Study of a functional equation associated to the Kummer’s equation of the trilogarithm.

Applications

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Abstract: In this paper we study a generalisation in 9 unknown functions of a homogeneous version of the Kummer’s equation for $L_{i3}$. We give an explicit description of the space of local holomorphic solutions at a generic point in $\mathbb{C}^2$. Then we apply this description firstly for obtaining new non linearisable maximal rank webs (confirming some results annonced about one year ago by G. Robert ([He1])). Secondly we show that under suitable conditions, the trilogarithm is the only function which verifies the Kummer equation.

1 Introduction

Since the works of Spence, Kummer and Abel, a big number of functional equations satisfied by low-order polylogarithms $\text{Li}_n$ are known ($n \leq 5$) (see [lew]). For example, it was proved by Spence and independently by Kummer that the trilogarithm $\text{Li}_3$ verifies the functional equation

$$2\text{Li}_3(x) + 2\text{Li}_3(y) - \text{Li}_3\left(\frac{x}{y}\right) + 2\text{Li}_3\left(\frac{1-x}{1-y}\right) + 2\text{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \text{Li}_3(xy)$$

$$+ 2\text{Li}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\text{Li}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \text{Li}_3\left(x\frac{(1-y)^2}{y(1-x)^2}\right) \quad (K_3)$$

$$= 2\text{Li}_3(1) - \log(y)^2 \log\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3} \log(y) + \frac{1}{3} \log(y)^3$$

for $x, y \in \mathbb{R}$ such that $0 < x < y < 1$. (from now on we note $E_3(x, y)$ the right member of $(K_3)$).
Let us introduce the interior functions which appear in $(K_3)$:

\begin{align*}
U_1(x, y) &= x \\
U_2(x, y) &= y \\
U_3(x, y) &= \frac{x}{y} \\
U_4(x, y) &= \frac{1 - y}{1 - x} \\
U_5(x, y) &= \frac{x(1 - y)}{y(1 - x)} \\
U_6(x, y) &= \frac{x(1 - y)}{y(1 - x)^2} \\
U_7(x, y) &= \frac{1 - y}{y(x - 1)} \\
U_8(x, y) &= \frac{1 - y}{y(x - 1)^2} \\
U_9(x, y) &= \frac{x(1 - y)^2}{y(1 - x)^2}
\end{align*}

They all are rational functions with real coefficients such that their level curves in $\mathbb{C}^2$ are lines, conics or cubics.

Associated to $(K_3)$, we can consider the following homogeneous functional equation in 9 unknown functions:

\[ F_1(U_1) + F_2(U_2) + F_3(U_3) + \ldots + F_9(U_9) = 0 \quad (\mathcal{E}) \]

In this paper our interest is in the local solutions of this equation, a local solution being a nine-uplet of function-germs satisfying the equation above. But we have to make it more precise.

Since we can consider equation $(\mathcal{E})$ in any neighbourhood of any $\omega \in \mathbb{C}^2 \setminus S'$, where $S'$ is the union of the polar locus of the functions $U_i$, we introduce “the space of local solutions in the class $\mathcal{F}$ of $(\mathcal{E})$ at $\omega$”:

\[ S^\mathcal{F}_\omega = \left\{ \mathbf{F} = (F_i) \in \prod_{i=1}^{9} \mathcal{F}_{\omega_i} \mid \sum_{i=1}^{9} F_i(U_i) = 0 \text{ in } \mathcal{F}_{\omega} \right\} \]

(In this definition, $\omega_i$ denotes $U_i(\omega)$ for $i = 1, \ldots, 9$, $\mathcal{F}$ is any sheaf of function-germs on $\mathbb{K}$ or $\mathbb{K}^2$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and $\mathcal{F}_\theta$ denotes the germ at $\theta \in \mathbb{K}, \mathbb{K}^2$).

We will deal with at least measurable functions and to have non pathologic situation, we will take $\omega$ generic in $\mathbb{C}^2 \setminus S'$ more precisely such that the level curves of the $U_i$’s are not tangent in $\omega$.

Let be $S := \left( \bigcup_{i < j} \left\{ \eta \in \mathbb{C}^2 \mid dU_i \wedge dU_j(\eta) = 0 \right\} \right) \cup S'$ : from now on, we will work with the sheaf of real measurable function-germs noted $\mathcal{M}$ and we will take $\omega \in \mathbb{C}^2 \setminus S$.

