THE FOURIER-MUKAI TRANSFORM IN K-THEORY

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Abstract. The Baum-Connes map for finitely generated free abelian groups is a K-theoretic analogue of the Fourier-Mukai transform from algebraic geometry. We describe this K-theoretic transform in the language of topological correspondences, and compute its action on K-theory (of tori) described geometrically in terms of Baum-Douglas cocycles, showing that it maps the class of a subtorus to the class of a suitably defined ‘dual torus.’ We deduce the Fourier-Mukai inversion formula. Combining Fourier-Mukai duality and spin duality for tori also results in a different KK-theoretic dual for a torus – its Baum-Connes dual. We use our methods to give a purely topological description of this Fourier-Mukai/Baum-Connes duality for free abelian groups, by a correspondence argument based on the Fourier-Mukai transform.

1. Introduction

Bott Periodicity in KK-theory for C*-algebras states that $C_0(\mathbb{R})$ is KK$_1$-equivalent to $\mathbb{C}$. The ingredients of one proof involve the operators $x$ and $\frac{d}{dx}$ that figure so prominently in the fundamentals of quantum mechanics. Multiplication by $x$ is a self-adjoint elliptic multiplier of $C_0(\mathbb{R})$ and determines a class $[x] \in KK_1(C_0(\mathbb{R}), C_0(\mathbb{R}))$, while $D := -i \frac{d}{dx}$ on the line determines a self-adjoint elliptic operator on the Hilbert space $L^2(\mathbb{R})$ and a class $[D] \in KK_1(C_0(\mathbb{R}), \mathbb{C})$ – the Dirac class of $\mathbb{R}$ with its standard K-orientation. Bott Periodicity follows from the equations

$$[x] \otimes C_0(\mathbb{R}) [D] = 1 \in KK_0(C_0(\mathbb{R}), C_0(\mathbb{R})),$$

in the relevant equivariant Kasparov groups. Moreover, Bott Periodicity

$$[x] \otimes C_0(\mathbb{R}) [D] = 1 \in KK_0(C_0(\mathbb{R}), C_0(\mathbb{R})),$$

remains true in this equivariant setting and implies that $C_0(\mathbb{R})$ is KK$_1^Z$-equivalent to $\mathbb{C}$. The mechanics of KK-theory then give that $A$ is KK$_1^Z$-equivalent to $C_0(\mathbb{R}) \otimes \mathbb{Z}$ for any $\mathbb{Z}$-C*-algebra $A$. Kasparov’s descent map

$$j_\mathbb{Z} : KK^Z(A, B) \to KK(A \rtimes \mathbb{Z}, B \rtimes \mathbb{Z})$$

when combined with $\mathbb{Z}$-equivariant Bott Periodicity then gives that $A \rtimes \mathbb{Z}$ is KK$_1$-equivalent to $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$ for any $A$. The second algebra is a very easy C*-algebra to understand, since $\mathbb{Z}$ acts...
properly and freely on \( \mathbb{R} \), and on \( X \times \mathbb{R} \), for any \( X \); for example if \( A = C_0(X) \) is commutative, then \( C_0(\mathbb{R}, A) \times \mathbb{Z} \) is (strongly Morita equivalent to) the mapping cylinder of the action, whose \( K \)-theory is of course easy to compute.

For any \( d \), the map
\[
\text{KK}_*^{\mathbb{Z}^d}(A, B) \to \text{KK}_*^{\mathbb{Z}^d}(C_0(\mathbb{R}^n) \otimes A, B)
\]
induced by external product with the class \([D] \in \text{KK}_{-d}(C_0(\mathbb{R}^n), C)\) of the Dirac operator on \( \mathbb{R}^d \) is a special case of the ‘Dirac map’ studied in \([3]\) for various classes of groups. In this note, we will show that the Dirac map for free abelian groups is a K-theoretic version of the Fourier-Mukai transform from algebraic geometry, and calculate its effect on K-homology of \( C^*(\mathbb{Z}^d) \cong C(\hat{\mathbb{Z}}^d) \) in geometric terms, using the theory of correspondences.

A topological correspondence between smooth manifolds \( X \) and \( Y \), a concept due to Connes and Skandalis \([3]\), is the content of a diagram
\[
X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y,
\]
where \( b \) is a smooth and proper map, \( f \) a K-oriented smooth map, and \( \xi \) a K-theory datum. A correspondence determines an element of
\[
\text{KK}_*(C_0(X), C_0(Y)),
\]
since \( b \) determines an element by ordinary functoriality \( b^* \in \text{KK}_0(C_0(M), C_0(X)) \), \( f : M \to Y \) a wrong way, or ‘shriek’ morphism \( f! \in \text{KK}_{\dim Y - \dim M}(C_0(M), C_0(Y)) \), and the K-theory datum can be integrated as a ‘twist,’ using the ring structure on topological K-theory.

The correspondence concept is very closely related to one in algebraic geometry, sometimes called a Fourier-Mukai transform. If \( X \) and \( Y \) are smooth projective varieties, then a suitable object \( \mathcal{E} \) in the derived category \( \mathcal{D}^b(X \times Y) \) of sheaves over \( X \times Y \), gives rise to a transformation
\[
\mathcal{D}^b(X) \xrightarrow{(Rp_X)^*} \mathcal{D}^b(X \times Y) \xrightarrow{\cdot \mathcal{F}} \mathcal{D}^b(X \times Y) \xrightarrow{(Rp_Y)_*} \mathcal{D}^b(Y).
\]
between derived categories of sheaves, where the maps \((Rp_X)^*\) the derived inverse image functor and \((Rp_Y)_*\) the derived direct image functor, for the coordinate projections \( p_X, p_Y \). For example, if \( X = Y \), the structure sheaf of the diagonal \( \Delta \subset X \times X \) can be used, and the induced map is the identity. Mukai proved that if \( T \) was an abelian variety, \( \hat{T} \) its dual, and \( \mathcal{F} \) the structure sheaf of sections of the Poincaré bundle \( \beta \) described above, then the resulting transform
\[
\mathcal{F} : \mathcal{D}^b(T) \to \mathcal{D}^b(\hat{T})
\]
is an isomorphism, with inverse the map induced by the obvious ‘dual’ Fourier-Mukai transform, obtained by flipping \( T \) and \( \hat{T} \). The isomorphism is called Fourier-Mukai duality. It is an instance of a \( T \)-duality in physics, relating two string theories with different space-time geometries.

Returning to the previous discussion, let \( \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d \), a torus, and \( \hat{\mathbb{T}}^d := \text{Hom}(\mathbb{Z}^d, \mathbb{T}) \) the Pontryagin dual of \( \mathbb{Z}^d \), another torus – the ‘dual torus.’ The algebras \( C_0(\mathbb{R}^d) \times \mathbb{Z}^d \) and \( C(\mathbb{T}^d) \) are strongly Morita equivalent. So applying the descent map to the Dirac class for \( \mathbb{R}^d \) gives a morphism
\[
j_{\mathbb{Z}^d}([D]) \in \text{KK}_{-d}(C_0(\mathbb{R}^d) \times \mathbb{Z}^d, C^*(\mathbb{Z}^d)) \cong \text{KK}_{-d}(C(\mathbb{T}^d), C^*(\mathbb{Z}^d)).
\]
Fourier transform \( C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^d) \) induces a map on KK and so we obtain a morphism
\[
[F] \in \text{KK}_{-d}(C(\mathbb{T}^d), C(\mathbb{Z}^d)).
\]
By \([3]\) and an inspection of the definitions, \([1.5]\) is equal to the class of the topological correspondence
\[
\mathbb{T}^d \xrightarrow{p_{\mathbb{T}^d}} (\mathbb{T}^d \times \mathbb{Z}^d, \mathbb{P}_d) \xrightarrow{p_{\mathbb{Z}^d}} \mathbb{Z}^n.
\]
where $\mathcal{P}_d$ is the Poincaré bundle over $\mathbb{T}^d \times \mathbb{Z}^d$, the complex line bundle over $\mathbb{T}^d \times \mathbb{Z}^d$ obtained by inducing a point $\chi \in \mathbb{Z}$ to a vector bundles $E_\chi$ over $\mathbb{T} \cong \mathbb{T} \times \{\chi\}$, and taking the union over $\chi$.

We will therefore refer to the correspondence (1.6) as the Fourier-Mukai correspondence, and denote by
\[ |F_d| \in KK_{-d}(C(\mathbb{T}^d), C(\mathbb{Z}^d)) \]
its class in $KK$ (the degree $-d$ is correct even in real $KK$-theory). The Fourier-Mukai correspondence induces a map, the ‘Fourier-Mukai transform’,
\[ F_d: K^* (\mathbb{T}^d) \to K^{*-d}(\mathbb{Z}^d), \]
shifting degrees by $-d$ (and likewise for $K$-homology), and as we will show, it has a very geometric description in terms of the canonical geometric (Baum-Douglas) cocycles generating $K$-theory for tori. If $T \subset \mathbb{T}^d$ is a $j$-dimensional torus subgroup of $T^d$, the inclusion $i: T \to T^d$ is $K$-orientable and defines a correspondence $\cdot \mapsto T \to T^d$, and corresponding class $[T]$ in $K^{d-j}(\mathbb{T}^d)$ (the Thom class of the normal bundle of $i$). Such a torus $T$ lifts to a linear subspace $L \subset \mathbb{R}^d$. Let $L^\perp$ denote the characters of $\mathbb{R}^d$ which vanish on $L$ and $\mathbb{Z}^d_\perp$ denote characters of $\mathbb{R}^d$ which vanish on $\mathbb{Z}^d$. Let $T^\perp \subset \mathbb{R}^d/\mathbb{Z}^d_\perp$ be the image of $L^\perp$ under the quotient map, the ‘dual’ (embedded) torus.

