RANDOM MATRICES WITH PRESCRIBED EIGENVALUES AND EXPECTATION VALUES FOR RANDOM QUANTUM STATES

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ABSTRACT. Given a collection $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ of real numbers, there is a canonical probability distribution on the set of real symmetric or complex Hermitian matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$. In this paper, we study various features of random matrices with this distribution. Our main results show that under mild conditions, when $n$ is large, linear functionals of the entries of such random matrices have approximately Gaussian joint distributions. The results take the form of upper bounds on distances between multivariate distributions, which allows us also to consider the case when the number of linear functionals grows with $n$. In the context of quantum mechanics, these results can be viewed as describing the joint probability distribution of the expectation values of a family of observables on a quantum system in a random mixed state. Other applications are given to spectral distributions of submatrices, the classical invariant ensembles, and to a probabilistic counterpart of the Schur–Horn theorem, relating eigenvalues and diagonal entries of Hermitian matrices.

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1. Introduction

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be real numbers, and let $M_n^\mathbb{R}(\lambda)$ denote the family of real symmetric $n \times n$ matrices with eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ (with multiplicity). The orthogonal group $O(n)$ acts transitively on $M_n^\mathbb{R}(\lambda)$ by conjugation, and from this action $M_n^\mathbb{R}(\lambda)$ inherits a canonical probability measure. A random matrix chosen according to this probability measure is distributed as $U\Lambda U^\dagger$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and $U$ is chosen according to
the Haar probability measure on $O(n)$. Likewise, the family $M_n^C(\Lambda)$ of complex Hermitian matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ possesses a canonical probability measure which is the distribution of the random matrix $UAU^*$, where $U$ is now chosen according to the Haar probability measure on the unitary group $U(n)$.

In this paper we consider the asymptotic behavior of $d$-dimensional marginals of these probability measures (sometimes referred to as isospectral distributions) when $n$ is large. Such marginals include in particular the joint distributions of collections entries of random matrices of the form $UAU^t$ or $UAU^*$. Our results give upper bounds on distances between these marginal distributions and multivariate Gaussian distributions. Such quantitative estimates hold for fixed, finite (but large) $n$, which in turn allows us to consider, as $n \to \infty$, how quickly $d$ may grow with $n$ such that the Gaussian behavior of $d$-dimensional marginals is preserved. In typical situations we will see that all such marginals are asymptotically Gaussian, as long as $d \ll \sqrt{n}$; for many choices of $\Lambda$ and many marginals, it is in fact only necessary that $d \ll n^{3/2}$.

Our results reverse the situation from classical random matrix theory, which begins by specifying the joint distributions of the entries of a random matrix and investigates the resulting joint distribution of the eigenvalues. This inverse approach reveals a new form of universality: marginals of high-dimensional random matrices with nearly any arrangement of prescribed eigenvalues are indistinguishable from marginals of the Gaussian orthogonal or unitary ensemble. This in turn puts limits on how far one can hope to extend classical universality to random matrix ensembles with dependent entries. Weakly correlated entries, even weakly correlated nearly-Gaussian entries, turn out to be consistent with almost any kind of spectral behavior.

If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are nonnegative and $\sum_{i=1}^{n} \lambda_i = 1$, then our results have an important interpretation in terms of quantum mechanics. In this case $\rho = UAU^*$ is a random density matrix, representing a mixed state with weights $\{\lambda_i\}$ of a quantum system with an $n$-dimensional state space $\mathcal{H} = \mathbb{C}^n$. (See for example [47, 3, 33] for general information on various models of random mixed quantum states, and [36, 35] for earlier work considering this precise model.) If $B_1, \ldots, B_d$ are $d$ linearly independent observables on $\mathcal{H}$, then the joint probability distribution of their expectation values in the state $\rho$ is a $d$-dimensional marginal of the distribution of $\rho$, and therefore, by our results, is distributed approximately as a $d$-dimensional Gaussian random vector. Moreover, if $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_d$ and each $B_i$ arises as an observable on $\mathcal{H}_i$ (that is, we are considering a compound system and separately observing the component systems), then the expectation values $\langle B_i \rangle$ are approximately distributed as uncorrelated jointly Gaussian random variables. Besides random density matrices with fixed eigenvalues, our results cover induced random density matrices, which arise as quantum marginals of random pure states on compound quantum systems.

Random matrices of the form $UAU^*$ are familiar in free probability, where stochastically independent matrices of this type are used to asymptotically model freely independent non-commutative random variables. Free probability is chiefly concerned with the asymptotic spectral distributions of functions of families of such random matrices (where $\Lambda = \Lambda_n$ has a known limiting spectral measure when $n \to \infty$), in contrast to our interest here in linear projections. These two lines of inquiry intersect in the study of the limiting spectral behavior of submatrices, sometimes referred to as free compressions (see e.g. [34]). By combining
our main result with a quantitative version of Wigner’s semicircle law, we are able to deduce that submatrices of a range of sizes have asymptotically semicircular spectral distributions.

In addition to deterministic $\Lambda$, we can by conditioning allow $\Lambda$ to be random and independent of $U$. Such a construction produces exactly that class of distributions on real symmetric (respectively, complex Hermitian) matrices which are invariant under orthogonal (respectively, unitary) conjugation, including the so-called unitarily invariant ensembles.

Finally, specializing our results to the diagonal entries of $U\Lambda U^*$ lets us investigate in a natural way the “typical” relationship between the eigenvalues and diagonal entries of a real symmetric or complex Hermitian matrix. In this way we find, in Theorem 11, a probabilistic counterpart of the Schur–Horn theorem, which characterizes pairs of $n$-tuples which occur as the eigenvalues and diagonal entries of some real symmetric matrix. Theorem 11 is thus a Hermitian analogue of the single ring theorem [16, 39], which can be viewed as a probabilistic counterpart of the Weyl–Horn theorem.

Our main technical tool is a multivariate version of Stein’s method for normal approximation developed in [5], which was used in that paper to prove similar results for marginals of the entries of Haar-distributed random matrices; see [12, 43, 22, 21] for additional applications of this method in random matrix theory. There are at least two other well-established approaches to studying random matrices of the form considered here. First, the so-called Harish-Chandra–Itzykson–Zuber integral formula [20] gives an exact expression for the Fourier transform of the distribution of $U\Lambda U^*$. Second, the Weingarten calculus developed in [6, 8, 7] gives combinatorial expressions for joint moments or joint cumulants of entries Haar-distributed random matrices, thus more indirectly of matrices of our form. The chief advantage of Stein’s method over these other approaches is that it automatically yields quantitative bounds on distances between distributions. We recall that the explicit dependence of our bounds on the projection dimension $d$ are a crucial aspect of our results. Such quantitative bounds are rather delicate work to deduce using Fourier-analytic methods (and the HCIZ formula is itself awkwardly suited for high-dimensional analysis; cf. [13, 17]), and are generally out of reach using moments. Nevertheless, it may be that our main results could be sharpened somewhat by combining our Stein’s method approach with the Weingarten calculus; see the comments after the proofs of Theorems 2 and 4 in Section 2.

In the remainder of this introduction we will state our results and expand on the above discussion, deferring the proofs to the later sections.

1.1. Main results. We first establish some notation and terminology.

We denote by $M_n(F)$ the space of $n \times n$ matrices with entries in the field $F$. We denote by $M_n^{sa}(\mathbb{R})$ the space of real symmetric matrices and by $M_n^{sa}(\mathbb{C})$ the space of complex Hermitian matrices. For either $F = \mathbb{R}$ or $\mathbb{C}$, recall that $M_n^{sa}(F)$ is a real vector space which is equipped with the (real) Hilbert–Schmidt inner product $\langle A, B \rangle = \text{tr}(AB^*) = \text{tr}(AB)$.

For $A \in M_n(\mathbb{C})$ and $1 \leq p \leq \infty$, $\|A\|_p$ denotes the Schatten $p$-norm of $A$, which is the $\ell_p$ norm of the sequence of singular values of $A$. If $A \in M_n^{sa}(\mathbb{C})$ then $\|A\|_p$ is equal to the $\ell_p$ norm of the sequence of eigenvalues of $A$. In the special cases $p = 2, \infty$ this reduces to the Hilbert–Schmidt and operator norms, which we also denote by

$$\|A\|_{HS} = \|A\|_2 = \sqrt{\text{tr}(A^*A)}$$

and

$$\|A\|_{op} = \|A\|_\infty$$
Let \( \Lambda \in M_n(\mathbb{R}) \) be a nonscalar diagonal matrix, and define \( A = U \Lambda U^t \), where \( U \) is a Haar-distributed random matrix from \( \mathbb{O}(n) \). Let \( B_1, \ldots, B_d \in M_n^{sa}(\mathbb{R}) \) satisfy \( \text{tr} B_i B_j = \delta_{ij} \) and \( \text{tr} B_i = 0 \), and define the random vector \( X \in \mathbb{R}^d \) by \( X_i = \text{tr} A B_i \).
Then
\[
W_1 \left( \frac{\sqrt{(n-1)(n+2)}}{\sqrt{2}\|\Lambda\|_{HS}} X, g \right) \leq 8\sqrt{2}\frac{\sqrt{n-1}(n+2)}{n} \frac{\|\tilde{\Lambda}\|_{op}^2}{\|\Lambda\|_{HS}^2} \sum_{i=1}^{d} \|B_i\|_{op}^2
\]
\[
= 8\sqrt{2}\frac{\sqrt{n-1}(n+2)}{n} \frac{1}{s_{r}(\tilde{\Lambda})} \sum_{i=1}^{d} \frac{1}{s_{r}(B_i)},
\]
where \( \tilde{\Lambda} = \Lambda - \frac{1}{n} (\text{tr}(\Lambda)) I_n \).

If \( d = 1 \) then also
\[
d_{TV} \left( \frac{\sqrt{(n-1)(n+2)}}{\sqrt{2}\|\Lambda\|_{HS}} X, g \right) \leq 16\sqrt{2}\frac{\sqrt{n-1}(n+2)}{n} \frac{\|\tilde{\Lambda}\|_{op}^2}{\|\Lambda\|_{HS}^2} \|B_1\|_{op}^2.
\]

We emphasize that in Theorem 1 and many of the results below, the bounds given are completely explicit and hold as soon as \( n \geq 2 \).

Theorem 1 shows that marginals of the distribution of the random matrix \( A \) are close to those of the GOE. Indeed, if \( V \) is a \( d \)-dimensional subspace of \( \{ B \in M_n^{sa}(\mathbb{R}) \mid \text{tr}\ B = 0 \} \) and \( G \) is an \( n \times n \) GOE random matrix, then \( \frac{1}{\sqrt{2}} \pi_V(G) \) has a standard Gaussian distribution on \( V \). Thus the left-hand side of the inequality (2) is precisely
\[
W_1 \left( \frac{\sqrt{(n-1)(n+2)}}{\sqrt{2}\|\Lambda\|_{HS}} \pi_V(A), \frac{1}{\sqrt{2}} \pi_V(G) \right) = \frac{1}{\sqrt{2}} W_1 \left( \frac{\sqrt{(n-1)(n+2)}}{\sqrt{2}\|\Lambda\|_{HS}} \pi_V(A), \pi_V(G) \right).
\]

The following application of Theorem 1 is illustrative. Suppose that \( n \) is even, and that \( \frac{n}{2} \) of the diagonal entries of \( \Lambda \) are equal to \( \sqrt{n} \), and \( \frac{n}{2} \) of them are equal to \( -\sqrt{n} \). Theorem 1 implies that
\[
W_1 \left( \pi_V(A), \pi_V(G) \right) \leq C \frac{d}{\sqrt{n}}
\]
for some absolute constant \( C \), and any \( d \)-dimensional subspace \( V \). Thus all the \( d \)-dimensional marginals of \( A \) (on the subspace of trace-zero matrices) are close to Gaussian as long as \( d \ll \sqrt{n} \), although the spectrum of \( A \) is very different from the spectrum of the GOE. (As mentioned above, the trace-zero restriction will be removed below.)

Alternatively, suppose that the \( B_i \) all have stable rank of the order of \( n \). In that case the same bound as in (3) applies for an arbitrary nonzero \( \Lambda \); for the specific \( \Lambda \) suggested above, the bound improves to
\[
W_1 \left( \pi_V(A), \pi_V(G) \right) \leq C \frac{d}{n^{3/2}}.
\]

This improvement in the normal approximation of the random vector \( X \) for essentially high-rank coefficient matrices \( B_i \) is a quadratic counterpart to a phenomenon uncovered in [23]. It is shown there that the best rate of convergence for \( \text{Re} \text{tr} BU \), where \( U \) is a Haar-distributed random matrix from \( U(n) \), depends on the asymptotic behavior of the singular values of \( B \). As in our case, the slowest convergence rate occurs for rank \( B = 1 \).

