Greedy regularized kernel interpolation

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Abstract
Kernel based regularized interpolation is a well known technique to approximate a continuous multivariate function using a set of scattered data points and the corresponding function evaluations, or data values. This method has some advantage over exact interpolation: one can obtain the same approximation order while solving a better conditioned linear system. This method is well suited also for noisy data values, where exact interpolation is not meaningful. Moreover, it allows more flexibility in the kernel choice, since approximation problems can be solved also for non strictly positive definite kernels. We discuss in this paper a greedy algorithm to compute a sparse approximation of the kernel regularized interpolant. This sparsity is a desirable property when the approximant is used as a surrogate of an expensive function, since the resulting model is fast to evaluate. Moreover, we derive convergence results for the approximation scheme, and we prove that a certain greedy selection rule produces asymptotically quasi-optimal error rates.

1 Kernels and regularized interpolation
Our goal is to construct an approximant on an input space Ω ⊂ R^d, d ≥ 1, of an unknown continuous function f : Ω → R provided the knowledge of arbitrary pairwise distinct data points X_n := {x_i}^n_{i=1} ⊂ Ω, n ∈ N, and data values {f(x_i)}^n_{i=1} ⊂ R.

The approximant is constructed via kernel interpolation. We recall here the basic facts required for our analysis, while we refer to [20] for further details.

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On $\Omega$ we consider a positive definite kernel $K : \Omega \times \Omega \rightarrow \mathbb{R}$, i.e., a symmetric function such that for any $n \in \mathbb{N}$ and any set $X_n := \{x_i\}_{i=1}^n \subset \Omega$ of pairwise distinct points the kernel matrix $A \in \mathbb{R}^{n \times n}$, $A_{ij} := K(x_i, x_j)$, is positive semidefinite. For a strictly positive definite $K$, $A$ is required to be positive definite.

Associated with the kernel $K$ there is a uniquely defined native Hilbert space $H := H_K(\Omega)$ of functions $\Omega \rightarrow \mathbb{R}$. The space is the unique Hilbert space of functions from $\Omega$ to $\mathbb{R}$ where $K$ acts as a reproducing kernel, i.e., $K(\cdot, x) \in H$ and $(f, K(\cdot, x))_H = f(x)$ for all $x \in \Omega$, $f \in H$. The elements of this space are of the form $f := \sum_{i \in I} \alpha_i K(\cdot, x_i)$ for $I$ a countable set, $\{\alpha_i\}_{i \in I} \subset \mathbb{R}$, and $\{x_i\}_{i \in I} \subset \Omega$, and for $g := \sum_{j \in J} \beta_j K(\cdot, y_j)$ it holds $(f, g)_H = \sum_{i \in I, j \in J} \alpha_i \beta_j K(x_i, y_j)$. Moreover, if the kernel has smoothness $K \in C^{2\tau}(\Omega \times \Omega)$ with $\tau \geq 0$ and $\Omega$ an open set, then it holds that $H_K(\Omega) \subset C^{\tau}(\Omega)$.

One of the main reasons of interest for positive definite kernels is that various approximation problems can be solved in $H$ for arbitrary pairwise distinct data points $X_n \subset \Omega$ and data values $\{f(x_i)\}_{i=1}^n \subset \mathbb{R}$, $f \in H$. Indeed, one can consider a loss and a regularization functional $L, R : H \rightarrow \mathbb{R}$, and a regularization parameter $\lambda \geq 0$ to define an approximant of $f \in H$ as

$$s^\lambda_n(f) := s^\lambda(f, X_n) := \arg \min_{s \in H} L(s) + \lambda R(s),$$

and to obtain a pointwise reconstruction of $f$ it is common to consider functionals defined as

$$L(s) := L(s, f, X_n) := \sum_{i=1}^n (f(x_i) - s(x_i))^2, \quad R(s) := \|s\|^2_H,$$

in which case a solution $s^\lambda_n(f)$ of eq. (1) is called a regularized interpolant of $f$. Other choices of the functionals lead, for example, to Support Vector Machines and Support Vector Regression (see e.g. [17]).

This approximation process is well known and characterized for a wide class of functionals by the Representer Theorem (see [18], and [16] for a general statement), and for the special case considered here the following holds.

**Theorem 1** (Representer Theorem for regularized interpolation). If $K$ is positive definite, the problem eq. (1) with functionals eq. (2) admits a solution of the form

$$s^\lambda_n(f) = \sum_{j=1}^n \alpha_j K(\cdot, x_j),$$

where the vector of coefficients $\alpha \in \mathbb{R}^n$ is the solution of the linear system

$$(A + \lambda I) \alpha = b, \quad b_i := f(x_i).$$

If $K$ is strictly positive definite, this is the unique solution for all $\lambda \geq 0$. 

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This approximant is well defined also for positive definite kernels, since the linear system eq. \( (1) \) has a unique solution as long as \( \lambda > 0 \). This is not the case for pure interpolation, i.e., \( \lambda = 0 \), since the matrix \( A \) can be singular in this case. On the other hand, in the case of strictly positive definite kernels the interpolant \( s^0(f) \) always exists and is unique, but it is still in general useful to consider a regularized interpolant. Indeed, \( \lambda \) is a tunable parameter which provides a trade-off between pointwise accuracy, since \( s^0(f) \) exactly interpolates \( f \) on \( X_n \), and stability, since the condition number of \( A + \lambda I \) is a strictly decreasing function of \( \lambda \). Moreover, in several applications the data values \( \{ f(x_i) \}_{i=1}^n \) may be affected by noise, thus it makes no sense to require exact interpolation.

We remark that this kind of approximation can be extended to deal with vector-valued functions \( f : \Omega \to \mathbb{R}^q, \quad q > 1 \). In this case an approximant can be easily obtained by applying the same scheme to each of the \( q \) components of \( f \), while keeping the set \( X_n \) fixed across them. The only required modification is to change the right hand side in eq. \( (1) \), which becomes a \( n \times q \) matrix with \( b_i := f(x_i)^T \). The resulting solution \( \alpha \) is now also a \( n \times q \) matrix, and each of its rows can be used as a coefficient vector in eq. \( (1) \) to obtain the desired vector-valued prediction. This construction corresponds to the use of a trivial matrix-valued kernel, but more sophisticated options are possible (see e.g. \([9, 24]\)). Nevertheless, we consider here only the case \( q = 1 \), while we will analyze the general matrix-valued case in full generality in a forthcoming work.

The goal of this paper is to describe an efficient way to compute \( s^\lambda_n(f) \) for an iteratively increasing set of points \( X_n \), which is adaptively enlarged at each iteration by selecting a new point from a set \( \Omega_h \subset \Omega \) in a greedy way. This method is a direct extension of the (Vectorial) Kernel Orthogonal Greedy Algorithm ((V)KOGA) \([22]\), which applies to the case of exact interpolation (i.e., \( \lambda = 0 \)) with strictly positive definite kernels. We will describe this extension and the resulting algorithm in section \( 2 \) and we will consider greedy selection rules of \( X_n \) which generalize the \( f \)- and \( P \)-greedy rules for interpolation (\([4, 15]\)).

When the points are selected freely inside \( \Omega \), i.e., \( \Omega_h := \Omega \), the process is a way to place suitable sampling points \( X_n \). If instead the selection is made from a large but finite set \( \Omega_h := X_N \subset \Omega \) of given data points or measure locations with \( N \gg n \), then \( s^\lambda_N(f) \) can be understood as a sparse approximation of \( s^\lambda_N(f) \), in the sense that in the sum eq. \((3)\) only the terms corresponding to points in \( X_n \) are nonzero (although the coefficients are in general not the same). In both cases, a good selection of \( X_n \) guarantees that only a small number \( n \) of points is sufficient to obtain a good accuracy.

The reduction of the number of non-zero terms in the expansion eq. \( (3) \) has different computational advantages, and it is mainly interesting in case \( s^\lambda_n(f) \) is used as a surrogate model of an expensive function \( f \) in a multi-query scenario (see e.g. \([7, 8, 23]\)). In this case, the time required to obtain
the evaluation $s_{\lambda}^n(f)(x)$ for a new input $x \in \mathbb{R}^d$ is a crucial measure of the usability of the surrogate, and it clearly depends on the size of the expansion eq. (3).

In some notable cases, also convergence rates can be derived for the regularized interpolation process using sampling inequalities [12,21]. They apply to translational invariant kernels such as the Gaussian or the Wendland kernels [19], and they prove that regularized interpolation has the same error rate of interpolation, provided $\lambda$ is chosen small enough, depending on the distribution of the interpolation points. We will adapt them to our generalized setting and, after proving some general error bounds in section 3, in section 4.1 we will show that, in the case of the generalization of $P$-greedy, the results of [13] can be extended to conclude that the greedy selected points provide the same convergence rate given by these sampling inequalities for optimally placed points.

2 Iterative computation and greedy algorithms

The general structure of the greedy algorithm is the following. We will come back in section 4 to good criteria to select the next point $x_n$, and for now we concentrate on the computation of $s_{\lambda}^n(f) \in \mathcal{H}$. We start from the empty set $X_0 := \emptyset$, the zero subspace $V(X_0) := \{0\}$, and the zero interpolant $s_{\lambda}^0(f) := 0 \in V(X_0)$. At every iteration $n > 0$ we select a new point $x_n \in \Omega_h \setminus X_{n-1}$ and define $X_n := X_{n-1} \cup \{x_n\}$ and $V(X_n) := \text{span} \{K(\cdot, x_i), x_i \in X_n\}$, and then compute $s_{\lambda}^n(f) \in V(X_n)$ by theorem 1 as the regularized interpolant with data points $X_n$ and values $\{f(x_i), x_i \in X_n\}$.

