ARI, GARI, ZIG and ZAG

An introduction to Ecalle’s theory of multiple zeta values

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The text has two goals. The first is to give an introduction to Ecalle’s work on mould theory, multiple zeta values and double shuffle theory and relate this work explicitly to the classical theory of multiple zeta values and double shuffle expressed in the usual terms of two non-commutative variables. The second is to provide complete proofs of those of his main statements and identities which are useful in the context of (non-colored) multiple zeta values. Many of these proofs were never written down by Écalle. Some of them are difficult, laborious and not enlightening, yet it is clearly necessary to have them in order to be able to apply with confidence a theory that, once in place, forms an astonishingly powerful toolbox with many applications. Of these laborious proofs, some have been relegated to appendices and others, which appear in full in separate publications, simply cited.

The emphasis in this text is to provide an easily approachable introduction to Ecalle’s language while placing it almost from the start in the context of multiple zeta value theory.

Disclaimer: This text is not final and is not submitted for publication. The intention is to continue to add to and complete it over time.
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Chapter 1

Real and formal multiple zeta values

In this first chapter, we introduce some of the basic objects of study in the classical theory; the algebras of real and formal multiple zeta values, the real and formal Drinfel’d associators, the double shuffle Lie algebra, and the weight grading and depth filtrations. Everything in this chapter is well-known and has been written in detail elsewhere, so we content ourselves with recalling the main definitions and facts without proof.

§1.1. Multiple zeta values and their regularizations

For every sequence \( k = (k_1, \ldots, k_r) \) of strictly positive integers with \( k_1 \geq 2 \), let \( \zeta(k_1, \ldots, k_r) \) be the multiple zeta value defined by

\[
\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}. \tag{1.1.1}
\]

For every word in \( \mathbb{Q}(x, y) \), we define a multiple zeta value \( \zeta(w) \) as follows. If \( w \) starts with \( x \) and ends with \( y \), we write \( w = x^{k_1 - 1} y \cdots x^{k_r - 1} y \) with \( k_1 \geq 2 \), and set \( \zeta(w) = \zeta(k_1, \ldots, k_r) \).

For general \( w \), we write \( w = y^r vx^s \) and set

\[
\zeta(w) = \sum_{a=0}^{r} \sum_{b=0}^{s} (-1)^{a+b} \zeta(\pi(sh(y^a, y^{r-a} v x^{s-b}, x^b)), \tag{1.1.2}
\]

where \( \pi \) is the projection of a polynomial onto the convergent words, i.e. those starting with \( x \) and ending with \( y \), and \( \zeta \) is considered to be additive. This way of extending the real multizeta values of convergent words (called convergent multizeta values) to all words is called the shuffle regularization, because of the following property that characterizes it.

Definition. The shuffle product of two words \( u \) and \( v \) in an alphabet \( \mathcal{X} \) is defined recursively by \( sh(u, 1) = sh(1, u) = u \) and \( sh(Xu, Yv) = X sh(u, Yv) + Y sh(Xu, v) \) for any letters \( X, Y \in \mathcal{X} \).

The path leading to the formula given in (1.1.2) is not a short one, starting as it does by using standard regularization techniques to give regularized values to the non-convergent multizeta values in the form of integrals over simplices ([LM]). The explicit formula (1.1.2) was established by H. Furusho in [F] (Prop. 3.2.3).

Examples. We use the notation in which the shuffle of two words is written as a formal sum of words. Taking \( \mathcal{X} = \{a, b, c, d\} \), we have

\[
sh((ab), (cd)) = abcd + acbd + acdb + cabd + cdab + cadb + cdab.
\]
Taking $X = \{x, y\}$, we thus have

\[ sh((x, y), (x, y)) = 4xyy + 2xyy. \]

**Theorem 1.1.1.** For all words $u, v \in \mathbb{Q}\{x, y\}$, the regularized $\zeta$ values defined in (1.1.2) satisfy the shuffle relations

\[ \zeta(sh(u, v)) = \zeta(u)\zeta(v) \] (1.1.3)

in the alphabet $X = \{x, y\}$.

Multiple zeta values possess a second interesting multiplicative property.

**Definition.** Let $\mathcal{Y}$ be an additive alphabet, i.e. a set equipped with an addition rule such that for every pair of letters $X, Y \in \mathcal{Y}$, $X + Y$ is also an element of $\mathcal{Y}$. The stuffle product in the additive alphabet $\mathcal{Y}$ is defined recursively by

\[ st(u, 1) = st(1, u) = u \]

and

\[ st(Xu, Yv) = Xst(u, Yv) + Yst(Xu, v) + (X + Y)st(u, v) \] (1.1.4)

for all letters $X, Y \in \mathcal{Y}$.

An equivalent formulation of the stuffle product is given by

\[ st(u, v) = \sum_{\sigma \in Sh_{\leq}(r, s)} c^\sigma(u, v) \] (1.1.5)

where $u$ is a word in $r$ letters and $v$ in $s$ letters, $Sh_{\leq}(r, s)$ is the set of surjective maps

\[ \sigma : \{1, \ldots, r + s\} \rightarrow \{1, \ldots, N\} \]

for all $1 \leq N \leq r + s$ such that

\[ \sigma(1) < \cdots < \sigma(r) \quad \text{and} \quad \sigma(r + 1) < \cdots < \sigma(r + s), \]

and for each $\sigma \in Sh_{\leq}(r, s)$, we set $c^\sigma(u, v) = (c_1, \ldots, c_N)$ with

\[ c_i = \sum_{k \in \sigma^{-1}(i)} a_k. \] (1.1.6)

By the definition of $Sh_{\leq}(r, s)$, $c_i$ is either a single letter $a_k$ or a sum of two letters $a_k + a_l$ with $k \leq r < l$.

**Examples.** Let $\mathcal{A}$ be an additive alphabet; then we have

\[ st(a, b) = (a, b) + (b, a) + (a + b) \]

\[ st((a, b), (c)) = abc + acb + cab + (a + b, c) + (a, b + c) \]

\[ st((a, b), (b)) = 2(a, b, b) + (b, a, b) + (a + b, b) + (a, 2b). \]
Considering the additive alphabet \( \mathbb{N}^+ \), we have for example

\[
\text{st}((2, 1), (2)) = 2(2, 2, 1) + (2, 1, 2) + (4, 1) + (2, 3).
\]

In a different notation that will be used often below, let \( \mathcal{Y} = \{y_1, y_2, y_3, \ldots \} \) with the addition rule \( y_i + y_j = y_{i+j} \). This is identical to considering the alphabet \( \mathbb{N}^+ \) except that the numbers now appear as indices. We have for example

\[
\text{st}((y_1), (y_2, y_3)) = (y_1, y_2, y_3) + (y_2, y_1, y_3) + (y_2, y_3, y_1) + (y_3, y_1, y_2) + (y_2, y_4).
\]  

(1.1.7)

For all convergent words \( u, v \), considered to be written in the variables \( y_i = x^{i-1}y \), the convergent multizeta values satisfy the \textit{stuffle relations} \( \zeta(\text{st}(u, v)) = \zeta(u)\zeta(v) \) in the alphabet \( \mathcal{Y} = \{y_i| i \geq 0\} \), considered to be additive via the rule \( y_i + y_j = y_{i+j} \). This result follows easily from the expression of \( \zeta(k_1, \ldots, k_r) \) as a power series. But there is a second regularization of the zeta values, called the \textit{stuffle regularization}, extending the stuffle relation to all words in the \( y_i \). It is defined as follows.

**Definition.** The \textit{Drinfel’d associator} \( \Phi_{KZ} \) is given by

\[
\Phi_{KZ} = 1 + \sum_{w \in \mathcal{Q}(x, y)} (-1)^{d(w)}\zeta(w)w,
\]

(1.1.8)

where for each monomial \( w \) in \( x, y \), \( d(w) \) denotes the \textit{depth} of \( w \), which is the number of \( y \)'s occurring in the word \( w \). Let \( \Phi \) denote the \textit{double shuffle power series} defined by \( \Phi(x, y) = \Phi_{KZ}(x, -y) \), so

\[
\Phi(x, y) = 1 + \sum_w \zeta(w)w.
\]

Let \( \pi_y \) denote the projection of power series onto their words ending in \( y \), rewritten in the \( y_i \). Set

\[
\Phi_* = \exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \zeta(y_n)y_1^n\right)\pi_y(\Phi),
\]

(1.1.9)

and for every word \( v \) in the \( y_i \), define \( \zeta^*(v) \) to be the coefficient of the word \( v \) in \( \Phi_* \), denoted \( (\Phi_*|v) \). Since the exponential “correction” factor is a power series in \( y_1 \), it follows that for any convergent word \( v \) (i.e. any word in the \( y_i \) not starting with \( y_1 \)), we have \( \zeta^*(v) = \zeta(v) \). Inversely, the stuffle-regularized values \( \zeta^*(1, \ldots, 1) \) come entirely from the correction factor and are all polynomials in the single zeta values \( \zeta(n) \); we see for instance that

\[
\zeta^*(1) = \zeta(1) = 0, \quad \zeta^*(1, 1) = -\frac{1}{2}\zeta(2), \quad \zeta^*(1, 1, 1) = \frac{1}{3}\zeta(3),
\]

\[
\zeta^*(1, 1, 1, 1) = -\frac{1}{4}\zeta(4) + \frac{1}{8}\zeta(2)^2 = -\frac{1}{4}\zeta(4) + \frac{5}{16}\zeta(4) = \frac{1}{16}\zeta(4);
\]

thus, we can write the correction factor as

\[
\exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \zeta(y_n)y_1^n\right) = \sum_{n \geq 1} \zeta^*(1, \ldots, 1)y_1^n.
\]

(1.1.10)
For words of the form $w = y_1^j v$ with $v$ a word in the $y_i$ not starting with $y_1$, the stuffle regularized multizeta values are given by the formula

$$
\zeta^*(w) = \left(\Phi_*|v\right) = \sum_{j=0}^{i} \zeta^*(1, \ldots, 1)(\Phi|y_1^{i-j}v).
$$

(1.1.11)

The values $\zeta^*(v)$ are called the stuffle regularization of the convergent multizeta values, because of the following theorem.

**Theorem 1.1.2.** For all words $u, v$ in the variables $y_i$, the values $\zeta^*(v)$ satisfy the stuffle relations

$$
\zeta^*(st(u, v)) = \zeta^*(u)\zeta^*(v).
$$

(1.1.12)

**Remark.** Theorems 1.1.1 and 1.1.2 are part of the classical theory of multizeta values, proved originally by Drinfel’d in the form of the two following statements on $\Phi_{KZ}$:

(i) $\Phi_{KZ} \in \mathbb{Q}\langle\langle x, y \rangle\rangle$ is group-like for the coproduct $\Delta$ defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\Delta(y) = y \otimes 1 + 1 \otimes y$.

(ii) $\Phi^* \in \mathbb{Q}\langle\langle y_1, y_2, \ldots \rangle\rangle$ is group-like for the coproduct $\Delta^*$ defined by

$$
\Delta^*(y_i) = \sum_{k+l=i} y_k \otimes y_l.
$$

Theorems 1.1.1 and 1.1.2 are direct translations of these two properties on power series into multiplicative properties of the coefficients of those power series (cf. [R] for a detailed exploration of these facts).

**Definition.** Let $Z$ denote the $\mathbb{Q}$-algebra generated by the convergent multizeta values under the multiplication law (1.1.3). By (1.1.2) and (1.1.11), $Z$ contains all the shuffle and stuffle regularized multizeta values. For every word $w \in \mathbb{Q}\langle x, y \rangle$ of length (i.e. degree) $n$ containing $r$ $y$’s, the corresponding multiple zeta value $\zeta(w)$ is said to be of weight $n$ and depth $r$. For each $n \geq 0$, let $Z_n$ denote the $\mathbb{Q}$-vector space generated by the convergent multiple zeta values of weight $n$. We have $Z_0 = \mathbb{Q}$, $Z_1 = \langle 0 \rangle$, $Z_2 = \langle \zeta(2) \rangle$.

The algebra $Z$ has a rich structure of which the shuffle and stuffle families of algebraic relations (known as the double shuffle relations) are only one aspect. There are many other known algebraic relations between elements of $Z$, and also, of course, difficult problems of transcendence and irrationality. Few results are known on the transcendence; the fundamental conjecture that all multiple zeta values are transcendent still seems far out of reach.

The transcendence conjecture can be subsumed into the following seemingly simple structural conjecture on $Z$.

**Main transcendence conjecture.** The weight provides a grading of the $\mathbb{Q}$-algebra $Z$; in other words, there are no linear relations between multizeta values of different weights.
This assumption indeed implies that every multizeta value is transcendent, since otherwise, if some \( \zeta \) of weight \( n \) were algebraic, there would be a minimal polynomial \( P(x) \) such that \( P(\zeta) = 0 \); each term of the polynomial would be a \( \zeta^i \), which when expanded out as a sum by the shuffle multiplication rule would yield a non-zero linear combination of multizetas of weight \( in \), and the sum of all these terms of different weights would be zero, contradicting the main conjecture.

The conjectures concerning transcendence seem unprovable for the time being, but the combinatorial/algebraic structure of the multizeta algebra is still a rich subject of study, with another conjecture specifically concerning algebraic relations.

**Main algebraic conjecture.** The “regularized” double shuffle relations (1.1.3) and (1.1.12) generate all algebraic relations between multizeta values.

This conjecture makes it natural to focus attention on the double shuffle relations. For this purpose, it is useful to define a *formal multiple zeta algebra* of transcendent symbols satisfying only the regularized double shuffle relations, and investigate its structure. This algebra, defined in the next section, is one of the main objects of study in the theory of multiple zeta values.

### §1.2. Formal multiple zeta values

For every word \( w \) in \( x \) and \( y \), let \( \overline{Z}(w) \) denote a formal symbol associated to \( w \), and let \( \mathbb{Q}[\overline{Z}(w)] \) be the commutative \( \mathbb{Q} \)-algebra generated as a vector space by these symbols, equipped with the multiplication law

\[
\overline{Z}(u)\overline{Z}(v) = \overline{Z}(sh(u, v)).
\]  

(1.2.1)

Let \( SH \) be the quotient of \( \mathbb{Q}[\overline{Z}(w)] \) by the linear relations analogous to (1.1.2)

\[
\overline{Z}(w) = \sum_{a=0}^{r} \sum_{b=0}^{s} (-1)^{a+b} \overline{Z}(\pi(sh(y^a, y^{r-a}v^ax^{a-b}, x^b)))
\]

(1.2.2)

for every non-convergent word \( w \). As in theorem 1.1, this definition ensures that the multiplication law (1.2.1) passes to the quotient \( SH \). We write \( \tilde{Z}(w) \) for the image of \( \overline{Z}(w) \) in \( SH \).

In analogy with (1.1.9), we define \( \tilde{Z}^*(1, \ldots, 1) \) to be the coefficient of \( y_1^n \) in the formal power series with coefficients in \( SH \)

\[
\exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \tilde{Z}(y_n)y_1^n\right),
\]

so they are polynomials in the \( \tilde{Z}(y_i) \); note that all polynomials in the \( \tilde{Z}(w) \) can be expressed as linear combinations of convergent multizetas by using the multiplication rule (1.2.1) and
then (1.2.2). In analogy with (1.1.11), we set

\[ \tilde{Z}^*(w) = \sum_{j=0}^{i} \tilde{Z}^*(1, \ldots, 1)(\Phi|_{y_1^{i-j}v}) = \sum_{j=0}^{i} \tilde{Z}^*(1, \ldots, 1)\tilde{Z}(y^{i-j}v), \]  

for every word \( w = y_1^i v \) where \( v \) is a word in the \( y_i \) not starting with \( y_1 \); thus these values can also be expressed as linear combinations of convergent \( \tilde{Z}(w) \). Therefore, \( SH \) is generated as a vector space by the \( \tilde{Z}(w) \) for convergent \( w \).

Let \( FZ \), the formal multizeta algebra, be the vector space quotient of \( SH \) by the relations

\[ \tilde{Z}^*(st(u,v)) = \tilde{Z}^*(u)\tilde{Z}^*(v), \]

which although they appear algebraic, can be written as above as linear relations between the convergent \( \tilde{Z}(w) \). The multiplication (1.2.1) passes to \( FZ \), making it into a \( Q \)-algebra. We write \( Z(w) \) for the image of \( \tilde{Z}(w) \) in \( FZ \).

By definition, we have a surjection \( FZ \to Z \). But the space \( FZ \) is easier to study than \( Z \) because the real multizeta values satisfy unknown numbers of other relations, including, as explained in 1.1, the fact that it is not even known whether they are transcendent, or whether there are any linear relations between real multizeta values of different weights. It is tempting to conjecture that \( FZ \cong Z \), but pending any kind of knowledge about the transcendence properties of real multizeta values, we adopt the strategy of replacing the real value algebra by the formal multizeta algebra \( FZ \) as the main object of study in the combinatorial/algebraic theory of multizetas.

By definition, \( FZ \) is a graded algebra, with \( FZ_0 = Q \), \( FZ_1 = 0 \) and \( FZ_2 \) a one-dimensional space generated by \( Z(2) = Z(xy) \) (as for real multizetas, we use the notation \( Z(k_1, \ldots, k_r) = Z(x^{k_1-1}y \cdots x^{k_r-1}y) \)). Let \( NFZ \) denote the quotient of \( FZ \) by the ideal generated by \( Z(2) \).

Let \( NFZ^2 \) denote the quotient of \( NFZ \) modulo the ideal generated by \( FZ_0 \) and products \( FZ^2_{>0} \). Known as the new formal zeta space, lifts of its generators to \( NFZ \) form a set of ring generators. In fact, \( NFZ^2 \) is more than just a vector space. An important and difficult theorem due to Racinet states that the dual of \( NFZ^2 \) is a Lie algebra, known as the double shuffle Lie algebra \( \mathfrak{ds} \) (see next section). Thus \( NFZ^2 \) is a Lie coalgebra, and \( NFZ \) is a Hopf algebra. In Chapter 4, we give the neat and simple theoretical proof of Racinet’s theorem that emerges easily from Ecalle’s theory.

The following section is devoted to the Lie algebra \( \mathfrak{ds} \), which is one of the main points of focus of the entire theory, thanks to the simplicity of its definition and the concrete nature of its elements, which make it into a valuable and attractive “way in” to the theory, accessible to explicit computation.

§1.3. The double shuffle Lie algebra \( \mathfrak{ds} \)

Definition. The Lie algebra \( \mathfrak{ds} \) is the dual of the Lie coalgebra \( NFZ^2 \) of new formal multizeta
values. It can be defined directly as the set of polynomials \( f \in \mathbb{Q}(x,y) \) having the two following properties.

(1) The coefficients of \( f \) satisfy the **shuffle relations**

\[
\sum_{w \in sh(u,v)} (f|w) = 0,
\]

where \( u, v \) are words in \( x, y \) and \( sh(u,v) \) is the set of words obtained by shuffling them. This condition is equivalent to the assertion that \( f \in \text{Lie}[x,y] \).

(2) Let \( f_\ast = \pi_y(f) + f_{\text{corr}} \), where \( \pi_y(f) \) is the projection of \( f \) onto just the words ending in \( y \), and

\[
f_{\text{corr}} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (f|x^{n-1}y)y^n.
\]

(When \( f \) is homogeneous of degree \( n \), which we usually assume, then \( f_{\text{corr}} \) is just the monomial \( \frac{(-1)^n}{n} (f|x^{n-1}y)y^n \).) The coefficients of \( f_\ast \) satisfy the **stuffle relations**:

\[
\sum_{w \in st(u,v)} (f_\ast|w) = 0,
\]

where now \( u, v \) and \( w \) are words ending in \( y \), considered as rewritten in the variables \( y_i = x^{i-1}y \), and \( st(u,v) \) is the stuffle of two such words.

For every \( f \in \text{Lie}[x,y] \), define a derivation \( D_f \) of \( \text{Lie}[x,y] \) by setting it to be

\[
D_f(x) = 0, \quad D_f(y) = [y,f]
\]
on the generators. Define the **Poisson bracket** on (the underlying vector space of) \( \text{Lie}[x,y] \) by

\[
\{f,g\} = [f,g] + D_f(g) - D_g(f).
\]

This definition corresponds naturally to the Lie bracket on the space of derivations of \( \text{Lie}[x,y] \); indeed, it is easy to check that

\[
[D_f,D_g] = D_f \circ D_g - D_g \circ D_f = D_{\{f,g\}}.
\]

**Definition.** Let \( \mathbf{L} \) denote the Lie algebra generated by the polynomials \( C_i = \text{ad}(x)^{i-1}(y), \quad i \geq 1 \) inside \( \mathbb{Q}(x,y) \). We have \( \text{Lie}[x,y] = \mathbb{Q}x \oplus \mathbf{L} \), and it is a standard result of Lazard elimination that the \( \text{ad}(x)^{i-1}(y) \) generate \( \mathbf{L} \) freely. The **twisted Magnus Lie algebra** \( \mathbf{mt} \) is defined to be the Lie algebra whose underlying vector space is \( \mathbf{L} \), but equipped with the Poisson bracket (1.3.4).

In his 2000 Ph.D. thesis, G. Racinet proved the following theorem, using a complicated series of arguments later condensed and reworked in the appendix to [Furusho]. In Chapter
of this text, we show how this result drops naturally and easily out of Ecalle’s theory once the basic machinery has been established.

**Theorem 1.3.1.** The double shuffle space $\mathfrak{ds}$ is a Lie algebra under the Poisson bracket, i.e. $\mathfrak{ds}$ is a Lie subalgebra of $\mathfrak{mt}$.

This theorem raises the question of the Lie algebra structure of $\mathfrak{ds}$, which has given rise to a great deal of conjectures and computations.

**Structure conjecture for $\mathfrak{ds}$.** The Lie algebra $\mathfrak{ds}$ is freely generated by one generator of weight $n$ for each odd $n \geq 3$.

In 2010, an important breakthrough by F. Brown concerning motivic multiple zeta values had, as a consequence, the result that the free Lie algebra on one generator in each odd weight $\geq 3$ does have a canonical injection into $\mathfrak{ds}$. For the rest, this is still a wide open question.

The double shuffle Lie algebra inherits a grading from $\text{Lie}[x, y]$, corresponding to the degree (weight) of polynomials. We write $\mathfrak{ds}_n$ for the graded part of weight $n$. It is also equipped with an increasing depth filtration

$$\mathfrak{ds}^1 \subset \mathfrak{ds}^2 \subset \cdots$$

where $f \in \mathfrak{ds}$ lies in $\mathfrak{ds}^d$ if the smallest number of $y$’s appearing in any monomial of $f$ is greater than or equal to $d$. The depth filtration is not a grading because there are known (so-called “period polynomial”) linear combinations of elements of depth $d$ which are themselves in depth $> d$. This filtration is dual to the decreasing filtration on $\mathbb{Z}$ given by letting the depth of $\zeta(k_1, \ldots, k_r)$ be equal to $r$. Again, this is a filtration rather than a grading since there can be linear relations mixing depths. The first example was already known to Euler: $\zeta(2,1) = \zeta(3)$.

The following theorem is more or less “folklore”, but the only published proof so far appears to be the one in [IKZ] (which actually proves the slightly stronger Theorem 1.4.1 in the next section), which uses some rather astute combinatorics.

**Theorem 1.3.2** Let $n \geq 3$, $d \geq 1$. Then the quotient space $\mathfrak{ds}_n^d/\mathfrak{ds}_n^{d+1}$ is equal to 0 if $d \not\equiv n \mod 2$.

In Chapter 3, §3.4, we show how the proof of this result (or rather, of Theorem 1.4.1 below) falls out as an easy consequence of Ecalle’s methods.

Theorem 1.3.2 is just one special case of another structure conjecture for $\mathfrak{ds}$, that is much finer than the previous one. Let $BK(X, Y)$ denote the Broadhurst-Kreimer function of two commutative variables defined by

$$BK(X, Y) = \frac{1}{1 - \mathcal{O}(X)Y + \mathcal{S}(X)Y^2 - \mathcal{S}(X)Y^4}, \quad (1.3.6)$$

where $\mathcal{O}(X) = X^3/(1 - X^2)$ and $\mathcal{S}(X) = X^{12}/(1 - X^4)(1 - X^6)$. Let $\mathcal{U}\mathfrak{ds}$ denote the universal enveloping algebra of $\mathfrak{ds}$. Then $\mathcal{U}\mathfrak{ds}$ is automatically equipped with a weight grading and depth filtration corresponding to those of $\mathfrak{ds}$. The following conjecture was
formulated by Broadhurst and Kreimer for real multiple zetas, but it applies just as well to formal ones.

**Broadhurst-Kreimer structure conjecture for \( ds \).** For all \( n \geq 3 \) and \( d \geq 1 \), the coefficient of \( X^n Y^d \) in the Taylor expansion of \( BK(X, Y) \) is the dimension of the graded quotient space \( Uds_n^d/Uds_n^{d+1} \).

Note in particular that all terms of the Taylor expansion of \( O(X) \) are of odd degree, so in the Taylor expansion of \( O(X)Y \) the coefficients of terms where \( n \not\equiv d \) mod 2 are all 0, and the same is even more obvious for the terms \( S(X)Y^2 \) and \( S(X)Y^4 \) which contain only monomials in which \( n \) and \( d \) are even. Thus Theorem 1.3.2 would be a corollary of the Broadhurst-Kreimer structure conjecture. Furthermore, ignoring the depth filtration comes down to setting \( Y = 1 \), so the Broadhurst-Kreimer conjecture can be simplified to a conjecture purely on the weight-grading of \( Uds \), namely the dimension of the graded piece \( Uds_n \) is given by the coefficient of \( X^n \) in the generating series

\[
\frac{1}{1 - O(X)Y} = \frac{1 - X^2}{1 - X^2 - X^3}.
\]

This is well-known to be the generating series for the graded dimensions of the free algebra on one generator in each odd weight \( n \geq 3 \), which is the universal enveloping algebra of the free Lie algebra on the same generators. Thus the Broadhurst-Kreimer conjecture also implies the free-generation structure conjecture on \( ds \) given above.

§1.4. **The linearized double shuffle space**

**Definition.** The linearized double shuffle space \( ls \) is defined to be the set of polynomials in \( x, y \) of degree \( \geq 3 \) satisfying the shuffle relations (1.3.1) (i.e. belonging to the free Lie algebra \( \text{Lie}[x, y] \)) and a second set of relations given by

\[
\sum_{w \in sh(u, v)} (\pi_y(f)|w) = 0,
\]

where \( \pi_y(f) \) is the projection of \( f \) onto the words ending in \( y \), rewritten in the variables \( y_i = x^{i-1}y \), \( u, v \) are words in the \( y_i \) and \( w \) belongs to their shuffle in the alphabet \( y_i \). However, we exclude from \( ls \) all (linear combinations of) the depth 1 even degree polynomials, namely \( ad(x)^{2n+1}(y), n \geq 1 \). Note that the condition (1.4.1) is empty on the depth 1 polynomials, so including or excluding them is essentially a convention.

The space \( ls \) is not only graded by weight, but also by depth, since unlike the shuffle relations (1.3.1), the shuffle relations (1.4.1) respect the depth. We write as usual \( ls_n \) for the graded part of weight \( n \) and \( ls^d \) for the graded part of depth \( d \).

**Proposition 1.4.1.** The associated graded for the depth filtration of \( ds \) is contained in \( ls \); i.e. in weight \( n \geq 3 \) and depth \( d \geq 1 \), we have

\[
ds_n^d/\mathcal{I}^d_n \subset ls^d.
\]
Proof. It is immediate that for any \( f \in \mathfrak{ds} \), if \( \overline{f} \) is obtained from \( g \) by taking only the terms of minimal depth (i.e. minimal number of \( y \)'s), then \( \overline{f} \in \mathfrak{ls} \). Indeed, if \( d \) is the (minimal) depth of \( g \), then the stuffle relations of depth \( d \) are actually shuffle relations since the additional terms in the stuffle where indices are “stuffed” together are words of smaller depth, and therefore have coefficient 0 in \( f \). Thus the truncations in minimal weight of elements \( f \in \mathfrak{ds} \) all satisfy the linearized double shuffle relations, showing (1.4.2).

The only point that needs some care is the case \( d = 1 \), where the odd degree polynomials \( ad(x)^{2n+1}(y) \) have been excluded from \( \mathfrak{ls} \). Therefore we need a separate argument in order to check (1.4.2) in the case \( d = 1 \); it is necessary to show that there is no element in \( \mathfrak{ds} \) of depth 1 and even weight. The proof we give here appears in complete detail in [C, Theorem 2.30 (i)]. By explicitly solving the depth 2 stuffle relations for \( f \in \mathfrak{ds} \), given by

\[
(f|x^iyx^{n-2-i}y) + (f|x^{n-2-i}yx^iy) + (f|x^{n-1}y) = 0,
\]

one finds that

\[
(f|x^{n-2}y^2) = \frac{n-1}{2}(f|x^{n-1}y).
\]

Now suppose that \( f \in \mathfrak{ds} \) is of even weight \( n \) and of depth 1, i.e. the coefficient \((f|x^{n-1}y)\neq 0\). Since every Lie polynomial satisfies \( f = (-1)^{n-1}\overline{f} \) where \( \overline{f} \) denotes the polynomial \( f \) written backwards (i.e. with each monomial in \( x \) and \( y \) written backwards), if \( n \) is even then \( f \) can contain no palindromic words. Therefore in particular \((f|yx^{n-2}y) = 0\), and so the relation (1.4.3) for \( i = n-2 \), given by

\[
(f|x^{n-2}yy) + (f|yx^{n-2}y) + (f|x^{n-1}y) = 0,
\]

simplifies to

\[
(f|x^{n-2}yy) = -(f|x^{n-1}y),
\]

contradicting (1.4.4). This concludes the proof that no depth 1 element of even weight can exist in \( \mathfrak{ds} \), and therefore \( \mathfrak{ds}_n^1/\mathfrak{ds}_n^2 \subset \mathfrak{ls}_n \). \( \diamond \)

The above result is actually the motivation for dropping the even depth 1 Lie polynomials from \( \mathfrak{ls} \). It is an open question whether the inclusion (1.4.2) is also a surjection, i.e. whether every element of the linearized double shuffle space is the lowest-depth part of some double shuffle element.

The stronger version of Theorem 1.3.2 also holds for \( \mathfrak{ls} \).

