The Hölder Inequality for KMS States

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Abstract
We prove a Hölder inequality for KMS States, which generalises a well-known trace-inequality. Our results are based on the theory of non-commutative $L_p$-spaces.

1 Introduction

Trace inequalities have played a key role both in mathematics and quantum statistical mechanics [4, 12, 13]. In recent years numerous trace inequalities have been generalised to σ-finite von Neumann algebras, for example the Golden-Thompson and Peierls-Bogolubov inequalities [2]. In this short note, we generalize the Hölder trace-inequality. The latter has been used, for example, by Ruelle to construct interacting Gibbs states [21][22] in a box and then control their thermodynamic limit. While trace inequalities are useful for quantum systems constrained to a finite volume, there are good reasons to abandon the boxes and study quantum statistical systems directly in infinite volume. As the generator of the time evolution will no longer have discrete spectrum, trace inequalities can not be applied. Thus the Hölder trace-inequality has to be replaced by the generalised inequality presented in Section 2. It was pointed out by Fröhlich [7] that the Hölder inequality given in Section 2 also plays a crucial role in the context of thermal quantum field theory.

The paper is organised as follows. In Section 2 we recall some basic notions of Tomita-Takesaki theory and state the main result. Section 3 contains an introduction to non-commutative $L_p$-spaces. Section 4 provides the proof of the main theorem.

2 The Main Result

In quantum statistical mechanics, thermal equilibrium states are characterised by the KMS condition [3], which is (a) a generalisation of the Gibbs condition

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to systems in infinite volume; (b) formulated in terms of analyticity properties of the correlation functions; and (c) can be derived from first principles, like passivity [20] or stability under small adiabatic perturbations of the dynamics [9].

**Definition 2.1.** Let $\mathcal{A}$ be a $C^*$-algebra and $\{\tau_t\}_{t \in \mathbb{R}}$ be a strongly continuous group of $*$-automorphisms of $\mathcal{A}$. A normalised positive linear functional $\omega_\beta$ over $\mathcal{A}$ is called a $(\tau, \beta)$-KMS state for the inverse temperature $\beta > 0$, if for all $A, B \in \mathcal{A}$ there exists a function $F_{A,B}$, which is continuous and bounded in the strip $0 \leq \Im z \leq \beta$ and analytic in the open strip $0 < \Im z < \beta$, with boundary values given by

$$F_{A,B}(t) = \omega_\beta(A \tau_t(B))$$

and $F_{A,B}(t + i\beta) = \omega_\beta(\tau(B)A)$ for all $t \in \mathbb{R}$.

The KMS-condition implies that $\omega_\beta$ is invariant under $\tau$ and therefore the latter can be unitarily implemented in the GNS representation $(\pi, H, \Omega)$ associated to the pair $(\mathcal{A}, \omega_\beta)$. Weak continuity of $\tau$ ensures the existence of a generator $L$, called the Liouvillean, such that $\pi(\tau_t(A))\Omega = e^{-itL}\pi(A)\Omega$ and $L\Omega = 0$.

As the vector $\Omega$ is cyclic and separating for the von Neumann algebra $M = \pi(\mathcal{A})''$, the algebraic operations on $M$ define maps on the dense set $M\Omega \subset H$. Tomita’s idea to study the $*$-operation on $M$ turned out to be especially fruitful. It leads to an anti-linear operator $S_o$,

$$S_o: A\Omega \mapsto A^*\Omega, \quad A \in M,$$

which is closable, and thus allows a polar decomposition for the closure $S = J\Delta^{1/2}$. The anti-linear involution $J$ is called the modular conjugation and the positive albeit in general unbounded operator $\Delta$ is called the modular operator. The modular conjugation $J$ satisfies $J^* = J$ and $J^2 = 1$, and induces a $*$-anti-isomorphism $j: A \mapsto \Delta^{1/2}A \Omega$ between the algebra $M$ and its commutant $M'$ (Tomita’s theorem).

More generally, an arbitrary normal faithful state over a von Neumann algebra $M$ is a $(\sigma, -1)$-KMS state with respect to the modular automorphisms $\sigma$ given by $A \mapsto \Delta^sA\Delta^{-is}$, $A \in M$, $s \in \mathbb{R}$, at temperature $\beta = -1$ (see, e.g., [5]). To be precise, the strong continuity assumption, which is part of Definition 2.1, holds on the restricted $C^*$-dynamical system [23] Proposition 1.18] associated to the $W^*$-dynamical system $(M, \sigma)$. Uniqueness of the modular automorphism ensures that $\Delta^{1/2} = e^{-\beta L/2}$.