Since $s_{\lambda}^n(f) \in V(X_n)$, for any basis $\{v_k\}_{k=1}^n$ of $V(X_n)$ we can write $s_{\lambda}^n(f) = \sum_{k=1}^n c_k v_k$ for suitable coefficients $\{c_k\}_{k=1}^n$. To have an efficient computation of $s_{\lambda}^n(f)$ and to avoid recomputing already computed quantities, we should employ a nested basis, i.e., span $\{v_k\}_{k=1}^n = V(X_n)$ for all $n$, and have that the coefficients $\{c_k\}_{k=1}^{n-1}$ do not change at step $n$.

In the case of non regularized interpolation with a strictly positive definite kernel $K$, the basis satisfying these properties is the Newton basis of [10,11], which can be obtained by a Gram-Schmidt orthonormalization of $\{K(\cdot, x_i), x_i \in X_n\}$ in $\mathcal{H}$, and for which it holds

$$v_k := \sum_{j=1}^n \beta_{jk} K(\cdot, x_j), \quad 1 \leq k \leq n,$$

with a matrix of coefficients $C_v := [\beta_{jk}]_{j,k=1}^n = L^{-T}$, where $A = LL^T$ is the Cholesky factorization of the kernel matrix $A$. This basis can be easily updated when adding a new point, since the leading principal submatrix of $L$ is the Cholesky factor of the corresponding leading principal submatrix of $A$. The resulting VKOGA algorithm uses this basis, suitable selection rules
for the new point, and the extension to vector-valued functions outlined in Section 1.

To extend this construction to the case of regularized interpolation for possibly non strictly positive definite kernels, from theorem 1 we see that the regularized interpolant is defined by coefficients which solve a linear system with matrix $A + \lambda I$. This matrix is in fact the kernel matrix of the kernel $K_\lambda(x, y) := K(x, y) + \lambda \mathbb{1}_{\{y\}}(x)$ on the points $X_n$, where $\mathbb{1}_{\{y\}}(x)$ is the indicator function of the set $\{x\}$, which is clearly symmetric and it is indeed strictly positive definite for $\lambda > 0$. In the following proposition we prove this fact and some related properties of the corresponding native spaces.

Proposition 2. Let $\Omega \subset \mathbb{R}^d$ have non empty interior, $K \in C(\Omega \times \Omega)$ be a positive definite kernel on $\Omega$, and $\lambda > 0$. Then

i) $K'(x, y) := \lambda \mathbb{1}_{\{y\}}(x)$ and $K_\lambda(x, y) := K(x, y) + K'(x, y)$ are strictly positive definite kernels on $\Omega$.

ii) The native spaces are related by $\mathcal{H}_{K_\lambda}(\Omega) = \mathcal{H}_K(\Omega) \oplus \mathcal{H}_{K'}(\Omega)$.

iii) For all $f \in \mathcal{H}_{K_\lambda}(\Omega)$ there exist unique $g \in \mathcal{H}_K(\Omega)$, $h \in \mathcal{H}_{K'}(\Omega)$ such that $f = g + h$, and it holds $\|f\|_{\mathcal{H}_{K_\lambda}(\Omega)}^2 = \|g\|_{\mathcal{H}_K(\Omega)}^2 + \|h\|_{\mathcal{H}_{K'}(\Omega)}^2$.

Proof. For any set $X_n \subset \Omega$ the kernel matrix of $K'$ is the scaled identity matrix $\lambda \cdot I$, so $K'$ is clearly positive definite and strictly positive definite if $\lambda > 0$. Its native space consists of functions $f(x) := \sum_{i \in I} \alpha_i K'(x, x_i) = \lambda \sum_{i \in I} \alpha_i \delta_{x_i}(x)$ for a countable set $I$, $\{x_i\}_{i \in I} \subset \Omega$, and $\{\alpha_i\}_{i \in I} \subset \mathbb{R}$. In particular, since $\{x_i\}_{i \in I}$ is countable and $\Omega$ is more than countable, $\mathcal{H}_{K'}(\Omega)$ contains no continuous functions except for $f := 0$.

Since $K$, $K'$ are positive definite, according to [1, Section 6] also $K_\lambda$ is positive definite, and it is strictly positive definite if at least one between $K$ and $K'$ is strictly positive definite, so in particular for $\lambda > 0$.

Since $K \in C(\Omega \times \Omega)$ it follows that $\mathcal{H}_K(\Omega) \subset C(\Omega)$ and thus $\mathcal{H}_K(\Omega) \cap \mathcal{H}_{K'}(\Omega) = \{0\}$. Then again [1, Section 6] guarantees that item i and item iii hold.

For simplicity we use from now on the notation $\mathcal{H} := \mathcal{H}_K(\Omega)$ and $\mathcal{H}_\lambda := \mathcal{H}_{K_\lambda}(\Omega)$. From the last proposition, item i, it follows that for $\lambda > 0$, $X_n \subset \Omega$, and $f \in \mathcal{H}_\lambda$, the $\mathcal{H}_\lambda$-interpolant of $f$ on $X_n$ is well defined. We denote it as

$$I^n_\lambda(f) = \sum_{j=1}^n \alpha_j K_\lambda(\cdot, x_j),$$

where $(A + \lambda I)\alpha = b$. Using item iii of proposition 2 and the definition of
$s_n^\lambda(f)$, we have a unique decomposition of $I_n^\lambda(f)$, which needs to satisfy

$$I_n^\lambda(f)(x) = \sum_{j=1}^{n} \alpha_j K(x, x_j) = \sum_{j=1}^{n} \alpha_j \lambda \mathbb{1}_{\{x_j\}}(x)$$

$$= s_n^\lambda(f)(x) + \sum_{j=1}^{n} \alpha_j \lambda \mathbb{1}_{\{x_j\}}(x), \ x \in \Omega.$$ 

Observe that this construction implies that the regularized interpolant is well defined also for $f \in \mathcal{H}_\lambda$. Moreover, for all $f \in \mathcal{H}_\lambda$ we get $I_n^\lambda(f)(x) = s_n^\lambda(f)(x)$ if $x \not\in X_n$.

The same decomposition remains valid if $I_n^\lambda(f)$ is expressed in terms of the Newton basis of $X_n$ in $\mathcal{H}_\lambda$, which we denote as $\{v_k^\lambda\}_{k=1}^{n}$, and which is defined, analogous to eq. (5), by coefficients

$$C_v := L^{-T}, \ A + \lambda J = L L^T.$$ (6)

Once again, we recall that this basis exists since $K^\lambda$ is strictly positive definite by item 3 of proposition 2. We recall that the interpolant $I_n^\lambda(f)$ is the orthogonal projection of $f \in \mathcal{H}_\lambda$ into $\text{span} \{K(\cdot, x_i), x_i \in X_n\}$, and

$$I_n^\lambda(f) = \sum_{k=1}^{n} (f, v_k^\lambda)_{\mathcal{H}_\lambda} v_k^\lambda.$$ (7)

Moreover, as elements of $\mathcal{H}_\lambda$, also the functions $v_k^\lambda$ have a unique decomposition, which is

$$v_k^\lambda(x) := \sum_{j=1}^{n} \beta_{jk} K^\lambda(x, x_j) = \sum_{j=1}^{n} \beta_{jk} K(x, x_j) + \lambda \sum_{j=1}^{n} \beta_{jk} \mathbb{1}_{\{x_j\}}(x), \ x \in \Omega.$$ (8)

For every $1 \leq k \leq n$, we denote as $v_k$ the first term in the right hand side, and as in the case of the interpolant above we have $v_k^\lambda(x) = v_k(x)$ if $x \not\in X_n$. The elements $\{v_k\}_{k=1}^{n}$ are clearly in $V(X_n)$. If $K$ is strictly positive definite they are linearly independent since the matrix $C_v$ from eq. (6) is invertible, so they are a basis, while they are at least a generating set for $V(X_n)$ if $K$ is only positive definite. This set of functions is what we need to have the efficient update of the regularized interpolant.

**Proposition 3.** Let $X_n := X_{n-1} \cup \{x_n\} \subset \Omega$ and $f \in \mathcal{H}_\lambda$. Then it holds

$$s_n^\lambda(f) = \sum_{k=1}^{n} (f, v_k^\lambda)_{\mathcal{H}_\lambda} v_k = s_{n-1}^\lambda(f) + (f, v_n^\lambda)_{\mathcal{H}_\lambda} v_n.$$ (9)

Moreover, if $f \in \mathcal{H}$ it holds $(f, v_k)_{\mathcal{H}} = (f, v_k^\lambda)_{\mathcal{H}_\lambda}$ so

$$s_n^\lambda(f) = \sum_{k=1}^{n} (f, v_k)_{\mathcal{H}} v_k = s_{n-1}^\lambda(f) + (f, v_n)_{\mathcal{H}} v_n.$$ (10)
Proof. For the first equality in eq. (9) we just need to prove that \( \sum_{k=1}^{n} (f, v_k^\lambda)_{\mathcal{H}_\lambda} v_k \) equals eq. (3) with \( \alpha \) that satisfies eq. (4). This holds since

\[
\sum_{k=1}^{n} (f, v_k^\lambda)_{\mathcal{H}_\lambda} v_k = \sum_{k=1}^{n} \sum_{i=1}^{n} \beta_{ik} (f, K_\lambda(\cdot, x_i))_{\mathcal{H}_\lambda} n \sum_{j=1}^{n} \beta_{jk} K(\cdot, x_j),
\]

and \( \beta_{ik} = (C_v)_{ik} = (L^{-T})_{ik} \), thus

\[
\sum_{i=1}^{n} \sum_{k=1}^{n} \beta_{ik} \beta_{jk} f(x_i) = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} (L^{-T})_{ik} (L^{-T})_{jk} \right) f(x_i)
\]

\[
= \sum_{i=1}^{n} \left( (A + \lambda I)^{-1} \right)_{ji} f(x_i) = \alpha_j.
\]

The second equality holds because the basis is nested since \( v_n^\lambda \) depends only on the points \( X_n \), being \( C_v \) upper triangular. Moreover, if \( f \in \mathcal{H} \subset \mathcal{H}_\lambda \) it holds \( (f, K(\cdot, x))_{\mathcal{H}} = (f, K_\lambda(\cdot, x))_{\mathcal{H}_\lambda} \) since both are the reproducing kernel of the corresponding space, so in particular \( (f, v_k^\lambda)_{\mathcal{H}} = (f, v_k^\lambda)_{\mathcal{H}_\lambda} \) holds by linearity and eq. (10) follows.

