A case of multivariate Birkhoff interpolation using high order derivatives

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Abstract

We consider a specific scheme of multivariate Birkhoff polynomial interpolation. Our samples are derivatives of various orders $k_j$ at fixed points $v_j$ along fixed straight lines through $v_j$ in directions $u_j$, under the following assumption: the total number of sampled derivatives of order $k, k = 0, 1, \ldots$ is equal to the dimension of the space homogeneous polynomials of degree $k$. We show that this scheme is regular for general directions. Specifically this scheme is regular independent of the position of the interpolation nodes. In the planar case, we show that this scheme is regular for distinct directions.

Next we prove a “Birkhoff-Remez” inequality for our sampling scheme extended to larger sampling sets. It bounds the norm of the interpolation polynomial through the norm of the samples, in terms of the geometry of the sampling set.

Keywords Norming set · Norming constant · Multivariate Hermite interpolation · Remez-type inequality · Multivariate Birkhoff interpolation · Poised
1 Introduction

In this paper we study a specific scheme of Birkhoff polynomial interpolation. As in many other cases (compare [11, 9, 11, 14]) our samples are derivatives of various orders $k_j$ at fixed points $v_j$ along fixed straight lines through $v_j$ in directions $u_j$. It is convenient to assume that for each $j$ exactly one derivative is sampled, but the points and the directions are allowed to coincide. However, we make the following additional assumption: as we consider sampling of polynomials of degree $d$, the total number of sampled derivatives of order $k_j$, $k = 0, 1, \ldots, d$, is equal to the dimension of the space of homogeneous polynomials of degree $k_j$.

We show that the necessary and sufficient conditions on $v_j, u_j$ for this Birkhoff interpolation problem to be well-posed take a rather simple form, they depend only on the sampling directions $u_j$, but are independent of the sampling points $v_j$.

Our second result is a “Birkhoff-Remez” (see [21]) inequality for the sampling scheme considered (extended to larger sampling sets). It bounds the norm of the interpolation polynomial on a compact set by the norm of the samples, in terms of the geometry of the sampling set.

Our results are motivated by the following two basic questions:

1. For a prescribed type of Hermite or Birkhoff interpolation problem, find the conditions by which there exists a unique solution. This problem is central in Approximation Theory (see [18, 22, 4] and references therein). It was traditionally considered, from a somewhat different point of view, also in Algebraic geometry (see [20, 1, 8, 9, 13] and references therein). In particular, one of central open problems in this direction is the general dimensionality problem asks, for which $n, d, r, m_1, \ldots, m_r$ the corresponding problem is almost regular on $P_n^d$.

2. Our results are motivated by the following two basic questions: for each $v_j \in \mathbb{R}^n, 1 \leq j \leq r$, determine the corresponding problem to be well-posed and independent of the sampling points $v_j$. Consider the following multivariate Hermite interpolation problem.

**Problem 1.1** (Hermite Interpolation). Let $\{v_1, \ldots, v_r\} \subset \mathbb{R}^n$ be a set of points and let $\{m_1, \ldots, m_r\}$ be a set of positive integers such that $\sum_{k=1}^r \binom{m_k+n}{n}$ is the dimension of the space of homogeneous polynomials of degree $m_k$. For each $k = 1, \ldots, r$, let $\{\psi_{a,k}\}_{|a| \leq m_k}$ be a given set of real values. Find $P \in P_n^d$ that satisfies

$$D^a P(v_k) = \psi_{a,k}, \quad |a| < m_k, \quad k = 1, \ldots, r.$$  \hfill (1.1)

Above and forth we use the standard multi-index notation. For $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha \in \{\mathbb{N} \cup 0\}^n$, we define: Absolute value, $|\alpha| = \alpha_1 + \ldots + \alpha_n$; Power, for $u \in \mathbb{R}^n, u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$; Partial derivative, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. We also define $D^0 P(v) = P(v)$.

For $P \in P_n^d$ given in monomial basis, $P = \sum_{|\alpha| \leq d} \alpha^n x^\alpha$, let $A = A(v_1, \ldots, v_r)$ be the left hand side matrix form of the linear system (1.1). Since the determinant of $A$ is polynomial in the points $v_1, \ldots, v_r$, $A$ is either singular for every set of points or it is regular for almost all sets. Consequently we say that an interpolation problem is almost regular if it is uniquely solved for almost all sets of points. The general dimensionality problem asks, for which $n, d, r, m_1, \ldots, m_r$, the corresponding problem is almost regular on $P_n^d$. In particular, this is not the case for $n = d = r = 2$ and $m_1, m_2 = 2$. Indeed, here the number of samples is 6 which is the dimension of the space of quadratic polynomials. However, for any two points $v_1, v_2 \in \mathbb{R}^2$ let $ax + by + c = 0$ be the equation of the straight line through $v_1$ and $v_2$. Then $P(x, y) = (ax + by + c)^2$ is a nonzero polynomial of degree 2, vanishing together with its first order derivatives both at $v_1$ and at $v_2$. See [8] for a stimulating discussion of this problem.
The case of multivariate Birkhoff interpolation is a generalization of Hermite interpolation. We are now allowed to take directional derivatives and these need not be consecutive at each point. This adds another complication to the regularity question of a specific instance of a Birkhoff interpolation problem. In this case regularity may depend on both the points and the directions.

