Ideals in the Goldman Algebra

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Abstract

The goal of this work is to study the ideals of the Goldman Lie algebra $S$. To do so, we construct an algebra homomorphism from $S$ to a simpler algebraic structure, and focus on finding ideals of this new structure instead. The structure $S$ can be regarded as either a $\mathbb{Q}$-module or a $\mathbb{Q}$-module generated by free homotopy classes. For $\mathbb{Z}$-module case, we proved that there is an infinite class of ideals of $S$ that contain a certain finite set of free homotopy classes. For $\mathbb{Q}$-module case, we can classify all the ideals of the new structure and consequently obtain a new class of ideals of the original structure. Finally, we show an interesting infinite chain of ideals that are not those ideals obtained by considering the new structure.

1. Lie algebra homomorphism from $\mathbb{Z}$-module of freely homotopic classes on compact orientable surface

Definition 1. Let $\hat{\pi}_0$ be the set of freely homotopic classes of curves on the surface and $\mathbb{Z}[\hat{\pi}_0]$ be the $\mathbb{Z}$-module generated by $\hat{\pi}_0$.

Recall that for closed surface with genus $g$, its fundamental group is:

$$\pi_1 = \langle a_1, a_2, \ldots, a_{2g-1}, a_{2g} | a_1a_2a_1^{-1}a_2^{-1} \ldots a_{2g-1}a_{2g}a_{2g-1}^{-1}a_{2g}^{-1} = 1 \rangle$$

For surface with boundary, its fundamental group is a free group generated by $n$ generators: $\pi_1 = \langle a_1, a_2, \ldots, a_n \rangle$

As a result, we can represent each element of $\pi_1$ uniquely as a linear reduced word in letter $a_i$, the word with letters $a_i$ so that its 2 consecutive letters do not cancel each other.
Now define **cyclic word** as a word whose letters are arranged on a circle instead of a line, and a reduced cyclic word is a cyclic word whose 2 consecutive letters do not cancel each other.

A **linear representative** $W$ of the cyclic word $W$ is the linear word obtained by making a cut to 2 consecutive word of $W$. From here, we can see that $W$, and its linear representative, $W$, have the same word length.

We have the $1-1$ correspondence between $\hat{\pi}_0$ and the conjugacy classes of $\pi_1$ so $x \in \hat{\pi}_0$ will correspond to a conjugacy class $\gamma_x$ of $\pi_1$. Therefore, we can represent each element $x$ of $\hat{\pi}_0$ uniquely as a cyclic reduced word $W$ so that if $W$ is a reduced linear word corresponding to an element of $\pi_1$ in the conjugacy class $\gamma_x$, then by repeatedly canceling the first and last letters of $W$, we will get a linear representative of $W$.

**Definition 2.** For closed compact surface $S$ with genus $g$ and with fundamental group $\pi_1 = \langle a_1, a_2, \ldots, a_{2g-1}, a_{2g} | a_1 a_2 a_1^{-1} a_2^{-1} \ldots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} = 1 \rangle$, we define the abelianization of its fundamental group, $A(n)$, as $\langle a_1, a_2, \ldots, a_n | a_i a_j = a_j a_i \rangle$.

For compact surface with boundary $S$ with the fundamental group $\pi_1 = \langle a_1, a_2, \ldots, a_n \rangle$, we define the abelianization of its fundamental group, $A(n)$, as $\langle a_1, a_2, \ldots, a_n | a_i a_j = a_j a_i \rangle$ instead.

Now let $\mathbb{Z}[A(n)]$ be the $\mathbb{Z}$-module generated by $A(n)$, and consider the map $\text{Ab}$

$$\text{Ab} : \mathbb{Z}[\hat{\pi}_0] \rightarrow \mathbb{Z}[A(n)]$$

$$\sum_{i=1}^{k} q_i u_i \mapsto \sum_{i=1}^{k} q_i \text{Re}(w_i)$$

Here $q_i \in \mathbb{Z}$, $u_i \in \hat{\pi}_0$. Also, $w_i$ is one chosen reduced linear word so that it corresponds to an element $v_i \in \pi_1$ which is in the conjugacy class of $\pi_1$ that corresponds to $u_i$. We call this $w_i$ the **linear element** of $u_i$. Note that there may be multiple linear elements of the same $u_i \in \hat{\pi}_0$, and each linear representative of the reduced cyclic word corresponding to $u_i$ is a linear element of $u_i$. Moreover, $\text{Re}(w) = \prod_{i=1}^{g} a_i^{\mu_i}$ is defined as an element in $A(n)$ that is obtained by reordering the letter $a_i$ in the reduced linear word $w$.

Now $\text{Ab}$ doesn’t depend on the choice of linear element $w_i$ of $u_i$. Indeed, if $w_i$
and \( w'_i \) are two different linear elements of \( u_i \), then \( w_i \) and \( w'_i \) must be conjugate to each other, i.e., there is some linear word \( w \) so that \( w_i = w w'_i w^{-1} \), and, therefore, \( \text{Re}(w_i) = \text{Re}(w'_i) \).

Finally, it is easy to see that \( A b \) is a \( \mathbb{Z} \)-module homomorphism and is surjective.

**Definition 3.** For any \( u_1 \) and \( u_2 \in \hat{\pi}_0 \), let \( m[w_1, w_2] \) be the sum of integer coefficients of their Goldman bracket. This means that if we take some representative curves \( \alpha \) and \( \beta \) of \( u_1 \) and \( u_2 \), then \( m[u_1, u_2] = \sum_{p \in \alpha \cap \beta} \text{sign}(p) \), the sum of signs over all intersection points \( p \) of curves \( \alpha \) and \( \beta \).

This definition will not depend on \( \alpha \) and \( \beta \) because Goldman bracket is the same for all curves in some same freely homotopic class of curves.

**Definition 4.** We now define a Lie bracket on \( \mathbb{Z}[A(n)] \) to make it a Lie algebra.

Since \( a_i \) and \( a_j \) has word-length 1, their preimages from the map \( A b \) is uniquely defined. As a result, we can define an symplectic product for \( a_i \) and \( a_j \):

\[
\langle a_i, a_j \rangle = m[Ab^{-1}(a_i), Ab^{-1}(a_j)] \quad \forall 1 \leq i, j \leq n.
\]

, \( m \) is defined in definition 3

Notice that \( \langle a_i, a_j \rangle = 1 \) or \( 0 \) or \( -1 \) since there is at most 1 linking pair and \( \langle a_i, a_j \rangle = -\langle a_j, a_i \rangle \) because of the anti-symmetric property of Goldman bracket.

Now for \( w_1 = \prod_{i=1}^{g} a_i^{a_{i}} \), \( w_2 = \prod_{i=1}^{g} a_i^{a_{i}} \in A(n) \), we define their symplectic product:

\[
\langle w_1, w_2 \rangle = \sum_{i,j=1}^{n} m_i n_j \langle a_i, a_j \rangle
\]

Therefore, \( \langle wv, ts \rangle = \langle w, t \rangle + \langle v, t \rangle + \langle w, s \rangle + \langle v, s \rangle \) \( \forall w, v, t, s \in A(n) \)

Now we define the bracket on elements of \( A(n) \):

\[
[w_1, w_2] = \langle w_1, w_2 \rangle w_1 w_2 = \langle w_1, w_2 \rangle \prod_{i=1}^{g} a_i^{a_{i}+a_{i}} \quad \forall w_1, w_2 \in A(n)
\]

Now the bracket in \( A(n) \) is extended to the bracket in \( \mathbb{Z}[A(n)] \) by \( \mathbb{Z} \)-bilinearity.
In particular, for any $\sum_{i=1}^{k} k_i w_i$ and $\sum_{j=1}^{l} l_j v_j \in \mathbb{Z}[A(n)]$ with $k_i, l_j \in \mathbb{Z}$, and $w_i, v_j \in A_n$, 

$$\sum_{i=1}^{k} k_i w_i, \sum_{j=1}^{l} l_j v_j = \sum_{1 \leq i \leq k, 1 \leq j \leq l} k_i l_j [w_i, v_j]$$

It is not difficult to check the anti-symmetry and Jacobi identities for this bracket on elements in $A(n)$ and, therefore, on elements in $\mathbb{Z}[A(n)]$. This makes $\mathbb{Z}[A(n)]$ become a Lie algebra.

**Theorem 1.** We prove that $m[u_1, u_2] = \langle \text{Ab}(u_1), \text{Ab}(u_2) \rangle \forall u_1, u_2 \in \hat{\pi}_0$. As a result, the map $\text{Ab}$ is a Lie algebra homomorphism.

**Proof.** Let $l(c)$ be the length of the linear or cyclic word $c$.

We prove theorem 1 by induction on the maximum $m = \max\{l(c_1), l(c_2)\}$, where $c_1$ and $c_2$ are cyclic words that correspond to $u_1$ and $u_2$.

If the maximum word-length $m$ is 1, then theorem 1 follows from the definition of symplectic product on $A(n)$. Assume that theorem 1 is true for all $m \leq k - 1$. Now we prove that it is also true for $m = k$.

