High spin, high derivatives.

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Abstract. This work is dedicated to honour the memory of Víctor Manuel Villanueva Sandoval. One of the academic interests shared with Victor was the disturbing fact that despite many years of work of the most talented people, there is no consistent quantum field theory formulation for interacting high spin particles. We propose here a formalism for the description of high spin particles by directly generalizing the structure of Dirac theory for spin $\frac{1}{2}$ to the $(j, 0) \oplus (0, j)$ representation. We find that this generalization leads naturally to a formalism of order $2j$ in the derivatives of the field. For $j > 1$, we conjecture that this construction evades the linear instabilities of the Ostrogradski theorem for theories with higher than two time derivatives due to the constrictions introduced by the projection onto subspaces of well defined parity.

1. Introduction.
By the end of the summer of 2013, Víctor Manuel Villanueva Sandoval sadly passed away, at the age of 43 and after spending almost twenty years doing physics. I met Victor for the first time in the summer of 1993 during a stay at the Institute of Physics of Guanajuato University. After completing his undergraduate studies at the UMSNH, Victor decided to come to Guanajuato University for his Master and Ph. D. and did work on gravitation and theoretical high energy physics.

It was during this period that I spent with Victor time enough to know him personally. It was a surprise to me to find someone who love fishing at 1800 meters above the sea level. I was born at a small town close to the beach on the pacific and for me fishing was hundred per cent an activity to be done in the sea. I learn from Victor the secrets of fishing in a lake during the many enjoyable mornings that he invited me to go to one of the many small lakes in the neighbourhood of León.

Among the many subjects that entered our long conversations inevitably we always landed on physics. Both of us got interested in the ancient problem of high spin and in the formal problems of constrained dynamics. He eventually evolved into the second subject and its applications to specific problems. I was entertained with hadron physics and trying to find a way to describe high spin hadrons. We did publish only one paper together -on hadron physics- but I enjoyed our discussions on many subjects which encouraged me to address the old problems from a new perspective. It droves me to try to understand the high spin problem from first principles which eventually lead to the conceptual ideas to be discussed here.

The quantum field theory of high spin particles is problematic as originally discussed by Johnson and Sudharsan in Ref. [1]. Later on, many authors have been addressed this problem from different perspectives [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].
In this short contribution honouring Víctor I will present the advances on a new way of dealing with the quantum field theory of high spin. This work was developed in collaboration with Selim Gómez Ávila and Rodolfo Ferro Hernández and its final form will be published elsewhere.

2. Free theory for spinning particles with well defined parity.

High spin particles are usually described using fields transforming under the HLG as tensors (for bosons) or spinor-tensors (for fermions). These representations contain additional spin components which are projected out of the theory. This elimination of the undesired spin components is done imposing constraints on the field which are lost or modified when interactions are introduced. These modifications in general spoils the consistency of the QFT.

The Dirac theory was conceived as a linearization of the Klein-Gordon equation but from a modern perspective it can be viewed as a simple projection onto subspaces of well defined parity in the \((1,0)\oplus (0,1)\) representation space. This representation of the HLG contains only spin 1/2 Poincaré sectors and there is no need to impose constraints to project out undesired components. This opens the possibility of using the direct single spin generalization of the Dirac matrix basis for the Dirac space. This was done in [15] based on the covariant properties of parity operator.

The covariant matrix basis for the \((j,0)\oplus (0,j)\) space in general contains \([2(2j+1)]^2\) matrices of dimension \(2(2j+1)\) which can be classified according to their HLG transformation properties as two Lorentz scalar operators, the unit matrix and the chirality operator \(\chi\); six matrix operators transforming in the \((1,0)\oplus (0,1)\) representation, \(M_{\mu\nu}\) the generators of the HLG; a pair of symmetric traceless tensors, \(S^{j\mu_1\mu_2...\mu_{2j}}\) and \(\chi S^{j\mu_1\mu_2...\mu_{2j}}\) transforming in the \((j,j)\) representation and higher rank matrix tensors transforming in the \((2,0)\oplus (0,2)\oplus (3,0)\oplus (0,3)\) \(\ldots\), \((2j,0)\oplus (0,2j)\) representation [15].

