Rigorous derivation of the full primitive equations by the scaled Boussinesq equations with rotation

Xueke Pu · Wenli Zhou

Received: 21 November 2022 / Revised: 21 November 2022 / Accepted: 15 February 2023 / Published online: 9 March 2023
© The Author(s), under exclusive licence to Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2023

Abstract
The primitive equations of large-scale oceanic dynamics form a fundamental model in geophysical flows. It is well-known that the primitive equations can be formally derived by the hydrostatic approximation. In this paper, we rigorously derive the full primitive equations from the mathematical point of view. More precisely, we prove that the scaled Boussinesq equations with rotation converge to the full primitive equations in a strong sense, globally in time, with the convergence rate $O(\varepsilon)$, as the aspect ratio $\varepsilon$ goes to zero.

Keywords Boussinesq equations · Primitive equations · Hydrostatic approximation · Strong convergence

Mathematics Subject Classification 35Q35 · 35Q86 · 86A05

1 Introduction

For large-scale oceanic dynamics, an important feature is that the vertical scale of the ocean is much smaller than the horizontal scale. From a physical point of view, this scale analysis means that we can use the hydrostatic approximation to simulate the motion of the ocean in the vertical direction. Due to this fact, the primitive equations of oceanic dynamics can be formally derived from the Boussinesq equations with rotation (see [9, 27]). Mathematically, there are already several results on the rigorous
derivation of the primitive equations or the rigorous justification of the hydrostatic approximation. The weak convergence from the Navier–Stokes equations to the primitive equations was first proved by Azérad-Guillén [1], while the strong convergence from the Navier–Stokes equations to the primitive equations is due to the work of Li-Titi [29], see also Furukawa et al.[13] in the $L^p$ settings. For the case where the limiting system is the primitive equations with only horizontal viscosity, we refer to the work of Li-Titi-Yuan [31]. Note that the limiting system they obtained was the primitive equations without temperature. Therefore, the aim of this paper is mathematically to derive the full primitive equations.

Let $\Omega_{\varepsilon} = G \times (-\varepsilon, \varepsilon)$ be a $\varepsilon$-dependent domain, where $G = (0, 1) \times (0, 1)$. Here, $\varepsilon = H/L$ is called the aspect ratio, which measures the ratio of the vertical scale $H$ to the horizontal scale $L$ of the ocean. For large-scale ocean circulation, the aspect ratio $\varepsilon \sim 10^{-3} \ll 1$. Consider the anisotropic Boussinesq equations with rotation

$$
\begin{cases}
\partial_t u + (u \cdot \nabla)u + \nabla p - \theta \vec{k} + f_0 \vec{k} \times v = \frac{1}{Re_1} \Delta_h u + \frac{\varepsilon^2}{Re_2} \partial_{zz} u, \\
\partial_t \theta + u \cdot \nabla \theta = \frac{1}{Rt_1} \Delta_h \theta + \frac{\varepsilon^2}{Rt_2} \partial_{zz} \theta, \\
\nabla \cdot u = 0,
\end{cases}
$$

(1.1)
in a thin domain $\Omega_{\varepsilon}$, where the velocity field $u = (v, w) = (v_1, v_2, w)$, the pressure $p$ and the temperature $\theta$ are the unknowns. $f_0$ is the Coriolis parameter. The unit vector $\vec{k} = (0, 0, 1)$ points to the $z$-direction. The parameters $Re_1$, $Re_2$, $Rt_1$ and $Rt_2$ are Reynolds numbers. Denote by $\nabla_h = (\partial_x, \partial_y)$ the horizontal gradient operator. Then, the horizontal Laplacian operator $\Delta_h$ is given by

$$
\Delta_h = \nabla_h \cdot \nabla_h = \partial_{xx} + \partial_{yy}.
$$

We first transform the $\varepsilon$-dependent domain $\Omega_{\varepsilon}$ into a fixed domain. To this end, we introduce some scaling transformations,

$$
u = (v, w), \quad v(x, y, z, t) = v(x, y, \varepsilon z, t),
$$

$$
w(x, y, z, t) = \frac{1}{\varepsilon} w(x, y, \varepsilon z, t), \quad p(x, y, z, t) = p(x, y, \varepsilon z, t),
$$

$$
\theta(x, y, z, t) = \varepsilon \theta(x, y, \varepsilon z, t),
$$

for any $(x, y, z) \in \Omega =: G \times (-1, 1)$ and for any $t \in (0, \infty)$. Under these scalings, system (1.1) defined on $\Omega_{\varepsilon}$ becomes the scaled Boussinesq equations with rotation

$$
\begin{cases}
\partial_t v_e - \frac{1}{Re_1} \Delta_h v_e - \frac{1}{Re_2} \partial_{zz} v_e + (v_e \cdot \nabla_h) v_e + w_e \partial_z v_e + \nabla_h p_e + f_0 \vec{k} \times v_e = 0, \\
\varepsilon^2 \left( \partial_t w_e - \frac{1}{Rt_1} \Delta_h w_e - \frac{1}{Rt_2} \partial_{zz} w_e + v_e \cdot \nabla_h w_e + w_e \partial_z w_e \right) + \partial_z p_e - \theta_e = 0, \\
\partial_t \theta_e - \frac{1}{Rt_1} \Delta_h \theta_e - \frac{1}{Rt_2} \partial_{zz} \theta_e + v_e \cdot \nabla_h \theta_e + w_e \partial_z \theta_e = 0, \\
\nabla_h \cdot v_e + \partial_z w_e = 0,
\end{cases}
$$

(SBE)
defined on the fixed domain $\Omega$, subject to

\begin{align}
 v_\varepsilon, w_\varepsilon, p_\varepsilon \text{ and } \theta_\varepsilon \text{ are periodic in } x, y, z, & \quad (1.2) \\
 (v_\varepsilon, w_\varepsilon, \theta_\varepsilon)|_{t=0} = (v_0, w_0, \theta_0), & \quad (1.3) \\
 v_\varepsilon, w_\varepsilon, p_\varepsilon \text{ and } \theta_\varepsilon \text{ are even, odd, even and odd in } z, & \quad (1.4)
\end{align}

We point out that the condition (1.4) is preserved by (SBE) provided that the initial data satisfy this condition. Owing to this fact and (1.2), in this paper, we always assume that the initial data satisfy the compatibility conditions

\begin{align}
 v_0, w_0 \text{ and } \theta_0 \text{ are periodic in } x, y, z, & \quad (1.5) \\
 v_0, w_0 \text{ and } \theta_0 \text{ are even, odd and odd in } z, & \quad (1.6)
\end{align}

Throughout the paper, we will use the same notation $L^p(\Omega)$ or $H^m(\Omega)$ to denote both a space itself and its finite product spaces. For simplicity, we denote by notation $\| \cdot \|_p$ the $L^p(\Omega)$ norm.

**Remark 1.1** The initial value $w_0$ for vertical velocity $w_\varepsilon$ is uniquely determined by the incompressible condition and the compatibility condition (1.6), which can be represented as

\begin{equation}
 w_0(x, y, z) = -\int_{-1}^{z} \nabla h \cdot v_0(x, y, \xi) d\xi, \quad (1.7)
\end{equation}

for any $(x, y) \in G$ and $z \in (-1, 1)$. From (1.6) and (1.7) it can be deduced that

\[ \int_{-1}^{1} \nabla h \cdot v_0(x, y, z) dz = 0. \]

When the aspect ratio $\varepsilon$ goes to zero, (SBE) formally converges to the full primitive equations

\begin{equation}
 (\text{FPE}) \quad \begin{cases}
 \partial_t v - \frac{1}{Re_1} \Delta_h v - \frac{1}{Re_2} \partial_{zz} v + (v \cdot \nabla_h) v + w \partial_z v + \nabla_h p + f_0 \vec{k} \times v = 0, \\
 \partial_z p - \theta = 0, \\
 \partial_t \theta - \frac{1}{Rt_1} \Delta_h \theta - \frac{1}{Rt_2} \partial_{zz} \theta + v \cdot \nabla_h \theta + w \partial_z \theta = 0, \\
 \nabla_h \cdot v + \partial_z w = 0,
\end{cases}
\end{equation}

corresponding to

\begin{align}
 v, w, p \text{ and } \theta \text{ are periodic in } x, y, z, & \quad (1.8) \\
 (v, \theta)|_{t=0} = (v_0, \theta_0), & \quad (1.9) \\
 v, w, p \text{ and } \theta \text{ are even, odd, even and odd in } z, & \quad (1.10)
\end{align}

In consequence, the aim of this paper is to prove the aspect ratio limit from (SBE) to (FPE). It is crucial to point out that the global well-posedness of strong solutions to
(FPE) with (1.8)–(1.10) can be established by a similar argument as that of Cao-Titi [9, Theorem 2], for any initial data \((v_0, \theta_0) \in H^1\), which will play an important role in proving the aspect ratio limit from (SBE) to (FPE). Results on the study of the hydrostatic approximation in different settings are available in [33, 34].

Next we want to recall some results concerning the primitive equations. The global existence of weak solutions to the primitive equations with full viscosity and diffusivity was first given by Lions-Temam-Wang [26–28], but uniqueness of weak solutions to this mathematical model is still unknown except for some special cases [2, 22, 30, 36]. Moreover, the existence of global strong solutions to the primitive equations was proved by Kobelkov [21] and Kukavica-Ziane [24, 25] in the case of mixed Dirichlet and Neumann boundary conditions, see also Hieber-Kashiwabara [19], Hieber et al.[18] and Giga et al.[14] in the \(L^p\) settings. In addition, for the initial data lying in the critical solenoidal Besov space, the unique global strong solution of primitive equations was obtained by Giga et al.[15].

Subsequently, the global strong solution to the primitive equations was naturally studied in the cases of partial dissipation. More details on these cases can be found in the work of Cao-Titi [10], Fang-Han [12], Li-Yuan [32] and Cao-Li-Titi [4–8]. We remark that the inviscid primitive equations are known to be ill-posed in Sobolev spaces [17, 35], and its smooth solutions may develop singularity in finite time [3, 20, 37]. However, the local well-posedness of the inviscid primitive equations can be established under the assumption of initial data belonging to the space of analytic function (see, e.g., [16, 23]).

