Spin liquids from Majorana Zero Modes in a Cooper Box

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We propose a path for constructing diverse interacting spin systems from topological nanowires in Cooper Boxes. The wires are grouped into a three-wire building block called an 'hexon', consisting of six Majorana zero modes. In the presence of a strong charging energy, the hexon becomes a Cooper box equivalent to two spin-1/2 degrees of freedom. By considering arrays of hexons and controlling the distances between the various wires, one can tune the Hamiltonian governing the low-energy spins, thus providing a route for controllably constructing interacting spin systems in one- and two-dimensions. We explicitly present realizations of the one-dimensional spin-1/2 XXZ chain, as well as the transverse field Ising model. We propose an experiment capable of revealing the nature of critical points in such effective spin systems by applying a local gate voltage and measuring the induced charge at a distance. To demonstrate the applicability of this approach to two-dimensions, we provide a scheme for realizing the topologically ordered Yao-Kivelson spin-liquid model, which has a collective Majorana edge mode, similar to the B-phase of Kitaev’s honeycomb model.

Introduction: Quantum spin models are of paramount importance in condensed matter physics. While spin-models were traditionally devised to study magnetically ordered materials, they are nowadays known to exhibit highly non-trivial behavior, such as diverse critical phenomena and topological order (see, e.g., [1–3]).

An important mathematical tool used to uncover these non-trivial properties is the fermionization of the spins to Majorana degrees of freedom, which in a few notable cases leads to exact solutions. Important examples are the Jordan-Wigner transformation [4], which allows for exact solutions of one-dimensional (1D) spin-1/2 models, such as the XXZ and Ising models [5–7], as well as two-dimensional (2D) ones, such as Yao-Kivelson (YK) model [3]. A more recent example is the Kitaev transformation [2], originally used to solve the Kitaev honeycomb model and demonstrate the emergence of non-Abelian spin-liquid behavior.

Recent strong evidence indicate the emergence of Majorana zero modes (MZMs) on the edges of semiconductor nanowires with spin-orbit coupling, which are in proximity to an s-wave superconductor [8–18]. When a few such MZMs are placed in a quantum dot with strong Coulomb interactions, a so called Majorana-Cooper-Box, or MZM island, is formed. The MZMs in the island can be mapped onto spin degrees of freedom. For example, considering four MZMs in an islands, each pair forms a fermion, thus generating four degenerate states. Including the constraint on the total number of particles in the box due to the strong Coulomb interactions, an effective two-level system—or a spin 1/2—is formed. Indeed, similar ideas have been used to study the so-called "topological Kondo effect" [19–24], and realize few spin-liquid models [25–27]. Such models are of interest due to their promise as platforms for fault tolerant quantum computing.

The approach of constructing effective spin systems from MZMs is reciprocal to the common fermionization of spin models: instead of starting with physical spins and mapping them to Majorana degrees of freedom through mathematical transformations, we begin with physical MZMs and map them onto spins. In some cases, the resulting spin models may then be solved through a distinct mathematical transformation to fictitious Majorana degrees of freedom, which are non-local with respect to the physical MZMs.

In this work we propose a different setup, where each box is made of three semiconducting wires, as shown in Fig. 1a, and demonstrate that in the presence of a strong charging energy, two effective spin degrees of freedom emerge at low energies. Due to the presence of six MZMs, we refer to our building block as an 'hexon' [28]. The hexon building blocks are shown to be highly tunable, and in fact, controlling the coupling between different MZMs (e.g., by tuning the local chemical potential) allows us to fully determine the coupling between different spins and the effective magnetic field they experience. If many such building blocks are arranged in a 1D line, or cover the 2D plane, this allows us to simulate a plethora of spin models in 1D and 2D using only tunneling and local charging terms.

We start by focusing on the 1D setup shown in Fig. 1b, and demonstrate that by controlling the couplings between different MZMs, one can simulate SU(2) invariant spin-1/2 chains. In particular, we realize the spin-1/2 Heisenberg chain, known to be described by a low-energy Luttinger Liquid (LL) fixed point. By modulating the distance between a specific pair of MZMs as a function of time and measuring the induced charge at a distance, we propose measurable imprints of this critical point.

