Quantitative Properties on the Steady States to
A Schrödinger-Poisson-Slater System

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Abstract

A relatively complete picture on the steady states of the following
Schrödinger-Poisson-Slater (SPS) system

\[
\begin{align*}
-\Delta Q + Q &= VQ - CSQ^2, \quad Q > 0 \text{ in } \mathbb{R}^3 \\
Q(x) &\to 0, \quad \text{as } x \to \infty, \\
-\Delta V &= Q^2, \quad \text{in } \mathbb{R}^3 \\
V(x) &\to 0 \quad \text{as } x \to \infty.
\end{align*}
\]

is given in this paper: existence, uniqueness, regularity and asymptotic
behavior at infinity, where $CS > 0$ is a constant. To the author’s knowl-
edge, this is the first uniqueness result on SPS system.

Keyword: SPS system; existence; uniqueness; regularity; asymptotic behavior;
Mathematics Subject Classification(2000) 35J50 35J60

1 Introduction

Due to its importance in various physical frameworks: gravitation, plasma
physics, semiconductor theory, quantum chemistry and so on (see, e.g. \[2\] \[15\] \[17\]
and the reference therein), the following Schrödinger-Poisson-Slater (SPS) sys-
tem in terms of the wave function $\psi : \mathbb{R}^3 \times [0,T) \to \mathbb{C}$

\[
\begin{align*}
\frac{i\partial \psi}{\partial t} &= -\Delta_x \psi + V(x,t)\psi + CS|\psi(x,t)|^{2\alpha}\psi, \quad \lim_{x \to \infty} \psi(x,t) \to 0 \\
\psi(x, t = 0) &= \psi_0(x), \\
-\Delta_x V &= \epsilon|\psi|^2, \quad \lim_{x \to \infty} V(x,t) \to 0
\end{align*}
\]

has been studied extensively in recent years, see \[1\] \[2\] \[3\] \[6\] \[17\] \[20\] \[21\] for in-
stance. Here $\alpha$, $\epsilon$, $CS$ are constants., such as $\epsilon = +1$ (repulsive case), $\epsilon = -1
(attractive case) depending on the type of interaction considered. The last term
(the Slater term $|\psi(x,t)|^{2\alpha} \psi$) is usually considered to be a correction to the
nonlocal term $V\psi$, for example, $\alpha = \frac{1}{3}$, which is called the Slater correction,
or $\alpha = \frac{2}{3}$, which is named as Dirac correction. The interested reader is recom-

Based on the fact that the total mass
$$M[\psi] := \int_{\mathbb{R}^3} |\psi(x,t)| dx$$

and the energy functional
$$E[\psi] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\psi|^2 + \frac{c}{4} \int_{\mathbb{R}^3} \left( \frac{1}{4\pi |x|} - 1 \right) |\psi|^2 + \frac{CS}{2\alpha + 2} \int_{\mathbb{R}^3} |\psi|^{2\alpha + 2}$$

are preserved along the time evolution, the standing wave solutions, whose in-

Let $I_2 = \frac{1}{4\pi |x|}^{-1}$ be the Newtonian potential in the Euclidean space $\mathbb{R}^3$. It is
well known that the self-consistent potential $V$ can be then rewritten explicitly
in the form of convolution by the Newtonian potential $I_2$ as
$$V(x) = \epsilon I_2 * |Q|^2(x) = \epsilon \int_{\mathbb{R}^3} \frac{|Q(y)|^2}{4\pi |x - y|} dy,$$

so that the SPS system (1.1) is reduced to a single nonlinear and nonlocal
Schrödinger-type equation
$$-\Delta Q + \lambda Q = -\epsilon (I_2 * Q^2) Q - CS|Q|^{2\alpha} Q,$$

which is a special case of Schrödinger-Maxwell equations (6).

The existence of standing waves has been studied from various perspectives
in the vast mathematical literature. For example, in [21], the author investigated
the existence of critical points of the functional $E[\psi] + \lambda M[\psi]$ on the Sobolev
space $H^1(\mathbb{R}^3)$, see also the reference therein. Nevertheless, due to the variational
structure of the equation (1.2), solutions of (1.2) are usually treated as kind of
critical points of some energy functionals. From a physical point of view, the
most interesting critical points, the so called steady states, are minimizers of
the problem
\[ I_M = \inf \{ E[u] : u \in H^1(\mathbb{R}^3) ; |u|_2 = M \}. \]

In the repulsive case \( \epsilon = +1, C_S < 0 \), many results have been known. We only mention some of them. In [10], a negative answer was given to \( \alpha = 0 \). In [4], a positive answer is given to \( \alpha \in (0, 1/2) \) with \( M > 0 \) small. In [21], as part of its results, a positive answer was obtained to the Slater correction case: \( \alpha = 1/3 \). Quite recently the authors in [3] applied the well known concentration-compactness method to study the existence of the minimizer of \( I_M \) systematically for all \( \alpha \in (0, 2) \), and a very complete answer on this question was given there. See also the reference therein to obtain more information on the known result.