**Theorem 1.7.** If $F_d$ is the Fourier-Mukai transform in $K$-theory, $i: T \to T^d$ a $j$-dimensional subtorus, $T^\perp$ its dual, as above, then
\[ F_d([T]) = (-1)^{\frac{d(d+1)}{2}} \cdot [T^\perp] \in K_*(\mathbb{Z}^d), \]
with $[T]$ and $[T^\perp]$ defined above.

We now discuss the ‘Fourier-Mukai Inversion Formula.’

The ‘dual’ Fourier Mukai correspondence
\[ \mathbb{Z}^d \xleftarrow{pr_1} (\mathbb{Z}^d \times T^d, \mathcal{P}_d) \xrightarrow{pr_2} T^d, \]
defines an element $[F_d] \in KK_{-d}(C(\mathbb{Z}^d), C(\mathbb{T}^d))$. Composing in $KK$ the Fourier-Mukai correspondence and its dual results in a $-2d$-dimensional $KK$-morphism, while now taking into account Bott Periodicity, identifies under $KK_{-2d} \cong KK_0$ with a zero-dimensional morphism. The following theorem asserts then that composing the Fourier-Mukai morphism and its dual is ‘Bott Periodic’ to the identity.

**Theorem 1.8.** The Fourier-Mukai correspondence and its dual are inverses in complex $KK$-theory:
\[ [F_d] \otimes_{C(\mathbb{Z}^d)} \mathbb{C} = 1_{C(\mathbb{T}^d)} \in KK_0(C(\mathbb{T}^d), C(\mathbb{Z}^d)), \]
\[ [F_d] \otimes_{C(\mathbb{T}^d)} \mathbb{C} = 1_{C(\mathbb{Z}^d)} \in KK_0(C(\mathbb{Z}^d), C(\mathbb{T}^d)). \]

Thus, Bott Periodicity and complex coefficients $\mathbb{C}$ is absolutely necessary for Fourier-Mukai duality to hold in $KK$-theory; the ‘Inversion Formula’ given above is obviously not true in real $KK$-theory for dimension reasons.

In the last part of the paper, we observe that the Fourier-Mukai correspondence can be used to describe Baum-Connes duality for free abelian groups purely topologically (using correspondences.) Baum-Connes duality refers to the $KK$-duality for discrete groups $G$ with finite $BG$, the natural system of isomorphisms for all $\mathbb{C}^*$-algebras $A, B$,
\[ KK_*(C(BG) \otimes A, B) \cong KK_*(A, C^*_r(G) \otimes B). \]
The unit of the duality for any $G$ is the class
\[ [\mathcal{P}] \in KK_0(\mathbb{C}, C^*_r(G) \otimes C(BG)). \]
The co-unit in $\text{KK}_0(C^*_r(G) \otimes C(BG), \mathbb{C})$ for this KK-duality can be in principal described using the Dirac-dual-Dirac method. For free abelian groups all of this can be carried out quite explicitly. The co-unit is a $K$-homology class built from the Dirac-Schrödinger operator(s) $x \pm \frac{d}{dx}$. See [12].

Using Fourier-transform, duality for free abelian groups has the form

$$\text{KK}^*(\mathbb{C}(\mathbb{T}^d) \otimes A, B) \cong \text{KK}^*(A, \mathbb{C}(\mathbb{Z}^d) \otimes B),$$

and therefore is a duality between the space $\mathbb{T}^d$ and the space $\mathbb{Z}^d$. Hence it is a duality which can be in principal expressed purely in terms of correspondences, and we show that in fact the Poincaré dual of the Fourier-Mukai correspondence, the class of the correspondence

$$\mathbb{T}^d \times \mathbb{Z}^d \xrightarrow{\text{id}} (\mathbb{T}^d \times \mathbb{Z}^d, \mathcal{P}_d) \to \cdot$$

represents the co-unit of the duality with $[\mathcal{P}_d]$ as unit.

**Theorem 1.9.** The class $[\mathcal{P}] \in \text{KK}_0(\mathbb{C}(\mathbb{T}^d \times \mathbb{Z}^d), \mathbb{C})$ is the unit of a duality in analytic $KK$-theory between $\mathbb{C}(\mathbb{T}^d)$ and $\mathbb{C}(\mathbb{Z}^d)$. The co-unit is the class

$$\Delta_{\text{FM}} = [\partial_d \cdot \mathcal{P}_d] \in \text{KK}_0(\mathbb{C}(\mathbb{T}^d \times \mathbb{Z}^d), \mathbb{C})$$

of the Dirac-Dolbeault operator $\partial_d$ on $\mathbb{T}^d \times \mathbb{Z}^d$, twisted by the Poincaré line bundle $\mathcal{P}_d$.

The theorem would seem to amount to be a form of index theorem for Dirac-Schrödinger operators, as we explain briefly at the end of the paper.

2. **Topological correspondences**

The theory of topological correspondences goes back to Connes and Skandalis and was developed further in [8], in particular, shown to form a category naturally isomorphic to Kasparov’s $KK$, when the arguments are smooth manifolds. The fact that the topological correspondences (of [8]) map to the analytically defined ones of [3] is a very general way of stating the Index Theorem of Atiyah and Singer.

In this paper we will be operating in the environment of of this ‘topological’, or correspondence picture, of KK-theory’ and so give a quick summary of it before proceeding to the application.

**Definition 2.1.** A smooth correspondence (or just correspondence) between smooth manifolds $X$ and $Y$ is a quadruple

$$\Phi = (X, b : (M, \xi) \xrightarrow{\text{b}} Y),$$

where

- $M$ is a smooth manifold,
- $\xi \in \text{RK}_X^*(M)$ is a representable $K$-theory class with $X$-compact support,
- $b : M \to X$ is a smooth map, and
- $f : M \to Y$ is smooth and $K$-oriented: that is, $T_f := TM \oplus f^*(TY)$ is a $K$-oriented (real) vector bundle.

We sometimes refer to $f$ as the forward map and $b$ the backward map of $\Phi$.

The sum of two correspondences is given by their disjoint union:

$$[X \xrightarrow{b_0} (M_0, \xi_0) \xrightarrow{f_0} Y] + [X \xrightarrow{b_1} (M_1, \xi_1) \xrightarrow{f_1} Y] := [X \xrightarrow{b_0 \sqcup b_1} (M_0 \sqcup M_1, \xi_0 \sqcup \xi_1) \xrightarrow{f_0 \sqcup f_1} Y],$$

where $f_0 \sqcup f_1$ is $K$-oriented by taking the direct sum of the two $K$-orientations. The degree of a correspondence is defined to be $\deg \xi - \dim M$ if this locally constant function is constant; any correspondence is the sum of correspondences with well defined degrees.
A proper map \( b: Y \to X \) can be represented as the correspondence

\[
X \xleftarrow{b} (Y, 1) \xrightarrow{\text{id}_Y} Y,
\]

since if \( b \) is proper, any representable K-theory class has \( X \)-compact support. If \( f: X \to Y \) is any smooth, K-oriented map it fits into the correspondence

\[
X \xrightarrow{\text{id}_X} (X, 1) \xrightarrow{f} Y.
\]

If \( \Psi = (M, \xi, b, f) \) is a correspondence between \( X \) and \( Y \) then we define \( -\Psi := (M, \xi, b, -f) \) where \(-f\) means the same map as \( f \) but with the opposite K-orientation.

Equivalence of correspondences is generated by three steps:

- Isomorphism,
- Thom modification, and
- Bordism.

**Definition 2.2.** Two correspondences

\[
(\ref{eq:correspondence})
\]

\[
X \xleftarrow{b_0} (M_0, \xi_0) \xrightarrow{f_0} Y \quad \text{and} \quad X \xleftarrow{b_1} (M_1, \xi_1) \xrightarrow{f_1} Y
\]

are isomorphic if there is a diffeomorphism \( \varphi: M_0 \to M_1 \) which fits into \( \ref{eq:correspondence} \) making it commute such that

- \( \xi_0 = \varphi^*(\xi_1) \)
- The K-orientations on \( \varphi^*(T_{f_1}) \) and on \( T_{f_0} \) agree after identifying these two vector bundles by the bundle isomorphism \( \varphi^*(TM_1) \cong TM_0 \) induced by the derivative of \( \varphi \).

A correspondence with boundary, or \( \partial \)-correspondence, is a correspondence

\[
X \xleftarrow{\partial} (M, \xi) \xrightarrow{f} Y,
\]

where \( M \) is a manifold with boundary. Suppose that \( \partial M = \partial_0 M \sqcup \partial_1 M \). Give the inward facing normal bundle at \( \partial_0 M \) the positive K-orientation and the inward facing normal bundle at \( \partial_1 M \) the negative K-orientation. Such a \( \partial \)-correspondence \( \Phi \) induces a correspondence on its boundary as

\[
\partial_i \Phi := \left[ X \xleftarrow{\partial_i} (\partial_i M, \xi_0 \partial_i M) \xrightarrow{f|_{\partial_i M}} Y \right].
\]

Here we give \( f|_{\partial_i M} \) the K-orientation coming from the 2-out-of-3 Lemma.

**Definition 2.4.** Two smooth correspondences \( \Phi_0 \) and \( \Phi_1 \) are called bordant if there is a \( \partial \)-correspondence \( \Phi \) such that

\[
(-1)^i \Phi_i = \partial_i \Phi \quad \text{for} \quad i = 0, 1.
\]

If \( \Phi \) and \( \Psi \) are bordant correspondences we write \( \Phi \sim_b \Psi \).

