phi | F \rangle \) have just such a symplectic structure, i.e., for \( F = E \), the energy representation in \( \mathcal{S} \cap \mathcal{K}_\pm^2 \) has a weak symplectic structure. The existence of non-trivial local invariants means the geodesics which represent the evolution of a state in the space of states can be non-trivial geometrically, and the dynamical physical evolution of the systems they idealize may be non-trivial as well.

### 3.1 Canonical Poisson and symplectic structures

This section is a brief recapitulation and reinterpretation of standard results for Lie algebras and their duals from the perspective of Lie-Poisson structures. See, e.g., \[30\]. It will serve as an indication of the structure of \( \Phi_\pm^\times \), which reflects the structure of \( g^\times \) in important regards.

There is a canonical symplectic structure on the direct product of a Lie algebra with its dual, i.e., on \( g \times g^\times \). There is a similar canonical symplectic structure on \( \Phi_\pm \times \Phi_\pm^\times \) and other pairings of locally convex vector spaces and duals, e.g., associated with the scalar product viewed as a Cartesian pair. These are categorized as weak symplectic structures. Note that the real Hilbert space \( \mathcal{H} \) is associated with a weak symplectic structure, but the complex Hilbert space \( \mathcal{H}^C \) is associated with a strong symplectic form defined by the Hermitean product on \( \mathcal{H}^C \times \mathcal{H}^C \), which coincides with the symplectic form on \( \mathcal{H}^C \times (\mathcal{H}^C)^\times \), so a complex Hilbert space is associated with a strong symplectic form while a real Hilbert space possesses a weak one. (See, e.g., \[30\], chapter 2.)

The dual of a Lie algebra is a Poisson manifold, e.g., \( g^\times \) is Poisson, and co-adjoint orbits in \( g^\times \) individually possess a symplectic manifold structure: \( g^\times \) can be thought of as a union of these co-adjoint-orbit-symplectic-manifolds (but need not be a symplectic manifold itself, since those unions may be disjoint, meaning there is no global symplectic form). On \( g^\times \), we have a Poisson structure associated with the Lie-Poisson brackets on \( g^\times \), and a symplectic structure associated with the Lie-Poisson structure on the co-adjoint orbits. We thus have symplectic sheaves in the Poisson manifold \( \Phi_\pm^\times \), e.g., “\( Ad_{sp(4,\mathbb{C})} \)” applied to the realization of \( sp(4,\mathbb{R})^C \pm \) on \( \Phi_{sp(4,\mathbb{R})^C} \) forms a series of symplectic sheaves (co-adjoint orbits) in \( \Phi_\pm^\times \). Inclusion is a Poisson mapping. We can also think of \( g^\times \) as \( T^G/G \), where \( G \) is
obtained by integration of $g$, and $g^\times$ must be even dimensional [30], page 293 and Chapter 13. For $g$ the value of the momentum mapping is always an element in $g^\times$, and under the momentum mapping a Poisson (e.g., including symplectic) action of a connected Lie group $G$ is taken as the co-adjoint action of $G$ on the dual algebra $g^\times$. See [31], Appendix 5, page 374.

We thus have an association of transitive action of a connected Lie group, the momentum map, Poisson action of a group on a space, and a topology change to a weak-$\times$ topology in the dual of the Lie algebra of that Lie group. The transitive symplectic action (momentum map) of a connected Lie group is taken as the co-adjoint action of that Lie group on the appropriate dual. We will see this structure mirrored every time when we address the issue of why what appear to be adjoint orbits on some representation space $\Phi^{sp}(4, R)\pm$ are in fact co-adjoint orbits in the representation space $\Phi^{\times}_{sp(4, R)}C\pm$. It is appropriate to begin addressing that issue now.

The scalar product $\langle \bullet | \bullet \rangle$ on $\Phi \times \Phi^\times$ is what matters here, both physically and mathematically. There is a canonical symplectic structure here (e.g., when viewed as a Cartesian pair), and it exists canonically on $g \times g^\times$ as well (provided those transformations have well defined adjoints). This means that, e.g., even a seemingly “adjoint transformation on $g$” can, if transitive, be thought of as an element of the group of symplectic transformations on $g \times g^\times$, and that group of (complex) symplectic transformations may include analytic continuation transformations (a form of complex symplectic transformation). For instance, even the mere choice of a ladder operator basis for a real group may amount to a complex symplectic transformation in precisely this sense. See, e.g., the $SO(3)$ and $SO(2,1)$ comparison, yielding a complex spectrum for $SO(2,1)$ in this regard, used as an example in [4], Section 2.