There are mainly one obstacle which brings difficulty in the repulsive case: the variational problem is translation invariant which causes the loss of compactness of minimizing sequence. Thus the concentration-compactness method supplies a suitable framework to tackle this problem. However, in the attractive case \( \epsilon = -1, C_S > 0 \), even though the problem is still translation invariant but with another good structure which makes the study easier: unlike in the repulsive case, there is no competition between the nonlocal term \( \int \int I_2(x - y)|u(x)||u(y)|^2 dxdy \) and the kinetic energy \( \int |\nabla u|^2 \), thus the powerful rearrangement method is also available.

In this paper, we shall limit ourselves to the attractive case with positive contribution given by the Slater term, i.e., \( \epsilon = -1, C_S > 0 \). The exact value of \( C_S \) is not important but since it cannot be normalized, we shall always leave it in the equation. Moreover, limited to the uniqueness argument on the minimizer, we will only consider the case \( \alpha = \frac{1}{2} \). Under these considerations we are able to draw a relatively complete picture about the steady states: existence, uniqueness, smoothness and asymptotic behavior. Most of the results are allowed to extend to general cases, which will be pointed out whenever it is possible. Our main result reads

**Theorem 1.1.** Let \( \epsilon = -1, C_S > 0, \alpha = \frac{1}{2}, M > 0 \). \( I_M \) is attained at a minimizing function \( Q \) with the following properties:

1. (radiality and monotonicity) there exists a decreasing function \( Q_0 : [0, \infty) \rightarrow (0, \infty) \) such that \( Q(x) = Q_0(|x - x_0|) \) for some \( x_0 \in \mathbb{R}^3 \);

2. (regularity) modulo some scaling transformations, \( Q \) satisfies the Euler-Lagrange equation for some constant \( C_S \)

\[
\begin{cases}
-\Delta Q + Q = \left( \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} Q(y)^2 dy \right) Q - C_S Q^2, \quad Q > 0 \text{ in } \mathbb{R}^3, \\
Q \in H^1(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3).
\end{cases}
\]

3. (uniqueness) modulo translations, the minimizer \( Q \) is unique.

As mentioned above, the existence result will be established in section 2 via a rearrangement argument combining together with concentration compactness method.
It is a tough job to study the uniqueness of minimizers or positive solutions (modulo translation) to both repulsive case and attractive case of SPS system. Very little is known. Since the energy functional is not convex, the conventional way to prove uniqueness does not apply. Another way to show uniqueness is by virtue of the Euler-Lagrange equation satisfied by minimizers, and ODE technique could be applied once the radially symmetric property of minimizers is shown. However, because of the competition between the kinetic term \( \int_{\mathbb{R}^3} |\nabla u|^2 \) and the nonlocal term \( \int_{\mathbb{R}^3} (I_2 * u^2) u^2 \), in the repulsive case it is not trivial to show the radiality of minimizers. See [8] for some part results on this.

Let us briefly discuss about the uniqueness of positive (radial) solutions of the SPS equation here. Without considering the nonlocal term, there has been a lot of results on the uniqueness of the scalar field equation

\[-\Delta u + u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n,\]

and its generalizations, we only mention the paper [11] here. For the SPS system, the difficulty mainly stems from the nonlocal term, for which requires the global information of the solutions always. Thus the local method, the commonly applied shooting method of ODE is not available in general. Under the consideration of the nonlocal term but without the Slater term, the first result is due to E.H.Lieb who derived the uniqueness of positive radially symmetric solution to the Choquard equation

\[-\Delta u + u = \left( \int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|} u(y)^2 dy \right) u, \quad \text{in } \mathbb{R}^3, \quad u(x) \to 0 \text{ as } |x| \to \infty \]

in [13]. His argument depends heavily on the special structure of the equation, and seems impossible to extend to the equation with general Choquard term \( \left( \int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|} |u(y)|^{p} dy \right) |u|^p - 2 u \) for \( p > 1 \). But his method is allowed to extend to some higher dimensions, say, 4 and 5 dimension, see [12, 24] for example. Quite recently, Lieb’s result was proved to be true for positive solutions by Ma and Zhao in [16].

Contrary to the attractive case, a big difference is shown to the repulsive case(\( \epsilon = +1, C_S < 0 \)). In [20] Ruiz showed the following nonexistence and multiplicity result on positive radial solutions

**Theorem.** Let \( 1 < p < 2, \lambda > 0 \) and consider the equation

\[
\begin{align*}
-\Delta u + u + \lambda \phi u &= |u|^{p-1} u, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \\
\phi &\to 0 \text{ as } |x| \to \infty.
\end{align*}
\]

Then (1) if \( \lambda \geq 1/4 \), there is no nontrivial solution in the space \( H^1 \times D^1 \);
(2) if \( \lambda \) small enough, there exist at least two positive radial solutions.

