ASYMPTOTIC DECAY FOR THE CLASSICAL SOLUTION OF THE CHEMOTAXIS SYSTEM WITH FRACTIONAL LAPLACIAN IN HIGH DIMENSIONS

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ABSTRACT. In this paper, we study the generalized chemotaxis system with fractional Laplacian. The existence and the uniqueness of global classical solution are proved under the assumption that the initial data are small enough. During the proof, with the help of the fixed point theorem, the asymptotic decay behaviors of \( u \) and \( \nabla v \) are also shown.

1. Introduction. This paper is devoted to the study of a parabolic-parabolic Keller-Segel system, involving the fractional Laplacian

\[
\begin{cases}
\partial_t u + (-\Delta)^s u + \nabla \cdot (u \nabla v) = 0, & x \in \mathbb{R}^n, \ t > 0, \\
\partial_t v + (-\Delta)^s v = u - v, & x \in \mathbb{R}^n, \ t > 0, \\
u(0, x) = u_0(x), \ v(0, x) = v_0(x), & x \in \mathbb{R}^n.
\end{cases}
\tag{1.1}
\]

Here, \( u \) represents the density of the cells, \( v \) stands for the concentration of the chemical. As usual, this model is developed to describe the biological phenomenon chemotaxis with anomalous diffusion. The action of the integro-differential operator \((-\Delta)^s\), \( s \in (0, 1) \), on a smooth bounded function \( v: \mathbb{R}^n \to \mathbb{R} \) is defined by

\[
(-\Delta)^s v(x) = \gamma_n \text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+2s}} \, dy \text{ with } \gamma_n := \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n+2s}/2},
\tag{1.2}
\]

and the notation P.V. means that the integral is taken in the Cauchy principle value sense (see for example [18]).

When \( s = 1 \), system (1.1) is the classical model

\[
\begin{cases}
\partial_t u = \nabla \cdot (\nabla u - u \nabla v), & x \in \mathbb{R}^n, \ t > 0, \\
\partial_t v = \Delta v + u - v, & x \in \mathbb{R}^n, \ t > 0, \\
u(0, x) = u_0(x), \ v(0, x) = v_0(x), & x \in \mathbb{R}^n.
\end{cases}
\tag{1.3}
\]

It was proposed by Keller and Segel [13] in 1970, which has played an increasing important role in the past four decades. Corrias and Perthame [9, 10] proved that small initial data give rise to the global weak solutions. In the integral sense, the weak solution vanish as the heat equation for large time and exhibit a regularizing effect of hyper-contractivity type was shown as well. Moreover, the review [1] by
Bellomo, Bellouquid, Tao and Winkler showed the qualitative analysis of a variety of chemotaxis models, such as the existence of weak solutions, blow-up and asymptotic behavior.

When \( s \in (0, 1) \), system (1.1) is the fractional chemotaxis model, which was first studied by Escudero in [11]. More precisely, the author focused on the Keller-Segel model as follows

\[
\begin{align*}
\frac{\partial u}{\partial t} + (-\Delta)^s u + \chi \nabla \cdot [u \nabla v] &= 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\
-\Delta v + v &= u, \quad x \in \mathbb{R}^n, \quad t > 0, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

(1.4)

and proved that the model has blowing-up solutions for large initial conditions in dimensions \( n \geq 2 \). In dimension \( n = 1 \), the author also obtained the global existence with the initial data \( u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \). This system has been widely studied by Biler et al. [2, 4, 5, 6, 7] and Li et al. [14, 15, 16], as well. For example, in [2], the conditions for local and global in time existence of positive weak solutions in dimensions \( n = 2, 3 \) were obtained. In [6], they deduced the existence of local in time mild solutions and global mild solutions under the small initial data, based on applications of the linear analytic semigroup theory to quasi-linear evolutions equations. In [7], the blow-up of solutions, with regard to the Keller-Segel model with either classical or fractional diffusion in two dimensions, in terms of suitable Morrey spaces norm is derived. Moreover, Li, Rodrigo and Zhang [16] obtained the local existence and uniqueness of solutions, and also attained mass conservation and non-negative solutions.

Compared with the former works, we study parabolic-parabolic Keller-Segel system and assume that the models both move by anomalous diffusion. Moreover, the purpose of this paper is to prove the global existence and the uniqueness of the classical solution to system (1.1), as well as to attain the asymptotic decay behavior. Specifically, with the aid of the fixed point theorem, under the small initial data, the existence, the uniqueness are proved and the asymptotic decay behavior of the solution is also obtained. More precisely, we clarify the asymptotic decay behaviors of the \( W^{m-n-3, \infty} \)-norm of \( u \) and \( \nabla v \) (see Theorem 3.1).

At last, we present the outline of the paper. In Section 2, we prove some preliminary results. Section 3 deals with the proof of the main results in the paper.

2. Preliminary. In this section, we will give some results which will be used in the proof of the existence of the solution to the problem (1.1) in the next section. As in [3, 8, 10], the solution of (1.1) can be written as

\[
u(t) := u(t, x) = K_t(x) * u_0(x) - \int_0^t \nabla K_{t-\tau}(x) * [u(\tau) \nabla v(\tau)] d\tau \quad \text{(2.1)}
\]

and

\[
v(t) := v(t, x) = e^{-t} K_t(x) * v_0(x) + \int_0^t e^{(t-\tau)} K_{t-\tau}(x) * u(\tau) d\tau. \quad \text{(2.2)}
\]

Here, \( K_t(x) \) is the heat kernel, which is defined by

\[
K(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-|\xi|^2} d\xi
\]

(2.3)

and

\[
K_t(x) = t^{-n/2} K(t^{-1/2} x).
\]

(2.4)
Lemma 2.1. (\cite{12}) $K(x)$ satisfies the following properties

(i) $|K(x)| \leq C(1 + |x|)^{-n-2\gamma}, \, K(x) \in L^p(\mathbb{R}^n), \, p \in [1, \infty]$, \quad \text{(2.5)}

(ii) $|\nabla K(x)| \leq C(1 + |x|)^{-n-1}, \, \nabla K(x) \in L^p(\mathbb{R}^n), \, p \in [1, \infty]$, \quad \text{(2.6)}

(iii) $|(-\Delta)^\gamma K(x)| \leq C(1 + |x|)^{-n-2\gamma}, \, (-\Delta)^\gamma K(x) \in L^p(\mathbb{R}^n), p \in [1, \infty]$. \quad \text{(2.7)}

Here, $C$ is a positive constant.

We perform the following $L^1$-norm of $K_t(x)$ combining with Lemma 2.1, which will be used later in the article.

Lemma 2.2. For $0 < \gamma < 1$, we have the following inequality for $K_t(x)$ that

$$\|(-\Delta)^\gamma K_t(x)\|_{L^1(\mathbb{R}^n)} \leq Ct^{-\frac{\gamma}{2}}. \quad \text{(2.8)}$$

Proof. Combining (2.7) with (2.4), we get

$$\|(-\Delta)^\gamma K_t(x)\|_{L^1(\mathbb{R}^n)} = t^{-\frac{\gamma}{2}} \int_{\mathbb{R}^n} |(-\Delta)^\gamma K(t^{-\frac{1}{2}}x)|dx$$

$$= t^{-\frac{\gamma}{2}} \cdot t^{-\frac{\gamma}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(-\Delta)^\gamma K(y)|dy$$

$$\leq Ct^{-\frac{\gamma}{2}} \int_{\mathbb{R}^n} (1 + |y|)^{-n-2\gamma}dy$$

$$\leq Ct^{-\frac{\gamma}{2}} \int_{\mathbb{R}^n} (1 + r)^{-n-2\gamma} \cdot r^{n-1}dr$$

$$\leq Ct^{-\frac{\gamma}{2}} \int_0^\infty (\frac{r}{1+r})^{n-1} \cdot (1 + r)^{-1-2\gamma}dr$$

where $C$ is a constant, not depending on $t$. \hfill \Box

Lemma 2.3. Consider

$$\begin{cases} \partial_t u + (-\Delta)^\gamma u = 0, & x \in \mathbb{R}^n, \, t > 0, \\ u(0, x) = \phi(x), \quad x \in \mathbb{R}^n. \end{cases} \quad \text{(2.10)}$$

Using Fourier transformation, the solution of (2.10) can be written as

$$u(t, x) = K_t(x) * \phi(x). \quad \text{(2.11)}$$

Let $N \geq 0$ be an integer. Then the following estimates hold

$$\|K_t(x) * \phi(x)\|_{W^{N, \infty}(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{\gamma}{2}} \|\phi(x)\|_{W^{N+\gamma, 1}(\mathbb{R}^n)}, \, \forall \, t \geq 0 \quad \text{(2.12)}$$

and

$$\|K_t(x) * \phi(x)\|_{W^{N, 1}(\mathbb{R}^n)} \leq C\|\phi(x)\|_{W^{N, 1}(\mathbb{R}^n)}, \, \forall \, t \geq 0 \quad \text{(2.13)}$$

where $C$ is a positive constant, not depending on $t$.

