NIL CLEAN DIVISOR GRAPH

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Abstract. In this article, we introduce a new graph theoretic structure associated with a finite commutative ring, called nil clean divisor graph. For a ring \( R \), nil clean divisor graph is denoted by \( G_{NC}(R) \), where the vertex set is \( \{ x \in R : x \neq 0, \exists y(\neq 0, \neq x) \in R \text{ such that } xy \text{ is nil clean} \} \), two vertices \( x \) and \( y \) are adjacent if \( xy \) is a nil clean element. We prove some interesting results of nil clean divisor graph of a ring.

1. Introduction

In this article, rings are finite commutative rings with non zero identity. Diesl [4], introduced the concept of nil clean ring as a subclass of clean ring in 2013. He defined that an element \( x \) of a ring \( R \) to be a nil clean element if it can be written as a sum of an idempotent element and a nilpotent element of \( R \). \( R \) is called nil clean ring if every element of \( R \) is nil clean. Also in 2015, Kosan and Zhou [8], developed the concept of weakly nil clean ring as a generalization of nil clean ring. An element \( x \) of a ring \( R \) is weakly nil clean if \( x = n + e \) or \( x = n - e \), where \( n \) is a nilpotent element and \( e \) is an idempotent element of \( R \). The set of nilpotent elements, set of unit elements, nil clean elements and weakly nil clean elements of a ring \( R \) are denoted by \( \text{Nil}(R) \), \( U(R) \), \( NC(R) \) and \( WNC(R) \) respectively. By graph, we consider simple undirected graph. For a graph \( G \), the set of edges and the set of vertices are denoted by \( E(G) \) and \( V(G) \) respectively. The concept of zero-divisor graph of a commutative ring was introduced by Beck in [3] to discuss the coloring of rings. In 1999, Anderson and Livingston [1], introduced zero divisor graph \( \Gamma(R) \) of a commutative ring \( R \). They defined, the vertex set of \( \Gamma(R) \) to be the set of all non-zero zero divisors of \( R \) and two vertices \( x \) and \( y \) are adjacent if \( xy = 0 \). Li et al. [9], developed a kind of graph structure of a ring \( R \), called nilpotent divisor graph of \( R \), whose vertex set is \( \{ x \in R : x \neq 0, \exists y(\neq 0) \in R \text{ such that } xy \in \text{Nil}(R) \} \) and two vertices \( x \) and \( y \) are adjacent if \( xy \in \text{Nil}(R) \). In 2018, Kimball and LaGrange [7], generalized the concept of zero divisor graph to idempotent divisor graph. For any idempotent \( e \in R \), they defined the idempotent divisor graph \( \Gamma_e(R) \) associated with \( e \), where \( V(\Gamma_e(R)) = \{ a \in R : \text{there exists } b \in R \text{ with } ab = e \} \) and two vertices \( a \) and \( b \) are adjacent if \( ab = e \).

In this article, we introduce nil clean divisor graph \( G_{NC}(R) \) associated with a finite commutative ring \( R \). We define the nil clean divisor graph \( G_{NC}(R) \) of a ring \( R \) by taking \( V(G_{NC}(R)) = \{ x \in R : x \neq 0, \exists y(\neq 0, \neq x) \in R \text{ such that } xy \in NC(R) \} \) as

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the vertex set and two vertices $x$ and $y$ are adjacent if and only if $xy$ is a nil clean element of $R$. Clearly nil clean divisor graph is a generalization of both idempotent divisor graph and nilpotent divisor graph. The properties like girth, clique number, diameter and dominating number etc. of $G_N(R)$ have been studied.