We then provide a recipe for constructing the transverse field Ising model, known to give rise to the Ising critical point. We propose imprints of this critical point, which are in particular capable of directly probing the properties of the so called σ-operator [29].

Finally, we describe the construction of the 2D YK
decorated honeycomb model (shown in Figs. 1c-1d), giving rise to a spin-liquid state with a chiral Ising CFT on the edge, and discuss the experimental consequences of this gapless edge.

The hexon: The basic building block in our construction is the so-called hexon [28], illustrated in Fig. 1a. Each hexon is composed of three semiconductor nanowires with strong spin-orbit coupling. The wires are proximity coupled to an s-wave superconductor. Applying a strong Zeeman field drives the wires into the topological regime, in which protected MZMs reside near the ends of each wire [8–17]. The Majorana zero modes are conveniently denoted by the operators \(a_\alpha, b_\alpha\), with the indices \(\alpha = x, y, z\), as illustrated in Fig. 1a. The presence of six Majorana zero modes leads to a degeneracy of 8.

Taking Coulomb blockade into account, and assuming that the charging energy \(E_C\) is the largest energy scale in play, we fix the charge, and therefore the parity, of the entire hexon by controlling a back-gate voltage. At low energies, the parity of the entire hexon can be written in terms of the MZMs as

\[
P = i a_1 a_2 a_3 b_1 b_2 b_3. \tag{1}
\]

Thus, by controlling the back-gate voltage, we can effectively apply the constraint \(P = 1\) (or similarly, \(P = -1\)), thus reducing the ground state degeneracy to 4.

To find a useful parametrization of the remaining 4-dimensional low energy subspace, we define spin-1/2 operators according to [30]

\[
S_\alpha^x = ia_y a_z, \quad S_\alpha^y = ia_x a_z, \quad S_\alpha^z = ia_x a_y \tag{2}
\]

\[
S_\beta^x = ib_y b_z, \quad S_\beta^y = ib_x b_z, \quad S_\beta^z = ib_x b_y. \tag{3}
\]

It can easily be checked that these are in fact spin-1/2 operators (i.e., they satisfy the relation \(S^i S^m = \epsilon_{ijk} \delta^m_k + \delta^m_j\)), which commute with the total parity [Eq. (1)], and therefore do not violate the parity-fixing constraint. The number of states indeed coincides with the degeneracy of a two-spin system, and we find that at low energies the six MZMs are reduced to two effective spin-1/2 degrees of freedom. In what follows, the effective spin degrees of freedom will be used to design non-trivial spin models by engineering the coupling between different MZMs.

Coupling the spins: We start by studying the terms that arise from coupling the MZMs within the hexon. The first of these arises when the lengths \(l_\alpha\) of the wires (see Fig. 1b) are made short enough such that the Majorana wavefunctions at the two ends overlap. In this case we get terms of the form

\[
H_1 = i \sum_{\alpha=x,y,z} J_\alpha a_\alpha b_\alpha. \tag{4}
\]

where the coupling constants \(J_\alpha\) are controlled by the lengths \(l_\alpha\). Notice that the sign of \(J_\alpha\) can also be tuned as the overlap between the MZM wave-functions generically changes sign as a function of \(l_\alpha\). Alternatively, by tuning the chemical potential in the wire, one controls the localization length of the MZMs and therefore their coupling. Taking the constraint \(P = 1\) into account, and using Eqs. (2) and (3), we can write these as

\[
H_1 = \sum_{\alpha=x,y,z} J_\alpha S_\alpha^a S_\alpha^b. \tag{5}
\]

One can also generate a different set of terms by coupling MZMs of the same type (\(a\) with \(a\) and \(b\) with \(b\)), e.g., by changing the distance between wires. This generates coupling terms of the form

\[
H_2 = i \sum_{\alpha\alpha'} (a_\alpha a_{\alpha'} + b_\alpha b_{\alpha'}). \tag{6}
\]

In terms of the spin operators, \(H_2\) can be written as

\[
H_2 = \sum_{\alpha} B_\alpha (S_\alpha^a + S_\alpha^b), \tag{7}
\]

with \(B_\alpha \propto \epsilon_{\alpha\beta\gamma} \tilde{\gamma}_{\beta\gamma}\).

