Hybrid Riemannian Conjugate Gradient Methods with Global Convergence Properties

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Abstract This paper presents new Riemannian conjugate gradient methods and global convergence analyses under the strong Wolfe conditions. The main idea of the new methods is to combine the good global convergence properties of the Dai-Yuan method with the efficient numerical performance of the Hestenes-Stiefel method. The proposed methods compare well numerically with the existing methods for the Rayleigh quotient minimization problem on the unit sphere. Numerical comparisons show that they perform better than the existing ones.

Keywords conjugate gradient method · Riemannian optimization · hybrid conjugate gradient method · global convergence · strong Wolfe conditions

1 Introduction

This paper focuses on the conjugate gradient method. Nonlinear conjugate gradient methods in Euclidean space are a class of important methods for solving unconstrained optimization problems. In [8], Hestenes and Stiefel developed a conjugate gradient method for solving linear systems with a symmetric positive-definite matrix of coefficients. In [6], Fletcher and Reeves extended the conjugate gradient method to unconstrained nonlinear optimization problems. Theirs is the first nonlinear conjugate gradient method in Euclidean space. Al-Baali [2] indicated that the Fletcher-Reeves method converges globally and generates the descent direction with an inexact line

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search when the step size satisfies the strong Wolfe conditions \[16,17].\] Polak and Ribière \[10]\] introduced a conjugate gradient method with good numerical performance. Dai and Yuan \[4]\] introduced a conjugate gradient method with a better global convergence property than that of the Fletcher-Reeves method. Hestenes-Stiefel and Polak-Ribière-Polyak methods do not always converge under the strong Wolfe conditions, and for this reason, hybrid conjugate gradient methods have been presented in \[5,9,15]. Touati-Ahmed and Storey \[15]\] and Hu and Storey \[9]\] proposed methods combining the Fletcher-Reeves and Polak-Ribière-Polyak method. Moreover, Dai and Yuan \[5]\] proposed the hybrid conjugate gradient method, which combines the Dai-Yuan method and the Hestenes-Stiefel method. These nonlinear conjugate gradient methods in Euclidean space are summarized by Hager and Zhang in \[7].\]

The conjugate gradient method in Euclidean space is applicable to a Riemannian manifold. In \[14]\], Smith introduced the notion of Riemannian optimization using the exponential map and parallel translation. However, using the exponential map or parallel translation on the Riemannian manifold is generally not computationally effective. Absil, Mahony, and Sepulchre \[1]\] proposed to use a mapping called a retraction that approximates the exponential map. Moreover, they introduced the notion of vector transport, which approximates parallel translation. In addition, Ring and Wirth \[11]\] introduced generalized line search methods (e.g., the Wolfe conditions \[16,17]\) on Riemannian manifolds.

Using the retraction and vector transport, Ring and Wirth \[11]\] presented the Fletcher-Reeves type nonlinear conjugate gradient method on Riemannian manifolds. They indicated that the Fletcher-Reeves methods have a global convergence property under the strong Wolfe conditions. However, their convergence analysis assumed that the vector transport does not increase the norm of the search direction vector, which is not the standard assumption (see \[13\] Section 5). To remove this unnatural assumption, Sato and Iwai \[13]\] introduced the notion of scaled vector transport \[13, Definition 2.2\]. They proved that by using scaled vector transport, the Fletcher-Reeves method on the Riemannian manifold generates a descent direction at every iteration and converges globally without impractical assumptions. Similarly, in \[12]\], Sato used scaled vector transport in a convergence analysis. He indicated that the Dai-Yuan type Riemannian conjugate gradient method generates a descent direction at every iteration and converges globally under the Wolfe conditions. This means that the Dai-Yuan method has a better global convergence property than that of the Fletcher-Reeves method on Riemannian manifolds.

In this paper, we propose hybrid Riemannian conjugate gradient methods exploiting the idea used in the paper \[5\]. Our methods combine the good numerical performance of the Hestenes-Stiefel method with the efficient global convergence property of the Dai-Yuan method. Moreover, we present convergence analyses of our methods. The proof is along the lines of \[5, Theorem 2.3\], except that the step-size assumption is stronger than that of the Euclidean case. This is due to the use of scaled vector transport. Our hybrid methods converge globally if the size of the parameter, which is used to determine the search direction, with respect to that of the Dai-Yuan method is in a certain range (Theorem 2). We provide two examples which satisfy such a condition. In numerical experiments, we show that our hybrid methods outperform the Dai-Yuan and Polak-Ribière-Polyak methods.
This paper is organized as follows. Section 2 reviews the fundamentals of Riemannian geometry and Riemannian optimization. Section 3 proposes the hybrid Riemannian conjugate gradient methods and presents global convergence analyses for them. Section 4 compares our methods with the existing Riemannian conjugate gradient methods through numerical experiments. Section 5 concludes the paper with mention of future work.