The multiplicity result has also been proved in [19, 21] in case \( 1 < p < 11/7 \). Note that by setting \( u = \frac{1}{\sqrt{\lambda}} v \), we get equation

\[
\begin{align*}
-\Delta v + v + \phi v &= \frac{1 - p}{2} |v|^{p-1} v, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= v^2, \\
\phi &\to 0 \text{ as } |x| \to \infty.
\end{align*}
\]
i.e., $\epsilon = 1, C_S = -\lambda^{1/4}$ in our context. Thus in general one does not expect uniqueness of positive radial solutions to the repulsive case of SPS system for large value of $C_S$. However, we still don’t know whether the uniqueness of minimizers if it exist hold or not.

In section 3, we shall follow Lieb’s idea to show the uniqueness of minimizers of $I_M$ by virtue of the equation

$$-\Delta u + u = \left( \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} u(y)^2 dy \right) u - C_S u^2, \quad u = u(|x|) > 0 \text{ in } \mathbb{R}^3.$$  

An interesting question is to extend the uniqueness result to positive solution of the equation under the mild condition that $u \rightarrow 0$ as $x \rightarrow \infty$. It is also expected to extend our uniqueness result to the problem with a general Slater term.

In the last section, we give a study on asymptotic behavior of the minimizer, which states that

**Theorem 1.2.** Let $Q \in H^1(\mathbb{R}^3)$ be a positive radially symmetric solution to the equation

$$
\begin{aligned}
-\Delta Q + Q &- (I_2 \ast Q^2) Q + C_S Q^2 = 0, \quad \text{in } \mathbb{R}^3 \\
Q(|x|) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty.
\end{aligned}
$$

Then

$$
\lim_{|x| \rightarrow \infty} Q(x)|x|^{1-\alpha/2} e^{\alpha|x|} \in (0, \infty),
$$

with

$$
\alpha = \frac{1}{4\pi} \int_{\mathbb{R}^3} |Q|^2 dx.
$$

A more general result on the asymptotic behavior of solutions to SPS system will also be given there in the last section.

All the notations in the paper are standard, constants will be denoted by $C, C_M, \ldots$ which may be different from line to line.

**2 Existence**

Without misunderstanding, in this section all the integrals will be taking in $\mathbb{R}^3$ unless specified. Our approach is variational. Since the value of $C_S$ does not enter the play, we shall always assume that $C_S = 1$ in the following.

Consider the following minimizing problem: let $M > 0$ and $A_M = \{ u \in H^1(\mathbb{R}^3) ; |u|_{L^2} = M \}$, define

$$
I_M = \inf \{ E(u) ; u \in A_M \},
$$

where

$$
E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{4} \int (I_2 \ast u^2) u^2 + \frac{1}{3} \int |u|^3.
$$

The first observation of $I_M$ is that
Proposition. $0 > I_M > -\infty$, and minimizing sequences are bounded.

In fact, for any $u \in H^1$, the Hardy-Littlewood-Sobolev inequality implies that there exists a positive constant $C$ such that
\[
\int_{\mathbb{R}^3} (I_2 * u^2) \, u^2 \leq C|u|_4^4,
\]
thus combining together with Sobolev inequality we have
\[
\int_{\mathbb{R}^3} (I_2 * u^2) \, u^2 \leq C|u|_4^4 \leq C \left( |u|_2^{3/4} |u|_6^{1/4} \right)^4
\leq C|u|_2^3 \nabla u_2
\leq C|u|_2^6 + \int |\nabla u|^2,
\]
therefore, for all $u \in A_M$ there holds
\[
E(u) \geq \frac{1}{4} \int |\nabla u|^2 - CM^6 \geq -CM^6 > -\infty,
\]
furthermore, the above inequality implies that any minimizing sequence of $I_M$ is a bounded sequence in $H^1(\mathbb{R}^3)$. Easy to note that the Slater term could be replaced by $C_S|Q|^{2\alpha+2}$ for any $\alpha \in (0, 2)$.

$I_M < 0$ is consequence of scaling property of the energy functional $E$. For any $t > 0$, let $u^t(x) = t^{3/2}u(tx)$ so that $|u^t|_2 = |u|_2$, then
\[
E(u^t) = \frac{t^2}{2} \int |\nabla u|^2 - \frac{t}{4} \int (I_2 * u^2) \, u^2 + \frac{t^{3/2}}{3} \int |u|^3.
\]
Easily to see that
\[
\inf_{t > 0} E(u^t) < 0,
\]
for any nonzero function $u$, from which follows that
\[
I_M = \inf \left\{ \inf_{t > 0} E(u^t); u \in A_M \right\} < 0.
\]

We remark that only if $2 > \alpha > 1/3$, we can show the same thing with the Slater term $C_S|Q|^{2\alpha+2}$.

