Classification of left octonion modules

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Abstract

It is natural to study octonion Hilbert spaces as the recently swift development of the theory of quaternion Hilbert spaces. In order to do this, it is important to study first its algebraic structure, namely, octonion modules. In this article, we provide complete classification of left octonion modules. In contrast to the quaternionic setting, we encounter some new phenomena. That is, a submodule generated by one element \( m \) may be the whole module and may be not in the form \( Om \). This motivates us to introduce some new notions such as associative elements, conjugate associative elements, cyclic elements. We can characterize octonion modules in terms of these notions. It turns out that octonions admit two distinct structures of octonion modules, and moreover, the direct sum of their several copies exhaust all octonion modules with finite dimensions.

Keywords: Octonion module; associative element; cyclic element; \( C\ell_7 \)-module.

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1 introduction

The theory of quaternion Hilbert spaces brings the classical theory of functional analysis into the non-commutative realm (see [10, 16, 17, 20, 21]). It arises some new notions such as spherical spectrum, which has potential applications in quantum mechanics (see [4, 6]). All these theories are based on quaternion vector spaces, or more precisely, quaternion modules, and quaternion bimodules. A systematic study of quaternion modules is given by Ng [15]. It turns out that the
category of (both one-sided and two-sided) quaternion Hilbert spaces is equivalent to the category of real Hilbert spaces.

It is a natural question to study the theory of octonion spaces. Goldstine and Horwitz in 1964 [8] initiated the study of octonion Hilbert spaces; more recently, Ludkovsky [12, 13] studied the algebras of operators in octonion Banach spaces and spectral representations in octonion Hilbert spaces. Although there are few results about the theory of octonion Hilbert spaces, it is not fully developed since it even lacks coherent definition of octonion Hilbert spaces.

In contrast to the complex or quaternion setting, some new phenomena occur in the setting of octonions (see Example 4.13, 4.14):

- If \( m \) is an element of an octonion module, then \( \mathcal{O}m \) is not an octonion sub-module in general.
- If \( m \) is an element of an octonion module, then the octonion sub-module generated by \( m \) maybe the whole module.

This means that the structure of octonion module is more involved. We point out that some gaps appear in establishing the octonionic version of Hahn-Banach Theorem by taking \( \mathcal{O}m \) as a submodule ([12, Lemma 2.4.2]). The submodule generated by a submodule \( Y \) and a point \( x \) is not of the form \( \{ y + px \mid y \in Y, \ p \in \mathcal{O} \} \), this is wrong even for the case \( Y = \{ 0 \} \). It means the proof can not repeat the way in canonical case. The involved structure of octonion module accounts for the slow developments of octonion Hilbert spaces.

In the study of octonion Hilbert space and Banach space, it heavily depends on the direct sum structure of the space under considered sometimes, which always brings the question back to the classic situation. For example, in the proof of [12, Theorem 2.4.1], it declares that every \( \mathcal{O} \)-vector space is of the following form:

\[
X = X_0 \oplus X_1 e_1 \oplus \cdots \oplus X_7 e_7.
\]

Note that therein the definition of \( \mathcal{O} \)-vector space is actually a left \( \mathcal{O} \)-module with an irrelevant right \( \mathcal{O} \)-module structure. We thus can only consider the left \( \mathcal{O} \)-module structure of it. We show that the assertion above does not always work. In order to study octonion Hilbert spaces, we need to provide its solid algebraic foundation by studying the deep structure of one-sided \( \mathcal{O} \)-modules and \( \mathcal{O} \)-bimodules. We only consider the left \( \mathcal{O} \)-modules in this paper, the bimodule case will be discussed in a later paper. In this paper, we characterize the structure of \( \mathcal{O} \)-modules completely.

We remark that Eilenberg [5] initiated the study of bimodules over non-associative rings. Jacobson studied the structures of bimodules over Jordan algebra and alternative algebra [11]. One-sided modules over octonion was investigated in [8] for studying octonion Hilbert spaces. However for the classification of \( \mathcal{O} \)-modules, the problem is untouched.

In this article, we shall give the classification of \( \mathcal{O} \)-modules. It turns out that the set \( \mathcal{O} \) admits two distinct \( \mathcal{O} \)-module structures. One is the canonical one, denoted by \( \mathcal{O} \); the other is denoted by \( \mathcal{O}^- \) (see Example 3.5). To characterize any \( \mathcal{O} \)-module \( M \), we need to introduce some new notions, called associative element and conjugate associative element. Their collections are denoted by \( \mathcal{A}(M) \) and \( \mathcal{A}^- (M) \) respectively. The ordered pair of their dimensions as real vector spaces is called the type of \( M \). These concepts are crucial in the classification of left \( \mathcal{O} \)-modules.

Our first main result is about the characterization of the left \( \mathcal{O} \)-modules.

**Theorem 1.1.** Let \( M \) be a left \( \mathcal{O} \)-module. Then

\[
M = \mathcal{O} \mathcal{A}(M) \oplus \mathcal{O} \mathcal{A}^- (M).
\]

If \( \dim_{\mathbb{R}} M < \infty \), then

\[
M \cong \mathcal{O}^{n_1} \oplus \mathcal{O}^{-n_2},
\]
where \((n_1, n_2)\) is the type of \(M\).

Its proof depends heavily on the isomorphism between the category \(O\text{-Mod}\) and the category \(Cl_7\text{-Mod}\). By the matrix realization of \(Cl_7\),

\[
Cl_7 \cong M(8, \mathbb{R}) \oplus M(8, \mathbb{R}),
\]

there are only two kinds of simple left \(O\)-module, namely, \(O\) and \(\mathbb{C}\) up to isomorphism. Hence the structure of finite dimension \(O\)-modules follows by Wedderburn’s Theorem for central simple algebras [18]. The general case relies on some elementary properties of \(A(M)\) and \(A^{-}(M)\), along with an important fact that every element of \(O\)-module generates a finite dimensional submodule. By the way, we found that the octonions \(O\) can be endowed with both \(Cl_7\)-module structure and \(Cl_6\)-structure, using it we get an irreducible complex representation of \(Spin(7)\).

Our second topic is about the cyclic elements. In contrast to the setting of complex numbers and quaternions, cyclic elements play key roles in the study of octonion sub-modules. An element \(m\) in a given octonion module \(M\) is called cyclic elements if the submodule generated by it is exactly \(O\cdot m\). The collection of these elements is denoted by \(C(M)\). It turns out that the cyclic elements are determined by the associative elements \(A(M)\) and the conjugate associative elements \(A^{-}(M)\) completely.

**Theorem 1.2.** For any left \(O\)-module \(M\), we have

\[
C(M) = \left( \bigcup_{p \in O} p \cdot A(M) \right) \bigcup \left( \bigcup_{p \in O} p \cdot A^{-}(M) \right).
\]

In view of Theorems 1.1 and Theorem 1.2, we find

\[
M = \text{Span}_R C(M).
\]

This means that the module \(M\) is determined completely by its cyclic elements in the form of real linear combination. For any element \(m\) in a left \(O\)-module \(M\), there exist \(m^\pm \in \text{Span}_R C^\pm(M)\) such that \(m = m^+ + m^-\). We can decompose \(m^\pm\) into a combination of real linearly independent cyclic elements. Denote by \(l_m^\pm\) the minimal length of the decompositions of \(m^\pm\). We conjecture that,

\[
\langle m \rangle_O \cong O^{l^+_m} \oplus O^{l^-_m}.
\]

If the conjecture is right, then the structure of the submodule generated by one element is completely clear.

## 2 Preliminaries

### 2.1 The algebra of the octonions \(O\)

The algebra of the octonions \(O\) is a non-associative, non-commutative, normed division algebra over the \(\mathbb{R}\). Let \(e_1, \ldots, e_7\) be its natural basis throughout this paper, i.e.,

\[
e_i e_j + e_j e_i = -2 \delta_{ij}, \quad i, j = 1, \ldots, 7.
\]
For convenience, we denote $e_0 = 1$.