Gadella has given the necessary and sufficient mathematical conditions for analytic continuation in the function space realizations of the RHS structure [15], and in particular shows that the analytic continuation proceeds from the realization of $\mathcal{H}^\times$ to the function space realization of $\Phi^\times$. Hörmander [9], gives a prescription for the analytic continuation of an elliptic Green’s function which demonstrates how one must proceed with the abstract spaces. The fundamental solutions are analytically continued, but a topology change is mandated since this analytic continuation is to the distributions. Hence, a complex symplectic transformation of a real algebra which works an extension to the complexes must be accompanied by a topology change to a weak-$\times$ topology. Analytic continuation is a momentum mapping (!) in the jargon of dynamical systems. Tied up in this general topology change shown by Hörmander are issues in harmonic analysis necessary for the existence of Green’s functions, resolvents (whose poles provide the spectrum), etc. All the mathematical machinery necessary for us to do physics depends on us making this topology change, and this topology change is not optional, but is necessary for well defined mathematics as well as dynamics as we have sought to describe them.
3.2 Symplectic structures and forms of algebras

A complex structure is a type of symplectic structure. It is important in the present context to think of the complex numbers in their symplectic form, i.e., as a real algebra \( \mathbb{R} \oplus i \circ \mathbb{R} \). This is because if we think of the field \( \mathbb{C} \) as an algebra, the operation of involution in the form of complex conjugation cannot be uniquely be defined without the addition of more structure than the axioms of a field provide. A space is an algebra plus a scalar product, and when there is an inclusion of the space in the dual, as there is in the RHS paradigm, the adjoint operation of the algebra must be an involution, which suggests we should think of the ring of scalars as an algebra and the adjoint operation as accompanied by an involution of the algebra of scalars. (This suggestion becomes compulsory when one also wishes to define transformations in such a way that their action is symplectic (=dynamical). See below.) For example, if one thinks only in terms of complex conjugation of the field of complex numbers on a conventional Hilbert space, the representation of a complex Lie group obtained from exponentiating a complex Lie algebra (using an esa realization of the generators of the algebra, so as to make them identifiable with physical observables) would make the scalar product of that space not be uniquely defined, since exponentials of complex operators (e.g., with mixed real and imaginary parts) would not have unique adjoints! Those operators of the form \( H = H_0 + iH_1 \) could no longer be characterized as either hermitean or anti-hermitean, and so any effort to accommodate complex operators and complex spectra on von Neumann’s Hilbert space is destined for very serious mathematical difficulties.

The usual insistence on hermitean (or anti-hermitean) representations does not stem from any structure of the Lie algebra, but from the need for unitarity in commonly met examples involving the von Neumann Hilbert space, with the hermitean scalar product. The hermitean conjugation there has a symplectic (=dynamical) action on the von Neumann Hilbert space. The extension of this form of hermitean scalar product to our complex group/algebra situation on our different spaces would not make the transformations represented complex symplectic, however, which may be even more serious than being non-hermitean and non-unitary, since this relates to integrability in a more general way, and would also be contrary to our current understanding of what is “dynamical”.

If, however, one thinks of \( \mathbb{C} \) as a real algebra (i.e., considers it in its symplectic form \( \mathbb{R} \oplus i \circ \mathbb{R} \)), then one may define an involution for that real algebra uniquely. There may be many possible alternative ways of defining involutions for a given real algebra, but you must choose one form of involution which is unique and stick to it \( [53] \). In the interests of proceeding uniquely in the present algebraically oriented treatment, it is critical to think of “analytic continuation” not in terms of “field extension” but in terms of (transitive) transformations extending a real algebra into the symplectic form of an enlarged algebra. We further insisted on defining the action of representations of groups of transformations on our spaces in such a
A complex manifold with a hermitean metric whose imaginary part is closed (e.g., symplectic) is called a Kähler manifold. On the complexification of the real phase space $T \times \mathbb{R}^n \cong \mathbb{R}_{p,q}^{2n}$, which will be denoted (abusively) $\tilde{T} \times \mathbb{R}^n \cong \mathbb{R}_{p,q}^{2n} \cong \mathbb{C}^n$, one has an hermitean metric:

$$h(\xi, \eta) = (\xi, \eta) - i\omega(\xi, \eta) = (\xi, \eta) - i[\xi, \eta].$$  \hfill (38)

That $\omega$ is closed on $\mathbb{R}_{p,q}^{2n}$, $d\omega = 0$, is a consequence of the existence of a symplectic (Darboux) coordinate system, i.e., Euclidean structure on $\mathbb{R}^{2n}$. In such a coordinate system the torsion vanishes, meaning (pseudo-)Riemannian connections can be defined for the manifold which possesses a complex structure \[33\], page 167f. We may take the vanishing of the torsion for as the integrability condition for integrating Lie algebras to the (connected part of the) Lie group, i.e., so that torsion free connections can be said to exist on the complex simple Lie group.

