Local simulation of singlet statistics for a restricted set of measurements

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Abstract
The essence of Bell’s theorem is that, in general, quantum statistics cannot be reproduced by a local hidden variable (LHV) model. This impossibility is strongly manifested when statistics collected by measuring certain local observables on a singlet state, violates the Bell inequality. In this work, we search for local POVMs with binary outcomes for which an LHV model can be constructed for singlet statistics. We provide various subsets of observables for which an LHV model can be provided for singlet statistics.

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1. Introduction

A violation of the Bell–CHSH inequality [1] by statistics generated from local measurements, performed on an entangled state shared between two spatially separated parties, certifies such quantum states as nonlocal. The singlet state of two qubits (an EPR state) exhibits maximum nonlocality [2] for a proper choice of local observables. Although, for pure entangled states, the degree of nonlocality is in direct proportion to the entanglement content of a quantum state; this is, in general, not true for mixtures of entangled states [3, 4]. Werner first gave a counter-intuitive example of mixed entangled states (popularly known as Werner states) [5] whose statistics, when subjected to projective measurements, can be generated by a local hidden variable (LHV) model. A similar example for a tripartite entangled state which can be simulated by an LHV model, was first provided in a work by Toth and Acin [6]. A good review of the research on hidden variable theories can be found in [7].

Interestingly, in 2003, Toner and Bacon [8] gave a twist to earlier studies, by providing a model for singlet simulation which requires only 1 cbit of communication, supplemented with
local variables. Soon after, Cerf et al [9] showed that 1 nl-bit (single PR-Box) is also sufficient for singlet simulation. Recently, another model has been provided for singlet simulation which uses (possibly) a signaling resource, namely $S^p$ correlations, which suggests a trade-off relation between the required communication and local randomness in measurement results [10, 11]. Deggore et al [12] could map the problem of simulating entangled states to distributed sampling problems. A more thorough review of the simulation of entangled state statistics, from a communication complexity point of view, can be found in [13]. Other recent works [14–17] show that lack of free will can also be considered as a resource for singlet simulation. There have also been some attempts at solving the difficult problem of simulating multipartite entanglement and non-maximally bipartite entangled states either by the use of communication or by nonlocal (no-signaling) resources [18, 19]. All these varied approaches have deepened our understanding of quantum correlation and its use as a physical resource in various information processing tasks.

As for providing local variable models for a class of entangled states, in a seminal work in 2002, Barrett [20] generalized the work of Werner [5], by constructing an LHV model for any positive-operator-valued measurements at the expense of the weight associated with a singlet in the Werner state. Motivated by these studies, we pose the problem from the opposite direction, i.e. rather than weakening the (singlet) state, we search for a class of (weakened) dichotomic observables (POVM)s for which a local model can be provided. In particular, here, we provide the subset of the most general two outcome measurements represented by positive operator value measures (POVM)s and present local models for the singlet statistics generated from them. We provide some sets of local observables which are optimal for the protocol we have suggested. First we show that, if an observable on any one side is sufficiently restricted (deviates from the ideal projective measurement), the resulting statistics for the singlet state have an LHV model. Next, we provide another model which is symmetric, in the sense that observables on both of sides are similarly restricted. Finally, we identify a more general set of observables for which an LHV models exists, with some further restrictions. Before we derive our results, in the following section, we give a mathematical description of a general two-outcome POVM.

2. A general two-outcome POVM

Generalized quantum observables are described by POVMs [21]. For finite, say $n$, outcome measurements on a $d$-dimensional state space, a POVM is a collection of self-adjoint operators $\{E_i\}$ acting on a complex Hilbert space $\mathbb{C}^d$ satisfying the following conditions: (i) $0 \leq E_i \leq I$ for all $i$, and (ii) $\sum_i E_i = I$, where $i \in \{1, 2, \ldots, n\}$. A measurement of such an observable $\{E_i\}$ on a quantum state $\rho$ results in any one of the $n$ possible outcomes; the probability of an occurrence of an $i$th outcome (termed as clicking of $i$th effect) is $\text{Tr}[\rho E_i]$. A subclass of this type of general measurement has an interesting physical interpretation, unsharp spin properties, introduced by Busch [22, 24].

