Affine Particles and Fields

Djordje Šijački*
Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

Abstract

The covering of the affine symmetry group, a semidirect product of translations and special linear transformations, in \( D \geq 3 \) dimensional spacetime is considered. Infinite dimensional spinorial representations on states and fields are presented. A Dirac-like affine equation, with infinite matrices generalizing the \( \gamma \) matrices, is constructed.

1 Introduction

Dmitri Ivanenko, one of the great gravitational physicists of the last century, and G. Sardanashvily begin their comprehensive review paper on gauge treatment of gravity [1] with the following statement: "At present Einstein’s General Relativity (GR) still remains the most satisfactory theory of classical gravitation for all now observable gravitational fields. GR successfully passed the test of recent experiments on the radiolocation of planets and on the laser-location of the Moon, which have put the end to some other versions of gravitation theory, e.g., the scalar-tensor theory. At the same time the conventional description of gravity by Einstein’s GR obviously faced a number of serious problems, and even some corner-stones of gravitation theory still remain disputable up to our day. This is reflected also in the rather curious uninterrupted flow of proposals for new designations of this theory”.

After more than twenty years, despite of important developments in the subject matter this statement is still fairly accurate. As for the gauge approach to gravity, the theory obtained by gauging the Poincaré group is in a mature stage. Here, there are definite results concerning the structure of

*email: sijacki@phy.bg.ac.yu
the theory, classical sector applications, coupling to tensorial and spinorial matter etc. However, there are still notable difficulties in the quantum sector. A sizable part of the above review paper [1] is devoted to the $GL(4, R)$ symmetry as "one of the most natural candidates to generalize the Lorentz gauge gravitation". The $SL(4, R) \subset GL(4, R)$ symmetry plays, in the Affine group case, the role of the Lorentz symmetry in the Poincaré case. A weak point of the metric-affine [2, 3] and/or the gauge-affine [4, 5] approaches to the gauge theories of gravity is still the one of the spinorial affine matter description. In particular, there are no yet candidates for a Dirac-like wave equation that would successfully describe spinorial affine particles and fields. This is related to the group theoretical properties of the quantum affine symmetries in $D \geq 3$, especially to the fact that the corresponding linear spinorial representations are necessarily infinite dimensional.

The aim of this paper is to shed some light on the description of the spinorial affine matter in $D \geq 3$ dimensional spacetime along the lines of a recent paper that was concerned primarily with the $D = 3$ case [6]. We study the algebraic structure and the construction of the spinorial representations of the following physically relevant groups,

$$T_D \wedge SL(D, R) \supset SL(D, R) \supset Spin(D)$$

Moreover, we consider a construction of a Dirac-like equation for infinite-component spinorial $SL(D, R)$ field. This construction is carried out by embedding the $sl(D, R)$ algebra as well as the corresponding $D$-vector $X$, that generalizes Dirac’s $\gamma$ matrices, into the $sl(D + 1, R)$ algebra. This $D$-dimensional flat-spacetime Dirac-like equations, for $D \geq 3$, are of significant importance for a construction of spinorial fields, "world spinors" [7, 8], in a generic non-Riemannian spacetime of arbitrary torsion and curvature.

2 Affine group and algebra

The general affine group $GA(D, R)$, in $D$-dimensional spacetime, is a semidirect product of the group $T_D$ of translations and the general linear group $GL(D, R)$, i.e. $GA(D, R) = T_D \wedge GL(D, R)$. The nontrivial dimensionality of the $GA(D, R)$ Hilbert space representations is determined by its special affine subgroup $SA(D, R) = T_D \wedge SL(D, R)$. Owing to the fact that we are
interested in nontrivial spinorial and tensorial affine-spacetime structure, we confine ourselves, in this paper, to the $SA(D, R)$ group and for the sake of brevity refer to it as to the ”affine” group. The commutation relations of the $sa(D, R)$ algebra of the $SA(D, R)$ group read

$$[P_a, P_b] = 0,$$
$$[Q_{ab}, P_c] = ig_{ac}P_b,$$
$$[Q_{ab}, Q_{cd}] = ig_{bc}Q_{ad} - ig_{ad}Q_{cb},$$

the structure constants $g_{ab}$ being either $\delta_{ab} = (+1, +1, \ldots, +1)$, $a, b, c, d = 1, 2, \ldots, D$ for the $SO(D)$ subgroup or $\eta_{ab} = (+1, -1, \ldots, -1)$, $a, b, c, d = 0, 1, \ldots, D - 1$ for the $D$-dimensional Lorentz subgroup $SO(1, D - 1)$ of the $SL(D, R)$ group.

The important $sl(D, R)$ subalgebras are as follows.

(i) $so(1, D - 1)$: The $M_{ab} = Q_{[ab]}$, for $g_{ab} = \eta_{ab}$, operators generate the Lorentz-like subgroup $SO(1, D - 1)$ with $J_{ij} = M_{ij}$ (angular momentum) and $K_i = M_{0i}$ (the boosts) $i, j = 1, 2, \ldots, D - 1$.