Because the functions $U_i$ are rational functions, we can consider that they are defined on $\mathbb{C}^2 \setminus S$ and so the equation $(\mathcal{E})$ can be seen as a complex equation in the complex field.

In the part 2 of this note, we will consider the space of holomorphic solutions of $(\mathcal{E})$ at $\omega \in \mathbb{C}^2 \setminus S$, i.e. the space

\[ S^\mathcal{O}_\omega = \left\{ \mathbf{F} = (F_i) \in \prod_{i=1}^{9} \mathcal{O}_{\omega_i} \mid \sum_{i=1}^{9} F_i(U_i) = 0 \text{ in } \mathcal{O}_{\omega} \right\} \]

By a classical result of web geometry it comes that $S^\mathcal{O}_\omega$ is a finite-dimensional $\mathbb{C}$-linear space and we have a bound $\dim_{\mathbb{C}}(S^\mathcal{O}_\omega) \leq 36$. 

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Next we will give a family $\Gamma$ of 36 linearly independent elements of $S^O$. Thus it comes that $\Gamma$ must be a basis and so it spans the whole space.

In part 3, we will apply the preceding results: the fact that $S^O$ is of maximal dimension 36 gives us some new examples of non linearisable maximal rank planar webs. We will discuss this more precisely in part 3.1.; in part 3.2. we will apply the explicit knowledge of $S^O$ given by $\Gamma$ to the problem of characterizing $L_i_3$ by the equation $(K_3)$ of Kummer, what was our initial goal.

**Remark:** 1. While I was working on the subject, I was told by G. Henkin that in a personnal communication to him ([Hé1]), A. Hénaut annoncèd that his colleague G. Robert had found that the Kummer’s web is of maximal rank by constructing an explicit basis of the space of abelian relations, what is equivalent to part 2. of this paper. But no additional informations about this were given until now.

2. This is a short version of a paper in preparation wich will display the results presented here in a more complete way as well as some new results: we will show that any local holomorphic solution of $(E)$ is “a priori” a global but multiform solution, and use this to construct a method to solve $(E)$ by considering the monodromy of those solutions.

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### 2 explicit resolution of $(E)$ in the holomorphic case

Let’s take $\omega_0 = (\frac{1}{4}, \frac{1}{2}) \in \mathbb{R}^2 \setminus S$. We will solve $(E)$ at $\omega_0$ ( but we will find later that the resolution we get gives a resolution at all $\omega' \in \mathbb{C}^2 \setminus S$).

From now on, we note $S := S^O_{\omega_0}$.

The way that we will use to explicitly solve $(E)$ is the following: we have this classical result of web geometry (see [Bl-Bd]):

**Proposition 1** Let $N$ be a positive integer and $V_1, V_2, ..., V_N$ be $N$ elements of $O_0(\mathbb{C}^2, \mathbb{C})$ such that we have the generic condition $dV_i \wedge dV_j(0) \neq 0$ ($i < j$).

Then the space \( \left\{ (G_i) \in (O_0)^N \mid \sum G_i(V_i) = 0 \right\} \) has a finite dimension less than \( \frac{N(N-1)}{2} \).

In the case of the equation $(E)$ we succeed to construct 36 linearly independant holomorphic soltions at $\omega_0$. It shows that $S$ is of maximal possible dimension what is exceptional (see part 3.1.).
First we have to consider the constant solutions of \((E)\). We can easily construct them. They form a space of complex dimension 8. Let \(\{C_i\}_{i=1,...,8}\) be a basis of it.

We have the 28 following 9-uplets of holomorphic germs. Verifying that they are 28 linearly independant elements of \(S\) would be easy, but tedious, so we will skip this part.

