THE GRADIENT DISCRETISATION METHOD FOR NONLINEAR POROUS MEDIA EQUATIONS

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Abstract. The gradient discretisation method (GDM) is a generic framework for designing and analysing numerical schemes for diffusion models. In this paper, we study the GDM for the porous medium equation, including fast diffusion and slow diffusion models. Using discrete functional analysis techniques, we establish a strong $L^2$-convergence of the approximate gradients and a uniform-in-time convergence for the approximate solution, without assuming non-physical regularity assumptions on the data or continuous solution. Being established in the generic GDM framework, these results apply to a variety of numerical methods, such as finite volume, (mass-lumped) finite elements, etc. The theoretical results are illustrated, in both fast and slow diffusion regimes, by numerical tests based on mass-lumped conforming $\mathbb{P}_1$ finite elements.

1. Introduction

In this paper, we study nonlinear porous media equations of the form

$$\begin{align*}
\partial_t u - \Delta \beta(u) &= f \quad \text{in } \Omega_T := (0,T) \times \Omega, \\
u(0,\cdot) &= u_0 \quad \text{in } \Omega, \\
\beta(u) &= 0 \quad \text{on } (0,T) \times \partial \Omega,
\end{align*}$$

where $\beta(u) = |u|^{m-1}u$ with $m > 0$, $\Omega$ is an open bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with boundary $\partial \Omega$, $f \in L^2(\Omega_T)$ and $u_0 \in L^{m+1}(\Omega)$. In the case $m > 1$, the equation (1) is the standard model of diffusion of a gas in porous media, which is also called the slow diffusion model. The case $m = 2$ describes the flow of an ideal gas in porous media while $m > 2$ that of a diffusion of a compressible fluid through porous media. Other choices of exponent appear in different physical situations, such as thermal propagation in plasma ($m = 6$) or plasma radiation ($m = 4$). The slow diffusion equation is not uniformly elliptic as it degenerates at unknown points where $u = 0$. An interesting feature is that, if $u_0$ is compactly supported, then at any $t > 0$ the support of $u(t)$ has a free boundary with a finite speed of propagation, see e.g. [5, 18]. In the fast diffusion case $0 < m < 1$, the equation (1) is relevant in the description of plasma physics, the kinetic theory of gas or fluid transportation in porous media [3, 6, 19]. Since the modulus of ellipticity $|u|^{m-1}$ blows up whenever $u$ vanishes, the fast diffusion equation is a singular equation. An essential difference between the fast and slow diffusion cases is that for fast diffusion the solution decays to zero in some finite time depending on the initial data while for slow diffusion the solution decays to zero in infinite time like an inverse power of $t$, see e.g. [4, 7].
If we formally represent (1) as
\[ \frac{\partial}{\partial t} u - \text{div}(\beta'(u) \nabla u) = f, \]
where \( \beta'(u) = m|u|^{m-1} \), then the above equation is a classical nonlinear diffusion equation. The term \( |u|^{m-1} \) in (2) induces the nonlinearity that raises many challenges in the analysis of the porous media equations. These problems have been studied extensively both in theory, see e.g. [4, 17, 19], numerical analysis, see e.g. [1, 11–13], as well as in numerical approximations (without convergence proof) [14].

In particular, authors in [11] study equation (2) with variable exponent of nonlinearity, i.e. \( \beta'(u) \) is replaced by \( |u|^{\gamma(x)} \) where \( \gamma > 1 \) (i.e. \( m > 2 \) in our case). In order to deal with the degeneracy in the problem, an approximate regularized problem is investigated. A space-time discretization scheme using the finite element method in space and the discontinuous Galerkin method in time is proposed for the regularized model (not the original problem). Furthermore, error estimates are obtained with strong regularity assumptions on the solution of the regularized model, which are not expected to be satisfied by classical solutions to (1) such as Barrenblatt solutions [15]. A space and time dependent exponent \( \gamma \) is studied in [1] using the same method as in [11]. The slow diffusion case (\( m > 1 \)) is studied in [12] where a fully discrete scheme is proposed and \( L^2 \) error estimates are proved with strong assumptions on the solution and the pressure \( \beta'(u) \). In [13], the author studies the fast and slow diffusion cases in which a fully discrete Galerkin approximation is considered for the transformed Richards model
\[ \frac{\partial}{\partial t} \psi(w) - \Delta w = 0, \]
where \( w := |u|^{m-1} u \) and \( \psi(s) := |s|^{(1-m)/m}s \). Error estimates in non-standard quasi norms and rates of convergence are proved with strong regularity assumptions on the solution of (1). In a recent preprint [16], a general nonlinear diffusion equation on the whole space \( \mathbb{R}^d \) is considered. The authors propose monotone schemes of finite difference type on Cartesian meshes and obtain \( L^r \text{loc} \)-convergence for these schemes.

In this paper, we use the gradient discretisation method to approximate (1) and discrete functional analysis techniques to obtain an \( L^\infty(0, T; L^{m+1}(\Omega)) \)-convergence result without assuming non-physical regularity assumptions on the data. The gradient discretisation method (GDM) is a generic framework for the design and analysis of numerical schemes for diffusion models. Using only a few discrete elements (a space, and a function and gradient reconstruction), it describes a variety of numerical schemes – such as finite volumes, finite elements, discontinuous Galerkin, etc. – and identifies three key properties of the discrete elements that ensure the convergence for linear and nonlinear models. Analyses carried out within the GDM therefore yield convergence theorems that directly apply to all the methods that fit into this framework. We refer to the monograph [10] for a detailed presentation of the GDM, and of the methods it contains.

The paper is organised as follows. Section 2.1 recalls notations of the GDM and introduce an implicit-in-time gradient scheme for (1). The definition of weak solutions and the main convergence result are stated in Section 2.2. We provide a priori estimates for the approximate solution in Section 3. Section 4 contains the initial weak convergence of the gradient scheme. The main convergence result, including
the uniform-in-time convergence, is proved in Section 5. Numerical results are provided in Section 6 to illustrate the generic convergence results; to this purpose, we choose a particular gradient scheme, the mass-lumped conforming $\mathcal{P}_1$ method, and we evaluate its accuracy when approximating the Barrenblatt solution. These tests are presented for a variety of exponents $m$, including both fast diffusion and slow diffusion ranges. A brief conclusion is presented in Section 7, and technical results are gathered in an appendix (Section 8).

2. The gradient discretisation method and the main convergence result

2.1. Gradient scheme. We recall here the notions of the gradient discretisation method. The idea of this general analysis framework is to replace, in the weak formulation of the problem, the continuous space and operators by discrete ones; the set of discrete space and operators is called a gradient discretisation (GD), and the scheme obtained after substituting these elements into the weak formulation is called a gradient scheme (GS). The convergence of the obtained GS can be established based on only a few general concepts on the underlying GD. Moreover, different GDs correspond to different classical schemes (finite elements, finite volumes, etc.). Hence, the analysis carried out in the GDM directly applies to all these schemes, and does not rely on the specificity of each particular method.

Definition 2.1. $\mathcal{D} = \left( X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{I}_{\mathcal{D}}, (t^{(n)})_{n=0, \ldots, N} \right)$ is a space-time gradient discretisation for homogeneous Dirichlet boundary conditions, with piecewise constant reconstruction, if

(i) the set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional real vector space,

(ii) the linear map $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^2(\Omega)^d$ is a piecewise constant reconstruction operator in the following sense: there exists a basis $(e_i)_{i \in I}$ of $X_{\mathcal{D},0}$ and a family $(\Omega_i)_{i \in I}$ of disjoint subsets of $\Omega$ such that, for all $u = \sum_{i \in I} u_i e_i \in X_{\mathcal{D},0}$, it holds $\Pi_{\mathcal{D}} u = \sum_{i \in I} u_i 1_{\Omega_i}$, where $1_{\Omega_i}$ is the characteristic function of $\Omega_i$,

(iii) the linear mapping $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^2(\Omega)^d$ gives a reconstructed discrete gradient. It must be chosen such that $\| \nabla_{\mathcal{D}} \|_{L^2(\Omega)}$ is a norm on $X_{\mathcal{D},0}$,

(iv) $\mathcal{I}_{\mathcal{D}} : L^{m+1}(\Omega) \rightarrow X_{\mathcal{D},0}$ is an interpolation operator,

(v) $t^{(0)} = 0 < t^{(1)} < \cdots < t^{(N)} = T$.

We then let $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$ and $\delta t = \max_{n=0, \ldots, N-1} \delta t^{(n+\frac{1}{2})}$. For any $(v^{(n)})_{n=0, \ldots, N} \subset X_{\mathcal{D},0}$, we define the piecewise-constant-in-time functions

$\Pi_{\mathcal{D}} v : [0, T] \rightarrow L^2(\Omega), \quad \nabla_{\mathcal{D}} v : [0, T] \rightarrow L^2(\Omega)^d$ and $\delta_{\mathcal{D}} v : (0, T] \rightarrow L^2(\Omega)$ by: For $n = 0, \ldots, N - 1$, for any $t \in (t^{(n)}, t^{(n+1)})$, for a.e. $x \in \Omega$

$\Pi_{\mathcal{D}} v(t, x) := \Pi_{\mathcal{D}} v^{(0)}(x), \quad \Pi_{\mathcal{D}} v(t, x) := \Pi_{\mathcal{D}} v^{(n+1)}(x), \quad \nabla_{\mathcal{D}} v(t, x) := \nabla_{\mathcal{D}} v^{(n+1)}(x), \quad \delta_{\mathcal{D}} v(t) = \delta_{\mathcal{D}}^{(n+\frac{1}{2})} v := \frac{\Pi_{\mathcal{D}} v^{(n+1)} - \Pi_{\mathcal{D}} v^{(n)}}{\delta t^{(n+\frac{1}{2})}} \in L^2(\Omega)$.

