Numerical Solution of System of Two Nonlinear Volterra Integral Equations

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ABSTRACT

In this paper, using the implicit trapezoidal rule in conjunction with Newton’s method to solve nonlinear system. We have used a Maple 17 program to solve the System of two nonlinear Volterra integral equations. Finally, several illustrative examples are presented to show the effectiveness and accuracy of this method.

Keywords

System of two nonlinear Volterra integral equations; Implicit trapezoidal rule; Newton’s method.

Academic Discipline And Sub-Disciplines

Mathematics, Integral Equation and Elasticity.

TYPE (METHOD/APPROACH)

Research Paper.
INTRODUCTION

In this paper, we consider the Volterra integral equation of the second kind

\[ x(t) = f(t) + \int_0^t k(t, s, x(s)) \, ds, \]  

(1)

Where \( x, f \) and \( k \) are vector-valued functions with \( m \) components. If \( f \) and \( k \) are continuous and \( k(t, s, x(s)) \) satisfies a Lipschitz condition with respect to \( x \), then a unique solution \( x(t) \) of (1) exists [1,4,7].

Volterra integral equations have been found to be effective to describe some application such as population dynamics, renewal equations, nuclear reactor dynamics, viscoelasticity, study of epidemics, super fluidity, damped vibrations, heat conduction and diffusion [1,7].

In this paper, we present the computation of numerical solution of system of two nonlinear Volterra integral equation of the second kind.

PRELIMINARIES

In this section, we recall the main theorems [7].

**Theorem 1.** Consider the equation

\[ x(t) = f(t) + \int_0^t p(t, s)k(t, s)x(s) \, ds \]  

(2)

Where

1) \( f(t) \) is continuous in \( 0 \leq t \leq T \),
2) \( k(t, s) \) is a continuous function in \( 0 \leq s \leq t \leq T \),
3) for each continuous function \( h \) and all the integrals

\[ \int_{t_1}^{t_2} p(t, s)k(t, s)h(s) \, ds \]

\[ \int_{0}^{t} p(t, s)k(t, s)h(s) \, ds \]

are continuous functions of \( t \),
4) \( p(t, s) \) is absolutely integrable with respect to \( s \) for all \( 0 \leq t \leq T \),
5) there exist points \( T_0 < T_1 < T_2 < \cdots < T_N = T \) such that with \( t \geq T_1 \)

\[ \min_{T_{i+1}} k \int_{T_i}^{T_{i+1}} |p(t, s)| \, ds \leq \alpha < 1, \]

Were \( k = \max_{0 \leq s \leq T} |k(t, s)| \),
6) for every \( t \geq 0 \) such that with \( t \geq T_1 \)

\[ \lim_{\delta \to 0} \int_{t}^{t+\delta} |p(t + \delta, s)| \, ds = 0. \]

Then (2) has a unique continuous solution in \( 0 \leq t \leq T \).

**Theorem 2.** Consider the equation

\[ x(t) = f(t) + \int_0^t p(t, s)k(t, s, x(s)) \, ds \]  

(3)

Where

1) \( f(t) \) is continuous in \( 0 \leq t \leq T \),
2) \( k(t, s, u) \) is a continuous function in \( 0 \leq s \leq t \leq T \),
3) the Lipschitz condition

\[ |k(t, s, y) - k(t, s, z)| \leq L|y - z| \]

is satisfies for \( 0 \leq s \leq t \leq T \) and all \( y \) and \( z \),
4) \( p(t, s) \) satisfies conditions (3)-(4) of Theroem 1 with \( k \) replaced by \( L \) and \( k(t, s, h(s)) \) instead of \( k(t, s)h(s) \).

Then (3) has a unique continuous solution in \( 0 \leq t \leq T \).
THE MATHEMATICS OF THE VOLterra PROCEDURE

In this section, we use the technique of the Volterra equation [2, 7] to find an approximate solution of (1) at the equally spaced points \( t_n = t_0 + nh \) for \( n = 1, \ldots, N \) where \( t_0 = 0 \) and \( N \) is the total number of steps of size \( h \). \( X_n \) denotes the approximation of \( x(t) \) at \( t = t_n \).

