Spontaneous Breaking of Parity in 2+1-Dimensional Thirring Model

Y.M. Ahn, B.K.Chung, J.-M.Chung and Q-Han Park

Research Institute for Basic Sciences and
Department of Physics, Kyunghee University
Seoul 130-701, Korea

ABSTRACT

A new aspect of the vacuum structure of 2+1-dimensional Thirring model is presented. Using the Fierz identity, we split the current-current four-Fermi interaction in terms of a matrix valued auxiliary scalar field and compute its effective potential. Energy consideration shows that contrary to earlier expectations, parity in general is spontaneously broken at any finite order of N, where N is the number of the two component spinors. In the large N limit, there does not exist a stable vacuum of the theory thereby making the application of the large N limit to Thirring model dangerous. A detailed analysis for parity breaking solutions in N=2,3 cases is given.

1 E-mail address; qpark@nms.kyunghee.ac.kr
It is well known that 2+1-dimensional QED admits dynamical mass generations.[1]-[3] Solving the Dyson-Schwinger gap equation of the theory, Appelquist et al. [4] have argued that in the large N limit, where N is the number of two-component fermions, masses are dynamically generated in such a way to preserve the overall parity even though each individual mass term violates parity symmetry. Recently, similar analysis has been applied to 2+1-dimensional Thirring model of vector-vector type four-Fermi interactions with a same type of dynamical mass generations.[5] The four-Fermi interaction model in 2+1 dimensions are also known to be renormalizable in the 1/N-expansion.[6]

In this Letter, we bring attention to the danger of this type of analysis, particularly in the case of 2+1-dimensional Thirring model. We analyze symmetry-breaking patterns in 2+1-dimensional Thirring model for finite N as well as for the large N limit. Using the Fierz identity, we split the current-current four-Fermi interaction in terms of a matrix valued auxiliary scalar field and instead of solving the Dyson-Schwinger gap equation, we compute the effective potential of the auxiliary field to study the dynamical mass generations. Energy consideration shows that contrary to earlier expectations, overall parity in general is spontaneously broken at any finite order of N. In the large N limit, there does not even exist a stable vacuum of the theory which brings doubts to the 1/N-analysis of 2+1-dimensional Thirring model. The discrepancy of our result with earlier works[4] based on the Dyson-Schwinger equation arises because in [4], the overall parity-breaking solution was discarded in the large N limit analysis since it exceeded the ultraviolet cut-off. However, our result shows that for finite N and for certain range of coupling constants the parity-breaking solution becomes a true vacuum. In the large N limit, it exceeds the ultraviolet cut-off as before which however makes other solutions semi-classically unstable. We argue that similar difficulty also arises for 2+1-dimensional QED.

Consider the 2+1-dimensional Thirring model given by the Lagrangian

\[ L = \bar{\psi}^i i\gamma^\mu \partial_\mu \psi^i - \frac{g}{2N} (\bar{\psi}^i \gamma_\mu \psi^i)^2 \]  

where \( \psi^i \) are two-component spinors and \( i \) runs over from 1 to N. In two-component representation, the Dirac \( \gamma \) matrices are given in terms of the Pauli matrices,

\[ \gamma^0 = \sigma_2 \ , \ \gamma^1 = i\sigma_3 \ , \ \gamma^2 = i\sigma_1 \ . \]
Note that the Lagrangian in Eq.(1) is invariant under the global $U(N)$ transformation as well as under the parity transformation $P$,

$$\begin{align*}
P : (x, y, t) &\rightarrow (-x, y, t) \\
\psi'(x') &\rightarrow \psi(x) = P\psi(x)P^{-1} = \sigma_1\psi(x) .
\end{align*}$$

(3)

It is believed that due to the strong coupling of fermions, the parity symmetry breaks down spontaneously at the quantum level which generates dynamical masses for fermions. However, it was argued that for even number of fermions $N = 2n$, half of the fermions $\psi^i ; i = 1, \ldots, n$ get mass $m$ while the other half $\psi^i ; i = n + 1, \ldots, 2n$ get mass $-m$ so as to preserve the overall parity $P_4 = PZ_2$ where $Z_2$ mixes fermions:[5]

$$
Z_2\psi^i(x) \rightarrow \psi^{n+i}(x) , \ Z_2\psi^{n+i}(x) \rightarrow \psi^i(x) ; i = 1, \ldots, n .
$$

(4)

