PERSISTENT TWO-DIMENSIONAL STRANGE ATTRACTORS
FOR A TWO-PARAMETER FAMILY OF
EXPANDING BAKER MAPS

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Abstract. We characterize the attractors for a two-parameter class of two-
dimensional piecewise affine maps. These attractors are strange attractors,
probably having finitely many pieces, and coincide with the support of an
ergodic absolutely invariant probability measure. Moreover, we demonstrate
that every compact invariant set with non-empty interior contains one of these
attractors. We also prove the existence, for each natural number n, of an open
set of parameters in which the respective transformation exhibits at least 2^n
non connected two-dimensional strange attractors each one of them formed by
4^n pieces.

1. Introduction. Analytic proofs on the existence of strange attractors are not
so much common in literature and this fact is even true in simpler scenarios like
two-dimensional piecewise affine transformations. Let us begin by introducing the
notion of strange attractor.

Definition 1.1. By an attractor for a transformation f defined in a compact man-
ifold M, we mean an f-invariant and transitive set A whose stable set
W^s(A) = \{ z ∈ M : d(f^n(z), A) → 0 as n → ∞ \}
has non-empty interior. An attractor is said to be strange if there exists a dense
orbit \{ f^n(z_1) : n ≥ 0 \} displaying exponential growth of the derivative: There exists
some constant c > 0 and a unit vector v such that, for every n ≥ 0,
\[ \| Df^n(z_1)v \| ≥ e^{cn}, \] (1)
where \( \| \cdot \| \) stands for the norm of a vector.

From the above definition, it follows that there always exists a dense orbit on
A with at least one positive Lyapounov exponent: There exists a dense orbit
\{ f^n(z_1) : n ≥ 0 \} and a unit vector v for which
\[ \limsup_{n→∞} \frac{1}{n} \log \| Df^n(z_1)v \| ≥ c > 0. \] (2)

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Throughout this paper we shall only consider two-dimensional transformations and we shall say that a strange attractor is two-dimensional (from now on, 2-D strange attractor) if it contains a dense orbit with two positive Lyapounov exponents. We also observe that our definition of 2-D strange attractor is stronger than the usual one which only requires (see, for instance, [7]) that the sum of the Lyapounov exponents is positive.

In this paper we shall consider Expanding Baker Maps (EBMs for short). Roughly speaking an EBM is a two-dimensional piecewise linear map which firstly folds (perhaps several times) its domain of definition and after that expands the folded domain. In this piecewise expanding linear setting the proof of the existence of a dense orbit with two positive Lyapounov exponents easily follows if we demonstrate, see Lemma 6.2, the existence of a dense orbit not visiting the respective critical set, i.e. the set in which the map is not smooth. This is enough to claim that (2) holds along any such orbit and for any unit vector \( v \).

In [4] certain two-parameter family \( F = \{ \Psi_{a,b} : (a,b) \in \mathcal{P} \} \) of EBMs defined on a triangle \( T \) was introduced. Let us briefly describe where the family \( F \) comes from: In [9] the author studies certain unfoldings of homoclinic tangencies for 3D diffeomorphisms. Dynamical properties of the associated limit return maps are described in [6] and [7]. Later, in [2] the notion of EBMs arises as piecewise linear models for these limit return maps. The study of these EBMs is carried on in [3], [4] and [5]. The family \( F \), as well as the general definition of EBMs, will be again described in Section 2. The main objective of this paper is to characterize the attractors exhibited by \( \Psi_{a,b} \) for every \((a,b) \in \mathcal{P}\).

**Theorem 1.2.** For every \((a,b) \in \mathcal{P}\), there exists a finite family \( A_{a,b} \) of 2-D strange attractors for \( \Psi_{a,b} \) with:

(i) If \( A \) is an attractor for \( \Psi_{a,b} \), then \( A \in A_{a,b} \).

(ii) If \( A \in A_{a,b} \), then there exists an ergodic absolutely continuous invariant measure \( \mu \) for \( \Psi_{a,b} \) supported on \( A \).

(iii) For every \( A \in A_{a,b} \) there exists a natural number \( p \) for which \( A \) can be decomposed \( A = X_0 \cup X_1 \cup \cdots \cup X_{p-1} \) in such a way that \( \Psi_{a,b}(X_i) = X_{i+1 \mod p} \).

The measure \( \mu \) supported on \( A \) is mixing (up to the eventual period \( p \)) from which the map \( \Psi_{a,b} \) is topologically mixing on every \( X_i \).

(iv) If \( A \in A_{a,b} \), then \( A \) traps almost every point in \( W^s(A) \), i.e., for almost every point \( x \in W^s(A) \), there exists \( j \in \mathbb{N} \) with \( \Psi_{a,b}^j(x) \in A \). Moreover, the set \( \bigcup_{A \in A_{a,b}} W^s(A) \) covers a full Lebesgue measure set of \( T \).

(v) If \( U \) is a compact \( \Psi_{a,b} \)-invariant set with non-empty interior then there exists \( A \in A_{a,b} \) such that \( A \subset U \). Moreover, if \( U_1 \) and \( U_2 \) are compact \( \Psi_{a,b} \)-invariant sets with disjoint non-empty interiors, then there exist two different 2-D strange attractors \( A_i \in A_{a,b} \), with \( A_i \subset U_i \), for \( i = 1,2 \).

As a consequence of Theorem 1.2 and using [5] we also shall prove the following result.

**Corollary 1.** For every \( n \in \mathbb{N} \) there exists a set of parameters \( \mathcal{P}^n \) with non-empty interior such that if \((a,b) \in \mathcal{P}^n \), then \( \Psi_{a,b} \) has at least \( 2^n \) 2-D strange attractors.

The proof of Theorem 1.2 is based on the results given in [1], [8] and [10]. Using a result of M. Tsujii, see [10] or Proposition 3, we shall conclude that, for any \((a,b) \in \mathcal{P} \), the map \( \Psi_{a,b} \) has a finite collection of ergodic absolutely continuous probability measures \( \mu_1, \mu_2, \ldots, \mu_\ell \). As we shall see in (11), the set of attractors...
announced in Theorem 1.2 coincides with
\[ A_{a,b} = \{ \text{supp}(\mu_i) : i = 1, \ldots, l \} \]
where \( \text{supp}(\mu) \) stands for the support of a measure \( \mu \).

