Numerical method for a boundary value problem for a linear singularly perturbed parabolic delay differential equation of reaction-diffusion type with discontinuous source term

Parthiban Saminathan, Franklin Victor
Department of Mathematics, Bishop Heber College
Tiruchirappalli-620017, Tamil Nadu, India.
E-mail: parthitsaminathan@gmail.com, franklinvicto@gmail.com

Abstract. A singularly perturbed boundary value problem for a linear parabolic second order delay differential equation of reaction-diffusion type with discontinuous source term is considered in the rectangle domain $\Omega = \{(x,t) : 0 < x < 2, 0 < t \leq T\}$. Discontinuity occurs in the source term at $(d,t) \in \Omega$. As the highest order space derivative is multiplied by a singular perturbation parameter, the components of the solution exhibit boundary layers at $(x,t) = (0,t)$ and $(x,t) = (2,t)$ and interior layers at $(x,t) = (1,t)$ and/or at $(x,t) = (d,t)$ and $(x,t) = (1 + d,t)$ with respect to the position of $(d,t)$ in $\Omega$. A numerical method which uses classical finite difference scheme on a Shishkin piecewise uniform mesh is suggested to approximate the solution. The method is proved to be first order convergent uniformly for all the values of singular perturbation parameters. Numerical illustrations are presented so that the theoretical results are supported.

1. Introduction
Singularly perturbed delay differential equations are used to model various phenomena in population dynamics, physiology, control systems and so on. But the restriction that the source function is continuous may not be practical in certain cases. In this paper, a singularly perturbed system of linear parabolic delay differential equations of reaction-diffusion type with discontinuous source terms is considered. The components of the solution of these systems exhibit boundary and interior layers due to the presence of the perturbation parameters, delay term and discontinuous source terms.

In [4], a singularly perturbed boundary value problem for a linear parabolic second order delay differential equation of reaction-diffusion type has been considered. As the highest order space derivative is multiplied by a singular perturbation parameter, the solution exhibits boundary layers. Also, the delay term that occurs in the space variable gives rise to interior layers. A numerical method which uses classical finite difference scheme on a Shishkin piecewise uniform mesh is suggested to approximate the solution. The method is proved to be first order convergent uniformly with respect to the singular perturbation parameter. Numerical illustrations are also presented. In the present paper we extend the work done in the above paper to a linear parabolic second order delay differential equation of reaction-diffusion type with discontinuous source term.
2. The continuous problem

A singularly perturbed linear parabolic second order delay differential equation of reaction-diffusion type with discontinuous source term is considered in the rectangle domain \( \Omega = \{(x,t) : 0 < x < 2, 0 < t \leq T\} \). Discontinuity occurs in the source term at \((d,t) \in \Omega\). The following notations are introduced \( \Omega^+ = (0,d) \times (0,T), \Omega^- = [d-2] \times [0,T], \Omega = (d,2) \times (0,T), \Omega^T = \Omega^+ \cup \Omega^- \) and the jump at \((d,t)\) in any function \( w \) is denoted by \( |w|(x,t) = w(d+,t) - w(d-,t) \). The corresponding two-point boundary value problem is

\[
Lu(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + a(x,t)u(x,t) + b(x,t)u(x-1,t) = f(x,t) \text{ on } \Omega^- \cup \Omega^+,
\]

\[
u \text{ given on } \Gamma, f(d-,t) \neq f(d+,t) \text{ and } u(x,t) = \chi(x,t), (x,t) \in [-1,0] \times [0,T],
\]

where \( \tilde{\Omega} = \{(0,1-) \times (0,T]\} \cup \{(1+,2) \times (0,T]\}, \tilde{\Omega}^- = \{(0,1-) \times [0,T]\} \cup \{(1+,2] \times [0,T]\}, \tilde{\Omega} = \{(0,d) \times (0,T]\} \cup \{(d,1-) \times (0,T]\}, \tilde{\Omega}_0 = \{(0,0) \times (0,T]\} \cup \{(0,2) \times (0,T]\}, \tilde{\Omega}_1 = \{(d,1-) \times [0,T]\} \cup \{(0,2) \times [0,T]\}, \tilde{\Omega}_1 = \{(1+,d) \times [0,T]\} \cup \{(d,2) \times [0,T]\}, \tilde{\Omega}_2 = \{(1+,d) \times [0,T]\} \cup \{(1+,d) \times [0,T]\} \cup \{(d,2) \times [0,T]\}, \Omega = \Omega_1 \cup \Gamma, \Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R \text{ with } u(0,t) = \chi(x,t) \text{ on } \Gamma_L = \{(0,t) : 0 \leq t \leq T\}, u(0,0) = \phi_b(x) \text{ on } \Gamma_B = \{(x,0) : 0 \leq x \leq 2\}, \text{ and } u(0,t) = \phi_R(t) \text{ on } \Gamma_R = \{(2,t) : 0 \leq t \leq T\}.

Assumption 2.1. The function \( \chi \) is sufficiently smooth on \([-1,0] \times [0,T]\).

Assumption 2.2. For all \((x,t) \in [0,2] \times [0,T]\), the functions \( a(x,t) \) and \( b(x,t) \) satisfy

\[
a(x,t) + b(x,t) > 2\alpha, \text{ for some real number } \alpha > 0
\]

and

\[
b(x,t) < 0
\]

Assumption 2.3. The functions \( a(x,t) \) and \( b(x,t) \) are in \( C^2([0,2] \times [0,T]) \).

Because \( f(x,t) \) is discontinuous at \((d,t)\), the solution \( u(x,t) \) need not necessarily have a continuous second order derivative at the point \((d,t)\). Thus \( u(x,t) \notin C^2([0,2] \times [0,T]) \), but the first derivative of the solution exists and is continuous on \((0,2) \times (0,T]\), as is shown in Theorem 3.1.

