A REMARK ON THE HERZLICH VOLUME OF ASYMPTOTICALLY COMPLEX HYPERBOLIC EINSTEIN MANIFOLDS

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ABSTRACT. We observe inequalities involving the Herzlich volume of a 4-dimensional asymptotically complex hyperbolic Einstein manifold and its Euler characteristic provided the metrics is either Kähler or selfdual. In the selfdual case we have to assume furthermore that the Kronheimer-Mrowka invariant is non vanishing.

1. STATEMENT OF RESULTS

In this short note, we observe inequalities involving the Herzlich volume of 4-dimensional asymptotically complex hyperbolic Einstein manifold \((X, g)\) and the Euler characteristic \(\chi(X)\), provided the metric is either Kähler or selfdual. In the selfdual case we have to assume that the Kronheimer-Mrowka invariant is moreover non vanishing.

The acronym \(ACH\) shall be used as a shorthand in the rest of this paper for asymptotically complex hyperbolic. Our main result is stated next, however the reader yearning for basic definitions may want to read Section 2 first.

Given an orientable 4-dimensional ACH Einstein manifold \((X, g)\), Herzlich introduces in [6] an invariant of the metric \(V(g)\), closely related to the renormalized volume \(V(g)\). An essential feature of this invariant proved by Herzlich, is the Gauss-Bonnet type formula

\[
\chi(X) = \frac{1}{8\pi^2} \int_X \left( |W_g|^2 - \frac{1}{24} s_g^2 \right) \text{vol}_g + \frac{1}{4\pi^2} V(g),
\]

where \(W_g\) denotes the Weyl curvature of \(g\) and \(s_g\) the scalar curvature.

**Theorem A.** Let \(\overline{X}\) be a 4-dimensional oriented manifold with boundary, such that its interior \(X\) carries an ACH Einstein metric \(g\).

If \(g\) is Kähler with respect to a complex structure compatible with the orientation of \(\overline{X}\), we have

\[
4\pi^2 \chi(X) \geq V(g)
\]

where \(V(g)\) is the Herzlich volume of \(g\). In addition equality holds if and only if the metric is complex hyperbolic.

If \(g\) is selfdual and \((\overline{X}, \xi)\) admits a monopole class, where \(\xi\) be the (positive cooriented) contact structure on \(\partial \overline{X}\) induced by the conformal infinity of \(g\), then

\[
4\pi^2 \chi(X) \leq V(g)
\]

and equality holds if and only if the metric is complex hyperbolic.

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Remark 1.1. The above result shows a clear disjunction between the selfdual Einstein metrics (1.3) and the Kähler Einstein metrics (1.2). One can compare Theorem A to a similar and yet dramatically different result due to Anderson in the real case: given a 4-dimensional orientable asymptotically real hyperbolic (ARH) Einstein manifolds \((X, g)\), we have

\[ V(g) \leq \frac{4\pi^2}{3} \chi(X), \]

where \(V(g)\) is the renormalized volume of \(g\). Moreover, equality holds if and only if the metric is real hyperbolic. The proof follows immediately from a Gauss-Bonnet formula similar to (1.1) in the real setting (cf. [1] for more details). However, the inequality holds for every ARH Einstein metric and no disjunction phenomenon appears in this case.

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2. Definitions

Let \(\overline{X}^4\) be an oriented manifold with oriented boundary \(Y^3\) and interior \(X \subset \overline{X}\) endowed with an ACH Einstein metric \(g\).

Recall that a cooriented contact distribution \(\xi \subset TY\) is given as the kernel of a globally defined 1-form \(\eta\), defined up to multiplication by a positive function, such that \(\eta \wedge d\eta\) is a volume form. The contact structure is positive if this volume form is compatible with the orientation of \(Y\).

A strictly pseudoconvex CR structure \(J\) is given by an almost complex structure \(J\) defined along a positive cooriented contact structure \(\xi = \ker \eta\) such that \(\gamma(\cdot, \cdot) = d\eta(\cdot, J\cdot)\) is a Hermitian metric defined along \(\xi\). With the above notations, \(\xi = \ker \eta\) is referred to as the (positive cooriented) contact structure induced by \(J\).

