Bending, Buckling and Vibrations Analysis of the Graphene Nanoplate Using the Modified Couple Stress Theory

Majid ESKANDARI SHAHRAKI*, Mahmoud SHARIATI**, Naser ASIABAN***, Jafar ESKANDARI JAM****

*Aerospace Engineering, Ferdowsi University of Mashhad, Mashhad, Iran, E-mail: mjdeskindari@gmail.com
**Mechanical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran, E-mail: mshariati44@gmail.com
***Mechanical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran, E-mail: naser.asiaban@mail.um.ac.ir
****Mechanical Engineering, Malek-Ashstar University of Technology, Tehran, Iran, E-mail: jejam@mail.com

1. Introduction

The atomic and molecular scale test is known as the safest method for the study of materials in small-scales. In this method, the nanostructures are studied in real dimensions. The atomic force microscopy (AFM) is used to apply different mechanical loads on nanoplates and measure their responses against those load in order to determine the mechanical properties of the nanoplate. The difficulty of controlling the test conditions at this scale, high economic costs and time-consuming processes are some setbacks of this method. Therefore, it is used only to validate other simple and low-cost methods.

Atomic simulation is another solution for studying small-scale structures. In this method, the behavior of atoms and molecules is examined by considering the intermolecular and interatomic effects on their motions, which eventually involves the total deformation of the body. In the case of large deformations and multi atomic scale the computational costs is too high, so this method is only used for small deformation problems.

Given the limitations of the aforementioned methods for studying nanostructures, researchers have been looking for simpler solutions for nanostructures. Modeling small-scale structures using continuum mechanics is another solution to this problem. There are a variety of size-dependent continuum theories that consider size effects, some of these theories are: micromorphic theory, micromorphic theory, micropolar theory, Kurt’s theory, non-local theory, modified couple stress theory and strain gradient elasticity. All of which are the developed notion of classical field theories, which include size effects.

2. Modified couple stress theory

In 2002 Yang et al. [1] proposed a modified couple stress model by modifying the theory proposed by Toppin [2], Mindlin and Thursten [3], Quitter [4] and Mindlin [5] in 1964. The modified couple stress theory consists of one material length scale parameter for projection of the size effect, whereas the classical couple stress theory has two material length scale parameters. In the modified couple stress theory, the strain energy density in the three-dimensional volume is expressed as the follow:

\[ U = \frac{1}{2} \int_V \left( \sigma_{ij} \epsilon_{ij} + m_{ij} \chi_{ij} \right) dV \quad i, j = 1, 2, 3 \]  

where:

\[ \epsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right) \]  

\[ \chi_{ij} = \frac{1}{2} \left( \theta_{ij} + \theta_{ji} \right) \]

\[ \sigma_{ij}, \text{ the stress tensor, and } m_{ij}, \text{ the deviatory part of the couple stress tensor, are defined as:} \]

\[ \sigma_{ij} = \lambda \epsilon_{ik} \delta_{kj} + 2 \mu \epsilon_{ij} \]  

\[ m_{ij} = 2 \mu l^2 \chi_{ij} \]

where: \( \lambda \) and \( \mu \) are the lame constants; \( \delta_{ij} \) is the Kronecker delta and \( l \) is the material length scale parameter. From Eqs. (3) and (6) it can be seen that \( \chi_{ij} \) and \( m_{ij} \) are symmetric.

3. Mindlin’s plate model

The displacement equations for the Mindlin’s plate model are defined as [8]:

Fig. 1 A schematic of the nanoplate and axes
where: $\varphi_{x}$ and $\varphi_{y}$ are the rotations of the normal vector around the \(x\) and \(y\) axis respectively, and \(w\) is the midpoint displacement of the plate in the \(z\)-axis direction. The strain and stress tensors, the symmetric part of the curvature tensor, and the rotational vector for the Mindlin's plate is obtained as follows:

\[
\varepsilon_{xx} = z \frac{\partial \varphi_{x}}{\partial x},
\]

\[
\varepsilon_{yy} = z \frac{\partial \varphi_{y}}{\partial y},
\]

\[
\varepsilon_{zz} = 0,
\]

\[
\varepsilon_{x y} = \frac{1}{2} \left( \frac{\partial \varphi_{x}}{\partial y} + \frac{\partial \varphi_{y}}{\partial x} \right),
\]

\[
\varepsilon_{x z} = \frac{1}{2} \left( \frac{\partial \varphi_{z}}{\partial x} + \frac{\partial \varphi_{x}}{\partial z} \right),
\]

\[
\varepsilon_{y z} = \frac{1}{2} \left( \frac{\partial \varphi_{y}}{\partial z} + \frac{\partial \varphi_{z}}{\partial y} \right),
\]

