Robust Wirtinger Flow for Phase Retrieval with Arbitrary Corruption

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Abstract

We consider the phase retrieval problem of recovering the unknown signal from the magnitude-only measurements, where the measurements can be contaminated by both sparse arbitrary corruption and bounded random noise. We propose a new nonconvex algorithm for robust phase retrieval, namely Robust Wirtinger Flow, to jointly estimate the unknown signal and the sparse corruption. We show that our proposed algorithm is guaranteed to converge linearly to the unknown true signal up to a minimax optimal statistical precision in such a challenging setting. Compared with existing robust phase retrieval methods, we improved the statistical error rate by a factor of $\sqrt{n/m}$ where $n$ is the dimension of the signal and $m$ is the sample size, provided a refined characterization of the corruption fraction requirement, and relaxed the lower bound condition on the number of corruption. In the noise-free case, our algorithm converges to the unknown signal at a linear rate and achieves optimal sample complexity up to a logarithm factor. Thorough experiments on both synthetic and real datasets corroborate our theory.

1 Introduction

In the fields of machine learning, signal processing and statistics, one important problem is solving a quadratic system of equations. Specifically, we are interested in solving the following system of $m$ quadratic equations:

$$y_i = |\langle a_i, x^* \rangle|, \quad 1 \leq i \leq m,$$

where $x^* \in \mathbb{R}^n$ or $\mathbb{C}^n$ is the unknown signal we try to recover. $a_i \in \mathbb{R}^n$ or $\mathbb{C}^n$ is the design/sensing vectors and $y = (y_1, y_2, \ldots, y_m)^\top$ is the observation vector. Equivalently, (1.1) can be written as quadratic form: $y_i^2 = |\langle a_i, x^* \rangle|^2$. Due to its combinatorial nature caused by the missing signs of $\langle a_i, x^* \rangle$, solving such a quadratic system of equations is generally considered as NP-hard (Pardalos and Vavasis, 1991).
In the literature of physical sciences, the problem of solving (1.1) is also known as phase retrieval (Fienup, 1978; Candès et al., 2015b), where the goal is to reconstruct the unknown signal vector from magnitude only measurements. There exists a large body of literature (Fienup, 1978, 1982; Gerchberg, 1972; Candès et al., 2013, 2015a; Netrapalli et al., 2013; Candès et al., 2015b; Goldfarb and Qin, 2014; Wei, 2015; Zhang and Liang, 2016; Wang and Giannakis, 2016; Goldstein and Studer, 2016; Sun et al., 2016; Wang et al., 2016; Zhang et al., 2016b; Huang et al., 2016) for phase retrieval in the noise-free and noisy cases. The applications of phase retrieval include X-ray crystallography (Harrison, 1993; Miao et al., 1999), microscopy (Miao et al., 2008), diffraction and array imaging (Bunk et al., 2007; Chai et al., 2010), optics (Millane, 1990) and so on.

In many applications, it is not uncommon that the measurements $|\langle a_i, x^* \rangle|$’s are corrupted by errors\textsuperscript{1}. The corruption arises due to various reasons such as illumination, occlusion, device malfunctioning, damage of measuring equipment or simply recording errors. These types of corruption are usually large in magnitudes and do not disappear by averaging the results. Hence it is of great importance for the phase retrieval algorithms to be able to handle these corruption that can be arbitrarily large, and if possible, identify the location of the corruption. Nevertheless, most of existing phase retrieval algorithms do not have an intrinsic mechanism to deal with arbitrary corruption, and would fail when the corruption is present.

In this paper, we aim to develop a new phase retrieval algorithm that is able to recover the unknown signal with arbitrary corruption. It is obvious that if all the measurements are corrupted, there is no hope to recover the unknown signal. Therefore, we assume the corruption is sparse, i.e., only a fraction of the measurements is corrupted. Our work is along the line of the Wirtinger flow (WF)-type approaches (Candes et al., 2015b; Chen and Candès, 2015; Zhang and Liang, 2016; Wang and Giannakis, 2016), which solves the problem by minimizing a nonconvex loss function with gradient descent algorithm and can be shown to converge to the unknown signal under good initialization. By using the “reshaped” amplitude-based loss function (Zhang and Liang, 2016; Wang and Giannakis, 2016), we propose a new robust Wirtinger Flow (Robust-WF) algorithm for phase retrieval, which is proved to be robust against arbitrarily large corruption. In particular, we address two important questions in this setting: (1) how many observations do we need; and (2) how many corruption can we tolerate. Experiments on both synthetic data and real data verify the advantages of our algorithm and corroborate our theory. The main contributions of this paper are highlighted as follows:

- Unlike existing algorithms (Zhang et al., 2016a; Hand and Voroninski, 2016) which only estimate the unknown signal, our proposed Robust-WF algorithm jointly estimates the unknown signal and the sparse corruption, and is proved to exactly recover the unknown signal in the noise-free setting at a linear rate from an optimal $O(n \log n)$ samples up to a logarithmic factor.

- We provide a much preciser characterization on the corruption fraction $\alpha$, and our algorithm allows the fraction of corrupted entries up to $\delta/(C \log^2 (m))$. In sharp contrast, previous work such as Zhang et al. (2016a); Hand and Voroninski (2016) requires the corruption fraction $\alpha$ to be a sufficiently small enough constant. A careful analysis of their proofs shows that their corruption tolerance must depend on the problem dependent quantities $(m, n$ and so on$)$. But due to the limitation of the concentration technique for sample median, they fail to provide a finer threshold.

\textsuperscript{1}It is important to distinguish corruption from random noise.
We propose a new proof framework, which unifies the optimization error and statistical error analysis, and covers both noisy and noise-free cases. Benefited from the new proof technique, we can show that statistical error achieved by our proposed algorithm is in the order of $O(\sqrt{n/m} \cdot \|\epsilon\|_\infty)$, which improves the previous best result by a factor of $\sqrt{n/m}$. We believe that our new proof technique is of independent interest, and is directly applicable to analyzing existing phase retrieval algorithms for proving sharper statistical rates.

Last but not least, the computational complexity of our algorithm is $O(mn \cdot \log(1/\epsilon))$, which matches the state-of-the-art result. In other words, our algorithm is able to recover the unknown signal from corrupted measurements without paying any additional computational price.

**Notation.** For a vector $x \in \mathbb{R}^n$, define vector norm as $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$, the infinity norm as $\|x\|_\infty = \max_i \{x_i\}$. And $\|x\|_0 = \sum_{i=1}^n 1\{x_i \neq 0\}$ denotes the number of nonzero entries in $x$. For a matrix $A \in \mathbb{R}^{n_1 \times n_2}$, we denote the spectral norm $\|A\|_2 = \max_{\|u\|_2 = 1} \|Au\|_2$. For two sets $S_1$ and $S_2$, we denote $S_1 \setminus S_2 = \{x \in S_1, x \notin S_2\}$ as the relative complement set and $S_1^c$ as the complement set of $S_1$. Further we denote the sign function as $\text{sgn}(t) = t/|t|$ indicators the sign of $t$.

**Organization.** The remainder of this paper is organized as follows: in Section 3, we review the problem formulation in detail. We present the algorithm in Section 4, and the main theory in Section 5. In Section 6, we sketch the proof of the main theory. In Section 7, we compare the proposed algorithm with existing algorithms on both synthetic data and real-world datasets. Finally, we conclude this paper in Section 8.

## 2 Related Work

Various techniques have been developed for solving the phase retrieval problem. They can be generally classified into two categories: convex approaches and nonconvex approaches. Convex methods like PhaseLift (Candes et al., 2013), PhaseCut (Waldspurger et al., 2015) adopt a so-called matrix-lifting technique to linearize the constraint by introducing a rank-one matrix and then relax the rank-one condition. Recently another convex method called PhaseMax (Goldstein and Studer, 2016) was proposed, which operates in the original signal space rather than lifting it to a higher dimensional space. While convex approaches do enjoy good recovery guarantee, their computational complexity are usually too large to afford especially when the dimension of signal is high.

On the other hand, nonconvex approaches including Gerchberg-Saxton (Gerchberg, 1972), Fienup (Fienup, 1982), AltMinPhase (Netrapalli et al., 2013), trust-region (Sun et al., 2016), choose to directly optimize the nonconvex problem. Recently, a method called Wirtinger flow (WF) (Candes et al., 2015b) was shown to work remarkably well by using a spectral method for initialization and gradient descent for refinement. It only requires $O(n \log n)$ measurements to recover the signal within $O(mn^2 \cdot \log(1/\epsilon))$ flops. The follow-up work, called Truncated Wirtinger Flow (TWF) (Chen and Candes, 2015), introduced the truncation to select a subset of samples, which reduces the sample complexity to $O(n)$ and computational complexity to $O(mn \cdot \log(1/\epsilon))$. More recently, Truncated Amplitude Flow (TAF) (Wang and Giannakis, 2016) and Reshaped Wirtinger Flow (RWF) (Zhang and Liang, 2016) went further along this line and used the magnitude of $\langle a_i, x^* \rangle$ instead of its square as the observations. Zhang and Liang (2016) proved that RWF enjoys the same sample complexity as TWF even without truncation in gradient steps. TAF (Wang and Giannakis, 2016) used an orthogonality-promoting initialization method, which returns better initial solutions compared with
the spectral counterparts. Many stochastic/incremental algorithms such as Incremental Truncated Wirtinger Flow (ITWF) (Kolte and Özgür, 2016), Incremental Reshaped Wirtinger Flow (IRWF) (Zhang et al., 2016b), Stochastic Truncated Amplitude Flow (STAF) (Wang et al., 2016) have also been developed. However, they fail to improve the computational complexity due to the higher requirements on the step size parameter. While the aforementioned variants of WF do improve robustness of the original WF algorithm by carefully selecting a subset of samples, they still fail in arbitrary corruption setting.

In a seminal work (Zhang et al., 2016a) for phase retrieval with arbitrary corruption, a Median-TWF algorithm was introduced to utilize the properties of the median estimator to enhance the robustness in such a corruption setting. Nevertheless, it requires $O(n \log n)$ samples in the gradient descent stage in order to recover the unknown signal. A careful examination of its initialization proof shows that it has an extra condition on the lower bound of the number of corruption, i.e., the number of corruption should not be too small, which is apparently very counterintuitive (See details in Section 4). Furthermore, Zhang et al. (2016a) did not provide a precise characterization of the corruption fraction in terms of the problem dependent parameters. Hand and Voroninski (2016) simply combined the initialization procedure of Median-TWF and the optimization procedure for PhaseMax to deal with arbitrary corruption and hence suffers from the drawbacks of both methods. Hand (2017) showed that PhaseLift is robust to corruption. Nevertheless, PhaseLift is based on convex optimization, which is time consuming.

Another line of research that is very related to our work is low-rank matrix/tensor estimation under corruption, that include robust principal component analysis (Candès et al., 2011; Chandrasekaran et al., 2011; Hsu et al., 2011; Chen et al., 2013; Netrapalli et al., 2014; Gu et al., 2016; Yi et al., 2016), robust low-rank matrix completion (Agarwal et al., 2012; Goldfarb and Qin, 2014; Klopp et al., 2014; Cherapanamjeri et al., 2016), robust tensor decomposition (Gu et al., 2014; Anandkumar et al., 2015). The key idea of these methods is to estimate the unknown low-rank matrix/tensor and the sparse corruption matrix/tensor simultaneously. At a high level, our work is inspired by this line of research.

