DEFORMATION OF SYMMETRIC FUNCTIONS AND THE RATIONAL STEENROD ALGEBRA

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ABSTRACT. In 1999, Reg Wood conjectured that the quotient of $\mathbb{Q}[x_1, \ldots, x_n]$ by the action of the rational Steenrod algebra is a graded regular representation of the symmetric group $\mathfrak{S}_n$. As pointed out by Reg Wood, the analog of this statement is a well known result when the rational Steenrod algebra is replaced by the ring of symmetric functions; actually, much more is known about the structure of the quotient in this case.

We introduce a non-commutative $q$-deformation of the ring of symmetric functions, which specializes at $q = 1$ to the rational Steenrod algebra. We use this formalism to obtain some partial results. Finally, we describe several conjectures based on an extensive computer exploration. In particular, we extend Reg Wood’s conjecture to $q$ formal and to any $q \in \mathbb{C}$ not of the form $-a/b$, with $a \in \{1, \ldots, n\}$ and $b \in \mathbb{N}$.

Contents

1. Introduction
1.1. Summary of the results
1.2. Organization of the paper
2. Preliminaries
2.1. Graded rings and Hilbert series
2.2. Representations of the symmetric group on polynomials
2.3. The Weyl algebra
2.4. Duality in the Weyl algebra and scalar product on polynomials
2.5. Hit and harmonic polynomials
2.6. Specialization lemmas
3. The $q$-Steenrod algebra
3.1. Structure and dimension of the $q$-Steenrod algebra
3.2. Action on polynomials; hit and harmonic polynomials
4. Specializations
5. The low degree and truncated cases
6. Changing the number of variables
6.1. Strings
6.2. Length of the strings
6.3. Complete analysis of the 2 variables case
7. Why is the $q$ case more difficult than the $q = 0$ case?
7.1. Elementary symmetric functions do not exist
7.2. Schubert Polynomials do not exist
References

1. Introduction

The rational Steenrod algebra is the subalgebra $\mathcal{A}$ of the Weyl algebra generated by the rational Steenrod squares $D_k := \sum_i x_i^{k+1} \partial_i$ [Woo97]. The Steenrod squares
are derivations, and satisfy the Lie relations $[D_k, D_l] = (l - k)D_{k+l}$. In particular, the Steenrod algebra is generated by just $D_1$ and $D_2$.

Let $\mathbb{C}[X_n] := \mathbb{C}[x_1, \ldots, x_n]$ be a ring of polynomials, and set $\pi := x_1 \cdots x_n$. Consider the action of the Steenrod algebra on $\mathbb{C}[X_n]$ through the algebraic Thom map, so that for $f \in \mathcal{A}$ and $p \in \mathbb{C}[X_n]$,

$$(1) \quad \rho(f).p := \frac{1}{\pi} f \pi.p.$$

This action seems to share properties with the natural action of the ring of symmetric polynomials on polynomials by multiplication.

**Conjecture 1** (Rational hit conjecture of Reg Wood [Woo98, Woo01]). The quotient $\mathbb{C}[X_n]/\mathcal{A}+\mathbb{C}[X_n]$ is a graded regular representation of the symmetric group $S_n$.

In this article, we present preliminary results from research in progress around this conjecture. We refer to [Woo97, Woo98, Woo01] for motivations.

It is actually easier to get rid of the algebraic Thom map, by conjugating once for all the Steenrod algebra by this map. So, we consider instead the isomorphic algebra denoted Steen generated by the Weyl operators

$$(2) \quad P_k := \frac{1}{\pi} D_k\pi = \sum_i x_i^k (1 + x_i \partial_i).$$

Note that those operators are no longer derivations.

Now, $P_k$ appears clearly as a usual symmetric power-sum, with a deformation. Since the goal is to transfer properties of the ring of symmetric polynomials to the Steenrod algebra, this suggests that we interpolate continuously between the two. Hence, we introduce the $q$-Steenrod algebra denoted $\text{Steen}_q$ which is generated by the Weyl operators

$$(3) \quad P_{q,k} := \sum_i x_i^k (1 + qx_i \partial_i).$$

The 0-Steenrod algebra is the ring of symmetric polynomials, while the 1-Steenrod algebra is the rational Steenrod algebra conjugated by the algebraic Thom map.

In the following, $q$ can be either a given complex number $q_0$, or a formal parameter. In the later case we take $\mathbb{K} := \mathbb{C}(q)$ as base field, or $\mathbb{K} := \mathbb{C}[q]$, when we just need a base ring. Roughly speaking, all the statements we consider are of algebraic nature, and hold for $q$ formal if, and only if, they hold for $q$ generic (that is in most cases for all but a countable set).

### 1.1. Summary of the results.

In [GH94], a concrete realization $\text{Harm}$ of the quotient $\mathbb{C}[X_n]/\text{Sym}^+\mathbb{C}[X_n]$ is constructed as a space of so-called *harmonics*. Following a similar approach, we construct a concrete realization $\text{Harm}_q$ of the quotient $\mathbb{C}[X_n]/\text{Steen}_q^+\mathbb{C}[X_n]$.

Both our theoretical results, and our computer exploration leads us to extend Reg Wood’s conjecture as follows:

**Conjecture 2.** Let $q$ be formal, or a complex number not of the form $-a/b$, with $a \in \{1, \ldots, n\}$, $b \in \mathbb{N}$ and $a \leq b$. Then, for any $n$, $\text{Harm}_q$ is isomorphic to $\text{Harm}$ as a graded $\mathfrak{S}_n$-module. In particular, it is isomorphic to the regular representation of $\mathfrak{S}_n$.

The main results of this article are summarized in the following two propositions. Recall that a monomial $x^K = x_1^{k_1} \cdots x_n^{k_n}$ in $\mathbb{K}[X_n]$ is called *staircase monomial* if $k_i \leq n - i$, for all $i$.

**Proposition 1.** The following are equivalent:

(a) Conjecture 2 holds for $q$ formal;
(b) For any $n$, $\dim \text{Harm}_q \geq \dim \text{Harm}$;

(c1) For any $n$, and any polynomial $p \in \text{Harm}_q$, all the variables have degree at most $n - 1$ in $p$.

(c2) For any $n$, the staircase monomials are the leading terms of the polynomials in $\text{Harm}_q$ for the lexicographic order.

(d) Any basis of $\mathbb{C}[X_n]/\text{Sym}^+ \mathbb{C}[X_n]$ is a basis of $\mathbb{K}[X_n]/\text{Steen}_q^+ \mathbb{K}[X_n]$.

**Proposition 2.** Assume that conjecture 2 holds for $q$ generic. Then, for $q_0 \in \mathbb{C}$, the following are equivalent:

(a) Conjecture 2 holds for $q_0$;

(b) For any $n$, $\dim \text{Harm}_{q_0} \leq \dim \text{Harm}$;

(c) For any $n$, the non-staircase monomials are the leading terms of the polynomials in $\text{Hit}_q$ for the lexicographic order.

1.2. **Organization of the paper.** After a background section, we define the $q$-deformed Steenrod algebra $\text{Steen}_q$ (Definition 5), together with $q$-hit and $q$-harmonic polynomials. We compute the dimension and the structure of $\text{Steen}_q$ (Theorems 3.1 and 3.2).

In the section 4, we describe the relations between the generic setting (when $q$ is a formal parameter), the complex setting (when $q$ in a complex number) and the classical setting (when $q = 0$).

Then, we analyze the space of polynomials that can be hit from harmonics by successive application of at most $n$ Steenrod operators. This provides information in small degrees. In particular we show that the submodules of degree smaller than $n$ of $\mathbb{K}[X_n]/\text{Sym}^+ \mathbb{K}[X_n]$ and $\mathbb{K}[X_n]/\text{Steen}_q^+$ are isomorphic (Corollary 1).

In the next section, we investigate the relation between the $q$-harmonics in $n$ variables and in $n + 1$ variables. We construct a trick to go from $n$ to $n + 1$: the so-called strings. The main result of this part is that the conjecture is equivalent to the fact that the harmonics in $n$-variables are of degree smaller than $n$ in each variable (Proposition 17). We also analyze completely the simple cases of 1 and 2 variables.

In the next-to-last section we describe some attempts to prove conjecture 2 by looking for $q$-analogs of elementary symmetric function (Subsection 7.1) and $q$-analogs of Schubert polynomials (Subsection 7.2), and we explain why they fail.

Finally, the last-section is devoted to a computer exploration which motivates our conjectures. In particular, we give, for small numbers of variables and small degrees, a tight superset of the values of $q$ for which conjecture 2 does not hold. Most computations were realized with the open source library MuPAD-Combinat [HT04] for the Computer Algebra System MuPAD [The96]. The extra tools we developed at this occasion are available upon request, and will eventually be integrated into MuPAD-Combinat, after appropriate cleanup and documentation.

2. **Preliminaries**

The main topic of this paper is the study of the $q$-deformation of a module. Hence we will work in two different settings: either $q := q_0$ is a complex number, in which case the base field is $\mathbb{K} := \mathbb{C}$, or $q$ is a formal complex number, and $\mathbb{K} := \mathbb{C}(q)$. Many of the results are true in both settings; in this case, we use $\mathbb{K}$ as base field without precision.

For an alphabet $X$ of commuting variables, we denote respectively by $\mathbb{K}[X]$, $\mathbb{K}(X)$, and $\text{Sym}(X)$ the ring of polynomials, the field of fractions, and the ring of symmetric functions in the variables in $X$. We denote by $X_n$ the alphabet of...

