Linear stability of magnetic vortex chains in a plasma in the presence of equilibrium electron temperature anisotropy

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Abstract. The linear stability of chains of magnetic vortices in a plasma is investigated analytically in two dimensions by means of a reduced fluid model assuming a strong guide field and accounting for equilibrium electron temperature anisotropy. The chain of magnetic vortices is modelled by means of the classical "cat’s eyes" solutions and the linear stability is studied by analysing the second variation of a conserved functional, according to the Energy-Casimir method. The stability analysis is based on a fluid model obtained from a gyrofluid model by means of a simple Hamiltonian reduction and is carried out on the domain bounded by the separatrices of the vortices. Two cases are considered, corresponding to a ratio between perpendicular equilibrium ion and electron temperature much greater or much less than unity, respectively. In the former case, equilibrium flows depend on an arbitrary function. Stability is attained if the equilibrium electron temperature anisotropy is bounded from above and from below, with the lower bound corresponding to the condition preventing the firehose instability. A further condition sets an upper limit to the amplitude of the vortices, for a given choice of the equilibrium flow. For cold ions, two sub-cases have to be considered. In the first one, equilibria correspond to those for which the velocity field is proportional to the local Alfvén velocity. Stability conditions imply: an upper limit on the amplitude of the flow, which automatically implies firehose stability, an upper bound on the electron temperature anisotropy and again an upper bound on the size of the vortices. The second sub-case refers to equilibrium electrostatic potentials which are not constant on magnetic flux surfaces and the resulting stability conditions correspond to those of the first sub-case in the absence of flow.

1. Introduction

The identification of coherent structures and the investigation of their stability is a classical subject in plasma physics. Among the various coherent structures that can form in plasmas, chains of magnetic vortices (also referred to as magnetic islands) are of considerable relevance for both laboratory and space plasmas, and the study of their stability began already a few decades ago [1, 2, 3, 4, 5]. Such stability analysis was often carried out in the context of a magnetohydrodynamic (MHD) description of a plasma and the modelling of the magnetic vortex chain took advantage from the existence of a well known solution of the Liouville equation (explicitly given later in Eq. (22)) which can be applied when investigating plasma equilibria with a symmetry [6]. This solution was adopted much earlier in fluid dynamics, where it is usually referred to as Kelvin-Stuart "cat’s eyes" solution [7, 8]. In plasma physics, such equilibrium solution proved to be a standard starting point for the investigation of problems related to island coalescence (see, for instance Refs. [9, 10, 11, 12] and references therein). To the
best of our knowledge, however, analytical investigations of the stability of magnetic island chains remained a minority, with respect to the vast amount of numerical results obtained on this subject. In particular, the impact of some two-fluid effects on the stability of magnetic vortex chains seems to lack a fully analytical description.

The purpose of this paper is to provide, by means of fully analytical methods, sufficient conditions for the linear stability of classes of equilibria with "cat's eyes" solutions for the magnetic field, in the framework of a reduced fluid model accounting for two-fluid effects. More precisely, we consider equilibrium solutions of the model, for which the magnetic field, in the plane perpendicular to a constant and uniform guide field, is described by the "cat's eyes" solution. The adopted reduced fluid model descends from the two-field Hamiltonian gyrofluid model considered in Ref. [13] (see Sec. 6 of such Reference), which in turn provides a slight generalization of the gyrofluid model for kinetic Alfvén wave turbulence presented in Ref. [14]. The model adopted in the present paper extends reduced MHD [15] [16], with the inclusion of equilibrium electron temperature anisotropy and magnetic perturbations in the direction parallel to the guide field. In its general formulation, the model accounts also for electron inertia and finite ion Larmor radius effects. In the present analysis, however, electron inertia will be neglected and ion temperature will be considered only in two extreme and opposite limits, i.e. when the ion temperature, referred to the plane perpendicular to the guide field, is much greater and much less than the corresponding electron temperature, respectively.

The investigation of the above mentioned two-fluid effects could shed some light, for instance, on instabilities driven by electron temperature anisotropy on magnetic vortex chains. This mechanism might be relevant for nearly collisionless plasmas, such as the solar wind, where the equilibrium distribution functions of particle populations are typically anisotropic. With regard to this, we remark a recent application of the "cat’s eyes" solutions, in the framework of reduced MHD [17], for the description of magnetic vortex chains observed in the solar wind by the Cluster spacecraft [18] [19].
Observational data proved indeed to yield structures compatible with those of the "cat’s eyes" solution. Such analysis, however, focused on scales larger than the ion thermal gyroradius, where two-fluid effects have little relevance.

With regard to ion temperature effects, although our analysis is limited to two extreme cases, it might provide a leading order indication of what configurations of electron gyrocenter density, electrostatic potentials and parallel magnetic perturbations can support magnetic vortex chains in plasmas with hot or cold ions at equilibrium (or, equivalently, at scales smaller or larger than the ion thermal gyroradius, given that the characteristic scale of the model is the perpendicular sonic Larmor radius).

The method adopted for the stability analysis is the Energy-Casimir method for determining formal stability, which implies linear stability [20, 21]. This method typically applies to Hamiltonian systems with a noncanonical Poisson bracket and is based on identifying conditions for which the second variation of a functional conserved by the model has a definite sign, when evaluated at the equilibrium point. This method is described in Refs. [20, 21] and examples of its application in the fluid and plasma physics literature can be found in Refs. [22, 23, 24, 25, 26, 27]. An application of this method to a plasma equilibrium with a "cat’s eyes" chain of vortices is provided in Ref. [28]. We also point out the description, in Ref. [29], of an MHD analytical investigation of the stability of magnetic vortex chains in the presence of flows, with application to tokamaks.

We mention that the steps of the Energy-Casimir method for linear stability analysis adopted here, can be extended to yield conditions for nonlinear stability, by carrying further estimates. This procedure is applied to fluids in Refs. [30, 31] and is described with various fluid and plasma examples in Ref. [21]. In Ref. [32], an analysis based on this method yields conditions for nonlinear stability of the "cat’s eyes" solution for the 2D Euler equation for an incompressible fluid. In the absence of results on the existence of solutions for the nonlinear system under investigation, which is the case for the present model, however, only conditional nonlinear stability can be proved [21].
Therefore, we content with deriving conditions for linear stability, which is also what is provided by usual analytical stability methods adopted in plasma physics.

Finally, we remark that a technical difficulty posed by the present problem concerns the two-dimensional (2D) domain where the stability analysis is carried out. We choose this domain to be the portion of space enclosed by the separatrices of the vortices, borrowing a procedure adopted in Ref. [32].

The paper is organized as follows. In Sec. 2 we introduce the general gyrofluid model of Ref. [13], review its main properties and recall its Hamiltonian structure. Subsequently, we carry out a simple Hamiltonian reduction, which yields a two-dimensional Hamiltonian version of the model, without electron inertia. The Casimir invariants of the resulting model are recalled and, together with the Hamiltonian, will form the starting point for the stability analysis. At the end of the Section we introduce the spatial domain where the stability analysis will be carried out. Sections 3 and 4 present the stability analysis in the limit of hot and cold ions, respectively. Both Sections begin with the introduction of the model equations in the corresponding limit, and of their conserved quantities. This is followed by the analysis of the first variation of the conserved functional, which leads to the classes of equilibria of interest. The two Sections end with the analysis of the second variation, yielding the stability conditions, which are discussed in the final part of each Section. We conclude in Sec. 5. Two Appendices are also provided. In Appendix A the main assumptions of the gyrofluid model are reviewed and its derivation from gyrokinetic equations is summarized in a qualitative way. Appendix B briefly reviews the adopted method for stability analysis.

2. The general gyrofluid model and its reduction

The starting point of our analysis consists of a nonlinear two-field gyrofluid model for collisionless plasmas, which assumes the presence of a strong component of the magnetic field (strong guide field assumption) along one direction. The model consists of the
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following two evolution equations

\[ \frac{\partial N_e}{\partial t} + [\phi, N_e] - [B\|, N_e] - [A\|, U_e] + \frac{\partial U_e}{\partial z} = 0, \]  

\[ \frac{\partial}{\partial t} (A\| - \delta^2 U_e) + [\phi - B\|, A\| - \delta^2 U_e] + \frac{1}{\Theta_e} [A\|, N_e] + \frac{\partial}{\partial z} \left( \phi - B\| - \frac{N_e}{\Theta_e} \right) = 0, \]  

complemented by the static relations

\[ N_e + (1 - \Gamma_0 i + \Gamma_1 i)B\| + (1 - \Gamma_0 i - \tau_{\perp i} \delta^2 \Delta_{\perp}) \frac{\phi}{\tau_{\perp i}} = 0, \]  

\[ U_e = b_x \Delta_{\perp} A\|, \]  

\[ B\| = -\frac{\beta_{\perp i}}{2} (N_e - (1 - \Gamma_0 i + \Gamma_1 i)\phi + (1 + 2\tau_{\perp i}(\Gamma_0 i - \Gamma_1 i))B\|), \]  

which permit to express \( \phi, A\| \) and \( B\| \) in terms of the dynamical variables \( N_e \) and \( A\| - \delta^2 U_e \).

Equation (1) is the continuity equation for electron gyrocenters, whereas Eq. (2) is the equation for the evolution of the momentum of electron gyrocenters in the direction parallel to the guide field (which will be referred to as parallel direction, in the following, as opposed to "perpendicular", which, as customary, refers to the plane perpendicular to the guide field). Alternatively, Eq. (2) can be seen as a generalized Ohm’s law in the parallel direction accounting for electron inertia and equilibrium electron temperature anisotropy. Eqs. (3), (4) and (5), on the other hand, correspond to the quasi-neutrality relation and to the components of Ampère’s law parallel and perpendicular, respectively, to the direction of the guide field. The model is formulated on a domain \( \mathcal{T}_n \) which, adopting a Cartesian reference frame \( xyz \), is given by \( \mathcal{T}_n = \{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 2\pi n, -L_y \leq y \leq L_y, -L_z \leq z \leq L_z \} \), with \( n \) a non-negative integer and \( L_y \) and \( L_z \) two positive constants. In Eqs. (1)-(5) the fields \( N_e, U_e, \phi, A\| \) and \( B\| \) are all functions of the independent variables \( x, y, z, t \), with \( t \) indicating time, and are all assumed to be periodic over the domain \( \mathcal{T}_n \). We indicated with \( N_e \) the fluctuations of the electron gyrocenter density, with \( U_e \), the fluctuations of the electron gyrocenter parallel velocity and with \( \phi \) the fluctuations of the electrostatic
potential, whereas $A_{∥}$ and $B_{∥}$ are related to the magnetic field $\mathbf{B}$ by

$$
\mathbf{B}(x, y, z, t) = \hat{z} + B_{∥}(x, y, z, t) \hat{z} + \nabla A_{∥}(x, y, z, t) \times \hat{z}.
$$

(6)

In Eq. (6), the first term on the right-hand side accounts for the (dimensionless) strong guide field, directed along the unit vector $\hat{z}$, whereas $B_{∥}$ indicates the perturbation of the magnetic field along $z$ and $A_{∥}$ is the fluctuation of the $z$ component of the vector potential (also referred to as magnetic flux function). The strong guide field assumption implies $B_{∥} \ll 1$ and $|\nabla A_{∥}| \ll 1$ in a sense that is made more precise in Appendix A.