Setting \( t = t_n \) in (1), we have

\[
x(t_n) = f(t_n) + \int_0^{t_n} k(t_n, t, x(t)) \, dt
\]

By the composite trapezoidal rule an\ approximation of the integral in (4) is

\[
\frac{1}{2} \left[ k(t_n, t_0, x(t_0)) + 2 \sum_{j=1}^{n-1} k(t_n, t_j, x(t_j)) + k(t_n, t_n, x(t_n)) \right]
\]

Replacing \( x(t_n) \) in (4) and (5) by \( X_n \), we obtain the implicit trapezoidal rule

\[
X_n = f(t_n) + h \left[ \frac{1}{2} k(t_n, t_0, X_0) + \sum_{j=1}^{n-1} k(t_n, t_j, X_j) + \frac{1}{2} k(t_n, t_n, X_n) \right]
\]

Where \( X_0 = f(0) \) since \( x(0) = f(0) \).

Defining \( \sigma_n \) by

\[
\sigma_n = f(t_n) + h \left[ \frac{1}{2} k(t_n, t_0, X_0) + \sum_{j=1}^{n-1} k(t_n, t_j, X_j) \right]
\]

We can rewrite (6) as

\[
X_n - \frac{h}{2} kk(t_n, t_n, X_n) - \sigma_n = 0,
\]

Where \( 0 \) denotes the zero vector. From (8), we see that \( X_n \) is the solution of the vector equation

\[
\phi(u) = 0,
\]

Where \( \phi \) is the vector-valued function

\[
\phi(u) = u - \frac{h}{2} kk(t_n, t_n, u) - \sigma_n
\]

We will obtain an approximation to the solution \( X_n \) of (9) by way of the matrix-valued function \( G \) defined in (11). If \( A(u) \) is an \( m \) by \( m \) matrix-valued function that is invertible in a neighborhood of \( X_n \), then \( X_n \) is a fixed point of

\[
G(u) = u - A(u) \phi(u).
\]

Assuming the components of \( G(u) \) have continuous first and second order partial derivatives and that the first order partial derivatives and that the first order partial derivatives at \( X_n \) are equal to zero, it can be shown that if \( A(u) \) is set equal to the Jacobian matrix of the function \( \phi \), the iterates \( X_n^{(p)} \) defined by (13) below will usually converge quadratically to \( X_n \) provided the starting value is sufficiently close to \( X_n \). The Jacobian matrix of \( \phi \) is the \( m \) by \( m \) matrix \( f(u) \) with the element

\[
j(u)_{ij} = \frac{\partial}{\partial u_j} \phi_i(u) = \delta_{ij} - \frac{1}{2} h \frac{\partial}{\partial u_j} k_i(t_n, t_n, u)
\]

In row \( i \) and column \( j \), where \( \delta_{ij} \) is the Kronecker delta. Details of the statements made here follow from the discussion of Newton's method for nonlinear systems in [2]. Linz gives a brief outline of the trapezoidal rule and Newton's method for Volterra integral systems of the second kind in Section of [7].

We obtain \( X_n \) from \( X_{n-1} \) by setting \( X_n^{(0)} = X_{n-1} \) and then generating the iterates \( X_n^{(p)} \) from

\[
X_n^{(p)} = G \left( X_n^{(p-1)} \right) = X_n^{(p-1)} - J^{-1} \left( X_n^{(p-1)} \right) \phi \left( X_n^{(p-1)} \right)
\]

For \( p = 1, 2, 3, \ldots \). (This is Newton's method for nonlinear systems.) Let \( y \) denote the solution of the matrix equation

\[
j \left( X_n^{(p-1)} \right) y = \phi \left( X_n^{(p-1)} \right).
\]

Then the iteration formula (13) becomes

\[
X_n^{(p)} = X_n^{(p-1)} - y.
\]
We compute the solution \( y = J^{-1} X_n^{(p-1)} \phi \left( X_n^{(p-1)} \right) \) using the command Linear Solve. The iterates \( X_n^{(p)} \) are computed until the infinity norm of the vector \( y \) is less than a prescribed tolerance Tol. Then \( X_n \) is assigned the value of the last iterate \( [2,7] \).

