QUARTIC K3 SURFACES AND CREMONA TRANSFORMATIONS

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Abstract. We prove that there is a smooth quartic K3 surface automorphism that is not derived from the Cremona transformation of the ambient three-dimensional projective space. This gives a negative answer to a question of Professor Marat Gizatullin.

1. Introduction

Throughout this note, we work over the complex number field \( \mathbb{C} \).

In his lecture “Quartic surfaces and Cremona transformations” in the workshop on Arithmetic and Geometry of K3 surfaces and Calabi-Yau threefolds held at the Fields Institute (August 16-25, 2011), Professor Igor Dolgachev discussed the following question with several beautiful examples supporting it:

**Question 1.1.** Let \( S \subset \mathbb{P}^3 \) be a smooth quartic K3 surface. Is any biregular automorphism \( g \) of \( S \) (as abstract variety) derived from a Cremona transformation of the ambient space \( \mathbb{P}^3 \)? More precisely, is there a birational automorphism \( \tilde{g} \) of \( \mathbb{P}^3 \) such that \( \tilde{g}_* (S) = S \) and \( g = \tilde{g}|S \)? Here \( \tilde{g}_* (S) \) is the proper transform of \( S \) and \( \tilde{g}|S \) is the, necessarily biregular, birational automorphism of \( S \) then induced by \( \tilde{g} \).

Later, Dolgachev pointed out to me that, to his best knowledge, Gizatullin was the first who asked this question. The aim of this short note is to give a negative answer to the question:

**Theorem 1.2.** (1) There exists a smooth quartic K3 surface \( S \subset \mathbb{P}^3 \) of Picard number 2 such that \( \text{Pic} (S) = \mathbb{Z} h_1 \oplus \mathbb{Z} h_2 \) with intersection form:

\[
(h_i, h_j) = \begin{pmatrix} 4 & 20 \\ 20 & 4 \end{pmatrix}.
\]

(2) Let \( S \) be as above. Then \( \text{Aut} (S) \) has an element \( g \) such that it is of infinite order and \( g^*(h) \neq h \). Here \( \text{Aut} (S) \) is the group of biregular automorphisms of \( S \) as an abstract variety and \( h \in \text{Pic} (S) \) is the hyperplane section class.

(3) Let \( S \) and \( g \) be as above. Then there is no element \( \tilde{g} \) of \( \text{Bir} (\mathbb{P}^3) \) such that \( \tilde{g}_* (S) = S \) and \( g = \tilde{g}|S \). Here \( \text{Bir} (\mathbb{P}^3) \) is the Cremona group of \( \mathbb{P}^3 \), i.e., the group of birational automorphisms of \( \mathbb{P}^3 \).

Our proof is based on a result of Takahashi concerning the log Sarkisov program (\cite{Ta}), which we quote as Theorem 3.1, and standard argument concerning K3 surfaces.

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Remark 1.3. (1) Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, i.e., a smooth curve of genus 1. It is classical that any element of $\text{Aut}(C)$ is derived from a Cremona transformation of the ambient space $\mathbb{P}^2$. In fact, this follows from the fact that any smooth cubic curve is written in Weierstrass form after a linear change of coordinates and the explicit form of the group law in terms of the coordinates.

(2) Let $n$ be an integer such that $n \geq 3$ and $Y \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+2$. Then $Y$ is an $n$-dimensional Calabi-Yau manifold. It is well-known that $\text{Bir}(Y) = \text{Aut}(Y)$, it is a finite group and any element of $\text{Aut}(Y)$ is derived from a biregular automorphism of the ambient space $\mathbb{P}^{n+1}$. In fact, the statement follows from $K_Y = 0$ in $\text{Pic}(Y)$ (adjunction formula), $H^0(T_Y) = 0$ (by $T_Y \simeq \Omega_{\mathbb{P}^n}^{n-1}$ together with Hodge symmetry), and $\text{Pic}(Y) = \mathbb{Z}h$, where $h$ is the hyperplane class (Lefschetz hyperplane section theorem).

