The Nielsen Identities for the
Two-Point Functions of QED and QCD

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Abstract  We consider the Nielsen identities for the two-point functions of full QCD and QED in the class of Lorentz gauges. For pedagogical reasons the identities are first derived in QED to demonstrate the gauge independence of the photon self-energy, and of the electron mass shell. In QCD we derive the general identity and hence the identities for the quark, gluon and ghost propagators. The explicit contributions to the gluon and ghost identities are calculated to one-loop order, and then we show that the quark identity requires that in on-shell schemes the quark mass renormalisation must be gauge independent. Furthermore, we obtain formal solutions for the gluon self-energy and ghost propagator in terms of the gauge dependence of other, independent Green functions.

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1. Introduction

The Nielsen identities\cite{1} deserve to be better known. They follow from a modification of Becchi-Rouet-Stora-Tyutin (BRST) symmetry in a similar way to how the usual Ward identities are obtained from BRST. The identities are, however, of a quite different nature to the Ward identities. This may be illustrated by (very) schematically writing them as
\[
\frac{\partial}{\partial \xi} G_1 = G_2 G_3,
\]
where $\xi$ is the gauge parameter and the $G_i$ are Green’s functions. This then, for example, clearly tells us that if the right hand side vanishes, then $G_1$ must be gauge independent. The aim of this paper is to investigate the consequences of the Nielsen identities for the two-point functions of QED and QCD.

We will not follow here the original derivation of Nielsen, but rather use an alternative approach due to Piguet and Sibold\cite{2}. The identities have been previously employed\cite{3} to study the gauge invariance of the Higgs mass and the role of gauge symmetry in the effective potential for models of spontaneous symmetry breaking.

It will become evident in what follows that the Nielsen identities are particularly useful for investigations of on-mass shell Green’s functions and on-shell renormalisation constants. The on-shell renormalisation scheme is widely used in QED, the electroweak theory and QCD with heavy quarks and it is of the greatest importance for studies of the S-matrix.

For completeness and clarity we begin by discussing the case of QED in Section 2. We show in particular that the pole mass of the electron is gauge independent and that the photon self-energy can be simply shown to be gauge parameter independent. We stress here the interplay between the usual (BRST) Ward identities and the Nielsen identities.

In Section 3 we present the Nielsen identities for QCD. We explicitly calculate the one-loop contributions in the identity for the gluon propagator, and then demonstrate that in on-shell renormalisation schemes the quark mass renormalisation must also be gauge independent to all orders. The Nielsen identity for the ghost propagator is constructed and its explicit content is determined to one-loop order. Formal solutions for the gluon
self-energy and ghost propagator appear from this analysis which deserve further study.

Section 4 provides a perturbative analysis of the Nielsen identities illustrating the
gauge parameter independence of the mass renormalisation in the mass-shell scheme. Some
conclusions are presented in Section 5.

2. The QED Identities

The standard path integral formulation of QED in covariant gauges is based upon the
following Lagrangian

\[ \mathcal{L}_{\text{QED}} = -\frac{1}{4} F^2 + \bar{\psi} (i \partial - m) \psi + \frac{\xi}{2} B^2 + B \partial \cdot A - \bar{c} \, \Box \, c, \]  

(2.1)

where \( \xi \) is the gauge parameter, \( F \) is the field strength and \( B \) is an auxiliary field\(^4\).

Although ghosts decouple in QED, one often includes them so as to demonstrate that the
Lagrangian is invariant under the following BRST transformations

\[ \delta A_\mu = \epsilon \partial_\mu c, \quad \delta \bar{\psi} = +i \epsilon c \bar{\psi}, \quad \delta c = 0, \]

\[ \delta B = 0, \quad \delta \psi = -i \epsilon c \psi, \quad \delta \bar{c} = \epsilon B. \]  

(2.2)

The Ward identities may then be obtained by exploiting this invariance in the usual
fashion\(^5\).

Following [2], the Nielsen identities are obtained by making the following addition to
the Lagrangian

\[ \mathcal{L}_{\text{QED}} \rightarrow \mathcal{L}_{\text{QED}} + \frac{\chi}{2} \bar{c} B, \]  

(2.3)

where \( \chi \) is a global Grassmannian variable, \( \chi^2 = 0 \). (In the following care must be taken
with the minus signs that are needed under the interchange of Grassmannian variables!) It
is clear upon a little reflection that the addition of this term cannot change the dynamics
of the theory. To see this most easily consider that the generating functional can be
expanded in \( \chi \) and, as a result of its Grassmannian nature, only two terms survive: \( \chi^0 \)
and \( \chi^1 \). The first term gives us the usual dynamics of QED and the second must vanish
by virtue of ghost number when we integrate over the ghost fields. This shows that we
have not changed any physics. In what follows we will employ the Lagrangian (2.3).
This modified Lagrangian is invariant under the following extended set of BRST transformations

\[ \delta^+ A_\mu = \epsilon \partial_\mu c, \quad \delta^+ \bar{c} = \epsilon B, \]
\[ \delta^+ B = 0, \quad \delta^+ c = 0, \quad (2.4) \]
\[ \delta^+ \psi = -i \epsilon c \psi, \quad \delta^+ \xi = \epsilon \chi, \]
\[ \delta^+ \bar{\psi} = +i \epsilon c \bar{\psi}, \quad \delta^+ \chi = 0. \]

where \( \epsilon \) is a Grassmann quantity. The \( F^2 \) and Dirac Lagrangians are, as usual, separately invariant and the remaining terms are readily seen to be invariant. The crucial point to note is that the gauge parameter is now transformed.

To exploit this invariance and so derive the Nielsen identities we now consider the following generating functional

\[ Z = \int [d\mu] \exp \left\{ i \int d^4 x \mathcal{L}_{\text{QED}} + J_\mu A_\mu + \bar{J}_\psi \psi + \bar{\psi} J_{\bar{\psi}} + B J_B \right\} , \quad (2.5) \]

The various sources denoted by \( J \) with a subscript are standard ones. We have not included sources for the ghost fields, for example, since we will not consider their Green’s functions, for QED they are anyway trivial. The purpose of the additional, rather exotic looking, sources denoted by \( K \)'s will become apparent in a moment. Note that we may rewrite this part of the Lagrangian as

\[ \bar{K}_\psi \frac{\delta^+ \psi}{\delta \epsilon} + \frac{\delta^+ \bar{\psi}}{\delta \epsilon} K_{\bar{\psi}}. \quad (2.6) \]

To study the gauge dependence of the electron and photon propagators, we now introduce the generating functional of proper Green functions.

\[ \Gamma(A^\mu, \psi, \bar{\psi}, B, c, \bar{c}, \chi, \xi, \bar{K}_\psi, K_{\bar{\psi}}) = \]
\[ W(J_\mu, J_{\bar{\psi}}, \bar{J}_\psi, J_B, K_{\bar{\psi}}, \bar{K}_\psi, \chi, \xi) - \int d^4 x J_\mu A_\mu + J_B B + \bar{J}_\psi \psi + \bar{\psi} J_{\bar{\psi}} \quad (2.7) \]