To start with, we recall some preliminaries about graph theory. For a graph $G$, the degree of a vertex $v \in G$ is the number of edges incident to $v$, denoted by $\text{deg}(v)$. The neighbourhood of a vertex $v \in G$ is the set of all vertices incident to $v$, denoted by $A_v$. A graph $G$ is said to be connected, if for any two distinct vertices of $G$, there is a path in $G$ connecting them. Number of edges on the shortest path between vertices $x$ and $y$ is called the distance between $x$ and $y$ and is denoted by $d(x, y)$. If there is no path between $x$ and $y$, then we say $d(x, y) = \infty$. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the maximum of distances of each pair of distinct vertices in $G$. If $G$ is not connected, then we say $\text{diam}(G) = \infty$. Also girth of $G$ is the length of the shortest cycle in $G$, denoted by $\text{gr}(G)$ and if there is no cycle in $G$, then we say $\text{gr}(G) = \infty$. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by an edge. A clique is a subset a of set of vertices of a graph such that its induced subgraph is complete. A clique having $n$ number of vertices is called an $n$-clique. The maximal clique of a graph is a clique such that there is no clique with more vertices. The clique number of a graph $G$ is denoted by $\omega(G)$ and defined as the number of vertices in a maximal clique of $G$.

2. Nil clean divisor graph

We introduce nil clean divisor graph as follows:

**Definition 2.1.** For a ring $R$, nil clean divisor graph, denoted by $G_N(R)$ is defined as a graph with vertex set \( \{ x \in R : x \neq 0, \exists y(\neq 0, \neq x) \in R \text{ such that } xy \in NC(R) \} \) and two vertices $x$ and $y$ are adjacent if $xy \in NC(R)$.

From the above definition, we observe that nil clean divisor graph is a generalization of nilpotent divisor graph, which is again a generalization of zero divisor graph. For any idempotent $e \in R$, nil clean divisor graph of $R$ is also a generalization of $\Gamma_e(R)$. As an example, the nil clean divisor graph $G_N(\mathbb{Z}_6)$ is shown below:

![Figure 1. Nil clean divisor graph of $\mathbb{Z}_6$.](image)

**Theorem 2.2.** The nil clean divisor graph $G_N(R)$ is complete if and only if $R$ is a nil clean ring.
Proof. Let \( G_N(R) \) is a complete and \( x \in R \). If \( x = 0 \), then \( x \) is nil clean, if \( x \neq 0 \) then \( x.1 = x \) is nil clean as \( 1 \in V(G_N(R)) \). Converse is clear from the definition of nil clean divisor graph.

If \( \mathbb{F} \) is a finite field of order \( n \), then clearly \( NC(\mathbb{F}) = \{0,1\} \). Hence for any \( x(\neq 0) \in \mathbb{F} \), \( x \) is adjacent to only \( x^{-1} \), provided \( x \neq x^{-1} \). Hence the nil-clean divisor graph of \( \mathbb{F} \) is as follows:

![Nil clean divisor graph of \( \mathbb{F} \).](image)

Note that \( x_i \neq x_i^{-1} \) and \( y_i \neq y_i^{-1} \), otherwise we may get some isolated point as well in the graph.

**Corollary 2.3.** For a field \( \mathbb{F} \) of order \( n \), where \( n > 2 \). If \( A = \{a \in \mathbb{F} : a = a^{-1}\} \) then the following hold.

1. Diameter of \( \mathbb{F} \) is infinite.
2. \( Gr(G_N(\mathbb{F})) = \infty \) and \( \omega(G_N(\mathbb{F})) = 2 \).
3. \( |V(G_N(\mathbb{F})))| = n - |A| - 1 \).

**Theorem 2.4.** If \( R \) has a non trivial idempotent or non trivial nilpotent element, then the girth of \( G_N(R) \) is 3.

**Proof.** If \( R \) has a non trivial idempotent \( e \), then \( \{0, 1, e, 1 - e\} \subset NC(R) \) and we get a cycle \( 1 - e - (1 - e) - 1 \). Also if \( R \) has a non trivial nilpotent \( n \), then \( \{0, 1, n, n + 1\} \subset NC(R) \). In this case \( 1 - n - (n + 1) - 1 \) is a cycle in \( G_N(R) \). \( \square \)

**Theorem 2.5.** If \( R \) has only trivial idempotents and trivial nilpotent, then girth of \( G_N(R) \) is infinite.

**Proof.** Since \( R \) has only trivial idempotents and trivial nilpotent so by Lemma 2.6 \[2\], \( R \) is a field. Hence the result. \( \square \)

**Theorem 2.6.** Let \( R \) be a ring. Then the following hold.

1. Either \( R \) is a field or \( G_N(R) \) is connected.
2. \( diam(R) = \infty \) or \( diam(R) \leq 3 \).
3. \( gr(G_N(R)) = \infty \) or \( gr(G_N(R)) = 3 \).

