Phase Transitions for One-Dimensional Lorenz-Like Expanding Maps

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Abstract
Given an one-dimensional Lorenz-like expanding map we describe a class $\mathcal{A}$ of potentials $\phi : [0, 1] \to \mathbb{R}$ admitting at most one equilibrium measure and we construct a family of continuous but not weak-Hölder continuous potentials for which we observe phase transitions. This give a certain generalization of the results proved in Pesin and Zhang (J Stat Phys 122(6):1095–1110, 2006), where the authors have proved this for a smaller class of potentials, that is, for uniformly expanding maps and weak-Hölder continuous potentials. Indeed, the class $\mathcal{A}$ form an open and dense subset of $C([0, 1], \mathbb{R})$, with the usual $C^0$ topology.

Keywords Equilibrium measure · Lorenz maps

Mathematics Subject Classification Primary 37D25; Secondary 37D30 · 37D20

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1 Introduction

An one-dimensional Lorenz-like expanding map is a map $\ell : [0, 1]\setminus \{d\} \to [0, 1]$, where $d \in (0, 1)$ is the discontinuity point, together with a “partition” $\mathcal{P} = \{(0, d), (d, 1)\}$. The boundary of such a system is $\mathcal{P} = \{(0, d), (d, 1)\}$ (see, Sect. 2.1, for more details).

Let $\phi : [0, 1] \to \mathbb{R}$ be a continuous function. As $d$ is a point of discontinuity of $\ell$ we could define alternatively $\phi : [0, 1]\setminus \{d\} \to \mathbb{R}$ so that $\phi(d^+) = \lim_{x \to d^+} \phi(x)$ and $\phi(d^-) = \lim_{x \to d^-} \phi(x)$ but for convenience, we consider $\phi$ defined at $[0, 1]$. A Borel invariant measure $\mu_\phi$ is called an equilibrium state for system $(\ell, \phi)$ if it is solution of the equation

$$P_{\text{top}}(\ell, \phi) = \sup \left\{ h_\mu(\ell) + \int_{[0,1]} \phi \, d\mu \mid \mu \in \mathcal{M}_{\text{inv}}(\ell) \right\},$$

where $P_{\text{top}}(\ell, \phi)$ denotes the topological pressure of $(\ell, \phi)$ (see Sect. 2.3 for more details for pressure) and the supremum is taken over all Borel invariant probability measures. We denote by $H^\gamma([0, 1], \mathbb{R})$, the set of Hölder-continuous potential, where by an element belonging to $H^\gamma([0, 1], \mathbb{R})$ we mean a function $\phi : [0, 1] \to \mathbb{R}$ satisfying $|\phi(x) - \phi(y)| \leq K|x - y|^\gamma$, for all $x, y \in [0, 1]$, and some constants $\gamma > 0$, $0 < K < \infty$.

In Bronzi and Oler (2018) using the results Buzzi and Sarig (see Buzzi and Sarig 2003) the authors proved that if

$$\max \left\{ \limsup_{n \to \infty} \frac{1}{n}(S_n\phi)(0), \limsup_{n \to \infty} \frac{1}{n}(S_n\phi)(1) \right\} < P_{\text{top}}(\phi, \ell)$$

then $\phi$ admits a unique equilibrium measure. The proof of this result is obtained by noting that the system is a particular case of topologically transitive countable Markov shifts. At this point, it is natural for us to ask the following question.

**Question 1** Is there a class of potentials admitting a unique equilibrium measure which is larger than the class of Hölder continuous potentials?

Let $V_n(\phi)$ be given $V_n(\phi) = \sup_{x, y \in C_n} \{|\phi(x) - \phi(y)|\}$ where $C_n$ are cylinders defined by partition $\mathcal{P}$ (see, Sect. 2.1, for more details). We consider $WH^\gamma([0, 1], \mathbb{R})$ the set of functions $\phi : [0, 1] \to \mathbb{R}$ satisfying

$$V_n(\phi) \leq A\gamma^n, \text{ for all } n \geq 1$$

with $A > 0$ and $0 < \gamma < 1$. An element $\phi \in WH^\gamma([0, 1], \mathbb{R})$ we call a weak-Hölder-continuous potential.
In this context, Pesin and Zhang (2006) gave a positive answer to this question. More precisely, authors describe a large class of potentials (denoted by $\mathcal{H}$) admitting unique equilibrium measures. This class $\mathcal{H}$ includes all Hölder continuous potentials but goes far beyond them, i.e., $WH^\gamma([0,1],\mathbb{R}) \subset \mathcal{H}$.

The first goal here is to build a class of potentials, admitting the unique equilibrium measure, substantially larger than the class of potentials $\mathcal{H}$ studies by Pesin and Zhang (2006). Such a class we want to build we will denote by $\mathcal{A}$. More precisely, we say that $\phi \in qWH^\gamma([0,1],\mathbb{R})$ is quasi-weak-Hölder-continuous if there exists $0 < \gamma < 1$ such that for all $n \geq 1$, $V_n(\phi) \leq A(n)\gamma^n$, for all $n \geq 1$, where $A$ is a function depending on $n$ and satisfying $0 < \lim_{n \to \infty} \frac{A(n+1)}{A(n)} < L$, for some positive constant $L$.

Observe that $WH^\gamma([0,1],\mathbb{R}) \subset qWH^\gamma([0,1],\mathbb{R})$. In this context, defining the set $\mathcal{A} = \{ \phi : [0,1] \to \mathbb{R} | \phi \in qWH^\gamma([0,1],\mathbb{R})$ and $P_{top}(\phi, \partial \mathcal{P}, \ell) < P_{top}(\phi, \ell) \}$

we announce our first theorem.

**Theorem A** Let $\ell : [0,1] \setminus \{d\} \to [0,1]$ be an one-dimensional Lorenz-like expanding map and consider $\phi \in \mathcal{A}$. Then $\phi$ admits a unique equilibrium measure.

We say that $\phi \in SV([0,1],\mathbb{R})$ has summable variation if

$$\sum_{n \geq 2} V_n(\phi) < +\infty.$$ We define $\Phi : \Sigma(\ell) \to \mathbb{R}$ by $\Phi(C) = \lim_{n \to \infty} \inf (\phi(C_n))$, where $\phi : [0,1] \to \mathbb{R}$ is a continuous potential. The proof of Theorem A is based on a reduction to the symbolic dynamics $\sigma$ of $\ell$ (see Sect. 2.4). First we show that if $\phi$ is a continuous potential then $\phi$ satisfies condition (1.1). Next, we show that $\Phi$ satisfies $\sup_{\Sigma(\ell)}(\Phi) < \infty$, $P(\Phi, \sigma) < \infty$ and $\sum_{n \geq 0} V_n(\Phi) < \infty$, where $\sup_{\Sigma(\ell)}(\Phi) = \sup \{|\Phi(C)| : C \in \Sigma(\ell)\}$. Thus by Theorem 2.2, $(\sigma, \Phi)$ admits a unique equilibrium measure. To complete the proof of Theorem A, we apply the Proposition 2.2 in order to obtain a map $\pi$ which is bi-measurable, injective, surjective and satisfies $\pi \circ \sigma = \ell \circ \pi$. Next, to finish the proof, it is enough to observe that if $\phi \in \mathcal{A}$, then $\phi \in SV([0,1],\mathbb{R})$.

Recall that in Pesin and Zhang (2006), based on Sarig’s results (see Sarig 2001), constructed a family of continuous (but not Hölder continuous) potentials $\varphi_c$ exhibiting phase transitions, i.e., there exists a critical value $c_0 > 0$ such that for every $0 < c < c_0$ there is a unique equilibrium measure for $\varphi_c$ which is supported on $(0,1]$ and for $c > c_0$ the equilibrium measure is the Dirac measure at 0. Here we also build a family of continuous potentials (but not Hölder continuous and not weak-Hölder continuous) $\phi_t \in \mathcal{A}$ where the phase transition phenomenon occurs. With this in mind we can establish our second result as follow.
Theorem B There exists a critical value $t_1 > 0$ such that the family of continuous potentials $\phi_t \in \mathcal{A}$ such that $\phi_t \notin (H^\alpha([0, 1], \mathbb{R}) \cap WH^\gamma([0, 1], \mathbb{R}))$ presents phase transitions, i.e:

1. if $t \in (-\infty, t_1)$, then there are at least two equilibrium measures for $\phi_t$. The Dirac measure at $p_k^+$ and $p_k^-$ are the equilibrium states;
2. if $t = t_1$, then there is no equilibrium state for $\phi_{t_1}$;
3. if $t \in (t_1, \infty)$, then there is a unique equilibrium measure for $\phi_t$. This measure is supported on $(0, 1)$.