**Theorem 1.4.1.** The subspace \( \mathfrak{ls}_n^d \) of \( \mathfrak{ls} \) is zero if \( n \not\equiv d \mod 2 \).

By (1.4.2), Theorem 1.3.2 is an immediate consequence of this one. As explained in the previous section, we give a simple proof of Theorem 1.4.1 using Ecalle’s methods in Chapter 3, §3.4.
Chapter 2

The Lie algebra ARI

§2.1. Moulds and bimoulds

We work over the field \( \mathbb{C} \) of complex numbers. Let \( u_1, u_2, \ldots \) and \( v_1, v_2, \ldots \) denote two infinite sequences of indeterminates. A \textit{bimould} \( M \) is a collection of functions

\[
M_r \left( \begin{array}{c}
  u_1 \\
  u_2 \\
  \vdots \\
  u_r \\
  v_1 \\
  v_2 \\
  \vdots \\
  v_r 
\end{array} \right)
\]

for each \( r \geq 0 \), where each \( M_r \) is a function of the \( 2r \) variables \( u_1, \ldots, u_r, v_1, \ldots, v_r \) (in particular \( M_0 \) is a constant). These functions are a priori arbitrary, but later, in the context of the study of multizeta values, we will restrict our attention to rational functions, polynomials, and constants. A \textit{mould} is a bimould that is actually only a function of the \( u_i \), and a \textit{v-mould} is a function only of the \( v_i \). Most of the time, when there is no risk of confusion, we drop the index \( r \) and write \( M \left( \begin{array}{c}
  u_1 \\
  u_2 \\
  \vdots \\
  u_r \\
  v_1 \\
  v_2 \\
  \vdots \\
  v_r 
\end{array} \right) \) for \( M_r \left( \begin{array}{c}
  u_1 \\
  u_2 \\
  \vdots \\
  u_r \\
  v_1 \\
  v_2 \\
  \vdots \\
  v_r 
\end{array} \right) \), the depth \( r \) being indicated automatically by the number of variables. However, on occasion when working with a specific mould it may be necessary to use the index for precision; for example the mould \( M_2 (u_1, u_2) = u_2 \) is different from the mould \( M_3 (u_1, u_2, u_3) = u_2 \). We write \( M (\emptyset) \) for \( M_0 \). The space of all bimoulds is denoted \( \text{BIMU} \).

Two moulds or bimoulds \( M, N \in \text{BIMU} \) can be added, multiplied and, if \( N (\emptyset) = 0 \), composed. Writing \( w_i = (u_i) \) (or considering the variables \( w_i \) as belonging to an arbitrary alphabet), we have

\[
(M + N)(w_1, \ldots, w_r) = M(w_1, \ldots, w_r) + N(w_1, \ldots, w_r)
\]

\[
mu(M, N)(w_1, \ldots, w_r) = \sum_{0 \leq i \leq r} M(w_1, \ldots, w_i)N(w_{i+1}, \ldots, w_r)
\]

\[
(M \circ N)(w_1, \ldots, w_r) = \sum_{w=w_1 \cdots w_s \neq \emptyset} M(|w_1|, \ldots, |w_s|)N(w_1) \cdots N(w_s).
\]

(2.1.1)

Here, \( |(w_1, \ldots, w_r)| \) denotes the single-letter word \( w_1 + \cdots + w_r \), which is \( (u_1 + \cdots + u_r) \) in the bimould case.

\textbf{Remark.} Moulds are generalizations of power series. If a mould \( M \) takes constant values on each word, then it can be identified with the power series

\[
M = \sum_{(w_1, \ldots, w_r)} M(w_1, \ldots, w_r)w_1 \cdots w_r.
\]

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Exercise. Check that in the power series case, the rules for addition, multiplication and composition are just the usual ones.

Examples. (1) The first examples are the Log and Exp moulds given by $\text{Exp}(\emptyset) = \text{Log}(\emptyset) = 0$,
\[
\begin{align*}
\text{Log}(w_1, \ldots, w_r) &= (-1)^{r+1} \frac{1}{r} \\
\text{Exp}(w_1, \ldots, w_r) &= \frac{1}{r!}.
\end{align*}
\]

(2) The identity mould for multiplication $1$ is given by $1(\emptyset) = 1$ and all other values are 0.

(3) The identity mould $\text{Id}$ for composition is given by
\[
\text{Id}(w_1, \ldots, w_r) = \begin{cases} 
0 & \text{for } r = 0 \text{ and all } r > 1 \\
1 & \text{for } r = 1.
\end{cases}
\]

Exercise. Show that on the one-letter alphabet $T = \{t\}$, $\text{Exp}$ is the mould corresponding to the power series $e^t - 1$, $\text{Log}$ to $\log(1+t)$ and $\text{Id}$ to $t$. Show that as expected, $\text{Exp} \circ \text{Log} = \text{Id}$.

§2.2. The Lie algebra ARI

Definition. Let $\overline{\text{BARI}}$ (resp. $\overline{\text{ARI}}$, $\overline{\text{ARI}}$) denote the set of bimoulds (resp. the subspace of moulds, resp. of $v$-moulds) satisfying $A(\emptyset) = 0$. These spaces are obviously vector spaces, and even Lie algebras under the Lie bracket $\limu$ defined by $\limu(A, B) = \mu(A, B) - \mu(B, A)$. But Ecalle introduces an alternative bracket, the $\text{ari}$-bracket, making the same underlying vector space into a different Lie algebra. In chapter 3, we will explore the analogy between the two brackets on $\text{ARI}$ and the two different Lie brackets on the free Lie algebra $\text{Lie}[x, y]$ seen in Chapter 1. Let us define some necessary notation for the $\text{ari}$-bracket and other operators in Ecalle’s theory.

Flexions. Let $w = \left( \begin{array}{c} u_1 \cdots u_r \\ v_1 \cdots v_r \end{array} \right)$. For every possible way of cutting the word $w$ into three (possibly empty) subwords $w = abc$ with
\[
a = \left( \begin{array}{c} u_1, \ldots, u_k \\ v_1, \ldots, v_k \end{array} \right), \quad b = \left( \begin{array}{c} u_{k+1}, \ldots, u_{k+l} \\ v_{k+1}, \ldots, v_{k+l} \end{array} \right), \quad c = \left( \begin{array}{c} u_{k+l+1}, \ldots, u_r \\ v_{k+l+1}, \ldots, v_r \end{array} \right),
\]
set
\[
\begin{cases} 
[c = c & \text{if } b = \emptyset \\
[a = a & \text{if } b = \emptyset \\
b = b & \text{if } c = \emptyset \\
[b = b & \text{if } a = \emptyset,
\end{cases}
\]
For every bimould $B \in \text{BARI}$, we define operators $amit(B)$ and $anit(B)$ on BARI as follows:

$$amit(B) \cdot A = \sum_{\substack{\omega=abc \\ b,c \neq \emptyset}} A(a|c)B(b|), \quad \text{(2.2.1)}$$

$$anit(B) \cdot A = \sum_{\substack{\omega=abc \\ a,b \neq \emptyset}} A(a|c)B([b]). \quad \text{(2.2.2)}$$

For every pair of moulds $B, C \in \text{BARI}$, we set

$$axit(B,C) \cdot A = amit(B) \cdot A + anit(C) \cdot A \quad \text{(2.2.3)}$$

and

$$arit(B) \cdot A = axit(B,-B) \cdot A = amit(B) \cdot A - anit(B) \cdot A. \quad \text{(2.2.4)}$$

We have the following explicit expression for $arit(B)$:

$$(arit(B) \cdot A)(w) = \sum_{\substack{\omega=abc \\ b,c \neq \emptyset}} A(a|c)B(b|) - \sum_{\substack{\omega=abc \\ a,b \neq \emptyset}} A(a|c)B([b]). \quad \text{(2.2.5)}$$

For $A \in \text{ARI}$ (resp. $\overline{\text{ARI}}$) we define the analogous operators on $\text{ARI}$ (resp. $\overline{\text{ARI}}$) by dropping the lower (resp. upper) flexion signs in (2.2.1), (2.2.2) and (2.2.5).

**Proposition 2.2.1.** For all bimoulds $B \in \text{BARI}$ (resp. moulds $B \in \text{ARI}$, resp. v-moulds $B \in \overline{\text{ARI}}$), the operators $amit(B)$, $anit(B)$ and $arit(B)$ are derivations for the $limu$-bracket.

The proof of this proposition is given in §A.1 of the Appendix.

Define a “pre-Lie” operation on BARI by

$$prearit(A,B)(w) = (arit(B) \cdot A + mu(A,B))(w) = \sum_{\substack{\omega=abc \\ b \neq \emptyset}} A(a|c)B(b|) - \sum_{\substack{\omega=abc \\ a,b \neq \emptyset}} A(a|c)B([b]), \quad \text{(2.2.6)}$$
Then the \emph{ari}-bracket is defined on BARI by the formula
\[
\text{ari}(A, B) = \text{preari}(A, B) - \text{preari}(B, A),
\]
so it is given explicitly by the formula
\[
\text{ari}(A, B) = \sum_{w = abc, b \neq \emptyset} \left( A(a[c]B(b)) - B(a[c]A(b)) \right) - \sum_{w = abc, a, b \neq \emptyset} \left( A(a[c]B(b)) - B(a[c]A(b)) \right).
\]

Notice that we then have the “Poisson bracket” type identity*
\[
\text{ari}(A, B) = \text{arit}(B) \cdot A - \text{arit}(A) \cdot B + \limu(A, B).
\]

This analogy with the situation of two non-commutative free variables \(x, y\) as in Chapter 1, and further analogies with the group laws in the next section, will be explained in Chapter 3. As above, the operators \(\text{preari}\) and \(\text{ari}\) can be defined on ARI resp. \(\overline{\text{ARI}}\) by dropping the lower resp. upper flexion signs from their defining formulas.

**Proposition 2.2.2.** The \emph{ari}-bracket is a Lie bracket, therefore ARI (resp. \(\overline{\text{ARI}}, \text{BARI}\)) are Lie algebras under \(\text{ari}\).

**Proof.** FIX 1
\[
[\text{arit}(A), \text{arit}(B)] = \text{arit}(B) \circ \text{arit}(A) - \text{arit}(A) \circ \text{arit}(B) = \text{arit}(\text{ari}(A, B)).
\]

\[\Box\]

\section*{§2.3. Symmetrality, alternality, symmetrility, alternility}

For the study of multizeta values, Ecalle introduces four fundamental symmetries.

**Symmetrality and alternality.** The first two symmetries are based on the shuffle product defined in §1.1.

**Definition.** A bimould (resp. mould resp. \(v\)-mould) is said to be \emph{symmetral} if it has constant term 1 and
\[
M(\text{sh}(u, v)) = M(u)M(v) \text{ for all words } u, v,
\]
and \emph{alternal} if it has constant term 0 and
\[
M(\text{sh}(u, v)) = 0 \text{ for all words } u, v.
\]

\[\text{cf. } \text{ARI/GARI et la décomposition des multizètes en irréductibles, p. 28 (75) and p. 29 (84)}.\]
Note that it suffices to check both relations for the pairs \((u, v) = (w_1, \ldots, w_s), (w_{s+1}, \ldots, w_r)\) for \(1 \leq s \leq [r/2]\) since all shuffle relations can be deduced from these by variable change.

**Examples.** The alternality condition in depth 2 is

\[
M\left(\text{sh}((u_1), (u_2))\right) = M(u_1, u_2) + M(u_2, u_1) = 0.
\]

In depth 3, there is again only one condition to check, namely

\[
M\left(\text{sh}((u_1), (u_2, u_3))\right) = M(u_1, u_2, u_3) + M(u_2, u_1, u_3) + M(u_2, u_3, u_1).
\]

The other shuffle condition \(M\left(\text{sh}((u_1, u_2), (u_3))\right) = 0\) is automatically satisfied if this one is, by the variable change \(u_3 \mapsto u_1, u_1 \mapsto u_2, u_2 \mapsto u_3\). In depth 4, there are two necessary conditions for alternality, namely

\[
M\left(\text{sh}((u_1, u_2), (u_3, u_4))\right) = M(u_1, u_2, u_3, u_4) + M(u_2, u_1, u_3, u_4) + M(u_2, u_3, u_1, u_4) + M(u_2, u_3, u_4, u_1) = 0
\]

and

\[
M\left(\text{sh}((u_1, u_2), (u_3, u_4))\right) = M(u_1, u_2, u_3, u_4) + M(u_1, u_2, u_3, u_4) + M(u_1, u_3, u_4, u_2) + M(u_3, u_1, u_4, u_2) + M(u_3, u_4, u_1, u_2) = 0.
\]

**Symmetrility and alternility.** In this text we only define the second set of symmetries for moulds in the \(v_i\), although Ecalle’s flexion unit definition works for all bimoulds (cf. *Flexion structure*, p. 64-68.). These relations are deduced from the stuffle product introduced in §1.1. Recall that on an additive alphabet \(X\) the stuffle product is given by (1.1.5). To establish the symmetrility/alternility relations, we do not need to work with actual sequences; only the lengths of the sequences count. Let us write \(u = (v_1, \ldots, v_r), v = (v_{r+1}, \ldots, v_{r+s})\) for indetermines \(v_i\), and set

\[st(r, s) = st(u, v)\]

Let \(M\) be a mould. For each stuffle sum \(st(r, s)\), we define a symmetrility/alternility sum of terms in \(M\), by associating a specific term to each word in (1.1.5) as follows. For each \(\sigma \in Sh_r^\leq (r, s)\), let \(I_\sigma \subset \{1, \ldots, N\}\) be the set of indices \(i\) such that \(|\sigma^{-1}(i)| = 2\). To each word \(c^\sigma(u, v)\) as in (1.1.6), we associate a set of \(2^{|I_\sigma|}\) words indexed by the subsets \(J \subset I_\sigma\) (including the empty set), defined as follows:

\[
C_J^\sigma = (d_1, \ldots, d_N)
\]

where we write \(\sigma^{-1}(i) = \{k_\sigma, l_\sigma\}\) with \(k_\sigma < l_\sigma\) for all \(i \in I_\sigma\), and

\[
d_i = \begin{cases} 
  v_{\sigma^{-1}(i)} & \text{if } |\sigma^{-1}(i)| = 1 \\
  v_{k_\sigma} & \text{if } |\sigma^{-1}(i)| = 2 \text{ and } i \notin I_\sigma \\
  v_{l_\sigma} & \text{if } |\sigma^{-1}(i)| = 2 \text{ and } i \in I_\sigma.
\end{cases}
\]
Note that if $I_\sigma = \emptyset$ then $C^\sigma_{I_\emptyset} = c^\sigma(u,v)$. We set
\[
M_{r,s} = \sum_{\sigma \in Sh^\leq(r,s)} M^\sigma_{r,s} \quad (2.3.3)
\]
where
\[
M^\sigma_{r,s} = \frac{1}{\prod_{i \in I_\sigma}(v_{k_\sigma} - v_{l_\sigma})} \sum_{J \subset I_\sigma} (-1)^{|J|} M(C^\sigma_J). \quad (2.3.4)
\]

**Low depth.** In depth 2, the set $Sh^\leq(r,s)$ contains only three maps: the identity map $\sigma_1$, the map $\sigma_2$ exchanging 1 and 2, the map $\sigma_3 : \{1,2\} \to \{1\}$ sending 1 and 2 to 1. The corresponding words are
\[
c^{\sigma_1}((v_1),(v_2)) = (v_1,v_2), \quad c^{\sigma_2}((v_1),(v_2)) = (v_2,v_1), \quad c^{\sigma_3}((v_1),(v_2)) = (v_1 + v_2),
\]
so the stuffle sum is $st(1,1) = st((v_1),(v_2)) = (v_1,v_2) + (v_2,v_1) + (v_1 + v_2)$. We have $I_{\sigma_1} = I_{\sigma_2} = \emptyset$, $I_{\sigma_3} = \{1\}$, and $\sigma_3^{-1}(1) = \{k_{\sigma_3}, l_{\sigma_3}\}$ with $k_{\sigma_3} = 1$, $l_{\sigma_3} = 2$. The words $C^\sigma_J$ corresponding to the two subsets $J = \emptyset$ and $J = I_{\sigma_3}$ of $I_{\sigma_3} = \{1\}$ are $C^\sigma_{\emptyset} = (v_1)$ and $C^\sigma_{I_{\sigma_3}} = (v_2)$. The corresponding alternity terms are
\[
\begin{cases}
M^\sigma_{r,s} = M(c^{\sigma_1}((v_1),(v_2))) = M(v_1,v_2) \\
M^\sigma_{r,s} = M(c^{\sigma_2}((v_1),(v_2))) = M(v_2,v_1) \\
M^\sigma_{r,s} = \frac{1}{(v_1-v_2)}(M(v_1) - M(v_2)),
\end{cases}
\]
so the alternity sum in depth 2 is given by
\[
M_{1,1}(v_1,v_2) = M(v_1,v_2) + M(v_2,v_1) + \frac{1}{v_1-v_2}(M(v_1) - M(v_2)). \quad (2.3.5)
\]
In depth 3 the condition corresponding to $st(1,2) = st((v_1),(v_2,v_3)) = (v_1,v_2,v_3) + (v_2,v_1,v_3) + (v_2,v_3,v_1) + (v_1 + v_2,v_3) + (v_2,v_1 + v_3)$ is given by
\[
M_{1,2}(v_1,v_2,v_3) = M(v_1,v_2,v_3) + M(v_2,v_1,v_3) + M(v_2,v_3,v_1)
+ \frac{1}{v_1-v_2}(M(v_1,v_3) - M(v_2,v_3)) + \frac{1}{v_1-v_3}(M(v_2,v_1) - M(v_2,v_3)).
\]
In depth 4, the term in $M_{2,2}$ corresponding to the word $(v_1 + v_3, v_2 + v_4)$ in the stuffle sum $st(2,2) = st((v_1,v_2),(v_3,v_4))$ is given by
\[
\frac{1}{(v_1-v_3)(v_2-v_4)}(M(v_1,v_2) - M(v_3,v_2) - M(v_1,v_4) + M(v_3,v_4)). \quad (2.3.6)
\]
**Definition.** The mould $M \in ARI$ is said to be **symmetril** if it has constant term 1 and for all pairs $1 \leq r \leq s$ we have
\[
M_{r,s}(v_1,\ldots,v_{r+s}) = M_r(v_1,\ldots,v_r)M_s(v_{r+1},\ldots,v_{r+s}), \quad (2.3.7)
\]
\[
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\]
and alternil if it has constant term 0 and for all pairs we have
\[ M_{r,s}(v_1, \ldots, v_{r+s}) = 0. \] (2.3.8)

**Remark.** If \( M \) is a polynomial-valued mould, then the alternility sums are polynomials. To see this, it suffices to note that setting \( v_{k,s} = v_{l,s} \) for any \( \sigma \in I_\sigma \), in the numerator of \( M'_{r,s} \) yields zero, canceling out the pole in (2.3.4).

§2.4. **Swap commutation in ARI**

We begin this section by defining some of the main mould operators. Let push, neg, anti, mantar and swap be the operators on bimoulds defined as follows:

\[
\begin{align*}
\text{push}(M) \left( \begin{array}{cccc}
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
\end{array} \right) &= M \left( \begin{array}{cccc} -u_1 - \cdots - u_r & u_1 & \cdots & u_{r-1} \\
u_r & v_1 - v_r & \cdots & v_{r-1} - v_r \\
\end{array} \right) \\
\text{neg}(M) \left( \begin{array}{cccc}
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
\end{array} \right) &= M \left( \begin{array}{cccc} -u_1 & -u_2 & \cdots & -u_r \\
u_1 & -v_2 & \cdots & -v_r \\
\end{array} \right) \\
\text{anti}(M) \left( \begin{array}{cccc}
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
\end{array} \right) &= M \left( \begin{array}{cccc} u_r & u_{r-1} & \cdots & u_1 \\
u_r & v_{r-1} & \cdots & v_1 \\
\end{array} \right) \\
\text{mantar}(M) \left( \begin{array}{cccc}
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
\end{array} \right) &= (-1)^{r-1} M \left( \begin{array}{cccc} u_r & \cdots & u_1 \\
u_r & v_{r-1} & \cdots & v_1 \\
\end{array} \right) \\
\text{swap}(M) \left( \begin{array}{cccc}
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
u_1 & v_2 & \cdots & v_r \\
\end{array} \right) &= M \left( \begin{array}{cccc} v_r & v_{r-1} & \cdots & v_2 - v_3 & v_1 - v_2 \\
u_1 + \cdots + u_r & u_1 + \cdots + u_{r-1} & \cdots & u_1 + u_2 & u_1 \\
\end{array} \right).
\end{align*}
\]

The first four operators can be considered as operators only on ARI (resp. ARI) by ignoring the \( v_i \) (resp. the \( u_i \)). The swap, however, exchanges the two spaces ARI and ARI. We will make use below of the following elementary identity, proved by simple application of the variables changes above:

\[ \text{neg} \circ \text{push} = \text{anti} \circ \text{swap} \circ \text{anti} \circ \text{swap}. \] (2.4.1)

The purpose of this section and the next one is to prove a set of fundamental identities expressing how swap commutes with the ARI operators amit, anit, arit, preari, ari and preawi (in this section) and with the GARI operators garit and gari (in the next one). These commutations yield a set of fundamental identities that lie at the heart of Ecalle’s theory.

Recall the definitions of the operators amit and anit given in §2.2, as well as the definitions of the operators axit and arit:

\[
\begin{align*}
\text{axit}(B, C) \cdot A &= \text{amit}(B) \cdot A + \text{anit}(C) \cdot A, \\
\text{arit}(B) \cdot A &= \text{axit}(B, -B) \cdot A = \text{amit}(B) \cdot A - \text{anit}(B) \cdot A
\end{align*}
\] (2.4.2, 2.4.3)
to which we now add the definition of \( \text{awit} \), as follows:

\[
\text{awit}(B) \cdot A = \text{axit}(B, \text{anti} \circ \text{neg}(B)) = \text{amit}(B) \cdot A - \text{anit}(\text{anti} \circ \text{neg}(B)) \cdot A. \quad (2.4.4)
\]

In analogy to the \( \text{preari} \) law

\[
\text{preari}(A, B) = \text{arit}(B) \cdot A + \text{mu}(A, B), \quad (2.4.5)
\]

we also now define the \( \text{preawi} \) law

\[
\text{preawi}(A, B) = \text{awit}(B) \cdot A + \text{mu}(A, B). \quad (2.4.6)
\]

The key identities are the following ones, which are proven in §A.2 of the Appendix:

\[
\begin{align*}
\text{swap}(\text{amit}(\text{swap}(B)) \cdot \text{swap}(A)) &= \text{amit}(B) \cdot A + \text{mu}(A, B) - \text{swap}(\text{mu}(\text{swap}(A), \text{swap}(B))), \quad (2.4.7) \\
\text{swap}(\text{anit}(\text{swap}(B)) \cdot \text{swap}(A)) &= \text{anit}(\text{push}(B)) \cdot A. \quad (2.4.8)
\end{align*}
\]

Using these two, it is quite easy to compute the \( \text{swap} \) commutations with \( \text{arit} \), \( \text{preari} \), \( \text{ari} \) and \( \text{preawi} \). Applying the identities (2.4.7) and (2.4.8) to (2.4.3) immediately yields

\[
\begin{align*}
\text{swap}(\text{arit}(\text{swap}(B)) \cdot \text{swap}(A)) \\
&= \text{swap}(\text{amit}(\text{swap}(B)) \cdot \text{swap}(A)) - \text{swap}(\text{anit}(\text{swap}(B)) \cdot \text{swap}(A)) \\
&= \text{amit}(B) \cdot A + \text{mu}(A, B) - \text{swap}(\text{mu}(\text{swap}(A), \text{swap}(B))) - \text{anit}(\text{push}(B)) \cdot A \\
&= \text{axit}(B, -\text{push}(B)) \cdot A + \text{mu}(A, B) - \text{swamu}(A, B) \quad (2.4.9)
\end{align*}
\]

where \( \text{swamu}(A, B) = \text{swap}(\text{mu}(\text{swap}(A), \text{swap}(B))) \). Applying (2.4.7) and (2.4.8) to (2.4.5) yields the following computation (\( \text{preira} \) is defined by the first equality):

\[
\begin{align*}
\text{preira}(A, B) : = \text{swap}(\text{preari}(\text{swap}(A), \text{swap}(B))) \\
&= \text{swap}(\text{arit}(\text{swap}(B)) \cdot A) + \text{swamu}(A, B) \\
&= \text{axit}(B, -\text{push}(B)) \cdot A + \text{mu}(A, B) \\
&= \text{amit}(B) \cdot A + \text{anit}(\text{push}(B)) \cdot A + \text{mu}(A, B) \\
&= \text{arit}(B) \cdot A + \text{anit}(B - \text{push}(B)) \cdot A + \text{mu}(A, B) \\
&= \text{preari}(A, B) + \text{anit}(B - \text{push}(B)) \cdot A \\
&= \text{irat}(B) \cdot A + \text{mu}(A, B), \quad (2.4.10)
\end{align*}
\]

where the last line introduces the operator \( \text{irat}(B) \cdot A \) given by

\[
\text{irat}(B) \cdot A = \text{axit}(B, -\text{push}(B)) \cdot A. \quad (2.4.11)
\]
Applying the same method to \( ari \) yields the operator \( ira \) computed as:

\[
ira(A, B) := \text{swap}(ari(\text{swap}(A), \text{swap}(B)))
\]

\[
= axit(B, -\text{push}(B)) \cdot A + \text{mu}(A, B) - axit(A, -\text{push}(A)) \cdot B - \text{mu}(B, A).
\]

Finally, we define and compute \( preiwa \) as follows:

\[
preiwa(A, B) := \text{swap}(preawi(\text{swap}(A), \text{swap}(B)))
\]

\[
= \text{swap}(\text{amit}(\text{swap}(B)) \cdot \text{swap}(A))
\]

\[
+ \text{swap}(\text{anit}(\text{anti} \cdot \text{neg}(\text{swap}(B))) \cdot \text{swap}(A))
\]

\[
+ \text{swap}(\text{mu}(\text{swap}(A), \text{swap}(B)))
\]

\[
= \text{amit}(B) \cdot A + \text{anit}(\text{push} \cdot \text{swap} \cdot \text{anti} \cdot \text{neg} \cdot \text{swap}(B)) \cdot A + \text{mu}(A, B)
\]

\[
= \text{amit}(B) \cdot A + \text{anit}(\text{anti}(B)) \cdot A + \text{mu}(A, B)
\]

\[
= \text{iwat}(B) \cdot A + \text{mu}(A, B)
\]  

(2.4.13)

where the last line introduces the definition

\[
\text{iwat}(B) \cdot A = axit(B, anti(B)) \cdot A.
\]

(2.4.14)

Note that an easy corollary of (2.4.12) is the following result.

**Lemma 2.4.1.** If \( A, B \) are push-invariant moulds in \( ARI \), then

\[
\text{swap}(ari(\text{swap}(A), \text{swap}(B))) = ari(A, B).
\]

(2.4.15)

**Proof.** By (2.2.4), we have \( arit(B) = axit(B, -B) \). If \( A \) and \( B \) are push-invariant, then by (2.4.12) we have

\[
\text{swap}(ari(\text{swap}(A), \text{swap}(B))) = ari(B) \cdot A + lm(\text{mu}(A, B) - ari(A) \cdot B,
\]

which is nothing but \( ari(A, B) \) by (2.2.9).

\[\diamondsuit\]

§2.5. Special subspaces of \( ARI \)

There are many interesting subspaces of \( ARI \), containing only moulds having special symmetry properties or *dimorphic symmetries* to use Ecalle’s term, which is to say moulds in \( ARI \) having a special symmetry property and whose swap, in \( ARI \), has another.

**Definition.** We write
• $\text{ARI}^{\text{pol}}$ (resp. $\overline{\text{ARI}}^{\text{pol}}$, $\text{BARI}^{\text{pol}}$) for the subspace of polynomial-valued (bi)moulds;

• $\text{ARI}_{al}$ (resp. $\overline{\text{ARI}}_{al}$, $\text{BARI}_{al}$) for the subspace of alternal (bi)moulds.

Following Ecalle, we also use the notation $\text{ARI}_{a/b}$ for moulds in ARI having the property $a$ and/or whose swap has the property $b$; for instance we may write $\text{ARI}_{al}$ for moulds in ARI with alternal swap. The most important dimorphy spaces we will consider are the following:

• $\text{ARI}_{al/al}$, the subspace of alternal moulds in ARI whose swap is alternal in ARI, and $\text{ARI}_{al/al}$, the subspace of $\text{ARI}_{al/al}$ of moulds that are even functions of $u_1$ in depth 1;

• $\text{ARI}_{al*al}$, the subspace of alternal moulds in ARI whose swap is alternal in ARI up to addition of a constant-valued mould, and the corresponding subspace $\text{ARI}_{al*al}$ of moulds that are even functions in depth 1;

• $\text{ARI}_{al/il}$, the subspace of alternal moulds in ARI whose swap is alternil;

• $\text{ARI}_{al*il}$, the subspace of alternal moulds in ARI whose swap is alternil up to addition of a constant-valued mould.

In this section we are concerned with studying the Lie algebra properties of some of these subspaces. In particular the following result follows immediately from the definition of the $\ari$-bracket, which is made up of operations and flexions that preserve polynomials.

**Proposition 2.5.1.** The subspace $\text{ARI}^{\text{pol}}$ is a Lie algebra under the $\ari$-bracket.

We also have the next, significantly more difficult result, whose detailed proof is given in [SS, Appendix A].

**Proposition 2.5.2.** $\text{ARI}_{al}$ and $\overline{\text{ARI}}_{al}$ are Lie algebras under the $\ari$ bracket. More generally, if $A$ and $B$ are alternal moulds, then $\ari(B) \cdot A$ is alternal.

The main result of this section is that $\text{ARI}_{al/al}$ and $\text{ARI}_{al*al}$ are Lie algebras under the $\ari$-bracket. This result is given in Theorem 2.5.6 below. We first need three lemmas.

**Lemma 2.5.3.** If $A \in \text{ARI}_{al}$, then

$$\text{anti}(A)(w_1, \ldots, w_r) = (-1)^{r-1}A(w_1, \ldots, w_r),$$

in other words, $A$ is mantar-invariant.