Given that the work herein involves the use of hermitean metrics on complex manifolds, i.e., involves (weak) Kähler metrics, the vanishing of the torsion is not a trivial matter, and depends on the fact that $\mathbb{R}^{2n}$ is torsion free, and that the complex semi-simple Lie algebras are torsion free. In a generalized Riemannian geometry setting, the skew symmetric part of the metric is associated with possible torsion (and some corrections to the curvature tensor arising from the torsion), and the hermitean and Kähler metrics have just such a skew symmetric part, as shown above. The torsion itself is not an especially serious obstruction to integrability (e.g., we are untroubled by holonomy, and indeed recognize it in physics as the geometric, or Berry, phase), but the conditions necessary for the existence of torsion also makes possible shear, and the formal possibility of essential discontinuities on a set of positive measure can present serious problems! Local integrability is possible because the possibility of shear on sets of positive measure is avoided through satisfaction of the integrability condition on the geodesic semigroup. We may view shear as being restricted to, at most, a set of zero measure, which we understand to be reflected in the branch cut (if any) and countable set of resonance poles associated with the analytic continuation. The branch cut represents a bifurcation of solution sets by discontinuities in the second and higher derivatives, and so does not represent shear in the ordinary sense. Hence, shear occurring during the dynamical time evolution in our spaces of generalized states, if present at all, exists only on a set of measure zero, the countable set of Gamow vectors, or their associated countable set of Breit-Wigner resonance poles \[55\].
3.3 Hamiltonian symplectic actions (integrability)

Because we have defined a new type of Lie algebra and Lie group representation, it is appropriate to take an aside to demonstrate that our constructions are non-pathological. We have constrained ourselves by insisting that the representation have a symplectic action on the representation spaces, and also that there be weak symplectic forms, etc., so some further explanation is necessary.

The representation of evolutionary flows generated by semisimple symmetry transformations are defined over a variety of spaces representing quantum mechanical states, e.g., all the spaces in the Gadella diagrams of installment one [1]. There are a variety of integrability conditions which must fully mesh everywhere throughout this structure. The conditions for the Lie algebra structure of the infinitesimal generators of these transformations to be integrable (e.g., into the connected part of a group structure) are reflected in their representation counterparts as conditions for the flows to have “single valued hamiltonian functions”.

For a semisimple symmetry group \( G \), by limiting consideration consideration to semi-groups and sub-semi-groups of \( G \) which are strictly infinitesimally generated, we require in the first instance that the group is the union of the two strictly infinitesimally generated semi-groups, which are unique: \( G = G_+ \cup G_- \) [16], p. 378. The representation of these strictly infinitesimally generated sub-semi-groups will be represented by the Lie algebra as a subscript, as in \( \mathcal{I}_{g\pm} \). The complete spaces \( \mathcal{I}_{g\pm} \) and \( \mathcal{I}_{g\times} \) are not necessarily restricted by this limitation of consideration, so that we must think \( \mathcal{I}_{g\pm} \subseteq \mathcal{I}_{g\times} \), since there may be sheet to sheet jumps within \( \mathcal{I}_{g\pm} \). We are interested in the intersection of the Schwartz space and space of Hardy class functions, \( \mathcal{I}_{g\pm} \cap \mathcal{H}_2^\pm \) and \( \mathcal{I}_{g\pm} \cap \mathcal{H}_2^\pm \), and with the \( \mathcal{H}^p \) spaces one cannot be assured that sufficiently many functionals exist to separate a point from a subspace [34]. The spaces of physical interest, \( \mathcal{I}_{g\pm} \) and \( \mathcal{I}_{g\times} \), are likely to differ at most on a set of measure zero.

A Lie algebra \( g \) when viewed as a linear vector space possesses a dual \( g^\times \), and one normally defines an inner product \( \langle g, \xi \rangle, \xi \in g^\times, g \in g \). (A linear space is an algebra plus a scalar product.) On representation spaces, one conventionally reflects the bi-duality of this product (reflexitivity of \( g \) and \( g^\times \) as spaces) by appropriate definition of product modules over the representation spaces, and there should also be reflexivity of the representation spaces and duals [56]. For semi-groups and their algebra of generators, a similar scalar product can be defined, although one works with semi-group rings rather than rings. See Section [2] and installment three of this series. See also, e.g., [36], page 398.

