The Riesz hull of a semisimple MV-algebra

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Dedicated to Prof. Antonio Di Nola on the occasion of his 65th birthday.

Abstract

MV-algebras and Riesz MV-algebras are categorically equivalent to abelian lattice-ordered groups with strong unit and, respectively, with Riesz spaces (vector-lattices) with strong unit. A standard construction in the literature of lattice-ordered groups is the vector-lattice hull of an archimedean lattice-ordered group. Following a similar approach, in this paper we define the Riesz hull of a semisimple MV-algebra.

1 Introduction

MV-algebras were first defined by Chang [3] as algebraic structures corresponding to the \(\infty\)-valued Lukasiewicz logic. An MV-algebra is a structure \((A, \oplus, *, 0)\), where \((A, \oplus, 0)\) is an abelian monoid and the following identities hold for all \(x, y \in A\): \((x^*)^* = x\), \(0^* \oplus x = 0^*\) and \((x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x\). One of the main engines of MV-algebra theory is the categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit [13]. As a consequence, any MV-algebra is isomorphic to the unit interval \([0, u]\) of an abelian lattice-ordered group \((G, u)\), with operations defined by \(x^* = u - x\) and \(x \oplus y = u \wedge (x + y)\). MV-algebras stand to Lukasiewicz logic as boolean algebras stand to classical logic: an equation holds in any MV-algebra if and only if it holds in the real interval \([0, 1]\) endowed with the following operations

\[x \oplus y = \min\{1, x + y\}\] and \(x^* = 1 - x\),

for every \(x, y \in [0, 1]\). The real interval \([0, 1]\) with the above operations is the standard MV-algebra and it is usually denoted by \([0, 1]_{MV}\).
Adding a product operation to the signature of MV-algebras was a natural step, which led to fruitful results, both in logic and algebra. Once the MV-algebra structure is enriched, categorical equivalences with particular lattice-ordered structures are proved.

*PMV-algebras* are defined in [10] as MV-algebras endowed with a product operation \( \cdot : A \times A \to A \), satisfying some particular identities. The category of PMV-algebras is equivalent with the category of lattice-ordered rings with strong unit. In [11] the internal product is replaced by a scalar multiplication with scalars from \([0,1]\), so MV-algebras are endowed with a map \( \cdot : [0,1] \times A \to A \). The structures obtained in this way are called *Riesz MV-algebras* and they are categorically equivalent with Riesz spaces (vector-lattices) with strong unit. The real interval \([0,1]\) endowed with the natural product generates the variety of Riesz MV-algebras. Note that, in the case of PMV-algebras, \([0,1]\) generates only a proper quasi-variety [17].

A standard construction in the literature of lattice-ordered groups is the *vector-lattice hull* of an archimedean lattice-ordered group, defined by Conrad in [7] and further analyzed by Bleier in [3]. We also refer to [16] for an extensive treatment of hull classes for archimedean lattice-ordered groups.

We briefly remind Conrad’s definition. If \( G \) is an archimedean lattice-ordered group, then the *v-hull* of \( G \) is a vector-lattice \( U \) such that \( G \) is an essential subgroup of \( U \) and no proper \( \ell \)-subspace of \( U \) contains \( G \). Assume \( G_d \) is the divisible hull of \( G \) and \( \hat{G}_d \) is the Dedekind-MacNeille completion of \( G_d \). Hence the vector-lattice generated by \( G_d \) in \( \hat{G}_d \), denoted by \( \mathbf{R}(G) \), is the v-hull of \( G \). Moreover, Bleier proved that the correspondence \( G \mapsto \mathbf{R}(G) \) is functorial.

In this paper we investigate a similar construction for semisimple MV-algebras and semisimple Riesz MV-algebras. If \( A \) is a semisimple MV-algebra we say that a Riesz MV-algebra \( U \) is the *Riesz hull* of \( A \) if \( A \) is essentially embedded in \( U \) and \( A \) is a set of generators for \( U \). In Section 4 we prove that any *semisimple MV-algebra has a Riesz hull*. Moreover, the Riesz hull of the free MV-algebra over a set \( X \) is the free Riesz MV-algebra over \( X \). In Section 5 we prove that the construction of the Riesz hull is functorial. Moreover, the hull functor commutes with the categorical equivalences between the corresponding classes of MV-algebras and lattice-ordered groups.

We chose to make direct proofs in the theory of MV-algebras. Alternative proofs can be given using Conrad’s construction and various preservation properties of the categorical equivalence between MV-algebras and lattice-ordered groups, but we find the direct approach more relevant for our purpose.

In Section 2 and 3 we recall the basic results on MV-algebras and Riesz MV-algebras that are required for our development. We refer to [8] for background knowledge on lattice-ordered groups, to [15] for Riesz spaces and to [6] for universal algebra.
2 MV-algebras

**Definition 2.1.** An **MV-algebra** is a structure \((A, \oplus, *, 0)\) of type \((2,1,0)\) which satisfies the following:

(MV1) \((A, \oplus, 0)\) is an abelian monoid,
(MV2) \((a*)* = a,\)
(MV3) \(0* \oplus a = 0*,\)
(MV4) \((a* \oplus b)* \oplus b = (b* \oplus a)* \oplus a,\)

for any \(a, b \in A.\)

We refer to [5] for all the unexplained notions related to MV-algebras.

In any MV-algebra \(A\) we can define the following:

\[\begin{align*}
1 & \overset{\text{def}}{=} 0*, \\
\overline{a \cdot b} & \overset{\text{def}}{=} (a* \oplus b*), \\
\overline{a \lor b} & \overset{\text{def}}{=} (a \cdot b*) \oplus b, \\
\overline{a \land b} & \overset{\text{def}}{=} (a + b*) \cdot b,
\end{align*}\]

for any \(a, b \in A.\) Hence \((A, \overline{\lor}, \overline{\land}, 0, 1)\) is a bounded distributive lattice such that \(a \leq b\) if and only if \(a \cdot b* = 0.\)

The notions of MV-homomorphism and MV-subalgebra are defined as usual.

