ON A SECOND-ORDER RATIONAL DIFFERENCE EQUATION AND A RATIONAL SYSTEM

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Abstract. We give a complete description of the qualitative behavior of the second-order rational difference equation \#166. We also establish the boundedness character for the rational system in the plane \#(8,30).

1. Introduction

In their book [7], Kulenović and Ladas initiated a systematic study of the general second-order rational difference equation,

\[ x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, \quad n = 0, 1, \ldots, \] (1)

with nonnegative parameters so that \( A + B + C > 0 \) and nonnegative initial conditions chosen to avoid division by zero. A main feature of this study was the subdivision of the problem into a large number of special cases. The study of these special cases has attracted a great deal of attention in the literature. A large amount of work has been directed toward developing a complete picture of the qualitative behavior of the difference equation (1). A detailed account of the progress up to 2007 can be found in [1] and [2]. A more recent account of the subsequent progress up to 2009 can be found in [4]. According to Ref. [4], there remain only two special cases of (1) for which the qualitative behavior has not been established yet. However, the authors of Ref. [4] claim that a change of variables found in Ref. [8] reduces the special case \#141 to the special case \#66 which was resolved in [11]. After carefully reading [8] we could not find the purported change of variables. So, as far as we know, there remain three special cases of (1) for which the qualitative behavior has not been established yet. These three remaining special cases are the cases numbered 68, 141 and 166 in the numbering system given in [6]. The special cases \#68 and \#141 are the two subcases of the following second-order rational difference equation

\[ x_{n+1} = \frac{\alpha + x_n}{A + x_n + C x_{n-1}}, \quad n = 0, 1, \ldots, \]
with \( A \geq 0 \), all other parameters positive, and nonnegative initial conditions. The special case \#166 is the second-order rational difference equation

\[
x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, \ldots,
\]

with positive parameters and nonnegative initial conditions. In this article, we prove that in all ranges of positive parameters the unique equilibrium of the difference equation (2) is globally asymptotically stable. Thus, the special cases \#68 and \#141 are the only remaining cases of (1) for which the qualitative behavior has not been established yet.

More recently, in [5], Camouzis, Kulenović, Ladas, and Merino have initiated a systematic study of the general rational system of difference equations in the plane,

\[
x_{n+1} = \frac{\alpha + \beta x_n + \gamma y_n}{A + Bx_n + Cy_n}, \quad y_{n+1} = \frac{p + \delta x_n + \epsilon y_n}{q + Dx_n + Ey_n}, \quad n = 0, 1, 2, \ldots,
\]

with nonnegative parameters and nonnegative initial conditions chosen to avoid division by zero. According to Ladas, there remain only two special cases of (3) for which the boundedness character has not been established yet. These two remaining special cases are the cases numbered \((6, 25)\) and \((8, 30)\) in the numbering system given in [5]. The special case \#(6, 25) is the system

\[
x_{n+1} = \frac{x_n}{y_n}, \quad y_{n+1} = x_n + \epsilon y_n, \quad n = 0, 1, 2, \ldots,
\]

with \( \epsilon > 0 \) and nonnegative initial conditions. The special case \#(8, 30) is the system

\[
x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{\alpha + \gamma y_n}{x_n + y_n} \quad n = 0, 1, \ldots,
\]

with positive parameters and nonnegative initial conditions. In this article, we prove that in all ranges of positive parameters every solution of the difference equation (4) is bounded. Thus, the special case \#(6, 25) is the only remaining case of (3) for which the boundedness character has not been established yet.

