Localized Mass and Spin in 2+1 Dimensional
Topologically Massive Gravity

A. Edery and M. B. Paranjape

Groupe de Physique des Particules, Département de Physique,
Université de Montréal, C.P. 6128,
succ. centreville, Montréal, Québec, Canada, H3C 3J7

Abstract
Stationary solutions to the full non-linear topologically massive gravity (TMG) are obtained for localized sources of mass \( m \) and spin \( \sigma \). Our results show that the topological term induces spin and that the total spin \( J \) (which is the spin observed by an asymptotic observer) ranges from 0 to \( \sigma + \frac{m}{\mu} \left( \frac{4\pi + m}{4\pi + 2m} \right) \) depending on the structure of the spin source (here \( \mu \) is the topological mass). We find that it is inconsistent to consider actual delta function mass and spin sources. In the point-like limit, however, we find no condition constraining \( m \) and \( \sigma \) contrary to a previous analysis [2].

I. Introduction
A few years ago Deser [1] obtained solutions to linearized TMG with point mass \( m \) and point spin \( \sigma \) as source. Among other things, he found that the topological term induced a spin \( m/\mu \). In our work we consider the full non-linear theory with a highly localized (approaching a point) mass source \( m \) and spin source \( \sigma \). Like the linear theory, we obtain an induced spin due to the presence of the topological term but it has a range of values (the value depends on the structure of the spin source \( \sigma \)). For the case where the total spin is at its maximum value, we obtain an induced spin of \( \frac{m}{\mu} \left( \frac{4\pi + m}{4\pi + 2m} \right) \). As \( m \) gets smaller this result approaches \( m/\mu \) which is the value obtained in the linear theory.

Clément [2] has also obtained solutions to the full non-linear theory with delta function mass \( m \) and delta function spin \( \sigma \) as sources. His results showed that the mass and spin must be constrained by the condition \( m + \mu \sigma = 0 \). We argue, however, that the use of a delta function spin source, while allowed in the linear theory [1], is not allowed in the full theory and renders the condition \( m + \mu \sigma = 0 \) invalid.
An alternative approach exists \cite{5} using dreibeins and the spin connection, where delta function spin and mass sources can be treated directly. This formulation facilitates the inclusion of torsion and one finds non-zero torsion localized at the position of the source. We will not pursue the possibility of allowing for torsion below. Existence of torsion is an additional structure on the manifold; one has to specify the equation governing the torsion field. We restrict our work to metrical theories of gravitation, where the torsion is zero and the metric is covariantly conserved.

II. Field Equations

We begin our work by writing down the well known field equations for TMG with energy-momentum tensor $T_{\mu\nu}$ as source. The Einstein field equations including a topological mass term is given by \cite{1,2} (in units where $8\pi G = 1$)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{\mu} C_{\mu\nu} = -\kappa^2 T_{\mu\nu} \]  

(1)

where $R_{\mu\nu}$ is the Ricci tensor, $R \equiv R_{\mu\nu} g^{\mu\nu}$ is the curvature scalar, and $C_{\mu\nu}$ is the 3 dimensional Weyl (Cotton) tensor defined by

\[ C_{\mu\nu} = \frac{1}{2} (det g_{\delta\sigma})^{-1/2} \left( \epsilon^{\alpha\beta}_{\mu} D_\alpha R_{\nu\beta} + \epsilon^{\alpha\beta}_{\nu} D_\alpha R_{\mu\beta} \right). \]

Note that the trace of $C_{\mu\nu}$ is identically zero. The reason for the negative sign in front of $T_{\mu\nu}$ in (1) has been discussed by Deser \cite{1}. We simplify the field equations (1) by choosing a rotationally symmetric, stationary metric. The most general form for such a metric is given by \cite{3}

\[ ds^2 = n^2 (dt + \omega_i dx^i)^2 + h_{ij} dx^i dx^j \]

(2)

with $det g_{\mu\nu} = n^2 det h_{ij}$. We find that, for the present purpose, it suffices to consider the case with $n = 1$ (i.e. we show that the metric with $n = 1$ does support solutions with point mass and point spin). In two space dimensions any metric is conformally flat so that we can express $h_{ij}$ as

\[ h_{ij} = -e^{\phi(r)} \delta_{ij}. \]

