Model Reduction Method based on Rational Canonical form of System Matrix and Krylov Subspace

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Abstract. For single input and single output time-invariant linear system, a new projection method to obtain reduced models is presented by making use of the rational canonical form of system matrix and the Krylov subspace. At first, the system matrix is transformed to its rational canonical form by use of linear transformation. And then both projection method and Krylov subspace method are used to reduce model. The advantage of this method is the poles of the reduced system are same as those of the original. Thus the reduced system remains the stability when the original system is. This method is more effective than simple Krylov subspace method. Numerical examples demonstrate the effectiveness of the method.

1. Introduction
Model reduction is a technique to simplify the simulation of dynamical systems described by differential equations. The idea is to project the original, high-dimensional, state-space onto a properly chosen low-dimensional subspace to arrive at a (much) smaller system having properties similar to the original system. Complex systems can thus be approximated by simpler systems involving fewer equations and unknown variables, which can be solved much more quickly than the original problem. Consider the linear, time-invariant system described by the state–space equations

\[
\begin{align*}
\dot{X} &= AX + Bu \\
Y &= CX 
\end{align*}
\]  

(1)

where \(X(t) \in \mathbb{R}^n\), \(A\), \(B\) and \(C\) are real matrices of appropriate sizes. \(u(t)\) and \(Y(t)\) are appropriate real function matrices. and The system (1) will be expressed as \(\{A, B, C\}\) in short. Let \(X_0 = Z^TX\) and \(Z^TV = E_m\), where \(Z\) and \(V\) are all \(n \times m\) matrices, called projection matrices, \(E_m\) is \(m \times m\) unit matrix. If for \(Z\) and \(V\), there exist an \(m \times m\) real matrix \(A_0\), an \(m \times n_1\) real matrix \(B_0\) and an \(n_2 \times m\) real matrix \(C_0\) such that

\[
\begin{align*}
\dot{X}_0 &= A_0X_0 + B_0u \\
Y_0 &= C_0X_0 
\end{align*}
\]  

(2)

with \(A_0 = Z^TAV\), \(B_0 = Z^TB\), \(C_0 = CV\), then (2) is a reduced model of (1).

The model reduction methods based on projection matrix got a wide range of applications in the last decade. Those gradually resolve the problems of numerical stability, matching accuracy, and maintaining RLC network passivity. These methods essentially make use of the greater redundancy in observability and controllability of the system.
This paper focuses on linear time-invariant state-space models with only single input and single output system in state space form

\[
\begin{align*}
\dot{X} &= AX + bu \\
Y &= cX
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n}, \ X(t), b, c \in \mathbb{R}^n \) and \( u(t), Y(t) \in \mathbb{R} \).

There have been several kinds of model reduction methods. They can be classified into three categories according to [1-5]:

(a) Singular value decomposition (SVD) based methods,
(b) Krylov (moment matching) based methods, and
(c) SVD-Krylov based methods.

In recent years, Krylov subspace methods have become popular tools for computing reduced models of high order linear time-invariant systems ([4-9]). The reduction can be done by applying a projection from high order to lower order space using the bases of some subspaces called input and output Krylov subspaces.

A Krylov subspace is a subspace spanned by a sequence of vectors generated by a given matrix and a vector as follows. Given a matrix \( A \) and a starting vector \( b \), the kth Krylov subspace \( \kappa_k(A,b) \) is spanned by a sequence of \( k \) column vectors:

\[
\kappa_k(A,b) = \text{span}\{b, Ab, A^2b, \ldots, A^{k-1}b\}
\]

where \( A \) is a constant \( n \times n \)-matrix, \( b \) is a constant \( n \times 1 \)-vector (the so-called starting vector) and \( k \) is some given positive integer. The vectors \( b, Ab, A^2b, \ldots, A^{k-1}b \) constructing the subspace are called basic vectors.

In this paper, a new method of model reduction for SISO large-scale dynamical systems is put forward. It is a kind of projection method, whose projection matrix depends on the rational canonical form of system matrix and the Krylov subspace. This method is more effective than simple Krylov subspace method.

For the use of the rational canonical form of square matrix to model reduction, we will explain it as the following theorem.

**Theorem** For any \( n \times n \) matrix \( A \) over a field \( F \), there exists a nonsingular matrix \( P \) over the field \( F \) such that \( A \) is similar to the unique following quasi-diagonal matrix

\[
PAP^{-1} = \begin{pmatrix}
A_1 & A_2 & \cdots & A_r \\
& A_1 & A_2 & \cdots \\
& & A_1 & A_2 & \cdots \\
& & & A_1
\end{pmatrix},
\]

where \( A_i \ (i = 1, 2, \ldots, r) \) are companion matrices of the polynomials \( d_i(\lambda) \) correspondingly, and satisfying \( d_1(\lambda)|\cdots|d_r(\lambda)|d_1(\lambda) \), in which the sum of the degrees of these polynomials \( d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda) \) is \( n \).

It is important to note that these \( A_i \ (i = 1, 2, \ldots, r) \) in the quasi diagonal matrix are of reverse order compared with conventional rational canonical form, which is still called a rational canonical form for convenience.

