A SHARP SCATTERING THRESHOLD LEVEL FOR
MASS-SUBCRITICAL NONLINEAR SCHRODINGER SYSTEM

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Abstract. In this paper, we consider the quadratic nonlinear Schrodinger system in three space dimensions. Our aim is to obtain sharp scattering criteria. Because of the mass-subcritical nature, it is difficult to do so in terms of conserved quantities. The corresponding single equation is studied by the second author and a sharp scattering criterion is established by introducing a distance from a trivial scattering solution, the zero solution. By the structure of the nonlinearity we are dealing with, the system admits a scattering solution which is a pair of the zero function and a linear Schrodinger flow. Taking this fact into account, we introduce a new optimizing quantity and give a sharp scattering criterion in terms of it.

1. Introduction. This paper is devoted to the study of the following quadratic Schrodinger system in three space dimensions:
\[
\begin{cases}
  i\partial_t u + \Delta u = -2v\bar{u} & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
  i\partial_t v + \frac{1}{2}\Delta v = -u^2 & (t, x) \in \mathbb{R} \times \mathbb{R}^3,
\end{cases}
\tag{NLS}
\]
where \( i = \sqrt{-1} \), \( u, v : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C} \) are unknown functions, \( \Delta = \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} \), and \( \bar{u} \) is the complex conjugate of \( u \). At least formally, the system \( \text{(NLS)} \) has the following two conserved quantities: One is mass
\[
M[u, v] := \int_{\mathbb{R}^3} (|u(x)|^2 + 2|v(x)|^2) dx
\]
and the other is energy
\[
E[u, v] := \int_{\mathbb{R}^3} \left( |\nabla u(x)|^2 + \frac{1}{2} |\nabla v(x)|^2 - 2 \text{Re}(u(x)^2\bar{v}(x)) \right) dx.
\]

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Our aim here is to obtain sharp scattering criteria for the system.

The equation (NLS) is a special case of the system

\[
\begin{aligned}
&i\partial_t u + \frac{1}{2m} \Delta u = \lambda v u, \\
&i\partial_t v + \frac{1}{2M} \Delta v = \mu u^2,
\end{aligned}
\]  

(1.1)

where \( m, M \) are positive constants and \( \lambda, \mu \) are real-valued constants. The system (1.1) describes the Raman amplification in a plasma (see [3] for more detail). When a so-called \( \) mass resonance \( \) condition \( M = 2m \) is met, it is also regarded as a non-relativistic limit of the nonlinear Klein-Gordon system

\[
\begin{aligned}
&\frac{1}{2c^2} \partial^2_t u - \frac{1}{2m} \Delta u + \frac{mc^2}{2} u = -\lambda v u, \\
&\frac{1}{2c^2} \partial^2_t v - \frac{1}{2M} \Delta v + \frac{Mc^2}{2} v = -\mu u^2
\end{aligned}
\]  

(see [10]). We refer the reader to [10] for the local well-posedness results of (NLS) in the mass space \( L^2(\mathbb{R}^3) \) and in the energy space \( H^1(\mathbb{R}^3) \) and the existence of ground states.

In this paper, we study (NLS) with mass resonance condition in a homogeneous weighted space. For \( s \in [0, \frac{3}{2}) \), we define the homogeneous weighted space \( \dot{F}^{s}_{H,1/2} \subset F^{s} \) \( \dot{H} \) by the norm

\[
\|f\|_{\dot{F}^{s}_{H,1/2}} := \|Ff\|_{\dot{F}^{s} \cap L^2(\mathbb{R}^3)} = \|x^s f\|_{L^2(\mathbb{R}^3)}.
\]

Here, \( F \) denotes the Fourier transform on \( \mathbb{R}^3 \), that is,

\[
Ff(\xi) := \hat{f}(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx.
\]

We consider the system (NLS) with the initial condition

\[
(u(0), v(0)) = (u_0, v_0) \in \dot{F}^{s}_{H,1/2} \times \dot{F}^{s}_{H,1/2}.
\]

(1.2)

The homogeneous weighted space is one of scaling critical spaces. The system (NLS) has the following scaling structure. If \( (u, v) \) is a solution (in a suitable sense) to (NLS) then the pair

\[
(u_{\lambda}(t, x), v_{\lambda}(t, x)) := (\lambda^2 u(\lambda^2 t, \lambda x), \lambda^2 v(\lambda^2 t, \lambda x))
\]

is also a solution for any \( \lambda > 0 \). Corresponding transform of initial data is as follows:

\[
(\phi(x), \psi(x)) \mapsto (\phi_{\lambda}(x), \psi_{\lambda}(x)) := (\lambda^2 \phi(\lambda x), \lambda^2 \psi(\lambda x))
\]

(1.3)

for \( \lambda > 0 \). The homogeneous space \( \dot{F}^{s}_{H,1/2} \times \dot{F}^{s}_{H,1/2} \) is scaling critical space in the sense that the norm of the initial data is invariant under the scaling (1.3). Notice that the scaling leaves the homogeneous Sobolev space \( H^s \) \( H^s \)-norm of the initial data unchanged if and only if \( s = -1/2 < 0 \). The critical regularity is negative and so our problem is mass-subcritical. The system is studied in the inhomogeneous weighted space \( \dot{F}^{1/2} \) \( \dot{F}^{1/2} \) and a small-data-scattering-type result of (NLS) is given in [9]. Large data scattering for the corresponding single equation is studied in the homogeneous weighted space in [20] (see also [6, 24] and references therein), Fourier-Lebesgue/Fourier-Morrey space in [22], and homogeneous Sobolev space under radial symmetry in [17]. Remark that the radial symmetry is crucial when
we work with the homogeneous Sobolev space with negative regularity since some ill-posedness results are known \cite{2}. In this paper, we choose the homogeneous Sobolev space since we want to work with a scaling critical space without radial symmetry.

Let us make the notion of the solution clear. We need a slight modification of the notion compared with $L^2$ or $H^1$ solutions because the Schrödinger flow is not unitary in the homogeneous weighted space $F\dot{H}^{\frac{1}{2}}$.

**Definition 1.1** (Solution). Let $I \subset \mathbb{R}$ be a nonempty time interval. We say that a pair of functions $(u,v) : I \times \mathbb{R}^3 \to \mathbb{C}^2$ is a solution to (NLS) on $I$ if $(e^{-it\Delta}u(t), e^{-\frac{i}{2}it\Delta}v(t)) \in (C(I; F\dot{H}^{\frac{1}{2}}))^2$ and the Duhamel formula

$$
\begin{align*}
\begin{cases}
e^{-it\Delta}u(t) = e^{-it\Delta}u(\tau) + 2i \int_\tau^t e^{-is\Delta}(u(\tau))(s)ds, \\
e^{-\frac{i}{2}it\Delta}v(t) = e^{-\frac{i}{2}it\Delta}v(\tau) + i \int_\tau^t e^{-\frac{i}{2}is\Delta}(v(\tau))(s)ds
\end{cases}
\end{align*}
$$

holds in $F\dot{H}^{\frac{1}{2}}$ for any $t, \tau \in I$, where $e^{it\Delta} := F^{-1}e^{-it|\xi|^2}F$ is the Schrödinger group. We express the maximal interval of existence of $(u,v)$ by $I_{\text{max}} = (T_{\text{min}}, T_{\text{max}})$. We say $(u,v)$ is forward time-global (resp. backward time-global) if $T_{\text{max}} = \infty$ (resp. if $T_{\text{min}} = -\infty$).

This definition of solutions is not time-translation invariant. That is, if $(u,v)$ is a solution to (NLS), then $(u(\cdot + \tau), v(\cdot + \tau))$ is not necessarily a solution for $\tau \in \mathbb{R}$.

To state the local well-posedness for (NLS), we introduce the function spaces $X_{m,r}^s(t)$, $W_1$, and $W_2$ defined by norms

$$
\|f\|_{X_{m,r}^s(t)} := \left\| \left( -\frac{t^2}{m^2} \Delta \right)^\frac{s}{2} e^{-\frac{im|\xi|^2}{2t}}f \right\|_{L_t^\infty}, \quad \|f\|_{W_j} := \|f\|_{L_t^{2j}X_{j_r}^\|t\|_{j_r}}.
$$

For a space $X_{m,r}^s(t)$, we omit the second exponent when $r = 2$, that is, $X_{m}^s(t) = X_{m,2}^s(t)$. We discuss these function spaces in more detail in Subsection 2.3 and Subsection 2.5, below. The following is our result on the local well-posedness. A more detailed version is given later as Theorem 3.2.

**Theorem 1.2** (Local well-posedness). For any initial time $t_0 \in \mathbb{R}$ and data $(u_0, v_0) \in X_{1/2}^{1/2}(t_0) \times \dot{X}_{1}^{1/2}(t_0)$ there exist an open interval $I \ni t_0$ and a unique solution $(u,v) \in (C_t(I; X_{1/2}^{1/2}) \cap W_1(I)) \times (C_t(I; \dot{X}_{1}^{1/2}) \cap W_2(I))$ to (NLS) with the initial condition $(u(t_0), v(t_0)) = (u_0, v_0)$. Moreover, the solution depends continuously on the initial data.

Now, we turn to the large time behavior of solutions to (NLS). In this paper, we are interested in scattering solutions defined as follows:

**Definition 1.3.** We say that a solution $(u,v)$ scatters if $(u,v)$ is forward time-global and there exists $(u_+, v_+) \in F\dot{H}^{\frac{1}{2}} \times F\dot{H}^{\frac{1}{2}}$ such that

$$
\lim_{t \to -\infty} \| (e^{-it\Delta}u(t), e^{-\frac{i}{2}it\Delta}v(t)) - (u_+, v_+) \|_{F\dot{H}^{\frac{1}{2}} \times F\dot{H}^{\frac{1}{2}}} = 0. \quad (1.4)
$$

We can consider scattering also backward in time. However, we restrict ourselves to the positive time direction in the sequel.

One has several equivalent characterizations of the scattering. For example, the scattering is equivalent to the boundedness of $\|u\|_{W_1((\tau,T_{\text{max}}))}$ or of $\|v\|_{W_2((\tau,T_{\text{max}}))}$ for some $\tau \in I_{\text{max}}$. See Proposition 3.3 for the details.
1.1. Criterion for scattering by conserved quantities. We are interested in obtaining a sharp condition for scattering. This subject is recently extensively studied based on a concentration compactness/rigidity type argument after Kenig and Merle [15]. As for the Schrödinger system (NLS), the first author treated five dimensions [8] and Inui, Kishimoto, and Nishimura treated four dimensions [12]. In these results, a sharp condition for scattering is given in terms of conserved quantities. For example, in the four dimensions, the equation is mass-critical and we have the following simple criterion in terms of the mass: a solution scatters for both time direction if the mass of a solution is smaller than that of the ground state solution. This is a natural extension of the single equation case [5, 18, 19]. In five dimensions, a condition for blowup is also studied.

Remark that if a solution \((u, v)\) scatters then any scaled solution \((u[\lambda], v[\lambda])\) scatters also. Consequently, any criterion for scattering is scaling invariant.\(^1\) Hence, if we look for a criterion given in terms of some characteristic quantity of a solution, it is natural that the quantity is scaling invariant. Recall that mass \(M\) is scaling invariant in four dimensions, and the product of two quantities \(ME\) is a scaling invariant in five dimensions. In the previous results [8, 12], these quantities play a crucial role in the criteria there.

However, in the three dimensional case, it would be difficult to give a criterion in terms of the mass and the energy. They are not scaling invariant:
\[
M[\phi_{(\lambda)}], \psi_{(\lambda)}] = \lambda M[\phi, \psi], \quad E[\phi_{(\lambda)}, \psi_{(\lambda)}] = \lambda^3 E[\phi, \psi]
\]
for \(\lambda > 0\). Furthermore, the right-hand sides both have a positive power of \(\lambda\). It means that one can make the both magnitudes of \(M\) and of \(E\) small or large at the same time by scaling. This is a feature of the mass-subcritical case. Thus, we may not have a criterion similar to those in four or five dimensional cases, as one may not construct a scaling invariant quantity by a combination or product of (positive powers of) these quantities.\(^2\)

Hence, in the sequel, we look for a criterion which is not given in terms of the conserved quantities, as in [20, 21, 22, 23].

1.2. Trivial scattering set and minimization of non-scattering solutions. It can be said that the main purpose of this paper is to investigate threshold phenomena between scattering solutions near a prescribed “trivial scattering set” and non-scattering solutions, taking a system nature into account.

For comparison, let us recall the single NLS equation with the gauge invariant quadratic nonlinearity:
\[
i\partial_t w + \Delta w = |w|w, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3
\]
(1.5)
with \(w(0) = w_0 \in \mathcal{F}H^{\frac{1}{2}}\). Let \(\mathcal{S}_{+, \text{single}}\) be the set of initial data for which the corresponding solution scatters. Since the zero solution is a scattering solution, we

\(^1\)Suppose that we have a criterion “if \((u, v)\) satisfies a condition \(P\) then \((u, v)\) scatters.” Then, it actually reads as “if there exists \(\lambda > 0\) such that \((u[\lambda], v[\lambda])\) satisfies a condition \(P\) then \((u, v)\) scatters.” The latter criteria is scaling invariant in such a sense that the validity of its assumption is left invariant under the scaling.

\(^2\)Remark that if we allow negative powers, we have scaling invariant quantities such as \(E/M^3\). Working with this quantity corresponds to looking at the energy of \(H^1 \times H^2\)-solutions under mass constraint: For any nontrivial solution \((u, v)\), there exists a unique normalized solution \((u_{(\lambda_0)}, v_{(\lambda_0)})\) which satisfies \(M[u_{(\lambda_0)}, v_{(\lambda_0)}] = 1\). Indeed, \(\lambda_0 = M[u, v]^{-1}\) is the choice. Then, \(E[u_{(\lambda_0)}, v_{(\lambda_0)}] = E[u, v]/M[u, v]^3\).
have

\[ \{0\} \subset S_{+ \text{single}}. \]

Then, it is natural to define a “size” of a solution by the distance from the zero solution. This fact leads us to a study of a quantity like

\[ \inf \left\{ \| w_0 \|_{F^{1/2}} : w_0 \notin S_{+ \text{single}} \right\}. \quad (1.6) \]

In [20, 21], it is known that this infimum value is strictly smaller than the size of the ground state solution for (1.5) and further that there exists a minimizer to this quantity.

Let us go back to the system case. We define \( S_+ \) as the set of initial data \((u_0, v_0) \in F^{1/2} \times F^{1/2}\) for which the corresponding solution scatters. A straightforward generalization of the quantity (1.6) is

\[ \inf \left\{ (\| u_0 \|_{F^{1/2}}^2 + \alpha \| v_0 \|_{F^{1/2}}^2)^{1/2} : (u_0, v_0) \notin S_+ \right\} \quad (1.7) \]

with some constant \( \alpha > 0 \). However, there may not be a strong motivation to study the distance from the trivial solution \((0, 0)\) other than the similarity to that in the single equation case, (1.6). We want to find a different way of sizing which is connected with a system nature. To this end, we look at the fact that not only the zero solution \((0, 0)\) but also all solutions of the form \((0, e^{it\Delta} v_0)\) can be also regarded as a trivial scattering solution for arbitrary \( v_0 \in F^{1/2} \). Taking this fact into account, one natural choice of the scattering threshold would be with respect to the distance of a data from the set \(\{0\} \times F^{1/2}\). This choice leads us to consider the following optimization problem:

\[ \ell_{v_0} := \inf \{ \| u_0 \|_{F^{1/2}} : (u_0, v_0) \notin S_+ \} \in (0, \infty]. \quad (1.8) \]

By using a stability type argument, we will show that \( \ell_{v_0} > 0 \) for any \( v_0 \in F^{1/2} \) (see, Proposition 3.4). The following criterion is obvious by the definition of \( \ell_{v_0} \).

**Proposition 1.4** (Sharp small data scattering). Let \((u_0, v_0) \in F^{1/2} \times F^{1/2}\) and let \((u, v) : I_{\text{max}} \times \mathbb{R}^3 \to C^2\) be a solution to (NLS) with the initial condition (1.2). If \( \| u_0 \|_{F^{1/2}} < \ell_{v_0} \) then \((u, v)\) scatters.

The above criterion “\( \| u_0 \|_{F^{1/2}} < \ell_{v_0} \)” is sharp in such a sense that the number \( \ell_{v_0} \) may not be replaced with any larger number. The questions which we address in this paper are the following two: (a) to obtain a condition which implies \( \ell_{v_0} \) is finite; (b) to show the existence of a minimizer to \( \ell_{v_0} \) (when \( \ell_{v_0} \) is finite).

