On Laplacian energy of non-commuting graphs of finite groups

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Abstract: In this paper, we compute Laplacian energy of the non-commuting graphs of some classes of finite non-abelian groups.

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1 Introduction

Let $\mathcal{G}$ be a graph. Let $A(\mathcal{G})$ and $D(\mathcal{G})$ denote the adjacency matrix and degree matrix of the graph respectively. Then the Laplacian matrix of $\mathcal{G}$ is given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$. Let $\beta_1, \beta_2, \ldots, \beta_m$ be the eigenvalues of $L(\mathcal{G})$ with multiplicities $b_1, b_2, \ldots, b_m$. Then the Laplacian spectrum of $\mathcal{G}$, denoted by $L-$Spec($\mathcal{G}$), is the set $\{\beta_1^{b_1}, \beta_2^{b_2}, \ldots, \beta_m^{b_m}\}$. The Laplacian energy of $\mathcal{G}$, denoted by $LE(\mathcal{G})$, is given by

$$LE(\mathcal{G}) = \sum_{\mu \in L-$Spec($\mathcal{G}$)} \left| \mu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|} \right|$$ (1.1)

where $v(\mathcal{G})$ and $e(\mathcal{G})$ are the sets of vertices and edges of the graph $\mathcal{G}$ respectively. A graph $\mathcal{G}$ is called L-integral if $L-$Spec($\mathcal{G}$) contains only integers. Various properties of L-integral graphs and $LE(\mathcal{G})$ are studied in [2, 15, 17].

Let $G$ be a finite non-abelian group with center $Z(G)$. The non-commuting graph of $G$, denoted by $A_G$ is a simple undirected graph such that $v(A_G) = G \setminus Z(G)$ and two vertices $x$ and $y$ are adjacent if and only if $xy \neq yx$. Various aspects of non-commuting graphs of different families of finite non-abelian groups are studied in [1, 3, 9, 13, 23]. Note that the complement of $A_G$ is the commuting graph of $G$ denoted by $\overline{A}_G$. Commuting graphs are studied extensively in [4, 14, 18, 21]. In [11], the authors have computed the Laplacian spectrum of the non-commuting graphs of several well-known families finite non-abelian groups. In this paper we study the Laplacian energy of those classes of finite groups.

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2 Some Computations

In this section, we compute Laplacian energy of some families of groups whose central factors are some well-known groups.

Theorem 2.1. Let $G$ be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^5 = b^3 = 1, b^{-1}ab = a^2 \rangle$. Then

$$LE(A_G) = \left( \frac{120}{19} |Z(G)| + 30 \right) |Z(G)|.$$  

Proof. It is clear that $|e(A_G)| = 19|Z(G)|$. Since $\frac{G}{Z(G)} = Sz(2)$, we have

$$\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^5Z(G) = b^3Z(G) = Z(G), b^{-1}abZ(G) = a^2Z(G) \rangle.$$  

Then

\[
C_G(a) = Z(G) \sqcup aZ(G) \sqcup a^2Z(G) \sqcup a^3Z(G) \sqcup a^4Z(G), \\
C_G(ab) = Z(G) \sqcup abZ(G) \sqcup a'b^2Z(G) \sqcup a'b^3Z(G), \\
C_G(a^2b) = Z(G) \sqcup a^2bZ(G) \sqcup a^3b^2Z(G) \sqcup ab^3Z(G), \\
C_G(a^2b^3) = Z(G) \sqcup a^2b^3Z(G) \sqcup a^3b^5Z(G) \sqcup ab^5Z(G), \\
C_G(b) = Z(G) \sqcup bZ(G) \sqcup b^2Z(G) \sqcup b^3Z(G) \\
\text{and} \\
C_G(a^2b) = Z(G) \sqcup a^2bZ(G) \sqcup a^3b^2Z(G) \sqcup a^4b^3Z(G)
\]

are the only centralizers of non-central elements of $G$. Since all these distinct centralizers are abelian, we have

$$\overline{A_G} = K_4|Z(G)| \sqcup 5K_3|Z(G)|$$

and hence $|e(A_G)| = 150|Z(G)|^2$. By Theorem 3.1 of [11], we have

$$L-\text{Spec}(A_G) = \{0, (15|Z(G)|)^4|Z(G)|^{-1}, (16|Z(G)|)^{15|Z(G)|^{-5}}, (19|Z(G)|)^5 \}.$$  

