ON THE GOTTLIEB GROUP, DRINFELD CENTRE AND THE CENTRE OF A CROSSED MODULE

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Abstract. The aim of this paper is to introduce the concept of the centre of a crossed module $G^* = (G_2 \to G_1)$. This centre is closely related to the Gottlieb group of the classifying space of a crossed module and also to the Drinfeld centre of a monoidal category introduced independently by Drinfeld and Joyal and Street. Our definition of the centre is based on certain crossed homomorphisms $G_1 \to G_2$, which makes it easy to relate it to group cohomology.

1. Introduction

The aim of this work is to introduce and prove essential properties of the centre of a crossed module. In particular, our results establish a connection between the Drinfeld-Joyal-Street centre of a monoidal category [12] and the Gottlieb group [8] of a pointed topological space. To the best of our knowledge, such a link between these classical objects has not been observed before.

For a CW-complex $X$ we define the centre of $X$ to be the connected component of the mapping space $Map(X, X)$ containing the identity map $id_X$. Here $Map(X, X)$ is the set of all continuous maps $X \to X$, equipped with the compact-open topology. The centre of $X$ will be denoted by $Z_X$. It has an $H$-space structure induced by the composition of maps. In particular, the group $\pi_1(Z_X, id_X)$ is abelian. These spaces are classical objects, although not known by this name. They have been extensively studied, see for example the survey paper [17] and the references given therein. We are interested in these spaces in the following context in which they arose in Gottlieb’s work [8].

Let $(X, x_0)$ be a pointed space. Then the evaluation at $x_0$ defines the pointed map $ev_{x_0} : (Z_X, id_X) \to (X, x_0)$, which induces a group homomorphism $\pi_1(Z_X, id_X) \to \pi_1(X, x_0)$. The image of this homomorphism is denoted by $G(X, x_0)$ and is called the Gottlieb group of $(X, x_0)$. Gottlieb, among other results, proved that if $X = B\pi$ is the classifying space of a discrete group $\pi$, then there is a homotopy equivalence $Z(B\pi) \simeq B(Z\pi)$ and an isomorphism of groups $G(B\pi, 1) \cong Z\pi$,

where $Z\pi$ is the centre of the group $\pi$, see [8] Theorem III.2 and [8] Corollary 1.13).

Our aim is to extend these results to spaces $X$ for which $\pi_i(X) = 0$ for $i \neq 1, 2$. It is well-known that algebraic models for such spaces are crossed modules, see for example [13], [1], [3]. In fact, any crossed module $G_* = (G_2 \to G_1)$ has a classifying space $BG_*$, which is connected and has vanishing homotopy groups in dimensions $\geq 3$. Conversely, any CW-complex of such type is homotopy equivalent to $BG_*$ for some crossed module $G_*$. Moreover, one can assume that $G_1$ is a free group.

By definition, a crossed module $G_*$ is a group homomorphism $\partial : G_2 \to G_1$ together with an action of $G_1$ on $G_2$ satisfying some properties (see Section 2.1). The most
important invariants of the crossed module $G_*$ are the group $\pi_1(G_*) = \text{Coker} (\partial)$ and the $\pi_1(G_*)$-module $\pi_2(G_*) = \text{Ker} (\partial)$.

One of the main results of this paper is to show that any crossed module $\partial : G_2 \to G_1$ fits in a commutative diagram

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\delta} & Z_1(G_*) \\
\downarrow{\text{id}} & & \downarrow{z_1} \\
G_2 & \xrightarrow{\partial} & G_1
\end{array}
\]

where the top horizontal $G_2 \xrightarrow{\delta} Z_1(G_*)$ and right vertical $Z_1(G_*) \xrightarrow{z_1} G_1$ arrows have again crossed module structures. In fact the former is even a braided crossed module, which we call the \textit{centre of the crossed module} $\partial : G_2 \to G_1$ and denote it by $Z_*(G_*)$. We denote the crossed module corresponding to the right vertical arrow by $G_*/Z_*(G_*)$. Conceptually, we want to think of the latter as a 2-mathematical analogue of the quotient of the crossed module by its centre, see Subsection 3.2.

The centre of a crossed module is closely related to the Drinfeld-Joyal-Street centre of a monoidal category \cite{12}. Namely, it is well-known that any (braided) crossed module defines a (braided) monoidal category \cite{11}. It turns out that the centre of a monoidal category \cite{12}. Namely, it is well-known that any (braided) crossed module defines a (braided) monoidal category \cite{11}. It turns out that the braided monoidal category corresponding to the braided crossed module $Z_*(G_*)$ \cite{11} is isomorphic to the centre of the monoidal category corresponding to $G_*$, see Proposition 12.

Our definition of $Z_*(G_*)$ is based on certain crossed homomorphisms $G_1 \to G_2$ and has some advantage compared to one based on monoidal categories. Namely, the description of $Z_*(G_*)$ in terms of crossed homomorphisms makes it easy to relate the centre of a crossed module to group cohomology. In fact, there are two interesting connections. Firstly, as we will show, $\pi_1(Z_*(G_*))$ is a subgroup of $H^1(G_1, G_*)$, the cohomology of $G_1$ with coefficients in the crossed module $G_*$, as defined in \cite{9}. Secondly, the essential invariants of $Z(G_*)$ are closely related to the low dimensional group cohomology. In fact, one has an isomorphism of groups

$$\pi_2(Z_*(G_*)) \cong H^1(\pi_1(G_*), \pi_2(G_*))$$

and the group $\pi_1(Z_*(G_*))$ fits in an exact sequence

$$0 \to H^1(\pi_1, \pi_2) \to \pi_1(Z_*(G_*)) \to Z_2(G_*)(\pi_1(G_*)) \xrightarrow{g} H^2(\pi_1, \pi_2),$$

where $\pi_1$ and $\pi_2$ denote the groups $\pi_1(G_*)$ and $\pi_2(G_*)$, see Lemma 11 and part iv) of Proposition 13 and $Z_2(G_*)(\pi_1(G_*))$ is the subgroup of the centre of $\pi_1(G_*)$ consisting of those elements which act trivially on $\pi_2(G_*)$.

The main applications of our centre are the following results in homotopy theory (see Proposition 24 and Theorem 19). Let $G_*$ be a crossed module with free $G_1$. Then there is a homotopy equivalence

$$Z(BG_*) \simeq B(Z_*(G_*)).$$

We then give an explicit description of $G(BG_*, 1)$ as a subgroup of $Z_2(G_*)(\pi_1(G_*))$, by identifying it with the kernel of the homomorphism $g$ in the above exact sequence, i.e. we show exactness of the sequence

$$0 \to G(X, x_0) \to Z_2(G_*)(\pi_1(G_*)) \xrightarrow{g} H^2(\pi_1(G_*), \pi_2(G_*)).$$

It should be pointed out that in the 80’s Norrie also introduced the notion of a centre of a crossed module \cite{15}, but our notion differs from hers. We show that there exists a comparison morphism from Norrie’s centre to ours, which induces an isomorphism on $\pi_2$ but not on $\pi_1$. The advantage of our definition is the exact sequence \cite{11} which shows that $\pi_1$ of our centre has a nice relation to group cohomology.
For any group $G$, one has the crossed module $\partial : G \to Aut(G)$, where $\partial(g)$ is the inner automorphism corresponding to $g \in G$. This crossed module is denoted by $\text{AUT}(G)$. For $G = D_4$ we compute both Norrie’s and our centre. Norrie’s centre is $C_2 \to \{1\}$, while our centre is more complicated and our computations show that $\pi_1(Z_\ast(\text{AUT}(D_4))) \cong C_2 \times C_2$ and hence they have nonisomorphic $\pi_1$, see Section 3.6.

The paper is organised as follows. After the preliminaries in Section 2 we introduce and prove the main properties of the centre of a crossed module in Section 3. The connection to the Gottlieb group is explored in Section 4.

As was demonstrated by Loday [13] there is a generalisation of crossed modules, known as $\text{cat}^n$-groups, which serve as algebraic models for connected spaces $X$ for which $\pi_i(X) = 0$ for all $i > n$. We believe that there exists an extension of our centre to $\text{cat}^n$-groups and they will have a connection to the centre of such $X$.

2. Preliminaries on crossed modules and monoidal categories

The material of this section is well known. We included it in order to fix terminology and notations.

2.1. Crossed modules. Recall that a crossed module $G_\ast$ is a group homomorphism $\partial : G_2 \to G_1$ together with another group homomorphism $\rho : G_1 \to Aut(G_2)$ such that

\begin{align*}
\partial(xa) &= x\partial(a)x^{-1}, \\
\rho(b)x &= \rho(ba)x^{-1},
\end{align*}

where $a, b \in G_2$ and $x \in G_1$. Here and elsewhere we write $\pi a$ instead of $\rho(x)(a)$.

We refer to [9] for an extensive study of crossed modules and their role in homotopy theory. Additionally, we recommend the recent article [10] for the history of crossed modules and applications.

It follows that $\text{Im}(\partial)$ is a normal subgroup of $G_1$ and thus

$$\pi_1(G_\ast) = G_1/\text{Im}(\partial)$$

is a group. Moreover,

$$\pi_2(G_\ast) = \text{Ker}(\partial)$$

is a central subgroup of $G_2$ and the action of $G_1$ on $G_2$ induces a $\pi_1(G_\ast)$-module structure on the abelian group $\pi_2(G_\ast)$. Thus one has an exact sequence of groups

$$1 \to \pi_2(G_\ast) \to G_2 \xrightarrow{\partial} G_1 \to \pi_1(G_\ast) \to 1.$$
category structure. The monoidal structure is denoted by $\cdot$ and the corresponding bifunctor

$$\cdot : \mathbf{Cat}(G_*) \times \mathbf{Cat}(G_*) \to \mathbf{Cat}(G_*)$$

is given on objects by $(x, y) \mapsto x \cdot y = xy$, and on morphisms it is given by

$$(x \xrightarrow{a} x') \cdot (y \xrightarrow{b} y') = (xy \xrightarrow{ab} x'y').$$

This is well-defined, because

$$\partial(a \cdot b)xy = \partial(a)x\partial(b)x^{-1}xy = \partial(a)x\partial(b)y = x'y'.$$

In particular, if $x \xrightarrow{a} x'$ is a morphism, and we act by the functor $\cdot (-) \cdot y$, we obtain the morphism $xy \xrightarrow{a} x'y$, while if we act by the functor $x \cdot (-)$ on a morphism $y \xrightarrow{b} y'$, we obtain the morphism $xy \xrightarrow{y} x'y$. In other words, $a \cdot 1_y = a$ and $1_x \cdot b = x'b$.

We always consider the groupoid $\mathbf{Cat}(G_*)$ as pointed, where the chosen object is $1 \in G_1$. In particular, we will write $\pi_1(\mathbf{Cat}(G_*))$ instead of $\pi_1(\mathbf{Cat}(G_*), 1)$. Then comparing the definitions we see that $\pi_{i+1}(G_*) = \pi_i(\mathbf{Cat}(G_*))$ for $i = 0, 1$.

### 2.2. Braided crossed modules.

**Definition 1.** A braided crossed module (BCM for short) $G_*$ is a group homomorphism $\partial : G_2 \to G_1$ together with a map $\{ -, - \} : G_1 \times G_1 \to G_2$ such that

1. $\partial\{x, y\} = [x, y]$,
2. $\{\partial a, \partial b\} = [a, b]$,
3. $\{\partial a, x\} = \{x, \partial a\}^{-1}$,
4. $\{x, yz\} = \{x, y\}\{x, z\}\{zxz^{-1}y^{-1}, y\}$,
5. $\{xy, z\} = \{x, y\}y^{-1}\{y, z\}$.

Here as usual $x, y \in G_1$ and $a, b \in G_2$. Recall also that $[x, y] = xyx^{-1}y^{-1}$ is the commutator.

A symmetric crossed module (SCM for short) is a BCM for which

$$\{y, x\} = \{x, y\}^{-1}$$

holds for all $x, y \in G_1$. This condition is much stronger than (4).

The definitions of BCM and SCM, except for the terminology, go back to Conduché [5, 2.12]. The relation with braided and symmetric monoidal categories was discovered by Joyal and Street [11].