Theorem 2 is our main result in the complex Hermitian case.
Theorem 2. Let $\Lambda \in M_n(\mathbb{R})$ be a nonscalar diagonal matrix, and define $A = U\Lambda U^*$, where $U$ is a Haar-distributed random matrix from $\mathbb{U}(n)$. Let $B_1, \ldots, B_d \in M_{n\times n}(\mathbb{C})$ satisfy $\text{tr} B_i B_j = \delta_{ij}$ and $\text{tr} B_i = 0$, and define the random vector $X \in \mathbb{R}^d$ by $X_i = \text{tr} A B_i$. 

Then

$$W_1\left(\frac{\sqrt{n^2 - 1}}{\|\Lambda\|_{HS}} X, g\right) \leq 8\sqrt{n} \frac{\|\tilde{\Lambda}\|_{op}^2}{\|\Lambda\|_{HS}^2} \sum_{i=1}^d \|B_i\|_{op}^2 = 8\sqrt{n} \frac{1}{\text{sr}(\Lambda)} \sum_{i=1}^d \frac{1}{\text{sr}(B_i)},$$

where $\tilde{\Lambda} = \Lambda - \frac{1}{n} (\text{tr}(\Lambda)) I_n$. 

If $d = 1$ then also

$$d_{TV}\left(\frac{\sqrt{n^2 - 1}}{\|\Lambda\|_{HS}} X, g\right) \leq 16\sqrt{n} \frac{\|\tilde{\Lambda}\|_{op}^2}{\|\Lambda\|_{HS}^2} \|B_1\|_{op}^2.$$

As above, Theorem 2 shows that marginals of the distribution of the complex Hermitian version of the random matrix $A$ are close to those of the GUE: if $V \in M_{n\times n}(\mathbb{C})$ and $G$ is now an $n \times n$ GUE random matrix, then the left-hand side of (4) is equal to

$$W_1\left(\frac{\sqrt{n^2 - 1}}{\|\Lambda\|_{HS}} \pi_V(A), \pi_V(G)\right).$$

The same specific example discussed above (eigenvalues of $A$ evenly split between $\pm \sqrt{n}$) serves as a useful prototype for Theorem 2 as well.

For the sake of brevity, from this point on we will explicitly state our results only in the complex Hermitian version, although all the results below have real symmetric counterparts which differ only in the constants which appear. We will also omit further estimates in total variation for the univariate case.

As mentioned above, the assumptions on the coefficient matrices $B_i$ can be removed by a suitable affine transformation.

Corollary 3. Let $\Lambda \in M_n(\mathbb{R})$ be a nonscalar diagonal matrix, and define $A = U\Lambda U^*$, where $U$ is a Haar-distributed random matrix from $\mathbb{U}(n)$. Let $B_1, \ldots, B_d \in M_{n\times n}(\mathbb{C})$ and let $\tilde{B}_j = B_j - \frac{1}{n} (\text{tr} B_j) I_n$. Define $\Sigma \in M_d(\mathbb{R})$ by $\Sigma_{ij} = \text{tr} \tilde{B}_i \tilde{B}_j$ and $v \in \mathbb{R}^d$ by $v_i = \frac{1}{n} (\text{tr} \Lambda)(\text{tr} B_i)$. Define the random vector $X \in \mathbb{R}^d$ by $X_i = \text{tr} A \tilde{B}_i$. Then

$$W_1\left(X, \frac{\|\tilde{\Lambda}\|_{HS}}{\sqrt{n^2 - 1}} \Sigma^{1/2} g + v\right) \leq \frac{8d}{\sqrt{n - 1}} \frac{\|\Sigma^{1/2}\|_{op}}{\|\tilde{\Lambda}\|_{HS}}.$$

In order to keep the statement of Corollary 3 simple, we have made use of the trivial estimate $\text{sr}(B_i) \geq 1$ (thus replacing the sum in the last expression in (4) with $d$). Corollary 3 is only applied in the present paper in the proof of Theorem 4 below, where $\text{sr}(B_i) \leq 2$ for each $i$, so there is no essential loss in making this simplification. For applications involving $B_i$ with large stable rank, modifying the proof of Corollary 3 to make use of any available special structure of the problem at hand seems likely to be more useful than attempting to formulate a general-purpose off-the-shelf result.
In the case that $\Lambda$ has rank 1 (in which case the results above only have interesting content when all of the coefficient matrices $B_j$ have large stable rank), the bound on normal approximation from Theorem 4 can be sharpened somewhat. By rotation invariance, we may assume in this case that $\Lambda = e_1e_1^*$, where $e_1$ is the first standard basis vector of $\mathbb{C}^n$. Then $Z = U e_1$ is uniformly distributed on the unit sphere of $\mathbb{C}^n$, and
\[
\text{tr } AB_j = \text{tr } ZZ^* B_j = \langle B_j Z, Z \rangle.
\]
The rank 1 case thus amounts to studying the joint distribution of quadratic forms on the unit sphere.

**Theorem 4.** Let $Z = (Z_1, \ldots, Z_n)$ be uniformly distributed on the complex unit sphere. Let $B_1, \ldots, B_d \in M_{sa}(\mathbb{C})$ satisfy $\text{tr } B_j B_k = \delta_{jk}$ and $\text{tr } B_j = 0$ for $j = 1, \ldots, d$, let $X_j := \langle B_j Z, Z \rangle$. There is a universal constant $C$ such that
\[
W_1 \left( \sqrt{n(n+1)} X, g \right) \leq C \sum_{j=1}^d \| B_j \|_4^2 \leq C \sum_{j=1}^d \frac{1}{\text{sr}(B_j)}.
\]

As in the proof of Corollary 3 if desired, one could remove the assumptions on the $B_j$ using standard linear algebraic techniques.

The second, weaker upper bound in Corollary 4 shows that in this setting, the asymptotic behavior of projections is Gaussian if each coefficient matrix $B_j$ has stable rank of larger order than $d^2$; in particular, if $d$ is fixed and the stable ranks of the $B_j$ grow without bound, Theorem 2 applied to this setting would require that the stable ranks be of larger order than $d \sqrt{n}$. Moreover, the first, stronger upper bound in Corollary 4 shows that the same conclusion holds if some of the $B_j$ have a small number of relatively large eigenvalues.

In principle, it should be possibly to modify our proofs to prove a result which encompasses both Theorems 2 and 4 as special cases, although there are technical challenges; see the discussion following the proof of Theorem 4.

The proofs of Theorem 2 (and indications of how to modify the proof for the real symmetric case in Theorem 1), Corollary 3, and Theorem 4 are given in Section 2 below.

### 1.2. Expectation values of observables for random quantum states.

As mentioned earlier, the results above have a natural interpretation in terms of random mixed states of quantum mechanical systems. We will briefly summarize some basic terminology for readers unfamiliar with quantum mechanics; see [1, 3] for more details. For consistency we will continue to use the same linear-algebraic notation as above, rather than switching to the bra-ket notation typically used in the context of quantum mechanics.

A **density matrix** is a matrix $\rho \in M_{sa}(\mathbb{C})$ with nonnegative eigenvalues such that $\text{tr } \rho = 1$. Equivalently, $\rho \in M_{sa}(\mathbb{C})$ is a density matrix if $\rho = \text{tr}_2(\psi \psi^*)$, where $\psi$ is a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^s \cong \mathbb{C}^{ns}$ for some $s$, and $\text{tr}_2 : M_n(\mathbb{C}) \otimes M_s(\mathbb{C}) \to M_n(\mathbb{C})$ is the partial trace defined by $\text{tr}_2(A \otimes B) = (\text{tr } B) A$. A density matrix $\rho \in M_n(\mathbb{C})$ represents a mixed state of a quantum mechanical system modeled on the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$.

A pure state corresponds to the special case of $\rho = \psi \psi^*$ for a unit vector $\psi \in \mathbb{C}^n$; the vector $\psi$ itself is often said to represent such a pure state. A mixed state is thus the partial trace over $\mathbb{C}^s$ of a pure state in some larger Hilbert space $\mathbb{C}^n \otimes \mathbb{C}^s$. In this case the factor spaces $\mathbb{C}^n$ and $\mathbb{C}^s$ represent interacting subsystems, and $\rho = \text{tr}_2(\psi \psi^*)$ represents the state
of the individual system modeled by $\mathbb{C}^n$, when the composite system is in the pure state $\psi$; $\rho$ is sometimes referred to as a \textit{quantum marginal} of $\psi$.

An observable of a quantum mechanical system modeled on $\mathbb{C}^n$ is represented by a Hermitian matrix $B \in M_{n^2}(\mathbb{C})$. If the system is in a mixed state represented by the density matrix $\rho$, then the expectation value of the observable $B$ is $\langle B \rangle = \text{tr}(\rho B)$; in a pure state $\psi$ this becomes $\langle B \rangle = \langle B \psi, \psi \rangle$.

We can thus interpret Theorems 2 and 4 and Corollary 3 as statements about the joint probability distributions of expectation values of observables of quantum systems in random states. Suppose $B_1, \ldots, B_d \in M_{n^2}(\mathbb{C})$ are observables on a quantum system modeled by the Hilbert space $\mathbb{C}^n$. We assume that the system is in a mixed state $\rho$ with known eigenvalues $\lambda$, but which is otherwise unknown; this is reasonably modeled by a random density matrix $\rho = U\Lambda U^*$ with $U \in \mathbb{U}(n)$ Haar-distributed. Theorem 2 and Corollary 3 show that, under certain hypotheses, the random vector

$$(\langle B_1 \rangle, \ldots, \langle B_d \rangle) \in \mathbb{R}^d$$

has a jointly Gaussian probability distribution. Theorem 4 does the same for a random pure state $Z$ uniformly distributed in the unit sphere of $\mathbb{C}^n$. (Note that the randomness here comes entirely from the uncertainty in the state $\rho$; there is no quantum mechanical randomness since we are considering expectation values of the observables.)

While it is natural to consider the case in which the eigenvalues, but nothing else, are known, even this level of certainty may not hold in practice. There are several well-studied probability measures on the space $n \times n$ of density matrices, among the most important of which are the so-called induced measures $\mu_{n,s}$ for integer $s \geq 1$ (see [1, 3, 33, 48]). If $Z$ is uniformly distributed on the unit sphere of $\mathbb{C}^n \otimes \mathbb{C}^s$, $\mu_{n,s}$ is the distribution of the random density matrix

$$\rho_{n,s} = \text{tr}^2(ZZ^*) \in M_n(\mathbb{C})$$

That is, $\rho_{n,s}$ is a quantum marginal on $\mathbb{C}^n$ of a uniform random pure state on the composite system modeled by $\mathbb{C}^n \otimes \mathbb{C}^s$. In the special case that $s = n$, $\mu_{n,n}$ coincides with normalized Lebesgue measure (usually referred to as Hilbert–Schmidt measure in this context) on the space of density matrices.

The following result is an easy application of Theorem 4. (Since $\mu_{n,s}$ is invariant under unitary conjugation, one could also approach this via Theorem 9 below; however, the approach via Theorem 4 gives a stronger result.)

**Theorem 5.** Let $\rho_{n,s} = \text{tr}^2(ZZ^*)$ be a quantum marginal on $\mathbb{C}^n$ of the uniform random pure state $Z$ on $\mathbb{C}^n \otimes \mathbb{C}^s$. Let $B_1, \ldots, B_d$ be traceless $n \times n$ Hermitian matrices with $\text{tr}(B_jB_k) = \delta_{jk}$. For $j = 1, \ldots, d$, let $X_j := \text{tr}(\rho_{n,s}B_j)$. There is a universal constant $C$ such that

$$W_1\left(\sqrt{n(ns+1)}X, g\right) \leq \frac{C}{\sqrt{s}} \sum_{j=1}^d \|B_j\|^2.$$  

Theorem 5 is proved in Section 3.

A particularly important special case is when $\mathbb{C}^n$ is itself a tensor product $\mathbb{C}^n = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ and the $B_j$ have the form

$$B_j = I_{n_1} \otimes \cdots \otimes I_{n_{j-1}} \otimes C_j \otimes I_{n_{j+1}} \otimes \cdots \otimes I_{n_k}$$
for some \( C_j \in M_{n_j}^\sa(\mathbb{C}) \). In that case \( \mathbb{C}^n \) itself models a composite system, and each \( B_j \) corresponds to an observable acting on a distinct component system; note that when the \( B_j \) are traceless they are automatically orthogonal with respect to the Hilbert–Schmidt inner product.