Proof. Using (2.4) and (2.5) with $p = \infty$, we have

$$\|K_t(x)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{\gamma}{2}}. \quad \text{(2.14)}$$

We obtain from (2.11) and (2.14) that

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{\gamma}{2}} \|\phi(x)\|_{L^1(\mathbb{R}^n)}, \, \forall \, t > 0 \quad \text{(2.15)}$$

and

$$\|u(t)\|_{L^1(\mathbb{R}^n)} \leq C\|\phi(x)\|_{L^1(\mathbb{R}^n)}, \, \forall \, t \geq 0 \quad \text{(2.16)}$$

where $C$ is independent of $t$.\hfill \Box
Multiplying the first equation of (2.10) by $u$, integrating on $[0,t]$ with respect to $\tau$, we find
\[ \|u(t)\|_{L^2(\mathbb{R}^n)} \leq \|\phi(x)\|_{L^2(\mathbb{R}^n)}, \quad \forall \, t \geq 0. \tag{2.17} \]
For multiindex $\alpha (|\alpha| \leq m)$, $w = D_\alpha u$ satisfies
\[ \begin{cases} \partial_t w + (-\Delta)^s w = 0, & x \in \mathbb{R}^n, \ t > 0, \\ w(0, x) = D_\alpha \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.18} \]
The same argument as in the proof of (2.17) gives
\[ \|u(t)\|_{H^s(\mathbb{R}^n)} \leq C\|\phi(x)\|_{H^s(\mathbb{R}^n)}, \quad \forall \, t \geq 0. \tag{2.19} \]
By Sobolev Embedding Theorem, we have
\[ \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C\|u(t)\|_{H^{\frac{n+1}{2}}(\mathbb{R}^n)} \leq C\|\phi(x)\|_{H^{\frac{n+1}{2}}(\mathbb{R}^n)} \leq C\|\phi(x)\|_{W^{n+1,1}(\mathbb{R}^n)}, \quad \forall \, t \geq 0 \tag{2.20} \]
where $C$ is a positive constant.
Combining (2.15) with (2.20) we obtain
\[ \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{\theta}{\theta - q}}\|\phi(x)\|_{W^{n+1,1}(\mathbb{R}^n)}, \quad \forall \, t \geq 0 \tag{2.21} \]
where $C$ is a constant, not depending on $t$. Again, applying (2.21), (2.16) for $w$, the conclusions (2.12), (2.13) follow \(\Box\).

**Lemma 2.4.** Let $N \geq 0$ be an integer. Assume $u = K_t(x) * \phi(x)$ is a solution of (2.10), then the following estimate holds
\[ \|K_t(x) * \phi(x)\|_{W^{N,q}(\mathbb{R}^n)} \leq Ct^{-\frac{\theta}{2}(\frac{1}{q} - \frac{1}{p})}\|\phi(x)\|_{W^{N,q}(\mathbb{R}^n)}, \quad \forall \, t > 0 \tag{2.22} \]
where $1 < q \leq p < \infty$ and $C$ is a positive constant, not depending on $t$.

**Proof.**

**Step 1.** Young’s inequality.

Let
\[ A(x) \in L^\rho(\mathbb{R}^n), \ \phi(x) \in L^q(\mathbb{R}^n) \tag{2.23} \]
and
\[ 1 < q < \rho', \ \frac{1}{\rho} + \frac{1}{\rho'} = 1, \tag{2.24} \]
then
\[ \|A(x) * \phi(x)\|_{L^\rho(\mathbb{R}^n)} \leq \|A(x)\|_{L^\rho(\mathbb{R}^n)}\|\phi(x)\|_{L^q(\mathbb{R}^n)} \tag{2.25} \]
where
\[ \frac{1}{\rho'} = 1 - \frac{1}{\rho} = \frac{1}{q} - \frac{1}{p}. \tag{2.26} \]

**Step 2.** The estimate of $L^\rho$-norm of $K_t(x) * \phi(x)$. From formulas (2.4) and (2.5), changing the variable $\xi = t^{-\frac{1}{\theta}} x$, we have
\[ \|K_t(x)\|_{L^\rho(\mathbb{R}^n)} = t^{-\frac{\theta}{\rho'}} \left( \int_{\mathbb{R}^n} K^\rho(t^{-\frac{1}{\theta}} x) dx \right)^{\frac{1}{\rho}} \]
\[ = t^{-\frac{\theta}{\rho'}(\frac{1}{\theta} - \frac{1}{p})} \left( \int_{\mathbb{R}^n} K^\rho(\xi) d\xi \right)^{\frac{1}{\rho}} \]
\[ \leq Ct^{-\frac{\theta}{\rho'}(1 - \frac{1}{p})}. \tag{2.27} \]
Replacing \(u\) with the same arguments as in the proof of (2.27), we get
\[
\|K_t(x)\|_{L^p(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2p}}. \tag{2.28}
\]
Combining (2.25) with (2.28), we obtain
\[
\|K_t(x) \ast \phi(x)\|_{L^p(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2p}}\|\phi(x)\|_{L^q(\mathbb{R}^n)}, \quad \forall \ t > 0. \tag{2.29}
\]

**Step 3.** Complete the proof.

For multiindex \(\alpha (|\alpha| \leq N)\), replacing \(u\) and \(\phi\) by \(D_x^\alpha u\) and \(D_x^\alpha \phi\) respectively, one can get (2.22).

**Lemma 2.5.** Let \(N \geq 0\) be an integer. Assume \(u = K_t(x) \ast \phi(x)\) is a solution of (2.10), then the following estimates hold
\[
\|\nabla (K_t(x) \ast \phi(x))\|_{W^{N+2,1}(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n+1}{2N+2}}\|\phi(x)\|_{W^{N+2,1}(\mathbb{R}^n)}, \quad \forall \ t \geq 0 \tag{2.30}
\]
and
\[
\|\nabla (K_t(x) \ast \phi(x))\|_{W^{N+1,1}(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n+1}{2N+1}}\|\phi(x)\|_{W^{N+1,1}(\mathbb{R}^n)}, \quad \forall \ t \geq 0 \tag{2.31}
\]
where \(C\) is a positive constant, not depending on \(t\).

**Proof.** By (2.11) we have
\[
u(t, x) = K_t(x) \ast \phi(x) \tag{2.32}
\]
and
\[
\nabla u(t, x) = \nabla (K_t(x) \ast \phi(x)) = \nabla (K_t(x)) \ast \phi(x). \tag{2.33}
\]
Using (2.4) and (2.6) with \(p = \infty\), we have
\[
\|\nabla K_t(x)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2N+1}}. \tag{2.34}
\]
By Young’s inequality and (2.34), we have
\[
\|\nabla u(t, x)\|_{L^\infty(\mathbb{R}^n)} \leq \|\nabla K_t(x)\|_{L^\infty(\mathbb{R}^n)}\|\phi(x)\|_{L^1(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2N+1}}\|\phi(x)\|_{L^1(\mathbb{R}^n)}, \quad \forall \ t > 0. \tag{2.35}
\]
With the same arguments as in the proof of (2.27), we get
\[
\|\nabla u(t, x)\|_{L^1(\mathbb{R}^n)} \leq C\|\phi(x)\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} t^{-\frac{n+1}{2}}|\nabla_x K((x - \xi)t^{-\frac{n}{2}})|d\xi
\]
\[= C t^{-\frac{n}{2}}\|\phi(x)\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla_y K(y)|dy \tag{2.36}
\]
\[\leq C t^{-\frac{n}{2}}\|\phi(x)\|_{L^1(\mathbb{R}^n)}, \quad \forall \ t > 0.
\]
Replacing \(u\) by \(\nabla u\) in (2.20) and (2.16), we get
\[
\|\nabla u(t, x)\|_{L^\infty(\mathbb{R}^n)} \leq C\|\phi(x)\|_{W^{N+2,1}(\mathbb{R}^n)}, \quad \forall \ t \geq 0 \tag{2.37}
\]
and
\[
\|\nabla u(t, x)\|_{L^1(\mathbb{R}^n)} \leq C\|\phi(x)\|_{W^{1,1}(\mathbb{R}^n)}, \quad \forall \ t \geq 0. \tag{2.38}
\]
Combining (2.35) with (2.37) we get
\[
\|\nabla u(t, x)\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n+1}{2N+2}}\|\phi(x)\|_{W^{N+2,1}(\mathbb{R}^n)}, \quad \forall \ t \geq 0. \tag{2.39}
\]
Combining (2.36) with (2.38) we get
\[
\|\nabla u(t, x)\|_{L^1(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{2}}\|\phi(x)\|_{W^{1,1}(\mathbb{R}^n)}, \quad \forall \ t \geq 0. \tag{2.40}
\]
For multiindex $\alpha (|\alpha| \leq N)$, replacing $u$ and $\phi$ by $D_x^\alpha u$ and $D^\alpha \phi$ respectively in (2.30)-(2.40), one can get (2.30)-(2.31).