3 Problem Setup

We consider the phase retrieval problem where the observations are contaminated by sparse corruption with arbitrarily large magnitudes and random noises. For concreteness, our analysis will focus on the real-valued Gaussian model. It is worth noting that our proposed algorithm can be directly extended to complex Gaussian model. More specifically, suppose the observations are generated from the following measurement model:

$$y_i = |a_i^\top x^*| + \eta_i^* + \epsilon_i, \quad 1 \leq i \leq m,$$

where $\{a_i\}_{i=1}^m$ are Gaussian random measurement vectors, each of which is independently drawn from a multivariate normal distribution $N(0, I_{n \times n})$, $x^* \in \mathbb{R}^n$ is the unknown true signal to be recovered, and $\eta^* \in \mathbb{R}^m$ is the arbitrarily large corruption vector that contains at most $\alpha m$ non-zero entries. Through the rest of this paper, we refer to $\alpha$ as the corruption fraction parameter of $\eta^*$. And $\epsilon \in \mathbb{R}^m$ is the noise vector with zero mean and each element $\epsilon_i$ is bounded by $|\epsilon_i| \leq \delta \|x^*\|_2$ for some universal constant $\delta$. Similar boundedness assumption on $\epsilon_i$ has been widely made in the literature (Chen and Candès, 2015; Zhang et al., 2016a; Zhang and Liang, 2016; Wang and Giannakis, 2016).
When there is no random noise, Model (3.1) will reduce to
\[ y_i = |a_i^\top x^*| + \eta_i^*, \quad 1 \leq i \leq m. \] (3.2)

Note that the same models have been investigated in Zhang et al. (2016a); Hand and Voroninski (2016). Next, following the convention of phase retrieval (Candes et al., 2015b; Chen and Candes, 2015; Zhang and Liang, 2016; Wang and Giannakis, 2016), we define the distance between an estimate \( x \) to the unknown signal \( x^* \) to be:
\[
\text{dist}(x, x^*) := \min_{\phi \in [0, 2\pi)} \| e^{-j\phi} x - x^* \|_2. \] (3.3)

For real valued data, it can be reduced to
\[
\text{dist}(x, x^*) := \min \{ \| x + x^* \|_2, \| x - x^* \|_2 \}.
\]

4 The Proposed Algorithm

In this section, we introduce our proposed Robust Wirtinger Flow algorithm in Algorithm 1. Our method is composed of two stages: the initialization stage and the gradient descent stage.

**Algorithm 1 Robust Wirtinger Flow**

1: **Input:** Observation vector \( y = \{y_i\}_{i=1}^m \), Measurement vectors \( \{a_i\}_{i=1}^m \), Thresholding parameter \( \tilde{\alpha} \), Stepsize \( \mu \).

   **Stage I: Initialization**
2: \( \eta^{(0)} = \mathcal{H}_{\tilde{\alpha}m}(y), \quad \hat{y} = y - \eta^{(0)} \)
3: \( \lambda_0 = \sqrt{\frac{1}{m} \sum_{i=1}^m \hat{y}_i^2} \)
4: \( x^{(0)} = \lambda_0 \tilde{x} \) where \( \tilde{x} \) is the leading eigenvector of
\[
Y := \frac{1}{m} \sum_{i=1}^m \hat{y}_i^2 a_i a_i^\top
\]

   **Stage II: Gradient Descent**
5: **for** \( t = 0 \) to \( T - 1 \) **do**
6: \( \eta^{(t+1)} = \mathcal{H}_{\tilde{\alpha}m}(y - A(x^{(t)})) \)
   where \( A(x^{(t)}) = [a_i^\top x^{(t)}]_i = |a_i^\top x^{(t)}| \)
7: Update \( x^{(t+1)} \) by
\[
x^{(t)} - \frac{\mu}{m} \sum_{i=1}^m (|a_i^\top x^{(t)}| + \eta_i^{(t+1)} - y_i) \text{sgn}(a_i^\top x^{(t)}) a_i
\]
8: **end for**
9: **Output:** \( x^{(T)} \)

The hard thresholding operator \( \mathcal{H} \) in Algorithm 1 is defined as follows:
\[
[\mathcal{H}_s(w)]_i := \begin{cases} 
  w_i, & \text{if } |w_i| \geq |w^{(s)}|, \\
  0, & \text{otherwise}
\end{cases}, \quad (4.1)
\]

where \( w^{(s)} \) denotes the element whose magnitude is the \( s \)-largest in \( w \in \mathbb{R}^m \). Note that the thresholding parameter \( \tilde{\alpha} \) is a tuning parameter in real world problems.
For the ease of presentation, throughout the rest of this paper, we denote by \( y_i^* = |a_i^\top x^*| \) the non-contaminated component (i.e., without the random noise and arbitrary corruption) of the \( i \)-th measurement. Let \( y^* = [y_1^*, \ldots, y_m^*]^\top \).

**Initialization Stage:** Our initialization procedure consists of two parts: estimating the magnitude of the true signal \( x^* \) by \( \lambda_0 \) and estimating the phase of \( x^* \) by \( \hat{x} \). The intuition here is to find a good estimator \( \hat{y} \) for the non-contaminated measurements \( y^* \) and show that our magnitude estimator \( \lambda_0 \) and phase estimator \( \hat{x} \) based on \( \hat{y} \) would be close to their corresponding values based on the unknown \( y^* \). By taking the hard thresholding operator in Step 2 of Algorithm 1, \( \eta^{(0)} \) contains the largest \( \tilde{\alpha}m \) elements in \( y \). Hence \( \hat{y} = y - \eta^{(0)} \) is the observation vector with largest \( \tilde{\alpha}m \) elements removed. Suppose the magnitude of the corruption is fairly large, it is reasonable to adopt hard thresholding operator to remove the large entries in \( y \) temporarily. Similar ideas have been used in Yi et al. (2016); Netrapalli et al. (2014).

For the magnitude estimation, in the arbitrary corruption setting, we can later show that, by conducting the hard thresholding operator as shown in Algorithm 1, as long as the corruption fraction \( \alpha \) satisfies certain upper bound condition, \( \lambda_0 := \sqrt{(1/m) \sum_{i=1}^m \tilde{y}_i^2} \) concentrates to \( \|x^*\|_2 \) with high probability. In terms of estimating the phase of the unknown signal, following the idea of Candes et al. (2015b), a spectral method is used by computing the leading eigenvector of \( Y \), where \( Y \) in (4.1) again to get our new corruption estimator \( \hat{x} \).

Given that the output \( x^{(0)} \) from the initialization stage is close enough to the true signal \( x^* \), we can show that the above objective function in the neighborhood of the \( x^* \) behaves like a strongly convex and smooth function. This gives rise to the idea of using a gradient descent algorithm in this stage. Due to the sparsity constraint on \( \eta \), we need to apply the hard thresholding operator in (4.1) again to get our new corruption estimator \( \eta^{(t+1)} \). According to our model in (3.1) we have \( y_i = |a_i^\top x^*| + \eta_i^* + \epsilon_i \), it is reasonable to use the following update for \( \eta^{(t+1)} \):

\[
\eta^{(t+1)} = \mathcal{H}_{\tilde{\alpha}m}(y - A(x^{(t)})),
\]

where \( [A(x^{(t)})]_i = |a_i^\top x^{(t)}| \). For updating \( x^{(t+1)} \), one can compute the gradient of \( L(x, \eta) \) with...
respect to \( \mathbf{x} \) as follows:

\[
\nabla_{\mathbf{x}} L(\mathbf{x}, \eta) = \frac{1}{m} \sum_{i=1}^{m} (|a_i^\top \mathbf{x}| + \eta_i - y_i) \text{sgn}(a_i^\top \mathbf{x}) a_i, \tag{4.3}
\]

which naturally leads to a gradient update for \( \mathbf{x}^{(t+1)} \) as:

\[
\mathbf{x}^{(t)} - \frac{\mu}{m} \sum_{i=1}^{m} (|a_i^\top \mathbf{x}^{(t)}| + \eta_i^{(t+1)} - y_i) \text{sgn}(a_i^\top \mathbf{x}^{(t)}) a_i,
\]

where \( \mu \) is the step size. Note that for our model in (3.1), \( y_i = |a_i^\top \mathbf{x}^*| + \eta_i^* + \epsilon_i \), when \( \mathbf{x}^{(t)} \) gets closer to \( \mathbf{x}^* \), our estimation for \( \eta^* \) also becomes more accurate. Thus by taking the gradient update for only \( \mathbf{x} \), we manage to get precise estimation results for both \( \mathbf{x}^* \) and \( \eta^* \).

5 Main Theory

In this section, we provide the main theory about Algorithm 1, including the linear rate convergence analysis and sharper statistical results for robust phase retrieval with arbitrary corruption.

**Theorem 5.1.** Consider the phase retrieval problem with both arbitrary corruption and bounded noises defined in (3.1). Let \( \{C_i\}_{i=1}^{5} \) be some universal constants. For any signal \( \mathbf{x}^* \in \mathbb{R}^n \), Gaussian measurement vectors \( \{\mathbf{a}_i\}_{i=1}^{m} \) drawn from i.i.d \( N(0, \mathbf{I}_{n \times n}) \), let \( \epsilon \in \mathbb{R}^n \) be a bounded noise vector with \( |\epsilon_i| \leq \delta \cdot \|\mathbf{x}^*\|_2 \), and \( \eta^* \) be the \( \alpha m \)-sparse corruption vector with arbitrary large magnitudes, suppose the sparse parameter \( \alpha \) satisfies \( \alpha \leq \delta/(C_0 \log^2(m)) \), if \( m \geq C_1 \cdot n \log n \), then \( \mathbf{x}^{(0)} \) generated by Stage I of Algorithm 1 satisfies

\[
\text{dist}(\mathbf{x}^{(0)}, \mathbf{x}^*) \leq \frac{1}{10} \|\mathbf{x}^*\|_2,
\]

with probability at least \( 1 - 10e^{-C_2 n} - 2e^{-C_3 m} - 8/n^2 - 6/m \). Furthermore, taking a constant step size \( \mu \leq \mu_0 \) and starting from \( \mathbf{x}^{(0)} \), if \( m \geq C_4 \cdot n \), the output \( \mathbf{x}^{(t)} \) from Stage II in Algorithm 1 satisfies

\[
\text{dist}(\mathbf{x}^{(t)}, \mathbf{x}^*) \leq \frac{1}{10} \left(1 - \frac{\mu}{2}\right)^t \cdot \|\mathbf{x}^*\|_2 + C_5 \sqrt{\frac{n}{m}} \cdot \|\epsilon\|_\infty,
\]

with probability at least \( 1 - 11e^{-C_2 n} - 8e^{-C_3 m} - 8/n^2 - 8/m \).

**Remark 5.2.** Theorem 5.1 provides theoretical guarantees on the performance of our algorithm for robust phase retrieval with arbitrary corruption and random noise. It provides a refined characterization of the corruption tolerance, i.e., \( \alpha \leq \delta/(C_0 \log^2(m)) \). Note that Median-TWF (Zhang et al., 2016a) only provides recovery guarantee when \( \alpha \) is a small constant. However, by a careful analysis of their proof, we can see that \( \alpha \) is not an absolute constant and should depend on the problem-dependent parameters. Yet it is difficult to explicitly derive such a dependence. Furthermore, as we can see, the distance between \( \mathbf{x}^{(t)} \) output by our algorithm and \( \mathbf{x}^* \) is upper bounded by two terms: the optimization error and the statistical error. Due to the linear convergence rate of our proposed algorithm, at most \( O(\log(1/\epsilon)) \) iterations are needed in order to achieve \( \epsilon \)-error in optimization. And the overall computational complexity for our algorithm is \( O(mn \log(1/\epsilon)) \), which matches the computational cost of state-of-the-art methods (Chen and Candès, 2015; Wang...
and Giannakis, 2016; Zhang and Liang, 2016). In terms of statistical error, Algorithm 1 achieves a shaper statistical rate in the order of $O(\sqrt{n/m} \cdot \|\epsilon\|_\infty)$, compared with previous algorithms whose error rate is typically in the order of $O(\|\epsilon\|_\infty)$ (Zhang et al., 2016a) or $O(\|\epsilon\|_2 / \sqrt{m})$ (Chen and Candes, 2015; Zhang and Liang, 2016; Wang and Giannakis, 2016).

