Antimagic Labeling for Unions of Graphs with Many Three-Paths

Angel Chavez†  Parker Le‡  Derek Lin§  Daphne Der-Fen Liu¶  Mason Shurman‖
November 28, 2022

Abstract

Let $G$ be a graph with $m$ edges and let $f$ be a bijection from $E(G)$ to $\{1, 2, \ldots, m\}$. For any vertex $v$, denote by $\phi_f(v)$ the sum of $f(e)$ over all edges $e$ incident to $v$. If $\phi_f(v) \neq \phi_f(u)$ holds for any two distinct vertices $u$ and $v$, then $f$ is called an antimagic labeling of $G$. We call $G$ antimagic if such a labeling exists. Hartsfield and Ringel [9] conjectured that all connected graphs except $P_2$ are antimagic. Denote the disjoint union of graphs $G$ and $H$ by $G \cup H$, and the disjoint union of $t$ copies of $G$ by $tG$. For an antimagic graph $G$ (connected or disconnected), we define the parameter $\tau(G)$ to be the maximum integer such that $G \cup tP_3$ is antimagic for all $t \leq \tau(G)$. Chang, Chen, Li, and Pan showed that for all antimagic graphs $G$, $\tau(G)$ is finite [3]. Further, Shang, Lin, Liaw [17] and Li [13] found the exact value of $\tau(G)$ for special families of graphs: star forests and balanced double stars, respectively. They did this by finding explicit antimagic labelings of $G \cup tP_3$ and proving a tight upper bound on $\tau(G)$ for these special families. In the present paper, we generalize their results by proving an upper bound on $\tau(G)$ for all graphs. For star forests and balanced double stars, this general bound is equivalent to the bounds given in [17] and [13] and tight. In addition, we prove that the general bound is also tight for every other graph we have studied, including an infinite family of jellyfish graphs, cycles $C_n$ with $3 \leq n \leq 9$, and the double triangle $2C_3$.

*Partially supported by the National Science Foundation grant DMS 1600778.
†California State University Los Angeles. Current address: University of Minnesota Twin Cities. Email: chave389@umn.edu.
‡California State University Los Angeles. Email: plea31@calstatela.edu.
§California State University Los Angeles. Email: dlin4@calstatela.edu
¶Corresponding author. California State University Los Angeles. Email: dliu@calstatela.edu.
‖Current Address: University of California, Irvine. Email: mshurman@uci.edu.
1 Introduction

The graphs considered in this article are not necessarily connected, unless otherwise indicated. Let $G$ be a graph with $m$ edges. For a bijection $f : E(G) \rightarrow \{1,2,\ldots,m\}$ and for any vertex $v$, denote by $\phi_f(v)$ the sum of $f(e)$ over all edges $e$ incident to $v$. We call $f$ an antimagic labeling of $G$ if for any pair of vertices $u$ and $v$, $\phi_f(u) \neq \phi_f(v)$. A graph is antimagic if it admits an antimagic labeling. When $f$ is clear in context, we shorten $\phi_f(v)$ to $\phi(v)$ and call it the $\phi$-value of $v$.

Antimagic labeling was introduced by Hartsfield and Ringel [9], in which the following conjecture was posed:

**Conjecture 1.** [9] Every connected graph except $P_2$ is antimagic.

Conjecture 1 has received much attention in the past years (cf. [1, 7, 11]), and many families of graphs are known to be antimagic. Alon, Kaplan, Lev, Roditty, and Yuster [1] proved that dense graphs are antimagic. Precisely, the authors showed that graphs of order $n$ with minimum degree $\delta(G) \geq c \log n$ for some constant $c$ or with maximum degree $\Delta(G) \geq n - 2$ are antimagic. Other families of graphs known to be antimagic include regular graphs [2, 5, 6], trees with at most one vertex of degree two and their subdivisions [11, 12], caterpillars [7, 15, 16], spiders [18], and double spiders [4].

While Conjecture 1 has been studied extensively, antimagic labelings for disconnected graphs have received less attention. It is known that there exist nontrivial disconnected non-antimagic graphs. For instance, it is easy to see that the union of two copies of $P_3$ is not antimagic (Figure 1).

![Figure 1: The graph 2P3 is not antimagic. The graphs above exhaust all possible labelings, and prove that none are antimagic. Circled numbers are \(\phi\)-values and uncircled numbers are edge labels. Twice circled \(\phi\)-values are identical.](image)

Further, it is clear that a graph containing $P_2$ as a component is not antimagic: both vertices adjacent to the isolated edge will have the same $\phi$-value. Similarly, a graph containing two isolated vertices as components is not antimagic: both vertices will have a $\phi$-value of 0. A graph containing exactly one isolated vertex as a component will be antimagic if and only if the graph induced by deleting that vertex is antimagic. Therefore, this paper will focus on graphs that do not contain isolated edges ($P_2$) or isolated vertices as components.

Throughout the paper, we denote the union of disjoint graphs $G$ and $H$ by $G \cup H$, and the union of $t$ copies of $G$ by $tG$. Chang, Chen, Li, and Pan (Theorem 3.5, [3]) showed that for any graph $G$, $G \cup tP_3$ is not antimagic for a sufficiently large $t$. It is natural to consider the following parameter for a graph $G$:
Definition 1. If $G$ is antimagic, define $\tau(G)$ as the maximum non-negative integer $t$ such that $G \cup tP_3$ is antimagic for all $0 \leq t' \leq t$. If $G$ is not antimagic, define $\tau(G) = -\infty$.

The aim of this article is to investigate

Question 1. What is $\tau(G)$ for a graph $G$?

The main result of this paper is the following general upper bound of $\tau(G)$. An edge is internal if both of its ends are non-leaf vertices; otherwise it is a pendant edge.

Theorem 1. Let $G$ be a graph with $n$ vertices and $m$ edges including $k$ internal edges. Assume $G$ does not contain isolated vertices nor $P_2$ as a component. Let $t'$ be the number of components of $G$ isomorphic to $P_3$. If $G \cup tP_3$ is antimagic, then

$$t \leq \min \left\{ (3 + 2\sqrt{2})(m - n) + (1 + \sqrt{2})(m + \frac{1}{2}), \ 2m + 5(k - t') + 1 \right\}.$$

Denote the floor of the right-side of the inequality in Theorem 1 by $\beta(G)$:

$$\beta(G) := \left\lfloor \min \left\{ (3 + 2\sqrt{2})(m - n) + (1 + \sqrt{2})(m + \frac{1}{2}), \ 2m + 5(k - t') + 1 \right\} \right\rfloor.$$

Corollary 2. Let $G$ be a graph with $n$ vertices and $m$ edges including $k$ internal edges. Assume $G$ does not contain isolated vertices nor $P_2$ as a component. Let $t'$ be the number of components of $G$ isomorphic to $P_3$. Then $\tau(G) \leq \beta(G)$. Moreover, if $\tau(G) = \beta(G)$, then the converse also holds. That is, $G \cup tP_3$ is antimagic if and only if $t \leq \tau(G)$.

The proof of Theorem 1 is presented in Section 2. As mentioned above, a result in [3] implies that $\tau(G) \leq 8m + 1$, where $m = |E(G)|$. Theorem 1 implies $\tau(G) \leq 7m + 1$.