**Lemma 2.6.** Let $N \geq 0$ be an integer and $u = K_t(x) \ast \phi(x)$ be a solution of (2.10), then the following estimate holds for any $1 < q \leq p < \infty$

$$\|D^\gamma (K_t(x) \ast \phi(x))\|_{W^{\gamma,p} (\mathbb{R}^n)} \leq C t^{-\frac{1}{p} (|\gamma| + \frac{n}{q} - \frac{1}{q})} \|\phi(x)\|_{W^{\gamma,p} (\mathbb{R}^n)}$$

(2.41)

where $\gamma$ is a multiindex with $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ and $D^\gamma = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$. Here, $C$ is a positive constant, not depending on $t$.

**Proof.** For multiindex $\gamma$, from (2.11), we have

$$D^\gamma (u(t,x)) = D^\gamma (K_t(x) \ast \phi(x)) = D^\gamma (K_t(x) \ast \phi(x)).$$

(2.42)

Note that

$$\|D^\gamma (K_t(x))\|_{L^p (\mathbb{R}^n)} \leq Ct^{-\frac{1}{p} (|\gamma| + \frac{n}{q} - \frac{1}{q})}, \forall t > 0,$$

(2.43)

by Young’s inequality, we deduce

$$\|D^\gamma (K_t(x) \ast \phi(x))\|_{L^p (\mathbb{R}^n)} \leq \|D^\gamma (K_t(x))\|_{L^p (\mathbb{R}^n)} \|\phi(x)\|_{L^q (\mathbb{R}^n)}$$

$$\leq Ct^{-\frac{1}{p} (|\gamma| + \frac{n}{q} - \frac{1}{q})} \|\phi(x)\|_{L^q (\mathbb{R}^n)}, \forall t > 0,$$

(2.44)

where $1 - \frac{1}{q} = \frac{1}{p} - \frac{1}{q}$.

For multiindex $\alpha (|\alpha| \leq N)$, replacing $u$ and $\phi$ by $D_x^\alpha u$ and $D^\alpha \phi$ respectively in (2.44) one can get (2.41).

**Lemma 2.7.** Consider the Cauchy problem

$$\begin{cases}
\partial_t u + (-\Delta)^s u = F(t,x), & x \in \mathbb{R}^n, \ t > 0, \\
       u(0, x) = \phi(x), & x \in \mathbb{R}^n.
\end{cases}$$

(2.45)

Given a positive number $T > 0$. Assume

$$\phi(x) \in H^s (\mathbb{R}^n), \ F(t,x) \in L^2 (0,T; L^2 (\mathbb{R}^n)),$$

(2.46)

then problem (2.45) has a unique weak solution satisfying

$$u \in L^2 (0,T; H^{2s} (\mathbb{R}^n)) \text{ and } \partial^\alpha u \in L^2 (0,T; L^2 (\mathbb{R}^n)).$$

(2.47)

Moreover, the following estimate holds

$$\int_0^T \|(-\Delta)^s u(t)\|_{L^2 (\mathbb{R}^n)}^2 \ dt \leq C \left( \|\phi(x)\|_{H^1 (\mathbb{R}^n)}^2 + \int_0^T \|F(t,\cdot)\|_{L^2 (\mathbb{R}^n)}^2 \right)$$

(2.48)

where the constant $C$ is independent of $T$.

**Proof.** Since $H^s (\mathbb{R}^n)$ is separable, we may take $\{w_k\}_{k=1}^\infty$ to be a basis of $H^s (\mathbb{R}^n)$. Given a positive integer $m$, we will look for a function $u_m(t)$ of the form

$$u_m(t) := \sum_{i=1}^m g_{im}(t) w_i$$

(2.49)

where we hope to select the coefficients $g_{ik}(t)$ $(0 \leq t \leq T, \ k = 1, 2, \cdots, m)$, so that

$$(u_m'(t), w_j) + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_m(t,x) - u_m(t,y))(w_j(x) - w_j(y))}{|x-y|^{n+2s}} \ dx \ dy = (F(t,x), w_j)$$

(2.50)

with $1 \leq j \leq m, \ t \in [0,T]$. Here, $(\cdot, \cdot)$ denotes the inner product in $L^2 (\mathbb{R}^n)$. 

Employing Cauchy inequality, we find
\[ u_m(0) = \sum_{i=1}^{m} g_{im}(0) w_i \] (2.51)
and
\[ u_m(0) = \sum_{i=1}^{m} g_{im}(0) w_i \rightarrow \phi(x) \text{ in } H^s(\mathbb{R}^n) \text{ as } m \to \infty. \] (2.52)

Using (2.49), (2.50) and (2.51) can be rewritten as
\[ \sum_{i=1}^{m} g'_{im}(t) (w_i, w_j) + \sum_{i=1}^{m} g_{im}(t) (\nabla w_i, \nabla w_j) = (F(t, x), w_j) \] (2.53)
with \( 1 \leq j \leq m, t \in [0, T]\). Linear independence of \( w_1, w_2, \ldots, w_m \) is equivalent to
\[ \det([w_i, w_j]) \neq 0. \] (2.54)

According to standard existence theory for ordinary differential equations, there exists a unique solution
\[ g_{im}(t) \in H^1((0, T]), \ 1 \leq i \leq m. \] (2.55)

And then \( u_m \) defined by (2.49) solves (2.53) for a.e. \( 0 \leq t \leq T \) and
\[ u_m(t) \in H^1(0, T; H^s(\mathbb{R}^n)). \] (2.56)

Next we will give some estimates of \( u_m(t) \).

Multiplying equation (2.50) by \( g_{im}(t) \), summing for \( j = 1, 2, \ldots, m \) and then recalling (2.49), we find
\[ \frac{1}{2} \frac{d}{dt} \| u_m(t) \|^2_{L^2(\mathbb{R}^n)} + \| u_m(t) \|^2_{H^s(\mathbb{R}^n)} = (F(t, x), u_m(t)). \] (2.57)

Here, we denote
\[ \| u_m(t) \|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u_m(t, x) - u_m(t, y)|^2}{|x - y|^{n+2s}} \, dx \, dy = C \| (-\Delta)^{\frac{s}{2}} u_m(t) \|^2_{L^2(\mathbb{R}^n)}. \] (2.58)

Employing Cauchy inequality, we find
\[ \frac{d}{dt} \| u_m(t) \|^2_{L^2(\mathbb{R}^n)} \leq \| u_m(t) \|^2_{L^2(\mathbb{R}^n)} + \| F(t, \cdot) \|^2_{L^2(\mathbb{R}^n)}. \] (2.59)

Set
\[ \eta(t) := \| u_m(t) \|^2_{L^2(\mathbb{R}^n)}, \ \xi(t) := \| F(t, \cdot) \|^2_{L^2(\mathbb{R}^n)}. \] (2.60)

Then (2.59) implies
\[ \eta'(t) \leq \eta(t) + \xi(t), \] (2.61)
for a.e. \( 0 \leq t \leq T \). Thus the Gronwall’s inequality yields the estimate
\[ \eta(t) \leq e^t \left( \eta(0) + \int_0^T \xi(t) \, dt \right), \ 0 \leq t \leq T. \] (2.62)