In the case of non regularized interpolation it is also possible to define a residual as the difference between the target function \( f \) and the current interpolant. It can be used to decide what point to select in the \( f \)-greedy variant of VKOGA, and it also has an efficient update formula.

The same can be obtained for regularized interpolation as follows. Observe that both the residual and the update rule are nothing but the ones of the interpolant \( I_n^\lambda(f) \), which are already known to satisfy the desired properties. We prove the statement only for completeness.

Proposition 4. Let \( f \in \mathcal{H}_\lambda \) and define the residual \( r_n \in \mathcal{H}_\lambda \) as

\[
0 := f, \quad r_n := f - \sum_{k=1}^{n} c_k v_k^\lambda, \quad n \geq 1.
\]

Then we have \( r_n(x_k) = 0 \) for \( 1 \leq k \leq n \) and

\[
s_n^\lambda(f) = \sum_{k=1}^{n} (r_{k-1}, v_k^\lambda)_{\mathcal{H}_\lambda} v_k.
\]

Proof. Since \( \{v_k^\lambda\} \) is \( \mathcal{H}_\lambda \)-orthonormal, we obtain

\[
(r_{k-1}, v_k^\lambda)_{\mathcal{H}_\lambda} = \left( f - \sum_{j=1}^{k-1} c_j v_j^\lambda, v_k^\lambda \right)_{\mathcal{H}_\lambda} = (f, v_k^\lambda)_{\mathcal{H}_\lambda}.
\]
and thus eq. (12) equals eq. (9). Moreover, by the form eq. (7) of the interpolant \( I_n^\lambda(f) \) we obtain

\[
\begin{align*}
    r_n(x_k) &= f(x_k) - \sum_{k=1}^{n} c_k v_k^\lambda(x_k) = f(x_k) - \sum_{k=1}^{n} (f, v_k^\lambda)_{\mathcal{H}} v_k^\lambda(x_k) \\
    &= f(x_k) - I_n^\lambda(f)(x_k) = 0.
\end{align*}
\]

\[\square\]

3 Approximation schemes and error estimation

We recall that a standard way to measure the pointwise interpolation error is via the power function. Although we do not consider the case here, we remark that it would be possible to have a proper definition also for positive definite kernels. Indeed, it would be sufficient to solve the linear system defining the interpolant using the pseudo-inverse of the kernel matrix.

Instead in the following, whenever we mention \( s_n^0 \) or its power function \( P_n \) we implicitly assume that \( K \) is strictly positive definite so that both objects are well defined, which is instead always the case for the interpolant \( I_n^\lambda \) if \( \lambda > 0 \).

For the interpolants \( s_n^0 \) and \( I_n^\lambda \) the power function can be defined and computed as

\[
P_n(x) := \sup_{f \in \mathcal{H}} \frac{|f(x) - s_n^0(f)(x)|}{\|f\|_{\mathcal{H}}} = \|K(\cdot, x) - s_n^0(K(\cdot, x))\|_{\mathcal{H}} \quad (13)
\]

\[
Q_n^\lambda(x) := \sup_{f \neq 0} \frac{|f(x) - I_n^\lambda(f)(x)|}{\|f\|_{\mathcal{H}}_{\lambda}} = \|K_\lambda(\cdot, x) - I_n^\lambda(K_\lambda(\cdot, x))\|_{\mathcal{H}_{\lambda}},
\]

and in both cases from the definition we obtain pointwise error bounds of the form

\[
\begin{align*}
    |f(x) - s_n^0(f)(x)| &\leq P_n(x)\|f\|_{\mathcal{H}}, \quad x \in \Omega, \; f \in \mathcal{H} \quad (14) \\
    |f(x) - I_n^\lambda(f)(x)| &\leq Q_n^\lambda(x)\|f\|_{\mathcal{H}_{\lambda}}, \quad x \in \Omega, \; f \in \mathcal{H}_{\lambda},
\end{align*}
\]

where the bounds can not be improved for a fixed \( x \in \Omega \), if they have to hold for all \( f \in \mathcal{H} \) or \( f \in \mathcal{H}_{\lambda} \).

To obtain the same kind of error bound as in eq. (14), we define the power function of regularized interpolation as

\[
P_n^\lambda(x) := \sup_{f \in \mathcal{H}} \frac{|f(x) - s_n^\lambda(f)(x)|}{\|f\|_{\mathcal{H}}}, \quad (15)
\]

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which immediately gives

\[ |f(x) - s_n^\lambda(f)(x)| \leq P_n^\lambda(x)\|f\|_\mathcal{H}, \ x \in \Omega, \ f \in \mathcal{H}. \]

By this definition it holds indeed \( P_n^\lambda = P_n \) if \( \lambda = 0 \), and we have the following result.

**Proposition 5.** For all \( x \in \Omega \) we have

\[
P_n^\lambda(x)^2 = \left\| K(\cdot,x) - s_n^\lambda(K(\cdot,x)) \right\|^2_\mathcal{H} = K(x,x) - 2 \sum_{k=1}^n v_k(x)^2 + \sum_{i,k=1}^n v_k(x)v_i(x)(v_k,v_i)_\mathcal{H}. \quad (16)
\]

**Proof.** First observe that, using eq. (10), for all \( f,g \in \mathcal{H} \) it holds

\[
\left( f, s_n^\lambda(g) \right)_\mathcal{H} = \left( s_n^\lambda(f), g \right)_\mathcal{H}.
\]

To simplify the notation we define \( v_x := K(\cdot,x) \). For any \( x \in \Omega \) and \( f \in \mathcal{H} \) we have

\[
\left| \left( f - s_n^\lambda(f) \right)(x) \right| = \left| (v_x,f)_\mathcal{H} - (v_x,s_n^\lambda(f))_\mathcal{H} \right| = \left| (v_x,f)_\mathcal{H} - (s_n^\lambda(v_x),f)_\mathcal{H} \right| = \left| (v_x - s_n^\lambda(v_x),f)_\mathcal{H} \right| \leq \left\| v_x - s_n^\lambda(v_x) \right\| \|f\|_\mathcal{H},
\]

so from eq. (15) it follows that \( P_n^\lambda(x) \leq \left\| v_x - s_n^\lambda(v_x) \right\|_\mathcal{H}. \)

The equality is reached by taking \( f := f_x := v_x - s_n^\lambda(v_x) \). Indeed, the norm of \( f_x \) is

\[
\|f_x\|_\mathcal{H} = \left\| v_x - s_n^\lambda(v_x) \right\|_\mathcal{H} = (v_x,v_x)_\mathcal{H} - 2(v_x,s_n^\lambda(v_x))_\mathcal{H} + (s_n^\lambda(v_x),s_n^\lambda(v_x))_\mathcal{H}
\]

\[
= (v_x,v_x)_\mathcal{H} - 2(v_x,s_n^\lambda(v_x))_\mathcal{H} + (v_x,s_n^\lambda(s_n^\lambda(v_x)))_\mathcal{H}
\]

\[
= v_x(x) - 2s_n^\lambda(v_x)(x) + s_n^\lambda(s_n^\lambda(v_x))(x),
\]

and by linearity of \( s_n^\lambda \) we obtain

\[
(f_x - s_n^\lambda(f_x))(x) = v_x(x) - s_n^\lambda(v_x)(x) - s_n^\lambda(v_x)(x) + s_n^\lambda(s_n^\lambda(v_x))(x)
\]

\[
= v_x(x) - 2s_n^\lambda(v_x)(x) + s_n^\lambda(s_n^\lambda(v_x))(x) = \|f_x\|_\mathcal{H}^2,
\]

thus \( |f_x(x) - s_n^\lambda(f_x)(x)|/\|f_x\|_\mathcal{H} = \|f_x\|_\mathcal{H} = \|v_x - s_n^\lambda(v_x)\|_\mathcal{H}. \)

Using the form eq. (10) of the regularized interpolant, the second equality easily follows. Observe that the terms can not be simplified for \( \lambda > 0 \) because the basis \( \{v_k\}_{k=1}^n \) is not orthogonal in \( \mathcal{H} \).
Thanks to eq. (14), upper bounds on the power function give upper bounds on the pointwise approximation error achieved by the corresponding approximation scheme. The $P$-greedy variant of VKOGA uses precisely this idea and selects at each iteration the new point $x_n$ that maximizes $P_n(x)$ over $\Omega_h \setminus X_{n-1}$. It can be proven that this selection strategy produces approximants that have a quasi-optimal convergence rate, i.e., $n$ greedily selected points provide, up to a different constant, the same convergence order of $n$ optimally placed ones (see [13]). We would like to achieve the same result here by defining a suitable $P$-greedy selection rule and prove optimality of the corresponding interpolant. In the case of interpolation, the actual use of this selection criterion is possible because also the power function has an efficient update rule. For example, in the case of $Q_{n}^{\lambda}$, for the Newton basis $\{v_{k}^{\lambda}\}_{k=1}^{n}$ it holds

$$Q_{n}^{\lambda}(x) = K_{\lambda}(x, x) - \sum_{k=1}^{n} v_{k}^{\lambda}(x)^{2} = Q_{n-1}^{\lambda}(x)^{2} - v_{n}^{\lambda}(x)^{2}, \quad (17)$$

and similarly for $P_{n}$ using the corresponding Newton basis. Moreover, the convergence results of [13] are possible because of the use of the general results of [5], which apply to the case of approximation schemes which are best approximations, like it is the case for interpolation in $H, H_{\lambda}$. Instead, it is clear from proposition 5 that both these properties are not realized by $P_{n}^{\lambda}$. Nevertheless, we will overcome the problem by relating $P_{n}^{\lambda}$ to $P_{n}$ and $Q_{n}^{\lambda}$ as follows.