### 1.3 Main Results

It will turn out that the following quantity \( \ell_{v_0}^\dagger \) plays an important role in the analysis of \( \ell_{v_0} \).

**Definition 1.5.** For \( v_0 \in F^{1/2} \) and \( 0 \leq \ell < \infty \), we let

\[ L_{v_0}(\ell) := \sup \left\{ \|(u, v)\|_{W_j([0,T_{\text{max}}]) \times W_j([0,T_{\text{max}}])} : (u, v) \text{ is a solution to (NLS) on } [0, T_{\text{max}}], \right. \]

\[ \left. v(0) = v_0, \; \|u(0)\|_{F^{1/2}} \leq \ell, \; u(0) \in F^{1/2} \right\}, \]

where \( W_j([0,T_{\text{max}}]) \) (\( j = 1, 2 \)) is a Strichartz-like function space defined in Subsection 2.5, below. Further, define

\[ \ell_{v_0}^\dagger := \sup \{ \ell : L_{v_0}(\ell) < \infty \} \in (0, \infty]. \]
We have \( \ell_v^\dagger \leq \ell_v \) by their definitions (see Lemma 4.4 for more detail). Intuitively, this can be seen by noticing that if \( \| u_0 \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} < \ell_v^\dagger \) then not only \((u,v)\) scatters but also we have a priori bound \( \|(u,v)\|_{W_1((0,\infty)) \times W_2((0,\infty))} \leq L_v \| u_0 \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} < \infty \). As for the single-equation (1.5), it is known that these two kinds of quantities coincide each other (see [22]).

Our first result is as follows.

**Theorem 1.6.** \( \ell_v^\dagger = \min(\ell_0, \ell_v) \) is true for any \( v_0 \in \mathcal{F} \dot{H}^{\frac{1}{2}} \), including the case where the both sides are infinite. In particular, \( \ell_v^\dagger = \ell_0 \) holds.

It is worth noting that \( \ell_v^\dagger = \infty \) guarantees \( \ell_v = \infty \) but the inverse is not necessarily true. Our interest in the sequel is to see what happens when \( \ell_v^\dagger < \infty \).

In the case \( v_0 = 0 \), we have \( \ell_0^\dagger = \ell_0 \), including the case both are infinite, as seen in Theorem 1.6.

**Theorem 1.7.** If \( \ell_0^\dagger < \infty \) then there exists a minimizer \((u^{(0)}(t),v^{(0)}(t))\) to \( \ell_0(= \ell_0^\dagger) \) such that

1. \( v^{(0)}(0) = 0 \) and \( \| u^{(0)}(0) \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} = \ell_0; \)
2. \((u^{(0)}(t),v^{(0)}(t))\) does not scatter.

So far, we do not know whether \( \ell_0^\dagger < \infty \) or not. It will turn out that this question is important to understand the attainability of \( \ell_v \) for all non-zero \( v_0 \). One quick consequence of \( \ell_0^\dagger = \infty \) is \( \ell_v = \ell_v^\dagger \) for all \( v_0 \), which follows from Theorem 1.6. We will resume this subject later.

Let us move on to the case \( v_0 \neq 0 \). Suppose \( \ell_v^\dagger < \infty \). Then, we have either

\[
\ell_v^\dagger = \ell_v \quad \text{or} \quad \ell_v^\dagger < \ell_v.
\]  

(1.9)

The following Theorem 1.8 is about the first case and Theorem 1.9 is about the second case.

**Theorem 1.8.** Fix \( v_0 \in \mathcal{F} \dot{H}^{\frac{1}{2}} \setminus \{0\} \). Suppose that \( \ell_v^\dagger = \ell_v < \ell_0 \). Then, there exists a minimizer \((u^{(v_0)}(t),v^{(v_0)}(t))\) to \( \ell_v \).

The case \( \ell_v^\dagger = \ell_v = \ell_0 < \infty \) is excluded in the above theorem. We consider this exceptional case in Remark 1.10, below.

Let us consider the second case of (1.9). In this case, the following strange thing occurs: Take \( u_0 \in \mathcal{F} \dot{H}^{\frac{1}{2}} \) with \( \| u_0 \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} = \ell_v^\dagger \) and consider the corresponding solution \((u(t),v(t))\) with the data \((u_0,v_0)\). Then, on one hand, the solution \((u(t),v(t))\) scatters for any choice of such \( u_0 \) since \( \| u_0 \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} \leq \ell_v \). However, on the other hand, for arbitrarily large number \( N > 0 \), one can choose \( u_0 \in \mathcal{F} \dot{H}^{\frac{1}{2}} \) so that the corresponding solution \((u(t),v(t))\) satisfies

\[
\|(u,v)\|_{W_1((0,\infty)) \times W_2((0,\infty))} \geq N.
\]

The next theorem tells us how this is “attained”. Notice that the second case of (1.9) occurs only when \( \ell_0 = \ell_v^\dagger < \infty \), thanks to Theorem 1.6. Consequently, there is a minimizer to \( \ell_0 \) in this case, by means of Theorem 1.7.

**Theorem 1.9.** Fix \( v_0 \in \mathcal{F} \dot{H}^{\frac{1}{2}} \setminus \{0\} \). Suppose that \( \ell_v^\dagger < \ell_v \). Pick a sequence \( \{u_{n,v_0}\}_n \subset \mathcal{F} \dot{H}^{\frac{1}{2}} \) satisfying \( \| u_{n,v_0} \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} < \ell_v^\dagger \) for all \( n \geq 1 \),

\[
\lim_{n \to \infty} \| u_{n,v_0} \|_{\mathcal{F} \dot{H}^{\frac{1}{2}}} = \ell_v^\dagger,
\]
and
\[ \lim_{n \to \infty} \| (u_n, v_n) \|_{W_1((0, \infty)) \times W_2((0, \infty))} = \infty, \]
where \((u_n(t), v_n(t))\) is a solution with the initial data \((u_n(0), v_n(0)) = (u_0, v_0)\). Then, there exist a subsequence of \(n\), a minimizer \((u^{(0)}, v^{(0)})\) to \(\ell_0\), and two sequences \(\{\xi_n\}_n \subset \mathbb{R}^3\) and \(\{h_n\}_n \subset 2^\mathbb{Z}\) such that
\[ |\log h_n| + |\xi_n| \to \infty \]
and
\[ e^{-i x \cdot h_n^\xi_n} (u_{0,n})_{\{h_n\}} \to u^{(0)}(0) \]
in \(\mathcal{F}H^{\frac{1}{2}}\)
hold along the subsequence. In particular, along the same subsequence, it holds for any \(\tau \in (0, T_{\text{max}}(u^{(0)}, v^{(0)}))\) that
\[
(u_n(t), v_n(t)) = \left( e^{-it|\xi_n|^2 + i x \cdot \xi_n} u^{(0)}_{|h_n|} \right)(t, x - 2\xi_n t), e^{-2it|\xi_n|^2 + 2ix \cdot \xi_n} v^{(0)}_{|\xi_n|^2} \right)(t, x - 2\xi_n t)
+ (0, e^{\frac{1}{2}tH} v_0) + o_{X^{1/2}}(1)
\]
for \(0 \leq t \leq \tau h_n^2\).

**Remark 1.10.** The special case \(\ell_{v_0} = \ell_0 = \ell_{v_0} < \infty\) \((v_0 \neq 0)\) is not included in the above two theorems. In this exceptional case, the conclusion of Theorem 1.8 or Theorem 1.9 holds. Namely, if there does not exist a minimizer to \(\ell_{v_0}\) as in Theorem 1.8, then the conclusion of Theorem 1.9 is true.

Let us summarize the above results. Let \(v_0 \in \mathcal{F}H^{\frac{1}{2}}\) be a given function. If \(\ell^1_{v_0} = \infty\) then we have \(\ell_{v_0} = \ell_{v_0} < \infty\) \((v_0 \neq 0)\) is not included in the above two theorems. In this exceptional case, the conclusion of Theorem 1.8 or Theorem 1.9 holds. Namely, if there does not exist a minimizer to \(\ell_{v_0}\) as in Theorem 1.8, then the conclusion of Theorem 1.9 is true.

**Question 1.** In \((\text{NLS})\), does \(v_0 = 0\) implies scattering of the corresponding solution for any \(u_0\)?

If it were true, that is, if \(\ell_0 = \ell^1_{v_0} = \infty\) then Theorem 1.6 tells us that \(\ell^1_{v_0} = \ell_{v_0}\) is true for any \(v_0\), as mentioned above.

Although we do not know the exact value of \(\ell_{v_0}\), we are able to have a condition which implies the finiteness of \(\ell_{v_0}\) and to give an upper bound for \(\ell_{v_0}\). A simple one is a condition in terms of the energy.

**Theorem 1.11.** Fix nontrivial \(u_0, v_0 \in \mathcal{F}H^{\frac{1}{2}} \cap H^1\). If \(E[u_0, v_0] \leq 0\) then the corresponding solution \((u, v)\) does not scatter. In particular, \(\ell_{v_0} \leq \|u_0\|_{\mathcal{F}H^{\frac{1}{2}}}\).

As a consequence, one sees that a standing wave solution, not only the ground state but also all excited states, is not a minimizer of the optimizing problem (1.8). Let \(\omega > 0\) and let \((Q_{1, \omega}, Q_{2, \omega})\) be a solution to the elliptic equation
\[ -\Delta Q_{1, \omega} + \omega Q_{1, \omega} = 2Q_{1, \omega}Q_{2, \omega}, \quad -\frac{1}{2}\Delta Q_{2, \omega} + 2\omega Q_{2, \omega} = Q_{1, \omega}^2. \]
(1.10)
Then, \((e^{i\omega t}Q_{1,\omega}(x), e^{2i\omega t}Q_{2,\omega}(x))\) is a standing wave solution to (\(\text{NLS}\))\). We have \(E[Q_{1,\omega}, Q_{2,\omega}] < 0\) (see [10, Theorem 4.1]). As a result of Theorem 1.11, there exists an open neighborhood \(\mathcal{N} \subset \mathbb{R}^2\) of \((1, 1)\) such that

\[
(c_1Q_{1,\omega}(x), c_2Q_{2,\omega}(x)) \notin \mathcal{S}_+
\]

for all \((c_1, c_2) \in \mathcal{N}\). Hence, any solution to (1.10) is not an optimizer to (1.8). In particular, \(\ell_{Q_{2,\omega}}\) is strictly smaller than \(\|Q_{1,\omega}\|_{F_{H}^{1/2}}\). Similarly, \((Q_{1,\omega}(x), Q_{2,\omega}(x))\) is not a solution to (1.7) for any \(\alpha > 0\). We can also find intuitively the fact from the orbital stability of a standing wave \((e^{i\omega t}Q_{1,\omega}(x), e^{2i\omega t}Q_{2,\omega}(x))\) in \(H^1 \times H^1\) (see [4] for more detail).

In our context, we want to find a condition which is stated in terms of \(v_0\) only. We give two criteria in this direction. The first one is for large data case:

**Corollary 1.12.** For any \(v_0 \in F_{H}^{1/2} \cap H^1\) with \(v_0 \neq 0\), there exists \(c_0 > 0\) such that the estimate \(\ell_{c_0} \leq c_0 \|v_0\|_{L^2}^{1/2}\) holds for any \(c \geq c_0\).

The second one is criterion for a specific \(v_0\):

**Corollary 1.13.** Pick \(v_0 \in F_{H}^{1/2} \cap H^1\). If there exists \(\theta \in \mathbb{R}\) such that a Schrödinger operator \(\Delta - 2\text{Re}(e^{i\theta}v_0)\) has a negative eigenvalue then \(\ell_{v_0} < \infty\). Moreover, if \(\varphi \in F_{H}^{1/2} \cap H^1\) is a real-valued eigenfunction associated with a negative eigenvalue \(\tilde{e} < 0\) of \(\Delta - 2\text{Re}(e^{i\theta}v_0)\) then the estimate

\[
\ell_{v_0} \leq \frac{\|\varphi\|_{F_{H}^{1/2}}}{\sqrt{2|\tilde{e}|\|\varphi\|_{L^2}}} \|\nabla v_0\|_{L^2}
\]

is true.

**Remark 1.14.** The estimate given in Corollary 1.13 is scaling invariant. Indeed, if \(\varphi(x)\) is an eigenfunction of \(\Delta - 2\text{Re}(e^{i\theta}v_0)\) associated with a negative eigenvalue \(\tilde{e}\) then \(\varphi_{\lambda}(x)\) is an eigenfunction of \(\Delta - 2\text{Re}(e^{i\theta}(v_0)_{\lambda})\) and the corresponding eigenvalue is \(\lambda^2\tilde{e}\), where \(f_{\lambda}\) denotes the scaling of \(f\) defined in (1.3).

**Remark 1.15.** It is possible to study the optimizing problem (1.7). Let us introduce a slightly different formulation: For \(\rho \geq 0\), we let

\[
B(\rho) := \inf \left\{ \|u_0\|_{F_{H}^{1/2}} : (u_0, v_0) \notin \mathcal{S}_+, \|v_0\|_{F_{H}^{1/2}} \leq \rho \right\}. \tag{1.11}
\]

Then, for any \(\rho \geq 0\) such that \(B(\rho)\) is finite, there exists a minimizer, say \((u_\rho, v_\rho)\), to \(B(\rho)\). The minimizer satisfies \(\ell_{v_\rho} = \ell_{u_\rho}\), and \(u_\rho\) is a minimizer to \(\ell_{u_\rho}\), i.e., \(\ell_{u_\rho} = \|u_\rho\|_{F_{H}^{1/2}}\) (Theorem 7.1). Further, it turns out that the analysis of \(B(\rho)\) is applicable to that of (1.7) (Theorem 7.2). Remark that \(B(0) = \ell_0\) holds and that \(B(\rho)\) is non-increasing in \(\rho\). Hence, \(\ell_0 < \infty\), which is our question, can be also phrased as “\(B(\rho) < \infty\) for any \(\rho \geq 0\).”

**Remark 1.16.** One can deduce similar results in the frame work of \(FH^1 \times FH^1\) or \((FH^1 \cap H^1) \times (FH^1 \cap H^1)\). The main difference of the results in these settings is that one can show that the minimizers belong to the corresponding space, and hence they are global-in-time solutions due to the mass conservation. We would remark that it is not clear our minimizers given in the paper coincide those obtained in the above setting. Since we are working with minimization at fixed time \(t = 0\) and there is no time translation invariance, we do not know whether the minimizers have compact orbit nor enjoy additional regularity.
The rest of the paper is organized as follows. In Section 2, we collect some notations and inequalities. Then, we define function spaces. In Section 3, we prove local well-posedness for (NLS) and give a necessary and sufficient condition for scattering. Then, we check that the solutions to (NLS) with nonpositive energy does not scatter (Theorem 1.11). In Section 4, we investigate properties of $L^1_{\mathfrak{t}_0}$ and $\ell^1_{\mathfrak{t}_0}$. In Section 5, we obtain linear profile decomposition theorem (Theorem 5.10). In Section 6, we prove Theorem 1.6, Theorem 1.8, and Theorem 1.9 and consider the optimizing problem $B(\rho)$. In Section 8, we prove corollaries of Theorem 1.11.

2. Preliminary. In this section, we prepare some notations and estimates used throughout the paper.

2.1. Notations. For non-negative $X$ and $Y$, we write $X \lesssim Y$ to denote $X \leq CY$ for a constant $C > 0$. If $X \lesssim Y \lesssim X$, we write $X \sim Y$. The dependence of implicit constants on parameters will be indicated by subscripts when necessary, e.g. $X \lesssim_u Y$ denotes $X \leq CY$ for some $C = C(u)$. We write $a^r \in [1, \infty]$ to denote the Hölder conjugate to $a \in [1, \infty]$, that is, $\frac{1}{r} + \frac{1}{a} = 1$ holds. For $s \in \mathbb{R}$, the operator $|\nabla|^s$ is defined as the Fourier multiplier operator with multiplier $|\xi|^s$, that is, $|\nabla|^s \xi = \mathcal{F}^{-1}|\xi|^s \mathcal{F}$. For a set $A \in \mathbb{R}^d$, $1_A(x)$ stands for the characteristic function of $A$.

