Branching of Automorphic Fundamental Solutions

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July 7, 2014
Outline

Context

Automorphic Fundamental Solutions

Branching
Solutions of Automorphic Differential Equations

- Poincaré series of Good, Diaconu-Goldfeld, Diaconu-Garrett
  - subconvexity of $GL_2$ automorphic $L$-functions in the $t$ aspect over an arbitrary number field
- higher rank: Diaconu-Garrett moment identities
- lattice point counting in symmetric spaces $G/K$ where $G$ is complex (D)
- eigenfunctions for pseudo-differential operators
  - meromorphic continuation of Eisenstein series (CdV)
Outline

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Automorphic Fundamental Solutions

Branching
Fundamental Solutions

Differential operator $D$. Fundamental solution for $D$ is solution to

$$Du = \delta \quad (\delta = \text{Dirac delta})$$
Fundamental Solutions

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Here we are interested in fundamental solutions for $(\Delta - \lambda)$ on $\Gamma \setminus G/K$ where

- $G$ is a reductive or semi-simple Lie group, $K \subset G$ maximal compact, $\Gamma \subset G$ discrete
- $\Delta$ is the Laplacian, the image of Casimir for $\mathfrak{g}$
- $\lambda$ is a complex parameter
- $\delta$ is Dirac delta at basepoint
Engineering Math

Spectral expansions immediate (heuristically).

Example: Fourier expansions

\[
\left( \frac{d^2}{dx^2} - 4\pi^2 w^2 \right) u_w = \delta
\]

\[
\mathcal{F} \left( \left( \frac{d^2}{dx^2} - 4\pi^2 w^2 \right) u_w \right) = \mathcal{F}(\delta)
\]

\[
-4\pi^2 (\xi^2 + w^2) \mathcal{F} u_w = 1
\]

Thus \( \mathcal{F} u_w = -1/(4\pi^2(\xi^2 + w^2)) \), and

\[
u_w = \int_{-\infty}^{\infty} \frac{-e^{2\pi i x \xi}}{4\pi^2(\xi^2 + w^2)} \, d\xi = \frac{-e^{2\pi w|x|}}{4\pi w} \quad (\text{Re}(w) > 0)
\]
Similarly . . .

Simplest automorphic case: $SL_2(\mathbb{Z}) \backslash \mathcal{H}$

$$u_w = \sum_f \frac{\bar{f}(z_0) f}{\lambda_f - \lambda_w} + \frac{\Phi_0(z_0) \Phi_0}{\lambda_0 - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{E_{\frac{1}{2}-it}(z_0) E_{\frac{1}{2}+it}}{\lambda_s - \lambda_w} \, dt \quad (\text{Re}(w) > \frac{1}{2})$$

where $z_0$ is the base point in $SL_2(\mathbb{Z}) \backslash \mathcal{H}$, $f$ ranges over an orthonormal basis of cusp forms, $\Phi_0$ is the constant automorphic form, $s = \frac{1}{2} + it$, $\lambda_w = w(w - 1)$, and $\lambda_f$, $\lambda_0$, and $\lambda_s$ are the eigenvalues of $f$, $\Phi_0$, and $E_s$, respectively.
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Convergence??
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$$+ \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{E_{1/2-\text{i}t}(z_0) E_{1/2+\text{i}t} dt}{\lambda_s - \lambda_w} \quad (\text{Re}(w) > \frac{1}{2})$$

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Convergence?? Global automorphic Sobolev theory! (D)
Poincaré series: wind-up corresponding fundamental solution on free space $G/K$.

$$P\epsilon_w(g) = \sum_{\gamma \in \Gamma} u^\text{free}_w(\gamma g)$$

Then $P\epsilon_w$ is an automorphic fundamental solution.

Perhaps more common to start with $P\epsilon_w$, find its spectral expansion, hoping for meromorphic continuation in $w$. 
The Geometric Side

Poincaré series: wind-up corresponding fundamental solution on free space $G/K$.

$$\text{Pé}_w(g) = \sum_{\gamma \in \Gamma} u^\text{free}_w(\gamma g)$$

Then $\text{Pé}_w$ is an automorphic fundamental solution.

Perhaps more common to start with $\text{Pé}_w$, find its spectral expansion, hoping for meromorphic continuation in $w$.

However: sometimes the automorphic fundamental solution exhibits branching!
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Automorphic Fundamental Solutions

Branching
To be more precise . . .

$$(\Delta - \lambda_w)^\nu u_w = \delta_{z_0} \quad (\nu \in \mathbb{N})$$

Global automorphic Sobolev theory ensures:

- solution $u_w$ exists and is unique in global automorphic Sobolev spaces
- automorphic spectral expansion converges in Sobolev topology for $\text{Re}(w) \gg 1$
- $\nu \gg 1$ ensures that the spectral expansion converges uniformly pointwise (or in any $C^k$-topology that we wish)

However:
- meromorphic continuations along different $w$-paths may differ by a term of moderate growth (branching)
- the resulting function may lie outside of global automorphic Sobolev spaces
To be more precise . . .

\[(\Delta - \lambda_\nu) u_\nu = \delta_{z_0} \quad (\nu \in \mathbb{N})\]

Global automorphic Sobolev theory ensures:

- solution \( u_\nu \) exists and is unique in global automorphic Sobolev spaces
- automorphic spectral expansion converges in Sobolev topology for \( \text{Re}(\nu) \gg 1 \)
- \( \nu \gg 1 \) ensures that the spectral expansion converges uniformly pointwise (or in any \( C^k \)-topology that we wish)

However:

- meromorphic continuations along different \( \nu \)-paths may differ by a term of moderate growth (branching)
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Hilbert-Maass Fundamental Solutions

Let $k$ be a totally real number field of degree $n > 1$ over $\mathbb{Q}$, and let $\mathfrak{o}$ be its ring of integers.

