BRANCHES IN THE BRUHAT-TITS TREE FOR LOCAL FIELDS OF EVEN CHARACTERISTIC

LUIS ARENAS-CARMONA & CLAUDIO BRAVO

Abstract. We extend our previous computations for the relative positions of branches of quaternions to the case of local fields of even characteristic. This is a key step to understand the set of maximal orders containing a given suborder, which is useful, for instance, to compute relative spinor images, thus solving the selectivity problem. In our previous work, the results were given in terms of the quadratic defect. In the present context, we introduce and characterize an analogous concept for Artin-Schreier extensions. It is no longer useful to restrict our attention to orders generated by pure quaternions, as a separable quadratic extension contains no non-trivial element of null trace. In this work we state our result for an arbitrary pair of generators, for which we discuss a more general version of the Hilbert symbol in this context.

1. Introduction

Let $K$ be a local field of characteristic two. It is a known fact that $K$ is isomorphic to the field of Laurent series $\mathbb{F}_{2r}((\pi))$ over the finite field $\mathbb{F}_{2r}$ for some positive integer $r$. We assume $K = \mathbb{F}_{2r}((\pi))$ in all that follows. In particular, the ring of integers is the power series ring $O = \mathbb{F}_{2r}[[\pi]]$, $\pi$ is a uniformizing parameter, and $\mathbb{K} = \mathbb{F}_{2r}$ is the residue field of $K$. We let $\nu : K \to \mathbb{Z} \cup \{\infty\}$ be the usual valuation and, for any fractional ideal $I \subset K$, we denote by $\nu(I)$ the valuation $\nu(a)$ of any element $a$ satisfying $I = aO$. We also use the absolute value $|a| = c^{\nu(a)}$, where $c < 1$ is a fix positive real number. We let $\mathbb{M}_2(K)$ denote the ring of 2-by-2 matrices with coefficients in $K$, and we identify the scalar $\lambda$ with the scalar matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ whenever confusion is unlikely or irrelevant.

The Bruhat-Tits tree $t(K)$, also called the BTT in the sequel, is defined as a graph whose vertices are the closed balls $B$ in $K$, with an edge joining two balls if and only if one is a maximal proper sub-ball of the other. To talk with ease of pictures, we define the level of any such vertex as the integer $-r \in \mathbb{Z}$ such that $B = B^{[r]}_a$ is the ball of center $a$ and radius $|\pi^r|$. We also write $\nu(B) = r$, which is an extension of our previous notation as fractional ideals are balls centered at 0.

There is a natural bijection between the vertex set $V_{t(K)}$ of this graph and the set of maximal orders in $\mathbb{M}_2(K)$ given by equation 1 in §1.1. Let $\Omega \subseteq \mathbb{M}_2(K)$ be an arbitrary order. The set $S_K(\Omega)$ of maximal orders in $\mathbb{M}_2(K)$ containing $\Omega$ can be describe as the vertex set of a full subgraph $s_K(\Omega) \subseteq t(K)$ (c.f. [5, §1]). The graph $s_K(\Omega)$, which we call the branch of $\Omega$ in all that follows, is usually a tubular neighborhood of a line $m_K(\Omega)$, i.e., a thick line with stem $m_K(\Omega)$ as defined in §1.1 except for some specific orders, namely (c.f. [2, Prop. 5.3] and [2, Prop. 5.4]):

1. The scalar ring $\Omega = O\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cong O$, denoted just by $O$ in what follows, is contained in every maximal order.
(2) The nilpotent order \( \Omega = \mathcal{O}[u] \), where \( u \in \mathbb{M}_2(K) \setminus \{0,1\} \) is nilpotent, has a branch \( \mathfrak{g}_K(\Omega) \) called an infinite foliage (c.f. [5]), which can be described as the union of a strictly increasing sequence of thick lines with a common leaf.

**Remark 1.1.** In the context of Diophantine approximation, infinite foliages are called horoballs (c.f. [7]).

An explicit description of the graph \( \mathfrak{g}_K(\Omega) \) simplifies the study of embeddings of the order \( \Omega \) into Eichler orders, or any other intersection of maximal orders, like rank-3 orders [1, Lemma 3.2]. It also allows us to compute the spinor image and, therefore, explicitly describing spinor class fields or representation fields. Both play a central role in solving the selectivity problem [1, §2].

The purpose of the present work is to extend the results of [5] to local field with even characteristic, i.e., we intend to characterize the graphs \( \mathfrak{g}_K(\Omega) \), when \( \Omega = \mathcal{O}[q_1, q_2] \) is the order generated by two elements \( q_1, q_2 \in \mathbb{M}_2(K) \). As in our previous work, our approach includes the use of an embedding of the graph \( t(K) \), or more precisely a suitable subdivision of \( t(K) \), into the graph \( t(L) \), for a suitable finite field extension \( L/K \) that depends on \( q_1 \) and \( q_2 \) (c.f. [5, §1]). One of our main results extends, to the present setting, the explicit formulas obtained in [3] and [5] to compute the numerical invariants that describe the branch \( \mathfrak{g}_K(\Omega) \) in terms of the relations satisfied by the generators \( q_1, q_2 \in \mathbb{M}_2(K) \). In [5] we restricted ourselves to the case \( \text{tr}(q_1) = \text{tr}(q_2) = 0 \), but this is too restrictive in the present setting, as a quadratic extension might contain no non-trivial trace-zero elements. For this reason, we assume throughout that \( q_1 \) and \( q_2 \) are not scalars and that they satisfy the equations

\[
q_1^2 + a_1q_1 + b_1 = q_2^2 + a_2q_2 + b_2 = 0,
\]

\[
\Lambda(q_1, q_2) := q_1(q_2 + a_2) + q_2(q_1 + a_1) = \lambda,
\]

for some elements \( a_1, a_2, b_1, b_2, \lambda \in K \). We call the matrix \( \Lambda(q_1, q_2) \) the symmetric product by analogy with the corresponding concept in odd characteristic (c.f. [5, §2]). By an explicit computation, using the quaternion involution

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \overline{A} = \begin{pmatrix} d & b \\ c & a \end{pmatrix},
\]

we can see that \( \Lambda(q_1, q_2) = q_1q_2' + q_2q_1' \) is a scalar matrix. In fact

\[
\Lambda \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = (ad' + bc' + cb' + da') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Note that \( \overline{A} = \det(A)A^{-1} \), when \( A \in \mathbb{M}_2(K) \) is invertible. The standard generators \( q_1 = i \) and \( q_2 = j \) of an even characteristic quaternion algebra are given by \( \lambda = a_1 = 0 \) and \( a_2 = 1 \) in [1], see [8].

In our previous work, the results are given in terms of the quadratic defect. For this reason, we need both, to study the quadratic defect in even characteristic and to introduce an analog for Artin-Schreier extensions, which we call the Artin-Schreier defect in this work. Both play a significant role in the new context.

1.1. **Conventions on graphs, orders and Moebius transformations.** In all of this work, a graph \( \mathfrak{g} \) is a set of vertices \( V_\mathfrak{g} \) endowed with a symmetric relation "\(-\)" that we call the neighborhood relation. Two vertices \( v \) and \( v' \) satisfying \( v - \mathfrak{g} v' \)
are called neighbors. A subgraph of \( g \) is any graph \( h \) with a vertex set \( V_h \subseteq V_g \) that satisfies the statement
\[
\forall v, v' \in V_h : \quad [(v - h v') \Rightarrow (v - g v')].
\]
When the converse holds for every pair \((v, v') \in V_h \times V_h\), the graph \( h \) is called a full subgraph of \( g \). All subgraphs in this work are assumed to be full. The intersection of a family of full subgraphs is well defined with these conventions, and it is also a full subgraph. The valency of a vertex \( v \in V_g \) is defined as the cardinality of its set of neighbors. When every vertex in a connected graph \( V_g \) has valency two or one, we call the valency-one vertices the endpoints of \( g \). A finite walk in \( g \) is a sequence of vertices \( w = v_0v_1 \ldots v_r \), satisfying the following conditions:
1. \( v_i \in g v_{i+1} \), for \( i = 0, \ldots, r - 1 \), and
2. \( v_i \neq v_{i+2} \), for \( i = 0, \ldots, r - 2 \).

The latter is called the no-backtracking condition. We often emphasize the vertices \( v_0 \), or initial vertex, and \( v_r \), the final vertex, by saying a walk from \( v_0 \) to \( v_r \). A graph \( g \) is called connected whenever there is a walk from every vertex \( v_0 \in V_g \) to every vertex \( v_r \in V_g \). A cycle is a walk \( v_0v_1 \ldots v_m \), for which \( v_r = v_0 \). A tree is a connected graph with no cycles. Equivalently, a graph \( g \) is a tree if there exists a unique walk from \( v_0 \) to \( v_r \) for any pair of vertices \((v_0, v_r) \in V_g \times V_g\). It is easy to see that walks in a tree have no repeated vertices. All graphs considered in this work are trees. The integer \( r \) above is called the length of the walk, and written \( r =: l(w) \). We consider the trivial sequence \( v_0 \) as a walk of length 0. We also define infinite walks of two types:

- A single infinite walk is a sequence of the form \( w = v_0v_1 \ldots \), with one vertex for each natural number, satisfying (1) and (2) as above. We define \( l(w) := \infty \).
- A double infinite walk is a sequence with one vertex for each integer, i.e., \( w = \ldots v_{-1}v_0v_1 \ldots \), also assuming (1) and (2). By convention, we write \( l(w) := 2\infty \).

