Abstract—High-resolution parameter estimation algorithms designed to exploit the prior knowledge of incident signals from strictly second-order (SO) non-circular (NC) sources allow for a lower estimation error and can detect twice as many sources. In this paper, we derive the $R$-D NC Standard ESPRIT and the $R$-D NC Unitary ESPRIT algorithms that provide a significantly better performance compared to their original versions for arbitrary source signals. They are applicable to shift-invariant $R$-D antenna arrays and do not require a centro-symmetric array structure. Moreover, we present a first-order asymptotic parameter estimation analysis of the proposed algorithms, which is based on the estimation error of the signal subspace arising from the noisy measurements. The derived expressions for the resulting parameter estimation error are explicit in the noise realizations and asymptotic in the effective signal-to-noise ratio (SNR), i.e., the results become exact for either high SNRs or a large sample size. We also provide mean squared error (MSE) expressions, where only the assumptions of a zero mean and finite SO moments of the noise are required, but no assumptions about its statistics are necessary. As a main result, we analytically prove that the asymptotic performance of both $R$-D NC ESPRIT-type algorithms is identical in the high effective SNR regime. Finally, a case study shows that no improvement from strictly non-circular sources can be achieved in the special case of a single source.

Index Terms—Unitary ESPRIT, non-circular sources, performance analysis, DOA estimation.

I. INTRODUCTION

ESTIMATING the parameters of $R$-dimensional ($R$-D) signals, e.g., their directions of arrival, frequencies, Doppler shifts, etc., has long been of great research interest, given its importance in a variety of applications such as radar, sonar, biomedical imaging, and wireless communications. Among other subspace-based parameter estimation schemes [1], [2], $R$-D Standard ESPRIT [3], $R$-D Unitary ESPRIT [4], [5], and their tensor extensions $R$-D Standard Tensor-ESPRIT and $R$-D Unitary Tensor-ESPRIT [6] are some of the most valuable estimators due to their high resolution and low complexity. However, these methods assume arbitrary source signals and do not take prior knowledge such as the second-order (SO) non-circularity of the received signals into account. With the growing popularity of subspace-based parameter estimation algorithms, their performance analysis has attracted considerable attention. The two most prominent performance assessment strategies have been proposed in [7] and [8]. The concept in [7] and the follow-up papers [9], [10], [11] analyze the eigenvector distribution of the sample covariance matrix, originally proposed in [12]. However, it requires Gaussianity assumptions on the source symbols and the noise, and is only asymptotic in the sample size $N$. In contrast, [8] and its extensions [13], [14] provide an explicit first-order approximation of the estimation error caused by the perturbed subspace estimate due to a small additive noise contribution. It directly models the leakage of the noise subspace into the signal subspace. Unlike [7], this approach is asymptotic in the effective signal-to-noise ratio (SNR), i.e., the results become accurate for either high SNRs or a large sample size $N$. Thus, it is even valid for the single snapshot case $N = 1$ if the SNR is sufficiently high. Furthermore, as it is explicit in the noise realizations, no assumptions about the statistics of the signals or the noise are necessary. However, for the mean squared error (MSE) expressions in [8], a circularly symmetric noise distribution is assumed. In [13], we have derived new MSE expressions that only require the noise to be zero-mean with finite SO moments regardless of its statistics and extended the framework of [8] to the case of $R$-D parameter estimation. Further extensions of these results for the perturbation analyses of Tensor-ESPRIT-type algorithms have been presented in [16] and [17], respectively. The special case of the performance assessment for a single source was considered in [21] and the asymptotic efficiency of MUSIC and Root-MUSIC was presented in [15] and [19], respectively. However, these results are asymptotic in the sample size $N$ or even in the number of sensors $M$. The results presented here are also accurate for small values of $M$ and asymptotic in the effective SNR.

Recently, a number of improved high-resolution subspace-based parameter estimation schemes have been proposed for non-circular (NC) sources. These include NC MUSIC [20], NC Root-MUSIC [21], 1-D NC Standard ESPRIT [22], and 2-D NC Unitary ESPRIT [23]. Unlike the original parameter estimation methods, they exploit prior knowledge about the signals’ SO statistics, i.e., their strict SO non-circularity [24]. Examples of such signals include BPSK, Offset-QPSK, PAM, and ASK-modulated signals. By applying a preprocessing

$\text{arXiv:1402.2936v1  [cs.IT]  12 Feb 2014}$
In this paper, we first present the \( R-D \) NC Standard ESPRIT and the \( R-D \) NC Unitary ESPRIT algorithms as an extension of \([22]\) and \([23]\). They exploit the strict SO non-circularity of \([22]\) and \([23]\). They facilitate design decisions on \( M \) to achieve a certain performance for specific SNRs and allow for the computation of the asymptotic efficiency. Note that in \([31]\), we have also incorporated structured least squares to solve the augmented shift invariance equation into the performance analysis.

This remainder of this paper is organized as follows: The data model and the preprocessing for strictly non-circular sources are introduced in Section II and Section III. In Section IV, the \( R-D \) NC Standard ESPRIT and \( R-D \) NC Unitary ESPRIT algorithms are derived. Their performance analysis is presented in Section V before the special case of a single source is analyzed in Section VI. Section VII illustrates and discusses the numerical results, and concluding remarks are drawn in Section VIII.

**Notation:** We use italic letters for scalars, lower-case boldface letters for column vectors, and upper-case boldface letters for matrices. The superscripts \( T, *, H, \) and \( ^{-1} \) denote the transposition, complex conjugation, conjugate transposition, matrix inversion, and the Moore-Penrose pseudo inverse of a matrix, respectively. The Kronecker product is denoted as \( \otimes \). The operator \( \text{vec} \{ A \} \) stacks the columns of the matrix \( A \in \mathbb{C}^{M \times N} \) into a column vector of length \( MN \times 1 \) and the operator \( \text{diag}\{ a \} \) returns a diagonal matrix with the elements of \( a \) placed on its diagonal. The matrix \( \Pi_M \) is the \( M \times M \) exchange matrix with ones on its antidiagonal and zeros elsewhere. Also, the matrices \( \mathbb{1}_M \) and \( 0_M \) denote the \( M \times M \) matrices of ones and zeros, respectively. Moreover, \( \text{Re} \{ \cdot \} \) and \( \text{Im} \{ \cdot \} \) extract the real and imaginary part of a complex number respectively, \( \| x \|_2 \) represents the 2-norm of the vector \( x \), and \( \mathbb{E} \{ \cdot \} \) stands for the statistical expectation.

**II. Data Model**

Let a noise-corrupted linear superposition of \( d \) undamped exponentials be sampled on an arbitrary \( R \)-dimensional (\( R-D \)) shift-invariant-structured grid\(^1\) of size \( M_1 \times \ldots \times M_R \) at \( N \) subsequent time instants \( \mathbb{1} \). The \( n \)-th snapshot of the observed \( R-D \) data sequence can be modeled as

\[
\begin{align*}
\mathbf{x}(n) = \sum_{i=1}^{d} s_i(n) \prod_{r=1}^{R} e^{j \omega_m r} + \mathbf{n}_m(n),
\end{align*}
\]

where \( m_r = 1, \ldots, M_r, \ n = 1, \ldots, N, \ s_i(n) \) denotes the complex amplitude of the \( i \)-th undamped exponential at time instant \( n \), \( \omega_m \) is the spatial frequency in the \( r \)-th mode for \( i = 1, \ldots, d \) and \( r = 1, \ldots, R \), and \( \mathbf{n}_m(n) \) contains the samples of the zero-mean additive noise component. In the array signal processing context, each of the \( R-D \) exponentials represents a narrow-band planar waveform emitted from stationary far-field sources and the complex amplitudes \( s_i(n) \) are the zero-mean source symbols. The objective is to estimate the spatial frequencies \( \omega_m \), \( \forall \ r, i \) from \( \mathbb{1} \).

\(^1\)The grid needs to be decomposable into the outer product of \( R \) one-dimensional sampling grids \([13]\).
In order to obtain a more compact formulation of (1), we collect the observed samples into a measurement matrix $X \in \mathbb{C}^{M \times N}$ with $M = \prod_{i=1}^{R} M_{i}$ by stacking the $R$ spatial dimensions along the rows and aligning the $N$ snapshots as the columns. We can then model $X$ as

$$X = AS + N \in \mathbb{C}^{M \times N},$$

where $A = [a_{1}(\mu_{1}), \ldots, a_{d}(\mu_{d})] \in \mathbb{C}^{M \times d}$ is the array steering matrix. It consists of the array steering vectors $a_{i}(\mu_{i})$ corresponding to the $i$-th spatial frequency defined by

$$a_{i}(\mu_{i}) = a^{(1)}(\mu_{1}) \otimes \cdots \otimes a^{(R)}(\mu_{R}) \in \mathbb{C}^{M \times 1},$$

where $a^{(r)}(\mu_{i}) \in \mathbb{C}^{M_{r} \times 1}$ is the array steering vector in the $r$-th mode. Furthermore, $S \in \mathbb{C}^{d \times N}$ represents the source symbol matrix and $N \in \mathbb{C}^{M \times N}$ contains the samples of the additive sensor noise. Due to the assumption of strictly SO non-circular sources, the complex symbol amplitudes of each element can be decomposed as $a_{i}(\mu_{i}) = a_{i}(\mu_{i})^\ast \otimes a_{i}(\mu_{i})$.