The cases (i) \((d,t) \in (0,1) \times (0,T]\), (ii) \((d,t) \in (1,2) \times (0,T]\) and (iii) \((d,t) = (1,1) \times (0,T]\) are considered separately. When \( d = (1,1) \), the problem (1) is same as in [4] and hence can be solved by using the same numerical method constructed in [4]. The cases (i) \( d \in (0,1) \times (0,T]\) and (ii) \( d \in (1,2) \times (0,T]\) are discussed elaborately in this section.

The problem (1) can be rewritten as follows for the case (i),

\[
L_1 u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + a(x,t)u(x,t) = g(x,t), \text{ on } \tilde{\Omega}_1
\]

where \( g(x,t) = f(x,t) - b(x,t)\chi(x-1,t) \)

\[
L_2 u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + a(x,t)u(x,t) + b(x,t)u(x-1,t) = f(x,t), \text{ on } \tilde{\Omega}_2
\]

and as follows for the case (ii),

\[
L_1 u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + a(x,t)u(x,t) = g(x,t), \text{ on } (0,1) \times (0,T]
\]
where \( g(x, t) = f(x, t) - b(x, t)\chi(x - 1, t) \)

\[
L_2 u(x, t) = \frac{\partial u}{\partial t}(x, t) - \varepsilon \frac{\partial^2 u}{\partial x^2}(x, t) + a(x, t)u(x, t) + b(x, t)u(x - 1, t) = f(x, t), \text{ on } \tilde{\Omega}_3
\]

\( u(x, t) = \chi(x, t) \) on \([-1, 0] \times [0, T], \) \( f(d-, t) \neq f(d+, t), u(0, t) = \chi(0, t), u(d-, t) = u(d+, t), \)

\[
\frac{\partial u}{\partial x}(d-, t) = \frac{\partial u}{\partial x}(d+, t), u(x, 0) = \phi_B(x) \text{ on } \Gamma_{B_1} = \{(x, 0) : 0 \leq x \leq 1\}, u(1-, t) = u(1+, t),
\]

\[
\frac{\partial u}{\partial t}(1-, t) = \frac{\partial u}{\partial t}(1+, t), u(x, 0) = \phi_B(x) \text{ on } \Gamma_{B_2} = \{(x, 0) : 1 \leq x \leq 2\}, u(2, t) = \phi_R(t) \text{ on } \Gamma_R.
\]

The reduced problem corresponding to (4) - (5) is defined by

\[
\frac{\partial u_0}{\partial t}(x, t) + a(x, t)u_0(x, t) = g(x, t), \text{ on } \tilde{\Omega}_1
\]

\[
\frac{\partial u_0}{\partial t}(x, t) + a(x, t)u_0(x, t) + b(x, t)u_0(x - 1, t) = f(x, t), \text{ on } \tilde{\Omega}_2.
\]

and reduced problem corresponding to (6) - (7) is defined by

\[
\frac{\partial u_0}{\partial t}(x, t) + a(x, t)u_0(x, t) = g(x, t), \text{ on } (0, 1) \times (0, T)
\]

\[
\frac{\partial u_0}{\partial t}(x, t) + a(x, t)u_0(x, t) + b(x, t)u_0(x - 1, t) = f(x, t), \text{ on } \tilde{\Omega}_3.
\]

In general as \( u_0(x, t) \) need not satisfy \( u_0(0, t) = u(0, t) \) and \( u_0(2, t) = u(2, t), \) the solution \( u(x, t) \) exhibits boundary layers at \( x = 0 \) and \( x = 2. \) In addition to that, as \( u_0(1-, t) \) need not be equal to \( u_0(1+, t), \) the solution \( u(x, t) \) exhibits interior layers at \( x = 1. \) Moreover \( f(d-, t) \neq f(d+, t), \) so that \( u_0(d-, t) \) need not be equal to \( u_0(d+, t), \) the solution \( u(x, t) \) exhibits additional layers at \( x = d \) and \( x = 1 + d \) in case (i) and at \( x = d \) in case (ii).

The norm and compatibility conditions fulfilled at the corners \((0,0)\) and \((2,0)\) of \( \Gamma \) are same as in [4].

3. Analytical results

**Theorem 3.1.** The given problem (1) has a solution

\[
u \in \mathcal{C} = C^4_\lambda([0, 2] \times [0, T]) \cap C^2_\lambda((0, 2) \times (0, T)) \cap C^1_\lambda(\overline{\Omega}\setminus\{d\}).
\]

**Proof:** The proof is by construction.

**Case (i):** Let \( y_1, z_1, y_2 \) and \( z_2 \) be the particular solutions of

\[
\frac{\partial y_1}{\partial t}(x, t) - \varepsilon \frac{\partial^2 y_1}{\partial x^2}(x, t) + a(x, t)y_1(x, t) = g(x, t), (x, t) \in (0, d) \times (0, T)
\]

\[
\frac{\partial z_1}{\partial t}(x, t) - \varepsilon \frac{\partial^2 z_1}{\partial x^2}(x, t) + a(x, t)z_1(x, t) = g(x, t), (x, t) \in (d, 1) \times (0, T)
\]

\[
\frac{\partial y_2}{\partial t}(x, t) - \varepsilon \frac{\partial^2 y_2}{\partial x^2}(x, t) + a(x, t)y_2(x, t) = f(x, t) - b(x, t)y_1(x - 1, t), (x, t) \in (1, 1 + d) \times (0, T)
\]

\[
\frac{\partial z_2}{\partial t}(x, t) - \varepsilon \frac{\partial^2 z_2}{\partial x^2}(x, t) + a(x, t)z_2(x, t) = f(x, t) - b(x, t)z_1(x - 1, t), (x, t) \in (1 + d, 2) \times (0, T).
\]