2 Riemannian Conjugate Gradient Methods

Let us start by reviewing the nonlinear conjugate gradient methods in Euclidean space. The search direction $\eta_k$ of the nonlinear conjugate gradient method is determined by

$$\eta_0 = -\nabla f(x_0)$$

and

$$\eta_{k+1} = -\nabla f(x_{k+1}) + \beta_{k+1} \eta_k,$$  \hspace{1cm} (1)

where $x_0 \in \mathbb{R}^n$, $\beta_0 = 0$, and $\beta_k$ is a parameter to be suitably defined. Well-known formulas for $\beta_k$ are the Fletcher-Reeves (FR) [6], Dai-Yuan (DY) [4], Polak-Ribière-Polyak (PRP) [10], and Hestenes-Stiefel (HS) [8] formulas, given by

$$\beta_{FR} = \frac{||\nabla f(x_k)||^2}{||\nabla f(x_{k-1})||^2},$$  \hspace{1cm} (2)

$$\beta_{DY} = \frac{\nabla f(x_k) - \nabla f(x_{k-1})}{\eta_{k-1}^T y_{k-1}},$$  \hspace{1cm} (3)

$$\beta_{PRP} = \frac{\nabla f(x_k)^T y_{k-1}}{||\nabla f(x_{k-1})||^2},$$  \hspace{1cm} (4)

$$\beta_{HS} = \frac{\nabla f(x_k)^T y_{k-1}}{\eta_{k-1}^T y_{k-1}}.$$  \hspace{1cm} (5)

respectively, where $y_{k-1} = \nabla f(x_k) - \nabla f(x_{k-1})$.

In the Euclidean space setting, a line search optimization algorithm updates the current iterate $x_k$ to the next iterate $x_{k+1}$ with the updating formula,

$$x_{k+1} = x_k + \alpha_k \eta_k,$$  \hspace{1cm} (6)

where $\alpha_k > 0$ is a positive step size. One often chooses a step size $\alpha_k > 0$ to satisfy the Wolfe conditions [16][17], namely,

$$f(x_k + \alpha_k \eta_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T \eta_k,$$  \hspace{1cm} (7)

$$\nabla f(x_k + \alpha_k \eta_k)^T \eta_k \geq c_2 \nabla f(x_k)^T \eta_k$$  \hspace{1cm} (8)

where $0 < c_1 < c_2 < 1$. When the step size satisfies the following condition, which is a substitute of (8):

$$|\nabla f(x_k + \alpha_k \eta_k)^T \eta_k| \leq c_2 |\nabla f(x_k)^T \eta_k|,$$  \hspace{1cm} (9)

we call (7) and (9), the strong Wolfe conditions.
In [5], Dai and Yuan proved that the method defined by (1) and (6) produces a descent search direction at every iteration and converges globally if the step size $\alpha_k > 0$ satisfies (7) and (8), and $\beta_k$ satisfies
\[-\sigma \leq \frac{\beta_k}{\beta_0} \leq 1,
\]
where $\sigma := (1 - c_2)/(1 + c_2)$ and $c_2$ is a constant in the second condition (8). In this paper, we extend these choices of the parameter $\beta_k$ to Riemannian manifolds.

Now we will briefly outline Riemannian optimization, especially the Riemannian conjugate gradient method, by summarizing [1]. Moreover, we will introduce relevant notation of Riemannian geometry.

Let $(M, g)$ be a Riemannian manifold with a Riemannian metric $g$, and let $T_x M$ be the tangent vector space of $M$ at a point of $x \in M$. In addition, let $T M$ be the tangent bundle of $M$, which is defined by $T M = \bigcup_{x \in M} T_x M$. Let $f : M \rightarrow \mathbb{R}$ be a smooth objective function. Throughout this paper, to simplify the notation, we will write the Riemannian metric $g(\cdot, \cdot)$ as $\langle \cdot, \cdot \rangle$.

An unconstrained optimization problem on a Riemannian manifold $M$ is expressed as follows:

**Problem 1** Let $f : M \rightarrow \mathbb{R}$ be smooth. Then, we would like to
\[
\text{Minimize } f(x) \quad \text{subject to } x \in M,
\]

In order to generalize line search optimization algorithms to Riemannian manifolds, we will use the notions of a retraction and vector transport (see [1]), which are defined as follows:

**Definition 1 (Retraction)** Let $M$ be a manifold and $T M$ be a tangent bundle of a manifold $M$. Any smooth map $R : T M \rightarrow M$ is called a retraction on $M$, if it has the following properties.

- $R_x(0_x) = x$, where $0_x$ denotes the zero element of $T_x M$;
- With the canonical identification $T_0_x T_x M \simeq T_x M$, $R_x$ satisfies $DR_x(0_x)[\xi] = \xi$ for all $\xi \in T_x M$,

where $R_x$ denotes the restriction of $R$ to $T_x M$, and $DR$ is the differential of $R$ (see [1] Section 3).

**Definition 2 (Vector transport)** Let $M$ be a manifold and $T M$ be a tangent bundle of $M$. Any smooth map $T : T M \oplus T M \rightarrow T M$, where $\oplus$ denotes the Whitney sum, is called a vector transport on $M$, if it has the following properties.

- There exists a retraction $R$, called the retraction associated with $T$, such that $T_{\eta}(\xi) \in T_{R_\eta(\xi)} M$ for all $x \in M$, and for all $\eta, \xi \in T_x M$;
- $T_{0_x}(\xi) = \xi$ for all $\xi \in T_x M$. 


where $T_{\eta}(\xi)$ denotes $T(\eta, \xi)$.