This argument is based on the Dyson-Schwinger equation method in the large N limit which gives rise to the same type of dynamical parity breaking for 2+1-dimensional QED.[4] In the following, we show that contrary to the above argument the overall parity $P_4$ is in fact broken. To do so, we first note that the action Eq.(1) can be written via Fierz transformation

$$L = \bar{\psi}i\gamma^\mu \partial_\mu \psi + \frac{g}{2N}(\bar{\psi}\psi)^2 + \frac{g}{N}\bar{\psi}^j\psi^j\bar{\psi}^i\psi^i$$

(5)

where the four-Fermi interactions can be splitted by introducing a matrix-valued auxiliary field $M_{ij}$

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu + M)\psi - \frac{N}{4g}trM^2 + \frac{N}{4g(N + 2)}(trM)^2 .$$

(6)

This reduces to the original Lagrangian when $M_{ij}$ is eliminated by integration which identifies $M_{ij}$ with $M_{ij} = \frac{g}{N}\bar{\psi}^k\psi^k\delta^{ij} + \frac{2g}{N}\bar{\psi}^j\psi^i$. In order to understand the structure of vacuum, we compute the effective potential for $M_{ij}$ which amounts to summing the one loop fermion diagrams at zero momentum. Since the matrix $M$ is hermitian, $M$ may be diagonalized with real components $M = diag(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and the effective potential, invariant under the diagonalization, is given by

$$V_{eff} = -\frac{\Lambda^3}{6\pi^2} \sum_{i=1}^N ln(1 + \frac{\lambda_i^2}{\Lambda^2}) - \frac{1}{3\pi^2} \sum_{i=1}^N |\lambda_i|^3(tan^{-1}\frac{\lambda_i}{\Lambda} - \frac{\pi}{2}) + (\frac{N}{4g} - \frac{\Lambda}{3\pi^2})\sum_{i=1}^N \lambda_i^2 - \frac{N}{4g(N + 2)}(\sum_{i=1}^N \lambda_i)^2$$

(7)
where a cut-off has been introduced, after Wick rotation, at $k^2 = \Lambda^2$ in order that the integral be well defined. In terms of the dimensionless quantity $x_i \equiv \lambda_i / \Lambda$ and the rescaled coupling constant $\tilde{g} \equiv 2\Lambda g / 3\pi^2$, Eq.(7) becomes

$$
\tilde{V}_{eff} = -\sum_{i=1}^{N} \ln(1 + x_i^2) - 2 \sum_{i=1}^{N} |x_i|^3(tan^{-1}|x_i| - \frac{\pi}{2}) + \left(\frac{N}{\tilde{g}} - 2\right) \sum_{i=1}^{N} x_i^2 - \frac{N}{\tilde{g}(N+2)} \sum_{i=1}^{N} x_i^2. \tag{8}
$$

In the following, we analyze various symmetry-breaking patterns of the potential $\tilde{V}_{eff}$ according to the value of coupling constant $\tilde{g}$. First consider the case $N = 2$. The classical stability of the perturbative vacuum ($x_1 = 0, x_2 = 0$) is governed by the second derivatives of the potential $\tilde{V}_{eff}$ at $(0, 0)$:

$$
A \equiv \frac{\partial^2 \tilde{V}}{\partial x_1^2}|_{(0,0)} = -6 + \frac{3}{\tilde{g}}, \quad B \equiv \frac{\partial^2 \tilde{V}}{\partial x_1 \partial x_2}|_{(0,0)} = -\frac{1}{\tilde{g}}, \quad C \equiv \frac{\partial^2 \tilde{V}}{\partial x_2^2}|_{(0,0)} = -6 + \frac{3}{\tilde{g}} \tag{9}
$$

which imply that the perturbative vacuum is a local maximum if $B^2 - AC < 0$ and $A + C < 0$, a local minimum if $B^2 - AC < 0$ and $A + C > 0$, or a saddle point if $B^2 - AC > 0$. Thus, the potential $\tilde{V}_{eff}(x_1, x_2)$ and the perturbative vacuum $(0, 0)$ have the following properties:

i) $\tilde{g} < 0$: $\tilde{V}_{eff}$ is bell-shaped and $(0, 0)$ is the absolute maximum of $\tilde{V}_{eff}$ so that there is no stable vacuum.

ii) $0 < \tilde{g} < 1/3$: $\tilde{V}_{eff}$ is cup-shaped and $(0, 0)$ is the absolute minimum of $\tilde{V}_{eff}$ so that $(0, 0)$ is a stable vacuum.

iii) $1/3 < \tilde{g} < 2/3$: There are two local minima at $(m, m)$ and $(-m, -m)$ for $0 < m < 1$. $\tilde{V}_{eff}$ is double-well shaped when restricted on the line $x_1 = x_2$ and U-shaped when restricted on the line $x_1 = -x_2$ and $(0, 0)$ is a saddle point of $\tilde{V}_{eff}$. Thus, $(\pm m, \pm m)$ become stable vacua which break parity symmetry spontaneously.