(ii) $so(D)$: The $J_{ab} = Q_{[ab]}$, for $g_{ab} = \delta_{ab}$, $J_{ij}$ and $N_i = Q_{(0i)}$ operators generate the maximal compact subgroup $SO(D)$.

(iii) $sl(D - 1)$: The $J_{ij}$ and $T_{ij} = Q_{(ij)}$ operators generate the subgroup $SL(D - 1, R)$ - the ”little group of the massive particle states”.

The $SL(D, R)$ commutation relations, in terms of the metric preserving antisymmetric operators $M_{ab} = Q_{[ab]}$ and the remaining traceless symmetric operators $T_{ab} = Q_{(ab)}$ that generate the (non-trivial) $D$-volume preserving transformations, are given as follows

$$[M_{ab}, M_{cd}] = -i\eta_{ac}M_{bd} + i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} - i\eta_{bd}M_{ac},$$
$$[M_{ab}, T_{cd}] = -i\eta_{ac}T_{bd} - i\eta_{ad}T_{bc} + i\eta_{bc}T_{ad} + i\eta_{bd}T_{ac},$$
$$[T_{ab}, T_{cd}] = +i\eta_{ac}M_{bd} + i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} + i\eta_{bd}M_{ac}.$$

The quantum mechanical symmetry group $G_{qm}$ is given as the $U(1)$ minimal extensions of the corresponding classical symmetry group $G_{cl}$,

$$1 \rightarrow U(1) \rightarrow G_{qm} \rightarrow G_{cl} \rightarrow 1.$$ 

In practice, finding $G_{qm}$ is accounted for by taking the universal covering group of the $G_{cl}$ group (topology changes), and by solving the algebra commutation relations for possible central charges (algebra deformation). There
are no nontrivial central charges of the $sa(D, R)$ and $sl(D, R)$ algebras, and the remaining important question for quantum applications is the one of the affine symmetry covering group. The translational part of the $SA(D, R)$ group is contractible to a point and thus irrelevant for the covering question. The $SL(D, R)$ subgroup is, according to the Iwasawa decomposition, given by $SL(D, R) = SO(D, R) \times A \times N$, where $A$ is a subgroup of Abelian transformations (e.g. diagonal matrices) and $N$ is a nilpotent subgroup (e.g., upper triangular matrices). Both $A$ and $N$ subgroups are contractible to point. Therefore, the covering features are determined by the topological properties of the maximal compact subgroup of the group in question. In our case, that is the $SO(D, R)$ group, i.e. more precisely its central subgroup.

The universal covering group of the $SO(D)$, $D \geq 3$ group is its double covering group isomorphic to $Spin(D)$. In other words $SO(D) \simeq Spin(D)/Z_2$ (for a detail account of $Spin(D)$ and $Pin(D)$ groups cf. [9]). The special affine and the special linear groups have double-coverings as their universal coverings as summarized by the following diagram of exact sequences

\[
\begin{array}{ccccccccc}
1 & \to & T_D & \to & \overline{SA}(D, R) & \to & \overline{SL}(D, R) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & T_D & \to & SA(D, R) & \to & SL(D, R) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & & & & 1 & & & & \\
\end{array}
\]

In the physically most interesting case, $D = 4$, there is a homomorphism between $SO(3) \times SO(3)$ and $SO(4)$. Since $SO(3) \simeq SU(2)/Z_2$, where $Z_2$ is the two-element center $\{1, -1\}$, one has $SO(4) \simeq [SU(2) \times SU(2)]/Z_2^d$, where $Z_2^d$ is the diagonal discrete group whose representations are given by $\{1, (-1)^{2j_1} = (-1)^{2j_2}\}$ with $j_1$ and $j_2$ being the Casimir labels of the two $SU(2)$ representations. The full $Z_2 \times Z_2$ group, given by the representations $\{1, (-1)^{2j_1}\} \otimes \{1, (-1)^{2j_2}\}$, is the center of $Spin(4) = SU(2) \times SU(2)$, which is thus the quadruple-covering of $SO(3) \times SO(3)$ and a double-covering of $SO(4)$. The groups $SO(3) \times SO(3)$, $SO(4)$ and $Spin(4) \simeq SU(2) \times SU(2)$ are thus the maximal compact subgroups of $SO(3, 3)$, $SL(4, R)$ and $\overline{SL}(4, R)$.
respectively. One can sum up these results by the following exact sequences

\[
\begin{array}{cccccc}
1 & 1 & \\
\downarrow & \downarrow & \\
1 & \rightarrow & Z_2^d & \rightarrow & Z_2 \times Z_2 & \rightarrow \ Z_2 & \rightarrow & 1 \\
\downarrow & \downarrow & \\
1 & \rightarrow & Z_2^d & \rightarrow & \overline{SL}(4, R) & \rightarrow & SL(4, R) & \rightarrow & 1 \\
\downarrow & \downarrow & \\
& & SO(3, 3) & \rightarrow & SO(3, 3) & \\
\downarrow & \downarrow & \\
1 & 1 & \\
\end{array}
\]