**Proof.** We first show the following equality on sums of shuffle relations:

$$\text{sh}((1), (2, \ldots, r)) - \text{sh}((2, 1), (3, \ldots, r)) + \text{sh}((3, 2, 1), (4, \ldots, r)) + \cdots$$

$$+(-1)^{r-1}\text{sh}((r-1, \ldots, 2, 1), (r)) = (1, \ldots, r) + (-1)^{r-1}(r, \ldots, 1).$$

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Indeed, using the recursive formula for shuffle, we can write the above sum with two terms for each shuffle, as

\[
(1, \ldots, r) + 2 \cdot \text{sh}((1), (3, \ldots, r))
- 2 \cdot \text{sh}((1), (3, \ldots, r)) - 3 \cdot \text{sh}((2, 1), (4, \ldots, r))
+ 3 \cdot \text{sh}((2, 1), (4, \ldots, r)) + 4 \cdot \text{sh}((3, 2, 1), (5, \ldots, r))
+ \cdots + (-1)^{r-2}(r - 1) \cdot \text{sh}((r - 2, \ldots, 1), (r))
+ (-1)^{r-1}(r - 1) \cdot \text{sh}((r - 2, \ldots, 1), (r)) + (-1)^{r-1}(r, r - 1, \ldots, 1)
= (1, \ldots, r) + (-1)^{r-1}(r, \ldots, 1).
\]

Using this, we conclude that if \(A\) satisfies the shuffle relations, then

\[
A(w_1, \ldots, w_r) + (-1)^{r-1}A(w_r, \ldots, w_1),
\]

which is the desired result. \(\diamondsuit\)

**Lemma 2.5.4.** \(\text{ARI}_{al^{\iota}al}^{\iota}\) is \((\text{neg} \circ \text{push})\)-invariant.

**Proof.** We first deal with the case \(A \in \text{ARI}_{al/\text{al}}\). Using (2.4.1) and (2.5.2), we have

\[
\text{neg} \circ \text{push}(A)(w_1, \ldots, w_r) = \text{anti} \circ \text{swap} \circ \text{anti} \circ \text{swap}(A)(w_1, \ldots, w_r)
= (-1)^{r-1}\text{anti} \circ \text{swap} \circ \text{swap}(A)(w_1, \ldots, w_r)
= (-1)^{r-1}\text{anti}(A)(w_1, \ldots, w_r)
= A(w_1, \ldots, w_r),
\]

which proves the result.

To extend the argument from \(\text{ARI}_{al/\text{al}}\) to \(\text{ARI}_{al^{\iota}al}^{\iota}\) takes some extra arguments, that we take here directly from [SS]. Suppose that \(A \in \text{ARI}_{al^{\iota}al}^{\iota}\), so \(A\) is alternal and \(\text{swap}(A) + A_0\) is alternal for some constant mould \(A_0\). By additivity, we may assume that \(A\) is concentrated in depth \(r\). First suppose that \(r\) is odd. Then \(\text{mantar}(A_0)(v_1, \ldots, v_r) = (-1)^{r-1}A_0(v_r, \ldots, v_1)\), so since \(A_0\) is a constant mould, it is mantar-invariant. But \(\text{swap}(A) + A_0\) is alternal, so it is also mantar-invariant by Lemma B.1; thus \(\text{swap}(A)\) is mantar-invariant, and the identity \(\text{neg} \circ \text{push} = \text{mantar} \circ \text{swap} \circ \text{mantar} \circ \text{swap}\) shows that \(A\) is \(\text{neg} \circ \text{push}\)-invariant as in (B.2).

Finally, we assume that \(A\) is concentrated in even depth \(r\). Here we have \(\text{mantar}(A_0) = -A_0\), so we cannot use the argument above; indeed \(\text{swap}(A) + A_0\) is mantar-invariant, but

\[
\text{mantar}(\text{swap}(A)) = \text{swap}(A) + 2A_0.
\]

Instead, we note that if \(A\) is alternal then so is \(\text{neg}(A) = A\). Thus we can write \(A\) as a sum of an even and an odd function of the \(u_i\) via the formula

\[
A = \frac{1}{2}(A + \text{neg}(A)) + \frac{1}{2}(A - \text{neg}(A)).
\]
So it is enough to prove the desired result for all moulds concentrated in even depth \( r \) such that either \( \text{neg}(A) = A \) (even functions) or \( \text{neg}(A) = -A \) (odd functions). First suppose that \( A \) is even. Then since \( \text{neg} \) commutes with \( \text{push} \) and \( \text{push} \) is of odd order \( r + 1 \) and \( \text{neg} \) is of order 2, we have

\[
(\text{neg} \circ \text{push})^{r+1}(A) = \text{neg}(A) = A. \tag{B.5}
\]

However, we also have

\[
\text{neg} \circ \text{push}(A) = \text{mantar} \circ \text{swap} \circ \text{mantar} \circ \text{swap}(A)
= \text{mantar} \circ \text{swap}(\text{swap}(A) + 2A_0) \text{ by (B.3)}
= \text{mantar}(A + 2A_0)
= A - 2A_0.
\]

Thus \( (\text{neg} \circ \text{push})^{r+1}(A) = A - 2(r + 1)A_0 \), and this is equal to \( A \) by (B.5), so \( A_0 = 0 \); thus in fact \( A \in \text{ARI}_{al/ol} \) and that case is already proven.

Finally, if \( A \) is odd, i.e. \( \text{neg}(A) = -A \), the same argument as above gives \( A - 2(r + 1)A_0 = -A \), so \( A = (r + 1)A_0 \), so \( A \) is a constant-valued mould concentrated in depth \( r \), but this contradicts the assumption that \( A \) is alternal since constant moulds are not alternal, unless \( A = A_0 = 0 \). Note that this argument shows that all moulds in \( \text{ARI}_{al*al} \) that are not in \( \text{ARI}_{al/ol} \) must be concentrated in odd depths.

**Lemma 2.5.5.** \( \text{ARI}_{al*al} \) is \( \text{neg} \)-invariant and \( \text{push} \)-invariant.

**Proof.** Let \( A \in \text{ARI}_{al*al} \). Because \( \text{neg}(A) = \text{push}(A) \) by Lemma 2.5.4, it is enough to prove that \( \text{neg}(A) = A \). As before, we may assume that \( A \) is concentrated in a fixed depth \( d \), meaning that \( A(w_1, \ldots, w_d) = 0 \) for all \( r \neq d \). If \( d = 1 \), then \( A = \text{neg}(A) \) is just the assumption on \( A \). If \( d = 2s \) is even, then since \( \text{neg} \) is of order 2 and commutes with \( \text{push} \) and \( \text{push} \) is of order \( d + 1 = 2s + 1 \), we have

\[
A = (\text{neg} \circ \text{push})^{2s+1}(A) = \text{neg}^{2s+1}(A) = \text{neg}(A).
\]

If \( d = 2s + 1 \) is odd, we can write \( A \) as a sum of an even and an odd part

\[
A = \frac{1}{2}(A(w_1, \ldots, w_{d}) + A(-w_1, \ldots, -w_{d})) + \frac{1}{2}(A(w_1, \ldots, w_{d}) - A(-w_1, \ldots, -w_{d}))
\]

so we may assume that \( A(w_1, \ldots, w_{d}) \) is odd, i.e. \( \text{neg}(A) = -A \). Then, since \( A \) is alternal, using the shuffle \( sh((w_1, \ldots, w_{2s})(w_{2s+1})) \), we have

\[
\sum_{i=0}^{2s} A(w_1, \ldots, w_i, w_{2s+1}, w_{i+1}, \ldots, w_{2s}) = 0.
\]

Making the variable change \( w_0 \leftrightarrow w_{2s+1} \) gives

\[
\sum_{i=0}^{2s} A(w_1, \ldots, w_i, w_0, w_{i+1}, \ldots, w_{2s}) = 0,
\]

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which we write out as

$$\sum_{i=0}^{2s} A \begin{pmatrix} u_1 & \ldots & u_i & u_0 & u_{i+1} & \ldots & u_{2s} \\ v_1 & \ldots & v_i & v_0 & v_{i+1} & \ldots & v_{2s} \end{pmatrix} = 0.$$  \tag{2.5.3}

Now consider the shuffle relation $sh((w_1)(w_2, \ldots, w_{2s+1}))$, which gives

$$\sum_{i=1}^{2s+1} A(w_2, \ldots, w_i, w_{i+1}, \ldots, w_{2s+1}) = 0. \tag{2.5.4}$$

Set $u_0 = -u_1 - \ldots - u_{2s+1}$. Since $neg \circ push$ acts like the identity on $A$, we can apply it to each term of (2.5.4) to obtain

$$\sum_{i=1}^{2s} -A \begin{pmatrix} u_0 & u_2 & \ldots & u_i & u_1 & u_{i+1} & \ldots & u_{2s} \\ v_{2s+1} & v_2 - v_{2s+1} & \ldots & v_i - v_{2s+1} & v_1 - v_{2s+1} & v_{i+1} - v_{2s+1} & \ldots & v_{2s} - v_{2s+1} \end{pmatrix} - A \begin{pmatrix} u_0 & u_2 & \ldots & u_{2s} & u_{2s+1} \\ -v_1 & v_2 - v_1 & \ldots & v_{2s} - v_1 & v_{2s+1} - v_1 \end{pmatrix} = 0.$$

We apply $neg \circ push$ again to the final term of this sum in order to get the $u_{2s+1}$ and $v_{2s+1}$ to disappear, obtaining

$$\sum_{i=1}^{2s} -A \begin{pmatrix} u_0 & u_2 & \ldots & u_i & u_1 & u_{i+1} & \ldots & u_{2s} \\ -v_{2s+1} & v_2 - v_{2s+1} & \ldots & v_i - v_{2s+1} & v_1 - v_{2s+1} & v_{i+1} - v_{2s+1} & \ldots & v_{2s} - v_{2s+1} \end{pmatrix} + A \begin{pmatrix} u_1 & u_0 & u_2 & \ldots & u_{2s-1} & u_{2s} \\ v_{1} - v_{2s+1} & -v_{2s+1} & v_2 - v_{2s+1} & \ldots & v_{2s-2} - v_{2s-1} & v_{2s-1} - v_{2s} \end{pmatrix} = 0.$$

Making the variable changes $u_0 \leftrightarrow u_1$ and $v_1 \leftrightarrow v_0 - v_1$, $v_i \leftrightarrow v_i - v_1$ for $2 \leq i \leq 2s$, $v_{2s+1} \rightarrow -v_1$ in this identity yields

$$\sum_{i=1}^{2s} -A \begin{pmatrix} u_1 & u_2 & \ldots & u_i & u_0 & u_{i+1} & \ldots & u_{2s} \\ v_1 & v_2 & \ldots & v_i & v_0 & v_{i+1} & \ldots & v_{2s} \end{pmatrix} + A \begin{pmatrix} u_0 & u_1 & u_2 & \ldots & u_{2s-1} & u_{2s} \\ v_0 & v_1 & v_2 & \ldots & v_{2s-1} & v_{2s} \end{pmatrix} = 0. \tag{2.5.5}$$

Finally, adding (2.5.3) and (2.5.5) yields

$$2A \begin{pmatrix} u_0 & u_1 & \ldots & u_{2s} \\ v_1 & v_2 & \ldots & v_{2s} \end{pmatrix} = 0,$$

so $A = 0$. This concludes the proof that if $A \in ARI_{al/al}$, then $A(w_1, \ldots, w_d)$ is an even function for all $d > 1$; thus if we assume in addition that $A$ is even for $d = 1$, then $neg(A) = A$, and by Lemma 2.5.4, we have $push(A) = A$. $\diamondsuit$
Finally, to prove Theorem 2.5.6, we will also need the following important identity that appears in Chapter 4 as Lemma 2.4.1. For all push-invariant moulds $A, B \in \text{ARI}$, we have

$$\text{swap}(\text{ari}(A, B)) = \text{ari}(\text{swap}(A), \text{swap}(B)),$$

(2.5.6)

**Theorem 2.5.6.** $\text{ARI}_{\text{al/al}}$ is a Lie algebra under the ari-bracket.

**Proof.** Let $A, B \in \text{ARI}_{\text{al/al}}$ and set $C = \text{ari}(A, B)$. The mould $C$ is alternal by Proposition 2.5.2. By Lemma 2.5.5, $A$ and $B$ are push-invariant, so by (2.5.6) we have $\text{swap}(C) = \text{swap}(\text{ari}(A, B)) = \text{ari}(\text{swap}(A), \text{swap}(B))$, which is also alternal by Proposition 2.5.2. It remains only to check that $C$ is even in depth 1. But in fact, $C\left(\frac{u_1}{v_1}\right) = 0$, as the depth 1 part of an ari-bracket is always zero, which follows directly from its definition in (2.2.8).

\[\Box\]

§2.6. The group GARI

The last two sections of this chapter are devoted to the group GARI. We begin by defining GARI to be the set of moulds in the variables $u_i$ with constant term 1; similarly, we define $\text{GARI}_{\text{al}}$ to be the set of moulds in the $v_i$ with constant term 1, and $\text{GBARI}$ the set of bimoulds with constant term 1. We will only consider GARI in this section, but every statement and definition is equally valid for $\text{GARI}_{\text{al}}$ and $\text{GBARI}$.

We can realize GARI as the exponential of ARI via the exponential map $\text{expari}$ defined by

$$\text{expari}(A) = \sum_{i \geq 0} \frac{1}{i!} \text{preari}(A, \ldots, i)_i = 1 + A + \frac{1}{2!} \text{preari}(A, A) + \frac{1}{3!} \text{preari}(A, A, A) + \cdots,$$

(2.6.1)

where $\text{preari}(A, \ldots, A)_i$ is understood to be taken from left to right, for example

$$\text{preari}(A, A, A) = \text{preari}(\text{preari}(A, A), A).$$

(Note that while in principle $\text{preari}$ is an operator on pairs of moulds from ARI, the definition (2.2.6) makes perfect sense even if only the second mould is in ARI and the first is an arbitrary mould.) Indeed, since the only condition on elements of GARI is to have constant term 1, moulds in the group $\text{expari}(\text{ARI})$ certainly satisfy this condition, and since like all exponentials $\text{expari}$ is an isomorphism, its inverse $\text{logari}$ takes moulds with constant term 1 to moulds with constant term 0, i.e. $\text{logari} : \text{GARI} \to \text{ARI}$. This shows that GARI is a group.

Naturally, GARI has subgroups corresponding to the interesting subalgebras of ARI. The most crucial definition is the following.

**Definition.** A mould $A \in \text{GARI}$ is *symmetral* if for all pairs of words $u, v$ in the $u_i$, we have

$$\sum_{w \in \text{sh}(u, v)} A(w) = A(u)A(v).$$

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We write $\text{GARI}_a$ for the set of symmetral moulds in GARI.

The following basic result will be useful later on.

**Proposition 2.6.1.** We have

$$\text{expari}(\text{ARI}_{al}) = \text{GARI}_a.$$

Proof. The proof was worked out completely by N. Komiyama in the appendix of [K] (Theorem A.7).

The following lemma is a good exercise, so we merely sketch the proof.

**Lemma 2.6.2.** If $B \in \text{GARI}_a$ and $A \in \text{ARI}_{al}$, then the composition $B \circ A$ is symmetral.

**Sketch of proof.** Consider the expression for $B \circ A$ in (2.1.1). Summing up the terms $(B \circ A)(w_1, \ldots, w_r)$ where $w = (w_1, \ldots, w_r)$ runs through the shuffles $sh(u, v)$ of two words $u$ and $v$, we obtain

$$\sum_{w \in sh(u, v)} \sum_{w = w_1 \cdots w_s} B(|w_1|, \ldots, |w_s|)A(w_1) \cdots A(w_s).$$

Let $u = (u_1, \ldots, u_l)$, $v = (u_{l+1}, \ldots, u_s)$. There are two types of decomposition $w = w_1 \cdots w_r$; for which $w_1 \cdots w_m$ is of length $l$ and $w_m \cdots w_r$ is of length $r - l$, which are called “compatible with $uv$”, and the “incompatible” ones for which there is no such division of the decomposition into two compatible chunks.

The proof essentially works as follows. We fix one decomposition of $uv$ into chunks $u_1 \cdots u_s$, and then consider the corresponding decompositions $w = w_1 \cdots w_s$ of all words $w \in sh(u, v)$. If the fixed decomposition is incompatible, then we can show that

$$\sum_{w \in sh(u, v)} B(|w_1|, \ldots, |w_s|)A(w_1) \cdots A(w_s) = 0,$$

simply because the different shuffles that give the same term $B(|w_1|, \ldots, |w_s|)$ factor out in front of a sum of terms of the form $A(w_1) \cdots A(w_s)$ that is in fact a product of sums of shuffles and is therefore zero, since $A$ is alternal.

If the fixed decomposition is compatible, then one can show what happens in two steps. To start with, all the terms in which $|w_1|, \ldots, |w_s|$ is not compatible with $u$ and $v$ in the sense that each $|w_i|$ is either a sum of consecutive letters of $u$ or consecutive letters of $v$ sum to zero as above, due to the alternality of $A$. Finally, the remaining terms in the sum are sums of shuffles of the $|u_i|$ in the decompositions $uv = u_1 \cdots u_s$, and thus they simplify to products due to the symmetrality of $M$.

\[\text{§2.7. The group law on GARI}\]
For each mould $B$ in GBARI we can associate an automorphism of GBARI denoted $\text{garit}_B$ by the formula:

$$\text{garit}_B \cdot A = \sum_{w = a_1b_1c_1 \cdots a_sb_sc_s, b_i \neq \emptyset, a_i c_i + 1 \neq \emptyset} A([b_1] \cdots [b_s])B(a_1) \cdots B(a_s)\text{invmu}(B)([c_1] \cdots \text{invmu}(B)([c_s])$$

for $s \geq 1$, where the flexions are as defined in §2.2 and invmu($B$) is of course the inverse of $B$ for the $\mu$-multiplication. Later, another automorphism will also be very useful:

$$\text{ganit}_B \cdot A = \sum_{w = b_1c_1 \cdots b_sc_s, \text{only } c_s \text{ can be } 0} A(b_1) \cdots B([c_1] \cdots B([c_s]).$$

The expressions for $\text{garit}_B$ and $\text{ganit}_B$ on GARI and $\text{GARI}$ are obtained as usual from (2.7.1) by ignoring the lower resp. upper flexions. In Chapter 3, §3.5, we will see the familiar expressions for these automorphisms when we consider the very restricted case of moulds that are power series in two non-commutative variables with constant term 1, forming the so-called twisted Magnus group.

The group law in GARI, denoted $\text{gari}$, is given by

$$\text{gari}(A, B) = \mu(\text{garit}_B \cdot A, B).$$

This law is linear in $A$, so that the product $\text{gari}(A, B)$ can be extended from pairs of moulds in GARI to pairs of moulds where $A$ is arbitrary and $B$ is in GARI. By linearizing $B$, we recover the $\text{preari}$ operator. The linearizing procedure works as follows: we set $B = 1 + \epsilon C$ for a mould $C \in \text{ARI}$, and consider coefficients in the field $k[[\epsilon]]/(\epsilon^2)$ if $k$ is the base field. Then we find that

$$\text{garit}(1+\epsilon C) \cdot A = A + \epsilon \text{arit}(C) \cdot A,$$

so

$$\text{gari}(A, 1+\epsilon C) = \mu(\text{garit}(1+\epsilon C) \cdot A, 1+\epsilon C)
= \mu(A + \epsilon \text{arit}(C) \cdot A, 1+\epsilon C)
= A + \epsilon \text{arit}(C) \cdot A + \epsilon \mu(A, C) = A + \epsilon \text{preari}(A, C).$$

The inverse of a mould $B$ for the $\text{gari}$-multiplication is written $\text{invgari}(B)$. Since $\text{ARI}$ is a Lie algebra for the Lie bracket $\text{ari}$, GARI is a pro-unipotent group. Then $\text{preari}$ is the pre-Lie law which expresses multiplication inside the universal enveloping algebra of $\text{ARI}$ of two elements in $\text{ARI}$ (or more generally one element in the enveloping algebra and one in $\text{ARI}$), and $\text{expari}$ is the usual Lie exponential map. Like exp of any Lie algebra, the group GARI acts on the Lie algebra via an adjoint action known as $\text{ad}_{\text{ari}}$ and defined by

$$\text{ad}_{\text{ari}}(A) \cdot B = \frac{d}{dt} \bigg|_{t=0} \text{gari}(A, \text{expari}(tB), \text{invgari}(A))
= B + \text{ari}(\logari(A), B) + \frac{1}{2} \text{ari}(\logari(A), \text{ari}(\logari(A), B) + \cdots$$

(2.7.5)
or equivalently, by

\[ \text{ad}_\text{ari}(A) \cdot B = \text{gari}(\text{preari}(A, B), \text{invgari}(A)). \] (2.7.6)

Writing \( \text{adgari} \) for the conjugation operator

\[ \text{adgari}(A) \cdot B = \text{gari}(A, B, \text{invgari}(B)), \] (2.7.7)

the following diagram then commutes (as for any Lie algebra):

\[
\begin{array}{ccc}
\text{GARI} & \xrightarrow{\text{adgari}(A)} & \text{GARI} \\
\text{expari} \uparrow & & \downarrow \text{logari} \\
\text{ARI} & \xrightarrow{\text{ad}_\text{ari}(A)} & \text{ARI},
\end{array}
\]

where \( \text{logari} \) is the inverse of the isomorphism \( \text{expari} \) (cf. [Pisa, p. 47]).

We conclude this section with the definition of the \( \text{gaxit} \) operator on \( \text{GBARI} \) and the \( \text{gaxi} \)-multiplication law on the group \( \text{GAXI} = \text{GBARI} \times \text{GBARI} \). The very general law \( \text{gaxit} \), which can be restricted to \( \text{GARI} \) and \( \text{GBARI} \) in the usual way, gives the action of a pair of moulds on a mould, whereas \( \text{gaxi} \) is a multiplication law on pairs of moulds. Following [Pisa, p. 42], set

\[ \text{gaxit}_{B,C} \cdot A = \sum_{w=a_1b_1c_1\cdots a_sb_sc_s, b_i \neq 0, a_i, c_{i+1} \neq 0} A([b_1]\cdots [b_s])B(a_1)\cdots B(a_s)]C([c_1]\cdots c([c_s]), \] (2.7.8)

and

\[ \text{gaxi}((A, B), (C, D)) = (\mu(\text{gaxit}_{C,D} \cdot A, C), \mu(D, \text{gaxit}_{C,D} \cdot B)). \] (2.7.9)

Thus we have \( \text{garit}_A = \text{gaxit}_{A, \text{invmu}(A)} \) and \( \text{ganit}_A = \text{gaxit}_{1,A} \). Then

\[ \text{gaxi}((A, \text{invmu}(A)), (C, \text{invmu}(C))) = \]
\[ = \left( \mu(\text{garit}_C \cdot A, C), \mu(\text{invmu}(C), \text{garit}_C \cdot \text{invmu}(A)) \right) \] (2.7.10)

since \( \text{garit}_C \) is a group automorphism for \( \mu \)-multiplication. This shows that \( \text{gaxi} \) of two pairs of the form \( (A, \text{invmu}(A)) \) is again of that form, and \( \text{gari}(A, B) \) is just the left-hand component of (2.7.10). In other words, \( \text{GARI} \) is identified with the subgroup of \( \text{GBARI} \times \text{GBARI} \) of pairs of the form \( (A, \text{invmu}(A)) \) and \( \text{gari} \) is just \( \text{gaxi} \) restricted to this subgroup. In later chapters, other specializations of \( \text{gaxit} \) and \( \text{gaxi} \) to specific subgroups will be useful for certain proofs. In Chapter 3, §3.5, we will also explain the connection between \( \text{GARI} \) and \( \text{gari} \) and the familiar twisted Magnus group with its twisted Magnus multiplication.
§2.8. Swap commutation in GARI

In this section we introduce Ecalle’s first fundamental identity (2.8.4), which expresses the commutation of swap with gari. Let \( \text{gira}(A, B) \) be the swapped gari-product, i.e.

\[
gira(A, B) := \text{swap}(\text{gari}(\text{swap} \cdot A, \text{swap} \cdot B)).
\]

By methods similar to those of §4.1, we can show that

\[
gira(A, B) = \text{gaxi}(\text{h}(A), \text{h}(B))
\]

(2.8.1)

with \( h = \text{push} \cdot \text{swap} \cdot \text{invmu} \cdot \text{swap} \).

We define two operators on moulds following Ecalle ([Pisa, p. 49]):

\[
\text{ras} \cdot B = \text{invgari} \cdot \text{swap} \cdot \text{invgari} \cdot \text{swap}(B)
\]

(2.8.2)

\[
\text{rash} \cdot B = \text{mu}(\text{push} \cdot \text{swap} \cdot \text{invmu} \cdot \text{swap}(B), B).
\]

(2.8.3)

**Theorem 2.8.1.** We have Ecalle’s first fundamental identity:

\[
gira(A, B) = \text{ganit}_{\text{rash}(B)} \cdot \text{gari}(A, \text{ras} \cdot B).
\]

(2.8.4)

The remainder of this chapter is devoted to proving this theorem. Recall the definitions of \( \text{gaxit}, \text{ganit}, \text{garit}, \text{gaxi} \) and \( \text{gari} \) from §2.6. We use the (perhaps slightly doubtful) notation \( \text{invgaxi}_{A,B}(A) \) to denote the left-hand component of the pair \( \text{invgaxi}(A, B) \).

**Lemma 2.8.2.** We have

\[
\text{gaxit}_{A,B} \cdot \text{garit}_{\text{invgaxi}_{A,B}(A)} = \text{ganit}_{\text{mu}(B,A)}.
\]

(2.8.5)

**Proof.** We have

\[
\text{garit}_{\text{invgaxi}_{A,B}(A)} = \text{gaxit}_{\text{invgaxi}_{A,B}(A), \text{invmu}-\text{invgaxi}_{A,B}(A)},
\]

and the composition of two gaxits is given by

\[
\text{gaxit}_{A,B} \cdot \text{gaxit}_{C,D} = \text{gaxit}_{\text{gaxit}_{A,B}(C),\text{A},B,\text{gaxit}_{A,B}(D)},
\]

(2.8.6)

so we can multiply the terms on the LHS of (2.8.5) to obtain

\[
\text{gaxit}_{\text{gaxit}_{A,B}(\text{invgaxi}_{A,B}(A)),\text{A},B,\text{gaxit}_{A,B}(\text{invmu}-\text{invgaxi}_{A,B}(A))}.
\]

(2.8.7)

But we have

\[
\text{gaxit}_{A,B}(\text{invgaxi}_{A,B}(A)) = \text{invmu} A,
\]

(2.8.8)
since by definition of the $gaxi$-multiplication, we have

$$\mu(gaxit_{A,B}(invgaxi_{A,B}(A)), A) = gaxi(invgaxi_{A,B}(A), A) = 1.$$ 

Thus we can substitute (2.8.8) into (2.8.7) to obtain

$$gaxit_{1,B}gaxit_{A,B}(invmu \cdot invgaxi_{A,B}(A)).$$

Similarly, by (2.8.8) and because $gaxit$ is an automorphism for $\mu$, we find that

$$gaxit_{A,B}(invmu \cdot invgaxi_{A,B}(A)) = invmu\left(gaxit_{A,B}(invgaxi_{A,B}(A))\right)$$

$$= invmu \cdot invmu \cdot A$$

$$= A,$$

and replacing this into (2.8.9) yields the desired result $gaxit_{1,\mu}(B,A)$, which is equal to $ganit_{\mu(B,A)}$. This concludes the proof of Lemma 2.8.2.

Let $h = push \cdot swap \cdot invmu \cdot swap$ as in (2.8.1), and let us introduce the notation $gaxit_{B,h} = gaxi((A, h(A)), (B, h(B)))$, i.e. $gaxit_{B,h}(A, B) = \mu(gaxit_{B,h} \cdot A, B)$ by (2.7.9). Finally, we write $invgaxi_{h}(A) = invgaxi_{A,h}(A)$, i.e. the left-hand component of the $gaxi$-inverse of the pair $(A, h(A))$.

**Lemma 2.8.3.** We have

$$invgaxi_{h}(B) = swap \cdot invgari \cdot swap \cdot B.$$  \hspace{1cm} (2.8.10)

**Proof.** We will show using (2.8.1) that the pair $(swap \cdot invgari \cdot swap \cdot B, h(swap \cdot invgari \cdot swap \cdot B))$ is the $gaxi$-inverse of $(B, h(B))$. We have

$$gaxi((swap \cdot invgari \cdot swap \cdot B, h(swap \cdot invgari \cdot swap \cdot B)), (B, h(B)))$$

$$= gira(swap \cdot invgari \cdot swap \cdot B, B)$$

$$= swap\left(gari(invgari \cdot swap \cdot B, swap \cdot B)\right)$$

$$= swap(1)$$

$$= 1,$$

where 1 is the identity mould (that takes the value 1 on the empty set and 0 elsewhere). Thus $swap \cdot invgari \cdot swap \cdot B$ is indeed the left-hand component of the $gaxi$-inverse of $(B, h(B))$, i.e. $invgaxi_{h}(B)$.