As usual, the double infinite walks \( \ldots v_{-1}v_0v_1 \ldots \) and \( \ldots v_{-1}v'_1v'_2 \ldots \) are considered as equal whenever there is a fixed \( m \in \mathbb{Z} \) satisfying \( v'_t = v_{t+m} \) for every \( t \in \mathbb{Z} \). We define an end of the graph as an equivalence class of single infinite walks, where \( v_0v_1 \ldots \) and \( v'_0v'_1 \ldots \) are equivalent when there is a fixed \( m \) satisfying \( v'_t = v_{t+m} \) for every sufficiently large positive integer \( t \). In pictures, we usually visualize an end as a star (⋆) at the border of the tree. When \( h \) is a full subgraph of \( g \), there is a natural identification between the set \( \partial(h) \) of ends of \( h \), on one hand, and, on the other, the set of ends of \( g \) for which a walk in \( h \) (i.e., a walk that contains only vertices in \( V_h \)) can be chosen as a representative. We exploit this identification by notational abuses of the type \( a \in \partial(h) \), for an end \( a \in \partial(g) \), or many of its verbal equivalents. It is easy to see that the ends of the BTT \( t(K) \) are naturally in correspondence with the \( K \)-points of the projective line \( \mathbb{P}^1 \) (c.f. [4] §4).

For any walk \( w \) in a tree \( g \), we define the line \( p_w \) as the smallest subtree of \( g \) containing the vertices in the sequence \( w \). If \( w \) is not explicit, we just say a line \( p \). If \( w = v_0v_1 \ldots v_r \), we also write \( p[v_0, v_r] = p_w \). Similarly, we denote by \( p(a, b) \) the graph whose vertices are precisely the vertices in a double infinite walk joining the ends \( a, b \in g \). The last one is called a maximal path, or sometimes simply a path, in the sequel. A ray \( p[v, a] \) is defined analogously. The length of a line is the length of the associated walk, and written analogously, e.g., \( l(p_w) = l(w) \). We say
that \( v_0 \) is an \( r \)-neighbor of \( v_r \) if there is a line of length \( r \) whose endpoints are precisely \( v_0 \) and \( v_r \). A tubular neighborhood \( \bar{p}^{(n)} \), of some line \( p \), is the subtree of \( \mathfrak{g} \) containing precisely the \( s \)-neighbors of vertices in \( p \), for all \( s \leq n \).

As a vertex in the BTT is a ball, which is fully determined by its radius and one center, it is immediate that \( V_{t(K)} \) can be identified with a subset of \( V_{t(L)} \), for any finite field extension \( L/K \). In what follows, a vertex in \( V_{t(L)} \) is said to be defined over \( K \), if it corresponds to a vertex in \( V_{t(K)} \). This definition extends naturally to intermediate fields. However, this identification cannot define a morphism of graphs unless \( L/K \) is an unramified extension. We fix this problem by a normalization of the distance function. For any pair of vertices \((v, v') \in V_{t(L)} \times V_{t(L)}\), we define their distance by \( \delta(v, v') = \frac{1}{e(L/K)} \ell(p[v, v']_L) \), where \( p[v, v']_L \) is the corresponding line in \( t(L) \), and \( e(L/K) \) denotes the ramification index. This distance is independent of the field \( L \). We also normalize the valuation on \( L \) in a similar fashion. For example, a uniformizing parameter \( \pi_L \) of \( L \) has the valuation \( \nu(\pi_L) = \frac{1}{e(L/K)} \). A similar convention applies to the absolute value. A real number is said to be defined over an intermediate field \( F \) if it is a multiple of the valuation of the corresponding uniformizer \( \nu(\pi_F) \).

We apply this definition to both, valuations of elements or distance between vertices or subgraphs. Next result is trivial but useful:

**Lemma 1.2.** Let \( v \) and \( v' \) be two vertices of \( t(L) \) that are both defined over an intermediate field \( F \). Then, a vertex \( v'' \) in the line \( p[v, v']_L \) is defined over \( F \) if and only if its distance to either endpoint is defined over \( F \).

A line with endpoints defined over \( F \) is said to be defined over \( F \). The same applies to maximal paths and rays, replacing endpoints by balls if needed. Let \( p(a, b)_L \) be the unique maximal path in \( t(L) \) whose ends are \( a, b \in \mathbb{P}^1(L) \). When the extension \( L/K \) is Galois, the group \( \text{Gal}(L/K) \) acts on either set \( \mathbb{P}^1(L) \) or \( t(L) \). These two actions are compatible, i.e., paths in \( t(L) \) satisfy the relation

\[
\sigma(p(a, b)_L) = p(\sigma(a), \sigma(b))_L, \quad \forall \sigma \in \text{Gal}(L/K), \forall a, b \in \mathbb{P}^1(L)
\]

(c.f. [5 §3]). There is also a natural action of the group \( \mathcal{M}(L) \) of Möbius transformations on the set of balls. For every element \( \mu \in \mathcal{M}(L) \) and every ball \( B \subseteq L \), we define:

1. \( \mu * B = \mu(B) \), if the latter is a ball, while
2. \( \mu * B \) is the smallest ball properly containing \( \mu(B^c) \), if \( \mu(B^c) \) is a ball.

The complement above is defined in \( \mathbb{P}^1(L) \), so \( \infty \in B^c \) for any ball \( B \). In fact, if we remove the ball \( B \), as a vertex, from the BBT, the remaining graph has \( \kappa + 1 \) connected components, where \( \kappa \) is the cardinality of the residue field of \( L \), and their corresponding sets of ends are the sets in the decomposition \( \mathbb{P}^1(L) = B^c \cup B_1 \cup \cdots \cup B_r \), where \( B_1, \ldots, B_r \) are all maximal proper sub-balls of \( B \). The action by Möbius transformation permute balls by permuting such decompositions. See [4] or [5] for details. We also have a compatibility property \( \sigma(\mu * B) = \sigma(\mu) * \sigma(B) \), for any element \( \sigma \in \text{Gal}(L/K) \), any transformation \( \mu \in \mathcal{M}(L) \) and any ball \( B \subseteq L \).

Following [5], vertices in \( t(L) \) that are not defined over \( K \) are called here ghost vertices. Any maximal path \( p(a, b)_L \subseteq t(L) \) defined over \( K \) is identified with the corresponding path in \( t(K) \). Maximal paths that are not of this form are called ghost paths.
To every ball $B = B^{|v(\rho)|}_a$ with $a, \rho \in L$, we associate the maximal order
\begin{equation}
\mathcal{D}_B = \text{End}_{\mathcal{O}_L} \left[ \left\{ \left( \begin{array}{c} a \\ 1 \\
\end{array} \right), \left( \begin{array}{c} \rho \\ 0 \\
\end{array} \right) \right\} \right].
\end{equation}

This defines a one-to-one correspondence between balls and maximal orders that is assumed in all that follows. It is compatible with field extensions when the order $\mathcal{D} \subseteq M_2(K)$ is identified with $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}_L \subseteq M_2(L)$. The branch $s_K(\Omega)$ is defined as the largest full subgraph whose vertices are balls corresponding to maximal orders containing $\Omega$. As usual, we write $\mathcal{s}_K(q_1, \ldots, q_n)$ instead of $s_K(\Omega)$ if $\Omega = \mathcal{O}[q_1, \ldots, q_n]$ is the order generated by $q_1, \ldots, q_n$. Clearly $\mathcal{s}_K(q_1, \ldots, q_n) = \bigcap_{i=1}^n s_K(q_i)$. With these conventions, the vertex set $V_{s_K(\Omega)}$ can be identified with $V_{s_L(\Omega)}(\Omega)$, where $\Omega = \mathcal{O}_L \otimes_{\mathcal{O}} \Omega$. The same applies to other notations like $s_K(q_1, \ldots, q_n)$ and $s_L(q_1, \ldots, q_n)$.

A thick line $p$ is defined over $K$ if its stem and its depth are defined over $K$. Similarly, an infinite foliage is defined over $K$ if its end and, at least, one leaf (valency-one vertex) is defined over $K$. The following results are easy consequence of Lemma 1.2.

**Lemma 1.3.** A thick line is defined over $K$ if and only if the stem and, at least, one leaf are defined over $K$.

**Lemma 1.4.** If two thick lines or infinite foliages, or one of each, are defined over $K$, then the (unique) smallest path from one to the other is defined over $K$.

### 2. Main results

For any element $a \in K$, we denote by $p_a(X) = X^2 + X + a$ the corresponding Artin-Schreier polynomial. Note that
\begin{equation}
p_{a+b}(x + y) = p_a(x) + p_b(y) \quad \text{and} \quad p_a(0) = a.
\end{equation}

We define the Artin-Schreier defect $\mathbb{D}(a)$ as the fractional ideal $\mathbb{D}(a) = \bigcap_{h \in K} (p_a(h))$. We define the quadratic defect $\delta(a)$, as usual, by the formula $\delta(a) = \bigcap_{h \in K} (h^2 + a)$ (c.f. [6 §63:A]). The following result is an analog for the Artin-Schreier defect of the classical characterization of the quadratic defect in terms of quadratic extensions (c.f. [6 §63:A & Theo. 63.4]).

**Theorem 2.1.** The image of the Artin-Schreier defect is the set:
\[ S = \{(0), \mathcal{O}\} \cup \{(\pi^{-2t+1}) : t > 0\}. \]

Furthermore, if an element $b$ in the algebraic closure $\overline{K}$ satisfies $p_0(b) = a$, then the following statements hold:

i.- $\mathbb{D}(a) = \{0\}$ if and only if $b \in K$.

ii.- $\mathbb{D}(a) = \mathcal{O}$ if and only if $b$ generates an unramified quadratic extension of $K$.

iii.- $\mathbb{D}(a) = (\pi^{-2t+1})$, for some $t > 0$, if and only if $b$ generates a ramified quadratic extension of $K$.

