Comments on $D$-branes on Orbifolds and K-theory

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Abstract

We systematically revisit the description of $D$-branes on orbifolds and the classification of their charges via K-theory. We include enough details to make the results accessible to both physicists and mathematicians interested in these topics. The minimally charged branes predicted by K-theory in $\mathbb{Z}_N$ orbifolds with $N$ odd are only BPS. We confirm this result using the boundary state formalism for $\mathbb{Z}_3$. For $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifolds with and without discrete torsion, we show that the K-theory classification of charges agrees with the boundary state approach, largely developed by Gaberdiel and collaborators, including the types of representation on the Chan-Paton factors.
1 Introduction

Much of the recent progress in string theory revolves around the concept of Dirichlet-branes. These are objects that source Ramond-Ramond bosonic massless fields of type II and type I string theories carrying one unit of charge [1, 2]. More precisely, D-branes are non-perturbative states that enter in the theory as boundaries for the closed string world-sheet, introducing open strings in a consistent way. This definition leads to a geometrical interpretation of D-branes as a surface where the ends of open strings are free to move; the ends of the open string satisfy Dirichlet boundary conditions along the normal directions to the surface, hence their name. If the surface has $p$ dimensions, the brane is denoted shortly as $D_p$-brane. The excitations of the $D_p$-branes are described by the open strings which end on it. Branes with the opposite charge are called anti-$D_p$-branes (or $D_{\bar{p}}$-branes for short).

For some branes supersymmetry plays a crucial role. In particular, some branes are states that preserve half of the spacetime supersymmetries, they are known as Bogomolnyi-Prasad-Sommerfield (BPS) branes. These $D$-branes are stable objects since they form short representations of the supersymmetry algebra. Subsequently, their mass is completely determined by their charge ($m = |Q|$ in the appropriate units). These BPS branes constitute key evidence in the formulation of various strong/weak duality conjectures among various string theories. An analysis of some duality conjectures beyond supersymmetry brought about the discovery of new solitonic states in string theory. These states are non-BPS branes as they do not preserve any spacetime supersymmetry and consequently the mass and charge relationship is generic ($m > |Q|$). Interestingly, these states can be stable if they are the lightest states of the theory [3, 4, 5].

D-branes figure prominently is string phenomenology. The goal of the string phenomenology program is to find $D$-brane configurations with world-volume field theories resembling the Standard Model of Particles. Some concrete models have been proposed based on: Calabi-Yau compactifications, orbifold and orientifold compactifications, orbifold singularities like $\mathbb{C}^3/\mathbb{Z}_N$ and $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$, among others. At the moment there is not a single model that could be considered completely realistic. The search for realistic models requires knowledge of the full spectrum of $D$-branes and their conserved charges in a wide range of string theories.

There are two main approaches towards the construction of the spectrum of $D$-brane charges. The first one is given by the boundary state formalism wherein a $D$-brane is a coherent state in the closed string theory that emits or absorbs closed string states [6, 7, 8].
The boundary state has to be a physical state of the closed string theory and therefore must satisfy several conditions (see section 2). These conditions imply that a $D$-brane is a linear combination of boundary states defined for the different sectors of the closed string theory. Crucially, this linear combination of boundary states encodes the information of the open string spectrum characterizing the $D$-brane. The consistent set of solutions of the physical conditions is the way the boundary state formalism answers the question of the spectrum of $D$-branes. This construction of $D$-branes does not rely on space-time supersymmetry and allows us to analyze the spectrum of $D$-branes in backgrounds where supersymmetry is broken or does not exist at all, as in type 0A and 0B theories.

Let us mention an important property of the boundary state formalism approach to the classification of D-brane charges. For a given set of Ramond-Ramond (R-R) charges, a $D$-brane configuration exist with these charges. The space of all R-R charges is a lattice. The basis vectors of this lattice are charges that corresponds to fundamental branes. By fundamental we mean branes that can not be written as linear combinations of others, hence carrying smaller charges and masses. Therefore, a general charge in the lattice can be generated by an integer linear combination of these fundamental branes.

The other approach for the classification $D$-branes, based largely on Sen’s construction [5], is K-theory. The spectrum of open strings with both ends on a $D$-brane contains, among other states, massless vectors. They are $U(1)$ gauge fields living on the $D$-brane world-volume. By considering a set of $N$ coincident branes, one obtains a non-Abelian group $G$; for Type II string theory, it is $U(N)$. Therefore, $N$ $D$-branes are roughly described by a $G$ gauge bundle whose base space is the world-volume of the branes. In this way, a system of $N$ coincident $D$-branes and $N$ coincident anti-$D$-branes is characterized by a pair of vector bundles (a $G_1$ gauge bundle for the $D$-branes and a $G_2$ gauge bundle for the anti-$D$-branes). This system of branes-antibranes is unstable since the spectrum of the open string stretched between the branes and the anti-branes contains a tachyon. For the case in which the two gauge bundles are the same, the system carries no net charge and the instability allows the possibility of brane-antibrane annihilation, recovering the closed string vacuum state and preserving charge conservation. This leads to introducing equivalence classes of pairs of bundles. Such classes are naturally elements of K-theory. The original suggestion of [9] stating that a natural framework for $D$-brane charges is K-theory was further tested in [10] for Type IIB string theory and Type I in ten dimensions, the IIA case was discussed in [11].

The initial statement that K-theory describes the complete space of conserved charges was generalized to various settings: string theory on orbifolds, orientifolds and backgrounds with
a B-field. The relevant K-theories were determined to be the equivariant, real and twisted K-theory, respectively. The general believe is that: for each string model there exists a generalization of K-theory that classifies the D-brane charges. The boundary state formalism allows to compute the spectrum of D-brane charges. The space of all conserved charges is a sublattice of the full lattice of charges. As we have mentioned above, the fundamental branes obtained by boundary states are the generators of this full lattice. Then, it is enough to construct the fundamental branes in the specific string models in order to compare the results with K-theory predictions. Some tests of this version of the K-theory conjecture have been realized using the boundary state formalism in several simple or bifold and orientifold models [12, 13, 14].

By scrutinizing more general physical situations it has become apparent that ordinary K-theory is not enough to characterize $D$-brane charges. For example, considerations of string theory in backgrounds with nontrivial torsion classes suggests that perhaps $D$-brane charges are calculated by differential K-theory [15]. Considerations of certain anomalies in IIA lead to the conjecture that the relevant object to classify $D$-brane charges is Elliptic Cohomology [16]. K-theory also has problems in classifying Ramond-Ramond fluxes. In such case, the classification is incompatible with S-duality in type IIB [17]. More generally, it is fair to state that the generalized cohomology theory that completely characterizes $D$-brane charges in the general case is still not completely understood.

1.1 Outline

In this paper we explicitly present the K-theory group for general Abelian orbifolds. We shall consider the simplest non-trivial examples of orbifolds like $\mathbb{C}^3/\mathbb{Z}_N$ and $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ and consider the inclusion of discrete torsion when appropriate.

Interestingly, an analysis of K-theory suggests an asymmetry in the K-theory of $\mathbb{Z}_N$ orbifolds depending on whether $N$ is even or odd. This result motivates us to consider boundary states with the intention of verifying this prediction. In particular, K-theory predicts that the $\mathbb{Z}_3$ orbifold should have only BPS branes. We proceed to verify this prediction using the boundary state formalism. We compare the K-theory predictions for the $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ without and with discrete torsion orbifolds with the results obtained by Gaberdiel and Craps [18] using boundary states. We find full agreement, including the type of representations (conventional or projective) acting on the Chan-Paton factors.

Some of the results presented here are known in the literature, however, we feel that
a systematic presentation of the mathematical and physical sides is lacking for orbifolds like $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_N$ without and with discrete torsion. There are various reviews that exhaustively present some particular aspects of the topics we touch briefly here. We refer the reader interested in details in the construction of boundary states to the excellent reviews [6, 7, 8, 19], and for K-theory aspects to [20, 21]. Our goal in reviewing part of the material on perturbative construction of $D$-branes is to allow for an easy introduction to the interested mathematical reader, likewise we present many explicit calculations of K-theory which should be accessible to interested physicists.

The organization of the paper is as follows. We start with a general discussion of $D$-branes. Section 2 contains a detailed description of the boundary states techniques. Section 3 discusses strings and branes on orbifolds including the case of discrete torsion. Section 4 contains a number of explicit examples including $\mathbb{Z}_N$ with even and odd $N$. This section contains an account of the relevant K-theory. Section 5 discusses the presence of discrete torsion in various types of orbifolds. We present some conclusions and point out to some interesting open questions in 6. In two appendices we collect a number of technical results used in the main text. Appendix A contains perturbative string theory results and appendix B contains results of from K-theory.

2 D-branes as boundary states

In this section we briefly present the construction of $D$-branes as boundary states. We refer the reader interested in the details to the excellent reviews [6, 7, 8, 19]. We begin with the open and closed string description of $D$-branes and follow with a review of the boundary state formalism. We will see that to describe $D$-branes, the boundary state has to satisfy various conditions such as invariance under all symmetries of the string theory and a consistent open/closed string interaction. Finally, we review the classification of $D$-branes in Type IIA/IIB in ten dimensions, using the boundary state and K-theory formalism.

2.1 Open and Closed string description of $D$-branes

We work in the R-NS formalism and the light-cone gauge quantization. We follow the notation of [3, 8]. A $Dp$-brane is a hyper plane, in ten dimensional space-time, that extend along $p$ spatial directions and where the endpoints of open strings can end. In other words, a $Dp$-brane is defined by the boundary conditions the endpoints of open strings satisfy. There
are Neumann boundary conditions along the $p + 1$-directions (including time):

$$\partial_\sigma X^\mu(\sigma, \tau)|_{\sigma=0,\pi} = 0, \quad \mu = 0, \ldots, p$$

(2.1)

and Dirichlet boundary conditions along $9 - p$ directions (the transversal directions to the $Dp$-brane):

$$X^\nu(\sigma, \tau)|_{\sigma=0,\pi} = a^\nu, \quad \nu = p + 1, \ldots, 9,$n

(2.2)

where $X^\mu$ are the bosonic world-sheet fields which are maps from the world-sheet into space-time. The constants $a^\nu$ denote the position of the $Dp$-brane in space-time and the parameters $\sigma$ and $\tau$ are the spatial and temporal coordinates on the world-sheet respectively. The end points of the strings are at $\sigma = 0, \pi$. The interaction between two $D$-branes is given by vacuum fluctuations of an open string beginning on one $D$-brane and ending on the other $D$-brane propagating in a loop with periodic time $\tau \in [0, 2\pi t]$. Graphically, the topology of the open string world-sheet is a cylinder ending on the two branes. The interaction process is described by the one-loop amplitude

$$\int_0^\infty \frac{dt}{2t} \text{Tr}(\hat{P} e^{-2tH_o}),$$

(2.3)

where $t$ is the modulus of the cylinder and runs over the range $0 < t < \infty$. The trace is taken over the open string spectrum and weighted by the exponential of the Hamiltonian $H_o$ denotes a partition function with a projector operator $\hat{P}$ inserted. We will return to the role of this operator in some concrete examples. The Hamiltonian of the open string $H_0$ is explicitly defined in appendix A.

Since the theory is conformal invariant, one can always find a conformal transformation such that the world-sheet coordinates are exchanged: $\sigma \leftrightarrow \tau$. After a conformal rescaling of the world-sheet coordinates the interaction between $D$-branes has the topology of a cylinder with length parameterized by $l = 1/2t$ [6, 8, 22]. It is drawn by a closed string state of length $2\pi$ propagating between the branes in a euclidean time $2\pi l$. The ends of the cylinder lie on the $D$-branes and represent boundary closed string states that are created or annihilated by the branes. In this way a $D$-brane is a boundary or a source of closed strings. The interaction process is described by the tree-level closed string amplitude

$$\int_0^\infty dl \langle Dp'| e^{-iH_c} |Dp \rangle,$$

(2.4)

where $H_c$ is the closed string Hamiltonian described in appendix A.

These two descriptions of the interaction of $D$-branes are physically different but they are equivalent in the sense that the interaction amplitudes are related to each other by a
conformal transformation. This equivalence is referred to as open/closed string duality or world-sheet duality.

2.2 Boundary States

Under the modular transformation, the open boundary conditions (2.1) and (2.2) become boundary conditions for closed strings states [6]:

$$
\begin{align*}
\partial_{\tau} X^\mu(\sigma, 0)\langle Bp \rangle &= 0 \quad \mu = 0, \ldots, p \\
X^\nu(\sigma, 0)\langle Bp \rangle &= a^\nu\langle Bp \rangle \quad \nu = p + 1, \ldots, 9.
\end{align*}
$$

These boundary conditions are defined at $\tau = 0$ but similar conditions are imposed at $\tau = 2\pi l$.