2. Model reduction based on rational canonical form of system matrix and Krylov subspace

For the model (3), doing nonsingular linear transformation \( \bar{X} = PX \), it is transformed to the following
\[
\begin{align*}
\dot{X} &= \bar{A}X + \bar{b}u \\
Y &= \bar{c}X
\end{align*}
\]

where

\[
\bar{A} = PAP^{-1} = \begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_j
\end{pmatrix}, \quad \bar{b} = Pb \quad \text{and} \quad \bar{c} = cP^{-1}.
\]

Next we do order reduction to the above model.

Let \(m \times n\) matrix \(Z = V\), \(Z^T = (E \ O \ \cdots \ O)\) satisfying \(Z^TV = (E \ O \ \cdots \ O)\) \(= E\) and

\[
Z^T\bar{A}V = (E \ O \ \cdots \ O)\begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_j
\end{pmatrix} = A_j.
\]

Let \(X_i = Z^TX\), \(b_i = Z^Tb\) and \(c_i = \bar{c}V\), where \(Z\) and \(V\) \((V=Z)\) are two \(m \times n\) matrices, called aggregation matrices. So the model (4) is transformed to the following reduced model

\[
\begin{align*}
\dot{X}_1 &= A_1X_1 + b_1u \\
Y_1 &= c_1X_1
\end{align*}
\]

Since the minimal polynomial of \(A_i\) is \(d_i(\lambda)\), which is also the minimal polynomial of the original system matrix \(A\). It means that all the poles of the original system (3) are included in the system (5). That is, the reduced model will remain to be stable if the original one is.

Now we go on to the further model reduction to the model (5) by the use of Krylov subspace.

For linear system (5), the probability that it is controllable is \(1^{\text{10}}\), which is means that the controllability matrix \(Q_c = \{b\ A b \cdots A^{m-2} b \ A^{m-1} b\}\) is nonsingular.

For any positive integer \(k\), \((1 < k \leq m - 1)\), column vectors of the matrix \(b \ A b \cdots A^{k-2} b \ A^{k-1} b\) are linearly independent. These vectors can be transformed into a set of normal orthogonal vector sets and then compose to matrix \(V\) and matrix \(Z = V\) satisfying \(Z^TV = E\).

Let \(\bar{A}_1 = Z_1^TA_1V_1\), \(\bar{b}_1 = Z_1^Tb_1\) and \(\bar{c}_1 = c_1V_1\), the projection matrices are \(Z_1\) and \(V_1\), then we obtain an order reduced model as (6).

\[
\{\bar{A}_1, \bar{b}_1, \bar{c}_1\}.
\]

### 3. Model Simulations

Suppose a state-space model is

\[
A = \begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 & 2 & -1 & 3 \end{pmatrix}^T, \quad c = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad
\]
where \( A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -3 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 \end{bmatrix} \).

Let \( Z = V = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \), we get \( A_1 = Z^T A V, \ b_1 = Z^T b, \ c_1 = c V \), then the reduced model is

\[
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}, \ b_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \ c_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \tag{8}
\]

Now we reduce the model (8) to order 2 model, the column vectors in matrix \( \begin{bmatrix} b_1 & A b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \\ -3 \end{bmatrix} \) can be transformed into a set of orthogonal vector sets and then compose to matrix \( V_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1/3 \\ 1/6 \end{bmatrix} \), let

\[
Z_1 = \frac{1}{5} \begin{bmatrix} 0 & -5 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}, \text{ which is satisfying } Z_1^T V_1 = E. \text{ We get the following reduced model }
\]

\[
\overline{A}_1 = Z_1^T A V = \begin{bmatrix} -1.6 & -0.3 \\ 1.2 & -0.4 \end{bmatrix}, \overline{b}_1 = Z_1^T b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \overline{c}_1 = c_1 V = \begin{bmatrix} 2 & -2/3 \end{bmatrix}, \tag{9}
\]

If we reduce the model (7) to order 2 model by the use of Krylov subspace directly, we get model (10).

\[
A_{k2} = \begin{bmatrix} -2.0534 & 0.9124 \\ -0.2903 & -1.0537 \end{bmatrix}, \ b_{k2} = \begin{bmatrix} -3.8026 \\ 0.7348 \end{bmatrix}, \ c_{k2} = \begin{bmatrix} -0.1422 & 0.6249 \end{bmatrix}, \tag{10}
\]

**Figure 1.** The unit response waveforms of three systems
The unit response waveforms of the original system (7) and of the reduction system models (8), (9) and (10) show as above. As can be seen from the figure, the graphics of reduction model (8) and of the original system (7) are entirely overlapping (the middle line shown in the Fig). The graphic of the reduction model (9) (the below line shown in the Fig) and of the original system (7) have error in a small range. The graphic of the reduction model (10) (the above line shown in the Fig) and of the original system (7) have error in a certain range.

4. Conclusions
This paper presents a new reduction models for SISO linear system by using the rational canonical form of system matrix and the Krylov subspace. Firstly, the system matrix is transformed to its rational canonical form by use of linear transformation. And then both projection method and Krylov subspace method are used to reduce model. The advantage of this method is the poles of the reduced system are same as those of the original. Thus the reduced system remains the stability when the original system is. This method is more effective than simple Krylov subspace method. Simulation results are show to verify the validity and feasibility of the methods.

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