ALTERNATIVE LAGRANGIANS FOR EINSTEIN METRICS \(^a\)

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We shall use the variational decomposition technique in order to calculate equations of motion and Noether energy-momentum complex for some classes of non-linear gravitational Lagrangians within the first-order (Palatini) formalism. In particular, a complex space-time appears as a solution of our variational problem.

1 Introduction

We report on recent results stating that some classes of non-linear gravitational Lagrangians give, in the first-order formalism, Einstein field equations and the Komar expression for the energy-momentum complex. Such Lagrangians are particularly important since, at the classical level, they are equivalent to General Relativity. However, their quantum contents and divergences could be slightly improved. This note is based on joint works with M. Ferraris (Torino) and I. Volovich (Moscow).

1.1 Variational Decomposition and Noether Theorems

It is well known that a variation (i.e. functional derivative) of an arbitrary-order Lagrangian \( L(\phi) \equiv L(\phi, \phi_\mu, \phi_{\mu\nu}, \ldots) \) decomposes into two parts according to the “first variation formula”

\[
\delta L = \frac{\delta L}{\delta \phi} \delta \phi + \partial_\mu \phi^\mu
\]  

(1)

Here \( \phi_\mu = \partial_\mu \phi, \ldots \) denotes the partial derivatives of \( \phi \) with respect to (local) space-time (independent) variables \( x^\mu, \mu = 1, \ldots, n \). The first term represents the Euler-Lagrange expression, i.e. field equations. The second part is a
divergence of \( q^\mu \equiv q^\mu(\phi, \delta \phi) \), where

\[
q^\mu = \left[ \frac{\partial L}{\partial \phi_\mu} - \partial_\nu \left( \frac{\partial L}{\partial \phi^\mu_{\nu}} \right) \right] \delta \phi + \frac{\partial L}{\partial \phi^\mu_{\nu}} \delta \phi_\nu + \ldots
\]

Although this second (boundary) term does not contribute to the equations of motion it is physically important since it does contribute to the conservation laws (Noether Theorems).

For the variation \( \delta \phi \) implemented by an (infinitesimal) symmetry transformation one has \( \delta_\tau L = \partial_\mu \tau^\mu \) without using the equations of motion. Therefore, equation (1) can be rewritten under the following form

\[
\frac{\delta L}{\delta \phi_\mu} \delta \phi = -\partial_\mu(q^\mu_\tau - \tau^\mu)
\]

where \( q^\mu_\tau = q^\mu(\phi, \delta_\tau \phi) \). A Noether current then arises

\[
E^\mu \equiv E^\mu(\phi, \delta_\tau \phi) = q^\mu_\tau - \tau^\mu
\]

which is conserved on shell, i.e. when the field equations are satisfied. One writes \( \partial_\mu E^\mu \approx 0 \) and calls it a weak conservation law. In the present paper we deal with so-called local symmetries (and second Noether’s Theorem).

In this case, there exists a skew-symmetric quantity \( U^{\mu \nu} = -U^{\nu \mu} \), called a superpotential (see e.g.

\[
\partial_\mu E^\mu \approx 0 \Rightarrow E^\mu \approx \partial_\nu U^{\mu \nu}
\]

i.e. \( E^\mu \) differs from the divergence \( \partial_\nu U^{\mu \nu} \) by a quantity which vanishes on shell.

### 1.2 Second-Order Einstein-Hilbert Lagrangian

Einstein metrics are extremals of the Einstein-Hilbert purely metric variational problem. Consider the Einstein-Hilbert (linear) gravitational Lagrangian

\[
L_H(g, \partial g, \partial^2 g) = |\det g|^\frac{1}{2} (R - c)
\]