Recall also the following definition [11].

**Definition 2.** A reduced quadratic module (RQM for short) is a group homomorphism $G_2 \xrightarrow{\partial} G_1$ together with a group homomorphism

$$G_1^a \otimes G_1^b \to G_2, \quad \bar{x} \otimes \bar{y} \to \{\bar{x}, \bar{y}\},$$

where $K^{ab} = K/[K, K]$ and $\bar{x}$ denotes the image of $x \in K$ in $K^{ab}$. One requires that $G_2$ and $G_1$ are nilpotent groups of class two and for all $a, b \in G_2$ and $x, y \in G_1$ the following identities hold:

$$1 = \{\partial(a), \bar{x}\}\{\bar{x}, \partial(a)\},$$

$$[a, b] = \{\partial(a), \partial(b)\},$$

$$[x, y] = \partial\{\bar{x}, \bar{y}\}.$$

It is clear that any RQM is a BCM.

A morphism of BCM is a pair of group homomorphisms that preserve the structure of BCM, ensuring compatibility with the boundary maps and the braiding operation.
Lemma 3. Let $G_\ast$ be a BCM. Define $\rho : G_1 \to \text{Aut}(G_2)$ by

\[ x_a := \{x, \partial a\}a. \]

Then one obtains a crossed module.

Proof. This fact is due to Conduché [5]. \qed

Lemma 4. Let $G_\ast$ be a BCM. Then $\pi_1(G_\ast)$ is an abelian group and the induced action of $\pi_1(G_\ast)$ on $\pi_2(G_\ast)$ is trivial.

Proof. By (4) we have $[G_1, G_1] \subseteq \text{Im}(\partial)$. Hence $\pi_1(G_\ast) = G_1/\text{Im}(\partial)$ is abelian. For the second part, observe that $a \in \pi_2(G_\ast)$ iff $\partial(a) = 1$. Hence $x^a = \{x, 1\}a$. Thus we only need to show that $\{x, 1\} = 1$. By taking $y = z = 1$ in (8) one obtains $\{1, 1\} = 1$. Next, we put $y = z = 1$ in (7) to obtain $\{x, 1\} = 1$. \qed

The following important result is due to Joyal and Street [11].

Lemma 5. Let $G_\ast$ be a crossed module. Then there is a one-to-one correspondence between the bracket operations $\{-,\} : G_1 \times G_1 \to G_2$ satisfying the conditions listed in Definition 7 and the braided monoidal category structures on $\text{Cat}(G_\ast)$. Under this equivalence, the element $\{x, y\}$ corresponds to the braid $yx \xrightarrow{[x,y]} xy$.

2.3. The centre of a monoidal category. Let $(C, \otimes)$ be a monoidal category. Recall that [12] the centre of $(C, \otimes)$ is the braided monoidal category $Z(C, \otimes)$ which is defined as follows. Objects of $Z(C, \otimes)$ are pairs $(x, \xi)$ where $x$ is an object of $C$ and $\xi : (-) \otimes x \to x \otimes (-)$ is a natural isomorphism of functors for which the diagram

\[
\begin{array}{ccc}
y \otimes z \otimes x & \xrightarrow{\xi_y \otimes \xi_z} & x \otimes y \otimes z \\
\downarrow 1_y \otimes \xi_z & & \downarrow \xi_y \otimes 1_z \\
y \otimes x \otimes z & \xrightarrow{\xi_y \otimes 1_z} & x \otimes y \otimes z
\end{array}
\]

commutes for all $y, z \in \text{Ob}(C)$. Here and in what follows we write $\xi_y : y \otimes x \to x \otimes y$ for the value of $\xi$ on $y \in \text{Ob}(C)$. A morphism $(x, \xi) \to (y, \eta)$ in $Z(C, \otimes)$ is a morphism $f : x \to y$ of $C$ such that the diagram

\[
\begin{array}{ccc}
z \otimes x & \xrightarrow{\xi_z} & x \otimes z \\
\downarrow 1_z \otimes f & & \downarrow f \otimes 1_z \\
z \otimes y & \xrightarrow{\eta_y} & y \otimes z
\end{array}
\]

commutes for all $z \in \text{Ob}(C)$. The monoid structure on $Z(C, \otimes)$ is given by

\[(x, \xi) \otimes (y, \eta) := (x \otimes y, \xi_z),\]

where $\xi_z : z \otimes x \otimes y \to x \otimes y \otimes z$ is the composite map

\[z \otimes x \otimes y \xrightarrow{\xi_z \otimes 1_y} x \otimes z \otimes y \xrightarrow{1_z \otimes \eta_y} x \otimes y \otimes z.\]

Finally, the braiding

\[c : (x, \xi) \otimes (y, \eta) \to (y, \eta) \otimes (x, \xi)\]

is given by the morphism $\eta_y : x \otimes y \to y \otimes x$.

The assignment $(x, \xi) \mapsto x$ obviously extends as a strict monoidal functor $Z(C, \otimes) \to C$, which is denoted by $z_c$. 
3. The centre of a crossed module

Let \( G \) be a crossed module. In this section we will construct a braided crossed module \( Z_*(G_1) \) such that the braided monoidal categories \( \text{Cat}(Z_*(G_1)) \) and \( Z(\text{Cat}(G_1)) \) are isomorphic. Because of this fact we will call the crossed module \( Z_*(G_1) \) the centre of \( G_1 \). It should be pointed out that \( Z_*(G_1) \) differs from the centre of \( G_1 \) in the sense of [13], though there is some relation, which will be discussed in Section 3.5.

The crucial step in the definition of the centre of a crossed module is the group \( Z_1(G_1) \). As a set, it is constructed in Definition 3 and in Lemma 7 we equip it with a group structure. The group \( G_1 \) acts on \( Z_1(G_1) \) (Lemma 8) and in this way we obtain a crossed module \( z_1 : Z_1(G_1) \to G_1 \) (see Proposition 9). The centre \( Z_*(G_1) \) has the form \( G_2 \to Z_1(G_1) \). The definition of the (braided) crossed module structure on \( Z_*(G_1) \) and checking the axioms is given in Lemma 11 and Corollary 13. The relation to the centre of a monoidal category is given in Proposition 12. The section 3.3 clarifies relations between \( \pi_2 \) and \( \pi_1 \). One needs to check three identities: (9), (10) and (11). As a set, it is constructed in Definition 6, and in Lemma 7 we equip it with a group structure. The group \( G_1 \) acts on \( Z_1(G_1) \) (Lemma 8) and in this way we obtain a crossed module \( z_1 : Z_1(G_1) \to G_1 \) (see Proposition 9). The centre \( Z_*(G_1) \) has the form \( G_2 \to Z_1(G_1) \). The definition of the (braided) crossed module structure on \( Z_*(G_1) \) and checking the axioms is given in Lemma 11 and Corollary 13. The relation to the centre of a monoidal category is given in Proposition 12. The section 3.3 clarifies relations between \( \pi_2 \) and \( \pi_1 \).

3.1. Definition. The following is the main definition of this paper.

**Definition 6.** Let \( G_1 \) be a crossed module. Denote by \( Z_1(G_1) \) the set of all pairs \( (x, \xi) \) where \( x \in G_1 \) and \( \xi : G_1 \to G_2 \) is a map satisfying the following identities

\[
\begin{align*}
\partial \xi(t) &= [x, t], \\
\xi(\partial a) &= x \cdot a^{-1}, \\
\xi(st) &= \xi(s) \cdot \xi(t).
\end{align*}
\]

Here \( s, t \in G_1 \) and \( a \in G_2 \).

Observe that the last condition says that \( G_1 \xrightarrow{\xi} G_2 \) is a crossed homomorphism.

In this section we construct two crossed modules involving \( Z_1(G_1) \). The first one is \( Z_1(G_1) \to G_1 \) (see Proposition 9 below) and the second one is \( G_2 \to Z_1(G_1) \) (see Corollary 13). The latter is called the centre of \( G_1 \) and is in fact a BCM.

We start with a group structure on \( Z_1(G_1) \).

**Lemma 7.** i) If \((x, \xi), (y, \eta) \in Z_1(G_1)\), then \((xy, \xi \cdot \eta) \in Z_1(G_1)\), where \( \xi \cdot \eta \) is a map \( G_1 \to G_2 \) given by

\[
t \mapsto \xi(t) \cdot \eta(t).
\]

ii) The set \( Z_1(G_1) \) is a group under the operation

\[
(x, \xi) \cdot (y, \eta) := (xy, \xi \cdot \eta).
\]

The unit element is \((1, 1)\), where \( 1 \) is the identity of \( G_1 \) and \( 1 : G_1 \to G_2 \) is the constant map with value 1.

**Proof.** i) Denote by \( \zeta \) the function \( \xi \cdot \eta \). One needs to check three identities: (9), (10) and (11). We have

\[
\partial \zeta(t) = \partial(x \cdot \xi(t)) \cdot \partial(\eta(t))
\]

\[
= x \partial(\eta(t)) \cdot a^{-1} \partial(\xi(t))
\]

\[
= x[y, t]a^{-1}x^{-1}[x, t]
\]

\[
= xyt^{-1}x^{-1}t^{-1}
\]

\[
= [xy, t]
\]

and the condition (11) holds.
We also have
\[ \zeta(\partial a) = \tilde{x}(\eta(\partial a))\xi(\partial a) = \tilde{x}(\eta(\partial a)\cdot a^{-1}) = x(\eta(\partial a)\cdot a^{-1}) = x\eta(\partial a)\cdot a^{-1}. \]

Thus the condition (10) also holds.

Finally, we also have
\[ \zeta(st) = \tilde{x}(\eta(st))\xi(st) = \tilde{x}(\eta(s)\cdot x\eta(t))\xi(s)\xi(t). \]

On the other hand, we also have
\[ \zeta(s)\xi(t) = \tilde{x}(\eta(s)\cdot x\eta(t))\xi(s)\xi(t). \]

To show these expressions are equal, we have to show that
\[ \tilde{x}(\eta(s)\cdot x\eta(t)) = \tilde{x}(\eta(s)\cdot x\eta(t)), \]

or equivalently that \( \tilde{x}(\eta(s)\cdot x\eta(t)) = \tilde{x}(\eta(t)). \) To this end, observe that
\[ \tilde{x}(\eta(s)\cdot x\eta(t)) = \tilde{x}(\eta(s)\cdot x\eta(t)) = \tilde{x}(\eta(t)) \]

and the identity (11) follows.

The proof of ii) is straightforward and left to the reader.

\[ \square \]

It is clear from the definition of the multiplication in \( Z_1(G_\ast) \), that the map
\[ z_1 : Z_1(G_\ast) \to G_1 \]

is a group homomorphism. Our next aim is to show that this is in fact a crossed module.

**Lemma 8.** For any \( z \in G_1 \) and any \( (x, \xi) \in Z_1(G_\ast) \) one has \( z^x = zxz^{-1} \) and
\[ \psi(t) = \tilde{z}(\xi(t)). \]

**Proof.** First of all we need to check that the pair \((z^x, \psi)\) satisfies the identities (9)–(11). In fact, we have
\[ \partial(\psi(t)) = \partial(\tilde{z}(\xi(t))) = z[x, z^{-1} t]z^{-1} = zxx^{-1}tx^{-1}z^{-1}(t^{-1})z^{-1}. \]

On the other hand, the last expression is equal to
\[ zxx^{-1}tzxx^{-1}z^{-1}t(zxx^{-1}) = (zzx^{-1})t(zxx^{-1})t^{-1}, \]

which is the same as \([z^x, t]\) and (9) follows.

Next, we have
\[ \psi(\partial a) = \tilde{z}(\xi^{-1})(\partial(a)) = \tilde{z}(\xi^{-1})(\partial(z^{-1}a)) = \tilde{z}(\xi^{-1})(\partial(z^{-1}a)) = zxx^{-1}a \cdot a^{-1} \]

and the equality (10) follows.

We also have
\[ \psi(st) = \tilde{z}(\xi^{-1} s \cdot z^{-1} t) = \tilde{z}(\xi^{-1} s \cdot z^{-1} s \cdot z^{-1} t) = \psi(s \cdot z^{-1} t) = \psi(s). \]

proving the equality (11). Hence, \((z^x, \psi) \in Z_1(G_\ast). \)

\[ \square \]
Proposition 9. The construction described in Lemma 8 defines an action of the group $G_1$ on $Z_1(G_s)$. In this way, the map

$$z_1 : Z_1(G_s) \rightarrow G_1$$

is a crossed module.