For the sake of uniformity, we set $\mathfrak{f}_K(\Omega) = s_K(\Omega)$ if $s_K(\Omega)$ is an infinite foliage. This is not perfect, but seems to be the alternative that makes statements simpler in Theorem 2.2 below. By the stem length of a thick line we mean the length of its stem. By convention, we set the stem length of an infinite foliage as $\infty$. As before, this choice is not perfect. For a matrix $q$ that is integral over $\mathcal{O}$ we write $l(q)$ for the stem length of its branch, which is always in the set $\{0, 1, \infty, 2\infty\}$ [2].
polynomials, and let $q_0$ and $A$ satisfying the hypotheses of the preceding theorem. This can be answered as follows:

Let $A$ and $B$ denote the largest integer not exceeding $q_0$. Then, if $q_0 = \nu(q) + 1$, it equals the distance between the stems, otherwise the length of their intersection is $\min\{-2d_f, l(q_1), l(q_2)\}$, except in the following cases:

- When $m_1$ and $m_2$ are both reducible inseparable polynomials, and $\lambda \neq 0$, the depth of $m_K(q_1) \cap m_K(q_2)$ is $\frac{\nu(\lambda)}{2}$. Here the stem of $m_K(q_1) \cap m_K(q_2)$ is a vertex if $\nu(\lambda)$ is even and an edge otherwise. If $\lambda = 0$, then either $m_K(q_1) \subseteq m_K(q_2)$ or $m_K(q_2) \subseteq m_K(q_1)$.

- When $m_1$ and $m_2$ are both reducible separable polynomials and $d_f = -\infty$, the formula predicts an intersection of length $2\infty$, a maximal path. However, if $q_1 q_2 \neq q_2 q_1$, we get a ray instead.

A natural question that arises at this point is whether there exist matrices satisfying the hypotheses of the preceding theorem. This can be answered as follows:

| $Y_1$ | $Y_2$ | $d_f$ |
|-------|-------|-------|
| $A^s$ | $A^s$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_0^2 a_1^2} \right)$ |
| $A^s$ | $A^t$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_1^2} \right) - t_2$ |
| $A^t$ | $A^s$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_0^2} \right) + t_2$ |
| $A^t$ | $B^s$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_1 a_2^2} \right) - t_2$ |
| $B^s$ | $B^s$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_1 a_2^2} \right) - t_1$ |
| $B^s$ | $A^t$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_1^2} \right) - t_1 - t_2$ |
| $B^t$ | $B^s$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_0^2} \right) - t_1$ |
| $B^t$ | $A^s$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_0^2} \right) + t_1 - t_2$ |
| $B^s$ | $B^t$ | $-\frac{1}{2} \nu \left( \frac{\Delta}{a_1 a_2^2} \right) + t_1$ |

Table 1. The value of $d_f$ for $m_1 \in Y_1$ and $m_2 \in Y_2$. 

We say that $f(X) \in K[X]$ is ramified (respectively, unramified) if its decomposition field is ramified (respectively, unramified) over $K$. Similarly, we say that $f(X) \in K[X]$ is separable if its roots in $\overline{K}$ are different. In any other case, we say that $f(X) \in K[X]$ is inseparable. For example, the polynomial $f(X) = X^2 + cX + d$ is separable if and only if $c \neq 0$. We define $A^s$ as the set of either reducible or unramified separable irreducible polynomials and $B^s$ as the set of ramified separable irreducible polynomials. The sets $A^t$ and $B^t$ are defined analogously in the inseparable case. Note, however, that there is no irreducible inseparable unramified polynomial. Let $[a]$ denote the largest integer not exceeding $a$.

**Theorem 2.2.** Let $m_i(X) = X^2 + a_i X + b_i \in \mathcal{O}[X]$, for $i \in \{1, 2\}$, be quadratic polynomials, and let $q_1, q_2 \in M_2(K) \backslash K$ be matrices satisfying $m_1(q_1) = 0$, $m_2(q_2) = 0$ and $q_1 m_2 + q_2 m_1 = \lambda$. Consider $\Delta = \Delta(\lambda, m_1, m_2) = \lambda^2 + a_1 a_2 \lambda + a_1^2 b_2 + a_2^2 b_1$, and let $d_f$ be the fake distance defined by cases in Table 1, where we use the following convention:

$$
I \equiv \left\lfloor \frac{\nu(I) - 1}{2} \right\rfloor, \quad \text{where } I = \left\lfloor \frac{b}{\nu(b)} \right\rfloor \delta(b_i) \quad \text{if } m_i \text{ is separable}
$$

Then, if $d_f > 0$, it equals the distance between the stems, otherwise the length of their intersection is $\min\{-2d_f, l(q_1), l(q_2)\}$, except in the following cases:

- When $1$ and $m_2$ are both reducible inseparable polynomials, and $\lambda \neq 0$, the depth of $m_K(q_1) \cap m_K(q_2) = \frac{\nu(\lambda)}{2}$. Here the stem of $m_K(q_1) \cap m_K(q_2)$ is a vertex if $\nu(\lambda)$ is even and an edge otherwise. If $\lambda = 0$, then either $m_K(q_1) \subseteq m_K(q_2)$ or $m_K(q_2) \subseteq m_K(q_1)$.

- When $m_1$ and $m_2$ are both reducible separable polynomials and $d_f = -\infty$, the formula predicts an intersection of length $2\infty$, a maximal path. However, if $q_1 q_2 \neq q_2 q_1$ we get a ray instead.
Theorem 2.3. Let $\lambda \in K$, $m_1$, $m_2$ and $\Delta$ be as in the Theorem 2.2. We define, for $x, y, z, w \in K$, with $y, w \neq 0$, the expression

$$C(x, y, z, w) = \frac{1}{y^w} \left[ y^2 m_1 \left( \frac{z}{y} \right) + w^2 m_2 \left( \frac{z}{w} \right) + a_1 z y + a_2 z w \right].$$

Then, there exist linearly independent elements $q_1, q_2 \in \mathbb{M}_2(K) \setminus K$ satisfying the identities in Equation (1) if and only if any of the following conditions holds:

i. $\Delta \neq 0$, and at least one polynomial, $m_1$ or $m_2$, has a zero in $K$.

ii. $\Delta \neq 0$, and there exist two pairs $(x, y)$ and $(z, w)$ in $K \times K^*$ satisfying $C(x, y, z, w) = \lambda$.

iii. $\Delta = 0$, $a_1 \neq 0$ and $m_1$ has a zero in $K$.

iv. $\Delta = 0$, $a_2 \neq 0$ and $m_2$ has a zero in $K$.

v. $\Delta = 0$, $a_1 = a_2 = 0$.

In the last case, however, the matrices $q_1$ and $q_2$ are contained in a two dimensional subalgebra.

If $K$ were not a perfect field, we would need an extra condition

$$[K(\sqrt{b_1}, \sqrt{b_2}) : K] \leq 2$$

in item (v) above (see §8). To prove this theorem, we need to study the semisimplicity of certain $K$-algebras. This is essentially a generalization of [8, Ch. II, Th. 1.3] that plays the role of the Hilbert symbol in even characteristic.

3. Artin-Schreier and quadratic defects

Let $a \in K$, and let $p_a(X) = p_0(X) + a$ be the Artin-Schreier polynomial (c.f. §2). We write $P_0 = p_0(K)$ for the image of $p_0$, which is an additive subgroup of $K$. Note that $b \mapsto p_a(b)$ is a continuous function on the locally compact space $K$ satisfying $\lim_{b \to \infty} p_a(b) = \infty$. Next result is straightforward from §5 and the observation that $\mathbb{D}(a) \subseteq \left( p_a(h) \right)$ for any $h \in K$:

Lemma 3.1. We have $\mathbb{D}(a + b) \subseteq \mathbb{D}(a) + \mathbb{D}(b)$, for any pair of elements $a, b \in K$.

In particular, $\mathbb{D}(a) = \mathbb{D}(a + c)$ for any $c \in P_0$. Also, for any element $a \in K$, there exists $h = h(a)$ satisfying $\mathbb{D}(a) = \left( p_a(h) \right)$.

Example 3.2. $p_\pi(X)$ has a zero in $K$, since the image $\overline{p}_\pi(X) = X(X + 1) \in \mathbb{K}[X]$ has two different roots in the residue field $\mathbb{K}$, and Hensel’s Lemma applies. We conclude that $\mathbb{D}(\pi) = \{0\}$.

Proof of Theorem 2.1. Let $s \in \mathbb{Z} \cup \{\infty\}$, and set $\pi^\infty = 0$. By Lemma 3.1 if $\mathbb{D}(a) = \langle \pi^s \rangle$, there exist $b \in K$ satisfying $p_a(b) = u\pi^s$, for some $u \in \mathcal{O}^*$. In particular $p_a + u\pi^s(b) = 0$, i.e., $\mathbb{D}(a + u\pi^s) = \{0\}$. If $s > 0$, or if $s = \infty$, we conclude, reasoning as in Example 3.2 that the polynomial $p_{a\pi^s}(X) = p_0(X) + u\pi^s$ has a root in $K$, in particular $\mathbb{D}(u\pi^s) = \{0\}$. It follows from Lemma 3.1 that $\mathbb{D}(a) = \{0\}$. Now, assume that $s = -2t$, for $t \in \mathbb{Z}_{>0}$, and write $u = b_0 + e$, where $b_0 \in \mathbb{F}_2^*$, and $|e| < 1$. As $\mathbb{F}_2^t$ is a perfect field, we have $b_0 = a_0^2$, for some $a_0 \in \mathbb{F}_2^t$. This implies that

$$p_a(b + a_0\pi^{-t}) = p_a(b) + p_0(a_0\pi^{-t}) = \pi^{-2t}e + a_0\pi^{-t}$$

has a larger valuation than $p_a(b)$. This contradicts the fact that $\mathbb{D}(a) = \left( p_a(b) \right)$. Now, we assume that $s = -2t + 1$, where $t > 0$. Let $L$ be the decomposition field
of

\[ p_\alpha(X) = p_0(X + b) + p_\alpha(b) = p_0(X + b) + u\pi^{-2t+1}, \]

and let \( \alpha \in L \) be one of its roots. Then \( \pi^t(\alpha + b) \) satisfies the Eisenstein polynomial

\[ q(X) = X^2 + \pi^tX + u\pi. \]

This implies that \( L \) ramifies over \( K \). On the other hand, for an arbitrary \( t > 0 \), we let \( a = \pi^{-2t+1} \). It is clear that \( \mathbb{D}(a) \subseteq (\pi^{-2t+1}) \). Suppose that equality fails to hold. In particular, \(|p_\alpha(b)| < |\pi^{-2t+1}|\), and by Dominance Principle we have \(|b(b + 1)| = |p_\alpha(b) + a| = |\pi^{-2t+1}| > 1\). This implies that \(|b| = |b + 1| > 1\), and then \(|b^2| = |p_0(b)| = |\pi^{-2t+1}|\), which is absurd. We conclude that \( \mathbb{D}(a) = (\pi^{-2t+1}) \). This proves that \((\pi^{-2t+1})\) is in the image of the Artin-Shreier defect for any positive integer \( t \). Finally, we assume that \( \mathbb{D}(a) = \mathcal{O} \) and choose \( L \) and \( \alpha \) as before. Then \( \alpha + b \) satisfy the polynomial \( p_\alpha(X) = X^2 + X + u \).

