The thermal and kinematic Sunyaev-Zel’dovich effects revisited

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Abstract

This paper shows that a simple convolution integral expression based on the mean value of the isotropic frequency distribution corresponding to photon scattering off electrons leads to useful analytical expressions describing the thermal Sunyaev-Zel’dovich effect. The approach, to first order in the Compton parameter is able to reproduce the Kompaneets equation describing the effect. Second order effects in the parameter $z = \frac{\omega_c}{m_e c}$ induce a slight increase in the crossover frequency.

1 Introduction

Ever since R. Sunyaev and Ya. B. Zel’dovich predicted the distortion of the spectral density of the Cosmic Microwave Background Radiation (CMBR) in 1969\textsuperscript{1} \textsuperscript{2} by the ionized plasma in globular clusters, whose main constituent is a relatively dense electron gas, dozens of papers have been published addressing various aspects of this phenomenon. This avalanche of works is mainly due to the undeniable importance of such effect, not only because it yields information on deviations from the isotropy and homogeneity of the Universe, but also provides an aid in determining other important cosmological parameters such as the baryonic density of matter, Hubble’s constant, age and velocity of
massive clusters, and others. These features have been thoroughly reviewed in recent literature so we shall not dwell with them here [3]-[6].

Although practically every cosmologist or astrophysicist believes that this effect is now “clearly” understood, there are reasons that we believe will help clarifying some of the still subtle details that remain unclear in the available treatments. Firstly, the early attempts to study this effect by visualizing the motion of the photons through the hot electron gas in the cluster as a diffusion process which, in the non-relativistic limit is described by the famous Kompaneets equation [7][8]. The analysis of the results obtained through the hot electron gas in the cluster through the use of this equation are well-known [3]-[5]. Nevertheless, it was soon realized by many authors, including Sunyaev himself, that due to the very small probability that an incoming photon is scattered even once in its passage through the gas, a Compton-like scattering of a photon off an electron was a much more suitable mechanism to explain the effect. This trend of ideas has been recently reviewed by Dolgov et al [4] where the reader may find more literature about the subject. The somewhat intriguing result is that both approaches lead to identical results, a fact that needs further clarification. We address this question in this paper.

Secondly, the role of the optical depth $\tau$, related to the distance a photon can travel in the plasma before it is scattered off an electron, plays in the way the broadening of the spectral lines of the scattered photon is modified by the Doppler effect, has not yet been justified in a convincing manner. Here we address this question form an entirely different point of view. And thirdly, we also show that this different interpretation of $\tau$ also leads to the kinematic Sunyaev-Zel’dovich (SZ) effect in a rather trivial way.

2 The method

We begin our discussion by establishing a general expression for the full distorted spectrum, $I(\nu)$ of the scattered radiation off the plasma. Let $\tau$ be the optical depth as discussed above. This quantity is a direct measure of the probability that a photon is scattered off an electron in the gas. Thus, since the effect is small, $(1-\tau)$ is the probability that the photon traverses the plasma unscattered. Therefore,

$$I(\nu) = (1-\tau) \int_{-\infty}^{\infty} I_o(\bar{\nu}) \delta(\bar{\nu} - \nu) d\bar{\nu} + \frac{\tau}{\sqrt{2\pi}\sigma(\nu)} \int_{-\infty}^{\infty} I_o(\bar{\nu}) \exp \left[-\left(\frac{\bar{\nu}-(1-2z)\nu}{2\sigma(\nu)}\right)^2\right] d\bar{\nu}$$

(1)

The first term in Eq. (1) is self-explanatory, the lower limit justified since $I_o(\nu)$, the incoming flux is zero for negative frequencies. The second term, the important one, defines a Gaussian probability function describing scattering of a photon with incoming frequency $\nu$ off an electron whose average kinetic energy in the gas, in the non-relativistic approximation is $\frac{1}{2}kT_e$. This function has been thoroughly discussed in earlier work [9] so we shall not discuss it here further.
In Eq.(1) the incoming flux is given, for $\nu > 0$ by:

$$I_o(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{\nu}{T}} - 1}$$  \hspace{1cm} (2)

where $T \equiv 2.726 \text{ K}$, the temperature of the cosmic background radiation, $k$ is Boltzmann’s constant, $h$ is Planck’s constant, and $c$ the speed of light. Further, $\sigma(\nu)$ is the width of the spectral line at frequency $\nu$ and its squared value reads [9]:

$$\sigma^2(\nu) = 2 \frac{kT}{mc^2} \nu^2 = 2z \nu^2$$  \hspace{1cm} (3)

In Eq.(2) $T_e$ is the temperature of the electron scatterers, $m$ is the electron mass and $I_o(\bar{\nu}) \equiv 0$ for $\bar{\nu} < 0$.