The continuous extensions of infinitesimal generators \( A \to A^\times \), which are taken to be essentially self adjoint, is adopted because the operator \( A^\times \) dual to a generator \( A \) is equal to the weak-\( ^\times \) generator of the dual semi-group.
When the $G$-action of a semi-simple group $G$ preserves the symplectic form $\omega$, all of the fundamental vector fields of the action of $\mathfrak{g}$ on $G$ are locally hamiltonian (locally integrable) [12], p. 47, see also [38,39]. The action of the strictly infinitesimally generated semi-simple (sub-)semi-groups thus preserve the symplectic form and all of the fundamental vector fields are thereby locally hamiltonian. This follows because $\mathfrak{g} \subset \mathfrak{g}^\times$, fixing the symplectic form on all of the symplectic sheaves of $\mathfrak{g}^\times$ (even though there may be no global symplectic form on $\mathfrak{g}^\times$ itself.)

The Lie algebra of generators of the strictly infinitesimally generated Lie semi-groups (as ray semi-groups [16]), therefore exponentiate to generate locally hamiltonian semi-flows on the (connected part of the semi-simple) semi-groups $G_\pm$. This locally hamiltonian structure should be carried forward by non-pathological representation mappings. There may even be global integrability, but at least local integrability is assured for all semi-simple semi-groups and their non-pathological representations.

At this juncture, we see torsion free structures on the complex semi-simple Lie algebras whose groups are therefore (locally path) connected (although not necessarily simply connected). We have symplectic structures on the direct product of (complex) spaces and their duals. We are thus ready to invoke the result that the symplectic actions of semi-simple Lie algebras on symplectic manifolds are Hamiltonian [31], Appendix 5, [12]. In fact, as to the symplectic transformations on $\mathfrak{g} \times \mathfrak{g}^\times$ (e.g., on $\mathfrak{sp}(4, \mathbb{C}) \times \mathfrak{sp}(4, \mathbb{C})^\times$) we have a textbook Hamiltonian action [31], page 346, example (e):

The co-adjoint action of $G$ on a co-adjoint orbit in $\mathfrak{g}^\times$ with the $\pm$ orbit symplectic structure has a momentum map which is the $\pm$ inclusion map. . . This momentum map is clearly equivariant, thus providing an example of a globally Hamiltonian action which is not an extended point transformation.

Equivariance of the mapping is a condition for integrability. See also Section [2] and installment three [2] of this series. The Singular Frobenius Theorem permits integration in systems involving distributions.

This globally Hamiltonian action in the abstract Lie algebra dual is reflected in the representations. In particular, the Hamiltonian vector fields on the (weak) symplectic representation manifold have symplectic flows upon which “energy” is conserved [31], page 566. This will enable us to, e.g., make statements of a thermodynamic character appropriate to an energetically isolated system of quantum resonances represented by Gamow states, as we will do in the third installment of this series. This also makes conservation principles available generally.

This globally Hamiltonian action may also exhibit the sensitive dependence on initial conditions (inherent in analytic continuation [40]). Thus, in the present instance the global integrability which the momentum map promises may not be usable as a practical matter, and in the case of analytic continuation, only local integrability
may be obtained as a useful result.

4 Predicted Observables

There are several general areas in which possible observable consequences of this model may emerge, and these will be pursued in detail separately from this paper. The most obvious is the quantization of the width (or, equivalently, the decay rate) as characterizing the resonances described here. Widths are usually difficult to measure, so the actual experimental resolution of this effect may not be easy. Note, however, that the $Z$ eigenvalue appearing in the width is also a characteristic of the $|j,m\rangle$ eigenstates prior to analytic continuation. The realization of the $SU(1,1)$ algebra used herein (and also the $SU(2)$ algebra) has an association with interferometry in quantum optics [43], and perhaps sufficient sensitivity can be attained by the use of non-linear optical phenomena, such as four wave mixing, etc. The indicated decay rate is inversely proportional to the total energy in the system (including the $\frac{\hbar}{2}$ vacuum zero point energy) and independent of the energy gap between the two oscillators (or fields).

Because the energy is analytically continued to include values from the negative real axis, the excitation numbers of the modes can be negative. (Recall that $m = n_A + n_B$ is a characteristic quantum number of the eigenstates of $SU(1,1)$ prior to the analytic continuation which produced the Gamow states.) For vacuum states, $n_A + n_B = m = 0$, there is still a finite decay probability for the process transferring energy from oscillator $A$ to oscillator $B$. This provides an explicit representation of, e.g., the well known vacuum fluctuations of quantum optics.

The present work also implies the possibility of observing quantization effects in the rate of decay of any two field resonant decay processes. If one interprets the two field interferometry constructions of [43] quite literally, then at the appropriate energy scales quite general combinations of fields should show analogues to the familiar interferometry process in quantum optics. Thus, one infers that there may be electroweak analogues to Fabry-Perrot interferometers, four wave mixing, etc., at the electroweak unification scale of energy.

