Multilevel Monte Carlo methods for the approximation of invariant distribution of Stochastic Differential Equations

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Abstract

We develop a framework that allows the use of the multi-level Monte Carlo (MLMC) methodology [5] to calculate expectations with respect to the invariant measures of ergodic SDEs. In that context, we study the (over-damped) Langevin equations with a strongly concave potential. We show that, when appropriate contracting couplings for the numerical integrators are available, one can obtain a uniform in time estimate of the MLMC variance in contrast to the majority of the results in the MLMC literature. As a consequence, one can approximate expectations with respect to the invariant measure in an unbiased way without the need of a Metropolis-Hastings step. In addition, a root mean square error of $O(\varepsilon)$ is achieved with $O(\varepsilon^{-2})$ complexity on par with Markov Chain Monte Carlo (MCMC) methods, which however can be computationally intensive when applied to large data sets. Finally, we present a multi-level version of the recently introduced Stochastic Gradient Langevin (SGLD) method [33] built for large datasets applications. We show that this is the first stochastic gradient MCMC method with complexity $O(\varepsilon^{-2} \log \varepsilon^{3})$, in contrast to the complexity $O(\varepsilon^{-3})$ of currently available methods. Numerical experiments confirm our theoretical findings.

1 Introduction

We consider a probability density $\pi(x) \propto e^{U(x)}$ on $\mathbb{R}^d$ with the unknown normalising constant. A typical task is the computation of the following quantity

$$\pi(g) := \mathbb{E}_\pi g = \int_{\mathbb{R}^d} g(x)\pi(dx), \quad g \in L^1(\pi). \quad (1)$$

Even if $\pi(dx)$ is given in an explicit form, quadrature methods, in general, are inefficient in high dimensions. On the other hand probabilistic methods scale very well with the dimension

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and are often the method of choice. With this in mind, we explore the connection between dynamics of stochastic differential equations (SDEs)

\[ dX_t = \nabla U(X_t)dt + \sqrt{2}dW_t, \quad X_0 \in \mathbb{R}^d, \]  

(2)

and the target probability measure \( \pi(dx) \). The key idea is that under appropriate assumptions on \( U(x) \) one can show that the solution to (2) is ergodic and has \( \pi(dx) \) as its unique invariant measure [8]. However, there exist a limited number of cases where analytical solutions to (2) are available and therefore numerical approximations are used [14].

The numerical analysis approach is to discretize (2). Extra care is required when \( U \) is not globally Lipschitz [21, 28]. The numerical analysis approach is that it might be the case that even though (2) is geometrically ergodic, the corresponding discretization might not be [24]. The numerical analysis approach also introduces bias because the numerical invariant measure does not coincide with the exact one in general [29], resulting hence in a biased estimation of \( \pi(g) \) in (1). Furthermore, if one uses the Euler-Maruyama method to discretize (2), then computational complexity of \( O(\varepsilon^{-3}) \) is required for achieving a root mean square error of order \( O(\varepsilon) \) in the approximation of (1). Furthermore, even if one mitigates the bias due to numerical discretization by a series of decreasing time steps in combination with an appropriate weighted time average of the quantity of interest [16], the computational complexity still remains \( O(\varepsilon^{-3}) \). [31]

An alternative way of sampling from \( \pi \) exactly, so that it does not face the bias issue introduced by pure discretization of (2), is by using the Metropolis-Hastings algorithm [9]. We will refer to this as the computational statistics approach. The fact that the Metropolis Hastings algorithm leads to asymptotically unbiased samples of the probability measure is one of the reasons why it has been the method of choice in computational statistics. Moreover, unlike the numerical analysis approach, computational complexity of \( O(\varepsilon^{-2}) \) is now required for achieving root mean square error of order \( O(\varepsilon) \) in the (asymptotically unbiased) approximation of (1) (see Table 1). I like the table but would delete comments, also we need to be careful with unbiasing.

We notice that MLMC [5] and the unbiasing scheme [22, 23] are able to achieve the \( O(\varepsilon^{-2}) \) complexity for computing expectations of SDEs on a fixed time interval \([0, T]\), despite using biased numerical discretizations. We are interested in extending this approach to the case of ergodic SDEs on the time interval \([0, \infty)\), see also discussion in [5].

A particular application of (2) is when one is interested in approximating the posterior expectations for a Bayesian inference problem. More precisely, if the data \( y_i \) are i.i.d \( U(x) \) becomes

\[ \nabla U(x) = \nabla \log \pi_0(x) + \sum_{i=1}^{N} \nabla \log \pi(y_i|x), \]  

(3)

with \( \pi_0(x) \) being the prior distribution of \( x \). When dealing with problems where the number of data items \( N \gg 1 \) is large, both the standard numerical analysis and the MCMC approaches suffer due to the high computational cost associated with calculating the likelihood terms \( \nabla \log \pi(y_i|x) \) over each data item \( y_i \). One way to circumvent this problem is the stochastic gradient Langevin algorithm (SGLD) introduced in [33], which replaces the sum of the \( N \) likelihood terms by an appropriately reweighted sum of \( s \ll N \) terms. This leads to the

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1In this paper the computational complexity is measured in terms of the expected number of random number generations and arithmetic operations.
Numerical analysis

| Name | Standard MCMC | Time average of $m$-order methods | MLMC for SGLD | MLMC framework for SDE |
|------|---------------|-----------------------------------|--------------|----------------------|
| Comment | e.g. RWM, HMC, MALA if geometrically ergodic | if discretised SDE is geometrically ergodic | If contracting coupling is available. | |
| Cost | $\varepsilon^{-2}$ | $\varepsilon^{-\frac{2(2m+1)}{2m}}$ | $\varepsilon^{-2} |\log \varepsilon|^3$ | $\varepsilon^{-2}$ |
| Bias | by choice of burn-in is asymptotically negligible | vanishing for appropriately decreasing stepsize otherwise | cannot be debiased | can be debiased by randomisation |
| Reference for Cost | [25] [17] | [20] [30] | present article | |
| Parallel computation | yes, but all chains are affected by burn-in | yes | |

Table 1: Summary of different numerical approaches for approximating expectations with respect to a probability measure.

Following recursion formula

$$x_{k+1} = x_k + h \left( \nabla \log \pi_0(x_k) + \frac{N}{s} \sum_{i=1}^{s} \nabla \log \pi(y_{\tau_k}^i|x_k) \right) + \sqrt{2h} \xi_k$$

(4)

where $\xi_k$ is a standard Gaussian random variable on $\mathbb{R}^d$ and $\tau_k = (\tau_k^1, \cdots, \tau_k^s)$ is a random subset of $[N] = \{1, \cdots, N\}$, generated for example by sampling with or without replacement from $[N]$. Notice, that this corresponds to a noisy Euler discretisation, which for fixed $N, s$ still has computational complexity $O(\varepsilon^{-3})$ as discussed in [31] [30]. In this article, we are able to show that careful coupling between fine and coarse paths allows the application of the MLMC framework and hence reduction of the computational complexity of the algorithm to $O(\varepsilon^{-2}(\log \varepsilon)^3)$.

In summary the main contributions of this paper are:

1. Extension of the MLMC framework to the time interval $[0, \infty)$ for (2) when $U$ is strongly concave.

2. A convergence theorem that allows the estimation of the MLMC variance using uniform in time estimates in the Wasserstein metric for a variety of different numerical methods.
3. A new method for (unbiasedly) estimating expectations with respect to the invariant measures without the need of accept/reject steps (as in MCMC). The methods we propose can be better parallelised than MCMC.

4. The application of this scheme to stochastic gradient Langevin dynamics (SGLD) which reduces the complexity of $\varepsilon^{-3}$ to $\varepsilon^{-2} |\log \varepsilon|^{3}$ much closer to the standard $\varepsilon^{-2}$ complexity of MCMC.

The rest of the paper is organised as follows. In Section 2 we describe the standard MLMC framework, discuss the contracting properties of the true trajectories of (2) and describe an algorithm for applying MLMC with respect to time $T$ for the true solution of (2). In Section 3 we present the new algorithm, as well as a framework that allows proving its convergence properties for a numerical method of choice. In Section 4 we present two examples of suitable numerical methods, while in Section 5 we describe a new version of SGLD with complexity $\varepsilon^{-2} |\log \varepsilon|^{3}$. We conclude in Section 6 where a number of relevant numerical experiments are described.

2 Preliminaries

In Section 2.1 we revise the classic, finite time, MLMC framework, while in Section 2.2 we state key the asymptotic properties of solutions of (2) when $U$ is strongly concave.

2.1 MLMC with fixed terminal time $T$.

Fix $T > 0$ and consider the problem of approximating $E[g(X_T)]$ where $X_T$ is a solution of the SDE (2) and $g : \mathbb{R}^d \to \mathbb{R}$. A classical approach to this problem consists of constructing a biased (bias arise due to time-discretization) estimator of the form

$$\frac{1}{N} \sum_{i=1}^{N} g((x^M_T)^{(i)})$$

(5)

where $\{x^M_T\}$, is a discrete time approximation of (2) over $[0,T]$ with $M$ number of time steps. A central limit theorem for the estimator (5) has been derived in [3], and it was shown its computational complexity is $O(\varepsilon^{-3})$, for the root-mean-square error $O(\varepsilon)$ (as opposed to $O(\varepsilon^{-2})$ that can be obtained if we could sample $X_T$ without the bias). The recently developed Multilevel Monte Carlo approach allows recovering optimal complexity $O(\varepsilon^{-2})$, despite the fact that the estimator used builds on biased samples. This is achieved by exploiting the following identity [5] [11]

$$E[g_L] = E[g_0] + \sum_{\ell=1}^{L} E[g_{\ell} - g_{\ell-1}],$$

(6)

where $g_{\ell} := g(x^M_T)$ with $\{x^M_T\}$, $\ell = 0 \ldots L$, is the discrete time approximation of (2) over $[0,T]$ with $M_\ell$ number of time steps, $M_\ell h_\ell = T$. This identity leads to the following unbiased estimator of $E[g(x^M_T)]$,

$$\frac{1}{N_0} \sum_{i=1}^{N_0} g^{(i,0)} + \sum_{\ell=1}^{L} \left\{ \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} (g^{(i,\ell)} - g^{(i,\ell-1)}) \right\},$$

(4)
where \( g_{\ell}^{(i,\ell)} = g((x_T^M)^{(i)}) \) are independent samples at level \( \ell \). The inclusion of the level \( l \) in the superscript \((i,\ell)\) indicates that independent samples are used at each level \( l \). The efficiency of MLMC lies in the coupling of \( g_{\ell}^{(i,\ell)} \) and \( g_{\ell-1}^{(i,\ell-1)} \) that results in small \( \text{Var}[g_{\ell} - g_{\ell-1}] \).