Here standard notation for the Riemann and Ricci tensor

\[
R^\alpha_{\mu \nu \rho} = R^\alpha_{\rho \mu \nu}(g) = \partial_\nu \Gamma^\alpha_{\rho \mu} - \partial_\rho \Gamma^\alpha_{\nu \mu} + \Gamma^\sigma_{\rho \mu} \Gamma^\alpha_{\sigma \nu} - \Gamma^\sigma_{\nu \rho} \Gamma^\alpha_{\sigma \mu}
\]

\[
R_{\mu \nu} = R_{\mu \nu}(g) = R^\alpha_{\mu \alpha \nu}
\]

of the Levi-Civita connection on a space-time manifold \( M \) (dim\( M = n \))

\[
\Gamma^\alpha_{\beta \mu}(g) = \frac{1}{2} g^{\alpha \sigma} \left( \partial_\gamma g_{\mu \sigma} + \partial_\mu g_{\sigma \beta} - \partial_\sigma g_{\beta \mu} \right)
\]
is in use. In the Lagrangian above $R = R(g) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ denotes the scalar curvature. In this way the metric $g$ becomes the only dynamical variable of the theory. According to well known formula

$$\delta \sqrt{g} = -\frac{1}{2} \sqrt{g} g_{\alpha\beta} \delta g^{\alpha\beta},$$

variation of $L_H$ with respect of an arbitrary variation of $g$ reads

$$\delta L_H = \sqrt{g} [R_{\alpha\beta} - 1/2(R-c)g_{\alpha\beta}] \delta g^{\alpha\beta} + \sqrt{g} g^{\alpha\beta} \delta R_{\alpha\beta}$$

Taking into account that from (4)

$$\delta R_{\alpha\beta} = \nabla_\mu \delta \Gamma^\mu_{\alpha\beta} - \nabla_\alpha \delta \Gamma^\alpha_{\beta\sigma}$$

and that covariant derivatives $\nabla_\alpha$ for the Levi-Civita connection of $g$ commutes with $\sqrt{g} g^{\alpha\beta}$, we get

$$\sqrt{g} g^{\alpha\beta} \delta R_{\alpha\beta} = \nabla_\mu [\sqrt{g} g^{\alpha\beta} (\delta \Gamma^\mu_{\alpha\beta} - \delta^\mu_{\alpha} \delta \Gamma^\alpha_{\beta\sigma})]$$

Quantity in the square brackets transforms as a vector density of weight 1. It allows to replace the covariant derivatives $\nabla_\mu$ in (7) by the partial one $\partial_\mu$.

Therefore, a variational decomposition for $L_H$ takes finally the form

$$\delta L_H = \sqrt{g} [R_{\alpha\beta} - 1/2(R-c)g_{\alpha\beta}] \delta g^{\alpha\beta} + \partial_\mu [\sqrt{g} g^{\alpha\beta} (\delta \Gamma^\mu_{\alpha\beta} - \delta^\mu_{\alpha} \delta \Gamma^\alpha_{\beta\sigma})]$$

This produces, of course, the Einstein field equations for the metric $g$

$$R_{\mu\nu}(g) = \Lambda g_{\mu\nu}$$

with the cosmological constant $\Lambda = c/(n-2)$.

As a symmetry transformation, consider now a 1-parameter group of diffeomorphisms generated by the vectorfield $\xi = \xi^\alpha \partial_\alpha$ on $M$. In this case one can utilize the well known expressions

$$\delta_\xi g = \delta_\xi g_{\alpha\rho} \equiv \nabla_\alpha \xi_\rho + \nabla_\rho \xi_\alpha$$

$$\delta_\xi \Gamma = \delta_\xi \Gamma^\rho_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - \nabla_\rho \xi_\alpha \nabla_\rho \xi_\beta$$

where $\mathcal{L}_\xi$ stands for the Lie derivative along $\xi$. Our Lagrangian is reparametrization invariant, in the sense that diffeomorphisms of $M$ transform $L_H$ as a scalar density of weight 1. This means that, at the infinitesimal level, one has

$$\delta_\xi L_H = \mathcal{L}_\xi L_H = \partial_\alpha (\xi^\alpha L_H)$$

$d$ We simply write $\sqrt{g}$ instead of $\sqrt{|\text{det}g|}$.

$e$ Since one deals with a symmetric connection.

$f$ In this letter we always assume $n > 2$. 