To recapitulate, we find that each hexon is equivalent to two spins degrees of freedom, and that the effective coupling between the two spins, as well as coupling to an external magnetic field, can be controlled by tuning the coupling between the MZMs (for example, with gate potentials). In what follows we use these hexon building blocks to form 1D and 2D interacting spin models.

Realizing SU(2)-invariant spin chains: Consider the array of hexons depicted in Fig. 1b. As we discussed above, these are equivalent to an array of spins, labeled by \(S_{j,\gamma}\), where \(j\) enumerates the different hexon unit cells and \(\gamma = a, b\) differentiates between the two spins in each unit cell.

We start by assuming that the distance between different wires is large such that the effective Zeeman field \([B_\alpha\text{ in Eq. (7)}]\) vanishes, yet the lengths \(l_\alpha\) are small enough to generate \(H_1\)-type terms of the form

\[
H_1 = \sum_j \sum_{\alpha=x,y,z} J_\alpha S_\alpha^a S_{j,b}^a. \tag{8}
\]

Coupling terms of the form \(S_{j,a}^a S_{j+1,a}^a\) can additionally be generated by bringing different hexons close to each other. This generates tunneling terms of the form

\[
H_{\text{tunneling}} = i \sum_{\alpha=x,y,z} \tilde{\gamma}_{\alpha} \sum_j b_{\alpha j} a_{\alpha j+1}. \tag{9}
\]

These terms, however, alter the parity of the hexons and therefore do not commute with the constraint. Under our assumption that the charging energy \(E_C\) is the largest
energy scale, the tunneling terms in Eq. (9) thus scale down to zero. Nevertheless, we can form combinations of these terms that commute with the constraint. The lowest order terms generated in perturbation theory take the form

$$H'_1 = \sum_{\alpha=x,y,z} J'_\alpha \sum_j S_{j,b}^\alpha S_{j+1,a}^\alpha$$

where $J'_\alpha \propto \frac{\Pi_{\alpha'=x,y,z} J'_{\alpha'}}{E_C}$.

At low energies, our model is therefore given by a combination of Eqs. (8) and (10). For simplicity, we start by assuming that the system was tuned to be $SU(2)$-invariant, i.e., $J_\alpha = J$, and $J'_{\alpha} = J'$. We further assume that $J,J' > 0$.

Clearly, if $J > J'$, we get a fully gapped dimerized phase, in which the two spins corresponding to each hexon form a singlet state. In the opposite regime where $J' > J$, the system is again in a dimerized phase, now with adjacent spins originating from different hexons forming singlet states.

The two above phases are topologically distinct, with the second state giving rise to a protected decoupled spin on each edge. As such, we expect to find a critical point if we tune $J = J'$. Indeed, at this point our model becomes the spin-$\frac{1}{2}$ Heisenberg model, known to be dual to a 1D model of interacting fermions. The latter is described by the Luttinger-liquid fixed point Hamiltonian

$$H_{LL} = \frac{v}{2\pi} \int dx \left[ K (\partial_x \theta)^2 + \frac{1}{K} (\partial_x \varphi)^2 \right],$$

with the Luttinger parameter $K = \frac{1}{2}$ [31], the spin operator $S_z(x) = \frac{1}{2} \partial_x \varphi$, and $[\theta(x), \varphi(x')] = i\pi \Theta(x - x')$. The Luttinger parameter can be varied if the $SU(2)$ symmetry is broken to $U(1)$, i.e., if one of the components $J_\alpha$ is not the same as the other two. Indeed, mutual capacitance terms generically renormalize the Luttinger parameter [32]. Notice that we neglected higher order tunneling terms as a renormalization group analysis indicates they are irrelevant.