The strain energy in the above equation is expressed as follows:

\[
\delta U = \int \left[ \sigma_{x x} \varepsilon_{x x} + \sigma_{y y} \varepsilon_{y y} + \sigma_{z z} \varepsilon_{z z} + 2\sigma_{x y} \varepsilon_{x y} + \sigma_{x z} \varepsilon_{x z} + \sigma_{y z} \varepsilon_{y z} \right] dV,
\]

For the sake of simplification, the coefficient of each variable in the above equation is named from \(F_{i}\) to \(F_{15}\) and this equation can be rewritten as shown below:

\[
\delta U = \int \left[ F_{1} \delta w_{x x} + F_{2} \delta w_{y y} + F_{3} \delta w_{z z} + F_{4} \delta w_{x y} + F_{5} \delta w_{x z} + F_{6} \delta w_{y z} \right] dV.
\]

where:

\[
F_{i} = \frac{1}{4} \mu l^{2} \left( \frac{\partial^{2} \varphi_{i}}{\partial y^{2}} + \frac{\partial \varphi_{i}}{\partial x} - \frac{\partial \varphi_{i}}{\partial y} \right),
\]

\[
F_{2} = \frac{1}{4} \mu l^{2} \left( \frac{\partial^{2} \varphi_{i}}{\partial x^{2}} + \frac{\partial \varphi_{i}}{\partial y} - \frac{\partial \varphi_{i}}{\partial x} \right),
\]

\[
F_{3} = \frac{1}{4} \mu l^{2} \left( \frac{\partial^{2} \varphi_{i}}{\partial x^{2}} + \frac{\partial \varphi_{i}}{\partial y} - \frac{\partial \varphi_{i}}{\partial x} \right),
\]

\[
F_{4} = \mu \left( \frac{\partial \varphi_{i}}{\partial x} \right),
\]
378

\[ F_s = \mu \left( \frac{\partial w}{\partial y} + \phi_s \right), \]  
\[ F_x = 4 \mu d^2 z^2 \left( \frac{\partial^2 \phi_x}{\partial x^2} - \frac{\partial^2 \phi_s}{\partial y^2} \right), \]  
\[ F_y = F_y = 4 \mu d^2 z^2 \left( \frac{\partial^2 \phi_s}{\partial x^2} - \frac{\partial^2 \phi_s}{\partial y^2} \right), \]  
\[ F_{10} = (\lambda + 2 \mu) \frac{\partial \phi_s}{\partial z} + \lambda z \frac{\partial \phi_s}{\partial y} + \frac{1}{4} \mu d^2 \left( \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} + \frac{\partial \phi_s}{\partial x} - \frac{\partial \phi_s}{\partial y} \right), \]  
\[ F_{12} = \mu z^2 \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_s}{\partial y} \right) + \mu d^2 \left( \frac{\partial \phi_x}{\partial y} + \frac{1}{2} \frac{\partial \phi_s}{\partial y} - \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} \right), \]  
\[ F_{13} = \mu z^2 \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_s}{\partial y} \right) + \mu d^2 \left( \frac{\partial \phi_x}{\partial y} - \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} - \frac{1}{2} \frac{\partial^2 \phi_s}{\partial y^2} \right), \]  
\[ F_{14} = \mu \left( \frac{\partial w}{\partial y} + \phi_s \right), \]  
\[ F_{15} = \mu \left( \frac{\partial w}{\partial y} + \phi_s \right). \]  

4. The buckling force

For a rectangular plate with length \( a \), width \( b \) and thickness \( h \), under the axial forces \( \{ P_x, P_y, P_z \} \), the buckling force is obtained as shown in equation (44) [7]:

\[ P_x \left( \frac{\partial^3 w}{\partial x^3} + 2 P_y \frac{\partial^3 w}{\partial x \partial y^2} + P_z \frac{\partial^2 w}{\partial y^2} \right) = q(x, y), \]  

where: \( P_x \) is the Axial force along the x axis; \( P_y \) is the Axial force along the y axis; \( P_z \) is the shearing force in the xy plane, and \( q(x, y) \) is the out-of-plane force.