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As a consequence, equation (2), in this case, can be written as follows:

\[ E^\mu(\xi) = \sqrt{g}(g^{\alpha\beta} \delta^\mu_\sigma - g^{\alpha\mu} \delta^\beta_\sigma) \nabla_\alpha \nabla_\beta \xi^\sigma + (2\sqrt{g}R^\mu_\sigma - \delta^\mu_\sigma L_H) \xi^\sigma \] (12)

This provides the global and covariant expression for the Noether energy-momentum flow of a gravitational field represented by the Einstein metric \( g \) and calculated along a vectorfield \( \xi \). The corresponding superpotential \[ U^\mu_\nu_H(\xi) = |detg|^\frac{1}{2}(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \] (13)
is known as the Komar superpotential. Problems with the definition of gravitational energy and momentum appear when one tries to make (12-13) independent of the vectorfield \( \xi \). An interesting application of the Komar expression to the black hole entropy has been presented in [18].

2 Non-Linear First-Order Lagrangians

It is known that the non-linear Hilbert type Lagrangians \( f(R) \sqrt{g} \), where \( f \) is a function of one real variable, lead to fourth order equation for \( g \), which are not equivalent to Einstein equations unless \( f(R) = R - c \) (linear case), or to appearance of additional matter fields. It is also known that the linear ”first order” Lagrangian \( r \sqrt{g} \), where \( r = r(g, \Gamma) = g^{\alpha\beta} r_{\alpha\beta}(\Gamma) \) is a scalar concomitant of the metric \( g \) and linear (symmetric) connection \( \Gamma \), leads to separate equations for \( g \) and \( \Gamma \) which turn out to be equivalent to Einstein equations for \( g \). In the sequel we shall use small letters \( r_{\beta\mu\nu} \) and \( r_{\beta\nu} = r_{\beta\alpha\nu} \) to denote the Riemann and Ricci tensor of an arbitrary (symmetric) connection \( \Gamma \) (still given by the same formulae (4)), i.e. without assuming that \( \Gamma \) is the Levi-Civita connection of \( g \).

2.1 Hilbert Type Lagrangians

As we explained above inequivalence with General Relativity could also hold for non-linear first-order Lagrangians \[ L_f(g, \Gamma) = \sqrt{g} f(r) \] (14)

Now, the scalar \( r(g, \Gamma) = g^{\alpha\beta} r_{\alpha\beta}(\Gamma) \) is not longer the scalar curvature, since \( \Gamma \) is not longer Levi-Civita connection of the metric \( g \). We choose a metric and a symmetric connection as independent dynamical variables (so-called Palatini method, see also [3]). Variation of \( L_f \) gives

\[ \delta L_f = \sqrt{g}(f'(r)r_{\alpha\beta} - 1/2 f(r)g_{\alpha\beta}) \delta g^{\alpha\beta} + \sqrt{g} f'(r)g^{\alpha\sigma} \delta r_{\alpha\beta} \] 10 Such Lagrangians have been investigated in [10].
Substituting $\delta r_{\alpha \beta}$ by an analog of (6) with $\nabla_{\alpha}$ being the covariant derivative with respect to $\Gamma$ and applying the covariant Leibniz rule ("integrating by parts") give rise to the variational decomposition

$$\delta L_f = \sqrt{g}(f'(r)r_{\alpha \beta} - 1/2 f(r)g_{\alpha \beta})\delta g^{\alpha \beta} - \nabla_{\beta}[(\sqrt{g}f'(r))(g^{\alpha \sigma} \delta^\beta_{\lambda})$$

$$- g^{\alpha \beta} \delta^\beta_{\lambda})]\delta g^{\alpha \beta}$$

$$- \nabla_{\beta}[\sqrt{g}f'(r)g^{\alpha \beta}(\delta \Gamma_{\mu}^{\alpha \beta} - \delta^\mu_{\beta} \delta \Gamma_{\alpha \sigma}^{\sigma})]$$

(15)

First observe that the boundary term in (15) apart of the factor $f'(r)$ is exactly the same as in the Einstein-Hilbert case (8). Field equations in this case are

$$f'(r)r_{(\mu \nu)} - \frac{1}{2} f(r)g_{\mu \nu} = 0$$

(16)

$$\nabla_{\alpha}[f'(r)\sqrt{g}g^{\mu \nu}] = 0$$

(17)

where () denotes symmetrization. In fact, variation of $L_f$ with respect to $\Gamma$ leads to the following equations (see also (15)):

$$\nabla_{\beta}[\sqrt{g}f'(r)(g^{\alpha \sigma} \delta^\beta_{\lambda} - g^{\beta \sigma} \delta^\alpha_{\lambda})] = 0$$

which due to the symmetry of $g^{\mu \nu}$ reduce to (17). Notice that (16) are not yet Einstein equations, even when $f(r) = r$. Equations (16-17) must be considered together with the consistency condition obtained by contraction of (16) with $g^{\mu \nu}$. It gives then

$$f'(r)r - \frac{n}{2} f(r) = 0$$

(18)