Also, evidently, the expression for $\mathbf{B}$ in Eq. (6) is not divergence-free. Indeed, it only represents the expression of the total magnetic field at the first order in the fluctuations. The higher-order contributions, which guarantee $\nabla \cdot \mathbf{B} = 0$, are negligible at the order retained in the model.

The dimensionless variables adopted in Eqs. (1)-(5) are defined by

$$
x = \frac{\bar{x}}{\rho_{s⊥}}, \quad y = \frac{\bar{y}}{\rho_{s⊥}}, \quad z = \frac{\bar{z}}{\rho_{s⊥}}, \quad t = \omega_{ci} \bar{t},
$$

$$
N_e = \frac{\tilde{N}_e}{n_0}, \quad U_e = \frac{\tilde{U}_e}{c_{s⊥}},
$$

$$
\phi = \frac{e \tilde{\phi}}{T_{0⊥e}}, \quad B_{∥} = \frac{\tilde{B}_{∥}}{B_0}, \quad A_{∥} = \frac{\tilde{A}_{∥}}{B_0 \rho_{s⊥}},
$$

(7)

where the tilde denotes the dimensional quantities. In Eq. (7) $B_0$ is the amplitude of the guide field, $n_0$ is the homogeneous equilibrium particle density (equal for both electrons and ions), $T_{0⊥e}$ is the equilibrium electron temperature in the plane perpendicular to the guide field, $e$ is the proton charge. Denoting with $m_i$ the mass of the ions present in the plasma and with $c$ the speed of light, we also made use of the quantities $\omega_{ci} = eB_0/(m_ic)$ indicating the ion cyclotron frequency, $c_{s⊥} = \sqrt{T_{0⊥e}/m_i}$ indicating the sound speed based on the perpendicular temperature and $\rho_{s⊥}$, which is the sonic Larmor radius, also based on the perpendicular temperature. Four independent constant parameters are present in the system, and are given by

$$
\delta = \sqrt{\frac{m_e}{m_i}}, \quad \beta_{⊥e} = 8\pi \frac{n_0 T_{0⊥e}}{B_0^2}, \quad \tau_{⊥i} = \frac{T_{0⊥i}}{T_{0⊥e}}, \quad \Theta_e = \frac{T_{0⊥e}}{T_{0∥e}},
$$

(8)
representing the mass ratio, the ratio between perpendicular electron pressure and guide field magnetic pressure, the ion-to-electron perpendicular temperature ratio, and the electron temperature anisotropy, respectively, at equilibrium. We also introduced the short-hand notation $b_\star$ defined by

$$b_\star = \frac{2}{\beta_{\perp_e}} + 1 - \frac{1}{\Theta_e}, \quad (9)$$

to indicate the modification due to electron temperature anisotropy in the parallel Ampère’s law $[4]$. Note that $b_\star = 2/\beta_{\perp_e}$ in the isotropic case. We indicated with $\Delta_\perp$ the Laplacian operator in the perpendicular plane, so that $\Delta_\perp f = \partial_{xx} f + \partial_{yy} f$ for a function $f$. The operators $\Gamma_{0i}$ and $\Gamma_{1i}$ represent the standard operators associated with ion gyroaverage. We can define them in the following way. Let us consider a function $f = f(x, y, z)$, periodic over $\mathcal{T}_n$ and indicate with $\mathcal{T}_n$ the lattice $\mathcal{T}_n = \{(l/n, \pi m/L_y, \pi p/L_z) : (l, m, p) \in \mathbb{Z}^3\}$. We write the Fourier representation of $f$ as

$$f(x, y, z) = \sum_{\mathbf{k} \in \mathcal{T}_n} f_\mathbf{k} \exp(i \mathbf{k} \cdot \mathbf{x}),$$

where $\mathbf{x}$ and $\mathbf{k}$ are vectors of components $(x, y, z)$ and $(k_x, k_y, k_z)$, respectively, with $k_x = l/n$, $k_y = m\pi/L_y$, $k_z = p\pi/L_z$, for $(l, m, p) \in \mathbb{Z}^3$. It is also convenient to introduce the quantity $b_i = \tau_{\perp_i} k_{\perp_i}^2$, where $k_{\perp_i} = \sqrt{k_x^2 + k_y^2}$ in adimensional variables is the perpendicular wave number (in dimensional variables one would have $b_i = \tilde{k}_{\perp_i}^2 \rho_{th_{\perp_i}}^2$, where $\tilde{k}_{\perp}$ is the dimensional perpendicular wave number and $\rho_{th_{\perp_i}} = \sqrt{T_{0_{\perp_i}}/m_i/\omega_{ci}}$ is the perpendicular thermal ion gyroradius). The action of the operators $\Gamma_{0i}$ and $\Gamma_{1i}$ on the function $f$ is defined by

$$\Gamma_{0i} f(x, y, z) = \sum_{\mathbf{k} \in \mathcal{T}_n} I_0(b_i) e^{-b_i} f_\mathbf{k} e^{i \mathbf{k} \cdot \mathbf{x}}, \quad (10)$$

$$\Gamma_{1i} f(x, y, z) = \sum_{\mathbf{k} \in \mathcal{T}_n} I_1(b_i) e^{-b_i} f_\mathbf{k} e^{i \mathbf{k} \cdot \mathbf{x}}, \quad (11)$$

with $I_0$ and $I_1$ indicating the modified Bessel functions of the first kind, of order 0 and 1, respectively. The canonical bracket $[,]$, on the other hand, is defined by $[f, g] = \partial_x f \partial_y g - \partial_y f \partial_x g$, for two functions $f$ and $g$.

In the light of the above definition, we can recognize in the first three terms of the continuity equation $[1]$, the material derivative of the electron gyrocenter density, which
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is advected by a generalized incompressible velocity field \( \mathbf{U}_\perp = \hat{z} \times \nabla (\phi - B_\parallel) \). The latter originates from the electron gyrocenter velocity in the perpendicular plane, induced by electromagnetic perturbations. The last two terms, on the other hand, express the gradient of the parallel electron gyrocenter velocity along the magnetic field. Equation (2) expresses, with its first two terms, the material derivative of the fluid version of the parallel canonical electron momentum \( \delta^2 U_e - A_\parallel \), advected by \( \mathbf{U}_\perp \). The third and last term of the equation express the force exerted by the parallel gradient of the electron parallel pressure and are affected by temperature anisotropy, whereas an additional force comes from the variation of \( \phi - B_\parallel \) along the guide field. Alternatively, one could think at the terms \( [A_\parallel, N_e/\Theta_e + B_\parallel] - \partial_z (N_e/\Theta_e + B_\parallel) \) as coming from the projection of the divergence of the anisotropic electron pressure tensor along the magnetic field. The terms \( \partial_t A_\parallel + [\phi, A_\parallel] + \partial_z \phi \), on the other hand, come from the projection of the electric field along the magnetic field, whereas the remaining terms, i.e. those containing the parameter \( \delta^2 \), are those associated with the projection of the electron inertia terms. In the quasi-neutrality relation (3), the first two terms and the last term indicate the electron density fluctuations (recall that, in the limit of small electron finite Larmor radius (FLR) corrections, which is the case here, the relation \( n_e = N_e + B_\parallel - \delta^2 \Delta_\perp \phi \) holds, where \( n_e \) indicates the normalized electron density fluctuations, which differ from the electron gyrocenter density fluctuations \( N_e \)). The remaining terms account for the contributions due to electromagnetic perturbations, and depend on ion finite Larmor radius effects, that arise when expressing the ion density fluctuations in terms of ion gyrocenter variables. Likewise, analogous contributions appear in Eq. (5), upon replacing \( N_e \) with \( n_e - B_\parallel \) (in Eq. (5), the above mentioned electron FLR contribution \( \delta^2 \Delta_\perp \phi \), appearing in the relation between \( n_e \) and \( N_e \), turn out to be negligible according to both scaling \((A.10)\) and \((A.11)\)). Parallel Ampère’s law (4) expresses the fact that the parallel current density is proportional to the parallel electron gyrocenter velocity, the contribution of the gyrocenter ion velocity being negligible in the present model. This relation is also affected by the temperature anisotropy.
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The present model extends the model derived in Ref. [14] by including also equilibrium electron temperature anisotropy. As explained in Ref. [14], it also extends other models adopted in the literature. For instance, in the limit of isotropic electron temperature $\Theta_e = 1$, for cold ions ($\tau_{\perp i} \ll 1$), and with $\beta_{\perp e}$ small enough to neglect parallel perturbations $B_\parallel$, the system reduces to the two-field model considered in a number of works on collisionless magnetic reconnection such as those of Refs. [34] and [35]. In turn, such limit reduces to low-$\beta$ reduced MHD [16, 15] when electron inertia and the parallel pressure term in Eq. (2) are neglected. The system (1)-(5) can also be seen as an extension of the model for inertial kinetic Alfvén turbulence described in Ref. [36], accounting also for ion finite Larmor radius effects, parallel electron pressure and equilibrium electron temperature anisotropy.

The model (1)-(5) can be derived from the gyrokinetic model described in Ref. [37]. The derivation procedure can be found in Ref. [13] (see, in particular Sec. 6 of the cited Reference) and is summarized in Appendix A.

2.1. Hamiltonian structure

As remarked in Ref. [13], the model is obtained by taking moments of gyrokinetic equations and imposing a closure (A.9) that guarantees the existence of a Hamiltonian structure. Such structure is of noncanonical type (see, e.g. Ref. [20]). The Hamiltonian functional is given by

$$H(N_e, A_e) = \frac{1}{2} \int_{\tau_n} d^3 x \left( \frac{N_e^2}{\Theta_e} + \delta^2 U_e^2 + b_\star |\nabla_{\perp} A_\parallel|^2 - N_e (\phi - B_\parallel) \right), \quad (12)$$

with $A_e = A_\parallel - \delta^2 U_e$. By means of the relations (3)-(5), it can be shown that the operators permitting to express the functions $A_\parallel, B_\parallel$ and $\phi$ in terms of the dynamical variables $N_e$ and $A_e$ are symmetric with respect to the inner product $\langle f \mid g \rangle = \int_{\tau_n} d^3 x f g$. This allows to show the conservation of $H$.

We introduce the notation $F_f = \delta F/\delta f$ to indicate the functional derivative of a functional $F$ with respect to a function $f$. With this notation, the noncanonical Poisson
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The evolution equations (1) and (2) can then be written in the Hamiltonian form

\[ \frac{\partial A_e}{\partial t} = \{A_e, H\}, \quad \frac{\partial N_e}{\partial t} = \{N_e, H\}, \tag{14} \]

with \(H\) and \(\{\,\,\}\) given by Eqs. (12) and (13), respectively.