Note that $\Pi_{\mathcal{D}} v$ is defined everywhere, including at $t = 0$. This will be required to state a uniform-in-time convergence result on this reconstructed function.

If $v = \sum_{i \in I} v_i e_i \in X_{\mathcal{D},0}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $g(0) = 0$, we define $g(v) := \sum_{i \in I} g(v_i) e_i \in X_{\mathcal{D},0}$. The piecewise constant feature of $\Pi_{\mathcal{D}}$ then shows that

$\forall v \in X_{\mathcal{D},0}, \quad \Pi_{\mathcal{D}} g(v) = g(\Pi_{\mathcal{D}} v).$ (3)
Once a GD has been chosen, an implicit-in-time gradient scheme for (1) is defined in the following way.

**Algorithm 2.2** (GS for (1)). Set \( u^{(0)} := T_D u_0 \) and let \( u = (u^{(n)})_{n=0,\ldots,N} \subset X_D,0 \) satisfy:

\[
\langle \delta_D u, \Pi_D \phi \rangle_{L^2(\Omega_T)} + \langle \nabla_D \beta(u), \nabla_D \phi \rangle_{L^2(\Omega_T)} = \langle f, \Pi_D \phi \rangle_{L^2(\Omega_T)},
\]

(4)

for all ‘test function’ \( \phi = (\phi^{(n)})_{n=0,\ldots,N} \subset X_D,0 \).

In order to establish the stability and convergence of the GS (4), sequences of space-time gradient discretisations \((D_l)_{l \geq 1}\) are required to satisfy consistency, limit-conformity and compactness properties [10]. The consistency is slightly adapted here to account for the nonlinearity we consider. In the following, we let \( \hat{m} = \max(1, 1/m) \).

**Definition 2.3** (Consistency). A sequence \((D_l)_{l \geq 1}\) of space-time gradient discretisations in the sense of Definition 2.1 is said to be consistent if

- for all \( \phi \in L^{1+m}(\Omega) \cap H^1_0(\Omega) \), letting
  \[
  \hat{S}_{D_l}(\phi) := \min_{w \in X_{D_l}} \left( \| \Pi_{D_l} w - \phi \|_{L^{1+m}(\Omega)} + \| \nabla_{D_l} w - \nabla \phi \|_{L^2(\Omega)} \right),
  \]

we have \( \hat{S}_{D_l}(\phi) \to 0 \) as \( l \to \infty \),

- for all \( \psi \in L^{m+1}(\Omega) \), \( \Pi_{D_l} T_{D_l} \psi \to \psi \) in \( L^{m+1}(\Omega) \) as \( l \to \infty \)

- \( \delta t_{D_l} \to 0 \) as \( l \to \infty \).

**Definition 2.4** (Limit-conformity). A sequence \((D_l)_{l \geq 1}\) of space-time gradient discretisations in the sense of Definition 2.1 is said to be limit-conforming if, for all \( \phi \in H_{\text{div}}(\Omega) := \{ \phi \in L^2(\Omega)^d : \text{div} \phi \in L^2(\Omega) \} \), letting

\[
W_{D_l}(\phi) := \max_{v \in X_{D_l}\setminus\{0\}} \frac{\int_{\Omega} \left( \nabla_{D_l} v(x) \cdot \phi(x) + \Pi_{D_l} v(x) \text{div} \phi(x) \right) dx}{\| \nabla_{D_l} v \|_{L^2(\Omega)}},
\]

we have \( W_{D_l}(\phi) \to 0 \) as \( l \to \infty \).

**Definition 2.5** (Compactness). A sequence \((D_l)_{l \geq 1}\) of space-time gradient discretisations in the sense of Definition 2.1 is said to be compact if

\[
\limsup_{\xi \to 0} T_{D_l}(\xi) = 0,
\]

where

\[
T_{D_l}(\xi) := \max_{v \in X_{D_l}\setminus\{0\}} \frac{\| \Pi_{D_l} v(\cdot + \xi) - \Pi_{D_l} v \|_{L^2(\mathbb{R}^d)}}{\| \nabla_{D_l} v \|_{L^2(\Omega)}}, \quad \forall \xi \in \mathbb{R}^d,
\]

and \( \Pi_{D_l} v \) has been extended by 0 outside \( \Omega \).

A sequence of GDs that is compact or limit-conforming also satisfies another important property: the coercivity [10, Lemmas 2.6 and 2.10].

**Lemma 2.6** (Coercivity of sequences of GDs). If a sequence \((D_l)_{l \geq 1}\) of space-time gradient discretisations in the sense of Definition 2.1 is compact or limit-conforming, then it is coercive: there exists a constant \( C_p \) such that

\[
C_{D_l} := \max_{v \in X_{D_l}\setminus\{0\}} \frac{\| \Pi_{D_l} v \|_{L^2(\Omega)}}{\| \nabla_{D_l} v \|_{L^2(\Omega)}} \leq C_p, \quad \forall l \geq 1.
\]
2.2. The main convergence result. To state the main result of this paper, we introduce a space of continuous functions for the weak topology of $L^{m+1}(\Omega)$, and we define a weak solution to (1).

$$C([0,T]; L^{m+1}(\Omega_w)) := \text{space of functions } v : [0,T] \to L^{m+1}(\Omega)$$

that are continuous for the weak topology of $L^{m+1}(\Omega)$.

For a given $s \in [0,T]$, we set $\Omega_s = (0,s) \times \Omega$. We also set

$$\zeta(z) := \int_0^z \beta(s)ds = \frac{1}{m+1}|z|^{m+1} \text{ for all } z \in \mathbb{R}.$$  \hspace{1cm} (5)

**Definition 2.7** (Weak solution to (1)). Assume that $m > 0$, $u_0 \in L^{m+1}(\Omega)$ and $f \in L^2(\Omega_T)$. A weak solution to (1) is a function $\bar{u}$ such that

(i) $\bar{u} \in C([0,T]; L^{m+1}(\Omega_w))$ and $\bar{u}(0,\cdot) = u_0$ in $L^{m+1}(\Omega)$,

(ii) $\beta(\bar{u}) \in L^2(0,T; H^1_0(\Omega))$, $\zeta(\bar{u}) \in L^2(0,T; L^1(\Omega))$,

(iii) $\partial_t \bar{u} \in L^2(0,T; H^{-1}(\Omega))$, and for any $\phi \in L^2(0,T; H^1(\Omega))$

$$\int_0^T -\langle \partial_t \bar{u}(t), \phi(t) \rangle_{H^{-1}_0} dt + \langle \nabla \beta(\bar{u}), \nabla \phi \rangle_{L^2(\Omega_T)} = \langle f, \phi \rangle_{L^2(\Omega_T)}.$$  \hspace{1cm} (6)

The existence of a weak solution to (1) will be obtained as a by-product of our convergence analysis.

**Theorem 2.8** (Convergence of the gradient scheme). Let $(\mathcal{D}_l)_{l \geq 1}$ be a sequence of gradient discretisations that is consistent, limit-conforming and compact. Then, for each $l \geq 1$ there exists $u_l$ solution to the gradient scheme (4) with $\mathcal{D} = \mathcal{D}_l$. Moreover, there exists a weak solution $\bar{u}$ to (1) in the sense of Definition 2.7 such that, up to a subsequence as $l \to \infty$,

- $\Pi_{\mathcal{D}_l} u_l \rightharpoonup \bar{u}$ strongly in $L^2(0,T; L^{m+1}(\Omega))$,
- $\nabla \beta(u_l) \rightharpoonup \nabla \beta(\bar{u})$ strongly in $L^2(\Omega_T)$.

3. A priori estimates

We first provide a priori estimates for the solution $u$ to (4), and then deduce its existence in Corollary 3.2. For legibility, we drop the index $l$ in sequences of gradient discretisations, and we simply write $\mathcal{D}$ instead of $\mathcal{D}_l$.

**Lemma 3.1.** Let $\zeta$ be defined by (5), and $u$ be a solution of (4). Then, for any integer number $k \in [1,N]$, we have

$$\int_{\Omega} \zeta(\Pi_{\mathcal{D}} u^{(k)})(x)dx + \|\nabla \beta(u)\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \zeta(\Pi_{\mathcal{D}} u^{(0)})(x)dx + \langle f, \Pi_{\mathcal{D}} \beta(u) \rangle_{L^2(\Omega)}.$$  \hspace{1cm} (7)

Consequently, there exists a constant $C > 0$ depending only on $f$, $C_p \geq C_D$ and

$$C_{ini} \geq \|\Pi_{\mathcal{D}} u_{0}\|_{L^{m+1}(\Omega)} \text{ such that}$$

$$\|\Pi_{\mathcal{D}} u\|_{L^2(0,T; L^{m+1}(\Omega))} + \|\Pi_{\mathcal{D}} \zeta(u)\|_{L^2(0,T; L^1(\Omega))} + \|\nabla \beta(u)\|_{L^2(\Omega_T)} \leq C.$$  \hspace{1cm} (8)

**Proof.** We choose the test function $\phi = (\beta(u^{(0)}), \cdots, \beta(u^{(k)}), 0, \cdots, 0) \in X_{\mathcal{D},0}$ in (4) to write

$$\langle \delta_{\mathcal{D}} u, \Pi_{\mathcal{D}} \beta(u) \rangle_{L^2(\Omega)} + \|\nabla \beta(u)\|_{L^2(\Omega)}^2 = \langle f, \Pi_{\mathcal{D}} \beta(u) \rangle_{L^2(\Omega)}.$$  \hspace{1cm} (9)
For any \( n = 0, \ldots, N - 1 \) and \( t \in (t^n, t^{n+1}] \), we estimate the first term in the left hand side of (9), starting from

\[
\delta_D u(t) \Pi_D \beta(u^{n+1}) = \frac{1}{\delta(t^{n+1})} (\Pi_D u^{n+1} - \Pi_D u^n) \beta(\Pi_D u^{n+1}).
\]  

