QUANTUM LIE ALGEBRAS OF TYPE $A_n$ POSITIVE, PBW BASES AND THE YANG-BAXTER EQUATION

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ABSTRACT. We show explicitly a generalised Lie algebra embedded in the positive and negative parts of the Drinfeld-Jimbo quantum groups of type $A_n$. Such a generalised Lie algebra satisfy axioms closely related to the ones found by S.L. Woronowicz. For the universal enveloping algebra of such generalised Lie algebras we establish several conditions in order to obtain bases of type Poincaré-Birkhoff-Witt. Besides a graded algebra is proposed and some relations with the quantum Yang-Baxter equation are studied.

1. INTRODUCTION

The applications of the theory of Lie algebras and groups to theoretical physics are well known. However theoretical problems have reached a level which requires more general algebraic objects. For instance, the objects called quantum groups [1] which contains Lie groups and Lie algebras as degenerate cases. Moreover, related to quantum groups appear different kinds of generalised Lie algebras [2, 3, 4]. In particular, inside of the Drinfeld-Jimbo quantum groups (or quantised universal enveloping algebras) several authors [5, 6, 7] have discovered generalised Lie algebras, or quantum Lie algebras. In this paper we deal with the Drinfeld-Jimbo quantum group $U_q(sl_{n+1})$.

In [5] is shown that there exists inside $U_q(sl_{n+1})$ a generalised color Lie algebra $(sl_{n+1}^+)_q$ such that its universal enveloping algebra is the positive part $U^+_q(sl_{n+1})$ of $U_q(sl_{n+1})$. Unfortunately the axioms of such a generalised Lie algebra depend on the chosen basis.

The aim of this paper is to propose generalised Lie algebra axioms for the module $(sl_{n+1}^+)_q$ independent of the choice of basis. We call such a structure $T$-Lie algebra. A part of such axioms (antisymmetry, Jacobi identity) are closely related to the ones found by Woronowicz [2].

Besides we shall prove that $(sl_{n+1}^+)_q$ satisfies an additional Jacobi identity in a similar manner to the balanced generalised Lie algebras of Lyubashenko-Sudbery [3]. This means, $(sl_{n+1}^+)_q$ is a balanced generalised Lie algebra. However, the universal enveloping algebra of $(sl_{n+1}^+)_q$ as a $T$-Lie algebra is not the same universal enveloping algebra as a balanced generalised Lie algebra.

Finally the relations with the quantum Lie algebra definition of Vybornov [4] are studied. Following the work of Vybornov we obtain that, in some sense, the quantum Yang-Baxter equation (QYBE) includes not only a generalised Jacobi identity, and a generalised antisymmetry, but an additional property called multiplicativity, which is a part of the conditions in order to obtain bases of type Poincaré-Birkhoff-

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Let $k$ be a commutative unitary ring.

**Definition 1.** A $T$-Lie algebra is a $k$-module $L$ with morphisms

$$\begin{align*}
\llbracket, \rrbracket &: L \otimes_k L \to L, & \text{bracket} \\
S &: L \otimes_k L \to L, & \text{presymmetry} \\
\langle, \rangle &: L \otimes_k L \to L \otimes_k L, & \text{pseudobracket}
\end{align*}$$

such that the following axioms are satisfied

1. **Stability**
   
   (a) There exist $L = \bigoplus_{n \in \mathbb{N}} L_n$ a strict gradation of $L$ related to $\llbracket, \rrbracket$, (this means $[L_{\eta_1}, L_{\eta_2}] \subseteq L_{\eta_1 + \eta_2 - 1}$).
   
   (b) $\langle L_{\eta_1}, L_{\eta_2} \rangle \subseteq (L \otimes_k L)_{\eta_1 + \eta_2 - 1}$. 