Now we are to state the main result of this paper. Suppose that initial data \((v_0, \theta_0) \in H^2(\Omega)\). Then, from (1.7), it follows that \((v_0, w_0, \theta_0) \in H^1(\Omega)\). By a similar argument as that of Lions-Temam-Wang [27, Part IV], there exists a unique local strong solution \((v_\varepsilon, w_\varepsilon, \theta_\varepsilon)\) to (SBE) with (1.2)–(1.4). Denote by \(T^*_\varepsilon\) the maximal existence time of the local strong solutions to (SBE). Let

\[
\begin{align*}
(U_\varepsilon, \Pi_\varepsilon, P_\varepsilon) &= (V_\varepsilon, W_\varepsilon, \Pi_\varepsilon, P_\varepsilon), \\
(V_\varepsilon, W_\varepsilon, \Pi_\varepsilon, P_\varepsilon) &= (v_\varepsilon - v, w_\varepsilon - w, \theta_\varepsilon - \theta, p_\varepsilon - p).
\end{align*}
\]

Subtracting (FPE) from (SBE), then the difference function \((V_\varepsilon, W_\varepsilon, \Pi_\varepsilon, P_\varepsilon)\) satisfies system

\[
\begin{align}
\partial_t V_\varepsilon - \frac{1}{Re_1} \Delta_h V_\varepsilon - \frac{1}{Re_2} \partial_{zz} V_\varepsilon + (U_\varepsilon \cdot \nabla) V_\varepsilon + (u \cdot \nabla) V_\varepsilon \\
+ (U_\varepsilon \cdot \nabla) w + \nabla_h P_\varepsilon + f_0 k \times V_\varepsilon = 0, & \quad (1.11) \\
\varepsilon^2 \left( \partial_t W_\varepsilon - \frac{1}{Re_1} \Delta_h W_\varepsilon - \frac{1}{Re_2} \partial_{zz} W_\varepsilon + U_\varepsilon \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w + u \cdot \nabla W_\varepsilon \right) \\
+ \varepsilon^2 \left( \partial_t w + u \cdot \nabla w - \frac{1}{Re_1} \Delta_h w - \frac{1}{Re_2} \partial_{zz} w \right) + \partial_z P_\varepsilon - \Pi_\varepsilon = 0, & \quad (1.12) \\
\partial_t \Pi_\varepsilon - \frac{1}{Rt_1} \Delta_h \Pi_\varepsilon - \frac{1}{Rt_2} \partial_{zz} \Pi_\varepsilon + U_\varepsilon \cdot \nabla \Pi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Pi_\varepsilon = 0, & \quad (1.13) \\
\nabla_h \cdot V_\varepsilon + \partial_z W_\varepsilon = 0, & \quad (1.14)
\end{align}
\]
defined on $\Omega \times (0, T_\varepsilon^\ast)$. For this case, we have the following strong convergence theorem.

**Theorem 1.1** Given a periodic function pair $(v_0, \theta_0) \in H^2(\Omega)$ with $\int_{-1}^{1} \nabla h \cdot v_0 \, dz = 0$. Denote by $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ and $(v, \theta)$ the unique local strong solution of (SBE) subject to (1.2)–(1.4) and the unique global strong solution of (FPE) corresponding to (1.8)–(1.10), respectively.

(i) For any $T > 0$, there exists a small positive constant $\varepsilon(T) = \frac{2\lambda_0}{3\sqrt{M_4(T)}}$ such that (SBE) has a unique strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ on the time interval $[0, T]$, and that the energy estimate holds

$$
\sup_{0 \leq t \leq T} \left( \|(V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon)\|_{H^1}^2 \right)(t)
+ \int_0^T \left( \frac{1}{Re_1} \|\nabla h V_\varepsilon\|_{H^1}^2 + \frac{1}{Re_2} \|\partial_z V_\varepsilon\|_{H^1}^2 + \frac{\varepsilon^2}{Re_1} \|\nabla h W_\varepsilon\|_{H^1}^2 \right) dt
+ \int_0^T \left( \frac{\varepsilon^2}{Re_2} \|\partial_z W_\varepsilon\|_{H^1}^2 + \frac{1}{Rt_1} \|\nabla h \Pi_\varepsilon\|_{H^1}^2 + \frac{1}{Rt_2} \|\partial_z \Pi_\varepsilon\|_{H^1}^2 \right) dt \leq \varepsilon^2 M_5(T),
$$

for every $\varepsilon \in (0, \varepsilon(T))$, where both $M_4(t)$ and $M_5(t)$ are the nonnegative continuously increasing functions that do not depend on $\varepsilon$;

(ii) As a consequence, we have the strong convergences

$$
w_\varepsilon \rightarrow w \text{ in } L^\infty \left(0, T; L^2(\Omega)\right), (v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon) \rightarrow (v, 0, \theta) \text{ in } L^\infty \left(0, T; H^1(\Omega)\right),$$

$$
\left( \frac{\nabla h v_\varepsilon}{\sqrt{Re_1}}, \frac{\partial_z v_\varepsilon}{\sqrt{Re_2}}, \frac{\varepsilon \nabla h w_\varepsilon}{\sqrt{Re_1}} \right) \rightarrow \left( \frac{\nabla h v}{\sqrt{Re_1}}, \frac{\partial_z v}{\sqrt{Re_2}}, 0 \right) \text{ in } L^2 \left(0, T; H^1(\Omega)\right),$$

$$
\left( \frac{w_\varepsilon}{\sqrt{Re_2}}, \frac{\nabla h \theta_\varepsilon}{\sqrt{Rt_1}}, \frac{\partial_z \theta_\varepsilon}{\sqrt{Rt_2}} \right) \rightarrow \left( w, 0, \frac{\nabla h \theta}{\sqrt{Rt_1}}, \frac{\partial_z \theta}{\sqrt{Rt_2}} \right) \text{ in } L^2 \left(0, T; H^1(\Omega)\right),$$

and the convergence rate is of the order $O(\varepsilon)$.

**Remark 1.2** (i) The convergence results in Theorem 1.1 imply a rigorous justification of hydrostatic approximation. In other words, (FPE) can be obtained by replacing the vertical momentum equation in (SBE) with the hydrostatic approximation

$$\partial_z p - \theta = 0.$$

(ii) We point out that the convergence results here are global in time but not uniform in time compared with [29]. The main reason is that the $H^1$ norm on the strong solutions to the primitive equations with full viscosity and diffusivity is controlled by a nonnegative continuously increasing function with respect to time [9].

(iii) This paper only focuses on the case where initial data $(v_0, \theta_0) \in H^2(\Omega)$, which allows us to deal with this problem in the framework of strong solutions. In this case, we can carry out the $L^2$ estimate on the difference function directly (see Proposition 2.2).
The rest of this paper is arranged as follows: in Sect. 2, we carry out the basic energy estimate on system (1.11)–(1.14); in Sect. 3, the first-order energy estimate on this system is established under some smallness condition; the proof of Theorem 1.1 is presented in Sect. 4.

2 $L^2$ estimate on $(V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon)$

Under the assumption of initial data $(v_0, \theta_0) \in H^1(\Omega)$, the global well-posedness of strong solutions to (FPE) with Neumann boundary conditions was established by Cao-Titi [9]. Similarly, we also have the well-posedness result for the case of periodic boundary conditions.

**Proposition 2.1** Suppose that $(v_0, \theta_0) \in H^1(\Omega)$ with \( \int_{-1}^{1} \nabla h \cdot v_0 dz = 0 \). Then, the following assertions hold true:

(i) For any $T > 0$, there exists a unique strong solution $(v, \theta)$ to (FPE) subject to (1.8)–(1.10) such that

$$(v, \theta) \in C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad (\partial_t v, \partial_t \theta) \in L^2(0, T; L^2(\Omega));$$

(ii) The global strong solution $(v, \theta)$ to (FPE) satisfies the energy estimate

$$\sup_{0 \leq s \leq t} \left( \| (v, \theta) \|_{H^1(\Omega)}^2 \right)(s) + \int_0^t \left( \| (\nabla v, \nabla \theta) \|_{H^1(\Omega)}^2 + \| (\partial_t v, \partial_t \theta) \|_2^2 \right) ds \leq \mathcal{M}_1(t),$$

for any $t \in [0, \infty)$, where $\mathcal{M}_1(t)$ is a nonnegative continuously increasing function.

The proof of Proposition 2.1 is similar to that of [9, Theorem 2], so we omit it. Next we present a crucial lemma (see [5, Lemma 2.1]) which will be frequently used in the rest of this paper.

**Lemma 2.1** The following inequalities hold

$$\int_G \left( \int_{-1}^{1} \varphi(x, y, z)dz \right) \left( \int_{-1}^{1} \psi(x, y, z)\phi(x, y, z)dz \right) dxdy \leq C \| \varphi \|_2^{1/2} \left( \| \varphi \|_2^{1/2} + \| \nabla h \varphi \|_2^{1/2} \right) \| \psi \|_2^{1/2} \left( \| \psi \|_2^{1/2} + \| \nabla h \psi \|_2^{1/2} \right) \| \phi \|_2,$$

$$\int_G \left( \int_{-1}^{1} \varphi(x, y, z)dz \right) \left( \int_{-1}^{1} \psi(x, y, z)\phi(x, y, z)dz \right) dxdy \leq C \| \psi \|_2^{1/2} \left( \| \psi \|_2^{1/2} + \| \nabla h \psi \|_2^{1/2} \right) \| \phi \|_2^{1/2} \left( \| \phi \|_2^{1/2} + \| \nabla h \phi \|_2^{1/2} \right) \| \varphi \|_2,$$

for every $\varphi, \psi, \phi$ such that the quantities on right-hand side make sense, where $C$ is a positive constant.
Based on Proposition 2.1, we will perform the basic energy estimate on system (1.11)–(1.14).

Proposition 2.2 Assume that \((v_0, \theta_0) \in H^1(\Omega)\) with \(\int_{-1}^{1} \nabla_h \cdot v_0 dz = 0\). Then, system (1.11)–(1.14) has the basic energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon) \|_2^2 \right) (s) + \int_0^t \left( \frac{1}{Re_1} \| \nabla_h V_\varepsilon \|_2^2 + \frac{1}{Re_2} \| \partial_z V_\varepsilon \|_2^2 \right) ds
+ \int_0^t \left( \frac{\varepsilon^2}{Re_1} \| \nabla_h W_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_2} \| \partial_z W_\varepsilon \|_2^2 + \frac{1}{Rt_1} \| \nabla_h \Pi_\varepsilon \|_2^2 + \frac{1}{Rt_2} \| \partial_z \Pi_\varepsilon \|_2^2 \right) ds
\leq \varepsilon^2 \mathcal{M}_2(t),
\]

for any \(t \in [0, T^*_e]\), where

\[
\mathcal{M}_2(t) = C e^{C t + C(t+1)[\mathcal{M}_1(t)+\mathcal{M}_1^2(t)]} \left[ \| (v_0, w_0, \theta_0) \|_2^3 (1 + t)^3 + \mathcal{M}_1(t) + \mathcal{M}_1^2(t) \right].
\]

Here, \(C\) is a positive constant depending only on \(Re_1, Re_2, Rt_1\) and \(Rt_2\).

