GENERATING SERIES OF INTERSECTION VOLUMES OF SPECIAL CYCLES ON UNITARY SHIMURA VARIETIES

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ABSTRACT. We form a generating series of regularized volumes of intersections of special cycles on a non-compact unitary Shimura variety with a fixed base change cycle. We show that it is a Hilbert modular form by identifying it with a theta integral, with parameters lying outside the classical convergence range of Weil, which we show converges. By applying the regularized Siegel-Weil formulas of Ichino and Gan-Qiu-Takeda, we show that modular form is the restriction of a hermitian modular form of degree $n$ related to Siegel Eisenstein series on $U(n,n)$. These results follow from a computation showing the Kudla-Millson Schwartz function vanishes under the Ikeda map.

CONTENTS

Introduction 1
1. Notation 4
2. Cycles on unitary Shimura varieties 5
2.1. Special cycles 6
2.2. Intersection volumes 6
3. Theta integrals 9
3.1. Weil representation 9
3.2. Seesaw dual-reductive pairs 9
3.3. Fourier coefficients of theta integrals 11
4. Restrictions of Eisenstein Series 14
4.1. Eisenstein series 14
4.2. Regularized theta integrals 15
4.3. Regularized Siegel-Weil formula 15
4.4. Refined regularization 16
4.5. The mixed model 17
Remarks 19
5. Appendix 19
5.1. The Kudla-Millson construction 19
5.2. Vanishing of the Ikeda map 23
References 26

INTRODUCTION

Let $E/F$ be a CM extension of number fields, with $n = [F : \mathbb{Q}]$, and $(V,Q)$ a non-degenerate hermitian space over $E$ of dimension $m$. Put

$$H = \text{Res}_{F/\mathbb{Q}} U(V,Q)$$

and assume $H(\mathbb{R})$ is non-compact. Denote by $D$ the set of maximal negative-definite subspaces of $V(\mathbb{R})$. Then $D$ is a hermitian symmetric domain on which $H(\mathbb{R})$ acts transitively.

Let $L \subset V$ be a maximal lattice (in the sense of [Shi64]), and let $\Gamma(L)$ denote its stabilizer in $H(\mathbb{Q})$. If $\Gamma \subset \Gamma(L)$ is a suitable arithmetic subgroup, the quotient

$$M = \Gamma \backslash D$$

is a complex algebraic variety.
The variety $M$ hosts a collection of special cycles $C_\beta$, indexed by $\beta \in \text{Herm}_r(E)$, the set of $E$-hermitian $r \times r$ matrices, for $1 \leq r \leq m$. We encode the cycles $C_\beta$ in a formal generating series as follows.

Recall the complex Siegel half-space of degree $r$

$$\mathbb{C}O_r = \{ \tau = u + iv \colon u, v \in \text{Herm}_r(\mathbb{C}), \ v > 0 \}.$$ 

For each $T = (\tau_1, \cdots, \tau_n) \in \mathbb{C}O_n$, $\beta = (\beta_1, \cdots, \beta_n) \in \text{Herm}_r(E)^n$, write

$$e_\ast(\beta T) = \exp(2\pi i \sum \text{Tr} \beta_i \tau_i).$$

A remarkable fact is that

$$F(T) = \sum_{\beta \in \text{Herm}_r(E)} C_\beta \cdot e_\ast(\beta T)$$

behaves like the Fourier expansion of a Siegel automorphic form on $U(r,r)^n$. More concretely, let $C \subset M$ be an algebraic cycle of dimension $r$. One may consider the intersection numbers $I(C,C_\beta)$ of $C$ and $C_\beta$, in some suitable sense. Then the series

$$F(T;C) = \sum_{\beta \in \text{Herm}_r(E)} I(C,C_\beta)e_\ast(\beta T)$$

should, under ideal conditions, be the Fourier expansion of an actual automorphic form.

The work of Hirzebruch and Zagier on intersection numbers of curves on Hilbert modular varieties, and that of Kudla [Kud78] on Picard modular varieties, are the first results of this type. The phenomenon was studied in great generality by Kudla and Millson in the 1980s [KM86, KM88, KM90], with $F(T;C)$ understood as a kind of geometric theta lift of the cycle $C$. If $V$ is anisotropic, so that $M$ is compact, their work proves all the expected properties of $F(T)$. One may also replace $(V,Q)$ by a quadratic space over $F$, and obtain cycles $C_\beta$ on arithmetic quotients of symmetric space that may not be algebraic. The geometric theta lift then produces automorphic forms on $\text{Sp}(2nr)$.

More recently, Funke and Millson have extended and generalized some of these results to the non-compact case [FM02, FM06, FM11, FM13]. Although they mainly focus on the orthogonal case, their methods are expected to also apply mutatis mutandis to the unitary case.

In this paper, we focus on a particular unitary case where $M$ is a non-compact, and $C$ is a base change cycle, obtained from a model $(V_0, Q_0)$ of $(V,Q)$ over an imaginary quadratic field $E_0/\mathbb{Q}$, so that

$$(V, Q) = (V_0, Q_0) \otimes_{E_0} E.$$ 

We assume that $(V_0, Q_0)$ has signature $(n,1)$, and take $r = 1$. We put

$$H_0 = U(V_0, Q_0).$$ 

If $D_0$ denotes the hermitian symmetric domain associated to $H_0$, $D$ may be identified with $D_0^n$. The injection $H_0 \hookrightarrow H$ then induces a diagonal embedding $D_0 \hookrightarrow D^n$. For a suitable arithmetic subgroup $\Gamma_0 \subset H_0(\mathbb{Q}) \cap \Gamma$, we put

$$M_0 = \Gamma_0 \backslash D_0,$$

and let $C_0$ denote the cycle given by the induced locally finite map

$$i_0 : M_0 \rightarrow M.$$ 

We will have reason to consider different lattices $L \subset V$, so we write $C_{\beta,L}$ to emphasize the dependence. For non-zero $\beta \in F = \text{Herm}_1(E)$, the cycles $C_{\beta,L}$ and $C_0$ have complementary codimension. By $I(M_0, C_{\beta,L})$ we denote the intersection number of $C_{\beta,L}$ and $M_0$, in a suitable sense.

Let $K(L) \subset H_0(\mathbb{A}_f)$ denote the stabilizer of $L \otimes \mathbb{Z} \subset V(\mathbb{A}_f)$, and fix a set of representatives $h_1, \cdots, h_l$ of $H_0(\mathbb{Q}) \backslash H_0(\mathbb{A}_f)/KH_0$.

Denote by $L_i \subset V_0^n \simeq V$ the lattices

$$L_i = h_i(L \otimes \mathbb{Z}) \cap V_0(\mathbb{Q})^n, \quad i = 1, \cdots, l.$$
For \( \tau = (\tau_1, \cdots, \tau_n) \in \mathbb{H}^n \), we write \( \tau_i = u_i + iv_i \), with \( u_i, v_i \in \text{Herm}_n(\mathbb{C}) \), and put
\[
F(\tau) = c_0 + i^{-n} \prod_{i=1}^{n} |v_i|^{(n+1)/4} \sum_{\beta \in F} \left( \sum_{i=1}^{r} I(C_0, C_{\beta, L_i}) \right) e_\star(\beta \tau),
\]
(0.7)
where \( c_0 \) is some explicit constant.

The aim of this paper is to show \( F(\tau) \) is the Fourier expansion of a Hilbert modular form of weight \( \frac{m+1}{2} \) on \( \mathbb{H}^n \), and to identify it with the restriction to \( U(1, 1)^n \) of an automorphic form on \( U(n, n) \) derived from Siegel Eisenstein series.

We now describe the results in more detail.

Let \( (W, J) \) be a split skew-hermitian space of signature \( (1, 1) \), and put
\[
(W_0, J_0) = (\text{Res}_{E/\mathbb{Q}} W, \text{tr}_{E/\mathbb{Q}} J).
\]
Define
\[
G_0 = U(W_0, J_0), \quad G = \text{Res}_{F/\mathbb{Q}} U(W, J),
\]
so that \( G_{0, \mathbb{R}} = U(n, n) \) and \( G_\mathbb{R} \simeq U(1, 1)^n \). One has a natural inclusion \( G \subset G_0 \) that “forgets” \( F \)-linearity.

After fixing a non-trivial additive character \( \psi \) on \( \mathbb{A}_\mathbb{Q}/\mathbb{Q} \) and a suitable Hecke character \( \chi \) on \( \mathbb{A}_{E_0}/E_0 \), one obtains a Weil representation
\[
\omega : G_0(\mathbb{A}) \times H_0(\mathbb{A}) \to \text{End}(\mathcal{S})
\]
where \( \mathcal{S} = \mathcal{S}(V_0(\mathbb{A})^n) \) is a space of adelic Schwartz functions. To each \( \phi \in \mathcal{S} \) one may associate the theta kernel
\[
\Theta(g, h; \phi) = \sum_{x \in V_0(\mathbb{A})^n} \omega(g, h) \phi(x).
\]
Here \( \omega(g, h) \phi(x) \) denotes \( \omega(g, h) \phi \) evaluated at \( x \). There is an associated theta integral
\[
I(g, \phi) = \int_{H_0(\mathbb{Q}) \backslash H_0(\mathbb{A})} \Theta(g, h; \phi) dh.
\]
(0.8)
The Witt index \( r \) of \( (V_0, Q_0) \) is equal to 1. Since \( V_0 \) is not anisotropic and \( m - r \leq n \), by a theorem of Weil [Wei60] the above integral will in general diverge. For a specific \( \varphi \in \mathcal{S} \) derived from a construction of Kudla and Millson, we will prove:

**Theorem A.** For \( g \in G(\mathbb{A}) \subset G_0(\mathbb{A}) \), the integral \( I(g, \varphi) \) is absolutely convergent and defines an automorphic form on \( G(\mathbb{A}) \). Its restriction to \( G(\mathbb{R}) \subset G(\mathbb{A}) \) can be identified with \( F(\tau) \).

To identify \( F(\tau) \) with the restriction of an automorphic form on \( U(n, n) \), we consider regularized theta integrals of the form
\[
B(g, s; \phi) = \int_{H_0(\mathbb{Q}) \backslash H_0(\mathbb{A})} \Theta(g, h; \omega(z) \phi) E_{H_0}(h, s) dh, \quad \text{Re}(s) > 0.
\]
Here \( E_{H_0}(h, s) \) is a particular auxiliary Eisenstein series on \( H_0 \), and \( \omega(z) \) is a regularizing Hecke operator acting on \( \mathcal{S} \). The integral converges absolutely for \( \text{Re}(s) \) large and has a meromorphic continuation to the \( s \)-plane. It has a pole of order at most two at \( s = s_0 = \frac{1}{2} \), and a Laurent expansion
\[
B(g, s; \phi) = B_{-2}(\phi)(s - s_0)^{-2} + B_{-1}(\phi)(s - s_0)^{-1} + B_0(\phi) + B_1(\phi)(s - s_0) + \cdots,
\]
where each coefficient \( B_k(\phi) \) is an automorphic form on \( G_0(\mathbb{A}) \).

On the other hand, to \( \phi \in \mathcal{S} \) one associates in a standard way a Siegel-Weil section \( \Phi^{(s)} \in \text{Ind}_{P_0(\mathbb{A})}^{G_0(\mathbb{A})} \chi |\det|^{\frac{s}{2}} \), which is a family of functions on \( G_0(\mathbb{A}) \) parametrized by \( s \in \mathbb{C} \). The corresponding Siegel Eisenstein series is
\[
E(g, s; \phi) = \sum_{\gamma \in P_0(\mathbb{Q}) \backslash G_0(\mathbb{Q})} \Phi^{(s)}(\gamma g), \quad g \in G_0(\mathbb{A}).
\]
The series has a meromorphic continuation to the \( s \)-plane, with at most a simple pole at \( \rho = \frac{m - r}{2} = 1 \). Its Laurent expansion at \( s = \rho \) is written
\[
E(g, s; \phi) = A_{-1}(\phi)(s - \rho)^{-1} + A_0(\phi) + A_1(\phi)(s - \rho) + \cdots,
\]
where each \( A_{-1}(\phi) \) is again an automorphic form on \( G_0(\mathbb{A}) \).
To state the result, we must mention that to \( \phi \in \mathcal{S} \) one may associate another \( \phi' \in \mathcal{S} \) by a certain procedure. It involves lifting it to \( \tilde{\phi} \) in the Weil representation of a larger group, applying the derivative of a normalized intertwining operator to \( \Phi^{(r)}(\tilde{\phi}) \), and restricting the corresponding Schwartz function back to \( \mathcal{S} \). When \( \phi = \varphi \) as before, we prove:

**Theorem B.** As automorphic forms on \( G_0(\mathfrak{a}) \),

\[
B_{-1}(\varphi) = A_0(\varphi) - \frac{1}{2} A_{-1}(\varphi').
\]

Furthermore, for \( g \in G(\mathfrak{a}) \) we have

\[
B_{-1}(\varphi)(g) = I(g, \varphi).
\]

The proof is by first applying the regularized Siegel–Weil formulas of Ichino [Ich04] to show that \( B_{-2}(\varphi) = 0 \). This allows us to simplify and refine the second-term identity of [GQT14] to obtain the precise expression for \( B_{-1}(\varphi) \).

1. **Notation**

We fix an ordering \( \lambda_1, \cdots, \lambda_9 \) of the distinct embeddings \( F \hookrightarrow \mathbb{R} \), and use both \( z^\sigma \) and \( \mathfrak{z} \) to denote complex conjugation on \( E \) and \( E_0 \). If \( U \) is a \( k \)-vector space and \( A \) is a \( k \)-algebra, we write \( U(A) \) for \( U \otimes_k A \). The finite adeles are denoted \( \mathbb{A}_f \). For a lattice \( L \subset V \), we put \( \hat{L} = L \otimes \mathbb{Z} \subset V(\mathbb{A}_f) \), where \( \mathbb{Z} = \varprojlim \mathbb{Z}/n\mathbb{Z} \).

\( (V, Q) \) will denote a non-degenerate hermitian space of dimension \( m \) over \( E \). We have

\[
V(\mathbb{R}) \simeq V^{(1)} \oplus \cdots \oplus V^{(g)},
\]

where each \( V^{(i)} = V \otimes_{F, \lambda_i} \mathbb{R} \) is a complex hermitian space of dimension \( m \).

We fix a maximal isotropic subspace \( X \subset V \), so that \( r = \dim X \) is the Witt index of \( (V, Q) \). We fix an ordered basis \( (x_1, \cdots, x_r) \) for \( X \). We fix a dual isotropic subspace \( X^* \subset V \) spanned by vectors \( (x_1^*, \cdots, x_r^*) \) satisfying \( Q(x_i, x_j^*) = \delta_{ij} \). We thus occasionally identify \( X^* \) with \( \text{Hom}(X, E) \). There’s an orthogonal decomposition

\[
V = X \oplus V_{an} \oplus X^*,
\]

where \( V_{an} \) is an anisotropic subspace of dimension \( m_0 = m - 2r \).

As in the introduction, we fix a maximal lattice \( L \subset V \). By the classification of such lattices in [Shi64], we can assume \( (x_1, \cdots, x_r) \) and \( (x_1^*, \cdots, x_r^*) \) have been chosen in such a way that

\[
L = a_1 x_1 + \cdots + a_r x_r + L' + a_1^* x_1^* + \cdots + a_r^* x_r^*.
\]

Here \( a_i \subset E \) are fractional ideals, \( a_i^* = \overline{a_i}^{-1} \delta_E^{-1} \), and \( L' \subset V_{an} \) is a maximal lattice.

We let \( P_H \subset H \) denote the stabilizer of \( X \), and put \( K_H = K_{H, \infty} K(L) \), where \( K(L) \subset H(\mathbb{A}_f) \) is the stabilizer of \( \hat{L} \subset V(\mathbb{A}_f) \), and \( K_{H, \infty} = H(\mathbb{R}) \cap U(m) \).