We recall that a **lattice-ordered group** (an \(\ell\)-group) is a structure \((G, +, 0, \leq)\) such that \((G, +, 0)\) is a group, \((G, \leq)\) is a lattice and any group translation is isotone [8]. An element \(u \in G\) is a **strong unit** if \(u \geq 0\) and for any \(x \in G\) there is a natural number \(n\) such that \(x \leq nu.\) An \(\ell u\)-group will be an abelian \(\ell\)-group which has a strong unit. If \((G, u)\) is an \(\ell u\)-group, we define

\[0, u] = \{ x \in G \mid 0 \leq x \leq u \}\]

and

\[x \oplus y = (x + y) \land u, \ x* = u - x, \text{ for any } x, y \in [0, u].\]

Then \([0, u]_G = ([0, u], \oplus, *, 0)\) is an MV-algebra.

We denote by \(\mathcal{MV}\) the category of MV-algebras and by \(\mathcal{AG}_u\) the category of unital abelian lattice-ordered groups with unit-preserving \(\ell\)-morphisms. In [18] the functor \(\Gamma : \mathcal{AG}_u \to \mathcal{MV}\) is defined as follows:

\[\Gamma(G, u) = [0, u]_G, \text{ for any unital } \ell\text{-group } (G, u),\]

\[\Gamma(f) = f|[0, u], \text{ for any } \ell\text{-morphism } f : (G, u) \to (G', u') \text{ from } \mathcal{AG}_u.\]

**Theorem 2.1.** [18] The functor \(\Gamma\) establishes a categorical equivalence between \(\mathcal{AG}_u\) and \(\mathcal{MV}\).

The standard MV-algebra is \([0, 1] = \Gamma(\mathbb{R}, 1)\).
Theorem 2.2. An equation holds in $[0,1]$ if and only if it holds in any MV-algebra. As a consequence, the variety of MV-algebras is generated by $[0,1]$.

Theorem 2.3. Any MV-algebra $A$ is isomorphic with an algebra of $*[0,1]$-valued functions, where $*[0,1]$ is the unit interval of the lattice-ordered group of nonstandard reals $*\mathbb{R}$.

If $A$ is an MV-algebra, $a \in A$ and $n \geq 0$ is a natural number, we define

$$0a \overset{df}{=} 0 \text{ and } na \overset{df}{=} (n-1)a \oplus a, \text{ if } n > 0.$$  

Definition 2.2. If $\iota : A \rightarrow B$ is an MV-embedding then we say that:

1. $\iota$ is order dense if for any $b > 0$ in $B$, there exists $a > 0$ in $A$ such that $\iota(a) \leq b$,
2. $\iota$ is essential if for any $b > 0$ in $B$, there exists $a > 0$ in $A$ such that $\iota(a) \leq nb$, for some natural number $n \geq 0$.

For any MV-algebra $A$, a nonempty set $I \subseteq A$ is an MV-ideal if the following hold:

1. $a \leq b$ and $b \in I$ implies $a \in I$,
2. $a, b \in I$ implies $a \oplus b \in I$.

Remark 2.1. An embedding $\iota : A \rightarrow B$ is essential if and only if for any ideal $I$ of $B$, $I \neq \{0\}$ implies $I \cap \iota(A) \neq \{0\}$.

Lemma 2.1. Let $\iota : A \rightarrow B$ be an essential embedding. If $C$ is an MV-algebra and $f_A : A \rightarrow C$, $f_B : B \rightarrow C$ are MV-homomorphisms such that $f_B \circ \iota = f_A$ and $f_A$ is an embedding then $f_B$ is an embedding.

Proof. Assume $b \in B$ such that $f_B(b) = 0$. If $b \neq 0$ there is $a > 0$ in $A$ such that $a \leq nb$, so $f_A(a) = f_B(\iota(a)) = 0$. Since $f_A$ is an embedding we infer that $a = 0$, which is a contradiction, so $b = 0$ and $f_B$ is an embedding. \qed

An ideal $I$ of $A$ is proper if $I \neq A$. A maximal ideal is a maximal element of the set of proper ideals ordered by inclusion. We denote by Max$(A)$ the set of all maximal ideals of $A$. Remember that, for any MV-algebra $A$, Max$(A)$ endowed with the spectral topology is a compact and Hausdorff space. An MV-algebra is semisimple if $\bigcap\{I \mid I \in \text{Max}(A)\} = \{0\}$.

Recall that an $}\ell$u-group $(G, u)$ is archimedean if, for any $x, y \in G$, we have

$$nx \leq y, \text{ for any } n \in \mathbb{N}, \text{ implies } x \leq 0.$$
Remark 2.2. [5] Let $A$ be an MV-algebra and $(G, u)$ an $\ell u$-group such that $A \simeq \Gamma(G, u)$. Then $A$ is semisimple if and only if $G$ is archimedean.

The semisimple MV-algebras are the algebras of $[0, 1]$-valued functions, i.e. for any semisimple MV-algebra $A$ there exists a set $X$ such that $A \simeq \Gamma(G, u)$. If $X$ is a topological space, we set $C(X) = \{ f : X \to [0, 1] \mid f \text{ continuous} \}$, which obviously is a semisimple MV-algebra.

Theorem 2.4. [5] Any semisimple MV-algebra $A$ is isomorphic with a separating subalgebra of $C(\text{Max}(A))$.