The bound of Theorem 1 is sharp for many graphs. An $n$-star $S_n$, $n \geq 3$, is a tree with a center vertex $v$ and $n$ pendant edges incident to $v$. A star forest is a forest whose components are stars. All edges in a star forest are pendant edges, so $k = 0$ in Theorem 1. A double star $S_{a,b}$ is a tree created by adding an edge between the centers of stars $S_a$ and $S_b$. That is, $S_{a,b}$ has $a + b + 1$ edges where only one is internal ($k = 1$). By Theorem 1, we obtain the following:

Corollary 3. Let $F$ be a non-trivial forest with $m$ edges, $q$ components, $t'$ components isomorphic to $P_3$, and $k$ internal edges, where each component has at least two edges. Then

$$\tau(F) \leq \min \left\{ -q(3 + 2\sqrt{2}) + (1 + \sqrt{2})(m + \frac{1}{2}), \ 2m + 5(k - t') + 1 \right\}.$$

Corollary 4. If $G$ is a star forest with $m$ edges, $q$ components and each component has at least three edges, then

$$\tau(G) \leq \min \{ -q(3 + 2\sqrt{2}) + (1 + \sqrt{2})(m + \frac{1}{2}), \ 2m + 1 \}.$$
Corollary 5. The double star $S_{a,b}$, $a, b \geq 1$, has
\[
\tau(S_{a,b}) \leq \min\{-3(2\sqrt{2}) + (1 + \sqrt{2})(a + b + \frac{3}{2}), 2(a + b + 4)\}.
\]
Shang, Lin, and Liaw [17] and Li [13] proved bounds for $\tau(G)$ when $G$ is a star forest and when $G$ is a double star, respectively. Their bounds for those graphs coincide with Corollary 4 and Corollary 5 respectively, but with different expressions.

By providing desired valid antimagic labelings, the authors of [17] and [13] also showed the reverse direction of this inequality for star forests and balanced double stars, giving us:

Theorem 6. [17] A star forest $G$ where each component has at least three edges has $\beta(G) = \tau(G)$.

Theorem 7. [13] For any $a \geq 2$, $\beta(S_{a,a}) = \tau(S_{a,a})$.

In [13, 17], the authors studied antimagic labelings of $G \cup tP_3$ for special trees and forests which contain no cycles. In Section 3 and Section 4, we explore families of graphs that do contain cycles: jellyfish graphs and 2-regular graphs.

For positive integers $r$ and $k$ with $k \geq 3$, the jellyfish graph $J(C_k, r)$ is obtained by taking a cycle $C_k$ and adding $r$ pendant edges (with leaves) to each vertex on $C_k$. By Theorem 1, we have:

Corollary 8. If $J(C_k, r) \cup tP_3$ is antimagic, then
\[
t \leq \min\left\{\left(1 + \sqrt{2}\right) \left(kr + k + \frac{1}{2}\right), 2kr + 7k + 1\right\}.
\]
Consequently, if $r \geq 11$, then $\tau(J(C_3, r)) \leq 6r + 22$.

We prove in Section 3 that the bound of Theorem 1 is sharp for infinitely many jellyfish graphs:

Theorem 9. For $r \geq 11$, $J(C_3, r) \cup tP_3$ is antimagic if and only if $t \leq 6r + 22$. Equivalently, $\beta(J(C_3, r)) = \tau(J(C_3, r))$, provided $r \geq 11$.

In Section 3 and the works discussed above, it seems simpler to find antimagic labelings for $G \cup tP_3$ for all $t \leq \tau(G)$ when the second bound in Theorem 1 is smaller. This occurs when $G$ has many leaves and few internal edges. On the other hand, it is more difficult to find such labelings when the first bound is smaller: so far to our knowledge these labelings have only been found for finitely many graphs.

Because $\tau(G)$ has not been well studied in graphs where the first bound in Theorem 1 is smaller, in Section 4, we investigate 2-regular graphs, where because $m = n = k$, the first bound is always smaller. We start by proving some general recursive properties for antimagic labelings of $G \cup tP_3$ for any graph $G$. These properties prove especially useful when $G$ is 2-regular. With these results, we show that the bound in Theorem 1 is sharp for $2C_3$ and $C_n$, $3 \leq n \leq 9$, and we conjecture this holds for all $C_n$. In Section 5, we ask whether the bound on $\tau(G)$ in Theorem 1 is tight and raise other questions for future research.
2 Proof of Theorem 1

Before we introduce Lemma 10, the main lemma in the proof of Theorem 1, we establish necessary notations. Let $G$ be a graph with $n$ vertices and $m$ edges. Let $t$ be a non-negative integer. Suppose $f$ is an antimagic labeling for $G' = G \cup tP_3$. Then $G'$ has $m' = m + 2t$ edges. Denote the centers (degree-2 vertices) of the 3-paths by $\{w_1, w_2, \ldots, w_t\}$.

Denote the sum of all the labels we can use, $[1, m' = m + 2t]$, by:

$$s(G, t) := \sum_{e \in E(G)} f(e) + \sum_{i=1}^{t} \phi_f(w_i) = \sum_{i=1}^{m+2t} i = 2t^2 + (2m + 1)t + \frac{m(m+1)}{2}. \quad (1)$$

Denote $V^*(G') = V(G) \cup \{w_1, w_2, \ldots, w_t\}$. For every $u \in V^*(G')$, $\phi_f(u) \leq m + 2t$ if and only if $\phi_f(u) = f(e)$ for some $e \in E(G)$. Because $|V^*(G')| = n + t$ and $|E(G)| = m$, it must be that at least $t + n - m$ vertices $u \in V^*(G')$ have $\phi_f(u) \geq m + 2t + 1$. Denote the least total vertex sums for these vertices by:

$$l(G, t) := \sum_{i=1}^{t+n-m} (m + 2t + i) = \frac{(t + n - m)(5t + n + m + 1)}{2}. \quad (2)$$

When $G$ and $t$ are clear in the context, we simply denote $s(G, t)$ and $l(G, t)$ by $s$ and $l$, respectively.

Lemma 10. Let $G$ be a graph with $n$ vertices and $m$ edges. If $G \cup tP_3$ is antimagic for some non-negative integer $t$, then $s \geq 1$.

Proof. Let $f$ be an antimagic labeling of $G' = G \cup tP_3$. Then $G'$ has $m' = m + 2t$ edges. For each vertex $v \in V(G)$, either $\phi(v) \geq m' + 1$ or $\phi(v) = f(e)$ for some $e \in E(G)$.

When $t \geq 1$, denote the degree-2 vertices of the $t$ 3-paths by $w_i$, $1 \leq i \leq t$, so that $\phi(w_1) < \phi(w_2) < \phi(w_3) < \ldots < \phi(w_t)$. Define the following:

$$B := \{v \in V(G) \mid \phi(v) \geq m' + 1\},$$
$$B' := \{v \in V^*(G') \mid \phi(v) \geq m' + 1\},$$
$$S := \{v \in V(G) \mid \phi(v) \leq m'\},$$
$$W := \{e \in E(G) \mid f(e) = \phi(w_i) \text{ for some } w_i\},$$
$$E(S) := \{e \in E(G) \mid f(e) = \phi(v) \text{ for some } v \in S\},$$
$$R := E(G) \setminus (W \cup E(S)).$$

If $t = 0$, then $W = \emptyset$ and $B = B'$. Observe that $W \cup E(S) \cup R$ is a partition of $E(G)$. Hence $m = |W| + |E(S)| + |R|$. Note that $n = |B| + |S|$ and $|S| = |E(S)|$. Therefore, we obtain $|B| = n - m + |W| + |R|$ and

$$\sum_{v \in B} \phi(v) + \sum_{v \in S} \phi(v) = 2 \sum_{e \in E(G)} f(e) = \sum_{e \in E(G)} f(e) + \sum_{e \in E(S)} f(e) + \sum_{e \in W} f(e) + \sum_{e \in R} f(e) = \sum_{e \in E(G)} f(e) + \sum_{v \in S} \phi(v) + \sum_{e \in W} f(e) + \sum_{e \in R} f(e).$$
Simplifying the above, we obtain
\[ \sum_{e \in E(G)} f(e) = \sum_{v \in B} \phi(v) - \sum_{e \in W} f(e) - \sum_{e \in R} f(e). \]

Note that \(|B'| = |B| + t - |W| = n - m + |W| + |R| + t - |W| = n - m + t + |R|.

