Matrix Games, Linear Programming, and Linear Approximation

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Abstract. The following four classes of computational problems are equivalent:
- solving matrix games,
- solving linear programs,
- best $l^\infty$ linear approximation,
- best $l^1$ linear approximation.

Key words Matrix games, linear programming, linear approximation, least absolute deviations.

Definitions

First we recall relevant definitions.

An affine function of variables $x_1, \ldots, x_n$ is $b_0 + c_1x_1 + \cdots + c_nx_n$ where $b_0, c_i$ are given numbers.

An $l^\infty$ linear approximation problem, also known as (discrete) Chebyshev approximation problem is the problem of minimization of the following function:

$$\max(|f_1|, \ldots, |f_m|) = \|(f_1, \ldots, f_m)\|_\infty,$$

where $f_1, \ldots, f_m$ are $m$ affine functions of $n$ variables. This objective function is piece-wise linear and convex.

An $l^1$ linear approximation problem, also known as finding the LAD (least-absolute-deviations) fit, is the problem of minimization of the following function:

$$\sum_{i=1}^{m} |f_i| = \|(f_1, \ldots, f_m)\|_1,$$

where $f_1, \ldots, f_m$ are $m$ affine functions of $n$ variables. This objective function is piece-wise linear and convex.

A matrix game is given by a (payoff) matrix $A$. To solve a matrix game is to find a row $p$ (an optimal strategy for the row player), a column $q$ (an optimal strategy for the column player), and a number $v$ such that $p = (p_i) \geq 0, \sum p_i = 1, q = (q_j) \geq 0, \sum q_i = 1, pA \geq v \geq Aq$. The number $v$ is known as the value of game. The pair $(p, q)$ is known as an equilibrium for the matrix game.

As usual, $x \geq 0$ means that every entry of the vector $x$ is $\geq 0$. We write $y \leq t$ for a vector $y$ and a number $t$ if every entry of $y$ is $\leq t$. We go even further in abusing notation, denoting by $y - t$ the vector obtaining from $y$ by subtracting $t$ from every entry. Similarly, we denote by $M + c$ the matrix obtained from $M$ by adding a number $c$ to every entry.
A matrix game is called *symmetric* if the payoff matrix is skew-symmetric. Recall that the value of any symmetric game is 0, and the transposition gives a bijection between the optimal strategies of the players.

A *linear constraint* is any of the following constraints: \( f \leq g, f \geq g, f = g \), where \( f, g \) are affine functions. A *linear program* is an optimization (maximization or minimization) of an affine function subject to a finite system of linear constraints.

**Statement of results**

It is well known, that solving a matrix game can be reduced to solving a pair of linear programs, dual to each other. It is also known that solving any linear program can be reduced to finding an optimal strategy with positive last component for a symmetric matrix game. In both reductions, the size of data (in terms of the number of given numbers or the number of given bits) may increase at most two times.

A subtle point here is: how can we compute an optimal strategy (for a symmetric game) with a positive last entry or prove that no such strategy exists? An answer is that for any vertex in the set of optimal strategy with positive last entry is a solution of a system of linear equations whose coefficients are the entries of the payoff matrix or 0,1, so a positive lower bound \( \alpha \) can be given for this entry (at least in the case when all given numbers are rational). Namely, let \( \beta \) be an upper bound for the absolute values of the numerators and denominators of the entries of the payoff matrix of size \( N \times N \). Then \( \alpha = \beta^{-2N} N^{-N/2} \) will work. Notice that \( 0 < \alpha < 1 \).

The mixed strategies for the column player with the last entry \( \geq \alpha \) in the symmetric game are the mixed strategies for the column player for the modified game obtained by adding the \( (\alpha/(1-\alpha)) \)-multiple of the last column to the other columns of the payoff matrix. The optimal strategies for a modified matrix game give optimal strategies with positive last entry for the original symmetric game provided that the value of the modified game stays 0 (otherwise, there are no optimal strategies with positive last entry for the original symmetric game hence the original linear program has no optimal solutions).

Given any \( l^\infty \) approximation problem with the objective function (1), here is a well-known reduction (Vaserstein, 2003) to a linear program with one additional variable \( t \):

\[
t \rightarrow \min, \text{ subject to } -t \leq f_i \leq t \text{ for } i = 1, \ldots, m.
\]

This is a linear program with \( n + 1 \) variables and \( 2m \) linear constraints. Since any linear program can be reduced to a matrix game (see above), we conclude that finding a Chebyshev fit can be reduced to solving a matrix game.

The converse reduction is a main goal of this paper:

**Theorem 1.** Solving any matrix game can be reduced to finding a Chebyshev fit. More precisely, when the game is given by an \( m \) by \( n \) matrix, we construct a Chebyshev approximation problem with \( 2m + 2n + 3 \) affine functions of \( m + n + 1 \) variables as well as a bijection between the equilibria for the matrix game and the solutions for the approximation problem.

Given any \( l^1 \) approximation problem with the objective function (2), here is a well-known reduction (Vaserstein, 2003) to a linear program with \( m \) additional variables \( t_i \):

\[
\sum_{i=1}^{m} t_i \rightarrow \min, \text{ subject to } -t_i \leq f_i \leq t_i \text{ for } i = 1, \ldots, m.
\]

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This is a linear program with \( n + m \) variables and \( 2m \) linear constraints. Since any linear program can be reduced to a matrix game (see above), we conclude that finding the best \( l^1 \)-fit can be reduced to solving a matrix game.

