The two-boundary sine-Gordon model

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We study in this paper the ground state energy of a free bosonic theory on a finite interval of length $R$ with either a pair of sine-Gordon type or a pair of Kondo type interactions at each boundary. This problem has potential applications in condensed matter (current through superconductor-Luttinger liquid-superconductor junctions) as well as in open string theory (tachyon condensation). While the application of Bethe ansatz techniques to this problem is in principle well known, considerable technical difficulties are encountered. These difficulties arise mainly from the way the bare couplings are encoded in the reflection matrices, and require complex analytic continuations, which we carry out in detail in a few cases.

I. INTRODUCTION

This paper is concerned with the study of 1 + 1 quantum field theories defined on a finite interval of length $R$. When the bulk is massless and conformal boundary conditions are chosen, such theories can be entirely understood using the formalism of boundary conformal field theory [1]. The case where one deviates from this situation either by adding an interaction in the bulk or at the boundaries (or both) is more difficult, and quite interesting as it involves crossovers depending on the bulk and boundary mass scales as well as the finite size $R$.

Since the “conformal revolution”, this topic was probably first considered in print in [2] (although it has been studied in unpublished work of A. Zamolodchikov) where the basic thermodynamic Bethe ansatz and Destri de Vega (DDV) approaches were delineated in the case of simple integrable theories, mostly massive in the bulk, with simple boundary conditions. Since then, the topic has attracted a fair amount of attention (see for instance [3], [4], [5], [6], [7]).

In the present paper, we discuss in details the limit complementary to [2] where the bulk is a massless theory - mostly a free boson - and the perturbations sit on the boundary. We shall have mostly two theories in mind. The first is of the boundary sine-Gordon model type

$$H = \frac{1}{2} \int_0^R dx [ (\partial_t \phi)^2 + (\partial_x \phi)^2 ] + \Delta_l \cos \left( \frac{\beta}{2} \phi(0) \right) + \Delta_r \cos \left( \frac{\beta}{2} \phi(R) - \chi \right),$$

and the second of Kondo type

$$H = \frac{1}{2} \int_0^R dx [ (\partial_t \phi)^2 + (\partial_x \phi)^2 ] + \Delta_l \left[ e^{i \frac{\pi}{2} \beta \phi(0)} S_l^- + e^{-i \frac{\pi}{2} \beta \phi(0)} S_l^+ \right] + \Delta_r \left[ e^{i \frac{\pi}{2} \beta \phi(R)} S_r^- + e^{-i \frac{\pi}{2} \beta \phi(R)} S_r^+ \right].$$

In this last equation, $S^{\pm}_{l,r}$ refer to $U_q sl(2)$ spins in representations of spin $j_l, j_r$ and $q = e^{-i \pi \beta^2}$. The case with $j_l = j_r = 1/2$ could be called a “double Kondo problem”. In that case, as usual, the $\beta$ dependence of the interaction can be traded for a coupling anisotropy in spin space - notice that we restrict to the case where the $l$ and $r$ sides see the same anisotropy, a condition necessary to preserve integrability.

This double Kondo theory mustn’t be confused with the two impurity Kondo model. Although there also one would expect results to depend on the couplings for each impurity and the distance between the impurities, the detailed set-up is entirely different [8].

There are plenty of physical motivations to tackle these problems. On the condensed matter side, they can be thought of as generalizations of quantum impurity problems, where the bulk of the theory gets modified by the addition of a single local defect, e.g. a boundary or a quantum degree of freedom. Now, instead of only one impurity, we have two different ones coupled by a finite-size quantum field theory. One could label such a situation as a “quantum bridge problem”. An experimental setup for such problems could involve an interacting device - maybe

\footnote{In [2] the results referred to as boundary sine-Gordon model concern mostly a bulk sine-Gordon theory with Dirichlet type boundary conditions in $0, R$. The terminology has somewhat changed since then, and boundary sine-Gordon theory nowadays refers rather to a free boson with a cosine perturbation at the boundary.}
a carbon nanotube or quantum wire - connected to two leads on either end. If the excitations of the device suffer from a gap when trying to infiltrate the leads, then in a first approximation all the action takes place at the contacts between the device and the leads. The excitations, upon hitting the contacts, will be reflected and mixed by the “boundary” according to some rules dictated by the specifics of the couplings, in other words the excitations will face some nontrivial, energy dependent boundary conditions, and the situation could be described by hamiltonians such as (1.2) in the low energy, universal limit.

Possibly the simplest example of such a situation occurs when one considers two different (s-wave) superconductors connected by a bridge made of a quantum wire, e.g. a carbon nanotube, a construction called a Josephson junction. Since both superconductors inherently possess a gap for electronic excitations within their bulk, an effective theory can be obtained by considering only the wire itself with two “impurities” on either side replacing the superconductors. Solving this effective model - which is essentially described by (1) leads to the determination of the Josephson current [9].

We do not know of any experimental condensed matter situation where the two boundary Kondo model would be relevant (although this does not seem impossible to imagine). The model has however appeared in string theory in the study of tachyon instabilities [10] where the ends of the open string are coupled to a background, and the boundary spin acts in the space of Chan-Paton factors (and the free boson corresponds to the $X^{25}$ coordinate). The double boundary sine-Gordon model could presumably also be interpreted in the context of tachyon condensation following [11].

Notice that in the hamiltonian (2) we have not introduced the phase difference $\chi$. This is because such a phase can always be absorbed into a (gauge) redefinition of the spin operators $S^\pm$. Alternatively, observe that a perturbative computation of the ground state energy of this model will only involve configurations of charges (the charges in the boundary vertex operators) which are independently neutral on the left and right sides, which leads to the possibility of shifting $\phi$ by a constant independently on each boundary (notice also that this ground state energy will then expand in powers of $\Delta_1^2, \Delta_2^2$. The situation for (1) is very different, as all charge configurations which are only neutral overall (ie by combining left and right sectors) do contribute. The dependence of the ground state energy on $\chi$ is actually one of the main concerns of this paper.

Let us now give some generalities about our approach. We consider a strip of width $R$ and length $L \to \infty$, with Euclidean coordinates $x \in [0,R]$ and $y \in \mathbb{R}$. We will mostly discuss the double boundary sine-Gordon model in this paper (1), which belongs to a more general class of bulk and boundary perturbed conformal field theories, whose Euclidean action can be schematically written

$$S = S_{CFT} + CBC$$

$$+ \int_0^R dx \int dy \Phi(x,y) + \int dy (\Phi_{B_1}(y) + \Phi_{B_2}(y)) .$$

Here, we have as a starting point a CFT with action $S_{CFT}$, perturbed by some relevant operator $\Phi(x,y)$ contained within the operator content of that particular CFT. In the presence of boundaries, one needs to specify conformal boundary conditions (CBCs) at $x = 0, R$, and include the nontrivial boundary effects as a perturbation on the CBCs by some relevant boundary operators (in which we have included coupling constants $g_{l, r}$ $\Phi_{B_{l, r}}(y) \equiv g_{l, r} \Phi(x,y)_{x=0, R}$ at the left and right boundaries.

The theory (1) turns out to be integrable. This can be shown by a simple generalization of the argument of [12], which we discuss here for completeness. The Lagrangian or perturbed CFT approaches both provide a symmetric stress-energy tensor $T_{\mu \nu}$ with components $T_{zz} \equiv T, T_{z \bar{z}} \equiv \Theta$ satisfying the continuity equations

$$\partial_z T = \partial_{\bar{z}} \Theta, \quad \partial_z \bar{T} = \partial_{\bar{z}} \Theta.$$

In the presence of boundaries, as was shown by Cardy [1], the choice of conformal boundary conditions ensures that no momentum can flow through the boundaries, in other words that the off-diagonal components of the stress-energy tensor vanish at the boundaries, $T_{xy}|_{x=0, R} = 0$. In the presence of boundary perturbations, however, this equation gets modified to

$$T_{xy}|_{x=0, R} = -i(T - \bar{T})|_{x=0, R} = \frac{d}{dy} \vartheta_{l, r}$$

where $\vartheta$ is some local boundary field, which can be related to the boundary fields $\Phi_{B_{l, r}}$.

Consider now the integrals over a closed contour $C$

$$P_1(C) = \int_C (T dz + \Theta d\bar{z}), \quad \bar{P}(C) = \int_C (\bar{T} d\bar{z} + \bar{\Theta} dz).$$

By the continuity equations (4), these do not change under deformations of the contour $C$: $P_1(C) = \bar{P}(C) = 0$. Let us now take $C$ to be the contour illustrated in figure 1.
\[ 0 = P_1(C) + \bar{P}_1(C) = \sum_{a=u,d,l,r} P_1(C_a) + \bar{P}_1(C_a). \]  

The contours \( C_l \) and \( C_r \), by equation (5), can be seen to give the contributions
\[ P_1(C_{l,r}) + \bar{P}_1(C_{l,r}) = \Theta_{l,r}(y_1) - \Theta_{l,r}(y_2). \]

If we take \( y \) to be the time direction of our quantization scheme, this then shows that the quantity
\[ H_{l,r}(y) = \int_0^R dx (T + \bar{T} + 2\Theta) + \Theta_l(y) + \Theta_r(y) \]

is \( y \)-independent, in other words is an integral of motion.

If the theory is integrable, these equations are but the first elements of an infinite series. Then, there are an infinite number of local fields \( T_{s+1}, \Theta_{s-1} \) obeying
\[ \partial_z T_{s+1} = \partial_z \Theta_{s-1}, \quad \partial_z \bar{T}_{s+1} = \partial_z \bar{\Theta}_{s-1} \]
in the bulk. Provided they obey the boundary conditions
\[ (T_{s+1} + \Theta_{s-1} - \bar{T}_{s+1} - \bar{\Theta}_{s-1})|_{x=0,R} = \mp i \frac{d}{dy} \Theta_{s,l,r}(y) \]
for some infinite subset \( \{s_B\} \subset \{s\} \), the theory will remain integrable despite the boundaries, with integrals of motion
\[ H^{(s)}_{l,r}(y) = \int_0^R dx \left[ T_{s+1} + \Theta_{s-1} + \bar{T}_{s+1} + \bar{\Theta}_{s-1} \right] + \Theta_{s,l}(y) + \Theta_{s,r}(y). \]

Equations (11) define the possible integrable boundary conditions. They formed the fundamental set of conditions used in the analysis of [12].

Integrability can be used in two different - and equivalent through crossing - ways to calculate the ground state energy. To see this, let us switch to a hamiltonian formalism; imaginary time can run along \( x \) or \( y \). For time flowing along \( y \), it is natural to identify the first element of the infinite sequence of conserved quantities, equation (9), with the Hamiltonian. The boundaries are thus true boundaries in space, and the bulk excitations satisfy appropriate boundary conditions. In this language, baptized the \( L \)-channel, the partition function is obtained by tracing over all such states, with \( L \to \infty \) as the inverse temperature:
\[ Z = \text{Tr} e^{-LH_{l,r}}. \]

On the other hand, in the \( R \)-channel, we take time to flow along \( x \), from an initial state to a final one (i.e. from \( x = R \) to \( x = 0 \)). The Hamiltonian in the bulk is thus the usual one (without boundary contributions), and the associated space of states is the same as in the absence of boundaries. The partition function now becomes a simple matrix element, obtained by sandwiching the time-evolution operator \( H \) along time \( R \) between the initial and final states (the “boundary” states) \( |B_{l,r}\rangle \), i.e.
This latter point of view leads to the TBA in the crossed channel, which we will use extensively in what follows. To handle (14) one uses the description of the integrable theory as a massive (if there is also a bulk perturbation) or massless (if the perturbation affects only the boundary) scattering theory. Such a theory is described by the spectrum of particles together with the bulk and boundary scattering matrices, and the constraint of factorized scattering.

Consider for instance the massive case, and parametrize energy and momentum of particles through a rapidity $\theta$. The explicit form of the boundary states is [12],

$$ |B\rangle = \exp\left(\int_0^\infty d\theta K^{ab}(\theta)A^+_a(\theta)A^b_b(\theta)\right)|0\rangle, $$

where the amplitudes appearing in the exponential are given by the analytic continuation of the reflection matrices:

$$ K^{ab}(\theta) = R^a_b\left(\frac{i\pi}{2} - \theta\right). $$

Using this basic ingredient, a straightforward method to compute the ground state energy was proposed in [2], in the case where bulk and boundary scattering are diagonal. The non diagonal cases require more work, and discussing them is part of our purpose here. It so turned out however that even in the diagonal case, considerable surprises are encountered - as was first discovered in [4] - due in part to the way the bare couplings are encoded in the boundary scattering description. Calculating the correct ground state energy requires analytical continuation of the results of [2], a process which is quite difficult.

The case of double Kondo model (2) is a bit different, as there are additional boundary degrees of freedom. The proof of integrability would proceed along similar lines, with appropriate modifications as discussed in [13]. The boundary state formalism can also be generalized to this case. We will, however, take a different approach based on the Destri-de Vega formalism.

Most of this paper is devoted to the double boundary sine-Gordon model (sections 2 to 6). We will also present results for the double Kondo model in section 7. We shall first discuss analytic continuation by considering the case of the Ising model in section 2. In sections 3 and 4, we extend the analysis of [2] to the case where the boundary scattering is non diagonal while the bulk scattering is still diagonal. The analytical continuation is carried out as a simple generalization of section 2 for the case $\beta^2 = 4\pi$ (many of the results of this section have been previously published in [9], but we include a more thorough explanation here, which helps to understand the generalizations we propose). The other “reflectionless cases” require considerably more work; only the case $\beta^2 = \frac{8\pi}{3}$ is fully discussed, in section 5. The double Kondo model is tackled in section 6; there, the bulk and boundary interactions are non diagonal, and we use a different method, based on the often called Destri-de Vega approach [14] (although the method has been used before in slightly different settings [15]).

II. A SCALAR THEORY: THE ISING MODEL

Let us now turn to a specific example, and consider the Ising model. As is well-known, the bulk action of this theory takes the form of a free action for Majorana fermions $\psi, \bar{\psi}$. At the left and right boundaries $x = 0, R$, there are two (in general different) values of the boundary magnetic field $h_{l,r}$ coupled to the local boundary spin operators

$$ \Phi_{l,r} = \frac{1}{2}(\psi + \bar{\psi})_{x=0,R} a_{l,r}(y). $$

where $a_{l,r}$ are boundary fermions, and we have chosen a convenient normalization. The total action is

$$ S = \int_0^R dx \int dy \left[ \psi \partial^2_x \psi - \bar{\psi} \partial^2_x \bar{\psi} \right] + \int dy (\mathcal{L}_l + \mathcal{L}_r), $$

$$ \mathcal{L}_{l,r} = \frac{1}{2} \psi \bar{\psi}|_{x=0,R} + a_{l,r} \partial_x a_{l,r} + h_{l,r} \Phi_{l,r}. $$

To proceed, let us do some elementary perturbative calculations, and adopt for a moment a more general setting. Consider thus a theory that is conformal in the bulk, and is subject to boundary perturbations at the left and right edges with coupling constants $g_{l,r}$. We can write the partition function of our system as

$$ Z = \langle 0 \vert T_y \exp \int_0^L dy \left[ g_l \Phi(0, y) + g_r \Phi(R, y) \right] \vert 0 \rangle $$
where \( |0\rangle \) is the ground-state of the unperturbed system, \( T_y \) is the \( y \)-ordering operator, and all the correlation functions are evaluated in the unperturbed CFT with the chosen CBCs. We introduce a conformal mapping by defining the new coordinates

\[
w = w_1 + i w_2 = e^{i \pi z/R}
\]

in such a way that the original strip maps to the half-disc, as illustrated in figure 2.

The boundary is the horizontal axis, the perturbation for \( x = 0 \) now sitting on the boundary \( w_1 > 0 \), and the one for \( x = R \) sitting on the one for \( w_1 < 0 \). Introducing \( d \) the dimension of the operators \( \Phi \) (such that the two point function on the boundary goes as \( 1/|w|^{2d} \)) and \( \rho = e^{-\pi L/R} \), we find

\[
Z_0 = \langle 0 | T \exp \int \frac{1}{\rho} \frac{dw}{w^{1-d}} \left( \frac{R}{\pi} \right)^{1-d} [g_r \Phi(w) + g_r \Phi(-w)] | 0 \rangle.
\]

The strategy is to evaluate \( \ln Z_0 \) for large \( L \), and extract the leading term proportional to \( L \), since in that limit one has \( \ln \frac{Z_0}{L} \approx -E_0 \). Elementary calculations give then

\[
E_0 = -\frac{\pi c}{24} - \frac{\pi}{R} \frac{R}{\pi} 2(1-d) \Gamma(1-2d) \Gamma(d) + 2 \frac{g_l g_r}{d} \Gamma(2d; d+1,-1) + \ldots
\]

This formula will be useful in what follows.

Let us get back now to the particular case of the Ising model and consider the simplest case where one of the boundary fields vanishes, the other being equal to \( h \). We then obtain

\[
E_0(h) = -\frac{\pi}{48R} - \frac{\pi}{R} C
\]

where

\[
C = \lim_{\rho \to 0} \frac{1}{\ln 1/\rho} \sum_{N=1}^{\infty} (\sqrt{2h})^{2N} \int_{\rho \leq |x_1| \leq \ldots \leq |x_{2N}| \leq 1} \langle \Phi(x_1) \ldots \Phi(x_{2N}) \rangle dx_1^{1/2} \ldots dx_{2N}^{1/2}
\]

in which \( \langle ... \rangle_c \) means that the correlators are evaluated in the unperturbed CFT. Evaluation to first two non trivial orders gives

\[
E_0(h) = -\frac{\pi}{48R} - \frac{h^2}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n + 1/2} - \frac{h^4 R^2}{8\pi^2} \sum_{n=0}^{\infty} \frac{1}{(n + 1/2)(p + 1/2)(n + p + 1)}
\]

The first term is manifestly divergent (as would be seen as well from (22)), and requires some extra regularization. Since it diverges logarithmically, it is natural to expect any dimensionally regularized version of this term (such as given by the TBA) to go as \( \ln(cst \; h^2 R) \). The next term, like all the subsequent ones, is well-defined. The sum can be evaluated in closed form:

\[
\sum_{n=0}^{\infty} \frac{1}{(n + 1/2)(p + 1/2)(n + p + 1)} = 7\zeta(3)
\]
giving rise to
\[ E_0(h) = -\frac{\pi}{48R} - \frac{h^2}{4\pi} \ln (\text{cst} h^2 R) + \frac{\zeta(3)}{\pi^2} h^4 R + O(h^6 R^2). \] (26)

Higher orders could be obtained without too much difficulty, but the foregoing expression will serve our purposes.