We note that $K_Y = 0$ implies that any birational automorphism of $Y$ is an isomorphism in codimension one, so that for any birational automorphism $g$ of $Y$, we have a well-defined group isomorphism $g^*$. This implies that $g$ is biregular and it is derived from an element of $\text{Aut}(\mathbb{P}^{n+1}) = \text{PGL}(\mathbb{P}^{n+1})$.

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2. Proof of Theorem (1.2) (1), (2)

In this section, we shall prove Theorem (1.2)(1)(2) by dividing it into several steps. The last lemma (Lemma (2.5)) will be used also in the proof of Theorem (1.2) (3).

Lemma 2.1. There is a projective K3 surface such that $\text{Pic}(S) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$ with

$$((h_i, h_j)) = \begin{pmatrix} 4 & 20 \\ 20 & 4 \end{pmatrix}.$$

Proof. Note that the abstract lattice given by the symmetric matrix above is an even lattice of rank 2 with signature $(1,1)$. Hence the result follows from [Mo], Corollary (2.9), which is based on the surjectivity of the period map for K3 surfaces (see e.g. [BHPV], Page 338, Theorem 14.1) and Nikulin’s theory ([Ni]) of integral bilinear forms.

From now on, $S$ is a K3 surface in Lemma (2.7).

Note that the cycle map $c_1 : \text{Pic}(S) \to \text{NS}(S)$ is an isomorphism for a K3 surface. So, we identify these two spaces. $\text{NS}(S)_{\mathbb{R}}$ is $\text{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}$. The positive cone $P(S)$ of $S$ is the connected component of the set

$$\{x \in \text{NS}(S)_{\mathbb{R}} \mid (x^2)_S > 0\},$$

containing the ample classes. The ample cone $\text{Amp}(S) \subset \text{NS}(S)_{\mathbb{R}}$ of $S$ is the open convex cone generated by the ample classes.

Lemma 2.2. $\text{NS}(S)$ represents neither 0 nor $-2$. In particular, $S$ has no smooth rational curve and no smooth elliptic curve and $(C^2)_S > 0$ for all non-zero effective curves $C$ in $S$. In particular, the positive cone of $S$ coincides with the ample cone of $S$. 
It is straightforward to see that \( \sigma \).

**Proof.** We have \((xh_1 + yh_2)^2)_S = 4(x^2 + 10xy + y^2)\). Hence there is no \((x, y) \in \mathbb{Z}^2\) such that \((xh_1 + yh_2)^2)_S \in \{-2, 0\}\). \(\square\)

**Lemma 2.3.** After replacing \(h_1\) by \(-h_1\), the line bundle \(h_1\) is very ample. In particular, \(\Phi_{|h_1|} : S \to \mathbb{P}^3\) is an isomorphism onto a smooth quartic surface.

**Proof.** \(h_1\) is non-divisible in \(\text{Pic}(S)\) by construction. It follows from Lemma (2.2) and \((h_1^2)_S = 4 > 0\) that one of \(\pm h_1\) is ample with no fixed component. By replacing \(h_1\) by \(-h_1\), we may assume that it is \(h_1\). Then, by [SD], Theorem 6.1, \(h_1\) is a very ample line bundle with the last assertion. \(\square\)

By Lemma (2.3), we may and will assume that \(S \subset \mathbb{P}^3\) and denote this inclusion by \(i\), and a general hyperplane section by \(h\). That is, \(h = H \cap S\) for a general hyperplane \(H \subset \mathbb{P}^3\), from now on. Note that \(h = h_1\) in \(\text{Pic}(S)\).

**Lemma 2.4.** There is an automorphism \(g\) of \(S\) such that \(g\) is of infinite order and \(g^*(h) \neq h\) in \(\text{Pic}(S)\).