**Proof.** Suppose \( R \) is a reduced ring.
Case (I): If \( R \) has no non trivial idempotent, then \( R \) is a field.
Case (II): If $R$ has a non trivial idempotent, say $e \in Idem(R)$, then for any $x, y \in V(G_N(R))$, there exist $x_1, y_1 \in V(G_N(R))$, such that $xx_1, yy_1 \in NC(R) = Idem(R)$. So, we have a path $x - x_1 e - y_1 (1 - e) - y$ from $x$ to $y$.

If $R$ is not a reduced ring, then there exists $n \in Nil(R)$, such that $x - n - y$ is a path from $x$ to $y$, for any $x, y \in V(G_N(R))$. Hence (1) and (2) follow from the above observations and Figure 2.

(3) If $R$ is reduced, then either $R$ is a field or there exists a non trivial idempotent $e \in R$, such that $1 - e - (1 - e) - 1$ is a cycle. So, $gr(G_N(R)) = \infty$ or $gr(G_N(R)) = 3$. If $R$ is a non reduced ring, then since nilpotent graph is a subgraph of nil clean divisor graph, so from Theorem 2.1 [9], $gr(G_N(R)) = 3$.

\[\square\]

**Corollary 2.7.** If $R$ is not a reduced ring, then $diam(R) \leq 2$.

**Corollary 2.8.** A ring $R$ is a field if and only if nil clean divisor graph of $R$ is bipartite.

**Proof.** $\Rightarrow$ Trivial.

$\Leftarrow$ If nil clean divisor graph of $R$ is bipartite then $gr(G_N(R)) \neq 3$. So from Theorem 2.7, $gr(G_N(R)) = \infty$ and hence $R$ is a field. \[\square\]

**Theorem 2.9.** For a ring $R$, the following are equivalent.

1. $G_N(R)$ is a star graph.
2. $R \cong \mathbb{Z}_3$.

**Proof.** The result follows from the fact that $gr(G_N(R)) = \infty$ if and only if $R$ is a field. \[\square\]

**Theorem 2.10.** For any ring $R$, $\omega(G_N(R)) \geq \max\{|\text{Nil}(R)|, |\text{Idem}(R)| - 1\}$.

**Proof.** From the definition of nil clean divisor graph, we observe that $\text{Nil}(R)$ and $\text{Idem}(R)$ respectively induces a complete subgraph of $G_N(R)$. \[\square\]

Next we study about nil clean divisor graph of weakly nil clean ring.

**Theorem 2.11.** Let $R$ be a weakly nil clean ring which is not nil clean. Then $\omega(G_N(R)) \geq |\mathbb{R}| - 1$ and $diam(R) = 2$ if $|R| > 3$ is even, where $[x]$ is the greatest integer function.

**Proof.** As $x \in WNC(R)$ implies $-x \in NC(R)$, so if $|R|$ is even, then $|NC(R)| \geq \frac{|R|}{2}$ and if $|R|$ is odd, then $|NC(R)| \geq \frac{|R|+1}{2}$. Since $R$ is commutative, so product of any two nil clean element is also a nil clean element. Hence $\omega(G_N(R)) \geq \frac{|R|}{2}$.

Since $|R| > 3$, so $R$ is not a field and hence $G_N(R)$ is connected. As $|R \setminus \{0\}|$ is odd, so there exists an element $a \in R$ such that $x \in NC(R) \cap WNC(R)$. Hence for any $x, y \in R$, $x - a - y$ is a path in $G_N(R)$ and $diam(G_N(R)) = 2$ as $R$ is not a nil clean ring. \[\square\]
3. NIL CLEAN DIVISOR GRAPH OF $\mathbb{Z}_{2p}$ AND $\mathbb{Z}_{3p}$, FOR ANY ODD PRIME $p$

In this section we study the structures of $G_N(\mathbb{Z}_{2p})$ and $G_N(\mathbb{Z}_{3p})$, for any odd prime $p$.