Proof. Take two elements $(x_i, \xi_i)$, $i = 1, 2$ in $Z_1(G_s)$. Assume that

$$(x, \xi) = (x_1, \xi_1) \cdot (x_2, \xi_2)$$

is the product of these elements and $z \in G_1$. We claim that

$$\tilde{z}(x, \xi) = \tilde{z}(x_1, \xi_1) \cdot \tilde{z}(x_2, \xi_2).$$

Recall that $x = x_1 x_2$ and $\xi(t) = \tilde{x}_1 \tilde{\xi}_2(t) \cdot \xi_1(t)$. It follows that

$$\tilde{z}(x, \xi) = (\tilde{x}_1 \tilde{z} x_2, \psi),$$

where

$$\psi(t) = \tilde{z}(\xi(t)) = \tilde{z}(\xi_1) \cdot \tilde{z}(\xi_2).$$

On the other hand, we have

$$\tilde{z}(x_1, \xi_1) = (\tilde{z} x_1, \psi_1)$$

and

$$\tilde{z}(x_2, \xi_2) = (\tilde{z} x_2, \psi_2),$$

where $\psi_i(t) = \tilde{z}\xi_i(z^{-1} t)$, $i = 1, 2$. Hence

$$\tilde{z}(x_1, \xi_1) \cdot \tilde{z}(x_2, \xi_2) = (\tilde{z} x_1 \tilde{z} x_2, \psi')$$

where

$$\psi'(t) = \tilde{z} x_1 \tilde{z}_2 (\xi(t)) \tilde{z}_1 (z^{-1} t) = \tilde{z} x_1 \tilde{z}_2 (z^{-1} t) \tilde{z}_1 (z^{-1} t)$$

and the claim follows.

We still need to show the following equalities:

$$\tilde{z}(u(x, \xi)) = \tilde{z} u(x, \xi),$$

$$z_0(\tilde{z}(x, \xi)) = \tilde{z} z_0(x, \xi)$$

and

$$z_0(x, \xi)(y, \eta) = (x, \xi)(y, \eta)(x, \xi)^{-1}.$$

Since the proofs of the first two identities are straightforward, we omit them and check only the validity of the third identity, which is equivalent to

$$\tilde{z}(y, \eta)(x, \xi) = (x, \xi)(y, \eta).$$

The RHS equals $(xy, t \mapsto x \eta(t) \xi(t))$, while the LHS equals

$$(x y x^{-1}, t \mapsto x \eta(x^{-1} t x))(x, \xi) = (x y, t \mapsto x y x^{-1} \xi(t) \cdot x \eta(x^{-1} t x)).$$

So we need to check

$$x \eta(t) \xi(t) = x y x^{-1} \xi(t) \cdot x \eta(x^{-1} t x).$$

The last equality is equivalent to

$$\eta(t) x^{-1} \xi(t) = y x^{-1} \xi(t) \cdot \eta(x^{-1} t x).$$

We set $t = xs$. Then the above relation is equivalent to

$$\eta(xs) x^{-1} \xi(xs) = y x^{-1} \xi(xs) \eta(xs).$$

So we need to check

$$\eta(xs) = y x^{-1} \xi(xs) \cdot \eta(xs) (z^{-1} \xi(xs))^{-1}. \tag{12}$$

According to the equation (12), we have
Proof. The pair $(\xi, \partial(x))$ is in the kernel of $z_1: Z_1(G_*) \to G_1$ if $x = 1$. Now Definition \ref{def:crossed-homomorphism} shows that the values of $\xi$ lie in $\pi_2(G_*)$ and $\xi$ vanishes on the image of $\partial: G_2 \to G_1$. Since it is a crossed homomorphism it factors in a unique way as a crossed homomorphism $\pi_1(G_*) \to \pi_2(G_*)$ and hence the result.

We put $a = x^{-1}(x(s))$, $b = \eta(sx)$. Since
$$\partial(a) = x^{-1}(\partial(x))x = x^{-1}[x, xs][x, s],$$
we obtain
$$y^{-1}(x(s)) \cdot \eta(sx) \left(x^{-1}(x(s)) \right)^{-1} = \eta([x, s]) \cdot aba^{-1}.$$

Thus the equality (12) holds and hence the result is proved. \qed

Lemma 10. One has an exact sequence
$$0 \to \text{Der}(\pi_1(G_*), \pi_2(G_*)) \to Z_1(G_*) \xrightarrow{\zeta_1} G_1,$$
where as usual $\text{Der}$ denotes the set of all crossed homomorphisms.

Proof. The pair $(x, \xi)$ is in the kernel of $z_1: Z_1(G_*) \to G_1$ iff $x = 1$. Now Definition \ref{def:crossed-homomorphism} shows that the values of $\xi$ lie in $\pi_2(G_*)$ and $\xi$ vanishes on the image of $\partial: G_2 \to G_1$. Since it is a crossed homomorphism it factors in a unique way as a crossed homomorphism $\pi_1(G_*) \to \pi_2(G_*)$ and hence the result. \qed

Our next goal is to relate $Z_1(G_*)$ to the centre of the monoidal category $\text{Cat}(G_*)$. We start with the following observation.

Lemma 11. i) Let $c \in G_2$. Then $(\partial(c), \zeta_c) \in Z_1(G_*)$, where $\zeta_c: G_1 \to G_2$ is the map given by
$$\zeta_c(t) = c(c)^{-1}.$$

ii) The map $\delta: G_2 \to Z_1(G_*)$ given by
$$\delta(c) = (\partial(c), \zeta_c)$$
is a group homomorphism.

iii) Define the action of $Z_1(G_*)$ on $G_2$ by
$$(x, \xi)^a := x^a.$$
Then $\delta: G_2 \to Z_1(G_*)$ is a crossed module.

Proof. i) We have to check that the pair $(\partial(c), \zeta_c)$ satisfies the identities (9) - (11).
In fact, we have
$$\partial(\zeta_c(t)) = \partial(c(c)^{-1}) = \partial(c(t \partial c^{-1})^{-1} = \partial c t \partial c^{-1} t^{-1} = [\partial c, t].$$
Thus (9) holds. Next, we also have
$$\zeta_c(\partial a) = c(\partial a)^{-1} = c(aca)^{-1} = [c, a] = (\partial a)^{-1}.$$
Hence \( \text{(10)} \) holds. Finally, we have
\[
\zeta_c(s)(\zeta_c(t)) = c(s) c^{-1} \cdot (c^t c)^{-1} \\
= c(s) c^{-1}(s) c^{-1} \\
= c(s)^{-1} \\
= \zeta_c(st).
\]

This finishes the proof.

ii) Clearly \((\partial b, \zeta_b)(\partial c, \zeta_c) = (\partial(bc), \partial b \cdot \zeta_b)\), where
\[
(\partial b \cdot \zeta_b)(t) = b(c^t(c^{-1})b^{-1}(b^t(b^{-1})) \\
= bc^t(b^{-1}) \\
= bc^t(c^{-1}b^{-1} \\
= \zeta_{bc}(t).
\]

Thus \((\partial b, \zeta_b)(\partial c, \zeta_c) = (\partial(bc), \zeta_{bc})\). Hence \(\delta\) is a group homomorphism.

iii) We have to check the identities \(\text{(2)}\) and \(\text{(3)}\). For the second, observe that
\[
\delta(b_2) = (\partial(b_2)b_2)a = \partial(b)a = bab^{-1}.
\]

For the first one, we need to check that
\[
\delta((x, \xi)a) = (x, \xi)(a(x, \xi)^{-1}).
\]

Equivalently,
\[
\delta(xa)(x, \xi) = (x, \xi)(a),
\]

which is the same as
\[
(\partial x, \zeta_x)(x, \xi) = (x, \xi)(\partial(a), \zeta_a).
\]

The first coordinate of the LHS is equal to \(x\partial(a)x^{-1} \cdot x = x\partial(a)\), the same as the first coordinate of the RHS. Hence we need to show that for all \(t \in G_1\) one has
\[
x\partial(a)x^{-1} \xi(t) \cdot x a(txa)^{-1} = x(a(txa)^{-1})\xi(t).
\]

Equivalently,
\[
\partial(a)x^{-1} \xi(t) \cdot a x^{-1} tx \cdot (a^{-1}) = a^t(a^{-1})x^{-1} \xi(t).
\]

Using the equality \(\partial(a)b = aba^{-1}\) and cancelling out \(a\), we see that the equality can be rewritten as
\[
x^{-1} \xi(t) \cdot x^{-1} tx \cdot (a^{-1}) = t(a^{-1})x^{-1} \xi(t).
\]

Now this is obviously equivalent to
\[
\xi(t) x^t (a^{-1}) = x (a^{-1}) \xi(t)
\]

and hence to
\[
\xi(t) x^t (a^{-1}) \xi(t)^{-1} = x (a^{-1}).
\]

Based on \(\text{(10)}\), we have \(x^t b = \xi(\partial(b))b\). Hence we can write
\[
x^t (a^{-1}) = \xi(\partial(a^{-1})) t^t (a^{-1}) \\
= \xi(t \partial(a)^{-1}) t^t (a^{-1}) \\
= \xi(t^t \partial(a^{-1}) t^{-1}) t^t (a^{-1}) \\
= \xi(t^t \partial(a^{-1}) \xi(t^{-1}) t^{-1} (a^{-1}) \\
= \xi(t^t x^t (a^{-1}) t^t a^t t^t (a^{-1}) \xi(t^{-1}) a^t (a^{-1}) \\
= \xi(t^t x^t (a^{-1}) \xi(t^{-1}) t^{-1}) (a^{-1})
\]

and we are done. □
Proposition 12. i) Let \( (x, \xi) \in Z_1(G_\ast) \). Then for any \( y \in G_1 \) we have an arrow
\[
yx \xymatrix{ \xi(y) \ar[r] & xy }
\]
in the category \( \text{Cat}(G_\ast) \).

ii) By varying \( y \) one obtains a natural isomorphism of functors
\[
(-) \cdot x \xymatrix{ \xi \ar[r] & x \cdot (-) }.
\]

iii) The pair \( (x, \bar{\xi}) \) is an object of \( Z(\text{Cat}(G_\ast)) \).

iv) The map
\[
Z_1(G_\ast) \to \text{Ob}(Z(\text{Cat}(G_\ast))), \quad (x, \xi) \mapsto (x, \bar{\xi})
\]
is a bijection.

v) If \( (x, \xi), (y, \eta) \in Z_1(G_\ast) \) and \( a \in G_2 \) is an element such that \( y = \partial(a)x \), then the arrow \( x \xymatrix{ \alpha \ar[r] & y } \) of \( \text{Cat}(G_\ast) \) defines a morphism in \( Z(\text{Cat}(G_\ast)) \) iff
\[
(y, \eta(t)) = \xi = \delta(a)(x, \xi).
\]

Hence the map \( Z_1(G_\ast) \to \text{Ob}(Z(\text{Cat}(G_\ast))) \) constructed in iv) extends to an isomorphism of monoidal categories
\[
\text{Cat}(Z_*(G_\ast)) \to Z(\text{Cat}(G_\ast)).
\]

Proof. i) We have to check that \( \partial(\xi(y))yx = xy \), which is a consequence of the equality \([1]\).

ii) Take a morphism \( y \xymatrix{ \alpha \ar[r] & z } \) in \( \text{Cat}(G_\ast) \). Consider the diagram
\[
\begin{array}{ccc}
yx & \xymatrix{ \xi(y) \ar[r] & xy } \\
\alpha \cdot 1_x & \xymatrix{ \xi(z) \ar[r] & xz } \\
1_x \cdot \alpha & \\
\end{array}
\]

Since \( a \cdot 1_x = a, 1_x \cdot a = a \) and \( z = \partial(a)y \), we have
\[
\xi(z) \circ (a \cdot 1_x) = \xi(\partial(a)y)a = \xi(\partial(a)) \partial a \xi(y)a = (a \cdot a^{-1}) (a \xi(y)a^{-1})a = x \xi(y) = (1_x \cdot a) \circ \xi(y).
\]

Thus the above diagram commutes and the result follows.

iii) We have to check the commutativity of the triangle in Section 2.3. It requires us to verify that
\[
(\xi(y) \cdot 1_z) \circ (1_y \cdot \xi(z)) = \xi(yz)
\]
which is an immediate consequence of the equality \([11]\), because \( \xi(y) \cdot 1_z = \xi(y) \) and \( 1_y \cdot \xi(z) = y \xi(z) \).

iv) Take any object \( (x, \bar{\xi}) \) of the category \( Z(\text{Cat}(G_\ast)) \). By definition \( x \) is an object of \( \text{Cat}(G_\ast) \), thus \( x \in G_1 \). Moreover \( \xi \) is a natural isomorphism \( (-) \cdot x \xymatrix{ \xi \ar[r] & x \cdot (-) } \).