We remark that a step of the following proof requires the use of equation eq. (27), which in turn follows from proposition 8. Both results are proven independently from the next proposition, so we postpone them to simplify the exposition of the results.

**Proposition 6.** For $x \in \Omega$ it holds $P_{n}^{\lambda}(x) \leq \sqrt{\lambda}$, while we have

$$P_{n}(x) \leq P_{n}^{\lambda}(x) \leq Q_{n}^{\lambda}(x) \quad \text{for all} \quad x \in \Omega \setminus X_{n}. \quad (18)$$

In particular

$$\left\|P_{n}^{\lambda}\right\|_{L_{\infty}(\Omega)} \leq \left\|Q_{n}^{\lambda}\right\|_{L_{\infty}(\Omega)}. \quad (19)$$

**Proof.** First, for $x \in X_{n}$ it holds $|f(x_{i}) - s_{n}^{\lambda}(x_{i})| \leq \sqrt{\lambda}\|f\|_{H}$ (see e.g. Proposition 3.1 in [21]), so $P_{n}^{\lambda}(x) \leq \sqrt{\lambda}$ thanks to the definition eq. (15).

The first inequality in eq. (18) follows from proposition 5 and the definitions eq. (13). Indeed, both $s_{n}^{0}$ and $s_{n}^{\lambda}$ are maps into $V(X_{n})$, but the interpolant is the best approximation operator in $H$, so

$$\left\|K(\cdot, x) - s_{n}^{0}(K(\cdot, x))\right\|_{H} \leq \left\|K(\cdot, x) - s_{n}^{\lambda}(K(\cdot, x))\right\|_{H}. \quad (18)$$
For the second inequality, and for \( x \notin X_n \) and \( f \in \mathcal{H}_\lambda \), we observed in section 2 that \( s_n^\lambda(f)(x) = I_n^\lambda(f)(x) \). Moreover, from Proposition 2, we have that the unit ball in \( \mathcal{H} \) is contained in the unit ball of \( \mathcal{H}_\lambda \), thus using a standard argument we can conclude that for \( x \notin X_n \)

\[
P_n^\lambda(x) = \sup_{\substack{f \in \mathcal{H} \atop f \neq 0}} \frac{|f(x) - s_n^\lambda(f)(x)|}{\|f\|_{\mathcal{H}}} \leq \sup_{\substack{f \in \mathcal{H}_\lambda \atop f \neq 0}} \frac{|f(x) - I_n^\lambda(f)(x)|}{\|f\|_{\mathcal{H}_\lambda}} \]

\[
= \sup_{\substack{f \in \mathcal{H}_\lambda \atop f \neq 0}} \frac{|f(x) - I_n^\lambda(f)(x)|}{\|f\|_{\mathcal{H}_\lambda}} = Q_n^\lambda(x).
\]

Finally, since \( Q_n^\lambda \) is the power function of the interpolation with the strictly positive definite kernel \( K_\lambda \), it holds \( Q_n^\lambda(x) = 0 \) if and only if \( x \in X_n \). In particular for all \( n \) the maximum of \( Q_n^\lambda \) is reached for \( x \notin X_n \), and for this point it holds \( P_n^\lambda(x) \leq Q_n^\lambda(x) \). Since \( P_n^\lambda(x) \leq \sqrt{\lambda} \) as just proved, we just need to show that \( Q_n^\lambda(x) \geq \sqrt{\lambda} \) for \( x \in \Omega \setminus X_n \), which follows from eq. (27).

An illustration of the relation between the three power functions is provided in Figure 1 in the case \( \Omega := [0,1] \), \( X_4 := \{0.1, 0.4, 0.7, 0.8\} \), the Gaussian kernel \( K(x,y) := \exp \left(-4\|x-y\|_2^2\right) \) and \( \lambda = 0.1 \).

![Figure 1: Power functions \( P_n \) (interpolation in \( \mathcal{H} \)), \( P_n^\lambda \) (regularized interpolation) and the discontinuous \( Q_n^\lambda \) (interpolation in \( \mathcal{H}_\lambda \)) for the Gaussian kernel in \( [0,1] \) and a given point set \( X_n \), \( n = 4 \).](image)

Using \( Q_n^\lambda \) as an upper bound for \( P_n^\lambda \) solves both issues, since \( Q_n^\lambda \) can be efficiently updated by eq. (15), and it is related to a best approximation.
operator, i.e., the orthogonal projection in $\mathcal{H}_\lambda$. To complete the analysis we need to estimate the decay rate of $Q^\lambda_n$, an we will do so by relating it to the one of $P_n$. To this end, we first state the following, which is an easy generalization of the case of interpolation.

**Proposition 7.** Let $X_n \subset \Omega$ and $\lambda > 0$ if $K$ is positive definite or $\lambda \geq 0$ if $K$ is strictly positive definite. Then there exists a Lagrange basis $\{\ell^\lambda_j\}_{j=1}^n$ of $V(X_n)$ s.t.

$$s^\lambda_n(f)(x) = \sum_{i=1}^n f(x_i)\ell^\lambda_j(x), \; x \in \Omega. \tag{20}$$

The basis is defined by

$$\ell^\lambda_j = \sum_{i=1}^n ((A + \lambda I)^{-1})_{ij}K(\cdot, x_i). \tag{21}$$

and $\ell^\lambda_j(x_i) = \delta_{ij}$ if $\lambda = 0$.

Moreover, the $\ell_2$-Lebesgue function $\Lambda^\lambda_{n,2}(x)$ can be computed for all $x \in \Omega$ as

$$\Lambda^\lambda_{n,2}(x) := \sup_{f \in \mathcal{H}_K(\Omega), f \neq 0} \frac{|s^\lambda_n(f)(x)|}{\|f\|_{\mathcal{L}_2(X_n)}} = \sqrt{\sum_{j=1}^n \ell^\lambda_j(x)^2}, \tag{22}$$

and it holds $\Lambda^\lambda_n(x) < \Lambda^\mu_n(x)$ if $\lambda > \mu \geq 0$.

**Proof.** It is clear that eq. (21) defines a basis of $V(X_n)$ since the coefficient matrix is invertible, and formula eq. (20) holds. In particular, it holds

$$|s^\lambda_n(f)(x)| \leq \sum_{j=1}^n |f(x_j)\ell^\lambda_j(x)| \leq \left(\sum_{j=1}^n f(x_j)^2\right)^{1/2} \left(\sum_{j=1}^n \ell^\lambda_j(x)^2\right)^{1/2},$$

and the equality is reached, for a fixed $x \in \Omega$, by considering $f := f_x$ with $f_x(x_j) := \ell^\lambda_j(x)$.

Defining $k_x := [K(x, x_1), \ldots, K(x, x_n)]^T$, for $x \in \Omega$ we have from eq. (21) that $\ell^\lambda_j(x) = ((A + \lambda I)^{-1}k_x)_j$, thus

$$\Lambda^\lambda_{n,2}(x)^2 = \sum_{j=1}^n \ell^\lambda_j(x)^2 = k_x^T(A + \lambda I)^{-2}k_x. \tag{23}$$

In particular for $\lambda > \mu \geq 0$ we have

$$\Lambda^\mu_n(x) - \Lambda^\lambda_n(x) = k_x^T((A + \mu I)^{-2} - (A + \lambda I)^{-2})k_x \geq 0,$$

since the matrix is positive semidefinite. Indeed, if $A = USU^T$ is an eigen-decomposition of $A$ with $\Sigma := \text{diag}(\sigma_i)$ and $\sigma_1 \geq \sigma_2 \geq \sigma_n \geq 0$, we have
\[ A + \lambda I = U(\Sigma + \lambda I)U^T \] and thus \( U((A + \mu I)^{-2} - (A + \lambda I)^{-2})U^T \) is a diagonal matrix with diagonal elements
\[
\frac{1}{(\sigma_i + \mu)^2} - \frac{1}{(\sigma_i + \lambda)^2},
\]
which are non negative if \( 0 \leq \mu \leq \lambda \).

Using \( \Lambda_{n,2}^\lambda \) we can now exactly quantify the difference between \( P_n^\lambda \) and \( Q_n^\lambda \).

**Proposition 8.** For all \( x \in \Omega \) and \( 0 \leq \mu \leq \lambda \) we have
\[
Q_n^\lambda(x)^2 \leq P_n^\mu(x)^2 + \lambda \left( 1 + \Lambda_{n,2}^\mu(x)^2 \right),
\]
and equality holds for \( \mu = \lambda \) and \( x \notin X_n \).

**Proof.** We use the formula eq. (16) for the power function \( P_n^\mu \), but we express the interpolant in terms of the Lagrange basis as in eq. (20). We use again the same notation \( k_x \) as in the previous proof and we obtain
\[
P_n^\mu(x)^2 = \| K(\cdot, x) - s_n^\mu(K(\cdot, x)) \|_H^2
\]
\[
= K(x, x) - 2 \sum_{j=1}^{n} \ell_j^\mu(x)K(x, x_j) + \sum_{i,j=1}^{n} \ell_i^\mu(x)\ell_j^\mu(x)K(x_i, x_j)
\]
\[
= K(x, x) - 2 k_x^T (A + \mu I)^{-1} k_x + k_x^T (A + \mu I)^{-1} A (A + \mu I)^{-1} k_x
\]
\[
= K(x, x) - k_x^T (A + \mu I)^{-1} (2(A + \mu I) - A)(A + \mu I)^{-1} k_x
\]
\[
= K(x, x) - k_x^T (A + \mu I)^{-1} (A + 2\mu I)(A + \mu I)^{-1} k_x.
\]
In particular, for \( \mu = 0 \) we obtain the usual formula
\[
P_n(x)^2 = P_n^0(x)^2 = K(x, x) - k_x^T A^{-1} A^{-1} k_x = K(x, x) - k_x^T A^{-1} k_x,
\]
and the same holds for the interpolatory power function \( Q_n^\lambda \), where instead we define \( k_x^\lambda := [K_\lambda(x, x_1), \ldots, K_\lambda(x, x_n)]^T \) and obtain
\[
Q_n^\lambda(x)^2 = K_\lambda(x, x) - (k_x^\lambda)^T (A + \lambda I)^{-1} k_x^\lambda.
\]
Moreover, we have \( \Lambda_{n}^\mu(x)^2 = k_x^T (A + \mu I)^{-2} k_x \) from eq. (23).

