A Social Influence Model Based On Circuit Theory

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ABSTRACT
Understanding the behaviors of information propagation is essential for the effective exploitation of social influence in social networks. However, few existing influence models are tractable and efficient for describing the information propagation process, especially when dealing with the difficulty of incorporating the effects of combined influences from multiple nodes. To this end, in this paper, we provide a social influence model that alleviates this obstacle based on electrical circuit theory. This model vastly improves the efficiency of measuring the influence strength between any pair of nodes, and can be used to interpret the real-world influence propagation process in a coherent way. In addition, this circuit theory model provides a natural solution to the social influence maximization problem. When applied to real-world data, the circuit theory model consistently outperforms the state-of-the-art methods and can greatly alleviate the computation burden of the influence maximization problem.

1. INTRODUCTION
A social network is a graph of relationships between individuals, groups, or organizations. As an effective tool in connecting people and spreading information, social networks have caught the attention of millions of small businesses as well as major companies. To exploit social networks to gain influence, it is essential to understand the behaviors of information propagation in social networks. However, given the large-scale of existing social networks, it is challenging to design a tractable and efficient model to describe the information propagation process in social networks. For instance, let us consider a scenario that an event (such as a rumor) initially happened on a node (denoted as the seed node), this event may happen again on some neighboring nodes and may further happen on some far-away nodes due to the information propagation through the network. Then, the question is how does the information propagate? Also, after sufficient propagation, what probability will this event happen on a random node? (this probability could be viewed as the influence strength from the seed node to this random node.) Indeed, there are several existing models to describe the above information propagation process, such as the Linear Threshold(LT) model [10] the Independent Cascade(IC) model [8]. However, these models are operational models, they are untractable and inefficient. Under these models, if we want to get the probability of one event happened on a node, we need to run Monte-Carlo simulations of these influence models for sufficiently many times (e.g. 20000 times) to obtain an accurate estimate of the probability for the event happen [5], and this approach is very time-consuming. Moreover, if the event initially happened on more than one node (seed), the probability of this event happened on a random node will be a combination of influences from all of these seeds. In this case, how to get the probability for the event happen and how to identify the independent probability (or independent influence) of each seed are challenging research issues.

To this end, in this paper, we propose a social influence model based on electrical circuit theory to simulate the information propagation process in social networks. With this model, we are able to obtain a probabilistic influence matrix to describe the influence strength between any pair of nodes. Also, based on this model, we provide a novel method to identify the independent influence of each seed node when there are more than one seed. This can lead to a natural solution to the social influence maximization problem [11] [5] [21], which targets on finding a small subset of influential nodes in a social network to influence as many nodes as possible. Specifically, the contributions of this paper can be summarized as follows.

- We propose the idea of exploiting the circuit network to simulate the information (or influence) propagation process in social networks. This simulation is straightforward and is easy to interpret and understand. To the best of our knowledge, this is the first work that exploits circuit theory for estimating the spread of social influence. What’s more, this model is tractable even for more than one seed node. With this model, we could provide a novel way to compute the independent influence of a seed naturally.
- We identify an upper bound of the node’s influence on the network. This upper bound property can help us to select the truly influential nodes and drastically reduce the search space. By exploiting these computational properties, we develop an efficient approximation method to compute the influences and successfully apply this circuit theory based model to solve the social influence maximization problem. Along this line, we first prove that the influence spread function is also submodular under the circuit model. Then, by the greedy strategy, we design an algorithm to select K influential nodes to maximize their influence. Finally, experimental results show that the circuit theory based methods have the advantages over the state-of-the-art algorithms for the social influence maximization problem in terms of both efficiency as well as the effectiveness.
2. A SOCIAL INFLUENCE MODEL BASED ON CIRCUIT THEORY

In this section, we propose a social influence model based on electrical circuit theory.

2.1 Social Influence Modeling

Social influence refers to the behavioral change of individuals affected by others in a network. Social influence is an intuitive and well-accepted phenomenon in social networks [1]. In this section, we will propose a quantitative definition of social influence to measure its amount.

Let $G = (V, E)$ be an information network, where the node set $V$ includes all of individuals, the arc set $E$ represents all the social connections. Let $F_{i \rightarrow j}$ denote the influence from node $i$ to node $j$, we propose three rules to define the influence between any pair of nodes.

1. The influence from oneself should be $1$, that is $F_{i \rightarrow i} = 1$.
2. The influence could transmit through the network arcs with certain probability.
3. The influence on one node should be related to the influences on the node’s neighbors, suppose node $j$’s neighbor set is $N_j = \{j_1, j_2, ..., j_m\}$ (i.e., $\forall k \in N_j$, $(k, j) \in E$), then

$$F_{i \rightarrow j} = f_j(t_{i,j}F_{i \rightarrow j_1}, t_{i,j}F_{i \rightarrow j_2}, ..., t_{i,j}F_{i \rightarrow j_m})$$

where $t_{i,j}$ is the transmission probability on arc $(k, j)$.

To the construction modeled by the above three rules, there are two factors could change its shape. The first one is the transmission probabilities on arcs. In this paper, we propose an assumption to confine the probability, that is

**Assumption 1.** The sum of transmission probabilities flowing into one node should be less than or equal to $1$. That is,

$$\theta_i = \sum_{j=1}^{n} t_{ji} \leq 1 \quad \text{for} \quad i = 1, 2, ..., n$$

or

$$T' e = \Theta \leq e$$

where $\Theta = \text{diag}(\theta_1, \theta_2, ..., \theta_n)$ and $T = \{t_{ij}\}_{n \times n}$ is a transmission matrix in which $t_{ij}$ is the transmission probability from node $i$ to node $j$. If $(j, i) \notin A$, then $t_{ij} = 0$.

Actually, this assumption is used for measuring the amount of information (e.g., with regard to an event or message) that will be accepted by each node. The corresponding value varies in the range of $[0, 1]$, where $0$ stands for the ignorance of the message and $1$ means this node totally believes in it. Notably, in the last section of this paper, we also discuss the way how to break through the confinement of this assumption.

