M-Theory and Two-Dimensional Effective Dynamics

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Abstract
We calculate from M-theory the two-dimensional low energy effective dynamics of various brane configurations. In the first part we study configurations that have a dual description in type IIA string theory as two-dimensional \((4,0)\) Yang-Mills theories with gauge group \(SU(N_1) \times SU(N_2)\) and chiral fermions in the bi-fundamental representation. In the second part we derive related equations of motion which describe the low energy internal dynamics of a supersymmetric black hole in four-dimensional \(N = 1\) supergravity, obtained as an M-fivebrane wrapped on a complex four-cycle.
1 Introduction

One of the most fascinating aspects of brane dynamics is their ability to predict sophisticated results in quantum field theories that had previously been studied for many years. In particular, it has been found that one can predict the low energy effective actions, including all perturbative and non-perturbative effects. Although these predictions can not be made for a general quantum field theory, the cases which can be treated in this way are of considerable interest. From the brane perspective the non-trivial effects often arise from their classical dynamics and can only be identified with the properties of a quantum field theory by using some kind of duality.

In this paper we will use classical brane dynamics to derive the low energy effective action for a variety of two dimensional quantum field theories. In the first part of this paper we focus on theories that have a interpretation within IIA string theory as D-brane configurations and hence have a Yang-Mills interpretation. In the second part of this paper we will use this technique to derive the low energy effective action for four-dimensional black holes which result from M-fivebranes wrapped on a four-cycle of a six-dimensional Calabi-Yau manifold.

Let us consider the situation in \([1]\) where brane configurations of the form

\[
\begin{align*}
\text{NS}5 : & \quad 1 \ 2 \ 3 \ 4 \ 5 \\
\text{D}4 : & \quad 1 \ 2 \ 3 \ 6 
\end{align*}
\]  

\(1\)

were analysed. Here the D-fourbranes end on the NS-fivebranes. From the standard rules of D-branes the low energy dynamics of this configuration is given by a four-dimensional \(N = 2\) gauge theory. The technique introduced in \([1]\) was to analyse the strong coupling picture of this configuration from M-theory. Here the entire configuration appears as a single M-fivebrane

\[
\begin{align*}
\text{M5} : & \quad 1 \ 2 \ 3 \ 4 \ 5 \\
\text{M}5 : & \quad 1 \ 2 \ 3 \ 6 \ 10 
\end{align*}
\]  

\(2\)

In this paper we will consider configurations with less supersymmetry obtained by adding additional D-fourbranes ending on the NS-fivebranes

\[
\begin{align*}
\text{NS}5 : & \quad 1 \ 2 \ 3 \ 4 \ 5 \\
\text{D}4 : & \quad 1 \ 2 \ 3 \ 6 \\
\text{D}4 : & \quad 1 \ 4 \ 5 \ 6 
\end{align*}
\]  

\(3\)
In M-theory this configuration lifts to

\[
\begin{align*}
M5 &: 1 \ 2 \ 3 \ 4 \ 5 \\
M5 &: 1 \ 2 \ 3 \ 6 \ 10 \\
M5 &: 1 \ 4 \ 5 \ 6 \ 10
\end{align*}
\] (4)

The above configurations can be viewed from the worldvolume of the first M-fivebrane. There (3) appears as a threebrane soliton and (4) appears as a string soliton. In this case these configurations have the interpretation of a single M-fivebrane wrapped around a calibrated sub-manifold \(\Sigma\) of eleven-dimensional Minkowski space [2, 4]. It was shown in [2, 4] that (2) and (4) will preserve one half and one quarter of the M-fivebrane supersymmetry respectively if the manifold they define is complex. By examining the form of this manifold appearing in (4) one can derive the Seiberg-Witten curve [5] for a large class of gauge theories [1]. In addition, from the equations of motion for the M-fivebrane [6] one can derive a four-dimensional \(N = 2\) effective theory for the threebrane soliton [8, 9, 10]. This is precisely the Seiberg-Witten low energy effective action [5], including all non-perturbative effects.

Here we will perform a similar analysis of the configuration (3). We will show that it has a two-dimensional chiral \((4, 0)\) effective Yang-Mills theory living on the D-fourbrane intersection. We will derive the general form for the M-theory surface (3) including all of its moduli. In addition we will explicitly calculate the effective dynamics for the string soliton (3) and argue that it contains the complete low energy effective dynamics of two-dimensional Yang-Mills theory. Note that M-theory analysis of two-dimensional Yang-Mills theories have already appeared in [11, 12, 13, 14]. However these papers consider different brane configurations, with \((2, 2)\), \((4, 4)\), \((4, 4)\) and \((0, 2)\) supersymmetry respectively, and in particular do not calculate the low energy effective action. The the effective action for two-dimensional models with \((2, 2)\) supersymmetry were obtained in [15, 16] using the techniques of geometric engineering in type II string theory.

One could also view the configuration (3) in M-theory without reference to ten-dimensional string theory. In this approach, the supersymmetry condition is interpreted as saying that the M-fivebrane is wrapped on a complex four-cycle of \(C^3\). If we consider surfaces that can be compactified then the string soliton can be reduced to five dimensions by compactifying M-theory. Wrapping the string soliton on an \(S^1\) and fur-
ther compactifying to four dimensions we obtain a black hole in $N = 1$ supergravity. Similar black holes were studied in [18, 19] from this point of view and it was found that their microscopic degrees of freedom correctly accounted for the black hole's entropy. In this paper we will consider surfaces of this type and derive the low energy dynamics of the corresponding string soliton. This provides a dynamical system of equations for the internal structure of a black hole at low energies.

The intersecting branes (4) have also been considered from the point of view of the AdS/CFT correspondence since the near horizon limit has an $AdS_3 \times S^2 \times E_6$ geometry [17]. One motivation for this work is to help elucidate the relation we between description of quantum field theory provided by the M-fivebrane and the description found by the AdS/CFT conjecture. In general the former is not expected to have exact quantitative agreement with quantum field theory whereas the later is. However in this paper we will argue that exact quantitative information can be obtained from the M-fivebrane. Thus this work should help explore a configuration where both approaches are valid.

The rest of this paper is organised as follows. In the next section we analyse the type IIA brane configuration (3) using D-branes. Then in section three we go to the strong string coupling limit of M-theory and analyse the corresponding configuration (4). Next we turn to the problem of calculating the low energy dynamics of this configuration. In section four we introduce the M-fivebrane equations of motion and in section five we derive the two-dimensional effective theory. In section six we consider a different family of surfaces, whose brane configurations can be interpreted as black holes in four-dimensions and derive their low energy effective actions.

2 The Type IIA Brane Configuration

Let us first recall the brane configuration (3) discussed in [4]. There one considers two parallel NS-fivebranes in type IIA string theory lying in the $(x^1, x^2, x^3, x^4, x^5)$ plane and separated along the $x^6$ direction by a distance $L$. One then introduces $N_1$ parallel D-fourbranes in the $(x^1, x^2, x^3, x^6)$ plane, suspended between the two NS-fivebranes. The worldvolume theory on $N_1$ parallel D-fourbrane can be described by open string perturbation theory and at low energy is a five-dimensional $U(N_1)$ gauge theory with sixteen
supersymmetries (the equivalent of $N = 4$ in $D = 4$). An overall $U(1)$ factor describes the centre of mass motion of the D-fourbranes and is trivial so we will ignore it. Now consider the presence of the NS-fivebranes. These are infinitely heavy as compared to the D-fourbranes (which are now finite in extent along $x^6$). Therefore they do not provide any low energy degrees of freedom. Instead they cause the low energy D-fourbrane theory to be reduced to four dimensions and project out half of the supersymmetry. Thus the effective theory for the configuration is a four-dimensional $N = 2$ $SU(N_1)$ Yang-Mills theory.