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1. 2007/06: as noted by Adriano Garsia, this statement is in fact incorrect

2. 2007/06: http://mupad-combinat.sf.net
the $n$ commuting variables $\{x_1, x_2, \ldots, x_n\}$. Unless explicitly stated, $X := X_\infty$ is the alphabet in the countably many variables $\{x_1, \ldots, x_n, \ldots\}$, and we use the shorthand $\text{Sym} = \text{Sym}(X_\infty)$.

### 2.1. Graded rings and Hilbert series.

In the following, most of the vector spaces we consider are graded, with finite dimensional homogeneous components, and comparing their dimensions will be one of our main tools. The Hilbert series of a graded vector space $A := \bigoplus_{d=0}^\infty A_d$ is the generating series of the dimensions of the homogeneous components of $A$:

\[
\text{Hilb}_t(A) := \sum_{d=0}^\infty \dim A_d t^d.
\]

Given two formal series $H(t) := \sum_d h_d t^d$ and $L(t) := \sum_d l_d t^d$, we say that $H(t)$ is dominated by $L(t)$, and write $H(t) \leq L(t)$, whenever $h_d \leq l_d$ for all $d$. The relation of domination is preserved by multiplication by series with nonnegative coefficients, such as Hilbert series and generating functions.

Let us recall some basic Hilbert Series.

**Proposition 3.** The Hilbert series of $\mathbb{K}[X_n]$ is

\[
\text{Hilb}_t(\mathbb{K}[X_n]) = \frac{1}{(1-t)^n}.
\]

The Hilbert series of the algebra $\text{Sym} := \text{Sym}(X_\infty)$ of symmetric functions on a countably many variables is

\[
\text{Hilb}_t(\text{Sym}) = \prod_{i>0} \frac{1}{1-t^i}.
\]

The Hilbert series of the algebra $\text{Sym}(X_n)$ of symmetric polynomials on $n$ variables is

\[
\text{Hilb}_t(\text{Sym}(X_n)) = \prod_{i=1}^n \frac{1}{1-t^i}.
\]

### 2.2. Representations of the symmetric group on polynomials.

The representation theory of the symmetric groups is a well known combinatorial subject. The goal of this subsection is to recall several basic useful facts. For more details, the reader can refer to [FH96, GH94, Sag91] or, in French, to [Kro95].

The irreducible representations $V_\lambda$ of the symmetric group $S_n$ are naturally indexed by partitions $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ of sum $n$ (denoted by $\lambda \vdash n$). There is a particular occurrence of $V_\lambda$, called the Specht module, in the natural representation of $S_n$ on polynomials. This module is spanned by the so-called Specht polynomials. Recall that the partition $\lambda$ is depicted (in the French way) as a diagram of boxes putting $\lambda_i$ boxes in the $i$-th row, starting from the bottom. A filling of the boxes by the numbers $(1, 2, \ldots, n)$ is called a standard filling $F$ of shape $\lambda$. A standard filling which is increasing from left to right along rows and from bottom to top along columns is called a standard tableau.

For example here is the diagram of the partition $\lambda := (5, 3, 2)$ together with a standard filling and a standard tableau of shape $\lambda$. 

\[
\begin{array}{c|c|c|c}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & 8 \\
\hline
9 & 10 & 11 & 12 \\
\hline
\end{array}
\]
Recall that the *Vandermonde determinant* of a finite alphabet of variables \( \{y_1, \ldots, y_k\} \) is defined by

\[
\Delta(y_1, \ldots, y_k) := \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{k-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_k & y_k^2 & \cdots & y_k^{k-1} \end{vmatrix} = \prod_{1 \leq i < j \leq k} (y_j - y_i).
\]

For each standard filling \( F \), the *Specht polynomial* \( \text{Sp}_F \) is defined as the product of the Vandermonde determinants of the variables indexed by the columns of \( F \).

For example the Specht polynomial associated with the previous filling is

\[
\text{Sp}_F := \Delta(x_{10}, x_9, x_8) \cdot \Delta(x_4, x_1, x_2) \cdot \Delta(x_3, x_5) \cdot \Delta(x_7) \cdot \Delta(x_6) = (x_9 - x_{10})(x_8 - x_9)(x_8 - x_9)(x_1 - x_4)(x_2 - x_4)(x_2 - x_1)(x_5 - x_3) \cdot 1 \cdot 1.
\]

It is easy to see that, for \( \sigma \in S_n \), and \( F \) a filling, \( \sigma \text{Sp}_F = \text{Sp}_{\sigma F} \). Hence, the span of \( \{\text{Sp}_F\} \), where \( F \) runs through the standard fillings of a given shape \( \lambda \vdash n \) is a representation \( V_\lambda \) of \( S_n \), which is called *Specht module* indexed by \( \lambda \).

**Proposition 4.** The Specht modules \( (V_\lambda)_{\lambda \vdash n} \) form a complete family of irreducible representations of \( S_n \).

The Specht polynomials indexed by fillings are not linearly independent. A basis of the Specht module \( V_\lambda \) is given by \( \{\text{Sp}_F\} \), where \( F \) runs through the standard tableaux of shape \( \lambda \).

Note that the polynomials \( \text{Sp}_F \) are homogeneous. Furthermore, each Specht module is the lowest degree occurrence of the corresponding irreducible representation; more precisely:

**Proposition 5.** Let \( d \) be the degree of the Specht module \( V_\lambda \). The multiplicity of the irreducible representation \( V_\lambda \) in \( K[X_n] \) in degree \( d \) is 1; Any occurrence of a representation isomorphic to \( V_\lambda \) in \( K[X_n] \) occurs in degree at least \( d \).

There is a construction for higher degree occurrences of representations isomorphic to \( V_\lambda \) by [TY93] using so-called *higher Specht polynomials*. They form an explicit basis for the quotient module \( K[X_n]/\text{Sym}^+(X_n)K[X_n] \).

Let \( W \) a representation of \( S_n \). Recall that the direct sum of all representations isomorphic to some \( V_\lambda \) is a sub-representation called the *isotypic component* indexed by \( \lambda \) of \( W \) and denoted \( W_\lambda \). Then \( W \) decomposes canonically as \( W = \bigoplus_{\lambda \vdash n} W_\lambda \). Moreover, in the group ring \( K[S_n] \) of \( S_n \), there are special elements \( e_\lambda \) called *central orthogonal idempotents* such that \( e_\lambda W = W_\lambda \) and \( e_i e_j = \delta_{i,j} \).

Consider a non zero polynomial \( f \) in \( V_\lambda \) and a representation \( W \) isomorphic to \( V_\lambda \). By Maschke’s theorem, the set of the images of \( f \) by all \( S_n \)-morphisms from \( V_\lambda \) to \( W \) is a line. Moreover, for two linearly independent polynomials \( f_1 \) and \( f_2 \), these lines are in direct sum.

**Definition 1.** Let \( f \in V_\lambda \) be a non zero polynomial, and \( W \) be a \( S_n \)-representation. The set of the images of \( f \) by all \( S_n \)-morphisms from \( V_\lambda \) to \( W \) is called the *Garnir component* of \( W \) associated to \( f \) denoted by \( W_\lambda(f) \).

\( W_\lambda(f) \) has naturally a structure of vector space isomorphic to \( \text{hom}(V_\lambda, W) \) of which it is a sort of concrete realization. Then we know that

**Proposition 6.** Let \( W \) be a representation of \( S_n \). Then

\[
W = \bigoplus_{\lambda \vdash n} W_\lambda = \bigoplus_{\lambda \vdash n} \left( \bigoplus_{\text{Shape}(T) = \lambda} W_\lambda(\text{Sp}_T) \right),
\]

where the last sum is on standard tableaux of shape \( \lambda \).
Note that this direct sum decomposition is dependent of the chosen basis of $V_\lambda$
(here the Specht basis).

2.3. The Weyl algebra. The following brief introduction to the Weyl algebra is essentially taken from \cite{Woo97}. For a variable $x_i$ define the partial derivative

$$\partial_i := \frac{\partial}{\partial x_i}.$$  

Following standard practice, we write abbreviated expressions for monomials:

$$x^K := \prod x_i^{k_i}, \quad \partial^L := \partial_i^{l_i},$$

where $K := (k_1, \ldots, k_n, \ldots)$ and $L := (l_1, \ldots, l_n, \ldots)$ are exponent vectors. The degree of $x^K$ is $\deg(x^K) := \sum_i k_i$, and the order of $\partial^L$ is $\ord(\partial^L) := \sum_i l_i$.

Weyl monomials are monomials in the variables $x_i$ and the partial derivatives $\partial_i$. They act by multiplication and derivation on $\mathbb{K}[X_n]$, and form an algebra under addition and composition. For example,

$$\partial_1 x_1^4 = 4x_1^3, \quad \partial_1 x_2^2 = 0, \quad \partial_1 \partial_2 p = \partial_2 \partial_1 p,$$

$$x_1 x_2, (1 + x_2 x_4) = x_2 x_1 (1 + x_2 x_4) = x_1 x_2 + 2x_1 x_2^2 x_4,$$

$$\partial_1 x_1, p = \partial_1 \cdot (x_1 p) = (\partial_1 x_1) p + x_1 (\partial_1 p) = (1 + x_1 \partial_1) p.$$

This is formulated more precisely as follows:

**Definition 2.** The Weyl algebra $\text{Weyl}(X) := \text{Weyl}(X)$ is the associative algebra, with unit, generated by $x_i, \partial_i$, subject to the relations

$$[x_i, x_j] = 0, \quad [\partial_i, \partial_j] = 0, \quad [x_i, \partial_j] = \delta_{ij},$$

where square brackets denotes the Lie product and $\delta_{ij}$ the Kronecker symbol.