For a semisimple MV-algebra $A$ we denote by $A$ the subalgebra of $C(\text{Max}(A))$ such that $A \simeq A$ and by $\varphi_A : A \to A$ the corresponding isomorphism.

Definition 2.3. An MV-algebra $A$ is divisible if for any element $a \in A$ and $n > 1$ in $\mathbb{N}$ there exists $x \in A$ such that $nx = a$ and $(n - 1)x \leq x^*$.

We refer to [14] for a systematic investigation of the divisible MV-algebras and their logic.

Remark 2.3. [14] An $\ell u$-group $G$ is divisible if for any element $g \in G$ and any $n > 1$ in $\mathbb{N}$ there exists $x \in G$ such that $nx = g$. One can easily see that an $\ell u$-group $(G, u)$ is divisible if and only if the MV-algebra $[0, u]_G$ is divisible.

If $X$ is a compact Hausdorff space then $C(X, \mathbb{R}) = \{ f : X \to \mathbb{R} \mid f \text{ continuous} \}$ is an $\ell u$-group and the constant function $1$ is a strong unit.

Remark 2.4. It is well-known that any MV-algebra can be embedded in a divisible one (see, for example, [12]). We provide the details of this embedding for the semisimple case, which is relevant for our paper.

Assume $(G, 1)$ is an $\ell u$-subgroup of $(C(X, \mathbb{R}), 1)$ and $A = [0, 1]_G \subseteq C(X)$. We define $G_d = \{ \frac{g}{n} \mid g \in G, n \in \mathbb{N}, n \neq 0 \}$ and $A_d = [0, 1]_{G_d}$. Hence $G_d$ is a divisible $\ell u$-group and $A_d$ is a divisible MV-algebra. Let $g \in G$ and $n \in \mathbb{N}$ such that $\frac{g}{n} \in A_d$. It follows that $0 \leq g \leq n 1 \in G$, so there are $a_1, \ldots, a_n \in A$ such that $g = a_1 + \cdots + a_n$. Hence $\frac{g}{n} = \frac{a_1}{n} + \cdots + \frac{a_n}{n}$. In consequence $A_d = \{ a \in C(X) \mid a = \frac{a_1}{n} + \cdots + \frac{a_n}{n} \text{ for some } n \in \mathbb{N}, n \neq 0 \text{ and } a_1, \ldots, a_n \in A \}$, and it is straightforward that $A \subseteq A_d$.

If $X$ is a compact Hausdorff space and $A \leq C(X)$ is a semisimple MV-algebra then we get an embedding $\iota_{A,d} : A \to A_d$.

We note that

Lemma 2.2. Under the above hypothesis, the following properties hold.

(a) The embedding $\iota_{A,d}$ is essential.
(b) If \( U \) is a semisimple divisible MV-algebra and \( f : A \to U \) is an MV-homomorphism then there exists a unique MV-homomorphism \( f_d : A_d \to U \) such that \( f_d(a) \circ_{I,A,d} = f \). Moreover, if \( f \) is an embedding then \( f_d \) is also an embedding.

Proof. (a) follows easily from the description of \( A_d \) from Remark 2.4.
(b) Assume that \( G \) and \( G_d \) are the \( \ell u \)-groups from Remark 2.4. One can easily see that whenever \((H, v)\) is a divisible \( \ell u \)-group and \( h : G \to H \) is an \( \ell u \)-morphism there exists a unique \( \ell u \)-morphism \( h^\# : G_d \to H \) extending \( h \), which is simply defined by \( h^\#(\frac{g}{n}) = \frac{h(g)}{n} \) for any \( g \in G \) and \( n \in \mathbb{N} \). Hence the extension result for MV-algebras follows using the functor \( \Gamma \). The embeddings are preserved by (a) and Lemma 2.1. 

Recall that an MV-algebra \( A \) is complete if any subset \( \{a_i | i \in I\} \) of \( A \) has infimum and supremum.

Definition 2.4. For a semisimple MV-algebra \( A \) we say that \( \hat{A} \) is the Dedekind-MacNeille completion of \( A \) if \( A \subseteq \hat{A} \), \( \hat{A} \) is complete and for any element \( \hat{a} \in \hat{A} \) there exists a family \( \{a_i | i \in I\} \subseteq A \) such that \( \hat{a} = \bigvee\{a_i | i \in I\} \).

Since any complete MV-algebra is semisimple [5, Proposition 6.6.2], only semisimple MV-algebras admit completions.

Remark 2.5. Any semisimple MV-algebra \( A \) has a Dedekind-MacNeille completion \( \hat{A} \), which is unique up to isomorphism. Moreover, \( A \) is order dense in \( \hat{A} \).

We refer to [1] for a study of completions in the theory of MV-algebras, with a special focus on the Dedekind-MacNeille completion.

3 Riesz MV-algebras

Any MV-algebra is isomorphic with the unit interval of an \( \ell u \)-group. If we consider a Riesz space with strong unit instead of an \( \ell u \)-group, then the unit interval is closed under the scalar multiplication with scalars from \([0,1]\). The structures obtained in this way are studied in [11].

Definition 3.1. [11] A Riesz MV-algebra is a structure \((V, \cdot, \oplus, 0)\), where \((V, \oplus, 0)\) is an MV-algebra and \( \cdot : [0,1] \times V \to V \) is a function such that:

\[
\begin{align*}
\text{(RMV1)} \quad & r \cdot (a \oplus b^*) = (r \cdot a) \oplus (r \cdot b)^*, \\
\text{(RMV2)} \quad & \max(r - q, 0) \cdot a = (r \cdot a) \oplus (q \cdot a)^*, \\
\text{(RMV3)} \quad & (r \cdot q) \cdot a = r \cdot (q \cdot a),
\end{align*}
\]
1 \cdot a = a,

for any \( r, q \in [0, 1] \) and \( a, b \in V \).

