FAN VALUATIONS AND SPHERICAL INTRINSIC VOLUMES

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Abstract. We generalize valuations on polyhedral cones to valuations on fans. For fans induced by hyperplane arrangements, we show a correspondence between rotation-invariant valuations and deletion-restriction invariants. In particular, we define a characteristic polynomial for fans in terms of spherical intrinsic volumes and show that it coincides with the usual characteristic polynomial in the case of hyperplane arrangements. This gives a simple deletion-restriction proof of a result of Klivans–Swartz.

The metric projection of a cone is a piecewise-linear map, whose underlying fan prompts a generalization of spherical intrinsic volumes to indicator functions. We show that these intrinsic indicators yield valuations that separate polyhedral cones. Applied to hyperplane arrangements, this generalizes a result of Kabluchko on projection volumes.

1. Introduction

Let $\mathcal{S}$ be a collection of sets closed under taking intersections. A map $\varphi$ from $\mathcal{S}$ into some abelian group $G$ is a valuation if

$$
\varphi(S \cup T) = \varphi(S) + \varphi(T) - \varphi(S \cap T),
$$

for any $S,T \in \mathcal{S}$ such that $S \cup T \in \mathcal{S}$. For geometric objects such as convex polytopes, polyhedra, or subspaces, valuations are a gateway between geometry and combinatorics, amply demonstrated in [2, 13]. In particular Ehrenborg and Readdy [6] showed how generalizations of Zaslavsky’s famous formula for the number of regions of a hyperplane arrangement can be easily inferred using valuations. The purpose of this note is to further the ties between geometry and combinatorics by studying valuations on more general arrangements of geometric objects.

A fan in $\mathbb{R}^d$ is a finite collection $\mathcal{N}$ of equi-dimensional polyhedral cones that pairwise meet in faces. We further require that all cones have the same linear span and thus speak of fans relative to subspaces of $\mathbb{R}^d$. Although the collection of fans $\text{Fans}_d$ is a meet-semilattice with respect to common refinement to which valuations can be generalized [8, 19], there is no natural join operation. To define valuations on fans, we adapt Sallee’s notion of weak valuations [20] on polytopes: A map $\varphi : \text{Fans}_d \to G$ is a fan valuation if for every fan $\mathcal{N} \in \text{Fans}_d$ and linear hyperplane $H$

$$
\varphi(\mathcal{N}) = \varphi(\mathcal{N} \cap H^\leq) + \varphi(\mathcal{N} \cap H^\geq) - \varphi(\mathcal{N} \cap H),
$$

where $H^\leq, H^\geq$ denote the two halfspaces induced by $H$. We defer precise definitions to Section 2. Any arrangement $\mathcal{A}$ of linear hyperplanes in some subspace $U \subseteq \mathbb{R}^d$ induces a fan...
\( N(A) \). In Proposition 2.2, we show that fan valuations on arrangements satisfy

\[
\varphi(N(A)) = \varphi(N(A \setminus H)) + \varphi(N(A/H)).
\]

Such invariants are closely related to deletion-contraction invariants on (simple, loopless) matroids ([23, Sect. 3.11], [7], [4]). The main difference is that we do not impose special treatment when \( H \) is a coloop of \( A \). The benefit is that these weak deletion-restriction invariants have the structure of an abelian group. More precisely, let \( \mathcal{L}(A) \) be the lattice of flats of \( A \) and write \( w_0(A), \ldots, w_d(A) \) for the Whitney numbers of the first kind. Then the group weak deletion-restriction invariants is spanned by the Whitney numbers.

We show that fan valuations invariant under rotation yield precisely the weak deletion-restriction invariants.

**Theorem 1.1.** Let \( \varphi : \text{Fans}_d \to G \) be a rotation-invariant fan valuation. Then there are \( g_0, \ldots, g_d \in G \) such that for any hyperplane arrangement \( A \)

\[
\varphi(N(A)) = g_0 w_0(A) + \cdots + g_d w_d(A).
\]

Conversely, for any deletion-restriction invariant \( \psi \) on hyperplane arrangements, there is a fan valuation \( \varphi \) with \( \psi(A) = \varphi(N(A)) \).

For the latter part, we consider fan valuations induced by spherical intrinsic volumes, also known as projection volumes. The \( k \)-th spherical intrinsic volume \( v_k(C) \) of a cone \( C \subseteq \mathbb{R}^d \) is the probability that the point of \( C \) closest to \( x \in B_d \) is contained in the relative interior of a face \( F_x \subseteq C \) of dimension \( k \). The definition is extended to fans by setting \( v_k(N) : = \sum_{C \in N} v_k(C) \).

With that, we define the (unsigned) characteristic polynomial of a fan by

\[
\overline{\chi}_N(t) := v_0(N) + v_1(N)t + \cdots + v_d(N)t^d.
\]

We argue that \( \overline{\chi}_N(t) \) is a suitable generalization of the characteristic polynomial of a hyperplane arrangement. We show that it satisfies Zaslavsky’s fundamental results:

\[
\overline{\chi}_N(1) = \# \text{ regions of } N \quad \text{ and } \quad \overline{\chi}_N(-1) = 0.
\]

We derive the latter from an identity of Hug–Kabluchko [11], for which we provide a self-contained proof (Theorem 3.12).

Most importantly, we show \( \overline{\chi}_N(A)(t) = (-1)^{\dim A} \chi_A(-t) \), where \( \chi_A(t) \) is the usual characteristic polynomial of an arrangement. This gives a simple deletion-restriction proof of the main result of Klivans–Swartz [14] that identifies projection volumes with the Whitney numbers of \( \mathcal{L}(A) \).

Whereas Hadwiger’s famous classification theorem [10] states that the linear space of continuous and rigid-motion valuations on convex bodies is spanned by the usual intrinsic volumes, there is no such result for spherical convex sets. McMullen [9, Problem 49] conjectured that the linear space of continuous and rotation-invariant valuations on spherical convex bodies is spanned by the spherical intrinsic volumes. From this perspective, Theorem 1.1 together with the result of Klivans–Swartz can be seen as an indication for this conjecture.