**Lemma 3.1.** If $a \in V(G_N(\mathbb{Z}_{2p}))$, where $p$ is an odd prime, then the following hold.

1. If $a = p$, then $\deg(a) = 2p - 2$.
2. If $a \in \{1, p - 1, p + 1, 2p - 1\}$, then $\deg(a) = 2$.
3. Otherwise $\deg(a) = 3$

**Proof.** Clearly $NC(\mathbb{Z}_{2p}) = \{0, 1, p, p + 1\}$.

1. If $a = p$, then for any $y \in V(G_N(\mathbb{Z}_{2p}))$, either $yp = p$ or $yp = 0$. Hence every element of $V(G_N(\mathbb{Z}_{2p}))$ is adjacent to $p$.
2. It is easy to observe that, $A_1 = \{p, p + 1\}$, $A_{p-1} = \{p, 2p - 1\}$, $A_{p+1} = \{1, p\}$ and $A_{2p-1} = \{p - 1, p\}$.
3. Let $a \in \mathbb{Z}_{2p} \setminus \{0, 1, p - 1, p, p + 1, 2p - 1\}$.

Case (I): Let $a$ be an even number. If $ax = 0$ in $\mathbb{Z}_{2p}$, then it has two solutions 0 and $p$. If $ax = 1$ in $\mathbb{Z}_{2p}$, then it has no solution, since $\gcd(2p, a) = 2 \nmid 1$. If $ax = p$ in $\mathbb{Z}_{2p}$, then also it has no solution, since $\gcd(2p, a) = 2 \nmid p$. If $ax = p + 1$ in $\mathbb{Z}_{2p}$, then it has two distinct solutions $x_1$ and $x_2$ in $\mathbb{Z}_{2p}$, since $\gcd(2p, a) = 2 \mid p + 1$. Hence we conclude that $A_a = \{p, x_1, x_2\}$.

Case (II): Let $a$ be an odd number. If $ax = 0$ in $\mathbb{Z}_{2p}$, then it has a unique solution $x = 0$. If $ax = 1$ in $\mathbb{Z}_{2p}$, then it has unique odd solution $x = y_1$ in $\mathbb{Z}_{2p}$, since $\gcd(2p, a) = 1 \mid 1$. If $ax = p$ in $\mathbb{Z}_{2p}$, then it has unique solution $x = p$, since $\gcd(2p, a) = 1 \mid p$. If $ax = p + 1$ in $\mathbb{Z}_{2p}$, then it has unique even solution $x = y_2$ in $\mathbb{Z}_{2p}$, since $\gcd(2p, a) = 1 \mid p + 1$. Hence $A_a = \{p, y_1, y_2\}$

From the above cases it follows $\deg(a) = 3$.

\[\square\]

**Remark 3.2.** In the proof of Lemma 3.1 (3), Case(I), since $ax_1 = ax_2$ in $\mathbb{Z}_{2p}$, so $x_1 - x_2 = 0$ or $p$, but $x_1 - x_2 \neq 0$ as $x_1$ and $x_2$ are distinct. Hence if $x_1$ is odd, then $x_2$ is even and if $x_1$ is even, then $x_2$ is odd.

From Lemma 3.1 and Remark 3.2 for any prime $p > 2$, the nil clean divisor graph of $\mathbb{Z}_{2p}$ is the following:

![Figure 3. Nil clean divisor graph of $\mathbb{Z}_{2p}$.](image-url)
In Figure 2, \(a_i\) and \(b_i\) are even numbers from \(\mathbb{Z}_{2p} \setminus \{0, 1, p-1, p, p+1, 2p-1\}\) such that \(a_i b_i = p + 1\), for \(1 \leq i \leq \frac{p-3}{2}\). Also \(c_i = a_i + p\) and \(d_i = b_i + p\), for \(1 \leq i \leq \frac{p-3}{2}\). From the above observations we conclude the following:

**Theorem 3.3.** The following hold for nil clean divisor graph \(G_N(\mathbb{Z}_{2p})\), for any odd prime \(p\).

1. Clique number of \(G_N(\mathbb{Z}_{2p})\) is 3.
2. Diameter of \(G_N(\mathbb{Z}_{2p})\) is 2.
3. Girth of \(G_N(\mathbb{Z}_{2p})\) is 3.
4. \(\{p\}\) is the unique smallest dominating set for \(G_N(\mathbb{Z}_{2p})\), that is, dominating number of the graph is 1.