(3)

The negative sign in (3) corresponds to Minkowski signature. The functions $\psi(r)$ and $\phi(r)$ completely determine the metric. We are particularly interested in the asymptotic behaviour of $\psi(r)$ since it is proportional to the total spin $J$ (see \cite{4}).
The scalar twist $\rho(r)$ is defined by

$$
\rho \equiv \frac{n}{\sqrt{\det h_{ij}}} \epsilon^{ij} \partial_i \omega_j
$$

$$
= e^{-\phi(r)} \left( \frac{\psi'(r)}{r} + 2\pi \psi(0) \delta^2(\vec{r}) \right)
$$

(4)

where $\psi'(r) \equiv \frac{d\psi}{dr}$. Then the Ricci tensor and curvature scalar are given by

$$
R_{00} = \frac{1}{2} \rho^2
$$

$$
R_i^j = \epsilon^{jk} e^{-\phi} \partial_k (\rho)
$$

$$
\tilde{R}^{ij} = \frac{1}{2} \tilde{R} h^{ij} - \frac{h^{ij} \rho^2}{2}
$$

$$
R = \tilde{R} - \frac{\rho^2}{2}
$$

(5)

and the Weyl (Cotton) tensor is given by

$$
C_{00} = \rho^3 - \frac{1}{2} \left( \tilde{\nabla}^2 \rho + \tilde{R} \rho \right)
$$

$$
C_i^j = \frac{\epsilon^{jk} e^{-\phi}}{2} \partial_k \left( \frac{3}{2} \rho^2 - \frac{1}{2} \tilde{R} \right)
$$

$$
C^{ij} = h^{ij} \left( \frac{-3}{8} \rho^3 + \frac{1}{2} \tilde{\nabla}^2 \rho + \frac{1}{4} \rho \tilde{R} \right) - \frac{1}{2} \tilde{\nabla}^i \tilde{\nabla}^j \rho
$$

(6)

All quantities in (3) and (4) are defined with respect to the spatial metric $h_{ij}$. Indices $i, j, k$ are raised and lowered by $h_{ij}$, $\tilde{\nabla}$ stands for covariant differentiation with respect to $h_{ij}$ and $\tilde{R}$ is the two dimensional curvature scalar given by

$$
\tilde{R} = e^{-\phi} \tilde{\nabla}^2 \phi \text{ where } \tilde{\nabla}^2 \text{ is the flat laplacian.}
$$

(7)

We now solve the field equations (1) for each component using (3), (4) and (7). Without loss of generality we set $\kappa = 1$.

The (0,0), (0,j) and (i,j) component equations are respectively

$$
\frac{3}{4} \rho^2 - \frac{1}{2} e^{-\phi} \tilde{\nabla}^2 \phi + \frac{1}{\mu} \rho^3 - \frac{1}{2\mu} \tilde{\nabla}^2 \rho - \frac{1}{2\mu} \rho e^{-\phi} \tilde{\nabla}^2 \phi = -T_{00}
$$

(8)
\[
\frac{\epsilon^{ij} e^{-\phi}}{2} \partial_k \left( \rho + \frac{3}{2\mu} \rho^2 - \frac{1}{2\mu} e^{-\phi} \nabla^2 \phi \right) = -T^j_0 \tag{9}
\]
\[
- e^{-\phi} \delta^{ij} \left( -\frac{\rho^2}{4} - \frac{1}{2\mu} \rho^3 + \frac{1}{2\mu} \nabla^2 \rho + \frac{1}{4\mu} \rho e^{-\phi} \nabla^2 \phi \right) - \frac{1}{2\mu} \hat{\nabla}^i \hat{\nabla}^j \rho = -T^{ij}. \tag{10}
\]

Note that the scalar twist \( \rho \) appearing in the above equations cannot contain a delta function or else quantities like \( \rho^2 \) and \( \rho^3 \) in those equations would be ill defined. This implies that \( \psi(0) \) appearing in the definition of \( \rho \) i.e. (4) must be zero. Therefore \( \rho \) reduces to
\[
\rho = e^{-\phi} \frac{\psi'(r)}{r}. \tag{11}
\]

We approach the problem of localized sources, not by actually specifying \( T_{\mu\nu} \) but by examining the metric dependent side of the field equations (8-10), and drawing conclusions on the scalar twist \( \rho \) and the function \( \phi \) if \( T_{\mu\nu} \) were localized. Hence \( T_{\mu\nu} \) is only defined via the field equations (8-10), and therefore automatically covariantly conserved (i.e. since the metric dependent side of the field equations obey the Bianchi identities).