If \( x \in X_n \) it holds \( Q_n^\lambda(x) = 0 \), so eq. (24) easily follows since the right hand side is non negative.

If instead \( x \notin X_n \) it holds \( k_x^\lambda = k_x \) since \( K_\lambda(x, x_i) = K(x, x_i) \) for \( 1 \leq i \leq n \), thus eq. (25) and eq. (26) imply that
\[
Q_n^\lambda(x)^2 - P_n^\mu(x)^2 - \lambda \Lambda_{n}^\mu(x)^2 - \lambda =
\]
\[
k_x^T (A + \lambda I)^{-1} + (A + \mu I)^{-1} (A + 2\mu I)(A + \mu I)^{-1} - \lambda(A + \mu I)^{-2} k_x.
\]
We denote as $B(\mu, \lambda)$ the matrix in the right hand side. Using matrices $U, \Sigma$ as in the proof of proposition 7, we have that $UB(\mu, \lambda)U^T$ has diagonal elements

$$
\rho_i := -\frac{1}{\lambda + \sigma_i} + \frac{2\mu + \sigma_i}{(\mu + \sigma_i)^2} \frac{\lambda}{(\mu + \sigma_i)^2} = -\frac{(\lambda - \mu)^2}{(\lambda + \sigma_i)(\mu + \sigma_i)^2},
$$

which are negative for all $0 \leq \mu < \lambda$, and exactly zero for $\mu = \lambda$, i.e., $B(\mu, \lambda)$ is negative definite for $\mu < \lambda$ and the zero matrix for $\mu = \lambda$, and thus the statements follows.

In proving the convergence of the algorithm we will only need the case $K$ strictly positive definite and $\mu = 0$ in eq. (18). Nevertheless, the case $\mu = \lambda > 0$ allows to conclude that the right hand side is well defined also when $K$ is positive definite, since $P^{\lambda}_n$ is still well defined in this case, and especially in this case for $x / \notin X_n$ eq. (18) implies that

$$
Q^{\lambda}_n(x)^2 = P^{\lambda}_n(x)^2 + \lambda \left(1 + \Lambda^{\lambda}_{n,2}(x)^2\right) \geq \lambda,
$$

i.e., we can expect $\|Q^{\lambda}_n\|_{L^\infty(\Omega)}$ to converge at most to $\sqrt{\lambda}$ for $n \to \infty$, and not to 0.

### 3.1 The case of translational invariant kernels

In some notable cases, convergence rates can be derived for the regularized interpolation process using sampling inequalities. They apply to translational invariant kernels $K(x, y) := \Phi(x - y)$ which are strictly positive definite on $\mathbb{R}^d$, and such that $\Phi$ has a continuous Fourier transform $\hat{\Phi}$ on $\mathbb{R}^d$. In this case, the native space on a set $\Omega \subset \mathbb{R}^d$ satisfying an interior cone condition can be described in terms of $\hat{\Phi}$. In particular, if there exist $c_\Phi, C_\Phi > 0$ and $\tau \in \mathbb{N}$, $\tau > d/2$, such that

$$
c_\Phi \left(1 + \|\omega\|_2^2\right)^{-\tau} \leq \hat{\Phi}(\omega) \leq C_\Phi \left(1 + \|\omega\|_2^2\right)^{-\tau},
$$

shortly $\hat{\Phi}(\omega) \sim (1 + \|\omega\|_2^2)^{-\tau}$, then $K \in C^{2\tau}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathcal{H}(\mathbb{R}^d)$ is norm equivalent to the Sobolev space $W^2_2(\mathbb{R}^d)$. Examples of kernels of this type are e.g. the Wendland kernels [19]. If instead the Fourier transform decays faster than any polynomial one has a native space of infinitely smooth functions, and this is the case e.g. of the Gaussian kernel or the inverse multiquadric (IMQ) kernel.

Sampling inequalities quantify the error in terms of the fill distance

$$
h_n := h_{X_n, \Omega} := \sup_{x \in \Omega} \min_{x_j \in X_n} \|x - x_j\|_2,
$$

and they bound the error in approximating the derivative $D^a(f)$, where $a := (a_1, \ldots, a_d) \in \mathbb{N}_0^d$ is a multi index and $|a| := a_1 + \cdots + a_d$.

We state only the particular case of the bounds in the $L^\infty$-norm, and refer to the cited papers for a more general version.
Choosing indeed, the resulting convergence rates are optimal in the sense that, by Theorem 9 there is a constant of the kernel matrix. It is known from [2, 3] that in the case of item i of which can be used to estimate a lower bound on the minimal eigenvalue equalities in terms of the $H(\Omega)$.

Moreover, the constant $C$ includes a factor depending on $\lambda$ over other findings, that there is an upper bound on the maximal condition.

We use these three to deduce convergence rates of $Q_n$ (see [14]), while solving a potentially much better conditioned linear system.

i) If $\hat{\Phi}(\omega) \sim (1 + ||\omega||^2)^{-\tau}$, $\tau > d/2$, there exist constants $C, h_0$ such that for all $X_n \subset \Omega$ with $h_n \leq h_0$, $0 \leq |a| < \tau - d/2$, and $f \in \mathcal{H}$ it holds

$$
\|D^\alpha(f)\|_{L^\infty(\Omega)} \leq Ch_n^{-|a|} \left(h_n^{-d/2} ||f||_H + \|f|_{X_n}||_{\infty(X_n)}\right), \quad (28)
$$

and, under the same hypotheses and for any $\lambda > 0$, it holds

$$
\|D^\alpha \left(f - \hat{s}_n^\lambda(f)\right)\|_{L^\infty(\Omega)} \leq Ch_n^{-|a|} \left(h_n^{-d/2} + \sqrt{\lambda}\right) ||f||_H. \quad (29)
$$

ii) Assume additionally that $\Omega$ has a Lipschitz boundary. If $K$ is the Gaussian or IMQ kernel there exist constants $C', C'', h_0^\prime$ such that for all $X_n \subset \Omega$ with $h_n \leq h_0^\prime$, $\alpha \in \mathbb{N}_0^d$, and $f \in \mathcal{H}$ it holds

$$
\|D^\alpha(f)\|_{L^\infty(\Omega)} \leq e^{-C'/\sqrt{\tau_n}} ||f||_H + C''h_n^{-|a|} ||f|_{X_n}||_{\infty(X_n)}, \quad (30)
$$

and, under the same hypotheses and for any $\lambda > 0$, it holds

$$
\|D^\alpha \left(f - \hat{s}_n^\lambda(f)\right)\|_{L^\infty(\Omega)} \leq \left(2e^{-C'/\sqrt{\tau_n}} + C''h_n^{-|a|}\right) ||f||_H. \quad (31)
$$

Observe that in the case of the Gaussian the exponential term in eq. (30) and eq. (31) can be improved to $e^{C'\log(h_n)/\sqrt{\tau_n}}$, with a different constant $C'$ ( [12, Theorem 3.5]). Moreover, the constant $C$ in eq. (28) and eq. (29) includes a factor depending on $c_\Phi, C_\Phi$, which is needed to express the inequalities in terms of the $H$-norm instead of the Sobolev norm.

Both bounds are used in the corresponding papers to conclude, among other findings, that there is an upper bound on the maximal $\lambda$ to be used. Indeed, the resulting convergence rates are optimal in the sense that, by choosing $\lambda \leq h_n^{2-d}$ in the first case or $\lambda \leq (C'')^{-2} \exp\left(2C'/\sqrt{\tau_n}\right) h_n^{2|a|}$ in the second one, one gets, up to constants, the same order of pure interpolation (see [14]), while solving a potentially much better conditioned linear system. We use these bounds to deduce convergence rates of $Q_n$ and $P_n$.

To quantify the decay rate of $Q_n^\lambda$ using proposition 8 we also need to control $\Lambda_{n,2}^\alpha$. This kind of stability is usually related to the separation distance

$$q_n := q_{X_n} := \frac{1}{2} \min_{x_i \neq x_j \subseteq X_n} \|x_i - x_j\|_2,
$$

which can be used to estimate a lower bound on the minimal eigenvalue of the kernel matrix. It is known from [2, 3] that in the case of item i of theorem 9 there is a constant $c > 0$ such that

$$
\|\Lambda_{n,2}^\alpha\|_{L^\infty(\Omega)} \leq c \left(\frac{h_n}{q_n}\right)^{\tau-d/2} + 1, \quad (32)
$$

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and a bound on $\Lambda_{n,2}^0$ would be sufficient in view of proposition 8. Nevertheless, the same is not true for infinitely smooth kernels, since in this case the lower bound on the smallest eigenvalue and the upper bound on the error have a significant gap (see e.g. [6]). Namely, $q_n$ and $h_n$ appear in the right hand side with different exponents, so the upper bound in the last equation is not bounded even for $h_n \approx q_n$. Instead, we can employ the same technique of [2] to obtain a similar result in the case of regularized interpolation.

