ON THE CLASSIFICATION OF QUADRATIC HARMONIC MORPHISMS BETWEEN EUCLIDEAN SPACES

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Abstract
We give a classification of quadratic harmonic morphisms between Euclidean spaces (Theorem 2.4) after proving a Rank Lemma. We also find a correspondence between umbilical (Definition 2.7) quadratic harmonic morphisms and Clifford systems. In the case $\mathbb{R}^4 \rightarrow \mathbb{R}^3$, we determine all quadratic harmonic morphisms and show that, up to a constant factor, they are all bi-equivalent (Definition 3.2) to the well-known Hopf construction map and induce harmonic morphisms bi-equivalent to the Hopf fibration $\mathbb{S}^3 \rightarrow \mathbb{S}^2$.

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1. Quadratic harmonic morphisms and their equations

Definition 1.1. A map \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is called a quadratic map if all of the components of \( \varphi \) are quadratic functions (i.e. homogeneous polynomials of degree 2) in \( x_1, \ldots, x_m \). By a quadratic harmonic map (respectively a quadratic harmonic morphism) we mean a harmonic map (respectively a harmonic morphism) which is also a quadratic map.

Note that any quadratic harmonic morphism is a non-constant map by our definition. From the theory of quadratic functions and bilinear forms we know that a quadratic map \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n \) can always be written as

\[
\varphi(X) = (X^t A_1 X, \ldots, X^t A_n X)
\]

where \( X \) denotes the column vectors in \( \mathbb{R}^m \), \( X^t \) the transpose of \( X \) and the \( A_i (i = 1, \ldots, n) \) are symmetric \( m \times m \) matrices (henceforth called component matrices).

Quadratic harmonic morphisms form a large class of harmonic morphisms between Euclidean spaces as the following examples show.

Example 1.2. All the following maps are quadratic harmonic morphisms:

(i) Quadratic harmonic morphisms from orthogonal multiplications.
It is well-known that the standard multiplications \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( n = 1, 2, 4, \) or \( 8 \), in the real algebras of real, complex, quaternionic and Cayley numbers are both orthogonal multiplications and harmonic morphisms. In fact, Baird [1] (Theorem 7.2.7) proves that these are the only possible dimensions for an orthogonal multiplication \( f : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^n \) to be a harmonic morphism.

(ii) (see [1]) The Hopf construction maps \( F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \). These are defined by

\[
F(X, Y) = (\|X\|^2 - \|Y\|^2, 2f(X, Y))
\]

where \( f \) is one of the orthogonal multiplications defined in (i).

(iii) Quadratic harmonic morphisms from Clifford systems.
Let \((P_1, \ldots, P_n)\) be a Clifford system on \( \mathbb{R}^{2m} \), i.e. an \( n \)-tuple of symmetric endomorphisms of \( \mathbb{R}^{2m} \) satisfying \( P_i P_j + P_j P_i = 2\delta_{ij} I \) for \( i, j = 1, \ldots, n \).
Then it follows from Baird [1] (Theorem 8.4.1) that
\[ F(X) = (\langle P_1X, X \rangle, \ldots, \langle P_nX, X \rangle) \]
(where \(\langle , \rangle\) denotes the inner product in Euclidean space) is a quadratic harmonic morphism with dilation \(\lambda^2(X) = 4\|X\|^2\) for each \(X \in \mathbb{R}^{2m}\).

(iv) Quadratic harmonic morphisms from the complete lifts.
Let \(\varphi : \mathbb{R}^m \supset U \rightarrow \mathbb{R}^n\) be a \(C^1\) map from an open connected subset of \(\mathbb{R}^m\) into \(\mathbb{R}^n\). The (real) complete lift (cf. [8] Definition 2.1) of \(\varphi\) is the map \(\varphi : \mathbb{R}^{2m} \supset U \times \mathbb{R}^m \rightarrow \mathbb{R}^n\), given by \(\varphi(X, Y) = J(\varphi(X))Y\), where \(J(\varphi(X))\) is the Jacobian matrix of \(\varphi\) at \(X \in U\). It follows from Ou [8] (Theorem 3.3) that the complete lift of any quadratic harmonic morphism is again a quadratic harmonic morphism.

For some further examples, see Loubeau [7]. In the rest of this section we will give equations that characterize quadratic harmonic morphisms between Euclidean spaces.