In terms of the natural basis, an element in octonions can be written as

$$x = x_0 + \sum_{i=1}^{7} x_i e_i, \quad x_i \in \mathbb{R}.$$ 

The conjugate octonion of $x$ is defined by $\overline{x} = x_0 - \sum_{i=1}^{7} x_i e_i$, and the norm of $x$ equals $|x| = \sqrt{x \overline{x}} \in \mathbb{R}$, the real part of $x$ is $\text{Re} x = x_0 = \frac{1}{2} (x + \overline{x})$.

The full multiplication table is conveniently encoded in the Fano mnemonic graph (see [3, 22]). In the Fano mnemonic graph, the vertices are labeled by $1, \ldots, 7$ instead of $e_1, \ldots, e_7$. Each of the 7 oriented lines gives a quaternionic triple. The product of any two imaginary units is given by the third unit on the unique line connecting them, with the sign determined by the relative orientation.

**Fig.1** Fano mnemonic graph

The associator of three octonions is defined as

$$[x, y, z] = (xy)z - x(yz)$$

for any $x, y, z \in \mathbb{O}$, which is alternative in its arguments and has no real part. That is, $\mathbb{O}$ is an alternative algebra and hence it satisfies the so-called R. Monfang identities [19]:

$$(xy)z = x(yxz), \quad z(xy) = (zxy)x, \quad x(yz)x = (xy)(zx).$$

The commutator is defined as

$$[x, y] = xy - yx.$$ 

### 2.2 Universal Clifford algebra

We shall use the Clifford algebra $\mathcal{C}l_7$ to study left $\mathbb{O}$-modules. In this subsection, we review some basic facts for universal Clifford algebras. The Clifford algebras are introduced by Clifford in 1882. For its recent development, we refer to [2, 7, 14].

**Definition 2.1.** Let $\mathbb{A}$ be an associative algebra over $\mathbb{R}$ with unit 1 and let $v : \mathbb{R}^n \to \mathbb{A}$ be an $\mathbb{R}$-linear embedding. The pair $(\mathbb{A}, v)$ is said to be a Clifford algebra over $\mathbb{R}^n$, if

(i). $\mathbb{A}$ is generated as an algebra by $\{v(x) \mid x \in \mathbb{R}^n\}$ and $\{\lambda 1 \mid \lambda \in \mathbb{R}\}$;
(ii). \((v(x))^2 = -|x|^2, \forall x \in \mathbb{R}^n\).

We need some notations and conventions.

- \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n, |x|^2 = \sum_{i=1}^{n} x_i^2\).
- Let \(\{f_i\}_{i=1}^{n}\) be the canonical orthonormal basis of \(\mathbb{R}^n\), \(g_i = v(f_i) \in \mathbb{A}\), and \(f_i = (0, \ldots, 0, 1, 0, \ldots)\) with 1 in the \(i\)-th slot.
- Let \(\mathcal{P}(n)\) be the collection of all the subsets of \(\{1, \ldots, n\}\).
- For any \(\alpha \in \mathcal{P}(n)\), if \(\alpha \neq \emptyset\), we write \(\alpha = \{\alpha_1, \ldots, \alpha_k\}\) with \(1 \leq \alpha_1 < \cdots < \alpha_k \leq n\) and we set \(g_\alpha = g_{\alpha_1} \cdots g_{\alpha_k}\). Otherwise, we denote \(g_{\emptyset} = 1\).

The Clifford algebra \(\mathbb{A}\) can be described alternatively with the above notations as

(i). \(\mathbb{A}\) is \(\mathbb{R}\)-linearly generated by \(\{g_\alpha \mid \alpha \in \mathcal{P}(n)\}\);

(ii). \(g_ig_j + g_jg_i = -2\delta_{ij}\) for any \(i, j = 1, \ldots, n\).

It is well-known that \(\dim_{\mathbb{R}} \mathbb{A} \leq 2^n\). The Clifford algebra \((\mathbb{A}, v)\) over \(\mathbb{R}^n\) may be not unique (up to isomorphism of algebras); see for example [7]. But the universal Clifford algebra \(C\ell_n\) over \(\mathbb{R}^n\) is unique up to isomorphism.

**Definition 2.2.** A Clifford algebra \((\mathbb{A}, v)\) is said to be a universal Clifford algebra, if for each Clifford algebra \((\mathbb{B}, \mu)\) over \(\mathbb{R}^n\), there exists an algebra homomorphism \(\beta : \mathbb{A} \rightarrow \mathbb{B}\), such that \(\mu = \beta \circ v\) and \(\beta(1_\mathbb{A}) = 1_\mathbb{B}\). Namely, the following diagram commutes.

\[
\begin{aligned}
\mathbb{A} &\xrightarrow{\beta} \mathbb{B} \\
\downarrow v &\quad &\downarrow \mu \\
\mathbb{R}^n &\quad &\mathbb{R}^n
\end{aligned}
\]

We recall some equivalent descriptions of universal Clifford algebra \(C\ell_n\).

**Theorem 2.3** ([7]). \((\mathbb{A}, v)\) is a Clifford algebra over \(\mathbb{R}^n\), the following are equivalent:

(i). \((\mathbb{A}, v)\) is a universal Clifford algebra \(C\ell_n\) over \(\mathbb{R}^n\);

(ii). \(\dim_{\mathbb{R}} \mathbb{A} = 2^n\);

(iii). \(g_1 \cdots g_n \notin \mathbb{R}\).

At the last of this subsection, we give the following algebra isomorphism of \(C\ell_n\) (see for example [2]).

- \(C\ell_{n+8} \cong C\ell_n \otimes M(16, \mathbb{R}) \cong M(16, C\ell_n)\);

- for \(n = 0, \ldots, 7\) we have table
Here, we denote by $M(k, \mathbb{F})$ the collection of all $k \times k$ matrices with each entry in the algebra $\mathbb{F}$.

## 3 $\mathbb{O}$-modules

We set up in this section some preliminary definitions and results on left $\mathbb{O}$-modules.

**Definition 3.1.** A real vector space $M$ is called a (left) $\mathbb{O}$-module, equipped with a scalar multiplication $\mathbb{O} \times M \to M$, denoted by

$$(q, m) \mapsto qm,$$

such that the following axioms hold for all $q, q_1, q_2 \in \mathbb{O}$, $\lambda \in \mathbb{R}$ and all $m, m_1, m_2 \in M$:

1. $(\lambda q)m = \lambda (qm) = q(\lambda m)$;
2. $(q_1 + q_2)m = q_1 m + q_2 m$, $q(m_1 + m_2) = qm_1 + qm_2$;
3. $[q_1, q_2, m] = -[q_2, q_1, m]$;
4. $1m = m$.

Here, the left associator is defined by

$$[q_1, q_2, m] := (q_1 q_2)m - q_1 (q_2 m).$$

Note that this definition is equivalent to the definition given in [12, 13], wherein the axiom (iii) is replaced by $q^2 m = q(qm)$, for all $q \in \mathbb{O}, m \in M$.

The proof is trivial by polarizing the above relation. It also agrees with the one given in [8] wherein $M$ needs to satisfy an additional axiom: $p(p^{-1}x) = x$, which can be deduced from the equation $p(px) = p^2 x$ directly.

Let $M$ be a left $\mathbb{O}$-module. The definition of the terms submodule, homomorphism, isomorphism, kernel of a homomorphism, which are familiar from the study of associative modules, do not involve associativity of multiplication and are thus immediately applicable to the case in general. Let $\text{Hom}_\mathbb{O}(M, M')$ denote the set of all $\mathbb{O}$-homomorphisms from $M$ to $M'$ as usual ($M'$ being arbitrary $\mathbb{O}$-module). So is the notation $JN$ for the subset of $M$ spaned by all products $rn$ with $r \in J$ and $n \in N$ ($J$ being arbitrary nonempty subset of $\mathbb{O}$ and $N$ being arbitrary nonempty subset of $M$), here we must of course distinguish between $J_1(J_2 N)$ and $(J_1 J_2)N$. Let $\langle N \rangle_\mathbb{O}$ denote the minimal submodule which contains $N$ as before. An element $m \in M$ is said to be associative if

$$[p, q, m] = 0, \quad \forall p, q \in \mathbb{O}.$$
Denote by $\mathcal{A}(M)$ the set of all associative elements in $M$:
$$\mathcal{A}(M) := \{ m \in M \mid [p, q, m] = 0, \forall p, q \in \mathbb{O} \}.$$  

One useful identity which holds in any $\mathbb{O}$-module $M$ is
$$[p, q, r]m + p[q, r, m] = [pq, r, m] - [p, qr, m] + [p, q, rm], \quad (3.1)$$
where $p, q, r \in \mathbb{O}$, $m \in M$. The proof is by straightforward calculations. It follows that:  

**Lemma 3.2.** For all associative element $m \in \mathcal{A}(M)$, we have
$$[p, q, rm] = [p, q, r]m, \quad \text{for all } p, q, r \in \mathbb{O}.$$  

The following elementary property will be useful in the sequel. The proof is trivial and will be omitted here.  