The rest frame parity operator turns out to be \(S^{00...0}\), i.e., the totally temporal part of the first symmetric tensor. The boost operator can be easily constructed for the \((j,0)\oplus (0,j)\) representation and it is easy to show that

\[
B(p)\Pi B^{-1}(p) = \frac{S^{\mu_1\mu_2...\mu_{2j}}p_{\mu_1}p_{\mu_2}...p_{\mu_{2j}}}{m^{2j}} \equiv S_j(p) \frac{1}{m^2}.
\]

The symmetric tensor \(S^{\mu_1\mu_2...\mu_{2j}}\) is then related to the form of parity operator as seen in the reference frame where the spin \(j\) particle has momentum \(p\).

For spin \(j = 1/2\) we obtain the following basis

\[
\{1, \chi, S^\mu, \chi S^\mu, M^{\mu\nu}\},
\]

where

\[
S^\mu = \Pi(g^{0\mu} - 2iM^{0\mu}).
\]

This is the conventional basis except for a \(1/2\) factor in \(M_{\mu\nu}\). The chirality operator \(\chi\) is the Dirac \(\gamma^5\) matrix and the two “symmetric” operators of rank \(2j = 1\) are simply \(\gamma^\mu\) and \(\gamma^5\gamma^\mu\).

The covariant form of parity operator in momentum space is

\[
B(p)\Pi B^{-1}(p) = \frac{S^\mu p_\mu}{m} = \frac{\gamma^\mu p_\mu}{m}.
\]

For \(j = 1\) we have a set of 36 covariant matrices denoted as

\[
\{1, \chi, S^{\mu\nu}, \chi S^{\mu\nu}, M^{\mu\nu}, C^{\mu\nu\alpha\beta}\}.
\]
The two scalars and the antisymmetric tensor are the direct generalization of the Dirac case. The first symmetric tensor is given by
\[ S_{\mu\nu} = \Pi \left( g^{\mu\nu} - ig_{\gamma\mu} M^{\nu\gamma} - \{ M^{\mu\nu}, M^{0\nu} \} \right). \] (6)

The 10 independent components of a symmetric tensor are further reduced to 9 by the traceless condition
\[ S_{\mu\mu} = 0, \] (7)
leaving only the components of the (1, 1) HLG representation. The second tensor transforming in the (1, 1) HLG representation is simply \( \chi S_{\mu\nu} \).

The last tensor in Eq. (5), transforms in the (2, 0) \( \oplus (0, 2) \) representation and is given by
\[ C_{\mu\nu\alpha\beta} = 4 \{ M^{\mu\nu}, M^{\alpha\beta} \} + 2 \{ M^{\mu\alpha}, M^{\nu\beta} \} - 2 \{ M^{\mu\beta}, M^{\nu\alpha} \} - 8(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}). \] (8)

It satisfies
\[ C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\alpha\beta\nu}, \quad C_{\mu\nu\alpha\beta} = C_{\alpha\beta\mu\nu}. \] (9)
In addition, this tensor satisfies the Bianchi identity
\[ C_{\mu\nu\alpha\beta} + C_{\mu\alpha\beta\nu} + C_{\mu\beta\nu\alpha} = 0, \] (10)
and contracting any pair of indices yields a vanishing result. It is easy to show that these conditions yield only 10 independent components transforming in the (2, 0) \( \oplus (0, 2) \) representation.

