SOME OVERDETERMINED PROBLEMS RELATED TO THE ANISOTROPIC CAPACITY

CHIARA BIANCHINI, GIULIO CIRAOLO, AND PAOLO SALANI

Abstract. We characterize the Wulff shape of an anisotropic norm in terms of solutions to overdetermined problems for the Finsler p-capacity of a convex set $\Omega \subset \mathbb{R}^N$, with $1 < p < N$. In particular we show that if the Finsler p-capacitary potential $u$ associated to $\Omega$ has two homothetic level sets then $\Omega$ is Wulff shape. Moreover, we show that the concavity exponent of $u$ is $q = -(p-1)/(N-p)$ if and only if $\Omega$ is Wulff shape.

AMS subject classifications. 35N25, 35B06, 35R25.

Key words. Wulff shape. Overdetermined problems. Capacity. Concavity exponent.

1. Introduction

The aim of this paper is to study some unconventional overdetermined problems for the Finsler p-capacity of a bounded convex set $\Omega$ associated to a norm $H$ of $\mathbb{R}^N$, $N \geq 3$.

Given a bounded convex domain $\Omega \subset \mathbb{R}^N$, the p-capacity of $\Omega$ is defined by

$$\text{Cap}_p(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} |D\varphi|^p \, dx, \, \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\}$$

with $1 < p < N$. When the Euclidean norm $|\cdot|$ is replaced by a more general norm $H(\cdot)$, one can consider the so called Finsler p-capacity $\text{Cap}_{H,p}(\Omega)$, which is defined by

$$\text{Cap}_{H,p}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} H^p(D\varphi) \, dx, \, \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\},$$

for $1 < p < N$. Under suitable assumptions on the norm $H$ and on the set $\Omega$, the above infimum is attained and

$$\text{Cap}_{H,p}(\Omega) = \frac{1}{p} \int_{\mathbb{R}^N} H^p(Du_\Omega) \, dx,$$

where $u_\Omega$ is the solution of the Finsler p-capacity problem

$$\begin{cases}
\Delta^H_p u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\
u = 1 & \text{on } \partial \Omega, \\
u \to 0 & \text{as } H(x) \to +\infty.
\end{cases}$$

(1.2)

Here $\Delta^H_p$ denotes the Finsler p-Laplace operator, i.e. $\Delta^H_p u = \text{div}(H^{p-1}(Du) \nabla H(Du))$. The function $u_\Omega$ is named (Finsler) p-capacitary potential of $\Omega$.

When $\Omega$ is Wulff shape, i.e. it is a sublevel set of the dual norm $H_0$

$$\Omega = B_{H_0}(r) = \{ x \in \mathbb{R}^N : H_0(x - \bar{x}) < r \}$$

(see Section 2 for definitions), the solution to (1.2) can be explicitly computed and it is given by

$$v_r(x) = \left( \frac{H_0(x - \bar{x})}{r} \right)^{\frac{q}{p}},$$

(1.3)

with

$$q = \frac{p-1}{N-p}.$$ 

(1.4)

It is straightforward to verify that the potential $v_r$ in (1.3) enjoys the following properties:

(i) the function $v_r^q$ is convex, i.e. $v_r$ is $q$-concave;

(ii) the superlevel sets of $v_r$ are homothetic sets and they are Wulff shapes;

(iii) $H(Dv_r)$ is constant on the level sets of $v_r$.

The aim of this paper is to show that each of the properties (i)-(iii) characterizes the Wulff shape under some regularity assumptions on the norm $H$ and on $\Omega$. In particular, we assume that $H \in J_p$, where

$$J_p = \{ H \in C^2_c(\mathbb{R}^N \setminus \{0\}), \, H^p \in C^{2,1}(\mathbb{R}^N \setminus \{0\}) \}.$$

(1.5)
Our first main result is related to property (i) and it is about concavity properties of the solution to (1.2). We recall that a nonnegative function $v$ with convex support is $\alpha$-concave, for some $\alpha \in [-\infty, +\infty]$, if

- $v$ is a positive constant in its support set, in case $\alpha = +\infty$;
- $v^\alpha$ is concave, in case $\alpha > 0$;
- $\log v$ is concave, in case $\alpha = 0$ (and $v$ is called log-concave);
- $v^\alpha$ is convex, in case $\alpha < 0$;
- all its super level sets $\{v > t\}$ are convex, in case $\alpha = -\infty$ (and $v$ is called quasi-concave).