Proof. Taking the \(L^2(\Omega)\) inner product of the third equation in (SBE) with \(\theta_\varepsilon\) and integrating the resulting differential equation in time between 0 to \(t\), we obtain

\[
\frac{1}{2} \| \theta_\varepsilon(t) \|_2^2 + \int_0^t \left( \frac{1}{Rt_1} \| \nabla_h \theta_\varepsilon \|_2^2 + \frac{1}{Rt_2} \| \partial_z \theta_\varepsilon \|_2^2 \right) ds \leq \frac{1}{2} \| \theta_0 \|_2^2.
\]

We multiply the first two equations in (SBE) by \(v_\varepsilon\) and \(w_\varepsilon\), respectively, and integrate over \(\Omega \times (0, t)\) to get

\[
\frac{1}{2} \left( \| v_\varepsilon \|_2^2 + \varepsilon^2 \| w_\varepsilon \|_2^2 \right) (t) + \int_0^t \frac{1}{Re_1} \| \nabla_h v_\varepsilon \|_2^2 ds
+ \int_0^t \left( \frac{1}{Re_2} \| \partial_z v_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_1} \| \nabla_h w_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_2} \| \partial_z w_\varepsilon \|_2^2 \right) ds
= \frac{1}{2} \left( \| v_0 \|_2^2 + \varepsilon^2 \| w_0 \|_2^2 \right) + \int_0^t \int_{\Omega} \theta_\varepsilon w_\varepsilon dxdydz ds.
\]

Owing to Hölder’s inequality and Young’s inequality, from (2.2) we deduce that

\[
\sup_{0 \leq s \leq t} \left( \| v_\varepsilon \|_2^2 + \varepsilon^2 \| w_\varepsilon \|_2^2 \right) (s) + \int_0^t \frac{1}{Re_1} \| \nabla_h v_\varepsilon \|_2^2 ds
+ \int_0^t \left( \frac{1}{Re_2} \| \partial_z v_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_1} \| \nabla_h w_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_2} \| \partial_z w_\varepsilon \|_2^2 \right) ds
\leq C \left( \| v_0 \|_2^2 + \varepsilon^2 \| w_0 \|_2^2 + t \| \theta_0 \|_2^2 \right).
\]
Taking the $L^2(\Omega)$ inner product of (1.11), (1.12) and (1.13) with $V_\epsilon$, $W_\epsilon$ and $\Pi_\epsilon$, respectively, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|V_\epsilon\|^2 + \epsilon^2 \|W_\epsilon\|^2 + \|\Pi_\epsilon\|^2 \right) + \left( \frac{1}{Re_1} \|\nabla_h V_\epsilon\|^2 + \frac{1}{Re_2} \|\partial_z V_\epsilon\|^2 \right) \\
+ \left( \frac{\epsilon^2}{Re_1} \|\nabla_h W_\epsilon\|^2 + \frac{\epsilon^2}{Re_2} \|\partial_z W_\epsilon\|^2 + \frac{1}{Rt_1} \|\nabla_h \Pi_\epsilon\|^2 + \frac{1}{Rt_2} \|\partial_z \Pi_\epsilon\|^2 \right) \\
= \int_\Omega [(U_\epsilon \cdot \nabla) V_\epsilon + (u \cdot \nabla) V_\epsilon + (U_\epsilon \cdot \nabla) v] \cdot (-V_\epsilon) dxdydz \\
+ \int_\Omega \epsilon^2 (U_\epsilon \cdot \nabla W_\epsilon + U_\epsilon \cdot \nabla w + u \cdot \nabla W_\epsilon + u \cdot \nabla w) (-W_\epsilon) dxdydz \\
+ \int_\Omega (U_\epsilon \cdot \nabla \Pi_\epsilon + U_\epsilon \cdot \nabla \theta + u \cdot \nabla \Pi_\epsilon) (-\Pi_\epsilon) dxdydz \\
+ \int_\Omega \left[ \Pi_\epsilon - \epsilon^2 \left( \partial_t w - \frac{1}{Re_1} \Delta_h w - \frac{1}{Re_2} \partial_{zz} w \right) \right] W_\epsilon dxdydz \\
=: I_1 + I_2 + I_3 + I_4. \tag{2.4}
\]

For the first integral term $I_1$, using Hölder’s inequality, Lemma 2.1 and Young’s inequality gives

\[
I_1 = \int_\Omega [(U_\epsilon \cdot \nabla) V_\epsilon + (u \cdot \nabla) V_\epsilon + (U_\epsilon \cdot \nabla) v] \cdot (-V_\epsilon) dxdydz \\
= \int_\Omega [(U_\epsilon \cdot \nabla) v] \cdot (-V_\epsilon) dxdydz = \int_\Omega [(U_\epsilon \cdot \nabla) v] \cdot v dxdydz \\
= \int_\Omega [(V_\epsilon \cdot \nabla_h) V_\epsilon \cdot v + (\nabla_h \cdot V_\epsilon) V_\epsilon \cdot v] dxdydz \\
+ \int_\Omega (V_\epsilon \cdot \partial_z v) \left( \int_{-1}^\zeta \nabla_h \cdot V_\epsilon d\xi \right) dxdydz \\
\leq C \|\partial_z v\|^1_2 \|\nabla_h \partial_z v\|^1_2 \|\nabla_h \partial_z v\|^1_2 \|\nabla_h V_\epsilon\|^1_2 \|\nabla_h V_\epsilon\|^1_2 \\
+ C \|v\|^1_2 \|\nabla_h V_\epsilon\|^1_2 \|\nabla_h V_\epsilon\|^1_2 \\
\leq C \left( \|v\|^2_{H^1} + \|v\|^2_2 + \|\nabla^2 v\|^2_2 + \|\nabla v\|^4_2 + \|\nabla v\|^2_2 \right) \|\nabla^2 v\|^2_2 \\
+ \frac{1}{8} \left( \frac{1}{Re_1} \|\nabla_h V_\epsilon\|^2_2 + \frac{1}{Re_2} \|\partial_z V_\epsilon\|^2_2 \right). \tag{2.5}
\]

Note that the incompressible condition, the integration by parts and the Sobolev embedding theorem have been used in the above estimate. Thanks to Lemma 2.1 and Young’s inequality, we obtain
\[ I_2 := \int_{\Omega} \varepsilon^2 (U_\varepsilon \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w + u \cdot \nabla W_\varepsilon + u \cdot \nabla w)(-W_\varepsilon) \, dx \, dy \, dz \]
\[ = \int_{\Omega} \varepsilon^2 (U_\varepsilon \cdot \nabla w + u \cdot \nabla w)(-W_\varepsilon) \, dx \, dy \, dz \]
\[ = \int_{\Omega} \varepsilon^2 (u_\varepsilon \cdot \nabla w)(-W_\varepsilon) \, dx \, dy \, dz = \int_{\Omega} \varepsilon^2 (u_\varepsilon \cdot \nabla W_\varepsilon) w \, dx \, dy \, dz \]
\[ = \int_{\Omega} \varepsilon^2 \left( w(v_\varepsilon \cdot \nabla h W_\varepsilon) - \omega w_\varepsilon (\nabla h \cdot V_\varepsilon) \right) \, dx \, dy \, dz \]
\[ = \int_{\Omega} \varepsilon^2 \left( -\int_{-1}^{z} \nabla h \cdot v d\xi \right) \left[ (v_\varepsilon \cdot \nabla h W_\varepsilon) - \omega w_\varepsilon (\nabla h \cdot V_\varepsilon) \right] \, dx \, dy \, dz \]
\[ \leq C \varepsilon^2 \left( \|v_\varepsilon\|_2^4 + \|v_\varepsilon\|_2^2 \|\nabla h v_\varepsilon\|_2^2 + \|\nabla v\|_2^2 \|\nabla^2 v\|_2^2 + \varepsilon^4 \|w_\varepsilon\|_2^4 \right) \]
\[ + C \varepsilon^6 \|w_\varepsilon\|_2 \|\nabla h w_\varepsilon\|_2^2 + \frac{1}{8} \left( \frac{1}{Re_1} \|\nabla h V_\varepsilon\|_2^2 + \frac{\varepsilon^2}{Re_1} \|\nabla h W_\varepsilon\|_2^2 \right) . \] (2.6)

A similar argument as that for the first integral term \( I_1 \) leads to

\[ I_3 := \int_{\Omega} (U_\varepsilon \cdot \nabla \Pi_\varepsilon + U_\varepsilon \cdot \nabla \theta + u \cdot \nabla \Pi_\varepsilon)(-\Pi_\varepsilon) \, dx \, dy \, dz \]
\[ = \int_{\Omega} \left[ (V_\varepsilon \cdot \nabla h \Pi_\varepsilon) \theta + (\nabla h \cdot V_\varepsilon) \Pi_\varepsilon \theta + \left( \int_{-1}^{z} \nabla h \cdot V_\varepsilon d\xi \right) \Pi_\varepsilon \partial_z \theta \right] \, dx \, dy \, dz \]
\[ \leq C \left( \|\theta\|_6 \|V_\varepsilon\|_3 \|\nabla h \Pi_\varepsilon\|_2 + \|\theta\|_6 \|\Pi_\varepsilon\|_3 \|\nabla h V_\varepsilon\|_2 \right) \]
\[ + C \|\partial_z \theta\|_2^{1/2} \|\nabla h \cdot V_\varepsilon\|_2^{1/2} \|\Pi_\varepsilon\|_2 \|\nabla h V_\varepsilon\|_2 \]
\[ \leq C \left( \|\theta\|_2^{H+1} + \|\theta\|_2^4 + \|\nabla \theta\|_2^2 + \|\nabla \theta\|_2^{1/2} \|\nabla h V_\varepsilon\|_2 \right) \left( \|V_\varepsilon\|_2^2 + \|\Pi_\varepsilon\|_2^2 \right) \]
\[ + \frac{1}{8} \left( \frac{1}{Re_1} \|\nabla h V_\varepsilon\|_2^2 + \frac{1}{Re_2} \|\partial_z V_\varepsilon\|_2^2 + \frac{1}{Re_1} \|\nabla h \Pi_\varepsilon\|_2^2 + \frac{1}{Re_2} \|\partial_z \Pi_\varepsilon\|_2^2 \right) . \] (2.7)