**Definition 2.5.** If \( V \) is a (real) K-oriented vector bundle over \( M \), and \( \tau_V^Y : \text{RK}_X(M) \to \text{RK}_X(V) \) is the Thom Isomorphism, then we define the Thom modification of

\[
X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y
\]

along \( V \) to be the correspondence

\[
X \xleftarrow{b \circ \pi_V} (V, \tau_V^Y(\xi)) \xrightarrow{f \circ \pi_V} Y.
\]

Here we are K-orienting \( f \circ \pi_V \) as the composition of K-oriented maps.

If \( \Phi \) is a Thom modification of \( \Psi \) along some vector bundle, then we write \( \Psi \sim_{T_m} \Phi \).
Putting isomorphism, Thom modification, and bordism together yields equivalence of correspondences.

**Definition 2.6.** For smooth manifolds \( X \) and \( Y \), we define \( \text{KK}_*(X, Y) \) to be the set of equivalence classes of correspondences from \( X \) to \( Y \), where equivalence of correspondences to be the equivalence relation generated by isomorphism, Thom modification, and bordism.

Equivalence preserves the degree and sum of correspondence, so \( \text{KK}_*(X, Y) \) is a graded monoid. It is in fact a group. For any correspondence \( \Phi = (M, \xi, b, f) \) from \( X \) to \( Y \), \( \Phi \cup -\Phi \) is bordant to the zero correspondence, using the \( \partial \)-correspondence

\[
\Phi^I = [X \xleftarrow{b \circ \text{pr}_1} (M \times I, \text{pr}_1^*\xi) \xrightarrow{f \circ \text{pr}_1} Y],
\]

where \( I = [0, 1] \) is the unit interval K-oriented in the obvious way, and \( f \circ \text{pr}_1 \) is the composition of K-oriented maps.

**Example 2.7.** Let \( \Phi = (M, \xi, b, f) \) be a correspondence from \( X \) to \( Y \). Let \( N \subset M \) be an open subset and suppose there is some \( \eta \in \text{RK}_X^*(N) \) which maps to \( \xi \) under the map \( \text{RK}_X^*(N) \rightarrow \text{RK}_X^*(M) \) induced by the open inclusion.

Then

\[
[X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y] \sim_b [X \xleftarrow{b \cup \infty} (N, \eta) \xrightarrow{f|_N} Y].
\]

Indeed, for out manifold with boundary we take

\[
\tilde{M} = N \times \{0\} \cup M \times (0, 1] \subset M \times [0, 1];
\]

this is an open subset of \( M \times [0, 1] \) whence is itself a smooth manifold. The forward and backward maps, \( \tilde{f} \) and \( \tilde{b} \) respectively, are defined via the commuting diagram

\[
\begin{array}{ccc}
M \times [0, 1] & \xrightarrow{\tilde{b}} & \tilde{M} \\
\downarrow{\text{bo} \circ \text{pr}_1} & & \downarrow{f \circ \text{pr}_1} \\
X & \xleftarrow{b} & Y \\
\end{array}
\]

Since the inclusion \( \tilde{M} \rightarrow M \times [0, 1] \) is a local diffeomorphism, it is canonically K-oriented, which K-orients \( \tilde{f} \). It remains to specify the K-theory data. The map \( \text{pr}_1 : N \times [0, 1] \rightarrow N \) is proper, hence \( \text{pr}_1^* (\eta) \in \text{RK}_X^*(N \times [0, 1]) \). Since \( N \times [0, 1] \) is open in \( M \), it extends to a class \( \xi \in \text{RK}_X^*(M) \) with the desired properties. Thus

\[
[X \xleftarrow{\tilde{b}} ([\tilde{M}, \xi]) \xrightarrow{\tilde{f}} Y]
\]

is the required \( \partial \)-correspondence.

**Definition 2.8.** Suppose that \( \Phi = (M, \xi, b_M, f_M) \) and \( \Psi = (N, \eta, b_N, f_N) \) are transverse if the map

\[
df_M - db_N : T_m M \oplus T_n N \rightarrow T_{f_M(m)} Y
\]

is surjective for all \((m, n) \in M \times Y \) \( N := \{(x, y) \in M \times N : f_M(x) = b_N(y)\} \).

Transversality ensures that the fibered product \( M \times_Y N \) is a smooth manifold. Let \( \text{pr}_M : M \times_Y N \rightarrow M \) (resp. \( \text{pr}_N \)) is the projection onto \( M \) (resp. \( N \)).

**Definition 2.9.** If \( \Phi = (M, \xi, b_M, f_M) \) and \( \Psi = (N, \eta, b_N, f_N) \) are transverse correspondences from \( X \) to \( Y \) and from \( Y \) to \( Z \) respectively, then their **intersection product** is the correspondence

\[
\Phi \otimes \Psi := [X \xleftarrow{b_M \circ \text{pr}_M} (M \times_Y N, \text{pr}_M^* (\xi) \circ \text{pr}_N^* (\eta)) \xrightarrow{f_N \circ \text{pr}_N} Z] \in \text{KK}_*(X, Z),
\]

where \( f_N \circ \text{pr}_N \) is given the K-orientation discussed below.
We endow $f_N \circ \text{pr}_N$ with the following K-orientation. Since $f_N$ is K-oriented, it is sufficient to K-orient $\text{pr}_N$ as the composition of K-oriented maps is K-oriented. To do this, first observe that transversality implies that we have an exact sequence

$$0 \to T(M \times Y N) \to T(M \times N)|_{M \times Y N} \xrightarrow{df_M - db_N} (f_M \circ \text{pr}_M)^*TY \to 0.$$  

In particular, we have an isomorphism $T(M \times Y N) \oplus (f_M \circ \text{pr}_M)^*TY \cong \text{pr}_M^*TM \oplus \text{pr}_N^*TN$. This implies the equality

$$T(M \times Y N) \oplus \text{pr}_N^*(TN) \oplus \text{pr}_M^*(TM \oplus f_M^*TY)$$

$$\cong \text{pr}_M^*(TM \oplus TM) \oplus \text{pr}_N^*(TN \oplus TN)$$

which, since $f_M$ is K-oriented, gives a K-orientation to $\text{pr}_N$ by the 2 out of 3 lemma. Two different choice of splitting will yield isomorphisms which are connected by a path, whence give the same K-orientation.

**Remark 2.10.** The ring structure on topological K-theory, from the correspondence point of view, can be described as follows. Let $\delta: X \to X \times X$ be the diagonal map. Suppose $i: M \to X$ and $i': M' \to X$ are two closed submanifolds with K-oriented normal bundles of dimensions $d, d'$, defining correspondences and classes

$$[i!]= \begin{pmatrix} * \leftarrow M \overset{i}{\to} X \end{pmatrix} \in \text{KK}_d(*, X), \quad [i'!]= \begin{pmatrix} * \leftarrow M' \overset{i'}{\to} X \end{pmatrix} \in \text{KK}_{d'}(*, X)$$

Then if $i$ and $i'$ are transverse, then the ring product $[i!] \wedge [i'!] \in \text{K}^{-d-d'}(X)$ is represented by the correspondence $* \leftarrow M \cap M' \overset{i''}{\to} X$, with $i''$ the inclusion of the smooth and K-oriented manifold $M \cap M'$.

**Definition 2.11.** Let $\Phi = (M, \xi, b_M, f_M) \in \text{KK}_* (X_1, Y_1)$ and $\Psi = (N, \eta, b_N, f_N) \in \text{KK}_* (X_2, Y_2)$. The exterior product of $\Phi$ and $\Psi$ is defined as

$$\Phi \times \Psi := \left[ X_1 \times X_2 \overset{b_M \times b_N}{\to} (M \times N), \text{pr}_M^*(\xi) \otimes \text{pr}_N^*(\eta) \right] \xrightarrow{f_M \times f_N} Y_1 \times Y_2 \in \text{KK}_* (X_1 \times X_2, Y_1 \times Y_2),$$

where $f_M \times f_N$ is given the product K-orientation.

Combining the intersection product with the exterior product we obtain natural cup-cap products “over” any auxiliary space $U$ by:

$$(2.12) \quad \hat{\circ}_U : \text{KK}_* (X_1, Y_1 \times U) \times \text{KK}_* (U \times X_2, Y_2) \to \text{KK}_* (X_1 \times X_2, Y_1 \times Y_2)$$

$$(\Phi, \Psi) \mapsto (\Phi \times \text{id}_{X_2}) \otimes (\text{id}_{Y_1} \times \Psi).$$

In the rest of the paper, we will abbreviate notation for correspondences involving point as targets or source: a correspondence $\cdot \leftarrow (M, \xi) \overset{f}{\to} Yf$ will be simply denoted $(M, \xi) \overset{f}{\to} Y$, and similarly, a correspondence $X \overset{b}{\to} (M, \xi) \to \cdot$ will be denoted $X \overset{b}{\to} (M, \xi)$.