\[
\begin{align*}
F_1 &= \left( \log(\bullet), -\log(\bullet), -\log(\bullet), 0, 0, 0, 0, 0, 0 \right) \\
F_2 &= \left( \log\left( \frac{1}{1-\bullet} \right), 0, \log(1-\bullet), -\log(1-\bullet), 0, 0, 0, 0, 0 \right) \\
F_3 &= \left( \log(1-\bullet), -\log(1-\bullet), 0, \log(\bullet), 0, 0, 0, 0, 0 \right) \\
F_4 &= \left( 0, 0, \log(\bullet), \log(\bullet), -\log(\bullet), 0, 0, 0, 0 \right) \\
F_5 &= \left( \log(1-\bullet), 0, -\log(1-\bullet), 0, \log(1-\bullet), 0, 0, 0, 0 \right) \\
F_6 &= \left( \log(\bullet), \log(\bullet), 0, 0, 0, -\log(\bullet), 0, 0, 0 \right) \\
F_7 &= \left( \log(\bullet), 0, 0, \log(\bullet), 0, 0, -\log(\bullet) + i\pi, 0, 0 \right) \\
F_8 &= \left( \frac{1}{\bullet}, 0, 0, 0, \frac{1}{\bullet}, 0, \frac{1}{\bullet} - 1, 0, 0 \right) \\
F_9 &= \left( \log(1-\bullet), 0, 0, 0, 0, -\log(1-\bullet), \log(1-\bullet), 0, 0 \right) \\
F_{10} &= \left( 0, \bullet, 0, \bullet, 0, 0, 0, -1, 0, 0 \right) \\
F_{11} &= \left( 0, 0, 0, 0, 0, \log(\bullet), -\log(\bullet), \log(\bullet), 0 \right) \\
F_{12} &= \left( 0, \log(\bullet), 0, 0, 0, 0, -\log(1-\bullet), \log(1-\bullet), 0 \right) \\
F_{13} &= \left( 0, 0, 0, 0, 0, \log(\bullet), \log(\bullet), -\log(\bullet) - 2i\pi \right) \\
F_{14} &= \left( 0, 0, 0, 0, \log(1-\bullet), 0, \log(1-\bullet), 0, -\log(1-\bullet) \right) \\
F_{15} &= \left( 0, \frac{1}{\bullet}, 0, 0, \bullet, 0, 0, -1, 0 \right) \\
F_{16} &= \left( \bullet, 0, 0, \frac{1}{\bullet}, 0, 0, 0, \frac{1}{\bullet} - 1, 0 \right)
\end{align*}
\]
\[ \mathbf{F}_{17} = \left( 0, 0, a(\bullet), 0, 0, -a(\bullet), 0, 0, -a(\bullet) \right) \]
\[ \mathbf{F}_{18} = \left( 2\log^2(\bullet), 2\log^2(\bullet), -\log^2(\bullet), 0, 0, -\log^2(\bullet), 0, 0, 0 \right) \]
\[ \mathbf{F}_{19} = \left( 0, 0, 0, 0, \log^2(\bullet), -2\log(\bullet)^2 - 2\log(\bullet)^2, \log^2(\bullet) + 4i\pi \log(\bullet) - 4\pi^2 \right) \]
\[ \mathbf{F}_{20} = \left( 0, 0, \log^2(\bullet), -2\log^2(\bullet), 0, 0, 0, \log^2(\bullet) \right) \]
\[ \mathbf{F}_{21} = \left( d(\bullet), -d(\bullet), -d(\bullet), -d(\bullet), d(\bullet), 0, 0, 0, 0 \right) \]
\[ \mathbf{F}_{22} = \left( d(\bullet), d(\bullet) - \frac{i\pi}{2} \log(\bullet), 0, 0, 0, -d(\bullet), d(\bullet), -d(\bullet), 0 \right) \]
\[ \mathbf{F}_{23} = \left( \pi^2, 0, 0, d(\bullet) - \frac{i\pi}{2} \log(\bullet), d(\bullet), 0, d(\bullet), d(\bullet) + \frac{i\pi}{2} \log(\bullet) - i\pi \log(1 - \bullet), -d(\bullet) \right) \]
\[ \mathbf{F}_{24} = \left( \log^3(\bullet) \log(\bullet) \log(\bullet), 0, -\log^3(\bullet) \log(\bullet), \log^3(\bullet) - \frac{1}{2} \log^2(\bullet) + i\pi \log(\bullet), \frac{\pi^2}{3} \right) \]
\[ \mathbf{F}_{25} = \left( 0, \frac{1}{2} \log^2(\bullet), 0, \log^3(\bullet) \log(\bullet) \log(\bullet), 0, \log^3(\bullet) \log(\bullet), \log^3(\bullet) \log(\bullet), -\log^3(\bullet) \log(\bullet) \right) \]
\[ \mathbf{F}_{26} = \left( 2\log^2(\bullet), 0, -\log^3(\bullet), 0, 2\log^2(\bullet), -\log^3(\bullet), 2\log^2(\bullet), 0, -\log^3(\bullet) \right) \]
\[ \mathbf{F}_{27} = \left( 2g(\bullet), 2g(\bullet), -g(\bullet), 2g(\bullet), 2g(\bullet), -g(\bullet), 2\bar{g}(\bullet), -g(\bullet) \right) \]
\[ \mathbf{F}_{28} = \left( 2h(\bullet), 2h(\bullet) - \frac{2\pi^2}{3} \log(\bullet), -h(\bullet), 2h(\bullet), 2h(\bullet), -h(\bullet), 2\bar{h}(\bullet), 2\bar{h}(\bullet), -h(\bullet) \right) \]