Let us now compare it with the non-perturbative results obtained from the TBA. The idea will be recalled in detail in the next section, so for now we just give the result, which can be found easily by taking the massless limit of [2]:

\[ E_0(h) = -\frac{1}{4\pi R} \int_0^\infty d\kappa \ln \left[ 1 + \frac{\kappa - h^2 R}{\kappa + h^2 R} e^{-\kappa} \right]. \] (27)

It is not entirely straightforward to expand it in powers of \( h^2 R \), in particular because the second order term actually involves a logarithmic piece. To proceed, if we write \( E = -\frac{1}{4\pi R} I(x) \), \( x \equiv h^2 R \), we have

\[ \Delta I = I(x) - I(0) = \int_0^\infty d\kappa \ln \left[ 1 - \frac{2x}{(1 + e^\kappa)(\kappa + x)} \right]. \]

It is then more convenient to expand the derivative as

\[ \frac{\partial}{\partial x} \Delta I = \int_0^\infty d\kappa \frac{1 - e^{-\kappa}}{\kappa} \sum_{n=1}^\infty (-1)^n \left( \frac{x}{\kappa} \right)^n \left( \frac{1 - e^{-\kappa}}{1 + e^{-\kappa}} \right)^n + \int_0^\infty d\kappa \left[ \frac{1}{\kappa} \left( \frac{1 - e^{-\kappa}}{1 + e^{-\kappa}} \right) - \frac{1}{\kappa + x} \right]. \]

Introducing the constant

\[ c = \int_0^\infty d\kappa \frac{1 - e^{-\kappa} - 2\kappa e^{-\kappa}}{\kappa(1 + \kappa)(1 + e^{-\kappa})} \approx 0.125633, \]

one finds then, after reintegration, that

\[ \Delta I = x(c - 1) + x \ln x - \sum_{n=2}^\infty (-1)^n \frac{(x/2)^n}{n} \int_{-\infty}^\infty dt \left( \frac{\tanh t}{t} \right)^n. \] (28)

These integrals cannot all be evaluated in closed form, but the lowest one can:

\[ \int_{-\infty}^\infty dt \left( \frac{\tanh t}{t} \right)^2 = \frac{28}{\pi^2} \zeta(3) \approx 3.41023, \]

from which it follows that

\[ E(h) = -\frac{\pi}{48R} - \frac{h^2}{4\pi} (c - 1 + \ln h^2 R) + \frac{\zeta(3)}{\pi^2} h^4 R + O(h^6 R^2). \] (29)

This agrees with the perturbative calculation. In fact, one can also check the validity of the result to all orders (27) by comparing it with the lattice calculations in appendix B of [19], so things are quite satisfactory here.

Surprises start when one considers the case of two non-vanishing applied fields \( h_1, h_r \neq 0 \). The TBA approach then gives

\[ E_0(h_1, h_r) = -\frac{1}{4\pi R} \int_0^\infty d\kappa \ln \left[ 1 + \frac{\kappa - h_1^2 R}{\kappa + h_1^2 R} \frac{\kappa - h_r^2 R}{\kappa + h_r^2 R} e^{-\kappa} \right]. \] (30)

This is confusing because it seems to have an expansion in powers of \( h_1^2 \) and \( h_r^2 \) only, a result in manifest contradiction with perturbative calculations. Related to this puzzle is the fact that (30) is the same for \( h_1 h_r > 0 \) and \( h_1 h_r < 0 \) while the two cases ought to be different based on even simpler arguments than perturbation theory: indeed, in the limit of large fields, the first case describes a partition function with fixed ++ boundary conditions, and \( E_{gs} = -\frac{\pi}{24\sqrt{R}} \), while the one describes a partition function with fixed -- boundary conditions, and \( E_{gs} = -\frac{\pi}{24\sqrt{R}} + \frac{\pi}{2\pi} \), the difference corresponding to the boundary dimension of the spin operator, \( d = \frac{1}{2} \).

To make things more precise, the perturbative expansion (22) leads, at second order, to

\[ E_0(h_1, h_r) = -\frac{\pi}{48R} - \frac{h_1^2}{4\pi} \ln(\text{cst} h_1^2 R) - \frac{h_r^2}{4\pi} R \ln(\text{cst} h_r^2 R) - \frac{1}{4} h_1 h_r + ... \] (31)
To solve the apparent contradiction with (30) we attempt to expand it in powers of the magnetic fields. Since we already have a controlled expansion for one vanishing field, e.g. \( h_r = 0 \), we consider

\[
E_0(h_l, h_r) - E_0(h_l, 0) = -\frac{1}{4\pi R} \int_0^\infty dk \ln \left[ \frac{1 + \frac{\kappa h_l^2 R}{\kappa + h_l^2 R} e^{-\kappa}}{1 + \frac{\kappa h_r^2 R}{\kappa + h_r^2 R} e^{-\kappa}} \right].
\]

This can be rewritten as

\[
E_0(h_l, h_r) - E_0(h_l, 0) = -\frac{1}{4\pi R} h_l h_r \int_0^\infty dx \left[ 1 - \frac{2h_r^2 R}{(\kappa + h_r^2 R)e^{\kappa} + \kappa - h_l^2 R} \right].
\]

This is not analytical in \( h_r^2 \), as can easily be demonstrated by trying to calculate the derivative with respect to \( h_r^2 \) at \( h_r = 0 \), which diverges. To proceed, we make the change of variables \( \kappa = h_l h_r Rx \) (assuming \( h_l h_r > 0 \)), and get after some simple manipulation

\[
E_0(h_l, h_r) - E_0(h_l, 0) = -\frac{1}{4\pi} h_l h_r \int_0^\infty dx \left[ 1 - \frac{2}{2x + (h_r x + h_l) \frac{h_l h_r R x}{h_r}} \frac{1}{h_l x + h_r} \right].
\]

Putting \( h_r = 0 \) in the bracket now gives a convergent integral, so

\[
E_0(h_l, h_r) - E_0(h_l, 0) = -\frac{h_l h_r}{4\pi} \int_0^\infty dx \ln \left[ 1 + \frac{2}{(2 + h_l^2 R)x^2} \right]
\]

and thus, the first non trivial term in \( h_l h_r \) is obtained by setting \( h_l = 0 \) in the integral

\[
E_0(h_l, h_r) - E_0(h_l, 0) = -\frac{h_l h_r}{4\pi} \int_0^\infty dx \ln \left[ 1 + \frac{1}{x^2} \right] = -\frac{1}{4} h_l h_r
\]

in agreement with the perturbative result.

We thus resolve in a simple way the first paradox, by recognizing that the TBA integral must expand in fact powers of \( h_l \) and \( h_r \), but not \( h_l^2 \), \( h_r^2 \), at least when both fields are positive (devising a scheme to perform this expansion is another matter).

There does remain a bigger problem, which is that the expansion of the TBA formula (30) is in powers of \( |h_l| \) and \( |h_r| \), and thus does not see the difference between the cases \( h_l h_r > 0 \) and \( h_l h_r < 0 \) (the cases \( h_l, h_r > 0 \) and \( h_l, h_r < 0 \) being meanwhile equivalent by spin reversal symmetry).

The reason why we have this problem seems interesting. We do not think we understand it fully, but an important hint can be found by going for a while to the initial description based on the ordered, massive phase (that is, before we take the massless limit). The \( R \) matrix is calculated with a single (say, right) boundary at \( x = 0 \), but of course, the problem is undefined until one has set proper boundary conditions at (left) \( x = -\infty \). Since we are in the ordered phase, there are two possible such conditions, spin fixed up or down. For a given magnetic field on the (right) boundary, the two (left) boundary conditions at \( x = -\infty \) should correspond to different problems, with presumably different states describing the right boundary. Ghoshal and Zamolodchikov in [12] argue that spin up and \( h_l > 0 \) correspond to the usual boundary state, while spin down and \( h_l > 0 \) correspond instead to an “excited boundary state”, obtained formally by some procedure of analytic continuation. We believe that a similar feature holds, the other boundary in our finite geometry playing the role of the boundary condition at \( x = -\infty \) in [12]. If this is the case, the TBA approach proposed in [2] lacks a crucial element, and is good only when, roughly speaking, the boundary couplings would not lead to existence of frustration in the integrable massive bulk theory compatible with the massless boundary perturbations. In the Ising case: when the two fields have the same sign.

How to handle the case \( h_l h_r < 0 \) thus requires some additional ingredient. To see what it may be, let us calculate the ground state energy of the Ising model on the strip by brute force, diagonalizing the hamiltonian \( H \). Following calculations initiated in [17] we have

\[
E_0 = \frac{1}{2} \sum_{k<0} k
\]

The modes \( k \) are determined from the quantization equation

\[
X = e^{2\pi R k} \frac{h_l^2 + 2ik \ h_l^2 + 2ik}{h_l^2 - 2ik \ h_l^2 - 2ik} = -1.
\]

One can easily show that this equation has only real \( k \) solutions, and that if \( k \) is a solution, so is \( -k \). Therefore, the sum in (36) runs only over the \( k > 0 \) values.
Now a detailed study of the quantization equation shows that all solutions are analytical functions of $h_t^2$ and $h_r^2$ except for one of them, which in particular goes as $k = \pm k_0 \approx \pm \frac{1}{2} |h_t h_r|$ at small magnetic fields.

Manipulations of (36) in the usual way lead to the TBA integral formula (30). Suppose now that we believe this result for $h_t h_r > 0$. To get the result for $h_t h_r < 0$, we could take the formal expansion of (30) in powers of $h_t, h_r$ and continue analytically in the variable $h_t h_r$. This, it seems from the previous analysis, involves only the modes $\pm k_0$. If the mode $+k_0$ is included in the sum for $h_t h_r > 0$, the analytically continued result will be obtained by discarding it, and using instead the mode $-k_0$: in other words, we suggest that the energy for negative $h_t h_r$ simply reads

$$E_0(h_t, h_r) = -\frac{1}{4\pi R} \int_0^\infty d\kappa \ln \left( 1 + \frac{\kappa - h_t^2 R}{\kappa + h_t^2 R} \frac{\kappa - h_r^2 R}{\kappa + h_r^2 R} e^{-\kappa} \right) + k_0,$$  \hspace{1cm} (38)

If $h_t h_r < 0$, this expression goes as $-\frac{1}{2} |h_t h_r| + \frac{1}{2} |h_t h_r| = \frac{1}{2} |h_t h_r| = -\frac{1}{4} h_t h_r$, as desired. We also observe that at large magnetic fields, it goes to $E_0 \approx -\frac{1}{2\pi} + \frac{1}{2\pi} k_0 \rightarrow \frac{1}{2\pi}$, in agreement with the conformal prediction for $+-$ boundary conditions.

Two tests are therefore satisfied. We also observe that (38) implies the simple result

$$E_0(h_t, h_r) - E_0(h_t, -h_r) = -k_0.$$  \hspace{1cm} (39)

Note that the idea of adding an extra contribution in different regimes of couplings appears first in [4].

To provide supplementary justification for this construction, we have performed some numerical checks on equation (39) using Mathematica. The plots were produced using exact diagonalization of a lattice Ising model at its bulk critical point, for system sizes up to ten lattice sites, and compared to the theoretical prediction in equation (39), i.e. the root $k_0$. Figure 3 shows our results.

![Ground-state energy difference](image)

**FIG. 3.** Ground-state energy difference for the two-boundary Ising model with boundary magnetic fields $h_t = h_r$ and $h_t = -h_r$. This is plotted as a function of the scaling parameter $R^{1/2} h$ for system sizes up to ten lattice sites, and compared to the theoretical prediction in equation (39), i.e. the root $k_0$.

Difficulties related with the relative values of the two boundary parameters will be generic - and worse - in the case of the double boundary sine-Gordon model, and require considerable efforts. On top of this, an additional difficulty arises due to the presence of the phase $\chi$: even if the bulk is chosen to be at the reflectionless points and hence have diagonal scattering, the boundary reflections cannot be simultaneously diagonalized, requiring substantial modification to the crossed channel TBA of [2]. We would like to illustrate this in the case of the free fermion point $\beta^2 = 4\pi$.

**III. ALL THE DIFFICULTIES IN A NUTSHELL: THE DOUBLE BOUNDARY SINE-GORDON MODEL AT $\beta^2 = 4\pi$.**

### A. The crossed channel TBA

At the free fermion point $\beta^2 = 4\pi$, the (massive) sine-Gordon theory simplifies considerably. The only fundamental excitations are two fermions, the soliton and the antisoliton. Their scattering matrix in the bulk is trivial, independent of rapidity - the creation and annihilation operators obey the canonical anticommutation relations
\{ A_a(\theta), A^\dagger_b(\theta') \} = \delta_{ab} \delta(\theta - \theta') \tag{40}

with all other anticommutators vanishing, \( a = \pm \) labelling soliton/antisoliton particle types. Scattering at the boundaries is encoded in the \( R \) matrix

\[
\begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix} \equiv \begin{pmatrix} P_+ & Q \\ Q & P_- \end{pmatrix} \tag{41}
\]

There will be one reflection matrix for each boundary, and its elements will depend on the rapidity of the incident particle, as well as the bulk and boundary parameters of the field theory. It is easier to obtain the reflection amplitudes formally by solving the boundary bootstrap equations; one is thus left with some undetermined relation between the parameters entering the \( R \) matrix, and the parameters in the field theory. We will have to tackle this issue in more details later. For now, we observe (as discussed in details in the appendix) that the point \( \beta^2 = 4\pi \) being equivalent to a free fermion theory, explicit calculations can easily be carried out, and one finds [22]

\[
\begin{align*}
P^+_{l,r}(\theta) &= [\cosh \theta - \frac{1}{2} \gamma_{l,r} \cosh(\theta \mp i\phi_{l,r})]/D_{l,r}(\theta), \\
Q^+_{l,r}(\theta) &= \frac{i \sinh 2\theta}{2 D_{l,r}(\theta)}, \\
D_{l,r}(\theta) &= -i\gamma_{l,r} \cosh(\frac{\theta - i\phi_{l,r} - i\pi/2}{2}) \sinh(\frac{\theta + i\phi_{l,r} + i\pi/2}{2}) - \cosh^2 \theta,
\end{align*}
\]

where \( \gamma_{l,r} \propto \Delta^2_{l,r}/m \) and \( \phi_l = 0, \phi_r = \phi_0 \).

In the \( R \)-channel description of the problem, the explicit form of the boundary states then reads

\[
|B_{l,r} > \propto \exp \left( \int_0^\infty d\theta K^\text{ab}_{l,r}(\theta) A^\dagger_a(-\theta) A^\dagger_b(\theta) \right) |0 >.
\tag{44}
\]

Here, the \( K \) matrices are given by the analytic continuation of the boundary R-matrix according to

\[
K_{l,r}(\theta) = \frac{i/2}{\sinh^2 \theta + \gamma_{l,r} \cosh \left( \frac{\theta + i\phi_{l,r}}{2} \right) \cosh \left( \frac{\theta - i\phi_{l,r}}{2} \right)} \times
\]

\[
\times \left( \begin{array}{c}
\sinh 2\theta \\
-2\sinh \theta + \gamma_{l,r} \sinh(\theta + i\phi_{l,r}) \\
\sinh 2\theta
\end{array} \right) \tag{45}
\]

As it should, this obeys the boundary cross-unitarity condition \( K_{ab}(\theta) = -K_{ba}(-\theta) \), in view of the fact that the scattering matrix is simply \(-1\).

It is now straightforward to take the massless limit through

\[
m \rightarrow 0, \quad \theta_0 \rightarrow \infty, \quad m e^\theta = me^{\theta_0 + \theta'} = me^{\theta'}, \quad \mu \text{ finite}. \tag{46}
\]

\( \theta' \) (which will we often denote by \( \theta \) when no ambiguity is possible) is then integrated from \(-\infty \) to \( \infty \). We will often denote \( \mu e^{\theta'} \equiv \kappa \), and in practice set \( \mu = 1 \) (sometimes the massless limit is taken with \( \mu = 2 \) instead, which does not affect any of the results. Also, we use the notation \( \mu \) briefly in appendix C, but no confusion should be possible).

Performing the massless limit on the \( K \)-matrix (45), we get

\[
K_{l,r}(\theta) = \frac{i}{\mu e^{\theta'} + \Delta^2_{l,r}} \frac{\mu e^{\theta'}}{\Delta^2_{l,r}e^{i\phi_{l,r}}} e^{-i\phi_{l,r}} \tag{47}
\]

(Note that some care has to be exercised in getting the \( K \) matrix in the massless limit directly from the \( R \) matrix in the same limit). The couplings \( \Delta \) are renormalized couplings, in this case \( \Delta = \sqrt{\pi}\Delta \).

Equipped with this data, we now consider the calculation of the ground state energy generalizing [2]. Although this was already carried out in [9], we include here some more extensive explanations, which will be useful in the following sections.

Let us now turn to the calculation of the partition function in the limit \( L \rightarrow \infty \), where the boundary states are well-defined. Our starting point is a version of equation (14):

\[
Z = \sum_\alpha \frac{\langle B_l|\alpha\rangle \langle \alpha|B_r\rangle}{\langle \alpha|\alpha\rangle} e^{-RE_\alpha}, \tag{48}
\]

where \( \alpha \) is an index running through all states with nonzero inner product with the boundary states. In view of the structure of the boundary states, we will thus need to trace over all states of the form
We have to be very careful here: in the case of a theory with only one type of particle, the rapidities can be ordered using the strict inequality \( \theta_N > ... > \theta_1 \), since the Bethe wave functions vanish identically for coinciding rapidities (see e.g. [2]). Now, however, we have more than one species of particles, so we can have coinciding rapidities without having vanishing Bethe wave functions, as long as the labels of particles at coinciding rapidities do not get reproduced.

A more economical strategy consists in introducing creation operators for particle pairs, and a quartet one [9]:

\[
P_{ab}^\dagger(\theta) \equiv A_{ab}^\dagger(-\theta)A_{ab}^\dagger(\theta), \quad Q_{ab}^\dagger(\theta) \equiv A_{ab}^\dagger(-\theta)A_{ab}^\dagger(\theta), \quad \theta > 0
\]

where the pair creation operators come in four flavours, labeled 11, 12, 21, and 22. We can now consistently restrict the rapidities to be positive and noncoincident in the set of states \( |\alpha\rangle \). Note that because of basic anticommutation relations and the Pauli principle, the structure of the boundary state dictates that the quartet appears with fugacity \( K^{11} \theta K^{22} \theta - K^{12} \theta K^{21} \theta = \det K \).