### 3. The Fourier-Mukai Transform in K-theory

We start with some remarks on the well-known K-theory of tori. The K-theory of the torus $T^d$ can be identified with an exterior algebra

$$K^* (T^d) \cong \Lambda^*_\mathbb{Z} (\mathbb{Z} \{x_1, \ldots, x_n\})$$

of a free abelian group $\mathbb{Z} \{x_1, \ldots, x_n\}$ on $d$ generators. The quickest way to see this is via the Künneth Theorem, which implies that

$$K^* (T^d) \cong K^* (T) \hat{\otimes}_\mathbb{Z} K^* (T) \hat{\otimes}_\mathbb{Z} \cdots \hat{\otimes}_\mathbb{Z} K^* (T),$$

with $\hat{\otimes}$ the graded tensor product of groups. As $K^* (T)$ has two generators $[1] \in K^0 (T)$ and $[u] \in K^{-1} (T)$. The external products putting $|u|$ in the $k$th factor and $|1|$’s otherwise
where we may for instance interpret the ring product
\[ x_i \wedge x_j \in K^{-r}(\mathbb{T}^d), \quad i_1 < \cdots < r, \]
for \( K^*(\mathbb{T}^d) \), and as \( x_i \wedge x_j = -x_j \wedge x_i \), this describes \( K^*(\mathbb{T}^d) \) as an exterior algebra.

We start by describing this K-theory ring in terms of correspondences.

In terms of correspondences, the unitary \( u(z) = z \) on \( \mathbb{T} \) represents the ‘Bott class’ of the circle, which is the class of the 1-point correspondence \( * \to \mathbb{T} \), by including, say, the point \( 1 \in \mathbb{T} \). So the \( x_k \) are represented by the compositions

\[
\begin{array}{ccc}
\pi_k^{-1}(1) & \xleftarrow{i_k} & \mathbb{T}^d \\
\downarrow & & \downarrow \\
* & \xrightarrow{i} & T \\
\uparrow & & \uparrow \\
T & \xrightarrow{i} & \mathbb{T}^d
\end{array}
\]

The subtori \( T_k := \pi_k^{-1}(1), k = 1, \ldots, d \) are hyperplanes in \( \mathbb{T}^d \) and the conclusion is that these hyperplanes represent the generators in the sense that

\[ x_k = \left[ \mathbb{T}^{d-1} \xrightarrow{i_k} \mathbb{T}^d \right] \in \text{KK}_{1}(\mathbb{T}, \mathbb{T}^d), \]

where \( i_k : \mathbb{T}^{d-1} \to \mathbb{T}^d \) the inclusion. The \( x_k \)’s generate \( K^*(\mathbb{T}^d) \) as a ring. However, the ring product \( \wedge \) on \( K^*(\mathbb{T}^d) \) corresponds to intersection of cocycles as explained in Remark 2.10. So we may for instance interpret the ring product \( x_r \wedge x_s \) geometrically by

\[ x_r \wedge x_s = \left[ T_r \cap T_s \xrightarrow{i_{r,s}} \mathbb{T}^d \right] \]

with \( i_{r,s} \) the inclusion, provided that \( r \neq s \), in which case \( r \) and \( s \) are clearly transverse, and \( T_r \cap T_s \) is now a co-dimension 2 subtorus, giving a 2-dimensional correspondence and K-theory class. One may iterate this construction in the obvious way: if \( T_{k_1}, \ldots, T_{k_r} \), are any \( r \) distinct coordinate hyperplanes in \( \mathbb{T}^d \), and if we partition \( \{ k_1, \ldots, k_r \} \) into two sets \( I \) and \( J \), then the inclusion of \( \cap_{k_i \in I} T_{k_i} \) in \( \mathbb{T}^d \) is transverse to the inclusion of \( \cap_{k_i \in J} T_{k_i} \) in \( \mathbb{T}^d \), provided that \( r \leq d \), which implies by transversality that the correspondence

\[ x_{k_1} \wedge \cdots \wedge x_{k_r} = (-1)^{\frac{(k+1)(k+2)}{2}} \left[ * \xleftarrow{i} \mathbb{T}^{d-1} \xrightarrow{i_{k}} \mathbb{T}^d \right] \]

equals the wedge \( x_k := x_{k_1} \wedge \cdots \wedge x_{k_r} \).

By an oriented subgroup of \( \mathbb{T}^d \) we will mean a Lie embedding

\[ i : T \to \mathbb{T}^d \]

of a closed subgroup of \( \mathbb{T}^d \), together with an orientation, whence K-orientation, on \( T \). Let \( k = \dim(T) \). We denote by \( [T]_* \in \text{KK}_{-k}(\mathbb{T}^d, \cdot) \) and \( [T]_! \in \text{KK}_{d-k}(: \mathbb{T}^d) \) respectively the classes of the correspondences

\[ \left[ \mathbb{T}^d \xleftarrow{i} T \right], \quad \left[ T \xrightarrow{i} \mathbb{T}^d \right]. \]

Now if \( i' : T' \to T^d \) is another (oriented) subtorus such that \( i, i' \) are transverse, then the composition diagram
describes the K-theory K-homology pairing between $[T]^*$ and $[T]_*$ in terms of a correspondence from a point to a point:

$$\langle [T], [T]^* \rangle = [T \cap T' \to \cdot] \in KK(\cdot, \cdot) \cong \mathbb{Z}.$$ 

Now any torus of dimension $> 0$ has exactly two K-orientations, corresponding to its two orientations, and with either of them, it is a boundary. Since every connected component of $T \cap T'$ is a torus, it is a boundary, the correspondence $\cdot \leftarrow T \cap T' \to \cdot$ is a boundary, and

$$\langle [T], [T'] \rangle = 0,$$

unless $T$ and $T'$ have exactly complementary dimension, which implies that $T \cap T'$ is zero-dimensional. To summarize

**Proposition 3.1.** If $T, T'$ are oriented closed subgroups of $T^d$, of dimensions $k, k'$, then the K-theory-K-homology pairing $\langle [T], [T'] \rangle$ between $[T]$ and $[T']$ is zero unless $k + k' = d$, and in this case,

$$\langle [T], [T'] \rangle = |T \cap T'|,$$

where $|T \cap T'|$ is the number of points in the intersection.

The Proposition together with an obvious guess supplies a natural dual basis is to the $x_i$’s, in the sense of the K-theory-K-homology pairing. Let

$$y_k := \left[ T^d \leftarrow T \right] \in KK_{-1} (T^d, \cdot).$$

Then computing $\langle x_k, y_k \rangle$ by transversality gives immediately that $\langle x_k, y_j \rangle = \delta_{ij}$, so that $y_1, \ldots, y_d$ is the dual basis.

We may thus identify the K-homology with the abelian group $K_* (T^d) \cong \Lambda^\infty (\mathbb{Z} \{ y_1, \ldots, y_n \})$.

The dual torus of $\hat{T}^d$ is by definition $\hat{\mathbb{Z}}^d$, the Pontryagin dual of $\mathbb{Z}^d$.

**Definition 3.2.** The Poincaré bundle $P_d$ is the complex line bundle over $T^d \times \hat{\mathbb{Z}}^d$ given by

$$P_d = \mathbb{R}^d \times \hat{\mathbb{Z}}^d \times \mathbb{C} / \sim$$

where $\sim$ is

$$\langle x, \chi, \lambda \rangle \sim \langle x + n, \chi(n) \lambda \rangle, \quad \text{for } n \in \mathbb{Z}^d,$$

and the bundle projection is induced by the coordinate projection $\mathbb{R}^d \times \hat{\mathbb{Z}}^d \times \mathbb{C} \to \mathbb{R}^d \times \hat{\mathbb{Z}}^d$.

We endow $T^d$ and $\hat{\mathbb{Z}}^d$ with their product K-orientations. The map $pr_{\hat{\mathbb{Z}}^d} : T^d \times \hat{\mathbb{Z}}^d \to \hat{\mathbb{Z}}^d$ is K-oriented by the K-orientation on $\mathbb{T}^d$.

**Definition 3.4.** The *Fourier-Mukai correspondence* is the correspondence

$$\mathbb{T}^d \xleftarrow{pr_1} (\mathbb{T}^d \times \hat{\mathbb{Z}}^d, P_d) \xrightarrow{pr_2} \hat{\mathbb{Z}}^d,$$

where $P_d$ is the class of the Poincaré line bundle.

We set

$$[\mathcal{F}_d] \in KK_{-d} (\mathbb{T}^d, \hat{\mathbb{Z}}^d)$$
its class in topological KK-theory.

The main purpose of this note is to compute the action of the Fourier-Mukai correspondence on geometric cycles and co-cycles parameterized by oriented torus subgroups.

Suppose \( \hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}^d \) is a \( k \)-dimensional oriented torus subgroup, defining a class \( [\hat{\mathbb{Z}}]_* \in \text{KK}_{-k}(\hat{\mathbb{Z}}^d, \ast) \). Let

\[
\hat{\varphi} : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}^d
\]

be the inclusion. By Pontryagin duality, there is a unique morphism \( \varphi : \mathbb{Z}^d \to \mathbb{Z}^k \) whose Pontryagin dual is \( \hat{\varphi} \). The morphism \( \varphi \) extends to an \( \mathbb{R} \)-linear map \( \varphi : \mathbb{R}^d \to \mathbb{R}^k \) which preserves the integer lattice, and then descends to a map \( \tilde{\varphi} : T^d \to T^k \). Its kernel is a \( d-k \)-dimensional subtorus of \( T^d \). The exact sequence of groups

\[
0 \to \ker(\tilde{\varphi}) \to T^d \to T^k \to 0
\]

differentiates to an exact sequence of vector bundles

\[
(3.5) \quad 0 \to T(\ker(\tilde{\varphi})) \to T(T^d)|_{\ker(\tilde{\varphi})} \xrightarrow{D\tilde{\varphi}} \tilde{\varphi}^* (T(T^k)) \to 0.
\]

By the 2-out-of-3 Lemma we obtain a canonical K-orientation on the subtorus \( \ker(\varphi) \subset T^d \).