with

\[ \bullet := \text{Id}_C \]
\[ a(\bullet) := \text{arcth} \left( \sqrt{\bullet} \right) \]
\[ d(\bullet) := \log^3(\bullet) - \frac{1}{2} \log(\bullet) \log(1 - \bullet) - \frac{\pi^2}{6} \]
\[ g(\bullet) := \log^3(\bullet) - \log(\bullet) \log^2(\bullet), \log(\bullet) - \frac{1}{3} \log^2(\bullet) \log(1 - \bullet) - \frac{2}{9} \log^3(\bullet) \log(1 - \bullet) \]
\[ h(\bullet) := 2\log(\bullet) \log^2(\bullet) + \log^3(\bullet) \log(1 - \bullet) \]
\[ \bar{g}(\bullet) := g(\bullet) - \frac{i\pi}{3} \log^2(\bullet) + \frac{4i\pi}{3} d(\bullet) + \frac{\pi^2}{3} \log(1 - \bullet) + \frac{2i\pi^3}{9} \]
\[ \bar{h}(\bullet) := h(\bullet) + 2i\pi \log^2(\bullet) - 4i\pi d(\bullet) - \pi^2 \log(1 - \bullet) - \frac{2i\pi^3}{3} \]

where all those functions are considered holomorphic functions on the whole simply connected domain \( \mathbb{C} \setminus \{0\} \times \mathbb{R}^- \cup \{1\} \times \mathbb{R}^+ \), functions which correspond to their usual definition on \([0,1[\).
If we consider the family $\Gamma = \{ C_i, F_j \mid 1 \leq i \leq 8, 1 \leq j \leq 28 \}$ we obtain a family of 36 linearly independent elements of $\mathcal{S}$. From the proposition 1, we know that $\dim_{\mathbb{C}}(\mathcal{S}) \leq \frac{9(9-1)}{2} = 36$, and as we have seen before, this implies that $\Gamma$ is a basis of $\mathcal{S}$.

So we have

$$\mathcal{S}_{\omega_0}^O(\mathcal{E}) = \text{Vect}_{\mathbb{C}}(\langle \Gamma \rangle)$$

**Remark:** Let $\omega' \in \mathbb{C}^2 \setminus S$ be different of $\omega_0$. There is a path $\gamma$ in $\mathbb{C}^2 \setminus S$ connecting $\omega_0$ to $\omega'$. If $F = (F_1, \ldots, F_9) \in \mathcal{S}$ we can easily see that each $F_i$ admits an analytic continuation along the path $\gamma_i := U_i \circ \gamma$ because this is verified for any element of the basis $\Gamma$. It gives a holomorphic germ at $\omega' := U_i(\omega')$ noted $F_i^{[\gamma_i]}$. Then by analytic continuation along $\gamma$ and by the unicity principle we get

$$\sum_{i=1}^{9} F_i^{[\gamma_i]}(U_i) = 0 \quad \text{in} \quad \mathcal{O}_{\omega'}$$

and so $F^{[\gamma]} := (F_i^{[\gamma_i]}) \in \mathcal{S}_{\omega'}^O$. It’s clear that the application $F \rightarrow F^{[\gamma]}$ is a linear isomorphism between $\mathcal{S}$ and $\mathcal{S}_{\omega'}^O$. This way, we can explicitly solve the equation $(\mathcal{E})$ at any point of $\mathbb{C}^2 \setminus S$.

### 3 Applications

The fact that the dimension of $\mathcal{S}$ is maximal and the explicit description of $\mathcal{S}$ both allow us to obtain some new results in two a priori distinct subjects: the theory of planar webs and the theory of polylogarithms.