Substituting the above into Eq. (1) and by Eq. (2) we get:
\[ s = \sum_{v \in B} \phi(v) + \sum_{i=|W|+1}^{t} \phi(w_i) - \sum_{e \in R} f(e) \]
\[ = \sum_{v \in B'} \phi(v) - \sum_{e \in R} f(e) \]
\[ \geq \sum_{i=1}^{t+m+n-t-|R|} (m' + i) - \sum_{e \in R} f(e) \]
\[ \geq \sum_{i=1}^{t+n-m} (m' + i) + |R|(m' + 1) - |R|m' \]
\[ \geq l. \] (3)

Hence, the proof is complete. ■

**Proof of the First Bound in Theorem 1** By Lemma 10, if \( G \cup tP_3 \) is antimagic we have \( s(G, t) \geq l(G, t) \), which is a quadratic inequality in \( t \). Solving this inequality, we find that:
\[ t \leq \left(4m - 3n + \frac{1}{2}\right) + \sqrt{(3\sqrt{2}m - 2\sqrt{2}n + \frac{1}{\sqrt{2}})^2 - \frac{1}{4}} \]
\[ = \left(4m - 3n + \frac{1}{2}\right) + \left(3\sqrt{2}m - 2\sqrt{2}n + \frac{1}{\sqrt{2}}\right) + O\left(\frac{1}{n + m}\right). \]

**Proposition 11.** For any integers \( m \) and \( n \), we have
\[ \left\lfloor \sqrt{(3\sqrt{2}m - 2\sqrt{2}n + \frac{1}{\sqrt{2}})^2 - \frac{1}{4}} \right\rfloor = \left\lfloor 3\sqrt{2}m - 2\sqrt{2}n + \frac{1}{\sqrt{2}} \right\rfloor. \]

*Proof.* Suppose to the contrary there exists an integer \( x \) such that
\[ \sqrt{(3\sqrt{2}m - 2\sqrt{2}n + \frac{1}{\sqrt{2}})^2 - \frac{1}{4}} < x \leq 3\sqrt{2}m - 2\sqrt{2}n + \frac{1}{\sqrt{2}}. \]

Square both sides and simplify to yield \( \frac{1}{4} < x^2 - (18m^2 + 8n^2 - 24mn - 4n + 6m) \leq \frac{1}{2} \), which is impossible since \( m \) and \( n \) are integers. This completes the proof of Proposition 11. ■

By Proposition 11, \( t \leq (3 + 2\sqrt{2})(m - n) + (1 + \sqrt{2})(m + \frac{1}{2}) \), completing the proof of the first bound in Theorem 1.

6
Theorem 1. We start with the following lemma:

Proof of the Second Bound in Theorem 1. Next we prove the second bound, \( \tau(G) \leq 2m + 5(k-t')+1 \). Suppose \( G' = G \cup tP_3 \) is antimagic. If \( t + t' \leq k \), then \( t \leq k - t' \leq 2m + 5(k-t')+1 \). Assume \( t + t' > k \). Let \( f \) be an antimagic labeling for \( G' \). Since \( G \) contains \( k \) internal edges, at least \( (t + t' - k) \) 3-paths must have \( \phi(w_i) \geq m'+1 = m + 2t + 1 \). Denote by \( y \) the sum of \( \phi(w_i) \) for these \( (t + t' - k) \) paths. Then we can bound \( y \) below by using the fact that every \( \phi(w_i) \) is distinct, and we can bound \( y \) above by using the fact that all the edges are given different labels:

\[
\sum_{i=1}^{t+t'-k} (m' + i) \leq y \leq \sum_{i=1}^{2(t+t'-k)} (m' + 1 - i).
\]

As \( m' = m + 2t \), using direct calculation and solving the above inequalities, we obtain \( t \leq 2m + 5(k-t')+1 \). This completes the proof of Theorem 1. \( \square \)

Our next result shows that \( \beta(G) \geq 0 \) if every component of \( G \) has at least 3 edges.

Proposition 12. Let \( G \) be a graph without isolated vertices and having no \( P_2 \) as a component. If \( \beta(G) < 0 \), then \( G \) contains at least one \( P_3 \) as a component and \( G \) is not antimagic.

Proof. We first prove that \( \beta(G) \geq 0 \) if \( G \) does not contain \( P_3 \) as a component. Assume \( G \) does not have \( P_3 \) as a component. Then every component of \( G \) contains at least 3 edges. If \( G \) has \( q \) components, then \( m - n \geq -q \), and \( m \geq 3q \). Therefore the first bound of \( \beta(G) \) is non-negative. As the second bound of \( \beta(G) \) is always positive, \( \beta(G) \geq 0 \). If \( G \) is antimagic, then \( \beta(G) \geq \tau(G) \geq 0 \). Hence, the second conclusion holds. \( \square \)

In general, there is no simple way to determine which of the two bounds in Theorem 1 is better (smaller), but we can make some estimates. Suppose \( m \to \infty \). Without loss of generality, suppose the graph does not contain \( P_3 \) as a component. Because \( k \leq m \), the second bound is smaller if \( \frac{k}{m} < 3(5\sqrt{2} - 7) \approx 0.21 \). Furthermore, if \( k \approx m \) then the second bound is better if and only if \( \frac{k}{m} < 3(5\sqrt{2} - 7) \approx 0.21 \). This ratio is derived by substituting \( m \) for \( k \) in the second bound and solving. Note that this if and only if statement does not hold without the assumption that \( k \approx m \). For example, in star forests, where \( k = 0 \neq m \), even though \( \frac{k}{m} \approx 1 > 3(5\sqrt{2} - 7) \), the second bound is smaller for sufficiently large \( m \). For the cases when \( n \approx m \), such as in trees, the second bound is smaller if and only if roughly \( \frac{k}{m} < \frac{\sqrt{2} - 1}{5} \approx 0.08 \). This ratio is derived by substituting \( n \) for \( m \) in the first bound. Note that in 2-regular graphs, \( \frac{k}{m} = 1 \), so the first bound is always smaller.

3 Proof of Theorem 9

Recall a jellyfish graph \( J(C_k, r) \) is established by attaching \( r \) pendant edges to every vertex of a \( k \)-cycle \( C_k \). Throughout this section we denote the \( k \) internal vertices on the jellyfish by \( v_i, i \in [1, k] \), and the \( t \) internal vertices on the 3-paths in the graph \( J(C_k, r) \cup tP_3 \) by \( w_i, i \in [1, t] \).

To prove Theorem 9, we start with the following lemma:
Lemma 13. Let $k, n$ be positive integers, where $n$ and $k$ are not both even. Let $a_1, a_2, \ldots, a_n$ be integers, and define sets of consecutive integers by $A_i = [a_i, a_i + k - 1]$, $i \in [1, n]$. Let $S$ be the multi-set-union $\cup_{i=1}^n A_i$ where repetitions are allowed (if the $A_i$'s are disjoint, then $S = \cup_{i=1}^n A_i$). Then $S$ can be partitioned into multi-sets $S_1, S_2, \ldots, S_k$ so that each $S_j$ contains exactly one element from each $A_i$, $i \in [1, n]$ and the set $\{\sum S_j : j \in [1, k]\}$ consists of $k$ consecutive integers, where $\sum S_j$ is the sum of elements in $S_j$. Formally,

$$\sum S_j = \sum_{x \in S_j} x = a_1 + j - 1 + \frac{1}{k} \left( \sum_{y \in S \setminus A_1} y \right).$$

Proof. We write $S$ as an $n \times k$ matrix $M$, where the $i^{th}$-row are numbers from $A_i = [a_i, a_i + k - 1]$. For each odd row $M_{2i+1}$ we write the elements from $A_{2i+1}$ in increasing order while for each even row $M_{2i}$ we write the elements from $A_{2i}$ in decreasing order. Observe that the column sums of the sub-matrix formed by any two consecutive rows $M_i$ and $M_{i+1}$ are identical. Explicitly, this means for any column index $j$, $M_{ij} + M_{i+1,j} = a_i + a_{i+1} + k - 1$.