Next we study about nil clean divisor graph of \(\mathbb{Z}_{3p}\). Here we study the graph theoretic properties of \(G_N(\mathbb{Z}_{3p})\).

**Lemma 3.4.** In \(G_N(\mathbb{Z}_{3p})\); where \(p \equiv 2(\text{mod}3)\), the following hold.

1. \(\deg(3k) = 5\) if \(3k \not\equiv \{p + 1, 2p - 1\}\), for \(1 \leq k \leq p - 1\).
2. \(\deg(p + 1) = \deg(2p - 1) = 4\).

**Proof.** Here \(NC(\mathbb{Z}_{3p}) = \{0, 1, p + 1, 2p\}\). Observe that \(3k.x \equiv 1(\text{mod}3p)\) and \(3k.x \equiv 2p(\text{mod}3p)\) has no solution, as \(gcd(3k, 3p) = 3\) does not divide 1 and 2p. The congruence \(3k.x \equiv 0(\text{mod}3p)\) has three incongruent solutions \(\{0, p, 2p\}\) in \(\mathbb{Z}_{3p}\). Also \(3k.x \equiv p + 1(\text{mod}3p)\) has three distinct incongruent solutions in \(\mathbb{Z}_{3p}\), as \(gcd(3k, 3p) = 3\) divides \(p + 1\).

1. As \(x^2 \equiv p + 1(\text{mod}3p)\), has two solutions \(p + 1\) and \(2p - 1\), hence if \(3k \not\equiv \{p + 1, 2p - 1\}\), then \(\deg(3k) = 6 - 1 = 5\), as \(0 \not\in V(G_N(\mathbb{Z}_{3p}))\).
2. If \(3k \in \{p + 1, 2p - 1\}\), then \(\deg(3k) = 6 - 2\), as \(0 \not\in V(G_N(\mathbb{Z}_{3p}))\) and we do not consider any loop.

**Lemma 3.5.** In \(G_N(\mathbb{Z}_{3p})\), where \(p \equiv 2(\text{mod}3)\) the following hold.

1. \(\deg(p) = \deg(2p) = 2p - 2\).
2. For \(x \in \{1, p - 1, 3p - 1, 2p + 1\}\), \(\deg(x) = 2\).
3. For \(x \in \mathbb{Z}_{3p} \setminus L\), \(\deg(x) = 3\), where \(L = \{3k : 1 \leq k \leq p - 1\} \cup \{1, p - 1, 2p + 1, 3p - 1, p, 2p\}\).