In this paper, we will focus on the differentiated retraction $T^R$ as a vector transport, defined by

$$T^R_{\eta}(\xi) := DR_x(\eta)[\xi] \quad (\xi \in T_xM).$$

where $x \in M$ and $\eta \in T_xM$. It is easy to prove that $T^R$ satisfies the properties of Definition 1 (see [1], Chapter 8).

In Riemannian optimization, by using a retraction $R$ and vector transport $T$ on $M$, we can generalize the updating formula (6) and the search direction of the conjugate gradient method (1) to, respectively

$$x_{k+1} = R_{x_k}(\alpha_k \eta_k),$$

$$\eta_{k+1} = -\langle \nabla f(x_{k+1}), \beta_{k+1}T_{\eta_k}(\eta_k) \rangle,$$

(see [1]), where $\alpha_k > 0$ is a positive step size. We call the search direction $\eta_k$ a descent direction, if $\eta_k$ satisfies

$$\langle \nabla f(x_k), \eta_k \rangle_{x_k} < 0.$$

Moreover, the line search conditions (7) and (8) can be generalized to Riemannian manifolds as follows:

$$f(R_{x_k}(\alpha_k \eta_k)) \leq f(x_k) + c_1 \alpha_k \langle \nabla f(x_k), \eta_k \rangle_{x_k},$$

(13)

$$\langle \nabla f(R_{x_k}(\alpha_k \eta_k)), DR_{x_k}(\alpha_k \eta_k)[\eta_k] \rangle_{R_{x_k}(\alpha_k \eta_k)} \geq c_2 \langle \nabla f(x_k), \eta_k \rangle_{x_k},$$

(14)

where $0 < c_1 < c_2 < 1$ (see [12, 13]). We call (13) the Armijo condition. Moreover, the second of the strong Wolfe conditions (9) can be rewritten as

$$\left| \langle \nabla f(R_{x_k}(\alpha_k \eta_k)), DR_{x_k}(\alpha_k \eta_k)[\eta_k] \rangle_{R_{x_k}(\alpha_k \eta_k)} \right| \leq c_2 \left| \langle \nabla f(x_k), \eta_k \rangle_{x_k} \right|.$$ (15)

We call (13) and (15) the strong Wolfe conditions.

Sato and Iwai [13] introduced the notion of scaled vector transport. A scaled vector transport of the $k$-th iterate $T^{(k)}$ associated with $T^R$ is defined by

$$T^{(k)}_{\alpha_k \eta_k}(\eta_k) := \begin{cases} T^R_{\alpha_k \eta_k}(\eta_k), \\ \|\eta_k\|_{x_k} \end{cases} \quad \text{if} \quad \|T^R_{\alpha_k \eta_k}(\eta_k)\|_{x_k+1} \leq ||\eta_k\|_{x_k},$$

$$T^R_{\alpha_k \eta_k}(\eta_k), \quad \text{otherwise.}$$

(16)

Note that scaled vector transport does not satisfy the properties of Definition 2. Thus, we cannot call this vector transport with mathematical exactitude; however, by using scaled vector transport, we often obtain good convergence properties for the Riemannian conjugate gradient methods.
Scaled vector transport $T^{(k)}$ satisfies the following inequalities:

$$\left| \left( \nabla f(x_{k+1}), T_{\alpha_k, \eta_k}^{(k)} (\eta_k) \right)_{x_{k+1}} \right| \leq \left| \left( \nabla f(x_{k+1}), T_{\alpha_k, \eta_k}^R (\eta_k) \right)_{x_{k+1}} \right|$$  \hspace{1cm} (17)

and

$$\left\| T_{\alpha_k, \eta_k}^{(k)} (\eta_k) \right\|_{x_{k+1}} \leq \eta_k \| x_k \|.$$  \hspace{1cm} (18)

Now, we would like to verify that inequality (17) holds. From the definition of scaled vector transport (16), we obtain

$$\left| \left( \nabla f(x_{k+1}), T_{\alpha_k, \eta_k}^{(k)} (\eta_k) \right)_{x_{k+1}} \right| = \left| \left( \nabla f(x_{k+1}), s^{(k)} T_{\alpha_k, \eta_k}^R (\eta_k) \right)_{x_{k+1}} \right|,$$

where $s^{(k)}$ denotes

$$s^{(k)} := \min \left\{ 1, \frac{\| \eta_k \|_x}{\| T_{\alpha_k, \eta_k}^R (\eta_k) \|_{x_{k+1}}} \right\} \leq 1.$$

Therefore, it follows that

$$\left| \left( \nabla f(x_{k+1}), T_{\alpha_k, \eta_k}^{(k)} (\eta_k) \right)_{x_{k+1}} \right| = s^{(k)} \left| \left( \nabla f(x_{k+1}), T_{\alpha_k, \eta_k}^R (\eta_k) \right)_{x_{k+1}} \right| \leq \left| \left( \nabla f(x_{k+1}), T_{\alpha_k, \eta_k}^R (\eta_k) \right)_{x_{k+1}} \right|,$$

which leads to (17). Obviously, (16) implies (18).