To prove Theorem B, we first define the one-parameter family of functions $\phi_t : [0, 1] \to \mathbb{R}$, by

$$
\phi_t(x) = \begin{cases} 
\frac{1}{[r(1 - \log(x))]^n}, & x \in \left[\bigcup_{n=0}^{\infty} \mathcal{C}_n\right] - \{0\} \\
0, & x = 0,
\end{cases}
$$

for all $t \in (t_0, +\infty)$, and some fixed $t_0 > 0$. Next, using the definition of Hölder-continuous and weak-Hölder-continuous potentials we show that we cannot find a parameter $t$, such that $\phi_t \in (H^\alpha([0, 1], \mathbb{R}) \cap WH^\gamma([0, 1], \mathbb{R}))$. Finally, the phenomenon of phase transitions occurs when we study the properties of $P_{\text{top}}(\ell, \phi)$.

We have already seen that $\mathcal{A}$ contains a large amount of continuous potentials that have a single equilibrium measure, but they are not Hölder neither weak-Hölder continuous. One of the main objectives of the theory of dynamic systems has been to study the typical properties of a system. Such studies can be directed toward understanding and discovering dynamic properties from both topological and ergodic points of view, whose are satisfied for a “large” set of dense or residual systems. In [2] the authors proved that if $\ell$ is an one-dimensional Lorenz-like expanding map and considering $H^\alpha([0, 1], \mathbb{R})$ with the usual $C^0$ topology, than there exists an open and dense subset $\mathcal{H}_0$ of $H^\alpha([0, 1], \mathbb{R})$ such that each $\phi \in \mathcal{H}_0$ admits exactly one equilibrium state. In this context we can establish our third theorem which can be seen as a generalization of Bronzi and Oler’s result from Bronzi and Oler (2018).

Theorem C Let $\ell : [0, 1]\{d\} \to [0, 1]$ be an one-dimensional Lorenz-like expanding map and consider $\mathcal{A} \subset C([0, 1], \mathbb{R})$ as defined at (1.2). Then $\mathcal{A}$ is an open and dense set, in the $C^0$ topology.

The basic idea in order to prove Theorem C is to consider

$$
\mathcal{A} = S(\phi) \cap D(\phi),
$$

where

$$
S(\phi) = \left\{ \phi \in C([0, 1], \mathbb{R}) \mid \sum_{n \geq 2} V_n(\phi) < \infty \right\}.
$$
and

\[ D(\phi) = \{ \phi \in C([0, 1], \mathbb{R}) \mid P_{top}(\phi, \partial P, \ell) < P_{top}(\phi, \ell) \} . \]

In order to prove that \( A \) is open and dense in \( C([0, 1], \mathbb{R}) \) it is enough to show that \( S(\phi) \) and \( D(\phi) \) are open and dense in \( C([0, 1], \mathbb{R}) \). By Theorem A in Bronzi and Oler (2018), \( D(\phi) \) is open and dense in \( C([0, 1], \mathbb{R}) \). Thus we only need to show that \( S(\phi) \) is open and dense in \( C([0, 1], \mathbb{R}) \).

**Organization**

In Sect. 2 we review some standard facts about one-dimensional Lorenz-like expanding map and equilibrium states. Sect. 3 is dedicated to the proof of Theorem A. Theorem B is proved in Sect. 4. Section 5 is devoted to proof Theorem C.

**2 Background and Preliminary Results**

**2.1 One-Dimensional Lorenz-Like Expanding Map**

Lorenz maps originally arise from the study of geometric models for the Lorenz equations (Guckenheimer 1976; Guckenheimer and Williams 1979; Lorenz 1963; Sparrow 1982; Williams 1979). This model induces an one-dimensional Lorenz-like expanding map. Here we are considering the maps studied by Glendinning (1990).

**Definition 2.1** An one-dimensional Lorenz-like expanding map is a function \( \ell : [0, 1] \to [0, 1] \) satisfying the following properties:

1. \( \ell \) has a unique discontinuity at \( x = d \) and \( \ell(d^+) = \lim_{x \to d^+} \ell(x) = 0, \ell(d^-) = \lim_{x \to d^-} \ell(x) = 1; \)
2. For any \( x \in [0, 1]\setminus\{d\} \), \( \ell'(x) > \sqrt{2} \)
3. Each inverse branch of \( \ell \) extends to a \( C^{1+\theta} \) function on \([\ell(0), 1]\) or on\([0, \ell(1)]\), for some \( \theta > 0 \), and if \( g \) denotes any of these inverse branches, then \( g'(x) \leq \lambda < 1 \) (Fig. 1).

We denote by \( \mathcal{P} \) the natural partition of \([0, 1]\setminus\{d\} \), i.e., \( \mathcal{P} = \{(0, d), (d, 1)\} \). The boundary of \( \mathcal{P} \) is \( \partial \mathcal{P} := \{0, d, 1\} \). Also, we define

\[ \mathcal{P}^{(n)} = \{C_n = P_0 \cap L^{-1}(P_1) \cap \cdots \cap L^{-n+1}(P_{n-1}) \neq \emptyset \mid P_i \in \mathcal{P}\}. \]

In order to find a periodic point for an one-dimensional Lorenz-like expanding map we define \( C^{\pm}_{n} \) and \( C^{-}_{n} \) as the cylinders on the right and left hand side of the discontinuity \( d \), respectively, i.e., \( d \in \partial C^{\pm}_{n} \). For this purpose we introduce an auxiliary family \( A_{n} \) by recursively as follows: let \( A_{0} := (d, 1) \),

\[ \text{if } d \notin T^i(A_0) \text{ we define } A_i = T(A_{i-1}) \text{ for each } 0 < i \leq n - 1 \]

or
Fig. 1 One-dimensional Lorenz-like expanding map

\[
\begin{align*}
\text{if } d \in T^i(A_0) \text{ we define } & \quad \begin{cases} 
A_n = T^n(A_0), \\
A_{n+1} = T^n(A_n^*), 
\end{cases} \\
\text{where } A_n^* & \text{ is the connected component of } A_n \setminus \{d\} \text{ which contains } T^n(d^+).
\end{align*}
\]

**Definition 2.2** (see Bronzi and Oler 2018; Graczyk and Swipolhk atek 1998) An integer \( N \) is a cutting time for \( T \) if \( d \in AN \).

We show that there are sequences of integers \( \{N_k^{\pm}\}_{k \in \mathbb{N}} \) and sequences of periodic points \( \{p_k^{\pm}\} \) such that \( p_k^{\pm} \rightarrow d^\pm \), where \( d^\pm \in \partial C_{N_k^{\pm}} \), and \( \ell_{N_k^{\pm}}(p_k^{\pm}) = p_k^{\pm} \) with \( p_k^{\pm} \in C_{N_k^{\pm}}, \forall k \geq 1 \). Moreover, we show that \( P(\phi, \partial \mathcal{P}, \ell) \) can be calculated by the average of \( p_k^{\pm} \).

**Lemma 2.1** (see Bronzi and Oler 2018) For every \( k \in \mathbb{N} \) there exist \( p_k^{\pm} \in C_{N_k^{\pm}} \) such that \( \ell_{N_k^{\pm}}(p_k^{\pm}) = p_k^{\pm} \).

### 2.2 Birkhoff Averages Properties

Given a map \( \phi \in C([0, 1], \mathbb{R}) \), we have a notion of Birkhoff averages defined by

\[
(S_n\phi)(x) = \phi(x) + \phi(\ell(x)) + \cdots + \phi(\ell^{n-1}(x)) = \sum_{j=1}^{n-1} \phi(\ell^j(x)).
\]

**Lemma 2.2** Let \( \ell : [0, 1]\setminus\{d\} \to [0, 1] \) be an one-dimensional Lorenz-like expanding map and \( \phi : [0, 1] \to \mathbb{R} \) a continuous map. Then there exists a constant \( C > 0 \) such that

\[
\left| \frac{1}{n} ((S_n\phi)(x) - (S_n\phi)(y)) \right| \leq C,
\]

for all \( x, y \in [0, 1] \).
Proof Note that, \( \phi \) is bounded. Then \( \frac{1}{n} \left| (S_n \phi)(x) - (S_n \phi)(y) \right| \leq 2|\phi|_\infty \), for all \( x, y \in [0, 1] \). Thus, it is sufficient to consider \( C = 2|\phi|_\infty \) where \( |\phi|_\infty \) is the norm of supremum.

As \( \ell \) is not continuous in \( d \in [0, 1] \) we make the following convention: \( S_n \phi(d^+) \) is the right-hand side limit of the function \( S_n \phi(z) \) at \( d \) and \( S_n \phi(d^-) \) is the left-hand side limit of \( S_n \phi(z) \) at \( d \). More precisely, for any \( n \in \mathbb{N} \) we define

\[
S_n \phi(d^\pm) = \lim_{z \to d^\pm} \sum_{i=0}^{n-1} \phi(\ell^i(z)).
\]

By definition \( \ell(d^+) = 0 \) and \( \ell(d^-) = 1 \), so we conclude that:

\[
\lim_{n \to \infty} \frac{1}{n} S_n \phi(d^+) = \lim_{n \to \infty} \frac{1}{n} S_n \phi(0) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} S_n \phi(d^-) = \lim_{n \to \infty} \frac{1}{n} S_n \phi(1).
\]

The following Lemma 2.3 guaranties that the above relations are well defined.