Lemma 1.3. Let \(\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n\) be a quadratic map with \(\varphi(X) = (X^tA_1X, \ldots, X^tA_nX)\). Then \(\varphi\) is harmonic if and only if
\[
(1) \quad \text{tr}A_i = 0, \ (i = 1, \ldots, n).
\]

Proof. The harmonicity of \(\varphi\) is equivalent to the statement that all components of \(\varphi\) are harmonic functions, which is easily seen to be equivalent to Equation (1). \(\square\)

Proposition 1.4. Let \(\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n\) be a quadratic map with \(\varphi(X) = (X^tA_1X, \ldots, X^tA_nX)\). Then \(\varphi\) is horizontally weakly conformal if and only if the following equations hold
\[
(2) \quad A_iA_j + A_jA_i = 0, \ (i, j = 1, \ldots n, \ i \neq j), \quad (3) \quad A_i^2 = A_j^2, \ (i, j = 1, \ldots n).
\]

Proof. For a map \(\varphi(X) = (\varphi^1(X), \ldots, \varphi^n(X))\) between Euclidean spaces, horizontal weakly conformality is equivalent to (See [4], [3])
\[
(4) \quad \langle \nabla\varphi^i(X), \nabla\varphi^j(X) \rangle = \lambda^2(X)\delta^{ij}
\]
where \(\delta^{ij}\) is the Kronecker delta and \(\nabla\varphi^i(X)\) denotes the gradient of the component function of \(\varphi^i(X)\).
Now for quadratic map $\varphi$, we can calculate its Jacobian matrix as

$$J(\varphi(X)) = \begin{pmatrix}
2X^tA_1 \\
\vdots \\
2X^tA_n
\end{pmatrix}.$$ 

It is easily seen that Equation (1) is equivalent to the following two equations

\[(5) \quad X^tA_iA_jX \equiv 0, \ (i, j = 1, \ldots, n, i \neq j)\]

\[(6) \quad X^tA_i^2X \equiv X^tA_j^2X, \ (i, j = 1, \ldots, n).\]

Since Equations (5) and (6) are identities of quadratic functions in $x_1, \ldots, x_m$, and noting that $A_iA_j$ is not symmetric in general we conclude that (5) and (6) are equivalent to (2) and (3) respectively. Thus we end the proof of the proposition. \[\square\]

It is well-known (see [4], [6]) that a map between Riemannian manifolds is a harmonic morphism if and only if it is both a harmonic map and a horizontal weakly conformal map. So by combining Lemma 1.3 and Proposition 1.4 we have

**Theorem 1.5.** A quadratic map $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ ($m \geq n$) with $\varphi(X) = (X^tA_1X, \ldots, X^tA_nX)$ is a harmonic morphism if and only if

1. $trA_i = 0$, $(i = 1, \ldots, n)$,
2. $A_iA_j + A_jA_i = 0$, $(i, j = 1, \ldots, n, \ i \neq j)$,
3. $A_i^2 = A_j^2$, $(i, j = 1, \ldots, n)$.

2. THE CLASSIFICATION

In this section we shall prove the Rank Lemma for quadratic harmonic morphisms which will be the basis for the classification theorems.

**Lemma 2.1.** (The Rank Lemma for quadratic harmonic morphisms)

Let $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a quadratic harmonic morphism with $\varphi(X) = (X^tA_1X, \ldots, X^tA_nX)$. Then

(a) All the component matrices $A_i$ have the same rank which is an even number.

(b) All the component matrices $A_i$ have the same spectrum.

**Proof.** Suppose that $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a quadratic harmonic morphism with $\varphi(X) = (X^tA_1X, \ldots, X^tA_nX)$. Then by Theorem 1.5 we have

$$A_i^2 = A_j^2, \ (i, j = 1, \ldots, n)$$
which implies that
\[ \text{rank} A^2_i = \text{rank} A^2_j. \]

The equality of rank\( A_i \) now follows from the following

**Claim.** For any symmetric matrix \( A \), \( \text{rank} A^2 = \text{rank} A \).

**Proof of Claim.** It is a standard fact that \( A \) can be diagonalized by an orthogonal matrix \( P \), so \( P^{-1}AP = D \) is a diagonal matrix. But
\[
\text{rank} A^2 = \text{rank} P^{-1} A^2 P = \text{rank} P^{-1} A P P^{-1} A P = \text{rank} A.
\]