We can now rewrite our trace over states in (48) as running over the set of all states of the form

\[
|\alpha\rangle \in \{ P_{ab}^\dagger b_N(\theta_N) ... P_{ab}^\dagger b_1(\theta_1) Q_{ab}^\dagger(\theta_{N_q}) ... Q_{ab}^\dagger(\theta_{1})|0\},
\]

\[
\forall N, N_q \in \mathbb{N}, \theta_N > ... > \theta_1, \theta_{N_q} > ... > \theta_1, \theta_i \neq \theta_q \}.
\]  

(51)

For a state with \( N \) pairs with rapidities \( \theta_1,...,\theta_N \) and \( N_q \) quartets with rapidities \( \theta_{1}^q,...,\theta_{N_q}^q \), we can easily show by explicitly expanding the exponentials in the expressions for the boundary states that

\[
\frac{\langle B|\alpha\rangle\langle\alpha|B\rangle}{\langle\alpha|\alpha\rangle} = \prod_{i=1}^{N} K_{b_i}^{b_i}(\theta_i) K_{r_i}^{a_i}(\theta_i) \prod_{j=1}^{N_q} \det[\bar{K}_{r_j}(\theta_i) K_{r_j}(\theta_j)]
\]

(52)

To construct the entropy of a field configuration, necessary to complete the TBA, we use the following argument. Imagine putting our system in a very large but finite domain of length \( L \) in the space-like direction of the R-channel picture. Since all bulk scattering is given by the trivial scattering matrix \( S = -1 \), the quantization rules dictate that the allowed rapidities appearing in the pair and quartet creation operators lie on a lattice of almost evenly spaced values (separated by intervals of order \( 1/L \), and symmetric with respect to sign exchange). Moreover, in view of the Pauli principle, no superpositions of creation operators can take place on a given rapidity. Thus, the entropy of a state configuration with \( N_{ab} \) pairs of type \( a,b \) and of \( N_{q} \) quartets is very simple to calculate, being that of a gas of mutually excluding particles of five different types. Explicitly, for \( N_T \) total allowed rapidities, we have the simple formula

\[
e^S = \frac{N_T!}{N_1!N_2!N_3!N_4!(N_T - N_1 - N_2 - N_3 - N_4)!}.
\]

(53)

Introducing the usual occupation densities \( \rho, \rho_{ab}, \rho_{q} \) as well as the hole density \( \rho_{h} = \rho - \sum_{ab} \rho_{ab} - \rho_{q} \) as we take the limite \( L \to \infty \), we can rewrite this in the Stirling approximation as

\[
S \approx L \int_{0}^{\infty} d\theta \left[ \rho \ln \rho - \sum_{ab} \rho_{ab} \ln \rho_{ab} - \rho_{q} \ln \rho_{q} - \rho_{h} \ln \rho_{h} \right].
\]

(54)

The energy is on the other hand given by

\[
E = L \int_{0}^{\infty} d\theta \left[ \sum_{ab} \rho_{ab} + 2\rho_{q} \right] 2m \cosh \theta,
\]

(55)

allowing us to put everything together to obtain the partition function

\[
Z \approx \int [d\rho] \exp \left\{ L \int_{0}^{\infty} d\theta \left[ \sum_{ab} (\ln \bar{K}_{i}^{ba} K_{r}^{ab} - 2m R \cosh \theta) \rho_{ab} + (\ln \det[\bar{K}_{r} K_{r}] - 4m R \cosh \theta) \rho_{q} + S[\rho] \right] \right\}.
\]

(56)

Introducing the quasienergies \( e^{-\epsilon_{ab}} = \rho_{ab}/\rho_{h} \) and a similar one for the quartets, we can obtain the TBA equations by using a saddle-point approximation on the partition function. From the quantization equation we get the constraint \( \rho_{h}(1 + \sum_{ab} e^{-\epsilon_{ab}} + e^{-\epsilon_{q}}) = \frac{m}{2} \cosh \theta \), which when substituted in the saddle-point equation leads (after some straightforward manipulations) to
\[
\ln Z = \frac{mL}{2\pi} \int_0^\infty d\theta \cosh \theta \ln \left(1 + \sum_{ab} e^{-\epsilon_{ab} + \epsilon_q}\right),
\]
\[
\epsilon_{ab} = 2mR \cosh \theta - \ln \tilde{K}_l^{bo} K_r^{ab},
\]
\[
\epsilon_q = 4mR \cosh \theta - \ln \det(\tilde{K}_l K_r).
\]

Reinterpreting this as the ground-state energy of the original problem, we arrive at the final formula
\[
E_0 = -\frac{1}{2\pi} \int_0^\infty d\theta m \cosh \theta \ln \left[1 + \text{tr} (\tilde{K}_l K_r) e^{-2mR \cosh \theta} + \det(\tilde{K}_l K_r) e^{-4mR \cosh \theta}\right].
\]

where the \(K\) matrices are given in equation (45).

We can now take the massless limit using the results (47) to obtain the final expression
\[
E_0 = -\frac{1}{4\pi} \int_0^\infty d\kappa \ln \left[1 + \frac{2 \kappa^2 + \Delta_l^2 \Delta_r^2 \cos 2\chi}{(\kappa + \Delta_l^2)(\kappa + \Delta_r^2)} e^{-\kappa R} + \frac{(\kappa - \Delta_l^2)(\kappa - \Delta_r^2)}{(\kappa + \Delta_l^2)(\kappa + \Delta_r^2)} e^{-2\kappa R}\right].
\]

In the first appendix, we provide the details of an independent derivation of this result using refermionization, confirming that the TBA construction that we have employed is justified.

B. Analytic structure of the ground-state energy

The study of the analytical properties of the ground-state energy of the double-boundary Ising model in the first part of this paper has shown how careful one has to be in interpreting the data coming from the TBA equations. It is here again very easy to miss out on the correct analytic continuation procedure to be imposed.

Let us thus carefully study our expression for the ground-state energy of the double-boundary sine-Gordon model at the free fermion point, equation (59), whose bulk term we reproduce here:
\[
E_0 = -\frac{1}{4\pi} \int_0^\infty d\kappa \ln \left[1 + \frac{2 \kappa^2 + \Delta_l^2 \Delta_r^2 \cos 2\chi}{(\kappa + \Delta_l^2)(\kappa + \Delta_r^2)} e^{-\kappa R} + \frac{(\kappa - \Delta_l^2)(\kappa - \Delta_r^2)}{(\kappa + \Delta_l^2)(\kappa + \Delta_r^2)} e^{-2\kappa R}\right].
\]

First of all, we expect the ground state energy to be an even function of the phase \(\chi\), as is obvious from the boundary sine-Gordon Hamiltonian. We can consequently concentrate on positive \(\chi\) from now on. The first crucial observation is that equation (60) is valid only in the domain \(\chi \in [0, \pi/2]\), since it exhibits a singularity at \(\chi = \pi/2\): for \(\chi \in [0, \pi/2]\), the argument of the logarithm has zeroes in the complex plane of the variable \(\kappa\) that hit the real axis at \(\kappa = 0\) when \(\chi\) reaches \(\pi/2\).

We can gain a bit more intuition about this formula by considering the limits \(\Delta_l, \Delta_r \rightarrow \infty\), where the integral reduces to
\[
E_0 = -\frac{1}{4\pi R} \int_0^\infty dx \ln \left[1 + 2 \cos 2\chi e^{-x} + e^{-2x}\right].
\]

This integral is tabulated, and we get
\[
E_0 = -\frac{\pi}{24R} + \frac{\chi^2}{2\pi R}, \quad \chi \in [0, \pi/2],
\]
\[
E_0 = -\frac{\pi}{24R} + \frac{(\chi - \pi)^2}{2\pi R}, \quad \chi \in [\pi/2, \pi].
\]

As a function of the phase difference \(\chi\), the basic integral representation (59) for the ground-state energy thus has a cusp at \(\chi = \pi/2\).

A simple computation shows that this should not be the case. Namely, let us consider the boundary Hamiltonian (A12) in the limit \(\Delta_l, \Delta_r \rightarrow \infty\). In that case, the field becomes pinned at \(x = 0\) to \(\phi(0, t) = 2\sqrt{n(t + 1/2)}\), while at \(x = R\) it gets pinned to \(2\sqrt{n(m + 1/2)} + \chi/\sqrt{\pi}\), where \(n, m \in \mathbb{Z}\). This immediately tells us that the zero mode operator \(\Pi_0\) has to have eigenvalue \(\Pi_0 = 2\sqrt{n(p + 1/2)}\), \(p \in \mathbb{Z}\). For the phase \(\chi\) in the fundamental domain \(|\chi| < \pi\), the lowest-energy state is thus the one with \(p = 0\), so the energy simply becomes (using (A6) and adding specifically the contribution from the dynamical modes)
\[
E_0 = -\frac{\pi}{24R} + \frac{\chi^2}{2\pi R}, \quad |\chi| < \pi
\]

exactly. This thus clearly demonstrates that the basic integral representation (59) works in the domain \(|\chi| < \pi/2\), but fails for \(\pi/2 < |\chi| < \pi\).
Let us now study the case of small $\Delta_{l,r}$. From the integral representation for the ground-state energy (59), we can write (after factorizing $(1 + e^{-\kappa})^2$ in the logarithm and a few more basic manipulations)

$$E_0 = -\frac{\pi}{24R} - \frac{1}{4\pi R} \int_0^\infty d\kappa \ln \left[ 1 - \frac{4\eta(\kappa(1 + e^{-\kappa}) + \eta \sin^2 \chi)e^{-\kappa}}{(1 + e^{-\kappa})^2(\kappa + \eta)^2} \right]$$

(64)

where we have for simplicity considered $\Delta_l = \Delta_r$ and defined the dimensionless parameter $\eta = \Delta^2_{l,r}R$. One sees that it is not possible to do a simple Taylor expansion in $\eta$ in this formula. Rather, to evaluate it in this limit, we observe that the main contributions come from the region of integration where $\kappa \sim \eta$. Defining $\xi$ such that $\eta \ll \xi \ll 1$, we can approximate the integral term as

$$I \approx -\frac{1}{4\pi R} \int_0^\xi d\kappa \ln \left[ 1 - \frac{\eta(2\kappa + \eta \sin^2 \chi)}{(\kappa + \eta)^2} \right] = -\frac{1}{4\pi R} \int_0^\xi d\kappa \ln \left[ \frac{\kappa^2 + \eta^2 \cos^2 \chi}{(\kappa + \eta)^2} \right].$$

(65)

This can in turn be approximated by rescaling by $\eta$ and replacing the upper bound on the integral by infinity. The resulting integral is tabulated, and we finally obtain

$$E_0 \approx -\frac{\Delta^2_{l,r}}{4} |\cos \chi| + \text{cst.}$$

(66)

The absolute value gives rise to the same kind of cusp as the one obtained in the limit $\Delta_{l,r} \to \infty$. Here, we can compare with straightforward perturbation theory which gives us $\cos \chi$ instead of its absolute value.

These considerations again indicate the presence of a problem with the integral representation (60) around the point $\chi = \pi/2$. In order to repair this, we can first use a procedure very similar to the one used in the Ising case. Namely, we identify the ground-state energy $E_0$ with the integral representation (60) only in the domain $|\chi| < \pi/2$. The domain $\pi/2 < |\chi| < \pi$ is accessed by analytic continuation, which is performed based on the explicit diagonalization (see appendix A). First, note that all the roots of the quantization condition (A25) are well-behaved as a function of $\chi$ except for the pair closest to the origin, which we label as $\pm k_0$. This is the root that we need to analytically continue, as in the Ising case. Thus, in the domain $\pi/2 < |\chi| < \pi$, the true ground-state energy should be given by the integral representation (60) plus $k_0$.

The validity of this procedure can again be checked in the limits of strong and weak boundary pairings. In the limit $\Delta_{l,r} \to \infty$, the root $k_0$ is given by $k_0 = \frac{\pi}{2R}$. This leads, in the domain $\pi/2 < |\chi| < \pi$, to a ground-state energy of $E_0 = \frac{-\pi}{24R} + \frac{(\chi - \pi)^2}{2\pi R} + \frac{(\pi/2)^2}{\pi R} = \frac{-\pi}{24R} + \frac{\pi^2}{2\pi R}$, which is the result required by the above considerations using the zero-mode operators. In the weak boundary pairing limit $\Delta_{l,r} \to 0$, the root $k_0$ behaves as $-\frac{\Delta^2_{l,r}}{4} \cos \chi$, leading in the domain $\pi/2 < |\chi| < \pi$ to $E_0 = -\frac{\Delta^2_{l,r}}{4} |\cos \chi| - \frac{1}{4} \Delta^2_{l,r} \cos \chi + \text{cst.} = -\frac{\Delta^2_{l,r}}{4} \cos \chi + \text{cst.}$, as required by simple perturbation theory.

To summarize, we write our final result as

$$E_0^{SG}(\Delta_l, \Delta_r, \chi) = E_0(\Delta_l, \Delta_r, \chi), \quad |\chi| \leq \pi/2,$n$$

$$E_0^{SG}(\Delta_l, \Delta_r, \chi) = E_0(\Delta_l, \Delta_r, \chi) + k_0, \quad \pi/2 \leq |\chi| < \pi,$n$$

(67)

with $2\pi$ periodicity in $\chi$, and where $E_0$ is the integral (60). Plots of this ground-state energy as a function of the phase difference $\chi$ can be found in [9] for various values of the boundary parameters.

To extend the method away from the free fermions point, it is necessary first to reinterpret the addition of $k_0$ directly within the TBA approach; this will help make the case $\lambda = 2$ more transparent. We start by writing down again our results from the TBA. For $\lambda = 1$, since the scattering in the bulk is trivial, the quasienergies obey the trivial equation

$$\epsilon(\kappa) = \kappa$$

(68)

where the ground-state energy has the integral representation (we have rescaled the boundary parameters according to $\Delta_{l,R} = R^{1/2} \Delta_{l,r}$)

$$E_0 = -\frac{1}{4\pi R} \int_0^\infty d\kappa \ln \left[ 1 + 2\kappa^2 + \Delta^2_{l,R} \cos 2\chi e^{-\epsilon(\kappa)} + \frac{(\kappa - \Delta^2_{l,R})^2(\kappa - \Delta^2_{R})}{(\kappa + \Delta^2_{l,R})(\kappa + \Delta^2_{R})} e^{-2\epsilon(\kappa)} \right].$$

(69)

and we have performed some simple rescaling. Now of course, the equation (68) defines the quasienergy everywhere in the complex plane of $\kappa$. In particular, $\epsilon(\kappa)$ is pure imaginary if and only if $\kappa$ is also pure imaginary. Moreover, this equation does not need to be modified as a function of $\chi$, i.e. there is no analytical continuation needed for the quasienergy.
If, however, we closely examine the ground-state energy, it is clear that something goes wrong. Looking at the integrand at \(\kappa = 0\) yields

\[
\ln \left[ 1 + 2 \cos 2\chi e^{-\epsilon(0)} + e^{-2\epsilon(0)} \right] = 2 \ln \cos \chi
\]

(70)

which clearly diverges when \(\chi \to \pi/2\), so we know that there is a singularity hitting \(\kappa = 0\) at \(\chi = \pi/2\). For general \(\chi\), the position of this singularity is found by selecting the proper solution of the singularity condition

\[
e^{2\epsilon(\kappa)} + 2\kappa^2 + \Delta_L^2 \Delta_R^2 \cos 2\chi e^{\epsilon(\kappa)} + \frac{(\kappa - \Delta_L^2)(\kappa - \Delta_R^2)}{(\kappa + \Delta_L^2)(\kappa + \Delta_R^2)} = 0.
\]

(71)

A careful look at this shows that we must look for such a singularity at values of \(\kappa\) for which \(\epsilon(\kappa)\) is purely imaginary, and therefore, according to what we wrote before, on the imaginary axis of \(\kappa\). We therefore note the singularity position as \(i\kappa_s\) (note that if \(i\kappa_s\) is a singularity, so is \(-i\kappa_s\)). The picture is then as follows: singularities at \(\pm i\kappa_s\) touch zero at \(\chi = \pi/2\), and the ground-state energy integral (69) must be modified to take this into account. Simple contour-integral manipulations then allow to write the modification to the ground-state energy, which should be added for \(\chi > \pi/2\), in the form \(\delta E_0 = \frac{\pi}{2\kappa}\) making contact with the previous mode expansion arguments.

This is how the analytical continuation should be understood for the case \(\lambda = 1\) in the language of the TBA, and the logic of this procedure will allow us to find the solution to the analytic continuation problem in the case \(\lambda = 2\). There, things will be more complicated: in particular, the self-consistency equations will themselves have to be analytically continued, and the singularity positions will lie in a different domain of the complex plane.

### IV. THE DOUBLE BOUNDARY SINE-GORDON MODEL: SOME GENERAL RESULTS AT THE REFLECTIONLESS POINTS.

#### A. The ground state energy

The massive sine-Gordon theory has its discrete symmetry \(\phi \to \phi + \frac{2\pi}{\beta} N, N \in \mathbb{N}\) spontaneously broken for \(\beta^2 < 8\pi\). The spectrum has a number of fundamental particles classified as a soliton-antisoliton pair, and breathers (bound states). Although the soliton and the antisoliton are always present, the number of breathers is dictated by the value of the interaction parameter \(\beta\). Namely, the breathers are labeled with the integer \(n\) s.t. \(n = 1, 2, ..., < \lambda\) where

\[
\lambda = \frac{8\pi}{\beta^2} - 1.
\]

(72)

Solitons and antisolitons carry respectively positive and negative topological charge, whereas the breathers are neutral. If the soliton mass is given by \(m_i\), then the breathers carry masses

\[
m_n = 2m \sin \frac{n\pi}{2\lambda}, \quad n = 1, 2, ..., < \lambda.
\]

(73)

Let us adopt like in the free fermion case the notation \(A_+^\dagger(\theta)\) and \(A_-^\dagger(\theta)\) for the creation operators at rapidity \(\theta\) of the soliton and antisoliton, respectively. The creation operators for the \(n\)-th breather at rapidity \(\theta\) will be denoted \(B_n^\dagger(\theta)\). Factorized scattering in terms of these is described by the fundamental scattering amplitudes

\[
A_+^\dagger(\theta_1)A_+^\dagger(\theta_2) = a(\theta_1 - \theta_2)A_+^\dagger(\theta_2)A_+^\dagger(\theta_1),
\]

\[
A_-^\dagger(\theta_1)A_-^\dagger(\theta_2) = b(\theta_1 - \theta_2)A_-^\dagger(\theta_2)A_-^\dagger(\theta_1) + c(\theta_1 - \theta_2)A_+^\dagger(\theta_2)A_-^\dagger(\theta_1).
\]

(74)

The amplitudes for these scattering processes are well-known [21], and take the form either of infinite products of gamma functions or of more compact integral representations. These are reproduced in appendix B.

The introduction of a boundary interaction is described like in the free fermion case by a reflection matrix on the left and right boundaries respectively. In addition, there is an amplitude \(B_\eta(\theta)\) for each breather - the scattering cannot mix particles, except solitons and antisolitons.

