A Stochastic Derivative-Free Optimization Method with Importance Sampling: Theory and Learning to Control

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Abstract
We consider the problem of unconstrained minimization of a smooth objective function in $\mathbb{R}^n$ in a setting where only function evaluations are possible. While importance sampling is one of the most popular techniques used by machine learning practitioners to accelerate the convergence of their models when applicable, there is not much existing theory for this acceleration in the derivative-free setting. In this paper, we propose the first derivative-free optimization method with importance sampling and derive new improved complexity results on non-convex, convex and strongly convex functions. We conduct extensive experiments on various synthetic and real LIBSVM datasets confirming our theoretical results. We test our method on a collection of continuous control tasks on MuJoCo environments with varying difficulty. Experiments show that our algorithm is practical for high dimensional continuous control problems where importance sampling results in a significant sample complexity improvement.

Introduction
In this paper, we consider the optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f: \mathbb{R}^n \to \mathbb{R}$ is a “smooth” but not necessarily convex function, bounded from below and it achieves its global minimum at some $x_\star \in \mathbb{R}^n$. In particular, we enforce throughout the paper the following smoothness assumption:

Assumption 1. The objective function $f$ has coordinate-wise Lipschitz gradient, with Lipschitz constants $L_1, \ldots, L_n > 0$. Moreover, $f$ is bounded from below by $f(x_\star) \in \mathbb{R}$. That is, $f$ satisfies

$$f(x_\star) \leq f(x + te_i) \leq f(x) + \nabla_i f(x)t + \frac{L_i t^2}{2}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where $\nabla_i f(x)$ is the $i$th partial derivative of $f$ at $x$.

DFO. We consider the Derivative-Free Optimization (DFO) (Conn, Scheinberg, and Vicente 2009; Kolda, Lewis, and Torczon 2003) setting. That is, we assume that the derivatives of $f$ are numerically impractical to obtain, unreliable (e.g., noisy function evaluations (Chen 2015)), or not available at all. In typical DFO applications, evaluations of $f$ are possible through runs/simulations of some black-box software only. Optimization problems of this type appear in many applications, including computational medicine (Marsden, Feinstein, and Taylor 2008), fluid-dynamics (Alaire 2001; Haslinger and Mckinen 2003), localization (Marsden et al. 2004; 2007) and continuous control (Mania, Guy, and Recht 2018; Salimans et al. 2017).

Literature on DFO methods for solving (1) has a long history. Some of the first approaches were based on deterministic direct search (DDS) (Hooke and Jeeves 1961). Subsequently, additional variants of DDS, including randomized approaches, were proposed in (Matyas 1965; Karmannov 1974a; 1974b; Baba 1981; Dorea 1983; Sarma 1990). However, complexity bounds for deterministic direct search methods have only been established recently by the works of (Vicente 2013; Garminjani and Vicente 2013; Konečný and Richtárik 2014; Dodangeh and Vicente 2016). Recently, complexity bounds have also been derived for randomized methods (Diniz-Ehrhardt, Martinez, and Raydan 2008; Stich, Muller, and Gartner 2013; Gratton et al. 2015). For instance, the work of (Diniz-Ehrhardt, Martinez, and Raydan 2008; Gratton et al. 2015) imposes a decrease condition on whether to accept or reject a step of a set of random directions. Moreover, (Nesterov and Spokoiny 2017; Dvurechensky, Gasnikov, and Gorbunov 2018) derived new complexity bounds for accelerated random search.

