Dynamic renormalization-group approach to diffusive flow in heterogeneous systems

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(February 8, 2020)

Conventional methods for the simulation of diffusive systems are quite slow when applied to strongly inhomogeneous systems. We present a new hierarchical approach based on dynamic renormalization-group ideas and on the Walsh transform (or Haar wavelet) of signal-processing theory. The method is very efficient for simulation of petroleum reservoirs or other strongly inhomogeneous diffusive or pressure-driven flow systems. In a test case, the hierarchical method is found to achieve 1.5% accuracy roughly 25 times faster than conventional finite difference methods.

Traditional finite-element and finite-difference methods for numerical solution of differential equations have a discretization error that depends on a power of the time or space increment, $\Delta t$ or $\Delta r$. In the case of a diffusive system, stability usually requires $\Delta t$ to be of order $\Delta r^2/D$ (where $D$ is the diffusivity), and the discretization error is proportional to a power of $\Delta r$. In highly inhomogeneous systems, parts of which require very small $\Delta r$ and/or very large diffusivity $D$, this requires a very small $\Delta t$.

The instability for large $\Delta t$ arises when material moves more than one cell diameter during the time interval $\Delta t$; in the usual explicit finite difference (EFD) algorithm, material is allowed to move from a cell only into its nearest neighbors. Our approach solves this problem by allowing material to move across as many cells as necessary. We describe the motion of the material by a discrete Green function or influence function $G_{\Delta t}(d, s)$ such that $G_{\Delta t}(d, s)c(s)$ is the amount of material that moves from a source cell $s$ (whose original material content is $c(s)$) to a destination cell $d$ during the time interval $\Delta t$. If $G_{\Delta t}(d, s)$ is nonzero only when $d$ and $s$ are nearest neighbor cells, this is equivalent to a conventional EFD algorithm. This is the case for sufficiently small $\Delta t$, so we may begin with such an algorithm and coarsen the time scale by doubling $\Delta t$. The Green function for the interval $2\Delta t$ is obtained from that for $\Delta t$ by self-convolution (see Eq. 6 below).

Of course, this repeated convolution process increases the spatial range of the influence function, which is stored in our computer implementation as a linked list; soon the number of destination cells $d$ that can be reached from each source cell $s$ becomes large and the method becomes very time-consuming. However, we can coarsen the space as well as the time scale, by lumping cells together into larger cells. This decreases the number of source cells, as well as the number of destination cells reached from each source cell, and hence the size of the Green-function list. This scheme is motivated by the renormalization-group method which has been so useful in the theory of critical phenomena [1], although our present description does not require prior knowledge of renormalization-group theory. It has been shown [2] that the diffusion problem in a homogeneous system has a fixed point with respect to a combined space-and-time renormalization transformation. That is, we can continue coarsening the space and time scales indeﬁnitely without inductively increasing the size of the Green-function list.

Such cell coarsening is even more useful in an inhomogeneous system, because we can use physical information to choose which cells to lump together. An inhomogeneous system such as an oil reservoir tends to be compartmentalized into compartments within which oil flows fairly freely, separated by relatively impermeable regions. If we lump cells between which there is relatively free flow (high effective diffusivity), when we reach a large scale the cells will be the compartments.

We can visualize the hierarchical lumping of cells by placing the cells on a binary tree as in Fig. 1.

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The disadvantage of this coarsening process, of course, is that we lose spatial resolution in our description of the system. To some extent such a contraction of the description is desired, but we would like to maintain control of our approximations and limit the loss of information. When we replace contents \( c(l) \) and \( c(l') \), of two cells \( l \) and \( l' \) by the coarse-grained content \( c(L) \equiv c(l) + c(l') \) where the larger cell \( L \) is the union of \( l \) and \( l' \), we can avoid losing any information if we also include as a variable the difference \( c(L, 1) \equiv c(l) - c(l') \) as well as the sum \( c(L, 0) \) (the arguments 0 and 1 simply indicate whether a sum or a difference is intended).

