Sparse Eigenvectors of the Discrete Fourier Transform

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Abstract

We construct a basis of sparse eigenvectors for the \( N \)-dimensional discrete Fourier transform. The sparsity differs from the optimal by at most a factor of four. When \( N \) is a perfect square, the basis is orthogonal.

1 Introduction

The \( N \)-dimensional discrete Fourier transform (DFT) can be viewed as an \( N \times N \) complex-valued matrix \( D \). If we set \( \omega = e^{-2\pi i/N} \), then we define the entry in the \( j \)-th row and \( k \)-th column of \( D \) to be

\[
(D)_{j,k} = \omega^{jk} / \sqrt{N}
\]

(where \( j \) and \( k \) run from 0 to \( N - 1 \)).

The DFT has a number of strange features. For instance, if we apply the DFT to a vector four times, we recover the original vector. In other words,

\[
D^4 = I.
\]

It follows that any eigenvalue of \( D \) must be a fourth root of unity, namely an element of \( \{ \pm 1, \pm i \} \).

If two eigenvectors share the same eigenvalue, then any linear combination of them is also an eigenvector (with the same eigenvalue). Therefore, there are four vector subspaces corresponding to each of the four possible eigenvalues. The dimension of each of these subspaces was worked out by McClellan and Park [7], and is determined by the value of \( N \) mod 4:

| \( N \) | \( \lambda = 1 \) | \( \lambda = -1 \) | \( \lambda = -i \) | \( \lambda = i \) |
|-------|----------------|----------------|----------------|----------------|
| \( 4m \) | \( m + 1 \) | \( m \) | \( m \) | \( m - 1 \) |
| \( 4m + 1 \) | \( m + 1 \) | \( m \) | \( m \) | \( m \) |
| \( 4m + 2 \) | \( m + 1 \) | \( m + 1 \) | \( m \) | \( m \) |
| \( 4m + 3 \) | \( m + 1 \) | \( m + 1 \) | \( m + 1 \) | \( m \) |

At this point, we might like to choose a basis of \( N \) linearly independent (and perhaps orthogonal) eigenvectors. But since we have entire subspaces full of eigenvectors, which should we choose?
McClellan and Park offered a simple but non-orthogonal basis in their original paper [4]. Grünbaum [5] discovered another matrix that commutes with the DFT—this preserves a set of eigenvectors but breaks the symmetry in the eigenvalues, so we can pick out individual (and orthogonal) eigenvectors. Dickinson and Steiglitz [3] simultaneously discovered a different commuting matrix and used it for the same purpose. Other authors have found orthogonal bases by examining orthogonal polynomials, such as Kravchuk [1] or Hermite [6] polynomials.

In this paper we take a slightly different tack—we focus on sparsity. The sparsity of a vector is the number of non-zero coordinates, i.e. the size of the support of the vector. For each dimension $N$, we will construct a basis of eigenvectors all of which are sparse—all have sparsity within a factor of four of the best (sparsest) possible. If $N$ is a perfect square, then the basis is orthogonal and the eigenvectors all have sparsity $O(\sqrt{N})$. It is possible to change to this basis efficiently by using part of a fast Fourier transform.

2 Eigenspace Projections

Let $\mathcal{E}_k$ be the vector subspace of $\mathbb{C}^N$ consisting of all eigenvectors of $D$ with eigenvalue $i^{-k}$. As we discussed in the introduction, all eigenvectors of $D$ lie in one of $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \text{ or } \mathcal{E}_3$. We are going to examine some functions that project onto these four eigenspaces. Let