5. Virtual work of the external forces

In these kind of problems, the virtual work of three kinds of external forces are included in the solutions, if the middle-plane and the middle-perimeter of the plate are shown as \( \Omega \) and \( \Gamma \) respectively, these virtual works are [8]:

1. The virtual work done by the body forces, which is applied on the volume \( V = \Omega \times (-h/2, h/2) \).
2. The virtual work done by the surface tractions on the upper and lower surfaces \( \Omega \).
3. The virtual work done by the shear tractions on the lateral surfaces, \( S = \Gamma \times (-h/2, h/2) \).

If \((f_x, f_y, f_z)\) are the body forces, \((r_x, r_y, r_z)\) are the forces acting on the \( \Omega \) plane, \((t_x, t_y, t_z)\) are the Cauchy’s tractions and \((S_x, S_y, S_z)\) are surface couples the Variations of the virtual work is expressed as:

\[ \delta w = -\left[ \int_{\Omega} (f_x \delta u + f_y \delta v + f_z \delta w + q_x \delta u + q_y \delta v + + q_z \delta w + c_x \delta \theta_x + c_y \delta \theta_y + c_z \delta \theta_z) \, dx \, dy \right] + \int_{\Gamma} (t_x \delta u + t_y \delta v + + t_z \delta w + s_x \delta \theta_x + s_y \delta \theta_y + s_z \delta \theta_z). \]  

Given that in this study only the external force \( q \) was applied, virtual work becomes:

\[ \delta w = \int_{\Omega} \int_{\Gamma} q(x, y) \delta w(x, y) \, dx \, dy, \]  

the variation of kinetic energy is obtained as:

\[ \delta T = \int_{\Omega} \int_{\Gamma} \rho \left( \dot{u} \dot{\delta u} \dot{v} \dot{\delta v} \dot{\delta w} \dot{u} \dot{\delta u} + \dot{u} \dot{\delta v} \dot{u} \dot{\delta v} + \dot{u} \dot{\delta w} \dot{u} \dot{\delta w} \right) \, dx \, dy, \]  

where: \( \rho \) is the density.

Finally using the Hamilton’s principle, it can be said that [9]:

\[ \int_{0}^{T} (\delta T - (\delta U - \delta w)) \, dt = 0, \]  

where: \( T \) is the kinetic energy; \( U \) is the strain energy, and \( w \) is the work of the external forces.

6. The final governing equations of the plate after applying the buckling and external forces

Using Hamilton’s principle, Eq. (48), and the Eqs. from (44) to (47), the governing equations of the plate including the buckling and external forces are obtained as follows:

\[ \begin{bmatrix} \chi_i \left( \frac{\partial^2 F_1}{\partial x_i^2} \frac{\partial F_1}{\partial x} + \frac{\partial^2 F_2}{\partial y^2} \frac{\partial F_1}{\partial y} + \frac{\partial^2 F_3}{\partial x \partial y} \frac{\partial F_1}{\partial x \partial y} \right) \right] = q(x, y) + \rho h \frac{\partial^2 w}{\partial t^2}, \]  

\[ \int_{\Omega} \int_{\Gamma} \left( \frac{\partial^2 F_1}{\partial x_i^2} + \frac{\partial^2 F_2}{\partial y^2} - \frac{\partial F_2}{\partial y} - \frac{\partial F_3}{\partial y} + \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial y} \right) \, dx \, dy = \rho h \frac{\partial^2 w}{\partial t^2}. \]
\[ \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F_{13}}{\partial x \partial y} + \frac{\partial^3 F_{3}}{\partial x^2 \partial y} + \frac{\partial^3 F_{11}}{\partial x \partial y} - F_{13} = \frac{\rho h^3}{12} \partial^3 \varphi_x. \] (51)