**Lemma 2.3** (see Bronzi and Oler 2018) Let \( \ell : [0, 1] \setminus \{d\} \to [0, 1] \) be an one-dimensional Lorenz-like expanding map and consider \( \phi \in H^\gamma ([0, 1], \mathbb{R}) \).

(i) If does not exist \( n_0 \in \mathbb{N} \) such that \( \ell^{n_0}(0) = d \), then

\[
\lim_{n \to \infty} \frac{1}{n} S_n \phi(d^+) = \lim_{n \to \infty} \frac{1}{n} S_n \phi(0).
\]

(ii) If there exists \( n_0 \in \mathbb{N} \) such that \( \ell^{n_0}(0) = d \), then

\[
\lim_{n \to \infty} \frac{1}{n} S_n \phi(d^+) = \frac{1}{n_0} S_{n_0} \phi(0).
\]

The same conclusion holds for \( d^- \) replacing 0 for 1.

**Remark 2.1** From now on we use \( \lim_{n \to \infty} \frac{1}{n} S_n \phi(0) \) or \( \lim_{n \to \infty} \frac{1}{n} S_n \phi(1) \) to refer one of the items in the above Lemma 2.3.

**Lemma 2.4** Let \( \ell : [0, 1] \setminus \{d\} \to [0, 1] \) be an one-dimensional Lorenz-like expanding map and a potential \( \phi \in C([0, 1], \mathbb{R}) \). If \( N_k^+ \in \mathbb{N} \) is such that \( d \in \partial C_{N_k^+}, p_k^+ \in C_{N_k^+} \) and \( \ell^{N_k^+}(p_k^+) = p_k^+ \), then

\[
\lim_{k \to \infty} \frac{1}{N_k^+} S_{N_k^+} \phi(p_k^+) = \lim_{n \to \infty} \frac{1}{n} S_n \phi(0).
\]

If \( N_k^- \in \mathbb{N} \) is such that \( d \in \partial C_{N_k^-}, p_k^- \in C_{N_k^-} \) and \( \ell^{N_k^-}(p_k^-) = p_k^- \), then

\[
\lim_{k \to \infty} \frac{1}{N_k^-} S_{N_k^-} \phi(p_k^-) = \lim_{n \to \infty} \frac{1}{n} S_n \phi(1).
\]
2.3 Topological Pressure

According to Buzzi and Sarig (2003), the pressure of a subset \( S \subset [0, 1] \) and a potential \( \phi \in C([0, 1], \mathbb{R}) \) is defined by

\[
P_{\text{top}}(\phi, S, \ell) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{C_n \in \mathcal{P}(n) : S \cap C_n \neq \emptyset} \sup_{x \in C_n} e^{S_n \phi(x)} \right),
\]

where the Birkhoff average is well defined.

The topological pressure of \( \ell \) for \( \phi \in C([0, 1], \mathbb{R}) \) is defined by

\[
P(\phi, \ell) = P_{\text{top}}(\phi, [0, 1], \ell).
\]

**Corollary 2.1** Let \( \ell : [0, 1] \setminus \{d\} \to [0, 1] \) be an one-dimensional Lorenz-like expanding map, \( \phi \in C([0, 1], \mathbb{R}) \), and

\[
M(\phi, \ell) = \max \left\{ \limsup_{n \to \infty} \frac{1}{n} (S_n \phi)(0), \limsup_{n \to \infty} \frac{1}{n} (S_n \phi)(1) \right\}.
\]

Then there exists a constant \( C > 0 \) such that

\[
M(\phi, \ell) - C \leq P_{\text{top}}(\phi, \partial \mathcal{P}, \ell) \leq M(\phi, \ell) + C.
\]

**Proof** We just use Lemma 2.2 and make the superficial modifications to the proof of Proposition 3.1 of Bronzi and Oler (2018).

**Proposition 2.1** (see Buzzi and Sarig 2003) Consider \( \ell \) a piecewise expanding Lorenz-like map and a piecewise uniformly continuous potential \( \phi \). Let \( \nu \) be an ergodic probability measure. If \( \nu(S) > 0 \), then

\[
P_{\text{top}}(\phi, S, \ell) \geq h_{\nu}(\ell) + \int \phi \, d\nu,
\]

where \( h_{\nu}(\ell) \) is the metric entropy of \( \nu \).

Furthermore, let \( \mathcal{M}_{\ell}([0, 1]) \) denote the collection of \( \ell \)-invariant Borel probability measures on \([0, 1]\). A measure \( \mu_{\phi} \in \mathcal{M}_{\ell}([0, 1]) \) is called an equilibrium state for \( \phi \) if

\[
\sup_{\mu \in \mathcal{M}_\ell([0,1])} \left\{ h_{\mu}(\ell) + \int \phi \, d\mu \right\} = h_{\mu_{\phi}}(\ell) + \int \phi \, d\mu_{\phi}.
\]

**Theorem 2.1** (Buzzi and Sarig 2003) Let \( \ell : [0, 1] \setminus \{d\} \to [0, 1] \) be an one-dimensional Lorenz-like expanding map be a piecewise expanding map such that for all non-empty open sets \( U \) we have \( \ell([0, 1]) \subset \bigcup_{k \geq 0} \ell^k(U) \), and consider a potential \( \phi \in C([0, 1], \mathbb{R}) \) satisfying \( P_{\text{top}}(\phi, \partial \mathcal{P}, \ell) < P_{\text{top}}(\phi, \ell) \). Then \( P_{\text{top}}(\phi, \ell) = \sup_{\mu \in \mathcal{M}_\ell([0,1])} \left\{ h_{\mu}(\ell) + \int \phi \, d\mu \right\} \) and there is a unique measure equilibrium state for potential \( \phi \).
2.4 Countable Markov Subshifts

Here we follow the notations, definitions and results of Buzzi and Sarig (2003). Let \( \text{dom}(\ell^n) \subset [0, 1] \) denote the domain of definition of \( \ell^n \). The symbolic dynamics of \(((0, 1), P, \ell)\) is the left-shift \( \sigma \) defined on the set:

\[
\Sigma(\ell) = \left\{ P = (P_0, P_1, \ldots) \in P^{\mathbb{N} \cup \{0\}} \mid \exists n \geq 0, x \in \text{dom}(\ell^n) \text{ and } \ell^n(x) \in P_n \right\},
\]

where \( \left\{ \cdot \right\} \) denotes the closure in the compact space \( P^{\mathbb{N} \cup \{0\}} \). We using the cylinder notation \( C_n = \bigcap_{i=0}^{n} \ell^{-i}(P_i) \). For all \( C \in \Sigma(\ell) \), the map \( \pi : \Sigma(\ell) \rightarrow [0, 1] \) is defined by \( \pi(C) = \bigcap_{n \geq 0} C_n \). Indeed, we define \( \Phi : \Sigma(\ell) \rightarrow \mathbb{R} \) to be \( \Phi(C) = \lim_{n \rightarrow \infty} \inf (\phi(C_n)) \), where \( \phi : [0, 1] \rightarrow \mathbb{R} \) is continuous potential. As \( \ell \) is piecewise expanding, \( \pi \) and \( \Phi \) are well defined.

**Proposition 2.2** (see Buzzi and Sarig 2003) Define \( \Delta = (\pi^{-1}(\partial P)) \). If \( P_{top}(\phi, \partial P, \ell) < P_{top}(\phi, \ell) \) then

\[
\pi : \Sigma(\ell) \setminus \left\{ \bigcup_{k \geq 0} \sigma^k(\Delta) \right\} \rightarrow [0, 1] \setminus \left\{ \bigcap_{k \geq 0} \ell^{-k}(\partial P) \right\}
\]

is a measure-theoretic isomorphism and satisfies \( \pi \circ \sigma = \ell \circ \pi \).

A shift invariant probability measure \( m \) is called an equilibrium measure for \( \Phi : \Sigma(\ell) \rightarrow \mathbb{R} \) if \( h_m(\sigma) + \int \Phi \, dm \) is well defined and maximal. The Gurevich pressure of \( \Phi \) is given by

\[
P_{G}(\Phi, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{\sigma^n(x) = x} e^{(S_n \Phi_n)(x)} [a](x) \right),
\]

where \( a \in S \) is fixed, \( [a] = \{ x \in \Sigma : x_0 = a \} \) and \( (S_n \Phi) = \sum_{i=0}^{n} \Phi \circ \sigma^i \).