The following proposition is easy but crucial, which will provide sufficient and necessary condition for the compactness of minimizing sequences of $I_M$. The index $\alpha = 1/2$ plays a role here. For other $\alpha$’s, there are some argument difficulty.

Proposition 2.1. $I_M < I_\alpha$ for any $0 < \alpha < M$. Furthermore, $I_M = M^6 I_1$.

Proof. The transform $u \mapsto u_M = M^4u(M^2x)$ is a 1-1 mapping from $A_1$ to $A_M$, direct computation yields that
\[
E(u_M) = M^6E(u),
\]
hence

\[ I_M = \inf \{ E(u); u \in A_M \} = \inf \{ M^6 E(u); u \in A_1 \} = M^6 I_1. \]

Since \(-\infty < I_1 < 0\), \(I_M\) is strictly decreasing with respect to \(M\). This gives the proof.

We are now on the position to show the existence result.

**Theorem 2.2. (Existence)** For any \(M > 0\) there exists a minimizer of \(I_M\). Moreover, for any minimizing sequence \(\{u_k\}\), the rearrangement sequence \(\{u_k^*\}\) is also a minimizing sequence containing a subsequence converging in \(H^1(\mathbb{R}^3)\), and the minimizer is nonnegative and nonincreasing radially symmetric.

**Proof.** The existence proof is standard, and we just give it for the reader’s convenience.

Suppose that \(\{u_k\} \subset A_M\) is a minimizing sequence so that \(E(u_k) \to I_M\). As observed above, this sequence is bounded in \(H^1\). Let \(u_k^*\) be the nonincreasing symmetric rearrangement of \(u_k\), then \(u_k^* \in H^1(\mathbb{R}^3)\) is radially symmetric, nonnegative, nonincreasing function satisfying

\[
\int \nabla u_k^*|^2 \leq \int \nabla u_k|^2, \\
\int (I_2 * u_k^2) u_k^2 \geq \int (I_2 * u_k^2) u_k^2, \\
\int |u_k^*|^p = \int |u_k|^p \quad \text{for } p = 2, 3,
\]

(see [14] for a proof), thus it follows that \(u_k^* \in A_M\) and

\[ E(u_k^*) \leq E(u_k), \]

hence \(\{u_k^*\}\) is also a minimizing sequence and thus bounded.

Therefore, we can always assume that the minimizing sequence \(\{u_k\}\) is nonincreasing nonnegative and radially symmetric. Thus there exists a function \(u \in H^1(\mathbb{R}^3)\) so that up to a subsequence there has

\[
u_k \rightharpoonup u \text{ in } H^1(\mathbb{R}^3), \\
u_k \to u \text{ in } L^2_{\text{loc}}(\mathbb{R}^3), \\
u_k \to u \text{ a.e. } x \in \mathbb{R}^3.
\]

It is easy to see that \(u\) is a nonincreasing nonnegative and radially symmetric function. Moreover, by the compact embedding theorem of [23] there holds that

\[ u_k \to u \text{ in } L^p(\mathbb{R}^3), \quad \text{for all } 2 < p < 6. \]
Let $v_k = u_k - u$, it follows from the above strong convergence in $L^p$ space that

\[
\int |\nabla u_k|^2 = \int |\nabla v_k|^2 + \int |\nabla u|^2 + o(1)
\]
\[
\int_{\mathbb{R}^3} (I_2 * u_k^2) u_k^2 = \int_{\mathbb{R}^3} (I_2 * u^2) u^2 + o(1),
\]
\[
\int |u_k|^p = \int |u|^p + o(1),
\]
where $o(1) \to 0$ as $k \to \infty$. Therefore

\[
E(u_k) = \frac{1}{2} \int |\nabla v_k|^2 + E(u) + o(1). \tag{2.3}
\]

By passing to limit $k \to \infty$, we obtain that

\[
E(u) \leq I_M < 0,
\]
which implies $u \neq 0$. Thus we only need to show that $u \in \mathcal{A}_M$, i.e. $|u|^2 = M$.

However, by the Fatou’s lemma $|u|^2 \leq \liminf_k |u_k|^2 = M$. Suppose that $u \notin \mathcal{A}_M$, then $\exists \alpha \in (0, M)$ s.t. $|u|^2 = \alpha$. Thus $E(u) \geq I_\alpha > I_M$ according to Proposition 2.1 which contradicts to $E(u) \leq I_M$. Thus $|u|^2 = M$. This shows that $u$ is a minimizer of $I_M$.