The first important property of the augmented steering matrix $A^{(nc)}$ is formuluated in the following theorem:

**Theorem 1.** If the array steering matrix $A$ is shift-invariant (7), then $A^{(nc)}$ is also shift-invariant and satisfies

$$J_{1}^{(nc)}A^{(nc)} = J_{2}^{(nc)}A^{(nc)}, \quad r = 1, \ldots, R,$$

where

$$J_{1}^{(nc)} = I_{\prod_{i=1}^{R} M_{i}} \otimes I_{\prod_{i=1}^{R} M_{i}}, \quad J_{2}^{(nc)} = I_{\prod_{i=1}^{R} M_{i}} \otimes I_{\prod_{i=1}^{R} M_{i}}.$$

**Proof:** See Appendix A.

If the physical array is centro-symmetric, i.e., it is symmetric with respect to its centroid, its array steering matrix $A_{c}$ satisfies

$$A_{c}^{\ast} = A_{c} \Delta_{c},$$

where $\Delta_{c} \in \mathbb{C}^{d \times d}$ is a unitary diagonal matrix. If (12) holds, we have $J_{1}^{(nc)} = \Pi_{M_{(sel)}^{(nc)}}^{(nc)} A_{c}^{(nc)}$ and hence the augmented selection matrices $J_{1}^{(nc)}A_{c}^{(nc)}$ and $J_{2}^{(nc)}A_{c}^{(nc)}$ simplify to

$$J_{k}^{(nc)}(r) = J_{2}^{(nc)}(r) = J_{1}^{(nc)}(r), \quad k = 1, 2.$$

Note that this special case was assumed in [22] and [23].

The second important property of $A^{(nc)}$ is stated in the following theorem:

**Theorem 2.** The augmented steering matrix $A^{(nc)}$ always exhibits centro-symmetry even if $A$ is not centro-symmetric.

**Proof:** Assuming that $A$ satisfies (7) but not necessarily (12), we have

$$\Pi_{2M}A^{(nc)} = \left[ \begin{array}{cc} 0 & \Pi_{M} A^{\ast} \\ \Pi_{M} & 0 \end{array} \right] A^{\ast},$$

where $\Delta_{c}$ becomes $\Delta_{c}^{\ast}$ which is unitary and diagonal. Therefore, $A^{(nc)}$ satisfies (12), which shows that it is centro-symmetric regardless of the centro-symmetry of $A$.

This result shows that $R$-D NC Unitary ESPRIT, derived in the next section, can be applied to a broader variety of array geometries than $R$-D Unitary ESPRIT, which requires a centro-symmetric array. An example is provided in Fig. 2 of Section VII.
IV. PROPOSED R-D NC ESPRIT-TYPE ALGORITHMS

In this section, we present the NC Standard ESPRIT and the NC Unitary ESPRIT algorithms for arbitrarily formed R-dimensional shift-invariant-structured array geometries, where centro-symmetry is not required. Furthermore, we summarize some important properties at the end.

A. R-D NC Standard ESPRIT Algorithm

Based on the augmented data model (5), we estimate the signal subspace $\hat{U}^{(nc)}_s \in \mathbb{C}^{2M \times d}$ by computing the $d$ dominant left singular vectors of $X^{(nc)}$. As $A^{(nc)}$ and $\hat{U}^{(nc)}_s$ span approximately the same column space, we can find a non-singular matrix $T \in \mathbb{C}^{d \times d}$ such that $A^{(nc)} \approx \hat{U}^{(nc)}_s T$. Using this relation, the overdetermined set of $R$ augmented shift invariance equations (3) can be expressed in terms of the estimated augmented signal subspace, yielding

$$J^{(nc)}_1 \hat{U}^{(nc)}_s \Gamma^{(r)} \approx J^{(nc)}_2 \hat{U}^{(nc)}_s, \quad r = 1, \ldots, R \tag{15}$$

with $\Gamma^{(r)} \in \mathbb{C}^{d \times d}$. Often, the $R$ unknown matrices $\Gamma^{(r)}$ are estimated using least squares (LS), i.e.,

$$\hat{\Gamma}^{(r)} = \left( J^{(nc)}_1 \hat{U}^{(nc)}_s \right)^{\dagger} J^{(nc)}_2 \hat{U}^{(nc)}_s \in \mathbb{C}^{d \times d}. \tag{16}$$

Finally, after solving (16) for $\hat{\Gamma}^{(r)}$ independently, the correctly paired spatial frequency estimates are obtained by $\hat{\mu}_i^{(r)} = \arg\{\hat{\lambda}_i^{(r)}\}$, $i = 1, \ldots, d$, where the eigenvalues $\hat{\lambda}_i^{(r)}$ of $\hat{\Gamma}^{(r)}$ are obtained by performing a joint eigendecomposition across all $R$ dimensions through the simultaneous Schur decomposition $[5]$.

The R-D NC Standard ESPRIT algorithm is summarized in Table I.

| 1) Estimate the augmented signal subspace $U^{(nc)}_s \in \mathbb{C}^{2M \times d}$ via the truncated SVD of the augmented observation $X^{(nc)} \in \mathbb{C}^{2M \times N}$. |
|---|
| 2) Solve the overdetermined set of augmented shift invariance equations $J^{(nc)}_1 \hat{U}^{(nc)}_s \Gamma^{(r)} \approx J^{(nc)}_2 \hat{U}^{(nc)}_s$, $r = 1, \ldots, R$, by using the LS, TLS or SLS algorithm, where $J^{(nc)}_k \in \mathbb{R}^{M \times (2M-k)}$, $k = 1, 2$, is defined in [4]. |
| 3) Compute the eigenvalues $\hat{\lambda}_i^{(r)}$, $i = 1, \ldots, d$ of $\hat{\Gamma}^{(r)}$ jointly for all $r = 1, \ldots, R$. Recover the correctly paired spatial frequencies $\hat{\mu}_i^{(r)}$ via $\hat{\mu}_i^{(r)} = \arg\{\hat{\lambda}_i^{(r)}\}$. |

B. R-D NC Unitary ESPRIT Algorithm

As a main feature, R-D Unitary ESPRIT involves forward-backward averaging (FBA) $[7]$ of the measurement matrix $X$, which results in a centro-Hermitian matrix, i.e., matrices $Z \in \mathbb{C}^{p \times q}$ that satisfy $\Pi_p Z^* \Pi_q = Z$. Therefore, it can be efficiently formulated in terms of only real-valued computations $[4]$. This is achieved by a bijective mapping of the set of centro-Hermitian matrices onto the set of real-valued matrices $[23]$. To this end, let us define left $\Pi$-real matrices, i.e., matrices $Q \in \mathbb{C}^{p \times q}$ satisfying $\Pi_p Q^* = Q$. A sparse and square unitary left $\Pi$-real matrix of odd order is given by

$$Q_{2n+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & 0_{n \times 1} & jI_n \\ 0_{n \times 1} & \sqrt{2} & 0 \\ I_n & 0_{n \times 1} & -jI_n \end{bmatrix}. \tag{17}$$

A unitary left $\Pi$-real matrix of even order is obtained from (17) by dropping its center row and center column. Moreover, left $\Pi$-real matrices can be constructed by post-multiplying a left $\Pi$-real matrix $Q$ by an arbitrary real matrix $R$ of appropriate size. Using this definition, any centro-Hermitian matrix $Z \in \mathbb{C}^{p \times q}$ can be transformed into a real-valued matrix through the transformation $[23]$:

$$\varphi(Z) = Q_r^H Z Q_q \in \mathbb{R}^{p \times q}. \tag{18}$$

In Unitary ESPRIT, the centro-Hermitian matrix obtained after FBA is given by $\hat{\Sigma}$

$$\hat{\Sigma} = [X \quad \Pi_M X^* \Pi_N] \in \mathbb{C}^{M \times 2N}. \tag{19}$$