Now suppose $w_1$ and $w_2$ are linear representatives of the cyclic reduced words corresponding to $u_1$ and $u_2$.

Now suppose $w_1 = x_1 y_1$ ($x_1$ is some letter $a_i$), and $w_2 = x_2 y_2$ ($x_2$ is also some letter $a_j$).

So $l(y_i) \leq k - 1$, where $y_i$ are some linear reduced word.

Choose the base point $P$ on the surface for the fundamental group $\pi_1$. For $i = 1, 2$, choose a loop $\beta_i$ based at $P$ that has the corresponding reduced linear word $x_i$ in the fundamental group, and a loop $\gamma_i$ that has the reduced linear word $y_i$ in fundamental group. Let $\alpha_i$ be the loop obtained by concatenating $\beta_i$ with $\gamma_i$. Then $\alpha_i$ will have the linear reduced word $w_i$ in the fundamental group, and therefore, $\alpha_i$ is in the freely homotopic class $u_i \in \hat{\pi}_0$.

Now we change the loop $\alpha_2$ to $\alpha_2'$, which is freely homotopic to $\alpha_2$, by slightly moving the point $P \in \alpha_2$ to a different point $P' \in \alpha_2'$ so that the two new sub-loops $\beta_2'$ and $\gamma_2'$ of $\alpha_2'$, which intersect at $P'$ and make up $\alpha_2'$, are homotopic to two original sub-loops $\beta_2$ and $\gamma_2$.

So we have $\beta_1$ and $\gamma_1$ are in some freely homotopic class $u_{\beta_1}$ and $u_{\gamma_1} \in \hat{\pi}_0$. 

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\(\beta_2\) and \(\beta'_2\) are in some freely homotopic class \(u_{\beta_2} \in \hat{\pi}_0\), and \(\gamma_2\) and \(\gamma'_2\) are in some freely homotopic class \(u_{\gamma_2} \in \hat{\pi}_0\).

Note that \(x_1, y_1, x_2, y_2\) are linear elements of \(u_{\beta_1}, u_{\gamma_1}, u_{\beta_2}, u_{\gamma_2}\) respectively. Moreover, lengths of cyclic words or linear elements of \(u_{\beta_i}\) and \(u_{\gamma_i}\) for \(i = 1, 2\) are at most \(k - 1\).

Also note that \(P \neq P'\) so the set of intersection points of \(\alpha_1\) and \(\alpha'_2\) is the disjoint union of sets of intersection points of \(\beta_1\) and \(\beta'_2\), \(\beta_1\) and \(\gamma'_2\), \(\gamma_1\) and \(\beta'_2\), \(\gamma_1\) and \(\beta_2\), \(\gamma_1\) and \(\gamma'_2\). Therefore,

\[
m[u_1, u_2] = m[\alpha_1, \alpha_2] = m[\alpha_1, \alpha'_2] = \sum_{p \in \alpha_1 \cap \alpha'_2} \text{sign}(p)
\]

\[
= \sum_{p \in \beta_1 \cap \beta'_2} \text{sign}(p) + \sum_{p \in \beta_1 \cap \gamma'_2} \text{sign}(p) + \sum_{p \in \gamma_1 \cap \beta'_2} \text{sign}(p) + \sum_{p \in \gamma_1 \cap \gamma'_2} \text{sign}(p)
\]

\[
= m[u_{\beta_1}, u_{\beta_2}] + m[u_{\beta_1}, u_{\gamma_2}] + m[u_{\gamma_1}, u_{\beta_2}] + m[u_{\gamma_1}, u_{\gamma_2}]
\]

By induction hypothesis on 4 pairs of elements in \(\hat{\pi}_0, (u_{\beta_1}, u_{\beta_2}), (u_{\beta_1}, u_{\gamma_2}), (u_{\gamma_1}, u_{\beta_2}), (u_{\gamma_1}, u_{\gamma_2})\), this sum \(m[u_1, u_2]\) equals to:

\[
= \langle \text{Ab}(u_{\beta_1}), \text{Ab}(u_{\beta_2}) \rangle + \langle \text{Ab}(u_{\beta_1}), \text{Ab}(u_{\gamma_2}) \rangle
\]

\[
+ \langle \text{Ab}(u_{\gamma_1}), \text{Ab}(u_{\beta_2}) \rangle + \langle \text{Ab}(u_{\gamma_1}), \text{Ab}(u_{\gamma_2}) \rangle
\]

\[
= \langle \text{Re}(x_1), \text{Re}(x_2) \rangle + \langle \text{Re}(x_1), \text{Re}(y_2) \rangle + \langle \text{Re}(y_1), \text{Re}(x_2) \rangle + \langle \text{Re}(y_1), \text{Re}(y_2) \rangle
\]

\[
= \langle \text{Re}(w_1), \text{Re}(x_2) \rangle \text{Re}(y_2) \rangle
\]

\[
= \langle \text{Ab}(u_1), \text{Ab}(u_2) \rangle
\]

This finishes the induction step and also proof for theorem 1. \(\square\)

**Remark 1.** Now we consider \(\mathbb{Q}[\hat{\pi}_0]\) as a \(\mathbb{Q}\)-module generated by \(\hat{\pi}_0\), and as a Lie algebra with Goldman bracket. We also consider \(\mathbb{Q}[A(n)]\), the \(\mathbb{Q}\)-module generated by \(A(n)\), instead, and define the map:

\[
\text{Ab}_\mathbb{Q} : \mathbb{Q}[\hat{\pi}_0] \to \mathbb{Q}[A(n)]
\]

\[
\sum_{i=1}^{k} q_i u_i \mapsto \sum_{i=1}^{k} q_i \text{Re}(w_i)
\]
in a similar way that we defined \( Ab \) in definition 2, but with \( q_i \in \mathbb{Q} \) instead.

Moreover, we can put a Lie bracket on \( \mathbb{Q}[A(n)] \) and make it a Lie algebra by extending bracket on \( A(n) \), which is defined in definition 4, by \( \mathbb{Q} \)-bilinearity.

Then, by theorem 1, \( Ab_{\mathbb{Q}} \) is also a Lie algebra homomorphism.

2. A class of ideals obtained from preimage of the Lie algebra homomorphism \( Ab \)

**Definition 5.** Denote by \( G \) the set of integer multiples of elements of \( A(n) \) (defined in definition 2), or

\[
G = \{ \alpha x, \text{ where } \alpha \in \mathbb{Z}, \text{ and } x = \prod_{j=1}^{n} a_{ij} \in A(n), \ i_j \in \mathbb{Z} \}.
\]

We say that a \( \mathbb{Z} \)-submodule of \( \mathbb{Z}[A(n)] \) is **geometric** if it is a \( \mathbb{Z} \)-submodule generated by a subset of \( G \).

Consider a \( \mathbb{Z} \)-submodule \( M \) of \( \mathbb{Z}[A(n)] \). For each \( n \)-tuple of integers \((i_1, \ldots, i_n)\), if there is non-zero \( \alpha_0 \) such that \( \alpha_0 w \) is in \( M \), where \( w = \prod_{j=1}^{n} a_{ij} \in A(n) \), then we define \( \alpha_{w,M} = \alpha_M(i_1, i_2, \ldots, i_n) \) as the smallest positive integer \( \alpha \) such that \( \alpha w = \alpha \prod_{j=1}^{n} a_{ij} \in M \). If there is no such multiple, then define \( \alpha_{w,M} = \alpha_M(i_1, i_2, \ldots, i_n) = 0 \).

**Lemma 1.** A geometric \( \mathbb{Z} \)-submodule \( I \) is an ideal of the Lie algebra \( \mathbb{Z}[A(n)] \) if and only if \( \alpha_{w,M} | \langle v, w \rangle \alpha_v \) for all \( v, w \in A(n) \). Here \( \alpha_w = \alpha_{w,I} \).

**Proof.** Suppose there is a geometric \( \mathbb{Z} \)-submodule \( I \) satisfying \( \alpha_w | \langle v, w \rangle \alpha_v \) for all \( v, w \in A(n) \). To prove that \( I \) is an ideal, we just need to prove that \( [\alpha_w, v, w] \in I \) for every \( v \in A(n) \)

By lemma’s assumption, \( \alpha_{wv} | \langle w, wv \rangle \alpha_w = \langle w, v \rangle \alpha_w \), so by definition of \( \alpha_{wv} \), we must have \( [\alpha_w, v, w] = \alpha_w | \langle w, v \rangle wv \in I \).

The other direction of the theorem is proved in a similar way.

**Theorem 2.** For each finite set \( Y \) of some freely homotopic classes on surface \( S \), there is an infinite class of nontrivial ideals that contain \( Y \).

**Proof.** Assume that \( \mathbb{Z}[A(n)] \) is the Lie algebra corresponding to the surface \( S \). We just need to prove that there is a class of non-trivial ideals of \( \mathbb{Z}[A(n)] \) that contain \( X \) for any finite subset \( X \) of \( A(n) \). Then the preimage of these ideals under the Lie algebra homomorphism \( Ab \) will be the ideals that we look for. Here \( X = Ab(Y) \)
Each element of $X$ is combination of some finite element in $A(n)$.