2.2. Hamiltonian reduction to the two-dimensional limit with no electron inertia

The analysis of chains of magnetic vortices based on Kelvin-Stuart "cat's eyes" solutions leads us to consider a 2D reduction of the system (1)-(5). In the 2D limit, the application of the Energy-Casimir method becomes also particularly fruitful, due to the abundance of Casimir invariants. Furthermore, chains of magnetic vortices observed, for instance in the solar wind, appear to have an essentially 2D structure \[19\]. Also, we introduce a further simplification by neglecting corrections due to electron inertia (i.e. by taking the limit \(\delta \to 0\)). This amounts to neglecting effects occurring on scales comparable to the electron skin depth. Magnetic chains on scales much larger than the electron skin depth are indeed those observed in the solar wind \[19, 17\].

We perform this reduction of the two-field gyrofluid model acting on its Hamiltonian structure, rather than directly on its equations of motion. This permits to preserve a Hamiltonian character in the reduced model. We assume then that in the Hamiltonian (12) and in the Poisson bracket (13) the field variables \(N_e\) and \(A_e\) are invariant with respect to the \(z\) coordinate. Also, we neglect in the Hamiltonian (12) the contributions proportional to electron inertia, which amounts to neglecting the second term on the right-hand side of Eq. (12) (parallel electron kinetic energy) as well as the electron FLR correction corresponding to the last term of Eq. (3), which intervenes when one expresses \(\phi\) and \(B_\parallel\) in terms of \(N_e\). Analogously, we act on the Poisson bracket (13) by
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neglecting the contribution \( \delta^2 U_e \) due to electron momentum, in the expression for \( A_e \) and suppress the term proportional to \( \delta^2 \) in the first line of Eq. (13). Note that we are allowed to carry out the latter operation because the Poisson bracket (13) maintains all the properties of a Poisson bracket (and in particular the Jacobi identity) for all values of \( \delta \), and in particular also for \( \delta = 0 \).

As a result, we obtain the following reduced Hamiltonian (where we reabsorbed in the definition of \( H \) a factor 2\( L_z \) due to the integration with respect to \( z \) between \( -L_z \) and \( L_z \))

\[
H(N_e, A^\parallel) = \frac{1}{2} \int_{D_n} d^2x \left( \frac{N_e^2}{\Theta_e} + b_e |\nabla A^\parallel|^2 - N_e(\phi - B^\parallel) \right),
\]

and the reduced Poisson bracket

\[
\{F, G\} = \int_{D_n} d^2x (N_e [F_{N_e}, G_{N_e}] + A^\parallel ([F_{A^\parallel}, G_{N_e}] + [F_{N_e}, G_{A^\parallel}])).
\]

We also introduced the 2D domain \( D_n = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi n, -L_y \leq y \leq L_y\} \).

Accordingly, the Fourier components for the fields in this model get defined over the 2D lattice \( \mathcal{D}_n = \{(l/n, \pi m/L_y) : (l, m) \in \mathbb{Z}^2\} \).

The equations of motion resulting from the Hamiltonian (15) and the Poisson bracket (16) are

\[
\frac{\partial N_e}{\partial t} + [\phi - B^\parallel, N_e] - b_e [A^\parallel, \Delta A^\parallel] = 0, \tag{17}
\]

\[
\frac{\partial A^\parallel}{\partial t} + [\phi - B^\parallel, A^\parallel] + \frac{1}{\Theta_e} [A^\parallel, N_e] = 0, \tag{18}
\]

where we also made use of Eq. (4). The expressions for \( B^\parallel \) and \( \phi \) in terms of \( N_e \) follow from Eqs. (3) and (5) which, in the limit \( \delta \to 0 \), become

\[
N_e + (1 - \Gamma_0 i + \Gamma_1 i) B^\parallel + (1 - \Gamma_0 i) \frac{\phi}{\tau_{\perp i}} = 0, \tag{19}
\]

\[
B^\parallel = -\frac{\beta_{1e}}{2} (N_e - (1 - \Gamma_0 i + \Gamma_1 i) \phi + (1 + 2\tau_{\perp i}(\Gamma_0 i - \Gamma_1 i))B^\parallel). \tag{20}
\]

We note that the Poisson bracket (16) is the same as that of 2D reduced MHD \[38\]. Therefore, Eqs. (17)-(18), in addition to the Hamiltonian (15), possess infinite conserved
Figure 1: The figure shows a surface plot and some contour lines of the "cat’s eyes" function $A_{eq}$. The domain $D_n$, enclosed by the separatrices, indicated with black dotted curves, and the domain $R_n$, corresponding to the rectangle enclosed by black solid lines, are also shown. The figure refers to the case $n = 3$ and $a = 1.12$.

functionals, given by

$$C_1 = \int_{D_n} d^2 x N_e F(A_{\parallel}), \quad C_2 = \int_{D_n} d^2 x G(A_{\parallel}),$$

(21)

where $F$ and $G$ are arbitrary functions. The functionals $C_1$ and $C_2$ are Casimir invariants of the Poisson bracket [16] and, as such, they satisfy $\{C_1, E\} = \{C_2, E\} = 0$ for every functional $E$. As discussed in Ref. [38], the Casimir $C_1$ includes, among others, the conservation of the integral of $N_e$ over an area bounded by contour lines of $A_{\parallel}$. The Casimir $C_2$, on the other hand, expresses, for $G(A_{\parallel}) = A_{\parallel}$, conservation of magnetic helicity at leading order. This arises as a consequence of removing electron inertia, which violates the frozen-in condition.

2.3. The domain of analysis

We intend to analyse the linear stability of equilibria such that the equilibrium solution for the magnetic flux function $A_{\parallel}$ corresponds to the "cat’s eyes" solution

$$A_{eq}(x, y) = -\log(a \cosh y + \sqrt{a^2 - 1} \cos x),$$

(22)

where $a > 1$.

As shown in Fig. 1, the contour lines of the function $A_{eq}$, in general describe chains of magnetic vortices in the plane $xy$. As $a \to 1^+$, the equilibrium configuration tends toward a uni-directional sheared magnetic field, with no magnetic vortices.
The function $A_{eq}$ is known to be a solution of the Liouville’s equation

$$\Delta_{\perp} A_{\parallel} = -e^{2A_{\parallel}}. \quad (23)$$

Inspired by the procedure followed in Ref. [32], we carry out the stability analysis of the magnetic vortex chains on the domain

$$D_n = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi n, \ |y| \leq \cosh^{-1} \left(1 + \frac{\sqrt{a^2 - 1}}{a} (1 - \cos x)\right)\right\}, \quad (24)$$

with $a > 1$.

An example of such domain is depicted in Fig. 1. One can see that the domain corresponds to the domain bounded by the separatrices of the vortex chain and the number $n$ indicates the number of vortices in the domain. This choice for the domain allows, with appropriate boundary conditions, for the application of the Poincaré inequality, which is crucial for carrying out some stability estimates that will be required later in the analysis. Of course, the choice of such domain rules out the effect of perturbations coming from outside the vortex chain. This can indeed be seen as a limitation of the present analysis. However, numerical simulations [39, 40, 41] show that, for instance, secondary instabilities due to colliding jets, can originate inside a magnetic island, and the subsequent turbulent evolution of the instability remains confined within the island. Therefore, in addition to the above mentioned technical argument related to the Poincaré inequality, restricting the analysis to the region enclosed by the separatrices does not appear to rule out all physically relevant processes. On the other hand, as pointed out in Ref. [32], this prevents from a direct comparison with classical results on stability of magnetic island chains, such as those of Refs. [1, 2, 5].

The domain $D_n$ differs from the domain $\mathcal{D}_n$ introduced in Sec. 2.2 and different boundary conditions will have to be adopted. In particular, the definitions (10) and (11) of ion gyroaverage operators $\Gamma_{0i}$ and $\Gamma_{1i}$, are valid for a periodic domain. Variants of gyroaverage operators, that permit to account for different boundary conditions (e.g. Dirichlet), have been discussed, for instance in Refs. [42, 43]. These variants are often based on Taylor expansions or Padé approximants, and on the identification between the
quantity \(b_i\) and the operator \(-\tau_{\perp i} \Delta_{\perp}\). We follow the same practice and, in the two cases that will be treated in the subsequent Sections, we will use (very rough) approximants of the ion gyroaverage operators in two opposite limits.

3. Hot-ion case : \(\tau_{\perp i} \gg 1\)

We consider the model (17)-(20) in the limit \(\tau_{\perp i} \gg 1\), corresponding to an equilibrium perpendicular ion temperature much larger than the corresponding electron temperature. This limit was adopted for instance in the model of Ref. [36] to describe turbulence at kinetic scales in the magnetosheath. Because of the relation 
\[
\rho_{th\perp i} = \sqrt{\tau_{\perp i} \rho_{s\perp i}},
\]
considering the limit \(\tau_{\perp i} \gg 1\) implies that the characteristic length of our model, corresponding to \(\rho_{s\perp i}\), is much smaller than the perpendicular thermal ion gyroradius \(\rho_{th\perp i}\). Therefore, in this sense, one can refer to this limit, also as to a sub-ion limit. Because 
\[
I_0(b_i)e^{-b_i} = I_0(\tau_{\perp i} k_{\perp i}^2)e^{-\tau_{\perp i} k_{\perp i}^2} \to 0, \text{ as } \tau_{\perp i} \to +\infty,
\]
for all \((k_x, k_y) \in \mathcal{D}_n \setminus (0, 0)\) and 
\[
I_1(b_i)e^{-b_i} = I_1(\tau_{\perp i} k_{\perp i}^2)e^{-\tau_{\perp i} k_{\perp i}^2} \to 0, \text{ as } \tau_{\perp i} \to +\infty,
\]
for all \((k_x, k_y) \in \mathcal{D}_n\) we simply take, for \(\tau_{\perp i} \gg 1\), the following approximated form for the operators \(\Gamma_0i\) and \(\Gamma_{1i}\):
\[
\Gamma_0i f(x, y) = 0, \quad \Gamma_{1i} f(x, y) = 0,
\]
for a function \(f(x, y)\) with \((x, y) \in D_n\) (for the mode \((k_x, k_y) = (0, 0)\), an agreement between the exact form for \(\Gamma_0i\) acting on functions over \(D_n\) and the approximated form, written in Eq. (25), for \(\tau_{\perp i} \gg 1\) and for functions over \(D_n\) can be obtained if, in the former case, one restricts to functions \(f(x, y, t) = \sum_{(k_x, k_y) \in \mathcal{D}_n} f_{(k_x, k_y)}(t) \exp(i(k_x x + k_y y))\) such that \(f_{(0,0)} = 0\), i.e. functions with zero spatial average).

In the limit \(\tau_{\perp i} \gg 1\), the model (17)-(20) thus reduces to
\[
\frac{\partial N_e}{\partial t} - b_\star [A_{\parallel}, \Delta_{\perp} A_{\parallel}] = 0, \quad \frac{\partial A_{\parallel}}{\partial t} - \kappa [N_e, A_{\parallel}] = 0,
\]
where
\[
B_{\parallel} = -N_e, \quad \phi = -\frac{2}{\beta_{\perp e}} N_e.
\]
In Eq. (27) we also introduced the parameter
\[ \kappa = \frac{2}{\beta_{\perp e}} + \frac{1}{\Theta_e} - 1. \] (29)

For \( \Theta_e = 1 \), i.e. for isotropic temperature, the model is analogous to the 2D version of the reduced electron MHD model discussed in Refs. [44] and [45].