Next, we extend the concept of Unitary ESPRIT to the augmented data model in (5) and derive the R-D NC Unitary ESPRIT algorithm. Therefore, the FBA step as well as the real-valued transformation have to be applied to $X^{(nc)}$. Here, FBA is performed by replacing the NC measurement matrix $X^{(nc)} \in \mathbb{C}^{M \times N}$ by the column-wise augmented measurement matrix $X^{(nc)} \in \mathbb{C}^{2M \times 2N}$ defined by

$$\hat{\Sigma}^{(nc)} = \begin{bmatrix} X^{(nc)} & \Pi_2 M X^{(nc)^*} \Pi_N \\ \Pi_M X^* & \Pi_2 M X^* \Pi_N \end{bmatrix} \tag{20}$$

$$\hat{\Sigma}^{(nc)} = \begin{bmatrix} X^{(nc)} & \Pi_2 M X^{(nc)^*} \Pi_N \\ \Pi_M X^* & \Pi_2 M X^* \Pi_N \end{bmatrix}. \tag{21}$$

Due to the fact that equivalently to (19), $\hat{\Sigma}^{(nc)}$ is centro-Hermitian, it can be transformed into a real-valued matrix that takes the simple form

$$\varphi(\hat{\Sigma}^{(nc)}) = Q_{2M}^H \hat{\Sigma}^{(nc)} Q_{2N} \tag{22}$$

$$= 2 \begin{bmatrix} \text{Re}\{X\} & 0_{M \times N} \\ \text{Im}\{X\} & 0_{M \times N} \end{bmatrix}. \tag{23}$$

The proof is given in Appendix E.

In the next step, we define the transformed augmented steering matrix as $D^{(nc)} = Q_{2M}^H A^{(nc)}$. Based on the R-D shift invariance property of $A^{(nc)}$ proven in Theorem 1, it can easily be verified that $D^{(nc)}$ obeys

$$K^{(nc)}_1 D^{(nc)} \Omega^{(r)} = K^{(nc)}_2 D^{(nc)}, \quad r = 1, \ldots, R \tag{24}$$

where the $R$ pairs of augmented selection matrices in (9) are transformed according to (9) as

$$K^{(nc)}_1(r) = 2 \text{Re}\left\{ Q_{M_2 M}^H M/M_r \tilde{J}^{(nc)}_1 \tilde{Q}_{2M} \right\} \tag{25}$$

$$K^{(nc)}_2(r) = 2 \text{Im}\left\{ Q_{M_2 M}^H M/M_r \tilde{J}^{(nc)}_2 \tilde{Q}_{2M} \right\} \tag{26}$$

and the real-valued set of diagonal matrices $\Omega^{(r)} = \text{diag}\{[\omega_1^{(r)}, \ldots, \omega_M^{(r)}]\} \in \mathbb{R}^{d \times d}$ with $\omega_i^{(r)} = \tan(\mu_i^{(r)}/2)$ contain the spatial frequencies in the $r$-th mode.
obtained by

Finally, the correctly paired spatial frequency estimates are summarized in this subsection. Firstly, both algorithms 

eigenvalues that are summarized in this subsection. Secondly, it will be shown in Section V-B that the performance of 

NC Unitary ESPRIT and R-D NC Unitary ESPRIT can both resolve up to

\[
\min\{\min_r(2M_r^{(sel)}M/M_r), N\}
\]

incoherent sources as compared to \(\min_r(M_r^{(sel)}M/M_r), N\) and \(\min_{r,s} M/M_r, 2N\) for R-D Standard ESPRIT and R-D Unitary ESPRIT, respectively. Thus, if \(N\) is large enough, we can detect twice as many incoherent sources. Fourth, due to the exchange matrix \(W_N\) in (6), the real-valued transformation in R-D NC Unitary ESPRIT can be efficiently computed by stacking the real part and the imaginary part of \(X\) on top of each other, cf. equation (23). Finally, the computational complexity of both algorithms is dominated by the signal subspace estimate via the SVD of \(X\), the pseudo inverse in (16) and (28) respectively, and the required matrix multiplications, which results in a computational cost of \(O((2M)^2 + (2M)^2N)\). However, the complexity of R-D NC Unitary ESPRIT is lower than that of R-D NC Standard ESPRIT as these operations are real-valued.

V. PERFORMANCE OF R-D NC ESPRIT-TYPE ALGORITHMS

In this section, we present the first-order analytical performance assessment of R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT. As will be shown in Subsection V-B the performance of R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT is asymptotically identical. Therefore, we first resort to the simpler derivation of the expressions for R-D NC Standard ESPRIT and then show their equivalence. In contrast to our previous results in [29], we here also include the real-valued transformation in R-D NC Unitary ESPRIT into the proof.

A. Performance of R-D NC Standard ESPRIT

To obtain a first-order perturbation analysis of the parameter estimates, we adopt the two-step procedure of the analytical framework proposed in [8]. We first develop a first-order subspace error expansion and then find a corresponding first-order expansion for the parameter estimation error. It is evident from [8] that the preprocessing does not violate the assumption of a small noise perturbation made in [8]. Hence, we can apply the concept of [8] to the augmented measurement matrix in [6]. The results are asymptotic in the high effective SNR and explicit in the noise term \(N^{(nc)}\).

Starting with the subspace error expression based on (6), we can express the SVD of the noise-free observations \(X^{(nc)}_0\) as

\[
X^{(nc)}_0 = \begin{bmatrix} U_1^{(nc)} & U_0^{(nc)} \end{bmatrix} \begin{bmatrix} \Sigma^{(nc)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^{(nc)} \\ V_0^{(nc)} \end{bmatrix}^H,
\]
where $U_n^{(nc)} \in \mathbb{C}^{2M \times d}$, $U_n^{(nc)} \in \mathbb{C}^{2M \times (2M-d)}$, and $V_n^{(nc)} \in \mathbb{C}^{N \times d}$ span the signal subspace, the noise subspace, and the row space respectively, and $\Sigma^{(nc)} \in \mathbb{C}^{d \times d}$ contains the non-zero singular values on its diagonal. Next, we write the perturbed signal subspace estimate of $U_n^{(nc)}$ from the previous section as $\tilde{U}_n^{(nc)} = U_n^{(nc)} + \Delta U_n^{(nc)}$, where $\Delta U_n^{(nc)}$ denotes the estimation error. From (8) and its application to (6), we obtain the first-order subspace error approximation

$$\Delta U_n^{(nc)} = U_n^{(nc)} \tilde{U}_n^{(nc)^T} N^{(nc)} V_n^{(nc)^*} \Sigma_n^{(nc)^{-1}} + O(\Delta^2),$$

where $\Delta = \| N^{(nc)} \|$, and $\| \cdot \|$ represents an arbitrary sub-multiplicative norm. Equation (30) models the leakage of the noise subspace into the signal subspace due to the effect of the noise. The perturbation of the particular basis for the noise subspace $U_n^{(nc)}$, which is taken into account in [13, 14] can be ignored as the choice of this basis is irrelevant for $\hat{R}$-D NC Standard ESPRIT.

For the parameter estimation error of the $i$-th spatial frequency in the $r$-th mode obtained by the LS solution in (16), we follow the lines of (8) to obtain

$$\Delta \mu_i^{(r)} = \text{Im} \left\{ p_i^T \left( J_i^{(nc)} r U_n^{(nc)} \right)^* + J_i^{(nc)} r \Delta U_n^{(nc)}^* q_i \right\} + O(\Delta^2),$$

where $\lambda_i^{(r)} = \omega^{\mu_i^{(r)}}$ is the $i$-th eigenvalue of $\Gamma^{(r)}$ in the $r$-th mode, $q_i$ represents the $i$-th eigenvector of $\Gamma^{(r)}$, i.e., the $i$-th column vector of the eigenvector matrix $Q$, and $p_i^T$ is the $i$-th row vector of $P = Q^{-1}$. Hence, the eigendecomposition of $\Gamma^{(r)}$ in the $r$-th mode is given by

$$\Gamma^{(r)} = Q \Lambda^{(r)} Q^{-1},$$

where $\Lambda^{(r)}$ contains the eigenvalues $\lambda_i^{(r)}$ on its diagonal. Then, by inserting (30) into (31), we can write the first-order approximation for the estimation errors $\Delta \mu_i^{(r)}$ explicitly in terms of the noise perturbation $N^{(nc)}$.