Since $X$ is finite, $X$ must be the subset of the $\mathbb{Z}$-submodule generated by $X_1 = \{a_1^{i_1}a_2^{i_2}...a_n^{i_n}|(i_1, i_2, \ldots, i_n) \in K_0\}$, for some finite subset $K_0 \subseteq \mathbb{Z}^n$.

Now consider the geometric $\mathbb{Z}$-submodule $I_K$ generated by
\[
\{\gamma_{i_1,i_2,...,i_n}a_1^{i_1}a_2^{i_2}...a_n^{i_n} \text{ such that } \gamma_{i_1,i_2,...,i_n} = 1 \text{ if } (i_1, i_2, \ldots, i_n) \in K \text{ and } \gamma_{i_1,i_2,...,i_n} = \gcd(i_1, i_2, \ldots, i_n) \text{ otherwise}\}.
\]

In this $\mathbb{Z}$-submodule $I_K$, $\alpha_{w,I_K} = \alpha_w = \gamma_{w_1,w_2,...,w_n}$ where $w = \prod_{j=1}^{n} a_j^{w_j}$.

Then $\alpha_{v}|\gcd(w_1, w_2, \ldots, w_n)|\sum_{j=1}^{n} w_j(\sum_{i=1}^{n} (a_j, a_i)v_i) = \langle w, v \rangle |\langle w, v \rangle \alpha_v$, for every $v = \prod_{i=1}^{n} a_i^{v_i} \in A(n)$.

By lemma 1, this submodule $I_K$ is an ideal of the Lie algebra $\mathbb{Z}[A(n)]$.

Because $K_0$ is finite, there is an infinite number of $n$-tuples $(i_1, i_2, \ldots, i_n) \notin K_0$ such that $\gcd(i_1, i_2, \ldots, i_n) > 1$.

Therefore, we have an infinite sequence of $n$ tuples $\{\alpha_j\}_{i=1}^{\infty}$ so that $\alpha_j = (j_1, j_2, \ldots, j_n) \notin K_0$ with $\gcd(j_1, j_2, \ldots, j_n) > 1$. Then we can choose $K_j = K_0 \cup \{\alpha_i\}_{i=1}^{\infty}$ so that we have an infinite sequence of distinct ideals $\{I_{K_j}\}$ that contains $K_0$ and therefore contains $X$.

By looking at its fundamental polygon, it is easy to see that a closed surface with genus $g$ has the corresponding $A(n) = A(2g) = \langle a_1, a_2, \ldots, a_{2g} \rangle$ with $n = 2g$ and $\langle a_{2i-1}, a_{2i} \rangle = 1$ and all other $\langle a_i, a_j \rangle = 0$ for $i \leq j$. Using this fact, we will prove the following proposition:

**Proposition 1.** Consider the algebra $\mathbb{Z}[A(n)]$ with $n = 2g$ that is associated with the closed surface $S$ with genus $g$. A geometric submodule $M$ of $\mathbb{Z}[A(n)]$ is an ideal if and only if for every $(i_1, \ldots, i_{2g})$ and $(k_1, \ldots, k_{2g}) \in \mathbb{Z}^{2g}$, $\alpha_M(k_1, k_2, \ldots, k_{2g})$ divides $\alpha_M(i_1, i_2, \ldots, i_{2g}) \cdot \gcd(\gcd(k_{2t-1}, k_{2t}) \cdot \gcd(i_{2i-1}, i_{2i}))$.

**Proof.** We prove the first direction of the theorem by assuming that $M$ is an ideal. By identifying $a_i$ with edges of the fundamental polygon of closed surface $S$, we have $\langle a_{2i-1}, a_{2i} \rangle = 1$ for all $i = \frac{1}{n}$ and $\langle a_i, a_j \rangle = 0$ for other $i \leq j$.
Using lemma 1, we have:

\[ \alpha_M(k_1, k_2, \ldots, k_{2g})|\alpha_M(i_1, i_2, \ldots, i_{2g}) \sum_{t=1}^{g} (i_{2t-1}k_{2t} - k_{2t-1}i_{2t}). \]  

(1)

From equation 1, we have: \( \alpha_M(1, 0, \ldots, 0) \) divides \( i_2\alpha_M(i_1, i_2, \ldots, i_{2g}) \).

Then \( \alpha_M(1, 0, \ldots, 0) \) divides \( \alpha_M(0, 1, \ldots, 0) \). Similarly, \( \alpha_M(0, 1, \ldots, 0) \) divides \( \alpha_M(1, 0, \ldots, 0) \) and therefore \( \alpha_M(1, 0, \ldots, 0) = \alpha_M(0, 1, \ldots, 0) \).

Then, again by equation 1, we get \( \alpha_M(1, 0, \ldots, 0) \) divides \( i_1\alpha_M(i_1, i_2, \ldots, i_{2g}) \). Therefore, \( \alpha_M(0, 1, \ldots, 0) \) divides \( \gcd(i_1, i_2)\alpha_M(i_1, i_2, \ldots, i_{2g}) \).

From equation (1), we also have \( \alpha_M(k_1, k_2, \ldots, k_{2g}) \) divides \( k_2\alpha_M(1, 0, \ldots, 0) \) and \( k_1\alpha_M(0, 1, \ldots, 0) \). So \( \alpha_M(k_1, k_2, \ldots, k_{2g}) \) divides \( \gcd(k_1, k_2)\alpha_M(1, 0, \ldots, 0) \).

Therefore,

\[ \alpha_M(k_1, k_2, \ldots, k_{2g})|\gcd(k_1, k_2)\alpha_M(1, 0, \ldots, 0)|\gcd(k_1, k_2)\gcd(i_1, i_2)\cdot\alpha_M(i_1, i_2, \ldots, i_{2g}) \]

Similarly, we can prove that, and so we finish the proof for the first direction.

The other direction that if \( \alpha_M(k_1, k_2, \ldots, k_{2g}) \) divides \( \alpha_M(i_1, i_2, \ldots, i_{2g}) \cdot \gcd_{t=1}^{g}(\gcd(k_{2t-1}, k_{2t}) \cdot \gcd(i_{2t-1}, i_{2t})) \), then \( M \) is an ideal is obvious from lemma 1.

Similarly, using lemma 6, we have a similar proposition for surface with boundary

**Proposition 2.** Consider the Lie algebra \( \mathbb{Z}[A(n)] \) with \( n = 2g + b - 1 \) that is associated with the surface \( S \) which has genus \( g \) and \( b \) boundary components. A geometric submodule \( M \) of \( \mathbb{Z}[A(n)] \) is an ideal if and only if for every \( (i_1, \ldots, i_n) \) and \( (k_1, \ldots, k_n) \in \mathbb{Z}^n \), \( \alpha_M(k_1, k_2, \ldots, k_n) \) divides \( \alpha_M(i_1, i_2, \ldots, i_n) \cdot \gcd_{t=1}^{g}(\gcd(k_{2t-1}, k_{2t}) \cdot \gcd(i_{2t-1}, i_{2t})) \).
3. Classification of symmetric ideals of $\mathbb{Q}$-module of freely homotopic classes on compact orientable surface

**Definition 6.** For a surface $S$, consider the Lie algebra $\mathbb{Q}[A(n)]$, with the corresponding abelianization of $\pi_1(S)$, $A(n)$, which is defined in definition 2. Consider the surjective Lie algebra homomorphism $\text{Ab}_\mathbb{Q}$. A ideal $I$ of $\mathbb{Q}[\hat{\pi}_0]$ is symmetric if it is the preimage of some ideal in $\mathbb{Q}[A(n)]$ under the map $\text{Ab}_\mathbb{Q}$.

Now we consider the surface $S$, and suppose the abelianization of its fundamental group, $A(n)$, is generated by $a_1, a_2, \ldots, a_n$. We will find all ideals of the Lie algebra $\mathbb{Q}[A(n)]$, and, therefore, all symmetric ideals of the Goldman algebra $\mathbb{Q}[\hat{\pi}_0]$ on surface $S$.

**Definition 7.** Let $C_S = \{ x \in A(n) | \langle x, y \rangle = 0 \ \forall y \in A(n) \} = \{ x \in A(n) | \langle x, a_i \rangle = 0 \ \forall i = 1, n \}$ be the center of $\mathbb{Q}[A(n)]$.

**Definition 8.** Let $A$ be the matrix with entries $A_{ji} = \langle a_i, a_j \rangle$ for $i, j = 1, n$. For $x = \prod_{i=1}^{n} a_i^{x_i} \in A_n$, let $X = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \subset \mathbb{Q}^n$, and let

$$M(x) = A.X = (M_1(x), M_2(x), M_3(x), \ldots, M_n(x)) \in \mathbb{Z}^n \subset \mathbb{Q}^n,$$

where $M_j(x) = \sum_{i=1}^{n} \langle a_i, a_j \rangle x_i$

Then we have:

$$\langle x, y \rangle = \sum_{i,j=1}^{n} \langle a_i, a_j \rangle x_i y_j = \sum_{j=1}^{n} y_j \sum_{i=1}^{n} \langle a_i, a_j \rangle x_i = \sum_{j=1}^{n} y_j M_j(x) = Y.M(x)$$

, for $x = \prod_{i=1}^{n} a_i^{x_i}, y = \prod_{i=1}^{n} a_i^{y_i}$

Here $Y.M(x)$ is normal dot product between 2 vectors $Y$ and $M(x)$ of the vector space $\mathbb{Q}^n$.