**Remark 5.3.** Theorem 5.1 suggests that Algorithm 1 achieves a sample complexity of $O(n \log n)$ in the initialization stage. This sample complexity is optimal up to a logarithm factor. Some previous work such as TWF (Chen and Candes, 2015), TAF (Wang and Giannakis, 2016) and RWF (Zhang and Liang, 2016) achieve sample complexity of $O(n)$. Nonetheless, these algorithms are only for phase retrieval without arbitrary corruption. Median-TWF method (Zhang et al., 2016a) achieves $O(n)$ sample complexity in its initialization stage, yet it requires an additional lower bound condition on the number of corruption. We believe that the extra $\log n$ factor in the sample complexity of our initialization stage is not an artifact of our proof. We will explore new initialization procedure to get rid of this extra logarithmic factor in our future work. In the gradient descent stage, our algorithm achieves the optimal sample complexity of $O(n)$, which beats the Median-TWF (Zhang et al., 2016a) whose sample complexity is $O(n \log n)$.

An immediate result of Theorem 5.1 is the following corollary, which suggests that our algorithm can exactly recover the unknown signal $x^*$ with arbitrary large corruption in the noise-free model.

**Corollary 5.4.** Consider the phase retrieval problem with arbitrary corruption defined in (3.2). Under the same condition as in Theorem 5.1, $x(0)$ generated by Stage I of Algorithm 1 satisfies

$$\text{dist}(x(0), x^*) \leq \frac{1}{10} \|x^*\|_2,$$

with probability at least $1 - 10e^{-C_2n} - 8/n^2 - 6/m$. Furthermore, taking a constant step size $\mu \leq \mu_0$ and starting from $x(0)$, if $m \geq C_4 \cdot n$, the output $x(t)$ from Stage II in Algorithm 1 satisfies

$$\text{dist}(x(t), x^*) \leq \frac{1}{10} \left(1 - \frac{\mu}{2}\right)^t \cdot \|x^*\|_2,$$

with probability $1 - 10e^{-C_2n} - 6e^{-C_3m} - 8/n^2 - 8/m$.

### 6 Proof Sketch of the Main Theorem

In this section, we highlight the proof sketch of the main theorem in Section 5. The detailed proofs can be found in the Appendix.

#### 6.1 Robust Initialization

In this subsection we briefly demonstrate how our initialization procedure showed in Stage I of Algorithm 1 will generate an initial solution close enough to the true signal.

First we investigate the effects of the hard thresholding operator. Since $\eta^{(0)} = H_{\tilde{\alpha}m}(y)$, where $H$ denotes for the hard thresholding operator defined in (4.1) and $\tilde{y} = y - \eta^{(0)}$ by our algorithm procedure. It can be shown that the following two claims hold:

**Claim 1:** $\hat{y} - y^* - \epsilon = \eta^* - \eta^{(0)}$,

**Claim 2:** $\|\hat{y} - y^* - \epsilon\|_\infty \leq 2\|y^* + \epsilon\|_\infty$. 

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Given those two claims, we have the following lemma to ensure the estimation accuracy of the magnitude of signal, i.e., $\|x^*\|_2$.

**Lemma 6.1.** Let $\eta^{(0)}, \hat{y}$ be defined as in Stage I of Algorithm 1, suppose $\eta^*$ is $\alpha m$-sparse, $\eta^{(0)}$ is $\tilde{\alpha}m$-sparse and the corruption sparse parameter $\alpha$ satisfies $\alpha \leq \delta/(c_1 \log(m))$, and the additional noise is bounded as $|\epsilon_i| \leq \delta \|x^*\|_2$, for the model defined in (3.1) we have

$$
(1 - 7\delta) \|x^*\|_2^2 \leq \frac{1}{m} \sum_{i=1}^m \hat{y}_i^2 \leq (1 + 8\delta) \|x^*\|_2^2
$$

with probability at least $1 - 2/m$ where $\delta$ is a problem dependent constant.

Next we evaluate the accuracy for estimating the phase of $x^*$. In Algorithm 1 we have $Y := \frac{1}{m} \sum_{i=1}^m \hat{y}_i^2 a_i a_i^\top$. Assume that $\|x^*\|_2 = 1$, define $\hat{Y} := \frac{1}{m} \sum_{i=1}^m (y_i^2 + 2y_i^* \epsilon_i) a_i a_i^\top$, with $E[\hat{Y}] = I + 2x^* x^*\top$. We further have the following lemma to characterize the difference between $Y$ and $E[\hat{Y}]$:

**Lemma 6.2.** Consider the model defined in (3.1), suppose $\eta^*$ is $\alpha m$-sparse and the sparse parameter $\alpha$ satisfies $\alpha \leq \delta/(c_2 \log^2(m))$, if $m \geq c_3 \cdot n \log n$, the $Y$ defined in Stage I of Algorithm 1 satisfies

$$
\|Y - E[\hat{Y}]\|_2 \leq 4\delta,
$$

with probability at least $1 - 10e^{-c_4 m} - 2e^{-c_5 m} - 8/n^2 - 4/m$ where $\delta$ is a problem dependent constant.

By matrix perturbation theory, specifically, the Davis-Kahan sin $\Theta$ Theorem (Davis, 1963; Yu et al., 2015), we obtain

$$
\sin(\Theta(\tilde{x}, x^*)) \leq \frac{\|Y - E[\hat{Y}]\|_2}{\lambda_{gap}/2} \leq 4\delta,
$$

where the second inequality follows from Lemma 6.2, $\lambda_{gap}$ is the eigengap between the largest and the second largest eigenvalues of $E[\hat{Y}]$, $\tilde{x}$ is the leading eigenvector for $Y$ and $\Theta(\tilde{x}, x^*)$ denotes the angle between $\tilde{x}$ and $x^*$. Hence we conclude that the distance between the leading eigenvector of $Y$ and $E[\hat{Y}]$, i.e., $\text{dist}(\tilde{x}, x^*)$ satisfies:

$$
\text{dist}(\tilde{x}, x^*) \leq \sqrt{2} \sin(\Theta(\tilde{x}, x^*)) \leq 4\sqrt{2}\delta. \quad (6.1)
$$

Finally consider the distance between $x^{(0)}$ and $x^*$ when $\|x^*\|_2 \neq 1$,

$$
\text{dist}(x^{(0)}, x^*) \leq \text{dist}(\lambda_0 \tilde{x}, \|x^*\|_2 \cdot \tilde{x}) + \text{dist}((\|x^*\|_2 \cdot \tilde{x}, x^*)
\leq \max\{\sqrt{1 + 8\delta} - 1, 1 - \sqrt{1 - 7\delta}\} \cdot \|x^*\|_2 + 4\sqrt{2}\delta \cdot \|x^*\|_2
\leq \frac{1}{10} \|x^*\|_2,
$$

where the second inequality follows from Lemma 6.2 and (6.1), and the last inequality can be easily achieved by choosing $\delta$ to be sufficiently small. This completes the proof.
6.2 Convergence Analysis

In this subsection, we show that once the initial solution falls in the neighborhood of \( x^* \), the Stage II of our algorithm guarantees a linear convergence towards the true signal \( x^* \). Denote the support for \( \eta^* \) as \( \Omega^* \) and the support for \( \eta^{(t+1)} \) as \( \Omega' \). By discussing the support each sample index \( i \) falls in, we decompose the gradient \( \nabla_x L(x^{(t)}, \eta^{(t+1)}) \) into two parts:

**Approximate Gradient:**

\[
g(x^{(t)}, \eta^{(t+1)}) = \frac{1}{m} \sum_{i \notin \Omega'} (|a_i^\top x^{(t)}| - |a_i^\top x^*|) \text{sgn}(a_i^\top x^{(t)}) a_i.
\]

**Residual Gradient:**

\[
\nabla_x L(x^{(t)}, \eta^{(t+1)}) - g(x^{(t)}, \eta^{(t+1)}) = -\frac{1}{m} \sum_{i \in \Omega \setminus \Omega'} \eta_i^* \cdot \text{sgn}(a_i^\top x^{(t)}) a_i - \frac{1}{m} \sum_{i \notin \Omega'} \epsilon_i \cdot \text{sgn}(a_i^\top x^{(t)}) a_i.
\]

By the gradient descent update rules we can easily have

\[
\text{dist}(x^{(t+1)}, x^*) \leq \|h - \mu \cdot g(x^{(t)}, \eta^{(t+1)})\|_2 + \mu \|\nabla_x L(x^{(t)}, \eta^{(t+1)}) - g(x^{(t)}, \eta^{(t+1)})\|_2.
\]

(6.2)

For the approximate distance term we have the following lemma:

**Lemma 6.3.** For the approximated distance defined in (6.2), if \( m \geq c_0 n \) and the corruption sparsity parameter \( \alpha \) satisfies \( \alpha \leq \delta/(c_2 \gamma \cdot \log m) \) then with probability at least \( 1 - e^{-c_1 m} - 1/m \), we have

\[
\|h - \mu \cdot g(x^{(t)}, \eta^{(t+1)})\|_2^2 \leq (1 - 2\mu(0.74 - 3\delta) + \mu^2(1 + \delta)^2) \|h\|_2^2,
\]

holds for all \( h \in \mathbb{R}^n \) satisfies \( \|h\|_2 \leq \|x^*\|_2/10 \).

For the residual gradient error term we have the following two lemmas characterizing the two terms corresponding to the two terms in the residual gradient:

**Lemma 6.4.** Denote the support for \( \eta^* \) as \( \Omega^* \) and the support for \( \eta^{(t+1)} \) as \( \Omega' \), suppose \( a_i \in \mathbb{R}^n, i = 1, \ldots, m \) are Gaussian vectors independently drawn from \( N(0, I_{n \times n}) \), if the corruption sparsity parameter \( \alpha \) satisfies \( \alpha \leq \delta/(c_2 (1 + \delta) \log m) \), then with probability at least \( 1 - 2e^{-c_3 m} - 1/m \), the following

\[
\left\| \frac{1}{m} \sum_{i \notin \Omega'} \eta_i^* \cdot \text{sgn}(a_i^\top x^{(t)}) a_i \right\|_2 \leq 2\sqrt{\delta} \cdot \|h\|_2 + 2\sqrt{\alpha(1 + \delta)} \cdot \|\epsilon\|_\infty.
\]

(6.3)

holds for all non-zero vectors \( h \in \mathbb{R}^n \).

**Lemma 6.5.** Denote the support for \( \eta^{(t+1)} \) as \( \Omega' \), suppose \( a_i \in \mathbb{R}^n, i = 1, \ldots, m \) are Gaussian vectors independently drawn from \( N(0, I_{n \times n}) \), we have

\[
\left\| \frac{1}{m} \sum_{i \notin \Omega'} \epsilon_i \cdot \text{sgn}(a_i^\top x^{(t)}) a_i \right\|_2 \leq c_4 \|\epsilon\|_\infty \cdot \sqrt{\frac{n}{m}}
\]

(6.4)

with probability at least \( 1 - \exp(-n/2) \).
Combining the results from Lemmas 6.3, 6.4 and 6.5, we obtain the following inequality describing the contraction between consecutive iterations:

$$\text{dist}(x^{(t+1)}, x^*) \leq (1 - \mu/2) \cdot \|h\|_2 + c_4\mu\sqrt{\frac{n}{m}} \cdot \|\epsilon\|_\infty,$$

where $\mu \leq \mu_0$ with $\mu_0$ as a positive universal constant, $\delta$ is chosen to be sufficiently small and $\alpha \leq c_2 \cdot n/m$. The final conclusion in Theorem 5.1 is obtained by iteratively conducting the above contraction formula with a mathematical induction argument.