(10)

Since \( \beta \) is increasing, \( \zeta \) is convex and thus above its tangent line, which implies \( \zeta(b) - \zeta(a) \leq (b-a) \beta(b) \) for all \( a, b \in \mathbb{R} \). Applying this inequality with \( a = \Pi_D u^n \) and \( b = \Pi_D u^{n+1} \), it follows from (10) that

\[
\frac{1}{\delta(t^{n+1})} (\zeta(\Pi_D u^{n+1}) - \zeta(\Pi_D u^n)) \leq \delta_D u(t) \Pi_D \beta(u^{n+1}).
\]  

(11)

Plugging that in (9) and using a telescopic sum yields (7). The \textit{a priori} estimates (8) follow from this relation, by writing

\[
\int_{\Omega} \zeta(\Pi_D u^{(k)})(x) d\mathbf{x} + \| \nabla_D \beta(u) \|_{L^2(\Omega_k)}^2 \leq \int_{\Omega} \zeta(\Pi_D u^{(0)})(x) d\mathbf{x} + \langle f, \Pi_D \beta(u) \rangle_{L^2(\Omega_k)} + \frac{C_D^2}{2} \| \Pi_D \beta(u) \|_{L^2(\Omega_k)}^2 + \frac{1}{2C_D} \| \Pi_D \beta(u) \|_{L^2(\Omega_k)}^2.
\]

Noting that \( \zeta(a) = \frac{|a|^{m+1}}{(m+1)} \geq 0 \) for all \( a \in \mathbb{R} \) and recalling that

\[
\| \Pi_D \beta(u) \|_{L^2(\Omega_k)} \leq C_D \| \nabla_D \beta(u) \|_{L^2(\Omega_k)} \leq C_p \| \nabla_D \beta(u) \|_{L^2(\Omega_k)},
\]

we obtain the estimates in (8).

The following corollary guarantees the existence of a solution to (4).

**Corollary 3.2.** If \( D \) is a gradient discretisation in the sense of Definition 2.1 then there exists at least one solution to the gradient scheme (4).

**Proof.** The result is obtained using (8), the properties \( |\beta(u)| = |u|^m \) and \( u\beta(u) = |u|^{m+1} \), and the same arguments as in [9, Corollary 4.2].

In the following lemma, we estimate the dual norm of the discrete time derivative \( \delta_D u \). The dual norm \( \| \cdot \|_{*,D} \) on \( \Pi_D(X_{D,0}) \subset L^2(\Omega) \) is defined by

\[
\forall v \in \Pi_D(X_{D,0}), \; \| v \|_{*,D} := \sup \left\{ \int_{\Omega} v(x) \Pi_D \phi(x) d\mathbf{x} : \phi \in X_{D,0}, \| \nabla_D \phi \|_{L^2(\Omega)} = 1 \right\}.
\]

**Lemma 3.3.** There exists a constant \( C \) depending on \( f, C_p \geq C_D \) and \( C_{ini} \geq \| \Pi_D I_D u_0 \|_{L^{m+1}(\Omega)} \) such that

\[
\int_0^T \| \delta_D u(t) \|_{*,D}^2 dt \leq C.
\]

**Proof.** We first fix \( k \in \{0, \ldots, N - 1\} \) and then, for any \( \xi \in X_{D,0} \), take \( \phi = (\phi(k))_{n=0,\ldots,N} \subset X_{D,0} \) in (4) such that \( \phi^{(k+1)} = \xi \) and \( \phi^i = 0 \) for all \( i \neq k + 1 \). Using the Cauchy–Schwarz inequality and the definition of \( C_D \) we deduce

\[
\delta(t^{k+\frac{1}{2}}) \langle \delta(t^{k+\frac{1}{2}}) u, \Pi_D \xi \rangle_{L^2(\Omega)} = -\langle \nabla_D \beta(u), \nabla_D \xi \rangle_{L^2(\Omega)} + \langle f, \Pi_D \xi \rangle_{L^2(\Omega)} \leq \delta(t^{k+\frac{1}{2}}) \| \nabla_D \beta(u^{k+1}) \|_{L^2(\Omega)} \| \nabla_D \xi \|_{L^2(\Omega)} + \delta(t^{k+\frac{1}{2}}) \| f^{(k+1)} \|_{L^2(\Omega)} C_D \| \nabla_D \xi \|_{L^2(\Omega)},
\]
where \( f^{(k+1)}(x) = \frac{1}{\delta t^{(k+\frac{1}{2})}} \int_{t^{(k)}}^{t^{(k+1)}} f(t, x) dt \). Taking the supremum over all \( \xi \in X_{\Delta, 0} \) satisfying \( \| \nabla_D \xi \|_{L^2(\Omega)} = 1 \), we deduce
\[
\delta t^{(k+\frac{1}{2})} |\beta^{(k+\frac{1}{2})}_D|_{\ast, \Delta} \leq \delta t^{(k+\frac{1}{2})} \| \nabla_D \beta(u^{(k+1)}) \|_{L^2(\Omega)} + \delta t^{(k+\frac{1}{2})} \| f^{(k+1)} \|_{L^2(\Omega)} C_D.
\]
Divide by \((\delta t^{(k+\frac{1}{2})})^{1/2}\), square, and sum over \( k \) (and use Jensen or Cauchy–Schwarz to estimate the term involving \( f^{(k+1)} \)) to get
\[
\int_0^T |\delta_D u(t)|^2 \ast, \Delta dt \leq 2 \| \nabla_D \beta(u) \|_{L^2(\Omega_T)}^2 + 2 C_D^2 \| f \|_{L^2(\Omega_T)}^2.
\]
This together with (8) complete the proof of the lemma.

To deal with the lack of global Lipschitz estimates on \( \beta \), we introduce cutoff functions. The definition of these functions is different in the case \( 0 < m < 1 \) (fast diffusion) and \( m > 1 \) (slow diffusion), because each of these cases correspond to a certain breakdown in the Lipschitz properties of \( \beta \): for fast diffusion, \( \beta \) is not Lipschitz-continuous at 0, whereas for slow diffusion, the Lipschitz constant of \( \beta \) explodes at \( \pm \infty \). We therefore define \( \beta^+_k \) and \( \beta^-_k \) for any \( k > 0 \) as
\[
\text{for } 0 < m < 1, \quad \beta_k^+(r) := \begin{cases} k^{1-m} r & \text{for } -1/k \leq r \leq 1/k, \\ \beta(r) & \text{for } r < -1/k \text{ or } r > 1/k, \end{cases}
\]
and
\[
\text{for } m > 1, \quad \beta_k^-(r) := \begin{cases} k^m & \text{for } r \geq k, \\ \beta(r) & \text{for } -k < r < k, \\ -k^m & \text{for } r \leq -k. \end{cases}
\]
The cutoff functions \( \beta^+_k \) and \( \beta^-_k \) are globally Lipschitz continuous with respective Lipschitz constants \( L_k^+ = k^{1-m} \) and \( L_k^- = m k^{m-1} \). Our goal here is to estimate the time-translates of \( \beta(u) \). To achieve this, we will be using the cutoff functions, as well as the following relation.

**Lemma 3.4.** Recalling that \( \hat{m} = \max(1, 1/m) \), for any \( a, b \in \mathbb{R} \) we have
\[
(\beta_k^+(b) - \beta_k^+(a))^2 \leq \hat{m} L_k^+(b-a) (\beta(b) - \beta(a)), \quad \text{for } i = s, f.
\]

**Proof.** By noting that \( \beta'(r) = m |r|^{m-1} \) for \( r \in \mathbb{R}\setminus\{0\} \) and
\[
(\beta_k^+_r)'(r) := \begin{cases} k^{1-m} r & \text{for } -1/k < r < 1/k, \\ \beta'(r) & \text{for } r < -1/k \text{ or } r > 1/k, \end{cases}
\]
\[
(\beta_k^-)'(r) := \begin{cases} 0 & \text{for } r < -k \text{ or } r > k, \\ \beta'(r) & \text{for } -k < r < k, \end{cases}
\]
we obtain
\[
0 \leq (\beta_k^+_r)'(r) \leq \frac{1}{m} \beta'(r) \quad \forall r \in \mathbb{R}\setminus\{-1/k, 0, 1/k\}
\]
\[
0 \leq (\beta_k^-)'(r) \leq \beta'(r) \quad \forall r \in \mathbb{R}\setminus\{-k, 0, k\}.
\]
The above inequalities imply that for any \( a, b \in \mathbb{R} \),
\[
\left| \int_a^b (\beta_k^+_r)'(s) ds \right| \leq \hat{m} \left| \int_a^b \beta'(s) ds \right|, \quad \text{for } i = s, f.
\]
or, equivalently,
\[ |\beta_k^i(b) - \beta_k^i(a)| \leq \hat{m}|\beta(b) - \beta(a)|, \quad \text{for } i = s, f. \] (14)
Together with the global Lipschitz property of \( \beta_k^i \) and the monotonicity of \( \beta \), this yields
\[
(\beta_k^i(b) - \beta_k^i(a))^2 \leq L_k^i |b - a| \beta_k^i(b) - \beta_k^i(a) \\
\leq L_k^i |b - a| \hat{m}|\beta(b) - \beta(a)| = \hat{m}L_k^i (b - a)(\beta(b) - \beta(a)),
\]
which completes the proof.

We can now estimate the time translates of \( \beta(u) \).