Furthermore, since $u_k$ converges weakly to $u$ in $L^2$ and also the $L^2$ norm is preserved, we see that $u_k \to u$ in $L^2$. Also, by the equation (2.3) we get that $\int |\nabla v_k|^2 \to 0$ as $k \to \infty$, i.e., $\nabla u_k \to \nabla u$ in $L^2$. Hence $u_k \to u$ in $H^1(\mathbb{R}^3)$.

This completes the proof. \qed

\section{Smoothness and Uniqueness}

In this section, we show that any minimizer of $I_M$ is $C^\infty$ smooth and modulo translations, it is unique. Remember that up to scaling, minimizers of $I_M$ satisfies the following Euler-Lagrange equation

\[
\begin{cases}
-\Delta Q + Q - (I_2 \ast Q^2) Q + C_S Q^2 = 0, & \text{in } \mathbb{R}^3 \\
Q = Q(|x|) \geq 0, & \\
Q \in H^1(\mathbb{R}^n)
\end{cases} \tag{3.1}
\]

Essentially, all results in this section are consequences of equation theory. Thus they are applicable to solutions of SPS equation, not only minimizers of $I_M$.

First we derive the following regularity result, which is only a consequence of the regularity theory of elliptic equations, see, e.g. \[9\].

\begin{theorem}
$u \in C^\infty(\mathbb{R}^3)$ with bounded derivatives of all orders.
\end{theorem}
Proof. Written by $V = I_2 * Q^2$, it follows from the Riesz potential theory (see [22]) that $I_2$ is a bounded linear operator from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ for any $1 < p < \frac{3}{2}$ and $q = \frac{3p}{3p - 2}$, thus $V \in L^q(\mathbb{R}^3)$ for any $3 < q < \infty$ since $Q^2 \in L^1 \cap L^{3/2}(\mathbb{R}^3)$, thus $VQ \in L^q(\mathbb{R}^3)$ for all $r \in (6/5, 6)$. As $Q^2 \in L^1 \cap L^3(\mathbb{R}^3)$ we see that $VQ + CSQ^2 \in L^3(\mathbb{R}^3)$. Hence by the Calderon-Zygmund theory, $Q \in W^{2,3}(\mathbb{R}^3)$, which implies that $Q \in C^\alpha(\mathbb{R}^3)$ for all $0 < \alpha < 1$, the Hölder space of order $\alpha$.

Hence $Q^2 \in C^{\alpha}(\mathbb{R}^3)$ for all $0 < \alpha < 1$. Since $-\Delta V = Q^2$, Schauder’s theory implies that $V \in C^{2,\alpha}(\mathbb{R}^3)$ for all $0 < \alpha < 1$. (This could be firstly done locally, say, on balls $B(x, 1)$, but since the estimates does not depend on the center $x$, we get $C^{2,\alpha}$ estimate on the whole space.) Again, we find that $VQ + Q^2$ is Hölder continuous, hence $Q \in C^{2,\alpha}(\mathbb{R}^3)$ for all $0 < \alpha < 1$.

In general, if $Q \in C^{k,\alpha}(\mathbb{R}^3)$, then $V \in C^{k+\alpha,\alpha}(\mathbb{R}^3)$ for all $0 < \alpha < 1$, which ensures that $VQ + Q^2 \in C^{k,\alpha}(\mathbb{R}^3)$, and apply Schauder’s theory again to see that $Q \in C^{2+k,\alpha}(\mathbb{R}^3)$. This finishes the proof.

Theorem 3.2. There exists at most one positive solution of the Schrodinger-Poisson-Slater equation (3.1).

Proof. We follow the method of [13]. Since $Q \geq 0$ and $Q \neq 0$, rewriting the equation as

$$-\Delta Q + c(x)Q = (I_2 * Q^2) Q \geq 0 \quad \text{in} \ \mathbb{R}^3,$$

where $c(x) = 1 + CSQ \geq 0$ for all $x \in \mathbb{R}^3$, then the strong maximum principle and Hopf lemma of elliptic equations (see [9] for instance) implies that $Q > 0$ and $Q'(|x|) = \frac{\partial Q}{\partial r}(x) < 0$ for $|x| > 0$.