$$
F_0 = (1/4) \left[ I + D + D^2 + D^3 \right]
$$

$$
F_1 = (1/4) \left[ I + iD - D^2 - iD^3 \right]
$$

$$
F_2 = (1/4) \left[ I - D + D^2 - D^3 \right]
$$

$$
F_3 = (1/4) \left[ I - iD - D^2 + iD^3 \right]
$$

Putting it more compactly, we have $F_k = (1/4) \sum_{j=0}^{3} i^j k D^j$. (Bose [2] and others have pointed out the utility of these functions; they can be viewed as factors of the characteristic polynomial.) We have constructed each $F_k$ so that it projects vectors into the $\mathcal{E}_k$ subspace. To see why this works, suppose we take any vector $v \in \mathbb{C}^N$. Then

$$
D(F_k v) = (1/4) \left[ Dv + i^k D^2 v + i^{2k} D^3 v + i^{3k} D^4 v \right]
$$

$$
= i^{-k} (1/4) \left[ i^k Dv + i^{2k} D^2 v + i^{3k} D^3 v + i^{4k} Iv \right]
$$

$$
= i^{-k} F_k v
$$

so $F_k v$ is an eigenvector with eigenvalue $i^{-k}$. Therefore, the image of $F_k$ is contained in $\mathcal{E}_k$.

Next, suppose we choose an arbitrary eigenvector $v \in \mathbb{C}^N$ with eigenvalue $i^{-k}$. Then

$$
F_k(v) = (1/4)[v + i^k Dv + i^{2k} D^2 v + i^{3k} D^3 v]
$$

$$
= (1/4)[v + i^k i^{-k} v + i^{2k} i^{-2k} v + i^{3k} i^{-3k} v]
$$

$$
= v
$$

2
Therefore, the image of $F_k$ contains $\mathcal{E}_k$, and hence

$$\text{Im}(F_k) = \mathcal{E}_k$$

The observation above provides us with a method of transforming a basis for $\mathbb{C}^N$ into an eigenvector basis:

**Lemma 1** Suppose that $\{\beta_1, ..., \beta_N\}$ is a basis for $\mathbb{C}^N$. Consider the set of vectors $\{F_k\beta_j\}$ for $k = 0, 1, 2, 3$ and $j = 1, ..., N$. These vectors are all eigenvectors of $D$, and it is possible to choose $N$ of them to form a basis for $\mathbb{C}^N$.

**Proof:** Fix $k$. Since $\{\beta_1, ..., \beta_N\}$ span $\mathbb{C}^N$, then $\{F_k\beta_1, ..., F_k\beta_N\}$ span $\text{Im}(F_k) = \mathcal{E}_k$. Therefore, some subset of them form a basis for $\mathcal{E}_k$. Repeating this process for each $k$ gives us a basis for all of $\mathbb{C}^N$. Since the $F_k$ produce only eigenvectors, we are done. \( \square \)

## 3 Modulated Delta Trains

In continuous time, a delta train consists of an infinite sequence of regularly spaced impulse functions. (This object is also called a “Dirac comb”.) We will examine a discrete time version of this function. Suppose $v = (v_0, ..., v_{N-1}) \in \mathbb{C}^N$ and that $N = d_1 \cdot d_2$. Then we call $v$ a delta train if it is of the form

$$v_j = \begin{cases} 1/\sqrt{d_2} & j \equiv 0 \ (\text{mod} \ d_1) \\ 0 & \text{else} \end{cases}$$

As with the continuous case, the discrete Fourier transform of a delta train is another delta train. For instance, if $Dv = w$, with $v$ as above, then

$$w_j = \begin{cases} 1/\sqrt{d_1} & j \equiv 0 \ (\text{mod} \ d_2) \\ 0 & \text{else} \end{cases}$$

We are interested in a slightly more general case— we want to be able to specify an offset in the non-zero entries of $v$ and $w$. We can accomplish this effect by translation and modulation. Suppose that $a$ and $b$ are integers. Then we can define the modulated delta train $g_{d_1}(a, b) \in \mathbb{C}^N$, where $(g_{d_1}(a, b))_j$ is the $j$-th coordinate, by:

$$(g_{d_1}(a, b))_j = \begin{cases} \omega^{-bj}/\sqrt{d_2} & j \equiv a \ (\text{mod} \ d_1) \\ 0 & \text{else} \end{cases}$$