**Proposition 3.3.** If $f \in \text{Hom}_R(M, N)$, then $f([p, q, x]) = [p, q, f(x)]$ for all $p, q \in \mathbb{O}$, $x \in M$. Therefore $f(\mathcal{A}(M)) \subseteq \mathcal{A}(N)$.  

We give several elementary left $\mathbb{O}$-module examples.  

**Example 3.4.** It is easy to see the real vector spaces $\mathbb{O}$, $\mathbb{O}^n$, $M(n, \mathbb{O})$ with the obvious multiplication are all left $\mathbb{O}$-module. Clearly, the sets of associative elements on these modules are $\mathbb{R}$, $\mathbb{R}^n$, $M(n, \mathbb{R})$ respectively.  

We can define a different $\mathbb{O}$-module structure on the octonions $\mathbb{O}$ itself.  

**Example 3.5 (\overline{\mathbb{O}}).** Define:
$$p \cdot x := px, \quad \forall p \in \mathbb{O}, x \in \mathbb{O}.$$  

It’s easy to check this is a left $\mathbb{O}$-module. Indeed
$$p^2 x = px = p(p \cdot x).$$
We shall denote this $\mathbb{O}$-module by $\overline{\mathbb{O}}$. By direct calculations, we obtain:
$$[p, q, x] = [p, q, x] + [\overline{p}, \overline{q}, \overline{x}].$$
This implies that $\mathcal{A}(\overline{\mathbb{O}}) = \{0\}$. Note that Proposition 3.3 ensures that $\mathcal{A}(M) \cong_R \mathcal{A}(N)$ when $M \cong_O N$, and therefore $\mathbb{O} \not\cong \overline{\mathbb{O}}$.  

However, there is a special subset in $\overline{\mathbb{O}}$, that is, the real subspace $\mathbb{R}$. To describe such elements, we introduce a new notion of conjugate associative element.  

**Definition 3.6.** An element $m \in M$ is said to be **conjugate associative** if
$$(pq)m = q(pm), \quad \forall p, q \in \mathbb{O}.$$  

Denote by $\mathcal{A}^{-}(M)$ the set of all conjugate associative elements.  

**Lemma 3.7.** $\mathcal{A}^{-}(\overline{\mathbb{O}}) = \mathbb{R}$.  

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Proof. Suppose $x \in \mathcal{A}^-(\mathbb{O})$, then for any $p, q \in \mathbb{O}$,

$$0 = (pq)^{\cdot}x - q^{\cdot}(p^{\cdot}x) = (pq)^{\cdot}x - (qp)^{\cdot}x + [q, p, x] = [p, q]^{\cdot}x - [p, q, x]$$

Hence we obtain

$$[p, q]^{\cdot}x - [p, q, x] - [p, q]^{\cdot}x = 0$$

this implies that $x \in \mathbb{R}$. Clearly $\mathbb{R} = \mathcal{A}^-(\mathbb{O})$, thus $\mathcal{A}^-(\mathbb{O}) = \mathbb{R}$. □

Lemma 3.8. For any left $\mathbb{O}$-module $M$, we have $\mathcal{A}(M) \cap \mathcal{A}^-(M) = \{0\}$.

Proof. Obviously $0 \in \mathcal{A}(M) \cap \mathcal{A}^-(M)$. Let $x \in \mathcal{A}(M) \cap \mathcal{A}^-(M)$, then for any $p, q \in \mathbb{O}$,

$$[p, q]^{\cdot}x = (pq)^{\cdot}x - (qp)^{\cdot}x = (pq)^{\cdot}x - (pq)^{\cdot}x = [p, q, x] = 0$$

This implies $x = 0$ since we can choose $p, q \in \mathbb{O}$ such that $[p, q] \neq 0$. This proves the lemma. □

Remark 3.9. Clearly, both $\mathcal{A}(M)$ and $\mathcal{A}^-(M)$ are real vector spaces. If $M$ is of finite dimension, we call the ordered pair $(\dim_{\mathbb{R}} \mathcal{A}(M), \dim_{\mathbb{R}} \mathcal{A}^-(M))$ type of $M$. We shall use these notions to describe the structure of left $\mathbb{O}$-modules. It turns out that type is a complete invariant in the finite dimensional case.

Let $M$ be a left $\mathbb{O}$-module. We shall establish some properties of associative elements and conjugate associative elements which will be used in the sequel.

Lemma 3.10. Let $\{x_i\}_{i=1}^n$ be an $\mathbb{R}$-linearly independent set of associative elements of $M$. If

$$\sum_{i=1}^n r_i x_i = 0, \quad r_i \in \mathbb{O} \text{ for each } i = 1, \ldots, n,$$

then $r_i = 0$ for each $i = 1, \ldots, n$.

Proof. The proof is by induction on $n$. For the case $n = 1$, we have $rx = 0$. If $r \neq 0$, since $x \in \mathcal{A}(M)$, it follows that

$$0 = r^{-1}(rx) = (r^{-1}r)x = x,$$

a contradiction with our assumption $x \neq 0$. Assume the lemma holds for degree $k$, we will prove it for $k + 1$. Suppose $\sum_{i=1}^{k+1} r_i x_i = 0$ and $r_{k+1} \neq 0$. Denote $s_i = -r_i r_{k+1}^{-1}$. Therefore

$$x_{k+1} = \sum_{i=1}^k s_i x_i.$$

Since $x_{k+1} \in \mathcal{A}(M)$, thus for all $p, q, s_i \in \mathbb{O}$,

$$0 = [p, q, x_{k+1}] = \sum_{i=1}^k [p, q, s_i x_i] = \sum_{i=1}^k [p, q, s_i] x_i$$

where we have used Lemma 3.2 in the last equation. Hence by induction hypothesis we conclude that,

$$[p, q, s_i] = 0, \text{ for each } i \in \{1, \ldots, k\} \text{ and for all } p, q \in \mathbb{O}.$$
This implies $s_i \in \mathbb{R}$. Note that
\[ \sum_{i=1}^{k} s_i x_i = x_{k+1}, \]
which contradicts the hypothesis that \( \{x_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly independent, we thus prove the lemma.

Note that we have actually proved the following property.

**Corollary 3.11.** Let \( S \subseteq \mathcal{A}(M) \). Then that \( S \) is \( \mathcal{O} \)-linearly independent if and only if it is \( \mathbb{R} \)-linearly independent.

**Lemma 3.12.** Under the assumptions of Lemma 3.10, if \( y = \sum_{i=1}^{n} r_i x_i \in \mathcal{A}(M) \), then we have \( r_i \in \mathbb{R} \) for each \( i \in \{1, \ldots, n\} \).

**Proof.** Since \( y \in \mathcal{A}(M) \), we have for all \( p, q \in \mathcal{O} \),
\[ 0 = [p, q, \sum_{i=1}^{n} r_i x_i] = \sum_{i=1}^{n} [p, q, r_i] x_i. \]
By Lemma 3.10, we conclude \( [p, q, r_i] = 0 \), \( i = 1, \ldots, n \). This yields \( r_i \in \mathbb{R} \). \qed

We next consider the properties of conjugate elements. It turns out that similar statements hold for \( S \subseteq \mathcal{A}^{-}(M) \).