The second one is the way how the influence on one node is related to the influences on its neighbors. In the previous works, Aggarwal et al. [11] proposed a way to describe the relation among the neighbors’ influence, that is

$$F_{i \rightarrow j} = 1 - \Pi_{k \in N_j}(1 - t_{kj}F_{j \rightarrow k})$$

which claims that the transmitted influence from different neighbors should be independent to each other. This way is a theoretically reasonable way but it is too complex to get its closed-form solution. So, in this paper, we propose a linear method to define it. That is,

$$F_{i \rightarrow j} = \frac{1}{1 + \lambda_j} \sum_{k \in N_j} t_{kj}F_{j \rightarrow k} \quad \text{for} \quad j \neq i$$

where $\lambda_j$ locates in the range $(0, +\infty)$ which is a damping coefficient of node $j$ for the influence propagation. The smaller the $\lambda_j$ is (i.e., approaching to $0$), the less the information will be blocked by node $i$. This number varies with respect to the topic that the propagating information belongs to. If node $i$ favors the topic of the propagating information, $\lambda_i$ should approach to $0$, otherwise, it should approach to $+\infty$. Thus, this way could be topic sensitive when the damping coefficients on each node are decided according to their favorite.

2.2 A Physical Implication For Linear Model

To the linear way proposed in Equation [5] there is a physical implication when the information network $G$ is undirected. We find that when we construct a circuit network $G'$ by the following way, the current flow in the circuit is running in the same way with the information propagation described by Equation [5] and the potential value on each node is an equivalent value to the influence on it.

For an undirected network $G$ in Figure 1(a), we construct its corresponding circuit network $G'$ in Figure 1(b) as follows.

**Figure 1:** An Illustration Example.

- First, construct a topologically isomorphic circuit network of $G$, where the conductance between node $i$ and $j$ is equal to $c_{ij}$ and guarantees that $\sum_{j=1}^{n} c_{ij} = 1$, where $d_j = \sum_{i=1}^{n} c_{ij}$;
- Second, connect each node $i$ with an external electrode $E_i$ through an additional electric conductor with conductance $c_{ij}$. The electric potential value on $E_i$ is $\frac{c_{ij}}{(1 + \lambda_j, \theta_j)}$, where $\nu_i$ is a real number which will be decided later.

In $G'$, Kirchhoff equations [13] tell us that the total current flowing into any node $j$ should sum up to zero, thus in the circuit network $G'$:

$$\sum_{k=1}^{n} c_{jk}(\tilde{U}_k - \tilde{U}_j) + (1 + \lambda_j - \theta_j)\tilde{U}_j = 0$$

where $\sum I_j$ is the total current flowing into node $j$ and $\tilde{U}_j$ is the electric potential value on node $j$. From Equation [5] we have

$$\tilde{U}_j = \frac{1}{d_j + \nu_i} \sum_{k=1}^{n} c_{jk}\tilde{U}_k + \nu_i d_j$$

$$= \frac{1}{1 + \lambda_j} \sum_{k=1}^{n} t_{kj}(\tilde{U}_k + \nu_j)$$

In an information network, the information will flow along with the direction of an arc. Any network could be an information network. For example, twitter’s information network should be its inverse, for the information flows from node $j$ to node $i$ on arc $(i, j)$ in twitter.
In this Equation, if we decide \( v_j \) by the following way
\[
v_j = \begin{cases} 
\text{a number to guarantee } U_j = 1 & j = i \\
0 & j \neq i
\end{cases}
\] (7)
then comparing with Equation\[ it is easy to get
\[
U_j = F_{i\rightarrow j}
\]
which means that the potential value on node \( j \) is an equivalent value to the influence \( F_{i\rightarrow j} \).

Notably, when \( j \neq i, v_j \neq 0 \), then the potential value on \( E_j \) will be equal to 0 also, all of these nodes could be viewed as ground nodes, thus this circuit network can work well.

### 2.3 Influence Matrix

In light of the circuit network, we amend our linear model as follows
\[
F_{i\rightarrow j} = \frac{1}{1 + \lambda_j} \sum_{k \in N_j} (t_{jk} F_{k\rightarrow i} + v_j) \quad \text{for } j = 1, 2, 3, \ldots n \tag{8}
\]
where \( v_j \) is a correction to guarantee that, when \( j = i, F_{i\rightarrow j} = 1 \). Thus, the value of \( v_j \) could be determined as
\[
v_j = \begin{cases} 
\text{a number to guarantee } F_{i\rightarrow j} & j = i \\
0 & j \neq i
\end{cases}
\] (9)

Equation\[ could be rewritten as
\[
F_i = (I + \Lambda - T)^{-1} T F_i + v
\]
where
\[
F_i = \begin{bmatrix} F_{i\rightarrow 1}, F_{i\rightarrow 2}, \ldots, F_{i\rightarrow n} \end{bmatrix}^T
\]
\[
T = \begin{bmatrix} t_{ij} \end{bmatrix}_{i=1}^n
\]
\[
v = [0, 0, \ldots v_i \neq 0, \ldots 0]^T
\]
\[
\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)
\]
which could be solved as
\[
F_i = (I + \Lambda - T)^{-1} v
\] (10)
\[
= \Gamma^{-1} v = P v
\] (11)
where the transpose of \( (I + \Lambda - T) \) is strictly diagonally dominant, thus it is invertible, and we denote \( (I + \Lambda - T)^{-1} = \Gamma \) and denote its inverse as \( P = \begin{bmatrix} p_{ij} \end{bmatrix}_{i=1}^n \). For \( v \) is a vector with only one nonzero entry \( v_i \), thus
\[
F_i = v_i P_i
\] (12)
and based on Rule 1, \( F_{i\rightarrow i} \) should be equal to 1, there is
\[
v_i p_i = 1
\]
thus,
\[
v_i = \frac{1}{p_i}
\] (13)

Similarly, we could get
\[
F_j = v_j P_j = \frac{1}{p_{jj}} P_j \quad \text{for } j = 1, 2, 3, \ldots n
\] (14)

This Equation could be rewritten as
\[
F = \begin{bmatrix} F_1, F_2, \ldots, F_n \end{bmatrix}^T = \begin{bmatrix} [1/p_{11}, 1/p_{22}, \ldots, 1/p_{nn}] P \end{bmatrix}^T = \text{diag}(P)^{-1} P
\]
\[
= \begin{bmatrix} f_{ij} \end{bmatrix}_{i=1}^n
\] (15)

Because the entries of matrix \( F \) describe the influences between any pairs of nodes (in \( F, f_{ij} = F_{i\rightarrow j} \)), in this paper we call \( F \) as influence matrix and call \( F_i \) as node \( i \)'s influence vector.

