Remarks on limit theorems for reversible Markov processes

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Abstract

We propose some backward-forward martingale decomposition for reversible Markov chains that are later used to prove the functional Central limit theorem for reversible Markov chains with asymptotically linear variance of partial sums. We also provide a proof of the equivalence between the variance been asymptotically linear and finiteness of the integral of $1/(1 - t)$ with respect to the associated spectral measure $\rho$. We show a result on uniform integrability of the supremum of the average sum of squares of martingale differences that is interesting by itself.

Key words: central limit theorem, stationary processes, reversible Markov chains, Martingales, forward-backward decomposition.

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1 Introduction

We review the functional central limit theorem for stationary Markov chains with self-adjoint operator and general state space. We investigate the case when the variance of the partial sum is asymptotically linear in $n$, and propose a new proof of the functional CLT for reversible Markov chains in Corollary 1.5 of Kipnis and Varadan (1986). We prove the equivalence of $\lim_{n \to \infty} Var(S_n)/n$ and convergence of $\int_{-1}^{1} \frac{\rho(dt)}{1 - t}$, for a mean zero function $f$ of a stationary reversible Markov chain. Here, $\rho$ is the spectral measure corresponding to $f$. This equivalence is used to provide a new forward-backward martingale decomposition for the given class of processes. Among new results of this paper, is a forward-backward martingale decomposition for stationary reversible Markov chains. In Proposition 3 we state a convergence theorem that helps establish a martingale convergence theorem in Lemma 4. A new proof of the central limit theorem based on Heyde (1974) is provided. Throughout this paper we use the spectral theory of bounded self-adjoint operators. In Section 1 we have the introduction, Section 2 is about the forward-backward martingale decomposition and Section 3 tackles the new proof of the functional central limit theorem for reversible Markov processes.
1.1 Definitions and notations

We assume that \((\xi_n, n \in \mathbb{Z})\) is a stationary reversible Markov chain defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in a general state space \((S, \mathcal{A})\). The marginal distribution is denoted by \(\pi(A) = \mathbb{P}(\xi_0 \in A)\). We also assume there is a regular conditional distribution for \(\xi_1\) given \(\xi_0\) denoted by \(Q(x, A) = \mathbb{P}(\xi_1 \in A | \xi_0 = x)\). Let \(Q\) also denote the Markov operator acting via 
\[
(Qf)(x) = \int_S f(s)Q(x, ds),
\]
and \(L^2(\pi) = \{f: \int_S f^2 d\pi < \infty, \int_S f d\pi = 0\}\). For \(f \in L^2(\pi)\), let
\[
X_i = f(\xi_i), \quad S_n = \sum_{i=1}^n X_i, \quad \sigma_n = (\mathbf{E}S_n^2)^{1/2}.
\]

Denote by \(\mathcal{F}_k\) the \(\sigma\)-field generated by \(\xi_i\) with \(i \leq k\). \(L^p = \{f: \int |f|^p d\pi < \infty\}\). For any integrable random variable \(X\) we denote \(\mathbb{E}_k(X) = \mathbb{E}(X | \mathcal{F}_k)\). \(\mathbb{E}_0(X_1) = (Qf)(\xi_0) = \mathbb{E}(X_1 | \xi_0)\), using this notation. We denote by \(||X||_p\) the norm of \(X\) in \(L^p(\Omega, \mathcal{F}, \mathbb{P})\). \(W(t)\) is the standard Brownian motion. All throughout the paper \(\Rightarrow\) denotes weak convergence, \([x]\) is the integer part of \(x\). For the proofs of the theorems, we need:

**Lebesgue’s dominated convergence theorem:** If the sequence of functions \(f_n\) is such that \(|f_n| < g\) \(\mu\)-almost everywhere, where \(g\) is \(\mu\)-integrable, and \(f_n \to f\) \(\mu\)-almost everywhere, then \(f_n\) and \(f\) are \(\mu\)-integrable and \(\int f_n d\mu \to \int f d\mu\).