Equations (26)-(27) are supplemented with the boundary conditions
\[ A_{\parallel}\big|_{\partial D_n} = a_A, \] (30)
\[ N_e\big|_{\partial D_n} = a_N, \] (31)
with \( a_A, a_N \in \mathbb{R} \) and where we indicated with \( \partial D_n \) the boundary of \( D_n \). The boundary condition (30) expresses the fact that the perpendicular magnetic field \( B_{\perp} = \nabla A_{\parallel} \times \hat{z} \) is tangent to the boundary, i.e. \( B_{\perp} \cdot n = 0 \), where \( n \) is the outward unit vector normal to the boundary \( \partial D_n \). The condition (31), on the other hand, implies \( U_{\perp e} \cdot n = 0 \), meaning that the incompressible flow \( U_{\perp e} = \hat{z} \times \nabla (\phi - B_{\parallel}) = (1 - 2/\beta_{\perp e}) \hat{z} \times \nabla N_e \) is tangent to the boundary.

The procedure we adopt to investigate the linear stability of magnetic vortex chains is summarized in Appendix B. Detailed descriptions of the method can be found in Refs. [20] and [21]. The first step consists of finding a functional \( F \) given by a combination of constants of motion of the system (26)-(27). To this purpose we can use the Hamiltonian (15) and the Casimir invariants (21), with \( \phi \) and \( B_{\parallel} \) given by Eq. (28). Indeed, such functionals are also conserved by the system (26)-(27) on the domain \( D_n \). This can be shown by direct computation making use of the identities
\[ \int_{D_n} d^2 x f \Delta_{\perp} g = - \int_{D_n} d^2 x \nabla f \cdot \nabla g + \int_{\partial D_n} f \frac{\partial g}{\partial n} ds, \] (32)
\[ \int_{D_n} d^2 x f [g, h] = \int_{D_n} d^2 x h [f, g] - \int_{\partial D_n} h f \nabla g \cdot d\mathbf{l}, \] (33)
for functions \( f, g \) and \( h \), and of the boundary conditions (30) and (31). In Eq. (32) we indicated with \( ds \) the scalar infinitesimal arc element and with \( \partial g/\partial n = \nabla g \cdot n \) the gradient normal to the boundary. In Eq. (33) we indicated with \( d\mathbf{l} \) the vectorial infinitesimal arc element.
3.1. First variation and equilibria

We consider then the conserved functional $F = H + C_1 + C_2$, explicitly given by

$$F(N_e, A_\parallel) = \int_{D_n} d^2x \left( b_s |\nabla A_\parallel|^2 + \kappa \frac{N_e^2}{2} + N_e \mathcal{F}(A_\parallel) + \mathcal{G}(A_\parallel) \right). \quad (34)$$

Adopting, for the variations $\delta A_\parallel$ and $\delta N_e$, the boundary conditions

$$\delta A_\parallel|_{\partial D_n} = 0, \quad \delta N_e|_{\partial D_n} = 0, \quad (35)$$

the first variation of $F$ is given by

$$\delta F(N_e, A_\parallel; \delta N_e, \delta A_\parallel) =$$

$$\int_{D_n} d^2x \left( (-b_s \Delta_\perp A_\parallel + \mathcal{F}'(A_\parallel) N_e + \mathcal{G}'(A_\parallel)) \delta A_\parallel + (\kappa N_e + \mathcal{F}(A_\parallel)) \delta N_e \right),$$

where the prime denotes derivative with respect to the argument of the function.

Setting the first variation $\delta F$ equal to zero for arbitrary perturbations, leads to the system

$$\Delta_\perp A_\parallel = \frac{\mathcal{F}'(A_\parallel) N_e}{b_s} + \frac{\mathcal{G}'(A_\parallel)}{b_s}, \quad (37)$$

$$\mathcal{F}(A_\parallel) = -\kappa N_e, \quad (38)$$

Solutions of Eqs. (37)-(38) are equilibrium solutions of the system (26)-(27). Eq. (37) can be seen as a Grad-Shafranov equation for the current density $-\Delta_\perp A_\parallel$, whereas Eq. (38) expresses the fact that the electron gyrocenter density fluctuations $N_e$ (and, by virtue of Eq. (28), the electrostatic potential and the parallel magnetic perturbations) are constant on perpendicular magnetic field lines identified by $A_\parallel = \text{constant}$. For such equilibria $\mathcal{F}(A_\parallel) = \kappa(\beta_{\perp_{ee}}/2) \phi$, and in particular $\mathcal{F}(A_\parallel) = \phi$ for isotropic temperature. Therefore, for $\mathcal{F} = 0$ we obtain an equilibrium with no perpendicular equilibrium flow. For $\mathcal{F}(A_\parallel) = \pm \sqrt{2/\beta_{\perp_{ee}}} A_\parallel$ and assuming isotropic temperature, on the other hand, one obtains Alfvénic solutions, in which the equilibrium $E \times B$ velocity field, given by $\hat{z} \times \nabla \phi$, equals, in dimensional units, the local Alfvén velocity field (or its opposite). In the more general case with $\Theta_e \neq 1$, the Alfvén velocity will be modified by an effect due
to temperature anisotropy. When $F$ is taken as a linear function of $A_{\parallel}$, clearly also the perpendicular equilibrium flow exhibits the "cat’s eyes" pattern.

The system is characterized by the two arbitrary functions $F$ and $G$. Because we are interested in solutions for $A_{\parallel}$ given by the "cat’s eyes" function (22), we constrain Eq. (37) to equal the Liouville equation (23) (we consider here non-propagating solutions but a generalization to account for a constant propagation velocity could be carried out). This occurs if the following condition on the function $G$ is fulfilled:

$$G(A_{\parallel}) = -\frac{b_0}{2}e^{2A_{\parallel}} + \frac{F^2(A_{\parallel})}{2\kappa} + c_1,$$

with $c_1$ arbitrary constant.

Our analysis will then focus on the class of equilibria given by

$$A_{\parallel} = A_{eq},$$

$$N_e = -\frac{F(A_{eq})}{\kappa},$$

for $\kappa \neq 0$, with $A_{eq}$ given by Eq. (22) and arbitrary $F$. The corresponding expressions for $\phi$ and $B_{\parallel}$ at equilibrium are given by $\phi = 2F(A_{eq})/(\beta_{\perp}\kappa)$ and $B_{\parallel} = F(A_{eq})/\kappa$, respectively. Therefore, we note that, for $\tau_{\perp} \gg 1$, equilibria obtained from the above variational principle and possessing a magnetic vortex chain, admit a whole class of flows (or, equivalently, of electron gyrocenter density or parallel magnetic perturbations) depending on an arbitrary function.

3.2. Second variation and stability conditions

The second variation of $F$, making use of the boundary conditions (35) and rearranging terms, can be written as

$$\delta^2 F(A_{\parallel}, N_e; \delta A_{\parallel}, \delta N_e) = \int_{D_\alpha} d^2x \left( b_0|\nabla \delta A_{\parallel}|^2 + (F''(A_{\parallel})N_e + G''(A_{\parallel}) - F'(A_{\parallel}))|\delta A_{\parallel}|^2 
+ (\kappa - 1)|\delta N_e|^2 + (F'(A_{\parallel})\delta A_{\parallel} + \delta N_e)^2 \right)$$

We intend to find conditions for which $\delta^2 F$, evaluated at the class of equilibrium of interest, is positive for arbitrary perturbations. If we impose $b_0 > 0$ and $\kappa > 1$, it is only
the coefficient of $|\delta A\parallel|^2$ that can provide a negative contribution, and thus indefiniteness, to $\delta^2 F$. Using the relation (39) one finds that, for the class of equilibria of interest, such coefficient is given by $-2b_\ast e^{2A_{eq}} + (1/\kappa - 1)F')(A_{eq})$. For $\kappa > 1$ this coefficient is always negative, so the second variation has no definite sign. This indefiniteness seems to reflect a feature of "cat’s eyes" equilibria that was already pointed out in Ref. [46] in the case of the 2D Euler equation for an incompressible flow. This difficulty can be overcome, as indicated in Ref. [46], by making use of a Poincaré inequality. In our specific case, the required Poincaré inequality reads

$$\int_{D_n} d^2x |\nabla \delta A\parallel|^2 \geq k^2_{\min} \int_{D_n} d^2x |\delta A\parallel|^2,$$

with $\delta A\parallel_{|\partial D_n} = 0$. In the inequality (43), $k^2_{\min}$ is the minimal eigenvalue of the operator $-\Delta_\perp$ acting on the functions defined over $D_n$ and vanishing on the boundary of $D_n$. The inequality (43) can be derived with a straightforward modification of the procedure followed in Ref. [32]. Following this same Reference, we make use of the fact that $k^2_{\min} > k^2_R$, where $k^2_R$ is the minimal eigenvalue of the operator $-\Delta_\perp$ on the functions defined over $R_n$ and vanishing on the boundary of $R_n$. The domain $R_n \supset D_n$ is defined by

$$R_n = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2n\pi, \ |y| \leq l = \cosh^{-1}\left(1 + 2\sqrt{a^2 - 1}/a\right)\}$$

(44)

and corresponds to the rectangle of width $2n\pi$ and height $2l$ equal to the magnetic island width. The rectangle $R_n$ is depicted in Fig. 1. For perturbations vanishing on the boundary of $R_n$, one has

$$k^2_R = \frac{1}{4n^2} + \frac{\pi^2}{4l^2}.$$  

(45)

With regard to this point, we remark that the expression for the minimal eigenvalue (45) differs by a factor 4 from the one used in Ref. [32] for the fluid case. The reason for this difference is due to the fact that in Ref. [32], in order to obtain the equilibrium equation, the perturbations of the stream function were assumed to vanish on the boundary and to have zero circulation along the boundary. In our case, in order to obtain the desired equilibrium equations for the magnetic field, it is sufficient to impose
that the perturbations of $A_\parallel$ and $N_e$ vanish on the boundary.