Therefore, 

\[
\left| 0 - \frac{2e(A_G)}{|A_G|} \right| = \left| \frac{300}{19} |Z(G)| \right|, \quad \left| 15|Z(G)| - \frac{2e(A_G)}{|A_G|} \right| = \frac{15}{19} |Z(G)|, \\
\left| 16|Z(G)| - \frac{2e(A_G)}{|A_G|} \right| = \frac{4}{19} |Z(G)|, \quad \left| 19|Z(G)| - \frac{2e(A_G)}{|A_G|} \right| = \frac{4}{19} |Z(G)|. \quad \text{By (1.1), we have}
\]

$$LE(A_G) = \frac{300}{19} |Z(G)| + (4|Z(G)| - 1) \left( \frac{15}{19} |Z(G)| \right) + (15|Z(G)| - 5) \left( \frac{4}{19} |Z(G)| \right)$$

$$\quad + 5 \left( \frac{61}{19} |Z(G)| \right).$$

Hence the result follows. \qed

Theorem 2.2. Let $G$ be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime integer. Then

$$LE(A_G) = 2p(p - 1)|Z(G)|.$$
Proof. It is clear that $|v(A_G)| = (p^2 - 1)|Z(G)|$. Since $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, we have $\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^p, aba^{-1}b^{-1} \in Z(G) \rangle$, where $a, b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$

$$C_G(a) = Z(G) \cup aZ(G) \sqcup \cdots \sqcup a^{p-1}Z(G) \text{ for } 1 \leq i \leq p-1 \text{ and}$$

$$C_G(a^ib) = Z(G) \cup a^ibZ(G) \sqcup \cdots \sqcup a^ib^{p-1}Z(G) \text{ for } 1 \leq j \leq p$$

are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Therefore

$$\mathfrak{A}_G = K_{C_G(a) \setminus Z(G)} \sqcup \bigcup_{j=1}^{p} K_{C_G(a^j) \setminus Z(G)}.$$ 

Since, $|C_G(a)|$ and $|C_G(a^j)|$ for $1 \leq j \leq p$, we have

$$\mathfrak{A}_G = (p + 1)K_{(p-1)|Z(G)|},$$

and hence $|v(A_G)| = \frac{p^2(p-1)(p-1)^2}{2} |Z(G)|^2$. By Theorem 3.2 of [1], we have

$$L\text{-Spec}(A_G) = \{0, ((p^2 - p)|Z(G)|)^{(p^2 - 1)|Z(G)|} - p - 1, ((p^2 - 1)|Z(G)|)^p \}. $$

Therefore, $0 < \frac{2|v(A_G)|}{|Z(G)|} = p(p-1)|Z(G)|, \ (p^2 - p)|Z(G)| - \frac{2|v(A_G)|}{|Z(G)|} = 0$ and $\left( (p^2 - 1)|Z(G)| - \frac{2|v(A_G)|}{|Z(G)|}\right) = (p-1)|Z(G)|$. By [1], we have

$$LE(A_G) = p(p-1)|Z(G)| + ((p^2 - 1)|Z(G)| - p - 1)0 + p((p-1)|Z(G)|).$$

Hence the result follows.

\[\square\]

**Corollary 2.3.** Let $G$ be a non-abelian group of order $p^3$, for any prime $p$, then

$$LE(A_G) = 2p^2(p-1).$$

**Proof.** Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.2 \[\square\]

**Theorem 2.4.** Let $G$ be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then

$$LE(A_G) = \frac{(2m^2 - 3m)(m - 1)|Z(G)|^2 + m(4m - 3)|Z(G)|}{2m - 1}.$$ 

**Proof.** Clearly, $|v(A_G)| = (2m - 1)|Z(G)|$. Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G) \rangle$, where $x, y \in G$ with $xy \neq yx$. It is easy to see that for any $z \in Z(G)$

$$C_G(xy^j) = C_G(xy^jz) = Z(G) \cup xy^jZ(G), 1 \leq j \leq m \text{ and}$$

$$C_G(y^j) = C_G(y^jz) = Z(G) \cup yZ(G) \sqcup \cdots \sqcup y^{m-1}Z(G), 1 \leq i \leq m - 1$$

are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$ and $|C_G(x^jy)| = 2|Z(G)| \text{ for } 1 \leq j \leq m \text{ and } |C_G(y^j|z(G)| \text{ for } 1 \leq j \leq m \text{ and } |C_G(y^j) = m|Z(G)|$. Hence

$$\mathfrak{A}_G = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}.$$
and \(|e(A_C)| = \frac{3m(m-1)|Z(G)|^2}{2}\). By Theorem 3.4 of [11], we have

\[
\text{L-Spec}(A_C) = \{0, (m|Z(G)|)^{(m-1)|Z(G)|-1}, (2(m - 1)|Z(G)|)^{m|Z(G)|-m}, ((2m - 1)|Z(G)|)^m\}.
\]