Notice that if $v$ is $\alpha$-concave for some $\alpha > -\infty$, then it is $\beta$-concave for every $\beta \in [-\infty, \alpha]$. Then quasi-concavity is the weakest among concavity properties.

Concavity properties of solutions to elliptic and parabolic equations are a popular field of investigation. Classical results in this framework are for instance the log-concavity of the first Dirichlet eigenfunction of the Laplacian (see [2]), the preservation of concavity by the heat flow (see again [2]), the $\frac{1}{2}$-concavity (i.e. the concavity of the square root) of the torsion function (see [16, 6, 7, 14]), the quasi-concavity of the Newton potential and of the $p$-capacitary potential (see [10, 15]). The latter results are especially related to the situation we consider in this paper. Indeed, it is proved in [1] (and it can be also obtained with the methods of [3]) that when $\Omega$ is a convex domain, its $p$-capacitary potential $u_\Omega$ is a quasi-concave function, i.e. all its superlevel sets are convex. Moreover, as we have seen, quasi-concavity is the weakest property in this context and one may expect and ask more than this. Then, following [14, 17, 13], it is natural to define the concavity exponent associated to the solution to (1.2) as

$$\alpha(\Omega, p) = \sup\{\beta \leq 1 : u_\Omega \text{ is } \beta\text{-concave} \}.$$  

(1.6)

In the Euclidean case, it was proved in [17] that the concavity exponent attains its maximum when $\Omega$ is a ball (and only in this case). In the following theorem, we characterize the Wulff shape in terms of property (i) above. More precisely, we generalize the results of [17] to the anisotropic setting and we prove that the exponent $q$ characterizes the Wulff shape.

**Theorem 1.1.** Let $H$ be a norm of $\mathbb{R}^N$ in the class (1.5) and let $\Omega$ be a bounded convex domain of $\mathbb{R}^N$ of class $C^2$. Then

$$\alpha(\Omega, p) \leq q,$$

with $q$ given by (1.4), and equality holds if and only if $\Omega$ is Wulff shape.

The proof of Theorem 1.1 is based on the Brunn-Minkowski inequality for Finsler $p$-capacity, recently proved in [1] (and here recalled in Proposition 2.1), and upon the fact $u$ can have a level set homothetic to $\Omega$ if and only if $\Omega$ is a ball. Clearly this property is related to property (ii) above. And indeed the characterization of the Wulff shape is achieved whenever just two superlevel sets of $u_\Omega$ are homothetic, as expressed in the following theorem.

**Theorem 1.2.** Let $H$ be a norm of $\mathbb{R}^N$ in the class (1.5). Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded convex domain with boundary of class $C^2$. If there exists a solution to (1.2) having two homothetic superlevel sets, then $\Omega$ is Wulff shape.

The Euclidean counterpart of Theorem 1.2 was proved in [17]. The proof of Theorem 1.2 in the anisotropic setting passes through the following theorem, which is related to property (iii).

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^N$ be a convex domain containing the origin and with boundary of class $C^{2,\alpha}$. Let $H \in \mathcal{J}_p$ and let $R > 0$ be such that $\overline{\Omega} \subset B_{H_0}(R)$. There exists a solution to

$$\begin{cases}
\Delta_p^H u = 0 & \text{in } B_{H_0}(R) \setminus \overline{\Omega} \\
u = 1 & \text{on } \partial \Omega \\
u = 0 & \text{on } \partial B_{H_0}(R) \\
H(Du) = C & \text{on } \partial B_{H_0}(R),
\end{cases}$$

(1.7)

for some constant $C > 0$ if and only if $\Omega = B_{H_0}(r)$, with

$$r \geq \frac{N - p}{p - 1}.$$  

(1.8)
Since two boundary conditions (Dirichlet and Neumann) are imposed on a prescribed part of the boundary, Theorem 1.3 clearly falls in the realm of \textit{overdetermined problems}: since the domain $\Omega$ is not prescribed, the unknown of the problem is in fact the couple $(\Omega, u)$, and by imposing that $u$ has some peculiar property (which is not commonly shared by all the solution of the involved PDE), one asks whether this is sufficient to uniquely determine the domain $\Omega$. In this sense, also Theorem 1.1 and Theorem 1.2 can be considered as overdetermined problems, since we ask for a solution of a Dirichlet problem satisfying some extra special condition (e.g., quasi-concavity or homothety of level sets, respectively).