NUMERICAL EXAMPLES

In this section, we solve some examples, and we can compare the numerical results with the exact solution.

**Example 1.** Consider the system of Volterra integral equations

\[
X_1(t) = 1 - \frac{t^2}{2} + \int_0^t (X_1(s) + s e^s X_2(s)) ds
\]

\[
X_2(t) = 1 + \frac{t^2}{2} + \int_0^t (se^{-s} X_1(s) - X_2(s)) ds
\]

With the exact solution \( X_1(t) = e^t \) and \( X_2(t) = e^{-t} \).

| \( t \) | \( X_1(t) \) | \( X_2(t) \) | Exact 1 \( = e^t \) | Exact 2 \( = e^{-t} \) |
|-------|-------------|-------------|-----------------|-------------------|
| 0.0   | 1.000000    | 1.000000    | 1.00000         | 1.00000           |
| 0.1   | 1.105263    | 0.904760    | 1.10517         | 0.90484           |
| 0.2   | 1.221603    | 0.818589    | 1.22140         | 0.81873           |
| 0.3   | 1.350137    | 0.740630    | 1.34986         | 0.74082           |
| 0.4   | 1.492318    | 0.670080    | 1.49182         | 0.67032           |
| 0.5   | 1.649369    | 0.606250    | 1.64872         | 0.60653           |
| 0.6   | 1.822949    | 0.548490    | 1.82212         | 0.54881           |
| 0.7   | 2.014098    | 0.496220    | 2.01375         | 0.49659           |
| 0.8   | 2.226825    | 0.448920    | 2.22554         | 0.44933           |
| 0.9   | 2.461134    | 0.406117    | 2.45960         | 0.40657           |
| 1.0   | 2.720068    | 0.367330    | 2.71828         | 0.36738           |
| 1.1   | 3.006234    | 0.332330    | 3.00417         | 0.33287           |
| 1.2   | 3.322446    | 0.300600    | 3.32012         | 0.30119           |
| 1.3   | 3.671900    | 0.271890    | 3.66930         | 0.27253           |
| 1.4   | 4.058043    | 0.245900    | 4.05520         | 0.24660           |
| 1.5   | 4.484743    | 0.222390    | 4.48169         | 0.22313           |
Fig. 1 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 1.

Example 2. Consider the system of Volterra integral equations

\[ X_1(t) = 1 + t^2 - \frac{t^3}{3} - \frac{t^4}{3} + \int_0^t \left( \frac{(t-s)^3}{3} X_1(s) + (t-s)^2 X_2(s) \right) ds \]

\[ X_2(t) = 1 - t - t^3 - \frac{t^4}{4} - \frac{t^5}{420} + \int_0^t \left( \frac{(t-s)^4}{4} X_1(s) + (t-s)^3 X_2(s) \right) ds \]

With the exact solution \( X_1(t) = 1 + t^2 \) and \( X_2(t) = 1 + t - t^3 \).