**Fatou lemma:** For any sequence of nonnegative functions \(f_n\), \(\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu\).

We also need to introduce here some notions from the spectral theory that are very useful.

1.2 Spectral Theory of self-adjoint operators

Self-adjoint operators have spectral families with certain regularity properties, beyond the properties shared by all spectral families, which are very important in the proof of the theorems in this paper. Recall that a linear vector space \(\mathbb{H}\) is a Hilbert space, if it is endowed with an inner product \(\langle \cdot, \cdot \rangle\), associated with a norm \(||\cdot||\) and metric \(d(\cdot, \cdot)\), such that every Cauchy sequence has a limit in \(\mathbb{H}\). Elements \(x, y\) of a Hilbert space are said to be orthogonal if \(\langle x, y \rangle = 0\). Suppose there is a nondecreasing family \((M(\lambda), \lambda \in \mathbb{R})\) of closed subspaces of \(\mathbb{H}\) depending on a real parameter \(\lambda\), such that the intersection of all the \(M(\lambda)\) is \(\{0\}\) and their union is dense in \(\mathbb{H}\). Recall that The family is “nondecreasing” if \(M(\lambda_1) \subset M(\lambda_2)\) for \(\lambda_1 < \lambda_2\). This property also extends to the associated family \((E(\lambda), \lambda \in \mathbb{R})\) of orthogonal projections on \(M(\lambda)\). The associated family of orthogonal projections is called spectral family or resolution of the identity if \(\lim_{\lambda \to -\infty} E(\lambda) = 0\) and \(\lim_{\lambda \to \infty} E(\lambda) = 1\).

**Spectral theorem for self-adjoint operators in Hilbert spaces:**

Every self-adjoint operator \(Q\) in a Hilbert space \(\mathbb{H}\) admits an expression \(Q = \int_{-\infty}^{\infty} \lambda dE(\lambda)\) by means of a spectral family \((E(\lambda), \lambda \in \mathbb{R})\) which is uniquely determined by \(Q\).

The family \((E(\lambda), \lambda \in \mathbb{R})\) yields valuable information on the spectral structure of \(Q\): the location of its singular or absolutely continuous spectrum and its eigenvalues. Also, it naturally leads to the definition of functions \(f(Q)\), for a wide family of functions \(f\). When the operator is bounded, the integral can be taken over the spectrum \(\sigma(Q)\) of the operator (set of points \(\lambda\) for which there is no bounded inverse to \(Q - \lambda I\), where \(I\) is the identity operator). This applies to Markov operators (a Markov operator is a unity-preserving positive contraction). The inner product in a Hilbert space allows to define \(Q^*\), the adjoint operator to \(Q\) by the formula \(\langle Qx, y \rangle = \langle x, Q^*y \rangle\), \(\forall x, y \in \mathbb{H}\). The operator \(Q\) is self-adjoint if the above yields \(Q^* = Q\). For more, see Conway (1990).
**Example of Markov operator:** Assume that \((\xi_n, n \in \mathbb{Z})\) is the Markov chain defined above. Q induces an operator acting via \((Qf)(x) = \int_{S} f(s) Q(x, ds)\) in the Hilbert space \(L^2(\pi)\). The defined operator Q is a Markov operator with spectrum on \([-1, 1]\). For such Q, the above representation becomes: \(Q = \int_{-1}^{1} \lambda dE(\lambda)\), leading to <Qf, f> = \(\int_{-1}^{1} \lambda d <E(\lambda)f, f> = \int_{-1}^{1} \lambda d\rho(\lambda)\), where \(\rho(\lambda)\) denotes the spectral measure of the operator applied to f.

Based on this example, for a reversible Markov chain generated by Q, \(\text{E}(\text{E}(X_k|\mathcal{F}_0)\text{E}(X_j|\mathcal{F}_0)) = \int_{-1}^{1} t^{k+j} \rho(dt)\). The following important by itself lemma holds.