**Proof.** Here \(NC(\mathbb{Z}_{3p}) = \{0, 1, p + 1, 2p\}\).

1. Clearly \(p.x \equiv 1(\text{mod}3p)\) and \(p.x \equiv p + 1(\text{mod}3p)\) have no solution as \(gcd(3p, p)\) does not divide 1 and \(p + 1\). Also \(p.x \equiv 0(\text{mod}3p)\) has \(p\) incongruent solutions \(\{3k : 0 \leq k \leq p - 1\}\) and \(p.x \equiv 2p(\text{mod}3p)\) has \(p\) incongruent solutions \(\{3k + 2 : 0 \leq k \leq p - 1\}\). Since \(0 \not\in V(G_N(\mathbb{Z}_{3p}))\) and \(p\) is of the form \(3i + 2\), for some \(0 \leq i \leq p - 1\), hence \(\deg(p) = 2p - 2\). Now \(2p.x \equiv 0(\text{mod}3p)\) has \(p\) incongruent solutions \(\{3k : 0 \leq k \leq p - 1\}\) and \(2p.x \equiv 2p(\text{mod}3p)\) has \(p\) incongruent solutions \(\{3k + 1 : 0 \leq k \leq p - 1\}\). But \(2p.x \equiv 1(\text{mod}3p)\) and \(2p.x \equiv p + 1(\text{mod}3p)\) have no solutions. Hence \(\deg(2p) = 2p - 2\), since \(2p\) is of the form \(3i + 1\), for some \(1 \leq i \leq p - 1\).
(2) Since \( x \equiv a(\text{mod } 3p) \), has only one solution \( a \), hence \( \deg(1) = 2 \). Also \((3p - 1)x \equiv c(\text{mod } 3p)\) has only one solution \((3p - 1)a\), hence \( \deg(3p - 1) = 2 \), as \( 0 \notin V(G_N(\mathbb{Z}_{3p})) \) and \( 3p - 1 \in U(\mathbb{Z}_{3p}) \). Equation \((p - 1)x \equiv 1(\text{mod } 3p)\) and \((2p + 1)x \equiv c(\text{mod } 3p)\) have a unique solutions, where \( c \in \{0, 1, 2p, p + 1\} \). Since \( p - 1, 2p + 1 \in U(\mathbb{Z}_{3p}) \), so \( \deg(p - 1) = \deg(2p + 1) = 2 \).

(3) Let \( a \in \mathbb{Z}_{3p} \setminus L \). As \( \gcd(a, 3p) = 1 \), so \( a.x \equiv 0(\text{mod } 3p) \) has a unique solution \( x = 0 \). Also \( a.x \equiv c(\text{mod } 3p) \), where \( c \in \{1, 2p, p + 1\} \) has a unique solution. Hence \( \deg(a) = 3 \).

\[\square\]

From Lemma \[3.3\] and Lemma \[3.5\] for any prime \( p > 3 \) with \( p \equiv 2(\text{mod } 3) \), the nil clean divisor graph of \( \mathbb{Z}_{3p} \) is the following:

![Figure 4: Nil clean divisor graph of \( \mathbb{Z}_{3p} \), where \( p \equiv 2(\text{mod } 3) \).](image)

In Figure \[4\] \( \{l_i, k_i\} \subseteq \{3k : 1 \leq k \leq p - 1\} \), \( a_i c_i \equiv 1(\text{mod } 3p) \), \( b_i d_i \equiv 1(\text{mod } 3p) \) and \( a_i k_i \equiv c_k l_i \equiv b_l k_i \equiv d_l i \equiv p + 1(\text{mod } 3p) \), for \( 1 \leq i \leq \frac{p - 3}{2} \). Also \( a_i \equiv c_i \equiv 1(\text{mod } 3) \) and \( b_i \equiv d_i \equiv 2(\text{mod } 3) \), for \( 1 \leq i \leq \frac{p - 3}{2} \).

**Theorem 3.6.** For any prime \( p \), where \( p \equiv 2(\text{mod } 3) \), the following hold for \( G_N(\mathbb{Z}_{3p}) \).

1. Girth of \( G_N(\mathbb{Z}_{3p}) \) is 3.
2. Clique number of \( G_N(\mathbb{Z}_{3p}) \) is 3.
3. Diameter of \( G_N(\mathbb{Z}_{3p}) \) is 3.
4. \( \{p, 2p\} \) is the unique smallest dominating set for \( G_N(\mathbb{Z}_{3p}) \), that is, dominating number of the graph is 2.