Note that \( \mathbb{D}(a) = \mathcal{O} \) is the smallest possible non trivial Artin-Shreier defect, and it is attained when the roots of \( p_\alpha(X) \) generate an unramified quadratic extension. These elements \( a \) play an analog role to that of units of minimal quadratic defect for a local field \( K \) with odd characteristic.

Let \( \delta(a) = \bigcap_{b \in \mathcal{O}(h^2 + a)} \bigcup_{i = \pm 2N} a_i\pi^i \), where \( a_i \in \mathbb{F}_{2^r} \). Set \( a_i = b_i^2 \), which can always be done, since \( \mathbb{F}_{2^r} \) is perfect. If \( \xi = \bigcup_{i = \pm 2N} b_i^2\pi^i \), then \( a + \xi^2 \) has a nonzero coefficient only for odd exponents \( \pi^i \). In particular, either \( a \) is a square of \( K \), or otherwise \(|a + \xi^2| = |\pi|^{2t+1}\), for some \( t \in \mathbb{Z} \). Note that this absolute value is optimal by the Dominance Principle, since squares cannot have an odd valuation. Next result follows:

**Proposition 3.3.** The image of the quadratic defect \( \delta \) is exactly the set \( \{ (\pi^{2t+1}) : t \in \mathbb{Z} \} \), and we have \( |\delta(a)| \leq |a| \), for any element \( a \in K \). Furthermore, there exists \( b \in K \) satisfying \( |\delta(a)| = |\delta(a + b^2)| = |a + b^2| \).

### 4. Branches and Stems in characteristic 2

Let \( q \in M_2(K) \) be a matrix that is integral over \( \mathcal{O} \), i.e., its irreducible polynomial has integral coefficients. We say that \( q \) is separable if \( m_q(X) = \text{irr}_{q,K}(X) = X^2 + cX + d \in \mathcal{O}[X] \) is separable, i.e: \( c \neq 0 \). Otherwise, we say that \( q \) is inseparable. Suppose that \( m_q(X) \) is separable and splits over \( K \). Let \( \alpha \) be a root of \( m_q(X) \). Note that \( \alpha + c \) is the other root. Note also that \( w = \frac{2 \alpha}{\pi} \in M_2(K) \) is an idempotent. Then \( s_{K}(q) = s_{K}(w)^{[\nu(c)]} \), and \( s_{K}(w) \) is a path in \( \mathbb{M}(K) \) (c.f. [5, §4]). This implies that \( s_{K}(w) \) is the stem of \( s_{K}(q) \).

We assume next that \( m_q(X) \in \mathcal{O}[X] \) is separable and irreducible. Let \( L = K(\alpha) \) be the splitting field, where \( \alpha \) is a root of \( m_q \). Define \( w \in M_2(L) \) as above. Since \( \text{Gal}(L/K) \) permutes transitively the ends of the path \( s_L(w) \) (c.f. [5, §5]), it follows that these ends are \( x + y\alpha \) and \( x + y(\alpha + c) \), for some \( x, y \in K \). Formula (4.1) in [5] tell us that the two idempotents satisfying \( s_{L}(\tau) = p(a, b)L \) are \( \tau = \frac{b}{b - a} \left( \begin{array}{cc} 1 & -ab \\ -1 & a \end{array} \right) \) and \( 1 - \tau \) (in any characteristic). From here, a straightforward computation show that the two possible choices for \( q \) are \( q = \beta(x, y, c, d) = \frac{1}{b - a} \left( \begin{array}{cc} x & x^2 + cy + dy^2 \\ 1 & x + yc \end{array} \right) \) and \( q = \beta(x, y, c, d) + c \). We define \( v(\xi) = p(\xi, \infty) \), for every \( \xi \in K \), and call it the \( K \)-vine of \( \xi \). The nearest \( K \)-vine \( v(\xi) \) from \( s_{L}(w) \), which we call the \( K \)-vine of \( w \), minimizes the expression \(|x + \xi + y\alpha| = |x + \xi + y(\alpha + c)| \). See Fig. III(A) and Fig.
\[ a = 0, \text{ since } N \]

Assume first that \( s \mapsto \eta \) is \( \epsilon \)-tent elements above is completely determined by the end \( a \) in (B). Distances are normalized.

In other words, \( \xi \in K \) satisfies the identity
\[
|x + \xi + y\alpha|^2 = |x + \xi + y\alpha||x + \xi + y(\alpha + c)|
\]
\[
= |yc|^2 \left( \frac{x + \xi}{cy} \right)^2 + \left( \frac{x + \xi}{cy} \right) + \frac{d}{c^2} \right| = |yc|^2 \left| d \left( \frac{d}{c^2} \right) \right|.
\]

If \( p \neq \pi \) is unramified, we have \( |x + \xi + y\alpha| = |yc| \), by Theorem 2.1. We conclude that the depth of \( s_K(q) \) is \( \nu(c) \), and its stem is the highest vertex in \( s_L(w) \), as in Fig. (A). On the other hand, if \( p = \pi \) is ramified, we have \( |x + \xi + y\alpha| = |yc| \cdot |\pi|^{-t + \frac{3}{2}} \), where \( D \left( \frac{d}{c^2} \right) = (\pi^{-2t+1}) \) and \( t > 0 \). Then, the stem of \( s_K(q) \) is the line \( p[v_0, v_1] \) in Fig. (B), and the depth of this branch is \( \nu(c) - t \), which is non negative since \( \nu \left( \frac{d}{c^2} \right) \leq -2t + 1 \), so \( \nu(c) - t \geq \frac{1}{2} \nu(d) - 1 \geq -\frac{1}{2} \), as \( d \in \mathcal{O} \). Note that, in the figure, \( v_0 \) and \( v_1 \) are defined over \( K \), but \( v_{1/2} \) is not.

Now, we assume that \( m_{q}(X) = X^2 + b \), and that there exist \( \alpha \in \mathcal{O} \) satisfying \( \alpha^2 = b \). In this case \( \eta = q + \alpha \in \mathbb{M}_2(K) \) is a nilpotent element. This implies that \( s_K(q) = s_K(\eta) \) is an infinite foliage (c.f. \( 2 \) \( \S \) 2). Let \( f \) be an infinite foliage whose end is \( a \in \mathbb{P}^1(K) \). If \( a \in K \), there is a unique leaf \( B \in V_f \) of the form \( B = B[^s_a] \). See Fig. (A). This is called the highest leaf of \( f \). Note that every infinite foliage in the BTT is determined by its unique end and one leaf, so the infinite foliage described above is completely determined by the end \( a \) and the number \( s \), and it is denoted \( f(a, s) \). When \( a = \infty \), every leaf has the form \( B = B[^s_a] \), with \( s \in K \). In the latter case we say \( f = f(\infty, s) \). See Fig. (A').

**Lemma 4.1.** The set \( \mathbb{N}_{a,t} = \mathcal{O}^* \left( \begin{array}{cc} a \pi^{-t} & a^2 \pi^{-t} \\ \pi^{-t} & a \pi^{-t} \end{array} \right) \) consists precisely of all nilpotent elements \( \eta \in \mathbb{M}_2(K) \) satisfying \( s_K(\eta) = f(a, t) \). On the other hand, the set \( \mathbb{N}_{\infty,t} = \mathcal{O}^* \left( \begin{array}{cc} \pi^{-t} & \pi^t \\ 0 & \pi^t \end{array} \right) \) consists precisely of all nilpotent elements \( \eta \in \mathbb{M}_2(K) \) satisfying \( s_K(\eta) = f(\infty, t) \).

**Proof.**
Assume first that \( a \neq \infty \). Applying a Moebius transformation \( \mu \) of the form \( z \mapsto z + a \), which corresponds to the matrix \( \mu = \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \), we can assume that \( a = 0 \), since \( \mu^{-1} \mathbb{N}_{a,t} \mu = \mathbb{N}_{0,t} \). In this case we have \( \eta \in \left( \begin{array}{cc} \mathcal{O}^* & \pi^t \mathcal{O}^* \\ \pi^{-t} \mathcal{O}^* & \mathcal{O}^* \end{array} \right) \), for every
satisfies \( q \) conjugate of the pair \((m, m)\) of polynomials \( z \) and \( m \). It follows that \( \eta = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \), for some \( u \in \mathcal{O}^* \). The case \( a = \infty \) is analogous. \( \square \)

We assume \( m_q(X) \) is inseparable and irreducible. Let \( L/K \) be the splitting field of \( m_q(X) \) and let \( \alpha \in L \) be a root. Observe that \( \mathfrak{s}_K(q) \) is finite, as \( \mathcal{O}[q] \) is an integral domain (c.f. [2, §4]). In particular, \( \mathfrak{s}_L(q) \) is an infinite foliage whose unique end is \( z = x + y \alpha \), as in Fig. 2(A), is in \( L/K \). Assume \( x \in K \) and \( y \in K^* \), so that \( \mathfrak{s}_L(q) = \{ (z, s) \} \), i.e., \( B = B^s \) is a leaf.