In the case that \( V = \{ B \in M_n^\sa(\mathbb{C}) \mid \text{tr } B = 0 \} \), \( \pi_V(A) = A - \frac{1}{n}(\text{tr } A)I_n = \tilde{A} \). Theorem 5 implies that

\[
W_1 \left( \sqrt{n(n+1)} \tilde{\rho}_{n,s}, \tilde{G} \right) \leq C \frac{n^2}{\sqrt{s}},
\]

where \( G \) is an \( n \times n \) GUE random matrix. This recovers the fact, apparently first observed in [1, Theorem 6.35(i)], that for fixed \( n \) and \( s \to \infty \), \( \tilde{\rho}_{n,s} \) converges, after appropriate rescaling, to the traceless GUE. Moreover, it adds to this observation a rate of convergence, which allows the earlier result to be meaningfully extended to the regime \( s \gg n^4 \).

### 1.3. Joint distributions of entries.

We now focus our attention on the matrix entries of \( A = U\Lambda U^* \). Our results below are stated only for traceless \( \Lambda \); extending to the general case is trivial but more complicated to state.

**Theorem 6.** Let \( \Lambda \in M_n(\mathbb{R}) \) be a nonzero diagonal matrix with \( \text{tr } \Lambda = 0 \), and define \( A = U\Lambda U^* \), where \( U \) is a Haar-distributed random matrix from \( \mathbb{U}(n) \). Let \( X \in \mathbb{R}^d \) be a random vector whose entries are distinct choices among the diagonal entries of \( A \), the real parts of the above-diagonal entries of \( A \) scaled up by \( \sqrt{2} \), and the imaginary parts of the above-diagonal entries of \( A \) scaled up by \( \sqrt{2} \). Then

\[
W_1 \left( \frac{\sqrt{n^2 - 1}}{\| \Lambda \|_{HS}} X, \rho \right) \leq 9d \frac{\sqrt{n}}{\| \Lambda \|_{\text{op}}} \frac{\| \Lambda \|_{HS}^2}{\text{sr}(\Lambda)}.
\]

This result in particular gives a direct comparison between principle submatrices of \( A \) and the GUE, as follows.

**Theorem 7.** Let \( \Lambda \in M_n(\mathbb{R}) \) be a nonzero diagonal matrix with \( \text{tr } \Lambda = 0 \), and define \( A = U\Lambda U^* \), where \( U \) is a Haar-distributed random matrix from \( \mathbb{U}(n) \). Let \( B \) be the upper-left \( k \times k \) truncation of \( A \), and let \( G \) be a \( k \times k \) GUE matrix. Then

\[
W_1 \left( \frac{\sqrt{n^2 - 1}}{\| \Lambda \|_{HS}} B, \rho \right) \leq 18k^2 \frac{\sqrt{n}}{\| \Lambda \|_{\text{op}}} \frac{\| \Lambda \|_{HS}^2}{\text{sr}(\Lambda)}.
\]

Via quantitative versions of the semicircle law for the GUE and concentration of measure arguments, this allows us to approximate the spectral measure of a suitably scaled version of \( B \) by the semicircle law.

**Theorem 8.** Let \( \Lambda \in M_n(\mathbb{R}) \) be a nonzero diagonal matrix with \( \text{tr } \Lambda = 0 \), and define \( A = U\Lambda U^* \), where \( U \) is a Haar-distributed random matrix from \( \mathbb{U}(n) \). Let \( B \) be the upper-left \( k \times k \) truncation of \( A \), and let \( M := \frac{\sqrt{n^2 - 1}}{\| \Lambda \|_{HS}} B \). Let \( \rho_{sc} \) denote the semicircular distribution,
with density \( \sqrt{4-t^2} \) on \([-2, 2]\). Then

\[
\mathbb{E}W_1(\mu_{k^{-1/2}M}, \rho_{sc}) \leq 18k^2 \sqrt{n} \|\Lambda\|_{op}^2 + C\frac{\sqrt{\log k}}{k}
= \frac{18k^2 \sqrt{n}}{sr(\Lambda)} + C\frac{\sqrt{\log k}}{k},
\]

and

\[
\mathbb{P}\left[ W_1(\mu_{k^{-1/2}M}, \rho_{sc}) \geq \frac{18k^2 \sqrt{n}}{sr(\Lambda)} + C\frac{\sqrt{\log k}}{k} + t \right] \leq \exp \left[ -\frac{nt^2}{12} \left( \frac{2 \sqrt{n^2 - 1} \|\Lambda\|_{op}}{k} \|\Lambda\|_{HS} \right)^{-2} \right]
\]

\[
\leq \exp \left[ -\frac{k^2 (sr(\Lambda)) t^2}{48n} \right].
\]

The typical situation of interest in free probability theory is that of a sequence of \( n \times n \) matrices \( \Lambda_n \) having a limiting spectral measure with bounded support. Indeed, suppose that \( \Lambda_n \) is a sequence of traceless diagonal \( n \times n \) matrices which have (as \( n \to \infty \)) a limiting spectral measure \( \mu \) with compact support, and let \( P_k = I_k \oplus 0_{n-k} \) denote orthogonal projection of \( \mathbb{C}^n \) onto the first \( k \) coordinates. Then in the regime \( k = o(n) \), \( \alpha P_k \Lambda UU^* P_k \) has a limiting spectral measure which can be expressed as

\[
\mu \boxtimes (\alpha \delta_\alpha + (1-\alpha) \delta_0) = (1-\alpha) \delta_0 + \alpha \mu^{1/\alpha}
\]

(see e.g. [34, Section 14]). Here \( \boxtimes \) denotes multiplicative free convolution, \( \delta_x \) denotes the point mass at \( x \), and \( \mu^{1/\alpha} \) denotes an additive free convolution power of order \( 1/\alpha \). It follows that, in the notation of Theorem 8,

\[
\mu_{k^{-1/2}B} \xrightarrow{n \to \infty} \sqrt{\alpha} \mu^{1/\alpha}.
\]

The free central limit theorem implies that

\[
\sqrt{\alpha} \mu^{1/\alpha} \xrightarrow{\alpha \to 0} \rho_{sc}.
\]

As pointed out to the authors by Ion Nechita, this suggests that \( \mu_{k^{-1/2}M} \to \rho_{sc} \) in the setting of Theorem 8 for the entire regime \( k = o(n) \) (corresponding to \( \alpha \to 0 \)). However, to make this argument rigorous, one would need to justify interchanging the limits in (6) and (7).

Theorem 8 rigorously shows by other means that this conclusion is indeed valid for at least part of the regime \( k \ll n \). Typically, one has \( \|\Lambda_n\|_{op} \leq C \); the existence of the limiting spectral measure for \( \Lambda_n \) then implies that \( \|\Lambda_n\|_{HS} \approx \sqrt{n} \), so that \( sr(\Lambda_n) \approx n \). In this situation, Theorem 8 shows that a semicircular limit holds in expectation and in probability if \( k \to \infty \) and \( k \ll n^{1/4} \), and almost surely (thanks to the Borell–Cantelli lemma) if also \( k \gg \sqrt{\log n} \). More generally, \( \mu_{k^{-1/2}M_n} \) converges to \( \rho_{sc} \) in probability if

\[
\sqrt{\frac{n}{sr(\Lambda_n)}} \ll k \ll \frac{\sqrt{sr(\Lambda_n)}}{n^{1/4}}.
\]
(note it is possible to choose such $k$ if $sr(\Lambda_n) \gg n^{3/4}$) and converges almost surely if

$$\sqrt{\frac{n \log n}{sr(\Lambda_n)}} \ll k \ll \sqrt{\frac{sr(\Lambda_n)}{n^{1/4}}}$$

(requiring $sr(\Lambda_n) \gg n^{3/4} \sqrt{\log n}$).

Proofs of Theorem 9 and Theorem 8 are given in Section 4.

1.4. Classical invariant ensembles. Suppose now that $A$ is a random matrix in $M_n^{sa}(\mathbb{C})$ whose distribution is invariant under conjugation by unitary matrices; such classes of random matrices occur frequently in mathematical physics. The random matrix $A$ has the same distribution as $UAU^*$, where $\Lambda$ is a random diagonal matrix with the same eigenvalues as $A$ and $U$ is a Haar-distributed random matrix in $\mathbb{U}(n)$ which is independent of $\Lambda$. This observation allows the marginals of $A$ to be analyzed by applying Theorem 2 conditionally on $\Lambda$.

**Theorem 9.** Let $A$ be a random matrix in $M_n^{sa}(\mathbb{C})$ whose distribution is invariant under unitary conjugation. Let $B_1, \ldots, B_d \in M_n^{sa}(\mathbb{C})$ satisfy $\text{tr} B_i B_j = \delta_{ij}$ and $\text{tr} B_i = 0$, and define the random vector $X \in \mathbb{R}^d$ by $X_i = \text{tr} A B_i$. Let $g = (g_1, \ldots, g_d)$ be a standard Gaussian random vector in $\mathbb{R}^d$, independent of $A$, and let $\bar{A} = A - \frac{1}{n} (\text{tr} A) I_n$.

Then

$$W_1 \left( X, \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) \leq \frac{8}{\sqrt{n}} \mathbb{E} \left( \frac{\|A\|_{op}^2}{\|A\|_{HS}} \right) \sum_{i=1}^d \frac{1}{sr(B_i)} ,$$

and

$$W_1 \left( \frac{\sqrt{n^2 - 1}}{\mathbb{E} \|A\|_{HS}} X, g \right) \leq \frac{8 \sqrt{n}}{\mathbb{E} \|A\|_{HS}} \mathbb{E} \left( \frac{\|A\|_{op}^2}{\|A\|_{HS}} \right) \sum_{i=1}^d \frac{1}{sr(B_i)} + \sqrt{d} \frac{\mathbb{E} \|\bar{A}\|_{HS} - \mathbb{E} \|\bar{A}\|_{HS}}{\mathbb{E} \|A\|_{HS}} .$$

A widely studied class of random matrices whose distributions are invariant under unitary conjugation are the unitarily invariant ensembles (sometimes referred to as matrix models); see e.g. [10, 11, 37]. These are random matrices with a density with respect to Lebesgue measure on $M_n^{sa}(\mathbb{C})$ proportional to $\exp(-n \text{tr} V)$ for some function $V : \mathbb{R} \to \mathbb{R}$, where $\text{tr} V(A)$ is understood in the sense of functional calculus. Up to the choice of normalization, the Gaussian Unitary Ensemble is the special case where $V(x) = x^2$. The following corollary is an easy consequence of Theorem 9 for a large class of potentials $V$; it is likely that the result holds in greater generality.

**Corollary 10.** Let $V : \mathbb{R} \to \mathbb{R}$ be twice-differentiable with $V''(x) \geq \alpha > 0$ for all $x$, and suppose that $A$ is a random matrix in $M_n^{sa}(\mathbb{C})$ with a density proportional to $\exp(-n \text{tr} V)$ with respect to Lebesgue measure on $M_n^{sa}(\mathbb{C})$.

Then, with the notations of Theorem 9

$$W_1 \left( \frac{\sqrt{n^2 - 1}}{\mathbb{E} \|A\|_{HS}} X, g \right) \leq \frac{\kappa}{\sqrt{n}} \sum_{i=1}^d \frac{1}{sr(B_i)} ,$$

where $\kappa$ depends only on $\alpha$.

Theorem 9 and Corollary 10 are proved in Section 5.
1.5. **A probabilistic perspective on the Schur–Horn theorem.** The Schur–Horn theorem characterizes pairs of sequences \((d_1, \ldots, d_n)\) and \((\lambda_1, \ldots, \lambda_n)\) of real numbers which can occur as the diagonal entries and eigenvalues, respectively, of a real symmetric or complex Hermitian matrix. Specifically, if \(A\) is real symmetric or Hermitian, with diagonal entries \(d_1, \ldots, d_n\) and eigenvalues \(\lambda_1, \ldots, \lambda_n\), then the sequence \((d_i)_{1 \leq i \leq n}\) is majorized by \((\lambda_i)_{1 \leq i \leq n}\) (written \((d_i)_{i=1}^n \prec (\lambda_i)_{i=1}^n\)); that is, \((d_i)_{i=1}^n\) is a convex combination of permutations of \((\lambda_i)_{i=1}^n\). Conversely, if \((d_i)_{i=1}^n \prec (\lambda_i)_{i=1}^n\), then there is a real symmetric matrix with diagonal entries \(d_1, \ldots, d_n\) and eigenvalues \(\lambda_1, \ldots, \lambda_n\). See [25, Section 9.B] for further discussion, proofs, and references.