The Weyl algebras $\text{Weyl}(X_n)$ are defined for each $n$ in a similar way by restricting to the finite set of variables $x_1, \ldots, x_n$ and the corresponding partial derivatives.

By repeated differentiation, any element of the Weyl algebra can be expressed as a linear combination of Weyl monomials where the polynomial part is on the left and the derivatives on the right. For example,

$$\partial_1 x_1 = x_1 \partial_1 + 1$$

$$\partial_i^k x_1 = x_1 \partial_i^k + k \partial_i^{k-1}$$

$$\partial_1 \partial_2 x_1 x_2 = x_1 x_2 \partial_1 \partial_2 + x_1 \partial_1 + x_2 \partial_2 + 1$$

We shall refer to such monomials as the standard monomials. As an Abelian group, Weyl is freely generated by the standard monomials $x^K \partial^L$, as $K$ and $L$ range over exponent vectors.

An element $f$ in the Weyl algebra lies in filtration $k$ if the maximum order of a term of $f$ in standard form does not exceed $k$. Then, $\text{Weyl}(X_n)$ and Weyl are filtered algebras in the sense that the composition product of two elements, in filtration $m$ and $k$ respectively lies in filtration $m + k$. The Weyl algebras are graded by assigning grading $\deg(x^K) - \ord(\partial^L)$ to $x^K \partial^L$.

Given a polynomial or a Weyl operator $p$, we denote by $o(p)$ the orbit sum of $p$, that is the sum of all the polynomials in the orbit of $m$ under the action of the symmetric group. Typically, if $\lambda$ is a partition, then $m_\lambda := o(x^\lambda)$ is the monomial symmetric function indexed by $\lambda$. Note that we have to be a little bit careful, since $o(p)$ might be undefined for an infinite alphabet $X$, as for example for $o(x_1 + x_2)$; however, for a monomial or Weyl monomial $m$, $o(m)$ is always defined. We will consider differential operators such as

$$P_k := \sum_i x_i^k (1 + x_i \partial_i) := o(x_i^k (1 + x_i \partial_i))$$
which are obtained by such symmetrization over an infinite alphabet. To be precise, such operators do not belong to the Weyl algebra, because they involve infinite sums. They do not even necessarily act properly on $\mathbb{K}[X]$; for example $P_k.1 = x_1^k + \cdots + x_n^k + \cdots$ involves an infinite sum. In general, an infinite sum makes sense as an operator on $\mathbb{K}[X]$ providing that, for each $n$, all but a finite number of the monomials $x^K \partial^L$ annihilate $\mathbb{K}[X_n]$.

In practice, this is not an issue; such symmetric operators can be manipulated in the same classical manner as, for example, symmetric functions on an infinite number of variables. They restrict to elements of the Weyl algebra $\text{Weyl}(X_n)$ for each $n$, and the notions of order, filtration and grading carry over properly. By extension, we also call them Weyl operators.

The wedge product (or formal product) of two differential operators is defined on standard monomials of the Weyl algebra by

$$x^K \partial^L \wedge x^M \partial^N := x^{K+L} \partial^{M+N}.$$  

In other words, the partial derivatives are made to commute with all the variables, and Weyl becomes polynomial in both sets of variables $x_i, \partial_i$ under the wedge product. In working out the composition product of two standard forms in the Weyl algebra, the wedge product is the term of top filtration. The wedge product is extended to infinite sums by linearity.

2.4. Duality in the Weyl algebra and scalar product on polynomials.

**Definition 3.** The dual $f^*$ of a Weyl operator $f$ is defined by sesquilinearity on the standard monomials:

$$(x^K \partial^L)^* := x^L \partial^K.$$  

For example,

$$(2x_1 x_2^2 \partial_2 \partial_3^2 + 3ix_2 \partial_1^4)^* = 2x_2^2 x_3 \partial_1 \partial_2^2 - 3ix_1 \partial_2.$$  

Note that in the literature, the dual of a polynomial $p$ is often written $p(\partial)$ instead of $p^*$ [GH94].

**Remark 1.** The dual is an anti-morphism with respect to composition:

$$(fg)^* = g^*f^*$$  

Denote by $p(0)$ the constant term of a polynomial $p$. For two polynomials $p$ and $q$, set

$$\langle p \mid q \rangle := (p^* . q)(0).$$  

It is easy to check that this defines a scalar product which makes the monomials $\{x^K\}$ into an orthogonal basis. In fact, we have

$$\langle x^K \mid x^L \rangle = \begin{cases} 0 & \text{if } K \neq L \\ K! & \text{if } K = L \end{cases},$$  

where $K! = k_1! k_2! \cdots k_n!$.

**Proposition 7.** Let $f$ be a Weyl operator, seen as a linear endomorphism of $\mathbb{K}[X]$. Then, the adjoint of $f$ with respect to the scalar product $\langle \cdot \mid \cdot \rangle$ is its dual Weyl operator $f^*$.

**Proof.** Note first that for a Weyl operator $f$ and a polynomial $p$, we have

$$(f^* . p)(0) = ((f.1)^* . p)(0);$$  

indeed, \((f - f.1)\) is a pure differential operator, hence \((f - f.1)^* . p\) is a pure polynomial with no constant term. Then,

\[
⟨f.p \mid q⟩ = ((f.p)^* . q)(0) = ((fp.1)^* . q)(0) = ((fp)^* . q)(0) = (p^* . f^* . q)(0) = (p^* . (f^* . q))(0) = (p \mid f^* . q) ,
\]

as desired.

2.5. Hit and harmonic polynomials. In this subsection, we fix a set \(S\) of homogeneous Weyl operators, and denote by \(S^+\) the operators in \(S\) of positive degree.

**Definition 4.** A polynomial is called \(S\)-hit if there exists some polynomials \(p_1, \ldots, p_k\) and operators \(f_1, \ldots, f_k \in S^+\) such that

\[
p = f_1.p_1 + \cdots + f_k.p_k .
\]

A polynomial \(p\) is called \(S\)-harmonic if it satisfies the differential equations

\[
f^* . p = 0, \text{ for all } f \in S^+ .
\]

The sub-spaces of \(S\)-hit and \(S\)-harmonic polynomials are denoted respectively by \(\text{Hit}\) and \(\text{Harm}\).

For example, if \(S = \{x_1^2 + \cdots + x_n^2\}\), the harmonics in \(\mathbb{K}[X_n]\) are the polynomials \(p\) which satisfy

\[
\partial_1^2 p + \cdots + \partial_n^2 p = 0 ;
\]

hence the name harmonic [GH94]. If \(S\) consists only of polynomials, as in the previous example, then the space of \(S\)-hits is simply the ideal generated by \(S^+\) in \(\mathbb{K}[X_n]\).

**Remark 2.** Consider an operator \(f \in S^+\), any Weyl operator \(g\), and a polynomial \(p\). Then, \(fg.p = f.(g.p)\) is hit. Dually, if \(p\) is harmonic then \(gf^* . p = g.(f^* . p) = g.0 = 0\): it is killed by any operator in the left ideal generated by \(S^+\) in the Weyl algebra. Hence, \(S\) and the right ideal \(S^+\text{Weyl}(X_n)\) generated by \(S^+\) in the Weyl algebra have the same hit and harmonic polynomials.

Furthermore, if \(A^+\) is any subspace of the Weyl algebra such that \(S^+ \subset A^+ \subset S^+\text{Weyl}(X_n)\), then Hit is the image \(A.\mathbb{K}[X_n]\) of \(A^+ \otimes \mathbb{K}[X_n]\) by the bilinear evaluation map \((f.p) \mapsto f.p\).

**Proposition 8.** A polynomial \(p\) is \(S\)-harmonic if, and only if, it is orthogonal to any \(S\)-hit polynomial \(q\). That is, the space of \(S\)-harmonic polynomials is the orthogonal complement of the space of \(S\)-hit polynomials.

**Proof.** This is a simple variation on the fact that the image of a linear morphism is the orthogonal complement of the kernel of its adjoint. Let \(p\) be a polynomial. By duality, for any polynomial \(q\) and operator \(f \in S^+\), we have

\[
⟨f^* . p \mid q⟩ = ⟨p \mid f . q⟩ .
\]

Assume that \(p\) is \(S\)-harmonic; then it is obviously orthogonal to any \(f . q\) and by linearity to any \(S\)-hit. Reciprocally, assume that \(p\) is orthogonal to any hit polynomial, and consider an operator \(f \in S^+\); then \(f^* . p = 0\) because it is orthogonal to all polynomials.

**Proposition 9.** Let \(A\) be the graded algebra generated by \(S^+\), acting on \(\mathbb{K}[X_n]\). Then, \(\text{Harm}\) generates \(\mathbb{K}[X_n]\) as an \(A\)-module. In particular,

\[
\text{Hilb}_t(A).\text{Hilb}_t(\text{Harm}) \supseteq \text{Hilb}_t(\mathbb{K}[X_n]),
\]

with equality if, and only if, \(\mathbb{K}[X_n]\) is a free \(A\)-module.
Proposition 10. Assume that the operators in $S$ are symmetric. Then, any Specht polynomial $Sp_F$ is $S$-harmonic. Hence, the space of $S$-harmonics contains at least one copy of each irreducible representation of the symmetric group $S_n$.

