**EVEN SYMMETRY OF SOME ENTIRE SOLUTIONS TO THE ALLEN-CAHN EQUATION IN TWO DIMENSIONS**

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**Abstract.** In this paper, we prove even symmetry and monotonicity of certain solutions of Allen-Cahn equation in a half plane. We also show that entire solutions with finite Morse index and four ends must be evenly symmetric with respect to two orthogonal axes. A classification scheme of general entire solutions with finite Morse index is also presented using energy quantization.

**Keywords:** Allen-Cahn equation, Hamiltonian identity, Level Set, Saddle solutions, Even symmetry, Monotonicity, Morse index.

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**1. Introduction**

We shall consider entire solutions of the following Allen-Cahn equation

\[ u_{xx} + u_{yy} - F'(u) = 0, \quad |u| \leq 1, \quad (x, y) \in \mathbb{R}^2, \]

where \( F \) is a balanced double-well potential, i.e., \( F \in C^2([-1, 1]) \) satisfies \( F(1) = F(-1) = 0 \) and

\[
\begin{aligned}
F'(-1) &= F'(1) = 0, \\
F''(-1) &> 0, \\
F''(1) &> 0;
\end{aligned}
\]

for some \( t_0 \in (0, 1) \). Without loss of generality, we may assume that \( t_0 = 0 \). A typical example of balanced double well potential is \( F(u) = \frac{1}{4}(1-u^2)^2, \quad u \in \mathbb{R} \).

It is well-known that there exists a unique transition layer solution \( g(y) \) (up to translation) to the one dimensional Allen-Cahn equation

\[
\begin{aligned}
g''(s) - F'(g(s)) = 0, \\
\lim_{s \to -\infty} g(s) = 1, \\
\lim_{s \to -\infty} g(s) = -1.
\end{aligned}
\]

We may assume that \( g(0) = 0 \). Indeed, \( g \) is a minimizer of the following energy functional

\[
E(v) := \int_{-\infty}^{\infty} \left( \frac{1}{2} |v'|^2 + F(v) \right) dx
\]

in \( \mathcal{H} := \{ \nu \in H^1_{\text{loc}}(\mathbb{R}) : -1 \leq \nu \leq 1, \lim_{s \to \pm \infty} \nu(s) = \pm 1 \} \) and

\[
e := E(g) = \int_{-1}^{1} \sqrt{2F(u)} du < \infty.
\]

The solution \( g \) is non-degenerate in the sense that the linearized operator has a kernel spanned only by \( g' \).
If \( u \) is an evenly symmetric solution in \( x \), we may regard \( u \) as a solution in the half plane \( \mathbb{R}^2_+ := \{ (x,y) | x \geq 0, \ y \in \mathbb{R} \} \),

\[
\begin{aligned}
&u_{xx} + u_{yy} - F'(u) = 0, \quad |u| < 1, \quad (x,y) \in \mathbb{R}^2_+ \\
&u_x(0,y) = 0, \quad y \in \mathbb{R}.
\end{aligned}
\]

(1.4)

We may also assume that \( u \) satisfies the monotone condition

\[
u_x(x,y) > 0, \quad x > 0, \quad y \in \mathbb{R}.
\]

(1.5)

Our main theorem states that \( u \) must be evenly symmetric with respect to \( y \) and monotone for \( y > 0 \).

**Theorem 1.1.** Assume that \( u(x,y) \) is even in \( x \) and satisfies (1.4) and (1.5). Then \( u \) is even in \( y \), i.e.,

\[
u(x,y) = u(x,-y), \quad (x,y) \in \mathbb{R}^2_+
\]

after a proper translation in \( y \). Moreover, \( u_y(x,y) < 0 \) for \( x > 0, y > 0 \), and the 0-level set of \( u \) for \( x > x_0 \) can be expressed as the graph of two \( C^{3,\beta} \) functions \( y = \pm k(x) \) which is asymptotically linear, i.e. \( k(x) = \kappa x + C + o(1) \) for some constants \( \kappa > 0, C \), as \( x \) goes to infinity.

In particular, we have

\[
limit_{x \to \infty} u(x,y) = 1, \quad \forall y \in \mathbb{R}.
\]

(1.7)

This symmetry result may be regarded as the counterpart of De Giorgi conjecture for a half plane.

We shall prove Theorem 1.1 in three main steps. First, we carry out a preliminary analysis of the 0-level set \( \Gamma \) of \( u \) and show that \( \Gamma \) can be regarded as graphs of two \( C^{3,\beta} \) functions \( y = \pm k(x) \) which is asymptotically linear, i.e. \( k(x) = \kappa x + C + o(1) \) for some constants \( \kappa > 0, C \), as \( x \) goes to infinity.

We shall also discuss the even symmetry of entire solutions whose asymptotically behavior at infinity are roughly prescribed. For example, we can show that an entire solution with **finite Morse index** and **four ends** must be evenly symmetric in both \( x \) and \( y \), after a proper translation and rotation. For a finite integer \( m \geq 0 \), we say that a solution \( u \) defined in \( \Omega \subset \mathbb{R}^n \) has **finite Morse index** \( m \) if \( m \) is the maximal dimension of any linear subspace of Sobolev space \( H^1(\Omega) \) contained in

\[
\mathcal{N} := \{ \phi \in H^1(\Omega) : \int_{\Omega} |\nabla \phi|^2 + F''(u)\phi^2dV < 0 \} \cap \{0\}.
\]

(1.8)

If \( m = 0 \), \( u \) is also called a **stable** solution in \( \Omega \). If an entire solution \( u \) has **finite Morse index**, then \( u \) must be **stable** outside a large enough ball \( B_{R_0} \) (see [6] and [7]).

An entire solution \( u \) is called a solution with **2k ends** for some positive integer \( k \) if the 0-level set \( \Gamma \) of \( u \) outside a large disc \( B_R(0) \) consists of \( 2k \) imbedded \( C^1 \) curves \( \Gamma_i := \{ (r(t), \theta(t)) : \forall t \geq 0 \} \) \( 1 \leq i \leq 2k \) in polar coordinates, and \( r_i(t) \to \infty \) as \( t \to \infty \),

\[
\Gamma_i \subset \{ (r, \theta) : r \geq R, \ \theta^-_i < \theta < \theta^+_i, \quad 1 \leq i \leq 2k \}
\]

where \( 0 \leq \theta^-_i < \theta^+_i < \theta^-_{i+1} < \theta^+_{i+1} < 2\pi, \quad 1 \leq i \leq 2k - 1 \).

We have the following symmetry result for entire solutions with **four ends**.
**Theorem 1.2.** Suppose that $u$ is an entire solution to (1.1) with finite Morse index and four ends. Assume also

\begin{equation}
0 < \theta^+_i - \theta^-_i < \pi, \quad 1 \leq i \leq 4.
\end{equation}

Then, after a proper translation and rotation, $u$ satisfies

\begin{equation}
(1.10) \quad u(x, y) = u(x, -y) = u(-x, y), \quad \forall (x, y) \in \mathbb{R}^2
\end{equation}

and

\begin{equation}
(1.11) \quad u_x(x, y) > 0, \quad u_y(x, y) < 0, \quad \forall x > 0, \quad y > 0
\end{equation}

and (1.7) holds. Moreover, there exists an angle $\Theta = 2\theta \in (0, \pi)$ such that the 0-level set of $u$ in the first quadrant is a graph of a $C^{3,\beta}$ function $y = k(x)$ for $x > X_0$ large enough, and

\begin{equation}
k(x) = x \tan \theta + o(1), \quad \text{as } x \to \infty.
\end{equation}

An entire solution $u$ with four ends may be called a *saddle* solution. The above theorem may be regarded as a form of De Giorgi conjecture for saddle solutions. The angle $\Theta$ may be called the *contact angle* of $u$ (see [11] for more discussion).