Because
\[ \eta(0) = \| u_m(0) \|^2_{L^2(\mathbb{R}^n)} \leq \| \phi(x) \|^2_{L^2(\mathbb{R}^n)}, \]
we obtain the following estimate by (2.59) and (2.62),
\[ \max_{0 \leq t \leq T} \| u_m(t) \|^2_{L^2(\mathbb{R}^n)} \leq C \| \phi(x) \|^2_{L^2(\mathbb{R}^n)} + \int_0^T \| F(t, \cdot) \|^2_{L^2(\mathbb{R}^n)} \, dt. \] (2.63)
Here, $C$ is a constant depending on $T$. Returning now to equality (2.58) we integrate from 0 to $t$ and employ the inequality above to find
\[
\int_0^t \|(-\Delta)\frac{\partial}{\partial t}u_m(t)\|^2_{L^2(\mathbb{R}^n)} \, dt = C \int_0^t [u_m(t)]^2_{H^s(\mathbb{R}^n)} \, dt
\]
\[
\leq C(\|\phi(x)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|F(t, \cdot)\|^2_{L^2(\mathbb{R}^n)} \, dt)(2.64)
\]
Multiplying equation (2.50) by $g_j(x)$, summing for $j = 1, 2, \cdots, m$, then recalling (2.49), we find
\[
\int_0^t \|u_m'(t)\|^2_{L^2(\mathbb{R}^n)} \, dt + \int_0^t \|\frac{\partial}{\partial t}u_m(t)\|^2_{L^2(\mathbb{R}^n)} \leq (\int_0^t \|F(t, \cdot)\|^2_{L^2(\mathbb{R}^n)} \, dt) + (\int_0^t \|\phi(x)\|^2_{H^s(\mathbb{R}^n)} \, dt)(2.65)
\]
We integrate (2.65) from 0 to $t$ and employ (2.51) to yield
\[
\int_0^t \|u_m'(t)\|^2_{L^2(\mathbb{R}^n)} \, dt + \int_0^t \|\frac{\partial}{\partial t}u_m(t)\|^2_{L^2(\mathbb{R}^n)} \leq [u_m(0)]^2_{H^s(\mathbb{R}^n)} + \int_0^t \|F(t, \cdot)\|^2_{L^2(\mathbb{R}^n)} \, dt(2.66)
\]
with $0 \leq t \leq T$.

By (2.46) and (2.52) we have
\[
\|u_m'(t)\|^2_{L^2(0; T; L^2(\mathbb{R}^n))} \leq C(\|\phi(x)\|^2_{H^s(\mathbb{R}^n)} + \|F(t, \cdot)\|^2_{L^2(0; T; L^2(\mathbb{R}^n)})(2.67)
\]
and
\[
\max_{0 \leq t \leq T} [u_m(t)]^2_{H^s(\mathbb{R}^n)} \leq C(\|\phi(x)\|^2_{H^s(\mathbb{R}^n)} + \|F(t, \cdot)\|^2_{L^2(0; T; L^2(\mathbb{R}^n)}).
\]
According to the energy estimates (2.63), (2.64), (2.67) and (2.68), there exists a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ and a function $u \in L^2(0; T; H^s(\mathbb{R}^n))$ with $u' \in L^2(0; T; L^2(\mathbb{R}^n))$, such that
\[
u_{m_l}(t) \rightharpoonup u(t), \text{ weakly in } L^2(0; T; H^s(\mathbb{R}^n)) \text{ as } l \to \infty (2.69)
\]
and
\[
u_{m_l}'(t) \rightharpoonup u'(t), \text{ weakly in } L^2(0; T; L^2(\mathbb{R}^n)) \text{ as } l \to \infty. (2.70)
\]
In order to show that $u$ is a solution of problem (2.45), upon passing to weak limits in $L^2(0; T; H^s(\mathbb{R}^n))$, we set $m = m_1$ in (2.50), by (2.69) and (2.70), to find
\[
<u'(t), w_j> + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(t,x) - u(t,y))(w_j(x) - w_j(y))}{|x - y|^{n+2s}} \, dx \, dy = <F(t,x), w_j>. \]
\[
(2.71)
\]
Note that $w_j \in H^{-s}(\mathbb{R}^n), (2.71)$ can be rewritten as
\[
<u'(t) + (-\Delta)^s u(t), w_j> = <F(t,x), w_j> \text{ in } L^2(0; T; H^s(\mathbb{R}^n)) (2.72)
\]
where $<\cdot, \cdot>$ denotes the pairing of $H^{-s}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$.

Since $\{u_m\}_{m=1}^\infty$ is a basis of $H^s(\mathbb{R}^n)$, (2.72) implies
\[
u'(t) + (-\Delta)^s u(t) = F(t,x). (2.73)
\]
According to (2.46) and (2.67), we have
\[
u'(t) = \partial_t u, F(t,x) \in L^2(0; T; L^2(\mathbb{R}^n)). (2.74)
\]
By (2.45), we deduce
\[
(-\Delta)^s u(t) \in L^2(0; T; L^2(\mathbb{R}^n)), (2.75)
\]
which implies
\[ u(t) \in L^2(0, T; H^{2s}(\mathbb{R}^n)). \]  
(2.76)

Hence, (2.47) follows. We need to prove that \( u \) satisfies initial data. In fact, from (2.67)-(2.68) we obtain that
\[ u_m(t) \to u(0, x), \text{ weakly in } L^2(\mathbb{R}^n) \text{ as } l \to \infty. \]  
(2.77)

Note that (2.52), we get
\[ \int \frac{\partial_t u}{\tau} \left| \frac{\phi(x)}{\tau} \right| \left| \frac{\partial_t \phi(x)}{\tau} \right| \]  
Multiplying (2.45) by \( u \) and \( \partial_t u \), respectively, and then integrating from 0 to \( T \) with respect to \( t \), we find for 0 \( \leq t \leq T \) that
\[ \|u(t)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|u(\tau)\|^2_{H^s(\mathbb{R}^n)} d\tau \leq C \left( \|\phi(x)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|F(\tau, \cdot)\|^2_{L^2(\mathbb{R}^n)} d\tau \right) \]  
(2.79)

and
\[ \int_0^t \|\partial_t u(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau + \|u(t)\|^2_{H^s(\mathbb{R}^n)} \leq \|\phi(x)\|^2_{H^s(\mathbb{R}^n)} + \int_0^t \|F(\tau, \cdot)\|^2_{L^2(\mathbb{R}^n)} d\tau. \]  
(2.80)

From (2.79), a weak solution of (2.45) is unique. Therefore, the sequence \( \{u_m\}_{m=1}^{\infty} \) converges. Using equation (2.45) and (2.80), we obtain the estimates (2.48).

\begin{corollary}
Assume
\[ \phi \in H^{m+1}(\mathbb{R}^n), \quad F(t, x) \in L^2(0, T; H^m(\mathbb{R}^n)) \]  
(2.81)

where \( m \in \mathbb{N}, s \in (0, 1) \), then problem (2.45) possesses a unique weak solution \( u = u(t, x) \) satisfying
\[ u \in L^2(0, T; H^{m+2s}(\mathbb{R}^n)) \]  
(2.82)

and
\[ \partial_t u \in L^2(0, T; H^m(\mathbb{R}^n)), \]  
(2.83)

with estimate
\[ \int_0^T \|(-\Delta)^s u(t)\|^2_{H^{m}(\mathbb{R}^n)} dt \leq C \left( \|\phi(x)\|^2_{H^{m+1}(\mathbb{R}^n)} + \int_0^T \|F(t, \cdot)\|^2_{H^{m}(\mathbb{R}^n)} dt \right) \]  
(2.84)

where the constant \( C \) is independent of \( T \).
\end{corollary}

\textbf{Proof.} Let \( v = D^\gamma u, |\gamma| = k (k = 1, 2, \ldots, m) \), then \( v \) satisfies
\[ \begin{cases} 
\partial_t v + (-\Delta)^s v = D^\gamma F(t, x), & x \in \mathbb{R}^n, t > 0, \\
v(0, x) = D^\gamma \phi(x), & x \in \mathbb{R}^n.
\end{cases} \]  
(2.85)

Applying estimates (2.45)-(2.48) for \( v \), we obtain (2.82)-(2.84).

\begin{lemma}
([17]) Assume \( b \geq a > 0 \) and \( b > 1 \), then the following inequality holds
\[ \int_0^\infty (1 + t - \tau)^{-a}(1 + \tau)^{-b} d\tau \leq C(1 + t)^{-a} \]  
(2.86)

where \( C \) is a positive constant.
\end{lemma}
3. Existence and asymptotic decay. In this section, we prove the main existence result of the system (1.1) applying the fixed point theorem, meanwhile, obtain the asymptotic decay behaviors of the $W^{m-n-3,\infty}$ norm of $u$ and $\nabla v$.