**Lemma 1**

Let \((X_i, i \in \mathbb{N})\) be defined by \(\mathbb{D}\). Then, \(\frac{\text{Var}(S_n)}{n} \to \sigma^2 < \infty \iff \int_{-1}^{1} \frac{1}{1-t} \rho(dt) < \infty\).

Moreover, \(\lim_{n \to \infty} \frac{\text{Var}(S_n)}{n} = \int_{-1}^{1} \frac{1+t}{1-t} \rho(dt)\).

**Proof.** It is well known that for a stationary reversible mean zero Markov chain \((X_i, i \in \mathbb{N})\), \(S_n = \text{E}(S_n|\mathcal{F}_0) + \sum_{j=1}^{n} [\text{E}(S_n - S_{j-1}|\mathcal{F}_j) - \text{E}(S_n - S_{j-1}|\mathcal{F}_{j-1})]\), where the summands are orthogonal in \(L^2\) as sequence of martingale differences. Therefore, we obtain \(\frac{\text{Var}(S_n)}{n} = \frac{1}{n} \text{E}(\text{Var}(S_n)|\mathcal{F}_0)^2 + \frac{1}{n} \sum_{j=1}^{n} [\text{E}(\text{Var}(S_n - S_{j-1}|\mathcal{F}_j) - \text{Var}(S_n - S_{j-1}|\mathcal{F}_{j-1})]^2\). Using the fact that for this sequence, \(\text{E}(\text{Var}(S_n - S_{j-1}|\mathcal{F}_j) - \text{Var}(S_n - S_{j-1}|\mathcal{F}_{j-1})^2 = \text{E}(\text{Var}(S_n - S_{j-1}|\mathcal{F}_j))^2 - \text{E}(\text{Var}(S_n - S_{j-1}|\mathcal{F}_{j-1})^2\), we obtain \(\frac{\text{Var}(S_n)}{n} = \frac{1}{n} \text{E}(\text{Var}(S_n|\mathcal{F}_0)^2 + \frac{1}{n} \sum_{j=1}^{n} [\text{E}(\text{Var}(S_n - S_{j-1}|\mathcal{F}_j))^2 - \text{E}(\text{Var}(S_n - S_{j-1}|\mathcal{F}_{j-1})^2]\). Therefore, by stationarity, \(\frac{\text{Var}(S_n)}{n} = \frac{1}{n} \text{E}(\text{Var}(S_n|\mathcal{F}_0)^2 + \frac{1}{n} \sum_{j=1}^{n} [\text{E}(\text{Var}(X_0 + S_k|\mathcal{F}_0))^2 - \text{E}(\text{Var}(S_{k+1}|\mathcal{F}_0))^2]\). So,

\[
\frac{\text{Var}(S_n)}{n} = \frac{1}{n} \int_{-1}^{1} (t + \cdots + t^n)^2 \rho(dt) + \frac{1}{n} \sum_{k=1}^{n-1} \int_{-1}^{1} [(1 + t + \cdots + t^k)^2 - (t^2 + \cdots + t^{k+1})]^2 \rho(dt).
\]

The last equality uses the spectral representation \(\text{E}(X_0X_k) = \text{E}(\text{E}(X_0|\mathcal{F}_0)\text{E}(X_k|\mathcal{F}_0)) = \int_{-1}^{1} t^k \rho(dt)\).