In the second part of the paper, we take a more refined look at spherical intrinsic volumes. The collection of points \( x \in \mathbb{R}^d \) such that the nearest point in \( C \) is contained in a fixed face \( F \subseteq C \) is a polyhedral cone \( \Pi_F(C) \) and \( M(C) := \{ \Pi_F(C) : F \subseteq C \text{ face} \} \) is a complete fan, which we call the Moreau fan of \( C \). The face lattice of \( M(C) \) is given by the interval poset of the face lattice of \( C \). Such fan structures were considered by Björner under the name of
anti-prisms in connection with a question of Lindström and our findings reconfirm results announced in [3].

It is known that \( C \mapsto v_k(C) \) is a cone valuation [17]. We prove a generalization that this holds on the level of simple indicator functions: Consider the set \( \Pi_k(C) = \bigcup_F \Pi_F(C) \), where the union is over all \( k \)-dimensional faces of \( C \). Its simple indicator is the function \( V_k(C) : \mathbb{R}^d \to \mathbb{Z} \) that disagrees with the indicator of \( \Pi_k(C) \) only on a set of measure zero. We show that \( C \mapsto V_k(C) \) is a valuation (Theorem 3.2) and that \( C \) can be recovered from \( V_k(C) \) for all \( \dim \text{lineal}(C) \leq k \leq \dim C \) with \( 2k \neq d \) (Theorem 3.5).

For the function \( V_k(\mathcal{N}(\mathcal{A})) = \sum_{C \in \mathcal{N}(\mathcal{A})} V_k(C) \) of a hyperplane arrangement \( \mathcal{A} \) it follows that \( V_k(\mathcal{N}(\mathcal{A}))(x) = w_k(\mathcal{A}) \) for all generic \( x \in \mathbb{R}^d \). Kabluchko [12] showed that the exceptional set \( \{ x : V_k(\mathcal{N}(\mathcal{A}))(x) \neq w_k(\mathcal{A}) \} \) coincides with the support of a hyperplane arrangement. We generalize Kabluchko’s result in Theorem 3.8 with a short proof that also allows us to give a simple interpretation for the associated arrangement.

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### 2. Fan valuations and deletion-restriction invariants

Throughout, we assume that polyhedral cones are nonempty and contain the origin. For a polyhedral cone \( C \subseteq \mathbb{R}^d \), the **linear hull** \( \text{lin}(C) \) is the inclusion-minimal linear subspace containing \( C \) and the **lineality space** \( \text{lineal}(C) \) is the inclusion-maximal linear subspace contained in \( C \). A **fan** in \( \mathbb{R}^d \) is a finite collection \( \mathcal{N} \) of polyhedral cones such that for all \( C, C' \in \mathcal{N} \)

(i) \( C \cap C' \) is a face of \( C \) and
(ii) \( \text{lin}(C) = \text{lin}(C') \).

It follows that all cones are of the same dimension. We set \( \dim \mathcal{N} := \dim \mathcal{N} \) and \( \text{lin}(\mathcal{N}) := \text{lin}(C) \) as well as \( \text{lineal}(\mathcal{N}) := \text{lineal}(C) \) for any \( C \in \mathcal{N} \). The **rank** of \( \mathcal{N} \) is \( r(\mathcal{N}) = \dim(\mathcal{N}) - \dim \text{lineal}(\mathcal{N}) \). Denote by \( \text{Fans}_d \) the collection of fans in \( \mathbb{R}^d \). We define for any set \( S \subseteq \text{lin}(\mathcal{N}) \)
\[
\mathcal{N} \cap S := \{ C \cap S : C \in \mathcal{N}, \ \text{relint}(C) \cap S \neq \emptyset \}.
\]

A map \( \varphi \) from \( \text{Fans}_d \) into some abelian group is a **fan valuation** if \( \varphi(\emptyset) = 0 \) and for any \( \mathcal{N} \in \text{Fans}_d \) and hyperplane \( H \subseteq \text{lin}(\mathcal{N}) \)
\[
\varphi(\mathcal{N}) = \varphi(\mathcal{N} \cap H^\leq) + \varphi(\mathcal{N} \cap H^\geq) - \varphi(\mathcal{N} \cap H),
\]
where \( H^\leq, H^\geq \) denote the two closed halfspaces induced by \( H \).

Let \( \text{Cones}_d \) be the intersectional family of polyhedral cones in \( \mathbb{R}^d \). Every cone \( C \in \text{Cones}_d \) gives rise to a fan \( \{ C \} \in \text{Fans}_d \) and a fan valuation gives rise to map \( \varphi' \) on \( \text{Cones}_d \) satisfying
\[
\varphi'(C) = \varphi'(C \cap H^\leq) + \varphi'(C \cap H^\geq) - \varphi'(C \cap H).
\]
Such a map is called a **weak valuation** [20]. Every valuation on cones is naturally a weak valuations and Sallee’s [20] arguments imply that every weak valuation on polyhedral cones
is a valuation. In particular, if \( \varphi' \) is a cone valuation, then

\[
\varphi(\mathcal{N}) := \sum_{C \in \mathcal{N}} \varphi'(C)
\]

is a fan valuation. The next result shows that every valuation is of that form.

**Proposition 2.1.** Let \( \varphi \) be a fan valuation. Then

\[
\varphi(\mathcal{N}) = \sum_{C \in \mathcal{N}} \varphi(\{C\}).
\]

Thus, we will write \( \varphi(C) \) instead of \( \varphi(\{C\}) \) from now on.

**Proof.** The claim follows trivially if \( \mathcal{N} \) is empty or if it consists of a single cone. If \( \dim(\mathcal{N}) = 1 \), then the only nontrivial case is \( \mathcal{N} = \{ \mathbb{R}_{\geq 0} c, -\mathbb{R}_{\geq 0} c \} \) for some \( c \in \mathbb{R}^d \setminus \{0\} \). Now \( H = \{0\} \) is the unique linear hyperplane in \( \text{lin}(\mathcal{N}) \) and \( \mathcal{N} \cap H = \emptyset \). The assertion follows from the definition of fan valuations.