#### 2.3.1 The Computation of \( F \)

Because \( F = \text{diag}(P)^{-1} P \), the computation of \( F \) is actually to get \( P = (I + \Lambda - T)^{-1} \). Because \( (I + \Lambda - T) \) is a strictly diagonally dominant matrix which satisfies the convergence condition of Gauss-Seidel method, its inverse \( P \) could be computed by a very fast way through a Gauss-Seidel iteration process.

Because
\[
(I + \Lambda - T) P_i = e_i,
\]
where \( P_i \) could be viewed as the variables of this linear system of equations. For \( (I + \Lambda - T) \) is strictly diagonally dominant, \( P_i \) could be solved by Gauss-Seidel method. Specifically, Gauss-Seidel method is an iterative method which is operated as the following procedures:
1. Set \( p_{ij}^{(0)} = 0 \) for \( j = 1, 2, \ldots n \);  
2. \( p_{ij}^{(k+1)} = \frac{1}{t_{ij} - \sum_{j \neq i} g_{ij} p_{ij}^{(k)} - \sum_{j \neq i} g_{ij} p_{ij}^{(k+1)}} \), for \( j = 1, 2, \ldots n \); 
3. continue Step 2 until the changes made by an iteration are below a given tolerance.

This procedures is efficient. To get \( P_i \) within a valid tolerance range, it often need only dozens of iterations. Thus, the time complexity of computing \( P_i \) is \( O(|E|) \) and the time complexity of \( F \) is \( O(|V||E|) \). Moreover, because the computation of \( P \) could be partitioned to \( n \) parts, we could compute \( P \) in \( n \) parts in parallel.

Even though, the computation of \( F \) is still too consuming to be applied to a large scale network. For real networks, it is not necessary to compute the influence vectors of each nodes since many of them is nobody in the network. In the following section, we will propose a fast method to estimate which nodes are important to the network.

#### 2.3.2 The Upper Bound of Node’s Influence On a Network

Suppose the influence vector of node \( i \) is \( F_i = [f_{i1}, f_{i2}, \ldots, f_{in}]^T \), then its total influence on the network should be \( \sum_{i=1}^n f_{ii} \) and we number this denominator as \( \mathcal{F}_i \) and this number could be viewed as the importance of node \( i \) in the network.

On the other hand, we denote \( P_i = P_i e = \sum_{j=1}^n p_{ji} \). Then, we could get the following property

**Property 1.**
\[
\mathcal{F}_i \leq (1 + \lambda_i) P_i
\] (16)

**Proof.** For
\[
\mathcal{F}_i = \sum_{j=1}^n f_{ji} = \frac{1}{p_{ii}} \sum_{j=1}^n p_{ji} = \frac{1}{p_{ii}} P_i
\] (17)

For \( P(I + \Lambda - T) \), the dot product of \( i \)-th row of \( P \) and \( i \)-th column of \( (I + \Lambda - T) \) should be 1, that is
\[
\sum_{j=1}^n p_{ji}(I + \Lambda - T)_{ij} = 1
\]
where \( (I + \Lambda - T)_{ii} = (1 + \lambda_i) \) and \( (I + \Lambda - T)_{ij} = -t_{ij} \) for any \( i \neq j \). Then, this equation is equivalent to \( p_{ii}(1 + \lambda_i) - \sum_{j=1,j\neq i}^n p_{ji} t_{ij} = 1 \). Thus, we have \( p_{ii}(1 + \lambda_i) = 1 + \sum_{j=1,j\neq i}^n p_{ji} t_{ij} \geq 1 \). And then,
\[
\frac{1}{p_{ii}} \leq (1 + \lambda_i)
\]
With Equation 17 there is
\[ \mathcal{F}_i \leq (1 + \lambda_i) \mathcal{P}_i \]

In Equation 16 the quantity \( \mathcal{P}_i \) is a value that could be got by fast method. Let’s denote \( \mathcal{P} = [\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n] \), for \( \mathcal{P}_i = \mathcal{P}^e \), there is \( \mathcal{P} = \mathcal{P}^e \) which could be rewritten as
\[ (I + \Lambda - T)\mathcal{P} = \mathcal{e} \]

This is a linear system of equations which satisfy the convergence condition of Gauss-Seidel method. As the variables of this system, \( \mathcal{P} \) could be computed in \( O(|E|) \) time by the similar way described in Section 2.3.1. Thus, thanks to Property 1, we could get the upper bound of nodes’ importance in Section 3.2.3.

Thus, we could summarize the independent influence into two conditions:

1. Each node in \( S \) will never be influenced by node \( k \);
2. Each node in \( S \) will never propagate the influence derived from node \( k \).

Because the nodes in \( S \) are the ones on which event \( e \) has already happened, they do not need to be influenced by \( k \) and cannot be influenced. Therefore, the above condition 1 needs to be satisfied. On the other hand, because each node in \( S \) itself will spread the influence on the network about event \( e \), it will block the same type of influence from \( k \). Therefore, we assume nodes in \( S \) will never propagate the influence from \( k \).

Based on the above two conditions, we could construct a circuit network \( G'(S, k) \) to model the independent influence from node \( k \) to the other nodes as follows:

1. First, construct the circuit network \( G' \);
2. Put an electric pole with voltage value 0 on each node in \( S \);
3. Set the voltage value on external electrode \( E_k \) to \( \frac{V_{(n+1) \to k}}{1+\lambda_k} \);
4. Set the voltage value on external electrode \( E_k(i \neq k) \) to 0;

For the graph in Figure 1(a), suppose the seed set \( S = \{3, 8\} \) and \( k = 4 \), then the corresponding \( G'(S, 4) \) is showed in Figure 2. From this figure, we can see that the above two conditions about independent influence are satisfied naturally.