2. **Antisymmetry**
   
   (a) $\llbracket, \rrbracket S = - \llbracket, \rrbracket$
   
   (b) $\langle, \rangle S = - \langle, \rangle$
   
   (c) $\llbracket, \rrbracket \langle, \rangle = 0$

3. **Jacobi identity**

   (1) $\llbracket, \rrbracket (1 \otimes \llbracket, \rrbracket) - \llbracket, \rrbracket (1 \otimes \langle, \rangle) - \llbracket, \rrbracket (1 \otimes \langle, \rangle) (S \otimes 1) = 0$

**Example 1.** A $G$-algebra $\Lambda$ is a graded $k$-module $\Lambda = \bigoplus_n \Lambda_n$ equipped with two multiplications, $(\lambda, \gamma) \mapsto \lambda \gamma$ and $(\lambda, \gamma) \mapsto [\lambda, \gamma]$ with some additional properties, we are interested just in the properties of the bracket $\llbracket, \rrbracket$. These are

(2) $[\lambda, \gamma] \in \Lambda_{m+n-1}$, if $\lambda \in \Lambda_m, \gamma \in \Lambda_n$

(3) $[\lambda, \gamma] = -(-1)^{(m-1)(n-1)}[\gamma, \lambda]$, if $\lambda \in \Lambda_m, \gamma \in \Lambda_n$, and if $\lambda \in \Lambda_m, \gamma \in \Lambda_n, \eta \in \Lambda_p$

(4) $(-1)^{(m-1)(p-1)}[\lambda, [\gamma, \eta]] + (-1)^{(n-1)(m-1)}[\gamma, [\eta, \lambda]] + (-1)^{(p-1)(n-1)}[\eta, [\lambda, \gamma]] = 0$

The property (4) says that $\Lambda = \bigoplus_n \Lambda_n$ is a strictly graded algebra related to $\llbracket, \rrbracket$. Therefore $\Lambda$ has a structure of $T$-Lie algebra with bracket $\llbracket, \rrbracket$, pseudobracket $\langle, \rangle = 0$, and presymmetry $S$ defined by

$$S(\lambda \otimes \gamma) = (-1)^{(m-1)(n-1)}\gamma \otimes \lambda, \text{ if } \lambda \in \Lambda_m, \gamma \in \Lambda_n.$$

Since,

$$S(1 \otimes \llbracket, \rrbracket) = (1 \otimes \llbracket, \rrbracket)(1 \otimes S)(S \otimes 1)$$

$$S(\llbracket, \rrbracket \otimes 1) = (\llbracket, \rrbracket \otimes 1)(S \otimes 1)(1 \otimes S)$$

then the Jacobi identity (4) is equivalent to the Jacobi identity (1) of $\Lambda$ as a $T$-Lie algebra.

Observe that we are not using the standard reduced degree of $\Lambda$ (see §8, p 89).
If $L$ is a $T$-Lie algebra we define its universal enveloping algebra $U(L)$ as the quotient

$$U(L) = L^\otimes / J$$

where $L^\otimes$ is the $k$-tensorial algebra of $L$ and $J$ is the two sided ideal generated by

$$x \otimes y - S(x \otimes y) - (x,y) - [x,y], \quad \forall x,y \in L.$$

Related to each $T$-Lie algebra $L$ there exist an abelian $T$-Lie algebra denoted $L^0$. Its structure is: $L^0 = L$ as $k$-module, $[,]^0 = 0$, $(,)^0 = 0$, $S^0 = S$. The universal enveloping algebra

$$S(L) = U(L^0)$$

is called $q$-symmetric algebra of $L$.

Now, let $B$ a totally ordered basis of $L$ a $T$-Lie algebra with strict gradation given by

$$L = \oplus_{\eta \in \mathbb{N}} L_\eta.$$

Let $x_\lambda \in B, \Sigma = (x_{\lambda_1}, \ldots, x_{\lambda_n})$ finite-non-decreasing sequence of elements of $B$. We write $\eta(\lambda) = \eta(x_\lambda) = \alpha$ if $x_\lambda \in L_\alpha$, $z_\lambda = x_\lambda \in S(L)$, $z_\Sigma = z_{\lambda_1} \cdots z_{\lambda_n} \in S(L)$, $z_\emptyset = 1 \in S(L)$, $\eta(\Sigma) = \eta(z_\Sigma) = \eta(x_{\lambda_1}) + \ldots + \eta(x_{\lambda_n})$. Besides we put $x_\lambda \leq \Sigma$ if $x_\lambda \leq x_{\lambda_1}$.

Let $S_\eta$ be the $k$-submodule of $S(L)$ generated by $z_\Sigma$ such that $\eta(\Sigma) \leq p$.