The value of \( \xi \) on an object \( y \) is a morphism \( yx \xymatrix{ \alpha \ar[r] & xy } \) of \( \text{Cat}(G_\ast) \) denoted by \( \xi_y \).

Thus \( \xi_y \in G_2 \) satisfies the condition \( \partial \xi_y yx = xy \). It follows that \( y \xymatrix{ \alpha \ar[r] & \xi_y } \) defines a map \( \xi : G_1 \to G_2 \) satisfying the condition \([9]\). By definition of the centre, we have
\[
\xi_{yz} = \xi_y \circ \xi_z
\]
and the condition \([11]\) follows. This implies that \( \xi_1 = 1 \). Finally, the condition \([11]\) follows from the naturality of \( \xi \). In fact, the commutative square in the proof of part ii) implies that for any \( y \xymatrix{ \alpha \ar[r] & z } \) we have
\[
\xi(z)a = x a \xi(y).
\]
We can take \( y = 1 \). Then \( z = \partial a \) and hence
\[
\xi(\partial a)a = x a
\]
proving (10). Thus we have constructed the inverse map
\[
\text{Ob}(\mathcal{Z}(\text{Cat}(\mathcal{G}_*))) \to \mathbf{Z}_1(\mathcal{G}_*)
\]
proving the statement.

v) The diagram
\[
\begin{array}{c}
\xymatrix{ z \otimes x & \xrightarrow{\xi(z)} & x \otimes z \\
1_z \otimes x & & a \otimes 1_z \\
z \otimes y & \xrightarrow{\eta(z)} & y \otimes z }
\end{array}
\]
commutes iff \( \eta(z) = a\xi(z)(\partial a)^{-1} \) and this happens for all \( z \) iff
\[
(y, \eta) = \delta(a)(x, \xi).
\]

In fact, we have
\[
\delta(a)(x, \xi) = (\partial(a)x, z \mapsto \partial(a)\xi(z)\zeta_a(z))
\]
\[
= (\partial(a)x, z \mapsto a\xi(z)(\partial a)^{-1}).
\]

This fact already implies that the functor \( \text{Cat}(\mathcal{Z}_*(\mathcal{G}_*)) \to \mathcal{Z}(\text{Cat}(\mathcal{G}_*)) \) is an isomorphism of categories. Hence, we only need to check that the monoidal structures in both categories are compatible under this isomorphism. Assume that \( (x, \xi), (y, \eta) \in \mathbf{Z}_1(\mathcal{G}_*) \). Denote by \( (x, \bar{\xi}) \) and \( (y, \bar{\eta}) \) the corresponding objects of \( \mathcal{Z}(\text{Cat}(\mathcal{G}_*)) \). According to the definition of the monoidal structure on \( \mathcal{Z}(\text{Cat}(\mathcal{G}_*)) \), we have
\[
(x, \bar{\xi}) \cdot (y, \bar{\eta}) = (xy, \bar{\zeta}),
\]
where \( \bar{\zeta}_z \) is the composite
\[
\begin{array}{c}
xzy & \xrightarrow{\xi(z)^{-1}y} & xzy \\
1_x \cdot \eta(y) & \xrightarrow{\xi(z)} & xyz
\end{array}
\]
which is the same as \( x \eta(y)\xi(z) \) and we are done.

\[
\square
\]

**Corollary 13.** The map \( \mathbb{G}_2 \xrightarrow{\delta} \mathbf{Z}_1(\mathcal{G}_*) \) together with the bracket
\[
\{(x, \xi), (y, \eta)\} := \xi(y)
\]
defines a braided crossed module structure on
\[
\mathcal{Z}_*(\mathcal{G}_*) := (\mathbb{G}_2 \xrightarrow{\delta} \mathbf{Z}_1(\mathcal{G}_*)).
\]

**Proof.** Since the monoidal structure on \( \mathcal{Z}(\text{Cat}(\mathcal{G}_*)) \) is braided, the same will be true for \( \mathcal{Z}_*(\mathcal{G}_*) \), if we transport the structure under the constructed isomorphism. Hence the result follows from Lemma [5]

\[
\square
\]

**3.2. 2-categorical meaning of the main construction.** To summarise, we see that the crossed module \( \partial : \mathbb{G}_2 \to \mathbb{G}_1 \) fits in a commutative diagram
\[
\begin{array}{ccc}
\mathbb{G}_2 & \xrightarrow{\delta} & \mathbf{Z}_1(\mathcal{G}_*) \\
\downarrow{id} & & \downarrow{z_1} \\
\mathbb{G}_2 & \xrightarrow{\partial} & \mathbb{G}_1
\end{array}
\]
where both $G_2 \xrightarrow{\delta} Z_2(G_2)$ and $Z_2(G_2) \xrightarrow{\zeta} \mathcal{G}_2$ are crossed modules, the first one is even a braided crossed module. The vertical arrows form a morphism of crossed modules

$$z_* : Z_*(G_*) \to G_*.$$ 

The crossed module $Z_2(G_2) \xrightarrow{\zeta} \mathcal{G}_2$ is denoted by $G_2/\mathcal{Z}_2(G_2)$. It is an easy exercise to see that one has an exact sequence

$$0 \to \pi_1(Z_*(G_*)) \to \pi_1(G_*) \to \pi_1(G_2/\mathcal{Z}_2(G_2)) \to$$

$$\to \pi_0(Z_*(G_*)) \to \pi_0(G_*) \to \pi_0(G_2/\mathcal{Z}_2(G_2)) \to 0.$$

This suggests that we can think of $Z_2(G_2)$, $G_*$ and $G_2/\mathcal{Z}_2(G_2)$ as fitting into a 2-mathematical analogue of a short exact sequence

$$0 \to Z_*(G_*) \to G_* \to G_2/\mathcal{Z}_2(G_2) \to 0.$$

However, in this paper, we will not give a formal definition of short exact sequences in the 2-mathematical sense.

### 3.3. On homotopy groups of $Z_*(G_*)$

Let $G_*$ be a crossed module. We have constructed a BCM $Z_*(G_*)$. In this section we investigate the homotopy groups $\pi_i(Z_*(G_*))$, $i = 1, 2$ of this crossed module. The case $i = 2$ is easy and the answer is given by the following lemma.

**Lemma 14.** Let $G_*$ be a crossed module. Then

$$\pi_2(Z_*(G_*)) \cong H^0(\pi_1(G_*), \pi_2(G_*)).$$

**Proof.** By definition $a \in \pi_1(Z_*(G_*))$ iff $\delta(a) = (1, 1)$, thus when $\partial(a) = 1$ and $\zeta_\alpha(t) = 1$ for all $t \in G_1$. These conditions are equivalent to the conditions $a \in \pi_2(G_*)$ and $t^a = a$ for all $t \in G_1$ and hence the result.

On the other hand, to obtain information on $\pi_1(Z_*(G_*))$, we need to fix some notation. The main result is formulated in terms of an exact sequence, see Proposition 17.

For a group $G$, we let $Z(G)$ denote the centre of $G$. Moreover, if $H$ is a $G$-group, we set

$$st_H(G) = \{g \in G| {}^g h = h \text{ for all } h \in H\}.$$

It is obviously a subgroup of $G$. The intersection of these two subgroups is denoted by $Z_H(G)$. Thus $g \in Z_H(G)$ iff $^g x = x$ and $^g h = h$ for all $x \in G$ and $h \in H$.

Let $G_*$ be a crossed module. In this case we have defined two groups

$$Z_{\pi_2(G_*)}(\pi_1(G_*)) \quad \text{and} \quad Z_{\pi_2(G_*)}(G_1).$$

We will come back to the second group in Section 3.5. Now we relate the first group to $\pi_1(Z_*(G_*))$. To this end we fix some notation. If $x \in G_1$, then we let $\mathrm{cl}(x)$ denote the class of $x$ in $\pi_1(G_*)$. Take now an element $(x, \xi) \in Z_1(G_*)$. Accordingly, $\mathrm{cl}(x, \xi)$ denotes the class of $(x, \xi)$ in $\pi_1(Z_*(G_*))$. I claim that the class $\mathrm{cl}(x) \in \pi_1(G_*)$ belongs to

$$Z_{\pi_2(G_*)}(\pi_1(G_*)),$$

in fact, the equation of Definition 10 implies that $[x, y] = \partial\xi(y)$ and hence $\mathrm{cl}([x, y]) = 0$ in $\pi_0(G_*)$. It follows that $\mathrm{cl}(x)$ is central in $\pi_1(G_*)$. Moreover, the equation of Definition 10 tells us that $\xi(\partial a) = a(\partial a)^{-1}$. In particular, if $a \in \pi_2(G_*)$ (i.e. $\partial(a) = 1$) then $a^a = a$ and the claim follows. Thus we have defined the group homomorphism

$$Z_1(G_*) \xrightarrow{\omega'} Z_{\pi_2(G_*)}(\pi_1(G_*))$$

by $\omega'(x, \xi) := \mathrm{cl}(x)$. 

Proposition 15.  

i) For any crossed module $G_*$, the composite map 
\[
G_2 \xrightarrow{\delta} Z_1(G_*) \xrightarrow{\omega'} Z_{\pi_2(G_*)}(\pi_1(G_*))
\]
is trivial. Hence the map $\omega'$ induces a group homomorphism 
\[
\omega : \pi_1(Z_*(G_*)) \to Z_{\pi_2(G_*)}(\pi_1(G_*)).
\]

ii) For a 1-cocycle $\phi : \pi_1(G_*) \to \pi_2(G_*)$, the pair $(1, \tilde{\phi}) \in Z_1(G_*)$, where $\tilde{\phi} : G_1 \to G_2$ is the composite map 
\[
G_1 \to \pi_1(G_*) \xrightarrow{\delta} \pi_2(G_*) \hookrightarrow G_2.
\]

Moreover, the assignment $\phi \mapsto \text{cl}(1, \tilde{\phi})$ induces a group homomorphism 
\[
f : H^1(\pi_1(G_*), \pi_2(G_*)) \to \pi_1(Z_*(G_*))
\]

iii) These maps fit in an exact sequence 
\[
0 \to H^1(\pi_1(G_*), \pi_2(G_*)) \xrightarrow{\tilde{f}} \pi_1(Z_*(G_*)) \xrightarrow{\omega} Z_{\pi_2(G_*)}(\pi_1(G_*)).
\]

Proof.  

i) Take $a \in G_2$. By construction $\delta(a) = (\partial(a), \zeta_a)$. Hence 
\[
\omega'(\delta(a)) = \text{cl}(\partial(a)) = 1.
\]

ii) Since $\phi$ is a 1-cocycle, $\tilde{\phi}$ satisfies the condition [11] of Definition 9. Next, the values of $\tilde{\phi}$ belong to $\pi_2(G_*)$, so $\partial \tilde{\phi} = 1$ and the condition [11] follows. Finally, 
\[
\tilde{\phi}(\partial a) = \phi([\partial a]) = \phi[1] = 1
\]
and the map 
\[
\tilde{\phi}(\partial a) = \phi([\partial a]) = \phi[1] = 1
\]
also holds. It remains to show that if $\phi(t) = b(t)b^{-1}$, for an element $b \in \pi_2(G_*)$, then $\text{cl}(1, \tilde{\phi}) = 1$, but this follows from the fact that $\delta(b) = (1, \phi)$.

iii) Exactness at $H^1(\pi_1(G_*), \pi_2(G_*))$: Assume $\phi : \pi_1(G_*) \to \pi_2(G_*)$ is a 1-cocycle such that $\text{cl}(1, \tilde{\phi})$ is the trivial element in $\pi_1(Z_*(G_*)$. That is, there exists a 
\[
c \in G_2
\]
such that $\delta(c) = (1, \tilde{\phi})$. Thus $\partial(c) = 1$ and $\phi(t) = c(t)c^{-1}$. So $c \in \pi_2(G_*)$ and the second equality implies that the class of $\phi$ is zero in $H^1(\pi_1(G_*), \pi_2(G_*))$, proving that $f$ is a monomorphism.