Now we show that rank\( A_i \) is even. It suffices to do the proof for quadratic harmonic morphism \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^2 \) with \( \varphi(X) = (X^t A_1 X, X^t A_2 X) \). After a suitable choice of orthogonal coordinates, \( A_1 \) assumes the diagonal form
\[
A_1 = \begin{pmatrix}
D_1 & 0 & 0 \\
0 & -D_2 & 0 \\
0 & 0 & 0_r
\end{pmatrix}
\]

where \( 0_r \) denotes the \( r \times r \) zero matrix, \( D_1 \) is the \( k \times k \) diagonal matrix with entries the positive eigenvalues \( S_+ = \{ \lambda_1, \ldots, \lambda_k \} \), and \( D_2 \) is the \( l \times l \) diagonal matrix with the entries the absolute values of the negative eigenvalues \( S_- = \{ \xi_1, \ldots, \xi_l \} \), where \( k + l + r = m \).

Using Equations (2) and (3) we see that \( A_2 \) must have the form
\[
A_2 = \begin{pmatrix}
0 & B_1 & 0 \\
B_1^t & 0 & 0 \\
0 & 0 & 0_r
\end{pmatrix}
\]

where \( B_1 \) denotes a \( k \times l \) matrix satisfying \( D_1 B_1 = B_1 D_2 \), which means
\[
\begin{pmatrix}
\lambda_1 b_{11} & \ldots & \lambda_1 b_{1l} \\
\vdots & \ddots & \vdots \\
\lambda_k b_{k1} & \ldots & \lambda_k b_{kl}
\end{pmatrix}
= \begin{pmatrix}
\xi_1 b_{11} & \ldots & \xi_1 b_{1l} \\
\vdots & \ddots & \vdots \\
\xi_1 b_{k1} & \ldots & \xi_1 b_{kl}
\end{pmatrix}
\]

Since rank\( A_1 = \text{rank} A_2 = k + l \), we see from Equation (3) that any \( \lambda_i \in S_+ \) must be equal to one of the numbers in \( S_- \) otherwise the \( i \)th row of \( B_1 \) would be zero vector and rank\( A_2 < k + l \). This means that \( S_+ \subset S_- \). A similar reasoning gives \( S_- \subset S_+ \). Thus we have \( S_+ = S_- \), which means that \( S_+ \) and \( S_- \) have equal numbers of the same elements, i.e. \( k = l \) and \( S_+ = S_- = \{ \lambda_1, \ldots, \lambda_k \} \). Thus rank\( A_1 = \text{rank} A_2 = 2k \) is even, which ends the proof of (a). For (b) we first note, from the above proof, that the eigenvalues of a component matrix of a quadratic harmonic morphism must
appear in pairs $\pm \lambda$. On the other hand, it is elementary that if a symmetric linear transformation $A^2$ has an eigenvalue $\lambda^2 > 0$ then $A$ must have one of eigenvalues $\pm \lambda$. Now the rest of the proof follows from the fact that all $A_i^2$ ($i = 2, \ldots, n$) have eigenvalues $\{\lambda_1^2, \ldots, \lambda_k^2\}$, where $\lambda_1, \ldots, \lambda_k$ are the positive eigenvalues of $A_1$.

**Definition 2.2.** Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a quadratic harmonic morphism. Then the **Q-rank** of $\varphi$, denoted by $Q\text{-}\text{rank}(\varphi)$, is defined to be the rank of its component matrices. $\varphi$ is said to be **Q-nonsingular** if $Q\text{-}\text{rank}(\varphi) = m$, otherwise it is said to be **Q-singular**.

We are now ready to give a characterization of quadratic harmonic morphisms to $\mathbb{R}^2$.

**Proposition 2.3.** Let $\varphi : \mathbb{R}^m \to \mathbb{R}^2$ ($m \geq 2$) be a quadratic harmonic morphism.

(i) If $\varphi$ is Q-nonsingular, then $m = 2k$ for some $k \in \mathbb{N}$ and, with respect to suitable orthogonal coordinates in $\mathbb{R}^m$, $\varphi$ assumes the normal form

$$\varphi(X) = \left( X^t \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} X, X^t \begin{pmatrix} 0 & B_1^t \\ B_1 & 0 \end{pmatrix} X \right)$$

where $D, B_1 \in GL(\mathbb{R}, k)$, with $D$ diagonal and satisfying

$$\begin{cases} DB_1 = B_1D \\ B_1^tB_1 = D^2. \end{cases}$$

(ii) Otherwise $Q\text{-}\text{rank}(\varphi) = 2k$, for some $k$, $0 \leq k < m/2$, and $\varphi$ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \to \mathbb{R}^{2k}$ followed by a Q-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \to \mathbb{R}^2$.