**Lemma 3.13.** Let \( S = \{x_i\}_{i=1}^{n} \subseteq \mathcal{A}^{-}(M) \) be an \( \mathbb{R} \)-linearly independent set. If
\[ \sum_{i=1}^{n} r_i x_i = 0, \quad r_i \in \mathcal{O} \text{ for each } i = 1, \ldots, n, \]
then \( r_i = 0 \) for each \( i = 1, \ldots, n \).
Moreover, if \( y = \sum_{i=1}^{n} r_i x_i \in \mathcal{A}^{-}(M) \), then \( r_i \in \mathbb{R} \) for each \( i = 1, \ldots, n \).

**Proof.** Induction on \( n \) as before. The case of \( n = 1 \) is trivial. Assume the lemma holds for degree \( k \), we will prove it for \( k+1 \). Suppose \( \sum_{i=1}^{k+1} r_i x_i = 0 \) and \( r_{k+1} \neq 0 \). Denote \( s_i = -r_i r_{k+1}^{-1} \). Therefore
\[ x_{k+1} = \sum_{i=1}^{k} s_i x_i. \]
Since \( x_{k+1} \in \mathcal{A}^{-}(M) \), thus for all \( p, q, \in \mathcal{O} \),
\[ 0 = (qp)x_{k+1} - p(qx_{k+1}) \]
\[ = \sum_{i=1}^{k} (qp)(s_i x_i) - p(q(s_i x_i)) \]
\[ = \sum_{i=1}^{k} (s_i (qp)) x_i - p((s_i q)x_i) \]
\[ = -\sum_{i=1}^{k} ([s_i, q, p]) x_i \]
Hence by induction hypothesis we obtain that \( s_i \in \mathbb{R} \). The rest of the proof runs much the same as in Lemma 3.10. \qed
4 The structure of left $\mathcal{O}$-moules

In this section, we are in a position to formulate the structure of any left $\mathcal{O}$-module. We will first concerned with the finite dimensional case and then the general case follows.

4.1 Finite dimensional $\mathcal{O}$-modules

It is well-known (for example, [3, 9]) that the octonions have a very close relationship with spinors in 7, 8 dimensions. In particular, multiplication by imaginary octonions is equivalent to Clifford multiplication on spinors in 7 dimensions. It turns out that the category of left $\mathcal{O}$-modules is isomorphic to the category of left $\mathcal{O}$-modules.

For any left $\mathcal{O}$-module $M$, according to $[e_i, e_j, x] = -[e_j, e_i, x]$, we get the left multiplication operator $L$ satisfies:

$$L_{e_i}L_{e_j} + L_{e_j}L_{e_i} = -2\delta_{ij}\text{Id}.$$  

Hence $\mathbb{A} := \text{Span}_\mathbb{R}\{L_{e_i} \mid i = 0, 1, \ldots, 7\}$ is a Clifford algebra over $\mathbb{R}^7$. This yields a $\mathcal{O}_7$-module structure on $M$, because from the universal properties of $\mathcal{O}_7$, we have a non-trivial ring homomorphism

$$\rho : \mathcal{O}_7 \to \mathbb{A} \to \text{End}_\mathbb{R}(M).$$

Denote this $\mathcal{O}_7$-module by $\mathcal{O}_7M$, or just $M$, and the $\mathcal{O}_7$-scalar multiplication is given by

$$g_{o}m := e_{\alpha_1}(e_{\alpha_2}(\cdots (e_{\alpha_k}m)))$$

for any $\alpha \in \mathcal{P}(n)$. Here $g_{o} = g_{\alpha_1} \cdots g_{\alpha_k} := L_{e_{\alpha_1}} \cdots L_{e_{\alpha_k}}$. Let $f \in \text{Hom}_\mathcal{O}(M, M')$ be a left $\mathcal{O}$-homomorphism, where $M, M'$ are two left $\mathcal{O}$-modules. Then

$$f(L_{e_i}x) = f(e_i x) = e_i f(x) = L_{e_i} f(x).$$

and hence $f(g_{o}x) = g_{o}f(x)$. This means $f \in \text{Hom}_{\mathcal{O}_7}(M, M')$. Conversely, for any left $\mathcal{O}_7$-module $M$, let $\{g_i\}_{i=1}^7$ be a basis in $\mathbb{R}^7$. Define:

$$e_i^* x := g_i x.$$

Then

$$e_i^*(e_j^* x) = g_i(g_j x) = (g_i g_j) x = (-2\delta_{ij} - g_i g_j)x = -2\delta_{ij}x - e_j^*(e_i^* x)$$

This implies that

$$e_i^*(e_j^* x) + e_j^*(e_i^* x) = (e_i e_j + e_j e_i)^* x$$

by transposition of terms, we obtain

$$[e_i, e_j, x] = -[e_j, e_i, x].$$

This yields for any $p, q \in \mathcal{O}$, $[p, q, x] = -[q, p, x]$. Consequently $M$ is a left $\mathcal{O}$-module. For any $f \in \text{Hom}_{\mathcal{O}_7}(M, M')$,

$$f(e_i^* x) = f(g_i x) = g_i f(x) = e_i^* f(x)$$

Therefore $f \in \text{Hom}_\mathcal{O}(M, M')$. In summary, we get the following important result:

**Theorem 4.1.** The category of left $\mathcal{O}$-module is isomorphic to the category of left $\mathcal{O}_7$-module. Moreover, the only two kinds of simple $\mathcal{O}$-module are $\mathcal{O}$ and $\mathcal{O}^\perp$ up to isomorphism.
Proof. Naturally we have two categories $\mathcal{O}\text{-Mod}$ and $\mathcal{C}l_{7}\text{-Mod}$.

$$T : \mathcal{O}\text{-Mod} \rightarrow \mathcal{C}l_{7}\text{-Mod}$$

and for any morphism $\varphi \in \text{Hom}_{\mathcal{O}}(M,N)$, it maps to $T(\varphi): \mathcal{C}l_{7}M \rightarrow \mathcal{C}l_{7}N$, which is given by 

$$T(\varphi) : m \mapsto \varphi(m).$$

Clearly, this is an isomorphism by above discussion. As is well known, $\mathcal{C}l_{7}$ is a semi-simple algebra and $\mathcal{C}l_{7} \cong M(8,\mathbb{R}) \oplus M(8,\mathbb{R})$, the only simple $\mathcal{C}l_{7}$-module is $(\mathbb{R}^8,0)$ and $(0,\mathbb{R}^8)$ up to isomorphism [7]. Hence there also only exist two kinds of simple $\mathcal{O}$-module. In view of Example 3.5, we thus conclude that $\mathcal{O}$ and $\overline{\mathcal{O}}$ are the two different kinds of simple $\mathcal{O}$-modules. This completes the proof.

\textbf{Corollary 4.2.} If $M$ is of finite real dimension, then 

$$M \cong \mathcal{O}^{n_1} \oplus \overline{\mathcal{O}}^{n_2}. \quad (4.1)$$

where $(n_1, n_2)$ is the type of $M$. In particular, $\dim_{\mathbb{R}}M = 8(n_1 + n_2)$.

Proof. By Theorem 4.1, we can regard $M$ as a $\mathcal{C}l_{7}$-module $\mathcal{C}l_{7}M$. It is known that $\mathcal{C}l_{7}$ is a semi-simple algebra, we have 

$$\mathcal{C}l_{7} \cong M(8,\mathbb{R})M \oplus M(8,\mathbb{R}).$$

This means that $\mathcal{C}l_{7}$ has exactly two isomorphism classes of simple $\mathcal{C}l_{7}$-modules (see for example [1, Chap5, Thm10]). In view of Wedderburn’s Theorem for central simple algebras therefore, we conclude 

$$\mathcal{C}l_{7}M \cong \mathcal{C}l_{7}S_1^{n_1} \oplus \mathcal{C}l_{7}S_2^{n_2}$$

where $S_1, S_2$ are the representation of the two isomorphism classes of simple $\mathcal{C}l_{7}$-modules. Thus by Theorem 4.1, 

$$\mathcal{O}M \cong \mathcal{O}S_1^{n_1} \oplus \mathcal{O}S_2^{n_2}.$$ 

In particular, let $S_1 = \mathcal{O}, S_2 = \overline{\mathcal{O}}$, then we get the conclusion as desired. We next show that $(n_1, n_2)$ is just the type. By definition, one can prove $\mathcal{A}(M + N) = \mathcal{A}(M) \oplus \mathcal{A}(N)$ and $\mathcal{A}^{-}(M + N) = \mathcal{A}^{-}(M) \oplus \mathcal{A}^{-}(N)$. Then it follows from Lemma 3.7 that $\dim_{\mathbb{R}}\mathcal{A}(\mathcal{O}^{n_1} \oplus \overline{\mathcal{O}}^{n_2}) = n_1$ and $\dim_{\mathbb{R}}\mathcal{A}^{-}(\mathcal{O}^{n_1} \oplus \overline{\mathcal{O}}^{n_2}) = n_2$. This completes the proof.