Notice should be made of the fact that the mathematical expression for $\sigma^2(\nu)$ corresponds to what in the literature is called the inverse Compton scattering [10]. In particular, this is the reason of the factor $2z$ in such equation. To show this fact we take into account the frequency distribution function for photons with incoming frequency $\tilde{\nu}$ isotropically scattered by electrons with velocities $u = \beta c$ [10],

$$P_{iso}(\nu, \tilde{\nu}, \beta) = \frac{\tilde{\nu}}{(2\beta \gamma)^2 \nu} \left\{ \begin{array}{l}
(1 + \beta) \frac{\tilde{\nu}}{\nu} - 1 + \beta, \frac{1-\beta}{1+\beta} \leq \frac{\tilde{\nu}}{\nu} \leq 1 \\
1 + \beta - (1 - \beta) \frac{\tilde{\nu}}{\nu}, 1 < \frac{\tilde{\nu}}{\nu} \leq \frac{1+\beta}{1-\beta}
\end{array} \right. \hspace{1cm} (4)
$$

where $\gamma = \left(1 - \beta^2\right)^{-1/2}$. The expected value for the photon frequency at fixed $\beta$ is then given by:

$$\langle \nu \rangle = \frac{\int_{\frac{1-\beta}{1+\beta}}^{1-\beta} \nu P_{iso}(\nu, \tilde{\nu}, \beta) d\tilde{\nu}}{\int_{\frac{1-\beta}{1+\beta}}^{1+\beta} P_{iso}(\nu, \tilde{\nu}, \beta) d\tilde{\nu}} = \frac{1 + \beta^2}{1 - \beta^2} \nu \approx (1 + 2\beta^2) \nu$$  \hspace{1cm} (5)

Direct application of the equipartition energy theorem allows us to identify $\langle \beta^2 \rangle = \frac{m\nu^2}{2mc^2} = \frac{kT}{2mc^2} = z$. Physically, this implies that a typical photon scattered off an electron will have a temperature dependent blue shift simply given by $2z\nu^2$. It is interesting to notice that a similar “frequency shift approach” leads to the analytical well known expression of the kinematic SZ effect. This is shown in the appendix.

Eq. (1) can be evaluated analytically leading to an expression which is identical to the one reported in the literature. Setting in Eq. (1)

$$\alpha = \frac{\tilde{\nu} - (1 - 2z)\nu}{\sigma(\nu)} \hspace{1cm} (6)$$

we get that

$$I(\nu) - I_o(\nu) = -\tau I_o(\nu) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} I_o(\nu + \Delta \nu(\alpha)) \exp\left[-\alpha^2\right] d\alpha \hspace{1cm} (7)$$

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where
\[ \Delta \nu (\alpha) = -2 \nu z + 2 z^{1/2} \nu \alpha \] (8)

Since \( z \) is a small number [3] and \( I_o \) is an analytical function of \( \nu \) we may expand the integrand in Eq. (7) in a Taylor series:
\[ I_o (\nu + \Delta \nu (\alpha)) = I_o (\nu) + \Delta \nu (\alpha) \frac{\partial I_o}{\partial \nu} + ... \] (9)

All odd powers of \( \alpha \) yield zero upon integration so, after a straightforward calculation, we get that
\[ I (\nu) - I_o (\nu) = -2 \nu \frac{\partial I_o}{\partial \nu} + y \nu^2 \frac{\partial^2 I_o}{\partial \nu^2} + 2 \tau z^2 \nu^2 \frac{\partial^2 I_o}{\partial \nu^2} - 2 \tau z^2 \nu^3 \frac{\partial^3 I_o}{\partial \nu^3} + \frac{\tau z^2 \nu^4}{2} \frac{\partial^4 I_o}{\partial \nu^4} \] (10)

To first order in \( y = \tau z \) we reproduce exactly the analytic expression for the distortion curve [1-2]. It is also interesting to notice that, taking into account the second order terms in \( z \) included in Eq. (10), the crossover frequency \( \nu_c \) corresponding to the solution of equation
\[ I (\nu) - I_o (\nu) = 0 \] (11)

is now clearly dependent on the value of \( z \) namely, on the electron temperature, a feature not ordinarily recognized for a non-relativistic intra-cluster gas. For typical values of \( \tau = 10^{-2} \), and \( kT_e = 3KeV \) a numerical calculation yields \( \nu_c = 221.34GHz \), while the value obtained by considering only linear terms in \( z \) yields \( \nu_c = 217.16 GHz \), a difference of nearly 1.74%. Fig. 1 shows the plot of the analytical non-relativistic expression up to second order in \( z^2 \), compared with its first order counterpart for a temperature of \( kT = 3KeV \). It is interesting to notice that, although a non-relativistic Maxwellian was used in the convolution integral, the Wien side of the distortion curve is practically identical to its relativistic counterpart, as presented in Ref. [11]. This remark is confirmed in Fig. 2 with \( kT = 15KeV \).

