Discrete maximum principles for nonlinear elliptic finite element problems on Riemannian manifolds with boundary

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The maximum principle forms an important qualitative property of second order elliptic equations, therefore its discrete analogues, the so-called discrete maximum principles (DMPs) have drawn much attention owing to their role in reinforcing the qualitative reliability of the given numerical scheme. In this paper DMPs are established for nonlinear surface finite element problems on Riemannian manifolds, corresponding to the classical pointwise maximum principles on surfaces in the spirit of Pucci et al. Various real-life examples illustrate the scope of the results.

Keywords: nonlinear elliptic problems on surfaces; maximum principles; discrete maximum principles; surface finite element method; simplicial mesh.

1. Introduction

The maximum principle forms an important qualitative property of second order elliptic equations Gilbarg & Trudinger (1983); Protter & Weinberger (1984); Pucci & Serrin (2007). We will refer to such results as continuous maximum principles (CMPs). Typical maximum principles often arise in a form stating that a solution attains a nonnegative maximum on the boundary, i.e.

\[
\max_S u \leq \max \{ 0, \max_{\partial S} u \}.
\]  

(1.1)

CMPs have also been studied on Riemannian manifolds. Thereby the main subject is often an Omori–Yau type maximum principle, but in this paper we are interested in classical pointwise versions, such as the ones studied by Antonini, Mugnai and Pucci in Antonini et al. (2007), which give analogous results to (1.1).

The discrete analogues of CMPs, the so-called discrete maximum principles (DMPs) have drawn much attention. The DMP is in fact an important measure of the qualitative reliability of the numerical scheme, otherwise one could get unphysical numerical solutions like negative concentrations, etc.

We are interested for DMPs on a surface with boundary, in the context of the surface finite element method (SFEM). Originally, the surface FEM was developed by Dziuk in Dziuk (1988) for elliptic problems on closed surfaces (and further developed for evolving surfaces by Dziuk and Elliott in Dziuk & Elliott (2007, 2013)). The theory was extended to elliptic problems on surfaces with boundaries in the preprint Burman et al. (2015). In such a case a DMP reproduces the above relation for the surface finite element solution \( u_h \) instead of \( u \). Various DMPs in Euclidean domains, including geometric conditions on the computational meshes for FEM solutions, have been given, e.g. in Ciarlet (1970); Ciarlet & Raviart (1973); Drăgănescu et.al. (2005); Xu & Zikatanov (1999). The
authors’ previous work, e.g. Karátson & Korotov (2005); Korotov & Křížek (2001); Korotov et al. (2001), involves various types of linear and nonlinear equations and systems, and typical geometric conditions are nonobtuseness or acuteness in the case of simplicial meshes.

To our knowledge, no DMP has been established for (even linear) elliptic problems on surfaces yet. In this paper we provide such results in the spirit of Antonini et al. (2007) for nonlinear finite element problems on manifolds with boundary. We suitably adapt a treatment of the finite element matrix from our earlier paper on the flat domain case Karátson & Korotov (2005). We work in the mentioned surface finite element setting, cf. Burman et al. (2015); Dziuk (1988), and use the properties of the lift operator to achieve the arising stability requirements, thus various discrete maximum-minimum principles are established. We note that discrete maximum principles for parabolic reaction–diffusion equations on closed surfaces have been treated in Frittelli et al. (2017a, 2016, 2017b).

In this paper the issue of generating meshes with suitable angle properties is also addressed. The theoretical results are supported by numerical tests.

2. Formulation and properties of the problem

Our formulation of the problem is based on Antonini et al. (2007). We consider a nonlinear elliptic problem which belongs to the class of problems studied in Antonini et al. (2007) on manifolds. Our goal is to adapt our approach from Karátson & Korotov (2005) developed for Euclidean domains. We first show that a proper CMP can be obtained from Karátson & Korotov (2005) developed for Euclidean domains. We suitably adapt a treatment of the finite element matrix from our earlier paper on the manifold and of the SFEM, respectively.

2.1 The nonlinear elliptic problem on a Riemannian manifold

2.1.1 The manifold. Following the setting and notations of Antonini et al. (2007), let \( \mathcal{M} \) be a smooth complete \( d \)-dimensional Riemannian manifold, and let \( S \subset \mathcal{M} \) be a closed bounded and regular subdomain of \( \mathcal{M} \), so that \( \overline{S} \) is a smooth manifold with boundary, with outward normal \( \nu \). In the sequel we shall write \( TS \times_\mathcal{M} \mathbb{R} \) in place of \( TS \times S (\mathcal{M} \times \mathbb{R}) \) to denote the fibered product bundle. In analogy with the Euclidean case, points of \( TS \times_\mathcal{M} \mathbb{R} \) will be denoted with \((x, z, \xi)\), where \((x, \xi) \in TS \) and \((x, z) \in S \times \mathbb{R} \). Integrals will be taken with respect to the natural Riemannian measure, denoted by \( d\mathcal{M} \). The surface gradient, or tangential gradient (i.e. the orthographic projection of the standard gradient onto the surface) of a function \( u \) on \( S \) will be simply denoted by \( \nabla_S u \), i.e. \( \nabla_S u = \nabla u - (\nabla u \cdot \nu) \nu \). Further, \( \Delta_S \) denotes the Laplace–Beltrami operator, and \( \text{div}_S \) the tangential divergence.

For \( p \geq 1 \), the Lebesgue and Sobolev norms are defined as

\[
\|u\|_{p,S} := \left( \int_S |u|^p d\mathcal{M} \right)^{1/p} \quad \text{and} \quad \|u\|_{1,p,S} := \|u\|_{p,S} + \|\nabla u\|_{1,p,S},
\]

respectively, where \( |u| := g(u, u)^{1/2} \) using the metric tensor \( g \), further, \( W^{1,p}(S) \) and \( W_0^{1,p}(S) \) are defined as the closure of \( C_c^\infty(S) \) and \( C_0^\infty(S) \), respectively, in the Sobolev norm \( \| \cdot \|_{1,p,S} \). We define \( p' := p/(p-1) \) if \( p > 1 \) and \( p' := 0 \) if \( p = 1 \) (see e.g. Dziuk (1988); Dziuk & Elliott (2013)).

There are similar Sobolev embedding estimates of \( W_0^{1,p}(S) \) available as in the Euclidean case. Let \( p^* := \frac{dp}{d-p} \) if \( d > p \) and \( p^* := \infty \) if \( d \leq p \). Here \( S \) is a closed bounded subdomain of the complete Riemannian \( d \)-manifold \( \mathcal{M} \), hence it is a compact Riemannian \( d \)-manifold itself. Therefore, by Aubin (1998), for all \( 1 \leq p_1 < p^* \) we have \( W_0^{1,p}(S) \subset L^{p_1}(S) \) and

\[
\|v\|_{p_1,S} \leq k_1 \|v\|_{1,p,S} \quad (\forall v \in W_0^{1,p}(S))
\]

for some constant \( k_1 > 0 \) depending on \( p \) and \( p_1 \) but independent of \( v \).

2.1.2 The problem. We consider a nonlinear boundary value problem of the following type, involving a scalar nonlinear coefficient in the principal part:

\[
\begin{cases}
-\text{div}_S \left( b(x, u, \nabla_S u) \nabla_S u \right) + q(x, u) = f(x) & \text{on } S, \\
u = g(x) & \text{on } \partial S,
\end{cases}
\]

with the bounded surface \( S \subset \mathcal{M} \), with boundary \( \partial S \), under the following conditions:

Assumptions 2.1.2.
(A1) Let \( p \geq 2 \) be a given parameter.

(A2) The scalar functions \( b : S \times \mathbb{R} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R} \), \( q : S \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous. There holds \( f \in L^p' (\Omega) \), further, \( g = g^* \mid \partial S \) for some \( g^* \in W^{1,\gamma} (S) \), where \( \gamma \geq p \) if \( p > d \) and \( \gamma > d \) if \( p \leq d \).

(A3) (Uniform ellipticity.) The function \( b \) satisfies

\[
\mu_0 + \mu_1 |\xi|^p \leq b(x, z, \xi) \leq M_0 + M_1 |\xi|^p
\]

with constants \( \mu_0, \mu_1, M_0, M_1 > 0 \) independent of \( (x, z, \xi) \in S \times \mathbb{R} \times \mathbb{R}^{d+1} \).

(A4) Let \( 2 \leq p_1 < +\infty \) if \( d \leq p \), or \( 2 \leq p_1 < \hat{p} := \frac{2d}{p} \) if \( d > p \). There exist constants \( \alpha, \beta \geq 0 \) such that for any \( x \in S \) and \( z \in \mathbb{R} \) the following growth conditions hold: for every \( z \geq 0 \),

\[
0 \leq q(x, 0) = q(x, z) \quad (\forall z \leq 0), \quad 0 \leq q(x, z) - q(x, 0) \leq \alpha z + \beta z^{p_1 - 1}.
\]

The weak solution \( u \in W^{1,p} (S) \) of problem (2.2) will be defined as follows:

\[
\int_S [b(x, u, \nabla_S u) \nabla_S u \cdot \nabla_S v + q(x, u)v] \, d\mathcal{M} = \int_S f v \, d\mathcal{M} \quad (\forall v \in W^{1,p}_0 (S)) \quad (2.3)
\]

and \( u - g^* \in W^{1,p}_0 (S) \).

