ABUNDANCE OF MATRICES IN GAUSSIAN INTEGERS

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Abstract. In [HLS], N. Hindman, I. Leader and D. Strauss proved the abundance for a matrix with rational entries. In this paper we proved it for the ring of Gaussian integers. We showed the result when the matrix is taken with entries from $\mathbb{Q}[i]$. The main obstacle is in the field of complex numbers, no linear order relation exists. We overcome that in a tactful way.

1. Introduction

Some problems in Ramsey Theory can be seen as follows:

Let $M$ be any $r \times s$ matrix, where $r, s \in \mathbb{N}$ and all entries of $M$ are from $\mathbb{N}$. Then, could it be possible for each and every partition (finite) of $\mathbb{N}$, one can find some $\vec{u} \in \mathbb{N}^s$ so that all entries from $M\vec{u}$ are in one cell from the partition?

At first, take van der Waerden’s Theorem. Then the elements from the arithmetic progression \( \{a, a + d, \ldots, a + ld\} \) is nothing but entries from

\[
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
\vdots & \vdots \\
1 & l
\end{pmatrix}
\cdot \begin{pmatrix} a \\ d \end{pmatrix}.
\]

So, we have this typical definition:

**Definition 1.1.** Let \((T, +)\) be a commutative monoid. Suppose \(r, s \in \mathbb{N}\) and \(M\) be any \(r \times s\) matrix which has entries in $\mathbb{N} \cup \{0\}$. Then \(M\) is called image partition regular on \(T\) (symbolised as \(IPR/T\)) if and only if for each \(k \in \mathbb{N}\) and \(T = \bigcup_{j=1}^{k} D_i\), then there must exist \(j \in \{1, 2, \ldots, k\}\) and \(\vec{u} \in (T \setminus \{0\})^s\) so that \(M\vec{u} \in D_j^r\).

Let \((T, +)\) be a commutative monoid, \(t \in T\) be any element and \(m \in \mathbb{N}\). Then we denote the \(m\) times sum of \(t\) with itself by \(mt\).

In 1993, N. Hindman and I. Leader has introduced image partition regularity of matrices in [1]. Next in [HLS] authors provided some new characterizations and consequences of image partition regularity. One of there characterizations is that the image partition regular matrices are precisely those that preserve a certain notion of largeness (“central sets”). In [DJ], authors completely accomplised the determination of image partition regular matrices over $\mathbb{Z}[i]$.

Furstenberg and Glasner have shown in [F1] that for a particular notion of largeness in a group, namely piecewise syndeticity, if a set \(B\) is a large subset $\mathbb{Z}$, then for any \(l \in \mathbb{N}\), the set of length \(l\) arithmetic progressions lying entirely in \(B\) is large among the set of all length \(l\)-arithmetic progressions. In [BH], authors extend this result to apply to infinitely many notions of largeness in arbitrary semigroups and to other partition regular structures besides arithmetic progressions. In [HLS], authors showed the abundance of image partition regular matrices. Also, they examined for
other well known notions of largeness. (Like IP*, Δ*, PS*, piecewise syndetic, thick etc.). To obtain these results, usual order relation of \( \mathbb{N} \) played an important role. We have extend these results of abundance of matrices on \( \mathbb{Z}[i] \), the ring of Gaussian integers. In \( \mathbb{Z}[i] \), there is no order relation. So, we need to derive these results using different technique.

First we need to prove this lemma before proceed to the main results:

**Lemma 1.2.** Let \( A \) be a \( u \times v \) image partition regular matrix over \( \mathbb{Q}[i] \). The following are equivalent:

(a) There exists \( \tilde{s} \in (\mathbb{Q}[i])^v \) such that \( A\tilde{s} = \tilde{1} \).

(b) There exists \( l \in \mathbb{Z}[i] \setminus \{0\} \) such that, if \( p \) is in the smallest ideal of \( \beta \mathbb{Z}[i] \) and \( l\mathbb{Z}[i] \in p \), then, for every \( P \in p \), there exists \( \bar{z} \in (\mathbb{Z}[i])^v \) such that \( A\bar{z} \in P^u \).