Proof. Consider an operator $f \in S^+$. Its dual $f^*$ is an $S_n$-morphism of negative degree. Hence, $f^* Sp_F$ lies in the same isotypic component as $Sp_F$, and $\deg(f^* Sp_F) < \deg( Sp_F )$. By minimality of the degree of $Sp_F$, $f^* Sp_F = 0$. □

As a final example, take for $S$ the $n$ symmetric power-sums on $n$ variables $(p_k := x_1^k + \cdots + x_n^k)_{k=1, \ldots, n}$. Then, the space $Hit$ of hit polynomials is the ideal generated by the symmetric polynomials. There exists an explicit Gröbner basis for the lexicographic term order, which shows that the leading terms of Hit are the non-staircase monomials (see e.g. [Stu93]).

The harmonics are the polynomials $p$ which satisfy the generalized harmonic equations

$$(22) \quad p_k^* p = \partial_1^k p + \cdots + \partial_n^k p = 0, \quad \text{for } k = 1, \ldots, n.$$ 

The Vandermonde $\Delta := \prod(x_j - x_i)$ is the Specht polynomial $Sp_{(1, \ldots, 1)}$, and as such is harmonic. Moreover, any partial derivative $\partial_i$ commute with the $p_k^*$, and it follows that it stabilizes the space Harm of harmonic polynomials. Actually, any harmonic is of the form $p^* \Delta$, for some polynomial $p$, and if a set $B$ of $n!$ polynomials is a basis of $\mathbb{K}[X_n]/\text{Sym} \cdot \mathbb{K}[X_n]$, then $(p^* \Delta)_{p \in B}$ is a basis of Harm. Finally, $\mathbb{K}[X_n]$ is a free $\text{Sym}(X_n)$-module, and the Hilbert series of Harm($X_n$) is given by

$$\text{Hilb}_t(\text{Harm}(X_n)) = \frac{\text{Hilb}_t(\mathbb{K}[X_n])}{\text{Hilb}_t(\text{Sym}(X_n))} = \prod_{i=1}^n \frac{1 - t^i}{1 - t} = (1 + t)(1 + t + t^2) \cdots (1 + t + \cdots + t^{n-1}).$$

For further details, we refer to [GH94]

2.6. Specialization lemmas. In this subsection, $U$ is a $\mathbb{C}$-vector space with a distinguished basis $B$. The goal here is to deal with $q$-deformations of finite dimensional sub-spaces of $\mathbb{C}(q) \otimes U$ and their specializations.

Lemma 2.1. Let $(u_1(q), \ldots, u_k(q))$ be a family of vectors of $\mathbb{C}(q) \otimes U$. For all but a finite set of values $q_0 \in \mathbb{C}$, namely the roots of the denominators of the coefficients of the $u_i$’s on the basis $B$, the specialization $q = q_0$ is well defined. In this case,

$$\text{Rank}_\mathbb{C}(u_1(q_0), \ldots, u_k(q_0)) \leq \text{Rank}_\mathbb{C}(q)(u_1(q), \ldots, u_k(q)).$$

Moreover, there is only a finite number of values $q_0 \in \mathbb{C}$ such that the inequality above is strict, namely the roots of the non-trivial principal minors.

Let $V(q)$ be a subspace of $\mathbb{C}(q) \otimes U$ of finite dimension. For any $q_0 \in \mathbb{C}$, let $V(q_0)$ be its specialization, that is the vector space of all meaningful specializations of elements of $V(q)$. Note that this definition is highly dependent on the basis $B$.

Lemma 2.2. There exists a basis $B'(q) = (v_1(q), \ldots, v_k(q))$ of $V(q)$ such that the coefficients of the elements of $B'(q)$ in the basis $B$ are polynomials in $q$, and for any specialization $q = q_0$, the family $B'(q_0)$ is a basis of $V(q_0)$ (in particular $\dim_{\mathbb{C}} V(q_0) = \dim_{\mathbb{C}(q)} V(q)$).

This is a classical result, but let us give a short effective proof.
Proof. First remark that, by the preceding lemma, the vector space $V(q_0)$ of all meaningful specializations of elements of $V(q)$ is of $C$-dimension at most the $C(q)$-dimension of $V(q)$. Consequently we just have to construct a basis of $V(q)$ without denominators such that all specializations are linearly independent. Let us construct such a basis.

Choose a $C(q)$-basis $A(q)$ of $V(q)$ and expand the vectors of $A(q)$ over $B$. By a suitable multiplication, we can remove the denominators and then suppose that the coefficients of the expansion are in $C[q]$. Now we apply the fraction-free Gram-Schmidt orthogonalization to $A(q)$, and get a basis $A'(q) = (a_1(q), \ldots, a_n(q))$. By dividing each $a_i(q)$ by the greatest common divisor of its coefficients, (it is often called the content of $a_i(q)$) we get a new basis $B'(q) = (b_i(q))$ of $V(q)$.

This basis is orthogonal on $C[q]$ and thus remains orthogonal for any specialization at $q = q_0$. Furthermore, an element $b_i(q)$ of $B'(q)$ is content-free, so $b_i(q_0)$ never vanishes. We conclude that, for any $q_0$, the family $B'(q_0)$ is a $C$-basis for $V(q_0)$. □

Moreover, since the character table of $S_n$ is with integer coefficients, the action of central idempotents commutes with specializations. Thus if $V(q)$ is an $S_n$-module then

- $V(q)$ is isomorphic as $S_n$-modules to $C(q) \otimes V(q_0)$ for all $q_0$;
- $C(q_0)$ and $C(q_1)$ are isomorphic as $S_n$-modules for all $q_0, q_1$.

3. The $q$-Steenrod algebra

The similarities between Sym and Steen leads us to define a new algebra to interpolate between the two.

Definition 5. Let $P_{q,k}$ be the following $q$-analogue of the $P_k$ operator:

$$P_{q,k} := \sum_i x_i^k (1 + qx_i \partial_i).$$

The $q$-Steenrod algebra Steen$_q$ is the algebra over $C(q)$ generated by the $P_{k,k}, k > 0$.

Note that the specializations at $q = 0$ and $q = 1$ yield respectively the symmetric functions and the rational Steenrod algebra (conjugated by the Thom map):

- $P_{0,k}$ is the usual operator of multiplication by the symmetric power-sum $p_k$.
- $P_{1,k} = P_k$.

Remark 3. It would be tempting to actually introduce two parameters $q$ and $q'$:

$$P_{q,q',k} := \sum_i x_i^k (q' + qx_i \partial_i).$$

Now, the rational Steenrod algebra itself is also a special case of the $(q, q')$-Steenrod algebra, with $q' = 0$ and $q = 1$: working in projective space versus $q$ removes the singularity at $q = \infty$. The price to pay is that $C[q, q']$ is not anymore a principal ideal domain; this renders computations, and especially linear algebra, farless practical. Since we are mainly interested in the cases $q = 0$ and $q = 1$, we stick in the following to a single parameter.

3.1. Structure and dimension of the $q$-Steenrod algebra. The $q$-Steenrod squares satisfy the following commutation rule:

Proposition 11. For $k, l \in \mathbb{N}$, and $q$ complex or formal,

$$[P_{q,k}, P_{q,l}] = q(l - k) P_{q,k+l}.$$
For a composition \( \mu := (\mu_1, \ldots, \mu_k) \), define \( P_{q,\mu} := P_{q,\mu_1} \cdots P_{q,\mu_k} \). Using the commutation rules, it is obvious that any \( P_{q,\mu} \) can be rewritten as a linear combination of \( P_{q,\lambda} \)'s indexed by partitions. For example,

\[
P_{q,(1,2)} = qP_{q,(3)} + P_{q,(2,1)}.
\]

**Theorem 3.1.** Let \( q \) be formal or a complex number.

(a) The operators \( P_{q,\lambda} \), where \( \lambda \) runs through all integer partitions form a vector space basis of \( \text{Steen}_q \);

(b) The Hilbert Series of the \( q \)-Steenrod algebra is given by

\[
\text{Hilb}_t(\text{Steen}_q) = \text{Hilb}_t(\text{Sym}) = \prod_{i>0} \frac{1}{1 - t^i};
\]

(c) The ideal of relations between the \( P_{q,k} \)'s is generated by the commutation relations

\[
[P_{q,k}, P_{q,l}] = q(l - k)P_{q,k+l}.
\]

(d) \( \text{Steen}_q \) is the enveloping algebra of the Lie algebra spanned by the \( P_{q,k} \).