In order to derive an analytical expression for the MSE of $\hat{R}$-D NC Standard ESPRIT, we resort to [15, 16], where we have derived an MSE expression that only depends on the SO statistics of the noise, i.e., the covariance matrix and the pseudo-covariance matrix, assuming the noise to be zero-mean. As the preprocessing in [16] does not violate the zero-mean assumption, [15] is applicable once the SO statistics are found. Therefore, for $n^{(nc)} = \text{vec}(N^{(nc)}) \in \mathbb{C}^{2MN \times 1}$, its covariance matrix $R_{nn}^{(nc)} = \mathbb{E}(n^{(nc)} n^{(nc)^T}) \in \mathbb{C}^{2MN \times 2MN}$ and its pseudo-covariance matrix $C_{nn}^{(nc)} = \mathbb{E}(n^{(nc)} n^{(nc)^T}) \in \mathbb{C}^{2MN \times 2MN}$, the MSE for the $i$-th spatial frequency in the $r$-th mode is given by

$$\mathbb{E}\left\{ |\Delta \mu_i^{(r)}|^2 \right\} = \frac{1}{2} \left( r_i^{(nc)^T} W^{(nc)^T} + r_i^{(nc)^T} W^{(nc)^T} r_i^{(nc)^T} \right) + O(\Delta^2),$$

where

$$r_i^{(nc)^T} = q_i \otimes \left( \left( J_i^{(nc)} r U_n^{(nc)} \right)^* \right)^T = \mathbb{C}^{2Md \times 1}.$$

A matrix norm is called sub-multiplicative if $\| A \cdot B \| \leq \| A \| \cdot \| B \|$ for arbitrary matrices $A$ and $B$.

Thus, the SO statistics of $n^{(nc)}$ can be expressed by means of the covariance matrix $R_{nn}^{(nc)} = \mathbb{E}(n^{(nc)} n^{(nc)^T})$ and the pseudo-covariance matrix $C_{nn}^{(nc)} = \mathbb{E}(n^{(nc)} n^{(nc)^T})$. We first expand $n^{(nc)}$ as

$$n^{(nc)} = \text{vec}\left\{ N^{(nc)} \right\} = \text{vec} \left\{ N \right\} \left( I_m \otimes \Pi_M \right)^*,$$

for arbitrary matrices $A \in \mathbb{C}^{m \times N}$. We further expand $n^{(nc)}$ as

$$n^{(nc)} = \text{vec}\left\{ N^{(nc)} \right\} = \text{vec} \left\{ N \right\} \left( I_m \otimes \Pi_M \right)^*,$$

for arbitrary matrices $A \in \mathbb{C}^{m \times N}$. We first expand $n^{(nc)}$ as

$$n^{(nc)} = \text{vec}\left\{ N^{(nc)} \right\} = \text{vec} \left\{ N \right\} \left( I_m \otimes \Pi_M \right)^*.$$
B. Performance of R-D NC Unitary ESPRIT

So far, we have only derived the explicit first-order parameter estimation error approximation and the MSE expression for R-D NC Standard ESPRIT. In this subsection, however, we show that the analytical performance of R-D NC Unitary ESPRIT and R-D NC Standard ESPRIT is identical in the high effective SNR. To this end, we recall that R-D NC Unitary ESPRIT includes forward-backward-averaging (FBA) (20) as well as the transformation into the real-valued domain (22) as preprocessing steps. We first investigate the influence of FBA and state the following theorem:

**Theorem 3.** Applying FBA to \( X^{(nc)} \) does not improve the signal subspace estimate.

**Proof:** To show this result, we simply use the FBA-processed augmented measurement matrix \( \tilde{X}^{(nc)} \) in (21) and compute the Gram matrix \( G = \tilde{X}^{(nc)\dagger}\tilde{X}^{(nc)} \), which yields

\[
G = [X^{(nc)} X^{(nc)}\Pi_N] [X^{(nc)} X^{(nc)}\Pi_N]^H = 2 X^{(nc)}X^{(nc)\dagger}.
\]

(42)

Thus, the matrix \( G \) reduces to the Gram matrix of \( X^{(nc)} \) and the column space of \( X^{(nc)} \) is the same as the column space of the Gram matrix of \( X^{(nc)} \). Consequently, \( X^{(nc)} \) already includes FBA. This completes the proof.

Next, we analyze the real-valued transformation as the second preprocessing step of R-D NC Unitary ESPRIT and formulate the theorem:

**Theorem 4.** The performance after the real-valued transformation (22) is asymptotically identical to the performance of the complex-valued case before (22) in the high effective SNR.

**Proof:** See Appendix C.

As a result of Theorem 3 and Theorem 4, we can conclude that the asymptotic performance of R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT is asymptotically identical in the high effective SNR.

VI. SINGLE SOURCE CASE

So far, we have derived an MSE expression for both R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT (33), which is deterministic and no Monte-Carlo simulations are required. However, this is only the first step as the derived MSE expression is formulated in terms of the subspaces of the unperturbed measurement matrix and hence, provides no explicit insights into the influence of the physical parameters, e.g., the SNR, the number of sensors, the sample size, etc. Knowing how the performance scales with these system parameters as a second step can facilitate array design decisions on the number of required sensors to achieve a certain performance for a specific SNR. Moreover, different parameter estimators can be objectively compared to find the best one for specific scenarios. Establishing a generally valid formulation is an intricate task given the complex dependence of the subspaces on the physical parameters. However, special cases can be considered to gain more insights by such an analytical performance assessment. Inspired by [15], we present results for the 1-D case of a single strictly SO non-circular source captured by a ULA under circularly symmetric white noise in this section. Furthermore, we obtain the same asymptotic estimation error for 1-D NC Standard ESPRIT and 1-D NC Unitary ESPRIT as proven in the previous section. We relate our results to the deterministic 1-D NC Cramér-Rao bound [14] to determine their asymptotic efficiency in closed-form. Note that an extension of these results to the R-D case is straightforward.

A. 1-D NC Standard ESPRIT and Unitary ESPRIT

As the asymptotic performance of both algorithms is the same, it is again sufficient to determine the MSE expressions for 1-D NC Standard ESPRIT. We have the following result:

**Theorem 5.** For the case of an \( M \)-element ULA (1-D), a single strictly non-circular source \((\ell = 1)\), and circularly symmetric white noise, the MSE of the spatial frequency for NC Standard and NC Unitary ESPRIT is given by

\[
\mathbb{E}(\{\Delta \mu\})^2 = \frac{1}{\hat{\rho} \cdot (M-1)^2} + O \left( \frac{1}{\hat{\rho}^2} \right),
\]

(43)

where \( \hat{\rho} \) represents the effective SNR \( \hat{\rho} = N \hat{P}_s/\sigma^2 \) with \( \hat{P}_s \) being the empirical source power given by \( \hat{P}_s = ||s||^2/N \) and \( s \in \mathbb{C}^N \times 1 \).

**Proof:** We start the proof by simplifying the MSE expression for 1-D NC Standard ESPRIT in (33). In the single source case the noise-free NC measurement matrix can be written as

\[
X_0^{(nc)} = a^{(nc)}(\mu)s^T,
\]

(44)

where \( a^{(nc)}(\mu) = [a^T(\mu), \Psi M a^H(\mu)]^T \in \mathbb{C}^{2M \times 1} \) is the augmented array steering vector with \( \Psi = \Psi^\ast \Psi^* = e^{-j2\phi} \), \( s \in \mathbb{C}^{N \times 1} \) contains the source symbols, and \( \hat{P}_s = ||s||^2/N \) is the empirical source power. In what follows, we drop the dependence of \( a^{(nc)} \) on \( \mu \) for notational convenience.

If we assume a ULA of isotropic elements, we have \( a = [1, e^{i\mu}, \ldots, e^{i(M-1)\mu}]^T \) and \( ||a^{(nc)}||^2_2 = 2M \). The selection matrices \( J_1^{(nc)} \) and \( J_2^{(nc)} \) are then chosen according to (13) with \( J_1 = [I_{M-1}, 0_{(M-1)\times 1}] \) and \( J_2 = [0_{(M-1)\times 1}, I_{M-1}] \) for maximum overlap, i.e., \( M^{(nc)} = M - 1 \). Note that (43) is a rank-one matrix and we can directly determine the subspaces from the SVD as

\[
U_s^{(nc)} = u_1^{(nc)} = \frac{a^{(nc)}}{||a^{(nc)}||_2} = \frac{a^{(nc)}}{\sqrt{2M}},
\]

\[
\Sigma_s^{(nc)} = \sigma_1^{(nc)} = \sqrt{2MN\hat{P}_s},
\]

\[
V_s^{(nc)} = v_1^{(nc)} = \frac{s^*}{||s||_2} = \frac{s^*}{\sqrt{N\hat{P}_s}}.
\]