Table.2 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 2.

| \( t \) | \( X_1(t) \) | \( X_2(t) \) | Exact1 = 1 + \( t^2 \) | Exact2 = 1 + \( t - t^3 \) |
|-------|-------|-------|----------------|----------------|
| 0.0   | 1.00000 | 1.00000 | 1.00000         | 1.00000         |
| 0.1   | 1.01018 | 0.89903 | 1.01000         | 1.09900         |
| 0.2   | 1.04020 | 0.79210 | 1.04000         | 1.19200         |
| 0.3   | 1.08945 | 0.67309 | 1.09000         | 1.27300         |
| 0.4   | 1.15693 | 0.53564 | 1.16000         | 1.33600         |
| 0.5   | 1.24125 | 0.37302 | 1.25000         | 1.37500         |
| 0.6   | 1.34060 | 0.17802 | 1.36000         | 1.38400         |
| 0.7   | 1.45276 | -0.05716 | 1.49000       | 1.35700         |
| 0.8   | 1.57509 | -0.34107 | 1.64000         | 1.28800         |
| 0.9   | 1.70450 | -0.68310 | 1.81000         | 1.17100         |
| 1.0   | 1.83743 | -1.09365 | 2.00000         | 1.00000         |
| 1.1   | 1.96978 | -1.58430 | 2.21000         | 0.76900         |
| 1.2   | 2.09686 | -2.16793 | 2.44000         | 0.47200         |
| 1.3   | 2.21327 | -2.85900 | 2.69000         | 0.10300         |
| 1.4   | 2.31281 | -3.67372 | 2.96000         | -0.34400        |
| 1.5   | 2.38830 | -4.63041 | 3.25000         | -0.87500        |
Fig. 2 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 2.

Example 3. Consider the system of Volterra integral equations

\[ X_1(t) = \cos t - t \sin t + \int_0^t (\sin(t - s) X_1(s) + \cos(t - s) X_2(s)) \, ds \]

\[ X_2(t) = \sin t - t \sin t + \int_0^t (\cos(t - s) X_1(s) - \sin(t - s) X_2(s)) \, ds \]

With the exact solution \( X_1(t) = \cos t \) and \( X_2(t) = \sin t \).

Table 3 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 3.

| \( t \) | \( X_1(t) \) | \( X_2(t) \) | Exact 1 = \( \cos t \) | Exact 2 = \( \sin t \) |
|-------|-------------|-------------|----------------|----------------|
| 0.0   | 1.00000     | 0.00000     | 1.00000        | 0.00000        |
| 0.1   | 0.99949     | 0.18957     | 0.99500        | 0.09983        |
| 0.2   | 0.99687     | 0.35534     | 0.98007        | 0.19867        |
| 0.3   | 0.98990     | 0.49396     | 0.95534        | 0.29552        |
| 0.4   | 0.97620     | 0.60260     | 0.92106        | 0.38942        |
| 0.5   | 0.95336     | 0.67899     | 0.87758        | 0.47943        |
| 0.6   | 0.91893     | 0.72143     | 0.82534        | 0.56464        |
| 0.7   | 0.87051     | 0.72884     | 0.76484        | 0.64422        |
| 0.8   | 0.80577     | 0.70076     | 0.69671        | 0.71736        |
| 0.9   | 0.72254     | 0.63737     | 0.62161        | 0.78333        |
| 1.0   | 0.61883     | 0.53949     | 0.54030        | 0.84147        |
| 1.1   | 0.49289     | 0.40857     | 0.45360        | 0.89121        |
| 1.2   | 0.34324     | 0.24665     | 0.36236        | 0.93204        |
| 1.3   | 0.16875     | 0.05636     | 0.26750        | 0.96356        |
| 1.4   | -0.03138    | -0.15913    | 0.16997        | 0.98545        |
| 1.5   | -0.25752    | -0.39615    | 0.07074        | 0.99749        |
**Example 4.** Consider the system of Volterra integral equations

\[
X_1(t) = e^t - 2t + \int_0^t (e^{-s}X_1(s) + e^sX_2(s)) \, ds
\]

\[
X_2(t) = e^{-t} + \sinh(2t) + \int_0^t (e^sX_1(s) + e^{-s}X_2(s)) \, ds
\]

With the exact solution \(X_1(t) = e^t\) and \(X_2(t) = e^{-t}\).