The standard positive cone $P^\sharp \subset H$ is defined as

$$P^\sharp = \overline{\{JA\Omega : A \in M\}} = \{\Delta^{1/4}A\Omega : A \in M^+\},$$

where the bar denotes norm closure [1]. Consequently, a KMS state on a $C^*$-dynamical system $(\mathcal{A}, \tau)$ gives rise to a von Neumann algebra in standard form, namely a quadruple $(\mathcal{H}, M, J, P^\sharp)$, where $\mathcal{H}$ is a Hilbert space, $M$ is a von Neumann algebra, $J$ is an anti-unitary involution on $\mathcal{H}$ and $P^\sharp$ is a self-dual cone in $\mathcal{H}$ such that:
\begin{enumerate}
\item $JMJ = M'$;
\item $JA^{\ast}J = A^{\ast}$ for $A$ in the center of $M$;
\item $J\Psi = \Psi$ for $\Psi \in \mathcal{P}^{\ast};$
\item $AJP^{\ast} \subset \mathcal{P}^{\ast}$ for $A \in M$.
\end{enumerate}

The vector state induced by $\Omega$ extends the KMS state $\omega_{\beta}$ from $A$ to $M$, and we denote this state by the same letter. Now set, for $p \in \mathbb{N}$ and $A \in M^{+},$

$$
\| A \|_{p} = \left( \frac{1}{p} \sum_{t=1}^{p} \omega_{\beta}(e^{itL/p}A \cdots e^{itL/p}A) \right)^{1/p},
$$

(2)

The subscript indicates the analytic continuation of the map $t \mapsto F(t) = \omega_{\beta}(e^{itL/p}A \cdots e^{itL/p}A)$ to $F(i\beta)$. To simplify the notation we will denote $F(i\beta)$ by $\omega_{\beta}(e^{-\beta L/p}A \cdots e^{-\beta L/p}A)$.

**Theorem 2.2** (Hölder inequality). \textit{Consider a $(\tau, \beta)$-KMS state $\omega_{\beta}$ over a $C^{\ast}$-dynamical system $(A, \tau)$. Let $(z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}$ be such, that $0 \leq \Re z_{j}, \sum_{j=1}^{n} \Re z_{j} \leq 1$, and let $p_{j}$ be the smallest, positive even integer such that $\frac{1}{p_{j}} \leq \min\{ \Re z_{j+1}, \Re z_{j} \}$, with $z_{n+1} = z_{n}$ and $z_{0} = z_{1}$. Then}

$$
\left| \omega_{\beta}(A_{n}e^{-z_{n}L} \cdots A_{1}e^{-z_{1}L}A_{0}) \right| \leq \| A_{0} \|_{p_{0}} \cdots \| A_{n} \|_{p_{n}}
$$

(3)

for all $A_{0}, \ldots, A_{n} \in M^{+}$.

**Remarks**

(i) Although the multi-boundary Poisson kernels \cite{23, Lemma 4.4.8} for the domain $f(n)$ can be computed explicitly (the computation can be traced back to Widder \cite{27}), it seems unlikely that the Hölder inequality (3) can be derived using only methods of complex analysis (unless $n = 2$).

(ii) Let $M_{0}$ denote a weakly dense sub-algebra of analytic elements in $M$. It follows that, for $p \in \mathbb{N}$ and $A \in M_{0}^{\ast},$

$$
\| A \|_{p} = \omega_{\beta}(\tau_{\beta/2p}(A) \cdots \tau_{(2p-1)\beta/2p}(A)\tau_{\beta}(A))^{1/p}.
$$

(4)

Thus Theorem 2.2 is a generalisation of the Hölder inequality for Gibbs states, as stated, for example, in \cite{16, 17}.

Two more aspects of Theorem 2.2 are notable. Firstly, it estimates a non-commutative expression in terms of essentially commutative ones, which can be evaluated using spectral theory, and secondly, the bounds are uniform in $\Im z_{j},$

\footnote{An element $A \in M$ is called analytic for $\tau_{t}$ if there exists a strip $I_{\lambda} = \{ z \in \mathbb{C} : |3z| < \lambda \}$ in $\mathbb{C}$, and a function $f : I_{\lambda} \rightarrow M$, such that (i) $f(t) = \tau_{t}(A)$ for $t \in \mathbb{R}$, and (ii) $z \mapsto \phi(f(z))$ is analytic for all $\phi \in M_{\ast}$.}

3
The proof of Theorem 2.2 relies on the theory of non-commutative $L^p$-spaces, but the appeal of the theorem may well be that knowledge of non-commutative integration theory is not required in order to apply the inequality.