For a given $\Theta \in (0, \pi)$, the uniqueness of four ends entire solutions with contact angle $\Theta$ is still unknown. It is stated in [8] that the formal dimension of the moduli space of entire solutions with $2k$ ends is $2k$. For $k = 2$, it means that formally there is local uniqueness of saddle solutions with a fixed contact angle, up to a translation and rotation. However, the global uniqueness is a very different and more difficult question.

The condition (1.9) is a technical condition and is believed to be unnecessary. However, we need it for the proof of an energy bound in Lemma 5.1 for a functional

\begin{equation}
E_R(u) := \int_{B_R} \frac{1}{2} |\nabla u|^2 + F(u)\, dx dy.
\end{equation}

If we assume the energy bound Lemma 5.1 directly instead of (1.9), the conclusion of Theorem 1.2 still holds. Indeed, we have the following general energy quantization result. Note that a different energy quantization phenomenon has been shown for Ginzburg-Landau equation (see [3]).

**Theorem 1.3.** Assume that $u$ is an entire solution of (1.1) with finite Morse index. Then there holds either

\begin{equation}
(1.13) \quad \lim_{R \to \infty} \frac{E_R(u)}{R} = \infty,
\end{equation}

or

\begin{equation}
(1.14) \quad \lim_{R \to \infty} \frac{E_R(u)}{R} = 2k e
\end{equation}

for some positive integer $k$.

In the latter case, $u$ must be an entire solution with $2k$ ends, and the 0-level set of $u$ must be asymptotically straight lines. Moreover, if we denote the directions of these lines by $\nu_i = \langle \cos \theta_i, \sin \theta_i \rangle$, $1 \leq i \leq 2k$, then

\begin{equation}
\sum_{i=1}^{2k} \nu_i = \langle 0, 0 \rangle.
\end{equation}
It is suspected that the first case in Theorem 1.3 may not happen at all. It would be interesting to show that only (1.14) holds and for a given configuration \( \nu_i, 1 \leq i \leq k \) there exist only two corresponding solutions with opposite signs after a proper translation. All entire solutions with finite Morse index could then be classified accordingly.

We note that the existence of entire solutions with finite Morse index and \( 2k \) ends has been shown in [8]. It was also pointed out in [8] that there may not be any symmetry for entire solutions with six or more ends. Note also that (1.15) implies (1.9) for \( k = 2 \).

The paper is organized as follows. In Section 2, some preliminary results for entire solutions of Allen-Cahn equation in all dimensions shall be stated. In Section 3, we will prove Theorem 1.1. In Section 4, a simpler version of Theorem 1.2 shall be proven. Theorem 1.2 and the energy quantization property will be proven in Section 5.

2. Some Basic Properties

In this section we shall state some useful properties of entire solutions to the Allen-Cahn equation.

We first state a gradient estimate (1.1) for all dimensions which was proven in [14].

**Proposition 2.1.** Assume that \( F(s) \geq 0, \forall s \in [-1, 1] \). Suppose that \( u \) is a solution to (1.1). Then

\[
|\nabla u|^2(x, y) \leq 2F(u(x, y)), \quad (x, y) \in \mathbb{R}^n.
\]  

It is also well-known that \( u \) has the following exponential decay with respect to distance from the level set.

**Proposition 2.2.** Assume that \( u \) is a solution to (1.1). Then there exist constants \( C \) and \( \nu > 0 \) such that

\[
|u^2 - 1| + |\nabla u| + |\nabla^2 u| \leq Ce^{-\nu d(x, y)}
\]

where \( d(x, y) \) is the distance to the 0-level set \( \Gamma \) of \( u \).

This property can be proven by comparing \( u \) with a solution \( u_R > 0 \) of the Allen-Cahn equation in a ball \( B_R \) centered at \( (x, y) \) with zero boundary condition, where \( R = d(x, y) \). (See, e.g., [9].)

The following monotonicity property of energy is shown in [15].

**Proposition 2.3.** Assume that \( u \) is a solution to (1.1). Then \( E_R(u)/R \) is increasing in \( R \).

3. Even Symmetry of Solutions in a Half Plane

We now consider an entire solution \( u \) which is even in \( x \). Note that \( u \) may be regarded as a solution of (1.4) in a half plane.

We first study the limit of \( u(\tau + x, y) \) as \( \tau \) goes to infinity. Define

\[
u^\tau(x, y) := u(\tau + x, y), \quad x \geq -\tau, \; \forall y \in \mathbb{R}.
\]
It is easy to see that \( u^\tau(x, y) \) converges to some function \( u^+(y) > -1 \) in \( C^3_{\text{loc}}(\mathbb{R}^2) \) as \( \tau \) goes to infinity, and \( u^+(y) \) satisfies one dimensional Allen-Cahn equation

\[
(3.1) \quad u_{yy} - F'(u) = 0, \quad y \in \mathbb{R}.
\]

Let

\[
\sigma^\tau(x, y) = \frac{u^\tau_x(x, y)}{u^\tau_x(0, 0)} > 0, \quad \forall x \geq -\tau, \ y \in \mathbb{R}.
\]

By the Harnack inequality and the gradient estimate for elliptic equations, we know that \( \sigma^\tau(x, y) \) converges to \( \sigma^*(x, y) > 0 \) in \( C^2_{\text{loc}}(\mathbb{R}^2) \) as \( \tau \) goes to infinity, and \( \sigma^*(x, y) \) satisfies the linearized equation of Allen-Cahn equation

\[
(3.2) \quad \sigma_{xx} + \sigma_{yy} - F''(u^+(y))\sigma = 0, \quad (x, y) \in \mathbb{R}^2.
\]

Hence \( u^+ \) is a stable solution of (3.1). Then there are three possibilities for \( u^+ \):

(i) \( u^+ \equiv 1 \);
(ii) \( u^+(y) = g(y - K) \) for some constant \( K \);
(iii) \( u^+(y) = g(K - y) \) for some constant \( K \).

The goal is to show that only (i) holds. To do so, we shall prove several basic properties for \( u \). The first property is an energy estimate of \( u \) on a line.

3.1. **Energy estimate.** We first show a simple but important lemma regarding the energy of \( u \) on \( y \)-axis.