We can give a complete description of the set of homomorphisms between two finite dimensional left $\mathcal{O}$-modules.

\textbf{Theorem 4.3.} Let $M, N$ be two left $\mathcal{O}$-modules of finite dimension. Suppose the type of $M$ and $N$ are $(m_1, m_2), (n_1, n_2)$ respectively. Then 

$$\text{Hom}_{\mathcal{O}}(M, N) \cong M_{n_1 \times m_1}(\mathbb{R}) \bigoplus M_{n_2 \times m_2}(\mathbb{R}). \quad (4.2)$$

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Proof. It follows from Corollary 4.2 that,
\[ M \cong \mathbb{O}^{m_1} \oplus \overline{\mathbb{O}}^{m_2}, \quad N \cong \mathbb{O}^{n_1} \oplus \overline{\mathbb{O}}^{n_2}. \] (4.3)

We claim each homomorphism of \( \text{Hom}_\mathbb{O}(M, N) \) is of the form
\[
\begin{pmatrix}
\theta_{11} & \cdots & \theta_{1m_1} & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\theta_{n_11} & \cdots & \theta_{n_1m_1} & 0 & 0 & 0 \\
0 & 0 & 0 & \psi_{11} & \cdots & \psi_{1m_2} \\
0 & 0 & 0 & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \psi_{n_21} & \cdots & \psi_{n_2m_2}
\end{pmatrix}.
\]

Indeed, we have \( \text{Hom}_\mathbb{O}(\mathbb{O}, \mathbb{O}) = \mathbb{R} \text{ Id}_\mathbb{O}, \text{Hom}_\mathbb{O}(\overline{\mathbb{O}}, \overline{\mathbb{O}}) = \mathbb{R} \text{ Id}_\mathbb{O}, \text{Hom}_\mathbb{O}(\mathbb{O}, \overline{\mathbb{O}}) \cong \{0\} \). Notice that for any \( \varphi \in \text{Hom}_\mathbb{O}(\mathbb{O}, \mathbb{O}) \) we have
\[ \varphi(q) = q \varphi(1), \]
we only need to show that \( \varphi(1) \in \mathbb{R} \). Proposition 3.3 then yields the conclusion as required. As for the case \( \text{Hom}_\mathbb{O}(\overline{\mathbb{O}}, \overline{\mathbb{O}}) \cong \mathbb{R} \). Given \( f \in \text{Hom}_\mathbb{O}(\overline{\mathbb{O}}, \overline{\mathbb{O}}) \), write \( f(1) = r \), then
\[ f(x) = f(\overline{x}:1) = \overline{x} f(1) = xr. \]
Since \( f \) is an \( \mathbb{O} \)-homomorphism, it follows that for all \( p, x \in \mathbb{O}, \)
\[ (p\overline{x}) r = f(p\overline{x}) = p \overline{x} f(x) = p \overline{x} r. \]
This yields \( [p, x, r] = 0 \) for all \( p, x \in \mathbb{O} \), consequently \( r \in \mathbb{R} \) as desired. Finally, it follows from Schur’s Lemma that
\[ \text{Hom}_\mathbb{O}(\mathbb{O}, \overline{\mathbb{O}}) \cong \text{Hom}_{C\ell}(C\ell(\mathbb{O}, \overline{\mathbb{O}})) = \{0\}. \]
This completes the proof.

At last, we point out another relation between \( \text{Spin}(7) \) and octonions. More precisely, we can provide a realization of spinor space over \( \text{Spin}(7) \) in terms of octonions \( \mathbb{O} \).

Here we refer to the concept of (Weyl) spinor spaces as the irreducible complex representations of \( \text{Spin}(p,q) \); see [14, Chap. I sec. 4]. By definition,
\[ \text{Spin}(7) = \{ v(x_1)v(x_2) \cdots v(x_{2k}) : x_i \in \mathbb{R}^7, |x_i| = 1, i = 1, \ldots, 2k \} \]
It is well-known that
\[ \text{Spin}(7) \subset \mathcal{C}\ell_7^+, \]
where \( \mathcal{C}\ell_7^+ \) is \( \mathbb{R} \)-linearly generated by \( \{ g_\alpha | \alpha \in \mathcal{P}(n), |\alpha| = 2k \} \).

Let \( \mathcal{C}\ell_7 = \mathcal{C}\ell_6 \otimes \mathbb{C} \) be the complexification of \( \mathcal{C}\ell_7 \). It is a complex Clifford algebra and there exists a \( \mathbb{C} \)-algebra isomorphism
\[ \mathcal{C}\ell_7 \cong \mathbb{M}(8, \mathbb{C}) \oplus \mathbb{M}(8, \mathbb{C}). \]
Notice that \( \mathcal{C}\ell_7^+ \) is \( \mathbb{C} \)-linearly generated by \( \text{Spin}(7) \) and
\[ \mathcal{C}\ell_7^+ \cong \mathcal{C}\ell_6 \cong \mathbb{M}(8, \mathbb{C}) \]
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as \( \mathbb{C} \)-algebra. This means that \( \mathbb{C}l_7^+ \) is a simple algebra. Therefore, \( \text{Spin}(7) \) has only one irreducible representation. We denote by \( S_6 \) the irreducible representation of \( \text{Spin}(7) \). It is also a \( \mathbb{C}l_6 \)-module.

As seen before, \( \mathbb{O} \) has a \( \mathbb{C}l_7 \)-module structure. Thanks to

\[
L_{e_1} \cdots L_{e_7} = -\text{Id}
\]

(4.4)

it admits a \( \mathbb{C}l_6 \)-module structure \( \mathbb{C}l_6 \mathbb{O} \), too. Indeed, let \( \mathbb{A} \) be the algebra generated by \( \{L_{e_i} \mid i = 1, \ldots, 7\} \). In virtue of (4.4), we know that \( \mathbb{A} \) is not the universal Clifford algebra over \( \mathbb{R}^7 \), so that it is isomorphic to \( \mathbb{C}l_6 \); see [7]. Consequently, we have a ring homomorphism

\[
\mathbb{C}l_6 \cong \mathbb{A} \hookrightarrow \text{End}_\mathbb{R}(\mathbb{O}),
\]

which provides a \( \mathbb{C}l_6 \)-module structure \( \mathbb{C}l_6 \mathbb{O} \) on \( \mathbb{O} \).

As a result, \( \mathbb{C}l_6 \mathbb{O} \otimes \mathbb{C} \) is a simple \( \mathbb{C}l_6 \)-module, which gives the realization of spinor space over \( \text{Spin}(7) \).

### 4.2 Structure of general left \( \mathbb{O} \)-modules

In this subsection, we proceed to study the characterization of the general left \( \mathbb{O} \)-modules. We first introduce the notion of basis on \( \mathbb{O} \)-modules.

**Definition 4.4.** Let \( M \) be a left \( \mathbb{O} \)-module. A subset \( S \subseteq M \) is called a **basis** if \( S \) is \( \mathbb{O} \)-linearly independent and \( M = \mathbb{O}S \).

**Theorem 4.5.** Each left \( \mathbb{O} \)-module \( M \) has a basis \( S \) included by \( \mathbb{A}(M) \cup \mathbb{A}^{-}(M) \). In particular, \( M = \mathbb{O}\mathbb{A}(M) \oplus \mathbb{O}\mathbb{A}^{-}(M) \).