**Proof.** (a) ⇒ (b). We can choose \( \bar{w} \in (\mathbb{Z}[i])^v \) and \( l \in \mathbb{Z}[i] \setminus \{0\} \) such that \( A\bar{w} = \bar{l} \), where \( \bar{l} = (l, l, \ldots, l)^T \in (\mathbb{Z}[i] \setminus \{0\})^u \). Suppose that \( p \) is in the smallest ideal of \( \beta \mathbb{Z}[i] \) and that \( l\mathbb{Z}[i] \in p \). Let \( P \in p \). There exists a minimal idempotent \( q \in \beta \mathbb{Z}[i] \) such that \( p = p + q \) \cite{HS}, Theorem 2.8 and Lemma 1.30. Let \( P = \{ b \in P : b \in P \} \). Then \( P' \in p \). Since \( l\mathbb{Z}[i] \in p \) we can choose \( t \in \mathbb{Z}[i] \) such that \( lt \in \mathbb{Z}[i] \). Let \( Q = -lt + P \in q \). By \cite{DJ}, Theorem 3.1(h), there exists \( \bar{x} \in (\mathbb{Z}[i] \setminus \{0\})^v \) such that \( A\bar{x} \in Q^v \) and for every \( i \in \{1, 2, \ldots, v\} \). Then \( t\bar{w} + \bar{x} \in (\mathbb{Z}[i] \setminus \{0\})^v \) and \( A(t\bar{w} + \bar{x}) = lt + A\bar{x} \in P^u \).

(b) ⇒ (a). We may suppose that the entries of \( A \) are in \( \mathbb{Z}[i] \) as we could replace \( A \) by \( nA \) for a suitable \( n \in \mathbb{N} \).

Suppose that \( \bar{l} \notin \{ A\bar{x} : \bar{x} \in (\mathbb{Q}[i])^v \} \). Then there exists \( \bar{u} \in (\mathbb{Z}[i])^v \) such that \( \bar{u} \cdot A\bar{x} = 0 \) for every \( \bar{x} \in (\mathbb{Q}[i])^v \), but \( \bar{u} \cdot \bar{l} \neq 0 \).

Choose a Gaussian prime number \( r \in \mathbb{Z}[i] \setminus \{0\} \) satisfying \( |r| > |l| \) and \( |r| > |l| \). Let \( q \) be a minimal idempotent in \( \beta \mathbb{Z}[i] \) and let \( p = l + q \). Then \( P = \{ t \in \mathbb{Z}[i] \setminus \{0\} : t \equiv l \pmod{r} \} \in p \). Then \( t \equiv l \pmod{r} \), a contradiction, as \( \bar{u} \cdot A\bar{x} \equiv 0 \pmod{r} \), but \( \bar{u} \cdot \bar{l} \neq 0 \pmod{r} \). \( \square \)

2. **Large sets in \( \mathbb{Z}[i] \)**

There are several notions of largeness that make sense in any semigroup. The notion of “central” sets is one of these. Among the others are the notions of “syndetic”, “piecewise syndetic”, “IP”, and “Δ” sets.

**Definition 2.1.** Let \( (S, +) \) be a commutative semigroup and let \( B \subseteq S \).

(a) The set \( B \) is syndetic if and only if there exists some \( G \in \mathcal{P}_f (S) = \{ H \subseteq S : H \text{ is finite and nonempty} \} \) such that \( S = \bigcup_{t \in G} t + B \).

(b) The set \( B \) is piecewise syndetic if and only if there exists some \( G \in \mathcal{P}_f (S) \) such that for every \( F \in \mathcal{P}_f (S) \) there exists \( x \in S \) such that \( F + x \subseteq \bigcup_{t \in G} t + B \).

(c) The set \( B \) is an IP set if and only if there exists a sequence \( \langle x_n \rangle_{n=1}^\infty \) in \( S \) such that \( FS (\langle x_n \rangle_{n=1}^\infty) \subseteq B \), where \( FS (\langle x_n \rangle_{n=1}^\infty) = \{ \sum_{n \in F} x_n : F \in \mathcal{P}_f (\mathbb{N}) \} \) and the sums are taken in increasing order of indices.