The covariant form of parity operator in momentum space in this case is given by
\[ B(p) \Pi B^{-1}(p) = \frac{S_{\mu\nu} p_{\mu} p_{\nu}}{m^2}. \] (11)

Similar calculations can be done for high spin \( j > 1 \) following the algorithm proposed in [15]. We now turn to the identification of states with well defined parity in the \((j, 0) \oplus (0, j)\) representation. In the rest frame and momentum space this can be done simply using the parity projector
\[ \mathbb{P}_\pm(0) u_\pm(0) = u_\pm(0), \] (12)
where
\[ \mathbb{P}_\pm(0) = \frac{1}{2} (1 \pm \Pi), \] (13)
and the sign \(+(-)\) denotes the corresponding parity.

In an arbitrary frame the projector is given by
\[ \mathbb{P}_\pm(p) = \frac{1}{2} \left( 1 \pm \frac{S_j(p)}{m^{2j}} \right). \] (14)

It can be shown that the symmetric operator satisfies
\[ (S_j(p))^2 = (p^2)^{2j}, \] (15)
which can be used to show that, *on-shell*, the following relations hold
\[ (\mathbb{P}_\pm(p))^2 = \mathbb{P}_\pm(p), \quad \mathbb{P}_+(p) \mathbb{P}_-(p) = \mathbb{P}_-(p) \mathbb{P}_+(p) = 0. \] (16)
In an arbitrary frame, the parity projection condition can be obtained simply boosting Eq. (14) to yield
\[ (\pm S_j(p) - m^{2j}) u_\pm(p, \lambda) = 0. \] (17)
In configuration space and for plane waves
\[ \psi_{\pm}(x) = u_{\pm}(p, \lambda)e^{-ip \cdot x}, \] (18)
we get
\[ (\pm S_j(i\partial) - m^{2j}) \psi_{\pm}(x) = 0. \] (19)
For \( j = 1/2 \) and positive parity we recover the Dirac theory
\[ [iS^\mu \partial_\mu - m] \psi(x) = 0. \] (20)

The suitable lagrangians for Eqs. (19) are
\[ \mathcal{L}_{\pm} = \overline{\psi}_{\pm}(x) \left( \pm S_j(i\partial) - m^{2j} \right) \psi_{\pm}(x), \] (21)
where the adjoint spinor is given by
\[ \overline{\psi} = \psi^\dagger \Pi. \] (22)

We will consider the positive parity case in the following in order to simplify the notation. The negative parity case is easily obtained with the appropriate sign changes.

Notice that the formalism involves derivatives of the field of order \( 2j \) at the level of the equation of motion. For spin \( (j > 1) \) we have more than two field derivatives. There is an ancient result concerning higher derivative theories due to Ostrogradski [16] stating that non-degenerate higher derivative theories necessarily contain linear instabilities. Non-degenerate higher derivative theories are thus inconsistent with the quantum field theory principles (see e.g. [17] for a pedagogical introduction).

We remark that Ostrogradski’s main assumption, namely non-degeneracy is not satisfied for a constrained system and degenerate theories can be free of the linear instabilities [18]. In our formalism there are constraints related to the parity projection. These constraints are of purely kinematical origin and we conjecture that the formalism is free of the linear instabilities.