As a consequence of invariance under (2.4) we have

\[ \delta^+ \Gamma \equiv 0 = \delta^+ A_\mu \frac{\delta \Gamma}{\delta A_\mu} + \delta^+ \psi \frac{\delta \Gamma}{\delta \psi} + \delta^+ \bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} + \delta^+ c \frac{\delta \Gamma}{\delta c} + \delta^+ \xi \frac{\delta \Gamma}{\delta \xi}, \quad (2.8) \]
where we have used $\delta^+ B = \delta^+ c = \delta^+ \chi = 0$. From (2.6) we have $\delta^+ \psi = \epsilon \frac{\delta \Gamma}{\delta K_\psi}$ etc. We may therefore rewrite (2.8) as

$$
0 = \partial_\mu c \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta \Gamma}{\delta K_\psi} \frac{\delta^2 \Gamma}{\delta \psi} + \frac{\delta \Gamma}{\delta \bar{K}_\psi} \frac{\delta \bar{K}_\psi}{\delta \psi} \frac{\delta \Gamma}{\delta \bar{K}_\psi} + B \frac{\delta \Gamma}{\delta \bar{c}} + \chi \frac{\partial \Gamma}{\partial \xi}.
$$

(2.9)

Note that implicit coordinate space integrations are understood in the above!

We are now in a position to obtain the Nielsen identities for QED. We differentiate (2.9) with respect to $\chi$ and then set $\chi$ to zero.

$$
0 = \frac{\partial \Gamma}{\partial \xi} - \partial_\mu c \frac{\delta \Gamma}{\delta \bar{K}_\psi} \frac{\delta \bar{c}}{\delta \psi} + \frac{\delta^2 \Gamma}{\delta \psi} \frac{\delta \bar{K}_\psi}{\delta \psi} \frac{\delta \Gamma}{\delta \bar{K}_\psi} + B \frac{\delta^2 \Gamma}{\delta \bar{c}}
$$

(2.10a)

In the special case when no further functional derivatives with respect to ghost fields will be applied to (2.10a) ghost number conservation implies a simplification. The so simplified result is then

$$
0 = \frac{\partial \Gamma}{\partial \xi} + B \frac{\delta^2 \Gamma}{\delta \bar{c}} + \frac{\delta \bar{K}_\psi}{\delta \psi} \frac{\delta \Gamma}{\delta \bar{K}_\psi} + \frac{\delta^2 \Gamma}{\delta \psi} \frac{\delta \bar{K}_\psi}{\delta \psi},
$$

(2.10b)

where we have used the result $\partial_\mu c = \partial_\mu \frac{\delta \Gamma}{\delta \psi} = 0$ for the one-point function. From these central results we can generate the QED Nielsen identities for the two-point functions.

### 2.1 The Mass of the Electron

The first application studied here is the electron mass. Although the the fermion field $\psi$ is not BRST-invariant and may not be naively identified with the electron$^{[6,7]}$, we will now show that its pole mass is gauge independent and may so be given a physical meaning. The inverse fermion propagator is given by $iS^{-1}(y-x) = \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \bar{\psi}(x)}$. Differentiating (2.10b) with respect to $\psi(y)$ and $\bar{\psi}(x)$ we obtain

$$
\frac{\partial}{\partial \xi} \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \bar{\psi}(x)} = + \frac{\delta^3 \Gamma}{\delta \psi(y) \delta \bar{K}_\psi \delta \psi(x) \delta \bar{\psi}(x)} + \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \psi(x) \delta \bar{K}_\psi \delta \chi} + \frac{\delta^3 \Gamma}{\delta \psi(y) \delta \psi \delta \psi(x) \delta \bar{K}_\psi \delta \chi},
$$

(2.11)
and all other terms will vanish, either by fermion conservation or by their relation to
one-point functions.

To complete the analysis of the Fermion propagator (2.11) must be transformed into
momentum space. Defining
\[
\frac{\delta^2 \Gamma}{\delta \chi K_{\bar{\psi}}(w) \delta \psi(y)} = \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} e^{-iq \cdot (y-z) - i\ell \cdot (w-z)} F(q, \ell, -q - \ell),
\]
and recalling that (2.11) contains implicit \(w\) and \(z\) integrations, it is found that
\[
\frac{\partial}{\partial \xi} \left. S^{-1}(p) = S^{-1}(p) \{ F(p, -p, 0) + \bar{F}(-p, p, 0) \} \right|_{p^2 = M^2}.
\]
This result is the Nielsen identity for the inverse fermion propagator. In particular, since
the right hand side vanishes at the mass shell, and since \(F(-p, p, 0)\) has no single particle
poles, we have
\[
\left. \frac{\partial S^{-1}(p)}{\partial \xi} \right|_{p^2 = M^2} = 0,
\]
where \(M\) is the pole mass.

From this result it is easy to see that \(M\) is gauge independent. Since we may, in a
covariant gauge, generally write the inverse propagator as
\[
S^{-1}(p) = A(p^2) p - B(p^2),
\]
then \(M\) is defined by
\[
A(M^2) M = B(M^2).
\]
If we differentiate this equation with respect to \(\xi\) and compare it with (2.14) rewritten in
the following way
\[
\left. \left( \frac{\partial A(p^2)}{\partial \xi} p - \frac{\partial B(p^2)}{\partial \xi} \right) \right|_{p^2 = M^2} = M \frac{\partial A(M^2)}{\partial \xi} - \frac{\partial B(M^2)}{\partial \xi} = 0,
\]
we see that we obtain the desired result
\[
\left[ A + \frac{\partial A}{\partial M} - \frac{\partial B}{\partial M} \right] \frac{\partial M}{\partial \xi} = \left[ \frac{\partial B(M^2)}{\partial \xi} - M \frac{\partial A(M^2)}{\partial \xi} \right] = 0 \rightarrow \frac{\partial M}{\partial \xi} = 0
\]
so that the pole mass is gauge parameter independent (note that the bracketed quantity on the left hand side of (2.18) is non-zero). In Section 4 this point will be studied more carefully in a perturbative fashion.

2.2 The Photon Propagator

The inverse photon propagator may be studied in a similar fashion to the above. By functional differentiation of (2.10b) with respect to $A_\nu(x)$ and $A_\lambda(y)$, we find

$$\frac{\partial}{\partial \xi} \frac{\delta^2 \Gamma}{\delta A_\nu(x) \delta A_\lambda(y)} = 0,$$

(2.19)

where many terms that must, from considerations of fermion and ghost number or relation to one-point functions, vanish have been neglected. At this stage it is important to recognize that the auxiliary field $B$ is independent of $A_\mu$. To make a direct connection between (2.19) and the photon vacuum polarization, it is necessary to consider some aspects of the auxiliary field formalism [4,8].