The fact that each one of the elements of \( A_{a,b} \) is a 2-D strange attractor follows from Lemma 6.1. The proof of the first statement in Lemma 6.1 uses Proposition 2. This result was proved by B. Saussol in [8]. We point out here that according to the first statement in Lemma 6.1, each attractor \( A = \text{supp}(\mu_i) \) traps every point of \( \text{Basin}(\mu_i) \), the basin of the measure \( \mu_i \). The key to obtain this is the fact that the interior of \( \text{supp}(\mu_i) \) is a set with, according to [8] (see also Proposition 2), full \( \mu_i \)-measure. All these arguments are crucial to prove statements (iv) and (v) in Theorem 1.2.

Moreover, we also use a result of J. Buzzi, see [1] or Proposition 1, to obtain the decomposition and mixing properties of each attractor of the family \( A_{a,b} \) announced in the third item of Theorem 1.2. Consequently, statement (iii) in Theorem 1.2 only depends on the results in [1]. Finally, statements (i), (ii) and (iv) in Theorem 1.2 are proved using Proposition 3. Therefore these statements depend on the results in [10].

This paper is organized as follows: In Section 2 we recall the notion of Expanding Baker Maps and the definition of the family \( F = \{ \Psi_{a,b} : (a,b) \in P \} \). In Section 3 we introduce the result of Buzzi announced before. Section 4 is devoted to demonstrate that for every \( (a,b) \in P \) there exists some natural number \( L = L(a,b) \) such that \( \Psi_{a,b}^L \) satisfies all the hypotheses needed to apply the abovementioned result of Saussol. Fruitfull properties of the basin of each ergodic a.c.i.m. of \( \Psi_{a,b} \) are obtained in Section 5 by using the aforementioned result of Tsujii. All these ingredients allow us to demonstrate Theorem 1.2 in Section 6. Finally, in Section 7 we prove Corollary 1 and close a conjecture stated in [4]. This conjecture, see Conjecture 3, deals with certain one-parameter family \( \{ \Lambda_t \} \) of EBM s. These maps \( \Lambda_t \) were introduced in [2] and, as we shall see at the beginning of Section 7.1, the maps \( \Lambda^y_t \) belong to the family \( F = \{ \Psi_{a,b} : (a,b) \in P \} \). We therefore end this paper by proving the existence, for each natural number \( n \), of an interval of parameters \( t \) in which the respective map \( \Lambda_t \) exhibits at least \( 2^n \) 2-D strange attractors. This result depends on Corollary 1 which in turn depends on the fifth statement of Theorem 1.2 and on the notion of renormalizable EBM. Renormalizable EBMs were firstly considered in [4], see also the beginning of Section 7.

2. The family \( F \) of Expanding Baker Maps. In [4] a two-parameter family \( F = \{ \Psi_{a,b} : (a,b) \in P \} \) of EBMs was introduced. Each map \( \Psi_{a,b} \) is defined on the triangle \( T \subset \mathbb{R}^2 \) with vertices \( (0,0) \), \( (2,0) \) and \( (1,1) \), i.e.
\[ T = \{ (x,y) : 0 \leq x \leq 1, 0 \leq y \leq x \} \cup \{ (x,y) : 1 \leq x \leq 2, 0 \leq y \leq 2 - x \} \]
The set of parameters \( P \) is given by
\[ P = \{ (a,b) : (1,2] \times [1,2] : ab \leq 2 \}. \]
Splitting \( T = T_0 \cup T_1 \) with
\[ T_0 = \{ (x,y) \in T : 0 \leq x \leq 1 \}, \quad T_1 = \{ (x,y) \in T : 1 \leq x \leq 2 \}, \]
the maps \( \Psi_{a,b} \) are defined by
\[ \Psi_{a,b}(x, y) = \begin{cases} 
(ax, ay), & \text{if } (x, y) \in T^{-}_0 \\
(a(b - y), a(b - x)), & \text{if } (x, y) \in T^{+}_0 \\
(a(2 - x), ay), & \text{if } (x, y) \in T^{-}_1 \\
(a(b - y), a(b - 2 + x)), & \text{if } (x, y) \in T^{+}_1 
\end{cases} \] (6)

where
\[ T^{-}_0 = \{(x, y) \in T_0 : x + y \leq b\}, \]
\[ T^{+}_0 = \{(x, y) \in T_0 : x + y \geq b\}, \]
\[ T^{-}_1 = \{(x, y) \in T_1 : x - y \geq 2 - b\}, \]
\[ T^{+}_1 = \{(x, y) \in T_1 : x - y \leq 2 - b\}. \] (7)

Therefore, as was already pointed out in [4] every map \( \Psi_{a,b} \) is an EBM defined on \( T \). In order to give a precise definition of EBM, let us start by recalling the definition of fold.

**Definition 2.1.** Let \( K \subset \mathbb{R}^2 \) be a compact and convex domain with non-empty interior, \( P \) a point in \( K \) and \( L \) a straight line with \( L \cap \text{int}(K) \neq \emptyset \) and \( P \notin L \). Then \( L \) divides \( K \) into two subsets denoted by \( K_0 \) and \( K_1 \) (\( K_0 \) the one containing \( P \)). We define the fold \( F_{L,P} \) as the map
\[ F_{L,P}(Q) = \begin{cases} Q, & \text{if } Q \in K_0 \\
\overline{Q}, & \text{if } Q \in K_1 \n\end{cases} \]
where \( \overline{Q} \) denotes the symmetric point of \( Q \) with respect to \( L \).

In the above conditions, the map \( F_{L,P} \) is said to be a **good fold** if \( F_{L,P}(K) = K_0 \).

Now, take \( F_{L_1,P} \) a good fold defined on \( K \) and let \( L_2 \) be a straight line with \( L_2 \cap \text{int}(K_0) \neq \emptyset \) and \( P \notin L_2 \). Then, \( L_2 \) divides \( K_0 \) into two subsets \( K_{00} \) and \( K_{01} \) (\( K_{00} \) denotes the one containing \( P \)). Let us assume that \( F_{L_2,P}(K_0) = K_{00} \) (i.e, \( F_{L_2,P} \) is a good fold). Repeating these arguments, we may successively define a sequence of good folds \( F_{L_1,P}, \ldots, F_{L_n,P} \) where
\[ F_{L_1,P} : K \to K_0, \]
\[ F_{L_i,P} : K_{0^{i-1}0} \to K_{0^{i}10}, \quad i = 2, \ldots, n \]
with \( K_{0^{i-1}0} \subset K_{0^{i-1}10} \) and \( P \in K_{0^{i}10} \) for every \( i = 1, \ldots, n \).