The universal covering group \( \overline{G} \) of a given group \( G \) is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering, \( \overline{SL}(D, R) \), exists provided one can embed \( SL(D, R) \) into a group of finite complex matrices that contain \( Spin(D) \) as subgroup. A scan of the Cartan classical algebras points to the \( SL(D, C) \) groups as a natural candidate for the \( SL(D, R) \) groups covering. However, there is no match of the defining dimensionalities of the \( SL(D, R) \) and \( Spin(D) \) groups for \( D \geq 3 \),

\[
\text{dim}(SL(D, C)) = D < 2^{\left[\frac{D}{2}\right]} = \text{dim}(Spin(D)),
\]

except for \( D = 8 \). In the \( D = 8 \) case, one finds that the orthogonal subgroup of the \( SL(8, R) \) and \( SL(8, C) \) groups is \( SO(8, R) \) and not \( Spin(8) \). For a detailed account of the \( D = 4 \) case cf. [10]. Thus, we conclude that there are no finite-dimensional covering groups of the \( SL(D, R) \) groups for any \( D \geq 3 \). An explicit construction of all spinorial, unitary and nonunitary multiplicity-free [11] and unitary non-multiplicity-free [12], \( SL(3, R) \) representations shows that they are all defined in infinite-dimensional spaces.

The universal (double) covering groups of the \( \overline{SL}(D, R) \) and \( \overline{SA}(D, R) \), \( D \geq 3 \) groups are groups of infinite complex matrices. All their spinorial representations are infinite-dimensional and when reduced w.r.t. \( Spin(D) \) subgroups contain representations of unbounded spin values.

### 3 Representations on states

The \( \overline{SA}(D, R) \) Hilbert space representations are, owing to the semidirect product group structure, induced as in the Poincaré case from the correspond-
ing little group (stability subgroup) representations. The correct quantum
mechanical interpretation requires these representations to be unitary.

The steps in the construction of the unitary irreducible \( \mathcal{SA}(D, R) \) Hilbert
space representations are, in the quantum physics terminology, as follows:
(i) determine the vectors characterized by the maximal set of good quan-
tum numbers of the Abelian subgroup \( T_D \) generators, (ii) determine the little
group as a subgroup of the \( SL(D, R) \) transformations that leaves these
vectors invariant, and (iii) induce the unitary irreducible \( \mathcal{SA}(D, R) \)
representations from the corresponding \( T_D \) and little group representations. In
contradistinction to the Poincaré case, the little groups that describe affine
particles are more complex in structure due to the fact that a orthogonal
type of group is enlarged here to the linear one.

The little group of the \( \mathcal{SA}(D, R) \) Hilbert-space particle states is of the
form \( T_{D-1} \wedge SL(D-1, R) \), where the Abelian invariant subgroup \( T_{D-1} \) of
the little group is generated by \( Q_{1j}, j = 2, 3, \ldots, D \). Owing to the fact that
the little group is itself given as a semidirect product, we have the following
possibilities:

(i) The whole little group is represented trivially corresponding to a scalar
state.

(ii) The \( T_{D-1} \) subgroup is represented trivially, \( D(T_{D-1}) \to 1 \), i.e. \( D(Q_{1j}) \)
\( \to 0 \), the remaining little group is \( SL(D-1, R) \), and the corresponding "affine
particle" is described by the unitary irreducible \( \mathcal{SL}(D-1, R) \) representations.
These representations are infinite dimensional, even in the tensorial case, due
to noncompactness of the \( SL(D, R) \) group.

(iii) The whole little group \( T_{D-1} \wedge SL(D-1, R) \) is represented nontrivially.
The corresponding "affine particles" are described by \( D-1 \) real additive
quantum numbers provided by \( Q_{1k}, k = 2, 3, \ldots, D \), and the representations
of a next step little group \( T_{D-2} \wedge SL(D-2, R) \) that is a subgroup of \( SL(D-1, R) \).
The \( T_{D-2} \) subgroup is generated by \( Q_{2k}, k = 3, 4, \ldots, D \). Here, we have
again the above branching situation, either we represent \( T_{D-2} \) trivially and
have an effective \( SL(D-2, R) \) little group, or we represent \( T_{D-2} \) nontrivially
and arrive at the next step little group \( T_{D-3} \wedge SL(D-3, R) \subset SL(D-2, R) \),
and so on.

\[ \overline{SL}(D, R) \]

\[ T_{D-1} \supset \overline{SL}(D - 1, R) \]

\[ T_{D-2} \supset \overline{SL}(D - 2, R) \]

\[ \ldots \]

\[ T_{3} \supset \overline{SL}(3, R) \]

\[ T_{2} \supset \overline{SL}(2, R) \]

Let \((a, \bar{A}) \in \overline{SA}(D, R), a \in T_{D}, \bar{A} \in \overline{SL}(D, R)\) be the group elements.

The group composition law is

\[(a_{1}, \bar{A}_{1})(a_{2}, \bar{A}_{2}) = (a_{1} + A_{1}a_{2}, \bar{A}_{1}\bar{A}_{2}),\]

where \(A \in SL(D, R)\) corresponds to \(\bar{A} \in \overline{SL}(D, R)\) through \(\overline{SL}(4, R)/Z_{2} \rightarrow SL(D, R)\).