**Lemma 3.5.** Let \( D \) be a gradient discretisation in the sense of Definition 2.1 and \( u \) be a solution to scheme (4). Then, there exists a constant \( C \) depending only on \( m, f, C_p \geq C_D \) and \( C_m \geq \| \Pi_D \hat{I} u_0 \|_{L^{m+1}(\Omega)} \) such that:

- for \( m \in (0, 1) \),
  \[
  \| \Pi_D \beta(u)(\cdot + \tau, \cdot) - \Pi_D \beta(u) \|^2_{L^2(\Omega_{T-\tau})} \leq C(\delta t_D + \tau)^{2m/(m+1)},
  \] (15)
- for \( m > 1 \),
  \[
  \| \Pi_D \beta(u)(\cdot + \tau, \cdot) - \Pi_D \beta(u) \|^2_{L^1(\Omega_{T-\tau})} \leq C(\delta t_D + \tau)^{2/(m+1)}. \] (16)

**Proof.** In this proof, \( C \) denotes a generic constant that has the same dependencies as in the lemma. For \( i = s, f \) and any \( k \in \mathbb{N} \) we have
\[
\Pi_D \beta(u)(\cdot + \tau, \cdot) - \Pi_D \beta(u) = \Gamma^i_{1,k} + \Gamma^i_{2,k} + \Gamma^i_{3,k}
\] (17)
with
\[
\Gamma^i_{1,k} := \Pi_D (\beta_k^i(u) - \beta(u)), \quad \Gamma^i_{2,k} := \Pi_D (\beta(u) - \beta_k^i(u)) (\cdot + \tau, \cdot),
\]
and
\[
\Gamma^i_{3,k} := \Pi_D \beta_k^i(u)(\cdot + \tau, \cdot) - \Pi_D \beta_k^i(u).
\]
To estimate \( \Gamma^i_{1,k} \), define
\[
\Omega_k^i := \{(t, x) \in \Omega_{T-\tau} : |\Pi_D u(t, x)| < 1/k \}, \\
\Omega_k^s := \{(t, x) \in \Omega_{T-\tau} : |\Pi_D u(t, x)| > k \}.
\]
The commutativity property (3) shows that \( |\Pi_D (|u|^{m-1}u)| = |\Pi_D |u|^{2m} = |\Pi_D u|^{2m} \) and thus, when \( i = f \),
\[
\| \Gamma^i_{1,k} \|^2_{L^2(\Omega_{T-\tau})} = \int_{\Omega_{T-\tau}} 1_{\Omega_k^i} |\Pi_D (|u|^{m-1}u - \beta_k^i(u))|^2(t, x)dxdt \\
= \int_{\Omega_k^i} |\Pi_D (|u|^{m-1}u - k^{1-m}u)|^2(t, x)dxdt \\
\leq 2 \int_{\Omega_k^i} |\Pi_D u|^{2m}(t, x) + k^{2-2m}||\Pi_D u||^2(t, x)dxdt \\
\leq 4k^{2-2m} |\Omega_k^i| \leq 4|\Omega_{T}|k^{-2m}.
\] (18)
By the Hölder and Chebyshev inequalities, (3) and (8), we estimate \( \Gamma^i_{1,k} \):
\[
\| \Gamma^i_{1,k} \|^2_{L^1(\Omega_{T-\tau})} = \int_{\Omega_{T-\tau}} 1_{\Omega_k^i} |\Pi_D (|u|^{m-1}u - \beta_k^i(u))|(t, x)dxdt
\]
For any $t$ we have

\[ \int_{\Omega_{T-r}} \mathbb{I}_{\Omega_k} \Pi_D (|u|^m + k^m) (t, x) dx \ dt \]

\[ \leq 2 \int_{\Omega_{T-r}} \mathbb{I}_{\Omega_k} |\Pi_D u|^m (t, x) dx \ dt \]

\[ \leq 2 \|\Pi_D u\|_{L^{m+1}(\Omega_{T-r})} \|\Omega^2_k\|^{1/(m+1)} \]

\[ \leq 2k^{-1} \|\Pi_D u\|_{L^{m+1}(\Omega_{T-r})} \leq Ck^{-1}. \quad (19) \]

Similarly, we also have estimates for $\Gamma^i_{2,k}$:

\[ \|\Gamma^i_{2,k}\|^2_{L^2(\Omega_{T-r})} \leq Ck^{-2m}, \quad \|\Gamma^i_{3,k}\|_{L^1(\Omega_{T-r})} \leq Ck^{-1}. \quad (20) \]

We estimate $\Gamma^i_{3,k}$ using the same arguments in [9, Lemma 4.4]. Thanks to Lemma 3.4 we have

\[ \|\Gamma^i_{3,k}\|^2_{L^2(\Omega_{T-r})} \leq \hat{m} L_k \int_0^{T-r} A(t, \tau) dt, \quad (21) \]

with

\[ A(t, \tau) := \int_\Omega (\Pi_D u(t + \tau, x) - \Pi_D u(t, x)) (\Pi_D \beta(u)(t + \tau, x) - \Pi_D \beta(u)(t, x)) dx. \]

For any $t \in (0, T)$ there exists $n_t \in \{0, \ldots, N - 1\}$ such that $t^{(n_t)} < t \leq t^{(n_t+1)}$. We rewrite $A$ by expressing $\Pi_D u(t + \tau, x) - \Pi_D u(t, x)$ as the sum of its jumps in time, and use the definition of dual semi-norm $| \cdot |_{*,D}$ to infer

\[ A(t, \tau) = \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} \int_{\Omega} \delta D^{(j+\frac{1}{2})} u(x) \Pi_D (\beta(u^{(n_{t+1}+1)}) - \beta(u^{(n_{t+1}+1)}))(x) dx \]

\[ \leq \|\nabla_D (\beta(u^{(n_{t+1}+1)}) - \beta(u^{(n_{t+1}+1)}))\|_{L^2(\Omega)} \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} |\delta D^{(j+\frac{1}{2})} u|_{*,D} \]

\[ = \|\nabla_D \beta(u)(t + \tau) - \nabla_D \beta(u)(t)\|_{L^2(\Omega)} \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} |\delta D^{(j+\frac{1}{2})} u|_{*,D}. \]

Together with the Cauchy–Schwarz inequality (on the sums and the integrals) and (8), this implies

\[ \int_0^{T-r} A(t, \tau) dt \leq \left[ \int_0^{T-r} \|\nabla_D \beta(u)(t + \tau) - \nabla_D \beta(u)(t)\|^2_{L^2(\Omega)} dt \right]^{1/2} \]

\[ \times \left[ \int_0^{T-r} \left( \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} |\delta D^{(j+\frac{1}{2})} u|_{*,D} \right)^2 dt \right]^{1/2} \]

\[ \leq C \left[ \int_0^{T-r} \left( \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} \right) \left( \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} |\delta D^{(j+\frac{1}{2})} u|_{*,D}^2 \right) dt \right]^{1/2}. \quad (22) \]

For any $t \in (0, T - \tau)$, we note that

\[ \sum_{j=n_t+1}^{n_{t+1}} \delta t^{(j+\frac{1}{2})} = t^{(n_{t+1}+1)} - t^{(n_t+1)} \]

\[ = \left[ t^{(n_{t+1}+1)} - (t + \tau) \right] + (t - t^{(n_t+1)}) + \tau \leq 2\delta t + \tau. \]
Hence, (22) and Lemma 3.3 yield
\[
\int_0^{T-\tau} A(t, \tau) dt \leq C(2\delta t_D + \tau)^{1/2} \left[ \int_0^{T-\tau} \left( \int_{t^{(n+\gamma)}}^{t^{(m+\gamma)}} |\delta_D u(s)|^2 \, ds \right)^{1/2} dt \right] \leq C(\delta t_D + \tau).
\]
Together with (21), this implies
\[
\|\Gamma_{i,k}\|_{L^2(\Omega_T)}^2 \leq C\ell_k^i(\delta t_D + \tau), \quad \text{for } i = s, f.
\] (23)
From (17)–(20) and (23), we deduce that
for \( m < 1 \): \[ \|\Pi_D\beta(u)(+ \tau, \cdot) - \Pi_D\beta(u)\|_{L^2(\Omega_T)}^2 \leq Ck^{-2m} + Ck^{1-m}(\delta t_D + \tau), \]
for \( m > 1 \): \[ \|\Pi_D\beta(u)(+ \tau, \cdot) - \Pi_D\beta(u)\|_{L^1(\Omega_T)}^2 \leq Ck^{-2} + Cmk^{m-1}(\delta t_D + \tau). \]
The results (15) and (16) follow from the above inequalities by choosing \( k = (\delta t_D + \tau)^{-1/(m+1)} \).

4. Initial convergence of gradient schemes

Let us start with a strong convergence result on \( \Pi_D\beta(u_i) \).

**Lemma 4.1.** Let \( r = 2 \) if \( m \in (0, 1) \) and \( r = 1 \) if \( m > 1 \). There exists a subsequence of \( \{\Pi_D\beta(u_i)\}_{i \geq 1} \) (still denoted by \( \{\Pi_D\beta(u_i)\}_{i \geq 1} \) and \( \tilde{\beta} \in L^r(\Omega_T) \) such that, as \( l \to \infty \),
\[
\Pi_D\beta(u_i) \to \tilde{\beta} \quad \text{strongly in } L^r(\Omega_T).
\]
**Proof.** The result is obtained using (8) and the compactness of \( (\mathcal{D}_l)_{l \geq 1} \) (to estimate the space translates of \( \Pi_D\beta(u_i) \)), Lemma 3.5 (to estimate the time translates of \( \Pi_D\beta(u_i) \)), Kolmogorov’s theorem and the same arguments as in [9, p.748].