By Lemma 4.1 we have \( q = \left( \frac{(x + y \alpha)u \pi_L^s + \alpha}{u \pi_L^s} \right) \left( \frac{u^2 + y^2 b}{u \pi_L^s + \alpha} \right) \), for some \( u \in \mathcal{O}_L \). As \( q \in \mathbb{M}_2(K) \), we conclude that \( u \pi_L^s \in \mathcal{O}_K \), and therefore also that \( y = u^{-1} \pi_L^s \). Hence \( q = \beta'(x, y, b) \) and \( B = B^{\upsilon(y)} \). On the other hand, the \( K \)-vine \( \upsilon(\xi) \) of \( \eta = q + \alpha \), which minimizes \(|x + \xi + y \alpha|\), by definition satisfies

\[
|x + \xi + y \alpha|^2 = |y|^2 \left( \frac{x + \xi}{y} \right)^2 + b = |y|^2 |\delta(b)|.
\]

Since \( \delta(b) = (\pi^{2t+1}) \), it follows that \( |x + \xi + y \alpha| = |y| \cdot |\pi|^{t+\frac{1}{2}} \), where \( t \in \mathbb{N} \). We conclude that the depth of \( \mathfrak{s}_K(q) \) equals \( t \), and its stem is the line \( p|v_0, v_1| \) in Fig. 2(B), whose midpoint \( v_{1/2} = B^s \left( \frac{v(y)^{t+\frac{1}{2}}}{s} \right) \) is a ghost vertex.

4.1. Computing symmetric products for split polynomials. The purpose of this subsection is to provide Table 2, which allows us to easily compute the symmetric product \( \lambda = \Lambda(q_1, q_2) \) and the discriminant \( \Delta \) from the branches of either quaternion in most cases. This can be done up to some arbitrary units \( u \) and \( v \), but they are irrelevant in our main proofs. We assume everywhere that the polynomials \( m_1, m_2 \in K[X] \) split, since we can otherwise compute in a suitable extension where they do. Let \( \alpha_i \) be a root of \( m_i \), for \( i \in \{1, 2\} \), and assume the matrices \( g_1, g_2 \in \mathbb{M}_2(K) \) satisfy \( m_1(q_1) = m_2(q_2) = 0 \). Assume everywhere that neither \( q_1 \) nor \( q_2 \) is a scalar. Recall that \( \lambda = \Lambda(q_1, q_2) \), as defined in (1), is a scalar matrix, and therefore a conjugation invariant. This allows us to choose a convenient conjugate of the pair \((q_1, q_2)\) to compute \( \lambda \), so in particular it allows us to apply Moebius transformations to move the stems to one of the pairs in the table. For
Table 2. The value of $\nu(\lambda)$ for the particular branches described in §4.1. Here, $u, v \in O^*$ are arbitrary.

| $m_1$ | $m_2$ | $m_K(q_1)$ | $m_K(q_2)$ | $\lambda$ | $\Delta$ |
|-------|-------|-------------|-------------|---------|---------|
| sep.  | sep.  | $p(0, \infty)$ | $p(1, \theta)$ | $q_1(a_2 + a_2 \alpha + \frac{a_2 \bar{\alpha}}{1+\alpha})$ | $\bar{\alpha}^2(\bar{\alpha} \bar{\alpha})$ |
| sep.  | sep.  | $p(0, \infty)$ | $p(0, \infty)$ | $a_2(a_2 + a_2 \alpha)$ | 0 |
| insep. | insep. | $f(\infty, 0)$ | $f(0, \theta)$ | $\alpha \eta + \alpha^2 \theta$ | 0 |
| insep. | insep. | $f(\infty, 0)$ | $f(\infty, \theta)$ | 0 | 0 |
| insep. | insep. | $p(0, 1)$ | $f(\infty, r)$ | $a_2(a_2 + \eta \bar{\alpha})$ | $\Delta^2(\alpha \alpha)$ |
| insep. | insep. | $p(0, \infty)$ | $f(\infty, r)$ | $a_2$ | 0 |

Assume that $m_1$ and $m_2$ are both separable. Let $w_i = \frac{a_i + \alpha i \bar{a}_i}{a_i}$ in $M_2(K)$, for $i \in \{1, 2\}$, be the corresponding idempotents. Note that the quaternion involution satisfies $\overline{q_1} = q_1 + a_2$, $\overline{w_i} = \frac{\overline{a_i} - \alpha i \bar{a}_i}{a_i} = 1 + w_i$ and $\Lambda(q_1, q_2) = \Lambda(q_1, q_2) = \lambda + a_1 a_2$. For any idempotent $w$, the only other idempotent with the same branch is $\overline{w} = 1 + w$. There is, therefore, some ambiguity in the choice of $q_1$ and $q_2$, but this coincides, in fact, with the ambiguity in the choice of the roots in the expression for $\lambda$. As $\Delta(\lambda, m_1, m_2) = \Delta(\lambda + a_1 a_2, m_1, m_2)$, this has no effect on the computation of $\Delta$.

If the stems $s_K(w_1)$ and $s_K(w_2)$ fail to coincide, we can assume that they are the maximal paths $s_K(w_1) = p(0, \infty)$ and $s_K(w_2) = p(1, \theta)$, respectively (c.f. §4), where $\theta$ could be $\infty$. As in the preceding reference, we can assume $w_1 = \left( \begin{array}{cc} a_2 & \alpha \\ 0 & 1 \end{array} \right)$ and $w_2 = \left( \begin{array}{cc} \frac{a_2 \bar{a}_2 - \alpha a_1}{1+\alpha} & \alpha a_1 \\ \alpha a_1 & \frac{a_2 \bar{a}_2 - \alpha a_1}{1+\alpha} \end{array} \right)$. This leads to the choice $q_1 = A(a_1, \alpha_1)$ and $q_2 = \left( \begin{array}{cc} \frac{\alpha a_1 + \alpha_2}{1+\alpha_1} & \alpha_2 \\ \alpha_2 & \frac{\alpha a_1 + \alpha_2}{1+\alpha_1} \end{array} \right)$. On the other hand, if $s_K(w_1) = s_K(w_2)$, we can assume this branch is the maximal path $p(0, \infty)$. In the latter case we assume $w_1 = w_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$, so that $q_1 = A(a_1, \alpha_1)$ and $q_2 = A(a_2, \alpha_2)$. Now a straightforward computation gives us the first two lines of the table.

Now, we assume that $m_1$ and $m_2$ are both inseparable. Let $\gamma_i = q_i + \alpha_i \bar{a}_i$ in $M_2(K)$, for $i \in \{1, 2\}$, be the associated nilpotent elements. Moebius transformations act transitively on pairs $(a, b) \in \mathbb{P}^1(K) \times \mathbb{P}^1(K)$ with $a \neq b$. Therefore, applying a Moebius transformation if we needed, we can assume that the ends of $s_K(\gamma_1)$ and $s_K(\gamma_2)$ are 0 and $\infty$, unless they coincide. In the latter case we assume both to be $\infty$. Hence, we assume $s_K(\gamma_1) = f(\infty, 0)$, in either case, while we assume either $s_K(\gamma_2) = f(0, s)$ or $s_K(\gamma_2) = f(\infty, s)$, according to the case. Equivalently, we assume $\eta_1 = A'(u, 0)$ and we assume either $\eta_2 = A'(v \pi^- s, 0)$ or $\eta_2 = A'(v \pi^+ s, 0)$, for some $u, v \in O^*$. This gives us the third and fourth lines.

Finally, assume that $m_1$ is separable and $m_2$ is inseparable. Let $w_1 = \frac{a_2 + \alpha_1}{a_1}$ as before. Suppose first that the end of $s_K(q_2)$ is not an end of $s_K(w_1)$. Then, as Moebius transformations act transitively on triples of different ends, we can assume $s_K(q_2) = f(\infty, r)$ and $s_K(w_1) = p(1, 0)$. This implies that $q_1 = \left( \begin{array}{cc} 0 & a_1 \\ a_1 & a_1 + a_1 \end{array} \right)$ and $q_2 = A'(u \pi^+ s, \alpha_2)$, for some $u \in O^*$. On the other hand, if $s_K(w_1)$ and $s_K(q_2)$ have a common end, then we can choose $q_2$ as before, while we assume $s_K(w_1) = p(0, \infty)$, and hence $q_1 = A(a_1, \alpha_1)$. The result follows.
5. Proof of Theorem 2.2 in the inseparable case

We denote, as usual, the depth and diameter of the branch $u$ (of an order) by $p(u)$ and $d(u)$, respectively (c.f. §5). In all of §5, $m_i$, $q_i$, $\alpha_i$ and $\lambda$ are as in §4.1, except that the polynomials $m_1$ and $m_2$ are assumed to be inseparable, whence $a_1 = a_2 = 0$ in the whole section. Denote by $\eta_i = q_i + \alpha_i$ the associated nilpotent for either value of $i \in \{1, 2\}$. We make frequent use of next result:

**Lemma 5.1.** [3] Prop. 2.4] Let $f_1, f_2$ be two infinite foliages, and let $d = d(f_1 \cap f_2)$. Then, if $d < \infty$, we have $p(f_1 \cap f_2) = \lfloor \frac{d}{2} \rfloor$ and the stem of $f_1 \cap f_2$ is a vertex when $d$ is even and an edge otherwise. If $d = \infty$, then $f_1 \subset f_2$ or $f_2 \subset f_1$.