7. Obtaining the general governing equation of the Mindlin's plate (including buckling, bending and vibrations)

Considering the following constants:

\[ C_1 = \frac{1}{4} \mu l^2 h, \] (52)
\[ C_2 = \mu h k_i, \] (53)
\[ C_3 = \frac{1}{4} \mu l^2 \Gamma^2, \] (54)
\[ C_4 = -\mu I_2 - \mu l^2 h, \] (55)
\[ C_5 = -\lambda I_2 - 2\mu I_2 - \frac{1}{4} \mu l^2 h, \] (56)
\[ C_6 = -\mu I_2 - \lambda I_2 + \frac{3}{4} \mu l^2 h, \] (57)
\[ C_7 = \rho h, \] (58)
\[ C_8 = \frac{\rho h^3}{12}, \] (59)
\[ \kappa = \frac{5}{6} = 0.8 \] (60)

where:

\[ I_i = \frac{1}{2} Z_i' dz \] (61)

the general governing equation of the Mindlin's plate will become:

\[ \begin{align*}
2C_1 \frac{\partial^2 w}{\partial x^2} + C_1 \frac{\partial^2 w}{\partial x \partial y} + C_1 \frac{\partial^2 w}{\partial y^2} - C_2 \frac{\partial^3 w}{\partial x^3} - C_2 \frac{\partial^3 w}{\partial x^2 \partial y} - C_2 \frac{\partial^3 w}{\partial x \partial y^2} - C_2 \frac{\partial^3 w}{\partial y^3} &= q(x, y) + C_7 \frac{\partial^2 w}{\partial t^2}, \\
C_3 \left( \frac{\partial^4 \varphi_x}{\partial x \partial y^3} + \frac{\partial^4 \varphi_y}{\partial x^3 \partial y} + \frac{\partial^4 \varphi_y}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi_y}{\partial x \partial y^3} \right) + C_4 \frac{\partial^3 \varphi_y}{\partial y^2} + C_3 \frac{\partial^3 \varphi_y}{\partial x \partial y} + C_4 \frac{\partial^3 \varphi_y}{\partial x \partial y^2} &= + C_1 \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + C_2 \frac{\partial^2 \varphi_y}{\partial x \partial y} + C_3 \frac{\partial^2 \varphi_y}{\partial x^2} + C_4 \frac{\partial \varphi_y}{\partial y}, \\
C_3 \left( \frac{\partial^4 \varphi_x}{\partial x^2 \partial y} + \frac{\partial^4 \varphi_y}{\partial x^3 \partial y} + \frac{\partial^4 \varphi_y}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi_y}{\partial x \partial y^3} \right) &+ C_4 \frac{\partial^3 \varphi_x}{\partial y^2} + C_3 \frac{\partial^3 \varphi_x}{\partial x \partial y} + C_4 \frac{\partial^3 \varphi_x}{\partial x \partial y^2} + C_1 \frac{\partial^3 \varphi_x}{\partial x^2 \partial y} + C_2 \frac{\partial^2 \varphi_x}{\partial x \partial y} + C_3 \frac{\partial^2 \varphi_x}{\partial x^2} &= + C_4 \frac{\partial \varphi_x}{\partial y}.
\end{align*} \] (62)

8. Solution of the governing equations using Navier's method

The Navier's solution is applicable to the rectangular plates which have simply supported boundary conditions on all edges. Since the boundary conditions are spontaneously satisfied in this method, the unknown functions of the plate's mid-plane were assumed to be double trigonometric series [8]:

\[ W(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \alpha_m \sin \beta_n y \sin \omega t, \] (65)
\[ \varphi_x(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn} \cos \alpha_m \sin \beta_n y \sin \omega t, \] (66)
\[ \varphi_y(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn} \sin \alpha_m \cos \beta_n y \sin \omega t. \] (67)

The force can also be calculated from the following relations:

\[ q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \alpha_m \sin \beta_n, \] (68)
\[ Q_{mn} = \frac{4}{ab} \int_{0}^{b} \int_{0}^{a} q(x, y) \sin \alpha_m \sin \beta_n y \sin \omega t \sin \omega t, \] (69)
\[ Q_{mn} = \begin{cases} \frac{16 Q_{mn}}{\pi \mu a^2} : \text{For sinusoidal force} \\ \frac{4 Q_{mn} \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{2} : \text{For uniform force} \\ \text{For point force in the plane center} \end{cases}, \] (70)

where:

\[ a = \frac{\pi m}{a}, \beta = \frac{\pi n}{b}, i = \sqrt{-1}. \] (71)

Simply-supported boundary conditions were also satisfied by the Navier's method according to the following equations:

\[ \begin{align*}
&x = 0, x = a, \quad \varphi_x(0, y) = \varphi_x(a, y) = 0 \\
&w(0, y) = w(a, y) = 0 \\
&\varphi_y(0, y) = \varphi_y(a, y) = 0.
\end{align*} \] (72)
9. The general equation matrix of a Mindlin’s plate