We now recall the concept of EBM.
Definition 2.2. Let \( K \subset \mathbb{R}^2 \) be a compact and convex domain with non-empty interior. Let \( P \) be a point in \( K \) and \( \{ \mathcal{F}_{L1, P}, \ldots, \mathcal{F}_{Ln, P} \} \) a sequence of good folds of \( K \) with \( P \in \mathcal{K}_{0,1,0} \) for every \( i = 1, \ldots, n \). Let \( \Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be an expanding linear map, i.e., \( |\det(\Lambda)| > 1 \). Let us consider

\[
\tilde{\Lambda} : Q \in \mathbb{R}^2 \mapsto \tilde{\Lambda}(Q) = P + \Lambda(Q - P)
\]

and assume that \( \tilde{\Lambda}(\mathcal{K}_{0,1,0}) \subset K \). We define the **Expanding Baker Map** associated to \( P, A, L_1, \ldots, L_n \) as the map

\[
\tilde{\Lambda} \circ \mathcal{F}_{Ln} \circ \ldots \circ \mathcal{F}_{L_1} : K \rightarrow K.
\]

We take the notation

\[
\tilde{\Lambda} \circ \mathcal{F}_{Ln} \circ \ldots \circ \mathcal{F}_{L_1} = \text{EBM}(K, L_1, \ldots, L_n, P, A).
\]

Now, it is not hard to check, see (6) and (7), that \( \mathcal{F} = \{ \Psi_{a,b} = \text{EBM}(T, C, L(b), O, B_a) \} : (a, b) \in \mathcal{P} \} \) where \( T \) and \( \mathcal{P} \) were defined in (3) and (4) and, see Figure 1,

\[
C = \{ (x, y) \in T : x = 1 \},
\]

\[
L(b) = \{ (x, y) \in T : x \leq 1 \text{ and } x + y = b \},
\]

\[
O = (0, 0),
\]

\[
B_a = \begin{pmatrix}
a & 0 \\
0 & a
\end{pmatrix}.
\]

The choice of \( \mathcal{P} \), see (4), guarantees that \( T \) is invariant by \( \Psi_{a,b} \) for every \( (a, b) \in \mathcal{P} \).

3. **Placing the family \( \mathcal{F} \) on Buzzi’s scenario.** Let us recover from [1] the following definitions.

An arc is a map \( \gamma : [0, 1] \rightarrow \mathbb{R}^2 \) which is one-to-one, at least of class \( C^1 \) and such that \( ||\gamma'(t)|| \neq 0 \) for each \( t \in [0, 1] \). An analytic piece of \( \mathbb{R}^2 \) is a non-empty and bounded open subset of \( \mathbb{R}^2 \), the boundary of which is a finite union of analytic arcs. A piecewise analytic map of the plane is a map \( f : Y \rightarrow \mathbb{Y} \) such that:

1. \( Y = \bigcup_{A \in \mathcal{P}} A \) where \( \mathcal{P} \) is a finite collection of pairwise disjoint analytic pieces of \( \mathbb{R}^2 \).
2. For each \( A \in \mathcal{P} \), the restriction \( f : A \rightarrow f(A) \) can be extended to \( f_A : U \rightarrow V \), with \( f_A \) an analytic diffeomorphism between neighbourhoods \( U, V \) of \( A \) and \( f(A) \).

Such a map is called expanding if \( \inf_{x \in Y} \min_{v \in S^1} \| Df(x)v \| > 1 \).

Taking

\[
\mathcal{P} = \{ \text{int}(T_0^+), \text{int}(T_0^-), \text{int}(T_1^+), \text{int}(T_1^-) \}
\]

the facts that \( T_i^+ \) and \( T_i^- \) are triangles and, see (6), \( \Psi_{a,b}|T_i \) are linear imply that, for every \( (a, b) \in \mathcal{P} \), the map \( \Psi_{a,b} \) is an expanding piecewise analytic map of the plane. In fact, defining \( Y = \bigcup_{A \in \mathcal{P}} A \) with the partition \( \mathcal{P} \) given in (8), it is easy to get

\[
\inf_{x \in Y} \min_{v \in S^1} \| D\Psi_{a,b}(x)v \| = a > 1.
\]

Therefore, according to the Main Theorem in [1], we have the following result (see in (4) the definition of the set of parameter \( \mathcal{P} \)).
Proposition 1. For every \((a, b) \in \mathcal{P}\), there exist absolutely continuous invariant measures for \(\Psi_{a,b}\). Moreover:

(i) Each one of these a.c.i.m.’s is a convex combination of a fixed, finite collection of ergodic ones.

(ii) For every ergodic measure \(\mu\) of \(\Psi_{a,b}\), there exist a constant \(\kappa < 1\), a natural number \(p\) and a decomposition of the support of \(\mu\), \(\text{supp}(\mu) = X = X_0 \cup X_1 \cup \cdots \cup X_{p-1}\) with \(\Psi_{a,b}(X_i) = X_{i+1} \mod p\) such that for all Lipschitz functions \(h, g : X \to \mathbb{R}\) and for all \(n \geq 0\):

\[
|\int_{X_0} h(g \circ \Psi_{a,b}^n)d\mu - \int_{X_0} h\mu \int_{X_0} gd\mu| \leq C(h, g)\kappa^n
\]

for some \(C(h, g) < \infty\) independent of \(n\).

Remark 1. It is well-known that statement (ii) in Proposition 1 implies that the measure \(\mu\) is mixing (up to the eventual period \(p\)) and consequently the map \(\Psi_{a,b}\) is topologically mixing on any \(X_i\).

In order to illustrate the second statement of Proposition 1, we show in Figure 2 two numerically obtained attractors (according to Lemma 6.1, these attractors coincide with the support of an ergodic absolutely continuous invariant measure of \(\Psi_{a,b}\)) for the map \(\Psi_{a,b}\) when \(a = 1.12\) and \(b = 1.35\). The respective map displays two different non-connected attractors, each one of them formed by 4 connected pieces which are dynamically defined by \(\Psi_{a,b}\).