Many specific Hermite and Birkhoff interpolation schemes were shown to be almost regular ([11, 12, 14, 15, 16, 19]). In our scheme regularity is achieved for any choice of the points, if the directions are generic (that is, for almost all sets of directions).

2 The problem and its regularity

Let us give now an accurate setting of the problem. Denote by \( P \) the set of pairs of a point \( \psi \) and a direction vector \( u \) such that \( \psi = 0 \) and \( u \) is generic (that is, for almost all sets of directions). The case of multivariate Birkhoff interpolation is a generalization of Hermite interpolation (or reconstruction) scheme. This leads to “Remez-type” (or “Norming”) inequalities (see [5, 10] and references therein). Our second main result is a “Birkhoff-Remez” inequality for the interpolation scheme.

## 2 The problem and its regularity

Let us give now an accurate setting of the problem. Denote by \( \mathcal{P}_n^d \) the space of polynomials of degree at most \( d \) on \( \mathbb{R}^n \) and by \( \mathcal{L}_n^d \subseteq \mathcal{P}_n^d \) the space of homogeneous polynomials of degree \( d \) on \( \mathbb{R}^n \). The dimension of \( \mathcal{P}_n^d \) is thus \( \binom{d+n}{n} \), that is, the number of distinct monomials in \( n \)-variables of degree at most \( d \), and the dimension of \( \mathcal{L}_n^d \) is \( \binom{d+n-1}{n-1} \). We define \( N_{n,d} = \dim(\mathcal{L}_n^d) \).

Note that \( \mathcal{P}_n^d \) is a direct sum of \( \mathcal{L}_n^0, \ldots, \mathcal{L}_n^d \), hence \( \dim(\mathcal{P}_n^d) = \sum_{k=0}^d N_{n,k} \).

Let \( P \in \mathcal{P}_n^d \). For a point \( v \) and a direction \( u \) in \( \mathbb{R}^n \), we denote by \( D_u^k P(v) \) the \( k \)-th directional derivative of \( P \) at \( v \) along the straight line in the direction \( u \). With a slight abuse of this notation, we will also define \( D_u^0 P(v) = P(v) \).

**Problem 2.1.** For each \( k = 0, 1, \ldots, d \): let \( Z_k = \{(v_{k,j}, u_{k,j})\} \), \( j = 1, \ldots, N_{n,k} \), be a given set of pairs of a point \( v_{k,j} \in \mathbb{R}^n \) and a direction vector \( u_{k,j} \in \mathbb{R}^n \). For each \( k = 0, 1, \ldots, d \): let \( \Psi_k = \{\psi_{k,j}\} \subseteq \mathbb{R} \), \( j = 1, \ldots, N_{n,k} \), be a given set of real values. We seek a polynomial \( P \in \mathcal{P}_n^d \) which satisfies

\[
D_{u_{k,j}}^k P(v_{k,j}) = \psi_{k,j}, \quad j = 1, \ldots, N_{n,k}, \quad k = 0, \ldots, d. \tag{2.1}
\]

Problem 2.1 is called regular on \( \mathcal{P}_n^d \), given \( Z_0, \ldots, Z_d \), if it has a unique solution \( P \in \mathcal{P}_n^d \), for all \( \Psi_k = \{\psi_{k,j}\} \subseteq \mathbb{R} \), \( j = 1, \ldots, N_{n,k} \), \( k = 0, \ldots, d \).

Here is our first main result:

**Theorem 2.1.** For any sample points \( v_{k,j} \), and for general directions \( u_{k,j} \), Problem 2.1 is regular. Moreover, sets of directions that do not define a unique solution are exactly those for which there exists at least one \( 1 \leq k \leq d \) such that the directions \( u_{k,1}, \ldots, u_{k,N_{n,k}} \) are roots of some nonzero, \( n \)-variate homogeneous polynomial of degree \( k \).

Note that Theorem 2.1 is independent of the configuration of the points.

Proving Theorem 2.1 we now consider the following intermediate problem.