If $m_1$ has a root $\alpha_1 \in K$, we assume $\eta_1 = q_1 + \alpha_1 = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$, whence $s_K(q_1) = s_K(\eta_1) = \tilde{f}(\infty, 0)$. Furthermore, if both $m_1$ and $m_2$ have roots in $K$, and if the branches $s_K(q_1)$ and $s_K(q_2)$ have different ends, we can assume also $s_K(q_2) = \tilde{f}(0, s)$. Then, if $s_K(q_1) \cap s_K(q_2) = \emptyset$, we have $s > 0$ and the distance between these branches is $s = -\nu(\lambda) = -\frac{1}{2} \nu(\Delta)$, by Table 2 which equals $d_f$ by Table 1. See Fig. 5(A). On the other hand, if $u = s_K(q_1) \cap s_K(q_2)$ is not empty, then its diameter is $d(u) = \nu(\lambda)$ (see Fig. 5(B)), so Lemma 5.1 implies that $p(u) = \lfloor \frac{\nu(\lambda)}{2} \rfloor$ and the stem of $u$ has one or two vertices depending on the parity of $\nu(\lambda)$. From Table 2 we know that $s_K(q_1)$ and $s_K(q_2)$ have the same end precisely when $\lambda = 0$. In the latter case one branch is contained in the other.

Now we assume that $m_1$ has a root in $K$ but $m_2$ does not. In this case we still assume that $s_K(q_1) = \bar{f}(\infty, 0)$. Let $L = K(\sqrt{t_2})$. We denote by $\mu_2 = x + y\sqrt{t_2}$ the unique end of $s_L(q_2) = \bar{f}(\mu_2, \nu(y))$ and set $\delta(y) = (t_2 + 1)$, where $t_2 \in \mathbb{Z}$. Note that, in this case, the $K$-stem $m_K(q_2)$ is the edge with endpoints $v_0$ and $v_1$ in Fig. 2 (B). By applying a Möbius transformation, namely a translation taking $\mu_2$ to 0, we are able to use Table 2 to conclude that $\frac{1}{2} \nu(\Delta) = \nu(\lambda)$ is the level of the highest leaf of $s_L(q_2)$, which according to Fig. 2(B) is $-\nu(y)$. In this case, if $m_K(q_1) \cap m_K(q_2) = \emptyset$, the distance between these branches is $\nu(y) + t_2 = -\frac{1}{2} \nu(\Delta) + t_2 = d_f$, since the stem $m_K(q_2)$ is a distance $t_2$ from every leaf of $s_L(q_2)$ (see Fig. 2(B) again). For the same reason, if $m_K(q_1) \cap m_K(q_2) \neq \emptyset$, this intersection is precisely one vertex, namely $v_0$, if $t_2 = -\nu(y)$, while $s_K(q_2) \subset s_K(q_1)$ if $t_2 < -\nu(y)$. The last two conditions are equivalent to $d_f = 0$ and $d_f < 0$ respectively, so the result follows in this case.

Finally, we assume that $m_1$ and $m_2$ are both irreducible. Let $L = k(\sqrt{b_1}, \sqrt{b_2})$. We denote by $\mu_i = x_i + y_i\sqrt{b_i}$ the end of $s_L(q_i)$, and we define the fake branch
both polynomials split over $K$. The result follows. In either case $d/\nu > 0$, midpoint. Note that this is, for either branch, the ghost vertex $v$.

We conclude that either $d/\nu > 0$, as we can see in Fig. 4, where the possible relative position of the $K$-stems are depicted. Now, as the distance between a leaf of the stem of $s_K(q_1)$ and $f(q_1)$ is 1, for $i \in \{1, 2\}$, if $s_K(q_1)$ fails to intersect $s_K(q_2)$, then $d(m_K(q_1), m_K(q_2)) = -\nu(\lambda) + t_1 + t_1 = d_f$ (Fig. 4(A)), while they intersect in a single point precisely when $d_f = 0$, as in Fig. 4(B). When the stems coincide, both fake branches have a vertex at distance $1/2$ from the common midpoint. Note that this is, for either branch, the ghost vertex $v_{1/2}$ in Fig. 2(B).

We conclude that either $d(f(q_1), f(q_2)) = 1$, as in Fig. 4(C), or $f(q_1)$ intersects $f(q_2)$ non-trivially. In either case $-\nu(\lambda) + t_1 + t_1 + 2 < 2$, whence $d_f = \nu(\lambda) + t_1 + t_1 < 0$. The result follows.

6. Proof of Theorem 2.2 in the separable case

In all of this section, we assume $m_1$ and $m_2$ are separable polynomials. We let $q_i$, $\alpha_i$, $\lambda$ and $\Delta$ be as in §4.1. As before, we first prove Theorem 2.2 in the case where both polynomials split over $K$. For $i \in \{1, 2\}$, let $w_i = \frac{\nu - \alpha_i}{\alpha_i}$ be the associated idempotent.

From the definition of the fake distance $d_f$ in Th. 2.2 we observe that $d_f = -\infty$ precisely when $\Delta = 0$. We need to prove that this is the case precisely when the length $l$ of the intersection $m_K(q_1) \cap m_K(q_2)$ is not finite. Furthermore, we need to prove that $l = 2\infty$ if and only if $q_1$ and $q_2$ commute, while $l = \infty$ when they fail to do so. Now, the value $2\infty$ is attained precisely when both stems coincide. By the arguments given in [4,1] this is the case precisely when one of the following alternatives hold:

- $w_1 = w_2$ or
- $w_1 = \overline{w_2} = 1 + w_2$.

As a commutative sub-algebra contains at most 2 idempotents, the condition “$q_1$ and $q_2$ commute” is equivalent to $l = 2\infty$. In this case, we do have $l = 2\infty = \max\{-2d_f, l(q_1), l(q_2)\}$, with the natural conventions. In the remainder of the split case we assume that $q_1$ and $q_2$ fail to commute.
We assume that the stems of the quaternions \( q_1, q_2 \in \mathbb{M}_2(K) \) are as shown in the first row of Table 2, i.e., \( m_K(w_1) = p(0, \infty) \) and \( m_K(w_2) = p(1, \theta) \), with \( \theta \notin \{0, 1\} \). In this case, we have \( \frac{\Delta}{a_1^2a_2^2} = \frac{\theta}{(1+\theta)^2} \). If \( \theta = \infty \), we have \( d_f = -\infty \), and also the stems intersect in a ray, as shown in Fig. 5(C). This takes care of the last infinite case. Suppose next that \( \theta \in K \). Assume first that \( m_K(q_1) \cap m_K(q_2) = \emptyset \). Then, as seen in Fig. 5(A), the smallest ball containing both 0 and \( \theta \) is \( B_{0}^{(\nu(\theta))} \), whence \( \nu(\theta) = 0 \) and \( d \left( m_K(q_1), m_K(q_2) \right) = \nu + \frac{\Delta}{(1+\theta)^2} = d_f \). If \( m_K(q_1) \cap m_K(q_2) \neq \emptyset \), using the Moebius transformation \( \sigma(z) = z^{-1} \) if needed, we can assume \( \theta \in \mathcal{O} \), as in Fig. 5(B). Note that \( \frac{\theta}{(1+\theta)^2} = \frac{\sigma(\theta)}{(1+\theta)} \) is invariant under this transformation. Then the length of \( m_K(q_1) \cap m_K(q_2) \) is \( l = \nu(\theta) = \nu \left( \frac{\theta}{(1+\theta)^2} \right) = -2d_f \), and the result follows.

Now we handle the cases where one or both roots \( \alpha_1 \) or \( \alpha_2 \) is not in \( K \). In any of the following cases:

- \( m_1 \) splits and \( m_2 \) is unramified,
- \( m_2 \) splits and \( m_1 \) is unramified or
- both \( m_1 \) and \( m_2 \) are unramified

the fake distance coincide with the the distance between the \( L \)-stems. Here we have the same cases depicted in Fig. 4. We can reason exactly as in §6 to finish the proof in this case.

Now, we assume that \( m_1 \) splits and \( m_2 \) is ramified. Then, by the split case, we know that the distance between the \( L \)-stems \( m_L(q_1) \) and \( m_L(q_2) \), where \( L \) is the decomposition field of \( m_2 \), is \(-\nu \frac{\Delta}{a_1^2a_2^2}\). The \( K \)-stem \( m_K(q_1) \) coincide with \( m_L(q_1) \), under the usual identification, and it is a maximal path in \( t = t(K) \), so it cannot be closer to \( m_L(q_2) \) than the corresponding \( K \)-stem, which contains \( m_K(q_2) \), as in Fig. 1(B). As the endpoints of the stem \( m_K(q_2) \) are at distance \( t_2 \) from the stem \( m_L(q_2) \), where \( D \left( \frac{b_2}{a_2} \right) = (\pi - t_2+1) \), as seen in Fig. 1(B), if the stem \( m_K(q_1) \) does not contain the stem \( m_K(q_2) \), then the distance between these stems is \(-\nu \frac{\Delta}{a_1^2a_2^2} - t_2 = d_f \), and the intersection is a vertex precisely when \( d_f = 0 \). Let \( \alpha_1 \) be a root of \( m_1 \). Note that

\[
\frac{\Delta}{a_1^2a_2^2} = \left( \frac{\lambda}{a_1a_2} + \alpha_1 \right)^2 + \left( \frac{\lambda}{a_1a_2} + \alpha_1 \right) + \frac{b_2}{a_2^2},
\]
satisfies $\nu \left( \frac{\Delta}{a_1^2 a_2^2} \right) \leq \nu \left( \frac{b_i}{a^2} \right) = 2t_i + 1$, whence $d_f = -\frac{1}{2} \nu \left( \frac{\Delta}{a_1^2 a_2^2} \right) - t_2 \geq -1/2$. Moreover, $m_K(q_1)$ contains $m_K(q_2)$ precisely when $m_L(q_1)$ contains the midpoint of $m_L(q_2)$, and this happens precisely when $d_f = -1/2$, since the distance between $m_L(q_2)$ and the $K$-vine of $q_2$ is $t_2 - 1/2$. Note that, in this case, one possible choice for the $K$-vine is the maximal path $m_L(q_1)$.