For \( n > 0 \), we let \( (W_n, J_n) \) denote the split skew-hermitian space of dimension \( 2n \) spanned by a basis \( e_1, \cdots, e_n, f_1, \cdots, f_n \) with respect to which \( J_n \) is given by

\[
J_n(x, y) = \frac{1}{\sqrt{n}} \begin{pmatrix} I_n & \end{pmatrix} x, \quad x, y \in E^{2n}.
\]

We put

\[
G_n = \text{Res}_{Q}^{E} U(W_n, J_n),
\]

and \( G = G_n \) when \( n \) is fixed.

Let \( Y = \text{Span}\{e_1, \cdots, e_n\} \), \( Y^* = \text{Span}\{f_1, \cdots, f_n\} \), so that \( Y \) and \( Y^* \) are dual isotropic subspaces, and \( W = Y \oplus Y^* \). We will sometimes identify \( Y^* \) with \( \text{Hom}(Y, E) \).

Let \( P \) denote the Siegel parabolic stabilizing \( Y \), \( M \) its Levi factor, and \( N \) the unipotent radical. We may identify these with

\[
M = \text{GL}(Y), \quad N = \text{Hom}_{F}(Y, Y^*), \quad P = NM.
\]
More explicitly, using the fixed basis for $W$ we can write

$$P(\mathcal{A}) = \left\{ p(a) = \begin{pmatrix} a & * \\ \mathbf{1} & 1 \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{A}_E) \right\},$$

$$M(\mathcal{A}) = \left\{ m(a) = \begin{pmatrix} a & * \\ \mathbf{1} & 1 \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{A}_E) \right\},$$

$$N(\mathcal{A}) = \left\{ n(b) = \begin{pmatrix} 1 & b \\ \mathbf{1} & 1 \end{pmatrix} : b \in \mathrm{Herm}_n(\mathbb{A}_E) \right\},$$

where $\mathrm{Herm}_n(k)$ denotes $n \times n$ hermitian matrices with entries in $k$, for any $E$-algebra $k$.

Let $K_G = G(\mathbb{Z}) \subset G(\mathbb{A})$ be the standard maximal compact of $G(\mathbb{A})$.

For $g \in G(\mathbb{A})$, we have a decomposition

$$g = n(b) \cdot m(a) \cdot k, \quad a \in \mathrm{GL}_n(\mathbb{A}_E), \quad b \in \mathrm{Herm}_n(\mathbb{A}_E), \quad k \in K_G. \quad (1.4)$$

We write $a(g) = a$ under this decomposition.

As in the introduction, we put

$$W_0 = \mathrm{Res}_{E/E_0} W, \quad J_0 = \mathrm{tr}_{E/E_0} J.$$ 

We explicitly identify $(W_0, J_0)$ with $(W_{ng}, J_{ng})$ over $E_0$ as follows.

Fix a $\mathbb{Z}$-basis $\{r_1, \ldots, r_g\}$ for $O_F$, and a trace-dual basis $\{s_1, \ldots, s_n\}$ for $\mathfrak{d}^{-1}_F$. We then have corresponding isomorphisms

$$\beta : \mathbb{Z}^g \to \mathfrak{d}_F^{-1}, \quad (x_1, \ldots, x_g) \mapsto \sum_{i=1}^g s_i x_i, \quad \beta' : \mathbb{Z}^g \to O_F, \quad (x_1, \ldots, x_g) \mapsto \sum_{i=1}^g r_i x_i. \quad (1.5)$$

The $E_0$-linear map

$$\beta_{E_0} : \mathfrak{d}_F \to \mathfrak{d}_F : \beta_{E_0}^n = \beta_{E_0}^{n_0} \oplus \beta_{E_0}^{n_g} \to E^n \oplus E^n \quad (1.6)$$

then identifies $(W_{ng}, J_{ng})$ over $E_0$ with $(W_0, J_0)$, using the fixed basis $\{e_i, f_i\}_{i=1}^n$ for $W$. Then For

$$G_0 = U(W_0, J_0), \quad (1.7)$$

as in the introduction, $G_0(\mathbb{R})$ may be identified with the complex unitary group $U(ng, ng)$.

The canonical inclusion $G \subset G_0$ that “forgets” $F$-linearity will be sometimes denoted

$$j : G \hookrightarrow G_0. \quad (1.8)$$

2. CYCLES ON UNITARY SHIMURA VARIETIES

To begin with, let $(V, Q)$ denote an arbitrary non-degenerate hermitian space over $E$ of dimension $m$.

Assume that each $V^{(i)} = V \otimes_{F, \lambda_i} \mathbb{R}$ has signature $(p_i, q_i)$, and that $p_i q_i \neq 0$ for some $i$. The hermitian symmetric domain $D$ associated to $H(\mathbb{R})$ has complex dimension $\sum_{i=1}^m p_i q_i$, and may be identified with

$$D = \{ Z \subset V(\mathbb{R}) : \dim Z = \sum_{i=1}^m q_i, \  Q|_Z < 0 \}.$$ 

For $Z_0 \in D$ fixed, let $K_\infty \subset H(\mathbb{R})$ denote its stabilizer, so that $D \simeq H(\mathbb{R})/K_\infty$.

An element $Z \in D$ consists of a maximal negative-definite subspace of $V \otimes \mathbb{R} \simeq V^{(1)} + \cdots + V^{(g)}$, where the $V^{(i)}$ are mutually orthogonal complex hermitian spaces. Let $D^{(i)}$ denote the set of maximal negative-definite subspaces in $V^{(i)}$ (consisting of a single point if $q_i = 0$). Since each $Z \in D$ can be written uniquely as $Z = Z^{(1)} + \cdots + Z^{(g)}$, with $Z^{(i)} \in D^{(i)}$, we can identify $D$ with $D^{(1)} \times \cdots \times D^{(g)}$.

Let $\Gamma(L) = K(L) \cap H(\mathbb{Q})$, and assume $\Gamma \subset \Gamma(L)$ is an arithmetic subgroup such that $\Gamma / Z(\Gamma)$ acts freely on $D$. Then as is well-known

$$M = D / \Gamma \quad (2.1)$$

da complex algebraic variety that admits a model over a number field.

We will now describe the family of special cycles on $M$. 

2.1. Special cycles. Let \( x = (x_1, \cdots, x_r) \subset L \) be a collection of lattice vectors spanning a non-degenerate subspace \( V_x' \) of \( V \). Let \( V_x = V_x^\perp \) be the orthogonal complement, and write \( Q_x = Q|_{V_x}, Q_x' = Q|_{V_x'} \).

Put
\[
H_x = \text{Res}_{F/Q} \left( U(V_x, Q_x) \times U(V_x', Q_x') \right).
\]

Then \( H_x \) injects into \( H \) canonically, and its symmetric space may then be identified with
\[
D_x = \{ Z \in D : Z = Z \cap V_x(\mathbb{R}) + Z \cap V_x'(\mathbb{R}) \}.
\]

If each \( V_x^{(i)} \) has signature \((r_i, s_i)\), then
\[
dim_{\mathbb{C}} D_x = \sum_{i=1}^{n} (r_i s_i + (p_i - r_i)(q_i - s_i)).
\]

Let \( \Gamma_x = H_x(\mathbb{Q}) \cap \Gamma \), and put \( M_x = D_x/\Gamma_x \). The basic cycle \( C_x \) associated to \( x \) is the image of the locally-finite map
\[
\iota_x : M_x \to M.
\]

Now let \( \beta \in \text{Herm}_n(E) \) be a hermitian \( g \times g \) matrix, and put
\[
\mathscr{I}_\beta = \{ x = (x_1, \cdots, x_g) \in V^g : Q(x, x) = \beta \}.
\]

Here \( Q(x, x) \) denotes the matrix \( (Q(x_i, x_j)) \). The group \( \Gamma \) acts on \( V^g \) diagonally, and preserves \( \mathscr{I}_\beta \). Let \( y_1, \cdots, y_l \) be a set of representatives for the (finite number of) orbits of the action of \( \Gamma \) on \( \mathscr{I}_\beta \).

The special cycle associated to \( \beta \in \text{Herm}_n(E) \) is defined to be
\[
C_\beta = \prod_{i=1}^{l} C_{y_i}.
\]

We remark that \( C_\beta \) is a disjoint union of locally finite maps to \( M \).

Now suppose \( (V, Q) = (V_0, Q_0) \otimes_{E_0} E \), where \((V_0, Q_0)\) is hermitian over \( E_0 \) of signature \((p_0, q_0)\), with \( pq \neq 0 \). For convenience we assume \( V_0 \subset V \). Composing the inclusion \( V_0(\mathbb{R}) \hookrightarrow V(\mathbb{R}) \simeq \prod_i V^{(i)} \) with each projection \( \prod_i V^{(i)} \to V^{(i)} \) we obtain isomorphisms \( V_0(\mathbb{R}) \simeq \prod_i V^{(i)} \) of complex hermitian spaces. Let \( H_0 = U(V_0, Q_0) \) be as in the introduction, and identify its symmetric space \( D_0 \) with the space of negative-definite \( q_0 \)-dimensional subspaces of \( V_0(\mathbb{R}) \). Then via the isomorphisms \( V_0(\mathbb{R}) \simeq \prod_i V^{(i)} \), each \( D^{(i)} \) is identified with \( D_0 \), and the map \( D_0 \hookrightarrow D \) induced by \( H_0(\mathbb{R}) \) with the diagonal \( D_0 \to D_0 \).

Take \( \Gamma_0 \subset H_0(\mathbb{Q}) \cap \Gamma \) to be an arithmetic subgroup such that \( \Gamma_0/Z(\Gamma_0) \) acts freely on the hermitian symmetric domain \( D_0 \) of \( H_0 \). Put
\[
M_0 = D_0/\Gamma_0.
\]

We assume furthermore that \( M_0 \) has finite volume.

The base change cycle associated to \( (V_0, Q_0) \) and \( \Gamma_0 \) is the image of locally finite map
\[
\iota_0 : M_0 \to M.
\]

2.2. Intersection volumes. Since \((V_0, Q_0)\) has signature \((p, q)\), the hermitian symmetric domain \( D_0 \) has complex dimension \( pq \), and \( D \) has dimension \( npq \).

For \( x = (x_1, \cdots, x_r) \in V^r \), let \((r_i, s_i)\) denote the signature of \( V_x^{(i)} \). Then \( D_x \) has dimension
\[
\sum_{i=1}^{n} (r_i s_i + (p - r_i)(q - s_i)) = npq - \sum_{i=1}^{n} (ps_i + qr_i - 2r_i s_i).
\]

Then we have
\[
\text{codim } C_0 = (g - 1)pq, \quad \text{codim } C_x = \sum_{i=1}^{n} (ps_i + qr_i - 2r_i s_i)
\]
and \( C_0, C_x \) have complementary codimension if and only if
\[
\sum_{i=1}^{n} s_i(p - r_i) + r_i(q - s_i) = pq.
\]

From now on we assume \((p, q) = (n, 1)\).
Lemma 2.1. Then $C_0$ and $C_x$ have complementary codimension if and only if either $(r_i, s_i) = (1, 0)$ for all $i$, or $(r_i, s_i) = (p - 1, 1)$ for all $i$.

Proof. Each term of the sum in (2.35) is non-negative, there are $n$ terms, and the RHS is $n$. Hence the codimensions are complementary if and only if every summand is exactly 1. Now $0 \leq s_i \leq q = 1$. If $s_i = 0$, we must have $r_i = 1$, and if $s_i = 1$, $r_i = p - 1$. Since $r_i + s_i = \dim V_x$ for all $i$ and $n > 1$, the two cases are mutually exclusive.

In fact the two cases in the lemma describe the same basic cycle $C_x$ coming from a decompositions $V = V_x + V'_x$, with $V_x$ and $V'_x$ switched. This is the reason for restriction to cycles with $r = 1$.

We therefore consider the intersection of $C_0$ with $C_b$, for $b \in F = \text{Herm}_1(E)$. By the lemma, the cycles have complementary codimension if and only if $b$ is totally positive. However, even in this case the intersection will not in general be transversal. Nevertheless it is a union of basic cycles on $M_0$, of varying codimensions, which all have finite volume since $M_0$ does, by assumption.

Let $\xi \in V$. For $\lambda : F \rightarrow \mathbb{R}$, let $\lambda(\xi)$ denote the image of $\xi$ in $V = V \otimes_{F, \lambda} \mathbb{R}$, and $\xi^\lambda$ the pre-image of $\lambda(\xi)$ under $V_0(\mathbb{R}) \cong V$. Write $V(\xi)$ for the subspace of $V_0(\mathbb{R})$ spanned by $\xi^\lambda$ for all $\lambda$, and $D_0(\xi)$ for $D_0 \cap D_\xi$.

Proposition 2.2. If $V(\xi)$ is a positive-definite subspace of $V_0(\mathbb{R})$, then $D_0(\xi)$ is isomorphic to a complex ball of dimension $n - \dim V(\xi)$. Otherwise it contains at most one point.

Proof. Suppose $z_0 \in D_0 \cap D(\xi)$. Then for each $\lambda$, the image of $z_0$ in $\lambda$ must be either orthogonal or proportional to $\xi^\lambda$. If $\lambda(Q(\xi, \lambda)) < 0$, $z_0$ must be the line spanned by $\xi^\lambda$, hence $D_0(\xi)$ is either empty or only contains only $z_0$. If $\lambda(Q(\xi, \lambda)) > 0$ for all $\lambda$, $z_0$ must be orthogonal to all $\xi^\lambda$, i.e. $V(\xi) \subset z_0^\perp$, which implies $V(\xi)$ is positive-definite since $V_0(\mathbb{R})$ has signature $(n, 1)$. If $V(\xi)$ is positive-definite, then $D_0(\xi)$ is the set of negative-definite lines in $V(\xi)^\perp$, which has signature $(n - \dim V(\xi), 1)$.

Suppose that $x = (x_1, \cdots, x_n) \in V_0^n$ corresponds to $\xi \in V$ under our fixed isomorphism. In other words, $\xi = \sum_{j=1}^n r_j x_j$, where $(r_j)_j$ is the fixed $\mathbb{Z}$-basis for $\mathfrak{d}^{-1}$. Let $V_{0,x}$ denote the subspace of $V_0$ spanned by $x_i$. Then $V_{0,x}(\mathbb{R}) \supset V(\xi)$, since each $\xi^{(i)} = \sum_{j=1}^n \lambda_i(r_j)x_j \in V_{0,x}(\mathbb{R})$. On the other hand, the matrix $U = (u_{ij}) \in \text{Herm}_n(\mathbb{C})$ with $u_{ij} = \lambda_i(r_j)$ is invertible, so that $V_{0,x}(\mathbb{R}) = V(\xi)$.

If $Q(\xi, \lambda)$ is positive-definite, the subspace $D_{0,x} \subset D_0$ associated to $x \in V_0^n$ coincides with $D(\xi)$ defined above.

Let $K(\xi)$ denote the fiber product of the morphisms $\iota_\xi : D_x/\Gamma_x \rightarrow D/\Gamma$ and $\iota_0 : D_0/\Gamma_0 \rightarrow D/\Gamma$ over $D/\Gamma$. In other words

$$K(\xi) = \{(\Gamma_0 z, \Gamma_\xi w) : z \in D_0, w \in D_\xi, \Gamma z = \Gamma w\}.$$

Proposition 2.3. The map

$$\prod_{[\gamma] \in \Gamma_0 \backslash \Gamma/\Gamma_\xi} D(\gamma\xi)/(\Gamma_0 \cap \Gamma\gamma\xi) \rightarrow K(\xi),$$

induced by $(z, \gamma) \mapsto (\Gamma_0 z, \Gamma_\xi \gamma^{-1} z)$ is a bijection.