Moreover, let \( \Lambda_1, \Lambda_2 \) be two index sets satisfying \( |\Lambda_1| = |S \cap \mathbb{A}(M)|, |\Lambda_2| = |S \cap \mathbb{A}^{-}(M)| \), here \( |S| \) stands for the cardinality of \( S \), then \( M \cong (\oplus_{i \in \Lambda_1} \mathbb{O}) \oplus (\oplus_{i \in \Lambda_2} \mathbb{O}) \).

Its proof will depend on the following lemma.

**Lemma 4.6.** Let \( M \) be a left \( \mathbb{O} \)-module, then \( \langle m \rangle_\mathbb{O} \) is finite dimensional for any \( m \in M \). More precisely, the dimension is at most 128.

**Proof.** \( \langle m \rangle_\mathbb{O} \) is such module generated by \( e_i (e_{i_2} (\cdots (e_{i_n} m)) \rangle \), where \( i_k \in \{1, 2, \ldots, 7\}, n \in \mathbb{N} \). Note that

\[
e_i (e_j m) + e_j (e_i m) = (e_i e_j + e_j e_i) m = -2\delta_{ij} m,
\]

hence the element defined by \( e_i (e_{i_2} (\cdots (e_{i_n} m)) \rangle \) for \( n > 7 \) can be reduced. Thus the vectors \( \{m, e_1 m, \ldots, e_7 m, e_1 (e_2 m), \ldots, e_1 (e_2 (\cdots (e_7 m)) \rangle \} \) will generate \( \langle m \rangle_\mathbb{O} \), we conclude that \( \dim_\mathbb{R} \langle m \rangle_\mathbb{O} \leq C_7^0 + C_7^1 + \cdots + C_7^7 = 128 \).

**Remark 4.7.** In fact, this property has already appeared in [8]. However, it is worth stressing the essentiality of this property. It enables us to characterize the structure of general left \( \mathbb{O} \)-modules in terms of finite case.
Proof of Theorem 4.5. For the case \( \dim \mathcal{M} < \infty \), by Corollary 4.2, \( \mathcal{M} \cong \mathbb{O}^n \oplus \mathbb{O}^{n'} \) for some nonnegative integers \( (n,n') \). It’s easy to see that there is a basis \( \{ \epsilon_i \}_{i=1}^{n+n'} \subseteq \mathcal{A}(\mathcal{M}) \cup \mathcal{A}^-(\mathcal{M}) \). As for general case, let \( S^+ \) be a basis of the real vector space \( \mathcal{A}(\mathcal{M}) \) and \( S^- \) a basis of \( \mathcal{A}^-(\mathcal{M}) \), let \( S := S^+ \cup S^- \subseteq \mathcal{C}(\mathcal{M}) \). It follows from Lemma 3.8 that \( \mathcal{A}(\mathcal{M}) \cap \mathcal{A}^-(\mathcal{M}) = \{ 0 \} \), \( S \) is clearly \( \mathbb{R} \)-linearly independent. In view of Lemma 3.10 and Lemma 3.13, we conclude that \( S \) is also \( \mathbb{O} \)-linearly independent. We next show that \( \mathbb{O}S = \mathcal{M} \). If not, there exists a nonzero element \( m \in \mathcal{M} \setminus \mathbb{O}S \). It follows by Lemma 4.6 that \( \langle m \rangle_\mathbb{O} \) has a basis \( \{ x_i(m) \}_{i=1}^n \subseteq \mathcal{A}(\langle m \rangle_\mathbb{O}) \cup \mathcal{A}^-(\langle m \rangle_\mathbb{O}) \subseteq \mathcal{A}(\mathcal{M}) \cup \mathcal{A}^-(\mathcal{M}) \), hence we can assume

\[
m = \sum_{i=1}^n r_i x_i(m), \quad r_i \in \mathbb{O}.
\]

Suppose \( x_i(m) = \sum_j r_{ij} s_j \), \( s_j \in S \), and it follows from Lemma 3.12 and Lemma 3.13 that \( r_{ij} \in \mathbb{R} \), then we obtain

\[
m = \sum (r_i r_{ij}) s_j, \quad r_i r_{ij} \in \mathbb{O}, \quad s_j \in S,
\]

which contradicts our assumption.

Let \( N \) denote the \( \mathbb{O} \)-module \( (\oplus_{i \in \Lambda_1} \mathbb{O}) \oplus (\oplus_{i \in \Lambda_2} \mathbb{O}) \). Canonically we can choose a basis \( \{ \epsilon_i \}_{i \in \Lambda_1 \cup \Lambda_2} \) and satisfies \( \epsilon_i \in \mathcal{A}(N) \) when \( i \in \Lambda_1 \), \( \epsilon_j \in \mathcal{A}^-(N) \) when \( j \in \Lambda_2 \). Let \( S = \{ s_i \}_{i \in \Lambda_1 \cup \Lambda_2} \) be a basis of \( \mathcal{M} \) satisfying the hypothesis in theorem. We define:

\[
f : \mathcal{M} \to N, \quad \sum r_i s_i \mapsto \sum r_i \epsilon_i.
\]

Then for any \( p \in \mathbb{O} \),

\[
f \left( p \sum_{\Lambda_1 \cup \Lambda_2} r_i s_i \right) = f \left( \sum_{\Lambda_1} p(r_i s_i) + \sum_{\Lambda_2} p(r_i s_i) \right) \\
= f \left( \sum_{\Lambda_1} (pr_i) s_i + \sum_{\Lambda_2} (r_i p) s_i \right) \\
= \sum_{\Lambda_1} (pr_i) \epsilon_i + \sum_{\Lambda_2} (r_i p) \epsilon_i \\
= \sum_{\Lambda_1} p(r_i \epsilon_i) + \sum_{\Lambda_2} p(r_i \epsilon_i) \\
= pf \left( \sum_{\Lambda_1 \cup \Lambda_2} r_i s_i \right)
\]

This shows \( f \in \text{Hom}_\mathbb{O}(\mathcal{M}, N) \), similarly we can also define \( g \in \text{Hom}_\mathbb{O}(N, \mathcal{M}) \) such that

\[
f g = \text{Id}_N, \quad g f = \text{Id}_\mathcal{M}.
\]

Hence we get the conclusion as desired.

\[ \square \]

Remark 4.8. In the study of octonion Hilbert space and Banach space, it heavily depends on the direct sum structure of the space under considered sometimes, which always brings the question back
to the classic situation. For example, in the proof of Hanh-Banach Theorem in [12, Theorem 2.4.1], it declares that every \(\mathcal{O}\)-vector space is of the following form:

\[
x = X_0 \oplus X_1 e_1 \oplus \cdots \oplus X_7 e_7.
\]

Note that therein the definition of \(\mathcal{O}\)-vector space is actually a left \(\mathcal{O}\)-module with an irrelevant right \(\mathcal{O}\)-module structure. We thus can only consider the left \(\mathcal{O}\)-module structure of it. In view of Theorem 4.5, of course the assertion does not always work.

### 4.3 Cyclic elements in left \(\mathcal{O}\)-module

Let \(M\) be a left \(\mathcal{O}\)-module throughout this subsection. The submodule \(\langle m \rangle_{\mathcal{O}}\) generated by one point will be very different from the classical case. As is known that \(\mathcal{O}m\) is not always a submodule (see Example 4.13, Example 4.14). We introduce the notion of cyclic element, which generates a simple submodule, to describe this phenomenon. It turns out that every element is a real linear combination of cyclic elements, although the quantity of cyclic elements is much less than others.

**Definition 4.9.** An element \(m \in M\) is said to be cyclic if \(\langle m \rangle_{\mathcal{O}} = \mathcal{O}m\). Denote by \(\mathcal{C}(M)\) the set of all cyclic elements in \(M\).

**Proposition 4.10.** An element \(m \in M\) is cyclic if and only if for all \(r, p \in \mathcal{O}\), there exists \(q \in \mathcal{O}\), such that \([r, p, m] = qm\).

**Proof.** Assume \(m\) satisfies the hypothesis above, then it’s easy to check that \(\mathcal{O}m\) is a submodule, that is, \(m\) is cyclic. Assume \(m\) is cyclic, thus for all \(r, p \in \mathcal{O}\), \(r(pm) \in \mathcal{O}m\) and hence there exists \(s \in \mathcal{O}\), such that \(r(pm) = sm\), then \([r, p, m] = (rp - s)m\). This proves the lemma.