(d) The set \( B \) is a Δ set if and only if there exists a sequence \( \langle x_n \rangle_{n=1}^\infty \) in \( S \) such that for every \( n, m \in \mathbb{N} \) with \( n < m \), \( x_m \in (x_n + B) \).

Notice that, if \( S \) can be embedded in a group \( G \), \( B \) is a Δ set if and only if there is a sequence \( \langle x_n \rangle_{n=1}^\infty \) such that \( \{ -x_n + x_m : m, n \in \mathbb{N} \text{ and } n < m \} \subseteq B \). Notice
also that any IP set is a $\Delta$ set. (Given $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq B$ and given $n \in \mathbb{N}$, let $y_n = \sum_{i=1}^n x_n$). Given any property $\mathcal{E}$ of subsets of a set $X$, there is a dual property $\mathcal{E}^*$ defined by specifying that a subset $B$ of $X$ is an $\mathcal{E}^*$ set if and only if $B \cap A \neq \emptyset$ for every $\mathcal{E}$ set $A$.

**Definition 2.2.** Let $(S,+)$ be a commutative semigroup and let $B \subseteq S$. Then $B$ is a central* set if and only if $B \cap A \neq \emptyset$ if for every central set $A$ in $S$. Also, $B$ is a PS* set if and only if $B \cap A \neq \emptyset$ for every piecewise syndetic set $A$ in $S$. $B$ is an IP* set if and only if $B \cap A \neq \emptyset$ for every IP set $A$ in $S$. $B$ is a syndetic* set if and only if $B \cap A \neq \emptyset$ for every syndetic set $A$ in $S$, and $B$ is a $\Delta^*$ set if and only if $B \cap A \neq \emptyset$ for every set $A$ in $S$.

The concept of “syndetic*” is more commonly referred to as “thick”, and we shall follow this practice.

The sets $\Delta$ and $\Delta^*$ are interesting because they arise as sets of recurrence, which in turn have significant combinatorial properties. (See [F].) The other notions discussed above have simple, and useful, algebraic characterizations in terms of $\beta S$.

**Lemma 2.3.** Let $(S,+)$ be a commutative semigroup and let $B \subseteq S$.
(a) $A$ is piecewise syndetic if and only if $A \cap K(\beta S) \neq \emptyset$.
(b) $A$ is IP if and only if there is some idempotent of $\beta S$ in $A$.
(c) $A$ is syndetic if and only if $A \cap L(\beta S) \neq \emptyset$.
(d) $A$ is central if and only if there is some minimal idempotent of $\beta S$ in $A$.
(e) $A$ is central* if and only if $A \cap \beta S = \emptyset$.
(f) $A$ is thick if and only if $A$ contains a left ideal of $\beta S$.
(g) $A$ is IP* if and only if every minimal idempotent of $\beta S$ is in $A$.
(h) $A$ is PS* if and only if every central set $B \subseteq S$.

**Proof.** Statement (a) is [HS Theorem 4.40], (b) is [HS Theorem 5.12], (c) is [BHM Theorem 2.9(d)], and (d) is the definition of central. Statements (e), (f), (g), and (h) follow easily from statements (d), (c), (b), and (a) respectively. \(\square\)

Now we will prove a lemma we need.

**Lemma 2.4.** Let $A$ be a $u \times v$ image partition regular matrix over $\mathbb{Q}[i]$. The following are equivalent:
(a) There exists $\tilde{s} \in \mathbb{Q}[i]^v$ such that $A\tilde{s} = \tilde{1}$.
(b) There exists $l \in \mathbb{Z}[i] \setminus \{0\}$ such that, if $p$ is in the smallest ideal of $\beta \mathbb{Z}[i]$ and $l\mathbb{Z}[i] \in p$, then, for every $P \in p$, there exists $\tilde{z} \in (\mathbb{Z}[i] \setminus \{0\})^u$ such that $A\tilde{z} \in P^v$.