Proof. Let $(\Gamma_0 z, \Gamma_\xi w) \in K(\xi)$, so that $z = \gamma w$ for some $\gamma \in \Gamma$. Then $z \in D_0 \cap D_D = D_0 \cap D_\gamma = D(\gamma\xi)$. Conversely, given any $z \in D(\gamma\xi)$, we have $(\Gamma_0 z, \Gamma_\xi \gamma^{-1} z) \in K(\xi)$. This shows that the map

$$\prod_{\gamma \in \Gamma} D(\gamma\xi) \rightarrow K(\xi), \quad (z, \gamma) \mapsto (\Gamma_0 z, \Gamma_\xi \gamma^{-1} z)$$

is surjective. Now suppose $(z_1, \gamma_1)$ and $(z_2, \gamma_2)$ map to the same point in $K(\xi)$, so that

$$(\Gamma_0 z_1, \Gamma_\xi \gamma_1^{-1} z_1) = (\Gamma_0 z_2, \Gamma_\xi \gamma_2^{-1} z_2).$$

Then for some $\gamma_0 \in \Gamma_0$, $\gamma \in \Gamma_\xi$, we have $z_2 = \gamma_0 z_1$ and $\gamma_1^{-1} z_1 = \gamma_1 \gamma_2^{-1} z_2$, hence $\gamma_1 \gamma_2^{-1} \gamma_0 z_1 = z_1$. Then since $\gamma$ acts without fixed points on $D$, we have $\gamma_1 \gamma_2^{-1} \gamma_0 = 1$, so that $\gamma_1 = \gamma_0^{-1} \gamma_2 \gamma_\xi^{-1}$ and $[\gamma_1] = [\gamma_2]$ as double cosets in $\Gamma_0 \backslash \Gamma/\Gamma_\xi$.

Conversely, if $\gamma_1 = \gamma_0 \gamma_2 \gamma_\xi$, then $D(\gamma_1\xi) = D_0 \cap D_0 \gamma_2 \gamma_\xi = D_0 \cap \gamma_0 D_\gamma \gamma_\xi = \gamma_0 D(\gamma_2\xi)$, so that $D(\gamma_1\xi)$ and $D(\gamma_2\xi)$ have the same image in $D_0/\Gamma_0$. This shows the images of $D(\gamma\xi)$ in $K(\xi)$ are disjoint, as $\gamma$ ranges over double coset representatives of $\Gamma_0 \backslash \Gamma/\Gamma_\xi$.

Now suppose $(z_1, \gamma)$ and $(z_2, \gamma)$ map to the same point in $K(\xi)$. Then $\gamma_0 z_1 = z_2$ for some $\gamma_0 \in \Gamma_0$, and $\gamma \gamma_0^{-1} z_1 = \gamma^{-1} z_2$, so that $z_2 = \gamma \gamma_0^{-1} \gamma^{-1} z_2$, hence $\gamma_0 = \gamma \gamma_0^{-1} \gamma^{-1} \in \Gamma \gamma^{-1} = \Gamma_\xi$. Then $\gamma_0 \in \Gamma_0 \cap \Gamma_\xi$, and
conversely for any such $γ_0$, the points $(γ_0z_1, γ)$ and $(z_1, γ)$ map to the same in $K(ξ)$. This shows that the surjective map

$$\prod_{[γ] ∈ Γ_0 \backslash Γ/Γ_ξ} D(γξ) \rightarrow K(ξ)$$

induces the bijection in the statement. □

Let $O_i = Γξ_i$, $i = 1, \cdots, t$ be the distinct orbits of $Γ$ acting on

$$I_b(L) = \{ξ ∈ V : Q(ξ, ξ) = b\},$$

and let $ι_b : \bigsqcup_{i} D_ξ/Γ_ξ \rightarrow D/Γ$ denote the disjoint union of the locally finite maps $ι_{ξ_i} : D_ξ/Γ_ξ \rightarrow D/Γ$. The special cycle $C_b$ is then the image of $ι_b$. By the proposition, the fiber product of $ι_b$ with $ι_0 : D_0/Γ_0 \rightarrow D/Γ$ is given by

$$\prod_{i=1}^{t} \prod_{[γ] ∈ Γ_0 \backslash Γ/Γ_ξ} D(γξ) / (Γ_0 ∩ Γξ) \rightarrow D/Γ, \quad (z, γ)i \mapsto (Γ_0z, Γξ_0γ^{-1}z).$$

Denote by $L_0 ⊂ V^n_0$ the lattice corresponding to $L ⊂ V$ under the isomorphism $V^n_0 \simeq V$, $(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} r_i x_i$. Let $r = (r_1, \cdots, r_n)$ considered as a column vector, and $β ∈ \text{Herm}_n(E_0)$ the matrix $Q_0(x_i, x_j)$. Then

$$Q(ξ, ξ) = ^t r β r.$$ 

Now let

$$I_β(L_0) = \{x ∈ L_0 : Q_0(x_i, x_j) = β\}.$$

Then map $x \mapsto ξ$ then establishes a bijection

$$\bigcup_{β ∈ \text{Herm}_n(E_0)} I_{0, β}(L_0) \rightarrow I_b(L).$$

For each $ξ ∈ I_b$, the orbit $Γξ \subset I_b$ consists of $Γ/Γ_ξ$ elements. On the other hand, the action of $Γ_0$ on $I_{0, β}$ splits each such $Γ$-orbit into $Γ_0$-orbits indexed by

$$Γ_0 \backslash Γ/Γ_ξ.$$

For each $x ∈ L_0$, let $Γ_x = \{γ ∈ Γ : γx = x\}$. Then we have a map

$$ι_{0,x} : D_0,x/(Γ_x ∩ Γ_0) \rightarrow D_0/Γ_0,$$

whose image is a basic cycle on $D_0/Γ_0$, that we denote by $C_{0,x}$. In fact $Γ_x = Γξ ∩ Γ_0$ and $D_{0,x} = D(ξ)$, so that these are the same cycles occurring in Prop. [233]

**Theorem 2.4.** For each $b ∈ F$,

$$C_0 ∩ C_b = \bigcap_{β ∈ \text{Herm}_n(E_0) \atop ^t r β r = b} C_β.$$

*Proof.* The cycle $C_0 ∩ C_b$ is a union of $C_0 ∩ C_ξ$ as $ξ$ runs through representatives of $Γ$-orbits in $I_b(L)$. By Proposition [233] each $C_0 ∩ C_ξ$ is a disjoint union of images of $C_{0,γx}$, with $[γ] ∈ Γ_0 \backslash Γ/Γ_ξ$. On the other hand, each such $C_{0,γx}$ is the basic cycle in $D_0/Γ_0$ associated to the $Γ_0$-orbit of $γx$. □

We wish to define an appropriate notion of volume for basic and special cycles on $D_0/Γ_0$, and correspondingly define the intersection volume of $C_0$ and $C_b$ by

$$I(C_0, C_b) = \sum_{β ∈ \text{Herm}_n(E_0) \atop ^t r β r = b} \text{vol}(C_β). \quad (2.9)$$

We do this by means of a particular Schwartz-Bruhat function $φ_{KM} ∈ S(V_0(\mathbb{R})^n)$ constructed by Kudla and Millson, detailed in the appendix.

For $x ∈ V^n_0$, Let $C_x = D(x)/Γ(x)$ be the associated basic cycle in $D_0/Γ_0$. Since $D(x)$ is totally geodesic in $D_0$, there is a geodesic fibration $D_0 → D(x)$, which induces a map $π_x : D_0/Γ(x) → D(x)/Γ(x)$ whose fibers
are geodesics. The function \( f_x(h) = \varphi_{\text{RM}}(h^{-1}x) \), \textit{a priori} defined on \( H_0(\mathbb{R}) \), descends to a function \( f_x(z) \) on \( D_0/\Gamma(x) \). It is integrable over \( D_0/\Gamma(x) \), and for each \( z \in D(x)/\Gamma(x) \), the fiber integral

\[
\int_{\pi^{-1}(z)} f_x(z) dz,
\]

is \textit{independent} of \( z \), and equal to \( i^{-n} \exp(-\pi \text{Tr} Q_0(x,x)) \). Now we define the (regularized) volume \( \text{vol}(C_x) \) by the relation

\[
\int_{D_0/\Gamma(x)} f_x(z) dz = \text{vol}(C_x)i^{-n} \exp(-\pi \text{Tr} Q_0(x,x)).
\]

(2.10)

Here \( dz \) denotes the canonical volume element induced by the Bergman metric on \( D(x) \). Note that if \( C_x \) has finite volume, then

\[
\int_{D_0/\Gamma(x)} f_x(z) dz = \int_{C_x} \int_{\pi^{-1}(w)} f_x(z) dz dw = i^{-n} \exp(-\pi \text{Tr} Q(x,x)) \int_{C_x} dw = \text{vol}(C_x)i^{-n} \exp(-\pi \text{Tr} Q_0(x,x)).
\]

3. Theta integrals

3.1. Weil representation. We at first let \((V, Q)\) be a general non-degenerate hermitian space over \( E \) of dimension \( m \), and \((W, J) = (W_0, J_0)\) the split skew-hermitian space over \( E \) of dimension \( 2n \). Let \( g = [F : Q] \). As before, associated unitary will be denoted by groups \( G \) and \( H \).

Let \( \psi : \mathbb{A}/Q \to \mathbb{C}^\times \) be a non-trivial additive character, and \( \chi : \mathbb{A}_E^\times /E^\times \to \mathbb{C}^\times \) a Hecke character such that \( \chi|_{\mathbb{A}_E^\times} = \epsilon^m \) where \( \epsilon : \mathbb{A}_E^\times /F^\times \to \mathbb{C}^\times \) is the quadratic character associated to \( E/F \).

Let

\[
S = \mathcal{S}(V(\mathbb{A})^n)
\]

be the space of Schwartz-Bruhat functions. We will occasionally identify it with \( \mathcal{S}(Y \otimes_E V \otimes \mathbb{A}) \) using the fixed isomorphism \( E^n \cong Y = \text{Span}_E(e_1, \cdots, e_n) \).

Associated to the data \((V, Q), (W, J), \psi, \text{ and } \chi\), is a model of the Weil representation \( \omega = \omega_{\psi, \chi} \) of \( G(\mathbb{A}) \times H(\mathbb{A}) \) acting on \( S \). The action of \( h \in H(\mathbb{A}) \) on \( \phi \in S \) is given by

\[
(\omega(h)\phi)(x) = \phi(h^{-1}x), \quad x \in V(\mathbb{A})^n.
\]

(3.1)

Let \( w_0 \in G(\mathbb{Q}) = U(W, J)(F) \) be defined by

\[
w_0(e_i) = -f_i, \quad w_0(f_j) = e_j.
\]

The action of \( G(\mathbb{A}) \) on \( S \) is determined by

\[
(\omega(m)\phi)(x) = \chi(\det(a)) | \det(a)|^{m_2/2} \phi(a^{-1}x), \quad m = m(a) \in M(\mathbb{A}), \quad a \in \text{GL}_n(\mathbb{A}_E),
\]

\[
(\omega(n)\phi)(x) = \psi(\text{tr}(x,x)b) \phi(x), \quad n = n(b) \in N(\mathbb{A}), \quad b \in \text{Herm}_n(\mathbb{A}_E),
\]

\[
(\omega(w_0)\phi)(x) = (\mathcal{F}_Q \phi)(x) = \int_{V(\mathbb{A})^n} \phi(y) \psi(y, x) dy,
\]

where \( dy \) is the Haar measure with respect to the Fourier transform \( \mathcal{F}_Q = \mathcal{F}_{Q, \psi} \) satisfies the Fourier inversion formula \( \mathcal{F}_Q(\mathcal{F}_Q \phi)(x) = \phi(-x) \).

3.2. Seesaw dual-reductive pairs. For \((V, Q)\) and \((W, J)\) as above, put

\[
\mathbb{W} = W \otimes_E V, \quad \langle \; , \; \rangle = \text{Tr}_{E/F}(J^* \otimes_E Q).
\]

Then \((\mathbb{W}, \langle \; , \; \rangle)\) is a symplectic \( F \)-vector space. The \( F \)-groups \( U(W, J) \) and \( U(V, Q) \) form a \textit{dual reductive pair} in \( \text{Sp}(\mathbb{W}) \). In other words, they are mutual centralizers under the natural map \( U(W, J) \times U(V, Q) \hookrightarrow \text{Sp}(\mathbb{W}) \). Applying \( \text{Res}_E^F \) we obtain

\[
G \times H \hookrightarrow \text{Res}_E^F \text{Sp}(\mathbb{W}).
\]

(3.2)

Now assume that \((V, Q) = (V_0, Q_0) \otimes_{E_0} E\), and put

\[
W_0 = \text{Res}_{E_0/E} W, \quad J_0 = \text{Tr}_{E_0/E} J.
\]

Similar to \((\mathbb{W}, \langle \; , \; \rangle)\) define

\[
\mathbb{W}_0 = W_0 \otimes_{E_0} V_0, \quad \langle \; , \; \rangle_0 = \text{Tr}_{E_0/Q_0}(J_0^* \otimes_{E_0} Q_0).
\]

(3.3)
Proposition 3.2. Left as linear algebra exercise.

There is a canonical homomorphism $\text{Res}^E_G \text{Sp}(\mathbb{W}) \to \text{Sp}(\mathbb{W}_0)$, and the inclusions $H, G \subset \text{Res}^E_G \text{Sp}(\mathbb{W})$, $H_0, G_0 \subset \text{Sp}(\mathbb{W}_0)$ are compatible with $i : H_0 \hookrightarrow H$ and $j : G \hookrightarrow G_0$. In the terminology of [Kud83], $(G, H)$ and $(G_0, H_0)$ are seesaw weakly dual pairs, represented by a diagram

$$
\begin{array}{ccc}
G_0 & \rightarrow & H \\
\downarrow & & \downarrow \\
G & \rightarrow & H_0.
\end{array}
$$

Here the oblique lines indicate mutual centralizers and the vertical lines are inclusions.

Suppose $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ is a non-trivial additive character and $\chi_0 : \mathbb{A}_E^0/\mathbb{E}_0^0 \to \mathbb{C}^\times$ be a Hecke character such that $\chi_0|\mathbb{A}^\times = \epsilon_E^0$, where $\epsilon_E^0$ is the quadratic character associated to $E_0/\mathbb{Q}$. Then again we have an associated Weil representation $\omega_0 = \omega_{\psi, \chi_0}$ of $G_0(\mathbb{A}) \times H_0(\mathbb{A})$ acting on $\mathcal{J}(V_0 \otimes V_0(\mathbb{A}))$, where $Y_0 = \text{Res}_{E_0}^E Y$ is a maximal isotropic subspace of $(W_0, J_0)$.

As $E_0$-vector spaces we have

$$V \otimes_E Y \cong (V_0 \otimes_{E_0} E) \otimes_E Y \cong V_0 \otimes_{E_0} Y_0$$

so that $\mathcal{J}(V_0 \otimes_{E_0} Y_0)$ is in fact canonically identified with $\mathcal{J}(V \otimes_E Y) = S$. In order to obtain compatible Weil representations $\omega$ and $\omega_0$, we assume

$$\psi = \psi_0 \circ \text{Tr}_{E/\mathbb{Q}}, \quad \chi = \chi_0 \circ \text{Nm}_{E/\mathbb{E}_0}.$$

For instance $\psi_0$ and $\chi_0$ may be chosen first and $\psi, \chi$ defined as above.