**Problem 2.2.** Let the set \( \hat{Z} = \{s_j\} \subseteq \mathbb{R}^n \), \( j = 1, \ldots, N_{n,k} \), and \( \Psi = \{\psi_j\} \subseteq \mathbb{R} \), \( j = 1, \ldots, N_{n,k} \), \( k \in \mathbb{N} \), be given. We seek for \( P \in \mathcal{L}_n^k \) that satisfies

\[
P(s_j) = \psi_j, \quad j = 1, \ldots, N_{n,k}. \tag{2.2}
\]
Problem 2.2 is called regular on $\mathcal{L}^k_n$, given $\tilde{Z}$, if it has a unique solution $P \in \mathcal{L}^k_n$ for each $\Psi$.

For $\tilde{Z} = \{s_j\} \subset \mathbb{R}^n$, $j = 1, \ldots, N_{n,k}$, consider the following multidimensional homogeneous Vandermonde matrix $A_{n,k}(\tilde{Z})$, $[A_{n,k}(\tilde{Z})]_{i,j} = s_i^{\alpha_j}$, $i,j = 1, \ldots, N_{n,k}$, where the multi-indices $\alpha, |\alpha| = k$, are ordered lexicographically. $A_{n,k}(\tilde{Z})$ is the matrix associated with Problem 2.2 written in the monomial basis. Consequently Problem 2.2 is regular on $\tilde{Z}$ if and only if the determinant of $A_{n,k}(\tilde{Z})$ is nonzero. Since this determinant is homogeneous in the coordinates of each of the vectors $s_i$, this property depends only on the directions of the vectors $s_i$, but not on their length.

Let us return to Problem 2.1. For a set $Z_k = \{(v_{k,j}, u_{k,j})\}, j = 1, \ldots, N_{n,k}$, of pairs of a point $v_{k,j}$ and a direction $u_{k,j}$ in $\mathbb{R}^n$, denote by $\tilde{Z}_k = (u_{k,1}, \ldots, u_{k,N_{n,d}}) \subset \mathbb{R}^n$, the set of the corresponding directions $u_{k,j}$.

**Proposition 2.1.** Problem 2.1 is regular on $\mathcal{P}^d_n$ given $Z_0, \ldots, Z_d$, if and only if, Problem 2.2 is regular on $\mathcal{L}^k_n$ given $\tilde{Z}_k$, for each $k = 1, \ldots, d$. Equivalently, the determinant of $A_{n,k}(\tilde{Z}_k)$ is nonzero for each $k = 1, \ldots, d$. In particular, the regularity of Problem 2.1 is determined only by the directions of the vectors $u_{k,j}$, and is invariant with respect to their length, and with respect to the position of the points $v_{k,j}$.

**Proof.** For each polynomial $P = \sum_{|\alpha| \leq d} a_\alpha x^{\alpha} \in \mathcal{P}^d_n$, and for each $k = 0, 1, \ldots, d$, denote by $P_k = \sum_{|\alpha| = k} a_\alpha x^{\alpha}$ the $k$-th homogeneous component of $P$, and put $\tilde{P}_k = \sum_{l=k}^d P_l$. Let us recall that for a point $v$ and a direction $u$ in $\mathbb{R}^n$, we denote by $D^k_u P(v)$ the $k$-th derivative $\frac{d^k P(v + tu)}{dt^k} \bigg|_{t=0}$ of $P$ at $v$ along the straight line in the direction $u$.

**Lemma 2.1.** For $P, P_k, \tilde{P}_k$ as above, for each $v, u \in \mathbb{R}^n$, and for each $k = 1, \ldots, d$, we have

$$D^k_u P_l(v) = 0, \quad l < k, \quad D^k_u P(v) = D^k_u \tilde{P}_k(v), \quad D^k_u P_k(v) = k! P_k(u). \quad (2.3)$$

**Proof.** The first equality is immediate, and the second follows directly from the first. The third one is Euler’s identity for homogeneous polynomials. It follows directly from the fact that for $P(x)$ being the monomial $x^\alpha$, $|\alpha| = k$, the highest $k$-th degree term in $P(v + tu)$ is $u^\alpha \cdot t^k$.

Assume now that Problem 2.2 is regular on $\mathcal{L}^k_n$ given $\tilde{Z}_k$, for each $k = 1, \ldots, d$. Consider the part of the interpolation equations (2.1) for Problem 2.1 with the highest order derivatives:

$$D^d_{u_{d,j}} P(v_{d,j}) = \psi_{d,j}, \quad j = 1, \ldots, N_{n,d}. \quad (2.4)$$