Let \( P_\sigma(\Sigma) \) denote the collection of \( \sigma \)-invariant Borel probability measure on \( \Sigma \). The pressure of \( m \in P_\sigma(\Sigma) \) is given by \( P_m(\Phi, \sigma) = h_m(\sigma) + \int \Phi \, dm \). Note that this is not always well-defined, \( \Phi \) might not be integrable, or it might happen that \( h_m(\sigma) = +\infty \) and \( \int \Phi \, dm = -\infty \). If \( \sigma \) is topologically mixing and \( \sup(\Phi) < \infty \), then

\[
P(\Phi) = \sup \{ P_m(\Phi) : m \in P_\sigma(\Sigma), \ P_m(\Phi) \text{ is well defined} \}.
\]

The condition \( \sup_{\Sigma(\ell)}(\Phi) < \infty \) guarantees that \( \int \Phi \, dm \) is well defined (possibly infinite), so the well defined condition reduces to a preclusion of the \( m \) for which \( h_m(\sigma) = \infty \) and \( \int \Phi \, dm = -\infty \). In this the authors prove the following:
Theorem 2.2 (see Buzzi and Sarig 2003) Let \((\Sigma, \sigma)\) be a topological transitive countable Markov shift and suppose \(\Phi: \Sigma \to \mathbb{R}\) satisfies \(\sup_{\omega \in \Sigma(\ell)} \Phi < \infty\), \(P(\Phi, \sigma) < \infty\) and \(\sum_{n \geq 0} V_n(\Phi) < \infty\). Then there exists at most one invariant probability measure \(m\) such that \(\int \Phi \, dm\) is well defined and maximal.

## 3 Proof of Theorem A

We have divided the proof into a sequence of propositions, lemmas and corollaries.

**Proposition 3.1** Let \(\ell : [0, 1] \setminus \{d\} \to [0, 1]\) be an one-dimensional Lorenz-like expanding map and consider \(\phi : [0, 1] \to \mathbb{R}\) a continuous potential. Then \(\phi\) satisfies \(P_{\text{top}}(\phi, \partial P, \ell) < P_{\text{top}}(\phi, \ell)\).

**Proof** Let \(\phi \in C([0, 1], \mathbb{R})\), be a continuous potential. The function \(\phi_k^\pm\) is defined by \(\phi_k^\pm(x) = \phi(x) + \phi_0^\pm(x)\), for all \(x \in \mathcal{P}\), where the behavior of the family \(\phi_0^\pm\) is defined as follows. Recall that by Corollary 2.1, there exists a subsequence \(N_k^\pm \to \infty\) such that \(d \in \partial C_{N_k^\pm}, p_k^\pm \in C_{N_k^\pm}\) and \(\ell^N(p_k) = p_k^\pm\). Let \(I_j^\pm = (\ell^j(p_k^\pm) - \delta_k^\pm, \ell^j(p_k^\pm) + \delta_k^\pm)\) be intervals, where \(0 \leq j \leq N_k^\pm - 1\). Since the orbit of \(p_k^\pm\) is a finite set, there exists \(\delta_k^\pm > 0\) such that \(I_i^\pm \cap I_j^\pm = \emptyset\), for all \(0 \leq j < i \leq N_k^\pm - 1\) with \(i \neq j\). Fix \(\delta_k^\pm > 0\) and consider a periodic point \(p_k^\pm \in [0, 1]\) with period \(N_k^\pm\), of the one-dimensional Lorenz-like expanding map \(\ell\).

The function \(\phi_{j,k}^\pm\) is defined by

\[
\phi_{j,k}^\pm(x) = \begin{cases} 
-1 \left(\frac{x - \ell^j(p_k^\pm) + \delta_k^\pm)(x - \ell^j(p_k^\pm) - \delta_k^\pm)}{(x - \ell^j(p_k^\pm) + \delta_k^\pm)(x - \ell^j(p_k^\pm) - \delta_k^\pm)}\right), & x \in I_j^\pm \\
0, & x \in (I_j^\pm)^c,
\end{cases}
\]

for all \(j \in \{0, 1, \ldots, N_k^\pm - 1\}\) and \(x \in [0, 1]\). Here for simplicity of notation, we write \(M = \exp\left(\frac{1}{\delta_k^\pm}\right)\). Note that for each \(j \in \{0, 1, \ldots, N_k^\pm - 1\}\), \(\ell^j(p_k^\pm)\) is a maximum local point of \(\phi_{j,k}\) in \(I_{j,k}\) and \(\phi_{j,k}(\ell^j(p_k^\pm)) = M\).

Let \((a_n)\) be a sequence of real numbers such that \(a_n\) tends monotonically to zero and \(\frac{1}{n} \sum_{j=0}^{\infty} a_j < \infty\) (e. g \(a_n = \frac{1}{n}\)). In this way we construct \(\phi_{j,k}^\pm\) by

\[
\phi_{j,k}^\pm(x) = \begin{cases} 
a_j M \cdot \phi_j(x), & x \in I_j^\pm \\
0, & x \in (I_j^\pm)^c.
\end{cases}
\]

Observe that the largest value assumed by \(\phi_{j,k}^\pm\) on \(I_j\) is \(a_j\). Finally we define
\[
\phi_0^\pm(x) = \begin{cases} 
\sum_{j=1}^{N_k^\pm-1} \phi_{j,k}^\pm(x), & x \in \bigcup_j I_j^\pm \\
0, & x \in \left( \bigcup_j I_j^\pm \right)^c 
\end{cases}
\]

**Remark 3.1** As the map \(\phi_0^\pm\) is \(C^\infty\) we have that \(\phi_k^\pm\) is uniformly continuous. Indeed, as each \(\phi_{j,k}\) is built on the orbit of the periodic point \(p_k^\pm\), we have \(\phi_k^\pm(\ell (p_k^\pm)) = \sum_{j=0}^{N_k^\pm-1} a_j\). Now by definition \(\phi_k^\pm(x) = \phi(x) + \phi_0^\pm(x)\), have that \(S_n \phi_k^\pm(x) = S_n \phi(x) + S_n \phi_0^\pm(x)\). Thus,

\[
S_n \phi_k^\pm(p_k^\pm) = S_n \phi(p_k^\pm) + \sum_{j=0}^{N_k^\pm-1} a_j.
\]

By Proposition 2.1 and Lemma 2.4, we have

\[
P_{top}(\phi, S, \ell) = \max \left\{ \limsup_{k \to \infty} \frac{1}{N_k^+} S_{N_k^+} \phi(p_k^+) + \limsup_{k \to \infty} \frac{1}{N_k^-} S_{N_k^-} \phi(p_k^-) \right\}.
\] (3.1)

where \(p_k^+ \in C_{N_k^+}, p_k^- \in C_{N_k^-}\) are such that \(\ell^N_k(p_k^+) = p_k^+, \ell^N_k(p_k^-) = p_k^-\) and \(d \in \partial C_{N_k^+} \cap \partial C_{N_k^-}\).

First we consider \(k\) fixed. Considering \(C_{N_k^+} \in \mathcal{P}(N_k^+ - 1)\) as in Corollary 2.1, then there exists \(p_k^+ \in C_{N_k^+}\) such that \(\ell^N_k(p_k^+) = p_k^+.\) Furthermore one can construct a measure \(\mu_k^+(\cdot) = \left( \frac{1}{N_k^+} \sum_{j=0}^{N_k^+ - 1} \delta_{\ell^j(p_k^+)}(\cdot) \right)\), where \(\delta_{\ell^j(p_k^+)}\) is the Dirac measure with \(\delta_{\ell^j(p_k^+)}(\ell^j(p_k^+)) = 1, j \in \{0, 1, \ldots, N_k^+ - 1\}\). As \(\mu_k^+(C_{N_k}) > 0\) by Proposition 2.1 we have

\[
P_{top}(\phi_k^+, \ell) \geq P_{top}(\phi_k^+, C_{N_k^+}, \ell) \geq h_{\mu_k^+}(\ell) + \int \phi_k^+ d\mu_k^+
\]

\[
= \frac{1}{N_k^+} \left( \sum_{j=0}^{N_k^+ - 1} \phi_k^+(\ell^j(p_k^+)) \right) + \frac{1}{N_k^-} \left( \sum_{j=0}^{N_k^- - 1} \phi_0^+(\ell^j(p_k^+)) \right)
\]

\[
= \frac{1}{N_k^+} \left( \sum_{j=0}^{N_k^+ - 1} (\phi(\ell^j(p_k^+)) + \phi_0^+(\ell^j(p_k^+))) \right) + \frac{N_k^- - 1}{N_k^-} \left( \sum_{j=0}^{N_k^- - 1} (\phi_0^+(\ell^j(p_k^+))) \right)
\]

\[\square\ Springer\]
\[
= \frac{1}{N_k^+} S_N \phi(p_k^+) + \frac{1}{N_k^+} \left( \sum_{j=0}^{N_k^+ - 1} \phi_0^+ (\ell^j (p_k^+)) \right)
= \frac{1}{N_k^+} S_N \phi(p_k^+) + \frac{1}{N_k^+} \sum_{j=0}^{N_k^+ - 1} a_j.
\] (3.2)