Finally, it remains to deal with the last integral term \( I_4 \). We apply the integration by parts, Hölder’s inequality and Young’s inequality to get

\[ I_4 := \int_{\Omega} \left[ \Pi_\varepsilon - \varepsilon^2 \left( \partial_t w - \frac{1}{Re_1} \Delta_h w - \frac{1}{Re_1} \partial_{zz} w \right) \right] w_\varepsilon \, dx \, dy \, dz \]
\[ = \int_{\Omega} \varepsilon^2 \left( -\int_{-1}^{z} \partial_t v d\xi - \frac{1}{Re_1} \nabla h w \right) \cdot \nabla h W_\varepsilon \, dx \, dy \, dz \]
\[ + \int_{\Omega} \left[ -\frac{\varepsilon^2}{Re_2} \partial_z w (\partial_z W_\varepsilon) - \Pi_\varepsilon \left( \int_{-1}^{z} \nabla h \cdot V_\varepsilon d\xi \right) \right] \, dx \, dy \, dz \]
\[ \leq C \|\Pi_\varepsilon\|_2^2 + C \varepsilon^2 \left( \|\partial_t v\|_2^2 + \|\nabla v\|_2^2 + \|\nabla^2 v\|_2^2 \right) \]
For the purpose of establishing the first-order energy estimate on system (1.11)–(1.14), we substitute (2.5)–(2.8) into (2.4) and apply Grönwall's inequality to the resulting differential equation, it follows from (2.1) and (2.3) that

\[
\left( \| (v, \varepsilon W, \Pi) \|_2^2 \right) (t) + \int_0^t \left( \frac{1}{Re_1} \| \nabla_h V \|_2^2 + \frac{1}{Re_2} \| \partial_z W \|_2^2 \right) ds
\]

\[
+ \int_0^t \left( \frac{\varepsilon^2}{Re_1} \| \nabla_h W \|_2^2 + \frac{\varepsilon^2}{Re_2} \| \partial_z W \|_2^2 + \frac{1}{Rt_1} \| \nabla_h \Pi \|_2^2 + \frac{1}{Rt_2} \| \partial_z \Pi \|_2^2 \right) ds
\]

\[\leq \exp \left\{ C \int_0^t \left( 1 + \| v \|_{H^1}^2 + \| \nabla \cdot v \|_2^2 + \| \nabla v \|_2^2 + \| \nabla^2 v \|_2^2 \right) ds \right\}
\]

\[
\times \left\{ \varepsilon^4 \int_0^t \left( \| v \|_2^4 + \| \partial_z v \|_2^2 \right) ds \right\}
\]

\[
+ \varepsilon^2 \int_0^t \left( \| v \|_2^4 + \| \partial_z v \|_2^2 \right) ds \right\}
\]

\[
\leq C \varepsilon^2 \exp \left\{ C \varepsilon (t + 1) [\mathcal{M}_1(t) + \mathcal{M}_2(t)] \right\} \left\| (v_0, w_0, \theta_0) \right\|_2^2 (1 + t)^3 + \mathcal{M}_1(t) + \mathcal{M}_2(t) \right),
\]

completing the proof. □

### 3 $L^2$ estimate on $(\nabla V, \varepsilon \nabla W, \nabla \Pi)$

For the purpose of establishing the first-order energy estimate on system (1.11)–(1.14), we need to carry out the second-order energy estimate on (FPE) corresponding to (1.8)–(1.10). To this end, we first employ (1.10) to rewrite (FPE). Integrating the second equation in (FPE) with respect to $z$ yields

\[
p(x, y, z, t) = p_v(x, y, t) + \int_{-1}^z \theta(x, y, \xi, t) d\xi,
\]

where $p_v(x, y, t)$ represents unknown surface pressure as $z = 0$. According to (3.1) and the incompressible condition, we can reformulate (FPE) as

\[
\partial_t v - \frac{1}{Re_1} \Delta_h v - \frac{1}{Re_2} \partial_z v + (v \cdot \nabla_h) v - \left( \int_{-1}^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \partial_z v + \nabla_h p_v(x, y, t) + \int_{-1}^z \nabla_h \theta(x, y, \xi, t) d\xi + f_0k \times v = 0,
\]

\[
\partial_t \theta - \frac{1}{Rt_1} \Delta_h \theta - \frac{1}{Rt_2} \partial_z \theta + v \cdot \nabla_h \theta - \left( \int_{-1}^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \partial_z \theta = 0.
\]
subject to

\( v \) and \( \theta \) are periodic in \( x, y, z \),

\( (v, \theta)|_{t=0} = (v_0, \theta_0) \).

\( v \) and \( \theta \) are even and odd in \( z \), respectively.

**Proposition 3.1** Suppose that \((v_0, \theta_0) \in H^2(\Omega)\) with \(\int_{-1}^{1} \nabla_h \cdot v_0 dz = 0\). Then, (FPE) has the second-order energy estimate

\[
\sup_{0 \leq s \leq t} \left( \|\Delta v\|_2^2 + \|\Delta \theta\|_2^2 \right)(s)
+ \int_0^t \left( \|\nabla \Delta v\|_2^2 + \|\nabla \partial_z v\|_2^2 + \|\nabla \Delta \theta\|_2^2 + \|\nabla \partial_z \theta\|_2^2 \right) ds \leq M_3(t),
\]

for any \( t \in [0, \infty) \), where

\[
M_3(t) = C e^{C(t+1)(1+M_1(t)+M_2(t))} \left[ \|v_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2 + M_1(t) \right].
\]

Here, \( C \) is a positive constant depending only on \( Re_1, Re_2, Rt_1, Rt_2 \) and \( |f_0| \).

**Proof.** Applying the gradient operator \( \nabla \) to equation (3.2), then taking the dot product of the resulting equation with \( \nabla (\partial_t v - \Delta v) \), and finally integrating over \( \Omega \), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta v\|_2^2 + \frac{1}{Re_1} \|\nabla \nabla_h v\|_2^2 + \frac{1}{Re_2} \|\nabla \partial_z v\|_2^2 \right)
+ \left( \frac{1}{Re_1} \|\nabla_h \Delta v\|_2^2 + \frac{1}{Re_2} \|\partial_z \Delta v\|_2^2 + \|\nabla \partial_z v\|_2^2 \right)
= \int_\Omega \left[ (v \cdot \nabla_h) v - \left( \int_{-1}^z \nabla_h \cdot v d\xi \right) \partial_z v \right] : \nabla (\Delta v - \partial_t v) dxdydz
+ \int_\Omega \left[ \nabla \left( \int_{-1}^z \nabla_h \theta d\xi \right) : \nabla \Delta v - \partial_t v - \nabla (f_0 \tilde{k} \times v) : \nabla \partial_z v \right] dxdydz,
\]

in which we have used the following facts that

\[
\int_\Omega \nabla \nabla_h p_v(x, y, t) : \nabla (\partial_t v - \Delta v) dxdydz = 0,
\]

\[
\int_\Omega \nabla (f_0 \tilde{k} \times v) : \nabla \Delta v dxdydz = 0.
\]

In order to estimate the first integral term on the right-hand side of (3.4), the gradient operator \( \nabla \) will be divided into two parts, \( \nabla_h \) and \( \partial_z \). Then, we use Hölder’s inequality,
the Sobolev embedding theorem, Lemma 2.1 and Young’s inequality to obtain

\[
\int_{\Omega} \nabla \left[ (v \cdot \nabla h) v - \left( \int_{\Omega} \nabla_h \cdot v d\xi \right) \partial_z v \right] : \nabla (\Delta v - \partial_t v) \, dx dy dz
\]

\[
= \int_{\Omega} \left[ (\partial_j v \cdot \nabla h) v - \left( \int_{\Omega} \nabla_h \cdot \partial_j v d\xi \right) \partial_z v \right] \cdot (\partial_j \Delta v - \partial_j \partial_t v) \, dx dy dz
\]

\[
+ \int_{\Omega} \left[ (v \cdot \nabla h) \partial_z v - \left( \int_{\Omega} \nabla_h \cdot v d\xi \right) \partial_j \partial_z v \right] \cdot (\partial_j \Delta v - \partial_j \partial_t v) \, dx dy dz
\]

\[
+ \int_{\Omega} \left[ (\partial_z v \cdot \nabla h) v - (\nabla_h \cdot v) \partial_z v \right] \cdot (\partial_z \Delta v - \partial_z \partial_t v) \, dx dy dz
\]

\[
\leq C \left( \|v\|_{H^1}^2 + \left\| \nabla^2 v \right\|_2^2 + \|v\|_{H^1}^4 + \|\nabla v\|_2 \left\| \nabla^2 v \right\|_2 \right)
\]

\[
\times \left( \frac{1}{Re_1} \|\nabla \nabla_h v\|_2 + \frac{1}{Re_2} \|\nabla \partial_z v\|_2 \right)
\]

\[
+ \frac{1}{6} \left( \frac{1}{Re_1} \|\nabla \Delta v\|_2 + \frac{1}{Re_2} \|\partial_z \Delta v\|_2 + \|\nabla \partial_t v\|_2 \right). \tag{3.5}
\]

Here, \( \partial_j \in \{\partial_x, \partial_y\} \). Using Hölder’s inequality and Young’s inequality yields

\[
\int_{\Omega} \left[ \nabla \left( \int_{\Omega} \nabla_h \theta d\xi \right) \cdot \nabla (\Delta v - \partial_t v) - \nabla (f_0 k \times v) \cdot \nabla \partial_t v \right] \, dx dy dz
\]

\[
= \int_{\Omega} \left[ \int_{\Omega} \partial_j \nabla_h \theta d\xi \right] \cdot (\partial_j \Delta v - \partial_j \partial_t v) \, dx dy dz
\]

\[
+ \int_{\Omega} \left[ \nabla_h \theta \cdot (\partial_z \Delta v - \partial_z \partial_t v) - \nabla (f_0 k \times v) \cdot \nabla \partial_t v \right] \, dx dy dz
\]

\[
\leq C \left( \|\nabla v\|_2^2 + \|\nabla \theta\|_2^2 \right) + \frac{C}{R t_1} \|\nabla \nabla_h \theta\|_2^2
\]

\[
+ \frac{1}{6} \left( \frac{1}{Re_1} \|\nabla \Delta v\|_2^2 + \frac{1}{Re_2} \|\partial_z \Delta v\|_2^2 + \|\nabla \partial_t v\|_2^2 \right). \tag{3.6}
\]

By substituting (3.5) and (3.6) into (3.4), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta v\|_2^2 + \frac{1}{Re_1} \|\nabla \nabla_h v\|_2^2 + \frac{1}{Re_2} \|\nabla \partial_z v\|_2^2 \right)
\]

\[
+ \left( \frac{1}{Re_1} \|\nabla \Delta v\|_2^2 + \frac{1}{Re_2} \|\partial_z \Delta v\|_2^2 + \|\nabla \partial_t v\|_2^2 \right)
\]

\[
\leq \frac{C}{R t_1} \|\nabla \nabla_h \theta\|_2^2 + \frac{1}{3} \left( \frac{1}{Re_1} \|\nabla \Delta v\|_2^2 + \frac{1}{Re_2} \|\partial_z \Delta v\|_2^2 + \|\nabla \partial_t v\|_2^2 \right)
\]

\[
+ C \left( \|v\|_{H^1}^2 + \|\nabla^2 v\|_2^2 + \|v\|_{H^1}^4 + \|\nabla v\|_2 \left\| \nabla^2 v \right\|_2 \right).
\]
Similarly, applying the gradient operator \( \nabla \) to equation (3.3), multiplying the resulting equation by \( \nabla (\partial_t \theta - \Delta \theta) \), and integrating over \( \Omega \), from Hölder’s inequality, the Sobolev embedding theorem, Lemma 2.1 and Young’s inequality it can be deduced that

\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta \theta \|_{H^1}^2 + \frac{1}{R_t} \| \nabla h \theta \|_{L^2}^2 + \| \nabla \partial_z \theta \|_{L^2}^2 \right) + \left( \frac{1}{R_t} \| \nabla h \Delta \theta \|_{L^2}^2 + \frac{1}{R_t} \| \partial_z \Delta \theta \|_{L^2}^2 + \| \nabla \partial_t \theta \|_{L^2}^2 \right) = \int_{\Omega} \nabla \left[ (\int_{-1}^{z} \nabla h \cdot v d \xi) \partial_z \theta \right] \cdot \nabla (\Delta \theta - \partial_t \theta) \, dx dy dz 
\]

\[
\leq C \left[ \left( \| v \|_{H^1}^2 + \| \nabla^2 v \|_{L^2}^2 + \| \nabla \|_{H^1}^2 \right) \left( \frac{1}{R_t} \| \nabla h \theta \|_{L^2}^2 + \frac{1}{R_t} \| \nabla \partial_z \theta \|_{L^2}^2 + \| \nabla \partial_t \theta \|_{L^2}^2 \right) + \frac{1}{6} \left( \frac{1}{R_t} \| \nabla h \Delta \theta \|_{L^2}^2 + \frac{1}{R_t} \| \nabla \partial_z \Delta \theta \|_{L^2}^2 + \| \nabla \partial_t \theta \|_{L^2}^2 \right) \right].
\] (3.8)

