Transport Reversal in a Thermal Ratchet

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Abstract

Transport of a Brownian particle moving in a periodic potential is investigated in the presence of symmetric unbiased external force. The viscous medium is alternately in contact with the two heat reservoirs. We present the analytical expression of the net current at quasi-steady state limit. It is found that the competition of the asymmetric parameter of potential with the temperature difference leads to the phenomena like current reversal. The competition between the two driving factors is a necessary but not a sufficient condition for current reversals.

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1. INTRODUCTION

Transport phenomena play a crucial role in many processes from physical, biological to social systems. There has been an increasing interest in transport properties of nonlinear systems which can extract usable work from unbiased nonequilibrium fluctuations [1–4]. This comes from the desire of understanding molecular motors [5], nanoscale friction [6], surface smoothening [7], coupled Josephson junctions [8], optical ratchets and directed motion of laser cooled atoms [9], and mass separation and trapping schemes at microscale [10].

The focus of research has been on the noise-induced unidirectional motion over the last decade. A ratchet system is generally defined as a system that is able to transport particles in a periodic structure with nonzero macroscopic velocity in the absence of macroscopic force on average. In these systems, directed Brownian motion of particles is generated by nonequilibrium noise in the absence of any net macroscopic forces and potential gradients. Typical examples are rocking ratchets [4, 11], flashing ratchets [12], diffusion ratchets [13], correlation ratchets [4, 14] and white-shot-noise ratchets [2]. In all these studies, the potential is taken to be asymmetric in space. It has also been shown that a unidirectional current can also appear for spatially symmetric potentials if there exits an external random force either asymmetric or spatially-dependent. If spatially periodic structures are exposed to additive Poissonian white shot noise of zero average, a macroscopic current occurs even in the absence of spatial asymmetry [2].

The current reversal is very important in new particle separation devices such as electrophoretic separation of micro-particles [15]. It is also of interest in biology [32]. Motions of macromolecules are probably responsible for the vesicle transport inside eukaryotic cells. A typical example is the motion of proteins along a microtubule, modelled usually by a ratchet...
It is well known that the two typical proteins, kinesins and dyneins, move along tubulin filaments in opposite directions. This can be explained by the current reversal.

The current reversal in ratchet systems can be engendered by varying system parameters. The current can be reversed, for example, by a noise of Gaussian force with non-white power spectrum in presence of stationary periodic potential. The current reversal can also be obtained in two-state ratchets if the long arm is kinked. Bier and Astumian have also found the current reversal in a fluctuating three-state ratchet. In the presence of a kangaroo process as the driving force, the current reversal can be triggered by varying the noise flatness, the ratio of the fourth moment to the square of the second moment. The current reversal can be induced by both an additive Gaussian white and an additive Ornstein-Uhlenbeck noise in a correlation ratchet. The current reversal also appear in forced inhomogeneous ratchets.

The previous works on the current reversal are limited to case of one heat reservoir. The present study extends the study of current reversal to the case of two heat reservoirs. When a positive driving factor competes with a negative one, the current may reverse its direction. The competition between the competitive driving factors is necessary but not sufficient for the current reversal. Our emphasis is on finding conditions of generating current reversal. This is achieved by using a quasi-steady state limit to solve the Fokker-Planck equation.

2. NET CURRENT OF THE THERMAL RATCHET

Consider a Brownian particle moving in a sawtooth potential with an unbiased external force where the medium is alternately in contact with the two heat reservoirs. This model was first proposed to describe molecular motor in biological system. The particle motion
satisfies the dimensionless Langevin equation of motion [30, 31]

\[ m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - \frac{dU(x)}{dx} + F(t) + \sqrt{2k_B T(x)} \beta \xi(t), \tag{1} \]

where \( x \) stands for the position of Brownian particle, \( m \) the mass of the particle, \( \beta \) the viscous friction drag coefficient, \( k_B \) Boltzmann constant, \( T(x) \) absolute temperature. \( \xi(t) \) is a randomly-fluctuating Gaussian white noise of zero mean and the autocorrelation function \(< \xi(t)\xi(s) > = \delta(t-s) \). Here \(< ... > \) denotes an ensemble average over the distribution of the fluctuating forces \( \xi(t) \). \( F(t) \) is an external periodic force (Fig. 1b), satisfy

\[ F(t + \tau) = F(t), \int_0^\tau F(t)dt = 0. \tag{2} \]

The geometry of symmetric potential \( U(x) = U(x + L) \) is displayed in Fig. 1a and \( U(x) \) within the interval \( 0 \leq x \leq L \) is described by

\[ U(x) = \begin{cases} \frac{\dot{u}}{L_1} x, & 0 \leq x < L_1; \\ \frac{\dot{u}}{L_2} (L - x), & L_1 \leq x \leq L, \end{cases} \tag{3} \]

where \( L = L_1 + L_2 \) is the period of the potential. The temperature \( T(x) \) has the same period as the potential \( U(x) \). Therefore, \( T(x) = T(x + L) \),

\[ T(x) = \begin{cases} T + \delta, & 0 \leq x < L_1; \\ T, & L_1 \leq x \leq L. \end{cases} \tag{4} \]