Consider the function,

\[
u(x, t) = \begin{cases}
y_1(x, t) + (u(0, t) - y_1(0, t))\psi_1(x, t) + A\psi_2(x, t), & (x, t) \in (0, d) \times (0, T) \\
z_1(x, t) + B\psi_3(x, t) + C\psi_4(x, t), & (x, t) \in (d, 1) \times (0, T) \\
y_2(x, t) + D\psi_5(x, t) + E\psi_6(x, t), & (x, t) \in (1, 1 + d) \times (0, T) \\
z_2(x, t) + (u(2, t) - z_2(2, t))\psi_8(x, t) + F\psi_7(x, t), & (x, t) \in (1 + d, 2) \times (0, T)
\end{cases}
\]
where $\psi'_i, i = 1, 2, \ldots, 8$, are the solutions of

$$\frac{\partial \psi_1}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_1}{\partial x^2}(x, t) + a(x, t)\psi_1(x, t) = 0, (x, t) \in (0, d) \times (0, T],$$

$$\psi_1(x, 0) = 0, \psi_1(0, t) = 1, \psi_1(d, t) = 0,$$

$$\frac{\partial \psi_2}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_2}{\partial x^2}(x, t) + a(x, t)\psi_2(x, t) = 0, (x, t) \in (0, d) \times (0, T],$$

$$\psi_2(x, 0) = 0, \psi_2(0, t) = 0, \psi_2(d, t) = 1,$$

$$\frac{\partial \psi_3}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_3}{\partial x^2}(x, t) + a(x, t)\psi_3(x, t) = 0, (x, t) \in (d, 1) \times (0, T],$$

$$\psi_3(x, 0) = 0, \psi_3(d, t) = 1, \psi_3(1, t) = 0,$$

$$\frac{\partial \psi_4}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_4}{\partial x^2}(x, t) + a(x, t)\psi_4(x, t) = 0, (x, t) \in (d, 1) \times (0, T],$$

$$\psi_4(x, 0) = 0, \psi_4(d, t) = 0, \psi_4(1, t) = 1,$$

$$\frac{\partial \psi_5}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_5}{\partial x^2}(x, t) + a(x, t)\psi_5(x, t) = 0, (x, t) \in (1, 1+d) \times (0, T],$$

$$\psi_5(x, 0) = 0, \psi_5(1, t) = 1, \psi_5(1+d, t) = 0,$$

$$\frac{\partial \psi_6}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_6}{\partial x^2}(x, t) + a(x, t)\psi_6(x, t) = 0, (x, t) \in (1, 1+d) \times (0, T],$$

$$\psi_6(x, 0) = 0, \psi_6(1, t) = 0, \psi_6(1+d, t) = 1,$$

$$\frac{\partial \psi_7}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_7}{\partial x^2}(x, t) + a(x, t)\psi_7(x, t) = 0, (x, t) \in (1+d, 2) \times (0, T],$$

$$\psi_7(x, 0) = 0, \psi_7(1+d, t) = 1, \psi_7(2, t) = 0,$$

$$\frac{\partial \psi_8}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_8}{\partial x^2}(x, t) + a(x, t)\psi_8(x, t) = 0, (x, t) \in (1+d, 2) \times (0, T],$$

$$\psi_8(x, 0) = 0, \psi_8(1+d, t) = 0, \psi_8(2, t) = 1.$$

Here, $A, B, C, D, E$ and $F$ are constants evaluated from the following equations which are derived using the conditions that $u(x, t)$ and $\frac{\partial u}{\partial x}(x, t)$ are continuous at $x = d, x = 1$ and $x = 1+d$.

$$A - B = z_1(d+, t) - y_1(d-, t), C - D = y_2(1+, t) - z_1(1-, t)$$

$$E - F = z_2((1+d)+, t) - y_2((1+d)-, t)$$

$$A \frac{\partial \psi_2}{\partial x}(d, t) - B \frac{\partial \psi_3}{\partial x}(d, t) - C \frac{\partial \psi_4}{\partial x}(d, t) = \frac{\partial z_1}{\partial x}(d+, t) - \left( \frac{\partial u}{\partial x}(0, t) - \frac{\partial y_1}{\partial x}(0, t) \right) \psi_1(d, t)$$

$$- (u(0, t) - y_1(0, t)) \frac{\partial \psi_1}{\partial x}(d+, t) - \frac{\partial y_1}{\partial x}(d-, t)$$

$$B \frac{\partial \psi_3}{\partial x}(1, t) + C \frac{\partial \psi_4}{\partial x}(1, t) - D \frac{\partial \psi_5}{\partial x}(1, t) - E \frac{\partial \psi_6}{\partial x}(1, t) = \frac{\partial y_2}{\partial x}(1+, t) - \frac{\partial z_1}{\partial x}(1-, t)$$

$$D \frac{\partial \psi_5}{\partial x}(1+d, t) + E \frac{\partial \psi_6}{\partial x}(1+d, t) - F \frac{\partial \psi_7}{\partial x}(1+d, t)$$

$$= \frac{\partial z_2}{\partial x}((1+d)+, t) - \frac{\partial y_2}{\partial x}((1+d)-, t) + (u(2, t) - z_2(2, t)) \frac{\partial \psi_8}{\partial x}(1+d, t)$$

$$+ \left( \frac{\partial u}{\partial x}(2, t) - \frac{\partial z_2}{\partial x}(2, t) \right) \psi_8(1+d, t)$$
Note that on $\Omega, 0 \leq \psi_1, \psi_2, \ldots, \psi_8 \leq 1$. Therefore $\psi_1, \psi_2, \ldots, \psi_8$ cannot have an internal maximum and thus $\frac{\partial \psi}{\partial x} (x, t) \neq 0, (x, t) \in \Omega, i = 1, 2, \ldots, 8$. Hence in the determinant corresponding to the above system does not vanish ensuring the existence of the constants $A, B, \ldots, F$.