**Lemma 1.** Let $L$ be a $T$-Lie algebra and $B$ a free basis of $L$ as $k$-module, $B$ totally ordered, such that

1. $S(x \otimes y) = q_{x,y} y \otimes x, \quad q_{x,y} \in k, \forall x,y \in B$;
2. $q_{x,y} = q_{y,x}^{-1}$.

Then, there exist a $k$-morphism

$$\eta: L \otimes_k S(L) \to S(L)$$

satisfying

(A) $x_\lambda \cdot z_\Sigma = z_\lambda z_\Sigma$ for $x_\lambda \leq \Sigma$;
(B) $x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma \in S(\eta(\lambda) + \eta(\Sigma) - 1)$.

**Proof.** See [1], proof of lemma V.1. \qed

### 3. Poincaré-Birkhoff-Witt bases and $T$-Lie algebras

In [2] we proved that the PBW theorem is not a general property for all the basic $T$-Lie algebras. We have to restrict them. Such a restriction is called adequate. Thanks to lemma [2], the condition adequate makes sense not only for basic $T$-Lie algebras but for $T$-Lie algebras.

**Definition 2.** An adequate $T$-Lie algebra is a $T$-Lie algebra with a totally ordered free basis for $L$ as a $k$-module such that they hold the condition adequate (see [2], definition V.1).

We do not have to confuse adequate basic $T$-Lie algebra (defined in [2]) with adequate $T$-Lie algebra (defined in this paper).

Let $L^3$ be the $k$-submodule of the $k$-tensorial algebra $L^\otimes$ generated by

$$x_i \otimes x_j \otimes x_k, \quad x_i < x_j < x_k \in B$$
and let $^3L$ be the $k$-submodule generated by
\[ x_k \otimes x_j \otimes x_i, \quad x_i < x_j < x_k \in \mathcal{B}. \]

Let us put $S_2, S_1 : L^3 \rightarrow L^3$, $S_1 = 1 \otimes S$, $S_2 = (1 \otimes S)$.

**Proposition 1.** Let $L$ be an adequate $T$-Lie algebra related to the basis $\mathcal{B}$ such that,
1. $S([.,] \otimes 1)|_{L^3} = (1 \otimes [.,])S_1S_2|_{L^3}$;
2. $S(1 \otimes [.,])|_{L^3} = ([.,] \otimes 1)S_2S_1|_{L^3}$;
3. $S([.,] \otimes 1)|_{L^3} = (1 \otimes [.,])S_1S_2|_{L^3}$;
4. $S(1 \otimes [.,])|_{L^3} = ([.,] \otimes 1)S_2S_1|_{L^3}$;

then $U(L)$ holds a basis of type PBW related to $\mathcal{B}$.

**Proof.** According with [2], it will suffice to prove that $L$ is a basic $T$-Lie algebra. The conditions (1), (2) ensure multiplicativity, while conditions (3), (4) say that Jacobi identity for $L$ as a basic $T$-Lie algebra follows from the Jacobi identity as $T$-Lie algebra. Therefore $L$ is a basic $T$-Lie algebra.

**Proposition 2.** Let $L$ be a basic $T$-Lie algebra with basis $\mathcal{B}$. If it holds
1. $S([.,] \otimes 1)|_{L^3} = (1 \otimes [.,])S_1S_2|_{L^3}$;
2. $S(1 \otimes [.,])|_{L^3} = ([.,] \otimes 1)S_2S_1|_{L^3}$;
3. $[.,]S((x_i, x_j) \otimes x_j) = ([.,](1 \otimes [.,])S_1S_2(x_i \otimes x_j \otimes x_j), \forall x_i, x_j \in \mathcal{B}$.

and a second Jacobi identity
\[ [.,]((1 \otimes [.,])S_1S_2 - ([.,] \otimes 1)S_2S_1 + ([.,] \otimes 1)S_2)|_{L^3} = 0 \]

then
\[ [.,](1 \otimes [.,]) - [.,]([.,] \otimes 1) - [.,](1 \otimes [.,])S_1 = 0 \]
\[ [.,](1 \otimes [.,]) - [.,]([.,] \otimes 1) + [.,]([.,] \otimes 1)S_2 = 0 \]