7 Experiments

In this section, we evaluate the performance of our Robust-WF algorithm against other state-of-the-art baseline algorithms. We conduct our experiments on both synthetic and real data set.

7.1 Baseline Methods

We compare our algorithm with the following baseline methods: RWF: Reshaped Wirtinger Flow (Zhang and Liang, 2016), TAF: Truncated Amplitude Flow (Wang and Giannakis, 2016), TWF: Truncated Wirtinger Flow (Chen and Candes, 2015), and Median-TWF: Median Truncated Wirtinger Flow (Zhang et al., 2016a).

Note that here we did not compare with the original Wirtinger Flow algorithm (Candes et al., 2015b) since both its initialization stage and gradient descent stage do not involve any truncation or robust estimation technique, thus does not work well when there is corruption.

7.2 Parameter Settings

For all algorithms, we run a fixed number of iterations $T = 250$ and the number of power iterations in initialization stage is also fixed to 200. We set $\beta = 1.2\alpha$ for our Robust-WF algorithm and all the truncation parameters in those baseline algorithms to the suggested values in respective papers. The step size is tuned for each algorithm.

7.3 Synthetic Data

In each setting, we generate a Gaussian measurement matrix $A \in \mathbb{R}^{m \times n}$ with rows drawn independently from a standard multivariate normal distribution $N(0, I)$. The true signal $x^*$ is also generated from an independent standard multivariate normal distribution $N(0, I)$. We generate a sparse corruption vector $\eta^*$ with at most $\alpha m$ non-zero entries. Each of the non-zero entry comes with a random corruption in the magnitude of $0.5\|x^*\|_2$. The random noise $\epsilon$ is generated independently from a uniform distribution $U(0, p)$ with $p$ ranging from 0 to 2.

We evaluate the performance of the algorithms based on relative errors, i.e., $\text{dist}(x^{(T)}, x^*)/\|x^*\|_2$. In noise-free setting, we also consider the empirical success rate, which is defined as the ratio of successful trials against the total number of trails. Specifically, a trial is declared successful if the output $x^{(T)}$ satisfies $\text{dist}(x^{(T)}, x^*) \leq 10^{-8}$. We test the performance of all algorithms by ranging $\alpha$ from 0 to 0.4 with an increment of 0.01 in two settings: $n = 100, m = 1000$ and $n = 200, m = 2000$.

Figure 1 shows the case where no corruption and no noise are involved. We can see that all baseline algorithms and our proposed algorithm achieve similar performance except for Median-TWF, which is less accurate compared with other approaches in this case.
Figure 2(a) illustrates the performances of all algorithms in the noise-free setting. We plot the empirical success rate of all five algorithms including ours against the corruption fraction under 20 replications. As we can see from the figure, our proposed algorithm clearly out-performs all the other baseline methods. Non-robust algorithms like TWF, RWF and TAF starts to fail quite early even when the corruption fraction is quite small. Median-TWF algorithm also fails when $\alpha$ goes to around 0.2, while our proposed Robust-WF can still successfully recover the signal when $\alpha$ is around 0.3. Figure 2(b) illustrates the performances of all algorithms in the noisy setting. We plot the relative error of all algorithms against the corruption fraction under 20 replications. Again, it is obvious from the figure that our proposed algorithm achieves the best relative error compared with other state-of-the-art algorithms. Figure 3 shows similar result for the other setting.

In Figure 4, we further investigate the convergence result of our algorithm and the baseline methods. We can see that our proposed algorithm indeed enjoys a linear rate of convergence. This is consistent with our theoretical result in Theorem 5.1. Moreover, our proposed method outperforms all the other baseline algorithms in different noise settings. It can be seen that when the magnitude of the uniformly distributed noise varies from 0 to 2, our algorithm always achieves smaller recovery error than the other baselines. Particularly, in noise-free case, even though both our algorithm and Median-TWF meet the criterion for exact recovery (i.e., $\text{dist}(x^{(T)}, x^{*}) \leq 10^{-8}$), the error achieved by our Robust-WF algorithm is significantly smaller than Median-TWF algorithm. Figure 5 illustrates similar results for the other setting with different dimension $n$ and sample size $m$.

![Relative error with respect to the iteration count for all baseline algorithms and our proposed algorithm under no corruption noise-free circumstances for two different settings.](image)

**Figure 1:** Relative error with respect to the iteration count for all baseline algorithms and our proposed algorithm under no corruption noise-free circumstances for two different settings.

7.4 Real Data

We also evaluate the performance of our algorithm on the recovery of real images from the Fourier intensity measurements (two dimensional Coded Diffraction Patterns model). The Coded Diffraction Patterns (CDP) model is a type of physically realizable measurements of images with random masks (see details in Candes et al. (2015a,b); Chen and Candes (2015)). Suppose $x^{*} \in \mathbb{R}^n$ is the vectorization of a real image matrix (consider only one color band for each $x^{*}$), the CDP model basically collects the magnitude of the discrete Fourier transform (DFT) of $K$ modulations of the
signal $\mathbf{x}^\ast$. Specifically,

$$y^{(k)} = |F D^{(k)} \mathbf{x}^\ast|, \ 1 \leq k \leq K,$$

where $\mathbf{F}$ stands for the discrete Fourier transform matrix and $\mathbf{D}^{(k)}$ stands for a diagonal phase delay matrix with its diagonal entries uniformly sampled from $\{1, -1, j, -j\}$. Under the above definition, a total number of $m = nK$ observations are collected. In this experiment, we set $K = 12$ and evaluate all the algorithms by three real world benchmark images. Details about the benchmark images can be found in Table 1. We consider the corrupted CDP model where additional random

![Figure 2](image1.png) ![Figure 3](image2.png)

**Figure 2:** (a) Empirical success rate of exact recovery against the corruption fraction $\alpha$ for all algorithms where $n = 100, m = 1000$ under 20 replications; (b) Relative error against the corruption fraction $\alpha$ for all algorithms where $n = 100, m = 1000$ under 20 replications.

**Figure 3:** (a) Empirical success rate of exact recovery against the corruption fraction $\alpha$ for all algorithms where $n = 200, m = 2000$ under 20 replications; (b) Relative error against the corruption fraction $\alpha$ for all algorithms where $n = 200, m = 2000$ under 20 replications.
corruption are imposed upon the CDP model to test the recovery performance of different phase retrieval algorithms. In detail, we randomly selected 5% of the total pixels and impose corruption with magnitudes up to the level of $\|x^*\|_2$.

Table 2 demonstrates the performance of our algorithm and other baseline methods on the recovery of the real images. We can see that for all three benchmark images, our proposed Robust-WF algorithm achieves the smallest relative recovery error. This clearly demonstrates the superiority of our algorithm over other baseline methods.

**Table 1: Summary of the Test Images.**

| Images  | Dimensions   | #Modulations (K) |
|---------|--------------|------------------|
| Lenna   | 512 $\times$ 512 $\times$ 3 | 12               |
| Stanford| 1280 $\times$ 320 $\times$ 3 | 12               |
| Galaxy  | 1920 $\times$ 1080 $\times$ 3 | 12               |

* The last coordinate 3 in dimensions refers to the three color bands (R/G/B) of the images.
8 Conclusions and Future Work

In this paper, we proposed a Robust-WF algorithm for phase retrieval with arbitrary corruption. Both theoretical analysis and experiments were conducted to show the superiority of our algorithm. In our future work, we aim to explore alternative ways that can get rid of the logarithmic factor in the sample complexity of the initialization stage.
A Proof of Theorem 5.1

We prove our main theorem in this section. For the ease of expression, we define \( \gamma := \tilde{\alpha}/\alpha \) in the following proof.

A.1 Robust Initialization

Here we prove that our initialization procedure showed in Stage I of Algorithm 1 will generate an initial solution close enough to the true signal as long as the fraction of corruption \( \alpha \) is small enough.

Let \( \eta^{(0)} = H_{\gamma\alpha m}(y) \), where \( H \) denotes for the hard thresholding operator defined in (4.1) and \( \hat{y} = y - \eta^{(0)} \) as our initial guess for the true non-contaminated observations \( y^* \). First we want to make two claims:

\begin{align}
\text{Claim 1: } \hat{y} - y^* - \epsilon &= \eta^* - \eta^{(0)}, \\
\text{Claim 2: } \|\hat{y} - y^* - \epsilon\|_\infty &\leq 2\|y^* + \epsilon\|_\infty.
\end{align}

Claim 1 can be shown since by model definition (3.1) we have \( \hat{y} = y - \eta^{(0)} = y^* + \eta^* + \epsilon - \eta^{(0)} \). To show Claim 2 we need to discuss the subset that each data sample belongs to. Denote the support for \( y^* \) as \( \Omega^* \) and the support for \( \eta^{(0)} \) as \( \Omega \) and we have the following three cases:

\begin{itemize}
  \item \( i \in \Omega \): Since \( i \) belongs to the support for \( \eta^{(0)} \), we have \( \eta_i^{(0)} = y_i \) and thus \( \hat{y}_i - y_i^* - \epsilon_i = -y_i^* - \epsilon_i \).
  \item \( i \notin \Omega, i \notin \Omega^* \): By model definition (3.1) we have \( y_i = y_i^* + \eta_i^* + \epsilon_i \). Since \( i \) does not belong to the support for \( \eta^{(0)} \), \( y_i \) is not in the \( \gamma\alpha m \)-largest magnitude elements in \( y \) in order to be outside of \( \Omega \), i.e., \( |y_i| \leq |y^{(\gamma\alpha m)}| \). Therefore we must have \( \eta_i^* \leq 2\|y^* + \epsilon\|_\infty \), otherwise we would have \( |y_i| = |y_i^* + \eta_i^* + \epsilon_i| \geq \|y^* + \epsilon\|_\infty \) for any \( \gamma > 1 \) which conflicts the previous conclusion. Also \( i \notin \Omega \) means \( \eta_i^{(0)} = 0 \), \( \hat{y}_i = y_i \) and hence \( \hat{y}_i - y_i^* - \epsilon_i = \eta_i^* \leq 2\|y^* + \epsilon\|_\infty \).
  \item \( i \notin \Omega, i \notin \Omega^* \): Since \( i \) does not belong to either the support for \( \eta^{(0)} \) or the support for \( \eta^* \), we immediately have \( \eta_i^* = 0, \hat{y}_i = y_i \). Note that by model definition (3.1) we have \( y_i = y_i^* + \epsilon_i \) and hence \( \hat{y}_i - y_i^* - \epsilon_i = 0 \).
\end{itemize}

According to the above discussion we immediately have \( \|\hat{y} - y^* - \epsilon\|_\infty \leq 2\|y^* + \epsilon\|_\infty \). Given the above two claims, now we first estimate the magnitude of \( x^* \) as

\[ \lambda_0 = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \hat{y}_i^2}. \]

By Lemma 6.1, we have

\[ (1 - 7\delta)\|x^*\|_2^2 \leq \frac{1}{m} \sum_{i=1}^{m} \hat{y}_i^2 \leq (1 + 8\delta)\|x^*\|_2^2, \]