III. Solving the Field Equations

We consider a localized spin source \( \sigma \) by allowing the scalar twist \( \rho(r) \) be a rapidly decreasing function of \( r \). The total spin \( J \), which is conserved and invariant under general coordinate transformations, is given by (see [4])
\[
J \equiv 2\pi \lim_{r \to \infty} \psi(r) = 2\pi \int_0^\infty \rho e^\phi r dr, \tag{12}
\]
where (11) was used. The scalar twist \( \rho \) can therefore be interpreted as the total spin density. The mass \( m \) is defined as the total energy when the total spin is zero (i.e. \( \rho = 0 \)) and when the topological term is absent (i.e. \( \mu \to \infty \)). It is therefore given by
\[
m = \int \frac{1}{2} \left( e^{-\phi} \nabla^2 \phi \right) e^\phi d^2 r = \int \frac{1}{2} \nabla^2 \phi d^2 r. \tag{13}
\]
Here \( \nabla^2 \phi \) can be arbitrarily localized. The above equation for \( m \) leads to the condition that
\[
r \phi' \bigg|_0^\infty = \frac{m}{\pi}. \tag{14}
\]
For reasons given in section IV we do not allow $\nabla^2 \phi$ to be a delta function (i.e. we exclude the possibility that $\phi(r) \propto \ln r$ as $r \to 0$). Therefore

$$\lim_{r \to 0} (r \phi') = 0$$

(15)
i.e. if $\lim_{r \to 0} (r \phi') = k$ where $k \neq 0$, then $\phi(r) \propto k \ln r$ as $r \to 0$.

We now want to find an expression for the total spin $J$. We first integrate the $T^i_0$ equation (9) and find that

$$\int \epsilon^{ij} x^i \left( -T^j_0 \right) e^{2\phi} d^2 r = -\frac{1}{2} \int \left( \rho + \frac{3}{2\mu} \rho^2 - \frac{1}{2\mu} e^{-\phi \nabla^2 \phi} e^{\phi} r d^2 r \right).$$

(16)
The integral of the third term can be readily evaluated using (14) and (15) and gives $-(m/\mu) (1 + m/4\pi)$. As we will show in section IV, this result is different from that obtained using naive manipulations with delta function sources. For the first two terms we take $\rho(r)$ to be a rapidly decreasing function of $r$ with the condition that $\lim_{r \to \infty} \rho(r) r^2 e^{\phi} = \lim_{r \to \infty} \rho(r) r^2 e^{\phi} = 0$. The integrals of these two terms are well defined for any regular mass distribution $\nabla^2 \phi$, but depend on the actual profile and there will be small corrections for any well localized mass. However, when evaluated in the point mass limit the result is

$$\left( 2\pi + m \right) \int_{0}^{\infty} \left( \rho + \frac{3}{2\mu} \rho^2 \right) e^{\phi} r dr.$$

(17)

Then using (12) for the total spin, one can rewrite (16) as

$$J = \sigma + \frac{m}{\mu} \left( \frac{4\pi + m}{4\pi + 2m} \right) - 2\pi \int_{0}^{\infty} \frac{3}{2\mu} \rho^2 e^{\phi} r dr.$$

(18)

where

$$\sigma \equiv \frac{2\pi}{2\pi + m} \int \epsilon^{ij} x^i \left( -T^j_0 \right) e^{2\phi} d^2 r.$$

(19)