**Lemma 4.2.** Let \( T > 0 \) and take a sequence \( (\mathcal{D}_l)_{l \geq 1} \) of space-time gradient discretisations, in the sense of Definition 2.1, that is consistent. Let \( u_l \) be a solution to (4) with \( \mathcal{D} = \mathcal{D}_l \). Then, the sequence \( \{\Pi_D u_l\}_{l \geq 1} \) is relatively compact uniformly-in-time and weakly in \( L^{m+1}(\Omega) \), i.e. there exists a subsequence of \( \{\Pi_D u_l\}_{l \geq 1} \) (still denoted by \( \{\Pi_D u_l\}_{l \geq 1} \) and a function \( \bar{u} : [0, T] \to L^{m+1}(\Omega) \) such that, for all \( \phi \in L^{1+m}(\Omega) \), the sequence of functions
\[
t \in [0, T] \mapsto L^{m+1}(\Omega_T) \to L^{m+1}(\Omega)
\]
converges uniformly on \([0, T]\) to the function
\[
t \in [0, T] \mapsto L^{m+1}(\Omega_T) \to L^{m+1}(\Omega_T).
\]
Moreover, \( \bar{u} \) is continuous \([0, T] \to L^{m+1}(\Omega) \) for the weak topology of \( L^{m+1}(\Omega) \).

**Proof.** The result is a consequence of the discontinuous Ascoli-Arzelà theorem [9, Theorem 6.2] (see also [10, Theorem C.11, p455]). Let us check the assumptions of this theorem.
Let \((\phi_i)_{i \in \mathbb{N}} \subset C^\infty_c(\Omega)\) be a dense sequence in \(L^{1+1/m}(\Omega)\) and equip the ball \(B\) of radius \(C\) (from (8)) in \(L^{m+1}(\Omega)\) with the following metric
\[
d_B(v, w) = \sum_{i \in \mathbb{N}} \min\left(1, \frac{1}{2^i} L^{m+1}(v - w, \phi_i)_{L^{1+1/m}}\right)
\text{ for } v, w \in B.
\]

The metric \(d_B\) defines the weak topology of \(L^{m+1}(\Omega)\) on \(B\), and the set \(B\) is metric compact and therefore complete for this weak topology. It follows from (8) that \(\Pi_{D_l} u_l(t, \cdot) \in B\) for \(t \in [0, T]\). It remains to estimate \(d_B(\Pi_{D_l} u_l(s), \Pi_{D_l} u_l(s'))\) for \(0 \leq s \leq s' \leq T\). In the following, \(C\) denotes a generic constant that may change from one line to the next but does not depend on \(l\) or \(i\).

We first define the interpolant \(P_{D_l} : H^0_0(\Omega) \to X_{D_l, 0}\) by
\[
P_{D_l} \phi := \arg\min_{w \in X_{D_l, 0}} \left(\|\Pi_{D_l} w - \phi\|_{L^{1+1/m}(\Omega)} + \|\nabla_{D_l} w - \nabla \phi\|_{L^2(\Omega)}\right).
\] (24)

We rewrite \(\Pi_{D_l} u_l(s') - \Pi_{D_l} u_l(s)\) as the sum of its jumps \(\delta_{t(k + \frac{l}{2})} u_l\) at points \(t(n)\) between \(s\) and \(s'\). Using the definition of \(\|\cdot\|_{L^1(\Omega)}\), the Cauchy–Schwarz inequality and Lemma 3.3 we obtain
\[
\left|\int_{\Omega} \left(\Pi_{D_l} u_l(s', x) - \Pi_{D_l} u_l(s, x)\right) \Pi_{D_l} P_{D_l} \phi_i(x) dx\right|
= \left|\int_{t(n)}^{t(n+1)} \int_{\Omega} \delta_{t} u_l(t, x) \Pi_{D_l} P_{D_l} \phi_i(x) dx dt\right|
\leq \int_{t(n)}^{t(n+1)} |\delta_{t} u_l(t)|_{L^{1+1/m}(\Omega)} \|\nabla_{D_l} P_{D_l} \phi_i\|_{L^2(\Omega)} dt
\leq C^{1/2} (t(n+1) - t(n))^{1/2} \|\nabla_{D_l} P_{D_l} \phi_i\|_{L^2(\Omega)}.
\] (25)

By noting that \(t(n+1) - t(n) \leq |s' - s| + \delta t_{D_l}\), we deduce from (8) and (25) that
\[
\left|\int_{\Omega} \left(\Pi_{D_l} u_l(s', x) - \Pi_{D_l} u_l(s, x)\right) \phi_i(x) dx\right|
\leq \left|\int_{\Omega} \left(\Pi_{D_l} u_l(s', x) - \Pi_{D_l} u_l(s, x)\right) \Pi_{D_l} P_{D_l} \phi_i(x) dx\right|
+ \left|\int_{\Omega} \left(\Pi_{D_l} u_l(s', x) - \Pi_{D_l} u_l(s, x)\right) \left(\Pi_{D_l} P_{D_l} \phi_i(x) - \phi_i(x)\right) dx\right|
\leq C \left(|s' - s| + \delta t_{D_l}\right)^{1/2} \|\nabla_{D_l} P_{D_l} \phi_i\|_{L^2(\Omega)} + C \|\Pi_{D_l} P_{D_l} \phi_i - \phi_i\|_{L^{1+1/m}(\Omega)}.
\] (26)

It follows from (24) and the consistency of \((D_l)_{l \geq 1}\) that \(\|\Pi_{D_l} P_{D_l} \phi_i - \phi_i\|_{L^{1+1/m}(\Omega)} \leq C S_{D_l}(\phi_i)\) and \(\|\nabla_{D_l} P_{D_l} \phi_i\|_{L^2(\Omega)} \leq S_{D_l}(\phi_i) + \|\nabla \phi_i\|_{L^2(\Omega)}\) Since \(S_{D_l}(\phi_i) \to 0\) as \(l \to \infty\), there exists a constant \(C_{\phi_i}\) depending only on \(\phi_i\) such that \(S_{D_l}(\phi_i)\) and \(\|\nabla \phi_i\|_{L^2(\Omega)} \leq C_{\phi_i}\). Hence, we estimate the right hand side of (26) to obtain
\[
\left|\int_{\Omega} \left(\Pi_{D_l} u_l(s', x) - \Pi_{D_l} u_l(s, x)\right) \phi_i(x) dx\right| \leq C \left(|s' - s| + \delta t_{D_l}\right)^{1/2} C_{\phi_i} + C S_{D_l}(\phi_i).
\]

Together with the definition of metric \(d_B\), this implies
\[
d_B(\Pi_{D_l} u_l(s), \Pi_{D_l} u_l(s')) \leq \sum_{i \in \mathbb{N}} \min\left(1, \frac{C |s' - s|^{1/2} C_{\phi_i}}{2^i}\right).
\]
Step 2: Passing to the limit in scheme (4) (i) and (ii) in Definition 2.7.

Using the dominated convergence theorem for series and the fact that, for any \( i \in \mathbb{N} \), \( \hat{S}_{D_i}(\phi_i) \rightarrow 0 \) and \( \delta t_{D_i} \rightarrow 0 \) as \( l \rightarrow \infty \), we see that

\[
\lim_{|s' - s| \rightarrow 0} \sum_{i \in \mathbb{N}} \frac{\min(1, C|s' - s|^{1/2}C_{\phi_i})}{2^i} = 0
\]

and

\[
\lim_{l \rightarrow \infty} \sum_{i \in \mathbb{N}} \frac{\min(1, C\delta t_{D_i}^{1/2}C_{\phi_i} + C\hat{S}_{D_i}(\phi_i))}{2^i} = 0.
\]

Hence the assumptions of the discontinuous Ascoli–Arzela theorem are satisfied and the proof is complete.

We recall the following lemma (see e.g. [10, Lemma 4.8]).

**Lemma 4.3** (Regularity of the limit). Let \( (D_l)_{l \geq 1} \) be a sequence of space-time gradient discretisations, in the sense of Definition 2.1, that is limit-conforming. Let \( v_l := (v_l^{(n)})_{n=0, \ldots, N_l} \subset X_{D_l} \) for any \( l \in \mathbb{N} \), be such that \( (\nabla_{D_l} v_l)_{l \in \mathbb{N}} \) is bounded in \( L^2(\Omega_T)^d \).

Then, there exists \( v \in L^2(0, T; H^1_0(\Omega)) \) such that, along a subsequence as \( l \rightarrow \infty \),

\[
\Pi_{D_l} v_l \rightarrow v \text{ weakly in } L^2(\Omega_T) \text{ and } \nabla_{D_l} v_l \rightarrow \nabla v \text{ weakly in } L^2(\Omega_T)^d.
\]

We can now prove a preliminary and weaker version of Theorem 2.8, in which the convergences are weak.

**Theorem 4.4.** Under the assumptions of Theorem 2.8, there exists a weak solution \( \tilde{u} \) to (1) in the sense of Definition 2.7 such that, up to a subsequence as \( l \rightarrow \infty \),

- \( \Pi_{D_l} \beta(u_l) \rightarrow \beta(\tilde{u}) \text{ weakly in } L^2(\Omega_T) \),
- \( \nabla_{D_l} \beta(u_l) \rightarrow \nabla \beta(\tilde{u}) \text{ weakly in } L^2(\Omega_T)^d \).

**Proof.**

Step 1: Convergence of discrete solutions.

Using the Minty trick [9, Lemma 3.5] recalled in Lemma 8.2, we deduce from Lemmas 4.2 and 4.1 that \( \hat{\beta} = \beta(\tilde{u}) \) a.e. on \( \Omega_T \), that \( \tilde{u} \in C([0, T]; L^{m+1}(\Omega)_w) \), and that \( \Pi_{D_l} u_l \rightarrow \tilde{u} \) uniformly-in-time and weakly in \( L^{m+1}(\Omega) \). Moreover, by consistency of the gradient discretisations, \( \Pi_{D_l} u_l(0) = \Pi_{D_l} \mathcal{I}_{D_l} u_0 \rightarrow u_0 \) in \( L^{m+1}(\Omega) \). Hence, the uniform-in-time weak-in-space convergence shows that \( \tilde{u}(0) = u_0 \). Owing to Lemma 4.1, the estimate (8) and Lemma 4.3 we have \( \beta(\tilde{u}) \in L^2(0, T; H^1_0(\Omega)) \) and

\[
\Pi_{D_l} \beta(u_l) \rightarrow \beta(\tilde{u}) \text{ weakly in } L^2(\Omega_T), \quad (27)
\]

\[
\nabla_{D_l} \beta(u_l) \rightarrow \nabla \beta(\tilde{u}) \text{ weakly in } L^2(\Omega_T)^d. \quad (28)
\]

Since \( \zeta \) is a convex continuous function, we deduce from Lemma 4.2 and Lemma 8.1 that for any \( t \in [0, T] \)

\[
\int_{\Omega} \zeta(\tilde{u})(t, x)dx \leq \lim_{l \rightarrow \infty} \inf \int_{\Omega} \zeta(\Pi_{D_l} u_l)(t, x)dx.
\]

Together with (8) this implies \( \zeta(\tilde{u}) \in L^\infty(0, T; L^1(\Omega)) \), which shows that \( \tilde{u} \) satisfies (i) and (ii) in Definition 2.7.