**Lemma 2.8.4.** We have

$$\begin{align*}
\mu(gaxit_{A,B}(invgaxi_{A,B}(A)), A) &= invmu \cdot A \\
\mu(garit_{C}(invgari(C)), C) &= invmu \cdot C \\
\mu(gaxit_{B,h}(invgaxi_{h}(C)), C) &= invmu \cdot C.
\end{align*}$$  \hspace{1cm} (2.8.11)
Proof. Writing \( \text{garit}_C = \text{gaxit}_{C, \text{invmu}} \) and \( \text{gaxit}^h_C = \text{gaxit}_{C, h(C)} \) shows that the first equality implies the second and third, so we only need to prove the first one. To prove it, we simply note that the left-hand component of \( \text{gaxi}(\text{invgaxi}(A, B), (A, B)) \) is the identity mould \( 1 \), and it is given by \( \text{mu}(\text{gaxit}_{A, B}(\text{invgaxi}_{A, B}(A)), A) \). This proves the result. \( \Box \)

Lemma 2.8.5. We have

\[ \text{ganit}_{rash \cdot C}(rash \cdot C) = C. \]  
\( (2.8.12) \)

Proof. Recall that \( rash \cdot B = \text{mu}(h(B), B) \). By (2.8.10) we have

\[ rash \cdot B = \text{invgari} \cdot \text{swap} \cdot \text{invgari} \cdot \text{swap} \cdot B = \text{invgari} \cdot \text{invgaxi}^h(B). \]  
\( (2.8.13) \)

Let us apply (2.8.5) with \( A = C \) and \( B = h(C) \), so that

\[ \text{gaxit}_{C, h(C)} \cdot \text{garit}_{\text{invgaxi}_{C, h(C)}(C)} = \text{ganit}_{\text{rash} \cdot C}. \]  
\( (2.8.14) \)

The LHS of (2.8.12) is the RHS of (2.8.14) applied to \( rash \cdot C \), so to compute it, we will study the LHS of (2.8.14) applied to \( rash \cdot C \). Using the fact that \( \text{gaxit} \) is a \( \text{mu} \)-automorphism, we obtain

\[ \text{gaxit}^h_C \cdot \text{garit}_{\text{invgaxi}^h(C)}(\text{invgari} \cdot \text{invgaxi}^h(C)) = \text{gaxit}^h_C \cdot \text{invmu} \cdot \text{invgaxi}^h(C) \]  
by (2.8.11)

\[ = \text{invmu} \cdot \text{gaxit}^h_C \cdot \text{invgaxi}^h(C) \]

\[ = \text{invmu} \cdot \text{invmu} \cdot C \]  
by (2.8.11)

\[ = C. \]

This completes the proof. \( \Box \)

We can now prove Theorem 2.8.1. By (2.8.1) we have

\[ \text{gira}(A, B) = \text{gaxi}^h(A, B). \]

With this, the desired (2.8.4) becomes

\[ \text{gaxi}^h(A, B) = \text{ganit}_{\text{rash} \cdot B} \cdot \text{gari}(A, rash \cdot B). \]  
\( (2.8.15) \)

By (2.8.5), we have

\[ \text{gaxit}_{A, B} \cdot \text{garit}_{\text{invgaxi}_{A, B}(A)} = \text{ganit}_{\text{mu}(B, A)}. \]

Replacing the couple \( (A, B) \) by \( (B, h(B)) \) and recalling that \( \text{mu}(h(B), B) = rash \cdot B \), this gives

\[ \text{gaxit}^h_B \cdot \text{garit}_{\text{invgaxi}^h(B)} = \text{ganit}_{\text{rash} \cdot B}, \]
which, given that the inverse automorphism of $\text{garit}_B$ is $\text{garit}_{\text{invgari}}(B)$, we can rewrite as

$$gaxi^h_B = \text{ganit}_{\text{rash} \cdot B} \cdot \text{garit}_{\text{invgari} \cdot \text{invga}x^h(B)} = \text{ganit}_{\text{rash} \cdot B} \cdot \text{garit}_{\text{ras} \cdot B} \quad (2.8.16)$$

since by definition of $\text{ras}$ and (2.8.10) we have

$$\text{ras} \cdot B = \text{invgari} \cdot \text{swap} \cdot \text{invgari} \cdot \text{swap} \cdot B = \text{invgari} \cdot \text{invga}x^h(B).$$

We will prove (2.8.4) by applying each side of (2.8.16) to a mould $A$, then $\mu$-multiplying the result with $B$.

The LHS of (2.8.16) yields

$$\mu(gaxi^h_B(A), B) = gaxi^h(A, B).$$

The RHS yields

$$\mu(\text{ganit}_{\text{rash} \cdot B} \cdot \text{garit}_{\text{ras} \cdot B}(A), B)$$
$$= \mu(\text{ganit}_{\text{rash} \cdot B} \cdot \text{garit}_{\text{ras} \cdot B}(A), \text{ganit}_{\text{rash} \cdot B}(\text{ras} \cdot B)) \quad \text{by (2.8.12)}$$
$$= \text{ganit}_{\text{rash} \cdot B} \cdot \mu(\text{garit}_{\text{ras} \cdot B}(A), \text{ras} \cdot B))$$
$$= \text{ganit}_{\text{rash} \cdot B} \cdot \text{gari}(A, \text{ras} \cdot B)).$$

This completes the proof of Theorem 2.8.1.

The following corollary of Theorem 2.8.1 containing the equality (2.8.17) will be useful in Chapter 4, when we come to prove Ecalle’s second fundamental identity. Let $\text{fragari}(A, B) = \text{gari}(A, \text{invgari}(B))$. Then (2.8.17) is proved simply by substituting $C = \text{invgari} \cdot \text{ras} \cdot B = \text{swap} \cdot \text{invgari} \cdot \text{swap} \cdot B$ into (2.8.4).

**Corollary 2.8.6.** We have

$$\text{swap} \cdot \text{fragari}(\text{swap} \cdot A, \text{swap} \cdot C) = \text{ganit}_{\text{crash} \cdot C} \cdot \text{fragari}(A, C), \quad (2.8.17)$$

where $\text{crash} \cdot C = \text{rash} \cdot \text{swap} \cdot \text{invgari} \cdot \text{swap} \cdot C$. 

33
Chapter 3

From double shuffle to ARI

In this chapter, we define a map from the twisted Magnus Lie algebra \( \mathfrak{mt} \) (introduced in §1.3) to \( \text{ARI}_{\alpha_1}^{\text{pol}} \), and prove that it is a Lie algebra isomorphism. We further show that the images of the two Lie subalgebras \( \mathfrak{i}_s \) and \( \mathfrak{ds} \) of \( \mathfrak{mt} \) defined in §1.3 and §1.4 map isomorphically onto \( \text{ARI}_{\alpha_1}^{\text{pol}} \) and \( \text{ARI}_{\alpha_1}^{\text{pol}^*} \). In §3.4 we use the results of Chapter 2 together with these isomorphisms to show how Ecalle’s methods give a simple proof of some basic results on double shuffle (Theorems 1.3.2 and 1.4.1), namely that \( \mathfrak{i}_s^d \) is zero if \( n \not\equiv d \mod 2 \), and hence also \( \mathfrak{ds}_n^d \) is zero if \( n \not\equiv d \mod 2 \).

§3.1. The ring \( \mathbb{Q}\langle C \rangle \)

Consider the ring of polynomials \( \mathbb{Q}\langle x, y \rangle \) in non-commutative variables \( x, y \). Let \( \partial_x \) denote the differential operator with respect to \( x \). Set \( C_i = \text{ad}(x)^{i-1}(y), i \geq 1 \), so \( C_1 = y, C_2 = [x, y], C_3 = [x, [x, y]], \ldots. \)

**Definition.** Let \( \mathbb{Q}\langle C \rangle \) denote the subspace of \( \mathbb{Q}\langle x, y \rangle \) of polynomials \( f \) such that \( \partial_x(f) = 0 \).

The following well-known result is just a standard application of Lazard elimination.

**Lemma 3.1.1.** The subspace \( \mathbb{Q}\langle C \rangle \subset \mathbb{Q}\langle x, y \rangle \) is equal to the subring generated by the \( C_i \), \( i \geq 1 \). Moreover the \( C_i \) are free generators of this ring.

Let \( \pi_y \) be the projector onto polynomials ending in \( y \) (i.e. \( \pi_y \) forgets all the monomials ending in \( x \)). The usefulness of the ring \( \mathbb{Q}\langle C \rangle \) is that \( \pi_y \) has a section on \( \mathbb{Q}\langle C \rangle \). Indeed, for any polynomial \( g \) ending in \( y \), define \( \text{sec}(g) \) by

\[
\text{sec}(g) = \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_x^i(g) x^i.
\]

**Lemma 3.1.2.** [R, Prop IV.2.8] (1) \( \text{sec} \circ \pi_y = \text{id} \) on \( \mathbb{Q}\langle C \rangle \).

(2) \( \pi_y \circ \text{sec} = \text{id} \) on \( \mathbb{Q}\langle x, y \rangle y \).

§3.2. Associating moulds to elements \( f \in \mathbb{Q}\langle C \rangle \)

**Definitions.** Let \( \mathbb{Q}\langle C \rangle_n \) denote the vector subspace of polynomials in \( \mathbb{Q}\langle C \rangle \) of homogeneous degree \( n \) in \( x \) and \( y \), \( \mathbb{Q}\langle C \rangle^r \) the subspace of polynomials of homogeneous degree \( r \) (i.e. linear combinations of monomials of the form \( C_{a_1} \cdots C_{a_{n}} \)), and \( \mathbb{Q}\langle C \rangle_n^r \) the intersection. The space \( \mathbb{Q}\langle C \rangle \) is bigraded, i.e. \( \mathbb{Q}\langle C \rangle = \bigoplus_{n, r \geq 0} \mathbb{Q}\langle C \rangle_n^r \). If \( f \in \mathbb{Q}\langle C \rangle \), we write \( f_n \) for its weight \( n \) part and \( f^r \) for its depth \( r \) part.

Let \( \pi_y(f) \) denote the projection of \( f \) onto the monomials ending in \( y \) as above, and let \( f_y \) denote \( \pi_y(f) \) rewritten in the variables \( y_i = x^{i-1} y, \ i \geq 1 \), and \( f_y^r \) the depth \( r \) part, i.e. \( \pi_y(f_y^r) \) written in the \( y_i \). Similarly, let \( \pi_Y(f) \) denote the projection of \( f \) onto the
monomials starting with $y$. Let $\text{ret}_X : \mathbb{Q}(x, y) \to \mathbb{Q}(x, y)$ denote the “backwards writing” map

$$\text{ret}_X (x^{a_0} y \ldots y x^{a_{r-1}} y x^{a_r^1} y \ldots y x^{a_r}) = x^{a_r} y x^{a_{r-1}} y \ldots y x^{a_0}. \quad (3.2.1)$$

Note that $\text{Lie}[x, y] \subset \mathbb{Q}(C)$. If $f \in \mathbb{Q}(C)_n$ is actually a Lie element, we have

$$\text{ret}_X (f) = (-1)^{n-1} f. \quad (3.2.2)$$

Finally, let $f_Y^r$ denote the polynomial $\text{ret}_X (\pi_Y (f^r))$ written in the variables $y_i$ and $f_Y = \sum_r f_Y^r$. We note here that by a result in [CS], the introduction of $f_Y$ gives an equivalent formulation of the definition of $\mathfrak{d}S$ that will be useful below.

We saw in Lemma 3.1.1 that $\mathbb{Q}(C)$ is the set of polynomials in $\mathbb{Q}(x, y)$ that can be written as polynomials in the $C_i$, and that such a writing is unique. Let $f_C$ denote $f$ written in this way.

Define three maps from monomials in the variables $x, y$ (resp. $y_1, y_2, \ldots$ resp. $C_1, C_2, \ldots$) to monomials in commutative variables $z_0, z_1, \ldots$ (resp. $u_1, u_2, \ldots$ resp. $v_1, v_2, \ldots$) as follows:

$$\iota_X : x^{a_0-1} y \ldots y x^{a_r-1} y x^{a_r^1} y \ldots y x^{a_r^1} y x^{a_r} \mapsto z_0^{a_0-1} \ldots z_r^{a_r-1}$$

$$\iota_C : C_{a_1} \ldots C_{a_r} \mapsto u_1^{a_1-1} \ldots u_r^{a_r-1} \quad (3.2.3)$$

$$\iota_Y : y a_1 \ldots y a_r \mapsto v_1^{a_1-1} \ldots v_r^{a_r-1}.$$

Then we define a mould in commutative variables $z_0, z_1, \ldots$ associated to $f \in \mathbb{Q}(C)_n$ as follows:

$$\text{vimo}_f (z_0, z_1, \ldots, z_r) = \iota_X (f^r), \quad (3.2.4)$$

and also a mould and a $v$-mould associated to $f$ by

$$\text{ma}_f (u_1, \ldots, u_r) = (-1)^{r+n} \iota_C (f_C^r), \quad \text{mi}_f (v_1, \ldots, v_r) = \iota_Y (f_Y^r). \quad (3.2.5)$$

All other values of these moulds are 0.

**Remark.** Note that by (3.2.2), if $f \in \text{Lie}[x, y]$, we have

$$\pi_y (f) = (-1)^{n-1} \text{ret}_X (\pi_Y (f)), \quad$$

so $f^r_Y = (-1)^{n-1} f^r_Y$. Thus, if $f \in \text{Lie}[x, y]$, the $v$-mould $\text{mi}$ can also be defined by

$$\text{mi}_f (v_1, \ldots, v_r) = (-1)^{n-1} \iota_Y (f_Y^r). \quad (3.2.6)$$

When we turn our attention to the twisted Magnus Lie algebra $\mathfrak{m}$t and its double shuffle and linearized double shuffle subspaces, in §§3.3-3.4, we will be in this situation.

Since the maps $\iota_X$, $\iota_C$ and $\iota_Y$ are obviously invertible, we recover $f$ from $\text{vimo}_f$, $f_C$ from $\text{ma}$ and $f_Y$ from $\text{mi}$. But of course, we easily recover $f$ from $f_C$ by expanding out
the $C_i$, and we also recover $f$ from $f_Y$ by setting $f = \sec(f_Y)$, as we have assumed that $f \in \mathbb{Q}(C)_n$. Thus, for any element $f \in \mathbb{Q}(C)_n$, $f$, $f_C$, $f_Y$ and $vimo_f$ are all different encodings of the same information. The moulds $ma$ and $mi$ are also equivalent encodings, related to $vimo_f$ as follows.

**Lemma 3.2.1.** The mould $ma$ and the $v$-mould $mi$ are obtained from $vimo_f$ by the formulas

\[
ma_f(u_1, \ldots, u_r) = vimo_f(0, u_1, u_1 + u_2, \ldots, u_1 + \cdots + u_r) \quad (3.2.7)
\]

\[
mi_f(v_1, \ldots, v_r) = vimo_f(0, v_r, v_r-1, \ldots, v_1). \quad (3.2.8)
\]

The proof of this lemma is given in §A.3 of the Appendix.

**Remark.** If $vimo(z_0, \ldots, z_r)$ for $r \geq 0$ is an arbitrary family of polynomials, then there is a unique $f \in \mathbb{Q}(x,y)$ associated to it by (3.2.3). It is natural to ask what condition on the family $vimo$ ensures that $f \in \mathbb{Q}(C)$. We leave the following answer as an exercise.

**Lemma 3.2.2.** If $f \in \mathbb{Q}(x,y)$ and $vimo_f$ is defined as in (3.2.4), then $f \in \mathbb{Q}(C)$ if and only if

\[
vimo_f(z_0, \ldots, z_r) = vimo_f(0, z_1 - z_0, z_2 - z_0, \ldots, z_r - z_0) \quad (3.2.9)
\]

for $r \geq 1$.

**Remarks.** (1) Observe that if we apply the variable change $u_1 = z_1 - z_0$, $u_2 = z_2 - z_1$, $u_3 = z_3 - z_2, \ldots, u_r = z_r - z_{r-1}$ to $ma_f(u_1, \ldots, u_r)$, obtaining $vimo_f(0, z_1 - z_0, \ldots, z_r - z_0)$. Thanks to (3.2.9), if $f \in \mathbb{Q}(C)$ then this is equal to $vimo_f(z_0, \ldots, z_r)$, so that $ma_f$ is yet another equivalent coding for $f \in \mathbb{Q}(C)$, and the same holds for $mi_f$ using the variable change $v_j = z_{r-j+1} - z_0$.

(2) From the expressions (3.2.7) and (3.2.8), it is immediate that for $f \in \mathbb{Q}(C)$, we have

\[
swap(ma_f) = mi_f. \quad (3.2.10)
\]

**Example.** Let $f$ be the degree 3 Lie polynomial

\[
f = [x, [x, y]] + [[x, y], y] = x^2y - 2xyx + yx^2 + xy^2 - 2yxy + y^2x. \quad (3.2.11)
\]

Then $\pi_y(f) = x^2y - 2xyx + yx^2$, $f_Y = y_3 - 2y_1y_2 + y_2y_1$ and $f_C = C_3 - C_1C_2 + C_2C_1$, and we have

\[
\begin{align*}
\begin{cases}
vimo_f(z_0) = 0 \\
vimo_f(z_0, z_1) = z_0^2 - 2z_0z_1 + z_1^2 \\
vimo_f(z_0, z_1, z_2) = z_0 - 2z_1 + z_2 \\
vimo_f(z_0, z_1, z_2, z_3) = 0,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
ma_f(\emptyset) = 0 \\
ma_f(u_1) = u_1^2 \\
ma_f(u_1, u_2) = -u_1 + u_2 \\
ma_f(u_1, u_2, u_3) = 0,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
mi_f(\emptyset) = 0 \\
mi_f(v_1) = v_1^2 \\
mi_f(v_1, v_2) = -2v_2 + v_1 \\
mi_f(v_1, v_2, v_3) = 0.
\end{cases}
\end{align*}
\]

The results of this section can be summarized by the following theorem. We write $\mathbb{Q}_0(C)$ for the subspace of polynomials in $\mathbb{Q}(C)$ with constant term 0.
Theorem 3.2.3. The map

\[ ma : \mathbb{Q}_0 \langle C \rangle \to ARI^{pol} \quad (3.2.12) \]

is a ring isomorphism, where \( \mathbb{Q}_0 \langle C \rangle \) is equipped with the ordinary (concatenation) multiplication of polynomials, and \( ARI^{pol} \) with the multiplication \( \mu \).

Proof. By Lemma 3.1.1 together with the definition of \( \iota_C \) in (3.2.3) and the definition of the map \( ma \) in (3.2.5), we see that \( ma \) is a vector space isomorphism from \( \mathbb{Q}_0 \langle C \rangle \) to the set of polynomial-valued moulds, so it restricts from \( \mathbb{Q}_0 \langle C \rangle \) to \( ARI^{pol} \). Thus it remains only to show that

\[ ma f g = \mu (ma f, ma g). \quad (3.2.13) \]

By additivity, it is enough to assume that \( f \) and \( g \) are monomials in the \( C_i \), say \( f = C_{a_1} \cdots C_{a_r} \) and \( g = C_{b_1} \cdots C_{b_s} \); then it is immediate that

\[ ma f g = u_1^{a_1-1} \cdots u_r^{a_r-1} u_{r+1}^{b_{r+1}-1} \cdots u_{r+s}^{b_{r+s}-1} = \mu (ma f, ma g). \]

This concludes the proof. \( \diamond \)

§3.3. The Poisson bracket and the ARI bracket

In this section we prove that the Poisson bracket is carried over to the \( ari \)-bracket under the isomorphism \( \mathbb{Q} \langle C \rangle \xrightarrow{ma} ARI^{pol} \) of (3.2.12). This result was originally proved in [R, Appendice A, §5]. After introducing the key result in Lemma 3.3.1 (due to Racinet), we then compare the derivations \( D_f \) and \( arit(ma f) \) in Proposition 3.3.3 and deduce the equality

\[ ma \{ f, g \} = arit(ma f, ma g) \]

in Corollary 3.3.4.

Observe that if \( f \in \mathbb{Q} \langle C \rangle_n \), then \( \partial_x ([x, f]) = 0 \), so by Lemma 3.1.1, \( [x, f] \in \mathbb{Q} \langle C \rangle_{n+1} \).

By Lemma 3.1.1, we can consider both \( f \) and \( [x, f] \) as being polynomials in the \( C_i \).

Lemma 3.3.1. [R] Let \( f \in \mathbb{Q} \langle C \rangle_n \). Then for \( 0 \leq r \leq n \), we have

\[ ma [x, f_r] = -(u_1 + \cdots + u_r) ma f_r. \quad (3.3.1) \]

Proof. Note first that \( a \mapsto [x, a] \) is a derivation, i.e. \( [x, ab] = [x, a] b + a [x, b] \). Thus, writing \( f^r = \sum_a c_a C_{a_1} \cdots C_{a_r} \), where \( a = (a_1, \ldots, a_r) \), we have

\[
[x, f^r] = \sum_a c_a [x, C_{a_1} \cdots C_{a_r}] = \sum_a c_a \sum_{i=1}^r C_{a_1} \cdots C_{a_{i-1}} [x, C_{a_i}] C_{a_{i+1}} \cdots C_{a_r} \\
= \sum_a \sum_{i=1}^r c_a C_{a_1} \cdots C_{a_{i-1}} C_{a_{i+1}} C_{a_{i+1}} \cdots C_{a_r}.
\]

Thus, the left-hand side of (3.3.1) is equal to

\[ (-1)^{r+n+1} \sum_a \sum_{i=1}^r c_a u_1^{a_1-1} \cdots u_i^{a_i} \cdots u_r^{a_r-1}. \quad (3.3.2) \]

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But since \( ma_f = (-1)^r \sum a_c u_1^{a_1-1} \cdots u_r^{a_r-1} \), (3.3.2) is equal to \( ma_f \) multiplied by \(- (u_1 + \cdots + u_r)\), proving (3.3.1).

\[ \Box \]

**Proposition 3.3.2.** For any mould \( A \), the operator \( \text{arit}(A) \) is a derivation for the multiplication.

The proof is given in the Appendix, §A.4.

**Proposition 3.3.3.** Let \( f \in \mathbb{Q}(C)_r \) be of homogeneous depth \( r \) and \( g \in \mathbb{Q}(C)_m \) of homogeneous depth \( s \). Let \( D_f \) be the derivation of \( \mathbb{Q}(C) \) defined by \( D_f(x) = 0 \), \( D_f(y) = [y, f] \). Then

\[
ma_{D_f(g)} = -\text{arit}(ma_f) \cdot ma_g. \tag{3.3.3}
\]

**Proof.** We have \( D_{f+g} = D_f + D_g \), so we may assume that \( f = C_{a_1} \cdots C_{a_r} \) is a monomial in the \( C_i \). Furthermore, a derivation of \( \mathbb{Q}(C) \) is defined by its action on the generators \( C_i \), so we may take \( g = C_m = \text{ad}(x)^{m-1}(y) \). Let \( F_0 = [y, f] \), and for \( i \geq 1 \), let \( F_i = \text{ad}(x)^i([y, f]) \).

In particular, we have

\[
D_f(g) = [x, [x, \cdots, [x, [y, f]] \cdots ]] = \text{ad}(x)^{m-1}([y, f]) = F_{m-1}.
\]

Then by Lemma 3.3.1, since all the \( F_i \) are in depth \( r + 1 \), we have

\[
ma_{F_i} = -(u_1 + \cdots + u_{r+1})ma_{F_{i-1}} \quad \text{for} \quad i > 0,
\]

so

\[
ma_{F_i} = (-1)^i(u_1 + \cdots + u_{r+1})ma_{F_0},
\]

so the left-hand side of (3.3.3) is equal to

\[
ma_{D_f(g)} = ma_{F_{m-1}}
\]

\[
= (-1)^{m-1}(u_1 + \cdots + u_{r+1})^{m-1}ma_{F_0}
\]

\[
= (-1)^{m-1}(u_1 + \cdots + u_{r+1})^{m-1}ma_{[y, f]}
\]

\[
= (-1)^{m+r+n+1}(u_1 + \cdots + u_{r+1})^{m-1}\text{arit}(C_1C_{a_1} \cdots C_{a_r} - C_{a_1} \cdots C_{a_r}C_1)
\]

\[
= (-1)^{m+r+n+1}(u_1 + \cdots + u_{r+1})^{m-1}(u_2^{a_1-1} - u_{r+1}^{a_r-1}) - u_1^{a_1-1} - u_{r+1}^{a_r-1}.
\]

since \( ma_{[y, f]} = (-1)^{r+n+2}\text{arit}([y, f]) \). Now consider the right-hand side of (3.3.3). By (3.2.5), we have

\[
ma_f(u_1, \ldots, u_r) = (-1)^{r+n}\text{arit}(f) = (-1)^{r+n}u_1^{a_1-1} \cdots u_r^{a_r-1},
\]

where \( n = a_1 + \cdots + a_r \), and

\[
ma_g(u_1) = (-1)^{m-1}u_1^{m-1}.
\]

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Since $ma_g$ has value zero on any word of length greater than 1, the defining formula for $arit(A)$ simplifies when $A = ma_f$, $B = ma_g$ to
\[
(arit(ma_f) \cdot ma_g)(u_1, \ldots, u_r) = ma_g(u_1)ma_f(u_2, \ldots, u_{r+1}) \\
+ ma_g(u_1 + \cdots + u_{r+1})ma_f(u_1, \cdots, u_r) \\
- ma_g(u_1 + \cdots + u_{r+1})ma_f(u_2, \ldots, u_{r+1}) \\
- ma_g(u_1)ma_f(u_2, \ldots, u_{r+1})
\]

\[= ma_g(u_1 + \cdots + u_{r+1})\left( ma_f(u_1, \cdots, u_r) - ma_f(u_2, \ldots, u_{r+1}) \right) \]

\[= (-1)^{m+r+n}(u_1 + \cdots + u_{r+1})^{m-1}(u_2^{a_1-1} \cdots u_{r+1}^{a_r-1} - u_1^{a_1-1} \cdots u_r^{a_r-1}).\]

This proves (3.3.3).  

\[\square\]

**Corollary 3.3.4.** Let $f \in \mathbb{Q}(C)_n$ be of homogeneous depth $r$ and $g \in \mathbb{Q}(C)_m$ of homogeneous depth $s$. Then
\[
ma_{\{f,g\}} = arit(ma_f, ma_g).
\]  

**Proof.** Recall that $\{f, g\} = D_f(g) - D_g(f) + fg - gf$. By (3.3.3) and (3.2.13), we then have
\[
ma_{\{f,g\}} = -arit(ma_f) \cdot ma_g + arit(ma_g) \cdot ma_f + mu(ma_f, ma_g) - mu(ma_g, ma_f) \\
= arit(ma_g) \cdot ma_f - arit(ma_f) \cdot ma_g + limu(ma_f, ma_g) \\
= arit(ma_f, ma_g)
\]

by (2.2.9). This concludes the proof.  

\[\square\]

### §3.4. From $\mathfrak{ds}$ to ARI

In this section we relate the special Lie subspaces $\mathfrak{mt}$, $\mathfrak{ls}$ and $\mathfrak{ds}$ of $\mathbb{Q}(C)$ to some of the special subspaces of ARI defined in §2.5. The proofs are based on the following explicit comparison of double shuffle properties of polynomials in $\mathbb{Q}(C)$ with symmetry properties on moulds.

**Lemma 3.4.1.** Let $f \in \mathbb{Q}(C)_n$. Then

(i) $f$ satisfies shuffle in $x, y$ if and only if $ma_f \in ARI_{al}^{pol}$;  
(ii) $f_Y$ satisfies shuffle in the $y_i$ if and only if $mi_f \in ARI_{al}^{pol}$;  
(iii) $f_Y$ satisfies stuffle in the $y_i$ if and only if $mi_f \in ARI_{il}^{pol}$;  
(iv) $f_Y$ satisfies stuffle in the $y_i$ in depth $1 \leq r < n$ if and only if $mi_f \in ARI_{stil}^{pol}$.

This Lemma is proved in the Appendix, §A.5.

**Theorem 3.4.2.** The isomorphism $ma: \mathbb{Q}(C) \leadsto ARI^{pol}$ restricts to an isomorphism of Lie algebras
\[
ma: \mathfrak{mt} \leadsto ARI_{al}^{pol}.
\]  

(3.4.1)
Proof. We first observe that \( mt \subset Q(C) \) since by definition, the underlying vector space of \( mt \) is the Lie algebra freely generated by the \( C_i, i \geq 1 \) (see \S 1.3). Indeed, we have \( \text{Lie}[x, y] \cap Q(C) = mt \).

Since \( ma \) is injective on \( Q(C) \), it is injective restricted to \( mt \). By \S 1.3 (1), a polynomial \( f \in Q(C) \) satisfies shuffle if and only if \( f \in \text{Lie}[x, y] \), which shows that every \( f \in mt \) satisfies shuffle. Then Lemma 3.4.1 (i) shows that \( ma_f \in \text{ARI}_{al}^{pol} \). Conversely, if \( A \in \text{ARI}_{al}^{pol} \), then since \( ma : Q(C) \to \text{ARI}_{al}^{pol} \) is an isomorphism, there exists \( f \in Q(C) \) such that \( A = ma_f \), and then again by Lemma 3.4.1 (i), \( f \) must satisfy shuffle, i.e. \( f \in \text{Lie}[x, y] \cap Q(C) = mt \).\( \diamond \)

We can now proceed to the first main result of this section.

Theorem 3.4.3. The map \( f \mapsto ma_f \) yields a Lie algebra isomorphism

\[
\mathfrak{ls} \sim \text{ARI}_{al/la}^{pol}.
\] (3.4.2)

Proof. Thanks to (3.3.5), which shows that the Poisson bracket on \( mt \) carries over to the \( ari \)-bracket, it suffices to show that (3.4.2) is a vector space isomorphism. Let \( f \in \mathfrak{ls} \); we may assume that \( f \) is homogeneous of degree \( n \). Recall that the definition of \( \mathfrak{ls} \) is that \( f \) must satisfy shuffle in \( x, y \) and \( \pi_y (f) \) must satisfy shuffle in the \( y_i \). Since \( f \) is a Lie polynomial, we have \( f_Y = ret_X (\pi_Y (f)) = (-1)^{n-1} \pi_y (f) \), so \( f_Y \) satisfies shuffle in the \( y_i \) if and only if \( \pi_y (f) \) (rewritten in the \( y_i \)) does. But by Lemma 3.4.1 (ii), \( f_Y \) satisfies the shuffle in the \( y_i \) if and only if \( mf_f \in \text{ARI}_{al} \). Thus the image of \( \mathfrak{ls} \) under the injective map \( f \mapsto ma_f \) lies in \( \text{ARI}_{al/la}^{pol} \). Recall that by the definition of \( \mathfrak{ls} \) (see \S 1.4), the even degree depth 1 polynomials \( ad(x)^{2i+1} (y) \) are excluded from \( \mathfrak{ls} \); thus the image of \( \mathfrak{ls} \) lies in \( \text{ARI}_{al/la}^{pol} \).

Conversely, if \( A \in \text{ARI}_{al/la}^{pol} \), then since \( ma : Q(C) \to \text{ARI}_{al}^{pol} \) is an isomorphism, there exists a unique \( f \in Q(C) \) such that \( ma_f = A \), and then by Lemma 3.4.1, \( f \) must satisfy shuffle and \( f_Y \) must satisfy shuffle in the \( y_i \), and if \( ma_f \) is of depth 1 then \( f \) is of odd degree, so \( f \in \mathfrak{ls} \).\( \diamond \)

From this we deduce the proofs of Theorem 1.4.1 (which then implies Theorem 1.3.2), which is essentially no more than a translation back into \( Q(C) \) of Theorem 2.7.6 stating that \( \text{ARI}_{al/la} \) is a Lie algebra under the \( ari \)-bracket.