After solving the governing equations and naming the coefficient of each variable, we have:

\[
U = 2C \alpha \beta + C \alpha + C \beta + C \alpha + C \beta ,
\]

(74)

\[
U = U = -C \alpha - C \alpha \beta + C \alpha ,
\]

(75)

\[
U_1 = U_2 = -C \beta_1 - C \alpha_2 \beta + C \beta ,
\]

(76)

\[
U_3 = -C \beta_4 - C \alpha_3 \beta^2 - C \beta^2 - C \alpha_3 \beta + C \beta_1 ,
\]

(77)

\[
U_4 = C \alpha \beta^3 + C \alpha \beta^3 - C \alpha \beta_2 - C \alpha \beta + C \beta_1 ,
\]

(78)

\[
U_5 = -C \alpha \beta^3 - C \alpha \beta^3 - C \alpha \beta_2 - C \alpha_2 \beta + C \beta_1 ,
\]

(79)

\[
U_6 = C \alpha \beta^3 + C \alpha \beta^3 - C \alpha \beta_2 - C \alpha_2 \beta + C \beta_1 ,
\]

(80)

\[
K_1 = -C_0 ,
\]

(81)

\[
K_2 = K_3 = K_4 = K_5 = K_6 = K_7 = K_8 = 0 ,
\]

(82)

\[
K_9 = K_0 = -K_9 .
\]

(83)

Finally, the general equation matrix of the Mindlin’s plate along with the auxiliary equations will be obtained as follows:

\[
\begin{bmatrix}
U_1 & U_2 & U_3 \\
U_4 & U_5 & U_6 \\
U_7 & U_8 & U_9
\end{bmatrix} - \alpha^2 \begin{bmatrix}
K_1 & K_2 & K_3 \\
K_4 & K_5 & K_6 \\
K_7 & K_8 & K_9
\end{bmatrix} \begin{bmatrix}
\omega_{xx} \\
\omega_{yy} \\
\omega_{xy}
\end{bmatrix} = \begin{bmatrix}
Q_{xx} \\
Q_{yy} \\
Q_{xy}
\end{bmatrix}_0.
\]

(84)

In this study, graphene is chosen as the plate’s material. A single-layer graphene plate has the following properties [9]: \( E = 1.06 \) TPa, \( v = 0.25, h = 0.34 \) nm, \( r = 2250 \) kg/m^2.

Also, the relationship between \( E, \mu \) and \( \nu \) can be expressed as:

\[
\lambda = \frac{vE}{(1 + v)(1 - 2v)}, \quad \mu = \frac{E}{2(1 + v)}.
\]

(85)

where: \( \mu \) and \( \lambda \) are the Lamé's coefficients; \( E \) is the Young's modulus [10]. The value of the distributed force was considered to be \( q = 1N/m \).

10. Results and discussion

Results were obtained using a computational program coded in the MATLAB software. The results have also been compared with the literature [11, 12] and good agreements between results were observed. The plate’s dimensional parameters are chosen as follows: \( A \) is plate’s length; \( b \) is plate’s width; \( h \) is plate’s thickness; \( t \) is material length scale parameter.

Table 1 shows the Mindlin’s nanoplate bending rate under sinusoidal load for different material length scale parameters to thickness \( h/b \) and length to width ratio \( ab/b \). As can be seen, as the length scale parameter to thickness ratio increases, the bending ratio decreases but it increases due to the increase in the plate’s length to width ratio.

| \( ah \) | \( h/b \) |
|---|---|
| 0 | 7.0630 | 1.6642 | 0.5104 | 0.1406 |
| 1.5 | 14.2905 | 3.3664 | 1.0306 | 0.2820 |
| 2 | 21.1039 | 4.9708 | 1.5205 | 0.4145 |

Table 2 compares the values of critical force for different nanoplates under a bi-axial surface loading for various length to thickness ratios. It was observed that, the Mindlin’s nanoplate has the highest, and the Third-order nanoplate has the lowest critical force values.

| \( ah \) | \( a/h \) |
|---|---|
| 5 | 1 | 0.4145 | 0.2820 |
| 10 | 0.8925 | 0.3304 | 0.2820 |
| 20 | 3.9522 | 0.2082 | 0.2820 |
| 30 | 2.2231 | 0.1695 | 0.2820 |
| 50 | 1.4228 | 0.1297 | 0.2820 |

Fig. 2 compares the bending values of different nanoplates under the uniform surface traction for different length to width ratios. As can be seen, the Kirchhoff’s nanoplate yielded the lowest values and the third-order nanoplate yielded the highest values for bending. Table 3 shows the dimensionless bending values of Mindlin’s nanoplate under the uniform surface traction and sinusoidal load for material length scale to thickness and length to width ratios. As shown in the table, except for the classical theory \( \nu = 0 \), the dimensionless bending values under sinusoidal load were higher than bending values obtained under the uniform surface traction. It was also found that with an increase in the material length scale parameter to thickness and length to width ratio of the nanoplate, the dimensionless bending value decreases.