Exactness at $\pi_1(Z_*(G_*))$: First take a cocycle $\phi : \pi_1(G_*) \to \pi_2(G_*)$. Then 
\[
\omega' \circ f([\phi]) = \omega(\text{cl}(1, \tilde{\phi})) = \text{cl}(1) = 1.
\]

Take now an element $(x, \xi) \in Z_1(G_*)$ such that $\text{cl}(x, \xi) \in \text{Ker}(\omega)$. Thus $x = \partial a$ for $a \in G_2$. Then $[(x, \xi)] = [(y, \eta)]$, where $(y, \eta) = (x, \xi)\delta(a)^{-1}$. Since 
\[
y = x\partial a^{-1} = 1,
\]
we see that $\partial \eta(t) = [1, t] = 1$ and $\eta(\partial c) = 1$. So $\eta = \tilde{\phi}$, where $\phi : \pi_1(G_*) \to \pi_2(G_*)$ is a 1-cocycle. Thus $f([\phi]) = [(1, \eta)] = [(x, \xi)]$ and exactness at $\pi_1(Z_*(G_*))$ follows.

Our next aim is to define the homomorphism 
\[
g : Z_{\pi_2(G_*)}(\pi_1(G_*)) \to H^2(\pi_1(G_*), \pi_2(G_*))
\]
and extend the exact sequence constructed in part iii) of Proposition 15. This will be based on the following result.

Proposition 16. Take an element $x \in G_1$ such that $\text{cl}(x)$ is in the centre of $\pi_1(G_*)$. 

Then there exists a map $\psi : G_1 \to G_2$ such that $\partial \psi(t) = [x, t]$ for all $t \in G_1$. Consider the expression 
\[
\tilde{\theta}(s, t) := \psi(s)^* \psi(t) \psi(st)^{-1}.
\]
Then $\tilde{\theta}(s, t) \in \pi_2(G_*)$ and the map $\tilde{\theta}$ satisfies the 2-cocycle condition: 
\[
\tilde{\theta}(s, t)\tilde{\theta}(s, tr) = \tilde{\theta}(s, t)\tilde{\theta}(st, r).
\]

Moreover, the class of $\tilde{\theta}$ in $H^2(G_1, \pi_2(G_*))$ is independent of the choice of $\psi$. 

Proof. By assumption, we have \([x, t] \in \text{Im}(\partial)\) for all \(t \in G_1\). Thus we can choose 
\(\psi(t) \in G_2\) such that \([x, t] = \partial(\psi(t))\). By definition we have the equality 
\[\bar{\theta}(s, t)\psi(st) = \psi(s^*\psi(t)).\]
Applying \(\partial\) to this equation, we obtain 
\[\partial(\bar{\theta}(s, t))[x, st] = [x, s] \cdot [x, t].\]
Since \([x, yz] = [x, y] \cdot [x, z]\) holds in any group, it follows that \(\partial(\bar{\theta}(s, t)) = 1\). Thus \(\bar{\theta}(s, t) \in \pi_2(G_\ast)\). In particular, \(\bar{\theta}(s, t)\) is a central element of \(G_2\).
To show that the 2-cocycle condition holds, we can write 
\[\bar{\theta}(s, t)\bar{\theta}(st, r)\psi(str) = \bar{\theta}(s, t)\bar{\theta}(st, r)\psi(st^*r)\]
\[= \psi(s^*\psi(t)^{st}\psi(r))\]
\[= \psi(s^*\psi(t)\psi(tr)).\]
Since \(^*\bar{\theta}\) is central, we obtain
\[\bar{\theta}(s, t)\bar{\theta}(st, r)\psi(str) = \theta(t, r)\bar{\theta}(s, tr)\psi(str).\]
After cancelling out \(\psi(str)\), we obtain the expected property.
If \(\psi_1\) also satisfies the condition \(\partial\psi_1(s) = [x, t]\), we see that \(\psi_1(s) = \psi(s)\phi(s)\), where \(\phi\) takes values in \(\pi_2(G_\ast)\). Since the last subgroup is closed under the action of \(G_1\) we also have \(^*\phi(t) \in \pi_2(G_\ast)\). In particular both \(\phi(s)\) and \(^*\phi(t)\) are central. It follows that 
\[\bar{\theta}_1(s, t) = \bar{\theta}(s, t)\phi(s)^*\phi(t)\phi(st)^{-1}\]
and hence the result.

Our next aim is to show that if one chooses the map \(\psi\) from Proposition \ref{prop:16} more carefully, then one can achieve that \(\bar{\theta}\) factors through \(\pi_1(G_\ast) \times \pi_1(G_\ast)\) and hence one obtains an element in \(H^2(\pi_1(G_\ast), \pi_2(G_\ast))\). Recall that \(cl\) denotes the canonical surjection 
\[G_1 \xrightarrow{cl} \pi_1(G_\ast) = \frac{G_1}{\text{Im}(\partial)}.\]

**Proposition 17.** Take an element \(x \in G_1\) such that \(cl(x) \in \mathbb{Z}(\pi_1(G_\ast), \pi_1(G_\ast))\) and choose a section \(\alpha : \pi_1(G_\ast) \to G_1\) of \(cl\) such that \(\alpha(1) = 1\). Then there is a map 
\[\beta : \pi_1(G_\ast) \to G_2\]
such that 
\[(13) \quad \partial(\beta(z)) = [x, \alpha(z)]\]
for all \(z \in \pi_1(G_\ast)\). Moreover, the formula 
\[(14) \quad \psi(s) = x(a)\beta(cl(s))^{-1}\]
gives rise to a well-defined map \(\psi : G_1 \to G_2\). Here \(a\) is an element in \(G_2\) for which 
\[(15) \quad s = \partial(a)\alpha(cl(s)).\]
Then \(\psi\) and the corresponding \(\bar{\theta}\) (see Proposition \ref{prop:17}) satisfy the following properties:
\begin{itemize}
  \item[i)] \(\partial\psi(t) = [x, t]\).
  \item[ii)] \(\psi(\partial a) = x(a) \cdot a^{-1}\).
  \item[iii)] \(\bar{\theta}(\partial(a), \alpha(z)) = 1\).
  \item[iv)] \(\bar{\theta}(\alpha(z), \partial(a)) = 1\).
  \item[v)] \(\bar{\theta}(\partial(a), \partial(b)) = 1\).
  \item[vi)] The function \(\bar{\theta}\) factors through \(\pi_1(G_\ast) \times \pi_1(G_\ast) \to \pi_2(G_\ast)\), and hence defines a class \(\theta\) in \(H^2(\pi_1(G_\ast), \pi_2(G_\ast))\).
  \item[vii)] The class \(\theta\) is independent of the choice of \(\psi\) and \(\alpha\).
\end{itemize}
viii) The assignment \( x \mapsto \theta \) defines a group homomorphism
\[
g : Z_{\pi_2(\mathcal{A}_s)}(\pi_1(\mathcal{G}_s)) \to H^2(\pi_1(\mathcal{G}_s), \pi_2(\mathcal{G}_s)),
\]
which fits in the exact sequence
\[
0 \to H^1(\pi_1(\mathcal{G}_s), \pi_2(\mathcal{G}_s)) \xrightarrow{\partial} \pi_1(\mathcal{Z}_0(\mathcal{G}_s)) \xrightarrow{\alpha} Z_{\pi_2(\mathcal{G}_s)}(\pi_1(\mathcal{G}_s)) \xrightarrow{\partial} H^2(\pi_1(\mathcal{G}_s), \pi_2(\mathcal{G}_s)).
\]

**Proof.** By the condition on \( x \) also have
\[
\partial(m) = \partial(m) \quad \text{for all } y \in \mathcal{G}_1. \quad \text{Thus} \quad [x, y] = \partial(m) \quad \text{for some } m \in \mathcal{G}_2.
\]
For a given \( y \in \pi_1(\mathcal{G}_s) \), we take \( y = \alpha(z) \) and denote the \( \mathcal{G}_s \) elements by \( \beta(z) \). Then we have \([x, y] = \partial(y)\).

Clearly any element \( s \in \mathcal{G}_1 \) can be written as in the formula (16) for some \( \alpha \in \mathcal{G}_2 \). Let us show that the expression \( x\alpha \psi(\alpha) \alpha^{-1} \) in the equality (16) does not depend on the choice of \( \alpha \) in the equality (16). In fact, if we have another decomposition \( \alpha = \partial(b) \alpha \), then \( \partial(b) = \partial(a) \). It follows that \( b = a \cdot c \), where \( \partial(c) = 1 \), and \( c = \pi_2(\mathcal{G}_s) \) and in particular it is central and since \( \partial(x) \in Z_{\pi_2(\mathcal{A}_s)}(\pi_1(\mathcal{G}_s)) \), we also have \( \beta(z) \).

Hence the function \( \psi \) is well-defined.

Now we verify the properties i)-viii).

i) Observe that
\[
\partial(\psi(z)) = \partial(x\alpha \psi(\alpha) \alpha^{-1}) = x\partial(a)x^{-1}[x, \alpha(\alpha)]\partial(a^{-1}) = x\partial(a)\alpha(\alpha)x^{-1}\alpha(\alpha^{-1})\partial(a^{-1}) = [x, \partial(a)\alpha(\alpha)]
\]
Since \( s = \partial(a)\alpha(\alpha) \), we see that \( \psi \) satisfies the condition i).

ii) If \( s = \partial(a) \), we have \( \alpha(\alpha) = 1 \) and thus \( \psi(\partial(a)) = \partial(a) \cdot a^{-1} \) and hence the condition ii) also holds.

iii) By taking \( a = 1 \) and \( s = \alpha(z) \) in (14), we obtain \( \psi(\alpha(z)) = \alpha(z) \). By property ii) we have \( \psi(\partial(a)) = \partial(a) \cdot a^{-1} \). Therefore
\[
\psi(\partial(a))\alpha(\alpha) = \alpha(\alpha)\cdot a^{-1}\alpha(\alpha) = \alpha(a) \cdot a^{-1} = \psi(\partial(a)\alpha(z)) \quad \text{and the condition iii) follows.}
\]

iv) We need to show that
\[
\psi(\alpha(z)\partial(a)) = \psi(\alpha(z))\alpha(z)\psi(\partial(a)).
\]

The RHS is equal to \( \beta(z) \alpha(z) \). Since \( \alpha(z)\partial(a) = \partial(b)\alpha(z) \), where \( b = \alpha(z) \), we can rewrite the LHS:
\[
\psi(\alpha(z)\partial(a)) = \psi(\partial(b)\alpha(z)) = \beta(b)\beta(z) = \alpha(\alpha) \beta(z) = \alpha(z) \alpha(z) \beta(z) \alpha(z).
\]

Comparing these expressions we see that the equality (14) is equivalent to
\[
x\alpha(z) \alpha(z) \alpha(z) = \beta(z) \alpha(z) \alpha(z) \alpha(z).
\]

We have
\[
\beta(z) \alpha(z) = \beta(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) = \beta(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z) \alpha(z)
\]
and part iv) is proved. Here we used the fact that \( \partial \beta(z) = [x, \alpha(z)] \).
v) We have $\psi(\partial(a)\partial(b)) = \psi(\partial(ab)) = x(ab)\alpha^{-1}$. On the other hand

$$\psi(\partial(a))^{\partial(a)}\psi(\partial(b)) = x(a) a^{-1} a x b b^{-1} a^{-1} = x(ab)\alpha^{-1}$$

and the result follows.