Therefore, \(|0 - \frac{2|e(A_C)|}{|Z(G)||e(A_C)|} = \frac{3m(m-1)|Z(G)|}{2(m-1)}|Z(G)| - \frac{2|e(A_C)|}{|Z(G)||e(A_C)|} = \frac{m(m-1)|Z(G)|}{2(m-1)}|Z(G)|\)

\[
\left|2(m - 1)|Z(G)| - \frac{2|e(A_C)|}{|Z(G)||e(A_C)|}\right| = \frac{(m-1)(m-2)|Z(G)|}{2m-1} \quad \text{and} \quad \left|2(m - 1)|Z(G)| - \frac{2|e(A_C)|}{|Z(G)||e(A_C)|}\right| = \frac{(m^2 - m + 1)|Z(G)|}{2m-1}.
\]

By (11), we have

\[
\text{LE}(A_C) = \frac{3m(m-1)|Z(G)|}{2m-1} + \left((m-1)|Z(G)| - 1\right) \left(\frac{m(m-1)|Z(G)|}{2m-1}\right)
\]

\[+ (m|Z(G)| - m) \left(\frac{(m-1)(m-2)|Z(G)|}{2m-1}\right) + m \left(\frac{(m^2 - m + 1)|Z(G)|}{2m-1}\right)
\]

and hence the result follows.

Using Theorem 2.4, we now compute the Laplacian energy of the non-commuting graphs of the groups \(M_{2mn}, D_{2m}\) and \(Q_{4m}\) respectively.

**Corollary 2.5.** Let \(M_{2mn} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle\) be a metacyclic group, where \(m > 2\).

\[
\text{LE}(A_{M_{2mn}}) = \begin{cases} 
\frac{m(2m-3)(m-1)n^2 + m(4m-3)n}{m-1}, & \text{if } m \text{ is odd} \\
\frac{m(m-2)(m-3)n^2 + m(2m-3)n}{m-1}, & \text{if } m \text{ is even}.
\end{cases}
\]

**Proof.** Observe that \(Z(M_{2mn}) = \langle b^2 \rangle \) or \(\langle b^2 \rangle \cup a \frac{2n}{2} \langle b^2 \rangle\) according as \(m\) is odd or even. Also, it is easy to see that \(\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}\) or \(D_m\) according as \(m\) is odd or even.

Hence, the result follows from Theorem 2.4.

As a corollary to the above result we have the following results.

**Corollary 2.6.** Let \(D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle\) be the dihedral group of order \(2m\), where \(m > 2\).

\[
\text{LE}(A_{D_{2mn}}) = \begin{cases} 
m^2, & \text{if } m \text{ is odd} \\
m(m^2 - 3m + 3), & \text{if } m \text{ is even}.
\end{cases}
\]

**Corollary 2.7.** Let \(Q_{4m} = \langle x, y : y^{2m} = 1, x^2 = y^m, yxy^{-1} = y^{-1} \rangle\), where \(m \geq 2\), be the generalized quaternion group of order \(4m\). Then

\[
\text{LE}(A_{Q_{4m}}) = \frac{2m(4m^2 - 6m + 3)}{2m - 1}.
\]

**Proof.** The result follows from Theorem 2.4 noting that \(Z(Q_{4m}) = \{1, a^m\}\) and \(\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}\).
3 Some well-known groups

Now we compute Laplacian energy of the non-commuting graphs of some well-known families of finite non-abelian groups.