By Lemma 2.1 these equations are reduced to $P_d(u_{d,j}) = \frac{1}{d!} \psi_{d,j}, \quad j = 1, \ldots, N_{n,d}$. This is an instance of Problem 2.2 and by our assumption this system is regular. Hence, the highest homogeneous component $P_d$ of a solution $P$, is uniquely determined by (2.4). Next we consider $\tilde{P}_d = P - P_d$. This is a polynomial of degree $d - 1$, and it satisfies the corrected system of equations (2.1), which for the derivatives of order $d - 1$ takes the form:

$$D^{d-1}_{u_{d-1,j}} \tilde{P}_d(v_{d-1,j}) = \psi_{d-1,j} - D^{d-1}_{u_{d-1,j}} P_d(v_{d-1,j}), \quad j = 1, \ldots, N_{n,d-1}. \quad (2.4)$$
As above, from this system we find the unique homogeneous component \( P_{d-1} \) of \( P \). Continuing in this way till the degree one and then recovering the constant term of \( P \) by setting \( P_0(v_{0,1}) = \psi_{0,1} - \sum_{k=1}^d P_k(v_{0,1}) \), we have uniquely reconstructed \( P \).

In the opposite direction, assume that given Problem 2.1 with the sets \( Z_1, \ldots, Z_d \), some of the associated Problems 2.2 are not regular. Fix the smallest index \( l \leq d \) for which this happens. Then, we can find a nonzero homogeneous polynomial \( P_l \) such that

\[
D_{u,j}^l P_l(v_{u,j}) = 0, \quad j = 1, \ldots, N_{n,l}.
\]

Now we construct the right hand side in Problem 2.1 for which it has a nonzero solutions. Start with the solution \( P = P_l \) and put

\[
\psi_{k,j} := D_{u,k,j}^l P_l(v_{k,j}), \quad j = 1, \ldots, N_{n,k}, \quad k = 0, \ldots, d.
\]

By the construction, \( \psi_{k,j} = 0 \) for \( k \geq l \). However, \( \psi_{k,j} \) may be nonzero for \( k = 0, 1, \ldots, l-1 \).

We consider now Problem 2.1 on the sets \( Z_0, Z_1, \ldots, Z_{l-1} \), that is, for polynomials of degree \( l-1 \). Since by construction Problem 2.2 is regular on each of the sets \( \tilde{Z}_0, \tilde{Z}_1, \ldots, \tilde{Z}_{l-1} \), we conclude, by the already proved part of Proposition 2.1 that Problem 2.1 is regular for \( \mathcal{P}_n^{l-1} \) on \( Z_0, Z_1, \ldots, Z_{l-1} \). So we can find (uniquely) a polynomial \( P' \) of degree \( l-1 \), such that

\[
D_{u,k,j}^l P'(v_{k,j}) = \psi_{k,j}, \quad j = 1, \ldots, N_{n,k}, \quad k = 0, \ldots, l-1.
\]

Therefore, \( P = P_l - P' \) is a nonzero polynomial such that

\[
D_{u,k,j}^l P(v_{k,j}) = 0, \quad j = 1, \ldots, N_{n,k}, \quad k = 0, \ldots, d.
\]

We conclude that Problem 2.1 on \( Z_0, Z_1, \ldots, Z_d \) is not regular. This completes the proof of Proposition 2.1.

We now prove Theorem 2.1

**Proof.** By Proposition 2.1, the regularity of Problem 2.1 on \( \mathcal{P}_n^d \) given the sets \( Z_0, Z_1, \ldots, Z_d \), is equivalent to the regularity of Problem 2.2 on \( \mathcal{L}_n^k \) given the set \( \tilde{Z}_k = (u_{k,1}, \ldots, u_{k,N_{n,k}}) \), for each \( k = 1, \ldots, d \). In turn, for each \( k = 1, \ldots, d \), Problem 2.2 on \( \mathcal{L}_n^k \) given \( \tilde{Z}_k = (u_{k,1}, \ldots, u_{k,N_{n,k}}) \), is the standard Lagrange interpolation on \( \mathcal{L}_n^k \) which is regular exactly when \((u_{k,1}, \ldots, u_{k,N_{n,k}}) \in \mathbb{R}^n \) are not the roots of some \( P \in \mathcal{L}_n^k \). \( \square \)

**Theorem 2.2.** In the planar case, for \( n = 2 \), Problem 2.1 is regular if and only if for each \( k = 1, \ldots, d \), the directions of the vectors \( u_{k,j} \) are pairwise linearly independent.