We can do this at any level of the tree shown in Fig. 1: the difference between the two halves of the small cell \( l \) can be denoted by \( c(l, 1) \). We can even define a difference of differences \( c(L, 1, 1) \equiv c(l, 1) - c(l', 1) \). Each "1" in this expression can be regarded as one bit of a "Walsh sequency index" \( w \). (The Walsh transform is a discrete signal transform used in electrical engineering \[3, 4\], which we here generalize to a hierarchical system.) We will lump the bits into a single binary integer (Walsh index) \( w \), so \( c(L, 1, 1) \) becomes \( c(L, w) \) with \( w = 3 \) (11 in binary). We can define a general "Walsh variable" recursively by

\[
\begin{align*}
c(L, w0) & \equiv c(l, w) + c(l', w) \\
c(L, w1) & \equiv c(l, w) - c(l', w)
\end{align*}
\]

whenever \( l \) and \( l' \) are the two halves of the cell \( L \). Here the Walsh index \( w0 \) is \( w \) with a zero bit appended at the right, i.e., \( w0 \equiv 2w \); similarly \( w1 \equiv 2w + 1 \). In this notation, our original \( c(l) \) is written \( c(l, 0) \) and serves to start the recursion. The Walsh index plays a role similar to that of the wavenumber in the Fourier transform, in that variables with a small Walsh index describe large-scale structure, whereas large Walsh indices describe small wavelength structure within a cell \( L \). For each "1" bit in the binary representation of \( w \), there is one subtraction in the construction of \( c(l, w) \).

If each cell-lumping replaces two cell contents by two Walsh variables, we never lose variables and our equations remain as complicated as ever (but exact). However, we are now in a position to selectively drop small terms in the Green function: the variables with large Walsh indices (i.e., many differences rather than sums) will be less important than those with small Walsh indices. Our algorithm drops terms from the Green function if they are less than some pre-set tolerance \( \delta \). Typically, terms involving differences are much smaller than those involving sums, so these are the ones dropped. This is the virtue of the hierarchical description: terms describing the effects of a cell content are never negligible compared to the terms for nearby cells, whereas terms describing the effects of differences may be negligible compared to terms for sums. These advantages are similar to those of spectral (Fourier transform) methods in homogeneous systems; in a sense one can regard the Walsh transform as the proper generalization of the Fourier transform to inhomogeneous systems.

Commercial reservoir simulation programs usually treat a reservoir as a three-component system \[5\] (oil, gas, water) in which flow is governed by Darcy’s law. However, to provide a straightforward test of the hierarchical method described above, we will consider only one component (oil). In this case, Darcy’s law reduces to the diffusion equation

\[
\frac{dP(r, t)}{dt} = \nabla \cdot [D(r) \nabla P(r, t)]
\]

where the pressure \( P \) plays the role of the diffusing density; the effective diffusivity \( D(r) \) is proportional to the permeability. So the problem we will actually solve is that of diffusion in a very inhomogeneous system; the inhomogeneity is contained in the function \( D(r) \).

FIG. 1. Sketch of a hierarchically subdivided system (left) and its representation as a binary tree (right). Cell \( L \) is subdivided into \( l \) and \( l' \) as described in the text.
To describe the evolution of the cell content $c_t(d)$ of a cell labeled $d$ (proportional to the "density" $P$), the most straightforward discretization of Eq. 3 is

$$\frac{c_t+\Delta t(d) - c_t(d)}{\Delta t} = \Delta r^{-2} \sum_f D(f)[c_t(d_+(f)) - c_t(d)]$$

(3)

where the sum is over faces $f$ of the cell $d$, and $d_+(f)$ is the cell in front of the (directed) face $f$ (the cell behind it is always $d$). When we lump cells, so some of our variables are Walsh variables $c(d, w)$, we can write the discretization (Eq. 3) in the form

$$c_t+\Delta t(d, w) = \sum_{s, v} G_{\Delta t}(d, w; s, v) c_t(s, v)$$

(4)

where $G_{\Delta t}(d, w; s, v)$ is a Green function or influence function describing the influence of a Walsh variable in the source cell $s$ on one in the destination cell $d$. Before any lumping has occurred, all Walsh variables have $w = 0$, and $G_{\Delta t}(d, w; s, v) = (\Delta t/\Delta r^2) D(f)$ if $d$ and $s$ are neighbors separated by the face $f$ and $w = v = 0$, and zero otherwise. Equation 4 is thus equivalent to an EFD algorithm, which requires a small $\Delta t$; we have used the practical limit of stability $\frac{\Delta t}{\Delta r^2/4D_{\text{max}}}$

(5)

where $D_{\text{max}}$ is the maximum diffusivity in the system.