**Proposition 3.1.** Let $G$ be a non-abelian group of order $pq$, where $p$ and $q$ are primes with $p \mid (q - 1)$. Then

$$LE(A_G) = \frac{2q(p^2 - 1)(q - 1)}{pq - 1}.$$

**Proof.** It is clear that $|v(A_G)| = pq - 1$. Note that $|Z(G)| = 1$ and the centralizers of non-central elements of $G$ are precisely the Sylow subgroups of $G$. The number of Sylow $q$-subgroups and Sylow $p$-subgroups of $G$ are one and $q$ respectively. Therefore, we have

$$\mathfrak{A}_G = K_{q-1} \sqcup qK_{p-1}$$

and hence $|e(A_G)| = \alpha_2(p^2 - 1)(q - 1)$. By Proposition 4.1 of [11], we have

$$L\text{-Spec}(A_G) = \{0, (pq - q)^{q-2}, (pq - p)^{pq-2q}, (pq - 1)^q\}.$$

Therefore, $\frac{|pq - p| - 2\alpha_2(A_G)|}{|v(A_G)|} = \frac{q^2 - p^2 - q^2 + q}{pq - 1}$ and $|pq - 1 - 2\alpha_2(A_G)| = \frac{pq - q}{pq - 1}$. By (1.1), we have

$$LE(A_G) = \frac{p^2q^2 - p^2q - q^2 + q}{pq - 1} + (pq - 2q) \left( \frac{q(p - p - 1)}{pq - 1} \right) + q \left( \frac{p^2q + q^2 - 2pq - q + 1}{pq - 1} \right)$$

and hence the result follows. \hfill \Box

**Proposition 3.2.** Let $QD_{2^n}$ denotes the quasidihedral group $(a, b : a^{2^n-1} = b^2 = 1, bab^{-1} = a^{2^n-2} - 1)$, where $n \geq 4$. Then

$$LE(A_{QD_{2^n}}) = \frac{2^{3n-3} - 2^{2n} + 3.2^n}{2^{n-1} - 1}.$$

**Proof.** It is clear that $Z(QD_{2^n}) = \{1, a^{2^n-2}\}$, so $|v(A_{QD_{2^n}})| = 2(2^n - 1)$. Note that

$$C_{QD_{2,n}}(a) = C_{QD_{2,n}}(a^i) = \langle a \rangle \text{ for } 1 \leq i \leq 2^{n-1} - 1, i \neq 2^{n-2} \text{ and }$$

$$C_{QD_{2,n}}(a^j b) = \{1, a^{2^n-2}, a^j b, a^{j+2^n-2} b\} \text{ for } 1 \leq j \leq 2^{n-2}$$

are the only centralizers of non-central elements of $QD_{2^n}$. Note that these centralizers are abelian subgroups of $QD_{2^n}$. Therefore, we have

$$\mathfrak{A}_{QD_{2^n}} = K_{|C_{QD_{2,n}}(a)\setminus Z(QD_{2^n})|} \sqcup \bigcup_{j=1}^{2^{n-2}} K_{|C_{QD_{2,n}}(a^j b)\setminus Z(QD_{2^n})|}.$$ 

Since $|C_{QD_{2,n}}(a)| = 2^{n-1}, |C_{QD_{2,n}}(a^j b)| = 4$ for $1 \leq j \leq 2^{n-2},$ we have $\mathfrak{A}_{QD_{2,n}} = K_{2^{n-1}-2} \sqcup 2^{n-2} K_2$. Hence

$$|e(A_{QD_{2^n}})| = \frac{3.2^{2n-2} - 6.2^n}{2}.$$
By Proposition 4.2 of [11], we have

\[ \text{L-Spec}(A_{QD_2^n}) = \{0, (2^{n-1})^2, (2^n - 4)^{3^n-2}, (2^n - 2)^{3^n-2}\}. \]

Therefore,

\[ |0 - \frac{2^k |(A_{QD_2^n})|}{|v(A_{QD_2^n})|} | = \frac{2^n - 4(2^{n-1} - 2)}{2^{2n-1} - 2}, \quad |2^n - 4 - \frac{2^k |(A_{QD_2^n})|}{|v(A_{QD_2^n})|} | = \frac{2^n - 4(2^{n-1} - 2)}{2^{2n-1} - 2}, \]

\[ |2^n - 2 - \frac{2^k |(A_{QD_2^n})|}{|v(A_{QD_2^n})|} | = \frac{2^n - 4(2^{n-1} - 2)}{2^{2n-1} - 2}. \]

By [11], we have

\[ LE(A_{QD_2^n}) = \frac{3 \cdot 2^{n-1} (2^n - 2) + (2^n - 3)\left(\frac{2^{2n-2} - 4}{2^{n-1} - 2}\right)}{2^{2n-1} - 2} + 2^n \left(\frac{2^{2n-2} - 6}{2^{n-1} - 2}\right) + 2^n \left(\frac{2^{2n-2} - 2}{2^{n-1} - 2}\right) + 2^n \left(\frac{2^{2n-2} - 2}{2^{n-1} - 2}\right) \]

and hence the result follows.