2. Equation \#166

**Theorem 1.** The unique equilibrium of the difference equation (2) is globally asymptotically stable.

**Proof.** The proof will proceed in three cases. The first case will be the case where \( A \leq \alpha \), which was proved in [7]. For the reader’s convenience, we will restate the proof here. Notice that for \( A \leq \alpha \),

\[
x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}} = \frac{A - A + \alpha + \beta x_n + x_{n-1}}{A + x_{n-1}} = 1 + \frac{\alpha - A + \beta x_n}{A + x_{n-1}}.
\]

Via the change of variables \( x_n = 1 + z_n \), this difference equation reduces to the following equation.

\[
z_{n+1} = \frac{\alpha - A + \beta + \beta z_n}{A + 1 + z_{n-1}}.
\]
It was shown in [11] that the unique equilibrium is globally asymptotically stable for the above equation, thus the unique equilibrium is globally asymptotically stable for the difference equation (2) in this case. Now we will address the case where \( A > \alpha \) and \( A \leq \beta + \alpha \). We will begin by showing that the interval \([1, \infty)\) is an invariant attracting interval. First notice that

\[
\frac{\partial}{\partial x} \left( \frac{\alpha + \beta y + x}{A + x} \right) = \frac{A - \alpha - \beta y}{(A + x)^2},
\]

(5)

Suppose \( x_n \geq 1 \), then

\[
x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}} \geq \frac{\alpha + \beta + x_{n-1}}{A + x_{n-1}},
\]

and, due to the fact that \( A \leq \beta + \alpha \),

\[
x_{n+1} \geq \frac{\alpha + \beta + x_{n-1}}{A + x_{n-1}} \geq 1.
\]

It follows by induction that whenever \( x_N \geq 1 \), then \( x_n \geq 1 \) for all \( n \geq N \). Thus, \([1, \infty)\) is an invariant interval. Let \( \{x_n\}_{n=1}^{\infty} \) be a nonnegative solution to the difference equation (2). Assume that \( x_n \notin [1, \infty) \) for all \( n \in \mathbb{N} \). Then \( x_n < \frac{A - \alpha}{\beta} \) for all \( n \in \mathbb{N} \), since if \( x_N \geq \frac{A - \alpha}{\beta} \) for some \( N \in \mathbb{N} \), then

\[
x_{N+1} = \frac{\alpha + \beta x_N + x_{N-1}}{A + x_{N-1}} \geq \frac{A + x_{N-1}}{A + x_{N-1}} = 1,
\]

yielding a contradiction to our prior assumption. Since \( x_n < \frac{A - \alpha}{\beta} \) for all \( n \in \mathbb{N} \) we get via (5),

\[
x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}} \geq \frac{\alpha + \beta x_n}{A}, \quad n \in \mathbb{N}.
\]

(6)

So, under these assumptions, our solution satisfies the difference inequality (6). Now, in the case where \( A \leq \beta \) any solution which satisfies the difference inequality (6) must be an unbounded solution, contradicting the assumption \( x_n \notin [1, \infty) \) for all \( n \in \mathbb{N} \). On the other hand, if \( A > \beta \) then, applying Theorem 3 from [13] or similar results, for each \( \epsilon > 0 \) there exists an \( N_\epsilon \) so that \( x_n \geq \frac{\alpha}{A - \beta} - \epsilon \) for all \( n \geq N_\epsilon \). Since \( A \leq \beta + \alpha \) in this case, \( \frac{\alpha}{A - \beta} \geq 1 \). So, in this case, for each \( \epsilon > 0 \) there exists an \( N_\epsilon \) so that \( x_n \in [1 - \epsilon, 1) \) for \( n \geq N_\epsilon \). In other words, \( x_n \to 1 \). Thus, in this case, an arbitrary solution either converges to 1, or enters the invariant interval \([1, \infty)\). Now, for a solution in \([1, \infty)\), we may make the change of variables \( x_n = 1 + z_n \) reducing the equation as follows.

\[
z_{n+1} = \frac{\alpha + 1 + z_{n-1} + \beta (1 + z_n)}{A + 1 + z_{n-1}} - 1 = \frac{\alpha - A + \beta + \beta z_n}{A + 1 + z_{n-1}}.
\]