In the case when \(T_{D-1}\) is represented trivially, the \(\overline{SA}(D, R)\) representations on states are given by the following expression

\[ D(a, \bar{A})f[j](p, [m]) = e^{ia \cdot (Ap)} \sum_{[m']} D_{[m']}[m] (L_{Ap}^{-1} \bar{A}L_{p}) f[j](Ap, [m']), \]

where \([j]\) are the \(\overline{SL}(D, R)\) quantum numbers, and \(L_{p}\) represents the action of an element \(C\) defined by \(\bar{A} = CH, H \in SL(D - 1, R)\) on the state \(p(0) = (p_{0}, 0, 0, 0)\), i.e., \(p = L_{p}p(0) = C_{p}(0)\). The \(\overline{SL}(D - 1, R)\) subgroup of the \(\overline{SA}(D, R)\) group is represented linearly, while the elements of the \(\overline{SL}(D, R)/\overline{SL}(D - 1, R)\) factor group are primarily realized non-linearly over \(SL(D - 1, R)\) and then represented linearly. Once again, the remaining \(\overline{SL}(D - 1, R)\) little group is non-compact, and in the quantum case one has to make use of its unitary irreducible representations (both spinorial and tensorial) that are necessarily infinite dimensional.
4 Representations on fields

The representations of the Poincaré group on fields are given by the following well known expressions

\[
(D(a, \Lambda)\Phi_m)(x) = (D(\Lambda))_m^n \Phi_n(\Lambda^{-1}(x - a))
\]

\[
(a, \Lambda) \in T_D \wedge Spin(1, D - 1),
\]

where \(m, n\) enumerate a basis of the representation space of the field components.

In the standard applications to gravity and/or particle physics, one makes use of the finite-component representations of the Lorentz group on fields. This is in agreement with experiment. For instance, boosted particles do not get spin excited. The fact that finite-dimensional representations, \(D(\Lambda)\), of the Lorentz subgroup are, due to its noncompactness, nonunitary is of no physical relevance. In fact, only the field components corresponding to the modes described by the unitary representation of the little group are allowed to propagate by means of field equations. In other words, unitarity is imposed in the Hilbert space of the representations on states only, while the field equations provide for a full Lorentz covariance, and restrict the field components in such a way that the physical degrees of freedom are as given by the corresponding particle states.

Representations of the affine group \(\overline{SA}(D, R)\) on fields are given by the same expression with the Lorentz group being replaced by the \(\overline{SL}(D, R)\) group. There are two physical requirements that have to be satisfied in the affine case: (i) representations of the Lorentz subgroup \(Spin(1, D)\) have to be finite-dimensional and thus nonunitary and (ii) representations of the affine-particle little group \(\overline{SL}(D - 1, R)\) have to be unitary and thus (due to little group’s noncompactness) infinite-dimensional.

The correct unitarity properties of the affine fields can be achieved by making use of the unitary (irreducible) representations and the so called “deunitarizing” automorphism of the \(\overline{SL}(D, R)\) group. The \(\overline{SL}(D, R)\) commutation relations are invariant under the “deunitarizing” automorphism [10],

\[
A : \overline{SL}(D, R) \to \overline{SL}(D, R)
\]

\[
J^A_{ij} = J_{ij}, \quad K^A_j = iN_j, \quad N^A_j = iK_j,
\]

\[
T^A_{ij} = T_{ij}, \quad T^A_{00} = T_{00}, \quad i, j = 1, 2, \ldots, D - 1,
\]
so that \((J_{ij}, iK_i)\) generate the new compact \(Spin(D)^A\) and \((J_{ij}, iN_i)\) generate \(Spin(1, D - 1)^A\).

For the (spinorial) particle states, we use the basis vectors of the unitary irreducible representations of \(\overline{SL}(D, R)^A\), so that the compact subgroup finite multiplets correspond to \(Spin(D)^A\), generated by \(\{J_{ij}, iK_i\}\), while \(Spin(1, D - 1)^A\), generated by \(\{J_{ij}, iN_j\}\), is represented by unitary infinite-dimensional representations. We now perform the inverse transformation and return to \(\overline{SL}(D, R)\) for our physical identification. \(\overline{SL}(D, R)\) is represented non-unitarily, the compact \(Spin(D)\) is represented by non-unitary infinite-dimensional representations while the Lorentz group is represented by non-unitary finite representations. These finite-dimensional non-unitary Lorentz group representations are precisely those that ensure a correct particle interpretation. Note that \(\overline{SL}(D - 1, R)\), the stability subgroup of \(\overline{SA}(D, R)\), is represented unitarily.