Thus, if $n$ is odd, the column sums of $M$ form a set of consecutive $k$ integers, and the proof is complete by letting $S_j$ be the elements in the $j^{th}$ column, $j \in [1, k]$.

Now assume $n$ is even. By our assumption, $k$ must be odd. We re-arrange the numbers in the second row $M_2$ to be:

$$M_2' = (a_2 + \frac{k+1}{2}, a_2 + \frac{k+3}{2}, \ldots, a_2 + k - 1, a_2, a_2 + 1, \ldots, a_2 + \frac{k-1}{2}).$$

The column sums of the sub-matrix formed by $M_1$ and $M_2'$ are a set of consecutive $k$ integers, $[a_1 + a_2 + \frac{k+1}{2}, a_1 + a_2 + k - 1 + \frac{k-1}{2}]$.

By the above discussion, the column sums of the remaining $n - 2$ rows (if $n \geq 4$) are identical. Hence, the column sums of $M'$ (where $M_2$ is replaced by $M_2'$) form a set of consecutive $k$ integers. The proof is complete by letting $S_j$ be the elements in the $j^{th}$ column of $M'$, $j \in [1, k]$.

Proof of Theorem 9) By Corollary 8 it suffices to show that $G' = J(C_3, r) \cup tP_3$ is antimagic for $t \leq 6r + 22$. Note $|E(G')| = m' = 3r + 2t + 3$.

Assume $t \leq 2$. Label the internal edges on the jellyfish with $m'$, $m' - 1$, $m' - 2$, so that the internal vertices on the jellyfish have partial sums $[2m' - 3, 2m' - 1]$. If $t = 1$ or $t = 2$, label a 3-path with $m' - 3$ and $m' - 4$. If $t = 2$, label the remaining 3-path with $m' - 5$ and $m' - 6$. We then proceed to label the remaining edges of the jellyfish by applying Lemma 13 to the collection $[2m' - 3, 2m' - 1] \cup [1, 3r] = [2m' - 3, 2m' - 1] \cup [1, 3] \cup [4, 6] \cup \cdots \cup [3r - 2, 3r]$ to obtain three sets with distinct sums. Assign the numbers in the set containing $2m' - 3$ to the edges incident to the vertex on the cycle with a partial sum $2m' - 3$. Do the same for $2m' - 1$ and $2m' - 2$. This will ensure the vertex sums of the three internal vertices of the jellyfish are distinct. Because $r \geq 11$, the resulting labeling is antimagic as $\phi(v_1), \phi(v_2), \phi(v_3) > \phi(w_i) > m'$ for any $w_i$. This completes the proof for $t \leq 2$.

Assume $t \geq 3$. In the following three steps, we (1) assign labels to the edges of the cycle $C_3$ and three 3-paths, (2) assign labels to the pendant edges of the jellyfish, and (3) assign labels to the unlabeled 3-paths.
(1) Edges on $C_3$ and three 3-paths: Regardless of $t$, we first fix the labels of three 3-paths with labels $\{2, 6\}, \{4, 5\},$ and $\{3, 7\}$ to obtain $\phi$-values $[8, 10]$ for the internal vertices of the three 3-paths. Next, we assign $8, 9, 10$ to $E(C_3)$ on the jellyfish, which gives us the partial $\phi$-values of $v_1, v_2, v_3$ as $[17, 19]$, as shown in Figure 2.

![Figure 2: Fixed labels for $J(C_3, r) \cup tP_3$. Circled numbers are (partial) $\phi$-values and other numbers are labels on edges.](image)

(2) Pendant edges on the jellyfish: This step is split into two cases. First, suppose that $t$ is even or $t = 3$. Label four pendant edges using labels in $\{1\} \cup [11, 21]$ to each $v_i$ as shown in Figure 3.

![Figure 3: Fixed labels for $J(C_3, r) \cup tP_3$, where $t$ is even or $t = 3$.](image)

As the sum of the four pendant edges incident to each $v_i$ is 59, the partial vertex sums of $v_i$ on the jellyfish are: $[76, 78]$. We then label the remaining pendant edges on the jellyfish by applying Lemma 13 with the sums $[76, 78]$ and the unused consecutive labels $[22, 3r + 9] = [22, 24] \cup [25, 27] \cup \cdots \cup [3r + 7, 3r + 9]$ such that the vertex sums of the 3 internal vertices of the jellyfish remain consecutive integers. If $t = 3$, the resulting labeling is antimagic because $\phi(v_1), \phi(v_2), \phi(v_3) > m'$ as 76, 77, and 78 are in different $S_j$’s. If $t \geq 4$ and $t$ is even, we calculate this set of vertex sums as:

$$76 + \frac{1}{3} \sum_{i=22}^{3r+9} i, \quad 78 + \frac{1}{3} \sum_{i=22}^{3r+9} i = \left[ \frac{1}{2}(3r^2 + 19r + 28), \quad \frac{1}{2}(3r^2 + 19r + 32) \right]. \quad (4)$$

Next, suppose $t$ is odd and $t \neq 3$. We label the pendant edges of the jellyfish by applying Lemma 13 to $[17, 19] \cup [11, 3r + 10] = [17, 19] \cup [11, 13] \cup [14, 16] \cdots \cup [3r + 8, 3r + 10]$ so that
the vertex sums of the three internal vertices of the jellyfish are the consecutive integers:

\[
\left[ 17 + \frac{1}{3} \left( \sum_{i=11}^{3r+10} i \right), \ 19 + \frac{1}{3} \left( \sum_{i=11}^{3r+10} i \right) \right] = \left[ \frac{1}{2}(3r^2 + 21r + 34), \ \frac{1}{2}(3r^2 + 21r + 38) \right]. \quad (5)
\]

(3) Remaining 3-paths: In steps (1) and (2), we have used the labels in \([2, 3r + 10]\) if \(t\) is odd and \(t \neq 3\), and labels in \([1, 3r + 9]\) if \(t\) is even or \(t = 3\).

In this step, if \(t\) is odd and \(t \neq 3\), we first label one of the remaining 3-paths by \(\{1, m'\}\). Then for all cases, there are an odd number of unlabeled 3-paths. The unused labels are:

\[
\begin{cases}
[3r + 11, 3r + t + 6] \cup [3r + t + 7, 3r + 2t + 2] & \text{if } t \text{ is odd and } t \neq 3; \\
[3r + 10, 3r + t + 6] \cup [3r + t + 7, 3r + 2t + 3] & \text{if } t \text{ is even or } t = 3.
\end{cases}
\]

Applying Lemma 13, we partition the above unused labels into pairs to the edges of the remaining 3-paths, so that their internal vertices have consecutive vertex sums as:

\[
\left[ 6r + t + 15 + \left\lceil \frac{t}{2} \right\rceil, \ 6r + 2t + 11 + \left\lfloor \frac{t}{2} \right\rfloor \right]. \quad (6)
\]

To show that \(f\) is an antimagic labeling, it suffices to verify:

(i) \(\phi(w_i) > m\) for \(i \in [4, t]\), and

(ii) \(\phi(w_i) < \min\{\phi(v_j)\}\) for \(i \in [1, t]\) and \(j \in [1, 3]\).