For an introduction to the web theory, we refer to the basic book of Blaschke and Bol “Geometrie der Gewebe” [Bla-Bol] and to [Ch-Gr1], [Che] or [Web] for a more modern point of view. As for polylogarithms, we refer to the books [Lew] and [Pol] and to the talk of J. Osterl´e at the s´ eminaire Bourbaki (see [Ost]).

#### 3.1 Applications to web theory

We suppose that the basic notions of web geometry are known. From Bol’s counterexample we know that not all the webs of maximal rank are linearisable (and so algebraic): his web noted $\mathcal{B}$ is the global singular 5-web on $\mathbb{C}P^2$, the 5 foliations of which are given by the level curves of the functions $U_1, U_2, \ldots$ and $U_5$. Let’s call $S_B$ the singular locus of $\mathcal{B}$: it is the union of the polar locus of the $U_i$’s ($i \leq 5$) with the algebraic set $\cup_{1 \leq i \leq 5} \{ w \in \mathbb{C}^2 \mid dU_i \wedge U_j(w) = 0 \}$.

It is a sub-web of the global singular 9-web noted $\mathcal{K}$ (for “Kummer”) defined by the level curves of the functions $U_i$ for $i = 1, \ldots, 9$. Its singular locus is $S_K = S$.
From the elements $F_1, F_2, F_3, F_4, F_5$ and $F_{21}$ of $S$ we can construct a base of the space $A(B)$ of the abelian relations of $B$ at $\omega_0$.

So we have $\dim S = 6$ and the web $B$ is of maximal rank 6, although it is not linearisable. From its discovery by Bol in the 30’s onwards, this was the single known counterexample to the problem of linearisation of planar webs of maximal rank.

From the fact that $\dim S = 36$, we easily get that $\dim A(K) = 28$, and because $K$ is not linearisable, $K$ is another example of this kind of web called “exceptional planar webs” by S.S. Chern.

According to Chern and Griffiths (see [Ch-Gr2] page 83), classifying the non linearisable maximal rank webs is the fundamental problem in web geometry. This explain the importance of this new example of exceptional web.

But the explicit knowledge of the basis $\Gamma$ of $S$ allows to study all the subwebs of $K$. For any subset $J \subset \{1, \ldots, 9\}$ we note $\hat{T}_{\bowtie J}$ the $|J|$-subweb of $K$ given by the level curves of the function $U_j$, with $j \in J$.

If $j_1, \ldots, j_p$ are $p$ distinct integers in $\{1, \ldots, 9\}$, then we note $\hat{T}_{\bowtie j_1 \ldots j_p} := \{1, \ldots, 9\} \setminus \{j_1, \ldots, j_p\}$.

Proposition 2 :

- $K$ is an exceptional 9-web
- $\hat{T}_{\bowtie 69}$ is an exceptional 7-web
- $\hat{T}_{\bowtie 679}$ is an exceptional 6-web
- $\hat{T}_{\bowtie 248}$ is an exceptional 6-web

Thoses two exceptional 6-webs are not equivalent (up to local diffeomorphism).

- $\hat{T}_{\bowtie 369}$ is a maximal-rank hexagonal 6-web

Remarks:

1. By Bol’s theorem (see [Bl-Bo]), the fact that $\hat{T}_{\bowtie 369}$ is hexagonal implies that it is linearisable in a web formed by 6 pencils of lines. So it is algebraic, and the associated algebraic curve is an union of 6 lines in $\mathbb{C}P^2$.

2. The subwebs $\hat{T}_{\bowtie 36}$ and $\hat{T}_{\bowtie 39}$ are exceptional too but equivalent to $\hat{T}_{\bowtie 69}$.

3. The subwebs $\hat{T}_{\bowtie 689}$, $\hat{T}_{\bowtie 239}$, $\hat{T}_{\bowtie 359}$, and $\hat{T}_{\bowtie 136}$ are exceptional too but equivalent to $\hat{T}_{\bowtie 679}$.

4. The subwebs $\hat{T}_{\bowtie 147}$, $\hat{T}_{\bowtie 257}$, and $\hat{T}_{\bowtie 158}$ are exceptional too but equivalent to $\hat{T}_{\bowtie 248}$.