Now, if \( \alpha + \beta > A \), then this reduced equation was resolved in [11], where it was shown that the unique equilibrium is globally asymptotically stable for the above equation. In the very special subcase where \( \alpha + \beta = A \), the above equation may be rewritten as,

\[
z_{n+1} = \frac{(A - \alpha)z_n}{A + 1 + z_{n-1}}.
\]
For the above equation, every solution converges to zero. Thus, $x_n \to 1$ in this very special case. So, we have shown that in the case where $A > \alpha$ and $A \leq \beta + \alpha$ that every nonnegative solution of the difference equation (2) converges to the unique positive equilibrium. The final case we must consider is the case where $A > \beta + \alpha$. We will begin our consideration of this case, by proving that the interval $[0, A - \alpha \beta]$ is invariant. Suppose that $x_n \leq A - \alpha \beta$, then

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}} \leq 1 < A - \alpha \beta.$$

Now we show that $[0, A - \alpha \beta]$ is attracting. Assume, for the sake of contradiction, that $x_n \not\in [0, A - \alpha \beta]$ for all $n \in \mathbb{N}$. Under this assumption, since $x_n > A - \alpha \beta$, we may use (5) to obtain the following difference inequality.

Thus, applying Theorem 2 from [13] or similar results, for each $\epsilon > 0$ there exists a $N_\epsilon$ so that $x_n \leq \frac{\alpha}{A - \beta} + \epsilon$ for all $n \geq N_\epsilon$. Now, since $\frac{\alpha}{A - \beta} < 1 < \frac{A - \alpha}{\beta}$, every solution with these properties must eventually enter the interval $[0, \frac{A - \alpha}{\beta}]$, contradicting our assumption that this does not occur. So, we have shown that in this case the interval $[0, \frac{A - \alpha}{\beta}]$ is an invariant interval which every solution must eventually enter. Since our difference equation is nondecreasing with respect to each argument and has a unique nonnegative equilibrium in this interval, the m-M theorem, see [9] and [12], implies that the unique equilibrium $\bar{x}$ is globally asymptotically stable in this case. The unique equilibrium of equation #166 is well known to be locally asymptotically stable in all cases, see [7] for local stability of the cases we have not yet shown. □

3. THE BOUNDEDNESS CHARACTER OF THE SPECIAL CASE #(8, 30)

Now we present the boundedness character of the following system numbered #(8, 30) in the numbering system developed in [5].

$$x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{\alpha + \gamma y_n}{x_n + y_n}, \quad n = 0, 1, \ldots,$$

with nonnegative parameters and nonnegative initial conditions. It turns out that the $x_n$ component of the system #(8, 30) can be reduced to the difference equation

$$x_n = \left( \frac{1}{x_{n-1} x_{n-2}} \right) \frac{\alpha + \gamma x_{n-1} x_{n-2}}{1 + x_{n-1}}, \quad n \geq 2,$$

through algebraic identities. This reduction proceeds as follows. The first equation of the system #(8, 30) gives us

$$y_n = x_{n+1} x_n, \quad n \geq 0.$$

Substituting this in for $y_n$ in the second equation gives us

$$y_{n+1} = \frac{\alpha + \gamma x_{n+1} x_n}{x_n + x_{n+1} x_n}, \quad n \geq 0.$$
Substituting this into the first equation gives us
\[ x_{n+2} = \left( \frac{1}{x_{n+1}x_n} \right) \frac{\alpha + \gamma x_{n+1}x_n}{1 + x_{n+1}}, \quad n \geq 0. \]

This yields the difference equation (7). So from now on it suffices to show that every solution of (7) is bounded.

**Theorem 2.** Every solution is bounded for the rational difference equation (7) with positive parameters and positive initial conditions.

**Proof.** Recall our difference equation (7),
\[ x_n = \left( \frac{1}{x_{n-1}x_{n-2}} \right) \frac{\alpha + \gamma x_{n-1}x_{n-2}}{1 + x_{n-1}}, \quad n \geq 2. \]

Iterating with respect to the leftmost \( x_{n-1} \) term in the denominator we get,
\[ x_n = x_{n-3} \left( \frac{1 + x_{n-2}}{\alpha + \gamma x_{n-2}x_{n-3}} \right) \frac{\alpha + \gamma x_{n-1}x_{n-2}}{1 + x_{n-1}}, \quad n \geq 3. \]