There are several ways to prove this fact. The following simpler proof was suggested by the referee.

**Proof.** Let us consider the following orthogonal transformation \(\sigma\) of \(\text{NS}(S)\):

\[ \sigma(h_1) = 10h_1 - h_2 , \quad \sigma(h_2) = h_1 . \]

It is straightforward to see that \(\sigma\) is certainly an element of \(\text{O}(\text{NS}(S))\) and preserves the positive cone of \(S\), which is also an ample cone of \(S\) by Lemma (2.2). Note also that \(\sigma\) is of infinite order, because one of the eigenvalues is \(5 + 4\sqrt{6} > 1\).

Let \(n\) be a positive integer such that \(\sigma^n = id\) on the discriminant group \((\text{NS}(S))^*/\text{NS}(S)\). Such an \(n\) exists as \((\text{NS}(S))^*/\text{NS}(S)\) is a finite set. Let \(T(S)\) be the transcendental lattice of \(S\), i.e., the orthogonal complement of \(\text{NS}(S)\) in \(H^2(S, \mathbb{Z})\). Then, by [Ni], Proposition 1.6.1, the isometry \((\sigma^n, id_{T(S)})\) of \(\text{O}(\text{NS}(S)) \times \text{O}(T(S))\) extends to an isometry \(\tau\) of \(H^2(S, \mathbb{Z})\). Since \(\tau\) also preserves the Hodge decomposition and the ample cone, there is then an automorphism \(g\) of \(S\) such that \(g^* \neq \tau\) by the global Torelli theorem for K3 surfaces (see eg. [BHPV], Chapter VIII). This \(g\) satisfies the requirement. \(\square\)

Let \(g\) be as in Lemma (2.4). Then the pair \((S \subset \mathbb{P}^3, g)\) satisfies all the requirements of Theorem (1.2)(1), (2).

**Lemma 2.5.** Let \((S \subset \mathbb{P}^3, g)\) be as in Theorem (1.2)(1), (2). Let \(C \subset S\) be a non-zero effective curve of degree \(< 16\), i.e.,

\[ (C \cdot h)_S = (C \cdot H)_{\mathbb{P}^3} < 16 . \]

Then \(C = S \cap T\) for some hypersurface \(T\) in \(\mathbb{P}^3\).

**Proof.** Recall that \(h = h_1\) in \(\text{Pic}(S)\). There are \(m, n \in \mathbb{Z}\) such that \(C = mh_1 + nh_2\) in \(\text{Pic}(S)\). Then

\[ (C \cdot h)_S = 4(m + 5n) > 0 , \quad (C^2)_S = 4(n^2 + 10mn + m^2) > 0 . \]

Here the last inequality follows from Lemma (2.2). Thus, if \((C \cdot h)_S < 16\), then \(m + 5n\) is either 1, 2 or 3 by \(m, n \in \mathbb{Z}\). Hence we have either one of

\[ m = 1 - 5n , , m = 2 - 5n , , m = 3 - 5n . \]
Substituting into \( n^2 + 10mn + m^2 > 0 \), we obtain one of either
\[
1 - 24n^2 > 0, \quad 4 - 24n^2 > 0, \quad 9 - 24n^2 > 0.
\]
Since \( n \in \mathbb{Z} \), it follows that \( n = 0 \) in each case. Therefore, in \( \operatorname{Pic}(S) \), we have \( C = mh \) for some \( m \in \mathbb{Z} \). Since \( H^1(P^3, \mathcal{O}_{P^3}(\ell)) = 0 \) for all \( \ell \in \mathbb{Z} \), the natural restriction map
\[
\iota^*: H^0(P^3, \mathcal{O}_{P^3}(m)) \to H^0(S, \mathcal{O}_S(m))
\]
is surjective for all \( m \in \mathbb{Z} \). This implies the result.