This is a straightforward generalization of results in [Woo97] about the rational Steenrod algebra, and follows from an easy triangularity property with respect to monomial functions. Namely, consider the expansion of

\[
P_{q,(2,1)} = o(x_1^2 x_2) + (q + 1) o(x_1^2)
\]

\[
+ q^2 o(x_1^5 \partial_1) + q^2 o(x_1^3 x_2 \partial_1 + q o(x_1^3 x_2 \partial_2) + 2q(q+1) o(x_1^3 \partial_1) + q o(x_1^2 x_2^2 \partial_1).)
\]

Its polynomial part is a symmetric function, and can be expressed in terms of monomial symmetric function: \( P_{q,(2,1)} = m_{(2,1)} + (q + 1) m_{(3)} \). Note that \( m_{(2,1)} \) has coefficient 1, while \( m_3 \) is indexed by a shorter partition. This generalizes immediately:

**Lemma 3.1.** The polynomial part \( P_{q,\lambda,1} \) of \( P_{q,\lambda} \) is a symmetric function of the form

\[
P_{q,\lambda,1} = m_{\lambda} + \sum_{\mu} c_{\mu} m_{\mu},
\]

where \( \mu \) runs through partitions of length \( \ell(\mu) < \ell(\lambda) \).

It follows that, except for \( q = 0 \), all the \( q \)-Steenrod algebras on an infinite alphabet are isomorphic independently of \( q \); only their action on polynomials differ:

**Theorem 3.2.** Let \( q \) be formal (respectively a non zero complex number). Then, the \( q \)-Steenrod algebra \( \text{Steen}_q \) is isomorphic as a graded algebra to \( \mathbb{C}(q) \otimes \text{Steen} \) (respectively \( \text{Steen} \)).

**Proof.** By renormalizing the \( q \)-Steenrod squares \( \tilde{P}_{q,k} := \frac{1}{q} P_{q,k} \), we obtain the commutation relations

\[
[\tilde{P}_{q,k}, \tilde{P}_{q,l}] = (l - k) \tilde{P}_{q,k+l}.
\]

Hence the algebra \( \text{Steen}_q \) has the same presentation with generators and relations as \( \mathbb{C}(q) \otimes \text{Steen} \). \( \square \)

We consider now the \( q \)-Steenrod algebra over a finite alphabet.

**Proposition 12.** For any finite alphabet \( X_n \),

\[
\prod_{i=1}^{n} \frac{1}{1 - t^i} = \text{Hilb}_t(\text{Sym}(X_n)) \leq \text{Hilb}_t(\text{Steen}_q(X_n)) \leq \frac{1}{(1 - t)^n} \prod_{i=1}^{n} \frac{1}{1 - t^i}
\]
Proof. The first inequality follows again from lemma \[5.1\] Note that, for \( k \geq 1 \), any Weyl monomial \( x^K \partial^L \) appearing in \( P_{q,k} \) satisfies \( k_i \geq 2l_i \) for all \( i \). This property is preserved by multiplication, and it follows that \( P_{q,\lambda} \) can be written as a linear combination of orbit-sums \( o(x^{2K+L} \partial^L) \), where \( \sum k_i + l_i = n \), and the \( k_i \) are decreasing. The second inequality follows. \( \square \)

It appears from computations that the first inequality is strict, starting from degree \( n + 1 \). Namely, for \( n \leq 5 \), we noticed that the polynomials \((P_{q,\lambda}(X_n),x_1)_{\lambda} \) are always linearly independent (see Sections \[?? \] and \[7.1\]), whereas there is a linear relation between the symmetric functions \((p_\lambda(X_n))_{\lambda} \).

The second inequality is very coarse, and could certainly be refined. For example, one can easily check that

\[
(30) \quad \text{Hilb}_t(\text{Steen}_q(X_1)) = \frac{1}{1-t} \ll \frac{1}{(1-t)^2}.
\]

Still, this inequality implies that \( \text{Hilb}_t(\text{Steen}_q(X_n)) < \text{Hilb}_t(\text{Steen}_q(X)) \); indeed the dimension of \( \text{Steen}_q \) is counted by the number \( p(d) \) of partitions whose asymptotic \[HR18\] is:

\[
(31) \quad p(d) \approx \frac{1}{4\sqrt{3} n} \exp \pi \sqrt{\frac{2n}{3}}.
\]

3.2. Action on polynomials; hit and harmonic polynomials. We consider now the natural action of \( \text{Steen}_q \) on \( K[X_n] \). Following subsection \[2.3\] with \( S = \text{Steen}^+_q \), we define the spaces \( \text{Hit}_q \) of \( q \)-hit polynomials and \( \text{Harm}_q \) of \( q \)-harmonic polynomials. For \( q \neq 0 \), the \( q \)-Steenrod algebra is generated by \( P_{q,1} \) and \( P_{q,2} \), and it follows that the \( q \)-harmonics can be described by just two differential equations:

\[
(32) \quad P_{q,1}^* p = 0 \quad \text{and} \quad P_{q,2}^* p = 0.
\]

Furthermore, it follows immediately from the triangularity property described in lemma \[5.1\] that any symmetric function is \( q \)-hit.

Proposition 13. Let \( q \) be formal or a complex number, and \( X \) be a commutative alphabet. Then, \( \text{Steen}_q(X).1 = \text{Sym}(X) \).

4. Specializations

The action of the \( q \)-Steenrod algebra behaves properly with respect to specialization. That is, if a polynomial \( p(q) \) specializes properly to \( p(q_0) \), then \( P_{q,k} p(q) \) and \( P_{q,k}^* p(q) \) specializes properly to \( P_{q_0,k} p(q_0) \) and \( P_{q_0,k}^* p(q_0) \) respectively. Hence, we can apply the specialization lemmas to obtain some properties of the \( q \)-harmonics and \( q \)-hits with respect to specialization. In particular we obtain the implication (b) \( \Rightarrow \) (a) of proposition \[1\]

Proposition 14. Let \( q_0 \in C \). Then,

(a) Let \( S(q) \) be a set of homogeneous polynomials in \( C(q)[X_n] \), which specializes properly at \( q = q_0 \). Then, the \( \text{Steen}_q \)-module (resp. \( \text{Steen}^+_q \)-module) \( M(q) \) generated by \( S \) in \( C(q)[X_n] \) specializes to the \( \text{Steen}_{q_0} \)-module (resp. \( \text{Steen}^+_{q_0} \)-module) \( M(q_0) \) generated by \( S(q_0) \).

(b) The Hilbert series of \( M(q) \) dominates the Hilbert series of \( M(q_0) \);

(c) Let \( p \in C[X_n] \). The Hilbert series of the \( \text{Steen}_q \)-module generated by \( p \) dominates the Hilbert series of the corresponding \( \text{Sym}(X_n) \)-module;

(d) Let \( B \) be a \( \text{Sym}(X_n) \)-module basis of \( C[X_n] \), e.g. the Schubert polynomials or the staircase monomials. Then \( B \) spans \( C(q)[X_n] \) as a \( \text{Steen}_q \)-module; note that this module is not necessarily free.
(c) Hit$_q$ specializes at $q = q_0$ to Hit$_{q_0}$; in particular the Hilbert series of Hit$_q$ dominates the Hilbert series of Hit$_{q_0}$; furthermore, for all but a countable number of $q_0 \in \mathbb{C}$, there is equality;

(f) Hit$_q +$ Harm = $\mathbb{C}(q)[X_n]$.

Proof. (a) Fix a degree $d$. Extract from $S(q)$ a finite subset $A(q)$ which generates linearly all the other elements of $S(q)$ of degree less than $d$, and specializes properly at $q = q_0$. Consider the set $B(q)$ of all $P_{q,\lambda}, p(q)$, where $p$ belongs to $A(q)$ and $\lambda$ is a partition of $d - \deg(p)$. Then, $B(q)$ generates $M(q)_d$, and specializes to $B(q_0)$ which generates $M(q_0)_d$. Hence, $M(q)_d$ specializes to $M(q_0)_d$.

The others are direct applications of (a).

Similarly, the following proposition provides the implication $(b) \Rightarrow (a)$ of proposition\footnote{2} and more:

**Proposition 15.** (a) If $p(q)$ is $q$-harmonic and specializes properly at $q = q_0$, then $p(q_0)$ is harmonic;

(b) There exists a basis of Harm$_q$ which specializes properly for any $q_0$;

(c) Harm$_q$ specializes to a subspace Harm$_q(q_0)$ of Harm$_{q_0}$

(d) The Hilbert series of Harm$_q$ is dominated by the Hilbert series of Harm;

(e) The orthogonal projector $\mathbb{C}(q)[X_n] \hookrightarrow$ Harm $\otimes \mathbb{C}(q)$ is injective on Harm$_q$.

Proof. (a) For any $k$, $P^*_{q,k}, p(q)$ specializes to $P^*_{q_0,k}, p(q)$ for all $k$; hence the latter is zero whenever the former is.

(b) Direct application of the specialization lemma\footnote{2}.

(c) Consequence of (a) and (b).

(d) Take $f \in$ Hit $\cap$ Harm$_q$. Then, for all $k$, $\sum_i (1 + qx_i \partial_i) \partial_i^k f = 0$.

Write $f = q^{\text{cal}}(f)(f_0 + qf_1)$, with $0 \neq f_0 \in \mathbb{C}[X_n]$ and $f_1 \in \mathbb{C}(q)[X_n]$ such that $f_1$ has no $\frac{4}{q}$ coefficient. Then, $f_0 + qf_1$ is also $q$-harmonic, and it follows that $f_0$ is harmonic. This is in contradiction with $f_0$ being hit. \hfill $\Box$

5. The low degree and truncated cases

One of the main difficulties is that the $q$-Steenrod algebra on a finite alphabet $X_n$ is much bigger than Sym($X_n$). In this section, we get around this difficulty by truncating the $q$-Steenrod algebra to make it share the dimension of Sym($X_n$). No truncation occurs in low degree, so all the results of this section also apply to the full $q$-Steenrod algebra in degree $d \leq n$.