**Lemma 2.** For $x, y \in A_n$, there is no $v \in A_n$ such that $\langle x, v \rangle \neq 0$ and $\langle y, v \rangle = 0$ if and only if $x^{k_1} = cy^{k_2}$ for some $c \in C_S, k_1 \neq 0$, and $k_2 \in \mathbb{Z}$.
Proof. Firstly, if there is no \( v \in A_n \) such that \( \langle x, v \rangle \neq 0 \) and \( \langle y, v \rangle = 0 \), then \( \langle x, v \rangle = 0 \) whenever \( \langle y, v \rangle = 0 \). But this means that for every \( V \in \mathbb{Z}^n \) and \( V.M(y) = 0, V.M(x) = 0 \). Hence, the subspace \( M(y)^\perp \) of \( \mathbb{Q}^n \subset \) the subspace \( M(x)^\perp \) of \( \mathbb{Q}^n \). If \( M(x) \) is non-zero vector in \( \mathbb{Q}^n \), then \( M(y) \) is also a non-zero vector, and both of the subspaces of \( \mathbb{Q}^n \), \( M(x)^\perp \) and \( M(y)^\perp \), have the same \((n-1)\) dimension. Hence they must be equal and therefore also equal to the subspace \( \{M(x), M(y)\}^\perp \). Then the set \( \{M(x), M(y)\} \) must be linearly dependent in \( \mathbb{Q}^n \).

Hence, \( M(x) = kM(y) \) for some \( k \in \mathbb{Q} \), or \( k_1M(x) = k_2M(y) \) for some \( k_1 \neq 0, k_2 \in \mathbb{Z} \). Therefore, \( Ak_1X = Ak_2Y \) or \( (k_1X - k_2Y) \in \text{Ker}A \), which means that \( M(x^{k_1}y^{-k_2}) = 0 \in \mathbb{Q}^n \). Let \( c = x^{k_1}y^{-k_2} \). Then \( \langle c, y \rangle = Y.M(c) = y, 0 = 0 \) for every \( y \in G \), and therefore \( c \in C \), completing the proof of lemma 4. (Since the other direction is trivial)

\[ D \]

Definition 9. For a finite sequence of fixed distinct \( k \) elements of \( C_S \), \( \{c_i\}_{i=1}^k \), and a finite sequence of non-zero rational numbers, \( \{q_i\}_{i=1}^k \), we define \( X_\alpha \) as \( \{\sum_{i=1}^k q_i c_i x, x \in A_n \setminus C_S\} \), where \( \alpha = \alpha(c_1, \ldots, c_k, q_1, \ldots, q_k) \) depends only on \( c_i \) and \( q_i \).

Lemma 3. Let \( \{q_i\}_{i=1}^k \) be a fixed sequence of non-zero rational numbers, and \( \{c_i\}_{i=1}^k \) be a sequence of \( k \) distinct elements in \( C_S \). Denote \( f(x) = \sum_{i=1}^k q_i c_i x \), and so \( X_\alpha = X_{\alpha(c_1, \ldots, c_k, q_1, \ldots, q_k)} = \{f(x)|x \in A_n \setminus C_S\} \). Now suppose \( I \) is an ideal of the Lie algebra \( \mathbb{Q}[A(n)] \). Then if \( I \) contains some element of the form \( f(x) \in X_\alpha \) \( (x \in A_n \setminus C_S) \), then \( X_\alpha \subset I \).

As a result, in the case for closed surface \( S \), where \( C_S = \{e\} \), if \( I \) contains some \( x \in A_n \setminus \{e\} \), then \( A_n \setminus \{e\} \subset I \). (Simply because in this case \( X_\alpha = A_n \) for all \( \alpha \))

Proof.

Lemma 3.1. If \( f(x) \in I \), and \( \langle x, y \rangle \neq 0 \), then \( f(y) \in I \)

Proof. Suppose that \( f(x) \in I \).

\[
[f(x), x^{-1}y] = \sum_{i=1}^k q_i \langle c_i x, x^{-1}y \rangle c_i y = \langle x, y \rangle \left( \sum_{i=1}^k q_i c_i y \right) = \langle x, y \rangle f(y) \in I
\]

because \( \langle c_i x, x^{-1}y \rangle = \langle c_i, x^{-1}y \rangle + \langle x, x^{-1}y \rangle = 0 + \langle x, x^{-1} \rangle + \langle x, y \rangle = \langle x, y \rangle \neq 0 \).

Hence \( f(y) \) is also in \( I \).\]

\[ \square \]
Lemma 3.2. Suppose that \( f(x) \in I \) with \( M_i(x) \neq 0 \) for some \( i \). Then for every \( z \in A_n \) with \( M_i(z) \neq 0 \), \( f(z) \) also \( \in I \).

Proof. Take \( y_0 \in A_n \) with \( Y_0 = (0, \ldots, 0, 1, 0, \ldots, 0) \), with 0 is in \( i \)th position.

We have \( \langle x, y_0 \rangle = X.M(x) = M_i(x) \neq 0 \). By lemma 3.1, \( f(y_0) \) is also \( \in I \). Now, for any \( z \in A_n \) with \( M_i(z) \neq 0 \), because \( y_0 \in I \), and \( \langle y_0, z \rangle = -\langle z, y_0 \rangle = -Y_0.M(z) = -M_i(z) \neq 0 \), \( f(z) \) also \( \in I \) by lemma 3.1.

Now come back to the proof of lemma 3. By the lemma’s assumption, there is some \( x_0 \not\in C_S \): \( f(x_0) \in I \). Since \( x_0 \not\in C_S \), there exists \( i \) such that \( M_i(x_0) \neq 0 \).

Now suppose that \( x \) is any element of \( A_n \) that is not in \( C_S \). Then \( M_j(x) \neq 0 \) for some \( j \). We will prove that \( f(x) \in I \).

Assume \( r_i \) is \( i \)th row of matrix \( A \). Then \( M_i(z) = r_i.Z \). Since \( M_i(x), M_j(y) \neq 0 \), \( r_i \) and \( r_j \) must be different from zero vectors. Therefore, \( r_i^\perp \) and \( r_j^\perp \) are two subspaces of \( \mathbb{Q}^n \) with dimension at most \( n - 1 \). Hence there is some \( Z \in \mathbb{Q}^n \) that is outside the union of the two subspaces. Choose \( z = \prod a_i^{z_i} \in A_n \), with \( Z = (z_1, z_2, \ldots, z_n) \). So we have \( M_i(z) = r_i.Z \), and \( M_j(z) = r_j.Z \) are both non-zero vectors of \( \mathbb{Q}^n \).

Since \( f(x_0) \in I \), and \( M_i(x_0), M_i(z) \neq 0 \), we have \( f(z) \in I \) by lemma 3.2. Since \( f(z) \in I \), and \( M_j(z), M_j(x) \neq 0 \), we get \( f(x) \in I \) by lemma 3.2 again. Therefore \( f(x) \in I \) for every \( x \in A_n \setminus C_S \), and so \( X_a \subseteq I \).

Theorem 3. Every ideal \( I \) of the Lie algebra \( \mathbb{Q}[A(n)] \) on closed (compact) surface \( S \) is \( \mathbb{Q}[\{e\}] \) or \( \mathbb{Q}[A_n \setminus \{e\}] \) or \( \mathbb{Q}[A_n] \). Here \( \mathbb{Q}[X] \) is the \( \mathbb{Q} \)-submodule of \( \mathbb{Q}[A(n)] \) generated by \( X \).

Proof. Suppose \( I \neq \mathbb{Q}[\{e\}] \). Among all elements \( e \in I \setminus \mathbb{Q}[\{e\}] \neq \emptyset \), choose \( \gamma_0 \) to be one of those elements that has smallest number of terms in \( A_n \).

Suppose that \( \gamma_0 = q_1u_1 + q_2u_2 + \ldots + q_ku_k, q_i \neq 0 \in \mathbb{Q} \), and \( u_i \in A_n \) are distinct.

If there is some \( u_i = e \), and, WLOG, that is \( u_1 = e \). Then \( k \geq 2 \), and \( q_2 \neq e \).