In order to simplify the notation, we shall frequently write \( ra \) instead of \( r \cdot a \), for any \( r \in [0, 1] \) and \( a \in V \). For a Riesz MV-algebra \( (V, \cdot, \oplus, ^*, 0) \) we denote by \( U(V) = (V, \oplus, ^*, 0) \) its MV-algebra reduct.

**Remark 3.1.** [11] If \( V \) is a Riesz MV-algebra and \( I \subseteq U(V) \) is an MV-algebra ideal, then \( r \cdot a \in I \) for any \( r \in [0, 1] \) and \( a \in I \). Hence a Riesz MV-algebra has the same theory of ideals as its MV-algebra reduct. In consequence, a Riesz MV-algebra is semisimple if and only if its MV-algebra reduct is semisimple.

**Proposition 3.1.** [11] If \( V_1 \) and \( V_2 \) are Riesz MV-algebra and \( f : U(V_1) \rightarrow U(V_2) \) is an MV-homomorphism, then \( f(ra) = rf(a) \), for any \( r \in [0, 1] \) and \( a \in V_1 \).

**Remark 3.2.** [11] By the previous proposition, it follows that Riesz MV-algebra homomorphisms are just MV-homomorphisms between Riesz MV-algebras, so we shall only state that a function is an MV-homomorphism, even if the domain and the codomain are Riesz MV-algebras.

We recall that a Riesz space (vector-lattice) [15] is a structure \( (L, \cdot, +, 0, \leq) \) such that \( (V, +, 0, \leq) \) is an abelian \( \ell \)-group, \( (V, \cdot, +, 0) \) is a real vector space and, in addition, \( x \leq y \) implies \( r \cdot x \leq r \cdot y \), for any \( x, y \in L \) and \( r \in \mathbb{R} \), \( r \geq 0 \). A Riesz space is unital if the underlaying \( \ell \)-group is unital. If \( (L, u) \) is a Riesz space with strong unit, then we denote by \( \Gamma_R(L, u) = ([0, u], \cdot, \oplus, ^*, 0) \), where \( \cdot \) is the scalar multiplication restricted to scalars from \( [0, 1] \).

**Remark 3.3.** [11] For any unital Riesz space \( (L, u) \), the structure \( \Gamma_R(L, u) \) is a Riesz MV-algebra.

In this way we can define a functor \( \Gamma_R : \mathcal{RS}_u \rightarrow \mathcal{RMV} \), where \( \mathcal{RS}_u \) is the category of unital Riesz spaces and \( \mathcal{RMV} \) is the category of Riesz MV-algebras. The categorial equivalence from Theorem 2.1 leads to the following one.

**Theorem 3.1.** [11] The functor \( \Gamma_R \) establishes a categorical equivalence.

The standard Riesz MV-algebra is \( ([0, 1], \cdot, \oplus, ^*, 0) \) where \( ([0, 1], \oplus, ^*, 0) \) is the standard MV-algebra and \( \cdot \) is the product of real numbers.

**Theorem 3.2.** [11] The variety of Riesz MV-algebras is generated by \( [0, 1] \).

**Lemma 3.1.** The Dedekind-MacNeille completion of a semisimple divisible MV-algebra \( D \) is a Riesz MV-algebra \( \hat{D} \) in which \( D \) is order dense.
Proof. It is a straightforward consequence of the fact that the functor $\Gamma$ preserves both divisibility $[14]$ and completeness $[13]$. Hence there exists a divisible $\ell$-group $(G, u)$ such that $D \simeq \Gamma(G, u)$ and $D = \Gamma(G, u)$, where $G$ is the Dedekind-MacNeille completion of $G$. Now we use the fact that the Dedekind-MacNeille completion of a divisible abelian $\ell$-group is a Riesz space $[8]$. The result can be directly proved by setting $ra = \bigvee \{qa \mid q \in [0, 1] \cap \mathbb{Q}, q \leq r\}$ for any $r \in [0, 1]$ and $a \in D$. 

Remark 3.4. By Theorem 2.3 for any MV-algebra $A$ there exists a set $X$ such that $A$ is embedded in the MV-algebra $\left(\ast [0, 1]\right)^X$. Since $\ast [0, 1]$ is obviously a Riesz MV-algebra, one can easily see that $\left(\ast [0, 1]\right)^X$ becomes a Riesz MV-algebra with the scalar multiplication defined componentwise. Hence any MV-algebra can be embedded in a Riesz MV-algebra.

In the following we prove that, for a semisimple MV-algebra $A$, we can define a unique (up to isomorphism) Riesz MV-algebra in which $A$ is essentially embedded and we will further analyze the properties of this embedding.

4 The Riesz MV-algebra hull

In the sequel, we follow closely the similar construction for archimedean $\ell$-groups from $[7]$ and $[3]$, but our proofs are made directly in the context of MV-algebras.

Due to Remark 3.2 in the rest of this paper we will make no distinction between MV-homomorphisms and Riesz MV-algebra homomorphisms. If $A$ is an MV-algebra and $X$ is a subset of $A$, we shall denote by $\langle X \rangle_{MV}$ the MV-subalgebra generated by $X$ in $A$. Similarly, if $V$ is a Riesz MV-algebra and $X$ is a subset of $V$, we shall denote by $\langle X \rangle_{RMV}$ the Riesz MV-subalgebra generated by $X$ in $V$.