As in [13], applying Newton’s theorem which states that

$$- (I_2 * Q^2)(r) = \int_{0}^{r} K(r, s)Q^2(s)ds - \int_{0}^{\infty} Q^2(s)ds =: A_Q(r) - \lambda_Q,$$

where

$$K(r, s) = s \left( 1 - \frac{s^{n-2}}{r^{n-2}} \right) \geq 0$$

for $r \geq s$, the equation can be read as

$$- \left( Q'' + \frac{n-1}{r} Q' \right) + A_Q(r)Q + CSQ^2 = (\lambda_Q - 1)Q.$$

We claim that $\lambda_Q - 1 > 0$. In fact, since $Q \geq 0 \geq Q'$, by multiplying $Q$ on both sides of the equation and integrating from 0 to $\infty$, one derives that

$$(\lambda_Q - 1) \int_{0}^{\infty} Q^2 = \int_{0}^{\infty} Q'^2 - \frac{n-1}{r} Q'Q + A_Q(r)Q^2 + Q^3dr > 0.$$ 

By a simple scaling argument ($Q(r) = \mu^2 P(\mu r)$, $\mu^2 = \lambda_Q - 1$), we only need to consider the initial value problem with normalized coefficient. The special power of Slater term is crucial here.

$$\begin{cases}
- (Q'' + \frac{n-1}{r} Q') + A_Q(r)Q + CSQ^2 = Q, & r > 0, \\
Q(0) = 0, \\
Q'(0) = 0.
\end{cases}$$
We are now able to prove the theorem.

Suppose on the contrary $Q, R$ are two such solutions. Then by local uniqueness result of ODE, we get that if $Q \neq R$ then $Q(0) \neq R(0)$. So we may assume that $Q(0) > R(0)$, it follows by continuity that $Q > R$ in a neighborhood of zero. We use Sturm-type comparison argument as follows. Multiplying the equation of $Q$ by $R$ and vice versa, and letting $S = Q'R - QR'$, then we get from the difference of the two equations that

$$(r^{n-1}S)' = r^{n-1} (A_Q - A_R) QR + r^{n-1} (Q - R) QR.$$ 

Since $Q > R$ in a neighborhood of 0, we see that $(r^{n-1}S)' > 0$ in some maximum interval $(0, r^*) \subset (0, \infty)$ s.t. $(r^{n-1}S)' > 0$ in $(0, r^*)$ and $(r^{n-1}S)'(r^*) = 0$. We shall show that $r^* = \infty$. Otherwise, suppose that $r^* < \infty$, then $(r^{n-1}S)' > 0$ in $(0, r^*)$ and $S(0) = 0$ implies that $S > 0$ in $(0, r^*)$. Equivalently, i.e.,

$$(Q/R)' = S/R > 0 \quad \text{in} \quad (0, r^*).$$ 

Hence

$$Q > \frac{Q(0)}{R(0)} R > R \quad \text{in} \quad (0, r^*].$$ 

But this turns out that

$$A_Q(r^*) > A_R(r^*),$$ 

then $(r^{n-1}S)'(r^*) > 0$, contradiction to the assumption. Hence $r^* = \infty$, and so $Q > R$ and $A_Q - A_R > 0$ in $(0, \infty)$. However, this is also impossible, since we have

$$0 < \int_0^\infty r^{n-1} (A_Q - A_R + Q - R) QR = \int_0^\infty (r^{n-1}Q')' R - (r^{n-1}R')' Q dr = 0.$$ 

Thus $Q \equiv R$. This finishes the proof. \(\square\)

4 Asymptotic behavior

In this last section, we study the asymptotic behavior of solutions. There should be some results already in the literature, but since we cannot give such a reference, and also under the consideration that it would be useful for a further study in the future, we'd like to give a precise enough result here. As both repulsive and attractive case of SPS system with different Slater terms are interesting, we'd better to allow our result to have some generality.

Theorem 1.2 becomes a special case of the following one

**Theorem 4.1.** Let $Q$ be a positive radially symmetric solution to the equation

$$\begin{cases}
-\Delta Q + Q = \epsilon (I_2 * Q^2) Q + f(Q)Q, & \text{in } \mathbb{R}^3 \\
Q(|x|) \to 0 & \text{as } |x| \to \infty.
\end{cases}$$
where, $\epsilon = \pm 1$, and for some $\beta > 0$, $f$ satisfies the condition that

$$\lim_{t \to 0^+} \frac{f(t)}{t^\beta} = 0.$$ 

Then

$$\lim_{|x| \to \infty} Q(x)|x|^{1-\epsilon \alpha/2}e^{|x|} \in (0, \infty),$$

with

$$\alpha = \frac{1}{4\pi} \int_{\mathbb{R}^3} |Q|^2 \, dx.$$ 

A typical example of $f$ is given by $f(t) = \sum c_i t^{-1}t_i$ for some constant $c_i$ and some positive constants $\beta_i$, especially, $f(t) = -C_S |t|^{2\alpha t}$ coincides with the Slater term.

In case $f \equiv 0$, this result has been contained in [18] Proposition 6.5, where the authors proved much more general results. Their method can be easily applied to our theorem, but for the reader’s convenience, we shall give a sketch of proof here. From the proof the readers will see that the last term $f(Q)Q$ is in essence a perturbation of the nonlocal term and thus nothing changes. There are still rooms to relax the restriction on $f$. In the same spirit, our theorem can be extended to general Choquard equation and higher dimensions as well, but which is not our interest here.