The result expressed in Eq. (11) is of significant importance. Eq. (10) contains the famous Kompaneets equation [5]-[8] describing the thermal SZ effect using a diffusive mechanism to account for the motion of photons in an optically thin electron gas where the electron temperature \( T_e >> T \), the photon temperature (see specifically Ref. [6]). One must notice that, since most photons are not scattered even once, a diffusion approximation would then hardly seem adequate [3]. Nevertheless, the diffusion mechanism is mathematically identical to the absorption-emission process of photons by electrons to first order in \( z \). The problem posed in Ref. [9] requiring an explanation of why two different mechanisms lead to the same result is now mathematically solved. Finally, we want to stress that it is now clear why the curves describing the thermal SZ effect obtained by numerical integration of different mathematical expressions turn out to be identical. We therefore expect that the relativistic SZ effect and
Figure 1: CMBR SZ effect distortion $\frac{\Delta I}{\tau}$ as computed by Eq.(10), with $kT = 3K$eV (solid curve). The intensity change is measured in units of $\frac{(hc)^2}{2k^2T_0^4}$ and the frequency is given in Hz. The short-dashed line next to it shows the intensity change corresponding to the Kompaneets-based expression. The thermal relativistic SZ effect is represented by the long-dashed curve.

Figure 2: The same as in Fig. 1, with $kT = 15K$eV.
other photon scattering problems in hot plasmas can be handled by a similar procedure.

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Appendix

The central idea of the convolution integral approach to CMBR distortions is that, in a dilute gas, a scattering law $G(\tilde{\nu}, \nu)$ is given by what in statistical physics is known as the dynamic structure factor, where $\tilde{\nu}$ and $\nu$ are the corresponding incoming/outcoming photon frequencies. Then, one computes the distorted occupation number through the convolution integral

$$n(\nu) = \int_0^{\infty} \, n_o(\tilde{\nu}) G(\tilde{\nu}, \nu) \, d\tilde{\nu}$$

with

$$n_o(\tilde{\nu}) = \frac{1}{e^{\frac{\tilde{\nu}}{kT}} - 1}$$

In the non-relativistic kinetic SZ effect, one can easily show that a suitable structure factor is given by:

$$G_k(\tilde{\nu}, \nu) = (1 - \tau) \delta(\tilde{\nu} - \nu) + \tau \delta(\tilde{\nu} - (1 - \frac{U}{c})\nu)$$

$U$ being the cluster velocity (approaching the observer). The optical depth $\tau$ measures essentially the proportion of photons being captured (scattered) by electrons. The intergalactic gas cloud in clusters of galaxies has an optical depth $\tau \sim 10^{-2}$.

The formalism basically states that a fraction $(1 - \tau)$ of photons passes through the cluster without being scattered, while a $\tau$ fraction of them shifts its frequency through the ordinary Doppler effect. The integrals can be evaluated leading to an expression which is identical to the one reported in the literature. Indeed, since

$$n(\nu) = (1 - \tau) \int_0^{\infty} \, n_o(\tilde{\nu}) \, \delta(\tilde{\nu} - \nu) \, d\tilde{\nu} + \tau \int_0^{\infty} \, n_o(\tilde{\nu}) \, \delta(\tilde{\nu} - (1 - \frac{U}{c})\nu) \, d\tilde{\nu}$$

then

$$n(\nu) = n_o(\nu) (1 - \tau) + \tau \int_0^{\infty} \, n_o(\tilde{\nu}) \, \delta(\tilde{\nu} - (1 - \frac{U}{c})\nu) \, d\tilde{\nu}$$

If we now perform the substitution

$$\tilde{\alpha} = \tilde{\nu} - (1 - \frac{U}{c})\nu$$

we may expand the integrand in a Taylor series. Straightforward calculation leads to the ordinary expression for the kinetic SZ effect:

$$n(\nu) - n_o(\nu) = -\frac{U}{c} \frac{\partial n_o}{\partial \nu}$$
If we now define $I_k(\nu)$ as the distorted kinematic SZ spectrum and consider the expressions $x = \frac{h\nu}{kT}$ and 

$$\frac{n(\nu) - n_o(\nu)}{n_o(\nu)} = \frac{I_k(\nu) - I_o(\nu)}{I_o(\nu)},$$

we finally get the desired expression:

$$I_k(\nu) - I_o(\nu) = \frac{2(kT)^3}{(hc)^2} \frac{x^4 e^x}{(e^x - 1)^2} \frac{U \tau}{c}$$

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