The truncated $q$-Steenrod algebra TSteen$_q(X_n)$ is the subspace of Steen$_q(X_n)$ spanned by $(P_{q,\lambda})_{\ell(\lambda) \leq n}$. Note that Steen$_q(X_n)$ and TSteen$_q(X_n)$ coincide in degree $d \leq n$; however TSteen$_q(X_n)$ is not an algebra.

Here also, applying lemma\footnote{3} yields TSteen$_q(X_n).1 =$ Sym, and it follows that Hilb$_t$(TSteen$_q(X_n)) =$ Hilb$_t$(Sym($X_n$)), for $q$ formal or a complex number.

**Problem 1.** For which polynomials $p$ do TSteen$_q.p$ and Steen$_q.p$ coincide?

For such a polynomial $p$, find a straightening algorithm for rewriting an element of Steen$_q.p$ as element of TSteen$_q$.

The proper analogue of Hit$_q(X_n)$ is the space of truncated $q$-hit polynomials THit$_q(X_n)$ which is spanned by the $f.p$, where $f \in$ TSteen$_q^+(X_n)$, and $p \in$ Harm($X_n$). The polynomials $h$ in the orthogonal THarm$_q(X_n) :=$ THit$_q(X_n)^\perp$ are called truncated $q$-harmonics. Note that they do not necessarily satisfy $P^*_{q,1}, h = 0$ and $P^*_{q,3}, h = 0$.

Reg Wood’s conjecture holds for the truncated $q$-Steenrod algebra, and for $q$ formal:
Theorem 5.1. Let \( q \) be formal. Then,

(a) \( \operatorname{Hilb}_t(\operatorname{THarm}_q(X_n)) = \operatorname{Hilb}_t(\operatorname{Harm}(X_n)) \);
(b) \( \operatorname{THarm}_q(X_n) \) is a graded regular representation of the symmetric group;
(c) \( \mathbb{K}[X_n] = \operatorname{THarm}_q(X_n) \oplus \operatorname{Hit}(X_n) = \operatorname{Harm}(X_n) \oplus \operatorname{THit}_q(X_n) \).

Furthermore, similar statements hold in the isotypic and Garsnir components of \( \mathbb{K}[X_n] \).

Proof. The injectivity of \( \tilde{\phi} \) is that the image \( \tilde{\phi}(\sum f_x \partial_x^k + 1 + qy \partial_y) \neq 0 \) of \( P_{q,k}^* \) is a complex number, that is \( \tilde{\phi}(\sum f_x \partial_x^k + 1 + qy \partial_y) \neq 0 \).

Furthermore, \( \mathbb{K}[X_n] \) is a free \( \operatorname{Sym}(X_n) \) module, so applying proposition \( 1 \) yields

\[
\begin{align*}
\operatorname{Hilb}_t(\mathbb{C}[X_n]) & \leq \operatorname{Hilb}_t(\operatorname{THarm}_q(X_n)) \operatorname{Hilb}_t(\operatorname{THarm}_q(X_n)) \\
& \leq \operatorname{Hilb}_t(\operatorname{Sym}(X_n)) \operatorname{Hilb}_t(\operatorname{Harm}) = \operatorname{Hilb}_t(\mathbb{C}[X_n]).
\end{align*}
\]

It follows that equality must hold, and that \( \operatorname{Hilb}_t(\operatorname{THarm}_q) = \operatorname{Hilb}_t(\operatorname{Harm}) \). The same reasoning can be carried over in any isotypic or Garsnir component, and all the statements of the theorem follow. \( \square \)

Corollary 1. Let \( \operatorname{Hilb}_{t, \leq d}(A) \) denote the truncation of the Hilbert series of \( A \) up to degree \( d \).

- If \( q \) is formal, then \( \operatorname{Hilb}_{t, \leq n}(\operatorname{Steen}_q(X_n)) = \operatorname{Hilb}_{t, \leq n}(\operatorname{Sym}(X_n)) \).
- If \( q \) is a complex number, then \( \operatorname{Hilb}_{t, \leq n}(\operatorname{Steen}_q(X_n)) \leq \operatorname{Hilb}_{t, \leq n}(\operatorname{Sym}(X_n)) \).

Proof. This is a reformulation of the preceding theorem by the fact that for \( d \leq n \) the algebra \( T\operatorname{Steen}_q \) and \( \operatorname{Steen}_q \) coincide. \( \square \)

6. Changing the number of variables

In the preceding section, we proved that conjecture holds when the number of the variable is greater than the degree. The goal here is to relate such \( q \)-harmonics on a large number of variable with \( q \)-harmonics on fewer variables, trying to take down as much information as possible. In particular we prove the equivalence \( (a) \iff (c1) \) of proposition \( 1 \).

Proposition 16. Let \( X \) be a set of variables and \( y \notin X \) be another variable. Let us denote by \( \operatorname{Harm}_q(X, y)_{\deg(y) \leq d} \) the subspace of \( \operatorname{Harm}_q(X, y) \) of polynomials of degree at most \( d \) in the variable \( y \). The isomorphism

\[
\pi_{y^d} : \left( \mathbb{K}(X, y)_{\deg(y) \leq d}/\mathbb{K}(X, y)_{\deg(y) \leq d-1} \right) \cong \mathbb{K}(X)
\]

defines an injective mapping:

\[
\tilde{\pi}_{y^d} : \operatorname{Harm}_q(X, y)_{\deg(y) \leq d}/\operatorname{Harm}_q(X, y)_{\deg(y) \leq d-1} \hookrightarrow \operatorname{Harm}_q(X).
\]

Proof. The injectivity of \( \tilde{\pi}_{y^d} \) follows from the definition. The only thing to prove is that the image \( \tilde{\pi}_{y^d} \) of a \( q \)-harmonic polynomial of degree \( d \) in \( \mathbb{K}[X, y] \) is in \( \operatorname{Harm}_q(X) \). It comes from the fact that the \( P_{q,k}^* \) are primitive, that is

\[
P_{q,k}^*(X, y) = P_{q,k}^*(X) + P_{q,k}^*(y).
\]

Write \( f = f_0 y^d + f_1 \) with \( \deg_y(f_1) < d \). The identification of the homogeneous components of degree \( d \) in \( y \) in the preceding equation proves that

\[
\left( \sum_{x \in X} (1 + qx \partial_x) \partial_x^k + (1 + qy \partial_y) \partial_y^k \right) \cdot f = 0.
\]
This ends the proof.

The following proposition yields as corollary the equivalence \((a) \Rightarrow (c1)\) and by an easy induction the implication \((a) \Rightarrow (c2)\) of proposition \(\Box\)

**Proposition 17.** Let \(q\) be generic. Then, the following are equivalent

(a) Conjecture \(\Box\) holds;
(b) For all \(n\), any \(q\)-harmonic on \(n\) variables has degree at most \(n-1\) on each variable;
(c) For all \(n\), the mapping

\[
\tilde{\pi}_y^d : \text{Harm}_q(X_n, y)_{\text{deg}(y) \leq d} / \text{Harm}_q(X_n, y)_{\text{deg}(y) \leq d-1} \rightarrow \text{Harm}_q(X_n),
\]

is an isomorphism for \(d \leq n\). For \(d > n\), \(\text{Harm}_q(X_n, y)_{\text{deg}(y) = d} = \{0\}\).

**Proof.**
(a) \(\Rightarrow\) (c): For any degree \(d\), the injectivity of \(\tilde{\pi}_y^d\) implies that

\[
\text{Hilb}_t(\text{Harm}_q(X_n, y)_{\text{deg}(y) \leq d} / \text{Harm}_q(X_n, y)_{\text{deg}(y) \leq d-1}) \leq t^d \text{Hilb}_t(\text{Harm}_q(X_n)),
\]

which is strict if, and only if, \(\tilde{\pi}_y^d\) is an isomorphism. On the other hand, we assumed that conjecture \(\Box\) holds. Identifying \(y\) with \(x_{n+1}\), it follows that

\[
\text{Hilb}_t(\text{Harm}_q(X_{n+1})) = (1 + t + \cdots + t^n) \text{Hilb}_t(\text{Harm}_q(X_n)).
\]

If for some \(d \leq n-1\), the inequality in equation \(\Box\) above is strict, the terms of degree \(d\) in equation \(\Box\) will differ. Then, by a similar reasoning, the existence of a non-trivial component \(\text{Harm}_q(X_n, y)_{\text{deg}(y) \leq d} / \text{Harm}_q(X_n, y)_{\text{deg}(y) \leq d-1}\) for \(d \geq n\) would contradict equation \(\Box\).

(b) \(\Rightarrow\) (a): By the assumption, and the injectivity of \(\tilde{\pi}_x^d\), for any \(n\),

\[
\text{Hilb}_t(\text{Harm}_q(X_{n+1})) \leq (1 + t + \cdots + t^n) \text{Hilb}_t(\text{Harm}_q(X_n)).
\]

By induction

\[
\text{Hilb}_t(\text{Harm}_q(X_{n+1})) \leq (1 + t) (1 + t + t^2) \cdots (1 + t + \cdots + t^n) = \text{Hilb}_t(\text{Harm}).
\]

Suppose now that, for a given \(n_0\), this inequality is strict:

\[
\text{Hilb}_t(\text{Harm}_q(X_{n_0+1})) < (1 + t) (1 + t + t^2) \cdots (1 + t + \cdots + t^{n_0}).
\]

Let \(d\) be the valuation of the difference. Applying again induction, we obtain that

\[
\text{Hilb}_t(\text{Harm}_q(X_{d+1})) < (1 + t) (1 + t + t^2) \cdots (1 + t + \cdots + t^d),
\]

with the inequality still appearing in degree \(d\). However, the number of variables now exceeds \(d\), and this is in contradiction with Corollary \(\Box\).