The boundary bootstrap method applied to this case yields the four amplitudes \(P_{\pm}, Q_{\pm}\) in terms of two free parameters \(\xi, k\) which we will call “IR” parameters following [6]. Their relationship to the “UV” parameters \(\Delta, \chi\) appearing in the boundary Lagrangian is a complicated issue in general, which we will address later on. Suffice to say now that the results of the boundary bootstrap for the boundary scattering amplitudes are summarized in appendix C. In particular, we have \(\beta_0/2 \equiv \chi\).

We will restrict in what follows to reflectionless points where the bulk scattering is diagonal (that is, \(c(\theta) = 0\). They occur at \(\lambda \in \mathbb{N}\). In these cases, there will be \(\lambda - 1\) breathers in the theory, with the original interaction parameter given
by $\beta^2 = \frac{2m}{\hbar}$. As in the free fermion case, the solitons and antisolitons have to be carefully treated when ordering the rapidities. As detailed before, it is better to move to a basis where only positive rapidities are used, and the states involved in the trace for the partition function are constructed using pair and quartet creation operators $P_{ab}^l, Q_l$ (see the discussion around equation (50)). Here, we will need to add the breather pair creation operator, defined as

$$R_n^l(\theta) \equiv B_n^l(-\theta)B_n^l(\theta), \quad \theta > 0. \quad (75)$$

Now, we simply need to add all the possible breather pair states to our set of states to trace over:

$$|\alpha\rangle \in \{ P_{anbN}(\theta_N) ... P_{ab1}(\theta_1) Q_l(\theta_{N_q}) ... Q_l(\theta_1) R_{N_n}^l(\theta_{N_b}) ... R_{n1}^l(\theta_1) \} |0\rangle,$$

$$\forall N, N_q, N_b \in \mathbb{N}, \theta_N > ... > \theta_1, \theta_{N_q}^2 > ... > \theta_1^2, \theta_i \neq \theta_j^2 \} \quad (76)$$

In order to compute the partition function in the thermodynamic limit, let us start with the quantization conditions for the allowed rapidities of the different types of excitations. For definiteness, let us consider a state with $N$ pairs with rapidities $\theta_1, ..., \theta_N$, $N_q$ quartets with rapidities $\theta_1^q, ..., \theta_N^q$ and $N_b$ breather pairs with rapidities $\theta_1^b, ..., \theta_N^b$. Let us imagine moving the particle in pair $i$ having rapidity $\theta_i$ around the system, and requiring periodicity of the wavefunction. This leads to the quantization condition

$$e^{imL \sinh \theta_i} S(2\theta_i) \prod_{j \neq i=1}^{N} S(\theta_i - \theta_j) S(\theta_i + \theta_j) \prod_{k=1}^{N_q} S^2(\theta_i - \theta_k^q) S^2(\theta_i + \theta_k^q) \prod_{l=1}^{N_b} S^{(n)}(\theta_i - \theta_l^b) S^{(n)}(\theta_i + \theta_l^b) = 1. \quad (77)$$

The other quantization condition is obtained by moving one member of the breather pairs around the system in a similar fashion:

$$e^{im_n \sinh \theta_i} S^{(n)}(\theta_i) \prod_{j=1}^{N} S^{(n)}(\theta_i - \theta_j) S^{(n)}(\theta_i + \theta_j) \prod_{l \neq i}^{N_q} S^{(n,q)}(\theta_i - \theta_l^q) S^{(n,q)}(\theta_i + \theta_l^q) \prod_{l=1}^{N_b} S^{(n)}(\theta_i - \theta_l^b) S^{(n)}(\theta_i + \theta_l^b) = 1. \quad (78)$$

As usual, we define densities of occupied and unoccupied rapidities but now for pairs, quartets and breather pairs. A complete set of states is obtained by considering positive rapidities only (remember: our pairs have one component at positive rapidity, and one at negative rapidity). For example, $[\rho_{ab} + \rho_{ab}^h] L d\theta$ gives the number of allowed rapidities in the interval $\theta, \theta + \delta\theta$ for pairs of type $ab$, with $\rho_{ab}$ the density of filled states. The symmetry properties $\rho_{ab}(-\theta) = \rho_{ab}(\theta)$, $\rho_q(-\theta) = \rho_q(\theta)$ and $\rho_n(-\theta) = \rho_n(\theta)$ can then be used to simplify the resulting equations. In logarithmic form, we get (using the usual notation $a*b(\theta) = \int_{-\infty}^{\infty} d\theta' a(\theta')b(\theta')$)

$$\sum_{ab} \rho_{ab}(\theta) + \rho_q(\theta) + \rho_h^h(\theta) = \frac{m}{2\pi} \cosh \theta - \Phi \left[ \sum_{ab} \rho_{ab}(\theta) + 2\rho_q(\theta) \right] - \sum_{n=1}^{\lambda=1} \Phi^{(n)} \ast \rho_n(\theta),$$

$$\rho_n(\theta) + \rho_h^h(\theta) = \frac{m_n}{2\pi} \cosh \theta - \Phi^{(n)} \left[ \sum_{ab} \rho_{ab}(\theta) + 2\rho_q(\theta) \right] - \sum_{m=1}^{\lambda=1} \Phi^{(n,m)} \ast \rho_m(\theta), \quad (79)$$

where the integration kernels are obtained from the scattering matrices according to

$$\Phi(\theta) = -\frac{1}{2\pi i} \frac{d}{d\theta} \ln S(\theta),$$

$$\Phi^{(n)}(\theta) = -\frac{1}{2\pi i} \frac{d}{d\theta} \ln S^{(n)}(\theta),$$

$$\Phi^{(n,m)}(\theta) = -\frac{1}{2\pi i} \frac{d}{d\theta} \ln S^{(n,m)}(\theta), \quad (80)$$

with the explicit form of the scattering matrices to be found in appendix B.

The boundary matrices will appear in an interesting fashion in the TBA as rapidity dependent fugacities. For example, for a state with $N$ pairs with rapidities $\theta_1, ..., \theta_N$, $N_q$ quartets with rapidities $\theta_1^q, ..., \theta_N^q$ and $N_b$ breather pairs with rapidities $\theta_1^b, ..., \theta_N^b$, the fugacity is

$$\langle B_l | \alpha \rangle \langle \alpha | B_r \rangle = \prod_{i=1}^{N} \overline{K}_{i}^{a_i^b_i}(\theta_i) K_{r_i}^{a_i}(\theta_i) \prod_{j=1}^{N_q} \det[K_l(\theta_j^q)K_r(\theta_j^q)] \prod_{k=1}^{N_b} \overline{K}_l^{(n)}(\theta_k) K_r^{(n)}(\theta_k) \quad (81)$$
In a similar way, the energy can be written as
\[
\ln \frac{\langle B|\alpha\rangle \langle \alpha|B^+ \rangle}{\langle \alpha|\alpha \rangle} = L \int_0^\infty d\theta \left[ \sum_{ab} \ln[\tilde{K}_i^{ab}(\theta)K_r^{ab}(\theta)]\rho_{ab}(\theta) + \lambda^{-1} \right.
\]
\[
+ \ln \det[\tilde{K}_i(\theta)K_r(\theta)]\rho_q(\theta) + \sum_{n=1}^{\lambda-1} \ln[\tilde{K}_i^{(n)}(\theta)K_r^{(n)}(\theta)]\rho_n(\theta) \right].
\]
(82)

In a similar way, the energy can be written as
\[
E = L \int_0^\infty d\theta \left[ 2m \cosh \theta \left( \sum_{ab} \rho_{ab}(\theta) + 2\rho_q(\theta) \right) + \sum_{n=1}^{\lambda-1} 2m_n \cosh \theta \rho_n(\theta) \right],
\]
(83)

whereas the entropy becomes in the Stirling approximation as \( L \to \infty \)
\[
S \approx L \int_0^\infty d\theta \left[ \sum_{ab} \rho_{ab} - \rho_q - \rho_q^h + \sum_{n=1}^{\lambda-1} \left( \rho_n + \rho_n^h \right) \ln \left( \rho_n + \rho_n^h \right) - \rho_n \ln \rho_n - \rho_n^h \ln \rho_n^h \right].
\]
(84)

Equations (82, 83, 84) can now be substituted in the expression for the partition function in the R-channel, (48), which now becomes a functional integral over the various densities. The TBA equations then arise from the saddle-point evaluation of this expression, subject to the constraints coming from the quantization conditions (79). Introducing the usual quasienergies, we obtain after standard manipulations
\[
\epsilon_{ab}(\theta) = 2mR \cosh \theta - \ln \tilde{K}_i^{ab}(\theta)K_r^{ab}(\theta) + \Phi \ln \left[ 1 + \sum_{cd} e^{-\epsilon_{cd}(\theta)} + e^{-\epsilon_q(\theta)} \right] + \sum_{n=1}^{\lambda-1} \Phi^{(n)} \ln \left[ 1 + e^{-\epsilon_n(\theta)} \right],
\]
\[
\epsilon_q(\theta) = 4mR \cosh \theta - \ln \det \tilde{K}_i(\theta)K_r(\theta) + 2\Phi \ln \left[ 1 + \sum_{cd} e^{-\epsilon_{cd}(\theta)} + e^{-\epsilon_q(\theta)} \right] + 2 \sum_{n=1}^{\lambda-1} \Phi^{(n)} \ln \left[ 1 + e^{-\epsilon_n(\theta)} \right],
\]
\[
\epsilon_n(\theta) = 2m_nR \cosh \theta - \ln \tilde{K}_i^{(n)}(\theta)K_r^{(n)}(\theta) + \Phi^{(n)} \ln \left[ 1 + \sum_{cd} e^{-\epsilon_{cd}(\theta)} + e^{-\epsilon_q(\theta)} \right] + \sum_{m=1}^{\lambda-1} \Phi^{(n,m)} \ln \left[ 1 + e^{-\epsilon_n(\theta)} \right].
\]
(85)

Substituting these back in the expression for the partition function, and absorbing the boundary factors into a redefinition of \( \epsilon \)'s, we arrive at the ground-state energy
\[
E_0 = -\frac{1}{2\pi} \int_0^\infty d\theta m \cosh \theta \ln \left[ 1 + tr\tilde{K}_i(\theta)K_r(\theta)e^{-\epsilon(\theta)} + \det \tilde{K}_i(\theta)K_r(\theta)e^{-2\epsilon(\theta)} \right] -
\]
\[
- \frac{1}{2\pi} \int_0^\infty d\theta \sum_{n=1}^{\lambda-1} m_n \cosh \theta \ln \left[ 1 + \tilde{K}_i^{(n)}(\theta)K_r^{(n)}(\theta)e^{-\epsilon_n(\theta)} \right]
\]
(86)

where the energies are self-consistent solutions to the system of equations
\[
\epsilon(\theta) = 2mR \cosh \theta + \Phi \ln \left[ 1 + tr\tilde{K}_i(\theta)K_r(\theta)e^{-\epsilon(\theta)} + \det \tilde{K}_i(\theta)K_r(\theta)e^{-2\epsilon(\theta)} \right] +
\]
\[
+ \sum_{n=1}^{\lambda-1} \Phi^{(n)} \ln \left[ 1 + \tilde{K}_i^{(n)}(\theta)K_r^{(n)}(\theta)e^{-\epsilon_n(\theta)} \right]
\]
\[
\epsilon_n(\theta) = 2m_nR \cosh \theta + \Phi^{(n)} \ln \left[ 1 + tr\tilde{K}_i(\theta)K_r(\theta)e^{-\epsilon(\theta)} + \det \tilde{K}_i(\theta)K_r(\theta)e^{-2\epsilon(\theta)} \right] +
\]
\[
+ \sum_{m=1}^{\lambda-1} \Phi^{(n,m)} \ln \left[ 1 + \tilde{K}_i^{(m)}(\theta)K_r^{(m)}(\theta)e^{-\epsilon_n(\theta)} \right], \quad n = 1, \ldots, \lambda - 1.
\]
(87)
B. Universal form

These equations can be put in a simpler, universal form, following [29]. The advantage of this reformulation is that only one integral kernel remains. For a given value of $\lambda$, we label the breathers by $b = 1, \ldots, \lambda - 1$, the soliton by $b = \lambda$, and the anti-soliton by $b = \lambda + 1$. We can then use the remarkable identity [29]

$$[\delta_{ab} + \Phi_{ab}(k)]^{-1} = \delta_{ab} - \frac{1}{2 \cosh k/h} L_{ab}$$

(88)

holding between the Fourier transform of the scattering kernels and the incidence matrix $L_{ab}$ of the relevant Dynkin diagram (for sine-Gordon at $\lambda$, this is the diagram of the $B_{\lambda+1}$ algebra). The Coxeter number is $h = 2\lambda$ in that case.

After basic manipulations, we obtain the universal form

$$\epsilon_a(\theta) = \nu_a(\theta) + \sum_b L_{ab} \Phi_a \ast [-\nu_b(\theta) + \epsilon_b(\theta) + \ln F_b(\theta)],$$

(89)

where the universal kernel is

$$\Phi_a(\theta) = \frac{\lambda}{2\pi \cosh \Lambda \theta}$$

(90)

and the boundary dependence appears in the $F$ functions

$$F_b(\theta) = \begin{cases} 
1 + \tilde{K}_1(b)(\theta) \tilde{K}_1(b) e^{-\epsilon_b(\theta)}, & b = 1, \ldots, \lambda - 1, \\
[1 + \text{tr} \tilde{K}_1(\theta) K_r(\theta) e^{-\epsilon(\theta)} + \det \tilde{K}_1(\theta) K_r(\theta) e^{-2\epsilon(\theta)}]^{1/2}, & b = \lambda, \lambda + 1.
\end{cases}$$

(91)

in which we have defined the kink and antikink quasienergies as $\epsilon_\lambda = \epsilon_{\lambda+1} = \epsilon$. Finally, the driving terms $\nu$ are redundant in the equations (the $\nu_a$ gets cancelled by the $\sum_b L_{ab} \Phi_a \ast [-\nu_b]$, but very useful in the numerics, where they provide the correct asymptotics at large rapidites. They are

$$\nu_a(\theta) = 2m_a R \cosh \theta.$$  

(92)

C. The massless limit

Taking the massless limit as described earlier leads to the following equations:

$$E_0 = -\frac{1}{4\pi} \int_0^\infty d\kappa \ln \left[ 1 + \text{tr} \tilde{K}_1(\kappa) K_r(\kappa) e^{-\epsilon(\kappa)} + \det \tilde{K}_1(\kappa) K_r(\kappa) e^{-2\epsilon(\kappa)} \right] -$$

$$-\frac{1}{4\pi} \sum_{n=1}^{\lambda-1} 2 \sin \frac{\pi n}{2\lambda} \int_0^\infty d\kappa \ln \left[ 1 + \tilde{K}_1(n)(\kappa) K_r(n)(\kappa) e^{-\epsilon_n(\kappa)} \right]$$

(93)

where the functions $\epsilon, \epsilon_n$ are self-consistent solutions to the system of equations

$$\epsilon(\kappa)/\kappa = R + \int_0^\infty d\kappa' \Phi(\kappa', \kappa') \ln \left[ 1 + \text{tr} \tilde{K}_1(\kappa') K_r(\kappa') e^{-\epsilon(\kappa')} + \det \tilde{K}_1(\kappa') K_r(\kappa') e^{-2\epsilon(\kappa')} \right] +$$

$$+ \sum_{n=1}^{\lambda-1} \int_0^\infty d\kappa' \Phi^{(n)}(\kappa', \kappa') \ln \left[ 1 + \tilde{K}_1^{(n)}(\kappa') K_r^{(n)}(\kappa') e^{-\epsilon_n(\kappa')} \right]$$

$$\epsilon_n(\kappa)/\kappa = 2 \sin \frac{\pi n}{2\lambda} R +$$

$$+ \int_0^\infty d\kappa' \Phi^{(n)}(\kappa, \kappa') \ln \left[ 1 + \text{tr} \tilde{K}_1(\kappa') K_r(\kappa') e^{-\epsilon(\kappa')} + \det \tilde{K}_1(\kappa') K_r(\kappa') e^{-2\epsilon(\kappa')} \right] +$$

$$+ \sum_{m=1}^{\lambda-1} \int_0^\infty d\kappa' \Phi^{(n,m)}(\kappa, \kappa') \ln \left[ 1 + \tilde{K}_1^{(m)}(\kappa') K_r^{(m)}(\kappa') e^{-\epsilon_m(\kappa')} \right], \quad n = 1, \ldots, \lambda - 1.$$  

(94)

The integral kernels are obtained from the massless limit of the known ones, and are given explicitly by
\[ \Phi(\kappa, \kappa') = \frac{1}{\pi} \sum_{j=1}^{\lambda-1} \frac{\sin \pi j / \lambda}{\kappa^2 + \kappa'^2 + 2\kappa\kappa' \cos \pi j / \lambda}, \]
\[ \Phi^{(n)}(\kappa, \kappa') = \frac{2}{\pi} \left[ \frac{\kappa^2 + \kappa'^2}{\kappa^4 + \kappa'^4 + 2\kappa^2\kappa'^2 \cos \pi n / \lambda} + \frac{\kappa^2 + \kappa'^2}{\kappa^4 + \kappa'^4 - 2\kappa^2\kappa'^2 \cos \pi n / \lambda} \right] + \frac{2}{\pi} \sum_{l=1}^{n-1} \frac{\cos (n - 2l) \pi \kappa}{\kappa^2 + \kappa'^2 - 2\kappa\kappa' \sin (n - 2l) \pi / \lambda}, \]
\[ \Phi^{(n,m)}(\kappa, \kappa') = \frac{2}{\pi} \left[ \frac{\kappa^2 + \kappa'^2}{\kappa^4 + \kappa'^4 + 2\kappa^2\kappa'^2 \cos \pi n / \lambda} + \frac{\kappa^2 + \kappa'^2}{\kappa^4 + \kappa'^4 - 2\kappa^2\kappa'^2 \cos \pi n / \lambda} \right] + \frac{2}{\pi} \sum_{l=1}^{m-1} \frac{\cos (m - n - 2l) \pi \kappa}{\kappa^2 + \kappa'^2 - 2\kappa\kappa' \sin (m - n - 2l) \pi / \lambda}, \quad n \geq m. \tag{95} \]

The boundary scattering matrices become in the massless limit
\[ K(\kappa) = \prod_{j=1}^{\lambda} \left( \lambda \kappa + \Delta^\frac{\lambda+1}{2} e^{-i \frac{\pi}{\lambda} \Delta^\lambda + i \frac{\pi}{\lambda}} \right)^{-1} \left( \Delta^\lambda + 1 e^{-i \pi \eta + i \frac{\pi}{\lambda}(\Delta^\lambda - 1)} \right)^{\lambda+1}. \tag{96} \]