In order to solve the equations (2.5), we expand the closed string coordinate operators $X^\mu$ in terms of the oscillator modes. The light-cone coordinates $X^0, X^9$ satisfy Dirichlet boundary conditions [23]. In this context we will be describing D-instantons, but performing an appropriate Wick rotation one can transform these states back to ordinary D-branes. The boundary conditions (2.5) become

$$
\begin{align*}
\hat{p}^\mu\langle Bp \rangle &= 0 \quad \mu = 1, \ldots, p + 1, \\
(\alpha^\mu_n + \tilde{\alpha}^{\nu}_{-n})\langle Bp \rangle &= 0 \quad \mu = 1, \ldots, p + 1, \\
(\alpha^{\nu}_n - \tilde{\alpha}^{\nu}_{-n})\langle Bp \rangle &= 0 \quad \nu = p + 2, \ldots, 8, \\
\hat{x}^\nu\langle Bp \rangle &= b^\nu\langle Bp \rangle \quad \nu = 0, 9, p + 2, \ldots, 8,
\end{align*}
$$

where $\hat{p}^\mu$ is the center of mass momentum operator, $\alpha_n$ and $\tilde{\alpha}_n$ are the left- and right-moving modes of the bosonic operator $X$ with $n \in \mathbb{Z}$ and $\hat{x}^\mu$ is the center of mass position operator. In supersymmetric string theories one has to include analogous boundary conditions for the fermions

$$
\begin{align*}
(\psi^\mu_r + i\eta \tilde{\psi}^{\nu}_{-r})\langle Bp, \eta \rangle &= 0 \quad \mu = 1, \ldots, p + 1, \\
(\psi^{\nu}_r - i\eta \tilde{\psi}^{\nu}_{-r})\langle Bp, \eta \rangle &= 0 \quad \nu = p + 2, \ldots, 8,
\end{align*}
$$

which define the fermionic part of the boundary state. Here, $\psi_r$ and $\tilde{\psi}_r$ are the left- and right-moving modes of the fermion operators; $r \in \mathbb{Z}$ for the R sector and $r \in \mathbb{Z} + \frac{1}{2}$ for the NS sector; $\eta = \pm 1$ denotes the spin structure. These boundary conditions can be solved separately for the different closed string sectors of the theory. Using the techniques of [24] one can easily find the solution to equations (2.6) and (2.7). Since left and right movers are related at the boundaries, only boundary states in the NS-NS and R-R sector are allowed.
The direct product of the solutions of these equations determines a boundary state in the Fock space of the closed superstring in the NS-NS and R-R sector

\[
|\tilde{B}_{p,k,\eta}\rangle_{\text{NS-NS R-R}} = \exp \left\{ \sum_{n>0} \frac{1}{n} \alpha_{-n}^\mu S_{\mu\nu} \tilde{\alpha}_{-n}^\nu + i \eta \sum_{r>0} \psi_{-r}^{\mu} S_{\mu\nu} \tilde{\psi}_{-r}^\nu \right\} |\tilde{B}_{p,k,\eta}\rangle^{(0)}_{\text{NS-NS R-R}},
\]

where \( S_{\mu\nu} \) is a diagonal matrix encoding the boundary conditions of the \( D_p \)-brane. It has entries equal to \(-1\) for the \( p+1 \) Neumann directions and \(+1\) for the \( 7-p \) Dirichlet boundary conditions. From the boundary conditions (2.6), the momentum on the Neumann directions is zero. So, the boundary state carries momentum \( k \) only along the Dirichlet directions. The state \( |\tilde{B}_{p,k,\eta}\rangle^{(0)} \) denotes the Fock vacuum. In the NS-NS sector it is the same as the ground state of the closed string. It is unique and independent of the spin structure. In the R-R sector the Fock vacuum needs special attention since it is defined by the zero modes and is degenerate. The precise definition of the R-R ground state is given in appendix A.

It is convenient to work with a localized boundary state for which one has to take the Fourier transform of equation (2.8)

\[
|\tilde{B}_{p,a,\eta}\rangle_{\text{NS-NS R-R}} = \int \prod_{\mu=0,9,p+2,...,8} dk^\mu e^{ik\cdot a} |\tilde{B}_{p,k,\eta}\rangle_{\text{NS-NS R-R}},
\]

where \( a \) denotes the position vector of the brane in the Dirichlet directions. This form of the boundary state is suggested by the consistency conditions described below.

### 2.3 Consistency conditions on the Boundary States

So far we have constructed boundary states as solutions to the closed boundary conditions (2.5). A \( D \)-brane will be described by a linear combination of boundary states (2.9) defined in the different sectors of the closed string and carrying different spin structures. The linear combinations are determined essentially by three requirements. First, a boundary state describing a \( D_p \)-brane has to be a physical state of the Hilbert space \( \mathcal{H} \) in the closed string which means that it has to be invariant under all symmetries and projection operators of the theory. In particular, the boundary state has to be GSO-invariant. If orbifold or orientifold symmetries are considered, the boundary state should be invariant under these symmetries. Secondly, a \( D \)-brane constructed by boundary states should contain all the information about the open string defining the \( D_p \)-brane. The open string spectrum can be determined by computing the interaction amplitude between two \( D_p \)-branes and after a
conformal transformation it should be expressed into a one-loop open string amplitude

\[ \int_0^\infty dl \langle Dp | e^{-lH_c} | Dp \rangle = \int_0^\infty \frac{dt}{2t} \text{Tr}(\hat{P} e^{-2tH_o}), \quad (2.10) \]

that is, the \( Dp \)-brane defined by boundary states should satisfy the open/closed string duality. Third, the open string introduced in this way must have consistent interactions with the closed string sector of the theory. It means that the end points of the open string lying on the \( D \)-brane should be able to join to form a physical closed string state of the theory.

These conditions are intrinsic of an interactive string theory and they do not rely in space-time supersymmetry. It allows one to construct branes in supersymmetric and non-supersymmetric string theories as well as supersymmetric (BPS) and non-supersymmetric (non-BPS) branes [25, 26].

We have mentioned in Sec.2.1 that a \( D \)-brane has a geometrical interpretation as an hyperplane where open strings can end and we have seen that consistency conditions force the boundary states to encode all the information about these open strings. In this sense, the interpretation of \( D \)-branes as boundary states described above relies in the space-time geometry. In a general context, one could be interested in analyzing \( D \)-branes in string theories without referring to the geometry of the space-time. There are several examples of string theories, for instance, Gepner models, WZW models, two-dimensional string theories, where space-time is partially replaced by a conformal field theory. In these cases the formalism of boundary states is more powerful.

Generically, a boundary state in a rational conformal field theory is a boundary that satisfy the gluing conditions \( (W_n - \tilde{W}_{-n})|i\rangle = 0 \) where \( W \) and \( \tilde{W} \) are the generator of the symmetry algebra of the theory. It means that the boundary state preserves a diagonal symmetry algebra. Solutions to these equations are called Ishibashi states [27, 28]. The boundary state is a linear combination of Ishibashi states: \( |\alpha\rangle = \sum_j B^j_i |j\rangle \) where the coefficients \( B^j_i \) are restricted by the Cardy condition, i.e.; by the modular transformation of the partition functions. This condition is equivalent to the open-closed string duality. If there are more symmetry algebras in the system the boundary also has to be a solution of the gluing conditions defined by the generators of such algebras [29, 30]. The Cardy condition produces \( D \)-branes as boundary states that are acceptable from the perspective of CFT. But when applying it to string theories, the Cardy condition is not enough to produce \( D \)-branes, since string theories include other physical considerations.
2.4 Review of D-branes in ten dimensions

*BPS branes*

As an example of $Dp$-branes described by boundary states we review the simplest case of $D$-branes in IIA or IIB in ten dimensions. In these theories there is only a projector operator, the GSO. So, the first task is to see if the the boundary state defined by equations (2.8) and (2.9) is GSO invariant. In the NS-NS, the ground state is taken to have negative eigenvalue under the action of the GSO operators $(-1)^F$ and $(-1)^{\tilde{F}}$. The world-sheet fermions $\psi^\mu$ and $\tilde{\psi}^\mu$ anti-commute with these operators. From these facts, it is easy to prove that

\begin{equation}
(-1)^F|Bp,a,\eta\rangle_{\text{NS-NS}} = -|Bp,a,-\eta\rangle_{\text{NS-NS}}, \quad (-1)^{\tilde{F}}|Bp,a,\eta\rangle_{\text{NS-NS}} = -|Bp,a,-\eta\rangle_{\text{NS-NS}}.
\end{equation}

The GSO operator change the spin structure of the boundary states. Therefore, the linear combination of boundary states in the NS-NS sector with opposite spin structure will be GSO invariant

\begin{equation}
|Bp,a\rangle_{\text{NS-NS}} = \frac{1}{2}(|Bp,a,\eta\rangle_{\text{NS-NS}} - |Bp,a,-\eta\rangle_{\text{NS-NS}}).
\end{equation}

In the R-R sector the GSO operator has the form $(1 + (-1)^F)(1 \pm (-1)^{\tilde{F}})$, with the positive sign corresponding to Type IIB and the negative to Type IIA. From equation (A.7) and (2.8) the action of $(-1)^F$ and $(-1)^{\tilde{F}}$ on the R-R boundary state is

\begin{align}
(-1)^F|Bp,a,\eta\rangle_{\text{R-R}} &= |Bp,a,-\eta\rangle_{\text{R-R}}, \\
(-1)^{\tilde{F}}|Bp,a,\eta\rangle_{\text{R-R}} &= (-1)^{p+1}|Bp,a,-\eta\rangle_{\text{R-R}},
\end{align}

and the GSO-invariant state in the R-R sector is

\begin{equation}
|Bp,a\rangle_{\text{RR}} = \frac{4i}{2}(|Bp,a,\eta\rangle_{\text{R-R}} + |Bp,a,-\eta\rangle_{\text{R-R}}).
\end{equation}

with $p$ even for IIA and $p$ odd for IIB.

Although the boundary states (2.12) and (2.14) are GSO invariant, one has to take a linear combination of these states in order to define a D-brane

\begin{equation}
|Dp,a\rangle = \mathcal{N}(|Bp,a\rangle_{\text{NS-NS}} + \epsilon|Bp,a\rangle_{\text{R-R}}).
\end{equation}

The parameter $\epsilon = \pm 1$ describes the R-R charge of the brane. It can be determined by saturating the boundary state with the corresponding R-R vertex operator [31]. By convention,
the positive sign corresponds to a $Dp$-brane and the negative sign to an anti-$Dp$-brane. The normalization constant $N$ is to be determined.

This linear combination, as well as the coefficients defining the NS-NS and R-R boundary states, is suggested by the conditions that the tree-level amplitude of the $D$-branes has to be equivalent to a one-loop open string amplitude. To see that, one replaces (2.12), (2.14) and (2.15) into (2.4) and after the transformation $l = 1/2t$ one obtains that each component of the $D$-brane produces an open string partition function

$$\int_0^\infty dl_{\text{NS-NS}} \langle Bp,a,\eta | e^{-lH_c} | Bp,a,\eta \rangle_{\text{NS-NS}} = \int_0^\infty \frac{dt}{2t} \text{Tr}_{\text{NS}} e^{-tH_0},$$

$$\int_0^\infty dl_{\text{NS-NS}} \langle Bp,a,\eta | e^{-lH_c} | Bp,a,-\eta \rangle_{\text{NS-NS}} = \int_0^\infty \frac{dt}{2t} \text{Tr}_{\text{R}} e^{-tH_0}.$$

The right-hand side of these equations are open string amplitudes and they can be expressed in terms of the Jacobi theta functions [22]. Expanding the functions around $t \to \infty$, one can see that the first and third open string amplitudes given by the right-hand side of equation (2.16) contain a tachyon in the spectrum. This result will be important when analyzing the stability of the $D$-branes. The equalities in (2.16) are satisfied if the normalization constant is

$$N^2 = \frac{V_{p+1}}{(2\pi)^{p+1}} \frac{1}{32}$$

(2.17)

where $V_{p+1}$ is the world-volume of the brane which together with the factor $(2\pi)^{-(p+1)}$ come from the momentum integration of the open string partition function. We recall that the boundary state (2.9) is the correct one that allows to set the equalities of these amplitudes.

One can read from these relations that the tree-level amplitude satisfy the open/closed string equivalence

$$\int_0^\infty dl \langle Dp,a | e^{-lH_c} | Dp,a \rangle = \int_0^\infty \frac{dt}{2t} \text{Tr}_{\text{NS-R}} \frac{1 + (-1)^F}{2} e^{-tH_0}$$

(2.18)

where $\text{Tr}_{\text{NS-R}}$ denotes the difference between the traces in the NS and R sector of the open string. The projector operator $\hat{P} = \frac{1 + (-1)^F}{2}$ inserted in the trace is the GSO operator. The open string tachyon is projected out giving rise to a stable brane. Since the open string spectrum is supersymmetric, this brane is BPS.
Non-BPS branes

Let us now consider the $Dp - D\bar{p}$ system that, in terms of boundary states, is represented by

$$|Dp, a\rangle + |D\bar{p}, a\rangle = 2|Bp, a\rangle_{\text{NS-NS}}, \quad (2.19)$$

where we have used (2.15) to obtain the right-hand side up to some normalization constant.

The combination of brane-anti-brane breaks supersymmetry, therefore it is a non-BPS brane. From the first and second equations in (2.16) it follows that the respective open string amplitude has a $\text{Tr}_{\text{NS-R}}$ with projector operator $\hat{P} = 1$. Therefore the tachyon in the open string spectrum, producing the instability of the $D$-brane, is preserved. To understand this fact let us write the tree-level amplitude of this brane

$$\langle Dp + D\bar{p}|e^{-iH_c}|Dp + D\bar{p}\rangle = \langle Dp, a|e^{-iH_c}|Dp, a\rangle + \langle Dp, a|e^{-iH_c}|D\bar{p}, a\rangle$$

the first and last terms are of the form (2.18) and represent an open string with both ends lying on a $Dp$-brane and an open string with both ends lying on an anti-$Dp$-brane, respectively. The second and third term give rise to a $\text{Tr}_{\text{NS-R}}$ with the projector operator $\hat{P} = \frac{1}{2} \frac{(-1)^F}{2}$. In this case the GSO projection is opposite to that of equation (2.18). The open string amplitude corresponds to an open string beginning on a $Dp$-brane and ending on an anti-$D$-brane and vice versa. The projector preserves the tachyon present in the NS sector and it makes the system unstable. The brane (2.19) is not independent as it relies directly on the existence of the BPS brane (2.15).