In quantum statistical mechanics the uniformity in imaginary time is useful for establishing the existence of real time Greens functions from the Schwinger functions. Beyond quantum statistical mechanics, inequality (3) is also useful in constructive quantum field theory. In [7] Fröhlich argued that the Hölder inequality will guarantee the existence of thermal Wightman functions for a certain class of models. A complete proof of this claim is given in [14]. Additionally, in a forthcoming work by M. Rouleux and the first author, the Hölder inequality is used to show that the Wightman distributions of the $P(\phi)_2$ model on the de Sitter space satisfy a micro-local spectrum condition.

3 Non-commutative $L^p$-spaces

Normal states over a von Neumann algebras provide a non-commutative extension of classical integration theory, i.e., commutative $L^p$-spaces, and one recovers the latter in case the algebra is abelian [18]. Among the many approaches to non-commutative $L^p$-spaces [2 10 11 15 24 26], Araki and Masuda’s approach [3] is best suited for our purposes. We start with a short introduction to relative modular operators for weights. A more elaborate discussion of relative modular operators can be found in [25].

3.1 Relative Modular Operators

Consider a general ($\sigma$-finite) von Neumann algebra $\mathcal{M}$ and let $\phi$ be a normal semi-finite weight on $\mathcal{M}$. The semi-cyclic representation\footnote{The semi-cyclic representation is a generalisation of the GNS representation to weights.} makes it possible to define an anti-linear operator $S_{\phi,\Omega}$ by

$$S_{\phi,\Omega}A \Omega = \xi_{\phi}(A^*), \quad A \in \mathcal{N}_{\phi}^*,$$

where $\mathcal{N}_{\phi} = \{A \in \mathcal{M} : \phi(A^*A) < \infty\}$, and $\xi_{\phi}(A)$ is the semi-cyclic representation of $A \in \mathcal{N}_{\phi}$ in

$$\mathcal{H}_{\phi} = \mathcal{N}_{\phi}/\ker \phi.$$

$S_{\phi,\Omega}$ is closable and the closure $\overline{S_{\phi,\Omega}}$ has a polar decomposition $\overline{S_{\phi,\Omega}} = J_{\phi,\Omega} \Delta_{\phi,\Omega}^{1/2}$. It is noteworthy that

$$\Delta_{\phi,\Omega} = S_{\phi,\Omega}^* \overline{S_{\phi,\Omega}},$$

is a positive, in general unbounded, operator on the original Hilbert space $\mathcal{H}$. If $\phi$ is a vector state associated to $\xi \in \mathcal{H}$ such that $\phi(x) = \langle \xi, x\xi \rangle$, then $\xi_{\phi}(A) = A\xi$ and we denote $\Delta_{\phi,\Omega}$ by $\Delta_{\xi,\Omega}$ and $J_{\phi,\Omega}$ by $J_{\xi,\Omega}$. In order to keep the notation simple, $e^{-\beta L/2}$ will from now on be written as $\Delta^{1/2} \equiv \Delta_{\Omega,\Omega}^{1/2}$. 

4
A key role in the proof of Theorem 2.2 will be played by the following estimate: define, for any \( \alpha > 0 \), a set

\[
I_\alpha^{(n)} = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |Rz_j| \leq \alpha, 0 \leq |Rz_j| \}
\]

Let \( z \in I^{(n)} \equiv I_1^{(n)} \) and \( z', z'' \in \mathbb{C} \) be such that \( |Rz'|, |Rz''| > 0 \), \( z' + z'' = z_m \) and

\[
Rz_1 + \ldots + Rz_{m-1} + Rz''_m \leq 1/2,
\]

\[
Rz_1 + \ldots + Rz_{m+1} + Rz''_m \leq 1/2.
\]

Under these conditions, Araki \([3\, \text{Lemma A}]\) has shown\(^3\) that for \( \phi_1, \ldots, \phi_n \in \mathcal{M}_+^+ \) and \( X_0, \ldots, X_n \in \mathcal{M} \)

\[
\left| \left( \Delta_{\phi_m, \Omega}^{z_m} X_m^* \Delta_{z_{m-1}, \Omega}^{z_{m-1}} \Delta_{z_{m-2}, \Omega}^{z_{m-2}} \cdots \Delta_{\phi_1, \Omega}^{z_1} X_0 \Omega \right) \right| \leq \left( \prod_{j=1}^n \|X_j\| \right) (\Omega, \mathcal{B})^{\alpha_j} \left( \prod_{j=1}^n \phi_j(\mathcal{B})^{Rz_j} \right),
\]

with \( z_0 = 1 - \sum_{j=1}^n Rz_j \).