**Proof.** Let $x, y, z \in \mathcal{B}$. If $x > y > z$ then the Jacobi identity of $L$ as a $T$-Lie algebra together with the hypothesis (1) and (2) say
\[ [x, [y, z]] - [[x, y], z] - q_{x,y}[y, [x, z]] = 0 \]
besides, the second Jacobi identity (3) can be written as
\[ [x, [y, z]] - [[x, y], z] + q_{y,z}[x, z], y] = 0 \]

First, we are going to prove
\[ [x, [y, z]] - [[x, y], z] - q_{x,y}[y, [x, z]] = 0 \]

There are several cases:

**Case** $x = y = z$: then (10) trivially holds.

**Case** Two of $x, y, z$ are the same: now there are several subcases;

**Subcase** $x = y, z$: the left side of (10) is $[x, [x, z]] - q_{x,x}[x, [x, z]] = 0$.

**Subcase** $x, y = z$: the left side of (10) is
\[-[[x, y], y] - q_{x,y}[y, [x, y]] = S([x, y] \otimes y) - q_{x,y}[y, [x, y]] = 0 \]
since hypothesis (3)
Subcase $x = z, y$: again, the left side of (11) is

$$[x, [y, x]] - [[x, y], x] = [x, [y, x]] + q_{x,y}[[y, x], x]$$

$$= [x, [y, x]] - q_{y,x}[S([y, x] \otimes x)]$$

$$= [x, [y, x]] - q_{y,x}[q_{x,y}[x, [y, x]]]$$

because hypothesis (3).

Case $x, y, z$ three different elements: let be the left side of (10). There are several subcases: $x > y > z, x > z > y, y > x > z, y > z > x, z > x > y, z > y > x$

Subcase $x > y > z$: equation (8)

Subcase $x > z > y$:

$$s = -q_{y,z}[x, [z, y]] - [[x, y], z] + q_{x,y}[y, [x, z]]$$

$$= -q_{y,z}[x, [z, y]] - [[x, y], z] - q_{x,y}q_{y,x}q_{y,z}[[x, z], y]$$

$$= -q_{y,z}([x, [z, y]] - [[x, y], z] + q_{x,y}([[x, y], z]) = 0$$

thanks to (3).

Subcase $y > x > z$:

$$s = [x, [y, z]] + q_{x,y}[[y, z], z] - q_{x,y}[y, [x, z]]$$

$$= -q_{x,y}([y, [x, z]] - [[y, x], z] - q_{y,x}[x, [y, z]]) = 0$$

since the Jacobi identity in $L$.

The remaining subcases are similar.

By similar computing (7) holds too.

4. A QUANTUM Lie algebra of type $A_n$ positive

Let $[,]$ be the usual bracket of the Lie algebra $sl_{n+1}$, $\mathcal{B} = \{ e_{ij} \mid 1 \leq i < j \leq n+1 \}$ canonical basis of the Lie subalgebra $sl_{n+1}^+$ form by upper triangular matrices. Put $h_{ij} = [e_{ij}, e_{ij}^+]$, $1 \leq i < j \leq n + 1$. Then $[h_{ab}, e_{ij}] = c_{ab} e_{ij}$, where $c_{ab,ij} \in \mathbb{Z}$. Let $k$ be an unitary commutative ring, $(sl_{n+1}^+)_q$ the $k$-free module with basis $\mathcal{B}$. Define: an order by

$$e_{ab} < e_{uv}, \text{ if } a + b < i + j \text{ or } (a + b = i + j \text{ and } b < j),$$

a $k$-morphism $S : (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q \rightarrow (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q$ by

$$S(e_{ab} \otimes e_{ij}) = q^{c_{ab,ij}}e_{ij} \otimes e_{ab}, \text{ if } e_{ab} < e_{ij}, S(e_{ab} \otimes e_{ab}) = e_{ab} \otimes e_{ab}, \quad S^2 = 1,$$

a $k$-morphism $[,] : (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q \rightarrow (sl_{n+1}^+)_q$ by

$$[e_{ab}, e_{ij}]q = [e_{ab}, e_{ij}] \text{ if } e_{ab} < e_{ij}, \quad [\cdot, ]qS = -[\cdot, ]q \text{ if } e_{ab} > e_{ij}$$

a $k$-morphism $\langle, \rangle : (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q \rightarrow (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q$ by

$$\langle e_{ab}, e_{ij} \rangle = \begin{cases} (q - q^{-1})e_{ib} \otimes e_{aj} & \text{if } a < i < b \text{ and } i < b < j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\langle, \rangle S = -\langle, \rangle.$$