The paper is organized as follows. In Section 2 we introduce some notation and basic properties of Finsler norms; then we recall some known fact about the Finsler capacity $\text{Cap}_{p}(\Omega)$ of a convex set and, in particular, the Brunn-Minkowski inequality from [1]. Theorems 1.3, 1.2 and 1.1 are proved in Sections 3, 4 and 5, respectively.

\textbf{Acknowledgements.} The authors have been partially supported by GNAMPA of INdAM and by a PRIN Project of Italian MIUR.

2. \textbf{Notations}

2.1. \textbf{Norms of $\mathbb{R}^{N}$.} We consider the space $\mathbb{R}^{N}$ endowed with a generic norm $H : \mathbb{R}^{N} \to \mathbb{R}$ such that:

(i) $H$ is convex;

(ii) $H(\xi) \geq 0$ for $\xi \in \mathbb{R}^{N}$ and $H(\xi) = 0$ if and only if $\xi = 0$;

(iii) $H(t\xi) = |t|H(\xi)$ for $\xi \in \mathbb{R}^{N}$ and $t \in \mathbb{R}$.

Then we identify the dual space of $\mathbb{R}^{N}$ with $\mathbb{R}^{N}$ itself via the scalar product $\langle ; ; \rangle$. Accordingly the space $\mathbb{R}^{N}$ turns out to be endowed also with the dual norm $H_{0}$ given by

$$H_{0}(x) = \sup_{\xi \neq 0} \frac{\langle x; \xi \rangle}{H(\xi)} \quad \text{for} \quad x \in \mathbb{R}^{N}. \quad (2.1)$$

We denote by $B_{H_{0}}(r)$ the anisotropic ball centered at $O$ with radius $r$ in the norm $H_{0}$, i.e.

$$B_{H_{0}}(r) = \{ x \in \mathbb{R}^{N} : H_{0}(x) < r \}.$$

Analogously, we define

$$B_{H}(r) = \{ \xi \in \mathbb{R}^{N} : H(\xi) < r \}.$$

The sets $B_{H_{0}}(r)$ and $B_{H}(r)$ are called Wulff shape of $H$ and $H_{0}$, respectively; in the special case $r = 1$ they are indicated by $B_{H_{0}}, B_{H}$, respectively. Notice that, in the language of the theory of convex bodies, $H$ is the support function of $B_{H_{0}}$ and $H_{0}$ is in turn the support function of $B_{H}$.

For a regular convex domain $\Omega$ the Finsler perimeter is defined by

$$P_{H}(\partial \Omega) = \int_{\partial \Omega} H(\nu) \, d\sigma,$$

where $\nu$ is the outer unit normal to $\partial \Omega$.

2.2. \textbf{Finsler capacity.} For a bounded convex domain $\Omega$ in $\mathbb{R}^{N}$ its Finsler $p$-capacity, denoted by $\text{Cap}_{H, p}(\Omega)$, is defined as follows:

$$\text{Cap}_{H, p}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^{N}} H^{p}(D\varphi) \, dx, \varphi \in C_{0}^{\infty}(\mathbb{R}^{N}), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\},$$

for $N \geq 3$ and $1 < p < N$. If $H$ is a norm in the class (1.5), the integral operator is strictly convex and hence (1.1) admits a unique solution $u_{\Omega}$, which satisfies

$$\begin{cases}
\Delta_{H}^{p} u = 0 & \text{in } \mathbb{R}^{N} \setminus \overline{\Omega}, \\
u = 1 & \text{on } \partial \Omega, \\
u \to 0 & \text{as } H(x) \to +\infty.
\end{cases}$$

The function $u_{\Omega}$ is called the Finsler $p$-capacitary potential of $\Omega$. As already noticed when $\Omega$ is a convex set the potential $u_{\Omega}$ is at least quasi-concave, that is its superlevel sets are convex sets (see Lemma 4.4 [1]).