The converse reduction is the second goal of this paper:

**Theorem 2.** Solving any matrix game can be reduced to solving an \( l^1 \) linear approximation problem. More precisely, when the game is given by an \( m \) by \( n \) matrix, we construct an \( l^1 \) approximation problem with \( 4m + 4n + 6 \) affine functions of \( m + n + 1 \) variables as well as a bijection between the equilibria for the matrix game and the solutions for the approximation problem.

**Proof of Theorem 1**

Consider any matrix game with the payoff matrix \( A \) with \( m \) rows and \( n \) columns. It can be reduced to the symmetric game with the payoff matrix

\[
M = \begin{pmatrix}
0 & A + C & -J \\
-A^T - C & 0 & J' \\
J^T & -J' & 0
\end{pmatrix},
\]

where \( J \) (rest. \( J' \)) is the column of \( m \) (resp., \( n \)) ones and the number \( C \) is such that \( A + C > 0 \). The skew-symmetric matrix \( M = -M^T \) has size \((m + n + 1) \times (m + n + 1)\). (J. von Neumann suggested another reduction resulting in a skew-symmetric matrix of size \((mn) \times (mn)\) which is not so good from computational point of view.)

The bijection between the solutions \((p, q, v)\) for the game with the matrix \( A \) and the optimal strategies for the row player in the symmetric game with the matrix \( M \) is given by

\[
(p, q) \mapsto (p, q^T, v + C)/(2 + v + C).
\]

Note that the last entry of any optimal strategy for the symmetric game above is positive because \( A + C > 0 \).

Now we start with any matrix game, with the payoff matrix \( M = -M^T \) of size \( N \) by \( N \). (In the situation above, \( N = m + n + 1 \).) Our problem is to find a column \( x = (x_i) \) (an optimal strategy) such that

\[
Mx \leq 0, \ x \geq 0, \sum x_i = 1.
\]  

This problem (3) (of finding an optimal strategy) is about finding a feasible solution for a system of linear constraints. It can be written as the following linear program with an additional variable \( t \) and the optimal value 0:

\[
t \rightarrow \min, \ Mx \leq t, \ x \geq 0, \sum x_i = 1.
\]

Now we find the largest entry \( c \) in the matrix \( M \). If \( c = 0 \), then \( M = 0 \) and the problem (1) is trivial (every mixed strategy \( x \) is optimal). So we assume that \( c > 0 \).
Adding the number $c$ to every entry of the matrix $M$, we obtain a matrix $M + c \geq 0$ (all entries $\geq 0$). The linear program (4) is equivalent to

$$t \rightarrow \min, (M + c)x \leq t, x \geq 0, \sum x_i = 1$$

(5)

in the sense that these two programs have the same feasible solutions and the same optimal solutions. The optimal value for (4) is 0 while the optimal value for (5) is $c$.

Now we can rewrite (5) as follows:

$$\| (M + c)x \|_\infty \rightarrow \min, x \geq 0, \sum x_i = 1$$

(6)

which is a Chebyshev approximation problem with additional linear constraints. We used that $M + c \geq 0$, hence $(M + c)x \geq 0$ for every feasible solution $x$ in (4). The optimal value is still $c$.

Now we rid off the constraints in (4) as follows:

$$\left\| \begin{pmatrix} (M + c)x \\ c - x \\ \sum x_i + c - 1 \\ -\sum x_i - c + 1 \end{pmatrix} \right\|_\infty \rightarrow \min .$$

(7)

Note that the optimization problems (6) and (7) have the same optimal value $c$ and every optimal solution of (6) is optimal for (7). Conversely, for every $x$ with a negative entry, the objective function in (7) is $> c$. Also, for every $x$ with $\sum x_i \neq 1$, the objective function in (7) is $> c$. So every optimal solution for (5) is feasible and hence optimal for (6).

Thus, we have reduced solving any symmetric matrix game with $N \times N$ payoff matrix to a Chebyshev approximation problem (7) with $2N + 2$ affine functions in $N$ variables.

**Proof of Theorem 2**

As in the proof of Theorem 1, we first reduce our game to a symmetric $N$ by $N$ game where $N = m + n + 1$ and set $c$ to be largest entry in the matrix $M$. The case $c = 0$ is trivial, so let $c > 0$.

We want to find a column $x$ such that

$$x \geq 0, \sum x_i = 1, Mx \leq 0.$$ 

Consider the $l^1$ approximation problem whose objective function is $f(x) =$

$$\left\| \begin{pmatrix} Mx \\ c + Mx \\ x \\ 1 - x \\ -1 + \sum x_i \\ 1 - \sum x_i \end{pmatrix} \right\|_1 = \| Mx \|_1 + \|c + Mx \|_1 + \|x \|_1 + \| 1 - x \|_1 + \| -1 + \sum x_i \|_1 + \| 1 - \sum x_i \|_1$$
with $4N + 2$ affine functions of $N$ variables.

Note that $f(x) = Nc + N$ for every optimal strategy $x$ and that $f(x) > Nc + N$ for every $x$ which is not an optimal strategy. So solving this approximation problem is equivalent to solving the matrix game.

**Remark.** Our result implies that every $l^1$ linear approximation problem can be reduced to a $l^\infty$ linear approximation problem and vice versa.

There is an obvious direct reduction of the $l^1$ approximation problem with the objective function (2) to

$$\max |f_1 \pm f_2 \pm \cdots \pm f_m| \to \min$$

which is a Chebyshev approximation problem with $2^{m-1}$ affine functions in $n$ variables. This reduction increases the size exponentially, while our reductions increases the size linearly.

**Remark.** There are methods for solving $l^1$ approximation problems alternative to the simplex method [Bloomfield–Steiger 1983]. Our reductions allows us to use these methods for solving arbitrary linear programs and matrix games.

**Remark.** A preprint with Theorem 1 appeared at arXiv [Vaserstein 2006].

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