With the help of the above reasoning, we can state that, for $b_\star > 0$

$$
\delta^2 F(A_{eq}, F(A_{eq}); \delta A_\parallel, \delta N_e) \geq \int_{D_n} d^2 x \left( (b_\star k_R^2 - 2b_\star e^{2A_{eq}} + (1/\kappa - 1) F^2(A_{eq}) ) |\delta A_\parallel|^2 
+ (\kappa - 1)|\delta N_e|^2 + (F'(A_{eq}) \delta A_\parallel + \delta N_e)^2 \right). 
$$

(46)
The coefficient of $|\delta A_\parallel|^2$ on the right-hand side of Eq. (46) can be made positive by choosing appropriate bounds for $F^2(A_{eq})$. In particular, noticing that

$$
\min_{(x,y) \in D_n} (-2b_\star e^{2A_{eq}(x,y)}) = -2b_\star e^{2A_{eq}(\pi,0)} = -\frac{2b_\star}{(a - \sqrt{a^2 - 1})^2},
$$

one can write that, for $(x, y) \in D_n$:

$$
b_\star k_R^2 - 2b_\star e^{2A_{eq}(x,y)} + \left( \frac{1}{\kappa} - 1 \right) F^2(A_{eq}(x,y)) 
\geq b_\star \left( k_R^2 - \frac{2}{(a - \sqrt{a^2 - 1})^2} \right) + \left( \frac{1}{\kappa} - 1 \right) F^2(A_{eq}(x,y)).
$$

(48)

Making use of the relations (46), (48), (45) as well as of the previously mentioned conditions $b_\star > 0$ and $\kappa > 1$, we can conclude that the linear stability of the family of equilibria (40)-(41) is attained if the following three conditions are satisfied:

$$
b_\star > 0,
$$

(49)

$$
\kappa > 1,
$$

(50)

$$
b_\star \left( \frac{1}{4n^2} + \frac{\pi^2}{4l^2} - \frac{2}{(a - \sqrt{a^2 - 1})^2} \right) \geq \max_{(x,y) \in D_n} \left( 1 - \frac{1}{\kappa} \right) F^2(A_{eq}(x,y)).
$$

(51)

Note that the right-hand side of Eq. (51) is not negative when the condition (50) is fulfilled.

In order to get some physical insight from these conditions we resort first to the definitions (9) and (29). In terms of the perpendicular electron beta parameter $\beta_{\perp e}$ and on the electron temperature anisotropy parameter $\Theta_e$, the conditions (49) and (50) imply

$$
\Theta_e > \frac{\beta_{\perp e}}{2 + \beta_{\perp e}}, \quad \text{if } 0 < \beta_{\perp e} \leq 1,
$$

(52)

$$
\frac{\beta_{\perp e}}{2 + \beta_{\perp e}} < \Theta_e < \frac{\beta_{\perp e}}{2(\beta_{\perp e} - 1)} \quad \text{if } 1 < \beta_{\perp e} < 4.
$$

(53)
From the relations (52) and (53) it emerges that the stability conditions imply an upper bound $\beta_{\perp e} = 4$ for the perpendicular electron plasma beta parameter. This bound appears not to be too restrictive for typical solar wind or magnetospheric parameters. We also observe that the condition $\Theta_e > \beta_{\perp e}/(2 + \beta_{\perp e})$, that emerges in our analysis in both Eq. (52) and (53) (and which corresponds to $b_\star > 0$), is the condition that suppresses the firehose instability in the stability analysis of spatially homogeneous equilibria based on linear waves (see, e.g. Ref. [47]). Although our conditions are sufficient but not necessary, we could argue that also magnetic vortex chains could be subject to the same instability. For $1 < \beta_{\perp e} < 4$ an upper bound for temperature anisotropy also appears. This is due to the condition (50). However, unlike the lower bound, this bound does not appear to be related to instability thresholds familiar from wave linear theory and in particular to those, such as mirror instability (see, e.g. Ref. [47]), occurring when the temperature anisotropy parameter $\Theta_e$ is too large.

The condition (51), on the other hand, involves directly the structure of the magnetic vortex chain and of the equilibrium electron gyrocenter density (or, equivalently, of the equilibrium electrostatic potential or of the parallel magnetic perturbations). Inserting the expression for the length $l$ in terms of $a$, which can be extracted from Eq. (44), the condition (51) can be reformulated as

$$b_\star \left( \frac{1}{4n^2} + \frac{\pi^2}{4 \left( \cosh^{-1} \left( 1 + 2 \sqrt{a^2 - 1} \right) \right)^2} - \frac{2}{(a - \sqrt{a^2 - 1})^2} \right) \geq \max_{(x,y) \in D_n} \left( 1 - \frac{1}{\kappa} \right) F'^2(A_{eq}(x,y)).$$

(54)

Obviously, this condition depends on the choice of the arbitrary function $F$. For the choice $F = 0$, which corresponds to $\phi = 0$ at equilibrium, and thus no perpendicular flow, the right-hand side of Eq. (54) vanishes. If we consider a single vortex ($n = 1$) in the absence of flow (i.e. the most favorable situation for stability), then, for $b_\star > 0$, one can verify numerically that the condition (54) is satisfied for

$$1 < a < 1.026.$$

(55)
From Eq. (54), it also transpires that considering longer chains of vortices by increasing \( n \), makes it more difficult to satisfy the stability condition. For instance, for \( n = 4 \), always in the absence of perpendicular flow, one has that the stability condition is satisfied for \( 1 < a < 1.023 \). Considering the expression (22), this implies a ratio \( \sqrt{a^2 - 1}/a \), between the amplitude of the vortices and that of the background sheared magnetic field, equal at most to approximately 0.21, in order to fulfill the stability condition.

When \( \mathcal{F}(A_{||}) \) is chosen as a linear function \( \mathcal{F}(A_{||}) = V_1 A_{||} \), with constant \( V_1 \), the condition Eq. (54) becomes

\[
 b_* \left( \frac{1}{4n^2} + \frac{\pi^2}{4 \left( \cosh^{-1} \left( 1 + 2 \sqrt{a^2 - 1}/a \right) \right)^2} - \frac{2}{(a - \sqrt{a^2 - 1})^2} \right) \\
\geq \left( 1 - \frac{1}{\kappa} \right) V_1^2.
\]  

(56)

Because, at equilibrium \( \mathcal{F}(A_{eq}) = \kappa(\beta_{\perp,i}/2) \phi \), from interpreting \( \phi \) as a stream function for the equilibrium flow, it follows that \( V_1^2 \) is proportional to the ratio between the square of the amplitude of the equilibrium flow and that of the local Alfvén velocity. The condition (56) can then be seen as an upper bound on the speed of the equilibrium flow. This condition is similar to the sub-Alfvénic condition emerging from the Energy-Casimir method applied to other plasma models [48, 21, 49].

4. Cold-ion case : \( \tau_{\perp,i} \ll 1 \)

In this Section we consider the opposite limit, i.e. \( \tau_{\perp,i} \ll 1 \). This limit is adopted mainly for laboratory plasmas [45]. In terms of scales, it implies that the characteristic scale \( \rho_{s,\perp} \) is much larger than the perpendicular ion thermal gyroradius \( \rho_{th,\perp} \).

Based on the relations

\[
 I_0(\tau_{\perp,i}k_{\perp}^2)e^{-\tau_{\perp,i}k_{\perp}^2} = 1 - \tau_{\perp,i}k_{\perp}^2 + \mathcal{O}(\tau_{\perp,i}^2)
\]

and

\[
 I_1(\tau_{\perp,i}k_{\perp}^2)e^{-\tau_{\perp,i}k_{\perp}^2} \to 0, \text{ as } \tau_{\perp,i} \to 0, \text{ for all } (k_x, k_y) \in \mathcal{P}_n,
\]

we consider the following approximations for the ion gyroaverage operators for the cold-ion limit :

\[
 \Gamma_{0i} f(x, y) = (1 + \tau_{\perp,i} \Delta_\perp) f(x, y) + \mathcal{O}(\tau_{\perp,i}^2), \quad \Gamma_{1i} f(x, y) = 0,
\]

(57)
for $f$ defined over the domain $D_n$. With this prescription, the model \((17)-(20)\) in the cold-ion limit becomes

\[
\frac{\partial N_e}{\partial t} + [\phi, N_e] - b_\star[A_\|, \Delta_\perp A_\|] = 0, \quad (58)
\]

\[
\frac{\partial A_\|}{\partial t} + [\phi, A_\|] + \lambda[N_e, A_\|] = 0, \quad (59)
\]

with

\[
B_\| = -\frac{\beta_{\perp e}}{2 + \beta_{\perp e}} N_e, \quad \Delta_\perp \phi = N_e \quad (60)
\]

and the parameter $\lambda$ defined by

\[
\lambda = \frac{\beta_{\perp e}}{2 + \beta_{\perp e}} - \frac{1}{\Theta_e}. \quad (61)
\]

The parameter $\lambda$ is associated with the terms coming from the divergence of the anisotropic electron pressure tensor. In the limit of isotropic temperature ($\Theta_e = 1$) and when $B_\|$ is negligible, this model can be seen as the two-field model studied in Ref. [35] in the limit of vanishing electron inertia. If, furthermore, the third term on the left-hand of Eq. (59) is also neglected, the model becomes analogous to 2D low-$\beta$ reduced MHD.

We adopt the following boundary conditions:

\[
A_\|\vert_{\partial D_n} = a_A, \quad (62)
\]

\[
\phi\vert_{\partial D_n} = a_\phi, \quad (63)
\]

with $a_A, a_\phi \in \mathbb{R}$. The boundary condition (62) is identical to Eq. (30) and implies $B_\perp \cdot n = 0$. Equation (63), analogously to Eq. (31), refers to a condition of a velocity field tangent to the boundary. However, in the hot-ion case, because of the proportionality between $\phi$ and $B_\|$, the condition applied to the entire field $U_\perp e = \hat{z} \times \nabla (\phi - B_\|)$. In the cold-ion case, $\phi$ and $B_\|$ are no longer proportional, so that the condition (63) expresses the fact that the normalized $E \times B$ velocity field, given by $U_{E \times B} = \hat{z} \times \nabla \phi$, is tangent to the boundary, i.e. $U_{E \times B} \cdot n = 0$.

With the help of the identities (32)-(33) and applying the boundary conditions (62)-(63), it is possible to show that the functionals given in (15) and (21), with $B_\|$ and
\( \phi \) related to \( N_e \) by Eq. (60), are conserved by the system (58)-(59) on the domain \( D_n \). Therefore, we can consider the constant of motion

\[
F(N_e, A_\parallel) = \int_{D_n} d^2x \left( b_* \frac{\left| \nabla A_\parallel \right|^2}{2} + \frac{\left| \nabla \phi \right|^2}{2} - \lambda \frac{N_e^2}{2} + N_e F(A_\parallel) + G(A_\parallel) \right).
\]

(64)

We remark that, although \( F \) is a functional of \( N_e \) and \( A_\parallel \), we also used, for convenience, the variable \( \phi \) for its expression on the right-hand side of Eq. (64). We point out that \( \phi \) has to be intended as the unique solution of the problem \( \Delta_{\perp} \phi = N_e \), with \( \phi|_{\partial D_n} = a_\phi \). In this way, the field \( \phi \) can be interpreted as \( \phi = \Delta_{\perp}^{-1} N_e \) and is unambiguously defined for a given \( N_e \).

4.1. First variation and equilibria

We impose the following boundary conditions for the perturbations of \( A_\parallel \) and \( \phi \):

\[
\delta A_\parallel|_{\partial D_n} = 0, \quad \delta \phi|_{\partial D_n} = 0, \quad \int_{\partial D_n} \frac{\partial \delta \phi}{\partial n} ds = 0.
\]

(65)

Analogously to the case of the field \( \phi \), also the perturbation \( \delta \phi \) has to be interpreted as the solution of the problem \( \Delta_{\perp} \delta \phi = \delta N_e \), with \( \delta \phi|_{\partial D_n} = 0 \), with \( \delta N_e \) indicating the perturbation of the dynamical variable \( N_e \). The two boundary conditions concerning \( \delta \phi \) correspond to those also adopted in Ref. [32]. Indeed, in the cold-ion case, the second term on the right-hand side of Eq. (64) is analogous to the kinetic energy term in the conserved functional of the 2D Euler equation for an incompressible fluid.