**Remark 3.1.** Consider the space of \( n \times n \)-matrices \( M_n(\mathbb{C}) \) \( \ni \xi, \eta \) equipped with the inner product \( \langle \xi, \eta \rangle = \text{Tr} \xi^* \eta \) and two positive matrices \( 0 < \nu, \omega \in M_n(\mathbb{C}) \). Moreover, assume that \( \text{Tr} \omega = 1 \). Now apply the Hölder trace inequality \([10]\)

\[
|\text{Tr} \omega AB| \leq \|A\|_{\omega,p} \|B\|_{\omega,q}, \quad p^{-1} + q^{-1} = 1,
\]

where \( \langle A \rangle_{\omega} \equiv \text{Tr} \omega A \) and \( \|A\|_{\omega,p} \equiv \text{Tr} (\omega^{1/2p} |A|^{1/2p})^p \), to the relative modular operator \( \Delta_{\nu,\omega} \), which satisfies \( \Delta_{\nu,\omega}^p \xi = \nu^{1/p} \xi \omega^{-1/p} \) for \( p \in \mathbb{N} \). Thus, for \( 1/p + 1/q = 1 \),

\[
|\langle A_2 \Delta_{\nu_2,\omega}^1 A_1 \Delta_{\nu_1,\omega}^{1/q}, A_0 \rangle_{\omega} | \leq \left( \prod_{j=0}^2 \|A_j\|_{\infty} \right) \|\Delta_{\nu_2,\omega}^1 \|_{\omega,p} \|\Delta_{\nu_1,\omega}^{1/q} \|_{\omega,q} = \left( \prod_{j=0}^2 \|A_j\|_{\infty} \right) (\mathcal{B})_{\nu_2}^{1/p} (\mathcal{B})_{\nu_1}^{1/q}.
\]

### 3.2 Positive Cones and \( L_p \)-Spaces for von Neumann Algebras

Consider a general (\( \sigma \)-finite) von Neumann algebra \( \mathcal{M} \) in standard form with cyclic and separating vector \( \Omega \). For \( 2 \leq p \leq \infty \), Araki and Masuda define \([3\, \text{Equ. (1.3)}, \text{p. 340}]\)

\[
L_p(\mathcal{M}, \Omega) \equiv \{ \zeta \in \bigcap_{\xi \in \mathcal{H}} D(\Delta_{\xi,\Omega}^p) : \|\zeta\|_p < \infty \},
\]

\(^3\text{Note that, in contrast to [3], our inner product is linear in the second entry.}\)
where
\[ \|\zeta\|_p = \sup_{\|\xi\|=1} \|\Delta^{\frac{1}{p}}_{\xi,\Omega} \zeta\|. \]

For \(1 \leq p < 2\), \(L_p(\mathcal{M}, \Omega)\) is defined as the completion of \(\mathcal{H}\) with respect to the norm
\[ \|\zeta\|_p = \inf\{\|\Delta^{\frac{1}{p}}_{\xi,\Omega} \zeta\| : \|\xi\| = 1, s_{\mathcal{M}}(\xi) \geq s_{\mathcal{M}}(\zeta)\}. \]

Here \(s_{\mathcal{M}}(\xi)\) denotes the smallest projection in \(\mathcal{M}\), which leaves \(\xi\) invariant. The cones \(\mathbb{P}_\alpha(\mathcal{M}, \Omega)\) [3, Equ. (1.13)]
\[ \mathbb{P}_\alpha = \{\Delta^\alpha A\Omega : A \in \mathcal{M}^+\}, \quad 0 \leq \alpha \leq 1/2, \]

can be used to define the positive part of \(L_p(\mathcal{M}, \Omega)\) [3, Equ. (1.14), p. 341]:
\[ L_p^+(\mathcal{M}, \Omega) = L_p(\mathcal{M}, \Omega) \cap \mathbb{P}_{1/(2p)}, \quad 2 \leq p \leq \infty. \quad (11) \]