As a result, \( q_2 \not\in C_S \), and there exists \( y \in A_n \) such that \( \langle q_2, y \rangle \neq 0 \). Because \( \langle q_2, q_2^{-1} \rangle = 0, y \neq q_2 \), and so \( [\gamma_0, y] = \sum_{i=2}^{k} q_i \langle u_i, y \rangle u_i y \not\in \mathbb{Q}[\{e\}] \), and is in the ideal \( I \). However, it has less than \( \gamma_0 \) at least 1 term in \( A_n \). Contradiction. Therefore, \( u_i \neq e \) for all \( i = \frac{1}{1}, k \)
Now suppose there is some \( v \in G \) such that \( \langle u_i, v \rangle \neq 0 \) and \( \langle u_j, v \rangle = 0 \). Then the element \( \gamma_1 = [\sum_{1 \leq t \leq k} q_t u_t, v] = \sum_{1 \leq t \neq j \leq k} q_t (u_t, v) u_t v \in I \setminus Q[\{e\}] \) has at most \( k - 1 \) terms in \( A_n \). Contradiction. Therefore, there is no \( v \in G \) such that \( \langle u_i, v \rangle \neq 0 \) and \( \langle u_j, v \rangle = 0 \). Hence, for any \( 1 \leq i \neq j \leq k, \langle u_i, v \rangle = 0 \) whenever \( \langle u_j, v \rangle = 0 \).

From here, we also have \( \langle u_i, u_j \rangle = 0 \forall 1 \leq i, j \leq k \).

Now consider any \( 1 \leq i_0 \neq j_0 \leq k \). Because \( u_{i_0} \notin C_S \), there exists some \( t \in G \) such that \( \langle u_{i_0}, t \rangle \neq 0 \). Then, by the above statement, \( \langle u_{i_0}, t \rangle = 0 \) whenever \( \langle u_{j_0}, t \rangle = 0 \).

As a result, \( \beta_0 = [\sum_{i=1}^k q_i u_i, t] = \sum_{i=1}^k (\langle u_i, t \rangle q_i) u_t t \) have exactly \( k \) terms in \( A_n \), and, therefore, also has the smallest number of terms in \( A_n \). By a similar argument as before, we get \( \langle u_{i_0}, t \rangle = 0 \) whenever \( \langle u_{j_0}, t \rangle = 0 \).

By lemma 2, \( (u_{i_0}, t)_{i_0} = (u_{j_0}, t)_{j_0} \) for some \( k_1 \neq 0, k_2 \in \mathbb{Z} \). Hence \( t^{k_1-k_2} = u_{j_0}^{k_2} u_{i_0}^{-k_1} \). As a result, \( \langle u_{i_0}, t^{k_1-k_2} \rangle = \langle u_{i_0}, u_{j_0}^{k_2} u_{i_0}^{-k_1} \rangle = k_2 \langle u_{i_0}, u_j \rangle - k_1 \langle u_{i_0}, u_i \rangle = 0 - 0 = 0 \).

If \( k_1 \neq k_2 \), then \( \langle u_{i_0}, t \rangle = \frac{1}{k_1-k_2} \langle u_{i_0}, t^{k_1-k_2} \rangle = 0 \). Contradiction. Therefore, \( k_1 = k_2 \neq 0 \). Hence, \( u_{i_0} = u_{j_0} \). Therefore, \( u_i = u_j \) with every \( i \neq j \). As a result, \( k \) has to be 1.

As a result, \( u_1 = \frac{1}{q_1} \gamma_0 \in I \) for some \( u_1 \in A_n \neq e \). By lemma 3, we have \( A_n \setminus e \subset I \). Therefore, \( I \) can only be \( Q[\{e\}], Q[A_n \setminus \{e\}] \) or \( Q[A_n] \).

\[ \text{Definition 10.} \] For \( x \in Q[A_n] \), suppose \( x = \sum_{i=1}^m q_i u_i \), where \( u_i \in A_n \), and \( q_i \) are non-zero rational numbers.

We say that \( u_i \sim u_j \) if \( u_i u_j^{-1} \in C_S \). Then the set \( \{u_1, u_2, \ldots, u_m\} \) is partitioned into equivalent classes \( R_i \) for \( 1 \leq i \leq p \).

Then \( x = \sum_{i=1}^p (\sum_{u_j \in R_i} q_j u_j) = \sum_{i=1}^p (\sum_{u_j \in R_i} q_j c_j u_{h_i}) \), where \( u_{h_i} \) is a representative of the equivalent class \( R_i \).

Note that each inner sum is either \( e \in Q[C_S] \), the \( Q \)-submodule of \( Q[A_n] \) generated by \( C_S \), or equal to \( f(u_{h_i}) \in X_{\alpha} \).

So \( x \) can be rewritten as \( \sum_{i=1}^{p_1} f(v_i) + c, \) for \( f(v_i) \) in some \( X_{\alpha_i} \), some \( c \in Q[C_S] \), and some \( p_1 \leq p \). Moreover, by using the previous process, this representation of \( x \), and we will call this the standard representation of \( x \in Q[A_n] \).

\[ \text{Lemma 4.} \quad \text{For any } y \in A_n, \text{ and } f_{\alpha}(x) \text{ that is in some } X_{\alpha}, \text{ we have } [f_{\alpha}(x), y] = \langle x, y \rangle f_{\alpha}(xy). \] As a result, \( Q[X_{\alpha}] \) is an ideal of \( Q[A(n)] \). We will call such ideal \( Q[X_{\alpha}] \) a primitive ideal.
Proof.

\[ [f_\alpha(x), y] = \left[ \sum_i q_i c_i x, y \right] = \sum_i q_i (c_i x, y) c_i x y = \sum_i (x, y) q_i c_i x y = (x, y) f_\alpha(xy) \]

Lemma 5. Suppose that an ideal \( I \) of the Lie algebra \( \mathbb{Q}[A(n)] \) has an element whose standard representation contains a term that is an element of some \( X_{\alpha_0} \). Then \( X_{\alpha_0} \subset I \).

Proof. Among all of elements of \( I \) that contains a term \( \in X_{\alpha_0} \) in its standard expression, choose \( \gamma_0 \) to be the element with the smallest number of terms in \( A_n \). Note that the set of elements of \( I \) containing a term \( \in X_{\alpha_0} \) in its standard expression is non-empty because of the lemma’s assumption.

Suppose \( \gamma_0 \) has the standard representation \( \gamma_0 = \sum_{i=0}^{k} f_{\alpha_i}(x_i) + c \), where \( x_i \) and \( x_i x_j^{-1} \in A_n \setminus C_S \) for \( 1 \leq i \neq j \leq k \), and \( c \in C_S \). In this representation, \( f_{\alpha_0}(x_0) \) is the term that belongs to \( X_{\alpha_0} \). We will prove that \( c = 0 \) and \( k = 0 \).

If \( c \neq 0 \), then take \( y \in A_n \) such that \( \langle x_0, y \rangle \neq 0 \). Using lemma 4, we get:

\[ \gamma_0' = \frac{1}{\langle x_0, y \rangle} [\gamma_0, y] = \sum_{i=0}^{k} \frac{1}{\langle x_0, y \rangle} [f_{\alpha_i}(x), y] \]

\[ = \sum_{i=0}^{k} \frac{\langle x_i, y \rangle}{\langle x_0, y \rangle} f_{\alpha_i}(x_i y) = \sum_{0 \leq i \leq k, \langle x_i, y \rangle \neq 0} f_{\beta_i}(x_i y) \in I \]

for some \( f_{\beta_i} \in X_{\beta_i} \).

Because for \( 1 \leq i \neq j \leq k \), \( x_i y (x_j y)^{-1} = x_i x_j^{-1} \notin C_S \), \( \sum_{0 \leq i \leq k, \langle x_i, y \rangle \neq 0} f_{\beta_i}(x_i y) \) is the standard representation of \( \gamma_0' \).

Also, in this presentation, the first term is \( f_{\beta_0}(x_0 y) = f_{\alpha_0}(x_0 y) \in X_{\alpha_0} \) (Note that \( \langle x_0, y \rangle = \langle x_0 y, y \rangle \neq 0 \), so \( x_0 y \notin C_S \)). However, \( \gamma_0' \) has at most \( k + 1 \) terms in the standard representation, and this is impossible because of the choice of \( \gamma_0 \), which has \( k + 2 \) terms. So \( c = 0 \).

Now suppose, by contradiction, that \( k > 0 \).

Now for each \( 1 \leq i_0 \leq k \), if there is some \( y \in G \) such that \( \langle x_0, y \rangle \neq 0 \) and
\[ \langle x_0, y \rangle = 0, \text{ then:} \]

\[
\gamma_1 = \frac{1}{\langle x_0, y \rangle} [\gamma_0, y] = \sum_{i=0}^{k} \frac{1}{\langle x_0, y \rangle} [f_{\alpha_i}(x), y] \\
= \sum_{i=0}^{k} \frac{\langle x_i, y \rangle}{\langle x_0, y \rangle} f_{\alpha_i}(x_i y) = \sum_{0 \leq i \leq k, \langle x_i, y \rangle \neq 0} f_{\beta_i}(x_i y) \in I
\]

for some \( f_{\beta_i} \in X_{\beta_i} \).