**Lemma 3.1.** Suppose that \( u \) is a solution to (1.4) and (1.5). Then

\[
(3.3) \quad \int_{\mathbb{R}} \left[ F(u(0, y)) + \frac{1}{2}u_x^2(0, y) \right] dy < 3e.
\]

**Proof.** Define

\[
h(y) = \int_{0}^{\infty} u_y u_x dx, \quad \forall y \in \mathbb{R}.
\]

In view of (2.1) and the positivity of \( u_x \), it is easy to see that \( h(y) \) is well-defined and

\[
|h(y)| < \int_{0}^{\infty} \sqrt{2F(u(x, y))} \cdot u_x dx \leq e - G(u(0, y)) < e, \quad \forall y \in \mathbb{R}
\]

where

\[
G(t) := \int_{-1}^{t} \sqrt{2F(s)} ds, \quad \forall t \in [-1, 1].
\]

Differentiating \( h(y) \) with respect to \( y \) and using (1.4), we obtain

\[
(3.4) \quad h'(y) = \int_{0}^{\infty} (u_y u_x + u_y u_{xy}) dx
\]

\[
= \int_{0}^{\infty} \frac{\partial}{\partial x} \left( F(u) - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 \right) dx
\]

\[
= [F(u^+(y)) + \frac{1}{2}(u^+_y)^2(y)] - [F(u(0, y)) + \frac{1}{2}u_y^2(0, y)].
\]
Here we have used the facts $u_x(0, y) = 0$ and $\lim_{x \to \infty} u_x(x, y) = 0, \forall y \in \mathbb{R}$. Then, we derive
\[
\int_a^b [F(u(0, y)) + \frac{1}{2} u_y^2(0, y)] dy \\
= \int_a^b [F(u^+(y)) + \frac{1}{2} (u_y^+)^2(y)] dy + (h(a) - h(b)).
\]

Define
\[
\rho(x) = \int_{\mathbb{R}} [F(u(x, y)) + \frac{1}{2} u_y^2(x, y) - \frac{1}{2} u_x^2(x, y)] dy
\]
and
\[
\rho^+ = \int_{\mathbb{R}} [F(u^+(y)) + \frac{1}{2} (u_y^+)^2(y)] dy.
\]

Then, letting $a \to -\infty$ and $b \to +\infty$ in (3.6), in view of the bound of $h(y)$ we obtain
\[
\rho(0) = \rho^+ + \lim_{a \to -\infty} h(a) - \lim_{b \to \infty} h(b) \leq 3e.
\]

Therefore, (3.3) is proven.

\[
\square
\]

3.2. A Hamiltonian identity. Next we shall show a Hamiltonian identity for solutions of (1.4).

Lemma 3.2. Assume that $u(x, y)$ satisfies (1.4) and (1.5). Then
\[
\rho(x) = \rho(0), \quad \forall x \in \mathbb{R}_+.
\]

Proof. By (3.3) and the boundedness of $u$ in $C^3(\mathbb{R}^n)$, we know that the following limits exist
\[
v^+ := \lim_{y \to \infty} u(0, y), \quad v^- := \lim_{y \to -\infty} u(0, y)
\]
and
\[
|v^+| = 1, \quad |v^-| = 1.
\]
Indeed, by the standard translation argument it can be shown that
\[
v^+(x, y) := \lim_{t \to \infty} u(x, y + t), \quad v^-(x, y) := \lim_{t \to -\infty} u(x, y + t)
\]
exist and are solutions to (1.4), and hence
\[
v^+(x, y) \equiv v^+, \quad v^-(x, y) \equiv v^-, \quad (x, y) \in \mathbb{R}^2.
\]

In particular,
\[
\lim_{|y| \to \infty} u_x(x, y) = 0, \quad \lim_{|y| \to \infty} u_y(x, y) = 0, \quad \forall x \geq 0.
\]

Define
\[
h(R, y) := \int_0^R u_y u_x dx, \quad \forall y \in \mathbb{R}.
\]

Then, in view of (3.11), we have
\[
\lim_{|y| \to \infty} h(R, y) = 0, \quad \forall R \geq 0.
\]
As before, differentiating \( h(R, y) \) with respect to \( y \) and using (1.4), we can obtain
\[
h'(R, y) = \int_0^R \left( u_{yy}u_x + u_yu_{xy} \right) dx
= \int_0^R \frac{\partial}{\partial x} \left( F(u) - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 \right) dx
= \left[ F(u(R, y)) + \frac{1}{2}u_y^2(R, y) - \frac{1}{2}u_x^2(R, y) \right] - \left[ F(u(0, y)) + \frac{1}{2}u_y^2(0, y) \right].
\]
Then, integrating the above with respect to \( y \) in \( \mathbb{R} \), we derive
\[
(3.12) \quad \rho(0) - \rho(R) = \lim_{a \to -\infty} h(R, a) - \lim_{b \to \infty} h(R, b) = 0.
\]
The lemma is proven.

We can indeed show the following limit.

**Lemma 3.3.**
\[
(3.13) \quad \lim_{|y| \to \infty} u(x, y) = -1, \quad \forall x \in \mathbb{R}.
\]

**Proof.** We shall show the lemma by considering different cases.

In Case (i), i.e., \( u^+ \equiv 1 \), there are four possibilities:

1. \( v^+ = 1, \quad v^- = 1; \)
2. \( v^+ = -1, \quad v^- = 1; \)
3. \( v^+ = 1, \quad v^- = -1; \)
4. \( v^+ = -1, \quad v^- = -1; \)

From (3.9) and the Hamiltonian identity (3.10) we have
\[
(3.14) \quad \lim_{a \to -\infty} h(a) - \lim_{b \to \infty} h(b) + \rho^+ = \rho(0) = \lim_{x \to \infty} \rho(x).
\]
In Subcase (1), we can estimate
\[
\lim_{a \to -\infty} |h(a)| \leq \lim_{a \to -\infty} [G(1) - G(u(0, a))] = 0
\]
and
\[
\lim_{b \to \infty} |h(b)| \leq \lim_{b \to \infty} [G(1) - G(u(0, b))] = 0.
\]
Then (3.9) becomes
\[
\rho(0) = \lim_{a \to -\infty} h(a) - \lim_{b \to \infty} h(b) = 0.
\]
This is a contradiction, and therefore Subcase (1) is excluded.

In Subcase (2), we can estimate
\[
\lim_{a \to -\infty} |h(a)| \leq \lim_{a \to -\infty} [G(1) - G(u(0, a))] \leq e
\]
and
\[
\lim_{b \to \infty} |h(b)| \leq \lim_{b \to \infty} [G(1) - G(u(0, b))] = 0.
\]
Then (3.9) becomes
\[
\rho(0) \leq e.
\]
On the other hand, by the definition of \( e \), we have \( \rho(0) \geq e \). Then we have \( u(0, y) = g(\pm y + K_1) \) for some \( K_1 \in \mathbb{R} \). Then \( u(x, y) - u(0, y) \) is nonnegative and satisfies a linearized equation of (1.1). By the Harnack inequality, we can derive
\( u(x, y) \equiv u(0, y) \). This contradicts with (1.5), and hence Subcase (2) is excluded. Subcase (3) is similar to Subcase (2). The lemma then follows easily from (3.11).

In Case (ii), i.e., \( u^+(y) = g(y - K) \), in view of the monotone condition (1.6) we know only Subcases (2) and (4) are possible. If Subcase (2) happens, then (3.9) becomes

\[
\rho(0) = e + \lim_{a \to -\infty} h(a) - \lim_{b \to \infty} h(b) = e.
\]

Since \( \rho(0) \ge e \), we get a contradiction immediately as in Case (i). Therefore Subcase (2) is excluded.