Remark 2.1 As mentioned before, related problems have been studied in the context of the DMP on usual Euclidean domains in our earlier paper Karátson & Korotov (2005), i.e. the above problem (2.2) is a surface analogue of the problems considered therein. Now in (2.2) we only consider Dirichlet boundary conditions, on the other hand, compared to Karátson & Korotov (2005), we impose more general growth conditions allowing \( p \neq 2 \) as well, which involves a more general setting using the Banach space \( W^{1,p}_0 (\Omega) \).

In the sequel we omit the sign \( d\mathcal{M} \) in the integrals for brevity, i.e. \( \int_S f := \int_S \mathcal{M} f \).

2.2 Continuous maximum principle on the manifold

2.2.1 Preliminaries. We will rely on Antonini et al. (2007), where CMPs are proved for a general class of elliptic inequalities on manifolds. We summarize the required background in a rewritten form involving elliptic equations with right-hand side of given sign.

Let us consider an elliptic equation

\[
- \text{div}_S \mathbf{K}(x, u, \nabla_S u) + F(x, u, \nabla_S u) = k(x)
\]

(2.5)

on the domain \( S \subset \mathcal{M} \) under the following conditions:

Assumptions 2.2.1.

(i) The functions \( \mathbf{K} : TS \times \mathbb{R} \rightarrow TS \) and \( F : TS \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous, further, \( \mathbf{K}(x, z, \xi) \in T_x \mathcal{M} \) for all \( (x, z) \in S \times \mathbb{R} \) and \( \xi \in T_z \mathcal{M} \).

(ii) There exist constants \( a_1 > 0 \) and \( b_1, a_2 \geq 0 \) such that for all \( (x, z, \xi) \in TS \times \mathbb{R} \) there holds

\[
\mathbf{K}(x, z, \xi) \cdot \xi \geq a_1 |\xi|^p - a_2 |z|^p, \quad F(x, z, \xi) \geq -b_1 |\xi|^p.
\]

The treatment in Antonini et al. (2007) involves \( p \)-regular weak solutions of (2.5), defined as functions \( u \in L^{1}_{\text{loc}} (S) \) satisfying

\[
\mathbf{K}(\cdot, u, \nabla_S u) \in L^{p'}_{\text{loc}} (S, TS), \quad F(\cdot, u, \nabla_S u) \in L^p_{\text{loc}} (S)
\]

and demanding (2.3) only for compactly supported \( v \in W^{1,p}_0 (S) \). The following result holds:

Theorem 2.1 Let Assumptions 2.2.1 hold with \( a_2 = 0 \) in item (ii). Let \( u \) be a \( p \)-regular weak solution of (2.5) such that \( u \in W^{1,p}_{\text{loc}} (S) \), and let the right-hand side \( k \leq 0 \) on \( S \). If \( u \leq M \) on \( \partial S \) for some constant \( M \geq 0 \), then \( u \leq M \) in \( S \).

In fact, this result is the reformulation of (Antonini et al., 2007, Theorem 3.3.) for equations instead of inequalities and with rearranged signs.
2.2.2 **CMP for the studied class.** We may now prove the CMP by reformulating the above result in the vein of inequality (1.1).

**Theorem 2.2** Let Assumptions 2.1.2 hold. Assume that 

$$f(x) - q(x,0) \leq 0 \quad (a.e. \forall x \in S).$$  \hfill (2.6)

(1) Then

$$\max_S u \leq \max \{0, \max_{\partial S} g\}. \quad \hfill (2.7)$$

(2) In particular, if 

$$g \geq 0$$

then

$$\max_S u = \max_{\partial S} g, \quad \hfill (2.8)$$

and, if 

$$g \leq 0$$

then we have the nonpositivity property

$$\max_S u \leq 0.$$  \hfill (3)

(3) If we do not assume 

$$u \in C(\overline{S})$$

but only 

$$g \in L^\infty(S),$$

then the above statements hold with replacing each max by ess sup, respectively.

**Proof.** We verify that problem (2.2) under Assumptions 2.1.2 is a special case of problem (2.5) under Assumptions 2.2.1. Throughout, the notation $(x, z, \xi)$ will stand for the general points of $S \times \mathbb{R} \times \mathbb{R}^{d+1}$. We first subtract $q(x,0)$ from both sides of (2.2), then

$$\begin{cases}
-\text{div}_S \left( b(x, u, \nabla_S u) \nabla_S u \right) + \tilde{q}(x, u) = \tilde{f}(x) & \text{on } S, \\
u = g(x) & \text{on } \partial S
\end{cases} \quad \hfill (2.9)$$

where

$$\tilde{q}(x, z) := \min_{-\partial S} q(x, z) - q(x,0), \quad \tilde{f}(x) := f(x) - q(x,0)$$

for all $x \in S$, $z \in \mathbb{R}$. Let

$$K(x, z, \xi) := b(x, z, \xi) \cdot \xi, \quad F(x, z, \xi) := \tilde{q}(x, z).$$

Owing to the continuity of $b$ and $q$, assumption (i) obviously holds. Further, for all arguments, assumptions (A3) and (A4) yield

$$|K(x, z, \xi)| \leq b(x, z, \xi) \cdot |\xi| \leq M_0|\xi| + M_1|\xi|^{p-1},$$

hence

$$|K(\cdot, u, \nabla_S u)|^{p'} \leq M_3|\nabla_S u|^{p'} + M_4|\nabla_S u|^{(p-1)p'} = M_3|\nabla_S u|^{p'} + M_3|\nabla_S u|^p.$$  \hfill (2.10)

Here the second term is integrable on $S$ since the weak solution $u \in W^{1,p}(S)$, and also the first term is integrable since $p' \leq 2 \leq p$. Hence

$$K(\cdot, u, \nabla_S u) \in L^{p'}(S, TS).$$

Similarly, from (A4), for all arguments

$$0 \leq \tilde{q}(x, z) \leq \alpha|z| + \beta|z|^{p-1},$$

hence

$$|F(\cdot, u, \nabla_S u)|^{p'} = |q(\cdot, u)|^{p'} \leq \alpha|u|^{p'} + \beta|u|^p$$

is integrable on $S$ since $u \in L^p(S)$ and $p' \leq 2 \leq p$. Also, since $u \in W^{1,p}(S)$, all local requirements are fulfilled even globally on $S$. Equation (2.3) holds for all $v \in W^{1,p}_0(S)$, hence especially for compactly supported $v \in W^{1,p}_0(S)$. Altogether, $u$ satisfies all criteria of a $p$-regular weak solution. Further, for all $x \in S$, now

$$k(x) := \tilde{f}(x) := f(x) - q(x,0) \leq 0.$$
That is, we can apply Theorem 2.1: if \( u \leq M \) on \( \partial S \) for some constant \( M \geq 0 \), then \( u \leq M \) in \( S \). Now we can already prove the desired statements.

(1) Let \( \mu := \max u = \max g \). If \( \mu \geq 0 \) then we can apply Theorem 2.1 with choice \( M := \mu \), namely, since \( u \leq \mu \) on \( \partial S \), therefore \( u \leq \mu \) in \( S \) too, i.e.

\[
\max_{\partial S} u \leq \mu = \max \{ 0, \mu \} = \max_{\partial S} \{ 0, \max g \}.
\]

If \( \mu < 0 \) then we can apply Theorem 2.1 with choice \( M := 0 \), namely, since \( u \leq 0 \) on \( \partial S \), therefore \( u \leq 0 \) in \( S \) too, i.e.

\[
\max_{\partial S} u \leq 0 = \max \{ 0, \mu \} = \max_{\partial S} \{ 0, \max g \}.
\]

In both cases we have obtained the desired statements (2.7).

(2)–(3) These are obvious consequences of statement (1). \( \square \)

In the special case \( q \equiv 0 \), the equality (2.8) holds without assuming \( g \geq 0 \):

**Theorem 2.3** Consider problem (2.2) with \( q \equiv 0 \), under the assumptions of Theorem 2.2. That is, let (A1)–(A3) hold, \( u \in C(\overline{S}) \), and assumption (2.6) now takes the form \( f(x) \leq 0 \) (\( \forall x \in S \)). Then

\[
\max_{\partial S} u = \max_{\partial S} g. \tag{2.10}
\]

**Proof.** If \( \max_{\partial S} g \geq 0 \) then (2.7) implies (2.10). Let \( \max_{\partial S} g < 0 \), say, \( \max_{\partial S} g = -K \) with some \( K > 0 \). Then the function \( w := u + K \) satisfies the same type of problem with right-hand side \( f \) and b.c. \( g + K \), respectively, hence Theorem 2.2 is valid for this problem as well, and (2.7) for \( w \) yields

\[
\max_{\partial S} w \leq \max \{ 0, \max_{\partial S} (g + K) \} = 0.
\]

Then

\[
\max_{\partial S} u \leq -K = \max_{\partial S} g.
\]

\( \square \)

**Remark 2.2** We note that the corresponding minimum principles and nonnegativity property hold if the sign conditions on \( f \) are reversed.