It is also predicted that the characterization of an irreversible process can take a particular form: pure exponential decay. In the derivation of the complex spectral resolution, Bohm obtained two terms [13,14], rather than the single sum over Gamow vectors obtained here. Thus, for the prepared state $\phi^+$, one has formally

$$\phi^+ = \sum_{i=1}^{n} \psi_i^G \langle \psi_i^G | \phi^+ \rangle + \int_{0}^{-\infty} dE \ |E^-\rangle \langle +E | \phi^+ \rangle . \quad (39)$$

The first term on the right hand side corresponds to the result of the present paper. The second term on the right hand side (sometimes called the “background
term”)) has the geometric meaning of a holonomy contribution: there is no observable holonomy in the present two oscillator system since the evolution does not generate a trajectory which closes in the curved state space. This, of course, is precisely the sort of complication we avoided by working locally and prescribing general inverses in the earlier chapters. The plain geometric implication is that even if one should manage to return to what might be called the initial configuration, the various symmetry operations representing the necessary preparation steps and the necessary evolution steps will be by geodesic transport over curved space and one expects that some geometric phase (holonomy) will be accumulated, making the “background term” no longer zero. Indeed, there may possibly be some general identification between simple irreversible dynamical time evolution and holonomy.

It is to be expected that careful examination of the “background term” should disclose the role of the history of the system (non-Markovian dynamical time evolution—discussed in part three [2]) in the irreversibility of the system as expressed by the semigroups of evolution of that system. One is thus led to expect deviations from exponential decay only in systems which have a history, e.g., correlations among events between subsystems. The formalism predicts, e.g., that “spontaneous” decay should be exponential, perhaps as the result of some stochastic external perturbation (see [2]), and further one might observe deviations from the exponential when there exists correlations (coherence) between the decay events. This agrees with physical intuitions, and the principles seem to have weak (non-quantitative) experimental support. See [57].

The gauge interpretation of the symplectic transformations of this paper which will be undertaken in the fourth installment of this series should also have physical observables [3].

Finally, in resonant quantum microsystems, geometric phase (holonomy) accumulating during system evolution seems inextricably associated with intrinsic microphysical irreversibility. For resonant quantum quantum microsystems, there is perhaps some intrinsic association between the accumulation of geometric phase (holonomy) and entropy growth.

A The $SU(1,1)$ Dissipative Oscillator

This section is a partial recapitulation of the $SU(1,1)$ dissipative oscillator, first described in the paper of Feshbach and Tikochinsky [11]. In addition to changes in emphasis and mathematical detail, in the present work there are changes of form, e.g., only operators which are “symmetric under time reversal” will be used (contrary to the original). From these operators which are time reversal symmetric, irreversible time evolution will emerge in the form of exponentially decaying Gamow vectors.
The equation of motion of the damped simple oscillator is:

\[ m\ddot{x} + R\dot{x} + kx = 0 \quad (A.1) \]

Canonical quantization requires a Lagrangian, and the appropriate Lagrangian requires an auxiliary variable \( y \):

\[ L = mx\dot{y} + \frac{1}{2}R(xy - \dot{x}y) - kxy \quad (A.2) \]

The equation of motion for \( y \) becomes:

\[ m\ddot{y} - R\dot{y} + ky = 0 \quad (A.3) \]

The Hamiltonian \( H \) is

\[ H = \frac{1}{m}p_x p_y + \frac{R}{2m}(yp_x - xp_y) + \left(k - \frac{R^2}{2m}\right)xy \quad (A.4) \]

The \([a, a^\dagger] = \mathbb{I} = [b, b^\dagger] = 0\), etc. commutation relations are equivalent to the \([p, q] = -i\hbar \mathbb{I}\), etc., Heisenberg relations, and canonical quantization is straightforward. After canonical quantization and translation of variables into destruction and creation operators for two modes, \( a = (\sqrt{m}\Omega x + ip_x)/\sqrt{2\Omega} \), and \( b = (\sqrt{m}\Omega y + ip_y)/\sqrt{2\Omega} \), rearrangement suggests the further change of variables:

\[ a = \frac{1}{\sqrt{2}}(A + B) \quad b = \frac{1}{\sqrt{2}}(A - B) \quad (A.5) \]

We then have a splitting of the Hamiltonian

\[ H = H_0 + H_1 \quad (A.6) \]

where

\[ H_0 = \Omega (A^\dagger A - B^\dagger B) \quad \Omega = k - \frac{R^2}{2m} \quad (A.7) \]

and

\[ H_1 = i\frac{\Gamma}{2}(A^\dagger B^\dagger - AB) \quad \Gamma = \frac{R}{m} \quad (A.8) \]

There is a natural system–reservoir interpretation of the above, and in the limit \( R \to 0 \) the eigenstates of \( H \) reduce to those of the simple undamped harmonic oscillator provided one considers only states annihilated by \( B \) [11].