Now we go to the strong coupling limit of the string theory given by eleven-dimensional M-theory. Here one must introduce an extra dimension $x^{10}$ which is compactified on a circle with radius $R$. While the NS-fivebranes are now simply M-fivebranes the D-fourbranes become M-fivebranes wrapped around the $x^{10}$ dimension. Thus we arrive at the M-theory configuration (2). Indeed the whole configuration can be viewed as a single M-fivebrane wrapped over a manifold $\Sigma_{SW}$. It was shown in [1] that $\Sigma_{SW}$ was precisely the known Seiberg-Witten elliptic curve appearing in the complete low energy effective action for $N = 2$ $SU(N_1)$ Yang-Mills theory [5] (see also [20] for a similar role of the M-fivebrane). In addition by studying the dynamics of the M-fivebrane one can determine the low energy effective action including an infinite number of instanton corrections [8]. Thus the classical dynamics of the M-fivebrane is capable of predicting exact coefficients in four-dimensional supersymmetric Yang-Mills theory.

In this paper we wish to study in a similar way the type IIA brane configuration (3) obtained by including $N_2$ additional D-fourbranes to (1) in the $(x^1, x^4, x^5, x^6)$ plane. Thus the intersection between the two D-fourbranes is a three-dimensional space. However the NS-fivebranes insure that the $x^6$ dimension is finite and hence the low energy theory is two-dimensional. In addition the extra D-fourbranes project out another half of the supersymmetries leaving only four supercharges. To see this consider the large string coupling constant limit where the configuration (3) is lifted to the M-theory configuration (4). The three M-fivebranes in (4) preserve the supersymmetries

$$\Gamma_{012345} \epsilon = \epsilon , \quad \Gamma_{0123610} \epsilon = \epsilon , \quad \Gamma_{0145610} \epsilon = \epsilon .$$

(5)
It follows that the remaining four supersymmetries are chiral

$$\Gamma_{01}\epsilon = -\epsilon,$$  \hfill (6)

and hence the two-dimensional low energy theory has \((4,0)\) supersymmetry.

Now consider the field theory living on the D-brane intersection. At weak coupling this is obtained by standard D-brane techniques and is given in [21]. As mentioned above the intersecting D-fourbranes reduce the five-dimensional field theory to three dimensions. Note also that there is a trivial \(SO(3) \cong SU(2)\) symmetry from the \(x^7, x^8, x^9\) dimensions. This leads to an \(SU(2)\) R-symmetry group. If we now add the NS-fivebranes we must dimensionally reduce the system to two dimensions. The counting of massless states gives the following modes in terms of two-dimensional \(N = (4,4)\) super-multiplets. From open strings that begin and end on the D-fourbranes parallel to \((x^1, x^2, x^3, x^6)\) we obtain a vector-multiplet with gauge group \(SU(N_1)\) (here \(\mu = 0, 1\) and \(\alpha = 1, 2\))

$$a, \chi_A, \lambda_{A\pm}^\alpha, A_{\mu}$$ \hfill (7)

where \(a = X^4 + iX^5\) and \(\chi_A = A_2 + iA_3\). There is also a hyper-multiplet

$$X^7, X^8, X^9, A_6, \rho_{A\pm}^\alpha$$ \hfill (8)

which is in the adjoint representation of \(SU(N_1)\) (since the over-all centre of mass degree of freedom naturally decouples we have \(SU(N_1)\) instead of \(U(N_1)\)). Here \(X^d\) is the scalar field representing the fluctuation of the D-fourbrane in the \(x^d\) direction and \(A_{m}\) is the component of the worldvolume gauge field in the \(x^m\) direction. There are similar multiplets

$$b, \chi_B, \lambda_{B\pm}^\alpha, B_{\mu}$$ \hfill (9)

where \(b = X^2 + iX^3, \chi_B = B_4 + iB_5\) and

$$X^7, X^8, X^9, B_6, \rho_{B\pm}^\alpha$$ \hfill (10)

in the adjoint of \(SU(N_2)\) found by examining the open strings that begin and end on the D-fourbranes parallel to the \((x^1, x^4, x^5, x^6)\) plane. So in total we have a vector-multiplet and a hyper-multiplet in the adjoint of \(SU(N_1) \times SU(N_2)\). Note that these states are
simply the dimensional reduction of ten-dimensional $N = 1$ super-Yang-Mills-multiplets reduced to two dimensions. These two multiplets are by themselves insensitive to each other and hence preserve the full $(8,8)$ supersymmetry (i.e. the same as a single D-fourbrane).

In addition there is a hyper-multiplet

$$\phi_1, \phi_2, \phi_3, \phi_4, \psi_\pm$$

coming from open strings stretching between the two types of D-fourbrane \[21\]. These states transform in the fundamental representation of $U(N_1) \times U(N_2)$. Again, since a $U(1) \times U(1)$ factor of this group is trivial, the interacting part of the hyper-multiplet is in the fundamental representation of $SU(N_1) \times SU(N_2)$. These states mediate the interaction between the two factors of the gauge group (i.e. the interactions between the two types of D-fourbranes) and break the supersymmetry to $(4,4)$.

The NS-fivebranes do not just dimensionally reduce the three-dimensional intersection to two dimensions. It was pointed out in \[22\] that they also impose boundary conditions where the D-fourbranes meet the NS-fivebranes which can eliminate zero modes and break additional supersymmetries. Following this argument let us consider the adjoint hyper-multiplets. These fields represent fluctuations of the D-fourbranes in the directions transverse to the NS-fivebranes and, since the D-fourbranes must end on the NS-fivebrane, we must impose Dirichlet boundary conditions so that the scalar fields $X^7, X^8, X^9$ vanish at $x^6 = 0, L$. This implies that they are massive modes (with massive of order the string scale) and hence they do not appear in the low energy Yang-Mills theory. This argument applies separately to each type of D-fourbrane, as in the construction of \[1\], and so it follows that the $(4,4)$ superpartners $A_6, B_6, \rho_A, \rho_B$ are massive too. On the other hand the vector multiplets represent the fluctuations of the D-fourbranes within the NS-fivebranes and need not vanish at $x^6 = 0, L$. Instead we simply impose Neumann boundary conditions which leave zero modes for all the fields in the $(4,4)$ multiplet.

The boundary conditions imposed on the bi-fundamental hyper multiplet are more difficult to analysis since they do not have a straightforward geometrical interpretation. To this end a more formal argument can be constructed which reproduces the above massless modes for the adjoint multiplets and also shows that only the two lowest helicity zero
modes of the bi-fundamental hyper multiplet remain and are singlets under supersymmetry \[23\], i.e. two helicity \(-\frac{1}{2}\) fermions. To summarise, the surviving two-dimensional zero modes are a \((4, 4)\) vector multiplet \((a, \chi_A, \lambda^\alpha_{A\pm}, A_\mu)\) with gauge group \(SU(N_1)\), a second \((4, 4)\) vector multiplet \((b, \chi_B, \lambda^\alpha_{B\pm}, B_\mu)\) with gauge group \(SU(N_2)\) and two right-handed fermions \(\psi^a\) in the fundamental of \(SU(N_1) \times SU(N_2)\). These right-handed fermions are singlets under spacetime supersymmetry leaving \((4, 0)\) supersymmetry on the two-dimensional intersection.

Note that the vector multiplet fields can carry momentum off the intersection (i.e. in the \(x^2, x^3, x^4, x^5\) directions), although the bi-fundamental fermions only propagate along \(x^0, x^1\). Without the bi-fundamental fermions the two gauge groups do not interact and can equally be viewed as a single four-dimensional \(N = 2\) \(SU(N_1) \times SU(N_2)\) gauge theory. Therefore the resulting effective dynamics we obtain more accurately describe a one-dimensional defect in a four-dimensional \(N = 2\) \(SU(N_1) \times SU(N_2)\) gauge theory. However, since the interactions between the two different gauge groups propagate only in two dimensions, the effective dynamics we obtain should contain the full description of an interacting \((4, 0)\) \(SU(N_1) \times SU(N_2)\) two-dimensional gauge theory.

