GRAVITOMAGNETIC RESONANCE SHIFT DUE TO A SLOWLY ROTATING COMPACT STAR

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Abstract

The effect of a slowly rotating mass on a forced harmonic oscillator with two degrees of freedom is studied in the weak field approximation. It is found that according to the general theory of relativity there is a shift in the resonant frequency of the oscillator which depends on the density and rotational frequency of the gravitational source. The proposed shift is quite small under normal physical situations however it is estimated that for compact x-ray sources such as white dwarfs, pulsars, and neutron stars the shift is quite appreciable.

1 Introduction

The general theory of relativity is logically one of the most compelling theories of modern physical science. However being a physical theory it is imperative that it should not only describe as many different physical phenomenon as possible but also make accurate predictions of physical quantities relevant to observations. As regards the accuracy in prediction the general theory of relativity has done extremely well in all its known tests such as the perihelion shift of mercury, deflection of light, and gravitational redshift experiments1,2. However the range of applicability where general theory of relativity can be compared with direct observations has remained small, owing to the fact that under normal physical situations on earth and inside the solar system it is very difficult to differentiate its predictions from other rival theories such as the Newton’s theory of gravitation, even when it is considered under its linear approximation. The discovery of compact x-ray sources in the universe has opened up

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new avenues to the test general relativity in naturally existing systems. In these astrophysical laboratories extreme physical conditions are encountered which are impossible to be reproduced on earth. For example densities of a usual compact x-ray source is of the order up to $10^{28}$ kg/m$^3$ and rotational periods $10^{10}$ Hz. Under such conditions general relativity provides not only a consistent picture of various physical aspects of the phenomenon but also predicts, with unprecedented accuracy, magnitudes of various physical parameters involved such as the timing of rotational periods and their decay with time. On the other hand all such systems are composed of atoms existing under a very strong gravitational field. Within the classical theory these atoms can be taken as forced harmonic oscillators whose physical behavior depends very much on the background gravitational field.

In this paper we study the effects of gravitational field on the behavior of such oscillators within the framework of the general theory of relativity. After giving a preliminary introduction, we write the equations of motion for a forced harmonic oscillator in section 2, making a plausible physical assumption that the star is rotating slowly (i.e., far below the speed of light limit). In section 3 we discuss the solution to the equations of motion and especially their behavior near the resonant frequency. In section 4, we give the behavior of the oscillators for compact astrophysical objects and also give a comparison. Lastly a brief summary and conclusion is given in section 5.

### 2 The Equations of Motion

#### 2.1 Preliminaries

According to the general theory of relativity the motion of a free particle of gravitational mass $m_g$ is given by the geodesic equation

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0; \quad \alpha, \beta, \gamma = 0, 1, 2, 3$$

(1)

where $x^\alpha \equiv (x^i, x^0) = (r, t)$ are the position coordinates of the test particle and $i = 1, 2, 3$. Here $s$ is the affine parameter and $\Gamma^\alpha_{\beta\gamma} = g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})/2$ is the Christoffel symbol, where $g^{\alpha\beta}$ is the metric tensor and $'_{,\alpha}$ denotes differentiation with respect to $x^\alpha$.

Let the metric tensor be expressed as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

(2)

where $\eta_{\alpha\beta} = \text{diag}(-1,-1,-1,1)$ is the Minkowski metric tensor and $h_{\alpha\beta}$ is the perturbation to the metric. Then under the linear approximation $h_{\alpha\beta} \ll 1$ we can take time $t$ as the affine parameter. Requiring rotation to be slow (i.e., neglecting square and higher powers of velocity vector $v \equiv dr/dt$) we obtain the following expression for acceleration for a slowly rotating sphere of homogeneous mass density
\[
\frac{d^2 \mathbf{r}}{dt^2} = \mathbf{G} + \mathbf{v} \times \mathbf{H}
\]  

(3)

where

\[ \mathbf{G} = -\nabla \varphi, \quad \mathbf{H} = \nabla \times \mathbf{a} \]  

(4)

and

\[
\varphi = -\iiint \frac{\rho}{r} dV, \quad \mathbf{a} = \iiint \frac{\rho \mathbf{v}}{r} dV
\]  

(5)

\(\rho\) being the mass density and \(V\) is the volume. Here the remarkable formal analogy of the above results with the classical electromagnetic theory must be noticed with the proviso that a factor of 4 is to be multiplied with the gravitomagnetic vector potential \(\mathbf{a}\). Pursuing this analogy, we find\(^7\) that for a slowly rotating sphere of homogeneous mass density first term in expression (3) is the gravitoelectric (GE) potential given by the Newtonian gravitational potential \(-M\hat{r}/r\) (where \(M\) is the mass of the gravitating source, and \(G = 1 = c\) unless mentioned otherwise). Moreover there is the gravitomagnetic (GM) potential \(\mathbf{v} \times \mathbf{H}\) where

\[
\mathbf{H} = -\frac{12}{5} MR^2 (\Omega r \frac{\mathbf{r}}{r^5} - \frac{1}{3} \frac{\Omega}{r^3})
\]  