**Proof.** (a) $\Rightarrow$ (b). We can choose $\tilde{v} \in (\mathbb{Z}[i] \setminus \{0\})^v$ and $l \in \mathbb{Z}[i] \setminus \{0\}$ such that $A\tilde{v} = \tilde{l}$, where $\tilde{l} = (ll \ldots l)^T \in (\mathbb{Z}[i] \setminus \{0\})^u$. Suppose that $p$ is in the smallest ideal of $\beta \mathbb{Z}[i]$ and that $l\mathbb{Z}[i] \in p$. Let $P \in p$. There exists a minimal idempotent $q \in \beta \mathbb{Z}[i]$ such that $p = p + q$ [HS Theorem 2.8 and Lemma 1.30]. Let $P' = \{v \in P : -v + p \in q\}$ and $P'' = P' + q$. Since $l\mathbb{Z}[i] \in p$ we can choose $m \in \mathbb{Z}[i] \setminus \{0\}$ such that $lm \in P''$. Let $Q = -lm + P \in q$. By [DD Theorem 3.1(h)], there exists $\tilde{z} \in (\mathbb{Z}[i] \setminus \{0\})^v$ such that $A\tilde{z} \in Q^v$ and $|z_i + mu_i| > 0$ for every $i \in \{1,2,\ldots,v\}$. (The fact that the entries of $\tilde{z}$ can be chosen to be arbitrary large follows from [DD Lemma 2.3].) For every $r \in \mathbb{N}$, $\{\tilde{z} \in (\mathbb{Z}[i] \setminus \{0\})^v : |z_i| > r\}$ for all $i \in \{1,2,\ldots,v\}$ is a member of every idempotent in $\beta (\mathbb{Z}[i]^v)$. Then $m\tilde{z} + \tilde{z} \in (\mathbb{Z}[i] \setminus \{0\})^v$ and $A(m\tilde{z} + \tilde{z}) = m\tilde{l} + A\tilde{z} \in P^v$.\(\square\)
Theorem 2.6. In a given set is large among the set of all images. The results apply in the current context in terms of when the set of images contained progressions (including the constant ones), then \( Z \in (\mathbb{Z} \times [i])^u \), such that \( \bar{A} \in \mathbb{P} \), and let \( p = l + q \). Then
\[
P = \{ v \in \mathbb{Z} \times [i] \setminus \{ 0 \} : v \equiv 1 \pmod{r} \} \in p \quad \text{(by [Lemma 2.14]).}
\]
It follows from (b) that there exists \( \bar{z} \in \mathbb{Z} \times [i] \) such that \( A \bar{z} \in \mathbb{P}^u \) and hence that \( A \bar{z} \equiv l \bar{I} \pmod{r} \). This is a contradiction, as \( \bar{u} \cdot A \bar{z} \equiv 0 \pmod{r} \), but \( \bar{u} \cdot l \bar{I} \) is not divisible by \( r \).

The following theorem relates image partition regular matrices and piecewise syndetic sets.

Theorem 2.5. Let \( A \) be a \( u \times v \) image partition regular matrix over \( \mathbb{Q} \times [i] \). The following statements are equivalent:

(a) For every piecewise syndetic subset \( P \) of \( \mathbb{Z} \times [i] \), there exists \( \bar{z} \in (\mathbb{Z} \times [i])^u \) such that \( A \bar{z} \in \mathbb{P}^u \);

(b) There exists \( \bar{z} \in (\mathbb{Z} \times [i])^u \) such that \( A \bar{z} \equiv \bar{I} \), where \( \bar{I} \) denotes the vector in \((\mathbb{Z} \times [i])^u\) whose entries are all equal to 1.

Proof. This follows easily from Lemma 2.3(a) and the proof of (a) \( \Leftrightarrow \) (b) in Lemma 2.3, taking \( l = 1 \).