By solving a constrained optimization problem (cost&accuracy) one can see that reduced computational cost (variance) arises since the MLMC method allows one to efficiently combine many simulations on low accuracy grids (at a corresponding low cost), with relatively few simulations computed with high accuracy and high cost on very fine grids. It is shown in Giles [5] that under the assumptions\(^2\)

\[
|\mathbb{E}[g - g_{\ell}]| \leq c_1 h_\ell^\alpha, \quad \text{Var}[g_{\ell} - g_{\ell-1}] \leq c_2 h_\ell^\beta,
\]

for some \( \alpha \geq 1/2, \beta > 0, c_1 > 0 \) and \( c_2 > 0 \), the computational complexity of the resulting multi-level estimator with the accuracy \( \varepsilon \) is proportional to

\[
\mathcal{C} = \begin{cases} 
\varepsilon^{-2}, & \beta > \gamma, \\
\varepsilon^{-2} \log^2(\varepsilon), & \beta = \gamma, \\
\varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < \gamma
\end{cases}
\]

where the cost of the algorithm is of order \( h^{-\gamma} \). Typically, constants \( c_1, c_2 \) grow exponentially as they follow from classical finite time weak and strong convergence analysis of the numerical schemes. The aim of this paper is to establish bounds \((7)\) uniformly in time, i.e.

\[
\sup_{T>0} |\mathbb{E}[g - g_{\ell}]| \leq c_1 h_\ell^\alpha, \quad \sup_{T>0} \text{Var}[g_{\ell} - g_{\ell-1}] \leq c_2 h_\ell^\beta.
\]

The reader may notice that in the regime when \( \beta > \gamma \), the computationally complexity of \( \varepsilon^{-2} \) coincides with that of an unbiased estimator. However, the MLMC estimator as defined here is still biased, with the bias being controlled by the choice of level parameter \( L \). In Section 3.2 we will demonstrate that in fact it is possible eliminate the bias by a clever randomisation trick.

### 2.2 Properties of ergodic SDEs with strongly concave drifts

Consider the SDE \((2)\) and let \( U \) satisfy the following condition

**HU0** For any \( x, y \in \mathbb{R}^d \) there exists constant \( m \) s.t

\[
\langle \nabla U(y) - \nabla U(x), y - x \rangle \leq -m|x - y|^2,
\]

which is also known as a one-side Lipschitz condition. Condition **HU0** is satisfied for strongly concave potential, i.e when for any \( x, y \in \mathbb{R}^d \) there exists constant \( m \) s.t

\[
U(y) \leq U(x) + \langle \nabla U(x), y - x \rangle - \frac{m}{2} |x - y|^2.
\]

\( ^2\)Recall \( h_\ell \) is the time step used in the discretization of the level \( l \).
Condition **HU0** ensures the contraction needed to establish uniform in time estimates. For the transparency of the exposition we introduce the following flow notation for the solution to (2), starting at $X_0 = x$

$$
\psi_{s,t,W}(x) := x + \int_s^t \nabla U(X_r)dr + \int_s^t \sqrt{2}dW_r, \quad x \in \mathbb{R}^d. \tag{10}
$$

The theorem below demonstrates that solutions to (2) driven with the same Brownian motion, but with different initial conditions enjoy a exponential contraction property.

**Theorem 2.1.** Let $(W(t))_{t \geq 0}$ be a standard $\mathbb{R}^d$ Brownian Motion. Take random variable $Y_0, X_0 \in \mathbb{R}^d$ and define $X_T = \psi_{0,T,W}(X_0)$ and $Y_T = \psi_{0,T,W}(Y_0)$. Then

$$
\mathbb{E}|X_T - Y_T|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{-2mT} \tag{11}
$$

**Proof.** The result follows from Itô’s formula. Indeed we have

$$
\frac{1}{2} e^{2mt}|X_t - Y_t|^2 = \frac{1}{2}|X_0 - Y_0|^2 + \int_0^t (me^{2ms}|X_s - Y_s|^2 + e^{2ms} \langle \nabla U(X_s) - \nabla U(Y_s), X_s - Y_s \rangle) ds.
$$

Assumption **HU0** yields

$$
\mathbb{E}|X_t - Y_t|^2 \leq e^{-2mT}\mathbb{E}|X_0 - Y_0|^2,
$$

as required. \qed

**Digression on s-Wasserstein norm.** The $s$-Wasserstein distance between measures $\nu_1$ and $\nu_2$ defined on a Polish space $E$, is given by

$$
W_s(\nu_1, \nu_2) = \left( \inf_{\pi \in \Gamma(\nu_1, \nu_2)} \int_{E \times E} d^s(x, y) \pi(dx, dy) \right)^{\frac{1}{s}},
$$

with $\Gamma(\nu_1, \nu_2)$ being the set of couplings of $\nu_1$ and $\nu_2$ (all measures on $E \times E$ with marginals $\nu_1$ and $\nu_2$) and $d(x, y) = |x - y|$. We denote $\mathcal{L}(\psi_{0,t,W}(x)) = P_t(x, \cdot)$. That is $P_t$ is the transition kernel of the SDE (2).

Since the choice of the same driving BM in (2.1) is an example of coupling (11) implies

$$
W_2(P_t(x, \cdot), P_t(y, \cdot)) \leq |x - y| \exp(-mt) \tag{12}
$$

Consequently $P_t$ only has one unique invariant distribution and thus the process is ergodic [7]. Although optimality of the coupling in the definition of Wasserstein norm is an appealing property, for the practical considerations one should only consider coupling that are feasible to implement. That is the philosophy we follow in this paper. For a more general discussion of implementable couplings we refer the reader to [2, 6].

For the MLMC in $T$, coupling with the same Brownian motion is not enough, as in general solutions to SDEs (2) are $1/2$-Hölder continuous, [15], i.e for any $t > s > 0$ there exists a constant $C > 0$ such that

$$
\mathbb{E}|X_t - X_s|^2 \leq C|t - s| \tag{13}
$$

and it is well known that this bound is sharp. As we shall see later this bound will not lead to an efficient MLMC implementation. However, by suitable coupling of the SDE solutions on
time intervals of length $T$ and $S$, $T > S$, respectively, we will be able to take advantage of the exponential contraction property obtained in Theorem 2.1. This idea is clearly demonstrated in Figure 1 and described for the context of simulating SDEs below. To couple processes with different terminal times $T_i$ and $T_j$, $i \neq j$, we exploit the time homogeneous Markov property of the flow \( (10) \). More precisely, we construct a pair of solutions to \( (2) \) \( (X^{(f,\ell)}, X^{(c,\ell)}), \ i \geq 0 \) which we refer to as fine and coarse paths, such that

\[
\mathcal{L}(X^{(f,\ell)}(T_{\ell})) = \mathcal{L}(X^{(c,\ell)}(T_{\ell-1})), \ \forall i \geq 0,
\]

and

\[
\mathbb{E}|X^{(f,\ell)} - X^{(c,\ell)}|^2 \leq \mathbb{E}|X^{(f,\ell)}(0) - X^{(c,\ell)}(0)|^2 e^{-2mT_{\ell-1}}.
\]

Following [22, 23, 2, 5] we propose to

- First obtain solution to \( (2) \) over \([0, T_{\ell-1}]\). We take \( X^{(f,\ell)}(0) = \psi_{0,T_{\ell-1},W}(X(0)) \)
- Next couple fine and coarse paths on the remaining time interval \([0, T_{\ell-1}]\) using the same Brownian motion \( (W) \) i.e

\[
X^{(f,\ell)}(T_{\ell-1}) = \psi_{0,T_{\ell-1},W}(X^{(f,\ell)}(0)), \quad X^{(c,\ell)}(T_{\ell-1}) = \psi_{0,T_{\ell-1},W}(X(0)).
\]

Since \( \nabla U(\cdot) \) in \( (2) \) is time homogenous and consequently so is corresponding transition kernel \( \mathcal{L}(\psi_{0,t,W}(x)) = P_t(x,\cdot) \), condition 14 holds.

Theorem 2.1 yields

\[
\mathbb{E}|X^{(f,\ell)}(T_{\ell-1}) - X^{(c,\ell)}(T_{\ell-1})|^2 \leq \mathbb{E}|X^{(f,\ell)}(0) - X(0)|^2 e^{-2mT_{\ell-1}}.
\]

Take \( \rho > 1 \) and define

\[
T_{\ell} := \frac{\log 2}{2m} \rho(\ell + 1) \quad \forall \ell \geq 0.
\]

In our case \( g^{(i,\ell)} = g((X^{(f,\ell)}(T_{\ell-1}))^{(i)}) \) and \( g^{(i,\ell-1)} = g((X^{(c,\ell)}(T_{\ell-1}))^{(i)}) \) and we assume that \( g \)
is globally Lipschitz. Hence
\[
\mathbb{E}[g(X^{(f,\ell)}(T_{\ell-1})) - g(X^{(c,\ell)}(T_{\ell-1}))]^2 \leq \text{Lip}(g)^2 \mathbb{E}[X^{(f,\ell)}(T_{\ell-1}) - X^{(c,\ell)}(T_{\ell-1})]^2 \\
\leq \text{Lip}(g)^2 \mathbb{E}[X^{(f,\ell)}(0) - X(0)]^2 e^{-2mT_{\ell-1}} \\
\leq \text{Lip}(g)^2 \mathbb{E}[X^{(f,\ell)}(0) - X_0]^2 2^{-\rho_\ell} \\
\leq \text{Lip}(g)^2 \mathbb{C}|T_\ell - T_{\ell-1}| 2^{-\rho_\ell}. \quad \forall \ell \geq 0,
\]
where the last inequality follows from (13).

3 MLMC in $T$ for SDEs with biased samples

Having described a coupling algorithm with good contraction properties, we now present the main algorithm in Section 3.1. We then in Section 3.2 discuss an approach that allows to obtain unbiased estimators for the quantities of interest. Finally, in Section 3.3 we present a general numerical analysis framework for proving the convergence of our algorithm.