Subject to the boundary conditions (65), the first variation of \( F \) reads

\[
\delta F(N_e, A_\parallel; \delta N_e, \delta A_\parallel) =
\int_{D_n} d^2x \left( (-b_* \Delta_{\perp} A_\parallel + F'(A_\parallel) N_e + G'(A_\parallel)) \delta A_\parallel + (F(A_\parallel) - \lambda N_e - \phi) \delta N_e \right).
\]

(66)

Setting the first variation equal to zero leads to the following equilibrium equations:

\[
\Delta_{\perp} A_\parallel = \frac{F'(A_\parallel) N_e}{b_*} + \frac{G'(A_\parallel)}{b_*},
\]

(67)

\[
F(A_\parallel) = \phi + \lambda N_e.
\]

(68)
Imposing that $A_\parallel$ satisfies Liouville’s equation implies that Eq. (67) becomes

$$G'(A_\parallel) = -N_e F'(A_\parallel) - b_\star e^{2A_\parallel}. \tag{69}$$

We consider first the case where $F'(A_\parallel) \neq 0$. In this case, from Eq. (69), one has

$$N_e = -\frac{b_\star e^{2A_\parallel} + G'(A_\parallel)}{F'(A_\parallel)}, \tag{70}$$

from which it follows that, at equilibrium, $N_e = N_e(A_\parallel)$. Equation (68) thus implies that also $\phi = \phi(A_\parallel)$, for the equilibria of interest. Using this fact in Eq. (69), together with the relation $\Delta_\perp \phi = N_e$, leads to the equation

$$\phi''(A_\parallel) |\nabla A_\parallel|^2 = -\frac{G'(A_\parallel)}{F'(A_\parallel)} + \frac{e^{2A_\parallel}}{F'(A_\parallel)}(\phi'(A_\parallel) F'(A_\parallel) - b_\star). \tag{71}$$

We specialize now to the solution of interest $A_\parallel = A_{eq}$. Because the right-hand side of Eq. (71) is a function of $A_\parallel$ only, so has to be the left-hand side. In particular, for $A_\parallel = A_{eq}$ one has to verify if $|\nabla A_{eq}|^2$ is a function of $A_{eq}$ only. In order to test this, we consider the function

$$\Upsilon(x, y) = |\nabla A_{eq}(x, y)|^2 = \frac{(a^2 - 1) \sin^2 x + a^2 \sinh^2 y}{(a \cosh^2 y + \sqrt{a^2 - 1} \cos x)^2}. \tag{72}$$

If $|\nabla A_{eq}|^2$ were a function of $A_{eq}$ only, then, upon the local change of coordinates $(x, y) \leftrightarrow (x', A_{eq})$ given by

$$x = x', \tag{73}$$

$$y = \cosh^{-1}\left(\frac{e^{-A_{eq}}}{a} - \frac{\sqrt{a^2 - 1}}{a} \cos x'\right), \tag{74}$$

(invertible, for instance, for $0 < x < \pi$ and $0 < y < \cosh^{-1}(1 + (\sqrt{a^2 - 1}/a)(1 - \cos x)))$ one would have $\Upsilon(x, y) = \Upsilon(x', A_{eq}) = \Upsilon(A_{eq})$, for every $x'$ in the domain of invertibility. However,

$$\Upsilon(x, y) = \Upsilon(x', A_{eq}) = 1 - e^{2A_{eq}} - 2\sqrt{a^2 - 1} e^{A_{eq}} \cos x'. \tag{75}$$

Because $\partial \Upsilon / \partial x' = 2\sqrt{a^2 - 1} \exp(A_{eq}) \sin x' \neq 0$ (for instance for $0 < x' < \pi$), we conclude that $\Upsilon$ is not constant with respect to $x'$ and thus $|\nabla A_{eq}|^2$ is not a function
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of $A_{eq}$ only on $D_n$. As a consequence, in order for Eq. (71) to hold for "cat’s eyes" equilibria, one has to set $\phi''(A_{eq}) = 0$, which implies

$$\phi = K_1 A_{eq} + K_2,$$

with $K_1$ and $K_2$ arbitrary constants. As a consequence, using $N_e = \Delta \phi$, we obtain that the equilibria supporting magnetic vortex chains are given by

$$A_\parallel = A_{eq},$$

$$N_e = K_1 \Delta A_{eq} = -\frac{K_1}{(a \cosh y + \sqrt{a^2 - 1} \cos x)^2};$$

with $K_1 \neq 0$. From Eqs. (68) and (69) one obtains that the corresponding choice for the arbitrary functions $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\mathcal{F}(A_\parallel) = -\lambda K_1 e^{2A_\parallel} + K_1 A_\parallel + K_2,$$

$$\mathcal{G}(A_\parallel) = -\frac{\lambda K_1^2}{2} e^{4A_\parallel} + \frac{K_1^2 - b_\star}{2} e^{2A_\parallel} + G_1,$$

with arbitrary constant $G_1$.

We recall that, in the case $\lambda = 0$, the problem of determining equilibrium solutions with flow can be circumvented [50, 51], in the case of sub-Alfvénic flows, by rewriting Eq. (67) in terms of the new variable

$$u(A_\parallel) = \int_0^{A_\parallel} dg \sqrt{1 - \mathcal{F}^2(g)/b_\star}.$$  

This transformation leads to a Grad-Shafranov equation (i.e. without flow) for the independent variable $u$. Once solutions for this equation are found, the corresponding equilibrium magnetic and velocity fields can be constructed. This procedure was applied also in Ref. [29]. However, it was applied to a magnetic field different from the one we obtain from Eq. (22), although it shares the same magnetic surfaces.

The expressions for $\phi$ and $B_\parallel$ at equilibrium, on the other hand, are given by Eq. (76) and by $B_\parallel = -K_1 \beta_\perp/(2 + \beta_\perp) \Delta A_{eq}$, respectively. For these equilibria, the $U_{E \times B}$ velocity is locally proportional to the perpendicular Alfvén velocity. The corresponding
streamlines, therefore, exhibit the same pattern of the magnetic vortex chain. The electron gyrocenter density and the parallel magnetic perturbations, on the other hand, are proportional to the equilibrium current density given by \(-\Delta_\perp A_{eq}\).

In the case \(F'(A_\parallel) = 0\) the equilibrium equations (67)-(68) decouple. The "cat’s eyes" solutions for the magnetic flux function are obtained with the choice

\[
G'(A_\parallel) = -b_*e^{2A_\parallel}.
\]

(82)

On the other hand, given that \(F'(A_\parallel) = 0\) implies \(F(A_\parallel) = F_1\), with \(F_1\) arbitrary constant, Eq. (68) yields

\[
\lambda \Delta_\perp \phi + \phi = F_1.
\]

(83)

Therefore, in this case, \(\phi\) and \(N_e\) are not constrained to be constant, at equilibrium, on the contour lines of \(A_{eq}\), as in the previous case. We remark that in this case, unlike low-\(\beta\) reduced MHD (formally retrieved by setting \(\lambda = 0\) and \(b_* = 2/\beta_\perp\)), the choice \(F'(A_\parallel) = 0\) does not necessarily lead to zero \(E \times B\) flow. Indeed, the presence of the additional contribution due to the first term on the left-hand side of Eq. (83), originated from the electron pressure tensor, makes it possible to obtain non trivial flows in the presence of magnetic vortex chains.

The equilibria considered in this case are thus given by

\[
A_\parallel = A_{eq},
\]

(84)

\[
N_e = \Delta_\perp \phi_{eq},
\]

(85)

where \(\phi_{eq}\) is a solution of Eq. (83) with boundary condition (63). Clearly, one can transform this problem into an equivalent problem for a homogeneous equation with Dirichlet boundary conditions, by introducing the new variable \(\bar{\phi} = \phi - F_1\) and imposing the boundary condition \(\bar{\phi}|_{\partial D_n} = a\phi - F_1\). Analytical solutions of this problem can be sought for, for instance with the method described in Ref. [52].
4.2. Second variation and stability conditions

The second variation of the functional \( \delta^2 F(A, N_e; \delta A, \delta N_e) = \) reads

\[
\int_{\mathcal{D}} d^2 x \left( b_\star |\nabla \delta A|^2 + |\nabla \delta \phi|^2 - \lambda |\delta N_e|^2 + 2 \mathcal{F}'(A) |\delta N_e \delta A| + (\mathcal{G}''(A) + N_e \mathcal{F}''(A)) |\delta A|^2 \right). \tag{86}
\]

Considering \( \delta (F(A)) = F'(A) \delta A \) and using the boundary conditions \( \mathcal{D} \), the expression \( (86) \) can be reformulated in the following way (see also Refs. \[32, 49\]):

\[
\delta^2 F(A, N_e; \delta A, \delta N_e) = \int_{\mathcal{D}} d^2 x \left( b_\star |\nabla \delta A|^2 - |\nabla \delta \phi - \nabla (\delta F(A))|^2 + (\mathcal{F}'(A) \Delta_\perp \mathcal{F}'(A) + \mathcal{G}''(A) + N_e \mathcal{F}''(A)) |\delta A|^2 - \lambda |\delta N_e|^2 \right). \tag{87}
\]

We specialize now to the equilibria of interest and consider first the case \( \mathcal{F}'(A, K_1 E_\perp A) \neq 0 \). Making use of the expressions \( (77)-(78) \), as well as of the relations \( (79)-(80) \), in Eq. \( (87) \), we obtain that the second variation, evaluated at the equilibrium of interest, can be rearranged to give

\[
\delta^2 F(A_{eq}, K_1 \Delta_\perp A_{eq}; \delta A, \delta N_e) = \int_{\mathcal{D}} d^2 x \left( (b_\star - K_1^2 (1 - 2 \lambda e^{2A_{eq}})^2) |\nabla \delta A|^2 + |\nabla \delta \phi - K_1 (1 - 2 \lambda e^{2A_{eq}}) \nabla \delta A|^2 \right) \tag{88}
\]

\[
+ (K_1^2 - b_\star + 4 \lambda K_1^2 |\nabla \delta A|^2 (2 \lambda e^{2A_{eq}} - 1) - 4 \lambda^2 K_1^2 e^{4A_{eq}}) 2 e^{2A_{eq}} |\delta A|^2 - \lambda |\delta N_e|^2 \right),
\]

where we also used the equilibrium relation \( \Delta_\perp A_{eq} = - \exp(2A_{eq}) \).