More recently, Bergou et. al. proposed a new randomized direct search method called Stochastic Three Points (STP) method. STP, in each iteration $k$, generates a random search direction $s_k$ according to a certain probability law, then compares the objective function at three points: current iterate $x_k$, a point $x_+ = x_k + \alpha_k s_k$ in the direction of $s_k$ and a point $x_- = x_k - \alpha_k s_k$ in the direction of $-s_k$. The method then chooses the best of these three points as the new iterate:

$$x_{k+1} = \arg\min \{ f(x_k), f(x_+), f(x_-) \}.$$ 

Notation: As for the notations, $\mathbb{E} [\cdot]$ denotes the expectation operator. The standard inner product is defined as $\langle x, y \rangle = x^\top y$. We also denote the $\ell_1$-norm and $\ell_2$-norm by...
Algorithm 1 Stochastic Three Points Method with Importance Sampling (STP$_{1S}$)

**Initialization**

Choose initial iterate $x_0 \in \mathbb{R}^n$, stepsize parameters $v_1, \ldots, v_n > 0$ and probabilities $p_1, \ldots, p_n > 0$ summing up to 1.

For $k = 0, 1, 2, \ldots$

1. Select $i_k = i$ with probability $p_i > 0$.
2. Choose stepsize $\alpha_{ik}$ proportional to $1/v_{i_k}$.
3. Let $x_+ = x_k + \alpha_{ik} e_{i_k}$ and $x_- = x_k - \alpha_{ik} e_{i_k}$.
4. $x_{k+1} = \arg \min \{ f(x_k), f(x_+), f(x_-) \}$

$\| \cdot \|_1$ and $\| \cdot \|_2$, respectively. We define $L = \max_i L_i$ for a given sequence of scalars $L_1, \ldots, L_n$.

**Paper Overview and Contributions**

While importance sampling, a term that typically refers to the nonuniform sampling of random directions in stochastic algorithms, has been widely investigated in gradient based methods (Zhao and Zhang 2015; Qu, Richtárik, and Zhang 2015; Richtárik and Takáč 2016; Stich, Raj, and Jaggi 2017), to the best of our knowledge there exist no work on importance sampling in the random direct search setting. To this end, we study STP and analyze its complexity with arbitrary probabilities. In particular, we restrict the random directions to be sampled from discrete distributions, i.e., in each iteration of STP a random direction $s_k$ from a finite set of independent directions $\{b_1, \ldots, b_n\} \subset \mathbb{R}^n$ is sampled. That is, we set $s_k = b_i$ with probability $p_i > 0$. We then propose new sampling strategies that are either optimal or at least improve the complexity bounds, i.e., importance sampling.

**Coordinate directions**

Without loss of generality, we only consider directions in the canonical basis of $\mathbb{R}^n$, i.e., $e_1, \ldots, e_n$. The general case can be recovered via a linear change of variables: $x = By$, where $B \in \mathbb{R}^{n \times n}$. Indeed, consider the problem

$$\min_{y \in \mathbb{R}^n} f_B(y) \overset{\text{def}}{=} f(By)$$

instead. A coordinate update $y_{k+1} = y_k + \alpha_{ik} e_i$ for the reparameterized problem (2) corresponds to updates of the form $x_{k+1} = x_k + \alpha_{ik} b_i$, where $b_i$ is the $i$th column of $B$, for the original problem (1). In light of the above discussion, the newly proposed algorithm dubbed STP$_{1S}$ is formally described as Algorithm 1.

**Complexity bounds**

To the best of our knowledge, ours are the first complexity bounds (bounds on the number of iterations) for a DFO method with importance sampling. We design importance sampling that improves the worst-case iteration complexity bounds compared to state-of-the-art algorithms. These bounds have the same dependence on the precision $\epsilon$ as classical bounds in the literature, i.e. $1/\epsilon^2$ for non-convex $f$, $1/\epsilon$ for convex $f$ and $\log(1/\epsilon)$ for strongly convex $f$; see for instance (Bergou, Gorbunov, and Richtárik 2019; Nesterov and Spokoiny 2017). However, the leading constant, which is often the bottleneck in practical performance, especially when low or medium accuracy solutions are acceptable, is improved and often dramatically so. Typically, the improvement is via replacing the maximum Lipschitz constant of the gradient by the average Lipschitz constants of all coordinates (see Theorems 1, 2, 3, and 4). The improvement we obtain is similar to the improvement obtained by importance sampling in stochastic coordinate (gradient) descent methods (Zhao and Zhang 2015; Qu, Richtárik, and Zhang 2015; Richtárik and Takáč 2016).