Recall the $\mathbb{Z}$-bases $S = \{ s_i \}_{i=1}^g$ and $R = \{ r_i \}_{i=1}^g$ fixed for $O_F$ and $O_{F^1}$ in (1.5). For $b \in F$, let $B \in M_g(F)$ be the matrix of the $\mathbb{Q}$-linear map $F \to F$, $x \mapsto bx$, with respect to $S$ in the source, and $R$ in the target. Let $\xi = r_1 x_1 + \cdots + r_g x_g \in V(F)$, with $x_i \in V_0$, and similarly $\eta = r_1 y_1 + \cdots + r_g y_g$. Denote by $Q(x, y) \in M_g(E)$ the matrix with entries $Q(x_i, y_j)$.

**Lemma 3.1.** $\text{tr}_{E/\mathbb{E}_0}(Q(\xi, \eta)b) = \text{Tr}(Q(x, y)B)$.

*Proof.* Left as linear algebra exercise. \hfill \Box

We fix the isomorphism

$$\mathcal{J}(V(\mathbb{A})^n) \cong \mathcal{J}(V_0(\mathbb{A})^n), \quad \varphi \mapsto \varphi_0 = \varphi \circ \beta^n_{\psi, \chi}(g),$$

with $\beta$ as in (1.5).

**Proposition 3.2.** The map $\varphi \mapsto \varphi_0$ intertwines the representations $\omega_{Q, \psi, \chi}$ and $\omega_{Q_0, \psi_0, \chi_0}$ of $G$. In other words for each $g \in G(\mathbb{A})$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{J}(V(\mathbb{A})^n) & \xrightarrow{\varphi} & \mathcal{J}(V_0(\mathbb{A})^n) \\
\mathcal{J}(V(\mathbb{A})^n) & \xrightarrow{\omega_{Q, \psi, \chi}(g)} & \mathcal{J}(V_0(\mathbb{A})^n) \\
\mathcal{J}(V_0(\mathbb{A})^n) & \xrightarrow{\omega_{Q_0, \psi_0, \chi_0}(g)} & \mathcal{J}(V_0(\mathbb{A})^n).
\end{array}
$$

*Proof.* It’s enough to verify this for $n = 1$, since taking direct sums implies the statement for higher $n$.

For $n = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in F$, and $\phi \in \mathcal{J}(V(\mathbb{A}))$ we have

$$\omega_Q(n) \phi(\xi) = \psi(Q(\xi, \xi)b) \phi(x) = \psi_0(\text{Tr}_{F/\mathbb{Q}} Q(x, x)b) \phi(x) = \psi_0(\text{Tr}(Q(x, x) \cdot B)) \phi(x),$$

the last identity by the lemma. The desired compatibility for $n \in N(\mathbb{A})$ follows, since $n$ corresponds to $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ in $G_0(\mathbb{A})$ under the identification (1.6) of split skew-hermitian spaces $E^2$ and $E_0^g$ over $E_0$. 


The element \( \omega_0 \in G \) acts on \( \mathcal{S}(V(\mathbb{A})) \) by Fourier transform. We have

\[
(F_{Q_0} \varphi_0)(y) = \int_{V_0(\mathbb{A})^\beta} \varphi_0(x) \psi_0(\text{tr} Q_0(y, x)) dx.
\]

For \( \eta = \beta_\lambda(y), \xi = \beta_\lambda(x) \), by the lemma we have \( \psi_F(Q(\eta, \xi)) = \psi_0(\text{tr} Q_0(y, x)) \). Changing variables by \( \beta_\lambda: V_0(\mathbb{A})^\beta \rightarrow V_\mathbb{A} \), we get

\[
(F_{Q_0} \varphi_0)(\beta^{-1}(\eta)) = \int_{V_\mathbb{A}} \varphi(\xi) \psi(\text{tr} Q(\xi, \eta)) C d\xi = C \cdot (F_{Q} \varphi)(\eta)
\]

where \( C \) is determined by \( (\beta_\lambda^{-1})^*(dx) = C \cdot d\xi \). We claim that \( C = 1 \), i.e. \( (\beta_\lambda^*)^*(d\xi) = dx \).

The product formula implies that the Haar measure on \( V_0(\mathbb{A})^\beta \) with respect to which \( F_{Q_0} \) is self-dual, in fact doesn’t depend on \( Q_0 \), even though self-dual local factors do. This essentially comes down to \(| \det Q_0|_{\mathbb{A}/E_0} = 1 \). It follows that to check \( C = 1 \) we can assume \((V_0, Q_0)\) is the standard split hermitian form on \( E_0^m \), and further that \( m = 1 \).

Write the self-dual measure \( d\xi \) on \( V(\mathbb{A}) = \mathbb{A}_E \) as a product \( d\xi = \prod_{p} d\xi_p \) of local self-dual measures \( d\xi_p \) on \( E_p \). For each rational prime \( p \), we have \( \beta \otimes \mathbb{Z}_p = \beta_p : \mathbb{Z}_p^2 \rightarrow \prod_{v|p} \mathbb{F}_v \). Since \( d\xi_p \) is the self-dual measure on \( \mathcal{O}_{F_v} \), \( \beta^*_p(d\xi_p) \) gives \( \mathbb{Z}_p^2 \) measure \(|D_{F_v}|^{1/2} \). Then \( \beta^*(d\xi_0) \) gives \( \mathbb{Z}_p^2 \) measure \(|D_F|^{1/2} \). It follows that if \( \{dx_{w_0}\} \) are self-dual measures on \( V_0(E_0, w_0) \) at all finite places \( w_0 \) of \( E_0 \), then \( (\beta_\lambda^*)^*(\prod_w d\xi_w) = |D_F|^{1/2} \prod_w dx_{w_0} \). On the other hand the self-dual measures \( d\xi_\infty \) and \( dx_\infty \) on \( E \otimes \mathbb{R} \) and \( E_0^0 \otimes \mathbb{R} \) both coincide with that on \( C^2 \). Then via \( \beta_\infty : E_0 \otimes \mathbb{R} \rightarrow E \otimes \mathbb{R} \) we have \( (\beta_\infty^*)^*(d\xi_\infty) = |\det \beta_\infty| dx_\infty = |D_F|^{-1/2} dx_\infty \). It follows that

\[
(\beta^*_\lambda)^*(d\xi) = |D_F|^{-1/2} dx_\infty \cdot |D_F|^{1/2} dx_f = dx.
\]

The restriction map \( \text{Gal}(F^{ab}/F) \rightarrow \text{Gal}(Q^{ab}/Q) \) is surjective and induces an isomorphism \( \text{Gal}(E/F) \rightarrow \text{Gal}(E_0/Q) \). Under the reciprocity map of global class field theory, that restriction map is induced by the norm \( N_{F/Q} : \mathbb{A}_F^\times \rightarrow \mathbb{A}_E^\times \) which is the same as \( N_{E/F} \) restricted to \( \mathbb{A}_F^\times \). This implies if \( \chi_0|_{\mathbb{A}_E^\times} = c_{E_0}^m \), then \( \chi_0 \circ N_{E/F_0}|_{\mathbb{A}_F^\times} = c_{E_0}^m \). The compatibility of \( \omega \) and \( \omega_0 \) for elements \( m(a) \in G(\mathbb{A}) \) follows from this. \( \square \)

3.3. Fourier coefficients of theta integrals. Using the Weil representation \( \omega \) of \( G(\mathbb{A}) \times H(\mathbb{A}) \) one associates to each \( \phi \in \mathcal{S} \) the theta kernel

\[
\Theta_{\phi}(g, h) = \sum_{x \in V(Q)^n} (\omega(g, h)\phi)(x), \quad g \in G(\mathbb{A}), \ h \in H(\mathbb{A}). \tag{3.7}
\]

One may consider the theta integral

\[
\int_{[H]} \Theta_{\phi}(g, h) dh, \tag{3.8}
\]

where \( dh \) is the invariant measure on \( |H| = H(\mathbb{Q}) \setminus H(\mathbb{A}) \) giving it total volume 1. Let \( r_V \) denote the Witt index of \( (V, Q) \). By a theorem of Weil, the theta integral converges absolutely for all \( \phi \) if and only if either \( V \) is anisotropic, or

\[ \dim V - r_V > \frac{1}{2} \dim W + 1. \]

Now we consider again the case \( (V, Q) = (V_0, Q_0) \oplus E_0 \), with \((V_0, Q_0)\) of signature \((n, 1)\), and \((W_0, J_0) = \text{Res}_{E/E_0}(W, J) \). For the pair of spaces \((V_0, Q_0)\), \((W_0, J_0)\), the numbers are

\[ \dim V_0 = n + 1, \ r_{V_0} = 1, \ \dim W_0 = 2n. \]

These fall just outside the convergence range of Weil, so there is no guarantee that the theta integrals associated to \( \omega_0 \) converge. However, for the pair \( (V, Q) \), \((W, J)\), we have

\[ \dim V = n + 1, \ \dim W = 2, \]

which is within the convergence range since \( n > 1 \). For \( \phi \in \mathcal{S} \), let \( \phi_0 \) denote its image in \( \mathcal{S} = \mathcal{S}(V(\mathbb{A})) \). The compatibility of the Weil representations \( (\omega, S) \) and \( (\omega_0, S_0) \) implies that for \( g \in G(\mathbb{A}), \ h_0 \in H_0(\mathbb{A}), \)

\[ \Theta_{\phi_0}(g(h_0), h_0) = \Theta_{\phi}(g, (\iota(h_0))) \tag{3.9} \]
which implies
\[ I(g; φ) = \int_{[H_0]} \Theta_φ(g, υ(h_0))dh_0 \]
converges absolutely for any \( φ \in S, g \in G(ℋ) \). For convenience we now drop \( υ \) and \( h_0 \) from the notation, and simply write \( \Theta_φ(g, h_0) \), understood to mean either side of (3.3).

For \( b \in F \), the \( b \)th Fourier coefficient of \( I(g, φ) \) with respect to the Siegel parabolic \( P(ℋ) \subset G(ℋ) \) is by definition
\[ I_b(g; φ) = \int_{N(ℋ)} I(n(g; φ)ψ(\text{tr}_{E/F} bb_0))dn, \]
where \( n = n(b_0) = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \in N(ℋ) \).

Now considering \( φ \) as an element of \( \mathcal{S}(V(ℋ)) \), we may write
\[ I_b(g; φ) = \int_{N(ℋ)} \int_{[H_0]} \sum_{ξ ∈ V(Q)} (ω(n(g; φ)φ(ξ)ψ(-\text{tr}_{E/F} bb_0))dh dn
\]
\[ = \int_{[H_0]} \sum_{ξ ∈ V(Q)} ω(g, h)φ(ξ) \int_{N(ℋ)} ψ(\text{tr}_{E/F}((Q(ξ, ξ) - h_0))dndh. \]
The integral then picks out the terms \( ξ ∈ V(Q) \) with \( Q(ξ, ξ) = b \). As the volume of \( N(ℋ) \) is 1, we obtain
\[ I_b(g, φ) = \int_{[H_0]} Θ_b(g, h; φ)dh \]
where
\[ Θ_b(g, h; φ) = \sum_{ξ ∈ V(Q) \ Q(ξ, ξ) = b} ω(g)φ(h^{-1}ξ). \]

Now assume that \( φ = φ_∞ ⊗ φ_L \), where \( φ_L \in \mathcal{S}(V(ℋ_f)) \) is the characteristic function of \( L ⊗ ℋ_f \subset V(ℋ_f) \), and that \( φ_∞ \in \mathcal{S}(V(R)) \) is invariant under a maximal compact subgroup \( K_0 \subset H_0(R) \)-invariant. Let \( K_0(L) \subset H_0(ℋ_f) \) be the stabilizer of \( L ⊗ ℋ_f \), considered as a lattice in \( V_0^n \), take representatives \( h_1, \ldots h_i \) for \( H_0(Q) \setminus H_0(ℋ_f)/K_0(L) \), and put
\[ L_i = h_i(L ⊗ ℋ_f) \cap V(Q), \quad K_i = h_iK_0(L)h_i^{-1}, \quad Γ_i = K_i \cap H_0(Q). \] (3.10)

Then for \( g_∞ ∈ G(R) \),
\[ I_b(g_∞, φ) = \sum_{i=1}^l \int_{Γ_i \setminus D_0} \sum_{ξ ∈ L_i \ Q(ξ, ξ) = b} ω(g_∞)φ_∞(h^{-1}ξ)dh, \]
where \( dh \) is the invariant measure on \( Γ_i \setminus D_0 \) induced from \( H_0(R) \).

Now let
\[ I_{L_i,b} = \{ ξ ∈ L_i : Q(ξ, ξ) = b \}. \]
For \( ξ ∈ I_{L_i,b} \) we write \( O_ξ \) for the Γ-orbit of \( ξ \). Then for some \( ξ_1, \ldots, ξ_t \),
\[ I_b(g_∞, φ) = \sum_{i=1}^t \sum_{ξ ∈ O_ξ} \omega(g_∞)φ_∞(h^{-1}ξ)dh. \]
Now let \( Γ_ξ \) denote the stabilizer of \( Γ \) acting on \( ξ \). Each orbit \( O_ξ \) is then in bijection with \( Γ/Γ_ξ \). Then
\[ I_b(g_∞, φ) = \sum_{i=1}^t \sum_{j=1}^\ell \sum_{g ∈ Γ/Γ_ξ} \omega(g_∞)φ_∞(h^{-1}γξ)dh \]
where \( Γ_ξ,i = Γ_ξ \cap Γ_i \). Now put
\[ C_ξ,i = Γ_ξ,i \setminus D_ξ,i, \quad M_i = Γ_i \setminus D_0, \quad E_i = Γ_ξ,i \setminus D_0. \]
Then \( C_ξ,i \) is a special cycle on \( M_i \) via a map
\[ γ_{ξ,i} : C_ξ,i \to M_i \].
induced by $D_\xi \subset D_0$. The natural inclusion $D_{\xi,i} \hookrightarrow D_0$ induces a fibration $D_0 \to D_{\xi,i}$ by geodesics, which provides a fibration $pr_i : E_i \to C_{\xi,i}$. If $f$ is an integrable function on $E_i$, we may compute the integral using Fubini’s theorem by first integrating over the fibers of $pr_i$ to get $(pr_i)_*f$, then integrating that on $C_{\xi,i}$. Now suppose $f = f_{g,\xi}$ is the function on $D_0$ induced by $h \mapsto \omega(g)\phi_g(h^{-1}\xi)$. If $\kappa_{g,\xi} = (pr_i)_*f_{g,\xi}$ is constant on $D_{\xi,i}$, we obtain

$$I_b(g_{\infty};\phi) = \sum_{i=1}^t \sum_{j=1}^t \sum_{\gamma \in \Gamma_{\xi,i}} \text{vol}(C_{\xi,i})\kappa_{g_{\infty},\gamma}.$$  

The Kudla-Millson Schwartz function $\phi_{KM} \in \mathcal{S}(V(\mathbb{R}))$ has this property and the corresponding constant, calculated in [K87], is determined by

$$\kappa(\xi) = \kappa_{1,\xi} = i^{-n} \exp(-\pi \text{tr}_{E/F} Q(\xi,\xi)).$$

(3.11)

Let $\tau = (\tau_i)_i \in \mathfrak{H}^n$, write $\tau_i = u_i + iv_i$, and put $a_i = \sqrt{v_i}$. Let $\alpha \in GL_1(E \otimes \mathbb{R})$ and $\nu \in \text{Herm}_1(F \otimes \mathbb{R})$ correspond to $a = (a_i)_i$ and $u = (u_i)_i$, and put

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & \nu^{-1} \\ \Gamma & 1 \end{pmatrix} \in G(\mathbb{R}).$$

Then for $\phi \in \mathcal{S}(V(\mathbb{R}))$, by definition

$$\omega(g_{\tau})\phi(\xi) = \chi(\alpha) |\alpha|_{\mathcal{A}_E}^{m/2} \phi(\xi) \psi(\text{tr}_{E/F} Q(\xi,\xi)\nu).$$