In [BHK] it was shown that if “large” meant any of “\( \Delta \),” “IP,” “central,” “central*,” “IP*,” or “\( \Delta \)”, and \( B \) is a large subset of \( \mathbb{N} \), then for every positive \( \alpha \in \mathbb{R} \) and every \( \gamma \in \mathbb{R} \) with \( 0 < \gamma < 1 \), \( \{\lfloor \alpha n + \gamma \rfloor : n \in B \} \) is also large (in the same sense). In [F1] it was shown that if \( B \) is a piecewise syndetic subset of \( \mathbb{Z} \times [i] \), and \( AP^l = \{(a, a + d, \ldots, a + (l - 1)d) : a, d \in \mathbb{Z} \} \), the group of length \( l \) arithmetic progressions (including the constant ones), then \( B^l \cap AP^l \) is piecewise syndetic in \( AP^l \). In [BH] a systematic study of this latter phenomenon was undertaken. These results apply in the current context in terms of when the set of images contained in a given set is large among the set of all images.

Theorem 2.6. Let \( A \) be a \( u \times v \) matrix with entries from \( \mathbb{Q} \times [i] \), let \( I = \{ A \bar{z} : \bar{z} \in (\mathbb{Z} \times [i])^u \} \cap (\mathbb{Z} \times [i])^u \), and let \( C \subseteq \mathbb{Z} \times [i] \).

(a) If \( I \neq \emptyset \), “large” is any of “IP*”, “\( \Delta \)”, “PS*”, or “central*”, and \( C \) is large in \( \mathbb{Z} \times [i] \), then \( I \cap C^u \) is large in \( I \).

(b) If \( \bar{I} \in I \), “large” is any of “piecewise syndetic”, “central”, or “thick”, and \( C \) is large in \( \mathbb{Z} \times [i] \), then \( I \cap C^u \) is large in \( I \).

Proof. (a) For IP* and \( \Delta \), [BH] Corollary 2.3 requires only that \( I \) be a subsemigroup of \((\mathbb{Z} \times [i])^u\). For PS* and central*, [BH] Corollary 2.7 requires in addition that for each \( i \in \{1, 2, \ldots, u\} \), the \( i \)-th projection \( \pi_i[I] \) be piecewise syndetic in \( \mathbb{Z} \times [i] \). This trivially holds because, if \( z \in \pi_i[I] \), then \( z \mathbb{Z} \times [i] \in \pi_i[I] \).

(b) Letting \( E = I \), we have that
\[
\begin{pmatrix}
a \\
a \\
a \\
\vdots \\
a
\end{pmatrix} : a \in \mathbb{Z} \times [i] \subseteq E
\]
so that [BH] Theorem 3.7 applies.

We shall be concerned for the rest of this section with establishing analogues of [DJ] Theorem 3.1(i) for the other notions of largeness. That is, we wish to determine conditions that guarantee that if a set C is “large” in \( \mathbb{Z}[i] \), then \( \{ \tilde{x} \in (\mathbb{Z}[i])^n : A\tilde{x} \in C^n \} \) is “large” in \((\mathbb{Z}[i])^n\).

**Lemma 2.7.** Let \( A \) be a \( u \times v \) matrix with entries from \( \mathbb{Z}[i] \), define \( \varphi : (\mathbb{Z}[i])^u \to (\mathbb{Z}[i])^n \) by \( \varphi(x) = A\tilde{x} \), and let \( \tilde{\varphi} : (\mathbb{Z}[i])^n \to (\beta\mathbb{Z}[i])^n \) be its continuous extension. Then \( \tilde{\varphi} \) is a homomorphism and \( (K((\beta\mathbb{Z}[i])^n))^u = (K(\beta\mathbb{Z}[i]))^n \).

(a) If there exists \( \tilde{z} \in (\mathbb{Z}[i])^v \) such that \( A\tilde{z} \in (\mathbb{Z}[i])^u \), then \( \tilde{\varphi}((\beta\mathbb{Z}[i])^n) \cap K(\beta\mathbb{Z}[i])^n \neq \emptyset \).

(b) If for all \( \tilde{z} \in (\mathbb{Z}[i])^v \), \( A\tilde{z} \in (\mathbb{Z}[i])^u \), then \( \tilde{\varphi}(K((\beta\mathbb{Z}[i])^n)) \subseteq K((\beta\mathbb{Z}[i])^n) \).