Fig. 3 shows the values of dimensionless critical force for Mindlin's nanoplate under a uniaxial force in the \( x \)-direction. It was found that this value increases due to an increase in length to thickness ratio of the nanoplate. Furthermore, when the effect of size parameter is neglected (classical theory), the value of dimensionless critical force becomes constant and reaches its lowest value, but with an increase in the size parameter the dimensionless critical force value increases.
The dimensionless bending values of Mindlin's nanoplate under uniform surface traction and sinusoidal load for various material length scale to thickness and length to width ratios \((a/h = 30, q = 1e-18N/mm)\)

| \(a/b\) | \(l/h\) | \(a/b\) | \(l/h\) | \(a/b\) | \(l/h\) | \(a/b\) | \(l/h\) |
|-------|-------|-------|-------|-------|-------|-------|-------|
|       | Uniform | Sinusoidal | Uniform | Sinusoidal | Uniform | Sinusoidal | Uniform | Sinusoidal |
| 1     | 1      | 1      | 0.235573 | 0.235626 | 0.072128 | 0.072264 | 0.019740 | 0.019913 |
| 1.5   | 1      | 1      | 0.235499 | 0.235569 | 0.071942 | 0.072121 | 0.019506 | 0.019735 |
| 2     | 1      | 1      | 0.235479 | 0.235541 | 0.071892 | 0.072049 | 0.019443 | 0.019643 |

Fig. 2 Comparison of bending values for different nanoplates under the uniform surface traction for different aspect ratios \((a/h = 30, l/h = 1, q = 1e-18N/mm)\)

Fig. 3 Values of dimensionless critical force for Mindlin's nanoplate under a uniaxial force in the \(x\)-direction for different material length scale to thickness and length to thickness ratio of the nanoplate \((a/b=1, m=1, n=1)\)

Fig. 4 shows the values of critical force for Mindlin's nanoplate under a uniaxial force in the \(x\)-direction for various material length scale to thickness and length to thickness ratio of the nanoplate \((a/b=1, m=1, n=1)\)

Fig. 5 shows the values of dimensionless critical force for Mindlin's nanoplate under a bi-axial surface force in \(x\) and \(y\) directions. As can be seen, this value increases due to an increase in, length scale parameter to thickness and length to thickness ratio of nanoplate.

Fig. 4 Values of critical force for Mindlin's nanoplate under a uniaxial force in the \(x\)-direction for various material length scale to thickness and length to thickness ratio of the nanoplate \((a/b=1, m=1, n=1)\)

Fig. 5 Values of dimensionless critical force for Mindlin's nanoplate under a bi-axial surface force in \(x\) and \(y\) directions for material length scale to thickness and length to thickness ratio of the nanoplate \((a/b=1, m=1, n=1)\)

Figs. 6 – 9 shows the dimensionless frequency of
different modes of Mindlin's nanoplate \( (\omega_1, \omega_2, \omega_3, \omega_4) \) (except for the classical theory \( l=0 \)). It was observed that this value increases due to an increase in length to thickness ratio. Also, for the classical theory (neglecting the effect of size parameter) the dimensionless frequency reaches its lowest value, but with an increase in the size effect, the dimensionless frequency values increase.

Fig. 6 Comparison of dimensionless frequencies of the first mode \( (\omega_{11}) \) for a Mindlin's nanoplate for various material length scale parameter to thickness and length to thickness ratios of the nanoplate \( (h=0.34, ab=1) \)

Table 4 shows that the dimensionless frequency of different modes of Mindlin's nanoplate increases due to an increase in material length scale parameter to thickness ratio.