**Proof.** In the planar case, for each \( k \), it is known that (but might not be easy to locate reference to, see for example [24]) the homogeneous Vandermonde determinants of the matrix \( A_{2,k}(\tilde{Z}_k) \) take the following convenient form:

\[
\det \left( A_{2,k}(\tilde{Z}_k) \right) = \prod_{1 \leq i < j \leq k+1} \det [u_{k,i}, u_{k,j}],
\]

where \( \det [u_{k,i}, u_{k,j}] \) denotes the determinant of the two by two matrix having \( u_{k,i} \) and \( u_{k,j} \) as its rows. It follows that \( \det \left( A_{2,k}(\tilde{Z}_k) \right) \) is nonzero is equivalent to the directions of \( k \)th Homogeneous Problem being pairwise independent. Finally, by Proposition 2.1, the regularity of the homogeneous problems is equivalent to the regularity of Problem 2.1. \( \square \)
3 Birkhoff-Remez inequality for Problem 2.1

3.1 Norming inequalities

The classical Remez inequality and its generalizations compare maxima of a polynomial $P$ on two sets $U \subset G$ (see [6, 11, 21, 25] and references therein). We would like to extend this setting, in order to include into sampling information on $U$ the derivatives of $P$. Such an extension is provided by a wider (and also classical) setting of “norming sets” and “norming inequalities”. Let $G \subset \mathbb{R}^n$ be a compact domain, and let $L \subset P_n^d$ be a normed linear subspace of the space of polynomials of degree at most $d$, equipped with the norm $||P||_G = \max_G |P|$. Let $L^*$ denote the dual space of all the linear functionals on $L$.

Definition 3.1. Let $U \subset L^*$ be a bounded set of linear functionals on $L$. A semi-norm $||P||_U$ on $L$ is defined as

$$||P||_U = \sup_{w \in U} |w(P)|.$$  \hspace{0.5cm} (3.1)

The set $U$ is said to be $L$-norming if the semi-norm $||P||_U$ is in fact a norm on $L$, that is, if for $P \in L$ we have that if $||P||_U = 0$ then $P = 0$. Equivalently, $U$ is $L$-norming if

$$N_L(G, U) := \sup_{P \in L, P \neq 0} \frac{||P||_G}{||P||_U} < \infty.$$  \hspace{0.5cm} (3.2)

$N_L(G, U)$ is called the $L$-norming constant of $U$ on $G$.

The usual Remez-type inequalities are included in the new setting by identifying $x \in U \subset G$ with the linear functional $\delta_x$ sampling a polynomial at the point $x$.

3.2 Robust polynomial reconstruction

To illustrate the role of the norming constant in estimating the robustness of polynomial reconstruction, let us consider the following reconstruction scheme: our goal is to find the “best” polynomial approximation of a given function $f$ on $G$ according to the norm $||\cdot||_G$. In other words, the “ideal approximation” is a polynomial $\bar{P} = \arg \min_{P \in P_n^d} ||f - P||_G$ (where $\arg \min_{P \in P_n^d}$ is the operator extracting the minimizing polynomial). Let us denote the error $||f - \bar{P}||_G$ of this ideal approximation by $E$. Now, let us assume that our input consists of noisy measurements of $f$ on a subset $U \subset G$ (this setting can be easily extended to the measurements being more general linear functionals). So we start with a function $\tilde{f} = f + \nu$ on $U$, where $\nu$ is the measurement error function, satisfying $\max_U |\nu(x)| \leq h$. As an output we take $\hat{P} = \arg \min_{P \in P_n^d} ||\tilde{f} - P||_U$. The following result shows that the output error $||\hat{P} - P||_G$ can be bounded in terms of the norming constant $N = N_{P_n^d}(G, U)$, $E$ and $h$.

Proposition 3.1.

$$||\hat{P} - P||_G \leq 2N(E + h).$$

Proof. By the construction of $\bar{P}$ we have that $||\tilde{f} - \bar{P}||_U \leq ||f - \bar{P}||_U + ||\nu||_U \leq E + h$. Since $\hat{P} = \arg \min_{P \in P_n^d} ||\tilde{f} - P||_U$, then $||\tilde{f} - \hat{P}||_U \leq ||\tilde{f} - \bar{P}||_U \leq E + h$. We conclude that $||P - \hat{P}||_U \leq 2(E + h)$, and hence $||\hat{P} - P||_G \leq 2N(E + h)$. \hfill \Box
3.3 Norming inequality for extended Problem 2.1

Let us recall (and extend) some notations introduced in the proof of Theorem 2.1. For each polynomial \( P = \sum_{|\alpha| \leq d} a_\alpha x^\alpha \in \mathcal{P}_n^d \), and for each \( k = 0, 1, \ldots, d \), we have denoted by \( P_k = \sum_{|\alpha| = k} a_\alpha x^\alpha \) the \( k \)-th homogeneous component of \( P \). We will denote by \( \mathcal{P}_n^{d,k} \subset \mathcal{P}_n^d \) the subspace of all the polynomials \( P \) of the form \( P = \sum_{k \leq |\alpha| \leq d} a_\alpha x^\alpha \). Recall that for \( P \in \mathcal{P}_n^d \) we have denoted \( \tilde{P}_k = \sum_{l=k}^d P_l \in \mathcal{P}_n^{d,k} \).