To formulate perturbation theory it is necessary to consider the free field case, corresponding to the quadratic part of the Lagrangian (2.1). The mixing between $B$ and $\partial \cdot A$ in the Lagrangian must be diagonalized after functional integration, leading to the following free field bosonic propagators.

$$\int d^4x e^{ip \cdot x} \langle O | T (B(x)B(0)) | O \rangle = 0$$

(2.20a)

$$\int d^4x e^{ip \cdot x} \langle O | T (B(x)A_\mu(0)) | O \rangle = \frac{p^\mu}{p^2}$$

(2.20b)

$$\int d^4x e^{ip \cdot x} \langle O | T (A_\mu(x)A_\nu(0)) | O \rangle = i \left[ -\frac{g^{\mu\nu}}{p^2} + (1 - \xi) \frac{p^\mu p^\nu}{p^4} \right]$$

(2.20c)

BRS symmetry implies that (2.20a) and (2.20b) are true to all orders in perturbation theory.

$$0 = \frac{\delta_{BRS}}{\delta \epsilon} \langle O | T (\bar{c}(x)B(y)) | O \rangle = \langle O | T (B(x)B(y)) | O \rangle$$

(2.21a)

$$0 = \frac{\delta_{BRS}}{\delta \epsilon} \langle O | T (\bar{c}(x)A_\mu(y)) | O \rangle = \langle O | T (B(x)A_\mu(y)) | O \rangle + \langle O | T (\partial_\mu c(x)\bar{c}(y)) | O \rangle$$

(2.21b)
By recognizing that the ghosts are decoupled, (2.21b) then implies that (2.20b) is valid to all orders in perturbation theory. This then implies that radiative corrections to the tree-level result (2.20b) are absent, so the photon vacuum polarization is necessarily transverse as illustrated in Figure 1.

\[ 0 = p^\mu \Pi_{\mu\lambda} \left( -\frac{g^{\nu\lambda}}{p^2} + (1 - \xi) \frac{p^\nu p^\lambda}{p^4} \right) \]  
(2.22)

Hence we have the usual result for the full photon propagator.

\[ \int d^4x \ e^{ip(x-y)} \langle O| T (A_\mu(x)A_\nu(y))|O \rangle = i \left[ -\frac{g^{\mu\nu}}{p^2} + \frac{p^\mu p^\nu}{p^4} \right] \frac{1}{1 + \Pi(p^2)} - i\xi \frac{p^\mu p^\nu}{p^4} \]  
(2.23)

The implications of the results for the Green functions (2.20a), (2.20b) and (2.23) for the generating functional \( \Gamma \) can be understood by consideration of the following functional identities.

\[ \frac{\delta B(x)}{\delta A_\mu(y)} = 0 = \frac{\delta A_\mu(x)}{\delta B(y)} \]

\[ \frac{\delta B(x)}{\delta B(y)} = \delta(x - y) \]

\[ \frac{\delta A_\mu(x)}{\delta A_\nu(y)} = \delta(x - y) \delta^\mu_\nu \]  
(2.24)

In terms of generating functionals these expressions become

\[ \delta(x - y) \delta^\mu_\nu = \int d^4z \frac{\delta^2 \Gamma}{\delta A_\lambda(z) \delta A_\nu(y)} \frac{\delta^2 W}{\delta J_\lambda(z) \delta J_\mu(x)} + \frac{\delta^2 \Gamma}{\delta B(z) \delta A_\nu(y)} \frac{\delta^2 W}{\delta J_B(z) \delta J_\mu(x)} \]

\[ 0 = \int d^4z \frac{\delta^2 \Gamma}{\delta A_\lambda(z) \delta B(y)} \frac{\delta^2 W}{\delta J_\lambda(z) \delta J_B(x)} + \frac{\delta^2 \Gamma}{\delta B(z) \delta B(y)} \frac{\delta^2 W}{\delta J_B(z) \delta J_B(x)} \]  
(2.25)

\[ \delta(x - y) = \int d^4z \frac{\delta^2 \Gamma}{\delta A_\lambda(z) \delta B(y)} \frac{\delta^2 W}{\delta J_\lambda(z) \delta J_B(x)} + \frac{\delta^2 \Gamma}{\delta B(z) \delta B(y)} \frac{\delta^2 W}{\delta J_B(z) \delta J_B(x)} \]

The functional derivatives of \( W \) are known from the Green functions discussed above, and hence (2.25) implies that the quadratic pieces of \( \Gamma \) simply diagonalize the various \( A_\mu, B \) two-point Green functions in momentum space.

\[ \int d^4x \ e^{ipx} \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\nu(0)} = p^2 \left[ -\sigma_{\mu\nu} + \sigma_{\mu\nu} \right] \left( 1 + \Pi(p^2) \right) \]  
(2.26a)
\[ \int d^4 x e^{ip \cdot x} \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta B(0)} = p_\mu \]  
\[ \int d^4 x e^{ip \cdot x} \frac{\delta^2 \Gamma}{\delta B(x) \delta B(0)} = \xi \]  

(2.26b) 

(2.26c)

The transversality of (2.26a) is easily seen to be consistent with the fundamental identity (2.9). † Upon functional differentiation of (2.9) with respect to \( A_\mu \) and \( c \), setting \( \xi = 0 \), and then Fourier transforming, we see that

\[ p^\mu \int d^4 x e^{ip \cdot x} \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\nu(0)} = 0 \]  

(2.27)

consistent with the analysis leading to (2.26a).

Finally, substituting the result of (2.26a) into the identity (2.19) we find that the polarization \( \Pi(p^2) \) must be independent of the gauge parameter\(^7\)

\[ \frac{\partial \Pi(p^2)}{\partial \xi} \equiv 0 \]  

(2.24)

This result shows the power of the Nielsen identities.

Since the ghosts decouple this concludes our survey of the Nielsen identities for the two-point functions of QED.

3. The QCD Identities

The reader who has closely followed the above will have no difficulty in finding the analogous identities for QCD\(^9\). We give the basic steps and so define our notation. (Note that colour indices are implicit.) The modified QCD Lagrangian is

\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^2 + \bar{\psi} (i D - m) \psi + \frac{\xi}{2} B^2 + B \partial \cdot A - \bar{c} \partial^\mu D_\mu c + \frac{\chi}{2} \bar{c} B \]  

(3.1)

which is invariant under the following augmented BRST transformations

\[ \delta^+ A_\mu = \epsilon D_\mu c , \quad \delta^+ \bar{c} = \epsilon B , \]

\[ \delta^+ B = 0 , \quad \delta^+ c = -\frac{1}{2} \epsilon [ c , c ] , \]

\[ \delta^+ \bar{\psi} = -i \epsilon c \psi , \quad \delta^+ \xi = \epsilon \chi , \]

\[ \delta^+ \bar{\psi} = +i \epsilon c \bar{\psi} , \quad \delta^+ \chi = 0 . \]  

(3.2)

† We are grateful to the referee for bringing this point to our attention.
As in the QED case, the extension of BRST includes a transformation of the gauge parameter.