Lemma 2. If $e_{ab} > e_{ij} > e_{uv}$ then

$$[[e_{uv}, e_{ab}], e_{ij}] = [[e_{uv}, e_{ab}]q, e_{ij}]q = 0.$$
Proof. Since \([,]_q\) is a deformation of the usual bracket \([,]_n\) of \(sl_n\), this amounts to showing that \([e_{uv}, e_{ab}], e_{ij}] = 0\).

Because \(e_{uv} < e_{ab}\) we can suppose \(v = a\) then we have to prove

\[\{e_{ab}, e_{ij}\} = 0.\]  (11)

Notice that if \(b \neq i\) and \(j \neq u\) then (11) holds. But if \(b = i\) then \(a < b = i < j\) implies \(a + i < j + i\) and \(e_{ab} = e_{ai} < e_{ij}\), a contradiction. And if \(j = u\) then \(e_{ua} < e_{iu}\) implies \(u + a < i + u\) or \(u + a = i + u\) and \(a < u\), it follows \(a < u\), again a contradiction because \(u < v = a\). Therefore (11) holds.

\[\text{Theorem 1.}\] The \(k\)-module \((sl_{n+1}^+)_q\) has a structure of \(T\)-Lie algebra.

\[\text{Proof.}\] We have to prove the Jacobi identity (3) of the definition 1 for the bracket \([,]_q\). Since proposition 1 it suffices to prove that its conditions hold. Let \(e_{ij}\), \(1 \leq i < j \leq n\) be canonical basis of \((sl_{n+1}^+)_q\). If \(e_{ab} < e_{ij} < e_{uv}\) then

\[\{e_{ab}, e_{ij}\}_q < e_{uv}, \text{ if } \{e_{ab}, e_{ij}\}_q \neq 0 \text{ and } e_{ab} < [e_{ij}, e_{uv}]_q, \text{ if } [e_{ij}, e_{uv}]_q \neq 0\]

besides, if \([,]_n\) denotes the classical bracket of \(sl_n\) and \(h_{ij} = [e_{ij}, h_{ij}]\) then \([h_{ij}, e_{uv}] = [h_{uv}, e_{ij}] = c_{ij, uv} e_{uv}\) for certain \(c_{ij, uv} \in k\), and from the Jacobi identity in \(sl_n\) it follows

\[\{h_{ab}, [e_{ij}, e_{uv}]\} = (c_{ab,ij} + c_{ab, uv})[e_{ij}, e_{uv}]\]

this implies the conditions (2) and (1) of proposition 1. The condition (1) of proposition 2 holds because \([e_{ij}, e_{ab}]_q, e_{ab}]_q = 0, \forall e_{ij}, e_{ab}\). Now only remains to prove the second Jacobi identity (3).

Let \(x, y, z\) basic elements such that \(x > y > z\), then the left side of (3) is,

\[-q_{xy}q_{xz}q_{yz}([z, [y, x]] - [[y, x], z]) - q_{yz}q_{xz}[[z, x], y]
\]

\[= q_{xz}q_{y}([x, y], [z, x]) - [[z, x], y)]
\]

\[= 0\]

since lemma 1.

\[\text{Proposition 3.}\] The \(T\)-Lie algebra \((sl_{n+1}^+)_q\) is a balanced generalised Lie algebra.

\[\text{Proof.}\] Define \(\gamma : (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q \to (sl_{n+1}^+)_q \otimes_k (sl_{n+1}^+)_q\) by \(\gamma = 1 - S\). Then, equations (3) and (2) of proposition 2 are the left Jacobi identity and the right Jacobi identity respectively, (see 1).

5. On the canonical graded algebra related to the universal enveloping algebra

For the \(T\)-Lie algebras we can repeat the classical construction (8) (or color 10) of the graded algebra related to a universal enveloping algebra.