Assume now that the statement holds for all fans consisting of less than \( k \) cones and whose dimension is smaller than \( e \) for some \( k \geq 2 \) and \( e \geq 2 \). Let \( \mathcal{N} \) be a fan with \( k \) cones and \( \dim(\mathcal{N}) = e \). For \( C, C' \in \mathcal{N} \), let \( H \) be a separating hyperplane. Then \( \mathcal{N} \cap H^\leq \) and \( \mathcal{N} \cap H^\geq \) have less than \( k \) cones and \( \dim(\mathcal{N} \cap H) < e \). Let \( X = \{ C \in \mathcal{N} : \text{relint } C \cap H \neq \emptyset \} \). Then

\[
\varphi(\mathcal{N}) = \varphi(\mathcal{N} \cap H^\leq) + \varphi(\mathcal{N} \cap H^\geq) - \varphi(\mathcal{N} \cap H)
\]

\[
= \sum_{C \in \mathcal{N} \setminus X} \varphi(\{C\}) + \sum_{C' \in X} (\varphi(\{C \cap H^\leq\}) + \varphi(\{C \cap H^\geq\}) - \varphi(\{C \cap H\}))
\]

\[
= \sum_{C \in \mathcal{N} \setminus X} \varphi(\{C\}) + \sum_{C' \in X} \varphi(\{C'\}) = \sum_{C \in \mathcal{N}} \varphi(\{C\}). \qedhere
\]

### 2.1. Hyperplane arrangements

Let \( \mathcal{A} \) be a finite collection of linear hyperplanes in a subspace \( L \) of \( \mathbb{R}^d \). The complement \( L \setminus \bigcup \mathcal{A} \) is a collection of open cones whose closures define a fan that we denote by \( \mathcal{N}(\mathcal{A}) \). For a hyperplane \( H \not\in \mathcal{A} \) observe that \( \mathcal{N}(\mathcal{A}) \cap H = \mathcal{N}(\mathcal{A}/H) \), where \( \mathcal{A}/H := \{ H' \cap H : H' \in \mathcal{A}, H' \neq H \} \) is the **restriction** of \( \mathcal{A} \) to \( H \). If \( H \in \mathcal{A} \), we write \( \mathcal{A} \setminus H = \{ H' \in \mathcal{A} : H' \neq H \} \) for the **deletion** of \( H \). A **k-singleton**, or simply, a singleton, is a hyperplane arrangement consisting of a single hyperplane in some \( k \)-dimensional subspace of \( \mathbb{R}^d \), where \( 1 \leq k \leq d \).

**Proposition 2.2.** Let \( \varphi \) be a fan valuation and \( \mathcal{A} \) a hyperplane arrangement which is not a singleton. For any \( H \in \mathcal{A} \)

\[
\varphi(\mathcal{N}(\mathcal{A})) = \varphi(\mathcal{N}(\mathcal{A}\setminus H)) + \varphi(\mathcal{N}(\mathcal{A}/H)).
\]

**Proof.** Let \( H \in \mathcal{A} \) and set \( \mathcal{A}' = \mathcal{A} \setminus H \). The valuation property yields

\[
\varphi(\mathcal{N}(\mathcal{A}')) = \varphi(\mathcal{N}(\mathcal{A}') \cap H^\leq) + \varphi(\mathcal{N}(\mathcal{A}') \cap H^\geq) - \varphi(\mathcal{N}(\mathcal{A}') \cap H)
\]

We infer from Proposition 2.1 that \( \varphi(\mathcal{N}(\mathcal{A})) = \varphi(\mathcal{N}(\mathcal{A}') \cap H^\leq) + \varphi(\mathcal{N}(\mathcal{A}') \cap H^\geq) \). By definition \( \varphi(\mathcal{N}(\mathcal{A}' \cap H)) = \varphi(\mathcal{N}(\mathcal{A}/H)) \), which now yields the claim. \( \square \)
The group $\text{SO}(\mathbb{R}^d)$ of rotations acts on $\text{Fans}_d$ by $g \cdot \mathcal{N} := \{gC : C \in \mathcal{N}\}$. We call a fan valuation $\varphi$ invariant if $\varphi(g \cdot \mathcal{N}) = \varphi(\mathcal{N})$ for all $g \in \text{SO}(\mathbb{R}^d)$ and $\mathcal{N} \in \text{Fans}_d$. Clearly, if $\varphi$ is invariant, it assigns the same value to all $k$-singletons.

We define the **unsigned characteristic polynomial** $\overline{\chi}_\mathcal{A}(t)$ of an arrangement $\mathcal{A}$ recursively as follows. If $\mathcal{A}$ is a $k$-singleton, then $\overline{\chi}_\mathcal{A}(t) := t^k + t^{k-1}$. If $\mathcal{A}$ consists of more than one hyperplane, then for $H \in \mathcal{A}$

$$
\overline{\chi}_\mathcal{A}(t) := \overline{\chi}_{\mathcal{A}\backslash H}(t) + \overline{\chi}_{H/\mathcal{A}}(t).
$$

The unsigned characteristic polynomial is related to the usual characteristic polynomial (cf. [23, Sect. 3.11]) by $\overline{\chi}_\mathcal{A}(t) = (-1)^{\dim A} \chi_{\mathcal{A}}(-t)$. The coefficients of $\overline{\chi}_\mathcal{A}(t)$ are the (unsigned) Whitney numbers of the first kind denoted by $w_i(\mathcal{A})$.

**Theorem 2.3.** Let $\varphi$ be an invariant fan valuation taking values in an abelian group $G$. Then there are $a_0, \ldots, a_d \in G$, such that for every arrangement $\mathcal{A}$

$$
\varphi(\mathcal{N}(\mathcal{A})) = a_0 w_0(\mathcal{A}) + \cdots + a_{d-1} w_{d-1}(\mathcal{A}) .
$$

Moreover, the $a_i$’s are determined by the values $\varphi(\mathcal{A}^k)$, where $\mathcal{A}^k$ is a $k$-singleton, $1 \leq k \leq d$.