Next we estimate the direction of \( x^* \). Without loss of generality, we assume \( \|x^*\|_2 = 1 \). Recall in
Algorithm 1 we define $Y$ as:

$$Y := \frac{1}{m} \sum_{i=1}^{m} \hat{y}_i^2 a_i a_i^\top = \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i + y_i^* - \epsilon_i)^2 a_i a_i^\top$$

$$= \frac{1}{m} \sum_{i=1}^{m} (y_i^2 + 2y_i^* \epsilon_i) a_i a_i^\top + \frac{1}{m} \sum_{i=1}^{m} \epsilon_i^2 a_i a_i^\top + \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 a_i a_i^\top$$

$$+ \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i) (\hat{y}_i - y_i^* - \epsilon_i) a_i a_i^\top.$$

Further we can compute the expectation of $\bar{Y}$ as:

$$E[\bar{Y}] = E\left[ \frac{1}{m} \sum_{i=1}^{m} (y_i^*)^2 a_i a_i^\top \right] + E\left[ \frac{2}{m} \sum_{i=1}^{m} y_i^* \epsilon_i \cdot a_i a_i^\top \right]$$

$$= I + 2x^* x^*^\top,$$

since $\epsilon_i$ is independent with $a_i, y_i^*$ and has zero mean. By Lemma 6.2 we obtain

$$\|Y - E[\bar{Y}]\|_2 \leq 4\delta.$$

Now applying Davis-Kahan $\sin \Theta$ Theorem (Davis, 1963; Yu et al., 2015) we obtain

$$\sin \Theta(x, x^*) \leq \frac{\|Y - E[\bar{Y}]\|_2}{\lambda_{gap}/2} \leq 4\delta,$$

where $\lambda_{gap}$ is the eigengap between the largest and the second largest eigenvalues of $E[\bar{Y}], \tilde{x}$ is the leading eigenvector for $Y$ and $\Theta(x, x^*)$ denotes the angle between $x$ and $x^*$. We further have

$$\text{dist}(x, x^*) \leq \sqrt{2} \sin \Theta(x, x^*) \leq 4\sqrt{2}\delta.$$

Finally consider the distance between $x^{(0)}$ and $x^*$ when $\|x^*\|_2 \neq 1$,

$$\text{dist}(x^{(0)}, x^*) \leq \text{dist}(\lambda_0 x, \|x^*\|_2 \cdot \tilde{x}) + \text{dist}(\|x^*\|_2 \cdot \tilde{x}, x^*)$$

$$\leq \max\{\sqrt{1 + 8\delta} - 1, 1 - \sqrt{1 - 7\delta}\} \cdot \|x^*\|_2 + 4\sqrt{2}\delta \cdot \|x^*\|_2$$

$$\leq \frac{1}{10} \|x^*\|_2,$$

where the last inequality can be easily achieved by choosing $\delta$ to be sufficiently small. This completes the proof.

**A.2 Convergence Analysis**

Here we consider the robust phase retrieval model with additional noise defined in (3.1), i.e., $y_i = |a_i^\top x^*| + \eta_i^* + \epsilon_i$ and analyze the stability guarantees for our proposed algorithm. The initialization for noisy model can be found in Subsection A.1. In this subsection, we focus on
analyzing the gradient descent stage in Algorithm 1 assuming that the initialization stage has already generated an estimation which is close to the true signal. The final objective is to bound:

\[ \text{dist}^2(x^{(t+1)}, x^*) = \text{dist}^2(x^{(t)} - \mu \cdot \nabla_x L(x^{(t)}, \eta^{(t+1)}), x^*) \].

Notice that for the gradient update it is obvious that

\[ -x^{(t)} - \mu \nabla_x L(-x^{(t)}, \eta^{(t+1)}) = -\left\{ x^{(t)} - \mu \nabla_x L(x^{(t)}, \eta^{(t+1)}) \right\}, \]

hence by definition we have

\[ \text{dist}^2\left((-x^{(t)}) - \mu \cdot \nabla_x L(-x^{(t)}, \eta^{(t+1)}), x^*)\right) = \text{dist}^2(x^{(t)} - \mu \cdot \nabla_x L(x^{(t)}, \eta^{(t+1)}), x^*), \]

despite that the global phase function is unrecoverable. Thus we can get rid of the global phase term in distance function by letting \( x \) to be \( e^{-j\phi(x)} \) for simplicity and directly set \( h = x^{(t)} - x^* \) for the analysis. Now we consider the (sub)-gradient of \( L(x, \eta) \) respect to \( x \) for the \( t \)-th iterate:

\[ \nabla_x L(x^{(t)}, \eta^{(t+1)}) = \frac{1}{m} \sum_{i=1}^{m} \left( |a_i^\top x^{(t)}| + \eta_i^{(t+1)} - y_i \right) \cdot \text{sgn}(a_i^\top x^{(t)}) \cdot a_i \]

\[ = \frac{1}{m} \sum_{i=1}^{m} (|a_i^\top x^{(t)}| - |a_i^\top x^*|) \text{sgn}(a_i^\top x^{(t)}) a_i - \frac{1}{m} \sum_{i=1}^{m} (\eta_i^* - \eta_i^{(t+1)} + \epsilon_i) \text{sgn}(a_i^\top x^{(t)}) a_i. \]

(A.3)

First we try to further decompose the second term in the above equation. We again discuss the subset that each data sample belongs to. Denote the support for \( \eta^* \) as \( \Omega^* \) and the support for \( \eta^{(t+1)} \) as \( \Omega' \) and we have the following three cases:

- \( i \in \Omega' \): Since \( i \) belongs to the support for \( \eta^{(t+1)} \), we have \( \eta_i^{(t+1)} = y_i - |a_i^\top x^{(t)}| \). By model definition (3.1) we have \( y_i = y_i^* + \eta_i^* + \epsilon_i \). It immediately implies that \( \eta_i^* - \eta_i^{(t+1)} + \epsilon_i = |a_i^\top x^{(t)}| - |a_i^\top x^*| \).

- \( i \notin \Omega', i \in \Omega^* \): Since \( i \) does not belong to the support for \( \eta^{(t+1)} \), \( \eta_i^{(t+1)} = 0 \). We have \( \eta_i^* - \eta_i^{(t+1)} + \epsilon_i = \eta_i^* + \epsilon_i \).

- \( i \notin \Omega', i \notin \Omega^* \): Since \( i \) does not belong to either the support for \( \eta^{(t+1)} \) or the support for \( \eta^* \), we immediately have \( \eta_i^* - \eta_i^{(t+1)} + \epsilon_i = \epsilon_i \).

According to the above discussion, we can separate the summation over \( 1 \) to \( m \) into three parts:

\[ \frac{1}{m} \sum_{i=1}^{m} (\eta_i^* - \eta_i^{(t+1)} + \epsilon_i) \cdot \text{sgn}(a_i^\top x^{(t)}) \cdot a_i \]

\[ = \frac{1}{m} \left[ \sum_{i \in \Omega'} (|a_i^\top x^{(t)}| - |a_i^\top x^*|) + \sum_{i \in \Omega \setminus \Omega'} (\eta_i^* + \epsilon_i) + \sum_{i \notin \Omega \cup \Omega'} \epsilon_i \right] \cdot \text{sgn}(a_i^\top x^{(t)}) \cdot a_i \]

\( = \frac{1}{m} \sum_{i \in \Omega'} (|a_i^\top x^{(t)}| - |a_i^\top x^*|) \text{sgn}(a_i^\top x^{(t)}) a_i + \frac{1}{m} \sum_{i \in \Omega^* \setminus \Omega'} \eta_i^* \cdot \text{sgn}(a_i^\top x^{(t)}) a_i + \frac{1}{m} \sum_{i \notin \Omega'} \epsilon_i \cdot \text{sgn}(a_i^\top x^{(t)}) a_i. \)

(A.4)
Note that the first term above can be merged with the first term in (A.3), thus we obtain
\[
\nabla_x L(x(t), \eta(t+1)) = \sum_{i \in \Omega^r} \left( |a_i^\top x(t)| - |a_i^\top x^*| \right) \text{sgn}(a_i^\top x(t)) a_i
\]
approximate gradient
\[- \frac{1}{m} \sum_{i \in \Omega^r} \eta_i^* \cdot \text{sgn}(a_i^\top x(t)) a_i - \frac{1}{m} \sum_{i \in \Omega^r} \epsilon_i \cdot \text{sgn}(a_i^\top x(t)) a_i.
\]

For convenience, we define the first term in the R.H.S of the above equality as the approximate gradient:
\[
g(x(t), \eta(t+1)) = \sum_{i \in \Omega^r} \left( |a_i^\top x(t)| - |a_i^\top x^*| \right) \text{sgn}(a_i^\top x(t)) a_i.
\]

Thus we have
\[
\text{dist}(x(t+1), x^*) \leq \|x(t) - \mu \cdot \nabla_x L(x(t), \eta(t+1)) - x^*\|_2
\]
approximated distance
\[= \|x(t) - x^* - \mu \cdot g(x(t), \eta(t+1))\|_2 + \mu \|\nabla_x L(x(t), \eta(t+1)) - g(x(t), \eta(t+1))\|_2
\]
residual gradient error
\[= \|h - \mu \cdot g(x(t), \eta(t+1))\|_2 + \mu \|\nabla_x L(x(t), \eta(t+1)) - g(x(t), \eta(t+1))\|_2. \quad (A.5)
\]

Note that for the approximated distance term, by Lemma 6.3 we have
\[
\|h - \mu \cdot g(x(t), \eta(t+1))\|_2 = \left( 1 - 2\mu(0.74 - 3\delta) + \mu^2(1 + \delta)^2 \right) \|h\|_2. \quad (A.6)
\]

For the residual gradient error term, note that we have
\[
\|\nabla_x L(x(t), \eta(t+1)) - g(x(t), \eta(t+1))\|_2
\]
\[= \left\| \sum_{i \in \Omega^r} \eta_i^* \cdot \text{sgn}(a_i^\top x(t)) a_i + \sum_{i \in \Omega^r} \epsilon_i \cdot \text{sgn}(a_i^\top x(t)) a_i \right\|_2
\]
\[\leq \left\| \sum_{i \in \Omega^r} \eta_i^* \cdot \text{sgn}(a_i^\top x(t)) a_i \right\|_2 + \left\| \sum_{i \in \Omega^r} \epsilon_i \cdot \text{sgn}(a_i^\top x(t)) a_i \right\|_2. \quad (A.7)
\]

By Lemma 6.4 and Lemma 6.5 we obtain
\[
\left\| \sum_{i \in \Omega^r} \eta_i^* \cdot \text{sgn}(a_i^\top x(t)) a_i \right\|_2 \leq 2\sqrt{\delta} \cdot \|h\|_2 + 2\sqrt{\alpha(1 + \delta)} \cdot \|\epsilon\|_\infty, \quad (A.8)
\]
and
\[
\left\| \sum_{i \in \Omega^r} \epsilon_i \cdot \text{sgn}(a_i^\top x(t)) a_i \right\|_2 \leq c_4 \|\epsilon\|_\infty \cdot \sqrt{\frac{n}{m}}. \quad (A.9)
\]