Let us consider first the theory on the first D-fourbrane in the \((x^1, x^2, x^3, x^6)\) plane. The NS-fivebranes cause the \(x^6\) dimension to be compact with length \(L\). Therefore the four-dimensional Yang-Mills coupling constant \(g_{YM}\) is given by the ratio \(g_{YM}^2 = g_s/L\) where \(g_s = R^{3/2}\) is the string coupling constant \([1]\). It follows that we may go to the large distance, small curvature limit \(R, L \to \infty\) keeping the couplings fixed. In this limit M-theory is well described by eleven-dimensional supergravity. Thus the effective description that we obtain from M-theory should be accurate for any value of the Yang-Mills coupling constant. In fact we will see that, as is the case for the \(N = 2\) theories, the parameter \(R\) does not appear in the low energy dynamics, which are therefore insensitive to the extra dimension. Another important restriction on the low energy terms is that they are holomorphic. One may therefore expect some kind of non-renormalisation theorem that ensures they are the correct terms for the Yang-Mills theory. Thus we can conjecture on the basis of type IIA/M-theory duality that we will arrive at the exact quantum low energy effective action.
3 The M-fivebrane Geometry

Let us now pass to the strong coupling M-theory picture. The D-fourbranes then lift to M-fivebrane configuration \((\text{4})\). Indeed, as in \([1]\), all branes are M-fivebranes which can in turn be described by a single self-intersecting M-fivebrane with a complicated worldvolume. The intersection \((\text{4})\) can then be viewed as a supersymmetric string soliton on the worldvolume of the first M-fivebrane. Our approach is to derive the dynamics of the D-fourbrane intersection gauge theory by calculating the classical low energy dynamics of this string soliton of the M-fivebrane.

To perform the M-theory analysis it is useful to introduce the complex coordinates
\[
s = X^6 + iX^{10}, \quad t = e^{-s/R}, \quad w = x^2 + ix^3, \quad z = x^4 + ix^5.
\] (12)

It was shown in \([4]\) that the soliton \((\text{4})\) is a solution to the M-fivebrane equations of motion and preserves one quarter of the sixteen supersymmetries if \(s(z, w)\) is a holomorphic function. This has the interpretation of wrapping the M-fivebrane around a four-dimensional complex submanifold \(\Sigma\) of \(\mathbb{C}^3\) with coordinates \(t, z, w\). Equivalently we may embed the M-fivebrane in spacetime by a function of the form \(F(t, z, w) = 0\). Our first task then is to generalise the construction in \([1]\) and identify the appropriate complex surface for \((\text{4})\) and its moduli.

We are interested here in manifolds \(\Sigma\) which correspond to two NS-fivebranes in the \(x^0, x^1, x^2, x^3, x^4, x^5\) plane in ten dimensions. In this case we need two possible values of \(t\) for every point \((z, w)\), i.e. we need a two-sheeted cover of the \((z, w)\) plane. Thus we assume that \(F(t, z, w)\) takes the form
\[
A(z, w)t^2 - 2B(z, w)t + C(z, w) = 0.
\] (13)

We do not want to consider here a configuration where there are semi-infinite D-fourbranes coming off the two NS-fivebranes. To ensure this we must set \(A\) and \(C\) to constants, so that \(t\) is neither zero nor infinite (corresponding to \(s = \pm \infty\)) for any finite value of \(z\) and \(w\). Without loss of generality this leads to \(A = 1\) and \(C = \Lambda\). By rescaling \(t\) and \(B\) we can also set \(\Lambda = 1\). To obtain \(N_1\) D-fourbranes in the \((x^1, w, x^0)\) plane and \(N_2\) D-fourbranes in the \((x^1, z, x^0)\) plane we must chose \(B\) to be a polynomial of degree \(N_1\) in
z and degree $N_2$ in $w$. The most general form for $B$ can be written as

$$B(z, w) = \alpha z^{N_1} w^{N_2} - p(w) z^{N_1} - q(z) w^{N_2}$$

$$-\beta z^{N_1-1} w^{N_2-1} - r_1(w) z^{N_1-1} - r_2(z) w^{N_2-1}$$

$$+ R(z, w).$$

(14)

Here $\alpha$ and $\beta$ are constants, $p$ and $r_1$ are polynomials of degree $N_2 - 1$ and $N_2 - 2$ in $w$ respectively, $q$ and $r_2$ are polynomials of degree $N_1 - 1$ and $N_1 - 2$ in $z$ respectively and $R(z, w)$ is a polynomial of degree $N_1 - 2$ in $z$ and $N_2 - 2$ in $w$.

We can set $\alpha = 1$ by rescaling $z$ and $w$. This leaves $(N_1 + 1)(N_2 + 1) - 1$ coefficients in $B$ to determine. To this end we imagine that $t$ is fixed. This gives us a surface in the $z, w$ plane representing the two types of intersecting D-fourbranes. For large $z$ with $w$ fixed this surface takes the form

$$z(w^{N_2} - p(w)) - (q_{N_1-1} w^{N_2} + bw^{N_2-1} + r_1(w)) = 0 ,$$

(15)

where $q_{N_1-1}$ is the leading coefficient in $q(z)$ and we have ignored lower order terms. To describe the intersecting brane configuration we require that for large $z$ we go to a fixed value of $w$, representing a D-fourbrane in the $(x^1, z, x^6)$ plane. In addition there should be a symmetry between $z \rightarrow \infty$ and $z \rightarrow -\infty$. This requires that both terms in (15) vanish separately. In particular we must be at a zero of $w^{N_2} - p(w)$ and if we write

$$w^{N_2} - p(w) = \prod_{i'=1}^{N_2} (w - w_{i'}) ,$$

(16)

then we can interpret the $w_{i'}$ as the locations of the $N_2$ D-fourbranes in the $(x^1, z, x^6)$ plane. We can also shift the $w$ coordinate so that the centre of mass is frozen at $w = 0$, thereby setting $\sum_{i'} w_{i'} = 0$. This means that $p(w)$ is in fact only of degree $N_2 - 2$.

Similarly, by examining the large $w$ limit, we write

$$z^{N_1} - q(z) = \prod_{i=1}^{N_1} (z - z_i) ,$$

(17)

and interpret the $z_i$ as the locations of the $N_1$ D-fourbranes in the $(x^1, w, x^6)$ plane. By a shift in the $z$ coordinate we can set the centre of mass to $z = 0$. Again this means that $\sum_i z_i = 0$ and $q(z)$ is only of degree $N_1 - 2$. 

9
It also follows from (15) that the polynomial (recall that now \( q_{N_1-1} = 0 \))
\[
\beta w^{N_2-1} + r_1(w) ,
\]
has \( N_2 \) roots precisely at the points \( w_{ij} \). But this is a polynomial of degree \( N_2 - 1 \) and therefore it can only have \( N_2 \) roots if it is identically zero. Hence we must have \( \beta = r_1(w) = 0 \). Similarly from the large \( w \) analysis we find that \( r_2(z) = 0 \). Thus \( B \) can be written as
\[
B(z, w) = z^{N_1} w^{N_2} - p(w) z^{N_1} - q(z) w^{N_2} + R(z, w) ,
\]
where now \( p(w) \) is of degree \( N_2 - 2 \) and \( q(z) \) is of degree \( N_1 - 2 \).