In the special case $\Omega = B_{H_{0}}(r)$ the capacitary potential is easily computed and is given by (1.3), but this is not possible for general convex domain. However, when $\Omega$ is a convex
set, asymptotic estimates for \( u_\Omega \) are known. In particular, it has recently been proved in [1] the following:

\[
\lim_{|x| \to \infty} u_\Omega(x)H_0(x) \frac{1}{|x|^\frac{n}{p-1}} = C \Cap_{H_p}(\Omega),
\]

where \( C = (N-2)P_{H_p}^+(\partial B_{H_0}) \). Moreover, one can prove that there exists a constant \( \gamma \) such that

\[
\gamma^{-1} H(x)^{\frac{1}{p-1}} \leq H(\nabla u(x)) \leq \gamma H(x)^{\frac{1}{p-1}},
\]

(see [5], [4]).

The \( p \)-capacity operator satisfies a Brunn-Minkowski inequality. In the Euclidean setting, this was proved in [8], such result has been recently extended to quite general operators in divergence form in [1]. Here, we recall the following from [1].

**Proposition 2.1** ([1]). Let \( K, D \) be compact convex sets in \( \mathbb{R}^N \) satisfying

\[
\Cap_{H_p}(K), \Cap_{H_p}(D) > 0.
\]

For \( 1 < p < N \) and \( \lambda \in [0, 1] \) it holds

\[
\Cap_{H_p}( (1 - \lambda)K + \lambda D) \geq (1 - \lambda)\Cap_{H_p}(K) + \lambda\Cap_{H_p}(D),
\]

and equality holds if and only if \( K \) and \( D \) are homothetic sets.

3. PROOF OF THEOREM 1.3

Let

\[
v(x) = \frac{H_0(x)^{\frac{1}{p}} - R^\frac{2}{p}}{r^\frac{2}{p} - R^\frac{2}{p}}, \quad x \in \mathbb{R}^n \setminus \{O\},
\]

with \( r \) and \( q \) given by (1.8) and (1.4), respectively. When \( \Omega = B_{H_0}(r) \) is Wulff shape of radius \( r \), a direct check shows that \( v \) is the solution to (1.7).

Now we prove the reverse assertion. Let

\[
\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \overline{B_{H_0}(R)} \setminus \Omega, \\ v(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{B_{H_0}(R)}. \end{cases}
\]

We notice that \( \hat{u} \in C^1(\mathbb{R}^n \setminus \Omega) \) and it satisfies \( \Delta_{H_p}^\mu \hat{u} = 0 \) in \( \mathbb{R}^n \setminus \overline{\Omega} \) (which follows from the weak formulation of the equation).

Fix any \( t > 1 \) and set \( E = B_{H_0}(tR) \setminus \overline{\Omega} \). For \( \tau \in [0, 1] \), we define

\[
u_\tau = \tau \hat{u} + (1 - \tau)v.
\]

in \( \overline{E} \); notice that \( u_1 = \hat{u} \) and \( u_0 = v \).

The function \( \hat{u} - v \) satisfies an elliptic equation. Indeed, for any \( \phi \in C^1_0(E) \) we have

\[
0 = \int_E H(Du_1)^{p-1}(\nabla H(Du_1); D\phi) \, dx - \int_E H(Du_0)^{p-1}(\nabla H(Du_0); D\phi) \, dx
\]

\[
\int_E \left( \int_0^1 \frac{d}{d\tau} (H(Du_\tau)^{p-1}\nabla H(Du_\tau)) \, d\tau \right) ; D\phi \right) \, dx
\]

\[
= \int_E A(x) (D(\hat{u} - v); D\phi) \, dx,
\]

where \( A(x) = a_{ij}(x) \) is given by

\[
a_{ij}(x) = \int_0^1 (p - 1)H(Du_\tau)^{p-2}H_{\xi_i}(Du_\tau)H_{\xi_j}(Du_\tau) + H(Du_\tau)^{p-1}H_{\xi_i\xi_j}(Du_\tau)) \, d\tau
\]

\[
= \frac{1}{p} \int_0^1 (\nabla H(Du_\tau))^2_{ij}d\tau.
\]