Returning to Problem 2.1, we now extend its setting, allowing larger sets \( Z_k \). So now \( Z_k \subset \mathbb{R}^n \times \mathbb{R}^n \) may be any bounded set of couples \( (v, u) \) of a point and a direction vector.

For any linear subspace \( L \subset \mathcal{P}_n^d \) the sets \( Z_k \) can be considered as subsets \( \tilde{Z}_k \subset L^* \), if we identify the couple \( (v, u) \in Z_k \) with the linear functional \( D^k_{u,v} \) on \( L \) defined by \( D^k_{u,v}(P) = D^k_u P(v) \). For the sampling sets \( Z_0, \ldots, Z_d \), we define \( U = U(Z_0, \ldots, Z_d) = \bigcup_{k=0}^d \tilde{Z}_k \subset L^* \), and \( U_k = \bigcup_{l=k}^d \tilde{Z}_l \subset L^* \).

On the other hand, extending the notations used in Theorem 2.1, we denote by \( \tilde{Z}_k = \{ u : \exists v, (v, u) \in Z_k \} \subset \mathbb{R}^n \) the set of the directions \( u \) that appear in \( Z_k \). As above, for any linear subspace \( L \subset \mathcal{P}_n^d \) the set \( \tilde{Z}_k \) can be considered as a subset \( \tilde{Z}_k \subset L^* \), via identifying \( u \in \tilde{Z}_k \) with the evaluation functional at the point \( u, \delta_u \in L^* \).

To simplify the presentation we fix \( G \) to be equal to the unit ball \( B = B^1_n \subset \mathbb{R}^n \), and assume that for each \( k = 0, \ldots, d \), the sets \( Z_k \) satisfy \( Z_k \subset B \times B \), that is, both the sampling points \( v \) and the directions \( u \) belong to the unit ball \( B \).

**Theorem 3.1.** For each \( k = 0, \ldots, d \), set \( L_k = \mathcal{L}_n^k \subset \mathcal{P}_n^d \) to be the subspace of homogeneous polynomials of degree \( k \). Let each of the directions sets \( \tilde{Z}_k \subset L_k^* \) be \( L_k \)-norming on \( B \), with norming constant \( \theta_k = N_{L_k}(B, \tilde{Z}_k) \). Then \( U = U(Z_0, \ldots, Z_d) \) is norming for \( \mathcal{P}_n^d \), with the norming constant \( N_{\mathcal{P}_n^d}(B, U) \) satisfying

\[
N_{\mathcal{P}_n^d}(B, U) \leq \sum_{l=0}^d \frac{\theta_l}{l!} \prod_{j=0}^{l-1} (1 + m_j \cdot \frac{\theta_j}{j!}),
\]

where \( m_l = m(d, l) = T_d^{(l)}(1) \). Here \( T_d(x) \) is the \( d \)-th Chebyshev polynomial, and \( T_d^{(k)}(x) \) is its \( k \)-th derivative.

**Proof.** Let us denote by \( \kappa_k \) the norming constant \( N_{L_k}(B, \tilde{Z}_k) \). Then for each \( k = 0, \ldots, d \), we have

\[
\kappa_k = \frac{\theta_k}{k!} \tag{3.3}
\]

Indeed, by Lemma 2.1, the sets of linear functionals on \( L_k, \tilde{Z}_k \) and \( \tilde{Z}_k \) satisfy \( \tilde{Z}_k = k! \tilde{Z}_k \).

Next we prove by induction (starting from \( k = d \) and going down) the following result:

**Lemma 3.1.** For each \( k = 0, \ldots, d \), the set \( U_k \) is norming for \( \mathcal{P}_n^{d,k} \) on \( B \). The norming constant \( \eta_k = N_{\mathcal{P}_n^{d,k}}(B, U_k) \) satisfies

\[
\eta_d = \kappa_d,
\eta_k \leq \kappa_k + (1 + m_k \cdot \kappa_k) \eta_{k+1}, \quad k < d. \tag{3.4}
\]
Lemma 3.2. For \( k = d \) our problem is reduced to Problem 2.2 on the space \( L_d = L^d_n \subset P^d_n \) of homogeneous polynomials of degree \( d \). By assumptions, and by (3.3), we have \( N_{L_d}(B, \tilde{Z}_d) = \kappa_d = \frac{\beta_d}{\kappa} > 0 \). Consequently Lemma 3.1 holds for the case \( k = d \). Assume that (3.4) is satisfied for \( k = l + 1 \leq d \) and prove it for \( k = l \). Let \( P \in P^d_n \) satisfy \(|w(P)| \leq 1\) for each \( w \in U_l \). In particular, \(|w(P)| \leq 1\) for all \( w \in U_{l+1} \). Since for \( w \in U_{l+1}, w(P) = w(\tilde{P}_{l+1}) \), we have that for \( w \in U_{l+1}, |w(\tilde{P}_{l+1})| \leq 1 \). By the induction assumption, and by definition of the norming constant we have that

\[
||\tilde{P}_{l+1}||_B \leq \eta_{l+1}.
\] (3.5)