If $A$ is a semisimple MV-algebra, then its divisible hull $A_d$ is also semisimple. If $X = Max(A_d)$ is the compact Hausdorff space of the maximal ideals of $A_d$, then

$$A \simeq A \subseteq A_d \subseteq C(X).$$

Let $\hat{A}_d$ be the Dedekind-MacNeille completion of $A_d$. By Lemma 3.1 $\hat{A}_d$ is a Riesz MV-algebra. We denote by $R(A)$ the Riesz MV-algebra generated by $A$ in $\hat{A}_d$.

For a semisimple MV-algebra $A$, we assume the following:

$\varphi_A : A \rightarrow A$ is the canonical MV-algebra isomorphism,

$\iota_{A,d} : A \rightarrow A_d$ is the embedding of $A$ in its divisible hull $A_d$,

$\hat{i}_{A,d} : A_d \rightarrow \hat{A}_d$ is the embedding of $A_d$ in its Dedekind-MacNeille completion.
Hence we denote by $\iota_A : A \to R(A)$ the co-restriction to $R(A)$ of the homomorphism $\iota_{A,d} \circ \iota_{A,d} \circ \varphi_A$.

**Theorem 4.1.** If $A$ is a semisimple MV-algebra and $R(A)$ is defined as above, then the following properties hold.

(a) There exists an embedding $\iota_A : A \to R(A)$ and $R(A) = \langle \iota_A(A) \rangle_{RMV}$.

(b) The embedding $\iota_A$ is essential.

(c) If $V$ is a semisimple Riesz MV-algebra and $f : A \to V$ is an MV-embedding then there exists an MV-embedding $f_R : R(A) \to V$ such that $f_R(\iota_A(a)) = f(a)$, for any $a \in A$.

\[ A \xleftarrow{\iota_A} R(A) \xrightarrow{f} V \]

Proof.

(a) follows by definition, since $A$ is embedded in $A_d$ and $A_d$ is embedded in $\hat{A}_d$.

(b) is a straightforward consequence of Lemmas 2.2 and 3.1.

(c) Let $V$ be a semisimple Riesz MV-algebra and $f : A \to V$ an MV-embedding. Since $V$ is also divisible, by Remark 2.4 there is a unique MV-embedding $f_d : A_d \to V$ such that

$$f_d \circ \iota_{A,d} \circ \varphi_A = f.$$ 

If $\iota_V : V \to \hat{V}$ is the inclusion of $V$ in its Dedekind-MacNeille completion, then there exists a unique MV-embedding $f_d : \hat{A}_d \to \hat{V}$ such that

$$f_d \circ \iota_{A,d} = \iota_V \circ f_d.$$

\[ A \xleftarrow{\iota_{A,d} \circ \varphi_A} A_d \xrightarrow{f_d} V \]

\[ A_d \xleftarrow{\iota_{A,d}} \hat{A}_d \xrightarrow{f_d} \hat{V} \]

It follows that
\[
\hat{f}_d \circ \iota_A = \hat{f}_d \circ \iota_{A,d} \circ \varphi_A = \iota_V \circ f_d \circ \iota_{A,d} \circ \varphi_A = \iota_V \circ f,
\]
and we get
\[
\hat{f}_d(R(A)) = \hat{f}_d(\langle \iota_A(A) \rangle_{RMV}) = \langle \hat{f}_d(\iota_A(A)) \rangle_{RMV} = \langle \iota_V(f(A)) \rangle_{RMV} = \langle f(A) \rangle_{RMV} \subseteq V.
\]

Therefore we define \( f_R : R(A) \to V \) as the co-restriction to \( V \) of the restriction \( \hat{f}_d|_{R(A)} \). If \( g : R(A) \to V \) is another MV-embedding such that \( g \circ \iota_A = f \), then \( g \) and \( f \) coincide on the generators of \( R(A) \), so they coincide on \( R(A) \).

Following [7], we define the Riesz hull of an MV-algebra.

**Definition 4.1.** We say that a Riesz MV-algebra \( U \) is a Riesz hull of \( A \) if there exists an essential embedding \( \eta : A \to U \) such that \( U = \langle \eta(A) \rangle_{RMV} \).

In consequence, Theorem 4.1 asserts that any semisimple MV-algebra has a Riesz hull which is unique, up to isomorphism.

**Corollary 4.1.** If \( A \) is a semisimple MV-algebra, then \( R(A) \simeq R(A_d) \).

**Proof.** It is a straightforward consequence of the construction.

**Corollary 4.2.** If \( A \) is a semisimple MV-algebra and \( V \) is a semisimple Riesz MV-algebra such that \( A \subseteq V \) and \( V = \langle A \rangle_{RMV} \), then \( V \simeq R(A) \).

**Proof.** By Theorem 4.1 (c), there exists an MV-embedding \( e : R(A) \to V \) such that \( e(\iota_A(a)) = a \) for any \( a \in A \). Hence
\[
e(R(A)) = e(\langle \iota_A(A) \rangle_{RMV}) = \langle e(\iota_A(A)) \rangle_{RMV} = \langle A \rangle_{RMV} = V,
\]
so \( e \) is an isomorphism.

**Corollary 4.3.** Let \( A \) be a semisimple MV-algebra and \( V \) be a semisimple Riesz MV-algebra such that \( A \subseteq V \) and \( \langle A \rangle_{RMV} = V \). Then the embedding \( A \to V \) is essential. If, in addition, \( A \) is divisible, then the embedding \( A \to V \) is order dense.

**Proof.** The first part follows by Corollary 4.2 and Theorem 4.1 (b). If \( A \) is divisible, then \( A \simeq A_d \). In this case, the conclusion follows by the fact that \( A_d \subseteq R(A) \subseteq A_d \) and Lemma 3.1.

**Corollary 4.4.** If \( V \) is a semisimple Riesz MV-algebra, then \( V \simeq R(V) \). In this case, \( \iota_V \) is an isomorphism.
Proof. It follows from Corollary 4.2.