However, inspired by this construction one can propose an independent brane by

$$|\hat{Dp}', a\rangle = \hat{N}^2|Bp', a\rangle_{\text{NS-NS}}. \quad (2.21)$$

This brane produces an open string partition function with a NS and R sector unprojected. Therefore it has a tachyon in the open string spectrum and is unstable. From the same reason, the open string spectrum is non-supersymmetric. The brane is non-BPS. It is an independent brane if $p'$ is odd in IIA or even in Type IIB. The normalization constant is

$$\hat{N}^2 = \frac{V_{p'+1}}{(2\pi)^{p'+1}} \frac{1}{16}. \quad (2.22)$$

The boundary states tell us how the branes couple to the R-R fields of the theory, then we have a way of classifying branes according to the R-R charges. In the case in question
there is only one kind of brane carrying a R-R charge. This result is in agreement with that given by classifying the R-R charges of the branes using K-theory [10]. In this case the analysis is given by the K-theory of the transversal space $S^n$ to the D$p$-brane, with $9 = p+n$. We summarize the results in the following tables

| Brane   | Charges | Type IIA | Type IIB |
|---------|---------|----------|----------|
| Stable  | 1       | $p$ even | $p$ odd  |
| Unstable| 0       | $p$ odd  | $p$ even |

Table 1: Classification of R-R charges using boundary states

| K-theory          | n even | n odd |
|-------------------|--------|-------|
| $\tilde{K}(S^n)$ in Type IIB | $\mathbb{Z}$ | 0     |
| $K^{-1}(S^n)$ in Type IIA    | 0      | $\mathbb{Z}$ |

Table 2: Classification of R-R charges by K-theory

Here $\tilde{K}^i(S^n)$ is the reduced $i$-th $K$-theory group of the $n$-dimensional sphere; reduced means that we factor out the corresponding $K$-theory group of a point. The group only depends on whether $i$ is even or odd. $\tilde{K}^{\text{odd}} = K^{\text{odd}}$, since $K^{\text{odd}}(\ast) = 0$ where $\ast$ denotes a single point.

We have presented shortly the methodology of boundary states to construct branes in string in ten dimensions but the construction of boundary states on other string vacuums is similar.

### 3 Strings and branes on orbifolds

In this section we briefly review strings and $D$-branes on orbifolds. This formalism will be used in the following chapters. We will first give a physical definition of orbifolds and then we will see the conditions $D$-branes must satisfy to exist in these theories. Much of the presentation conforms to the previous works [32, 33, 34, 35, 36]. We also give an introduction to orbifolds with discrete torsion [37, 38, 39, 40]. Finally, we introduce the mathematical ingredients needed for the K-theoretic computations.
3.1 Orbifold theories

Mathematically, an orbifold is locally a quotient of a manifold by a finite group. We will study special kinds of orbifolds which are quotients of Euclidean spaces (for example, \( \mathbb{R}^6 \)) by finite groups which act linearly (real vector spaces on which the group acts by linear isomorphisms will also be called real representations). One can also take a torus \( \mathbb{R}^6/L \) where \( L \) is some six-dimensional lattice and other quotient by a finite group acting linearly on that space, which is equivalent to a finite group \( G \) on \( \mathbb{R}^6 \) which preserves the lattice \( L \). Clearly, this imposes an additional restriction on the group \( G \). Additionally, in this paper, we will only discuss the case of \( G \) Abelian (\( \mathbb{Z}_N \) and \( \mathbb{Z}_N \times \mathbb{Z}_N \)).

An orbifold \( \mathcal{O} \) is defined as a space which is locally the quotient of a manifold \( M \) by the action of a discrete group \( \Gamma \) with finite stabilizers. In the present paper, we shall only discuss examples which are \textit{global} quotients, i.e. \( \mathcal{O} = M/\Gamma \). The discrete group should be symmetry preserving with respect the metric\(^1\) of \( M \). In the construction of \( \mathcal{O} \) one has to identify the point \( X \in M \) with all points \( hX \), with \( h \in \Gamma \). The orbifold may fail to be a manifold at points with non-trivial stabilizer subgroups, but it is possible to repair these singularities by removing these fixed points and replacing them with a smooth non-compact manifold with appropriate asymptotic behavior. This is called \textit{blowing up} or \textit{resolving the singularities}.

However, preserving the singularities is not a problem, since one can still have consistent strings propagating on orbifolds \([32, 33, 34]\). Because points on \( M \) are identified under elements of the discrete group \( \Gamma \), a string closes only up to an element of \( \Gamma \). It means that the string fields should satisfy the boundary condition

\[
X(\sigma + 2\pi, \tau) = hX(\sigma, \tau) .
\]

This requirement will factorize the Hilbert space into subspaces\(^2\) \( \mathcal{H}_h \) for each \( h \in \Gamma \). Each sector \( \mathcal{H}_h \) is the Hilbert space for strings twisted by \( h \). For any non-trivial element \( h \), such sectors are referred to as \textit{twisted sectors} and these states are only closed on the orbifold space \( \mathcal{O} \). The center of mass of twisted sector string states is located at the fixed points while the oscillators obtain fractional quantum numbers according to the order of \( \Gamma \).

\( ^1 \)In the example of lattices defining toroidal compactifications, the orbifold group has to preserve the inner product between the vectors basis of the lattice. That is, the action of the orbifold on any vector of the lattice, has to be an element of the lattice.

\( ^2 \)The decomposition of the Hilbert space is into subspaces characterized by conjugacy classes; for Abelian groups they coincide.
For the identity element of $\Gamma$ the boundary condition for the string field is

$$X(\sigma + 2\pi, \tau) = X(\sigma, \tau).$$

The Hilbert space in this case is referred to as the *untwisted sector* and the string closes on the original space $M$.

In each subspace $\mathcal{H}_h$ one has to project onto those states which are invariant under the action of $g \in \Gamma$. For an Abelian group $\Gamma$, the partition function of the orbifold theory is given by

$$Z = \sum_h \text{Tr}_{\mathcal{H}_h} \hat{P} q^{L_0} \bar{q}^{\bar{L}_0}, \quad (3.2)$$

where the trace is taken on each sector twisted by $h$, $q = e^{2\pi i t}$ with $t$ a complex parameter. The projection operator onto the group invariant states is $\hat{P} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$. One can write this partition function as $Z = \frac{1}{|\Gamma|} \sum_{g, h \in \Gamma} Z(g, h)$ with

$$Z(g, h) = \text{Tr}_{\mathcal{H}_h} g q^{L_0} \bar{q}^{\bar{L}_0}. \quad (3.3)$$

From the point of view of the two-dimensional field theory on the world-sheet, $Z(g, h)$ represents the amplitude in the sector $\mathcal{H}_h$ for a string to propagate in an Euclidean time $2\pi Im t$ and being translated in $\sigma$ by $2\pi Re t$.

It is possible to interpret $Z(g, h)$ as the partition function over a world-sheet torus with modular parameter $t$ and boundary conditions on the string coordinates twisted by the group elements $g$ and $h$ in the $\tau$ and $\sigma$ directions respectively: $X(\sigma + 2\pi, \tau) = hX(\sigma, \tau)$, $X(\sigma, \tau + 2\pi) = gX(\sigma, \tau)$.

Since the partition function (3.3) is equivalent to a partition function of a world-sheet torus, it is easy to see that twisted sectors can be obtained from the untwisted sector by modular transformations. Then the twisted sectors are required by modular invariance of the torus partition function.

### 3.2 Branes in orbifolds and Representations

Given a string theory on a space $M$, one can analyze the behavior of the string on the orbifold quotient space $\mathcal{O}$ following the discussion above. If the original string theory on $M$ contains $D$-branes, it will be interesting to study the action of the orbifold group on the system of

---

3The complex parameter $t$ is usually denoted by $\tau$ in the literature, we have chosen this notation to avoid confusion with the temporal parameter of the world-sheet.
such $D$-branes. In particular, we would like to understand the action of the orbifold group $\Gamma$ on the gauge theory living on the world-volume of the system of $D$-branes. There are two choices one must make. To take a representation of $\Gamma$ on the space-time, $R(g)$ and the action of $\Gamma$ on the Chan-Paton factors $\gamma(g)$ [41] (expanded in [42]). In this way the action of $\Gamma$ on the fields $\phi = (A_\mu(x), X^i(x))$ living on the brane is

$$\gamma(g)^{-1}\phi\gamma(g) = R(g)\phi.$$  \hfill (3.4)

Invariant states under these projections give the gauge theory of the $D$-branes on orbifolds. From the point of view of open strings, there are as many branes as representations of $\Gamma$ on the Chan-Paton indices. The various $D$-branes differ physically in their R-R charges. For instance, the regular representation corresponds to $D$-branes with untwisted R-R charge. Such branes are called bulk branes since they can be localized out of the fixed points of the orbifold and can move freely on the orbifold space. Irreducible representations correspond to $D$-branes carrying untwisted and twisted R-R charges. These branes are called fractional branes [43] since they carry only a fraction of the charge with respect the the untwisted RR field of a bulk brane and they are stuck at the orbifold fixed points. In terms of boundary states, the bulk brane contains only boundary states in the untwisted R-R sector of the closed string. Fractional branes contain both untwisted and twisted boundary states. These two kinds of branes are BPS branes. There is also another kind of branes in orbifold theories which are non-BPS. In [44] they were termed truncated branes as they could be seen as a cut off of the fractional branes.

### 3.3 Discrete torsion

In the search for more general solutions to string theory satisfying modular invariance, Vafa [37] realized that it is possible to introduce a phase multiplying the different terms in the partition function of string theory on orbifolds. The name of discrete torsion refers to the phase $\epsilon(g,h)$ introduced to define a new orbifold theory. Modular invariance and higher loop factorization naturally restrict the form of the phases. The freedom of introducing this phase can, in certain cases, be directly related to the B-field. In [38] Vafa and Witten discussed some geometrical implications of introducing nontrivial phases to weight differently certain terms in the string partition function. Various aspects of the spacetime implications of discrete torsion have been considered in the literature. In particular, the effects of discrete torsion on the world volume theory of $D$-branes have been discussed by Douglas and others [35, 36, 39].
The modular invariant partition function on the torus, mentioned above [37], can be written as

\[ Z'(g, h) = \sum_{g, h \in \Gamma} \epsilon(g, h)Z(g, h) , \]  

(3.5)

where \( \epsilon(g, h) \) are phases called discrete torsion. The possible inequivalent phases are determined from modular invariance and factorization at higher loops. They should satisfy

\[ \begin{align*}
\epsilon(gh, k) &= \epsilon(g, k)\epsilon(h, k), \\
\epsilon(g, h) &= \epsilon(h, g)^{-1}, \\
\epsilon(g, g) &= 1,
\end{align*} \]  

(3.6)

for \( g, h, k \in \Gamma \).

These results can be interpreted in terms of group cohomology. The inequivalent different torsion theories are classified by the second cohomology group of \( \Gamma \) with values in \( U(1) \), \( H^2(\Gamma, U(1)) \) [38]. This cohomology group consists of the two-cocycles \( c(g, h) \in U(1) \) satisfying the cocycle condition

\[ c(g_1, g_2g_3)c(g_2, g_3) = c(g_1g_2, g_3)c(g_1, g_2) . \]  

(3.7)

Equivalence classes are constructed by the equivalence relation

\[ c'(g, h) = \frac{c_g c_h}{c_{gh}}c(g, h). \]  

(3.8)

where \( c_g \) and \( c_h \) are phases, that is, \( c_g \in U(1) \) for \( g \in \Gamma \). Defining

\[ \epsilon(g, h) = \frac{c(g, h)}{c(h, g)}, \]  

(3.9)

this discrete torsion phase is the same for cocycles in the same conjugacy class. The equivalence classes of cocycles of \( \Gamma \) are determined by the second cohomology group \( H^2(\Gamma, U(1)) \).

Discrete torsion can be implemented into the gauge theory of \( D \)-branes on orbifolds by requiring \( \gamma(g) \) (the representation of \( \Gamma \) on the Chan-Paton indices) to be a projective representation such that

\[ \gamma(g)\gamma(h) = c(g, h)\gamma(gh) , \]  

(3.10)

in this way, any projective representation determines a two-cocycle and in this sense we are incorporating enough information to describe discrete torsion. Recall that the conventional representation is defined as \( \gamma(g)\gamma(h) = \gamma(gh) \).
We will concentrate on orbifolds of the type \( \Gamma = \mathbb{Z}_n \times \mathbb{Z}_n \), the simplest group such that \( H^2(\Gamma, U(1)) \) is not trivial. The generators of this group are \( g_1 \) and \( g_2 \). A generic element of this discrete groups is of the form \( g_1^a g_2^b \) and will be denoted by \((a, b)\).

The 2-cocyle classes of \( H^2(\mathbb{Z}_n \times \mathbb{Z}_n, U(1)) \cong \mathbb{Z}_n \) are represented by \( c^m(g, h) : \mathbb{Z}_n \times \mathbb{Z}_n \to U(1) \)

\[
((a, b), (a', b')) \to \zeta^{m(ab' - a'b)}
\]

where \( m = 0, 1, \ldots, n - 1 \) denotes the \( m \) different elements of \( \mathbb{Z}_n \) and \( \zeta = e^{(2\pi i/n)} \) for \( n \) odd, \( \zeta = e^{(\pi i/n)} \) for \( n \) even. The discrete torsion phase can be obtained by replacing the cocycle (3.11) into (3.9). Then,

\[
\epsilon(g, h) = \zeta^{2m(ab' - a'b)}. \tag{3.12}
\]

The theory without discrete torsion correspond to \( m = 0 \). Minimal discrete torsion corresponds to the case when \((m, n) = 1\), it means, that \( \zeta^{2m} \) is a primitive \( n \)-th root of unity. In this case, the group \( \mathbb{Z}_n \times \mathbb{Z}_n \) has a unique projective representation [36, 45].