The generating functional $Z$ with sources (including ghosts) is

$$Z = \int [d\mu] \exp \left\{ i \int d^4x \, \mathcal{L}_{QCD} + J_\mu A^\mu + J_B B + \bar{\psi} J_\psi \bar{\psi} + \bar{J}_c c + \bar{c} J_\bar{c} \\
+ K_\mu (D^\mu c) - \frac{1}{2} \bar{K}_c [c, c] + \bar{K}_\psi (-ic\psi) + ic\bar{\psi} K_\bar{\psi} \right\}.$$  \hfill (3.3)

and the generating functional $\Gamma$ for the proper Green functions is

$$\Gamma(A^\mu, \psi, \bar{\psi}, B, c, \bar{c}, \chi, \xi, K_\mu, \bar{K}_\psi, K_\bar{\psi}) = W(J_\mu, J_B, J_\psi, \bar{J}_\psi, \bar{J}_c, J_\bar{c}, K_\mu, K_\bar{\psi}, \bar{K}_\psi, \chi, \xi) - \int d^4x J_\mu A^\mu + J_B B + \bar{\psi} J_\psi \bar{\psi} + \bar{J}_c c + \bar{c} J_\bar{c}$$  \hfill (3.4)

From this last result it is simple to obtain the nonabelian version of (2.9).

$$0 = \frac{\delta \Gamma}{\delta K_\mu} \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta \Gamma}{\delta K_\psi} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta \Gamma}{\delta K_\bar{\psi}} \frac{\delta \Gamma}{\delta \bar{\psi}} + B \frac{\delta \Gamma}{\delta c} + \frac{\delta \Gamma}{\delta K_c} \frac{\delta \Gamma}{\delta \bar{c}} + \chi \frac{\partial \Gamma}{\partial \xi} $$  \hfill (3.5a)

After differentiation with respect to $\chi$ and then setting $\chi = 0$ we find

$$0 = \frac{\partial \Gamma}{\partial \xi} + \frac{\delta^2 \Gamma}{\delta \chi K_\mu} \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta^2 \Gamma}{\delta \chi K_\psi} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta^2 \Gamma}{\delta \chi K_\bar{\psi}} \frac{\delta \Gamma}{\delta \bar{\psi}} + B \frac{\delta^2 \Gamma}{\delta \chi c} + \frac{\delta^2 \Gamma}{\delta K_c c}$$  \hfill (3.5b)

which is the QCD equivalent of (2.10a). As before, further functional differentiation with respect to fundamental non-ghost fields can be obtained from the following simpler result using ghost number conservation in (3.5b).

$$0 = \frac{\partial \Gamma}{\partial \xi} + \frac{\delta^2 \Gamma}{\delta \chi K_\mu} \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta^2 \Gamma}{\delta \chi K_\psi} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta^2 \Gamma}{\delta \chi K_\bar{\psi}} \frac{\delta \Gamma}{\delta \bar{\psi}} + B \frac{\delta^2 \Gamma}{\delta \chi c}$$  \hfill (3.5c)

Although expressions (3.5c) and (2.10b) are superficially very similar, the distinction between abelian and non-abelian theories shows up in two important ways. First, there is now the current $K_\mu$ which couples ghosts and gauge fields through the covariant derivative and secondly the ghosts are no longer decoupled.
3.1 Nielsen Identity for the Gluon Propagator

The distinction between non-abelian and abelian theories is most evident when considering the Nielsen identity for the gauge field propagators. To our knowledge, these identities have not been previously studied in an explicit calculation. Proceeding as in Section 2.2, one finds

\[ \frac{\partial}{\partial \xi} \frac{\delta^2 \Gamma}{\delta A_\nu(x) \delta A_\lambda(y)} = \frac{\delta^2 \Gamma}{\delta K_\mu \delta \chi \delta A_\nu(x) \delta A_\mu \delta A_\lambda(y)} + \nu \leftrightarrow \lambda \].

(3.6a)

Defining the Green functions,

\[ F_{\mu\nu}(p, q, -p - q) = \int d^4 x e^{ip \cdot x + iq \cdot y} \frac{\delta^3 \Gamma}{\delta A_\mu(x) \delta K_\nu(y) \delta \chi} \].

(3.6b)

\[ \Gamma_{\mu\nu}(p) = \int d^4 x e^{ip \cdot (x - y)} \frac{\delta^2 \Gamma}{\delta A_\mu(x) \delta A_\mu(y)} \].

(3.6c)

we have the following result after Fourier transforming (3.5a).

\[ \frac{\partial}{\partial \xi} \Gamma_{\nu\lambda}(p) = \Gamma_{\lambda\mu}(p) F_{\mu\nu}(-p, p, 0) + \nu \leftrightarrow \lambda \]

(3.6d)

As in the photon analysis, it is necessary to relate the gluon vacuum polarization to \( \Gamma_{\mu\nu}(p) \) before studying (3.6d). The analysis is only slightly more complicated than in the abelian case. The free field results (2.20) clearly are identical, but the BRS symmetry is somewhat different.

\[ 0 = \frac{\delta_{\text{BRS}}}{\delta \epsilon} \langle O \left| T \left( \bar{c}(x) B(y) \right) \right| O \rangle = \langle O \left| T \left( B(x) B(y) \right) \right| O \rangle \]

(3.7a)

\[ 0 = \frac{\delta_{\text{BRS}}}{\delta \epsilon} \langle O \left| T \left( \bar{c}(x) A_\mu(y) \right) \right| O \rangle = \langle O \left| T \left( B(x) A_\mu(y) \right) \right| O \rangle - \langle O \left| T \left( \bar{c}(x) D_\mu c(y) \right) \right| O \rangle \]

(3.7b)

The relevant Green function in (3.7b) is

\[ \int d^4 x e^{ip \cdot x} \langle O \left| T \left( \bar{c}(0) D_\mu c(x) \right) \right| O \rangle = \frac{p_\mu}{p^2} \]

(3.7c)

as can be verified by contracting both sides with \( p^\mu \) then applying the ghost equation of motion and the canonical commutation relations.

Thus we see from (3.7) that the abelian results of (2.20a) and (2.20b) apply to all orders in QCD. The same argument as outlined in Section 2.2 then implies that the gluon vacuum polarization must be transverse, and so (2.26a) is also valid in QCD.
In contrast to the case of QED, (3.6d) cannot be solved exactly since the Green function \( F_{\mu\nu}(p, -p, 0) \) must be determined perturbatively. However, we can explicitly display the perturbative contributions in this identity, specifically to one-loop order. Since the vacuum polarization \( \Pi(p^2) \) is well known (see, e.g., [8]) we only have to calculate \( F_{\mu\nu}(p, -p, 0) \).