Submit (A.8), (A.9) back into (A.7) we have the bound for the residual gradient error term:
\[
\|\nabla_x L(x(t), \eta(t+1)) - g(x(t), \eta(t+1))\|_2 \leq 2\sqrt{\delta} \cdot \|h\|_2 + 2\sqrt{\alpha(1 + \delta)} \cdot \|\epsilon\|_\infty + c_4 \|\epsilon\|_\infty \cdot \sqrt{\frac{n}{m}}. \quad (A.10)
\]
Combine the above result by submitting (A.10) and (A.6) into (A.5) we have
\[
\text{dist}(\mathbf{x}^{(t+1)}, \mathbf{x}^*) \leq \|\mathbf{h} - \mu \cdot \mathbf{g}(\mathbf{x}^{(t)}, \eta^{(t+1)})\|_2 + \mu \|\nabla_x L(\mathbf{x}^{(t)}, \eta^{(t+1)}) - \mathbf{g}(\mathbf{x}^{(t)}, \eta^{(t+1)})\|_2 \\
\leq (\sqrt{1 - 2\mu(0.74 - 2\delta)} + \mu^2(1 + \delta)^2 + 2\mu\delta) \cdot \|\mathbf{h}\|_2 + \left(2\mu\sqrt{\alpha(1 + \delta)} + 4\mu \sqrt{n/m}\right) \cdot \|\mathbf{\epsilon}\|_\infty \\
\leq (1 - \mu/2) \cdot \|\mathbf{h}\|_2 + \left(2\mu\sqrt{\alpha(1 + \delta)} + 4\mu \sqrt{n/m}\right) \cdot \|\mathbf{\epsilon}\|_\infty,
\]
where the last inequality holds when choosing \(\delta\) to be sufficiently small provided \(\mu \leq \mu_0\) with \(\mu_0\) as a positive universal constant. Note that if we have \(\alpha \leq c_2 \cdot n/m\), the second term on the R.H.S would be dominated by \(O(\sqrt{n/m})\) term. Thus we have
\[
\text{dist}(\mathbf{x}^{(t+1)}, \mathbf{x}^*) \leq (1 - \mu/2) \cdot \|\mathbf{h}\|_2 + c_4 \mu \sqrt{n/m} \cdot \|\mathbf{\epsilon}\|_\infty,
\]
for some universal constant \(c_3\). Next we are going to show the convergence result based on mathematical induction. For the first step in gradient descent stage, we have
\[
\text{dist}(\mathbf{x}^{(1)}, \mathbf{x}^*) \leq (1 - \mu/2) \cdot \text{dist}(\mathbf{x}^{(0)} - \mathbf{x}^*) + c_4 \mu \delta \sqrt{n/m} \cdot \|\mathbf{x}^*\|_2 \leq \frac{1}{10} \|\mathbf{x}^*\|_2, \quad (A.11)
\]
as long as \(m\) satisfies \(m \geq c_4 n\) for some universal constant \(c_4\). For the \((t + 1)\)-th iteration, suppose we have
\[
\text{dist}(\mathbf{x}^{(t)}, \mathbf{x}^*) \leq \frac{1}{10} \|\mathbf{x}^*\|_2,
\]
then it follows that
\[
\text{dist}(\mathbf{x}^{(t+1)}, \mathbf{x}^*) \leq (1 - \mu/2) \cdot \text{dist}(\mathbf{x}^{(t)} - \mathbf{x}^*) + c_4 \mu \delta \sqrt{n/m} \cdot \|\mathbf{x}^*\|_2 \leq \frac{1}{10} \|\mathbf{x}^*\|_2,
\]
as long as \(m\) satisfies \(m \geq c_5 n\) for some universal constant \(c_5\). Thus we proved that for all iterations we have
\[
\text{dist}(\mathbf{x}^{(t)}, \mathbf{x}^*) \leq \frac{1}{10} \|\mathbf{x}^*\|_2.
\]
Given this fact, direct computation leads to the following conclusion:
\[
\text{dist}(\mathbf{x}^{(t)}, \mathbf{x}^*) \leq \frac{1}{10} \left(1 - \frac{\mu}{2}\right)^t \cdot \|\mathbf{x}^*\|_2 + c_4 \sqrt{n/m} \cdot \|\mathbf{\epsilon}\|_\infty.
\]

**B Proof of Technical Lemmas in Section 6**

**B.1 Proof of Lemma 6.1**

*Proof.* Note that \(y_i^* = |a_i^T \mathbf{x}^*| = |\sum_{j=1}^n a_{ij} x_j|\) where \(a_{ij}\) follows \(N(0,1)\) distribution. Apply Hoeffding type inequality (Lemma D.3) and union bound to all \(i\) we have with probability at least \(1 - 1/m\),
\[
\|\mathbf{y}^*\|_\infty \leq \sqrt{c_0 \log(m)} \cdot \|\mathbf{x}^*\|_2,
\]

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where \( c_0 \) is a universal constant. It is easy to see \( \| \epsilon \|_\infty \leq \delta \| x^* \|_2 \leq \sqrt{c_0 \log(m)} \cdot \| x^* \|_2 \) with small \( \delta \). Thus we have
\[
\| y^* + \epsilon \|_\infty \leq \| y^* \|_\infty + \| \epsilon \|_\infty \leq 2\sqrt{c_0 \log(m)} \cdot \| x^* \|_2 \leq \sqrt{c_1 \log(m)} \cdot \| x^* \|_2,
\]
(B.1)

By definition we have
\[
\lambda_0^2 = \frac{1}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)^2 = \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i + y_i^* - \epsilon_i)^2 \\
= \frac{1}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)^2 + \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i + y_i^* - \epsilon_i)^2 + \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i).
\]
(B.2)

First we try to give the upper bound for \( \lambda_0 \). For term \( I_1 \), by lemma D.1 we have
\[
\frac{1}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)^2 = \frac{1}{m} \sum_{i=1}^{m} (y_i^*)^2 + \frac{2}{m} \sum_{i=1}^{m} y_i^* \cdot \epsilon_i + \frac{1}{m} \sum_{i=1}^{m} (\epsilon_i)^2 \\
\leq \frac{1}{m} \| y^* \|_2^2 + \frac{2}{m} \| y^* \|_2 \cdot \| \epsilon \|_2 + \frac{1}{m} \| \epsilon \|_2^2 \\
\leq (1 + \delta + 2\delta \sqrt{1 + \delta + \delta^2}) \| x^* \|_2^2 \leq (1 + 6\delta) \| x^* \|_2^2.
\]
(B.3)

where the first inequality follows from Cauchy-Schwarz inequality. For term \( I_2 \) we have
\[
\frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 = \frac{1}{m} \| \hat{y} - y^* - \epsilon \|_2^2 \\
\leq \alpha(\gamma + 1) \| \hat{y} - y^* - \epsilon \|_2^2 \leq 4\alpha(\gamma + 1) \| y^* + \epsilon \|_\infty^2,
\]
(B.4)

where the first inequality is implied by Claim 1 in Section A.1 and the second inequality follows from Claim 2 in Section A.1. By plug in (B.1) and the condition on \( \alpha \) we can further have
\[
\frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 \leq 4c_1 \alpha(\gamma + 1) \log(m) \| x^* \|_2^2 \leq \delta.
\]
(B.5)

For term \( I_3 \), similarly we have
\[
\frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i) \leq \frac{2}{m} \| y^* + \epsilon \|_\infty \cdot \| \hat{y} - y^* - \epsilon \|_1 \leq 2\alpha(\gamma + 1) \| y^* + \epsilon \|_\infty \cdot \| \hat{y} - y^* - \epsilon \|_\infty \\
\leq 4\alpha(\gamma + 1) \| y^* + \epsilon \|_\infty^2 \leq 4c_1 \alpha(\gamma + 1) \log(m) \cdot \| x^* \|_2^2 \\
\leq \delta,
\]
(B.6)

where the first inequality is due to Hölder inequality, the last inequality follows from the condition that \( \alpha \) and the rest is due to the same reason as for term \( I_2 \). By submitting (B.3), (B.5), (B.6) into (B.2) we have the following upper bound:
\[
\lambda_0^2 \leq (1 + 8\delta) \| x^* \|_2^2.
\]
Next we try to give the lower bound for $\lambda_0$. For term $I_1$, by lemma D.1 we have

$$\frac{1}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)^2 = \frac{1}{m} \sum_{i=1}^{m} (y_i^*)^2 + \frac{2}{m} \sum_{i=1}^{m} y_i^* \cdot \epsilon_i + \frac{1}{m} \sum_{i=1}^{m} (\epsilon_i)^2$$

$$\geq \frac{1}{m} \|y^*\|_2^2 - \frac{2}{m} \|y^*\|_2 \cdot \|\epsilon\|_2$$

$$\geq (1 - \delta - 2\delta \sqrt{1 + \delta}) \|x^*\|_2^2 \leq (1 - 5\delta) \|x^*\|_2^2.$$  \hspace{1cm} (B.7)

where the first inequality follows from Cauchy-Schwarz inequality. For term $I_2$ we obviously have

$$\frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 \geq 0.$$  \hspace{1cm} (B.8)

For term $I_3$, note that

$$\frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i) \geq - \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i)$$

$$\geq - \frac{2}{m} \|y^* + \epsilon\|_2 \cdot \|\hat{y} - y^* - \epsilon\|_2,$$

where the second inequality follows from Cauchy-Schwarz inequality. Following the same route as in (B.6) we have

$$\frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i) \geq -\delta.$$  \hspace{1cm} (B.9)

By submitting (B.7), (B.8), (B.9) into (B.2) we have the following lower bound:

$$\lambda_0^2 \geq (1 - 7\delta) \|x^*\|_2^2.$$  \hspace{1cm} \(\Box\)

This completes the proof.

**B.2 Proof of Lemma 6.2**

**Proof.** Note that according to the definition in Section A.1 we have

$$\|Y - \mathbb{E}[\bar{Y}]\|_2 \leq \|Y - \mathbb{E}[\bar{Y}]\|_2 + \|\bar{Y} - \bar{Y}\|_2$$

$$\leq \|\bar{Y} - \mathbb{E}[\bar{Y}]\|_2 + \|Y_{e1}\|_2 + \|Y_{e2}\|_2 + \|Y_{e3}\|_2.$$  \hspace{1cm} (B.10)

By applying the proof for the Gaussian model of Lemma 7.4 in Candes et al. (2015b) we have that for the first term in the R.H.S of (B.10), if $m \geq c_1 \cdot n \log n$,

$$\|\bar{Y} - \mathbb{E}[\bar{Y}]\|_2 \leq \delta,$$  \hspace{1cm} (B.11)

with probability at least $1 - 10e^{-c_2 n} - 8/n^2$. For term $Y_{e1}$, by model assumption $|\epsilon_i| \leq \delta \|x^*\|_2 = \delta$ we have

$$\|Y_{e1}\|_2 \leq \delta^2 . \left\| \frac{1}{m} \sum_{i=1}^{m} a_i a_i^\top \right\|_2 \leq \delta^2 (1 + \delta) \leq 2\delta.$$  \hspace{1cm} (B.12)
as long as $\delta < 1$ where the second inequality follows from Lemma D.1. For term $Y_{e2}$, note that by definition of the spectral norm we have

$$\left\| \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 a_i a_i^\top \right\|_2 = \max_{\|v\|_2 = 1} \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 v^\top a_i a_i^\top v,$$