**Theorem 3.1.** Assume $n \geq 2$ and $s \in \left(\frac{3}{2}, 1\right)$. For every integer $m \geq n + 4$, there exists a constant $E \in (0, 1)$, such that if initial data satisfy

$$
\|u_0\|_{W^{m,1}(\mathbb{R}^n)} + \|u_0\|_{H^{m+1}(\mathbb{R}^n)} + \|\nabla u_0\|_{W^{m-1,1}(\mathbb{R}^n)} + \|\Delta u_0\|_{H^{m}(\mathbb{R}^n)} \leq E^2, \quad (3.1)
$$

then the system (1.1) admits a unique global classical solution $(u, v)$. Moreover, the temporal decay estimates hold for every positive $\alpha := m - n - 3$

$$
(1 + t)^{-\frac{\alpha}{2}} \|u\|_{W^{\alpha,\infty}(\mathbb{R}^n)} < \infty, \quad t > 0 \quad (3.2)
$$

and

$$
(1 + t)^{-\frac{\alpha}{2}} \|\nabla v\|_{W^{\alpha,\infty}(\mathbb{R}^n)} < \infty, \quad t > 0. \quad (3.3)
$$

**Proof.** We divide the proof into five steps.

**Step 1.** Preliminary.

Let

$$
\alpha := m - n - 3. \quad (3.4)
$$

Define

$$
X_{m,E} = \{u := u(t,x) : D_m(u) \leq E\}
$$

and

$$
D_m(u) := \sup_{t \geq 0} \{(1 + t)^{\frac{\alpha}{2}} \|u(t)\|_{W^{\alpha,\infty}(\mathbb{R}^n)} \}
$$

$$
+ \sup_{t \geq 0} \|u(t)\|_{W^{m,1}(\mathbb{R}^n)} + \left(\int_0^\infty \|(-\Delta)^s u(t)\|_{H^m(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \quad (3.5)
$$

where $E$ will be chosen later.

Set the distance of any $u_1, u_2 \in X_{m,E}$ as

$$
d_{m,E}(u_1, u_2) := D_m(u_1 - u_2). \quad (3.6)
$$

Then, $(X_{m,E}, d_{m,E})$ is a nonempty complete metric space.

Given the initial data $u_0$ and $v_0$, for any fixed $\bar{u} \in X_{m,E}$, combining with (2.2), we set

$$
\bar{v}(t) = e^{-t} K_t(x) * v_0(x) + \int_0^t e^{(\tau-t)} K_{t-\tau}(x) * \bar{u}(\tau) d\tau. \quad (3.7)
$$

Define

$$
\mathcal{T}(\bar{u})(t) = K_t(x) * u_0(x) - \int_0^t \nabla K_{t-\tau}(x) * [\bar{u}(\tau) \nabla \bar{v}(\tau)] d\tau. \quad (3.8)
$$

**Step 2.** Essential inequalities.

As an application, we obtain some estimates on $\bar{v}$ defined in (3.7), which will be fundamental throughout the sequel.

At the beginning, applying (2.12) in Lemma 2.3 and Young’s inequality, we obtain

$$
\|\nabla \bar{v}(t)\|_{W^{\alpha,\infty}(\mathbb{R}^n)} \leq e^{-t}\|K_t(x) * \nabla v_0(x)\|_{W^{\alpha,\infty}(\mathbb{R}^n)} + \int_0^t e^{(\tau-t)} \|\nabla K_{t-\tau}(x) * \bar{u}(\tau)\|_{W^{\alpha,\infty}(\mathbb{R}^n)} d\tau.
$$
\[ \leq Ce^{-t}(1 + t)^{-\frac{1}{L}} \| \nabla \psi_0 \|_{W^{-2,1}(\mathbb{R}^n)} \]

Using (2.4) and (2.6) with \( p = 1 \), we have
\[ \| \nabla K_t(x) \|_{L^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{L}}. \] Combining (3.10) with (3.9), we obtain
\[ \| \nabla K_t(x) \|_{L^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{L}}. \]

Then we estimate the integration in the last term of (3.11),
\[
\int_0^t e^{(\tau-t)}(1 + \tau)^{-\frac{1}{L}}(1 + \tau)^{-\frac{1}{L}} \, d\tau
\]
\[ \leq C e^{-\frac{1}{2s}} (1 + t)^{-\frac{1}{L}} \left( (1 + \tau)^{-\frac{1}{L}} \right) \int_0^t e^{(\tau-t)}(1 + \tau)^{-\frac{1}{L}} \, d\tau
\]
\[ \leq C (1 + t)^{-\frac{1}{L}} \left( \int_0^t (1 + \tau)^{-\frac{1}{L}} \, d\tau + \Gamma(1 - \frac{1}{2s}) \right)
\]
\[ \leq C (1 + t)^{-\frac{1}{L}}. \]

Here, \( \Gamma(\cdot) \) denotes Gamma function and the following inequality has been used
\[ \Gamma(1 - \frac{1}{2s}) = \int_0^\infty e^{(\tau-t)}(t - \tau)^{-\frac{1}{L}} \, d\tau \leq C \]

with \( 1 - \frac{1}{2s} > 0 \).

Consequently, inserting (3.12) into (3.11) implies
\[ \| \nabla \psi(t) \|_{W^{-2,1}(\mathbb{R}^n)} \leq C (1 + t)^{-\frac{1}{L}} (e^{-t} \| \nabla \psi_0 \|_{W^{-2,1}(\mathbb{R}^n)} + E). \]

Next, by (2.13) in Lemma 2.3 and (2.31) in Lemma 2.5, we obtain
\[ \| \nabla \psi(t) \|_{W^{-1,1}(\mathbb{R}^n)} \]
\[ \leq e^{-t} \| K_t(x) * \nabla \psi_0 \|_{W^{-1,1}(\mathbb{R}^n)} + \int_0^t e^{(\tau-t)} \| \nabla K_{t-t}(x) * \psi(t) \|_{W^{-1,1}(\mathbb{R}^n)} \, d\tau
\]
\[ \leq C e^{-t} \| \nabla \psi_0 \|_{W^{-1,1}(\mathbb{R}^n)} + C \int_0^t e^{(\tau-t)}(1 + \tau) \| \psi(t) \|_{W^{-1,1}(\mathbb{R}^n)} \, d\tau
\]
\[ \leq C e^{-t} \| \nabla \psi_0 \|_{W^{-1,1}(\mathbb{R}^n)} + C \Gamma(1 - \frac{1}{2s}) E
\]
\[ \leq C e^{-t} \| \nabla \psi_0 \|_{W^{-1,1}(\mathbb{R}^n)} + CE. \]
Obviously
\[
\|\nabla \tilde{u}(t)\|_{W^{m-2,1}(\mathbb{R}^n)} \leq \|\nabla \tilde{u}(t)\|_{W^{m-1,1}(\mathbb{R}^n)} \leq C e^{-t} \|\tilde{v}_0\|_{W^{m-1,1}(\mathbb{R}^n)} + CE. \quad (3.15)
\]
Using (2.24) in Lemma 2.4 provided \( p = q = 2 \) and Young’s inequality, we have
\[
\|\Delta \tilde{u}(t)\|_{H^m(\mathbb{R}^n)} \leq e^{-t} \|K_t(x) * \Delta v_0\|_{H^m(\mathbb{R}^n)} + \int_0^t e^{(\tau-t)} \|\Delta(K_{t-\tau}(x) * \tilde{u}(\tau))\|_{H^m(\mathbb{R}^n)} d\tau
\leq C e^{-t} \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + \int_0^t e^{(\tau-t)} \|(-\Delta)^{(1-s)} K_{t-\tau}(x) * (-\Delta)^s \tilde{u}(\tau)\|_{H^m(\mathbb{R}^n)} d\tau
\leq C e^{-t} \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + \int_0^t e^{(\tau-t)} \|(-\Delta)^{(1-s)} K_{t-\tau}(x)\|_{L^1(\mathbb{R}^n)} \|(-\Delta)^s \tilde{u}(\tau)\|_{H^m(\mathbb{R}^n)} d\tau.
\] (3.16)

Applying (2.8) in Lemma 2.2 provided \( \gamma = 1-s \) and \( s > \frac{2}{3} \), we deduce
\[
\|\Delta \tilde{u}(t)\|_{H^m(\mathbb{R}^n)} \leq C e^{-t} \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + C \int_0^t e^{(\tau-t)} (t-\tau)^{-\frac{1}{2}} \|(-\Delta)^s \tilde{u}(\tau)\|_{H^m(\mathbb{R}^n)} d\tau
\leq C e^{-t} \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + C (\Gamma(3-\frac{2}{s})^\frac{3}{2} \int_0^t \|(-\Delta)^s \tilde{u}(\tau)\|_{H^m(\mathbb{R}^n)}^2 d\tau)^{\frac{1}{2}}
\leq C e^{-t} \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + CE.
\] (3.17)

**Step 3.** Proof of the map \( T: X_{m,E} \to X_{m,E} \).

Applying (2.12) in Lemma 2.3 and (2.30) in Lemma 2.5, we find that
\[
\|T(\tilde{u}(t))\|_{W^{m,\infty}(\mathbb{R}^n)} \leq \|K_t(x) * u_0\|_{W^{m,\infty}(\mathbb{R}^n)} + \int_0^t \|
abla K_{t-\tau}(x) * [\tilde{u}(\tau) \nabla \tilde{v}(\tau)]\|_{W^{m,\infty}(\mathbb{R}^n)} d\tau
\leq C (1+t)^{-\frac{n}{2s}} \|u_0\|_{W^{m-2,1}(\mathbb{R}^n)} + C \int_0^t (1+t-\tau)^{-\frac{n+1}{2s}} \|\tilde{u}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} \|
abla \tilde{v}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} d\tau.
\] (3.18)