2.3 **Computational scheme: the surface finite element method**

From now on, we are interested in manifolds that can be embedded in \( \mathbb{R}^{d+1} \) admitting an associated signed distance function \( \delta \), satisfying \( S = \{ x \in \mathbb{R}^{d+1} | \delta(x) = 0 \} \) and \( \nabla \delta = v \). This enables us to use the SFEM setting of Dziuk (1988); Dziuk & Elliott (2013). We note that this embedding assumption is not a very strong restriction in the practically most relevant case \( d = 2 \): as described e.g. in Han & Hong (2006), a wide class of two-dimensional manifolds can be embedded in \( \mathbb{R}^3 \) (globally and isometrically) under some proper conditions on their smoothness and Gauss curvature.

The smooth surface \( S \) with boundary \( \partial S \) is approximated by a triangulated one, denoted by \( S_h \), whose vertices \( B_1, B_2, \ldots, B_n \) lie on the surface, including boundary nodes of \( S_h \) lying on \( \partial S \), and it is given as

\[
S_h = \bigcup_{T \in \mathcal{T}_h} T.
\]

We always assume that the simplices \( T \) form an admissible triangulation (i.e. partition) \( \mathcal{T}_h \), with \( h \) denoting the maximum diameter. Admissible triangulations were introduced in (Dziuk & Elliott, 2007, Section 5.1): \( S_h \) is a uniform triangulation, i.e. every \( T \in \mathcal{T}_h \) satisfies that the inner radius \( \delta_T \) is bounded from below by \( ch \) with \( c > 0 \), and \( S_h \) is not a global double covering of \( S \). The discrete tangential gradient on the discrete surface \( S_h \) is given by

\[
\nabla_{S_h} \phi = \nabla \phi - (\nabla \phi \cdot v_h) v_h,
\]

understood in a piecewise sense, with \( v_h \) denoting the normal to \( S_h \), cf. Dziuk (1988). The boundary of the approximation surface is simply \( \partial S_h \), which is an analogous approximation of \( \partial S \).
We define the surface finite element discretisation of our problem, following Dziuk (1988) and Burman et al. (2015). We use simplicial elements and continuous piecewise linear functions. The finite element subspace $V_h$ is spanned by the continuous, piecewise linear basis functions $\chi_j$, satisfying

$$\chi_j (B_i) = \delta_{ij} \quad \text{for all } i, j = 1, 2, \ldots, \tilde{n},$$

therefore

$$V_h = \text{span} \{ \chi_1, \chi_2, \ldots, \chi_{\tilde{n}} \}.$$ 

Now, let $n < \tilde{n}$ be such that $B_1, B_2, \ldots, B_n$ are the vertices that lie on $S_h$ but not on $\partial S_h$, and let $B_{n+1}, \ldots, B_{\tilde{n}}$ be the vertices that lie only on $\partial S_h$. Then the basis functions $\chi_1, \chi_2, \ldots, \chi_n$ satisfy the homogeneous Dirichlet boundary condition on $\partial S_h$. Hence, we can define

$$V_h^0 := \text{span} \{ \chi_1, \chi_2, \ldots, \chi_n \} \subset W^{1,\rho}_0(S_h).$$

Further, let

$$g_h = \sum_{j=n+1}^{\tilde{n}} g_j \chi_j \in V_h$$ (2.11)

(with $g_j \in \mathbb{R}$) be the piecewise linear approximation of the function $g$ on $\partial S_h$ (and on the neighbouring elements).

2.3.1 The lift operator. In the following we recall the so-called lift operator, which was introduced in Dziuk (1988) and further investigated in Dziuk & Elliott (2007). The lift operator projects a finite element function on the discrete surface onto a function on the smooth surface.

Using the oriented distance function $\delta$ (cf. (Dziuk & Elliott, 2007, Section 2.1)), for a continuous function $\eta_h : S_h \to \mathbb{R}$ its lift is defined as

$$\eta^l_h(P) := \eta_h(x), \quad x \in S_h,$$

where for every $x \in S_h$ the value $P = P(x) \in S$ is uniquely defined via

$$x = P + v(P) \delta(x).$$ (2.12)

By $\eta^l$ we mean the function whose lift is $\eta$. Further, we have the lifted finite element spaces

$$V_h^l := \{ \phi_h = \phi^l_h | \phi_h \in V_h \},$$

$$V_h^{0,l} := \{ \phi_h = \phi^l_h | \phi_h \in V_h^0 \}.$$

The following equivalence between discrete and continuous norms has been shown in Demlow (2009); Dziuk (1988) for functions $\eta_h : S_h \to \mathbb{R}$ with lift $\eta^l_h : S \to \mathbb{R}$. This will help to ensure stability of the discrete problem using the surface approximation.

**Lemma 2.1** (equivalence of norms) The original and lifted norms are equivalent in both the $L^p$ and $W^{1,p}$ case, independently of sufficiently small mesh sizes $h$. That is, there exist constants $c, h_0 > 0$ such that for all $0 < h \leq h_0$,

$$c^{-1} ||\eta_h||_{L^p(S_h)} \leq ||\eta^l_h||_{L^p(S)} \leq c ||\eta_h||_{L^p(S_h)} \quad (\forall \eta_h \in L^p(S_h)),$$

$$c^{-1} ||\eta_h||_{W^{1,p}(S_h)} \leq ||\eta^l_h||_{W^{1,p}(S)} \leq c ||\eta_h||_{W^{1,p}(S_h)} \quad (\forall \eta_h \in W^{1,p}(S_h)).$$

2.3.2 The discrete problem. In order to find the approximate solution of (2.2), we have to solve the counterpart of (2.3)–(2.4) in $V_h$. We note here that in order to transfer the assumptions on the coefficient functions (A1)–(A4) to the discrete level we have to use negative lifts in the spatial variable. For any $x \in S_h$ with corresponding lift $P \in S$, cf. (2.12), we set

$$b^{-l}(x, \cdot, \cdot) = b(P, \cdot, \cdot),$$

and similarly using the negative lift in the first argument of all coefficient functions ($f, q, r$, etc.), as well. No lift is used in the other arguments. By this construction all the properties of (A1)–(A4) are clearly transferred to these lifted functions, with $S_h$ instead of $S$. 

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Following (2.9), we first subtract $q^{-l}(x,0)$ from both sides and consider the following discrete problem: find $u_h \in V_h$ such that

$$
\int_{S_h} \left( b^{-l}(x, u_h, \nabla S_h u_h) \nabla S_h u_h \cdot \nabla S_h v_h + \tilde{q}^{-l}(x, u_h) v_h \right) = \int_{S_h} \hat{f}^{-l} v_h \quad (\forall v_h \in V^0_h),
$$

(2.13)

and

$$
u_h - g_h \in V^0_h
$$

(2.14)

where

$$
\tilde{q}^{-l}(x, z) := q^{-l}(x, z) - q^{-l}(x, 0), \quad \hat{f}^{-l}(x) := f^{-l}(x) - q^{-l}(x, 0)
$$

for all $x \in S_h$, $z \in \mathbb{R}$. Further, denote

$$
r^{-l}(x, z) := \begin{cases} \frac{q^{-l}(x, z) - q^{-l}(x, 0)}{z}, & \text{if } z > 0, \\ 0, & \text{if } z \leq 0. \end{cases}
$$

Then, using also assumption (A4),

$$
\tilde{q}^{-l}(x, z) = r^{-l}(x, z) z, \quad 0 \leq r^{-l}(x, z) \leq \alpha + \beta|z|^{p_l - 2} \quad (\forall x \in S_h, \ z \in \mathbb{R}).
$$

(2.15)

In what follows, we will rewrite problem (2.13) as

$$
\int_{S_h} \left( b^{-l}(x, u_h, \nabla S_h u_h) \nabla S_h u_h \cdot \nabla S_h v_h + r^{-l}(x, u_h) u_h v_h \right) = \int_{S_h} \hat{f}^{-l} v_h \quad (\forall v_h \in V^0_h).
$$

(2.16)

We set

$$
u_h = \sum_{j=1}^{\tilde{n}} c_j \chi_j,
$$

(2.17)

and look for the coefficients $c_1, c_2, \ldots, c_{\tilde{n}}$.