Inequality (i) is true since by Eq. (6) and the assumptions that \(r \geq 11\) and \(t \leq 6r + 22\), we have:

\[6r + t + 15 + \left\lceil \frac{t}{2} \right\rceil \geq m' + 1 = 3r + 2t + 4.\]

Inequality (ii) is true since by the assumptions \(r \geq 11\) and \(t \leq 6r + 22\) together with Eq. (4), Eq. (5), and Eq. (6), we obtain:

\[6r + 2t + 11 + \left\lfloor \frac{t}{2} \right\rfloor < \begin{cases} \frac{1}{2}(3r^2 + 21r + 34) & \text{if } t \text{ is odd}; \\ \frac{1}{2}(3r^2 + 19r + 28) & \text{if } t \text{ is even}. \end{cases}\]

This completes the proof of Theorem 9. ■

We remark here that it has been shown in [14] that the bound in Corollary 8 is also tight for other jellyfish graphs \(J(C_k, r)\), including the following cases: (i) \(k = 3\) and \(r = 10\), (ii) \(k \leq 7\) and \(r \geq 11\), (iii) \(k \geq 8\) and \(r \geq 12\).

4 Two-Regular Graphs

In this section, we discuss the values of \(\tau(G)\) for 2-regular graphs, which are disjoint unions of cycles of lengths at least 3. For a 2-regular graph \(G\) with \(m\) edges we have \(m = n = k\) in Theorem 1, implying that the first bound in Theorem 1 is always smaller than the second.

Corollary 14. Let \(G\) be a 2-regular graph with \(n\) vertices. Then

\[\tau(G) \leq \left(1 + \sqrt{2}\right)(n + \frac{1}{2}).\]
In this section, we prove that the above bound is tight for $C_n$, $3 \leq n \leq 9$, and for $2C_3$. To this end, we start by establishing some general recursive properties for antimagic labelings of $G \cup tP_3$ for any graph $G$.

**Remark.** Recall $s = s(G, t)$ and $l = l(G, t)$ defined in Eq. (1) and Eq. (2) in the proof of Theorem 1. The calculation in Eq. (3) indeed shows that $s$ is at least the sum of the $t + n - m$ largest $\phi$-values of $V(G \cup tP_3)$. This fact can be used to prove the following results.

**Theorem 15.** Let $G$ be an $m$-edge $n$-vertex graph. Assume $f$ is an antimagic labeling for $G' = G \cup tP_3$ for some $t$. Then $\max\{\phi(v) : v \in V(G')\} \leq n + 3t + s - l$, where $s = s(G, t)$ and $l = l(G, t)$ are defined in Eq. (1) and Eq. (2).

**Proof.** Assume to the contrary that there exists some vertex $v \in V(G')$ with $\phi(v) \geq n + 3t + s - l + 1$. By Lemma 10, $s \geq l$. From the calculation of Eq. (3) the following contradiction emerges (recall $m' = m + 2t$):

\[
\begin{align*}
    s &= \sum_{v \in B'} \phi(v) - \sum_{e \in R} f(e) \\
    &\geq (n + 3t + s - l + 1) + \sum_{i=1}^{t+n-1} (m' + i) + |R|(m' + 1) - |R|m' \\
    &\geq (n + 3t + s - l + 1) + l - (m + 2t + t + n - m) \\
    &= s + 1.
\end{align*}
\]

Thus, the proof is complete. ■

**Lemma 16.** Let $p$ be a positive integer. Let $G'$ be an $m'$-edge graph that has a degree-2 vertex $v$ which is incident to edges $e_{m'}$ and $e_{m'-1}$. Let $G^*$ be the graph obtained by subdividing $e_{m'}$ into $p$ edges. If $G'$ admits an antimagic labeling $f$ such that $f(e_{m'}) = m'$, $f(e_{m'-1}) = m' - 1$, and $\phi_f(u) \leq 2m' - 1$ for all $u \in V(G')$, then there exists an antimagic labeling for $G^*$.

**Proof.** We prove the result by induction on $p$. Assume $p = 2$. Let $f$ be an antimagic labeling of $G'$ such that $f(e'_m) = m'$, $f(e'_{m'-1}) = m' - 1$, and $\phi(v) = 2m' - 1 = \max\{\phi(u) : u \in V(G')\}$. Let $G^*$ be obtained from $G'$ by subdividing $e_m$ into two edges, called $e_{m'}$ and $e_{m'+1}$, where $e_{m'+1}$ is incident to $e_{m'-1}$. Let $f'$ be a labeling for $G^*$ defined by $f'(e) = f(e)$ if $e \notin e_{m'+1}$, and $f'(e_{m'+1}) = m' + 1$. By the assumption, all vertices $u$ not incident to $e_{m'+1}$ have $\phi_f(u) \leq 2m' - 1$, while the two vertices incident to $e_{m'+1}$ have distinct vertex sums and both are greater than $2m' - 1$. Thus, $f'$ is an antimagic labeling for $G^*$.

Furthermore, under $f'$, the vertex incident to $e_{m'+1}$ and $e_{m'}$ is a degree-2 vertex in $G^*$ which is incident to the largest labels $m'$ and $m'+1$ and has the maximum $\phi$-value. Therefore, the result follows by induction on $p$. ■

**Lemma 17.** Let $p \geq 2$ and $t \geq 0$ be integers. Let $G$ be an $m$-edge $n$-vertex graph that has a degree-2 vertex $v$, where $e_m$ and $e_{m-1}$ are the edges incident to $v$. Let $s = s(G, t)$ and $l = l(G, t)$, as defined in Eq. (1) and Eq. (2). Let $G^*$ be the graph obtained by subdividing $e_m$ into $p$ edges. If $m \geq n$ and $G \cup tP_3$ admits an antimagic labeling $f$ such that $f(e_m), f(e_{m-1}) \geq t + s - l$, then $G^* \cup tP_3$ is antimagic.
Proof. Assume \( p = 2 \). Let \( f \) be an antimagic labeling of \( G \cup tP_3 \) such that \( f(e_m), f(e_{m-1}) \geq t + s - l \). Let \( G^* \) be obtained from \( G \) by subdividing \( e_m \) into two edges, \( e_m \) and \( e_{m+1} \), where \( e_{m+1} \) is incident to \( e_{m-1} \). Let \( f^* \) be a labeling for \( G^* \cup tP_3 \) defined by \( f^*(e) = f(e) \) if \( e \neq e_{m+1} \), and \( f^*(e_{m+1}) = m + 2t + 1 \). By Theorem 15 all vertices \( u \) not incident to \( e_{m+1} \) have \( \phi(u) \leq n + 3t + s - l \). Since \( m \geq n \), the two degree-2 vertices \( v \) and \( w \) incident to \( e_{m+1} \) have \( \phi(v), \phi(w) > n + 3t + s - l \) and \( \phi(v) \neq \phi(w) \).

Assume \( p = 3 \). Let \( G^* \) be obtained from \( G^* \) by subdividing \( e_{m+1} \) into two edges, \( e_{m+1} \) and \( e_{m+2} \), where \( e_{m+2} \) is incident to \( e_{m-1} \). Let \( f^{**} \) be a labeling for \( G^{**} \) defined by \( f^{**}(e) = f^*(e) \) if \( e \neq e_{m+2} \), and \( f^{**}(e_{m+2}) = m + 2t + 2 \). Similar to the above, it is not difficult to show that \( f^{**} \) is an antimagic labeling for \( G^{**} \). In addition, under \( f^{**} \) the degree-2 vertex incident to \( e_{m+2} \) and \( e_{m+1} \) is incident to the largest labels, \( m + 2t + 1 \) and \( m + 2t + 2 \), and has the maximum \( \phi \)-value. By applying Lemma 16 to \( G^{**} \), Lemma 17 follows.