From (2.3) we have that

\[
A \leq H(Du) \leq B
\]

in \( E \) for some constant \( A, B > 0 \). Notice that, since the super level sets of \( u \) are convex sets (see [1]) and those of \( v \) are Wulff shapes centered at the origin, we have

\[
\langle Du; \frac{x}{|x|} \rangle \geq \varepsilon > 0, \quad \varepsilon \leq \langle Dv; \frac{x}{|x|} \rangle \leq M,
\]

in \( E \) for some positive constants \( \varepsilon \) and \( M \). Hence, we can find \( \tau_0 \in (0, 1) \) such that

\[
(1 - \tau_0)|Du| \leq \tau_0|Du|/2,
\]
which implies that $|Du_\tau| \geq \tau_0 |Du|/2$ for every $\tau \in [\tau_0,1]$. From (3.2) and (3.1) we obtain

$$|Du_\tau| \geq \min \left( (1 - \tau_0)\varepsilon, \frac{\tau_0}{2}A \right) > 0 \quad \text{in } E,$$

which finally gives that $a \leq |Du_\tau| \leq b$ in $E$ for some constants $a,b > 0$. Such estimates imply that the operator

$$Lw = \text{div} (A(x)Dw)$$

is uniformly elliptic. Moreover, since $H^p \in C^{2,1}(\mathbb{R}^n \setminus \{O\})$, we also have that $a_{ij}$ are locally Lipschitz. Hence, $L$ satisfies the assumptions of Theorem 1.1 in [12] (see also [11]) and we have the analytic continuation for $\bar{u} - v$ in $E$, whence $\bar{u} - v \equiv 0$ in $E$ which implies that $u \equiv v$ in $B_{\bar{R}}(\bar{R}) \setminus \Omega$ and we conclude.

4. Proof of Theorem 1.2

For any $t \in (0,1)$ we set $U(t) = \{ x \in \mathbb{R}^N : u(x) \geq t \}$ and let $u_{U(t)}$ be the p-capacity potential of $U(t)$. Hence

$$u_{U(t)}(x) = \frac{1}{t} u(x), \quad (4.1)$$

for every $x \in \mathbb{R}^N \setminus U(t)$, where $u$ is the Finsler p-capacity potential of $\Omega$.

Let $t,s \in (0,1)$, with $t < s$, be the levels of $u$ such that $U(t), U(s)$ are homothetic, that is: there exist $\xi \in \mathbb{R}^N$ and $\rho > 1$ such that $U(t) = \rho U(s) + \xi$. Up to a translation we can assume $\xi = 0$ and hence

$$u_{U(t)}(x) = u_{U(s)} \left( \frac{x}{\rho} \right). \quad (4.2)$$

Step 1: $\rho \frac{x-t}{\rho-t} = t/s$.

From (2.2), (4.1) and (4.2), we have

$$\text{C Cap}_{\text{H},p}(\Omega) \frac{\rho}{t} = \lim_{|x| \to \infty} t u_{U(s)} \left( \frac{x}{\rho} \right) H_0^{\frac{N-p}{p}}(x).$$

By using again (2.2) and (4.1), and from the homogeneity of $H_0$, we find

$$\text{C Cap}_{\text{H},p}(\Omega) \frac{\rho}{t} = \frac{s}{t} \text{C Cap}_{\text{H},p}(\Omega) \frac{\rho}{s} \rho \frac{x-t}{\rho-t}$$

which implies that $\rho \frac{x-t}{\rho-t} = t/s$.

Step 2: Let $r_k = k \frac{x-t}{s-t}$ for $k \geq 0$. Then $U(r_0) = U(s)$ and $U(r_k) = \rho^k U(s)$ for $k \in \mathbb{N}$.

Indeed notice that for every $z < t$ the set $U(z)$ is homothetic to $U(z\hat{\xi})$ since by (4.1), (4.2) we have

$$U(z) = \{ u(x) \geq z \} = \{ u_{U(s)}(x) \geq \frac{z}{s} \} = \{ u(x) \geq \frac{z}{s+t} \},$$

that is $U(z) = \rho U(z\hat{\xi})$. Hence, recalling that $U(s) = U(r_0)$ and that $r_k = \frac{s}{t} r_{k-1}$, we obtain

$$U(r_k) = \rho^k U(s)$$

for every $k \geq 0$.