We denote by \( [\hat{\mathbb{Z}}]_* \in \text{KK}_{-k}(\hat{\mathbb{Z}}^d, \ast) \), \( [\ker(\hat{\mathbb{Z}})]_* \in \text{KK}_{d-k}(T^d, \ast) \), the classes of the Baum-Douglas cycles determined by \( \hat{\mathbb{Z}} \) and \( \ker(\hat{\mathbb{Z}}) \), as above.

**Theorem 3.6.** In the above notation,

\[
[\mathcal{F}_d] \otimes_{\hat{\mathbb{Z}}^d} [\hat{\mathbb{Z}}] = (-1)^{dk+k(k-1)}[\ker(\hat{\mathbb{Z}})],
\]

where \( [\mathcal{F}] \in \text{KK}_{-d}(\mathbb{T}^d, \hat{\mathbb{Z}}^d) \) is the class of the Fourier-Mukai correspondence.

The proof proceeds by several Lemmas.

**Lemma 3.8.** The map \( e_1 : K^0((0,1)^2) \to K^0(\mathbb{T} \times \hat{\mathbb{Z}}) \) induced by the embedding

\[
e : (0,1)^2 \to \mathbb{T} \times \hat{\mathbb{Z}}
\]

\[
(x,y) \mapsto (-e^x, -e^y),
\]

where \( e^x := \exp(2\pi ix) \) and \( e_y(n) := \exp(2\pi i n y) \), maps the Bott class \( \beta \in K^0((0,1)^2) \) to \( P_1 - 1 \in K^0(\mathbb{T} \times \hat{\mathbb{Z}}) \).

**Proof.** We first note that, with \( ev_1 \) the diffeomorphism \( ev_1(\chi) := \chi(1) \) for \( \chi \in \hat{\mathbb{Z}} \), we have a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathbb{Z}} & \xrightarrow{ev_1} & \mathbb{T} \\
\downarrow{e^x} & & \downarrow{e^x} \\
(0,1) & & (0,1)
\end{array}
\]

which yields a commutative diagram

\[
\begin{array}{ccc}
K^{-1}(\mathbb{T}) & \xrightarrow{ev_1^*} & K^{-1}(\hat{\mathbb{Z}}) \\
\downarrow{(-e^x)_1} & & \downarrow{(-e^x)_1} \\
K^0((0,1)^2) & & K^0((0,1)^2).
\end{array}
\]
Since \((e^x)_1(\beta) = [z] \in K^{-1}(\mathbb{Z})\) and \(\text{ev}_1^*([z]) = [\chi] \in K^{-1}(\hat{\mathbb{Z}})\), where \(\beta\) is the Bott class and \([\chi]\) denotes the class of the unitary \(\chi(1)\), it follows that \((-e^x)_1(\beta) = [\chi]\). Thus, in order to prove the claim it is sufficient to show \((-e^x, \text{id})_1([\chi]) = P_1 - 1\), because the following diagram commutes

\[
\begin{array}{ccc}
(0,1) \times \hat{\mathbb{Z}} & \xrightarrow{(-e^x, \text{id})} & T \times \hat{\mathbb{Z}} \\
(id, -e^x) & & \downarrow (-e^x, -e^x) \\
(0,1)^2 & & 
\end{array}
\]

We identify the 1-point compactification of \((0,1) \times \hat{\mathbb{Z}}\) with the space \((T \times \hat{\mathbb{Z}})/\{-1\} \times \hat{\mathbb{Z}}\). It follows then that the map \((-e^x, \text{id})_1\) is the restriction of the quotient map

\[q : T \times \hat{\mathbb{Z}} \to (T \times \hat{\mathbb{Z}})/\{-1\} \times \hat{\mathbb{Z}}\]

to the kernel of the augmentation map \(\{\infty\} \to [(0,1) \times \hat{\mathbb{Z}}]^+\). The class of \([\chi] \in K^0((0,1) \times \hat{\mathbb{Z}})\) is represented in \(K^0((0,1) \times \hat{\mathbb{Z}})^+\) by the class \([E_\chi] - [1]\), where \(E_\chi\) is the trivial line bundle modulo the relation

\[(0, \chi, \lambda) \sim (1, \chi, \chi(1)\lambda)\]

It is easy to see that \(q^*(E_\chi) = [(0,1) \times \hat{\mathbb{Z}}] \times \mathbb{C}/\sim\), where \(\sim\) is the relation in \(\mathbb{R}^3\), so

\[[0,1] \times \hat{\mathbb{Z}} \times \mathbb{C}/\sim \to P_1\]

is an isomorphism, which concludes the proof. \(\square\)

**Lemma 3.9.** The class \([T \overset{pr}{\to} (T \times \hat{\mathbb{Z}}, P_1)]\) equals the class \([T \leftarrow (*, 1)]\) in \(K_{-2}(\mathbb{T}) \cong K_0(\mathbb{T})\).

**Proof.** Since \(T \times \hat{\mathbb{Z}}\) is a boundary we have that \([T \overset{pr}{\to} (T \times \hat{\mathbb{Z}}, 1)] = 0\), whence

\[\big{[}T \overset{pr}{\to} (T \times \hat{\mathbb{Z}}, P_1)\big{]} = \big{[}T \overset{pr}{\to} (T \times \hat{\mathbb{Z}}, P_1 - 1)\big{]}\]

By Lemma [5.3] and Example [2.7]

\[\big{[}T \overset{pr}{\to} (T \times \hat{\mathbb{Z}}, P_1 - 1)\big{]} = \big{[}T \overset{e^x}{\leftarrow} ((0,1)^2, \beta)\big{]} \quad \text{in} \quad K_*(\mathbb{T})\]

Next, we note that the correspondence with boundary \([T \overset{F}{\leftarrow} ((0,1)^2 \times [0,1], \beta \hat{\circ} 1)]\), with \(F((x,y),t) = -e^{tx}\), provides a bordism between

\[\big{[}T \overset{pr}{\to} ((0,1)^2, \beta)\big{]} \quad \text{and} \quad \big{[}T \overset{e^x}{\leftarrow} ((0,1)^2, \beta)\big{]}\]

where 1 denotes the constant map at 1. Since \((0,1)^2\) is a tubular neighbourhood of \((\frac{1}{2}, \frac{1}{2})\) in \(\mathbb{R}^2\) and \(\beta\) is the Thom class, Thom modification shows that

\[\big{[}T \overset{e^x}{\leftarrow} ((0,1)^2, \beta)\big{]} \sim_{\text{Tm}} \big{[}T \overset{e^x}{\leftarrow} (*, 1)\big{]}\]

which was to be shown. \(\square\)

An immediate corollary of this result is

**Lemma 3.10.** For any \(d\),

\[\big{[}T^d \overset{pr}{\to} (T^d \times \hat{\mathbb{Z}}^d, P_\mathbb{A})\big{]} = (-1)^{\frac{d(d-1)}{2}} \big{[}T^d \leftarrow (*, 1)\big{]}\]

**Proof.** The canonical isomorphism \(T^d \times \hat{\mathbb{Z}}^d \to (T \times \hat{\mathbb{Z}})^d\) has determinant \((-1)^{\frac{d(d-1)}{2}}\), so that

\[\big{[}T^d \overset{pr}{\to} (T^d \times \hat{\mathbb{Z}}^d, P_\mathbb{A})\big{]} \to * = (-1)^{\frac{d(d-1)}{2}} \big{[}T^d \leftarrow ((T \times \hat{\mathbb{Z}})^d, P_\mathbb{A} \hat{\circ} P_\mathbb{A})\big{]}\]

while \([T^d \leftarrow ((T \times \hat{\mathbb{Z}})^d, P_\mathbb{A} \hat{\circ} P_\mathbb{A})]\) is the exterior product of \(d\) copies of \([T \leftarrow (T \times \mathbb{Z}, P_1)]\), each of which is equivalent to a point correspondence, whence the result follows. \(\square\)
We can now prove Theorem 3.6.

Proof of Theorem 3.6. The spirit of the proof is to decompose $\mathbb{T}^d$ into the product $\ker(\hat{\phi}) \times \mathbb{T}^k$, and apply Lemma 3.10 to the $\mathbb{T}^k$ factor.