3.1 Description of the algorithm

We now focus on the numerical discretization of the Langevin equation (2). In particular, we are interested in coupling the discretizations of (2) based on step size $h_\ell$ and $h_{\ell-1}$ with $h_\ell = h_0 2^{-\ell}$. Furthermore, as we are interested in computing expectations with respect to the invariant measure $\pi(dx)$ we also increase the time endpoint $T_\ell \uparrow \infty$ which is chosen such that $\frac{T_0}{h_0}, \frac{T_1}{h_1} \in \mathbb{N}$. We illustrate the main idea using two generic discrete time stochastic processes $(x^k_h)_{k \in \mathbb{N}}, (y^k_h)_{k \in \mathbb{N}}$ which can be defined as
\[
x^h_{k+1} = S^f_{h,\xi_k}(x^h_k), \quad y^h_{k+1} = S^c_{h,\xi_k}(y^h_k) \tag{18}
\]
where the operators $S^f, S^c : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^{d \times m} \to \mathbb{R}^d$ are Borel measurable, and $\xi, \tilde{\xi}$ are random input to the algorithms (in the simplest case Gaussian random variables). For example for the Euler discretisation we have
\[
S^\text{Euler}_{h_\ell}(x) = x - h_\ell \nabla U(x) + \sqrt{2h_\ell} \xi.
\]
We also use the notation $P_h(x, \cdot) = \mathcal{L}(S_{h,\xi}(x))$ with $\xi \sim \mathcal{N}(0, I)$ for the corresponding Markov kernel.

The coupling arises by evolving both fine and course paths jointly, over a time interval of length $T_{\ell-1}$, by doing two steps for the finer level (with the time step $h_\ell$) and one on the coarser level (with the time step $h_{\ell-1}$) using the discretization of the same Brownian path. In particular, let
\[
K_{h_\ell}(x, y, \cdot) = \mathcal{L}\left(S^f_{h_\ell,\xi_2} \circ S^f_{h_\ell,\xi_1}(x), S^c_{h_{\ell-1},\frac{1}{\sqrt{\ell}}(\xi_1 + \xi_2)}(y)\right), \tag{19}
\]
Notice that as in Section 2.1, the coupling arises due to the same Gaussian random variables being used on the fine and coarse level. From now on when the processes $(x^h), (y^h)$ are coupled we will use notation $(x^f_k), (x^c_k)$ for
\[
x^f_{k+\frac{1}{2}} = S^f_{h,\xi_{k+\frac{1}{2}}}(x^f_k), \tag{20}
\]
\[
x^c_{k+1} = S^c_{h,\xi_{k+1}}(x^c_k). \tag{21}
\]
The algorithm generating \((x_k^{(f)})_{2k\in\mathbb{N}}\) and \((x_k^{(c)})_{k\in\mathbb{N}}\) is presented in Algorithm 1.

1. Set \(x_0^{(f,\ell)} = x_0\), then simulate according to \(P_{h_\ell}\) up to \(x_{T_\ell-1}^{(f,\ell)}\).

2. Set \(x_0^{(c,\ell)} = x_0\) and \(x_0^{(f,\ell)} = x_{T_\ell-1}^{(f,\ell)}\), then simulate \((x^{(f,\ell)}, x^{(c,\ell)})\) jointly as

\[
(x_{k+1}^{(f,\ell)}, x_{k+1}^{(c,\ell)}) = \left(S^{(f,\ell)}_{h_\ell,\xi_{k,2}} \circ S^{(f,\ell)}_{h_\ell,\xi_{k,1}} (x_k^{(f,\ell)}), S^{(c,\ell)}_{h_\ell-1,\frac{1}{\sqrt{2}}(\xi_{k,1}+\xi_{k,2})} (x_k^{(c,\ell)})\right).
\] (22)

3. Set

\[
\Delta^{(i)}_\ell := g \left( x_{T_\ell-1}^{(f,\ell)} \right)^{(i)} - g \left( x_{T_\ell-1}^{(c,\ell)} \right)^{(i)}
\]

**Algorithm 1:** Coupling Langevin discretisations for \(T_\ell \uparrow \infty\)

### 3.2 Unbiasing Scheme

We are interested in calculating

\[
\lim_{\ell \to \infty} E(g_\ell) = \pi(g),
\]

where \(g_\ell^{(i,\ell)} := g \left( x_{T_\ell-1}^{(i,\ell)} \right)\). As in the MLMC construction we take

\[
E(\Delta_\ell) = \begin{cases} E(g_\ell) - E(g_{\ell-1}) & \ell \geq 1 \\ E(g_\ell) & \ell = 0 \end{cases}
\] (23)

It is clear that

\[
\pi(g) = \sum_{\ell=1}^{\infty} E(\Delta_\ell).
\]

MLMC methods would result in truncating this sum at some level \(L\), where the choice of deterministic \(L\) corresponds to time \(T_L\) and is dictated by the error/bias tolerance. It turns out that this truncation can be randomised. This idea goes back to von Neumann and has been systematically analysed in \[22, 23, 32\] and further extended in \[2\]. The construction goes as follows, consider

\[
Z = \sum_{j=0}^{N} \frac{\Delta_j}{P(N \geq j)}
\]

with \(N\) an independent integer valued random variable. Provided that we can apply Fubini’s theorem, \(Z\) is unbiased which can be seen in the following calculation

\[
E \left[ \sum_{j=0}^{N} \frac{\Delta_j}{P(N \geq j)} \right] = E \left[ \sum_{j=0}^{\infty} \frac{1_{(N \geq j)} \Delta_j}{P(N \geq j)} \right] = \sum_{j=0}^{\infty} E(\Delta_j) = \pi(g).
\]
The aim is to choose the distribution of $N$ such that $Z$ has finite second moment (Fubini’s theorem applies and the estimator has finite variance) and finite expected computing time (it is viable in the computational setting). Let $c_l$ be the expected computing time of $\Delta_l$, then the expected computing time $c$ of $Z$ satisfies

$$E(c) = E\left(\sum_{j=0}^{N} c_j\right) = \sum_{j=0}^{\infty} c_j P(N \geq j). \quad (24)$$

The second moment of $Z$ is finite as long as

$$\sum_{i \leq l} |\Delta_i|^2 |\Delta_l|^2 P(N \geq i) < \infty. \quad (25)$$

see Proposition 6, [22][23]. This method can also be applied in the present setting. Taking $h_L = 2^{-\ell}$ in (8) one can see that for (24) to be finite and (25) to hold one needs $\beta > \gamma$ (MLMC variance vs cost).

### 3.3 Convergence analysis

We will now present a general theorem for estimating the bias and the variance between levels given by (7) in the MLMC set up. We refrain from prescribing the exact dynamics of $(x^h_k)_{k \geq 0}, (y^h_k)_{k \geq 0}$ in (18), as we seek general conditions allowing the construction of uniform in time approximations of (2) in the Wasserstein norm. The advantage of working in this general setting is that if we wish to work with more advanced numerical scheme than the Euler-Maruyama method (implicit, projected, adapted, randomise etc) it will be enough to verify simple conditions to see how the performance of the complete algorithm will be affected without the need of reproofing all estimates again. We discuss specific methods that satisfy our assumptions in Section 4.

#### 3.3.1 Uniform estimates in time

**Definition 3.1 (Bias).** Let $g \in C^r_K(\mathbb{R})$ ($r$-times differentiable function with compact support). We say that process $(x^h_k)_{k \in \mathbb{N}}$ converges weakly uniformly in time with order $\alpha > 0$ to the solution of SDEs (2), if there exists a constant $c > 0$ such that

$$\sup_{t \geq 0} |E[g(X_t)] - E[g(x^h_{\lfloor t/h \rfloor})]| \leq ch^\alpha. \quad (26)$$

The difference in Definition 3.1 from the normal weak convergence definition is that here the weak error estimate needs to hold uniformly in time. The assumption of smoothness of $g$ could be dropped by using Malliavin calculus techniques.

We are ready to extend the definition for the MLMC variance. However, as operators $S^f$ and $S^c$ in (20)-(21) need not be the same, extra care is required. This extra flexibility allows analysing various coupling ideas. Furthermore, for notational simplicity we have dropped the dependence of the numerical scheme on $h$ as it should be clear that for a fixed $h$, $x^f_k$ runs with time-step $h/2$ and $x^c_k$ runs with time-step $h$. We now have the following definition for the MLMC variance.
Consider two processes $\mathcal{A}$ and $\mathcal{B}$. Assumption 3.1.

We now discuss the necessary conditions imposed on a numerical method applied to (2) that ensure that the estimates (7) hold uniformly in time as described above. In doing so, it would be instructive to decompose the global error analysis to the one step error and the regularity of the scheme. In order to do so, we introduce the notation $x^h_{k,x}$ for the process at time $k$ with initial condition $x_s$ at time $s < k$. If it is clear from the context which initial condition is used we just write $x^h_k$. We also define the conditional expectation operator as $E_n[\cdot] := E[\cdot | \mathcal{F}_n]$.

We now have the following definition

**Definition 3.4 (L$^2$-regularity).** We will say that one step operator $S: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d \times m} \to \mathbb{R}^d$ is $L^2$-regular uniformly in time if for any $x_n, y_n \in \mathbb{R}^d$, $\mathcal{F}_n$-measurable random variables there exists constants $K, C > 0$ and random variables $Z_{n+1}, R_{n+1} \in \mathcal{F}_{n+1}$ and $\mathcal{H}_n \in \mathcal{F}_{n+1}$, such that

$$S_{h,\xi_{n+1}}(x_n) - S_{h,\xi_{n+1}}(y_n) = x_n - y_n + Z_{n+1}$$

and

$$E_n[|S_{h,\xi_{n+1}}(x_n) - S_{h,\xi_{n+1}}(y_n)|^2] \leq (1 - Kh)|x_n - y_n|^2 + R_{n+1}$$

$$E_n[|Z_{n+1}|^2] \leq \mathcal{H}_n|x_n - y_n|^2h,$$

where

$$|R_n| \leq C(1 + |x_n|^q + |y_n|^q)h^{2\beta+1}, \quad q > 0$$

$$|\mathcal{H}_n| \leq C(1 + |x_n|^q + |y_n|^q)^{1/2}, \quad q > 0$$

and $C > 0$.