Now, for the homogeneous component \( P_l \) of \( P \) we have \( P_l = P - \tilde{P}_{l+1} \), and thus for \( w \in \tilde{Z}_l, |w(P_l)| = |w(P - \tilde{P}_{l+1})| \leq 1 + |w(\tilde{P}_{l+1})| \). To estimate the values \( w(\tilde{P}_{l+1}) \), which are the directional derivatives of order \( l \) of \( \tilde{P}_{l+1} \), we apply the classical Markov inequality, in the form presented in [17, 23].

Theorem 3.2. For \( P \in P^d_n \), and for any direction vector \( u \in \mathbb{R}^n, ||u|| \leq 1 \),

\[
||D^k_uP||_B \leq m_k ||P||_B,
\]

where \( m_k = T^{(k)}_d(1) \).

Applying this result to \( \tilde{P}_{l+1} \) we conclude, using (3.3) that for \( w \in \tilde{Z}_l \) the bound \(|w(\tilde{P}_{l+1})| \leq m_l \cdot \eta_{l+1} \) holds. Therefore, for \( w \in \tilde{Z}_l \) we get \(|w(P_l)| \leq 1 + m_l \cdot \eta_{l+1} \).

By the assumptions of the theorem the set \( \tilde{Z}_l \) is norming for \( L_l \), with the norming constant \( N_{L_l}(B, \tilde{Z}_l) = \theta_l \). Therefore, by (3.3), the set \( \tilde{Z}_l \) is also norming for \( L_l \), with the norming constant \( \kappa_l = N_{L_l}(B, \tilde{Z}_l) = \frac{\theta_l}{\kappa_l} \). We conclude that

\[
||P_l||_B \leq \kappa_l(1 + m_l \cdot \eta_{l+1}).
\]

Finally, since \( \tilde{P}_l = P_l + \tilde{P}_{l+1} \), we obtain

\[
||P||_B \leq ||\tilde{P}_{l+1}||_B + ||P_l||_B \leq \eta_{l+1} + \kappa_l(1 + m_l \cdot \eta_{l+1}).
\]

This inequality is true for each polynomial \( P \in P^d_n \), and hence

\[
\eta_l \leq \eta_{l+1} + \kappa_l(1 + m_l \cdot \eta_{l+1}) = \kappa_l + (1 + m_l \cdot \kappa_l)\eta_{l+1}.
\]

This completes the proof of Lemma 3.1.

To complete the proof of Theorem 3.1 it remains to solve explicitly the recurrence inequality (3.4).

Lemma 3.2. Let \( \tau_k, k = 0, \ldots, d, \) satisfy recurrence relation

\[
\begin{align*}
\tau_d &= \kappa_d, \\
\tau_k &= \kappa_k + (1 + m_k \cdot \kappa_k)\tau_{k+1}, \quad k < d.
\end{align*}
\] (3.6)

Then for each \( k = 0, \ldots, d, \) we have

\[
\eta_k \leq \tau_k = \sum_{l=k}^{d} \kappa_l \prod_{j=k}^{l-1} (1 + m_j \cdot \kappa_j).
\] (3.7)
Here the empty product (for \( l = k \)) is assumed to be equal to one.

**Proof.** First we prove by induction the expression for \( \tau_k \). For \( k = d \) we have \( \tau_d = \kappa_d \), which is the right hand side of (3.7). Now, for \( k < d \), using induction assumption, we have

\[
\tau_k = \kappa_k + (1 + m_k \cdot \kappa_k)\tau_{k+1} = \kappa_k + (1 + m_k \cdot \kappa_k) \sum_{l=k+1}^{d} \kappa_l \cdot \prod_{j=k+1}^{l-1} (1 + m_j \cdot \kappa_j)
\]

\[
\tau_k = \kappa_k + \sum_{l=k+1}^{d} \kappa_l \cdot \prod_{j=k}^{l-1} (1 + m_j \cdot \kappa_j) = \sum_{l=k}^{d} \kappa_l \cdot \prod_{j=k}^{l-1} (1 + m_j \cdot \kappa_j).
\]

This completes the proof of Lemma 3.2.

To complete the proof of Theorem 3.1 it remains to substitute into (3.7) the values \( \kappa_k = \theta_k k! \).