iv) $2/3 < \tilde{g}$: There are four local minima at $(m_1, m_1), (-m_1, -m_1)$ and $(m_2, -m_2), (-m_2, m_2)$. $\tilde{V}_{eff}$ is double-well shaped when restricted on the line $x_1 = x_2$ as well as on the line $x_1 = -x_2$. $m_1 > m_2$ and $\tilde{V}_{eff}(\pm m_1, \pm m_1) < \tilde{V}_{eff}(\pm m_2, \mp m_2)$. Thus $(\pm m_1, \pm m_1)$ become stable vacua which break parity symmetry spontaneously. However numerical computation shows that for $\tilde{g} \gtrsim 1.5, m_1 > 1$. This implies that the dynamically generated mass exceeds the cut-off $\Lambda$ and therefore $(\pm m_1, \pm m_1)$ are not sensible vacua. Nevertheless, this makes $(\pm m_2, \mp m_2)$ semi-classically unstable so that there is no stable vacuum in this case.

For $N = 3$, similar analysis leads to the following properties of the potential;
i) $\tilde{g} < 0$: $(0, 0, 0)$ is the absolute maximum of $\tilde{V}_{\text{eff}}$ so that there is no stable vacuum.

ii) $0 < \tilde{g} < 2/5$: $(0, 0, 0)$ is the absolute minimum of $\tilde{V}_{\text{eff}}$ so that $(0, 0, 0)$ is a stable vacuum.

iii) $2/5 < \tilde{g} < 1$: $(0, 0, 0)$ is a saddle point and there are two local minima at $(\pm m, \pm m, \pm m)$ which become stable vacuua breaking parity spontaneously.

iv) $1 < \tilde{g}$: $(0, 0, 0)$ is a local maximum and there are six local minima with same potential value and two absolute minima at $(\pm m, \pm m, \pm m)$ breaking parity spontaneously which become sensible vacua for $\tilde{g} \lesssim 1.9$.

In order to understand the large $N$ behavior, we first consider the even $N$ case and then take the large $N$ limit. For $N = 2n$, we look at the special sector of the domain of the potential: $x_1 = x_2 = \cdots = x_n = x$ and $x_{n+1} = x_{n+2} = \cdots = x_{2n} = y$ where the symmetry breaking vacuua are expected to arise. Then the effective potential becomes

$$
\frac{2}{N} \tilde{V}_{\text{eff}}(x, y) = -[\ln(1 + x^2) + \ln(1 + y^2)] - 2[|x|^3(tan^{-1}|x| - \pi/2) + |y|^3(tan^{-1}|y| - \pi/2)]
+ (\frac{N}{\tilde{g}} - 2)(x^2 + y^2) - \frac{N^2}{2\tilde{g}(N + 2)}(x + y)^2.
$$

(10)

In terms of a rescaled coupling constant $g' \equiv 2\tilde{g}/N$, the structure of the effective potential $\frac{2}{N} \tilde{V}_{\text{eff}}(x, y)$ is the same as that of the $N = 2$ case with the replacement of range of $g'$: i) $g' < 0$ , ii) $0 < g' < 4/(2 + N)$ , iii) $4/(2 + N) < g' < 2/3$ , iv) $2/3 < g'$. For large $N$, the potential $\frac{2}{N} \tilde{V}_{\text{eff}}$ possesses local minima along the line $x = -y$ when $g' > 2/3$. However, in the limit $N \to \infty$ the potential becomes unbounded below when restricted on the line $x = y$ for all values of $g'$. Therefore, there does not exist a stable vacuum in the large $N$ limit! At first sight, this result seems to be in contradiction to earlier works based on the Dyson-Schwinger equation method. In the Dyson-Schwinger type analysis with the $1/N$-approximation, local minima arise along the line $x = -y$ which agrees with our result. However another set of solutions to the Dyson-Schwinger equation, which arise on the line $x = y$ and also in agreement with our results, were discarded because they exceed the momentum cut-off range. Our results show that these solutions can not be simply discarded since they make other local minima semi-classically unstable. Finally, we note that even though our analysis is only for the case of 2+1 dimensional Thirring model, similar vacuum instability in the large $N$ limit may exist in other 2+1-dimensional model as
well. In particular the vacuum structure of 2+1-dimensional QED was essentially the same as that of the Thirring model in the Dyson-Schwinger approach which suggests that the 1/N-approximation of 2+1-dimensional QED is as well dangerous.

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