3. Discrete symmetries.
Using the gauge principle we obtain the equation of motion satisfied by \( \psi \) interacting with an external electromagnetic field as
\[ (S_j(i\partial - qA) - m^{2j}) \psi = 0. \] (23)
Complex conjugating Eq. (19) and multiplying by a matrix \( \Gamma \) in the \((j, 0) \oplus (0, j)\) space we get
\[ \left[ (-1)^{2j} \Gamma S_j^\dagger \Gamma^{-1} (i\partial + qA) - m^{2j} \right] \psi^c = 0. \] (24)
with
\[ \psi^c = \Gamma \psi^*. \] (25)
If we require \( \psi^c \) to satisfy the same equation but with the opposite charge we get
\[ \Gamma(S^\mu_{\mu_1\mu_2...\mu_{2j}})^* \Gamma^{-1} = (-1)^{2j} S^\mu_{\mu_1\mu_2...\mu_{2j}}. \] (26)
It can be shown that the matrix
\[ \Gamma = \begin{pmatrix} 0 & U \\ U^{-1} & 0 \end{pmatrix} \] (27)
with
\[ U = e^{-i\pi J_2}. \] (28)
satisfies Eq.(26). The square of the $U$ matrix is proportional to the unit matrix
\[ U^2 = e^{-i2\pi J_2} = (-1)^{2j}. \] (29)
This matrix also satisfies
\[ \Gamma \chi^* \Gamma^{-1} = -\chi \] (30)
which can be rewritten as
\[ \{C, \chi\} = 0. \] (31)
Also a straightforward calculation yields
\[ \Gamma (M_{\mu\nu})^* \Gamma^{-1} = -M_{\mu\nu}. \] (32)
Finally, it is easy to show that
\[ \Pi^* \Gamma^{-1} = (-1)^{2j} \Pi. \] (33)
This relation when written in terms of charge conjugation yields the expected result
\[ [C, \Pi] = 0 \quad \text{for bosons,} \]
\[ \{C, \Pi\} = 0 \quad \text{for fermions}. \] (34)

4. Chiral decomposition.
The $(j, 0) \oplus (0, j)$ representation has a natural chiral structure. The chirality operator $\chi$ satisfies
\[ \{\chi, S^{\mu_1 \cdots \mu_2j}\} = 0, \quad \chi^2 = 1 \] (35)
\[ [\chi, \mathcal{O}] = 0 \] (36)
where $\mathcal{O}$ denotes any other operator in the covariant basis. The chiral components corresponding to the $(j, 0)$ and $(0, j)$ subspaces can be obtained via the projections
\[ \psi_R = P_R \psi \quad \text{and} \quad \psi_L = P_L \psi, \] (37)
where
\[ P_R = \frac{1}{2} (1 + \chi), \quad P_L = \frac{1}{2} (1 - \chi). \] (38)
These operators satisfy
\[ P_R + P_L = 1, \quad P_R P_L = 0, \quad P_R^2 = P_R, \quad P_L^2 = P_L, \] (39)
and from the commutation relations in Eqs. (36) we get
\[ \mathcal{O} P_{R,L} = P_{R,L} \mathcal{O}, \quad S^{\mu_1 \cdots \mu_2j} P_R = P_L S^{\mu_1 \cdots \mu_2j}. \] (40)
The positive parity Lagrangian
\[ \mathcal{L} = \bar{\psi} \left[ S(i\partial) - m^{2j} \right] \psi \] (41)
can be decomposed into chiral components as
\[ \mathcal{L}_I = \bar{\psi}_R S(i\partial) \psi_R + \bar{\psi}_L S(i\partial) \psi_L - m^{2j} (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R). \] (42)
Clearly, in the massless limit, we get a chiral symmetry under the following transformations
\[ \psi_R' = \exp (i\alpha_R) \psi_R \quad \psi_L' = \exp (i\alpha_L) \psi_L. \] (43)
The chiral fields decouple in the massless limit allowing for different interactions of these fields. This is the very same structure of spin 1/2 fermions in the standard model.
5. Conclusions.
Using the recent construction of a covariant basis for the $(j,0) ⊕ (0,j)$ representation we propose here a formalism for the construction of a QFT for spin $j$ particles, based on the projection onto subspaces of well defined parity. For $j = 1/2$ our formalism recover the Dirac theory. For higher $j$ it has a chiral structure analogous to the Dirac field.

We construct the charge conjugation operator in the formalism and show that in the case of bosons it commutes with parity while in the case of fermions it anti-commutes as expected. The equation of motion is of order $2j$ in the derivatives of the field. However, the constraints associated to parity projection makes the theory degenerate and we conjecture that linear instabilities associated to higher derivative theories are not present in this case.

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