Table 4

| Mode    | \( l/h \) | 0  | 0.5 | 1   | 2   |
|---------|-----------|----|-----|-----|-----|
| \( \omega_{11} \) | 13.9429 | 28.7266 | 51.9052 | 99.1252 |
| \( \omega_{12} \) | 34.6425 | 71.3140 | 128.0217 | 237.9174 |
| \( \omega_{21} \) | 34.6425 | 71.3140 | 128.0217 | 237.9174 |
| \( \omega_{22} \) | 55.0918 | 113.3246 | 202.1703 | 365.8010 |
| \( \omega_{33} \) | 121.5505 | 249.5297 | 436.5378 | 722.2380 |

Table 5

| Mode    | \( l/h \) | 0  | 0.5 | 1   | 2   |
|---------|-----------|----|-----|-----|-----|
| \( \omega_{11} \) | 10.0816 | 20.7245 | 37.5829 | 72.1427 |
| \( \omega_{12} \) | 19.3340 | 39.8248 | 71.8354 | 136.2116 |
Continuation of Table 5

| Mode  | $a/b$ | 0     | 0.5   | 1     | 2     |
|-------|-------|-------|-------|-------|-------|
| $0_{21}$ | 0.5   | 30.8284 | 63.4718 | 114.0783 | 213.0666 |
| $0_{22}$ | 1     | 39.9678 | 82.2599 | 147.4292 | 272.0827 |
| $0_{31}$ | 2     | 88.6398 | 182.1338 | 321.6951 | 556.8022 |

Comparison of dimensionless frequencies of different modes of Mindlin’s nanoplate for various length to thickness ratios ($a/b=0.5$, $l/h=1$)

| Mode  | $a/h$ | 20     | 30     | 40     | 50     |
|-------|-------|--------|--------|--------|--------|
| $0_{21}$ | 0.5   | 280.4153 | 128.0217 | 72.7219 | 46.7575 |
| $0_{12}$ | 1     | 436.5378 | 202.1703 | 115.4757 | 74.4444 |
| $0_{22}$ | 2     | 860.2980 | 413.9252 | 240.0504 | 155.9272 |
| $0_{31}$ | 3     | 988.5087 | 481.2484 | 280.4153 | 182.5827 |
| $0_{32}$ | 4     | 1844.9056 | 988.5087 | 596.8069 | 395.7091 |

Comparison of dimensionless frequencies of different modes of Mindlin’s nanoplate for length to thickness ratio ($a/b=1$, $l/h=1$)

| Mode  | $a/h$ | 20     | 30     | 40     | 50     |
|-------|-------|--------|--------|--------|--------|
| $0_{11}$ | 0.5   | 115.4757 | 51.0952 | 29.3145 | 18.7965 |
| $0_{12}$ | 1     | 280.4153 | 128.0217 | 72.7219 | 46.7575 |
| $0_{21}$ | 2     | 436.5378 | 202.1703 | 115.4757 | 74.4444 |
| $0_{22}$ | 3     | 860.2980 | 413.9252 | 240.0504 | 155.9272 |
| $0_{31}$ | 4     | 903.7909 | 436.5378 | 253.5674 | 164.8397 |

Comparison of dimensionless frequencies of different modes of various nanoplates for length to thickness ratio ($a/b=1$, $l/h=1$)

| Mode  | $l/h$ | 20     | 30     | 40     |
|-------|-------|--------|--------|--------|
| Mindlin plate | 0.5   | 280.4153 | 128.0217 | 72.7219 |
| Kirchhoff plate | 1     | 436.5378 | 202.1703 | 115.4757 |
| Third order shear deformation plate | 2     | 860.2980 | 413.9252 | 240.0504 |

11. Conclusion

In this study, the bending, buckling and vibration of a graphene Mindlin’s nanoplate were investigated using the modified couple stress theory. As observed in the tables and figures, the Mindlin’s nanoplate bending rate under sinusoidal load, decreases with an increase in length to thickness ratio of the nanoplate, but, this value increases with an increase in the aspect ratio of the nanoplate. Furthermore, by comparing different nanoplates under uniform surface traction it was found that the Kirchhoff’s nanoplate yields the lowest and the third-order nanoplate yields the highest values for bending.

The buckling analysis showed that the dimensionless critical force increases due to an increase in material length scale parameter to thickness ratio and decreases due to an increase in length to thickness ratio of the nanoplate. But when the size effect parameter is neglected (classical theory), the value of dimensionless critical force becomes constant and reaches its lowest value, but with an increase in the size parameter the dimensionless critical force value increases.