Using the same arguments for \( \mu_k^-(\cdot) = \left( \frac{1}{N_k^-} \sum_{j=0}^{N_k^- - 1} \delta_{\ell^j (p_k^-)} \right)(\cdot) \), where \( \delta_{\ell^j (p_k^-)} \) is the Dirac measure with \( \delta_{\ell^j (p_k^-)}(\ell^j (p_k^-)) = 1 \), \( j \in \{0, 1, \ldots, N_k^- - 1\} \), we obtain

\[
P_{top}(\phi_k^-, \ell) \geq \frac{1}{N_k^-} S_N \phi(p_k^-) + \frac{1}{N_k^-} \sum_{j=0}^{N_k^- - 1} a_j.
\] (3.3)

By using (3.2) and (3.3), we get

\[
P_{top}(\phi_k^\pm, \ell) \geq \max \left\{ \frac{1}{N_k^+} (S_N^+ \phi)(p_k^+) + \frac{1}{N_k^+} \sum_{j=0}^{N_k^+ - 1} a_j, \right.
\[
\left. \frac{1}{N_k^-} (S_N^- \phi)(p_k^-) + \frac{1}{N_k^-} \sum_{j=0}^{N_k^- - 1} a_j \right\}.
\] (3.4)

As

\[
P_{top}(\phi, \ell) + P_{top}(\phi_0^\pm, \ell) \geq P_{top}(\phi + \phi_0^\pm, \ell) = P_{top}(\phi_k^\pm, \ell).
\]

we have that

\[
P_{top}(\phi, \ell) + P_{top}(\phi_0^\pm, \ell) \geq \max \left\{ \frac{1}{N_k^+} (S_N^+ \phi)(p_k^+) + \frac{1}{N_k^+} \sum_{j=0}^{N_k^+ - 1} a_j, \right.
\[
\left. \frac{1}{N_k^-} (S_N^- \phi)(p_k^-) + \frac{1}{N_k^-} \sum_{j=0}^{N_k^- - 1} a_j \right\}.
\]

Now we need to compute \( P_{top}(\phi_0^\pm, \ell) \). To do this, observe that

\[
Z_n(\phi_0^\pm, S) = \sum_{C_n \in \mathcal{P}(n-1)} \sup_{x \in C_n} e^{S_n(\phi_0^\pm)(x)} = \sum_{C_n \in \mathcal{P}(n-1)} \sup_{x \in C_n} e^{S_n(\phi_0^\pm)(x)}
+ \sum_{C_n \in \mathcal{P}(n-1)} \sup_{x \in C_n} e^{S_n(\phi_0^\pm)(x)}.
\]
By definition of $\phi_0^\pm$, we have $\phi_0^\pm(x) = 0$, for all $x \notin O(p^\pm)$. Thus

$$ \sum_{C_n \in \mathcal{P}(n-1) : C_n \cap O(p^\pm) = \emptyset} \sup_{x \in C_n} e^{S_n(\phi_0^\pm)(x)} = 0. $$

So

$$ Z_n(\phi_0^\pm, S) = \sum_{C_n \in \mathcal{P}(n-1) : C_n \cap O(p^\pm) \neq \emptyset} \sup_{x \in C_n} e^{S_n(\phi_0^\pm)(x)}. $$

By Remark 3.1, as $\phi_0^\pm(\ell^j(p_k^\pm)) = \sum_{j=0}^{N_k^\pm-1} a_j$, we obtain that

$$ \frac{1}{n} \log(Z_n(\phi_0^\pm, S)) = \frac{1}{n} \log \left( \sum_{C_n \in \mathcal{P}(n-1) : C_n \cap O(p^\pm) \neq \emptyset} \sup_{x \in C_n} e^{S_n(\phi_0^\pm)(x)} \right) \leq \frac{1}{n} \log \left( N_k^\pm e^n \sup_{k \in \mathbb{N}} \{a_k\} \right) = \sup_{k \in \mathbb{N}} \{a_k\}. $$

(3.5)

Letting $n \to \infty$ in the inequation (3.5), for all $k \geq 1$ we get

$$ P_{top}(\phi_0^\pm, \ell) \leq \sup_{k \in \mathbb{N}} \{a_k\}. $$

(3.6)

Combining inequalities (3.4) and (3.6), we obtain

$$ P_{top}(\phi, \ell) + \sup_{k \in \mathbb{N}} \{a_k\} \geq \max \left\{ \frac{1}{N_k^+} (S_{N_k^+} \phi)(p_k^+) + \frac{1}{N_k^+} \sum_{j=0}^{N_k^+-1} a_j, \frac{1}{N_k^-} (S_{N_k^-} \phi)(p_k^-) + \frac{1}{N_k^-} \sum_{j=0}^{N_k^--1} a_j \right\}. $$

(3.7)

Letting $k \to \infty$ at inequality (3.7), we have that

$$ P_{top}(\phi, \ell) \geq \max \left\{ \limsup_{k \to \infty} \frac{1}{N_k^+} S_{N_k^+} \phi(p_k^+), \limsup_{k \to \infty} \frac{1}{N_k^-} S_{N_k^-} \phi(p_k^-) \right\} + \frac{3c}{2}. $$

(3.8)

Combining (3.1) and (3.8) we obtain $P_{top}(\phi, S, \ell) < P_{top}(\phi, \ell)$. \qed
Lemma 3.1 Let $\ell : [0, 1] \setminus \{d\} \rightarrow [0, 1]$ be an one-dimensional Lorenz-like expanding map. Then $\ell$ is topologically transitive.

**Proof** Consider $U, V \subset [0, 1]$ non-empty open sets. Being $\ell$ uniformly piecewise expanding map we could find $k_0 \in \mathbb{N}$, such that $[0, 1] \subset \ell^{k_0}(U)$. As $V \subset [0, 1] \subset \ell^{k_0}(U)$ then there exists $k_0 \in \mathbb{N}$, such that $V \cap \ell^{k_0}(U) \neq \emptyset$. Thus we have proved that $\ell$ is topologically transitive. \qed

Lemma 3.2 Let $\Phi_1 : \Sigma(\ell) \rightarrow \mathbb{R}$ defined as in Sect. 2.4. Then $\sup_{\Sigma(\ell)}(\Phi_1) < \infty$.

**Proof** Remember that $\Phi_1 : \Sigma(\ell) \rightarrow \mathbb{R}$ to be $\Phi_1(C) = \lim_{n \rightarrow \infty} \inf (\phi(C_n))$, where $\phi : [0, 1] \rightarrow \mathbb{R}$ is continuous potential. Thus

$$|\Phi(C)| = \left| \lim_{n \rightarrow \infty} \inf (\phi(C_n)) \right| \leq \lim_{n \rightarrow \infty} |\inf (\phi(C_n))| \leq \lim_{n \rightarrow \infty} |\sup (\phi(C_n))|.$$

As $\sup_{[0,1]}(\phi) = \sup \{|\phi(x)| : x \in [0, 1]\} < \infty$, we have that $|\Phi(C)| < \infty$ and obtain $\sup_{\Sigma(\ell)}(\Phi_1) < \infty$. \qed

Now, we will prove that $P_G(\Phi, \sigma) < \infty$.

Lemma 3.3 Let $\Phi : \Sigma(\ell) \rightarrow \mathbb{R}$ defined as in Sect. 2.4. Then $P_G(\Phi, \sigma) < \infty$.

**Proof** To do this, observe that by Proposition 2.2 we have that $P_G(\Phi, \sigma) = P_{top}(\phi, \ell)$. Hence, we have

$$Z_n(\phi, [0, 1]) = \sum_{C_n \in P^{(n-1)} : S \cap \overline{C_n} \neq \emptyset} \sup_{x \in C_n} e^{S_n \phi(x)} \leq \# \left\{ C_n \in P^{(n-1)} : S \cap \overline{C_n} \neq \emptyset \right\} e^{n \sup_{[0,1]}(\phi)} \leq 2^n e^{n \sup_{[0,1]}(\phi)}.$$

Thus,

$$P_G(\Phi, \sigma) = P_{top}(\phi, \ell) = \lim_{n \rightarrow \infty} \sup \frac{1}{n} \log Z_n(\phi, [0, 1]) \leq \lim_{n \rightarrow \infty} \sup \frac{1}{n} \log \left(2^n e^{n \sup_{[0,1]}(\phi)}\right) = \log(2) + \sup_{[0,1]}(\phi).$$

As $\sup_{[0,1]}(\phi) < \infty$ we obtain $P_G(\Phi, \sigma) < \log(2) + \sup_{[0,1]}(\phi) < \infty$. \qed

The next step, is to prove that $\sum_{n \geq 0} V_n(\Phi) \leq \infty$. \hfill \&
Lemma 3.4 Let \( \Phi : \Sigma(\ell) \to \mathbb{R} \) defined as in Sect. 2.4. Then \( \sum_{n \geq 0} V_n(\Phi) \leq \infty \).