Throughout this paper, we will replace vector transport $T$ by scaled vector transport $T^{(k)}$ in (12). Therefore, the search direction of the Riemannian conjugate gradient method is determined by

$$\eta_{k+1} = -\nabla f(x_{k+1}) + \beta_k T_{\alpha_k, \eta_k}^{(k)} (\eta_k).$$  \hspace{1cm} (19)

In (19), $\beta_k$ is also given by generalizations of the formulas (2), (3), (4), and (5), i.e.,

$$\beta_k^{FR} = \frac{\| \nabla f(x_k) \|_{x_k}^2}{\| \nabla f(x_{k-1}) \|_{x_{k-1}}^2},$$  \hspace{1cm} (20)

$$\beta_k^{DY} = \frac{\| \nabla f(x_k) \|_{x_k}^2}{\left( \nabla f(x_k), T_{\alpha_{k-1}, \eta_{k-1}}^{(k-1)} (\eta_{k-1}) \right)_{x_k} - (\nabla f(x_{k-1}), \eta_{k-1})_{x_{k-1}}},$$  \hspace{1cm} (21)

$$\beta_k^{PRP} = \frac{\left( \nabla f(x_k), \nabla f(x_k) - T_{\alpha_{k-1}, \eta_{k-1}}^{(k-1)} (\nabla f(x_{k-1})) \right)_{x_k}}{\| \nabla f(x_{k-1}) \|_{x_{k-1}}^2},$$  \hspace{1cm} (22)

$$\beta_k^{HS} = \frac{\left( \nabla f(x_k), \nabla f(x_k) - T_{\alpha_{k-1}, \eta_{k-1}}^{(k-1)} (\nabla f(x_{k-1})) \right)_{x_k}}{\left( \nabla f(x_k), T_{\alpha_{k-1}, \eta_{k-1}}^{(k-1)} (\eta_{k-1}) \right)_{x_k} - (\nabla f(x_{k-1}), \eta_{k-1})_{x_{k-1}}}. $$  \hspace{1cm} (23)
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We call these formulas the Fletcher-Reeves, Dai-Yuan, Polak-Ribière-Polyak and Hestenes-Stiefel formulas, respectively. In the next section, we propose a new choice of $\beta_k$.

In [13], Sato and Iwai proved that using the scaled vector transport $T^{(k)}$ substitute of $T$ in [12] and the step size which satisfies the strong Wolfe conditions (13) and (15), the Fletcher-Reeves type conjugate gradient method defined by (11), (19), and (20) generates sequences that converge globally. Similarly, in [12], Sato indicated that if we use scaled vector transport, with a step size satisfying the Wolfe conditions, (13) and (14), the Dai-Yuan type conjugate gradient method defined by (11), (19), and (21) generates globally convergent sequences.

3 Riemannian Hybrid Conjugate Gradient Method and Its Global Convergence Analysis

3.1 Proposed hybrid Riemannian conjugate gradient method

This section describe the new Riemannian conjugate gradient descent method using a hybrid $\beta_k$, exploiting the idea used in [5].

Let $r_k$ be the size of $\beta_k$ with respect to $\beta_{DY}^k$ defined by (21), namely,

$$r_k := \frac{\beta_k}{\beta_{DY}^k}.$$  \hspace{1cm} (24)

We will prove that, for the method defined by (11) and (19), the search direction $\eta_k$ is the descent direction at every iteration and the method converges globally if the step size $\alpha_k > 0$ satisfies the strong Wolfe conditions (13) and (15), and the scalar $\beta_k$ is such that

$$-\sigma \leq r_k \leq 1,$$  \hspace{1cm} (25)

where $\sigma := (1 - c_2)/(1 + c_2) > 0$ and $c_2$ denotes the constant in the second of the strong Wolfe conditions (15). Furthermore, since the following two choices of $\beta_k$:

$$\beta_k = \max \{0, \min \{\beta_{DY}^k, \beta_{HS}^k\}\}$$  \hspace{1cm} (26)

and

$$\beta_k = \max \{-\sigma \beta_{DY}^k, \min \{\beta_{DY}^k, \beta_{HS}^k\}\}$$  \hspace{1cm} (27)

satisfy the condition (25), we can use either of these hybrid formulas $\beta_k$ defined by (26) and (27) as the scalar in (19). The above two choices of $\beta_k$ are examples of the hybrid methods in Euclidean space [5]. This implies that our hybrid method is a generalization of the method in [5]. The hybrid methods using (26) and (27) combine the good global convergence properties of the Dai-Yuan method (21) with the efficient numerical performance of the Hestenes-Stiefel method (23). Now, we note that, in
Euclidean space, the hybrid methods using (26) and (27) converge globally under the Wolfe conditions (7) and (8), whereas, on a Riemannian manifold, the hybrid methods need the strong Wolfe conditions (13) and (15) to converge globally. In Section 4, we provide a numerical evaluation showing that the Riemannian conjugate gradient methods with the hybrid \( \beta_k \) defined by (26) and (27) perform better than the Polak-Ribière-Polyak method.