We have $\chi(\alpha) = 1$, since the assumption $\chi_{\mathcal{A}_E} = e_{\mathcal{A}_E}$ implies $\chi$ has trivial infinity type. Also

$$|\alpha|_{\mathcal{A}_E} = |\text{Nm}(\det(\alpha))|_{\mathcal{A}_E} = \prod_i |a_i|^2 = \prod_i v_i,$$

and

$$\psi(Q(\xi,\xi)\nu) = \psi_0(\text{tr}_{E/F} Q(\xi,\xi)\nu) = \psi_0(\text{Tr} \beta u) = \exp(2\pi i \sum_j \lambda_j(b)u_j) = e_*(bu),$$

where $u$ is considered as a column vector. Therefore

$$\omega(g_{\tau})\phi(\xi) = \phi(h^{-1}\xi) \prod_i \text{Im}(\tau_i)^{m/2} e_*(bu).$$

For $\phi = \phi_{KM}$, it follows from (3.11) that

$$\int_{E_{\xi,i}} \omega(g_{\tau})\phi_{KM}(h^{-1}\xi)dh = \text{vol}(C_{\xi,i}) \prod_i \text{Im}(\tau_i)^{m/2} e_*(bu)i^{-n} \exp(-\pi \text{tr}_{E/F} Q(\xi a,\xi a)).$$

Now

$$\exp(-\pi \text{tr}_{E/F} Q(\xi a,\xi a)) = \exp(2\pi i \sum_j \lambda_j(b)iv_j) = e_*(biv)$$

so in fact

$$\int_{E_{\xi,i}} \omega(g_{\tau})\phi_{KM}(h^{-1}\xi)dh = \prod_i \text{Im}(\tau_i)^{m/2} e_*(br).$$

We then obtain the formula

$$I_b(g_{\tau};\phi_{KM}) = i^{-n} \prod_j \text{Im}(\tau_j)^{m/2} \sum_{i=1}^t \sum_{j=1}^t \sum_{\gamma \in \Gamma_{\xi,i}} \text{vol}(C_{\xi,i})e_*(br).$$

(3.12)

For $\tau \in \mathfrak{H}^n$, put

$$F_1(\tau) = \sum_{b \in F} I(C_{\xi,i},C_b)e_*(br)$$

where $C_{\xi} = D_0/\Gamma_1$, and $I(C_{\xi},C_b)$ is defined analogously to $I(C_{0},C_b)$. Put

$$F(\tau) = \sum_{i=1}^t F_1(\tau).$$

We have thus proven:

**Theorem 3.3.** $F(\tau) = i^n |\text{Im} \tau_i|^{-m/2} I(g_{\tau};\phi_{KM})$. In particular, $F(\tau)$ is a modular form of weight $\frac{m}{2}$. 
Proof. From \([3.12]\) we get
\[
I(g; \phi_{KM}) = \sum_{b \in F} i^{-n} \prod_{j} |\text{Im } \tau_j|^{m/2} \sum_{t=1}^{l} \sum_{i=1}^{t} \text{vol}(C_{\xi,i}) \epsilon_{*}(br).
\]

By Theorem \([2.4]\) and the discussion preceding it, we have
\[
\sum_{j=1}^{t} \sum_{\gamma \in \Gamma \setminus \Gamma_{t}} \text{vol}(C_{\xi,i}) = I(C_{i}, C_{b}),
\]
so that
\[
I(g_{\tau}; \phi_{KM}) = \sum_{b \in F} I_{b}(g_{\tau}, \phi_{KM}) = i^{-n} \prod_{j} |\text{Im } \tau_j|^{m/2} \sum_{i=1}^{t} \sum_{b \in F} I(C_{i}, C_{b}) \epsilon_{*}(br).
\]

In the second part of the article we show that \(I(g; \phi_{KM})\), and hence \(F(\tau)\), can be identified with the restriction of automorphic forms on \(U(n, n)\) related to Siegel Eisenstein series.

4. Restrictions of Eisenstein Series

We recall the construction of Eisenstein series associated to sections of parabolically induced representations. The Siegel-Weil formula identifies theta integrals in Weil’s convergence range with values of such Eisenstein series at particular points. Outside this range the regularized-Siegel Weil formula gives an analogous statement obtained by regularizing the theta integral.

4.1. Eisenstein series. We extend the Hecke character \(\chi : \mathbb{A}_{K} / E^{\times} \to \mathbb{C}\) to a character of \(P(\mathbb{A})\) by
\[
\chi \left( \begin{pmatrix} a & g \\ \alpha & 1 \end{pmatrix} \right) = \chi(\det a).
\]

We also have the determinant character \(p(a) \in P(\mathbb{A}) \to |\det(a)|^{s}, \) for \(s \in \mathbb{C}\). There is then a normalized induced representation
\[
I(s, \chi) = \text{Ind}_{P(\mathbb{A})}^{(4.2)}/\chi|_{\text{det}(a)|^{s}}.
\]

A section of \(I(s, \chi)\) is a function \(f^{(s)} : G(\mathbb{A}) \to \mathbb{C}\) depending on a parameter \(s \in \mathbb{C}\) such that
\[
f^{(s)}(pg) = \chi(a) \det |d^{s + \frac{n}{2}} f(g), \quad \text{for } g \in G(\mathbb{A}), p \in P(\mathbb{A}).
\]

The section \(f^{(s)}\) is called holomorphic if it’s \(K\)-finite and holomorphic in the variable \(s\). A standard section is a holomorphic section whose restriction to \(K\) is independent of \(s\).

Put
\[
s_{0} = \frac{m - n}{2}.
\]

The Siegel Eisenstein series associated to a holomorphic section \(f^{(s)}\) of \(I(s, \chi)\) is the series
\[
E(g, f^{(s)}) = \sum_{\gamma \in P(F) \setminus G(F)} f^{(s)}(\gamma g).
\]

It is convergent for \(s \geq n/2\), and has meromorphic continuation to the \(s\)-plane.

Associated to each \(\varphi \in \mathcal{S}(V(\mathbb{A})^{n})\) is a standard Siegel-Weil section \(f_{\varphi}^{(s)}\) given by
\[
f_{\varphi}^{(s)}(g) = |\det a(g)|^{s - s_{0}} \omega(g) \varphi(0).
\]

The Eisenstein series \(E(g, f_{\varphi}^{(s)})\) is holomorphic for \(\text{Re}(s) \geq 0\), except at possibly at \(s = s_{0}\), where it has a pole of order at most 1. Its Laurent expansion is written
\[
E(g, f_{\varphi}^{(s)}) = \frac{A_{-1}(\varphi)}{s - s_{0}} + A_{0}(\varphi) + A_{1}(\varphi) \cdot (s - s_{0}) + \cdots.
\]
By a result of Weil, the theta integral converges absolutely for all \( \phi \in \mathcal{S}(V^n(\mathbb{A})) \) if and only if either \( r = 0 \) or \( m - n > r \). If furthermore \( m > 2n \), the Siegel-Weil formula holds:

\[
E(g, f_\varphi^{(s)}) = I_Q(g, \varphi).
\]

### 4.2. Regularized theta integrals

Beginning with the work of [KR88], the Siegel-Weil formula has been extended outside the Weil range. For this the theta integral on the right hand side of the formula must be regularized.

An ingredient used in the regularization is the auxiliary Eisenstein series \( E_H(s, h) \) on \( H(\mathbb{A}) \) defined by

\[
E_H(s, h) = \sum_{\gamma \in P_H(F) \setminus H(F)} f_0^{(s)}(\gamma h),
\]

where \( f_0^{(s)} \) is the standard \( K_H \)-spherical section determined by \( f_0^{(s)}(1) = 1 \). \( E_H(s, h) \) has a pole of order at most 1 at \( s = \rho_H \), where

\[
\rho_H = \frac{m - r}{2},
\]

and the residue

\[
\kappa = \text{Res}_{s = \rho_H} E_H(s, h)
\]

is a constant function.

The regularization in [KR88] is achieved using a particular element \( z_G \) in the central enveloping algebra of \( \text{Lie}(G(\mathbb{R}))_c \), such that \( \Theta(\omega(\varphi); g, h) \) is rapidly decreasing on \( H \) for all \( \varphi \in \mathcal{S}(V^n(\mathbb{A})) \) [KR88,GQT14]. By Howe duality one may choose a corresponding element \( z_H \) in the universal enveloping algebra of \( \text{Lie}(H(\mathbb{R}))_c \), such that \( \omega(z_G) \) and \( \omega(z_H) \) coincide as operators on \( \mathcal{S}(V(\mathbb{A})^n) \). Then one has

\[
\omega(z_H) \cdot E_H(s, h) = P(s) \cdot E_h(s, h),
\]

for an explicitly computable function \( P(s) \) that depends in general on the choice of \( z_H \).

The regularized theta integral is then defined to be

\[
B(s, \varphi) = \frac{1}{\kappa \cdot P(s)} \int_{[H]} \Theta(\omega(\varphi); g, h) E_H(s, h) dh.
\]

We will assume the parameters \( r, m, n \) satisfy

\[
0 < m - n \leq r \leq n.
\]

In that case \( P(s) \) has a simple zero at \( s = \rho_H \), and \( B(s, \varphi) \) has a pole of order at most 2 [GQT14] Lemma 3.8. By convention, the Laurent expansion of \( B(s, \varphi) \) at \( s = \rho_H \) is written

\[
B(s, \varphi) = \frac{B_{-2}(\varphi)}{(s - \rho_H)^2} + \frac{B_{-1}(\varphi)}{(s - \rho_H)} + B_0(\varphi) + \cdots.
\]

### 4.3. Regularized Siegel-Weil formula

Since \( m - n > 0 \) by the assumption \ref{4.13}, the parameters fall in the so-called second-term range. In this case the regularized Siegel-Weil formulas of [Ich04] and [GQT14] apply, which we now briefly state.

Let \( V'' \subset V \) be a split hermitian subspace of dimension \( 2(m - n) \), and denote its orthogonal complement by \( V' \). Put \( Q' = Q|_{V'} \) and \( H' = U(V', Q') \).

Writing \( V'' \) as a direct sum \( X \oplus X^* \) of (dual) isotropic subspaces of dimension \( r_0 = m - n \), we have \( V = X \oplus V' \oplus X^* \). Using this decomposition, we define the parabolic subgroup \( P_H \subset H \) by

\[
P_H(R) = \left\{ \begin{pmatrix} a & * & * \\ g & * \\ & & \alpha^{-1} \end{pmatrix} : a \in \text{GL}_{r_0}(R \otimes X), \ g \in H'(R) \right\}, \ \ R \in \mathbb{Q} - \text{Alg}.
\]

Fix a good maximal compact subgroup \( K \subset H(\mathbb{A}) \) such that \( H(\mathbb{A}) = P_H(\mathbb{A}) \cdot K \). Let \( dk \) be the Haar measure on \( K \) that gives it volume 1, and define

\[
\pi_K : \mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n(\mathbb{A})), \ \pi_K(\phi)(x) = \int_K \phi(kx) dx.
\]
The Ikeda map is

\[ \text{Ik} : \mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n(\mathbb{A})), \quad \text{Ik}(\varphi)(x) = \int_{X^n} \phi \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} dy, \]

where the coordinates inside the integral correspond to \( V^n = X^n \oplus V^n \oplus X^{*n} \).

The regularized Siegel-Weil formula of Ichino then states

\[ A_{-1}(\phi) = c_K I_Q \langle g, \text{Ik} \circ \pi_K(\phi) \rangle. \quad (4.16) \]

Furthermore, by the main results of [GQT14], Theorem 8.1,

\[ A_{-1}(\phi) = B_{-2}(\phi) \quad \text{(first-term identity)} \quad (4.17) \]
\[ A_0(\phi) = B_{-1}(\phi) - \kappa' \{ B_0'(\text{Ik}(\pi_K(\phi))) \} \pmod{\text{Im} A_{-1}}. \quad \text{(second-term identity)} \quad (4.18) \]

Here \( \kappa' \) is an explicit constant, and \( B_0' \) is the (possibly regularized) theta integral associated to the dual reductive pair \((G, H')\). If \( V' \) is anisotropic, the term in \( \{ \cdots \} \) is to be interpreted as zero. We note the second-term identity is modulo the entire image of the map \( A_{-1} \) from \( \mathcal{S}(V(\mathbb{A})^n) \) to automorphic forms on \( G(\mathbb{A}) \).

### 4.4. Refined regularization.

The second-term identity simplifies when \( \phi \in \mathcal{S} \) satisfies certain conditions, and \( m = n + 1, r = 1 \). Note these values fall in the second-term range. By [IK13] the value of \( E(g, f^{(s)}_\varphi) \) at \( s = s_0 \) is in that case

\[ A_0(\phi) = B_{-1}(\phi) \pmod{\text{Im} A_{-1}}. \]

The specific conditions we impose on \( \phi \) are the following:

1. \( \phi \) is factorizable: \( \phi = \otimes_v \phi_v \), for \( \phi_v \in \mathcal{S}(V(\mathbb{Q}_v)^n) \).
2. For some place \( v_0 \), \( \text{Ik}_{v_0} \circ \pi_{K_{v_0}}(\phi_{v_0}) = 0 \), where \( \text{Ik}_{v_0} \) and \( \pi_{K_{v_0}} \) are the local components of \( \text{Ik} \) and \( \pi_K \).

In particular (2) implies \( \text{Ik} \circ \pi_K(\phi) = 0 \). By (4.16) and (4.17) then

\[ B_{-2}(\phi) = A_{-1}(\phi) = 0. \]

In other words, \( B(\phi) \) has a pole of order at most 1 at \( s = s_0 \).

Recall the global Siegel principal series representations \( I_n(s, \chi) \) of \( G_n(\mathbb{A}) \) and the \( H(\mathbb{A}) \)-invariant function

\[ \Phi_n^{(s)} : \mathcal{S}(V(\mathbb{A})^n) \to I_n(s, \chi), \quad \Phi_n^{(s)}(g) = (\omega_n(g)\psi)(0). \]

There’s a factorization \( \Phi_n^{(s)} = \otimes_v \Phi_n^{(s)}_{\mathbb{A}/v} \) into local components. We put

\[ \Phi_n = \Phi_n^{(s_0)}. \quad (4.19) \]

Then \( \Phi_n \) is \( G(\mathbb{A}) \)-intertwining and maps \( \psi \in \mathcal{S}(V(\mathbb{A})^n) \) to \( f^{(s_0)}_\psi \), the standard Siegel-Weil section introduced before.

There are standard global intertwining operators

\[ M_n(s, \chi) : I_n(s, \chi) \to I_n(-s, \chi), \quad (M_n(s, \chi)f)(g) = \int_{N(\mathbb{A})} f(wng)dn \]

which factor into a tensor product of normalized local intertwining operators

\[ M_n(s, \chi) = \otimes_v M_{n,v}^{s}(s, \chi). \quad (4.20) \]

The normalization we use is the one given by Lapid-Rallis [LR05], which is the same as in [GQT14], [GI14]. It’s known that the derivative \( M_{n,v}'(0, \chi) \) commutes with \( M_{n,v}^{s}(0, \chi) \) and preserves the irreducible submodules of \( I_n(v, \chi) \). The derivative \( M'(0, \chi) \) appears in the regularized Siegel-Weil formula via the following construction.

Choose and fix a factorizable \( \phi_1 = \otimes_v \phi_{1,v} \in \mathcal{S}(V(\mathbb{A})) \) such that

\[ \phi_1(0) = 1, \quad \pi_K(\phi_1) = \phi_1. \quad (4.21) \]

For \( \phi \in \mathcal{S}(V(\mathbb{A})^n) \), consider \( \phi_1 \otimes \phi \) as an element of \( \mathcal{S}(V(\mathbb{A})^{n+1}) \). Then there exists \( \phi_M \in \mathcal{S}(V(\mathbb{A})^{n+1}) \) such that

\[ \Phi_{n+1}(\phi_M) = M_{n+1}'(0, \chi)\Phi_{n+1}(\phi_1 \otimes \phi). \quad (4.22) \]
Let $\phi' \in \mathcal{S}(V(\mathbb{A})^n)$ be the restriction of $\phi_M$ to $V(\mathbb{A})^n$ via $\phi'(x) = \phi_M(0,x)$.