Here $\sigma$ is identified as the spin source (or bare spin) i.e. it is equal to the total spin when the topological term is absent ($\mu \to \infty$). Clearly, as $m$ approaches zero equation (19) reduces to the usual definition of spin in the linear theory. The $m$ appearing in the denominator of the spin source $\sigma$ arises because the spatial metric describes a cone with a negative angular defect $m$ and therefore the angle ranges from $0$ to $2\pi + m$ instead of $2\pi$ (see [1]).
The last two terms on the right hand side of (18) can be regarded as the induced spin, which has a dependence on $\rho(r)$. The total spin $J$ can therefore range from 0 to its maximum value of $\sigma + \frac{m}{\mu} \left( \frac{4\pi + m}{4\pi + 2m} \right)$. The induced spin reduces to the linearized result $m/\mu$ in the limit where $m$ and $\rho$ are small.

We do not obtain any specific equation relating the mass $m$ to the spin $\sigma$ as in ref. [2] where the condition $m + \mu \sigma = 0$ is obtained. For the specific case $\rho = 0$ (which implies $J = 0$) we obtain $m \left( \frac{4\pi + m}{4\pi + 2m} \right) + \mu \sigma = 0$ in disagreement with the result $m + \mu \sigma = 0$ of Ortiz [3]. We now discuss the reasons for this discrepancy.

IV. The Point Like Limit

We now give an analysis of the use of delta functions for sources. A delta function spin source for $T_{0}^{j}$ was used in the work of Clément [2]

\begin{equation}
T_{0}^{j} = \frac{-1}{2} \sigma e^{-\phi} e^{k} \partial_{k} \left( e^{-\phi} \delta^{2}(\vec{r}) \right)
\end{equation}

where $\sigma$ is equivalent to the spin source defined in (19). Then (9) becomes

\begin{equation}
\partial_{k} \left( \rho + \frac{3}{2\mu} \rho^{2} - \frac{1}{2\mu} e^{-\phi} \nabla^{2} \phi \right) = \sigma \partial_{k} \left( e^{-\phi} \delta^{2}(\vec{r}) \right).
\end{equation}

The $\rho$ and $\rho^{2}$ terms on the left hand side of (21) cannot match the delta function on the right hand side (see comment above (11)) and for the purposes of our forthcoming argument will simply be dropped). The term $e^{-\phi} \nabla^{2} \phi$, with $\phi$ proportional to $\ln r$, cannot match the delta function because the non-linearity in (21) imposes the products of derivatives of $\ln r$ which are ill defined at the origin. Consider the contraction of (21) with $x^{k}$. This implies

\begin{equation}
- \frac{1}{2\mu} e^{-\phi} \left( x^{k} \partial_{k} \left( \nabla^{2} \phi \right) - r\phi' \nabla^{2} \phi \right) = \sigma e^{-\phi} \left( x^{k} \partial_{k} \delta^{2}(\vec{r}) - r\phi' \delta^{2}(\vec{r}) \right)
\end{equation}

which requires, for $\phi = - (\mu \sigma / \pi) \ln r$,

\begin{equation}
r\phi' \nabla^{2} \phi = -2 \mu \sigma \ r\phi' \delta^{2}(\vec{r})
\end{equation}

where the first term in (22) is properly matched and the $e^{-\phi}$ is canceled with the corresponding $e^{\phi}$ in any volume element. Equation (23) is not sensible
for the imposed choice \( \phi \propto \ln r \). We cannot simply cancel the \( r\phi' \) from both sides since it is ill defined exactly at the point where the delta function has all its weight. It is not consistent to take \( r\phi' \) to be identically constant while taking \( \nabla^2 \phi \equiv (r\phi')' / r \) to be a delta function. The product \( r\phi' \nabla^2 \phi \) is clearly ill defined for \( \phi \propto \ln r \). The right hand side of (23) is also ill defined for \( \phi \propto \ln r \) because \( r\phi' \) does not lie in the space of functions on which \( \delta^2(\vec{r}) \) acts.