Corollary 3.4.4. The weight \( n, \) depth \( d \) space \( \mathfrak{ls}_n^d \) is zero if \( n \not\equiv d \mod 2 \); thus in particular the graded quotient \( \mathfrak{ls}_n^d / \mathfrak{ls}_n^{d+1} \) which lies inside it is zero if \( n \not\equiv d \mod 2 \).

Proof. Using the translation into moulds (3.4.2), the statement is equivalent to the fact that if \( A \in \text{ARI}_{al/la}^{pol} \) is a homogeneous polynomial mould \( A(u_1, \ldots, u_d) \) of of odd degree \( n - d \), then \( A = 0 \). But this follows immediately from Lemma 2.5.5 which says that elements of \( \text{ARI}_{al/la} \) are \( neg \)-invariant, i.e. \( A(u_1, \ldots, u_d) = A(-u_1, \ldots, -u_d) \); indeed if \( A \) is homogeneous of odd degree, then \( A \) must be zero.\( \diamond \)

This proof, or rather the proof of Lemma 2.5.5, is a perfect example of the real simplicity and magic of Ecalle’s methods.
Our next step is to prove the analogue of (3.4.2) for $\mathfrak{d}s$. We first need a lemma that slightly rephrases the definition of $\mathfrak{d}s$.

**Lemma 3.4.5.** The Lie algebra $\mathfrak{d}s$ is equal to the set of $f \in \text{Lie}[x,y]$ of degree $\geq 3$ such that $f_Y$, rewritten in the variables $y_i$, satisfies all the stuffle relations (1.3.3) except for those where both words in the pair $(u,v)$ are powers of $y$.

**Proof.** Let the depth of a stuffle relation as in (1.3.3) be equal to the sum of the depths of the two words $(u,v)$. Let $f \in \mathfrak{d}s$; we may assume that $f$ is homogeneous of degree $n$. Suppose that $f_Y$ satisfies all the stuffle relations of depths $< n$. Since $f$ is Lie, we have $\text{ret}_X(f) = (-1)^{n-1}f$, so in particular

$$f_Y = \text{ret}_X(\pi_Y(f)) = (-1)^{n-1}\pi_y(f);$$

thus $\pi_y(f)$ satisfies the same stuffle relations. Then [CS, Theorem 2] shows that there exists a unique constant, namely $a = \frac{(-1)^{n-1}}{n}(\pi_y(f)|x^{n-1}y)$, such that $\pi_y(f) + ay^n$, rewritten in the $y_i$, satisfies all of the stuffle relations. But the term $ay^n$ is equal to $f_{\text{corr}}$ as in (1.3.2), so this is equivalent to the original definition of $\mathfrak{d}s$ given in §1.3.  

**Theorem 3.4.4.** The isomorphism $ma : \mathbb{Q}\langle C \rangle \to \text{ARI}^{\text{pol}}_{\text{al}*\text{il}}$ restricts to a Lie algebra isomorphism

$$\mathfrak{d}s \simeq \text{ARI}^{\text{pol}}_{\text{al}*\text{il}}$$

**Proof.** We saw above that $f \mapsto ma_f$ maps $\mathfrak{d}s$ injectively into $\text{ARI}^{\text{pol}}_{\text{al}}$. Let $f \in \mathfrak{d}s$, and assume that $f$ is homogeneous of degree $n$. Then as in the proof of Lemma 3.4.5, $\pi_y(f)$ satisfies all the stuffle relations of depth $< n$, and $f_\star = \pi_y(f) + ay^n$ satisfies all the stuffle relation, where $a = \frac{(-1)^{n-1}}{n}(\pi_y(f)|x^{n-1}y)$.

Now, let $mi_f = \iota_Y(f_Y)$ as in (3.2.5), and let $mi'_f = \iota_Y(f_\star)$. Then since $f_\star$ satisfies the stuffle relations, by Lemma 3.4.1 (iii) we know that $mi'_f$ is alternil. But since (apart from the sign) $f_Y$ differs from $f_\star$ only by the depth $n$ term $ay^n$, the two moulds $mi_f$ and $mi'_f$ differ (up to sign) only by the depth $n$ component, which is a constant due to the homogeneity of $f$, which in terms of moulds means that each $mi_f(v_1, \ldots, v_r)$ is a polynomial of degree $n-r$. This means that it suffices to modify $mi_f$ by a constant in depth $n$ to make it fully alternil, which is the definition of $\text{ARI}^{\text{il}}_{\text{al}*\text{il}}$. Thus $ma_f \in \text{ARI}^{\text{pol}}_{\text{al}*\text{il}}$. The surjectivity holds as before, since surjectivity of $ma$ means that there exists a polynomial in $\mathbb{Q}\langle C \rangle$ such that $ma_f = A$ for any $A \in \text{ARI}^{\text{pol}}_{\text{al}*\text{il}}$, and then by Lemma 3.4.1, $f$ must satisfy shuffle and $f_Y$ stuffle for depths $< n$; then using Lemma 3.4.5 proves that $f \in \mathfrak{d}s$.  

**Example.** We take the same example as in (3.2.11), and check that $ma_f/mi_f$ is al * il (i.e. $ma_f \in \text{ARI}_{\text{al}}$ and $mi_f \in \text{ARI}_{\text{il}}$). Recall that

$$\begin{align*}
\{ & ma_f(u_1) = u_1^2 \\
& ma_f(u_1, u_2) = -u_1 + u_2, \\
\{ & mi_f(v_1) = v_1^2 \\
& mi_f(v_1, v_2) = v_1 - 2v_2.
\end{align*}$$

To show that $ma_f$ is alternal, the only condition to check is that $ma_f(u_1, u_2) + ma_f(u_2, u_1) = 0$, which is immediate. To show that $mi_f$ is alternil, we only have to check the alternility
relation corresponding to the stuffle relation for depth $r = 2$, given in (2.3.5):

$$
(v_{1} - 2v_{2}) + (v_{2} - 2v_{1}) + \frac{1}{v_{1} - v_{2}} v_{1}^2 + \frac{1}{v_{2} - v_{1}} v_{2}^2 = (-v_{1} - v_{2}) + (v_{1} + v_{2}) = 0.
$$

§3.5. The group GARI and the twisted Magnus group.

In this section we establish the isomorphism between the twisted Magnus group (defined below) and $GARI_{\text{pol}}$ which is the group analog of Theorem 3.4.2. The proof is basically a corollary of Theorem 3.4.2 using the exponential, but it is useful to recall the objects and definitions that are the translations of $GARI_{\text{pol}}$ and its associated operators $(ganit, garit, gari$ etc.) so as to clarify the fact that in this familiar context they are in fact familiar operators, on the one hand, and to emphasize the power of Ecalle’s theory in extending from polynomial-valued moulds to rational-valued moulds on the other. We end the section by explaining the meaning of some of the main identities from §2.7 in the twisted Magnus situation.

**Definition.** Let $f, g \in \mathfrak{m}t$, and define $p(f, g) = fg - D_{g}(f)$ to be the pre-Lie law associated to $\mathfrak{m}t$. Obviously $p(f, g) - p(g, f) = \{f, g\}$, and thanks to (3.3.3), we have

$$
ma_{p(f, g)} = mu(ma_{f}, ma_{g}) + arit(ma_{g}) \cdot ma_{f} = preari(ma_{f}, ma_{g}).
$$

The expression $p(f, g) = fg - D_{g}(f)$ actually expresses the multiplication rule on the universal enveloping algebra $U\mathfrak{m}t$ for all $g \in \mathfrak{m}t, f \in U\mathfrak{m}t$, not only when $f \in \mathfrak{m}t$.

Define the twisted Magnus exponential on $\mathfrak{m}t$ by

$$
exp^{\circ}(f) = 1 + f + \sum_{n \geq 2} \frac{1}{n!} p(f^n),
$$

Then by (2.6.1) we have

$$
ma_{exp^{\circ}(f)} = expari(ma_{f}).
$$

where $p(f^n) = p(p(f^{n-1}), f), p(f^3) = p(p(f, f), f)$ etc.

The *twisted Magnus group* $MT$ is the pro-unipotent group $exp^{\circ}(\mathfrak{m}t)$.

By the Milnor-Moore theorem, we have an isomorphism of vector spaces

$$
U\mathfrak{m}t \simeq \mathbb{Q}\langle C \rangle
$$

where both sides are Hopf algebras with the multiplication on the right-hand ring being different than the usual concatenation, but the coproduct being the restriction to $\mathbb{Q}\langle C \rangle$ of the standard coproduct defined by

$$
\Delta(C_i) = C_i \otimes 1 + 1 \otimes C_i, \quad i \geq 1.
$$
Indeed, the primitive elements of $\mathbb{Q}(C)$ for $\Delta$ are well-known to be the Lie polynomials in the $C_i$, which form the underlying vector space $L$ of $\mathfrak{mt}$ (see §1.3). Since the ring $\mathbb{Q}(C)$ is a graded polynomial ring (where the grading can be considered to be degree in $x, y$ or else the weight in the $C_i$ where each $C_i$ is of weight $i$) with $\mathbb{Q}(C)_0 = \mathbb{Q}$ and each graded part is finite-dimensional, Milnor-Moore applies and yields the isomorphism (3.5.3).

As in the general case of Lie algebras, we have the inclusion of the exponential group into the completion of the enveloping algebra, namely

$$\exp(\mathfrak{mt}) \subset \hat{U}\mathfrak{mt} \simeq \hat{\mathbb{Q}(C)},$$

where the right-hand ring is included (as vector spaces) in the power series ring on $x$ and $y$.

The group $\exp(\mathfrak{mt})$ consists of the power series in $x, y$ that have constant term 1 and no linear term in $x$, and are group-like, i.e. such that

$$\Delta(f) = f \otimes f.$$  \hspace{1cm} (3.5.7)

The expression for product of two elements of the subgroup $\exp(\mathfrak{mt})$ is the twisted Magnus multiplication law

$$f(x, y) \otimes g(x, y) = f(x, gyg^{-1})g(x, y).$$  \hspace{1cm} (3.5.8)

This multiplication corresponds to identifying $f \in \exp(\mathfrak{mt})$ with the endomorphism $R_f$ of $\hat{\mathbb{Q}(C)}$ given by $x \mapsto x, y \mapsto fyf^{-1}$. The twisted Magnus multiplication then simply corresponds to anticomposition of endomorphisms; indeed, we have

$$R_g \circ R_f (y) = R_g(fyf^{-1}) = f(x, gyg^{-1})gyg^{-1}f(x, gyg^{-1}),$$

so

$$R_g \circ R_f = R_{f(x, gyg^{-1})}g = R_{f \circ g}.$$  \hspace{1cm} (3.5.9)

We have

$$gari(ma_g) \cdot maf = ma_{R_f(f)}$$ \hspace{1cm} (3.5.10)

and

$$gari(maf, ma_g) = f \circ g.$$ \hspace{1cm} (3.5.11)

The group $MT$ is the set of all group-like power series in $\hat{\mathbb{Q}(C)}$ with constant term 1, equipped with the twisted Magnus multiplication $\circ$ given in (3.5.8). Let $\mathbb{Q}(C)_1$ denote the set of all power series in $\hat{\mathbb{Q}(C)}$ with constant term 1, equipped with the multiplication $\circ$ of (3.5.8). Then (3.5.11) shows that $ma$ gives rise to an isomorphism

$$ma : \mathbb{Q}(C)_1 \simeq \text{GARI}^{pol}.$$  \hspace{1cm} (3.5.12)

Restricting this isomorphism to the subgroup of group-like power series $MT = \exp(\mathfrak{mt})$ yields an isomorphism

$$ma : MT \simeq \text{GARI}_{as}^{pol},$$  \hspace{1cm} (3.5.13)
where GARI\textsubscript{as} is the group of symmetrical moulds, i.e. moulds \(A\) satisfying
\[
\sum_{w \in sh(u, v)} A(w) = A(u)A(v). \tag{3.5.14}
\]

With this background situation established, let us now explain one of the identities from §2.7 in the power series situation. We consider the equality of automorphisms (2.7.6).

For \(f, g, g' \in \hat{\mathbb{Q}}\langle C \rangle\), we define endomorphisms \(X_{(g, g')}, R_{f}\) and \(N_{f}\) of \(\hat{\mathbb{Q}}\langle C \rangle\) as follows: each one sends \(x \mapsto x\), and
\[
\begin{align*}
X_{(g, g')}(y) &= ggy' \\
R_{f}(y) &= fyf^{-1} \\
N_{f}(y) &= yf,
\end{align*}
\]
i.e. \(R_{f} = X_{(f, f^{-1})}\) and \(N_{f} = X_{(1, f)}\). We have
\[
\begin{align*}
\max_{X_{(g, g')}(f)} &= gaxit(ma_{g}, ma_{g'}) \cdot ma_{f} \\
\max_{R_{f}(f)} &= garit(ma_{g}) \cdot ma_{f} \\
\max_{N_{f}(f)} &= ganit(ma_{g}) \cdot ma_{f},
\end{align*}
\tag{3.5.15}
\]
where the second equality is (3.5.10) above and the others are analogous. Just as \(X_{(g, g')}, R_{g}\) and \(N_{g}\) are automorphisms of the group (under the usual multiplication) of power series with constant term 1, so \(gaxit, garit\) and \(ganit\) are automorphisms of \(\text{GARI}^{\text{pol}}\) equipped with the multiplication \(mu\).

We have
\[
X_{g} \circ X_{f} = X_{X_{(g, g')}(f)}, \tag{3.5.16}
\]
so if \(f\) is such that \(X_{(g, g')}(f)g = 1\), then \(ma_{f} = \text{invgaxi}(ma_{g})\). Thus, the translation of the equality (2.7.6) back to the twisted Magnus situation is given by
\[
X_{(g, g')} \circ R_{f} = N_{g'g}, \tag{3.5.17}
\]
where \(g^{-1} = X_{(g, g')}(f)\), i.e. \(ma_{f} = \text{invgaxi}(ma_{g})\). But it is easy to prove (3.5.17). Indeed, the automorphisms on both sides fix \(x\), so we only need to compare their images on \(y\). The RHS yields \(N_{g'g}(y) = yg'g\), and the LHS yields
\[
\begin{align*}
X_{(g, g')}R_{f}(y) &= X_{(g, g')}(fyf^{-1}) \\
&= X_{(g, g')}(f)gyg'X_{(g, g')}(f^{-1}) \\
&= yg'X_{(g, g')}(f^{-1}) \\
&= yg'g,
\end{align*}
\]
which proves that they are equal.

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Chapter 4

The mould pair \( pal / pil \) and its properties

§4.1. Diffeomorphisms and the mould \( pil \)

The passage from the space \( DIFF(x) \) of diffeomorphisms \( f(x) = x(1 + \sum_{r \geq 1} a_r x^r) \) to \( GARI \) is one of Ecalle’s key discoveries. Given \( f(x) \), he defines an associated mould \( p_f \) in \( GARI \), in fact giving two equivalent definitions for \( p_f \). These stem from two functions associated to \( f(x) \), namely the infinitesimal dilator \( f_\#(x) \), defined by

\[
f_\#(x) = x - \frac{f(x)}{f'(x)} = \sum_{r \geq 1} \gamma_r x^{r+1}, \tag{4.1.1}
\]

and the infinitesimal generator \( f_*(x) \) defined by

\[
f_*(x) = \sum_{r \geq 1} \epsilon_r x^{r+1} \tag{4.1.2}
\]

where the coefficients \( \epsilon_r \) are determined by the identity

\[
\exp(f_*(x)) \frac{d}{dx} \cdot x = f(x).
\]

Let \( re_1 = \frac{1}{v_1} \), and for \( r > 1 \) define the mould \( re_r \) recursively by \( re_r = arit(re_{r-1}) \cdot re_1 \). The mould \( re_r \) is concentrated in depth \( r \), and it is easy to show by induction that it has explicit expression

\[
re_r(v_1, \ldots, v_r) = \frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)v_r}. \tag{4.1.3}
\]

Let \( lop_f \) denote the mould in \( ARI \) defined by

\[
lop_f(v_1, \ldots, v_r) = \epsilon_r re_r(v_1, \ldots, v_r) = \epsilon_r \frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)v_r} \quad \text{for} \quad r \geq 1. \tag{4.1.4}
\]

The first definition of the mould \( p_f \) associated to \( f(x) \) comes from the infinitesimal generator of \( f(x) \) and is given by

\[
p_f = expari(lop_f). \tag{4.1.5}
\]

By construction, the moulds \( p_f \) associated to \( f \) satisfy

\[
p_{f \circ g} = gari(p_f, p_g). \tag{4.1.6}
\]
The second definition comes from the infinitesimal dilator, via the mould \( df \in \mathcal{ARI} \) defined by

\[
d_f(v_1, \ldots, v_r) = \gamma_r r v(v_1, \ldots, v_r) = \gamma_r \frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)v_r} \quad \text{for } r \geq 1; \quad (4.1.7)
\]

we define the mould \( p_f \) recursively by setting \( p_f(\emptyset) = 1 \) and

\[
der \cdot p_f = \text{preari}(p_f, df), \quad (4.1.8)
\]

where \( der \) is the operator on moulds such that

\[
(der \cdot A)(w_1, \ldots, w_r) = r A(w_1, \ldots, w_r).
\]

Indeed, note that since \( df(\emptyset) = 0 \), the depth \( r \) term of \( p_f \) can be deduced from the parts of \( p_f \) up to depth \( r - 1 \) via the right-hand side of (4.1.8).

**Proposition 4.1.1.** The two definitions of \( p_f \) are equivalent.

**Proof.** The main fact is that if we apply the linearization procedure, working in \( k[[\epsilon]]/\langle \epsilon^2 \rangle \), then the linearized dilator \( 1 + \epsilon f_\#(x) \) satisfies the identity

\[
(f \circ (1 + \epsilon f_\#))(x) = f(x) + \epsilon \sum_{n \geq 1} na_n x^{n+1}.
\]

Passing to the associated moulds by (4.1.6), using (2.7.4), the left-hand side maps to

\[
gari(p_f, p_1 + \epsilon f_\#) = p_f + \epsilon \text{preari}(p_f, df).
\]

We also see that the sum \( \sum_{n \geq 1} na_n x^{n+1} \) maps to \( der \cdot p_f \) since each term is multiplied by its degree, so the right-hand side altogether maps to

\[
p_f + \epsilon \text{der} \cdot p_f.
\]

This shows that \( p_f \) satisfies (4.1.8). \( \diamond \)

**Proposition 4.1.2.** The moulds \( p_f \) are symmetral.

**Proof.** By Proposition 2.6.1, since \( p_f = \text{expari}(\text{lopar}) \), it is enough to show that \( \text{lopar} \) is alternal. But \( re_1 \) is trivially alternal since it is concentrated in depth 1. Assuming as an induction hypothesis that \( re_{r-1} \) is alternal, we see by Proposition 2.5.2 that \( re_r = \text{arit}(re_{r-1}) \cdot re_1 \) is also alternal, which proves that \( \text{lopar} \) is alternal. \( \diamond \)

**Definition.** Let \( pil \) be the mould \( p_f \) constructed as above, where \( f(x) = 1 - e^{-x} \), and let \( dipil \) denote the mould \( df \) for this \( f \). In low depths, we have

\[
pil(v_1) = -\frac{1}{2v_1},
\]

\[
pil(v_1, v_2) = \frac{1}{12} \frac{2v_1 - v_2}{v_1 (v_1 - v_2)v_2},
\]

\[
pil(v_1, v_2, v_3) = -\frac{1}{27} \frac{(v_1 - v_2)v_2v_3}{v_1 v_2 v_3(v_1 - v_2)(v_2 - v_3)v_3 v_4},
\]

\[
pil(v_1, v_2, v_3, v_4) = \frac{1}{720} \frac{6v_1 v_2 v_3 - 10v_1 v_4 + v_2 v_3 + 5v_2 v_4 - 4v_3^2 + v_3 v_4}{v_1 v_2 v_3 (v_1 - v_2)(v_2 - v_3)(v_3 - v_4)}.
\]
Remarks. Ecalle gives some very pretty results on moulds associated to diffeomorphisms that we cite here without proof.

1. A mould \( A \in \text{GARI} \) lies in the image of \( \text{DIFF}(x) \) if and only if there exist constants \( c_r, r \geq 1 \) such that

\[
\mu(\text{anti} \cdot \text{swap}(A), \text{swap}(A)) = c_r \frac{1}{u_1 \cdots u_r},
\]

and if this is the case, then \( A = pf \) where \( f(x) = x + \sum_{r \geq 1} \frac{c_r}{r+1} x^{r+1} \).

2. If a mould \( A \in \text{GARI} \) is symmetrical, then \( \mu(\text{anti} \cdot A, A) \) is also symmetrical. Therefore, setting \( A = \text{swap}(pf) \), it is a necessary condition for the bisymmetry of \( pf \) that \( \mu(\text{anti} \cdot A, A) \) be symmetrical, i.e. that the mould defined by the right-hand side of (4.1.9) be symmetrical. One can show directly that the only mould of this form which is symmetrical is the one where \( c_r = (-1)^r/r! \), i.e. \( \mu(\text{anti} \cdot A, A) = \exp \mu(O) \) where \( O \) is the mould concentrated in depth 1 defined by \( O(u_1) = 1/u_1 \). Thus, since we can get the diffeomorphism \( f \) back from the \( c_r \) by setting \( a_r = c_r/(r+1) = (-1)^r/(r+1)! \), we find that the only diffeomorphism \( f \) for which \( pf \) could be bisymmetrical is

\[
f(x) = x + \sum_{r \geq 1} \frac{(-1)^r}{(r+1)!} x^{r+1} = 1 - e^{-x}.
\]

The next two sections will be devoted to giving Ecalle’s direct proof, not relying on this property, that \( pil \) is indeed bisymmetrical.

§4.2. Two definitions of the mould \( pal \)

The mould pair \( pal/pil \) is undoubtedly one of Ecalle’s most beautiful and powerful discoveries. In this chapter we give the most recent definition that Ecalle has given for the mould \( pal \) (cf. [Eupolars]), and then give the complete proof that \( pal = \text{swap}(pil) \).

Let \( dipil \in \text{ARI} \) be the mould \( d_f \) of the previous section, with \( f(x) = 1 - e^{-x} \).

Explicitly,

\[
dipil(v_1, \ldots, v_r) = \frac{-1}{(r+1)!} \text{re}_r(v_1, \ldots, v_r) = \frac{-1}{(r+1)!} \frac{v_1 + \ldots + v_r}{v_1(v_1-v_2) \cdots (v_{r-1} - v_r)v_r},
\]

and by (4.1.8), we have

\[
der \cdot pil = \text{preari}(pil, dipil).
\]

Definition. The definition of \( pal \) is an interesting analog of (4.2.2), but based on an auxiliary mould in \( \text{ARI} \) rather than \( \text{ARI} \) and constructing \( pal \) recursively using \( \mu \) where \( pil \) used \( \text{preari} \).
Let \( \text{dupal} \in \text{ARI} \) be the mould defined explicitly as follows: \( \text{dupal}(\emptyset) = 0 \) and for each \( r \geq 1 \),

\[
\text{dupal}(u_1, \ldots, u_r) = \frac{B_r}{r!} \sum_{i=0}^{r-1} (-1)^i \frac{C_{r-1}^i}{u_1 \cdots u_{i-1} \cdots u_r} \frac{1}{u_1 \cdots u_r} \\
= \frac{B_r}{r!} \frac{1}{u_1 \cdots u_r} \left( \sum_{i=0}^{r-1} (-1)^i C_{r-1}^i u_{i+1} \right). 
\] (4.2.4)

Note in particular that \( \text{dupal}(u_1, \ldots, u_r) = 0 \) for all odd \( r > 1 \). The mould \( \text{pal} \in \text{GARI} \) is defined by \( \text{pal}(\emptyset) = 1 \) and then, recursively depth by depth, by the formula

\[
\text{dur} \cdot \text{pal} = \text{mu}(\text{pal}, \text{dupal}), 
\] (4.2.5)

where \( \text{dur} \) is the operator on (bi)moulds defined by

\[
\text{dur} \cdot A(w_1, \ldots, w_r) = (w_1 + \cdots + w_r)A(w_1, \ldots, w_r). 
\] (4.2.6)

Up to depth 4, we have

\[
\begin{align*}
\text{pal}(u_1) &= -\frac{1}{2u_1} \\
\text{pal}(u_1, u_2) &= \frac{1}{12} \frac{u_1 + 2u_2}{u_1 u_2} \\
\text{pal}(u_1, u_2, u_3) &= -\frac{1}{24} \frac{1}{u_1(u_1 + u_2)}u_3 \\
\text{pal}(u_1, u_2, u_3, u_4) &= -\frac{1}{720} \frac{u_4^2 - 2u_1 u_2 - 2u_1 u_3 - 4u_1 u_4 - 3u_2 u_3 - 7u_2 u_4 + 6u_2 u_4}{u_1 u_2 u_3 u_4 (u_1 + u_2)(u_1 + u_2 + u_3 + u_4)},
\end{align*}
\]

**Theorem 4.2.1.** We have \( \text{pal} = \text{swap} (\text{pil}) \).

**Proof.** We need two lemmas.

**Lemma 4.2.2.** The derivations \( \text{dur} \) and \( \text{der} \) commute, and for any mould \( B \in \text{ARI} \), \( \text{dur} \) commutes with \( \text{amit}(B), \text{anit}(B), \text{arit}(B) \) and \( \text{irat}(B) \).

**Proof.** The commutation of \( \text{der} \) and \( \text{dur} \) is obvious since \( \text{der} \cdot \text{dur} \) and \( \text{dur} \cdot \text{der} \) both come down to multiplying the mould \( A \) by \( r(u_1 + \cdots + u_r) \) in depth \( r \). The commutation of \( \text{dur} \) with \( \text{arit}(B) \) and \( \text{irat}(B) \) follow immediately from the commutation with \( \text{amit}(B) \) and \( \text{anit}(B) \) since \( \text{arit}(B) = \text{amit}(B) - \text{anit}(B) \) by (2.2.4) and \( \text{irat}(B) = \text{amit}(B) - \text{anit}(\text{push}(B)) \) by (2.4.11). Looking at the definition of \( \text{amit}(B) \) in (2.2.1), we see that

\[
\text{amit}(B) \cdot \text{dur} \cdot A(w) = \sum_{w=abc} \sum_{b, c \neq \emptyset} (\text{dur} \cdot A)(a[c])B(b). 
\]

But if \( a = (u_1, \ldots, u_i), b = (u_{i+1}, \ldots, u_{i+k}) \) and \( c = (u_{i+k+1}, \ldots, u_r) \), we have

\[
a[c] = (u_1, \ldots, u_i, u_{i+1} + \cdots + u_{i+k+1}, u_{i+k+2}, \ldots, u_r), 
\] (4.2.7)

we see that \( (\text{dur} \cdot A)(a[c]) = (u_1 + \cdots + u_r)A(u_1, \ldots, u_r) \), so the same factor \( (u_1 + \cdots + u_r) \) occurs in every term of the sum over \( w = abc \) and therefore can be taken outside the sum,
leaving exactly \( \text{dur} \cdot \text{amit}(B) \cdot A \). The exact same argument holds for \( \text{amit}(B) \) (defined in (2.2.2)), with \( a \mid c \) instead of \( a \lceil c \). This concludes the proof. \( \diamond \)

**Proposition 4.2.3.** Set \( \text{dapal} = \text{swap}(\text{dipil}) \). Then

\[
\text{der} \cdot \text{dupal} = \text{dur} \cdot \text{dapal} + \text{irat}(\text{dapal}) \cdot \text{dupal} - \text{lu}(\text{dapal}, \text{dupal}).
\] (4.2.8)

The detailed proof of this identity is given in the Appendix, §A.6.