When calculating \( F_{\mu\nu}(p, -p, 0) \) it is important to recognize that the mixing between the fields \( B \) and \( A_\mu \) implies that we do not simply have a truncated Green function. This can be seen in the following way:

\[
\delta^3 \varGamma \frac{\delta^3 W}{\delta A_\mu(x) \delta K_\nu(y) \delta \chi} = \int d^4 s \frac{\delta^3 W}{\delta K_\nu(y) \delta \chi \delta J_\lambda(s)} \frac{\delta J_\lambda(s)}{\delta A_\mu(x)} + \frac{\delta^3 W}{\delta K_\nu(y) \delta \chi \delta J_B(s)} \frac{\delta J_B(s)}{\delta A_\mu(x)} \tag{3.8a}
\]

When the relevant Fourier transform is taken and the result of (2.26b) is applied, the second term becomes

\[
\frac{1}{2} p^\mu \int d^4 x d^4 y e^{ip \cdot x - ip \cdot y} \langle O| T (D_\nu c(y) B(x) \bar{c}(0) B(0)) |O \rangle = \frac{1}{2} \int d^4 x e^{ip \cdot x} \langle O| T (B(x) B(0)) |O \rangle = 0 \tag{3.8b}
\]

where the ghost equation of motion, commutation relations, and (2.20a) have been applied.

Thus after Fourier transforming (3.8a) we have

\[
F_{\mu\nu}(p, -p, 0) = \mathcal{F}_{\mu\nu}(p, -p, 0) \Gamma_{\lambda\nu}(p) \tag{3.8c}
\]

where

\[
\mathcal{F}_{\nu\lambda}(p, -p, 0) = \frac{1}{2} \int d^4 x d^4 y e^{ip \cdot x - ip \cdot y} \langle O| T (D_\nu c(y) A_\lambda(x) \bar{c}(0) B(0)) |O \rangle \tag{3.8d}
\]

is the full Green function.

To one loop order, the contributions to this Green function are given by the diagrams of Fig. 2. The results of the calculation for the contribution to \( \mathcal{F}_{\mu\nu} \) from individual diagrams
of Fig. 2 are

\[2 \mathcal{F}^{(i)}_{\mu\nu} = - \frac{g^2 N_c}{64\pi^2 p^4} P_\mu P_\nu \left[ -\frac{3\xi}{\tilde{\epsilon}} + 4\xi \right] \]

\[2 \mathcal{F}^{(ii)}_{\mu\nu} = \frac{g^2 N_c}{64\pi^2 p^4} \left[ p^2 g_{\mu\nu} \left( -2\xi - \frac{3}{\tilde{\epsilon}} \right) + P_\mu P_\nu \left( -\frac{3\xi}{\tilde{\epsilon}} + 6\xi + \frac{3}{\tilde{\epsilon}} \right) \right] \]

\[2 \mathcal{F}^{(iii)}_{\mu\nu} = \frac{g^2 N_c}{64\pi^2 p^4} \left[ p^2 g_{\mu\nu} \left( \frac{1}{\tilde{\epsilon}} - 2 \right) + P_\mu P_\nu \left( \frac{\xi}{\tilde{\epsilon}} - 1 + 2 \right) \right] \]

\[2 \mathcal{F}^{(iv)}_{\mu\nu} = \frac{g^2 N_c}{64\pi^2 p^4} P_\mu P_\nu \left[ \frac{\xi}{\tilde{\epsilon}} - \frac{3}{\tilde{\epsilon}} + 4 \right] \]

\[2 \mathcal{F}^{(v)}_{\mu\nu} = - \frac{g^2 N_c}{64\pi^2 p^4} P_\mu P_\nu \left[ \frac{\xi}{\tilde{\epsilon}} - \frac{3}{\tilde{\epsilon}} + 4 \right] \]

\[2 \mathcal{F}^{(vi)}_{\mu\nu} = - \frac{g^2 N_c}{64\pi^2 p^4} P_\mu P_\nu \left[ \frac{\xi}{\tilde{\epsilon}} \right] \]

The sum of these diagrams, including the tree level contribution is

\[2 \mathcal{F}_{\mu\nu} = \frac{p^\mu p^\nu}{p^4} - \frac{g^2 N_c}{64\pi^2 p^4} \left( p^2 g_{\mu\nu} - P_\mu P_\nu \right) \left( \frac{2}{\tilde{\epsilon}} + 2 + 2\xi \right) . \] (3.9b)

Note that we have introduced

\[\frac{1}{\tilde{\epsilon}} = \frac{1}{\epsilon} - \log 4\pi + \gamma + \log \left( -\frac{p^2}{\nu^2} \right) \quad D = 4 + 2\epsilon , \] (3.9c)

to be consistent with\(^{10}\), and included a factor of 2 to take care of the “crossed” term in (3.6d).

It is important that this Green function is explicitly transverse beyond tree-level as required by conservation of the ghost current \( K_\mu = D_\mu c \), which follows from ordinary BRST symmetry. This is easily seen by considering the Green function \( \mathcal{F}_{\mu\nu} \).

\[
\frac{1}{2} q^{\nu'} \int d^4 x d^4 y e^{ip\cdot x + iq\cdot y} \langle 0 \vert T ( A_\mu (x) D_\nu c(y) \bar{c}(0) B(0) ) \vert 0 \rangle
= \frac{1}{2} \int d^4 x e^{ip\cdot x} \langle 0 \vert T ( A_\mu (x) B(0) ) \vert 0 \rangle = \frac{p^\mu}{2p^2} ,
\] (3.9d)

where as usual, the ghost equations of motion and commutation relations have been applied. Clearly the identity (3.9d) is satisfied by the tree level contribution \( p^\mu q^{\nu'}/(2p^2 q^2) \), so the higher-loop contributions to \( \mathcal{F}_{\mu\nu}(p, -p, 0) \) must be transverse, serving as a consistency check on our calculation.
Returning to the identity (3.6d) and including terms up to one-loop we have the following result for $\Gamma_{\lambda\mu}(p)$ \[10\]

$$
\Gamma_{\lambda\mu}(p) = (g_{\mu\nu} p^2 - p_\mu p_\nu) \left( 1 + \Pi(p^2) \right)
$$

where

$$
\Pi(p^2) = \frac{g^2 N_c}{16\pi^2} \left[ \left( \frac{13}{6} - \frac{\xi}{2} \right) \frac{1}{\hat{\epsilon}} - \frac{97}{36} - \frac{1}{2} \xi - \frac{\xi^2}{4} \right], \quad (3.10a)
$$

and some terms independent of the gauge parameter have been neglected. These neglected terms in (3.10a) come from quarks, and are irrelevant for the analysis of (3.6d). Combining (3.10a) with the results of (3.9) to one-loop order yields

$$
2F_{\mu\nu}(p, -p, 0) = -\frac{g^2 N_c}{64\pi^2 p^2} \left( p^2 g_{\mu\nu} - p_\mu p_\nu \right) \left( \frac{2}{\hat{\epsilon}} + 2 + 2\xi \right) \quad (3.10b)
$$

which confirms the QCD Nielsen identity (3.6d) for the gluon propagator.