The case where $m_1$ is unramified and $m_2$ is ramified, is analogous, except that we replace the path $m_K(q_1)$ by a maximal path $p$ defined over an unramified extension $F/K$, which can be converted into an $F$-vine by a suitable Moebius transformation. In this case $d_f = -\frac{1}{2}$ is not possible, since $v_{1/2}$ in Fig. 1(B) is a point in a $K$-vine that is defined over a ramified quadratic extension of $K$, so it cannot belong to $p$, which contains a unique vertex defined over $K$. The cases where $m_1$ is ramified but $m_2$ is not are also similar.

In the remaining case, either stem, $m_K(q_1)$ or $m_K(q_2)$, is an edge located in the $K$-vine of the corresponding idempotent, either $w_1$ or $w_2$, as in Fig. 1(B). Let $t_1, t_2 \in \mathbb{N}$ be the integers defined by $\nu \left( \frac{b_i}{a^2} \right) = (\pi^{-2t_i+1})$, for $i \in \{1, 2\}$, so that $m_L(q_i)$ is at a distance $t_i$ from either endpoint of $m_K(q_i)$. Then the $K$-stems are located as shown in [6] Fig. 7, and we can reason precisely as in the ramified case of [6] §6.

7. **Proof of Theorem 2.2 in the mixed case**

In this section we assume that $m_1$ is a separable polynomial while $m_2$ is inseparable. Let $q_i, \alpha_i$ and $\lambda$ be as in §4.1. Here $\Delta = \lambda^2 + a_1^2 b_2$.

Again, we consider first the case where both polynomials split, so $d_f = -\frac{1}{2} \nu \left( \frac{\Delta}{a_1^2} \right)$. As before, we can assume that $q_1, q_2$ are as given in Table 2, this time in row 6 or 5, according to whether the branches have a common end (as in Fig. 6(A-B)) or not (as in Fig. 6(C)). Note that $\Delta = 0$, and hence $d_f = \infty$, only in row 6, which is precisely the case where the intersection, depicted as a double line in Fig. 6(C), is a ray. In any other case we have $d_f = -\frac{1}{2} \nu \left( \frac{\Delta}{a_1^2} \right) = -\nu(\pi^t) = -t$. If $m_K(q_1) \cap m_K(q_2) = 0$, then $-t > 0$ is the distance between the stems, as it is the length of the dashed line in Fig. 6(A). In the remaining case, the length of the intersection, denoted by a double line in Fig. 6(B), is $2t = -2d_f$.

In all that follows, $L$ is an algebraic extension of $K$ where both $m_1$ and $m_2$ split. First we consider the case where $m_1$ is unramified and $m_2$ splits in $K$. In this case, $m_K(q_1)$ consists only on the highest vertex in $m_L(q_1)$, and this is the
unique vertex in \( m_L(q_1) \cap t(K) \), as in Fig. 1(A) (c.f. 3). Assume first that \( m_L(q_1) \) and \( m_L(q_2) \) fail to intersect. The minimal path \( p \) from \( m_K(q_1) \) to \( m_L(q_2) \) is defined over \( K \) by Lemma 1.3. We claim that \( p \) is also the minimal path from \( m_L(q_1) \) to \( m_L(q_2) \). In fact, as \( L/K \) is unramified, every vertex in \( p \) is defined over \( K \) by Lemma 1.1. Since \( m_L(q_1) \cap t(K) \) has a unique vertex, the claim follows, whence \( d(m_K(q_1), m_K(q_2)) = d(m_L(q_1), m_L(q_2)) = d_f \). When \( m_L(q_1) \) and \( m_L(q_2) \) do intersect, a similar argument shows that \( m_L(q_1) \cap m_L(q_2) \) contains at most one vertex, so that \( d_f = 0 \).

Next assume \( m_1 \) ramifies and \( m_2 \) factors. Here \( m_K(q_1) \) consists in an edge whose midpoint \( v_1/2 \) is at distance \( t_1 - \frac{1}{2} \) from \( m_L(q_1) \), where \( \Delta \left( \frac{b_1}{\alpha_1} \right) = \langle \pi^{-2t_1+1} \rangle \), again as in Fig. 1(B). The end of the infinite foliage is defined over \( K \), whence it can be assumed to be \( \infty \) by applying a suitable Moebius transformation defined over \( K \). The ends of the maximal path, however, are conjugates, and defined over a ramified separable extension. We can replace the infinite foliage \( m_L(q_2) \) with a suitable fake branch \( m_L(\pi^{-n} q_2) \), as in §5, that lies at a distance \( -\frac{1}{2} \nu \left( \frac{\Delta}{\alpha_1} \right) + n \) from \( m_L(q_1) \), for \( n \) big enough. Let \( B \) be the leaf of this fake branch that is closest to \( m_K(q_1) \), as in Fig. 2(A). It is clear from the picture that the distance from \( B \) to \( m_K(q_1) \) is \( -\frac{1}{2} \nu \left( \frac{\Delta}{\alpha_1} \right) + n - t_1 \). Now the following facts are apparent:

- If \( m_K(q_1) \) does not intersect \( m_K(q_2) \), so we can set \( n = 0 \), we have
  \[
  d(m_K(q_1), m_K(q_2)) = -\frac{1}{2} \nu \left( \frac{\Delta}{\alpha_1} \right) - t_1 = d_f.
  \]
- If \( d_f = 0 \), the intersection is a vertex.
- \( d_f \) cannot be \( -\frac{3}{2} \), as \( B \) is defined over \( K \).
- If \( d_f \leq -1 \), then \( m_K(q_1) \subset m_K(q_2) \).

The result follows in this case.

Now, we assume that \( m_1 \) splits and \( m_2 \) is ramified. So that \( L = K(o_2) \) is an inseparable (ramified) extension. Then, again, the distance from the edge \( m_K(q_2) \) to a leaf of \( m_L(q_2) \) is \( t_2 \), where \( \delta(b_2) = (\pi^{2t_2+1}) \). In this case the (ghost) midpoint \( v'_{1/2} \) of \( m_K(q_2) \) is a leaf of a fake branch \( f(q_2) = m_L(q'_2) \), for \( q'_2 = \pi_L^{-1-2t_2}(q_2 - o_2) \), as in §5. The distance from \( f(q_2) \) to \( m_L(q_1) \) is, by applying the split case to \( q_1 \) and \( q'_2 \), the following:

\[
-\frac{1}{2} \nu \left( \frac{\Delta}{\alpha_1} \right) + t_2 + \frac{1}{2} = d_f + \frac{1}{2}.
\]

Furthermore, if \( p \) is the maximal path in \( L \) identified with \( m_K(q_1) \), then the path \( p_1 \) from \( v'_{1/2} \) to \( p \) cannot pass through any other vertex of the fake branch \( f(q_2) \), as the latter contains no point defined over \( K \), while the vertex after \( v'_{1/2} \) in \( p_1 \) must be defined over \( K \) by Lemma 1.1 and Lemma 1.3, as \( v'_{1/2} \) has neighbors defined over \( K \) at distance \( 1/2 \). Therefore, \( p_1 \) is either trivial, as in Fig. 7(B), or it contains an endpoint of \( m_K(q_2) \). The following facts are now apparent:

- If \( m_L(q_1) = m_K(q_1) \) does not contain the edge \( m_K(q_2) \), then the distance between these stems is \( d_f \geq 0 \).
- \( m_L(q_1) \) contains \( m_K(q_2) \) precisely when \( v'_{1/2} \) is in \( m_L(q_1) \), as in Fig. 7(B), and in this case \( d_f = -1/2 \).

Note that \( d_f = -1/2 \) is the minimum possible value in this case. The case where \( m_1 \) is unramified and \( m_2 \) is ramified, is analogous, except that in this case \( d_f = -\frac{1}{2} \).
is not possible, as both, the midpoint $v_{1/2}'$, in an edge defined over $K$, and the path $m_L(q_1)$, with a unique vertex defined over $K$, are defined over different quadratic extensions.

In the remaining case, either $K$-stem, $m_K(q_1)$ or $m_K(q_2)$, consist in an edge located in the $K$-vine minimizing the fake distance to the corresponding $L$-stem, as in Fig. 7(B) and Fig. 7(B), respectively. Let $t_1, t_2 \in \mathbb{N}$ the integers satisfying $D\left(\frac{b_1}{a_1}\right) = (\pi^{-2t_1+1})$ and $\delta(b_2) = (\pi^{2t_2+1})$. We define the fake branch of $s_K(q_2)$ by $f(q_2) = s_L(q_2')$, where, $q_2' = \pi^{-t_2-1}(q_2 - \alpha_2)$, as before. Recall from §5 that a leaf of this fake branch is located as one of the white circles in Fig. 4. By applying the computation for the split case, this time to $q_1$ and $q_2'$, we conclude that the distance from $f(q_2)$ to $m_L(q_1)$ is $-\frac{1}{2} + t_2 + 1 = d_f + t_2 + 1$, unless this amount is negative, and in the latter case $m_L(q_1)$ intersects $f(q_2)$ non-trivially. The upcoming reasoning is analog to the inseparable case, as depicted in Fig. 4, except that in this case the $L$-stem of the separable quaternion $q_1$ is at distance $t_1 - 1/2$ behind the corresponding $K$-stem. If the two $K$ stems $m_K(q_1)$ and $m_K(q_2)$ are different, even if they intersect at one point, the path from $f(q_2)$ to $m_L(q_1)$ must pass though one endpoint of each $K$-stem. The branch $f(q_2)$ is at distance 1/2 from the midpoint of $m_K(q_2)$, so the distance from this white point to the midpoint of $m_K(q_1)$ is larger than the distance between the $K$-stems by 3/2. The distance between the $K$-stems is therefore given by $(d_f + t_1 + 1) - (t_1 - 1/2) - 3/2 = d_f$, whence the the result follows. If the $K$-stems coincide and $d_f + t_1 + 1 \geq 0$, Fig. 4(C) shows that the path from $f(q_2)$ to $m_L(q_1)$ cannot be longer than $1/2 + (t_1 - 1/2) = t_1$, so $d_f < 0$. The same conclusion holds if $d_f + t_1 + 1 < 0$. This finishes the proof.