Given a sequence \(\lambda_1, \ldots, \lambda_n\) of eigenvalues, the Schur–Horn theorem identifies exactly which sequences of diagonal entries are possible. We now consider this question probabilistically: given a sequence \(\lambda_1, \ldots, \lambda_n\), what are the diagonal entries of a Hermitian matrix with these eigenvalues typically like? This is analogous to the single ring theorem considered in \([13, 16, 39, 19, 2]\), which can likewise be viewed as a probabilistic counterpart of the Weyl–Horn theorem which relates eigenvalues and singular values. The natural model of a random Hermitian matrix with the given eigenvalues is of course \(A = U \Lambda U^*\), with \(U\) distributed according to Haar measure on \(\mathbb{U}(n)\). (The joint distribution of diagonal entries was also considered in \([31]\); see also \([15, \text{Section 2.2}]\).

**Theorem 11.** For each \(n \in \mathbb{N}\), let \(\Lambda_n = \text{diag}(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})\), be a fixed diagonal matrix, and let \(\mu_n\) be the spectral measure of \(n^{-1/2} \Lambda_n\):

\[
\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\frac{1}{n^{1/2}} \Lambda_n^{(n)}}.
\]

Suppose that there is a probability measure \(\mu\) with mean \(m\) and variance \(\sigma^2 > 0\), such that \(W_2(\mu_n, \mu) \to 0\).

Let \(A_n = U_n \Lambda_n U_n^*\) with \(U_n\) Haar-distributed in \(\mathbb{U}(n)\), and let \(\nu_n\) be the empirical measure of the diagonal entries of \(A_n\):

\[
\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\Lambda_n^{(n)}}.
\]

If \(\|\tilde{\Lambda}_n\|_{\text{op}} = o(n)\), then \(\nu_n \to \mathcal{N}(m, \sigma^2)\) weakly in probability. If moreover \(\|\tilde{\Lambda}_n\|_{\text{op}} = \sigma(n)\), then \(\nu_n \to \mathcal{N}(m, \sigma^2)\) weakly almost surely.

Furthermore, if for some constant \(K\) and for all \(n\), \(\|\tilde{\Lambda}_n\|_{\text{op}} \leq K \sqrt{n}\), then there are constants \(\kappa_1, \kappa_2, \kappa_3 > 0\) depending only on \(K\) such that

\[
\kappa_1 \sqrt{\log n} \leq \mathbb{E} \left( \max_{1 \leq i \leq n} a_{ii}^{(n)} - \frac{1}{n} \text{tr} \Lambda_n \right) \leq \mathbb{E} \left( \max_{1 \leq i \leq n} \left| a_{ii}^{(n)} - \frac{1}{n} \text{tr} \Lambda_n \right| \right) \leq \kappa_2 \sqrt{\log n}
\]

for every \(n\), and with probability 1,

\[
\left( \max_{1 \leq i \leq n} \left| a_{ii}^{(n)} - \frac{1}{n} \text{tr} \Lambda_n \right| \right) \leq \kappa_3 \sqrt{\log n}
\]

for all sufficiently large \(n\).
Theorem 11 is proved in Section 6.

2. PROOFS OF THE MAIN RESULTS

Our main technical tool is the multivariate version of Stein’s method of exchangeable pairs introduced in [5]. (The method was extended and refined in [38, 27, 12], though we will not particularly make use of those improvements here.) The following essentially restates [5, Theorem 5] (cf. also [27, Theorem 4]) and (for the final statement) [26, Theorem 1].

**Theorem 12.** Suppose that $X$ be a random vector in $\mathbb{R}^d$, and for each $\epsilon \in (0, 1)$ there exists a random vector $X_\epsilon$ such that $(X, X_\epsilon)$ is exchangeable. Suppose there exist constants $\alpha, \sigma > 0$, a function $s(\epsilon)$, and a random $d \times d$ matrix $F$ such that

1. \[ \frac{1}{s(\epsilon)} \mathbb{E} \left[ X_\epsilon - X \bigg | X \right] \xrightarrow{\ell_1} -\alpha X, \]
2. \[ \frac{1}{s(\epsilon)} \mathbb{E} \left[ (X_\epsilon - X)(X_\epsilon - X)^T \bigg | X \right] \xrightarrow{\ell_1} 2\alpha \sigma^2 I_d + \mathbb{E}[F \mid X], \]
3. for each $\rho > 0$, \[ \frac{1}{s(\epsilon)} \mathbb{E} \left[ \|X_\epsilon - X\|^2 I_{\|X_\epsilon - X\|^2 > \rho} \right] \xrightarrow{\ell_1} 0. \]

If $g = (g_1, \ldots, g_d)$ is a standard Gaussian random vector, then

\[ W_1(X, \sigma g) \leq \frac{1}{2\alpha \sigma^2} \mathbb{E} \|F\|_{HS}. \]

Moreover, if $d = 1$ then

\[ d_{TV}(X, \sigma g) \leq \frac{1}{\alpha \sigma^2} \mathbb{E} |F|. \]

We will also use these bounds in the equivalent forms

\[ W_1 \left( \frac{1}{\sigma} X, g \right) \leq \frac{1}{2\alpha \sigma^2} \mathbb{E} \|F\|_{HS}. \]

and

\[ d_{TV} \left( \frac{1}{\sigma} X, g \right) \leq \frac{1}{\alpha \sigma^2} \mathbb{E} |F|. \]

As mentioned above, Theorem 12 is one version of Stein’s method for normal approximation. The basic idea of this version is as follows. Suppose that the random vector $X$ has “continuous symmetries” which allow one to make a small (parametrized by $\epsilon$) random change to $X$ which preserves its distribution. If $X$ were exactly Gaussian and this small random change could be made so that $(X, X_\epsilon)$ were jointly Gaussian, then we would have that $X_\epsilon = \sqrt{1-\epsilon^2} X + \epsilon Y$ for $X$ and $Y$ independent. The conditions of the theorem are then approximate versions of what happens, up to third order, in this jointly Gaussian case; the random matrix $F$ is thought of as a small error. The theorem then says that these conditions are enough to guarantee approximate Gaussian behavior.

**Proof of Theorem 12** Since $\text{tr} B_i = 0$, $X_i = \text{tr} \tilde{A} B_i$. We may therefore assume without loss of generality that $\text{tr} \Lambda = 0$.

To apply Theorem 12 we must construct an appropriate family of random vectors $X_\epsilon$; our construction is an adaptation of one first used by Stein in [42], and later applied in [26, 5].
Define
\[
R_\varepsilon := \left( \frac{\sqrt{1-\varepsilon^2}}{-\varepsilon} \begin{pmatrix} \varepsilon & \varepsilon^2 \\ \sqrt{1-\varepsilon^2} & 1 \end{pmatrix} \right) \oplus I_{n-2} = I_n + \varepsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0_{n-2} - \frac{\varepsilon^2}{2} I_2 \oplus 0_{n-2} + O(\varepsilon^3).
\]
Let \( V \in \mathbb{U}(n) \) be Haar-distributed independently of \( U \), and define \( V_\varepsilon := VR_\varepsilon V^* \) and \( A_\varepsilon := UV_\varepsilon \Lambda V_\varepsilon^* U^* \). Note that \((U,UV_\varepsilon)\) is exchangeable by the translation invariance of Haar measure. For each \( i \), let \((X_\varepsilon)_i = \text{tr}(A_\varepsilon B_i)\).

For notational convenience, define the \( n \times 2 \) matrix \( K = [v_1 \, v_2] \), where \( v_i \) are the columns of \( V \), and let \( Q = K \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K^* = v_1 v_2^* - v_2 v_1^* \). Then
\[
V_\varepsilon = I_n + \varepsilon Q - \frac{\varepsilon^2}{2} KK^* + O(\varepsilon^3)
\]
and so (using that \( Q^* = -Q \)).

\[
A_\varepsilon - A = U \left[ \varepsilon(Q\Lambda - \Lambda Q) - \varepsilon^2 \left( Q\Lambda Q + \frac{1}{2} KK^* \Lambda + \frac{1}{2} \Lambda KK^* \right) \right] U^* + O(\varepsilon^3).
\]

It is easy to show that \( \mathbb{E}Q = 0 \) (by conditioning on \( v_1 \), say) and \( \mathbb{E}KK^* = \mathbb{E}[v_1 v_1^* + v_2 v_2^*] = \frac{2}{n} I_n \). Moreover, from [5, Lemma 14] it follows that
\[
\mathbb{E}Q\Lambda Q = \frac{2}{(n-1)n(n+1)} \Lambda
\]
and
\[
\mathbb{E}[\text{tr}(QF)\text{tr}(QG)] = \frac{2}{(n-1)n(n+1)} ((\text{tr} F)(\text{tr} G) - n \text{tr}(FG))
\]
for \( F, G \in M_n(\mathbb{C}) \).

Therefore,
\[
\mathbb{E}[A_\varepsilon - A \mid U] = -\varepsilon^2 U \left[ \frac{2}{(n-1)n(n+1)} \Lambda + \frac{2}{n} \Lambda \right] U^* + O(\varepsilon^3) = -\frac{2n\varepsilon^2}{n^2 - 1} \Lambda + O(\varepsilon^3),
\]
and consequently
\[
\mathbb{E}[X_\varepsilon - X \mid X] = -\frac{2n\varepsilon^2}{n^2 - 1} X + O(\varepsilon^3),
\]
so that Theorem 1 applies with \( s(\varepsilon) = \varepsilon^2 \) and \( \alpha = \frac{2n}{n^2 - 1} \).