**Proof.** Clearly \( NC(\mathbb{Z}_{3p}) = \{0, 1, p + 1, 2p\} \).

1. Since \( p - (p + 1) - (2p + 1) - p \) is a cycle of \( G_N(\mathbb{Z}_{3p}) \), so girth of \( G_N(\mathbb{Z}_{3p}) \) is 3.
2. If possible, let \( \omega((G_N(\mathbb{Z}_{3p}))) = 4 \). Then there exists \( A = \{a_i : 1 \leq i \leq 4\} \subseteq V(G_N(\mathbb{Z}_{3p})) \) such that \( A \) forms a complete subgraph of \( G_N(\mathbb{Z}_{3p}) \). If \( x \in \mathbb{Z}_{3p} \setminus \{p, 2p, 3k : 1 \leq k \leq p - 1\} \), then \( \deg(x) \leq 3 \). Also \( x \) is adjacent to either \( p \) or \( 2p \), \( x^{-1} \) and \( 3i \), for some \( 1 \leq i \leq p - 1 \) (provided \( x \notin \{1, p - 1, 2p + 1, 3p - 1\} \)). But \( x^{-1} \) is also adjacent to \( 3j \), for some \( 1 \leq j \leq p - 1 \) such that \( i \neq j \). So \( A \subseteq \{p, 2p, 3k : 1 \leq k \leq p - 1\} \). Suppose \( a_1 = 3k \), for some \( 1 \leq k \leq p - 1 \). From Figure \[4\] \( A_{a_1} = \{p, 2p, 3i + 1, 3j + 2, 3s\} \), where \( 1 \leq i, j, s \leq p - 1 \), also \( 3s \notin A_{3i + 1}, 3s \notin A_{3j + 2}, 3i + 1 \notin A_{3j + 2}, p \notin A_{2p}, 2p \notin A_{3j + 2} \) and
Lemma 3.7. In $G_N(\mathbb{Z}_{3p})$; where $p \equiv 1 \pmod{3}$, the following hold.

1. $\deg(3k) = 5$ if $3k \notin \{p - 1, 2p + 1\}$, for $1 \leq k \leq p - 1$.
2. $\deg(p - 1) = \deg(2p + 1) = 4$.

Proof. Proof is similar to the proof of Lemma 3.4.

Lemma 3.8. In $\mathbb{Z}_{3p}$, where $p \equiv 1 \pmod{3}$, the following hold.

1. $\deg(p) = \deg(2p) = 2p - 2$.
2. For $x \in \{1, p + 1, 3p - 1, 2p - 1\}$, $\deg(x) = 2$.
3. For $x \in \mathbb{Z}_{3p} \setminus L$, $\deg(x) = 3$, where $L = \{3k : 1 \leq k \leq p - 1\} \cup \{1, p, 2p, p + 1, 2p - 1, 3p + 1\}$.

Proof. Proof is similar to the proof Lemma 3.5.

From Lemma 3.7 and Lemma 3.8, the nil clean divisor graph of $\mathbb{Z}_{3p}$, where $p \equiv 1 \pmod{3}$ is the following:

![Diagram of nil clean divisor graph of $\mathbb{Z}_{3p}$](image)

Figure 5. Nil clean divisor graph of $\mathbb{Z}_{3p}$, where $p \equiv 1 \pmod{3}$.

In Figure 5, $\{l_i, k_i\} \subseteq \{3k : 1 \leq k \leq p - 1\}$, $a_i c_i \equiv 1 \pmod{3p}$, $b_i d_i \equiv 1 \pmod{3p}$ and $a_i k_i \equiv c_i l_i \equiv b_i k_i \equiv d_i l_i \equiv 2p + 1 \pmod{3p}$, for $1 \leq i \leq \frac{p - 3}{2}$. Also $a_i \equiv c_i \equiv 2 \pmod{3}$ and $b_i \equiv d_i \equiv 1 \pmod{3}$, for $1 \leq i \leq \frac{p - 3}{2}$. Hence we get the following theorem:

Theorem 3.9. The following hold for $G_N(\mathbb{Z}_{3p})$, for any prime $p$, where $p \equiv 1 \pmod{3}$.

1. Girth of $G_N(\mathbb{Z}_{3p})$ is 3.
2. Clique number of $G_N(\mathbb{Z}_{3p})$ is 3.
3. Diameter of $G_N(\mathbb{Z}_{3p})$ is 3.
4. $\{p, 2p\}$ is the unique smallest dominating set for $G_N(\mathbb{Z}_{3p})$, that is, dominating number of the graph is 2.
Proof. Since Figure 4 and Figure 5 are similar, hence the proof is similar to the proof of Theorem 3.6.

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