Our first observation is the following lemma which is useful sometimes.

**Lemma 4.11.** Let \(M\) be a left \(\mathcal{O}\)-module, then \(\bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}(M) \subseteq \mathcal{C}(M)\).

**Proof.** Let \(0 \neq x \in \mathcal{A}(M)\), we want to prove that \(px \in \mathcal{C}(M)\). To see this, take \(r, s \in \mathcal{O}\) arbitrarily, then using Lemma 3.2, we obtain:

\[
[r, s, px] = [r, s, p]x = [r, s, p]((p^{-1}p)x) = ([r, s, p]p^{-1})(px) = ([r, s, p]p^{-1})(px) = ([r, s, p]p^{-1})(px) = ([r, s, p]p^{-1})(px)
\]

In view of Proposition 4.10, we thus get \(px \in \mathcal{C}(M)\) as desired.

**Remark 4.12.** Note that the set \(\bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}(M)\) is not \(\mathcal{O}\mathcal{A}(M)\). Actually,

\[
\mathcal{O}\mathcal{A}(M) = \left\{ \sum_{i=1}^{n} p_i x_i \mid p_i \in \mathcal{O}, x_i \in \mathcal{A}(M), n \in \mathbb{N} \right\}.
\]

In fact, it is easy to know the sum of two cyclic elements may be not a cyclic element anymore.
Now let us first consider the cyclic elements in case of finite dimensional \( \mathcal{O} \)-modules. It had been seen that \((e_1, e_2, 0)\) will generate a 16 dimensional real space and \((e_1, e_2, e_3)\) will generate a 24 dimensional real space, which means that both are not cyclic elements (see [8]). In fact, this is a general phenomenon. In case \( M = \mathcal{O}^2 \), we actually have:

**Example 4.13.** Suppose \( x = (x_1, x_2) \in M \), then \((x)_{\mathcal{O}} = M\) if and only if \( x_1, x_2 \) are real linearly independent.

In particular, in view of the structure of finite dimensional \( \mathcal{O} \)-modules (see Corollary 4.2), we obtain:

(i). \( \mathcal{E}(\mathcal{O}^2) = \bigcup_{p \in \mathcal{O}} p \cdot \mathbb{R}^2 \).

(ii). An element in \( \mathcal{O}^2 \) generates the whole space if and only if it is not cyclic.

**Proof.** Let \( x = (x_1, x_2) \notin \mathcal{E}(M) \), it is easy to see \( x_1, x_2 \) are real linearly independent. Indeed, if \( x_1 = r x_2 \) for some \( r \in \mathbb{R} \), then \( x = x_2(r, 1) \). Note that \((r, 1)\) is an associative element of \( \mathcal{O}^2 \), it follows from Lemma 4.11 that \( x \in \mathcal{E}(M) \), a contradiction. Hence both \( x_1, x_2 \) are not zero and \((x_1)^{-1} x_2 \notin \mathbb{R} \). Therefore,

\[
(x_1)^{-1} x = (1, (x_1)^{-1} x_2), \quad (x_1)^{-1} x_2 \notin \mathbb{R}.
\]

Thus we can choose \( p, q \in \mathcal{O} \), such that \([p, q, (x_1)^{-1} x_2] \neq 0\) (if not, we would have \([p, q, (x_1)^{-1} x_2] = 0\) for any \( p, q \), which means \((x_1)^{-1} x_2 \in \mathbb{R} \)). However,

\[
[p, q, (x_1)^{-1} x] = ([p, q, 1], [p, q, (x_1)^{-1} x_2]) = (0, [p, q, (x_1)^{-1} x_2]) \in (x)_{\mathcal{O}},
\]

we thus obtain

\[
\{(0, p) \mid p \in \mathcal{O}\} \subseteq (x)_{\mathcal{O}}.
\]

Similar arguments apply to \((x_2)^{-1} x\), we get the conclusion as desired. \(\square\)

An analogous statement holds for \( \overline{\mathcal{O}}^2 \). We next consider the case \( M = \mathcal{O} \oplus \overline{\mathcal{O}} \).

**Example 4.14.** \(((1, 1))_{\mathcal{O}} = \mathcal{O} \oplus \overline{\mathcal{O}}\)

Since

\[
e_1(1, 1) = (e_1, -e_1), \quad e_2(e_3(1, 1)) = e_2(e_3, -e_3) = (e_1, e_1),
\]

we conclude that both \((e_1, 0)\) and \((0, e_1)\) lie in \(((1, 1))_{\mathcal{O}}\), which yields that the submodules \(\{(0, p) \mid p \in \mathcal{O}\}\) and \(\{(p, 0) \mid p \in \mathcal{O}\}\) both lie in \(((1, 1))_{\mathcal{O}}\), too. Therefore \(((1, 1))_{\mathcal{O}} = \mathcal{O} \oplus \overline{\mathcal{O}}\).

In fact, Lemma 4.11, Example 4.13 and Example 4.14 are all specific to a general result that will be proved later. The following lemma is crucial to set up this result.

**Lemma 4.15.** Let \( x \) be any given nonzero element in \( M \). Then

\[
x \in \mathcal{E}(M) \iff \dim_{\mathbb{R}} (x)_{\mathcal{O}} = 8 \iff (x)_{\mathcal{O}} \cong \mathcal{O} \oplus \overline{\mathcal{O}}.
\]

**Proof.** Let \( x \in \mathcal{E}(M) \), then \((x)_{\mathcal{O}} = \mathcal{O} x\) and hence \( \dim_{\mathbb{R}} (x)_{\mathcal{O}} \leq 8 \). On the other hand, since \((x)_{\mathcal{O}}\) is a nonzero \( \mathcal{O} \)-module of finite dimension, thus \( \dim_{\mathbb{R}} (x)_{\mathcal{O}} \geq 8 \), therefore \( \dim_{\mathbb{R}} (x)_{\mathcal{O}} = 8 \), this means \((x)_{\mathcal{O}}\) is a simple \( \mathcal{O} \)-module and hence \((x)_{\mathcal{O}} \cong \mathcal{O} \) or \( \overline{\mathcal{O}} \). Suppose \((x)_{\mathcal{O}} \cong \mathcal{O} \) or \( \overline{\mathcal{O}} \). Assume \((x)_{\mathcal{O}} \cong \mathcal{O} \).
first. Let \( \varphi \) denote an isomorphism: \( \varphi : \langle x \rangle_\mathcal{O} \to \mathcal{O} \). For any \( m \in \langle x \rangle_\mathcal{O} \), suppose \( \varphi(x) = p, \varphi(m) = q \). Then
\[
\varphi(m) = q = qp^{-1}p = qp^{-1} \varphi(x) = \varphi((qp^{-1})x),
\]
according to that \( \varphi \) is isomorphism, we get \( m = (qp^{-1})x \), hence \( \langle x \rangle_\mathcal{O} \cong \mathcal{O}x \) which means \( x \in \mathcal{C}(M) \). If \( \langle x \rangle_\mathcal{O} \cong \overline{\mathcal{O}} \), still let \( \varphi \) denote the isomorphism: \( \varphi : \langle x \rangle_\mathcal{O} \to \overline{\mathcal{O}} \). For any \( m \in \langle x \rangle_\mathcal{O} \), suppose \( \varphi(x) = p, \varphi(m) = q \). Then
\[
\varphi(m) = q = (qp^{-1})p = (qp^{-1}) \varphi(x) = \varphi((qp^{-1})x),
\]
then we get \( m = (qp^{-1})x \), hence \( x \in \mathcal{C}(M) \).

According to above lemma, we define
\[
\mathcal{C}^+(M) := \{ x \in \mathcal{C}(M) \mid \langle x \rangle_\mathcal{O} \cong \mathcal{O} \} \cup \{ 0 \},
\]
\[
\mathcal{C}^-(M) := \{ x \in \mathcal{C}(M) \mid \langle x \rangle_\mathcal{O} \cong \overline{\mathcal{O}} \} \cup \{ 0 \}.
\]
Therefore \( \mathcal{C}(M) = \mathcal{C}^+(M) \cup \mathcal{C}^-(M) \). We shall show that all the cyclic elements are determined by the associative subset \( \mathcal{A}(M) \) and the conjugate associative subset \( \mathcal{A}^-(M) \).