Note that these are not operator spaces. The connection to the operator algebra \(\mathcal{M}\) is made through auxiliary spaces \(\mathcal{L}_p(\mathcal{M}, \Omega)\), which consist of formal expressions \(A = u\Delta^\beta_{\phi,\Omega}\) with \(\phi \in \mathcal{M}^+_r\) and \(u\) a partial isometry satisfying \(u^*u = s(\phi)\) (the support projection of \(\phi\)). The set of formal products
\[ X_0\Delta_{\phi_1,\Omega} \ldots \Delta_{\phi_n,\Omega} X_n, \quad (12) \]
is denoted by \(\mathcal{L}_r^*(\mathcal{M}, \Omega)\). Here is \(X_j \in \mathcal{M}\) (\(j = 0, \ldots, n\)), \(\phi_j \in \mathcal{M}^+_r\) (\(j = 1, \ldots, n\)) and \(\varepsilon = (z_1, \ldots, z_n) \in \mathcal{L}_r^{(n)}\). On the subset \(\mathcal{L}_{p,0}^+(\mathcal{M}, \Omega) \subset \mathcal{L}_p^*(\mathcal{M}, \Omega)\), characterized by the condition \(\sum_{j=1}^n \Re z_j = 1 - (1/p)\), it is possible to implement the star operation. The adjoint of a generic element \(\mathcal{L}_{p,0}^*(\mathcal{M}, \Omega)\) is defined to be
\[ X_0^*\Delta_{\phi_n,\Omega}^* \ldots X_1^* \Delta_{\phi_1,\Omega}^* X_n^* \quad (13) \]
A multiplication can be defined, using the product in \(\mathcal{M}\) to connect the formal expressions: \(BC \in \mathcal{L}_r^*(\mathcal{M}, \Omega)\) for \(B \in \mathcal{L}_{p,0}^+(\mathcal{M}, \Omega)\), \(C \in \mathcal{L}_q^*(\mathcal{M}, \Omega)\) and \(r^{-1} = p^{-1} + q^{-1} - 1\).

If \(r^{-1} = \sum_{j=1}^n (p_j)^{-1}\), \(r^{-1} + r^{-1} = 1\), \(\xi_j \in L_{p_j}(\mathcal{M}, \Omega)\), \(X_j \in \mathcal{M}\) (\(j = 0, \ldots, n\)), and \(\xi_j = u_j\phi_j^{1/p_j}\) (\(j = 1, \ldots, n\)) is the polar decomposition, then the product
\[ \xi = X_0\xi_1 X_1\xi_2 \ldots \xi_n X_n \in L_r(\mathcal{M}, \Omega) \quad (= L_{r'}(\mathcal{M}, \Omega)^*) \]
is defined by
\[ \langle \xi, \xi' \rangle = \omega(\Delta^{1/r'}_{\phi',\Omega} u^* X_0 u_{1}\Delta^{1/p_1}_{\phi_1,\Omega} X_1 u_{2}\Delta^{1/p_2}_{\phi_2,\Omega} \ldots u_{n}\Delta^{1/p_n}_{\phi_n,\Omega} X_n) \in L_r(\mathcal{M}, \Omega) \]
where \(\xi' \in L_{r'}(\mathcal{M}, \Omega)\) and \(\xi' = u'\phi'^{1/r'}\) is its polar decomposition.

Araki’s inequality [3] now entails a Hölder inequality: let \(\zeta_1 \in L_p(\mathcal{M}, \Omega)\) and \(\zeta_2 \in L_{p'}(\mathcal{M}, \Omega)\) for \(p^{-1} + p'^{-1} = r^{-1}\), then
\[ \|\zeta_1 \zeta_2\|_r \leq \|\zeta_1\|_p \|\zeta_2\|_{p'}. \quad (14) \]
Thus the product $\zeta_1\zeta_2$ is in $L_r(\mathcal{M}, \Omega)$ and, as the case $p^{-1} + p'^{-1} = 1$ suggests, the topological dual $L_p(\mathcal{M}, \Omega)^*$ of $L_p(\mathcal{M}, \Omega)$ is $L_{p'}(\mathcal{M}, \Omega)$. For $A \in L_p(\mathcal{M}, \Omega)$ and $B \in L_{p'}(\mathcal{M}, \Omega)^*$, the corresponding duality bracket is given by

$$
\langle A, B \rangle = (A\Omega, B\Omega),
$$

(15)

if $\Omega$ is in the domain of $A$ and $B$. According to [3] Notation 2.3 (4) $A$ and $B$ in $L_p^*(\mathcal{M}, \Omega)$ are said to be equivalent, if (i) $1 \leq p \leq 2$ and $A\Omega = B\Omega$; (ii) if $2 \leq p \leq \infty$ and

$$
(C, A) = (C, B)
$$

(16)

for all $C$ in $L_p(\mathcal{M}, \Omega)$.

Another important property is, that for $1 \leq p \leq \infty$, $x \in \mathcal{M}$ and $\zeta \in L_p(\mathcal{M}, \Omega)$, the following inequality holds:

$$
\|x\zeta\|_p \leq \|x\| \|\zeta\|_p.
$$

(17)

It is evident from the definition of the $L_p$-spaces, that $\mathcal{H}$ and $L_2(\mathcal{M}, \Omega)$ are equal. It is proven in [3] that $\mathcal{M} \cong L_\infty(\mathcal{M}, \Omega)$ as well as $\mathcal{M}_* \cong L_1(\mathcal{M}, \Omega)$.