### 2.3.3 The nonlinear algebraic system

Now we turn to the nonlinear algebraic system corresponding to (2.13). For any $\tilde{c} = (c_1, \ldots, c_{\tilde{n}})^T \in \mathbb{R}^\tilde{n}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, \tilde{n}$, we set

$$
b_{ij}(\tilde{c}) = \int_{S_h} b^{-l} \left( x, \sum_{k=1}^{n} c_k \chi_k, \sum_{k=1}^{n} c_k \nabla S_h \chi_k \right) \nabla S_h \chi_j \cdot \nabla S_h \chi_i,
$$

$$
r_{ij}(\tilde{c}) = \int_{S_h} r^{-l} \left( x, \sum_{k=1}^{n} c_k \chi_k \right) \chi_j \chi_i,
$$

$$
d_i = \int_{S_h} \hat{f}^{-l} \chi_i,
$$

$$
a_{ij}(\tilde{c}) = b_{ij}(\tilde{c}) + r_{ij}(\tilde{c}).
$$

Putting (2.17) and $v_h = \chi_j$ into (2.16), we obtain the $n \times \tilde{n}$ system of algebraic equations

$$
\sum_{j=1}^{\tilde{n}} a_{ij}(\tilde{c}) c_j = d_i, \quad i = 1, 2, \ldots, n.
$$

(2.18)

Using the notations

$$
A(\tilde{c}) = \{a_{ij}(\tilde{c})\}, \quad \text{and} \quad d = \{d_j\}, \quad e = \{e_j\}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n,
$$

$$
\tilde{A}(\tilde{c}) = \{a_{ij}(\tilde{c})\}, \quad \text{and} \quad \tilde{e} = \{e_j\}, \quad i = 1, 2, \ldots, n, \quad j = n+1, \ldots, \tilde{n},
$$

the system (2.18) turns into

$$
A(\tilde{c}) e + \tilde{A}(\tilde{c}) \tilde{e} = d.
$$

(2.19)

Defining further

$$
\check{A}(\tilde{c}) = \begin{bmatrix} A(\tilde{c}) & \tilde{A}(\tilde{c}) \end{bmatrix}, \quad \check{e} = \begin{bmatrix} e \ \tilde{e} \end{bmatrix},
$$

(2.20)

we rewrite (2.19) as follows

$$
\check{A}(\tilde{c}) \check{e} = d.
$$
In order to obtain a system with a square matrix, we enlarge our system to an \( \bar{n} \times \bar{n} \) one. Namely, since \( u_h = g_h \) on \( \partial S_h \), the coordinates \( c_i \) with \( n + 1 \leq i \leq \bar{n} \) satisfy automatically \( c_i = g_i \), i.e.
\[
\tilde{c} = \tilde{g},
\]
where
\[
\tilde{g} = \{g_j\}, \quad j = n + 1, \ldots, \bar{n}.
\]
That is, we can replace (2.19) by the equivalent system
\[
\begin{bmatrix}
A(\tilde{c}) & \tilde{A}(\tilde{c}) \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\tilde{c} \\
\tilde{c}
\end{bmatrix}
= \begin{bmatrix} d \\
\tilde{g}
\end{bmatrix}.
\]

**Remark 2.3** The solution of the arising nonlinear finite element problems can rely on various efficient methods, see e.g. Faragó & Karátson (2002).

3. **Discrete maximum principles**

3.1  **Classical matrix maximum principles**

Let us consider a linear algebraic system of equations of order \((n + m) \times (n + m)\):
\[
\bar{A} \bar{c} = \bar{b},
\]
where the matrix \( \bar{A} \) has the following structure:
\[
\bar{A} = \begin{bmatrix}
A & \tilde{A} \\
0 & I
\end{bmatrix}.
\]

In the above, \( I \) is an \( m \times m \) identity matrix, \( 0 \) is a \( m \times n \) zero matrix. In a (surface) finite element setting, such a partitioning arises corresponding to interior and boundary points.

We first recall some classical definitions and results, see, e.g. Ciarlet (1970); Varga (1962). We follow the terminology of Faragó (2010). Throughout, inequalities for matrices or vectors are understood elementwise, and \( c, \tilde{c} \) and \( \bar{c} \) denote vectors consisting of \( n \), \( m \) or \( n + m \) numbers, respectively.

**Definition 3.1** The matrix \( \tilde{A} \) in (3.1) satisfies

(a) the *discrete weak maximum principle (DwMP)* if for an arbitrary vector \( \tilde{c} = (c_1, \ldots, c_{n+m})^T \in \mathbb{R}^{n+m} \) satisfying \( (\tilde{A}\tilde{c})_i \leq 0 \), \( i = 1, 2, \ldots, n \), one has
\[
\max_{i=1,2,\ldots,n+m} c_i \leq \max\{0, \max_{i=n+1,\ldots,n+m} c_i\};
\]

(b) the *discrete strict weak maximum principle (DWMP)* if for any vector \( \tilde{c} = (c_1, \ldots, c_{n+m})^T \in \mathbb{R}^{n+m} \) satisfying \( (\tilde{A}\tilde{c})_i \leq 0 \), \( i = 1, 2, \ldots, n \), one has
\[
\max_{i=1,2,\ldots,n+m} c_i = \max_{i=n+1,\ldots,n+m} c_i.
\]

(DMPs without the term 'weak' also assert that only constant vectors may attain a maximum for 'interior' indices, but we do not address this property here.)

Based on Ciarlet & Raviart (1973); Faragó (2010) we have the following result, formulated in this way in Karáton & Korotov (2015):

**Theorem 3.2** If the matrix \( \tilde{A} \) in (3.1) satisfies the following conditions:

(i) \( a_{ij} \leq 0 \quad (\forall i = 1, \ldots, n, \; j = 1, \ldots, n + m; \; i \neq j) \),

(ii) \( \sum_{j=1}^{n+m} a_{ij} \geq 0 \quad (\forall i = 1, \ldots, n) \),

(iii) \( A \) is positive definite,

then \( \tilde{A} \) possesses the DwMP.

If the inequality in condition (ii) is replaced by equality, then \( \tilde{A} \) possesses the DWMP.
3.2 The discrete maximum principle for the nonlinear elliptic problem

3.2.1 The main results. The derivation of our main results needs the following lemmas on the stability of the boundary data and then on the norm-boundedness of the SFEM solutions. This means that we can achieve stability of the discrete problem, thanks to exploiting the lift properties of subsection 2.3.1 to control the geometric error due to the surface approximation.

**Lemma 3.1** Let $g^*$ and $\gamma$ be as given in Assumption 2.1.2 (A2). Then there exists a constant $c > 0$ such that for all $h \leq h_0$ with $h_0 > 0$ sufficiently small, the discrete function $g_h$, from (2.11), satisfies

$$
\| g_h \|_{1,p, S_h} \leq c \| g^* \|_{1,\gamma, S}.
$$

*Proof.* The assumption on $\gamma$ in (A2) implies that in each case we have $\gamma \geq p$ and $\gamma > d$. Since the compact manifold $S_h$ has bounded measure, the relation $\gamma \geq p$ implies that

$$
\| g_h \|_{1,p, S_h} \leq c_1 \| g_h \|_{1,\gamma, S_h}
$$

for some constant $c_1 > 0$ independent of $h$. Further, the relation $\gamma > d$ implies that $W^{1,\gamma}(S_h) \subset C(S_h)$, see, e.g., Adams (1975). Hence, by (Ciarlet, 1978, Thm. 3.1.6), the stability of the Lagrange interpolation holds for $(g^*)^{-1}$ in $W^{1,\gamma}$-norm on each triangle and hence also on all $S_h$, therefore

$$
\| g_h \|_{1,\gamma, S_h} \leq c_2 \| (g^*)^{-1} \|_{1,\gamma, S_h}
$$

for some constant $c_2 > 0$ independent of $h$. Finally, Lemma 2.1 implies that

$$
\| (g^*)^{-1} \|_{1,\gamma, S_h} \leq c_3 \| g^* \|_{1,\gamma, S}
$$

for some constant $c_3 > 0$ independent of $h$. These three estimates together yield the desired result. \( \square \)

**Lemma 3.2** The norms $\| u_h \|_{1,p, S_h}$ are bounded independently of $h$.

*Proof.* We will use the shorthand notation

$$
|v_h|_{1,p, S_h} := \| \nabla_{S_h} v_h \|_{p, S_h}
$$

Since (2.14) shows that $v_h := u_h - g_h \in V_h^0$, we use it as a test function in the discrete weak problem (2.16). Rearranging the terms, with $g_h$ on the right, we obtain

$$
\int_{S_h} \left( b^{-1}(x, u_h, \nabla_{S_h} u_h) \ | \nabla_{S_h} u_h \|^2 + r^{-1}(x, u_h) |u_h|^2 \right) = \int_{S_h} b^{-1}(x, u_h, \nabla_{S_h} u_h) \nabla_{S_h} u_h \cdot \nabla_{S_h} g_h + \int_{S_h} \left( r^{-1}(x, u_h) u_h g_h + \hat{f}^{-1}(u_h - g_h) \right).
$$