We now face the problem of constructing the (unitary) infinite-dimensional spinorial and tensorial representations of the \(\overline{SL}(D, R)\) group. The \(\overline{SL}(D, R)\) group can be contracted (a la Wigner-Inönü) w.r.t. its \(Spin(D)\) subgroup to yield the semidirect-product group \(T' \oplus Spin(D)\). \(T'\) is a \(\frac{1}{2}(D + 2)(D - 1)\) parameter Abelian group generated by operators \(U_{ab} = \lim_{\varepsilon \to 0}(\varepsilon T_{ab})\), which form a \(Spin(D)\) second rank symmetric operator obeying the following commutation relations,

\[
\begin{align*}
[J_{ab}, J_{cd}] &= -i\eta_{ad}J_{bd} + i\eta_{bd}J_{ad} - i\eta_{bd}J_{ac}, \\
[J_{ab}, U_{cd}] &= -i\eta_{ac}U_{bd} - i\eta_{bd}U_{ac} + i\eta_{bd}U_{ac}, \\
[U_{ab}, U_{cd}] &= 0.
\end{align*}
\]

An efficient way of constructing explicitly the \(\overline{SL}(D, R)\) infinite-dimensional representations consists in making use of the so called ”decontraction” formula, which is an inverse of the Wigner-Inönü contraction. According to the decontraction formula, the following operators

\[
T_{ab} = pU_{ab} + \frac{i}{2\sqrt{U \cdot \overline{U}}} [C_2(Spin(D)), U_{ab}],
\]

together with \(J_{ab}\) form the \(\overline{SL}(D, R)\) algebra. The parameter \(p\) is an arbitrary complex number, \(p \in \mathbb{C}\), and \(C_2(Spin(D))\) is the \(Spin(D)\) second-rank Casimir operator.

For the representation Hilbert space we take the homogeneous space of \(L^2\) functions of the maximal compact subgroup \(Spin(D)\) parameters.
The $\text{Spin}(D)$ representation labels are given either by the Dynkin labels $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ or by the highest weight vector which we denote by $\{j\} = \{j_1, j_2, \ldots, j_r\}$, $r = \left[ \frac{D-3}{2} \right]$. The $\overline{\text{SL}}(D, R)$ commutation relations are invariant w.r.t. an automorphism defined by:

$$s(J) = +J, \quad s(T) = -T.$$ 

This allows us to associate an 's-parity' to each $\text{Spin}(D)$ representation of an $\overline{\text{SL}}(D, R)$ representation. In terms of Dynkin labels we find:

$$s(D_2) = (-)^{\frac{1}{2}(\lambda_1+\lambda_2-\epsilon)},$$

$$s(D_{n\geq 3}) = (-)^{\lambda_1+\lambda_2+\ldots+\lambda_{n-2}+\frac{1}{2}(\lambda_n-\lambda_{n-1}-\epsilon)},$$

$$s(B_1) = (-)^{\frac{1}{2}(\lambda_1-\epsilon)},$$

$$s(B_{n\geq 2}) = (-)^{\lambda_1+\lambda_2+\ldots+\lambda_{n-1}+\frac{1}{2}(\lambda_n-\epsilon)}$$

where $\epsilon = 0$ and $\epsilon = 1$ for $\lambda$ even and odd, respectively.

The s-parity of the $\frac{1}{2}(D+2)(D-1)$-dimension representation $\{20 \ldots 0\} = \Box \Box$ of $\text{Spin}(D)$ is $s(\Box \Box) = +1$. A basis of an $\text{Spin}(D)$ irreducible representation is provided by the Gel'fand-Zetlin pattern characterized by the maximal weight vectors of the subgroup chain $\text{Spin}(D) \supset \text{Spin}(D-1) \supset \cdots \supset \text{Spin}(2)$. We write the basic vectors as $\{j\} \{m\}$, where $\{m\}$ corresponds to $\text{Spin}(D-1) \supset \text{Spin}(D-2) \supset \cdots \supset \text{Spin}(2)$ subgroup chain weight vectors.

The Abelian group generators $\{U\} = \{U^{\Box \Box}_{\mu}\}$ can be, in the case of multiplicity free representations, written in terms of the $\text{Spin}(D)$-Wigner functions as follows $U^{\Box \Box}_{\mu} = D^{\Box \Box}_{\mu}(\phi)$. It is now rather straightforward to determine the noncompact operators matrix elements, which are given by the following expression [13]:

$$\langle \{j'\} \{m'\} | T^{(\Box \Box)}_{\{\mu\}} | \{j\} \{m\} \rangle = \left(\begin{array}{ccc} \{j'\} & \{\Box \Box\} & \{j\} \\ \{m'\} & \{\mu\} & \{m\} \end{array}\right) \langle\{j'\}||T^{(\Box \Box)}||\{j\}\rangle,$$

$$\langle\{j'\}||T^{(\Box \Box)}||\{j\}\rangle = \sqrt{\text{dim}\{j'\}\text{dim}\{j\}} \left\{ p + \frac{1}{2}(C_2(\{j'\}) - C_2(\{j\})) \right\}$$

$$\times \left(\begin{array}{ccc} \{j'\} & \{\Box \Box\} & \{j\} \\ \{0\} & \{0\} & \{0\} \end{array}\right).$$
\begin{align*}
\begin{pmatrix} \cdot & \cdot & \cdot \end{pmatrix}
\end{align*}

is the appropriate "3j" symbol for the \textit{Spin}(D) group. The (unitary) infinite-dimensional representations of the \textit{SL}(D, R) algebra are given by these expressions of the non-compact generators together with the well known expressions for the maximal compact \textit{Spin}(D) algebra representations. Finally, we apply the deunitarizing automorphism \( \mathcal{A} \) for a correct physical interpretation.