The coefficients of \( |\nabla \delta A|^2, |\delta A|^2 \) and \( |\delta N_e|^2 \) in the integrand have indefinite sign. Identifying conditions for which they are positive will make the integrand, and in turn \( \delta^2 F \), positive, thus providing stability conditions for the equilibria under consideration. We begin by noticing that \( \lambda < 0 \) makes the coefficient of \( |\delta N_e|^2 \) positive. With regard to the coefficient of \( |\nabla \delta A|^2 \), we observe that it is positive, on the domain, if

\[
b_\star > \max_{(x,y) \in \mathcal{D}} K_1^2 (1 - 2 \lambda e^{2A_{eq}(x,y)})^2. \tag{89}
\]

For \( \lambda < 0 \), the maximum of the function on the right-hand side of Eq. \( (89) \) is attained at \( x = \pi \) and \( y = 0 \). Evaluating the function on the right-hand side of Eq. \( (89) \) at this
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point, yields the condition

\[ b_* > K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})} \right)^2. \]  

(90)

For \( \lambda < 0 \) the coefficient of \( |\delta A_\parallel|^2 \) is positive if \( b_* < K_1^2(1 - 4\lambda^2 \exp(4A_{eq})) \). This condition, however, is in conflict with the condition (89). Again, we can resort to the Poincaré inequality (43) which, if the condition (89) holds, when applied to the first term on the right-hand side of Eq. (88), provides the following bound:

\[
\delta^2 F(A_{eq}, K_1 \Delta_{\perp} A_{eq}; \delta A_\parallel; \delta N_e) \geq \int_{D_n} d^2 x \left( (\nabla \delta \phi - K_1(1 - 2\lambda e^{2A_{eq}}) \nabla \delta A_\parallel)^2 
\right.
\]

\[
+ k_R^2 (b_* - K_1^2(1 - 2\lambda e^{2A_{eq}})^2 + (K_1^2 - b_*) + 4\lambda K_1^2 |\nabla A_{eq}|^2(2e^{2A_{eq}} - 1)
\]

\[
- 4\lambda^2 K_1^2 e^{4A_{eq}} |\nabla A_\parallel|^2 - \lambda |\delta N_e|^2 \right). \]

(91)

The coefficient of \( |\delta A_\parallel|^2 \) on the right-hand side of Eq. (91) can be made positive considering that, for \( \lambda < 0 \), the terms proportional to \( |\nabla A_{eq}|^2 \) are non-negative and noticing that

\[
k_R^2 (b_* - K_1^2(1 - 2\lambda e^{2A_{eq}})^2)
\]

\[
\geq k_R^2 (b_* - \max_{(x,y) \in D_n} \left\{ K_1^2(1 - 2\lambda e^{2A_{eq}(x,y)})^2 \right\})
\]

\[
+ \min_{(x,y) \in D_n} \left\{ (K_1^2 - b_*) + 4\lambda K_1^2 e^{4A_{eq}} \right\}
\]

\[
= k_R^2 \left( b_* - K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})} \right)^2 \right) - 2 \frac{b_* - K_1^2}{(a - \sqrt{a^2 - 1})^2} - 8 \frac{\lambda^2 K_1^2}{(a - \sqrt{a^2 - 1})^6}. \]

(92)

We can therefore conclude that the second variation is positive, and consequently the equilibria (77)-(78) are linearly stable, if the following three conditions are satisfied:

\[ b_* > K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})} \right)^2, \]

(94)

\[ \lambda < 0, \]

(95)

\[
\left( \frac{1}{4n^2} + \frac{\pi^2}{4l^2} \right) \left( b_* - K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})} \right)^2 \right)
\]

\[ > 2 \frac{b_* - K_1^2}{(a - \sqrt{a^2 - 1})^2} + 8 \frac{\lambda^2 K_1^2}{(a - \sqrt{a^2 - 1})^6}. \]

(96)
The condition (94) can be seen as an upper limit, depending on $\beta_{\perp e}$, $\Theta_e$ and $a$, on the amplitude $K_1$ of the $E \times B$ flow. This condition also suppresses the firehose instability. The condition (95), on the other hand, can be reformulated as

$$\Theta_e < 1 + \frac{2}{\beta_{\perp e}}$$

and, analogously to Eq. (50) of the hot-ion case, provides an upper bound on electron temperature anisotropy. One can note that, for $\Theta_e = 1$, this condition is always satisfied.

In this limit, the term $-\lambda|\delta N_e|^2$ in Eq. (88) always provides a positive contribution to the second variation. This suggests that, for isotropic electron temperature, the electron pressure term associated with $\lambda$ has a stabilizing role, with respect to the reduced MHD case where $\lambda = 0$. The condition (96) can be fulfilled by sufficiently reducing the width of the islands letting the parameter $a$ approach 1. Indeed, the left-hand side of Eq. (96) can be made arbitrarily large letting $a \to 1^+$, in which limit $l \to 0^+$ and the term $\pi^2/(4l^2)$ goes to infinity. In the same limit, on the other hand, the denominators on the right-hand side tend to 1, so that the right-hand side remains bounded.

In the case $F'(A) = 0$, the second variation, evaluated at the equilibria (84)-(85), and using Eq. (82), reads

$$\delta^2 F(A_{eq}, \Delta \phi_{eq}; \delta A_\parallel, \delta N_e) = \int_{D_\alpha} d^2x \left( b_e |\nabla \delta A_\parallel|^2 + |\nabla \delta \phi|^2 - 2b_e e^{2A_{eq}} |\delta A_\parallel|^2 - \lambda |\delta N_e|^2 \right).$$

(98)

The second variation (98) actually corresponds to the second variation (88) in the limit $K_1 = 0$, i.e. with no $E \times B$ flow. However, as we pointed out in Sec. 4.1 for $F'(A) = 0$, the potential $\phi_{eq}$ can correspond to non-trivial flows. Nevertheless, stability conditions for this case can be directly obtained from Eqs. (94)-(96) by setting $K_1 = 0$ and can be formulated as

$$\frac{\beta_{\perp e}}{2 + \beta_{\perp e}} < \Theta_e < 1 + \frac{2}{\beta_{\perp e}},$$

$$\left( \frac{1}{4n^2} + \frac{\pi^2}{4l^2} \right) > \frac{2}{(a - \sqrt{a^2 - 1})^2}.$$
The condition (99) comes from the requirements $b_\star > 0$ and $\lambda < 0$ and prevents instabilities due to temperature anisotropy. The condition (100), on the other hand, implies restrictions on $a$ and is amenable to the same considerations discussed for the condition (54) in the case with no perpendicular flow.

5. Concluding remarks

In this work we studied the existence and the stability of stationary solutions, of a reduced fluid model, describing chains of magnetic vortices. The formation of chains of magnetic vortices, due to the reconnection of magnetic field lines, is a frequent phenomenon in laboratory and space plasmas. Observational evidence shows, in particular, the existence of chains of magnetic vortices, for instance in the plasma of the solar wind. The presence of such structures can have a strong impact on the turbulent spectrum of magnetic and kinetic plasma energy.

We first reduced the general gyrofluid model, by acting on its Hamiltonian structure, to a 2D version without electron inertia effects. Subsequently, we considered the resulting model in the asymptotic limit in which the equilibrium ion temperature, referred to the plane perpendicular to the direction of a strong magnetic guide field, is much greater than the electron one, i.e. $\tau_{\perp i} \gg 1$. In this limit we found equilibrium equations admitting solutions describing magnetic vortex chains supporting a class of non-trivial perpendicular flows, constant on the magnetic flux function contour lines, and depending on an arbitrary function. We obtained that such magnetic vortex chains equilibria are linearly stable if three conditions are fulfilled. Two of these conditions impose bounds on the electron temperature anisotropy, which, as expected, can be a source for instabilities. Depending on the range of values for $\beta_{\perp e}$, the temperature anisotropy has only a lower bound or is bounded from above and from below. Interestingly, the lower bound corresponds to the bound for firehose instability known for homogeneous equilibria according to linear wave stability analysis. Upper and lower bounds depend on the electron beta parameter. The third condition depends explicitly on the choice
of the equilibrium flow. For a given flow and for fixed $\beta_{\perp e}$ and $\Theta_e$, it can be seen as a condition on the maximum island width and on the length of the chain. Shorter chains with thin islands favor stability.

In the opposite, cold-ion case, with $\tau_{\perp i} \ll 1$, a slightly more intricate situation occurs, presenting two sub-cases. In one sub-case, the magnetic vortex chain supports an electrostatic potential $\phi$ linear with respect to the magnetic flux function. This restricts the equilibrium $E \times B$ velocity to be proportional to the local Alfvén velocity. The electron gyrocenter density $N_e$ and the parallel magnetic perturbations $B_\parallel$, on the other hand, are proportional to the current density associated with the vortex chain. In this sub-case, one stability condition suppresses the firehose instability but is stronger than the aforementioned condition, due to the presence of the equilibrium flow. A second condition sets an upper bound to temperature anisotropy and a third condition, again concerns also the size and the length of the chain. In the second sub-case, the fields $\phi$ and $N_e$ are no longer constrained to be constant on contour lines of the magnetic flux function and satisfy the relation $N_e = (-\phi + F_1)/\lambda$. In principle this can provide non-trivial flows. Stability conditions bound temperature anisotropy from above and from below, with the lower bound again corresponding to the firehose stability condition. The third condition, on the other hand, turns out to correspond to the one found for the hot-ion case in the absence of flows. If a non-trivial solution for the flow can be found in this case, the characteristics of such solution appear not to be crucial for stability.

Our analysis suggests that, in both hot and cold-ion regimes, several parameters of the system have to be controlled to attain the stability conditions. Such conditions appear to be rather compelling, and favor short chains with thin vortices and moderate anisotropy. The condition on the maximum vortex width is analogous to the condition for nonlinear stability of "cat’s eyes" vortex chains derived in Ref. [32]. We point out again, that our analysis is carried out over the domain enclosed by the separatrices and thus rules out external perturbations. It is well known that magnetic island chains are actually unstable on larger domains including regions outside the separatrices [1, 2, 5].
This seems to indicate that magnetic vortex chains might persist as coherent structures when perturbations coming from outside the chain are negligible.

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Appendix A. Summary of the derivation of the model

In this Section we summarize some assumptions underlying the model, which determine also its limits of validity, and we describe qualitatively the main steps of its derivation.