We estimate this equality appropriately from both above and below. For the lower bound of the left-hand side we use (A3), which yields $b^{-1}(x, \cdot, \xi) |\xi|^2 \geq 1$ for $x \in S_h$, and since $r^{-1}(x, \cdot) \geq 0$ on $S_h$, we obtain the lower bound

$$
\mu_1 |u_h|^p_{1,p, S_h} = \mu_1 \int_{S_h} \nabla_{S_h} u_h |u_h|^p \leq \int_{S_h} b^{-1}(x, u_h, \nabla_{S_h} u_h) \ | \nabla_{S_h} u_h \|^2 + r^{-1}(x, u_h) |u_h|^2.
$$

(3.3)

For the first term on the right-hand side, we have from (A3) that

$$
\int_{S_h} \left( b^{-1}(x, u_h, \nabla_{S_h} u_h) \nabla_{S_h} u_h \cdot \nabla_{S_h} g_h \right) = \int_{S_h} \left( (M_0 |\nabla_{S_h} u_h| + M_1 |\nabla_{S_h} u_h|^{p-1}) |\nabla_{S_h} g_h| \right) \leq \int_{S_h} \left( M_0 |u_h|_{1,p, S_h} g_h |g_h|_{1,p, S_h} + M_1 |u_h|_{1,p, S_h}^{p-1} |g_h|_{1,p, S_h} \right) \leq c \left( M_0 |u_h|_{1,p, S_h} + M_1 |u_h|_{1,p, S_h}^{p-1} \right) \| g_h \|_{1,p, S_h}
$$

using H"{o}lder’s inequality, with $\frac{1}{p} + \frac{1}{p'} = 1$, $p' \leq 2 \leq p$, and the boundedness of the surface. For the second term on the right-hand side, using (2.15), we have

$$
\int_{S_h} \left( r^{-1}(x, u_h) u_h g_h + \hat{f}^{-1}(u_h - g_h) \right) = \int_{S_h} \left( r^{-1}(x, v_h + g_h) (v_h + g_h) g_h + \hat{f}^{-1} v_h \right) \leq \int_{S_h} \left( (\alpha + \beta |v_h + g_h|^{p-2}) (v_h + g_h) g_h + \hat{f}^{-1} v_h \right) \leq \left( k_1 |v_h|_{1,p, S_h} + |g_h|_{1,p, S_h} \right) \| g_h \|_{1,p, S_h}
$$

$$
+ \left( k_1 |v_h|_{1,p, S_h} + |g_h|_{1,p, S_h} \right)^{p-1} \| g_h \|_{1,p, S_h}
$$

$$
+ c \hat{f}^{\frac{1}{p'}} |v_h|^{p-1}_{1,p, S_h},
$$

where $\alpha$ and $\beta$ are positive constants.
where the last estimates is obtained, similarly as before, using repeatedly Hölder’s inequality (with \( p' \leq 2 \leq p \)) and again the boundedness of the surface, and the equivalent version of (2.1) on \( W_0^{1,p}(S_h) \):
\[
\| v \|_{p, S_h} \leq k_1 \| v \|_{1, p, S_h}, \quad (\forall v \in W_0^{1,p}(S_h)),
\]

then the boundedness of the surface, and the equivalent version of (2.1) on \( \| f \|_{p', S} \). This can be further estimated using
\[
|v_h|_{1,p,S_h} \leq |u_h|_{1,p,S_h} + |g_h|_{1,p,S_h} \leq |u_h|_{1,p,S_h} + \| g_h \|_{1,p,S_h}
\]
in each term. Altogether, we obtain
\[
|u_h|_{1,p,S_h} \leq c \left( (|u_h|_{1,p,S_h} + 2 |g_h|_{1,p,S_h}) \| g_h \|_{1,p,S_h}
+ (|u_h|_{1,p,S_h} + |g_h|_{1,p,S_h} + \| g_h \|_{1,p,S_h})^{p-1} \| g_h \|_{1,p,S_h}
+ c \| f \|_{p', S} (|u_h|_{1,p,S_h} + \| g_h \|_{1,p,S_h}) \right),
\]
a polynomial of degree \( p-1 \) of \( |u_h|_{1,p,S_h} \) on the right-hand side, whose coefficients, containing \( \| f \|_{p', S} \) and \( \| g_h \|_{1,p,S_h} \), are bounded independently of \( h \), owing to (3.2). Further, we have a power \( p \) of \( |u_h|_{1,p,S_h} \) on the left-hand side. Since the latter is bounded by the former, this implies that \( |u_h|_{1,p,S_h} \) is bounded. Then \( |v_h|_{1,p,S_h} \leq |u_h|_{1,p,S_h} + |g_h|_{1,p,S_h} \), and and
\[
K = c \| g \|_{1,\gamma,S}.
\]
i.e. it is also bounded independently of \( h \).

Now we can verify the underlying matrix maximum principle for the stiffness matrix of our elliptic problem.

**Theorem 3.3** Let (A1)–(A4) hold and let us consider a family of simplicial triangulations \( \mathcal{T}_h \) \((h > 0)\) satisfying the following property: for any \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \bar{n} \) \((i \neq j)\)
\[
\nabla S_h \chi_i \cdot \nabla S_h \chi_j \leq -\frac{\sigma_0}{h^2} < 0
\]
(3.4)
on supp \( \chi_i \cap supp \chi_j \subset S_h \) with \( \sigma_0 > 0 \) independent of \( i, j \) and \( h \). Let the simplicial triangulations \( \mathcal{T}_h \) be regular, i.e. there exist constants \( m_1, m_2 > 0 \) such that for any \( h > 0 \) and any simplex \( T_h \in \mathcal{T}_h \)
\[
m_1 h^d \leq \text{meas}(T_h) \leq m_2 h^d
\]
(3.5)
(\text{where meas}(T_h) \text{ denotes the } d\text{-dimensional measure of } T_h).\

Then for sufficiently small \( h \), the matrix \( \tilde{A}(\tilde{c}) \) defined in (2.20) has the following properties:

(i) \( a_{ij}(\tilde{c}) \leq 0, \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \bar{n} \) such that \( i \neq j \).

(ii) \( \sum_{j=1}^{\bar{n}} a_{ij}(\tilde{c}) \geq 0, \quad i = 1, 2, \ldots, n. \)

(iii) \( A(\tilde{c}) \) is positive definite.

**Proof.** Let us recall that for any \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \bar{n}, \)
\[
a_{ij}(\tilde{c}) = \int_{S_h} \left( b^{-1}(x,u_h) \nabla S_h \chi_i \cdot \nabla S_h \chi_j + r^{-1}(x,u_h) \chi_i \chi_j \right).
\]

Now we prove properties (i)–(iii).

(i) Let \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, \bar{n} \) with \( i \neq j \) and let \( S_h^{ij} \) denote the interior of \( \text{supp } \chi_i \cap \text{supp } \chi_j \subset S_h \). If \( S_h^{ij} = \emptyset \) then
\[
a_{ij}(\tilde{c}) = 0.
\]
If $S^i_j \neq \emptyset$ then we use property (A3), namely that $b(\cdot, u_h, \nabla S^i_j u_h) \geq \mu_0$, and (3.4), further, $r(\cdot, u_h) \geq 0$ and the fact $0 \leq \chi_i \leq 1$ ($i = 1, 2, \ldots, n$), which imply

$$a_{ij}(\bar{e}) \leq -\frac{\sigma_0}{h^2} \mu_0 \text{meas}(S^i_j) + \int_{S^i_j} r^{-1}(x, u_h).$$  \hspace{1cm} (3.6)

Here, from (2.15),

$$\int_{S^i_j} r^{-1}(x, u_h) \leq \int_{S^i_j} (\alpha + \beta |u_h|^{p_i-2})�.$$  

We now further estimate the integral from above. From assumption (A4) we have $2 \leq p_1 \leq \hat{\rho}$, where $\rho = \frac{2d}{d-p}$ (if $d > p$) and $\hat{\rho} = +\infty$ (if $d \leq p$). Let us also consider the Sobolev embedding estimate (2.1), where $p^* = \frac{pd}{d-p}$ (if $d > p$) and $p^* = \infty$ (if $d \leq p$) were defined. Since $2 \leq p$ from (A1), we clearly have

$$\hat{\rho} \leq p^*. \hspace{1cm} (3.7)$$

Hence $p_1$ also satisfies $p_1 < p^*$, which implies that the Sobolev embedding estimate (2.1) holds with this $p_1$. Now, assume first that $p_1 > 2$ and let us fix a real number $r$ satisfying

$$\frac{d}{2} < r \leq \frac{p^*}{p_1-2}. \hspace{1cm} (3.8)$$

Such a number $r$ always exists, since for $d \leq p$ there is no upper bound ($p^* = \infty$), and for $d > p$ the assumption $p_1 < \hat{\rho}$ implies

$$p_1 - 2 < \hat{\rho} - 2 = \frac{2p}{d-p} = \frac{2p^*}{d}$$

which yields

$$\frac{d}{2} < \frac{p^*}{p_1-2}.$$  

Further, the assumption $d \geq 2$ on the dimension of the manifold implies $r > 1$. (In theory the trivial case $d = 1$ is also allowed, then we must assume that $r > 1$ is chosen, which is allowed since $\frac{p^*}{p_1-2} > 1$.)