3.2 Global convergence analysis

Zoutendijk’s theorem is described on Riemannian manifolds as follows:

**Theorem 1 (Zoutendijk [11])** Let \((M, g)\) be a Riemann manifold and \(R\) be a retraction on \(M\). Let \(f : M \to \mathbb{R}\) be a smooth, bounded below function with the following property: there exists \(L > 0\) such that

\[
|D(f \circ R)(\eta)[\eta] - D(f \circ R_x)(0)[\eta]| \leq Lt \quad (\eta \in T_xM, \quad \|\eta\|_x = 1, \quad x \in M, \quad t \geq 0).
\]

Suppose that in the line search optimization algorithm (11), each step size \(\alpha_k > 0\) satisfies the strong Wolfe conditions (13) and (15), and each search direction \(\eta_k\) is a descent direction. Then the following series converges:

\[
\sum_{k=0}^{\infty} \frac{\langle \text{grad} f(x_k), \eta_k \rangle_{x_k}^2}{\|\eta_k\|^2_{x_k}} < \infty. \tag{28}
\]

The proof of this theorem is along the lines of Zoutendijk’s theorem in Euclidean space (see [11, Theorem 3.3]). Next, we will prove the main convergence theorem.

**Theorem 2** Let \(f : M \to \mathbb{R}\) be a function satisfying the assumptions of Zoutendijk’s theorem. If each \(\alpha_k > 0\) satisfies the strong Wolfe conditions (13) and (15), and if \(\beta_k\) is such that \(-\sigma \leq r_k \leq 1\), then any sequence \(\{x_k\}\) generated by the Riemannian conjugate gradient method defined by (11) and (19) satisfies

\[
\liminf_{k \to \infty} \|\text{grad} f(x_k)\|_{x_k} = 0. \tag{29}
\]

**Proof** If \(\text{grad} f(x_{k_0}) = 0\) for some \(k_0\), (29) in such a case. Thus, it is sufficient to prove (29) only when \(\text{grad} f(x_k) \neq 0\) for all \(k \geq 0\).

First, we prove that each search direction \(\eta_k\) is a descent direction by induction. For \(\eta_0 = -\text{grad} f(x_0)\), it is obvious that \(\eta_0\) is a descent direction.

\(^3\) The formulas defined by (26) and (27) satisfy \(-\sigma \leq r_k \leq 1\).
Assume that η_{k-1} is a descent direction. Then, we find that

\[
\langle \nabla f(x_k), \eta_k \rangle_{x_k} = \left\langle \nabla f(x_k), \nabla f(x_k) + \beta_k \nabla T_{\alpha_{k-1} \eta_{k-1}}^{(k-1)}(\eta_{k-1}) \right\rangle_{x_k} \\
= -\|\nabla f(x_k)\|^2 + r_k \left[ \left\langle \nabla f(x_k), \nabla T_{\alpha_{k-1} \eta_{k-1}}^{(k-1)}(\eta_{k-1}) \right\rangle_{x_k} - \langle \nabla f(x_k-1), \eta_{k-1} \rangle_{x_{k-1}} \right] \\
= \left\langle \nabla f(x_k-1), \eta_{k-1} \right\rangle_{x_{k-1}} + (r_k - 1) \left\langle \nabla f(x_k), \nabla T_{\alpha_{k-1} \eta_{k-1}}^{(k-1)}(\eta_{k-1}) \right\rangle_{x_k} \beta_k^{DY},
\]

where the first equation comes from (19) and the second equation comes from \( \beta_k = r_k \beta_k^{DY} \) and (21). Accordingly, (21) ensures that

\[
\langle \nabla f(x_k), \eta_k \rangle_{x_k} = \left\langle \nabla f(x_k-1), \eta_{k-1} \right\rangle_{x_{k-1}} + (r_k - 1) \left\langle \nabla f(x_k), \nabla T_{\alpha_{k-1} \eta_{k-1}}^{(k-1)}(\eta_{k-1}) \right\rangle_{x_k} \beta_k^{DY},
\]

which, together with (24), implies that

\[
\beta_k = r_k \beta_k^{DY} \\
= \frac{r_k \langle \nabla f(x_k), \eta_k \rangle_{x_k}}{\langle \nabla f(x_k-1), \eta_{k-1} \rangle_{x_{k-1}} + (r_k - 1) \left\langle \nabla f(x_k), \nabla T_{\alpha_{k-1} \eta_{k-1}}^{(k-1)}(\eta_{k-1}) \right\rangle_{x_k}}.
\]

Let \( l_k \) and \( \xi_k \) be

\[
l_k := \frac{\left\langle \nabla f(x_k), \nabla T_{\alpha_{k-1} \eta_{k-1}}^{(k-1)}(\eta_{k-1}) \right\rangle_{x_k}}{\langle \nabla f(x_k-1), \eta_{k-1} \rangle_{x_{k-1}}}, \quad \xi_k := \frac{r_k}{1 + (r_k - 1)l_k}.
\]
Using (31) and (32), we obtain
\begin{align*}
\beta_k &= r_k \beta^D_k \\
&= r_k \frac{\langle \nabla f(x_k), \eta_k \rangle_{x_k}}{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}} \\
&= \frac{r_k}{1 + (r_k - 1)\frac{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}}{\langle \nabla f(x_k), \eta_k \rangle_{x_k}}} \\
&= \frac{r_k}{1 + (r_k - 1)\frac{\langle \nabla f(x_k), \eta_k \rangle_{x_k}}{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}}} = \eta_{k-1} \frac{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}}{\langle \nabla f(x_k), \eta_k \rangle_{x_k}}.
\end{align*}