We recall the standard Littlewood-Paley projection operators. Let $\phi$ be a radial cut-off function satisfies $1_{\{|\xi|\leq 4/3\}} \leq \phi \leq 1_{\{|\xi|\leq 5/3\}}$. For $N \in 2\mathbb{Z}$, the operators $P_N$ is defined as
\[
\hat{P}_N f(\xi) := \hat{f}_N(\xi) := \psi_N(\xi) \hat{f}(\xi),
\]
where $\psi_N(x) = \phi(x/N)$ and
\[
\psi_N(x) = \phi_N(x) - \phi_{N/2}(x).
\]

2.2. The Galilean transform and the Galilean operator. The identities
\[
[e^{it\Delta}(e^{it\xi_0} f)](x) = e^{-it|\xi_0|^2 + ix\cdot\xi_0} (e^{it\Delta} f)(x - 2t\xi_0),
\]
\[
[e^{\frac{it}{2}\Delta}(e^{2it\xi_0} f)](x) = e^{-2it|\xi_0|^2 + 2ix\cdot\xi_0} (e^{\frac{it}{2}\Delta} f)(x - 2t\xi_0)
\]

imply that the class of solutions to the linear Schrödinger equation is invariant under Galilean transform:
\[
(u(t, x), v(t, x)) \mapsto (e^{-it|\xi_0|^2 + ix\cdot\xi_0} u(t, x - 2\xi_0 t), e^{-2it|\xi_0|^2 + 2ix\cdot\xi_0} v(t, x - 2\xi_0 t))
\]
for $\xi_0 \in \mathbb{R}^3$. The invariance is inherited in the nonlinear equation (NLS).

The Galilean operator
\[
J_m(t) := x + i \frac{t}{m} \nabla,
\]
which is a multiple of the infinitesimal operator for transforms appearing in (2.2), plays an important role in the scattering theory for mass-subcritical nonlinear Schrödinger equation.

We define the multiplication operator
\[
[M_m(t)] f(x) := e^{\frac{m|x|^2}{2t}} f(x) \quad (t \neq 0)
\]
and the dilation operator
\[
[D(t)] f(x) := (2it)^{-\frac{1}{2}} f \left( \frac{x}{2t} \right) \quad (t \neq 0).
\]
It is known that the Schrödinger group is factorized as \( e^{it\Delta} = M_\frac{1}{2}^1(t)D(t) FM_\frac{1}{2}^1(t) \) by using these operators. This factorization deduces the identity 
\[
e^{it\Delta} \Phi(x) e^{-it\Delta} = M_\frac{1}{2}^1(t) \Phi(2it\nabla) M_\frac{1}{2}^1(-t)
\]
for suitable multiplier \( \Phi \), where \( \Phi(i\nabla) \) denotes the Fourier multiplier operator with multiplier \( \Phi(\xi) \), that is, \( \Phi(i\nabla) := \mathcal{F}^{-1}\Phi(\xi)\mathcal{F} \). The Galilean operator is written as follows:
\[
J_m(t) = e^{\frac{it}{m}\Delta} xe^{-\frac{it}{m}\Delta} = M_m(t) \frac{t}{m} \nabla M_m(-t),
\]
where the second equality holds for \( t \neq 0 \). We define a fractional power of \( J_m \) by
\[
J_m^s(t) := e^{\frac{it}{m}\Delta}|x|^s e^{-\frac{it}{m}\Delta} = M_m(t) \left( -\frac{t^2}{m^2}\Delta \right)^\frac{s}{2} M_m(-t) \quad \text{for} \quad s \in \mathbb{R}.
\]
Remark that the second formula is valid for \( t \neq 0 \).

2.3. Function spaces. We define a time-dependent spaces \( \dot{X}_m^{s,r} = \dot{X}_m^{s,r}(t) \) by using the norm
\[
\| f \|_{\dot{X}_m^{s,r}} := \| J_m^s(t)f \|_{L_x^r(\mathbb{R}^3)} \sim \| t^s \|_{\mathcal{M}_m(-t)f} \|_{L_x^r(\mathbb{R}^3)}.
\]
When \( r = 2 \), we omit the exponent \( r \), that is, \( \dot{X}_m^s = \dot{X}_m^{s,2} \). We can see immediately by the definition of \( J_m^s \) that the equivalence of norms in (2.3) for \( t \neq 0 \). It is natural to write
\[
f \in e^{\frac{it}{m}\Delta} \mathcal{F} \mathcal{H}_s \iff e^{-\frac{it}{m}\Delta} f \in \mathcal{F} \mathcal{H}_s.
\]
Then, we have \( e^{\frac{it}{m}\Delta} \mathcal{F} \mathcal{H}_s = \dot{X}_m^s(t) \).

We use Lorentz-modified space-time norms. For an interval \( I, 1 \leq q < \infty \), and \( 1 \leq \alpha \leq \infty \), the Lorentz space \( L_t^{q,\alpha}(I) \) is defined by using the quasi-norm
\[
\| f \|_{L_t^{q,\alpha}(I)} := \| |\lambda| \{ t \in I : |f(t)| > \lambda \} \|_{L^{q,\alpha}((0,\infty), \mathbb{R})}.
\]
For a Banach space \( X \), \( L_t^{q,\alpha}(I; X) \) is defined as the whole of functions \( u : I \times \mathbb{R}^3 \to \mathbb{C} \) satisfying
\[
\| u \|_{L_t^{q,\alpha}(I; X)} := \| |u(t)| x \|_{L_t^{q,\alpha}(I)} < \infty.
\]
The following equivalence is useful:
\[
\| f \|_{L_t^{q,\alpha}(I)} \sim \| f \cdot 1_{\{ 2^{k-1} \leq |f| \leq 2^k \}} \|_{L_t^{q,\alpha}(I)} \| e^{\alpha(k \in \mathbb{Z})}.
\]

2.4. Strichartz estimates. The standard Strichartz estimates for \( e^{it\Delta} \) were proved in [7, 14, 26]. We also need Strichartz’ estimates for the spaces \( L_t^{q,\alpha} \dot{X}_m^{s,r} \), which were proved in [22, 24].

Definition 2.1. If a pair \((q, r)\) satisfies
\[
2 < q < \infty, \quad 2 < r < 6, \quad \text{and} \quad \frac{2}{q} + \frac{3}{r} = \frac{3}{2},
\]
then \((q, r)\) is an admissible pair.

Remark that we do not include two end points \((\infty, 2)\) and \((2, 6)\) to admissible pairs. It is because they require exceptional treatments sometimes.

Proposition 2.2 (Strichartz estimates, [24]). Let \( s \geq 0 \) and \( t_0 \in I \subset \mathbb{R} \).

(1) For any admissible pair \((q, r)\), we have
\[
\| e^{\frac{it}{m}\Delta} f \|_{L_t^q \dot{X}_m^s \cap L_t^r \dot{X}_m^3} \lesssim \| f \|_{\mathcal{F} \mathcal{H}_s}.
\]
(ii) For any admissible pairs \((q, r)\) and \((\alpha, \beta)\), we have
\[
\left\| \int_{t_0}^t e^{\frac{1}{3}i((t-s)\Delta} F(s)ds \right\|_{L_t^\infty(\mathbb{R}^d) \cap L_t^{\frac{2}{q}}(\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\alpha'}(\mathbb{R}^d) \cap L_t^{\beta'}(\mathbb{R}^d)}.
\]

2.5. Specific function spaces. Throughout this paper, we use the following concrete choice of function spaces. The same exponents were used in [16, 20].

We define
\[
\left( \frac{1}{q_1}, \frac{1}{r_1} \right) := \left( \frac{1}{6}, \frac{7}{18} \right), \quad \left( \frac{1}{q}, \frac{1}{r} \right) := \left( \frac{2}{3}, \frac{2}{9} \right).
\]
The pair \((q_1, r_1)\) is admissible. The pair \((\tilde{q}, \tilde{r})\) satisfies the critical scaling relation \(\frac{2}{q} + \frac{3}{r} = 2\), and is not a admissible pair. These exponents satisfy the following relations:
\[
\frac{q}{q_1} = 1, \quad \frac{r}{r_1} = 2, \quad \text{is not a admissible pair.}
\]

We define the spaces
\[
S^{\text{weak}} := L_t^{\frac{2}{q}} L_x^{\frac{2}{r}}, \quad S := L_t^{\frac{2}{q}} L_x^{\frac{2}{r}},
\]
\[
W_j := L_t^{q_j,2} \dot{X}_{2,j-2}^{\frac{1}{2},\alpha}, \quad \alpha, \beta
\]
for the solutions and the spaces
\[
N_j := L_t^{q_{j-2},2} \dot{X}_{2,j-2}^{\frac{1}{2},\alpha},
\]
for the nonlinear terms. We use a notation \(S^{\text{weak}}(I)\) to indicate that the norm is taken over the space-time slab \(I \times \mathbb{R}^3\), and similarly for the other spaces.

2.6. Some estimates. In this section, we collect some estimates. They easily follow as in [22] (see also [16]).

**Lemma 2.3** (Embeddings). The following inequalities hold.
\[
\|u\|_{S^{\text{weak}}} \lesssim \|u\|_S \lesssim \|u\|_{W_j},
\]
where \(j = 1, 2\).

**Lemma 2.4** (Nonlinear estimates). The following inequalities hold.
\[
\|v\|_{X_1} \lesssim \|u\|_{S^{\text{weak}}} \|v\|_{W_2} + \|u\|_{W_1} \|v\|_{S^{\text{weak}}} \lesssim \|u\|_{W_1} \|v\|_{W_2},
\]
\[
\|u_1 u_2\|_{X_2} \lesssim \|u_1\|_{W_1} \|u_2\|_{S^{\text{weak}}} + \|u_1\|_{S^{\text{weak}}} \|u_2\|_{W_1} \lesssim \|u_1\|_{W_1} \|u_2\|_{W_1},
\]
\[
\|v\|_{L_t^{\frac{2}{q}} \dot{X}_{2,j}^{\frac{1}{2},\alpha} \dot{X}_{2,j}^{\frac{1}{2},\beta}} \lesssim \|v\|_{L_t^{\frac{2}{q}} \dot{X}_{2,j}^{\frac{1}{2},\alpha}} \|u\|_S + \|v\|_S \|u\|_{L_t^{\frac{2}{q}} \dot{X}_{2,j}^{\frac{1}{2},\beta}},
\]
\[
\|u_1 u_2\|_{L_t^{\frac{2}{q}} \dot{X}_{2,j}^{\frac{1}{2},\alpha}} \lesssim \|u_1\|_{L_t^{\frac{2}{q}} \dot{X}_{2,j}^{\frac{1}{2},\alpha}} \|u_2\|_S + \|u_1\|_S \|u_2\|_{L_t^{\frac{2}{q}} \dot{X}_{2,j}^{\frac{1}{2},\beta}},
\]
\[
\left\| \int_{t_0}^t e^{\frac{1}{3}(t-s)\Delta} (v \Pi)(s)ds \right\|_S \lesssim \|u\|_S \|v\|_S,
\]
\[
\left\| \int_{t_0}^t e^{\frac{2}{3}(t-s)\Delta} (u_1 u_2)(s)ds \right\|_S \lesssim \|u_1\|_S \|u_2\|_S.
\]

Remark that the last two are consequences of inhomogeneous Strichartz estimates for non-admissible pairs by Kato [13].
Lemma 2.5 (Interpolation in $\dot{X}^{s,r}_m$). Let $I \subset \mathbb{R}$. The following inequality holds.
\[
\|f\|_{L^{\rho,\gamma}(I,\dot{X}^{s,r}_m)} \lesssim \|f\|_{L^{\rho_1,\gamma_1}(I,\dot{X}^{s_1,r_1}_m)}^{1-\theta} \|f\|_{L^{\rho_2,\gamma_2}(I,\dot{X}^{s_2,r_2}_m)}^\theta
\]
for $1 \leq \rho, \rho_1, \rho_2 < \infty$, $1 \leq \gamma, \gamma_1, \gamma_2 \leq \infty$, $1 < r, r_1, r_2 < \infty$, $0 < \theta < 1$ with
\[
\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{\theta}{\rho_2}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{\theta}{\gamma_2}, \quad s = (1-\theta)s_1 + \theta s_2, \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{\theta}{r_2}.
\]

Lemma 2.6. Let $1 \leq r < \infty$, $0 < s < \frac{3}{r}$ and let $\chi \in S(\mathbb{R}^3)$, where $S(\mathbb{R}^3)$ denotes Schwartz space. Then, a multiplication operator $\chi \times$ is bounded on $\dot{X}^{s,r}_m$.

Lemma 2.7 (Square function estimate). For $0 \leq s \leq 2$ and $1 < p < \infty$, we have
\[
\left\| \nabla^s f \right\|_{L^p_e} \sim \left\| \left( \sum_{N \in 2^\mathbb{Z}} |P_N (\nabla^s f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p_e}.
\]

The following Hölder’s inequality for Lorentz spaces holds.

Lemma 2.8 (Hölder in Lorentz spaces, [11, 25]). Let $1 \leq q, q_1, q_2 < \infty$ and $1 \leq \alpha, \alpha_1, \alpha_2 < \infty$ satisfy
\[
\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \quad \text{and} \quad \frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}.
\]
Then, the following estimate holds.
\[
\|fg\|_{L^{q,\alpha}_e} \lesssim \|f\|_{L^{q_1,\alpha_1}_e} \|g\|_{L^{q_2,\alpha_2}_e}.
\]

3. Local well-posedness and Stability.

3.1. Local well-posedness. In this subsection, we establish a local theory in $(C_t \dot{X}^{1/2}_{1/2} \cap W_1) \times (C_t \dot{X}^{1/2}_{1/2} \cap W_2)$ for (NLS). The result is given as a consequence of Strichartz estimate (Proposition 2.2) and the estimates of the previous subsection (Lemma 2.3 and Lemma 2.4).

Let us first establish a weak version of the local well-posedness.

Proposition 3.1. There exists a universal constant $\delta > 0$ with the following property: Let $\tau \in \mathbb{R}$. If a pair of functions $(u_\tau, v_\tau) \in S'(\mathbb{R}^3)^2$ satisfies
\[
\|e^{i(t-\tau)}\Delta u_\tau, e^{\frac{i}{2}i(t-\tau)}\Delta v_\tau\|_{S(I) \times S(I)} \leq \delta
\]
for some interval $I \ni \tau$ then there exists a unique pair of functions $(u, v) \in S(I) \times S(I)$ such that
\[
\begin{align*}
u(t) &= e^{\frac{i}{2}i(t-\tau)}\Delta v_\tau + i \int^{\tau}_{t} e^{\frac{i}{2}i(s-\tau)}\Delta (u^2)(s)ds, \\
u(t) &= e^{i(t-\tau)}\Delta u_\tau + 2i \int^{t}_{t} e^{i(t-s)}(\nu^2)(s)ds,
\end{align*}
\]
holds in $S(I) \times S(I)$ sense and $(u, v)$ satisfies
\[
\|u, v\|_{S(I) \times S(I)} \leq 2\|e^{i(t-\tau)}\Delta u_\tau, e^{\frac{i}{2}i(t-\tau)}\Delta v_\tau\|_{S(I) \times S(I)}.
\]

This follows by a standard contraction mapping argument with the last two estimates of Lemma 2.4. We refer a pair of functions $(u, v)$ in this proposition to as an S-solution to (NLS) on $I$. 
Another use of Strichartz’ estimate then shows

Furthermore, the solution depends continuously on the initial data, that is, for any

For this interval, we have a unique

and any compact time interval\(I\) \(\ni t_0\) and a unique solution

Suppose\((u(t_0),v(t_0))= (u_0,v_0)\). Moreover, there exists a universal constant \(\delta > 0\) such that if the data satisfies

then the solution satisfies

Furthermore, the solution depends continuously on the initial data, that is, for any

\((u_0,v_0)\) \(\ni (u_0,v_0)\) \(\rightarrow (u,v)\) in \(W_1^1(\mathbb{T}) \times W_1^1(\mathbb{T})\) as \(n \rightarrow \infty\) and any compact time interval \(I \subset \mathbb{R}\), there exists \(n_0 \in \mathbb{N}\) such that \((\text{NLS})\) with initial data \((u_0,v_0)\) has a unique solution \((u_n,v_n) \in (C_t(\mathbb{T}:X^1_1) \cap W_1(\mathbb{T})) \times (C_t(\mathbb{T}:X^1_1) \cap W_1(\mathbb{T}))\) for any \(n \geq n_0\) and \((u_n,v_n) \rightarrow (u,v)\) in \((C_t(\mathbb{T}:X^1_1) \cap W_1(\mathbb{T})) \times (C_t(\mathbb{T}:X^1_1) \cap W_1(\mathbb{T}))\) as \(n \rightarrow \infty\).

Proof. The strategy of the proof is as follows: We first obtain a \(S\)-solution. Then, we show it is a solution in the sense of Definition 1.1 by a persistence-of-regularity type argument.

By Lemma 2.3 and Proposition 2.2, we have

Hence, we can choose an open interval \(I \ni t_0\) so that

For this interval, we have a unique \(S\)-solution \((u,v) \in S(I) \times S(I)\) by Proposition 3.1.

