About Stability of Irreducibility for Germs of Holomorphic Functions

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Abstract

This survey is about irreducibility for germs of a holomorphic function $f$. I will show that when the dimension of the domain $U$ of this holomorphic function $f$ is greater than 2, the irreducibility of germs are not necessary to be stable. That means, if the germ of $f$ at point $p$ is irreducible in the stalk of holomorphic functions at $p$, this does NOT means there exists an open neighborhood $V \subset U$ of this point $p$, such that for any point $q \in V$, the germ of $f$ at $q$ is irreducible at the stalk of holomorphic functions at $q$

1 Introduction

Let $U$ be an open set in $\mathbb{C}^n$ which contains 0, $f$ be a holomorphic function defined on $U$, $f_p$ is the germ of $f$ at point $p \in U$.

For any two holomorphic functions $g, h$ defined on $U$, if $g_0, h_0$ are relatively prime with each other, then with the help of resultants, we know that $g, h$ are relatively prime with each other nearby. Precisely to say, that means their exists an open neighborhood $V \subset U$ of 0, such that for any point $q \in V$, $g_q$ and $h_q$ are relatively prime with each other. In this sense, we can say that Being co-prime is a stable property.

Can we say Irreducibility is a stable property? In the case of dimension 2, the answer is positive, and the proof is easy. But in the case of dimension 3, I will present a polynomial as counter-example.

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2 Proof for the Case of Dimension 2

**Statement**: For any holomorphic function \( f = f(z_1, z_2) \) on \( U \subset \mathbb{C}^2(0 \in U) \), and the germ of \( f \) at origin is irreducible, then there exists an open neighborhood \( V \subset U \) of 0, such that for any point \( q \in V, f_q \) is irreducible. (**Remark**: If \( f(p) \neq 0 \), the \( f \) is irreducible at \( p \). So we only need to care about zero points of \( f \).)

**Proof**: Without the loss of generality, we can assume \( f(0, z_2) \) is not identically 0 near the origin, and \( f(0, 0) = 0 \).

let \( w = z_2^d + e_1(z_1)z_2^{d-1} + \cdots + e_{d-1}(z_1) + e_d(z_1) \) be a Weierstrass polynomial of \( f \) near 0.

Because \( w \) is irreducible at 0, so \( w \) and \( \frac{\partial w}{\partial z_2} \) are relatively prime near 0. Then the resultant of \( w \) and \( \frac{\partial w}{\partial z_2} \) is not zero. Then the common zero loci of \( w \) and \( \frac{\partial w}{\partial z_2} \) are discrete near 0.

From above, we know that there exists an open set \( V(0 \in V \subset U) \), such that in \( U \), \((0,0)\) is the only zero point of \( w \) which is POSSIBLE to be singular. (since for other points in \( q \in U, \frac{\partial w}{\partial z_2}(p) \neq 0 \). We can conclude that at any zero point \( p(p \neq 0) \) of \( w \) in \( V, w \) is a local complex parameter near \( p \). Since \( w \) is a local complex parameter near \( p \), then the germ of \( w \) at \( p \) is irreducible.

Finally, because \( w \) is a Weierstrass polynomial of \( f \) at 0, then we know that in \( V \), the irreducibility of \( f \) is as the same as that of \( w \). □

3 A Counter Example in Dimension 3

In the case of dimension 3, the statement should be:

**Statement**: For any holomorphic function \( f = f(z_1, z_2, z_3) \) on \( U \subset \mathbb{C}^3(0 \in U) \), and the germ of \( f \) at origin is irreducible, then there exists an open neighborhood \( V \subset U \) of 0, such that for any point \( q \in V, f_q \) is irreducible.

But unfortunately, this statement is not true. In this section, I will present, a polynomial of three variables, as a counter example.

This polynomial is \( f = z_3^2 - z_1z_2^2 \).
3.1 Irreducibility of $f$ at origin

Obviously, near 0, $f$ is a Weierstrass polynomial of itself (we choose $z_3$ as the polynomial variable). Now, we will show the irreducibility at origin by means of contradiction.

If $f$ is not irreducible at origin, then its Weierstrass polynomial is decomposable at origin as a Weierstrass Polynomial. Assume that, near origin, $f = (z_3 - g(z_1, z_2))(z_3 - h(z_1, z_2))$, here $g, h$ are holomorphic functions of variable $z_1, z_2$ near 0, and $g(0,0)=h(0,0)=0$.

From the factorization $f = (z_3 - g(z_1, z_2))(z_3 - h(z_1, z_2))$, we know that $g + h = 0, gh = -z_1z_2^2$, which implies $g^2 = z_1z_2^2$ near 0.

But if $g^2 = z_1z_2^2$ near 0. Then for some $\varepsilon \in \mathbb{C}$ whose norm is small enough, $g^2(z_1, \varepsilon) = \varepsilon^2 z_1$ near 0. But just from elementary knowledge of functions of one complex variable, we know this is not possible.

From argument above, we know $f$ is irreducible at origin.

3.2 Further Argument

At point $p = (z, 0, 0)(z \neq 0)$, we know that $f(p) = 0$, and easily we can factorize $f$ as $f = (z_3 + z_2r)(z_3 - z_2r)$ near $p$, here $r$ is a one-variable holomorphic function such that $r^2 = z_1$ near $(z, 0, 0)$(Because $z$ is not 0, so we can take square-root of $z_1$ near by.).

From the argument in 3.2, we know that, in any neighborhood $U$ of origin, there EXISTS some point $p$ such that $f$ is not irreducible at $p$. This fact can destroy our statement at the beginning of this section.