We now introduce the set of the assumptions needed for the proof of the main convergence theorem

**Assumption 3.1.** Consider two processes $(x^f_k)_{k \in \mathbb{N}}$ and $(x^c_k)_{k \in \mathbb{N}}$ obtained from the recursive application of the one step operators $S^f_{h,\xi}(\cdot)$ and $S^c_{h,\xi}(\cdot)$ as defined in [18]. We assume that

**H0** For $q > 1$ there exist functions $H^f_k := H(k,f,q)$ and $H^c_k := H(k,c,q)$

$$E|x^f_k|^q \leq H^f_k \quad \text{and} \quad E|x^c_k|^q \leq H^c_k,$$

with $\sup_{k \geq 0} H^f_k < \infty$ and $\sup_{k \geq 0} H^c_k < \infty$. 

\[ \text{Definition 3.2 (MLMC variance).} \] Let the operators in Equation (20) satisfy that for all $x$

$$\mathcal{L}(S^f_{h,\xi}(x)) = \mathcal{L}(S^c_{h,\xi}(x)).$$

We say that the MLMC variance is of order $\beta > 0$ if there exists a constant $c^\beta > 0$ s.t.

$$\sup_{t \geq 0} E|g(x^c_{t|h}) - g(x^f_{t|h})|^2 \leq c^\beta h^\beta.$$

**Remark 3.3.** Notice that the condition (26) trivially holds if $S^f = S^c$ i.e. the case where the same numerical method is applied both in the fine and the coarse level.

3.3.2 Statement of sufficient conditions

We now discuss the necessary conditions imposed on a numerical method applied to (2) that ensure that the estimates (7) hold uniformly in time as described above. In doing so, it would be instructive to decompose the global error analysis to the one step error and the regularity of the scheme. In order to do so, we introduce the notation $x^h_{k,x}$ for the process at time $k$ with initial condition $x_s$ at time $s < k$. If it is clear from the context which initial condition is used we just write $x^h_k$. We also define the conditional expectation operator as $E_n[\cdot] := E[\cdot | \mathcal{F}_n]$.

We now have the following definition
H1 For any \( x \in \mathbb{R}^d \)
\[
\mathcal{L}\left(S_{h,\xi}^f(x)\right) = \mathcal{L}\left(S_{h,\xi}^c(x)\right).
\]

H2 The operator \( S_{h,\xi}^f(\cdot) \) is \( L^2 \) regular uniformly in time.

Remark 3.5 (On condition H2). In the proof of the theorem (3.6) condition H2 is applied with \( x_{n} = x^f_{n} \) and \( y_{n} = x^c_{n} \) (see definition (3.4)). In this context, random variables 
\( Z_{n+1}(x^c_{n}, x^f_{n}), R_{n+1}(x^c_{n}, x^f_{n}) \) and \( H_{n+1}(x^c_{n}, x^f_{n}) \) are (measurable) functions of \( (x^c_{n}, x^f_{n}) \).

Consequently, under the condition H0 we have
\[
\sup_{n \geq 1} \mathbb{E}\left[\sum_{i=1}^{n} R_i\right] \leq C_R h^{2\beta} \quad (31)
\]
\[
\sup_{n \geq 1} \mathbb{E}\left[H_n^2\right] \leq C_H.
\]

Furthermore the roles of \( S_{h,\xi}^f(\cdot) \) and \( S_{h,\xi}^c(\cdot) \) in H2 can be swapped.

Below we present the main convergence result of this section. Using derived here estimates we can immediately estimate the rate of decay of MLMC variance.

Theorem 3.6. Take \((x^f_n)_{2n \in \mathbb{N}}\) and \((x^c_n)_{n \in \mathbb{N}}\) with \( h \in (0, 1] \) and assume that H0-H2 hold. Moreover, assume that there exists constants \( c_s > 0, c_w > 0 \) and \( \alpha \geq \frac{1}{2}, \beta \geq 0, p \geq 1 \) with \( \alpha \geq \frac{\beta}{2} \) such that for all \( n \geq 1 \)
\[
|\mathbb{E}_{n-1}(x^c_{n,x^c_{n-1}} - x^f_{n,x^c_{n-1}})| \leq c_w(1 + |x^c_{n-1}|^p)h^{\alpha+1},
\]
(32)
and
\[
\mathbb{E}_{n-1}|x^c_{n,x^c_{n-1}} - x^f_{n,x^c_{n-1}}|^2 \leq c_s(1 + |x^c_{n-1}|^{2p})h^{\beta+1}.
\]
(33)

Fix \( \xi \in (0, 1) \). Then the global error is bounded by
\[
\mathbb{E}[|x_{T/h,x_0}^c - x_{T/h,y_0}^f|^2] \leq |x_0 - y_0|^2 e^{-K\xi T} + \sum_{j=1}^{n} e^{j-(n-1))K\xi h\mathbb{E}(R_{j-1})} + h^\beta+1 \sum_{j=1}^{n} C_j(\xi) e^{j-(n-1))K\xi h},
\]
where \( C_k(\xi) : = \left(c_s H_{1,n} + \frac{4(c_w+2)|c_w H_{1,n} + c_w(H_{3,n}+2H_{2,n})|}{4(1-\xi)K}\right) \) with \( H_{1,n} := (1 + \mathbb{E}[|x^c_{n-1}|^{2p}]), H_{2,n} := (1 + \mathbb{E}[|x^c_{n-1}|^{4p}]), H_{3,n} := \mathbb{E}[H_{n-1}] \).

Proof. We begin using the following identity
\[
x_{n,x_0}^c - x_{n,y_0}^f = x_{n,x_{n-1}^c}^c - x_{n,x_{n-1}^c}^f + (x_{n,x_{n-1}^c}^c - x_{n,x_{n-1}^f}^c) + (x_{n,x_{n-1}^c}^f - x_{n,x_{n-1}^f}^f).
\]
We will be able to deal with the first term in the sum by using equations (32) and (33), while the second term will be controlled because of the \( L^2 \) regularity of the numerical scheme. In particular by squaring both sides in the equality above we have
\[
|x_{n,y_0}^c - x_{n,x_0}^f|^2 = |x_{n,x_{n-1}^c}^c - x_{n,x_{n-1}^c}^f|^2 + |x_{n,x_{n-1}^c}^f - x_{n,x_{n-1}^f}^c|^2 + 2(x_{n,x_{n-1}^c}^c - x_{n,x_{n-1}^c}^f)(x_{n,x_{n-1}^c}^c - x_{n,x_{n-1}^f}^c + Z_n).
\]
where in the last line we have used Assumption H2. Applying conditional expectation operator to both sides of the above equality

\[
\mathbb{E}_{n-1}[|x^{c}_{n,y_0} - x^{f}_{n,x_0}|^2] = \mathbb{E}_{n-1}[|x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}|] + \mathbb{E}_{n-1}[|x^{f}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}|] \\
+ 2\langle x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}, \mathbb{E}_{n-1}[x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}] \rangle + 2\mathbb{E}_{n-1}(\mathcal{Z}_n, x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}})
\]

Applying Cauchy-Schwarz inequality, and using the weak error estimate (32) leads to

\[
\mathbb{E}_{n-1}[|x^{c}_{n,y_0} - x^{f}_{n,x_0}|^2] \leq \mathbb{E}_{n-1}[|x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}|^2] + \mathbb{E}_{n-1}[|x^{f}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}|^2] \\
+ 2c_h h^{\alpha+1} |x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}|((1 + |x^{c}_{n,x_{n-1}}|^p) \\
+ 2\mathbb{E}_{n-1}[|\mathcal{Z}_n|^{1/2}] \mathbb{E}_{n-1}[|x^{c}_{n,x_{n-1}} - x^{f}_{n,x_{n-1}}|^2]^{1/2}.
\]

By assumptions H0-H2, and the strong error estimate (33) we have

\[
\mathbb{E}_{n-1}[|x^{c}_{n,y_0} - x^{f}_{n,x_0}|^2] \leq c_s(1 + |x^{c}_{n-1}|^2)p h^{\beta+1} + |x^{c}_{n-1} - x^{f}_{n-1}|^2(1 - K h) + \mathcal{R}_{n-1} \\
+ 2c_h h^{\alpha+1} |x^{c}_{n-1} - x^{f}_{n-1}|((1 + |x^{c}_{n-1}|^p) \\
+ 2\left(\mathbb{E}_{n-1}[|\mathcal{H}_n|] |x^{c}_{n-1} - x^{f}_{n-1}|^2 h\right)^{1/2} \left(c_s(1 + |x^{c}_{n-1}|^2)p h^{\beta+1}\right)^{1/2} \\
\leq c_s(1 + |x^{c}_{n-1}|^2)p h^{\beta+1} + |x^{c}_{n-1} - x^{f}_{n-1}|^2(1 - K h) + \mathcal{R}_{n-1} \\
+ 2c_h h^{\alpha+1} |x^{c}_{n-1} - x^{f}_{n-1}|((1 + |x^{c}_{n-1}|^p) \\
+ 2\left(|x^{c}_{n-1} - x^{f}_{n-1}|^2 h\right)^{1/2} \left(c_s\mathbb{H}_{n-1}(1 + |x^{c}_{n-1}|^2)h^{\beta+1}\right)^{1/2}.
\]

while taking expected values and applying Cauchy-Schwarz inequality and the fact that \(\alpha \geq \frac{\beta}{2}\) and \(h < 1\) (and hence \(h^{\alpha+1} \leq h^{\beta+1}\)) gives

\[
\mathbb{E}[|x^{c}_{n,y_0} - x^{f}_{n,x_0}|^2] \leq c_s(1 + \mathbb{E}[|x^{c}_{n-1}|^2)p h^{\beta+1} + \mathbb{E}[|x^{c}_{n-1} - x^{f}_{n-1}|^2](1 - K h) + \mathbb{E}[\mathcal{R}_{n-1}] \\
+ 4c_h \mathbb{E}[|x^{c}_{n-1} - x^{f}_{n-1}|^2 h]^{1/2} \mathbb{E}[(1 + |x^{c}_{n-1}|^2)p h^{\beta+1}]^{1/2} \\
+ 2\mathbb{E}\left(|x^{c}_{n-1} - x^{f}_{n-1}|^2 h\right)^{1/2} \mathbb{E}\left[c_s\mathcal{H}_{n-1}(1 + |x^{c}_{n-1}|^2)h^{\beta+1}\right]^{1/2}.
\]