**Proposition 4.1.** For $\phi \in \mathcal{S}(V(\mathbb{A})^n)$ satisfying (1), (2), and $\phi'$ as above, we have

$$A_0(\phi) - \frac{1}{2} A_{-1}(\phi') = B_{-1}(\phi).$$

**Proof.** Since $n - m = 1$, the pair of spaces $(W_n, J_n)$, $(V, Q)$ lie in the so-called boundary range, i.e. $n$ is the smallest value outside Weil’s convergence. Then $G_{n+1}(\mathbb{A})$, $H(\mathbb{A})$ are inside the classical convergence range, and so

$$A_0(\phi_1 \otimes \phi) = 2B_{-1}(\phi_1 \otimes \phi) \quad (4.23)$$

for $\phi_1 \otimes \phi \in \mathcal{S}(V(\mathbb{A})^{n+1})$. The second-term identity (4.18) of [GQT14] is obtained for the boundary case by computing the constant terms of both sides of the above with respect to a parabolic subgroup of $G_{n+1}$ with Levi component $GL_1 \times G_n$, then taking residues at $s = 0$. In fact following through their computations we see the precise identity is

$$A_0(\phi) - \frac{1}{2} A_{-1}(\phi') = B_{-1}(\phi) - C'_r \cdot B_0(\text{Ik} \circ \pi_{K_H}(\phi)) + C_r \cdot B_{-1}(\text{Ik} \circ \pi_{K_H}(\phi)) \quad (4.24)$$

for some explicit constants $C_r, C'_r$.

The statement then follows from the assumption $\text{Ik} \circ \pi_{K}(\phi) = 0$. \hfill \qed

In the next section, we will show that under the same assumptions the (unregularized) theta integral $I(g; \phi)$ in fact converges, and is equal to $B_{-1}(\phi)$.

**4.5 The mixed model.** For the moment let us again allow any general pair $(V, Q)$, $(W, J)$, where $(V, Q)$ has Witt index $r$. Recall the decomposition $V = X \oplus V_{an} \oplus X^*$ from [11] and $W = Y \oplus Y^*$. The model of the Weil representation of $\text{Sp}(W)(\mathbb{A})$ described before acts on $S = \mathcal{S}(Y \otimes_E V(\mathbb{A}))$. There is a mixed model of the same representation where the space acted on is $\tilde{S} = \mathcal{S}(Y \otimes_E V_{an} + W \otimes_X X^*)(\mathbb{A})$. We have

$$Y \otimes_E V = Y \otimes_E X + Y \otimes_E V_{an} + Y \otimes_E X^*$$

and

$$Y \otimes_E V_{an} + W \otimes_X X^* = Y \otimes_E V_{an} + Y \otimes_E X^* + (Y \otimes_E X)^*$$

where $Y^* \otimes E X^*$ has been identified with $(Y \otimes_E X)^*$. In these coordinates the intertwining map $\mathcal{S} \to \tilde{S}$, $\phi \mapsto \tilde{\phi}$ is given by the partial Fourier transform

$$\tilde{\phi}(v_0, u', v') = \int \phi \left( \begin{array}{c} u \\ v_0 \\ u' \end{array} \right) \psi_F(v'(u)) du. \quad (4.25)$$

Again for convenience we make the identifications

$$Y \otimes_E V_{an} = V_{an}^n, \quad W \otimes_X X^* = W^r$$

so that $\tilde{S} = \mathcal{S}(V_{an}(\mathbb{A})^n \oplus W(\mathbb{A})^r)$. Then the theta kernel associated to $\phi \in \tilde{S}$ may be written as

$$\Theta(g, h; \phi) = \sum_{v_0 \in V_{an}(F)^r, w \in W(F)^r} \omega(g, h)\tilde{\phi}(v_0, w). \quad (4.26)$$

Let $\omega(z)$ be the operator on $\mathcal{S}$ corresponding to the regularizing elements $z_G$ and $z_H$. For the moment we identify $W(F)^r$ with the matrix group $M_{n,r}(F)$. Two essential facts that enable the regularization procedure are that $\omega(z)\phi(v_0, w) = 0$ if $\text{rank}(w) < r$, and that for all $\phi \in \mathcal{S}$, the function

$$\Theta'(g, h, \phi) = \sum_{v_0 \in V_{an}(F)^r, w \in M_{n,r}(F)} \omega(g, h)\tilde{\phi}(v_0, w)$$

is rapidly decreasing on $H(F) \backslash H(\mathbb{A})$.

**Proposition 4.2.** Assume $r = 1$. If $\phi \in \mathcal{S}(V(\mathbb{A})^n)$ is $K_H$-invariant and satisfies $\text{Ik}(\phi) = 0$, then $h \mapsto \Theta(g, h; \phi)$ is rapidly decreasing on $H(F) \backslash H(\mathbb{A})$. 
**Proof.** We show that in the expansion (4.26), all the terms with \(\text{rank}(w) < r\) vanish. From (4.26), it’s easy to see that

\[
I_k(\phi)(v_0) = \hat{\phi}(v_0, 0).
\]

On the other hand if \(r = 1\), the condition \(\text{rank}(w) = r\) is simply \(w \neq 0\), so that \(\hat{\phi}(v_0, 0) = 0\) for all \(v_0 \in V_n(F)^n\). Now for \(h \in H(\mathbb{A})\) write \(h = pk\) with \(k \in K_H\) and

\[
p^{-1} = \begin{pmatrix} a & r & s \\
0 & t & u \end{pmatrix} \in P_H(\mathbb{A}).
\]

Since \(\phi\) is \(K_H\)-invariant,

\[
\omega(h)\phi \begin{pmatrix} u \\
v_0 \\
0 \end{pmatrix} = \omega(p)\phi \begin{pmatrix} u \\
v_0 \\
0 \end{pmatrix} = \phi \begin{pmatrix} au + rv_0 \\
h_0v_0 \\
0 \end{pmatrix}
\]

so that \(\omega(h)\hat{\phi}(v_0, 0) = I_k(\phi)(hv_0v_0) = 0\) for all \(h \in H(\mathbb{A})\). From the fact that the Ikeda map is \(G(\mathbb{A})\)-intertwining, it also follows that

\[
(\omega(g)\hat{\phi})(v_0, 0) = \omega(g)\hat{\phi}(v_0, 0) = I_k(\omega(g)\phi)(v_0) = \omega(g)I_k(\phi)(v_0) = 0.
\]

Then we have \(\omega(g, h)\hat{\phi}(v_0, 0) = 0\) for all \(v_0\), so that

\[
\Theta(g, h; \phi) = \Theta'(g, h; \phi).
\]

Since the right-hand side is rapidly decreasing, the proposition follows.

Now let us restrict again to the case where \((V, Q) = (V_0, Q_0) \otimes_{E_0} E\), with \((V_0, Q_0)\) having signature \((n, 1)\), \(n = [F : \mathbb{Q}]\), and \((W_0, J_0) = (\text{Res}_{E/E_0} W_1, \text{tr}_{E/E_0} J_1) \simeq (W_n, J_n)\). If \(\phi \in S\) is \(K_{H_0}\)-invariant, and \(I_k(\phi) = 0\), by the proposition the theta integral

\[
I(g; \phi) = \int_{[H_0]} \Theta(g, h; \phi)dh, \quad g \in G_0(\mathbb{A}),
\]

converges, even though \((V_0, Q_0)\), \((W_0, J_0)\) are outside Weil’s convergence range. On the other hand, so does the regularized theta integral

\[
I_{\text{reg}}(g, s; \phi) = \int_{[H_0]} \Theta(g, h; \omega(z)\phi)E_{H_0}(h, s)dh
\]

for large enough \(\text{Re}(s)\), and it has meromorphic continuation to the \(s\)-plane.

**Proposition 4.3.** Suppose that \(r = 1\), \(\phi\) is \(K_{H_0}\)-invariant, and \(I_k(\phi) = 0\). For \(\rho = \frac{1}{2}\), we have

\[
\text{Res}_{s=\rho} \frac{1}{P(s)\kappa} I_{\text{reg}}(g, s; \phi) = I(g; \phi).
\]

**Proof.** By Proposition (4.2), \(\Theta(g, h; \phi)\) is rapidly decreasing on \([H_0]\), so we can write

\[
\int_{[H_0]} \Theta(g, h; \omega(z)\phi)E_{H_0}(h, s)dh = \int_{[H_0]} \Theta(g, h; \phi)(z_{H_0}H_0(h, s))dh
\]

\[
= P(s) \int_{[H_0]} \Theta(g, h; \phi)E_{H_0}(h, s)dh.
\]

Then

\[
\text{Res}_{s=\rho} \frac{1}{P(s)\kappa} I_{\text{reg}}(g, s; \phi) = \text{Res}_{s=\rho} \frac{1}{\kappa} \int_{[H_0]} \Theta(g, h; \phi)E_{H_0}(h, s)dh
\]

\[
= \frac{1}{\kappa} \int_{[H_0]} \Theta(g, h; \phi)(\text{Res}_{s=\rho} E_{H_0}(h, s))dh
\]

\[
= \int_{[H_0]} \Theta(g, h; \phi)dh.
\]

\(\square\)
Corollary 4.4. Suppose \( \phi \) satisfies the assumptions of Proposition 4.1. Then
\[
I(g; \phi) = E(g; f^{(s_0)}) - \frac{1}{2} \text{Res}_{s=s_0} E'(g; f^{(s)}).
\]

Proof.

Now we apply the above to \( \phi = \varphi_{\text{KM}} \otimes \varphi_L \in \mathcal{S}(V_0(\mathbb{A})^n) \), which was used previously in \( \S 3 \) to prove the modularity of
\[
F(\tau) = \sum_{i=1}^I F_i(\tau) = \sum_{i=1}^I \sum_{b \in F} I(C_i, C_b) e_s(b\tau).
\]
By construction, \( \phi \) is factorizable. Theorem 5.6 from the appendix will show that \( \phi_{\text{KM}} \) is killed by the Ikeda map. Then the corollary therefore applies to \( \phi \).

Let \( T = \mathbb{C}^n \) be an element of the hermitian upper-half plane of degree \( n \), so that \( T = U + iv \) with \( U, V \in \text{Herm}_n(\mathbb{C}) \), and \( V \) positive-definite. Write \( T = \mathcal{A} \mathcal{A}^{-1} \), and put
\[
g_T = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & \mathcal{A}^{-1} \\ 0 & 1 \end{pmatrix} \in G_0(\mathbb{R}), \quad \tilde{F}(T) = i^n | \text{det} T |^{-m/2} I(g_T; \phi).
\]

Then \( F(T) \) is a hermitian modular form of degree \( n \), and essentially a Siegel Eisenstein series. The map \( G(\mathbb{R}) \to G_0(\mathbb{R}) \) then induces an injection \( i_{\mathbb{R}} : \mathcal{S}^n \to \mathbb{C}^n \). We have thus proved:

Theorem 4.5. For \( \phi = \varphi_{\text{KM}} \otimes \varphi_L \),
\[
\tilde{F}(T) = E(g_T; f^{(s)}) - \frac{1}{2} \text{Res}_{s=s_0} E'(g_T; f^{(s)}).
\]

In particular, \( F(\tau) \) is the restriction of the right-hand side above to \( \mathbb{C}^n \subset \mathbb{C}^n \).

Remarks. The same argument as in the computation of \( I_b(g, \phi) \), identifies the non-degenerate Fourier coefficients \( I(g, \phi) \), taken with respect to \( I_b(\mathbb{A}) \subset G_0(\mathbb{A}) \), with intersection numbers \( I(C_b, C_0) \), i.e. those indexed by \( \beta \in \text{Herm}_n(\mathbb{E}_0) \) with \( \text{det}(\beta) \neq 0 \). The restriction map to \( \mathbb{C}^n \) then groups together all such coefficients \( \beta \) with \( \mathcal{A} \mathcal{A}^{-1} b \).

It’s possible, but we have not been able to show, that the term \( \text{Res}_{s=s_0} E'(g, f^{(s)}) \) occurring in the refined regularization vanishes. This would be the case for instance, if the operation \( \phi \mapsto \phi' \) preserves the kernel of the Ikeda map. In that case the refined formula would say that the Siegel-Weil formula in fact holds outside the convergence range for particular \( \phi \), such as \( \varphi_{\text{KM}} \otimes \varphi_L \).

The “generalized” intersection volumes may have geometric interpretations in terms of the spectacle cycles of Funke and Millson [FMT1].

5. Appendix

5.1. The Kudla-Millson construction. First we recall the definition of the Kudla-Millson Schwartz function \( \varphi_{q,q}^{\dagger} \), following [KM86, K87, KM90] closely. We define an adelic Schwartz function \( \varphi_{\text{KM}} \in \mathcal{S}(V(\mathbb{A})^n) \) using \( \varphi_{q,q}^{\dagger} \), and we show it is annihilated by a regularizing Ikeda map
\[
I_k^{q,q} : \mathcal{S}(V(\mathbb{A})^n) \to \mathcal{S}(V'(\mathbb{A}))^n,
\]
for some complementary subspace \( V' \) of \( V \). That will imply the corresponding Eisenstein series \( E(s, \varphi_{\text{KM}}) \) has a simple pole at \( p_H = \frac{m-r}{r} \).

Let \( (U, (\ , \ )) \) be a non-degenerate complex hermitian space of dimension \( m \) and signature \( (p,q) \), with \( pq \neq 0 \) and \( p < q \), to be thought of as one of the archimedean completions of \( (V, Q) \) from the introduction. Put
\[
G_U = \text{SU}(U, (\ , \ )) \tag{5.1}
\]
Let \( D_0 \) be the set of negative-definite \( q \)-dimensional subspaces of \( U \), \( Z_0 \in D_0 \) the span of \( u_{p+1}, \ldots, u_m \). It may be identified with the hermitian symmetric domain associated to \( G_U \) as follows.

Fix a basis \( u_1, \ldots, u_m \) with dual \( z_1, \ldots, z_m \) with respect to which \( (\ , \ ) \) has the standard form
\[
(u, u) = \sum_{i=1}^p |z_i|^2 - \sum_{j=p+1}^m |z_j|^2, \quad u \in U.
\]
Let $Z_0 = \text{Span}\{u_{p+1}, \ldots, u_m\} \in D_0$. Then $D_0 \simeq G_U/K$, where $K$ is the maximal compact subgroup stabilizing $Z_0$.

Let $\mathfrak{k} = \text{Lie}(K)_{\mathbb{C}}$ and $\mathfrak{g} = \text{Lie}(G_U)_{\mathbb{C}}$ be the complexified Lie algebras, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the orthogonal decomposition with respect to the Killing form on $\mathfrak{g}$. Then $\mathfrak{p}$ can be identified with the complexified tangent space $T_{Z_0}(D_0)$, and its complex dual $\mathfrak{p}^*$ with left $G_U$-invariant 1-forms $\Omega^1(D_0)$ on $D_0$. The invariant $k$-forms $\Omega^k(D_0)$ are then identified with $\bigwedge^k \mathfrak{p}^*$.