For $\text{Re}(w) > \frac{1}{2}$, there is a unique solution $u_w$ to the automorphic differential equation $(\Delta - \lambda_w)u_w = \delta$:

$$u_w = \sum_{F} \frac{F(z_0) \cdot F}{\lambda_F - \lambda_w} + \frac{1}{(\lambda_1 - \lambda_w)\langle 1, 1 \rangle} + \sum_{\chi} \frac{1}{4\pi i} \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{1-s, \chi}(z_0) \cdot E_{s, \chi}}{\lambda_{s, \chi} - \lambda_w} \, ds$$

For each nontrivial unramified grossencharacter $\chi$, the corresponding integral $I_\chi$ has two branch points on the critical line.
Fix a grossencharacter $\chi$. Let $\sigma_1, \ldots, \sigma_n$ be the archimedean places of $k$. Take real parameters $t_\chi = (t_1, \ldots, t_n)$ with $t_1 + \cdots + t_n = 0$ such that

$$\chi(\alpha) = \sigma(\alpha)^{it_1} \cdots \sigma_n(\alpha)^{it_n}$$

where $\alpha \in (k \otimes \mathbb{Q} \mathbb{R})^\times$, and let

$$\|t_\chi\|^2 = \frac{1}{n} (|t_1|^2 + \ldots + |t_n|^2)$$

Writing the eigenvalue in terms of $s$ and $t_\chi$,

$$\lambda_{s,\chi} = \frac{1}{n} \left( (s + it_1)(s + it_1 - 1) + \ldots + (s + it_n)(s + it_n - 1) \right)$$

Thus the integrand has poles at

$$s = \frac{1}{2} \pm \sqrt{(w - \frac{1}{2})^2 + \|t_\chi\|^2}$$
Figure: Pathwise meromorphic continuation along these two paths in the $w$-plane yields functions that differ by a term of moderate growth. The dotted vertical line is the critical line $\text{Re}(w) = \frac{1}{2}$. The dashed horizontal lines are $\text{Im}(w) = \pm \|t_X\|$. 
Theorem

Let $\gamma_1$ and $\gamma_2$ be $\nu$-paths in $\mathbb{C}$, each originating at a point $\nu_0$ in the right half plane $\text{Re}(\nu) > \frac{1}{2}$, crossing the critical line once, and terminating at a point $\nu'_0$ in the left half plane $\text{Re}(\nu) < \frac{1}{2}$, with $\gamma_1$ crossing the critical line at a height greater than $\|t_\chi\|$ and $\gamma_2$ crossing at a height less than $\|t_\chi\|$. Then pathwise meromorphic continuations of $I_\chi(\nu)$ along the paths $\gamma_1$ and $\gamma_2$ differ by a term of moderate growth, namely by

$$4\pi i \cdot E_{1-s(\chi,\nu),\bar{\chi}}(z_0) \cdot E_{s(\chi,\nu),\chi} \cdot \frac{1 - 2s(\chi,\nu)}{1 - 2s(\chi,\nu)}$$

where, $s(\chi,\nu)$ is defined as follows. For fixed $\nu$ in $\text{Re}(s) > \frac{1}{2}$, $s(\chi,\nu)$ is the pole of the integrand of $I_\chi(\nu)$ in $\text{Re}(s) > \frac{1}{2}$. As $\nu$ crosses the critical line, $s(\chi,\nu)$ is defined by analytic continuation.
We sketch the proof.

Regularize:

\[
J_\chi(w) = \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s}(z_o)E_{s,\chi} - E_{1-s(\chi,w)}(z_o)E_{s(\chi,w),\chi}}{\lambda_{s,\chi} - \lambda_w} ds
\]

\[
+ \int_{\frac{1}{2}+i\mathbb{R}} \frac{ds}{\lambda_{s,\chi} - \lambda_w}
\]

By design the integrand of the first integral on the right side is continuous. The second integral can be evaluated by residues:

\[
2\pi i \times \text{Res}_{s = 1-s(\chi,w)} \frac{1}{(s - s(\chi,w))(s - (1-s(\chi,w)))} = \frac{2\pi i}{1 - 2s(\chi,w)}
\]
Consider $\chi = 1$. Then $s(\chi, w) = s(1, w) = w$, and

$$J_1(w) = \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{1-s,1}(z_0) E_{s,1} - E_{1-w,1}(z_0) E_{w,1}}{\lambda_{s,1} - \lambda_w} \, ds$$

$$+ \ E_{1-w,1}(z_0) E_{w,1} \cdot \frac{2\pi i}{1 - 2w} \quad (\text{Re}(w) > \frac{1}{2})$$