The operator \( H_0 \) is simply related to the \( \mathcal{C}^2 \) Casimir of \( SU(1,1) \). The operator \( H_1 \) together with two other operators form a realization of the algebra of \( SU(1,1) \), useful in determining the eigenvalues of the Hamiltonian:

\[ iX = \frac{1}{2}(A^\dagger B^\dagger + AB) \quad (A.9) \]
\[ iY = \frac{i}{2} (A^\dagger B^\dagger - AB) \quad (A.10) \]
\[ iZ = \frac{1}{2} (A^\dagger A + BB^\dagger) \quad (A.11) \]

obeying the algebra
\[ [X,Y] = Z \quad (A.12) \]
\[ [Z,Y] = X \quad (A.13) \]
\[ [X,Z] = Y \quad (A.14) \]

\( iZ \) is essentially the Hamiltonian for the two mode simple oscillator, with eigenvalues \( 2m + 1, m = (1/2)(n_A + n_B) \), while we may label the eigenvalues of \( H_0 \) by \( 2\Omega j \), where \( j = 1/2(n_A - n_B) \).

The Baker-Campbell-Hausdorf relation can be applied to \( B = i\mu X \in SU(1,1), A = Y \in SU(1,1), \mu \in \mathbb{R} \), to yield:
\[ e^{i\mu X} iYe^{-i\mu X} = iY \cos \mu - [X,Y] \sin \mu. \quad (A.15) \]

For the semi-simple (e.g., non-solvable) group \( SU(1,1) \) and its semi-simple algebra \( su(1,1) \), it follows that:
\[ e^{i\mu X} iYe^{-i\mu X} = iY \cos \mu - Z \sin \mu \quad (A.16) \]

so that
\[ e^{i(\pi/2)X} iYe^{-i(\pi/2)X} = -Z \quad (A.17) \]

or
\[ Y = i e^{-i(\pi/2)X} Z e^{i(\pi/2)X} \quad Z = -i e^{i(\pi/2)X} Y e^{-i(\pi/2)X} \quad (A.18) \]

Because the non-unitary dynamical transformations (analytic continuation) work a complexification of the algebra \( su(1,1) \longrightarrow su(1,1)^\mathbb{C} = sl(2,\mathbb{C}) \), the adjoint transformations of \( (iY) \) exit the realization of \( su(1,1) \) to become a realization of \( sl(2,\mathbb{C}) \); the transformed eigenvectors have left the representation space for \( su(1,1)^\mathbb{C} \) into a representation space of \( su(1,1)^\mathbb{C} = sl(2,\mathbb{C})_{\pm} \) as well. \( H_0 \) of the Hamiltonian is in fact proportional to the angular momentum operator \( J^2 \) of \( su(2) \), and there is also the “dangerous” so-called analytic continuation of \( su(2) \longrightarrow su(1,1) \) given by \( J_1 \mapsto iJ_1 = X, J_2 \mapsto iJ_2 = Y, \) and \( J_3 \mapsto Z \), so that \( Z \) can also be thought of as an \( su(2) \) generator, and thereby the source of a discrete spectrum; because \( Z \in sl(2,\mathbb{C}) \), so is \( iZ \), and hence the appropriateness of complex eigenvalues for \( iZ \) on the eigenvector of \( J_3 \). The complex eigenvalue is appropriate in \( sl(2,\mathbb{C}) \) because it is a complex algebra.

If the eigenstates of \( Z \) are \( |j,m\rangle \in \mathcal{S} \), the Schwartz space, as above, the eigenstates of \( Y \) resulting from the extension of these vectors to \( \Phi^\times \) using the \( Ad_{exp(\mu X)} \) map corresponding to \( \mu = -\pi/2 \) are \( e^{-i(\pi/2)X} |j,m\rangle \):
\[ (iY) e^{-i(\pi/2)X} |j,m\rangle = i(m + 1/2) e^{-i(\pi/2)X} |j,m\rangle \quad (A.19) \]
Because $m \geq 0$, this is a positive pure imaginary eigenvalue. There is also a negative pure imaginary eigenvalue corresponding to an eigenstate $e^{i(\pi/2)X} |j, m\rangle$ associated with $\mu = +i\pi/2$:

$$
(iY) e^{i(\pi/2)X} |j, m\rangle = -i (m + 1/2) e^{i(\pi/2)X} |j, m\rangle \tag{A.20}
$$