Write $U = U^+ \oplus U^-$, where $U^+ = \text{Span}\{u_1, \ldots, u_p\}$ and $U^- = Z_0$. Using the basis $v_1, \ldots, v_m$ to identify $\mathfrak{gl}(U)$ with $M_m(\mathbb{C})$, an explicit basis for $\mathfrak{g} \subset \mathfrak{gl}(U)$ is given by

$$\{X_{jk} : j < k\} \cup \{Y_{jk} : j \leq k\},$$

where

$$X_{jk} = -E_{jk} + E_{kj}, \quad Y_{jk} = -i(E_{jk} + E_{kj}). \quad (5.2)$$

Let $\{X_{jk}^*, Y_{jk}^*\}$ denote the dual basis for $\mathfrak{g}^*$. Then a basis for $\mathfrak{p}^*$ is

$$\{X_{jk}^*, Y_{jk}^* : 1 \leq j \leq p, \ p + 1 \leq k \leq m\}.$$

Put

$$\xi_{jk} = X_{j,k+p}^* + i Y_{j,k+p}, \quad 1 \leq j \leq p, \ 1 \leq k \leq q. \quad (5.3)$$

Under the identification $\mathfrak{p}^* \simeq \Omega^1(D)$, $\{\xi_{jk}\}$ is a basis for the subspace $(\mathfrak{p}^*)^+ \simeq \Omega^{1,0}(D)$ of $G_U$-invariant $(1,0)$-forms on $D$. Let $A_{jk}$ denote left-multiplication by $\xi_{jk}$ in the exterior algebra $\bigwedge \mathfrak{p}^*$. The unitary Howe operator $D^+ : \bigwedge \mathfrak{p}^* \otimes \mathcal{H}(V) \to \bigwedge \mathfrak{p}^* \otimes \mathcal{H}(V)$ is defined by

$$D^+ = \frac{1}{2^{2q}} \left\{ \prod_{k=1}^{q} \sum_{j=1}^{p} A_{jk} \otimes \left( z_j - \frac{1}{\pi} \frac{\partial}{\partial z_j} \right) \right\}. \quad (5.4)$$

Write

$$\varphi_0 = \exp \left( -\pi \sum_{i=1}^{m} |z_i|^2 \right) \quad (5.5)$$

for the standard Gaussian on $U$. The Kudla-Millson form of type $(q,q)$ is

$$\varphi_{+}^q = D^+ \overline{D^+} \varphi_0 \in \bigwedge \mathfrak{p}^* \otimes \mathcal{H}(U). \quad (5.6)$$

To unwind the expression for $\varphi_{+}^q$, we employ multi-index notation. For a set of integers

$$\alpha = (a_1, \ldots, a_q), \quad 1 \leq a_1 \cdots a_q \leq p \quad (5.7)$$

put

$$z_{\alpha} = z_{a_1} \cdots z_{a_q} \quad (5.8)$$

and

$$B_{\alpha} = A_{a_1,1} \wedge A_{a_2,2} \wedge \cdots \wedge A_{a_q,q}, \quad (5.9)$$

which is a $(q,0)$-form in $\bigwedge \mathfrak{p}^*$. We have a differential operator

$$D_i = z_i - \frac{1}{\pi} \frac{\partial}{\partial z_i}, \quad (5.10)$$

and its multi-index iteration

$$D_\alpha = D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_r}. \quad (5.11)$$

**Lemma 5.1.**

$$\varphi_{+}^q = 2^{-q} \sum_{\alpha, \alpha'} B_{\alpha} \wedge \overline{B_{\alpha'}} \otimes D_\alpha D_{\alpha'} \varphi_0,$$

where $\alpha, \alpha'$ range over all possible $q$-tuples satisfying (5.7).
Proof. By definition
\[ \varphi^*_{q,q} = \frac{1}{2^{q}} \left\{ \prod_{k=1}^{q} A_{j,k} \otimes \left( z_j - \frac{1}{\pi} \frac{\partial}{\partial z_j} \right) \right\} \left\{ \prod_{j=1}^{p} \sum_{k=1}^{q} A_{j,k} \otimes \left( z_j - \frac{1}{\pi} \frac{\partial}{\partial z_j} \right) \right\} \varphi_0. \]

Expanding the product, it can be written as
\[ \frac{1}{2^{2q}} \sum_{1 \leq a_1, \ldots, a_q \leq p} \sum_{1 \leq a_1', \ldots, a_q' \leq p} A_{a_1,1} \wedge \cdots \wedge A_{a_q,q} \wedge A_{a_1',1} \cdots A_{a_q',q} \otimes \left( z_{a_1} - \frac{1}{\pi} \frac{\partial}{\partial z_{a_1}} \right) \cdots \left( z_{a_q'} - \frac{1}{\pi} \frac{\partial}{\partial z_{a_q'}} \right) \varphi_0 \]

which is the same as
\[ \varphi^*_{q,q} = \frac{1}{2^{2q}} \sum_{\alpha, \alpha'} B_{\alpha} \wedge B_{\alpha'} \otimes D_\alpha D_{\alpha'} \varphi_0. \]

The function \( \varphi^*_{q,q} \) lies in \( \Lambda^{\otimes q} p^* \otimes \mathcal{S}(U) \) on which \( (1,1) \times G_U \) acts, as described in [KM86]. Let \( K_U = G_U(\mathbb{R}) \cup U_{2m}(\mathbb{C}) \), with the intersection taking place in \( GL_{2n}(\mathbb{C}) \simeq GL(U \otimes \mathbb{R}) \). Corresponding to the decomposition \( U = U_+ \oplus U_- \) we have \( K_U = K_+ \times K_- \), with \( K_\pm \) acting on \( V_\pm \). Then
\[ p^* \simeq (V_+ \otimes_E V_-)_\mathbb{C} \cong Hom_{E}(V_-, V_+) \otimes \mathbb{C} \]

and
\[ \mathcal{S}(V) \otimes \bigwedge^* p^* \simeq \mathcal{S}(V_-) \otimes \mathcal{S}(V_+) \otimes \bigwedge^* Hom_{E}(V_-, V_+) \mathbb{C}. \]

The action of \( K_+ \) is trivial on the Gaussian \( \varphi_0 \) and commutes with the Howe operator \( \nabla \), so it leaves \( \varphi^*_{q,q} \) invariant. The action of \( K_- \) is through both factors \( \mathcal{S}(V^\vee) \) and \( Hom_{E}(V_-, V_+) \mathbb{C} \). Of these, the action on \( \mathcal{S}(V^\vee) \) evidently leaves \( \varphi^*_{q,q} \) invariant, and the action on \( \bigwedge^* Hom_{E}(V_-, V_+) \mathbb{C} \) is by the determinant of \( K_- \subset GL(V_+ \otimes \mathbb{C}) \) [KM86 Theorem 3.1].

For \( n > 1 \), we put
\[ \varphi^*_{nq,nq} = \varphi^*_{q,q} \wedge \cdots \wedge \varphi^*_{q,q} \quad (n \text{ times}) \quad (5.12) \]
considered as an element in \( \Lambda^* \mathbb{p}^* \otimes \mathcal{S}(U^n) \) the following way. Consider \( \varphi^*_{nq,nq} \) as a Schwartz function on \( U \) with values in \( \Lambda^* p^* \). Then \( \varphi^*_{nq,nq} \) corresponds in the same way to the function
\[ U^n \to \bigwedge^* p^*, \quad (u_1, \ldots, u_n) \mapsto \varphi^*_{q,q}(u_1) \wedge \cdots \wedge \varphi^*_{q,q}(u_n). \]

We now consider the case \( n = p \), so that \( \varphi^+_{pq,pq} \) is a sum of terms of the form
\[ B_{\alpha} \wedge B_{\alpha'} \wedge B_{\alpha_2} \wedge B_{\alpha_p} \wedge B_{\alpha_p} \wedge \prod_{j=1}^{p} D_{\alpha_j} D_{\alpha_j'} \varphi_{0,(j)} \]

(5.13)
where \( \alpha_1, \ldots, \alpha_p \) are each a \( q \)-tuple of integers between \( 1 \) and \( p \), and \( D_{\alpha_j}, D_{\alpha_j'} \) are defined by \( \ref{} \) acting on \( \varphi_{0,(j)} \), the Gaussian on the \( j \)th copy of \( U \) in \( U^p \). Since \( \Lambda^{pq,pq} p^* \) is one-dimensional, we can write
\[ \varphi^+_{pq,pq} = (\omega \wedge \overline{\omega}) \otimes \varphi_{KM} \quad (5.14) \]
for some \( \varphi_{KM} \in \mathcal{S}(U^p) \),

\[ \omega = \xi_{1,1} \wedge \xi_{2,1} \wedge \cdots \wedge \xi_{p,1} \wedge \xi_{1,2} \wedge \cdots \wedge \xi_{p,2} \wedge \cdots \wedge \xi_{1,q} \wedge \cdots \wedge \xi_{p,q}. \]

Let \( \{ z_{i}^{(j)} : 1 \leq i \leq m, 1 \leq j \leq p \} \) denote the coordinates on \( U^p = \prod_{j=1}^{p} U^{(j)} \), where for each \( j \), \( \{ z_{1}^{(j)}, \ldots, z_{m}^{(j)} \} \) are the standard coordinates for \( U^{(j)} \). Then
\[ \Phi_0 = \exp \left( -\pi \sum_{i=1}^{m} \sum_{j=1}^{p} |z_{i}^{(j)}|^2 \right) \]
is the Gaussian on \( U^p \).
Lemma 5.2. The following formula holds:

$$\varphi_{\mathsf{KM}} = (-1)^{pq(p-1)/2} \sum_{\sigma, \sigma' \in S_p} D_{\sigma, \sigma'} \Phi_0 \in \mathcal{J}(U^p),$$

(5.16)

where $S_p$ denotes the symmetric group on $\{1, \ldots, p\}$ and $\sigma = (\sigma_1, \ldots, \sigma_q)$, $\sigma$ range over all pairs of $q$-tuples of elements of the symmetric group $S_p$, and

$$D_{\sigma, \sigma'} = \prod_{j=1}^{p} \prod_{k=1}^{q} D_{\sigma_k(j)} D_{\sigma'_k(j)}. \tag{5.17}$$

Proof. In the expression (5.13), writing $\alpha_j = (a_{j1}, \ldots, a_{jq})$ and $\alpha'_j = (a'_{j1}, \ldots, a'_{jq})$ for each $j$, the wedge product factor on the left is

$$(\xi_{a_{11}} \wedge \cdots \wedge \xi_{a_{1q}}) \wedge (\xi_{a'_{11}} \wedge \cdots \wedge \xi'_{a'_{1q}}) \wedge \cdots \wedge (\xi_{a_{q1}} \wedge \cdots \wedge \xi_{a_{qq}}) \wedge (\xi'_{a'_{q1}} \wedge \cdots \wedge \xi'_{a'_{qq}}). \tag{5.18}$$

For this expression not to vanish there must be no repeated wedge factors, so that it coincides with $\omega \wedge \overline{\omega}$ up to sign. This amounts to the condition

$$\{a_{1k}, \ldots, a_{pk}\} = \{a'_{1k}, \ldots, a'_{pk}\} = \{1, \ldots, p\}, \quad 1 \leq k \leq q.$$ 

In other words, $\sigma_k(j) = a_{jk}$ and $\sigma'_k(j) = a'_{jk}$ define elements $\sigma_k, \sigma'_k$ in $S_p$. Then writing $\xi_{\sigma, \sigma'}$ for (5.17), we get

$$\varphi_{pq,pq}^+ = \sum_{\sigma, \sigma' \in S_p} \xi_{\sigma, \sigma'} \otimes D_{\sigma, \sigma'} \Phi_0.$$ 

It’s enough to prove that $\xi_{\sigma, \sigma'} = (-1)^{pq(p-1)/2} \omega \wedge \overline{\omega}$. To show this we perform a sequence of transpositions that transforms (5.18) to $\omega \wedge \overline{\omega}$ and count the total sign change. We do this in two stages.

In the first stage, we sort the terms $\xi_{i,j}$. We find the last term in the desired order among them, i.e. $\xi_{p,q}$, and shift it left until it is in the leftmost spot, where $\xi_{1,1}$ is in (5.13). Then we take the second last term $\xi_{p-1,q}$ and also shift it all the way to the left, so that now $\xi_{p,q}$ becomes the second term from the left. We continue with $\xi_{p-2,q}$, etc. shifting the terms in strictly reverse order to the leftmost spot until at the end we move $\xi_{1,1}$ to the leftmost spot. At that point all the $\xi_{i,j}$ terms will be in the same order as in (5.13), while the $\xi_{i,j}$ terms remain unsorted.

In the second stage, we sort the $\xi_{i,j}$ terms. We take the term $\xi_{p,q}$ and shift it to the left until it is in position $pq + 1$ from the left, to the right of $\xi_{p,q}$. Then we take the second last term $\xi_{p-1,q}$ and shift it also to position $pq + 1$, between $\xi_{p,q}$ and $\xi_{p,q}^{-1}$. We continue again in reverse order shifting all terms $\xi_{i,j}$ to position $pq + 1$ until at the end we shift $\xi_{1,1}$ into that spot. At this point all terms are in the correct order and the expression is $\omega \wedge \overline{\omega}$ up to sign.

Now we count how many sign changes were made in each stage. In the first stage, when moving a term $\xi$ we distinguish two kinds of transpositions of adjacent terms: if $\xi$ is transposed with one of the terms $\xi_{i,j}$, we consider it of the first kind. If it is transposed with one of $\xi_{i,j}$, we consider it of the second kind.

We count the transpositions of the first kind first. Note that each term $\xi_{i,j}$ in $B_{\alpha_p}$ has to pass every term $\xi_{i,j}$ occurring in $B_{\alpha'_{i1}}, B_{\alpha'_{i2}}, \ldots, B_{\alpha'_{ip-1}}$, therefore it makes $q(p - 1)$ passes of the first kind. Altogether the $q$ terms of $B_{\alpha_p}$ make $q^2(p - 1)$ passes of the first kind. The terms $\xi_{i,j}$ in $B_{\alpha_{p-1}}$ have to pass every $\xi_{i,j}$ term in $B_{\alpha'_{i1}}, B_{\alpha'_{i2}}, \ldots, B_{\alpha'_{p-2}}$, exactly once, so altogether they make $q^2(p - 2)$ passes of the first kind. Terms in $B_{\alpha_{p-2}}$ make $q^2(p - 3)$ passes of the first kind, etc., and altogether the number of passes of the first kind in the first stage is $q^2(p - 1) + q^2(p - 2) + \cdots + q^2 = q^2(p - 1)p/2$, which is $pq(p - 1)/2$ modulo 2.

Now we observe that the number of transpositions of the second kind in the first stage is equal to the total number of transpositions in the second stage. Thus their sign contributions cancel out, and the total sign change is $(-1)^{pq(p-1)/2}$ as claimed. \(\square\)

Recall the hermitian space $(V, Q) = (V_0, Q_0) \otimes_{E_0} E$ from the introduction and put $U = V(\mathbb{R})$. Let $p_i : D = D_0^2 \to D_0$ denote the $i$th projection and put

$$\varphi_{\infty} = p_1^*(\tau_{\eta_0, \eta_0}^+) \wedge \cdots \wedge p_q^*(\tau_{\eta_0, \eta_0}^+) \in \Omega^{\eta_0, \eta_0}(D) \otimes \mathcal{J}(V_0(\mathbb{R})^{\eta_0}),$$

where $\eta_0$ denotes the symmetric group on $\{1, \ldots, \eta_0\}$.
where identify $\mathcal{S}(V(\mathbb{R})^n)$ and $\mathcal{S}(V_0(\mathbb{R})^n)$ using the isomorphism from \cite{KMS6}. If $\Delta : D_0 \to D$ is the diagonal map, then as explained in \cite{KMS6}

$$\Delta^* \varphi_\infty = \varphi_{\eta q, \eta q}^+. \quad (5.18)$$

In particular, if $\eta q = p$, then

$$\Delta^* \varphi_\infty = \omega \wedge \pi \otimes \varphi_{\text{KM}} \in \Omega^{1,1}(D_0) \otimes \mathcal{S}(V_0(\mathbb{R})^n). \quad (5.19)$$

5.2. **Vanishing of the Ikeda map.** For $z$ one of the variables $z_1, \cdots, z_m$ and non-negative integers $a$ and $b$ we define the function $F_{a,b}(z)$ as a polynomial in $z$ and $\pi$ by the relation

$$\left(\pi - \frac{1}{\pi} \frac{\partial}{\partial z}\right)^a \left(\pi - \frac{1}{\pi} \frac{\partial}{\partial \pi}\right)^b \varphi_0 = F_{a,b}(z) \cdot \varphi_0. \quad (5.20)$$

For a rapidly decreasing function $f(z)$ on $\mathbb{C}$, put

$$I(f) = \int_C f(z)e^{-2\pi \mid z \mid^2} dz. \quad (5.21)$$

For non-negative integers $a$ and $b$, $a \neq b$ it’s easy to verify that

$$I(P_{a,b}) = 0, \quad P_{a,b}(z) = \pi^a z^b \quad (5.22)$$

by sign considerations.