**Proposition 3.3.** Let \( G \) denotes the projective special linear group \( PSL(2, 2^k) \), where \( k \geq 2 \). Then

\[ LE(A_G) = \frac{3 \cdot 2^{3k} - 2 \cdot 2^{5k} - 7 \cdot 2^{4k} + 3 \cdot 2^{3k} + 4 \cdot 2^{2k} + 2^k}{2^{3k} - 2^k - 1}. \]

**Proof.** Clearly, \( |v(A_G)| = 2^{3k} - 2^k - 1 \). Since \( G \) is a non-abelian group of order \( 2^k (2^{2k} - 1) \) and its center is trivial. By Proposition 3.21 of [11], the set of centralizers of non-trivial elements of \( G \) is given by

\[ \{xPx^{-1}, xAx^{-1}, xBx^{-1} : x \in G\} \]

where \( P \) is an elementary abelian \( 2 \)-subgroup of \( A \) and \( B \) are cyclic subgroups of \( G \) having order \( 2^k \) and \( 2^k - 1 \) and \( 2^k + 1 \) respectively. Also the number of conjugates of \( P, A \) and \( B \) in \( G \) are \( 2^k + 1, 2^{k-1}(2^k + 1) \) and \( 2^{k-1}(2^k - 1) \) respectively. Hence \( \overline{A_G} \) of \( G \) is given by

\[ (2^k + 1)K_{xPx^{-1}} \cup 2^{k-1}(2^k + 1)K_{xPx^{-1}} \cup 2^{k-1}(2^k - 1)K_{xBx^{-1}}. \]

That is, \( \overline{A_G} = (2^k + 1)K_{xPx^{-1}} \cup 2^{k-1}(2^k + 1)K_{xPx^{-1}} \cup 2^{k-1}(2^k - 1)K_{xBx^{-1}}. \) Therefore,

\[ e(A_G) = \frac{2^{6k} - 3 \cdot 2^{4k} - 2^{3k} + 2 \cdot 2^{2k} + 2^k}{2}. \]

By Proposition 4.3 of [11], we have

\[ \text{L-Spec}(A_G) = \{0, (2^{3k} - 2^{k+1} - 1)^{2^{3k-1} - 2^{2k+1} - 2^k}, (2^{3k} - 2^{k+1} + 1)^{2^{3k-1} - 2^{2k+1} - 2^k}, \}

\[ (2^{3k} - 2^{k+1} + 1)^{2^{3k-1} - 2^{2k+1} - 2^k} \}

\[ (2^{3k} - 2^{k+1} - 1)^{2^{3k-1} - 2^{2k+1} - 2^k}, (2^{3k} - 2^{k+1} + 1)^{2^{3k-1} - 2^{2k+1} - 2^k}. \]

Now,

\[ |0 - \frac{2^k |(A_G)|}{|v(A_G)|} | = \frac{2^{6k} - 3 \cdot 2^{4k} - 2^{3k} + 2 \cdot 2^{2k} + 2^k}{2^{3k} - 2^{k+1}}, \quad |2^{3k} - 2^{k+1} - 1 - \frac{2^k |(A_G)|}{|v(A_G)|} | = \frac{2^{6k} - 2^{4k}}{2^{3k} - 2^{k+1}}, \]

\[ |2^{3k} - 2^{k+1} + 1 - \frac{2^k |(A_G)|}{|v(A_G)|} | = \frac{2^{6k} + 2^{4k}}{2^{3k} - 2^{k+1}}. \]
and $|^{2^{3k}} - 2^{k} - 1 - \frac{2\varphi(A_G)}{|v(A_G)|} = \frac{2^{4k} - 2^{3k} - 2^{2k} + 2^{k+1}}{2^{3k} - 2^{k} - 1}$. By (11), we have

$$LE(A_G) = \frac{2^{4k} - 3.2^{4k} - 2^{2k} + 2^{2k} + 2^{k} + (2^{3k-1} - 2^{2k} + 2^{k-1}) \left(\frac{2^{3k} - 2^{2k} - 1}{2^{3k} - 2^{k} - 1}\right)}{2^{3k} - 2^{k} - 1} + (2^{2k} - 2^{k} - 2) \left(\frac{2^{k}}{2^{3k} - 2^{k} - 1}\right) + (2^{2k} - 2^{k} - 3.2^{k-1}) \left(\frac{2^{k} - 1}{2^{3k} - 2^{k} - 1}\right) + (2^{2k} + 2^{k}) \left(\frac{2^{3k} - 2^{3k} + 2^{2k} + 2^{k+1}}{2^{3k} - 2^{k} - 1}\right)$$

and hence the result follows.