We can now complete the proof of Theorem 4.2.1. We first apply the \( \text{swap} \) to (4.2.2), obtaining

\[
\text{der} \cdot \text{swap}(\text{pil}) = \text{swap}(\text{preari}(\text{pil}, \text{dipil}))
= \text{swap}(\text{preari}(\text{swap}(\text{swap}(\text{pil})), \text{dapal}))
= \text{preira}(\text{swap}(\text{pil}), \text{dapal}).
\] (4.2.9)

Given that \( \text{swap}(\text{pil})(\emptyset) = 1 \), (4.2.9) can actually be used as a recursive depth-by-depth definition for \( \text{swap}(\text{pil}) \); i.e. we have two equivalent ways to compute \( \text{swap}(\text{pil}) \), either by swapping the terms of \( \text{pil} \) or by (4.2.9). Therefore, if \( \text{pal} \) is the mould defined in (4.2.5), to show that \( \text{pal} = \text{swap}(\text{pil}) \), it suffices to prove that \( \text{pal} \) satisfies (4.2.9), i.e. that

\[
\text{der} \cdot \text{pal} = \text{preira}(\text{pal}, \text{dupal}).
\] (4.2.10)

Set

\[
A = \text{der} \cdot \text{pal} - \text{preira}(\text{pal}, \text{dupal})
= \text{der} \cdot \text{pal} - \text{irat}(\text{dapal}) \cdot \text{pal} - \text{mu}(\text{pal}, \text{dapal}).
\] (4.2.11)

We apply \( \text{der} \) to the left hand side of (4.2.5). Using the fact that \( \text{irat}(\text{dapal}) \) is a \( \text{mu} \)-derivation, we have

\[
\text{der} \cdot \text{dur} \cdot \text{pal} = \text{der} \cdot \text{mu}(\text{pal}, \text{dupal}) \quad \text{by (4.2.5)}
= \text{mu}(\text{der} \cdot \text{pal}, \text{dupal}) + \text{mu}(\text{pal}, \text{der} \cdot \text{dupal})
= \text{mu}(\text{der} \cdot \text{pal}, \text{dupal}) + \text{mu}(\text{pal}, \text{irat}(\text{dapal}) \cdot \text{dupal})
+ \text{mu}(\text{pal}, \text{dur} \cdot \text{dupal}) - \text{mu}(\text{pal}, \text{dapal}, \text{dupal}) + \text{mu}(\text{pal}, \text{dupal}, \text{dapal}) \quad \text{by (4.2.8)}
= \text{mu}(\text{der} \cdot \text{pal}, \text{dupal}) + \text{irat}(\text{dapal}) \cdot \text{mu}(\text{pal}, \text{dupal}) - \text{mu}(\text{irat}(\text{dapal}) \cdot \text{pal}, \text{dupal})
+ \text{mu}(\text{pal}, \text{dur} \cdot \text{dupal}) - \text{mu}(\text{pal}, \text{dapal}, \text{dupal}) + \text{mu}(\text{pal}, \text{dupal}, \text{dapal})
= \text{mu}(\text{der} \cdot \text{pal}, \text{dupal}) - \text{mu}(\text{irat}(\text{dapal}) \cdot \text{pal}, \text{dupal}) - \text{mu}(\text{pal}, \text{dapal}, \text{dupal})
+ \text{irat}(\text{dapal}) \cdot \text{mu}(\text{pal}, \text{dupal}) + \text{mu}(\text{pal}, \text{dur} \cdot \text{dupal}) + \text{mu}(\text{pal}, \text{dupal}, \text{dupal})
= \text{mu}(A, \text{dupal}) + \text{irat}(\text{dapal}) \cdot \text{mu}(\text{pal}, \text{dupal}) + \text{mu}(\text{pal}, \text{dur} \cdot \text{dupal}) + \text{mu}(\text{pal}, \text{dupal}, \text{dupal})
= \text{mu}(A, \text{dupal}) + \text{irat}(\text{dapal}) \cdot \text{dur} \cdot \text{pal} + \text{mu}(\text{pal}, \text{dur} \cdot \text{dupal}) + \text{mu}(\text{pal}, \text{dupal}, \text{dupal})
= \text{mu}(A, \text{dupal}) + \text{irat}(\text{dapal}) \cdot \text{dur} \cdot \text{pal} + \text{mu}(\text{pal}, \text{dupal}, \text{dupal})
= \text{mu}(A, \text{dupal}) + \text{dur} \cdot \text{irat}(\text{dapal}) \cdot \text{pal} + \text{dur} \cdot \text{mu}(\text{pal}, \text{dupal}) \quad \text{by Lemma 4.2.2.}
\]
By Lemma 4.2.2, we also have \( \text{der} \cdot \text{dur} \cdot \text{pal} = \text{dur} \cdot \text{der} \cdot \text{pal} \), and the equality of \( \text{der} \cdot \text{dur} \cdot \text{pal} \) 
The last line above can thus be rewritten as

\[
\text{dur} \cdot \text{der} \cdot \text{pal} - \text{dur} \cdot \text{irat}(\text{dapal}) \cdot \text{pal} - \text{dur} \cdot \text{mu}(\text{pal}, \text{dapal}) = \text{mu}(A, \text{dupal}),
\]

i.e.

\[
\text{dur} \cdot A = \text{mu}(A, \text{dupal}). \tag{4.2.12}
\]

Now, although this looks like the defining equation (4.2.5) for \( \text{pal} \), in fact the defining equation (4.2.11) for \( A \) shows that \( A(\emptyset) = 0 \). But it is easy to show that if a mould \( A \) satisfies \( A(\emptyset) = 0 \) and (4.2.12), then \( A \) is identically 0. Indeed, suppose by induction that \( A(u_1, \ldots, u_i) = 0 \) for \( 0 \leq i < r \). Then

\[
(u_1 + \cdots + u_r)A(u_1, \ldots, u_r) = \sum_{i=0}^{r} A(u_1, \ldots, u_i)\text{dupal}(u_{i+1}, \ldots, u_r)
\]

\[
= A(u_1, \ldots, u_r)\text{dupal}(\emptyset)
\]

\[
= 0,
\]

so \( A(u_1, \ldots, u_r) = 0 \). Thus the expression (4.2.11) is equal to 0, proving the desired identity (4.2.10). This concludes the proof of Theorem 4.2.1. \( \Box \)

\section*{4.3. Symmetrality of \( \text{pal} \)}

Let

\[
P aj(r_1, \ldots, r_s) = \frac{1}{r_1(r_1 + r_2) \cdots (r_1 + \cdots + r_s)}.
\]

**Lemma 4.3.1.** The mould \( P aj \) is symmetral.

**Proof.** Let us proceed by induction on the length of the shuffles \( sh(u, v) \), i.e. the total length of the two words \( u \) and \( v \). When \( u \) and \( v \) are both of length 1, i.e. \( u = (r_1) \), \( u = (r_2) \), we have

\[
\sum_{w \in sh(u, v)} P aj(w) = P aj(r_1, r_2) + P aj(r_2, r_1) = \frac{1}{r_1(r_1 + r_2)} + \frac{1}{r_2(r_1 + r_2)} = \frac{1}{r_1r_2},
\]

so \( P aj \) is symmetral in length 2. Assume it is symmetral up to length \( s - 1 \), and let \( u = (r_1, \ldots, r_l) \), \( v = (r_{l+1}, \ldots, r_s) \) be two words of total length \( s \). We use the recursive definition

\[
sh(u, v) = sh(u', v) \cdot r_l + sh(u, v') \cdot r_s,
\]

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where \( u' = (r_1, \ldots, r_{l-1}) \) and \( v' = (r_{l+1}, \ldots, r_{s-1}) \). Letting \( R = \sum_{i=1}^{s} r_i \), we have

\[
\sum_{w \in sh(u, v)} Paj(w) = \sum_{w \in sh(u', v)} Paj(w, r_l) + \sum_{x \in sh(u, v')} Paj(x, r_s)
\]

\[
= \sum_{w \in sh(u', v)} Paj(w_1, \ldots, w_{s-1}, r_l) + \sum_{x \in sh(u, v')} Paj(x_1, \ldots, x_{s-1}, r_s)
\]

\[
= \sum_{w \in sh(u', v)} \frac{1}{w_1(w_1 + w_2) \cdots (w_1 + \cdots + w_{s-1}) R}
\]

\[
+ \sum_{x \in sh(u, v')} \frac{1}{x_1(x_1 + x_2) \cdots (x_1 + \cdots + x_{s-1}) R}
\]

\[
= \frac{1}{R} \sum_{w \in sh(u', v)} \frac{1}{w_1(w_1 + w_2) \cdots (w_1 + \cdots + w_{s-1})}
\]

\[
+ \frac{1}{R} \sum_{x \in sh(u, v')} \frac{1}{x_1(x_1 + x_2) \cdots (x_1 + \cdots + x_{s-1})}
\]

\[
= \frac{1}{R} \sum_{w \in sh(u', v)} Paj(w) + \frac{1}{R} \sum_{x \in sh(u, v')} Paj(x)
\]

\[
= \frac{1}{R} Paj(u') Paj(v) + \frac{1}{R} Paj(u) Paj(v') \quad \text{by the induction hypothesis}
\]

\[
= \frac{1}{R} \frac{1}{r_1(r_1 + r_2) \cdots (r_1 + \cdots + r_{l-1})} \frac{1}{r_{l+1}(r_{l+1} + r_{l+2}) \cdots (r_{l+1} + \cdots + r_s)}
\]

\[
+ \frac{1}{R} \frac{1}{r_1(r_1 + r_2) \cdots (r_1 + \cdots + r_l)} \frac{1}{r_{l+1}(r_{l+1} + r_{l+2}) \cdots (r_{l+1} + \cdots + r_{s-1})}
\]

\[
= \left( \frac{1}{R} \right) \left( (r_1 + \cdots + r_l) + (r_{l+1} + \cdots + r_s) \right) Paj(u) Paj(v)
\]

\[
= Paj(u) Paj(v).
\]

This proves that \( Paj \) is symmetric.

\[\square\]

**Lemma 4.3.2.** Let \( S \) be a mould such that \( S(\emptyset) = 1 \). Then the defining formula

\[
dur \cdot S = mu(S, duS)
\]

(4.3.1)

is equivalent to the inversion formula

\[
S(u) = 1 + \sum_{u_1, u_2, \ldots, u_s \neq u} Paj(|u_1|, \ldots, |u_s|) duS(u_1) \cdots duS(u_s),
\]

(4.3.2)

where if \( u = (r_1, \ldots, r_l) \) then \( |u| = r_1 + \cdots + r_l \).
Proof. We prove the equivalence of (4.3.1) and (4.3.2) by induction on the length of \( u \). When \( u = \emptyset \), the constant term 1 on the right-hand side of (4.3.2) ensures equality. For \( u = (u_1) \), we have

\[
S(u_1) = Paj(u_1)duS(u_1) = \frac{1}{u_1}duS(u_1)
\]

from (4.3.2), and from (4.3.1) we have

\[
u_1S(u_1) = S(\emptyset)duS(u_1) = duS(u_1)
\]

so they are equivalent. This settles the base case. Now assume the induction hypothesis that (4.3.1) and (4.3.2) give the same formula for \( S(u_1, \ldots, u_i) \) for \( i < r \). From (4.3.1), and using the induction hypothesis on each term in \( S \), we have

\[
(u_1 + \cdots + u_r)S(u_1, \ldots, u_r) = \sum_{i=0}^{r-1} S(u_1, \ldots, u_i)duS(u_{i+1}, \ldots, u_r)
\]

so writing \( u_s+1 = (u_{i+1}, \ldots, u_r) \) in each term and dividing both sides by \( R = (u_1 + \cdots + u_r) \), we find

\[
S(u_1, \ldots, u_r) = \sum_{1 \leq |u_{s+1}| \leq r} \sum_{u = u_1 \cdots u_{s+1}} \frac{1}{R} Paj(|u_1|, \ldots, |u_{s}|)duS(u_1) \cdots duS(u_{s})duS(u_{s+1})
\]

\[
= \sum_{u = u_1 \cdots u_{s+1}} Paj(|u_1|, \ldots, |u_{s+1}|)duS(u_1) \cdots duS(u_{s+1}).
\]

This proves that (4.3.1) is equivalent to (4.3.2).

\[\Box\]

Proposition 4.3.3. Let \( S \) and \( duS \) be two moulds related as in (4.3.1). If \( duS \) is alternal, then \( S \) is symmetric.

Proof. We will use the equivalent formula (4.3.2) for \( S \). Indeed, by formula (2.1.1) for mould composition, we see that (4.3.2) is equivalent to the statement that the definition \( dur \cdot S = mu(S, duS) \) is equivalent to \( S = 1 + Paj \circ duS \). Assume that \( duS \) is alternal. From Lemma 2.6.2, we know that for any alternal mould \( A \) and symmetrical mould \( B \), the composition \( B \circ A \) is symmetrical, and from Lemma 4.3.1 we know that \( Paj \) is symmetrical. This concludes the proof.

\[\Box\]

Theorem 4.3.4. The mould \( pal \) is symmetrical.

Proof. Thanks to Proposition 4.3.3, it is enough to show that \( dupal \) is alternal. But this reduces in fact to an easy exercise, namely showing that the only linear alternal moulds \( a_1u_1 + \cdots + a_r u_r \) are, up to scalar multiple, the binomial moulds \( \sum_{i=1}^{r} (-1)^i C_{r-1}^{i-1} u_i \).
§4.4. The identity $crash(pal) = pac$

Let $pac$ be the mould defined by $pac(u_1, \ldots, u_r) = \frac{1}{u_1 \cdots u_r}$. In this section we show two identities (4.4.1) and (4.4.6) that are essential to the proof of the second fundamental identity (4.5.2) stated and proved below in §4.5.

Lemma 4.4.1. We have

$$ crash(pal) := mu(push \cdot swap \cdot invmu \cdot invpil, swap \cdot invpil) = pac. \quad (4.4.1) $$

Proof. Since $pil$ is symmetral, we have

$$ mu(pari \cdot anti(pil), pil)) = 1, \quad (4.4.2) $$

and it’s easy to see by the homogeneous degrees of $pil$ that

$$ anti \cdot neg(pil) = pari \cdot anti(pil), \quad (4.4.3) $$

so we find that

$$ anti \cdot neg(pil) = invmu(pil). \quad (4.4.4) $$

Now, because of (4.4.4), we find that $pil \in GARI \cap GAWI$ (see [E,p. 44] for definition of $GAWI$), and thus the $gari$ and $gawi$ inverses are the same, so it makes sense to write $invpil \in GARI \cap GAWI$. This means that for $pil$ and $invpil$ we have

$$ \begin{cases} push \cdot swap \cdot invmu \cdot swap \cdot swap(pil) = anti \cdot swap(pil) \\ push \cdot swap \cdot invmu \cdot swap \cdot swap(invpil) = anti \cdot swap(invpil). \end{cases} \quad (4.4.5) $$

Thus the LHS of (4.4.1) is equal to

$$ crash(pal) = mu(anti \cdot swap(invpil), swap(invpil)), $$

which is nothing other than $gepar(invpil)$, so we can use §4.1.3 for $f(x) = -log(1 - x)$ which shows that

$$ gepar(invpil) = pic, $$

proving (4.4.1).

Lemma 4.4.2. We have

$$ ganit_{pic} \cdot invpil = swap \cdot invpil. \quad (4.4.6) $$

Proof. From (2.8.17) applied to $A = 1, B = pal$, we have

$$ swap \cdot invgari \cdot swap \cdot pal = swap \cdot invpil = ganit_{crash, pal}(invpal). \quad (4.4.7) $$
Using (2.8.12), from (4.4.1) we also know that
\[ ganit_{pac} \cdot invpal = swap \cdot invpil. \]

We need to use the elementary result
\[ invgani(pac) = pari \cdot anti \cdot paj, \]

where
\[ paj(u_1, \ldots, u_r) = \frac{1}{(u_1 + u_2)(u_1 + u_2 + u_3) \cdots (u_1 + \cdots + u_r)}. \]

This gives
\[ invpal = ganit_{pari-anti-paj} \cdot swap \cdot invpil, \]

so
\[ swap \cdot invpal = swap \cdot ganit_{pari-anti-paj} \cdot swap \cdot invpil. \]

It remains only to prove that the following two automorphisms of GARI are equal:
\[ ganit_{pic} = swap \cdot ganit_{pari-anti-paj} \cdot swap. \]

Now, every mould \( C \) in the \( v_i \) such that \( C(v_1, \ldots, v_r) \) is actually a rational function \( B \) of the variables \( v_2 - v_1, \ldots, v_r - v_1 \) satisfies the identity \( C = ganit_B(Y) \), by the calculation
\[
\begin{align*}
\text{ganit}_B(Y)(v_1, \ldots, v_r) &= \sum_{b_1c_1 \cdots b_sc_s} Y(b_1 \cdots b_s)B([c_1]) \cdots B([c_2]) \\
&= \sum_{b_1=(v_1), c_1=(v_2, \ldots, v_r)} Y(v_1)B(v_2-v_1, \ldots, v_r-v_1) \\
&= B(v_2-v_1, \ldots, v_r-v_1) \\
&= C(v_1, \ldots, v_r).
\end{align*}
\]

Let us write \( swap(Y) = Y \) a little abusively, since although the values in depths 0 and 1 are still 1, \( swap(Y) \) is considered a mould in the \( u_i \). We start to compute the right-hand side of (4.4.9) explicitly as
\[
\text{ganit}_{pari-anti-paj} \cdot Y(u_1, \ldots, u_r) = \frac{(-1)^{r-1}}{u_r(u_r-1+u_r) \cdots (u_2 + \cdots u_r)}
\]
(with \( ganit_{pari-anti-paj} \cdot Y(\emptyset) = 1, ganit_{pari-anti-paj} \cdot Y(u_1) = 1 \). Swapping this, we obtain for the RHS of (4.4.9):
\[
\text{swap} \cdot \text{ganit}_{pari-anti-paj} \cdot Y(u_1, \ldots, u_r) = \frac{1}{(v_2-v_1)(v_3-v_1) \cdots (v_r-v_1)}.
\]

Letting
\[ C(v_1, \ldots, v_r) = \frac{1}{(v_2-v_1)(v_3-v_1) \cdots (v_r-v_1)}, \]

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we see by (4.4.10) that \( C = ganit_B(Y) \) where

\[
B(v_1, \ldots, v_r) = \frac{1}{v_1 \cdots v_r},
\]

(4.4.11)
i.e. \( B = pic. \)

Note that we have not shown that \( crash(pil) = pic \), although it seems to be true. However, the above result is enough for our purposes, together with the important result stated by Écalle concerning the automorphism \( ganit_{pic} \) given in the corollary to Theorem 4.4.3 below.

**Proposition 4.4.3** Let \( A, B \in \overline{ARI} \) be such that \( A = ganit_{pic} \cdot B \). Then \( A \) satisfies the stuffle relations if and only if \( B \) satisfies the shuffle relations, i.e. \( A_{r,s}(v_1, \ldots, v_r) = 0 \) for all pairs \((r, s)\), where \( A_{r,s} \) defined as in (2.3.3), if and only if \( B \) is alternal.

**Proof.** The full and complex proof of this fundamental statement has been worked out by N. Komiyama in [K], Theorem 3.24.

**Corollary.** Let \( A = ganit_{pic} B \). Then \( A \) satisfies the stuffle relations, i.e. \( A_{r,s}(v_1, \ldots, v_r) = 0 \) for all \((r, s)\), if and only if \( B \) satisfies the shuffle relations.

§4.5. Écalle’s second fundamental identity

In this section we use Écalle’s first fundamental identity (2.8.4) and the results of §4.4 to prove another formula that is one of the main tools in his theory, namely the second fundamental identity, given in Theorem 4.5.2. It will be deduced from an initial version given in the following proposition.

**Proposition 4.5.1.** We have

\[
swap \cdot fragari(swapp \cdot A, pal) = ganit_{pic} \cdot fragari(A, pil).
\]

(4.5.1)

**Proof.** Applying the fundamental identity (2.8.4) to \( A = swap \cdot M \) and \( B = pal \) and using Lemma 4.4.1 yields

\[
swap \cdot fragari(M, swap \cdot pal) = ganit_{crash,pal} \cdot fragari(swapp \cdot M, pal) \\
= ganit_{pac} \cdot fragari(swapp \cdot M, pal).
\]

Thus by (4.4.8) we have

\[
ganit_{inv ganipac} \cdot swap \cdot fragari(M, pil) = ganit_{pari-anti-paj} \cdot swap \cdot fragari(M, pil) \\
= fragari(swapp \cdot M, pal).
\]

Applying swap to both sides and (4.4.9), we have

\[
swap \cdot ganit_{pari-anti-paj} \cdot swap \cdot fragari(M, pil) = ganit_{pic} \cdot fragari(M, pil) \\
= swap \cdot fragari(swapp \cdot M, pal),
\]

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which proves the desired (4.5.1).

\[ \text{Theorem 4.5.2. For every push-invariant mould } M, \text{ we have Ecalle’s second fundamental identity:} \]
\[ \text{swap} \cdot \text{ad}_{ari}(\text{pal}) \cdot M = \text{ganit}_{pic} \cdot \text{ad}_{ari}(\text{pil}) \cdot \text{swap}(M). \quad (4.5.2) \]

\[ \text{Proof. We use the defining identity} \]
\[ \text{ad}_{ari}(A) \cdot B = \text{fragari}(\text{preari}(A, B), A) \quad (4.5.3) \]

and equation (2.4.10) given by
\[ \text{swap}(\text{preari}(\text{swap} \cdot A, \text{swap} \cdot B)) = \text{axit}(B, -\text{push}(B)) \cdot A + \mu(A, B). \quad (4.5.4) \]

Using this for \( A = \text{pal} \) and \( B = M \), we find in particular that
\[ \text{preari}(\text{pil}, \text{swap} \cdot M) = \text{swap}(\text{axit}(M, -\text{push}(M)) \cdot \text{pal} + \mu(\text{pal}, M)) \]
\[ = \text{swap}(\text{arit}(M) \cdot \text{pal} + \mu(\text{pal}, M)) \quad \text{because } M \text{ is push-inv} \]
\[ = \text{swap} \cdot \text{preari}(\text{pal}, M). \quad (4.5.5) \]

Using (2.7.6) for \( A = \text{pal}, B = M \), we have
\[ \text{swap} \cdot \text{ad}_{ari}(\text{pal}) \cdot M = \text{swap} \cdot \text{fragari}(\text{preari}(\text{pal}, M), \text{pal}) \]
\[ = \text{swap} \cdot \text{fragari}(\text{swap} \cdot \text{preari}(\text{pal}, M), \text{pal}) \]
\[ = \text{ganit}_{pic} \cdot \text{fragari}(\text{swap} \cdot \text{preari}(\text{pal}, M), \text{pil}) \quad \text{by (4.5.1)} \]
\[ = \text{ganit}_{pic} \cdot \text{fragari}(\text{preari}(\text{pil}, \text{swap} \cdot M), \text{pil}) \quad \text{by (2.7.6)} \]
\[ = \text{ganit}_{pic} \cdot \text{ad}_{ari}(\text{pil}) \cdot \text{swap} \cdot M, \]

proving (4.5.2). \]

\[ \text{§4.6. Double shuffle is a Lie algebra} \]

Recall that by Theorem 3.4.4, the double shuffle Lie algebra \( \mathfrak{ds} \) is isomorphic to \( \text{ARI}_{al}^{pol}_{al+il} \). In this section we give Ecalle’s proof that the latter is a Lie algebra for the \( \text{ari} \)-bracket, thus giving a complete different proof of Racinet’s well-known theorem 1.3.1. Our proof comes directly from the paper [SS], and was indicated to us in a personal communication from Ecalle.

\[ \text{Theorem 4.6.1. The action of the operator } \text{ad}_{ari}(\text{pal}) \text{ on the Lie subalgebra } \text{ARI}_{al}^{pol}_{al+il} \subset \text{ARI} \text{ yields a Lie isomorphism of subspaces} \]
\[ \text{ad}_{ari}(\text{pal}) : \text{ARI}_{al}^{pol}_{al+il} \xrightarrow{\sim} \text{ARI}_{al}^{pol}_{al+il}. \quad (4.6.1) \]
Thus in particular $\text{ARI}_{al}$ forms a Lie algebra under the $\text{ari}$-bracket.

**Proof.** Let $A \in \text{ARI}$ be an even function in depth 1. Note first that $ad_{\text{ari}}(\text{pal})$ preserves the depth 1 component of moulds in $\text{ARI}$, so $ad_{\text{ari}}(\text{pal}) \cdot A$ is also even in depth 1.

We first consider the case where $A \in \text{ARI}_{al}$, i.e. $\text{swap}(A)$ is alternal without addition of a constant correction. By Proposition 2.6.1, $\text{GARI}_{as} = \text{expari}(\text{ARI}_{al})$, so in particular $\text{GARI}_{as}$ acts by the adjoint action on $\text{ARI}_{al}$, and therefore since $\text{pal}$ is symmetral by Theorem 4.3.4, the mould $ad_{\text{ari}}(\text{pal}) \cdot A$ is alternal. By Lemma 2.5.5, $A$ is push-invariant, so we can apply Ecalle’s second fundamental identity (4.5.2) and find that

$$\text{swap}(ad_{\text{ari}}(\text{pal}) \cdot A) = \text{ganit}(\text{pic}) \cdot (ad_{\text{ari}}(\text{pil}) \cdot \text{swap}(A)).$$

Since $A \in \text{ARI}_{al}$, $\text{swap}(A)$ is alternal, and thus again by Proposition 2.6.1, $ad_{\text{ari}}(\text{pil}) \cdot \text{swap}(A)$ is again alternal; thus $\text{ganit}(\text{pic}) \cdot ad_{\text{ari}}(\text{pil}) \cdot \text{swap}(A)$ is alternal, and finally by (4.6.2), $\text{swap}(ad_{\text{ari}}(\text{pal}) \cdot A)$ is alternal, which proves that $ad_{\text{ari}}(\text{pal}) \cdot A \in \text{ARI}_{al}$ as desired.

We now consider the general case where $A \in \text{ARI}_{al}$. Let $C$ be the constant-valued mould such that $\text{swap}(A) + C$ is alternal. We will need the following result to deal with the constant mould $C$.

**Lemma 4.6.2.** [B, Corollary 4.43] *If $C$ is a constant-valued mould, then*

$$\text{ganit}(\text{pic}) \cdot ad_{\text{ari}}(\text{pil}) \cdot C = C.$$  

**Proof.** We apply the fundamental identity (4.5.2) in the case where $A = \text{swap}(A) = C$ is a constant-valued mould, obtaining

$$\text{swap}(ad_{\text{ari}}(\text{pal}) \cdot C) = \text{ganit}(\text{pic}) \cdot (ad_{\text{ari}}(\text{pil}) \cdot C).$$

So it is enough to show that the left-hand side of this is equal to $C$, i.e. that $ad_{\text{ari}}(\text{pal}) \cdot C = C$. Directly from the definitions, we see that if $A \in \text{ARI}$, then $\text{ari}(C) \cdot A = 0$ and $\text{ari}(A) \cdot C = \text{lu}(C, A)$. Thus

$$\text{ari}(A, C) = \text{lu}(A, C) + \text{ari}(A) \cdot C - \text{ari}(C) \cdot A = 0.$$  

Now, by (2.7.5) we see that $ad_{\text{ari}}(\text{pal}) \cdot C$ is a linear combination of iterated $\text{ari}$-brackets of $\text{logari}(\text{pal})$ with $C$, but since $\text{pal} \in \text{GARI}$, $\text{logari}(\text{pal}) \in \text{ARI}$, so (4.6.4) shows that $\text{ari}(\text{logari}(\text{pal}), C) = 0$, i.e. all the terms in (2.7.5) are 0, which concludes the proof.

Returning to the case $A \in \text{ARI}_{al}$, we again have that $ad_{\text{ari}}(\text{pal}) \cdot A$ is alternal, so to conclude the proof of the theorem it remains only to show that its swap is alternal up to addition of a constant mould, and we will show that this constant mould is exactly $C$. As before, since $\text{swap}(A) + C \in \text{ARI}$ is alternal, the mould

$$ad_{\text{ari}}(\text{pil}) \cdot (\text{swap}(A) + C) = ad_{\text{ari}}(\text{pil}) \cdot \text{swap}(A) + ad_{\text{ari}}(\text{pil}) \cdot C$$
is also alternal. Thus applying $ganit(pic)$ to it yields the alternil mould

$$ganit(pic) \cdot ad_{ari}(pil) \cdot swap(A) + ganit(pic) \cdot ad_{ari}(pil) \cdot C.$$ 

By Lemma 4.6.2, this is equal to

$$ganit(pic) \cdot ad_{ari}(pil) \cdot swap(A) + C,$$

which is thus alternil. Now, since $A$ is push-invariant by Lemma 2.5.5, we can apply (4.5.2) and find that (4.6.5) is equal to

$$swap(ad_{ari}(pal) \cdot A) + C,$$

which is thus also alternil. Therefore $swap(ad_{ari}(pal) \cdot A)$ is alternil up to a constant, which precisely means that $ad_{ari}(pal) \cdot A \in ARI_{al*il}$ as claimed. Since $ad_{ari}(pal)$ is invertible (with inverse $ad_{ari}(invgari \cdot pal)$), we can use all of these arguments in the other direction to show that $ad_{ari}(invgari \cdot pal)$ maps $ARI_{al*il}$ to $ARI_{al*al}$. Thus (4.6.1) is a Lie algebra isomorphism.

§4.7. Denominators

In this section we use Écalle’s second fundamental identity to prove the following result on denominators. Let $\Delta$ be the mould operator defined by

$$\Delta(A)(u_1, \ldots, u_r) = (u_1 + \cdots + u_r)u_1 \cdots u_r A(u_1, \ldots, u_r),$$

and let $ARI^\Delta$ denote the space of rational-valued moulds $P \in ARI$ such that $\Delta(P) \in ARI^{pol}$, i.e. such that the denominator of the rational function $P(u_1, \ldots, u_r)$ is “at worst” $(u_1 + \cdots + u_r)u_1 \cdots u_r$.

**Lemma 4.7.1.** (Baumard, thesis) Let $poc$ be the mould defined by $poc(\emptyset) = 1$ and

$$poc(v_1, \ldots, v_r) = \frac{1}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)}$$

for $r \geq 1$. Then $ganit_{pic} \circ ganit_{poc} = id$.

Proof. Lemme 4.37 in Baumard’s thesis.

**Lemma 4.7.2.** Let $A \in ARI^{pol}$. Then

$$swap \cdot ganit_{poc} \cdot A \in ARI^\Delta.$$ 

Proof. The explicit expression for $ganit$ in (2.7.2) shows that the only denominators that can occur in $ganit_{poc} \cdot A$ come from the factors

$$poc(|b_1| \cdots poc(|b_s|)$$

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for all decompositions $d_v = a_1 b_1 \cdots a_s b_s$ of $v = (v_1, \ldots, v_r)$ into chunks, where if the chunk $b_i$ is given by $(v_k, \ldots, v_{k+l})$, then

$$[b_i] = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1})$$

(note that $a_1 \neq \emptyset$ and therefore $k > 1$). By the definition of poc, only factors that can appear are $(v_l - v_{l-1})$ where $v_l$ is a letter in one of $b_i$, and these factors appear in each term with multiplicity one. Since the sum ranges over all possible decompositions, the only letter of $v$ that never belongs to any $b_i$ is $v_1$; all the other factors $(v_i - v_{i-1})$ appear.

Thus $(v_1 - v_2)(v_2 - v_3) \cdots (v_{r-1} - v_r)$ is a common denominator for all the terms in the sum defining $ganit_{poc} \cdot A$. The swap of this common denominator is equal to $u_2 \cdots u_r$, so this term is a common denominator for $swap \cdot ganit_{poc} \cdot A$, which proves the result.  

\[\square\]

**Theorem 4.7.3.** Let $N \in ARI_{pol}$ be such that $ad_{ari}(invpal) \cdot N$ is push-invariant. Then $ad_{ari}(invpal) \cdot N \in ARI^\Delta$.

**Proof.** If $M = ad_{ari}(invpal) \cdot N$ is push-invariant, then we can rewrite the second fundamental identity (4.5.2) in terms of $N$ as follows:

$$swap \cdot ad_{ari}(invpal) \cdot ganit_{poc} \cdot swap(N) = ad_{ari}(invpal) \cdot N,$$  \hspace{1cm} (4.7.1)

valid whenever $M = ad_{ari}(invpal) \cdot N$ is push-invariant. Applying Lemma 4.7.2 with $A = swap(N) \in \overline{ARI}_{pol}$ shows that the denominator of $ganit_{poc} \cdot swap(N)$ in depth $r$ is “at worst” $v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)$ for each $r \geq 1$.