The composition diagram for $[\mathcal{F}_d] \otimes \hat{\mathbb{T}} = (\mathbb{Z}^k, 1) \to \ast$ is

The K-orientation $\mathbb{T}^d \times \hat{\mathbb{T}}^k$ receives from the intersection product is given by the short exact sequence

$$0 \xrightarrow{} T(\mathbb{T}^d \times \hat{\mathbb{T}}^k) \xrightarrow{} T(\mathbb{T}^d \times \hat{\mathbb{T}}^d) \xrightarrow{} T(\hat{\mathbb{T}}^d) \xrightarrow{} 0.$$  

It canonically splits, yielding an isomorphism $T(\mathbb{T}^d \times \hat{\mathbb{T}}^k) \to T(\mathbb{T}^d \times \hat{\mathbb{T}}^d) \times T(\hat{\mathbb{T}}^d)$ given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & d\hat{\phi} & 1_d \\ 0 & 1_k & 0 \end{bmatrix}.$$  

The path

$$t \mapsto \begin{bmatrix} 1_d & 0 & 0 \\ 0 & td\hat{\phi} & 1_d \\ 0 & 1_k & 0 \end{bmatrix}$$

connects $\begin{bmatrix} 1_d & 0 & 0 \\ 0 & d\hat{\phi} & 1_d \\ 0 & 1_k & 0 \end{bmatrix}$ to $\begin{bmatrix} 1_d & 0 & 0 \\ 0 & 0 & 1_d \\ 0 & 1_k & 0 \end{bmatrix}$ in $O(dk)$,

whence the K-orientation that $\mathbb{T}^d \times \hat{\mathbb{T}}^k$ receives is $(-1)^{dk}$ times the canonical K-orientation, since the right most matrix has determinant $(-1)^{dk}$. Thus,

$$\mathcal{F}_d \otimes \hat{\mathbb{T}}^k \xrightarrow{\phi} (\mathbb{Z}^k, 1) \to \ast = (-1)^{dk}[\mathbb{T}^d \xrightarrow{id} \mathbb{T}^d \times \mathbb{Z}^k, \mathcal{P}_d|\hat{\phi} \to \ast],$$

where $\mathcal{P}_d|\hat{\phi}$ is the line bundle over $\mathbb{T}^d \times \hat{\mathbb{T}}^k$ given by the relation

$$(x, \chi, \lambda) \sim (x + n, \chi, \hat{\phi}(n)\lambda), \quad \text{for } x \in \mathbb{R}^d \text{ and } n \in \mathbb{Z}^d.$$  

Using our orientation preserving isomorphism $\phi : \ker(\hat{\phi}) \times \mathbb{T}^k \to \mathbb{T}^d$, this is equivalent to

$$[\mathbb{T}^d \xrightarrow{\phi} (\ker(\hat{\phi}) \times \mathbb{T}^k \times \hat{\mathbb{T}}^k, \phi^*\mathcal{P}_d|\hat{\phi} \to \ast],$$

and one can check that $\phi^*\mathcal{P}_d|\hat{\phi} = 1 \otimes \mathcal{P}_k$. Applying the correspondence $[(\phi^{-1})^*]$ and using Lemma 3.10 we have

$$[\phi^{-1}] \otimes [\mathbb{T}^d \xrightarrow{\phi} (\ker(\hat{\phi}) \times \mathbb{T}^k \times \hat{\mathbb{T}}^k, \phi^*\mathcal{P}_d|\hat{\phi} \to \ast)] = (-1)^d[\ker(\hat{\phi}) \times \mathbb{T}^k \xrightarrow{\phi^*} (\ker(\hat{\phi}) \times \mathbb{T}^k \times \hat{\mathbb{T}}^k, 1 \otimes \mathcal{P}_d)]$$

$$= (-1)^d[\ker(\hat{\phi}) \xrightarrow{\phi} \mathbb{T}^k \xrightarrow{\hat{\mathbb{T}}^k \xrightarrow{\phi^*} (\mathbb{T}^k \times \hat{\mathbb{T}}^k, \mathcal{P}_k)))]$$

$$= (-1)^d[\ker(\hat{\phi}) \xrightarrow{\phi} \mathbb{T}^k \xrightarrow{\hat{\mathbb{T}}^k \xrightarrow{\phi^*} (\mathbb{T}^k \xrightarrow{\hat{\mathbb{T}}^k \xrightarrow{\phi^*} (\mathbb{T}^k \times \hat{\mathbb{T}}^k, \mathcal{P}_k)))]$$

$$= (-1)^d[\ker(\hat{\phi}) \xrightarrow{\phi} \mathbb{T}^k \xrightarrow{\hat{\mathbb{T}}^k \xrightarrow{\phi^*} (\mathbb{T}^k \xrightarrow{\hat{\mathbb{T}}^k \xrightarrow{\phi^*} (\mathbb{T}^k \times \hat{\mathbb{T}}^k, \mathcal{P}_k)))]$$

$$= (-1)^d[\ker(\hat{\phi}) \xrightarrow{\phi} \mathbb{T}^k \xrightarrow{\hat{\mathbb{T}}^k \xrightarrow{\phi^*} (\mathbb{T}^k \times \hat{\mathbb{T}}^k, \mathcal{P}_k))].$$
Applying $[\phi]$ to $(-1)^d[\ker(\tilde{\varphi}) \times \mathbb{T}^k \leftarrow (\ker(\tilde{\varphi}), 1)]$ gives $[\mathbb{T}^d \leftarrow (\ker(\tilde{\varphi}), 1)]$, hence

$$\mathcal{F}_d \otimes [\mathbb{T}^d \leftarrow (\mathbb{T}^k, 1)] = (-1)^{dk} \mathbb{T}^d \leftarrow (\ker(\tilde{\varphi}), 1)],$$

which was to be shown.

\[\Box\]

Example 3.11. If $\tilde{\varphi} : * \rightarrow \mathbb{Z}^d$ is the embedding of a point, then $\ker(\tilde{\varphi}) = \mathbb{T}^d$ oriented in the canonical way so that

$$[\mathcal{F}_d] \otimes [\mathbb{T}^d \leftarrow (\mathbb{T}^d, 1)] = [\mathbb{T}^d \leftarrow (\mathbb{T}^d, 1)] = \mathbb{T}^d.$$

Thus, the Fourier-Mukai transform of the class of a point in $\mathbb{Z}^d$ was to be shown.

Example 3.12. If $\varphi : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is an isomorphism, then $\ker(\varphi) = 1$ oriented if $\varphi$ is K-orientation preserving or not. Thus

$$[\mathcal{F}_d] \otimes [\mathbb{Z}^d \leftarrow (\mathbb{Z}^d, 1)] = \epsilon(\varphi)(-1)^d[\mathbb{T}^d \leftarrow (\mathbb{T}^d, 1)],$$

where $\epsilon(\varphi)$ is $\pm 1$ if $\varphi$ K-orientation preserving or not. In particular,

$$[\mathcal{F}_d] \otimes [\mathbb{T}^d \leftarrow (\mathbb{T}^d, 1)] = (-1)^d[\mathbb{T}^d \leftarrow (\mathbb{T}^d, 1)].$$

Let $X$ be a compact, $d$-dimensional, K-oriented manifold. Spin duality for $X$ gives a natural system of isomorphisms

$$KK_*(X \times U, V) \cong KK_{*+d}(U, X \times V).$$

These isomorphisms are determined by a pair of classes

$$\tilde{\Delta} := \{ \cdot \leftarrow X \delta \rightarrow X \times X \} \in KK_n(\cdot, X \times X)$$

$$\Delta := [X \times X \delta \rightarrow X \rightarrow *] \in KK_{*n}(X \times X, \cdot),$$

where $\delta : X \rightarrow X \times X$ is the diagonal map. The classes $\Delta$ and $\tilde{\Delta}$ are called the co-unit and unit. They determine the above isomorphism(s) by setting

$$PD_{\text{spin}} : KK_*(X \times U, V) \rightarrow KK_{*+d}(U, X \times V),$$

$$PD_{\text{spin}}(f) := \tilde{\Delta} \otimes_{X \times X} (1_X \times f).$$

using the notation of \((2.12)\). Let $\sigma : X \times X \rightarrow X \times X$ be the coordinate flip. This is an instance of a pair of adjoint functors: namely

$$L_X \dashv L_X$$

where $L_X : KK \rightarrow KK$ is the functor of cartesian product with $X$. As in the theory of adjoint functors, the map $PD_{\text{spin}}$ can be inverted by a similar construction involving the class $\Delta$ provided the zig-zag equations hold:

$$\sigma \Delta \otimes 1_X \otimes_{X \times X} (1_X \otimes \sigma \Delta) = 1_X \quad (\sigma \Delta \otimes 1_X) \otimes_{X \times X} (1_X \otimes \Delta) = 1_X.$$

In this case, since $\sigma_* = \Delta$, and $\sigma^*(\Delta) = (-1)^d$ it follows that these two equations reduce to a single one, which is easily check by hand, using composition of correspondences by transversality.

$$\sigma \Delta \otimes 1_X \otimes_{X \times X} (1_X \otimes \sigma \Delta) = 1_X \quad (\sigma \Delta \otimes 1_X) \otimes_{X \times X} (1_X \otimes \Delta) = 1_X.$$

Given by flipping the legs of a correspondence. Here $pr_X \circ b \times f$ is K-oriented using the isomorphism

$$TM \oplus (pr_X \circ b \times f)^* \cong T_f \oplus (pr_X \circ b)^*(TX)$$

and the given K-orientation on $X$. 

Theorem 3.16. Let $X$ be a compact, $K$-oriented smooth manifold. Then on geometric cycles, $\text{PD}_{\text{spin}}$ is given by (3.15).

We apply this to $T^d$ and $\widehat{Z}^d$, which are both spin$^c$ manifolds. Then Poincaré duality provides an isomorphism $\text{KK}_*(T^d, \widehat{Z}^d) \cong \text{KK}_*(\widehat{Z}^d, T^d)$. The Poincaré dual of the Fourier-Mukai transform is

$$[\widehat{Z}^d]_{\text{PT}}^{-1} (T^d \times \widehat{Z}^d, \mathcal{P}_d) \xrightarrow{\text{PT}} T^d \in \text{KK}_{-d}(\widehat{Z}^d, T^d),$$

which, abusing notation, we also denote by $[F_d]$. Given a Lie embedding $\varphi : T^k \to T^d$, we would like to compute $[F_d] \otimes [T^d \times \varphi^* (T^k, 1)]$ as in Theorem 3.6 which we are now going to do.