We shall show this is a solution in the sense of Definition 1.1. By Proposition 2.2 and Lemma 2.4, one has

We subdivide the interval \(I\) into \(\bigcup_{j=0} I_j\) so that we have \(c\|\|(u,v)\|S(I_j)\times S(I_j) \leq \frac{1}{2}\) in each interval. Suppose \(t_0 \in I_0\). We have

Another use of Strichartz’ estimate then shows

Repeat the argument to obtain \((u,v) \in (L^\infty \cap X_{1/2}^1)(I) \times (L^\infty \cap X_{1/2}^1)(I)\) for \((u_0,v_0)\). The continuous dependence on initial data is a special case of Proposition 3.6. We omit the details.
3.2. Scattering criterion. In this subsection, we derive a necessary and sufficient condition for scattering (Proposition 3.3). We also give a scattering result for small data (Proposition 3.4).

Proposition 3.3 (Scattering criterion). Let \((u(t), v(t))\) be a unique solution to \((\text{NLS})\) given in Theorem 1.2. Let \(I_{\text{max}} = (T_{\text{min}}, T_{\text{max}})\) be the maximal interval of \((u(t), v(t))\). Then, the following seven statements are equivalent.

1. \((u, v)\) scatters;
2. \(\| (u, v) \|_{W_1(\tau, T_{\text{max}}) \times W_2(\tau, T_{\text{max}})} < \infty, \exists \tau \in I_{\text{max}};\)
3. \(\| (u, v) \|_{S(\tau, T_{\text{max}}) \times S(\tau, T_{\text{max}})} < \infty, \exists \tau \in I_{\text{max}};\)
4. \(\| u \|_{W_1(\tau, T_{\text{max}})} < \infty, \exists \tau \in I_{\text{max}};\)
5. \(\| v \|_{W_2(\tau, T_{\text{max}})} < \infty, \exists \tau \in I_{\text{max}};\)
6. \(\| u \|_{S(\tau, T_{\text{max}})} < \infty, \exists \tau \in I_{\text{max}};\)
7. \(\| v \|_{S(\tau, T_{\text{max}})} < \infty, \exists \tau \in I_{\text{max}}.\)

Proof. (1) \(\Leftrightarrow\) (2) \(\Leftrightarrow\) (3) is standard (see, for instance, [22]). Let us prove they are also equivalent to from (4) to (7). To this end, it suffices to show that (6) and (7) are equivalent. It is because (3) is equivalent to “(6) and (7)”.

Further, once the above equivalence is established, the rest (4) and (5) are handled easily: We have

\[ (4) \Rightarrow (6) \Leftrightarrow (2) \Rightarrow (4) \text{ and } (5) \Rightarrow (7) \Leftrightarrow (2) \Rightarrow (5). \]

Suppose (6). Then, for any \(T \in (0, T_{\text{max}})\), one deduces from Proposition 2.2 and Lemma 2.4 that

\[ \| v \|_{S(0, T)} \lesssim \| v_0 \|_{F H^{1/2}} + \| u \|_{S(0, T)}^2. \]

Here the implicit constant is independent of \(T\). Hence, by letting \(T \uparrow T_{\text{max}}\) we obtain (7).

Next, suppose (7). Take \(\tau \in (T_{\text{min}}, T_{\text{max}})\) to be chosen later. For any \(T \in (\tau, T_{\text{max}})\), we see

\[ \| u \|_{S(\tau, T)} \leq C \| u(\tau) \|_{X^{1/2}((\tau, T))} + C \| u \|_{S(\tau, T)} \| v \|_{S(\tau, T)}, \]

where the constant \(C\) is independent of \(\tau\) and \(T\). We now choose \(\tau\) so that

\[ C \| v \|_{S(\tau, T_{\text{max}})} \leq \frac{1}{2}. \]

This is possible by the property (7). Then, the above inequality implies that

\[ \| u \|_{S(\tau, T)} \leq 2C \| u(\tau) \|_{X^{1/2}((\tau, T))}. \]

Since \(T \in (\tau, T_{\text{max}})\) is arbitrary, we obtain the result. \(\square\)

We turn to a sufficient condition for scattering. One of the simplest conditions is due to smallness of the data.

Proposition 3.4 (Small data scattering). Let \((u_0, v_0) \in F H^{1/2} \times F H^{1/2}\) and let \((u, v)\) be a corresponding unique solution given in Theorem 1.2. Then, we have the following.

1. There exists \(\eta_1 > 0\) such that if \(\| (e^{it\Delta}u_0, e^{it\Delta}v_0) \|_{S \times S} \leq \eta_1\), then \((u, v)\) scatters.
2. There exists \(\eta_2 > 0\) such that if \(\| (e^{it\Delta}u_0, e^{it\Delta}v_0) \|_{W_1 \times W_2} \leq \eta_2\), then \((u, v)\) scatters.
3. There exists \(\eta_3 > 0\) such that if \(\| (u_0, v_0) \|_{F H^{1/2} \times F H^{1/2}} \leq \eta_3\), then \((u, v)\) scatters.

This follows from Proposition 3.1, Proposition 3.3, and Proposition 2.2.
3.3. Nonpositive energy implies failure of scattering. In this subsection, we give a proof of Theorem 1.11. To begin with, we will prove that if a data belongs \( H^1 \times H^1 \), in addition, then the corresponding solution given in Theorem 1.2 stays in \( H^1 \times H^1 \) and the mass and the energy make sense and are conserved. Furthermore, as is well-known, since our equation is mass-subcritical, the conservation of mass implies the solution is global.

**Proposition 3.5.** For any \( t_0 \in \mathbb{R} \) and \((u_0, v_0) \in (X_{1/2}^{1/2}(t_0) \cap H^1) \times (X_{1/2}^{1/2}(t_0) \cap H^1)\) there exists a unique time global solution \((u, v) \in (C_t(\mathbb{R}; X_{1/2}^{1/2} \cap H^1) \cap W_{1,loc}(\mathbb{R})) \times (C_t(\mathbb{R}; X_{1/2}^{1/2} \cap H^1) \cap W_{2,loc}(\mathbb{R}))\) to \((\text{NLS})\) with the initial condition \((u(t_0), v(t_0)) = (u_0, v_0)\). The solution have conserved mass and energy:

\[
M[u(t), v(t)] = M[u_0, v_0], \quad E[u(t), v(t)] = E[u_0, v_0].
\]

Furthermore, if the solution scatters then (1.4) holds also in \( H^1 \times H^1 \) sense.

This is done by a persistence-of-regularity argument. We omit the details of the proof. Now, we prove Theorem 1.11.

**Proof of Theorem 1.11.** Suppose that a solution \((u, v)\) given in Proposition 3.5 scatters. Then, the limit (1.4) holds also in \( H^1 \times H^1 \) sense. One sees from scattering in \( H^1 \times H^1 \) that

\[
\|u(t)\|_{L^2_x} + \|v(t)\|_{L^2_x} \to 0
\]

as \( t \to \infty \). Hence,

\[
2 \int_{\mathbb{R}^3} \Re(u(t)^2 \overline{v(t)}) \, dx \leq 2 \|u(t)\|_{L^2_x}^2 \|v(t)\|_{L^2_x} \to 0
\]

as \( t \to \infty \). We deduce that

\[
E[u_0, v_0] = \lim_{t \to \infty} E[u(t), v(t)] = \|\nabla u_+\|_{L^2_x}^2 + \frac{1}{2} \|\nabla v_+\|_{L^2_x}^2 \geq 0.
\]

Further, \( E[u_0, v_0] = 0 \) implies \((u_+, v_+) = (0, 0)\). By (1.4) in \( H^1 \) and the mass conservation implies \((u_0, v_0) = (0, 0)\).

3.4. Stability. In this subsection, we establish a stability result. Roughly speaking, the proposition implies that two solutions are also close each other if their initial data are close and the equations for them are close.

**Proposition 3.6** (Long time perturbation). Let \( I \) be a time interval with \( t_0 \in I \) and \( M > 0 \). Let \((\tilde{u}, \tilde{v}) : I \times \mathbb{R}^3 \to \mathbb{C}^2\) satisfy

\[
\begin{aligned}
&i\partial_t \tilde{u} + \Delta \tilde{u} = -2\overline{\tilde{v}} + e_1, \\
i\partial_t \tilde{v} + \frac{1}{2} \Delta \tilde{v} = -\tilde{u}^2 + e_2
\end{aligned}
\]

and

\[
\|((\tilde{u}, \tilde{v})\|_{W_1(I) \times W_2(I)} \leq M
\]

for some functions \(e_1, e_2\). Let \((u_0, v_0) \in X_{1/2}^{1/2}(t_0) \times X_{1/2}^{1/2}(t_0)\) and let \((u, v)\) be a corresponding solution to \((\text{NLS})\) with \((u(t_0), v(t_0)) = (u_0, v_0)\) given by Theorem 1.2. There exist \(\varepsilon_1 = \varepsilon_1(M) > 0\) and \(c = c(M) > 0\) such that if

\[
\|((\tilde{u}(t_0), \tilde{v}(t_0)) - (u_0, v_0)\|_{X_{1/2}^{1/2}(t_0) \times X_{1/2}^{1/2}(t_0)} + \|((e_1, e_2)\|_{N_1(I) \times N_2(I)} \leq \varepsilon
\]

for all \(t \in I\).
for some \(0 \leq \varepsilon < \varepsilon_1\) then the maximal existence interval of \((u, v)\) contains \(I\) and the solution satisfies

\[
\|(u, v) - (\bar{u}, \bar{v})\|_{(L^\infty_t(I; L^2_x) \cap W^1(I)) \times (L^\infty_t(I; L^2_x) \cap W^2(I))} \leq c\varepsilon.
\]

For the proof, see [20].

4. Properties of \(L_{v_0}\) and \(\ell_{v_0}^\dagger\). In this section, we collect properties of \(L_{v_0}\) and \(\ell_{v_0}^\dagger\).

Proposition 4.1. For any \(v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}\), there exist \(\varepsilon_1 > 0\) and \(\delta > 0\) such that

\[
L_{v_0}(\varepsilon) \leq \|e^{\frac{1}{2}\eta \Delta}v_0\|_{W^2([0, \infty))} + \delta \varepsilon
\]

holds for \(0 \leq \varepsilon < \varepsilon_1\). Here, the constants \(\varepsilon_1 > 0\) and \(\delta > 0\) depend only on \(\|e^{\frac{1}{2}\eta \Delta}v_0\|_{W^2([0, \infty))}\). In particular, \(\ell_{v_0}^\dagger > 0\) for any \(v_0 \in \mathcal{F}\dot{H}^{1/2}\).

Proof. Apply Proposition 3.6 with \((\bar{u}, \bar{v}) = (0, e^{\frac{1}{2}\eta \Delta}v_0)\), for which \(\varepsilon_1 = e_2 = 0\). \(\square\)

Proposition 4.2 (Properties of \(L_{v_0}\)). For each fixed \(v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}\), the function \(L_{v_0}\) is a non-decreasing continuous function defined on \([0, \infty)\).

Proof. It is clear that \(L_{v_0}\) is a non-decreasing function defined on \([0, \infty)\).

We prove the continuity. It is obvious that

\[
L_{v_0}(0) = \|e^{\frac{1}{2}\eta \Delta}v_0\|_{W^2([0, \infty))} < \infty.
\]

The continuity of \(L_{v_0}(\ell)\) at \(\ell = 0\) holds by Proposition 4.1.

Fix \(\ell_0 \in (0, \infty)\) such that \(L_{v_0}(\ell_0) < \infty\). Let us prove right continuity of \(L_{v_0}(\ell)\) at \(\ell = \ell_0\). Pick \(\varepsilon > 0\). Take \(\delta > 0\) so that \(\delta < \varepsilon_1\) and \(c\delta < \varepsilon\), where \(\varepsilon_1 = \varepsilon_1(L_{v_0}(\ell_0))\) and \(c = c(L_{v_0}(\ell_0))\) are the constants given in Proposition 3.6 with the choice \(M = L_{v_0}(\ell_0)\). Fix \(\ell \in (\ell_0, \ell_0 + \delta)\). Then, for any \(u_{0,1} \in \mathcal{F}\dot{H}^{\frac{1}{2}}\) satisfying \(\|u_{0,1}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell\), the function

\[
u_{0,2} = \frac{\ell_0}{\ell_0 + \delta}u_{0,1}
\]

satisfies \(\|u_{0,2}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell_0\) and \(\|u_{0,1} - u_{0,2}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \delta\). Let \((u_1, v_1)\) and \((u_2, v_2)\) be two solutions to (NLS) with initial data \((u_{0,1}, v_0)\) and \((u_{0,2}, v_0)\), respectively. Note that

\[
\|(u_2, v_2)\|_{W^1([0, \infty)) \times W^2([0, \infty))} \leq L_{v_0}(\ell_0)
\]

since \(\|u_{0,2}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell_0\). Hence, we have

\[
\|(u_1, v_1) - (u_2, v_2)\|_{W^1([0, \infty)) \times W^2([0, \infty))} \leq c\delta < \varepsilon
\]

by Proposition 3.6. Thus, it follows that

\[
\|(u_1, v_1)\|_{W^1([0, \infty)) \times W^2([0, \infty))} < \|(u_2, v_2)\|_{W^1([0, \infty)) \times W^2([0, \infty))} + \varepsilon \leq L_{v_0}(\ell_0) + \varepsilon.
\]

Taking the supremum over such \(u_{0,1} \in \mathcal{F}\dot{H}^{\frac{1}{2}}\), we obtain

\[
L_{v_0}(\ell) \leq L_{v_0}(\ell_0) + \varepsilon
\]

for \(\ell \in (\ell_0, \ell_0 + \delta)\). This shows the right continuity of \(L_{v_0}(\ell)\) at \(\ell = \ell_0\) together with non-decreasing property. The left continuity is a consequence of the continuous dependence on initial data in Theorem 3.2. We omit the details.
Lemma 4.4. \( \ell \) follows:

\[
\ell_v := \inf \{ \ell : L_v(\ell) = \infty \}
\]

for contradiction. Let \( \varepsilon_1 = c_1(\ell) \) be the constant given in Proposition 3.6. Fix \( 0 < \varepsilon < 1 \) so that \( \varepsilon \ell_0 < \varepsilon_1 \). Then, for any fixed \( u_{0,1} \in F \cdot \hat{H}^{0} \) with \( \| u_{0,1} \| \leq \ell_0 \), the function

\[
u_{0,2} := (1 - \varepsilon) u_{0,1}
\]

satisfies \( \| u_{0,2} \| \leq (1 - \varepsilon) \ell_0 \). Let \( (u_1, v_1) \) and \( (u_2, v_2) \) be two solutions to \( (NLS) \) with initial data \( (u_{0,1}, v_0) \) and \( (u_{0,2}, v_0) \), respectively. One sees that

\[
\| (u_2, v_2)\|_{W_1([0,\infty))xW_2([0,\infty))} \leq L_v ((1 - \varepsilon) \ell_0) \leq C < \infty.
\]

In addition, we have

\[
\| u_{0,1} - u_{0,2} \| \leq \varepsilon \| u_{0,1} \| \leq \varepsilon \ell_0.
\]

Applying Proposition 3.6, we obtain

\[
\| (u_1, v_1)\|_{W_1([0,\infty))xW_2([0,\infty))} \leq (1 - \varepsilon) \ell_0 + c \varepsilon \ell_0 < \infty,
\]

where \( c = c(\ell) \) is a constant. Taking supremum over \( u_{0,1} \), it follows that

\[
L_v \ell_0 \leq L_v ((1 - \varepsilon) \ell_0) + c \varepsilon \ell_0 < \infty.
\]

This is a contradiction.

By using the non-decreasing property of \( L_v \), we have the following:

**Proposition 4.3** (Another characterization of \( \ell_v \)). The following identity holds.

\[
\ell_v^1 = \inf \{ \ell : L_v(\ell) = \infty \}
\]

for any \( v \in F \cdot \hat{H}^{0} \).

**Proof.** When \( L_v(\ell) \) is finite for any \( \ell > 0 \), we see that the both sides are infinite. Otherwise, the two sets \( \{ \ell : L_v(\ell) < \infty \} \) and \( \{ \ell : L_v(\ell) = \infty \} \) give us a Dedekind cut of a totally ordered set \( [0,\infty) \), by means of Propositions 4.1 and Proposition 4.2.