![Figure 2](image)

**Figure 2.** Two numerically obtained attractors for \(\Psi_{a,b}\) when \(a = 1.12\) and \(b = 1.35\).

4. Placing the family \(\mathcal{F}\) on Saussol’s scenario. In [8] the author works with certain class of multi-dimensional piecewise expanding maps (not necessarily piecewise linear). This set of maps is defined at Section 2 in [8]. This definition contains five conditions, \((PE1) - (PE5)\). Our maps \(\Psi_{a,b}\) do not satisfy condition \((PE5)\). Nevertheless, we may still use Lemma 2.2 in [8]. This lemma ensures that, if some map \(f\) satisfies \((PE1) - (PE3)\) and certain extra condition (involving the so-called weighted multiplicity and the dilatation coefficient) holds, then some iterate of \(f\) satisfies \((PE1) - (PE5)\) and therefore we may apply all the results obtained in [8] to this iterate of \(f\). This will be enough for our purposes. Hence, let us begin by only describing conditions \((PE1) - (PE3)\) from Section 2 in [8] (we restrict ourselves to the two-dimensional framework).
Let $M \subset \mathbb{R}^2$ be a compact subset with $\text{int}(M) = M$ and $f : M \to M$. For $A \subset M$ and $\varepsilon > 0$ let $B_\varepsilon(A) = \{x \in \mathbb{R}^2 : d(x, A) \leq \varepsilon\}$, where $d$ is the Euclidean distance in $\mathbb{R}^2$. Assume that there exist at most countably many disjoint open sets $U_i$ and $V_i$ such that $V_i \supset U_i$ and maps $f_i : V_i \to \mathbb{R}^2$ such that $f_i|_{U_i} = f|_{U_i}$ for each $i$. Suppose that there are constants $c, \varepsilon > 0$ and $0 < \alpha \leq 1$ such that the following conditions hold:

(PE1) $f_i(V_i) \supset B_{\varepsilon_0}(f(U_i))$ for each $i$;
(PE2) for each $i$, $f_i \in C^1(V_i)$, $f_i$ is one-to-one and $f_i^{-1} \in C^1(f_i(V_i))$. Moreover, for all $i$ and every $\varepsilon \leq \varepsilon_0$ we assume that

$$|\det Df_i^{-1}(x) - \det Df_i^{-1}(y)| \leq c|\det Df_i^{-1}(z)|\varepsilon^\alpha,$$

whenever $z \in f_i(V_i)$ and $x, y \in B_{\varepsilon_0}(z) \cap f_i(V_i)$;
(PE3) $\text{Leb}(M \setminus \bigcup U_i) = 0$, where $\text{Leb}$ stands for the Lebesgue measure in $\mathbb{R}^2$.

To check that $\Psi_{a,b}$ satisfies (PE1) – (PE3) for every $(a, b) \in P$, it suffices to consider $M = T$, $U_1 = \text{int}(T_0^-)$, $U_2 = \text{int}(T_0^+)$, $U_3 = \text{int}(T_1^-)$, $U_4 = \text{int}(T_1^+)$ (see (7)), $V_i$ small neighbourhoods of $U_i$ and $(\Psi_{a,b})_i : V_i \to \mathbb{R}^2$ defined by, see (6),

$$(\Psi_{a,b})_1(x, y) = (ax, ay),$$

$$(\Psi_{a,b})_2(x, y) = (a(b - y), a(b - x)),$$

$$(\Psi_{a,b})_3(x, y) = (a(2 - x), ay),$$

$$(\Psi_{a,b})_4(x, y) = (a(b - y), a(b - 2 + x)).$$

The existence of $\varepsilon_0$ satisfying (PE1) is evident and, since $(\Psi_{a,b})_i$ are affine, (PE2) holds. Condition (PE3) is obviously fulfilled.

Now, let us deal with the extra condition announced before. We begin by defining the weighted multiplicity of a map $f$. Let $P$ be a finite (finite will be enough for our purposes) collection of pairwise disjoint bounded open subsets of $\mathbb{R}^2$. Set $Y = \bigcup_{A \in P} A$ and let $f : Y \to Y$ be a piecewise linear (piecewise linear will be enough for our purposes) map with respect to $P$ (this means that $f$ is linear in each subset $A \in P$). Let us take the sequence of partitions $P^n$ for the $n$th iterate of the partition $P$, i. e. the elements in $P^n$ are given by

$$A_0 \cap f^{-1}A_1 \cap \cdots \cap f^{-n+1}A_{n-1}, \quad A_0, \ldots, A_{n-1} \in P.$$

Then one may define for each $n \in \mathbb{N}$, the number

$$\text{mult}(P^n) = \max_{x \in Y} \text{card}\{A \in P^n : x \in A\}.$$

The weighted multiplicity of $f$ is defined by

$$h_{\text{mult}}(P, f) = \lim_{n \to \infty} \frac{1}{n} \log \text{mult}(P^n).$$

On the other hand, the dilatation coefficient of $f$ is defined according to [8] by

$$\delta(f) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in f^n(Y)} \|Df^{-n}(x)\|,$$

where the norm of the derivative is taken along each smooth branch of $f^{-n}$.

We point out that for $\Psi_{a,b}$, see (6), the collection $P$ of pairwise disjoint and bounded open subsets of $\mathbb{R}^2$ is obtained by considering, once again,

$$P = \{\text{int}(T_0^-), \text{int}(T_0^+), \text{int}(T_1^-), \text{int}(T_1^+)\}.$$
Therefore, \( \overline{Y} = T \) and, for every \((a, b) \in \mathcal{P}\), the elements in \(P^n\) are polygonal domains whose boundaries are made of straight segments which only can be horizontal, vertical or with slope 1 or −1. Thus, for each \((a, b)\) the numbers \(\text{mult}(P^n)\) remain bounded (for instance by eight). Then, in our case, \(h_{\text{mult}}(P, \Psi_{a,b}) = 0\) for every \((a, b) \in \mathcal{P}\). It is also easy to see that, for our case, \(\delta(\Psi_{a,b}) = -\log a\). Thus, for every \((a, b) \in \mathcal{P}\), we have \(h_{\text{mult}}(P, \Psi_{a,b}) + \delta(\Psi_{a,b}) < 0\). Hence, we may apply the following result (see Lemma 2.2 in \([8]\)).