We have found it convenient to use a renormalized parameter \( \Delta \) in this expression. What we call \( \Delta \) phase in the hamiltonian: one finds (see the appendix)
\[ \chi \left[ \frac{\sqrt{\kappa^2 + \kappa'^2 - 2\kappa\kappa' \cos \pi n / \lambda} + \sqrt{\kappa^2 + \kappa'^2 - 2\kappa\kappa' \sin \pi n / \lambda}}{\sqrt{\kappa^2 + \kappa'^2 - 2\kappa\kappa' \cos \pi n / \lambda} + \sqrt{\kappa^2 + \kappa'^2 - 2\kappa\kappa' \sin \pi n / \lambda}} \right], \]
\[ \text{tr} K_l(\kappa) K_r(\kappa) = 2 \left[ \prod_{j=1}^{\lambda} \left( \kappa + \Delta^\lambda e^{-i \frac{\pi}{\lambda} \Delta^\lambda + i \frac{\pi}{\lambda}} \right) \left( \kappa + \Delta^\lambda e^{-i \frac{\pi}{\lambda} \Delta^\lambda + i \frac{\pi}{\lambda}} \right) \right]^{1/2} \times \left[ \kappa^2 + \Delta^\lambda + 1 \Delta^\lambda + 1 \kappa \lambda \right], \]
\[ \text{det} K_l(\kappa) K_r(\kappa) = \left[ \prod_{j=1}^{\lambda} \left( \kappa + \Delta^\lambda e^{-i \frac{\pi}{\lambda} \Delta^\lambda + i \frac{\pi}{\lambda}} \right) \left( \kappa + \Delta^\lambda e^{-i \frac{\pi}{\lambda} \Delta^\lambda + i \frac{\pi}{\lambda}} \right) \right]^{1/2} \times \left[ \kappa^2 + (-1)^\lambda \Delta^2(\lambda+1) \right] \left[ \kappa^2 + (-1)^\lambda \Delta^2(\lambda+1) \right]. \tag{99} \]

The breather \( K \) matrices are
\[ K_{l,r}^{(2n)}(\kappa) = -\frac{1}{n} \prod_{l=1}^{n} \frac{\kappa^2 - 2\kappa \Delta^\lambda}{\kappa^2 + 2\kappa \Delta^\lambda} \cos \left( \frac{(l-1/2)\pi}{\lambda} \right) \Delta^\lambda + 1 \Delta^\lambda + 1 \kappa \lambda \]
\[ K_{l,r}^{(2n-1)}(\kappa) = \frac{\kappa - \Delta^\lambda}{\kappa + \Delta^\lambda} \prod_{l=1}^{n-1} \frac{\kappa^2 - 2\kappa \Delta^\lambda}{\kappa^2 + 2\kappa \Delta^\lambda} \cos \frac{l\pi}{\lambda} \Delta^\lambda + 1 \Delta^\lambda + 1 \kappa \lambda. \tag{100} \]

Notice that the phase \( \chi \) appears only in the \( K \) matrices for solitons and antisolitons, and for a single boundary, could easily be gauged away by a redefinition of the soliton and antisoliton.
V. THE DOUBLE BOUNDARY SINE-GORDON MODEL FOR $\lambda = 2$

A. Generalities

Let us now turn to a precise investigation of what the TBA equations entail. For the particular case $\lambda = 2$ (corresponding to $\beta^2 = 8\pi/3$, which is the next simplest case to tackle after the free fermion point $\lambda = 1$, $\beta^2 = 4\pi$), we can use the preceding results to rewrite the TBA in a more compact form. We have explicitly (after rescaling to the more convenient dimensionless parameters $R^{3/2} = \Delta_{L,R}^{3/2}$)

$$E_0 = -\frac{1}{4\pi R} \int_0^\infty dk \ln[1 + W_2(\kappa)] - \frac{\sqrt{2}}{4\pi R} \int_0^\infty dk \ln[1 + B_2(\kappa)],$$

$$W_2(\kappa) = \left[\frac{2(\kappa^2 + \Delta_L^3 \Delta_R^3 \cos 3\chi \epsilon^{-\epsilon_2(\kappa)} + (\kappa^2 + \Delta_L^3 - \sqrt{2\kappa} \Delta_L^{3/2})(\kappa^2 + \Delta_R^3 - \sqrt{2\kappa} \Delta_R^{3/2})e^{-2\epsilon_2(\kappa)}}{\kappa^2 + \Delta_L^3 + \sqrt{2\kappa} \Delta_L^{3/2})(\kappa^2 + \Delta_R^3 + \sqrt{2\kappa} \Delta_R^{3/2})} \right],$$

$$B_2(\kappa) = \frac{\kappa - \Delta_L^{3/2}\kappa - \Delta_R^{3/2}\kappa + \Delta_L^{3/2}(\Delta_R^3 + \sqrt{2\kappa} \Delta_L^{3/2}) e^{-\epsilon_1(\kappa)}}{\kappa + \Delta_L^{3/2}\kappa + \Delta_R^{3/2}(\Delta_L^3 + \sqrt{2\kappa} \Delta_R^{3/2})}$$

(101)

with the quasienergies determined from the universal form of the massless TBA equations

$$\epsilon_1(\kappa) = \frac{2}{\pi} \int_0^\infty dk' \frac{\kappa' \kappa^2}{\kappa^4 + \kappa'^4} \left[2\epsilon_2(\kappa') + \ln[1 + W_2(\kappa')]} \right] \equiv U \ast \{2\epsilon_2(\kappa) + \ln[1 + W_2(\kappa)]\},$$

$$\epsilon_2(\kappa) = \frac{2}{\pi} \int_0^\infty dk' \frac{\kappa' \kappa^2}{\kappa^4 + \kappa'^4} \left\{\epsilon_1(\kappa') + \ln[1 + B_2(\kappa')]} \right\} \equiv U \ast \{\epsilon_1(\kappa) + \ln[1 + B_2(\kappa)]\},$$

(102)

Here $U$ is the universal massless scattering kernel, and $\ast$ is the “massless convolution” in the $\kappa$ variable.

The solutions are required to obey the asymptotic conditions at $\kappa \to \infty$

$$\lim_{\kappa \to \infty} \frac{\epsilon_1(\kappa)}{\kappa} = \sqrt{2}, \quad \lim_{\kappa \to \infty} \frac{\epsilon_2(\kappa)}{\kappa} = 1.$$  

(103)

Numerically, the solution of the equations is straightforward in the domain $0 \leq \chi < \pi/3$ (the sign of $\chi$ is not important). In particular, all solutions $\epsilon_{1,2}(\kappa)$ are then real on the positive real axis of $\kappa$.

Note that these TBA equations, from a knowledge of the functions on the real line, trivially define $\epsilon_{1,2}(\kappa)$ in the domain $|\text{Arg}(\kappa)| \leq \pi/4$, which we call the first fundamental domain. This is due to the particular form of the integration kernel in the TBA equations, whose poles lie on the aforementioned lines. To obtain the TBA functions outside of the first fundamental domain for a given value of $\chi$ from a knowledge of their values within the first fundamental domain, one needs to perform an analytical continuation procedure (the details of which are to be found below). Note that as a function of $\chi$, things will get even more complicated. As we will see below, this will require another type of analytical continuation, with multiple steps, generalizing the continuation proposed for the case $\lambda = 1$ in [9].

The first bit of information we can gain comes from looking at the limit $\kappa \to 0$ of the universal form. Defining $\lim_{\kappa \to 0} \epsilon_i = L_i$, we get the equations

$$L_1 = \frac{1}{2} \ln \left[1 + 2 \cos \chi \epsilon^{L_2} + \epsilon^{2L_2} \right], \quad L_2 = \frac{1}{2} \ln \left[1 + \epsilon^{L_1} \right].$$

(104)

Using the notation $x_i = \epsilon^{L_i}$, we can rewrite these as

$$x_1 = x_2^2 - 1, \quad x_2(x_2^2 - 3) = 2 \cos 3\chi.$$  

(105)

These are useful equations allowing us to check the numerics. However, these give us the first indication that not everything is fine. The third-order equation for $x_2$ yields three different solutions: $x_2 = 2 \cos \chi, -\cos \chi \pm \sqrt{3}$. For $|\chi| \leq \pi/3$, the appropriate one to pick is

$$x_2 = 2 \cos \chi.$$  

(106)

At $\chi = \pi/3$, this therefore means that $x_1$ vanishes, and therefore $L_1 \to -\infty$. A naive solution to the TBA equations would produce a cusp in $\chi$ here. If one were to naively solve the TBA for $\chi$ beyond this value, one would simply regenerate previous results: for $\chi = \pi/3 + \eta$, the curves would be identical to those for $\chi = \pi/3 - \eta$, which is patently incorrect in view of the cusp. The same kind of thing happened for $\lambda = 1$ at $\chi = \pi/2$, and we refer the reader to the discussion in section 3 for a description of the problems and their proper solution.
As we expect analytic behaviour as a function of $\chi$, some form of analytic continuation will have to be performed. Note also that $x_2$ vanishes at $\chi = \pi/2$: this further problem is not repaired by the surgery at $\chi = \pi/3$, so the analytical continuation will involve more than one step.

To make progress on the problem, let us start by reformulating the TBA equations into a more convenient form. Introduce the following functions ($\omega = e^{i\pi/4}$)

$$P(\kappa) = \frac{(\kappa - \Delta^3/2_L)/(\kappa - \Delta^3/2_R)}{(\kappa + \Delta^3/2_L)/(\kappa + \Delta^3/2_R)},$$
$$Q(\kappa) = \left[\frac{(\omega - \Delta^3/2_L)/(\kappa - \omega \Delta^3/2_L)}{(\kappa + \omega \Delta^3/2_L)/(\kappa + \omega \Delta^3/2_L)}\right]^{1/2},$$

and

$$\cos \theta(\kappa, \chi) = \frac{\kappa^4 + \Delta^3_L \Delta^3_R \cos 3\chi}{(\kappa^4 + \Delta^3_L)^{1/2}(\kappa^4 + \Delta^3_R)^{1/2}}.$$

The eigenvalues of $\tilde{K}_x K_r$ are equal to $Q(\kappa)e^{\pm i\theta(\kappa, \chi)}$. Introducing then the $Y$ functions through

$$e^{x_1}(\kappa) \equiv P(\kappa)Y_1(\kappa), \quad e^{x_2}(\kappa) \equiv Q(\kappa)e^{\mp i\theta(\kappa, \chi)}Y_2^\pm(\kappa)$$

the universal form of the TBA can be rewritten

$$\ln Y_1(\kappa) = -\ln P(\kappa) + U \mp \{\ln Q^2(\kappa) + \ln[1 + Y_2^+(\kappa')] + \ln[1 + Y_2^-(\kappa')]\},$$
$$\ln Y_2^\pm(\kappa) = \pm i\theta(\kappa, \chi) - \ln Q(\kappa) + U \mp \{\ln P(\kappa) + \ln[1 + Y_1(\kappa)]\}.$$  

We can in fact verify the following integrals, for $d, \kappa \in \mathbb{R}^+$:

$$\frac{2}{\pi} \int_0^{\infty} dk' \frac{k'k^2}{k'^4 + k^4} \ln \left(\frac{k' - \omega d k' - \omega d}{k' + \omega d k' + \omega d}\right) = \ln \frac{\kappa^2 + d^2}{(\kappa + d)^2},$$
$$\frac{2}{\pi} \int_0^{\infty} dk' \frac{k'k^2}{k'^4 + k^4} \ln \left(\frac{k' - d k' + d}{k' + d k' - d}\right) = \frac{1}{2} \ln \left(\frac{\kappa - \omega d k - \omega d}{\kappa + \omega d k + \omega d}\right) \pm i \arctan \frac{d^2}{k^2}(111)$$

where we have to make a branch choice in the last equation. In fact, knowing that the phase of $Y_2^\pm$ is given by $\pm \theta$, we can see that the appropriate choice should yield a simplification of the TBA equations with the identities

$$U \mp \ln Q^2(\kappa) = \ln P(\kappa) + \ln R(\kappa),$$
$$U \mp \ln P(\kappa) = \ln Q(\kappa),$$

where

$$R(\kappa) = \frac{\kappa^2 + \Delta^3_L \kappa^2 + \Delta^3_R}{\kappa^2 - \Delta^3_L \kappa^2 - \Delta^3_R}. $$

The final form of the TBA is therefore

$$\ln Y_1(\kappa) = \ln R(\kappa) + U \mp \{\ln[1 + Y_2^+(\kappa)] + \ln[1 + Y_2^-(\kappa)]\},$$
$$\ln Y_2^\pm(\kappa) = \pm i\theta(\kappa, \chi) + U \mp \ln[1 + Y_1(\kappa)].$$

The ground state energy then reads

$$E_0 = \frac{-1}{4\pi R} \int_0^{\infty} d\kappa \left\{\sqrt{2} \ln[1 + Y_1^{-1}(\kappa)] + \ln[1 + Y_2^{+1}(\kappa)] + \ln[1 + Y_2^{-1}(\kappa)]\right\}. $$

Note that, one has to be careful along the lines $\arg(\kappa) = \pm \pi/4$ where the universal kernel leads to singular integrals, which have to be evaluated by contour deformation. First, note that simple manipulations yield

$$P(\omega \kappa)P(\tilde{\omega} \kappa) = Q^2(\kappa), \quad Q(\omega \kappa)Q(\tilde{\omega} \kappa) = P(\kappa).$$

and also $R(\omega \kappa) = R^{-1}(\tilde{\omega} \kappa)$. Observing that $\theta(\omega \kappa, \chi) = \theta(\tilde{\omega} \kappa, \chi)$ for real $\kappa$, and considering the evaluation of the TBA functions on these lines, yields the $Y$-system
\[
Y_1(\omega \kappa)Y_1(\bar{\omega} \kappa) = [1 + Y_2^+(\kappa)][1 + Y_2^-(\kappa)],
Y_2^\pm(\omega \kappa)Y_2^\mp(\bar{\omega} \kappa) = 1 + Y_1(\kappa)
\]  
(117)

which coincides with the \(D_3\) \(Y\)-system in [29]. In particular, the nontrivial periodicity in the complex plane of \(\kappa\) can be read from those. Namely, one can show by direct substitution that

\[
Y_1(\omega^6 \kappa) = Y_1(\kappa), \quad Y_2^\pm(\omega^6 \kappa) = Y_2^\pm(\kappa).
\]  
(118)

This means that the \(Y\)-functions naively have Laurent expansions in powers of \(\kappa^{4/3}\). In particular, the boundary pairings appear in this expansion in powers of \(\Delta_{L,R}^2\). Once again, like at the free fermion point, the TBA does not see clearly the term proportional to \(\Delta_L \Delta_R\) in the perturbative expansion of the ground-state energy at weak boundary pairing.

In view of the periodicity of the \(Y\) functions, and the consequence this has on the structure of their Laurent expansions, we see that the ground-state energy has a naïve expansion in powers of \(\Delta_{L,R}^2\). However, as we have stated above, problems are encountered at \(\chi = \pi/3\) which require modifying the TBA in a nontrivial way. We now discuss how this surgery is to be performed to ensure analyticity in the boundary phase difference \(\chi\).

### B. Scenario for the analytic continuation

As we have argued above, we encounter various types of singularities in the (solutions to the) TBA equations as a function of \(\chi\). Let us define the following terminology: types \(K_1, K_2^\pm\) singularities are defined for \(\kappa\) obeying

\[
1 + Y_1(\kappa) = 0, \quad 1 + Y_2^\pm(\kappa) = 0.
\]  
(119)

We also define types \(F_1, F_2^\pm\) by

\[
Y_1(\kappa) = 0, \quad Y_2^\pm(\kappa) = 0,
\]  
(120)

and types \(G_1, G_2^\pm\) by

\[
Y_1^{-1}(\kappa) = 0, \quad (Y_2^\pm)^{-1} = 0.
\]  
(121)

Types \(K_1, K_2^\pm\) and \(F_1, F_2^\pm\) have consequences on the ground-state energy integral, whereas \(G_1, G_2^\pm\) types are harmless.

We now have to ask ourselves what singularities appear in our equations, and how we have to take them into account. From the numerical solution of the TBA equations, we know that at \(\chi = \pi/3\), there are singularities of types \(K_2^\pm\) and \(F_1\) at \(\kappa = 0\). We propose the following scenario for the analytical continuation of the TBA beyond \(\chi = \pi/3\).

First of all, notice that a singularity of type \(K_2^\pm\) (as we know appears at \(\chi = \pi/3\)) requires \(\ln Y_2^\pm = i\pi \text{ mod}(2\pi)\). In particular, the real part of \(\ln Y_2^\pm\) must vanish (this is equivalent to saying that \(c_2\) must be purely imaginary, as was argued for the case \(\lambda = 1\) at the end of section 3). By examining the form of the universal scattering kernel, and in view of the comments made earlier about the use of the TBA equations to write the functions \(Y\) in the complex plane of \(\kappa\) from a knowledge of their value on the real line, we can easily convince ourselves that the condition \(\Re \ln Y_2^\pm = 0\) could only be satisfied on the axes \(\omega \kappa, \bar{\omega} \kappa\) with \(\kappa \in \mathbb{R}\). We therefore state

Proposition 1: at \(\chi = \pi/3\), singularities of type \(K_2^\pm\) and \(F_1\) occur at \(\kappa = 0\). Increasing \(\chi\) beyond \(\pi/3\) moves the \(K_2^\pm\) singularities to \(\omega \kappa_s, \bar{\omega} \kappa_s\), with \(\kappa_s \in \mathbb{R} > 0\). The \(F_1\) singularity moves to \(\kappa_s\) (that is, the \(K_2^\pm\) singularities are at angles \(\pm \pi/4\) in the complex plane of \(\kappa\), and the \(F_1\) singularity sits on the real line). Note that the relative position of the singularities is dictated by the \(Y\)-system.

If such a singularity structure occurs, the first TBA equation needs to be modified to take the pole contribution into account:

\[
\ln Y_1(\kappa) = \ln T(\kappa, \kappa_s) + \ln R_1(\kappa) + U \ast \{\ln[1 + Y_2^+(\kappa)] + \ln[1 + Y_2^-(\kappa)]\},
\]  
(122)

where we have defined the function

\[
\ln T(\kappa, \kappa_s) = \ln \frac{\kappa^2 - \kappa_s^2}{\kappa^2 + \kappa_s^2}
\]  
(123)

as the residue that pops out of the universal kernel. Note that this modification ensures the presence of a \(F_1\) singularity on the real line at \(\kappa_s\), as required. Note also that the extra term does not change the \(Y\)-system structure, since \(T(\omega \kappa, \kappa_s)T(\bar{\omega} \kappa, \kappa_s) = 1\).