Move $w$ across the critical line and reverse the regularization:

$$J_1(w) = \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{1-s,1}(z_0) E_{s,1}}{\lambda_{s,1} - \lambda_w} \, ds$$

$$- \ E_{1-w,1}(z_0) E_{w,1} \times$$

$$\left( \int_{\frac{1}{2} + i\mathbb{R}} \frac{1}{\lambda_{s,1} - \lambda_w} \, ds - \frac{2\pi i}{1 - 2w} \right) (\text{Re}(w) < \frac{1}{2})$$

Since $s = w$ is now the pole to the left of the critical line, residue calculus yields

$$\int_{\frac{1}{2} + i\mathbb{R}} \frac{ds}{\lambda_{s,\chi} - \lambda_w} = 2\pi i \times \text{Res}_{s = w} \frac{1}{(s - w)(s - (1 - w))} = \frac{2\pi i}{2w - 1}$$
Thus the integral corresponding to $\chi = 1$ is

$$J_1(w) = \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{1-s,1}(z_0) E_{s,1}}{\lambda_{s,1} - \lambda_w} \, ds \quad (\Re(w) > \frac{1}{2})$$

$$J_1(w) = \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{1-s,1}(z_0) E_{s,1}}{\lambda_{s,1} - \lambda_w} \, ds$$

$$+ E_{1-w,1}(z_0) E_{w,1} \cdot \frac{4\pi i}{1 - 2w} \quad (\Re(w) < \frac{1}{2})$$

Thus we see that the pathwise meromorphic continuation has an additional term when $w$ is left of the critical line.
Now: $\chi$ nontrivial.

- If $w$ crosses the critical line with imaginary part greater in magnitude than $\|t_\chi\|$, the radicand, $(w - \frac{1}{2})^2 + \|t_\chi\|^2$, in the expression for $s(\chi, w)$ moves around the branch point of the square root, the origin. As above, get additional term.
Now: $\chi$ nontrivial.

- If $w$ crosses the critical line with imaginary part greater in magnitude than $\|t_\chi\|$, the radicand, $(w - \frac{1}{2})^2 + \|t_\chi\|^2$, in the expression for $s(\chi, w)$ moves around the branch point of the square root, the origin. As above, get additional term.

- If $w$ crosses the critical line with imaginary part within a distance of $\|t_\chi\|$ of the real axis, the radicand, $(w - \frac{1}{2})^2 + \|t_\chi\|^2$, stays strictly in the right half plane and thus does not travel around the origin. No additional term.
Now: $\chi$ nontrivial.

- If $\omega$ crosses the critical line with imaginary part greater in magnitude than $\|t_\chi\|$, the radicand, $(\omega - \frac{1}{2})^2 + \|t_\chi\|^2$, in the expression for $s(\chi, \omega)$ moves around the branch point of the square root, the origin. As above, get **additional term**.

- If $\omega$ crosses the critical line with imaginary part within a distance of $\|t_\chi\|$ of the real axis, the radicand, $(\omega - \frac{1}{2})^2 + \|t_\chi\|^2$, stays strictly in the right half plane and thus does *not* travel around the origin. **No additional term**.

Thus **branching** is evident: pathwise meromorphic continuations of $J_\chi(\omega)$ depend non-trivially on the path, the branch points being $\omega = \frac{1}{2} \pm i \|t_\chi\|$. 
GL₃ Automorphic Fundamental Solution

Let \( G = \text{SL}_3(\mathbb{R}) \), \( K = \text{SO}(3) \) and \( \Gamma = \text{SL}_3(\mathbb{Z}) \).

\[
    u_w = \sum_{\text{cfm } F} \frac{\overline{F}(x_0)}{(\lambda_F - \lambda_w)^\nu} \cdot F + \frac{1}{\langle 1, 1 \rangle (\lambda_1 - \lambda_w)^\nu}
\]

\[
    + \frac{1}{|W|} \int_{\rho + i\alpha^*} \frac{E_{\chi_\mu}(x_0)}{(\lambda_\chi - \lambda_w)^\nu} \cdot E_{\chi_\mu} \, d\mu
\]

\[
    + \sum_{\text{GL}_2 \text{ cfms } f} \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{f,1-s}(x_0)}{(\lambda_{f,s} - \lambda_w)^\nu} \cdot E_{f,s} \, ds
\]

For each \( \text{GL}_2 \) cusp form \( f \) in the chosen orthonormal basis, the corresponding integral has \textit{two branch points} on the critical line.
Acknowledgements

The author thanks Paul Garrett, for posing this problem, Jeffrey Lagarias, for suggesting further work on the problem, and both of them for helpful questions and conversations. The author was partially supported by a research grant from the University of St. Thomas.

For a preprint, see arXiv:1401.2015 [math.NT] or visit

http://personal.stthomas.edu/dece4515