Table 1 summarizes complexity results obtained in this paper for STP and for STP$_{1S}$. The assumptions in Table 1 are in addition to Assumption 1.

**Empirical results**

In addition to our theoretical analysis, we conduct extensive testing to show the efficiency of the proposed method in practice. We use both synthetic and real datasets for ridge regression and squared SVM problems. In the non-convex case, we use continuous control tasks from the MuJoCo (Todorov, Erez, and Tassa 2012) suite following the recent success of DFO compared to model-free RL (Mania, Guy, and Recht 2018; Salimans et al. 2017). Results show that our approach leads to huge speedups compared against uniform sampling, the improvement can reach several orders of magnitude and comparable or better than state-of-art policy gradient methods.

**Non-Convex Case**

This section describes our complexity results for Algorithm 1 in the case when $f$ is allowed to be non-convex. We show that this method guarantees complexity bounds with the same order in $\epsilon$ as classical bounds in the literature, i.e., $1/\epsilon^2$ with an improved dependence on the Lipschitz constant. All proofs are left for the appendix.

**Theorem 1.** Let Assumption 1 be satisfied. Choose $\alpha_{ik} = \frac{\alpha_0}{v_{i_k} \sqrt{k+1}}$ where $\alpha_0 > 0$. If

$$K \geq \frac{2 \left( \sqrt{2(f(x_0) - f(x_+) + \alpha_0 \sum_{i=1}^n \frac{L_i}{\sqrt{i}}} \right)^2}{\left( \min_i \frac{p_i}{v_i} \right)^2 \epsilon^2}$$

then

$$\min_{k=0,1,\ldots,K} \mathbb{E} [\| \nabla f(x_k) \|_1] \leq \epsilon.$$

Note that the complexity depends on $\alpha_0$. The optimal choice of $\alpha_0$ minimizing (3) is $\alpha_0^* = 8^{1/4} \sqrt{\frac{f(x_0) - f(x_+)}{\sum_{i=1}^n \frac{L_i}{\sqrt{i}}}}$ in which case the complexity bound (3) takes the form

$$4 \sqrt{2} (f(x_0) - f(x_+)) \sum_{i=1}^n \frac{p_i L_i}{\sqrt{i}} \left( \min_i \frac{p_i}{v_i} \right)^2 \epsilon^2.$$

\footnote{We use several LIBSVM datasets (Chang and Lin 2011).}
**Theorem**

\[ p_i = \frac{\sqrt{L_i}}{\sum_{i=1}^{n} \sqrt{L_i}} \]

**Importance Sampling**

The complexity depends on the choice of the probabilities \( \{p_i\}_{i=1}^{n} \) and the quantities \( \{v_i\}_{i=1}^{n} \). For instance, if \( p_i = \frac{L_i}{\sum_{i=1}^{n} L_i} \) and \( v_i = \sqrt{L_i} \), then the complexity becomes

\[ 4\sqrt{2}(f(x_0) - f(x_*) + \sum_{i=1}^{n} \sqrt{L_i})^2 \]

On the other hand, if \( p_i = \frac{1}{n} \) and \( v_i = L_i \) then the complexity becomes

\[ 4\sqrt{2}(f(x_0) - f(x_*))n \sum_{i=1}^{n} L_i \]

Under the choice of uniform sampling, i.e. \( p_i = \frac{1}{n} \) and the choice \( v_i = L_i \), we recover the uniform sampling complexity of (Bergou, Gorbunov, and Richtárik 2019)

\[ 4\sqrt{2}(f(x_0) - f(x_*))n^2L \]

Table 1: Summary of the new derived complexity results as opposed to uniform sampling where \( r_0 = f(x_0) - f(x_*) \). The assumptions listed are in addition to Assumption 1. \( R_0 < \infty \) indicates a bounded level set where the exact definition is given in Assumption 2. The key differences in complexity between the uniform and importance sampling are detailed in text.