Since $H_1 = \Gamma Y$, we therewith have complex eigenvalues for the Hamiltonian $H = H_0 + H_1$, and the eigenvectors of $H$ are in fact Gamow vectors (belonging to $\Phi^\times$ [44[13][14][15]) which exhibit pure exponential growth or decay, depending on the sign of $\pm m$:

$$
\psi_{G}^{\pm} (t) = e^{-i(\pi/2)X} |j, m\rangle e^{-2i\Omega j (\Gamma/2)(2m+1)}t \tag{A.21}
$$

Because $iY \sim H_1$ is in the familiar form of a symmetric (hermitean) operator on Hilbert space, $e^{\pm i(\pi/2)X} |j, m\rangle$ cannot belong to the Hilbert space $\mathcal{H}$ domain of the hermitean operator $H_1$, as proven by these complex eigenvalues of the hermitean $H_1 = iY$. (One of the lessons of the main body of this paper is that these Gamow vectors are associated with semigroups of time evolution, with restricted time domains of definition. Thus, $\psi_{G}^{+} \equiv \psi^G$ is defined only for $t \leq 0$ and $\psi_{G}^{-} \equiv \psi^G$ is defined only for $t \geq 0$.)

These complex eigenvalues are totally unacceptable in the real algebra of a real group such as $SU(1, 1)$, or in a representation of the same. This is because $i\lambda \not\in \mathfrak{su}(1, 1)$, since the real algebra (and its associated group) is defined over the field of real scalars only.

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[45] Note the use of the symplectic form on $\Phi \times \Phi^\times$ and $\mathcal{H} \times \mathcal{H}^\times$ is associated with a scalar product. In the usual physical notation for the scalar product, $(\bullet|\bullet)$ or $(\langle\bullet|\bullet\rangle)$, the ket $|\bullet\rangle$ would be identified with the space and the bra $\langle\bullet|$ identified with the dual space. Hence, the scalar product of physics is identified with $\Phi^\times \times \Phi$, and so involves the transpose of the symplectic form adopted by mathematicians. We shall blissfully ignore this distinction, since our physical concern is with the magnitude of the resulting scalar product. Note that when working with dual pairs of spaces, $(\chi,\chi^\times)$, many things are freed from dependence on the particular choice of topology on $\chi$, meaning that many of the properties we will deal with by proceeding in this way are topologically invariant. See, e.g., [34]. Thus, the choice of seminorm topology
for \( \Phi \) or the choice of the norm topology for \( \mathcal{H} \) may actually have less effect on the final results than one might at first suspect. The use of the RHS formalism for the description of resonances has the advantage of a transparent mathematical rigor, whereas alternative treatments—such as using Hilbert space operators outside of their apparent domain of definition—often seem lacking in justification even if mathematical care is used. It is probable that careful examination of the RHS formalism would lead justification of many heuristic devices used to describe resonances over the years.

[46] The essential purpose of making an alternative choice to the field \( \mathbb{C} \) is the need for a geometric structure (in numerous places) which reflects the structure of the complex plane, and the requirement for a real algebra structure in order to have well defined adjoints, as will be discussed below. The commutative real algebra \( \mathbb{C}(1,i) \equiv \mathbb{R} \oplus i \mathbb{R} \) has such a structure. The algebra \( \mathbb{C}(1,i) \) is obtained by adding further structure to the field \( \mathbb{C} \), in order to obtain an algebraic representation of the complex plane.

[47] Since we use operators for which the notion of hermiticity is not governing, our dynamical analogue of unitarity is defined using the dual rather than the hermitean conjugate: that transformation \( T \) have a symplectic action on a space requires that \( T^\ast T = TT^\ast = I \) [23], which replaces for us the familiar rule in Hilbert space \( U^\dagger U = UU^\dagger = I \). Because we work with spaces having a complex symplectic structure, this form of adjoint transformation insures the transformations have a symplectic (=dynamical) action on our spaces of states, i.e., \( T \) so defined is a dynamical transformation on \( \Phi \) of \( \Phi \subset \mathcal{H} \subset \Phi^\ast \).

[48] Technically, the present procedure deals with a generalization of the gaussian pure states and the transformations between these types of states by (inhomogeneous) symplectic transformations. For a description of the gaussian pure state formalism this present work generalizes, see [10], and references therein.

[49] This procedure can be given the physical interpretation of representing the preparation procedure of an experiment as a combination of continuous canonical transformations, and the continuation along similar lines to represent the decay and measurement processes as canonical transformations is straightforward. There could ultimately be free oscillators at the outgoing end of the process. The equation (29) can be interpreted as the (active) canonical transform, relating to the measurement process, or as merely setting up a formal evaluation of the constant \( \Gamma \) which characterizes the random decay process.