**Proposition 3.4.** Let $G$ denotes the general linear group $GL(2, q)$, where $q = p^n > 2$ and $p$ is a prime. Then

$$LE(A_G) = q^3 - 2q^2 + q^2\quad q^4 - q^3 + 2q^2 + q^2\quad q^3 - q^2 + q + 1$$

$\frac{q^3}{q^2}$ and $\frac{q^2}{q}$. Hence $e(A_G) = \frac{q^3 - 2q^2 + q^2 - q^2}{2}$. By Proposition 4.4 of (11), we have

$$\text{L-Spec}(A_G) = (0, q^3 - q^2 + 2q^2 + 2q) + q^3 - q^2 - 2q + 1, (q^3 - q^2 + q + 1) + q^4 - q^3 + 2q^2 + q^2 + q + 1$$

Now, $0 - \frac{2\varphi(A_G)}{|\nu(A_G)|} = \frac{q^3 - 2q^2 + 5q^2 + q^2 - 4q^3 + q}{q^4 - q^3 + 2q^2 + q + 1} = \frac{q^3 - 2q^2 + q^2 + 2q^2 - q^2 - 1}{q^4 - q^3 + 2q^2 + q^2 - q^2 + 1} = \frac{q^3 - 2q^2 + q^2 + 2q^2 - q^2 + 1}{q^4 - q^3 + 2q^2 + q^2 - q^2 + 1}$ and $\frac{q^3 - 2q^2 + q^2 + 2q^2 - q^2}{q^4 - q^3 + 2q^2 + q^2 - q^2 + 1}$. By (11), we have

$$LE(A_G) = \frac{q^3 - 2q^2 + 5q^2 + q^2 - 4q^3 + q}{q^4 - q^3 + 2q^2 + q^2 - q^2 + 1} + (q^3 - q^2 - 2q^2 + q) \left(\frac{q^3 - 2q^2 + q}{q^4 - q^3 - q^2 + 1}\right) + \left(\frac{q^3 - 2q^2 + q}{2}\right) \left(\frac{q^3 - 2q^2 + q - 3q^2 - 1}{q^4 - q^3 - q^2 + 1}\right) + \left(\frac{q^3 - 2q^2 + q}{2}\right) \left(\frac{q^3 - 2q^2 + q - 3q^2 - 1}{q^4 - q^3 - q^2 + 1}\right)$$
and hence the result follows.

**Proposition 3.5.** Let \( F = GF(2^n) \), \( n \geq 2 \) and \( \vartheta \) be the Frobenius automorphism of \( F \), i.e., \( \vartheta(x) = x^2 \) for all \( x \in F \). If \( G \) denotes the group

\[
\left\{ U(a, b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}
\]

under matrix multiplication given by \( U(a, b)U(a', b') = U(a + a', b + b' + a'\vartheta(a)) \), then

\[
LE(A_G) = 2^{2n+1} - 2^{n+2}.
\]

**Proof.** It is clear that \( \nu(A_G) = 2^n(2^n - 1) \). Note that \( Z(G) = \{ U(0, b) : b \in F \} \) and so \( |Z(G)| = 2^n \). Let \( U(a, b) \) be a non-central element of \( G \). The centralizer of \( U(a, b) \) in \( G \) is \( Z(G) \cup U(a, 0)Z(G) \). Hence \( A_G = (2^n - 1)K_{2^n} \) and \( |c(A_G)| = 2^{3n - 3}2^{n+2} \).

By Proposition 4.5 of [11], we have

\[
\text{L-Spec}(A_G) = \{0, (2^{2n} - 2^{n+1})(2^n - 1)^2, (2^{2n} - 2^n)^{2^{n-2}}\}.
\]

Therefore, \( 0 - 2\nu(A_G) = 2^{2n} - 2^{2n}, \quad 2^{2n} - 2^{n+1} - 2\nu(A_G) = 0 \) and \( 2^{2n} - 2^n = 2^n \). By [11], we have

\[
LE(A_G) = 2^{2n} - 2^{2^n} + ((2^n - 1)^2)0 + (2^n - 2)2^n
\]

and hence the result follows.