3. Proof of Theorem (1.2) (3)

In his paper [1a], Theorem 2.3 and Remark 2.4, N. Takahashi proved the following remarkable theorem as a nice application of the log Sarkisov program (For terminologies, we refer to [KMM]):

**Theorem 3.1.** Let \( X \) be a Fano manifold of dimension \( \geq 3 \) with Picard number 1, \( S \in | - K_X | \) be a smooth hypersurface. Let \( \Phi: X \to X' \) be a birational map to a \( \mathbb{Q} \)-factorial terminal variety \( X' \) with Picard number 1, which is not an isomorphism, and \( S' := \Phi_* S \). Then:

1. If \( \operatorname{Pic}(X) \to \operatorname{Pic}(S) \) is surjective, then \( K_{X'} + S' \) is ample.
2. Let \( X = P^3 \) and \( H \) be a hyperplane of \( P^3 \). Note that then \( S \) is a smooth quartic \( K3 \) surface. Assume that any irreducible reduced curve \( C \subset S \) such that \( (C \cdot H)_{P^3} < 16 \) is of the form \( C = S \cap T \) for some hypersurface \( T \subset P^3 \). Then \( K_{X'} + S' \) is ample.

Applying Theorem (3.1)(2), we shall complete the proof of Theorem (1.2)(3) in the following slightly generalized form:

**Theorem 3.2.** Let \( S \subset P^3 \) be a smooth quartic \( K3 \) surface. Then:

1. Any automorphism \( g \) of \( S \) of infinite order is not the restriction of a birational automorphism of the ambient space \( P^3 \), i.e., the restriction of an element of \( \text{PGL}(P^3) \).
2. Assume further that \( S \) contains no curves of degree \( < 16 \) which are not cut out by a hypersurface. Then, any automorphism \( g \) of \( S \) of infinite order is not the restriction of a Cremona transformation of the ambient space \( P^3 \).

Recalling Lemma (2.5), we see that the pair \( (S \subset P^3, g) \) in Theorem (1.2)(1)(2) satisfies all the requirements of Theorem (3.2)(2). So, Theorem (1.2)(3) follows from Theorem (3.2)(2). We prove Theorem (3.2).

**Proof.** Let us first show (1). Consider the group \( G := \{ g \in \text{PGL}(P^3) \mid g(S) = S \} \). Let \( H \) be the connected component of \( \text{Hilb}(P^3) \) containing \( S \). Then \( G \) is the stabilizer group of the point \( [S] \in H \) under the natural action of \( \text{PGL}(P^3) \) on \( H \). In particular, \( G \) is a Zariski closed subset of the affine variety \( \text{PGL}(P^3) \). In particular, \( G \) has only finitely many irreducible components. Note that the natural map \( G \to \text{Aut}(S) \) is injective and \( H^0(S, T_S) = 0 \). Thus \( \dim G = 0 \). Hence \( G \) is a finite set.

Let \( g \in \text{Aut}(S) \). If there is an element \( \tilde{g} \in \text{PGL}(P^3) \) such that \( g = \tilde{g}|S \), then \( g \in G \), and therefore \( g \) is of finite order. This proves (1).

Let us show (2). We argue by contradiction, i.e., assuming to the contrary that there would be a birational map \( \tilde{g}: P^3 \to P^3 \) such that \( \tilde{g}_*(S) = S \) and that \( g = \tilde{g}|S \), we shall derive a contradiction.
We shall divide it into two cases:
(i) $\tilde{g}$ is an isomorphism, (ii) $\tilde{g}$ is not an isomorphism.

Case (i). By (1), $g$ would be of finite order, a contradiction.

Case (ii). By the case assumption, our $S$ would satisfy all the conditions of Theorem (3.1)(2). Recall also that $\tilde{g}_* S = S$. However, then, by Theorem (3.1)(2), $K_{P^3} + S$ would be ample, a contradiction to $K_{P^3} + S = 0$ in Pic ($P^3$).

This completes the proof. □

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