**Case (ii) :** Let $y, z_1$ and $z_2$ be the particular solutions of

$$
\frac{\partial y}{\partial t} (x, t) - \varepsilon \frac{\partial^2 y}{\partial x^2} (x, t) + a(x, t)y(x, t) = g(x, t), (x, t) \in (0, 1) \times (0, T]
$$

$$
\frac{\partial z_1}{\partial t} (x, t) - \varepsilon \frac{\partial z_1}{\partial x^2} (x, t) + a(x, t)z_1(x, t) = f(x, t) - b(x, t)y(x - 1, t), (x, t) \in (1, d) \times (0, T]
$$

$$
\frac{\partial z_2}{\partial t} (x, t) - \varepsilon \frac{\partial^2 z_2}{\partial x^2} (x, t) + a(x, t)z_2(x, t) = f(x, t) - b(x, t)y(x - 1, t), (x, t) \in (d, 2) \times (0, T].
$$

Consider the function,

$$
u(x, t) = \begin{cases} 
  y(x, t) + (u(0, t) - y(0, t))\psi_1(x, t) + A\psi_2(x, t), & (x, t) \in (0, 1) \times (0, T) \\
  z_1(x, t) + B\psi_3(x, t) + C\psi_4(x, t), & (x, t) \in (1, d) \times (0, T] \\
  z_2(x, t) + (u(2, t) - z_2(2, t))\psi_5(x, t) + D\psi_5(x, t), & (x, t) \in (d, 2) \times (0, T] 
\end{cases}
$$

where $\psi_i's, i = 1, 2, \ldots, 6$, are the solutions of

$$
\frac{\partial \psi_1}{\partial t} (x, t) - \varepsilon \frac{\partial^2 \psi_1}{\partial x^2} (x, t) + a(x, t)\psi_1(x, t) = 0, (x, t) \in (0, 1) \times (0, T], \\
\psi_1(x, 0) = 0, \psi_1(0, t) = 1, \psi_1(1, t) = 0,
$$

$$
\frac{\partial \psi_2}{\partial t} (x, t) - \varepsilon \frac{\partial^2 \psi_2}{\partial x^2} (x, t) + a(x, t)\psi_2(x, t) = 0, (x, t) \in (0, 1) \times (0, T], \\
\psi_2(x, 0) = 0, \psi_2(0, t) = 0, \psi_2(1, t) = 1,
$$

$$
\frac{\partial \psi_3}{\partial t} (x, t) - \varepsilon \frac{\partial^2 \psi_3}{\partial x^2} (x, t) + a(x, t)\psi_3(x, t) = 0, (x, t) \in (1, d) \times (0, T], \\
\psi_3(x, 0) = 0, \psi_3(1, t) = 1, \psi_3(d, t) = 0,
$$

$$
\frac{\partial \psi_4}{\partial t} (x, t) - \varepsilon \frac{\partial^2 \psi_4}{\partial x^2} (x, t) + a(x, t)\psi_4(x, t) = 0, (x, t) \in (1, d) \times (0, T], \\
\psi_4(x, 0) = 0, \psi_4(1, t) = 0, \psi_4(d, t) = 1,
$$

$$
\frac{\partial \psi_5}{\partial t} (x, t) - \varepsilon \frac{\partial^2 \psi_5}{\partial x^2} (x, t) + a(x, t)\psi_5(x, t) = 0, (x, t) \in (d, 2) \times (0, T], \\
\psi_5(x, 0) = 0, \psi_5(d, t) = 1, \psi_5(2, t) = 0,
$$

$$
\frac{\partial \psi_6}{\partial t} (x, t) - \varepsilon \frac{\partial^2 \psi_6}{\partial x^2} (x, t) + a(x, t)\psi_6(x, t) = 0, (x, t) \in (d, 2) \times (0, T], \\
\psi_6(x, 0) = 0, \psi_6(d, t) = 0, \psi_6(2, t) = 1.
$$

Here $A, B, C$ and $D$ are constants evaluated from the following equations which are derived
using the conditions that $u(x, t)$ and $\frac{\partial u}{\partial x}(x, t)$ are continuous at $x = 1$ and $x = 1 + d$.

$$\frac{A}{x} \frac{\partial \psi_2(x, t)}{\partial x} - B \frac{\partial \psi_3(x, t)}{\partial x} - C \frac{\partial \psi_4(x, t)}{\partial x} = \frac{\partial z_1(x, t)}{\partial x} + \left( \frac{\partial u}{\partial x}(x, t) - \frac{\partial y}{\partial x}(x, t) \right) \psi_1(x, t)$$

$$- (u(0, t) - y(0, t)) \frac{\partial \psi_1(x, t)}{\partial x} - \frac{\partial y(1-, t)}{\partial x}$$

$$B \frac{\partial \psi_3(x, t)}{\partial x} + C \frac{\partial \psi_4(x, t)}{\partial x} - D \frac{\partial \psi_5(x, t)}{\partial x} = \frac{\partial z_2(x, t)}{\partial x} + \left( \frac{\partial u}{\partial x}(x, t) - \frac{\partial y(2, t)}{\partial x} \right) \psi_6(x, t)$$

$$+ (u(2, t) - z_2(2, t)) \frac{\partial \psi_6(x, t)}{\partial x} - \frac{\partial z_1(x, t)}{\partial x}$$

Note that on $\Omega, 0 \leq \psi_1, \psi_2, \ldots, \psi_6 \leq 1$. Therefore $\psi_1, \psi_2, \ldots, \psi_6$ cannot have an internal maximum and thus $\frac{\partial \psi_i(x, t)}{\partial x} \neq 0, (x, t) \in \Omega, i = 1, 2, \ldots, 6$. Hence in the determinant corresponding to the above system does not vanish ensuring the existence of the constants $A, B, C, D$.

