Maximal and Calderón–Zygmund operators on the local variable Morrey–Lorentz spaces and some applications

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\section*{ABSTRACT}

In this paper, we give the definition of local variable Morrey–Lorentz spaces \(M_{p\cdot,q\cdot,\lambda}(\mathbb{R}^n)\) which are a new class of functions. Also, we prove the boundedness of the Hardy–Littlewood maximal operator \(M\) and Calderón–Zygmund operators \(T\) on these spaces. Finally, we apply these results to the Bochner–Riesz operator \(B^\delta\), identity approximation \(A_\varepsilon\) and the Marcinkiewicz operator \(\mu/\Omega\) on the spaces \(M_{p\cdot,q\cdot,\lambda}(\mathbb{R}^n)\).

\section*{1. Introduction}

The study of function spaces with variable exponent has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [1–5]). Various results on non-weighted and weighted boundedness in variable exponent Lebesgue spaces have been proved for maximal, singular and fractional-type operators (we refer to surveying papers [6,7]). In [8], variable exponent Lorentz spaces \(L_{p\cdot,q\cdot}(\mathbb{R}^n)\) are introduced, and the boundedness of the singular integral and fractional-type operators and corresponding ergodic operators is proved in these spaces.

The classical Morrey spaces were introduced by Morrey [9] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equations, the reader is referred to [10,11]. Some important results in the harmonic analysis have been extended to some new types of Morrey spaces. Recently, the study of Morrey spaces had been extended to the Morrey–Lorentz spaces [11,12], the Orlicz–Morrey spaces [13] and the Morrey spaces with variable exponents [14–16]. The study of these Morrey-type spaces has applications on partial differential equations, for example, they are related to the viscosity solutions of some fully nonlinear elliptic equations [17].
The Lorentz version of Morrey spaces, i.e. Lorentz–Morrey space $L_{p,q}(\mathbb{R}^n)$, was first defined in [12] and also considered in [18–20]. Later, the local Morrey–Lorentz spaces $M_{p,q}^{loc}(\mathbb{R}^n)$ are introduced and the basic properties of these spaces are given in [21]. These spaces are a very natural generalization of the Lorentz spaces such that $M_{p,q}^{loc}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. Recently, in [22–24], the authors have studied the boundedness of the Hilbert transform, the Hardy–Littlewood maximal operator $M$ and the Calderón–Zygmund operators $T$, and the Riesz potential $I_\alpha$ on the local Morrey–Lorentz spaces $M_{p,q}^{loc}$ by using related rearrangement inequalities, respectively. In [25], the authors give the definition of central Lorentz–Morrey space of variable exponent by the symmetric decreasing rearrangement. They prove the boundedness of the maximal operator in these spaces and establish Sobolev’s inequality for Riesz potentials.

This paper aims to establish the mapping properties of the Hardy–Littlewood maximal operator $M$ and Calderón–Zygmund operators $T$ on Morrey spaces built on variable Lorentz spaces. We define the local variable Morrey–Lorentz spaces $M_{p(q)}^{loc}(\mathbb{R}^n)$ and prove the boundedness of the maximal operator $M$ and Calderón–Zygmund operators $T$ on these spaces. We also give some applications of our results.

The paper is organized as follows. In Section 2, we give some notation and definitions. We introduce local variable Morrey–Lorentz spaces $M_{p(q)}^{loc}(\mathbb{R}^n)$. In Section 3, we prove the boundedness of the maximal operator and the Calderón–Zygmund operators $T$ in the spaces $M_{p(q)}^{loc}(\mathbb{R}^n)$. In Section 4, finally, as applications, we get the boundedness of the Bochner–Riesz operator $B^\delta_r$, Marcinkiewicz operator $\mu_\Omega$ and the sublinear operator $\sup_{\varepsilon>0} |A_\varepsilon f(x)|$ on the spaces $M_{p(q)}^{loc}(\mathbb{R}^n)$, where $A_\varepsilon$ is the identity approximation.