**Experimental signature:** The above constitutes an example of realizing a critical spin model from the physical MZMs. It is natural to ask whether one can measure imprints of the gapless spin model in the current realization. Such an imprint is required to distinguish between gapless and gapped states, as well as between gapless states described by different conformal field theories (CFTs).
Clearly, given that the charges of freedom are gapped, one cannot use electronic transport measurements. A possible route is then to use thermal conductance measurements instead. While such measurements are possible, and were in fact used recently to detect imprints of the non-Abelian nature of the quantum Hall plateau at filling 5/2 \cite{33-35}, they are difficult in practice.

Instead, we propose an alternative experiment in which a time-dependent gate modulating the coupling between two specific MZMs is applied. If we choose these to be $a_{xj_0}$ and $a_{yj_0}$ (or similarly $b_{xj_0}$ and $b_{yj_0}$) in a specific unit cell, we obtain a time-dependent Hamiltonian of the form $H_{pert} = f(t)S^z(x_0)$, where for simplicity we assume that $f(t) = V_0 \cos(\omega t + \phi_0)$.

To find imprints of the gapless nature of the underlying state, we propose to measure the expectation value of the time-ordered correlation function takes the form $\langle \chi(t-t', x-x_0) \rangle$ with $\chi$ being the dynamic susceptibility: $\chi(t-t', x-x') = \langle [S^z(x,t), S^z(x',t)] \rangle \Theta(t-t')$. As we demonstrate in the Appendix, in a non-chiral critical point, where the (time-ordered) correlation function takes the form $G \sim \alpha^{4h}/(\Delta^2-v^2)^{2h}$ (with $\alpha$ being the short distance cutoff, and $h$ the conformal dimension), we obtain

$$\langle S^z(x,t) \rangle = \int dt' f(t') \chi(t-t', x-x_0), \quad (12)$$

where $\chi$ is a constant phase. By measuring the induced parity on the distance and frequency provides a direct imprint of the critical point, where the state is not in a symmetry broken phase, and is connectable to the state in which all the spins form an eigenstate of $S^z$ with eigenvalue -1. The above two phases are separated by a gapless critical point at $h = J$, in which case the effective spin chain is described by an Ising fixed point with central charge $c = 1/2$ \cite{36}.

To probe this critical point, we can repeat the above experiment where a local time dependent gate modulates the $z$ component of the magnetic field at point $x_0$, and a charge probe at point $x$ effectively measures the induced $S^z(x,t)$. Since $S^z$ can be identified with the $\sigma$ primary field of the Ising CFT at low energies, its correlation function scales with $h = 1/16$. We can find the induced parity by plugging this into Eq. (13). The dependence of the parity on the distance and frequency provides a direct imprint of the non-trivial CFT.

The transverse field Ising model: The flexibility of altering the various length scale in our setup allows us to realize a large set of spin models which goes beyond the above SU(2) invariant chains. In what follows we provide an explicit construction of another prominent spin chain - the transverse field Ising model - defined by the Hamiltonian

$$H_{\text{Ising}} = \sum_j [-JS^z_j S^z_{j+1} + hS^z_j]. \quad (14)$$

The first term can be generated similarly to the above: by making the length $l_z$ of the $z$-type wires short enough and simultaneously bringing $x$ and $y$ type wires coming from adjacent hexons closer to each other. If these terms are taken to have identical amplitudes, they generate the first term in Eq. (14). In addition, assuming the distance between the $y$- and $z$-wires in each hexon is made short, we generate $H_{2}$-type terms, giving rise to the second term in Eq. (14).

As is well known, the transverse field Ising model possesses two different phases. For $J > h$, the ground state of the system spontaneously breaks the $S^z \rightarrow -S^z$ symmetry and the spins collectively point in the $\pm z$ direction. In the opposite regime, where $h > J$, the state is not in a symmetry broken phase, and is connectable to the state in which all the spins form an eigenstate of $S^z$ with eigenvalue -1. Above these two phases are separated by a gapless critical point at $h = J$, in which case the effective spin chain is described by an Ising fixed point with central charge $c = 1/2$ \cite{36}.

As we argue now, the same ideas can be applied to 2D spin models. To demonstrate this, we explicitly construct the so-called Yao-Kivelson model \cite{3}, which realizes a non-Abelian spin-liquid state.