**Proposition 10.** Under the same assumptions of the two cases of theorem 9, and with the same constants $C, C', C''$, we have the following:

i) For finitely smooth kernels it holds

$$
\| \Lambda_{n,2}^\lambda \|_{L_\infty(\Omega)} \leq C \left( \frac{h_n^{d/2}}{\sqrt{\lambda}} + 2 \right).
$$

(33)

ii) For infinitely smooth kernels it holds

$$
\| \Lambda_{n,2}^\lambda \|_{L_\infty(\Omega)} \leq \frac{2e^{-C'/\sqrt{h_n}}}{\sqrt{\lambda}} + 2C''.
$$

(34)

**Proof.** The regularized interpolant $s_n^\lambda(f)$ is a minimizer of $J(s) := \sum_{j=1}^n (s(x_j) - f(x_j))^2 + \lambda \| s \|_H^2$, so in particular we have

$$
\| s_n^\lambda(f) - f \|_{\ell_2(X_n)} + \lambda \| s_n^\lambda(f) \|_H^2 = J(s_n^\lambda(f)) \leq J(0) = \| f \|_{\ell_2(X_n)}^2.
$$

It follows that $\| s_n^\lambda(f) \|_H \leq \frac{1}{\sqrt{\lambda}} \| f \|_{\ell_2(X_n)}$ and, by the triangle inequality,

$$
\| s_n^\lambda(f) \|_{\ell_2(X_n)} \leq \| s_n^\lambda(f) - f \|_{\ell_2(X_n)} + \| f \|_{\ell_2(X_n)} \leq 2 \| f \|_{\ell_2(X_n)}.
$$

We can use these two bounds in the sampling inequalities eq. (28), eq. (30) and the fact that $\| f \|_{\ell_\infty(X_n)} \leq \| f \|_{\ell_2(X_n)}$ for any function. In the first case we obtain

$$
\| s_n^\lambda(f) \|_{L_\infty(\Omega)} \leq C \left( \frac{h_n^{d/2}}{\sqrt{\lambda}} \| s_n^\lambda(f) \|_H + \| s_n^\lambda(f) \|_{\ell_\infty(X_n)} \right)
$$

$$
\leq C \left( \frac{h_n^{d/2}}{\sqrt{\lambda}} + 2 \right) \| f \|_{\ell_2(X_n)},
$$

and with the second one to obtain

$$
\| s_n^\lambda(f) \|_{L_\infty(\Omega)} \leq 2e^{-C'/\sqrt{h_n}} \| s_n^\lambda(f) \|_H + C'' \| s_n^\lambda(f) \|_{\ell_\infty(X_n)}
$$

$$
\leq \left( \frac{2e^{-C'/\sqrt{h_n}}}{\sqrt{\lambda}} + 2C'' \right) \| f \|_{\ell_2(X_n)}.
$$

These two bounds give the result using the definition of $\Lambda_{n,2}^\lambda$. □
Remark 11. We remark that these bounds do not depend on the separation distance, and the first one provides asymptotically better bounds than eq. (32).

Moreover, although not used in this paper, we remark that the same argument of the last proof, and the fact that \( \|f\|_{\ell_2(X_n)} \leq \sqrt{n}\|f\|_{\ell_\infty(X_n)} \), allow to conclude that the standard \( \ell_\infty \) Lebesgue constant of regularized interpolation satisfies

\[
\left\| \Lambda_{n,\infty}^\lambda \right\|_{L_\infty(\Omega)} \leq C \left( \sqrt{n} \frac{h_n^{\tau-d/2}}{\sqrt{\lambda}} + 2 \right)
\]

in the first case, and similarly in the second one. In particular, this means that the Lebesgue constant is asymptotically bounded for all points \( X_n \) such that \( h_n^{\tau-d/2} \leq c \, n^{-1/2} \).

Combining theorem 9 and proposition 10 we have the following.

Proposition 12. In the setting of theorem 9, we have the following cases:

i) If \( h_n \leq h_0 \) it holds

\[
\left\| P_n^\lambda \right\|_{L_\infty(\Omega)} \leq C \left( h_n^{\tau-d/2} + \sqrt{\lambda} \right)
\]

\[
\left\| Q_n^\lambda \right\|_{L_\infty(\Omega)} \leq 2Ch_n^{\tau-d/2} + (3C + 1) \sqrt{\lambda}.
\]

ii) If \( h_n \leq h'_0 \) it holds

\[
\left\| P_n^\lambda \right\|_{L_\infty(\Omega)} \leq 2e^{-C'\sqrt{\lambda}} + C'' \sqrt{\lambda}
\]

\[
\left\| Q_n^\lambda \right\|_{L_\infty(\Omega)} \leq 4e^{-C'\sqrt{\lambda}} + (3C'' + 1) \sqrt{\lambda}.
\]

Proof. The bounds on \( P_n^\lambda \) are just the application of the bounds eq. (29) and eq. (31) with \( a = 0 \) to the definition eq. (15) of \( P_n^\lambda \). Moreover, from proposition 8 with \( \mu = \lambda \) we have

\[
\left\| Q_n^\lambda(x) \right\|_{L_\infty(\Omega)} \leq \left\| P_n^\lambda \right\|_{L_\infty(\Omega)} + \sqrt{\lambda} \left( 1 + \left\| \Lambda_{n,2}^\lambda \right\|_{L_\infty(\Omega)} \right),
\]

and we can use the bounds on \( \left\| P_n^\lambda \right\|_{L_\infty(\Omega)} \) and proposition 10.

In the first case we obtain

\[
\left\| Q_n^\lambda(x) \right\|_{L_\infty(\Omega)} \leq C \left( h_n^{\tau-d/2} + \sqrt{\lambda} \right) + \sqrt{\lambda} \left( 1 + C \left( \frac{h_n^{\tau-d/2}}{\sqrt{\lambda}} + 2 \right) \right)
\]

\[
\leq 2Ch_n^{\tau-d/2} + (3C + 1) \sqrt{\lambda},
\]
while in the second one it holds
\[
\|Q_n^\lambda(x)\|_{L_\infty(\Omega)} \leq 2e^{-C'/\sqrt{n}} + C''\sqrt{\lambda} + \sqrt{\lambda} \left(1 + \frac{2e^{-C'/\sqrt{n}}}{\sqrt{\lambda}} + 2C''\right) \\
\leq 4e^{-C'/\sqrt{n}} + (3C'' + 1) \sqrt{\lambda}.
\]

4 Greedy selection rules and convergence

Using proposition 4 and proposition 6 we can define two selection rules which generalize the \(f\)- and \(P\)-greedy selections of interpolation as follows: The regularized version of \(f\)-greedy is defined by selecting the new point \(x_n\) as

\[
x_n := \arg \max_{x \in \Omega} |r_{n-1}(x)|.
\]

This selection can be performed efficiently thanks to the update rule eq. (11), and it holds \(r_n(x_k) = 0\) for \(1 \leq k \leq n\), so no point is selected more than once.

The regularized version of \(P\)-greedy, instead, selects

\[
x_n := \arg \max_{x \in \Omega} Q_{n-1}^\lambda(x),
\]

which is just the standard \(P\)-greedy selection, but applied to the kernel \(K\). In particular \(Q_n^\lambda(x) = 0\) for \(x \in X_n\), so again no point is selected more than once, and the power function can be updated efficiently using eq. (17). Moreover, thanks to Proposition 6 any upper bound on \(\|Q_n^\lambda\|_{L_\infty(\Omega)}\) provides an upper bound on \(\|P_n^\lambda\|_{L_\infty(\Omega)}\), so it makes sense to select points to minimize \(Q_n^\lambda\) in order to minimize \(P_n^\lambda\).

We remark that both selection strategies are well defined also for \(K\) positive definite if \(\lambda > 0\), and they are nothing but the standard \(f\)- and \(P\)-greedy selections applied to \(K\). In particular, the selection of the points and the construction of the regularized interpolants can be obtained just by running VKOGA with kernel \(K\), and replacing \(K\) with \(K\) after the computation to obtain the desired regularized interpolant.

4.1 Convergence rates for \(P\)-greedy selection

We can now prove rates of convergence for the new \(P\)-greedy selection rule. In particular, we prove that \(n\) points selected by this criterion and \(n\) optimally chosen points give power functions such that \(\|Q_n^\lambda\|_{L_\infty(\Omega)}\) decays with the same rate.

The result is obtained by applying the theory of [13], which holds for the power function of a strictly positive definite kernel. We refer to this
paper for the details of the proof. The idea is the following: If there exists a certain placement of \( n \) points such that the corresponding power function has a given decay rate in terms of \( n \), then \( n \) points selected by the \( P \)-greedy algorithm give, up to constants, a power function with the same decay.

To apply this result here, we need first a decay rate on \( \|Q_n^\lambda\|_{L_\infty(\Omega)} \) in terms of \( n \), and this is obtained by a standard technique. Indeed, since the bounds of both theorem 9 and proposition 12 hold for any \( X_n \) provided \( h_n \) is small enough, one can choose in particular a sequence \( \{X_n\}_{n \in \mathbb{N}} \) of quasi uniform points, i.e., such that there exists a uniformity constant \( \gamma > 1 \) such that

\[
h_n \leq \gamma q_n, \quad n \in \mathbb{N},
\]

and this can be shown to imply the existence of a constant \( C_{\Omega, \gamma} \) such that

\[
h_n \leq C_{\Omega, \gamma} n^{-1/d}, \quad n \in \mathbb{N}.
\]

Combining this observation with proposition 12 we immediately obtain the following.