In this work, we consider general two-outcome POVMs $\{E, I - E\}$ acting on $\mathbb{C}^2$ (state space of a qubit). Effect $E$ is characterized by some parameters, say, $a_0 \in \mathbb{R}$ (a scalar) and $\vec{a} \in \mathbb{R}^3$ (a vector). We denote the norm of $\vec{a}$ by $\mu$. Then, the self-adjoint property, along with the condition $0 \leq E \leq I$, implies that $E$ can be expressed as

$$E = \frac{1}{2}[a_0 I + \mu \vec{a} \cdot \vec{\sigma}],$$

$$0 \leq a_0 \leq 2,$$

$$0 \leq \mu \leq \min\{a_0, 2 - a_0\}.$$
where $\hat{a} \cdot \vec{\sigma} = a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z$. Then, the corresponding operator, $I - E$, is also self-adjoint and satisfies the requirement $0 \leq I - E \leq I$. Thus, equation (1), along with the conditions (2) and (3), supplemented with an arbitrary direction $\hat{a}$, completely determine a two-outcome POVM $\{E, I - E\}$ acting on $\mathbb{C}^2$. The region feasible for the parameters $a_0$ and $\mu$, for defining such an effect $E_{a_0, \mu}$, is illustrated in figure 1.

An interesting application of the general two-outcome measurements considered here is in the study of the spin properties of spin-$\frac{1}{2}$ systems. In this context, P Busch [22, 24] first showed that a subclass of general two-outcome POVMs can be interpreted as a measurement of the unsharp-spin property of spin-$\frac{1}{2}$ particles. Under the condition of rotation covariance, the parameter $\{a_0, \mu\}$ is decoupled from $\hat{a}$ which can then be interpreted as an orientation of the measuring device. Further, the condition of symmetry under a rotation $\pi$ of the measuring device, gives $a_0 = 1$. Thus, the effect operator for an unsharp spin observable is in the form of $E_{a_0}^\mu (\hat{a}) = \frac{1}{2}[I \pm \mu \hat{a} \cdot \vec{\sigma}]$. The spectral decomposition of the positive operators $E_{a_0}^\mu (\hat{a})$ is

$$E_{a_0}^\mu (\hat{a}) = \left(\frac{1 \pm \mu}{2}\right) \frac{1}{2}[I \pm \hat{a} \cdot \vec{\sigma}] + \left(\frac{1 \mp \mu}{2}\right) \frac{1}{2}[I - \hat{a} \cdot \vec{\sigma}],$$

where $\frac{1}{2}[I + \hat{a} \cdot \vec{\sigma}]$ and $\frac{1}{2}[I - \hat{a} \cdot \vec{\sigma}]$ are one-dimensional spin projection operators on the Hilbert space $\mathbb{C}^2$. Now, the quantity $\frac{1\mp \mu}{2}$ can be suitably interpreted as degree of the reality (unsharpness) of the outcomes obtained from a spin measurement along direction $\hat{a}$. From this representation it is clear that the POVM $\{E_{a_0}^\mu (\hat{a}), E_{a_0}^\mu (\bar{\hat{a}})\}$ is a smeared version of the projective measurement $\{\frac{1}{2}[I + \hat{a} \cdot \vec{\sigma}], \frac{1}{2}[I - \hat{a} \cdot \vec{\sigma}]\}$—and in the case of the projective measurements, the unsharp parameter $\mu = 1$.