**Proof** Our proof starts with the observation that if \( \phi \in q W^r([0, 1], \mathbb{R}) \) then

\[
\sum_{n \geq 2} V_n(\phi) \leq \sum_{n \geq 2} A(n) \gamma^n < \infty
\]

since the series \( \sum_{n \geq 2} A(n) \gamma^n \) is convergent.

Now if \( C_1, C_2 \in \Sigma(\ell) \), then:

\[
|\Phi(C_1) - \Phi(C_2)| = \left| \lim_{n \to \infty} \inf \left( \phi\left(C_1^n\right) \right) - \lim_{n \to \infty} \inf \left( \phi\left(C_2^n\right) \right) \right|
\]

\[
\leq \lim_{n \to \infty} \left| \inf \left( \phi\left(C_1^n\right) \right) - \inf \left( \phi\left(C_2^n\right) \right) \right|
\]

On the other hand,

\[
\left| \inf \left( \phi\left(C_1^n\right) \right) - \inf \left( \phi\left(C_2^n\right) \right) \right| \leq |\phi(x_1) - \phi(x_2)|, \ x_1 \in C_1^n, x_2 \in C_2^n.
\]

Thus,

\[
|\Phi(C_1) - \Phi(C_2)| \leq |\phi(x_1) - \phi(x_2)|, \ x_1 \in C_1^n, x_2 \in C_2^n.
\]

Therefore,

\[
\sum_{n \geq 2} V_n(\phi) = \sum_{n \geq 2} \sup_{C_1, C_2 \in \Sigma(\ell)} \{|\Phi(C_1) - \Phi(C_2)|\}
\]

\[
\leq \sum_{n \geq 2} \sup_{x_1 \in C_1, x_2 \in C_2} \{|\phi(x_1) - \phi(x_2)|\}
\]

\[
\leq \sum_{n \geq 2} \sup_{x_1, x_2 \in C_n} \{|\phi(x_1) - \phi(x_2)|\}
\]

\[
= \sum_{n \geq 2} V_n(\phi).
\]

Now since, \( \sum_{n \geq 2} V_n(\phi) \leq \infty \), we get \( \sum_{n \geq 2} V_n(\Phi) \leq \infty \).

Now to finish the proof, it is enough to observe that if \( \phi \in \mathcal{A} \), then \( \phi \in SV([0, 1], \mathbb{R}) \).

If \( \phi \in \mathcal{A} \), then \( \phi \in q WH([0, 1], \mathbb{R}) \). \( \square \)

### 3.1 Proof of Theorem A

As \( \phi : [0, 1] \to \mathbb{R} \) is a continuous potential, then by Lemma 3.1 and Proposition 3.1, we have that \( \ell \) is topologically transitive and \( \phi \) satisfies \( P_{\text{top}}(\phi, \partial P, \ell) < P_{\text{top}}(\phi, \ell) \).
Thus by Proposition 2.2 there exists a measure-theoric isomorphism \( \pi \) such that
\[
\pi \circ \sigma = \ell \circ \pi \quad \text{and} \quad P_{top}(\Phi, \ell) = P_{top}(\phi, \ell). \tag{3.9}
\]

On the other hand, it follows from the Lemmas 3.1, 3.2, 3.3 and 3.4 that \( \Phi \) satisfies the hypotheses of the Theorem 2.2, i.e, \( \Phi \) admits a unique equilibrium states. Therefore, by (3.9) we conclude that \( \phi \) admits a unique equilibrium states, which proves the theorem.

4 Proof of Theorem B

Assume that \( C_n \) are the cylinders defined in Sect. 2.1. For all \( t \in (t_0, +\infty) \), for some fixed \( t_0 > 0 \), define the one-parameter family of functions \( \phi_t : [0, 1] \to \mathbb{R} \), by
\[
\phi_t(x) = \begin{cases} 
\frac{1}{[t(1 - \log(x))]^n}, & x \in \left( \bigcup_{n=0}^{\infty} C_n \right) \setminus \{0\} \\
0, & x = 0.
\end{cases}
\]

We have divided the proof of the Theorem B into a sequence of Lemmas.

**Lemma 4.1** \( \phi_t \notin H^\alpha([0, 1], \mathbb{R}) \).

**Proof** Suppose that there exists \( s \in \mathbb{R} \) such that \( \phi_s \) is a Hölder continuous map. Thus
\[
|\phi_s(x) - \phi_s(y)| \leq [\phi_s]_\alpha |x - y|^\alpha,
\]
where \([\phi_s]_\alpha\) is a positive constant. In particular, since \( \phi_s(0) = 0 \), we have that \([\phi_s]_\alpha \geq |\phi_s(x)|\). Therefore,
\[
[\phi_s]_\alpha \geq \frac{|\phi_s(x)|}{x^\alpha} = \frac{1}{x^\alpha[t(1 - ln(x))]^n}
\]

This is absurd, since the right-hand side diverges as \( x \to 0 \). \( \square \)

**Lemma 4.2** \( \phi_t \notin WH^\gamma([0, 1], \mathbb{R}) \).

**Proof** Suppose that there exists \( r \in \mathbb{R} \) such that \( \phi_r \) is a weak Hölder continuous map. Thus \( V_n(\phi_r) \leq A\gamma^n \), where \( A \) is a positive constant. Therefore,
\[
A \geq V_n(\phi_r) \gamma^n = \sup_{x, y \in C_n} |\phi_r(x) - \phi_r(y)|
\]

In particular, since \( \phi_r(0) = 0 \), we have that
\[
A \geq \sup_{x \in C_n} |\phi_r(x)| \gamma^n \geq \frac{1}{\gamma^n[t(1 - ln(x))]^n}
\]

This is absurd, since the right-hand side diverges as \( x \to 0 \). \( \square \)
**Lemma 4.3** \( \phi_t \in qWH^\gamma ([0, 1], \mathbb{R}) \).

**Proof** The derivative of \( \phi_t \) is given by

\[
\left( \frac{d\phi_t}{dx} \right) (x) = \frac{d}{dx} \left( \frac{1}{t(1 - \log(x))^n} \right) = \frac{n}{xt^n(1 - \ln(x))^{n+1}}.
\]

Thus,

\[
V_n(\phi_t) = \sup_{x,y \in C_n} \{|\phi_t(x) - \phi_t(y)|\}
\]

\[
= \sup_{x,y \in C_n} \left\{ \left\| \left( \frac{d\phi_t}{dx} \right)(\bar{x})(x - y) \right\| \right\}
\]

\[
= \sup_{x,y \in C_n} \left\{ \left\| \frac{n}{\bar{x}t^n(1 - \ln(\bar{x}))^{n+1}}(x - y) \right\| \right\}
\]

\[
\leq \sup_{\bar{x} \in C_n} \left\{ \left\| \frac{n}{\bar{x}t^n(1 - \ln(\bar{x}))^{n+1}} \right\| \right\} \sup_{x,y \in C_n} \{ |(x - y)| \}
\]

\[
= \sup_{\bar{x} \in C_n} \left\{ \left\| \frac{n}{\bar{x}t^n(1 - \ln(\bar{x}))^{n+1}} \right\| \right\} \text{diam}(C_n)
\]

where \( \bar{x} \) is a point between \( x \) and \( y \) given by the Mean Value Theorem. By definition 2.1, item 3 we get \( \text{diam}(C_n) \leq \lambda^n \), where \( \lambda \in (0, 1) \). Thus,

\[
V_n(\phi_t) \leq \sup_{\bar{x} \in C_n} \left\{ \left\| \frac{n}{\bar{x}t^n(1 - \ln(\bar{x}))^{n+1}} \right\| \right\} \lambda^n
\]

\[
= A(n)\lambda^n,
\]

where \( A(n) = \sup_{x \in C_n} \left\{ \left\| \frac{n}{x^n(1 - \ln(x))^{n+1}} \right\| \right\} \).