vi) In the 2-cocycle condition from Proposition 10, we first put $s = \partial(a)$, $t = \partial(b)$ $r = \alpha(\zeta)$ to obtain

$$\delta(a)\delta(b), \alpha(\zeta)\delta(\partial(a), \partial(b)\alpha(\zeta)) = \delta(\partial(a), \partial(b)\delta(\partial(a), \partial(\zeta)))$$

Use the relations iii)-v) to obtain $\delta(\partial(a), \partial(b)\alpha(\zeta)) = 1$. Thus $\delta(\partial(a), s) = 1$ for all $s \in G$. Quite similarly $\delta(s, \partial(a)) = 1$ for all $s$. Now we put $r = \partial(a)$ to obtain $\delta(s, t\partial(a)) = \delta(s, t)$, showing that the map $\delta(s, -)$ factors through the group $\pi_1(G)$. Similarly for the first argument.

vii) Now we prove that the class $\theta$ is independent of the choices which we made, namely of $\alpha$, $\beta$ and $\alpha$ in equality (15). We already proved that for chosen $\alpha$ and $\beta$ the function $\psi$ (and hence $\delta$) is independent of the choice of $\alpha$. Assume $\alpha$ is chosen and we have $\beta_1$ and $\beta$ for which the equality (15) holds. Then there is a map $\gamma : \pi_1(G) \to G_2$ such that $\beta_1(x) = \gamma(x)\beta(x)$. Since $\delta(\gamma(x)) = 1$, we see that $\gamma(x)$ and $\gamma(x)$ are central. It follows that

$$\psi_1(s) = x a \beta_1(\text{cl}(s)) a^{-1} = \psi(s) \gamma(\text{cl}(s)).$$

It follows that

$$\bar{\delta}_1(s, t) = \delta(s, t) \gamma(\text{cl}(s)) \gamma(\text{cl}(t)) \gamma(\text{cl}(s) \cdot \text{cl}(t))^{-1}$$

and thus both $\delta$ and $\bar{\delta}_1$ define the same class in $H^2(\pi_1(G), G_2(G))$. Consider now the case when we have chosen another section of $\text{cl}$, say $\alpha_1$. Then there exists a function $\eta : \pi_1(G) \to G_2$ for which

$$\alpha_1(z) = \delta(\eta(z)) \alpha(z).$$

As a function $\beta_1$ satisfying the relation $\partial(\beta_1(z)) = [x, \alpha_1(z)]$ we can choose

$$\beta_1(z) = \gamma(z) \beta(z) \eta(z)^{-1}.$$

In fact, we have

$$\partial(\beta_1(z)) = \partial(x \eta(z) \beta(z) \eta(z)^{-1})$$

$$= x \partial(\eta(z)) x^{-1} z \alpha(z) \partial(\eta(z))^{-1}$$

$$= x \partial(\eta(z)) \alpha(z) x^{-1} z \alpha(z)^{-1} \partial(\eta(z))^{-1}$$

$$= x a \alpha(z) x^{-1} \alpha(z)^{-1} \partial(\eta(z))^{-1}$$

$$= [x, \alpha_1(z)].$$

Next, we set $a_1 = a \eta(\text{cl}(s))^{-1}$. Then we have $s = \partial(a_1) \alpha_1(\text{cl}(s))$. Now we can write

$$\psi_1(s) = x a_1 \beta_1(\text{cl}(s)) a_1^{-1}$$

$$= x a_1^2 \eta(\text{cl}(s)) \beta(\text{cl}(s)) \eta(\text{cl}(s))^{-1} a_1^{-1}$$

$$= x (a_1 \eta(\text{cl}(s))) \beta(\text{cl}(s)) (a_1 \eta(\text{cl}(s)))^{-1}$$

$$= \psi(s)$$

and part vii) is proved.

viii) Exactness at $Z_2(G_2(\pi_1(G)))$: Take $(x, \xi) \in Z_1(G)$. Since $\omega'(x, \xi) = \text{cl}(x)$, we can choose $\psi = \xi$ for $g(\text{cl}(x))$. Clearly $\delta = 1$ for this $\psi$ and hence $g \circ \omega = 1$. Take now $x \in G$ such that $\text{cl}(x) \in Z_2(G_2(\pi_1(G)))$. Assume $g(\text{cl}(x)) = 1$. Thus there exists a function $\phi : \pi_1(G) \to \pi_2(G_2)$ for which

$$\bar{\delta}(\text{cl}(s), \text{cl}(t)) = \psi(\text{cl}(s)) \phi(\text{cl}(t)) \phi(\text{cl}(st))^{-1}. $$
If $\phi$ takes values in $\pi_2(G_*)$, we see that $\phi(\text{cl}(s))$ is central for all $s \in G_1$. As $x$ acts trivially on $\pi_2(G_*)$, it follows that the function $\psi'(t) = \psi(t)\phi(t)$ also satisfies the conditions in i) and ii) and, moreover, $\partial' = 1$, meaning that $\psi'$ is a 1-cocycle. Thus $(x, \psi') \in Z_1(G_*)$ and exactness follows.

\[ \square \]

3.4. Relation to nonabelian cohomology. Crossed modules can be used to define low dimensional non-abelian cohomologies of groups. This was first observed by Dedecker in the 60’s (see for instance [9] and references therein) and then developed by D. Guin [9, Breen, Borovoi [2], Noohi [14]. See also [16].

Let $G_*$ be a crossed module. According to Borovoi, the group $H^0(G_1, G_*)$ is defined by

\[ H^0(G_1, G_*) = \{ a \in G_2 \mid \partial a = 1 \text{ and } x a = a \text{ for all } x \in G_1 \}. \]

Clearly, $H^0(G_1, G_*) = H^0(\pi_1(G_*), \pi_2(G_*))$. It is a central subgroup of $G_2$.

In order to define the first cohomology group $H^1(G_1, G_*)$, we first introduce the group $\text{Der}_G(G_1, G_1)$. Elements of $\text{Der}_G(G_1, G_1)$ are pairs $(g, \gamma)$ (see [9, Definition 1.1]), where $g \in G_1$ and $\gamma : G_1 \to G_2$ is a function for which two conditions hold:

\[ \gamma(g h) = \gamma(g)g_\gamma(h) \quad \text{and} \quad \partial \gamma(t) = [g, t]. \]

Here $g, h, t \in G_1$. Comparing with Definition 6 we see that these are exactly the conditions [1] and [11] of Definition 6. Hence $Z_1(G_*) \subset \text{Der}_G(G_1, G_2)$. In fact, $Z_1(G_*)$ is a subgroup of $\text{Der}_G(G_1, G_2)$, where the group structure on $\text{Der}_G(G_1, G_2)$ is defined as follows.

If $(g, \gamma), (g', \gamma') \in \text{Der}_G(G_1, G_1)$ then $(gg', \gamma * \gamma') \in \text{Der}_G(G_1, G_2)$ (see [9, Lemme 1.2.1]), where $\gamma * \gamma'$ is defined by

\[ (\gamma * \gamma')(t) = \gamma_g(t) \cdot \gamma(t). \]

Thus $\text{Der}_G(G_1, G_2)$ is equipped with a binary operation

\[ (g, \gamma)(g', \gamma') \mapsto (gg', \gamma * \gamma'). \]

Thanks to [9, Lemme 1.2.2] in this way one obtains a group structure on $\text{Der}_G(G_1, G_2)$.

The group $H^1(G_1, G_*)$ is defined as the quotient $\text{Der}_G(G_1, G_2) / \sim$ where

\[ (g, \gamma) \sim (g', \gamma') \]

iff there exists $a \in G_2$ such that $\gamma'(t) = a^{-1} \gamma(t) \partial(a)$ for all $t \in G_1$ and $g' = \partial(a)^{-1}g$.

The following fact is a direct consequence of the definition.

Lemma 18. One has a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & H^0(G_1, G_*) & \to & G_2 & \to & \text{Der}_G(G_1, G_2) & \to & H^1(G_1, G_*) & \to & 1 \\
& \downarrow{\cong} & \uparrow{\text{id}} & \downarrow{} & \downarrow{} & \downarrow{} & \downarrow{} & \downarrow{} & \downarrow{} & \downarrow{} & \downarrow{} \\
0 & \to & \pi_2(Z_*(G_*)) & \to & G_2 & \to & Z_1(G_*) & \to & \pi_1(Z_*(G_*)) & \to & 1 \\
\end{array}
\]

3.5. Comparison with Norrie’s centre. Let $G_*$ be a crossed module. Then we have

\[ H^0(G_1, G_2) = \{ a \in G_2 \mid x a = a \text{ for all } x \in G_1 \}. \]

Take $a \in H^0(G_1, G_2)$. I claim that

\[ \partial(a) \in Z_2(G_1) = Z(G_1) \cap st_{G_2}(G_1). \]

In fact, we have

\[ x \partial(a)x^{-1} = \partial(x a) = \partial(a) \]

for all $x \in G_1$. Hence $\partial(a) \in Z(G_1)$. For any $b \in G_2$ we have $\partial b a = a$, thus $a \in Z(G_2)$.

It follows that $\partial a b = b$ and hence

\[ \partial(a) \in st_{G_2}(G_2) = \{ x \in G_1 \mid x a = a \text{ for all } a \in G_2 \} \]
and the claim follows.

Hence we have a homomorphism of abelian groups
\[ \mathcal{H}^0(\mathcal{G}_1, \mathcal{G}_2) \xrightarrow{\partial} \mathcal{Z}_2(\mathcal{G}_1), \]
which can be considered as a crossed module with trivial action of the target group on the source. This crossed submodule is denoted by \( \mathcal{Z}_2^{N\text{or}}(\mathcal{G}_1) \) and is called Norrie’s centre of \( \mathcal{G} \) [15, p. 133]. Take \( x \in \mathcal{Z}_2(\mathcal{G}_1) \). Then \((x, 1) \in \mathcal{Z}_1(\mathcal{G}_1)\), where \( 1 \) is the constant map \( \mathcal{G}_1 \to \mathcal{G}_2 \) with value 1. Clearly \( j_1(x) = (x, 1) \) defines an injective homomorphism \( j_1 : \mathcal{Z}_2(\mathcal{G}_1) \to \mathcal{Z}_1(\mathcal{G}_1) \). In fact, if we let \( j_2 \) denote the inclusion \( \mathcal{H}^0(\mathcal{G}_1, \mathcal{G}_2) \subset \mathcal{G}_2 \), one obtains an injective morphism of crossed modules \( j_* : \mathcal{Z}_2^{N\text{or}}(\mathcal{G}_1) \to \mathcal{Z}_2(\mathcal{G}_1) \) which induces an isomorphism on \( \pi_2 \) and a monomorphism on \( \pi_1 \). In general \( j_* \) is not a weak equivalence, see Section [5.6].

3.6. **The centre of the crossed module** \( D_4 \to \text{Aut}(D_4) \). For any group \( G \) there is a crossed module \( \partial : G \to \text{Aut}(G) \), where \( \partial(g) \) is the inner automorphism corresponding to \( g \in G \). This crossed modules is denoted by \( \text{AUT}(G) \). The centre of \( \text{AUT}(G) \) is in a sense the “2-dimensional centre” of \( G \). We compute this centre for \( G = D_4 \), the dihedral group of order 8. Recall that \( D_4 \) is generated by \( a, b \) modulo the relations \( a^4 = 1 = b^2 \) and \( bab = a^3 \). Denote by \( \bar{D}_4 \) the second copy of the same group. To distinguish it from the previous one, we use \( \alpha \) and \( \beta \) for the same generators, but now considered as elements of \( \bar{D}_4 \). Define the homomorphism
\[ \partial : D_4 \to \bar{D}_4 \]
by \( \partial(a) = a^2 \), \( \partial(b) = \beta \). The group \( \bar{D}_4 \) acts on \( D_4 \) by
\[ \alpha a = a, \quad \alpha b = ab, \]
\[ \beta a = a^{-1}, \quad \beta b = b. \]
Then \( \partial : D_4 \to \bar{D}_4 \) is a crossed module and it is easy to check that it is isomorphic to \( \text{AUT}(D_4) \).