Analysis of frequencies of different modes showed that this value increases due to an increase in length to thickness ratio. Also, for the classical theory (neglecting the effect of size parameter) the dimensionless frequency reaches its lowest value, but with an increase in the size effect, the dimensionless frequency values increase. It was also found that the Mindlin’s nanoplate yields the highest and the third-order nanoplate yields the lowest values for frequency.

References

1. Yang, F. A. C. M.; Chong, A. C. M.; Lam, D. C. C.; Tong, P. 2002. Couple stress based strain gradient theory for elasticity, International Journal of Solids and Structures 39(10): 2731-2743. https://doi.org/10.1016/S0020-7683(02)00152-X.
2. Toupin, R.A. 1962. Elastic materials with couple stresses, Arch. Rational Mech. Anal. 11: 385–414.
3. Mindlin, R. D.; Tiersten, H. F. 1962. Effects of couple-stresses in linear elasticity, Columbia Univ., New York CU-TR-48.
4. Koiter, W. T. 1969. Couple-stresses in the theory of elasticity, I & II. Proc. K. Ned. Akad. Wet. (B) 67: 17–44.
5. Mindlin, R. D. 1963. Microstructure in linear elasticity, C. Columbia Univ., New York Dept. of Civil Engineering and Engineering Mechanics TR-50.
6. Tsitias, G. C. 2009. A new Kirchhoff plate model based on a modified couple stress theory, International Journal of Solids and Structures 46(13): 2757-2764. https://doi.org/10.1016/j.ijsolstr.2009.03.004.
7. Wang, B.; Zhou, S.; Zhao, J.; Chen, X. 2011. A size-dependent Kirchhoff micro-plate model based on strain gradient elasticity theory, European Journal of Mechanics-A/Solids 30(4): 517-524. https://doi.org/10.1016/j.euromechsol.2011.04.001.
8. Thai, H. T.; Choi, D. H. 2013. Size-dependent functionally graded Kirchhoff and Mindlin plate models based on a modified couple stress theory, Composite Structures 95: 142-153. https://doi.org/10.1016/j.compstruct.2012.08.023.
9. Akgöz, B.; Civalek, Ö. 2012. Free vibration analysis for single-layered graphene sheets in an elastic matrix via modified couple stress theory, Materials & Design 42: 164-171. https://doi.org/10.1016/j.matdes.2012.06.002.
10. Roque, C. M. C.; Ferreira, A. J. M.; Reddy, J. N. 2013. Analysis of Mindlin micro plates with a modified couple stress theory and a meshless method, Applied Mathematical Modelling 37(7): 4626-4633. https://doi.org/10.1016/j.apm.2012.09.063.
11. Zhang, B.; He, Y.; Liu, D.; Shen, L.; Lei, J. 2015. an efficient size-dependent plate theory for bending, buck-
ling and free vibration analyses of functionally graded microplates resting on elastic foundation, Applied Mathematical Modelling 39: 3814-3845. https://doi.org/10.1016/j.apm.2014.12.001.

12. Jung, W. Y; Han, S. C; Park, W. T. 2014. A modified couple stress theory for buckling analysis of S-FGM nanoplates embedded in Pasternak elastic medium, Composites Part B: Engineering 60: 746-756. https://doi.org/10.1016/j.compositesb.2013.12.058.

M. Eskandari Shahraki, M. Shariati, N. Asiaban, J. Eskandari Jam

BENDING, BUCKLING AND VIBRATIONS ANALYSIS OF THE GRAPHENE NANOPLATE USING THE MODIFIED COUPLE STRESS THEORY

S u m m a r y

In this paper a Mindlin's plate model is developed for the Bending, buckling and vibration analysis of a graphene nanoplate based on a modified couple stress theory. The bending rates and dimensionless bending values under uniform surface traction and sinusoidal load, the dimensionless critical force under a bi-axial surface force in \( x \) and \( y \) directions and dimensionless frequencies of different modes are all obtained for various plate's dimensional ratios and material length scale to thickness ratios. The results are presented and discussed in details.

Keywords: modified couple stress theory, mindlin plate, rectangular nanoplate, Navier type solution.

Received August 23, 2020
Accepted October 04, 2021

This article is an Open Access article distributed under the terms and conditions of the Creative Commons Attribution 4.0 (CC BY 4.0) License (http://creativecommons.org/licenses/by/4.0/).