Again, \( \sum_{0 \leq i \leq k, \langle x_i, y \rangle \neq 0} f_{\beta_i}(x_i y) \) is a standard representation of \( \gamma_1 \) that contains \( f_{\beta_0}(x_0 y) = f_{\alpha_0}(x_0 y) \in X_{\alpha_0} \). However, \( \gamma_1 \) has at most \( k \) terms, while \( \gamma_0 \) has \( k + 1 \) terms in their standard presentation, and this is impossible.

As a result, \( \langle x_0, y \rangle = 0 \) whenever \( \langle x_i, y \rangle = 0 \) for all \( 1 \leq i \leq k \).

Therefore, for any \( 1 \leq i \leq k, \langle x_0, x_i \rangle = 0 \). Moreover, by lemma 2, we get \( x_0^{k_i^{(i)}} = c_i x_i^{k_i^{(i)}} \) for some \( c_i \in C_S \) and \( k_i^{(i)} \neq 0, k_i^{(i)} \in \mathbb{Z} \).

Since \( x_0 \not\in C_S \), there exists some \( y_0 \in G \) such that \( \langle x_0, y_0 \rangle \neq 0 \). Consider

\[
\gamma_2 = \frac{1}{\langle x_0, y_0 \rangle} [\gamma_0, y_0] = \sum_{i=0}^{k} \frac{1}{\langle x_0, y_0 \rangle} [f_{\alpha_i}(x_i), y_0] \\
= \sum_{i=0}^{k} \frac{\langle x_i, y_0 \rangle}{\langle x_0, y_0 \rangle} f_{\alpha_i}(x_i y_0) = \sum_{0 \leq i \leq k, \langle x_i, y_0 \rangle \neq 0} f_{\beta_i}(x_i y_0) \in I
\]

for some \( f_{\beta_i} \in X_{\beta_i} \).

Again, \( \sum_{0 \leq i \leq k, \langle x_i, y_0 \rangle \neq 0} f_{\beta_i}(x_i y_0) \) is the standard representation of \( \gamma_2 \) with at most \( k + 1 \) terms and contains the term \( f_{\alpha_0}(x_0 y_0) \in X_{\alpha_0} \). Therefore, \( \gamma_2 \) must also have the smallest number of terms in its standard representation.

Using the same argument, which we used for \( \gamma_0 \), for \( \gamma_2 \), we have for \( 1 \leq i \leq k \),

\[
(x_0 y_0)^{k_3^{(i)}} = d_i (x_i y_0)^{k_i^{(i)}} \text{ for some } d_i \in C_S \text{, and } k_3^{(i)} \neq 0, k_i^{(i)} \in \mathbb{Z}. \]

Therefore, \( y_0^{k_3^{(i)} - k_i^{(i)}} = x_i^{k_3^{(i)}} x_0^{-k_i^{(i)}} d_i \). Because \( \langle x_i, x_0 \rangle = 0 \) and \( c \in C_S \), \( \langle x_0, y_0 \rangle^{k_3^{(i)} - k_i^{(i)}} = 0 \).

If \( k_3^{(i)} \neq k_i^{(i)} \), we would have \( \langle x_0, y_0 \rangle = \frac{1}{k_3^{(i)} - k_i^{(i)}} \langle x_0, y_0 \rangle^{k_3^{(i)} - k_i^{(i)}} = 0 \). Contradiction.

So \( k_3^{(i)} = k_i^{(i)} \neq 0 \), and, therefore, for \( 1 \leq i \leq k \), \( x_i x_0^{-1} \in C \). Contradiction
because $x_ix_j^{-1} \not\in C_S$ for $1 \leq i \neq j \leq k$.

As a result, $f_{\alpha_0}(x_0) = \gamma_0 \in I$, so $I$ contains some element $f_{\alpha_0}(x_0)$ of $X_{\alpha_0}$, and, by lemma 3, $X_{\alpha_0} \subset I$.

Theorem 4. (Theorem for classification of ideals)

All ideals $I$ of the Lie algebra $Q[A_n]$ is the sum, as vector space, of $\{Q[X_{\alpha_i}]\}_{i \in T}$ and $C_1$, where $C_1$ is a subspace of $Q[C_S]$ and can be empty, and some index set $T$. Here $Q[T]$ is the $Q$-submodule of $Q[A(n)]$ generated by $T$.

Proof. Any subspace $C_1$ of $Q[C_S]$ is obviously an ideal. Therefore, if a $Q$-submodule (or subspace) $I$ is the sum of some of primitive ideals (defined in lemma 4), and $C_1$, then $I$ must also be an ideal of $Q[A_n]$.

Now we will prove that if $I$ is an ideal, then $I$ must be the sum of primitive ideals $Q[X_{\alpha}]$ and some subspace $C_1$ of $Q[C_S]$.

Let $\Gamma$ be the set of all $\alpha$ such that in the standard representation of some element of $I$, there is a term $f_{\alpha}(x)$ which is an element of $X_{\alpha}$. Let $C$ be the set of all terms that are in $Q[C_S]$ and appears in the standard representation of some elements of $I$. Clearly, $I$ is the subset of $\bigoplus_{\alpha \in \Gamma} Q[X_{\alpha}] \bigoplus Q[C_1]$.

Now, by lemma 5, $X_{\alpha}$ is a subset of $I$ for every $\alpha \in \Gamma$. Therefore, $\bigoplus_{\alpha \in \Gamma} Q[X_{\alpha}]$ is also a subset of $I$.

Now, for each element $c$ of $C$, there exists $x \in I$, with the standard representation $x = c + d$, where $c \in C \subset Q[C_S]$, and $d = \sum_{i=1}^{k} f_{\alpha_i}(x_i)$. Each $\alpha_i \in \Gamma$ because of the definition of $\Gamma$, so $d \in \bigoplus_{\alpha \in \Gamma} Q[X_{\alpha}]$. Therefore, $d \in I$, and hence, $c \in I$.

Therefore, $Q[C] \subset I$. Hence, $\bigoplus_{\alpha \in \Gamma} Q[X_{\alpha}] \bigoplus Q[C_1] \subset I$

As a result, $I = \bigoplus_{\alpha \in \Gamma} Q[X_{\alpha}] \bigoplus Q[C_1]$. □

Now the following lemma will help us compute the center $C_S$ explicitly.

Lemma 6. For the surface $S$ with boundary with genus $g$ and $b$ boundary components, there exists generators $a_1, a_2, ..., a_n$ with $n = 2g + b - 1$ such that the abelianization of the fundamental group of $S$, $A_n = \langle a_1, a_2, ..., a_n | a_i a_j = a_j a_i \rangle$ and $\langle a_{2i-1}, a_{2i} \rangle = 1 \forall i = \overline{1, g}$ and $\langle a_i, a_j \rangle = 0 \forall i \leq j$. 
Proof. We consider a $4n$-gon with $2n$ labels $a_1, a_2, ..., a_n, A_1, A_2, ..., A_n$ such that only odd edges (the first, the third, and so on) have labels. Then we mark the $2n$ edges with $2n$ labels $a_1, a_2, A_1, A_2, ..., a_{2i−1}, a_{2i}, A_{2i−1}, A_{2i}$, and then $A_{2g+1}, a_{2g+1}, ..., a_{2g+b−1}, a_{2g+b−1}$ in this order. The following picture describe the above process for surface with genus 2 and with 1 boundary component:

We then glue $a_i$ and $A_i$ for $i=1$ to $g$ without creating Mobius band to obtain a surface. Denote $u_i$, and $v_i$ be the vertices of the side $a_i$ and $A_i$ for $i=1$ to $g$. We define that 2 vertices $a$ and $b$ are equivalent: $a \sim b$ if $a$ and $b$ are in the same small arcs, which are even edges of the $4n$-gon that make up the boundaries. Then we will have cycles of equivalent vertices. The number of cycles will be the number of boundaries. In fact, we have the cycle $u_1 \sim v_2 \sim v_1 \sim u_2 \sim u_3 \sim v_4 \sim v_3 \sim u_4 \sim \ldots \sim u_{2g−1} \sim v_{2g} \sim v_{2g−1} \sim u_{2g} \sim v_{2g+1} \sim v_{2g+2} \sim \ldots \sim v_{2g+b−1}$ and $b−1$ other cycles of 1 vertex $\{u_{2g+i}\}_{i=1}^{b−1}$.

Therefore the obtained surface is an oriented surface with $b$ boundary component and with genus $(n+1−b)/2=g$. So this surface is homeomorphic to $S$ by the classification theorem for surface (with boundary). So we can identify $S$ with this glued surface.