**Corollary 4.5.** Assume $V_1$ and $V_2$ are semisimple Riesz MV-algebras with the same MV-algebra reduct. Then $V_1 \simeq V_2$.

Proof. If $A$ is the MV-algebra reduct of $V_1$ and $V_2$ then, by Corollary 4.2, we get $V_1 \simeq R(A) \simeq V_2$.

The above result asserts that, given an MV-algebra $A$, there is at most one structure, up to isomorphism, of Riesz MV-algebra with the MV-algebra reduct $A$.

In the sequel we prove that the Riesz MV-algebra hull preserves freeness. For a nonempty set $X$, we shall denote by $\text{Free}_{\text{MV}}(X)$ the free MV-algebra over $X$ and by $\text{Free}_{\text{RMV}}(X)$ the free Riesz MV-algebra over $X$. The free algebras exist in the classes of MV-algebras and Riesz MV-algebras since both classes are varieties.

**Proposition 4.1.** For any nonempty set $X$, $R(\text{Free}_{\text{MV}}(X)) \simeq \text{Free}_{\text{RMV}}(X)$. Therefore, the free MV-algebra generated by $X$ is essentially embedded in the free Riesz MV-algebra generated by $X$. Moreover, the embedding can be chosen to be an inclusion.

Proof. If $T = [0, 1]^{[0,1]^X}$ then $T$ is a Riesz MV-algebra with the operations defined component-wise. For any $x \in X$ we denote by $\pi_x \in T$ the corresponding projection function and we set $\hat{X} = \{\pi_x \mid x \in X\}$. Since the variety of MV-algebras is generated by $[0, 1]_{\text{MV}}$ and the variety of Riesz MV-algebras is generated by $[0, 1]_{\text{RMV}}$, by general properties in universal algebra, $\text{Free}_{\text{MV}}(X)$ is the MV-algebra generated by $\hat{X}$ in $T$ and $\text{Free}_{\text{RMV}}(X)$ is the Riesz MV-algebra generated by $\hat{X}$ in $T$. We have that $\text{Free}_{\text{MV}}(X) = \langle \hat{X} \rangle_{\text{MV}}$ and $\text{Free}_{\text{RMV}}(X) = \langle \hat{X} \rangle_{\text{RMV}} = \langle \text{Free}_{\text{MV}}(X) \rangle_{\text{RMV}}$.

The conclusion follows from Corollary 4.2.

**5 Categorical setting: the functor $R$**

The main step for obtaining a functorial setting is to prove a general extension result for morphisms, as which we do in Proposition 5.2. The results of this section follow closely the ideas from [3].

**Remark 5.1.** Let $A$ be a semisimple MV-algebra and $X \subset A$ such that $\langle X \rangle_{\text{MV}} = A$. Using Proposition 4.1, the free MV-algebra generated by $X$ is essentially included in the free Riesz MV-algebra generated by $X$ and we denote this inclusion by $\iota_X : \text{Free}_{\text{MV}}(X) \to \text{Free}_{\text{RMV}}(X)$. Let $\alpha : \text{Free}_{\text{MV}}(X) \to A$ be the unique
MV-homomorphism such that $\alpha(x) = x$ for any $x \in X$ and $\overline{\pi}: \text{Free}_{RMV}(X) \to R(A)$ be the unique MV-homomorphism such that $\overline{\pi}(x) = \iota_A(x)$ for any $x \in X$.

$$\text{Free}_{MV}(X) \xrightarrow{\iota_X} \text{Free}_{RMV}(X) \xleftarrow{\alpha} \text{Free}_{RMV}(X) \xrightarrow{\overline{\pi}} R(A)$$

**Proposition 5.1.** Under the above hypothesis, the following properties hold:

(a) $\overline{\pi} \circ \iota_X = \iota_A \circ \alpha$,

(b) $\alpha$ and $\overline{\pi}$ are surjective,

(c) $\ker \overline{\pi} = \bigcap \{J \mid J \in \mathcal{J}\}$, where

\[ \mathcal{J} = \{J \subseteq \text{Free}_{RMV}(X) \mid J \text{ ideal, } \iota_X(ker \alpha) \subseteq J, \text{ and } \text{Free}_{RMV}(X)/J \text{ is semisimple}\}. \]

**Proof.**

(a) $(\overline{\pi} \circ \iota_X)(x) = \iota_A(x) = (\iota_A \circ \alpha)(x)$ for any $x \in X$, so the morphisms coincide on generators.

(b) $\alpha(\text{Free}_{MV}(X)) = \alpha(\langle X \rangle_{MV}) = \langle \alpha(X) \rangle_{MV} = A$ and

\[ \overline{\pi}(\text{Free}_{RMV}(X)) = \overline{\pi}(\text{Free}_{MV}(X))_{RMV} = \langle \iota_A(\alpha(Free_{MV}(X))) \rangle_{RMV} = \langle \iota_A(A) \rangle_{RMV} = R(A). \]

(c) If $z \in ker \alpha$ then $\iota_A(\alpha(z)) = 0$, so $\overline{\pi}(\iota_X(z)) = 0$. It follows that $\iota_X(ker \alpha) \subseteq ker \overline{\pi}$. In fact, we have $\iota_X(ker \alpha) = ker \overline{\pi} \cap \iota_X(\text{Free}_{MV}(X))$. We set $\mathcal{J} = \bigcap \{J \mid J \in \mathcal{J}\}$ and $F = \text{Free}_{RMV}(X)/\mathcal{J}$. By a general result of universal algebra [6, Proposition 7.1], $F$ is isomorphic with a subdirect product of the family $\{\text{Free}_{RMV}(X)/J \mid J \in \mathcal{J}\}$, so $F$ is a subalgebra of a direct product of semisimple MV-algebras. Therefore, $F$ is a semisimple MV-algebra. If we set $M = \{y/\mathcal{J} \mid y \in \iota_X(\text{Free}_{MV}(X))\}$ then $\langle M \rangle_{RMV} = F$, so $R(M) = F$ by Corollary 4.3 and the inclusion $M \subseteq F$ is essential.