A reasonable way to handle the product \( r\phi' \nabla^2 \phi \) is to use the equivalence between a delta function \( \delta^2(\vec{r}) \) and the limit of a sequence of functions \( \delta_n(r) \) (see [7]) where

\[
\int \delta_n(r) \, d^2r = 1 \quad \forall n \quad \text{and} \quad \lim_{n \to \infty} \int \delta_n(r) f(r) \, d^2r = f(0).
\]

(24)

Here \( \delta_n(r) \) can be any smooth function, like a gaussian, that peaks a \( n \to \infty \). Then to satisfy (13) we let

\[
\nabla^2 \phi_n(r) = 2m\delta_n(r).
\]

(25)

Clearly (14)

\[
r \phi_n' \bigg|_0^\infty = \frac{m}{\pi}
\]

(26)

is valid for each \( n \). Since \( \nabla^2 \phi_n(r) \) is a smooth function for every \( n \), it follows that \( \phi_n(r) \) for any given \( n \) cannot be proportional to \( \ln r \) as \( r \to 0 \) (since \( \nabla^2 \ln r \) is not smooth at \( r=0 \)). Hence \( \lim_{r \to 0} (r\phi_n') = 0 \). Then the moment

\[
M = -\frac{1}{2} \int x^k \partial_k \left( -\frac{1}{2\mu} e^{-\phi} \nabla^2 \phi \right) e^\phi \, d^2r
\]

\[
= -\frac{1}{4\mu} \int \left( 2 + r\phi' \right) \nabla^2 \phi \, d^2r
\]

(27)

which was used to calculate the third term of (16) yields

\[
M = -\frac{\pi}{2\mu} \lim_{n \to \infty} \int_0^\infty \frac{1}{r} \left( r\phi_n'' \right) \left( 2 + r\phi_n' \right) r \, dr
\]

\[
= -\frac{\pi}{2\mu} \lim_{n \to \infty} \left( 2 \left( r\phi_n'' \right) \bigg|_0^\infty + \frac{1}{2} \left( r\phi_n' \right)^2 \bigg|_0^\infty \right)
\]

\[
= -\frac{\pi}{2\mu} \lim_{n \to \infty} \left( 2 \frac{m}{\pi} + \frac{1}{2} \left( \frac{m}{\pi} \right)^2 \right)
\]

\[
= -\frac{m}{\mu} \left( 1 + \frac{m}{4\pi} \right)
\]

(28)
where the limits on $r\phi'_n$ i.e.

$$\lim_{r \to \infty} r\phi'_n = \frac{m}{\pi} \quad \text{and} \quad \lim_{r \to 0} r\phi'_n = 0$$

was used (also note that the result (28) is valid for any $n$ and is independent of the limit $n \to \infty$). The right hand side of (23) would be equal to zero if the functions $\phi_n(r)$ were used, showing clearly the impossibility of satisfying equation (23) (and similarly (21)). A naive attempt to satisfy this equation by having $\phi = (m/\pi) \ln r$, $r\phi' = m/\pi$ and $\nabla^2 \phi = 2m\delta^2(\vec{r})$, which is not a consistent treatment of $r\phi'$, leads to the incorrect conclusion that $m + \mu \sigma = 0$ (see [2]) and to a wrong value for the moment $M$, namely

$$M = -\frac{m}{\mu} \left( 1 + \frac{m}{2\pi} \right).$$

The factor $\left( \frac{4\pi + m}{4\pi + 2m} \right)$ in (18) was obtained using the result (28) and would not appear if the result (30) were used. This is why for the specific case $\rho = 0$ we obtain the condition $m\left( \frac{4\pi + m}{4\pi + 2m} \right) + \mu \sigma = 0$ and not the condition $m + \mu \sigma = 0$ of Ortiz [4].

We have shown that the correct procedure in the non-linear theory is to take the source to be a function $\delta_n(r)$ that peaks as $n \to \infty$ instead of starting directly with a delta function. In the linearized theory [1] such a procedure is not necessary because products like $e^{-\phi} \nabla^2 \phi$ do not appear in the equations and one can begin directly with a delta function source. These products actually reflect the non-linearity of the field equations and this is why the factor $\left( \frac{4\pi + m}{4\pi + 2m} \right)$ in (18) does not appear in the linearized theory.

We thank Nourredine Hambli for useful discussions and NSERC of Canada and FCAR du Québec for financial support.

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