\(\Box\)

6.1. **Strings.** We introduce a little bit more formalism to analyze when \(q\)-harmonic polynomials in \(\mathbb{K}[X_{n+1}]\) have variables of degree at most \(n\).

**Proposition 18.** Consider \(f \in \mathbb{K}[X, y]\), and write its expansion on the variable \(y\) as

\[
f(X, y) = \sum_{i \geq 0} \frac{1}{i!} f_i(X) y^i.
\]

Then, \(f\) is \(q\)-harmonic if, and only if, for any \(i \geq 0\) and \(k \geq 1\),

\[
P_{q,k} f_i = -(1 + qi) f_{i+k}.
\]
Definition 7. A string

Setting

Equating the coefficients of

y

i > d

all

16 FLORENT HIVERT AND NICOLAS M. THIÉRY

Proof. Since \( f \) is \( q \)-harmonic, \( P_{q,k}^* f = 0 \). Hence,

\[
\sum_{i \geq 0} \frac{1}{i!} (P_{q,k}^* f_i) y^i + \sum_{i \geq 0} \frac{1}{i!} f_i (P_{q,k}^* f_i) = 0 ,
\]

and then

\[
\sum_{i \geq 0} \frac{1}{i!} (P_{q,k}^* f_i) y^i + \sum_{i \geq k} \frac{1}{(i-k)!} f_i (1 + q(i-k)) y^{i-k} = 0 .
\]

Setting \( j := i - k \) in the second sum, it follows that

\[
\sum_{i \geq 0} \frac{1}{i!} (P_{q,k}^* f_i) y^i + \sum_{j \geq 0} \frac{1}{j!} f_{j+k} (1 + qj) y^j = 0 .
\]

Equating the coefficients of \( y \) yields the desired equality, for all \( i \).

A remarkable consequence of proposition 18 is that a \( q \)-harmonic polynomial \( f \) can be reconstructed from its tail \( f_0 \). This motivates the following definition:

**Definition 6.** Let \( g \in \mathbb{K}(X) \) be a homogeneous polynomial. The string generated by \( g \) is the sequence \( F = \text{String}(g) := (f_i)_{i \geq 0} \) defined by

\[
f_0 := g, \quad \text{and for all } i > 0, \quad f_i := -P_{q,i}^* g.
\]

The length of the string \( F = (f_i) \) is defined by

\[
\ell(F) := 1 + \max\{i \mid f_i \neq 0\}.
\]

The head of the string is \( f_{\ell(F)} \).

The length of a string is nothing but the degree in the variable \( y \) of the associated polynomial \( f(X, y) := \sum \frac{1}{i!} f_i(X) y^i \). Obviously, if \( g \) is of degree \( d \) then \( f_i = 0 \) for all \( i > d \), consequently \( \ell(\text{String}(g)) < \deg(g) \) and the length is well defined.

**Definition 7.** A string \( F = (f_i) \) is called harmonic if

\[
P_{q,k}^* f_i = -(1 + qi) f_{i+k}, \quad \text{for all } i, k .
\]

Thanks to the relations in the \( q \)-Steenrod algebra, it is sufficient in the above definition to check that \( P_{q,1}^* \) acts properly:

**Proposition 19.** Let \( F = (f_i) \) a string. Then \( F \) is harmonic if, and only if,

\[
P_{q,1}^* f_i = -(1 + qi) f_{i+1}, \quad \text{for all } i .
\]

**Proof.** This is a simple consequence of the commutation relation. Namely, we have

\[
P_{q,k}^* f_i = -P_{q,k}^* P_{q,l}^* f_0 = q(l-k)P_{q,(k+1)}^* f_0 - P_{q,l}^* P_{q,k}^* f_0 = -q(l-k)f_{k+1} + P_{q,k}^* f_k .
\]

The proposition follows by an easy induction. \( \square \)

For example,

\[
P_{q,2}^* f_1 = -P_{q,2}^* P_{q,1}^* f_0 = -qP_{q,2}^* f_0 - P_{q,1}^* P_{q,2}^* f_0 = qf_3 - (1 + 2q)f_3 = -(1 + q)f_3 .
\]

Here are some basic properties of harmonic strings:

**Proposition 20.** Let \( F = (f_i) \) be an harmonic string. Then,

(a) The head \( f_{\ell(F)} \) of the string is harmonic;

(b) \( f_i \neq 0 \) whenever \( 0 \leq i \leq \ell(F) \) and \( q \neq -1/i \).

**Proof.** These are obvious consequences of the definitions: for \( k > 0 \), one has

\[
P_{q,k}^* (f_{\ell(F)}) = f_{\ell(F)+k} = 0 \quad \text{and} \quad P_{q,\ell(F)-1}^* (f_i) = -(1 + qi) f_{\ell(F)} \neq 0 .
\]

\( \square \)
6.2. **Length of the strings.** Proving that no $q$-harmonic polynomial in $\mathbb{K}[X_{n+1}]$ has variable degree $\leq n$ is obviously equivalent to proving that all strings on $\mathbb{K}[X_n]$ have length $\leq n - 1$. The rest of this section aims at some partial results in this direction.

**Proposition 21.** (a) Let $q = 0$, and $F$ be a 0-harmonic string on $\mathbb{K}[X_n]$. Then, the length of $F$ is at most $n$.

(b) Let $q$ be formal, and $F$ be a $q$-harmonic string on $\mathbb{K}[X_n]$ of length $d$, so that $f_0$ is of the form $f_{d,p}$, where $p$ is symmetric. Then, the length of $F$ is at most $n$.

(a) is a classical result, and (b) is a little extension to certain $q$-harmonic strings using a similar proof. Note that a $q$-harmonic string $F$ for $q$ generic will specialize at $q_0$ to a string $F(q_0)$ of same or smaller length. In particular, the above proposition shows that the string $F(0)$ will be of length at most $n - 1$, so all the $f_i(q)$, $i \geq n$ are divisible by $q$.

**Proof.** (a) Let $F$ be a 0-harmonic string on $\mathbb{K}[X_n]$ of length $k > n$ and degree $d = n + 1$. On $n$ variables, there is an explicit relation between symmetric polynomials: There exists a symmetric function $c\lambda$ of the form $\sum_{\lambda}(-1)^{f(\lambda)}c\lambda P^*_{0,\lambda} = 0$ where the $c\lambda$ are positive constants (cf. [Mac95]). This comes from the fact that, on $n$ variables, the $n$ first power sums generate $\text{Sym}(X_n)$ as a $\mathbb{Q}$-algebra. Applying $P^*_{0,\lambda}$ to $f_{k-d}$ yields $(-1)^{f(\lambda)}d\lambda f_k$ where $d\lambda$ are polynomials in $k$. Hence a contradiction.

(b) Let $F$ be a string. Thanks to the string relations,

$$P^*_{q,\lambda}(f_{k-|\lambda|}) = (-1)^{f(\lambda)}R\lambda(q)f_k$$

for some polynomial $R\lambda(q)$ with positive coefficients and constant term 1.

Assume now that $d$ is the length of $F$, and that $f_0$ is of the form $f_{d,p}$ with $p$ symmetric. Write the matrix $(P^*_{q,\lambda}(m\mu f_0))_{|\lambda|=|\mu|=n+1,f(\mu)\leq n}$. This matrix has one more column than rows. Furthermore, for $q = 0$ it has maximal rank. Hence, for $q$ generic it is also of maximal rank, and there exists a unique linear combination of the rows, up to a scalar coefficient in $\mathbb{C}(q)$. We pick up such a linear combination whose coefficients $C\lambda$ are polynomials in $\mathbb{C}[q]$ which are relatively prime in their set:

$$\forall \mu, \sum_{\lambda}(-1)^{f(\lambda)}C\lambda(q)P^*_{q,\lambda}(m\mu f_0) = 0.$$ 

At $q = 0$, this relation specializes, up to a coefficient $c \neq 0$, to the relation between the $p\lambda$’s described in equation [58]. We cancel out this coefficient, so that $C\lambda(0) = c\lambda$.