As

\[
\frac{A(n+1)}{A(n)} = \sup_{\bar{x} \in C_{n+1}} \left\{ \left| \frac{n+1}{\bar{x}t^{n+1}(1 - \ln(\bar{x}))^{n+2}} \right| \right\} \leq \sup_{\bar{x} \in C_n} \left\{ \left| \frac{n+1}{\bar{x}t^{n+1}(1 - \ln(\bar{x}))^{n+2}} \right| \right\}
\]

\[
\leq \sup_{\bar{x} \in C_n} \left\{ \left| \frac{n+1}{\bar{y}t^{n+1}(1 - \ln(\bar{y}))^{n+2}} \right| \right\} = \sup_{\bar{x} \in C_n} \left\{ \left| \frac{(n + 1)\bar{y}t^{n+1}(1 - \ln(\bar{y}))^{n+1}}{n\bar{y}t^{n+1}(1 - \ln(\bar{y}))^{n+2}} \right| \right\}
\]
\[
\sup_{y \in C_n} \left\{ \frac{n + 1}{n} \left( 1 - \ln(y) \right) \right\} = \frac{n + 1}{n} \left( 1 - \ln(\bar{y}) \right) \\
\leq \frac{n + 1}{n} \frac{1}{|t|},
\]

we have \( 0 < \lim_{n \to \infty} \frac{A(n+1)}{A(n)} \leq \frac{1}{|t|} \) and, therefore, \( \phi_t \in qWH([0, 1], \mathbb{R}) \), for all \( t > t_0 \). \qed

**Lemma 4.4** If \( t \in (t_1, +\infty) \), where \( t_1 = \frac{\lambda}{1 - \ln(\bar{x})} \) and \( \bar{x} \in (0, 1) \), then \( \phi_t \in SV(P, \mathbb{R}) \).

So, there is a unique equilibrium measure for \( \phi_t \). This measure is supported on \((0, 1)\).

**Proof** Note that

\[
\sum_{n=2}^{\infty} |V_n(\phi_t)| \leq \sum_{n=2}^{\infty} \sup_{y \in C_n} \left\{ \frac{n}{\bar{x} t^n (1 - \ln(\bar{x}))^{n+1}} \right\} \lambda^n,
\]

On the other hand

\[
\lim_{n \to \infty} \left| \sup_{y \in C_{n+1}} \left\{ \frac{n + 1}{\bar{x} t^{n+1} (1 - \ln(\bar{x}))^{n+2}} \right\} \lambda^{n+1} \right| \\
\leq \lim_{n \to \infty} \left| \sup_{y \in C_n} \left\{ \frac{n}{\bar{y} t^n (1 - \ln(\bar{y}))^{n+1}} \right\} \lambda^n \right| \lambda,
\]

\[
= \lim_{n \to \infty} \left| \sup_{y \in C_n} \left\{ \frac{(n+1)}{n} \frac{\bar{x} t^n (1 - \ln(\bar{x}))^{n+1}}{\bar{y} t^{n+1} (1 - \ln(\bar{y}))^{n+2}} \right\} \lambda \right|
\]

\[
= \lim_{n \to \infty} \left( \sup_{y \in C_n} \left\{ \frac{n + 1}{n} \frac{\lambda}{t \left( 1 - \ln(\bar{y}) \right)} \right\} \right).
\]
As \( t > \frac{\lambda}{1-\ln(\lambda)} \) we have that
\[
\lim_{n \to \infty} \sup_{y \in C_n} \left| \frac{n}{N^+ + 1} \left( n + 1 \right) \right| < 1
\]
and, therefore, \( \phi_t \in SV(P, \mathbb{R}) \). Thus \( \phi_t \in \mathcal{A} \) and by Theorem A, we conclude that \( \phi_t \) admits a unique equilibrium measure. \( \square \)

**Lemma 4.5** If \( t \in (-\infty, t_1) \), where \( t_1 = \frac{\lambda}{1-\ln(\lambda)} \) and \( \lambda \in (0, 1) \), then there are at least two equilibrium measures for \( \phi_t \). The Dirac measure at \( p_k^+ \) and \( p_k^- \) are the equilibrium states.

**Proof** Note that if \( t \in (-\infty, t_1) \), then \( \phi_t \notin SV(P, \mathbb{R}) \), i.e, \( \sum_{n=2}^{\infty} V_n(\phi_t) = \infty \) because
\[
\sum_{n=2}^{\infty} V_n(\phi_t) \geq \sum_{n=2}^{\infty} \left| \frac{n}{\lambda n + 1} \right| \left( n + 1 \right)
\]
and
\[
\lim_{n \to \infty} \left( \left| \frac{n + 1}{\lambda n + 1} \right| \left( n + 1 \right) \right) \geq \frac{1}{|t|} \frac{1}{1-\ln(\lambda)} > 1.
\]

On the other hand, we fixed \( k \) and consider \( p_k^\pm \in C^\pm_{N_k} \) such that \( \ell_{N_k}^\pm (p_k^\pm) = p_k^\pm \) (see Corollary 2.1). Furthermore one can construct a measure \( \mu_k^\pm (\cdot) = \left( \frac{1}{N_k^\pm} \sum_{j=0}^{N_k^\pm - 1} \delta_{\ell_j(p_k^\pm)} \right) (\cdot) \), where \( \delta_{\ell_j(p_k^\pm)} \) is the Dirac measure with \( \delta_{\ell_j(p_k^\pm)}(\ell_j(p_k^\pm)) = 1, j \in \{0, 1, \ldots, N_k^\pm - 1\} \). Indeed we can write \( n = q_n^+ N_k^+ + r_n^+, 0 \leq r_n^+ \leq N_k^+ - 1 \). Hence,
\[
P_{top}(\phi_t, \ell) = P_{top}(\phi_t, [0, 1], \ell) \geq P_{top}(\phi_t, C^\pm_{N_k}, \ell)
\]

\[
\geq \lim_{n \to \infty} \frac{1}{n} \log \left( \sup_{x \in C^\pm_{N_k}} e^{S_n \phi_t(x)} \right)
\]

\[
\geq \lim_{n \to \infty} \frac{1}{n} \log \left( e^{S_n \phi_t(p_k^\pm)} \right)
\]

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\[
\limsup_{n \to \infty} \frac{1}{n} S_n \phi_t(p_k^\pm) = \lim_{n \to \infty} \left( \frac{q_n^\pm}{q_n^\pm {N_k^\pm} + N_k^\pm} S_{N_k^\pm} \phi_t(p_k^\pm) + \frac{1}{q_n^\pm N_k^\pm} S_{r_n^\pm} \phi_t(p_k^\pm) \right) = \frac{1}{N_k^\pm} S_{N_k^\pm} \phi_t(p_k^\pm) = \sum_{j=0}^{N_k^\pm-1} \frac{1}{N_k^\pm} = \infty.
\]

Thus,

\[
h_{\mu^\pm}(\ell) + \int \phi_t \, d\mu^\pm = \frac{1}{N_k^\pm} \left( \sum_{j=0}^{N_k^\pm-1} \phi_t(\ell^j(p_k^\pm)) \right) = \frac{1}{N_k^\pm} \sum_{j=0}^{n-1} \frac{1}{N_k^\pm} = \infty
\]

So,

\[
P_{\text{top}}(\phi_t, \ell) = \infty = h_{\mu^\pm}(\ell) + \int \phi_t \, d\mu^\pm.
\]

Lemma 4.6 If \( t = \frac{\lambda}{1 - \ln(x)} \), then there is no equilibrium state.

**Proof** If \( t = \frac{\lambda}{1 - \ln(x)} \), then \( \phi(x) = \left( \frac{1 - \ln(x)}{\lambda} \right)^n \left( \frac{1}{1 - \ln(x)} \right)^n \). Indeed,

\[
\sum_{n=2}^{\infty} V_n(\phi_t) = \sum_{n=2}^{\infty} \frac{n}{\lambda(1 - \ln(x))} = \infty.
\]

Suppose that

\[
P_{\text{top}}(\phi, \ell) = \sup_{\mu \in \mathcal{M}_\ell(X)} \left\{ h_{\mu}(\ell) + \int \phi \, d\mu \right\}.
\]

As \( \phi \) is continuous, we get \( \sup_{[0,1]}(\phi) < \infty \). Thus there exists \( M > 0 \) such that \( P_{\text{top}}(\phi, \ell) \leq M \). Repeating the same arguments used in the proof of Lemma 4.5, we obtain

\[
P_{\text{top}}(\phi, \ell) \geq h_{\mu^\pm}(\ell) + \int \phi \, d\mu^\pm
\]

\[
= \frac{1}{N_k^\pm} \left( \sum_{j=0}^{N_k^\pm-1} \phi_t(\ell^j(p_k^\pm)) \right)
\]
\[
\frac{1}{N_k^\pm} \sum_{j=0}^{N_k^\pm-1} \left( \frac{1 - \ln(x)}{\lambda} \right)^{N_k^\pm} \left( \frac{1}{1 - \ln(\ell j (p_k^\pm)))} \right)^{N_k^\pm}
\]

So, doing \( k \to \infty \) we have \( M \geq P_{top}(\phi_t, \ell) \geq \infty \) which is a contradiction. Therefore, the variational principle cannot occurs. This shows that the potential \( \phi \) has no equilibrium state.

\[ \square \]