Step 2: Passing to the limit in scheme (4).
Let $\varphi \in C_{c}^{1}(\mathbb{R}, \mathbb{R})$ and $\psi \in H^{1}_{\text{loc}}(\Omega) \cap L^{1+1/m}(\Omega)$. Recalling the definition (24) of $P_{\Delta}$, we take $\phi := (\varphi(t^{(n-1)}))P_{\Delta}\psi$ for $n = 0, \ldots, N$ as test function in (4) (with $t^{(-1)} = t^{(0)} = 0$). This gives $T_{1}^{(l)} + T_{2}^{(l)} = T_{3}^{(l)}$ where, dropping the indices $l$ for legibility,

$$
T_{1}^{(l)} := \sum_{n=0}^{N-1} \varphi(t^{(n)})\delta t^{(n+\frac{1}{2})} \int_{\Omega} \delta \Omega \Pi_{\Delta} P_{\Delta} \psi(x) \, dx
$$

$$
T_{2}^{(l)} := \sum_{n=0}^{N-1} \varphi(t^{(n)})\delta t^{(n+\frac{1}{2})} \int_{\Omega} \nabla D \beta(u^{(n+1)}(x)) \cdot \nabla D P_{\Delta} \psi(x) \, dx
$$

$$
T_{3}^{(l)} := \sum_{n=0}^{N-1} \varphi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} f(t, x) \Pi_{\Delta} P_{\Delta} \psi(x) \, dx \, dt.
$$

Using the following equality (discrete integrate-by-parts, see [10, Eq. (D.15)])

$$
\sum_{n=0}^{N-1} \varphi(t^{(n)})\delta t^{(n+\frac{1}{2})} u = \sum_{n=0}^{N-1} \varphi(t^{(n)}) \left( \Pi_{\Delta} u^{(n+1)}(x) - \Pi_{\Delta} u^{(n)}(x) \right)
$$

$$
= \sum_{n=0}^{N-1} \left( \varphi(t^{(n+1)}) - \varphi(t^{(n)}) \right) \Pi_{\Delta} u^{(n+1)}(x) - \varphi(0) \Pi_{\Delta} u^{(0)}(x),
$$

we transform $T_{1}^{(l)}$ into

$$
T_{1}^{(l)} = -\int_{0}^{T} \varphi'(t) \int_{\Omega} \Pi_{\Delta} u(t, x) \Pi_{\Delta} P_{\Delta} \psi(x) \, dx \, dt - \varphi(0) \int_{\Omega} \Pi_{\Delta} u^{(0)}(x) \Pi_{\Delta} P_{\Delta} \psi(x) \, dx.
$$

By setting $\varphi_{\Delta}(t) := \varphi(t^{(n)})$ for $t \in (t^{(n)}, t^{(n+1)})$, we have

$$
T_{2}^{(l)} = \int_{0}^{T} \varphi_{\Delta}(t) \int_{\Omega} \nabla D \beta(u)(t, x) \cdot \nabla D P_{\Delta} \psi(x) \, dx \, dt,
$$

$$
T_{3}^{(l)} = \int_{0}^{T} \varphi_{\Delta}(t) \int_{\Omega} f(t, x) \Pi_{\Delta} P_{\Delta} \psi(x) \, dx \, dt.
$$

Since $\varphi_{\Delta} \to \varphi$ uniformly on $[0, T]$, $\Pi_{\Delta} P_{\Delta} \psi \to \psi$ in $L^{2}(\Omega) \cap L^{1+1/m}(\Omega)$ and $\nabla D P_{\Delta} \psi \to \nabla \psi$ in $L^{2}(\Omega)^{d}$, letting $l \to \infty$ in $T_{1}^{(l)} + T_{2}^{(l)} = T_{3}^{(l)}$ we see that $\bar{u}$ satisfies

$$
-\int_{0}^{T} \varphi'(t) \int_{\Omega} \bar{u}(t, x) \psi(x) \, dx \, dt + \varphi(0) \int_{\Omega} u_{0}(x) \psi(x) \, dx
$$

$$
+ \int_{0}^{T} \varphi(t) \int_{\Omega} \nabla \beta(\bar{u})(t, x) \cdot \nabla \psi(x) \, dx \, dt = \int_{0}^{T} \varphi(t) \int_{\Omega} f(t, x) \psi(x) \, dx \, dt.
$$

The above equality also holds with $\varphi(t) \psi(x)$ replaced by a tensorial function in $C_{c}^{\infty}(\Omega_{T})$. Hence, from the density of tensorial functions in $L^{2}(0, T; H^{1}_{\text{loc}}(\Omega))$ [8, Corollary 1.3.1] and noting that $\beta(\bar{u}) \in L^{2}(0, T; H^{1}(\Omega))$ and $f \in L^{2}(\Omega_{T})$, we deduce that $\partial_{t} \bar{u}$ belongs to $L^{2}(0, T; H^{-1}(\Omega))$ and that $\bar{u}$ satisfies (6).

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5. The uniform-in-time convergence result

\textbf{Lemma 5.1.} Let $\bar{u}$ be a weak solution of (1). Then
(i) for any \( T_0 \in [0, T] \)
\[
\int_\Omega \zeta(\bar{u})(T_0, \mathbf{x}) d\mathbf{x} + \int_0^{T_0} |\nabla \beta(\bar{u})(t)|^2_{L^2(\Omega)} dt = \int_\Omega \zeta(u_0)(\mathbf{x}) d\mathbf{x} + \int_0^{T_0} \langle f(t), \beta(\bar{u})(t) \rangle_{L^2(\Omega)} dt.
\]

(ii) the function
\[
t \in [0, T] \mapsto \int_\Omega \zeta(\tilde{u})(t, \mathbf{x}) d\mathbf{x} \in [0, \infty)
\]
is continuous and bounded,

(iii) \( \tilde{u} \) is continuous \([0, T] \to L^{m+1}(\Omega)\) for the strong topology of \( L^{m+1}(\Omega) \).

Proof.
Proof of (i): Take \( \phi = 1_{(0, T_0)} \beta(\bar{u}) \in L^2(0, T; H^1_0(\Omega)) \) in (6), where \( 1_A \) denotes the characteristic function of \( A \). This gives
\[
\int_0^{T_0} H^{-1}(\partial_t \bar{u}(t), \beta(\bar{u})(t))_{H^1_0} dt + \|\nabla \beta(\bar{u})\|_{L^2(\Omega T_0)}^2 = \langle f, \beta(\bar{u}) \rangle_{L^2(\Omega T_0)}.
\]

It remains to prove that
\[
\int_0^{T_0} H^{-1}(\partial_t \tilde{u}(t), \beta(\bar{u})(t))_{H^1_0} dt = \int_\Omega \zeta(\bar{u})(T_0, \mathbf{x}) d\mathbf{x} - \int_\Omega \zeta(u_0)(\mathbf{x}) d\mathbf{x}.
\]

Let \( \tilde{u} \) be an extension of \( \bar{u} \) outside \([0, T_0]\) obtained setting
\[
\tilde{u}(t) = \begin{cases} 
\bar{u}(T_0) & \text{for } t > T_0 \\
\bar{u}(t) & \text{for } t \in [0, T_0].
\end{cases}
\]

Considering the pointwise values of \( \tilde{u} \) makes sense owing to the weak continuity of \( \bar{u} : [0, T] \to L^{m+1}(\Omega) \). We recast the left hand side of (30) using the discrete time derivative of \( \tilde{u} \), defined by: for \( h > 0 \) and \( t > 0 \),
\[
d_h \tilde{u}(t) := \frac{\tilde{u}(t+h)-\tilde{u}(t)}{h}.
\]

By weak continuity of \( \tilde{u} \) it is easily checked that \( \partial_t \tilde{u} = 1_{(0, T_0)} \partial_t \bar{u} \in L^2(0, x; H^{-1}(\Omega)) \).

Hence \( d_h \tilde{u} \to \partial_t \tilde{u} \) weakly in this space as \( h \to 0 \) and thus
\[
\int_0^{T_0} H^{-1}(\partial_t \tilde{u}(t), \beta(\bar{u})(t))_{H^1_0} dt = \lim_{h \to 0} \int_0^{\infty} H^{-1}(d_h \tilde{u}(t), 1_{(0, T_0)} \beta(\bar{u})(t))_{H^1_0} dt
\]
\[
= \lim_{h \to 0} \frac{1}{h} \int_0^{T_0} \int_\Omega \left( \tilde{u}(t+h, \mathbf{x}) - \tilde{u}(t, \mathbf{x}) \right) \beta(\bar{u})(t, \mathbf{x}) d\mathbf{x} dt.
\]

Since \( \beta \) is increasing, \( \zeta \) is convex and above its tangent line, which means that \( \zeta(b) - \zeta(a) \geq (b-a)\beta(a) \) for all \( a, b \in \mathbb{R} \). Apply this inequality for the right hand side of (31) to get
\[
\int_0^{T_0} H^{-1}(\partial_t \tilde{u}(t), \beta(\bar{u})(t))_{H^1_0} dt \leq \lim_{h \to 0} \frac{1}{h} \int_0^{T_0} \int_\Omega \zeta(\tilde{u})(t+h, \mathbf{x}) - \zeta(\tilde{u})(t, \mathbf{x}) d\mathbf{x} dt.
\]
The continuity and boundedness of \( \bar{u} \). This implies the strong convergence in the weak topology of \( \bar{u} \). Let \( s \) be a straightforward consequence of (i).