Throughout the paper, we use the letter $C$ for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

2. Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at $x$ of radius $r$ and $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. Note that $|B(x, r)| = \omega_n r^n$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Let $f$ be a locally integrable function on $\mathbb{R}^n$. Hardy–Littlewood maximal function $Mf$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$  

Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas. For the operator $M$, the rearrangement inequality

$$c^{**}(t) \leq (Mf)^*(t) \leq C f^{**}(t), \quad t \in (0, \infty),$$  

holds, where $c$ and $C$ are independent of $f$. Here $f^*$ denotes the right continuous non-increasing rearrangement of $f$:

$$f^*(t) := \inf \{ \lambda > 0 : \mu_f(\lambda) \leq t \}, \quad t \in (0, \infty),$$

and

$$\mu_f(\lambda) := | \{ y \in \mathbb{R} : |f(y)| > \lambda \} |$$

is the distribution function of the function $f$.

Let $T$ be a Calderón–Zygmund operator, i.e. a linear operator bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions $f$ with compact support to the functions
$Tf \in L^1_{1 \text{loc}}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y) \, dy, \quad x \notin \text{supp} \, f,$$

provided it exists almost everywhere. Here $K(x, y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $C > 0$ and $0 < \varepsilon \leq 1$ such that

$$|K(x, y)| \leq C|x - y|^{-n}$$

for all $x, y \in \mathbb{R}^n$, $x \neq y$, and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \left( \frac{|x - x'|}{|x - y|} \right) \varepsilon |x - y|^{-n},$$

whenever $2|x - x'| \leq |x - y|$. More information about such operators can be found in [26, 27].

The following rearrangement inequality

$$(Tf)^*(t) \leq C \left( \int_0^t f^*(s) \, ds + \int_t^\infty f^*(s) \frac{ds}{s} \right)$$

(2)

is valid for the Calderón–Zygmund operators $T$, where $C$ is independent of $T$ (see, e.g. [28]). Similar sharp rearrangement estimates are of great importance in the study of operators on rearrangement-invariant function spaces as well as in interpolation theory.

Let $p(t)$ be a measurable function on $(0, \infty)$. We suppose that

$$1 < p_- \leq p(t) \leq p_+ < \infty,$$

where

$$p_- := \text{ess inf}_{0 < t < \infty} p(t), \quad p_+ := \text{ess sup}_{0 < t < \infty} p(t).$$

We denote by $p'(\cdot) = \frac{p(t)}{p(t) - 1}$. We will use the following decay conditions:

$$|p(t) - p(0)| \leq \frac{A_0}{|\ln t|}, \quad 0 < t \leq \frac{1}{2},$$

(3)

$$|p(t) - p(\infty)| \leq \frac{A_\infty}{\ln t}, \quad t \geq 2,$$

(4)

where $A_0, A_\infty > 0$ do not depend on $t$.

By $p \in \mathcal{P}_{0, \infty}(0, \infty)$, we denote the set of bounded measurable functions (not necessarily with values in $[1, \infty)$), which satisfy decay conditions (3) and (4). Also, by $L_{p'(\cdot)}(0, \infty)$ we denote the variable exponent Lebesgue space of measurable functions $\varphi$ on $(0, \infty)$ such that

$$\mathcal{J}_{p'(\cdot)}(\varphi) = \int_0^\infty |\varphi(s)|^{p'(s)} \, ds < \infty.$$