Now Young’s inequality gives that for any \(\varepsilon > 0\)

\[
\mathbb{E}[|x^{c}_{n-1} - x^{f}_{n-1}|^2 h]^{1/2} \mathbb{E}[(1 + |x^{c}_{n-1}|^2)p h^{\beta+1}]^{1/2} \leq \varepsilon \mathbb{E}[|x^{c}_{n-1} - x^{f}_{n-1}|^2] h + \frac{1}{4\varepsilon} \mathbb{E}[(1 + |x^{c}_{n-1}|^2)p] h^{\beta+1} \\
\mathbb{E}\left(|x^{c}_{n-1} - x^{f}_{n-1}|^2 h\right)^{1/2} \mathbb{E}\left[c_s\mathcal{H}_{n-1}(1 + |x^{c}_{n-1}|^2)h^{\beta+1}\right]^{1/2} \leq \varepsilon \mathbb{E}\left[|x^{c}_{n-1} - x^{f}_{n-1}|^2\right] h + \frac{1}{4\varepsilon} \mathbb{E}\left[c_s\mathcal{H}_{n-1}(1 + |x^{c}_{n-1}|^2)\right] h
\]

while

\[
\mathbb{E}\left[\mathcal{H}_{n-1}(1 + |x^{c}_{n-1}|^2)\right] \leq \frac{1}{2} \mathbb{E}\left[|\mathcal{H}_{n-1}|^2\right] + \mathbb{E}\left[1 + |x^{c}_{n-1}|^4\right].
\]

We now denote \(\gamma_n := \mathbb{E}[|x^{c}_{n,y_0} - x^{f}_{n,x_0}|^2]\) and \(H_{1,n} := (1 + \mathbb{E}[|x^{c}_{n-1}|^2]), H_{2,n} := (1 + \mathbb{E}[|x^{c}_{n-1}|^4]), H_{3,n} := \mathbb{E}[|\mathcal{H}_{n-1}|].\) Whence

\[
\gamma_n \leq \left(c_s H_{1,n} + \frac{4c_h H_{1,n} + c_s(H_{3,n} + 2H_{2,n})}{4\varepsilon}\right) h^{\beta+1} + \mathbb{E}[\mathcal{R}_{n-1}] + \gamma_{n-1}(1 - [K - (4c_w + 2\varepsilon)h])
\]

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We now choose  as \( \varepsilon = \frac{(1-\zeta)K}{(4c_w+2)}, \zeta \in (0,1) \) and define \( C_n(\zeta) := \left(c_sH_{1,n} + \frac{(4c_w+2)[4c_wH_{1,n}+c_s(H_3,n+2H_2,n)]}{4(1-\zeta)K} \right) \) which gives

\[
\gamma_n \leq (1 - K\zeta h)\gamma_{n-1} + C_n(\zeta)h^{\beta+1} + \mathbb{E}(R_{n-1}).
\]

We complete the proof by Lemma (3.7) below.

\[\square\]

**Lemma 3.7.** Let \( a_n, g_n, c \geq 0, n \in \mathbb{N} \) be given. Moreover, assume that \( 1 + \lambda > 0 \). Then, if \( a_n \in \mathbb{R}, n \in \mathbb{N} \), satisfies

\[a_{n+1} \leq a_n(1+\lambda) + g_{n+1} + c, \quad n = 0,1,\ldots,\]

then

\[a_n \leq a_0e^{n\lambda} + \frac{e^{n\lambda} - 1}{\lambda} + \sum_{j=0}^{n-1}g_{j+1}e^{((n-1)-j)\lambda}, \quad n = 1,\ldots.\]

### 3.3.3 Optimal choice of parameters

Theorem 3.6 is fundamental in terms of applying the MLMC as it guarantees that the estimate for the variance in (7) holds. In particular, we have the following Lemma.

**Lemma 3.8.** Assume all the assumption from 3.6 hold. Let \( g(\cdot) \) be a Lipschitz function. Define

\[h_\ell = 2^{-\ell}, \quad T_\ell = -\frac{\beta}{K\zeta}(\log h_0 + \ell \log 2), \quad \forall \ell \geq 0.\]

Then resulting MLMC variance is given by.

\[\text{Var}[\Delta_{\ell}] \leq 2^{-2\beta}, \quad \Delta_{\ell} = g \left( \frac{x^{(f,\ell)}}{T_{\ell-1}} \right) - g \left( \frac{x^{(c,\ell)}}{T_{\ell-1}} \right)\]

\[\text{Var}[\Delta_{\ell}] \leq 2^{-2\beta} = 2^{-\beta}, \quad \Delta_{\ell} = g \left( \frac{x^{(f,\ell)}}{T_{\ell-1}} \right) - g \left( \frac{x^{(c,\ell)}}{T_{\ell-1}} \right)\]

**Proof.** Since \( g \) is a Lipschitz function it is enough to prove the bound for the mean square difference between the coarse and the fine path. In particular Theorem 3.6 implies

\[
\mathbb{E} \left[ \left( \frac{x^{(f,\ell)}}{T_{\ell-1}} - \frac{x^{(c,\ell)}}{T_{\ell-1}} \right)^2 \right] \leq \mathbb{E} \left[ h_{T_{\ell-1}}^{x_{T_{\ell-1}}^{(f,\ell)} - x_0} - \exp \{ -K\zeta T_{\ell-1} \} + C_R h_{T_{\ell-1}}^\beta \right]
\]

\[+ h_{\beta} \mathbb{E} \left[ c_sH_1 + \frac{(4c_w+2)[4c_wH_1 + c_s(H_3 + 2H_2)]}{4(1-\zeta)K} \right) \frac{2h}{K} (1 - \exp \{ -K\zeta T_{\ell-1} \})\]

where we have used \( H_1 := \sup_{n \geq 0} \mathbb{E}[H_{1,n}], H_2 := \sup_{n \geq 0} \mathbb{E}[H_{2,n}], H_3 := \sup_{n \geq 0} \mathbb{E}[H_{3,n+2H_2,n}] \). Assumption H0 implies that \( \mathbb{E}[x_{T_{\ell-1}}^{(f,\ell)} - x_0]^2 \leq C \), for all \( \ell \), which together with the fact that the numerical moments are bounded implies that there exist constants such that

\[
\mathbb{E} \left[ \left( \frac{x^{(f,\ell)}}{T_{\ell-1}} - \frac{x^{(c,\ell)}}{T_{\ell-1}} \right)^2 \right] \leq C_1 \exp \{ -K\zeta T_{\ell-1} \} + C_2 h_{\ell}^\beta.
\]

In order to balance terms we choose \( T_{\ell-1} = -\frac{1}{K\zeta}\log h_{\ell}^\beta = -\frac{\beta}{K\zeta}(\log h_0 + \ell \log 2) \) and by ignoring the dependence on the constants \( C_1, C_2 \) we obtain

\[
\mathbb{E} \left[ \left( \frac{x^{(f,\ell)}}{T_{\ell-1}} - \frac{x^{(c,\ell)}}{T_{\ell-1}} \right)^2 \right] \leq C 2^{-\ell \beta},
\]

which completes the proof. \[\square\]
Remark 3.9. Unlike in the standard MLMC complexity theorem [3] where the cost of simulating single path is of order $O(h^{-1})$, here we have $O(h^{-1}|\log h|)$. This is due to the fact that terminal times are increasing with levels. For the case $h_1 = 2^{-\ell}$ this results in cost per path $O(2^{\ell})$ and does not exactly fit the complexity theorem in [3]. Clearly in the case when MLMC variance decays with $\beta > 1$ we still recover the optimal complexity of order $O(\varepsilon^{-2})$. However, in the case $\beta = 1$ following the proof by Giles one can see that the complexity becomes $O(\varepsilon^{-2}|\log \varepsilon|)$. 

Remark 3.10. In the proof above we have assumed that $K$ is independent of $h$, while we have also used crude bounds in order not to deal directly with all the individual constants, since these would be dependent on the numerical schemes used. In the example that follows we discuss an explicit calculation in the case of the Euler method.

Example 3.11. In the case of the Euler-Maruyama method as we see from the analysis\footnote{As we will see there $m' \leq m$ depending on the size of $\nabla U(0)$} in Section 4.1 \(K = 2m' - L^2h_\ell\), while $R_n = 0$, $H_n = L$. Here $L$ is the Lipschitz constant of the drift $\nabla U(x)$. This implies that

$$
E[(x_{T/h,y_0}^\ell - x_{T/h,x_0}^\ell)^2] \leq |x_0 - y_0|^2 e^{-K\zeta T} + h^{\beta+1} \sum_{j=1}^n C_j(\zeta)e^{(j-(n-1))K\zeta h},
$$

where $C_n(\zeta) := \left(c_n + (4c_u+2)(4c_u+2\frac{c_u L}{(1-\zeta)\zeta})\right) H_{1,n}$ with $H_{1,n} := (1 + \E[|x_{n-1}^2|])$. Furthermore, as we can see in Lemma 3.8 we have that

$$
E[|x_n|^2] \leq \E|x_0|^2 \exp\{-2m' - L^2h\} + 2(b|\nabla U(0)|^2 + h) \frac{1 - \exp\{-2m' - L^2h\}}{(2m' - L^2h)h}
$$

which gives

$$
E[(x_{T/h,y_0}^\ell - x_{T/h,x_0}^\ell)^2] \leq Ce^{-K\zeta T} + 2Ch^\beta(b|\nabla U(0)|^2 + h) \sum_{j=1}^n \frac{1 - e^{-Kj\zeta h}}{K} e^{(j-(n-1))K\zeta h}
$$

We now have that

$$
\sum_{j=1}^n \frac{1 - e^{-Kj\zeta h}}{K} e^{(j-(n-1))K\zeta h} = \frac{1}{K} \left( \frac{e^{2hK\zeta(1 - e^{K\zeta T})}}{e^{hK\zeta} - 1} - \frac{e^{hK(2\zeta-1)} - e^{-K(\zeta-1)T}}{e^{hK(\zeta-1)} - 1} \right)
$$

The equation above is illustrative of the fact that as $T \to \infty$ the term in the sum converges to an explicit constant. Furthermore, it is trivial to see that with the same choice of $T$ as in the Lemma 3.8 one obtains the desirable variance decay.

4 Examples of suitable methods

In this section we present two (out of many) numerical schemes that fulfil the conditions of Theorem 3.6. In particular, we need to verify that our scheme is $L^2$ regular in time, it has bounded numerical moments as in $H0$ and finally that it satisfies the one-step error estimates [32, 33].