Note that
\[
\|\tilde{u}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} \leq \|\tilde{u}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} \|
abla \tilde{v}(\tau)\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{u}(\tau)\|_{L^\infty(\mathbb{R}^n)} \|
abla \tilde{v}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)},
\] (3.19)
we deduce
\[
\|T(\tilde{u}(t))\|_{W^{m,\infty}(\mathbb{R}^n)} \leq C [(1+t)^{-\frac{n}{2s}} \|u_0\|_{W^{m-2,1}(\mathbb{R}^n)}
\] \[
+ \int_0^t (1+t-\tau)^{-\frac{n+1}{2s}} \|\tilde{u}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} \|
abla \tilde{v}(\tau)\|_{W^{m,\infty}(\mathbb{R}^n)}
\]
Hence, inserting (3.22) into (3.21) yields
\[
\|\mathcal{T}(\bar{u})(t)\|_{W^{\infty,\infty}(\mathbb{R}^n)} \\
\leq C(1+t)^{-\frac{n}{2}}\|u_0\|_{W^{m-2,1}(\mathbb{R}^n)} \\
+ C\int_0^t E(1+t-\tau)^{-\frac{n+1}{2}}(1+\tau)^{-\frac{n}{2}}(e^{-\tau}\|\nabla v_0\|_{W^{m-2,1}(\mathbb{R}^n)} + E)d\tau \\
+ C\int_0^t E(1+t-\tau)^{-\frac{n+1}{2}}(1+\tau)^{-\frac{n}{2}}(\|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)} + E)d\tau.
\]
(3.21)

By Lemma 2.9, choosing \(a = b = \frac{n}{2}\), we have
\[
\int_0^t (1+t-\tau)^{-\frac{n+1}{2}}(1+\tau)^{-\frac{n}{2}}d\tau \leq \int_0^t (1+t-\tau)^{-\frac{n+1}{2}}(1+\tau)^{-\frac{n}{2}}d\tau \leq C(1+t)^{-\frac{n}{2}}.
\]
(3.22)

Hence, inserting (3.22) into (3.21) yields
\[
\|\mathcal{T}(\bar{u})(t)\|_{W^{\infty,\infty}(\mathbb{R}^n)} \\
\leq C(1+t)^{-\frac{n}{2}}\|u_0\|_{W^{m-2,1}(\mathbb{R}^n)} + CE(1+t)^{-\frac{n}{2}}\|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)} + CE^2(1+t)^{-\frac{n}{2}}.
\]
(3.23)

Consequently,
\[
\sup_{t \geq 0} \left( (1+t)^{-\frac{n}{2}}\|\mathcal{T}(\bar{u})(t)\|_{W^{\infty,\infty}(\mathbb{R}^n)} \right) \leq C(\|u_0\|_{W^{m-2,1}(\mathbb{R}^n)} \\
+ \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)}) + CE^2.
\]
(3.24)

Next, using (2.13) in Lemma 2.3 and (2.31) in Lemma 2.5, we have
\[
\|\mathcal{T}(\bar{u})(t)\|_{W^{m,1}(\mathbb{R}^n)} \\
\leq \|K_t(x) \ast u_0\|_{W^{m,1}(\mathbb{R}^n)} + \int_0^t \|\nabla K_{t-\tau}(x) \ast [\bar{u}(\tau)\nabla \bar{v}(\tau)]\|_{W^{m,1}(\mathbb{R}^n)}d\tau \\
\leq C\|u_0\|_{W^{m,1}(\mathbb{R}^n)} + C\int_0^t (1+t-\tau)^{-\frac{n}{2}}\|\bar{u}(\tau)\nabla \bar{v}(\tau)\|_{W^{m+1,1}(\mathbb{R}^n)}d\tau \\
\leq C\|u_0\|_{W^{m,1}(\mathbb{R}^n)} + C\int_0^t (1+t-\tau)^{-\frac{n}{2}}\left[\|\bar{u}(\tau)\nabla \bar{v}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} \\
+ \sum_{i=0}^{1} \|D^{m+i}\bar{u}(\tau) \cdot \nabla \bar{v}(\tau)\|_{L^1(\mathbb{R}^n)} + \sum_{i=1}^{2} \|\bar{u}(\tau)D^{m+i}\bar{v}(\tau)\|_{L^1(\mathbb{R}^n)}\right]d\tau \\
:= C(\|u_0\|_{W^{m,1}(\mathbb{R}^n)} + A_1 + A_2 + A_3).
\]
(3.25)

We calculate \(A_1, A_2\) and \(A_3\) respectively.

Note that
\[
\|\bar{u}(\tau)\nabla \bar{v}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} \\
\leq \|\bar{u}(\tau)\|_{L^\infty(\mathbb{R}^n)}\|\nabla \bar{v}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)} + \|\bar{u}(\tau)\|_{W^{m-1,1}(\mathbb{R}^n)}\|\nabla \bar{v}(\tau)\|_{L^\infty(\mathbb{R}^n)},
\]
(3.26)
using (3.13) and (3.14), we have
\[ A_1 = \int_0^t (1 + t - \tau)^{-\frac{s}{2}} \| \bar{u}(\tau) \nabla \bar{v}(\tau) \|_{W^{m-1,1}(\mathbb{R}^n)} d\tau \]
\leq CE \int_0^t (1 + t - \tau)^{-\frac{s}{2}} (1 + \tau)^{-\frac{s}{2}} (e^{-\tau} \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)} + E) d\tau
\leq CE \int_0^t (1 + t - \tau)^{-\frac{s}{2}} (1 + \tau)^{-\frac{s}{2}} (e^{-\tau} \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)} + E) d\tau
\leq C \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)} + CE^2
\]

where we have used the inequality (2.86) in Lemma 2.9 with \( a = \frac{1}{2s} \) and \( b = \frac{n}{2s} \). Then by Hölder inequality, (3.13) and (3.14), we obtain
\[
A_2 = \int_0^t (1 + t - \tau)^{-\frac{s}{2}} \sum_{i=0}^1 \| D^{m+i} \bar{u}(\tau) \cdot \nabla \bar{v}(\tau) \|_{L^1(\mathbb{R}^n)} d\tau
\leq C \int_0^t (1 + t - \tau)^{-\frac{s}{2}} \| (-\Delta)^s \bar{u}(\tau) \|_{H^m(\mathbb{R}^n)} \| \nabla \bar{v}(\tau) \|_{L^2(\mathbb{R}^n)} d\tau
\leq C \left( \int_0^t \| (-\Delta)^s \bar{u}(\tau) \|_{H^m(\mathbb{R}^n)}^2 d\tau \right)^{\frac{1}{2}}
\times \left( \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \| \nabla \bar{v}(\tau) \|_{L^1(\mathbb{R}^n)} \| \nabla \bar{v}(\tau) \|_{L^\infty(\mathbb{R}^n)} d\tau \right)^{\frac{1}{2}}
\leq CE \left( \int_0^t (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{s}{2}} (e^{-2\tau} \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)}^2 + E^2) d\tau \right)^{\frac{1}{2}}
\leq C \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)} + CE^2
\]

where the inequality (2.86) with \( a = \frac{1}{s} \) and \( b = \frac{n}{2s} \) has been used in (3.28).