**Case (iii):** Let $y$ and $z$ be the particular solutions of

$$\frac{\partial y}{\partial t}(x, t) - \varepsilon \frac{\partial^2 y}{\partial x^2}(x, t) + a(x, t)y(x, t) = g(x, t), (x, t) \in (0, 1) \times (0, T]$$

$$\frac{\partial z}{\partial t}(x, t) - \varepsilon \frac{\partial^2 z}{\partial x^2}(x, t) + a(x, t)z(x, t) = f(x, t) - b(x, t)y(x - 1, t), (x, t) \in (1, 2) \times (0, T].$$

Consider the function,

$$u(x, t) = \begin{cases} y(x, t) + (u(0, t) - y(0, t))\psi_1(x, t) + A\psi_2(x, t), & (x, t) \in (0, 1) \times (0, T] \\ z(x, t) + (u(2, t) - z(2, t))\psi_4(x, t) + B\psi_3(x, t), & (x, t) \in (1, 2) \times (0, T] \end{cases}$$

where $\psi_i, i = 1, 2, 3, 4$, are the solutions of

$$\frac{\partial \psi_1}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_1}{\partial x^2}(x, t) + a(x, t)\psi_1(x, t) = 0, (x, t) \in (0, 1) \times (0, T],$$

$$\psi_1(x, 0) = 0, \psi_1(0, t) = 1, \psi_1(1, t) = 0,$$

$$\frac{\partial \psi_2}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_2}{\partial x^2}(x, t) + a(x, t)\psi_2(x, t) = 0, (x, t) \in (0, 1) \times (0, T],$$

$$\psi_2(x, 0) = 0, \psi_2(0, t) = 0, \psi_2(1, t) = 1,$$

$$\frac{\partial \psi_3}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_3}{\partial x^2}(x, t) + a(x, t)\psi_3(x, t) = 0, (x, t) \in (1, 2) \times (0, T],$$

$$\psi_3(x, 0) = 0, \psi_3(1, t) = 1, \psi_3(2, t) = 0,$$

$$\frac{\partial \psi_4}{\partial t}(x, t) - \varepsilon \frac{\partial^2 \psi_4}{\partial x^2}(x, t) + a(x, t)\psi_4(x, t) = 0, (x, t) \in (1, 2) \times (0, T],$$

$$\psi_4(x, 0) = 0, \psi_4(1, t) = 0, \psi_4(2, t) = 1.$$

Here $A$ and $B$ are constants evaluated from the following equations which are derived using the conditions that $u(x, t)$ and $\frac{\partial u}{\partial x}(x, t)$ are continuous at $x = 1$. 

---

**IOP Publishing**

Journal of Physics: Conference Series 1850 (2021) 012063 doi:10.1088/1742-6596/1850/1/012063
\[ A - B = z(1+, t) - y(1-, t) \]
\[ A \frac{\partial \psi_2}{\partial x}(1, t) - B \frac{\partial \psi_3}{\partial x}(1, t) = \frac{\partial z}{\partial x}(1+, t) + \left( \frac{\partial u}{\partial x}(2, t) - \frac{\partial z}{\partial x}(2, t) \right) \psi_4(1, t) \]
\[ + (u(2, t) - z(2, t)) \frac{\partial \psi_4}{\partial x}(1, t) - \frac{\partial y}{\partial x}(1-, t) \]
\[ - \left( \frac{\partial u}{\partial x}(0, t) - \frac{\partial y}{\partial x}(0, t) \right) \psi_1(1, t) - (u(0, t) - y(0, t)) \frac{\partial \psi_1}{\partial x}(1, t) \]

Note that on \( \Omega, 0 \leq \psi_1, \psi_2, \psi_3, \psi_4 \leq 1 \). Therefore \( \psi_1, \psi_2, \psi_3, \psi_4 \) cannot have an internal maximum and thus \( \frac{\partial \psi_2}{\partial x}(x, t) \neq 0, (x, t) \in \Omega, i = 1, 2, 3, 4 \). Hence in the determinant corresponding to the above system does not vanish ensuring the existence of the constants \( A, B \).

The decomposition of \( u \) into smooth and singular components, the estimates on the bounds of the derivatives of \( u \), its smooth and singular components are derived analogous to the methods used in [4]. In the following section, the piecewise uniform Shishkin mesh used for the discretisation of the BVPs for the different cases are presented.

4. The Shishkin mesh

Case (i):

A piecewise uniform Shishkin mesh with \( M \times N \) mesh-intervals is now constructed.

Let \( \Omega^M = \{ t_k \}_{k=1}^M, \Omega^N = \{ t_k \}_{k=0}^N \). \( \Omega^M \) and \( \Omega^N \) are mesh-intervals on \([0, T]\). The mesh \( \Omega^M \) is chosen to be a uniform mesh with \( M \) mesh-intervals on \([0, T]\). The mesh \( \Omega^N \) is chosen to be a piecewise-uniform mesh with \( N \) mesh-intervals on \([0, T]\). The interval \([0, d]\) is divided into 3 sub-intervals as follows \([0, \tau], (\tau, d - \tau) \) and \((d - \tau, d]\). The parameter \( \tau \), which determines the points separating the uniform meshes, is defined by

\[ \tau = \min \left\{ \frac{d}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N \right\}. \quad (12) \]

Then, on the sub-interval \((\tau, d - \tau]\) a uniform mesh with \( \frac{N}{8} \) mesh points is placed and on each of the sub-intervals \([0, \tau]\) and \((d - \tau, d]\), a uniform mesh of \( \frac{N}{8} \) mesh points is placed.