In the case of the multiplicity free \textit{SL}(D, R) representations, each \textit{Spin}(D) sub-representation appears at most once and has the same \( s \)-parity. This feature is especially useful for the task of reducing infinite-dimensional spinorial and tensorial representations of the \textit{SL}(D, R) group to the corresponding \textit{SL}(D - 1, R) sub-representations.

\section{Spinorial wave equations}

Let us consider the question of constructing a Dirac-like equation for an infinite-component spinorial affine field \( \Psi(x) \),

\begin{align*}
(iX^a \partial_a - M)\Psi(x) &= 0, \\
\Psi(x) &\sim D_{\text{spin}}(\textit{SL}(D, R)).
\end{align*}

The \( X^a \), \( a = 0, 1, \ldots, D - 1 \) vector operator, acting in the space of the \( \Psi \) field components, is an appropriate generalization of the Dirac \( \gamma \) matrices to the affine case. The \textit{SL}(D, R) affine covariance requires that the following commutation relations are satisfied

\begin{align*}
[M_{ab}, X_c] &= i\eta_{ac}X_a - i\eta_{ac}X_b \\
[T_{ab}, X_c] &= i\eta_{bc}X_a + i\eta_{bc}X_b.
\end{align*}

The first relation ensures Lorentz covariance, and is an easy one to fulfill. The second relation, required by the full affine covariance, turns out to be rather difficult to accomplish (cf. [14]).

One can obtain the matrix elements of the generalized Dirac matrices \( X_a \) by solving the above commutation relations for \( X_a \) in the Hilbert space of a suitable spinorial \( \textit{SL}(D, R) \) representation. Alternatively, one can embed both the \textit{SL}(D, R) algebra and \( X_a \) into the \( \textit{SL}(D + 1, R) \) algebra, and make use of representations of the embedding algebra to solve for \( X_a \). Let us denote the generators of \( \textit{SL}(D + 1, R) \) by \( Q_{AB}^{(D+1)} \), \( A, B = 0, \ldots, D \). Now, there are
two natural $D$-vector candidates for $X_a$ in $SL(D + 1, R)$, i.e. $A_a$, and $B_a$ defined by

$$A_a = Q^{(D+1)}_a, \quad B_a = Q^{(D+1)}_{Da}, \quad a = 0, 1, \ldots, D - 1.$$  

The operators $A_a$ and $B_a$, obtained in this way, fulfill the required $SL(D, R)$ $D$-vector commutation relations by construction. It is interesting to note that the operator $G_a = \frac{1}{2}(A_a - B_a)$ satisfies

$$[G_a, G_b] = -iM_{ab},$$

thereby generalizing the corresponding property of Dirac’s $\gamma$-matrices. Since $X_a$, $M_{ab}$ and $T_{ab}$ form a closed algebra, the $X_a$ operator connects only those $SL(D, R)$ representation states that are contained in the $SL(D+1, R)$ representation Hilbert space. By reducing a spinorial $\tilde{SL}(D+1, R)$ representation to the $\tilde{SL}(D, R)$ sub representations, we obtain a set of these representations that is closed w.r.t. an $X_a$ action. Moreover, an explicit form of the $X_a$ matrix elements is provided by the $\tilde{SL}(D+1, R)$ representation expressions.

There are quite a number of substantial changes when going from the Poincaré to the affine symmetry: spinorial representations are in finite dimensional, unitarity requirements are different, tensor algebra relevant for the wave equation questions is more restrictive etc. In order to have an impression about the general structure of the $X_a$ matrix, let us consider a toy model, where we make use of the finite-dimensional tensorial $SL(D, R)$ representations. As an example, let as start with the following $\frac{1}{6}(D+1)(D+2)(D+3)$-dimensional tensorial irreducible representation of $SL(D+1, R)$ that reduces to four $SL(D, R)$ representations as follows,

$$SL(D+1, R) \supset SL(D, R)$$

$$\begin{array}{l}
\varphi_{ABC} \supset \varphi_{a} \oplus \varphi_{ab} \oplus \varphi_{abc} \oplus 0,
\end{array}$$

where ”box” is the Young tableau for an irreducible vector representation of $SL(D, R)$. The effect of the action of the $SL(D, R)$ vector $X_a$ on the fields $\varphi$, $\varphi_a$ and $\varphi_{ab}$ and $\varphi_{abc}$ is

$$X_a \otimes \varphi \rightarrow \varphi_a,$$

$$X_a \otimes \varphi_{ab} \rightarrow \varphi_{abc},$$

$$X_a \otimes \varphi_{abc} \rightarrow 0.$$
Other possible Young tableaux do not appear due to the tensor algebra of the chosen \( \overline{SL}(D+1, R) \) representation. Gathering these fields in a vector \( \Phi = (\varphi, \varphi_a, \varphi_{ab}, \varphi_{abc})^T \), we can read off the matrix structure of \( X_a \).