This is a Banach function space with respect to the norm (see, e.g. [29])

$$\|\varphi\|_{L_{p'(\cdot)}} = \inf \left\{ \lambda > 0 : \mathcal{J}_{p'(\cdot)} \left( \frac{\varphi}{\lambda} \right) \leq 1 \right\}. $$
Definition 2.1 ([30]): Let $1 < q_- \leq q_+ < \infty$, $0 \leq \lambda < 1$. We denote by $LM_{q(-),\lambda} = LM_{q(-),\lambda}(0,\infty)$ the variable local Morrey space with finite norm
\[
\|\varphi\|_{LM_{q(-),\lambda}} = \sup_{r>0} r^{-\frac{\lambda}{q_+}} \|\varphi\|_{L_{q(-)}(0,r)}
\]
\[
= \sup_{r>0} \inf_{\eta > 0} \left\{ \eta > 0 : \int_0^r \left| \frac{\varphi(s)}{s^{q_-}} \right|^\frac{q(s)}{q_-} \frac{ds}{s} \leq 1 \right\},
\]
where $q_+(r) = q(0), 0 < r < 1$, and $q_+(r) = q(\infty), r \geq 1$.

Definition 2.2 ([8]): Let $1 \leq p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$. We denote by $L_{p(-),q(-)}(\mathbb{R}^n)$ variable Lorentz space, the space of functions $f$ on $\mathbb{R}^n$ such that $t^{\frac{1}{p_-} - \frac{1}{p_+}} f^*(t) \in L_{q(-)}(0,\infty)$, i.e.
\[
J_{p(-),q(-)}(f) = \int_0^\infty t^{\frac{q(t)}{p_-} - \frac{1}{p_+}} (f^*(t))^{q(t)} \frac{dt}{t} < \infty
\]
and denote
\[
\|f\|_{L_{p(-),q(-)}(\mathbb{R}^n)} = \inf \left\{ \sigma > 0 : J_{p(-),q(-)} \left( \frac{f}{\sigma} \right) \leq 1 \right\} = \left\| t^{\frac{1}{p_-} - \frac{1}{p_+}} f^*(t) \right\|_{L_{q(-)}(0,\infty)},
\]
where $f^*$ denotes the non-increasing rearrangement of $f$ such that
\[
f^*(t) = \inf \left\{ \lambda > 0 : \left| \{ y \in \mathbb{R}^n : |f(y)| > \lambda \} \right| \leq t \right\}, \quad \forall t \in (0,\infty)
\]
and
\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \frac{ds}{s}
\]
(see [31]). More information about variable Lorentz spaces can be found in [8,32].

We find it convenient to define the local variable Morrey–Lorentz spaces in the form as follows.

Definition 2.3: Let $0 < p_- \leq p_+ < \infty$, $1 < q_- \leq q_+ < \infty$ and $0 \leq \lambda < 1$. We denote by $\mathcal{M}_{p(-),q(-),\lambda}^{loc}(\mathbb{R}^n)$ the local variable Morrey–Lorentz space, the space of all measurable functions with finite quasi-norm
\[
\|f\|_{\mathcal{M}_{p(-),q(-),\lambda}^{loc}} := \sup_{r>0} r^{-\frac{\lambda}{q_+}} \left\| t^{\frac{1}{p_-} - \frac{1}{p_+}} f^*(t) \right\|_{L_{q(-)}(0,r)}.
\]
These spaces generalize variable Lorentz spaces such that $\mathcal{M}_{p(-),q(-),\lambda}^{loc} = L_{p(-),q(-)}$ when $\lambda = 0$, given in [8]. Also, if $\lambda = 0$ and $q(-) = p(-)$, then $\mathcal{M}_{p(-),p(-),0}^{loc} = L_{p(-)}$ are variable Lebesgue spaces defined in [33].

3. The maximal operator $M$ and the Calderón–Zygmund operators $T$ in the spaces $\mathcal{M}_{p(-),q(-),\lambda}^{loc}(\mathbb{R}^n)$