Let $f \in V_{krv}$, i.e. $f$ is push-invariant and divergence. Let $\nu : V_{krv} \to W_{krv}$ map $f(x, y)$ to $f(-x - y, y)$. Then $W_{krv}$ consists of Lie polynomials that are alternal and senary with circ-constant (i.e. $*$-circ-neutral) swap.

Let $B \in W_{krv}$, so that $swap(B)$ is $*$-circ-neutral and polynomial. Let $N = swap \cdot pari(B)$. By Lemma 4.7.2, we have

$$swap \cdot ganit_{poc} \cdot N \in ARI^\Delta.$$

By Proposition 30 of krvell1, the mould $swap \cdot ganit_{poc} \cdot swap(N)$ is circ-neutral. By Corollary 35 of krvell1, the mould

$$swap \cdot ad_{ari}(invpal) \cdot ganit_{poc} \cdot swap(N) \in ARI^\Delta.$$
APPENDIX

§A.1. Proof of Proposition 2.2.1.

Let \( A \in \text{BARI} \). We prove that \( \text{amit}(A) \) is a derivation for \( \text{mu} \). The case for \( \text{anit}(B) \) is analogous and we leave it as an exercise. It follows immediately from (2.2.3) and (2.2.4) that \( \text{axit}(B) \) and \( \text{arit}(B) \) are derivations.

For \( \text{amit} \), we need to prove the identity

\[
\text{amit}(A) \cdot \text{mu}(B, C) = \text{mu}(\text{amit}(A) \cdot B, C) + \text{mu}(B, \text{amit}(A) \cdot C).
\]

Since \( A, B, C \) all lie in \( \text{BARI} \) and therefore 0-valued on the emptyset, we can remove \( b \neq \emptyset \) from the definition of \( \text{amit} \); we have

\[
\text{amit}(A) \cdot \text{mu}(B, C) = \sum_{w=abc, c \neq \emptyset} \text{mu}(B, C)(a[c]A(b))
\]

\[
= \sum_{w=abc, d_1d_2=a[c]} B(d_1)C(d_2)A(b)
\]

\[
= \sum_{w=abc, a_1a_2=a} B(a_1)C(a_2[c]A(b)) + \sum_{w=abc, c_1c_2=[c]} B(ac_1)C(c_2)A(b)
\]

\[
= \sum_{w=abc, c \neq \emptyset} B(a_1)C(a_2[c]A(b)) + \sum_{w=abc, c_1 \neq \emptyset} B(a[c_1])C(c_2)A(b)
\]

\[
= \sum_{w=ad, d \neq \emptyset} B(a_1)(\text{amit}(A) \cdot C)(d) + \sum_{w=dc, d \neq \emptyset} (\text{amit}(A) \cdot B)(d)C(c_2).
\]

Noting that for \( A, B, C \in \text{ARI} \) we always have \( (\text{amit}(A) \cdot B)(\emptyset) = (\text{amit}(A) \cdot C)(\emptyset) = 0 \),

we can drop the requirement \( d \neq \emptyset \) under the sum, and therefore obtain exactly

\[
\text{mu}(B, \text{amit}(A) \cdot C) + \text{mu}(\text{amit}(A) \cdot B, C),
\]

as desired.

Exercise. Show similarly that \( \text{anit} \) is a derivation.

§A.2. Proofs of (2.4.7) and (2.4.8)

To prove these two key identities, we need the following explicit expressions for the flexions occurring in the definitions of the derivations, and the effect of \( \text{swap} \):

\[
a[c] = \begin{pmatrix} u_1 & \cdots & u_k \\ v_1 & \cdots & v_k \end{pmatrix} \begin{pmatrix} u_{k+1} + \cdots + u_{k+l+1} & \cdots & u_r \\ v_{k+l+1} & \cdots & v_r \end{pmatrix},
\]

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\[
\begin{pmatrix}
  u_{k+1} & \cdots & u_k & u_{k+1} + \cdots + u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\
  v_{k+1} - v_{k+l+1} & \cdots & v_k - v_{k+l+1} & v_{k+l+1} & \cdots & v_r
\end{pmatrix}
\]

\[
|b| = \begin{pmatrix}
  u_{k+1} & \cdots & u_k & u_{k+l} \\
  v_{k+1} - v_k & \cdots & v_k - v_{k+l} & v_k
\end{pmatrix}.
\]

Setting \( SC = \text{swap}(C) \) for any mould \( C \), we have

\[
SC(a|c) = SC\left(\begin{pmatrix}
  u_1 & \cdots & u_k & u_{k+1} + \cdots + u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\
  v_1 & \cdots & v_k & v_{k+l+1} & \cdots & v_r
\end{pmatrix}\right)
\]

\[
= C\begin{pmatrix}
  v_r & v_{r-1} - v_r & \cdots & v_{k+l+1} - v_{k+l+2} & v_k - v_{k+l+1} & v_{k-1} - v_k & u_{k+1} + \cdots + u_{k+l+1} + u_1 + \cdots + u_{k-1} \\
  u_1 + \cdots + u_r & u_1 + \cdots + u_{r-1} & \cdots & u_1 + \cdots + u_{k+l+1} & u_1 + \cdots + u_k & u_1 + \cdots + u_{k-1}
\end{pmatrix}
\]

\[
SC(b|c) = SC\left(\begin{pmatrix}
  u_{k+1} & \cdots & u_k & v_k + \cdots + v_{k+l} \\
  v_{k+1} - v_{k+l+1} & \cdots & v_k - v_{k+l+1} & u_{k+l+1} + \cdots + u_{k+l-1} + u_1 + \cdots + u_{k-1}
\end{pmatrix}\right)
\]

\[
= C\begin{pmatrix}
  v_{k+l} - v_k & v_{k+l-1} - v_{k+l} & \cdots & v_{k+1} - v_{k+2} \\
  u_{k+1} + \cdots + u_{k+l} & u_{k+1} + \cdots + u_{k+l-1} & \cdots & u_{k+1} + \cdots + u_{k-1}
\end{pmatrix}
\]

\[
SC([b]c) = SC\left(\begin{pmatrix}
  u_{k+1} & \cdots & u_k & u_{k+l} \\
  v_{k+1} - v_k & \cdots & v_{k+l} - v_k & v_k + \cdots + v_{k+l+1}
\end{pmatrix}\right)
\]

\[
= C\begin{pmatrix}
  v_{k+l} - v_k & v_{k+l-1} - v_{k+l} & \cdots & v_{k+1} - v_{k+2} \\
  u_{k+1} + \cdots + u_{k+l} & u_{k+1} + \cdots + u_{k+l-1} & \cdots & u_{k+1} + \cdots + u_{k-1}
\end{pmatrix}
\]

Applying the swap

\[
\begin{pmatrix}
  u_1 & u_2 & \cdots & u_r \\
  v_1 & v_2 & \cdots & v_r
\end{pmatrix} \mapsto \begin{pmatrix}
  v_r & v_{r-1} - v_r & \cdots & v_1 - v_2 \\
  u_1 + \cdots + u_r & u_1 + \cdots + u_{r-1} & \cdots & u_1
\end{pmatrix},
\]

i.e.

\[
\begin{cases}
  u_1 \mapsto v_r \\
  u_i \mapsto v_{r-i+1} - v_{r-i+2}, \text{ if } i > 1 \\
  u_1 + \cdots + u_i \mapsto v_{r-i+1} \\
  u_i + \cdots + u_j \mapsto -v_{r-i+2} + v_{r-j+1} \text{ if } i < j \\
  v_i \mapsto u_1 + \cdots + u_{r-i+1} \\
  v_i - v_{i+1} \mapsto u_{r-i+1} \\
  v_i - v_j \mapsto u_{r-j+2} + \cdots + u_{r-i+1} \text{ if } i < j \\
  v_i - v_j \mapsto -u_{r-i+2} - \cdots - u_{r-j+1} \text{ if } i > j
\end{cases}
\]
to these four terms yields
\[ C\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-k-l} & u_{r-k-l+1} + \cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{r-k-l} & v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix} \]
\[ C\begin{pmatrix} u_{r-k-l+1} & u_{r-k-l+2} & \cdots & u_{r-k} \\ v_{r-k-l+1} - v_{r-k+1} & v_{r-k-l+2} - v_{r-k} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix} \]
\[ C\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-k-l} & u_{r-k-l+1} + \cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{r-k-l} & v_{r-k-l+1} + v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix} \]
\[ C\begin{pmatrix} -u_{r-k-l+2} - \cdots - u_{r-k+1} & u_{r-k-l+2} & \cdots & u_{r-k} \\ v_{r-k-l+1} - v_{r-k+1} & v_{r-k-l+2} - v_{r-k+1} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix} \]

Setting \( m = r - k - l \), they can be written as
\[ C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} + \cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix} \]
\[ C\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{r-k} \\ v_{m+1} - v_{r-k+1} & v_{m+2} - v_{r-k} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix} \]
\[ C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} + \cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix} \]
\[ C\begin{pmatrix} -u_{m+2} - \cdots - u_{r-k+1} & u_{m+2} & \cdots & u_{r-k} \\ v_{m+1} - v_{r-k+1} & v_{m+2} - v_{r-k+1} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix} \]

Now putting \( r - k = m + l \) gives
\[ C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} + \cdots + u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+l+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix} \]
\[ C\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1} - v_{m+l+1} & v_{m+2} - v_{m+l+1} & \cdots & v_{m+l} - v_{m+l+1} \end{pmatrix} \]
\[ C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} + \cdots + u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix} \]
\[ C\begin{pmatrix} -u_{m+2} - \cdots - u_{m+l+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1} - v_{m+l+1} & v_{m+2} - v_{m+l+1} & \cdots & v_{m+l} - v_{m+l+1} \end{pmatrix} \]

Using all these, we can now prove (2.4.7) and (2.4.8).

**Proof of (2.4.7).** We have
\[ \text{swap}(\text{amit}(\text{swap}(B)) \cdot \text{swap}(A)) = \text{swap}\left( \sum_{w=abc} S(A[a|c]S(B[b])) \right) \]

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\[
\begin{align*}
A = \text{swap}\left[\sum_{l=1}^{r-1} \sum_{m=1}^{r-l} \left( \begin{array}{cccccccc}
v_r & v_{r-1} - v_r & \cdots & v_{k+l+1} - v_{k+l+2} & v_k - v_{k+l+1} & v_k - v_k & \cdots & v_{1} - v_k \\
u_1 + \cdots + u_r & u_1 + \cdots + u_{r-1} & \cdots & u_1 + \cdots + u_{k+l} & u_1 + \cdots + u_k & u_1 + \cdots + u_{k-1} & \cdots & u_1 + \cdots + u_2 \\
v_1 & v_2 & \cdots & v_{m} & u_{m+1} + \cdots + u_{m+1} & u_{m+2} & \cdots & u_r \\
v_1 & v_2 & \cdots & v_m & v_{m+1} & \cdots & u_{r} \\
\end{array} \right) \right] \\
= \sum_{l=1}^{r-1} \sum_{m=1}^{r-l} A \left( u_1 \ u_2 \ \cdots \ u_m \ u_{m+1} + \cdots + u_{m+l+1} \ u_{m+l+2} \ \cdots \ u_r \right) \cdot B \left( \begin{array}{cccc}
u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\
v_{m+1} - v_{m+l+1} & v_{m+2} - v_{m+l+1} & \cdots & v_{m+l} - v_{m+l+1} \\
u_k & u_k & \cdots & u_{k+l+1} \\
u_k & v_k & \cdots & v_{k+l+1} \\
u_k & \cdots & u_{k+l+2} \ u_k & \cdots & u_r \\
u_k & \cdots & u_{k+l+2} \ v_k & \cdots & v_r \\
u_k & \cdots & u_{k+l+2} \ v_k & \cdots & v_r \\
u_k & \cdots & u_{k+l+2} \ v_k & \cdots & v_r \\
\end{array} \right)
\end{align*}
\]

\[= \sum_{l=1}^{r-1} \sum_{m=1}^{r-l} A \left( u_1 \ u_2 \ \cdots \ u_k \ u_{k+1} + \cdots + u_{k+l+1} \ u_{k+l+2} \ \cdots \ u_r \right) \cdot B \left( \begin{array}{cccc}
u_k & u_k & \cdots & u_{k+l+1} \\
u_k & v_k & \cdots & v_{k+l+1} \\
u_k & \cdots & u_{k+l+2} \ v_k & \cdots & v_r \\
u_k & \cdots & u_{k+l+2} \ v_k & \cdots & v_r \\
u_k & \cdots & u_{k+l+2} \ v_k & \cdots & v_r \\
\end{array} \right)
\]

\[- \sum_{l=1}^{r-1} A \left( u_1 + \cdots + u_{l+1} \ u_{l+2} \ \cdots \ u_r \right) \cdot B \left( \begin{array}{cccc}
u_1 & u_2 & \cdots & u_l \\
u_1 & v_2 & \cdots & v_l \\
u_1 & \cdots & v_l \ u_1 & \cdots & v_1 \\
u_1 & \cdots & v_l \ u_1 & \cdots & v_1 \\
\end{array} \right)
\]

\[+ \sum_{l=1}^{r-1} A \left( u_1 \ u_2 \ \cdots \ u_{r-l} \right) \cdot B \left( \begin{array}{cccc}
u_{r-l+1} & u_{r-l+2} & \cdots & u_r \\
u_{r-l+1} & v_{r-l+2} & \cdots & v_r \\
\end{array} \right)
\]

\[= \text{amit}(B) \cdot A - \text{swap}\left( \text{mu}(\text{swap}(A), \text{swap}(B)) \right) + \text{mu}(A, B).
\]

**Proof of (2.4.8).** We have

\[
\text{swap}\left( \text{amit}(\text{swap}(B)) \cdot \text{swap}(A) \right) = \text{swap}\left( \sum_{w=\text{abc}}^{a,b\neq\emptyset} SA(a|c)SB([b]) \right)
\]

\[= \text{swap}\left[ \sum_{l=1}^{r-1} \sum_{k=1}^{r-l} \right]
\]

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A\left( 
\begin{array}{cccccccc}
  v_r & v_{r-1} - v_r & \cdots & v_{k+1} + 1 - v_{k+1} + 2 & v_k - v_{k+1} + 1 & v_{k-1} - v_k & \cdots & v_1 \\
  u_1 + \cdots + u_r & u_1 + \cdots + u_{r-1} & \cdots & u_1 + \cdots + u_{k+1} & u_1 + \cdots + u_{k+1} & u_1 + \cdots + u_{k-1} & \cdots & u_1 \\
\end{array}\right)
\cdot B\left( 
\begin{array}{cccc}
  v_{k+1} - v_k & v_{k+1} - v_{k+1} & \cdots & v_{k+1} - v_{k+2} \\
  u_{k+1} + \cdots + u_{k+1} & u_{k+1} + \cdots + u_{k+1} & \cdots & u_{k+1} + \cdots + u_{k+1} \\
\end{array}\right)
\]

= \sum_{l=1}^{r-1} \sum_{m=0}^{r-l-1} A\left( 
\begin{array}{cccc}
  u_1 & u_2 & \cdots & u_m \\
  v_1 & v_2 & \cdots & v_m \\
\end{array}\right)
\cdot B\left( 
\begin{array}{cccc}
  u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\
  v_{m+1} - v_{m+1} & v_{m+2} - v_{m+1} & \cdots & v_{m+l} - v_{m+l} \\
\end{array}\right)
\]

= \sum_{l=1}^{r-1} \sum_{m=1}^{r-l} A\left( 
\begin{array}{cccc}
  u_1 & u_2 & \cdots & u_m \\
  v_1 & v_2 & \cdots & v_m \\
\end{array}\right)
\cdot B\left( 
\begin{array}{cccc}
  u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\
  v_{m+1} - v_m & v_{m+2} - v_m & \cdots & v_{m+l} - v_m \\
\end{array}\right)
\]

= \sum_{l=1}^{r-1} \sum_{k=1}^{r-l} A\left( 
\begin{array}{cccc}
  u_1 & u_2 & \cdots & u_k \\
  v_1 & v_2 & \cdots & v_k \\
\end{array}\right)
\cdot B\left( 
\begin{array}{cccc}
  u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\
  v_{k+1} - v_k & v_{k+2} - v_k & \cdots & v_{k+l} - v_k \\
\end{array}\right)
\]

§A.3. Proof of Lemma 3.2.1.

We first prove (3.2.8), then (3.2.7). By (3.2.5), we have \( m_i(f_1, \ldots, v_r) = t_Y(f_Y) \). Since \( mi \) is additive, we may assume that \( f \) is a monomial, \( f = x^{a_0-1} y \cdots y x^{a_r-1} \). Then

\[
\pi_Y(f) = \begin{cases} 
  f & \text{if } a_0 = 1 \\
  0 & \text{otherwise.} 
\end{cases}
\]

So

\[
\ret_X(\pi_Y(f)) = \begin{cases} 
  x^{a_0-1} y \cdots x^{a_1-1} y & \text{if } a_0 = 1 \\
  0 & \text{otherwise.} 
\end{cases}
\]

and

\[
f_Y = \begin{cases} 
  y_{a_r} \cdots y_{a_1} & \text{if } a_0 = 1 \\
  0 & \text{otherwise.} 
\end{cases}
\]

Thus

\[
m_i(f_1, \ldots, v_r) = t_Y(f_Y) = \begin{cases} 
  v_1^{a_0-1} \cdots v_r^{a_1-1} & \text{if } a_0 = 1 \\
  0 & \text{otherwise.} 
\end{cases}
\]

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Now by (3.2.4), we have

\[ vimo_f(z_0, \ldots, z_r) = z_0^{a_0-1} z_1^{a_1-1} \cdots z_r^{a_r-1}, \]

so as desired, we have

\[
m_i f(v_1, \ldots, v_r) = vimo_f(0, v_r, \ldots, v_1) = \begin{cases} v_r^{a_1-1} \cdots v_1^{a_r-1} & \text{if } a_0 = 1 \\ 0 & \text{otherwise.} \end{cases}
\]

This settles the proof of (3.2.8) for \( m_i \).

The case of \( m_a \) is a little more complicated. Again, by additivity, we can assume that \( f \) is a monomial \( C_{a_1} \cdots C_{a_r} \) in the \( C_i \). We will prove it by induction on \( r \) (though there might be a better way). For the base case, \( r = 1 \), we have \( n = a_1 \) and

\[ f = C_{a_1} = \sum_{i=0}^{a_1-1} (-1)^i C_{a_1-1}^i z_0^{a_1-1-i} y x^i, \]

\[ vimo_f(z_0, z_1) = \sum_{i=0}^{a_1-1} (-1)^i C_{a_1-1}^i z_0^{a_1-1-i} z_1^i, \]

\[ vimo_f(0, u_1) = (-1)^{a_1-1} u_1^{a_1-1} = (-1)^{r+n} u_1^{a_1-1} = m_a f(u_1) \]

using Ecalle’s definition, and comparing with (3.2.5), we also have

\[ m_a f(u_1) = (-1)^{r+n} C_{a_1} = (-1)^{r+n} u_1^{a_1-1}, \]

which settles the base case.

Now make the induction hypothesis that (3.2.7) holds up to depth \( r - 1 \), and let \( f = C_{a_1} \cdots C_{a_{r-1}} C_{a_r} \). Using (3.2.5), we have

\[ m_a f(u_1, \ldots, u_r) = (-1)^{r+n} C_{a_r} f = (-1)^{r+n} u_1^{a_1-1} \cdots u_r^{a_r-1}. \]

Let us write \( g = C_{a_1} \cdots C_{a_{r-1}} \). Then again from (3.2.5), we have

\[ m_a f(u_1, \ldots, u_r) = m_g(u_1, \ldots, u_{r-1}) m_{a_{a_r}}(u_r). \]

By the induction hypothesis, we have

\[
\begin{align*}
mac_{a_r}(u_r) &= vimo_{a_r}(0, u_r) = (-1)^{a_r-1} u_r^{a_r-1} \\
ma_g(u_1, \ldots, u_{r-1}) &= vimo_g(0, u_1, \ldots, u_1 + \cdots + u_{r-1}).
\end{align*}
\]

So to prove (3.2.7), we have to show that

\[
vimo_f(0, u_1, \ldots, u_1 + \cdots + u_r) = vimo_g(0, u_1, \ldots, u_1 + \cdots + u_{r-1}) vimo_{a_r}(0, u_r) = (-1)^{a_r-1} vimo_g(0, u_1, \ldots, u_1 + \cdots + u_{r-1}) u_r^{a_r-1}.
\]

(A.3.1)
Write
\[ g = \sum_{a=(a_0, \ldots, a_{r-1})} c_a a_0^{a_0-1} y \ldots y x^{a_{r-1}-1}. \]

Then
\[ \text{vimo}_g(z_0, \ldots, z_{r-1}) = \sum_{a=(a_0, \ldots, a_{r-1})} c_a z_0^{a_0} z_1^{a_1-1} \ldots z_{r-1}^{a_{r-1}-1}, \]

and
\[ \text{vimo}_g(0, u_1, \ldots, u_1+\cdots+u_{r-1}) = \sum_{a=(a_1, \ldots, a_{r-1})} c_a u_1^{a_1-1}(u_1+u_2)^{a_2-1} \cdots (u_1+\cdots+u_{r-1})^{a_{r-1}-1}. \]

Thus the second term in (A.3.1) is given by
\[ \text{vimo}_g(0, u_1, \ldots, u_1+\cdots+u_{r-1}) \text{vimo}_{C_{a_r}}(0, u_r) \]
\[ = (-1)^{a_r-1} \sum_{a=(1,a_1,\ldots,a_{r-1})} c_a u_1^{a_1-1}(u_1+u_2)^{a_2-1} \cdots (u_1+\cdots+u_{r-1})^{a_{r-1}-1} u_r^{a_r-1}. \quad (A.3.2) \]

But also
\[ f = g C_{a_r} = \sum_{a=(0,\ldots,a_{r-1})} \sum_{j=0}^{a_{r-1}} (-1)^j C_{a_{r-1}}^j c_a x_0^{a_0-1} y \ldots y x_{r-1}^{a_{r-1}-1} x_r^{a_r-1-j} y x^j, \]

so
\[ \text{vimo}_f(z_0, \ldots, z_r) = \sum_{a=(0,\ldots,a_{r-1})} \sum_{j=0}^{a_{r-1}} (-1)^j C_{a_{r-1}}^j c_a z_0^{a_0} z_1^{a_1-1} \ldots z_{r-1}^{a_{r-1}-2} z_r^{a_r-j} z_r^j, \]

so
\[ \text{vimo}_f(0, z_1, \ldots, z_r) = \sum_{a=(1,a_1,\ldots,a_r)} \sum_{j=0}^{a_{r-1}} (-1)^j C_{a_{r-1}}^j c_a z_1^{a_1-1} z_2^{a_2-2} \ldots z_{r-1}^{a_{r-1}-2} z_r^{a_r-j} z_r^j, \]

so finally the first term in (A.3.1) is given by
\[ \text{vimo}_f(0, u_1, \ldots, u_1+\cdots+u_r) = \]
\[ \sum_{a=(1,a_1,\ldots,a_r)} \sum_{j=0}^{a_{r-1}} (-1)^j C_{a_{r-1}}^j c_a u_1^{a_1-1}(u_1+u_2)^{a_2-1} \cdots (u_1+\cdots+u_{r-1})^{a_{r-1}-2} u_r^{a_r-j}(u_1+\cdots+u_r)^j \]
\[ = \sum_{a=(1,a_1,\ldots,a_{r-1})} c_a u_1^{a_1-1}(u_1+u_2)^{a_2-1} \cdots (u_1+\cdots+u_{r-1})^{a_{r-1}-1}. \]
\[
\left( \sum_{j=0}^{a_r-1} (-1)^j C^j_{a_r-1} (u_1 + \cdots + u_{r-1})^{a_r-j}(u_1 + \cdots + u_r)^j \right)
= (-1)^{a_r-1} \sum_{\mathbf{a} = (1,a_1,\ldots,a_{r-1})} \mathbf{c}_a u_1^{a_1-1} (u_1 + u_2)^{a_2-1} \cdots (u_1 + \cdots + u_{r-1})^{a_{r-1}-1} \cdot u_r^{a_r-1}
\]

since the factor between large parenthesis is just the binomial expansion of
\[
\left( (u_1 + \cdots + u_{r-1}) - (u_1 + \cdots + u_r) \right)^{a_r-1} = (-1)^{a_r-1} u_r^{a_r-1}.
\]

But this is equal to the second term as given in (A.3.2), so (A.3.1) holds, thus proving (3.2.7).

§A.4. Proof of Proposition 3.3.2

We need to show that
\[
\text{arit}(A)(BC) = \text{arit}(A)(B)C + \text{Barit}(A)(C).
\]

(A1)

Using the definition of \( S_A(B) \) from (4.1),
\[
(S_A(B))(w) = \sum_{w=abc} B(ac')A(b) - \sum_{w=abc \ a \neq \emptyset} B(a''c)A(b),
\]

and \( \text{arit}(A)(B) = S_A(B) - BA \), we write
\[
(\text{arit}(A)(B))(w) = \sum_{w=abc} B(ac')A(b) - \sum_{w=abc \ a \neq \emptyset} B(a''c)A(b) - \sum_{w=ab} B(a)A(b).
\]

Splitting the first sum over \( c = \emptyset \) and \( c \neq \emptyset \), and recalling that \( c' = \emptyset \) when \( c = \emptyset \), this is equal to
\[
(\text{arit}(A)(B))(w) = \sum_{w=abc \ c \neq \emptyset} B(ac')A(b) + \sum_{w=ab} B(a)A(b) - \sum_{w=abc \ a \neq \emptyset} B(a''c)A(b) - \sum_{w=ab} B(a)A(b)
= \sum_{w=abc \ c \neq \emptyset} B(ac')A(b) - \sum_{w=abc \ a \neq \emptyset} B(a''c)A(b).
\]

(A2)

Thus we can write the right-hand side of (A1) as
\[
(\text{arit}(A)(B)C + \text{Barit}(A)(C))(w) = \sum_{w=uv} \left( \sum_{u=abc \ c \neq \emptyset} B(ac')A(b)C(v) - \sum_{u=abc \ a \neq \emptyset} B(a''c)A(b)C(v) \right)
\]

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or again as
\[
(arit(A)(B)C + Barit(A)(C))(w) = \sum_{w=abc} B(ac')A(b)C(v) - \sum_{w=abc} B(a''c)A(b)C(v)
\]

\[
+ \sum_{w=uabc} B(u)C(ac')A(b) - \sum_{w=uabc} B(u)C(a''c)A(b).
\]

(A3)

By (A2), the left-hand side of (A1) can be written
\[
arit(A)(BC) = \sum_{w=abc} BC(ac')A(b) - \sum_{w=abc} BC(a''c)A(b)
\]

\[
= \sum_{w=abc} \sum_{ac'=uv} B(u)C(v)A(b) - \sum_{w=abc} \sum_{a''c=uv} B(u)C(v)A(b) \quad (A4)
\]

\[
= \sum_{w=a_1a_2bc} B(a_1)C(a_2c')A(b) + \sum_{w=abc} B(ac_1'c)C(c_2)A(b) - \sum_{w=a_1a_2bc} B(a_1)C(a''c)A(b) - \sum_{w=abc} B(a''c_1)C(c_2)A(b). \quad (A5)
\]

The passage from (A4) to (A5) is obtained by separating the first term into two terms according to whether the decomposition \(ac' = uv\) is of the form \(u = a_1, v = a_2c'\) or of the form \(u = ac_1', v = c_2\) with \(c_1 \neq \emptyset\) (otherwise the case \(u = a, v = c'\) is counted twice). The second term is separated into two terms according to whether the decomposition \(a''c = uv\) is of the form \(u = a''c_1, v = c_2\) or of the form \(u = a_1, v = a_2c\) with \(a_2 \neq \emptyset\) (otherwise the term \(u = a, v = c\) is counted twice).

Relabeling the indices in the first term of (A5) by \(a_1 \mapsto u, a_2 \mapsto a\), we see that this term is equal to the third term of (A3).

Relabeling the indices in the second term of (A5) by \(c_1 \mapsto c, c_2 \mapsto v\), we see that this term is equal to the first term of (A3).

Relabeling the indices in the third term of (A5) by \(a_1 \mapsto u, a_2 \mapsto a\), we see that this term is equal to the fourth term of (A3).

Relabeling the indices in the fourth term of (A4) by \(c_1 \mapsto c, c_2 \mapsto v\), we see that this term is equal to the second term of (A3).

So (A3) is equal to (A5), i.e. \(arit(A)(B)(C) + Barit(A)(C) = arit(A)(BC)\), proving that \(arit(A)\) is a derivation.

\[
= anit(push(B)) \cdot A.
\]
§A.5. Proof of Lemma 3.4.1

(i) Let $f \in \mathbb{Q}\langle C \rangle_n$. We show that $f$ satisfies shuffle if and only if $ma(f)$ is alternal. We know that $f$ satisfies shuffle if and only if $f \in \text{Lie}[x, y]$, so $f$ satisfies shuffle if and only if

$$f \in \mathbb{Q}\langle C \rangle_n \cap \text{Lie}[x, y] = \text{Lie}[C_1, C_2, \ldots]$$

where $C_i = \text{ad}(x)^{i-1}(y)$. Thus the shuffle relations on $f$ written in $x, y$ are equivalent to the shuffle conditions written in the $C_i$. I.e., assuming by additivity that $f$ is of homogeneous depth $r$, we can write

$$f = \sum_{a=(a_1, \ldots, a_r)} c_a C_{a_1} \cdots C_{a_r}, \quad (A.5.1)$$

and the shuffle relations are

$$\sum_{w \in sh((C_{a_1}, \ldots, C_{a_i}),(C_{a_{i+1}}, \ldots, C_{a_r}))} (f|w) = 0. \quad (A.5.1)$$

It is convenient to write the shuffle using the set $Sh(i, r) \subset S_r$ of permutations $\sigma$ of $\{1, \ldots, r\}$ satisfying

$$\sigma(1) < \cdots < \sigma(i) \quad \text{and} \quad \sigma(i+1) < \cdots < \sigma(r).$$

Then (A.5.1) can be rewritten

$$\sum_{\sigma \in Sh(i, r)} (f|C_{a_{\sigma(1)-1}} \cdots C_{a_{\sigma(r)-1}}) = \sum_{\sigma \in Sh(i, r)} c_{a_{\sigma(1)-1} \cdots a_{\sigma(r)-1}} = 0. \quad (A.5.2)$$

Let us compare this property with the alternality condition on

$$ma(f) = \sum_a c_a u_1^{a_1-1} \cdots u_r^{a_r-1}.$$ 

The alternality conditions are given by

$$0 = \sum_{w \in sh((u_1, \ldots, u_i),(u_{i+1}, \ldots, u_r))} ma(f)(w)$$

$$= \sum_{\sigma \in Sh(i, r)} \sum_a c_{a_1 \cdots a_r} u_1^{a_{\sigma(1)-1}} \cdots u_r^{a_{\sigma(r)-1}}$$

$$= \sum_{\sigma \in Sh(i, r)} \sum_a c_{a_{\sigma(1)-1} \cdots a_{\sigma(r)-1}} u_1^{a_{\sigma(1)-1}} \cdots u_r^{a_{\sigma(r)-1}},$$

which monomial by monomial implies that

$$\sum_{\sigma \in Sh(i, r)} c_{a_{\sigma(1)-1} \cdots a_{\sigma(r)-1}} = 0,$$

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We saw in \(\S\) series which is identical to (A.5.2).