**Proposition 3.6.** Let \( F = GF(p^n) \) where \( p \) is a prime. If \( G \) denotes the group

\[
\left\{ V(a, b, c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}
\]

under matrix multiplication \( V(a, b, c)V(a', b', c') = V(a + a', b + b' + ca', c + c') \), then

\[
LE(A_G) = 2(p^{3n} - p^{2n}).
\]

**Proof.** Clearly, \( \nu(A_G) = p^n(p^{2n} - 1) \). We have \( Z(G) = \{ V(0, b, 0) : b \in F \} \) and so \( |Z(G)| = p^n \). The centralizers of non-central elements of \( A(n, p) \) are given by

(i) If \( b, c \in F \) and \( c \neq 0 \) then the centralizer of \( V(0, b, c) \) in \( G \) is

\( \{ V(0, b', c') : b', c' \in F \} \) having order \( p^{2n} \).

(ii) If \( a, b \in F \) and \( a \neq 0 \) then the centralizer of \( V(a, b, 0) \) in \( G \) is

\( \{ V(a', b', 0) : a', b' \in F \} \) having order \( p^{2n} \).

(iii) If \( a, b, c \in F \) and \( a \neq 0, c \neq 0 \) then the centralizer of \( V(a, b, c) \) in \( G \) is

\( \{ V(a', b', ca'a^{-1}) : a', b' \in F \} \) having order \( p^{2n} \).

It can be seen that all the centralizers of non-central elements of \( A(n, p) \) are abelian.

Hence,

\[
\overline{A_G} = K_{p^{2n} - p^n} \cup K_{p^{2n} - p^n} \cup (p^n - 1)K_{p^{2n} - p^n} = (p^n + 1)K_{p^{2n} - p^n}.
\]
and $e(A_G) = \frac{6n^2}{2} = \frac{5n^2}{2} - \frac{2n^2}{2} + \frac{n^2}{2}$. By Proposition 4.6 of \[11\], we have
\[
\text{L-Spec}(A_{n,p}) = \{0, (p^{3n} - p^{2n}) + 2^{n-1}, (p^{3n} - p^{2n}) \}.
\]
Therefore, $\left| 0 - \frac{2|e(A_G)|}{|\text{L-Spec}(A_{n,p})|} \right| = p^{3n} - p^{2n}$, $\left| p^{3n} - p^{2n} - \frac{2|e(A_G)|}{|\text{L-Spec}(A_{n,p})|} \right| = 0$ and
$\left| p^{3n} - p^{2n} - \frac{2|e(A_G)|}{|\text{L-Spec}(A_{n,p})|} \right| = p^{2n} - p^{n}$. By \[11\], we have
\[
\text{LE}(A_G) = p^{3n} - p^{2n} + (p^{3n} - 2p^{2n} - 1)0 + p^n(p^{2n} - p^n)
\]
and hence the result follows. \hfill \Box

4 Some consequences

For a finite group $G$, the set $C_G(x) = \{ y \in G : xy = yx \}$ is called the centralizer of an element $x \in G$. Let $|\text{Cent}(G)| = |\{C_G(x) : x \in G\}|$, that is the number of distinct centralizers in $G$. A group $G$ is called an $n$-centralizer group if $|\text{Cent}(G)| = n$. The study of these groups was initiated by Belcastro and Sherman \[6\] in the year 1994. The readers may conf. \[10\] for various results on these groups. In this section, we compute Laplacian energy of the non-commuting graphs of non-abelian $n$-centralizer finite groups for some positive integer $n$. We begin with the following result.

Proposition 4.1. If $G$ is a finite 4-centralizer group, then
\[
\text{LE}(A_G) = 4|\text{Z}(G)|.
\]

Proof. Let $G$ be a finite 4-centralizer group. Then, by \[6\] Theorem 2], we have $\frac{G}{\text{Z}(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.2 the result follows. \hfill \Box

Further, we have the following result.