Adding (3.7) and (3.8), and using Grönwall’s inequality, we obtain

\[
\left( \| \Delta v \|_{L^2}^2 + \frac{1}{R_t} \| \nabla h v \|_{L^2}^2 + \frac{1}{R_t} \| \nabla \partial_z v \|_{L^2}^2 + \| \Delta \theta \|_{L^2}^2 + \frac{1}{R_t} \| \nabla h \theta \|_{L^2}^2 \right) (t) + \frac{1}{R_t} \| \nabla \partial_z \theta \|_{L^2}^2 (t) + \int_{0}^{t} \left( \frac{1}{R_t} \| \nabla h \Delta \theta \|_{L^2}^2 + \frac{1}{R_t} \| \partial_z \Delta \theta \|_{L^2}^2 + \| \nabla \partial_t \theta \|_{L^2}^2 \right) ds + \\
\int_{0}^{t} \left( \frac{1}{R_t} \| \nabla h \Delta \theta \|_{L^2}^2 + \frac{1}{R_t} \| \partial_z \Delta \theta \|_{L^2}^2 + \| \nabla \partial_t \theta \|_{L^2}^2 \right) ds 
\leq \exp \left\{ C \int_{0}^{t} \left( \| v \|_{H^1}^2 + \| \nabla^2 v \|_{L^2}^2 + \| \nabla \|_{H^1}^2 \right) ds \right\} \left( \frac{1}{R_t} \| \nabla h \theta \|_{L^2}^2 + \| \nabla \partial_t \theta \|_{L^2}^2 \right) \
\times \left\{ \| \Delta v_0 \|_{L^2}^2 + \frac{1}{R_t} \| \nabla h v_0 \|_{L^2}^2 + \frac{1}{R_t} \| \nabla \partial_z v_0 \|_{L^2}^2 + \frac{1}{R_t} \| \nabla h \theta_0 \|_{L^2}^2 + \| \Delta \theta_0 \|_{L^2}^2 + \| \nabla \partial_z \theta_0 \|_{L^2}^2 + C \int_{0}^{t} (\| \nabla v \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2) ds \right\}
\]

which yields the result in Proposition 3.1 by (2.1). The proof is thus completed. \( \square \)

By means of Proposition 2.1, 2.2 and 3.1, we can perform the first-order energy estimate on system (1.11)–(1.14) under some smallness condition.
Proposition 3.2 Assume that \((v_0, \theta_0) \in H^2(\Omega)\) with \(\int_{-1}^1 \nabla_h \cdot v_0 dz = 0\). Then, there exists a small positive constant \(\lambda_0\) such that system (1.11)–(1.14) has the first-order energy estimate

\[
\sup_{0 \leq \tau \leq t} \left( \|\nabla (V_\epsilon, \epsilon W_\epsilon, \Pi_\epsilon)\|_2^2 \right) (s) \\
+ \int_0^t \left( \frac{1}{Re_1} \|\nabla \nabla_h V_\epsilon\|_2^2 + \frac{1}{Re_2} \|\nabla \nabla_h V_\epsilon\|_2^2 + \frac{\epsilon^2}{Re_1} \|\nabla \nabla_h W_\epsilon\|_2^2 \right) ds \\
+ \int_0^t \left( \frac{\epsilon^2}{Re_2} \|\nabla \nabla_\tau W_\epsilon\|_2^2 + \frac{1}{R_{t_1}} \|\nabla \nabla_h \Pi_\epsilon\|_2^2 + \frac{1}{R_{t_2}} \|\nabla \nabla_\tau \Pi_\epsilon\|_2^2 \right) ds \leq \epsilon^2 \mathcal{M}_4(t),
\]

for any \(t \in [0, T_\epsilon^*]\), provided that

\[
\sup_{0 \leq \tau \leq t} \left( \|\nabla (V_\epsilon, \Pi_\epsilon)\|_2^2 + \epsilon^2 \|\nabla W_\epsilon\|_2^2 \right) (s) \leq \lambda_0^2,
\]

where

\[
\mathcal{M}_4(t) = C \exp \left\{ C(t + 1) \left[ 1 + \mathcal{M}_1(t) + \mathcal{M}_2(t) + \mathcal{M}_3(t) + \mathcal{M}_1^2(t) + \mathcal{M}_2^2(t) \right] \right\} \\
\times \left\{ \mathcal{M}_1(t) + \mathcal{M}_3(t) + (t + 1) \left[ \mathcal{M}_1^2(t) + \mathcal{M}_2^2(t) + \mathcal{M}_3^2(t) \right] \right\}.
\]

Here \(C\) is a positive constant depending only on \(Re_1, Re_2, R_{t_1}\) and \(R_{t_2}\).

**Proof.** Taking the \(L^2(\Omega)\) inner product of (1.11), (1.12) and (1.13) with \(-\Delta V_\epsilon, -\Delta W_\epsilon\) and \(-\Delta \Pi_\epsilon\), respectively, then from the integration by parts it follows that

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla (V_\epsilon, \Pi_\epsilon)\|_2^2 + \epsilon^2 \|\nabla W_\epsilon\|_2^2 \right) + \left( \frac{1}{Re_1} \|\nabla \nabla_h V_\epsilon\|_2^2 + \frac{1}{Re_2} \|\nabla \nabla_\tau V_\epsilon\|_2^2 \right) \\
+ \left( \frac{\epsilon^2}{Re_1} \|\nabla \nabla_h W_\epsilon\|_2^2 + \frac{\epsilon^2}{Re_2} \|\nabla \nabla_\tau W_\epsilon\|_2^2 + \frac{1}{R_{t_1}} \|\nabla \nabla_h \Pi_\epsilon\|_2^2 + \frac{1}{R_{t_2}} \|\nabla \nabla_\tau \Pi_\epsilon\|_2^2 \right) \\
= \epsilon^2 \int_\Omega (U_\epsilon \cdot \nabla W_\epsilon + U_\epsilon \cdot \nabla w + u \cdot \nabla W_\epsilon + u \cdot \nabla w) \Delta W_\epsilon dx dy dz \\
+ \int_\Omega \left[ \epsilon^2 \left( \partial_t w - \frac{1}{Re_1} \Delta_h w - \frac{1}{Re_2} \partial_{zz} w \right) - \Pi_\epsilon \right] \Delta W_\epsilon dx dy dz \\
+ \int_\Omega (U_\epsilon \cdot \nabla \Pi_\epsilon + U_\epsilon \cdot \nabla \theta + u \cdot \nabla \Pi_\epsilon) \Delta \Pi_\epsilon dx dy dz \\
+ \int_\Omega [(U_\epsilon \cdot \nabla) V_\epsilon + (u \cdot \nabla) V_\epsilon + (U_\epsilon \cdot \nabla) v] \cdot \Delta V_\epsilon dx dy dz \\
=: R_1 + R_2 + R_3 + R_4,
\]

where we have used the following fact that

\[
\int_\Omega f_0(\vec{k} \times V_\epsilon) \cdot \Delta V_\epsilon dx dy dz = 0.
\]
Note that $|v| \leq \frac{1}{\bar{R}} \int_{-1}^{1} |v|dz + \int_{-1}^{1} |\partial_z v|dz$. We use the incompressible condition, Lemma 2.1 and Young’s inequality to obtain

\[
R_1 := \varepsilon^2 \int_\Omega \left( U_\varepsilon \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w + u \cdot \nabla W_\varepsilon + u \cdot \nabla w \right) dxdydz
\]

\[
= \varepsilon^2 \int_\Omega \left[ V_\varepsilon \cdot \nabla W_\varepsilon - (\partial_z W_\varepsilon) \int_{-1}^{1} \nabla h \cdot V_\varepsilon d\xi \right] dxdydz
\]

\[
+ \varepsilon^2 \int_\Omega \left[ (\nabla h \cdot v) \int_{-1}^{1} \nabla h \cdot V_\varepsilon d\xi - V_\varepsilon \cdot \int_{-1}^{1} \nabla h (\nabla h \cdot v) d\xi \right] dxdydz
\]

\[
+ \varepsilon^2 \int_\Omega \left[ v \cdot \nabla h W_\varepsilon + (\nabla h \cdot V_\varepsilon) \int_{-1}^{1} \nabla h \cdot v d\xi \right] dxdydz
\]

\[
+ \varepsilon^2 \int_\Omega \left[ (\nabla h \cdot v) \int_{-1}^{1} (\nabla h \cdot v) d\xi - v \cdot \int_{-1}^{1} \nabla h (\nabla h \cdot v) d\xi \right] dxdydz
\]

\[
\leq \varepsilon^2 \int_G \left( \int_{-1}^{1} (|V_\varepsilon| + |\partial_z V_\varepsilon|)dz \right) \left( \int_{-1}^{1} |\nabla h W_\varepsilon||\Delta W_\varepsilon|dz \right) dxdy
\]

\[
+ \varepsilon^2 \int_G \left( \int_{-1}^{1} |\nabla h V_\varepsilon|dz \right) \left( \int_{-1}^{1} |\partial_z W_\varepsilon||\Delta W_\varepsilon|dz \right) dxdy
\]

\[
+ \varepsilon^2 \int_G \left( \int_{-1}^{1} |\nabla h V_\varepsilon|dz \right) \left( \int_{-1}^{1} |\nabla h v||\Delta W_\varepsilon|dz \right) dxdy
\]

\[
+ \varepsilon^2 \int_G \left( \int_{-1}^{1} |\nabla^2 v|dz \right) \left( \int_{-1}^{1} |\nabla h ||\Delta W_\varepsilon|dz \right) dxdy
\]

\[
+ \varepsilon^2 \int_G \left( \int_{-1}^{1} |\nabla^2 v|dz \right) \left( \int_{-1}^{1} |\nabla v||\Delta W_\varepsilon|dz \right) dxdy
\]

\[
+ \varepsilon^2 \int_G \left( \int_{-1}^{1} |\nabla^2 v|dz \right) \left( \int_{-1}^{1} |v||\Delta W_\varepsilon|dz \right) dxdy
\]

\[
\leq C \left\{ \left( \frac{1}{Re_1} \| \nabla \nabla h V_\varepsilon \|_2^2 + \frac{1}{Re_2} \| \nabla \partial_z V_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_1} \| \nabla h W_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_2} \| \nabla \partial_z W_\varepsilon \|_2^2 \right) \right.
\]

\[
+ \left[ (1 + \varepsilon^2) \| V_\varepsilon \|_2^2 + \| \nabla h V_\varepsilon \|_2^2 + \| \partial_z V_\varepsilon \|_2^2 + \| V_\varepsilon \|_4^2 + \| V_\varepsilon \|_2^2 \| \nabla h V_\varepsilon \|_2^2 \right]
\]