The 2D Yao-Kivelson spin liquid: In the above analyses, we have demonstrated that the hexon building blocks provide a fruitful playground for realizing 1D spin chains. As we argue now, the same ideas can be applied to 2D spin models. To demonstrate this, we explicitly construct the so-called Yao-Kivelson model \cite{3}, which realizes a non-Abelian spin-liquid state.

To do that, we sort the hexons in structures similar to Fig. 1c. Notice that the labels $x, y, z$ of the MZMs are now alternating. In each hexon, we assume that the colored wire is made short and therefore induces $S^z_{\alpha'} S^z_{\beta}$-type terms. Correlated tunneling terms between different hexons also generate $S^z_{\alpha'} S^z_{\beta}$-type terms, with $\alpha$ determined by geometry - i.e., $\alpha$ is chosen such that $\Pi_{\alpha' \neq \alpha} \Pi'_{\alpha' \neq \alpha}$ is maximized. The resulting dominating terms are shown in Fig.
in terms of the MZMs and in terms of the spin degrees of freedom in the inset.

If many such building blocks are connected in a way that covers the 2D plane, we obtain the decorated honeycomb lattice geometry, shown in Fig 1d, where each link is given a label $\alpha$, stating the dominating $S^\alpha S^\alpha$ term. The resulting spin Hamiltonian is identical to the YK Hamiltonian, known to generate a non-Abelian spin liquid state in the so-called B-phase (as long as the coupling at the $x', y'$, and $z'$ links is not too large), which in addition spontaneously breaks time reversal symmetry. The Abelian A-phase of the Kitaev honeycomb model can also be realized, for example, if we take the $z'$ coupling to be much larger than $x', y'$. Other proposals for realizing this phase were given in Refs. [26, 27]. The advantage of the current proposal is the ability to control all the coupling terms with gate potentials.

Within the B-phase, the edge of the sample gives rise to a chiral Ising CFT, similar to the edge of a $p + ip$ superconducting state, which can be constructed from arrays of Majorana wires as well [37, 38]. As opposed to the $p + ip$ superconducting state, however, the resulting state is topologically ordered, with the $\sigma$-particle being deconfined. We note that one can obtain the above spin-liquid from the $p + ip$ superconducting state by condensing $\hbar/e$ vortices [39].

In order to measure imprints of the gapless edge, we again repeat the experiment above on the edge. As shown in the Appendix, the induced $S_z$ is given by

\[
\langle S_z(x, t) \rangle = V_0 \frac{\omega^{2h-1} \alpha^{2h}}{v^{2h}} \cos \left( \omega \left( t - \frac{\Delta x}{v} \right) + \varphi_0 \right),
\]

with $\hbar$ being the smallest dimension among the operators excited by $S_z$. In contrast to the 1D case, here the perturbation operates on a chiral edge, and therefore cannot act as the $\sigma$ primary field on that edge alone.

**Conclusions:** To summarize, in this manuscript we use MZMs as a basic building block for constructing nontrivial spin models. We show that a system containing six MZMs in a Majorana-Cooper box—an hexon—is equivalent to two spin 1/2 degrees of freedom. By changing the coupling between the different MZMs, one can controllably simulate spin models and tune them to criticality. We provided explicit examples for the XXZ model and the transverse field Ising model in 1D, as well as the YK model in 2D. We have discussed possible physical measurements capable of revealing the nature of these spin models at critical points.

We note that in 1D models, disorder can generally have drastic effects, and in particular change the character of critical points. Two-dimensional topologically ordered systems, such as the YK spin liquid, are protected against weak disorder. In particular, the gapless edges are generically protected by virtue of their chiral nature.

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of perturbations of the form the response of chiral and non-chiral one-dimensional critical systems. The time ordered propagators of one-dimensional CFTs can generally be written as

\[ G(t, x) = \frac{\alpha^{2(h + \bar{h})}}{[x - vt + i\text{sign}(t)]^{2h} [x + vt - i\text{sign}(t)]^{2\bar{h}}} \]

where \( h, \bar{h} \) are the conformal dimensions of the corresponding field. For a non-chiral field, we have \( h = \bar{h} \). For a chiral field, one of these vanishes.