After establishing the above two recursive results for general graphs, in the remaining of this section we shall focus on 2-regular graphs. Denote a cycle \( C_n \) by \( V(C_n) = \{v_1, \ldots, v_n\} \) and \( E(C_n) = \{e_1, e_2, \ldots, e_n\} \), where \( e_i = v_iv_{i+1} \) for \( 1 \leq i \leq n - 1 \), and \( e_n = v_nv_1 \).

Lemma 18. There exist antimagic labelings for \( C_n \cup tP_3 \), \( 0 \leq t \leq 6 \) and \( n \geq 3 \), such that the two largest labels, \( 2t + n \) and \( 2t + n - 1 \), are assigned to incident edges on \( C_n \). Consequently, for all \( n \geq 3 \), \( \tau(C_n) \geq 6 \).

Proof. By Lemma 16, it suffices to show the existence of antimagic labelings for \( C_3 \cup tP_3 \), \( 0 \leq t \leq 6 \), such that the two largest labels, \( 2t + 3 \) and \( 2t + 2 \), are assigned to incident edges on \( C_3 \). Such labelings are given in Table 1. In the table, for each \( t \), the labeling \( f_t \) consists of a 3-tuple \( (f_t(e_1), f_t(e_2), f_t(e_3)) \) for \( E(C_3) \) (where the two largest labels, \( 2t + 3 \) and \( 2t + 2 \), are underlined) and \( t \) pairs of labels for the 3-paths. An example is illustrated in Figure 4.

| \( t \) | \( f_t(e_1, e_2, e_3) \) | Pairs of labels on \( P_3 \) with their sums |
|-------|-----------------|------------------|
| 0     | \{1, 2, 3\}     | (1, 2, 3)        |
| 1     | \{3, 4, 5\}     | (1, 2, 3)        |
| 2     | \{4, 6, 7\}     | (1, 3, 4)        |
| 3     | \{5, 8, 9\}     | (1, 4, 5)        |
| 4     | \{6, 10, 11\}   | (1, 5, 6)        |
| 5     | \{6, 12, 13\}   | (1, 5, 6)        |
| 6     | \{13, 14, 15\}  | (1, 2, 3)        |

Table 1: Antimagic labelings for \( C_3 \cup tP_3 \), \( 0 \leq t \leq 6 \).

Remark. Consider the graph \( G \cup tP_3 \) where \( G \) is a 2-regular graph with \( n \) vertices. Then \( |E(G)| = n \) and \( m' = |E(G \cup tP_3)| = n + 2t \). Suppose the assumptions of Lemma 16 hold for \( G \cup tP_3 \), that is, there exists an antimagic labeling \( f \) for \( G \cup tP_3 \). Let \( \phi(v) = 2m' - 1 = 2n + 4t - 1 \). By Theorem 15, \( \phi(v) \leq n + 3t + s - l \) where \( s = s(G, t) \) and \( l = l(G, t) \). Hence, we obtain \( s - l + 1 \geq n + t \), a quadratic inequality in \( t \) with a positive solution:

\[
t \leq n - \frac{1}{2} + \sqrt{(\sqrt{2n} - \frac{1}{\sqrt{2}})^2 + \frac{7}{4}}
\]
Figure 4: An antimagic edge labeling of $C_3 \cup 6P_3$. Circled numbers are $\phi$-values while other numbers are edge labels. The two largest labels are underlined.

\[ f = (1 + \sqrt{2})(n - \frac{1}{2}) + O\left(\frac{1}{n}\right). \]  

Now suppose the assumptions of Lemma 17 hold for $G \cup tP_3$. Then there exist an antimagic labeling $f$ for $G \cup tP_3$ and an edge $e \in E(G)$ with $t + s - l + 1 \leq f(e) \leq n + 2t$, where $s = s(G, t)$ and $l = l(G, t)$. We then obtain $t + s - l + 1 \leq n + 2t$, the reverse of Eq. (7). Note that when the equality in Eq. (7) holds then $m' = t + s - l + 1$, implying the assumptions of Lemma 17 and Lemma 16 are equivalent. In conclusion, for a 2-regular graph $G$, to investigate possible $t$ values, $0 \leq t \leq \beta(G) = (1 + \sqrt{2})(n + \frac{1}{2})$, one might consider the following two sub-intervals:

\[ [0, (1 + \sqrt{2})(n - \frac{1}{2})] = [0, (1 + \sqrt{2})(n - \frac{1}{2})] \cup [(1 + \sqrt{2})(n - \frac{1}{2}), (1 + \sqrt{2})(n + \frac{1}{2})]. \]

In the first sub-interval, $0 \leq t \leq (1 + \sqrt{2})(n - \frac{1}{2})$, it might be possible to find a labeling satisfying the conditions of Lemma 16 (i.e., the largest two labels are assigned to incident two edges). Likewise for the second sub-interval it might be possible to find a labeling satisfying the conditions of Lemma 17.

From the above discussion, for a 2-regular graph $G$, if there exists an antimagic labeling $f$ for $G \cup tP_3$ satisfying the assumptions of Lemma 17, then $t \geq (1 + \sqrt{2})(n - \frac{1}{2})$. In the following result we prove that the converse of this also holds for small values of $n$.

**Lemma 19.** Let $n$ and $t$ be integers such that $3 \leq n \leq 9$ and

\[ (1 + \sqrt{2})(n - \frac{1}{2}) \leq t \leq (1 + \sqrt{2})(n + \frac{1}{2}). \]

Then $C_n \cup tP_3$ admits an antimagic labeling such that there exist two incident edges on $C_n$ receiving labels that are at least $t + s - l$, where $s$ and $l$ are defined in Eq. (1) and Eq. (2), respectively.

**Proof.** For $3 \leq n \leq 9$ and $(1 + \sqrt{2})(n - \frac{1}{2}) \leq t \leq (1 + \sqrt{2})(n + \frac{1}{2})$, we give a list of such labelings in Table 2 in the Appendix. These labelings were constructed with computer assistance. For each $n$ and $t$, the labeling $f_t$ consists of an $n$-tuple $(f_t(e_1), f_t(e_2), \cdots, f_t(e_n))$ for $E(C_n)$ (where two adjacent labels greater than or equal to $t + s - l$ are underlined) and $t$ pairs of labels for the 3-paths. See Figure 5 as an example.

Combining Lemma 17, Lemma 18, and Lemma 19, we obtain:
Corollary 20. Let \( n \) and \( t \) be integers such that \( t \leq \min\{22, \lfloor (1 + \sqrt{2})(n + \frac{1}{2}) \rfloor \} \). Then \( C_n \cup tP_3 \) is antimagic.

As \( \beta(C_9) = 22 \), Corollary 20 implies:

Theorem 21. For \( 3 \leq n \leq 9 \), \( \tau(C_n) = \beta(C_n) \). Equivalently, for \( 3 \leq n \leq 9 \), \( C_n \cup tP_3 \) is antimagic if and only if

\[
0 \leq t \leq \left\lfloor (1 + \sqrt{2})(n + \frac{1}{2}) \right\rfloor.
\]

We conjecture that the result of Theorem 21 holds for all \( n \).

Conjecture 2. For any \( n \), \( \tau(C_n) = \beta(C_n) \).

To confirm Conjecture 2 for \( n \geq 10 \), it is sufficient to extend Lemma 19 for all \( n \).

We conclude this section with the following two results.

Theorem 22. Let \( G \) be a 2-regular graph. If \( G \cup tP_3 \) is antimagic, then \( C_q \cup G \cup tP_3 \) is also antimagic for any \( q \geq 3 \).