**Proof.** Let $\varphi : \mathbb{R}^m \to \mathbb{R}^2$ be given by $\varphi(X) = (X^tA_1X, X^tA_2X)$. Then from the Rank Lemma we know that $Q\text{-}\text{rank}(\varphi)$ is even. If $\varphi$ is Q-nonsingular then $Q\text{-}\text{rank}(\varphi) = m = 2k$. As in the proof of the Rank Lemma, after a suitable choice of orthogonal coordinates, $A_1$ assumes the normal form

$$A_1 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

where $D$ denotes the $k \times k$ diagonal matrix having the positive eigenvalues of $A_1$ as diagonal entries. Then $A_2$ must have the form

$$A_2 = \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix}$$
with $B_1 \in GL(\mathbb{R}, k)$ satisfying $DB_1 = B_1 D$. This, together with $B_i^t B_1 = D^2$ given by (3) of Theorem 1.3, gives (i). Now (ii) follows from the fact that if $\varphi$ is $Q$-regular with rank $A = 2k < m$, then after a suitable choice of orthogonal coordinates, $A$ takes the form (5) and consequently $B$ the form (8).

Now we give the Classification Theorem for general quadratic harmonic morphisms.

**Theorem 2.4.** Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic harmonic morphism.

(I) If $\varphi$ is $Q$-nonsingular, then $m = 2k$ for some $k \in \mathbb{N}$ and, with respect to suitable orthogonal coordinates in $\mathbb{R}^m$, $\varphi$ assumes the normal form

$$\varphi(X) = \left( X^t \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} X, X^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X, \ldots, X^t \begin{pmatrix} 0 & B_{n-1} \\ B_{n-1}^t & 0 \end{pmatrix} X \right).$$

where $D, B_i \in GL(\mathbb{R}, k)$ with $D$ diagonal having the positive eigenvalues as its diagonal entries satisfy

$$\begin{cases} DB_i = B_i D \\ B_i^t B_i = D^2 \\ B_i^t B_j = -B_j^t B_i. \quad (i, j, = i, \ldots, n - 1, i \neq j). \end{cases}$$

(II) Otherwise $Q$-rank($\varphi$) = $2k$ for some $k$, $0 \leq k < m/2$, and $\varphi$ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{2k}$ followed by a $Q$-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$.

**Proof.** As in the proof of the Rank Lemma, after a suitable choice of orthogonal coordinates the first component matrix has the form (5), and all the other component matrices $A_n$ has the form

$$A_{i+1} = \begin{pmatrix} 0 & B_i \\ B_i^t & 0 \end{pmatrix}, \quad i = 1, \ldots, n - 1.$$

Now if $\varphi$ is $Q$-singular then $Q$-rank($\varphi$) = $2k < m$, for some $k$, $0 \leq k < m/2$. It is easily seen that $\varphi$ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{2k}$ followed by a $Q$-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$. Otherwise $\varphi$ is $Q$-nonsingular in which case $r = 0$. Thus we have the normal form (11). Note that for $n > 2$ Equation (2) gives the additional Equation (12).
Corollary 2.5. Any quadratic harmonic morphism is the composition of an orthogonal projection followed by a $Q$-nonsingular quadratic harmonic morphism from an even-dimensional space.

Remark 2.6. Thus to study quadratic harmonic morphisms it suffices to consider $Q$-nonsingular ones from even-dimensional spaces.

Definition 2.7. A quadratic harmonic morphism $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ with $\varphi(X) = \left(X^t A_1 X, \ldots, X^t A_n X\right)$ is said to be umbilical if all the positive eigenvalues of one (and hence all by The Rank Lemma) of its component matrices are equal.

There do exist quadratic harmonic morphisms which are not umbilical as the following example shows.