Because the motion of the ratchet is highly overdamped in general [30], the inertia term can be neglected. Hence, Eq. (1) reduces to, when \( \beta = 1 \) and \( k_B = 1 \),

\[ \frac{dx}{dt} = -\frac{dU(x)}{dx} + F(t) + \sqrt{2T(x)} \xi(t). \tag{5} \]

The probability density satisfies the associated Fokker-Planck equation [30, 31],

\[ \frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x} [(U'(x) - F(t))P(x,t) + \frac{\partial}{\partial x}(T(x)P(x,t))] = -\frac{\partial j(x,t)}{\partial x}, \tag{6} \]
FIG. 1: Potential and Driving force: (a) potential $U(x) = U(x + L)$; $U(x)$ is a piecewise linear and periodic potential; the period of the potential is $L = L_1 + L_2$; $\Delta = L_1 - L_2$; the temperature profiles is also shown. (b) Driving force $F(t)$ which preserved the zero mean $<F(t)>=0$; $F(t+\tau) = F(t)$; $F_0$ is amplitude of $F(t)$. 
\[ j(x, t) = -[U'(x) - F(t)]P(x, t) - \frac{d}{dx}[T(x)P(x, t)], \]  

(7)

here \( j \) is the probability current density. The prime stands for the derivative with respect to the space variable \( x \). \( P(x, t) \) is the probability density for the particle at position \( x \) and at time \( t \). It satisfies the normalization condition and the periodicity condition,

\[ P(x, t) = P(x + L, t), \]  

(8)

\[ \int_0^L P(x, t)dx = 1. \]  

(9)

If \( F(t) \) changes very slowly with respect to \( t \), namely, its period is longer than any other time scale of the system, there exists a quasi-steady state. In this case, by following the method in [30, 31], we can obtain the current \( j(F(t)) \) from Eq. (7)-Eq. (9),

\[ j(F(t)) = \frac{-Q}{G_1G_2 + HQ}, \]  

(10)

where \( Q, G_1, G_2 \) and \( H \) are

\[ Q = e^{a-b} - 1, \]

\[ G_1 = \frac{L + \Delta}{2a(T + \delta)}(1 - e^{-a}) + \frac{L - \Delta}{2bT}e^{-a}(e^b - 1), \]  

(11)

\[ G_2 = \frac{L + \Delta}{2a}(e^a - 1) + \frac{L - \Delta}{2b}e^a(1 - e^{-b}), \]

\[ H = A + B + C, \]

\[ A = \frac{1}{T + \delta}(\frac{L + \Delta}{2a})^2(a + e^{-a} - 1), \]
\[ B = \frac{L^2 - \Delta^2}{4abT}(1 - e^{-a})(e^b - 1), \]
\[ C = \frac{1}{T}(\frac{L - \Delta}{2b})^2(e^b - 1 - b), \]
\[ a = \frac{2U_0 - F(t)(L + \Delta)}{2(T + \delta)}, \]
\[ b = \frac{2U_0 + F(t)(L - \Delta)}{2T}. \] (12)

The average current is
\[ J = \frac{1}{\tau} \int_0^\tau j(F(t))dt, \] (14)
where \( \tau \) is the period of the driving force \( F(t) \). \( \tau \) is assumed to be longer than any other time scale of the system at quasi-steady state. For the external force \( F(t) \) shown in Fig. 1b,
\[ J = \frac{1}{2}[j(F_0) + j(-F_0)]. \] (15)

When both the potential and the temperature are symmetric (\( \delta = 0, F_0 = 0 \)), the current \( J \) reduces to:
\[ J = \frac{1}{2(2T + \delta)}(\frac{U_0}{2L})^2(\frac{1}{e^{U_0/T + \delta} - 1} - \frac{1}{e^{U_0/T} - 1}). \] (16)

Therefore, the net current is not zero even when both the potential and the temperature are spatially symmetric. The direction of the current is determined by the sign of \( \delta \). The particle tends to move from high temperature region to low temperature region. In fact, this agrees with the diffuse law.

3. RESULTS AND DISCUSSION

Figure 2 shows the current \( J \) as a function of the asymmetric parameter \( \Delta \) of the potential at \( \delta = 0 \). The current is negative for \( \Delta < 0 \), zero at \( \Delta = 0 \) and positive for \( \Delta > 0 \). Therefore, we can have the current reversal by changing the sign of \( \Delta \), the asymmetry of the potential.
FIG. 2: Current $J$ versus asymmetric parameter $\Delta$ of the potential at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$, $T = 1.0$ and $\delta = 0$.

FIG. 3: Current $J$ versus temperature difference $\delta$ at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$, $T = 10$ and $\Delta = 0$. 
Figure 3 shows the current $J$ versus temperature difference $\delta$ in a symmetric potential $\Delta = 0$. The temperature difference $\delta$ controls not only the magnitude but also the direction of the current. When $\delta = 0$ and $\Delta = 0$, there is no current. For the asymmetric potentials, varying temperature difference is another way of inducing a net current.