To describe the centre of \( \partial : D_4 \to \bar{D}_4 \), we first observe that there are unique crossed homomorphisms \( \xi, \eta, \theta : \bar{D}_4 \to D_4 \) for which
\[ \xi(\alpha) = a^2, \quad \xi(\beta) = a; \quad \eta(\alpha) = a, \quad \eta(\beta) = 1; \quad \theta(\alpha) = a^2, \quad \theta(\beta) = 1. \]
Then one checks that the pairs
\[ A = (\alpha, \xi), \quad B = (\beta, \eta), \quad C = (1, \theta) \]
belong to \( \mathcal{Z}_1(\partial) \). Here, for simplicity, we write \( \mathcal{Z}_1(\partial) \) instead of \( \mathcal{Z}_1(\partial : D_4 \to \bar{D}_4) \). One easily checks the following equalities
\[ (18) \quad C^2 = 1 = B^2 = A^4, \quad AC = CA, \quad BC = CB, \quad BAB = A^3. \]
It follows that in the exact sequence in Lemma [10] the last map is surjective, because the image contains generators \( \alpha, \beta \) and also it splits. It has the form
\[ 0 \to C_2 \to \mathcal{Z}_1(\partial) \xrightarrow{\partial} D_4 \to 0 \]
where the image of the nontrivial element of \( C_2 \) is \( C = (1, \theta) \). It follows that \( \mathcal{Z}_1(\partial) \cong C_2 \times D_4 \). In other words, \( \mathcal{Z}_1(\partial) \) is a group generated by \( A, B, C \) modulo the relations listed in [13].

With these notations, one easily checks that for \( \delta : D_4 \to \mathcal{Z}_1(\partial) \) one has
\[ \delta(a) = A^2 \text{ and } \delta(b) = BC. \]
Moreover, the corresponding BCM structure on
\[ \mathcal{Z}_1(\partial) = (\delta : D_4 \to \mathcal{Z}_1(\partial)) \]
is in fact a QRM (see Definition [2]) and uniquely determined by
\[ \{ A, A \} = a^2, \quad \{ A, B \} = a, \quad \{ A, C \} = 1, \]
\[ \{ B, A \} = a, \quad \{ B, B \} = 1, \quad \{ B, C \} = 1. \]
\[ \{ \bar{C}, \bar{A} \} = a^2, \quad \{ \bar{C}, \bar{B} \} = 1, \quad \{ \bar{C}, \bar{C} \} = 1. \]

It follows from Lemma 18 that the underlying crossed module structure on \( \delta : D_A \to Z_1(\partial) \) is completely described by \( \delta \) and
\[
A_a = a, \quad A_b = ab, \quad B_a = a^{-1}, \quad B_b = b, \quad C_a = a, \quad C_b = b.
\]

It follows from our description of \( Z_1(\partial) \) and \( \delta \) that
\[
\pi_1(Z_*(\partial)) \cong \begin{cases} C_2 \times C_2, & i = 1, \\ C_2, & i = 2. \end{cases}
\]

It is easy to see that
\[
\{1, a^2\} \to \{1\}
\]
is the Norrie’s centre of \( \partial : D_A \to D_A \) and hence the inclusion \( Z_*^{Nor}(\partial) \to Z_*(\partial) \) is not a weak equivalence.

Now we turn to the precrossed module \( Z_1(\partial) \xrightarrow{\pi_0} D_A \). Based on part i) of Lemma 8 we obtain
\[
^\alpha A = A, \quad ^\alpha B = A^2 B, \quad ^\alpha C = C, \quad ^\beta A = A^{-1}, \quad ^\beta B = B, \quad ^\beta C = C.
\]

4. Application in topology

4.1. The Whitehead centre and the main theorem. Let \( ZX \) be the centre of a CW-complex \( X \) as it is defined in the introduction. We also need the notion of the Whitehead centre of a pointed space \( (X, x_0) \), which is denoted by \( P(X, x_0) \), see [8]. By definition it is the subgroup of elements of \( \pi_1(X, x_0) \) which act trivially on \( \pi_k(X, x_0) \) for all \( k \geq 2 \).

Recall also that the evaluation at \( x_0 \) defines the homomorphism
\[
\pi_1(ZX, \text{id}_X) \to \pi_1(X, x_0),
\]
whose image is a central subgroup of \( \pi_1(X, x_0) \) known as the Gottlieb group \( G(X, x_0) \).

Gottlieb proved that \( G(X, x_0) \subset P(X, x_0) \), see [8, Theorem I.4]. For general \( X \), the inclusion \( G(X, x_0) \subset P(X, x_0) \) is strict, and the computation of \( G(X, x_0) \) is an interesting problem.

The following theorem identifies the group \( G(X, x_0) \) with an explicit subgroup of \( P(X, x_0) \) for \( X \) a connected CW-complex with \( \pi_1(X) = 0 \) for all \( i \geq 3 \).

**Theorem 19.** Let \( G_* = (G_2 \to G_1) \) be a crossed module such that \( G_1 \) is free and let \( X = BG_* \). Then we have an exact sequence
\[
0 \to G(X, x_0) \to Z_{\pi_2(G_*)}(\pi_1(G_*)) \xrightarrow{g} H^2(\pi_1(G_*), \pi_2(G_*)).
\]

Here \( P(X, x_0) \) coincides with \( Z_{\pi_2(G_*)}(\pi_1(G_*)) \) and \( g \) was constructed in Proposition 17.

The proof of Theorem 19 is given in Subsection 4.2 which is based on the technology of crossed complexes over groupoids [3].

4.2. Crossed complexes over groupoids. To define crossed complexes we first need to introduce the notion of a crossed module over a groupoid.

**Remark 20.** We are now following the convention of [3], where actions are defined on the right. This does not cause any problems, as we can translate a right action \( a^x \) to an equivalent left action via \( x_a := a^{x\mathbf{1}} \). Thus all the notions and results that we proved for left action have an equivalent formulation using right action.

A crossed module over a groupoid
\[
G_* = (G_2 \xrightarrow{\partial} G_1 \rightrightarrows G_0)
\]
consists of a groupoid \( (G_1 \rightrightarrows G_0) \), which will be denoted by \( \mathcal{G}_* \), together with a covariant functor \( G_2 : \mathcal{G} \rightarrow \text{Groups} \) and the collection of group homomorphisms

\[
\partial : G_2(g) \rightarrow G_1(g, g).
\]

Here \( g \in G_0 \) and \( G_2(g) \) denotes the value of \( G_2 \) on \( g \). Moreover, these data must satisfy the following identities:

\[
\partial(a^x) = x^{-1}\partial(a)x \quad \text{and} \quad a^{\partial(b)} = b^{-1}ab.
\]

Here \( x \in G_1(g, h) \), \( a, b \in G_2(g) \) and \( a^x \) as above denotes the image of \( a \) under the map \( G_2(x) : G_2(g) \rightarrow G_2(h) \).

So when \( G_0 \) is a one-element set, we recover the usual definition of a crossed module.

If \( g, h \in G_0 \) and \( x, y : x \rightarrow y \) are morphisms in \( \mathcal{G}_* \), then we write \( x \sim y \) if \( x = y\partial_2(z) \) for some \( z \in G_2(g) \). One easily sees that \( \sim \) is a congruence and hence we can form the quotient category, which is denoted by \( \pi_1(\mathcal{G}_*) \). Clearly \( \text{Ob}(\pi_1(\mathcal{G}_*)) = G_0 \). As usual with groupoids, \( \pi_1(\mathcal{G}_*, g) \) denotes the group of automorphisms of \( g \) in \( \pi_1(\mathcal{G}_*) \). Moreover, \( \pi_2(\mathcal{G}_*, g) \) denotes the group \( \text{Ker}(\partial : G_2(g) \rightarrow G_1(g, g)) \).

The set of connected components of \( \pi_1(\mathcal{G}_*) \) and \( \mathcal{G}_* \) are the same, which is denoted by \( \pi_0(\mathcal{G}_*) \). Varying \( g \) we see that the mappings \( g \mapsto \pi_2(\mathcal{G}_*, g) \) and \( g \mapsto \pi_1(\mathcal{G}_*, g) \) are functors \( \pi_1(\mathcal{G}_*) \rightarrow \text{Ab} \) and \( \pi_1(\mathcal{G}_*) \rightarrow \text{Groups} \) respectively.

A crossed complex is an algebraic structure that generalises the notion of crossed modules, extending them to a sequence of higher homotopical objects. Specifically, a crossed complex \( \mathcal{G}_* \) over a groupoid \( \mathcal{G}_* = (G_1 \rightrightarrows G_0) \) is defined by a sequence

\[
\cdots \rightarrow G_4 \xrightarrow{\partial} G_3 \xrightarrow{\partial} G_2 \xrightarrow{\partial} G_1 \rightrightarrows G_0,
\]

where \( G_2 \rightarrow G_1 \rightrightarrows G_0 \) is a crossed module over the groupoid \( \mathcal{G}_* \).

For dimensions \( n \geq 3 \), the components \( G_n \) are functors from \( \mathcal{G}_* \) to the category of abelian groups. Each \( \partial : G_n \rightarrow G_{n-1} \), \( n \geq 3 \) is a morphism of functors such that \( \partial \circ \partial = 0 \). The images of \( G_2 \) under \( \partial \) act trivially on all \( G_n \) for \( n \geq 3 \), which simplifies their algebraic structure by making \( G_n \) factor through a simpler groupoid \( \pi_1(\mathcal{G}_*) \).

One can define morphisms between crossed complexes that respect this hierarchy, creating the category \( \text{Crs} \) of crossed complexes. Important invariants associated with a crossed complex include:

- The Fundamental Groupoid \( \pi_1(\mathcal{G}_*) \), capturing path components and fundamental group-like properties.
- Higher Homology Groups \( H_n(\mathcal{G}_*, g) = \text{Ker}(\partial_n) / \text{Im}(\partial_{n+1}) \), defined for \( n \geq 2 \).

If the object set \( G_0 \) is a single point, the crossed complex \( \mathcal{G}_* \) is called reduced, or a crossed complex over a group. In this case, the complex simplifies, with \( \pi_1(\mathcal{G}_*) \) becoming a single group rather than a groupoid.

4.3. Homotopy relations of morphisms in \( \text{Crs} \). Let \( \alpha_\ast \) and \( \beta_\ast \) be morphisms between crossed complexes \( \mathcal{G}_\ast \rightarrow \mathcal{K}_\ast \). A homotopy \footnote{Definition 7.1.38, p. 218} from \( \alpha_\ast \) to \( \beta_\ast \) is the following data:

- a map \( \xi_0 : G_0 \rightarrow K_1 \) such that if \( g \in G_0 \) then \( \xi_0(g) \) is a morphism from \( \alpha_0(g) \) to \( \beta_0(g) \) in the groupoid \( K = (K_1 \rightrightarrows K_0) \); in particular we have

\[
\alpha_0(g) = s\xi(g),
\]

where \( s : G_1 \rightarrow G_0 \) is the source map;
• a function $\xi_1$ which assigns to each morphism $x : g \to h$ in $G_1$ an element $\xi_1(x) \in K_2(\beta_0(h))$ such that the diagram

\[
\begin{array}{ccc}
\alpha_0(g) & \xrightarrow{\alpha_1(x)} & \alpha_0(h) \\
\downarrow{\xi_0(g)} & & \downarrow{\xi_0(h)} \\
\beta_0(g) & \xrightarrow{\beta_1(x)} & \beta_0(h)
\end{array}
\]

commutes up to $\xi_1(x)$, that is

\[
\alpha_1(x) = \xi_0(g)^{-1} \partial \xi_1(x) \beta_1(x) \xi_0(g),
\]

and one also requires that

\[
\xi_1(xy) = \xi_1(x)^{\beta_1(y)} \cdot \xi_1(y)
\]

where $x$ and $y$ are composable morphisms in the groupoid $\bar{G}_*$;

• a function $\xi_n$, $n \geq 2$ which assigns to an object $g \in G_0$ a group homomorphism

\[
\xi_n(g) : G_n(g) \to K_{n+1}(\beta_0(g))
\]

such that for any arrow $x : g \to h$ of the groupoid $\bar{G}_*$ and any element $a \in G_n(g)$, $n \geq 2$ one has

\[
\xi_n(a^x) = \xi_n(a)^{\beta_1(x)}.
\]

Furthermore, one also has

\[
\alpha_n(a) = \begin{cases} 
\{\beta_n(a) \cdot \xi_{n-1}(\partial(a) \cdot \partial \xi_n(a))\xi_0(a)^{-1}, & n = 2 \\
\{\beta_n(a) + \xi_{n-1}(\partial(a) + \partial \xi_n(a))\xi_0(a)^{-1}, & n > 2.
\end{cases}
\]

Having defined a homotopy, one can talk about homotopy equivalences. As expected, any homotopy equivalence is also a weak equivalence (see [3, Exercise 7.1.45]).