Example 2.8. It can be checked that $\varphi : \mathbb{R}^8 \rightarrow \mathbb{R}^3$ given by

$$
\varphi = (2x_1^2 + 2x_2^2 + 3x_3^2 + 3x_4^2 - 2x_5^2 - 2x_6^2 - 3x_7^2 - 3x_8^2,
4x_1x_5 + 4x_2x_6 + 6x_3x_8 - 6x_4x_7,
-4x_1x_6 + 4x_2x_5 + 6x_3x_7 + 6x_4x_8)
$$

is a quadratic harmonic morphism which is not umbilical since its component matrices have two distinct positive eigenvalues.

For more results on constructions of harmonic morphisms into Euclidean spaces see Ou [9].

3. Quadratic harmonic morphisms and Clifford systems

Definition 3.1. i) The $(n + 1)$-tuple $(P_0, \ldots, P_n)$ of symmetric endomorphisms of $\mathbb{R}^{2m}$ is called a Clifford system on $\mathbb{R}^{2m}$ if

$$
P_iP_j + P_jP_i = 2\delta_{ij}I \quad (i, j = 0, 1, \ldots, n).
$$

ii) Let $(P_0, \ldots, P_n)$ and $(Q_0, \ldots, Q_n)$ be Clifford systems on $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q}$ respectively, then $(P_0 \oplus Q_0, \ldots, P_n \oplus Q_n)$ is a Clifford system on $\mathbb{R}^{2p+2q}$, the so-called direct sum of $(P_0, \ldots, P_n)$ and $(Q_0, \ldots, Q_n)$.

iii) A Clifford system $(P_0, \ldots, P_n)$ on $\mathbb{R}^{2m}$ is called irreducible if it is not possible to write $\mathbb{R}^{2m}$ as a direct sum of two non-trivial subspaces which are invariant under all $P_i$.

iv) Two Clifford systems $(P_0, \ldots, P_n)$ and $(Q_0, \ldots, Q_n)$ on $\mathbb{R}^{2m}$ are said to be algebraically equivalent if there exists $A \in O(\mathbb{R}^{2m})$ such that $Q_i = AP_iA^t$ for all $i = 0, 1, \ldots, n$. 

From the representation theory of Clifford algebras (see [3]) we have the following results:

**Theorem A.** (See [3])
(a) Each Clifford system is algebraically equivalent to a direct sum of irreducible Clifford systems.
(b) An irreducible Clifford system \((P_0, \ldots, P_n)\) on \(\mathbb{R}^{2m}\) exists precisely for the following values of \(n\) and \(m = \delta(n)\):

| \(n\)  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \ldots | \(n+8\) |
|-------|---|---|---|---|---|---|---|---|-------|--------|
| \(\delta(n)\) | 1 | 2 | 4 | 4 | 8 | 8 | 8 | 8 | \ldots | 16 \(\delta(n)\) |

(c) For \(n \not\equiv 0 \mod 4\), there exists exactly one algebraically equivalent class of irreducible Clifford systems. If \(n \equiv 0 \mod 4\), there are two.

**Definition 3.2.** Let \(\varphi, \tilde{\varphi} : \mathbb{R}^m \to \mathbb{R}^n\) be two quadratic harmonic morphisms. Then
1. \(\varphi\) and \(\tilde{\varphi}\) are said to be **domain-equivalent** if there exists an isometry \(P\) of \(\mathbb{R}^m\) such that \(\varphi = \tilde{\varphi} \circ P\). They are said to be **bi-equivalent** if there exist isometries \(P\) of \(\mathbb{R}^m\) and \(G\) of \(\mathbb{R}^n\) such that \(\varphi = G^{-1} \circ \tilde{\varphi} \circ P\).
2. The concepts of domain-equivalence and bi-equivalence can be defined similarly for harmonic morphisms between spheres (or, indeed any Riemannian manifolds).

Baird has proved ([4] Theorem 8.4.1) that any Clifford system \((P_0, \ldots, P_n)\) on \(\mathbb{R}^{2m}\) gives rise to a quadratic harmonic morphism \(\varphi : \mathbb{R}^{2m} \to \mathbb{R}^{n+1}\) defined by
\[
\varphi(X) = (\langle P_0 X, X \rangle, \langle P_1 X, X \rangle, \ldots, \langle P_n X, X \rangle).
\]
It is easy to see that two Clifford systems \((P_0, \ldots, P_n)\) and \((Q_0, \ldots, Q_n)\) on \(\mathbb{R}^{2m}\) are algebraically equivalent if and only if they give rise to domain-equivalent quadratic harmonic morphisms \(\varphi : \mathbb{R}^{2m} \to \mathbb{R}^{n+1}\). It is easy to see that any quadratic harmonic morphism given by a Clifford system is umbilical. We shall prove that up to a constant factor all umbilical quadratic harmonic morphisms arise this way.