**Proof.** Let $b_k := \varphi(\mathcal{N}(\mathcal{A}^k))$ for any $k$-singleton $\mathcal{A}^k$ and set $a_{k-1} := \sum_{i=k}^d (-1)^{k-i} b_i$ for $1 \leq k \leq d$. We proceed by induction. Since $a_{k-1} + a_k = b_k$, clearly (3) holds for $k$-singletons. Otherwise, let $H \in \mathcal{A}$. Then

$$
\varphi(\mathcal{N}(\mathcal{A})) = \varphi(\mathcal{N}(\mathcal{A}\backslash H)) + \varphi(\mathcal{N}(H/\mathcal{A}))
$$

$$
= \sum_{i=0}^{d-1} a_i w_i(\mathcal{A}\backslash H) + \sum_{i=0}^{d-1} a_i w_i(\mathcal{A}/H) = \sum_{i=0}^{d-1} a_i w_i(\mathcal{A}) \quad \square
$$

2.2. **Spherical intrinsic volumes.** Every weak deletion-restriction invariant arises from an invariant fan valuation, more precisely, from a combination of spherical intrinsic volumes. To show this, we introduce a characteristic polynomial for fans.

Given a polyhedral cone $C \subseteq \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, there is a unique point $\pi_C(x) \in C$ minimizing the Euclidean distance $\|x - \pi_C(x)\|_2$. The map $\pi_C : \mathbb{R}^d \to C$ is called the **metric projection** or **nearest-point map** of $C$; cf. [21, Sect. 1.2]. Let us denote by $F_x$ the unique face of $C$ that contains $\pi_C(x)$ in its relative interior and $\Pi_k(C) := \{x \in \mathbb{R}^d : \dim F_x = k\}$. The $k$-th **spherical intrinsic volume** is given by

$$
v_k(C) := \frac{\text{vol}(\Pi_k(C) \cap B_d)}{\text{vol}(B_d)}.
$$

In the next section we will consider the sets $\Pi_k(C)$ more closely and in particular deduce the known fact that $C \mapsto v_k(C)$ is a cone valuation (Corollary 3.3). It is apparent that $v_k$ is $\text{SO}(\mathbb{R}^d)$-invariant. We write $v_k(\mathcal{N})$ for the induced $k$-th spherical intrinsic volume of a fan $\mathcal{N}$ and we define its **characteristic polynomial**

$$
\overline{\chi}_\mathcal{N}(t) := v_0(\mathcal{N}) + v_1(\mathcal{N})t + \cdots + v_d(\mathcal{N})t^d .
$$

The characteristic polynomial shares a number of similarities with that of a hyperplane arrangement. Since the $\Pi_k(C)$ cover $\mathbb{R}^d$ and $\Pi_k(C) \cap \Pi_l(C)$ has measure zero, it follows that

$$
\overline{\chi}_\mathcal{N}(1) = |\mathcal{N}|
$$
and it follows from Theorem 3.12 in the next section that
\[ \chi_N(-1) = 0, \]
which is a counterpart to Zaslavsky’s famous results concerning characteristic polynomials of hyperplane arrangements.

For the case \( N = N(A) \), the spherical intrinsic volumes \( v_k(N) = \sum_{C \in N} v_k(C) \) were studied by Klivans and Swartz and the following is the main result of [14].

**Corollary 2.4.** Let \( A \) be a linear hyperplane arrangement. Then
\[ \chi_{N(A)}(t) = \chi_A(t). \]

**Proof.** In light of Proposition 2.2 and (2), it suffices to compute \( \chi_{N(A^k)}(t) \), where \( A^k \) is a \( k \)-singleton in some linear subspace \( L \). More specifically, if \( H^\leq \subset L \) is a halfspace, then \( v_k(H^\leq) = v_{k-1}(H^\leq) = \frac{1}{2} \) and \( v_j(H^\leq) = 0 \) for all other \( j \). This shows \( \chi_{N(A^k)}(t) = t^k + t^{k-1} \). Hence \( \chi_{N(A)}(t) \) and \( \chi_A(t) \) satisfy the same deletion-restriction recurrence with identical starting conditions. \( \square \)

### 3. Anti-prism fans and intrinsic volumes

In this section, we take a closer look at the geometric combinatorics of spherical intrinsic volumes by way of associated indicator functions.

#### 3.1. Moreau fans and anti-prisms

Let \( C \subseteq \mathbb{R}^d \) be a polyhedral cone. The **metric projection** \( \pi_C : \mathbb{R}^d \to C \) associates to any point \( x \in \mathbb{R}^d \) the unique point \( \pi_C(x) \in C \) with \( \|x - \pi_C(x)\| \) minimal. It is straightforward to verify that if \( x, y \in \mathbb{R}^d \) such that \( \pi_C(x), \pi_C(y) \in F \) for some face \( F \subseteq C \), then \( \pi_C(x + y) \in F \). Hence \( \Pi_F(C) := \pi_C^{-1}(F) \) is a closed, full-dimensional polyhedral cone. Moreau [18] considered the decomposition of space into the collection of cones
\[ \mathcal{M}(C) := \{ \Pi_F(C) : F \text{ is a nonempty face of } C \} \]
and we call \( \mathcal{M}(C) \) the **Moreau fan** of \( C \).

The combinatorics of Moreau fans can be nicely described in terms of Lindström’s interval posets [15]. Let \( (\mathcal{L}, \preceq) \) be a partially ordered set. The **interval poset** \( I(\mathcal{L}) \) is the collection of nonempty intervals \([a, c] = \{ b : a \preceq b \preceq c \}\) ordered by reverse inclusion. The maximal elements are precisely \([a, a]\) and if \( \mathcal{L} \) has a top and bottom element \( \hat{1} \) and \( \hat{0} \), respectively, then \([0, 1]\) is the unique minimum of \( I(\mathcal{L}) \). Lindström [15] asked if \( I(F(P)) \) is the face lattice of a polytope whenever \( F(P) \) is the face lattice of a polytope \( P \). Björner affirmatively answered Lindström’s question for 3-dimensional polytopes. The complete question was resolved in the negative by Dobbins [5].