Step 3: $U(s)$ is Wulff shape.

Let $x,y \in \partial U(s)$ and define

$$x_k = \rho^k x, \quad y_k = \rho^k y.$$

Notice that

$$\lim_{k \to \infty} |x_k| = \lim_{k \to \infty} |y_k| = +\infty. \quad (4.3)$$

From Step 2 the points $x_k,y_k$ belong to $\partial U(r_k)$, so that $u(x_k) = u(y_k) = r_k$. From (5.1) and (2.2) we obtain

$$\lim_{k \to \infty} u(x_k) H_0^{\frac{N-p}{p}}(x_k) = \text{C Cap}_{\text{H},p}(\Omega) \frac{\rho}{t} = \lim_{k \to \infty} u(y_k) H_0^{\frac{N-p}{p}}(y_k),$$

i.e.

$$\lim_{k \to \infty} r_k H_0^{\frac{N-p}{p}}(x_k) = \lim_{k \to \infty} r_k H_0^{\frac{N-p}{p}}(y_k).$$

By recalling the definition of $x_k$ and $y_k$ and Step 1, we have

$$\lim_{k \to \infty} H_0^{\frac{N-p}{p}}(x) = \lim_{k \to \infty} H_0^{\frac{N-p}{p}}(y),$$
which implies that
\[ H_0(x) = H_0(y) \]
for every \( x, y \in \partial U(s) \), i.e. \( U(s) \) is Wulff shape.

**Conclusion:** from Step 2, we obtain that \( U(r_k) \) is Wulff shape for any \( k \geq 0 \), which implies that the super level sets \( U(s) \) are concentric Wulff shapes. In particular there exists \( \beta > 0 \) such that \( u = \beta H_0(x)^{1/\beta} \) for any \( x \in \mathbb{R}^N \setminus U(s) \). From Theorem 1.3 we conclude.

5. **Proof of Theorem 1.1**

Let \( q \) be given by (1.4). Notice that for every \( x_0 \in \mathbb{R}^N \) and every \( R > 0 \), the concavity exponent of \( B_{H_0}(R, x_0) \) can be explicitly computed thanks to (1.3) and it holds \( \alpha(B_{H_0}(R, x_0), p) = q \).

We are going to prove that if the capacitary potential \( u \) of the set \( \Omega \) is \( q \)-concave then \( \Omega \) is Wulff shape and this entails the desired result. Indeed if \( u \) is \( q \)-concave, then \( u \) is \( s \)-concave too, for every \( s < q \).

Assume that the function \( u \) is \( q \)-concave. Since \( q < 0 \), then \( u^q \) is a convex function. We denote by \( V(t) \) the sublevel sets of the function \( u^q \), i.e. \( V(t) = \{ u^q \leq t \} \); the superlevel sets of \( u \) will be denoted by \( U(t) \). Hence
\[ U(t) = V(t^q). \]
Since \( u^q \) is convex, for every \( t_0, t_1 \in \mathbb{R} \) and every \( \lambda \in [0, 1] \) we have
\[ V((1 - \lambda)t_0 + \lambda t_1) \supseteq (1 - \lambda)V(t_0) + \lambda V(t_1). \] (5.1)
Let \( 0 < r < s < 1 \). By choosing \( t_0 = r^q, t_1 = s^q \) and defining
\[ t = ((1 - \lambda)r^q + \lambda s^q)^{1/\lambda}, \] (5.2)
(5.1) can be written as
\[ V(t^q) \supseteq (1 - \lambda)V(r^q) + \lambda V(s^q), \]
and hence
\[ U(t) \supseteq (1 - \lambda)U(r) + \lambda U(s). \]
From the monotonicity of the capacity and from Brunn-Minkowski inequality (2.4) it follows
\[ \text{Cap}_{H_0}(U(t)) \geq \text{Cap}_{H_0}((1 - \lambda)U(r) + \lambda U(s)) \]
\[ \geq \left( (1 - \lambda)\text{Cap}_{H_0}r^{\lambda q}U(r) + \lambda \text{Cap}_{H_0}s^{\lambda q}U(s) \right)^{1 - p}, \] (5.3)
and, since for every \( r \in (0, 1) \)
\[ \text{Cap}_{H_0}(U(r)) = r^{1 - p}\text{Cap}_{H_0}(\Omega), \]
inequality (5.3) gives
\[ \text{Cap}_{H_0}(\Omega)t^{1 - p} \geq \text{Cap}_{H_0}(\Omega) \left( (1 - \lambda)r^{\frac{\lambda q}{1 - p}} + \lambda s^{\frac{\lambda q}{1 - p}} \right)^{N - p}. \] (5.4)
The definition of \( t \) in (5.2) implies that the equality case holds in (5.4) and this entails that the equality sign in the Brunn-Minkowski inequality (2.4) is attained. Hence the superlevel set \( U(r) \) is homothetic to \( U(s) \) and Theorem 1.2 yields the conclusion.