**Theorem 3.3.** Up to a homothetic change of coordinates in \(\mathbb{R}^m\), any umbilical quadratic harmonic morphism \(\varphi : \mathbb{R}^m \to \mathbb{R}^n\) arises from a Clifford system.

**Proof.** We need only to show that, up to a homothetic change of the coordinates in \(\mathbb{R}^{2k}\), the component matrices of any \(Q\)-nonsingular umbilical
quadratic harmonic morphism $\varphi : \mathbb{R}^{2k} \to \mathbb{R}^n$ represent a Clifford system. Indeed it follows from Theorem 2.4 that with respect to suitable orthogonal coordinates in $\mathbb{R}^{2k}$, $\varphi$ assumes the normal form (11) with $D = \lambda \text{Id}$, and it is easily seen that after a change of scale in $\mathbb{R}^{2k}$ the component matrices become

$$A_1 = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \tilde{B}_1 \\ \tilde{B}_1^t & 0 \end{pmatrix}, \quad \ldots, \quad A_n = \begin{pmatrix} 0 & \tilde{B}_{n-1} \\ \tilde{B}_{n-1}^t & 0 \end{pmatrix}$$

with $\tilde{B}_i \in O(k)$ satisfying $\tilde{B}_i^t \tilde{B}_j = -\tilde{B}_j^t \tilde{B}_i$. $(i, j, = i, \ldots, n-1, i \neq j)$. It can be checked that

$$A_\alpha A_\beta + A_\beta A_\alpha = 2\delta_\alpha\beta I, \quad (\alpha, \beta = 1, \ldots, n).$$

Which means that the $A_\alpha$ represent a Clifford system. This ends the proof of the theorem.

**Example 3.4.** It is easy to check that $\varphi : \mathbb{R}^8 \to \mathbb{R}^5$ given by

$$\varphi(x, y) = (3|x|^2 - 3|y|^2, 6x_1y_1 - 6x_2y_2 - 6x_3y_3 - 6x_4y_4,$$

$$6x_1y_2 + 6x_2y_1 + 6x_3y_4 - 6x_4y_3, 6x_1y_3 + 6x_3y_1 + 6x_4y_2 - 6x_2y_4, 6x_1y_4 + 6x_4y_1 - 6x_2y_3 - 6x_3y_2)$$

is an umbilical quadratic harmonic morphism with all positive eigenvalues equal to 3. It is also easy to see that it arises from a Clifford system.

In the rest of this section we will determine all quadratic harmonic morphisms from $\mathbb{R}^4$ to $\mathbb{R}^3$ and show that they are all bi-equivalent to some constant multiple $\lambda \varphi_0$ of the standard Hopf construction map and that, up to a change of scale, they all restrict to $S^3 \to S^2$ and hence induce bi-equivalent harmonic morphisms. We thus recover, by simple means, part of a result of Eells and Yiu [2]. For further results on the existence of quadratic harmonic morphisms see Ou [10].

**Theorem 3.5.** Up to domain-equivalence, all quadratic harmonic morphisms $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$ are of the form

$$\varphi_t = \lambda (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2x_1x_3 \cos t + 2x_1x_4 \sin t - 2x_2x_3 \sin t + 2x_2x_4 \cos t, 2x_1x_3 \sin t - 2x_1x_4 \cos t + 2x_2x_3 \cos t + 2x_2x_4 \sin t)$$

where $\lambda \neq 0$ and $t \in [0, 2\pi)$. They are all bi-equivalent to a constant multiple of the standard Hopf construction map:

$$\lambda \varphi_0 = \lambda (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2x_1x_3 + 2x_2x_4, -2x_1x_4 + 2x_2x_3).$$
Proof. First we note that any quadratic harmonic morphism $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is $Q$-nonsingular since otherwise, $\varphi$ would be of the form 
\[ \mathbb{R}^4 \xrightarrow{\pi} \mathbb{R}^2 \xrightarrow{\varphi_1} \mathbb{R}^3 \]
where $\varphi_1$ is a non-constant quadratic harmonic morphism which is impossible. Next we

Claim: All quadratic harmonic morphisms $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ are umbilical.