It follows from (32) and (33) that \( \int_0^T \int_\Omega \zeta(\bar{u})(t, x) dx dt = \lim_{h \to 0} \frac{1}{h} \int_0^h \int_\Omega \zeta(\bar{u})(t, x) dx dt \). (32)

Since \( \bar{u} \in C([0, T]; L^{m+1}(\Omega)_w) \), we have

\[
\frac{1}{h} \int_0^h \bar{u}(t) dt \to \bar{u}(0) \text{ weakly in } L^{m+1}(\Omega) \text{ as } h \to 0.
\]

This together with the convexity of \( \zeta \) and Jensen’s inequality gives

\[
\int_\Omega \zeta(\bar{u})(0, x) dx \leq \liminf_{h \to 0} \int_\Omega \zeta \left( \frac{1}{h} \int_0^h \bar{u}(t, x) dt \right) dx
\]

\[
\leq \liminf_{h \to 0} \int_\Omega \frac{1}{h} \int_0^h \zeta(\bar{u})(t, x) dt dx.
\] (33)

It follows from (32) and (33) that

\[
\int_0^T \int_\Omega \zeta(\bar{u})(t, x) dx dt \leq \int_\Omega \zeta(\bar{u})(T_0, x) dx - \int_\Omega \zeta(\bar{u})(0, x) dx.
\] (34)

The reverse inequality of (34) is obtained by reversing the time. Indeed, applying (34) with \( \bar{u} \) replaced by \( v(t) := \bar{u}(T_0 - t) \) for \( t \in [0, T_0] \) and noting that \( \partial_t v(t) = -\partial_t \bar{u}(T_0 - t) \), we deduce

\[
\int_0^T \int_\Omega \zeta(\bar{u})(t, x) dx dt \geq \int_\Omega \zeta(\bar{u})(T_0, x) dx - \int_\Omega \zeta(\bar{u})(0, x) dx.
\] (35)

Hence, equality (30) follows immediately from (34) and (35), which completes the proof of (i).

Proof of (ii) and (iii): The continuity and boundedness of

\[
t \in [0, T] \mapsto \int_\Omega \zeta(\bar{u})(t, x) dx \in [0, \infty)
\]

is straightforward consequence of (i).

Let \( s \in [0, T] \) and \( (s_n)_{n \geq 1} \) be a sequence in \([0, T]\) converging to \( s \). Since \( \zeta(z) = \frac{1}{m+1}|z|^{m+1} \), it follows from (ii) and the continuity of \( \bar{u} : [0, T] \to L^{m+1}(\Omega) \) for the weak topology of \( L^{m+1}(\Omega) \) stated in Lemma 4.2 that

\[
\lim_{n \to \infty} \| \bar{u}(s_n) \|_{L^{m+1}(\Omega)} = \| \bar{u}(s) \|_{L^{m+1}(\Omega)} \text{ and}
\]

\[
\bar{u}(s_n) \to \bar{u}(s) \text{ weakly in } L^{m+1}(\Omega) \text{ as } n \to \infty.
\]

This implies the strong convergence \( \bar{u}(s_n) \to \bar{u}(s) \) in \( L^{m+1}(\Omega) \). Hence, \( \bar{u} \) is continuous \([0, T] \to L^{m+1}(\Omega) \), which complete the proof of the lemma.

We can now prove our main convergence result.

**Proof of Theorem 2.8.**

We consider the subsequence provided by Theorem 4.4.

Proof of the uniform-in-time convergence.
Let \( T_0 \in [0, T] \) and \( (T_l)_{l \geq 1} \) be a sequence in \([0, T]\) converging to \( T_0 \). We note that for any \( l \geq 1 \) there exists an integer number \( k \in [1, N] \) such that \( T_l \in (t^{(k-1)}, t^{(k)}) \). It follows from (7) that

\[
\int_{\Omega} \zeta(\Pi_{D_l}u_l)(T_l, x)dx + \| \nabla_{D_l}\beta(u_l) \|^2_{L^2(\Omega_{T_l})} \leq \int_{\Omega} \zeta(\Pi_{D_l}u_l(0))(x)dx + \langle f, \Pi_{D_l}\beta(u_l) \rangle_{L^2(\Omega_{T_l})}. \tag{36}
\]

Moving the term \( \| \nabla_{D_l}\beta(u_l) \|^2_{L^2(\Omega_{T_l})} \) to the right-hand side, taking the lim sup as \( l \) tends to infinity of the above inequality and noting that, by weak convergence of \( \nabla_{D_l}\beta(u_l) \) to \( \nabla\beta(\bar{u}) \) and strong convergence in \( L^2(0, T) \) of \( 1_{(0,T_l)} \) to \( 1_{(0,T_0)} \),

\[-\| \nabla\beta(\bar{u}) \|^2_{L^2(\Omega_{T_0})} \geq -\lim\inf_{l \to \infty} \| \nabla_{D_l}\beta(u_l) \|^2_{L^2(\Omega_{T_l})} = \lim\sup_{l \to \infty} (\| \nabla_{D_l}\beta(u_l) \|^2_{L^2(\Omega_{T_l})}),\]

we deduce

\[
\lim\sup_{l \to \infty} \int_{\Omega} \zeta(\Pi_{D_l}u_l)(T_l, x)dx \\
\leq -\| \nabla\beta(\bar{u}) \|^2_{L^2(\Omega_{T_0})} + \lim\sup_{l \to \infty} \int_{\Omega} \zeta(\Pi_{D_l}u_l(0))(x)dx + \lim\sup_{l \to \infty} \langle f, \Pi_{D_l}\beta(u_l) \rangle_{L^2(\Omega_{T_l})}.\]

As \( l \to \infty \) we have \( t^{(k)} \to T_0 \) since \( |T_l - t^{(k)}| \leq \delta t_{D_l} \). Together with (27) and the convergence \( \Pi_{D_l}I_{D_l}u_0 \to u_0 \) in \( L^{m+1}(\Omega) \) (consistency of \( (D_l)_{l \geq 1} \)), this implies

\[
\lim\sup_{l \to \infty} \int_{\Omega} \zeta(u_0)(x)dx \leq \int_{\Omega} \zeta(\bar{u})(x)dx + \langle f, \beta(\bar{u}) \rangle_{L^2(\Omega_{T_0})}.\]

Using the energy inequality in Lemma 5.1 we infer

\[
\lim\sup_{l \to \infty} \int_{\Omega} \zeta(\Pi_{D_l}u_l)(T_l, x)dx \leq \int_{\Omega} \zeta(\bar{u})(T, x)dx,
\]

or equivalently, since \( \zeta(z) = \frac{1}{m+1} |z|^{m+1} \),

\[
\lim\sup_{l \to \infty} \| \Pi_{D_l}u_l(T_l) \|_{L^{m+1}(\Omega)} \leq \| \bar{u}(T) \|_{L^{m+1}(\Omega)}.
\]

By Lemma 4.2 and the uniformly convexity of the space \( L^{m+1}(\Omega) \), this implies the strong convergence \( \Pi_{D_l}u_l(T_l) \to \bar{u}(T_0) \) in \( L^{m+1}(\Omega) \). The convergence of \( \Pi_{D_l}u_l \) in \( L^\infty(0, T; L^{m+1}(\Omega)) \) then follows by [10, Lemma C.13].

Proof of the strong convergence of gradients.

By taking the lim sup as \( l \to \infty \) of (36) with \( T_l = T \) and noting from the \( L^\infty(0, T; L^{m+1}(\Omega)) \) convergence of \( (\Pi_{D_l}u_l)_{l \geq 1} \) that

\[
\lim_{l \to \infty} \int_{\Omega} \zeta(\Pi_{D_l}u_l)(T, x)dx = \int_{\Omega} \zeta(\bar{u})(T, x)dx,
\]

we obtain

\[
\lim\sup_{l \to \infty} \| \nabla_{D_l}\beta(u_l) \|^2_{L^2(\Omega_T)} \leq \int_{\Omega} \zeta(u_0)(x)dx + \langle f, \beta(\bar{u}) \rangle_{L^2(\Omega_T)} - \int_{\Omega} \zeta(\bar{u})(T, x)dx.
\]

Together with the energy equality in Lemma 5.1 for \( T_0 = T \), this implies

\[
\lim\sup_{l \to \infty} \| \nabla_{D_l}\beta(u_l) \|^2_{L^2(\Omega_T)} \leq \| \nabla\beta(\bar{u}) \|^2_{L^2(\Omega_T)}.
\]
The strong convergence of $\nabla \beta(u)$ follows from the above inequality and the weak convergence (28).