8. ON THE REPRESENTATIONS OF SOME ALGEBRAS

In this section we prove Theorem 2 using tools from representation theory. Let $\lambda \in K$, let $m_i(X) = X^2 + a_iX + b_i \in \mathcal{O}[X]$, for $i \in \{1, 2\}$, and define $\Delta$ as in Theorem 2.2. In all of §8, we consider the $K$-algebra $A = A(\lambda, m_1, m_2)$, defined in terms of generators and relations as follows:

\[ A = K[ q_1, q_2 | m_1(q_1) = m_2(q_2) = 0, \ q_1(q_2 + a_2) + q_2(q_1 + a_1) = \lambda ] \]

Lemma 8.1. The algebra $A$ defined in (7) is a 4-dimensional $K$-algebra. It is a quaternion algebra if and only if $\Delta \neq 0$. 

\[ \text{Figure 7. In (A), } B \text{ is a leaf of } m_K(\pi_K^n q_2) \text{ and } \{v_0, v_1\} \text{ are the vertices } m_K(q_1). \text{ In (B) the vertices of } m_K(q_2) \text{ are } v_0' \text{ and } v_1'. \]
Proof. First, we assume that \( a_1 \neq 0 \) and \( \Delta \neq 0 \). Let \( q_2' = \lambda + a_2 q_1 + a_1 q_2 \) and \( q_1' = \frac{b_1}{a_1} \). It is apparent that

\[
A = K \left[ q_1', q_2' q_1'^2 + q_1' + \frac{b_1}{a_1} = q_2'^2 + \Delta = q_2' q_1' + (q_1' + 1)q_2' = 0 \right]
\]

is a cyclic algebra, and therefore a quaternion algebra (c.f. [8 Ch. 1, §1, Ex. 1.6]). Now, we assume that \( a_1 = a_2 = 0 \) and \( \Delta = \lambda^2 \neq 0 \). If \( b_2 = b_1 = 0 \), we can replace \( q_2 \) by \( \hat{q}_2 = q_2 + 1 \), and with the new generator \( A = A(\lambda, X^2, X^2 + 1) \), so we assume \( b_2 \neq 0 \). In the latter case, \( q_2'' = \frac{4q_2 a_1}{\Delta} \) gives the presentation

\[
A = K \left[ q_1'', q_2 | q_1'' + q_1' + \frac{b_1 b_2}{\lambda^2} = q_2'' + b_2 = q_1'' q_2 + q_2 (q_2'' + 1) = 0 \right],
\]

which is again a cyclic algebra.

In the rest of the proof we assume \( \Delta = 0 \). First consider the case \( a_1 \neq 0 \). Define \( q_1' \) and \( q_2' \) as above, and let \( \alpha' \in K \) be a root of the irreducible polynomial \( m'_1(X) = X^2 + X + \frac{b_1}{a_1} \) of \( q_1' \). Set \( L = K(1) \), so that there exists a representation \( \phi : A_L = A \otimes_K L \to M_4(L) \) defined by

\[
\phi(q_1' \otimes 1) = \begin{pmatrix} \alpha' + 1 & 0 & 0 & 0 \\ 0 & \alpha' & 0 & 0 \\ 0 & 0 & \alpha' & 0 \\ 0 & 0 & 0 & \alpha' + 1 \end{pmatrix}, \quad \phi(q_2' \otimes 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

whose image is 4-dimensional, and has a non-trivial radical \( R(A) \). This proves that \( A \) is a quaternion algebra and \( \dim_K A \geq 4 \). The converse inequality follows from the fact that \( \{1, q_1, q_2, q_1 q_2\} \) spans \( A \) as a vector space. The same argument holds if \( a_2 \neq 0 \). On the other hand, if \( a_1 = a_2 = 0 \), then \( 0 = \Delta = \lambda^2 \), and therefore \( \lambda = 0 \). Replacing these values in (7), we obtain that \( A \) is a 4-dimensional commutative algebra.

\[
\begin{proof}
\end{proof}
\]

Proposition 8.2. Assume \( \Delta \neq 0 \), so \( A \) is a quaternion algebra. Then \( A \cong M_2(K) \) if and only if at least one of the following conditions holds:

i. \( m_1(X) \) or \( m_2(X) \) has a zero in \( K \),

ii. \( \lambda = C(x, y, z, w) \) for some pairs \((x, y), (z, w) \in K \times K^*\), as in (6).

\[
\begin{proof}
\end{proof}
\]

Example 8.3. We claim that, if \( m_2 = x^2 + \pi \) and the splitting field of \( m_1 \) is an unramified quadratic extension of \( K \), then \( A = A(0, m_1, m_2) \) is a division algebra.
Note that $\lambda = a_2 = 0$ implies $\Delta = a_1^2 \pi \neq 0$. In fact, if $A$ were a matrix algebra, then there would exist pairs $(x, y), (z, w) \in K \times K^*$ satisfying $C(x, y, z, w) = 0$, or equivalently

\[(yw)C(x, y, z, w) = x^2 + a_1 xy + b_1 y^2 + z^2 + \pi w^2 + a_1 z y = 0.\]

Rearranging, we obtain $y^2m_1 \left(\frac{x + z}{y}\right) = |\pi w^2|$, which is impossible by the properties of the Artin-Schreier defect, or equivalently, since $m_1$ has no roots in the residue field. We conclude that $A$ is a division algebra. This is a characteristic-2 analog of [6] §63 B, Theo. 63.11 B.

**Proof of Theorem 2.3.** Assume first $\Delta \neq 0$, and assume the existence of elements $q_1, q_2 \in M_2(K)$ satisfying (1). Let $\varphi : A = A(\lambda, m_1, m_2) \to M_2(K)$ be the representation defined by $q_i \mapsto q_i$. As $A$ is a simple algebra and $\dim_K A = \dim_K M_2(K)$, $\varphi$ must be an isomorphism. Reciprocally, if $\varphi : A \to M_2(K)$ is an isomorphism, then the images $q_1 = \varphi(q_1)$ and $q_2 = \varphi(q_2)$ satisfy (1). The results follows in this case from Prop. 8.2.

In all that follows we assume $\Delta = 0$. Now suppose $a_1 \neq 0$ and assume the existence of $q_1, q_2 \in M_2(K)$ satisfying (1). Define $q_2'' = a_2 q_1 + a_1 q_2 = q_2' + \lambda$, where $q_2'$ corresponds to the element $q_2$ in the proof of Lemma 8.1 so that $q_2'' = \lambda^2$ and

\[(8) \quad q_1 q_2'' + q_2'' (q_1 + a_1) = a_1 \lambda.\]

Changing the base if needed, we can assume $q_2'' = \left(\begin{array}{cc} \lambda & 0 \\ 1 & \lambda \end{array}\right)$. Set $q_1 = \left(\begin{array}{cc} u & y \\ z & w \end{array}\right)$, so that identity (5) gives $y = 0$ and $w = u + a_1$. Then, the condition $m_1(q_1) = 0$ implies $m_1(u) = 0$. Conversely, if $m_1$ has a root $\alpha \in K$, then the quaternions $q_1 = \left(\begin{array}{cc} \alpha + u & 0 \\ 0 & \alpha \end{array}\right)$ and $q_2 = \left(\begin{array}{cc} \lambda + a_1 & 0 \\ 0 & \lambda + a_2 \end{array}\right)$ satisfy (1). The same holds if $a_2 \neq 0$.

Finally, assume $a_1 = a_2 = 0$, so that $\lambda = 0$, and set $L = K(\sqrt{b_1}, \sqrt{b_2})$, as in the proof of Lemma 8.1. It is not hard to see, as $\sqrt{b_1}$ and $\sqrt{b_2}$ satisfy the relations defining $A$, that $L$ is isomorphic to a quotient of $A$. If $[L : K] = 4$, so that $A \cong L$, the algebra $A$ cannot have a two dimensional representation. If $[L : K] \leq 2$, such representations do exists. Note however that any pair of matrices $(q_1, q_2)$ satisfying (1) must be contained in a commutative subalgebra, and in $M_2(K)$ such algebras are always two dimensional. The result follows if we prove that $[L : K] \leq 2$ in our case. In fact, any Laurent series can be written in the form

\[f(\pi) = \sum_{m \in \mathbb{Z}} a_m \pi^m = \sum_{n \in \mathbb{Z}} b_{2n} \pi^{2n} + \pi \sum_{n \in \mathbb{Z}} b_{2n+1} \pi^{2n},\]

where $b^n_m = a_m$, so $K$ has a unique inseparable quadratic extension. Note that this is the only point in the proof where we use the fact that the residue field is perfect.

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Luis Arenas-Carmona
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Chile, Casilla 653, Santiago, Chile
learenas@u.uchile.cl

Claudio Bravo
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Chile, Casilla 653, Santiago, Chile
claudio.bravo.c@ug.uchile.cl