Proof of Claim: Let $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a quadratic harmonic morphism. Then from Theorem 2.4 we have
\[
D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in GL(\mathbb{R}, 2)
\]
satisfying Equation (12). Now suppose that $\lambda_1 \neq \lambda_2$, then by using the first equation of (12) we have
\[
B_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \in GL(\mathbb{R}, 2).
\]
But then the third equation of (12) gives $a_1 b_1 = 0$ and $a_2 b_2 = 0$, which is impossible since $B_1, B_2$ are invertible. Thus we must have $\lambda_1 = \lambda_2$.

Now any $Q$-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ can be assumed to be of the form
\[
\varphi = \lambda \left( x_1^2 + x_2^2 - x_3^2 - x_4^2, \ x^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X, \ x^t \begin{pmatrix} 0 & B_2 \\ B_2^t & 0 \end{pmatrix} X \right)
\]
where $B_1, B_2 \in O(2)$ satisfy
\[
B_1^t B_2 = -B_2^t B_1. \tag{16}
\]
without loss of generality, we may assume that $B_1 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2)$, and $B_2 = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2)$, or $B_2 = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} \in O(2) \setminus SO(2)$. It can be checked that for the second possibility, Equation (16) has no solution. whilst for the first possibility Equation (16) is equivalent to
\[
\begin{cases} 
\cos(t - \theta) = -\cos(\theta - t) \\
\sin(t - \theta) = -\sin(\theta - t)
\end{cases} \tag{17}
\]
which has solutions $t - \theta = \theta - t \mod 2\pi$ i.e.,
\[
\theta = t - \frac{\pi}{2} \mod 2\pi = t \pm \frac{\pi}{2} \mod 2\pi \tag{18}
\]
Inserting (18) into (15) we have two families

\[(a) \quad \varphi_t = \lambda(x_1^2 + x_2^2 - x_3^2 - x_4^2),\]
\[2x_1 x_3 \cos t + 2x_1 x_4 \sin t - 2x_2 x_3 \sin t + 2x_2 x_4 \cos t,\]
\[2x_1 x_3 \sin t - 2x_1 x_4 \cos t + 2x_2 x_3 \cos t + 2x_2 x_4 \sin t)\]

and

\[(b) \quad \varphi_t = \lambda(x_1^2 + x_2^2 - x_3^2 - x_4^2),\]
\[2x_1 x_3 \cos t + 2x_1 x_4 \sin t - 2x_2 x_3 \sin t + 2x_2 x_4 \cos t,\]
\[-2x_1 x_3 \sin t + 2x_1 x_4 \cos t - 2x_2 x_3 \cos t - 2x_2 x_4 \sin t).\]

However, family (b) can be obtained from family (a) by an orthogonal change of coordinates in \(\mathbb{R}^3\). Thus any \(Q\)-nonsingular quadratic harmonic morphism \(\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3\) is domain-equivalent to some \(\varphi_t\), whilst \(\varphi_t = G^{-1} \circ \lambda \varphi_0\) for \(G\) given by

\[G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \end{pmatrix} \in SO(3).\]  

therefore any \(Q\)-nonsingular quadratic harmonic morphism \(\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3\) is bi-equivalent to a multiple of the Hopf construction map \(\lambda \varphi_0\). This ends the proof of the theorem.

**Corollary 3.6.** *Up to homothety of \(\mathbb{R}^4\), all quadratic harmonic morphisms \(\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3\) arise from algebraically equivalent irreducible Clifford systems on \(\mathbb{R}^4\), and they induce harmonic morphisms \(S^3 \rightarrow S^2\) bi-equivalent to the standard Hopf fibration given by (14) with \(\lambda = 1\).*

**Proof.** It is trivial to check that for all \(t\), the Clifford systems on \(\mathbb{R}^4\) represented by

\[A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \\ \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & \sin t & -\cos t \\ 0 & 0 & \cos t & \sin t \\ \sin t & \cos t & 0 & 0 \\ -\cos t & \sin t & 0 & 0 \end{pmatrix},\]

are irreducible and are clearly algebraically equivalent. We have seen that, after a possible change of scale in \(\mathbb{R}^4\), \(\varphi_t = G^{-1} \circ \varphi_0\) for \(G\) given by (19).
Thus for all $t$, $\varphi_t$ restricts to $S^3 \to S^2$ and is bi-equivalent to the classical Hopf fibration $S^3 \to S^2$, which ends the proof of the Corollary.

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