**Lemma 4.1.** Let \(f\) be a piecewise invertible \(C^1\) map with a partition into smooth components \(P\) such that (PE1) – (PE3) hold for some \(0 < \alpha \leq 1\). Suppose that the boundary of the partition is included in a finite number of \(C^1\) compact embedded submanifolds.

If \(h_{\text{mult}}(P, f) + \delta(f) < 0\) then some iterate of the map satisfies (PE1) – (PE5).

Therefore we have that, for every \((a, b) \in \mathcal{P}\), there exists some \(L = L(a, b) \in \mathbb{N}\), such that the map \(\Psi_{a,b}^L\) satisfies (PE1) – (PE5). In this way we have the following result.

**Proposition 2.** For every \((a, b) \in \mathcal{P}\) and for every a.c.i.m. \(\mu\) of \(\Psi_{a,b}\) the interior of the support of \(\mu\) has full \(\mu\)-measure. Moreover, each a.c.i.m. \(\mu\) of \(\Psi_{a,b}\) is finite.

*Proof.* Let \(L\) be the natural number for which \(\Psi_{a,b}^L\) satisfies (PE1) – (PE5). Since \(\mu\) is an a.c.i.m. for \(\Psi_{a,b}\) it is also an a.c.i.m. for \(\Psi_{a,b}^L\). Therefore, the first part of the result follows from Proposition 5.1 in \([8]\). In order to prove that \(\mu\) is finite, it is enough to apply Proposition 3.4 and Theorem 5.1 (ii) of \([8]\). \(\square\)

5. Placing the family \(\mathcal{F}\) on Tsujii’s scenario. Let us recall from \([10]\) some definitions and results. Let us put ourselves in the two-dimensional setting.

Each connected component of the complement of a straight line in \(\mathbb{R}^2\) is called a half-plane. A closed subset \(A \subset \mathbb{R}^2\) is said to be a polyhedron if it is obtained from closures of half-planes by taking intersections and unions for finite times.

Let \(U\) be a polyhedron in \(\mathbb{R}^2\) with non-empty interior. A piecewise linear map on \(U\) is a pair \((T, P)\) of a map \(T : U \to U\) and a family \(P = \{P_k\}_{k=1}^L\) of polyhedra satisfying:

(i) The interiors of polyhedra \(P_k\) are mutually disjoint.

(ii) It holds that \(\bigcup_{k=1}^L P_k = U\).

(iii) The restriction of \(T\) to the interior of each \(P_k\) is an affine map.

Let \(U^o = \bigcup_{k=1}^L \text{int}(P_k)\). A piecewise linear map \((T, P)\) is said to be expanding if there exists a constant \(\rho > 1\) such that, for every \(x \in U^o\) and all \(v \in T_x(\mathbb{R}^2)\),

\[\|DT(x)v\| \geq \rho\|v\|,\]

where \(DT(x)\) stands, as usual, for the differential map of \(T\) at \(x\).

Taking \(P = \{T_0^+, T_0^-, T_1^+, T_1^-\}\) the facts that \(T_i^+\) and \(T_i^-\) are triangles and, see (6), \(\Psi_{(a,b)|T_i^+}\) and \(\Psi_{(a,b)|T_i^-}\) are affine imply that, for every \((a, b) \in \mathcal{P}\), the map \(\Psi_{a,b}\) is a expanding piecewise linear map on the polyhedron \(T\). In fact, it is easy to see that

\[\|D\Psi_{a,b}(x)v\| \geq a\|v\|,\]

for every \(x \in T = \bigcup_{k=0}^L (\text{int}(T_k^-) \cup \text{int}(T_k^+))\) and all \(v \in T_x(\mathbb{R}^2)\).

Therefore, from Theorem 3 in \([10]\) we have the following result.
Proposition 3. For every \((a,b) \in \mathcal{P}\), there exist finitely many absolutely continuous ergodic probability measures \(\mu_1, \mu_2, \ldots, \mu_l\) for \(\Psi_{a,b}\). Moreover, the basin of each measure \(\mu_i\) is an open set modulo sets with null Lebesgue measure, and the union \(\bigcup_{i=1}^{l} \text{Basin}(\mu_i)\) has full Lebesgue measure in \(\mathcal{T}\).

We recall that the basin of an a.c.i.m. \(\mu\) for \(\Psi_{a,b}\) is given by
\[
\text{Basin}(\mu_i) = \{ x \in \mathcal{T} : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\Psi_{a,b}^j(x)} \rightarrow \mu_i \text{ weakly} \}.
\]

Remark 2. We may assume that for every \(i = 1, \ldots, l\) the basin of every measure \(\mu_i\) is a non-empty open set modulo sets with null Lebesgue measure. In other case, it follows that the basin of the measure should have zero Lebesgue measure and therefore the respective measure could be removed from the statement of Proposition 3 in such a way that the result still holds.

Remark 3. From Proposition 3 it is clear that the set \(\bigcup_{i=1}^{l} \text{int}(\text{Basin}(\mu_i))\) has full Lebesgue measure in \(\mathcal{T}\).

6. Proof of Theorem 1.2. Fix \((a,b) \in \mathcal{P}\). From Proposition 3 there exists a finite number of absolutely continuous invariant ergodic probabilities \(\mu_1, \mu_2, \ldots, \mu_l\) for \(\Psi_{a,b}\). Let us denote by \(Z_i\) the support of \(\mu_i\), \(i = 1, \ldots, l\). From now on, let us consider the family of subset of \(\mathcal{T}\) given by
\[
\mathcal{A}_{a,b} = \{ Z_i : i = 1, \ldots, l \}.
\]

This set \(\mathcal{A}_{a,b}\) is the one announced in the statement of Theorem 1.2.

To begin with, let us prove the following result

Lemma 6.1. For every \(i = 1, \ldots, l\) it holds that:

(i) The sets \(\text{int}(Z_i)\) trap every point in \(\text{Basin}(\mu_i)\), i.e., for every \(x \in \text{Basin}(\mu_i)\), there exists some natural \(j \in \mathbb{N}\) such that \(\Psi_{a,b}^j(x) \in \text{int}(Z_i)\).