A consequence of the alternative characterization is that

\[
L_v \ell_v^1 = \infty
\]

holds for any \( v \in F \cdot \hat{H}^{0} \). This follows from the continuity of \( L_v \). We also have the following:

**Lemma 4.4.** \( \ell_v \geq \ell_v^1 \) for any \( v \in F \cdot \hat{H}^{0} \).

**Proof.** If \( \ell_v = \infty \), then Lemma 4.4 holds. Let \( \ell_v < \infty \). By the definition of \( \ell_v \), for any \( \varepsilon > 0 \), there exists \( u_0 \in F \cdot \hat{H}^{0} \) such that

\[
\| u_0 \| < \ell_v + \varepsilon
\]
holds and the corresponding solution \((u(t), v(t))\) with data \((u(0), v(0)) = (u_0, v_0)\) does not scatter. Since Proposition 3.3 deduces \(\| (u, v) \|_{W_1([0, T_{\text{max}}]) \times W_2([0, T_{\text{max}}])} = \infty\) from the failure of scattering, we obtain

\[ L_{v_0}(\ell_{v_0} + \varepsilon) = \infty. \]

This implies the relation \(\ell_{v_0}^1 \leq \ell_{v_0} + \varepsilon\), thanks to Proposition 4.3. Since \(\varepsilon > 0\) is arbitrary, we have the desired conclusion.

The following is one of the key property to prove Theorem 1.6.

**Proposition 4.5.** \(\ell_0^1 \geq \ell_{v_0}^1 \) for any \(v_0 \in \mathcal{F}\mathcal{H}_{\frac{1}{2}}\).

**Proof.** Fix \(v_0 \in \mathcal{F}\mathcal{H}_{\frac{1}{2}}\). We assume that \(\ell_0^1 < \ell_{v_0}^1\) for contradiction. Then, we have

\[ L_0\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right) = \infty, \quad L_{v_0}\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right) < \infty. \]

Using the fact that \(L_0\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right) = \infty\) and the scaling argument, one can take data \(\{(U_{0,n}, 0)\}\) so that the corresponding solution \((U_n(t), V_n(t))\) to \((\text{NLS})\) satisfies

\[ \|U_{0,n}\|_{\mathcal{F}\mathcal{H}_{\frac{1}{2}}} \leq \frac{\ell_0^1 + \ell_{v_0}^1}{2} \quad (4.1) \]

and

\[ \| (U_n, V_n) \|_{W_1([0, n^{-1}]) \times W_2([0, n^{-1}])} \geq n \quad (4.2) \]

for all \(n \geq 1\). Let \((u_n, v_n)\) be another solution to \((\text{NLS})\) with the initial data \((U_{0,n}, v_0)\). Since \(L_{v_0}\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right) < \infty\), one sees from (4.1) that \((u_n, v_n)\) is global in time and

\[ \|(u_n, v_n)\|_{W_1([0, \infty)) \times W_2([0, \infty])} \leq L_{v_0}\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right) < \infty. \]

We now set \((\tilde{u}_n, \tilde{v}_n) = (u_n, v_n) - (0, e^{\frac{1}{2}it\Delta}v_0)\). Then, \((\tilde{u}_n, \tilde{v}_n)\) solves

\[
\begin{align*}
&i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + 2\tilde{v}_n \tilde{u}_n = -2(e^{\frac{1}{2}it\Delta}v_0)\tilde{u}_n, \\
i\partial_t \tilde{v}_n + \frac{1}{2}\Delta \tilde{v}_n + \tilde{u}_n^2 = 0, \\
&(\tilde{u}_n(0), \tilde{v}_n(0)) = (U_{0,n}, 0)
\end{align*}
\]

and so it is an approximate solution to \((\text{NLS})\) with an error

\[ e_1 = -2(e^{\frac{1}{2}it\Delta}v_0)\tilde{u}_n, \quad e_2 = 0. \]

Take \(\tau > 0\) and set \(I = [0, \tau]\). We have

\[ \|(e_1, e_2)\|_{N_1(I) \times N_2(I)} \lesssim \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2(I)} \|u_n\|_{W_1(I)} \leq \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2(I)} L_{v_0}\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right). \]

The right hand side is independent of \(n\), and tends to zero as \(\tau \downarrow 0\).

Now we apply the Proposition 3.6 with \(M = L_{v_0}\left(\frac{\ell_0^1 + \ell_{v_0}^1}{2}\right) + \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2([0, \infty])}\). Choose \(\tau\) sufficiently small so that the above upper bound of the error becomes smaller than the corresponding \(\varepsilon_1\). Since \((U_n, V_n)\) is a solution with the same initial data as \((\tilde{u}_n, \tilde{v}_n)\), we see from Proposition 3.6 that \((U_n, V_n)\) extends up to time \(\tau\) and obeys the bound

\[ \|(U_n, V_n)\|_{W_1(I) \times W_2(I)} \leq \|(\tilde{u}_n, \tilde{v}_n)\|_{W_1(I) \times W_2(I)} + C\varepsilon_1 \]
However, this contradicts with (4.2) for large $n$. \hfill $\Box$

5. **Linear profile decomposition.** In this section, we obtain a linear profile decomposition (Theorem 5.10).

5.1. **Linear profile decomposition.** Let us first introduce several operators and give a notion of deformation, which is a specific class of bounded operator.

**Definition 5.1 (Operators).** We define the following operators.

1. A dilation
   \[(D(h)(f,g))(x) = (f\{h\},g\{h\}) = (h^2 f(hx),h^2 g(hx)), \quad h \in 2\mathbb{Z},\]
2. A translation in Fourier space
   \[(T(\xi)(f,g))(x) = (e^{ix\cdot\xi} f(x), e^{2ix\cdot\xi} g(x)), \quad \xi \in \mathbb{R}^3.\]

**Definition 5.2.** We say that a bounded operator
\[\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) = T(\xi)D(h), \quad (\xi,h) \in \mathbb{R}^3 \times 2\mathbb{Z}\]
on $\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$ is called a deformation in $\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$. Let a set $G \subset \mathcal{L}(\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}})$ be composed of all deformations.

**Remark 5.3.** $G$ is a group with the functional composition as a binary operation. \[\text{Id} = T(0)D(1) \in G\] is the identity element. For $\mathcal{G} = T(\xi)D(h)$, the inverse element is $\mathcal{G}^{-1} = T(-\frac{\xi}{h})D(\frac{1}{h}) \in G$.

Next, we introduce a class of families of deformations.

**Definition 5.4 (A vanishing family).** We say that a family of deformations $\{\mathcal{G}_n = T(\xi_n)D(h_n)\}_n \subset G$ is vanishing if $|\xi_n| + |\log h_n| \to \infty$ as $n \to \infty$ holds.

**Lemma 5.5.** A family $\{\mathcal{G}_n\}_n \subset G$ is vanishing if and only if a family of inverse elements $\{\mathcal{G}_n^{-1}\}_n$ is vanishing.

**Proof.** It is clear from $T(\xi_n)D(h_n))^{-1} = T(-\frac{\xi_n}{h_n})D(\frac{1}{h_n}).$ \hfill $\Box$

The following characterization of the vanishing family is useful.

**Proposition 5.6.** For a family $\{\mathcal{G}_n\}_n \subset G$ of deformations, the following three statements are equivalent.

1. $\{\mathcal{G}_n\}_n$ is vanishing.
2. For any $(\phi,\psi) \in \mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$, $\mathcal{G}_n(\phi,\psi) \rightharpoonup (0,0)$ weakly in $\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$ as $n \to \infty$.
3. For any subsequence $\{\mathcal{G}_{n_k}\}_k$, there exist a subsequence $\{\mathcal{G}_{n_{k_l}}\}_l$ and a bounded sequence $\{(f_l,g_l)\}_l \subset \mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$ such that $(f_l,g_l) \rightharpoonup (0,0)$ and $\mathcal{G}_{n_{k_l}}^{-1}(f_l,g_l) \rightharpoonup (\phi,\psi) \neq (0,0)$ weakly in $\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$ as $l \to \infty$.

**Proof.** We mimic the argument in [22, 23]. (2) $\Rightarrow$ (3) holds by taking $k_l = k$ and $(f_k,g_k) = \mathcal{G}_{n_k}(\phi,\psi)$ for some $(\phi,\psi) \neq (0,0)$.

Next, we prove the contraposition of (3) $\Rightarrow$ (1). If $\mathcal{G}_n$ is not vanishing then the corresponding sequence of parameters is bounded. Hence, there exists a subsequence $\{\mathcal{G}_{n_k}\}_k$ and $\mathcal{G} \in G$ such that $\mathcal{G}_{n_k} \to \mathcal{G}$ in $\mathcal{L}(\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}})$ as $k \to \infty$. Then,
for any subsequence \( \{G_{n_k}\}_k \) and for any bounded sequence \( \{(f_l, g_l)\}_l \) such that \( (f_l, g_l) \to (0, 0) \) in \( \mathcal{F} \hat{H}^{\frac{1}{2}} \times \mathcal{F} \hat{H}^{\frac{1}{2}} \) as \( l \to \infty \), one has
\[
\left| \langle G_{n_k,1}^{-1}f_l, \phi \rangle_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \right| = \left| \langle f_l, G_{n_k,1} \phi \rangle_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \right| \\
\leq \left| \langle f_l, G_1 \phi \rangle_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \right| + \left| \langle f_l, G_{n_k,1} \phi - G_1 \phi \rangle_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \right| \\
\leq \left| \langle f_l, G_1 \phi \rangle_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \right| + \|f_l\|_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \|G_{n_k,1} \phi - G_1 \phi\|_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \\
\to 0 \quad \text{as} \quad l \to \infty
\]
for any \( \phi \in \mathcal{F} \hat{H}^{\frac{1}{2}} \). Similarly, \( \left| \langle G_{n_k,2}g_l, \psi \rangle_{\mathcal{F} \hat{H}^{\frac{1}{2}}} \right| \to 0 \) as \( k \to \infty \) for any \( \psi \in \mathcal{F} \hat{H}^{\frac{1}{2}} \).

Hence,
\[
(G_{n_k})^{-1}(f_{k_l}, g_{k_l}) \to (0, 0)
\]
in \( \mathcal{F} \hat{H}^{\frac{1}{2}} \times \mathcal{F} \hat{H}^{\frac{1}{2}} \) as \( k \to \infty \). Hence, (3) fails.

Finally, (1) \( \Rightarrow \) (2) is proved by a standard argument. We omit the details. \( \square \)

We now introduce a notion of orthogonality.

**Definition 5.7** (Orthogonality). We say two families of deformations \( \{G_n\}, \{\tilde{G}_n\} \subset G \) are orthogonal if \( \{G_n^{-1}\tilde{G}_n\} \) is vanishing.

**Remark 5.8.** Let \( \{G_n^1 = T(\xi_n^1)D(h_n^1)\} \subset G \) \((j = 1, 2)\) be two families of deformations. \( \{G_n^1\} \) and \( \{G_n^2\} \) are orthogonal if and only if
\[
\frac{h_n^1}{h_n^2} + \frac{h_n^2}{h_n^1} + \frac{\xi_n^1 - \xi_n^2}{h_n^1} \to \infty
\]
as \( n \to \infty \). This equivalence holds from the identity
\[
(G_n^1)^{-1}G_n^2 = T \left( \frac{\xi_n^2 - \xi_n^1}{h_n^1} \right) D \left( \frac{h_n^2}{h_n^1} \right)
\]

The following is immediate from the definition. For the details, see [22, 23].

**Proposition 5.9.** Let \( \{G_n\}, \{\tilde{G}_n\} \subset G \). Define the relation \( \sim \) as follows: If \( \{G_n\} \) and \( \{\tilde{G}_n\} \) are not orthogonal then \( \{G_n\} \sim \{\tilde{G}_n\} \). Then, \( \sim \) is an equivalent relation.

Let us now state the linear profile decomposition result.

**Theorem 5.10** (Linear profile decomposition). Let \( \{(\phi_n, \psi_n)\} \) be a bounded sequence in \( \mathcal{F} \hat{H}^{\frac{1}{2}} \times \mathcal{F} \hat{H}^{\frac{1}{2}} \). Passing to a sequence if necessary, there exist profile \( \{(\phi^j, \psi^j)\} \subset \mathcal{F} \hat{H}^{\frac{1}{2}} \times \mathcal{F} \hat{H}^{\frac{1}{2}}, \{(\Phi^j_n, \Psi^j_n)\} \subset \mathcal{F} \hat{H}^{\frac{1}{2}} \times \mathcal{F} \hat{H}^{\frac{1}{2}}, \) and pairwise orthogonal families of deformations \( \{G_n^j = T(\xi_n^j)D(h_n^j)\} \subset G \) \((j = 1, 2, \ldots)\) such that for each \( J \geq 1 \),
\[
(\phi_n, \psi_n) = \sum_{j=1}^{J} G_n^j(\phi^j, \psi^j) + (\Phi^j_n, \Psi^j_n)
\]
for any \( n \geq 1 \). Moreover, \( \{(\Phi^j_n, \Psi^j_n)\} \) satisfies
\[
(G_n^j)^{-1}(\Phi^j_n, \Psi^j_n) \longrightarrow \begin{cases} (\phi^j, \psi^j) & (J < j), \\
(0, 0) & (J \geq j) \end{cases}
\]


in $\mathcal{F} \dot{H}^{\frac{1}{2}} \times \mathcal{F} \dot{H}^{\frac{1}{2}}$ as $n \to \infty$ for any $j \geq 0$, where we use the convention $(\Phi^0_n, \Psi^0_n) = (\phi_n, \psi_n)$, and
\[
\limsup_{n \to \infty} \|(e^{it\Delta} \Phi^j_n, e^{rac{it}{t} \Delta} \Psi^j_n)\|_{L^q_t L^r_x L^q_t L^r_x} \to 0 \quad (5.1)
\]
as $J \to \infty$ for any $1 < q, r < \infty$ such that $\frac{1}{q} \in (\frac{1}{2}, 1)$ and $\frac{2}{q} + \frac{3}{r} = 2$. Furthermore, we have Pythagorean decomposition:
\[
\|\phi_n\|_{\mathcal{F} \dot{H}^{\frac{1}{2}}}^2 = \sum_{j=1}^{\infty} \|\phi^j_n\|_{\mathcal{F} \dot{H}^{\frac{1}{2}}}^2 + \|\Phi^j_n\|_{\mathcal{F} \dot{H}^{\frac{1}{2}}}^2 + o_n(1),
\]
\[
\|\psi_n\|_{\mathcal{F} \dot{H}^{\frac{1}{2}}}^2 = \sum_{j=1}^{\infty} \|\psi^j_n\|_{\mathcal{F} \dot{H}^{\frac{1}{2}}}^2 + \|\Psi^j_n\|_{\mathcal{F} \dot{H}^{\frac{1}{2}}}^2 + o_n(1),
\]
where $o_n(1)$ goes to 0 as $n \to \infty$.

**Proof.** We define
\[
\nu((\phi_n, \psi_n)) := \left\{(\phi, \psi) \in \mathcal{F} \dot{H}^{\frac{1}{2}} \times \mathcal{F} \dot{H}^{\frac{1}{2}} \mid \begin{array}{l}
\text{There exist } \xi_n \in \mathbb{R}^3 \text{ and } h_n \in 2^{Z_n} \text{ such that } \\
(G_n)^{-1}(\phi_n, \psi_n) \to (\phi, \psi) \text{ in } \mathcal{F} \dot{H}^{\frac{1}{2}} \times \mathcal{F} \dot{H}^{\frac{1}{2}} \end{array} \right\}
\]
and
\[
\eta((\phi_n, \psi_n)) := \sup_{(\phi, \psi) \in \nu((\phi_n, \psi_n))} \|(\phi, \psi)\|_{\mathcal{F} \dot{H}^{\frac{1}{2}} \times \mathcal{F} \dot{H}^{\frac{1}{2}}}.
\]
Then, a standard argument shows the theorem. However, the smallness (5.1) is replaced by $\eta((\Phi^j_n, \Psi^j_n)) \to 0$ as $J \to \infty$. The following Proposition 5.12 shows that this smallness is stronger. \qed

**Remark 5.11.** It is possible to prove Theorem 5.10 by combining similar decomposition in the framework of $H^{1/2} \times H^{1/2}$, pseudo-conformal transformation, and the fact that the rapid quadratic oscillation implies scattering. For the third respect, we refer the reader to [20, Proposition 4.1] (see also [1]).

5.2. **Control of vanishing.** To complete the proof of Theorem 5.10, we show the following in this subsection.