The map $\varphi$ lifts to a linear injection $\varphi : \mathbb{R}^k \to \mathbb{R}^d$ which maps $\mathbb{Z}^k$ into $\mathbb{Z}^d$. Its Pontryagin dual $\hat{\varphi} : \mathbb{R}^d \to \mathbb{R}^k$ is then a surjection which maps $\mathbb{Z}^d_1$ to $\mathbb{Z}^k_1$, where $\mathbb{Z}^d_1$ denotes the characters which vanish on $\mathbb{Z}^d$. Thus, $\hat{\varphi}$ descends to a surjection $\hat{\varphi} : \widehat{Z}^d \to \widehat{Z}^k$, which gives rise to an exact sequence

$$0 \to \text{ker}(\hat{\varphi}) \to \widehat{Z}^d \to \widehat{Z}^k \to 0,$$

which endows $\text{ker}(\hat{\varphi})$ with a $K$-orientation. With this set-up we have, as before, the following theorem.

Theorem 3.17. Suppose that $\varphi : T^k \to T^d$ is a Lie embedding. Then

$$[F_d] \otimes [T^d \times \varphi^* (T^k, 1)] = (-1)^{dk + k(k-1)} [\text{ker}(\hat{\varphi})].$$

For the proof we use the Pontryagin dual of the Fourier-Mukai Transform.

Definition 3.18. The Pontryagin dual of the Fourier-Mukai transform is the correspondence

$$(\chi, z, \lambda) \sim (\chi + \eta, z, \eta(z)\lambda) \quad \text{for } \eta \in \mathbb{Z}^d_1;$$

note that $\mathbb{Z}^d_1 \cong \mathbb{T}^d$, so that $\eta(z)$ makes sense.

The proof of Theorem 3.16 works verbatim to show that

$$[\widehat{F}_d] \otimes [T^d \times \varphi^* (T^k, 1)] = (-1)^{dk + k(k-1)} [\text{ker}(\hat{\varphi})],$$

so in order to prove 3.17 it suffices to prove the following lemma.

Lemma 3.19. $[F_d] = [\widehat{F}_d]$ in $\text{KK}_{-d}(\widehat{Z}^d, T^d)$.

Proof. For simplicity we take $d = 1$, the general case being completely analogous. Consider the map

$$\theta : Z \times T \to T \times \widehat{Z} \times T \times \widehat{Z}$$

$$\theta_z(\chi, z) \mapsto (\chi(1), \chi_z),$$

where $\chi_z(1) := z$. Then $\theta$ provides an isomorphism of correspondences between $[F_d]$ and

$$[\widehat{Z} \times \theta^* (\mathcal{P}_d)] \xrightarrow{\theta_z^*} T] = [\widehat{Z} \times \theta^* (\mathcal{P}_d) \xrightarrow{\theta_z^*} T],$$

where $\theta_z(\chi, z) = \chi_z$ and $\theta_z(\chi, z) = \chi(1)$. 


Now, consider then embeddings $e_x : (0, 1) \to \hat{\mathbb{Z}}$ and $e^y : (0, 1) \to \mathbb{T}$ given in the proof of Lemma 3.8. Using the embedding $e_x \times e^y : (0, 1)^2 \to \hat{\mathbb{Z}} \times \mathbb{T}$, it follows from Examples 2.7 and ?? that

$$\left[ \hat{\mathbb{Z}} \leftrightarrow \left( \hat{\mathbb{Z}} \times \mathbb{T}, \hat{\mathcal{P}}_d \right) \overset{\theta}{\longrightarrow} \mathbb{T} \right] = \left[ \hat{\mathbb{Z}} \leftrightarrow \left( \hat{\mathbb{Z}} \times \mathbb{T}, \hat{\mathcal{P}}_d - 1 \right) \overset{\theta}{\longrightarrow} \mathbb{T} \right]$$

The $\partial$-correspondence

$$\left[ \hat{\mathbb{Z}} \leftrightarrow ((0, 1)^2, \hat{\mathcal{P}}_d - 1) \overset{\theta}{\longrightarrow} \mathbb{T} \right]$$

shows that

$$\left[ \hat{\mathbb{Z}} \leftrightarrow ((0, 1)^2, \hat{\mathcal{P}}_d - 1) \overset{\theta}{\longrightarrow} \mathbb{T} \right] \sim_b \left[ \hat{\mathbb{Z}} \leftrightarrow ((0, 1)^2, \hat{\mathcal{P}}_d - 1) \overset{\theta}{\longrightarrow} \mathbb{T} \right].$$

Since

$$\left[ \hat{\mathbb{Z}} \leftrightarrow ((0, 1)^2, \hat{\mathcal{P}}_d - 1) \overset{\theta}{\longrightarrow} \mathbb{T} \right] = [\hat{\mathcal{F}}_d]$$

we are done. \qed

Finally, by combining Theorems 3.6 and 3.17 with Poincaré duality, we can show that $[\mathcal{F}_d]$ is invertible. Define

$$[\mathcal{F}_d] = \left[ \hat{\mathbb{Z}} \leftrightarrow (\hat{\mathbb{Z}} \times \mathbb{T}, \hat{\mathcal{P}}_d) \overset{pr_2}{\longrightarrow} \mathbb{T} \right],$$

where $\hat{\mathcal{P}}_d$ is the line bundle over $\hat{\mathbb{Z}} \times \mathbb{T}$ given by the relation $(\chi, x, \lambda) \sim (\chi, x + n, \chi(n) \lambda)$ for $x \in \mathbb{R}_d$ and $n \in \mathbb{Z}_d$. Then we have the following.

**Theorem 3.20.** For any $d$, we have

$$[\mathcal{F}_d] \otimes [\bar{\mathcal{F}}_d] = 1_{\mathbb{T}} \in \text{KK}_0(\mathbb{T}, \mathbb{T}^d) \quad \text{and} \quad [\bar{\mathcal{F}}_d] \otimes [\mathcal{F}_d] = 1_{\hat{\mathbb{Z}}_d} \in \text{KK}_0(\hat{\mathbb{Z}}_d, \hat{\mathbb{Z}}_d).$$

Thus, the class $[\mathcal{F}_d]$ is an invertible element of $\text{KK}_0(\mathbb{T}_d, \hat{\mathbb{Z}}_d)$

**Proof.** We compute directly that

$$[\mathcal{F}_d] \otimes [\bar{\mathcal{F}}_d] = \left[ \mathbb{T}^d \overset{pr_1}{\longrightarrow} (\hat{\mathbb{Z}} \times \mathbb{T}^d, \mathcal{P}_d \otimes \mathcal{P}_d) \overset{pr_2}{\longrightarrow} \mathbb{T}^d \right].$$

The Poincaré dual of $[\mathcal{F}_d] \otimes [\bar{\mathcal{F}}_d]$ is the class

$$[\mathbb{T}^{2d} \overset{pr_1}{\longrightarrow} (\hat{\mathbb{Z}} \times \mathbb{T}^d, \mathcal{P}_d \otimes \mathcal{P}_d) \in K_*(\mathbb{T}^{2d}),$$

where $\mathcal{P}_d$ is the line bundle over $\mathbb{T}^d \times \hat{\mathbb{Z}}_d$ determined by the relation

$$(x, y, \chi, \lambda) \sim (x + n, y + m, \chi(m - n) \lambda) \quad \text{for} \quad x, y \in \mathbb{R}_d \text{ and } n, m \in \mathbb{Z}_d.$$

One observes that this is equal to $[\mathcal{F}_d] \otimes [\mathcal{F}_d]$, where $\mathcal{F}_d : \hat{\mathbb{Z}}^d \to \hat{\mathbb{Z}}^{2d}$ is the map which sends $\chi$ to $(\chi, x)$.

We observe that the Pontryagin dual of $\mathcal{F}_d$ is the map

$$\hat{\Delta} : \mathbb{R}^d \to \mathbb{R}^d$$

$$(x, y) \mapsto y - x.$$ We identify the kernel of $\hat{\Delta}$ with $\mathbb{R}^d$ using the diagonal map $\Delta : \mathbb{R}^d \to \mathbb{R}^{2d}$. Thus, by Theorem 3.6, the Poincaré Dual of $[\mathcal{F}_d] \otimes [\bar{\mathcal{F}}_d]$ is

$$[\mathbb{T}^{2d} \overset{\Delta}{\longrightarrow} (\mathbb{T}, 1)].$$

The Poincaré dual of this element is the unit in $\text{KK}(\mathbb{T}^d, \mathbb{T}^d)$, which concludes the proof. \qed
4. SOME REMARKS ON BAUM-CONNES DUALITY

**Lemma 4.1.** Suppose $X$ is a smooth manifold, with two duals $X'$ and $X''$ in $\text{KK}$, of dimensions $k'$ and $k''$. Then there is a unique $\text{KK}$-equivalence

$$f \in \text{KK}_{k'-k''}(X, X')$$

making the diagram

(4.2) $$\text{KK}_*(X \times U, V) \xrightarrow{\text{PD}'} \text{KK}_{k+k'}(U, X' \times V) \xrightarrow{\hat{\Theta}_X f} \text{KK}_{k+k''}(U, X'' \times V)$$

commute.

Conversely, if $f: \text{KK}_d(X', X'')$ is a $\text{KK}$-equivalence, where $X'$ is a dual for $X$ with unit $\Delta'$ and co-unit $\hat{\Delta}'$, then $X''$ is a dual for $X$ with unit $\Delta'': = (1_X \otimes f^{-1}) \otimes_{X \times X'} \Delta'$ and co-unit $\hat{\Delta}'': = \hat{\Delta}' \otimes_{X \times X'} (1_X \otimes f)$.

The proof is routine.

Since the Fourier-Mukai transform $[F] \in \text{KK}_{-d}(T^d, \widehat{T^d})$ is an equivalence, by Lemma 4.1 we can ‘twist’ spin duality for $T^d$ by the Fourier-Mukai transform to obtain an even-dimensional $\text{KK}$-duality now between $T^d$ and the dual torus $\widehat{T^d}$.