(ii) The sets \(Z_i\) are 2-D strange attractors for \(\Psi_{a,b}\).

Proof. Let \(x\) be any point in \(\text{Basin}(\mu_i)\). Then, see (10), we have that
\[
\frac{1}{n} \sum_{j=0}^{n-1} \delta_{\Psi_{a,b}^j(x)} \rightarrow \mu_i
\]
weakly. Let us also recall that a sequence of measures \(\{\mu_n\}\) is said to be weakly convergent to a measure \(\mu\) if for any bounded and continuous function \(f : \mathcal{T} \rightarrow \mathbb{R}\), it holds that
\[
\int_{\mathcal{T}} fd\mu_n \rightarrow \int_{\mathcal{T}} fd\mu.
\]
By applying Portmanteau’s Theorem we may assert that, if every \(\mu_n\) and \(\mu\) are finite, then for any set \(A \subset \mathcal{T}\) with \(\mu(\partial A) = 0\), (these sets are called sets of continuity of \(\mu\)) one has
\[
\mu_n(A) \rightarrow \mu(A).
\]
Since \(\mu_i\) is an a.c.i.m. for \(\Psi_{a,b}\), Proposition 2 implies that \(\mu_i(\text{int}(Z_i)) = 1\) and hence \(\text{int}(Z_i)\) are sets of continuity of \(\mu_i\). Therefore
\[
\frac{1}{n} \sum_{j=0}^{n-1} \chi_{\text{int}(Z_i)}(\Psi_{a,b}^j(x)) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\Psi_{a,b}^j(x)}(\text{int}(Z_i)) \rightarrow 1.
\]
Lemma 6.2. If an orbit \( \{\Psi_{a,b}^n(x) : n \geq 0\} \) does not visit the critical set \( C_{a,b} \), then, for every unit vector \( v \), one has

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_v(\Psi_{a,b}^j(x)) = \log a > 0.
\]

Lemma 6.3. Let \( (a,b) \in \mathcal{P} \) and let \( U \) be any open set contained in \( T \). Then, \( \Psi_{a,b}(U) \) has non-empty interior.

This fact clearly implies the existence of \( j \in \mathbb{N} \) such that \( \Psi_{a,b}^j(x) \in \text{int}(Z_i) \). This proves the first statement of the lemma.

To demonstrate the second statement we begin by observing that, by the definition of support, it is clear that \( Z_i \) are compact sets (we may even claim, applying Proposition 2, that they have non-empty interior). Moreover, since \( \mu_i \) is invariant by \( \Psi_{a,b} \) it follows that each \( Z_i \) is (forward) invariant for \( \Psi_{a,b} \). In order to show that \( \Psi_{a,b} \) is transitive on \( Z_i \) let us take two open sets \( U \) and \( V \) in \( Z_i \). By the definition of support we have that \( \mu_i(U) > 0 \) and \( \mu_i(V) > 0 \). Hence, using also that \( \mu_i \) is ergodic, Birkhoff’s Theorem implies the existence of some point \( x \in U \) with

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_v(\Psi_{a,b}^j(x)) = \mu_i(V) > 0.
\]

Therefore, it is clear that \( \Psi_{a,b}^j(x) \in V \) for some \( j \in \mathbb{N} \) and consequently \( \Psi_{a,b}^j(U) \cap V \neq \emptyset \).

Now it remains to prove the existence of a dense orbit on \( Z_i \) exhibiting two positive Lyapunov exponents. To this end, we begin by showing the existence of a dense orbit in \( Z_i \) not visiting the critical set \( C_{a,b} \) of \( \Psi_{a,b} \). Let us denote by

\[
\tilde{C}_{a,b} = \{ x \in T : \Psi_{a,b}^j(x) \in C_{a,b} \text{ for some } j \in \mathbb{N} \}.
\]

Let us denote by \( \text{Leb} \) the Lebesgue measure in \( \mathbb{R}^2 \). Since \( \text{Leb}(\tilde{C}_{a,b}) = 0 \) and \( \mu_i \) is absolutely continuous with respect to \( \text{Leb} \) it follows that \( \mu_i(\tilde{C}_{a,b}) = 0 \).

On the other hand, in the same way as Lemma 4 in [6] was proved we may demonstrate that there exists a set \( S \) with \( \mu_i(S) = 1 \), in such a way that, if \( x_0 \) belongs to \( S \), then its \( \Psi_{a,b} \)-orbit is dense in \( Z_i \). The second statement of the lemma is then proved by applying the following result whose proof is evident.

**Lemma 6.2.** If an orbit \( \{\Psi_{a,b}^n(x) : n \geq 0\} \) does not visit the critical set \( C_{a,b} \), then, for every unit vector \( v \), one has

\[
\lim_{n \to \infty} \frac{1}{n} \log \|D\Psi_{a,b}^n(x)v\| = \log a > 0.
\]

\[\square\]

In the proof of Theorem 1.2 we shall also use the following result.

**Lemma 6.3.** Let \( (a,b) \in \mathcal{P} \) and let \( U \) be any open set contained in \( T \). Then, \( \Psi_{a,b}(U) \) has non-empty interior.
assume that there exists some point \( \bar{x} \in \mathcal{Z}_i \) with \( \bar{x} \notin U \). Since \( U \) is compact, there exists some open neighbourhood \( V_{\bar{x}} \) of \( \bar{x} \) with \( V_{\bar{x}} \cap U = \emptyset \). Now, from the fact that \( \Psi_{a,b} \) is transitive on \( \mathcal{Z}_i \), see Lemma 6.1, there must exists some natural number \( M \) such that \( \Psi_{a,b}^M(V_{\bar{x}}) \cap V_{\bar{x}} \neq \emptyset \). This contradicts the invariance of \( U \). Therefore we have proved that \( \mathcal{Z}_i \subset U \).

Finally, let us take two compact \( \Psi_{a,b} \)-invariant sets \( U_1 \) and \( U_2 \) with disjoint non-empty interiors. From the above arguments, it holds that there exist two 2-D strange attractors \( A_j \) and \( A_k \) (with \( j, k \in \{1, \ldots , l\} \)) with \( A_j \subset U_1 \) and \( A_k \subset U_2 \). These attractors must be different (i.e., \( j \neq k \)) because they have non-empty interior. The fifth statement of Theorem 1.2 is then proved.