Appendix

In this Appendix, we provide an explicit calculation of the response to the oscillating magnetic field. We study the the response of chiral and non-chiral one-dimensional critical systems.

The Response Function

In the main text, we proposed that the critical nature of our 1D spin models can be revealed by studying the effect of perturbations of the form \( H_{\text{pert}} = f(t)S^z(x_0) \), with \( f(t) = V_0 \cos(\omega t + \phi_0) \). To do that, we would like to compute the average value of \( S^z \) at a distant point \( x \) and later times.

Performing linear response, this can be written as

\[ \langle S^z(x, t) \rangle = \int dt' f(t') \chi(t - t', x - x_0), \tag{16} \]

with \( \chi \) being the dynamic susceptibility: \( \chi(t - t', x - x') = i \langle [S^z(x, t), S^z(x', t')] \rangle \Theta(t - t') \). In our case, the function \( f \) is harmonic, meaning we may write

\[ \langle S^z(x, t) \rangle = V_0 \Re \left\{ e^{i\phi_0} \int dt' e^{i\omega t'} \chi(t - t', x - x_0) \right\} = V_0 \Re \left\{ e^{i\omega t} e^{i\phi_0} \chi(\omega, x - x_0) \right\}, \tag{17} \]

where \( \chi(\omega, x) = \int dt e^{-i\omega t} \chi(t, x) \) is the frequency domain form of the dynamic susceptibility. The susceptibility can be written in terms of the time-ordered propagator as

\[ \chi(t, x) = -2\Theta(t) \text{Im} \{ G(t, x) \}. \]

The time ordered propagators of one-dimensional CFTs can generally be written as

\[ G(t, x) = \frac{\alpha^{2(h + \bar{h})}}{[x - vt + i\text{sign}(t)]^{2h} [x + vt - i\text{sign}(t)]^{2\bar{h}}}, \]

where \( h, \bar{h} \) are the conformal dimensions of the corresponding field. For a non-chiral field, we have \( h = \bar{h} \). For a chiral field, one of these vanishes.

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Non-chiral fields

For non-chiral fields, the susceptibility is given by

\[
\chi(t, x) = -2 \Theta(t) \alpha^4 \text{Im} \left\{ \frac{1}{(x - vt + i\epsilon)(x + vt - i\epsilon)} \right\}^{2h}
\]

\[
= -2 \Theta(t) \alpha^4 \text{Im} \left\{ e^{-2h \log A} \right\},
\]

where we follow Ref. [31] in defining

\[
A = (x - vt + i\epsilon) (x + vt - i\epsilon) = x^2 - v^2 t^2 + 2i\epsilon t.
\]

If we put the branch cut of the log along the negative real axis, we get an imaginary part only for \(x^2 - v^2 t^2 < 0\), i.e. \(t > \frac{|x|}{v}\), and we get

\[
e^{-2h \log A} = e^{-2h \log|x^2 - v^2 t^2|} - 2\pi i h \Theta(t + \frac{\pi}{4}) \Theta(t - \frac{\pi}{4}).
\]

We therefore obtain

\[
\chi(t, x) = -2 \sin(2\pi h)\alpha^4 \Theta(t) \Theta(t + \frac{\pi}{2}) \Theta(t - \frac{\pi}{2}) \frac{\Theta(1 - v t)}{(v^2 t^2 - x^2)^{2h}}.
\] (18)

If \(h = 1/2\), this expression vanishes. This result is non-physical, and indeed, the case \(h = 1/2\) requires special attention. In this case, we can explicitly write

\[
\chi(t, x) = -2 \Theta(t) \alpha^2 \text{Im} \left\{ \left( \frac{1}{x - vt + i\epsilon} \right) \left( \frac{1}{x + vt - i\epsilon} \right) \right\}^{2h}
\]

\[
= -2 \Theta(t) \alpha^2 \text{Im} \left\{ \mathcal{P} \frac{1}{x - vt} - i\pi \delta(x - vt) \right\} \mathcal{P} \frac{1}{x + vt} + i\pi \delta(x + vt) \right\}
\]

\[
= -2\pi \Theta(t) \alpha^2 \left( \frac{1}{x - vt} \delta(x + vt) - \frac{1}{x + vt} \delta(x - vt) \right),
\] (19)

where \(\mathcal{P}\) denotes the principal value.