Finally, applying (3.17) and Hölder inequality, we deduce
\[
A_3 = \int_0^t (1 + t - \tau)^{-\frac{s}{2}} \sum_{i=1}^2 \| \bar{u}(\tau) D^{m+i} \bar{v}(\tau) \|_{L^1(\mathbb{R}^n)} d\tau
\leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \| \bar{u}(\tau) \|_{L^2(\mathbb{R}^n)} \| \Delta \bar{v}(\tau) \|_{H^m(\mathbb{R}^n)} d\tau
\leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \| \bar{u}(\tau) \|_{L^2(\mathbb{R}^n)} \| \bar{u}(\tau) \|_{L^\infty(\mathbb{R}^n)} \| \Delta \bar{v}(\tau) \|_{H^m(\mathbb{R}^n)} d\tau
\leq CE \int_0^t (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{s}{2}} (e^{-\tau} \| \Delta v_0 \|_{H^m(\mathbb{R}^n)} + E) d\tau.
\]

For \( n \geq 2 \), direct calculation shows that
Inserting (3.30) into (3.29) yields

\[
\int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} (1 + \tau)^{\frac{n}{2}} d\tau \\
\leq \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} (1 + \tau)^{\frac{n}{2}} d\tau + \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} (1 + \tau)^{\frac{n}{2}} d\tau \\
\leq \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} (1 + \tau)^{\frac{n}{2}} d\tau + \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} (1 + \tau)^{\frac{n}{2}} d\tau \\
\leq \frac{4s}{8s - n - 2} \left[ (1 + \frac{t}{2})^{\frac{4s-n-2}{4}} - 1 \right] - \frac{4s}{8s - n - 2} \left[ 1 - (1 + \frac{t}{2})^{\frac{4s-n-2}{4}} \right] \\
\leq \frac{8s}{n + 2 - 4s} \leq C.
\]

Inserting (3.30) into (3.29) yields

\[
A_3 \leq C \left( E^2 + E \|\Delta v_0\|_{L^\infty(\mathbb{R}^n)} \right).
\]

Inseting (3.27), (3.28) and (3.31) into (3.25) yields

\[
\|T(\bar{u})(t)\|_{W^{m-1}(\mathbb{R}^n)} \leq C \left( \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)} + \|u_0\|_{W^{m,1}(\mathbb{R}^n)} + CE^2 \right).
\]

Obviously, \(T(\bar{u})(t)\) satisfies the problem

\[
\begin{align*}
\frac{\partial}{\partial t} T(\bar{u})(t) + (-\Delta)^s T(\bar{u})(t) &= -\nabla \cdot (\bar{u} \nabla \bar{v}), & x \in \mathbb{R}^n, & t > 0, \\
T(\bar{u})(0) &= u_0(x), & x \in \mathbb{R}^n.
\end{align*}
\]

By Corollary 2.8, we also obtain that

\[
\int_0^\infty \|(-\Delta)^s T(\bar{u})(t)\|^2_{H^m(\mathbb{R}^n)} dt \\
\leq C \left( \|u_0\|^2_{H^{m+1}(\mathbb{R}^n)} + \int_0^\infty \|\nabla \cdot (\bar{u} \nabla \bar{v})\|^2_{H^m(\mathbb{R}^n)} dt \right) \\
\leq C \left( \|u_0\|^2_{H^{m+1}(\mathbb{R}^n)} + \int_0^\infty \|\nabla \bar{u}(t) \cdot \nabla \bar{v}(t)\|^2_{H^m(\mathbb{R}^n)} dt + \int_0^\infty \|\bar{u}(t) \Delta \bar{v}(t)\|^2_{H^m(\mathbb{R}^n)} dt \right) \\
:= C \left( \|u_0\|^2_{H^{m+1}(\mathbb{R}^n)} + B_1 + B_2 \right).
\]

Here we perform \(B_1\) and \(B_2\), respectively.

Firstly, note that

\[
\|\nabla \bar{u}(t) \cdot \nabla \bar{v}(t)\|_{H^m(\mathbb{R}^n)} \\
\leq \|\nabla \bar{u}(t)\|_{H^\infty(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \bar{u}(t)\|_{L^\infty(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|_{H^\infty(\mathbb{R}^n)},
\]

we deduce

\[
B_1 \\
\leq \int_0^\infty \|\nabla \bar{u}(t) \cdot \nabla \bar{v}(t)\|^2_{H^m(\mathbb{R}^n)} dt \\
\leq C \int_0^\infty \left( \|\nabla \bar{u}(t)\|^2_{H^\infty(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|^2_{W^{\alpha-1,\infty}(\mathbb{R}^n)} + \|\nabla \bar{u}(t)\|^2_{W^{\alpha-1,\infty}(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|^2_{H^m(\mathbb{R}^n)} \right) dt \\
\leq C \int_0^\infty \left( \|\bar{u}(t)\|^2_{H^{m+1}(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|^2_{W^{\alpha,\infty}(\mathbb{R}^n)} + \|\bar{u}(t)\|^2_{W^{\alpha,\infty}(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|^2_{H^m(\mathbb{R}^n)} \right) dt \\
\leq C \int_0^\infty \left( \|\bar{u}(t)\|^2_{H^{m}(\mathbb{R}^n)} + \|(-\Delta)^s \bar{u}(t)\|^2_{H^m(\mathbb{R}^n)} \|\nabla \bar{v}(t)\|^2_{W^{\alpha,\infty}(\mathbb{R}^n)} \right) dt
\]
Note that
\[ \| \tilde{u}(t) \|_{H^1(\mathbb{R}^n)} \leq \| \tilde{u}(t) \|_{W^{1,1}(\mathbb{R}^n)} \| \tilde{u}(t) \|_{W^{1,\infty}(\mathbb{R}^n)} \]  
and
\[ \| \nabla \tilde{v}(t) \|_{L^2(\mathbb{R}^n)} \leq \| \nabla \tilde{v}(t) \|_{L^1(\mathbb{R}^n)} \| \nabla \tilde{v}(t) \|_{L^\infty(\mathbb{R}^n)}, \]

inserting (3.13) and (3.14) into (3.38), we have
\[ \| \nabla \tilde{v}(t) \|_{L^2(\mathbb{R}^n)}^2 \leq (1 + t)^{-\frac{n}{2}} (e^{-2t} \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)}^2 + E^2). \]

Inserting (3.13), (3.17), (3.37) and (3.39) into (3.36), we obtain
\[ B_1 \leq CE^2 \int_0^\infty (1 + t)^{-\frac{n}{2}} (1 + t)^{-\frac{n}{2}} (e^{-2t} \| \nabla v_0 \|_{W^{m-2,1}(\mathbb{R}^n)}^2 + E^2)dt + \]
\[ + CE^2 \int_0^\infty (1 + t)^{-\frac{n}{2}} (1 + t)^{-\frac{n}{2}} (e^{-2t} \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)}^2 + E^2)dt \]
\[ \leq C(\| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)}^2 + \| \Delta v_0 \|_{H^m(\mathbb{R}^n)}^2) + CE^4. \]

Similarly, note that
\[ \| \tilde{u}(t) \Delta \tilde{v}(t) \|_{H^{m}(\mathbb{R}^n)} \leq \| \tilde{u}(t) \|_{L^\infty(\mathbb{R}^n)} \| \Delta \tilde{v}(t) \|_{H^m(\mathbb{R}^n)} + \| \tilde{u}(t) \|_{H^m(\mathbb{R}^n)} \| \Delta \tilde{v}(t) \|_{L^\infty(\mathbb{R}^n)}, \]

we also obtain
\[ B_2 = \int_0^\infty \| \tilde{u}(t) \Delta \tilde{v}(t) \|^2_{H^m(\mathbb{R}^n)} dt \]
\[ \leq C \int_0^\infty \| \tilde{u}(t) \|_{W^{m-1,\infty}(\mathbb{R}^n)} \| \Delta \tilde{v}(t) \|^2_{H^m(\mathbb{R}^n)} + \| \tilde{u}(t) \|^2_{H^m(\mathbb{R}^n)} \| \Delta \tilde{v}(t) \|^2_{W^{m-1,\infty}(\mathbb{R}^n)} dt \]
\[ \leq C \int_0^\infty \| \tilde{u}(t) \|^2_{W^{m,\infty}(\mathbb{R}^n)} \| \Delta \tilde{v}(t) \|^2_{H^m(\mathbb{R}^n)} + \| \tilde{u}(t) \|^2_{H^m(\mathbb{R}^n)} \| \nabla \tilde{v}(t) \|^2_{W^{m,\infty}(\mathbb{R}^n)} dt \]
\[ \leq C \int_0^\infty \| \tilde{u}(t) \|^2_{W^{m,\infty}(\mathbb{R}^n)} \| \Delta \tilde{v}(t) \|^2_{H^m(\mathbb{R}^n)} dt \]
\[ + C \int_0^\infty (\| \tilde{u}(t) \|^2_{H^m(\mathbb{R}^n)} + \| (-\Delta) \tilde{u}(t) \|^2_{H^m(\mathbb{R}^n)}) \| \nabla \tilde{v}(t) \|^2_{W^{m,\infty}(\mathbb{R}^n)} dt. \]

Using (3.13), (3.17) and (3.37), we deduce
\[ B_2 \leq CE^2 \int_0^\infty (1 + t)^{-\frac{n}{2}} (e^{-2t} \| \nabla v_0 \|_{H^m(\mathbb{R}^n)}^2 + E^2) dt \]
\[ + CE^2 \int_0^\infty (1 + t)^{-\frac{n}{2}} (1 + t)^{-\frac{n}{2}} (e^{-2t} \| \nabla v_0 \|_{W^{m-2,1}(\mathbb{R}^n)}^2 + E^2)dt \]
\[ \leq C(\| \nabla v_0 \|_{W^{m-2,1}(\mathbb{R}^n)}^2 + \| \Delta v_0 \|_{H^m(\mathbb{R}^n)}^2) + CE^4. \]