So,

\[
\frac{\text{Var}(S_n)}{n} = \int_{-1}^{1} \frac{1}{n} [(t + \cdots + t^n)^2 + \sum_{k=1}^{n-1} ((1 + t + \cdots + t^k)^2 - (t + t^2 + \cdots + t^{k+1})]^2] \rho(dt).
\]

If \(\int_{-1}^{1} \rho(dt) < \infty\), then we can apply the Lebesgue’s dominated convergence theorem to \(f_n(t) = \frac{1}{n} [(t + \cdots + t^n)(t - t^n + 2) + \sum_{k=1}^{n-1} ((1 - t^{k+1})(1 + t)]\). For this sequence, we have \(|f_n| \leq 10/(1 - t)\) \(\rho\)-almost everywhere, and \(f_n \to (1 + t)/(1 - t) \rho\)-almost everywhere. Moreover, \((1 + t)/(1 - t)\) is \(\rho\)-integrable provided that \(1/(1 - t)\) is \(\rho\)-integrable. Therefore, by the Lebesgue’s dominated convergence theorem,

\[
\lim_{n \to \infty} \frac{\text{Var}(S_n)}{n} = \int_{-1}^{1} \frac{1 + t}{1 - t} \rho(dt).\]

Applying Fatou lemma to the functions above,

\[
\lim_{n \to \infty} \frac{\text{Var}(S_n)}{n} \geq \lim_{n} \inf \int_{-1}^{1} f_n(t) \rho(dt) \geq \int_{-1}^{1} \lim_{n} \inf f_n(t) \rho(dt) = \int_{-1}^{1} \frac{1 + t}{1 - t} \rho(dt).\]

if $\frac{\text{Var}(S_n)}{n}$ is convergent, then $\int_{-1}^{1} \frac{1 + t}{1 - t} \rho(dt) < \infty$; leading to $\int_{-1}^{1} \frac{\rho(dt)}{1 - t} < \infty$. So, $\lim_{n \to \infty} \frac{\text{Var}(S_n)}{n} = \int_{-1}^{1} \frac{1 + t}{1 - t} \rho(dt) = 2 \int_{-1}^{1} \frac{1}{1 - t} \rho(dt) - \mathbb{E}(X_0^2)$. This leads to the conclusion of the lemma. ■

2 Forward-Backward martingale decomposition

Martingale decomposition of sequences of random variables is a very important in probability theory. The proof of central limit theorems is often based on this decomposition. One shows that the variable can be represented as a sum of a martingale and a “remainder” with suitable properties. For more on this topic, see Wu (1999), Wu and Woodroofe (2004) and Zhao and al. (2010). For stationary reversible Markov chains, we obtain more flexibility to form martingale differences for triangular arrays. This allows to obtain in the limit (convergence in $\mathbb{L}^2$) martingale differences that sum up to martingales.

From Longla and al. (2012), for triangular arrays of random variables, we have

$$X_k + X_{k+1} = D_k^n + \hat{D}_k^n + \sum_{i=0}^{n-1} \mathbb{E}(S_{k+i} - S_{k+i+1}),$$

where $\hat{D}_k^n$ is the equivalent of $D_k^n$ for the reversed martingale, and

$$D_k^n = \theta_k^n - \mathbb{E}_{k-1} (\theta_k^n) = \frac{1}{n} \sum_{i=0}^{n-1} [\mathbb{E}(S_{k+i}) - \mathbb{E}_{k-1}(S_{k+i})].$$ (2)

Denoting $B_{n,k} = \frac{1}{n} \mathbb{E}(S_n - S_k)$, from the above formula we obtain

$$X_k + X_{k+1} = D_k^n + \hat{D}_k^n + B_{n,k} + B_{n,k+1}.$$ (3)

We shall show that $B_{n,k}$ and $B_{n,k+1}$ converge to 0 in $\mathbb{L}^2$.

**Proposition 2** Under the assumption of asymptotic linearity of the variance of partial sums, $B_{n,k} \to 0$ in $\mathbb{L}^2$ uniformly in $k$ as $n \to \infty$.

**Proof.** To show that $B_{n,k}$ converges uniformly in $k$ to 0 in $\mathbb{L}^2$, it is enough to show that the variance converges to 0 uniformly in $k$, and the expected value is equal to 0. The mean zero assumption solves the problem of the expected value, and we have $\text{Var}(B_{n,k}) = \frac{1}{n^2} \mathbb{E}(\mathbb{E}_k(S_n - S_k))^2$.