A formal solution for the gluon self-energy $\Pi(p^2)$ can be obtained from (3.6) and from the transverse nature of $F_{\mu\nu}(p, -p, 0)$. Defining

$$
2F_{\mu\nu}(p, -p, 0) = \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) F^{(1)}(p^2) \quad (3.11a)
$$

then using (3.10) and (3.6) we have

$$
\frac{\partial \Pi(p^2)}{\partial \xi} = \left[ 1 + \Pi(p^2) \right] F^{(1)}(p^2)
$$

$$
\frac{\partial}{\partial \xi} \log \left[ 1 + \Pi(p^2) \right] = F^{(1)}(p^2) \quad (3.11b)
$$

$$
1 + \Pi(p^2) = \exp \left[ \int F^{(1)}(p^2) d\xi \right]
$$

This formal solution relating the gluon self-energy to another Green function could be of interest in other contexts, but at present we do not see any immediate implications. However, this result could provide a different approach for studying questions of confinement.

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### 3.2 Gauge Dependence of the Quark Propagator

The Nielsen identity for the quark propagator can be obtained directly from (3.5b) by functional differentiation with respect to $\psi(x)$, $\bar{\psi}(y)$ and using quark number conservation.

$$
\frac{\partial}{\partial \xi} \frac{\delta^2 \Gamma}{\delta \psi(y) \delta \psi(x)} = \frac{\delta^2 \Gamma}{\delta K_\mu \delta \chi} \frac{\delta^3 \Gamma}{\delta \psi(y) \delta \psi(x) \delta A_\mu} + \frac{\delta^4 \Gamma}{\delta \psi(y) \delta \psi(x) \delta K_\mu \delta \chi} \frac{\delta \Gamma}{\delta A_\mu}.
$$

As in Section 2, the first and second terms are zero because of Lorentz invariance with operator insertions at zero momentum. This leads to a result identical to (2.13)\(^9\).

$$
\frac{\partial}{\partial \xi} S^{-1}(p) = S^{-1}(p) \left\{ F(p, -p, 0) + \bar{F}(p, -p, 0) \right\}.
$$

Again, since $S^{-1}(p)$ is zero at the mass shell, and since $F(p, -p, 0)$ has no single particle pole, we have the following Nielsen identity for the quark propagator.

$$
\left. \frac{\partial S^{-1}}{\partial \xi} \right|_{p^2 = M^2} = 0.
$$

Just as in Section 2, this result may be shown to imply that the quark mass shell is gauge independent in QCD for on-shell renormalisation schemes. We will return to a perturbative consideration of this idea in Section 4.

### 3.3 Nielsen Identity for the Ghost Propagator

To construct the Nielsen identity for the ghost propagator we need to differentiate (3.5a) with respect to $c(x)$ and $\bar{c}(0)$. After applying ghost and fermion number conservation the following identity is obtained:

$$
\frac{\partial}{\partial \xi} \frac{\delta^2 \Gamma}{\delta c(x) \delta \bar{c}(0)} = \frac{\delta^2 \Gamma}{\delta \chi \delta K_\mu(w)} \frac{\delta^3 \Gamma}{\delta c(x) \delta \bar{c}(0) \delta A_\mu(w)} - \frac{\delta^2 \Gamma}{\delta K_\mu(w) \delta c(x)} \frac{\delta^3 \Gamma}{\delta \chi \delta \bar{c}(0) \delta A_\mu(w)}
$$

$$
+ \frac{\delta^4 \Gamma}{\delta \chi \delta \bar{c}(w) \delta c(x) \delta \bar{c}(0)}.
$$

The dependence on the variable $w$ (which has an integration associated with it) has been made explicit to avoid possible confusion with $c(x)$. The first term on the right hand
side of (3.15) is zero since after Fourier transforming we have a zero momentum insertion with a vector operator. The third term is also zero since the one point function has the property \( B = \delta W/\delta J_B = 0 \).

To analyze the remaining terms we first recognize that

\[
\int d^4(x - y) e^{ip \cdot (x - y)} \frac{\delta^2 \Gamma}{\delta c(x) \delta K_\mu(y)} = \frac{p_\mu}{p^2} \tilde{D}^{-1}(p^2)
\]  

(3.16)

where \( \tilde{D}^{-1}(p^2) \) is the inverse ghost propagator. After Fourier transforming (3.15) we have the following Nielsen identity for the ghost propagator.

\[
\frac{\partial}{\partial \xi} \tilde{D}^{-1}(p^2) = \frac{p_\mu}{p^2} G_\mu(p, -p, 0) \tilde{D}^{-1}(p^2) + G^{(1)}(p, -p, 0) \tilde{D}^{-1}(p^2) \]  

(3.17a)

\[
G^{(1)}(p, q, -p - q) = \int d^4xd^4ye^{ip \cdot x + iq \cdot y} \frac{\delta^3 \Gamma}{\delta K_c(x) \delta c(y) \delta \chi} \]  

(3.17b)

\[
G^\mu(p, q, -p - q) = \int d^4xd^4ye^{ip \cdot x + iq \cdot y} \frac{\delta^3 \Gamma}{\delta A_\mu(x) \delta c(y) \delta \chi} \]  

(3.17c)

Now consider the Green function \( G^\mu \) to take into account the mixing between \( A_\mu \) and \( B \).

\[
\frac{\delta^3 \Gamma}{\delta \chi \delta \bar{c}(y) \delta A_\mu(x)} = \int d^4s d^4w \frac{\delta^3 W}{\delta \chi \delta J_c(w) \delta J_\lambda(s)} \frac{\delta J_c(w)}{\delta \bar{c}(y)} \frac{\delta J_\lambda(s)}{\delta A_\mu(x)} + \int d^4s d^4w \frac{\delta^3 W}{\delta \chi \delta J_c(w) \delta J_B(s)} \frac{\delta J_c(w)}{\delta \bar{c}(y)} \frac{\delta J_B(s)}{\delta A_\mu(x)} \]  

(3.18)

\[
= \int d^4s d^4w \frac{\delta^3 W}{\delta \chi \delta J_c(w) \delta J_\lambda(s)} \frac{\delta \bar{c}(y) \delta c(w)}{\delta A_\mu(x) \delta A_\lambda(s)} + \int d^4s d^4w \frac{\delta^3 W}{\delta \chi \delta J_c(w) \delta J_B(s)} \frac{\delta \bar{c}(y) \delta c(w)}{\delta A_\mu(x) \delta B(s)} \frac{\delta^2 \Gamma}{\delta^2 \Gamma} \]

After Fourier transforming we find that the first term in (3.18) does not contribute to \( p^\mu G_\mu(p, -p, 0) \) since \( \Gamma_{\mu \lambda}(p) \) is transverse. Thus we have

\[
\frac{p_\mu}{p^2} G_\mu(p, -p, 0) = \tilde{D}^{-1}(p^2) \frac{1}{2} \int d^4xd^4y e^{ip \cdot x - ip \cdot y} \langle O|T(\bar{c}(0)B(0)c(x)B(y))|O \rangle
\]  

(3.19)

where (2.26b) has been used.