To prove the first statement we observe that if \( A \) is an attractor for \( \Psi_{a,b} \), then from Remark 3, there exists \( i \in \{1, \ldots , l\} \) such that the set \( W^s(A) \cup \text{Basin}(\mu_i) \) has non-empty interior. From Lemma 6.1 it is evident that \( A = \mathcal{Z}_i \).

From the definition of \( A_{a,b} \) given at (11) the second statement of Theorem 1.2 is obvious.

The third statement of Theorem 1.2 is a direct consequence of the definition of \( A_{a,b} \), the second statement of Proposition 1 and Remark 1.

To prove the fourth statement let us choose some attractor \( A \in A_{a,b} \). From Lemma 6.1 we know that \( \text{Basin}(\mu_i) \subset W^s(\mathcal{Z}_i) \) where \( A = \mathcal{Z}_i = \text{supp}(\mu_i) \). Moreover, also from Lemma 6.1, we also know that if \( x \in \text{Basin}(\mu_i) \), then there exists some natural number \( j \) for which \( \Psi_{a,b}^j(x) \in A \). Therefore, the result is proved by bearing in mind that, from Proposition 3 the set \( W^s(\mathcal{Z}_i) \setminus \text{Basin}(\mu_i) \) has null Lebesgue measure. Hence, the set \( \bigcup_{i=1}^l W^s(\mathcal{Z}_i) \) covers a full Lebesgue measure set of \( T \).

7. Proof of Corollary 1. To prove Corollary 1 we need to recover from [4] the definition of renormalizable EBM.

**Definition 7.1.** Let \( \Gamma \) be a map defined in certain domain \( \mathcal{K} \). We said that \( \mathcal{D} \subset \mathcal{K} \) is a **restrictive domain** if \( \mathcal{D} \neq \mathcal{K} \) and there exists \( k = k(\mathcal{D}) \in \mathbb{N} \) such that

i) \( \Gamma^j(\mathcal{D}) \cap \mathcal{D} = \emptyset \) for every \( j = 1, \ldots , k-1 \),

ii) \( \Gamma^k(\mathcal{D}) \subset \mathcal{D} \).

**Definition 7.2.** An EBM \( \Gamma \) defined on certain domain \( \mathcal{K} \) is said to be **renormalizable** if there exists a restrictive domain \( \mathcal{D} \) (with an associated natural number \( k = k(\mathcal{D}) \)) such that \( \Gamma^k|_{\mathcal{D}} \) is, up to an affine change in coordinates, an EBM defined on \( \mathcal{K} \).

**Definition 7.3.** Let \( \Gamma \) be a renormalizable EBM with restrictive domain \( \mathcal{D} \) (with an associated natural number \( k = k(\mathcal{D}) \)). Let us denote \( \Gamma_1 = \Gamma^k|_{\mathcal{D}} \). If \( \Gamma_1 \) is a renormalizable EBM, we call \( \Gamma \) **twice renormalizable** EBM. In this way \( \Gamma \) renormalizable EBMs, for every \( n \in \mathbb{N} \), or even infinitely renormalizable EBMs may also be defined.

In [5] a subset \( \mathcal{P}_3 \subset \mathcal{P} \) of parameters was constructed in such a way that, if \( (a,b) \in \mathcal{P}_3 \), then \( \Psi_{a,b} \) is renormalizable (see Definition 7.2) in \( \mathcal{F} \). In this particular case, this means that, for every \( (a,b) \in \mathcal{P}_3 \), the restriction of \( \Psi_{a,b}^4 \) to each one of two different restrictive domains (denoted by \( \Delta_{a,b} \) and \( \Pi_{a,b} \)) is conjugate by means of an affine change in coordinates to an EBM which belongs to the family \( \mathcal{F} \). These restrictive domains, \( \Delta_{a,b} \) and \( \Pi_{a,b} \), have, see details in [5], disjoint interiors.

More precisely, see Theorem 3.5 in [5], there exist two (renormalization) operators
\[ H_\Delta : (a, b) \in \mathcal{P}_3 \rightarrow H_\Delta(a, b) = \left( a^4, \frac{-2 + b + ab}{a^2(1 + a - ab)} \right) \in \mathcal{P} \]

and

\[ H_{\Pi} : (a, b) \in \mathcal{P}_3 \rightarrow H_{\Pi}(a, b) = \left( a^4, \frac{-2 + b + ab}{a(1 + a - ab)} \right) \in \mathcal{P} \]

such that \( \Psi_{a,b}^4 \) restricted to \( \tilde{\Delta}_{a,b} \) (respectively to \( \Pi_{a,b} \)) is conjugate by means of an affine change in coordinates to \( \Psi_{H_\Delta(a,b)} \) (respectively to \( \Psi_{H_{\Pi}(a,b)} \)).

The main properties of the renormalization operators \( H_\Delta \) and \( H_{\Pi} \) have been described in Proposition 6 and Proposition 7 in \([5]\), respectively. In particular, as was done in Section 3.4 in \([5]\), if we define the set of parameters

\[ \mathcal{P} = \{(a, b) \in [1, 2] \times [1, 2) : ab < 2\} \]

then both \( H_\Delta \) and \( H_{\Pi} \) are expanding diffeomorphisms on any small enough neighbourhood of \( \mathcal{P} \) and they have a unique fixed point \( P^* = (1, \sqrt{2}) \). This fixed point is a global repeller for \( H_\Delta \) and \( H_{\Pi} \) and furthermore, see Remark 3.6 and Remark 3.7 in \([5]\), it also follows that \( \mathcal{P}_3 \subset H_\Delta(\mathcal{P}_3) \subset \mathcal{P} \) and \( \mathcal{P}_3 \subset H_{\Pi}(\mathcal{P}_3) \subset \mathcal{P} \), see also Figure 3.