4 Proof of the Main Result

**Lemma 4.1.** Let $A_1, \ldots, A_n \in \mathcal{M}^+$. Then there exist unique $\phi_j \in \mathcal{M}_+^*$ such that for $0 \leq p_j^{-1} \leq 1/2$

$$
\Delta_{\phi_j, \Omega}^{1/p_j} \Omega = \Delta^{1/2p_j} A_j \Omega \quad (j = 1, \ldots, n)
$$

(18)

and $\phi_j(I)^{1/p_j} = \|\Delta^{1/2p_j} A_j \Omega\|_{p_j}$. If also $\sum_{j=1}^n 1/p_j = 1/2$ holds, then

$$
\Delta_{\phi, \Omega}^{1/p, \Omega} = \Delta^{1/2p} A_n \Delta^{1/2p} \ldots \Delta^{1/2p} A_1 \Omega \in \mathcal{H}.
$$

(19)

**Proof.** Let $A_1, \ldots, A_n \in \mathcal{M}^+$ and $0 \leq p_j^{-1} \leq 1/2$, $j = 1, \ldots, n$. Then, by definition $\zeta_j = \Delta^{1/2p_j} A_j \Omega \in P_{p_j}^{1/2p_j}$. An application of inequality [3] yields

$$
\|\zeta_j\|_{p_j}^2 = \sup_{\|\xi\|=1} \|\Delta_{\xi, \Omega}^{(1/2)-1/(p_j)} \zeta_j\|^2
$$

(20)

$$
= \sup_{\|\xi\|=1} \left( \Delta_{\xi, \Omega}^{(1/2)-1/(p_j)} A_j \Omega, \Delta_{\xi, \Omega}^{(1/2)-1/(p_j)} A_j \Omega \right)
$$

(21)

$$
\leq \sup_{\|\xi\|=1} \sup_{\|\omega\|=1} \Delta_{\xi, \Omega}^{(1/2)-1/(p_j)} A_j \Omega, \Delta_{\xi, \Omega}^{(1/2)-1/(p_j)} A_j \Omega = A_j, \Omega, \|A_j\| < \infty,
$$

(22)

which establishes, that $\zeta_j \in L_{p_j}(\mathcal{M}, \Omega)$. Thus, according to [11], $\zeta_j \in L_{p_j}^*(\mathcal{M}, \Omega)$. By [3] Theorem 3 (4), p. 342] there exists a unique $\phi_j \in \mathcal{M}_+^*$ such that

$$
\zeta_j = \Delta_{\phi_j, \Omega}^{1/p_j} \Omega \quad \text{and} \quad \phi_j(I)^{1/p_j} = \|\zeta_j\|_{p_j} = \|\Delta^{1/2p_j} A_j \Omega\|_{p_j}.
$$

Thus, by definition [3] Notation 2.3 (4), $\Delta^{1/2p_j} A_j \Delta^{1/2p_j} = \Delta^{1/2p_j} A_j \Omega$ as elements in $L_{p_j}^*(\mathcal{M}, \Omega)$, where $p_j^{-1} + p_j^{-1} = 1$. Even though $\Delta_{\phi_j, \Omega}^{1/p_j} and \Delta^{1/2p_j} A_j \Omega$.
may not be equal as operators, Lemma 7.7 (2) in [3] ensures, that their composition as elements of the spaces \( L^p \) is well-defined: setting \( B_1 = \Delta^{1/p}_{\phi_2, \Omega} \), 
\[ B_2 = -\Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \quad \text{and} \quad C_2 = \Delta^{1/p_1}_{\phi_1, \Omega}, \]
there holds \( \sum_{i=1}^2 B_i = 0 \) as elements in \( L_{p_2}(\mathcal{M}, \Omega) \), and therefore, using the lemma cited,
\[
\Delta^{1/p_2}_{\phi_2, \Omega} \Delta^{1/p_1}_{\phi_1, \Omega} \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1, \Omega} \quad \text{(23)}
\]
as elements in \( L_{r_1}(\mathcal{M}, \Omega) = L_{r_1'}(\mathcal{M}, \Omega)^* \), where \( r_1^{-1} + r_1' = 1, r_1' = p_2^{-1} + p_2^{-1} - 1 \) and \( 1 \leq r_1' \leq 2 \) (in comparison to [3] indices and primed indices have swapped places). Note that this means \( r_1^{-1} = p_2^{-1} + p_1^{-1} \). Using the same lemma once more (with the appropriate choices of \( C_2 \) and \( B_3, B_4 \)) gives
\[
\Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1, \Omega} \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1, \Omega} \quad \text{(24)}
\]
as elements in \( L_{r_1'}(\mathcal{M}, \Omega)^* \). Together (23) and (24) imply
\[
\Delta^{1/p_2}_{\phi_2, \Omega} \Delta^{1/p_1}_{\phi_1, \Omega} \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1, \Omega} \quad \text{(25)}
\]
as elements in \( L_{r_1'}(\mathcal{M}, \Omega)^* \). Consequently,
\[
\Delta^{1/p_2}_{\phi_2, \Omega} \Delta^{1/p_1}_{\phi_1, \Omega} \equiv \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1, \Omega} \quad \text{(26)}
\]
as elements in \( L_{r_1'}(\mathcal{M}, \Omega)^* \). Iteration of this procedure results in
\[
\Delta^{1/p_n}_{\phi_n, \Omega} \cdots \Delta^{1/p_1}_{\phi_1, \Omega} \equiv \Delta^{1/2p_n} A_n \Delta^{1/2p_n} \cdots \Delta^{1/2p_2} A_2 \Delta^{1/2p_2} \Delta^{1/p_1}_{\phi_1, \Omega} \quad \text{(27)}
\]
as elements in \( L_2(\mathcal{M}, \Omega)^* \), because of \( \sum_{j=1}^n 1/p_j = 1/2 \). But since \( \mathcal{H} = \mathcal{H}^* = L_2(\mathcal{M}, \Omega)^* \) the proof is finished.