It is interesting to observe here that \( X_a \) has zero matrices on the block-diagonal which implies that the mass operator \( M \) in an affine invariant equation vanishes. Consider now an action of the \( X_a \) vector operator on an arbitrary irreducible representation \( D(g) \) of \( SL(D, R) \) labeled by \( [\nu_1, \nu_2, \ldots, \nu_{D-1}] \), \( \nu_i \) being the number of boxes in the \( i \)-th raw,

\[
[\nu_1, \nu_2, \ldots, \nu_{D-1}] \otimes [1, 0, \ldots, 0] = [\nu_1 + 1, \nu_2, \ldots, \nu_{D-1}] \oplus [\nu_1, \nu_2 + 1, \ldots, \nu_{D-1}] \oplus \ldots \\
\oplus [\nu_1, \nu_2, \ldots, \nu_{D-1} + 1] \oplus [\nu_1 - 1, \nu_2 - 1, \ldots, \nu_{D-1} - 1],
\]

where one counts, on the right hand side, the allowed representations only. None of the resulting representations is isomorphic to the starting representation \( D(g) \). This implies zero matrices on the block-diagonal of \( X_a \), in the Hilbert space of an arbitrary \( SL(D, R) \) irreducible representation. Let the representation space of an arbitrary reducible representation be spanned by \( \Phi = (\varphi_1, \varphi_2, \ldots)^T \) with \( \varphi_i \) irreducible. Now we consider the Dirac-type equation in the rest frame \( p(0) = (p_0, 0, \ldots, 0) \), restricted to the subspaces spanned by \( \varphi_i \), \( (i = 1, 2, \ldots) \),

\[
p_0 < \varphi_i | X^0 | \varphi_j > - < \varphi_i | M | \varphi_i > \delta_{ij} = 0,
\]

where we assumed the operator \( M \) to be diagonal. It follows that the \( M \) operator vanishes since \( < \varphi_i | X^0 | \varphi_i > = 0 \).

Let us now turn to the proper spinorial case of infinite-dimensional spinorial representations of the \( \overline{SL}(D, R) \) group. We embed the \( \overline{SL}(D, R) \) algebra, as well as the Dirac-like wave equation \( D \)-vector operator \( X^a \), \( a = 0, 1, \ldots, D - 1 \), into the \( \overline{SL}(D+1, R) \) algebra, and thus satisfy the \( [Q_{ab}, X_c] \) commutation relations by construction. Moreover, this embedding puts a constraint on the set of \( \overline{SL}(D, R) \) spinorial representations which define a Hilbert space of field components that is invariant w.r.t. \( X_a \) action.

An explicit construction consists of: (i) a construction of the unitary spinorial \( \overline{SL}(D+1, R) \) representations, (ii) an application of the deunitarizing \( A \) operator, (iii) an identification of the relevant physical operators in the \( \overline{SL}(D+1, R) \) algebra, (iv) a reduction of the chosen \( \overline{SL}(D+1, R) \) spinorial representation down to the corresponding \( \overline{SL}(D, R) \) sub-representations, and
(v) an evaluation of the $X^a$ matrix elements in the $SL(D, R)$ representation basis starting with the $STL(D + 1, R)$ representation matrix elements.

We repeat first the construction procedure, developed for the $SL(D, R)$ representations on fields, but this time in the $SL(D + 1, R)$ case. In this way, we arrive at the following expressions for the compact, $J_{AB}^{(D+1)} = Q_{[AB]}^{(D+1)}$, and the noncompact, $T_{AB}^{(D+1)} = Q_{\{AB\}}^{(D+1)}$, generators of the $SL(D + 1, R)$ group in the $Spin(D + 1)$ representations basis,

\[
\begin{align*}
\langle \{j\} \{m\} | J_{\{\mu\}}^{(D+1)\{\Box\}} | \{j\} \{m\} \rangle &= \sqrt{\text{dim}\{j\}} \left( \begin{array}{ccc}
\{j\} & \{\Box\} & \{j\} \\
\{m\} & \{\mu\} & \{m\}
\end{array} \right) \delta_{\{j\}\{j\}}, \\
\langle \{j\} \{m\} | T_{\{\mu\}}^{(D+1)\{\Box\}} | \{j\} \{m\} \rangle &= \left( \begin{array}{ccc}
\{j\} & \{\Box\} & \{j\} \\
\{m\} & \{\mu\} & \{m\}
\end{array} \right)
\end{align*}
\]

\[
\times \sqrt{\text{dim}\{j\}\text{dim}\{j\}} \left\{ p + \frac{1}{2} (C_2(\{j\}) - C_2(\{j\})) \right\} \left( \begin{array}{ccc}
\{j\} & \{\Box\} & \{j\} \\
\{0\} & \{0\} & \{0\}
\end{array} \right),
\]

with all representation labels changed properly as required by the $D \to D + 1$ replacement.