Now take $O$ inside the polygon. $P = Q$ are inside edges labeled by $a_i, A_i$ respectively. Let $a_i$ be the curve $O \rightarrow P = Q \rightarrow O$. Denote $a_i$ as the homotopy
classes of curve base at \( O \) which has the curve \( \alpha_i \) as an representation. Then

the fundamental group of the glued surface, and also, \( S \), will be the free group

generated by \( \{a_i\}_{i=1}^n \). Moreover, based on our labeling of the \( 4n \)-gon, we can choose

the curves \( \{\alpha_i\}_{i=1}^n \), which are freely homotopic to curves that are represented by

\( a_i \) in \( \pi_1(S) \), so that \( \alpha_i \) for \( i = 2g + 1, 2g + b - 1 \) intersects no other curve, and

\( \alpha_{2i-1} \) only intersects \( \alpha_{2i} \) for \( i = 1, g \) and no other curve. Hence, we will have the

abelianization of \( \pi_1(S) \), \( A_n = \langle a_1, a_2, \ldots, a_n | a_i a_j = a_j a_i \rangle \) such that \( \langle a_{2i-1}, a_{2i} \rangle = 1 \) \( \forall i = 1, g \) and \( \langle a_i, a_j \rangle = 0 \) \( \forall i \neq j \) because of definition of \( \langle a_i, a_j \rangle \)

\[ \Box \]

**Lemma 7.** (Lemma for computing \( C_S \))

- For closed surface \( S \), \( C_S = \{e\} \).

- By lemma 6, for surface \( S \) with boundary, there exists generators \( a_1, a_2, \ldots, a_n \)
  
  with \( n = 2g + b - 1 \) such that the abelianization of the fundamental group of
  
  \( S \), \( A_n = \langle a_1, a_2, \ldots, a_n | a_i a_j = a_j a_i \rangle \) and \( \langle a_{2i-1}, a_{2i} \rangle = 1 \) \( \forall i = 1, g \) and other
  
  \( \langle a_i, a_j \rangle = 0 \) \( \forall i \neq j \). Then \( C_S \) is the subgroup generated by \( \{a_i\}_{i=2g+1}^n \).

**Proof.** For the closed surface \( S \), and let \( a_j \) for \( j \in \overline{1,n} \) with \( n = 2g \) be generators

of \( A_n \) that corresponds to the \( i^{th} \) sides of the fundamental polygon of \( S \). Then

since only \( (2i-1)^{th} \) and \( (2i)^{th} \) sides of the fundamental polygon intersects, we have

\( \langle a_{2i-1}, a_{2i} \rangle = 1 \), and other \( \langle a_i, a_j \rangle = 0 \).

Now suppose \( x = \prod_{j=1}^n a_j^{k_j} \) for some integers \( k_j \). Then \( 0 = [x, a_{2i-1}] = k_{2i} \), and

\( 0 = [x, a_{2i}] = k_{2i-1} \) for each \( i \) from 1 to \( r \). Hence, \( C_S = \{e\} \).

For surface with genus \( g \) and \( b \) boundary components and with \( n = 2g + b - 1 \),

by lemma 6, \( A_n = \langle a_1, a_2, \ldots, a_n | a_i a_j = a_j a_i \rangle \) and \( \langle a_{2i-1}, a_{2i} \rangle = 1 \) \( \forall i = 1, g \) and other

\( \langle a_i, a_j \rangle = 0 \) \( \forall i \neq j \). Then \( C_S \) is the subgroup generated by \( \{a_i\}_{i=2g+1}^n \).

Now suppose \( x = \prod_{j=1}^n a_j^{k_j} \) is some element of \( C_S \). Then \( 0 = [x, a_{2i}] = k_{2i-1} \),

and \( 0 = [x, a_{2i-1}] = k_{2i} \) with \( i \in \overline{1,n} \). Therefore, \( x = \prod_{j=2g+1}^n a_j^{k_j} \). Hence \( C_S = \langle a_{2g+1}, \ldots, a_n \rangle \).

\[ \Box \]

Below is a corollary from the classification theorem of ideals and the lemma for computing the set \( C_S \) for surface \( S \) with more than 1 boundary components.
Corollary 1. There are infinitely many non-trivial symmetric ideals (defined in definition 6) of the Goldman algebra $Q[\hat{\pi}_0]$ on the surface with more than 1 boundary components.

4. An infinite descending chain of ideals in the Goldman algebra

Definition 11. Consider the surface $S$ with at least 1 boundary component. Choose a fixed boundary. Then choose a curve (the red curve) going around the boundary exactly 1 time. Suppose we have a curve $\alpha$ that touched the fixed red curve only at point $P$.

We define $T_0^+$ operation be the operation that transforms the curve $\alpha$ into a new curve $\beta$ that goes around the red curve 1 time in counter-clockwise direction and then goes around the original curve $\alpha$.

Note that by another homotopy operation, $\beta$ can further be transformed into $\gamma$. 

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Now we define the inverse operation of $T_0^+$. If we have a loop $\beta$ that goes around the red curve exactly one, and then goes around another $\alpha$ curve which only touches the red curve at $P$ then we define $T_0^-$ to be the operation that transforms $\beta$ into $\alpha$. Finally, we can define the operation $T_0$ to be either $T_0^+$ or $T_0^-$.

**Lemma 8.** Suppose that $\alpha$ is transformed into $\beta$ by an operation $T_0$. Moreover, suppose $\zeta$ is another curve that doesn’t intersect the red curve going around the boundary, then each term in the Goldman bracket $[\alpha, \zeta]$ can be transformed into a corresponding term in the bracket $[\beta, \zeta]$ by the operation $T_0$. (This correspondence is one-to-one).

**Proof.** WLOG, assume that $\alpha$ is transformed into $\beta$ by $T_0^+$. Note that all intersection points of $\alpha$ and $\zeta$ are the same as those of $\beta$ and $\zeta$ since $\zeta$ doesn’t intersect the red curve. Assume $Q$ is one of those intersection. Then clearly, the curve $\alpha_p\zeta$ can be transformed into $\beta_p\zeta$ by $T_0^+$.

**Definition 12.** Now suppose we have a new red curve going around the boundary $2^1 = 2$ times. We define $T_1$ to be an operation that is similar to $T_0$, and the only difference is the new red curve.
Now instead of letting the red curve go around the boundary 1 or 2 times, we can let the red curve go around the boundary $2^n$ times for any non-negative integer $n$. Then we can define $T_n$ in a similar way that we defined $T_0$ and $T_1$.

Then we have another version of lemma 8

**Lemma 9.** Suppose that $\alpha$ can be transformed into $\beta$ by an operation $T_n$. Moreover, suppose $\zeta$ is another curve that doesn’t intersect the red curve going around the boundary, then each term in the Goldman bracket $[\alpha, \zeta]$ can be transformed into a corresponding term in the bracket $[\beta, \zeta]$ by the operation $T_0$. (This correspondence is one-to-one).

**Definition 13.** Two curves on $S$, $\alpha$ and $\beta$, are equivalent under equivalence relation $\sim_n$ if $\alpha$ can be transformed into $\beta$ by operations $T_m$ for some $m \geq n \in \mathbb{Z}$ or by homotopy operations. We can then extend this equivalence relation $\sim_n$ for linear combinations of curves instead of just single curves.
Lemma 10. If two curves $\alpha \sim_n \beta$, then $[\alpha, \zeta] \sim_n [\beta, \zeta]$

Proof. The proof follows from lemma 9, and the fact that for any $\zeta$ there always exists a curve $\zeta'$, which is homotopic to $\zeta$ and doesn’t intersect the curve that going around the boundary a certain number of times. \hfill \Box

Definition 14. We define $X_n$ as the set of all equivalence classes of curves obtained from the equivalence relation $\sim_n$. Now let $S_n$ be the set of all linear combinations of elements of $X_n$ with coefficients in $\mathbb{Q}$. $S_n$ is a $\mathbb{Q}$-module.

Let the $\mathbb{Q}$-module homomorphism $\delta_n$ from the Goldman algebra to $S_n$ be the map that maps each freely homotopy class of curves to the equivalence classes in $X_n$ of one of the representation curve of that freely homotopy class. This map is well-defined since any two freely homotopic curves are equivalent under relation $\sim_n$.

By lemma 10, we can define a Lie bracket on $S_n$: $[\bar{\alpha}, \bar{\beta}] = [\alpha, \beta]$, where $\bar{\pi}$ is the $\sim_n$ equivalence classes of $\gamma$. This definition doesn’t depend on the representative curves $\alpha$ or $\beta$.

Lemma 11. By definition 14, $S_n$ is a Lie algebra, and $\delta_n$ is obviously a Lie algebra homomorphism. Let $I_n$ be the kernel of the map $\delta_n$. Then we have a descending chain of ideals $I_n$ because any two curves that are $\sim_{n+1}$-equivalent are also $\sim_n$-equivalent.

Lemma 12. Let $\pi_1(S, P)$ be the fundamental group of $S$ with base point $P$, and $c$ be an element of $\pi_1(S, P)$ that is the homotopy class of curves containing the chosen boundary (in counter-clockwise direction) (the red curve defined in definition 11).

Then any closed curve that is in the homotopy class of curves $cxe^{-1}x^{-1} \in \pi_1(S, P)$ is $\sim_0$ to the trivial loop.

