The representation formula for solutions of some class Hamilton–Jacobi equations

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Abstract. The lower semicontinuous solutions of Hamilton–Jacobi equation are constructed by Hopf formula, when Hamiltonian is maximum of linear functions.

Keywords: Hamilton-Jacobi equations, lower semicontinuous solutions, Hopf formula.

1 Introduction

We consider the Cauchy problem for Hamilton–Jacobi equation of the form

\begin{equation}
  u_t + H(u_x) = 0,
\end{equation}

\begin{equation}
  u(0, x) = \varphi(x)
\end{equation}

in domain \( S = \{(t, x): t > 0, x \in \mathbb{R}^n\} \) with the lower semicontinuous (lsc) initial function \( \varphi \).

For Hamilton \( H \) is convex with respect to \( u_x \), A. Douglis, S.N. Kruzkov first defined the notion of the generalized (semiconcave) solution of (1), (2).

Definition 1. The Lipschitz continuous function \( u(t, x) \) in \( S_T \) is called the generalized (semiconcave) solution of (1), (2) if \( u(t, x) \) solves (1) a.e. on \( S_T \), satisfies (2), and for \( \forall L \in \mathbb{R}^n, \exists C_\delta > 0 \), that the inequality

\begin{equation}
  u(t, x + l) - 2u(t, x) + u(t, x - l) \leq C_\delta |l|^2,
\end{equation}

holds, when \( (t, x) \in S_T^\delta = \{(t, x): 0 < \delta \leq t \leq T, x \in \mathbb{R}^n\} \).

E. Hopf gave \cite{H} the representation formula for the semiconcave (1), (2) solutions.

Theorem 1. Suppose \( H(p) \) is convex and satisfies the coercivity condition

\begin{equation}
  \lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty.
\end{equation}

Let \( \varphi \in \text{Lip}(\mathbb{R}^n) \), then the semiconcave solution of (1), (2) can be represented by formula

\begin{equation}
  u(t, x) = \min_{\xi \in \mathbb{R}^n} \left[ \varphi(\xi) + t\Phi\left(\frac{x - \xi}{t}\right) \right],
\end{equation}

where \( \Phi(q) = \sup_{p \in \mathbb{R}^n} [(p, q) - H(p)] \) is the Legendre transform of \( H(p) \).
If is $H(p)$ strictly convex, S.N. Kruzkov proved [3], that formula (5) gives the semiconcave solution, when $\varphi(x)$ is bounded and lsc on $R^n$. The solution in this case satisfies initial condition in the sense

$$\lim_{t \to 0} u(t, x) = \varphi(x).$$

The function

$$F(t, x, \xi) = t\Phi\left(\frac{x - \xi}{t}\right)$$

satisfied the initial condition

$$F(0, x, \xi) = \begin{cases} 0, & x = \xi, \\ +\infty, & x \neq \xi \end{cases}$$

is called the fundamental solution of (1).

For example

$$u_t + a|u|_2^2 = 0,$$

$a > 0$, the Legendre transform of $H(p) = a|p|^2$ is $\Phi(q) = \frac{|q|^2}{4a}$, and the fundamental solution of (7) is

$$F(t, x, \xi) = \frac{|x - \xi|^2}{4at}.$$

2 The calculation of fundamental solutions

In order to define a function $\Phi(q)$ we need to solve the equation

$$x = H_p\varphi'(y)t + y$$

with respect $y$. In general we can not do it. It can be done when hamiltonian has the form

$$H(p) = \max_{i=1,\ldots,m} \left( (a^i, p) + b_i \right),$$

where $a^i, p \in R^n, b_i \in R$. We define the fundamental solution and prove the representation formula (5) for solutions of

$$u_t + \max_{i=1,\ldots,m} \left( (a^i, u_x) + b_i \right) = 0.$$  

(9)

Notice, that the coercivity condition (4) for the hamiltonian (8) is not satisfied.

Let $x \in R$. For the linear equation

$$u_t + a_i u_x + b_i = 0,$$

where $a_i = \text{const}$, the Legendre tranform of $H(p) = a_i p + b_i$ is

$$\Phi(q) = \begin{cases} -b_i, & q = a_i, \\ +\infty, & q \neq a_i, \end{cases}$$
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and the fundamental solution

\[ F(t, x, \xi) = \begin{cases} -b_i t, & \xi = x - a_i t, \\ +\infty, & \xi \neq x - a_i t. \end{cases} \]

The solution can be represented by formula

\[ u(t, x) = \min_{\xi \in \mathbb{R}^n} [\varphi(\xi) + F(t, x, \xi)] = \varphi(x - a_i t) - b_i t. \]

This solution does not satisfy the semiconcave property (3), when \( \varphi(x) = |x| \). Thus we need to consider the other class of generalized solutions of (1), (2), which has been defined in [1].

**Definition 2.** A lsc function \( u \) on \( S \) with values in \( \mathbb{R} \cup \{ +\infty \} \) is a lsc solution of (1), (2), if

\[ p_t + H(p_x) = 0, \]

for all \( (p_t, p_x) \in D^- u(t, x) \) (superdifferential), when \( u(t, x) < +\infty \), and

\[ \lim_{(t, y) \to (t, x)} u(t, y) = \varphi(x). \]

We use the theorem which was proved in this paper.