Let \(L = \bigoplus_{\eta}L_\eta\) strictly gradation of the \(T\)-Lie algebra \(L\). Define the \(k\)-submodules of \(L^\otimes\) the tensorial algebra of \(L\),

\[T_m = \bigoplus_{\eta_1 + \ldots + \eta_k \leq m} (L_{\eta_1} \otimes \ldots \otimes L_{\eta_k}) , m \geq 0\]


Proof. Corollary 1 and let \( \pi : L^\otimes \to U(L) \) natural morphism. Put \( U_m = \pi(T_m) \), \( U_{-1} = 0 \). Set the \( k \)-module \( G^m = U_m/U_{m-1} \), and let the multiplication on \( U(L) \) define a bilinear map \( G^m \times G^p \to G^{m+p} \). This extends at once to a bilinear map \( G \times G \to G \) making \( G = \prod_{m \geq 0} G^m \) a graded associative algebra with 1.

Since \( \pi \) maps \( T^m \) into \( U_m \), the composite linear map \( \varphi_m : T^m \to U_m \to G^m \) makes sense. It is surjective because from \( \pi(T_m - T_{m-1}) = U_m - U_{m-1} \). Using the universal property of the \( k \)-module coproduct \( L^\otimes = \prod_{m \geq 0} T^m \) we get a \( k \)-morphism \( \varphi : L^\otimes \to G \), which is surjective.

Let \( I \) be the \( k \)-bilateral ideal of \( L^\otimes \) generated by \( x \otimes y - S(x \otimes y), \forall x, y \in L \).

**Proposition 4.** The map \( \varphi : L^\otimes \to G \) is a \( k \)-algebra morphism. Moreover, \( \varphi(I) = 0 \), so \( \varphi \) induces a \( k \)-algebra epimorphism \( \omega : S(L) \to G \).

**Proof.** First, notice that \( \varphi : L^\otimes \to G \) is a \( k \)-algebra morphism because the product definition of \( G \).

Let \( x \otimes y - S(x \otimes y), (x, y \in L) \) be a generator of \( I \) such that \( x \in L_{\eta_1}, y \in L_{\eta_2} \). Then \( \pi(x \otimes y - S(x \otimes y)) \in U_\eta \) where \( \eta = \eta_1 + \eta_2 \). On the other hand \( \pi(x \otimes y - S(x \otimes y)) = \pi([x, y] + (x, y)) \in U_{\eta-1} \), whence \( \pi(x \otimes y - S(x \otimes y)) \in U_{\eta-1}/U_{\eta-1} = 0 \). It follows \( I \subseteq \text{Ker} \varphi \).

In the classical case \( \omega \) is an isomorphism. However, this is not true in general, because if we take the \( T \)-Lie algebra \( L = (sl_4)_q \) (see [3], examples III.5, IV.4 and definition IV.1) then \( x_2 x_6 = 0 \) in \( U(L) \). Besides if \( p : L^\otimes \to S(L) \) is the natural projection then \( \omega(p(x_2 \otimes x_4)) = \pi(x_2 \otimes x_4) = 0 \). Therefore \( \omega \) it is not injective. Although if \( L \) is a basic \( T \)-Lie algebra, the symmetric algebra \( S(L) \) is isomorphic to the graded algebra \( G \).

**Corollary 1.**
1. If \( L \) is a basic \( T \)-Lie algebra then \( S(L) \simeq G \) as algebras.
2. If \( L \) is a basic \( T \)-Lie algebra then \( U(L) \) has no zero divisors \( \neq 0 \).
3. The algebras \( U(sl_{n+1}^+)_q \) and \( M_{p,q,e}(n) \) which is a multiparametric deformation of \( GL(n) \), have no zero divisors \( \neq 0 \).

**Proof.**
1. Just copy word by word the classical proofs [3], p 94 or [3], p 166.
2. See [3], theorem 4, p 164.
3. The algebras \( U(sl_{n+1}^+)_q \) and \( M_{p,q,e}(n) \) are both universal enveloping algebras of basic \( T \)-Lie algebras, (see [3]).