From stationarity, we obtain $\text{Var}(B_{n,k}) = \frac{1}{n^2} \int_{-1}^{1} (t + \cdots + t^{n-k})^2 \rho(dt) = \frac{1}{n^2} \int_{-1}^{1} f_n(t) \rho(dt)$, where $0 \leq f_n(t) = \frac{1}{n^2} (t + \cdots + t^{n-k})^2 \leq \frac{2}{n^2} \int_{-1}^{1} \frac{1}{1 - t} \rho(dt)$ almost surely. Applying Lemma 1, $\frac{\text{Var}(S_n)}{n} \leq \frac{2}{n} \int_{-1}^{1} \frac{\rho(dt)}{1 - t} \to 0$ as $\int_{-1}^{1} \frac{\rho(dt)}{1 - t} < \infty$. So, $B_{n,k} \to 0$ uniformly in $\mathbb{L}^2$. ■

**Proposition 3** (An $\mathbb{L}^2$ convergence theorem)

Let $(X_i, i \in \mathbb{Z})$ be a reversible stationary mean zero Markov chain with finite second moments.

If $\text{Var}(S_n)/n \to \sigma^2 \neq 0$, then $\sum_{i=0}^{n} (\mathbb{E}(X_i|\mathcal{F}_1) - \mathbb{E}(X_i|\mathcal{F}_0))$ converges in $\mathbb{L}^2$, where $\mathcal{F}_i$ is the $\sigma$-field generated by $(X_j, j \leq i)$.
Proof. To prove Proposition 3 we shall show that the sequence is a Cauchy sequence in $L_2$. Define $A_{n,p} = \mathbb{E}(\sum_{i=n}^{p} (X_i|\mathcal{F}_1) - (X_i|\mathcal{F}_0))^2 = \mathbb{E}(\mathbb{E}(S_p - S_{n-1}|\mathcal{F}_1) - (S_p - S_{n-1}|\mathcal{F}_0))^2$, $\forall p > n$. Squaring the quantity and computing the expected value by conditioning on $\mathcal{F}_0$ for the cross term, taking into account the Markov property and the fact that $\mathcal{F}_0 \subset \mathcal{F}_1$, we obtain $A_{n,p} = \mathbb{E}(\mathbb{E}(S_p - S_{n-1}|\mathcal{F}_1))^2 - \mathbb{E}(\mathbb{E}(S_p - S_{n-1}|\mathcal{F}_0))^2 = \mathbb{E}(\mathbb{E}(S_p - S_{n-2}|\mathcal{F}_0))^2 - \mathbb{E}(\mathbb{E}(S_p - S_{n-1}|\mathcal{F}_0))^2$. Recalling from spectral calculus that for a reversible Markov chain we have the representation of the second part of the lemma is similar.

The formula of Proposition 3 is defined above converges in $L_2$ respectivly to a martingale difference sequence and a reversed martingale difference.

Proposition 5 (Forward-backward martingale decomposition)

Let $(X_i)$ be defined by formula (7). Let $\text{Var}(S_n)/n \sigma^2 < \infty$. Then, $2S_n = M_n^d + M_n^r + X_n - X_0$, where $M_n^d$, $M_n^r$ are direct and reversed martingales respectively.

Proof. Recalling that $B_{n,k} \rightarrow 0$, $B_{n,k+1} \rightarrow 0$ in $L_2$, using Lemma 4 and the representation of $X_k + X_{k+1}$ by formula (3), we obtain as $n \rightarrow \infty$, $X_k + X_{k+1} = D_{k+1} + \tilde{D}_k$. It follows that

$$2S_n = \sum_{i=1}^{n} D_i + \sum_{i=1}^{n} \tilde{D}_i + X_n - X_0.$$  (4)
3 Central limit theorem

Theorem 6
Let \( (X_i, i \in \mathbb{N}) \) be a reversible mean zero Markov chain. If \( \mathbb{E}X_0 < \infty \) and \( \text{Var}(S_n)/n \rightarrow \sigma^2 \neq 0 \), then \( S_n/\sqrt{n} \Rightarrow N(0,1) \).