In this section, we prove the boundedness of the maximal operator $M$ and the Calderón–Zygmund operators $T$ in the local variable Morrey–Lorentz spaces. We need the following two definitions about Hardy operators which are used in the proof of our main theorems. These operators are very important in analysis and have been widely studied.
Definition 3.1 ([8]): Let \( \varphi \) be a measurable function on \((0, \infty)\). The weighted Hardy operators \( H_{\beta(t)} \) and \( \mathcal{H}_{\beta(t)} \) with power weight acting on \( \varphi \) are defined by

\[
H_{\beta(t)} \varphi(t) = t^{\beta(t) - 1} \int_0^t \frac{\varphi(s)}{s^{\beta(s)}} \, ds
\]

and

\[
\mathcal{H}_{\beta(t)} \varphi(t) = t^{\beta(t)} \int_t^\infty \frac{\varphi(s)}{s^{\beta(s) + 1}} \, ds.
\]

The following lemma provides some minimal assumptions on the function \( r_{\frac{q}{q(t)}} \) under which the so-defined spaces contain 'nice' functions.

Lemma 3.1 ([30]): Let \( 1 < q_- \leq q_+ < \infty, q \in \mathcal{P}_0(0, \infty), [r]_1 = \min\{1, r\} \text{ and } 0 \leq \lambda < 1. \) Then, the assumption

\[
\sup_{r > 0} [r]_1^{\frac{1}{q(t)}} r^{-\frac{1}{q_+(t)}} < \infty
\]

is sufficient for bounded functions \( f \) with compact support to belong to the local variable Morrey spaces \( LM_{q(t), \lambda}(\mathbb{R}^n) \).

Lemma 3.2 ([30]): Let \( 1 < q_- \leq q_+ < \infty, q \in \mathcal{P}_{0, \infty}(0, \infty), 0 \leq \lambda < 1, \lim_{t \to 0} t^{\beta(t)} \) exist and finite and condition (5) satisfies. Suppose that the following conditions hold:

(i) \( t^{\beta(t) - a} \) and \( \frac{1}{t^{\frac{a}{q(t)}}} - \beta(t) - a \) are almost decreasing for some \( a \in \mathbb{R} \), in the case of operator \( H_{\beta(t)} \).

(ii) \( t^{\beta(t) + b} \) and \( t^{\frac{1}{t^{\frac{b}{q(t)}}} - \beta(t) - b} \) are almost increasing for some \( b \in \mathbb{R} \), in the case of operator \( \mathcal{H}_{\beta(t)} \).

Then, the conditions

\[
\beta(t) < \frac{\lambda}{q_+(t)} + \frac{1}{q'(0)} , \quad \beta(t) > \frac{\lambda}{q_+(t)} - \frac{1}{q(\infty)}
\]

are sufficient for the Hardy operators \( H_{\beta(t)} \) and \( \mathcal{H}_{\beta(t)} \), respectively, to be defined on the space \( LM_{q(t), \lambda}(\mathbb{R}^n) \).

Lemma 3.3 ([30]): Let \( 1 < q_- \leq q_+ < \infty, q \in \mathcal{P}_{0, \infty}(0, \infty), 0 \leq \lambda < 1. \) Suppose also that conditions (5) and Lemma 3.2 are satisfied. Then the operators \( H_{\beta(t)} \) and \( \mathcal{H}_{\beta(t)} \) are bounded in the space \( LM_{q(t), \lambda}(0, \infty) \) if \( \beta(t) < \frac{\lambda}{q_+(t)} + \frac{1}{q'(0)}, \beta(t) > \frac{\lambda}{q_+(t)} - \frac{1}{q(\infty)} \), respectively.

The following theorem is one of the main results of our paper in which we give the boundedness of the maximal operator in the local variable Morrey–Lorentz spaces.