Now let $s > 1$ be chosen such that $\frac{1}{r} = \frac{1}{s} = 1$. Then Hölder’s inequality implies

$$\int_{S^i_j} |u_h|^{p_i-2} \leq ||1||^1_{L^1(S^i_j)} ||u_h|^{p_i-2}||_{L^1(S^i_j)} = \text{meas}(S^i_j)^{1/s} ||u_h|^{p_i-2}||_{L^{p_i-2}(S^i_j)}. \hspace{1cm} (3.9)$$

Here $(p_1-2)r \leq p^*$ and (2.1) imply

$$||u_h||_{L^{p_i-2}(S^i_j)} \leq ||u_h||_{L^{p_i-2,p,S}(S)} \leq k_1^{p_i-2} ||u_h||_{L^{p_i-2,p,S}}.$$  

Here, by Lemma 3.2, $||u_h||_{1,p,S}$ is bounded independently of $h$. Hence, (3.9) is bounded as

$$\int_{S^i_j} |u_h|^{p_i-2} \leq K_1 \text{meas}(S^i_j)^{1/s} \hspace{1cm} (3.10)$$

with some constant $K_1 > 0$ independent of $h$. Finally, if $p_1 = 2$ then the corresponding equality (3.10) holds trivially with $s = 1$.

The integral of the positive constant $\alpha$ is simply

$$\int_{S^i_j} \alpha = \alpha \text{meas}(S^i_j).$$

Substituting the two estimates into (3.6), we obtain

$$a_{ij}(\bar{e}) \leq \left( -\frac{\sigma_0 \mu_0}{h^2} + \alpha \right) \text{meas}(S^i_j) + \beta K_1 \text{meas}(S^i_j)^{1/s}. \hspace{1cm} (3.11)$$

For sufficiently small $h$, we can write

$$a_{ij}(\bar{e}) \leq A^j(h) := -\frac{C_0}{h^2} \text{meas}(S^i_j) + C_1 \text{meas}(S^i_j)^{1/s},$$

where $A^j(h) \geq 0$ and $C_1 > 0$ is a constant dependent on $\mu_0$ and $\sigma_0$.
more precisely, there exist positive constants \( h_0, C_0 \) and \( C_1 \) independently of \( h \) and \( i,j \) such that the above inequality holds for \( h < h_0 \). Then, using that \( \frac{1}{r} + \frac{1}{s} = 1 \) and the regularity (3.5) of the mesh, we have

\[
A^{ij}(h) = \text{meas}(S_h^j) \left( -\frac{C_0}{h^2} \text{meas}(S_h^i)^{1/r} + C_1 \right)
\leq \text{meas}(S_h^j) \left( -C_2 h^{-2+\frac{d}{d/r}} + C_1 \right).
\]

Since (3.8) implies \( \frac{d}{2} < 2 \), the term in brackets tends to \( -\infty \) as \( h \to 0 \) and hence \( A^{ij}(h) < 0 \) for \( h < h_0 \).

 Altogether, we obtain that there exists \( h_0 > 0 \) such that for \( h \leq h_0 \) and all \( i,j \)

\[
a_{ij}(\bar{\mathbf{c}}) < 0.
\]

(ii) For any \( i = 1,2,\ldots,n \),

\[
\sum_{j=1}^{\bar{n}} a_{ij}(\bar{\mathbf{c}}) = \int_{S_h} \left( b^{-I}(x,u_h) \nabla S_h \chi_i \cdot \nabla S_h \left( \sum_{j=1}^{\bar{n}} \chi_j \right) + r^{-I}(x,u_h) \chi_i \left( \sum_{j=1}^{\bar{n}} \chi_j \right) \right)
= \int_{S_h} r^{-I}(x,u_h) \chi_i \geq 0,
\]

using the fact that \( \sum_{j=1}^{\bar{n}} \chi_j \equiv 1 \) and that \( r, \chi_i \) are nonnegative.

(iii) For any vector \( \mathbf{d} \in \mathbb{R}^n \), \( \mathbf{d} \neq 0 \) and corresponding finite element function \( v_h = \sum_{j=1}^{\bar{n}} d_j \chi_j \neq 0 \), using that \( b \geq \mu_0 \) and \( r \geq 0 \), we have

\[
A(\bar{\mathbf{c}}) \mathbf{d} \cdot \mathbf{d} = \sum_{i,j=1}^{\bar{n}} a_{ij}(\bar{\mathbf{c}}) d_i d_j = \int_{S_h} \left( b^{-I}(x,u_h) \nabla S_h v_h \right) \left( \nabla S_h v_h \right)^2 + r^{-I}(x,u_h) v_h^2 \geq \mu_0 \int_{S_h} \left| \nabla S_h v_h \right|^2 > 0
\]

hence \( A(\bar{\mathbf{c}}) \) is positive definite.

Now it is straightforward to derive the analogue of Theorem 2.2 for system (2.19), i.e. the discrete maximum principle.

**Theorem 3.4 (Discrete maximum principle)** Let problem (2.2) satisfy

\[
f(x) - q(x,0) \leq 0 \quad (x \in S)
\]

and consider the discretization given in subsection 2.3. Under the conditions of Theorem 3.3, we have

\[
\max_{\bar{S}} u_{h}^I \leq \max \left\{ 0, \max_{\partial S_h} g^I \right\}
\]

In particular, if \( g \geq 0 \) then

\[
\max_{\bar{S}} u_{h}^I = \max_{\partial S_h} g^I
\]

and if \( g \leq 0 \) then we have the discrete nonpositivity property

\[
\max_{\bar{S}} u_{h}^I \leq 0.
\]

**Proof.** It follows from Theorem 3.3 and the properties of piecewise linear functions. Namely, Theorem 3.3 states that the matrix \( \hat{A}(\bar{\mathbf{c}}) \) satisfies the conditions of Theorem 3.2, further, (3.12) implies that \( \hat{A}(\bar{\mathbf{c}}) \chi_i = d_i = \int_{S_h} \hat{J}^{-I} \chi_i \leq 0 \). Hence, for our system \( \hat{A}(\bar{\mathbf{c}}) \bar{\mathbf{c}} = \mathbf{d} \), Theorem 3.2 provides the algebraic matrix maximum principle DwMP, which is equivalent to

\[
\max_{B_i \in \bar{S}_h} u_{h}(B_i) \leq \max \left\{ 0, \max_{B_i \in \partial \bar{S}_h} g_{h}(B_i) \right\}
\]

for the node points. Since \( u_{h} \) is a piecewise linear function, hence (3.16) is also true in the elements between the node points, i.e. on the whole domain \( \bar{S}_h \):

\[
\max_{\bar{S}_h} u_{h} \leq \max \left\{ 0, \max_{\partial \bar{S}_h} g_{h} \right\}
\]

The lift preserves extreme values, hence (3.17) implies (3.13). Finally, (3.14) and (3.15) are trivial consequences of (3.13).
3.2.2 Related results. One can verify in the same way the discrete minimum principle for system (2.19), in analogy with Theorem 3.4:

**THEOREM 3.5** Let problem (2.2) satisfy

\[ f(x) - q(x,0) \geq 0 \quad (x \in S) \]

and consider the discretisation given in subsection 2.3. Under the conditions of Theorem 3.3, we have

\[ \min_S u_h^I \geq \min \{ 0, \min_{\partial S} g_h^I \}. \]

In particular, if \( g \leq 0 \) then

\[ \min_S u_h^I = \min_{\partial S} g_h^I, \]

and if \( g \geq 0 \) then we have the discrete nonnegativity property

\[ \min_S u_h^I \geq 0. \]

In the special case \( g \equiv 0 \), the counterpart of Theorem 2.3 is valid, i.e. equality holds without assuming any sign condition on \( g \). We formulate this for both the maximum and minimum principles. Moreover, the strict negativity in (3.4) can be replaced by a weaker nonnegativity condition, and no regularity condition on the mesh like (3.5) needs to be assumed.

**THEOREM 3.6** Let us consider the following special case of problem (2.2):

\[
\begin{cases}
- \text{div}_S \left( b(x,u,\nabla_S u) \nabla_S u \right) = f(x) & \text{on } S, \\
u = g(x) & \text{on } \partial S.
\end{cases}
\]

(3.18)

under assumptions (A1)–(A3). Let the triangulation \( \mathcal{T}_h \) satisfy the following property: for any \( i = 1,2,\ldots,n, \ j = 1,2,\ldots,\bar{\eta} \ (i \neq j) \)

\[ \nabla_S \chi_i \cdot \nabla_S \chi_j \leq 0. \]

(3.19)

Then the following results hold:

(1) If \( f \leq 0 \) then \( \max_S u_h^I = \max_{\partial S} g_h^I. \)

(2) If \( f \geq 0 \) then \( \min_S u_h^I = \min_{\partial S} g_h^I. \)

(3) If \( f = 0 \) then the ranges of \( u_h^I \) and \( g_h^I \) coincide, i.e. we have \( [\min_S u_h^I, \max_S u_h^I] = [\min_{\partial S} g_h^I, \max_{\partial S} g_h^I] \) for the corresponding intervals.