It is clear that $\iota_X(ker \alpha) \subseteq \mathcal{J} \subseteq ker \overline{\pi}$. In order to prove that $\mathcal{J} = ker \overline{\pi}$, we assume that there exists an element $z \in ker \overline{\pi} \setminus \mathcal{J}$. Hence $z/\mathcal{J} \neq 0$ in $F$. Since the inclusion $M \subseteq F$ is essential it follows that there exists an element $y \in \iota_X(\text{Free}_{MV}(X))$ such that $0 < y/\mathcal{J} \leq nz/\mathcal{J}$. Note that $y/\mathcal{J} \neq 0$ in $F$. 12
We denote \( w = y \odot (nz)^* \), so \( w \in J \) and \( y \leq (nz) \lor (y) = (nz) \oplus w \). Note that \( w \in J \subseteq \ker \alpha \) and \( z \in \ker \alpha \), so we get \( y \in \ker \alpha \). But \( y \in l_X(Free_{MV}(X)) \), so \( y \in l_X(Free_{MV}(X)) \cap \ker \alpha = l_X(\ker \alpha) \). Since \( l_X(\ker \alpha) \subseteq J \), it follows that \( y/J = 0 \) in \( F \), which is a contradiction.

\[ \]

**Proposition 5.2.** Let \( A \) be a semisimple MV-algebra. For any semisimple Riesz MV-algebra \( V \) and for any MV-homomorphism \( f : A \rightarrow V \) there exists a unique MV-homomorphism \( f_R : R(A) \rightarrow V \) such that \( f_R(\iota_A(a)) = f(a) \), for any \( a \in A \).

\[
\begin{array}{c}
A \xrightarrow{\iota_A} R(A) \xleftarrow{f_R} V \\
\end{array}
\]

**Proof.** Assume \( V \) is a semisimple Riesz MV-algebra and \( f : A \rightarrow V \) is an MV-homomorphism. We consider \( X \subseteq A \) such that \( \langle X \rangle_{MV} = A \) and we define \( \alpha : Free_{MV}(X) \rightarrow A \) and \( \overline{\alpha} : Free_{RMV}(X) \rightarrow R(A) \) as in Remark 5.1. Let \( \overline{f} : Free_{RMV}(X) \rightarrow V \) be the unique MV-homomorphism such that \( \overline{f}(x) = f(x) \) for any \( x \in X \). By Proposition 5.1 (b), we infer that \( Free_{RMV}(X)/\ker \overline{\alpha} \cong R(A) \) and we can safely identify them.

\[
\begin{array}{c}
Free_{MV}(X) \xleftarrow{\overline{\alpha}} R(A) \xrightarrow{f_R} Free_{RMV}(X) \\
A \xleftarrow{\iota_A} R(A) \xrightarrow{f_R} V \\
\end{array}
\]

We note that \( \overline{f}(Free_{RMV}(X)) \) is a Riesz MV-subalgebra of \( V \), so it is semisimple. Therefore, by Proposition 5.1 (c) it follows that \( \ker \overline{\alpha} \subseteq \ker \overline{f} \), so there exists a unique MV-homomorphism \( f_R : R(A) \rightarrow V \) such that \( f_R \circ \overline{\alpha} = \overline{f} \). It follows that

\[
f_R \circ \iota_A \circ \alpha = f_R \circ \overline{\alpha} \circ \iota_X = \overline{f} \circ \iota_X = f \circ \alpha.
\]
Since $\alpha$ is surjective, we get $f_R \circ \iota_A = f$.

In order to prove the uniqueness, assume that $g : R(A) \to V$ is an MV-homomorphism such that $g \circ \iota_A = f$. It follows that $g \circ \iota_X = g \circ \iota_A \circ \alpha = f \circ \alpha = f \circ \iota_X$ and we get $g = f_R$, since they coincide on the generators of $\text{Free}_{RMV}(X)$. We proved that $g$ satisfies the property that uniquely defines $f_R$, so $g = f_R$. 

**Lemma 5.1.** Let $A$ and $B$ be semisimple MV-algebras. For any homomorphism $h : A \to B$, there is a unique homomorphism $R(h) : R(A) \to R(B)$ such that

$$R(h) \circ \iota_A = \iota_B \circ h.$$ 

In addition, if $h$ is an embedding, then $R(h)$ is also an embedding.

**Proof.** We apply Proposition 5.2 for $V = R(B)$ and for $f = \iota_B \circ h$. Therefore $R(h) = f_R$. 

We consider the forgetful functor between the category $\mathcal{RMV}_s$ of semisimple RMV-algebras and the category $\mathcal{MV}_s$ of semisimple MV-algebras:

$$U : \mathcal{RMV}_s \to \mathcal{MV}_s,$$

which forgets the scalar multiplication.

We also define the functor

$$R : \mathcal{MV}_s \to \mathcal{RMV}_s$$

as follows:

- for any semisimple MV-algebra $A$, $R(A)$ is the Riesz hull of $A$.
- for any MV-homomorphism $h : A \to B$, $R(h)$ is the unique homomorphism such that $U(R(h)) \circ \iota_A = \iota_B \circ h$.