(B.13)

where $v$ is any $n$-dimensional vector. Define the event $\mathcal{E}$ for simplicity:

$$\mathcal{E} := \{\|v^\top a_i\| \leq \tau, \text{ for all } \|v\|_2 = 1, i = 1, \ldots, m\},$$

where $\tau > 0$ which will be specified later. Let $\mathcal{E}^c$ be the compliment of $\mathcal{E}$ we have

$$\mathbb{P}[\mathcal{E}^c] = \mathbb{P}\left[ \max_i |v^\top a_i| > \tau \right] \leq m \cdot \mathbb{P}[|v^\top a_i| > \tau] \leq m \cdot \exp(-c_0\tau^2) = \delta,$$

where the last inequality follows from Hoeffding type inequality (Lemma D.3). This implies $\tau = \sqrt{c_0 \log(m/\delta)}$ is a feasible choice. Now submit the value of $\tau$ back and try upper bounding the first term on the R.H.S of (B.14) with $\tilde{\delta}$:

$$\mathbb{P}\left[ \mathcal{E}^c \right] = \mathbb{P}\left[ \max_i |v^\top a_i| > \tau \right] \leq m \cdot \mathbb{P}[|v^\top a_i| > \tau] \leq m \cdot \exp(-c_0\tau^2) = \delta,$$

where the last inequality follows from Hoeffding type inequality (Lemma D.3). This implies $\tau = \sqrt{c_0 \log(m/\delta)}$ is a feasible choice. Now submit the value of $\tau$ back and try upper bounding the first term on the R.H.S of (B.14) with $\tilde{\delta}$:

$$\mathbb{P}\left[ \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 \geq \frac{t}{\tau^2} \right] \leq \mathbb{P}\left[ \|y^* + \epsilon\|_\infty \geq \sqrt{\frac{t}{4c_0\alpha(\gamma + 1)\log(m/\delta)}} \right]$$

$$\leq m \cdot \exp\left( -\frac{t}{16\alpha(\gamma + 1)\log(m/\delta)} \right) = \tilde{\delta},$$

where the first inequality follows the same reasoning as in (B.4) and the second inequality follows the same reasoning as in (B.1). This implies to choose $t = 4\alpha(\gamma + 1)\log^2(m/\tilde{\delta})$. Thus we have with probability at least $1 - 2\delta$,

$$\left\| \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)^2 a_i a_i^\top \right\|_2 \leq 16\alpha(\gamma + 1)\log^2(m/\tilde{\delta}) \leq \delta,$$

(B.15)

where the last inequality is obtained by choosing $\tilde{\delta} = 1/m$ and the condition on $\alpha$. For term $Y_{e3}$, using similar conditioning argument as we did for term $y_{e2}$:

$$\left\| \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i) (\hat{y}_i - y_i^* - \epsilon_i) a_i a_i^\top \right\|_2 = \max_{\|v\|_2 = 1} \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i) (\hat{y}_i - y_i^* - \epsilon_i) v^\top a_i a_i^\top v,$$

(B.16)
and
\[
\mathbb{P} \left[ \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i)\mathbf{v}^\top \mathbf{a}_i \mathbf{v} \geq t \right]
\]
\[
\leq \mathbb{P} \left[ \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i)\mathbf{v}^\top \mathbf{a}_i \mathbf{v} \geq t, \mathcal{E} \right] + \mathbb{P} [\mathcal{E}^c].
\]
\[
\leq \mathbb{P} \left[ \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i) \geq t \right] + \mathbb{P} [\mathcal{E}^c].
\] (B.17)

Submit the value of \( \tau \) back and try upper bounding the first term on the R.H.S of (B.17) with \( \tilde{\delta} \):
\[
\mathbb{P} \left[ \frac{2}{m} \sum_{i=1}^{m} (y_i^* + \epsilon_i)(\hat{y}_i - y_i^* - \epsilon_i) \geq \frac{t}{\tau^2} \right] \leq \mathbb{P} \left[ \|\mathbf{y}^* + \epsilon\|_\infty \geq \sqrt{\frac{t}{4\alpha\gamma + 1 \log(m/\delta)}} \right]
\]
\[
\leq m \cdot \exp \left( -\frac{t}{16\alpha(\gamma + 1 \log(m/\delta))} \right) = \tilde{\delta},
\]
where the first inequality follows the same reasoning as in (B.6) and the second inequality follows the same reasoning as in (B.1). This implies to choose \( t = 4\alpha(\gamma + 1 \log(m/\delta)) \). Thus we have with probability at least \( 1 - 2\delta \),
\[
\left\| \frac{1}{m} \sum_{i=1}^{m} (\hat{y}_i - y_i^* - \epsilon_i)\mathbf{a}_i \mathbf{v} \right\|_2 \leq 16\alpha(\gamma + 1 \log(m/\delta)) \leq \delta,
\] (B.18)
where the last inequality is obtained by choosing \( \tilde{\delta} = 1/m \) and the condition on \( \alpha \). Combine the results from (B.10), (B.11), (B.12), (B.15) and (B.18) we have
\[
\|\mathbf{Y} - \mathbb{E}[\mathbf{Y}]\|_2 \leq 4\delta.
\] (B.19)

This completes the proof. \( \square \)

### B.3 Proof of Lemma 6.3

**Proof.** Note that we have
\[
\|\mathbf{h} - \mu \cdot \mathbf{g}(\mathbf{x}^{(t)}, \mathbf{y}^{(t+1)})\|_2^2 = \|\mathbf{h}\|_2^2 - 2\mu \langle \mathbf{g}(\mathbf{x}^{(t)}, \mathbf{y}^{(t+1)}), \mathbf{h} \rangle + \mu^2 \|\mathbf{g}(\mathbf{x}^{(t)}, \mathbf{y}^{(t+1)})\|_2^2.
\] (B.20)

For the inner product term we have
\[
\langle \mathbf{g}(\mathbf{x}^{(t)}, \mathbf{y}^{(t+1)}), \mathbf{h} \rangle = \frac{1}{m} \sum_{i \notin \mathcal{S}'} \left( \mathbf{a}_i^\top \mathbf{x}^{(t)} - (\mathbf{a}_i^\top \mathbf{x}^*) \cdot \text{sgn}((\mathbf{a}_i^\top \mathbf{x}^{(t)}) \cdot (\mathbf{a}_i^\top \mathbf{x}^*)) \right) \cdot (\mathbf{a}_i^\top \mathbf{h})
\]
\[
= \frac{1}{m} \left( \sum_{i \notin \mathcal{S}'} (\mathbf{a}_i^\top \mathbf{h})^2 + 2 \sum_{i \in \mathcal{S}} (\mathbf{a}_i^\top \mathbf{x}^*) \cdot (\mathbf{a}_i^\top \mathbf{h}) \right)
\]
\[
\geq \frac{1}{m} \sum_{i \notin \mathcal{S}'} (\mathbf{a}_i^\top \mathbf{h})^2 - \frac{2}{m} \sum_{i \in \mathcal{S}} \left| (\mathbf{a}_i^\top \mathbf{x}^*) \cdot (\mathbf{a}_i^\top \mathbf{h}) \right|
\]
\[
\geq \frac{1}{m} \sum_{i=1}^{m} (\mathbf{a}_i^\top \mathbf{h})^2 - \frac{1}{m} \sum_{i \notin \mathcal{S}'} (\mathbf{a}_i^\top \mathbf{h})^2 - \frac{2}{m} \sum_{i \in \mathcal{S}} \left| (\mathbf{a}_i^\top \mathbf{x}^*) \cdot (\mathbf{a}_i^\top \mathbf{h}) \right|,
\] (B.21)
where \( S := \{i : (a_i^\top x^*) \cdot (a_i^\top x^{(t)}) < 0, i \notin \Omega'\} \) and \( S' := \{i : (a_i^\top x^*) \cdot (a_i^\top x^{(t)}) < 0, 1 \leq i \leq m\} \).

Further we have with probability at least \( 1 - 2 \exp(-c_1\delta^2m) \) that

\[
\frac{2}{m} \sum_{i \in S'} |(a_i^\top x^*) \cdot (a_i^\top h)| \leq \frac{1}{m} \sum_{i=1}^{m} [(a_i^\top x^*)^2 + (a_i^\top h)^2] \mathbb{I} \{ (a_i^\top x^*) \cdot (a_i^\top x^{(t)}) < 0 \}
\]
\[
\leq \frac{1}{m} \sum_{i=1}^{m} [(a_i^\top x^*)^2 + (a_i^\top h)^2] \mathbb{I} \{ (a_i^\top x^*)^2 + (a_i^\top x^*) \cdot (a_i^\top h) < 0 \}
\]
\[
\leq \frac{1}{m} \sum_{i=1}^{m} [(a_i^\top x^*)^2 + (a_i^\top h)^2] \mathbb{I} \{ |a_i^\top x^*| \leq |a_i^\top h| \}
\]
\[
\leq \frac{2}{m} \sum_{i=1}^{m} (a_i^\top x^*)^2 \mathbb{I} \{ |a_i^\top x^*| \leq |a_i^\top h| \}
\]
\[
\leq (0.26 + 2\delta) \|h\|_2^2, \tag{B.22}
\]

where the last inequality follows from Lemma 3 in Zhang and Liang (2016) (presented in Lemma D.2). Moreover, note that \( |\Omega'| = \gamma m \). Apply Lemma D.3 and union bound to all \( i \) we have \( \max_i |a_i^\top h| \leq \sqrt{c_2 \log(m)} \cdot \|h\|_2 \) with probability at least \( 1 - 1/m \). Thus we obtain

\[
\frac{1}{m} \sum_{i \in \Omega'} (a_i^\top h)^2 \leq c_2 \gamma \alpha \cdot \log m \cdot \|h\|_2^2 \leq \delta \cdot \|h\|_2^2, \tag{B.23}
\]

where the last inequality due to that \( \alpha \leq \delta/(c_2 \gamma \cdot \log m) \). Combine (B.22) and (B.23) with (B.21) we have

\[
\langle g(x^{(t)}, \eta^{(t+1)}), h \rangle \geq (0.74 - 3\delta) \|h\|_2^2. \tag{B.24}
\]

For the square term \( \|g(x^{(t)}, \eta^{(t+1)})\|_2^2 \), denote each element of \( v \) as \( v_i = (|a_i^\top x^{(t)}| - |a_i^\top x^*|) \cdot \text{sgn}(a_i^\top x^{(t)}) \) and \( |(\Omega')^C| \) as \( s \) we have

\[
\|g(x^{(t)}, \eta^{(t+1)})\|_2^2 = \left\| \frac{1}{m} \sum_{i : \notin \Omega'} \left( |a_i^\top x^{(t)}| - |a_i^\top x^*| \right) \text{sgn}(a_i^\top x^{(t)}) a_i \right\|_2^2 = \frac{1}{m^2} \|A_s v_s\|_2^2
\]
\[
\leq \frac{1}{m^2} \|A_s\|_2^2 \cdot \|v_s\|_2^2 \leq \frac{1}{m^2} \|A\|_2^2 \cdot \|v\|_2^2,
\]

where \( A_s \in \mathbb{R}^{s \times n} \) is a matrix with each row being \( a_i^\top, i \notin \Omega' \), \( v_s \in \mathbb{R}^s \) is a vector with each element being \( v_i, i \notin \Omega' \), \( v \in \mathbb{R}^m \) is a vector with each element being \( v_i \) and the first inequality is due to Cauchy-Schwarz inequality. By Theorem 5.32 in Vershynin (2010) we have \( \|A\|_2 \leq \sqrt{(1 + \delta)m} \) with probability at least \( 1 - 2 \exp(-c_1\delta^2m) \). Consider

\[
|v_i|^2 = \left| (|a_i^\top x^{(t)}| - |a_i^\top x^*|) \cdot \text{sgn}(a_i^\top x^{(t)}) \right|^2 \leq \left| |a_i^\top x^{(t)}| - |a_i^\top x^*| \right|^2 \leq (a_i^\top h)^2.
\]