If we take the discrete Fourier transform of this vector, we find that we get

$$(Dg_{d_1}(a, b))_j = \begin{cases} \omega^{-ab}\omega^{aj}/\sqrt{d_1} & j \equiv b \ (\text{mod} \ d_2) \\ 0 & \text{else} \end{cases}$$

If we factor out the $\omega^{-ab}$ term, we can rewrite the transformed vector as:

$$Dg_{d_1}(a, b) = \omega^{-ab}g_{d_2}(b, -a)$$  \hspace{1cm} (1)
So, up to a unit-length constant, the discrete Fourier transform of a modulated delta train is another modulated delta train.

**Lemma 2** If we let \(a\) and \(b\) vary over the ranges \(0 \leq a < d_1\) and \(0 \leq b < d_2\), then the set of vectors \(\{g_{d_1}(a, b)\}\) forms an orthogonal basis for \(\mathbb{C}^N\).

**Proof:** There are \(d_1 \cdot d_2 = N\) vectors, so orthogonality will imply that the vectors span and are linearly independent. To see orthogonality, suppose that \(a \neq a'\). Then \(g_{d_1}(a, b)\) and \(g_{d_1}(a', b')\) have disjoint support, and are therefore orthogonal. On the other hand, if \(a = a'\) but \(b \neq b'\), then their discrete Fourier transforms have disjoint support, i.e.

\[
D(g_{d_1}(a, b)) = \omega^{-ab}g_{d_2}(b, -a) \quad \text{and} \quad D(g_{d_1}(a, b')) = \omega^{-ab'}g_{d_2}(b', -a)
\]

have disjoint support, and therefore the images are orthogonal. Since \(D\) is a unitary transformation, that implies that the original vectors were also orthogonal. \(\square\)

## 4 A Basis of Sparse Eigenvectors

We are now in a position to define our basis of sparse eigenvectors. Let \(\eta_1\) be the greatest divisor of \(N\) that is at most \(\sqrt{N}\), and let \(\eta_2\) be the least divisor of \(N\) that is at least \(\sqrt{N}\). (In other words, \(\eta_1\) and \(\eta_2\) are the closest divisors below and above \(\sqrt{N}\), respectively.) Note that \(\eta_1 \cdot \eta_2 = N\). We can now present our main theorem.

**Theorem 1** Take \(\eta_1, \eta_2\) as above. Then there exists a basis of eigenvectors of the discrete Fourier transform where every basis vector has at most \(2(\eta_1 + \eta_2)\) non-zero coordinates.

**Proof:** Lemma 2 shows that the set of vectors \(\{g_{\eta_1}(a, b)\}\) forms a basis, where \(0 \leq a < \eta_1\) and \(0 \leq b < \eta_2\). Lemma 4 shows that we can select a basis of eigenvectors from the set of all \(F_k g_{\eta_1}(a, b)\), where \(k = 0, 1, 2, 3\). If we consider an arbitrary vector \(F_k g_{\eta_1}(a, b)\) for some fixed \(k, a, b\), we can write it as

\[
F_k g_{\eta_1}(a, b) = \sum_{j=0}^{3} \omega^{jk} D_j g_{\eta_1}(a, b)
\]

\[
= g_{\eta_1}(a, b) + \omega^{-ab} \cdot k g_{\eta_2}(b, -a) + \omega^{-ab} \cdot 2k g_{\eta_2}(-a, -b) + \omega^{-ab} \cdot 3k g_{\eta_2}(-b, a)
\]