First, we recall the gyrokinetic system from which the gyrofluid model \((1)-(5)\) can be derived. This gyrokinetic system corresponds, in turn, to the gyrokinetic model derived in Ref. [37] when equilibrium drifts are neglected and a bi-Maxwellian distribution is chosen as equilibrium distribution function. Such gyrokinetic system, in dimensional variables, reads

\[
\frac{\partial \tilde{g}_s}{\partial t} + \frac{c}{B_0} \left[ J_{0s} \tilde{\phi} - \frac{v}{c} J_{0s} \tilde{A}_|| + 2 \frac{\mu_0 B_0}{q_s} J_{1s} \frac{\tilde{B}||}{B_0}, \tilde{g}_s \right] + v \parallel \frac{\partial}{\partial z} \left( \tilde{g}_s + \frac{q_s}{T_{0||s}} \mathcal{F}_{0s} \left( J_{0s} \tilde{\phi} - \frac{v}{c} J_{0s} \tilde{A}_|| + 2 \frac{\mu_0 B_0}{q_s} J_{1s} \frac{\tilde{B}||}{B_0} \right) \right) = 0, \quad (A.1)
\]

\[
\sum_s q_s \int dW_s J_{0s} \tilde{g}_s = \sum_s \frac{q_s^2}{T_{0\perp s}} \int dW_s \mathcal{F}_{0s} \left( 1 - J_{0s}^2 \right) \tilde{\phi}, \quad (A.2)
\]

\[
\sum_s q_s \int dW_s v_\parallel J_{0s} \left( \tilde{g}_s - \frac{q_s}{T_{0||s}} \frac{v}{c} \mathcal{F}_{0s} J_{0s} \tilde{A}_|| \right) = - \frac{c}{4\pi} \Delta_\perp \tilde{A}_|| + \sum_s \frac{q_s^2}{m_s} \int dW_s \mathcal{F}_{0s} \left( 1 - \frac{1}{\Theta_s} \frac{v^2}{v^2_{th||s}} \right) \left( 1 - J_{0s}^2 \right) \frac{\tilde{A}_||}{c}, \quad (A.3)
\]

\[
\sum_s \frac{\beta_{1s}}{n_0} \int dW_s 2 \frac{\mu_0 B_0}{T_{0\perp s}} J_{1s} \tilde{g}_s = - \sum_s \frac{\beta_{1s}}{n_0} \frac{q_s}{T_{0\perp s}} \int dW_s \mathcal{F}_{0s} \left( 2 \frac{\mu_0 B_0}{T_{0\perp s}} J_{1s} \right)^2 \frac{\tilde{B}_||}{B_0}, \quad (A.4)
\]
The index $s$ indicates the particle species ($s = e$ for electrons and $s = i$ for ions, when assuming a single ion species). Equations (A.2)-(A.4) express quasi-neutrality, parallel and perpendicular components of Ampère’s law, respectively. The gyrokinetic equation (A.1) describes the evolution of the function

$$
\tilde{g}_s(x, y, z, v_\parallel, \mu_0, t) = \tilde{f}_s(x, y, z, v_\parallel, \mu_0, t) + \frac{q_s}{T_{0s}} \frac{v_\parallel}{c} \mathcal{F}_{0s}(v_\parallel, \mu_0) J_{0s} \tilde{A}_\parallel(x, y, z, t). \tag{A.5}
$$

In Eq. (A.5) $\tilde{f}_s$ is the perturbation of the distribution function, whereas $\mathcal{F}_{0s}$ is the bi-Maxwellian equilibrium distribution function defined by

$$
\mathcal{F}_{0s}(v_\parallel, \mu_0) = \left( \frac{m_s}{2\pi} \right)^{3/2} \frac{n_0}{T_{0\parallel}^{1/2} T_{0\perp}} \frac{m_s v_\parallel^2}{2 T_{0\parallel}} e^{-\frac{m_s v_\parallel^2}{2 T_{0\parallel}}} e^{-\frac{\mu_0 B_0}{T_{0\perp}}}, \tag{A.6}
$$

The independent variables are the time coordinate $t \in [0, +\infty)$, the spatial coordinates $(x, y, z) \in \mathcal{T}_n$, the velocity coordinate parallel to the guide field $v_\parallel \in (-\infty, +\infty)$ and the magnetic moment $\mu_0 \in [0, +\infty)$ of the particle of species $s$, in the presence of the unperturbed magnetic guide field. We indicated with $q_s$ the charge of the particle of species $s$ and with $dW_s = (2\pi B_0/m_s) d\mu_0 d v_\parallel$ the volume element in velocity space, integrated over the particle gyration angle. The gyroaverage operators $J_{0s}$ and $J_{1s}$ are defined as

$$
J_{0s} f(x, y, z) = \sum_{\mathbf{k} \in \mathcal{F}_n} J_0(a_s) f_{\mathbf{k}} \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x}), \tag{A.7}
$$

$$
J_{1s} f(x, y, z) = \sum_{\mathbf{k} \in \mathcal{F}_n} \frac{J_1(a_s)}{a_s} f_{\mathbf{k}} \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x}), \tag{A.8}
$$

where $J_0$ and $J_1$ are zero and first order Bessel functions of the first kind and $a_s = k_\perp \sqrt{2\mu_0 B_0/m_s/\omega_{cs}}$ is the perpendicular wave number multiplied times the gyroradius of the particle of species $s$.

For the gyrofluid model we consider a plasma composed by electrons and a single ionized species of ions. Because the model was conceived mainly as an extended model for kinetic Alfvén waves, the coupling with ion gyrocenter fluctuations is removed by neglecting contributions of the perturbations of all ion gyrocenter moments in the quasi-neutrality relation and parallel as well as perpendicular projections of Ampère’s law.
The ion equilibrium distribution function is also assumed to be isotropic. Two small expansion parameters are adopted, given by $\delta \ll 1$ and $\epsilon \ll 1$, the latter corresponding to the normalized characteristic frequency of the fluctuations under consideration. Assuming these two parameters to be small, amounts to considering electron inertia as a small perturbation and to considering phenomena with frequencies much lower than the ion cyclotron frequency, the latter being one of the fundamental hypotheses in the gyrokinetic ordering.

A Laguerre-Hermite expansions is adopted for the perturbation of the electron gyrocenter distribution functions. This expansion is truncated by imposing the relations

$$T_{\parallel e} = 0, \quad T_{\perp e} = -B_{\parallel},$$

where $T_{\parallel e}$ and $T_{\perp e}$ correspond to the electron gyrocenter parallel and perpendicular temperature fluctuations, respectively. The closure relations (A.9), correspond, in terms of particle temperature fluctuations, to isothermal closures for both the parallel and perpendicular electron temperatures (recall that, when electron FLR are neglected, as in this case, the perpendicular temperature fluctuations of particles $t_{\perp e}$ are related to those of gyrocenters $T_{\perp e}$ by $t_{\perp e} = T_{\perp e} + B_{\parallel}$, as explained in Ref. [33]). Such truncated expansion for $\tilde{f}_e$ is inserted into Eqs. (A.2)-(A.4), whereas all the fluctuations of the ion gyrocenter moments are neglected.

Two different scalings are then adopted, one of them valid for $\tau_{\perp i} = \mathcal{O}(1)$ and a second one valid for $\tau_{\perp i} = \mathcal{O}(1/\delta)$, as $\delta \to 0$. The two scalings read

$$\tau_{\perp i} \sim \Theta_e \sim \nabla_\perp = \mathcal{O}(1), \quad \beta_{\perp e} = \mathcal{O}(\delta),$$

$$A_{\parallel} \sim \partial_z = \mathcal{O}(\delta^{1/2} \epsilon), \quad U_e = \mathcal{O} \left( \frac{\epsilon}{\delta^{1/2}} \right),$$

$$\partial_t \sim \phi \sim N_e = \mathcal{O}(\epsilon), \quad B_{\parallel} = \mathcal{O}(\delta \epsilon)$$

(A.10)
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and

\[ \tau_{\perp i} = \mathcal{O}(1/\delta), \quad \nabla_{\perp} \sim \Theta_e = \mathcal{O}(1), \quad \beta_{\perp e} = \mathcal{O}(\delta), \]

\[ A_{\parallel} \sim \partial_z = \mathcal{O}(\delta^{1/2}/\epsilon), \quad U_e = \mathcal{O}\left(\frac{\epsilon}{\delta^{1/2}}\right), \]

\[ \partial_t \sim \phi = \mathcal{O}(\epsilon), \quad B_{\parallel} \sim N_e = \mathcal{O}(\delta \epsilon), \quad (A.11) \]

respectively. Such two scalings are applied to the quasi-neutrality relation and to the parallel and perpendicular projections of Ampère’s law \((A.2)-(A.4)\), after inserting for \(\tilde{f}_e\) its truncated Laguerre-Hermite expansion, integrating with the help of orthogonality relations for Hermite and Laguerre polynomials and neglecting the perturbation of the ion gyrocenter distribution function. For each scaling, all terms at leading order, as well as corrections of order \(\delta\), are retained. This yields Eqs. (3)-(5).

On the other hand, from the zero and first order moments, with respect to \(v_{\parallel}\), of the evolution equation for the electron gyrocenter distribution function (Eq. (A.1)), one obtains Eqs. (1)-(2), upon imposing the closure relation \((A.9)\) and considering only the leading order terms in the expansion of the electron gyroaverage operators as \(\delta \to 0\) (the next order terms being of order at least \(\delta^2\) smaller, and thus negligible).

Appendix B. The stability algorithm

In this Section we briefly summarize the steps required for determining linear stability conditions according to the Energy-Casimir method.

We consider a dynamical system

\[ \frac{\partial \chi_i}{\partial t} = X_i(\chi_1, \cdots, \chi_N), \quad i = 1, \cdots, N, \quad (B.1) \]

evolving \(N\) fields \(\chi_1, \cdots, \chi_N\) all of which depend on time and on space variables \(x_1, \cdots, x_m\) belonging to some domain \(U \subset \mathbb{R}^m\), with \(m\) and \(N\) positive integers.

We suppose the system admits a family of \(s\) constants of motion \(\mathcal{C}_1, \cdots, \mathcal{C}_s\), i.e. functionals \(\mathcal{C}_1(\chi_1, \cdots, \chi_N), \cdots, \mathcal{C}_s(\chi_1, \cdots, \chi_N)\) such that \(d\mathcal{C}_i/dt = 0\), for \(i = 1, \cdots, s\). The functional \(F = \sum_{i=1}^s \mathcal{C}_i\) is then a constant of motion as well. For noncanonical
Hamiltonian systems, a natural choice for $F$ is given by $F = H + \sum_{i=1}^{s-1} C_i$, where $H$ is the Hamiltonian of the system and $C_1, \cdots, C_{s-1}$ are Casimir invariants. This is why the method is referred to as Energy-Casimir method.

Solutions of the equation

$$\delta F(\chi_1, \cdots, \chi_N; \delta \chi_1, \cdots, \delta \chi_N) = 0,$$

where $\delta F$ is the first variation of $F$, correspond to equilibria of the system (B.1). Such equilibrium points, denoted as $(\chi_{e1}, \cdots, \chi_{eN})$, can then be related to constants of motion by requiring that $(\chi_{e1}, \cdots, \chi_{eN})$ be a point where $\delta F$ vanishes. In this way, classes of equilibria (although in general not all the equilibria of the system) can be associated with different choices of constants of motion.

An equilibrium $(\chi_{e1}, \cdots, \chi_{eN})$ solution of $\delta F(\chi_1, \cdots, \chi_N; \delta \chi_1, \cdots, \delta \chi_N) = 0$ is formally stable (which implies linearly stable) if the second variation of $F$, evaluated at such equilibrium, i.e.

$$\delta^2 F(\chi_{e1}, \cdots, \chi_{eN}; \delta \chi_1, \cdots, \delta \chi_N)$$

has a definite sign. If this is the case, in fact, the expression (B.3) (or its opposite) can be taken as a conserved norm for the system (B.1) linearized about the equilibrium $(\chi_{e1}, \cdots, \chi_{eN})$.

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