**Lemma 4.2.** Let \( p \in \mathbb{N} \) be even and \( A \in \mathcal{M}^+ \). Then there exists \( \phi \in \mathcal{M}^+_\beta \) such that
\[
\|\Delta^{1/2p} A \Omega\|_p = \phi(\Omega)^{1/p} = \omega_\beta(A \Delta^{1/p} A \cdots A^{1/p} A)^{1/p}. \quad \text{(28)}
\]
On the r.h.s. we have used Araki’s symbolic notation introduced in the sentence following Eqn. [2].

**Proof.** As proved in Lemma 4.1 there exists \( \phi \in \mathcal{M}^+_\beta \), such that \( \|\Delta^{1/2p} A \Omega\|_p = \phi(\Omega) \), and \( \Delta^{1/2p} A \Delta^{1/2p} = \Delta^{1/p}_{\phi, \Omega} \) as elements in \( L^*_{p,0}(\mathcal{M}, \Omega) \). Thus, by [13] and [5],
\[
\omega_\beta(\Delta^{1/2p} A \Delta^{1/2p} \cdots A^{1/2p} A \Delta^{1/2p}) = (\Delta^{1/p}_{\phi, \Omega} \cdots \Delta^{1/p}_{\phi, \Omega} \Delta^{1/p}_{\phi, \Omega} \cdots \Delta^{1/p}_{\phi, \Omega}) \leq \phi(\Omega) = \|\Delta^{1/2p} A \Omega\|_p. \quad \text{(29)}
\]
Since \( \phi \in \mathcal{M}^+_\beta \), there exists [5] a vector \( \xi \in \mathcal{P}^\beta \) such that \( \phi(X) = (\xi, X \xi) \) for \( X \in \mathcal{M} \). Using \( \xi = J_{\phi, \Omega} \Delta^{1/2}_{\phi, \Omega} = J_{\xi, \Omega} \Delta^{1/2}_{\xi, \Omega} \), there holds
\[
\phi(X) = (\xi, X \xi) = (\Delta^{1/2}_{\phi, \Omega}, J_{\phi, \Omega} J_{\phi, \Omega} J_{\phi, \Omega} J_{\phi, \Omega} X^* \Omega),
\]
where $J^*_{\phi,\Omega}J_{\phi,\Omega} = s_M(\xi) s_M'(\Omega)$ is a projection [3, p. 396]. Therefore

$$\phi(\mathbb{I}) \leq (\Delta_{\phi,\Omega}^{1/2}, \Delta_{\phi,\Omega}^{1/2}) = \omega_{\beta}(A \Delta^{1/p} A \cdots \Delta^{1/p} A),$$

which finishes the proof. \qed

**Proof of Theorem 2.2.** Assuming the requirements of Theorem 2.2, Lemma 4.1 together with inequality (8), relation (28) and $w_j = z_j - (2p_j)^{-1} - (2p_{j-1})^{-1}$ imply

$$\left| \omega_{\beta}(A_n \Delta^{z_n} \cdots A_1 \Delta^{z_1} A_0) \right| = \left| \omega_{\beta}(\Delta^{1/2p_n} A_n \Delta^{1/2p_n} \Delta^{w_n} \cdots \Delta^{1/2p_0} A_0 \Delta^{1/2p_0}) \right|$$

$$\leq \omega_{\beta}(\mathbb{I})^{1 - \sum_{j=0}^n (p_j)^{-1}} \prod_{j=0}^n \phi_j(\mathbb{I})^{1/p_j} = \prod_{j=0}^n ||A_j||_{p_j}.$$

Again we have used Araki’s symbolic notation introduced in the sentence following Equ. (2). \qed

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**References**

[1] Araki, H., Some properties of modular conjugation operators of a von Neumann algebra and a non-commutative Radon-Nikodym derivative with a chain rule, Pac. J. Math. 50 (1974) 309–354.