Thus we find that there are \( (N_1 - 1) + (N_2 - 1) \) moduli coming from the polynomials \( p \) and \( q \) and also \( (N_1 - 1)(N_2 - 1) \) moduli from the polynomial \( R \) which we write in the form
\[
R(z, w) = \sum_{i=1}^{N_1-1} \sum_{\nu=1}^{N_2-1} u_{i\nu} z^{i-1} w^{\nu-1} .
\]

We have seen above that we can identify the moduli \( z_i \) and \( w_{ij} \) with the positions of the D-fourbranes. However the moduli in \( R \) determine subtle features of the hypersurface \( \Sigma \) which are hard to see in the large \( z, w \) limit where we can clearly identify distinct D-fourbranes. It follows from M-theory/type IIA duality that we can associate to the D-brane Yang-Mills theory the complex hyper-surface in \( \mathbb{C}^3 \) given by
\[
y^2 = \left[ z^{N_1} \prod_{i'} (w - w_{i'}) + w^{N_2} \prod_{i} (z - z_i) - z^{N_1} w^{N_2} + \sum_{i'\nu} u_{i\nu} z^{i-1} w^{\nu-1} \right]^2 - 1 ,
\]
where \( y = t - B \). Clearly for \( w (z) \) fixed to a constant this surface reduces to the Seiberg-Witten curve for \( N = 2 \) \( SU(N_1) (SU(N_2)) \) gauge theory after a trivial rescaling.

It will be helpful to re-derive this surface by considering only features of the two four-dimensional \( SU(N) \) gauge theories. From this point of view we want to construct a complex surface of the form
\[
y^2 = B^2 - 1 ,
\]
with \( B \) a polynomial of degree \( N_1 \) in \( z \) and \( N_2 \) in \( w \). Furthermore we require that the coefficients in \( B \) are constructed from scalar gauge invariant operators of the four-dimensional Yang-Mills theories. It is easy to see that for the groups \( SU(N_1) \) and \( SU(N_2) \),
the only independent such operators are \((i = 1, ..., N_1 - 1, \ i' = 1, ..., N_2 - 1)\)

\[
\begin{align*}
\tilde{z}_i & = \text{Tr}(a^{N_1+1-i}), \\
\tilde{w}_{i'} & = \text{Tr}(b^{N_2+1-i'}), \\
\tilde{u}_{i,i'} & = \text{Tr}(a^{N_1+1-i}b^{N_2+1-i'}).
\end{align*}
\]

\[(23)\]

In the brane configuration \([3]\) the \(SO(1, 9)\) Lorentz group is broken down to \(SO(1, 1) \times SO(3) \times SO(2) \times SO(2)\) corresponding to the two-dimensional Lorentz group and rotations in the \((x^7, x^8, x^9)\), \((x^4, x^5)\) and \((x^2, x^3)\) planes respectively. These last three transformations appear in the gauge theory as an \(SU(2) \times U(1) \times U(1)\) \(R\)-symmetry. If we consider the four-dimensional \(SU(N)\) gauge theory on each type of D-fourbrane (and ignore the presence of the other type of D-fourbrane) then the corresponding \(U(1)\) \(R\)-symmetry is broken by quantum effects to \(\mathbb{Z}_{2N}\). This group is generated by

\[
a \to e^{\frac{i\pi}{N_1}a}, \quad b \to e^{\frac{i\pi}{N_2}b}.
\]

\[(24)\]

Thus if we denote the weights of a field under \(\mathbb{Z}_{2N_1} \times \mathbb{Z}_{2N_2}\) by \((p, q)\) then the operators transform as

\[
\begin{align*}
\tilde{z}_i & = (N_1 + 1 - i, 0), \\
\tilde{w}_{i'} & = (0, N_2 + 1 - i'), \\
\tilde{u}_{i,i'} & = (N_1 + 1 - i, N_2 + 1 - i').
\end{align*}
\]

\[(25)\]

We require that \(\mathbb{Z}_{2N_1} \times \mathbb{Z}_{2N_2}\) must be a symmetry of the surface. Let us write

\[
B = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} c_{k,l} z^k w^l
\]

and normalise the leading term \(c_{N_1,N_2} = 1\). Noting that \(B \to -B\) is a symmetry of the surface we see that we can assign the variables \(z\) and \(w\) the weights \((1, 0)\) and \((0, 1)\) respectively under \(\mathbb{Z}_{2N_1} \times \mathbb{Z}_{2N_2}\). Now consider an arbitrary term in \(B\). Invariance of \(B\) (up to a minus sign) then tells us that \(c_{k,l}\) must have weight \((N_1 - k, N_2 - l)\). Thus we
identify
\[ c_{i-1,N_2} = \tilde{z}_i, \]
\[ c_{N_1,i' - 1} = \tilde{w}_{i'}, \]
\[ c_{i-1,i' - 1} = \tilde{u}_{ii'} . \]  
(27)

However the coefficients of the sub-leading terms \( c_{N_1-1,l} \) and \( c_{k,N_2-1} \) have weights \((1,N_2-l)\) and \((N_1-k,1)\) respectively. Since there are no gauge invariant operators with these dimensions we must set \( c_{N_1-1,l} = c_{k,N_2-1} = 0 \). Thus we find
\[ B = z^{N_1} w^{N_2} + \sum_i \tilde{z}_i z^{i-1} w^{N_2} + \sum_{i'} \tilde{w}_{i'} z^{N_1} w^{i'-1} + \sum_{ii'} \tilde{u}_{ii'} z^{i-1} w^{i'-1} . \]  
(28)

This is precisely the same surface that we obtained from the geometry of the intersecting branes with \( \tilde{u}_{ii'} = u_{ii'} \), \( \tilde{z}_i \) identified with the symmetric polynomial of degree \( N_1 + 1 - i \) in \( z_i \) and \( \tilde{w}_{i'} \) identified with the symmetric polynomial of degree \( N_2 + 1 - i' \) in \( w_{i'} \).

Just as was the case for the \( N = 2 \) models this picture also suggests that the moduli of surface are the scalar modes in the low energy effective description of the Yang-Mills theory. However one can easily see that not all of these scalar modes are dynamical in the low energy M-fivebrane theory. To see this consider the dynamics of the scalar modes alone (i.e. ignore for now the three-form field \( H \)). In this case the M-fivebrane dynamics are given by the standard \( p \)-brane action
\[ S = \int d^6 x \sqrt{-\det g} , \]  
(29)

where \( g_{mn} \) is the induced metric
\[ g_{mn} = \eta_{mn} + \frac{1}{2} \partial_m s \partial_n \bar{s} + \frac{1}{2} \partial_m \bar{s} \partial_n s . \]  
(30)

Expanding this expression out to second order in two-dimensional derivatives gives, ignoring a surface term,
\[ S = \frac{1}{2} \int d^2 x d^2 w d^2 z \partial_\mu s \partial^\mu \bar{s} . \]  
(31)

To evaluate the low energy scalar dynamics we promote the moduli of the surface to two-dimensional scalar fields so that
\[ \partial_\mu s = \sum_i \partial_\mu z_i \frac{\partial s}{\partial z_i} + \sum_{i'} \partial_\mu w_{i'} \frac{\partial s}{\partial w_{i'}} + \sum_{ii'} \partial_\mu u_{ii'} \frac{\partial s}{\partial u_{ii'}} . \]  
(32)
Substituting this into the action $S$ one finds non-convergent integrals (in the large $z, w$ limit) coming from the kinetic terms for the $z_i$ and $w_{i'}$ moduli. Thus we are forced to set the positions of the D-fourbranes $z_i(x)$ and $w_{i'}(x)$ to constants in the low energy theory. This is not unexpected since the D-fourbranes are infinitely heavy from the point of view of the two-dimensional intersection.

From the alternative derivation of the surface (21) above we also see that we must identify the moduli with the vacuum expectation values of the gauge invariant operators (23). In particular the $z_i$ and $w_{i'}$ are the $N_1-1$ and $N_2-1$ components of $a$ and $b$ in the Cartan subalgebra of $SU(N_1)$ and $SU(N_2)$ respectively. Using supersymmetry it follows that the superpartners of these fields, namely $\chi_A, \chi_B, A_\mu, B_\mu$, are also non-dynamical. This leaves us with $(N_1-1)(N_2-1)$ complex moduli $u_{i'i'}$ and their superpartners which we expect to have smooth finite energy behaviour at low energy. From the above analysis we see that these are identified with the composite gauge theory operators

$$u_{i'i'} = \text{Tr}(a^{N_1+1-i} b^{N_2+1-i'}) .$$

(33)

4 The M-Fivebrane Equations and Soliton Dynamics

Before proceeding to analyse the dynamics of the configuration (4) we must first discuss the equations of motion for the the M-fivebrane in flat eleven-dimensional spacetime. We then use these to find the equations of motion for the low energy motion of our string soliton solution on the M-fivebrane worldvolume. We will use the equations of motion found in [6] from the superembedding approach applied to the M-fivebrane [7].