Now, note that the support of \(g_{\eta_1}\) (i.e. the number of non-zero coordinates in the vector) is exactly \(\eta_2\). Also, note that multiplying a vector by a non-zero constant (such as \(\omega^{-ab} \cdot k\)) does not change the size of the support. Therefore

\[
|\text{supp}(F_k g_{\eta_1}(a, b))| \leq |\text{supp}(g_{\eta_1}(a, b))| + |\text{supp}(g_{\eta_2}(b, -a))| + |\text{supp}(g_{\eta_2}(-a, -b))| + |\text{supp}(g_{\eta_2}(-b, a))|
\]

\[
= 2(\eta_1 + \eta_2)
\]

\(^1\) We are using negative indices to simplify notation; because the index \(a\) is interpreted modulo \(d_1\), we could have written \(g_{d_2}(b, d_1 - a)\) for \(g_{d_2}(b, -a)\).
as desired. □

Let us refer to the basis of sparse eigenvectors as $\mathcal{B}$. We will show that $\mathcal{B}$ is nearly optimal from a sparsity perspective. We begin with some fundamental theorems on sparsity and the discrete Fourier transform.

**Theorem 2** Take any $v \in \mathbb{C}^N$. Then

$$|\text{supp}(v)| \cdot |\text{supp}(Dv)| \geq N$$

(2)

Also, suppose that $d_1 < d_2$ are consecutive divisors of $N$, and that $d_1 \leq |\text{supp}(v)| \leq d_2$. Then

$$|\text{supp}(Dv)| \geq \frac{N}{d_1 d_2} (d_1 + d_2 - |\text{supp}(v)|)$$

(3)

**Proof:** Equation (2) was proved by Donoho and Stark [4]. Equation (3) was proved by Meshulam [8], based on a theorem of Tao’s [9]. □

We can construct a lower bound on sparsity by applying Theorem 2 to eigenvectors.

**Corollary 1** If $v \in \mathbb{C}^N$ is an eigenvector of the discrete Fourier transform and $\eta_1 \leq \eta_2$ are the divisors of $N$ defined above, then

$$|\text{supp}(v)| \geq (1/2)(\eta_1 + \eta_2)$$

(4)

**Proof:** Whenever we apply a matrix to one of its eigenvectors (with non-zero eigenvalue), the number of non-zero entries in the resulting vector remains constant. In particular, $|\text{supp}(v)| = |\text{supp}(Dv)|$ for all eigenvectors of the DFT. Therefore, if $v$ is an eigenvector, Equation (2) reduces to

$$|\text{supp}(v)|^2 \geq N$$

and hence

$$|\text{supp}(v)| \geq \sqrt{N}$$

(5)

We split into two cases. Suppose $N$ is a perfect square. Then $\eta_1 = \eta_2 = \sqrt{N}$. Therefore, since $\sqrt{N} = (1/2)(\eta_1 + \eta_2)$, Equation (5) implies Equation (4).

On the other hand, suppose that $N$ is not a perfect square. If $|\text{supp}(v)| \geq \eta_2$, then Equation (4) holds immediately. Therefore, let us assume that $|\text{supp}(v)| < \eta_2$. Combined with Equation (5), we can conclude that

$$\eta_1 < \sqrt{N} \leq |\text{supp}(v)| < \eta_2$$

Since $\eta_1$ and $\eta_2$ are consecutive divisors of $N$, we can apply Equation (3) and conclude that

$$|\text{supp}(v)| \geq (\eta_1 + \eta_2 - |\text{supp}(v)|)$$

which simplifies to Equation (4). □

Combining Theorem 2 and Corollary 1 gives us the following result, which tells us that the basis $\mathcal{B}$ is within a factor of four of the sparsest possible.

**Corollary 2** Let $v$ be any eigenvector in the basis $\mathcal{B}$, and let $w$ be any eigenvector of $D$. Then

$$\frac{|\text{supp}(v)|}{|\text{supp}(w)|} \leq 4$$

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5 Orthogonality

Is the basis $\mathcal{B}$ orthogonal? The answer depends on the dimension $N$. We begin by stating the following standard result from linear algebra.