**Theorem 5.1.** Under the above settings, $(R, U)$ is an adjoint pair.

**Proof.** One can easily see that $R$ is a functor. If $A$ is a semisimple MV-algebra we define $\eta_A : A \to U(R(A))$, $\eta_A(a) = \iota_A(a)$ for any $a \in A$. If $V$ is a semisimple Riesz MV-algebra, let $\varepsilon_V = \iota_V^{-1} : R(U(V)) \to V$. By Corollary 4.4 $\varepsilon_V$ is an isomorphism.

In order to prove that $R$ is a left adjoint to $U$, we have to prove the following properties, for any MV-algebra $A$ and Riesz MV-algebra $V$:
(1) for any \( f \in \mathcal{M}_s(A, U(V)) \), there exists \( g \in \mathcal{R}\mathcal{M}_s(R(A), V) \) such that \( U(g) \circ \eta_A = f \),

(2) for any \( g \in \mathcal{R}\mathcal{M}_s(R(A), V) \) there exists \( f \in \mathcal{M}_s(A, U(V)) \) such that \( \varepsilon_V \circ R(f) = g \).

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & U(R(A)) \\
\downarrow & & \downarrow U(g) \\
U(V) & \xrightarrow{g} & V \\
\end{array}
\]

\[
\begin{array}{ccc}
R(U(V)) & \xrightarrow{R(f)} & V \\
\downarrow & & \downarrow \varepsilon_V \\
R(A) & \xrightarrow{\varepsilon_V} & V \\
\end{array}
\]

The property (1) follows by Proposition 5.2 with \( g = f_R \) whenever \( f \in \mathcal{M}_s(A, U(V)) \). In order to prove (2), assume that \( g \in \mathcal{R}\mathcal{M}_s(R(A), V) \) and set \( f = U(g) \circ \iota_A \). Hence \( R(f) \) is the unique homomorphism such that

\[ U(R(f)) \circ \iota_A = \iota_{U(V)} \circ f. \]

Therefore we have \( U(R(f)) \circ \iota_A = \iota_{U(V)} \circ U(g) \circ \iota_A \). Since \( \iota_A \) is an embedding, we get that \( U(R(f)) = \iota_{U(V)} \circ U(g) \).

We note that \( \iota_{U(V)} = U(\iota_V) = U(\varepsilon_V^{-1}) \). It follows that \( U(\varepsilon \circ R(f)) = U(g) \), so \( \varepsilon \circ R(f) = g \).

In the following we prove that the hull functor \( R \) and the functor \( \Gamma \) commute.

**Remark 5.2.** Let \( A \) be a semisimple MV-algebra and \((G, u)\) an \( \ell u \)-group such that \( A = \Gamma(G, u) \). If \( X \subseteq A \) and \( \langle X \cup \{u\} \rangle_\ell \) is the \( \ell \)-group generated by \( X \cup \{u\} \) in \( G \), then we note that \( \langle X \cup \{u\} \rangle_\ell \) is an \( \ell u \)-subgroup of \((G, u)\). As a consequence, we infer that \( \Gamma(\langle X \cup \{u\} \rangle_\ell, u) \) is an MV-subalgebra of \( A \). It is now straightforward that \( \langle X \rangle_{MV} = \Gamma(\langle X \cup \{u\} \rangle_\ell, u) \). Assume now that \( V \) is a Riesz MV-algebra, \((H, v)\) a unital Riesz space such that \( \Gamma_R(H, v) = V \) and \( X \subseteq V \). It is straightforward that \( \langle X \rangle_{R_MV} = \Gamma_R(\langle X \cup \{u\} \rangle_{\ell v}, u) \), where \( \langle X \cup \{u\} \rangle_{\ell v} \) is the Riesz space generated by \( X \cup \{u\} \) in \( H \).

**Proposition 5.3.** If \( A \) is a semisimple MV-algebra and \((G, u)\) an \( \ell u \)-group such that \( A = \Gamma(G, u) \), then \( R(A) = \Gamma_R(R(G), u) \).

**Proof.** We recall that \( A_d = \Gamma(G_d, u) \), so \( A_d \subseteq G_d \subseteq \hat{G}_d \). Moreover, \( \hat{G}_d \) is a Riesz space and \( R(G) = \langle G_d \rangle_{\ell v} \), i.e. \( R(G) \) is the Riesz space generated by \( G_d \) in \( \hat{G}_d \). Following Remark 5.2 we have \( R(A) = \langle A_d \rangle_{R_MV} = \Gamma_R(\langle A_d \cup \{u\} \rangle_{\ell v}, u) \). Since \( u \in A_d \) and \( \langle A_d \rangle_{\ell v} = \langle G_d \rangle_{\ell v} \), we get
\[ \mathbf{R}(A) = \Gamma_R((G_d)_{v\ell}, u) = \Gamma_R(\mathbf{R}(G), u). \]

We denote by \( \mathcal{AG}_{ua} \) the category of archimedean \( \ell_u \)-groups and by \( \mathcal{RS}_{ua} \) the category of archimedean Riesz spaces with strong unit. By [3], the correspondence \( G \mapsto \mathbf{R}(G) \), which associates to an \( \ell \)-group its \( v \)-hull, is functorial. If \( G \) has a strong unit \( u \), following Conrad’s construction, one can easily see that \( u \) is also a strong unit of \( \mathbf{R}(G) \). Hence we get a functor \( \mathbf{R} : \mathcal{AG}_{ua} \rightarrow \mathcal{RS}_{ua} \).

**Theorem 5.2.** The following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{AG}_{ua} & \xrightarrow{\Gamma} & \mathcal{MV}_s \\
\downarrow{\mathbf{R}} & & \downarrow{\mathbf{R}} \\
\mathcal{RS}_{ua} & \xrightarrow{\Gamma_R} & \mathcal{RMV}_s
\end{array}
\]

**Proof.** It is a straightforward consequence of Proposition [3].

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