This equation (except the case it is identically satisfied) forces $r$ to take a set of constant values $r = c$, with $c$ being solution of (18). In the generic case (simple roots, with $f'(c) \neq 0$, $n > 2$) equation (17) gives

$$\nabla_{\alpha}(\sqrt{g}g^{\mu \nu}) = 0$$

which, in turn, forces $\Gamma$ to be the Levi-Civita connection of $g$. Replacing back into (16) we find

$$R_{\mu \nu}(g) = \Lambda(c)g_{\mu \nu}$$

Einstein equations for the metric $g$ with $\Lambda(c) = f(c)/2f'(c) = c/n$. As we observed above, the boundary term in (15) is proportional with the factor $f'(c)$ to that of (8). Therefore, energy-momentum flow as well as superpotential are proportional to already known from the standard Einstein-Hilbert formalism (12-13). It shows universality of Einstein equations and Komar superpotential, i.e. their independence on the choice of the Lagrangian (represented by the function $f$). These properties hold true in any dimension $n > 2$. See\(^\text{11}\) for $n = 2$ case where non-generic cases have been also considered.
2.2 Ricci Squared Lagrangians

As the next examples consider the family of non-linear gravitational Lagrangians

\[ \hat{L}_f(g, \Gamma) = \sqrt{g}f(s) \]  

(19)

parameterized by the real function \( f \) of one variable. Now, the scalar \((\text{Ricci squared})\ concomitant \)

\[ s = s(g, \Gamma) = g^{\alpha \mu}g^{\beta \nu}s_{\alpha \beta}g_{\mu \nu}, \]

where \( s_{\mu \nu} = r(\mu \nu)(\Gamma) \) is the symmetric part of the Ricci tensor of \( \Gamma \). Variational decomposition formula reads

\[
\delta \hat{L}_f = \sqrt{g}(2f'(s)g^{\mu \nu}s_{\alpha \mu}g_{\beta \nu} - \frac{1}{2}f(s)g_{\alpha \beta})\delta g^{\alpha \beta} - \nabla_\nu[2\sqrt{g}f'(s)(s^{\alpha \beta}\delta^\nu_\lambda)
- s^{\alpha \beta}\delta^{\nu}_\lambda]\delta \Gamma_{\alpha \beta} + \partial_\nu[2\sqrt{g}f'(s)g^{\alpha \mu}g^{\beta \nu}s_{\nu \lambda}(\delta \Gamma^{\mu}_{\alpha \beta} - \delta^{\mu}_{\beta}\delta \Gamma^{\sigma}_{\alpha \sigma})] \]

(20)

where for short \( s^{\alpha \beta} = g^{\alpha \mu}g^{\beta \nu}s_{\mu \nu} \). Observe again that an essential part of the boundary term in (20) coincides with the previous cases (8, 15). Euler-Lagrange field equations are

\[ f'(s)g^{\mu \nu}s_{\alpha \mu}g_{\beta \nu} - \frac{1}{4}f(s)g_{\alpha \beta} = 0 \]  

(21)

\[ \nabla_\lambda(\sqrt{g}f'(s)g^{\alpha \mu}g^{\beta \nu}s_{\mu \nu}) = 0 \]  

(22)

Contraction of (21) with \( g^{\alpha \beta} \) gives the consistency equation

\[ f'(s)s - \frac{n}{4}f(s) = 0 \]  

(23)

Restricting our attention again to the generic case we find that for regular solutions \( s = c \neq 0 \) of (23) \( f'(c) \neq 0, n > 2 \) equation (21) can be rewritten in the following (matrix) form

\[ (g^{-1}h)^2 = \frac{c}{|c|}I \]  

(24)

where \( c/|c| = \pm 1 \) and

\[ h_{\alpha \beta} = \sqrt{\frac{n}{|c|}}s_{\alpha \beta}(\Gamma). \]

(25)