Similarly, the interval \([d, 1]\) is also divided into 3 sub-intervals \((d, d + \eta], (d + \eta, 1 - \eta]\) and \((1 - \eta, 1]\), where

\[ \eta = \min \left\{ \frac{1 - d}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N \right\}. \quad (13) \]

the interval \([d, 1 + d]\) is divided into 3 sub-intervals \([d, d + \eta], (d + \eta, 1 - \eta]\) and \((1 - \eta, 1]\) and the interval \((1 + d, 2]\) is divided into 3 sub-intervals \((1 + d, 1 + d + \eta], (1 + d + \eta, 2 - \eta]\) and \((2 - \eta, 2]\) having the same mesh pattern as in \([0, 1]\).

In practice, it is convenient to take

\[ N = 16k, k \geq 2. \quad (14) \]
From the above construction of $\Omega^{M,N}$, it is clear that the transition points $\{\tau, d - \tau, d + \eta, 1 - \eta, 1 + \tau, 1 + d - \tau, 1 + d + \eta, 2 - \eta\}$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}, h_{j+1} = x_{j+1} - x_j$ and if $x_j$ is a transition point, then $h_j^- = x_j - x_{j-1}, h_j^+ = x_{j+1} - x_j, J = \{x_j : h_j^- \neq h_j^+\}$.

**Case (ii):**

In this case, a piecewise uniform Shishkin mesh with $M \times N$ mesh-intervals is now constructed.

Let $\Omega^M = \{t_k\}_{k=1}^M, \Omega^N = \{t_k\}_{k=0}^N$, $\Omega^{N}_x = \Omega_{x1}^N \cup \Omega_{x2}^N \cup \Omega_{x3}^N$ where $\Omega_{x1}^N = \{x_j\}_{j=1}^{N-1}, \Omega_{x2}^N = \{x_j\}_{j=\frac{N}{4}}^{\frac{N}{4}+1}, \Omega_{x3}^N = \{x_j\}_{j=\frac{N}{4}+1}^{\frac{N}{2}}$ and $\frac{N}{2} = \Omega_{x3}^N \cap \Omega_{x4}^N \cap \Omega_{x5}^N$. Then $\Omega^M \times \Omega^N = \Omega^M_{x1} \times \Omega_{x1}^N \cup \Omega^M_{x1} \times \Omega_{x2}^N \cup \Omega^M_{x1} \times \Omega_{x3}^N = \Omega^M_{x4} \times \Omega_{x4}^N \cup \Omega^M_{x4} \times \Omega_{x5}^N \cup \Omega^M_{x5} \times \Omega_{x5}^N$, and if $\Omega^M \times \Omega^N = \Omega^M_{x1} \times \Omega_{x1}^N \cup \Omega^M_{x1} \times \Omega_{x2}^N \cup \Omega^M_{x1} \times \Omega_{x3}^N$, and if $\Omega^M \times \Omega^N = \Omega^M_{x4} \times \Omega_{x4}^N \cup \Omega^M_{x4} \times \Omega_{x5}^N \cup \Omega^M_{x5} \times \Omega_{x5}^N$. The interval $[0, 1]$ is divided into 3 sub-intervals as follows $[0, \tau], (\tau, 1 - \tau]$ and $(1 - \tau, 1)$. The parameter $\tau$, which determines the points separating the uniform meshes, is defined by

$$\tau = \min \left\{ \frac{1}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N \right\}. \quad (15)$$

Then, on the sub-interval $(\tau, 1 - \tau]$ a uniform mesh with $\frac{N}{6}$ mesh points is placed and on each of the sub-intervals $[0, \tau]$ and $(1 - \tau, 1)$, a uniform mesh of $\frac{N}{12}$ mesh points is placed.

Similarly, the interval $(1, d]$ is also divided into 3 sub-intervals $(1, 1 + \eta], (1 + \eta, d - \eta]$ and $(d - \eta, d]$, where

$$\eta = \min \left\{ \frac{d - 1}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N \right\}, \quad (16)$$

the interval $(d, 2]$ is divided into 3 sub-intervals $(d, d + \gamma], (d + \gamma, 2 - \gamma]$ and $(2 - \gamma, 2]$, where

$$\gamma = \min \left\{ \frac{2 - d}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N \right\}, \quad (17)$$

In practice, it is convenient to take

$$N = 12k, k \geq 2. \quad (18)$$

From the above construction of $\Omega^{M,N}$, it is clear that the transition points $\{\tau, d - \tau, 1 + \eta, d - \eta, d + \gamma, 2 - \gamma\}$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}, h_{j+1} = x_{j+1} - x_j$ and if $x_j$ is a transition point, then $h_j^- = x_j - x_{j-1}, h_j^+ = x_{j+1} - x_j, J = \{x_j : h_j^- \neq h_j^+\}$.

**5. The discrete problem**

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be first order parameter-uniform in time and essentially first order parameter-uniform in the space variable.

The discrete initial boundary value problem is now defined on any mesh by the finite difference method

$$L^{M,N}U(x_j, t_k) = D^- \frac{U(x_j, t_k)}{\varepsilon} + a(x_j, t_k)U(x_j, t_k) + b(x_j, t_k)U(x_j - 1, t_k) = f(x_j, t_k) \quad \text{on} \quad \Omega^{M,N} \quad (19)$$
Lemma 5.1. We now state and prove the main theoretical result of this section.