Another important property is that under suitable conditions, two POVMs can be jointly measurable [25]. Two POVMs of the form $\{E_1, I - E_1\}$ and $\{E_2, I - E_2\}$ are jointly measurable if there exits a four-outcome POVM $\{E_{12}, E_{12'}, E_{12}, E_{12'}\}$ such that it can reproduce the correct marginals, i.e. $E_1 = E_{12} + E_{12'}$ and $E_2 = E_{12} + E_{12'}$. For an unsharp spin observable it has been shown that [22] (also see the review [23]) two observables parameterized by, say, $(\mu_1, \hat{a}_1)$ and $(\mu_2, \hat{a}_2)$ are jointly measurable if, and only if, $\|\mu_1 \hat{a}_1 + \mu_2 \hat{a}_2\| + \|\mu_1 \hat{a}_1 - \mu_2 \hat{a}_2\| \leq 2$. On considering the unsharp parameter for both the spin observables to be same, i.e. $\mu_1 = \mu_2$, along with the fact $\|\hat{a}_1 + \hat{a}_2\| + \|\hat{a}_1 - \hat{a}_2\| \leq 2\sqrt{2}$ for any pair of unit vectors $\hat{a}_1$ and $\hat{a}_2$, it is easy to conclude that if the unsharp parameters $\mu_1 = \mu_2 \leq \frac{1}{2\sqrt{2}}$, then the joint measurement of an unsharp spin property can be realized for any such pair of directions.

**Figure 1.** The parameter $a_0 (\mu)$ varies along the horizontal (vertical) axis. Any point $(a_0, \mu)$ lying in the shaded triangular region POI (together with an arbitrary parameter $\hat{a}$), determines a two-outcome POVM $\{E, I - E\}$. Points on the dashed line, PU, represent unsharp spin measurements. Point P(1, 1) represents the ideal projective measurements.
3. An LHV model for singlet statistics for two outcome POVMs

Suppose, two spatially separated parties, Alice and Bob, share one qubit each from a singlet state

$$\rho_{AB} = \frac{1}{4}[I \otimes I - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z].$$

Let Alice’s (Bob’s) observable be a most general two-outcome POVM $E_A[a_0, \mu_A, \hat{a}]$ ($E_B[b_0, \mu_B, \hat{b}]$), defined by equation (1). If the effect $E_{A(B)}$ clicks, we denote the outcome by ‘yes’ otherwise ‘no’. Then the joint outcome probabilities are the following:

$$P_{AB}(\text{yes}, \text{yes}) = \frac{1}{4}[a_0 b_0 - \mu_A \mu_B \hat{a} \cdot \hat{b}],$$

$$P_{AB}(\text{yes}, \text{no}) = \frac{1}{4}[a_0(2 - b_0) + \mu_A \mu_B \hat{a} \cdot \hat{b}],$$

$$P_{AB}(\text{no}, \text{yes}) = \frac{1}{4}[(2 - a_0)b_0 + \mu_A \mu_B \hat{a} \cdot \hat{b}],$$

$$P_{AB}(\text{no}, \text{no}) = \frac{1}{4}[(2 - a_0)(2 - b_0) - \mu_A \mu_B \hat{a} \cdot \hat{b}].$$

3.1. Models for two-outcome measurements

The violation of the Bell–CHSH inequality [1], implies that there can be no LHV model for the singlet statistics generated by the projective measurements by both parties. Therefore, the statistics of the singlet can have an LHV model only if the general two outcome POVMs considered here are restricted (deviate from ideal projective measurements) in some way or other. Following Werner’s local model for some mixed entangled states [5], we provide two LHV models for a singlet state under certain restrictions on the parameters of two outcome POVMs. In both types of model, the vectors $\hat{\lambda} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, uniformly distributed over the unit sphere, are the local variables pre-shared between Alice and Bob.

3.1.1. A fully biased model $M_{fb}$. Let Bob’s observable $E_B(b_0, \mu_B, \hat{b})$ satisfy restriction $\mu_B \leq \frac{1}{2}\min\{b_0, 2 - b_0\}$ but there is no restriction on Alice’s observables, see figure 2.

Alice, for her observable $E_A[a_0, \mu_A, \hat{a}]$, declares ‘yes’ with a probability

$$P^A_{\lambda}(\text{yes}) = \frac{a_0}{2} + \frac{1}{2}\mu_A \cos \alpha.$$
Figure 3. Alice’s (Bob’s) parameters can take values from the dark gray triangular region on the left (right). Alice’s parameters, \(a_0\) and \(\mu_A\), as well as Bob’s parameters, \(b_0\) and \(\mu_B\), are restricted in the same way and come from the triangular region MOI on the left and right, respectively.

where \(\alpha\) is the angle between direction \(\hat{a}\) and \(\hat{\lambda}\). On the other hand, for the observable \(E_B[b_0, \mu_B, \hat{b}]\), Bob declares ‘yes’ with a probability

\[
P_{\hat{B}}(\text{yes}) = \frac{b_0}{2} - \mu_B \text{sgn}(\cos \beta),
\]

where \(\beta\) is the angle between the direction \(\hat{b}\) and \(\hat{\lambda}\), and \(\text{sgn}(x) = +1 (-1)\) for \(x \geq 0 (x < 0)\).