We now increase $\Delta t$ to $2\Delta t$; the new Green function is the convolution

$$G_{2\Delta t}(d, w; s, v) = \sum_{e, u} G_{\Delta t}(d, w; e, u) G_{\Delta t}(e, u; s, v)$$

(6)

After several such convolutions, the spatial range of the Green function is increased, especially in regions of high diffusivity. Here the pressure (i.e., density) in nearby cells equalizes rapidly—the contents $c$ cells contribute nearly equally to future contents $c$: $G(d, 0; l, 0) \approx G(d, 0; l', 0)$ (the zeroes here are the Walsh indices). To decide whether two cells $l$ and $l'$ should be lumped together, we look at the ratio (using $d = l$)

$$r = \frac{G(l, 0; l', 0)}{G(l, 0; l, 0)}$$

(7)

Generally $r < 1$; we have used $r > 0.91$ as our lumping criterion (see Fig. 3).

When we decide to lump two cells $l$ and $l'$ into a large cell $L$ (as in Fig. 3), we can calculate the new Green function in two stages. In the first stage we calculate elements $G'(d, w; s, v)$ in which the destination cell $d$ takes coarse values (including $L$) but $s$ takes values including $l$ and $l'$. These are the same as the old $G$s unless $d$ is $L$, in which case we obtain from Eq. 3

$$G'(L, w0; s, v) = G(l, w; s, v) + G(l', w; s, v)$$
$$G'(L, w1; s, v) = G(l, w; s, v) - G(l', w; s, v)$$

(8)

where as before $w0$ means $w$ with a zero bit appended. In the second stage, we calculate $G''(d, w; s, v)$ where both $d$ and $s$ take values $L$ and not $l$ or $l'$: we coarsen the source cell. Again, $G''(d, w; s, v) = G'(d, w; s, v)$ unless $s = L$, in which case

$$G''(d, w; L, v0) = \frac{1}{2}[G'(d, w; s, v) + G'(d, w; s', v)]$$
$$G''(d, w; L, v1) = \frac{1}{2}[G'(d, w; s, v) - G'(d, w; s', v)]$$

(9)

Although we have implemented our algorithm in three dimensions, we use a 2D system for our test calculation. The test system has four control parameters: $N$, $I$, $\delta$, and $r$. The first two describe the complexity of the system: $N$ is the system size ($16 \times 16$, $32 \times 32$, or $64 \times 64$) and the inhomogeneity parameter $I$ is a normalized standard deviation: the standard deviation of the permeability divided by the mean permeability. The other two parameters are the error tolerance $\delta$, which we choose to give an acceptable overall truncation error, and the lumping threshold $r$ (Eq. 5), which we tune to maximize the speed of the algorithm.

The diffusivity (i.e., permeability) distribution we have used is a realization of a log-normal distribution with fractal spatial correlations, obtained by exponentiating a correlated Gaussian distribution described elsewhere [5]. We adjust the normalized standard deviation $I$ by scaling the Gaussian distribution before exponentiating it. The prefactor that
governs the overall scale of the diffusivity or permeability can be removed from the problem by rescaling time. The system shown in Fig. 2 is $64 \times 64$; we specify the permeability in the smaller-$N$ systems by coarse-graining (averaging over $2 \times 2$ or $4 \times 4$ cells). The permeability at a face is taken to be the average of that in the adjoining cells.

As a test initial condition, we use a delta-function density concentrated in the lower left cell of the system. After an infinite time, the density takes a uniform value $P_\infty$; we evolve the system until the density in the source cell is $2P_\infty$, as shown in Fig. 2.

![Image](image3)

FIG. 2. The final density distribution in the $64 \times 64$ system with inhomogeneity $I = 20$, showing boundaries of lumped cells.

To compare our scheme with an EFD algorithm, we look first at the $64 \times 64$ system with a substantial inhomogeneity ($I = 20$), and use $r = 0.9$ (justified by Fig. 4 below). We vary the remaining parameter, the tolerance $\delta$, and plot in Fig. 3 the required CPU time (on a Silicon Graphics R4000PC Indy, 133 MHz) against the accuracy achieved, defined by the fractional rms error

$$\text{error}^2 \equiv N^{-1} \sum_c \left[ \frac{P(c) - P_{\text{exact}}(c)}{P_\infty} \right]^2. \quad (10)$$
FIG. 3. Logarithmic plot of the CPU time required by our hierarchical Green function algorithm, compared to that required by an explicit finite difference (EFD) algorithm, for a 64 × 64 system with inhomogeneity \( I = 20 \). The rightmost data point for each system size has tolerance parameter \( \delta = 10^{-2} \), and those for 32 × 32 system decrease by factors of 10. For 64 × 64 there are points at \( 5 \times 10^{-n} \) as well as at \( 10^{-n} \).