3. THE INFLUENCE FROM A NODE SET

In real applications, we may need to compute the influence from a node set \( S \) to the nodes of the network (Consistent with the notation \( F_{S \to j} \), we denote the influence from set \( S \) to node \( j \) as \( F_{S \to j} \).)

In [9], Goyal et al assumed that the influences of different nodes are independent of each other. Hence, the joint influence \( F_{S \to j} \) can be defined as
\[ F_{S \to j} = 1 - \prod_{k \in S} (1 - F_{k \to j}) \]

However, the influences from different nodes are obviously not mutually independent. Then, how can we get the independent influence of a node when it belongs to a node set \( S \)?

3.1 Independent Influence Inference

To simplify the discussion, we will first turn to solve an equivalent problem defined as follows.

PROBLEM 1. Given a seed set \( S \) and a node \( k \not\in S \), how to evaluate the independent influence \( F^{(S)}_{k \to j} \) (independent from the nodes of \( S \))?

For this problem, if we could work out \( F^{(S)}_{k \to j} \), then we could work out \( F^{(S \cup \{k\}) \to j} \) for any \( s \in S \) in the same way. Then, for the event source set \( S' = S + \{k\} \), we could work out the independent influence for any element in \( S' \), and
\[ F_{S' \to j} = 1 - \prod_{i \in S'} (1 - F^{(S' \cup \{k\})}_{i \to j}) \] (18)

Because \( F_{i \to j} = f_{ij} \), for ease of writing and consistent with the previous notation, in the following text, we will use \( f_{ij} \) to denote \( F_{i \to j} \), use \( f_{S \to j} \) to denote \( F_{S \to j} \), and use \( f^{(S)}_{ij} \) to denote \( F^{(S)}_{i \to j} \). By Equation 18 the happening probability \( f^{(S)}_{ij} \) can be computed. Thus, Problem 1 is the basis for computing the influence from a node set.

In order to get the independent influence \( f^{(S)}_{ij} \), let us think about the essence of independent influence in light of the circuit perspective. We could summarize the independent influence into two conditions:

1. Each node in \( S \) will never be influenced by node \( k \);
2. Each node in \( S \) will never propagate the influence derived from node \( k \).

Because the nodes in \( S \) are the ones on which event \( e \) has already happened, they do not need to be influenced by \( k \) and cannot be influenced. Therefore, the above condition 1 needs to be satisfied. On the other hand, because each node in \( S \) itself will spread the influence on the network about event \( e \), it will block the same type of influence from \( k \). Therefore, we assume nodes in \( S \) will never propagate the influence from \( k \).

Based on the above two conditions, we could construct a circuit network \( G'(S, k) \) to model the independent influence from node \( k \) to the other nodes as follows:

1. First, construct the circuit network \( G' \);
2. Put an electric pole with voltage value 0 on each node in \( S \);
3. Set the voltage value on external electrode \( E_k \) to \( \frac{V_{(n+1) \to k}}{1+\lambda_k} \);
4. Set the voltage value on external electrode \( E_k(i \neq k) \) to 0;

For the graph in Figure 1(a), suppose the seed set \( S = \{3, 8\} \) and \( k = 4 \), then the corresponding \( G'(S, 4) \) is showed in Figure 2. From this figure, we can see that the above two conditions about independent influence are satisfied naturally.

3.2 The Deduction of Independent Influence

In the previous subsection, we have constructed the circuit network for simulating the independent influence. In this subsection, we first analyze the potential value on each node which could be viewed as the influence on it. Then, we provide several properties of independent influence and the efficient way for computing it.

**The Electric Potential on Each Node for** \( G'(S, k) \). Based on the similar discussion as shown in subsection 2.3.1, we know that the electric potential on node \( j \) (except for \( S \)) is
\[ \bar{U}_j = \frac{1}{1+\lambda_j} \left( \sum_{l \in S} t_{lj} \bar{U}_l + v^{(S)}_j \right) \quad \text{for } j \not\in S. \] (19)

where
\[ v^{(S)}_j = \begin{cases} \text{a number to make } \bar{U}_j = 1 & \text{if } j = k \\
0 & \text{otherwise} \end{cases} \] (20)

Because \( \bar{U}_j = 0 \), for \( l \in S \), then Equation 19 can be rewritten as
\[ (1 + \lambda_j) \bar{U}_j = \sum_{l \in S} t_{lj} \bar{U}_l + v^{(S)}_j \quad \text{for } j \not\in S. \] (21)

Without loss of generality, we can set \( S = [n-|S| + 1, n-|S| + 2, ..., n] \) (i.e., the last \( |S| \) nodes in \( G' \)), then Equation 21 can be rewritten as
\[ (1 + \lambda_j) \bar{U}_j = \sum_{l \in S} t_{lj} \bar{U}_l + v^{(S)}_j \quad \text{for } j = 1, 2, ..., n-|S|, \] which is equivalent to
\[
\Gamma_{\mathbf{v}} \mathbf{v}_{\mathbf{n}} = \mathbf{v}_{\mathbf{n}}
\]
where \(U_{\mathbf{n}} = [U_1, U_2, \ldots, U_{|\mathbf{s}|}]^T\), \(\mathbf{v}_{\mathbf{n}} = [y_1, y_2, \ldots, y_{|\mathbf{s}|}]^T\) and \(\Gamma_{\mathbf{n}}\) is a matrix which is cut down from \(\Gamma\) by removing the columns and rows from \(n - |\mathbf{s}| + 1\) to \(n\).

Based on the previous discussion, the potential on node \(\mathbf{k}\) is a strictly diagonally dominant matrix, it satisfies the convergence condition of Gauss-Seidel method. And from \(\Gamma_{\mathbf{n}} \mathbf{v}_{\mathbf{n}} = \mathbf{v}_{\mathbf{n}}\), there is \(\Gamma_{\mathbf{n}}^{-1} \mathbf{v}_{\mathbf{n}} \leq \mathbf{v}_{\mathbf{n}}\).