The bosonic fields of the M-fivebrane consist of a closed three-form $H_{mnp}$ and five scalars $X^{a'}, a' = 6, 7, 8, 9, 10$, representing the transverse fluctuations. We denote the worldvolume coordinates by $x^m$, $m = 0, 1, 2, ..., 5$ and tangent frame indices by $a, b, c = 0, 1, 2, ..., 5$. The two-dimensional coordinates $x^0, x^1$ are denoted by $x^\mu$. The complex coordinates $z, w$ and $\bar{z}, \bar{w}$ are denoted by $z^\alpha$ and $\bar{z}^\alpha$ respectively. The scalar fields define a worldvolume metric and vielbein

$$g_{mn} = \eta_{mn} + \partial_m X^{a'} \partial_n X^{a'} ,$$

$$= \eta_{ab} e_m^a e_n^b ,$$
obtained through the pull-back of the flat eleven-dimensional metric.

In the full non-linear theory the three-form $H_{mnp}$ satisfies a complicated self-duality constraint. This is obtained by first considering a (linearly) self-dual three-form $h_{mnp}$

$$h_{mnp} = \frac{1}{3!} \epsilon_{mnpqrs} h^{qrs},$$  

(35)

which is not assumed to be closed. We can then construct the tensor

$$m^b_a = \delta^b_a - 2h_{acd}h^{bcd},$$  

(36)

and define

$$H_{mnp} = e^a_m e^b_n e^c_p (m^{-1})^d_c h_{abd}.$$  

(37)

Thus $H_{mnp}$ satisfies a non-linear self-duality constraint inherited from the (linear) self-duality of $h_{mnp}$.

We may now write the equations of motion for the bosonic fields in the form [6]

$$G^{mn} \nabla_m \nabla_n X^{a'} = 0,$$

$$G^{mn} \nabla_m H_{npq} = 0,$$

(38)

where

$$G^{mn} = e^m_a e^n_b m^a_c m^b_d \eta^{cd},$$  

(39)

is a second metric incorporating the three-form field. If $H_{mnp}$ is constructed as above and hence satisfies its non-linear self-duality constraint then the second equation in (38) is equivalent to the closure of $H_{mnp}$ [6].

To find the low energy motion of the string soliton it is sufficient to consider only terms up to second order in spacetime derivatives $\partial_\mu$ and the three-form $H_{mnp}$. In this case we may simply write

$$H_{mnp} = e^a_m e^b_n e^c_p h_{abc}.$$  

(40)

It is understood in (40) that $H$ appears in the worldvolume frame and $h$ in the tangent frame.
Once the self-duality condition on $H$ is satisfied the low energy equations of motion for the M-fivebrane are found from (38) to be
\[
\partial_{\mu} \partial^\mu s - \partial_{\alpha} \left[ \frac{\partial_{\mu} s \partial^\mu \bar{s} \partial_{\alpha} \bar{s}}{1 + |\partial s|^2} \right] - 4 H_{mn}^\alpha H^{mn\beta} \partial_{\alpha} \partial_{\beta} s = 0 ,
\]
for the scalar fields, where $|\partial s|^2 = |\partial_z s|^2 + |\partial_{\alpha} s|^2$. For the three-form we may simply use
\[
\partial_{[m} H_{n pq]} = 0 .
\]
Here we have assumed that only the two scalars $X^6$ and $X^{10}$ are non-constant. In particular the closure of $H_{mnp}$ gives several equations
\[
\begin{align*}
\partial_{[\mu} H_{\nu\alpha\beta]} &= 0 , \\
\partial_{[\mu} H_{\nu\alpha\bar{\beta}]} &= 0 , \\
\partial_{[\mu} H_{\alpha\beta\bar{\gamma}]} &= 0 , \\
\partial_{[\alpha} H_{\beta\gamma\delta]} &= 0 ,
\end{align*}
\]
and their complex conjugates.

To obtain the low energy motion for the soliton we let its moduli depend on $x^\mu$ and substitute the form for $s(z, w)$ into these equations. We must then dimensionally reduce the scalar equations to two-dimensions. Following the methods of [9] we consider
\[
\int_{\Sigma} \left\{ \partial_{\mu} \partial^\mu s - \partial_{\alpha} \left[ \frac{\partial_{\mu} s \partial^\mu \bar{s} \partial_{\alpha} \bar{s}}{1 + |\partial s|^2} \right] - 4 H_{mn}^\alpha H^{mn\beta} \partial_{\alpha} \partial_{\beta} s \right\} dz \wedge dw \wedge \theta^I = 0 ,
\]
where the $\theta^I$ are a basis for $H^2(\Sigma)$. For the three-form we consider only those equations which are second order in spacetime indices $\mu$ to be equations of motion. For these we consider
\[
\begin{align*}
\int_{\Sigma} \partial_{[\mu} H_{\nu\alpha\beta]} dz^\alpha \wedge dz^\beta \wedge \theta^I &= 0 , \\
\int_{\Sigma} \partial_{[\mu} H_{\nu\alpha\bar{\beta}]} dz^\alpha \wedge dz^\beta \wedge \theta^I &= 0 .
\end{align*}
\]
This leaves us with the last two equations in (43) which we view as constraints (along with self-duality) on $h_{mnp}$.
It is helpful at this point to list the non-vanishing components of the vielbein
\[ e_\alpha^\beta = \delta_\alpha^\beta + \left( \frac{\sqrt{1+|\partial s|^2} - 1}{|\partial s|^2} \right) \partial_\alpha s \bar{\partial}_\beta \bar{s} + \mathcal{O}((\partial_\mu)^2), \]
\[ e_\mu^\alpha = \frac{\partial_\mu s \bar{\partial}_\alpha \bar{s}}{\sqrt{1+|\partial s|^2}} + \mathcal{O}((\partial_\mu)^2), \]
\[ e_\mu^\nu = \delta_\mu^\nu + \mathcal{O}((\partial_\mu)^2), \]
(46)

plus their complex conjugates. In this paper we will only consider ansätze for the three-form which satisfy
\[ H_{\mu\nu\alpha} = 0. \]
(47)

From (40) we find that
\[ h_{\mu\nu\alpha} = \mathcal{O}((\partial_\mu)^2), \]
and it follows from (35) that \( h_{\alpha\beta\gamma} = \mathcal{O}((\partial_\mu)^2) \). Thus we simply find
\[ H_{\mu\alpha\beta} = e_\alpha^\gamma e_\beta^\delta h_{\mu\gamma\delta} , \quad H_{\mu\alpha\bar{\beta}} = e_\alpha^\gamma e_\bar{\beta}^\delta h_{\mu\gamma\delta} , \]
(49)

where \( H \) is in the world frame and \( h \) is in the tangent frame. We may now consider ansätze for \( H_{\mu\alpha\beta} \) and \( H_{\mu\alpha\bar{\beta}} \) of the form
\[ H_{\mu\alpha\beta} = U_{\mu} \kappa_{\alpha\beta} , \quad H_{\mu\alpha\bar{\beta}} = V_{\mu}^+ \kappa_{\alpha\bar{\beta}}^+ + V_{\mu}^- \kappa_{\alpha\bar{\beta}}^- , \]
(50)

where we have chosen \( \kappa_{\alpha\beta}^\pm dz^\alpha \wedge d\bar{z}^\beta \) to be (anti-) self-dual forms on \( \Sigma \). Substituting this into the self-duality condition we learn that \( U_{\mu} = \epsilon_{\mu\nu} U^\nu \) and \( V_{\mu}^\pm = \pm \epsilon_{\mu\nu} V_{\mu}^\nu \) are two-dimensional chiral fields.