To identify \( \sigma^2 \) and \( F \) from Theorem 1, we first compute expectations conditional on \( U \). Writing \( C_i := U^* B_i U \) and using \( \sim \) to denote equality to top order in \( \varepsilon \),
\[
\mathbb{E}[(X_\varepsilon - X)_{i}(X_\varepsilon - X)_{j} \mid U] \sim \varepsilon^2 \mathbb{E}[\text{tr}(U(Q\Lambda - \Lambda Q)U^* B_i)\text{tr}(U(Q\Lambda - \Lambda Q)U^* B_j) \mid U]
\]
\[
= \varepsilon^2 \mathbb{E}[\text{tr}((Q\Lambda - \Lambda Q)C_i)\text{tr}((Q\Lambda - \Lambda Q)C_j) \mid U].
\]
By (10),
\[
E[\text{tr}((Q\Lambda - \Lambda Q)C_i)\text{tr}((Q\Lambda - \Lambda Q)C_j) | U] = E[\text{tr}(Q\Lambda C_i)\text{tr}(Q\Lambda C_j) + \text{tr}(Q\Lambda C_i)\text{tr}(Q\Lambda C_j) - \text{tr}(Q\Lambda C_i)\text{tr}(Q\Lambda C_j) - \text{tr}(Q\Lambda C_i)\text{tr}(Q\Lambda C_j) | U]
\]
\[
= \frac{2}{(n-1)(n+1)} \text{tr}[-\Lambda C_i\Lambda C_j - C_i\Lambda C_j\Lambda + \Lambda C_i C_j \Lambda + C_i \Lambda \Lambda C_j]
\]
\[
= \frac{2}{(n-1)(n+1)} \text{tr}[(A^2 C_i C_j + \Lambda^2 C_j C_i - 2\Lambda C_i \Lambda C_j]
\]
\[
= \frac{2}{(n-1)(n+1)} \text{tr}[(A^2 B_i B_j + A^2 B_j B_i - 2AB_i AB_j].
\]
Now
\[
E[A^2] = E[UA^2U^*] = \sum_{i=1}^{n} \lambda_i^2 u_i u_i^* = \sum_{i=1}^{n} \lambda_i^2 \frac{1}{n} I_n = \frac{\|\Lambda\|^2_{HS}}{n} I_n.
\]
Supposing that \(D\) is diagonal,
\[
E[\text{tr}(ADAC)] = E[\text{tr}(UAU^*DU^*U^*C)]
\]
\[
= \sum_{i,j,l} u_{ij} \lambda_j \bar{u}_{kj} d_{kk} u_{\ell m} \lambda_{\ell m} c_{mi}
\]
\[
= \sum_{i,j,l} \lambda_j \lambda_{\ell} d_{kk} c_{mi} \bar{r}_{uij} u_{\ell k} u_{ij} u_{\ell k}.
\]
The latter expectation is nonzero only if \(i = m\), and then by [5, Lemma 14],
\[
E[\text{tr}(ADAC)] = E[\text{tr}(UAU^*DU^*U^*C)]
\]
\[
= \frac{1}{(n-1)n(n+1)} \sum_{i,j,l} \lambda_j \lambda_{\ell} d_{kk} c_{ii} \left[ n \delta_{ik} + n \delta_{j\ell} - \delta_{ik} \delta_{j\ell} - 1 \right]
\]
\[
= \frac{1}{(n-1)n(n+1)} \left[ n \|\Lambda\|^2_{HS} (\text{tr} D)(\text{tr} C) - \|\Lambda\|^2_{HS} \text{tr}(DC) \right].
\]
If \(B\) is Hermitian, we may write \(B = YDY^*\) for \(Y\) unitary and \(D\) diagonal. By the translation invariance of Haar measure, \(U\) is equal in distribution to \(YU\). Making this substitution inside the expectation and then using (13) yields
\[
E[\text{tr}(ABAC)] = E[\text{tr}(UAU^*DY^*U^*C)]
\]
\[
= E[\text{tr}(YUYU^*DU^*Y^*C)]
\]
\[
= E[\text{tr}(ADAY^*CY)]
\]
\[
= \frac{\|\Lambda\|^2_{HS}}{(n-1)n(n+1)} \left[ n (\text{tr} D)(\text{tr} Y^*C) - \text{tr}(DY^*CY) \right]
\]
\[
= \frac{\|\Lambda\|^2_{HS}}{(n-1)n(n+1)} \left[ n (\text{tr} B)(\text{tr} C) - \text{tr}(BC) \right].
\]
By (10), (11), (12), and (13), (and the facts that \( \text{tr}(B_iB_j) = \delta_{ij} \) and \( \text{tr} B_i = 0 \),
\[
\mathbb{E}[(W_\varepsilon - W)_i(W_\varepsilon - W)_j] \sim \varepsilon^2 \mathbb{E}(\text{tr}[(Q\Lambda - \Lambda Q)C_i]\text{tr}[(Q\Lambda - \Lambda Q)C_j])
\]
\[
= \frac{2\varepsilon^2}{(n-1)(n+1)} \mathbb{E} \text{tr}[A^2B_iB_j + A^2B_jB_i - 2AB_iAB_j]
\]
\[
= \frac{2\|\Lambda\|_{\text{HS}}^2\varepsilon^2}{(n-1)(n+1)} \left( \frac{2\delta_{ij}}{n} + \frac{2\delta_{ij}}{(n-1)n(n+1)} \right)
\]
\[
= \frac{4n\|\Lambda\|_{\text{HS}}^2\varepsilon^2}{(n^2-1)^2} \delta_{ij}.
\]
Based on this we take
\[
\sigma^2 = \frac{\|\Lambda\|_{\text{HS}}^2}{n^2-1}
\]
and
\[
F_{ij} = \frac{2}{n^2-1} \text{tr}[A^2B_iB_j + A^2B_jB_i - 2AB_iAB_j] - \frac{4n\|\Lambda\|_{\text{HS}}^2}{(n^2-1)^2}\delta_{ij}.
\]

We will estimate the variances in (15) using a Poincaré inequality. As is well known, if \( \lambda_1 \) is the smallest nonzero eigenvalue of \(-\Delta\) (where \( \Delta \) is the Laplace–Beltrami operator) on a compact Riemannian manifold \( \Omega \), then
\[
\text{Var } f(x) \leq \frac{1}{\lambda_1} \mathbb{E} \| \nabla f(x) \|^2
\]
for any smooth function \( f : \Omega \to \mathbb{R} \), where \( x \) is a random point distributed according to normalized volume measure on \( \Omega \) (see e.g. [23, Section 3.1]). An argument in the proof of [45, Theorem 3.9] shows that if \( \Omega = \mathbb{U}(n) \), then \( \lambda_1 = n \). It follows that if \( f : \mathbb{U}(n) \to \mathbb{R} \) is \( L \)-Lipschitz with respect to the geodesic distance \( d_g \) on \( \mathbb{U}(n) \), then \( \text{Var } f(U) \leq \frac{1}{n}L^2 \). So it suffices to estimate the Lipschitz constant of functions of the form
\[
f(U) = \text{tr}([U\Lambda U^*, B][U\Lambda U^*, C]^*).
\]

Using \( U \) and \( V \) for the moment to stand for arbitrary matrices in \( \mathbb{U}(n) \) and \( A, A', B, C \) to stand for arbitrary matrices, we observe first that
\[
\|[A,B]\|_{\text{HS}} \leq 2\|A\|_{\text{HS}}\|B\|_{\text{op}}
\]
and hence
\[
\|[A,B] - [A',B]\|_{\text{HS}} \leq 2\|B\|_{\text{op}}\|[A-A']\|_{\text{HS}}.
\]

Also,
\[
\|U\Lambda U^* - V\Lambda V^*\|_{\text{HS}} = \|(U - V)\Lambda U^* + V\Lambda(U - V)^*\|_{\text{HS}} \leq 2\|\Lambda\|_{\text{op}}\|U - V\|_{\text{HS}}.
\]
Now writing $A = UAU^*$ and $A' = VAV^*$, it follows from the Cauchy–Schwarz inequality, (17), (16), and (18) that

$$|f(U) - f(V)| = |\text{tr}([A, B][A, C] - [A', B][A', C]^*)| + |\text{tr}([A, B] - [A', B])|$$

$$\leq \|A, B\|_{HS} \|A, C - [A', B]\|_{HS} + \|A'\|_{HS} [A, B] - [A', B]\|_{HS}$$

$$\leq 16 \|B\|_{op} \|C\|_{op} \|\Lambda\|_{op}^2 \|U - V\|_{HS}$$

where the last estimate follows since $\|U - V\|_{HS} \leq d_g(U, V)$ (see e.g. [29, Lemma 1.3]).

The Poincaré inequality now implies that

$$\text{Var tr}([A, B][A, B]^*) \leq \frac{16^2}{n} \|B\|_{op}^2 \|B\|_{op}^2 \|\Lambda\|_{op}^4,$$

and so by (15),

$$\mathbb{E}\|F\|_{HS} \leq \frac{32}{\sqrt{n}(n^2 - 1)} \sum_{i=1}^{d} \|B_i\|_{op}^2.$$ (19)

The theorem now follows directly from Theorem 12.

The rather soft approach to the bound in (19) used above, based on Poincaré inequalities, yields an optimal bound in general. However, it is in principle possible (though unwieldy) to compute the variances appearing in (15) explicitly using the Weingarten calculus, and obtain a better result in certain cases. We discuss this point further following the proof of Theorem 4 below.

The proof of Theorem 4 is a straightforward modification of the proof above. The required mixed moments of entries of random orthogonal matrices can also be found in [5]. For the Poincaré inequality estimate, one must condition on the coset of the orthogonal group, and obtain a better result in certain cases. We discuss this point further following the proof of Theorem 4 below.

The proof of Theorem 4 is a straightforward modification of the proof above. The required mixed moments of entries of random orthogonal matrices can also be found in [5]. For the Poincaré inequality estimate, one must condition on the coset of the orthogonal group, and obtain a better result in certain cases. We discuss this point further following the proof of Theorem 4 below.

Proof of Corollary 3 As in the statement of the corollary, let $\Lambda \in M_n(\mathbb{R})$ be diagonal and let $B_1, \ldots, B_d \in M_n^{sa}(\mathbb{C})$. The random matrix $A$ is defined by $A = U\Lambda U^*$, where $U$ is Haar-distributed in $\mathbb{U}(n)$, and for each $j$, $X_j = \text{tr}(AB_j)$.

Recall that for any $B \in M_n(\mathbb{C})$, we denote by $\tilde{B}$ the traceless recentering of $B$:

$$\tilde{B} = B - \frac{1}{n} \text{tr}(B)I_n.$$

Note that $\tilde{A} = A - \frac{1}{n} \text{tr}(\Lambda)I_n = U\tilde{\Lambda} U^*$. Also, for each $j$,

$$X_j = \text{tr} AB_j = \text{tr}(\tilde{A} + \frac{1}{n} \text{tr}(\Lambda)I_n)(\tilde{B}_j + \frac{1}{n} \text{tr}(B_j)I_n) = \text{tr} \tilde{A} \tilde{B}_j + \frac{1}{n} \text{tr}(\Lambda) (\text{tr} B_j).$$

Recall that the matrix $\Sigma$ is given by

$$\Sigma_{ij} = \text{tr} \tilde{B}_i \tilde{B}_j = \text{tr} B_i B_j - \frac{1}{n} \text{tr} B_i \text{tr} B_j;$$

it is nonnegative definite, and positive definite if the $\tilde{B}_i$ are linearly independent.
If we define $C_i = \sum_{j=1}^d \Sigma^{-1/2}_{ij} \tilde{B}_j$, then
\[
\text{tr} C_i C_j = \sum_{\ell,m=1}^d [\Sigma^{-1/2}]_{\ell m} (\text{tr} \tilde{B}_\ell \tilde{B}_m) [\Sigma^{-1/2}]_{mj} = [\Sigma^{-1/2} \Sigma^{-1/2}]_{ij} = \delta_{ij}
\]
and
\[
\sum_{j=1}^d [\Sigma^{1/2}]_{ij} C_j = \sum_{j=1}^d [\Sigma^{1/2}]_{ij} [\Sigma^{-1/2}]_{j \ell} \tilde{B}_\ell = \tilde{B}_i.
\]

Now let $W_j = \text{tr} \tilde{A} C_j$ and $v_j = \frac{1}{n} (\text{tr} \Lambda) (\text{tr} B_j)$, so that $X = \Sigma^{1/2} W + v$.

Theorem 2 applied to $W$ gives that
\[
W_1 \left( \frac{\sqrt{n^2 - 1}}{n} \sqrt{v}, g \right) \leq 8d \sqrt{n} \frac{\| \Lambda \|_{HS}}{\| \Lambda \|_{op}},
\]
using the trivial estimate $\sigma_r (C_j) \geq 1$. Note that for a matrix $M$, multiplication by $M$ is $\| M \|_{op}$-Lipschitz, and so
\[
W_1 (MX, MY) = \sup_{\| f \|_r \leq 1} | \mathbb{E} f (MX) - \mathbb{E} f (MY) | \leq \| M \|_{op} W_1 (X, Y).
\]

It thus follows from above that
\[
W_1 \left( X, \frac{\sqrt{n^2 - 1}}{n} \Sigma^{1/2} g + v \right) \leq 8d \frac{\| \Sigma^{1/2} \|_{op} \| \tilde{A} \|_{op}^2}{\| \Lambda \|_{HS} \sqrt{n^2 - 1}}.
\]

For the proof of Theorem 3 we will make use of the fact that, when restricted to the sphere, traceless Hermitian matrices acting as bilinear forms on Euclidean space define eigenfunctions of the Laplacian. This fact is used in conjunction with the following theorem from [27]. We note that this theorem is also proved via Theorem 12 so that ultimately, the proofs of Theorems 2 and 3 rely on the same underlying ideas.

Theorem 13. Let $\Omega$ be a compact Riemannian manifold. Let $f_1, \ldots, f_d$ be eigenfunctions of the Laplace-Beltrami operator on $\Omega$, with eigenvalues $-\mu_1, \ldots, -\mu_d$, and suppose that the $f_i$ are orthonormal in $L^2(\Omega)$ (with the volume measure normalized to have total mass 1). If $Y$ is distributed uniformly (i.e., according to volume measure) on $\Omega$ and $X = (f_1(Y), \ldots, f_d(Y))$, then
\[
W_1 (X, g) \leq \left( \max_{1 \leq i \leq d} \frac{1}{\mu_i} \right) \mathbb{E} \sqrt{d} \sum_{i,j=1}^d [\nabla f_i(Y), \nabla f_j(Y)]^2 - \mathbb{E} \langle \nabla f_i(Y), \nabla f_j(Y) \rangle^2.
\]

Making use of the theorem involves integrating various polynomials over the complex sphere. The proof of the following lemma is a standard exercise; see, e.g., Section 2.7 of [14].