Case (iii) is similar to Case (ii). In all cases, we have proven that only Subcase (4) holds. Hence (3.13) is proven.

\[\Box\]

In the level set analysis below, we shall focus on Case (i): \( u^+ \equiv 1 \). The other two cases can be discussed similarly with minor modifications, and can be excluded eventually at the end of this section.

In view of (3.9) and (3.13), the 0-level set \( \Gamma \) of \( u \) can be represented by the graph of a function \( x = \gamma(y) \) which is defined for \( y \le K_1 \), and \( y \ge K_2 \) with \( K_1 \le K_2 \) and is \( C^3 \). By Lemma (3.3), we also know

\[
\lim_{|y| \to \infty} \gamma(y) = \infty.
\]

3.3. **The slope of the level set has a limit.** First we show the limits of \( \gamma'(y) \) exist as \( y \to \pm \infty \).

**Lemma 3.4.** There exist \( \theta_1 \in [0, \pi/2] \) and \( \theta_2 \in [-\pi/2, 0] \) such that

\[
\lim_{y \to \infty} \gamma'(y) = \tan \theta_1, \quad \lim_{y \to -\infty} \gamma'(y) = \tan \theta_2.
\]

Here we use the convention that \( \tan(\pi/2) = \infty \), \( \tan(-\pi/2) = -\infty \).

**Proof.** For any sequence \( \{y_m\} \) and constant \( \theta \in [-\pi/2, \pi/2] \) with \( |y_m| \to \infty \) and

\[
\lim_{m \to \infty} \gamma'(y_m) = \tan \theta,
\]

we define

\[
u^m(x, y) := u(x + \gamma(y_m), y + y_m), \quad x \ge -\gamma(y_m), \ y \in \mathbb{R}.
\]

Then \( \nu^m \) converges to \( u^* \) in \( C^3_{\text{loc}}(\mathbb{R}^2) \) after taking a subsequence if necessary, where \( u^* \) is a solution of (1.1) with \( \frac{\partial \nu^*}{\partial x}(x, y) \ge 0, \ (x, y) \in \mathbb{R}^2 \). By the Harnack inequality, we know that either \( \frac{\partial u^*}{\partial x}(x, y) \equiv 0 \), or \( \frac{\partial u^*}{\partial x}(x, y) > 0 \), \ (x, y) \in \mathbb{R}^2 \). In the first case, we define

\[
\sigma^m(x, y) = \frac{\nu^m(x, y)}{\nu^m(x, 0, 0)} > 0, \quad \forall x \ge -\gamma(y_m), \ y \in \mathbb{R}.
\]

By the Harnack inequality and the gradient estimate for elliptic equations, we know that \( \sigma^m(x, y) \) converges along a subsequence to \( \sigma^*(x, y) > 0 \) in \( C^2_{\text{loc}}(\mathbb{R}^2) \) as \( m \) goes to infinity. Furthermore, \( \sigma^*(x, y) \) satisfies the linearized equation of Allen-Cahn equation at \( u^* \)

\[
\sigma_{xx} + \sigma_{yy} - F''(u^*)\sigma = 0, \quad (x, y) \in \mathbb{R}^2.
\]

Hence \( u^* \) is stable in both cases. By the De Giorgi conjecture for \( n = 2 \) (9), we know that \( u^* \) depends only on one direction. Since \( u^*(0, 0) = 0 \), we conclude

\[
u^m(x, y) = g(x \cos \theta - y \sin \theta), \quad \forall (x, y) \in \mathbb{R}^2.
\]
Note that straightforward computations can lead to
\[
\rho^*(\theta) := \int_{\mathbb{R}} \left[ F(u^*(x, y)) + \frac{1}{2} (u_y^*)^2(x, y) - \frac{1}{2} (u_x^*)^2(x, y) \right] dy = e^{\sin \theta}.
\]
(See, e.g., [11].)

Next we shall show the first limit in (3.16).

Let
\[
\limsup_{y \to \infty} \gamma'(y) = \tan \theta_1
\]
for some \( \theta_1 \in [0, \pi/2] \).

If \( \liminf_{y \to \infty} \gamma'(y) = \tan \theta_0 < \tan \theta_1 \) for some \( \theta_0 \in [-\pi/2, \theta_1] \), then, for any fixed \( \theta \in (\theta_0, \theta_1) \) there exists a sequence \( \{y_m\} \) with \( \lim_{m \to \infty} \gamma'(y_m) = \tan \theta \) and \( y_m \to \infty \) as \( m \to \infty \).

For any fixed \( R > 0 \), by the monotone condition (1.5) and (3.18) we have
\[
\lim_{m \to \infty} h(y_m) = \lim_{m \to \infty} \int_{\gamma(y_m)+R}^{\gamma(y_m)-R} u_x u_y dx
+ O(1) \cdot \lim_{m \to \infty} \left[ G(1) - G(u(\gamma(y_m) + R, y_m)) \right]
+ O(1) \cdot \lim_{m \to \infty} \left[ G(u(\gamma(y_m) - R, y_m) - G(u(0, y_m)) \right]
= -\sin \theta \left[ G(g(R \cos \theta)) - G((g(-R \cos \theta)) + O(1)[G(g(-R))] \right]
\]
where \( G \) is defined in (3.4) and \( O(1) \) is with respect to \( R \to \infty \). Letting \( R \) go to infinity, we obtain
\[
\lim_{m \to \infty} h(y_m) = -e^{\sin \theta}.
\]
By (3.10), we know that \( \lim_{a \to \infty} h(a) \) exists and hence
\[
\lim_{y \to \infty} h(y) = -e^{\sin \theta}.
\]
This leads to
\[
\lim_{y \to \infty} \gamma'(y) = \tan \theta
\]
which contradicts (3.20). Therefore the first limit in (3.16) is proven.

Similarly, we can show the second limit in (3.16).

\[\Box\]

Furthermore, by (3.14) we have
\[
e^{(\sin \theta_1 - \sin \theta_2)} = \rho(0) = \lim_{R \to \infty} \rho(R)
\]

We note that in Case (ii), the above discussion can be modified with \( \theta_2 = -\pi/2 \) and \( y \to -\infty \) being replaced by \( y \to K \). Similar modifications can be done for Case (iii) with \( \theta_1 = \pi/2 \).

3.4. **The limits of slopes differ by a sign.** We shall show that the limits of the slopes of the level set differ only by a sign, i.e., \( \theta_1 = -\theta_2 \in (0, \pi/2) \).

**Lemma 3.5.** There holds
\[
\theta_1 = -\theta_2.
\]
Proof. Recall that $u$ is an even solution in $\mathbb{R}^2$ with respect to $x$.

Let us choose an angle $\theta \in (0, \pi/2)$, $\theta \neq \theta_1, -\theta_2$ and a Cartesian coordinate system $(z_1, z_2)$ such that $z_1$-axis and $y$ axis form an angle $\theta$. In other words, we have $x = z_1 \sin \theta + z_2 \cos \theta, y = z_1 \cos \theta - z_2 \sin \theta$. By (3.24), we know that

\begin{equation}
|u^2(z_1, z_2) - 1| + |\nabla u(z_1, z_2)| + |\nabla^2 u(z_1, z_2)| \leq C e^{-\nu_1 |z_1|}, \quad \forall z_1 \in \mathbb{R}
\end{equation}

for some positive constants $\nu_1 > 0$ and $C$.