Inserting (3.40) and (3.43) into (3.34), we get
\[ \left( \int_0^\infty \| (-\Delta)^s \mathcal{T}(\tilde{u})(t) \|^2_{H^m(\mathbb{R}^n)} dt \right)^{\frac{1}{2}} \]
\[ \leq C(\| u_0 \|_{H^{m+1}(\mathbb{R}^n)} + \| \Delta v_0 \|_{H^m(\mathbb{R}^n)} + \| \nabla v_0 \|_{W^{m-1,1}(\mathbb{R}^n)}) + CE^2. \]
Combining (3.24), (3.32) with (3.44), we have

$$D_m(\mathcal{T}(\bar{u}(t)))$$

$$\leq CE^2 + C(\|u_0\|_{W^{m,1}(\mathbb{R}^n)} + \|u_0\|_{H^{m+1}(\mathbb{R}^n)} + \|\Delta v_0\|_{H^m(\mathbb{R}^n)} + \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)})$$

$$\leq CE^2.$$

(3.45)

If we choose $E \leq \min\{1, \frac{1}{C}\}$ then we find that

$$D_m(\mathcal{T}(\bar{u}(t))) \leq E.$$  

We have proved that $\mathcal{T}$ is a mapping from $X_{m,E}$ to $X_{m,E}$.

**Step 4.** Some inequalities.

For any fixed $\bar{u}_1$, $\bar{u}_2 \in X_{m,E}$, from (3.7) we have $\bar{v}_1$, $\bar{v}_2$. According to the same arguments as in Step 2, we deduce

$$\|\nabla \bar{v}_1(t) - \bar{v}_2(t)\|_{W^{m,\infty}(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{m}{2}} d_{m,E}(\bar{u}_1, \bar{u}_2),$$

(3.46)

$$\|\nabla \bar{v}_1(t) - \bar{v}_2(t)\|_{W^{m-2,1}(\mathbb{R}^n)} \leq \|\nabla \bar{v}_1(t) - \bar{v}_2(t)\|_{W^{m-1,1}(\mathbb{R}^n)} \leq C d_{m,E}(\bar{u}_1, \bar{u}_2),$$

(3.47)

and

$$\|\Delta \bar{v}_1(t) - \Delta \bar{v}_2(t)\|_{H^m(\mathbb{R}^n)} \leq C d_{m,E}(\bar{u}_1, \bar{u}_2).$$

(3.48)

**Step 5.** The proof of a strict contraction.

Now, we will prove that $\mathcal{T} : X_{m,E} \to X_{m,E}$ is a strict contraction. Firstly, with the same arguments as in the proof of (3.18)-(3.23), we have

$$\|\mathcal{T}(\bar{u}_1(t)) - \mathcal{T}(\bar{u}_2(t))\|_{W^{m,\infty}(\mathbb{R}^n)}$$

$$\leq C d_{m,E}(\bar{u}_1, \bar{u}_2) E(1 + t)^{-\frac{m}{2}} + C d_{m,E}(\bar{u}_1, \bar{u}_2)(1 + t)^{-\frac{m}{2}} \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)}.$$

(3.49)

Hence

$$\sup_{t \geq 0} \left( (1 + t)^{\frac{m}{2}} \|\mathcal{T}(\bar{u}_1(t)) - \mathcal{T}(\bar{u}_2(t))\|_{W^{m,\infty}(\mathbb{R}^n)} \right)$$

$$\leq C d_{m,E}(\bar{u}_1, \bar{u}_2)(E + \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)}).$$

(3.50)

By the same arguments as in the proof of (3.25)-(3.32), we obtain that

$$\|\mathcal{T}(\bar{u}_1(t)) - \mathcal{T}(\bar{u}_2(t))\|_{W^{m,1}(\mathbb{R}^n)}$$

$$\leq C d_{m,E}(\bar{u}_1, \bar{u}_2)(E + \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)} + \|\Delta v_0\|_{H^m(\mathbb{R}^n)}).$$

(3.51)

Finally, using the same arguments as in the proof of (3.33)-(3.44), we have

$$\left( \int_0^\infty \|(-\Delta)^s(\mathcal{T}(\bar{u}_1(t)) - \mathcal{T}(\bar{u}_2(t)))\|^2_{H^m(\mathbb{R}^n)} dt \right)^\frac{1}{2}$$

$$\leq C d_{m,E}(\bar{u}_1, \bar{u}_2)(E + \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)} + \|\Delta v_0\|_{H^m(\mathbb{R}^n)}).$$

(3.52)

Combining (3.50), (3.51) with (3.52), we deduce

$$d_{m,E}(\mathcal{T}(\bar{u}_1(t)), \mathcal{T}(\bar{u}_2(t))) = D_m(\mathcal{T}(\bar{u}_1(t)) - \mathcal{T}(\bar{u}_2(t)))$$

$$\leq C d_{m,E}(\bar{u}_1, \bar{u}_2)(E + \|\nabla v_0\|_{W^{m-1,1}(\mathbb{R}^n)} + \|\Delta v_0\|_{H^m(\mathbb{R}^n)})$$

$$\leq C d_{m,E}(\bar{u}_1, \bar{u}_2)(E + E^2)$$

(3.53)
where $C$ is constant, not depending on $t$. If we choose $E < \min\{\frac{1}{C}, 1\}$, such that $C(E + E^2) \leq \frac{1}{2}$, then
\[
d_{m,E}(T(\bar{u}_1)(t), T(\bar{u}_2)(t)) = D_m(T(\bar{u}_1)(t) - T(\bar{u}_2)(t)) \leq \frac{1}{2} d_{m,E}(\bar{u}_1, \bar{u}_2).
\]
The map $T : X_{m,E} \to X_{m,E}$ is a strict contraction.

Combining the above five steps, applying fixed point theorem, we prove that the problem (1.1) has a unique global classical solution. Moreover, (3.24) and (3.13) imply (3.2)-(3.3). This completes the proof of Theorem 3.1. □

REFERENCES

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.*, 25 (2015), 1663–1763.

[2] P. Biler, T. Funaki and W. A. Woyczyński, Interacting particle approximation for nonlocal quadratic evolution problems, *Probab. Math. Statist.*, 19 (1999), 267–286.

[3] P. Biler, Local and global solvability of some parabolic systems modelling chemotaxis, *Adv. Math. Sci. Appl.*, 8 (1998), 715–743.

[4] P. Biler and W. A. Woyczyński, Global and exploding solutions for nonlocal quadratic evolution problems, *SIAM J. Appl. Math.*, 59 (1999), 845–869.

[5] P. Biler and W. A. Woyczyński, Nonlocal quadratic evolution problems, *Banach Center Publ.*, 52 (2000), 11–24.

[6] P. Biler and G. Karch, Blowup of solutions to generalized Keller-Segel model, *J. Evol. Equ.*, 10 (2010), 247–262.

[7] P. Biler, T Cieślak, G. Karch and J. Zienkiewicz, Local criteria for blowup in two-dimensional chemotaxis models, *Discrete & Continuous Dynamical Systems - A*, 37 (2017), 1841–1856, arXiv: 1410.7870.

[8] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, *J. Anal. Math.*, 68 (1996), 277–304.

[9] L. Corrias and B. Perthame, Critical space for the parabolic-parabolic Keller-Segel model in $\mathbb{R}^n$, *C. R. Acad. Sci. Paris, Ser. I*, 342 (2006), 745–750.

[10] L. Corrias and B. Perthame, Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces, *Math. Comput. Modelling*, 47 (2008), 755–764.

[11] C. Escudero, The fractional Keller-Segel model, *Nonlinearity*, 19 (2006), 2909–2918.

[12] B. L. Guo, X. K. Pu and F. H. Huang, Fractional Partial differential Equations and their Numerical Solutions, Originally published by Science Press in 2011, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[13] E. F. Keller and L. A. Segel, Initiation of smile mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399–415.

[14] D. Li and J.L. Rodrigo, Finite-time singularities of an aggregation equation in $\mathbb{R}^n$ with fractional dissipation, *Comm. Math. Phys.*, 287 (2009), 687–703.

[15] D. Li and J. L. Rodrigo, Refined blowup criteria and nonsymmetric blowup of an aggregation equation, *Adv. in Math.*, 220 (2009), 1717–1738.

[16] D. Li, J. L. Rodrigo and X. Zhang, Exploding solutions for a nonlocal quadratic evolution problem, *Rev. Mat. Iberoamericana*, 26 (2010), 295–332.

[17] D. Q. Li and Y. M. Chen, *Nonlinear Evolution Equation*, Science Press, 1999.

[18] E. D. Nezza, G. Palatucci and E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.

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