## 4 K-theory for charges of D-branes on flat \( \mathbb{Z}_N \) orbifolds

In this section we begin our K-theoretical description for groups of charges of D-branes in type IIA/IIB string theories on orbifolds \( \mathcal{O} \). The K-theory groups and boundary states for some simple classes of orbifolds in Type II string theories where analyzed in [44]. We revisit the boundary states for the non-compactlyfied orbifold \( \mathbb{Z}_2 \) presented in this work and we express the K-theory results in our notation in order to be consistent with the next section. Later we shall study the K-theory for orbifolds of flat spacetime by linear action by the finite abelian groups \( \mathbb{Z}_N \), leaving other orbifold models for the next section. In order to corroborate our results for \( N \) odd we consider the special case of orbifold \( \mathbb{Z}_3 \) and construct the boundary states for this model.

### 4.1 Generalities of K-theory on orbifolds

We will think of spacetime as

\[
V \times \mathbb{R}^{1,1} \tag{4.1}
\]

where \( \mathbb{R}^{1,1} \) is Minkowski 2-space (where the light-cone coordinates are defined) and \( V \) is a complex representation of \( G \) of complex dimension 4. The representation must be complex
to preserve supersymmetry, i.e. the reason for this is physical. $G$ acts trivially on $\mathbb{R}^{1,1}$. We have in general the orbifold

$$\mathcal{O} = (V \times \mathbb{R}^{1,1})/G.$$ 

The $D$-branes $M$ in (4.1) whose images in $\mathcal{O}$ we will consider will be real subrepresentations of $V \times \mathbb{R}^{1,1}$. This generality is needed to account for all the branes we consider. The relevant $K$-groups of interest are the equivariant $K$-groups with compact support

$$K^{i,c}_G(M^\perp),$$

(4.2)

where $M^\perp$ is the orthogonal complement of $M$ in $V \times \mathbb{R}^{1,1}$, and $i = 1$ or 0 depending on whether we are in IIA or IIB.

A basic feature of equivariant $K$-theory is Bott periodicity, which asserts that for any $G$-space $X$,

$$K^{i,c}_G(X \times W) \cong K^{i,c}_G(X)$$

(4.3)

for any complex $G$-representation $W$. Now a sum of two copies of any real representation has complex structure and conversely, an irreducible (hence, since $G$ is abelian, 1-dimensional) complex representation can be either irreducible also as an underlying real representation, or can be a sum of two copies of the same irreducible real representation. It already follows from this that all that matters for the $K$-group are the numbers of copies of the individual 1-dimensional irreducible real representations on $M$ mod 2, so we always have

$$K^{i,c}_G(M^\perp) \cong K^{i,c}_G(M).$$

(4.4)

Additionally, when $M$ itself is a complex representation of $G$, then we already know that [46]

$$K^{i,c}_G(M) \cong K^{i,c}_G(\ast) = R(G) = \mathbb{Z}^{|G|}$$

for $i$ even

$$= 0$$

for $i$ odd

(4.5)

The same analysis holds when the world volume of the brane $M$ is a sum of an $m$-dimensional complex $G$-representation and additional $\ell$ real dimensions on which $G$ acts trivially (assuming the time-like dimension is included, then $M$ is a $2m + \ell - 1$-brane). The only difference between this case and (4.5) is that then the dimensions $i$ have to be replaced by $i + \ell$. In physical language, then, the group of charges in this case is $\mathbb{Z}^{|G|}$ when $\ell$ is odd in IIA and even in IIB, and 0 otherwise.

In boundary states, if we denote a $Dp$-brane by $p = r + s'$, with $r$ the number of direction tangential to the brane where the orbifold acts trivially and $s'$ the number of Neumann directions transformed by the group, then $r \equiv \ell - 1$ and $s' \equiv 2m$. 

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In the case of $|G|$ even, we may encounter real representations. A general calculation of the groups (4.2) was given by Max Karoubi [47] (see also [48]). All the groups we will need in this paper however can be calculated from first principles by elementary means.

**4.2 Branes in $\mathbb{Z}_N$ orbifolds with $N$ even**

Let us now see the simple case when $G = \mathbb{Z}/N$ where $N$ is even. Then there exists a unique onto homomorphism

$$\phi : \mathbb{Z}/N \to \mathbb{Z}/2,$$

and therefore $\mathbb{Z}/N$ has a unique non-trivial 1-dimensional real representation $\alpha$, obtained by composing the sign representation of $\mathbb{Z}/2$ with the map $\phi$. Suppose the brane $M$ is a product of an $m$-dimensional complex representation of $\mathbb{Z}/N$, $s$ copies of the representation $\alpha$, and $\ell$ other dimensions on which $\mathbb{Z}/N$ acts trivially (when the time-like dimension is included, it is therefore a $(2m + s + \ell - 1)$-brane). This case reduces to the previous case when $s$ is even, so let us assume $s$ is odd.

Before proceeding with our K-theory discussion it is necessary to set the notations used in the mathematical and physical language. A complex coordinate on $\mathbb{C}^3$ is defined by the real coordinates $(x^{2i+1}, x^{2i+2})$. Then the 1-dimensional real representation corresponds to the case when the couple of real coordinates have mixed boundary conditions, i.e., Neumann-Dirichlet(ND) or DN. The complex representation will corresponds to the case when both coordinates have the same boundary conditions, NN or DD. We will be shifting from one notation to the other along these notes.

The key observation is that we have a cofibration sequence

$$\begin{align*}
(Z/N)/(Z/(N/2))_+ & \xrightarrow{f} S^0 \\
\xrightarrow{\cong} S^\alpha.
\end{align*} \tag{4.6}
$$

The map $f$ is the only non-trivial (non-constant) map, and the fact that we have a cofibration sequence (4.6) is readily verified by definition. Thus, from (B.4), we obtain a long exact sequence

$$\begin{align*}
\tilde{K}^{i-1}_{Z/(N/2)}(X \wedge (Z/N)/(Z/(N/2))_+) & \xrightarrow{f^*} \tilde{K}^{i}_{Z/N}(X \wedge S^0) \\
\xrightarrow{\cong} \tilde{K}^{i}_{Z/N}(X) \\
\xrightarrow{f^*} \tilde{K}^{i}_{Z/(N/2)}(X) \wedge (Z/N)/(Z/(N/2))_+).
\end{align*} \tag{4.7}
$$

(4.7) becomes

$$\begin{align*}
\tilde{K}^{i-1}_{Z/(N/2)}(X) & \xrightarrow{f^*} \tilde{K}^{i}_{Z/N}(X \wedge S^0) \\
\xrightarrow{\cong} \tilde{K}^{i}_{Z/N}(X) \\
\xrightarrow{f^*} \tilde{K}^{i}_{Z/(N/2)}(X) \wedge (Z/N)/(Z/(N/2))_+).
\end{align*} \tag{4.8}
$$
In the case $X = S^0$, $f^*$ in $K^0$ is the reduction map

$$R(\mathbb{Z}/N) \to R(\mathbb{Z}/(N/2))$$

(4.9)

induced by the inclusion $\mathbb{Z}/(N/2) \subset \mathbb{Z}/N$. (We denote the complex representation ring of $G$ by $R(G)$, the real representation ring by $RO(G)$). It then follows that (4.9) is an onto map

$$\mathbb{Z}^N \xrightarrow{f^*} \mathbb{Z}^{(N/2)}.$$  

(4.10)

In more detail, the $N$ summands correspond to irreducible representations of $\mathbb{Z}/N$, and (4.10) restricts the representation to $\mathbb{Z}/(N/2)$. Thus, the kernel is generated freely by elements of the form

$$\gamma - \gamma\alpha$$

where $\gamma$ is an irreducible (complex) representation of $\mathbb{Z}/N$. This group is isomorphic to $\mathbb{Z}^{N/2}$. Thus, we have computed for our brane $M$ in case of $s$ odd

$$K^\ell_c(\mathbb{Z}/N)(M) = \mathbb{Z}^{N/2},$$

$$K^{\ell+1}_c(\mathbb{Z}/N)(M) = 0.$$  

(4.11)

Summarizing, the K-theory group (4.5) and (4.11) classify all stable $D$-brane charges in Type IIA and Type IIB on a flat $\mathbb{Z}_N$ orbifold for $N$ even.

The spectrum of $D$-branes in flat and toroidal $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds was analyzed systematically using boundary states and the K-theory formalism in [44]. We briefly review the results of this construction for the flat $\mathbb{Z}_2$ orbifold. To begin with, we consider the space-time of the form

$$\mathcal{O} = \mathbb{R}^{1,5} \times \mathbb{R}^4/\mathbb{Z}_2.$$  

The generator $g$ of $\mathbb{Z}_2$ acts as a reflection $I_4$ on the coordinates of $\mathbb{R}^4$, and it acts trivially or $\mathbb{R}^{1,5}$. As before $x^0$ and $x^9$ denotes the light-cone coordinates. To describe a $Dp$-brane we use the notation $D(r; s)$, where $p = r + s$ where $r$ is the number of Neumann directions on which $\mathbb{Z}_2$ acts trivially and $s$ denotes the number of Neumann directions reflected by the group. In the mathematical language, this is a natural basis for the decomposition in terms of the eigenspaces or action of the representations on the coordinates (see section 4.1).

The construction of boundary states follows section 2 with the added feature that now there will also be boundary states $|B(r, s)\rangle_{\text{NS-NS,T}}$ and $|B(r, s)\rangle_{\text{R-R,T}}$, constructed from the NS-NS twisted and R-R twisted sectors, respectively. To construct consistent $D$-branes, physical boundary states must be invariant under the combined action of GSO- and orbifold-
projection. This restricts the values of \( r \) and \( s \). The boundary states in the different sector of the theory transforms as

This information allows one to construct the invariant \( D \)-branes as linear combination of the different invariant boundary states. The orbifold theory has stable BPS branes (fractional branes) of the form

\[
|D(r,s)\rangle = |B(r,s)\rangle_{\text{NS-NS}} + \epsilon_1 |B(r,s)\rangle_{\text{R-R}} + \epsilon_2 (|B(r,s)\rangle_{\text{NS-NS,T}} + \epsilon_1 |B(r,s)\rangle_{\text{R-R,T}}),
\]

defined up to normalization constants deduced by the open/closed string duality. This brane carries two charges due to the coupling of the brane with the fields in the untwisted and twisted R-R sectors. The sign of the charge is determined by \( \epsilon_{1,2} = \pm 1 \). The tree-level amplitude of this brane gives rise to an open string amplitude, as required by Eq.(2.10), with trace \( \text{Tr}_{\text{NS-R}} \frac{1+(-1)^F}{2} \). The GSO operator in this partition function projects out the tachyon and makes the spectrum of the open string supersymmetric. Then the brane is stable and BPS. From Table 3 one can see that this brane exist in Type IIB string theory for \( r \) odd and \( s \) even and in Type IIA, \( r \) and \( s \) should be both even. Note that the corresponding K-group is given by (4.5).

The other kind of branes are non-BPS (called truncated brane in [44]). They couple only to the NS-NS untwisted and R-R twisted sector and are defined by

\[
|\hat{D}(r,s)\rangle = |B(r,s)\rangle_{\text{NS-NS}} + \epsilon |B(r,s)\rangle_{\text{R-R,T}}.
\]

This brane carries only one charge represented by \( \epsilon \) and satisfy the relation (2.10) if the open string amplitude has the projection operator \( \hat{P} = \frac{1+(-1)^F}{2} \). This is a slight modification of the GSO condition since it incorporates the element \( g \). In Type IIB this brane is GSO and orbifold invariant if \( r \) and \( s \) are both odd and in Type IIA \( r \) should be even and \( s \) odd. The open string spectrum has tachyons coming from the term \( \text{Tr}_N \) and \( \text{Tr}_N(-1)^F g \) respectively.
and they cancel if and only if \( s = 0 \). Since the physical conditions restrict \( s \) to be odd in both Type IIA and Type IIB, the non-BPS brane is unstable and therefore decays to another brane with the same charge. The K-theory group is given in (4.11).

In [44] it was shown that non-BPS branes (4.13) can be stable in \( \mathbb{Z}_2 \) orbifolds with generator \( g = \mathcal{I}_4(-1)^{F_L} \) where \( \mathcal{I}_4 \) is a reflection of the coordinates in \( \mathbb{R}^4 \) and \( (-1)^{F_L} \) acts as \( \pm 1 \) on the left-moving space-time bosons and fermions, respectively. It comes form the fact that in this kind of orbifolds the consistency conditions put both \( r \) and \( s \) to be even for non-BPS branes in Type IIB while \( r \) should be odd and \( s \) even in Type IIA.

### 4.3 Branes in \( \mathbb{Z}_N \) orbifolds with \( N \) odd

It is important to note in 4.1 that when \( |G| \) is odd, every non-trivial irreducible real representation has a complex structure. This means that it is obtained from a complex representation by forgetting the complex structure. Accordingly, in the computations below, the allowed branes are those with all \( s_i \) even. So equation (4.5) classifies all orbifold \( D \)-brane charge groups in our setting in the case of \( |G| \) odd.

We would like to test this prediction analyzing the spectrum of \( D \)-branes in the particular orbifold \( \mathbb{C}^3/\mathbb{Z}_3 \) using boundary states. \( D0 \)-branes and their T-duals in this model were analyzed in [40] and \( D0 \) and \( D3 \) branes in the compactified version in [49]. We set the construction of these works in a systematic form that allows one to classify all branes in this \( \mathbb{Z}_3 \) orbifold.