Therefore, there holds a Hamiltonian identity like (3.10) with respect to $z$. Namely,

\begin{equation}
\rho(\theta, z_2) := \int_{\mathbb{R}} [F(u(z_1, z_2)) + \frac{1}{2} u^2_{z_1}(z_1, z_2) - \frac{1}{2} u^2_{z_2}(z_1, z_2)] dz_1 = \rho(\theta, 0) < \infty.
\end{equation}

(The proof is similar to (3.10); See also Theorem 1.1 in [11].) When $\theta > \theta_1, \theta > -\theta_2$, a straightforward computation can lead to

\begin{equation}
\begin{cases}
\lim_{z_2 \to \infty} \rho(\theta, z_2) = \rho(\theta, 0) = 0 \\
\lim_{z_2 \to -\infty} \rho(\theta, z_2) = 2 \rho(\theta, 0) = 2 \rho(\theta, \pi/2 - \theta).
\end{cases}
\end{equation}

Then we have

\[
\sin(\theta - \theta_1) + \sin(\theta + \theta_1) = \sin(\theta - \theta_2) + \sin(\theta + \theta_2)
\]

and hence $\theta_1 = -\theta_2$.

The same conclusion can be reached if $\theta$ is in other range compared to $\theta_1, -\theta_2$, with only slight difference in the expression in (3.26). The details is left to the reader. See also (2.12) in [11].

\hfill \Box

Since $\rho(0) > 0$, an easy consequence of Lemma 3.5 and (3.22) is $\theta_1 = -\theta_2 > 0$. Next we shall show $\theta_1 < \pi/2$.

If $\theta_1 = \pi/2$, we choose $\theta \in (0, \pi/2)$ and carry out the same computation as (3.26) to obtain

\begin{equation}
\rho(\theta, 0) = \lim_{z_2 \to \infty} \rho(\theta, z_2) = 2 \rho(\theta, \pi/2 - \theta).
\end{equation}

Letting $\theta \to \pi/2$, we obtain

\[
\lim_{\theta \to \pi/2} \rho(\theta, 0) = 0.
\]

On the other hand, by (2.11) we have

\[
\lim_{\theta \to \pi/2} \rho(\theta, 0) \geq \int_{\mathbb{R}} \frac{1}{2} u^2(x, 0) dx > 0.
\]

This is a contradiction, and hence proves $\theta_1 < \pi/2$.

3.5. The level set is asymptotically a straight line. Below we quote a lemma from [11] on the asymptotical behavior of the level set.

Lemma 3.6. Suppose that $u(y_1, y_2)$ is a solution of (1.4) in a cone $C := \{ y \in \mathbb{R}^2 : |y_1| \leq y_2 \tan \alpha_0, y_2 \geq M > 0 \}$ for some $0 < \alpha_0 < \pi/2$. The 0-level set of $u$ in $C$ is given by the graph of a function $y_1 = k(y_2)$. Assume

\begin{equation}
\lim_{y_2 \to \infty} k'(y_2) = 0.
\end{equation}
Then there is a finite number $A$ such that

$$\lim_{y_2 \to -\infty} k(y_2) = A.$$ \hfill (3.29)

The lemma can be shown in three steps. First, we show that an energy of $u$ on a line segment $[-y_2 \tan \alpha, y_2 \tan \alpha_0]$, $\alpha \in (0, \alpha_0)$ is exponentially close to $e$ as $y_2$ tends to $-\infty$. Second, we construct an optimal approximation of $u(\cdot, y_2)$ by a shift of the one-dimensional solution $g(y_1 - l(y_2))$, and show that the error is exponentially small in $L^2$ norm as $y_2$ goes to infinity. Finally, we deduce that the shift $l(y_2)$ has a finite limit, and then conclude that $k(y_2)$ has a finite limit. For the details of the proof, the reader is referred to [11].

Now we choose the coordinate system $(y_1, y_2)$ so that $y_2$-axis form an angle $\theta_1$ with $y$-axis, and $\alpha_0 < \min\{\pi/2 - \theta_1, \theta_1\}$. Using Lemma 3.4 and Lemma 3.6, we conclude that

$$\gamma(y) = (\tan \theta_1)y + A_2 + o(1), \quad \text{as } y \to -\infty.$$ \hfill (3.30)

Similarly, we can show

$$\gamma(y) = -(\tan \theta_1)y + A_3 + o(1), \quad \text{as } y \to -\infty.$$ \hfill (3.31)

Then, for $Y_0$ large enough, the inverse functions of $\gamma(y)$ for $y > Y_0$ and $y < Y_0$ exist, and may be written as $y = k_1(x), y = k_2(x)$ respectively. Moreover,

$$k_1(x) = \kappa x + B_1 + o(1), \quad k_2(x) = -\kappa x + B_2 + o(1)$$

as $x \to -\infty$, where $\kappa = \cot \theta_1$ is a positive (finite) constant, and $B_1, B_2$ are constants.

3.6. The moving plane method. Next we shall use the moving plane method to show the even symmetry of $u$ with respect to $y$. Due to the fact that the asymptotical behavior of $u$ is not homogeneous near infinity, in particular, there is a transition layer along the 0-level set, the classic moving plane method has to be carefully modified. Indeed, we have to use the exact asymptotical formulas of the 0-level sets $y = k_i(x), i = 1, 2$ near infinity as well the asymptotical behavior of $u$ along these curves.

For this purpose, we define $u_\lambda(x, y) := u(x, 2\lambda - y)$ and $w_\lambda := u_\lambda - u$ in $D_\lambda := \{(x, y) : x \geq 0, y \geq \lambda\}.$

**Lemma 3.7.** When $\lambda$ is sufficiently large, there holds $w_\lambda > 0$ in $D_\lambda$.

**Proof.** We first fix $X_0$ sufficiently large so that $k_1(x), k_2(x)$ are well defined. By the property of double well potential (1.2), there exists a sufficiently small constant $\delta > 0$ such that $F''(t) > 0, t \in [-1, -1 + \delta] \cup [1 - \delta, 1]$. There is also a sufficiently large constant $R_1 > 0$ such that $-1 < g(s) \leq -1 + \delta/2, \forall s < -R_1$ and $1 - \delta/2 \leq g(s) < 1, \forall s > R_1$, where $g$ is the one dimensional solution in (1.3). By (3.32) and (3.15), there exist $X_1, R_2$ sufficiently large such that for $x > X_1$,

$$\begin{cases}
    u(x, y) < -1 + \delta, \quad \text{if } y > k_1(x) + R_2, \text{ or } y < -k_2(x) - R_2, \\
    u(x, y) > 1 - \delta, \quad \text{if } 0 < y < k_1(x) - R_2, \text{ or } -k_2(x) + R_2 < y < 0, \\
    |u(x, y) + g(y \sin \theta_1 - x \cos \theta_1 - B_1 \sin \theta_1)| \leq \delta/2, \quad & \text{if } k_1(x) - R_2 < y < k_1(x) + R_2, \\
    |u(x, y) - g(y \sin \theta_1 + x \cos \theta_1 - B_2 \sin \theta_1)| \leq \delta/2, \quad & \text{if } k_2(x) - R_2 < y < k_2(x) + R_2. 
\end{cases}$$ \hfill (3.33)
When $\lambda > \lambda_1$ is sufficiently large, by (3.32) we have

$$k_2^\lambda(x) := 2\lambda - k_2(x) \geq k_1(y) + R_2, \quad \forall x \geq X_1.$$ 