(6)

\(R\) being the radius and \(\Omega\) is the angular velocity of the gravitational source. The second part of equation (3) has a non-Newtonian origin. It is regarded to exhibit typically Einsteinian effects and therefore plays an important role in testing Einstein’s theory of gravitation in the weak field and slow rotation approximation\(^8,9\). The independence of the GM potential from a particular frame and from a particular coordinate system used has been demonstrated thus making it physically significant\(^10\). This effect can be interpreted as ‘gravitomagnetic current’ induced in the vicinity of the gravitational source due to its rotation. For its analogy with the Lorentz force law for the electromagnetic field the force \(m_g(\mathbf{G} + \mathbf{v} \times \mathbf{H})\) is often referred to as gravitoelectromagnetic (GEM) force. However to measure the effects of this force on a given physical system high accuracy in experiments is required where it is necessary to isolate the experimental setup from contingencies\(^11,12\).

### 2.2 Forced Harmonic Oscillations in Weak Gravitational Field and Slow Rotation Approximation

Let us consider a forced damped harmonic oscillator having two degrees of freedom placed in a GEM field. If \(m_I\) is the inertial mass of the test particle then in addition to the binding force \(-m_I \omega_0^2 \mathbf{r}\), the damping force \(-m_I \gamma \mathbf{v}\), and an external force \(F_0 \exp i\omega t\) acting along, say, the \(x\) direction, there also acts on the particle a GEM force \(m_g(\mathbf{G} + \mathbf{v} \times \mathbf{H})\). Considering the GE field of magnitude \(-g\) to be along \(y\)-axis and the GM field of magnitude \(H\) in the direction normal
to the xy-plane, we find that the GEM force has a component \( m_y H v_y \) in the x direction and a component \(-m_x H v_x \) in the y direction; where \( v_x \) and \( v_y \) are the components of velocity in the x and y directions respectively. With these additional terms and the fact that \( m_y = m_1 (= m) \), in accordance with the general theory of relativity, we get the equations of motion as a coupled system of ordinary differential equations

\[
\frac{d^2 x}{dt^2} = -\omega_0^2 x - \gamma \frac{dx}{dt} + \frac{F_0}{m} \exp i\omega t + H \frac{dy}{dt},
\]

\[
\frac{d^2 y}{dt^2} = -\omega_0^2 y - \gamma \frac{dy}{dt} - g - H \frac{dx}{dt}.
\]

Now let us introduce the auxiliary variable \( \Re = x + iy \), then the equations of motion give a single equation

\[
\frac{d^2 \Re}{dt^2} + (\gamma + iH) \frac{d\Re}{dt} + \omega_0^2 \Re = \frac{F_0}{m} \exp i\omega t - ig
\]

Notice that the GM parameter \( H \) has a coupling to the damping parameter \( \gamma \) in the complex plane \((x, y)\). Dimensional analysis of equations (7) and (8) shows that \( H \) must have dimensions of cycles per second i.e., of frequency. These considerations give rise to the question whether these 'gravitomagnetic oscillations' produce significant effects on the physical behavior of an oscillatory system. As we shall see in the following that the answer to this question must be given in affirmative. However these effects are extremely small except in those cases where density and rotational frequency of the gravitational source are extremely large, such as those found in compact astrophysical x-ray sources.

### 3 Solution to the Equations of Motion and Interpretation

The exact solution to equation (9) can be obtained, however we are interested in studying the behavior of amplitude especially near the resonant frequency. To do so we express \( \Re \) as \( A \exp i\omega t \) where \( A \) is the complex amplitude. Substitution in equation (9) gives

\[
A(\omega) = \frac{E_0}{\omega_0^2 - \omega^2 - \omega H + i\omega \gamma}
\]

Multiplying the last expression with its complex conjugate and taking square root we have the following real expression for the amplitude

\[
|A(\omega)| = \sqrt{(F_0/m)^2 + g^2 - 2(F_0/m)g \sin \omega t \over (\omega_0^2 - \omega^2 - \omega H)^2 + \omega^2 \gamma^2}
\]

Comparing this expression for the amplitude with the expression for the amplitude of a forced damped harmonic oscillator having two degrees of freedom
we notice that there are two main differences, one the addition of \(g\) dependent terms in the numerator, two an addition of the term \(-\omega H\), in the denominator, to the square of eigenfrequency \(\omega_0\) of the system. The second difference is of particular importance since it involves the effect of GM field on the resonant frequency of the system. To estimate this effect we notice that in equation (11) for small damping i.e., \(\gamma \ll \omega\) and given values of the parameters \(F_0/m\), \(g\), and \(\omega_0\), the amplitude \(A\) is maximum when \(\omega_0^2 - \omega^2 - \omega H \approx 0\) i.e., when the applied frequency is approximately \((-H \pm \sqrt{H^2 + 4\omega_0^2})/2\). Expanding in powers of \((H/2\omega_0)^2\) and requiring that \(H \ll \omega_0\) we obtain the following approximate result for resonant frequency

\[
\omega \approx \omega_0 - \frac{H}{2}, \quad \gamma \ll 1, \quad H \ll \omega_0
\]

This result shows that there is a shift in the resonant frequency by an amount \(H/2\).