**Theorem 4.16.** Let \( M \) be a left \( \mathcal{O} \)-module, then:

(i). \( \mathcal{C}^+(M) = \bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}(M) \);

(ii). \( \mathcal{C}^-(M) = \bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}^-(M) \).

**Proof.** We prove assertion (i). We first show \( \bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}(M) \subseteq \mathcal{C}^+(M) \). Given any \( x \in \mathcal{A}(M) \).

Without loss of generality we can assume \( x \neq 0 \). Define a map \( \phi : \langle x \rangle_\mathcal{O} \to \mathcal{O} \) such that \( \phi(px) = p \) for \( p \in \mathcal{O} \). This is a homomorphism in \( \text{Hom}_{\mathcal{O}}(\langle x \rangle_\mathcal{O}, \mathcal{O}) \), since
\[
\phi(q(px)) = \phi((qp)x) = qp = q\phi(px).
\]
Define \( \varphi : \mathcal{O} \to \langle x \rangle_\mathcal{O} \) by \( \varphi(p) = px \). Then
\[
\varphi(pq) = (pq)x = p(qx) = p\varphi(q).
\]
Hence \( \varphi \in \text{Hom}_{\mathcal{O}}(\mathcal{O}, \langle x \rangle_\mathcal{O}) \) and \( \phi \varphi = id, \varphi \phi = id \) and thus \( \langle x \rangle_\mathcal{O} \cong \mathcal{O} \). This proves \( x \in \mathcal{C}^+(M) \).

Because \( px \in \langle x \rangle_\mathcal{O} \) and \( x = p^{-1}(px) \in \langle px \rangle_\mathcal{O} \) for \( p \neq 0 \), that is, \( \langle x \rangle_\mathcal{O} = \langle px \rangle_\mathcal{O} \) whenever \( p \neq 0 \). This implies \( \bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}(M) \subseteq \mathcal{C}^+(M) \). On the contrary, let \( 0 \neq x \in \mathcal{C}^+(M) \), hence there is an isomorphism \( \phi \in \text{Hom}_{\mathcal{O}}(\mathcal{O}, \langle x \rangle_\mathcal{O}) \). Suppose \( \phi(1) = y \in \langle x \rangle_\mathcal{O} \), since \( \phi \) is an isomorphism, there is \( 0 \neq r \in \mathcal{O} \) such that \( y = rx \). However in view of Proposition 3.3, \( y = \phi(1) \in \mathcal{A}(\langle x \rangle_\mathcal{O}) \subseteq \mathcal{A}(M) \), thus \( x = r^{-1}y \in \bigcup_{p \in \mathcal{O}} p \cdot \mathcal{A}(M) \). This proves assertion (i).

We prove assertion (ii). Easy to show that \( \langle x \rangle_\mathcal{O} = \mathcal{O}x \) for \( x \in \mathcal{A}^-(M) \). Let \( x \in \mathcal{A}^-(M) \).

Without loss of generality we can assume \( x \neq 0 \). Define a map \( \phi : \langle x \rangle_\mathcal{O} \to \overline{\mathcal{O}} \) such that \( \phi(px) = \overline{p} \) for \( p \in \mathcal{O} \). This is a homomorphism in \( \text{Hom}_{\mathcal{O}}(\langle x \rangle_\mathcal{O}, \overline{\mathcal{O}}) \), since
\[
\phi(q(px)) = \phi((pq)x) = \overline{pq} = q \overline{p} = q\phi(px).
\]
Define \( \varphi : \mathfrak{O} \to \langle x \rangle_{\mathfrak{O}} \) by \( \varphi(p) = px \). Then
\[
\varphi(pq) = \varphi(pq) = \overline{pq}x = p(\overline{pq}) = p\varphi(q).
\]
Hence \( \varphi \in \text{Hom}_{\mathfrak{O}}(\mathfrak{O}, \langle x \rangle_{\mathfrak{O}}) \) and \( \varphi \varphi = id, \varphi \phi = id \) and thus \( \langle x \rangle_{\mathfrak{O}} \cong \mathfrak{O} \). This proves \( x \in \mathcal{C}^{-}(M) \).

Hence we conclude from the fact \( \langle x \rangle_{\mathfrak{O}} = \langle px \rangle_{\mathfrak{O}} \) that \( \bigcup_{p \in \mathfrak{O}} p \cdot \mathcal{A}^{+}(M) \subseteq \mathcal{A}^{-}(M) \). On the contrary, let \( 0 \neq x \in \mathcal{A}^{-}(M) \), hence there is an isomorphism \( \phi \in \text{Hom}_{\mathfrak{O}}(\mathfrak{O}, \langle x \rangle_{\mathfrak{O}}) \). Suppose \( \phi(1) = y \in \langle x \rangle_{\mathfrak{O}} \), since \( \phi \) is an isomorphism, there is \( 0 \neq r \in \mathfrak{O} \) such that \( y = rx \). Note that for any \( p, q \in \mathfrak{O} \),
\[
(pq)y = (pq)\phi(1) = \phi((pq)\cdot1) = \phi(q\overline{p}) = q\phi(p\cdot1) = q(py).
\]
This shows \( y \in \mathcal{A}^{-}(M) \) and thus \( x = r^{-1}y \in \bigcup_{p \in \mathfrak{O}} p \cdot \mathcal{A}^{-}(M) \). This proves assertion (ii).

In view of Theorem 4.5, we conclude an important consequence of the above Theorem:

**Corollary 4.17.** For each left \( \mathfrak{O} \)-module \( M \), we have
\[
M = \text{Span}_{\mathbb{R}} \mathcal{C}^{+}(M) \oplus \text{Span}_{\mathbb{R}} \mathcal{C}^{-}(M) = \text{Span}_{\mathbb{R}} \mathcal{C}(M).
\]

**Remark 4.18.** The corollary shows that for any element \( m \in M \), there exist some real linear independent cyclic elements \( m_{i} \in \mathcal{C}(M), i = 1, \ldots, n, \) such that
\[
m = \sum_{i=1}^{n} m_{i}.
\]
However, this decomposition is not unique. For example, let \( M = \mathbb{O}^{3} \) and \( m = (1, 1 + e_{1}, e_{1}) \). Then choose \( m_{1} = (1, 0, 0), m_{2} = (0, 1 + e_{1}, 0), m_{3} = (0, 0, e_{1}) \) in \( \mathcal{C}(M) \), it clearly holds \( m = \sum_{i=1}^{3} m_{i} \) and they are real linear independent. On the other hand, we can choose \( m_{1}' = (1, 1, 0), m_{2}' = e_{1}(0, 1, 1) \) in \( \mathcal{C}(M) \), this is also a decomposition of \( m \). It is worth noticing the length of a decomposition of \( m \) must be no less than 2 and no more than 3. Loosely speaking, the length of the decomposition of one element reflects the size of the submodule generated by it. The accurate relation deserves further study.

For any element \( m \) in a left \( \mathfrak{O} \)-module \( M \), it follows from Corollary 4.17 that there exist \( m_{\pm} \in \text{Span}_{\mathbb{R}} \mathcal{C}_{\pm}(M) \) such that \( m = m^{+} + m^{-} \). We can decompose \( m_{\pm} \) into a combination of real linearly independent cyclic elements. Denote by \( l_{\pm} \) the minimal length of the decompositions of \( m_{\pm} \). We conjecture that,
\[
\langle m \rangle_{\mathfrak{O}} \cong \mathfrak{O}^{l^{+}_{\pm}} \oplus \mathfrak{O}^{l^{-}_{\pm}}.
\]

It holds true in above example. In fact, in a similar manner as in Example 4.13, we can prove
\[
\langle (1, 1 + e_{1}, e_{1}) \rangle_{\mathfrak{O}} = \mathfrak{O} \cdot (1, 1, 0) \oplus \mathfrak{O} \cdot e_{1}(0, 1, 1) \cong \mathfrak{O}^{2}.
\]

If the conjecture is right, then the structure of the submodule generated by one element is completely clear.
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