\( h_{\alpha \beta} \) is a symmetric, twice-covariant and due to (21) non-degenerate tensor field on \( M \) i.e., it is simply a metric. By making use of the Ansatz (25), equations (22) can be converted into the form

\[ \nabla_\lambda(\sqrt{h}h^{\alpha \beta}) = 0 \]
with $h^{\alpha\beta}$ being the inverse of $h_{\alpha\beta}$. Therefore, the connection $\Gamma$ has to be a Levi-Civita connection for the metric $h$ and as a consequence, (25) becomes an Einstein equation for $h$ with the cosmological constant $\Lambda = \sqrt{|c|/n}$. Substituting further (25) into the boundary term in (20) we find that, up to a constant multiplier, the energy-momentum flow and the superpotential are given by the same expressions as (12-13) with the metric $g$ replaced by $h$. This extends a notion of universality also to the class of Ricci squared Lagrangians.

The algebraic constraints (24) are of special interest by their own. They provide on space-time some additional differential-geometric structures, namely a Riemannian almost-product structure and/or an almost-complex anti-Hermitian ($\equiv$ Norden) structure.

In the (pseudo-)Riemannian almost-product case one equivalently deals with an almost-product structure given by the $(1,1)$ tensor field $P = g^{-1}h (P^2 = I)$ as well as with a compatible metric $h$ satisfying the condition

$$h(PX, PY) = h(X, Y)$$

which is also encoded in the simple algebraic relation (24). Here $X, Y$ denote two arbitrary vectorfields on $M$.

There is a wide class of integrable almost-product structures, namely so called warped product structures, which are an intrinsic property of some well know exact solutions of Einstein equations: these include e.g. Schwarzschild, Robertson-Walker, Reissner-Nordström, de Sitter, etc. (but not Kerr!). Some other examples are provided by Kaluza-Klein type theories, $3+1$ decompositions and more generally so called split structures. The explicit form of the zeta function on product spaces and of the multiplicative anomaly has been derived recently in.

In the anti-Hermitian case one deals with $2m$-dimensional manifold $M$, an almost complex structure $J = g^{-1}h (J^2 = -I)$ and an anti-Hermitian metric $h$:

$$h(JX, JY) = -h(X, Y)$$

This implies that the signature of $h$ should be $(m, m)$. In the Kählerian case ($\nabla J = 0$ for the Levi-Civita connection of $h$) the almost-complex structure is automatically integrable. We have proved that in fact the metric $h$ has to be a real part of certain holomorphic metric on a complex (space-time) manifold $M$.

It should be however remarked that a theory of complex manifolds with holomorphic metric (so called complex Riemannian manifolds) has become one

\footnote{In our case the metric $h$ should be in addition Einsteinian.}

\footnote{Recall that for a Hermitian metric $h(JX, JY) = h(X, Y)$.}
of the corner-stone of the twistor theory\footnote{23}. This includes a non-linear graviton\footnote{22}, theory of $H$-spaces\footnote{6} and ambitwistor formalism\footnote{21}.

2.3 Conclusions

We showed that the use of Palatini formalism leads to results essentially different from the metric formulation when one deals with non-linear Lagrangians: with the exception of special (“non-generic”) cases we always obtain the Einstein equations as gravitational field equations and Komar complex as a Noether energy-momentum complex. In this sense non-linear (matter-free)\footnote{23} theories are equivalent to General Relativity: they admit two families of alternative Lagrangians (14, 19) for the Einstein equations with a cosmological constant. In $n = 2$ dimensions, they provide a general mechanism for governing topology change.

Moreover, in the case of Ricci squared Lagrangians (19), besides the initial metric $g$ one gets the Einstein metric $h$. Both metrics are related by algebraic equation (24). These aspects have been considered in\footnote{4}. A characterization and examples of anti-Kähler Einstein manifolds as well as almost-product Einstein manifolds has been obtained.\footnote{4}

Our results can be relevant for quantum gravity. In fact, in order to remove divergences one has to add counterterms to the Lagrangian which depend not only on the scalar curvature but also on the Ricci and Riemann tensor invariants. It follows from our results that in the first order formalism, such counterterms do not change the semiclassical limit, since genericly we still have the standard Einstein equation.

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$^k$ See e.g. how to include matter.
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