**Proposition 5.12** (Control of vanishing). If a sequence $\{(\Phi_n, \Psi_n)\}_n \subset \mathcal{F} \dot{H}^{\frac{1}{2}} \times \mathcal{F} \dot{H}^{\frac{1}{2}}$ satisfies
\[
\|(\Phi_n, \Psi_n)\|_{\mathcal{F} \dot{H}^{\frac{1}{2}} \times \mathcal{F} \dot{H}^{\frac{1}{2}}} \leq M
\]
and
\[
\|(e^{it\Delta} \Phi_n, e^{rac{it}{t} \Delta} \Psi_n)\|_{L^q_t L^r_x L^q_t L^r_x} \geq \varepsilon_0
\]
for some $M > 0$, $\varepsilon_0 > 0$, and $1 < q, r < \infty$ with $\frac{1}{q} \in (\frac{1}{2}, 1)$ and $\frac{2}{q} + \frac{3}{r} = 2$, then
\[
\eta((\Phi_n, \Psi_n)) \geq M, \varepsilon_0, q, r, 1.
\]
To prove the proposition, we will need the following.
Remark that by Lemma 2.7, we have

\[ \|e^{it\Delta}f\|_{L^2_t([0,\infty);L^2_x)} \lesssim \|f\|_{\mathcal{F}H^1_1} \sup_{N \in \mathbb{N}} \left( \|e^{it\Delta}\psi_Nf\|_{L^2_t([0,\infty);L^2_x)} \right)^{1/2}, \]

where \( \psi_N \) is defined as (2.1).

Proof. By Lemma 2.7, we have

\[ \|e^{it\Delta}f\|_{L^2_x} = \|\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}f\|_{L^2_x} \sim \left( \|\sum_{N \in \mathbb{N}} |P_N\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}f|^2 \right)^{1/2} \|\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}f\|_{L^2_x} \]

for \( t > 0 \), where the implicit constant is independent of \( t \) by virtue of the scaling. Denote \( v_N = P_N\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}f \) for simplicity. By a convexity argument, one has

\[ \left( \sum_{N \in \mathbb{N}} |v_N|^2 \right)^{3/2} \left( \int_{L^1_t([0,\infty);L^2_x)} \sum_{N \in \mathbb{N}} |v_N|^2 \right)^{3/2} dx dt \]

\[ \lesssim \sum_{M,N \in \mathbb{N}, N \leq M} \int_{L^1_t([0,\infty);L^2_x)} |v_N|^2 \|v_M\|^3 dx dt, \]

where we have used the symmetry in the last line to reduce the matter to the case \( N \leq M \). Take \( r_1 \) and \( r_2 \) so that \( \frac{2}{3} < r_1 < 3 < r_2 < \frac{10}{3} \) and \( \frac{2}{3} = \frac{1}{r_1} + \frac{1}{r_2} \). By the Hölder inequality,

\[ \int_{L^1_t([0,\infty);L^2_x)} |v_N|^2 \|v_M\|^3 dx dt \]

\[ \leq \|v_N\|_{L^r_1([0,\infty);L^{r_1}_x)} \|v_N\|_{L^2_t([0,\infty);L^2_x)} \|v_M\|_{L^2_t([0,\infty);L^2_x)} \|v_M\|_{L^r_2([0,\infty);L^{r_2}_x)}. \]

Hence,

\[ \|e^{it\Delta}f\|_{L^2_t([0,\infty);L^2_x)}^3 \]

\[ \lesssim \left( \sup_{N \in \mathbb{N}} \|v_N\|_{L^r_1([0,\infty);L^{r_1}_x)} \right) \sum_{M,N \in \mathbb{N}, N \leq M} \|v_N\|_{L^r_1([0,\infty);L^{r_1}_x)} \|v_M\|_{L^r_2([0,\infty);L^{r_2}_x)}. \]

Remark that

\[ v_N = \mathcal{F}\psi_N \mathcal{F}^{-1}D(t)\mathcal{F}\mathcal{M}_\frac{1}{2}(t)f = D(t)\mathcal{F}\mathcal{M}_\frac{1}{2}(t)\psi_Nf = \mathcal{M}_\frac{1}{2}(t)^{-1}e^{it\Delta}\psi_Nf. \]

By the Strichartz estimate,

\[ \|v_N\|_{L^r_1([0,\infty);L^{r_1}_x)} \]

\[ = \|\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}\psi_Nf\|_{L^r_1([0,\infty);L^{r_1}_x)} \lesssim \||\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}\psi_Nf\|_{L^r_1([0,\infty);L^{\frac{10r}{12r-3r}}_x)} \]

\[ \lesssim \|\psi_Nf\|_{\mathcal{F}H^{\frac{10r}{12r-3r}}(\mathbb{R}^3)} \lesssim \||f|^{\frac{10r}{12r-3r}}|\mathcal{M}_\frac{1}{2}(-t)e^{it\Delta}\psi_Nf|\|_{L^r_1([0,\infty);L^{\frac{10r}{12r-3r}}_x)} \]

\[ \lesssim \|\psi_Nf\|_{\mathcal{F}H^{\frac{10r}{12r-3r}}(\mathbb{R}^3)} \lesssim \||f|^{\frac{10r}{12r-3r}}\psi_Nf|\|_{L^r_1([0,\infty);L^{\frac{10r}{12r-3r}}_x)} \]

Thus,

\[ \sum_{N \leq M, M,N \in \mathbb{N}} \|v_N\|_{L^{r_1}_1([0,\infty);L^{r_1}_x)} \|v_M\|_{L^{r_2}_2([0,\infty);L^{r_2}_x)} \]

\[ \leq \sum_{R \geq 1} \sum_{N \in \mathbb{N}} \|v_N\|_{L^{r_1}_1([0,\infty);L^{r_1}_x)} \|v_{NR}\|_{L^{r_2}_2([0,\infty);L^{r_2}_x)} \]

where \( |x|^{\frac{10r}{12r-3r}} \psi_Nf \).
One can choose a sequence $N$ using the estimate $\tau$ one can choose infinitely many holds for infinitely many $\tau$. Since the scaling property and Strichartz' estimate give us
\[
\|e^{it\Delta} \Phi_n\|_{L^1_t L^\infty_x} \geq \frac{\varepsilon_0}{2},
\]
or
\[
\|e^{it\Delta} \Psi_n\|_{L^1_t L^\infty_x} \geq \frac{\varepsilon_0}{2},
\]
holds for infinitely many $n$. We only consider the case where the former holds for infinitely many $n$. The proof for the other case is similar.

By interpolation and boundedness lemma, there exists $\delta$ such that
\[
\|e^{it\Delta} \Phi_n\|_{L^1_t L^\infty_x} \lesssim \|e^{it\Delta} \Phi_n\|_\theta \|\Phi_n\|^{1-\theta}_{\mathcal{F}^\delta},
\]
By means of Lemma 5.13 and the assumption, we have
\[
\sup_{N \in \mathbb{Z}} \|e^{it\Delta} x^\frac{1}{2} \psi_N \Phi_n\|_{L^1_t((0,\infty);L^2_x)} \gtrsim_{M,\varepsilon_0,q,r} 1.
\]
One can choose a sequence $N_n$ so that
\[
\|e^{it\Delta} x^\frac{1}{2} \psi_N \Phi_n\|_{L^1_t((0,\infty);L^2_x)} \gtrsim 1. \tag{5.2}
\]
Since the scaling property and Strichartz' estimate give us
\[
\|e^{it\Delta} x^\frac{1}{2} \psi_N \Phi_n\|_{L^1_t([0,\tau];L^2_x)} = N_n^\frac{\delta}{4} \|e^{it\Delta} (N_n x) \times \frac{1}{2} \psi \Phi_n(N_n)\|_{L^1_t([0,\tau];L^2_x)}
\]
\[
\leq N_n^\frac{\delta}{4} \|\|_{L^1_t([0,\tau])} \|e^{it\Delta} (N_n x) \times \frac{1}{2} \psi \Phi_n(N_n)\|_{L^1_t([0,\tau];L^2_x)}
\]
\[
\lesssim N_n^\frac{\delta}{4} \tau^\frac{1}{\delta} \|\|_{L^1_t([0,\tau])} \|\|_{L^1_t([0,\tau];L^2_x)}
\]
\[
\gtrsim \tau^\frac{1}{\delta} \|\|_{\mathcal{F}^\delta},
\]
one can choose $\tau_0 = \tau_0(M,\varepsilon_0,q,r) > 0$ small so that (5.2) is improved as
\[
\|e^{it\Delta} |x|^{\frac{1}{2}} \psi_N \Phi_n\|_{L^1_t([\tau_0 N^2_2,\infty);L^2_x)} \gtrsim 1
\]
for all $n \geq 1$. Hölder's inequality gives us
\[
\|e^{it\Delta} |x|^{\frac{1}{2}} \psi_N \Phi_n\|_{L^1_t([\tau_0 N^2_2,\infty);L^2_x)}
\]
\[
\leq \|\| \|e^{it\Delta} |x|^{\frac{1}{2}} \psi_N \Phi_n\|_{L^1_t([\tau_0 N^2_2,\infty);L^2_x)}\|\| t^{-\frac{1}{\delta}} e^{it\Delta} |x|^{\frac{1}{2}} \psi N \Phi_n\|_{L^\frac{12}{5}_t L^\frac{12}{5}_x}.
\]
Using the estimate
\[
\|\| t^{-\frac{1}{\delta}} e^{it\Delta} |x|^{\frac{1}{2}} \psi N \Phi_n\|_{L^\frac{12}{5}_t L^\frac{12}{5}_x} \lesssim \|\| t^{-\frac{1}{\delta}} e^{it\Delta} |x|^{\frac{1}{2}} \psi N \Phi_n\|_{L^\frac{12}{5}_t L^\frac{12}{5}_x}
\]
\[
\lesssim \|\| \| e^{it\Delta} |x|^{\frac{1}{2}} \psi N \Phi_n\|_{L^\frac{12}{5}_t L^\frac{12}{5}_x}.
\]
This completes the proof. □

**Proof of Proposition 5.12.** In what follows we denote various subsequences of $n$ again by $n$. By the pigeon hole principle,
\[
\|e^{it\Delta} \Phi_n\|_{L^1_t L^\infty_x} \gtrsim \frac{\varepsilon_0}{2}
\]
holds for infinitely many $n$. We only consider the case where the former holds for infinitely many $n$. The proof for the other case is similar.
we reach to the estimate
\[ N_n^{-\frac{3}{4}} |||t\|^{\frac{3}{4}} e^{it\Delta} |x|^{\frac{3}{4}} \psi_N(x)\|_{L^\infty(\tau_0 N^2, \infty; L^\infty)} \gtrsim 1 \]
for all \( n \geq 1 \). There exist \( t_n \geq \tau_0 N^2 \) and \( y_n \in \mathbb{R}^3 \) such that
\[ N_n^{-\frac{3}{4}} |||t\|^{\frac{3}{4}} e^{it\Delta} (|x|^{\frac{3}{4}} \psi_N(x)) (y_n) \| \gtrsim 1. \] (5.3)
By the integral representation of the Schrödinger group, we obtain
\[
N_n^{-\frac{3}{4}} |||t\|^{\frac{3}{4}} e^{it\Delta} (|x|^{\frac{3}{4}} \psi_N(x)) (y_n) | = N_n^{-\frac{3}{4}} \left| t_n^{\frac{3}{2}} (4\pi it_n)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i\frac{1}{2} \frac{\eta_n^2}{N^2}} |x|^{\frac{3}{4}} \psi_N(x) \Phi_n(x) dx \right| \\
\lesssim N_n^{-\frac{3}{4}} \left| \int_{\mathbb{R}^3} e^{-i \frac{2\eta_n}{N} x} e^{i \frac{|x|^2}{4N^2}} (N_n^{\frac{3}{4}} |x|^{-\frac{1}{3}} \psi(x)) |x|^{\frac{3}{4}} \Phi_n(x) dx \right| \\
= \left| \int_{\mathbb{R}^3} e^{i \frac{N^2 |x|^2}{4N^2}} (|x|^{-\frac{1}{3}} \psi(x)) |x|^{\frac{3}{4}} e^{-i \frac{2\eta_n}{N} x} N^2 \Phi_n (N_n x) dx \right|. \] (5.4)
Let
\[ \xi_n := - \frac{N_n y_n}{2t_n} \in \mathbb{R}^3, \quad h_n := N_n \in 2^\mathbb{N}. \]
Define a deformation \( G_n \in G \) so that \( G_n^{-1} = T(\xi_n) D(h_n). \)
Since \( \{ G_n (\Phi_n, \Psi_n) \}_{n} \) is a bounded sequence in \( \mathcal{F} \mathcal{H}^\frac{1}{2} \times \mathcal{F} \mathcal{H}^\frac{1}{2} \), it weakly converges to a pair \( (\Phi, \Psi) \in \mathcal{F} \mathcal{H}^\frac{1}{2} \times \mathcal{F} \mathcal{H}^\frac{1}{2} \) along a subsequence. It is obvious that \( (\Phi, \Psi) \in \nu(\{ (\Phi_n, \Psi_n) \}) \). Notice that \( 0 < \frac{N^2}{4t_n} \leq \frac{1}{4\eta_0} \). Hence, by extracting a subsequence if necessary, one has
\[
\int_{\mathbb{R}^3} e^{i \frac{N^2 |x|^2}{4N^2}} (|x|^{-\frac{1}{3}} \psi(x)) |x|^{\frac{3}{4}} (e^{-i \frac{2\eta_n}{N} x} N^2 \Phi_n (N_n x) dx) \\
\rightarrow \int_{\mathbb{R}^3} e^{i \frac{|x|^2}{4N^2}} (|x|^{-\frac{1}{3}} \psi(x)) |x|^{\frac{3}{4}} \Phi(x) dx
\]
as \( n \to \infty \), where \( a \in \mathbb{R} \) is the limit of \( \frac{N^2}{4t_n} \) along the (sub)sequence. Plugging this with (5.3) and (5.4), we conclude that
\[
1 \lesssim \int_{\mathbb{R}^3} e^{i |x|^2} (|x|^{-\frac{1}{3}} \psi(x)) |x|^{\frac{3}{4}} \Phi(x) dx \lesssim \| (\Phi, \Psi) \|_{\mathcal{F} \mathcal{H}^\frac{1}{2} \times \mathcal{F} \mathcal{H}^\frac{1}{2}} \leq \eta(\{ (\Phi_n, \Psi_n) \}).
\]
This is the desired estimate. \( \square \)

6. Proof of Theorem 1.6, Theorem 1.8, and Theorem 1.9. In this section, we prove Theorem 1.6, Theorem 1.8, and Theorem 1.9. The following proof shows all these theorems.
Proof of Theorems 1.6, Theorems 1.8, and Theorem 1.9. Fix $v_0 \in \mathcal{F}H^{\frac{1}{2}}$. First, we consider the case $\ell^\uparrow v_0 = \infty$. In this case, we can obtain $\ell^\uparrow v_0 = \ell v_0 = \ell_0 = \infty$. Indeed, we have $\infty = \ell^\uparrow v_0 \leq \ell v_0$ by Lemma 4.4. On the other hand, we have $\infty = \ell^\uparrow v_0 \leq \ell^\uparrow 0 \leq \ell_0$ by Proposition 4.5 and Lemma 4.4.