In the proof of theorem 3.1 we applied Markov inequality (3.2), to upper bound the \( l^{th} - 1 \) derivatives of polynomials \( P \in \mathcal{P}_{n,l}^d \), \( l = 1, \ldots, d \). These are incomplete polynomials and one may expect sharper Markov inequalities. Indeed approximation with incomplete polynomials and, in general, Markov-type inequalities for constrained polynomials are a subject of research in both approximation theory and general analysis (for the univariate case see, for example, [12, 7, 2]). For the multivariate case, we are unaware of a general result which improves the upper bound in (3.2). We suggest here that Theorem 3.1 can be improved using such a result.

### 3.3.1 Birkhoff-Remez inequality

The classical Remez inequality [21] bounds the maximum of a univariate polynomial \( P \) on the interval \([-1, 1]\) through its maximum on a subset \( Z \subset [-1, 1] \) of a positive measure. This theorem was extended to several variables in [6], and then further generalized in [25], where the Lebesgue measure \( \mu_n(Z) \) was replaced by a certain quantity \( \omega_{n,d}(Z) \), expressed through the metric entropy of \( Z \):

**Theorem 3.3.** ([22], Theorem 2.3.) If \( \omega_{n,d}(Z) = \omega > 0 \), then for each polynomial \( P \in \mathcal{P}^d_n \)

\[
\sup_{x \in B} |P(x)| \leq T_d \left( \frac{1 + (1 - \omega)^{\frac{1}{n}}} {1 - (1 - \omega)^{\frac{1}{n}}} \right) \sup_{x \in Z} |P(x)|.
\]

For any measurable \( Z \) we have \( \omega_{n,d}(Z) \geq \mu_n(Z) \), and \( \omega_{n,d}(Z) \) may be positive for discrete and finite \( Z \). We will not give here an accurate definition of \( \omega_{n,d}(Z) \), referring the reader to [25]. Replacing in Theorem 3.3 \( \omega_{d,n}(Z) \) with a smaller value \( \mu_n(Z) \) we obtain the result of [6], and putting \( n = 1 \) we get back the classical Remez inequality.

**Theorem 3.1** reduces estimation of the norming constant in our setting of Birkhoff interpolation to the norming constants \( \theta_k \) of the direction sets \( \mathcal{Z}_k \subset \mathbb{R}^n \), in the spaces of homogeneous polynomials of degree \( k \). In this situation the Remez-type inequality of Theorem 3.1 is applicable. We obtain the following bound:
Corollary 3.1. Assume that for each $k = 0, \ldots, d$ we have $\omega_k = \omega_{n,k}(\tilde{Z}_k) > 0$. Then we can put in Theorem 3.1 the values $\theta_k = T_d \left( \frac{1+(1-\omega_k)^{\frac{1}{n}}}{1-(1-\omega_k)^{\frac{1}{n}}} \right)$.

3.4 The univariate case

We now shortly discuss aspects of the reconstruction for the most basic case of $n = 1$ and the domain of approximation being $G = [0, 1] \subseteq \mathbb{R}$. In this case, and if the number of measurements is equal to $d + 1$, the approximation scheme is reduced to an instance of a single-variate (classic) Birkhoff interpolation. For each $k = 0, \ldots, d$, we have a single measurement of the $k$th derivative of $P$ at the point $v_k$ (where the points are not necessarily distinct or ordered). As before, $U = \{w_1, \ldots, w_{d+1}\}$ is the set of the corresponding linear functionals, $w_k(P) = \frac{d^{k-1}}{2^{d-k}} P(v_k)$.

If the points all coincide at a point $v$, then the interpolant polynomial is given by the Taylor approximation at $v$. In this case, a direct calculation shows that the norming constant $N = N_{P_d^1((0, 1), U)}$, is bounded by a constant not depending on $d$. Our preliminary calculation and numerical experiments indicate that $N$ remains bounded for equidistant points in $[0, 1]$.

Another natural problem which can be considered is the accuracy of the approximation when the function to be approximated, $f$, is smooth to some degree. In this setting, for a set of points $v_1, \ldots, v_{d+1}$, and for $f \in C^d[0, 1]$ our measurements are $\frac{d^{k-1}}{2^{d-k}} f(v_k)$, $k = 0, \ldots, d$. In general, the reconstruction error of the scheme will depend on the position of the points and can be studied using Birkhoff reminder Theorem (see [3]). Specifically, assume we start with a sequence of points $v_1, \ldots, v_{d+1}$ and then randomly permute them to get a new sequence $\tilde{v}_1, \ldots, \tilde{v}_{d+1}$ made out from the same set of points. Interpolating $f$ with the permuted sequence of measurements may have a significant effect on accuracy of the reconstruction comparing to the non permuted sequence.

We consider it an interesting and challenging problem attaining bounds for certain configurations of the points for the aforementioned problem. We would like to thank the referee for these observations and for suggesting this direction.

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