The problem (19) can be rewritten as,

\[ L_1^{M,N} U(x_j, t_k) = D_t^k U(x_j, t_k) - \varepsilon \delta^2 U(x_j, t_k) + a(x_j, t_k) U(x_j, t_k) = g(x_j, t_k) \quad \text{on } \Omega_1^{M,N} \cup \Omega_2^{M,N} \]

(20)

where \( g(x_j, t_k) = f(x_j, t_k) - b(x_j, t_k) \chi(x_j - 1, t_k) \)

\[ L_2^{M,N} U(x_j, t_k) = D_t^k U(x_j, t_k) - \varepsilon \delta^2 U(x_j, t_k) + a(x_j, t_k) U(x_j, t_k) + b(x_j, t_k) U(x_j - 1, t_k) = f(x_j, t_k) \quad \text{on } \Omega_3^{M,N} \]

(21)

in case (i) and

\[ L_1^{M,N} U(x_j, t_k) = D_t^k U(x_j, t_k) - \varepsilon \delta^2 U(x_j, t_k) + a(x_j, t_k) U(x_j, t_k) = g(x_j, t_k) \quad \text{on } \Omega_1^{M,N} \]

(22)

where \( g(x_j, t_k) = f(x_j, t_k) - b(x_j, t_k) \chi(x_j - 1, t_k) \)

\[ L_2^{M,N} U(x_j, t_k) = D_t^k U(x_j, t_k) - \varepsilon \delta^2 U(x_j, t_k) + a(x_j, t_k) U(x_j, t_k) + b(x_j, t_k) U(x_j - 1, t_k) = f(x_j, t_k) \quad \text{on } \Omega_3^{M,N} \]

(23)

in case (ii).

We now state and prove the main theoretical result of this section.

**Lemma 5.1.** Let \( u(x_j, t_k) \) denote the exact solution of (1) and \( U(x_j, t_k) \) the solution of (19). Then, for \( 0 \leq j \leq N, 0 \leq k \leq M \),

\[ ||U(x_j, t_k) - u(x_j, t_k)|| \leq C(M^{-1} + N^{-1} \ln N). \]

(24)

6. Numerical Illustration

The \( \varepsilon \)-uniform convergence of the numerical method proposed in this section is illustrated through examples presented in below.

**Example:** Consider the following problem

\[
\frac{\partial u}{\partial t}(x,t) - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + (3 + t)u(x,t) - u(x-1,t) = f(x,t), \quad \text{for } (x,t) \in (0, 2) \times [0, T],
\]

\[ u(x,t) = 1 \quad \text{for } x \in [-1, 0] \times [0, T], u(0,t) = 1, u(x,0) = 1, u(2,t) = 1. \]

(25)

where \( f(x,t) = \begin{cases} 5, & (x,t) \in (0,d) \times [0,T] \\ 0, & (x,t) \in (d,2) \times [0,T]. \end{cases} \)

We first investigate the robustness of the temporal discretization. In results shown in Tables 1 and 3 we have fixed the number of intervals in spatial (Shishkin) mesh to be \( N = 96 \), and present results for various \( M \) and \( \varepsilon \). Note the fully first-order convergence as predicted in Theorem 5.1. In Tables 2 and 4, we fix the number of time steps to be \( M = 16 \) and allow \( N \) to vary. Now we observe almost first-order convergence, again consistent with Theorem 5.1.
Case (i) \((d, t) = (0.4, t)\)

### Table 1. Values of \(D_{M,N}, p_{M,N}, p^*, C^{M,N}_{p^*}\) and \(C^*_{p^*}\) for \(\varepsilon = \eta/64, N = 32\) and \(\alpha = 1.9\)

| \(\eta\) | Number of mesh points | \(M\) | \(256\) | \(512\) | \(1024\) | \(2048\) |
|---|---|---|---|---|---|---|
| \(2^{-3}\) | 0.178E-03 | 0.457E-04 | 0.116E-04 | 0.292E-05 | 0.733E-06 |
| \(2^{-6}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |
| \(2^{-9}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |
| \(2^{-12}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |
| \(2^{-15}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |

\(D_{M,N}\): \(\varepsilon\)-uniform maximum point-wise errors, \(p_{M,N}\): \(\varepsilon\)-uniform order of local convergence, \(p^*\): \(\varepsilon\)-uniform order of convergence, \(C^{M,N}_{p^*} = D_{M,N}^{N_{p^*}}\) and \(C^*_{p^*}\): error constant.

- t-order of convergence, \(p^* = 1.96\)
- The error constant, \(C^*_{p^*} = 3.21\)

Where \(D_{M,N}\) - the \(\varepsilon\)-uniform maximum point-wise errors, \(p_{M,N}\) - the \(\varepsilon\)-uniform order of local convergence, \(p^*\) - the \(\varepsilon\)-uniform order of convergence, \(C^{M,N}_{p^*} = D_{M,N}^{N_{p^*}}\) and \(C^*_{p^*}\) - error constant.

### Table 2. Values of \(D_{M,N}, p_{M,N}, p^*, C^{M,N}_{p^*}\) and \(C^*_{p^*}\) for \(\varepsilon = \eta/64, M = 16\) and \(\alpha = 1.9\)

| \(\eta\) | Number of mesh points | \(N\) | \(128\) | \(256\) | \(512\) | \(1024\) | \(2048\) |
|---|---|---|---|---|---|---|---|
| \(2^{-3}\) | 0.2040E-01 | 0.1100E-01 | 0.5611E-02 | 0.2820E-02 | 0.1412E-02 |
| \(2^{-6}\) | 0.6204E-02 | 0.4041E-02 | 0.2399E-02 | 0.1367E-02 | 0.7622E-03 |
| \(2^{-9}\) | 0.6204E-02 | 0.4041E-02 | 0.2399E-02 | 0.1367E-02 | 0.7622E-03 |
| \(2^{-12}\) | 0.6204E-02 | 0.4041E-02 | 0.2399E-02 | 0.1367E-02 | 0.7622E-03 |
| \(2^{-15}\) | 0.6204E-02 | 0.4041E-02 | 0.2399E-02 | 0.1367E-02 | 0.7622E-03 |

\(D_{M,N}\): \(\varepsilon\)-uniform maximum point-wise errors, \(p_{M,N}\): \(\varepsilon\)-uniform order of local convergence, \(p^*\): \(\varepsilon\)-uniform order of convergence, \(C^{M,N}_{p^*} = D_{M,N}^{N_{p^*}}\) and \(C^*_{p^*}\) - error constant.