The Properties of Independent Influence. We have two properties about the independent influence as follows.

**Property 2.** \(f_{jk}^{(S)} \leq f_{jk}\).

**Property 3.** If define \(F_k^{(S)} = \sum_{j=1}^{n} y_j f_{jk}^{(S)}\), then \(F_k^{(S)} < F_k < (1 + A_k)P_{\mathbf{k}}\).

Property 2 shows that \(f_{jk}^{(S)}\) cannot be greater than \(f_{jk}\). If \(S\) is not empty, the influence from node \(k\) will be diluted by the influence from nodes in \(S\). Property 3 provides the upper bound of \(F_k^{(S)}\) which can help us reduce the unnecessary computation for those nodes with small independent influence in the network.

The Computation of Independent Influence. Based on Theorem 1, given the seed set \(S\), if we want to compute the independent influence of node \(k\), we just need to compute the \(k\)-th column of \(\Gamma_{\mathbf{n}}^{-1}\). Because \(\Gamma_{\mathbf{n}}\) is a strictly diagonally dominant matrix, it satisfies the convergence condition of Gauss-Seidel method. And from \(\Gamma_{\mathbf{n}} \mathbf{v}_{\mathbf{n}} = \mathbf{v}_{\mathbf{n}}\), there is \(\Gamma_{\mathbf{n}}^{-1} \mathbf{v}_{\mathbf{n}} = \mathbf{e}_k\). Taking the similar procedures in Section 2.3.1 we could compute the \(k\)-th column of \(\Gamma_{\mathbf{n}}^{-1}\) in \(O(|\mathcal{E}|)\) time.

4. SOCIAL INFLUENCE MAXIMIZATION

In this section, we show how to exploit the circuit model for handling the social influence maximization problem, which targets on finding a set of seeds in a way that these seeds will influence the maximal number of individuals in the network.

4.1 Problem Formulation and Algorithm

The social influence maximization problem can be formalized as

\[
S = \arg \max_{S \subseteq V} \sum_{j=1}^{n} F_{S \rightarrow j}\quad \text{subject to } |S| = K
\]

where \(F_{S \rightarrow j}\) follows the meaning in Equation 18 which is the influence from \(S\) to node \(j\). And in the following text, we denote \((\sum_{j=1}^{n} F_{S \rightarrow j})\) as \(\sigma(S)\) which is the influence spread function of \(S\) (i.e. the number of individuals will be influenced by \(S\)).

**Properties of \(\sigma(S)\).** Under the circuit model, we could prove that the influence spread function \(\sigma(S)\) is a submodular function. This shows in the following theorem.

**Theorem 2.** For all the seed sets, \(S_1 \subseteq S_2 \subseteq V\) and node \(s\), it holds that

\[
\sigma(S_1 \cup \{s\}) - \sigma(S_1) \geq \sigma(S_2 \cup \{s\}) - \sigma(S_2)
\]

**Proof.** For \(\sigma(S) = \sum_{j=1}^{n} F_{S \rightarrow j} = \sum_{j=1}^{n} s_j\), and then,

\[
\sigma(S \cup \{s\}) = \sum_{j=1}^{n} (1 - (1 - f_{sj})(1 - f_{sj}^{(S)}))
\]

\[
= \sum_{j=1}^{n} (f_{sj}^{(S)} + f_{sj} - f_{sj}^{(S)})f_{sj})
\]

thus

\[
\sigma(S \cup \{s\}) - \sigma(S) = \sum_{j=1}^{n} (f_{sj}^{(S)}(1 - f_{sj}))
\]

For \(S_1 \subseteq S_2\), it is easy to get that

\[
f_{sj}^{(S_1)} \leq f_{sj}^{(S_2)} \leq f_{sj}
\]

and then,

\[
\sum_{j=1}^{n} (f_{sj}^{(S_1)}(1 - f_{sj})) \leq \sum_{j=1}^{n} (f_{sj}^{(S_2)}(1 - f_{sj}))
\]

which is,

\[
\sigma(S_1 \cup \{s\}) - \sigma(S_1) \geq \sigma(S_2 \cup \{s\}) - \sigma(S_2)
\]

For simplicity, we denote \(\sigma(S \cup \{s\}) - \sigma(S) = \delta_{ij}^{(S)}\) in the following text. From the proof of Theorem 2, \(\delta_{ij}^{(S)} = \sum_{j=1}^{n} (f_{sj}^{(S)}(1 - f_{sj}))\), specifically, \(\delta_{ij}^{(S)} = \mathcal{P}_{\mathbf{j}}\). Thus, Equation 27 could be rewritten as

\[
\delta_{ij}^{(S)} \geq \delta_{ij}^{(S)}
\]

Proposed Algorithm. By a greedy strategy framework, we always choose the node which can produce the most increment on influence spread when adding it into \(S\). The greedy algorithm starts with an empty set \(S_0 = \emptyset\), and iteratively, in step \(k\), adds the node \(s_k\) which maximizes the increment on influence spread into \(S_{k-1}\)

\[
s_k = \arg \max_{s \in V \setminus S_{k-1}} \delta_{ij}^{(S_{k-1})}
\]

In this process, there is

**Property 4.** \(\delta_{ij}^{(S_{k-1})} \geq \delta_{ij}^{(S_k)} \geq \delta_{ij}^{(S_{k+1})} \geq \delta_{ij}^{(S_{k+2})} \geq \cdots \geq \delta_{ij}^{(S_{k'})}\).

Based on these properties, we design an algorithm to solve the social influence maximization problem, as shown in Algorithm 1. In this algorithm, we use \(\delta_{ij}\) to store the upper bound of the node
5. EXPERIMENTAL RESULTS

Here, we evaluate the performances of the Circuit method on several real-world social networks. Specifically, we demonstrate:

(a) the scalability and the ability to maximize the influence propagation comparing to benchmark algorithms; (b) the impact of the damping coefficient \( \lambda \); and (c) the effectiveness of the upper bounds.