5. The two 6-webs $\hat{T}_{\bowtie 679}$ and $\hat{T}_{\bowtie 248}$ are not equivalent because one can prove that $\hat{T}_{\bowtie 679}$ contains an exceptional 5-subweb (the Bol’s web $B$!), contrarily to $\hat{T}_{\bowtie 248}$, 5-subwebs of which have rank 5 and so are not exceptional.

5. We have a beautiful functional equation for $\text{Li}_2$ associated to $\hat{T}_{\bowtie 248}$ which is given by the element $F_{26}$ of $\Gamma$:

$$2\text{Li}_2(x) - \text{Li}_2\left(\frac{x}{y}\right) + 2\text{Li}_2\left(\frac{x(1-y)}{y(1-x)}\right) - \text{Li}_2(xy) + 2\text{Li}_2\left(-\frac{x(1-y)}{1-x}\right) - \text{Li}_2\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0$$

(I have not seen an equation of this form in the bibliography).
3.2 application to the caracterisation of $\text{Li}_3$ by the equation $(\mathcal{K}_3)$

Our objective here is to study the function which satisfies the equation $(\mathcal{K}_3)$. This kind of problem has been studied for a long time for the Cauchy equation $(\mathcal{C})$: we know that any non-constant measurable local solution of $(\mathcal{C})$ is constructed from the logarithm.

There is similar results for the dilogarithm (see [Kie] and [Blo]).

In his paper [Gon], A. Goncharov obtains some results of the same kind for the logarithm.

He considers the real single-valued cousin of $\text{Li}_3$ introduced by Ramakrishnan and Zagier:

$$\mathcal{L}_3(z) := \Re \left( \text{Li}_3(z) - \log |z| \text{Li}_2(z) + \frac{1}{3} \log |z|^2 \text{Li}_1(z) \right)$$

defined on the whole $\mathbb{CP}^1$ and extended to $\mathbb{R}[\mathbb{CP}^1]$ by linearity.

When it is well defined, he considers the following element of $\mathbb{Q}[\mathbb{CP}^1]$:

$$R_3(\alpha_1, \alpha_2, \alpha_3) := \sum_{i=1}^{3} \left( \{ \alpha_{i+2} \alpha_i - \alpha_i + 1 \} + \{ \frac{\alpha_{i+2} \alpha_i - \alpha_i + 1}{\alpha_{i+2} \alpha_i} \} + \{ \alpha_{i+2} \} 
+ \{ \frac{\alpha_{i+2} \alpha_{i+1} - \alpha_{i+1} + 1}{(\alpha_{i+2} \alpha_i - \alpha_i + 1) \alpha_{i+1}} \} - \{ \frac{\alpha_{i+2} \alpha_i - \alpha_i + 1}{\alpha_{i+2}} \} - \{ 1 \}
- \{ \frac{\alpha_{i+2} \alpha_{i+1} - \alpha_{i+1} + 1}{(\alpha_{i+2} \alpha_i - \alpha_i + 1) \alpha_{i+1} \alpha_{i+2}} \} + \{ \frac{\alpha_{i+2} \alpha_{i+1} - \alpha_{i+1} + 1}{\alpha_{i+2} \alpha_i - \alpha_i + 1} \}
+ \{ -\alpha_{i+2} \alpha_3 \} \right)$$

for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{CP}^1$. (The indices $i$ are taken modulo 3).

Next he proves that we have the functional equation in 22 terms

$$(**) \quad \mathcal{L}_3(R_3(a, b, c)) = 0 \quad a, b, c \in \mathbb{C}$$

Then he shows (part (a) of Theorem 1.10 in [Gon]) that

"the space of real continuous functions on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ that satisfy the functional equation (**) is generated by the functions $\mathcal{L}_3(z)$ and $D_2(z). \log(|z|)"$

where $D_2$ is the Bloch-Wigner function attached to $\text{Li}_2$ defined for $z \neq 0, 1, \infty$ by $D_2(z) := \Im (\text{Li}_2(z) + \log(1 - z). \log(|z|))$.

He had remarked before that if we specialize this equation by setting $a = 1, b = x$, and $c = \frac{1}{1-x}$, the equation (**) simplifies and by using the inversion relation $\mathcal{L}_3(x^{-1}) = \mathcal{L}_3(x)$, $x \in \mathbb{CP}^1$, it gives us exactly a homogeneous version (i.e. without the second member $E_3(x, y)$) of the equation $(\mathcal{K}_3)$.