If we assume that the three fields \( U_{\mu} \) and \( V_{\mu}^\pm \) are independent then the third equation in (13) implies that \( \kappa_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta \) and \( \kappa_{\alpha\bar{\beta}}^\pm dz^\alpha \wedge d\bar{z}^\beta \) are closed \((2, 0)\) and \((1, 1)\) forms on \( \Sigma \) respectively. It is not hard to see that a non-zero \( H_{\mu\nu\alpha} \) would correspond to \((1, 0)\) forms on \( \Sigma \). Finally we must consider the fourth equation in (13). We expect that this equation leads to no additional constraints on \( H \). The reason for this is that if we integrate it over the manifold \( \Sigma \) we obtain the constraint
\[ \int_{\Sigma} \partial_{\alpha} H_{\beta\gamma\delta} dz^\alpha \wedge d\bar{z}^\beta \wedge dz^\gamma \wedge d\bar{z}^\delta = \int_{\Sigma} dH = 0 , \]
(51)
which is identically true if $\Sigma$ has no boundary or if the fields which vanish at infinity. Thus the ansatze for chiral the fields on a general $\Sigma$ takes the form

$$H_{\mu\alpha\beta} = \sum_{I=1}^{b^{(2,0)}} U_{\mu I} \omega^I_{\alpha\beta}, \quad H_{\mu\omega^I_{\alpha\beta}} = \sum_{I=1}^{b^{+(1,1)}} V^+_\mu \omega^+_I + \sum_{I=1}^{b^{-(1,1)}} V^-_\mu \omega^-_I.$$  

(52)

Here $\omega^I, \omega^+_I$ and $\omega^-_I$ form basis of $H^{(2,0)}(\Sigma), H^{+(1,1)}(\Sigma)$ and $H^{-(1,1)}(\Sigma)$ respectively, where $H^{\pm(1,1)}(\Sigma)$ is the (anti-) self-dual subgroup of $H^{(1,1)}(\Sigma)$. The index $I$ is understood to run over the dimension of the appropriate cohomology group. The first two equations in (43) now give the following three equations for $U_{\mu I}, V^+_\mu I$ and $V^-_\mu I$

$$0 = \partial_{[\mu} U_{\nu I]} \omega^I + \partial_{[\mu} u^A U_{\nu I]} \frac{\partial \omega^I}{\partial u^A},$$
$$0 = \partial_{[\mu} V^+_{\nu I]} \omega^+_I + \partial_{[\mu} u^A V^+_{\nu I]} \frac{\partial \omega^+_I}{\partial u^A},$$
$$0 = \partial_{[\mu} V^-_{\nu I]} \omega^-_I + \partial_{[\mu} u^A V^-_{\nu I]} \frac{\partial \omega^-_I}{\partial u^A}. \quad (53)$$

With these ansatze for the chiral fields we also find the full scalar equation

$$0 = \partial_{[\mu} u^A \partial^\mu u^A \frac{\partial s}{\partial u^A} + \partial_{[\mu} u^A \partial^\mu u^B \frac{\partial^2 s}{\partial u^A \partial u^B} \left[ \frac{\partial s}{\partial u^A} \frac{\partial s}{\partial u^B} \frac{\partial s}{1 + |\partial s|^2} \right]$$
$$-4 \bar{U}_{\mu I} V^-_K \left( g^{\alpha\beta} g^{\gamma\delta} \bar{\omega}^J_{\delta\bar{\beta}} \bar{\omega}^K_{\delta\gamma} \partial_{\alpha} \partial_{\beta} s \right). \quad (54)$$

Finally we may now write the two-dimensional equations of motions. For the scalars we find

$$0 = \partial_{[\mu} u^A \left( \frac{\partial s}{\partial u^A} \right) \int_{\Sigma} dz \wedge dw \wedge \theta^I + \partial_{[\mu} u^A \partial^\mu u^B \left( \frac{\partial^2 s}{\partial u^A \partial u^B} \right) \int_{\Sigma} dz \wedge dw \wedge \theta^I$$
$$-4 \bar{U}_{\mu I} V^-_K \int_{\Sigma} \left( g^{\alpha\beta} g^{\gamma\delta} \bar{\omega}^J_{\delta\bar{\beta}} \bar{\omega}^K_{\delta\gamma} \partial_{\alpha} \partial_{\beta} s \right) dz \wedge dw \wedge \theta^I, \quad (55)$$

where $u^A$ are the moduli for the surface $s(z, w)$. It is easy to see that only those $\theta^I$ in $H^{(0,2)}(\Sigma)$ give a non-vanishing contribution to the integrals in (55). We also note that if if $U_\mu = V^\pm_\mu = 0$ then the equations of motion for the scalars $v^A$ can be derived from the
action (31). However this only leads to the first two terms in (55). Thus we expect that the third integral in (55) always vanishes. This is not surprising since, for (0, 2) forms $\theta^I$, the integrand is a total derivative. We also find the following equations for the chiral fields

$$
0 = \partial_{[\mu} U_{\nu]} J^I \int_{\Sigma} \omega^J \wedge \theta^I + \partial_{[\mu} u^A U_{\nu]} J^I \int_{\Sigma} \frac{\partial \omega^J}{\partial u^A} \wedge \theta^I ,
$$

$$
0 = \partial_{[\mu} V_{\nu]}^+ J^I \int_{\Sigma} \omega^{+J} \wedge \theta^I + \partial_{[\mu} u^A V_{\nu]}^+ J^I \int_{\Sigma} \frac{\partial \omega^{+J}}{\partial u^A} \wedge \theta^I ,
$$

$$
0 = \partial_{[\mu} V_{\nu]}^- J^I \int_{\Sigma} \omega^{-J} \wedge \theta^I + \partial_{[\mu} u^A V_{\nu]}^- J^I \int_{\Sigma} \frac{\partial \omega^{-J}}{\partial u^A} \wedge \theta^I .
$$

(56)

Since the wedge product of a self-dual and an anti-self-dual form vanishes, it follows that the only non-vanishing integrals in the equations for $U_\mu$, $V_\mu^+$ and $V_\mu^-$ come from $\theta^I$ in $H^{(0,2)}(\Sigma)$, $H^{+(1,1)}(\Sigma)$ and $H^{-(1,1)}(\Sigma)$ respectively.

## 5 Effective Dynamics of Two-Dimensional Yang-Mills Theories

In this section we will explicitly derive the low energy effective dynamics for an M-fivebrane wrapped around a four-cycle given by (21). This can then be interpreted as the M-theory prediction for the low energy effective theory of the two-dimensional $(4, 0)$ Yang-Mills theory.