Björner [3] also announced that \( I(F(P)) \) is the face poset of a complete fan (or star-convex sphere), the **anti-prism** fan of \( P \). We briefly reconfirm this result by showing that the Moreau fan of \( C = \text{cone}(P \times \{1\}) \) realizes \( I(F(P)) \). Let \( \mathcal{L}^{\text{op}} \) be the dual (or opposite) poset of \( \mathcal{L} \). Then
\[ I(\mathcal{L}) \cong \{(a, b) : a \preceq b \} \subseteq \mathcal{L} \times \mathcal{L}^{\text{op}}. \]
The face lattice $\mathcal{F}(C)$ of $C$ is the collection of nonempty faces of $C$ partially ordered by inclusion. This is a graded poset ranked by dimension and we denote by $\mathcal{F}_k(C)$ the $k$-dimensional faces of $C$. For $F \in \mathcal{F}(C)$ let

$$N_F C := \{ c \in \mathbb{R}^d : \langle c, x \rangle \leq \langle c, y \rangle \text{ for all } x \in C, y \in F \}$$

be the normal cone of $F$ at $C$. In particular the polar to $C$ is $C^\vee = N_{\text{lineal}(C)} C$ and hence $F \mapsto N_F C$ is an isomorphism from $\mathcal{F}(C)$ to $\mathcal{F}(C^\vee) = \mathcal{F}(C)^{\text{op}}$.

Let $x \in \mathbb{R}^d$. It follows from $\|x - \pi_C(x)\| \leq \|x - z\|$ for all $z \in C$ such that $x - \pi_C(x) \in N_F C$. Hence $\Pi_F(C) = F + N_F C$ for all nonempty faces $F \subseteq C$. Moreover

$$(4) \quad (F + N_F C) \cap (G + N_G C) = (F \cap G) + (N_G C \cap N_F C),$$

which shows the following.

**Proposition 3.1.** Let $C \subseteq \mathbb{R}^d$ be a polyhedral cone. The face lattice of the Moreau fan $\mathcal{M}(C)$ is isomorphic to the interval poset $I(\mathcal{F}(C))$.

### 3.2. Conical functions

For a subset $S \subseteq \mathbb{R}^d$, we denote its indicator function by $[S] : \mathbb{R}^d \to \{0, 1\}$, which is defined by $[S](x) = 1$ if and only if $x \in S$. Let $\mathfrak{C}_d \subseteq \text{Fun}(\mathbb{R}^d, \mathbb{Z})$ be the abelian subgroup spanned by $[C]$ for $C \in \text{Cones}_d$. Since $[C] = [C \cap H^\perp] + [C \cap H^\perp] - [C \cap H]$, the map $C \mapsto [C]$ is a valuation. Moreover, any homomorphism $\varphi : \mathfrak{C}_d \to G$ gives a valuation on cones by $\varphi(C) := \varphi([C])$ and Groemer [8] showed every cone valuation arises that way.

For a cone $C \subseteq \mathbb{R}^d$ define

$$V_k(C) := \sum_{F \in \mathcal{F}_k(C)} [\Pi_F(C)] \in \mathfrak{C}_d.$$

This is an indicator generalization of the spherical intrinsic volumes and $v_k(C)$ is recovered from $V_k$ by taking the spherical volume. Note that $V_k : \text{Cones}_d \to \mathfrak{C}_d$ is not a valuation: For the augmentation $\epsilon : \mathfrak{C}_d \to \mathbb{Z}$ with $\epsilon([C]) = 1$ for all nonempty cones $C$, we see that $\epsilon(V_k(C))$ is the number of $k$-dimensional faces of $C$, which is not a valuation. Nonetheless, we can view $V_k$ as a valuation taking values in the group of simple indicator functions

$$\mathfrak{S}_d := \mathfrak{C}_d/([C] : C \in \text{Cones}_d, \dim C < d).$$

**Theorem 3.2.** Let $C \subseteq \mathbb{R}^d$ be a cone and $H$ a hyperplane. Then

$$V_k(C) = V_k(C \cap H^\perp) + V_k(C \cap H^\perp) - V_k(C \cap H)$$

as simple indicator functions.

**Proof.** Observe that if $f = g$ for $f, g \in \mathfrak{S}_d$ if $f(x) = g(x)$ for almost all $x \in \mathbb{R}^d$. Thus, let $x \in \mathbb{R}^d$ a generic point. Let $C^\leq, C^\geq$, and $C^=\leq$ denote $C \cap H^\perp, C \cap H^\perp$, and $C \cap H$, respectively. Let $\pi_C(x) = y$ and $F \subseteq C$ the unique face with $y \in \text{relint}(F)$. We may assume that $y \in H^\leq$, that is, $y = \pi_{C \leq}(x)$ and $F^\leq = F \cap F^\leq$. Define $y^\leq, y^\geq$ with corresponding faces $F^\leq, F^\geq$.

If $y = y^\leq$, then $y \in H$ and $F \cap H \neq \emptyset$. If $F \subseteq H$, then $F = F^\leq = F^\leq = F^\geq$ and we are done. The case $F \not\subseteq H$ is not relevant, as $x$ is generic: perturbing $x$ parallel to $\text{lin}(F)$ moves $y$ away from $H$.

If $y^\geq = y^\leq$, then $y^\geq = y^\leq$. Indeed, any point on the segment $z \in [y, y^\geq]$ satisfies $\|x - z\| < \|x - y^\leq\|$ and $[y, y^\geq]$ meets $H$. It follows that $F^\geq = F^\leq$ and $\dim F = \dim F^\leq$. This proves the claim. \[\square\]
The spherical volume \( \sigma_{d-1}(C) := \frac{\text{vol}_{d}(C \cap B_{d})}{\text{vol}_{d}(B_{d})} \) is a simple valuation and hence extends to a linear function \( \sigma_{d-1} : \mathcal{S}_{d} \to \mathbb{R} \).

**Corollary 3.3.** The spherical volumes \( v_{k}(C) = \sigma_{d-1}(V_{k}(C)) \) are valuations.

Similar to the situation for polytopes, the valuations \( V_{k} \) separate \( \mathcal{C}_{d} \). The proof of the next result needs the following observation.

**Lemma 3.4.** Let \( C \subseteq \mathbb{R}^{d} \) be a cone and \( 0 \leq k \leq d \). Then \( V_{k}(C) = V_{d-k}(C^{\vee}) \).