Theorem 3.1: Let \( 1 \leq p_- \leq p_+ < \infty, 1 < q_- \leq q_+ < \infty, p, q \in \mathcal{P}_{0, \infty}(0, \infty), 0 \leq \lambda < 1 \) and \( f \in \mathcal{M}^{loc}_{p(t), q(t), \lambda}(\mathbb{R}^n) \). Suppose that conditions (5) and Lemma 3.2 are satisfied. Then the maximal operator \( M \) is bounded on the local variable Morrey–Lorentz spaces \( \mathcal{M}^{loc}_{p(t), q(t), \lambda}(\mathbb{R}^n) \).
Proof: Let $1 \leq p_- \leq p_+ < \infty$, $1 \leq q_- \leq q_+ < \infty$, $0 \leq \lambda < 1$, condition (5) satisfies and $f \in \mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)$. From the definition of local variable Morrey–Lorentz spaces and inequality (1), we get

$$
\|Mf\|_{\mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)} = \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} (Mf)^*(t) \right\|_{L_q(t)}(0, r)
\leq C \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \right\|_{L_q(t)}(0, r)
= C \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)} - 1} \int_0^t f^*(s) \, ds \right\|_{L_q(t)}(0, r)
= C \|H_\beta g\|_{L_q(t)}(0, \infty),
$$

where $g(t) = t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t)$, $eta(t) = \frac{1}{p(t)} - \frac{1}{q(t)}$. For $eta(t) = \frac{1}{p(t)} - \frac{1}{q(t)}$, the inequality $eta(t) < \frac{1}{q(t)} + \frac{\lambda}{q_+(\lambda)}$ holds. Therefore, by Lemma 3.3, we get

$$
\|H_\beta g\|_{L_q(t)}(0, \infty) \leq C \|g\|_{L_q(t)}(0, \infty)
= C \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \right\|_{L_q(t)}(0, r)
= C \|f\|_{\mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)}. \tag{6}
$$

From (6), we obtain the boundedness of the maximal operator $M$ in the space $\mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)$. 

In the case $\lambda = 0$, from Theorem 3.1, we get the boundedness of the maximal operator $M$ in the variable Lorentz spaces $L_{p, q}(\mathbb{R}^n)$ which is proved in [8].

The following theorem is the other main result of our paper in which we give the boundedness of Calderón–Zygmund operators in the local variable Morrey–Lorentz spaces.

**Theorem 3.2:** Let $1 \leq p_- \leq p_+ < \frac{q_+(\lambda)}{\lambda}$, $1 \leq q_- \leq q_+ < \infty$, $p, q \in \mathcal{P}_{0, \infty}(0, \infty)$, $0 \leq \lambda < 1$ and $f \in \mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)$. Suppose that conditions (5) and Lemma 3.2 are satisfied. Then the Calderón–Zygmund operator $T$ exists almost everywhere $x \in \mathbb{R}^n$. Moreover, $T$ is bounded on the local variable Morrey–Lorentz spaces $\mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)$.

**Proof:** Let $1 \leq p_- \leq p_+ < \frac{q_+(\lambda)}{\lambda}$, $1 \leq q_- \leq q_+ < \infty$, $0 \leq \lambda < 1$, condition (5) satisfies and $f \in \mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)$. From the definition of norm in local variable Morrey–Lorentz spaces and by using inequality (2), we get

$$
\|Tf\|_{\mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)} = \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} (Tf)^*(t) \right\|_{L_q(t)}(0, r)
\leq C \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)} - 1} \int_0^t f^*(s) \, ds \right\|_{L_q(t)}(0, r)
+ C \sup_{r > 0} r^{-\frac{\lambda}{q_+(\lambda)}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)} - 1} \int_t^\infty \frac{f^*(s)}{s} \, ds \right\|_{L_q(t)}(0, r)
= I_1 + I_2.
$$

$I_1$ can be estimated using the same method as in the proof of the boundedness of the maximal operator on $\mathcal{M}_{p, q, \lambda}^\text{loc}(\mathbb{R}^n)$ in Theorem 3.1.
Lemma 4.1 ([36, Lemma 2, p. 668]): Where $M$ Bochner–Riesz operator $B$ in this section, we give some applications of our main results. We obtain the boundedness of the Bochner–Riesz operator is defined by (see [34,35])

\[ B_{\delta}(f) := \frac{r^{\lambda}}{q_{\lambda}(t)} \int_{t}^{r} f^*(s) \, ds \]

Therefore, we get $I_2 \leq C\|f\|_{M_{p,q}^{\text{loc}}(\mathbb{R}^n)}$. Consequently, we obtain the boundedness of $T$ in $M_{p,\lambda}^{\text{loc}}(\mathbb{R}^n)$ from inequalities (6) and (7).