[50] Because \( g \subset g^\times \), there is a canonical \( \pm \)-inclusion of \( g \) into \( g^\times \). To respect the canonical symplectic form on \( g \times g^\times \) demands we chose the minus-inclusion, because on the representation space, in order for our transformations to be symplectic in their action on the representation space (e.g., be dynamical transformations on the representation space), it is necessary that they must satisfy the test analogous to the test for “unitarity” of group transformations. See [23]. (Recall also that the unitary transforms comprise a subgroup of the symplectic transforms.) This, in turn, leads quite canonically to the dynamical group representation indicated below simultaneously being defined on the Lie algebra representation space! The representation space is complete, so that Cauchy sequences defined by \( \exp \) of the realization of the algebra acting on elements of the representation space also yields elements of the space. Our asd Lie algebra representation with esa generators yields proper dynamical semigroup
representations as well. The action of our group of symplectic transformations is
symplectic on the spaces as well, which are Hausdorff spaces, establishing that they are
topologically transitive, so that one of the prerequisites for their association with chaos
is satisfied. Chaos issues will be addressed in the third installment of this series [2].
Conversely, since the flows of von Neumann’s definition of unitary transformations
on the conventional Hilbert space do not define a complex symplectic structure on
that space (i.e., if \( e^{-iHt} \) is a unitary transformation, then \( e^{-Ht} \) is not), their action is
not necessarily symplectic as to that space (and so the Schrödinger equation does not
necessarily reflect a proper transitive dynamical transformation of the conventional
Hilbert space).

[51] This means there is really no exponential catastrophe for the Gamow vectors when
the scalar product as a “length function” is properly defined. The “alternative
normalization that always seems to work” of [24], is in fact mathematically principled
and proper. Note this also involves identification of the scalar product conjugate of a
Gamow vector with the Schwartz reflection principle conjugate.

[52] The chosen forms mean that the \( H \) in equation (31) may differ from that in equations
(32) and (33). These equations must be understood as indicating a way to think in
different contexts, and emphasize that different rules may apply in different spaces.
One major point is that on \( \Phi \) (and its fully complex function space representation, as
constructed herein), both \( H \) and \( iH \) (and their representations on the fully complex
function spaces) may both be esa generators simultaneously. We will follow the form
of (33) and the definition of the adjoint involution operation, with the understanding
that \( H = H_0 + iH_1 \), reflecting the fact that the Hamiltonian of our analytically continued
system now is an element of a realization of the real form of a complex Lie algebra.
The Hilbert space obtained of the RHS containing the \( \Phi \) and \( \Phi^\times \) constructed using
the present definitions of the adjoint involution is certainly a different space from the
usual Hilbert space of the von Neumann formalism! Also note that in the exponential
mapping, the time parameter \( t \) does not change as \( H \longrightarrow H_0 + iH_1 \), and the complex
energy eigenvalues occur.

[53] For a discussion of the subtleties of complex conjugation versus involution see
Lounesto [32].

[54] It will be seen that the uniqueness requirement for the adjoint involution of our
complex Lie algebra/group representation means we must work with multicomponent
vectors which are in fact spinors. This issue will be addressed in the fourth installment
of this series [3].

[55] The fact that these Gamow vectors make up a set of zero measure has physical
significance for the ergodicity discussions in the third installment of this series [2].
If they had formed a set of positive measure, they and the resonances they represent
could not be associated with entropy growth, for instance. See, e.g., [41].

[56] We show in [51] that there is no exponential catastrophe for properly defined Gamow
states. There is a scalar product well defined between the Gamow state arising during
t \( \leq 0 \), \( |\tilde{\psi}^G\rangle \) and the Gamow state decaying during \( t \geq 0 \), \( |\Psi^G\rangle \). This is an instance
of the Schwartz reflection principle, and is well defined because of the reflexivity
of the spaces \( \Phi \) and \( \Phi^\times \)--one may add a countable set of \( |\tilde{\psi}^G\rangle \) to \( \Phi \) without adverse
consequence, and this is what has, in effect, occurred. This takes us out of the domain of strictly infinitesimally generated semigroups, which we should point out here, since our immediate concern is precisely the strictly infinitesimally generated semigroups, and no more.

[57] L.A. Khalfin, JETPh. Lett., 25 (1972) 349; L. Fonda, G.C. Ghirardi, A. Rimini, Repts. on Prog. in Phys., 41 (1978) 587, and references thereof. Within experimental limits, no deviation from exponential decay is observed in simple decay processes, E.B. Norman, et al, Phys. Rev. Lett. 60 (1988) 2246, although deviation has been observed during quantum tunnelling of a correlated system, S.R. Wilkinson, et al, Nature, 387 (1997) 575.