\[
+ \left[ \| v \|_H^2 + \| \nabla v \|_2^2 + \| v \|_2^4 + (1 + \varepsilon^4) \| v \|_H^2 \| \nabla v \|_2^2 \right] \right\} \| \nabla (V_\varepsilon, \varepsilon W_\varepsilon) \|_2^2
\]

\[
+ C \varepsilon^2 \left( \| v \|_H^4 + \| v \|_H^2 \| \nabla v \|_2^2 + \| \nabla v \|_2^4 + \| \nabla v \|_2^2 \| \nabla \Delta v \|_2^2 + \| V_\varepsilon \|_2^2 \right)
\]

\[
+ \frac{1}{48} \left( \frac{1}{Re_1} \| \nabla \nabla h V_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_1} \| \nabla \nabla h W_\varepsilon \|_2^2 + \frac{\varepsilon^2}{Re_2} \| \nabla \partial_z W_\varepsilon \|_2^2 \right). \tag{3.9}
\]
Due to the incompressible condition, Hölder’s inequality and Young’s inequality, we get

\[
R_2 := \int_\Omega \left[ \varepsilon^2 \left( \partial_t \omega - \frac{1}{Re_1} \Delta_h \omega - \frac{1}{Re_2} \partial_{zz} \omega \right) - \Pi_{\varepsilon} \right] \Delta_{\varepsilon} \omega \, dx \, dy \, dz
\]

\[
= \varepsilon^2 \int_\Omega \left( - \int_1^z \nabla_h \cdot \partial_t \omega d\xi + \frac{1}{Re_1} \int_1^z \Delta_h (\nabla_h \cdot \omega) d\xi + \frac{1}{Re_2} \nabla_h \cdot \partial_z \omega \right) \Delta_{\varepsilon} \omega \, dx \, dy \, dz
\]

\[
+ \int_\Omega \left[ -\nabla_h \Pi_{\varepsilon} \cdot \left( \int_1^z \nabla_h (\nabla_{\varepsilon} \cdot \omega) d\xi \right) - \partial_z \Pi_{\varepsilon} (\nabla_h \cdot \omega) \right] \, dx \, dy \, dz
\]

\[
\leq C \varepsilon^2 \left( \| \nabla \partial_t \omega \|_2 + \| \nabla^2 \|_2 + \| \nabla \Delta \|_2 \right) + C \left( \| \nabla \omega \|_2 \| \nabla_{\varepsilon} \|_2 \right)
\]

\[
+ \frac{1}{48} \left( \frac{1}{Re_1} \| \nabla \nabla_{\varepsilon} \|_2 + \frac{\varepsilon^2}{Re_1} \| \nabla \nabla_{\varepsilon} \|_2 + \frac{\varepsilon^2}{Re_2} \| \nabla \partial_z \omega \|_2 \right).
\]  

(3.10)

By a similar argument as that for the integral term \( R_1 \), the integral term \( R_3 \) can be bounded as

\[
R_3 := \int_\Omega \left( U_e \cdot \nabla \Pi_{\varepsilon} + U_e \cdot \nabla \theta + u \cdot \nabla \Pi_{\varepsilon} \right) \Delta_{\varepsilon} \Pi_{\varepsilon} \, dx \, dy \, dz
\]

\[
= \int_\Omega \left[ V_e \cdot \nabla \Pi_{\varepsilon} - (\partial_z \Pi_{\varepsilon}) \int_1^z \nabla_h \cdot \omega d\xi + V_e \cdot \nabla \theta \right] \Delta_{\varepsilon} \Pi_{\varepsilon} \, dx \, dy \, dz
\]

\[
+ \int_\Omega \left[ v \cdot \nabla \Pi_{\varepsilon} - (\partial_z \Pi_{\varepsilon}) \int_1^z \nabla_h \cdot \omega d\xi - (\partial_z \Pi_{\varepsilon}) \int_1^z \nabla_h \cdot \omega d\xi \right] \Delta_{\varepsilon} \Pi_{\varepsilon} \, dx \, dy \, dz
\]

\[
\leq C \left( \frac{1}{Re_1} \| \nabla \nabla_{\varepsilon} \|_2 + \frac{1}{Re_2} \| \nabla \partial_z \omega \|_2 + \frac{1}{Rt_1} \| \nabla \nabla_{\varepsilon} \|_2 + \frac{1}{Rt_2} \| \nabla \partial_z \Pi_{\varepsilon} \|_2 \right)
\]

\[
+ \left( \| v \|_2^4 + \| \nabla^2 \|_2 + \| \nabla \|_2 \right) \| \nabla \theta \|_2 + \| \theta \|_2 \| \nabla \theta \|_2 \right)
\]

\[
+ C \| V_e \|_2 \| \nabla \theta \|_2 + \| \nabla \Pi_{\varepsilon} \|_2 \right)
\]

\[
+ \frac{1}{48} \left( \frac{1}{Re_1} \| \nabla \nabla_{\varepsilon} \|_2 + \frac{1}{Re_2} \| \nabla \partial_z \omega \|_2 \right).
\]  

(3.11)

Similarly, we have

\[
R_4 := \int_\Omega [(U_e \cdot \nabla) V_e + (u \cdot \nabla) V_e + (U_e \cdot \nabla) v] \cdot \Delta V_e \, dx \, dy \, dz
\]

\[
= \int_\Omega \left[ (V_e \cdot \nabla_h) V_e - (\partial_z V_e) \int_1^z \nabla_h \cdot \omega d\xi \right] \Delta V_e \, dx \, dy \, dz
\]

\[
+ \int_\Omega \left[ (V_e \cdot \nabla_h) v - (\partial_z v) \int_1^z \nabla_h \cdot \omega d\xi \right] \Delta V_e \, dx \, dy \, dz
\]
Rigorous derivation of the full primitive equations by the scaled...

\[ + \int_{\Omega} \left[ (v \cdot \nabla h) V_\epsilon - (\partial_z V_\epsilon) \int_{-1}^{z} \nabla h \cdot v d\xi \right] \cdot \Delta V_\epsilon \, dx
dy
dz \]

\[ \leq C \left( \| v \|_{H^1}^2 \| \nabla v \|_{H^1}^2 + \| V_\epsilon \|_{H^2}^2 + \| \partial_z V_\epsilon \|_{H^2}^2 + \| \nabla h V_\epsilon \|_{H^2}^2 \right) \| \nabla V_\epsilon \|_2^2 + C \left( \| v \|_2^2 + \| \nabla v \|_{H^1}^2 + \frac{1}{Re_1} \| \nabla \nabla h V_\epsilon \|_2^2 + \frac{1}{Re_2} \| \nabla \partial_z V_\epsilon \|_2^2 \right) \| \nabla V_\epsilon \|_2^2 \]

\[ + C \| \nabla^2 v \|_2^2 \| V_\epsilon \|_2^2 + \frac{1}{48} \left( \frac{1}{Re_1} \| \nabla \nabla h V_\epsilon \|_2^2 + \frac{1}{Re_2} \| \nabla \partial_z V_\epsilon \|_2^2 \right) \] . \hspace{1cm} (3.12)

Adding (3.9), (3.10), (3.11) and (3.12) yields

\[ \frac{1}{2} \frac{d}{dt} \left( \| \nabla(V_\epsilon, \Pi_\epsilon) \|_2^2 + \epsilon^2 \| \nabla W_\epsilon \|_2^2 \right) + \frac{11}{12} \left( \frac{1}{Re_1} \| \nabla \nabla h V_\epsilon \|_2^2 + \frac{1}{Re_2} \| \nabla \partial_z V_\epsilon \|_2^2 \right) \]

\[ + \frac{11}{12} \left( \frac{\epsilon^2}{Re_1} \| \nabla \nabla h W_\epsilon \|_2^2 + \frac{\epsilon^2}{Re_2} \| \nabla \partial_z \Pi_\epsilon \|_2^2 + \frac{1}{Rt_1} \| \nabla \nabla h \Pi_\epsilon \|_2^2 + \frac{1}{Rt_2} \| \nabla \partial_z \Pi_\epsilon \|_2^2 \right) \]

\[ \leq C_\sigma \left[ \left( \frac{1}{Re_1} \| \nabla \nabla h V_\epsilon \|_2^2 + \frac{1}{Re_2} \| \nabla \partial_z V_\epsilon \|_2^2 + \frac{\epsilon^2}{Re_1} \| \nabla \nabla h V_\epsilon \|_2^2 + \frac{\epsilon^2}{Re_2} \| \nabla \partial_z V_\epsilon \|_2^2 \right) \]

\[ + \left( \frac{1}{Rt_1} \| \nabla \nabla h \Pi_\epsilon \|_2^2 + \frac{1}{Rt_2} \| \nabla \partial_z \Pi_\epsilon \|_2^2 \right) \| (1 + \| v \|_{H^1}^2 + \| \partial_z V_\epsilon \|_{H^2}^2 + \| V_\epsilon \|_{H^2}^2 + \| V_\epsilon \|_{H^2}^2 \| \nabla \nabla h V_\epsilon \|_{H^1}^2 + \| \nabla \nabla h \Pi_\epsilon \|_{H^1}^2 \] \]

\[ + C_\sigma \epsilon^2 \left( \| v \|_{H^1}^2 + \| \nabla v \|_{H^1}^2 + \| \nabla \nabla v \|_{H^1}^2 + \| \nabla \partial_z \Pi_\epsilon \|_{H^1}^2 \right) \| \nabla V_\epsilon \|_{H^1}^2 + \| \nabla \nabla v \|_{H^1}^2 + \| \nabla \delta \Pi_\epsilon \|_{H^1}^2 \] \]

\[ + C_\sigma \epsilon^2 \left( \| \nabla \nabla h V_\epsilon \|_2^2 + \| \nabla v \|_{H^1}^2 + \| \nabla \nabla h \Pi_\epsilon \|_2^2 + \| \nabla \partial_z \Pi_\epsilon \|_2^2 \right) \] \hspace{1cm} (3.13)