Lemma 14. Let $Z = (Z_1, \ldots, Z_n)$ be uniformly distributed on the complex unit sphere $\{ z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 = 1 \}$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+$, and define $\beta_j := \frac{\alpha_j}{2} + 1$ and $\beta = \sum_{j=1}^n \beta_j$. Then
\[
\mathbb{E} [ |Z_1|^{\alpha_1} \cdots |Z_n|^{\alpha_n} ] = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_n) \Gamma(n)}{\Gamma(\beta)}.
\]

The following compact expression for the mixed moments puts Lemma 14 into a form better suited to our purposes.
**Proposition 15.** Let \( Z = (Z_1, \ldots, Z_n) \) be uniformly distributed on the complex unit sphere. The only non-zero mixed moments of the entries of \( Z \) and their conjugates are those in which each entry appears the same time as its conjugate; these moments are given by the following formula:

\[
\mathbb{E}[Z_{i_1} \cdots Z_{i_k} \overline{Z}_{j_1} \cdots \overline{Z}_{j_k}] = \frac{1}{n(n+1) \cdots (n+k-1)} \sum_{\pi \in S_k} \delta_{i_1 j_{\pi(1)}} \cdots \delta_{i_k j_{\pi(k)}}.
\]

**Proof.** If any entry does not appear the same number of times as its conjugate, then by the invariance of Haar measure under the multiplication of a single coordinate by any unit modulus complex number, the expectation must vanish. It follows, then, that the expectation in the statement of the Proposition must in fact be a mixed absolute moment as in the statement of Lemma 14, with \( \beta = k + n \), and by the functional equation \( \Gamma(x + 1) = x \Gamma(x) \), it follows that

\[
\frac{\Gamma(n)}{\Gamma(\beta)} = \frac{1}{(n+k-1) \cdots n}.
\]

Next, observe that for a given \( j \), \( \Gamma(\beta_j) = \frac{(\alpha_j)!}{2^{\alpha_j}} \) (since \( \alpha_j \) is necessarily even), and thus \( \Gamma(\beta_j) \) is exactly the number of matchings of the \( \frac{\alpha_j}{2} \) of the \( Z_i \) with \( i \) with those \( \overline{Z}_{j_i} \) with \( j_i = j \). It follows that \( \Gamma(\beta_1) \cdots \Gamma(\beta_n) \) is the number of matchings of the \( Z_i \) with the \( Z_{j_i} \) so that each \( Z_j \) is always matched with \( \overline{Z}_j \). This is exactly the expression given by the sum over permutations formula in the statement of the Proposition. \( \square \)

**Proof of Theorem 4** First observe that \( \|B_j\|_4^4 \leq \|B_j\|_2 \|B_j\|_{\text{op}} \|B_j\|_{\text{HS}} = \|B_j\|_{\text{op}}^2 \), and so the second bound of the Theorem follows immediately from the first.

Turning to the proof of the first bound, first note that for \( z \in \mathbb{C}^n \), \( \langle B_j z, z \rangle \) is necessarily real, since \( B_j \) is Hermitian. For a Hermitian matrix \( B \), write \( B = B_r + iB_i \) with \( B_r \) real and symmetric, and \( B_i \) real and anti-symmetric. Then letting \( z = x + iy \),

\[
\langle Bz, z \rangle = \langle B_r x, x \rangle + \langle B_r y, y \rangle - \langle B_i y, x \rangle + \langle B_i x, y \rangle.
\]

That is, \( \langle Bz, z \rangle \) viewed as a function on \( S^{2n-1} \) (associating \( z = x + iy \) with \( (x, y) \in \mathbb{R}^{2n} \)) corresponds to the bilinear form with symmetric traceless matrix

\[
\tilde{B} = \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix},
\]

which is an eigenfunction of the spherical Laplacian with eigenvalue \(-4n\). (See [28] for details on this statement and facts about gradients needed below).

While it is necessary to view \( \langle Bz, z \rangle \) as a function on \( S^{2n-1} \) in order to apply Theorem 13 evaluating the integrals needed is generally simpler in the complex setting; this is justified since the push-forward of uniform measure on the complex unit sphere to \( S^{2n-1} \) is again the uniform measure.

By Proposition 15 for \( B \) Hermitian and traceless,

\[
\mathbb{E} \langle BZ, Z \rangle = \sum_{j,k=1}^n b_{jk} \mathbb{E} [Z_k \overline{Z}_j] = \frac{1}{n} \text{tr}(B) = 0,
\]
and for $B, C$ Hermitian and traceless,
\[
\mathbb{E} \left( \langle BZ, Z \rangle \langle CZ, Z \rangle \right) = \sum_{j,k,\ell,m} b_{jk\ell m} \mathbb{E} \left[ Z_k Z_m \overline{Z}_j Z_\ell \right] = \frac{1}{n(n+1)} \left[ \sum_{j,\ell} \left( b_{jj\ell\ell} + b_{j\ell j\ell} \right) \right] = \frac{\text{tr}(BC)}{n(n+1)},
\]
and so if $f_j(z) = \sqrt{n(n+1)} \langle B_j z, z \rangle$, then $f_1(Z), \ldots, f_d(Z)$ are orthonormal eigenfunctions of the Laplacian on $\mathbb{S}^{2n-1}$.

Now, the gradient $\nabla f_j$ appearing in Theorem 13 is the gradient defined by the Riemannian metric; in this case, it is the spherical gradient, which is given by
\[
\nabla_{\mathbb{S}^{2n-1}} f(z) = \nabla_{\mathbb{R}^{2n}} f(z) - \langle z, \nabla_{\mathbb{R}^{2n}} f(z) \rangle z.
\]
(Abusing notation, we are treating $z$ as a vector in $\mathbb{R}^{2n}$: $(z_1, \ldots, z_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.) For $f_j$ defined as above and $B$ as in (20),
\[
\nabla_{\mathbb{S}^{2n-1}} f_j(z) = \sqrt{n(n+1)} \left[ 2 \tilde{B}_j z - 2 \left\langle \tilde{B}_j z, z \right\rangle z \right] = 2\sqrt{n(n+1)} \tilde{B}_j z - 2zf_j(z),
\]
and so
\[
\langle \nabla_{\mathbb{S}^{2n-1}} f_j(z), \nabla_{\mathbb{S}^{2n-1}} f_k(z) \rangle = 4n(n+1) \left\langle \tilde{B}_j z, \tilde{B}_k z \right\rangle - 4f_j(z)f_k(z).
\]
Now, $\tilde{B}_j z = (\text{Re}(B_j z), \text{Im}(B_j z))$, and so
\[
\left\langle \tilde{B}_j z, \tilde{B}_k z \right\rangle = \langle \text{Re}(B_j z), \text{Re}(B_k z) \rangle + \langle \text{Im}(B_j z), \text{Im}(B_k z) \rangle = \text{Re} \left( \langle B_j z, B_k z \rangle \right).
\]
That is,
\[
(21) \quad \langle \nabla_{\mathbb{S}^{2n-1}} f_j(z), \nabla_{\mathbb{S}^{2n-1}} f_k(z) \rangle = 4n(n+1) \text{Re} \left( \langle B_j z, B_k z \rangle \right) - 4f_j(z)f_k(z).
\]
Taking the expectation using Proposition 15,
\[
\mathbb{E} \left( \langle B_j Z, B_k Z \rangle \right) = \sum_{\ell,p,q} [B_j]_{\ell p} \overline{[B_k]_{\ell q}} \mathbb{E} \left[ Z_p Z_q \overline{Z}_r Z_s \right] = \frac{1}{n} \text{tr} \left( B_j B_k^* \right) = \delta_{jk} n,
\]
and by the orthonormality of the $f_j$, this means that
\[
\mathbb{E} \left( \langle \nabla_{\mathbb{S}^{2n-1}} f_j(z), \nabla_{\mathbb{S}^{2n-1}} f_k(z) \rangle \right) = 4n\delta_{jk}.
\]

We now estimate the variance of this expression. For notational convenience, write $B_j = A$ and $B_k = C$. Then
\[
\mathbb{E} \left( \langle Az, Cz \rangle \right)^2 = \sum_{\ell,m,p,q,r,s} a_{\ell p} a_{mr} \overline{c_{\ell p}} \overline{c_{mr}} \mathbb{E} \left[ Z_p Z_r \overline{Z}_q Z_s \right] = \frac{1}{n(n+1)} \sum_{\ell,m,p,r} \left( a_{\ell p} a_{mr} c_{\ell p} c_{mr} + a_{\ell p} a_{mr} c_{lr} c_{mp} \right) = \frac{1}{n(n+1)} \left( \text{tr}(AC^*)^2 + \text{tr} \left( (AC^*)^2 \right) \right).
\]
and
\[
E |\langle Az, Cz \rangle|^2 = \sum_{\ell,m,p,q,r,s} a_{\ell p} a_{m r} c_{\ell q} c_{m s} E \left[ Z_p Z_r Z_q Z_s \right]
\]
\[
= \frac{1}{n(n+1)} \sum_{\ell,m,p,r,s} (a_{\ell p} a_{m r} c_{\ell q} c_{m s} + a_{\ell p} a_{m q} c_{\ell s} c_{m s})
\]
\[
= \frac{1}{n(n+1)} \left( \text{tr}(AA^*CC^*) + \text{tr}(AC^*)^2 \right),
\]
and so
\[
16n^2(n+1)^2 E \left[ \text{Re} \left( \langle B_j z, B_k z \rangle \right) \right]^2 = 8n^2(n+1)^2 E \left( \text{Re} \left[ \langle B_j z, B_k z \rangle^2 + |\langle B_j z, B_k z \rangle|^2 \right] \right)
\]
\[
= 8n(n+1) \text{Re} \left[ 2\delta_{jk} + \text{tr} \left( (B_j B_k)^2 \right) + \text{tr}(B_j^2 B_k^2) \right].
\]
By Hölder’s inequality for unitarily invariant norms (see [4, Corollary IV.2.6]) and the Cauchy–Schwarz inequality,
\[
\text{tr} \left( (B_j B_k)^2 \right) \leq \|B_j B_k\|_{HS} \|B_k B_j\|_{HS} \leq \|B_j^2\|_{HS} \|B_k^2\|_{HS} = \|B_j\|_4^2 \|B_k\|_4^2.
\]
Similarly,
\[
\text{tr} \left( B_j^2 B_k^2 \right) \leq \|B_j^2\|_{HS} \|B_k^2\|_{HS} = \|B_j\|_4^2 \|B_k\|_4^2,
\]
and so
\[
16n^2(n+1)^2 E \left[ \text{Re} \left( \langle B_j z, B_k z \rangle \right) \right]^2 \leq 16n(n+1) \left[ \delta_{jk} + \|B_j\|_4^2 \|B_k\|_4^2 \right].
\]
Next, regarding \( \pi \in S_3 \) as a bijection from \( \{p, r, t\} \) to itself, using Proposition 15 and assuming \( B \) and \( C \) are Hermitian and traceless,
\[
E \left[ \langle B z, C z \rangle \langle B z, z \rangle \right] = \sum_{\ell,p,q,r,s,t,u} b_{\ell p} b_{r q} c_{t u} c_{t u} E \left[ Z_p Z_r Z_t Z_u \right]
\]
\[
= \frac{1}{n(n+1)(n+2)} \sum_{\ell,p,r,t} \sum_{\pi \in S_3} b_{\ell p} b_{r \pi(p)} c_{t \pi(t)}
\]
\[
= \frac{1}{n(n+1)(n+2)} \left[ \text{tr}(BC^*)^2 + \text{tr}(B C^T B^T C) + \text{tr}(B B^T C^T C) \right],
\]
and so
\[
32n(n+1) E \left[ \text{Re} \left( \langle B_j z, B_k z \rangle \right) f_j(Z) f_k(Z) \right] = \frac{32n(n+1)}{n+2} \text{Re} \left[ \delta_{jk} + \text{tr}(B_j B_k^T B_j^T B_k) + \text{tr}(B_j^T B_k^T B_k B_j) \right].
\]
Bounding the traces as above, we have
\[
|32n(n+1) E \left[ \text{Re} \left( \langle B_j z, B_k z \rangle \right) f_j(Z) f_k(Z) \right]| \leq 32n[\delta_{jk} + 2 \|B_j\|_4^2 \|B_k\|_4^2].
\]
Lastly, letting $B$ and $C$ be traceless Hermitian matrices and using Proposition 15 as above,
\[
\mathbb{E}\left[ (BZ, Z)^2 (CZ, Z)^2 \right] = \sum_{\ell,m,p,q,r,s,t,u} b_{\ell m} b_{pq} c_{rs} c_{tu} \mathbb{E}\left[ Z_m Z_q Z_s Z_t Z_p Z_r Z_t \right] = \frac{1}{n(n+1)(n+2)(n+3)} \sum_{\ell,p,r,t} b_{\ell p(t)} b_{p\pi(t)} c_{r \pi(t)} c_{t \pi(t)} = \frac{1}{n(n+1)(n+2)(n+3)} \left[ \text{tr}(B^2) \text{tr}(C^2) + 4 \text{tr}(B^2 C^2) + 2 \text{tr}(BC)^2 + 2 \text{tr}((BC)^2) \right].
\]
It follows that
\[
16 \mathbb{E} f_j^2(Z) f_k^2(Z) = \frac{16n(n+1)}{(n+2)(n+3)} \left[ 1 + 2 \delta_{jk} + 4 \text{tr}(B_j^2 B_k^2) + 2 \text{tr}((B_j B_k)^2) \right],
\]
and thus
\[
|16 \mathbb{E} f_j^2(Z) f_k^2(Z)| \leq 16 \left[ 1 + 2 \delta_{jk} + 6 \|B_j\|_4^2 \|B_k\|_4^2 \right].
\]
All together then, there is an absolute constant $C$ such that
\[
\sum_{j,k=1}^d \text{Var} \langle \nabla_{\mathbb{S}^{2n-1}} f_j(z), \nabla_{\mathbb{S}^{2n-1}} f_k(z) \rangle \leq C \left[ n^2 \left( \sum_{j=1}^d \|B_j\|_4^2 \right)^2 + nd \right].
\]
For each $j$, $\|B_j\|_4^2 \geq n^{-1/2} \|B_j\|_{HS} = n^{-1/2}$, so the second term in the last estimate above is bounded by the first. This completes the proof. 