**Lemma 3** If $v$ and $w$ are eigenvectors of a unitary matrix with different eigenvalues, then the inner product $\langle v, w \rangle$ is zero.

This lemma implies that the $\mathcal{E}_k$ are orthogonal subspaces. We now turn to the general question of orthogonality.

**Theorem 3** If $N$ is a perfect square, then the basis $\mathcal{B}$ is orthogonal.

**Proof:** Because of Lemma 3, we can restrict our attention to eigenvectors in $\mathcal{E}_k$ for a fixed $k$; eigenvectors with different eigenvalues are automatically orthogonal. Since $N$ is a perfect square, we will set $\eta = \sqrt{N}$ (since $\eta_1 = \eta_2 = \sqrt{N}$ in this case).

Recall, from Equation 1, that the discrete Fourier transform of a modulated delta train is another modulated delta train:

$$Dg_\eta(a,b) = \omega^{-ab}g_\eta(b,-a)$$

Repeating this process, we find that

$$D^2 g_\eta(a,b) = g_\eta(-a,-b)$$
$$D^3 g_\eta(a,b) = \omega^{-ab}g_\eta(-b,a)$$
$$D^4 g_\eta(a,b) = g_\eta(a,b)$$

Recall that the $F_k g_\eta(a,b)$ are just weighted sums of these terms. Therefore, for some values of $a$ and $b$, the resulting vectors $F_k g_\eta(a,b)$ are multiples each other (i.e. are collinear). For instance, $F_k g_\eta(a,b)$ and $F_k g_\eta(b,-a)$ are collinear for all $a, b$.

More generally, if there exist $j, j'$ such that $D^j_\eta(a,b)$ and $D^j'_\eta(a',b')$ are collinear then

$$F_k g_\eta(a,b) and F_k g_\eta(a',b') are collinear.$$ Conversely, suppose that $F_k g_\eta(a,b)$ and $F_k g_\eta(a',b')$ are linearly independent. Then for any $j, j'$, we can conclude that

$$D^j_\eta(a,b) and D^j' \eta(a',b') are linearly independent.$$ From Lemma 2 we know that linearly independent modulated delta trains are orthogonal; in other words, if $F_k g_\eta(a,b)$ and $F_k g_\eta(a',b')$ are linearly independent, then

$$\langle D^j_\eta g(a,b), D^{j'}_\eta g(a',b') \rangle = 0 \quad (6)$$

We will return to this equation in a moment.

Suppose we take two distinct (and hence linearly independent) vectors from the basis $\mathcal{B}$ that both lie in $\mathcal{E}_k$, say $F_k g_\eta(a,b)$ and $F_k g_\eta(a',b')$, and
consider their inner product. We get

$$\langle F_k g(a, b), F_k g(a', b') \rangle = \left\langle \sum_{j=0}^{3} i^j D^j g(a, b), \sum_{j'=0}^{3} i^{j'} D^{j'} g(a', b') \right\rangle$$

$$= \sum_{j=0}^{3} \sum_{j'=0}^{3} i^{k(j-j')} \left\langle D^j g(a, b), D^{j'} g(a', b') \right\rangle$$

Using the assumed linear independence of our initial vectors $F_k g(a, b)$ and $F_k g(a', b')$, and plugging in Equation 6, we get

$$= \sum_{j=0}^{3} \sum_{j'=0}^{3} i^{k(j-j')} 0 = 0$$

which proves orthogonality. □

In addition to the perfect squares, there are several other exceptional values of $N$ for which $B$ is orthogonal in $N$ dimensions.

**Theorem 4** If $N = 2, 3$ or $8$, $B$ is an orthogonal basis.

**Proof:** From Lemma 3 we know that the $\mathcal{E}_k$ are orthogonal. From McClellan and Park’s table of dimensions in Section 1 we know that if $N < 4$, then $\dim(\mathcal{E}_k) \leq 1$. Therefore, the eigenvector basis is automatically orthogonal (and unique) for $N = 2$ and 3. For $N = 8$, the orthogonality can be verified by direct computation (which we will not repeat here). □

Are there more of these sporadic orthogonal bases? We are not sure, but for every other $N \leq 256$, there exist $(a, b)$ and $(a', b')$ that produce vectors that are neither orthogonal nor collinear.