**Table 4** Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 4.

| \(t\) | \(X_1(t)\) | \(X_2(t)\) | Exact 1 \(= e^t\) | Exact 2 \(= e^{-t}\) |
|---|---|---|---|---|
| 0.00 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.01 | 1.01025 | 1.03025 | 1.01005 | 0.99005 |
| 0.02 | 1.02103 | 1.06102 | 1.02020 | 0.98020 |
| 0.03 | 1.03233 | 1.09232 | 1.03045 | 0.97045 |
| 0.04 | 1.04419 | 1.12417 | 1.04081 | 0.96079 |
| 0.05 | 1.05662 | 1.15658 | 1.05127 | 0.95123 |
| 0.06 | 1.06965 | 1.18957 | 1.06184 | 0.94176 |
| 0.07 | 1.08328 | 1.22316 | 1.07251 | 0.93239 |
| 0.08 | 1.09754 | 1.25737 | 1.08329 | 0.92312 |
| 0.09 | 1.11246 | 1.29221 | 1.09417 | 0.91393 |
| 0.10 | 1.12805 | 1.32771 | 1.10517 | 0.90484 |
| 0.11 | 1.14434 | 1.36389 | 1.11628 | 0.89583 |
| 0.12 | 1.16134 | 1.40076 | 1.12750 | 0.88692 |
| 0.13 | 1.17909 | 1.43835 | 1.13883 | 0.87810 |
| 0.14 | 1.19760 | 1.47669 | 1.15027 | 0.86936 |
| 0.15 | 1.21690 | 1.51578 | 1.16183 | 0.86071 |
Fig. 4 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 4.

**Example 5.** Consider the system of Volterra integral equations

\[
X_1(t) = \sec^2 t - 2 \tan t + t + \int_0^t (X_1(s) + X_2(s))ds
\]

\[
X_2(t) = \tan^2 t - t + \int_0^t (X_1(s) - X_2(s))ds
\]

With the exact solution \(X_1(t) = \sec^2 t\) and \(X_2(t) = \tan^2 t\).

**Table 5 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 5.**

| \(t\) | \(X_1(t)\) | \(X_2(t)\) | \(\text{Exact 1} = \sec^2 t\) | \(\text{Exact 2} = \tan^2 t\) |
|-------|-------------|-------------|-----------------|-----------------|
| 0.00  | 1.00000     | 0.00000     | 1.00000         | 0.00000         |
| 0.01  | 1.00010     | 0.00010     | 1.00010         | 0.00010         |
| 0.02  | 1.00040     | 0.00040     | 1.00040         | 0.00040         |
| 0.03  | 1.00090     | 0.00090     | 1.00090         | 0.00090         |
| 0.04  | 1.00160     | 0.00160     | 1.00160         | 0.00160         |
| 0.05  | 1.00251     | 0.00250     | 1.00250         | 0.00250         |
| 0.06  | 1.00361     | 0.00361     | 1.00361         | 0.00361         |
| 0.07  | 1.00492     | 0.00492     | 1.00492         | 0.00492         |
| 0.08  | 1.00643     | 0.00643     | 1.00643         | 0.00643         |
| 0.09  | 1.00815     | 0.00814     | 1.00814         | 0.00814         |
| 0.10  | 1.01007     | 0.01007     | 1.01007         | 0.01007         |
| 0.11  | 1.01220     | 0.01220     | 1.01220         | 0.01220         |
| 0.12  | 1.01454     | 0.01454     | 1.01454         | 0.01454         |
| 0.13  | 1.01710     | 0.01709     | 1.01709         | 0.01709         |
| 0.14  | 1.01986     | 0.01986     | 1.01986         | 0.01986         |
| 0.15  | 1.02285     | 0.02284     | 1.02284         | 0.02284         |
Conclusion

In this paper, we compute the numerical solution of some examples and compare it with their exact solution. The computed values and graphics, illustrated by the results, agree well with the exact solution.

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