Identify a collar neighborhood of \(Y\) in \(\overline{X}\) with \(\{0, T\} \times Y\), with coordinate \(u\) on the first factor. On this collar neighborhood we have a model metric

\[ g_0 = du^2 + \eta^2 + \frac{\gamma}{u^2} \]

We say that a metric Riemannian \(g\) on \(X\) is ACH, with conformal infinity the strictly pseudoconvex CR structure \(J\), if near the boundary one has

\[ g = g_0 + \kappa, \]

where \(\kappa\) is a symmetric 2-tensor, such that \(|\nabla^k \kappa| = O(u^\delta)\) for some \(\delta > 0\) and all \(k \geq 0\) (here, all the norms and derivatives are taken with respect to the metric \(g_0\)).

Kronheimer and Mrowka introduced in [7] a suitable version of Seiberg-Witten theory for compact oriented manifolds with positive cooriented contact boundary. Therefore the theory applies to \((\overline{X}, \xi)\). The contact structure \(\xi\) induces a canonical \(\text{Spin}^c\)-structure on \(Y\) denoted \(s_\xi\). Recall that the space \(\text{Spin}^c(\overline{X}, \xi)\) is the set of equivalence classes of pairs \((s, h)\), where \(s\) is a \(\text{Spin}^c\)-structure on \(\overline{X}\) and \(h : s|_Y \rightarrow s_\xi\) is an isomorphism. Then, the Kronheimer-Mrowka invariant is given by a map

\[ s\text{w}_{\overline{X}, \xi} : \text{Spin}^c(\overline{X}, \xi) \rightarrow \mathbb{Z} \]

defined up to an overall sign, and obtained by “counting” the solutions of Seiberg-Witten equations. If \(s\text{w}_{\overline{X}, \xi}(s, h) \neq 0\) for some \((s, h) \in \text{Spin}^c(\overline{X}, \xi)\), we say that \((s, h)\) is a monopole class.

3. Examples of Kähler and selfdual ACH Einstein metrics

The only known examples of selfdual ACH Einstein metric at the moment consist of

(1) the complex hyperbolic plane, and, more generally, ACH manifolds obtained by taking suitable quotients of the complex hyperbolic plane by a group of isometries,
the Calderbank-Singer examples [5], defined in a neighborhood of the exceptional divisor of the minimal resolution of certain isolated orbifold singularities.

whereas the essential source of ACH Kähler-Einstein metrics in addition to hyperbolic examples (1) is provided by the Cheng-Yau metrics on strictly pseudoconvex domains of \( \mathbb{C}^2 \) (and more generally of \( \mathbb{C}^n \)). Other examples due to Biquard are obtained by deformation of the complex hyperbolic metric [2, 3] in the selfdual Einstein and the Kähler-Einstein cases. Finally further ACH Kähler-Einstein examples are obtained by drilling wormholes in their conformal infinity [4].

Remark 3.1. In each example given above, the boundary contact structure \( \xi \) induced by the conformal infinity of the metric is symplectically fillable. Therefore \( (\overline{X}, \xi) \) admits a monopole class by [7, Theorem 1.1]. It follows that Theorem A applies.

We shall give now an application of Theorem A, in the case of the ball. Consider the moduli space

\[ \mathcal{M} = \{ \text{ACH Einstein metrics on } B^4 \}/\{ \text{diffeomorphisms of } B^4 \} \]

Let \( \mathcal{M}_K^K \subset \mathcal{M} \) be the subspace of Kähler metrics and \( \mathcal{M}_{SD} \) the subspace of selfdual metric. Notice that \( \mathcal{M}_K^K \) contains all the Cheng-Yau metrics on strictly pseudoconvex domains of \( \mathbb{C}^2 \) diffeomorphic to a ball. Let \( \mathcal{M}_{std}^\bullet \) be the component of the moduli space \( \mathcal{M}^\bullet \) consisting of metrics with conformal infinity inducing the standard contact structure \( \xi_{std} \) on the sphere \( S^3 \), up to diffeomorphism. Since contact structure are stable under small deformation, it follows that \( \mathcal{M}_{std}^\bullet \) is a neighborhood of \( g_{hyp} \) in \( \mathcal{M}^\bullet \).