Before we proceed we must construct a suitable ansatz for $H_{mnp}$. A non-zero $H_{\mu\nu z}$ would lead to vector modes in the low energy dynamics and since these contain no propagating degrees of freedom we set them to zero. In addition, in this section it will be sufficient to consider $H_{\mu\alpha\bar{\beta}} = 0$, which we will justify below. Thus we are only concerned with the first equation in (50) corresponding to $(2, 0)$ forms on $\Sigma$. It is not hard to check that a basis of normalisable $(2, 0)$ forms is given by

$$
\omega^{ij} \equiv \frac{z^{-1} w^{j-1}}{y} dz \wedge dw = -\frac{1}{R} \frac{\partial s}{\partial u_{ij}} dz \wedge dw ,
$$

(57)
where $y$ is given by (21). Thus the general form for $H_{\mu\alpha\beta}$ is

$$H_{\mu\alpha\beta} = \sum_{ii'} U_{ii'} \omega_{ii'}^{\alpha\beta}. \quad (58)$$

Since we have set $H_{\mu\nu\alpha} = H_{\mu\alpha\bar{\beta}} = 0$ one finds that the scalar equation of motion simplifies to

$$\partial_\mu \partial^\mu s - \partial_\alpha \left[ \partial_\beta s \partial^\mu s \bar{\partial}_\alpha \bar{s} \right] \frac{1}{1 + |\partial s|^2} = 0. \quad (59)$$

Performing the reduction of the equations of motion given in the last section yields

$$\partial_\mu \partial^\mu u_{ii'} I^{ii'} jj' + \partial_\mu u_{ii'} \partial^\mu u_{kk'} \frac{\partial I^{ii'} jj'}{\partial u_{kk'}} + \partial_\mu u_{ii'} \partial^\mu u_{kk'} K^{ii' kk' jj'} = 0, \quad (60)$$

where

$$I^{ii'} jj' = \int_\Sigma \omega^{ii'} \wedge \bar{\omega}^{jj'}, \quad (61)$$

and

$$K^{ii' kk' jj'} = \int_\Sigma \partial_\alpha \left[ \omega_2^{ii'} \omega_2^{kk'} \bar{\partial}_\alpha \bar{s} \right] \frac{1}{1 + |\partial s|^2} dz \wedge dw \wedge \bar{\omega}^{jj'}. \quad (62)$$

Since the integrand is a total derivative, and the singularities are mild, it is not hard to see that $K^{ii' kk' jj'} = 0$, as we expected in the previous section. Thus we find the equations of motion

$$\partial_\mu \partial^\mu u_{ii'} I^{ii'} jj' + \partial_\mu u_{ii'} \partial^\mu u_{kk'} \frac{\partial I^{ii'} jj'}{\partial u_{kk'}} = 0, \quad (63)$$

for the $(N_1 - 1)(N_2 - 1)$ complex scalars $u_{ii'}$. Indeed the above equations of motion are then precisely those of the action obtain by dimensionally reducing (31). In this way we find the the two-dimensional action

$$S = \int d^2 x \partial_\mu u_{ii'} \partial^\mu \bar{u}_{jj'} I^{ii'} jj', \quad (64)$$

for the scalars $u_{ii'}$.

Next we must consider the equation of motion for $H_{\mu\alpha\bar{\beta}}$. This is obtained using (33) to give

$$\partial_\mu U_{\nu ii'} I^{ii'} jj' + \partial_\mu u_{kk'} U_{\nu ii'} \frac{\partial I^{ii'} jj'}{\partial u_{kk'}} = 0. \quad (65)$$

Here we find chiral equations of motion for $(N_1 - 1)(N_2 - 1)$ complex bosons. We have yet to account for the zero modes of $H_{\mu\alpha\bar{\beta}}$. These are obtained from $(1,1)$-forms on $\Sigma$. However, since we have accounted for all of the low energy fields of the D-fourbranes,
these fields, if any exist, are not zero modes of our worldvolume string soliton and so we are justified in ignoring them.

Before ending this section we will make some observations about the effective action that we have derived. For concreteness let us consider the simplest case where $N_1 = N_2 = 2$, i.e. for the group $SU(2) \times SU(2)$. We therefore have only one complex scalar mode $u$ and the surface (21) takes the form

$$y^2 = \left[ z^2 w^2 - w_0^2 z^2 - z_0^2 w^2 + u \right]^2 - 1.$$  \hspace{1cm} (66)

The equations of motion for the fields are now

$$\partial_\mu \partial^\mu u I + \partial_\mu u \partial^\mu u \frac{\partial I}{\partial u} = 0,$$

$$\partial_\mu \mathcal{U}_{\nu} I + \partial_\nu u \mathcal{U}_{\nu} \frac{\partial I}{\partial u} = 0,$$

with

$$I = \int_\Sigma \omega \wedge \bar{\omega} = \int \frac{d^2 w d^2 z}{y \bar{y}}.$$  \hspace{1cm} (68)

One can show by a straightforward change of variables that $I$ can be written as

$$I = \int \frac{d^2 \xi}{|\xi^2 - z_0^2 w_0^2|} I_{SW}(\xi^2 - u),$$  \hspace{1cm} (69)

where

$$I_{SW}(x) \equiv \int d^2 \xi \frac{1}{|((\xi^2 - x)^2 - 1)|},$$  \hspace{1cm} (70)

is the standard four-dimensional Seiberg-Witten elliptic integral for $SU(2)$ (in the notation of [3], $I_{SW}(u) = |da/du|^2 \text{Im} \tau$). From the known form for this function [3] one sees that for large $x$, $I_{SW}(x) \propto \ln x / |x|$ so that $I$ is convergent for large $\xi$. In addition one can see that the singular points of $I_{SW}(\xi^2 - z_0^2 w_0^2)$ are only logarithmic. Therefore, even if they occur at a zero of $\xi^2 - z_0^2 w_0^2$, $I$ remains finite and well defined.

Thus we obtain smooth low energy effective dynamics. Furthermore all these quantities are holomorphic and so one is strongly lead to believe that they are in fact the correct low energy dynamics, including all the quantum corrections. Note also that there is no value for $u$ where $I$ is singular. This is radically different to the solution of Seiberg and Witten [3] where there are points on the $u$ plane where the low energy effective
action is singular (i.e. points where $F_{SW}$ has a logarithmic singularity). As is well known these points are associated with BPS states becoming massless. Thus we conclude here that there are no points on the $u$ plane where extra light degrees of freedom need to be included. Note though that $I$ is singular if $z_0 = 0$ or $w_0 = 0$. This corresponds to putting two of the parallel D-fourbranes on top of each other, leading to a restoration in the non-Abelian gauge symmetry.

6 Black Hole Effective Field Theory

Let us now consider a different form for the surface (4). In particular, consider M-theory compactified on $\mathcal{M} \times S^1$, where $\mathcal{M}$ is a six-dimensional Calabi-Yau manifold. This results in four-dimensional $N = 1$ supergravity for the M-theory low energy effective action. If we wrap the M-fivebrane on a four-cycle in $\mathcal{M}$ and then the resulting string soliton over the $S^1$ we obtain a black hole in four dimensions, which is a BPS solution of the supergravity theory. The entropy of this black hole was found in [18, 19] by computing the number of degrees of freedom of the black hole. The latter arise from the moduli of the embedded cycle, the three-form field of the fivebrane, as well as the momenta on the $S^1$. These are the same fields that we discussed in section four above. Therefore in this section we will use the analysis described in section four to compute the precise effective low energy dynamics of the wrapped M-fivebrane. Just as for the previous case we obtain the low energy dynamics by letting the moduli of the M-fivebrane string soliton depend on the $x^\mu$ coordinates. Although we do not carry out the remaining $S^1$ reduction, this is straightforward and yields the low energy dynamics of the back hole. Again the resulting scalar degrees of freedom posses $(4,0)$ supersymmetry. Effective actions for the conformal field theory of black hole internal degrees of freedom have also been constructed in [24, 25, 26] in other contexts.

A key difference with the work in the previous section is that we now consider four-dimensional hypersurfaces $\Sigma$ of a six-dimensional complex space that can be compactified. In fact, we will consider any such $\Sigma$ which has no $(1,0)$ or $(0,1)$ forms. However we will continue to use the M-fivebrane equations of motion in a flat background. It is hoped that the resulting system of equations provide a good description of an M-fivebrane
wrapped on a four-cycle of a six-dimensional Calabi-Yau manifold. In principle it is straightforward to modify our results by using the M-fivebrane equations of motion in a curved background.