Applying this relation on $f_{k-d}$ yields

$$0 = \sum_{\lambda}(-1)^{f(\lambda)}C\lambda P^*_{q,\lambda}(f_0) = \left(\sum_{\lambda}C(q)\lambda R\lambda(q)\right)f_d.$$ 

However,

$$\sum_{\lambda}C\lambda(0)R\lambda(0) = \sum_{\lambda}c\lambda \neq 0.$$ 

Hence, $f_d = 0$, which is a contradiction. \qed
6.3. Complete analysis of the 2 variables case. We work over the alphabet \(X = \{x\}\), and make use of the strings machinery. For simplicity, let us use the notation \(x[d] := x^d/d!\). The Steenrod operators act by the rule:

\[
P_{q,k}^* x[d] = (1 + q(d - k)) x^{(d-1)}.
\]

Let \((f_i)\) be a harmonic string. Then,

\[
-f_{q,1}^* f_k = -P_{q,1}^* (-P_{q,k}^* f_0) = -(1 + kq) P_{q,k+1}^* f_0.
\]

Define \(A_k\) so that

\[
A_k(f_0) = (P_{q,1}^* P_{q,k}^* + (1 + kq) P_{q,k+1}^* ) f_0 = 0,
\]

and suppose that \(f_0 = x[d]\). Then, \(A_k(f_0)\) is of the form \(a_k x^{[d-k]}\), with

\[
a_k = (1 + q(d - k - 1))(1 + q(d - k) + 1 + kq).
\]

Hence, \(F = \text{String}(x[d])\) is harmonic if, and only if,

\[
\text{for all } 0 < k < d \quad (2 + qd)(1 + q(d - k - 1)) = 0.
\]

So, there are two possible solutions, namely \(d < 2\) and \(q = -2/d\).

Going back from strings to harmonic polynomials, we get the proposition:

**Proposition 22.** Let \(q\) be formal or a complex number. Then, the \(q\)-harmonic polynomials in \(\mathbb{K}[x, y]\) are spanned by

\[
1 \quad \text{and} \quad (x - y).
\]

if \(q\) is not of the form \(q = -2/d\) for \(d\) integer, \(d \geq 2\), and by

\[
1, \ (x - y), \ (x - y)(x + y)^{d-1}.
\]

otherwise.

In particular, if \(n \geq 2\) and \(d > n(n - 1)/2\) the polynomial \(f := (x_1 - x_2)(x_1 + x_2)^{d-1}\) is \(q_0\)-harmonic for \(q_0 := -2/d\). However, the highest degree of a harmonic polynomial in \(n\) variables is \(n(n - 1)/2\). This shows that conjecture \(\Box\) would not hold without condition on \(q\).

**Proposition 23.** Take \(d > 2\), and \(n \geq 2\) such that \(n(n - 1)/2 < d\). Then, the statement of conjecture \(\Box\) would not hold for \(q_0 := -2/d\).

7. Why is the \(q\) case more difficult than the \(q = 0\) case?

The difficulties when trying to prove conjecture \(\Box\) arises from the fact that the algebra \(\text{Steen}_q(X_n)\) is much bigger than \(\text{Sym}(X_n)\). We cite here some consequences of this; in this section we will expand more on the last two items.

- \(\mathbb{K}[X_n]\) is not a free \(\text{Steen}_q(X_n)\) module;
- The \(\text{Steen}_q\)-modules generated by two polynomials \(p_1\) and \(p_2\) may be non-isomorphic, even when \(p_1\) and \(p_2\) are of the same degree, or in the same isotypic or Garnir component;
- There are no proper \(q\)-analogs of elementary symmetric functions \(e_k\) which vanishes on \(X_n\) when \(k > n\);
- Their are very few operators which commute with the action of \(\text{Steen}_q\).

With this in view, conjecture \(\Box\) almost comes as a surprise: why isn’t the increased size of \(\text{Steen}_q(X_n)\) counterbalanced by fewer \(q\)-harmonics?
7.1. Elementary symmetric functions do not exist. As mentioned in section 2.5, one way to treat the classical case \( q = 0 \) is to construct an explicit Gröbner basis of the ideal \( H/\mathfrak{t} \). This construction relies on the existence of the elementary \( e_k \) and complete \( h_k \) symmetric functions which satisfy for any \( k \)

\[
\sum_{i=1}^{k} e_i h_{k-i} = 0, \quad e_k(X + Y) = \sum_{i=1}^{k} e_i(X) e_{k-i}(Y),
\]

and, last but not least,

\[
e_k(X) = 0, \quad \text{whenever } k > |X|.
\]

Finding \( q \)-analogs of the elementary and complete symmetric functions which satisfy the first two conditions is feasible, for example by considering Steen\(q\) (69) and, last but not least, Steen\(d\)pendent (see Sections ??). Hence, no linear combination of them vanishes on \( X_n \), and can serve as a reasonable \( q \)-analogue of \( e_{n+1} \).

7.2. Schubert Polynomials do not exist. In the classical case, the quotient \( \mathbb{K}[X]/\text{Sym}^+(X) \) is not only isomorphic to the regular representation of \( S_n \) but also to many deformations of it; among them one can find Hecke algebras. The more general construction has been described by Lascoux and Schützenberger in [LS87], with a 5-parameters deformation.

Recall that the symmetric group is generated by the elementary transpositions \( \sigma_i \) which exchange \( i \) and \( i + 1 \) for \( i < n \). A presentation is given by the relations

\[
\begin{align*}
\sigma_i^2 &= 1, & \text{for } 1 \leq i \leq n - 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, & \text{for } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & \text{for } 1 \leq i \leq n - 2.
\end{align*}
\]

The last two relations are called the braid relations, since they give a presentation of the braid group. A decomposition \( \sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \) of minimal length is called a reduced word for \( \sigma \).

The Hecke algebras \( H_n(v_1, v_2) \) are a 2-parameters family of quotients of dimension \( n! \) of the braid algebra, generated by \( T_i \) with the relations

\[
\begin{align*}
(T_i - v_1)(T_i - v_2) &= 0, & \text{for } 1 \leq i \leq n - 1, \\
T_i T_j &= T_j T_i, & \text{for } |i - j| > 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq n - 2.
\end{align*}
\]

Note that, as for the \((q, q')\)-Steenrod algebra, the structure of the algebra depends only on the quotient \( v_1/v_2 \).

Here, the useful special case is \( v_1 = v_2 = 0 \). Indeed, \( H_n(0, 0) \) acts on polynomials by the divided differences operators

\[
d_i(f) := \frac{f - \sigma_i f}{x_i - x_{i+1}}.
\]

To avoid confusion with the derivations we use \( d_i \) instead of the usual notation \( \partial_i \).

Note that the \( d_i \)'s are homogeneous operators of degree \(-1\) on polynomials.

Given a reduced word \( \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \) for a permutation \( \sigma \), define \( d_\sigma := d_{i_1} d_{i_2} \cdots d_{i_k} \).

Thanks to the braid relations, \( d_\sigma \) is independent of the choice of the reduced word for \( \sigma \). The family \( (d_\sigma)_{\sigma \in S_n} \) happens to be a basis of \( H_n(0, 0) \).

The key point here is the following proposition:
Proposition 24. The divided differences commute with the action of symmetric functions: for any \( p \in \text{Sym}(X_n) \), any \( i < n \) and any polynomial \( f \),
\[
(73) \quad d_i(p f) = p d_i(f) .
\]
The quotient \( \mathbb{K}[X_n]/\text{Sym}^+(X_n) \) is isomorphic to the regular representation of the divided differences algebra \( H_n(0,0) \).

This construction was first defined in geometry, where the quotient \( \mathbb{K}[X_n]/\text{Sym}^+(X_n) \) is interpreted as the cohomology ring \( H^*(F, \mathbb{Z}) \) of the flag manifold \( F \) [Dem74, BGG73].

Using the divided differences, one can construct a basis of \( \mathbb{K}[X_n]/\text{Sym}^+(X_n) \) composed of the so-called Schubert polynomials:

Definition 8. Let \( \rho := (n-1, n-2, \ldots, 1, 0) \), so that \( X^\rho = x_1^{n-1}x_2^{n-2}\ldots x_{n-1} \). For each permutation \( \sigma \in S_n \) the Schubert polynomial \( S_\sigma \) is defined by
\[
(74) \quad S_\sigma := d_\omega(x^\rho) ,
\]
where, for technical reasons, \( \omega \) has been chosen to be the maximal permutation of \( S_n \).

For more details about Schubert polynomials, we refer to [Mac91]. Note that they are denoted by \( S_\sigma \) in [Mac91], and by \( X_\sigma \) in Lascoux and Schützenberger work [LS82, LS87].

The main theorem is

Theorem 7.1 (Lascoux-Schützenberger, 1982). The family of Schubert polynomials \( (S_\sigma)_{\sigma \in S_n} \) is a basis of \( \mathbb{K}[X_n] \) as a free \( \text{Sym} \)-module. In particular, the family of Schubert polynomial \( (S_\sigma)_{\sigma \in S_n} \) is a basis of \( \mathbb{K}[X_n]/\text{Sym}^+ \).

Moreover the divided difference machinery provides a very efficient algorithm to decompose a polynomial in this basis.

This machinery relies heavily on the commutation property of proposition 24.

One strategy to prove the conjecture 2 is to search for a family of linear operators of degree \(-1\) which commute with the action of \( \text{Steen}_q \), hoping that they will generate a \( q \)-analog of the divided differences algebra. Unfortunately, we obtained by a computation that no such operator exists for, for example, \( n = 1, 2, 3, 4 \).

Maybe this machinery could still be made to work with some weaker skew commutation condition. A precise statement of this might be:

Problem 2. Does there exist an operator \( T \) of degree \(-1\) on polynomials such that for any \( f \in \text{Steen}_q \), there exists \( g \in \text{Steen}_q \) such that
\[
(75) \quad f T = T g .
\]
What is the structure of the algebra generated by such operators ?

For a similar reason, we could not generalize the fact that \( \text{Harm} \) is spanned by the derivatives of the Vandermonde. Indeed, there are no reasonable \( q \)-analogs of the derivations which commute with the operators \( P^*_{q,k} \)’s.
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