For the MSE expression in (33), we also require \( P_{a^{(nc)}} = U_s^{(nc)}U_s^{(nc)\dagger} = I_{2M} - \frac{1}{M} a^{(nc)}a^{(nc)\dagger} \), which is the projection matrix onto the noise subspace, and we have \( \Psi = e^{i\mu} \) and hence, \( p_1 = q_1 = 1 \) for the eigenvectors. Furthermore, \( R_{nn}^{(nc)} \) and \( C_{nn}^{(nc)} \) are given by (41).
Inserting these expressions into (33), we get
\[
\mathbb{E}\{(\Delta \mu)^2\} = \frac{\sigma_n^2}{2} \left( \left\| r^{(nc)} \right\|_2^2 \right) - \mathbb{E}\left\{ \left( r^{(nc)T} W^{(nc)} (I_N \otimes \Pi_{2M}) (r^{(nc)T} W^{(nc)})^T \right) \right\}
\] (45)
with
\[
r^{(nc)} = \begin{bmatrix} J_1^{(nc)} \frac{a^{(nc)}}{\sqrt{2M}} + (J_2^{(nc)} / \mu - J_1^{(nc)}) \end{bmatrix}^T \in \mathbb{C}^{2M \times 1},
\]
\[
W^{(nc)} = \left( \frac{1}{\sqrt{2MNp}} - \frac{s^{H}}{\sqrt{NP_{s}}} \right) \otimes P_{a^{(nc)}}^{\perp} \in \mathbb{C}^{2M \times 2MN}.
\] (46)

Note that \(r^{(nc)T} W^{(nc)}\) can also be written as \(r^{(nc)T} W^{(nc)} = \tilde{s}^T \otimes \tilde{a}^T\), where
\[
\tilde{s}^T = \frac{1}{\sqrt{2MNp}} s^H,
\]
\[
\tilde{a}^T = \begin{bmatrix} J_1^{(nc)} \frac{a^{(nc)}}{\sqrt{2M}} + (J_2^{(nc)} / \mu - J_1^{(nc)}) \end{bmatrix} P_{a^{(nc)}}^{\perp}. \quad (49)
\]
Thus, after straightforward calculations, the MSE in (45) can be expressed as
\[
\mathbb{E}\{(\Delta \mu)^2\} = \frac{\sigma_n^2}{2} \left( \left\| \tilde{s}^T \right\|_2^2 \right) - \mathbb{E}\left\{ \left( \tilde{s}^T \tilde{a} \tilde{T} \Pi_{2M} \tilde{a} \right) \right\}.
\] (50)

The first term \(\left\| \tilde{s}^T \right\|_2^2\) of (50) can be conveniently expressed as \(\left\| \tilde{s}^T \right\|_2^2 = \frac{1}{2M^2 N_{P_{s}}}\). For the second term \(\left\| \tilde{a}^T \right\|_2^2\) of (50), we simplify \(\tilde{a}^T\) and expand the pseudo-inverse of \(J_2^{(nc)} a^{(nc)}\) using the relation \(x^+ = x^H / \|x\|_2^2\). Then, taking the shift invariance equation \(J_2^{(nc)} a^{(nc)} / \mu - J_1^{(nc)} a^{(nc)} = 0\) into account, we arrive at
\[
\tilde{a}^T = \sqrt{2M} \left( \tilde{a}_1^T - \tilde{a}_2^T \right), \quad \text{where}
\]
\[
\tilde{a}_1^T = a^{(nc)} J_1^{(nc)} J_2^{(nc)} / \mu, \quad \tilde{a}_2^T = a^{(nc)} J_1^{(nc)} J_1^{(nc)}.
\] (51)

Next, as we have \(a^{(nc)} = a^{H} \tilde{\Psi} a^T \Pi_M\), it is easy to verify that
\[
\tilde{a}_1^T = [0, e^{-j \mu_0}, \ldots, e^{-j(M-1) \mu_0}, \tilde{\Psi}^*], \quad \tilde{a}_2^T = [1, e^{-j \mu_0}, \ldots, e^{-j(M-2) \mu_0}, \tilde{\Psi}^*, e^{j(M-1) \mu_0}, \ldots, e^{j \mu_0}, 1].
\]

Consequently, we obtain \(\left\| \tilde{a}^T \right\|_2^2 = \frac{8M}{(2M-2)^2}\). The third term \(\tilde{s}^T \tilde{a}\) can be simplified as \(\tilde{s}^T \tilde{a} = \frac{1}{2M^2 N_{P_{s}}} \tilde{\Psi}_{s_0}\), where we have used the equality \(s = \tilde{\Psi}_{s_0}\) and the fact that \(s_0^T = N_{P_{s}}\). Moreover, using (51), the last term of (50) can be reduced to \(\tilde{a}^T \Pi_{2M} \tilde{a} = -\frac{8M}{(2M-2)^2}\). Inserting these results into (50), we finally obtain for the MSE of 1-D NC Standard ESPRIT
\[
\mathbb{E}\{(\Delta \mu)^2\} = \frac{\sigma_n^2}{N_{P_{s}}} \frac{1}{(M-1)^2}, \quad (52)
\]
which is the desired result.

In a similar fashion, it can be shown that we arrive at the same result for the MSE (52) when this analysis is conducted for 1-D NC Unitary ESPRIT. Moreover, the expression (52) is equivalent to the ones obtained in [15] for their non-NC counterparts. Thus, no improvement in terms of the estimation accuracy can be achieved by applying 1-D NC Standard ESPRIT or 1-D NC Unitary ESPRIT for a single strictly non-circular source. This can also be seen from (53) derived in the next subsection, which is also the same as in the non-NC case [15].

B. Deterministic 1-D NC Cramér-Rao Bound and Asymptotic Efficiency of 1-D NC Standard and Unitary ESPRIT

In this part, we simplify the deterministic 1-D NC Cramér-Rao Bound derived in (54) for the special case of a single source. The result is shown in the next theorem:

**Theorem 6.** For the case of an \(M\)-element ULA (1-D) and a single strictly non-circular source \((d = 1)\), the deterministic 1-D NC Cramér-Rao Bound can be simplified to
\[
C^{(nc)} = \frac{1}{\rho} \frac{6}{M(M^2-1)}. \quad (53)
\]

**Proof:** See Appendix [D]

Under the stated assumptions, the asymptotic efficiency of 1-D NC Standard and Unitary ESPRIT can be explicitly computed as
\[
\eta = \lim_{\rho \to \infty} \frac{\mathbb{E}\{(\Delta \mu)^2\}}{\mathbb{E}\{(\Delta \mu)^2\}} = \frac{6(M-1)}{M(M+1)}. \quad (54)
\]
Again, the asymptotic efficiency (54) is equivalent to the one derived in [15], i.e., no gains are obtained from non-circular sources. It should be noted that \(\eta\) is only a function of the array geometry, i.e., the number of sensors \(M\). The outcome of this result is that 1-D NC ESPRIT-type algorithms using LS are asymptotically efficient for \(M = 2\) and \(M = 3\) as a single source. However, they become less efficient when the number of sensors grows, in fact, for \(M \to \infty\) we have \(\eta \to 0\). A possible explanation could be that an 1-D NC ULA offers not only one shift invariance with maximum overlap used in LS, but multiple invariances that are not exploited by LS.

VII. SIMULATION RESULTS

In this section, we provide simulation results to evaluate the performance of the proposed R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT algorithms along with the asymptotic behavior of the presented performance analysis. We compare the square root of the analytical MSE expression ("ana") in (33) to the root mean squared error (RMSE) of the empirical estimation error ("emp") of R-D NC Standard ESPRIT (NC SE) and R-D NC Unitary ESPRIT (NC UE) obtained by averaging over 5000 Monte Carlo trials. The RMSE is defined as
\[
\text{RMSE} = \frac{1}{Rd} \mathbb{E} \left\{ \sum_{r=1}^{R} \sum_{i=1}^{d} \left( \hat{\mu}_i^{(r)} - \mu_i^{(r)} \right)^2 \right\}, \quad (55)
\]
where \(\hat{\mu}_i^{(r)}\) is the estimate of \(i\)-th spatial frequency in the \(r\)-th mode. Furthermore, we compare our results to R-D Standard
Moreover, we assume zero-mean circularly symmetric white noise with rotation phases $\varphi_1 = 0$, $\varphi_2 = \pi/2$. In Fig. 2, we depict the RMSE versus the number of snapshots $N$ for the non-centro-symmetric 2-D array with $M = 20$ given in Fig. 4 where we also provide the subarrays in both dimensions. The SNR is fixed at 10 dB and we have $d = 3$ uncorrelated sources with the spatial frequencies $\mu_1^{(1)} = 0.25$, $\mu_2^{(1)} = 0.5$, $\mu_3^{(1)} = 0.75$, $\mu_1^{(2)} = 0.25$, $\mu_2^{(2)} = 0.5$, and $\mu_3^{(2)} = 0.75$. The rotation phases are given by $\varphi_1 = 0$, $\varphi_2 = \pi/4$, and $\varphi_3 = \pi/2$. Note that 2-D NC Unitary ESPRIT cannot be applied as the array is not centro-symmetric. It is apparent from Fig. 1 and Fig. 2 that in general, the NC schemes perform better than their non-NC counterparts. Specifically, 2-D NC Unitary ESPRIT provides a lower estimation error than 2-D NC Standard ESPRIT for low SNRs and a low sample size. Moreover, the analytical results agree well with the empirical estimation errors for high effective SNRs, i.e., when either the SNR or the number of samples becomes large. This also validates that the asymptotic performance of R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT is identical as both coincide with the analytical curves.