We provide a new proof of Theorem 6 based on the following result of Heyde (1974):

Theorem 7 (Heyde)
Let \( (X_i, i \in \mathbb{Z}) \) be a stationary and ergodic mean zero sequence of random variables with finite second moments. Assume that the following two conditions hold:

\[
\sum_{i=0}^{n} (\mathbb{E}(X_i|\mathcal{F}_1) - \mathbb{E}(X_i|\mathcal{F}_0)) \text{ converges in } L^2, \tag{5}
\]
\[
\text{Var}(S_n)/n \rightarrow \sigma^2 \neq 0, \tag{6}
\]
where \( \mathcal{F}_i \) is the \( \sigma \)-field generated by \( (X_j, j \leq i) \). Then, \( n^{-1/2}S_n \Rightarrow N(0,\sigma^2) \).

Proof. To prove Theorem 6, we shall verify the assumptions of Theorem 7. The second assumption (6) is common to both theorems. As for the first assumption (5), we can apply Proposition 3 to obtain convergence in \( L^2 \). So, all the assumptions are satisfied. □

Note that the only assumption of reversibility in Theorem 6 drops all assumptions on mixing rates that are usually imposed on the Markov chain. Proposition 1 of Dedecker and Rio (2000), reformulated for stationary martingales is as follows:

Proposition 8
Let \( (D_i, i \in \mathbb{Z}) \) be a stationary sequence of martingale differences or reversed martingale differences. Let \( S_n \) be the partial sums of any of the sequences. Let \( \lambda \) be a nonnegative real number and \( \Gamma_k = (S_k^* > \lambda) \), where \( S_k^* = \max_{1 \leq i \leq k} (0, S_1, \ldots, S_k) \). Then,

\[
\mathbb{E}((S_n^* - \lambda)^2) \leq 4 \sum_{i=1}^{n} \mathbb{E}(D_k^2|\Gamma_k), \tag{7}
\]
and \( n^{-1} \max_{1 \leq i \leq n} S_i^2 \) is uniformly integrable.

The proof of the first part of the conclusion of this proposition can be found in Dedecker and Rio (2000). The second part concerning uniform integrability follows from the inequality (7). Denoting \( M_n^* = \max_{1 \leq i \leq n} |S_i| \), from inequality (7) applied to \( (D_i) \) and \( (-D_i) \), we have \( n^{-1} \mathbb{E}((M_n^* - \lambda)^2) \leq 8n^{-1} \sum_{i=1}^{n} \mathbb{E}D_k^2 = 8\mathbb{E}D_0^2 \). Thus, taking \( \lambda = 0 \), we get \( n^{-1} \max_{1 \leq i \leq n} S_i^2 \) is uniformly bounded in \( L^1 \). The proof of uniform integrability of \( n^{-1} \max_{1 \leq i \leq n} S_i^2 \) reduces to showing that \( \limsup_n \int_{A} n^{-1} \max_{1 \leq i \leq n} S_i^2 dP \to 0 \) as \( \mathbb{P}(A) \to 0 \). This follows from

\[
\int_{A} n^{-1} \max_{1 \leq i \leq n} S_i^2 dP \leq 2 \int_{A} n^{-1} ((\max_{1 \leq i \leq n} |S_i| - \lambda \sqrt{n})^2 + (\lambda \sqrt{n})^2) dP = \\
= 2 \int_{A} n^{-1} (\max_{1 \leq i \leq n} |S_i| - \lambda \sqrt{n})^2 dP + 2\lambda^2 \mathbb{P}(A).
\]
So, \( \int_A n^{-1} \max_{1 \leq i \leq n} S_i^2 dP \leq 2n^{-1} \mathbb{E}(M_n^r - \lambda \sqrt{n})^2 + 2\lambda^2 \mathbb{P}(A) \leq 16n^{-1} \sum_{i=1}^{n} \mathbb{E}(D_i^2 \mathbb{1}_{(S_i > \lambda \sqrt{n})}) + 2\lambda^2 \mathbb{P}(A). \)