The joint probability of the outcome (yes, yes) can be calculated from

\[
P_{AB}^{\text{lhv}}(\text{yes}, \text{yes}) = \int \rho(\hat{\lambda}) P_{\hat{A}}(\text{yes}) P_{\hat{B}}(\text{yes}) d\hat{\lambda},
\]

where \(\rho(\hat{\lambda})\) is the considered (uniform) distribution of the hidden variable \(\hat{\lambda}\). Evaluating the above integral gives

\[
P_{AB}^{\text{lhv}}(\text{yes}, \text{yes}) = \frac{1}{4}[a_0 b_0 - \mu_A \mu_B \hat{a} \cdot \hat{b}],
\]

which exactly matches the quantum mechanical prediction for the outcome (yes, yes). The desired quantum mechanical probabilities for the other possible outcomes easily follow; for example, \(P_{AB}^{\text{lhv}}(\text{yes}, \text{no})\) is obtained simply by replacement \(P_{\hat{B}}(\text{yes}) \rightarrow P_{\hat{B}}(\text{no}) = 1 - P_{\hat{B}}(\text{yes})\) in the integrand of equation (6).

3.1.2. A fully symmetric model \(M_{\text{fs}}\). Let Alice’s and Bob’s observables satisfy the following restriction (see figure 3):

\[
\mu_A \leq \frac{1}{\sqrt{2}} \min\{a_0, 2 - a_0\},
\]

\[
\mu_B \leq \frac{1}{\sqrt{2}} \min\{b_0, 2 - b_0\}.
\]

Alice declares ‘yes’, for her observable \(E_A[a_0, \mu_A, \hat{a}]\), with a probability

\[
P_{\hat{A}}(\text{yes}) = \frac{a_0}{2} + \frac{1}{\sqrt{2}} \mu_A \cos \alpha,
\]

where \(\alpha\) is the angle between direction \(\hat{a}\) and \(\hat{\lambda}\).

Where else Bob declares ‘yes’, for his observable, \(E_B[b_0, \mu_B, \hat{b}]\), with a probability

\[
P_{\hat{B}}(\text{yes}) = \frac{b_0}{2} - \frac{1}{\sqrt{2}} \mu_B \text{sgn}(\cos \beta),
\]
where $\beta$ is the angle between the direction $\hat{b}$ and $\hat{\lambda}$, and $\text{sgn}(x) = +1(-1)$ for $x \geq 0(x < 0)$. As in the fully biased model $M_{fb}$, we find that this model, $(M_{fs})$, also simulates the correct statistics for the singlet.

If we consider the unsharp spin properties on both sides with a uniform value of $\mu_A$ and $\mu_B$ and also assume any pair are jointly measurable [22, 23] on both sides, then the conditions of the model $M_{fs}$ are automatically satisfied and hence this LHV model $M_{fs}$ can simulate the singlet statistics for any arbitrary pair of respective directions $\hat{a}$ and $\hat{b}$ for Alice and Bob.

3.2. A measure of the restriction on observables

By considering that the observables of Alice and Bob are picked from a uniform distribution of all possible two-outcome POVMs, we can define the measure $r$ for a restriction on the observables of any of the two parties in the following way (see figures 2 and 3):

$$r = \left[ 1 - \frac{\text{Area(MOI)}}{\text{Area(POI)}} \right] \times 100. \quad (11)$$

Now, one can easily calculate that in the model $M_{fb}$ ($M_{fs}$), there is a 0% (29.3%) restriction on Alice’s observables whereas Bob’s observables are restricted by 50% (29.3%).