We divide the proof into several propositions. Hereafter we write $V = I_2 * Q^2$.

Obviously $V(x) = V(|x|) > 0$ and $V(r) \to 0$ as $r \to \infty$. The first proposition is about exponential decay of $Q$.

**Proposition 4.2.** $Q(r) = O(e^{-r/2})$ as $r \to \infty$.

**Proof.** The proof is based on an easy comparison argument. Let $G(x) = |x|^{-1}e^{-|x|/2}$ so that $-\Delta G + G/2 = 0$ for $x \neq 0$. Since $Q(x) = o(1)$ as $|x| \to \infty$, for $|x|$ large enough, $\epsilon V(x) - f(Q(x)) \leq 1/2$, thus $-\Delta Q + Q/2 \leq 0$ for $|x|$ large. Therefore by multiplying $Q$ to the equation of $G$, integrating on the exterior ball $B_r^c(0)$ and vice versa, we get that $Q'G - QG' \leq 0$ as $r \to \infty$, i.e., $Q/G$ is a nonincreasing function, hence $Q(r) \leq (Q(r_0)/G(r_0))G(r)$ for all $r > r_0$ with $r_0$ large enough. This gives the proof. 

Secondly we estimate the nonlocal term $V$, which, unlike the solution $Q$, decays polynomially at infinity and gives contribution to the decay of $Q$.

**Proposition 4.3.** $V(r) = \frac{\alpha}{r} + O(r^{-2})$ and $V'(r) = -\frac{\alpha}{r^2} + O(r^{-3})$ as $r \to \infty$, where $\alpha = \frac{1}{4\pi} \int_{\mathbb{R}^3} |Q|^2 \, dx$.

**Proof.** This is rooted in the fact that $-\Delta V = Q^2$. Since $Q$ is radially symmetric, so is $V$, we have that

$$(r^2V'(r))' = -r^2Q^2(r) \leq 0,$$
which implies that $r^2V'(r)$ is nonincreasing and hence $V' \leq 0$ since $V'(0) = 0$ (by smoothness of $V$). Therefore

$$V'(r) = -r^{-2} \int_0^r s^2Q^2(s)ds = -\frac{1}{4\pi r^2} \int_{B_r(0)} Q^2 dx,$$

which implies that

$$V(r) = \int_r^\infty \frac{1}{4\pi s^2} \left( \int_{B_s(0)} Q^2 dx \right) ds = \frac{1}{4\pi r} \int_{\mathbb{R}^3} Q^2 dx - \int_r^\infty \frac{1}{4\pi s^2} \left( \int_{|x| \geq s} Q^2 dx \right) ds.$$

According to the exponential decay of $Q$ at infinity we see that $\int_r^\infty \frac{1}{4\pi s^2} \left( \int_{|x| \geq s} Q^2 dx \right) ds = O(r^{-2})$. This finishes the proof. \[\square\]

Finally to prove the asymptotic behavior of $Q$ at infinity we apply the following lemma which given by [18] Lemma 6.4

**Lemma.** Let $\rho \geq 0$ and $W \in C^1((\rho, \infty), \mathbb{R})$. If

$$\lim_{s \to \infty} W(s) > 0$$

and for some $\beta > 0$

$$\lim_{s \to \infty} W'(s)s^{1+\beta} = 0,$$

then there exists a nonnegative radial function $v : \mathbb{R}^N \setminus B_\rho \to \mathbb{R}$ such that

$$-\Delta v + Wv = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_\rho,$$

and for some $\rho_0 \in (\rho, \infty)$,

$$\lim_{|x| \to \infty} v(x)|x|^\frac{N-1}{2} \exp \left( \int_{|x| \rho_0} \sqrt{W} \right) = 1.$$

Our theorem now is a consequence of the above lemma and the previous estimates on $V$ and $Q$.

**Proof of theorem 4.1.** Let $W = 1 - \epsilon V - f(Q)$, from the estimates on $V$ and $Q$ one sees $f(Q(x))$ owns exponential decay by the assumption on $f$. Thus $W = 1 - \frac{\alpha}{r^2} + O(r^{-2})$ as $r \to \infty$ with $\alpha = \frac{1}{4\pi} \int_{\mathbb{R}^3} |Q|^2 dx$ and $\lim_{r \to \infty} W'(s)s^{1+1/2} = 0$. Therefore, according the above lemma, for $\rho$ large enough, there exists a nonnegative radial function $v : \mathbb{R}^3 \setminus B_\rho \to \mathbb{R}$ such that