**References**

[1] M. Akman, J. Gong, J. Hineman, J. Lewis, A. Vogel, The Brunn-Minkowski inequality and a Minkowski problem for nonlinear capacity, preprint.
[2] H. J. Brascamp, E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22 (1976), 366-389.
[3] C. Bianchini, M. Longinetti, P. Salani, *Quasiconcave solutions to elliptic problems in convex rings*, Indiana Univ. Math. J. 58 no. 4 (2009), 1565-1589.
[4] C. Bianchini, G. Ciraolo, *Wulff shape characterization in overdetermined anisotropic problems*, to appear on Comm. Partial Differential Equations.
[5] C. Bianchini, G. Ciraolo, P. Salani, *An overdetermined problem for the anisotropic capacity*, Calc. Var. Partial Differential Equations, 55:84 (2016).
[6] Ch. Borell, *Greenian potentials and concavity*, Math. Anal. 272 (1985), 155-160.
[7] L.A. Caffarelli, A. Friedman, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. 52 (1985), 431-456.
[8] A. Colesanti and P. Salani, *The Brunn-Minkowski inequality for p-capacity of convex bodies*, Math. Ann. 327 (2009), 459-479.
OVERDETERMINED PROBLEMS FOR THE ANISOTROPIC CAPACITY

[9] V. Ferone, B. Kawohl, Remarks on a Finsler-Laplacian, Proc. Am. Math. Soc. 137 (2009), 247-253.
[10] M. Gabriel, A result concerning convex level-surfaces of three-dimensional harmonic functions, London Math. Soc. J. 32 (1957), 286-294.
[11] N. Garofalo and F.-H. Lin, Monotonicity properties of variational integrals, $A_p$ weights and unique continuation, Indiana Univ. Math. J., 35 (1986), 245-268.
[12] N. Garofalo, N. and F.-H. Lin, Unique continuation for elliptic operators: a geometric-variational approach, Comm. Pure Appl. Math., 40 (1987), 347-366.
[13] A. Henrot, C. Nitsch, C. Trombetti, P. Salani, Optimal concavity of the torsion function, to appear in J. Optim. Theory Appl.
[14] A. U. Kennington, Power concavity and boundary value problems, Indiana Univ. Math. J. 34 (1985), 687-704.
[15] J. Lewis, Capacitary functions in convex rings, Arch. Rational Mech. Anal. 66 (1977), 201–224.
[16] L.G. Makar-Limanov, The solution of the Dirichlet problem for the equation $\Delta u = -1$ in a convex region, Mat. Zametki 9 (1971) 89-92 (Russian). English translation in Math. Notes 9 (1971), 52-53.
[17] P. Salani, A characterization of balls through optimal concavity for potential functions, Proc. AMS 143 (1) (2015), 173–183.

C. Bianchini, Dipartimento di Matematica e Informatica “U. Dini”, Università degli Studi di Firenze, Viale Morgagni 67/A, 50134 Firenze - Italy
E-mail address: chiara.bianchini@unifi.it

G. Ciraolo, Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi 34, 90123, Palermo - Italy
E-mail address: giulio.ciraolo@unipa.it

P. Salani, Dipartimento di Matematica e Informatica “U. Dini”, Università degli Studi di Firenze, Viale Morgagni 67/A, 50134 Firenze - Italy
E-mail address: paolo.salani@unifi.it