Obviously, any groupoid is a disjoint union of its connected components. This implies that any crossed complex over a groupoid is also a disjoint union of crossed complexes over connected groupoids. Next, it is also well-known that any connected groupoid is equivalent to a group considered as a one object category. This also has an implication for crossed complexes over groupoids. Having a crossed complex over a groupoid and an object $g$ we can form the following reduced crossed complex:

\[
\cdots \to G_4(g) \overset{\partial_4}{\to} G_3(g) \overset{\partial_3}{\to} G_2(g) \overset{\partial_2}{\to} G_1(g, g) \to \{g\}.
\]

Call it the fibre over $g$ and denote it by $G_*[g]$. Clearly, $\pi_1(G_*[g]) = \pi_1(G_*[g], g)$ and $H_n(G_*[g]) = H_n(G_*[g], g)$ for all $n \geq 2$. Thus for connected $G_*$ the inclusion $G_*[g] \to G_*$ is a weak equivalence. In fact it is even a homotopy equivalence thanks to [3, Proposition 7.1.46, pp. 220-221].

4.4. Symmetric monoidal closed category structure. It is an important fact that the category $\text{Crs}$ has a symmetric monoidal closed category structure [3, Chapter 9], which means that for any crossed complexes $G_*$ and $K_*$ there is a “functional object” $\text{CRS}_*(G_*, K_*)$ and a “tensor object” $G_* \otimes K_*$ such that

\[
\text{Hom}_{\text{Crs}}(G_* \otimes K_*, K_*) \cong \text{Hom}_{\text{Crs}}(G_*, \text{CRS}_*(G_*, K_*)).
\]

We will only use $\text{CRS}_*(G_*, K_*)$. Therefore we recall the corresponding construction following [3, p. 281].

In dimension $0$, $\text{CRS}_0(G_*, K_*)$ is the set of crossed complex morphisms from $G_*$ to $K_*$. Elements in $\text{CRS}_1(G_*, K_*)$ are homotopies. More explicitly, if $\alpha_*$ and $\beta_*$ are two such morphisms considered as two objects in the crossed complex $\text{CRS}_*(G_*, K_*)$, then the morphisms from $\alpha_*$ to $\beta_*$ are homotopies $\xi_*$ from $\alpha_*$ to $\beta_*$ as defined in Section 4.4.
One can show that in this way one obtains a groupoid. Next, elements of degree $k$ in $\text{CRS}_k(G_*, K_*)$ are $k$-homotopies [3 Definitions 9.3.3 p. 281]. Recall that if $\alpha_* : G_* \to K_*$ is a morphism of crossed complexes as above, then a $k$-fold homotopy, $k \geq 2$, from $G_*$ to $K_*$ over $\alpha_*$ is a collection of maps $\xi_n : G_n \to H_{n+k}$, $n \geq 0$, satisfying the following conditions:

- for $n \geq 2$, $a, b \in G_n$ and $x : g \to h$, one has $\xi_n(a x) = \xi_n(a)^{\alpha_1(x)}$,
- if $n \geq 2$, then $\xi_n$ is additive,
- for $n = 1$, the map $\xi_1$ satisfies the relation $\xi_1(x y) = \xi_1(x)^{\beta_1(y)} \times \xi_1(y)$.

Thus, for each morphism $\alpha_* : G_* \to K_*$ considered as an object of the groupoid $\text{CRS}(G_*, K_*)$, one denotes by $\text{CRS}_k(G_*, K_*)(\alpha_*)$ the collection of all $k$-fold homotopies over $\alpha_*$. Then the assignment $\alpha_* \mapsto \text{CRS}_k(G_*, K_*)(\alpha_*)$ is a part of the crossed complex $\text{CRS}_*(G_*, K_*)$, see details in [3, Definition 9.3.5, p. 282].

We will need the following fact, which is a straightforward consequence of the definition.

Let $n \geq 2$ and $G_*$ a crossed complex over a groupoid. We write $\ell(G_*) \leq n$ if $G_k = 0$ for all $k > n$. If $G_*$ is a crossed complex over a groupoid, we let $\tau_nG_*$ be the following quotient of $G_*:

$$
\tau_nG_* = \begin{cases} 
G_m, & m < n \\
G_n/\text{im}(\partial), & m = n \\
0, & m > n 
\end{cases}
$$

**Lemma 21.** Let $G_*$ and $K_*$ be crossed complexes over groupoids. Assume $\ell(K_*) \leq n$, where $n \geq 2$. Then $\text{CRS}_k(G_*, K_*) = \text{CRS}_k(\tau_nG_*, K_*)$.

The situation when both $G_*$ and $K_*$ are crossed modules is especially transparent. In this case we have $G_0 = \{1\} = K_0$ and $G_n = 0 = K_n$ for all $n \geq 3$. Hence $\text{CRS}_*(G_*, K_*)$ has the form $\cdots \to 0 \to \text{CRS}_2(G_*, K_*) \to \text{CRS}_1(G_*, K_*) \to \text{CRS}_0(G_*, K_*)$ where $\text{CRS}_0(G_*, K_*) = \text{Hom}_{\text{Cr}}(G_*, K_*)$.

Therefore objects are given by $\alpha_* = (\alpha_2, \alpha_1)$, where $\alpha_2 : G_2 \to K_2$ and $\alpha_1 : G_1 \to K_1$ are group homomorphisms such that the diagram

$$
\begin{array}{ccc}
\cdots & 0 & G_2 \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\cdots & 0 & K_2 \end{array}
$$

commutes and

$$
\alpha_2(\tau a) = \alpha_1(\tau) \alpha_1(a).
$$

If $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are two morphisms $G_* \to K_*$, then a homotopy $\alpha \Rightarrow \beta$ is a pair $\xi_* = (\xi_0, \xi_1)$, where $\xi_0 : G_0 \to K_1$ and $\xi_1 : G_1 \to K_2$ are maps satisfying the following properties (since $G_0 = \{1\}$ we will assume that $\xi_0 \in K_1$).

$$
\xi_1(x y) = \xi_1(x)^{\beta_1(x)} \xi_1(y),
$$
\[
\alpha_1(x) = \xi_0^{-1}\partial(\xi_1(x))\beta_1(x)\xi_0.
\]
\[
\alpha_2(a) = (\beta_2(a)\xi_1(\partial a))^{\xi_0(\partial a)^{-1}}.
\]

Here \(x, y \in G_1\) and \(a \in G_2\). Thus \(\text{CRS}_1(G_*, K_*)\)(\(\alpha, \beta\)) is the set of all pairs \((\xi_0, \xi_1)\) satisfying the above three conditions. Finally, \(\text{CRS}_2(G_*, K_*)\)(\(\alpha\)) is the set of all maps \(\tau : G_0 \to K_2\). Since \(G_0 = \{1\}\), we see that \(\text{CRS}_2(G_*, K_*) = G_2\).

**Lemma 22.** Let \(G_*\) be a crossed module. Then the fibre of \(\text{CRS}_*(G_*, G_*)\) at id = \((\text{id}_G, \text{id}_G)\) is equivalent to \(Z_*(G_*)\).

**Proof.** If \(\alpha = \beta = \text{id}\), then we see that the fibre in dimension 1 consists of pairs \((h_0, h_1)\), where \(h_0 \in G_1\). These pairs must satisfy the conditions

\[
h_1(xy) = h_1(x)^{y} \cdot h_1(y),
\]

\[
x = h_0^{-1}\partial(h_1(x))xh_0,
\]

\[
a = (ah_1(\partial a))^{(h_0^{-1})}.
\]

One easily sees that the first equality simply says that \(h_1 : G_1 \to G_2\) is a crossed homomorphism, the second one says that \(\partial(h_1(x)) = h_0xh_0^{-1}x^{-1} = [h_0, x]\), while the third one says \(h_1(\partial a) = a^{-1}a^{h_0}\) and the result follows.

\[\square\]

### 4.5. Relation to mapping spaces.

The category \(\text{Crs}\) of crossed complexes and the category \(\text{CW}\) of CW-complexes and cellular maps are related by the pair of functors

\[
\Pi_* : \text{CW} \to \text{Crs} \quad \text{and} \quad B : \text{Crs} \to \text{CW},
\]

such that on homotopy classes of maps one has a binatural bijection:

\[
[X, BG_*] \cong [\Pi_* X, G_*].
\]

Actually more is true, see [3] Theorem 11.4.19. p.378.

**Theorem 23.** Let \(X\) be a CW-complex and let \(G_*\) be a crossed complex. Then there is a natural weak homotopy equivalence

\[
B(\text{CRS}_*(\Pi_* X, G_*)) \to \text{Maps}(X, BG_*).
\]

The proof of Theorem 19 is based on the following result, which is of independent interest.

**Proposition 24.** If \(G_* = (G_2 \overset{\partial}{\to} G_1)\) is a crossed module such that \(G_1\) is a free group, then \(BZG_*\) and \(ZBG_*\) are homotopy equivalent.

**Proof.** Take \(X = BG_*\) in Theorem 23 to obtain a weak equivalence

\[
B(\text{CRS}_*(\Pi_* BG_*, G_*)) \to \text{Maps}(X, BG_*).
\]

By Lemma 21 we have that the first space can be replaced by \(B(\text{CRS}_*(\tau_2 \Pi_* BG_*, G_*))\). It is well-known that \(\Pi_* BG_* \to G_*\) is a weak equivalence (see for example, [4] p.100)). It follows that \(\tau_2(\Pi_* BG_*) \to G_*\) is also a weak equivalence. Observe that \(\tau_2(\Pi_* BG_*)\) is a crossed module with free group in dimension one, same for \(G_*\). It follows that these crossed modules are actually weak equivalent and cofibrant object in an appropriate model category structure, see [7] Corollary 2.13. Hence they are homotopy equivalent and as a consequence \(B(\text{CRS}_*(\tau_2 \Pi_* BG_*, G_*))\) and \(B(\text{CRS}_*(G_*, G_*))\) are homotopy equivalent. Hence we obtain the weak equivalences

\[
B(\text{CRS}_*(G_*, G_*)) \to B(\text{CRS}_*(\Pi_* BG_*, G_*)) \to \text{Maps}(X, BG_*).
\]

Looking at the component containing the identity map, we obtain a homotopy equivalence

\[
BZ_* G_* \to ZBG_*,
\]

proving the proposition.

\[\square\]
4.6. Proof of Theorem 19 Let $G = (G_2 \xrightarrow{\partial} G_1)$ be a crossed module such that $G_1$ is a free group. Consider the evaluation map

$$ZX = BZ_{*}G_{*} \to BG_{*} = X$$

whose algebraic model is the crossed module morphism $(\text{id}_{G_{2}}, z_{1}): Z_{*}(G_{*}) \to G_{*}$. Hence, the homomorphism between the homotopy groups

$$\pi_{1}(ZX, \text{id}_{X}) \to \pi_{1}(X, x_{0})$$

is induced by $z_{1}: (x, \xi) \mapsto x$ and thus $\pi_{1}(ZX, \text{id}_{X}) \to \pi_{1}(X, x_{0})$ factors through the map $\omega$ from Proposition 15. Hence,

$$G(X, x_{0}) = \text{Im}(\pi_{1}(ZX, \text{id}_{X}) \to \pi_{1}(X, x_{0}))$$

$$= \text{Im}(\pi_{1}(Z_{*}(G_{*})) \xrightarrow{\omega} Z_{\pi_{2}(G_{*})}(\pi_{1}(G_{*})))$$

$$= \text{Ker}(g),$$

where $g$ is the homomorphism defined in Proposition 17.

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