Hence, there exist chains

\[ A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \] \hspace{1cm} (12)

and

\[ A'_0 \supseteq A'_1 \supseteq A'_2 \supseteq \cdots \supseteq A'_n \supseteq \cdots \] \hspace{1cm} (13)

of subsets of \( \mathcal{P} \) such that

1. \( A_0 = H_\Delta(\mathcal{P}_3), A'_0 = H_{\Pi}(\mathcal{P}_3), A_1 = A'_1 = \mathcal{P}_3 \).
2. \( H_\Delta(A_n) = A_{n-1} \) and \( H_{\Pi}(A'_n) = A'_{n-1} \) for every \( n \in \mathbb{N} \).
3. \( \bigcap_{n \in \mathbb{N}} \text{closure}(A_n) = \bigcap_{n \in \mathbb{N}} \text{closure}(A'_n) = \{P^*\} \).

This means that, according to Definition 7.3, the map \( \Psi_{a,b} \) is \( n \)-times renormalizable (using \( n \)-times the operator \( H_\Delta \)) whenever the parameter \((a, b)\) belongs to \( A_n \) and \( \Psi_{a,b} \) is \( n \)-times renormalizable (using \( n \)-times the operator \( H_{\Pi} \)) if \((a, b)\) belongs to \( A'_n \).

Figure 3. (a) Filled in black, the set \( \mathcal{P}_3 \); encircled in a dashed black line, the set \( H_\Delta(\mathcal{P}_3) \). (b) Filled in black, the set \( \mathcal{P}_3 \); encircled in a dashed black line, the set \( H_{\Pi}(\mathcal{P}_3) \).
For every \( n \in \mathbb{N} \), we denote by
\[
P^n = A_n \cap A'_n. \tag{14}
\]
Then \( \Psi_{a,b} \) is \( n \)-times renormalizable in \( 2^n \) restrictive domains with pairwise disjoint interiors whenever \( (a, b) \in P^n \). Let us denote these restrictive domains by \( \{U_{n,j}\}_{j=1}^{2^n} \) and observe that, for every \( j = 1, \ldots, 2^n \), it holds that \( \Psi_{a,b}^j(\bigcup_{j=1} U_{n,j}) \subset U_{n,j} \).

Thus, if we define, for every \( j = 1, \ldots, 2^n \), the sets
\[
V_{n,j} = \bigcup_{i=0}^{4^n-1} \Psi_{a,b}^i(\bigcup_{j=1} U_{n,j})
\]
it follows that \( \{V_{n,j}\}_{j=1}^{2^n} \) is a family of \( 2^n \) compact \( \Psi \)-invariant sets with pairwise disjoint interiors. Therefore Corollary 1 easily follows from the fifth statement of Theorem 1.2.

**Remark 4.** Let us note that we are giving a large set of examples of 2-D strange attractors with several pieces. In particular, recovering the notation of the second statement of Proposition 1, attractors for which \( p = 4^n \), with \( n \) any arbitrarily large natural number.

7.1. Closing a conjecture. As we said in the Introduction, a one-parameter family \( \{\Lambda_t\}_{t \in [0, 1]} \) of EBM.s was originally introduced in [2]. These maps are defined in \( \mathcal{T} \), see (3), by
\[
\Lambda_t(x, y) = \begin{cases} \left(t(x + y), t(x - y)\right) & \text{if } (x, y) \in \mathcal{T}_0 \\ \left(t(2 - x + y), t(2 - x - y)\right) & \text{if } (x, y) \in \mathcal{T}_1 \end{cases}, \tag{15}
\]
where \( \mathcal{T}_0 \) and \( \mathcal{T}_1 \) were defined in (5). Hence, one may see that, for every \( t \in (1/\sqrt{2}, 1/\sqrt{4}) \), \( \Lambda_t^5 = \Psi_{a(t), b(t)} \in \mathcal{F} \) by taking \( a(t) = 16t^5 \) and \( b(t) = \frac{1}{2t} \), see Theorem 4.2 and Remark 5 in [4]. Let us observe that \( (a(t), b(t)) \) converges to the fixed point \( P^* \) of the renormalization operators \( H_\Lambda \) and \( H_{\Pi_1} \), whenever \( t \to \frac{1}{\sqrt{2}} \). This last claim, in particular, means that the curve of parameters \( \{(a(t), b(t)) : t \in (1/\sqrt{2}, 1/\sqrt{4})\} \) intersects all the sets \( A_n \) and \( A'_n \) constructed in (12) and (13). Therefore it also holds that \( \{(a(t), b(t)) : t \in (1/\sqrt{2}, 1/\sqrt{4})\} \) intersects all the sets \( P^n \), see (14), for which Corollary 1 holds.

In [4], three conjectures were established for the family of maps \( \{\Lambda_t\}_{t \in [0, 1]} \).

**Conjecture 1.** There exists a decreasing sequence \( \{t_n\}_{n \in \mathbb{N}} \), convergent to \( \frac{1}{\sqrt{2}} \) such that \( \Lambda_{t_n} \) is a \( n \)-times renormalizable EBM for every \( t \in (1/\sqrt{2}, t_n) \).

**Conjecture 2.** There is no value of \( t \) for which \( \Lambda_t \) is infinitely many renormalizable.

**Conjecture 3.** For each natural number \( n \) there exists an interval \( I_n \subset (1/\sqrt{2}, t_n) \) such that \( \Lambda_t \) displays, at least, \( 2^n-1 \) different strange attractors.

The first two were proven in [5]. Moreover, see Theorem B in [5], it was also demonstrated that there exists an interval \( I_n \) of parameters \( t \) such that \( \Lambda_t \) displays at least \( 2^n \) different attractors, whenever \( t \in I_n \). However, we did not know if, for every \( t \in I_n \), the parameter \( (a(t), b(t)) \) belongs to the region in which \( \Psi_{a,b} \) is a \( n \)-times renormalizable EBM. Now, we may prove even something stronger: Recalling the subsets \( P^n \) for which Corollary 1 holds and redefining the set of parameters \( I_n \) by the condition \( t \in I_n \) if and only if \( (a(t), b(t)) \in P^n \), we may conclude that, for every \( t \in I_n \), \( \Lambda_t^5 \) exhibits at least \( 2^n \) different 2-D strange attractors placed in
certain restrictive domain $\Delta_0 \subset \mathcal{T}$, see Lemma 4.1 in [4]. This fact is the key to claim that, if $t \in I_n$, then $\Lambda_t$ also has, at least, $2^n$ different 2-D strange attractors and therefore Conjecture 3 is proved.

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