Furthermore, let \( \zeta_k \) be
\begin{equation}
\zeta_k := \frac{1 + (r_k - 1)\eta_k}{\eta_k - 1}.
\end{equation}

Then, (30) guarantees that
\begin{align*}
\langle \nabla f(x_k), \eta_k \rangle_{x_k} \\
&= 1 + (r_k - 1)\frac{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}}{\langle \nabla f(x_k), \eta_k \rangle_{x_k}} ||\nabla f(x_k)||^2_{x_k} \\
&= 1 + (r_k - 1)\frac{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}}{\langle \nabla f(x_k), \eta_k \rangle_{x_k}} ||\nabla f(x_k)||^2_{x_k} - 1 \\
&= \zeta_k ||\nabla f(x_k)||^2_{x_k}.
\end{align*}

On the other hand, since \( \alpha_k \) satisfies the strong Wolfe conditions, (15) implies that
\begin{equation}
| \langle \nabla f(x_{k-1}), D R_{x_{k-1}}(\alpha_{k-1} \eta_{k-1}) \rangle_{x_{k-1}} | \leq c_2 | \langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}} |
\end{equation}
which, together with (31), (17) and (10), implies that
\begin{align*}
|\eta_k| &= \frac{\langle \nabla f(x_k), \nabla R^{(k-1)}_{x_{k-1}}(\eta_{k-1}) \rangle_{x_{k-1}}}{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}} \\
&\leq \frac{\langle \nabla f(x_k), \nabla R_{x_{k-1}}(\alpha_{k-1} \eta_{k-1}) \rangle_{x_{k-1}}}{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}} \\
&= \frac{\langle \nabla f(x_k), D R_{x_{k-1}}(\alpha_{k-1} \eta_{k-1}) \rangle_{x_{k-1}}}{\langle \nabla f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}} \leq c_2.
\end{align*}
This means $|l_k| \leq c_2 < 1$. From the inequality $l_k \leq c_2 < 1$, and the assumption $-\sigma \leq r_k \leq 1$, it follows that
\[
1 + (r_k - 1)l_k \geq 1 + (-\sigma - 1) c_2
\]
\[
= 1 + \frac{1 - c_2}{1 + c_2} - 1)c_2
\]
\[
= \frac{1 - c_2}{1 + c_2} > 0,
\]
which leads to
\[
\begin{cases}
l_k - 1 < 0, \\
1 + (r_k - 1)l_k > 0.
\end{cases}
\]
Hence,
\[
\zeta_k := \frac{1 + (r_k - 1)l_k}{l_k - 1} < 0,
\]
which, together with (35), implies that $\eta_k$ is a descent direction. Thus, induction shows that each $\eta_k$ is a descent direction.

Finally, we prove (29) by contradiction. Assume that
\[
\lim_{k \to \infty} \|\text{grad} f(x_k)\|_{x_k} > 0.
\]
Then, noting $\|\text{grad} f(x_k)\|_{x_k} \neq 0$ for all $k$, there exists $\gamma > 0$ such that
\[
\|\text{grad} f(x_k)\|_{x_k} \geq \gamma > 0.
\]
for all $k$. Since (19) means that
\[
\eta_k + \text{grad} f(x_k) = \beta_k \tau_{(k-1)}^{\alpha_{k-1}}(\eta_{k-1}).
\]
Taking the norms of the above equation and its square, it follows that
\[
\|\eta_k\|_{x_k}^2 = \beta_k^2 \|\tau_{(k-1)}^{\alpha_{k-1}}(\eta_{k-1})\|_{x_k}^2 - 2 \text{scal}(\text{grad} f(x_k), \eta_k)_{x_k} - \|\text{grad} f(x_k)\|_{x_k}^2.
\]
Dividing both sides of the above equation by $\langle \text{grad} f(x_k), \eta_k \rangle_{x_k}^2 \neq 0$, it follows that from (35) and (33),
\[
\frac{\|\eta_k\|_{x_k}^2}{\langle \text{grad} f(x_k), \eta_k \rangle_{x_k}^2} = \beta_k^2 \frac{\|\tau_{(k-1)}^{\alpha_{k-1}}(\eta_{k-1})\|_{x_k}^2}{\langle \text{grad} f(x_k), \eta_k \rangle_{x_k}^2}
\]
\[
- \frac{2}{\langle \text{grad} f(x_k), \eta_k \rangle_{x_k}} - \frac{\|\text{grad} f(x_k)\|_{x_k}^2}{\langle \text{grad} f(x_k), \eta_k \rangle_{x_k}^2}
\]
\[
= \xi_k^2 \frac{\|\tau_{(k-1)}^{\alpha_{k-1}}(\eta_{k-1})\|_{x_k}^2}{\langle \text{grad} f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}^2}
\]
\[
- \frac{1}{\|\text{grad} f(x_k)\|_{x_k}^2} \left(1 + \frac{2}{\xi_k^2} + \frac{1}{\xi_k^2}\right)
\]
\[
= \xi_k^2 \frac{\|\tau_{(k-1)}^{\alpha_{k-1}}(\eta_{k-1})\|_{x_k}^2}{\langle \text{grad} f(x_{k-1}), \eta_{k-1} \rangle_{x_{k-1}}^2}
\]
\[
+ \frac{1}{\|\text{grad} f(x_k)\|_{x_k}^2} \left(1 - \left(1 + \frac{1}{\xi_k^2}\right)^2\right).
\]
Namely,
\[
\|\eta_k\|_{x_k}^2 \leq \xi_k^2 \frac{\|T_{x_{k-1}}^{(k-1)}(\eta_{k-1})\|_{x_k}^2}{\|\nabla f(x_{k-1}), \eta_{k-1}\|_{x_{k-1}}^2} + \frac{1}{\|\nabla f(x_k)\|_{x_k}^2} \left\{ 1 - \left( 1 + \frac{1}{\xi_k} \right)^2 \right\}
\]  
(37)