4 Application to Compact Stars

The outer atmosphere of a white dwarf consists mainly of fully ionized \(^4\)He, \(^{12}\)C, and \(^{16}\)O atoms forming a crystalline structure in which electrons can be taken as forced harmonic oscillators, moving under the influence of lattice vibrations. Along the radial direction the motion of these electrons is very much limited by the GE attraction and the Fermi pressure, therefore in any thin layer enveloping the star the electrons can be taken as oscillating with two degrees of freedom. As the star of mass greater than the Chandrasekhar limit exhausts its thermonuclear fuel the GE force not only contracts the core (increasing the mass density per unit area), its angular frequency also is increased from 1 Hz (= 1 cycle per second) to 1 MHz or above. To study the resonant behavior of these electrons we begin with relatively less severe case of white dwarf of mass \(1M_\odot = 1.989 \times 10^{30} \text{kg}\) and radius \(7 \times 10^6 \text{m}\) rotating at frequency 1 Hz. Then the case of the pulsar of mass \(1.4M_\odot\) and radius \(3 \times 10^7 \text{m}\) rotating at frequency 30 Hz is studied. Finally we consider a neutron star of mass \(2M_\odot\) and radius \(1 \times 10^4 \text{m}\) rotating at frequency 1 kHz. In these cases \(g\) (= \(GM/R^2\), where \(G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2\)) ranges from 2.7075 \times 10^6 \text{m/ sec}^2 (for the white dwarf) to 2.0636 \times 10^{11} \text{m/ sec}^2 (for the pulsar) to 2.6533 \times 10^{12} \text{m/ sec}^2 (for the neutron star); consequently \((F_0/m)^2 + g^2 + 2(F_0/m)g \sin \omega t\) varies from order \(10^{12} \text{m/ sec}^2\) to \(10^{24} \text{m/ sec}^2\) where \(\theta = \omega t\) is the phase. Now from equation (6) we have in the equatorial plane at the surface of the star i.e., \(r = R\):

\[
H \equiv |\mathbf{H}| = \mu \frac{M}{R} |\Omega|
\]

where \(\mu = 4G/5c^2 \equiv 0.5940 \times 10^{-27} \text{m/kg}\). For the above given values of the relevant physical parameters we have from expression (13), \(H = 0.1687 \times 10^{-3} \text{Hz}\), \(1.6540 \text{Hz}\), and \(236.2932 \text{Hz}\) for the white dwarf, the pulsar and the neutron star respectively. With these values of parameters we plot the resonance curves,
using expression (11), for the three cases in figure (1) and (2) for electron with
eigenfrequency $1 \text{kHz}$, $\theta = \theta_1 = 0$, and $\gamma \approx 0.1 \text{Hz}$.

We notice that for a slow rotating pulsar the resonance shift $\Delta \omega = \omega - \omega_0 = H/2$ is not very large (about $0.8269 \text{Hz}$) as compared with the shift for the case
of the neutron star (about $117.3196 \text{Hz}$).

5 Summary and Conclusion

The above considerations can be summed up as follows: under the influence of
gravitomagnetic field of a rotating compact star there is a shift in the resonant
frequency of an oscillator lying in its vicinity. The shift depends on the density
and on the rotational frequency of the star.

Inside the solar system and especially in terrestrial laboratories the detection
of gravitomagnetic resonance shift is very unlikely (for Earth $H$ is approximately
$0.6440 \times 10^{-14} \text{Hz}$ and for the sun it is $0.6597 \times 10^{-13} \text{Hz}$). However for compact
gravitational objects the shift lies in the domain of observations. These observa-
tions may not be directly possible at present, but with further theoretical
as well as observational developments, it may well become possible to provide
evidence for such an effect.

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### 5.1 Figure Captions

Figure 1: Amplitude plots for an oscillating test particle exhibiting the resonance shift for a white dwarf (mass $1M_\odot$, radius $7 \times 10^6 m$, rotational frequency $1Hz$) and a pulsar (mass $1.4M_\odot$, radius $3 \times 10^4 m$, rotational frequency $30Hz$) as the gravitational source. The frequency $\omega$ is measured in $Hz$ and the amplitude is in units $m/Hz/sec^2$.

Figure 2: Amplitude plots for an oscillating test particle exhibiting the resonance shift for a pulsar (mass $1.4M_\odot$, radius $3 \times 10^4 m$, rotational frequency $30Hz$) and a neutron star (mass $2M_\odot$, radius $10^4 m$, rotational frequency $1kHz$) as the gravitational source. The frequency $\omega$ is measured in $Hz$ and the amplitude is in units $m/Hz/sec^2$. 
This figure "BMMFigure1.GIF.gif" is available in "gif" format from:

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