**Theorem 2.** Let \( \varphi : \mathbb{R}^n \to (-\infty, +\infty] \) be lsc and satisfy

\[ \varphi(x) \geq -C(|x| + 1), \quad C > 0, \quad x \in \mathbb{R}. \]

Let \( H \) be finite, continuous and convex. Then \( u \) defined by formula (5) is the unique lsc solution of (1), (2), that is bounded from below by a function of linear growth.

For the hamiltonians (8), suppose \( a_{i+1} > a_i \), the Legendre transform is

\[ \Phi(q) = \begin{cases} \frac{b_i - b_{i+1}}{a_{i+1} - a_i} (q - a_i) - b_i, & q \in [a_i, a_{i+1}], \\ +\infty, & q < a_i, \quad q > a_m. \end{cases} \]

Then the function

\[ F(t, x, \xi) = \begin{cases} \frac{b_i - b_{i+1}}{a_{i+1} - a_i} (x - \xi - a_i t) - b_i t, & \xi \in [x - a_{i+1} t, x - a_i t], \quad i = 1, \ldots, m - 1, \\ +\infty, & \xi < x - a_m t, \quad \xi > x - a_1 t, \end{cases} \]

is convex, satisfies a.e. (9) in \( \{(t, x) : x \in [\xi + a_1 t, \xi + a_m t]\} \) and the initial condition (6), thus, from the last theorem we have, that it is the unique fundamental solution of (9).

**Example 1.** Suppose we have the Cauchy problem

\[ u_t + |u_x| = 0, \]

\[ u(0, x) = \sin x. \]
Then
\[ \Phi(q) = \begin{cases} 
0, & q = [-1, 1], \\
+\infty, & q < -1, \; q > 1,
\end{cases} \]
\[ F(t, x, \xi) = \begin{cases} 
0, & \xi \in [x-t, x+t], \\
+\infty, & \xi < x-t, \; \xi > x+t,
\end{cases} \]
and the viscosity solution can be represented by formula
\[ u(t, x) = \min_{\xi \in [x-t, x+t]} \sin(\xi). \]

It is clear, that if we construct the Legendre transform of hamiltonian (8), then we easy define the fundamental solution. Next we explain, how we can define the Legendre transform, when \( x \in \mathbb{R}^n, \; n > 1 \).

Let \( x \in \mathbb{R}^2 \). Then the Legendre transform of
\[ H(p_1, p_2) = \max_{i=1,\ldots,m} \left( (a^1_ip_1 + a^2_ip_2) + b_i \right) \]
can be constructed in such way:

if \( m = 1 \), then
\[ \Phi(q) = \begin{cases} 
-b_i, & q = a^i, \\
+\infty, & q \neq a^i,
\end{cases} \]

if \( m = 2 \), then \( \Phi(q) \) is defined in the parametric form
\[
\begin{align*}
\Phi(s) &= (b_1 - b_2)s - b_1, \\
q_1 &= (a^2_1 - a^1_1)s + a^1_1, \\
q_2 &= (a^2_2 - a^1_2)s + a^1_2,
\end{align*}
\]
where \( s \in [0, 1] \), in other points of \( \mathbb{R}^2 \) the function \( \Phi(q) = +\infty \),

if \( m \geq 3 \), then define \( Q = \text{co}\{a^i\} \)-convex hull of set \( \{a^i, \; i = 1,\ldots,m\} \) and \( Q_k = \text{co}\{a^{k_1}, a^{k_2}, a^{k_3}\} \), where \( k_1, k_2, k_3 \in \{1,\ldots,m\} \), and \( a^i \notin Q_k \), when \( i \notin \{k_1, k_2, k_3\} \). Then \( \Phi(q) = \max_k \{ (\alpha_k, q) + \beta_k \} \), when \( q \in Q \), and \( \Phi(q) = +\infty \), if \( q \notin Q \). The coefficients \( \alpha_k, \beta_k \) are determined from the identity
\[
\begin{vmatrix}
q_1 - a^1_{k_1} & q_2 - a^2_{k_1} & (\alpha_k, q) + \beta_k + b_{k_3} \\
q_1 - a^1_{k_2} & q_2 - a^2_{k_2} & b_{k_1} - b_{k_2} \\
q_1 - a^1_{k_3} & q_2 - a^2_{k_3} & b_{k_1} - b_{k_3}
\end{vmatrix} = 0.
\]

The similar structure of the Legendre transform for the hamiltonians (7) may be realized in \( \mathbb{R}^n \).
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References

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REZIUMĖ

**Apie Hamiltono-Jakobi lygčių sprendinių išraiškas**

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Straipsnyje analizuojamos Hamiltono-Jakobi lygčių sprendinių išraiškos, kai hamiltonianas užduodamas kaip tiesinių funkcijų gaubiamoji.

**Raktiniai žodžiai:** Hamiltono-Jakobi lygtyms, pusiautolydžiai iš apačios sprendinio, Hopfo formulė.