The assumption \((-\sigma \leq) r_k \leq 1\) and \(l_k < 1\) lead to
\[
1 + (r_k - 1)l_k \geq -r_k, \quad \text{i.e.,} \quad -\{1 + (r_k - 1)l_k\} \leq r_k.
\]  
(38)

In addition, from \(l_k < 1\) and \(r_k \leq 1\), we have
\[
(1 - r_k)(1 - l_k) \geq 0, \quad \text{i.e.,} \quad 1 + (r_k - 1)l_k \geq r_k.
\]  
(39)

From (38) and (39), \(r_k\) satisfies
\[
|1 + (r_k - 1)l_k| \geq |r_k|.
\]

Which, together with (32), implies
\[
|\xi_k| \leq 1.
\]

From the above inequality with (37) and (38), we obtain
\[
\|\eta_k\|_{x_k}^2 \leq \|T_{x_{k-1}}^{(k-1)}(\eta_{k-1})\|_{x_k}^2 + \frac{1}{\|\nabla f(x_k)\|_{x_k}^2} \leq \|\eta_{k-1}\|_{x_{k-1}}^2 + \frac{1}{\|\nabla f(x_k)\|_{x_k}^2}.
\]

Using the above inequality recursively and noting hypothesis, \(\|\nabla f(x_k)\|_{x_k} \geq \gamma > 0\), and \(\|\eta_0\|_{x_0} = \|\nabla f(x_0)\|_{x_0}\), it follows
\[
\|\eta_k\|_{x_k}^2 \leq \sum_{i=0}^{k} \frac{1}{\|\nabla f(x_i)\|_{x_i}^2} \leq \sum_{i=0}^{k} \frac{1}{\gamma^2} = \frac{k + 1}{\gamma^2}.
\]

This means
\[
\frac{\|\nabla f(x_k), \eta_k\|_{x_k}^2}{\|\eta_k\|_{x_k}^2} \geq \frac{\gamma^2}{k + 1},
\]
which indicates
\[
\sum_{k=0}^{\infty} \frac{\|\nabla f(x_k), \eta_k\|_{x_k}^2}{\|\eta_k\|_{x_k}^2} \geq \sum_{k=0}^{\infty} \frac{\gamma^2}{k + 1} = \infty.
\]

This contradicts (23) in Zoutendijk’s theorem and completes the proof. \(\square\)
4 Numerical Experiments

This section compares the performances of the existing Riemannian conjugate gradient methods with those of the proposed methods. In our experiments, we consider the objective function called the Rayleigh quotient, which is described as

$$f : \mathbb{R}^n_* \to \mathbb{R} : x \mapsto \frac{x^T Ax}{x^T x}.$$ 

where $\mathbb{R}^n_*$ denotes an origin removed $n$-dimensional Euclidean space and $A \in \mathbb{R}^{n \times n}$ is a symmetric positive-definite matrix. We consider the following Rayleigh quotient minimization problem on the unit sphere $S^{n-1} := \{ x \in \mathbb{R}^n \| x \| = 1 \}$ (see [1, Chapter 4]), which is defined as follows:

**Problem 2**

Minimize $x^T Ax$ subject to $x \in S^{n-1}$.

On the unit sphere, the Rayleigh quotient is written as $x^T Ax$, since the numerator in the Rayleigh quotient is identical to 1. The optimal solutions of Problem 2 are the unit eigenvectors of $A$ associated with the smallest eigenvalue (see [1, Chapter 2]). We assume that the unit sphere is endowed with the natural Riemannian metric, namely,

$$\langle \xi, \eta \rangle_x := \xi^T \eta \quad (x \in S^{n-1}, \xi, \eta \in T_x S^{n-1}).$$

Next, the gradient of the objective function $f$ can be written as

$$\nabla f(x) = 2(I_n - xx^T)Ax \quad (x \in S^{n-1}),$$

where $I_n$ is an $n \times n$ identity matrix. We use a retraction $R$ defined by

$$R_x(\xi) := \frac{x + \xi}{\|x + \xi\|} \quad (x \in S^{n-1}, \xi \in T_x S^{n-1})$$

and the associated differentiated retraction written as

$$T^R_R(\xi) = \frac{1}{\|x + \eta\|} \left( I_n - \frac{(x + \eta)(x + \eta)^T}{\|x + \eta\|} \right) \xi \quad (x \in S^{n-1}, \eta, \xi \in T_x S^{n-1}).$$

In our experiments, we used the scaled vector transport [16] associated with $T^R$, to determine the search direction [19] in the Riemannian conjugate gradient method. We set $n = 100$ and used line search algorithms for the strong Wolfe conditions [13] and [15] with $c_1 = 0.0001$ and $c_2 = 0.9$. We set the initial point $x_0$ as

$$x_0 = \frac{1}{\sqrt{n}}(1, \cdots, 1)^T \in S^{n-1}.$$ 

We determined that a sequence had converged to an optimal solution if the stopping condition

$$\| \nabla f(x_k) \|_{x_k} < 10^{-6}$$

was satisfied.
was satisfied.