Proof. Suppose \( q = 3 \) and let \( G \) be a 2-regular graph with \( n \) vertices. Let \( f \) be an antimagic labeling of \( G \cup tP_3 \). Since the degree of every vertex \( v \) in \( V(G \cup tP_3) \) is at most 2, and \( m' = |E(G \cup tP_3)| = n + 2t \), we have \( \phi(v) \leq 2n + 4t - 1 \). We extend \( f \) to \( C_3 \cup G \cup tP_3 \) by assigning the largest three labels, \( n + 2t + i \), \( 1 \leq i \leq 3 \), to \( E(C_3) \) and keeping the same labels for other edges. Then we have \( \phi(v) > 2n + 4t \) for every \( v \in V(C_3) \). Therefore, \( f \) is an antimagic labeling for \( C_3 \cup G \cup tP_3 \). As the largest two labels are assigned to incident edges on \( C_3 \), the result for \( q \geq 4 \) follows by Lemma 16.

Theorem 23. For any \( n \geq 3 \), \( \tau(C_n \cup C_3) \geq 15 \). Moreover, \( \tau(2C_3) = \beta(2C_3) = 15 \).
Proof. Combining Theorem 22 with the antimagic labelings for \( C_3 \cup tP_3 \), \( 0 \leq t \leq 8 \), given in Table 1, we obtain \( \tau(C_n \cup C_3) \geq 8 \) for \( n \geq 3 \). In addition, Table 3 gives antimagic labelings for \( 2C_3 \cup tP_3 \) for \( 9 \leq t \leq 15 \), where the labelings satisfy the following properties:

- For each \( 9 \leq t \leq 13 \), the labeling assigns the two largest labels, \( m' \) and \( m' - 1 \), to incident edges on \( C_3 \), satisfying the hypotheses of Lemma 16.

- For each \( t = 14, 15 \), the labeling assigns two labels greater than or equal to the value of \( t + s - l \) to incident edges on \( C_3 \), satisfying the hypotheses of Lemma 17.

Thus, \( C_n \cup C_3 \cup tP_3 \) is antimagic for \( n \geq 3 \) and \( 9 \leq t \leq 15 \), implying \( \tau(C_n \cup C_3) \geq 15 \) for \( n \geq 3 \). The “moreover” part follows by Corollary 14, which shows \( \tau(2C_3) \leq 15 \). ■

See Figure 6 for an example.

![Figure 6: An antimagic labeling of 2C3 ∪ 13P3.](image_url)

5 Open Problems and Future Work

To study the general properties of \( \tau(G) \), the triangle inequality emerges:

**Conjecture 3.** For any antimagic graphs \( G \) and \( H \), it holds that \( \tau(G \cup H) \leq \tau(G) + \tau(H) \).

We partially confirm Conjecture 3 with additional conditions. Let \( G \) be a graph with \( m_G \) edges and \( n_G \) vertices. Denote the first bound in Theorem 1 by \( b_G^1 = (3 + 2\sqrt{2})(m_G - n_G) + (1 + \sqrt{2})(m_G + 1/2) \), and define \( \Delta_G^1 = b_G^1 - \tau(G) \). If \( G \) and \( H \) are graphs with \( \Delta_G^1 + \Delta_H^1 \leq \frac{1 + \sqrt{2}}{2} \), then \( \tau(G \cup H) \leq \tau(G) + \tau(H) \). By Theorem 1,

\[
\tau(G \cup H) \leq b_{G\cup H}^1
= b_G^1 + b_H^1 - \frac{1 + \sqrt{2}}{2} \quad (\because m_{G\cup H} = m_G + m_H \text{ and } n_{G\cup H} = n_G + n_H)
= \tau(G) + \Delta_G^1 + \tau(H) + \Delta_H^1 - \frac{1 + \sqrt{2}}{2}
\leq \tau(G) + \tau(H) \quad (\because \Delta_G^1 + \Delta_H^1 \leq \frac{1 + \sqrt{2}}{2})
\]
A similar statement can be made using the second bound of Theorem 1. Let $G$ be a graph with $n_G$ vertices, $m_G$ edges, $k_G$ internal edges, and $t'_G$ components isomorphic to $P_3$. Denote the second bound of Theorem 1 by $b^2_G = 2m_G + 5(k_G - t'_G) + 1$, and denote $\Delta^2_G = b^2_G - \tau(G)$. If $G$ and $H$ are graphs with $\Delta^2_G + \Delta^2_H \leq 1$, then $\tau(G \cup H) \leq \tau(G) + \tau(H)$.

For a graph $G$, we have established in Theorem 1 upper bounds $\beta(G)$ for $\tau(G)$, and we have shown that many graphs have $\tau(G) = \beta(G)$. On the other hand, if there exists a graph $G$ satisfying the hypotheses in Theorem 1 but has $\tau(G) < \beta(G)$, then for any positive integer $t$, where $\tau(G) < t \leq \beta(G)$, it remains to determine whether $G \cup tP_3$ is antimagic or not. We conjecture that $G \cup tP_3$ is not antimagic for those values of $t$ if they exist:

**Conjecture 4.** For a graph $G$, $G \cup tP_3$ is antimagic if and only if $t \leq \tau(G)$. That is, $\tau(G)$ is the maximum integer $t$ such that $G \cup tP_3$ is antimagic.

Conjecture 4 is equivalent to the following statement: If $G \cup tP_3$ is antimagic for some positive integer $t$, then $G \cup (t - 1)P_3$ is antimagic.

Thus far, we have not found a graph with $\tau(G) \neq \beta(G)$. Therefore, we ask if our bound is always tight:

**Question 2.** Does there exist a graph $G$ without isolated vertices nor $P_2$ as components with $\beta(G) \geq 0$ and $\tau(G) < \beta(G)$?

With computer aid, we checked various small graphs, and found the following graphs have $\tau(G) = \beta(G)$: the clique $K_4$, the graph induced by deleting an edge from $K_4$, and $C_3$ with at most one pendant edge extended from each of three the vertices on $C_3$.

Because $\beta(G) \geq 0$ when every component of $G$ has at least 3 edges, a weaker question arises:

**Question 3.** Are all graphs without isolated vertices nor $P_2$ nor $P_3$ as components antimagic?

An affirmative answer to Question 3 or a negative answer to Question 2 would imply that Conjecture 1 is true.

**Acknowledgement.** The authors would like to thank the three anonymous referees for their careful reading of the manuscript and for their insightful suggestions.

**References**

[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster, Dense graphs are antimagic, J. Graph Theory, 47 (2004), 297–309.

[2] K. Bérezi, A. Bernáth, and M. Vizer, Regular graphs are antimagic, Electric Journal of Combinatorics, 22 (2015), paper P3.34.

[3] F.-H. Chang, H.-B. Chen, W.-T. Li, and Z. Pan, Shifted-antimagic labelings for graphs, Graphs and Combinatorics, 37 (2021), 1065–1182.
[4] F.-H. Chang, P. Chin, W.-T. Li and Z. Pan, The strongly antimagic labelings of double spiders, Indian J. Discrete Math., 6 (2020), 43–68.

[5] F.-H. Chang, Y.-C. Liang, Z. Pan, X. Zhu, Antimagic labeling of regular graphs, J. Graph Theory, 82 (2016), 339–349.

[6] D. W. Crasen, Regular bipartite graphs are antimagic, J. Graph Theory, 60 (2009), 179–182.

[7] K. Deng, Y. Li, Caterpillars with maximum degree 3 are antimagic, Discrete Math., 342 (2019), 1799–1801.

[8] K. D. E. Dhanajaaya, and W.-T. Li, Antimagic labeling of forests with sets of consecutive integers, Discrete Applied Math., 309 (2022), 75–84.

[9] N. Hartsfield and G. Ringel, Pearls in Graph Theory, Academic Press, INC, Boston, 1990, pp. 108–109, Revised version 1994.