### 6 A Quick Note on Efficiency

If we are given an arbitrary vector, we may wish to represent it using our new sparse basis $B$. How efficiently can we change bases?

Suppose we have a vector $v \in \mathbb{C}^N$ in the standard basis, and we wish to represent it in an arbitrary basis. Generally speaking, this involves multiplying the vector by an $N \times N$ matrix, which takes time $O(N^2)$. However, because all the vectors in our basis $B$ are sparse, this corresponds to multiplication by a sparser matrix (unless $N$ is prime). For instance, if $N$ is a perfect square, the matrix multiplication only takes time $O(N \log d_2)$. However, we can do quite a bit better than that. If we consider the inner product of $g_{d_1}(a, b)$ with $v$, we are taking a sum of every entry whose index is equivalent to $a \mod d_1$, modulated by a complex exponential. This value is an entry of the $d_2$-dimensional DFT applied to those $d_2$ values. If we let $b$ vary, we can calculate all the $g_{d_1}(a, \cdot)$ inner products in time $O(d_2 \log d_2)$ by using a FFT algorithm. There are $d_1$ different possible values for $a$, so we can calculate all $g_{d_1}(a, b)$ in time $O(d_1 d_2 \log d_2) = O(N \log d_2)$. We can similarly calculate the $g_{d_2}(a, b)$ in time $O(N \log d_1)$. In an additional $O(N)$ steps we can combine them to
form the $F_k(g_{ab}(a, b))$ for all $a$ and $b$. Therefore, we can change basis to $\mathcal{B}$ in time $O(N \log N)$.

If we consider calculating the full FFT along a butterfly graph, then what we are really doing is halting the computation early—by reading off the partial results of the FFT and combining them, we calculate the projection onto the basis $\mathcal{B}$. In the case that $N$ is a perfect square, the butterfly splits exactly into two equal pieces and we calculate half of the FFT. Otherwise, we find the $\eta_1$ and $\eta_2$ that get as close as possible to splitting the FFT in half.

References

[1] N. M. Atakishiyev and K. B. Wolf. Fractional fourierkravchuk transform. *J. Opt. Soc. Am. A*, 14:1467–1477, 1997.
[2] N. K. Bose. Eigenvectors and eigenvalues of 1-d and n-d dft matrices. *International Journal of Electronics and Communications*, 55(2):131–133, 2001.
[3] Bradley W. Dickinson and Kenneth Steiglitz. Eigenvectors and functions of the discrete fourier transform. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 30(1):25–31, February 1982.
[4] David Donoho and Philip Stark. Uncertainty principles and signal recovery. *SIAM Journal of Applied Mathematics*, 49(3):906–931, June 1989.
[5] F. Alberto Grünbaum. The eigenvectors of the discrete fourier transform: A version of the hermite functions. *Journal of Mathematical Analysis and Applications*, 88:355–363, 1982.
[6] M. T. Hanna, N. P. A. Seif, and W. A. E. M. Ahmed. Hermite-gaussian-like eigenvectors of the dft matrix generated by the eigenanalysis of an almost tridiagonal matrix. *IEEE International Symposium on Circuits and Systems*.
[7] James H. McClellan and Thomas W. Parks. Eigenvalue and eigenvector decomposition of the discrete fourier transform. *IEEE Transactions on Audio and Electroacoustics*, 30(1):66–74, March 1972.
[8] Roy Meshulam. An uncertainty inequality for finite abelian groups. *European Journal of Combinatorics*, 27:63–67, January 2006.
[9] Terence Tao. An uncertainty principle for cyclic groups of prime order. 2004. arXiv:math.CA/0308286.