5.1 The Experimental Setup

Experimental Data. Four real-world networks have been used in the experiments. The first one is a Wikipedia voting network in which nodes represent wikipedia users and a directed edge from node \( i \) to node \( j \) represents that user \( i \) voted on user \( j \). The network contains all the Wikipedia voting data from the inception of Wikipedia till January 2008. The second one, denoted as ca-HepPh, is a collaboration network which is from the e-print arXiv, and it covers scientific collaborations between authors whose papers have been submitted to High Energy Physics - Phenomenology category. The third one is an even larger collaboration network, the DBLP Computer Science Bibliography Database, which is the same as in [1]. The fourth one is another large directed network collected by crawling the Amazon website. It is based on Customers Who Bought This Item Also Bought feature of the Amazon website, where if a product \( i \) is frequently co-purchased with product \( j \), then a directed edge from \( i \) to \( j \) will be added. We chose these networks since they can cover a variety of networks with sizes ranging from 103K edges to 2M edges and include two directed networks and two undirected networks. Some basic statistics about these networks are shown in Table 1.

| Networks     | Wiki-Vote | ca-HepPh | DBLP  | Amazon  |
|--------------|-----------|----------|-------|---------|
| #Node        | 7,115     | 22,698   | 655K  | 262K    |
| #Edge/Arc    | 103,689   | 237,010  | 2.0M  | 1.2M    |
| Type         | directed  | undirected| undirected| directed |

Table 1: Statistics of five selected real-world networks.

Benchmark Algorithms. The benchmark algorithms are as follows. First, Circuit is our Algorithm 1 where each node’s influence/independent-influence will be computed as the average result of 10 iterations and we set damping coefficient matrix \( \Lambda = I + \Delta \) and \( \lambda \) ranges in (0, 1]. CELF(2000) is the original greedy algorithm with the CELF optimization of [15], where \( R = 20000 \).

![Figure 4: The computational performances.](http://snap.stanford.edu/data/wiki-Vote.html)

![Figure 4: The computational performances.](http://snap.stanford.edu/data/ca-HepPh.html)

![Figure 4: The computational performances.](http://snap.stanford.edu/data/amazon0302.html)

---

Algorithm 1: Circuit\((G, K, \Lambda, T)\)

```
input : G(V, A, C), K, \Lambda
output: S
Compute \( P = [P_1, P_2, ... P_n]' = (I + \Lambda - T)^{-1}e \) (see it in Section 3.2).
for each vertex \( i \) in \( G \)
\[ S_i = (1 + \lambda_i)P_i; \]
while \(|S| < K\)
re-arrange the order of vertex to make \( S_i \geq S_{i+1}; \)
\[ \hat{\delta}_{\max} = 0; \]
for int \( i = 1 \) to \( n - |S| \)
if \( S_i > \hat{\delta}_{\max} \),
\[ //update \hat{\delta}_i \]
\[ \hat{\delta}_i = UpdateIncrement(i, S, \Lambda, T, G, F_S); \]
if \( S_i > \hat{\delta}_{\max} \),
\[ \hat{\delta}_{\max} = \hat{\delta}_i; \]
\[ s = i; \]
else
\[ break; \]
for int \( i = 1 \) to \( n - |S| \)
\[ f_{s+1} = 1 - (1 - f_s)(1 - f_{s+1}) //update F_S \]
\[ S = S \cup \{s\}; \]
\[ \hat{\delta}_i = 0; \]
return \( S; \)
```

Function UpdateIncrement\((i, S, \Lambda, T, G, F_S)\)

```
input : \( i, S, \Lambda, T, G, F_S \)
output: \( \hat{\delta}_i \)
Compute the \( i \)-th column of \( \Gamma^{-1} \) by the methods in Section 3.2 and then get \( f_{s+1} \) based on Equation 26.
for int \( j = 1 \) to \( n - |S| \)
\[ //Based on the discussion in Theorem 2\]
\[ \hat{\delta}_j + f_{s+1} \]
\[ return \hat{\delta}_i; \]
```
A performance comparison

In the following, we present a performance comparison of both effectiveness and efficiency between Circuit and the benchmarks. For the purpose of comparison, we record the best performance of each algorithm by tuning their parameters. We run tests on the five networks under the WC model to obtain the results of influence spread. The seed set size $K$ ranges from 1 to 50. Figure 3 shows the final results of influence spread. For easy to read, we paint tokens at each 5 points. Figure 4 shows the computational performance comparison for selecting 50 seeds.

In Figure 3(a) we can observe that CELF, Circuit and PMIA are three best algorithms for Wiki-Vote. PageRank performs well at the beginning but deteriorate when the seed set size is large (e.g., larger than 35). DegreeDiscountIC and Degree perform worse than others and DegreeDiscountIC performs a little bit better than Degree. Similar results can also be observed from the rest networks.

In summary, in most cases, Circuit and CELF perform much better than other methods. We believe there are two reasons. First, for these algorithms, their focus on dealing with the independent influence are very different. Degree and PageRank do not care whether the node’s influence is independent from other nodes or not. While DegreeDiscountIC and PMIA try to remove the influence that is diluted by other nodes, their methods are too simple to produce the real independent influence. In contrast, by a reasonable and tractable method, Circuit is able to compute each node’s independent influence, and please note that CELF may also get each node’s real independent influence if we set the parameter $R$ to an unlimited number. Thus, we can say that the more attention the algorithm pays on the independent influence, the better results the algorithm may get. Second, the structure of the networks are quite different (e.g. clustering coefficient), and these differences may also lead to different performances for each algorithm. For example, we find that Degree performs well for the networks where the cluster coefficient are comparably small (in such networks, the most influential nodes often belong to different clusters, and their influence will almost independent from each other, so Degree can work well).

Figure 4 shows the running time comparison for selecting 50 seeds on the five networks. We can observe that the order of the algorithms in running time is Degree > DegreeDiscountIC > PageRank > PMIA > Circuit > CELF, where “>” means “is more efficient than”. We can see that CELF is not scalable to large networks with million edges; while Circuit is scalable, and it takes less than one hour to run on the DBLP network with 2.0M edges. By comparing Figure 3 and Figure 4, we can observe that there are some kind of correlations between the effectiveness and efficiency of each algorithm. Thus, the differences in running time of these algorithms should also mainly lie in their solutions to independent influence. Generally speaking, the more attention they pay on the independent influence, the more complex the algorithm, and more time will take for running the results.