For this model the space-time has the form

\[
\mathcal{O} = \mathbb{R}^{1,1} \times \mathbb{R}^2 \times \mathbb{C}^3/\mathbb{Z}_3.
\]

The light-cone coordinate \( x^0 \) and \( x^9 \) define the two dimensional Minkowski space. The couple \((x^1, x^2)\) describe the coordinates on \( \mathbb{R}^2 \) where \( \mathbb{Z}_3 \) acts trivially. On \( \mathbb{C}^3 \) we have the complex coordinates

\[
z^i = \frac{1}{\sqrt{2}}(x^{2i+1} + i x^{2i+2}) \quad i = 1, 2, 3.
\]

Non-trivial elements of \( \mathbb{Z}_3 \) are denoted by \( g^m \) with \( m = 1, 2 \) and \( g \) the generator acting on the complex coordinates as

\[
g : (z^1, z^2, z^3) \rightarrow (e^{2\pi i \nu_1} z^1, e^{2\pi i \nu_2} z^2, e^{2\pi i \nu_3} z^3)
\]

where \( \nu_i = \frac{a_i}{3} \) with \( a_i \in \mathbb{Z} \). To preserve some supersymmetry, \( a_1 + a_2 + a_3 = 0 \) mod 3. We will be interested in the case \( \nu = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) \). The world-sheet fields along \( \mathbb{C}^3 \) will be
also complexified in the same way. As usual, in the untwisted sector the NS-NS has half-integer modes and the R-R sector the modes are integer. The untwisted R-R ground state is degenerated due to the zero-modes along the directions $x^1, x^2, z^1, z^2, z^3$.

In the twisted NS-NS sector fermion variables are modded as $Z + \frac{1}{2} + m \nu_i$. The twisted R-R sector has modes $Z + m \nu_i$ along the directions on which the orbifold acts, and there are zero-modes only on the directions not affected by the orbifold. The twisted R-R ground state is therefore two-degenerate.

We say that a $D^p$-brane is of type $(r, s)$ where $s = s_1 + s_2 + s_3$, it means that the brane has $r$ number of Neumann boundary conditions along the directions $(x^1, x^2)$ and $s_1, s_2, s_3$ Neumann directions along $(x^3, x^4), (x^5, x^6), (x^7, x^8)$ directions, respectively. The case $s_i = 1$ corresponds to mixed boundary conditions along any of these couple of coordinates.

A detailed analysis of the boundary states and the action of the orbifold and the GSO operator in the different boundary states of the theory is given in A. The action of GSO on the untwisted sector is the same as in the case discussed in 2.4. In the twisted sectors, non-trivial solution to the boundary conditions restrict the values of all $s_i$ to be even. The conditions for the invariance of the boundary states have been collected in Table 4.

| $|B(r, s)\rangle_{\text{NS-NS}}$ | GSO invariant | Orbifold invariant |
|-------------------------------|---------------|--------------------|
| $|B(r, s)\rangle_{\text{R-R}}$ | for any $r, s$ | for any $r, s$ |
| $|B(r, s)\rangle_{\text{NS-NS,T}}$ | if $r + s$ is even/odd in IIA/IIB | for all $s_i$ even |
| $|B(r, s)\rangle_{\text{R-R,T}}$ | for any $r$ | for any $r$ |

Table 4: Conditions for GSO and orbifold invariant of the different boundary states

We are now ready to describe the spectrum of $D$-branes. The only BPS brane is

$$
|D(r, s)\rangle = |B(r, s), \eta\rangle_{\text{NS-NS}} + \epsilon_0 |B(r, s), \eta\rangle_{\text{R-R}} + \sum_{m=1,2} \epsilon_m \left( |B(r, s), \eta\rangle_{\text{NS-NS,T} g^m} + \epsilon_0 |B(r, s), \eta\rangle_{\text{R-R,T} g^m} \right)
$$

(4.14)

where $\epsilon_0$ denotes the sign of the charge with respect to the untwisted R-R fields and $\epsilon_m$ gives the sign of the charge of the brane when coupling to the twisted R-R sector. This brane is stable and supersymmetric as can be seen computing the closed string interaction between itself. The interaction amplitudes in the closed and open string sectors were computed in
and we will not repeat them here. The open string spectrum is invariant under the projector $\frac{1+(-1)^F}{2} \frac{1+r+g^2}{3}$. Such branes exist for all $s_i$ even and $r$ has to be odd in Type IIB or even in Type IIA. This is the only fundamental brane with R-R charge in this model.\(^4\) This result agrees with the K-theory prediction.

### 4.4 D-brane charges in $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$

To give a mathematical discussion for the $(\mathbb{Z}/2)^n$ case, we refer the reader to the Appendix B for background. We know by (B.10) that since at most 4 non-trivial irreducible real representations can occur on $M$, it suffices to consider $n \leq 4$. Moreover, in the only case of $n = 4$ which cannot be reduced to $n \leq 3$, we have 4 irreducible real representations independent in the character group, so we can use (B.11) (with $n = k = 4$). Similarly, in the case of $n = 1$ (which was previously treated in the literature), only one non-trivial real representation exists, so this case again can be handled by (B.11) with $n = 1$. Therefore, we are now reduced to $n = 2$ or 3.

The first nontrivial case which cannot be settled by (B.10) or (B.11) occurs when $n = 2$, and $V$ is the sum of the three nontrivial 1-dimensional real representations $\alpha, \beta, \gamma$ (of course, $\gamma \cong \alpha \otimes \beta$). Let $A = \text{Ker}(\alpha)$, $B = \text{Ker}(\beta)$, $C = \text{Ker}(\gamma)$, so $A, B, C$ are the three subgroups of $G$ of order 2.

To tackle this case, consider the cofibration

$$G/C_+ \wedge S^{\alpha+\beta} \rightarrow S^{\alpha+\beta} \rightarrow S^{\alpha+\beta+\gamma}.$$  \(4.15\)

The equivariant $K_G$-theory of the first two summands can be calculated by (B.11) and (B.5): we have $\epsilon = 0$, and

$$\tilde{K}_G^0(G/C_+ \wedge S^{\alpha+\beta}) = \tilde{K}_C^0(S^{2\alpha}) = \tilde{K}_C^0(S^0) = \mathbb{Z} \oplus \mathbb{Z},$$ \(4.16\)

while

$$\tilde{K}_G^0(S^{\alpha+\beta}) = \mathbb{Z}$$ \(4.17\)

by (B.11). So we are done if we can calculate the map from (4.17) to (4.16) induced by the first arrow (4.15). In fact, note that the interesting information is just the image of that map, which can be calculated in $C \cong \mathbb{Z}/2$-equivariant $K$-theory. When restricted to $C$,\(^4\) There is a non-BPS brane with the same values of $(r,s)$ as that for the BPS brane. This is unstable. The interaction between the BPS and the non-BPS ensures that the BPS brane is the fundamental brane in the sense that it is of smaller mass and charge and is stable \([44]\).
\[ \alpha \cong \beta = \omega, \]  
which will denote the sign representation of \( C \). Now in \( K_C^0(S^\omega) \), we have the element \( c \) which, under the inclusion

\[ S^0 \subset S^\omega, \tag{4.18} \]

restricts to

\[ 1 - \omega \in R(C) = \tilde{K}_C^0(S^0). \tag{4.19} \]

Then the restriction of the generator of \( \tilde{K}_G^0(S^{\alpha+\beta}) \) to \( \tilde{K}_C^0(S^{2\omega}) \) is \( c^2 \). Our question is thus equivalent to finding the image of \( c^2 \) in \( \tilde{K}_C^0(S^0) \) under Bott periodicity. Now what happens is that the Bott element

\[ u \in \tilde{K}_C^0(S^{2\omega}) \]

under the map induced by the inclusion

\[ S^\omega \subset S^{2\omega} \]

maps to \( c \). (This is seen directly by the construction of the Bott element as \( 1 - H \) where \( H \) is the tautological line bundle on \( \mathbb{C}P^1 \) where \( \mathbb{Z}/2 \) acts by minus on \( \mathbb{C} \subset \mathbb{C}P^1 \).) So, we need to find the image of \( u^2 \in \tilde{K}_C^0(S^{4\omega}) \) under the composition \( \beta f^* \) where

\[ f : S^{2\omega} \to S^{4\omega} \]

is the inclusion (all such inclusions are homotopic) and \( \beta \) is Bott periodicity. But we already know that

\[ f^*(u^2) = u(1 - \gamma), \]

so

\[ \beta f^*(u^2) = 1 - \gamma. \tag{4.20} \]

This generates a direct summand in (4.16). In other words, the first map (4.15) induces in \( G \)-equivariant \( K \)-theory the inclusion of a direct \( \mathbb{Z} \) summand, and hence for \( V = \alpha + \beta + \gamma \), we have \( \epsilon = 1 \) and

\[ \tilde{K}_G^1(S^{\alpha+\beta+\gamma}) = \mathbb{Z}. \tag{4.21} \]

As already remarked, (B.11) implies that for \( V \subseteq \alpha + \beta \), \( \epsilon = 0 \) and

\[ \tilde{K}_G^0(S^{\alpha+\beta}) = \mathbb{Z}, \quad \tilde{K}_G^0(S^\alpha) = \mathbb{Z} \oplus \mathbb{Z}, \quad \tilde{K}_G^0(S^\omega) = \mathbb{Z}^4. \tag{4.22} \]

All other cases for \( n = 2 \) are related to these by symmetry, so the case of \( n = 2 \) is completely settled.
In the case of \( n = 3 \), let us denote \( \Gamma = (\mathbb{Z}/2)^3 \) and let \( \alpha_1, \alpha_2, \alpha_3 \) be three 1-dimensional real representations of \( \Gamma \) which are independent in the character group. Then similarly to (4.22), again, by (B.11),

\[
\tilde{K}^0_\Gamma(S^0) = \mathbb{Z}^8, \quad \tilde{K}^0_\Gamma(S^{\alpha_1}) = \mathbb{Z}^4, \quad \tilde{K}^0_\Gamma(S^{\alpha_1+\alpha_2}) = \mathbb{Z}^2, \quad \tilde{K}^0_\Gamma(S^{\alpha_1+\alpha_2+\alpha_3}) = \mathbb{Z}
\]  

(4.23)

and \( \epsilon = 0 \) in all these cases. In the first non-trivial case, we see by (B.10) that

\[
\tilde{K}^1_\Gamma(S^{\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2}) = \mathbb{Z}
\]  

(4.24)

and \( \epsilon = 1 \) in this case. In the other case

\[
V = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3,
\]

we use the cofibration sequence

\[
S^{\alpha_1\alpha_2\alpha_3} \wedge \Gamma/Ker(\alpha_1\alpha_2\alpha_3) \rightarrow S^{\alpha_1+\alpha_2+\alpha_3} \rightarrow S^{\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2\alpha_3}.
\]  

(4.25)

By (B.5), the \( K \)-theory of the first term is

\[
K^*_{Ker(\alpha_1\alpha_2\alpha_3)}(S^{\alpha_1+\alpha_2+\alpha_3}) \cong K^*_G(S^{\alpha+\beta+\gamma})
\]

which, as we have seen, is \( \mathbb{Z} \) located in odd dimension. On the other hand, the \( K \)-theory of the middle term of (4.25) is calculated by (4.23), giving \( \mathbb{Z} \) in even dimension. We therefore see that for dimensional reasons, the first arrow of (4.25) must induce 0 in \( K_\Gamma \), thus giving

\[
\tilde{K}^0_\Gamma(S^{\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2\alpha_3}) = \mathbb{Z}, \quad \epsilon = 0.
\]  

(4.26)

Now all cases of \( K_\Gamma \)-theories of 1-point compactifications of representations of \( \Gamma \) which contain at most 4 different 1-dimensional real representations are related to one of the cases (4.23), (4.24) or (4.25) by symmetry, so the case \( n = 3 \) is also completely settled.

The orbifold theories of type \((\mathbb{Z}/N)^n\) are very interesting since they accept discrete torsion. But so far we have been analyzing orbifolds without discrete torsion. In the following we shall revisit the D-brane charge classification with the boundary state formalism. The case with discrete torsion will be presented in the next section. In the discussion below we will concentrate in the non-compact orbifold \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) without discrete torsion. Boundary states and K-theory classifying D-brane charges in non-compact and compactified orbifolds \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) were analyzed extensively in [12].

The generators of this group are given by \( g_1 \) and \( g_2 \) and they act on the coordinates \( x^3, \ldots, x^8 \) as
\[
g_1(x^0, \ldots, x^9) = (x^0, x^1, x^2, x^3, x^4, -x^5, -x^6, -x^7, -x^8, x^9)
\]
\[
g_2(x^0, \ldots, x^9) = (x^0, x^1, x^2, -x^3, -x^4, x^5, x^6, -x^7, -x^8, x^9)
\]

with \( g_3 = g_1g_2 \). To describe a \( Dp \)-brane we use the notation \((r; s) = (r; s_1, s_2, s_3)\), where \( p = r + s_1 + s_2 + s_3 \). This means that the brane has Neumann boundary conditions along \( r + 1, s_1, s_2, s_3 \) of the \((x^0, x^1, x^2, x^3), (x^3, x^4), (x^5, x^6), (x^7, x^8)\), respectively. To connect with our mathematical notation above, the coordinates \( x^0, x^1, x^2 \) and \( x^9 \) are copies of the representation 1, the coordinates \( x^3, x^4 \) are copies of \( \alpha \), the coordinates \( x^5, x^6 \) are copies of \( \beta \) and \( x^7, x^8 \) are copies of \( \gamma \) (although \( \alpha, \beta, \gamma \) are notationally interchangeable).

There are several kinds of branes. We will write down only the fundamental branes that carry the smaller charge and mass. It was noticed in [51] that in the kind of orbifold in question, the open string endpoints can carry the conventional and projective representations of the orbifold group; and that the presence of discrete torsion does not change the results\(^5\). The boundary states carrying the projective representation were analyzed in [18, 51].