Another interesting observation is that models $M_{fb}$ and $M_{fs}$ belong to a general class of LHV models, $\{M_\kappa : \kappa \geq 0\}$. Under the restrictions,

$$\mu_A \leq \kappa \min\{a_0, 2 - a_0\}, \quad (12)$$

$$\mu_B \leq \frac{1}{2\kappa} \min\{b_0, 2 - b_0\}. \quad (13)$$

Alice declares ‘yes’ with a probability

$$P_A^\theta(\text{yes}) = \frac{a_0}{2} + \frac{1}{2\kappa} \mu_A \cos \alpha. \quad (14)$$

Bob declares ‘yes’ with a probability

$$P_B^\theta(\text{yes}) = \frac{b_0}{2} - \kappa \mu_B \text{sgn}(\cos \beta). \quad (15)$$

For any non-negative value of $\kappa$, we get an LHV model—$M_{fb}$ ($M_{fs}$) corresponding to $\kappa = 1$ ($\kappa = \frac{1}{\sqrt{2}}$). In figure 4 the two curves show the % restriction on Alice’s and Bob’s observables for the LHV models corresponding to different values of $\kappa$. The intersection point of the two curves corresponds to the symmetric model $M_{fs}$. Notice that $\kappa = \frac{1}{2}$ corresponds to another fully biased model, say $M_{fb}'$, which is the same as $M_{fb}$ except that the conditions on Alice’s and Bob’s observables are interchanged. In fact all the models for which $\kappa \in (0, \frac{1}{2}] \cup [1, \infty)$ are fully biased models; however, one can immediately observe that within this subclass, either $M_{fb}$ or $M_{fb}'$ is sufficient to simulate any other fully biased model. Thus only the subclass $\{M_\kappa : \kappa \in [1/2, 1]\}$ contains tight LHV models in the sense that they can capture any varying degree of restrictions on Alice’s and Bob’s observables.

3.3. A different class of model

In the previous cases, we put a restriction on the observables separately, on both sides. Now we put the following restriction on the observables, where one of the restrictions involves the parameters of both sides:

$$\frac{1}{\eta} \leq a_0 \leq 2 - \frac{1}{\eta}. \quad (16)$$
Figure 4. The black (gray) curve shows the percentage restriction on Alice’s (Bob’s) observables for LHV models corresponding to different values of $\kappa$. The intersection point of the two curves at $\kappa = \frac{1}{\sqrt{2}}$ corresponds to the completely symmetric LHV model $M_{fs}$. The fully biased model $M_{fb}$ ($M'_{fb}$) corresponds to $\kappa = \frac{1}{2}$ ($\kappa = 1$). All the models for $\kappa \in (0, \frac{1}{\sqrt{2}}] \cup [1, \infty)$ are fully biased.

\[ \mu_A \mu_B \leq \frac{1}{2\eta} \min\{b_0, 2 - b_0\}, \quad (17) \]

where $\eta \geq 1$. Alice and Bob can now simulate singlet statistics according to the following protocol:

\[ P^A(\text{yes}) = \frac{a_0}{2} + \frac{1}{2\eta} \cos \alpha, \quad (18) \]

\[ P^B(\text{yes}) = \frac{b_0}{2} - \eta \mu_A \mu_B \text{sgn}(\cos \beta). \quad (19) \]

But this model is obviously nonlocal as Bob’s output involves the parameters of observables on both sides. But the model can be made local for a given $\eta$ and fixed $\mu_A$. In this case there is no restriction on the direction $\hat{a}$ of Alice’s POVM. It might seem that by increasing the value of $\eta$, the range of $a_0$ can be extended but then, due to the condition (16), the range of $\mu_B$ is also restricted. So in some sense, in this model, $a_0$ and $\mu_B$ maintain a complementary relation for a given $b_0$.

4. Conclusion

The simulation of quantum statistics for a Werner state by LHV, has been an interesting area for understanding the physics of entanglement [3–5, 20]. We have studied cases where LHV simulation is possible for a singlet state. We find the optimal set of two outcome observables, for which singlet simulation by an LHV is possible under the suggested protocol. It is also interesting that for a uniform, unsharp parameter, the joint measurability of an unsharp spin property on both sides, implies an LHV model for a singlet. It will be interesting to study whether the set can be enlarged with respect to different LHV models.
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