Setting \( \lambda_0 = \sqrt{\frac{5}{12C_\sigma}} \), and then using the smallness condition,

\[ \sup_{0 \leq s \leq t} \left( \| \nabla(V_\epsilon, \Pi_\epsilon) \|_2^2 + \epsilon^2 \| \nabla W_\epsilon \|_2^2 \right) \leq \lambda_0^2, \]

we get from (3.13) that

\[ \frac{d}{dt} \left( \| \nabla(V_\epsilon, \Pi_\epsilon) \|_2^2 + \epsilon^2 \| \nabla W_\epsilon \|_2^2 \right) + \left( \frac{1}{Re_1} \| \nabla \nabla h V_\epsilon \|_2^2 + \frac{1}{Re_2} \| \nabla \partial_z V_\epsilon \|_2^2 \right) \]

\[ + \left( \frac{\epsilon^2}{Re_1} \| \nabla \nabla h W_\epsilon \|_2^2 + \frac{\epsilon^2}{Re_2} \| \nabla \partial_z W_\epsilon \|_2^2 + \frac{1}{Rt_1} \| \nabla \nabla h \Pi_\epsilon \|_2^2 + \frac{1}{Rt_2} \| \nabla \partial_z \Pi_\epsilon \|_2^2 \right) \]

\[ \leq C_\sigma \left[ \left( \| \nabla \nabla h V_\epsilon \|_{H^1}^2 + \| \nabla \nabla h \Pi_\epsilon \|_{H^1}^2 + \| \nabla \nabla h \Pi_\epsilon \|_{H^1}^2 + \right) \| \nabla \nabla h V_\epsilon \|_{H^1}^2 + \| \nabla \nabla h \Pi_\epsilon \|_{H^1}^2 \] \]

\[ + \left( \| \nabla \nabla h V_\epsilon \|_{H^1}^2 + \| \nabla \partial_z V_\epsilon \|_{H^1}^2 + \| \nabla \nabla h \Pi_\epsilon \|_{H^1}^2 + \right) \| \nabla \nabla h V_\epsilon \|_{H^1}^2 + \| \nabla \partial_z \Pi_\epsilon \|_{H^1}^2 \]
With the help of Proposition 2.2 and 3.2, we give the proof of Theorem 1.1. Note that the energy estimate holds we also need to eliminate the effect of the smallness condition in Proposition 3.2. This Denote by $T^*$ the maximal existence time of the strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ to (SBE) with (1.2)–(1.4). Then, for any $T > 0$, there is a small positive constant $\varepsilon(T) = \frac{2a_0}{3\sqrt{\mathcal{M}_4(T)}}$ such that $T^*>T$ as long as $\varepsilon \in (0, \varepsilon(T))$. Furthermore, the energy estimate holds

$$
\sup_{0 \leq s \leq t} \left\{ \| (V_\varepsilon, W_\varepsilon, \Pi_\varepsilon) \|_{H^1}^2 \right\} (s)
+ \int_0^t \left( \frac{1}{Re_1} \| \nabla v_\varepsilon \|_{H^1}^2 + \frac{1}{Re_2} \| \partial_z v_\varepsilon \|_{H^1}^2 + \frac{\varepsilon^2}{Re_1} \| \nabla W_\varepsilon \|_{H^1}^2 \right) ds
\leq C_\sigma \exp \left\{ C_\sigma \int_0^t \left( \| \nabla \theta \|_{H^1}^2 + \| \partial_z \Pi_\varepsilon \|_{H^1}^2 \right) ds \right\}
$$

By virtue of Grönwall’s inequality, from Proposition 2.1, 2.2 and 3.1, it follows that

$$
\begin{align*}
&\left\{ \| \nabla (V_\varepsilon, \Pi_\varepsilon) \|_{H^1}^2 + \varepsilon^2 \| \nabla W_\varepsilon \|_{H^1}^2 \right\} (t) + \int_0^t \left( \frac{1}{Re_1} \| \nabla v_\varepsilon \|_{H^1}^2 + \frac{1}{Re_2} \| \partial z \Pi_\varepsilon \|_{H^1}^2 \right) ds \\
&\quad + \int_0^t \left( \frac{\varepsilon^2}{Re_1} \| \nabla v_\varepsilon \|_{H^1}^2 + \varepsilon^2 \| \partial_z v_\varepsilon \|_{H^1}^2 + \frac{1}{Rt_1} \| \nabla v_\varepsilon \|_{H^1}^2 + \frac{1}{Rt_2} \| \partial z \Pi_\varepsilon \|_{H^1}^2 \right) ds \\
&\quad \leq C_\sigma \exp \left\{ C_\sigma \int_0^t \left( \| \nabla \theta \|_{H^1}^2 + \| \partial z \Pi_\varepsilon \|_{H^1}^2 \right) ds \right\}
\end{align*}
$$

Here, we have used the fact that $(V_\varepsilon, W_\varepsilon, \Pi_\varepsilon)|_{t=0} = 0$. This completes the proof. □

### 4 Proof of Theorem 1.1

With the help of Proposition 2.2 and 3.2, we give the proof of Theorem 1.1. Note that we also need to eliminate the effect of the smallness condition in Proposition 3.2. This can be achieved by the following proposition.

**Proposition 4.1** Denote by $T^*_\varepsilon$ the maximal existence time of the strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ to (SBE) with (1.2)–(1.4). Then, for any $T > 0$, there is a small positive constant $\varepsilon(T) = \frac{2a_0}{3\sqrt{\mathcal{M}_4(T)}}$ such that $T^*_\varepsilon > T$ as long as $\varepsilon \in (0, \varepsilon(T))$. Furthermore, the energy estimate holds

$$
\sup_{0 \leq s \leq T} \left\{ \| (V_\varepsilon, W_\varepsilon, \Pi_\varepsilon) \|_{H^1}^2 \right\} (s)
+ \int_0^T \left( \frac{1}{Re_1} \| \nabla v_\varepsilon \|_{H^1}^2 + \frac{1}{Re_2} \| \partial z v_\varepsilon \|_{H^1}^2 + \frac{\varepsilon^2}{Re_1} \| \nabla W_\varepsilon \|_{H^1}^2 \right) ds
\leq C_\sigma \exp \left\{ C_\sigma \int_0^T \left( \| \nabla \theta \|_{H^1}^2 + \| \partial z \Pi_\varepsilon \|_{H^1}^2 \right) ds \right\}
$$

□ Springer
\[
\begin{align*}
+ \int_0^t \left( \frac{\varepsilon^2}{\Re e_2} \| \partial_z W_\varepsilon \|_{H^1}^2 + \frac{1}{\Re t_1} \| \nabla_h \Pi_\varepsilon \|_{H^1}^2 + \frac{1}{\Re t_2} \| \partial_z \Pi_\varepsilon \|_{H^1}^2 \right) ds \\
\leq \varepsilon^2 \left( M_2(t) + M_4(t) \right),
\end{align*}
\]

for any \( t \in [0, T] \), where both \( M_2(t) \) and \( M_4(t) \) are nonnegative continuously increasing functions that do not depend on \( \varepsilon \).

**Proof.** For any \( T > 0 \), we set \( T'_\varepsilon = \min\{ T^*, T \} \). Then, it follows from Proposition 2.2 that

\[
\sup_{0 \leq s \leq t} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon) \|_{L^2}^2 \right)(s)
\]

\[
+ \int_0^t \left( \frac{1}{\Re t_1} \| \nabla h V_\varepsilon \|_2^2 + \frac{1}{\Re e_2} \| \partial_z V_\varepsilon \|_2^2 + \frac{\varepsilon^2}{\Re e_1} \| \nabla h W_\varepsilon \|_2^2 \right) ds
\]

\[
+ \int_0^t \left( \frac{\varepsilon^2}{\Re e_2} \| \partial_z W_\varepsilon \|_2^2 + \frac{1}{\Re t_1} \| \nabla h \Pi_\varepsilon \|_2^2 + \frac{1}{\Re t_2} \| \partial_z \Pi_\varepsilon \|_2^2 \right) ds \leq \varepsilon^2 M_2(t), \quad (4.1)
\]

for every \( t \in [0, T'_\varepsilon] \), where

\[
M_2(t) = C \exp \{ C(t+1) \left[ M_1(t) + M_2(t) + M_3(t) + M_4^2(t) + M_5^2(t) \right] \}
\]

\[
\times \left[ M_1(t) + M_3(t) + (t+1) \left[ M_4^2(t) + M_5^2(t) + M_6^2(t) \right] \right].
\]
Let $\varepsilon(T) = \frac{2\lambda_0}{3\sqrt{M_4(T)}}$. Owing to (4.2), we get
\[
\sup_{0 \leq s \leq t} \left( \|\nabla(V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon)\|_{H^1}^2 \right)(s) + \int_0^t \left( \frac{1}{Re_1} \|\nabla V_\varepsilon\|_{H^1}^2 + \frac{1}{Re_2} \|\nabla_\varepsilon V_\varepsilon\|_{H^1}^2 \right) ds \\
+ \int_0^t \left( \frac{\varepsilon^2}{Re_1} \|\nabla V_\varepsilon\|_{H^1}^2 + \frac{\varepsilon^2}{Re_2} \|\nabla_\varepsilon V_\varepsilon\|_{H^1}^2 + \frac{1}{Rt_1} \|\nabla_\varepsilon \Pi_\varepsilon\|_{H^1}^2 + \frac{1}{Rt_2} \|\nabla_\varepsilon \Pi_\varepsilon\|_{H^1}^2 \right) ds \\
\leq \frac{4\lambda_0^2 M_4(t)}{9 M_4(T)} \leq \frac{4\lambda_0^2}{9} < \lambda_0^2,
\]
for every $t \in [0, t'_\varepsilon)$ and for every $\varepsilon \in (0, \varepsilon(T))$, which leads to
\[
\sup_{0 \leq s < t'_\varepsilon} \left( \|\nabla(V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon)\|_{H^1}^2 \right)(s) < \lambda_0^2.
\]
(4.3)

The definition of $t'_\varepsilon$ and (4.3) imply that $t'_\varepsilon = T'_\varepsilon$. On account of this, combining (4.1) with (4.2) yields
\[
\sup_{0 \leq s \leq t} \left( \|\nabla(V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon)\|_{H^1}^2 \right)(s) \\
+ \int_0^t \left( \frac{1}{Re_1} \|\nabla V_\varepsilon\|_{H^1}^2 + \frac{1}{Re_2} \|\nabla_\varepsilon V_\varepsilon\|_{H^1}^2 + \frac{\varepsilon^2}{Re_1} \|\nabla_\varepsilon V_\varepsilon\|_{H^1}^2 \right) ds \\
+ \int_0^t \left( \frac{\varepsilon^2}{Re_2} \|\nabla_\varepsilon W_\varepsilon\|_{H^1}^2 + \frac{1}{Rt_1} \|\nabla_\varepsilon \Pi_\varepsilon\|_{H^1}^2 + \frac{1}{Rt_2} \|\nabla_\varepsilon \Pi_\varepsilon\|_{H^1}^2 \right) ds \\
\leq \varepsilon^2 \left( M_2(t) + M_4(t) \right),
\]
(4.4)
for every $t \in [0, T'_\varepsilon)$ and for every $\varepsilon \in (0, \varepsilon(T))$.

Next, it suffices to prove that $T^*_\varepsilon > T$ for every $\varepsilon \in (0, \varepsilon(T))$. Assume that $T^*_\varepsilon \leq T$, i.e., $T'_\varepsilon = \min\{T^*_\varepsilon, T\} = T^*_\varepsilon$. Then, it is clear that
\[
\limsup_{t' \to (T^*_\varepsilon)^-} \left( \|v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon\|_{H^1}^2 \right)(t') = \infty,
\]
(4.5)
since $T^*_\varepsilon$ is the maximal existence time of the strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ to (SBE).