6. Numerical results

In this section, we present numerical results using the mass-lumped $\mathbb{P}^1$ finite element method. All source files were written in Python 2.7.12 using FEniCS 1.5 [2]. Letting $T_0 = 0.1$ and $T = 1.0$, we consider Problem (1) for $t \in (T_0, T)$ and $x \in \Omega := (-0.5; 0.5) \times (-0.5; 0.5)$. Let $\mathcal{T}_h$ be a regular triangulation of the domain $\Omega$ into triangles of maximal mesh-size $h$. The mass-lumped $\mathbb{P}^1$ finite element scheme on $\mathcal{T}_h$ corresponds to the following gradient discretisation [10, Section 8.4]: letting $V_h \subset H^1_0(\Omega)$ be the space of continuous piecewise linear functions on $\mathcal{T}_h$, we define $(X_\mathcal{D}, \Pi_\mathcal{D}, \nabla \mathcal{D})$ such that $X_\mathcal{D} = V_h$ and, for $v \in V_h$, $\nabla \mathcal{D}v = \nabla v$ and, for all vertex $s$ of $\mathcal{T}_h$, $(\Pi_\mathcal{D}v)|_{E_s} = v(s)$ where $E_s$ is the dual cell around $s$, constructed by joining the centers of mass of the triangles and edge around $s$. The interpolant $\mathcal{I}_\mathcal{D} : L^{m+1}(\Omega) \to V_h$ can be taken as the $L^2$-projection on $V_h$, or the nodal interpolant if applied to continuous functions (which will be the case for us).

An analytical solution to (1) with $f = 0$ is the so-called the Barrenblatt solution [15]. It has the following explicit form

$$u_B(t, x) = t^{-\alpha}[C_B - \gamma |x|^2 t^{-2\beta}]^{1/(m-1)},$$

(37)

where $[s]_+ = \max\{s, 0\}$, $\alpha = \frac{d}{\alpha(m-1)+2}$, $\beta = \frac{\alpha}{2}$, $\gamma = \frac{\alpha(m-1)}{2md}$, and $C_B$ is an arbitrary constant chosen small enough such that $u$ satisfies the homogeneous Dirichlet boundary condition. We take the initial condition $u_0 = u_B(T_0, \cdot)$.

In Table 1, we compute the relative errors

$$E_{L^{m+1}} := \frac{\|u_B(T) - u^{(N)}\|_{L^{m+1}}}{\|u_B(T)\|_{L^{m+1}}}$$

in the slow diffusion case ($m > 1$) for several values of the space steps $h$ and corresponding uniform time steps $\delta t^{(m+\frac{1}{2})} = h^2$ for all $n$. The results suggest a convergence rate of at least order one with respect to the meshsize, for the exponents $m$ considered here. The apparent decrease of rate of convergence when $m$ increases is in part due to the fact that the norm in which the error is measured changes with $m$, and is more stringent as $m$ increases.

Table 2 shows results in the fast diffusion case ($m < 1$). Due to the fact that $|u|m^{-1}u$ is not Lipschitz-continuous in that case, we first rewrite (4) by letting $v = \beta(u)$ and then solve the corresponding nonlinear equation in the new variable $v$ using the Newton method. Although the relative errors are of a reasonable magnitude, the scheme converges very slowly in this fast diffusion case; this could be explained by the fact that this model is numerically particularly severe due to its singularity. To our best knowledge, the results presented here are the firsts, in the literature on porous medium equation, obtained with a direct simulation of the fast diffusion regime.

In Table 3, the error $E_{L^{m+1}}$ is computed for several values of uniform time steps $\delta t^{(m+\frac{1}{2})}$ and a fixed space step $h = 2^{-7}$. For moderate $m$, the rates in this table are close to 1, which is expected given that we use an implicit first-order time stepping. The decay of convergence rate for higher $m$ is due to a saturation of the space errors.
Table 1. Errors $E_{L^{m+1}}$ and convergence rates w.r.t. the space discretisation. $C_B = 0.005$ for all values of $m$.

| $h$ | $m = 1.5$ | $m = 2.0$ | $m = 2.5$ | $m = 3.0$ |
|-----|-----------|-----------|-----------|-----------|
|     | error     | rate      | error     | rate      | error     | rate      |
| 1/8 | $1.13e-01$|           | $1.31e-01$|           | $1.42e-01$|           | $1.31e-01$|           |
| 1/16| $4.08e-02$| $1.47$    | $5.92e-02$| $1.15$    | $5.41e-02$| $1.39$    | $7.02e-02$| $0.9$   |
| 1/32| $1.26e-02$| $1.70$    | $1.55e-02$| $1.93$    | $2.64e-02$| $1.04$    | $3.36e-02$| $1.06$  |
| 1/64| $3.56e-03$| $1.82$    | $6.49e-03$| $1.26$    | $1.36e-02$| $0.96$    | $2.84e-02$| $0.24$  |
| 1/128| $9.3e-04$ | $1.94$    | $2.52e-03$| $1.36$    | $6.62e-03$| $1.04$    | $1.14e-02$| $1.32$  |

Table 2. Errors $E_{L^{m+1}}$ and $\text{error}_x := (\|u^N(x)\|_x - \|u_B(T(x)\|_x$. $C_B = 0.1$ for $m = 0.3$ or 0.5, and $C_B = 0.01$ for $m = 0.7$.

| $h$ | $m = 0.3$ | $m = 0.5$ | $m = 0.7$ |
|-----|-----------|-----------|-----------|
|     | error     | error$_x$ | error     | error$_x$ | error     | error$_x$ |
| 1/8 | $2.23e-01$| $4.5e-01$ | $6.04e-02$| $6.9e-02$ | $7.74e-02$| $1.4e-01$ |
| 1/16| $6.74e-02$| $1.58e-01$| $5.43e-02$| $1.78e-02$| $1.56e-02$| $4.64e-02$|
| 1/32| $5.24e-02$| $3.77e-02$| $5.46e-02$| $6.78e-03$| $5.86e-03$| $1.3e-02$ |
| 1/64| $5.05e-02$| $1.32e-02$| $5.56e-02$| $2.24e-03$| $2.71e-03$| $4.56e-03$|

Table 3. Errors $E_{L^{m+1}}$ and convergence rates w.r.t. the time discretisation. $C_B = 0.005$ for all values of $m$.

| $k$ | $m = 1.5$ | $m = 2.0$ | $m = 2.5$ | $m = 3$ |
|-----|-----------|-----------|-----------|--------|
|     | error     | rate      | error     | rate  |
| 1/4 | $9.73e-02$|           | $8.15e-02$|       |
| 1/8 | $4.9e-02$ | $0.99$    | $4.19e-02$| $0.96$ |
| 1/16| $2.45e-02$| $1.00$    | $2.12e-02$| $0.98$ |
| 1/32| $1.21e-02$| $1.02$    | $1.06e-02$| $1.0$  |
| 1/64| $5.82e-03$| $1.06$    | $5.28e-03$| $1.01$ |

In Table 4, we compute the front distances at the final time $T = 1$ of the Barrenblatt and approximate solutions, with fixed space and time steps $h = \delta t^{(\frac{m+1}{2})} = 2^{-7}$. In this table, we set

$$d_v := \max_{x \in \Omega} \{|x| : v(x) \neq 0\}.$$

We also provide the relative errors for these front distances. The front is relatively well approximated, despite the usage of a low-order mass-lumped method. The error increases with $m$, which is consistent with the results in Table 1, and indicates that the Barrenblatt solution is more challenging to approximate for larger values of $m$. The surface plots of the numerical solution for $m = 2.5$, using $\delta t^{(\frac{m+1}{2})} = 1e-3$ and $h = 2^{-7}$, are presented in Figure 1. We notice the preservation of symmetry of the solution, and the expected expansion combined with diminution of the maximal value of the solution.
We presented and analysed the gradient discretisation method for the porous medium equation, in both slow and fast diffusion regimes. Using discrete functional analysis techniques, provided by the GDM framework and involving in particular a discrete Ascoli–Arzela theorem, we obtained a strong $L^2$ convergence result for the approximate gradients, and a uniform-in-time strong $L^{m+1}(\Omega)$ convergence result for the approximate solutions. These results apply to all methods that fall into the GDM framework. We illustrated the theoretical convergence using the mass-lumped conforming $P_1$ method to approximate the Barrenblatt solution. Even though the scheme’s performance is reduced for small or large values of the exponent $m$, the overall numerical approximations, including the location of the front, remain reasonably good in both regimes.

| $m$ | $d_u$ | $d_{u,0}$ | $\frac{|d_u - d_{u,0}|}{d_u}$ |
|-----|-------|----------|-----------------------------|
| 2.0 | 0.401 | 0.404    | 0.8%                        |
| 2.2 | 0.408 | 0.406    | 0.6%                        |
| 2.5 | 0.422 | 0.412    | 2.25%                       |
| 2.7 | 0.431 | 0.418    | 3.0%                        |
| 3.0 | 0.446 | 0.428    | 4.11%                       |

**Table 4.** The front distance of the approximate solution and the Barrenblatt solution.

8. Appendix

The proofs of following lemmas can be found in [9] (see Lemma 3.4 and 3.5 in this reference).

**Lemma 8.1.** Let $I \subset \mathbb{R}$ be a closed interval and $F : I \to (-\infty, +\infty]$ be a convex continuous function. Let $v \in L^2(\Omega; I)$ and $(v_n)_{n \in \mathbb{N}} \subset L^2(\Omega; I)$ be such that $(v_n)_{n \in \mathbb{N}} \to v$ weakly in $L^2(\Omega)$. Then

$$\int_{\Omega} F(v(x)) dx \leq \liminf_{n \to \infty} \int_{\Omega} F(v_n(x)) dx.$$

**Lemma 8.2** (Minty’s trick). Let $F \in C^0(\mathbb{R})$ be a nondecreasing function. Let $(X, \mu)$ be a measurable set with finite measure and let $(u_n)_{n \in \mathbb{N}} \subset L^p(X)$, with $p > 1$ satisfy

(1) there exists $u \in L^p(X)$ such that $(u_n)_{n \in \mathbb{N}} \to u$ weakly in $L^p(X),$

(2) $(F(u_n))_{n \in \mathbb{N}} \subset L^1(X)$ and there exists $v \in L^1(X)$ such that $(F(u_n))_{n \in \mathbb{N}} \to u$ strongly in $L^1(X).

Then $v = F(u)$ a.e. on $X$.

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Figure 1. Surface plot of the solution at several values of time $t$ with $m = 2.5$ and $C_B = 0.005$.

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