In Fig. 4, we show the RMSE as a function of the separation (“sep”) between $d = 2$ uncorrelated sources located at $\mu_1^{(1)} = -\text{sep}/2$, $\mu_2^{(1)} = 0$, $\mu_1^{(2)} = \text{sep}/2$, $\mu_2^{(2)} = \text{sep}$ with the rotation phases $\varphi_1 = 0$, $\varphi_2 = \pi/2$. We employ a $5 \times 6$ uniform rectangular array (URA), $N = 5$ snapshots, and the SNR is fixed at 30 dB. Fig. 4 demonstrates the RMSE as a function of the non-circularity phase separation $\Delta \varphi$ of the $d = 2$ uncorrelated sources with the spatial frequencies $\mu_1^{(1)} = 1$, $\mu_2^{(1)} = 1$, $\mu_1^{(2)} = 0.8$, and $\mu_2^{(2)} = 0.8$. The remaining parameters are kept the same. Again, it can be seen from Fig. 4 and Fig. 5 that the analytical results match the empirical ones. But more importantly, the gain of the NC ESPRIT-type methods increases if the sources approach each other. Also, 2-D NC Unitary ESPRIT outperforms 2-D NC Standard ESPRIT. Furthermore, as an important feature of strictly non-circular sources, it is shown that two sources decouple if their phase separation is equal to $\Delta \varphi = \pi/2$. This is also verified by Fig. 5 where the gain is largest for a maximum phase separation of $\Delta \varphi = \pi/2$.

In the final simulation, we consider the single source case, which was used in Section 7A to simplify the analytical MSE equations for 1-D NC Standard ESPRIT and 1-D NC Unitary ESPRIT only in terms of the physical parameters, i.e., the array $\mu_1^{(3)} = 0.25$, and $\mu_2^{(3)} = 0.5$, and a real-valued pair-wise correlation of $\rho = 0.9$. The rotation phases contained in $\Psi$ are given by $\varphi_1 = 0$ and $\varphi_2 = \pi/2$. In Fig. 6, we depict the RMSE versus the number of snapshots $N$ for the non-centro-symmetric 2-D array with $M = 20$ given in Fig. 4 where we also provide the subarrays in both dimensions. The SNR is fixed at 10 dB and we have $d = 3$ uncorrelated sources with the spatial frequencies $\mu_1^{(1)} = 0.25$, $\mu_2^{(1)} = 0.5$, $\mu_3^{(1)} = 0.75$, $\mu_1^{(2)} = 0.25$, $\mu_2^{(2)} = 0.5$, and $\mu_3^{(2)} = 0.75$. The rotation phases are given by $\varphi_1 = 0$, $\varphi_2 = \pi/4$, and $\varphi_3 = \pi/2$. Note that 2-D NC Unitary ESPRIT cannot be applied as the array is not centro-symmetric. It is apparent from Fig. 1 and Fig. 2 that in general, the NC schemes perform better than their non-NC counterparts. Specifically, 2-D NC Unitary ESPRIT provides a lower estimation error than 2-D NC Standard ESPRIT for low SNRs and a low sample size. Moreover, the analytical results agree well with the empirical estimation errors for high effective SNRs, i.e., when either the SNR or the number of samples becomes large. This also validates that the asymptotic performance of R-D NC Standard ESPRIT and R-D NC Unitary ESPRIT is identical as both coincide with the analytical curves.

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sources and shift-invariant arrays that are not necessarily algorithms specifically designed for strictly SO non-circular \( R \) ESPRIT and strictly non-circular source. D Standard ESPRIT. Hence, no gain is achieved from a single \( \varphi_1 = 0, \varphi_2 = \pi/2 \).

Fig. 4. Analytical and empirical RMSEs versus the separation (“sep”) of \( d = 2 \) uncorrelated sources at \( \mu_1^{(1)} = -\text{sep}/2, \mu_2^{(1)} = 0, \mu_1^{(2)} = \text{sep}/2, \mu_2^{(2)} = \text{sep} \) for a \( 5 \times 6 \) URA, \( N = 5, \text{SNR} = 30 \) dB, with rotation phases \( \varphi_1 = 0, \varphi_2 = \pi/2 \).

Fig. 5. Analytical and empirical RMSEs versus the phase separation for a \( 5 \times 6 \) URA, \( N = 5, \text{SNR} = 30 \) dB, \( d = 2 \) uncorrelated sources at \( \mu_1^{(1)} = 1, \mu_2^{(1)} = 1, \mu_1^{(2)} = 0.8, \mu_2^{(2)} = 0.8 \).

size \( M \) and the SNR. Fig. 4 shows the asymptotic efficiency (54) versus the number of sensors \( M \) for a ULA. The effective SNR is set to 25 dB, where \( P_s = 0 \) dBW, \( N = 10, \) and \( \sigma_n^2 = 0.032 \). This plot validates the fact that 1-D NC Unitary ESPRIT and 1-D NC Standard ESPRIT using LS become increasingly inefficient for \( M > 3 \). It should be stressed that the same curves are obtained for 1-D Unitary ESPRIT and 1-D Standard ESPRIT. Hence, no gain is achieved from a single strictly non-circular source.

VIII. CONCLUSION

In this paper, we have presented the \( R \)-D NC Standard ESPRIT and \( R \)-D NC Unitary ESPRIT parameter estimation algorithms specifically designed for strictly SO non-circular sources and shift-invariant arrays that are not necessarily centro-symmetric. We have also derived a first-order approximation of the analytical performance of both algorithms. Our results are based on a first-order expansion of the estimation error in terms of the explicit noise perturbation, which is required to be small compared to the signals but no assumptions about the noise statistics are needed. We have also derived MSE expressions that only depend on the finite SO moments of the noise and merely assume the noise to be zero-mean. All the resulting expressions are asymptotic in the effective SNR, i.e., they become accurate for either high SNRs or a large sample size. Furthermore, we have analytically proven that \( R \)-D NC Standard ESPRIT and \( R \)-D NC Unitary ESPRIT have the same asymptotic performance in the high effective SNR regime. However, \( R \)-D NC Unitary ESPRIT should be preferred due to its real-valued operations and its better performance at low effective SNRs. We have also computed the asymptotic efficiency in closed-form for a single source and found that no gain from non-circular sources is achieved in this case. Simulations demonstrate that for more than one strictly non-circular source, the NC gain is largest for closely-spaced sources and a rotation phase separation of \( \pi/2 \).

APPENDIX A

PROOF OF THEOREM 1

We start by inserting (10) and (11) into the augmented shift invariance relation (8), which yields

\[
\begin{bmatrix}
J_1 \mathbf{A} \\
\Pi_{M(\text{sel})} J_2 \mathbf{A} \Pi_{M(\text{sel})} \Phi \Psi^* \\
\Pi_{M(\text{sel})} J_1 \mathbf{A} \Psi \Psi^* \\
\end{bmatrix} = \begin{bmatrix}
J_2 \mathbf{A} \\
\Pi_{M(\text{sel})} J_1 \mathbf{A} \Phi \Psi^* \\
\Pi_{M(\text{sel})} J_2 \mathbf{A} \Psi \Psi^* \\
\end{bmatrix}
\]

The first \( M(\text{sel}) \) rows are given by \( J_1 \mathbf{A} \Phi = J_2 \mathbf{A} \), which was assumed for the theorem. The second \( M(\text{sel}) \) rows can be simplified by multiplying the left from the left with \( \Pi_{M(\text{sel})} \) and then using the fact that \( \Pi_M \Pi_M = I_M \). We obtain

\[
J_2 \mathbf{A} \Psi \Psi^* \Phi = J_1 \mathbf{A} \Psi \Psi^*.
\]

As \( \Psi \) and \( \Phi \) are diagonal, they commute. Then, multiplying twice by \( \Psi \) from the right-hand side cancels \( \Psi \) as \( \Psi \Psi^* = I_d \).
and we are left with

\[ J_2 A^* \Phi = J_1 A^* \]
\[ J_2 A^* = J_1 A^* \Phi^*, \]

(57)

where in the last step we have multiplied with \( \Phi^* \) from the right-hand side and used the fact that \( \Phi^* \Phi = I \). Finally, conjugating (57) shows that this expression is equivalent to \( J_1 A \Phi = J_2 A \), which was again assumed for the theorem. This concludes the proof. \( \square \)