(ii) The proof is identical to (i), with \(u_i\) replaced by \(v_i\) and \(C_{a_i}\) replaced by \(y_{a_i}\).

(iii) As in \(\S\)2.3, we write \(st(r,s)\) for the set of words in the stuffle sum

\[ st((a_1, \ldots, a_r), (a_{r+1}, \ldots, a_{r+s})). \]

We saw in \(\S\)2.3 that each stuffle sum \(st(r,s)\) corresponds to an alternility sum associated to a mould \(A\), containing one term for each word in the stuffle set. Let \(A_{r,s}\) denote the alternality sum associated to \(A\) corresponding to \(st(r,s)\) as in \(\S\)2.3; recall for example that \(st(1,2) = (a,b,c) + (b,a,c) + (b,c,a) + (a+b,c) + (b+a,c)\) and

\[ A_{1,2}(v_1, v_2, v_3) = A(v_1, v_2, v_3) + A(v_2, v_1, v_3) + A(v_2, v_3, v_1) + \frac{1}{(v_1 - v_2)}(A(v_1, v_3) - A(v_2, v_3)) + \frac{1}{(v_1 - v_3)}(A(v_2, v_1) - A(v_2, v_3)). \]

Assume that \(A\) is a polynomial-valued mould, i.e. \(A = m\hat{i}(f) = \text{swap}(ma(f))\) for a power series \(f \in \mathbb{Q}\) with constant term 1. We will show that \(A\) is symmetril if and only if \(f\) satisfies the stuffle relations in the sense of (1.3.3). To do this, we write

\[ A_{r}(v_1, \ldots, v_r) = \sum_{a=(a_1, \ldots, a_r)} c_{a}v_{1}^{a_1-1} \cdots v_{r}^{a_r-1}, \]

and compute the coefficient of a given monomial \(w = v_{1}^{b_1-1} \cdots v_{r+s}^{b_{r+s}-1}\) in each term of the alternility sum \(A_{r,s}\). For the shuffle-type terms in the alternility sum

\[ A(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(r+s)}) = \sum_{a=(a_1, \ldots, a_{r+s})} c_{a}v_{\sigma^{-1}(1)}^{a_1-1} \cdots v_{\sigma^{-1}(r+s)}^{a_{r+s}-1}, \]

the coefficient of \(w\) is the single coefficient \(c_{b_{\sigma^{-1}(1) \cdots b_{\sigma^{-1}(r+s)}}}\) of \(A\). But also in the case of the terms with denominators in the alternility sum, the coefficient of the monomial \(w\) is a single coefficient of \(A\). Indeed, since \(A\) is polynomial-valued, these terms simplify into polynomials whose monomials each have one coefficient from \(A\) as coefficient. We give the example of the depth 4 term corresponding to \((a+c, b+d)\):

\[
\frac{1}{(v_1 - v_3)(v_2 - v_4)}(A(v_1, v_2) - A(v_3, v_2) - A(v_1, v_4) + A(v_3, v_4))
\]

\[
= \frac{1}{(v_1 - v_3)(v_2 - v_4)} \sum_{a,b} c_{a,b}(v_{1}^{a} - v_{3}^{a})(v_{2}^{b} - v_{4}^{b})
\]

\[
= \sum_{a,b} c_{a,b}(v_{1}^{a-1} + v_{1}^{a-1}v_{3} + \cdots + v_{3}^{a-1})(v_{2}^{b-1} + v_{2}^{b-1}v_{4} + \cdots + v_{4}^{b-2});
\]
thus, the coefficient of a given monomial \( w = v_1^{b_1-1} \cdots v_4^{b_4-1} \) is equal to \( c_{b_1+b_3,b_2+b_4} \). Thus, the coefficient of a single monomial in the alternility sum \( A_{r,s} \) is exactly equal to the stuffle sum on the coefficients of the power series \( f \) such that \( A = m_i(f) \).

(iv) This assertion follows directly from the fact that if a polynomial \( f \in \text{Lie}_n[x,y] \) is such that \( f_Y \) satisfies the stuffle relations in depths \( 1 \leq r < n \), then there exists a unique term in \( y^n \), namely \( a_y = \frac{-1}{n}(f|x^{n-1}y)y^n \), such that \( f_Y + a_y \) satisfies the stuffle relations in all depths \( 1 \leq r \leq n \). (Cf. [SC, Theorem 2]).

§A.6. Proof of Proposition 4.2.3.

The proof consists in putting together a bunch of niggly lemmas, following Ecalle’s indications in [Eupolars]. Let \( I \) be the mould concentrated in depth 1 defined by \( I(u_1) = 1 \), and \( Pa \) the mould concentrated in depth 1 defined by \( Pa(u_1) = 1/u_1 \).

Lemma A.6.1. We have \( \text{dupal}(u_1) = I \), and for \( r \geq 1 \),

\[
\text{dupal}(u_1, \ldots, u_r) = \frac{B_r}{r!} lu(\cdots lu(I, Pa), \ldots, Pa, Pa). \tag{A.6.1}
\]

Proof. Let us use the notation \( lu^r(I, Pa, \ldots, Pa) \) for the bracket \( lu(\cdots lu(I, Pa), \ldots, Pa, Pa) \) where \( lu \) is iterated \( r \) times. By the definition (4.2.4) of \( \text{dupal} \), we certainly have \( \text{dupal}(u_1) = 1 \). Let us use induction on \( r \). Assume that

\[
\frac{(r-1)!}{B_{r-1}} \text{dupal}(u_1, \ldots, u_{r-1}) = lu^{r-2}(I, Pa, \ldots, Pa). \tag{A.6.2}
\]

We then have

\[
lu^{r-1}(I, Pa, \ldots, Pa)(u_1, \ldots, u_r)
= \frac{(r-1)!}{B_{r-1}} \left( \text{dupal}(u_1, \ldots, u_{r-1})Pa(u_r) - Pa(u_1)\text{dupal}(u_2, \ldots, u_r) \right)
= \frac{1}{u_1 \cdots u_r} \left( \sum_{i=0}^{r-1} (-1)^i C_{r-1}^i (u_{i+1} - u_{i+2}) \right)
= \frac{1}{u_1 \cdots u_r} \left( \sum_{i=0}^{r} (-1)^i C_{r}^i u_{i+1} \right)
= \frac{r!}{B_r} \text{dupal}(u_1, \ldots, u_r).
\]

This concludes the proof. \( \diamond \)

Since \( \text{dapal} = \text{swap}(\text{dipil}) \), it is given by

\[
\text{dapal}(u_1, \ldots, u_r) = -\frac{1}{(r+1)!} \text{swap}(re_r), \tag{A.6.3}
\]

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where we see explicitly from the definition of $re_r$ in (4.1.3) that

$$\text{swap}(re_r)(u_1, \ldots, u_r) = \frac{ru_1 + (r - 1)u_2 + \cdots + 2u_{r-1} + u_r}{u_1 \cdots u_r(u_1 + \cdots + u_r)}. \quad (A.6.4)$$

Let $mu_q(Pa) = mu(Pa, \ldots, Pa)$. The following lemma concerns the mould $\text{swap}(re_r)$.

**Lemma A.6.2.** For $r \geq 1$, the mould $\text{swap}(re_r)$ satisfies

(i) $\text{swap}(re_r) + \text{anti} \cdot \text{swap}(re_r) = (r + 1) \, mu_r(Pa)$

and

(ii) $-\text{push} \cdot \text{swap}(re_r) = \text{anti} \cdot \text{swap}(re_r)$. \quad (A.6.5)

**Proof.** (i) By (A.6.4), we have

$$\text{swap}(re_r) + \text{anti} \cdot \text{swap}(re_r) = (r + 1) \frac{1}{u_1 \cdots u_r},$$

and this is nothing other than $r + 1$ times $mu_r(Pa)$.

(ii) This is trivial; indeed the right-hand side is just

$$\frac{u_1 + 2u_2 + \cdots + ru_r}{u_1 \cdots u_r(u_1 + \cdots + u_r)}, \quad (A.6.7)$$

whereas $\text{push} \cdot \text{swap}(re_r)$ is given by

$$\frac{r(-u_1 - \cdots - u_r) + (r - 1)u_1 + \cdots + 2u_{r-2} + u_{r-1}}{u_1 \cdots u_r(u_1 + \cdots + u_r)},$$

which is nothing but the negative of (A.6.7). \hfill \Box

We need one more lemma that will help us compute the key term $\text{irat}(\text{dapal}) \cdot \text{dupal}$ of (4.2.8).

**Lemma A.6.3.** We have

$$\text{irat}(\text{swap}(re_r)) \cdot mu_q(Pa) = -(r - q + 1)mu_{r+q}(Pa) +$$

$$mu(\text{swap}(re_r), mu_q(Pa)) + mu(mu_q(Pa), \text{anti} \cdot \text{swap}(re_r)). \quad (A.6.8)$$

**Proof.** Thanks to (4.2.3), we can replace $\text{irat}$ by $\text{iwat}$ in (A.6.8), since by definition $\text{irat}(B) = \text{iwat}(B)$ whenever $B$ is a mould such that $\text{anti}(B) = -\text{push}(B)$. Using $\text{iwat}$ makes it easier to prove (A.6.8). We will do it by induction on $q$. 72
**Base case** $q = 1$. We first compute the mould $iwat(swap(re_r)) \cdot Pa$, which is concentrated in depth $r + 1$. By definition, we have $iwat(swap(re_r)) = amit(swap(re_r)) + anit(anti(swap(re_r)))$. We check directly using (2.2.1) that

\[
amit(swap(re_r)) \cdot Pa(u_1, \ldots, u_{r+1}) = \frac{1}{u_1 + \cdots + u_{r+1}} \cdot \frac{ru_1 + \cdots + 2u_{r-1} + u_r}{u_1 \cdots u_r(u_1 + \cdots + u_r)(u_1 + \cdots + u_{r+1})}. \tag{A.6.9}
\]

Similarly, we check directly from (2.2.2) that

\[
anit(anti(swap(re_r))) \cdot Pa(u_1, \ldots, u_{r+1}) = \frac{ru_{r+1} + \cdots + 2u_3 + u_2}{u_2 \cdots u_{r+1}(u_2 + \cdots + u_{r+1})(u_1 + \cdots + u_{r+1})}. \tag{A.6.10}
\]

Putting (A.6.10) and (A.6.11) together immediately yields

\[
wat(swap(re_r)) \cdot Pa(u_1, \ldots, u_{r+1}) = \frac{u_1u_2 + 2u_1u_3 + \cdots + (r - 1)u_1u_r + ru_1u_{r+1} + (r - 1)u_2u_{r+1} + \cdots + 2u_{r-1}u_{r+1} + u_r u_{r+1}}{u_1 \cdots u_r(u_1 + \cdots + u_{r-1})(u_2 + \cdots + u_r)}. \tag{A.6.11}
\]

Now, the right-hand of (A.6.8) for $q = 1$ is given by

\[
\frac{-r}{u_1 \cdots u_{r+1}} + \frac{ru_1 + \cdots + u_r}{u_1 \cdots u_{r+1}(u_1 + \cdots + u_r)} + \frac{u_2 + \cdots + ru_{r+1}}{u_1 \cdots u_{r+1}(u_2 + \cdots + u_{r+1})},
\]

and putting this over a common denominator yields exactly (A.6.11). This settles the base case.
**Induction step.** Assume that (A.6.8) holds up to $q$. We compute

\[
\text{irat}(\text{swap}(re_r)) \cdot \text{mu}_{q+1}(Pa)(u_1, \ldots, u_{r+q+1})
= \text{mu}(\text{irat}(\text{swap}(re_r)) \cdot \text{mu}_q(Pa), Pa) + \text{mu}(\text{mu}_q(Pa), \text{irat}(\text{swap}(re_r)) \cdot Pa)
= -(r - q + 1) \frac{1}{u_1 \cdots u_r + r + 1} + \text{mu}(\text{swap}(re_r), \text{mu}_q(Pa)) \frac{1}{u_r + q + 1}
+ \text{mu}(\text{mu}_q(Pa), \text{anti} \cdot \text{swap}(re_r)) \frac{1}{u_r + q + 1}
+ \frac{1}{u_1 \cdots u_q} \left( \text{irat}(\text{swap}(re_r)) \cdot Pa \right)(u_{q+1}, \ldots, u_{q+r+1})
= -(r - q + 1) \frac{1}{u_1 \cdots u_r + q + 1} + \frac{ru_1 + \cdots + u_r}{u_1 \cdots u_r + q + 1}
+ \frac{u_{q+1} + \cdots + ru_{q+r}}{u_1 \cdots u_r + q + 1} + \frac{-r}{u_1 \cdots u_r + q + 1}
+ \frac{u_{q+2} + \cdots + ru_{q+r+1}}{u_1 \cdots u_r + q + 1} + \frac{u_1 u_{q+1} + \cdots + u_{q+r+1}}{u_1 \cdots u_r + q + 1}
= -(r - q) \frac{1}{u_1 \cdots u_r + q + 1} + \frac{ru_1 + \cdots + u_r}{u_1 \cdots u_r + q + 1}
+ \frac{u_{q+2} + \cdots + ru_{q+r+1}}{u_1 \cdots u_r + q + 1} + \frac{u_1 u_{q+1} + \cdots + u_{q+r+1}}{u_1 \cdots u_r + q + 1}
= -(r - q) \text{mu}_{r+q+1}(Pa) + \text{mu}(\text{swap}(re_r), \text{mu}_{q+1}(Pa)) + \text{mu}(\text{mu}_{q+1}(Pa), \text{anti} \cdot \text{swap}(re_r))
\]

proving the induction step. This concludes the proof of Lemma A.6.3. $\diamond$

We will now compute the term $\text{irat}(\text{dupal}) \cdot \text{dupal}$ of (4.2.8). We have

\[
\text{irat}(\text{dupal}) \cdot \text{dupal} = \text{irat} \left( \sum_{r \geq 1} \frac{-1}{(r + 1)!} \text{swap}(re_r) \right) \cdot \left( \sum_{s \geq 1} \frac{B_s}{s!} \text{lu}^{s-1}(I, Pa, \ldots, Pa) \right)
= \sum_{r, s \geq 1} \frac{-1}{(r + 1)!} \frac{B_s}{s!} \text{irat}(\text{swap}(re_r)) \cdot \text{lu}^{s-1}(I, Pa, \ldots, Pa).
\]

Writing $\text{lu}^{s-1}(I, Pa, \ldots, Pa) = \sum_{i=0}^{s} (-1)^i C_{s-1}^i \text{mu}(\text{mu}_i(Pa), I, \text{mu}_{s-1-i}(Pa))$, this gives

\[
\sum_{r, s \geq 1} \sum_{i=0}^{s-1} \frac{-1}{(r + 1)!} \frac{B_s}{s!} (-1)^i C_{s-1}^i \text{irat}(\text{swap}(re_r)) \cdot \text{mu}(\text{mu}_i(Pa), I, \text{mu}_{s-1-i}(Pa))
\]

Since $\text{irat}(\text{swap}(re_r))$ is a derivation for $\text{mu}$, this is equal to
where \( E_{r,s,i} = \frac{(-1)^i}{(r+1)!} B_s \cdot (-1)^i C^i_{s-1} \). Using (A.6.8), this becomes

\[
\sum_{r,s \geq 1} \sum_{i=0}^{s-1} E_{r,s,i} \left( -(r - i + 1) \mu\left( \text{mu}_i(Pa), I, \mu_{s-1-i}(Pa) \right) \\
+ \mu\left( \text{mu}_i(Pa), \text{mu}_s(Pa), I, \mu_{s-1-i}(Pa) \right) \\
+ \mu\left( \text{mu}_i(Pa), \text{anti} \cdot \text{swap}(re_r), I, \mu_{s-1-i}(Pa) \right) \\
- \left(r - s + i + 2\right) \mu\left( \text{mu}_i(Pa), I, \mu_{r+s-1-i}(Pa) \right) \\
+ \mu\left( \text{mu}_i(Pa), I, \text{swap}(re_r), \mu_{s-1-i}(Pa) \right) \\
+ \mu\left( \text{mu}_i(Pa), I, \text{mu}_{s-1-i}(Pa), \text{anti} \cdot \text{swap}(re_r) \right) \\
+ \mu\left( \text{mu}_i(Pa), \text{irat}(\text{swap}(re_r)), I, \mu_{s-1-i}(Pa) \right) \right),
\]

(A.6.12)

Let us use the following substitution in the two terms containing \( \text{anti} \cdot \text{swap}(re_r) \):

\[
\text{anti} \cdot \text{swap}(re_r) = (r + 1) \mu_r(Pa) - \text{swap}(re_r).
\]

Then (A.6.12) becomes

\[
\sum_{r,s \geq 1} \sum_{i=0}^{s-1} E_{r,s,i} \left( -(r - i + 1) \mu\left( \text{mu}_r(Pa), I, \mu_{s-1-i}(Pa) \right) \\
+ \mu\left( \text{swap}(re_r), \text{mu}_i(Pa), I, \mu_{s-1-i}(Pa) \right) \\
+ (r + 1) \mu\left( \text{mu}_{r+i}(Pa), I, \mu_{s-1-i}(Pa) \right) \\
- \mu\left( \text{mu}_i(Pa), \text{swap}(re_r), I, \mu_{s-1-i}(Pa) \right) \\
- \left(r - s + i + 2\right) \mu\left( \text{mu}_i(Pa), I, \mu_{r+s-1-i}(Pa) \right) \\
+ \mu\left( \text{mu}_i(Pa), I, \text{swap}(re_r), \mu_{s-1-i}(Pa) \right) \\
+ (r + 1) \mu\left( \text{mu}_i(Pa), I, \mu_{r+s-1-i}(Pa) \right) \\
- \mu\left( \text{mu}_i(Pa), I, \mu_{s-1-i}(Pa), \text{swap}(re_r) \right) \\
+ \mu\left( \text{mu}_i(Pa), \text{irat}(\text{swap}(re_r)), I, \mu_{s-1-i}(Pa) \right) \right).
\]

(A.6.13)
Putting like terms together, this becomes

\[ \sum_{r,s \geq 1} \sum_{i=0}^{s-1} E_{r,s,i} \left( i \mu(u_{r+i}(Pa), I, u_{s-1-i}(Pa)) \right) \]

\[ + (s - i - 1) \mu(u_i(Pa), I, u_{r+s-1-i}(Pa)) \]

\[ + \mu(swap(re), u_i(Pa), I, u_{s-1-i}(Pa)) \]

\[ - \mu(u_i(Pa), swap(re), I, u_{s-1-i}(Pa)) \]

\[ + \mu(u_i(Pa), I, swap(re), u_{s-1-i}(Pa)) \]

\[ - \mu(u_i(Pa), I, u_{s-1-i}(Pa), swap(re)) \]

\[ + \mu(u_i(Pa), irat(swap(re)) \cdot I, u_{s-1-i}(Pa)) \].

We will compare (A.6.14)=iram (dapal) · dupal with the other crucial term lu (dapal, dupal) from (4.2.8). We have

\[ lu(dapal, dupal) = \mu(dapal, dupal) - \mu(dupal, dapal) \]

\[ = \sum_{r,s \geq 1} \left( \frac{-1}{(r+1)!} \frac{B_s}{s!} \mu(swap(re), lu^{s-1}(I, Pa, \ldots), Pa) \right) \]

\[ - \frac{-1}{(r+1)!} \frac{B_s}{s!} \mu(lu^{s-1}(I, Pa, \ldots), swap(re)) \) \]

\[ = \sum_{r,s \geq 1} \sum_{i=0}^{s-1} E_{r,s,i} \left( \mu(swap(re), u_i(Pa), I, u_{s-1-i}(Pa)) \right) \]

\[ - \mu(u_i(Pa), I, u_{s-1-i}(Pa), swap(re)) \). \]

Let us rewrite (4.2.8) as

\[ iram (dapal) \cdot dupal - lu(dapal, dupal) = der \cdot dupal - dur \cdot dapal. \]

We note that this equality holds in depth \( d = 1 \) since the depth 1 part of the left-hand side is zero, and the depth one part of \( der \cdot dupal \) is equal to that of \( dur \cdot dapal \), namely \(-1/2\). Thus from now on we work in depth \( d > 1 \).

The left-hand side is (A.6.14) - (A.6.15), which we compute as

\[ \sum_{r,s \geq 1} \sum_{i=0}^{s-1} E_{r,s,i} \left( i \mu(u_{r+i}(Pa), I, u_{s-1-i}(Pa)) \right) \]

\[ + (s - i - 1) \mu(u_i(Pa), I, u_{r+s-1-i}(Pa)) \]

\[ - \mu(u_i(Pa), swap(re), I, u_{s-1-i}(Pa)) \]

\[ + \mu(u_i(Pa), I, swap(re), u_{s-1-i}(Pa)) \]

\[ + \mu(u_i(Pa), irat(swap(re)) \cdot I, u_{s-1-i}(Pa)) \). \]
Setting \( d = r + s \) and \( ru_r = swap(re_r) \), we rewrite the sum as

\[
\sum_{d \geq 1} \sum_{s=1}^{d-1} \sum_{i=0}^{s-1} E_{d-s,s,i} \left( imu(mu_{d-s+i}(Pa), I, mu_{s-1-i}(Pa))
+ (s - i - 1) mu(mu_i(Pa), I, mu_{d-1-i}(Pa))
- mu(mu_i(Pa), ru_{d-s}, I, mu_{s-1-i}(Pa))
+ mu(mu_i(Pa), I, ru_{d-s}, mu_{s-1-i}(Pa))
+ mu(mu_i(Pa), irmu(riu_{d-s}) \cdot I, mu_{s-1-i}(Pa)) \right),
\]

which is useful because \( d \) gives the depth of the mould. Let us consider the first two lines of (A.6.18), whose simple expressions are easy to compute directly. For given indices \( d, s, i \), we have

\[
i mu(mu_{d-s+i}(Pa), I, mu_{s-1-i}(Pa)) + (s - i - 1) mu(mu_i(Pa), I, mu_{d-1-i}(Pa))
= \frac{(s - i - 1)u_{i+1} + iu_{d-s+i+1}}{u_1 \cdots u_d}.
\]  

(A.6.19)

The next three lines taken together are even simpler, since for given \( d, s, i \) we have

\[
- mu(mu_i(Pa), ru_{d-s}, I, mu_{s-1-i}(Pa)) + mu(mu_i(Pa), I, ru_{d-s}, mu_{s-1-i}(Pa))
+ mu(mu_i(Pa), irmu(riu_{d-s}) \cdot I, mu_{s-1-i}(Pa)) = \frac{(d - s + 1)u_{i+1}}{u_1 \cdots u_d}.
\]  

(A.6.20)

Using (A.6.19) and (A.6.20) we see that in given depth \( d \), (A.6.18) is equal to

\[
\sum_{s=1}^{d-1} \sum_{i=0}^{s-1} E_{d-s,s,i} \frac{(d - i)u_{i+1} + iu_{d-s+i+1}}{u_1 \cdots u_d}
= \frac{1}{u_1 \cdots u_d} \sum_{s=1}^{d-1} \sum_{i=0}^{s-1} (-1)^{i+1} \frac{1}{(d - s + 1)!} \frac{B_s}{s!} \binom{s - 1}{i} ((d - i)u_{i+1} + iu_{d-s+i+1})
= \frac{1}{u_1 \cdots u_d} \sum_{s=1}^{d-1} \sum_{i=0}^{s-1} (-1)^{i+1} \frac{1}{(d - s + 1)!} \frac{B_s}{s!} \binom{s - 1}{i} ((d - i)u_{i+1} + iu_{d-s+i+1})
= \frac{1}{u_1 \cdots u_d} \sum_{s=1}^{d-1} \sum_{i=0}^{s-1} (-1)^{i+1} \frac{1}{(d - s + 1)!} \frac{B_s}{s!} \binom{s - 1}{i} ((d - i)u_{i+1} + iu_{d-s+i+1})
= \frac{1}{u_1 \cdots u_d} \sum_{s=1}^{d-1} \sum_{i=0}^{s-1} (-1)^{i+1} \frac{1}{(d - s + 1)!} \frac{B_s}{s!} \binom{s - 1}{i} ((d - i)u_{i+1} + iu_{d-s+i+1})
\]
The coefficient of a given \( u_j \) for \( j \in \{1, \ldots, d\} \) in the linear factor is thus given by

\[
\sum_{s=j}^{d-1} (-1)^s \frac{B_s}{(d-s+1)!} \frac{d-j+1}{s(j-1)!(s-j)!} + \sum_{s=d-j+2}^{d-1} (-1)^{j+s-d} \frac{B_s}{(d-s+1)!} \frac{1}{s(j-d+s-2)!(d-j)!}
\]

(A.6.21)

Let us compare this with the depth \( d \) part of \( \text{der}(\text{dupal}) - \text{dur}(\text{dapal}) \), which is explicitly given by

\[
\frac{1}{u_1 \cdots u_d} \left( \frac{B_d}{(d-1)!} \left( \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} u_{i+1} \right) + \frac{1}{(d+1)!} (du_1 + (d-1)u_2 + \cdots + 2u_{d-1} + u_d) \right).
\]

In particular the coefficient of \( u_j \) in the linear factor for \( j \in \{1, \ldots, d\} \) is given by

\[
\frac{B_d}{(d-1)!} (-1)^{j-1} \binom{d-1}{j-1} + \frac{d-j+1}{(d+1)!}.
\]

(A.6.22)

Let us show that (A.6.21)=(A.6.22). Recall from the remark after (A.6.16) that we may assume that \( d > 1 \). We first assume that \( d \) is odd, so (A.6.22) reduces to \((d-j+1)/(d+1)!\). The equality with (A.6.22) can be simplified (thanks to H. Gangl) to the equality

\[
\sum_{n=0}^{d} \binom{d+1}{n} B_n \left( \binom{n-1}{n-d} + (-1)^{n-1} \binom{n-1}{n-d+j} \right) = (-1)^j
\]

(A.6.23)

for odd \( d > 1 \) and \( 1 \leq j \leq d \). The remarkable, elegant proof of (A.6.23) was provided to us by D. Zagier.

\[
(n, k) = (-1)^k (-n + k - 1, k) \text{ if } k \text{ nonnegative}
\]

\[
(n, k) = (-1)^k (-n - k, -k - 1, n - k) \text{ if } k \leq n
\]

\[
(n, k) = 0 \text{ otherwise}
\]

Interlude: \( d = 3, j = 1 \)

\[
n = 0: \binom{-1}{3} - \binom{-1}{\frac{3}{3}} = -(3,3) - (2,2) = -2
\]

\[
n = 1: \binom{0}{3} + \binom{0}{\frac{3}{3}} = 0
\]

\[
n = 2: \binom{1}{1} - \binom{1}{0} = 0
\]

\[
n = 3: 0
\]

Total 0

Interlude: \( d = 5, j = 2 \)

\[
n = 0: \binom{-1}{5} - \binom{-1}{\frac{5}{3}} = 2
\]

\[
n = 1: \binom{0}{5} + \binom{0}{\frac{5}{3}} = 0
\]

\[
n = 2: (15)(1/6) \cdot \left( \binom{5}{1} - \binom{5}{0} \right) = 5
\]

\[
n = 3: 0
\]

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\( n = 4: \) 
\((15)(-1/30)((3)/(2)) - ((3)/(1))) = 0 \)

\( n = 5: 0 \)

Total 4+4=8

Let \( f_{d,j} = \sum_{n=0}^{d} \left( \frac{d+1}{n} \right) B_{n} \left( \frac{n-1}{j} \right) \) for \( 0 \leq j \leq d \). Then the desired expression follows from the following more general equality, valid for all \( d \geq 0 \):

\[
f_{d,j} + (-1)^{n-1} f_{d,d-j} = (-1)^{j} + \delta_{d,0}. \tag{A.6.24}
\]

\[
F(x, y) = \sum_{d \geq j \geq 0} \frac{1}{(d+1)!} f_{d,j} x^{d-j} y^{j}.
\]

**Claim.** We have

\[
F(x, y) + F(-y, -x) = 1 + \sum_{i,j \geq 0} (-1)^{j} \frac{1}{(i+j+1)!} x^{i} y^{j}, \tag{A.6.25}
\]

and thus in particular

\[
f_{d,j} + (-1)^{d} f_{d,d-j} = \delta_{d,0} + (-1)^{j}.
\]

**Proof.** We have

\[
F(x, y) = \sum_{d=0}^{\infty} \frac{x^{d}}{(d+1)!} \left( \sum_{n=0}^{d} \left( \frac{d+1}{n} \right) B_{n} \left( 1 + \frac{y}{x} \right)^{n-1} - \left( \frac{-y}{x} \right) \left( 1 + \frac{y}{x} \right)^{-1} \right)
\]

\[
= \sum_{n \geq 0, r \geq 1} \frac{1}{n! r!} B_{n} (x+y)^{n-1} x^{r} + \frac{1}{x+y} (1 - e^{-y})
\]

\[
= \frac{e^{x} - 1}{e^{x+y} - 1} + \frac{1 - e^{-y}}{x+y}.
\]

This expression makes the calculation of \( F(x, y) + F(-y, -x) \) trivial and proves (A.6.24). \hspace{1cm} \diamond
The main references are of course ALL WORKS BY ECALLE CONCERNING ARI/GARI, cf. Écalle’s web page.

We also have refereed to a few other articles, some of which contain the only written proofs of some of Écalle’s basic statements.

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