4.1 Euler-Maryama method

We start by considering the explicit Euler scheme

\[ S^{f}_{h,\xi}(x) = x + h\nabla U(x) + \sqrt{2}h\xi, \]  

(34)

while \( S^{f} = S^{c} \), \( i.e. \) we are using the same numerical method for the fine and coarse paths. In order to be able to recover the integrability and regularity conditions we will need to impose further assumptions on the potential \( U \). In particular, additionally to assumption \( \text{HU0} \) we assume that

**HU1** There exists constant \( L \) such that for any \( x, y \in \mathbb{R}^{d} \)

\[ |\nabla U(x) - \nabla U(y)| \leq L|x - y| \]

As a consequence of this assumption we have

\[ |\nabla U(x)| \leq L|x| + |\nabla U(0)| \]

(35)

We can now prove the \( L^{2} \)-regularity in time of the scheme.

**\( L^{2} \)-regularity** Since regularity is a property of the numerical scheme itself and it doesn’t relate with the coupling between fine and coarse levels, for simplicity of notation we prove things directly for

\[ x_{n+1,x} = S^{f}_{h,\xi}(x_{n}). \]

In particular, the following Lemma holds.

**Lemma 4.1** (\( L^{2} \)-regularity). Let \( \text{HU0} \) and \( \text{HU1} \) hold. Then the explicit Euler scheme is \( L^{2} \) regular, \( i.e. \) and there exists constants \( K \in \mathbb{R} \) and \( L > 0 \) s.t

\[ \mathbb{E}_{n-1}[|x_{n,x_{n-1}} - x_{n,y_{n-1}}|^{2}] \leq (1 - (2m - L^{2}h)h)|x_{n-1} - y_{n-1}|^{2} \]

(36)

\[ \mathbb{E}_{n}[|Z_n|^{2}] \leq h^{2}L^{2}|x_{n-1} - y_{n-1}|^{2} \]

(37)

**Proof.** The difference between the Euler scheme taking values \( x_{n-1} \) and \( y_{n-1} \) at time \( n - 1 \) is given by

\[ x_{n,x_{n-1}} - x_{n,y_{n-1}} = x_{n-1} - y_{n-1} + h(\nabla U(x_{n-1}) - \nabla U(y_{n-1})). \]

This, along with \( \text{HU0} \) and \( \text{HU1} \) leads to

\[ \mathbb{E}_{n-1}[|x_{n,x_{n-1}} - x_{n,y_{n-1}}|^{2}] = |x_{n-1} - y_{n-1}|^{2} + 2h \langle \nabla U(x_{n-1}) - \nabla U(y_{n-1}), x_{n-1} - y_{n-1} \rangle \]

\[ + |\nabla U(x_{n-1}) - \nabla U(y_{n-1})|^{2}h^{2} \]

\[ \leq |y_{n-1} - x_{n-1}|^{2}(1 - 2mh + L^{2}h^{2}) \]

\[ = |y_{n-1} - x_{n-1}|^{2}(1 - (2m - L^{2}h)h). \]

This proofs the first part of the lemma. Next, due to \( \text{HU1} \)

\[ \mathbb{E}_{n-1}[|Z_n|^{2}] = h^{2}\mathbb{E}_{n-1}[|\nabla U(x_{n-1}) - \nabla U(y_{n-1})|^{2}] \]

\[ \leq h^{2}L^{2}|x_{n-1} - y_{n-1}|^{2}. \]

\[ \square \]
Integrability  In the Lipschitz case we only require mean-square integrability This will become apparent when we analyse one-step error and (32) and (33) will hold with $p = 1$

**Lemma 4.2 (Integrability).** Let $HU0$ and $HU1$ hold. Then,

$$\mathbb{E}[|x_n|^2] \leq \mathbb{E}[|x_0|^2] \exp\{-(2m' - L^2h)nh\} + 2(b|\nabla U(0)|^2 + h) \frac{1 - \exp\{-(2m' - L^2h)nh\}}{(2m' - L^2h)h}$$

(38)

**Proof.** We have

$$|x_n|^2 = |x_{n-1}|^2 + |\nabla U(x_{n-1})|^2 h^2 + 2h\xi^T \xi + 2hx_{n-1}^T \nabla U(x_{n-1}) + \sqrt{2h}hx_{n-1}^T \xi + 2h\sqrt{2h} \xi^T \nabla U(x_{n-1})$$

and hence

$$\mathbb{E}[|x_n|^2] \leq \mathbb{E}[|x_{n-1}|^2](1 - 2m'h + L^2h^2) + 2b|\nabla U(0)|^2 + 2dh.$$
Proof. We calculate
\[
x_{1,x}^{t} - x_{1,x}^{c} = x + \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 + \frac{h}{2} \nabla U \left( x + \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right) + \sqrt{h} \xi_2 - \left( x + h \nabla U(x) + \sqrt{h} (\xi_1 + \xi_2) \right)
\]
\[
= \frac{h}{2} \nabla U \left( x + \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right) - \frac{h}{2} \nabla U(x)
\]
(42)

It then follows from HU1 that
\[
|\mathbb{E}[x_{1,x}^{t} - x_{1,x}^{c}]| \leq \frac{h^{3/2}}{2} L \mathbb{E} \left| \frac{\sqrt{h}}{2} \nabla U(x) + \xi_1 \right|.
\]
Furthermore, if we use (35), a triangle equality and the fact that \( \mathbb{E} |\xi_1| = \sqrt{\frac{2d}{\pi}} \), we obtain (39). If we now assume that \( U \in C^3 \), then for \( \delta = \alpha x + (1 - \alpha)(x + \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1) \), \( \alpha \in [0, 1] \), we write
\[
\nabla U \left( x + \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right) = \nabla U(x) + \sum_{|\alpha|=2} \partial^\alpha U(x) \left( \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right)^\alpha
\]
\[
+ \frac{1}{2} \sum_{|\alpha|=3} \partial^\alpha U(\delta) \left( \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right)^\alpha,
\]
where we used multi-index notation. Consequently
\[
\mathbb{E} \left[ \nabla U \left( x + \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right) - \nabla U(x) \right]
\]
\[
\leq C h^2 \mathbb{E} \left[ (|x| + h|x|^2 + |\nabla U(0)| + h|\nabla U(0)|^2 + |\xi_1|^2) \right] .
\]
which together with \( \mathbb{E}|\xi_1|^2 = d \) gives (41). Equation (40) trivially follows from (42) by observing that
\[
\mathbb{E}|x_{1,x}^{t} - x_{1,x}^{c}|^2
\]
\[
\leq L^2 h^2 \mathbb{E} \left| \frac{h}{2} \nabla U(x) + \sqrt{h} \xi_1 \right|^2
\]
\[
\leq h^3 L^2 \left( \frac{h}{2} (|x|^2 + |\nabla U(0)|^2) + d \right)
\]
\[
\square
\]

Remark 4.4. In the case of log-concave target the bias of MLMC using the Euler method can be explicitly quantified using the results from [4].

4.2 Non-Lipschitz setting

In the previous subsection we found out that in order to analyse the regularity and the one-step error of the explicit Euler approximation, we had to impose an additional assumption about \( \nabla U(x) \) being globally Lipschitz. This is necessary since in the absence of this condition
Euler method is shown to be transient or even divergent [24, 12]. However, in many applications of interest this is a rather restricting condition. An example of this is the potential

\[ U(x) = -\frac{x^4}{4} - \frac{x^2}{2}. \]

A standard way to deal with this is to use either an implicit scheme or specially designed explicit schemes [10, 27].

### 4.2.1 Implicit Euler method

Here we will focus on the implicit Euler scheme

\[ x_n = x_{n-1} + h\nabla U(x_n) + \sqrt{2h}\xi_n \]

We will assume that Assumption HU0 holds and moreover replace HU1 with HU1’ Let \( k \geq 1 \). For any \( x, y \in \mathbb{R}^d \) there exists constant \( L \) s.t

\[ |\nabla U(x) - \nabla U(y)| \leq L(1 + |x|^{k-1} + |y|^{k-1})|x - y| \]

As a consequence of this assumption we have

\[ |\nabla U(x)| \leq L|x|^k + |\nabla U(0)| \]  

(43)

Observe that HU0 implies that

\[ \langle \nabla U(x), x \rangle \leq -\frac{m}{2}|x|^2 + \frac{1}{2}|\nabla U(0)|^2 \quad \forall x \in \mathbb{R}^d \]  

(44)

**Integrability** The integrability uniformly in time can be easily deduced from the results in [19, 18]. Nevertheless, for the convenience of the reader we will present the analysis of the regularity of the scheme, where the effect of the implicitness of the scheme on the regularity should become quickly apparent.

**L^2-regularity**

**Lemma 4.5** (\( L^2 \)-regularity). Let HU0 and HU1’ hold. Then an implicit Euler scheme is \( L^2 \) regular, i.e. and there exists constants \( K \in \mathbb{R} \) and \( L_Z > 0 \) s.t

\[ \mathbb{E}_{n-1}[(x_{n,x_{n-1}} - x_{n,y_{n-1}})^2] \leq (1 - 2mh)(y_{n-1} - x_{n-1})^2 + \mathcal{R}_{n-1}, \]  

(45)

and

\[ \sum_{k=0}^{\infty} \mathcal{R}_k \leq 0. \]

Moreover

\[ \mathbb{E}_{n-1}[|Z_n|^2] \leq h^2(1 - 2mh) L(1 + \mathbb{E}_{n-1}[|x_n|^{k-1}] + \mathbb{E}_{n-1}[|y_n|^{k-1}])^2|x_{n-1} - y_{n-1}|^2. \]