Corollary 4.2. If $G$ is a finite $(p + 2)$-centralizer group, then
\[
\text{LE}(A_G) = 2p(p - 1)|\text{Z}(G)|.
\]

Proof. Let $G$ be a finite $(p + 2)$-centralizer group. Then, by \[5\] Lemma 2.7], we have $\frac{G}{\text{Z}(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, by Theorem 2.2 the result follows. \hfill \Box

Proposition 4.3. If $G$ is a finite 5-centralizer group, then
\[
\text{LE}(A_G) = 12|\text{Z}(G)| - \frac{18|\text{Z}(G)|^2 + 27|\text{Z}(G)|}{5}.
\]

Proof. Let $G$ be a finite 5-centralizer group. Then by \[6\] Theorem 4], we have $\frac{G}{\text{Z}(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $D_6$. Now, if $\frac{G}{\text{Z}(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then by Theorem 2.2 we have
\[
\text{LE}(A_G) = 12|\text{Z}(G)|.
\]
If $\frac{G}{\text{Z}(G)} \cong D_6$, then by Theorem 2.2 we have
\[
\text{LE}(A_G) = \frac{18|\text{Z}(G)|^2 + 27|\text{Z}(G)|}{5}.
\]
\hfill \Box
Let \( G \) be a finite group. The commutativity degree of \( G \) is given by the ratio

\[
\Pr(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.
\]

The origin of the commutativity degree of a finite group lies in a paper of Erdős and Turán (see [12]). Readers may conf. [7, 8, 19] for various results on \( \Pr(G) \). In the following few results we shall compute various energies of the commuting graphs of finite non-abelian groups \( G \) such that \( \Pr(G) = r \) for some rational number \( r \).

**Proposition 4.4.** Let \( G \) be a finite group and \( p \) the smallest prime divisor of \( |G| \). If \( \Pr(G) = \frac{p^2 + p - 1}{p^3} \), then

\[
LE(A_G) = 2p(p-1)|Z(G)|.
\]

**Proof.** If \( \Pr(G) = \frac{p^2 + p - 1}{p^3} \), then by [16, Theorem 3], we have \( \frac{G}{Z(G)} \isomorphic Z_p \times Z_p \). Hence the result follows from Theorem 2.2. \( \square \)

As a corollary we have

**Corollary 4.5.** Let \( G \) be a finite group such that \( \Pr(G) = \frac{5}{14} \). Then

\[
LE(A_G) = 4|Z(G)|.
\]

**Proposition 4.6.** If \( \Pr(G) \in \{ \frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2} \} \), then

\[
LE(A_G) = 9, \frac{28}{3}, 25 \text{ or } \frac{126}{5}.
\]

**Proof.** If \( \Pr(G) \in \{ \frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2} \} \), then as shown in [22, pp. 246] and [20, pp. 451], we have \( \frac{G}{Z(G)} \) is isomorphic to one of the groups in \( \{ D_6, D_8, D_{10}, D_{14} \} \). Hence the result follows from Corollary 2.6. \( \square \)

**Proposition 4.7.** Let \( G \) be a group isomorphic to any of the following groups

(i) \( \mathbb{Z}_2 \times D_8 \)

(ii) \( \mathbb{Z}_2 \times Q_8 \)

(iii) \( M_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle \)

(iv) \( \mathbb{Z}_4 \times \mathbb{Z}_4 = \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle \)

(v) \( D_8 \times \mathbb{Z}_4 = \langle a, b : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2 c b \rangle \)

(vi) \( SG(16, 3) = \langle a, b : a^4 = b^4 = 1, ab = b^{-1} a^{-1}, ab^{-1} = ba^{-1} \rangle \).

Then

\[
LE(A_G) = 16.
\]

**Proof.** If \( G \) is isomorphic to any of the above listed groups, then \( |G| = 16 \) and \( |Z(G)| = 4 \). Therefore, \( \frac{G}{Z(G)} \isomorphic \mathbb{Z}_2 \times \mathbb{Z}_2 \). Thus the result follows from Theorem 2.2. \( \square \)

Recall that genus of a graph is the smallest non-negative integer \( n \) such that the graph can be embedded on the surface obtained by attaching \( n \) handles to a sphere. A graph is said to be planar if the genus of the graph is zero. We conclude this paper with the following result.
Theorem 4.8. Let \( \Gamma_G \) be the commuting graph of a finite non-abelian group \( G \). If \( \Gamma_G \) is planar then
\[
LE(A_G) = \frac{28}{3} \text{ or } 9.
\]

Proof. From \([3, Proposition 2.3]\), if \( A_G \) is planner then \( G \cong S_3, D_8 \) or \( Q_8 \). From Corollary \([2,9]\) and Corollary \([2,7]\) if \( G \cong D_8 \) or \( Q_8 \) then \( LE(A_G) = \frac{28}{3} \) and if \( G \cong S_3 \) then \( LE(A_G) = 9 \). \( \square \)

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