The ground state energy changes under this continuation: we will treat this after solving the singularity problem of the TBA equations.
The major remaining problem is to determine the value of $\kappa_s$. This comes from solving the singularity condition obtained by explicitly requiring e.g. $Y_2^{+}(\omega \kappa_s) = -1$ under Proposition 1. This translates into

$$\pi \pmod{2\pi} = \theta(\omega \kappa_s, \chi) + \frac{2}{\pi} \int_0^\infty d\kappa' \frac{\kappa' \kappa_s^2}{\kappa'^4 - \kappa_s^4} \ln[1 + Y_1(\kappa')].$$

(124)

Note that the fact that $Y_1$ has a type $F_1$ singularity at $\kappa_s$ makes the principal part integral well-behaved. The TBA system now comprises three equations, i.e. the two basic TBA equations, with the addition of the first singularity condition.

The first thing to notice is that this modification cures our problems at $\chi = \pi/3$ with the $\kappa \to 0$ asymptotics. That is, the relevant formulas become

$$L_1 = \ln(-1) + \frac{1}{2} \ln(1 + 2 \cos 3\chi e^{L_2} + e^{2L_2}), \quad L_2 = \frac{1}{2} \ln(1+ e^{L_1})$$

(125)

Again using the notation $x_i = e^{L_i}$, we can rewrite these as

$$x_1 = 1 - x_2^2, \quad x_2(x_2^2 - 3) = 2 \cos 3\chi,$$

(126)

providing a smooth curve for $x_1$ (note the change of sign) around the singular point.

Now as we tune $\chi$ up beyond this, we encounter the possibility of a type $K_1$ singularity, so this is by far not the end of the story. From the asymptotics at $\kappa \to 0$, we expect this to happen at $\chi = \pi/2$, where $x_2$ is bound to change sign (meaning that the function $Y_2$ must become negative on at least part of the real axis of $\kappa$). We therefore expect a $K_1$ singularity at $\kappa = 0$ for $\chi = \pi/2$.

Moreover, a careful study of the first singularity condition above shows that one can solve it for $\kappa_s$ real only provided (here, for $\Delta_L = \Delta_R$)

$$\kappa_s \leq \Delta^{3/2} \sqrt{|\cos 3\chi/2|}.$$

(127)

The immediate consequence of this is that the window of possible values of the first singular point grows between $\chi = \pi/3$ and $\pi/2$, but decreases again afterwards. We therefore expect the first singularity to collapse back to zero before $\chi$ reaches $\pi$.

If such a second singularity crosses the real axis, then we have to modify the TBA again. Namely, we now get a modified second TBA equation:

$$\ln Y_2^+(\kappa) = \pm i \theta(\kappa, \chi) + \ln T(\kappa, \kappa_{s2}) + U \ln[1 + Y_1(\kappa)]$$

(128)

where the parameter $\kappa_{s2}$ is a solution to the second type of singularity condition

$$\pi = i \ln T(\omega \kappa_{s2}, \kappa_s) - i \ln R(\omega \kappa_{s2}) = \frac{2}{\pi} \int_0^\infty d\kappa' \frac{\kappa' \kappa_{s2}^2}{\kappa'^4 - \kappa_{s2}^4} \left[\ln[1 + Y_2^+(\kappa')] + \ln[1 + Y_2^-(\kappa')]\right]$$

(129)

The final modification from the second singularity occurs in the first singularity condition: namely, the integral kernel appearing in it again diverges, and has to be continued. This produces the same kind of additional contribution as appeared in the TBA equations, corresponding to the residue of the universal scattering kernel evaluated at appropriate points. For clarity, we rewrite the full TBA, including the appropriate branch choices, for the case $\lambda = 2$.

In this, we have found it convenient to reabsorb the phases $\pm i \theta$ and the kernels $T$ in the $Y$-functions. The full TBA finally reads

$$\ln Y_1(\kappa) = \ln R(\kappa) + U \ln \left[1 + 2 \cos \theta(\kappa) \frac{\kappa^2 - \kappa_{s2}^2}{\kappa^2 + \kappa_{s2}^2} Y_2^{+}(\kappa) + \left(\frac{\kappa^2 - \kappa_{s2}^2}{\kappa^2 + \kappa_{s2}^2}\right)^2 Y_2^{-}(\kappa)\right],$$

$$\ln Y_2(\kappa) = U \ln \left[1 + \frac{\kappa^2 - \kappa_{s2}^2}{\kappa^2 + \kappa_{s2}^2} Y_1(\kappa)\right],$$

(130)

supplemented by the two singularity conditions

$$\pi = \arccos \frac{\Delta_L^3 \Delta_R^3 \cos 3\chi - \kappa_{s2}^4}{\sqrt{(\Delta_L^4 - \kappa_{s2}^4)(\Delta_R^4 - \kappa_{s2}^4)}} + 2 \arccos \frac{\kappa_{s2}^4}{\sqrt{\kappa_{s2}^4 + \kappa_{s2}^4}} + \frac{2}{\pi} \int_0^\infty d\kappa' \frac{\kappa' \kappa_{s2}^2}{\kappa'^4 - \kappa_{s2}^4} \ln \left[1 + \frac{\kappa'^2 - \kappa_{s2}^2}{\kappa'^2 + \kappa_{s2}^2} Y_1(\kappa')\right],$$

$$\pi = -2 \arccos \frac{\kappa_{s2}^2}{\sqrt{\kappa_{s2}^4 + \kappa_{s2}^4}} + 2 \arccos \frac{\kappa_{s2}^2}{\sqrt{\kappa_{s2}^4 + \Delta_L^4}} + 2 \arccos \frac{\kappa_{s2}^2}{\sqrt{\kappa_{s2}^4 + \Delta_R^4}} -$$

$$-\frac{2}{\pi} \int_0^\infty d\kappa' \frac{\kappa' \kappa_{s2}^2}{\kappa'^4 - \kappa_{s2}^4} \ln \left[1 + 2 \cos \theta(\kappa') \frac{\kappa'^2 - \kappa_{s2}^2}{\kappa'^2 + \kappa_{s2}^2} Y_2^{+}(\kappa') + \left(\frac{\kappa'^2 - \kappa_{s2}^2}{\kappa'^2 + \kappa_{s2}^2}\right)^2 Y_2^{-}(\kappa')\right],$$

(131)
with $\kappa_{s1} = 0$ for $\chi < \pi/3$ and $\chi > 2\pi/3$, and $\kappa_{s2} = 0$ for $\chi < \pi/2$. The evolution of the singularities is as follows: $\kappa_{s1}$ first appears at $\chi = \pi/3$, and moves up. At $\chi = \pi/2$, $\kappa_{s2}$ makes its appearance, and starts moving up. As it does so, it pulls $\kappa_{s1}$ back down, until the latter vanishes again at $\chi = 2\pi/3$. After that point, only $\kappa_{s2}$ remains. This behaviour is illustrated in figures (4, 5), where the two singularities are plotted as a function of the phase difference $\chi$ for various values of the boundary parameters $\Delta_L = \Delta_R$.

![FIG. 4](image1.png)

**FIG. 4.** The singularity $\kappa_{s1}$ plotted as a function of the phase difference $\chi$ for seven values of the boundary parameters $\Delta_L = \Delta_R$.

![FIG. 5](image2.png)

**FIG. 5.** The singularity $\kappa_{s2}$ plotted as a function of the phase difference $\chi$ for seven values of the boundary parameters $\Delta_L = \Delta_R$.

The ground state energy is given in these notations by the integral expression (115) analytically continued for the singularities of type $K_2^{\pm}$ and $F_1$ at $\chi = \pi/3$, and those of type $K_1$ and $F_2^{\pm}$ at $\chi = \pi/2$. The extra kernels from
these respective divergences cancel pairwise, and one is left after a few basic manipulations with the modified integral expression for the ground state energy (valid for all values of $\chi$

\[ E_0 = -\frac{1}{4\pi R} \int_0^\infty d\kappa \left\{ \ln \left[ 1 + 2 \cos(\theta(\kappa)) \frac{\kappa^2 - \kappa_{s2}^2 Y_2(\kappa)}{\kappa^2 + \kappa_{s2}^2} \right] + \left( \frac{\kappa^2 - \kappa_{s2}^2}{\kappa^2 + \kappa_{s1}^2} \right)^2 Y_2^2(\kappa) - 2 \ln Y_2(\kappa) + \sqrt{2} \ln \left[ 1 + \frac{\kappa^2 - \kappa_{s1}^2}{\kappa^2 + \kappa_{s1}^2} Y_1(\kappa) \right] - \sqrt{2} \ln Y_1(\kappa) \right\}. \]  \hspace{1cm} (132)

Due to the fact that we have reabsorbed the singularity contributions into the $Y$ functions, they are always real and positive on the real line.

The numerical solution to the full TBA equations (including the singularity conditions) over the full interval of $\chi$ from 0 to $\pi$ turns out to be a considerable challenge. The evaluation of the principal part integrals requires many sampling points, but this isn’t the main problem. The main problem is that the numerics tend to be unstable within a small domain around $\chi \approx 1.9$. This is due to the fact that the right-hand side of the singularity conditions are nonmonotonous functions, from which it is therefore difficult to isolate the correct root. Four integral equations have to be simultaneously obeyed, so the iteration procedure in general has many unstable directions. A simple root finding algorithm is insufficient, and one has to make the code rather elaborate to obtain sensible curves.

The curves for the ground-state energy as a function of $\chi$ for various values of the boundary parameters $\Delta$ are presented in figure (6). From these, it is transparent that our construction provides a completely interpolating solution between the $\sim \cos \chi$ behaviour at small $\Delta$ to the $\sim \chi^2$ one for large $\Delta$, as expected from the appropriate conformal limits. For large $\Delta$, the fit is perfect with the expected functional form (easily derived from mode expansions)

\[ E_0(\Delta \to \infty) = -\frac{\pi}{24R} + \frac{2\chi^2}{\beta^2 R} = \frac{\pi}{24R} + \frac{3\chi^2}{4\pi} \]  \hspace{1cm} (133)

with the last equality valid for the present case $\lambda = 2$.

FIG. 6. Ground-state energy of the two-boundary sine-Gordon model plotted as a function of the phase difference $\chi$ for seven values of the boundary parameters $\Delta_L = \Delta_R$. The interpolation between the conformal limits is clearly seen.

C. Some remarks on the general singularity structure

We are not sure what the general singularity structure will be for other values of $\lambda$. A generalization of the Ising model argument given in section II indicates that there always will be a singularity at $\frac{\pi}{2}$. Indeed, following [12], boundary states are properly understood by considering a semi-infinite system, and choosing at infinity either the boundary condition $\phi = 0$ or $\phi = \frac{2\pi}{\beta}$ (the equivalent of spin up or down in the Ising model). For $\chi \in \left[ 0, \frac{\pi}{2} \right]$, the
first condition defines the true ground state (equivalent to $h > 0$ in the Ising model) while for $\chi \in \left[\frac{\pi}{3}, \pi\right]$, the first condition corresponds to an excited state. Equivalently, if one sets always $\phi = 0$ at infinity, then the case $\chi \in \left[\frac{\pi}{3}, \pi\right]$ should correspond to an excited boundary state.

We pause here to stress that the ground state energy does not have any singularity, and that there is no level crossing the ground state at any of the singularity values in the massless, two boundary problem. Indeed, the foregoing argument assumed values at infinity fixed by the bulk interaction term, with periodicity $\phi \to \phi + \frac{2\pi}{\beta}$. In the massless two boundary problem, the periodicity is $\phi \to \phi + \frac{2\pi}{\beta}$, so the first level crossing is expected to occur at $\chi = \pi$.

The existence of the singularities we are struggling with translates into interesting behaviours for the excited levels of the hamiltonian. This can be seen quite clearly in the case of the Ising model, for instance with boundary fields $h_1 = h$ fixed, $h_2 = h'$ varying. Since the ground state is analytic, the first gap behaves as $\frac{1}{2} |hh'|$ at small magnetic fields, and pinches the ground state in a V shape around the symmetric point $h' = 0$.

Technically, the existence of singularities in the TBA approach follows from the dependence of the scattering matrices on $(\lambda + 1)\chi$ instead of $\chi$. This dependence is a consequence of the fact that the scattering matrices are in essence IR defined. Starting from the UV action (1) the Dirichlet IR boundary conditions are approached along the operators “dual” to $e^{i\beta_d/2}$, i.e. $e^{i\beta_d\phi_d/2}$ where $\phi_d = \phi - \bar{\phi}$ and $\frac{\beta_d}{\beta} = \frac{\pi}{R\chi}$. It follows that the dual coupling $\tilde{\Delta}_d \propto \left(\tilde{\Delta}\right)^{\lambda - 1}$ and therefore, if $\chi$ is the phase of the UV coupling, the phase of the IR coupling becomes $(\lambda + 1)\chi$. Clearly, the IR action (or the scattering matrix) by itself is not sufficient to describe all values of $\chi \mod 2\pi$, and it is natural to expect that the reduced information contained in $(\lambda + 1)\chi \mod 2\pi$ has to be supplemented with specification of a boundary state. While it is possible to associate a conformal Dirichlet boundary state with any value of the field $\phi(0)$, it is well known (see eg [27], whose conventions we follow) that for a given radius $R = \frac{2}{\sqrt{\lambda}}$, there are, when $R^2 = \frac{\lambda + 1}{2\pi}$, $\lambda + 1$ “special” boundary states. Presumably, the IR action (or the scattering matrix) supplemented by the choice of one of these boundary states would allow to explore the whole domain $\chi \mod 2\pi$, which would correspond in the TBA approach to analytic continuation at the values $\chi = \frac{\pi}{3\lambda}$. It may also be that more values require continuation. This can be inferred from the limit $\kappa \to 0$, which gives the following values of the $x_j$ for general $\lambda$: $x_j = \frac{\sin(\lambda + 2 - j)\chi}{\sin \chi}$. We thus expect from this to observe singularities also at values $\frac{\pi}{3\lambda}, \frac{\pi}{\lambda}, \frac{\pi}{2}, \frac{\pi}{3}$, and multiples (the pattern is similar but different from the one of excited boundary states discussed in [6]). More work is required to clarify this question.

VI. THE DOUBLE KONDO MODEL

The TBA approach to the double Kondo model is bound to be rather intricate, as the boundary scattering involves now (except for spin 1/2) boundary degrees of freedom, whose inclusion into the boundary state and the subsequent R-channel TBA is not obvious. Fortunately, the problem is perfectly well suited, on the other hand, to the Destri de Vega approach.

The general framework to apply the DDV approach to a problem with two boundaries has been discussed in details in [2]. There, the starting point was the inhomogeneous 6 vertex model with boundary fields. The role of the inhomogeneities was to introduce a bulk mass scale, while the boundary fields constrained the value of the sine-Gordon field on the boundaries. The net result was an expression for the ground state energy of the sine-Gordon model with double Dirichlet boundary conditions.

It is well known how to introduce a different kind of inhomogeneities in the 6 vertex model to get a theory that is massless, with free L movers and R movers interacting with a Kondo type impurity. It is similarly possible to obtain a theory with free L movers and R movers interacting with two different Kondo type impurities. Forgetting about the L movers, one can then transform this problem by folding into the problem we are interested in. It is a straightforward calculation to then obtain expressions for the ground state energy - these were actually almost already written in [2]. Since the most interesting limit is the isotropic one (XXX model), we give results in the repulsive regime of the sine-Gordon model, parametrizing $\beta^2 = 8\pi \frac{\Delta}{|\Delta|}$. One finds then

$$f(\theta) = i\mu e^{\theta} + iP_{\text{dry}}(\theta) - 2i \int_{-\infty}^{\infty} d\theta' \Phi(\theta - \theta') \text{Im} \ln \left[1 - e^{f(\theta' + i0)}\right]$$

$$E = -\frac{2}{\pi R} \int_{-\infty}^{\infty} \mu Re^{\theta} \text{Im} \ln \left[1 - e^{f(\theta + i0)}\right]$$

(134)

\(^2\)There is a misprint in this paper, as equation (9.3) corresponds to Neumann, not Dirichlet boundary conditions on the other boundary.
where $\Phi$ is the kernel already encountered in section IV

$$
\Phi(\theta) = -\int_{-\infty}^{\infty} \frac{dx}{2\pi^2} \frac{\sinh(t - 2)x}{\cosh x \sinh(t - 1)x} e^{2ix\theta/\pi}
$$

(135)

and the boundary kernel is

$$
iP_{bdry}(\theta) = i\int_{-\infty}^{\infty} \frac{dx}{2\cosh x \sinh(t - 1)x} \sin(2x(\theta - \theta_B)) / \pi - \frac{i\pi}{2} \frac{t - j_t}{t - 1} + (l \to r)
$$

(136)

The phase is such that at small coupling $\theta_B \to -\infty$ the boundary term vanishes while at large coupling $\theta \to \infty$ it reads $iP_{bdry} = \pi \left( \frac{l - j_t}{l - 1} + \frac{l - j_t}{l - t} \right)$, this maybe modulo $2\pi$.

Note the deep similarity of these equations with the ones usually written in the bulk - in the latter case, there would be a $i\omega$ term instead of a $iP_{bdry}$ term, the $\omega$ corresponding to a soliton fugacity.

While bulk equations involve a constant $\omega$, the variation of $P_{bdry}$ with the rapidity leads to some obvious difficulties. Let us consider for instance the simplest case of a spin 1/2 Kondo impurity at one boundary, Neumann conditions at the other boundary, and restrict also to the isotropic case $t = \infty$. As discussed in [8] going from UV to IR in this case amounts to fusing the identity (Kac Moody) representation that initially goes through the system with the spin 1/2 (Kac Moody) representation, so the ground state energy in the IR should correspond to the conformal weight $h = 1/2(1/2 + 1)/1 + 2 = 1/4$. This is in agreement with our formula, as the usual dilogarithm analysis gives

$$
h_{IR} = \frac{1}{4} \left( \frac{P_{bdry}(\infty)}{\pi} \right)^2 = \frac{1}{4}
$$

(137)

If however we put two Kondo impurities, we see that our formula gives $h_{IR} = 1$, while fusing with the Kac Moody twice gives the Kac Moody identity representation back, with lowest conformal weight $h = 0$. The $h = 1$ weight is instead an excited weight, and therefore our DDV equation, while it captures the ground state at small coupling, captures instead an excited state at large coupling.

It was also argued by Affleck and Ludwig in [26] that the picture for $j = \frac{1}{2}$ essentially carries over to the case of arbitrary $j$: in either case, flow from UV to IR is described by the absorption of an electron by the impurity, and a phase shift of $\pi/2$ for the fermion wave function. Hence all the foregoing remarks extend to the case of higher values of the spin as well, which agrees with our general formulas, as the value of $P_{bdry}(\infty)$ does not depend on the spins in the isotropic limit $t \to \infty$. In terms of the free boson, the boundary condition corresponding to the Kondo strong coupling fixed point is $\phi_l - \phi_r = \sqrt{\pi/2}$ modulo $2\pi$.

What happens in the anisotropic case is somewhat less clear, as the boundary condition left over in the strong coupling limit now seems to depend on the impurity $sl(2)_q$ spin.