This leads him to ask if this specialization characterizes the solutions of (**).
The explicit determination of a basis of $S$ done in part 2 allows us to give a positive answer to this question, in term of the function $L_i$.

Let’s first begin by a result of regularity for the measurable solutions of $(E)$:

**Proposition 3** Let $\omega \in \mathbb{R}^2 \setminus S$ and $F = (F_1, \ldots, F_9) \in S^M(\mathcal{E})$. Then each $F_i$ is in fact an analytic germ at $\omega_i$. Its complexification gives a germ $F^c_i \in O_{\omega_i}$ such that $F^c := (F^c_1, \ldots, F^c_9)$ is a holomorphic solution of $(\mathcal{E})$ at $\omega$.

**Sketch of the proof:** Because the level curves of the $U_i$’s are in generic position near $\omega$, it comes from the paper of A. Jarai that the $F_i$’s are continuous germs (see Theorem 3.3. in [Jar]). By elementary tools of integration you get next that they are smooth germs. Then, similarly as Joly and Rauch in [Jo-Ra], one formulates the equation $(E)$ in a differential form. By an argument of ellipticity and by using Petrowsky’s theorem (see [Pet]), you finally get that the $F_i$’s are analytic germs.

So we have two $\mathbb{R}$-linear morphisms:
the first is just the restriction to $\mathbb{R}^2$ with taking real part

$$
\rho : \frac{S^0(\mathcal{E})}{G} \longrightarrow \frac{S^M(\mathcal{E})}{\mathbb{R}e(G|_{\mathbb{R}^2})}
$$

and the second is given by the proposition 3

$$
\varphi : \frac{S^M(\mathcal{E})}{F} \longrightarrow \frac{S^0(\mathcal{E})}{F^c}
$$

It is clear that $\varphi \circ \rho = \text{Id}_{S^M(\mathcal{E})}$ and so the study of measurable solutions of $(\mathcal{E})$ at $\omega$ amounts to the study of the holomorphic solutions done in part 2.

We have this real semi-local characterization of $L_i$ by the equation $(K_3)$:

**Proposition 4** Let $\epsilon_0$ be a real such that $\frac{\sqrt{5} - 1}{2} < \epsilon_0 < 1$ and $F : -\infty, 1 \rightarrow \mathbb{R}$ be a measurable function such that for $0 < x < y < \epsilon_0$ we have

\[
2F(U_1(x, y)) + 2F(U_2(x, y)) - F(U_3(x, y)) + 2F(U_4(x, y)) + 2F(U_5(x, y)) - F(U_6(x, y)) + 2F(U_7(x, y)) + 2F(U_8(x, y)) - F(U_9(x, y)) = E_3(x, y)
\]

- If $F$ is continuous at 0 then there exists $a \in \mathbb{R}$ such that
  \[
  F := L_i + a (L_i - \frac{2}{g}L_i(1))
  \]
- If $F$ is derivable at 0 then $F := L_i$.

**Proof:** With our results of part 2 and the preceding remark, it is just a tedious exercise of linear algebra. The following statement, precisely related to Goncharov’s question, is equivalent to the preceding:
corollary 1 Let $\varepsilon_0$ be a real such that $\frac{\sqrt{5} - 1}{2} < \varepsilon_0 < 1$ and $\mathcal{G} : (-\infty, 1] \rightarrow \mathbb{R}$ be a measurable function such that for $0 < x < y < \varepsilon_0$ we have 

$$2\mathcal{G}(x) + 2\mathcal{G}(y) - \mathcal{G}\left(\frac{x}{y}\right) + 2\mathcal{G}\left(\frac{1 - y}{1 - x}\right) + 2\mathcal{G}\left(\frac{x(1 - y)}{y(1 - x)}\right) - \mathcal{G}(xy) + 2\mathcal{G}\left(\frac{x(1 - y)}{x - 1}\right) + 2\mathcal{G}\left(\frac{y - 1}{y(1 - x)}\right) - \mathcal{G}\left(\frac{x(1 - y)^2}{y(1 - x)^2}\right) = 2\mathcal{L}_3(1)$$

Then if we suppose $\mathcal{G}$ continuous at 0, then there exists $\alpha \in \mathbb{R}$ such that 

$$\mathcal{G} = \alpha L_3 + \frac{2}{9}(1 - \alpha) L_3(1)$$

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