**Proof.** By [HS] Corollary 4.22 we have that \( \tilde{\varphi} \) is a homomorphism, and by [HS] Theorem 2.23 \( K((\beta\mathbb{Z}[i])^n) = (K(\beta\mathbb{Z}[i]))^n \).

(a). Since \( \tilde{\varphi}((\beta\mathbb{Z}[i])^n) = \{ A\tilde{z} : \tilde{z} \in (\mathbb{Z}[i])^v \} \), we need to show that

\[
\{ A\tilde{z} : \tilde{z} \in (\mathbb{Z}[i])^v \} \cap (K(\beta\mathbb{Z}[i]))^n \neq \emptyset.
\]

Pick \( \tilde{z} \in (\mathbb{Z}[i])^v \) such that \( A\tilde{z} \in (\mathbb{Z}[i])^u \). Pick any minimal idempotent \( p \) in \( \beta\mathbb{Z}[i] \). Then by Lemma \([DJ]\) Lemma 2.2 \( \tilde{p} = \tilde{\varphi}(p) \in (K(\beta\mathbb{Z}[i]))^n \). To see that \( \tilde{p} \in \{ A\tilde{w} : \tilde{w} \in (\mathbb{Z}[i])^v \} \), let \( U \) be a neighborhood of \( \tilde{p} \) and for each \( i \in \{1,2,\ldots,u\} \), pick \( D_i \subseteq p \) such that \( x_i \in \mathbb{Z}[i] \). Pick \( a \subseteq \cap_i D_i \). Then \( A(a\tilde{z}) = a\tilde{p} \in U \).

(b). By part (a), \( \tilde{\varphi}((\beta\mathbb{Z}[i])^n) \cap K((\beta\mathbb{Z}[i])^n) \neq \emptyset \), so by [HS] Theorem 1.65, \( K(\tilde{\varphi}(\beta(\mathbb{Z}[i])^n)) = \tilde{\varphi}((\beta(\mathbb{Z}[i])^n) \cap K((\beta\mathbb{Z}[i])^n) \neq \emptyset \) (because \( \varphi(\beta(\mathbb{Z}[i])^n) \subseteq (\beta\mathbb{Z}[i])^n) \).

Also by [HS] Exercise 1.7.3, \( K(\tilde{\varphi}(\beta(\mathbb{Z}[i])^n)) = K(\beta(\mathbb{Z}[i])^n) \).

**Theorem 2.8.** Let \( A \) be a \( u \times v \) matrix with entries from \( \mathbb{Q}[i] \) and assume that for all \( \tilde{z} \in (\mathbb{Z}[i])^v \), every entry of \( A\tilde{z} \) has both positive real and imaginary parts. If “large” is any of “IP*”, “\( \Delta \)**, or “central**”, and \( C \) is large in \( \mathbb{Z}[i] \), then \( W = \{ \tilde{z} : \tilde{z} \in (\mathbb{Z}[i])^v : A\tilde{z} \in C^n \} \) is large in \( \mathbb{Z}[i] \).

**Proof.** We show first that it suffices to prove the theorem under the additional assumption that all entries of \( A \) are in \( \mathbb{Z}[i] \). Indeed, suppose we have done so and pick \( d \in \mathbb{N} \) such that all entries of \( nA \) are in \( \mathbb{Z}[i] \). We claim that \( dC \) is large in \( \mathbb{Z}[i] \), which we check individually.

Assume first that “large” is “IP*” and let a sequence \( (\langle z_n \rangle_{n=1}^\infty) \subseteq \mathbb{Z}[i] \) be given. By [HS] Theorem 5.14 and [DJ] Lemma 2.1 pick a subsystem \( (\langle w_n \rangle_{n=1}^\infty) \) of \( (\langle z_n \rangle_{n=1}^\infty) \) such that \( z \mid w_n \) for each \( n \). Pick \( a \subseteq C \cap \mathbb{N}((\langle \frac{w_n}{d} \rangle_{n=1}^\infty)) \). Then \( da \subseteq dC \cap \mathbb{N}((\langle \frac{z_n}{d} \rangle_{n=1}^\infty)) \).