**VII. CONCLUSION**

It is tempting to carry out a little bit further and propose a DDV equation for the double boundary sine-Gordon model.

To do so, we would like first to get back to the free fermion case. We introduce the two eigenvalues of the matrix $\hat{K}_l \hat{K}_r$, which we call $\Lambda_{\pm}$. Their expressions are (setting $\mu = 1$)

$$
\Lambda_{\pm} = \frac{1}{(e^\theta + \Delta_+^2)(e^\theta + \Delta_-^2)} \left[ e^{2\theta} + \Delta_+^2 \Delta_-^2 \cos 2\chi \pm i \sqrt{\Delta_+^2 \Delta_-^2 \sin^2 2\chi - e^{2\theta}(\Delta_+^2 + \Delta_-^2 + 2\Delta_+^2 \Delta_-^2 \cos 2\chi)} \right]
$$

(138)

and we rewrite the ground state energy (60) as (recall $\kappa = e^\theta$)

$$
E = -\frac{1}{4\pi} \int_{-\infty}^{\infty} d\theta e^\theta \ln \left( 1 + \Lambda_+(\theta)e^{-Re^\theta} \right) + -\frac{1}{4\pi} \int_{-\infty}^{\infty} d\theta e^\theta \ln \left( 1 + \Lambda_-(\theta)e^{-Re^\theta} \right)
$$

(139)

We now perform a shift of the variable of integration, $\theta \to \theta + \frac{i\pi}{2} - i\epsilon$ in the first integral, $\theta \to \theta - \frac{i\pi}{2} + i\epsilon$ in the second integral. Here, the $\epsilon$'s are necessary to avoid the zeroes of the logarithms in the integral. Care must also be exercised in moving the contours because of the branch cuts in the definitions of the square roots in $\Lambda_{\pm}$. This gives rise to the new result

$$
E = \frac{1}{4\pi} \left[ \int_{C_1} e^\theta \ln \left( 1 - \frac{1}{\Lambda_+}e^{Re^\theta} \right) + \int_{C_2} e^\theta \ln \left( 1 - \Lambda_+e^{-Re^\theta} \right) \right]
$$

(140)
where we have introduced the eigenvalues of the product $R_l R_r$ (no complex conjugation, see later),

$$
\lambda_{\pm} = \frac{1}{(e^{\theta} - i\Delta_l^2)(e^{\theta} - i\Delta_r^2)} \left[ -e^{2\theta} + \Delta_l^2 \Delta_r^2 \cos 2\chi \pm i\sqrt{\Delta_l^2 \Delta_r^2 \sin^2 2\chi + e^{2\theta}(\Delta_l^2 + \Delta_r^2 + 2\Delta_l^2 \Delta_r^2 \cos 2\chi)} \right]
$$

(141)

and the contours are represented in the figure 7.

![Contour plot](image)

**FIG. 7.** Contours used in equation (141).

Similar manipulations could lead to the same result with $\lambda_+$ replaced with $\lambda_-$, and the two are in fact equal. After some simple redefinitions we thus obtain the result

$$
E = \frac{1}{4i\pi} \left[ \int_{C_1} e^{\theta} \ln \left( 1 - e^{f(\theta)} \right) + \int_{C_2} e^{\theta} \ln \left( 1 - e^{-f(\theta)} \right) \right]
$$

(142)

where

$$
f(\theta) = Rie^\theta - \ln \lambda_{\pm}
$$

(143)

We conjecture that the DDV expression in the general case is very similar, with the exact same form for the energy, and as for the source equation

$$
f(\theta) = Rie^\theta - \ln \lambda_+ + \int_{C_1} \Phi(\theta - \theta') \ln \left( 1 - e^{f(\theta')} \right) + \int_{C_2} \Phi(\theta - \theta') \ln \left( 1 - e^{-f(\theta')} \right)
$$

(144)

and $\Phi$ is the kernel already used in the DDV equations of the previous paragraph.

Here, $\lambda_{\pm}$ are eigenvalues of $R_l R_r$, which have a structure entirely similar to the one for the case $\lambda = 1$,

$$
\lambda_{\pm} = \frac{1}{\prod_{j=1}^{\lambda} \left( e^{\theta} - \Delta_l^{\lambda+1/\lambda} e^{i\pi/2\lambda} e^{-i\pi j/\lambda} \right) \left( e^{\theta} - \Delta_r^{\lambda+1/\lambda} e^{i\pi/2\lambda} e^{-i\pi j/\lambda} \right)} \times
$$

$$
\times \left[ e^{2\lambda\theta} - \Delta_l^{\lambda+1} \Delta_r^{\lambda+1} \cos \eta \pm i \left[ \Delta_l^{2\lambda+2} \Delta_r^{2\lambda+2} \sin^2 \eta + e^{2\lambda\theta} \left( \Delta_l^{2\lambda+2} \Delta_r^{2\lambda+2} + 2 \cos \eta \Delta_l^{\lambda+1} \Delta_r^{\lambda+1} \right) \right]^{1/2} \right]
$$

(145)

(and recall $\eta = (\lambda + 1)\chi$). An important property is that $\lambda_+(\theta + i\pi) = \overline{\lambda_-}(\theta)$.

It is interesting to see what our conjecture gives in the case where $\beta^2 \to 8\pi$. There one finds

$$
\lambda_{\pm} = \frac{-1 - \Delta_l \Delta_r \cos \chi \pm i \sqrt{\Delta_l^2 \Delta_r^2 \sin^2 \chi + \Delta_l^2 + \Delta_r^2 - 2\Delta_l \Delta_r \cos \eta}}{\sqrt{(1 + \Delta_l^2)(1 + \Delta_r^2)}}
$$

(146)

In this limit, the dependence on the energy of the extra soliton fugacity has disappeared, and the ground state energy follows from the usual dilogarithm calculation

$$
E = -\frac{\pi}{24R} \left( 1 - \frac{6}{\pi} \alpha^2 \right)
$$

(147)

The eigenvalues being pure phases, one has

$$
\alpha = \pm \arg \lambda_{\pm} = \mp \text{arctan} \frac{\sqrt{\Delta_l^2 \Delta_r^2 \sin^2 \chi + \Delta_l^2 + \Delta_r^2 - 2\Delta_l \Delta_r \cos \chi}}{1 + \Delta_l \Delta_r \cos \chi}
$$

(148)
The relation between the coupling $\Delta$ and the bare coupling is simply $\Delta = \pi \tilde{\Delta}$.

We check this expression by comparing the resulting “current” in applications to Josephson junctions to the one obtained in [25]. The current follows

$$I \propto \alpha d\alpha d\chi \propto \Delta_l \Delta_r \sin \chi \sqrt{\Delta_l^2 + \Delta_r^2 - 2\Delta_l \Delta_r \cos \chi}$$

and $\alpha$ can be written

$$\alpha = \arccos \frac{1 + \Delta_l \Delta_r \cos \chi}{\sqrt{(1 + \Delta_l^2)(1 + \Delta_r^2)}}$$

It is easy to check that this coincides with the formula in [25] but for the redefinition

$$\Delta_{\text{here}} = \frac{\Delta_{\text{there}}}{1 - \Delta_{\text{there}}^2}$$

Getting back to arbitrary values of $\lambda$, in the limit $R \to \infty$, it is easy to check from the same kind of calculations that the correct value

$$E_0 = -\frac{\pi}{24R} \left(1 - 6i\frac{\chi^2}{\pi^2}\right)$$

is recovered.

Of course, these few tests are far from enough to completely justify our proposal, and we expect to get back to it in more details in the future.

As we were completing this paper, a preprint appeared [31] where the same problem is discussed, and the problem of analytic continuation is also claimed to be solved. We believe that the solution proposed in [31] is actually incorrect, for technical reasons we now discuss.

In this preprint, the authors claim to find the solutions to the singularity conditions on the imaginary axis of $\kappa$, for any value of $\lambda$. While this is certainly true for $\lambda = 1$ as shown in our earlier work [9], we have shown that for $\lambda = 2$, the singularities should be found on the lines with angles $\pm \pi/4$ in the complex plane of $\kappa$, instead of the imaginary axis. Moreover, the authors of [31] argue that only one singularity shows up for any value of $\lambda$, whereas we have shown (using for example of $\kappa \to 0$ asymptotics and the $Y$-system arguments) that two singularities have to be treated for $\lambda = 2$ to ensure analyticity in $\chi$. As an illustration, consider equation (3-12) of [31], which is the universal form of the TBA corrected for the first singularity. The result for the correction is the residue $\ln S_{a1}$, whereas we correct with the residue of the universal scattering kernel (see our equation (122)). The reason for the difference seems to be that the authors of [31] have not taken into account the fact that the function $\epsilon_1$ diverges at $\kappa = 0$ when $\chi = \pi/3$, in addition to the first logarithmic kernel involving $\epsilon_2$ (the notations are different: $\epsilon_0$ in their work is what we have called $\epsilon_2$ here). These two divergences each modify the TBA equations, and correcting for only one of them leads to the wrong results. This is transparent if the whole analytical continuation procedure is done using the universal for the TBA equations from the start, as we have done.

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APPENDIX A: THE FREE FERMION POINT

At $\beta^2 = 4\pi$, the sine-Gordon action has as is well-known a direct correspondence to the action of free fermions. In order to correctly treat our double-boundary theory at this free fermion point, we have to be very careful with our bosonization/refermionization identities. These are much simpler in the bulk, where we can simply toss away the finite-size terms, and for simple periodic boundary conditions, for which the quantizations rules of the zero modes can be directly read out. We will start from the very basics to illustrate the procedure forced upon us by the two-boundary geometry we are considering, and give every detail of the derivation from the canonical quantization of the free boson to the final formula for the ground-state energy.

Consider thus the interval $I : x \in [0, R]$ on the real line. We want to study a massless bosonic field $\phi(x)$ which is free in the bulk (i.e. for $x \in [0, R]$) but has some boundary contribution to the action at the points $x = 0, R$. If we
intend to develop $\phi(x)$ in a mode expansion, we might very well be tempted to use periodicity with period $R$. As we do not wish to identify $\phi(x)$ at the points 0 and $R$, in order to accommodate different boundary effects on both sides, this is not flexible enough. Instead, we will extend the definition of $\phi(x)$ to the interval $x \in [0, 2R]$ by using

$$\phi(2R - x) = \phi(x), \quad x \in [0, R]. \quad (A1)$$

We can then consistently extend the definition of $\phi(x)$ to the entire real line by using

$$\phi(x + 2R) = \phi(x), \quad x \in \mathbb{R}. \quad (A2)$$

This is important: the requirement that the fields on both ends of the original interval were not identified with one another, has required us to define the field as periodic on an interval with double the original length.

Let us consider the free Hamiltonian

$$H_0 = \frac{1}{2} \int_0^R dx [(\partial_t \phi)^2 + (\partial_x \phi)^2]. \quad (A3)$$

Imposing the properties of $\phi$ in (A1,A2), we can write the mode expansion

$$\phi(x, t) = \phi^0 + \Pi_0 \frac{t}{R} + \frac{i}{\sqrt{\pi}} \sum_{n \neq 0} \frac{1}{n} a_n \cos \frac{\pi n t}{R} e^{-i\pi nt/R} \quad (A4)$$

where the commutation relations of the modes read

$$[\phi^0, \Pi_0] = i, \quad [a_n, a_m] = n \delta_{n+m,0} \quad (A5)$$

with all others vanishing. In terms of these, the Hamiltonian reads

$$H_0 = \frac{\Pi_0^2}{2R} + \frac{\pi}{2R} \sum_{n \neq 0} a_n a_{-n}. \quad (A6)$$

Alternately, we can play another game and define a chiral left-moving boson as

$$\phi_L(t + x) = \phi^0_L + \Pi_0 \frac{t + x}{2R} + \frac{i}{4\pi} \sum_{n \neq 0} \frac{1}{n} a_n e^{i\pi n(t+x)/R} \quad (A7)$$

obeying the quasi-periodicity relation

$$\phi_L(x + 2R) = \phi_L(x) + \Pi_0. \quad (A8)$$

We can without contradiction do a nonlocal identification between our original bosonic field and this newly invented left-mover. Namely, we may at leisure impose the operator identity

$$\phi(x, t) = \phi_L(t + x) + \phi_L(t - x) + \phi^0 - 2\phi^0_L \quad (A9)$$

where the periodicity requirements of the original boson are automatically fulfilled given the quasi-periodicity of the chiral one. Note that under these circumstances, the Hamiltonian (A6) becomes

$$H_0 = \int_0^{2R} dx (\partial_x \phi_L)^2. \quad (A10)$$

Moreover, it is a trivial exercise to show that taking $\phi^0_L = \phi^0/2$ leads to consistent canonical commutation relations for the chiral field:

$$[\phi_L(x), \partial_{x'} \phi_L(x')] = \frac{i}{2} \sum_{n \in \mathbb{Z}} \delta(x - x' + 2Rn). \quad (A11)$$

Let us now move on to the model we are interested in. At the free fermion point, the boundary contributions to the Hamiltonian read

$$H_B = \tilde{\Delta}_l \cos \sqrt{\pi} \phi(0, t) + \tilde{\Delta}_r \cos (\sqrt{\pi} \phi(R, t) - \chi). \quad (A12)$$

To refermionize this at $\beta^2 = 4\pi$, we start by noting that the original boson at $x = 0, R$ is simply expressed in terms of the chiral boson:
\[ \phi(0, t) = 2\phi_L(t), \quad \phi(R, t) = 2\phi_L(R + t) - \Pi_0 \]  

Progress is now simply a matter of using the definition of the chiral fermion from our favourite bosonization/refermionization rules,

\[ \Psi = \frac{1}{\sqrt{4\pi a}} e^{-i\sqrt{4\pi} \phi_L}, \]  

to obtain the equivalent fermionic boundary Hamiltonian

\[ H_B = \frac{\Delta}{\sqrt{2}}[\Psi(0) + \Psi^\dagger(0)] + \frac{\Delta}{\sqrt{2}}[f\Psi(R)e^{i\chi} + f^\dagger\Psi^\dagger(R)e^{-i\chi}] \]  

where we have defined \( f = e^{-i\sqrt{\pi} \phi_L} \) and rescaled our boundary parameters as \( \Delta = \sqrt{\pi} \tilde{\Delta} \), setting the cut-off \( a \) equal to one. Note that we now face a linear fermionic term at \( x = 0 \), and that the behaviour of the operator \( \Pi_0 \), and thus that of \( f \), is still unspecified. To resolve this is simply a matter of noting that the operator \( \Pi_0 \) becomes quantized through the periodicity requirement that we imposed on the original theory. Writing the boundary contributions in terms of the left-mover, we get the \( \Pi_0 \) quantization relation and (being very careful with noncommuting operators) its fermionic implication

\[ \Pi_0 = \sqrt{\pi}n, \quad \Psi(x + 2R) = -\Psi(x) \]  

with \( n \in \mathbb{Z} \). These then tell us that the operator \( f \) in fact obeys the Majorana relations

\[ f = f^\dagger, \quad \{ f, f \} = 2, \quad \{ f, \Psi \} = 0. \]  

For aesthetic reasons, we would like our Hamiltonian to be quadratic in fermions, and somehow more symmetric with respect to the left and right components. In order to do this, we perform the simple transformation

\[ \Psi \to -a\Psi, \quad f \to i\ a\ b \]  

where \( a, b \) are two new Majorana fermions, living respectively on the left and right boundaries. One can check that all anticommutators are preserved under this transformation.

We thus finally obtain a better-shaped real-time action,

\[
S = \int dt \int_0^{2R} dx \left[ \frac{1}{2} \partial_i \Psi^\dagger(\partial_i - \partial_x)\Psi + \frac{\Delta}{\sqrt{2}}[f\Psi(R)e^{i\chi} + f^\dagger\Psi^\dagger(R)e^{-i\chi}] \right] + \\
+ \int dt \left[ \frac{i}{2} (a\partial_t a + b\partial_t b) + \frac{\Delta}{\sqrt{2}} a[\Psi(0) - \Psi^\dagger(0)] + \frac{\Delta}{\sqrt{2}} b[\Psi(R)e^{i\chi} - \Psi^\dagger(R)e^{-i\chi}] \right].
\]  

Our theory has thus boiled down to something rather obviously tractable, namely one that is quadratic in fermions, and which we can consequently solve exactly. Varying the action and eliminating the boundary fermions, we obtain two sets of boundary conditions at \( x = 0 \) and \( R \):

\[
\Psi(0^+, t) + \Psi^\dagger(0^+, t) = \Psi(0^-, t) + \Psi^\dagger(0^-, t)
\]  

\[
\partial_t \Psi(0^+, t) - \partial_t \Psi^\dagger(0^+, t) - \partial_t \Psi(0^-, t) + \partial_t \Psi^\dagger(0^-, t) = \\
= \frac{\Delta^2}{2} \left[ \Psi(0^+, t) - \Psi^\dagger(0^+, t) + \Psi(0^-, t) - \Psi^\dagger(0^-, t) \right]
\]  

\[
\Psi(R^+, t) + e^{-2i\chi}\Psi^\dagger(R^+, t) = \Psi(R^-, t) + e^{-2i\chi}\Psi^\dagger(R^-, t)
\]  

\[
\partial_t \Psi(R^+, t) - e^{-2i\chi}\partial_t \Psi^\dagger(R^+, t) - \partial_t \Psi(R^-, t) + e^{-2i\chi}\partial_t \Psi^\dagger(R^-, t) = \\
= \frac{\Delta^2}{2} \left[ \Psi(R^+, t) - e^{-2i\chi}\Psi^\dagger(R^+, t) + \Psi(R^-, t) - e^{-2i\chi}\Psi^\dagger(R^-, t) \right]
\]  

The fermion is a free left-mover in the bulk, with possible discontinuities at \( x = 0, R \). We choose the general mode expansions

\[
\Psi(x, t) = \sum_c c_k e^{-ik(x+t)} \quad 0 < x < R
\]  

\[
\Psi(x, t) = \sum_k d_k e^{-ik(x+t)} \quad R < x < 2R
\]
and substitute them back into (A23), keeping in mind the antiperiodicity of the fermions under \( x \rightarrow x + 2R \). This scheme is consistent provided the momenta \( k \) are quantized according to
\[
1 + W = 0, \quad W = 2 \left[ \frac{4(ik)^2 + \Delta^2 \Delta^2 \cos 2\chi}{2ik + \Delta^2} \right] e^{-2ikR} + \left[ \frac{2ik - \Delta^2}{2ik + \Delta^2} \right] e^{-4ikR} \tag{A25}
\]

The ground-state energy is then given by
\[
E_0 = \frac{1}{2} \sum_{k<0} k = \frac{1}{4\pi i} \int_C dk k \frac{dW}{1 + W} = -\frac{1}{4\pi i} \int_C dk \ln [1 + W]. \tag{A26}
\]

where the anti-clockwise contour \( C \) surrounds all the roots of our quantization condition (A25) lying on the negative half-line \( k < 0 \). This is illustrated in figure 8.