Since we consider Σ to have no one-cycles we set again set $H_{\mu\nu\alpha} = 0$. However now we need to consider the full M-fivebrane dynamics, including zero modes of $H_{\mu\alpha\bar{\beta}}$. Again we follow the discussion in section four. The scalar fields that arise from the moduli of Σ obey the field equation arising from the scalar field $s = X^6 + iX^{10}$. In two-dimensions this is reduced to

$$0 = \partial_\mu \partial^\mu u^A \int_\Sigma \frac{\partial s}{\partial u^A} dz \wedge dw \wedge \theta^I + \partial_\mu u^A \partial^\mu u^B \int_\Sigma \frac{\partial^2 s}{\partial u^A \partial u^B} dz \wedge dw \wedge \theta^I$$

$$-\mathcal{U}_{\mu J} \mathcal{Y}_{\alpha K}^{I\mu} \int_\Sigma \left( g^{\alpha \bar{\beta}} g^{\bar{\gamma} \bar{\delta}} g^{\bar{\epsilon} \bar{\eta}} g^{\bar{\omega} \bar{\theta}} \partial_\alpha \partial_\beta \partial_\gamma \partial_{\bar{\gamma}} \partial_\delta \partial_{\bar{\delta}} \partial_\epsilon \partial_{\bar{\epsilon}} \partial_\eta \partial_{\bar{\eta}} \partial_\omega \partial_{\bar{\omega}} \partial_{\theta} \partial_{\bar{\theta}} \right) dz \wedge dw \wedge \theta^I,$$

where we have dropped the total derivative term in (55). As is clear from this equation, if there are anti-self-dual (1, 1) forms on Σ then we see that there will be quadratic terms in $H$ in the scalar equation. The three-form field is again expanded as in (52) and we find the equations of motion

$$\partial_\mu \mathcal{U}_{\nu I} K^{I\mu} + \partial_\mu u^A \mathcal{U}_{\nu I} \frac{\partial K^{I\mu}}{\partial u^A} = 0,$$

$$\partial_\mu \mathcal{Y}_{\nu J}^{I\mu} + \partial_\mu u^A \mathcal{Y}_{\nu J}^{I\mu} \frac{\partial J^{I\mu}}{\partial u^A} = 0,$$

$$\partial_\mu \mathcal{Y}_{\nu}^{I\mu} - L^{I\mu} + \partial_\mu u^A \mathcal{Y}_{\nu}^{I\mu} - \frac{\partial L^{I\mu}}{\partial u^A} = 0,$$

where $u^A$ are the moduli of the surface (which are the scalars in the two-dimensional theory) and

$$K^{I\mu} = \int_\Sigma \omega^I \wedge \omega^I, \quad J^{I\mu} = \int_\Sigma \omega^{+I} \wedge \omega^{+I}, \quad L^{I\mu} = \int_\Sigma \omega^{-I} \wedge \omega^{-I}.$$

The above equations for the three-form field moduli do not apparently depend on many of the detailed features of the fivebrane dynamics. However, this can not be said of the equation of motion for the $u^A$ as a result of the last term in (71). These introduce complicated non-holomorphic integrals which are difficult to evaluate in general (although it seems likely that they can be evaluated along the lines used in [10]). Let us therefore concentrate here on the equations of motion for the chiral fields.
The $J, K$ and $L$ integrals in (73) can in general be evaluated using a Riemann Bilinear relation of the form

$$
\int_{\Sigma} \omega \wedge \lambda = (\Omega^{-1})^{I\!J} \int_{A_I} \omega \int_{A_J} \lambda ,
$$

(74)

where $\Omega_{I\!J}$ is the intersection form of $\Sigma$ and $A_I$ are a basis of homology 2-cycles, $I, J = 1, 2, ..., b^2(\Sigma)$.

Since the space of two-forms in the second de Rham cohomology and the space of two-cycles of $\Sigma$ are isomorphic, we can choose the two-cycles to be dual to the two-forms inherited from the underlying complex structure. Thus there exists a division of cycles into $C^I, \bar{C}^I$ and $C^I_{\pm}$, with the index range understood to depend on which cycles are referred to, such that

$$
\int_{C^I} w^J = \delta^J_I, \quad \int_{\bar{C}^I} \bar{w}^J = \delta^J_I, \quad \int_{C^I_{\pm}} w^{\pm J} = \delta^J_I
$$

(75)

with all other integrals zero. With this choice of basis the Riemann bilinear identity implies that the integrals in equation (73) are just given by the inverse of the intersection matrix, for example $K_{IJ} = (I^{-1})_{I\!J}$ (note that the range of the indices varies so $J, K$ and $L$ are distinct). We note that this choice of basis is not that usually used in the physics literature. Often the intersection matrix is assumed to have a simple form, which does not depend on the moduli of the surface, whereas the self-duality condition on the forms contains a moduli dependent matrix. The integral of the forms over the two-cycles then leads to the period matrix of the manifold. Given an arbitrary set of forms and their self-duality properties it is a matter of linear algebra to compute the transformation on the cycles to find a basis that satisfies (75).

To illustrate this point, let us consider a Riemann surface of genus one which has a canonical set of one cycles $A$ and $B$ with the intersection matrix $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and a holomorphic one-form $\lambda$. We can always normalise $\lambda$ such that

$$
\int_A \lambda = 1, \quad \int_B \lambda = \tau
$$

(76)

Changing to the one cycles $C, \bar{C}$ by

$$
C = \frac{\bar{\tau}}{\tau - \bar{\tau}} A + \frac{1}{\tau - \bar{\tau}} B, \quad \bar{C} = \frac{\tau}{\tau - \bar{\tau}} A - \frac{1}{\tau - \bar{\tau}} B
$$

(77)
the integrals take the simple form

\[ \int_C \lambda = 1, \int_C \bar{\lambda} = 0, \int_{\bar{C}} \lambda = 0, \int_{\bar{C}} \bar{\lambda} = 1. \]  

(78)

However, the intersection matrix becomes

\[ \Omega = -\frac{1}{(\tau - \bar{\tau})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]  

(79)

In light-cone coordinates, \( X^\pm = X^0 \pm X^1 \), the self-duality conditions on \( U_{\mu I} \) and \( V^\pm_\mu \) are readily analysed. Indeed the only non-vanishing components of \( U_{\mu I} \) are \( U_{-I} \) and the only non-vanishing components of \( V^\pm_\mu \) are \( V^-_I \) and \( V^+_I \). The above equation of motions then take a very simple form with the above choice of cycles. For example, the equation for \( U_{\mu I} \) becomes

\[ \partial_+ U_{-K} + \partial_+ u^A U_{-I} (I^{-1})^{IJ} \frac{\partial I^{JK}}{\partial u^A} = 0, \]  

(80)

and there are similar equations for \( V^\pm_\mu \).

Thus we find a system of equations which describe the two-dimensional motion of scalar degrees of freedom. The \( u^A \) scalars which arise from the moduli of the four-cycle are neither left nor right moving, but the scalars the arise from the three-form field are either left handed or right handed, depending on the (anti-) self-duality of the two-cycle they correspond to. These equations describe the motion of the degrees of freedom of a black string, but making a trivial reduction on \( S^1 \) we find equations that describe the internal dynamics of a black hole.

Clearly, these equations are classically superconformally invariant, but their \((4,0)\) supersymmetry is not enough to ensure that they are superconformally invariant at the quantum level due to anomalies (see for example [27]). However, since these equations arise from an Abelian M-fivebrane which should have a consistent quantum theory, we might expect these equations to be superconformally invariant in the full quantum theory. This is also expected since the near horizon limit has an \( AdS^3 \) structure [17]. One can also enquire if the above system of dynamics is integrable. Clearly, their conformal invariance implies that they possess an infinite number of conserved quantities, whose currents are polynomials and moments of the energy momentum tensor. However, this not necessarily sufficient for integrability, since they may not distinguish between all the physical states.
Finally, since the fivebrane equations of motion are known in the presence of a non-trivial background \[3\], one could repeat the derivation in the presence of the background to find the low energy dynamics of the black hole in the presence of other matter. In this system one could study the interaction of external matter and a black hole in detail.

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Note Added:

While this paper was being written up, \[28\] appear on the hep-th archive, which provides a similar discussion to that given in section six.

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