By Lemma 3.8, we can also choose $\lambda_1$ so that

$$u(x, y) < -1 + \delta, \quad 0 < x < X_1, \ y > \lambda_1.$$ 

We claim that $w_\lambda \geq 0$ in $D_\lambda$ for $\lambda > \lambda_1$, and shall show this claim in the following three subsets of $D_\lambda$ respectively:

$$D_\lambda^+ := \{(x, y) : 0 < x < X_1, \ y > \lambda, \text{ or } x > X_1, \ y > k_2^\lambda(x)\},$$

$$D_\lambda^- := \{(x, y) : x > X_1, \ y < k_1(x)\},$$

$$D_\lambda^0 := \{(x, y) : x > X_1, \ k_1(x) < y < k_2^\lambda(x)\}.$$

If the claim is not true in $D_\lambda^+$, then there exists a sequence of points $\{(x_m, y_m)\}_{m=1}^{\infty} \in D_\lambda^+$ such that

$$\lim_{m \to \infty} w_\lambda(x_m, y_m) = \lim_{m \to \infty} (u_\lambda(x_m, y_m) - u(x_m, y_m)) = \inf_{D_\lambda^+} w_\lambda(x, y) < 0.$$ 

It can be seen from (3.33) that $u_\lambda(x_m, y_m) < u(x_m, y_m) < -1 + \delta$ when $m$ is large enough. Then we can use the standard translating arguments to obtain a contradiction as follows. Define $w_\lambda^m(x, y) := w_\lambda(x + x_m, y + y_m)$ in $D_\lambda^+ - (x_m, y_m)$. Then $w_\lambda^m$ converges to $w_\lambda^\infty(x, y)$ in $C_{loc}^3(D^\infty)$ for some piecewise Lipschitz domain $D^\infty$ in $\mathbb{R}^2$ which contains a small ball centered at the origin. Furthermore, $w_\lambda^\infty$ attains its negative minimum at the origin and satisfies a linearized equation

$$w_{xx} + w_{yy} - F''(\xi(x, y))w = 0, \ (x, y) \in D^\infty$$

where $\xi(x, y) = su(x, y) + (1-s)u_\lambda(x, y)$ for some $s(x, y) \in (0, 1)$ and $F''(\xi(0, 0)) > 0$. This is a contradiction, which leads to the claim in $D_\lambda^+$. Similarly, the claim can be shown in $D_\lambda^-$ by the strong maximum principle, due to the fact that $u_\lambda > 1 - \delta$ in $D_\lambda^-$ as in (3.33). The claim is also true in $D_\lambda^0$ when $\lambda$ is large enough, due to the last two estimates in (3.33).

Then, using the strong maximum principle (or the Harnack inequality) to an elliptic equation satisfied by $w_\lambda$ which is similar to (3.34), the lemma is proven.

Now we define

$$\Lambda = \inf\{\lambda : u_\lambda(x, y) > u(x, y), (x, y) \in D_\lambda\}.$$ 

**Lemma 3.8.** There holds

$$\Lambda = (B_1 + B_2)/2$$

where $B_1, B_2$ are as in (3.32).

**Proof.** We shall prove this lemma by contradiction. Suppose the lemma does not hold. By (3.32), we can easily see that $\Lambda > (B_1 + B_2)/2$ and $w_\Lambda > 0, \forall (x, y) \in D_\Lambda$. Then there exists a sequence of numbers $\{\lambda_m\}$ such that $\lambda_m < \Lambda$, and

$$\lim_{m \to \infty} \lambda_m = \Lambda \quad \text{and the infimum of } w_{\lambda_m} \text{ in } D_{\lambda_m} \text{ is negative. Using (3.18) and the}$$

translating arguments as above, we can show that the infimum of $w_{\lambda_m}$ in $D_{\lambda_m}$ is achieved at a point $(x_m, y_m)$, i.e.,

$$w_{\lambda_m}(x_m, y_m) = \inf_{D_{\lambda_m}} w_{\lambda_m} < 0.$$
Since \( w_{\lambda_m} \) satisfies an elliptic equation similar to (3.34) with \( \xi(x_m, y_m) = su(x_m, y_m) + (1 - s)w_{\lambda_m}(x_m, y_m) \) for some \( s \in (0, 1) \), by the strong maximum principle we know that \( u(x_m, y_m) > -1 + \delta \) and hence \( y_m - k_1(x_m) < R_2 \) if \( x_m > X_1 \). By (3.18) and the assumption \( \Lambda > (B_1 + B_2)/2 \), we know \( x_m < X_2 \) for some constant \( X_2 \) independent of \( m \). Therefore there exists a subsequence of \( \{m\} \) (still denoted by itself) such that \( (x_m, y_m) \) converges to \( (x_0, y_0) \in D_\Lambda \) and \( w_{\lambda_m} \) converges to \( w_\lambda \) in \( C^{3,1}_{\loc}(D_\Lambda) \) as well as in \( C^3(B_1(x_0, y_0) \cap D_\Lambda) \). It is easy to see that \( \nabla w_\lambda(x_0, y_0) = 0 \).

Furthermore, \( w_\lambda \) is an even function in \( x \) and satisfies an elliptic equation similar to (3.34) in \( D_\Lambda \), by the Harnack inequality we can see that \( (x_0, y_0) \) is not on the \( y \)-axis. Hence \( (x_0, y_0) \) must be on the portion of boundary \( \{(x, y) : y = \Lambda \} \) of \( D_\Lambda \).

Then by the Hopf Lemma, we have \( \partial_n w_\lambda(x_0, y_0) > 0 \). This is a contradiction, which proves the lemma. \( \square \)

We note that \( u_\lambda \geq u \) in \( D_\Lambda \) and \( u_\lambda(x, \lambda) = \frac{1}{2} \frac{\partial}{\partial \theta} w_\lambda(x, \lambda) < 0, \forall x \in \mathbb{R} \) when \( \lambda > \Lambda \). Similarly, we can use the moving plane method from below, i.e., repeating the above procedure for \( w_\lambda := \text{in} D^\lambda_\Lambda := \{(x, y) : x > 0, y < \Lambda\} \), and conclude \( u_\lambda \geq u \) in \( D^\lambda_\Lambda \). Therefore, Theorem 1.1 is proven.

4. Even symmetry of entire solutions with four ends

We shall show that certain entire solutions of (1.1) with four ends must be evenly symmetric with respect to both \( x \)-axis and \( y \)-axis after a proper translation and rotation. First we consider the case that the four ends are asymptotically straight lines, i.e., on each \( 0 \)-level set \( \Gamma_i \) there holds

\[
y = \tan(\theta_i)x + A_i + o(1) \quad \text{as} \quad x \to \infty, \quad 1 \leq i \leq 4
\]

where \( 0 < \theta_i < \theta_{i+1} < 2\pi \), and \( \theta_i \neq \pi/2, \theta_i \neq \pi/2, i = 1, 2, 3, 4 \). Without loss of generality, after a proper rotation we may also assume that \( 0 < \theta_1 = 2\pi - \theta_4 < \pi/2 \) and \( \theta_2 \neq \pi, \theta_3 \neq \pi \).