**Proof.** To verify (1), we rely on Theorem 3.2 again, whose conditions follow similarly as in Theorem 3.3. The difference in the proof arises in proving property (i), i.e.

\[ a_{ij}(\vec{e}) \leq 0, \]

(3.20)

which now follows trivially: since the assumption \( q \equiv 0 \) implies \( r \equiv 0 \), we simply have

\[ a_{ij}(\vec{e}) = \int_{S_h} b^{-\frac{1}{2}}(x,u_h,\nabla_S u_h) \nabla_S \chi_i \cdot \nabla_S \chi_j \]

and thus (3.19) and the condition \( b \geq \mu_0 > 0 \) readily yield (3.20). Further, in property (ii), (3.11) is simply replaced by

\[ \sum_{j=1}^{\bar{\eta}} a_{ij}(\vec{e}) = \int_{S_h} b^{-\frac{1}{2}}(x,u_h,\nabla_S u_h) \nabla_S \chi_i \cdot \nabla_S \left( \sum_{j=1}^{\bar{\eta}} \chi_j \right) = 0, \]

hence we can apply the final statement of Theorem 3.2 to obtain that DWMP holds. This implies, similarly to the argument of Theorem (3.5) on piecewise linear functions, that \( \max_S u_h^I = \max_{\partial S} g_h^I. \)

Statement (2) follows from (1) by replacing \( u \) by \( -u \), and (3) is a direct consequence of (1) and (2).
3.2.3 On geometric mesh properties. Conditions (3.4) and (3.19) are easy to check, since the values \( \nabla_{S_i} \chi_i \cdot \nabla_{S_j} \chi_j \) are constant on each element. These conditions have a straightforward geometric interpretation: the angles must be uniformly acute for (3.4) and nonobtuse for (3.19).

The currently available constructions of acute triangulations on surfaces of various types and close issues have been recently surveyed in the paper (Zamfirescu, 2013, Section 3), see also the literature therein. We may notice as well that the acuteness property is stable with respect to small perturbations of the vertices involved, therefore e.g. constructions of acute triangulations on planes (see Zamfirescu (2013); Brandts et al. (2009)) can be easily adapted for producing acute triangulations for certain (not very curved) surfaces in 3D.

We underline that many real-life problems appearing in biology, biophysics and biochemistry are posed on sphere-like surfaces. In typical cases the surface at hand there is a membrane, a surface of a cell, or of a nucleous, a surface of a crystal, or some other spherical object, see for instance Dziuk & Elliott (2007, 2013) and the references therein. Acute triangulations and refinement on a spherical surface will be discussed in Section 4.

For general curved surfaces, the issue of refinements preserving acuteness (and the construction of families of acute triangulations) is a more involved task than the case of nonobtuse triangulations (similarly to the case of Euclidean domains). For the latter, for example, a constructive proof of existence of a family of nonobtuse triangulations of surfaces of cylindric-type 3D domains is given in Korotov (2012). It is based on the construction of conforming nonobtuse tetrahedral meshes and the fact that faces of nonobtuse tetrahedra are nonobtuse triangles.

A way of general remedy in such mesh generation is improving mesh properties using arbitrary Lagrangian Eulerian maps. Such an approach has been developed in Kovács (2016) to generate meshes with acute angles for general closed surfaces. The algorithm there is based on a constraint system and on arbitrary Lagrangian Eulerian (ALE) maps, see in particular Section 5.3 therein for meshes with angle conditions. This technique can be adapted to surfaces with boundary, see Section 4 below where we have done this for our numerical tests.

3.3 Examples

We may mention some important real-life examples from the paper Antonini et al. (2007) where the corresponding CMP has been proved. The arising equations include the following models:

(i) gas dynamics:
\[-\text{div}_S \left( \rho (|\nabla_S u|^2) \nabla_S u \right) = 0\]
where the function \( \rho \), which describes the relation of the velocity and the density, is determined by Bernoulli’s law;

(ii) surface \( p \)-Laplacian:
\[-\text{div}_S \left( |\nabla_S u|^{p-2} \nabla_S u \right) = 0\]
which minimizes the \( p \)-Dirichlet norm on \( S \);

(iii) radiative cooling:
\[-\text{div}_S \left( \kappa(x, u) |\nabla_S u|^{p-2} \nabla_S u \right) + \sigma u^4 = 0\]
where \( \kappa \) is the coefficient of heat conduction and \( \sigma \) is the radiation, assumed to be constant. (Here \( u \geq 0 \) is of physical interest Keller (1969), hence the nonlinearity is defined as \( q(x, z) := z^4 \) for \( z \geq 0 \) only and as \( q(x, z) \equiv 0 \) for \( z \leq 0 \).)

According to our results, if the finite element discretization of the corresponding boundary value problem satisfies the angle conditions described in subsection 3.2.3, then the numerical solution satisfies the DMP. The typical situation is that the boundary function \( g \) describes a nonnegative physical quantity: \( g \geq 0 \). Then Theorem 3.5 ensures the discrete nonnegativity property, i.e. that the numerical solution satisfies (3.15):
\[ \min_{S} u_h^l \geq 0 \]
in accordance with the physical reality.

4. Numerical experiments

In this section we present illustrative results of numerical tests performed for radiative cooling and \( p \)-Laplacian models from Section 3.3 posed on different surfaces with boundary. As a conclusion, we may observe the validity of the discrete maximum-minimum principle in each numerical example.
4.1 Generating acute meshes

As described in Subsection 3.2.3, the generation of acute or nonobtuse surface triangulations need special care. We first show examples where the particular surface properties can be exploited to generate such meshes directly, and then we also demonstrate how to apply the recently developed generation approach Kovács (2016). The studied surfaces are thus a hemisphere, a semi-torus, and an elaborate surface with four holes.

For the case of the hemisphere, the surface was approximated by subsequent refinements of the initial grid, taken to be the half of an icosahedron. In the refinement step each triangle was first divided into four similar triangles and then newly generated vertices were projected back to the surface.

For the case semi-torus, the meshes were generated via an initial subdivision into round strips, such that the vertices on one side of each strip lie on equal distances and on the other side similarly but in "chess-order", and finally linking the vertices as in Figure 2 (middle).

For the elaborate surface with four holes (a variant with boundary of an example in Elliott & Venkataraman (2015)), we adapted the techniques of Kovács (2016) to surfaces with boundary, also handling the extra difficulty of pairs of triangles with a joint edge (almost) perpendicular to the boundary, which are not accounted by the original algorithm. Using this method we generated triangular meshes with acute angles. The surface with four holes is associated with the distance function

$$\delta(x) = G(x_1^2) + G(x_2^2) + \frac{x_3^2}{0.1^2} - 1,$$

with $G(s) = 31.25(s - 0.36)(s - 0.95)$, then the surface is defined by

$$S = \{x \in \mathbb{R}^3 \mid \delta(x) = 0, x_3 \geq 0\}.$$

The quality of all three meshes, in terms of minimal and maximal angles, is shown in Figure 1.

![Figure 1](image_url)

**Fig. 1.** The minimum and maximum angles of the meshes generated for the test problems posed on the hemisphere (left), the semi-torus (middle) and the surface with four holes (right).

4.2 Radiative cooling on a hemisphere

First, we carry out some numerical experiments for the radiative cooling model from (iii) of subsection 3.3 with $\kappa = 1$ and $p = 2$:

$$-\Delta u + \sigma u^4 = 0 \quad \text{in } S, \quad u = g \quad \text{on } \partial S,$$

with the nonlinearity defined as $q(x,z) := z^4$ for $z \geq 0$ and $q(x,z) \equiv 0$ for $z \leq 0$. We choose $\sigma = 5$, and select the boundary function as

$$g(x,y,z) = 1 + xy.$$
We assume that \( S \) is a hemisphere of the radius 1, therefore \( \text{Ran}(g) = [0.5, 1.5] \), in particular, \( g \) is a nonnegative function over the boundary. This problem, as well as the ones below, were solved with a damped Newton iteration.