From now on, we assume $\ell^\uparrow v_0 < \infty$. By definition of $\ell^\uparrow$, we have $L_{v_0}(\ell^\uparrow v_0 - \frac{1}{n}) < \infty$ for each $n \in \mathbb{N}$, that is,

$$\sup \left\{ \| (u,v) \|_{W_{1,1}(0,\infty) \times W_2(0,\infty))} \bigg| \begin{array}{l} (u,v) \text{ is the solution to (NLS) on } [0,\infty), \\ v(0) = v_0, \quad \| u(0) \|_{\mathcal{F}H^{\frac{1}{2}}} \leq \ell^\uparrow v_0 - \frac{1}{n} \end{array} \right\} < \infty.$$ 

We note that $T_{\text{max}} = \infty$ because of Proposition 3.3. Since $L_{v_0}(\ell) < \infty$ for any $0 \leq \ell < \ell^\uparrow$, $L_{v_0}(\ell^\uparrow v_0) = \infty$, and $L_{v_0}$ is non-decreasing, we can take a sequence $\{m_n\}$ of $\mathbb{N}$ such that

$$L_{v_0}(\ell^\uparrow v_0 - \frac{1}{m_n}) < L_{v_0}(\ell^\uparrow v_0 - \frac{1}{m_{n+1}})$$

for each $n \in \mathbb{N}$. We take a sequence $\{u_{0,n}\} \in \mathcal{F}H^{\frac{1}{2}}$ satisfying

$$\ell^\uparrow v_0 - \frac{1}{m_n} < \| u_{0,n} \|_{\mathcal{F}H^{\frac{1}{2}}} \leq \ell^\uparrow v_0 - \frac{1}{m_{n+1}} \quad (6.1)$$

and

$$L_{v_0}(\ell^\uparrow v_0 - \frac{1}{m_n}) < \| (u_n,v_n) \|_{W_{1,1}(0,\infty) \times W_2(0,\infty))} \leq L_{v_0}(\ell^\uparrow v_0 - \frac{1}{m_{n+1}}),$$

where $(u_n,v_n)$ is the solution to (NLS) with the initial data $(u_{0,n},v_0)$. Since $\{(u_{0,n},v_0)\} \subset \mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$ is a bounded sequence, we apply Theorem 5.10 to this sequence. Then, there exists profile $\{ (\phi^j, \psi^j) \} \subset \mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$, remainder $\{ (R^J_n, L^J_n) \} \subset \mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}$, and pairwise orthogonal families of deformations $\{ \mathcal{G}^J_{h_n} = T(\xi^J_{h_n})D(h^J_{h_n}) \}_{n \in \mathbb{N}} \subset G \ (j = 1, 2, \ldots)$ such that

$$ (u_{0,n},v_0) = \sum_{j=1}^J \mathcal{G}^J_{h_n}(\phi^j, \psi^j) + (R^J_n, L^J_n) \quad (6.2)$$

for any $J \geq 1$. Since $v_0$ is independent of $n$, there exists unique $j_0$ such that $\psi^{j_0} = v_0$ and $\mathcal{G}^{j_0}_{h_n} = \text{Id}$. Furthermore, the remainder for $v$-component is zero: $L^J_n = 0$. Rearranging the profile $(\phi^j, \psi^j)$, we may let $j_0 = 1$. Then, the above decomposition reads as

$$(u_{0,n},v_0) = (\phi^1, v_0) + \sum_{j=2}^J \mathcal{G}^J_{h_n}(\phi^j, 0) + (R^J_n, 0).$$

From Theorem 5.10, we have Pythagorean decomposition:

$$\| u_{0,n} \|_{\mathcal{F}H^{\frac{1}{2}}}^2 = \sum_{j=1}^J \| \phi^j \|_{\mathcal{F}H^{\frac{1}{2}}}^2 + \| R^J_n \|_{\mathcal{F}H^{\frac{1}{2}}}^2 + o_n(1) \quad (6.3)$$

for each $J \geq 1$. The parameters are asymptotically orthogonal: if $j \neq k$, then

$$\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + \frac{| \xi^j_n - \xi^k_n |}{h_n^j h_n^k} \longrightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

The remainders satisfy

$$\mathcal{G}^J_{h_n}^{-1} R^J_n \longrightarrow 0 \quad \text{in} \quad \mathcal{F}H^{\frac{1}{2}} \quad \text{as} \quad n \rightarrow \infty.$$
Lemma 6.1. For any $1 \leq j \leq J$,
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|e^{it\Delta} R^J_n\|_{L^\infty_t L^r_x} = 0
\]
for any $1 < q, r < \infty$ with $\frac{1}{r} \in \left(\frac{1}{2}, 1\right)$ and $\frac{2}{q} + \frac{2}{r} = 2$.

We will prove that there exists only one $j_1$ satisfying $\phi^{j_1} \neq 0$, and it satisfies
\[
\|\phi^{j_1}\|_{\mathcal{F}^{1/2}_t} = \ell^J_{t_0}. \]
From (6.3), we have
\[
\sum_{j=1}^{\ell^J_{t_0}} \|\phi^j\|^2_{\mathcal{F}^{1/2}_t} \leq (\ell^J_{t_0})^2,
\]
and hence, $\|\phi^j\|_{\mathcal{F}^{1/2}_t} \leq \ell^J_{t_0}$ holds for any $j \geq 1$. Let $(\Phi_j, \Psi_j)$ be the solution to (NLS) with initial data $(\phi^j, \psi^j)$. We assume for contradiction that all $(\Phi_j, \Psi_j)$ scatter, that is,
\[
\|(\Phi_j, \Psi_j)\|_{W^1_1([0,\infty)) \times W^1_2([0,\infty))} < \infty
\]
is true for any $j \geq 1$. We set
\[
(\tilde{w}_n^J, \tilde{z}_n^J) := \sum_{j=1}^{J} \left( (\Phi_j)^{[h_n^j, \xi_n^j]}(t, x), (\Psi_j)^{[h_n^j, \xi_n^j]}(t, x) \right)
\]
and
\[
(\tilde{\bar{w}}_n^J, \tilde{\bar{z}}_n^J) := (\tilde{w}_n^J, \tilde{z}_n^J) + (e^{it\Delta} R^J_n, 0),
\]
where
\[
(\Phi_j)^{[h_n^j, \xi_n^j]}(t, x) := h_n^j e^{i x \cdot \xi_n^j} e^{-it|\xi_n^j|^2} \Phi_j(h_n^j t, h_n^j (x - 2t \xi_n^j)),
\]
\[
(\Psi_j)^{[h_n^j, \xi_n^j]}(t, x) := h_n^j e^{i x \cdot \xi_n^j} e^{-it|\xi_n^j|^2} \Psi_j(h_n^j t, h_n^j (x - 2t \xi_n^j)).
\]
We note that $(\Phi_j)^{[h_n^j, \xi_n^j], (\Psi_j)^{[h_n^j, \xi_n^j]})$ is a solution to (NLS) with initial data $G^j_n(\phi^j, \psi^j)$. Then, $(\tilde{w}_n^J, \tilde{z}_n^J)$ solves
\[
i \partial_t \tilde{w}_n^J + \Delta \tilde{w}_n^J = \sum_{j=1}^{J} \left( i \partial_t (\Phi_j)^{[h_n^j, \xi_n^j]} + \Delta (\Phi_j)^{[h_n^j, \xi_n^j]} \right) = -2 \sum_{j=1}^{J} (\Psi_j)^{[h_n^j, \xi_n^j]} (\Phi_j)^{[h_n^j, \xi_n^j]},
\]
\[
i \partial_t \tilde{z}_n^J + \frac{1}{2} \Delta \tilde{z}_n^J = \sum_{j=1}^{J} \left( i \partial_t (\Psi_j)^{[h_n^j, \xi_n^j]} + \frac{1}{2} \Delta (\Psi_j)^{[h_n^j, \xi_n^j]} \right) = - \sum_{j=1}^{J} (\Phi_j)^{2 [h_n^j, \xi_n^j]}.
\]
We also set
\[
\tilde{e}_{1,n}^J := i \partial_t \tilde{w}_n^J + \Delta \tilde{w}_n^J + 2 \tilde{w}_n^J \tilde{z}_n^J,
\]
\[
\tilde{e}_{2,n}^J := i \partial_t \tilde{z}_n^J + \frac{1}{2} \Delta \tilde{z}_n^J + (\tilde{w}_n^J)^2.
\]
Here, we introduce the following two lemmas.

**Lemma 6.1.** For any $\varepsilon > 0$, there exists $J_0 = J_0(\varepsilon)$ such that
\[
\limsup_{n \to \infty} \|(\tilde{w}_n^J, \tilde{z}_n^J) - (\tilde{w}_n^{J_0}, \tilde{z}_n^{J_0})\|_{W^1_1([0,\infty)) \times W^1_2([0,\infty))} \leq \varepsilon
\]
for any $J \geq J_0$.

**Lemma 6.2.** It follows that
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|(\tilde{e}_{1,n}^J, \tilde{e}_{2,n}^J)\|_{N^1_1([0,\infty)) \times N^2_2([0,\infty))} = 0.
\]
These are shown as in [22]. Using Lemma 6.1 with $\varepsilon = 1$, it follows that there exists $J_0$ such that
\[
\|\left(\bar{u}_n^J, \bar{v}_n^J\right)\|_{W_1(0, \infty) \times W_2(0, \infty)} \\
\leq \|\left(\bar{u}_n^J, \bar{v}_n^J\right)\|_{W_1(0, \infty) \times W_2(0, \infty)} \\
+ \|\left(\bar{u}_n^J, \bar{v}_n^J\right)\|_{W_1(0, \infty) \times W_2(0, \infty)} + \|e^{it\Delta} R_n^J\|_{W_1(0, \infty)} \\
\leq \sum_{j=1}^{J_0} \|\left(\Phi_j, \Psi_j\right)\|_{W_1(0, \infty) \times W_2(0, \infty)} + c\|R_n^J\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}} + 1 \\
\leq \sum_{j=1}^{J_0} \|\left(\Phi_j, \Psi_j\right)\|_{W_1(0, \infty) \times W_2(0, \infty)} + c\|\mathcal{F}\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}} + 1 =: M
\] (6.5)
for any $J \geq J_0$ and $n \geq 1$. Let $\varepsilon_1$ be given in Proposition 3.6. Then,
\[
\|(u_{0,n} - \bar{u}_n^J(0), v_0 - \bar{v}_n^J(0))\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}} \times \mathcal{F}\mathcal{H}^{\frac{1}{2}}} = 0.
\] (6.6)
Lemma 6.2 implies that there exists $J_1$ such that
\[
\limsup_{n \to \infty} \frac{\|\left(\bar{e}_1^J, \bar{e}_2^J\right)\|_{N_1(0, \infty) \times N_2(0, \infty)}}{\|\left(\bar{e}_1^J, \bar{e}_2^J\right)\|_{N_1(0, \infty) \times N_2(0, \infty)}} \leq \frac{\varepsilon_1}{4},
\]
for any $J \geq J_1$. Choose $J$ with $J \geq \max\{J_0, J_1\}$. There exists $n_0$ such that
\[
\|\left(\bar{e}_1^J, \bar{e}_2^J\right)\|_{N_1(0, \infty) \times N_2(0, \infty)} \leq \frac{\varepsilon_1}{2}
\] (6.7)
for any $n \geq n_0$. By (6.5), (6.6), (6.7), and Proposition 3.6, we deduce that a solution $(u_n, v_n)$ to (NLS) with initial data $(u_{0,n}, v_0)$ satisfies
\[
\|(u_n, v_n)\|_{W_1(0, \infty) \times W_2(0, \infty)} \leq C(M, \varepsilon_1) < \infty
\]
for any $n \geq n_0$. However, this contradicts with the definition of $(u_{n}, v_{n})$. Therefore, there exists $j_1 \geq 1$ such that
\[
\left\|\left(\Phi_{j_1}, \Psi_{j_1}\right)\right\|_{W_1(0, T_{\max}) \times W_2(0, T_{\max})} = \infty.
\]
By (6.4), another characterization of $\ell_{v_0}^1$ (Proposition 4.3), and Proposition 4.5, we have $\|\phi^1\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}} = \ell_{v_0}^1$ and $\phi^j = 0$ for all $j \neq j_1$. We encounter a dichotomy, $j_1 = 1$ or $j_1 = 2$.

Now, we suppose that $j_1 = 1$. Since a solution $(\Phi_1, \Psi_1)$ to (NLS) with initial data $(\phi^1, v_0)$ does not scatter, we have $\ell_{v_0} \leq \|\phi^1\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}} = \ell_{v_0}^1$ by the definition of $\ell_{v_0}$. Combining this inequality and Lemma 4.4, we obtain $\ell_{v_0} = \ell_{v_0}^1 = \|\phi^1\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}}$. This shows that $\phi^1$ is a minimizer to $\ell_{v_0}$. Moreover, it follows from Proposition 4.5 and Lemma 4.4 that $\ell_{v_0} = \ell_{v_0}^1 \leq \ell_0^1 \leq \ell_0$. Therefore, we have the identity $\ell_{v_0} = \min\{\ell_0, \ell_{v_0}\}$.

Let us move on to the case $j_1 = 2$. In this case, it follows that $(\phi^1, \psi^1) = (0, v_0)$ and $(\phi^2, \psi^2) = (\phi^2, 0)$. Since $(\Phi_2, \Psi_2)$ does not scatter, we have $\ell_{v_0} \leq \|\phi^2\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}} = \ell_{v_0}^1$ by the definition of $\ell_{v_0}$. Using Proposition 4.5 and Lemma 4.4, we obtain
\[
\ell_0 \leq \|\phi^2\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}} = \ell_{v_0}^1 \leq \ell_0^1 \leq \ell_0.
\]
In particular, we have $\ell_{v_0}^1 = \ell_0 = \|\phi^2\|_{\mathcal{F}\mathcal{H}^{\frac{1}{2}}}$. This shows that $\phi^2$ is a minimizer to $\ell_0$. In addition, we have
\[
(u_{0,n}, v_0) = \sum_{j=1,2} G_{n}^j(\phi^j, \psi^j) + (R_{n}^j, 0) = (0, v_0) + G_{n}^j(\phi^2, 0) + (R_{n}^j, 0),
\]
Then, by (6.1), (6.2), (6.3), and \(\|\phi^{h_0}\|_{\mathcal{F}H^{\frac{1}{2}}} = \ell^t_{v_0}\). Remark that we have the identity \(\ell^t_{v_0} = \min\{\ell_0, \ell_{v_0}\}\) also in this case.

In both cases, we have the identity \(\ell^t_{v_0} = \min\{\ell_0, \ell_{v_0}\}\), hence we have Theorem 1.6. If we assume that \(\ell_0 > \ell^t_{v_0}\) then the second case is precluded. This is nothing but Theorem 1.8.

Similarly, the assumption \(\ell_{v_0} > \ell^t_{v_0}\) precludes the case \(j_1 = 1\). This shows Theorem 1.9. Indeed, the above argument applies to the minimizing sequence satisfying the assumption of Theorem 1.9 and leads us to the same conclusion in the case \(j_1 = 2\). Let \(T_{\text{max}}\) denote the maximal existence time of a solution to (NLS) with initial data \((\phi^2, 0)\). Fix \(0 \leq \tau < T_{\text{max}}\). Recall that \((\Phi_j, \Psi_j)\) denotes the solution to (NLS) with initial data \((\phi^j, \psi^j)\), and \((\Phi_j)_{[h^j, \xi^j]}, (\Psi_j)_{[h^j, \xi^j]}\) does the solution to (NLS) with initial data \(G_{h^j}^{\phi^j, \psi^j}\). We set

\[
(\bar{u}_n, \bar{v}_n) := \sum_{j=1,2} \left((\Phi_j)_{[h^j, \xi^j]}, (\Psi_j)_{[h^j, \xi^j]}\right) = (0, e^{\frac{i}{2} t \Delta} v_0) + \left((\Phi_2)_{[h^2, \xi^2]}, (\Psi_2)_{[h^2, \xi^2]}\right).
\]

Then, \((\bar{u}_n, \bar{v}_n)\) solves

\[
i\partial_t \bar{u}_n + \Delta \bar{u}_n = \sum_{j=1,2} \left(i\partial_t (\Phi_j)_{[h^j, \xi^j]} + \Delta (\Phi_j)_{[h^j, \xi^j]}\right) = -2(\Phi_2)_{[h^2, \xi^2]} \overline{(\Phi_2)_{[h^2, \xi^2]}},
\]

\[
i\partial_t \bar{v}_n + \frac{1}{2} \Delta \bar{v}_n = \sum_{j=1,2} \left(i\partial_t (\Psi_j)_{[h^j, \xi^j]} + \frac{1}{2} \Delta (\Psi_j)_{[h^j, \xi^j]}\right) = -(\Phi_2)_{[h^2, \xi^2]}^2.
\]

We also set

\[
\bar{e}_{1,n} := i\partial_t \bar{u}_n + \Delta \bar{u}_n + 2\bar{v}_n \bar{u}_n = 2(\Phi_1)_{[h^1, \xi^1]} \overline{(\Phi_2)_{[h^2, \xi^2]}},
\]

\[
\bar{e}_{2,n} := i\partial_t \bar{v}_n + \frac{1}{2} \Delta \bar{v}_n + (\bar{u}_n)^2 = 0.
\]

We check the assumptions of Proposition 3.6. One has

\[
\| (\bar{u}_n, \bar{v}_n) \|_{W^1_{1, \infty}([0, \tau/h^2_n)] \times W^2_{1, \infty}([0, \tau/h^2_n])) \leq \|(0, e^{\frac{i}{2} t \Delta} v_0)\|_{W^1_{1, \infty}([0, \infty)) \times W^2_{1, \infty}([0, \infty))} + \| (\Phi_{j_0}, \Psi_{j_0}) \|_{W^1_{1, \infty}([0, \tau) \times W^2_{1, \infty}([0, \tau))} =: M < \infty,
\]

\[
\|(u_0, v_0) - (\bar{u}_n(0), \bar{v}_n(0))\|_{\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}} = \|(R_{n, \ell}^t 0)\|_{\mathcal{F}H^{\frac{1}{2}} \times \mathcal{F}H^{\frac{1}{2}}} \longrightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

and

\[
\|(\bar{e}_{1,n}, \bar{e}_{2,n})\|_{N_1([0, \tau/h^2_n)] \times N_2([0, \tau/h^2_n])} = \|\bar{e}_{1,n}\|_{N_1([0, \tau/h^2_n])} \longrightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

where the last estimate is shown as in the same spirit of Lemma 6.2 with a help of the first estimate. Therefore, we obtain

\[
(u_n, v_n) - (0, e^{\frac{i}{2} t \Delta} v_0) - \left((\Phi_2)_{[h^2, \xi^2]}, (\Psi_2)_{[h^2, \xi^2]}\right) \longrightarrow 0
\]

in \(L^\infty_t([0, \tau/h^2_n]]: X^{1/2}_1) \times L^\infty_t([0, \tau/h^2_n]): X^{1/2}_1\) as \(n \rightarrow \infty\). \(\square\)
7. Study of related optimization problems. We next consider the optimizing problem $\mathcal{B}(\rho)$ defined in (1.11).