[2] Araki, H., Golden-Thompson and Peierls-Bogolubov inequalities for a general von Neumann algebra, Commun. Math. Phys. 34 (1973) 167–178.

[3] Araki, H., Masuda, T., Positive cones and $L_p$-spaces for von Neumann algebras, Publ. RIMS, Kyoto Univ. 18 (1982) 339–411.

[4] M.B. Ruskai, Inequalities for traces on von Neumann algebras, Commun. Math. Phys. 26 (1972) 280–289.

[5] Bratteli, O. and Robinson, D.W., Operator Algebras and Quantum Statistical Mechanics I, II, Springer-Verlag, New York-Heidelberg-Berlin (1981).

[6] Diximier, N., Formes linéaires sur un anneau d’opérateurs, Bull. Soc. Math. France 81 (1953) 9–39.

[7] Fröhlich, J., The reconstruction of quantum fields from Euclidean Green’s functions at arbitrary temperature, Helv. Phys. Acta 48 (1975) 355–363.
[8] Haag, R., Hugenholtz, N.M. and Winnink, M., On the equilibrium states in quantum statistical mechanics, Commun. Math. Phys. 5 (1967) 215–236.

[9] Haag, R., Kastler, D. and Trych-Pohlmeyer, E.B., Stability and equilibrium states, Commun. Math. Phys. 38 (1974) 173–193.

[10] Haagerup, U., $L^p$-spaces associated with an arbitrary von Neumann algebra, in Algèbres d’opérateurs et leurs applications en physique mathématique, Colloques internationaux du CNRS, No. 274, Marseille 20-24 juin 1977, Éditions du CNRS, Paris (1979) 175–184.

[11] Hislum, M., Les espaces $L^p$ d’une algèbre de von Neumann définies par la dérivée spatiale, J. Funct. Analysis 40 (1981) 151–169.

[12] E.H. Lieb, Convex trace functions and the Wigner-Yanase-Dyson conjecture, Adv. Math. 11 (1973) 267–288.

[13] B. Simon, Trace Ideal and their Applications, London and New York, Cambridge Univ. Press (1979).

[14] Jäkel, C., and Robl, F., The relativistic KMS condition for the thermal $n$-point functions of the $P(\phi)_2$ model, to appear in Commun. Math. Phys..

[15] Kosaki, H., Application of the complex interpolation method to a von Neumann algebra (Non-commutative $L^p$-space), J. Funct. Anal. 56 (1984) 29–78.

[16] Majewski, A.W. and Zegarlinski, B., On quantum stochastic dynamics and noncommutative $L_p$ spaces, Lett. Math. Phys. 36 (1996) 337–349.

[17] Majewski, A.W. and Zegarlinski, B., On stochastic dynamics I: Spin systems on a lattice, Math. Phys. Elect. J. 1 (1995) 1–37.

[18] Murphy, G.J., $C^*$-Algebras and Operator Theory, Academic Press (1990) ISBN-10: 0125113609, ISBN-13: 978-0125113601.

[19] Nelson, E., Notes on non-commutative integration, J. Funct. Anal. 15 (1974) 103–116.

[20] Pusz, W., and Woronowicz, S.L., Passive states and KMS states for general quantum systems, Commun. Math. Phys. 58 (1978) 273–290.

[21] Ruelle, D., Analyticity of Green’s functions of dilute quantum gases, J. Math. Phys. 12 (1975) 901–903.

[22] Ruelle, D., Definition of Green’s functions for dilute Fermi gases, Helv. Phys. Acta 45 (1972) 215–219.

[23] Sakai, S., Operator Algebras in Dynamical Systems, Cambridge University Press (1991).
[24] Segal, I., A non-commutative extension of abstract integration, Ann. of Math. 57 (1953) 401–457; Correction to the Paper “A non-commutative extension of abstract integration”, Ann. of Math. 58 (1953) 595–596.

[25] Takesaki, M., Theory of Operator Algebras II, Springer.

[26] Terp, M., $L^p$-spaces associated with von Neumann algebras, Københavns Universitet, Matematisk Institut, Rapport No.3 (1981).

[27] Widder, D.V., Functions harmonic in a strip, Proceedings of the American Mathematical Society 12 (1961), 67–72.