The minimum and maximum values of the numerical solutions for the radiative cooling problem on different meshes are presented in Table 1 (along the minimum and maximum angles in the mesh).

| dof  | min\{\alpha\} | max\{\alpha\} | min\{u_h\} | max\{u_h\} |
|------|----------------|----------------|------------|------------|
| 91   | 54.39          | 71.20          | 0.5041     | 1.4755     |
| 341  | 54.09          | 71.80          | 0.5        | 1.5        |
| 1321 | 54.02          | 71.95          | 0.5        | 1.5        |
| 5201 | 54.00          | 71.98          | 0.5        | 1.5        |

Table 1. Minimum and maximum values of the numerical solutions for the radiative cooling problem on the hemisphere

4.3 \( p \)-Laplacian on a semi-torus

On the semi-torus we performed the experiment for the \( p \)-Laplacian with \( p = 4 \), i.e.

\[
-\text{div}_S(|\nabla_S u|^2 \nabla_S u) = 0 \quad \text{in} \ S, \\
\]  

\[
u = g \quad \text{on} \ \partial S. 
\]

Homogeneous \( p \)-Laplacian equations with \( p = 4 \) arise, e.g. in rheology, see Busuioc & Cioranescu (2000). We prescribe the following boundary data:

\[ g(x,y,z) = 10 + x, \quad \text{hence} \quad \text{Ran}(g) = [3, 17]. \]

The discrete maximum principle for the \( p \)-Laplace problem is illustrated by Table 2, similarly as in the previous table before.

| dof  | min\{\alpha\} | max\{\alpha\} | min\{u_h\} | max\{u_h\} |
|------|----------------|----------------|------------|------------|
| 36   | 29.39          | 89.82          | 3.13397    | 17         |
| 136  | 31.24          | 80.16          | 3.00629    | 17         |
| 528  | 31.72          | 76.37          | 3.00034    | 17         |
| 2080 | 31.85          | 74.92          | 3.00002    | 17         |
| 8256 | 31.88          | 74.39          | 3.00000    | 17         |

Table 2. Minimum and maximum values of the numerical solutions for the \( p \)-Laplace problem on the semi-torus

4.4 Radiative cooling on a surface with four holes

Let us consider again the radiative cooling problem (4.1) on the surface

\[
S = \{x \in \mathbb{R}^3 \mid \delta(x) = 0, \ x_3 \geq 0\},
\]

(4.2)

with the distance function \( \delta(x) = G(x_1^2) + G(x_2^2) + \frac{x_3^2}{0.17} - 1 \), and where \( G(s) = 31.25(s - 0.36)(s - 0.95) \). We prescribe the following boundary data on \( \partial S \):

\[ g(x,y,z) = 1 + xy, \quad \text{hence} \quad \text{Ran}(g) = [0.02345468, 1.97653755]. \]

Here the minimum and maximum values over the boundary were obtained by numerically computing the extrema of the constrained problem.

Similarly to the previous tables before, the minimum and maximum values of the numerical solutions for the radiative cooling problem on different meshes approximating the surface with four holes are presented in Table 3.

Finally, Figure 2 reports on the surface meshes and on the numerical solutions for all three examples. The plotted results correspond to the last rows in each table, respectively.
Table 3. Minimum and maximum values of the numerical solutions for the radiative cooling problem on the surface with four holes.

| dof | min{α} | max{α} | min{u_h} | max{u_h} |
|-----|--------|--------|----------|----------|
| 375 | 32.68  | 88.34  | 0.05137  | 1.94184  |
| 464 | 33.66  | 87.26  | 0.04997  | 1.94978  |
| 672 | 37.45  | 87.99  | 0.02776  | 1.97317  |
| 1025| 34.30  | 88.10  | 0.02886  | 1.97553  |

**Fig. 2.** Surface meshes and the numerical solutions for the three nonlinear problems (from top to bottom).
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REFERENCES

ADAMS, R.A., Sobolev spaces, Academic Press, New York-London, 1975.
ANTONINI, P., MUGNAI, D. AND PUCCI, P., Quasilinear elliptic inequalities on complete Riemannian manifolds, J. Math. Pures Appl. (6) 87 (2007), 582-600.
AUBIN, T., Some Nonlinear Problems in Riemannian Geometry, 1998, Springer-Verlag.
BRANDTS, J., KOROTOV, S., KRÍŽEK, M., AND ŠOLC, J., On nonobtuse simplicial partitions, SIAM Rev. 51 (2009), 317–335.
BURMAN, E., HANSSO, P., LARSSON, M.G., LARSSON, K. AND MASSING, A., Finite element approximation of the Laplace–Beltrami operator on a surface with boundary, arXiv: 11509.08597 (2015)
BUSUIOC, V., CIORANESCU, D., On a class of electrorheological fluids. Contributions in honor of the memory of Ennio De Giorgi, Ricerche Mat. 49 (2000), suppl., 29–60.
CIARLET, Ph., The finite element method for elliptic problems, North-Holland, Amsterdam, 1978
CIARLET, P. G., Discrete maximum principle for finite-difference operators, Aequationes Math. 4 (1970), 338–352.
CIARLET, P. G., RAVIART, P.-A., Maximum principle and uniform convergence for the finite element method, Comput. Methods Appl. Mech. Engrg. 2 (1973), 17–31.
DEMLOW, A., Higher–order finite element methods and pointwise error estimates for elliptic problems on surfaces, SIAM. J. Num. Anal. 47(2) (2009), 805–807.
DRAGANESCU, A., DUPONT, T. F., SCOTT, L. R., Failure of the discrete maximum principle for an elliptic finite element problem, Math. Comp. 74 (2005), no. 249, 1–23 (electronic).
DZIUJ, G., Finite elements for the Beltrami operator on arbitrary surfaces. Partial differential equations and calculus of variations, 142155, Lecture Notes in Math., 1357, Springer, Berlin, (1988).
DZIUJ, G. AND ELLIOTT, C.M., Finite elements on evolving surfaces. IMA Journal of Numerical Analysis, 27(2):262–292,, 2007.
DZIUJ, G. AND ELLIOTT, C.M., Finite element methods for surface PDEs. Acta Numerica, 22:289–396., 2013.
FARAGÓ, I., Matrix and Discrete Maximum Principles, in: LSSC 2009, LNCS 5910, 563–570 (2010).
FARAGÓ, I., KARÁTSON, J., Numerical solution of nonlinear elliptic problems via preconditioning operators. Theory and applications. Advances in Computation, Volume 11, NOVA Science Publishers, New York, 2002.
M. FRITTELLI, A. MADZVAMUSE, I. SGURA, AND C. VENKATARAMAN., Preserving invariance properties of reaction–diffusion systems on stationary surfaces. IMA Journal of Numerical Analysis, 2017. doi:10.1093/imanum/drx058.
M. FRITTELLI, A. MADZVAMUSE, I. SGURA, AND C. VENKATARAMAN., Lumped finite element method for reaction–diffusion systems on compact surfaces. arXiv:1609.02741, 2016.
M. FRITTELLI, A. MADZVAMUSE, I. SGURA, AND C. VENKATARAMAN., Lumped finite elements for reaction–cross–diffusion systems on stationary surfaces. Computers & Mathematics with Applications, 74(12):3008–3023, 2017.
C.M. ELLIOTT AND C. VENKATARAMAN., Error analysis for an ALE evolving surface finite element method. Numerical Methods for Partial Differential Equations, 31(2):459–499, 2015.
GILBARG, D., TRUDINGER, N. S., Elliptic partial differential equations of second order (2nd edition), Grundlehren der Mathematischen Wissenschaften 224, Springer, 1983.
HAN Q., HONG J-X., Isometric Embedding of Riemannian Manifolds in Euclidean Spaces, Mathematical Surveys and Monographs, vol. 130, AMS, 2006.
KARÁTSON, J., KOROTOV, S., Discrete maximum principles for finite element solutions of nonlinear elliptic problems with mixed boundary conditions, Numer. Math. 99 (2005), 669–698.
KARÁTSON, J., KOROTOV, S., Some discrete maximum principles arising for nonlinear elliptic finite element problems, Comput. Math. Appl. 70 (2015), no. 11, 2732-2741.
KELLER, H. B., Elliptic boundary value problems suggested by nonlinear diffusion processes, Arch. Rational Mech. Anal. 35 (1969), 363–381.
KOVÁCS, B., Computing arbitrary Lagrangian Eulerian maps for evolving surfaces, arXiv:1612.01701, (2016).
KOROTOV, S., KRÍŽEK, M., Acute type refinements of tetrahedral partitions of polyhedral domains, SIAM J. Numer. Anal. 39 (2001), 724–733.
KOROTOV, S., KRÍŽEK, M., NEITTAANMÄKI, P., Weakened acute type condition for tetrahedral triangulations and the discrete maximum principle, Math. Comp. 70 (2001), 107–119.
KOROTOV, S., On conforming nonobtuse tetrahedralizations of some cylindrical-type domains with curved boundaries, in: Proceedings of the International Conference on Applications of Mathematics (AM2012), Institute of Mathematics, Prague, Czech Republic (dedicated to the 60th birthday of M. Krížek), ed. by J. Brandts et al., 2012, 131–138.
PROTTER, M. H., WEINBERGER, H. F., Maximum principles in differential equations, Springer-Verlag, New York, 1984.
PUCCI, P., SERRIN, J. B., *The Maximum Principle*, Springer, 2007.

XU, J., ZIKATANOV, L., A monotone finite element scheme for convection-diffusion equations, *Math. Comp.* 68 (1999), 1429–1446.

VARGA, R., *Matrix iterative analysis*, Prentice Hall, New Jersey, 1962.

ZAMFIRESCU C. T., Survey of two-dimensional acute triangulations, *Discrete Mathematics* 313 (2013), 35–49.