In order to evaluate Eq. 17 for a general \(h\), we wish to get the frequency domain form of \(\chi(t, x)\) in Eq. 18:

\[
\chi(\omega, x) = 2\alpha^4 \sin(2\pi h) \int_{\frac{|x|}{v}}^{\infty} dt \frac{e^{-i\omega t}}{(v^2 t^2 - x^2)^{2h}}
\]

\[
= 2\alpha^4 \sin(2\pi h) \frac{v}{|x|^{4h-1}} \int_{1}^{\infty} dT \frac{e^{-i\omega |T|}}{(T^2 - 1)^{2h}},
\]

where we have defined \(T = \frac{vt}{|x|}\). Performing the integral, we obtain

\[
\chi(\omega, x) = B \alpha^4 v^{-2h-\frac{1}{2}} \left( \frac{\omega}{|x|} \right)^{2h-\frac{1}{2}} K_{\frac{1}{2}-2h} \left( \frac{\omega |x|}{v} \right),
\] (20)

with \(B = \frac{2\sin(2\pi h)\Gamma(1-2h)(-2i)^{\frac{1}{2}-2h}}{\sqrt{\pi}}\). Notice that while Eq. 18 vanishes for \(h = 1/2\), Eq. 20 has a finite limit for \(h \rightarrow 1/2\). In this case, since \(\sin(2\pi h) \Gamma(1-2h) \rightarrow \text{const}\) as \(h \rightarrow 1/2\), and \(K_{\frac{1}{2}-2h} |z| = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{1}{2}-2h)}{\sqrt{z}}\), we get

\[
\chi(\omega, x) \propto \alpha^2 \frac{e^{-i\omega |x|}}{v |x|}.
\]

We can obtain this result directly from Eq. 19. In this case

\[
\chi(\omega, x) = -2\pi \alpha^2 \int_{0}^{\infty} dt \left[ \frac{1}{x - vt} \delta(x + vt) - \frac{1}{x + vt} \delta(x - vt) \right] e^{-i\omega t}
\]

\[
= \frac{\pi \alpha^2}{v} \left[ -\frac{1}{x} \Theta(-x) e^{i\omega \frac{|x|}{v}} + \frac{1}{x} \Theta(x) e^{-i\omega \frac{|x|}{v}} \right] = \frac{\pi \alpha^2}{v} e^{-i\omega \frac{|x|}{v}} |x|.
\]
Chiral fields

For chiral (right moving) fields, the susceptibility is given by

$$\chi(t, x) = -2\alpha^{2h} \Theta(t) \text{Im} \left\{ \frac{1}{[x - vt + i\epsilon]^{2h}} \right\} = -2\alpha^{2h} \Theta(t) \text{Im} \left\{ e^{-2h \log A} \right\},$$

where now

$$A = x - vt + i\epsilon.$$ 

The same analysis as in the non-chiral case indicates that

$$e^{-2h \log A} = e^{-2h \log |x - vt| - 2pi h \Theta \left( t - \frac{x}{v} \right)},$$

and therefore

$$\chi(t, x) = -2 \sin \left( 2\pi h \right) \alpha^{2h} \frac{\Theta(t) \Theta \left( t - \frac{x}{v} \right)}{(vt - x)^{2h}}. \quad (21)$$

Calculating the Fourier transform, we obtain

$$\chi(\omega, x) = \frac{2\alpha^{2h} \sin \left( 2\pi h \right)}{v \cdot x^{2h-1}} \int_1^\infty dT \frac{e^{-i\omega T}}{(T - 1)^{2h}} = C \frac{\omega^{2h-1} \alpha^{2h}}{v^{2h}} e^{-i\omega x},$$

with $C = 2 \sin \left( 2\pi h \right) i^{2h-1} \Gamma(1 - 2h)$. 