[10] T.-Y. Huang, Antimatic Labeling on Spiders, Master’s Thesis, Department of Mathematics, National Taiwan University, 2015.

[11] G. Kaplan, A. Lev, Y. Roditty, On zero-sum partitions and anti-magic trees, Discrete Math., 309 (2009), 2010–2014.

[12] Y. Liang, T. Wong and X. Zhu, Anti-magic labeling of trees, Discrete Math., 331 (2014), 9–14.

[13] C. H. Li, The Antimagic Labeling of Balanced Double Star Union $cP_3$, Master Thesis, National Chung Hsing University, Taiwan, 2019.

[14] D. Lin, Antimagic and strongly antimagic labeling for sea urchins, Master Thesis, California State University Los Angeles, May 2022.

[15] A. Lozano, M. Mora, and C. Seara, Antimatic labeling of caterpillars, Applied Math and Computation, 347 (2019), 734–740.

[16] A. Lozano, M. Mora, C. Seara, and J. Tey, Caterpillars are antimagic, Mediterr. J. Math., 18 (2): Paper No. 29 (2021).

[17] J.-L. Shang, C. Lin, S.C. Liaw, On the antimagic labeling of star forests, Util. Math. 97 (2015), 373–385.

[18] J.-L. Shang, Spiders are antimagic, Ars Combinatoria, 118 (2015), 367–372.

[19] T. Wong and X. Zhu, Antimagic labelling of vertex weighted graphs, J. Graph Theory, 70 (2012), 348–350.
## Appendix

| $n$ | $t$ | $m'$ | $t + s - l$ | $P_{(e_1, e_2, \ldots, e_n)}$ \{ $\phi(V(C_n))$ \} | Pairs of labels on $P_3$ and their sums |
|---|---|---|---|---|---|
| 3 | 7 | 17 | 13 | \{6, 13, 17\}, \{19, 23, 30\} | (2, 4 | 6) | (1, 12 | 13) | (7, 10 | 17) | (3, 15 | 18) | (9, 11 | 20) | (5, 16 | 21) | (8, 14 | 22) |
| 4 | 9 | 22 | 19 | \{12, 7, 19\}, \{16, 26, 39\} | (3, 4 | 7) | (2, 10 | 12) | (5, 15 | 20) | (1, 22 | 23) | (6, 18 | 24) | (9, 16 | 25) | (13, 14 | 27) | (11, 17 | 28) | (8, 21 | 29) |
| 6 | 14 | 34 | 28 | \{12, 32, 8, 20, 28, 30\}, \{28, 40, 42, 44, 48, 58\} | (1, 7 | 8) | (3, 9 | 12) | (2, 18 | 20) | (4, 26 | 30) | (5, 27 | 32) | (6, 29 | 35) | (13, 23 | 36) | (15, 22 | 37) | (17, 21 | 38) | (14, 25 | 39) | (10, 31 | 41) |
| 8 | 19 | 46 | 36 | \{15, 16, 17, 19\}, \{31, 33, 36, 50, 52, 64, 69, 73\} | (1, 14 | 15) | (3, 13 | 16) | (5, 12 | 17) | (8, 11 | 19) | (2, 35 | 37) | (4, 34 | 43) | (10, 38 | 47) | (19, 26 | 45) | (12, 34 | 46) | (18, 29 | 47) | (10, 38 | 48) | (41, 39 | 50) |
| 9 | 22 | 53 | 34 | \{18, 3, 21, 24, 22, 30, 23, 45, 46\}, \{21, 24, 45, 46, 52, 53, 64, 69, 71\} | (1, 2 | 3) | (6, 12 | 18) | (4, 30 | 34) | (7, 47 | 54) | (9, 40 | 59) | (5, 16 | 21) | (8, 14 | 22) | (27, 32 | 59) | (26, 34 | 60) | (25, 36 | 61) | (19, 43 | 62) | (21, 42 | 63) | (24, 40 | 64) | (29, 37 | 66) | (28, 39 | 67) | (30, 41 | 71) |

Table 2: Antimagic labelings for Lemma 19.
| $t$ | $m'$ | $t + s - l$ | $g_t(E(2C_3)), \{\phi_{2C_3}(V(2C_3))\}$ | Pairs of labels on $P_3$ and their sums |
|-----|------|---------|-------------------------------------------------|----------------------------------|
| 9   | 24   | 48      | (20, 23, 21), (12, 7, 19), (19, 26, 31, 43, 44, 47) | (3, 4 | 7) (2, 10 | 12) (5, 15 | 20) (1, 22 | 23) (6, 18 | 24) (9, 16 | 25) (13, 14 | 27) (11, 17 | 28) (8, 21 | 29) |
| 10  | 26   | 46      | (8, 25, 20), (15, 16, 24), (31, 33, 34, 39, 40, 51) | (1, 7 | 8) (2, 23 | 25) (3, 12 | 15) (4, 22 | 26) (5, 11 | 16) (6, 18 | 24) (9, 19 | 28) (10, 20 | 30) (13, 14 | 27) (17, 21 | 38) |
| 11  | 28   | 43      | (3, 27, 25), (11, 23, 26), (30, 31, 34, 37, 49, 55) | (1, 2 | 3) (4, 24 | 28) (5, 6 | 11) (7, 20 | 27) (8, 15 | 23) (9, 17 | 26) (10, 19 | 29) (12, 21 | 33) (13, 22 | 35) (14, 18 | 32) (16, 25 | 41) |
| 12  | 30   | 39      | (3, 29, 30), (11, 23, 26), (32, 33, 34, 37, 49, 59) | (1, 2 | 3) (4, 25 | 29) (5, 6 | 11) (7, 19 | 26) (8, 22 | 30) (9, 14 | 23) (10, 28 | 38) (12, 24 | 36) (13, 27 | 40) (15, 16 | 31) (17, 18 | 35) (20, 21 | 41) |
| 13  | 32   | 34      | (3, 31, 32), (15, 16, 17), (31, 32, 33, 34, 35, 63) | (1, 2 | 3) (4, 11 | 15) (5, 12 | 17) (6, 10 | 16) (7, 29 | 36) (8, 30 | 38) (9, 28 | 37) (13, 26 | 39) (14, 27 | 41) (18, 22 | 40) (19, 23 | 42) (20, 24 | 44) (21, 25 | 46) |
| 14  | 34   | 28      | (10, 18, 34), (6, 28, 29), (28, 34, 35, 44, 52, 57) | (2, 4 | 6) (1, 9 | 10) (3, 16 | 18) (7, 22 | 29) (5, 31 | 36) (16, 21 | 37) (11, 27 | 38) (13, 26 | 39) (17, 23 | 40) (8, 33 | 41) (12, 30 | 42) (19, 24 | 43) (20, 25 | 45) (14, 32 | 46) |
| 15  | 36   | 21      | (5, 9, 35), (14, 21, 30), (14, 35, 40, 44, 50, 57) | (1, 4 | 5) (2, 7 | 9) (3, 18 | 21) (6, 30 | 36) (8, 29 | 37) (10, 28 | 38) (11, 31 | 42) (12, 27 | 39) (13, 33 | 46) (15, 34 | 49) (16, 32 | 48) (17, 24 | 41) (19, 26 | 45) (20, 23 | 43) (22, 25 | 47) |

Table 3: Antimagic labelings $g_t$ of $2C_3 \cup tP_3$ when $9 \leq t \leq 15$. The underlined numbers for $9 \leq t \leq 13$ are $m'$ and $m' - 1$ (satisfying the hypotheses of Lemma 16); while for $t = 14, 15$, they are labels greater than or equal to $t + s - l$ (satisfying the hypotheses of Lemma 17).