**Theorem 3.5.** Let \( C \subseteq \mathbb{R}^{d} \) be a cone and the dimension of \( \text{lineal}(C) \) is \( k \leq d \). If \( 2k \neq d \), then \( C \) can be recovered from \( V_{k}(C) \).

By the previous lemma, we can directly see that for \( 2k = d \) we have \( V_{k}(C) = V_{k}(C^{\vee}) \), making this assumption necessary.

**Proof.** First note that \( V_{k}(C) = 0 \) whenever \( k > \text{dim} \, C \) or, by Lemma 3.4, \( k < \text{dim} \, \text{lineal}(C) \).

Now, let \( S \subseteq \mathbb{R}^{d} \) be the collection of points for which \( V_{k}(C) \) does not vanish on a small neighborhood. This is the union of the interiors of \( \Pi_{F}(C) = F + N_{F}C \), where \( F \) ranges over the \( k \)-dimensional faces of \( C \). It follows from (4) that \( \dim \Pi_{F}(C) \cap \Pi_{G}(C) < d \) for any two distinct \( F, G \in \mathcal{F}_{k}(C) \) and hence we can recover the cones \( \Pi_{F}(C) \) from \( S \).

We may use Lemma 3.4 to assume that \( k < \frac{d}{2} \). Every \( k \)-face \( E \) of \( \Pi_{F}(C) = F + N_{F}C \) is of the form \( E = E' + N_{E'}C \), where \( E' \subseteq F \subseteq E'' \) are faces with \( d - k = \dim E'' - \dim E' \). We say that \( E \) is **free**, if there exists no \( k \)-face \( G \) of \( C \), such that \( E = \Pi_{F}(C) \cap \Pi_{G}(C) \). Equivalently, \( E \) is not free, if and only if there exist another \( k \)-faces \( G \neq F \) such that \( E' \supseteq G \supseteq E'' \). Thus \( E \) is free, if and only if \( F = E' \) or \( F = E'' \), and since \( \dim E'' - \dim E' = d - k > k \), the case \( F = E'' \) is impossible. Note that if \( F = E' \), then \( E'' = C \), so \( E = F + N_{C}C = F \). Therefore, the set of all free faces is precisely the set \( k \)-faces of \( C \), from which we can recover \( C \). \( \square \)

### 3.3. Characteristic indicators.

Corollary 2.4 can be generalized to the setting of indicator functions. Let \( \rho : \mathcal{C}_{d} \to \mathcal{S}_{d} \) be the canonical projection. Then we define \( \overline{V} : \mathbf{Fans}_{d} \to \mathcal{S}_{d}[t] \) as

\[
\overline{V}_{N}(t) := \sum_{k=0}^{d} \rho(V_{k}(N))t^{k} \in \mathcal{S}_{d}[t].
\]

This is a natural generalization of \( \overline{\chi}_{N}(t) \).

**Corollary 3.6.** Let \( A \) be an arrangement and \( 0 \leq k \leq d \). Then as elements of \( \mathcal{S}_{d}[t] \)

\[
(5) \quad \overline{V}_{\mathcal{N}(A)}(t) = \overline{\chi}_{A}(t) \cdot \rho([\mathbb{R}^{d}]).
\]

**Proof.** The proof of Corollary 2.4 applies verbatim on noting that

\[
\overline{V}_{\mathcal{N}(A^{k})}(t) = (t^{k} + t^{k-1}) \cdot \rho([\mathbb{R}^{d}]) = \overline{\chi}_{A^{k}}(t) \cdot \rho([\mathbb{R}^{d}])
\]

for all \( k \)-singletons \( A^{k} \), \( 1 \leq k \leq d \). \( \square \)

Already in dimension 2 one can see that (5) holds only for generic points. It was shown in [12] that the **exceptional set** of points \( x \in \mathbb{R}^{d} \) where \( V_{k}(A)(x) \neq w_{k}(A) \) is a hyperplane arrangement. We will slightly generalize the results of [12] with a simpler proof that prompts a simple interpretation for the exceptional set.
For the proof, as well as the precise statement of our results, we recall two well-known ring structures on $\mathcal{C}_d$. First, note that $(\mathcal{C}_d, +, \cdot)$ is a commutative ring with unit $1 = [R^d]$ with respect to pointwise multiplication of functions. For $C, C' \in \text{Cones}_d$ we have

$$(\{C\} \cdot \{C'\})(x) := \{C\}(x) \cdot \{C'\}(x) = \{C \cap C'\}(x).$$

A second ring structure $(\mathcal{C}_d, +, *)$ is obtained with respect to taking conical hulls $C \vee C' := \text{cone}(C \cup C')$:

$$(\{C\} \ast \{C'\})(x) := \{C \vee C'\}(x).$$

These two ring structures are related via polarity: The polarity map given by $[C] \mapsto [\text{relint}(C)]$ is linear and since $(C \cap D)^\vee = C^\vee + D^\vee$ for all $C, D \in \text{Cones}_d$ for which $C \cup D$ is convex, we see that polarity gives an isomorphism of rings $(\mathcal{C}, +, \cdot) \cong (\mathcal{C}, +, *)$.

Let $A$ be a hyperplane arrangement in a subspace $U$ of $\mathbb{R}^d$. The lattice of flats $\mathcal{L}(A)$ is the collection of subspaces formed by intersections of hyperplanes in $A$ ordered by reverse inclusion. The minimal element is $\hat{0} = U$ and $\hat{1} = \text{lineal}(A)$ is the maximal element. The Möbius function $\mu_{\mathcal{L}}$ of a finite partially ordered set $(\mathcal{L}, \preceq)$ is recursively defined for $x \preceq y \in \mathcal{L}$ as follows: If $x = y$, then $\mu_{\mathcal{L}}(x, y) = 1$, otherwise

$$\mu_{\mathcal{L}}(x, y) = - \sum_{x \preceq z \preceq y} \mu_{\mathcal{L}}(x, z).$$

We will suppress the subscript of $\mu$ if $\mathcal{L}$ is clear from the context. It is well known that $\mu = \mu_{\mathcal{L}(A)}$ alternates in sign, or, more precisely, $|\mu(L, M)| = (-1)^{\dim L - \dim M} \mu(L, M)$. Let $\delta(L, K) = 1$ if $L = K$ and $= 0$ otherwise. Denote by $\mathcal{F}(\mathcal{N})$ the set of all nonempty faces of all cones of $\mathcal{N}$. We will first show the following lemma, which will be essential to our generalization of Corollary 3.6:

**Lemma 3.7.** Let $A$ be a hyperplane arrangement with lattice of flats $\mathcal{L}(A)$. Then

$$\sum_{C \in \mathcal{N}(A)} \{C\} = \sum_{L \in \mathcal{L}(A)} |\mu(\hat{0}, L)| \cdot \{L\}$$

as elements in $\mathcal{C}_d$.