$$-\Delta v + Wv = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B_\rho,$$

and for some $\rho_0 \in (\rho, \infty)$,

$$\lim_{|x| \to \infty} v(x)|x|^\frac{N-1}{2} \exp \left( \int_{\rho_0} |x| \sqrt{W} \right) = 1.$$
We show that \( Q(x) = cv(x) \) for some constant \( c > 0 \). In fact, since both \( Q \) and \( v \) solves the same equation in \( \mathbb{R}^3 \setminus B_\rho \), we see that

\[
\int_{\mathbb{R}^N \setminus B_\rho} (-\Delta Q + WQ) v \, dx = \int_{\mathbb{R}^N \setminus B_\rho} (\nabla Q \nabla v + WQv) \, dx + 4\pi \rho^2 Q'(\rho)v(\rho)
\]

and

\[
\int_{\mathbb{R}^N \setminus B_\rho} (-\Delta v + Wv) Q \, dx = \int_{\mathbb{R}^N \setminus B_\rho} (\nabla Q \nabla v + WQv) \, dx + 4\pi \rho^2 Q(\rho)v'(\rho).
\]

Hence \( Q'v = v'Q \), which implies that \( (v/Q)' = 0 \). Hence \( Q = cv \) for some positive constant \( c \). Therefore

\[
\lim_{|x| \to \infty} Q(x)|x| \exp\left(\int_{\rho_0}^{|x|} \sqrt{W}\right) \in (0, \infty).
\]

Substituting \( W = 1 - \alpha r + O(r^{-2}) \) one gets that

\[
\lim_{|x| \to \infty} Q(x)|x|^{1-\alpha/2}e^{|x|} \in (0, \infty).
\]

This finishes the proof.

References

[1] O. Bokanowski, J. López, Ó. Sánchez, and J. Soler. Long time behaviour to the Schrödinger-Poisson-\( X^\alpha \) systems. In *Mathematical physics of quantum mechanics*, volume 690 of *Lecture Notes in Phys.*, pages 217–232. Springer, Berlin, 2006.

[2] O. Bokanowski, J. López, and J. Soler. On an exchange interaction model for quantum transport: the Schrödinger-Poisson-Slater system. *Math. Models Methods Appl. Sci.*, 13(10):1397–1412, 2003.

[3] I. Catto, J. Dolbeault, O. Sánchez, and J. Soler. Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the concentration-compactness principle. *Math. Models Methods Appl. Sci.*, 23(10):1915–1938, 2013.

[4] I. Catto and P.-L. Lions. Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. I. A necessary and sufficient condition for the stability of general molecular systems. *Comm. Partial Differential Equations*, 17(7-8):1051–1110, 1992.

[5] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.
[6] T. D’Aprile and D. Mugnai. Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(5):893–906, 2004.

[7] T. D’Aprile and J. Wei. On bound states concentrating on spheres for the Maxwell-Schrödinger equation. *SIAM J. Math. Anal.*, 37(1):321–342 (electronic), 2005.

[8] V. Georgiev, F. Prinari, and N. Visciglia. On the radiality of constrained minimizers to the Schrödinger-Poisson-Slater energy. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(3):369–376, 2012.

[9] D. Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[10] R. Illner, P. F. Zweifel, and H. Lange. Global existence, uniqueness and asymptotic behaviour of solutions of the Wigner-Poisson and Schrödinger-Poisson systems. *Math. Methods Appl. Sci.*, 17(5):349–376, 1994.

[11] M.K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.

[12] E. Lenzmann. Uniqueness of ground states for pseudorelativistic Hartree equations. *Anal. PDE*, 2(1):1–27, 2009.

[13] E. H. Lieb. Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation. *Studies in Appl. Math.*, 57(2):93–105, 1976/77.

[14] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[15] E. H. Lieb and B. Simon. The Thomas-Fermi theory of atoms, molecules and solids. *Advances in Math.*, 23(1):22–116, 1977.

[16] L. Ma and L. Zhao. Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Ration. Mech. Anal.*, 195(2):455–467, 2010.

[17] N. J. Mauser. The Schrödinger-Poisson-Xα equation. *Appl. Math. Lett.*, 14(6):759–763, 2001.

[18] J. Moroz, V.and Van Schaftingen. Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. *J. Funct. Anal.*, 265(2):153–184, 2013.

[19] D. Ruiz. Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere. *Math. Models Methods Appl. Sci.*, 15(1):141–164, 2005.
[20] D. Ruiz. The Schrödinger-Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.*, 237(2):655–674, 2006.

[21] Ó. Sánchez and J. Soler. Long-time dynamics of the Schrödinger-Poisson-Slater system. *J. Statist. Phys.*, 114(1-2):179–204, 2004.

[22] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[23] W. A. Strauss. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.*, 55(2):149–162, 1977.

[24] I. M. Tod, P. and Moroz. An analytical approach to the Schrödinger-Newton equations. *Nonlinearity*, 12(2):201–216, 1999.