Table 2: Detailed Comparison Between Circuit and CELF.

| Network | Effectiveness | Speedup | Search Rate |
|---------|---------------|---------|-------------|
| Wiki-Vote | 0.996 | 2.03 | 1.00 |
| ca-HepPh | 1.006 | 4.03 | 0.035 |

A Comparison with CELF. Here, we provide a more detailed comparison between Circuit and CELF, the best two algorithms in
we set time of Circuit and the result of its influence spread. Specifically, sponding influence spread and running time. And, for a clear view for both Degree and DegreeDiscountIC.

and Amazon in this paper), and similar results can be also observed to the e that the Circuit’s e

number of search nodes, CELF needs to search all nodes in the net-

mization problem, Circuit and CELF perform consistently well on their performances are not stable for many networks (e.g., DBLP and 4303 times faster than CELF respectively. With respect to the forms CELF significantly, in these three networks, it is 1578, 835, and 4303 times faster than CELF respectively. Similar to Figure 3, this table, “E

ctiveness and the running time of Circuit as the change of

λ

ranges from 0.05 to 1, step by 0.05, and then get the corre-

sponding effectiveness. Specifically, we compare their average per-

formances on the three networks, and the results are shown in Ta-

ble 2 where the performance of CELF is chosen as the baseline. In this table, “Effectiveness” is the ratio of Circuit’s influence spread result relative to CELF’s, “Speedup” is the ratio of Circuit’s running time relative to CELF’s and “Search Ratio” is the ratio of the number of nodes which have been investigated by Circuit relative to the number of nodes investigated by CELF. Similar to Figure 5 we can see that the Circuit’s effectiveness is very close to CELF in three moderate networks: 0.4% worse than CELF in Wiki-Vote but 0.6% better than CELF in ca-HepPh. For efficiency, Circuit outperforms CELF significantly, in these three networks, it is 1578, 835, and 4303 times faster than CELF respectively. With respect to the number of search nodes, CELF needs to search all nodes in the net-

work, while Circuit just need to search less than 5% of nodes due to the effect of Property 1

Summary. Generally, for solving the social influence maxi-

mization problem, Circuit and CELF perform consistently well on each network, but CELF is not scalable to large networks. Though PMIA and PageRank are two good scalable heuristic algorithms, their performances are not stable for many networks (e.g., DBLP and Amazon in this paper), and similar results can be also observed for both Degree and DegreeDiscountIC.

5.3 The Impact of λ

We investigate the effect of tuning parameter λ on the running time of Circuit and the result of its influence spread. Specifically, we set λ ranges from 0.05 to 1, step by 0.05, and then get the corre-

sponding influence spread and running time. And, for a clear view of the influence spread results, we use the ratio of their influence spread result relative to CELF’s to indicate their effectiveness.

Figures 5(a) and 5(b) show the effectiveness of Circuit with different λ on Wiki-Vote, ca-HepPh network respectively. In these figures, the x axis is the λ value; the red long dash line is y = 1 which indicates the results of CELF; the last red bar at the right side is the effectiveness of the best algorithm among PMIA, PageRank, DegreeDiscountIC, Degree. From these figures, we can obtain the following observations:

• The performance of of Circuit is very stable. No matter what value λ is, the difference of effectiveness is less than 0.04, and for most of λ values, the effectiveness of Circuit is larger than 0.97 (against CELF) and better than the best one among PMIA, PageRank, DegreeDiscountIC and Degree.

• When the λ value increases from 0 to 1, the effectiveness of Circuit ascends at the beginning and then descends starting from a certain point. This observation could be explained by our assumption in Section 2.2—there exists a certain damping coefficient in the real-world information propagation process. The father the manually set number is from the real
damping value, the worse our model describes the real inform-

ation propagation, and vice versa. Experimentally, the real λ located in the range [0.1, 0.4].

Figures 5(c) and 5(d) show the running time of Circuit with different λ (for a better view, we show the results of λ = 0.1, 0.2, 0.25, 0.3, 0.4, 0.6, 0.9 and remove the others) on Wiki-Vote, ca-HepPh respectively. On these figures, we can observe that the running time of Circuit is descending with the ascending of λ. When the λ = 0.6, the running times of Circuit are 22.2s, 56.1s respectively, which are comparable with PMIA’s (10.8s, 56.3s respectively), while the influence spread results of Circuit with λ = 0.6 are all better than PMIA’s. From the above observations, we can know that if we want to get a better effectiveness, we should set λ to be a number in [0.1, 0.4] and if we want to get the result efficiently, we should set λ to be a comparable large value.

5.4 The Effectiveness of Upper Bound

To demonstrate the effectiveness of the upper bound proposed in Property 1 we take a test in the following two networks: Wiki-Vote, ca-HepPh. We first select top-100 nodes with the highest indegree from each network, then compute their importance (Fj = \( \sum_{i \in U_j} f_i \)) and their corresponding upper bound (1 + λFj (with parameter λ = 0.25). We show the value of indegree, importance, upper bound of the top-100 nodes in Figure 6 where the green line is the indegree value, the blue long-dash line is the upper bound value, and the red line is the importance value. In Figure 6 we can observe that, for each network, the blue line is always very close to the red line which demonstrates that (1 + λFj) is a very compact upper bound of importance Fj. This is a very important reason why Property 1 can help us to reduce the search space to less than 5% (as illustrated in Table 4).

6. RELATED WORK

Related work can be grouped into two categories. In the first category, we describe some existing social influence models. The second category includes the existing works for the social influence maximization problem.