Observe that the map $C \mapsto (-1)^{\dim C}[\text{relint}(C)] \in \mathcal{C}_d$ is a valuation. The Euler map is the induced homomorphism $\mathcal{E} : \mathcal{C}_d \to \mathcal{C}_d$ and one checks $\mathcal{E} \circ \mathcal{E} = \text{id}$.

**Proof.** Let $A$ be an arrangement in the $d$-dimensional subspace $U$ and let $L \in \mathcal{L}(A)$. Since

$$\{L\} = \sum_{F \in \mathcal{F}(A), F \subseteq L} [\text{relint}(F)]$$

It follows that

$$\{L\} = (-1)^{\dim L} \mathcal{E}([L]) = \sum_{F \in \mathcal{F}(A), F \subseteq L} (-1)^{\dim L - \dim F}[F].$$
We calculate:

\[
\sum_{L \in \mathcal{L}(A)} |\mu(\hat{0}, L)| \cdot [L] = \sum_{L \in \mathcal{L}(A)} (-1)^{d-\dim L} \mu(\hat{0}, L) \sum_{F \in F(A) \subseteq L} (-1)^{\dim L - \dim F} [F]
\]

\[
= \sum_{F \in F(A)} (-1)^{d-\dim F} \sum_{L \in \mathcal{L}(A) \subseteq F} \mu(\hat{0}, L)[F]
\]

\[
= \sum_{F \in F(A)} (-1)^{d-\dim F} \delta(\hat{0}, \text{lin}(F))[F] = \sum_{C \in \mathcal{N}(A)} [C].
\]

\[\text{Theorem 3.8.} \quad \text{For all } 0 \leq k \leq d, \text{ as elements in } \mathcal{C}_d:
\]

\[
V_k(A) = \sum_{L \in \mathcal{L}_k(A)} \left( \sum_{K \in \mathcal{L}(A) \subseteq L} |\mu(L, K)| \cdot [K] \right) \ast \left( \sum_{M \in \mathcal{L}(A) \subseteq L} |\mu(\hat{0}, M)| \cdot [M^\perp] \right)
\]

The proof is inspired by the arguments leading to [14, Theorem 5]. The additional bookkeeping is delegated to the ring structure on \(\mathcal{C}_d\).

\textbf{Proof.} We can rewrite the left-hand side:

\[
V_k(A) = \sum_{P \in \mathcal{N}(A)} \sum_{F \in F_k(P)} [F] \ast [N_F P] = \sum_{F \in F_k(A)} [F] \ast \sum_{P \in \mathcal{N}(A) \subseteq F} [N_F P].
\]

Let \(L = \text{lin}(F)\) and denote by \(A_L = \{H \in A : L \subseteq H\}\) the \textbf{localization} of \(A\) at \(L\). Note that there is a one-to-one correspondence between the regions \(C\) of \(A_L\) and \(P \in \mathcal{N}(A)\) with \(F \subseteq P\) and that under this correspondence \(C^\vee = N_F P\). Thus

\[
\sum_{F \in F_k(A)} [F] \ast \sum_{P \in \mathcal{N}(A) \subseteq F} [N_F P] = \sum_{L \in \mathcal{L}_k(A)} \left( \sum_{F \in \mathcal{N}(A_L)} [F] \right) \ast \left( \sum_{C \in \mathcal{N}(A_L)} [C^\vee] \right).
\]

Using Lemma 3.7 on the inner sums gives the result:

\[
\sum_{F \in \mathcal{N}(A_L)} [F] = \sum_{K \in A_L} |\mu_{A_L}(L, K)| \cdot [K] = \sum_{K \in \mathcal{L}(A_L)} |\mu_{A_L}(\hat{0}, K)| \cdot [K],
\]

and:

\[
\sum_{C \in \mathcal{N}(A_L)} [C^\vee] = \left( \sum_{C \in \mathcal{N}(A_L)} [C] \right)^\vee = \left( \sum_{M \in \mathcal{A}_L} |\mu_{A_L}(\hat{0}, M)| \cdot [M^\perp] \right)^\vee
\]

\[
= \sum_{M \in \mathcal{L}(A) \subseteq L \subseteq M} |\mu_{A}(\hat{0}, M)| \cdot [M^\perp].
\]

\[\text{For direct comparison with Theorem 1.4 of [12], note that for } K \subseteq L \subseteq M \in \mathcal{L}(A) \text{ we have } \dim(K + M^\perp) = d \text{ if and only if } K = L = M. \text{ Thus for a generic point } x \in \mathbb{R}^d \text{ the evaluation } ([K] \ast [M^\perp])(x) = 0 \text{ if } K \neq L \text{ or } L \neq M. \text{ This proves Corollary 3.6 coefficientwise:}
\]

\[
V_k(A)(x) = \sum_{L \in \mathcal{L}_k(A)} |\mu(L, L)| \cdot |\mu(\mathbb{R}^d, L)| \cdot ([L] \ast [L^\perp])(x) = w_k(A),
\]
where we used that the terms $|\mu(\mathbb{R}^d, L)|$ sum up to $w_k$ (cf. [23, Section 3.10]). Since all summands in (6) are positive, this also shows that $V_k(\mathcal{A})(x) \geq w_k$ for (nongeneric) points $x \in \mathbb{R}^d$. We have $V_k(\mathcal{A})(x) > w_k$ precisely when $x \in K + M^\perp$ for $K \subseteq L \subseteq M$ such that $\dim L = k$ and $\dim(K + M^\perp) = d - 1$. This arrangement has a simple interpretation. Assume that $\mathcal{A}'$ is an arrangement of hyperplanes in some subspace $U \subseteq \mathbb{R}^d$, then $\mathcal{A}' + U^\perp = \{H + U^\perp : H \in \mathcal{A}'\}$ is the corresponding arrangement in $\mathbb{R}^d$ with lineality space $U^\perp$.