Moreover, for any $x_i \in \pi_1(S, P)$, and integers $m_i \geq n$, the curve that is in the homotopy class

$$\gamma = c^{m_1}x_1c^{-m_1}x_1^{-1}c^{m_2}x_2c^{-m_2}x_2^{-1} \cdots c^{m_k}x_kc^{-m_k}x_k^{-1} \in \pi_1(S, P)$$

is $\sim_n$ to the trivial loop.
Therefore, if

\[ C_n = \{ gc^{2m_1} x_1^{2^{-m_1}} x_1^{-1} c^{2m_2} x_2^{2^{-m_2}} x_2^{-1} \ldots c^{2m_k} x_k^{2^{-m_k}} x_k^{-1} g^{-1} \} \]

such that \( x_i, g \in \pi_1(S, P), m_i \geq n \}

then \([x] - [y] \in I_n\) with every \(x, y\) in the normal subgroup \(C_n\) of \(\pi_1(S, P)\), where \([x]\) is the freely homotopic class of curves so that one of the curve in this class is in the homotopy class \(x \in \pi_1(S, P)\). As a result, \(I_n\) is not one of the ideals that we considered in section 3.

Proof. Consider the following curve represented by \(cxc^{-1}x^{-1}\) in the fundamental group.

This curve is homotopic to the curve \(\alpha_0\)

We can perform the following operations on \(\alpha_0\): \(T_0\) operation, homotopy operation, then another \(T_0\) operation, and finally a homotopy operation to transform \(\alpha_0\) into the trivial loop.
Similarly, we can prove the more general statement for the equivalence relation \( \sim_n \) instead of \( \sim_0 \).

Lemma 13. The freely homotopic class \( \gamma \) containing the curve going around the boundary \( 2^n \) times belongs to \( I_n \) and doesn’t belong to \( I_{n+1} \) for \( n \in \mathbb{Z} \geq 0 \). Therefore, \( I_n \) is a strictly descending chain of ideals.

Proof. By a single \( T_n \) operation, we can transform any curve in the freely homotopic class \( \gamma \) into the trivial loop, so \( \gamma \in I_n \). Now we will prove that \( \gamma \notin I_{n+1} \).

Attaching a disk to the surface \( S \) by gluing the disk’s boundary with the curve that goes around the chosen boundary (of the surface \( S \)) \( 2^{n+1} \) times to obtain a new topological space \( S' \). If we can transform one curve into another curve on the original surface \( S \) by an operation \( T_m \) for \( m \geq n + 1 \), then we can also transform one curve into the other by a homotopy in \( S' \) by moving the curve through the disk attached instead.
Let $c \in \pi_1(S)$ be the homotopy class of curves containing the boundary that we have chosen. Then, by Van Kampen’s theorem, the fundamental group of the new space is $\pi_1(S) / N$, where $N$ is the normal subgroup generated by $c^{2^n}$. If $\gamma \in I_{n+1}$, then, by our previous argument, $c^{2^n}$ must be homotopic to the trivial loop $S'$. Hence, $c^{2^n} \in N$ or $c^{2^n} = gc^{2^{n+1}}g^{-1}$, for some $g \in \pi_1(S)$. Contradiction. Therefore, $\gamma \notin I_{n+1}$.

**Theorem 5.** For compact surface $S$ with more than one boundary component, we can choose our fixed boundary so that the homotopy class of curves containing this boundary is a generator of the fundamental group of $S$, which is a free group. In this case, the intersection of ideals $\bigcap_{n \geq 0} I_n$ is $\{0_{\mathbb{Q}/[\pi_0]}\}$.

**Proof.** Suppose by contradiction that there is some non-zero element in $\bigcap_{n \geq 0} I_n$. Then there will be 2 curves $\alpha$ and $\beta$ with different freely homotopic classes that are $\sim_n$ equivalent for each $n \in \mathbb{Z} \geq 0$. Suppose $\alpha$ and $\beta$ are in homotopy classes $a$ and $b$, which are elements of the fundamental group based at $P$, $\pi_1(S) = \pi_1(S, P)$. Then choose $n$ big enough such that the sum of absolute value of all exponents $k$ of subwords of the form $c^k$ in both $a$ and $b$ less than $2^{n-1}$. (Note that elements in $\pi_1(S)$ are reduced words).

Again, we attach to $S$ the disk whose boundary is the curve going around the boundary of the original surface $2^n$ times. Then $\alpha$ must be homotopic to $\beta$ in the new topological space $S'$.

For every reduced word in $\pi_1(S)$, $\ldots c^{m_1} \ldots c^{m_2} \ldots \in \pi_1(S)$, we consider the corresponding word $\ldots c^{m_1 \mod 2^n} \ldots c^{m_2 \mod 2^n} \ldots$, where we take mod $2^n$ of the exponents of subwords of the form of $c^i$. Let $G_n$ be the group of all of these corresponding words. We then have a natural surjective group homomorphism (or projection) $p$ from $\pi_1(S)$ to $G_n$. Every element in normal subgroup $N$ generated by $c^{2^n}$ is mapped to $\tilde{e}$, unit element in $G_n$. As a result, there exist a group homomorphism $f$ from $\pi_1(S) / N$ to $G_n$ so that the following diagram commutes:

$$\begin{array}{ccc}
\pi_1(S) / N & \xrightarrow{f} & G_n \\
p \downarrow & & \downarrow p \\
\pi_1(S) & \xrightarrow{p} & G_n
\end{array}$$

Let $p_1(x) = \overline{x}$, and $f(\overline{x}) = \tilde{x} = p(x)$. Because $\alpha$ and $\beta$ are homotopic in the
new space $S'$, $\overline{a}$ and $\overline{b}$ are conjugate in $\pi_1(S)/N$. Therefore, $\overline{agbg^{-1}} = \overline{e}$ for some $g \in \pi_1(S)$.

Now we can assume that $g$ and therefore $g^{-1}$ only have subword $c^m$ in the $g$ with $m \in \mathbb{Z}$, and $|m| \leq 2^{n-1}$ because we can replace $m$ by $m' = m \mod 2^n$, with $0 \leq m' \leq 2^n - 1$ and if $m'$ is still bigger than $2^{n-1}$, we then can replace $m$ by $m'' = m' - 2^n$. With these replacements, the identity $\overline{agbg^{-1}} = \overline{e} \in \pi_1(S)/N$ stays the same.

Now project two sides of the identity $\overline{agbg^{-1}} = \overline{e} \in \pi_1(S)/N$ onto $G_n$ by the map $f$, we got: $\overline{agbg^{-1}} = \overline{e}$. Note that $agbg^{-1}$ cannot be $e$ because the free homotopy classes of $\alpha$ and $\beta$ are different.

Therefore, $agbg^{-1} = y$ so that $y = x_1c^{2^{m_1}}y_1c^{2^{m_2}}z_1$ is a reduced word in $\pi_1(S)$ with $p(y) = \overline{e}$, and $c^{2^{m_1}}$ is the first subword of the form $c^{2^i}$ of $y$, and $c^{2^{m_2}}$ is the last subword of the form $c^{2^i}$ of $y$ if $y$ has at least 2 subwords of the form $c^{2^i}$ or $c^{2^{m_2}} = e$ otherwise. Because $y \neq e$ and $p(y) = \overline{e}$, $2^n$ must divide $2^{m_1}$, and $m_1 \geq n$. Similarly, if $c^{2^{m_2}} \neq e$, then we also have $m_2 \geq n$.

Now we have $e = g^{-1}a^{-1}x_1c^{2^{m_1}}y_1c^{2^{m_2}}z_1gb^{-1}$, where the words $x_1$ and $z_1$ have no letter $c$. Because sum of the exponents of any subwords of the form $c^i$ in $a$ and in $g^{-1}$ is less than $2^{n}$, $c^{2^{m_1}}$ will never be cancelled by $g^{-1}a^{-1}x_1$. If $c^{2^{m_2}} \neq e$, then, by a similar argument, $c^{2^{m_2}}$ also cannot be cancelled by $z_1gb^{-1}$.

If, however, $c^{2^{m_2}} = e$, then our previous identity must have the form $e = g^{-1}a^{-1}x_1c^{2^{m_1}}y_1gb^{-1}$, and $x_1$ and $y_1$ contain no $c$ letter. Now consider the abelianization of both sides, we got: $e = Ab(a)^{-1}Ab(x_1)Ab(c^{2^{m_1}})Ab(y_1)Ab(b)^{-1}$ ($g$ and $g^{-1}$ will have the abelianizations cancelled in the identity).

Now if we compare the exponents with base $c$ in both sides, we will get $0 = -\exp(a) + 2^{m_1} - \exp(b)$, where $\exp(t)$ is the exponent with base $c$ of the abelianization of the word $t$. By the choice of $n$, we have $2^{m_1} \geq 2^n > 2^{n-1} > \exp(a) + \exp(b)$. Contradiction.

As a result, the identity $e = g^{-1}a^{-1}x_1c^{2^{m_1}}y_1c^{2^{m_2}}z_1gb^{-1}$ cannot hold. Therefore, we finish our proof by contradiction. \hfill \Box
Theorem 6. There is a strictly descending chain of ideals that haven’t been considered in section 3. For surface with more than one boundary component, the intersection of these ideals is zero.

Proof. The proof follows from lemma 11, lemma 12, and lemma 13, and theorem 5.
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