In all branes given below \( r \) is even or odd in Type IIA or Type IIB, respectively. There are two types of BPS branes. One is a fractional brane with projective representation of the orbifold group on the Chan-Paton factors. It is defined for all \( s_i = 1 \) and has the form

\[
|D(r; s); a\rangle = |B(r; s); a\rangle + |B(r; s); -a\rangle ,
\]

where

\[
|B(r; s); a\rangle = |B(r; s); a\rangle_{NS-NS; U} + \epsilon|B(r; s); a\rangle_{R-R; U}
\]

\[
+ \epsilon'|B(r; s); a\rangle_{NS-NS; T_{g_i}} + \epsilon|B(r; s); a\rangle_{R-R; T_{g_i}} .
\]

We have to stress that the moduli space of this brane consist of the different fixed planes of \( g_i \) with \( i = 1, 2, 3 \). The position of the brane along the directions on which the orbifold acts trivially have been dropped out and \( a \) is the position of the brane in the directions on the planes fixed by \( g_i \). The orbifold acts on the Chan-Paton factors by a projective representation. This brane carries charge only with respect to the untwisted R-R charge. Therefore the respective K-group is given by (4.21).

\(^5\)We recall that the conventional representation corresponds to the case when \( \gamma(g)\gamma(h) = \gamma(gh) \).
Next, one has a *fractional* brane with conventional representation. It is defined for all \( s_i \) even and is given by

\[
|D(r, s)| = |B(r, s)|_{\text{NS-NS}} + \epsilon |B(r, s)|_{\text{R-R}} + \sum_{i=1}^{3} \epsilon_i (|B(r, s)|_{\text{NS-NS}, T_{g_i}} + \epsilon_1 |B(r, s)|_{\text{R-R}, T_{g_i}}) \tag{4.30}
\]

where \( \epsilon = \pm 1 \) determines the sign of the charge with respect to the untwisted R-R sector, \( \epsilon_i = \pm 1, i = 1, 2, 3 \) denote the sign of the charge with respect to the R-R twisted by \( g_i \). Such brane is stuck at all fixed point \( x^3 = \ldots = x^8 = 0 \) of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). This brane carries four charges, \( \epsilon \) and \( \epsilon_i, i = 1, 2, 3 \) and its K-theory group is given by the last relation in (4.22).

For the non-BPS case, there are also two kind of branes, one is also a fractional brane with conventional representation

\[
|\hat{D}(r, s)| = |B(r, s)|_{\text{NS-NS}} + \epsilon |B(r, s)|_{\text{R-R}, T_{g_i}}, \tag{4.31}
\]

This \( D \)-brane is charged under a massless R-R field in the twisted sector by \( g_i \). The corresponding K-theory is determined by the first relation in (4.22).

The last brane is a *fractional brane* with conventional representation of the group on the endpoints of the open string. It has one \( s_i \) odd and the rest even. It couples to two of the three R-R twisted sectors. Say for instance \( g_i, \) and \( g_j \) for \( i \neq j \). The respective boundary state is

\[
|\hat{D}(r, s)| = |B(r, s)|_{\text{NS-NS}} + \epsilon_i |B(r, s)|_{\text{R-R}, T_{g_i}} + \epsilon_j |B(r, s)|_{\text{R-R}, T_{g_j}} + \epsilon_i \epsilon_j |B(r, s)|_{\text{NS-NS}, T_{g_k}} \tag{4.32}
\]

This brane is stuck at all fixed points of the orbifold group. The K-theory group is given by the second relation in (4.22).

### 4.5 Branes in Type II on \( T^6/\mathbb{Z}_2 \times \mathbb{Z}_2 \)

So far we have analyzed the \( D \)-brane charge spectrum of the non-compactified \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold. The compactified case can be obtained straightforwardly. From the point of view of K-theory, the equivariant K-theory groups relevant here are determined by the cases discussed above: Suppose we have a transverse torus \( T^{a;b,c,d} \) where the numbers \( a, b, c, d \) are as above. Then we have

\[
K^*_\mathbb{Z}/2 \times \mathbb{Z}/2, c(T^{a;b,c,d}) = \bigoplus \left( \begin{array}{c} a \\ a' \end{array} \right) \left( \begin{array}{c} b \\ b' \end{array} \right) \left( \begin{array}{c} c \\ c' \end{array} \right) \left( \begin{array}{c} d \\ d' \end{array} \right) K^*_\mathbb{R}^{d',b',c',d'}. \]

where the sum is over all \( 0 \leq a' \leq a, 0 \leq b' \leq b, 0 \leq c' \leq c, 0 \leq d' \leq d \). The corresponding boundary states are given as those above in the uncompactified case. However in this case, one has to put particular attention to the stability radius.
5 K-theory and discrete torsion

Mathematically, discrete torsion in not an intrinsic property of the orbifold but its $K$-theory, which becomes twisted $K$-theory. This means that $K$-theory varies as we move around the orbifold, i.e. we have a “bundle of $K$-theories” on the orbifold. From a spacetime point of view, an $H_3$-flux can cause a twisting of $K$-theory in this sense. From the world-sheet point of view, on the other hand, $K$-theory twisting corresponds to an automorphism of the category of vector spaces in which Chan-Paton bundles take place.

It is proved in [52] that on a $G$-space $X$, $G$-equivariant $K$-theory $H_3$-twistings are classified by elements of

$$H^3_{Borel}(X, \mathbb{Z}).$$

(5.1)

Borel cohomology of a $G$-space $X$ is obtained by taking a space $EG$ which is contractible but has a free $G$-action (such space is unique up to homotopy equivalence under some minimal topological assumptions which we do not discuss here). Then Borel cohomology simply means cohomology of the Borel construction

$$EG \times X/(y, x) \sim (gy, gx) \text{ for } x \in X, y \in EG, g \in G.$$ 

For our purposes, we must answer the question as to what kind of twistings are possible in equivariant $K_G$-theory with compact supports of a space $X$. However, the answer turns out to be the same, there are twistings corresponding to all elements of (5.1), in other words no compact supports are needed in the twisting group. The reason is that generalized cohomology with compact supports is a direct limit of relative cohomology groups of pairs $(X, U)$ where $X - U$ is compact. All those groups are consistently twisted by a given element of (5.1), hence so is the direct limit.

In our case, the space $X$ is a representation, hence is contractible without compact supports. In this case, the Borel cohomology group (5.1) is simply the cohomology of the group

$$H^3(G, \mathbb{Z}).$$

(5.2)

5.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

We shall perform our mathematical calculation in only one case, namely

$$G = (\mathbb{Z}/2)^2.$$
In this case, we may write
\[ H^*(G, \mathbb{Z}/2) = \mathbb{Z}/2[x, y], \quad \text{dim}(x) = \text{dim}(y) = 1. \] (5.3)

Recall that the Bockstein homomorphism
\[ b : H^i(G, \mathbb{Z}/2) \rightarrow H^{i+1}(G, \mathbb{Z}) \] (5.4)
is the connecting homomorphism of the long exact sequence associated with the short exact sequence on coefficients
\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0. \]
(Note: the Bockstein is usually denoted by \( \beta \), but that would conflict with some of our other notation.)

Now one has
\[ H^3(G, \mathbb{Z}) \cong \mathbb{Z}/2 \] (5.5)
where the non-trivial element is
\[ b(xy), \] (5.6)
using the notation of (5.3), (5.4). For the twisting \( \tau \) associated with this element, the relevant twisted \( K \)-groups are
\[ K_{G,\tau}^0(\ast) = \mathbb{Z}, \quad (\epsilon \text{ even}) \]
\[ K_{G,\tau}^1(\alpha) = \mathbb{Z}, \quad (\epsilon \text{ odd}) \]
\[ K_{G,\tau}^1(\alpha + \beta) = \mathbb{Z} \oplus \mathbb{Z}, \quad (\epsilon \text{ odd}) \]
\[ K_{G,\tau}^1(\alpha + \beta + \gamma) = \mathbb{Z}^4, \quad (\epsilon \text{ odd}). \] (5.7)

To prove (5.7), the first group is \( K \)-theory of projective representations of \( G \) with the cocycle given by (5.6). Its elements can be thought of, for example, as virtual representations of the quaternionic group \( \{ \pm 1, \pm i, \pm j, \pm k \} \) where \( -1 \) acts by \( -1 \); these are just sums of the quaternionic representation. Projective equivariant bundles on \( S^1 \) are trivial, hence the \( K^1 \)-group vanishes in this case. For the three remaining groups, we smash again, in order, with the familiar cofibration sequences
\[ G/A_+ \rightarrow S^0 \rightarrow S^\alpha, \]
\[ G/B_+ \rightarrow S^0 \rightarrow S^\beta, \]
\[ G/C_+ \rightarrow S^0 \rightarrow S^\gamma. \]
The key point is that twisting disappears on proper subgroups, so we get long exact sequences

\[ K_G^i, c(\alpha) \to K_G^i, c(\tau(\beta)) \to K_G^{i, c}(\tau(\alpha)) \to K_G^{i, c}(\tau(\alpha + \beta)) \to K_G^{i, c}(\tau(\alpha + \beta + \gamma)) \to K_G^{i, c}(\tau(\alpha + \beta + \gamma + \delta)) \to K_G^{i, c}(\tau(\alpha + \beta + \gamma + \delta + \epsilon)) \to \cdots \]

The last term in (5.10) is by Bott periodicity. If we assume that the calculation of \( K_G^{i, c}(\tau(\alpha)) \) is correct, then in both (5.9) and (5.10) the last two terms shown are known and occur in different dimensions, so the remaining calculations follow.

In (5.8), the last two terms shown are \( \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \) and occur both in even dimensions, so it remains to show that the map between them is the inclusion of a direct summand. There are several ways of showing this. One is to map into the respective \( K \)-groups which we get when we smash all spaces involved with \( EG_+ \). Both groups inject, and in the target, the groups reduce to non-equivariant \( K \)-groups:

\[ K^i_\tau(BG) \to K^i(B(G/A)). \]

Both groups are known, in fact (5.11) has the form

\[ \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \]

(\( \mathbb{Z}_2 \) means 2-adic numbers). There are Atiyah-Hirzebruch spectral sequences converging to both groups. In the case of \( K^*(B(G/A)) \), the AHSS collapses, but the generator \( \iota \) of \( H^0(B(G/A), \mathbb{Z}) \) supports an extension to higher filtration degrees. Thus, \( 2\iota \) generates a direct summand \( \mathbb{Z} \). In the twisted case, the Atiyah-Hirzebruch spectral sequence is described in [52], and the first differential is calculated as

\[ d_3 = bSq^2 + H_3 \]

(recall that we denoted the Bockstein by \( b \) to avoid conflict with other notation). In the case of \( K^*_\tau(BG) \), \( H_3 \) is multiplication by (5.6), so the generator \( \iota \) of \( H^0 \) actually supports a differential. However, the odd degree elements of the \( E_2 \) Atiyah-Hirzebruch spectral sequence (which are all copies of \( \mathbb{Z}/2 \)) are then entirely wiped out by \( d_3 \), so no further differentials are possible, showing that \( 2\iota \) is a permanent cycle which must generate \( K^*_\tau(BG) \) because it is in filtration degree 0. It maps to \( 2\iota \) in (5.11), thus showing that (5.11) is an inclusion of a direct summand \( \mathbb{Z} \), and hence so is the second map (5.8).
There is another, more conceptual reason why (5.7) holds. The reader may notice that the twisted groups (5.7), in the homotopy-theoretical language, appear shifted from the corresponding groups (4.21), (4.22) by the dimension
\[ 1 + \alpha + \beta + \alpha\beta. \] (5.13)
This is indeed the case and is part of a general pattern. For any space \( X \) and any real even-dimensional vector bundle \( \eta \) on \( X \), we have the induced non-equivariant bundle on the Borel construction, whose Stiefel-Whitney classes can be thought of as the equivariant Stiefel-Whitney classes of \( \eta \) in Borel cohomology. In particular, there is the class
\[ W_3(\eta) = bw_2(\eta) \in H^3_{\text{Borel}}(X, \mathbb{Z}) \]
which is the obstruction to \( \eta \) being \( \text{Spin}^c \). But in addition to that, if we denote by \( V \) the total space of \( \eta \), we have generalized Bott periodicity in twisted \( K \)-theory
\[ K^*_{G,\tau}(V) \cong K^*_{G,\tau+W_3(\eta)}(X). \] (5.14)
The proof just mimics the classical index-theoretical proof of Bott periodicity in the world of twisted bundles (see also [53]).

In the current setting, \( X = \ast \) and \( V \) is given by (5.13), and one easily sees that \( W_3(V) \) is equal to (5.6); recall that the total Stiefel-Whitney class is
\[ w(V) = (1 + x)(1 + y)(1 + x + y). \]
Thus, (5.14) implies the shift indicated. It is worth remarking that all twistings of \( (\mathbb{Z}/2)^n \)-equivariant \( K \)-theory over a point are of this form, and hence this method can be used to find all twisted \( K(\mathbb{Z}/2)_n \)-theory groups with compact support of representations. Details will be given elsewhere. The boundary states for this case are those described in the case of orbifold without discrete torsion but the role of the \( s_i \) is exchanged. The \( K \)-theory corresponding to these branes is given (5.7).

In [8, 18] it was noticed that there is a “T-duality” relating the theory without and with discrete torsion. On the D-branes, this duality leaves \( r \) invariant while if the \( s_i \) are even, they becomes odd and vice versa. From the \( K \)-theory point of view, the \( K \)-group (5.7) is suggestive because of “T-duality” with the picture (4.21), (4.22). \( K \)-theoretically, this T-duality results by adding the representation \( 1 + \alpha + \beta + \gamma \) (as elsewhere, we denote a trivial representation by the same symbol as its dimension, so 1 is the 1-dimensional real representation), which
is orientable, but its $W_3$ is the non-trivial class in $H^3(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z})$. In particular, the shift $1 + \alpha + \beta + \gamma$ gives an isomorphism between K-group $\tilde{K}_0^G(S^0)$ and $K^{1,c}_{G,\tau}(\alpha + \beta + \gamma)$. The conventional representations of the brane classified by $\tilde{K}_0^G(S^0)$ is preserved under this shift. The reason is that suspension by one of the representations $\alpha, \beta, \gamma, \alpha + \beta$, or $\alpha + \beta + \gamma$ in this case corresponds roughly to replacing the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ by the subgroup. But on subgroups, the central extension given by this cocycle becomes trivial (the cocycle becomes a coboundary), which is why the representation becomes conventional. A similar discussion follows for the BPS brane with projective representation.