The natural choices for the $D$-vector $X_a$ operator is either $A_a$ ($a = 0, 1, \ldots, D - 1$),

\[
X_a = Q_{aD}^{(D+1)} = J_{aD}^{(D+1)} + T_{aD}^{(D+1)},
\]

or $B_a$ ($a = 0, 1, \ldots, D - 1$),

\[
X_a = Q_{Da}^{(D+1)} = J_{Da}^{(D+1)} - T_{Da}^{(D+1)}.
\]

The above expressions for the $J_{aD}^{(D+1)}$ and $T_{aD}^{(D+1)}$ operators matrix elements, provide us with an explicit form of the $X^a$ operator in the space of a spinorial field $\Psi(x)$ that transforms w.r.t selected spinorial $SL(D + 1, R)$ representation,

\[
(iX_a)^B_D \partial_a - M) \Psi_B(x) = 0, \quad A, B = 1/2, \ldots, \infty, \\
\Psi(x) = (\Psi^{(1)}(x), \Psi^{(2)}(x), \ldots)^T, \quad \Psi^{(i)}(x) \to D^{(i)}_{\text{spin}}(STL(D, R)) \Psi^{(i)}(x), \\
D^{(i)}_{\text{spin}}(STL(D + 1, R)) \supset \sum_{(i)} D^{(i)}_{\text{spin}}(STL(D, R)).
\]

Let us finally consider the question of the mass operator $M$. We make use here of the $s$-parity of the $STL(D, R)$ algebra. As stated above, the $s$-parity of the $Spin(D)$ second-rank tensor representation, $(20 \ldots 0) = \Box$, is $s(\Box) = \ldots = -1$.
+1, while the s-parity of the vector representation, \( (10\ldots0) = \Box \), is \( s(\Box) = -1 \). Now, all the states \( \Psi_A^{(i)} (A = 1/2, \ldots \infty) \) of a given spinorial irreducible representation \( D_{\text{spin}}^{(i)}(\mathcal{SL}(D, R)) \) are obtained by consecutive applications of the noncompact \( T^{(\Box)} \) operators and therefore have the same s-parity. The s-parity of the Dirac-like wave equation D-vector, \( X^a = X^{(\Box)} \), is \( s(X^a) = -1 \), and thus we find

\[
\langle \Psi^{(i)} | X^a | \Psi^{(i)} \rangle = 0, \quad i = 1, 2, \ldots .
\]

We conclude that the mass of an \( \overline{\mathcal{SL}}(D, R) \)-covariant Dirac-like wave equation can only be of a dynamical origin, i.e. a result of an interaction. This agrees with the fact that the Casimir operator of the special affine group \( \overline{\mathcal{SA}}(4, R) \) vanishes [15] leaving the masses unconstrained.

**Acknowledgments**

This work was supported in part by MSE, Belgrade, Project-101486.

**References**

[1] D. Ivanenko and G. Sardanashvily, The Gauge Treatment of Gravity, *Phys. Rep.* 94 (1983) 1-45.

[2] F.W. Hehl, G.D. Kerlick and P. von der Heyde, On a New Metric Affine Theory of Gravitation, *Phys. Lett.* B 63 (1976) 446-448.

[3] F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne’eman, Metric Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance, *Phys. Reports* 258 (1995) 1-171.

[4] Y. Ne’eman and Dj. Šijački, Unified Affine Gauge Theory of Gravity and Strong Interactions with Finite and Infinite \( \overline{GL}(4, R) \) Spinor Fields, *Ann. Phys. (N.Y.)* 120 (1979) 292-315.

[5] Y. Ne’eman and Dj. Šijački, Gravity from Symmetry Breakdown of a Gauge Affine Theory, *Phys. Lett.* B 200 (1988) 489-494.
[6] Dj. Šijački, $\mathbb{SL}(4, R)$ Embedding for a 3D World Spinor Equation, *Class. Quant. Grav.* **21** (2004) 4575-4593.

[7] Y. Ne’eman and Dj. Šijački, $\mathbb{SL}(4, R)$ World Spinors and Gravity, *Phys. Lett. B* **157** (1985) 275-279.

[8] Dj. Šijački, World Spinors Revisited, *Acta Phys. Polonica B* **29** (1998) 1089-1097.

[9] M. Berg, C. DeWitt-Morette, S. Gwo and E. Kramer, The Pin Groups in Physics: C, P, and T, *Rev. Math. Phys.* **13** (2001) 953-1034.

[10] Y. Ne’eman and Dj. Šijački, $\mathbb{GL}(4, R)$ Group-Topology, Covariance and Curved-Space Spinors, *Int. J. Mod. Phys. A* **2** (1987) 1655-1668.

[11] Dj. Šijački, All $\mathbb{SL}(3, R)$ Ladder Representations, *J. Math. Phys.* **31** (1990) 1872-1876.

[12] Dj. Šijački, The Unitary Irreducible Representations of $\mathbb{SL}(3, R)$, *J. Math. Phys.* **16** (1975) 298-311.

[13] Dj. Šijački, Generic Curved Space Superextendon Theories, in *Supermembranes and Physics in 2+1 Dimensions*, eds. M. Duff, C. Pope and E. Sezgin (World Scientific Pub., 1990) 213-223.

[14] I.Kirsch and Dj. Šijački, From Poincaré to Affine Invariance: How does the Dirac Equation Generalize?, *Class. Quant. Grav.* **19** (2002) 3157-3178.

[15] J. Lemke, Y. Ne’eman and J. Pecina-Cruz, Wigner Analysis and Casimir Operators of SA(4,R), *J. Math. Phys.* **33** (1992) 2656-2659.