The current $J$ as the function of $T$ is shown in Fig. 4 for different combinations of $\Delta$ and $\delta$. The curve is observed to be bell-shaped, which shows the feature of resonance. When $T \to 0$, $J$ tends to zero for all values of $\delta$ and $\Delta$. Therefore, the particle cannot pass the barrier and there is no current. When $T \to \infty$ so that the thermal noise is very large, the ratchet effect disappear and $J \to 0$, also. There is an optimized value of $T$ at which the current $J$ takes its maximum value. There is no current reversal at $\delta = 0.1, \Delta = 0.9$; $\delta = 0.1, \Delta = 0.1$; $\delta = -0.1, \Delta = -0.1$; and $\delta = -0.1, \Delta = -0.9$. In fact, the temperature cannot lead to the current reversal if $\Delta \delta > 0$ (Fig. 2-Fig. 4).
FIG. 5: (a) Current $J$ versus temperature $T$ for different values of the asymmetric parameters $\Delta$ at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$. (b) Current $J$ versus temperature $T$ at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$, $\delta = 0.1$ and $\Delta = -0.4$. 
In Fig. 5a, we plot the current $J$ as a function of temperature $T$ for different combinations of $\Delta$ and $\delta$. When temperature difference ($\delta = 0.1$) is positive, the current may reverse its direction as increasing temperature for negative $\Delta$ ($\Delta = -0.8$). It is observed that the current reversal may occur for negative $\delta$ and positive $\Delta$ ($\delta = -0.1$, $\Delta = 0.8$). We can also have the current reversal twice at $\delta = 0.1$, $\Delta = -0.4$ (Fig. 5b). Therefore, there may exist current reversal for $\Delta \delta < 0$. However, $\Delta \delta < 0$ is not sufficient condition for current reversal. For example, the current is always positive for $\delta = 0.1$, $\Delta = -0.3$ and negative for $\delta = -0.1$, $\Delta = 0.2$ (Fig. 5a).

In order to illustrate the current reversal in detail, the current contours are shown in Figs. 6 and 7, respectively. When $T > T_a$ or $\Delta > \Delta_c$, the current is always positive there is no current reversal (Fig. 6, also see the case $\delta = 0.1$, $\Delta = -0.3$ in Fig. 5a). The current

**FIG. 6:** Current contours on $\Delta$-$T$ plane at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$ and $\delta = 0.1$: $\Delta_c (-0.3987)$ is maximum $\Delta$ for the curve $J = 0$; $\Delta_b (-0.4913)$ is asymmetric parameter of potential at the cross point of $J = 0$ and $T = 0$; $T_a$ is temperature at the cross point of $J = 0$ and $\Delta = -1.0$. 

In order to illustrate the current reversal in detail, the current contours are shown in Figs. 6 and 7, respectively. When $T > T_a$ or $\Delta > \Delta_c$, the current is always positive there is no current reversal (Fig. 6, also see the case $\delta = 0.1$, $\Delta = -0.3$ in Fig. 5a). The current

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reversal may, however, occur by varying $T$ or $\Delta$ when $T < T_a$ or $\Delta < \Delta_c$ (Fig. 6, also see the case $\delta = 0.1$, $\Delta = -0.8$ in Fig. 5a). In particular, the current may reverse its direction twice if $\Delta_b < \Delta < \Delta_c$ (Fig 6, also see the case $\delta = 0.1$, $\Delta = -0.4$ in Fig 5b).

FIG. 7: Current contours on $\delta$-$T$ plane at $U_0 = 5$, $F = 3.0$, $L = 1.0$ and $\Delta = -0.6$: $\delta_a = 0; \delta_b$ (0.138) is temperature difference at the cross point of $J = 0$ and $T = 0; \delta_c$ (0.151) is maximum temperature difference for $J = 0$.

The current is always negative and there is no current reversal for $\delta \leq 0$ and $\Delta = -0.6$ (Fig. 7). When $\delta_a < \delta < \delta_c$, the current may reverse its direction as increasing temperature. The current may changes its direction twice, in particular, when $\delta_b < \delta < \delta_c$.

Therefore, we can not have the current reversal when $\delta \Delta \geq 0$. When $\delta \Delta < 0$, the current may reverse its direction. However, $\delta \Delta < 0$ is not the sufficient condition for current reversal.
4. CONCLUDING REMARKS

The transport of a Brownian particle moving in a periodic potential is studied in the presence of an unbiased fluctuation and two heat reservoirs. In a quasi-steady state limit, we obtain the current analytically. It is found that the asymmetric parameter $\Delta$ of the potential and the temperature difference $\delta$ are the two pivotal factors for obtaining a net current. For the two positive or the two negative driving factors such that $\Delta \delta > 0$, current cannot reverse its direction. The current reversal cannot occur either if there is only one driving factor such that $\Delta \delta = 0$. For the two opposite driving factors so that $\Delta \delta < 0$, the current reversal may occur. The current reversal can also occur twice at certain conditions. The condition $\Delta \delta < 0$ is a necessary but not a sufficient condition for the current reversal.

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