**Proposition 13.** Assume the hypotheses of theorem 9 hold, and let \( \{X_n\}_{n \in \mathbb{N}} \subset \Omega \) be a sequence of quasi uniform points with uniformity constant \( \gamma > 1 \). Then the following hold.

i) For any \( n \in \mathbb{N} \) with \( C_{\Omega, \gamma} n^{-1/d} \leq h_0 \) we have

\[
\|Q_n^\lambda\|_{L_\infty(\Omega)} \leq C_0 \left( n^{-\tau/d + 1/2} + \sqrt{\lambda} \right),
\]

with \( C_0 := \max \left( 2CC_{\Omega, \gamma}^{-d/2}, 3C + 1 \right) \).

ii) For any \( n \in \mathbb{N} \) with \( C_{\Omega, \gamma} n^{-1/d} \leq h'_0 \) we have

\[
\|Q_n^\lambda\|_{L_\infty(\Omega)} \leq C'_0 \left( e^{-c'_0 n^{-1/2d}} + \sqrt{\lambda} \right),
\]

with \( C'_0 := \max (4, 3C'' + 1), \quad c'_0 := C'C_{\Omega, \gamma}^{-1/2} \).

Second, the fact that a given decay rate is carried over to the decay rate of the greedy selected points is proven in [13] by using the results of [5]. To do so, we first prove the following slight generalization of [5, Corollary 3.3] in order to deal with the present case, where the convergence is to \( \sqrt{\lambda} \), and not to zero. The proof is postponed to Appendix appendix A as it is mainly unrelated to the content of this paper.

**Proposition 14.** Let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{V} \subset \mathcal{H} \) a subset, and

\[
d_n(\mathcal{V}, \mathcal{H}_K(\Omega)) := \inf_{\mathcal{V}_n \subset \mathcal{H}_K(\Omega)} \sup_{\mathcal{V}_n \subset \mathcal{H}_K(\Omega)} \|f - \Pi_{\mathcal{V}_n}(f)\|
\]

be the Kolmogorov width of \( \mathcal{V} \) in \( \mathcal{H} \). Let \( \sigma_n := \sup_{f \in \mathcal{V}} \|f - \Pi_{\bar{V}_n}(f)\| \), where \( \bar{V}_n \) is selected by the greedy algorithm of [5]. Then
i) \( \sigma_n \leq \sqrt{2\sigma_0} \min_{1 \leq m < n} \frac{d_{m+n}}{d_m} \) for all \( n \in \mathbb{N} \).

ii) If there are constants \( C_0, \eta > 0 \) such that \( d_n \leq C_0(n^{-\alpha} + \eta) \) for all \( n \in \mathbb{N} \), then \( \sigma_n \leq C_1(n^{-\alpha} + \eta) \) for all \( n \in \mathbb{N} \), with \( C_1 := 2^{1+5\alpha}C_0 \).

iii) If there are constants \( c_0, C_0, \eta > 0 \) such that \( d_n \leq C_0(e^{-c_0n^{-\alpha}} + \eta) \) for all \( n \in \mathbb{N} \), then \( \sigma_n \leq C_1(e^{-c_1n^{-\alpha}} + \eta) \) for all \( n \in \mathbb{N} \), with \( C_1 := \sqrt{2C_0\sigma_0}, c_1 := 2^{-1-2\alpha}c_0 \).

Using proposition 13, proposition 14, and [13], we finally obtain the following.

**Theorem 15.** Assume the hypotheses of theorem \([\ref{thm:regularization}]) hold, and let \( \{X_n\}_{n \in \mathbb{N}} \subset \Omega \) be a sequence of points selected by the regularized \( P \)-greedy algorithm. Then the following hold.

i) For any \( n \in \mathbb{N} \) with \( C_1\gamma^{-n-1/d} \leq h_0 \) we have

\[
\left\| Q_n^\lambda \right\|_{L_\infty(\Omega)} \leq C_1 \left( n^{-\tau/d + 1/2} + \sqrt{\lambda} \right),
\]

with \( C_1 := 2^{1+5\alpha}C_0 \) and \( C_0 \) as in proposition 13.

ii) For any \( n \in \mathbb{N} \) with \( C_1\gamma^{-n-1/d} \leq h'_0 \) we have

\[
\left\| Q_n^\lambda \right\|_{L_\infty(\Omega)} \leq C'_0 \left( e^{-c'_0n^{-1/d}} + \sqrt{\lambda} \right),
\]

with \( C_1 := \sqrt{2C'_0\sqrt{K}(x,x)}, c_1 := 2^{-1-2\alpha}c'_0 \) and \( C'_0, c'_0 \) as in proposition 13.

5 Experiments

We conclude this paper by demonstrating the decay rates of the power function for translational invariant kernels of different smoothness. We remark that the \( f \)-greedy variant of the algorithm has been recently used to construct a data-based surrogate from simulation data in [8]. We point to this paper for a practical application scenario.

In the following, we use the Wendland kernel \( W_{4,d} \) (see [19]) and the Gaussian kernel \( G(x,y) := \exp(\varepsilon^2\|x-y\|^2) \), which are respectively members of the two classes of kernels considered in theorem 15. In both cases, the shape parameter is fixed to \( \varepsilon = 1 \). The set \( \Omega \) is the unit ball in \( \mathbb{R}^2 \), which is represented by a discretization \( \Omega_h \) obtained by restricting a regular grid in \([-1,1]^2\) to \( \Omega \), so that the number of points is \( N \approx 20000 \). Both the greedy selection and the computation of the \( L_\infty(\Omega) \) norms are performed on this set.
\[ \lambda = 10^{-14} \quad \lambda = 10^{-12} \quad \lambda = 10^{-10} \quad \lambda = 10^{-8} \quad \lambda = 10^{-6} \]

| Kernel    | \( \lambda = 10^{-14} \) | \( \lambda = 10^{-12} \) | \( \lambda = 10^{-10} \) | \( \lambda = 10^{-8} \) | \( \lambda = 10^{-6} \) |
|-----------|----------------|----------------|----------------|----------------|----------------|
| Wendland  | 1.00           | 1.00           | 1.02           | 1.12           | 1.07           |
| Gaussian  | 1.36           | 1.48           | 1.46           | 1.54           | 1.44           |

Table 1: Coefficients \( c_\lambda \) relating the power functions \( P_n \) and \( Q^\lambda_n \) as described in section 5.

Figure 2: Decay of the supremum norm of the regularized power functions \( Q^\lambda_n \) (solid lines) obtained with the \( P \)-greedy algorithm for different values of the regularization parameters, for the Wendland (left) and Gaussian kernel (right). The dotted lines show the curves \( c_\lambda \left( \| P_n \|_{L_\infty(\Omega_h)} + \sqrt{\lambda} \right) \), where \( c_\lambda \) is computed as in section 5.

For both kernels, we compare the decay of the standard power function (i.e., \( \lambda = 0 \)) with \( Q^\lambda_n \) for \( \lambda = 10^{-14}, 10^{-12}, \ldots, 10^{-6} \).

The \( P \)-greedy algorithm is stopped when the maximum of the power function is below the tolerance of \( 10^{-16} \) (which happens only for \( \lambda = 0 \)) or when \( n = 1000 \) points are selected.

The results are in fig. 2 for \( W_{4,d} \) (left) and \( G \) (right). For each value of \( \lambda \), we compute the minimal coefficient \( c \) such that

\[ \left\| Q^\lambda_n \right\|_{L_\infty(\Omega_h)} \leq c_\lambda \left( \| P_n \|_{L_\infty(\Omega_h)} + \sqrt{\lambda} \right). \]

The plots show both the decay of the power functions (in solid lines) and the curves \( c_\lambda \left( \| P_n \|_{L_\infty(\Omega_h)} + \sqrt{\lambda} \right) \). Observe that, in the case \( \lambda = 0 \), the algorithm stops much earlier, say at \( n < 1000 \), so the dotted curves are limited to the first \( n \) iterations. The computed coefficients are in table 1.

Both kernels confirm the expected decay rate of theorem 15, and indeed the numerically computed constant, at least for this very particular setting, seem to be very small.
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A Proof of proposition 14

Proposition 16. Let $H$ be a Hilbert space, $\mathcal{V} \subset H$ a subset, and

$$d_n(\mathcal{V}, H) := \inf_{V_n \subset \mathcal{V}, \dim(V_n) = n} \sup_{f \in V} \| f - \Pi_{V_n}(f) \|$$

be the Kolmogorov width of $\mathcal{V}$ in $H$. Let $\sigma_n := \sup_{f \in \mathcal{V}} \| f - \Pi_{\bar{V}_n}(f) \|$, where $\bar{V}_n$ is selected by the greedy algorithm of [5]. Then

i) $\sigma_n \leq \sqrt{2\sigma_0} \min_{1 \leq m < n} \frac{n-m}{m} d_m^{n-m}$ for all $n \in \mathbb{N}$.

ii) If there are constants $C_0, \eta > 0$ such that $d_n \leq C_0(n^{-\alpha} + \eta)$ for all $n \in \mathbb{N}$, then $\sigma_n \leq C_1(n^{-\alpha} + \eta)$ for all $n \in \mathbb{N}$, with $C_1 := 2^{1+5\alpha} C_0$.

iii) If there are constants $c_0, C_0, \eta > 0$ such that $d_n \leq C_0(e^{-c_0n^{-\alpha}} + \eta)$ for all $n \in \mathbb{N}$, then $\sigma_n \leq C_1(e^{-c_1n^{-\alpha}} + \eta)$ for all $n \in \mathbb{N}$, with $C_1 := \sqrt{2C_0\sigma_0}$, $c_1 := 2^{-1-2\alpha} c_0$.