---

\(^7\)One also may consider the case of products of distribution functions, where after taking the log one ends up with a polynomial in the different variables.
Proof. The difference between the implicit Euler scheme taking values \( x_{n-1} \) and \( y_{n-1} \) time \( n - 1 \) is given by
\[
x_{n,x_{n-1}} - x_{n,y_{n-1}} = x_{n-1} - y_{n-1} + h(\nabla U(x_n) - \nabla U(y_n)).
\]
This, along with HU0 and HU1 leads to
\[
|x_{n,x_{n-1}} - x_{n,y_{n-1}}|^2 = |x_{n-1} - y_{n-1}|^2 + 2h(\nabla U(x_n) - \nabla U(y_n), x_n - y_n)
- |\nabla U(x_n) - \nabla U(y_n)|^2h^2 \\
\leq |x_{n-1} - y_{n-1}|^2 - 2mh|x_{n,x_{n-1}} - x_{n,y_{n-1}}|^2
\]
This implies
\[
|x_{n,x_{n-1}} - x_{n,y_{n-1}}|^2 \leq |x_{n-1} - y_{n-1}|^2 \frac{1}{1 + 2mh} \leq |x_{n-1} - y_{n-1}|^2 \left(1 - \frac{2mh}{1 + 2mh}\right).
\]
Next we take
\[
|x_{n,x_{n-1}} - y_{n,y_{n-1}}|^2 \leq |x_{n-1} - y_{n-1}|^2 - 2mh|x_n - y_n|^2 \\
= (1 - 2mh)|x_{n-1} - y_{n-1}|^2 - 2mh(|x_n - y_n|^2 - |x_{n-1} - y_{n-1}|^2).
\]
In view of (3.4) we define
\[
\mathcal{R}_k := -2mh(|x_k - y_k|^2 - |x_{k-1} - y_{k-1}|^2),
\]
and noticed that
\[
\sum_{k=1}^n \mathcal{R}_k = -2mh|x_n - y_n|^2 \leq 0.
\]
Hence by remark (3.5) the proof of the first statement in the lemma is complete. Now, due to HU1
\[
||Z_n||^2 = h^2|\nabla U(x_n) - \nabla U(y_n)| \leq h^2L(1 + |x_n|^{k-1} + |y_n|^{k-1})^2|x_n - y_n|^2 \\
\leq h^2 \left(1 - \frac{2mh}{1 + 2mh}\right) L(1 + |x_n|^{k-1} + |y_n|^{k-1})^2|x_{n-1} - y_{n-1}|^2.
\]
Observe that
\[
\mathbb{E}_{n-1}[|x_{n}^2|] = \mathbb{E}_{n-1}[|x_{n-1}|^2 + 2h(\nabla U(x_n), x_n) - |\nabla U(x_n)|^2h^2] \leq |x_{n-1}|^2 - mh|x_n|^2 + h|\nabla U(0)|^2.
\]
Consequently
\[
\mathbb{E}_{n-1}[|x_{n}^2|] = \frac{1}{1 + mh} \left(|x_{n-1}|^2 + h|\nabla U(0)|^2\right).
\]
This in turn implies that
\[
\mathcal{H}_{n-1} = \mathbb{E}_{n-1} \left(1 - \frac{2mh}{1 + 2mh}\right) (1 + |x_n|^{k-1} + |y_n|^{k-1}) \leq C_H(1 + |x_{n-1}|^{k-1} + |y_{n-1}|^{k-1}).
\]
Due to uniform integrability of the implicit Euler scheme, (31) holds. □
5 Bayesian inference using MLMC SGLD

The main computational tasks in Bayesian statistics is the approximation of expectations with respect to the posterior. The a priori uncertainty in a parameter $x$ is modelled using a probability density $\pi_0(x)$ called the prior. Here, we consider the where for a fixed parameter $x$ the data $\{d_i\}_{i=1,...,N}$ is supposed to be i.i.d. with density $\pi(d|x)$. By Bayes’ rule the posterior

$$\pi(x) \propto \pi_0(x) \prod_{i=1}^{N} \pi(d_i|x)$$

this distribution is invariant for the Langevin equation with the choice

$$\nabla U(x) = \nabla \log \pi_0(x) + \frac{N}{s} \sum_{i=1}^{s} \nabla \log \pi(d_{i,x}|x_k) + \sqrt{2h}\xi_k$$

where we take $\tau_k \sim U(\{1,...,N\})$ for $j = 1,...,s$ the uniform distribution on $1,...,N$ which corresponds to sampling $s$ items with replacement from $1,...,N$. Notice that each step only costs $s$ instead of $N$. We introduce a notation for the SGLD step as

$$x_{k+1} = S_{h,\xi_k,\tau_k}(x_k).$$

Since the SGLD is an explicit scheme, we need to make the assumptions:

**Assumption 5.1.** Lipschitz conditions for prior and likelihood.

$$|\nabla \log \pi(d_i | x) - \nabla \log \pi(d_i | y)| \leq L|x - y| \text{ for all } i, x, y$$

$$|\nabla \log \pi_0(x) - \nabla \log \pi_0(y)| \leq L_0|x - y| \text{ for all } x, y.$$  

Convexity conditions for prior and likelihood: There are $m_0 \geq 0$ and $m(d) \geq 0$

$$\log \pi_0(y) \leq \log \pi_0(x) + \langle \nabla \log \pi_0(x), y - x \rangle - \frac{m_0}{2}|x - y|^2$$

$$\log \pi(d_i | y) \leq \log \pi(d_i | x) + \langle \nabla \log \pi(d_i | x), y - x \rangle - \frac{m_{di}}{2}|x - y|^2$$

with $\inf_i m_0 + m_{di} > 0$. Let $m = m_0 + N \inf_i m_{di}$.  

---

**One-step errors estimates** Having established integrability, estimating the one-step error follows exactly the same line of the argument as in Lemma 4.3 and therefore we skip it.
We note that these conditions imply that $U$ in Equation (46) satisfies the assumption we posed on the Euler method.

**Lemma 5.1.** The recursion of the SGLD is uniformly $L^2$-regular

$$
\mathbb{E}[|x_n,x_{n-1}-x_{n-1},y_{n-1}|^2] \leq (1 - (2m - (L_0 + NL)^2 h)b)|x_{n-1} - y_{n-1}|^2
$$

and has uniformly bounded moments

$$
\sup_{n \geq 0} \mathbb{E}[|x_n|^q] < \infty.
$$

**Proof.** This follows straight from Lemmas 4.2 and 4.3. □

The only ingredient that changes is the one step difference estimate. We consider

$$
x_{1,x}^f = S_{h,\xi,x,\tau^F,2} \circ S_{h,\xi_1,\tau^F,1}(x)
$$

$$
x_{1,x}^c = S_{h,\frac{1}{\sqrt{2}}(\xi_1+\xi_2),x}(x).
$$

We note that we have already coupled the noise between the fine and coarse path in the traditional MLMC way. One question that naturally occurs now is if and how should one choose to couple between the subsampling of data. In particular, in order for the telescopic sum to be respected one needs to have that

$$
L(\tau^F,1) = L(\tau^F,2) = L(\tau^C).
$$

We first take $s$ independent samples $\tau^F,1$ on the first fine-step and another $s$ independent $s$-samples $\tau^F,2$ on the second fine-step. The following three choices of $\tau^C$ ensure that equation (48) holds.

- an independent sample of $\{1, \ldots, N\}$ without replacement denoted as $\tau^C_{\text{ind}}$ called independent coupling;
- a draw of $s$ samples without replacement from $(\tau^F,1, \tau^F,2)$ denoted as $\tau^C_{\text{union}}$ called union coupling;
- the concatenation of a draw of $\frac{s}{2}$ samples without replacement from $\tau^F,1$ and a draw of $\frac{s}{2}$ samples without replacement from $\tau^F,2$ (provided that $s$ is even) $\tau^C_{\text{strat}}$ called stratified coupling.

We stress that any of these couplings can be used in Algorithm 2 and hence we have use the generic symbol $\tau^C$ to denote this. In order to simplify the one step proof we need to introduce the following notation. In particular, we set $(\tau^F,1, \tau^F,2) = (\tau^F_{1:s}, \tau^F_{(s+1):2s})$, while

$$
c(x,i) = \begin{cases} 
\nabla \log \pi_0(x) & i = 0 \\
\nabla \log \pi(d_i | x) & \text{otherwise}
\end{cases}
$$

$$
b(x,\tau) : = c(x,0) + \frac{N}{|\tau|} \sum_{i=1}^{N} c(x,\tau_i).
$$
Lemma 5.2. (One step error bound) Suppose Assumption 5.1 is satisfied, then $\alpha$.

We stress here that in terms of the estimate (32) nothing changes and one obtains $\alpha = 1$. However, the subsampling has an important effect on the value of the parameter $\beta$ and hence this is the one we focus here. Furthermore, we prove our estimate in the case of the union and independent coupling but the same result would hold in terms of $\beta$ for the stratified coupling, with the only difference being the constant multiplying the $h^2$ term.

Lemma 5.2. (One step error bound) Suppose Assumption 5.1 is satisfied, then

$$
E|S_{\frac{x}{2}, \tau} \circ S_{\frac{y}{2}, \tau} - S_{h, \frac{1}{2}(\xi_1 + \xi_2), \tau_c}(x)|^2 
\leq M + 4 \left( \frac{h}{2} \right)^2 \left( \frac{1}{s} \sum_{i,j=1}^{N} |c(x,i)\rangle \langle c(x,j)| + \frac{c N}{s} \sum_{i=1}^{N} |c(x,i)|^2 \right)
$$

with $M = 2 \left( \frac{h}{2} \right)^2 (NL + L_0)^2 \left( h + \frac{4}{N} \sum_{i=1}^{N} |c(x,i)|^2 + \frac{c N}{N^2} \sum_{i,j=1}^{N} \langle c(x,i)\rangle \langle c(x,j)| \right)$.