- x-order of convergence, \(p^* = 0.8910\)
- The error constant, \(C^*_{p^*} = 3.340\)
Case (ii) \((d, t) = (1.4, t)\)

Table 3. Values of \(D^{M,N}, p^{M,N}, p^*, C_{p^*}^{M,N}\) and \(C_{p^*}^*\) for \(\varepsilon = \eta/64, N = 32\) and \(\alpha = 1.9\)

| \(\eta\) | Number of mesh points \(M\) | \(128\) | \(256\) | \(512\) | \(1024\) | \(2048\) |
|---|---|---|---|---|---|---|
| \(2^{-3}\) | 0.178E-03 | 0.457E-04 | 0.116E-04 | 0.292E-05 | 0.733E-06 |
| \(2^{-6}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |
| \(2^{-9}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |
| \(2^{-12}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |
| \(2^{-15}\) | 0.172E-03 | 0.442E-04 | 0.112E-04 | 0.282E-05 | 0.707E-06 |

\(D^{M,N}\) | 0.178E-03 | 0.457E-04 | 0.116E-04 | 0.292E-05 | 0.733E-06 |

\(p^{M,N}\) | 1.96 | 1.98 | 1.99 | 1.99 |

\(C_{p^*}^{M,N}\) | 3.21 | 3.21 | 3.16 | 3.09 | 3.02 |

\(p^*\) order of convergence, \(p^* = 1.96\)

The error constant, \(C_{p^*}^* = 3.21\)

Table 4. Values of \(D^{M,N}, p^{M,N}, p^*, C_{p^*}^{M,N}\) and \(C_{p^*}^*\) for \(\varepsilon = \eta/64, M = 16\) and \(\alpha = 1.9\)

| \(\eta\) | Number of mesh points \(N\) | \(128\) | \(256\) | \(512\) | \(1024\) | \(2048\) |
|---|---|---|---|---|---|---|
| \(2^{-3}\) | 0.6002E-02 | 0.3963E-02 | 0.2132E-02 | 0.1081E-02 | 0.5409E-03 |
| \(2^{-6}\) | 0.6058E-03 | 0.3762E-03 | 0.2972E-03 | 0.1746E-03 | 0.8596E-04 |
| \(2^{-9}\) | 0.6058E-03 | 0.3762E-03 | 0.2972E-03 | 0.1746E-03 | 0.8596E-04 |
| \(2^{-12}\) | 0.6058E-03 | 0.3762E-03 | 0.2972E-03 | 0.1746E-03 | 0.8596E-04 |
| \(2^{-15}\) | 0.6058E-03 | 0.3762E-03 | 0.2972E-03 | 0.1746E-03 | 0.8596E-04 |

\(D^{M,N}\) | 0.6002E-02 | 0.3963E-02 | 0.2132E-02 | 0.1081E-02 | 0.5409E-03 |

\(p^{M,N}\) | 0.5990 | 0.8942 | 0.9799 | 0.9989 |

\(C_{p^*}^{M,N}\) | 0.3231 | 0.3231 | 0.2633 | 0.2022 | 0.1533 |

\(p^*\) order of convergence, \(p^* = 0.5990\)

The error constant, \(C_{p^*}^* = 0.3231254\)
Figure 1. The Figure displays the numerical solution for the problem (25) for Case (i) \((d, t) = (0.4, t)\), computed for \(M = 16, N = 128\) and \(\varepsilon = 2^{-15}\). The solution \(u(x, t)\) has boundary layers at \((0, t)\) and \((2, t)\) and interior layers at \((0.4, t)\), \((1, t)\) and \((1.4, t)\).

Figure 2. The Figure displays the numerical solution for the problem (25) for Case (ii) \((d, t) = (1.4, t)\), computed for \(M = 16, N = 128\) and \(\varepsilon = 2^{-15}\). The solution \(u(x, t)\) has boundary layers at \((0, t)\) and \((2, t)\) and interior layers at \((1, t)\) and \((1.4, t)\).

7. Conclusion
In this paper a first order convergent numerical method has been suggested and analyzed for a linear parabolic second order delay differential equation of reaction-diffusion type with discontinuous source term. The solution profile depends on the location of the point of discontinuity. Further the delay term influences the occurrence of interior layers. If the point of discontinuity \(d\) is located in the interval \((0, 1)\), an additional interior layer occurs at the point \(1 + d\), due to the presence of the delay term. The numerical results support the convergence analysis established.

References
[1] V Franklin M Paramasivam J J H M and Valarmathi S 2013 *International Journal of Numerical Analysis and Modeling* **10** 178
[2] Valarmathi S and Miller J J H 2010 *International Journal of Numerical Analysis and Modeling* **7** 535
[3] M Paramasivam J J H M and Valarmathi S 2010 *Mathematical Communications* **15** 587
[4] S Parthiban S V and Franklin V 2016 *Numerical Method for a Singularly Perturbed Boundary Value Problem for a Linear Parabolic Second Order Delay Differential Equation* (India: Springer) p 117
[5] N Shivaranjani J J H M and Valarmathi S 2016 *A parameter uniform first order convergent numerical method for an initial value problem for a system of singularly perturbed delay differential equations* (India: Springer) p 135
[6] M Manikandan N Shivaranjani J J H M and Valarmathi S 2014 *A parameter uniform first order convergent numerical method for a boundary value problem for a singularly perturbed delay differential equation* (Switzerland: Springer) p 71
[7] Farrell P, Hegarty A, Miller J, Riordan E and Shishkin G 2000 *Robust computational techniques for boundary layers* (Boca Raton, Florida,USA: Chapman and hall/CRC)