5.2 $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold with $N$ odd

To check the $K$-theory prediction here, let us note again that the group of discrete torsions is $\mathbb{Z}/N$. We discuss the case when the torsion is given by the generator of that group. In this case, the relevant twisted $K$-group is $\mathbb{Z}$. To see this, we can use the twisted Atiyah-Hirzebruch spectral sequence converging to the corresponding completed $K$-group. The $E_2$-term is

$$Z[x, y]/(Nx, Ny) \oplus Z/N[x, y]\omega, \dim(x) = \dim(y) = 2, \dim(\omega) = 3. \quad (5.15)$$

By [52], there is a $d_3$-differential

$$d_3 : 1 \mapsto \omega,$$

so

$$d_3(x^m y^n) = \omega x^m y^n,$$

and the $E_3$-term is $\mathbb{Z}$, generated by $N \cdot 1$. The numbers $s_1, s_2, s_3$ are even here.

Our results above agree with those found in [18] where the boundary states for this kind of orbifold were constructed. The brane corresponding to the K-theory group is a fractional brane with projective representation where all $s_i$ are even. It has the form

$$|D(r, s_1, s_2, s_3); a, \epsilon, \epsilon_1\rangle = \sum_{m=0}^{N-1} \epsilon_1^m \sum_{n=0}^{N-1} w^{-mn} |D(r; s_1, s_2, s_3); g_2^n a, \epsilon\rangle_{T^q}^{\rho}$$

This brane carries a charge under the untwisted R-R charge. But in addition to this brane, there is a bulk brane with the same form as the projective fractional brane but with all $s_i$ equal to 1. Because of the different boundary conditions for each couple of coordinates, it is hard to predict the action of the orbifold on the Chan-Paton factors. However this bulk brane carries charge under the untwisted R-R field, too.
5.3 $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold with $N$ even

In this case, too, the $K$-theory prediction can be calculated. However, in some sense, the case considered here is the opposite to the case of $\mathbb{Z}/N \times \mathbb{Z}/N$ with $N$ odd, since in the present case we do not consider minimal torsion, but torsion corresponding to the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/N$. In this case, the torsion has the form of a $W_3$-class of a bundle, so we can use the Thom isomorphism of Donovan-Karoubi (see above and [53]): The predicted group of charges is then

$$\mathbb{Z}^{N^2/2^m}$$

where $0 \leq m \leq 2$. The number $m$ is the same as the corresponding number for the untwisted group when we add 1 to each of the numbers $r, s_1, s_2, s_3$.

6 Conclusions

Using perturbative string theory we have discussed the construction of branes in various orbifolds including with discrete torsion. We have shown full agreement with the corresponding equivariant and twisted K-theory results. Although a large part of our discussion is present in the literature we have aimed at presenting a comprehensive picture of both the physical and the mathematical sides. We have also obtained new results, most prominently the full analysis of the $\mathbb{Z}_3$ orbifold and the K-theory of the corresponding models of $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifolds without and with discrete torsion. We have also presented a full K-theoretic treatment of $(\mathbb{Z}_2)^n$ orbifolds with $n \leq 4$.

The asymmetry for $N$ even or odd predicted by K-theory presents interesting implications for the spectrum of $D$-brane charges and deserves further study. For example, in the study of discrete symmetries in quiver gauge theories dual to $D3$-branes on orbifolds [54] and asymmetry for $\mathbb{Z}_N$ orbifolds was also found depending on whether $N$ is odd or even; this asymmetry is arguably related to the structures of section 4, in particular, to the homomorphism of subsection 4.2.

In the context of perturbative string theory a lot of progress can be made in the classification of $D$-brane charges because one can directly compute them. However, there are many situations where a perturbative description is not available. There are many interesting questions that arise in this context, here we list some of those we hope to address in the near future.

The AdS/CFT correspondence has proved useful in attacking some questions related to
D-brane charges\cite{54,55} since it provides a dual description of string theory in the presence of Ramond-Ramond fluxes. We plan to extend part of our discussion to that situation. A particularly interesting situation, raised in\cite{56}, is the structure of string theory in nonabelian orbifolds and its relation to the decoupling limit and the quiver gauge theory.

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**A Perturbative string theory**

In this section we collect a number of relations that we used in the derivation of the main results of the text.

**A.1 Hamiltonians**

Along these notes, we will work with $\alpha' = 1$. The open string Hamiltonian is given as

$$H_0 = \pi p^2 + \frac{1}{4\pi} w^2 + \pi \sum_{\mu=1}^9 \left( \sum_{n=1}^{\infty} \alpha_{\mu}^n \alpha_{\mu}^n + \sum_{r>0} r \psi_{-r}^\mu \psi_{r}^\mu \right) + \pi C_0,$$

where $\mathbf{p}$ is the momentum of the endpoint open strings along the Neumann directions and $w$ denotes the difference between the two endpoints. The zero-point energy $C_0$ is zero in the R sector while in the NS-sector it is equal to $-\frac{1}{2} + \frac{s}{8}$ with $s$ the number of Dirichlet-Neumann boundary conditions.

The closed string Hamiltonian is

$$H_c = \pi K^2 + 2\pi \sum_{\mu=1}^8 \left( \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_n^\mu + \bar{\alpha}_{-n}^\mu \bar{\alpha}_n^\mu) + \sum_{r>0} r (\psi_{-r}^\mu \psi_r^\mu + \bar{\psi}_{-r}^\mu \bar{\psi}_r^\mu) + 2\pi C_c \right),$$

where $C_c$ is equal to -1 in the NS-NS sector and 0 in the R-R sector; and $K$ is the closed string momentum.
A.2 The R-R ground states

The boundary conditions defining the R-R vacuum are given by (2.7) for \( r = 0 \)

\[
(\psi_0^\mu + i\eta \tilde{\psi}_0^\mu)|Bp, \eta\rangle_{\text{R-R}}^0 = 0, \quad \mu = 1, \ldots, p + 1
\]
\[
(\psi_0^\nu - i\eta \tilde{\psi}_0^\nu)|Bp, \eta\rangle_{\text{R-R}}^0 = 0, \quad \nu = p + 2, \ldots, 8.
\]  
(A.2)

Let us to introduce

\[
\psi_{\pm}^\mu = \frac{1}{\sqrt{2}} (\psi_0^\mu \pm i\tilde{\psi}_0^\mu),
\]  
(A.3)

in these variables, equations (A.2) define the R-R ground state \(|Bp, k, \eta\rangle_{\text{R-R}}^0\) by the conditions

\[
\psi_{\eta}^\mu|Bp, k, \eta\rangle_{\text{R-R}}^0 = 0 \quad \mu = 1, \ldots, p + 1
\]
\[
\psi_{\eta}^\nu|Bp, k, \eta\rangle_{\text{R-R}}^0 = 0 \quad \nu = p + 2, \ldots, 8
\]  
(A.4)

where \( \eta = \pm \) and \( \psi_{\eta}^\mu \) and \( \psi_{\eta}^\nu \) can be seen as annihilation operators in the Neumann and Dirichlet directions respectively. The state \(|Bp, k, -\eta\rangle\) is created by applying consecutively creation operators on the R-R ground state

\[
|Bp, k, -\eta\rangle_{\text{R-R}}^0 = \prod_{\mu=1}^{p+1} \psi_{-\eta}^\mu \prod_{\nu=p+2}^{8} \psi_{\eta}^\nu|Bp, k, \eta\rangle.
\]  
(A.5)

The representation of the GSO operator in the R-R zero modes is given as [57]

\[
(-1)^F = \prod_{\mu=1}^{9} (\sqrt{2}\psi_0^\mu) = \prod_{\mu=1}^{9} (\psi_+^\mu + \psi_-^\mu),
\]
\[
(-1)^{\bar{F}} = \prod_{\mu=1}^{9} (\sqrt{2}\tilde{\psi}_0^\mu) = \prod_{\mu=1}^{9} (\psi_+^\mu - \psi_-^\mu).
\]  
(A.6)

The action of these operators on the R-R ground state is

\[
(-1)^F|Bp, k, \eta\rangle_{\text{R-R}}^0 = |Bp, k - \eta\rangle_{\text{R-R}}^0,
\]
\[
(-1)^{\bar{F}}|Bp, k, \eta\rangle_{\text{R-R}}^0 = (-1)^{p+1}|Bp, k - \eta\rangle_{\text{R-R}}^0.
\]  
(A.7)

A.3 Boundary states in \( \mathbb{Z}_3 \) orbifold

A.3.1 The untwisted sector

The world-sheet fields along \( \mathbb{C}^3 \) are defined as
\[ Z^i = \frac{1}{\sqrt{2}}(X^{2i+1} + iX^{2i+2}) \quad \bar{Z}^i = \frac{1}{\sqrt{2}}(X^{2i+1} - iX^{2i+2}) \quad i = 1, 2, 3 \]

\[ \lambda^i = \frac{1}{\sqrt{2}}(\psi^{2i+1} + i\psi^{2i+2}) \quad \bar{\lambda}^i = \frac{1}{\sqrt{2}}(\psi^{2i+1} - i\psi^{2i+2}) . \] (A.8)

The mode expansion of these complex fields are

\[ Z^i = z^i + 2\pi p^i \tau + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \left( \beta_n^i e^{-2\pi in(\tau + \sigma)} + \bar{\beta}_n^i e^{-2\pi in(\tau - \sigma)} \right) , \]

\[ \lambda^i = \sqrt{2\pi} \sum_r \lambda_r^i e^{-2\pi ir(\tau - \sigma)} , \] (A.9)

with similar relations for the conjugates. The complex oscillator modes are defined in terms of the real modes as

\[ \beta^i = \frac{1}{\sqrt{2}}(\alpha^{2i+1} + i\alpha^{2i+2}) \quad \bar{\beta}^i = \frac{1}{\sqrt{2}}(\alpha^{2i+1} - i\alpha^{2i+2}) \]

\[ \lambda^i = \frac{1}{\sqrt{2}}(\psi^{2i+1} + i\psi^{2i+2}) \quad \bar{\lambda}^i = \frac{1}{\sqrt{2}}(\psi^{2i+1} - i\psi^{2i+2}) \quad \] (A.10)

with similar relations for the left-modes. Under quantization the (anti)-commutation relations are

\[ [\beta_n^i, \bar{\beta}_m^j] = [\bar{\beta}_n^i, \beta_m^j] = n\delta_{n+m}\delta^{ij} , \] (A.11)

\[ \{\lambda^i_r, \bar{\lambda}^j_s\} = \{\bar{\lambda}^i_r, \lambda^j_s\} = \delta_{r+s}\delta^{ij} . \] (A.12)

The generator of the orbifold \( \mathbb{Z}_3 \) acts on world-sheet fields as

\[ g : Z^i \rightarrow e^{2\pi i\nu} Z^i \]

\[ g : \bar{Z}^i \rightarrow e^{-2\pi i\nu} \bar{Z}^i \] (A.13)

Under the action of the \( \mathbb{Z}_3 \) generator they are transform as

\[ \beta^i \rightarrow e^{2\pi i\nu} \beta^i \quad \bar{\beta}^i \rightarrow e^{-2\pi i\nu} \bar{\beta}^i \]

\[ \lambda^i \rightarrow e^{2\pi i\nu} \lambda^i \quad \bar{\lambda}^i \rightarrow e^{-2\pi i\nu} \bar{\lambda}^i \] (A.14)

The boundary conditions for the boundary states can be written explicitly given the the boundary conditions it satisfies along each pair of coordinates \((x^{2i+1}, x^{2i+2})\) defining the plane \(z^i\). If a couple of real coordinates has mixed boundary conditions (corresponding to any
\( s_i = 1 \) the gluing conditions are

\[
(\beta^i_n \pm \tilde{\beta}^i_n) |\eta\rangle = 0 \\
(\bar{\beta}^i_n \pm \tilde{\bar{\beta}}^i_n) |\eta\rangle = 0 \\
(\lambda^i_r \pm i\eta \tilde{\lambda}^i_r) |\eta\rangle = 0 \\
(\bar{\lambda}^i_r \pm i\eta \tilde{\bar{\lambda}}^i_r) |\eta\rangle = 0
\]

where the positive sign corresponds to Neumann-Dirichlet boundary conditions and negative sign for Dirichlet-Neumann conditions. The boundary state solving these equations is

\[
|\eta\rangle_{N-D}^0 = \exp \left\{ \mp \sum_{n=1}^{\infty} \frac{1}{n} (\beta^i_n \tilde{\beta}^i_n + \tilde{\beta}^i_n \bar{\beta}^i_n) \mp i\eta \sum_{r>0} \left( \lambda^i_r \tilde{\lambda}^i_r + \tilde{\lambda}^i_r \bar{\lambda}^i_r \right) \right\} |\eta\rangle_{NS-NS}^0 \tag{A.16}
\]

On the other hand, if the couple has the same boundary conditions (when \( s_i = 0, 2 \) corresponding to Dirichlete-Dirichlet or Neumann-Neumann, respectively) the boundary equations are

\[
(\beta^i_n \pm \tilde{\beta}^i_n) |\eta\rangle = 0 \\
(\bar{\beta}^i_n \pm \tilde{\bar{\beta}}^i_n) |\eta\rangle = 0 \\
(\lambda^i_r \pm i\eta \tilde{\lambda}^i_r) |\eta\rangle = 0 \\
(\bar{\lambda}^i_r \pm i\eta \tilde{\bar{\lambda}}^i_r) |\eta\rangle = 0
\]

with positive sign for Neumann-Neumann and negative sign for Dirichlet-Dirichlet. The boundary state solving the equations (A.17) for each par of coordinates is

\[
|\eta\rangle_{D-D}^0 = \exp \left\{ \mp \sum_{n=1}^{\infty} \frac{1}{n} (\beta^i_n \tilde{\beta}^i_n + \tilde{\beta}^i_n \bar{\beta}^i_n) \mp i\eta \sum_{r>0} \left( \lambda^i_r \tilde{\lambda}^i_r + \tilde{\lambda}^i_r \bar{\lambda}^i_r \right) \right\} |\eta\rangle_{NS-NS}^0 \tag{A.18}
\]

negative and positive signs are associated to Neumann-Neumann and Dirichlet-Dirichlet directions, respectively. Note that we have dropped out the dependence of \( r, s \) and the momentum but it will be restored latter. The R-R Fock vacuum is defined solving equations (A.15) and (A.17) for the zero-modes.