**Proof.** We will first perform the calculations that both approaches above have in common. First we introduce an intermediate term at $x_1 = x + \frac{1}{2} b(x, \tau_{t_{i+1}}) + \sqrt{h} \xi$

$$
E|S_{\frac{x}{2}, \tau} \circ S_{\frac{y}{2}, \tau} - S_{h, \frac{1}{2}(\xi_1 + \xi_2), \tau_c}(x) - \frac{1}{2} b(x, \tau_{t_{i+1}}) + \sqrt{h} \xi
= \left( \frac{h}{2} \right)^2 E|b(x_1, \tau_{\tau_{t_{i+1}}}) + b(x, \tau_{t_{i+1}}) - 2b(x, \tau_c)|^2
\leq 2 \left( \frac{h}{2} \right)^2 E|b(x, \tau_{\tau_{t_{i+1}}}) + b(x, \tau_{t_{i+1}}) - 2b(x, \tau_c)|^2 + 2 \left( \frac{h}{2} \right)^2 E|b(x_1, \tau_{\tau_{t_{i+1}}}) - b(x, \tau_{\tau_{t_{i+1}}})|^2.
$$

Algorithm 2: Coupling SGLD for $t_i \uparrow \infty$, different choices of $\tau^c$ are considered.
Putting everything together yields the distribution of the vector $\tau$. We focus first on (50) using Assumption 5.1

$$
\mathbb{E}[b(x, \tau_{(s+1):2s}^F) - b(x, \tau_{(s+1):2s})]^2 \\
\leq (NL + L_0)^2 \mathbb{E}|x - x_\|_2^2 \\
\leq (NL + L_0)^2 \left(h + \frac{h^2}{4} \mathbb{E}_r|b(x, \tau_{(s+1):2s}^F)|^2\right)
$$

where we introduced operator $\mathbb{E}_r[\cdot]$ to highlight the fact that expectation is taken with respect to the distribution of the vector $\tau$.

$$
\mathbb{E}_r|b(x, \tau_{(s+1):2s}^F)|^2 = |c(x, 0)|^2 + \sum_{i=1}^N 2 \langle c(x, 0), c(x, i) \rangle + \left(\frac{N}{s}\right)^2 \mathbb{E}_r\left|\sum_{i=1}^s c(x, \tau_i^F)\right|^2
$$

Now we focus on (49)

$$
\mathbb{E}[b(x, \tau_{(s+1):2s}^F) + b(x, \tau_{1:s}^F) - 2b(x, \tau^C)]^2 \\
= \left(\frac{N}{s}\right)^2 \left|\sum_{j=1}^s c(x, \tau_j^F)\right|^2 + \left|\sum_{j=s+1}^{2s} c(x, \tau_j^F)\right|^2 - 2 \sum_{i=1}^{T_3} c(x, \tau_i^C) | \\
= \left(\frac{N}{s}\right)^2 \mathbb{E}(|T_1|^2 + |T_2|^2 + |T_3|^2 + 2 \langle T_1, T_2 \rangle - 2 \langle T_1, T_3 \rangle - 2 \langle T_2, T_3 \rangle)
$$

We find

$$
\mathbb{E}[T_1]^2 + |T_2|^2 = 2 \left(\frac{s}{N} \sum_{i=1}^N |c(x, i)|^2 + \frac{s(s-1)}{N^2} \sum_{i,j=1}^N \langle c(x, i), c(x, j) \rangle\right) \\
\mathbb{E}[T_3]^2 = 4 \left(\frac{s}{N} \sum_{i=1}^N |c(x, i)|^2 + \frac{s(s-1)}{N^2} \sum_{i,j=1}^N \langle c(x, i), c(x, j) \rangle\right) \\
\mathbb{E} \langle T_1, T_3 \rangle + \mathbb{E} \langle T_2, T_3 \rangle = 2 \left(\frac{s}{N} \sum_{i=1}^N |c(x, i)|^2 + \frac{s(s-1)+s^2}{N^2} \sum_{i,j=1}^N \langle c(x, i), c(x, j) \rangle\right)
$$

Putting everything together yields

$$
\mathbb{E}(|T_1|^2 + |T_2|^2 + |T_3|^2 + 2 \langle T_1, T_2 \rangle - 2 \langle T_1, T_3 \rangle - 2 \langle T_2, T_3 \rangle) = \frac{N}{s} \sum_{i,j=1}^N \langle c(x, i), c(x, j) \rangle \left(6s(s-1) - 4s(s-1) + s^2 + \frac{2s^2}{N^2}\right) \\
+ \frac{(1+1+4+2-4)}{2} \frac{N}{s} \sum_{i=1}^N |c(x, i)|^2 .
$$
For the **independent coupling**, the only terms that change are $E \langle T_1, T_3 \rangle + E \langle T_2, T_3 \rangle$

$$E \langle T_1, T_3 \rangle + E \langle T_2, T_3 \rangle = 4 \left( \frac{s^2}{N^2} \sum_{i,j=1}^{N} \langle c(x,i), c(x,j) \rangle \right)$$

So overall we get for Equation (51)

$$E (|T_1|^2 + |T_2|^2 + |T_3|^2 + 2 \langle T_1, T_2 \rangle - 2 \langle T_1, T_3 \rangle - 2 \langle T_2, T_3 \rangle)$$

$$= \sum_{i,j=1}^{N} \langle c(x,i), c(x,j) \rangle \left( 6s(s-1) - 8s^2 - 2s^2 \right)$$

$$+ \left( \frac{6s}{N} \sum_{i=1}^{N} |c(x,i)|^2 \right)$$

**Remark 5.3.** Notice that the term $M$ is quite similar to the corresponding term for the Euler Method. However, in contrast to the Euler method the additional term is of order $h^2$. This seems sharp because of line (49) irrespective of the choice coupling no additional $h$ will appear. This is also verified by simulation. However, as already discuss in terms of the estimate (32) nothing changes in terms of $\alpha$, since if we were to take the expectation inside in (49) we would obtain 0.

**Corollary 5.4.** The Algorithm 2 based on the coupling given in Equation (47) and subsampling coupling $\tau_{\text{union}}$ or $\tau_{\text{strat}}$ with appropriately chosen $t_i$ has complexity $\varepsilon^{-1} \log(\varepsilon)^3$.

**Proof.** Because of Lemma 5.2 and Lemma 5.1 we can apply the results of Section 3.3. In particular, if we choose $T_i$ according to Lemma 3.8 we thus have $\beta = 1$ in Theorem 3.6 and then the complexity follows from Remark 3.9.

6 Numerical Investigations

In this Section we perform numerical simulations that illustrate our theoretical findings. We start by studying an Ornstein-Uhlenbeck process in Section 6.1 using the explicit Euler method, while in Section 6.2 we study a Bayesian logistic regression model using the SGLD.

6.1 Ornstein Uhlenbeck process

We consider the SDE

$$dX_t = -\kappa X_t dt + \sqrt{2} dW_t,$$

and its dicretization using the Euler method

$$x_{n+1} = S_{h,\xi}(x_n), \quad S_{h,\xi}(x) = x - h\kappa x + \sqrt{2h\xi}.$$  

Equation (52) is ergodic with its invariant measure being $N(0,\kappa^{-1})$. Furthermore, it is possible to show that the Euler method (53) is similarly ergodic with its invariant measure
being \( N \left( 0, \frac{2}{2 \kappa - \kappa h} \right) \). In Figure 2, we plot the outputs of our numerical simulations using Algorithm 1. The parameter of interest here is the variance of the invariant measure \( \kappa^{-1} \) which we try to approximate for different mean square error tolerances \( \varepsilon \).

More precisely, in Figure 2 we see allocation of samples for various levels with respect to \( \varepsilon \), while in Figure 2b we compare the computational cost of the algorithm as a function of the parameter \( \varepsilon \). As we can see the computational complexity grows as \( O(\varepsilon^{-2}) \) as predicted by our theory (Here \( \alpha = \beta = 2 \) in (32) and (33)). Finally, in Figure 2c we plot the approximation of the variance \( \kappa^{-1} \) from our algorithm. Note that this coincides with the choice \( g(x) = x^2 \) since the mean of the invariant measure is 0. As we can see as \( \varepsilon \) becomes smaller we get perfect agreement with the true value of the variance as expected.

6.2 Bayesian logistic regression

In the following we present numerical simulations for a binary Bayesian logistic regression model. In this case the data \( d_i \in \{-1, 1\} \) is modelled by

\[
p(d_i | t_i, x) = f(y_i x^T t_i)
\]

where \( f(z) = \frac{1}{1 + \exp(-z)} \in [0, 1] \) and \( t_i \in \mathbb{R}^d \) are fixed covariates. We put a Gaussian prior \( \mathcal{N}(0, C_0) \) on \( x \), for simplicity we use \( C_0 = I \) subsequently. By Bayes’ rule the posterior \( \pi \) satisfies

\[
\pi(x) \propto \exp \left( -\frac{1}{2} |x|_C^2 \right) \prod_{i=1}^{N} f(y_i x^T t_i).
\]

We consider \( d = 3 \) and \( N = 100 \) data points and choose the covariate to be

\[
L = \begin{pmatrix}
\ell_{1,1} & \ell_{1,2} & 1 \\
\ell_{2,1} & \ell_{2,2} & 1 \\
\vdots & \vdots & \vdots \\
\ell_{100,1} & \ell_{100,2} & 1
\end{pmatrix}
\]

for a fixed sample of \( \ell_{i,j} \overset{i.i.d.}{\sim} \mathcal{N}(0,1) \) for \( i = 1, \ldots, 100 \).

In Algorithm 2 we can choose the starting position \( x_0 \). It is reasonable to start the path of the individual SGLD trajectories at the mode of the target distribution (heuristically this

Figure 2: MLMC results for (52) for \( g(x) = x^2 \) and \( \kappa = 0.4 \).
makes the distance $E| x_0^{(c,\ell)} - x_0^{(f,\ell)} |$ in step 2 in Algorithm 2 small. That is we set the $x_0$ to be the maximum a posteriori estimator (MAP)

$$x_0 = \arg\max \exp \left( -\frac{1}{2} |x|^2 \right) \prod_{i=1}^{N} f(y_i x^T \delta_i)$$

which is approximated using the Newton-Raphson method. Our numerical results are described in Figure 3. In particular, in Figure 3a we illustrate the behaviour of the coupling by plotting an estimate of the average distance during the joint evolution in step 2 of Algorithm 2. The behaviour in this figure agrees qualitatively with the statement of Theorem 3.6, as $T$ grows there is an initial exponential decay up to an additive constant. For the simulation we used $h_0 = 0.02$, $T_\ell = 3(\ell + 1)$ and $s = 20$. Furthermore, in Figure 3b we plot CPU-time $\times \varepsilon^2$ against $\varepsilon$ for the estimation of the mean. The objective here is to estimate the mean square distance from the MAP estimator $x_0$ and the posterior that is $\int |x - x_0|^2 \pi(x) dx$. Again, after some initial transient where CPU-time $\times \varepsilon^2$ decreases, we see that we get a quantitative agreement with our theory since the CPU-time $\times \varepsilon^2$ increases in a logarithmic way in the limit of $\varepsilon$ going to zero.

Figure 3: (a) Illustration of the joint evolution in step 2 of Algorithm 2 for the union coupling, (b) Cost of MLMC (sequential CPU time) SGLD for Bayesian Logistic Regression for decreasing accuracy parameter $\varepsilon$ and different couplings

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