The improvements in Theorem 4 relative to applying Theorem 2 in the case that $\text{rank } \Lambda = 1$, result from more explicitly computing a variance, which here involves integrating polynomials of degree 8 on the complex sphere. It appears to be possible to improve Theorem 4, in such a way that Theorem 2 can be recovered as a special case, by integrating polynomials of degree 8 on the unitary group at a corresponding step, using the Weingarten calculus. (This is circumvented in the proof of Theorem 2 above using a Poincaré inequality and Lipschitz estimate.) Details will appear in future work.

3. Random quantum states: proof of Theorem 5

Proof of Theorem 5 First observe that
\[
X_j = \text{tr} \left( \text{tr}_1(ZZ^*) B_j \right) = \text{tr} \left( ZZ^* (B_j \otimes I_s) \right) = \langle (B_j \otimes I_s) Z, Z \rangle.
\]
Now,
\[
\langle B_j \otimes I_s, B_k \otimes I_s \rangle = \text{tr}(B_j B_k \otimes I_s) = s \text{tr}(B_j B_k) = s \delta_{jk},
\]
and so if $Y_j = \frac{1}{\sqrt{s}} \langle (B_j \otimes I_s) Z, Z \rangle$, then Theorem 4 applies, and
\[
W_1(\sqrt{ns(ns+1)} Y, g) \leq C \sum_{j=1}^d \left\| \frac{1}{\sqrt{s}} (B_j \otimes I_s) \right\|_4^2.
\]
Proof of Theorem 8.

For the Hoffman–Wielandt inequality [4, Theorem VI.4.1] implies that the form \( B \) is a constant \( C \) standard basis of the space of Hermitian matrices: let

\[
\| \frac{1}{\sqrt{s}} (B_j \otimes I_s) \|_4^2 = \frac{1}{s} \sqrt{\text{tr}((B_j \otimes I_s)^4)} = \frac{1}{\sqrt{s}} \sqrt{\text{tr}(B_j^4)},
\]

this completes the proof. \( \Box \)

4. Joint distributions of entries: proofs of Theorem 6 and Theorem 8

Proof of Theorem 6. Let \( E_{jk} \in M_n(\mathbb{R}) \) denote the matrix with \( j, k \) entry equal to 1 and all other entries 0. We apply Corollary 3, choosing all of the matrices \( B_i \) to be elements of the standard basis of the space of Hermitian matrices: let \( r \leq d \) be the number of \( B_i \) of the form \( B_{jj}^D = E_{jj} \), with the remaining \( d - r \) having the form

\[
B_{jk}^R = \frac{1}{\sqrt{2}} (E_{jk} + E_{kj}) \quad \text{or} \quad B_{jk}^I = \frac{i}{\sqrt{2}} (E_{jk} - E_{kj})
\]

for \( 1 \leq j < k \leq n \). Thus \( \text{tr}(AB_{jj}^D) = a_{jj}, \text{tr}(AB_{jj}^R) = \sqrt{2} \text{Re}(a_{jk}), \) and \( \text{tr}(AB_{jk}^I) = \sqrt{2} \text{Im}(a_{jk}) \). Observe that \( B_{jk}^R \) and \( B_{jk}^I \) are traceless, while \( B_{jj}^D = B_{jj}^P - \frac{1}{n} I_n \). It follows that in the setting of Corollary 3 if we order the coefficient matrices so that those of the form \( B_{jj}^D \) are listed first, then \( \Sigma = I_d - \frac{1}{n} J_r \), where \( J_r \in M_d(\mathbb{R}) \) consists of an \( r \times r \) block of 1s in the upper-left corner, with all other entries 0. Corollary 3 then implies that

\[
W_1 \left( \frac{\sqrt{n^2 - 1}}{\|\Lambda\|_{HS}} X, g \right) \leq W_1 \left( \frac{\sqrt{n^2 - 1}}{\|\Lambda\|_{HS}} X, \Sigma^{1/2} g \right) + W_1 \left( \Sigma^{1/2} g, g \right)
\]

\[
\leq 8d \sqrt{n} \| \Sigma^{1/2} \|_{op}^2 + W_1 \left( \Sigma^{1/2} g, g \right).
\]

Now

\[
W_1 \left( \Sigma^{1/2} g, g \right) \leq \sup_{\|f\|_1 \leq 1} \left| \mathbb{E} f(\Sigma^{1/2} g) - \mathbb{E} f(g) \right| \leq \left\| \Sigma^{1/2} - I_d \right\|_{op} \| f \| \leq \sqrt{d} \left\| \Sigma^{1/2} - I_d \right\|_{op}.
\]

From the description given above, it is immediate that \( \Sigma \) has eigenvalues 1 (with multiplicity \( d-1 \)) and \( 1 - \frac{4}{n} \) (with multiplicity 1), so that \( \left\| \Sigma^{1/2} \right\|_{op} = 1 \) and \( \left\| \Sigma^{1/2} - I_d \right\|_{op} = 1 - \sqrt{1 - \frac{4}{n}} \leq \frac{d}{n} \). We thus obtain

\[
W_1 \left( \frac{\sqrt{n^2 - 1}}{\|\Lambda\|_{HS}} X, g \right) \leq 8d \sqrt{n} \| \Lambda \|_{op}^2 + \frac{\sqrt{d}r}{n}.
\]

The stated bound now follows, since \( \|\Lambda\|_{HS}^2 \leq n \|\Lambda\|_{op}^2 \) and \( r \leq \max\{n, d\} \leq \sqrt{dn} \). \( \Box \)

Proof of Theorem 8. For \( G \) a \( k \times k \) GUE matrix, it was proved by Dallaporta [9] that there is a constant \( C \), independent of \( k \), such that

\[
\mathbb{E} W_1(\mu_{k-1/2G}, \rho_{sc}) \leq C \frac{\sqrt{\log k}}{k}.
\]

The Hoffman–Wielandt inequality [4, Theorem VI.4.1] implies that that \( C \mapsto \mu_C \) is \( \frac{1}{\sqrt{k}} \)-Lipschitz for \( k \times k \) normal matrices (taking \( W_1 \) as the metric on probability measures and

Since

\[
\left\| \frac{1}{\sqrt{s}} (B_j \otimes I_s) \right\|_4^2 = \frac{1}{s} \sqrt{\text{tr}((B_j \otimes I_s)^4)} = \frac{1}{\sqrt{s}} \sqrt{\text{tr}(B_j^4)},
\]

this completes the proof. \( \Box \)
the Hilbert-Schmidt distance on matrices), so for any coupling of normal random matrices $M_1$ and $M_2$,
\[
\mathbb{E} W_1(\mu_{M_1}, \mu_{M_2}) \leq \frac{1}{\sqrt{k}} \mathbb{E} \|M_1 - M_2\|_{HS},
\]
and by taking infimum over couplings,
\[
\mathbb{E} W_1(\mu_{M_1}, \mu_{M_2}) \leq \frac{1}{\sqrt{k}} W_1(M_1, M_2).
\]
Writing $M = \sqrt{n^2 - 1} B$, it follows from Theorem 7 that
\[
\mathbb{E} W_1(\mu_{k^{-1/2} M}, \mu_{G}) \leq 18k^2 \sqrt{n} \frac{\|\Lambda\|^2_{op}}{\|\Lambda\|^2_{HS}},
\]
Combining this with the estimate (22) yields
\[
\mathbb{E} W_1(\mu_{k^{-1/2} M}, \rho_{sc}) \leq 18k^2 \sqrt{n} \frac{\|\Lambda\|^2_{op}}{\|\Lambda\|^2_{HS}} + C \frac{\sqrt{k} \|\Lambda\|_{op}}{\|\Lambda\|_{HS}},
\]
which is the first statement of the Theorem.

To prove the second statement, consider the mapping $U \mapsto A \mapsto B$, where $A = U \Lambda U^*$ and $B$ is the upper-left $k \times k$ submatrix of $A$. Observe that
\[
\left| \|U_1 \Lambda U_1^*\|_{HS} - \|U_2 \Lambda U_2^*\|_{HS} \right| \leq \|(U_1 - U_2) \Lambda U_1^*\|_{HS} + \|U_2 \Lambda (U_1^* - U_2^*)\|_{HS}
\]
\[
\leq 2 \|\Lambda\|_{op} \|U_1 - U_2\|_{HS},
\]
and $A \mapsto B$ is a projection, so $B$ is a $2 \|\Lambda\|_{op}$-Lipschitz function of $U$. It follows that $W_1(\mu_{k^{-1/2} M}, \rho_{sc})$ is a $2 \|\Lambda\|_{op} \frac{\sqrt{n^2 - 1}}{k}$-Lipschitz function of $U$. Lipschitz functions on $U(n)$ satisfy the sub-Gaussian concentration inequality
\[
\mathbb{P}[F(U) \geq \mathbb{E} F(U) + t] \leq \exp \left[ \frac{1}{12} \frac{nt^2}{|F|_L^2} \right],
\]
(see [31, Corollary 17]), and so
\[
\mathbb{P}[W_1(\mu_{k^{-1/2} M}, \rho_{sc}) \geq \mathbb{E} W_1(\mu_{k^{-1/2} M}, \rho_{sc}) + t] \leq \exp \left[ \frac{nt^2}{12} \left( \frac{2 \sqrt{n^2 - 1}}{k} \frac{\|\Lambda\|_{op}}{\|\Lambda\|_{HS}} \right)^{-2} \right]
\]
\[
\leq \exp \left[ -\frac{k^2 \|\Lambda\|_{op}}{48n} \frac{(s_r(\Lambda))t^2}{48n} \right].
\]

5. INVARIANT ENSEMBLES: PROOF OF THEOREM 9 AND COROLLARY 10

Proof of Theorem 4 As discussed prior to the statement of the theorem, the random matrix $A$ has the same distribution as $UAU^*$, where $\Lambda$ is a real diagonal random matrix with the same eigenvalues as $A$ and $U$ is Haar-distributed in the unitary group, independent from $\Lambda$.  