In the case $\lambda = 0$, from Theorem 3.2, we get the boundedness of the Calderón n–Zygmund operator $T$ in the variable Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ which is proved in [8].

4. Some applications

In this section, we give some applications of our main results. We obtain the boundedness of the Bochner–Riesz operator $B_{r}^\lambda$, identity approximation $A_\xi$ and Marcinkiewicz operator $\mu_\Omega$ on the local variable Morrey–Lorentz spaces $M_{p,\lambda}^{\text{loc}}(\mathbb{R}^n)$.

4.1. Bochner–Riesz operator

Let $\delta > (n - 1)/2$, $B_{r}^\lambda(f)(\xi) = (1 - r^2|\xi|^2)^{\delta/2} f(\xi)$ and $B_{r}^\lambda(x) = r^{-n}B_{r}^\lambda(x/r)$ for $r > 0$. The maximal Bochner–Riesz operator is defined by (see [34,35])

\[ B_{\delta,\ast}(f)(x) = \sup_{r > 0} |B_{r}^\lambda(f)(x)|. \]

If $\delta > (n - 1)/2$, $B_{\delta,\ast}$ is pointwise majorized by the Hardy–Littlewood maximal operator $M$.

Lemma 4.1 ([36, Lemma 2, p. 668]): Let $\delta > (n - 1)/2$. Then the inequality

\[ B_{\delta,\ast}(f)(x) \leq CMf(x) \]

holds for all $f \in L_{1}^{\text{loc}}(\mathbb{R}^n)$, where $C$ is independent of $f$.

Therefore, the boundedness of the operator $B_{\delta,\ast}$ imply from Theorem 3.1.

Corollary 4.1: Let $1 < p_ - \leq p_ + < \infty$, $1 \leq q_ - \leq q_ + < \infty$, $p, q \in P_{0,\infty}(0,\infty)$, $0 \leq \lambda < 1$ and $f \in M_{p,\lambda}^{\text{loc}}(\mathbb{R}^n)$. Suppose that conditions (5) and Lemma 3.2 are satisfied. Then the Bochner–Riesz operator $B_{r}^\lambda$ is bounded on the local variable Morrey–Lorentz spaces $M_{p,\lambda}^{\text{loc}}(\mathbb{R}^n)$.

In the case $\lambda = 0$, from Theorem 3.1, the boundedness of the operator $B_{r}^\lambda$ in the variable Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ is obtained.
4.2. Identity approximation

It is known that the identity approximation (see [37])

\[ A_\varepsilon f(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} a\left(\frac{x-y}{\varepsilon}\right) f(y) \, dy, \]

where \( \int_{\mathbb{R}^n} a(y) \, dy = 1 \) and \( a(x) \) has a radial decreasing integrable majorant, is dominated by the maximal operator

\[ |A_\varepsilon f(x)| \leq CMf(x), f \in L^p(\mathbb{R}^n), \quad 1 \leq p \leq \infty, \]

with an absolute constant \( C > 0 \) not depending on \( x \) and \( \varepsilon \).

Since the maximal operator \( M \) is bounded on the spaces \( M_{p,q}^{loc}(\mathbb{R}^n) \), then we get the following.

**Corollary 4.2:** Let \( 1 < p_- \leq p_+ < \infty, 1 \leq q_- \leq q_+ < \infty, p,q \in \mathcal{P}_{0,\infty}(0,\infty), 0 \leq \lambda < 1 \) and \( f \in M_{p,q}^{loc}(\mathbb{R}^n) \). Suppose that conditions (5) and Lemma 3.2 are satisfied. Then, the sublinear operator \( \sup_{\varepsilon>0} |A_\varepsilon f(x)| \) is bounded on the local variable Morrey–Lorentz spaces \( M_{p,q}^{loc}(\mathbb{R}^n) \).