By Proposition 2.2, we know that Hamiltonian identity (3.10) holds. Moreover, in view of (3.18), on a fixed cone \( \{(r, \theta) = (x, y) : \theta_{i-1} + \delta < \theta < \theta_{i+1} - \delta\} \) with a sufficiently small \( \delta > 0 \) there holds

\[
|u(x, y) - g(x \sin \theta_i - y \cos \theta_i + A_i \cos \theta_i)| \to 0, \quad \text{uniformly as} \quad r \to \infty
\]

As in (3.19), by Hamiltonian identity (3.10) we can easily obtain that

\[
\rho(x) = e(\cos \theta_1 + \cos \theta_4) = e(- \cos \theta_2 - \cos \theta_3).
\]

Similarly, when \( x \)-axis is replaced by \( y \)-axis in Hamiltonian identity (3.10), we obtain

\[
e(\sin \theta_1 + \sin \theta_2) = e(- \sin \theta_3 - \sin \theta_4).
\]

We can easily derive that

\[
\pi - \theta_2 = \theta_1 = \theta_3 - \pi.
\]

Now we follow the moving plane procedure as in the proof of Theorem 1.1. It can be shown that Lemma 3.4 still holds with \( D_\Lambda \) being modified as \( \{(x, y) : y \geq \Lambda\} \). Furthermore, Lemma 3.5 also holds with

\[
\Lambda = \max\{(A_1 + A_4)/2, (A_2 + A_3)/2\}.
\]

Without loss of generality, after proper translation in \( y \) we may assume that \( \Lambda = A_1 + A_4 = 0 \geq A_2 + A_3 \).
Next we shall show
\begin{equation}
A_2 + A_3 = 0.
\end{equation}

For this purpose, let us now state another Hamiltonian identity for \( u \), which was used in [4] and [13] for solutions of nonlinear Schrodinger equation before. A similar identity for certain parabolic equations is also used in [5] and may be regarded as conservation of moment.

Define
\begin{equation}
E(x) = \int_{\mathbb{R}} y[F(u(x, y))] + \frac{1}{2} u_y^2(x, y) - \frac{1}{2} u_x^2(x, y) \, dy.
\end{equation}

Then, by (2.2), \( E(x) \) is well defined. We have
\begin{proposition}
(4.5)
\[ E(x) \equiv C, \quad x \in \mathbb{R}. \]
\end{proposition}

The proof of this Hamiltonian identity is based on (2.2) and is similar to those in [4] and [13]. The details is left to the reader.

Now, using (4.2), straightforward computations can lead to
\[ \lim_{x \to \infty} E(x) = (A_1 + A_4)e^{\cos \theta_1} = 0 \]
and
\[ \lim_{x \to -\infty} E(x) = (A_2 + A_3)e^{\cos \theta_1}. \]

Therefore, (4.3) is proven.

The moving plane method then leads to the even symmetry and monotonicity of \( u \) in \( y \). Repeating the above arguments with \( x \) and \( y \) switched, we can show the even symmetry and monotonicity of \( u \) in \( x \). Therefore, we have shown
\begin{theorem}
Assume that \( u \) is an entire solution with four ends satisfying (4.1). Then, after a proper translation and rotation, \( u \) satisfies (1.10) and (1.11).
\end{theorem}

\section{Energy quantization of entire solutions}

In this section we shall show that (4.1) holds under very mild conditions on \( u \). Indeed, we shall consider entire solutions with \( 2k \) ends in general and show some energy quantization properties for entire solutions with finite Morse index.

\begin{lemma}
Suppose \( u \) is an entire solution of (1.1) with \( 2k \) ends. Assume
\begin{equation}
\theta_i^+ - \theta_i^- < \pi, \quad 1 \leq i \leq 2k.
\end{equation}
Then
\begin{equation}
\mathcal{E}_R(u) \leq CR, \quad \forall R
\end{equation}
for some positive constant \( C \).
\end{lemma}

\begin{proof}
We only need to focus on conic region \( C_1 \) and show
\[ \int_{B_{R} \cap C_1} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) \, dx \, dy \leq CR, \quad \forall R. \]

Without loss of generality, we may assume
\begin{equation}
0 < \theta_1^- < \pi/2, \quad \pi/2 < \theta_1^+ < \pi.
\end{equation}

Choose $0 < \alpha^- < \theta_1^- < \alpha^+ < \pi$ and let $C_1^+ = \{(r, \theta) : \alpha^- < \theta < \alpha^+\}$. Define
\[
\rho_1(y) := \int_{y \cot \alpha^-}^{y \cot \alpha^+} [F(u) + \frac{1}{2}(u_x^2 - u_y^2)] \, dx.
\]
Then, in view of (2.2), it is easy to see that
\[
|\rho_1'(y)| = |\left[ F(u) + \frac{1}{2}(u_x^2 - u_y^2) + u_x y u_y \right]_{x = y \cot \alpha^-}^{x = y \cot \alpha^+} | 
\leq C e^{-\mu y}, \quad \forall y \geq R_0
\]
for some positive constants $C, \mu_1$. Hence we have
\begin{equation}
|\rho_1(R_1) - \rho_1(R_2)| \leq C e^{-\mu_1 R}, \quad \forall R_1 \leq R_2
\end{equation}
for some constant $C > 0$. In particular, we have
\[
|\rho_1(y)| \leq C, \quad \forall y \geq R_0.
\]
By (2.4), we have
\[
F(u) + \frac{1}{2}(u_x^2 - u_y^2) \geq \frac{1}{2} u_x^2.
\]
Hence
\begin{equation}
\int_{B_R \cap C_1^+} u_x^2 \, dx \, dy \leq C R < \infty
\end{equation}
for some constant $C > 0$.

Now we choose another Cartesian coordinates $(x', y')$ so that the $x'$-axis is a small rotation of $x$-axis and (5.5) and (5.6) still hold. Then we can obtain
\[
\int_{B_R \cap C_1^+} u_x^2 \, dx \, dy = \int_{B_R \cap C_1^+} u_x^2 \, dx' \, dy' \leq C < \infty
\]
Therefore we obtain
\[
\int_{B_R \cap C_1^+} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) \, dx \, dy 
\leq \int_{B_R \cap C_1^+} (F(u) + \frac{1}{2}(u_x^2 - u_y^2)) \, dx \, dy + C \int_{B_R \cap C_1^+} (u_x^2 + u_y^2) \, dx \, dy 
\leq CR, \quad \forall R > 0.
\]
Similarly, we can show that this estimate holds for all $i \in [1, 2k]$.

In view of (2.2), it is easy to see that
\[
\int_{\mathbb{R}^2 \setminus \cup_{i=1,2k} C_1^+} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) \, dx \, dy \leq \int_0^\infty C e^{-\mu r} \, dr < C
\]
for some constant $C > 0$. Hence (5.7) is proven.

In [15], Modica showed Proposition 2.3 which says that $E_R(u)/R$ is increasing in $R$. It follows immediately that $\lim_{R \to \infty} E_R(u)/R$ exists. Indeed, we can show the following energy quantization property for entire solutions with finite Morse index.
Lemma 5.2. Assume that $u$ is an entire solution of (1.1) with finite Morse index and $2k$ ends. Assume also the technical condition (5.1). Then the 0-level sets $\Gamma$ of $u$ are asymptotically straight lines, i.e., there exist $\theta_i \in [\theta_i^-, \theta_i^+]$, $1 \leq i \leq 2k$ such that on $\Gamma_i$

$$y = \tan(\theta_i)x + A_i + o(1) \quad \text{as} \quad x \to \infty, \quad 1 \leq i \leq 2k$$

where $\theta_i \neq \pi/2, \theta_i \neq 3\pi/2, \forall i \in [1, 2k]$ after a proper rotation. Moreover, (1.14) holds.