Again, we can explicitly display the perturbative content of (3.17a) to one-loop by calculating the diagrams in Figures 3 & 4 and noting that the tree level contribution in
each case is zero. The results are
\[
\frac{p^\mu}{p^2} G_\mu(p, -p, 0) = -\frac{g^2 N_c}{64\pi^2} \left[ \frac{1}{\epsilon} - 2 \right]
\]
\[
G^{(1)}(p, -p, 0) = -\frac{g^2 N_c}{64\pi^2} \left[ -\frac{2}{\epsilon} + 2 \right]
\] (3.20)

The result for the ghost propagator is \cite{10}
\[
\tilde{D}^{-1}(p^2) = p^2 \left[ 1 + \Pi(p^2) \right]
\]
\[
\Pi(p^2) = -\frac{g^2 N_c}{16\pi^2} \left[ \frac{3 - \xi}{4\epsilon} - 1 \right]
\] (3.21)

Substituting (3.20) and (3.21) into (3.17a) verifies the Nielsen identity for the ghost propagator to one-loop.

Similarly to the case for the gluon propagator, (3.17a) has a formal solution. Defining
\[
G^{(2)}(p, -p, 0) = G^{(1)}(p, -p, 0) + \frac{p^\mu}{p^2} G_\mu(p, -p, 0)
\] (3.22)

and noting that there is no dependence on \(\tilde{D}(p^2)\) in (3.22), we have the following formal solution to the Nielsen identity.
\[
\tilde{D}^{-1}(p^2) = p^2 \exp \left[ \int d\xi G^{(2)}(p, -p, 0) \right].
\] (3.23)

4. On-Shell Fermion Mass Renormalisation Constant

In this section we wish to investigate the consequences of the Nielsen identities [see (2.14) and (3.14)] for the fermion propagator in a perturbative analysis of on-shell renormalisation constants. The key result from both QED and QCD is
\[
\left. \frac{\partial S^{-1}_{F}}{\partial \xi} (p) \right|_{p^2 = M^2} = 0.
\] (4.1)

Consider the following definition of the fermion mass renormalisation constant (valid for both QED and QCD) in an on-shell scheme\cite{11, 12}
\[
Z_m \equiv \frac{m_0}{M},
\] (4.2)
where $p = M$ is a zero of the inverse propagator. We will prove that $Z_m$ (and hence $M$) must be gauge parameter independent to all orders in perturbation theory.

To construct this direct link between the results of perturbation theory and our formal proof of the gauge independence of the mass-shell we will show that the coefficients $M_i$ in the expansion of the mass renormalisation constant (we are using $D = 4 - 2\omega$ to be consistent with [9])

$$Z_m \equiv \frac{m_0}{M} = 1 + \sum_{i=1}^{\infty} \left( \frac{\alpha_s}{M^{2\omega}} \right)^i M_i,$$  \hspace{1cm} (4.3)

are gauge independent to all orders in perturbation theory. This has been explicitly observed to two loops in perturbation theory [11–13].

Now, the Feynman propagator is written, using the conventions of [11], as

$$S^{-1}_F(p) = p - m_0 - \Sigma(p),$$ \hspace{1cm} (4.4)

$$\Sigma(p) = \sum_{i=1}^{\infty} \left( \frac{\alpha_s}{p^{2\omega}} \right)^i \left[ m_0 A_i \left( \frac{m_0^2}{p^2} \right) + (p - m_0) B_i \left( \frac{m_0^2}{p^2} \right) \right].$$ \hspace{1cm} (4.5)

We must also have, in addition to the above Nielsen identity, that the inverse propagator itself must vanish on the mass shell ($p^2 = M^2$). This is sufficient to define the coefficients $M_i$ in the expansion of $Z_m$ in the following way. Working to the one-loop level, we will disregard anything of the order of $\alpha_s^2$ or higher. Thus we have the expression for $S^{-1}_F(p)$ at one-loop to be

$$S^{-1}_F(p) = p - m_0 - \frac{\alpha_s}{p^{2\omega}} \left[ m_0 A_1 \left( \frac{m_0^2}{p^2} \right) + (p - m_0) B_1 \left( \frac{m_0^2}{p^2} \right) \right].$$ \hspace{1cm} (4.6)

We then substitute for $m_0$ from (4.3), obtaining, to order $\alpha_s$

$$S^{-1}_F(p) = p - M + M - M \left[ 1 + \frac{\alpha_s}{M^{2\omega}} M_1 \right]$$

$$- \frac{\alpha_s}{p^{2\omega}} \left[ M A_1 \left( \frac{M^2}{p^2} \right) + (p - M) B_1 \left( \frac{M^2}{p^2} \right) \right].$$ \hspace{1cm} (4.7)

This expression is then evaluated on the mass shell and set equal to zero:

$$S^{-1}_F(p) \bigg|_{p^2 = M^2} = - \frac{\alpha_s}{M^{2\omega}} M M_1 - \frac{\alpha_s}{M^{2\omega}} M A_1 (1) = 0,$$ \hspace{1cm} (4.8)
thus we must have \( M_1 = -A_1(1) \) in order for equation (4.8) to be satisfied.

If one now considers the Nielsen identity (4.1), then one obtains to one loop the relation

\[
\frac{\partial S_F^{-1}}{\partial \xi}(p) = -\frac{\alpha_s}{\rho^{2\omega}} \left( m_0 \frac{\partial A_1}{\partial \xi} \left( \frac{m_0^2}{\rho^2} \right) + (\rho - m_0) \frac{\partial B_1}{\partial \xi} \left( \frac{m_0^2}{\rho^2} \right) \right) .
\]  

(4.9)

Upon going on-shell and substituting for \( m_0 \), we get

\[
\frac{\partial S_F^{-1}}{\partial \xi}(p) \bigg|_{\rho^2 = M^2} = -\frac{\alpha_s}{M^{2\omega}} M \frac{\partial A_1}{\partial \xi}(1) = 0 .
\]  

(4.10)

Thus we obtain the result that

\[
\frac{\partial A_1}{\partial \xi}(1) = 0 ,
\]  

(4.11)

and thus that

\[
\frac{\partial M_1}{\partial \xi} = 0 ,
\]  

(4.12)

ie., that \( M_1 \) is gauge independent.

It is this result that we wish to argue can be continued to all orders in perturbation theory. That this is possible can be seen by comparing the structure of the Nielsen identity above with that of the determining equation for \( M_1 \), before one goes on-shell. The structure is identical, with the exception of the leading term containing \( M_1 \). What remains when one goes on-shell, in (4.7) defines \( M_1 \), and what remains after going on-shell in (4.9) shows \( M_1 \) to be gauge independent.