6. The Yang-Baxter equation and \( T \)-Lie algebras

Now we are going to apply some remarks of Vybornov [3] to the theory of \( T \)-Lie algebras in order to obtain solutions of the QYBE.
Let $L$ be a free $k$-module with basis $B$ totally ordered and $S : L \otimes_k L \rightarrow L \otimes_k L$ such that $S(x \otimes y) = q_{x,y} y \otimes x$ for all $x, y \in B$ and certain $q_{x,y} \in k$. Let us put $L = L \oplus k$. We may extend $S$ to $\tilde{L}$ by means of defining
\[
S(x \otimes 1) = 1 \otimes x, \quad S(1 \otimes x) = x \otimes 1, \quad S(1 \otimes 1) = 1 \otimes 1, \quad \forall x \in L, 1 \in k.
\]
Define two families of linear maps
\[
R(\lambda), R(\lambda) : \tilde{L} \otimes \tilde{L} \rightarrow \tilde{L} \otimes \tilde{L}
\]
where $p : \tilde{L} \rightarrow L$ is the natural projection.
As usual, we define $R_1(\lambda) = R(\lambda) \otimes 1$, etc.

**Proposition 5.** Let $V$ be a $k$-submodule of $L^{3\otimes}$.
1. $R(\lambda) R(\lambda)' = R(\lambda)' R(\lambda) = 1$, for any $\lambda$ if and only if $[\cdot, \cdot] S = -[\cdot, \cdot]$;
2. $R_1(\lambda) R_2(\lambda) R_1(\lambda)|_V = R_2(\lambda) R_1(\lambda) R_2(\lambda)|_V$, for any $\lambda$ if and only if the Jacobi identity $\square$ holds on $V$ and the following multiplicity conditions hold also
\[
S(1 \otimes [\cdot, \cdot])|_V = ([\cdot, \cdot] \otimes 1) S_1 S_2|_V \quad S([\cdot, \cdot] \otimes 1)|_V = (1 \otimes [\cdot, \cdot]) S_1 S_2|_V
\]
\[
S(1 \otimes [\cdot, \cdot]) S_1|_V = ([\cdot, \cdot] \otimes 1) S_1|_V
\]
3. $R(\lambda)' R_2(\lambda)' R_1(\lambda)'|_V = R_2(\lambda)' R_1(\lambda)' R_2(\lambda)'|_V$, for any $\lambda$ if and only if the Jacobi identity $\square$ holds on $V$ and the following multiplicity conditions hold also
\[
S(1 \otimes [\cdot, \cdot])|_V = ([\cdot, \cdot] \otimes 1) S_1 S_2|_V \quad S([\cdot, \cdot] \otimes 1)|_V = (1 \otimes [\cdot, \cdot]) S_1 S_2|_V
\]
\[
S([\cdot, \cdot] \otimes 1)|_V = (1 \otimes [\cdot, \cdot]) S_1|_V.
\]

**Proof.** By straightforward computations on the basic elements. \hfill \Box

**Corollary 2.** Let $L$ be $(sl^+_{n+1})_q$ with canonical basis $B$.
1. Let us put $x < 0$, $\forall x \in B$. Let $V_1$ be the $k$-submodule of $L^3$ generated by $x \otimes y \otimes z$ such that $x, y, z \in B$, $x < y < z$ and $y < [x, y]_q$. Then, for any $\lambda$,
\[
R_1(\lambda) R_2(\lambda) R_1(\lambda)|_{V_1} = R_2(\lambda) R_1(\lambda) R_2(\lambda)|_{V_1}.
\]
2. Let us put $x > 0$, $\forall x \in B$. Let $V_2$ be the $k$-submodule of $L^3$ generated by $x \otimes y \otimes z$ such that $x, y, z \in B$, $x < y < z$ and $y > [x, y]_q$. Then, for any $\lambda$,
\[
R_1(\lambda)' R_2(\lambda)' R_1(\lambda)'|_{V_2} = R_2(\lambda)' R_1(\lambda)' R_2(\lambda)'|_{V_2}.
\]
Since the QYBE is equivalent to the braid equation (see \textsuperscript{12} p 3316), we have obtained restricted solutions to the QYBE.

Besides, for $L = (sl^+_{n+1})_q$, we have $V_1 = V_2 = L^{3\otimes}$. So, $(sl^+_{n+1})_q$ is a quantum Lie algebra according with Vybornov $\square$.

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