**Theorem 4.3.** Define

(4.4) $$\tilde{\Delta}_{\text{FM}} := [P_d] \in \text{KK}_0(\cdot, T^d \times \widehat{T^d}),$$

(4.5) $$\Delta_{\text{FM}} := \left[ T^d \times \widehat{T^d} \xleftarrow{\text{id}} (T^d \times \widehat{T^d}, P) \rightarrow \right] \in \text{KK}_0(T^d \times \widehat{T^d}, \cdot).$$

Then $\Delta_{\text{FM}}$ and $\tilde{\Delta}_{\text{FM}}$ are the unit and co-unit (respectively) of a 0-dimensional $\text{KK}$-duality between $T^d$ and $\widehat{T^d}$.

We call the duality of Theorem 4.3 Baum-Connes duality – it is, thus, an instance of a $\text{KK}$-duality, but it is clearly different from spin duality.

**Proof.** Spin duality for $T^d$ has unit and co-unit

(4.6) $$\Delta_{\text{Spin}} := \left[ T^d \times T^d \xleftarrow{\text{id}} T^d \rightarrow \cdot \right],$$

(4.7) $$\hat{\Delta}_{\text{Spin}} := \left[ \cdot \xleftarrow{T^d \delta} T^d \times T^d \rightarrow \right].$$

By Lemma 4.1 ‘twisting’ spin duality by $[F]$ results in the unit and co-unit

(4.8) $$\tilde{\Delta}_{\text{FM}} = \text{PD}_{\text{spin}}([F]^{-1}) = \text{PD}_{\text{spin}}([F]) = \left[ \ast \xleftarrow{\text{id}} (T^d \times \widehat{T^d}, P) \rightarrow T^d \times \widehat{T^d} \right],$$

(4.9) $$\Delta_{\text{FM}} = \text{PD}_{\text{spin}}([FM]) = \left[ T^d \times \widehat{T^d} \xleftarrow{\text{id}} (T^d \times \widehat{T^d}, P) \rightarrow \ast \right]$$

for another duality. The first correspondence defines the class $[P_d] \in K^0(T^d \times \widehat{T^d})$, of the Poincaré line bundle, and the second is as in the statement of the Proposition.

The theorem has the following consequence in analytic $\text{KK}$-theory.
Corollary 4.10. The class $[\mathcal{P}_d] \in \text{KK}_0(\mathbb{C}, C(\mathbb{T}^d \times \mathbb{Z}_d))$ is the unit of a duality in analytic KK-theory between $C(\mathbb{T}^d)$ and $C(\mathbb{Z}_d)$. The co-unit is the class

$$\Delta_{\text{FM}} = [\partial_d \cdot \mathcal{P}] \in \text{KK}_0(C(\mathbb{T}^d \times \mathbb{Z}_d), \mathbb{C})$$

of the Dirac-Dolbeault operator $\partial_d$ on $\mathbb{T}^d \times \mathbb{Z}_d$, twisted by the Poincaré line bundle $\mathcal{P}_d$.

Proof. Since $\Delta_{\text{FM}}$ and $\tilde{\Delta}_{\text{FM}}$ satisfy, by design, the zig-zag equations in topological KK, their images also satisfy them in analytic KK. The class $\Delta_{\text{FM}}$ is represented by the correspondence

$$\mathbb{T}^d \times \mathbb{Z}_d \xleftarrow{\text{id}} (\mathbb{T}^d \times \mathbb{Z}_d, \mathcal{P}_d) \to \mathbb{C},$$

which in analytic KK-theory maps to the class of the Dirac operator on $\mathbb{T}^d \times \mathbb{Z}_d$, twisted by the line bundle $\mathcal{P}$.

This has the following interesting consequence relating the Fourier-Mukai correspondence to Dirac-Schrödinger type operators. Suppose first that $d = 1$. Consider the even-graded Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and operator

$$\mathcal{D} = \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix}.$$ 

Diagonalization of the harmonic oscillator $-\frac{d^2}{dx^2} + x^2$, which has discrete spectrum the odd integers, all with multiplicity 1, leads to a coordinate representation of $x + \frac{d}{dx}$ (called an annihilation operator) as a weighted shift with weights growing like $\sqrt{n}$ (and Fredholm index 1.)

We will call such operators Dirac-Schrödinger operators.

Definition 4.11. The Dirac-Schrödinger cycle in dimension $d$, is the following spectral triple over $C^\infty(\mathbb{T}^d \times \mathbb{Z}_d)$.

a) If $d$ is even, then the Hilbert space is $H := L^2(\mathbb{R}^d, S)$, $L^2$-functions valued in the spin representation space for $\text{Cliff}(\mathbb{R}^d)$, with $\mathbb{Z}/2$-grading induced by the spin grading.

If $d$ is odd, we take the positive irreducible spin representation $S$ (ungraded) for $\text{Cliff}(\mathbb{R}^d)$, and the Hilbert space $L^2(\mathbb{R}^d, S) \oplus L^2(\mathbb{R}^d, S)$, with standard even grading.

b) The representation $\pi: C(\mathbb{T}^d) \otimes C(\mathbb{Z}_d) \to \mathcal{L}(H)$ determined by the commuting representations

$$\pi_1(f) = M_f, \quad \pi_2(g) = \lambda(\hat{g}),$$

where $\hat{f} \in C_0(\mathbb{R}^d)$ is the $\mathbb{Z}^d$-periodic lift of $f$, $\hat{g}$ denotes the Fourier transform of $g$, an element of $C^*(\mathbb{Z}^d)$, and $\lambda: C^*(\mathbb{Z}^d) \to \mathcal{L}(H)$ is induced by the unitary representation of $\mathbb{Z}^d$ on $H$ by translation on $\mathbb{R}^d$. If $d$ is odd, we use the direct sum of two copies of this representation.

b) If $d$ is even, the operator is $D_{\mathbb{R}^d} + c_X$, where $X: \mathbb{R}^d \to \text{Cliff}(\mathbb{R}^d)$ is the canonical inclusion, and $D_{\mathbb{R}^d} = \sum_i c(e_i) \nabla_i$ is the Dirac operator on sections of $S$. If $d$ is odd, the operator is

$$\begin{bmatrix} 0 & D_{\mathbb{R}^d} - c_X \\ -D_{\mathbb{R}^d} + c_X & 0 \end{bmatrix}.$$

We denote by

$$\Delta_{\text{DS}} \in \text{KK}_0(C(\mathbb{T}^d \times \mathbb{Z}_d), \mathbb{C})$$

the class of the Dirac-Schrödinger cycle in KK-theory.
Remark 4.12. It is well-known that $D := D_{\mathbb{R}^n} + c_X$ extends to a self-adjoint operator with $(1 + D^2)^{-1}$ compact. If $\phi \in C^\infty_c(\mathbb{R}^d)$ is a smooth function whose gradient $\nabla \phi$ is bounded, then

$$[\phi, D_{\mathbb{R}^n} + c_X] = [\phi, D_{\mathbb{R}^n}] = c \nabla \phi$$

is a bounded operator. This holds in particular for smooth, periodic functions $\phi = \tilde{f}$, for $f \in C^\infty(\mathbb{T}^d)$.

Secondly, if $n \in \mathbb{Z}^d$, $\lambda(g)$ the corresponding translation operator on $L^2(\mathbb{R}^d, S)$, then $D_{\mathbb{R}^n}$ commutes with $\lambda(g)$, while

$$[\lambda(n), c_X] = c_n,$$

is a constant Clifford multiplication operator, whence is bounded. It follows that

$$[\lambda(\hat{g}), D_{\mathbb{R}^n} + c_X]$$

is bounded for polynomials on $\mathbb{Z}^d$.

This shows that Definition 4.11 is a spectral triple.

Theorem 4.13. The Dirac-Schrödinger cycle determines the same class in $KK_0(\mathbb{T}^d \times \hat{\mathbb{Z}}^d, \mathbb{C})$ as the Dirac-Dolbeault operator on $\mathbb{T}^d \times \hat{\mathbb{Z}}^d$, twisted by the Poincaré line bundle $\mathcal{P}_d$.

That is:

$$\Delta_{FM} = \Delta_{DS} \in KK_0(\mathbb{T}^d \times \hat{\mathbb{Z}}^d, \mathbb{C}).$$

This theorem is a kind of index theorem for Dirac-Schrödinger operators, equating the classes in KK of the analytically defined $x + \frac{d}{dx}$ cycle, with the purely topologically defined $\mathcal{P}$-twisted Dirac-Dolbeault cycle.

For example, let us deduce from it the fact that the Dirac-Schrödinger cycle – that is, the operator $x + \frac{d}{dx}$ – has Fredholm index +1. The correspondence representation of the index of $\Delta_{DS}$ equals the index of $\Delta_{FM}$, which is the class of the correspondence

$$* \leftrightarrow (\mathbb{T} \times \hat{\mathbb{Z}}, \mathcal{P}_1) \rightarrow \cdot.$$

As argued in the previous section, this is bordant to the Thom modification of the 1-point correspondence

$$\cdot \leftrightarrow \cdot \rightarrow \cdot$$

corresponding to +1.

Proof. Theorem 4.13 follows since $\hat{\Delta}_{DS}$ is known to be a co-unit for the ‘Baum-Connes’ duality

$$KK(\mathbb{T}^d \times U, V) \cong KK(U, \hat{\mathbb{Z}}^d \times V)$$

with unit $[\mathcal{P}_d]$, by the paper [12].

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