**Theorem 7.1.** $\mathcal{B}(\rho)$ is non-increasing and right continuous. Suppose $\rho \geq 0$ is such that $\mathcal{B}(\rho) < \infty$ is true. Then, there exists a minimizer $(u_{\rho}, v_{\rho})$ to $\mathcal{B}(\rho)$ with the following properties:

1. $\|u_{\rho}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} = \mathcal{B}(\rho)$ and $\|v_{\rho}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho$;
2. $(u_{\rho}, v_{\rho}) \not\in S_{+}$.

Moreover, the identity

$$\mathcal{B}(\rho) = \inf\{\ell_{\nu_{0}} : \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\} = \inf\{\ell_{\nu_{0}} : \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\}$$

holds and the minimizer satisfies $\ell_{\nu_{0}} = \ell_{\nu_{0}}^{\dagger} = \|u_{\rho}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}}$. Furthermore,

$$\sup\{\ell_{\nu_{0}}(\ell) : \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\} \lesssim \rho, \ell 1.$$ for any $\ell \in [0, \mathcal{B}(\rho))$.

**Proof.** Non-increasing property of $\mathcal{B}(\rho)$ follows by definition. The identity

$$\mathcal{B}(\rho) = \inf\{\ell_{\nu_{0}} : \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\}$$

is also immediate by definition.

Introduce $\mathcal{B}^{\dagger}(\rho)$ as follows:

$$L(\ell, \rho) := \sup\{L_{\nu_{0}}(\ell) : \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\},$$

and

$$\mathcal{B}^{\dagger}(\rho) := \sup\{\ell : L(\ell, \rho) < \infty\} = \inf\{\ell : L(\ell, \rho) = \infty\} \in (0, \infty].$$

By Proposition 4.1 and Strichartz’ estimate, one has $L(\ell, \rho) \lesssim 1$ for $\ell \lesssim \rho 1$. Hence, $\mathcal{B}^{\dagger}(\rho) > 0$ for any $\rho > 0$. Mimicking the argument in Proposition 4.2, we see that, for each fixed $\rho \geq 0$, $L(\ell, \rho)$ is a non-decreasing continuous function of $\ell$ defined on $[0, \infty)$. Further, we see that $\mathcal{B}^{\dagger}(\rho)$ is right continuous by a standard argument.

We now claim that

$$\mathcal{B}^{\dagger}(\rho) \leq \mathcal{B}(\rho).$$

Indeed, we have

$$\mathcal{B}^{\dagger}(\rho) = \inf\{\ell : L(\ell, \rho) = \infty\} \leq \inf\{\ell : \exists \nu_{0} \in \mathcal{F}H^{\frac{1}{2}}_{x}, L(\ell) = \infty, \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\} \leq \inf\{\ell_{\nu_{0}} : \|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho\} = \mathcal{B}(\rho),$$

where the first inequality follows from fact that the existence of $\nu_{0} \in \mathcal{F}H^{\frac{1}{2}}_{x}$ such that $L(\nu_{0}) = \infty$ and $\|\nu_{0}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho$ implies $L(\ell, \rho) = \infty$.

Fix $\rho > 0$ such that $\mathcal{B}(\rho) < \infty$. Take an optimizing sequence $(u_{n}(t), v_{n}(t))$ for $\mathcal{B}^{\dagger}(\rho)$ satisfying

$$\mathcal{B}^{\dagger}(\rho) - \frac{1}{n} \leq \|u_{0,n}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \mathcal{B}^{\dagger}(\rho), \quad \|v_{0,n}\|_{F^{\frac{1}{2}}_{t}H^{\frac{1}{2}}_{x}} \leq \rho$$

and

$$n \leq \|(u_{n}, v_{n})\|_{W_{1,1}(0, \infty) 	imes W_{2,2}(0, \infty)} < \infty,$$

where $(u_{0,n}, v_{0,n}) = (u_{n}, v_{n})(0)$. Then, by a similar argument to the proof of Theorem 1.6, we obtain a minimizer $(u_{\rho}(t), v_{\rho}(t))$ to $\mathcal{B}^{\dagger}(\rho)$, which completes the proof.
of $\mathcal{B}^1(\rho) = \mathcal{B}(\rho)$. We omit the details of the proof but point out different respects compared with an optimizing sequence for $\ell_{v_0}$. The biggest difference is that the second component $v_{0,n}$ of the optimizing sequence may vary in $n$. As a result, we do not have a priori information about the second component in the profile decompositions, hence the decomposition takes the form

$$(u_{0,n}, v_{0,n}) = \sum_{j=1}^{J} G_n^j(\phi^j, \psi^j) + (R_n^j, L_n^j).$$

A contradiction argument shows there exists at least one $j$ such that $(\phi^j, \psi^j) \notin S_+$. We may let $j = 1$. One has

$$\|\phi^1\|_{\mathcal{F}^1} \leq \lim_{n \to \infty} \|u_{0,n}\|_{\mathcal{F}^1} = \mathcal{B}^1(\rho)$$

and

$$\|\psi^1\|_{\mathcal{F}^1} \leq \lim_{n \to \infty} \|v_{0,n}\|_{\mathcal{F}^1} \leq \rho$$

by the Pythagorean decomposition. Since $(\phi^1, \psi^1) \notin S_+$ and $\|\psi^1\|_{\mathcal{F}^1} \leq \rho$, one has

$$\|\phi^1\|_{\mathcal{F}^1} \geq \mathcal{B}(\rho)$$

by definition of $\mathcal{B}(\rho)$. Together with (7.2) and (7.3), we see that $\mathcal{B}(\rho) = \mathcal{B}^1(\rho) = \|\phi^1\|_{\mathcal{F}^1}$ holds and that the solution corresponds to the data $(\phi^1, \psi^1)$ is a minimizer to both of them. By (7.1) and $(\phi^1, \psi^1) \notin S_+$,

$$\mathcal{B}(\rho) = \inf \{ \ell_{v_0} : \|v_0\|_{\mathcal{F}^1} \leq \rho \} \leq \ell_{\psi^1} \leq \|\phi^1\|_{\mathcal{F}^1} = \mathcal{B}(\rho).$$

Hence, $\|\phi^1\|_{\mathcal{F}^1} = \ell_{\psi^1} = \mathcal{B}(\rho)$. Similarly, we have $\|\phi^1\|_{\mathcal{F}^1} = \ell_{\psi^1}$.

We now turn to the study of optimizing problem (1.7). Let us formulate the problem in an abstract setting. Let $f(x, y)$ be a function on $[0, \infty) \times [0, \infty)$ satisfying the following three conditions:

- Non-decreasing with respect to the both variables, i.e.,
  $$0 \leq x_1 \leq x_2, 0 \leq y_1 \leq y_2 \Rightarrow f(x_1, y_1) \leq f(x_2, y_2).$$

- Continuous, i.e., for any $(x_0, y_0) \in [0, \infty) \times [0, \infty)$,
  $$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).$$

- $f(0, 0) = 0$.

- No leaking to the infinity, i.e.,
  $$\inf \left\{ f\left(\|u_0\|_{\mathcal{F}^{1/2}}, \|v_0\|_{\mathcal{F}^{1/2}}\right) : (u_0, v_0) \notin S_+ \right\} < \min \left( \lim_{x \to \infty} f(x, 0), \lim_{y \to \infty} f(0, y) \right).$$

Let

$$\ell_f := \inf \left\{ f\left(\|u_0\|_{\mathcal{F}^{1/2}}, \|v_0\|_{\mathcal{F}^{1/2}}\right) : (u_0, v_0) \notin S_+ \right\}.$$

**Theorem 7.2.** Let $f$ satisfy the condition above. Then, it follows that

$$\ell_f = \inf_{v_0 \in \mathcal{F}^{1/2}} f(\ell_{v_0}, \|v_0\|_{\mathcal{F}^{1/2}}) = \inf_{v_0 \in \mathcal{F}^{1/2}} f(\ell_{v_0}, \|v_0\|_{\mathcal{F}^{1/2}}).$$

Furthermore, there exists a minimizer $(u^{(f)}(t), v^{(f)}(t))$ to $\ell_f$ such that

1. $f(\|u^{(f)}(0)\|_{\mathcal{F}^{1/2}}, \|v^{(f)}(0)\|_{\mathcal{F}^{1/2}}) = \ell_f$;
2. $(u^{(f)}(t), v^{(f)}(t))$ does not scatter;
3. \( \|u^{(f)}(0)\|_{\mathcal{F}H^{1/2}} = \ell_{u^{(f)}(0)} = \ell_1^{(f)}(0) \).

The minimizer is not a ground state.

**Remark 7.3.** The condition (7.4) is hard to check for general \( f \) since one does not know much about the set \( S_+ \). Following two are examples of a sufficient condition for (7.4) that does not involve \( S_+ \):

- \( \lim_{x \to \infty} f(x, 0) = \lim_{y \to \infty} f(0, y) = +\infty; \)
- there exists a solution \( (Q_{1,1}, Q_{2,1}) \) to (1.10) with \( \omega = 1 \) such that

\[
\frac{3}{4} \|Q_{1,1}\|_{\mathcal{F}H^{1/2}} + \frac{3}{4} \|Q_{2,1}\|_{\mathcal{F}H^{1/2}} < \min \left( \lim_{x \to \infty} f(x,0), \lim_{y \to \infty} f(0,y) \right).
\]

The problem (1.7) corresponds to the choice \( f(x, y) = (x^2 + ay^2)^{1/2} \). The function satisfies the first sufficient condition.

**Proof.** Since \( S_+ \neq \emptyset \), we have \( \ell_f < \infty \). Take a minimizing sequence for \( \ell_f \), i.e., take a sequence of initial data such that \( (u_0, v_0) \notin S_+ \) and

\[
\ell_f \leq f \left( \|u_0\|_{\mathcal{F}H^{1/2}}, \|v_0\|_{\mathcal{F}H^{1/2}} \right) \leq \ell_f + \frac{1}{n}.
\]

We claim that \( \|u_0\|_{\mathcal{F}H^{1/2}}, \|v_0\|_{\mathcal{F}H^{1/2}} \) is bounded. If not then \( \|u_0\|_{\mathcal{F}H^{1/2}} \) or \( \|v_0\|_{\mathcal{F}H^{1/2}} \) is not bounded. Let us consider the case \( \|u_0\|_{\mathcal{F}H^{1/2}} \) is not bounded. Take a subsequence so that \( \|u_0\|_{\mathcal{F}H^{1/2}} \to \infty \) as \( n \to \infty \). Then, by the nondecreasing assumption,

\[
f(\|u_0\|_{\mathcal{F}H^{1/2}}, 0) \leq f \left( \|u_0\|_{\mathcal{F}H^{1/2}}, \|v_0\|_{\mathcal{F}H^{1/2}} \right).
\]

Letting \( n \to \infty \), one obtains \( \lim_{x \to \infty} f(x,0) \leq \ell_f \), a contradiction. Hence, the claim is proved.

Take a subsequence so that \( (\|u_0\|_{\mathcal{F}H^{1/2}}, \|v_0\|_{\mathcal{F}H^{1/2}}) \) converges to a point, say \((x_\infty, y_\infty)\). By the continuity of \( f \), we have

\[
f(x_\infty, y_\infty) = \ell_f.
\]

On the other hand, \( (u_0, v_0) \notin S_+ \) gives us \( B(\|u_0\|_{\mathcal{F}H^{1/2}}) \leq \|u_0\|_{\mathcal{F}H^{1/2}} \). So, by the continuity of \( B \), we have

\[
B(y_\infty) \leq x_\infty.
\]

Let \( (u_\infty, v_\infty) \notin S_+ \) be the initial data of a minimizer to \( B(y_\infty) \) given in Theorem 7.1. It satisfies

\[
\|u_\infty\|_{\mathcal{F}H^{1/2}} = \ell_{u_\infty} = \ell_1^{(f)} = B(y_\infty) \leq x_\infty, \quad \|v_\infty\|_{\mathcal{F}H^{1/2}} \leq y_\infty.
\]

The minimizer is what we desired because

\[
\ell_f \leq f \left( \|u_\infty\|_{\mathcal{F}H^{1/2}}, \|v_\infty\|_{\mathcal{F}H^{1/2}} \right) \leq f(x_\infty, y_\infty) = \ell_f,
\]

which implies that \( f \left( \|u_\infty\|_{\mathcal{F}H^{1/2}}, \|v_\infty\|_{\mathcal{F}H^{1/2}} \right) = \ell_f \).

8. **Proof of corollaries of Theorem 1.11.** We have proven Theorem 1.11 in Subsection 3.3. Let us show its corollaries.

**Proof of Corollary 1.12.** For given \( v_0 \in \mathcal{F}H^{1/2} \cap H^1 \) with \( v_0 \neq 0 \), we take

\[
u_0 = v_0(x)^{1/2} |v_0(x)|^{1/2} \in \mathcal{F}H^{1/2} \cap H^1.
\]
Then, we have
\[
E[c^2 du_0, cu_0] \leq cd^2 \|\nabla v_0\|_{L^2}^2 + \frac{c^2}{2} \|\nabla v_0\|_{L^2}^2 - 2c^2 d^2 \|v_0\|_{L^3}^3
\]
for \( c > 0 \) and \( d = \|\nabla v_0\|_{L^2} \|v_0\|_{L^3}^{-3/2} \). There exists \( c_0 = c_0(v_0) > 0 \) such that the right side is negative for any \( c \geq c_0 \). For such \( c \), the corresponding solution does not scatter by virtue of Theorem 1.11. This also shows the bound
\[
\ell_{c v_0} \leq \|c^2 du_0\|_{\mathcal{F}H^{\frac{1}{2}}} = c^2 \|v_0\|_{\mathcal{F}H^{\frac{1}{2}}} \|\nabla v_0\|_{L^2} \|v_0\|_{L^3}^{-3/2}
\]
We have the desired result. \( \square \)

**Proof of Corollary 1.13.** We have
\[-\Delta \varphi - 2(\text{Re} \ e^{i\theta} v_0) \varphi = \partial \varphi.
\]
Remark that \( \varphi \) is real-valued. Multiplying this identity by \( \varphi \), and integrating, we have
\[
(-\Delta \varphi, \varphi)_{L^2} - (2(\text{Re} \ e^{i\theta} v_0) \varphi, \varphi)_{L^2} = \partial(\varphi, \varphi)_{L^2}.
\]
This can be rearranged as
\[
\|\nabla \varphi\|_{L^2}^2 + 2 \text{Re} \int (-e^{-i\theta} \varphi^2) \varphi dx = e \|\varphi\|_{L^2}^2.
\]
Here, we take \( u_0 = e^{-i\theta/2} \varphi \). Then,
\[
E[cu_0, v_0] = c^2 \|\nabla \varphi\|_{L^2}^2 + \frac{1}{2} \|\nabla v_0\|_{L^2}^2 - 2 \text{Re} \int (c^2 e^{-i\theta/2} \varphi^2) v_0 dx
\]
\[
= c^2 e \|\varphi\|_{L^2}^2 + \frac{1}{2} \|\nabla v_0\|_{L^2}^2.
\]
From \( \partial < 0 \), the choice \( c^2 = \frac{\|\nabla v_0\|_{L^2}^2}{2e \|\varphi\|_{L^2}^2} \) gives us \( E(u_0, v_0) = 0 \). Therefore, \((cu_0, v_0) \notin S_+ \) by Theorem 1.11. This also implies the bound
\[
\ell_{v_0} \leq \|cu_0\|_{\mathcal{F}H^{\frac{1}{2}}} = \frac{\|\varphi\|_{\mathcal{F}H^{\frac{1}{2}}} \|\nabla v_0\|_{L^2}}{\sqrt{2|\partial|}} \|\varphi\|_{L^2}.
\]
We complete the proof. \( \square \)

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