**Lemma 5.3.** $I(F_{a,b}) = 0$ if $a \neq b$.

**Proof.** Assume $b > a$ without loss of generality. First, a simple induction argument shows

$$F_{0,b} = (2z)^b.$$

Then $F_{a,b}$ is determined by

$$\left(\pi - \frac{1}{\pi} \frac{\partial}{\partial z}\right)^a (2z)^b \varphi_0 = F_{a,b} \varphi_0.$$

Note that for $\alpha \geq 0, \beta \geq 1$, we have

$$\left(\pi - \frac{1}{\pi} \frac{\partial}{\partial z}\right)^a \pi^\alpha z^\beta \varphi_0 = (2\pi^{\alpha+1} z^\beta - \frac{\beta}{\pi} \pi z^\beta - \frac{1}{\pi} \pi z^\beta) \varphi_0.$$

Now let $f(z)$ be a polynomial in $z$ and $\pi$ such that the coefficient of $\pi^\alpha z^\beta$ is non-zero only if $\beta > \alpha$. Let $\mu(f)$ be the minimum of $\beta - \alpha$ among all terms $\pi^\alpha z^\beta$ of $f$ with non-zero coefficients. The above identity shows that if $g(z)$ is defined by

$$g(z) \varphi_0 = \left(\pi - \frac{1}{\pi} \frac{\partial}{\partial z}\right) f(z) \varphi_0,$$

then

$$\mu(g) \geq \mu(f) - 1, \quad \text{if } g \neq 0.$$

Since $\mu(F_{0,b}) = b$, it follows that if $F_{a,b} \neq 0$, then $\mu(F_{a,b}) \geq b - a > 0$. Then every non-zero term of $F_{a,b}$ is of the form $\pi^\alpha z^\beta$, with $\beta > \alpha$. The lemma then follows from (5.22). \hfill \square

Next we set out to compute $I(f_k(z))$, for $f_k(z) = F_{k,k}(z)$, $k \geq 0$. The functions $f_k$ are determined by

$$E^k \varphi_0 = f_k(z) \varphi_0, \quad f_0(z) = 1, \quad k \geq 0 \quad (5.23)$$

where

$$E = \left(\pi - \frac{1}{\pi} \frac{\partial}{\partial z}\right) \left(\pi - \frac{1}{\pi} \frac{\partial}{\partial \pi}\right). \quad (5.24)$$

For any smooth function $f(z, \pi)$, we have

$$E(f \varphi_0) = \left\{ \left(4|z|^2 - \frac{2}{\pi}\right) f - \frac{2}{\pi} \left(\pi^2 \frac{\partial f}{\partial \pi} + z \frac{\partial f}{\partial z}\right) + \frac{1}{\pi^2} \frac{\partial^2 f}{\partial z^2}\right\} \varphi_0.$$

We normalize by changing variables $z = \frac{w}{\sqrt{2\pi}}$ and setting

$$g_k(w) = f_k \left(\frac{w}{\sqrt{2\pi}}\right) \frac{\pi^k}{2^k} \quad (5.25)$$
to obtain the recursive relation
\[ g_{k+1}(w) = \left\{ (|w|^2 - 1) g_k - \left( \frac{\partial g_k}{\partial w} + w \frac{\partial g_k}{\partial w^2} \right) + \frac{\partial^2 g_k}{\partial w^2} \right\}, \quad g_0(w) = 1 \] (5.26)

Then \( g_k(w) \) is the \( k \)th Laguerre polynomial in \(|w|^2\), normalized to be monic and have integer coefficients, e.g. \( g_1(w) = |w|^2 - 1 \).

**Proposition 5.4.** For \( k \geq 1 \),
\[ g_k(w) = \sum_{r=0}^{k} (-1)^{r+k} |w|^{2r} \frac{(k)!^2}{(r!)^2 (k-r)!}. \]

**Proof.** The formula is evidently valid for \( k = 1 \). Assume it holds for some \( k \). Noting that for \( r > 0, \)
\[ \frac{\partial}{\partial w} |w|^{2r} = w \frac{\partial}{\partial w} |w|^{2r} = r |w|^{2r}, \quad \frac{\partial^2}{\partial w^2} |w|^{2r} = r^2 |w|^{2r-2} \]
we have
\[ g_{k+1}(w) = (-1)^k (|w|^2 - 1)(k!) + \sum_{r=1}^{k} \frac{k!}{((r-1)!)^2 (k-r+1)!} \left\{ (|w|^{2r+2} - |w|^{2r} - 2r |w|^{2r} + r^2 |w|^{2r-2} \right\} \]
For \( 0 < r < k \), the coefficient of \(|w|^{2r}\) in the sum is equal to
\[ (-1)^{k+r+1} \left\{ \frac{k!}{((r-1)!)^2 (k-r+1)!} + (1+2r) \frac{k!}{((r+1)!)^2 (k-r-1)!} \right\} \]
\[ = (-1)^{k+r+1} \frac{k!}{((r-1)!)^2 (k-r+1)!} \left( r^2 + (1 + 2r)(k-r+1) + (k-r)(k-r+1) \right) \]
\[ = (-1)^{k+r+1} \frac{k!}{((r-1)!)^2 (k-r+1)!} \]
The leading coefficient of \( g_{k+1}(w) \) is \( \frac{(k!)^2}{(k!)^2 (k-k)!} = 1 \), and the constant term is
\[ (-1)^{k+1} (k!) + (-1)^{k+1} \frac{k!}{((k+1)!)^2 (k+1-1)!} \cdot 1^2 = (-1)^{k+1} (k+1)!. \]
All coefficients match those in the formula, which is therefore valid for \( k+1 \), and by induction for all \( k \). \( \square \)

Consider now a two-dimensional complex hermitian space of signature \((1, 1)\), with standard coordinates \((x_1, x_2)\). Put
\[ F_k(x_1, x_2) = f_k(x_1) e^{-\pi (|x_1|^2 + |x_2|^2)}. \]
It can be split into a sum of two isotropic lines via the change of variables \((w_1, w_2) = (\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}})\). In these coordinates \( F_k \) is given by
\[ F_k(x_1, x_2) = f_k \left( \frac{w_1 + w_2}{\sqrt{2}} \right) e^{-\pi (|w_1|^2 + |w_2|^2)}. \] (5.27)

The computation of the integral of \( F_k \) along the isotropic line \((w_1, 0)\) is the key lemma of this section.

**Lemma 5.5.**
\[ \int_C f_k \frac{z}{\sqrt{2}} e^{-\pi |z|^2} dz = 0 \]

**Proof.** Using the relation \( f_k(z) = g_k(\sqrt{2\pi z}) \frac{e^{\frac{z^2}{2}}}{\pi^k} \) and the formula for \( g_k \) from Proposition 5.4 we express the integral as
\[ \frac{2^k (-1)^k \cdot (k!)^2}{\pi^k} \sum_{r=0}^{k} \frac{k!}{(r!)^2 (k-r)!} \int_C \int_C |z|^{2r} e^{-\pi |z|^2} |z|^{2r} dz. \]
Since
\[ 2\pi \int_0^\infty x^{2r+1} e^{-\pi x^2} dx = \frac{r!}{\pi^r} \]

the integral is

$$\frac{2^k(-1)^k k!}{\pi^k} \sum_{r=0}^{k} \binom{k}{r} (-1)^r = 0.$$  

We now assume $ng = p$ and $q < p$. Let $\varphi_L$ denote the characteristic function of $\hat{L} \subset V(A_\mathbb{A})$, where $L \subset V$ is a maximal lattice. Put

$$\varphi = \varphi_{KM} \otimes \varphi_L \in \mathcal{S}(V(A)^n).$$  

(5.28)

Since $q < p$, the Weil index $r$ of $(V_0, Q_0)$ is $q$. Then the numbers $r, n_0 = \dim W_0 = ng$, and $m = p + q = \dim V_0$, satisfy

$$n_0 \leq m \leq 2n_0, \quad m - r \leq n_0.$$  

These are parameters for the pair $(W_0, J_0)$ and $(V_0, Q_0)$ falling outside the classical convergence range for the Siegel-Weil formula [Wei65], and inside the range for the regularized formula of Ichino [Ich04].

Let $r_0 = m - n_0 = q$. Suppose that $\{u_1, \ldots, u_m\}$ is the fixed standard basis of $V_0(\mathbb{R})$ used to define $\varphi_0$. for $j = 1, \ldots, r_0$, put

$$e_j = \frac{u_j + u_{m+1-j}}{\sqrt{2}}, \quad f_j = \frac{u_j - u_{m+1-j}}{\sqrt{2}}.$$  

Then $\{e_i, f_j : 1 \leq i, j \leq r_0\}$ form a basis for a subspace $V_0''$ of $V_0(\mathbb{R})$ such that

$$Q_0(e_i, f_j) = \delta_{i,j}, \quad Q_0(f_i, f_j) = Q_0(e_i, e_j) = 0.$$  

Put $Q'' = Q_0|_{V_0''}$, and

$$(V_0', Q_0') = (V_0'', Q''_0)^{\perp},$$  

(5.29)

so that $(V_0, Q_0) = (V_0', Q_0') + (V_0'', Q''_0)$ is an orthogonal decomposition.

Let $\beta' = \{v'_1, \ldots, v'_{p-q}\}$ be any basis for $(V_0', Q_0')$ and put $\beta = \{e_1, \ldots, e_{r_0}, v'_1, \ldots, v'_{p-q}, f_1, \ldots, f_{r_0}\}$. Then

$$[Q_0]_{\beta} = \begin{pmatrix} [Q''_0]_{\beta'} & I_p \\ I_p & [Q''_0]_{\beta'} \end{pmatrix}.$$  

Using this matrix presentation we take $P_0$ to be following parabolic subgroup of $H_0$:

$$P_0(R) = \left\{ \begin{pmatrix} A & * & * \\ X & * & (\overline{A})^{-1} \end{pmatrix} : A \in GL_{r_0}(R \otimes E_0), \ X \in H'_0(R) \right\}, \quad R \in \mathbb{Q}-\text{Alg}$$  

(5.30)

where $H'_0 = U(V'_0, Q'_0) \approx U(p - q; E_0/\mathbb{Q})$.

We choose a maximal compact subgroup $K_0 = \prod_v K_v$ of $H(A)$ as follows. For $v$ a rational prime, $K_v$ is the stabilizer of $L_v \subset V_0(\mathbb{Q}_v)$. For $v = \infty$, we take $K_\infty = H_0(\mathbb{R}) \cap U_{2m}(\mathbb{C})$, with $U_{2m}(\mathbb{C})$ and $H_0(\mathbb{R})$ considered as subgroups of $GL_{2m}(\mathbb{C}) \approx GL(V_0(\mathbb{R}))$.

The regularization procedure in [Ich04] involves applying to a function $\varphi_0 \in \mathcal{S}(V_0(\mathbb{R})^{ng})$ the $(g$-fold tensor power of) maps

$$\pi : \mathcal{S}(V_0(\mathbb{A})^n) \to \mathcal{S}(V_0(\mathbb{A})^n), \quad \pi = \text{Ik} \circ \pi_K,$$

(5.31)

where

$$\pi_K : \mathcal{S}(V_0(\mathbb{A})^n) \to \mathcal{S}(V_0(\mathbb{A})^n), \quad \pi_K(\psi)(x) = \int_{K_0} \psi(kx)dk$$  

(5.32)

and $\text{Ik}$ is the Ikeda map

$$\text{Ik} : \mathcal{S}(V_0(\mathbb{A})^n) \to \mathcal{S}(V_0(\mathbb{A})^n), \quad \text{Ik}(\psi)(x) = \pi_K^0(\psi)(x) = \int_{V''(\mathbb{A})^n} \Psi \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} dy$$  

(5.33)

with

$$V''_e = \text{Span}_E\{e_1, \ldots, e_{r_0}\} \subset V_0''; \quad V''_f = \text{Span}_E\{f_1, \ldots, f_{r_0}\}; \quad V_0 = V''_e \oplus V_0' \oplus V''_f.$$  

Theorem 5.6. $\varphi_{KM}$ lies in the kernel of the Ikeda map $\mathcal{S}(V_0(\mathbb{A})^p) \to \mathcal{S}(V_0(\mathbb{A})^p)$.  

Proof. First we note that $\varphi_{KM}$ is $K_0(\mathbb{R})$-invariant, i.e.
\[(k_\infty x) = \varphi_{KM}(x), \quad k_\infty \in K_0(\mathbb{R}).\]
This follows from the description of the action of $K_0(\mathbb{R}) \subset H_0(\mathbb{R})$ on $\mathcal{S}(V_0(\mathbb{R}))$ given in the previous section. The essential facts are that the Gaussian $\varphi_0$ is $K_0(\mathbb{R})$-invariant, and the Howe operators $D_\sigma$ commute with the action of $K_0(\mathbb{R})$.

Since $\varphi$ is factorizable, the adelic integral in $I_k(\varphi)$ vanishes if any of the corresponding local integral factors do. Then it suffices to show
\[
\int_{V''_c(\mathbb{R})^p} \varphi_{KM} \left( \begin{array}{c} y \\ x \end{array} \right) dy = 0.
\]

By Lemma 5.2, $\varphi_{KM}$ is up to a sign a sum of terms of the form
\[
\prod_{j=1}^p \left( \prod_{k=1}^q D_{\sigma_k(j)}^{(j)} D_{\sigma_k'(j)}^{(j)} \varphi_0 \right).
\]

We claim that for such terms, for each $j$,
\[
\int_{V''_c(\mathbb{R})} \left( \prod_{k=1}^q D_{\sigma_k(j)} D_{\sigma_k'(j)} \varphi_0 \right) \left( \begin{array}{c} y \\ x \end{array} \right) = 0
\]
from which the theorem follows. Writing $a_k = \sigma_k(j)$, $a'_k = \sigma'_k(j)$, the integrand as a function of standard coordinates $(z_1, \cdots, z_m)$ on $V_0(\mathbb{R})$ is
\[
\prod_{k=1}^q \left( z_{a_k} - \frac{1}{\pi} \frac{\partial}{\partial z_{a_k}} \right) \left( z_{a'_k} - \frac{1}{\pi} \frac{\partial}{\partial z_{a'_k}} \right) \varphi_0.
\]
The domain $V''_c(\mathbb{R})$ is a direct sum of $q$ isotropic lines spanned by $e_k$, for $k = 1, \cdots, q$. Fix some $k$, and let $a$, resp. $b$ denote the multiplicity of $k$ in $(a_1, \cdots, a_q)$, resp. $(a_1, \cdots, a'_q)$. We show the factor of the integral corresponding to the isotropic line $\text{Span}\{e_k\}$ vanishes by distinguishing two cases:

Case 1: $a \neq b$. This integral was considered in Lemma 5.3. It vanishes by argument considerations.

Case 2: $a = b$. This integral was explicitly computed and shown to vanish in Lemma 5.5. \hfill $\square$

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