**APPENDIX B**

**PROOF OF EQUATION (23)**

The real-valued transformation is carried out using sparse left \( \Pi \)-real matrices of even order according to (17). Expanding (23) yields

\[
\varphi(\tilde{X}^{(nc)}) = Q^{H}_{2Mj} \tilde{X}^{(nc)} Q_{2N}
\]
\[
= \frac{1}{2} \begin{bmatrix}
I_M & \Pi_M \\
-jI_M & j\Pi_M
\end{bmatrix} \begin{bmatrix}
X^{(nc)} & X^{(nc)} \Pi_N \\
0_{M \times N} & 0_{M \times N}
\end{bmatrix} \begin{bmatrix}
I_N & jI_N \\
-I_N & -jI_N
\end{bmatrix}
\]
\[
= \begin{bmatrix}
X + X^* & 0_{M \times N} \\
-jX + jX^* & 0_{M \times N}
\end{bmatrix} = 2 \begin{bmatrix}
\text{Re} \{X\} & 0_{M \times N} \\
\text{Im} \{X\} & 0_{M \times N}
\end{bmatrix},
\]

where we have used the fact that \(-jx + jx^* = 2 \text{Im} \{ x \} \) \( \forall x \in \mathbb{C} \). This completes the proof. \( \square \)

**APPENDIX C**

**PROOF OF THEOREM 4**

For simplicity, we only present the proof for the 1-D case, but the approach adopted here carries over to the \( R \)-D case straightforwardly. The estimated parameters after the real-valued transformation are extracted in a different manner as in the forward-backward-averaged complex-valued case, i.e., using the arctangent function. Hence, we develop a first-order perturbation expansion for the real-valued shift invariance equations and then show the equivalence of both cases. To this end, let \( \tilde{X}^{(nc)}_0 \in \mathbb{C}^{2M \times 2N} \) be the noise-free forward-backward averaged measurement matrix defined by decomposing according to

\[
\tilde{X}^{(nc)} = \begin{bmatrix} X^{(nc)}_0 & X^{(nc)}_0 \Pi_N \end{bmatrix} + [N^{(nc)} N^{(nc)} \Pi_N]
= \tilde{X}^{(nc)}_0 + \tilde{N}^{(nc)}.
\]

(58)

Its SVD can be expressed as

\[
\tilde{X}^{(nc)}_0 = \begin{bmatrix} \tilde{U}^{(nc)}_s & \tilde{U}^{(nc)}_h \end{bmatrix} \begin{bmatrix} \Sigma^{(nc)} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} \tilde{V}^{(nc)}_s & \tilde{V}^{(nc)}_h \end{bmatrix}^H,
\]

such that the complex-valued shift invariance equation for the forward-backward averaged data has the form

\[
J^{(nc)}_1 \tilde{U}^{(nc)}_s \Psi = J^{(nc)}_2 \tilde{U}^{(nc)}_h,
\]

(59)

where \( \Psi = Q^{(hba)} A Q^{(hba)}^{-1} \) and \( A = \text{diag} \{ \lambda_1, \ldots, \lambda_d \} \) with \( \lambda_i = e^{j\mu_i}, i = 1, 2, \ldots, d \). Performing the same steps as in Section 5(4) the first-order approximation of the estimation error after the application of FBA is given by

\[
\Delta \mu_i = \text{Im} \left\{ p_i^{(hba)} \left( J^{(nc)}_2 \tilde{U}^{(nc)}_h \right)^T \left( J^{(nc)}_2 \tilde{U}^{(nc)}_h \right)^+ \left[ J^{(nc)}_1 \tilde{J}^{(nc)}_1 / \lambda_i \right. \\
\left. - J^{(nc)}_1 \tilde{J}^{(nc)}_1 \right] \Delta \tilde{U}^{(nc)}_h q_i^{(hba)} + O(\Delta^2) \right\},
\]

(60)

where we have simply replaced the corresponding quantities in (31) by their FBA versions. Next, we show that the estimation error expansion for the real-valued case is equivalent to (60).

As the 1-D real-valued shift-invariance equation

\[
K^{(nc)}_1 E_s^{(nc)} Y = K^{(nc)}_2 E_s^{(nc)} H_s^{(nc)} + \mathbf{Y},
\]

(61)

where \( \mathbf{Y} = V \Omega V^{-1} \) and \( \Omega = \text{diag} \{ \omega_1, \ldots, \omega_d \} \) with \( \omega_i = \tan(\mu_i / 2), i = 1, 2, \ldots, d \), has the same algebraic form as its complex-valued counterpart, in (59), the same procedure from (3) can be applied to develop a first-order perturbation expansion. In fact, following the three steps discussed in (3), we find that the perturbation of \( \omega_i \) in terms of \( \mathbf{Y} \) and the perturbation of \( \mathbf{Y} \) in terms of the signal subspace estimation error \( \Delta \tilde{U}^{(nc)}_h \) lead to the same result, where \( J^{(nc)}_1, J^{(nc)}_2, U^{(nc)}_s, U^{(nc)}_h, \) and \( \Psi \) are consistently exchanged by \( K^{(nc)}_1, K^{(nc)}_2, E_s^{(nc)} H_s^{(nc)}, \) and \( \mathbf{Y} \), respectively. Thus, only the perturbation of \( \mu_i \) in terms of \( \omega_i = \tan(\mu_i / 2) \) is to be derived. Therefore, we compute the Taylor series expansion of \( \omega_i \), which is given by

\[
\omega_i + \Delta \omega \approx \tan(\mu_i / 2) + \Delta \mu \left( \frac{\tan^2(\mu_i / 2)}{2} + \frac{1}{2} \right)
\]
\[
= \omega_i + \Delta \mu \frac{\omega_i^2 + 1}{2 \omega_i^2 + 1},
\]

(62)

Combining (62) with the corresponding real-valued expressions for the perturbations of \( \omega_i \) and \( \mathbf{Y} \), we obtain

\[
\Delta \mu_i = p_i^{(hba)} \left( K^{(nc)}_1 E_s^{(nc)} \right)^+ \left( K^{(nc)}_2 - \omega_i K^{(nc)}_1 \right) \cdot \Delta E^{(nc)} q_i^{(hba)} \frac{2}{\omega_i^2 + 1},
\]

(63)

where \( q_i^{(hba)} \) is the \( i \)-th column of \( Q^{(hba)} \) and \( p_i^{(hba)} \) is the \( i \)-th row of \( P^{(hba)} = Q^{(hba)} \). Moreover, the perturbation of the real-valued subspace \( E_s^{(nc)} \) is expanded in terms of the transformed noise contribution \( \varphi(N^{(nc)}) \) as

\[
\Delta E_s^{(nc)} = E_s^{(nc)} E_s^{(nc)}^H \varphi(N^{(nc)}) W_s^{(nc)} (\Sigma_s^{(nc)})^{-1},
\]

(64)

where the required subspace are obtained from the SVD of the transformed real-valued measurement matrix \( \varphi(\tilde{X}^{(nc)}_0) = Q^{H}_{2M} \tilde{X}^{(nc)}_0 \in \mathbb{R}^{2M \times 2N} \) expressed as

\[
\varphi(\tilde{X}^{(nc)}_0) = \begin{bmatrix} E_s^{(nc)} & E_n^{(nc)} \end{bmatrix} \begin{bmatrix} \Sigma_s^{(nc)} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} W_s^{(nc)} & W_n^{(nc)} \end{bmatrix}^H.
\]

(65)

To simplify (63), it is easy to see that due to the fact that the matrices \( Q_{sp} \) are unitary, the subspaces of \( \varphi(\tilde{X}^{(nc)}_0) \) are also given by

\[
E_s^{(nc)} = Q^{H}_{2M} \tilde{U}^{(nc)}_s, E_n^{(nc)} = Q^{H}_{2M} \tilde{U}^{(nc)}_h, \Sigma_s^{(nc)} = \Sigma_s^{(nc)}, W_s^{(nc)} = Q^{H}_{2N} \tilde{V}^{(nc)}_s, W_n^{(nc)} = Q^{H}_{2N} \tilde{V}^{(nc)}_h.
\]
Moreover, we find that the transformed selection matrices $K_1^{(nc)}$ and $K_2^{(nc)}$ defined in (55) and (56) can be reformulated as

$$K_1^{(nc)} = Q_{2M}^H \left( J_1^{(nc)} + J_2^{(nc)} \right) Q_{2M},$$

$$K_2^{(nc)} = jQ_{2M}^H \left( J_1^{(nc)} - J_2^{(nc)} \right) Q_{2M},$$

which follows from expanding the real part and the imaginary part according to $2 \Re \{ x \} = x + x^*$ and $2 \Im \{ x \} = -x - x^*$. The conjugated term $Q_{2M}^H J_1^{(nc)} Q_{2M}$ can be simplified to $Q_{2M}^H J_1^{(nc)}$ because $J_1^{(nc)} = \Pi_{2M} J_1^{(nc)} \Pi_{2M}$ holds since the virtual array is always centrosymmetric as shown in Theorem 3 and the fact that $Q_p$ is left-$\Pi$-real.