The experiments used a MacBook Air (2017) with a 1.8 GHz Intel Core i5, 8 GB 1600 MHz DDR3 memory, and macOS Mojave version 10.14.5 operating system. The algorithms were written in Python 3.7.6 with the NumPy 1.17.3 package and the Matplotlib 3.1.1 package. We modified the strong Wolfe line search provided as scipy.optimize.line_search in the SciPy package, to compute the step size in (11).

For comparison, we chose two Riemannian conjugate gradient methods, i.e., the Dai-Yuan method (21) and the Polak-Ribiére-Polyak method (22). Below, we call the hybrid methods using (26) and (27), Hybrid1 and Hybrid2, respectively. The experiments generated 10 symmetric positive definite matrices with randomly chosen elements by using sklearn.datasets.make_spd_matrix in the scikit-learn 0.21.3 package. Note that the DY method (21) generates a descent direction at every iteration (see [12, Proposition 4.1]). In addition, in the proof of Theorem (2) we proved that the hybrid methods using (26) and (27) also generate a descent direction at every iteration. Therefore, with the DY method and the proposed hybrid methods, a step size which satisfies the strong Wolfe conditions can be computed at every iteration (see [11, Proposition 1]). However, with the PRP method, uphill search directions, namely the search direction $\eta_k$ satisfying $\langle \nabla f(x_k), \eta_k \rangle_{x_k} > 0$, may be generated (see (3)). If the search direction is uphill, a step size that satisfies the strong Wolfe conditions is sometimes not found. In such a case, we set the step size to $\alpha_k = 0.001$.

First, let us examine the results of Problem 2 with the first generated symmetric positive-definite matrix. Fig. 1 and Fig. 2 show the numerical results. Fig. 1 indicates that the hybrid methods converge after less than 600 iterations, while the DY and PRP methods do not converge even after 1000 iterations. Fig. 2 indicates that the proposed hybrid methods converge faster than the existing ones. These results thus show that they perform better than the other Riemannian conjugate gradient methods.

Table 1: Averages of the iterations and the elapsed time of Riemannian conjugate gradient methods until the stopping condition is satisfied.

|          | DY    | PRP   | Hybrid1 | Hybrid2 |
|----------|-------|-------|---------|---------|
| iteration| 2718.5| 1376.1| 321.6   | 319.5   |
| time     | 2.96  | 1.48  | 0.41    | 0.41    |
| time per iteration | $1.08 \times 10^{-3}$ | $1.07 \times 10^{-3}$ | $1.27 \times 10^{-3}$ | $1.28 \times 10^{-3}$ |

Second, we compared the average values of the quantities of the methods. Table 1 lists the average values of the iterations and the computational time required for convergence. The two hybrid methods using (26) and (27) are pretty much equal in performance. The hybrid methods converge to an optimal solution faster than the DY and PRP methods. On average, the hybrid methods require about one-fourth the number of iterations that the PRP method needs. Moreover, on average, the hybrid methods converge after only about one-ninth the number of iterations taken by the DY method. Furthermore, the average computational time required for convergence of the hybrid methods is shorter than that of the DY and PRP methods. We also observe that the hybrid methods require more computational time per iteration than
Fig. 1 Norm of gradient and number of iterations.

Fig. 2 Norm of gradient and elapsed time
the other methods. This is because the hybrid methods using (26) and (27) have to compute both $\beta_k^{DY}$ and $\beta_k^{HS}$ at every iteration, while the DY and PRP methods compute only $\beta_k^{DY}$ or $\beta_k^{PRP}$, respectively. Table 2 shows the results of the t-test between the existing methods and our hybrid methods, where the significance level is set at 5%. All values in the table are less than 0.05. These results indicate that the performances of the proposed hybrid methods are significantly different from those of the existing conjugate gradient methods.

Finally, let us consider the correspondence of our hybrid methods on Riemannian manifolds with the exiting hybrid methods in Euclidean space. In [5], Dai and Yuan indicated that the hybrid methods using (26) and (27) perform better than the PRP method in Euclidean space. Similarly, the above discussion shows that the hybrid methods are better than the PRP method on Riemannian manifolds. This means that our methods constitute a generalization of the Euclidean case.

### 5 Conclusion and Future Work

This paper presented hybrid Riemannian conjugate gradient methods and showed their global convergence properties. It compared them numerically with the existing Riemannian conjugate gradient methods with respect to Rayleigh quotient minimization problems on the unit sphere. The results of the numerical experiments demonstrated the efficiency of the hybrid methods.

Various hybrid conjugate methods have been proposed for Euclidean space, such as

$$\beta_k = \{0, \max\{\beta_k^{PRP}, \beta_k^{FR}\}\}.$$  

The hybrid conjugate methods in Euclidean space are summarized in [7]. We will present more hybrid methods and convergence analyses in a future paper.

### Declarations

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The authors declare that they have no conflict of interest.

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