Due to stationarity of the martingale (or reversed martingale) difference \( D_i \), the sequences \( D_i^2, 1 \leq i \leq n \) is uniformly integrable. Therefore, for all \( \varepsilon \) and for all \( k \), there exists \( n \) such that \( \mathbb{E}(D_k^2\mathbb{1}_{S_k > \lambda \sqrt{n}}) < \varepsilon. \) Thus, \( \int_A n^{-1} \max_{1 \leq i \leq n} S_i^2 dP \leq 16\varepsilon + 2\lambda^2 \mathbb{P}(A). \) Finally, we obtain

\[
\lim_{\mathbb{P}(A) \to 0} \limsup_{n \to \infty} \int_A n^{-1} \max_{1 \leq i \leq n} S_i^2 dP \leq 16\varepsilon. \]

Taking \( \varepsilon \to 0 \) leads to the conclusion of the proposition.

Now we are ready to propose a new proof of Corollary 1.5 of Kipnis and Varadhan (1986).

**Theorem 9 (Kipnis, Varadhan)**

For any reversible stationary Markov chain \( (\xi_j, j \in \mathbb{Z}) \) defined on a space \( \mathcal{X} \), and for any mean zero function \( f \) satisfying the following conditions:

1. \( \int f^2(x) \pi(dx) < \infty, \)
2. \( \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(f(\xi_1) + \cdots + f(\xi_n))^2 = \sigma^2 < \infty, \)

the reversible Markov chain defined by (7) satisfies, \( \frac{S_{[nt]}}{\sqrt{n}} \Rightarrow \sigma f |W(t). \)

**Proof.** To prove Theorem 9, we need to show tightness of \( W_n(t) = S_{[nt]} / \sqrt{n}. \) This means, show that \( \forall \varepsilon > 0, \lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}( \sup_{|s-t| < \delta} |W_n(t) - W_n(s)| > \varepsilon) = 0. \) Convergence of finite dimensional distributions repeats the steps of Theorem 1 of Longla and al. (2012). By Billingsley’s Theorem 8.3 (1968) formulated for random elements of \( D \) (see page 137 or formula (8.16) in Billingsley, 1968), taking into account stationarity of the process, this condition is satisfied if

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}( \max_{1 \leq j \leq [nt]} |S_j| > \varepsilon \sqrt{n}) = 0. \]

Therefore, it is enough to have

\[
\frac{\max_{1 \leq j \leq n} S_j^2}{n} \quad \text{is uniformly integrable.} \quad (8)
\]

The chain being reversible, we have by Proposition 5:

\[
2S_n = M_n^d + M_n^r + X_n - X_0, \quad (9)
\]

where \( M_n^d = \sum_{i=1}^{n} D_i^d \) and \( M_n^r = \sum_{i=1}^{n} D_i^r \) are respectively a direct and a reversed martingales. The sequences \( (D_i^d) \) and \( (D_i^r) \) are respectively stationary martingale differences and stationary reversed martingale differences. Due to the representation (5), the condition (8) is satisfied if

\[
\frac{\max_{1 \leq i \leq n} (M_i^d)^2}{n} \quad \text{is uniformly integrable for } \quad s = d, r, \quad (10)
\]

and

\[
\frac{\max_{1 \leq i \leq n} (X_i)^2}{n} \quad \text{is uniformly integrable.} \quad (11)
\]

The condition (10) is satisfied due to Proposition 8 and (11) is satisfied due to stationarity of the process \( (X_i, i \in \mathbb{Z}) \). This concludes the proof of the theorem. ■
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