Inserting (65) into (63) and applying the identities (68)-(70), we have

$$\Delta \mu_i = p_i^r_{(fba)} \left( (J_1^{(nc)} + J_2^{(nc)}) \hat{U}_s^{(nc)} + j(J_1^{(nc)} - J_2^{(nc)}) \right) \hat{U}_s^{(nc)} + \frac{2}{\omega_i^2 + 1},$$

where $\Delta \hat{U}_s^{(nc)} = \hat{U}_s^{(nc)} - \hat{U}_s^{(nc)} = \hat{A} \hat{U}_s^{(nc)} - \hat{U}_s^{(nc)} \hat{V}_s^{(nc)} \hat{S}_s^{(nc)}$.

In order to further simplify (68), we require the following two lemmas:

**Lemma 1.** The following identities are satisfied

$$\left( J_1^{(nc)} + J_2^{(nc)} \right) \hat{U}_s^{(nc)} = J_1^{(nc)} \hat{U}_s^{(nc)} \hat{\Psi},$$

$$\left( J_1^{(nc)} - J_2^{(nc)} \right) \hat{U}_s^{(nc)} = J_2^{(nc)} - \hat{U}_s^{(nc)} \hat{\Psi},$$

where $\hat{\Psi} = \hat{I}_d + \hat{\Psi} = Q_{(fba)} \left( I_d + \Lambda \right) Q_{(fba)}^{-1}$ and $\hat{\Psi} = -I_d + \Lambda = Q_{(fba)} \left( I_d - \Lambda + 1 \right) Q_{(fba)}^{-1}$.

**Proof:** The identities follow straightforwardly from $J_1^{(nc)} \hat{U}_s^{(nc)} \Psi = J_2^{(nc)} \hat{U}_s^{(nc)}$ by adding $J_1^{(nc)} \hat{U}_s^{(nc)}$ to both sides of the equation for the first identity, and subtracting $J_1^{(nc)} \hat{U}_s^{(nc)}$ and substituting $J_1^{(nc)} \hat{U}_s^{(nc)}$ by $J_2^{(nc)} \hat{U}_s^{(nc)} \Psi$ for the second identity.

**Lemma 2.** In the noiseless case, the solution $\Psi$ to (59) and the solution $\Omega$ to (61) have the same eigenvectors, i.e., $Q_{(fba)} = V$. Moreover, their eigenvalues are related as $\omega_i = \frac{1}{\omega_i - \Lambda}$.

**Proof:** Starting from $\Psi = \left( K_1^{(nc)} E_s^{(nc)} + K_2^{(nc)} E_s^{(nc)} \right)$ and replacing $E_s^{(nc)}$ with (65) and $K_n^{(nc)}$ with (66) and (67), we get

$$\Psi = \left( J_1^{(nc)} + J_2^{(nc)} \right) \hat{U}_s^{(nc)} + j(J_1^{(nc)} - J_2^{(nc)}) \hat{U}_s^{(nc)} = \psi^{-1} \psi = J Q_{(fba)} \left( I_d + \Lambda \right) \left( I_d - \Lambda \right) Q_{(fba)}^{-1} = Q_{(fba)} \Omega Q_{(fba)}^{-1},$$

where $\Omega = \operatorname{diag} \left( \frac{1}{\omega_i - \Lambda} \right)$ and we have used Lemma 1 in the first step.

Next, we consider the term $j(J_1^{(nc)} - J_2^{(nc)}) \omega_i(J_1^{(nc)} + J_2^{(nc)})$ in (68) and apply the relation $\omega_i = \frac{1}{\omega_i - \Lambda}$ from Lemma 2. We can then rewrite this term as $j(J_1^{(nc)} - J_2^{(nc)} \omega_i^{2})$. Moreover, the term $\Delta \hat{U}_s^{(nc)}$ in (68) can be expressed in terms of $\Lambda_i$ via Lemma 3. We obtain

$$\Delta \hat{U}_s^{(nc)} = \frac{\Lambda_i}{\omega_i \Lambda_i}.$$ Inserting these relations into (68) and replacing $(J_1^{(nc)} + J_2^{(nc)}) \hat{U}_s^{(nc)}$ via (69), yields

$$\Delta \mu_i = j p_i^r_{(fba)} \psi^{-1} \left( J_1^{(nc)} \hat{U}_s^{(nc)} + J_2^{(nc)} \right) \hat{U}_s^{(nc)} + \frac{2}{\omega_i^2 + 1},$$

where we used $p_i^r_{(fba)} \psi^{-1} = p_i^r_{(fba)} \left( \Lambda_i + \Lambda_i \right)^{-1}$ from Lemma 1 in the first equation.

As a final step, we notice that (72) must be real-valued as we have started from the purely real-valued expansion (63) and only used equivalence transforms to arrive at (72). However, if $-j \in \mathbb{R}$ for $z \in \mathbb{C}$ this implies that $\Re \{ z \} = 0$ and hence $-j = \Im \{ z \}$. Consequently, (72) can also be written as (68) and is therefore equivalent to the first-order expansion for R-D NC Standard ESPRIT with FBA. This concludes the proof of the theorem.

**Appendix D**

**Proof of Theorem 6**

We first state the expression for the $C^{(nc)}$ derived in (34), which is given by

$$C^{(nc)} = \frac{\sigma^2}{2N} \left( (G_2 - G_1 G_1^T) \odot \hat{R}_S \right) + \left( (G_1 G_1^T H_0) \odot \hat{R}_S \right) \left( (G_0 - H_0^T G_1^T H_0) \odot \hat{R}_S \right)^{-1} \cdot \left( (H_1^T - H_0^T G_0^T H_1) \odot \hat{R}_S \right) + \left( H_1 \odot \hat{R}_S \right) \left( (G_0 - H_0^T G_1^T H_0) \odot \hat{R}_S \right)^{-1} \cdot \left( H_1 \odot \hat{R}_S \right) \left( (G_0 - H_0^T G_1^T H_0) \odot \hat{R}_S \right)^{-1} \cdot \left( H_1 \odot \hat{R}_S \right) \left( (G_0 - H_0^T G_1^T H_0) \odot \hat{R}_S \right)^{-1} \cdot \left( H_1 \odot \hat{R}_S \right),$$

where $\hat{R}_S = \hat{S}_S^T / N$ and the matrices $G_n, H_n, n = 0, 1, 2,$ are defined as

$$G_0 = \Re \left\{ \Psi^H A^H \Psi \right\}, \quad H_0 = \Im \left\{ \Psi^H A^H \Psi \right\},$$

$$G_1 = \Re \left\{ \Psi^H D^H A \Psi \right\}, \quad H_1 = \Im \left\{ \Psi^H D^H A \Psi \right\},$$

$$G_2 = \Re \left\{ \Psi^H D^H D \Psi \right\}, \quad H_2 = \Im \left\{ \Psi^H D^H D \Psi \right\},$$

where $D = \begin{bmatrix} \frac{\partial a_{(m1)}}{\partial a_{(m2)}} & \cdots & \frac{\partial a_{(mN)}}{\partial a_{(m2)}} \end{bmatrix}$. In the case $d = 1$, the array matrices $A$ and $D$ become vectors $a, d \in \mathbb{C}^{M \times 1}, \Psi = \psi^{\psi}$, and $\hat{R}_S = s_{0} s_{0}^T / N = P$. For a ULA, we have $a = [1, e^{j \mu}, \ldots, e^{j(M-1)\mu}]^T$ and $d = \frac{\alpha}{\beta} = j [0, e^{j \mu}, \ldots, (M - 1) e^{j(M-1)\mu}]^T$. Consequently, the terms $a^H a, d^H d$, and $d^H a$ in (44) become $a^H a = M, d^H d = (M - 1)(2M - 1)$, and $d^H a = -j M(M - 1)$. 

Thus, the terms (74)–(76) simplify to
\[
    G_0 = M, \quad G_2 = \frac{M(M-1)(2M-1)}{6},
\]
\[
    H_1 = -\frac{M(M-1)}{2}, \quad G_1 = H_0 = H_2 = 0.
\]

After inserting (77) and (78) into (73) and straightforward calculations, we arrive at
\[
    C^{(nc)} = \frac{\sigma_n^2}{NP_n} \cdot \frac{6}{M(M^2-1)},
\]
which is the desired result.

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