In the case \( \lambda = 0 \), from Theorem 3.1, we get the boundedness of the operator \( \sup_{\varepsilon>0} |A_\varepsilon f(x)| \) in the variable Lorentz spaces \( L_{p,q}(\mathbb{R}^n) \) which is proved in [8].

4.3. Marcinkiewicz operator

In the case \( n \geq 2 \), we denote by \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) the unit sphere in \( \mathbb{R}^n \) equipped with the normalized Lebesgue measure \( d\sigma \). Suppose that \( \Omega \) satisfies the following conditions.

(i) \( \Omega \) is the homogeneous function of degree zero on \( \mathbb{R}^n \setminus \{0\} \), that is,

\[ \Omega(tx) = \Omega(x), \quad \text{for any } t > 0, x \in \mathbb{R}^n \setminus \{0\}. \]

(ii) \( \Omega \) has mean zero on \( S^{n-1} \), that is,

\[ \int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0. \]

(iii) \( \Omega \in \text{Lip}_\gamma(S^{n-1}), 0 < \gamma \leq 1 \), that is, there exists a constant \( C > 0 \) such that

\[ |\Omega(x') - \Omega(y')| \leq C|x' - y'|^{\gamma} \quad \text{for any } x', y' \in S^{n-1}, \]

where \( x' = \frac{x}{|x|} \) for any \( x \neq 0 \). In 1958, Stein [38] defined the Marcinkiewicz integral of higher dimension \( \mu_{\Omega} \) satisfying conditions (8)–(10) by

\[ \mu_{\Omega}(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(f)(x)|^2 \, \frac{dt}{t^3} \right)^{1/2}, \]

where

\[ F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy. \]

The continuity of Marcinkiewicz operator \( \mu_{\Omega} \) has been extensively studied in [37,39]. Let \( H \) be the space \( H = \{ h : \|h\| = (\int_0^\infty |h(t)|^2 \, dt/t^3)^{1/2} < \infty \} \). Then, it is clear that \( \mu_{\Omega}(f)(x) = \|F_{\Omega,t}(f)(x)\|_H \).
By the Minkowski inequality and conditions (8)–(10), we get
\[
\mu f (x) = \left( \int_0^{\infty} \left| \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-1}} \chi_{B(x,t)}(y) f(y) \, dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \leq \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \left( \int_0^{\infty} \chi_{B(x,t)}(y) \, dt \right)^{1/2} \, dy \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \, dy.
\]

Since the Calderón–Zygmund operator $T$ is bounded on the spaces $M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)$, then we get the following corollary.

**Corollary 4.3:** Let $1 < p_- \leq p_+ < \infty$, $1 \leq q_- \leq q_+ < \infty$, $p, q \in \mathcal{P}_{0,\infty}(0, \infty)$, $0 \leq \lambda < 1$ and $f \in M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)$. Suppose that conditions (5) and Lemma 3.2 are satisfied. Then, the Marcinkiewicz operator $\mu f$ is bounded on the local variable Morrey–Lorentz spaces $M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)$.

In the case $\lambda = 0$, from Theorem 3.2, we get the boundedness of the operator $\mu f$ in the variable Lorentz spaces $L_{p,q}(\mathbb{R}^n)$.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

The research of Kucukaslan was supported by the grant from The Scientific and Technological Research Council of Turkey [TUBITAK Grant-1059B191600675]. The research of Guliyev was partially supported by the grant from Elmin Inkişaf Fondu [Agreement No. EIF-BGM-4-RFTF-1/2017-21/01/1-M-08]. The research of Guliyev and Serbetci was partially supported by the grant from Cooperation Program 2532 TUBITAK-RFBR (Russian Foundation for Basic Research) with Agreement Number No. 119N455.

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