Proof. It is easy to see that $u_\epsilon(x) := u(x/\epsilon)$ is a critical point of functional

$$\mathcal{E}_{\epsilon,R}(u) = \int_{B_R \setminus B_1(2R)} \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \right) dx dy.$$  

Fix $R = 1$, $u_\epsilon$ is a stable critical point of (5.7) with $\mathcal{E}_{\epsilon,1}(u_\epsilon) < C < \infty$. By a Gamma-convergence result of Tonegawa (Theorem 5 in [18]), there exists a sequence $\epsilon_n$ and a union $L$ of $N$ non-intersecting lines of $B_1 \setminus B_1/2$ such that

$$\epsilon_n \cdot (\Gamma \cap (B_R/\epsilon_n \setminus B_1(2R))) \to L \quad \text{in Hausdorff distance as} \quad n \to \infty.$$

Now fix $R = 2, 3, \cdots$ and repeat the argument above for a subsequence of $\{\epsilon_n\}$ in the previous step, by the diagonal procedure we can find a subsequence, still denoted by $\epsilon_n$, such that (5.8) holds for all $R = 1, 2, \cdots$. Therefore $L$ must be the union of $N$ different rays starting from the origin, and

$$\lim_{R \to \infty} \mathcal{E}_R(u)/R = Ne.$$

Fix a ray in $L$. Without loss of generality, we may assume it to be the positive $x$-axis which belongs to $C_1$ after some rotation. Then, for any fixed small angles $\alpha_2 > \alpha_1 > 0$, there exists a sequence of conic regions $C_{R_n,M_n,\alpha_i} := \{(x, y) : R_n \leq x \leq M_n, \ |y| \leq \tan \alpha_i\}$, $i = 1, 2$ such that $R_n \to \infty, M_n/R_n \to \infty$ and

$$C_{R_n,M_n,\alpha_2} \cap \Gamma \subset C_{R_n,M_n,\alpha_1}.$$

On the other hand, thanks to the stability of $u$ in $\mathbb{R}^2 \subset B_{R_0}$ when $R_0$ is large enough, by similar arguments to the proof of (3.18) we can show that

$$C_{R_n,M_n,\alpha_2} \cap \Gamma = \{(x, y) : y = k(x), \ R_n \leq x \leq M_n\}$$

for some $C^2$ function $k(x)$ and

$$\max_{x \in [R_n,M_n]} |k'(x)| < \tan \alpha_1, \quad \max_{x \in [R_n,M_n]} |k''(x)| \to 0, \quad \text{as} \quad n \to \infty.$$

Moreover,

$$||u(x, y) - g(y - k(x))||_{C^2(C_{R_n,M_n,\alpha_2})} \to 0, \quad \text{as} \quad n \to \infty.$$

We may also assume that $k'(R_n) \to 0$. We claim that when $n$ is large enough, $M_n$ can be chosen as any number $R > R_n$ and (6.10) still holds. If this is not true, we can choose $M_n$ such that (5.10) holds but $k'(M_n) = \tan \alpha_1$. We claim that $(C_{R_n,M_n,\alpha_2} \setminus C_{R_n,2M_n,\alpha_1+\alpha_2/2}) \cap B_{M_n}(M_n,k(M_n))$ is empty. If we assume otherwise, without loss of generality, we may assume that $M_n$ is the first such sequence related to a ray in $L$. Now we use $\epsilon_n = 1/M_n$ as in (5.8), and obtain the limit as $L'$ which is the union of at least $N + 1$ rays. This is a contradiction to (5.9). Hence the claim is true. Then, using the modified Hamiltonian identity in
$\mathcal{C}_{R_n, M_n, \alpha_2}$ as in [5.4] with the $y$-axis being replaced by the tangential direction of $k(x)$ at $(M_n, k(M_n))$, we obtain
\[ e \leq e \cos \alpha_1 + o(1), \quad n \to \infty. \]
This is a contradiction, and hence proves that $M_n$ can be chosen as any $R > R_n$ when $n$ large enough. Therefore
\[ \mathcal{C}_{R_n, \infty, \alpha_2} \cap \Gamma = \{(x, y) : y = k(x), \ x > R_n\} \]
and
\[ |k'(x)| < \tan \alpha_1, \ x > R_n. \]
Since $\alpha_1 > 0$ is arbitrary, we obtain that
\[ \lim_{x \to \infty} k'(x) = 0. \]
Now use Lemma 3.6, we conclude that $\Gamma \cap \mathcal{C}_1$ is asymptotically straight line. The lemma then follows.

\[ \square \]

Remark 5.3. Given that $u$ satisfies the condition in Theorem 1.2. If we assume further that, after a proper rotation, the level set in $\mathcal{C}_i$ outside a large ball $B_R$ is a graph of a $C^2$ function $k(x)$, i.e.,
\[ (5.11) \quad \Gamma \cap \mathcal{C}_i \cap B_R^c = \{(x, y) : y = k(x), \ x > R\}, \ 1 \leq i \leq 2k, \]
then the conclusion of Lemma 5.2 can be shown directly without using the result in [18]. We just start the proof from (5.10) with $M_n = \infty$ and exploits the modified Hamiltonian identity. The details is omitted.

Theorem 1.2 follows from Lemma 5.2 and Theorem 4.2 directly. If we replace (1.9) in Theorem 1.2 by (5.2), the conclusion of Theorem 1.2 still holds.

Proof of Theorem 1.3
If (1.13) does not hold, by the monotonicity formula of Modica we know that (5.2) must be true. Using the $\Gamma$- convergence result of Tonegawa as in the proof of Lemma 5.2, we know that there exists a sequence $\{R_n\}$ such that $R_n \to \infty$ and
\[ (5.12) \quad \frac{1}{R_n} \cdot (\Gamma \cap B_{MR_n}) \to L \quad \text{in Hausdorff distance as} \quad n \to \infty \]
for any $M > 0$, where $L$ is the union of $N$ rays from the origin. Moreover, (5.13) holds. It follows that $\Gamma$ must be asymptotically straight lines at infinity, as in the proof of Lemma 5.2. Note that $\Gamma$ is a union of $C^2$ curves except at singular points where $u$ and $\nabla u$ both vanish, and $u$ $u$ behaves like harmonic function near these singular points. Therefore $N$ must be an even positive integer $2k$. We denote the directions of these lines by $\nu_i = (\cos \theta_i, \sin \theta_i), 1 \leq i \leq 2k$ with $0 < \theta_i < \theta_{i+1} < 2\pi, 1 \leq i \leq 2k - 1$, after a proper rotation. Using Hamiltonian identity similar to (3.26) but with more terms (see also [11]), we obtain
\[ (5.13) \quad \sum_{i=1}^{2k} e \sin(\theta_i + \theta) = 0 \]
for almost all $\theta$. Hence (1.15) holds. The proof of Theorem 1.3 is complete.

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