By Lemma D.1 we have

\[
\|v\|_2^2 = \sum_{i=1}^{m} |v_i|^2 = \sum_{i=1}^{m} (a_i^\top h)^2 = \sum_{i=1}^{m} h^\top a_i a_i^\top h \leq (1 + \delta)m \|h\|_2^2. \tag{B.25}
\]
Therefore combine the above results we obtain
\[ \|g(x(t), \eta(t+1))\|_2^2 \leq (1 + \delta)^2 \|h\|_2^2. \] \hspace{1cm} (B.26)

Submit (B.26) and (B.24) back into (B.20) and let \( \delta \) to be small enough we have
\[ \|h - \mu \cdot g(x(t), \eta(t+1))\|_2^2 = (1 - 2\mu(0.74 - 3\delta) + \mu^2(1 + \delta)^2) \|h\|_2^2. \]

This completes the proof. \( \square \)

B.4 Proof of Lemma 6.4

Proof. Denote \( u_i := \eta_i^* \cdot \text{sgn}(a_i^T x(t)) \), we have
\[
\left\| \frac{1}{m} \sum_{i \in \Omega \setminus \Omega'} \eta_i^* \cdot \text{sgn}(a_i^T x(t)) a_i \right\|_2 = \left\| \frac{1}{m} \sum_{i \in \Omega' \setminus \Omega} \eta_i^* \cdot \text{sgn}(a_i^T x(t)) a_i \right\|_2 = \frac{1}{m} \|A_s^T u_s\|_2 \\
\leq \frac{1}{m} \|A_s\|_2 \cdot \|u_s\|_2 \leq \frac{1}{m} \|A\|_2 \cdot \|u_s\|_2, \]
\hspace{1cm} (B.27)

where \( A_s \in \mathbb{R}^{s \times n} \) is a matrix with each row being \( a_i^T \), \( i \notin \Omega' \); \( u_s \in \mathbb{R}^s \) is a vector with each element being \( u_i, i \notin \Omega' \), the first equality is due to the fact that \( \eta_i^* \) is nonzero only when \( i \) belongs to the support set \( \Omega^* \) and the first inequality follows from Cauchy-Schwarz inequality. Again, by Theorem 5.32 in Vershynin (2010) we have \( \|A\|_2 \leq \sqrt{(1 + \delta)m} \) with probability at least \( 1 - 2 \exp(-c_1\delta^2m) \).

Further note that
\[
\|u_s\|_2^2 = \sum_{i \notin \Omega'} |u_i|^2 = \sum_{i \notin \Omega'} |\eta_i|^2 = \sum_{i \in \Omega^* \setminus \Omega'} |\eta_i|^2, \]
\hspace{1cm} (B.28)

where the last equality is again due to the fact that \( \eta_i^* \) is nonzero only when \( i \) belongs to the support set \( \Omega^* \). Note that when samples are from the support set \( \Omega^* \setminus \Omega' \), it implies that \( |y_i - |a_i^T x(t)|| \leq |(y - A(x(t)))|/\gamma am | \) where \( A(x(t)) \) \( i = |a_i^T x(t)|. \) Since \( \eta^* \) has at most \( am \) nonzero entries, we claim that
\[
|\eta_i^*| \leq 2\|y^* - A(x(t)) + \epsilon\|_{\infty}. \]
\hspace{1cm} (B.29)

The reason behind this is simple: if otherwise, we would have
\[
|y_i - |a_i^T x(t)|| = |\eta_i^* + \epsilon_i - |a_i^T x(t)|| \\
\geq \|y^* - A(x(t)) + \epsilon\|_{\infty} \\
= \|y - A(x(t)) - \eta^*\|_{\infty} \\
\geq \|(y - A(x(t)))|/\gamma am, \]

where the equality is due to the model defined in (3.1) and the last inequality follows that \( \eta^* \) has at most \( am \) nonzero entries. Specifically, suppose in the worst case, all \( am \) entries' of \( \eta^* \) have large magnitude which would be hard thresholded in the first place, then \( \gamma am \)-largest element would be
one of the elements in $y - A(x^{(t)})$, and thus less than or equal to $\| y - A(x^{(t)}) - \eta^* \|_{\infty}$ which causes the contradiction. Thus combining (B.28) and (B.29) we have

$$\| u_s \|_2 \leq 2\sqrt{\alpha m} \cdot \| y^* - A(x^{(t)}) + \epsilon \|_{\infty} \leq 2\sqrt{\alpha m} \cdot \| y^* - A(x^{(t)}) \|_{\infty} + 2\sqrt{\alpha m} \cdot \| \epsilon \|_{\infty},$$

(B.30)

where the last inequality is due to triangle inequality. Since by Hoeffding type inequality (Lemma D.3) we have

$$\| y^* - A(x^{(t)}) \|_{\infty} = \max_i \| a_i^T x^{(t)} - |a_i^T x^*| \| \leq \max_i \| a_i^T h \| \leq \sqrt{c_0 \log m \cdot \| h \|_2},$$

(B.31)

(B.30) can be further written as

$$\| u_s \|_2 \leq 2\sqrt{c_0 \alpha m \cdot \log m \cdot \| h \|_2} + 2\sqrt{\alpha m} \cdot \| \epsilon \|_{\infty}.$$  

(B.32)

Submit the above result back into (B.27) we have

$$\frac{1}{m} \sum_{i \not\in \Omega^c} \eta_i^* \cdot \text{sgn}(a_i^T x^{(t)}) a_i \|_2 \leq 2\sqrt{c_0 \alpha (1 + \delta) \cdot \log m \cdot \| h \|_2} + 2\sqrt{\alpha (1 + \delta) \cdot \| \epsilon \|_{\infty}} \leq 2\sqrt{\delta} \cdot \| h \|_2 + 2\sqrt{\alpha (1 + \delta) \cdot \| \epsilon \|_{\infty}},$$

where the last inequality holds as long as $\alpha \leq \delta/[c_0 (1 + \delta) \log m]$. This completes the proof.

B.5 Proof of Lemma 6.5

To prove Lemma 6.5, we need the following lemma.

Lemma B.1. Suppose $a_i \in \mathbb{R}^n, i = 1, \ldots, m$ are Gaussian vectors independently drawn from $N(0, I_{n \times n})$ and $w \in \mathbb{R}^m$ is independent with \{a_i\}_{i=1}^m, we have

$$P\left(\left\| \frac{1}{m} \sum_{i=1}^m w_i a_i \right\|_2 \geq c_4 \frac{\| w \|_2}{\sqrt{m}} \sqrt{\frac{n}{m}} \right) \leq \exp(-n/2),$$

where $c_4$ is a universal constant.

Proof of Lemma 6.5. Set $w_i = \epsilon_i \cdot \text{sgn}(a_i^T x^{(t)})$ and $s_3 := |(\Omega')^c|$. Since we have $\| w \|_2 \leq \| \epsilon \|_{\infty} \cdot \sqrt{s_3}$, by Lemma B.1, with probability at least $1 - \exp(-n/2)$ we get

$$\left\| \frac{1}{m} \sum_{i \not\in \Omega'} \epsilon_i \cdot \text{sgn}(a_i^T x^{(t)}) a_i \right\|_2 \leq c_4 \| \epsilon \|_{\infty} \cdot \sqrt{\frac{n}{m}} \sqrt{\frac{s_3}{m}} \leq c_4 \| \epsilon \|_{\infty} \cdot \sqrt{\frac{n}{m}}.$$
C Proof of Technical Lemmas in Section B

C.1 Proof of Lemma B.1

Proof. Since \( \{a_i\}_{i=1}^{m} \) are i.i.d. Gaussian vectors drawn from \( N(0, I_{n \times n}) \), conditioning on \( \{w_i\}_{i=1}^{m} \), the random vector \( \frac{1}{m} \sum_{i=1}^{m} w_i a_i \) is still a Gaussian vector with zero mean and covariance

\[
\Sigma = \frac{\|w\|^2}{m^2} \cdot I_{n \times n}.
\]

Therefore, suppose we denote \( g \) as the Gaussian vector \( \frac{1}{m} \sum_{i=1}^{m} w_i a_i \), by definition of \( L_2 \) norm we have

\[
\|g\|_2 = \sup_{\|w\|_2=1} \langle w, g \rangle.
\]

Thus \( \|g\|_2 \) is a supremum of a Gaussian process, hence by Theorem 7.1 in Ledoux (2005) we have

\[
P(\|g\|_2 \geq \mathbb{E}[\|g\|_2] + t) \leq \exp \left( \frac{-t^2 m^2}{2 \|w\|^2_2} \right).
\]

Now we use a covering argument to bound the expectation of \( \|g\|_2 \). Suppose \( \{w^1, w^2, \ldots, w^N\} \) is a 1/2-covering set of sphere \( S^{n-1} \), i.e., for any \( w \in S^{n-1} \), there exist \( w^i \) such that \( \|w - w^i\|_2 \leq 1/2 \). Note that we have

\[
\langle w, g \rangle = \langle w^i, g \rangle + \langle w - w^i, g \rangle.
\]

Taking suprema on both sides we have

\[
\|g\|_2 \leq \max_i \langle w^i, g \rangle + \frac{1}{2} \|g\|_2,
\]

which give rise to

\[
\|g\|_2 \leq 2 \max_i \langle w^i, g \rangle.
\]

Since each \( \langle w^i, g \rangle \) is zero-mean Gaussian with variance \( \|w\|^2_2/m^2 \), by standard Gaussian maxima bound we have

\[
\mathbb{E}[\|g\|_2] \leq \frac{4 \|w\|^2_2}{m} \sqrt{N} \leq \frac{4 \|w\|^2_2}{m} \sqrt{n \cdot \log 5}.
\]

Thus by combining the above results and choose \( t = O(\sqrt{n} \cdot \|w\|^2_2/m) \) we have

\[
P\left( \|g\|_2 \geq c_0 \frac{\|w\|_2}{\sqrt{m}} \sqrt{n/m} \right) \leq \exp \left( - \frac{n}{2} \right).
\]

This completes the proof.
D  Additional Auxiliary Lemmas

Lemma D.1 (Lemma 3.1 in Candes et al. (2013)). For any $0 < \delta < 1$, $a_i \sim N(0, I_{n \times n})$ independently, if $m > c_0 \delta^{-2} n$, then for all $h \in \mathbb{R}^n$, with probability at least $1 - 2 \exp(-c_1 \delta^2 m)$ we have

$$(1 - \delta) \|h\|_2^2 \leq \frac{1}{m} \sum_{i=1}^{m} (a_i^T h)^2 \leq (1 + \delta) \|h\|_2^2,$$

where $c_0, c_1$ are universal constants.

Lemma D.2 (Lemma 3 in Zhang and Liang (2016)). For any $0 < \delta < 1$, if $m > c_0 \delta^{-2} n \log \delta^{-1}$, then for all $h \in \mathbb{R}^n$ satisfying $\|h\|_2 \leq \|x^*\|_2/10$, with probability at least $1 - 2 \exp(-c_1 \delta^2 m)$ we have

$$\frac{1}{m} \sum_{i=1}^{m} (a_i^T h)^2 \cdot 1\{(a_i^T x^*)(a_i^T h) < 0\} \leq (0.13 + \delta) \|h\|_2^2,$$

where $c_0, c_1$ are universal constants.

Theorem D.3 (Proposition 5.10 in Vershynin (2010)). Suppose $X_1, X_2, \ldots, X_n$ are independent centered sub-Gaussian random variables, and $K = \max_i \|X_i\|_\psi_2$, then for every $a = [a_1, a_2, \ldots, a_n]^T \in \mathbb{R}^n$ and for every $t > 0$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} a_i X_i\right| > t\right) \leq \exp \left(- \frac{Ct^2}{K^2 \|a\|_2^2}\right),$$

where $C > 0$ is a constant.

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