Let us see now if these boundary states are GSO and orbifold invariant. For the untwisted NS-NS the action of the GSO operator is the same as that presented in 2.4. The orbifold action on the NS-NS vacuum is trivial. The boundary state defined in (A.18) has the combination \( \beta^i \tilde{\beta}^i + \bar{\beta}^i \tilde{\bar{\beta}}^i \) (also for fermions) in the exponential. It is invariant under the
action of the orbifold since each oscillator mode is multiplied by a complex one. However the term $\beta_i \tilde{\beta}_i + \tilde{\beta}_i \beta_i$ in (A.16) in not orbifold invariant. Only the combination $\frac{1}{3}(1 + g|\eta| + g^2|\eta|)$ will be invariant in this case.

In the untwisted R-R sector, the GSO has a complex representation on the zero-modes. The analysis discussed in A.2 is straightforward in this case and we will not repeat it again. The representation of the orbifold on the zero-modes is determined by the cubic roots of unity. However, only one of the three possibilities will leave invariant R-R ground states. The relevant representation is

$$g = \prod_{i=1,2,3} \left( e^{2\pi i/3} - i\sqrt{3} \lambda_i \lambda_i^\dagger \right) \left( e^{2\pi i/3} - i\sqrt{3} \tilde{\lambda}_i \tilde{\lambda}_i^\dagger \right)$$

with $g^2$ being the complex conjugate of $g$. It can be verified easily using (A.15), (A.17) and (A.19) that only the R-R vacuum with the same boundary conditions is orbifold invariant. Then only the boundary state defined in equation (A.18) will be preserved by the orbifold projection.

### A.3.2 Twisted sector

In the twisted sector the modes are shifted by $mv_i$. The annihilation operators are $\beta_{n-mv_i}$ and $\tilde{\beta}_{n-mv_i}$ for the left modes and $\tilde{\beta}_{n-mv_i}$ and $\beta_{n-mv_i}$ for the right modes. The respective fermionic operators are defined in a similar way. The boundary conditions in the sector twisted by $g^m$ are those as (A.15) and (A.17) with the modes shifted as indicated above. For the case of mixed boundary conditions the only possible solution is trivial. The other case, when the pair of real coordinates satisfy the same boundary conditions, has the non-trivial solution

$$|\eta, g^m\rangle_{\text{D-D}} = \exp \left( \mp \sum_{n=1}^\infty \frac{1}{n-mv_i} \beta^n_{-n-mv_i} \tilde{\beta}^n_{n-mv_i} + \frac{1}{n+mv_i} \tilde{\beta}^n_{-n-mv_i} \beta^n_{n-mv_i} \right)$$

$$\mp i\eta \sum_{r>0}^\infty (\lambda^i_{r-mv_i} \tilde{\lambda}^j_{r-mv_i} + \tilde{\lambda}^i_{r-mv_i} \lambda^j_{r-mv_i}) \right) |\eta, g^m\rangle_{\text{NS-NS}}$$

Boundary states on $\mathbb{C}^3/\mathbb{Z}_3$ are a tensor product of the boundary states along the three complex directions $z^i$. Therefore we conclude that in the twisted sector only boundary states with all $s_i$ even exist in this model.

Let us now restore the $(r, s)$ notation. The twisted sector has zero-modes on the $(x^1, x^2)$ directions. Therefore the condition of GSO and orbifold invariance will restrict only the values of $r$. The gluing conditions in the R-R twisted sector are
\[ \psi^\mu_\eta |B(r, s), \eta\rangle^0_{R-R,T} = 0 \quad \mu = 1, \ldots, r + 1 \]
\[ \psi^\nu_-\eta |B(r, s), \eta\rangle^0_{R-R,T} = 0 \quad \nu = r + 2, \ldots, 2 \]  
(A.21)

The ground states are defined as

\[ |B(r, s), +\rangle^0_{R-R,T} = a \prod_{\mu=1}^{r+1} \psi^\mu_+ \prod_{\nu=r+2}^{2} \psi^\nu_- |B(r, s), -\rangle^0_{R-R,T} \]
\[ |B(r, s), -\rangle^0_{R-R,T} = b \prod_{\mu=1}^{r+1} \psi^\mu_+ \prod_{\nu=r+2}^{2} \psi^\nu_- |B(r, s), +\rangle^0_{R-R,T} \]  
(A.22)

with \( a \) and \( b \) normalization constants. They are related by \( b = -\frac{1}{a} \).

The representation of the GSO operator on the zero-modes is

\[ (-1)^F = \pm 2i \psi^1_0 \psi^2_0 = \pm i (\psi^1_+ + \psi^1_-) (\psi^2_+ + \psi^2_-) \]
\[ (-1)^{\tilde{F}} = \pm 2i \tilde{\psi}^1_0 \tilde{\psi}^2_0 = \mp i (\psi^1_- - \psi^1_+) (\psi^2_+ - \psi^2_-) \]  
(A.23)

where the phases are determined by the conditions \((-1)^{2\tilde{F}} = (-1)^{2F} = 1\). The action of these operator onto the boundary states is

\[ (-1)^F |B(r, s), +\rangle = \mp ia |B(r, s), -\rangle \]
\[ (-1)^F |B(r, s), -\rangle = \mp ib |B(r, s), +\rangle \]
\[ (-1)^{\tilde{F}} |B(r, s), +\rangle = \pm (-1)^{r+1} ia |B(r, s), -\rangle \]
\[ (-1)^{\tilde{F}} |B(r, s), -\rangle = \pm (-1)^{r+1} ib |B(r, s), +\rangle \]  
(A.24)

The combination \(|B(r, s), +\rangle + |B(r, s), -\rangle\) will be \((-1)^F\) invariant if \( a = b = \pm i \). The action of \((-1)^{\tilde{F}}\) on this linear combination has eigenvalues \( \kappa (-1)^{r+1} \), where \( \kappa = \pm \). If we consider that at least one fractional brane, which couples to the untwisted and twisted sectors, exists then we have to fix \( \kappa = + \).

For IIB, \((-1)^F = (-1)^{\tilde{F}}\) while for IIA \((-1)^F = -(-1)^{\tilde{F}}\). From the last two equations in (A.24) and the values for \( a \) and \( b \) one finds that the linear combination of boundary states in the NS-NS twisted sector with opposite spin structure is GSO invariant if \( r \) is odd in IIB or \( r \) even in IIA.
Since the twisted R-R does not have zero-modes along the direction on which the orbifold acts, there are no further restrictions on the twisted R-R boundary states.

If we consider that at least one fractional brane, which couples to the untwisted and twisted sectors, exists then we have to fix $\kappa = +$.

For IIB, $(-1)^F = (-1)^\tilde{F}$ while for IIA $(-1)^F = -(1)^\tilde{F}$. From the last two equations in (A.24) and the values for $a$ and $b$ one finds that the linear combination is GSO invariant if $r$ is odd in IIB or $r$ even in IIA.

Since the twisted R-R does not have zero-modes along the direction on which the orbifold acts, there are no further restrictions on the twisted R-R boundary states. Putting all together one has the results in Table 4.

### B  Mathematical material

In this paper, we use the fact that the $K$-theory with compact supports of a space $X$ (satisfying some minimal topological assumptions which are true here) is equal to the reduced $K$-theory of the 1 point compactification $X^c$ of $X$: formulaically, we write

$$K_G(X) = \tilde{K}_G(X).$$

Recall that a reduced $G$-equivariant generalized cohomology theory (such as equivariant $K$-theory) is applied to a based $G$-space $X$, which means a space with a distinguished fixed point called the base point, and usually denoted by $\ast$. It is useful to recall also the topological operation of “smash product” of based spaces $X, Y$:

$$X \wedge Y = (X \times Y) / ((X \times \{\ast\}) \cup (\{\ast\} \times Y)).$$

Here we use the topological operation of quotient, which means that all the points in the set following the / sign are identified to a single point. It is worthwhile to note that the operation $\wedge$ is a commutative associative unital operation, whose unit is $S^0$, the 2-point (fixed) set with one point chosen as base point. Now we have

$$(X \times Y)^c = X^c \wedge Y^c.$$

It is useful for a $G$-space $X$ to denote

$$X_+ = X \amalg \{\ast\}.$$
This is in general a different construction than the 1-point compactification $X^c$, although they are equal when $X$ is compact. One has

$$K^i_G(X) = \widetilde{K}^i_G(X_+),$$

and when $Y$ is based,

$$\widetilde{K}^i_G(Y) = K^i_G(Y, \{\ast\}).$$

For a $G$-representation $V$, the 1-point compactification $V^c$ is often denoted by $S^V$.

A based map

$$f : X \to Y$$

is a map of based spaces such that $f(\ast) = \ast$. The based mapping cone of (B.1) is the based space

$$Cf = (Y \amalg (X \times [0,1]))/(x,1) \sim f(x), (x,0) \sim (\ast,t) \text{ for } x \in X, t \in [0,1].$$

The notation after the / sign means that we pass to equivalence classes of the smallest equivalence relation $\sim$ containing the relation specified. There is a canonical inclusion $Y \to Cf$, and one often refers to the sequence

$$\begin{array}{c}
X \\
\downarrow f \\
Y \\
\downarrow \downarrow \downarrow \\
Cf
\end{array}$$

as a cofibration sequence. The point of considering such sequences is that they lead to long exact sequences in reduced (equivariant) generalized cohomology. For example, we have

$$\to \widetilde{K}^i_G(Cf) \to \widetilde{K}^i_G(Y) \to K^i_G(X) \to \widetilde{K}^{i+1}_G(Cf) \to \ldots$$

(B.4)

Recall that by a basic property of equivariant generalized cohomology (sometimes referred to as the Wirthmüller isomorphism), for a subgroup $H \subset G$, we always have

$$\widetilde{K}^i_G(G/H_+ \wedge X) \cong \widetilde{K}^i_H(X),$$

(B.5)

notation. Now for a based space $X$, $S^n \wedge X$ is the (based) $n$-fold suspension of $X$, so we have, by a general property of generalized cohomology theories,

$$\widetilde{K}^\ell_G(S^n \wedge X) = \widetilde{K}^{\ell-n}_G(X).$$

(B.6)

Because of this, one generalizes this notation to finite-dimensional dimensional real $G$-representations $V$ as follows:

$$\widetilde{K}^{\ell-V}_G(X) = \widetilde{K}^\ell_G(S^V \wedge X)$$

(B.7)
(where $S^V$ was defined above). Now equivariant Bott periodicity asserts that the sign in (B.7) does not matter:

$$\tilde{K}^\ell_{G}(S^V) = \tilde{K}^\ell_{G}(X).$$

We can therefore take as “dimension” of the equivariant $K$-theory group any element of the real representation ring $RO(G)$. It is worthwhile to note that we essentially now reviewed the entire definition of an equivariant cohomology theory: the basic properties are the stability under suspension (B.7), the long exact sequence (B.4), and the indexing by elements of $RO(G)$; the last property is sometimes deleted or modified, but it holds for $K$-theory. For details on equivariant stable homotopy theory, we refer the reader to [58].

In fact, however, in the case of equivariant $K$-theory, by equivariant Bott periodicity, a simplification occurs. Dimensions belonging to the complex representation ring $R(G)$ can be identified with 0, and the group of non-trivial dimensions is

$$D(G) = RO(G)/R(G).$$  \hspace{1cm} (B.9)

In (B.9), the embedding $R(G) \subset RO(G)$ takes a complex representation to the underlying real representation. This is not a map of rings; rather, the image is an ideal of $RO(G)$.

There are two basic elementary principles which aid us in the calculation. First of all, assume that the representation $V$ is trivial when restricted to some subgroup $A \subset G$. Then, by a general principle of equivariant cohomology sometimes referred to as the Adams isomorphism,

$$\tilde{K}^i_{G}(S^V) \cong \tilde{K}^i_{G/A}(S^V) \otimes R(A) \cong \tilde{K}^i_{G/A}(S^V) \otimes \mathbb{Z}^{|A|}. \hspace{1cm} (B.10)$$

The other principle is that when $V = \gamma_1 \oplus ... \oplus \gamma_k$ where $\gamma_j$ are 1-dimensional real representations independent in the character group, then $\epsilon = 0$ and

$$\tilde{K}^0_{G}(S^V) \cong \mathbb{Z}^{2^{n-k}}. \hspace{1cm} (B.11)$$

To see this, one just takes the smash product of cofiber sequences of the form

$$G/Ker(\gamma_j)_+ \to S^0 \to S^{\gamma_j} \hspace{1cm} (B.12)$$

(where for a representation $\gamma$, $Ker(\gamma)$ is the maximal subgroup restricted to which $\gamma$ becomes trivial), using (B.5).

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