Let us illustrate by continuing on to two loops. To the two-loop level, we have

\[
S_F^{-1}(p) = \rho - m_0 - \frac{\alpha_s}{\rho^{2\omega}} \left( m_0 A_1 \left( \frac{m_0^2}{\rho^2} \right) + (\rho - m_0) B_1 \left( \frac{m_0^2}{\rho^2} \right) \right) - \frac{\alpha_s^2}{\rho^{4\omega}} \left( m_0 A_2 \left( \frac{m_0^2}{\rho^2} \right) + (\rho - m_0) B_2 \left( \frac{m_0^2}{\rho^2} \right) \right) ,
\]  

(4.13)
again we substitute for \( m_0 \) and keep all terms of order \( \alpha_s^2 \) or less. This gives

\[
S_F^{-1}(p) = \frac{p^2}{\psi - M} - M - \left( \frac{\alpha_s}{M^{2\omega}} M M_1 + \frac{\alpha_s^2}{M^{4\omega}} M M_2 \right) - \frac{\alpha_s}{p^{2\omega}} \left( M A_1 \left( \frac{M^2}{p^2} \right) \right) + \frac{\alpha_s}{M^{2\omega}} M M_1 A_1 \left( \frac{M^2}{p^2} \right) + \frac{\alpha_s}{M^{2\omega}} \frac{2M^3 M_1}{p^2} A'_1 \left( \frac{M^2}{p^2} \right) + \left( \psi - M - \frac{\alpha_s}{M^{2\omega}} M M_1 \right) B_1 \left( \frac{M^2}{p^2} \right) \tag{4.14}
\]

\[
- \frac{\alpha_s^2}{p^{4\omega}} \left( M A_2 \left( \frac{M^2}{p^2} \right) + (\psi - M) B_2 \left( \frac{M^2}{p^2} \right) \right).
\]

Going on-shell, it is easy to see that the terms in (4.14) of order \( \alpha_s \) are simply the one-loop result, and that the order \( \alpha_s^2 \) terms will define \( M_2 \) to be:

\[
M_2 = -(A_2(1) + M_1 (A_1(1) + 2A'_1(1) - B_1(1))). \tag{4.15}
\]

On the other hand, we can see that application of the identity (4.1) to the two-loop expression (4.13) will result in

\[
\frac{\partial S_F^{-1}}{\partial \xi}(p) = - \frac{\alpha_s}{p^{2\omega}} \left( M \frac{\partial A_1}{\partial \xi} \left( \frac{M^2}{p^2} \right) + \frac{\alpha_s}{M^{2\omega}} M \frac{\partial}{\partial \xi} \left( M A_1 \left( \frac{M^2}{p^2} \right) \right) \right) + \frac{\alpha_s}{M^{2\omega}} \frac{2M^3}{p^2} \frac{\partial}{\partial \xi} \left( M A'_1 \left( \frac{M^2}{p^2} \right) \right) + \left( \psi - M \right) \frac{\partial B_1}{\partial \xi} \left( \frac{M^2}{p^2} \right), \tag{4.16}
\]

and here we can see that when we go on shell the order \( \alpha_s \) part reproduces equation (4.11) and the order \( \alpha_s^2 \) part will give

\[
\left. \frac{\partial S_F^{-1}}{\partial \xi}(p) \right|_{p^2 = M^2} = - \frac{\partial}{\partial \xi} \left( M_1 (A_1(1) + 2A'_1(1) - B_1(1)) + A_2(1) \right) = 0, \tag{4.17}
\]

which obviously guarantees that \( M_2 \) is gauge independent. Note also that we made no use of one-loop results in obtaining equations (4.15) and (4.17), and thus the gauge independence
of $M_2$ in no way depends upon the gauge independence of $M_1$, although the one-loop results will obviously make things simpler.

One can see, then, that the relationship between the defining equation for $M_n$ and the $n$-loop Nielsen identity (4.1), which shows the gauge independence of $M_n$, will be maintained to all orders in perturbation theory.

Having demonstrated that $Z_m$ is gauge independent ($\partial Z_m / \partial \xi = 0$) it then follows from (4.2) that $\partial M / \partial \xi = 0$.

\[
\frac{\partial Z_m}{\partial \xi} + \frac{\partial M}{\partial \xi} \frac{\partial Z_m}{\partial M} = \frac{Z_m}{M} \frac{\partial M}{\partial \xi} \frac{\partial M}{\partial \xi} \left[ \frac{Z_m}{M} \frac{\partial Z_m}{\partial M} \right] = 0 \quad \Rightarrow \quad \frac{\partial M}{\partial \xi} = 0 \quad (4.18)
\]

This completes our perturbative analysis of the gauge independence of the mass shell and mass renormalisation constant in on-shell renormalisation schemes.

5. Conclusions

In this paper we have studied the Nielsen identities for the two-point functions of QED and QCD in the covariant formalism. It was demonstrated in the case of QED that the identities lead to results complementary to those of the usual Ward identities. As with the Ward identities, the Nielsen identities offer possibilities to check one’s calculations, however, they also allow us to see where physical meaning may be found in apparently gauge dependent Green’s functions. In particular it was proven that the electron pole mass, the photon polarization and the on-shell mass renormalisation constant $Z_m$ are all independent of the gauge parameter.

For QCD it was demonstrated that the quark pole mass and on-shell mass renormalisation constant are gauge independent. This is a formal property and we are not sure as to its correct interpretation in relation to quark confinement. In particular the generating function used above to describe QCD, does not take the Gribov ambiguity\cite{14} into account. It has recently been shown that this ambiguity prevents the construction outside
of perturbation theory of a globally BRST invariant field with quark number one\cite{15} which could be identified with a physical quark. *Perturbatively* invariant quark fields do exist, however, and we stress the need to clarify the connection between the invariance of the pole mass, shown here using functional methods, and the perturbatively gauge invariant solutions with quark number one found in \cite{15}. In this context we note that the gauge independence of the pole mass has already been shown\cite{13} to hold in perturbation theory up to two loops and appears to be true in the operator product expansion (OPE) of the quark propagator\cite{9,16}, although care must be taken there since gauge-dependent condensates appear in the OPE of the gauge-dependent propagator\cite{17}.

We have calculated the explicit one-loop content of the Nielsen identities obtained for the gluon and ghost propagators. Formal solutions to the ghost and gluon Nielsen identities were constructed for the propagators in terms independent Green functions. Since this gives a different view of the gluon propagator this solution may be of interest for studies of confinement.

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Figure Captions

**Figure 1.** Radiative corrections to the mixed propagator \( \langle O | T (B(x)A_\mu(y)) | O \rangle \). The dotted line represents the field \( B \).

**Figure 2.** One-loop contributions to the Green function \( \mathcal{F}_{\mu\nu}(p,-p,0) \). The representation of \( A_\mu \) and the ghosts is conventional, while the dotted line represents the field \( B \).

**Figure 3.** One-loop contributions to the Green function \( \frac{p^\mu}{p^2} G^\mu(p,-p,0) \). Diagram c leads to massless tadpoles and is thus zero in dimensional regularization.

**Figure 4.** One-loop contributions to the Green function \( G^{(1)}(p,-p,0) \). Diagram b leads to massless tadpoles and is thus zero in dimensional regularization.
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