Social Influence Models. In the literature, many works about social influence have been published. For instance, Anagnostopoulos et al. [3] proved the existence of social influence by statistical tests. Also, Goyal et al. [2] studied how to learn the true probabilities of social influence between individuals. In addition, there are several models to infer how the influence propagates through the network. For example, Granovetter et al. [10] proposed the Lin-

ear Threshold(LT) model to describe it, while Goldenberg et al. [1]...
proposed the Independent Cascade (IC) model. To the best of our knowledge, they are two most widely used models. Since these two models are not tractable, Kimura et al. [12] proposed a comparatively tractable model SPM and Aggarwal et al. [2] proposed a stochastic model to address this issue. In addition, Tang et al. [20] proposed a Topical Affinity Propagation approach, a graphical probabilistic model, to describe the topic-based social influence analysis problem. Recently, Easley et al. [7] and Aggarwal et al. [1] summarized generalized many existing studies on social influence and some other research aspects of social networks. More importantly, they demonstrate that by carefully study, the information exploited from social influence can be leveraged for dealing with the real-world problems (e.g., the problems from markets or social security) effectively and efficiently.

**Social Influence Maximization.** Domingo and Richardson proposed to exploit social influence for the marketing application, which is called viral marketing [5] [19], or social influence maximization. The goal is to find a set of seeds which will influence the maximal number of individuals in the network.

Kempe et al. formulated the influence maximization problem as the discrete optimization problem, and they considered three cascade models (i.e., IC model, Weighted Cascade (WC) model [11], and LT model). Also, they proved that the optimization problem is NP-hard, and presented a greedy approximation (GA) algorithm applicable to all three models, which guarantees that the influence spread result is within (1 - 1/e) of the optimal result. To address the efficiency issue, Leskovec et al. [15] presented an optimized greedy algorithm, which is referred to as the "Cost-Effective Lazy Forward" (CELF) scheme. The CELF optimization uses the modularity property of the influence maximization objective to reduce the number of evaluations on the influence spread of nodes. To address the scalability issue, Chen et al. proposed several heuristic methods includes DegreeDiscountIC [5] and PMIA [4] which uses local arborescence structures of each individual to approximate the social influence propagation. Alternatively, Wang et al. [21] presented a community-based greedy algorithm to find the Top-K influential nodes. They first detect the communities in social network and then find influential nodes from the selected potential communities. Among the aforementioned methods, the best algorithms in seeking the most influential seed set are those based on a Monte-Carlo simulations to compute the influence spread for each given seed set. The GA, CELF are all this type of algorithms.

7. CONCLUSION

In this paper, we developed a social influence model based on circuit theory for describing the information propagation in social networks. This model is tractable and flexible for understanding patterns of information propagation. Under this model, several upper bound properties were identified. These properties can help us to quickly locate the nodes to be considered during the information propagation process. This can drastically reduce the search space, and thus vastly improve the efficiency of measuring the influence strength between any pair of nodes. In addition, the circuit theory based model provides a new way to compute the independent influence of nodes and leads to a natural solution to the social influence maximization problem. Finally, experimental results showed the advantages of the circuit theory based model over the existing models in terms of efficiency as well as the effectiveness for measuring the information propagation in social networks.

8. REFERENCES

[1] C. Aggarwal. Social network data analytics. Springer-Verlag New York Inc, 2011.
[2] C. Aggarwal, A. Khan, and X. Yan. On flow authority discovery in social networks. 2011.
[3] A. Anagnostopoulos, R. Kumar, and M. Mahdian. Influence and correlation in social networks. In SIGKDD 2008, pages 7–15. ACM, 2008.
[4] W. Chen, C. Wang, and Y. Wang. Scalable influence maximization for prevalent viral marketing in large-scale social networks. In SIGKDD 2010, pages 1029–1038. ACM.
[5] W. Chen, Y. Wang, and S. Yang. Efficient influence maximization in social networks. In SIGKDD 2009, pages 199–208. ACM, 2009.
[6] P. Domingos and M. Richardson. Mining the network value of customers. In SIGKDD 2001, pages 57–66. ACM, 2001.
[7] D. Easley and J. Kleinberg. Networks, crowds, and markets: Reasoning about a highly connected world. Cambridge Univ Pr, 2010.
[8] J. Goldenberg, B. Libai and E. Muller. Talk of the network: A complex systems look at the underlying process of word-of-mouth. Marketing letters, 12(3):211–223. Springer, 2001.
[9] A. Goyal, F. Bonchi, and L. Lakshmanan. Learning influence probabilities in social networks. In WSDM 2010, pages 241–250. ACM, 2010.
[10] M. Granovetter. Threshold models of collective behavior. American journal of sociology, pages 1420–1443. JSTOR, 1978.
[11] D. Kempe, J. Kleinberg, and É. Tardos. Maximizing the spread of influence through a social network. In SIGKDD 2010, pages 137–146. ACM, 2003.
[12] M. Kimura, K. Saito. Tractable models for information diffusion in social networks. In PKDD 2006, pages 259–271. Springer, 2006.
[13] G. Kirchhoff. Vorlesungen uber mathematische physik, mechanik. Leipzig: Teubner, 1877.
[14] A. Langville and C. Meyer. Deeper inside pagerank. Internet Mathematics, 1(3):335–380, 2004.
[15] J. Leskovec, A. Krause, C. Guestrin, C. Faloutsos, J. VanBriesen, and N. Glance. Cost-effective outbreak detection in networks. In SIGKDD 2007, pages 420–429. ACM, 2007.
[16] C. Long, R.C.W. Wong. Minimizing Seed Set for Viral Marketing. In ICDM 2011, pages 427–436. IEEE, 2011.
[17] R. Narayanan, Y. Narahari. A shapley value-based approach to discover influential nodes in social networks. IEEE Transactions on Automation Science and Engineering, 99:1–18, IEEE, 2010.
[18] L. Page, S. Brin, R. Motwani, and T. Winograd. The pagerank citation ranking: Bringing order to the web. 1999.
[19] M. Richardson and P. Domingos. Mining knowledge-sharing sites for viral marketing. In SIGKDD 2002, pages 61–70. ACM, 2002.
[20] J. Tang, J. Sun, C. Wang, and Z. Yang. Social influence analysis in large-scale networks. In SIGKDD 2009, pages 807–816. ACM, 2009.
[21] Y. Wang, G. Cong, G. Song, and K. Xie. Community-based greedy algorithm for mining top-k influential nodes in mobile social networks. In SIGKDD 2010, pages 1039–1048. ACM, 2010.