Observe that since $\text{tr} B_i = 0$, $\text{tr} AB_i = \text{tr} \tilde A B_i$, and that $\|A\|_{HS} = \|\tilde A\|_{HS}$ and $\|A\|_{op} = \|\tilde A\|_{op}$. Now

$$W_1 \left( X, \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) = \sup_{|f|_{L^1} \leq 1} \left| \mathbb{E} f(X) - \mathbb{E} f \left( \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) \right| = \sup_{|f|_{L^1} \leq 1} \left| \mathbb{E} \left( \mathbb{E} f(X) - f \left( \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) \right) \right| \leq \mathbb{E} \sup_{|f|_{L^1} \leq 1} \left| \mathbb{E} f(X) - f \left( \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) \right| \leq \mathbb{E} \left( \frac{8}{\sqrt{n}} \left\| A \right\|_{HS}^2 \right) \sum_{i=1}^{d} \frac{1}{sv(B_i)},$$

by Theorem \[2\]

Next note that

$$W_1 \left( \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g, \frac{\mathbb{E}\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) = \sup_{|f|_{L^1} \leq 1} \left| \mathbb{E} f \left( \frac{\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) - \mathbb{E} f \left( \frac{\mathbb{E}\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) \right| \leq \mathbb{E} \left| \frac{\|A\|_{HS} g - \mathbb{E}\|A\|_{HS} g}{\sqrt{n^2 - 1}} \right| = \mathbb{E} \left( \left\| \frac{\|A\|_{HS} - \mathbb{E}\|A\|_{HS}}{\sqrt{n^2 - 1}} g \|g\| \right) \leq \sqrt{\mathbb{E} \left( \left\| \frac{\|A\|_{HS} - \mathbb{E}\|A\|_{HS}}{\sqrt{n^2 - 1}} g \|g\| \right) \leq \frac{\sqrt{\mathbb{E} \left( \left\| \frac{\|A\|_{HS} - \mathbb{E}\|A\|_{HS}}{\sqrt{n^2 - 1}} g \right) \|g\|}}{\sqrt{n^2 - 1}} \right)},$$

where we have used the independence of $g$ and $A$ in the last equality. The second statement of the theorem now follows from the triangle inequality for $W_1$ together with renormalization of $X$ and $\frac{\mathbb{E}\|A\|_{HS}}{\sqrt{n^2 - 1}} g$ by $\frac{\sqrt{n^2 - 1}}{\mathbb{E}\|A\|_{HS}}$. \hfill $\square$

**Proof of Corollary \[7\]** The assumptions on the distribution of $A$ imply that the distribution of satisfies a logarithmic Sobolev inequality, and hence a strong concentration of measure property; cf. \[2\] Section 5.1. In particular, for any 1-Lipschitz function $F : M_n^{sa}(\mathbb{C}) \to \mathbb{R}$ (with respect to the Hilbert–Schmidt norm),

$$P \left( F(A) - \mathbb{E} F(A) \geq t \right) \leq e^{-nt^2/2}$$

for all $t > 0$. From this it can be proved that

$$\beta_1 \sqrt{n} \leq \mathbb{E} \left\| \tilde A \right\|_{HS} \leq \beta_2 \sqrt{n} \quad \text{and} \quad \gamma_1 \leq \mathbb{E} \left\| \tilde A \right\|_{op} \leq \gamma_2,$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ depend only on $\alpha$ (see \[32\]). (For simplicity of exposition, in the remainder of of this proof, all constants may depend on $\alpha.$)
It follows directly from (25) that
\[ E \left| \| \bar{A} \|_{HS} - E \| \bar{A} \|_{HS} \right| \leq C. \]
It therefore suffices to show that
\[ E \left( \frac{\| \bar{A} \|^2_{op}}{\| \bar{A} \|_{HS}} \right) \leq \frac{C}{\sqrt{n}}. \]

Firstly, for \( t \geq \frac{8 \gamma^2}{\beta_1 \sqrt{n}} \),
\[
P \left[ \frac{\| \bar{A} \|^2_{op}}{\| \bar{A} \|_{HS}} \geq t \right] = P \left[ \| \bar{A} \|^2_{op} \geq t \| \bar{A} \|_{HS} \right] \leq P \left[ \| \bar{A} \|^2_{op} \geq \frac{\beta_1 \sqrt{n} t}{2} \right] + P \left[ \| \bar{A} \|_{HS} \leq \frac{\beta_1 \sqrt{n}}{2} \right] \leq P \left[ \| \bar{A} \|_{op} - E \| \bar{A} \|_{op} \geq \left( \frac{\beta_1 \sqrt{n}}{2} \right)^{1/2} - \gamma_2 \right] + P \left[ -\| \bar{A} \|_{HS} + E \| \bar{A} \|_{HS} > \frac{\beta_1 \sqrt{n}}{2} \right] \leq e^{-\alpha n^{3/2} t/8} + e^{-\alpha^2 n^2/8}. \]

Next, since \( \| \bar{A} \|_{op} \leq \| \bar{A} \|_{HS} \),
\[
P \left[ \frac{\| \bar{A} \|^2_{op}}{\| \bar{A} \|_{HS}} \geq t \right] \leq P \left[ \| \bar{A} \|_{op} \geq t \right] \leq P \left[ \| \bar{A} \|_{op} - E \| \bar{A} \|_{op} \geq t - \gamma_2 \right] \leq e^{-\alpha n^2/8} \]
for \( t \geq \frac{1}{2} \gamma_2 \).

We now estimate
\[
E \left( \frac{\| \bar{A} \|^2_{op}}{\| \bar{A} \|_{HS}} \right) = \int_0^\infty P \left[ \frac{\| \bar{A} \|^2_{op}}{\| \bar{A} \|_{HS}} \geq t \right] dt
\]
using (27) to bound the integrand for \( \frac{8 \gamma^2}{\beta_1 \sqrt{n}} \leq t \leq \frac{1}{2} \gamma_2 \), (28) for \( t \geq \frac{1}{2} \gamma_2 \), and the trivial upper bound of 1 for \( 0 \leq t \leq \frac{8 \gamma^2}{\beta_1 \sqrt{n}} \). \( \square \)

6. Diagonal entries: proof of Theorem 11

Proof of Theorem 11

Assume without loss of generality that for all \( n \), \( \text{tr} \Lambda_n = 0 \); this only amounts to writing \( \Lambda_n \) instead of \( \bar{\Lambda}_n \). In this case \( \int x \, d\mu_n(x) = 0 \) for each \( n \) (and so \( m = 0 \)). Let
\[
\sigma_n^2 = \int x^2 \, d\mu_n(x) = \frac{1}{n^2} \| \Lambda_n \|^2_{HS} \quad \text{and} \quad \sigma^2 = \int x^2 \, d\mu(x);
\]
because we have assumed that \( \mu_n \to \mu \) in \( W_2 \), we have that \( \sigma_n \to \sigma \).
First consider the mean measure $E\nu_n$. Given any test function $f : \mathbb{R} \to \mathbb{R}$,

$$E \int f \, d\nu_n = \frac{1}{n} \sum_{i=1}^{n} E f(a_{ii}^{(n)}).$$

Now for any $i$, $a_{ii}^{(n)} = \langle A_n e_i, e_i \rangle = \langle A_n U_n^* e_i, U_n^* e_i \rangle$, and $U_n^* e_i$ is distributed uniformly on the unit sphere in $\mathbb{C}^n$. Therefore,

$$E \int f \, d\nu_n = E f(\langle A_n Z, Z \rangle),$$

where $Z$ is uniformly distributed on the unit sphere in $\mathbb{C}^n$; that is, $E\nu_n$ is precisely the distribution of $\langle A_n Z, Z \rangle$. It follows immediately from the $d = 1$ case of Theorem 4 that (29)

$$W_1 \left( \langle A_n Z, Z \rangle, \sigma_n \sqrt{n} \right) \leq C \frac{\|A_n\|_2^2}{n^2 \sigma_n^2},$$

making use of the fact that $\|A_n\|_2^2 = n^2 \sigma_n^2$.

We now apply the concentration of measure phenomenon on $U(n)$. Note that if $A = U\Lambda U^*$, $B = V\Lambda V^*$ for $U, V \in U(n)$, then

$$\left| \sum_{i=1}^{n} a_{ii} - b_{ii} \right|^2 \leq \|A - B\|_{HS} = \|U\Lambda(U - V)^* + (V - U)\Lambda V^*\|_{HS} \leq 2 \|\Lambda\|_{op} \|U - V\|_{HS}.$$ 

If $f : \mathbb{R} \to \mathbb{R}$ is a 1-Lipschitz test function, then it follows that

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(a_{ii}) - \frac{1}{n} \sum_{i=1}^{n} f(b_{ii}) \right| \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |a_{ii} - b_{ii}| \leq \frac{2}{\sqrt{n}} \|\Lambda\|_{op} \|U - V\|_{HS};$$

that is, if $\nu = \frac{1}{n} \sum_{j=1}^{n} \delta_{a_{jj}}$ for $A = U\Lambda U^*$, then $U \mapsto \int f \, d\nu$ is a $\frac{2}{\sqrt{n}} \|\Lambda\|_{op}$-Lipschitz function of $U$. Then (24) implies that

$$P \left[ \left| \int f \, d\nu_n - E \int f \, d\nu_n \right| \geq t \right] \leq 2 \exp \left[ -\frac{n^2 t^2}{48 \|\Lambda_n\|_{op}^2} \right].$$

Suppose now that $\|\Lambda_n\|_{op} = o(n)$. Then $\|\Lambda_n\|_2^2 \leq \|\Lambda_n\|_{HS} \sqrt{n} \sigma_n = n \sigma_n \|\Lambda_n\|_{op}$, and so by (29) and the fact that $\sigma_n \to \sigma$,

$$\int f \, d\nu_n \to E f(\sigma g)$$

for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$. It follows from (31) that for some $\varepsilon(n) \to 0$,

$$P \left[ \left| \int f \, d\nu_n - E f(\sigma Z) \right| \geq t + \varepsilon(n) \right] \to 0$$

for each fixed $t > 0$, so that $\nu_n \to N(0, \sigma)$ weakly in probability. Moreover, if $\|\Lambda_n\|_{op}^2 = o\left( \frac{n}{\log n} \right)$, then

$$P \left[ \left| \int f \, d\nu_n - E f(\sigma Z) \right| \geq \frac{7 \|\Lambda_n\|_{op}}{n} \sqrt{n \log n + \varepsilon(n)} \right] \leq 2n^{-49/48},$$

for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$. It follows from (31) that for some $\varepsilon(n) \to 0$,
so \( \nu_n \to N(0, \sigma^2) \) weakly almost surely by the Borel–Cantelli Lemma.

Next, by (\ref{eq:brm}), for each \( i \), \( U \mapsto a_{ii} \) and \( U \mapsto -a_{ii} \) are \( 2 \| \Lambda \|_{op} \)-Lipschitz functions on \( \mathbb{U}(n) \), and so \( \ref{eq:brm} \) implies that

\[
P \left[ |a_{ii}| \geq t \right] \leq 2e^{-nt^2/48\|\Lambda_n\|_{op}^2}.
\]

Therefore

\[
P \left[ \max_{1 \leq i \leq n} |a_{ii}| \geq t \right] \leq 2ne^{-nt^2/48\|\Lambda_n\|_{op}^2},
\]

which implies that

\[
\mathbb{E} \max_{1 \leq i \leq n} |a_{ii}| \leq \int_0^\infty \min \left\{ 1, 2ne^{-nt^2/48\|\Lambda_n\|_{op}^2} \right\} \, dt \leq C \|\Lambda_n\|_{op} \sqrt{\frac{\log n}{n}},
\]

and by the Borel–Cantelli lemma, with probability 1,

\[
\max_{1 \leq i \leq n} |a_{ii}| \leq 10 \|\Lambda_n\|_{op} \sqrt{\frac{\log n}{n}}.
\]

for all sufficiently large \( n \).

On the other hand, given \( 1 \leq d \leq n \), by Theorem \ref{thm:brm}

\[
W_1 \left( \frac{\sqrt{n^2 - 1}}{\|\Lambda_n\|_{HS}} (a_{11}, \ldots, a_{dd}), (g_1, \ldots, g_d) \right) \leq 9d \sqrt{n} \frac{\|\Lambda_n\|_{HS}^2}{n \sigma_n^2} = 9d \|\Lambda_n\|_{op}^2 \frac{\sigma_n^2}{n^{3/2}}.
\]

Since \( \max_{1 \leq i \leq d} x_i \) is a 1-Lipschitz function of \( x \in \mathbb{R}^d \), it follows that

\[
\mathbb{E} \max_{1 \leq i \leq d} a_{ii} \geq \sigma_n \left( \mathbb{E} \max_{1 \leq i \leq d} g_i - \frac{9d \|\Lambda_n\|_{op}^2}{n^{3/2} \sigma_n^2} \right).
\]

It is well known that \( \mathbb{E} \max_{1 \leq i \leq d} g_i \geq c\sqrt{\log d} \). If we assume now that \( \|\Lambda_n\|_{op} \leq K \sqrt{n} \) then choosing \( d = \lceil \sqrt{n} \rceil \) completes the proof. \( \square \)

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