Remark 3.2. The contact structure \( \xi \) induced by the conformal infinity of every metric in \( \mathcal{M}_K^K \) is automatically symplectically fillable by the Kähler form. It follows by a theorem of Eliashberg that \( \xi \) is tight. Since \( \xi_{std} \) is the only tight contact structure on \( S^3 \), it turns out that \( \mathcal{M}_{std}^K = \mathcal{M}_K^K \). By contrast, it is an open question whether \( \mathcal{M}_{SD}^K \) is a proper subset of \( \mathcal{M}_{SD} \).

Remark 3.3. Small deformations of the conformal infinity of \( g_{hyp} \) are in 1 : 1 correspondence with small ACH Einstein deformations of the metric [2]. Moreover the moduli spaces \( \mathcal{M}_K^K \) and \( \mathcal{M}_{SD} \) intersect transversely at \( g_{hyp} \) [3]. Corollary B says that in some sense \( \mathcal{V} : \mathcal{M}_{std} \to \mathbb{R} \) admits a saddle point at \( g_{hyp} \). Thus, there are certain ACH Einstein deformations \( g \) of the complex hyperbolic metric, which are neither Kähler nor selfdual and such that \( \mathcal{V}(g) = \mathcal{V}(g_{hyp}) \). More generally, it would be interesting to understand precisely, how deformations of the conformal infinity act on the Herzlich volume of the corresponding ACH Einstein metric.

Corollary B. The Herzlich volume \( \mathcal{V} : \mathcal{M}_{std}^SD \to \mathbb{R} \) is bounded below, and admits a unique minimum at the complex hyperbolic metric \( \mathcal{V}(g_{hyp}) = 4\pi^2 \), whereas \( \mathcal{V} : \mathcal{M}^K \to \mathbb{R} \) is bounded above, and admits a unique maximum at the complex hyperbolic metric.

A similar version of the Corollary holds manifolds obtained by hyperbolic quotient as in examples (1).

4. PROOFS

Proof of Corollary B. Theorem A applies automatically to every metric in \( \mathcal{M}_K^K \).

Notice that the standard symplectic form on \( B^4 \) is a filling of the contact structure \( \xi_{std} \). It follows, according to Remark 3.1, that Theorem A applies to every metric in \( \mathcal{M}_{std}^SD \) as well, and the Corollary is proved.

Proof of Theorem A. If the metric is Kähler, we have the pointwise identity \( s_g^2 = 24|W_g|^2 \). Using the Gauss-Bonnet formula (1.1), we have

\[
0 \geq \int_X \left( \frac{s_g^2}{24} - |W_g|^2 \right) \text{vol}_g = 2\mathcal{V}(g) - 8\pi^2 \chi(X)
\]
with equality if and only if the metric is selfdual and inequality (1.2) is proved.

The inequality (1.3) is way less trivial. However the most difficult part of the proof is carried out in [9]. The reader may want to consult a short note in English as well, stating the result without the technicalities [8].

Given any oriented manifold $\bar{X}^4$, endowed with an ACH Einstein metric $g$, let $\xi$ be the cooriented positive contact structure induced by the conformal infinity. Assume that there exists a monopole class $((s, h) \in \text{Spin}^c(\bar{X}, \xi)$. Then one can construct a solution of Seiberg-Witten equations w.r.t $(s, h)$ and $g$, with a suitable decay at the boundary (cf. [9, Théorème 2]). Once a solution of Seiberg-Witten equations is obtained, one can prove the following result (cf. Proposition 29 and the end of the proof of Corollaire 31 in [9]),

**Proposition 4.1.** Let $\bar{X}^4$ be an oriented manifold with boundary endowed with an ACH Einstein metric and let $\xi$ be the contact structure induced by the conformal infinity on the boundary. If $(\bar{X}, \xi)$ admits a monopole class then

$$0 \leq \int_X \left( \frac{s_g^2}{24} - |W_g^\pm|^2 \right) \text{vol}_g$$

with equality if and only if the metric is Kähler-Einstein.

In particular, if the metric is selfdual, we have

$$0 \leq \int_X \left( \frac{s_g^2}{24} - |W_g|^2 \right) \text{vol}_g = 2\mathcal{V}(g) - 8\pi^2 \chi(X).$$

Moreover equality holds if and only if the metric is Kähler. This proves the inequality (1.3).

Finally inequalities (1.2) and (1.3) are saturated if and only if the metric $g$ is both selfdual and Kähler-Einstein, and therefore, complex hyperbolic. □

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