Next assume that “large” is “\( \Delta \)**. Let a set \( B \) in \( \mathbb{Z}[i] \) be given and choose a sequence \( (\langle z_n \rangle_{n=1}^\infty) \subseteq \mathbb{Z}[i] \) such that for every \( n, m \in \mathbb{N} \) with \( n < m, z_m \in z_n + B \). (In particular, for each \( n < m, |z_n| < |z_m| \) .) By passing to a subsequence, we may assume that for each \( n < m, z_n \equiv z_m \) (mod \( d \)). Pick \( j \subseteq \{1,2,\ldots,d\} \) such that for each \( n \in \mathbb{N}, z_n + j \equiv 0 \) (mod \( d \)). Then \( \langle \frac{z_n + j}{d} \rangle_{n=1}^\infty \) is a sequence in \( \mathbb{Z}[i] \) such that \( da \subseteq dC \cap \mathbb{N}((\langle \frac{z_n + j}{d} \rangle_{n=1}^\infty)) \).

Finally assume that “large” is “central**”. Let \( p \) be a minimal idempotent in \( \beta\mathbb{Z}[i] \). By Lemma \([DJ]\) Lemma 2.2, \( \frac{1}{d}p \) is a minimal idempotent so \( C \subseteq \frac{1}{d}p \) and consequently \( dC \subseteq p \). Since we have established that \( dC \) is large, we have that \( \{ \tilde{z} \in (\mathbb{Z}[i])^v : A\tilde{z} \in (dC)^n \} \) is large, and

\[
\{ \tilde{z} : \tilde{z} \in (\mathbb{Z}[i])^v : A\tilde{z} \in (dC)^n \} = \{ \tilde{z} : \tilde{z} \in (\mathbb{Z}[i])^v : A\tilde{z} \in C^n \}
\]
Thus we assume that all entries of $A$ are in $\mathbb{Z}[i]$. Define $\varphi : \mathbb{Z}[i]^v \to \mathbb{Z}[i]^u$ by $\varphi(z) = A\tilde{z}$, and let $\tilde{\varphi} : \beta(\mathbb{Z}[i])^u \to (\beta\mathbb{Z}[i])^u$ be its continuous extension. Assume first that “large” is “IP*”. Let $p$ be an idempotent in $\beta(\mathbb{Z}[i]^u)$. We need to show that $W \subseteq p$. Since $\varphi$ is a homomorphism, $\tilde{\varphi}(p)$ is an idempotent in $(\beta\mathbb{Z}[i])^u$ and so $\overline{\tilde{C}^u}$ is a neighborhood of $\tilde{\varphi}(p)$, and hence $W \subseteq p$ as required.

Next assume that “large” is “$\Delta^*$”. Let $B$ be a set in $\mathbb{Z}[i]^u$ and pick a sequence $\langle z_n^m \rangle_{n=1}^\infty$ in $(\mathbb{Z}[i] \setminus \{0\})^v$ such that for every $n, m \in \mathbb{N}$ with $n < m$, $z_n^m \in z_n^m + B$. In particular, for each $n < m$, we have $z_n^m - z_n^m \in (\mathbb{Z}[i] \setminus \{0\})^v$. We need to show that there exists $n < m$ such that $z_n^m - z_n^m \in W$.

For each $n \in \mathbb{N}$, let $y_n^m = A\tilde{z}_n^m$. Notice that, for $n < m$ we have that all real and imaginary entries of $A(\tilde{z}_n^m - \tilde{z}_n^m)$ are positive, and consequently modulus of each entry of $y_n^m$ is larger than the modulus of corresponding entry of $y_n^m$. By Ramsey’s Theorem ([GRS, Theorem 1.5]), pick an infinite subset $D_1$ of $\mathbb{N}$ such that for all $n < m$ in $D_1$, $y_n^m - y_n^m \subseteq C$ or for all $n < m$ in $D_1$, $y_n^m - y_n^m \in (\mathbb{Z}[i] \setminus \{0\}) \setminus C$. Since $C$ is a $\Delta^*$ set, the latter alternative is impossible, so the former must hold. Inductively, given $i \in \{1, 2, \ldots, u - 1\}$, choose by Ramsey’s Theorem an infinite subset $D_{i+1}$ of $D_i$ such that for all $n < m$ in $D_{i+1}$, $y_n^{i+1} - y_n^{i+1} \subseteq C$. Having chosen $D_u$ pick $n < m$ in $D_u$. Then $z_n^m - z_n^m \in W$. Finally assume that “large” is “central*”, and let $p$ be a minimal idempotent in $\beta(\mathbb{Z}[i]^u)$. By Lemma 2.7, $\tilde{\varphi}(p)$ is an idempotent and $\tilde{\varphi}(p) \in (K(\beta\mathbb{Z}[i]^u))^u$ so that $\tilde{\varphi}(p) \overline{C^u}$. □