5 Proof of Theorem C

Consider \( \phi \in \mathcal{A} \). By definition of \( \mathcal{A} \) we have that

\[ \mathcal{A} = \{ \phi \in C([0, 1], \mathbb{R}) : \phi \in SV([0, 1], \mathbb{R}) \text{ and } P_{top}(\phi, \partial P, \ell) < P_{top}(\phi, \ell) \} = S(\phi) \cap D(\phi), \]

where

\[ S(\phi) = \left\{ \phi \in C([0, 1], \mathbb{R}) \left| \sum_{n \geq 2} V_n(\bar{\phi}) < \infty \right. \right\}, \]

and

\[ D(\phi) = \left\{ \phi \in C_P([0, 1], \mathbb{R}) \left| P_{top}(\phi, \partial P, \ell) < P_{top}(\phi, \ell) \right. \right\}. \]

In order to prove that \( \mathcal{A} \) is open and dense in \( C([0, 1], \mathbb{R}) \) it is sufficient to show that \( S(\phi) \) and \( D(\phi) \) are open and dense in \( C([0, 1], \mathbb{R}) \). By Theorem A in Bronzi and Oler (2018), \( D(\phi) \) is open and dense in \( C([0, 1], \mathbb{R}) \). Thus we only need to show that \( S(\phi) \) is open and dense in \( C([0, 1], \mathbb{R}) \). For this we observe that

\[ S(\phi) = \left\{ \phi \in C([0, 1], \mathbb{R}) \left| \sum_{n \geq 2} V_n(\bar{\phi}) < \infty \right. \right\} \]

\[ = \left\{ \phi \in C([0, 1], \mathbb{R}) \left| \lim_{n \to \infty} \sum_{j=2}^{n} V_j(\phi) = s \right. \right\} \]

\[ \subset \left\{ \phi \in C([0, 1], \mathbb{R}) \left| \forall n \geq 0, \exists M > 0 : \sum_{j=2}^{n} V_j(\phi) < M \right. \right\}. \]
Fix $n \geq 0$ and define $F, G : C([0, 1], \mathbb{R}) \to \mathbb{R}$, by $F(\phi) = V_n(\phi)$ and $G(\phi) = M - \left[ \sum_{j=2}^{n} V_j(\phi) \right]$. If $\phi_0, \phi_1, \phi_2 \in C([0, 1], \mathbb{R})$ then

$$
\| F(\phi_1) - F(\phi_2) \|_{C^0} = \| V_n(\phi_1) - V_n(\phi_2) \|_{C^0} \\
= \sup_{x,y \in C_n} \{ |\phi(x) - \phi(y)| \} - \sup_{x,y \in C_n} \{ |\phi(x) - \phi(y)| \} \\
= \| (\phi_1)_{C_n} - (\phi_2)_{C_n} \|_{C^0} \leq \| (\phi_1 - \phi_2)_{C_n} \|_{C^0}
$$

and

$$
\lim_{\phi \to \phi_0} G(\phi) = \lim_{\phi \to \phi_0} \left( M - \left[ \sum_{j=2}^{n} V_j(\phi) \right] \right) = M - \lim_{\phi \to \phi_0} \left( \sum_{j=2}^{n} V_j(\phi) \right) \\
= M - \left[ \sum_{j=2}^{n} \lim_{\phi \to \phi_0} V_j(\phi) \right] = M - \left[ \sum_{j=2}^{n} V_j(\phi_0) \right] \\
= G(\phi_0).
$$

Thus

$$
S(\phi) \subset \left\{ \phi \in C([0, 1], \mathbb{R}) \mid \forall n \geq 0, \exists M > 0 : \sum_{j=2}^{n} V_j(\phi) < M \right\} = G^{-1}(\mathbb{R}^+). \tag{2.4}
$$

Recall that by Corollary 2.4, there exists a subsequence $N_k^+ \to \infty$ such that $d \in \partial C_{N_k^+}$, $p_k^+ \in C_{N_k^+}$ and $L_{N_k^+}(p_k) = p_k^+$. Let $I_j^\pm = (L_j(p_k^+) - \delta_k^\pm, L_j(p_k^+) + \delta_k^\pm)$ be intervals, where $0 \leq j \leq N_k^-$ - 1. Since the orbit of $p_k^+$ is a finite set, there exists $\delta_k^+ > 0$ such that $I_i^\pm \cap I_j^\pm = \emptyset$, for all $0 \leq i, j \leq N_k^-$ - 1 with $i \neq j$. Consider

$$
B_{\varepsilon,k}^\pm(x) = \begin{cases}
N_k^+ - 1 \\
\sum_{j=0}^{N_k^+ - 1} B_{\varepsilon,k,j}^\pm(x), & x \in \bigcup_{j=0}^{N_k^+ - 1} I_j^\pm \\
0, & \text{otherwise}
\end{cases}
$$

where $B_{\varepsilon,k,j}^\pm$ is a bump function defined by Fig. 2.

Consider $\phi_{\varepsilon,k}^\pm(x) = \phi(x) + B_{\varepsilon,k}^\pm(x)$, where $\phi \in S(\phi)$. Then we have the following properties:

1. $\phi_{\varepsilon,k}^\pm$ is continuous;
2. $\| \phi_{\varepsilon,k}^\pm - \phi \|_{C^0} < \varepsilon$, for all $\phi \in H^\prime([0, 1], \mathbb{R})$. 

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We need show that \( \phi_{\epsilon,k}^\pm \in S(\phi) \). First, note that

\[
\sum_{n \geq 2} V_n(\phi_{\epsilon,k}^\pm) = \sum_{n \geq 2} \sup_{x, y \in C_n} \left\{ \left| \phi_{\epsilon,k}^\pm(x) - \phi_{\epsilon,k}^\pm(y) \right| \right\} \\
= \sum_{n \geq 2} \sup_{x, y \in C_n} \left\{ \left| (\phi(x) - \phi(y)) - \left( B_{\epsilon,k}^\pm(x) - B_{\epsilon,k}^\pm(y) \right) \right| \right\} \\
= \sum_{n \geq 2} \sup_{x, y \in C_n} \left\{ |\phi(x) - \phi(y)| \right\} + \sum_{n \geq 2} \sup_{x, y \in C_n} \left\{ \left| B_{\epsilon,k}^\pm(x) - B_{\epsilon,k}^\pm(y) \right| \right\} \\
= \sum_{n \geq 2} V_n(\phi) + \sum_{n \geq 0} V_n(B_{\epsilon,k}^\pm).
\]

On the other hand, by definition of \( B_{\epsilon,k,l}^\pm \), we get

\[
V_n(B_{\epsilon,k,l}^\pm) = \sup_{x, y \in C_n} \left\{ \left| B_{\epsilon,k}^\pm(x) - B_{\epsilon,k}^\pm(y) \right| \right\} \\
\leq \sup_{x, y \in C_n} \left\{ \left| B_{\epsilon,k}^\pm(x) \right| \right\} + \sup_{x, y \in C_n} \left\{ \left| B_{\epsilon,k}^\pm(y) \right| \right\} \\
\leq 2\epsilon.
\]

So we have that \( \sum_{n \geq 2} V_n(\phi_{\epsilon,k}^\pm) < \infty \), which completes the proof.

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References

Bronzi, M. A., Oler, J. G.: Equilibrium state for one-dimensional Lorenz-like expanding maps. Bull. Braz. Math. Soc. (N.S.) 49(4), 873–892 (2018)

Buzzi, J., Sarig, O.: Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. Ergod. Theory Dyn. Syst. 23(5), 1383–1400 (2003)

Glendinning, P.: Topological conjugation of Lorenz maps by $\beta$-transformations. Math. Proc. Camb. Philos. Soc. 107(2), 401–413 (1990)

Graczyk, J., Swiapolhek, G.: The Real Fatou Conjecture, Annals of Mathematics Studies, vol. 144. Princeton University Press, Princeton (1998)

Guckenheimer, J.: A strange, strange attractor. In: Marsden, J., McCracken, M. (eds.) The Hopf Bifurcation Theorem and Its Applications, pp. 368–381. Springer, Berlin (1976)

Guckenheimer, J., Williams, R.F.: Structural stability of Lorenz attractors. Inst. Hautes Études Sci. Publ. Math. 50, 59–72 (1979)

Lorenz, E.N.: Deterministic non-periodic flow. J. Atmos. Sci 20, 130–141 (1963)

Pesin, Y., Zhang, K.: Phase transitions for uniformly expanding maps. J. Stat. Phys. 122(6), 1095–1110 (2006)

Sarig, O.M.: Phase transitions for countable Markov shifts. Commun. Math. Phys. 217(3), 555–577 (2001)

Sparrow, C.: The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors, Applied Mathematical Sciences, vol. 41. Springer, New York (1982)

Williams, R.F.: The structure of Lorenz attractors. Inst. Hautes Études Sci. Publ. Math. 50, 73–99 (1979)

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