![FIG. 8. Contour of integration for performing the summation over occupied energy states when calculating the ground-state energy. The crosses represent allowed values of the quasimomentum, which are symmetrically distributed around \( k = 0 \).](image)

The contour \( C \) can be split in two different parts, \( C_{1,2} \), corresponding respectively to the upper and lower branches. The first integral (over \( C_1 \)) then converges after performing a Wick rotation \( \kappa = ik \), whereas the second integral (over \( C_2 \)) converges after extracting a multiplicative factor \( \frac{2ik - \Delta^2}{2ik + \Delta^2} \) \( e^{-4ikR} \) from within the argument of the logarithm while performing the Wick rotation \( \kappa = -ik \). Setting the nonuniversal term \( \int_0^\infty dk k \) to zero, we then obtain the final formula
\[
E_0 = -\frac{1}{4\pi} \int_0^\infty d\kappa \ln \left[ 1 + \frac{2 \kappa^2 + \Delta^2 \Delta^2 \cos 2\chi}{(\kappa + \Delta^2)(\kappa + \Delta^2)} e^{-\kappa R} + \frac{(\kappa - \Delta^2)(\kappa - \Delta^2)}{(\kappa + \Delta^2)(\kappa + \Delta^2)} e^{-2\kappa R} \right] + \epsilon_l + \epsilon_r \tag{A27}
\]

where \( \epsilon_{l,r} \) are the boundary intensive energies
\[
\epsilon_{l,r} = -\frac{1}{4\pi} \int_0^\infty dk \frac{4k \Delta^2_{l,r}}{4k^2 + \Delta^4_{l,r}}. \tag{A28}
\]

We have thus obtained the ground-state energy of the \( \beta^2 = 4\pi \) sine-Gordon model in a finite width \( R \), in the presence of two integrable boundary interactions. Although the integral is not expressible in terms of elementary functions, it gives us the dependence of \( E_0 \) in terms of the boundary parameters \( \Delta_l, \Delta_r \) and phase \( \chi \).

**APPENDIX B: BULK SINE-GORDON SCATTERING MATRICES**

In the bulk, the scattering of the soliton and antisoliton is described by the general processes
\[
A_1^l(\theta_1)A_1^l(\theta_2) = a(\theta_1 - \theta_2)A_1^l(\theta_2)A_1^l(\theta_1),
A_1^l(\theta_1)A_1^r(\theta_2) = a(\theta_1 - \theta_2)A_1^r(\theta_2)A_1^l(\theta_1),
A_1^r(\theta_1)A_1^l(\theta_2) = b(\theta_1 - \theta_2)A_1^l(\theta_2)A_1^r(\theta_1) + c(\theta_1 - \theta_2)A_1^l(\theta_2)A_1^l(\theta_1). \tag{B1}
\]

The amplitudes for these scattering processes are well-known [21]. When \( \beta^2 = \frac{2\pi}{N} \), \( \lambda \in \mathbb{N} \), the theory contains the following fundamental particles: the soliton, the antisoliton, and \( \lambda - 1 \) breathers of mass \( m_n = 2m \sin \frac{n\pi}{2\chi} \). The bulk scattering simplifies since the backscattering amplitude \( c(\theta) \) vanishes; the other two amplitudes read
At the reflectionless points \( \beta \) as backscattering amplitude. A useful integral representation for this is [23].

The bootstrap solution yields these amplitudes in terms of two parameters, the original Lagrangian was not specified in [12]. For now, we state the results in terms of

\[
\rho(\theta) = \frac{1}{\pi} \Gamma(\lambda) \Gamma(1 + i \frac{\lambda \theta}{\pi}) \frac{\Gamma(1 - \lambda - i \frac{\lambda \theta}{\pi})}{\Gamma((2l + 1)\lambda + i \frac{\lambda \theta}{\pi})} \prod_{i=1}^{\infty} \frac{F_i(\theta) F_i(i \pi - \theta)}{F_i(0) F_i(i \pi)},
\]

\[
F_i(\theta) = \frac{\Gamma(2l \lambda + i \frac{\lambda \theta}{\pi}) \Gamma(1 + 2(\lambda + i \frac{\lambda \theta}{\pi}))}{\Gamma((2l + 1)\lambda + i \frac{\lambda \theta}{\pi}) \Gamma((2l - 1)\lambda + i \frac{\lambda \theta}{\pi})}
\]

A useful integral representation for this is [23]:

\[
\rho(\theta) = \frac{1}{\sin(\lambda(\pi + i \theta))} \exp\left(i \int_{-\infty}^{\infty} \frac{dy}{2y} \frac{2 \lambda \theta y \sinh[(1 - \lambda) y]}{\sin \theta \cosh \lambda y} \right).
\]

At the reflectionless points \( \beta^2 = \frac{8 \pi}{\lambda^2}, \lambda \in \mathbb{N} \), the bulk scattering amplitudes simplify considerably. Namely, the backscattering amplitude \( c(\theta) \) vanishes identically, while the others simplify according to

\[
a(\theta) = -\prod_{j=1}^{\lambda} \cos\left(\frac{2j \pi}{\lambda} + i \frac{\theta}{2}\right), \quad b(\theta) = \frac{\sin(-i \lambda \theta)}{\sin(\lambda(\pi + i \theta))} a(\theta) = (-1)^{\lambda+1} a(\theta).
\]

These have to be supplemented with the scattering amplitudes for processes involving breathers, which are defined as

\[
A_1(\theta_1) B_1^*(\theta_2) = S^{(n)}(\theta_1 - \theta_2) B_1^*(\theta_2) A_1(\theta_1),
B_1(\theta_1) B_1^*(\theta_2) = S^{(n,m)}(\theta_1 - \theta_2) B_1^*(\theta_2) B_1(\theta_1),
\]

\[
S^{(n)}(\theta) = \sinh \theta + i \cos \left(\frac{\pi}{2\lambda} + \frac{n \theta}{2}\right) \prod_{l=1}^{n-1} \sin^2 \left(\frac{(n-2l)\pi}{4\lambda} - \frac{\pi}{4} + i \frac{\theta}{2}\right),
\]

\[
S^{(n,m)}(\theta) = \sinh \theta + i \cos \left(\frac{(n+m)\pi}{2\lambda}\right) \sin \theta + i \sin \left(\frac{(n-m)\pi}{2\lambda}\right) \times \prod_{l=1}^{m-1} \sin^2 \left(\frac{(m-n-2l)\pi}{2\lambda} + i \frac{\theta}{2}\right) \cosh \left(\frac{m-n-2l\pi}{2\lambda} - i \frac{\theta}{2}\right) + i \frac{\theta}{2},
\]

\[
S^{(n,m)}(\theta) = \exp\left(-i \int_{-\infty}^{\infty} \frac{dy}{2y} \frac{2 \lambda \theta y \sinh((1 - \lambda) y)}{\sinh \lambda y \sin \theta y} \right),
\]

\[
S^{(n,m)}(\theta) = \exp\left(i \int_{-\infty}^{\infty} \frac{dy}{2y} \frac{2 \lambda \theta y}{\sinh \lambda y} \left[\delta_{n,m} - 2 \frac{\cosh \theta y \sinh((1 - \lambda) y)}{\sinh \lambda y \sin \theta y} \right] \right),
\]

\[
n \geq m = 1, ..., \lambda - 1.
\]

**APPENDIX C: BOUNDARY SINE-GORDON SCATTERING MATRICES**

1. Solution to the boundary bootstrap

The amplitudes for the scattering of the soliton and antisoliton on the boundaries were computed in [12]. The bootstrap solution yields these amplitudes in terms of two parameters \( \eta, \vartheta \) whose relation to the parameters in the original Lagrangian was not specified in [12]. For now, we state the results in terms of \( \eta, \vartheta \). For general \( \lambda \), the amplitudes for soliton and antisoliton boundary scattering read

\[
P_{\pm}(\theta) = \cos(-i \lambda \theta) \cos \eta \cosh \vartheta \mp \sin(-i \lambda \theta) \sin \eta \sinh \vartheta \cos R(\theta),
\]

\[
Q_{\pm}(\theta) = -\frac{1}{2} \sin(-2i \lambda \theta) R(\theta),
\]

\[
R(\theta) = R_0(\theta) R_1(\theta).
\]
The first of these functions is given by

\[ R_0(\theta) = \frac{F_0(\theta)}{F_0(-\theta)}, \quad (C3) \]

where

\[ F_0(\theta) = \frac{\Gamma(1 + i \frac{2 \lambda \theta}{\pi}) \prod_{k=1}^{\infty} \Gamma(4k + 1 + i \frac{2 \lambda \theta}{\pi}) \Gamma((4k + 1) + i \frac{2 \lambda \theta}{\pi}) \Gamma(1 + (4k - 1) + i \frac{2 \lambda \theta}{\pi})}{\Gamma(\lambda + i \frac{2 \lambda \theta}{\pi}) \prod_{k=1}^{\infty} \Gamma((4k + 1) + i \frac{2 \lambda \theta}{\pi}) \Gamma(1 + (4k - 1) + i \frac{2 \lambda \theta}{\pi}) \Gamma(1 + 4k) \Gamma(4k)} . \quad (C4) \]

Its integral representation reads

\[ R_0(\theta) = \exp \left( -i \int_{-\infty}^{\infty} \frac{dy}{y} \sin \left( \frac{2 \lambda \theta y}{\pi} \right) \frac{\sinh((\lambda - 1)y/2) \sinh(3\lambda y/2)}{\sinh(y/2) \sinh(2\lambda y)} \right) . \quad (C5) \]

The second function reads

\[ R_1(\theta) = \frac{\sigma(\eta, \theta) \sigma(i \partial, \theta)}{\cos \eta} \frac{\sigma(x, \theta)}{\cosh \theta} \quad (C6) \]

where

\[ \sigma(x, \theta) = \frac{\Pi(x, -\theta + i \frac{\pi}{2}) \Pi(-x, -\theta + i \frac{\pi}{2}) \Pi(x, \theta - i \frac{\pi}{2}) \Pi(-x, \theta - i \frac{\pi}{2})}{\Pi^2(x, i \frac{\pi}{2}) \Pi^2(-x, -i \frac{\pi}{2})} \]

\[ \Pi(x, \theta) = \prod_{l=0}^{\infty} \frac{\Gamma \left( \frac{1}{2} + (2l + \frac{1}{2}) \lambda + \frac{\pi}{\theta} + i \frac{2 \lambda \theta}{\pi} \right) \Gamma \left( \frac{1}{2} + (2l + \frac{1}{2}) \lambda + \frac{\pi}{\theta} + i \frac{2 \lambda \theta}{\pi} \right)}{\Gamma \left( \frac{1}{2} + (2l + \frac{3}{2}) \lambda + \frac{\pi}{\theta} + i \frac{2 \lambda \theta}{\pi} \right) \Gamma \left( \frac{1}{2} + (2l + \frac{3}{2}) \lambda + \frac{\pi}{\theta} + i \frac{2 \lambda \theta}{\pi} \right)} . \quad (C7) \]

This obeys the property

\[ \sigma(x, \theta) \sigma(x, -\theta) = \frac{\cos^2 x}{\cos(x + i \lambda \theta) \cos(x - i \lambda \theta)} \quad (C8) \]

(correcting the equation in [12]). The integral representation for these is

\[ \sigma(x, \theta) = \exp \left( i \int_{-\infty}^{\infty} \frac{dy}{y} \sinh(\lambda y (1 + i \theta/\pi)) \frac{\sin(\lambda \theta y/\pi)}{\sinh y \sinh(\lambda y)} \right) . \quad (C9) \]

Again, at the reflectionless points \( \beta^2 = \frac{8 \pi}{\lambda + 1} \), \( \lambda N \), these simplify considerably:

\[ R_0(\theta) = \prod_{j=1}^{\lambda - 1} \cos \left( \frac{\pi}{\lambda} - \frac{j}{\lambda} \right) \prod_{j=1}^{\lambda} \cos \left( \frac{\pi}{\lambda} - \frac{j}{\lambda} \right) \cos \left( \frac{\pi}{\lambda} + \frac{j}{\lambda} \right) \cos \left( \frac{\pi}{\lambda} - \frac{j}{\lambda} \right) . \quad (C10) \]

In the massless limit, we get

\[ R_0 \left( \frac{\pi}{2} - \theta \right) = e^{-\frac{\pi}{2} \lambda (\lambda - 1)} \]

\[ \frac{\sigma(\eta, i \frac{\pi}{2} - \theta)}{\cos \eta} = 2 e^{-\lambda \theta - i \pi \lambda / 2} \]

\[ \frac{\sigma(i \partial, i \frac{\pi}{2} - \theta)}{\cosh \theta} = \frac{2 - \lambda + 1 e^{-\lambda \theta / 2} - \theta / 2}{\prod_{j=1}^{\lambda} \cosh \left( \frac{\pi}{\lambda} - \frac{j}{\lambda} \right) \cosh \left( \frac{\pi}{\lambda} + \frac{j}{\lambda} \right) \cosh \left( \frac{\pi}{\lambda} - \frac{j}{\lambda} \right) \cosh \left( \frac{\pi}{\lambda} + \frac{j}{\lambda} \right)} \]

\[ R \left( \frac{\pi}{2} - \theta \right) = \frac{4 e^{-\frac{\pi}{2} \lambda (\lambda - 1) - \lambda \theta - i \frac{3 \lambda}{2} \theta}}{\prod_{j=1}^{\lambda} e^{\theta} + e^{\frac{\pi}{\lambda} + i \frac{\pi}{\lambda} - \frac{3 \lambda}{2} \theta} + i \frac{\pi}{\lambda} + i \frac{\pi}{\lambda}} . \quad (C11) \]

The breathers also scatter off the boundary, according to

\[ B^\lambda_B(\theta) B = R_B^{(\alpha)}(\theta) B^\lambda_B(\theta - \theta) . \quad (C12) \]
Since the breathers are bound states of (multiple) solitons and antisolitons, their boundary scattering amplitudes can be worked out systematically. This procedure was carried out in [24], with the result

\[ R_B^{(n)}(\theta) = R_0^{(n)}(\theta)R_1^{(n)}(\theta), \]  

(C13)

where

\[ R_0^{(n)}(\theta) = -\cos(\frac{\pi}{4n} - i\frac{\theta}{2}) \cos(\frac{\pi}{4n} + \frac{\pi}{4} + i\frac{\theta}{2}) \sin(\frac{\pi}{4n} - i\frac{\theta}{2}) \prod_{l=1}^{n-1} \sin\left(\frac{l\pi}{4n} - i\theta\right) \cos^2\left(\frac{\pi}{4n} + \frac{\pi}{4} + i\frac{\theta}{2}\right) \]  

(C14)

and the function \( R_1^{(n)}(\theta) \) depends on the parity of \( n \), i.e.

\[ R_1^{(2n)}(\theta) = S^{(2n)}(\eta, \theta)S^{(2n)}(i\eta, \theta), \quad n = 1, 2, \ldots < \lambda/2, \]

\[ S^{(2n)}(x, \theta) = \prod_{l=1}^{n} \frac{\cos\left(\frac{x}{l} - \frac{(l-1)\pi}{\lambda}\right) + i\sin\theta \cos\left(\frac{x}{l} + \frac{(l-1)\pi}{\lambda}\right) + i\sinh\theta}{\cos\left(\frac{x}{l} - \frac{(l-1)\pi}{\lambda}\right) - i\sin\theta \cos\left(\frac{x}{l} + \frac{(l-1)\pi}{\lambda}\right) - i\sinh\theta} \]  

(C15)

or

\[ R_1^{(2n-1)}(\theta) = S^{(2n-1)}(\eta, \theta)S^{(2n-1)}(i\eta, \theta), \quad n = 1, 2, \ldots < (\lambda + 1)/2, \]

\[ S^{(2n-1)}(x, \theta) = \frac{\cos\left(\frac{x}{n} + \frac{i\pi}{4}\right) + i\sin\theta \cos\left(\frac{x}{n} + \frac{i\pi}{4}\right) + i\sinh\theta}{\cos\left(\frac{x}{n} - \frac{i\pi}{4}\right) - i\sin\theta \cos\left(\frac{x}{n} - \frac{i\pi}{4}\right) - i\sinh\theta} \]  

(C16)

The unified integral representation for this is worked out to be

\[ S^{(n)}(x, \theta) = \exp\left(i\pi - i \int_{-\infty}^{\infty} \frac{dy}{y} \sin\left(\frac{2\lambda y}{\pi}\right) \frac{\cosh\left(\frac{2\pi y}{\lambda}\right) \sinh ny}{\cosh \lambda y \sinh y}\right). \]  

(C17)

2. Parameter correspondence

A crucial ingredient missing in the formulas above is the exact relationship between the boundary parameters \( \xi, k \) coming out of the bootstrap, and the physical variables \( \Delta_{1,r}, \chi \) appearing in the boundary Hamiltonian we started from. This correspondence was worked out by A. Zamolodchikov [28] (see also the semiclassical treatment in [30]). For a problem where the bulk sine-Gordon action has a term \( 2\mu \cos \beta \phi \) and a boundary term \( 2\mu_B \cos \beta \phi / 2 \), one has

\[ \cosh\left(\frac{\beta^2}{8\pi}(\vartheta \pm i\eta)\right) = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\frac{\beta^2}{8}} e^{\pm i\chi}. \]  

(C18)

In the massless limit, we will in general thus get

\[ \vartheta = (\lambda + 1) \ln \left[2\frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin \frac{\pi}{\lambda + 1}}\right], \quad \eta = (\lambda + 1) \chi. \]  

(C19)

Now from the paper [18] we have that, in the massless limit, \( \vartheta - \vartheta_B = \theta' - \theta_B \) so \( T_B = \frac{1}{\lambda} e^{\theta_B} = m e^{\theta' / \lambda} \) (in the last formula, the factor \( \frac{1}{\lambda} \) comes from your normalizations in the massless limit, different from the ones in [18]). Here \( m \) is the soliton mass, related with the bare sine-Gordon parameter \( \mu \) by [16]

\[ \mu = \frac{\Gamma(1/\lambda + 1)}{\pi \Gamma(\lambda + 1)} \left[ m \sqrt{\frac{\pi \Gamma (1/2 + 1/2\lambda)}{2 \Gamma(1/2\lambda)}} \right]^{2\lambda/\lambda + 1} \]  

(C20)

Putting everything together just reproduces the relation between \( T_B \) and \( \mu_B \) mentioned in the text.

[1] J. Cardy, “Conformal invariance and surface critical behavior”, in Conformal Invariance and Applications to Statistical Mechanics, eds. C. Itzykson, H. Saleur and J. B. Zuber (World Scientific, 1988).