The requirement in Theorem 2.8 that for all , every $\bar{z} \in \mathbb{Z}[i]^v$ entry of $A\bar{z}$ be positive may not be omitted ( see [HLS]).

Theorem 2.9. Let $A$ be a $u \times v$ matrix with entries from $\mathbb{Z}[i]$ and assume that for all $\bar{z} \in (\mathbb{Z}[i] \setminus \{0\})^v$, every entry of $A\bar{z} \in (\mathbb{Z}[i] \setminus \{0\})^u$. If $C$ is PS* in $\mathbb{Z}[i]$, then $W = \{\bar{z} \in (\mathbb{Z}[i] \setminus \{0\})^v : A\bar{z} \in \overline{C^u}\}$ is PS* in $\mathbb{Z}[i]^v$.

Proof. Define $\varphi : (\mathbb{Z}[i] \setminus \{0\})^v \to (\mathbb{Z}[i] \setminus \{0\})^v$ by $\varphi(\bar{z}) = A\bar{z}$, and let $\tilde{\varphi} : \beta(\mathbb{Z}[i]^u) \to (\beta\mathbb{Z}[i]^u)$ be its continuous extension. Let $p \in K(\beta(\mathbb{Z}[i]^u))$. By Lemma 2.7, $\tilde{\varphi}(p) \in (K(\beta\mathbb{Z}[i]^u))^u$ and thus $\tilde{\varphi}(p) \overline{C^u}$ and thus $W \subseteq p$ as required. □

Theorem 2.10. Let $A$ be a $u \times v$ image partition regular matrix with entries from $\mathbb{Z}[i]$. If $C$ is thick in $\mathbb{Z}[i]$, then $W = \{\bar{z} \in (\mathbb{Z}[i] \setminus \{0\})^v : A\bar{z} \in \overline{C^u}\}$ is thick in $\mathbb{Z}[i]^v$.

Proof. Since $C$ is thick, pick a left ideal $L$ of $\beta\mathbb{Z}[i]$ such that $L \subseteq \overline{C}$. Pick by [HLS] Corollary 2.6] a minimal idempotent $p \in L$. Define $\varphi : \mathbb{Z}[i]^v \to \mathbb{Z}[i]^u$ by $\varphi(\bar{z}) = A\bar{z}$, and let $\tilde{\varphi} : \beta(\mathbb{Z}[i]^u) \to (\beta\mathbb{Z}[i]^u)$ be its continuous extension. Let $\bar{p} = (p, p, \ldots, p)^T$ and pick by [DJ] Lemma 2.3 a minimal idempotent $q \in \beta(\mathbb{Z}[i]^u)$ such that $\tilde{\varphi}(q) = p$. (By [DJ] Theorem 3.1(h), $p$ satisfies the hypotheses of [DJ] Theorem 2.3.) We claim that $\tilde{\varphi}(\beta(\mathbb{Z}[i]^u) + q) \subseteq \overline{C^u}$ so that $\beta(\mathbb{Z}[i]^u) + q \subseteq \overline{W}$ as required. To this end, let $r \in \beta(\mathbb{Z}[i]^u)$ and let $i \in \{1, 2, \ldots, u\}$. Then $\pi_i(\tilde{\varphi}(r + q)) = \pi_i(\tilde{\varphi}(r)) + p \in \beta(\mathbb{Z}[i] + p) \subseteq L \subseteq \overline{C}$. □

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