By virtue of Proposition 2.1 and 3.1, the estimate holds
\[
\sup_{0 \leq t' \leq t} \left( \|v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon\|_{H^1}^2 \right)(t') \\
\leq \sup_{0 \leq t' \leq t} \left( \|v, \varepsilon w, \theta\|_{H^1}^2 \right)(t') + \sup_{0 \leq t' \leq t} \left( \|v_\varepsilon, \varepsilon w_\varepsilon, \Pi_\varepsilon\|_{H^1}^2 \right)(t') \\
\leq M_1(t) + \varepsilon^2 (M_1(t) + M_3(t)) + \varepsilon^2 (M_2(t) + M_4(t)),
\]
for every $t \in [0, T^*_\varepsilon)$, where the incompressible condition and estimate (4.4) are used.

It is obvious that (4.5) contradicts to the above estimate. This contradiction deduces that $T^*_\varepsilon > T$ and hence $T'_\varepsilon = T$, completing the proof of Proposition 4.1. □
The proof of Theorem 1.1 is as follows:

**Proof of Theorem 1.1.** For any $T > 0$, from Proposition 4.1, it follows that (SBE) corresponding to (1.2)-(1.4) exists a unique strong solution $(v_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ on the time interval $[0, T]$ for every $\varepsilon \in (0, \varepsilon(T))$. Moreover, the energy estimate holds

\[
\sup_{0 \leq t \leq T} \left( \| (V_\varepsilon, \varepsilon W_\varepsilon, \Pi_\varepsilon) \|_{H^1}^2 \right)(t) \\
+ \int_0^T \left( \frac{1}{Re_1} \| \nabla_h V_\varepsilon \|_{H^1}^2 + \frac{1}{Re_2} \| \partial_z V_\varepsilon \|_{H^1}^2 \right) dt \\
+ \int_0^T \left( \frac{\varepsilon^2}{Re_1} \| \nabla_h W_\varepsilon \|_{H^1}^2 + \frac{\varepsilon^2}{Re_2} \| \partial_z W_\varepsilon \|_{H^1}^2 + \frac{1}{Rt_1} \| \nabla_h \Pi_\varepsilon \|_{H^1}^2 + \frac{1}{Rt_2} \| \partial_z \Pi_\varepsilon \|_{H^1}^2 \right) dt \\
\leq \varepsilon^2 (M_2(T) + M_4(T)) =: \varepsilon^2 M_5(T),
\]

where $M_5(t)$ is a nonnegative continuously increasing function that does not depend on $\varepsilon$. By virtue of the above estimate, we obtain the strong convergences

\[
(v_\varepsilon, \varepsilon w_\varepsilon, \theta_\varepsilon) \to (v, 0, \theta) \text{ in } L^\infty \left( 0, T; H^1(\Omega) \right),
\]

\[
\left( \frac{\nabla_h v_\varepsilon}{\sqrt{Re_1}}, \frac{\partial_z v_\varepsilon}{\sqrt{Re_2}}, \frac{\varepsilon \nabla_h w_\varepsilon}{\sqrt{Re_1}} \right) \to \left( \frac{\nabla_h v}{\sqrt{Re_1}}, \frac{\partial_z v}{\sqrt{Re_2}}, 0 \right) \text{ in } L^2 \left( 0, T; H^1(\Omega) \right),
\]

\[
\left( \frac{\varepsilon \partial_z w_\varepsilon}{\sqrt{Re_2}}, \frac{\nabla_h \theta_\varepsilon}{\sqrt{Rt_1}}, \frac{\partial_z \theta_\varepsilon}{\sqrt{Rt_2}} \right) \to \left( 0, \frac{\nabla_h \theta}{\sqrt{Rt_1}}, \frac{\partial_z \theta}{\sqrt{Rt_2}} \right) \text{ in } L^2 \left( 0, T; H^1(\Omega) \right).
\]

Due to the incompressible condition, we deduce from $\frac{\nabla_h v_\varepsilon}{\sqrt{Re_1}} \to \frac{\nabla_h v}{\sqrt{Re_1}}$ in $L^2 \left( 0, T; H^1(\Omega) \right)$ and $v_\varepsilon \to v$ in $L^\infty \left( 0, T; H^1(\Omega) \right)$ that

\[
w_\varepsilon \to w \text{ in } L^2 \left( 0, T; H^1(\Omega) \right)
\]

and

\[
w_\varepsilon \to w \text{ in } L^\infty \left( 0, T; L^2(\Omega) \right),
\]

respectively. In addition, it is obvious that the convergence rate is of the order $O(\varepsilon)$. Theorem 1.1 is thus proved.

\[\square\]

**Acknowledgements**  The work of X. Pu was supported in part by the National Natural Science Foundation of China (No. 11871172) and the Science and Technology Projects in Guangzhou (No. 202201020132). The work of W. Zhou was supported by the Innovation Research for the Postgraduates of Guangzhou University (No. 2021GDJC-D09).

**Declarations**

**Conflict of interest**  The authors declare that there is no conflict of interest.
References

1. Azérad, P., Guillén, F.: Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. SIAM J. Math. Anal. 33, 847–859 (2001)

2. Bresch, D., Guillén-González, F., Masmoudi, N., Rodríguez-Bellido, M.A.: On the uniqueness of weak solutions of the two-dimensional primitive equations. Differ. Integral Equ. 16, 77–94 (2003)

3. Cao, C., Ibrahim, S., Nakanishi, K., Titi, E.S.: Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. Commun. Math. Phys. 337, 473–482 (2015)

4. Cao, C., Li, J., Titi, E.S.: Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity. Arch. Ration. Mech. Anal. 214, 35–76 (2014)

5. Cao, C., Li, J., Titi, E.S.: Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity. J. Differ. Equ. 257, 4108–4132 (2014)

6. Cao, C., Li, J., Titi, E.S.: Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion. Commun. Pure Appl. Math. 69, 1492–1531 (2016)

7. Cao, C., Li, J., Titi, E.S.: Strong solutions to the 3D primitive equations with horizontal dissipation: near $H^1$ initial data. J. Funct. Anal. 272, 4606–4641 (2017)

8. Cao, C., Li, J., Titi, E.S.: Global well-posedness of the 3D primitive equations with horizontal viscosity and vertical diffusivity. Phys. D 412, 132606 (2020)

9. Cao, C., Titi, E.S.: Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. Ann. of Math. 166, 245–267 (2007)

10. Cao, C., Titi, E.S.: Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion. Commun. Math. Phys. 310, 537–568 (2012)

11. Cao, C., Titi, E.S.: Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model. Commun. Pure Appl. Math. 56, 198–233 (2003)

12. Fang, D., Han, B.: Global well-posedness for the 3D primitive equations in anisotropic framework. J. Math. Anal. Appl. 484, 123714 (2020)

13. Furukawa, K., Giga, Y., Hieber, M., Hussein, A., Kashiwabara, T., Wrona, M.: Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations. Nonlinearity 33, 6502–6516 (2020)

14. Giga, Y., Gries, M., Hieber, M., Hussein, A., Kashiwabara, T.: The hydrostatic Stokes semigroup and well-posedness of the primitive equations on spaces of bounded functions. J. Funct. Anal. 279, 108561 (2020)

15. Giga, Y., Gries, M., Hieber, M., Hussein, A., Kashiwabara, T.: Analyticity of solutions to the primitive equations. Math. Nachr. 293, 284–304 (2020)

16. Ghoul, T.-E., Ibrahim, S., Lin, Q., Titi, E.S.: On the effect of rotation on the life-span of analytic solutions to the 3D inviscid primitive equations. Arch. Ration. Mech. Anal. 243, 747–806 (2022)

17. Han-Kwan, D., Nguyen, T.: Ill-posedness of the hydrostatic Euler and singular Vlasov equations. Arch. Ration. Mech. Anal. 221, 1317–1344 (2016)

18. Hieber, M., Hussein, A., Kashiwabara, T.: Global strong $L^p$ well-posedness of the 3D primitive equations with heat and salinity diffusion. J. Differ. Equ. 261, 6950–6981 (2016)

19. Hieber, M., Hussein, A., Kashiwabara, T.: Global well-posedness of the three dimensional primitive equations in $L^p$-spaces. Arch. Ration. Mech. Anal. 221, 1077–1115 (2016)

20. Ibrahim, S., Lin, Q., Titi, E.S.: Finite-time blowup and ill-posedness in Sobolev spaces of the inviscid primitive equations with rotation. J. Differ. Equ 286, 557–577 (2021)

21. Kobelkov, G.M.: Existence of a solution in the large for the 3D large-scale ocean dynamics equations. C. R. Math. Acad. Sci. Paris 343, 283–286 (2006)

22. Kukavica, I., Pei, Y., Rusin, W., Ziane, M.: Primitive equations with continuous initial data. Nonlinearity 27, 1135–1155 (2014)

23. Kukavica, I., Temam, R., Vicol, V.C., Ziane, M.: Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain. J. Differ. Equ. 250, 1719–1746 (2011)

24. Kukavica, I., Ziane, M.: The regularity of solutions of the primitive equations of the ocean in space dimension three. C. R. Math. Acad. Sci. Paris 345, 257–260 (2007)

25. Kukavica, I., Ziane, M.: On the regularity of the primitive equations of the ocean. Nonlinearity 20, 2739–2753 (2007)

26. Lions, J.L., Temam, R., Wang, S.: New formulations of the primitive equations of atmosphere and applications. Nonlinearity 5, 237–288 (1992)
27. Lions, J.L., Temam, R., Wang, S.: On the equations of the large scale ocean. Nonlinearity 5, 1007–1053 (1992)
28. Lions, J.L., Temam, R., Wang, S.: Mathematical theory for the coupled atmosphere-ocean models. J. Math. Pures Appl. 74, 105–163 (1995)
29. Li, J., Titi, E.S.: The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation. J. Math. Pures Appl. 124, 30–58 (2019)
30. Li, J., Titi, E.S.: Existence and uniqueness of weak solutions to viscous primitive equations for a certain class of discontinuous initial data. SIAM J. Math. Anal. 49, 1–28 (2017)
31. Li, J., Titi, E.S., Yuan, G.: The primitive equations approximation of the anisotropic horizontally viscous Navier-Stokes equations. J. Differ. Equ. 306, 492–524 (2022)
32. Li, J., Yuan, G.: Global well-posedness of \( z \)-weak solutions to the primitive equations without vertical diffusivity. J. Math. Phys. 63, 24 (2022)
33. Pu, X., Zhou, W.: On the rigorous mathematical derivation for the viscous primitive equations with density stratification, preprint. (2022) arXiv: org/2203.10529v1
34. Pu, X., Zhou W.: The hydrostatic approximation of the Boussinesq equations with rotation in a thin domain, preprint. (2022) arXiv: 2203.11418v1
35. Renardy, M.: Ill-posedness of the hydrostatic Euler and Navier-Stokes equations. Arch. Ration. Mech. Anal. 194, 877–886 (2009)
36. Tachim Medjo, T.: On the uniqueness of \( z \)-weak solutions of the three-dimensional primitive equations of the ocean. Nonlinear Anal. Real World Appl. 11, 1413–1421 (2010)
37. Wong, T.K.: Blowup of solutions of the hydrostatic Euler equations. Proc. Amer. Math. Soc. 143, 1119–1125 (2015)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.