Proof. We use [5] Theorem 3.2, which states that for all $N \geq 0, K \geq 1$, $1 \leq m < K$ it holds

$$\prod_{i=1}^{K} \sigma_{N+1}^{2} \leq \left( \frac{K}{m} \right)^{m} \left( \frac{K}{K-m} \right)^{K-m} \sigma_{N+1}^{2} d_{m}^{2K-2m}. \quad (39)$$

We see the three points separately.

item i The proof is exactly as in [5] Corollary 3.3], except that $\sigma_0 < 1$ does not hold in general, so it is not simplified in the upper bound.

item iii Again as in [5] Corollary 3.3, but using item i.

item ii This property is the only one that requires a slight modification in the estimation of the constant $C_1$, even if the other steps of the proof are not modified. Since $\sigma_n$ is non increasing we have $\sigma_{2n}^{2n} \leq \prod_{j=1}^{2n} \sigma_{n+1}^{2}$, and using eq. (39) with $N := K := n$ and $1 \leq m < n$ we obtain

$$\sigma_{2n}^{2n} \leq \prod_{j=n+1}^{2n} \sigma_{j}^{2} = \prod_{i=1}^{n} \sigma_{n+i}^{2} \leq \left( \frac{n}{m} \right)^{m} \left( \frac{n}{n-m} \right)^{n-m} \sigma_{n+1}^{2m} d_{m}^{2n-2m} \leq \left( \frac{n}{m} \right)^{m} \left( \frac{n}{n-m} \right)^{n-m} \sigma_{n}^{2m} d_{m}^{2n-2m}.$$
For $n := 2s$, $m := s$ we get $\sigma_{4s}^4 \leq 2^{2s} \sigma_{2s}^2 d_s^2$, i.e.,

$$\sigma_{4s} \leq \sqrt{2} \sqrt{\sigma_{2s} d_s}. \quad (40)$$

Assume it is false that $\sigma_n \leq C_1(n^{-\alpha} + \eta)$, and assume $M$ is the first index such that $\sigma_M > C_1(M^{-\alpha} + \eta)$.

We first assume $M := 4s$, $s \geq 1$. Since the claim is true for all $n \leq M$, using eq. (40) we obtain

$$\sigma_{4s} \leq \sqrt{2} \sqrt{\sigma_{2s} d_s} \leq \sqrt{2} \sqrt{C_1((2s)^{-\alpha} + \eta) C_0(s^{-\alpha} + \eta)}. \quad (41)$$

Since we have $\sigma_M > C_1(M^{-\alpha} + \eta)$ for $M := 4s$, it follows that

$$C_1((4s)^{-\alpha} + \eta) < \sqrt{2} \sqrt{C_1((2s)^{-\alpha} + \eta) C_0(s^{-\alpha} + \eta)},$$

and dividing by $\sqrt{C_1}$ and squaring the result gives

$$C_1 < \frac{2C_0((2s)^{-\alpha} + \eta) (s^{-\alpha} + \eta)}{(4s)^{-\alpha} + \eta} = \frac{2C_0(2^{-\alpha} + \eta s^\alpha) (1 + \eta s^\alpha)}{(4^{-\alpha} + \eta s^\alpha)^2}.$$

Denoting as $f(s) := f_{\lambda, C_0, \alpha}(s)$ the right hand side of the last inequality, we have that $C_1 < \min_{s \geq 1} f(s)$. Since

$$f'(s) = -\frac{2^{3\alpha+1} (2\alpha - 1) \alpha C_0 \sqrt{\lambda s^{\alpha-1}} \left(2^{\alpha+1} + 2^\alpha (2^\alpha + 2) \sqrt{\lambda s^{\alpha}} + 1\right)}{\left(4^\alpha \sqrt{\lambda s^{\alpha^\alpha}} + 1\right)^3},$$

which is negative for $s \geq 0$ since $\alpha > 0$, we can guarantee that

$$C_1 < f(0) = 2C_0 \cdot 2^{-\alpha} 4^{2\alpha} = 2^{1+3\alpha} C_0.$$

It follows that

$$C_1 < 2^{1+3\alpha} C_0 < 2^{1+5\alpha} C_0,$$

which is a contradiction to our choice for $C_1$.

All the other possible cases can be covered by assuming $M := 4s + q$ with $q \in \{1, 2, 3\}$, $s \geq 0$. Using again eq. (40) and the monotonicity of $\sigma_n$ we get

$$\sigma_{4s+q} \leq \sigma_{4s} \leq \sqrt{2} \sqrt{C_1((2s)^{-\alpha} + \eta) C_0(s^{-\alpha} + \eta)}.$$

On the other hand, since we assumed that the bound is not valid for $n = M := 4s + q$ we have (if $s \geq 1$)

$$\sigma_{4s+q} > C_1((4s + q)^{-\alpha} + \eta) > C_1(2^{-\alpha}(4s)^{-\alpha} + \eta),$$

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i.e.,

\[ C_1(2^{-\alpha}(4s)^{-\alpha} + \eta) < \sqrt{2} \sqrt{C_1((2s)^{-\alpha} + \eta)} C_0(s^{-\alpha} + \eta), \]

i.e.,

\[ C_1 < \frac{2C_0((2s)^{-\alpha} + \eta)(s^{-\alpha} + \eta)}{(2^{-\alpha}(4s)^{-\alpha} + \eta)^2} = \frac{2C_0(2^{-\alpha} + \eta s^\alpha)(1 + \eta s^\alpha)}{(2^{-3\alpha} + \eta s^\alpha)^2}. \]

In this case the derivative of the right hand side \( f(s) \) is

\[ f'(s) = -\frac{2^{5\alpha+1}\alpha (2^{\alpha} - 1) \eta s^{\alpha-1} (2^{\alpha+1}(1 + 2^\alpha) + \eta 2^\alpha(2^{2\alpha} + 2 + 2^{\alpha+1})s^\alpha + 1)}{(8^{\alpha}\eta s^\alpha + 1)^3} C_0 \]

which is again negative, so again we have that

\[ C_1 < f(0) = \frac{2^{2^{-\alpha}}C_0}{(2^{-3\alpha})^2} = 2^{1+5\alpha}C_0, \]  

which is a contradiction.

\[ \square \]

References

[1] N. Aronszajn. Theory of reproducing kernels. Transactions of the American Mathematical Society, 68:337–404, 1950.

[2] S. De Marchi and R. Schaback. Stability constants for kernel-based interpolation processes. Technical report, Dipartimento di Informatica, Università degli Studi di Verona, 2008.

[3] S. De Marchi and R. Schaback. Stability of kernel-based interpolation. Adv. Comput. Math., 32(2):155–161, 2010.

[4] S. De Marchi, R. Schaback, and H. Wendland. Near-optimal data-independent point locations for radial basis function interpolation. Adv. Comput. Math., 23(3):317–330, 2005.

[5] R. DeVore, G. Petrova, and P. Wojtaszczyk. Greedy algorithms for reduced bases in Banach spaces. Constr. Approx., 37(3):455–466, 2013.

[6] B. Diederichs and A. Iske. Improved estimates for condition numbers of radial basis function interpolation matrices. Journal of Approximation Theory, 2017.

[7] M. Köppel, F. Franzelin, I. Kröker, S. Oladyshkin, G. Santin, D. Wittwar, A. Barth, B. Haasdonk, W. Nowak, D. Pflüger, and C. Rohde. Comparison of data-driven uncertainty quantification methods for a carbon dioxide storage benchmark scenario. Technical report, 2018.
[8] T. Köppl, G. Santin, B. Haasdonk, and R. Helmig. Numerical modelling of a peripheral arterial stenosis using dimensionally reduced models and kernel methods. *International Journal for Numerical Methods in Biomedical Engineering*, 0(ja):e3095. e3095 cnm.3095.

[9] C. A. Micchelli and M. Pontil. On learning vector-valued functions. *Neural Comput.*, 17(1):177–204, 2005.

[10] S. Müller and R. Schaback. A Newton basis for kernel spaces. *J. Approx. Theory*, 161(2):645–655, 2009.

[11] M. Pazouki and R. Schaback. Bases for kernel-based spaces. *J. Comput. Appl. Math.*, 236(4):575 – 588, 2011.

[12] C. Rieger and B. Zwicknagl. Sampling inequalities for infinitely smooth functions, with applications to interpolation and machine learning. *Adv. Comput. Math.*, 32(1):103, 2008.

[13] G. Santin and B. Haasdonk. Convergence rate of the data-independent P-greedy algorithm in kernel-based approximation. *Dolomites Res. Notes Approx.*, 10:68–78, 2017.

[14] R. Schaback. Error estimates and condition numbers for radial basis function interpolation. *Adv. Comput. Math.*, 3(3):251–264, 1995.

[15] R. Schaback and H. Wendland. Adaptive greedy techniques for approximate solution of large RBF systems. *Numer. Algorithms*, 24(3):239–254, 2000.

[16] B. Schölkopf, R. Herbrich, and A. J. Smola. *A Generalized Representer Theorem*, pages 416–426. Springer Berlin Heidelberg, Berlin, Heidelberg, 2001.

[17] I. Steinwart and A. Christmann. *Support Vector Machines*. Information Science and Statistics. Springer, New York, 2008.

[18] G. Wahba. Support vector machines, reproducing kernel Hilbert spaces and the randomized GACV. In B. Schölkopf, C. Burges, and A. Smola, editors, *Advances in Kernel Methods, Support Vector Learning*, pages 69–88. MIT Press, 1999.

[19] H. Wendland. Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. *Adv. Comput. Math.*, 4(1):389–396, 1995.

[20] H. Wendland. *Scattered Data Approximation*, volume 17 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2005.

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[21] H. Wendland and C. Rieger. Approximate interpolation with applications to selecting smoothing parameters. *Numerische Mathematik*, 101(4):729–748, 2005.

[22] D. Wirtz and B. Haasdonk. A vectorial kernel orthogonal greedy algorithm. *Dolomites Res. Notes Approx.*, 6:83–100, 2013.

[23] D. Wirtz, N. Karajan, and B. Haasdonk. Surrogate modelling of multiscale models using kernel methods. *International Journal of Numerical Methods in Engineering*, 101(1):1–28, 2015.

[24] D. Wittwar, G. Santin, and B. Haasdonk. Interpolation with uncoupled separable matrix-valued kernels. Technical report, University of Stuttgart, 2017. In preparation.