Equation (8) yields the following inequality,
\[ x_n < \left( \frac{x_{n-3}}{\alpha} + \frac{1}{\gamma} \right) \frac{\alpha + \gamma x_{n-1}x_{n-2}}{1 + x_{n-1}}, \quad n \geq 3. \]

Iteration gives us the following inequality,
\[ \frac{1}{\alpha + \gamma x_{n-2}x_{n-3}} = \frac{1}{\alpha + \gamma x_{n-3} \left( \frac{1}{x_{n-3}x_{n-4}} \right) \frac{\alpha + \gamma x_{n-3}x_{n-4}}{1 + x_{n-3}}} < \frac{1}{\alpha + \gamma x_{n-3} \left( \frac{1}{x_{n-3}x_{n-4}} \right) \frac{\gamma x_{n-3}x_{n-4}}{1 + x_{n-3}}} = \frac{1}{\alpha + \gamma x_{n-3}}. \]

Using the above Equation (8) yields
\[ x_n < x_{n-3} \left( \frac{1}{\alpha + \gamma^2 x_{n-3}} + \frac{1}{\gamma x_{n-3}} \right) \frac{\alpha + \gamma x_{n-1}x_{n-2}}{1 + x_{n-1}}, \quad n \geq 3. \]

The proof proceeds by contradiction. Suppose that there is an unbounded solution \( x_n \). Take a subsequence \( x_{n_i} \to \infty \) so that \( x_{n_i} > x_{n_i-3} \) for all \( i \in \mathbb{N} \). Then equation (7) gives us that
\[ x_{n_i-1}x_{n_i-2} \to 0, \]
and so, from the inequality (9),
\[ x_{n_i-3} \to \infty. \]

Thus, for sufficiently large \( n_i \), it follows from the inequality (10) that
\[ x_{n_i} < x_{n_i-3}, \]
which contradicts our earlier assumption. \( \square \)

**Corollary 1.** Every solution of the system \#(8, 30) is bounded.
Proof. Let \( \{(x_n, y_n)\}_{n=1}^{\infty} \) be a solution of the system \#(8, 30). From Theorem 2 and the earlier reduction we know that \( \{x_n\}_{n=1}^{\infty} \) is bounded. Since
\[
y_n = x_{n+1} x_n, \quad n \geq 0,
\]
for all solutions of the system \#(8, 30), we get that \( \{y_n\}_{n=1}^{\infty} \) is bounded. \( \square \)

4. Conclusion

We have shown that the unique equilibrium of the difference equation \#166 is globally asymptotically stable and we have shown that every solution of the difference equation \#(8, 30) is bounded. We leave the reader with three crucial conjectures pertaining to four special cases.

The special cases \#68 and \#141 are the two subcases of the following second-order rational difference equation
\[
x_{n+1} = \frac{\alpha + x_n}{A + x_n + C x_{n-1}}, \quad n = 0, 1, \ldots,
\]
with \( A \geq 0 \), all other parameters positive, and nonnegative initial conditions. It is conjectured in [7] that the unique positive equilibrium is globally asymptotically stable for the difference equations \#68 and \#141. The difference equations \#68 and \#141 are now the only second-order rational difference equations for which the qualitative behavior has not been established yet.

The special case \#70 is the only remaining third-order rational difference equation whose boundedness character is yet to be determined. Special case \#70 is as follows,
\[
x_{n+1} = \frac{\alpha + x_n}{C x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots,
\]
with positive parameters and nonnegative initial conditions. It is conjectured in [6] that there exist unbounded solutions for some choice of nonnegative initial conditions for the difference equation \#70.

The special case \#(6, 25) is the following system of rational difference equations,
\[
x_{n+1} = \frac{x_n}{y_n}, \quad y_{n+1} = x_n + \epsilon y_n, \quad n = 0, 1, 2, \ldots
\]
with positive parameters and nonnegative initial conditions. There is a conjecture in [3], originating in [10], which claims that for each solution of the system \#(6, 25) the sequence \( \{x_n\}_{n=0}^{\infty} \) arising from the \( x \) component of the solution is bounded. The system \#(6, 25) is now the only rational system in the plane for which the boundedness character has not been established yet.

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