Note that the speedup factor of our algorithm compared to the EFD algorithm increases rapidly as the allowable error is increased. It is indicated by a double arrow at the error value of 1.5%, where it is about 25. Using this factor as a figure of merit for our algorithm, we plot it in Fig. 4 as a function of the lumping parameter \( r \); evidently performance improves as we increase \( r \) toward 1.0. Placing \( r \) very close to 1.0 risks magnifying the effects of small numerical errors, so we used \( r = 0.9 \) in the other figures.

FIG. 4. Speedup factor as a function of the lumping threshold \( r \) (Eq. 3), in a 32 × 32 system with inhomogeneity \( I = 20 \) and tolerance \( \delta = 10^{-4} \).
We are using the explicit finite difference algorithm as our point of comparison, not because it is the most efficient existing algorithm, but because it is the simplest. However, the use of more efficient implicit algorithms increases the allowed $\Delta t$ only by factors of order 2 and does not affect our conclusions.

The dependence of the speedup factor on the system properties is shown in Table I. Evidently the Green function algorithm is most advantageous in highly inhomogeneous systems.

| $I$ | System Size | $8 \times 8$ | $16 \times 16$ | $32 \times 32$ | $64 \times 64$ |
|-----|-------------|--------------|--------------|--------------|--------------|
| 10  | 0.2         | 1.6          | 15           | 18           |
| 20  | 0.3         | 3.0          | 32           | 43           |
| 40  | 0.3         | 5.2          | 63           | 71           |

TABLE I. Speedup factor as a function of system size $N$, for three values of the inhomogeneity $I$. Tolerance is $\delta = 10^{-4}$.

Although the method described here has some features in common with methods already in common use in grid-based numerical simulation, none of these older methods approaches its efficiency for inhomogeneous systems. The idea of coarsening cells is used in "multigrid" methods [8]. For example, the solution of Laplace’s equation by the relaxation method is very slow on a fine grid. It can be speeded up by doing a few iterations of relaxation on a larger grid to get the coarse features of the solution correct, and then returning to the fine grid to improve the finer features. Computational fluid dynamics codes often use an "adaptive grid" method [9], wherein larger grid sizes are used in regions where fields do not vary rapidly in space, and finer grids are used where there are fine-scale variations in the fields. These regions are typically rectangular and cannot follow compartment shapes as closely as ours. Unlike in our approach, the time increment cannot be increased above what is stable on the finest grid.

In conclusion, we have shown that a hierarchical algorithm based on the dynamic renormalization group and the Walsh transform can simulate diffusive flow in an inhomogeneous system much more efficiently than conventional finite-difference algorithms. This occurs because the rapid power-law dependence of CPU time on the fundamental scales $\Delta r$ and $\Delta t$ is replaced in the hierarchical method by a logarithmic dependence.

The work described here was partially supported by the DOE under Cooperative Agreement DE-FC02-91ER75678.

[1] K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 75 (1974).
[2] P. B. Visscher, "Exact Fixed Points in Discrete Hydrodynamics", J. Stat. Phys. **25**, 211-227 (1981).
[3] Beauchamp, K. G., "Applications of Walsh and related functions, with an introduction to sequency theory", Academic Press, London, Orlando 1984.
[4] M. Holschneider, "Wavelets", Clarendon Press, Oxford, 1995
[5] Khalid Aziz, "Petroleum reservoir simulation", Applied Science Publishers, London, 1979.
[6] Oran, E. S. and Boris, J. P., "Numerical Simulation of Reactive Flow", Elsevier Science Publishing, New York, 1987, p. 102.
[7] P. B. Visscher, Judy Dye, and Jen-Ho Fang, "Hierarchical Simulation of Fractal Distributions", Computers in Physics **7**, 217-225, 1993.
[8] W. L. Briggs, "A Multigrid Tutorial", SIAM, Philadelphia, 1987. elt meth.
[9] B. A. Boyett, M. S. El-Mandouh, and R. E. Ewing, "Local Grid Refinement for Reservoir Simulation" in "Computational Methods in the Geosciences", ed. by W. E. Fitzgibbon and M. F. Wheeler, SIAM, Philadelphia, 1992.