**Corollary 3.9.** Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$. Then $V_k(\mathcal{A})(x) > w_k$ for some $0 \leq k \leq d$ if and only if $x$ is contained in the arrangement

$$\Pi(\mathcal{A}) := \bigcup_{L \in \mathcal{L}(\mathcal{A})} (\mathcal{A}/L) + L^\perp.$$

A point $x \in \mathbb{R}^d$ is contained in $(\mathcal{A}/L) + L^\perp$ if and only if the orthogonal projection of $x$ onto $L$ is contained in $\mathcal{A}/L$. The exceptional set of Theorem 3.8 was also considered by Lofano–Paolini [16]. For a subspace $U \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, let $d_U(x) := \|x - \pi_U(x)\|$ be the distance of $x$ to $U$. Lofano–Paolini call a point $x \in \mathbb{R}^d$ generic with respect to an arrangement $\mathcal{A}$ if $d_L(x) \neq d_{L'}(x)$ for all distinct $L, L' \in \mathcal{L}(\mathcal{A})$. The set of non-generic points lies on a collection of quadrics and is in general not a hyperplane arrangement. A characterization of the points away from $\Pi(\mathcal{A})$ is as follows; cf. Lemma 5.2 of [16].

**Lemma 3.10.** Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then $x \notin \bigcup \Pi(\mathcal{A})$ if and only if $d_{L'}(x) > d_L(x)$ for all flats $L, L' \in \mathcal{L}(\mathcal{A})$ with $L' \subseteq L$. Equivalently, this is the case if and only if $d_H(x) > d_L(x)$ for all flats $L$ and $H \in \mathcal{A}/L$.

**Remark 3.11.** The proof of Theorem 3.8 and Lemma 3.7 applies ad verbatim to affine hyperplane arrangements. For those, one has to work in the ring $(\mathcal{X}_d, +, *)$ of indicator functions of polyhedra instead of polyhedral cones, where $*$ is defined via the Minkowski-sum, $[Q] * [Q'] = [Q + Q']$. The only problem is our use of polarity in (7). Here we use that (7) holds in $(\mathcal{X}_d, +, *)$, which canonically embeds into $(\mathcal{X}_d, +, *)$, since $C \vee C' = C + C'$ for $C, C' \in \text{Cones}_d$.

Finally, we also want to give a proof that $\chi_N(-1) = 0$, which we also do on the level of indicator functions. For this, we give a simple (algebraic) proof of a result of Schneider [22] and its generalization to polyhedra by Hug–Kabluchko [11]. Recall that $\mathcal{F}(Q)$ denotes the set of nonempty faces of a polyhedron $Q$. The **Euler-Characteristic** of $Q$, is defined as

$$\varepsilon(Q) := \sum_{F \in \mathcal{F}(Q)} (-1)^{\dim F}.$$

Recall that for a polyhedron $Q$ with $L = \text{lineal}(Q)$ we have $\varepsilon(Q) = 0$ if $L = 0$ and $Q$ is unbounded and $\varepsilon(Q) = (-1)^{\dim L} \varepsilon(L^{-\perp}(Q))$ otherwise.

**Theorem 3.12.** Let $Q \subseteq \mathbb{R}^d$ be a polyhedron. Then, as elements in $\mathcal{X}_d$:

$$\varepsilon(Q) \cdot |\mathbb{R}^d| = \sum_{F \in \mathcal{F}(Q)} (-1)^{\dim F} [F - N_F Q].$$

**Proof.** Note that $(-1)^{\dim Q - \dim F} = \mu(F, Q)$, where $\mu = \mu_{\mathcal{F}(Q)}$. Denote by $T_F C := (N_F C)^\perp$ the tangent cone of $C$ at $Q$. By the Sommerville-relation (cf. [1, Lem. 4.1]) we have for any
cone $D \in \text{Cones}_d$: 
\[
\sum_{G \in \mathcal{F}(D)} (-1)^{\dim G} \cdot [T_G D] = (-1)^{\dim D} [-\text{relint } D].
\]
Applying $\mathcal{E}$ on both sides and using $\dim T_G D = \dim D$ we get 
\[
\sum_{G \in \mathcal{F}(D)} (-1)^{\dim G} \cdot [\text{relint } T_G D] = (-1)^{\dim D} [-D].
\]
If we set $D := N_F Q$, we have $T_{G'}(N_F Q) = N_{G'} Q$ for $G' := N_G Q$, so this reads:
\[
(-1)^{\dim Q - \dim F} [-N_F Q] = \sum_{G' \in \mathcal{F}(N_F Q)} (-1)^{\dim G'} \cdot [\text{relint } T_{G'}(N_F Q)]
\]
\[
= \sum_{F \subseteq G \in \mathcal{F}(Q)} (-1)^{\dim Q - \dim G} \cdot [\text{relint } N_F G].
\]
Now, we can simply compute:
\[
\sum_{F \in \mathcal{F}(Q)} (-1)^{\dim F} \cdot [F] \ast [-N_F Q] = \sum_{F \in \mathcal{F}(Q)} [F] \ast \sum_{F \subseteq G \in \mathcal{F}(Q)} (-1)^{\dim G} \cdot [\text{relint } N_F G]
\]
\[
= \sum_{G \in \mathcal{F}(Q)} (-1)^{\dim G} \sum_{F \in \mathcal{F}(G)} [F] \ast [\text{relint } N_F G]
\]
\[
= \sum_{G \in \mathcal{F}(Q)} (-1)^{\dim G} \cdot [\mathbb{R}^d] = \varepsilon(Q) \cdot [\mathbb{R}^d].
\]
That $\chi_N(-1) = 0$ follows now immediately by noting that
\[
\sigma_{d-1}(F + N_F C) = \sigma_{d-1}(F - N_F C)
\]
for all cones $C \in \mathcal{N}$ and faces $F$ of $C$.

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