**Convex Case**

This section describes our complexity results for Algorithm 1 when the objective function \( f \) is convex. We show that this method guarantees complexity bounds with the same order in \( \epsilon \) as classical bounds in the literature, i.e., \( 1/\epsilon \) with an improved dependence on the Lipschitz constant. We will need the following additional assumption in the sequel.

**Assumption 2.** The function \( f \) is convex and has a bounded level set at \( x_0 \). That is, \( f \) satisfies:

\[ R_0 \overset{\text{def}}{=} \max_x \{ \|x - x_*\|_\infty : f(x) \leq f(x_0) \} < \infty \]

Note that if \( f \) is convex and has bounded level sets, the following holds:

\[ f(x) - f(x_*) \leq \langle \nabla f(x), x - x_* \rangle \leq \|\nabla f(x)\|_1 \|x - x_*\|_\infty \leq R_0 \|\nabla f(x)\|_1. \]

**Theorem 3.** Let Assumptions 1 and 2 be satisfied. Choose \( \alpha_{ik} = \frac{|f(x_{k+1} + tv_i) - f(x_k)|}{t v_i \|v_i\|} \) and sufficiently small \( t \) (see the appendix for the bound on \( t \)). If

\[ k \geq \frac{8 R_0^3 n}{\min_i \frac{p_i}{v_i} \left( 1 - \frac{\sum_{i=1}^{n} \frac{p_i}{v_i}}{2n} \right)^2 \epsilon^2} \]

then \( \min_{k=0,1,\ldots,K} \mathbb{E} \|\nabla f(x_k)\|_1 \leq \epsilon \).

Under the choice of importance sampling \( p_i = \frac{L_i}{\sum_{i=1}^{n} L_i} \) and \( v_i = L_i \), the complexity (6) becomes

\[ 4(f(x_0) - f(x_*))n \sum_{i=1}^{n} L_i \]

Similar to Theorem 1, the uniform sampling complexity of (Bergou, Gorbunov, and Richtárik 2019) can be recovered with \( p_i = \frac{1}{n} \) and \( v_i = L_i \). Note that the uniform sampling complexity is proportional to \( n^2L \) and since \( n \sum_{i=1}^{n} L_i \leq n^2L \), the worst case complexity of the number of iterations is also improved for this choice of importance sampling.
This section describes our complexity results for Algorithm 1 when objective function \( f \) is \( \lambda \)-strongly convex. We show that this method guarantees complexity bounds with the same order in \( \epsilon \) as classical bounds in the literature, i.e., \( \log(1/\epsilon) \) with an improved dependence on the Lipschitz constant. First, we define \( \lambda \)-strongly convexity functions.

**Assumption 3.** The function \( f \) is \( \lambda \)-strongly convex. That is, for some \( \lambda > 0 \), the following holds

\[
f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\lambda}{2} \| x - y \|^2.
\]

**Theorem 4.** Let Assumptions 1 and 3 be satisfied. Choose \( \alpha_{ik} = \frac{|f(x_k + \epsilon_i t_k) - f(x_k)|}{\epsilon_i t_k} \) and a sufficiently small \( t \) (see the appendix for the bound on \( t \)). If

\[
k \geq \frac{\max_i \frac{\| \nabla f(x_0) - \nabla f(x_i) \|}{\lambda}}{\log \left( \frac{2(f(x_0) - f(x_i))}{\epsilon} \right)},
\]

then \( \mathbb{E} [f(x_k) - f(x_\ast)] \leq \epsilon. \)

The complexity bound in (10) is minimized, in \( p_i \) for \( v_i = L_i \), with \( p_i = \frac{L_i}{\sum_{i=1}^{n} L_i} \). Importance sampling improves over uniform sampling, since \( \sum_{i=1}^{n} L_i \leq nL. \)

**Strongly Convex Case**

**Experiments**

We conduct extensive experiments on synthetic and real datasets comparing the uniform sampling \( \text{STP} \) against the importance sampling version \( \text{STP}_{IS} \). The experiments are conducted on several choices of the function \( f \). In particular, we perform experiments on regularized ridge regression on synthetic data and squared SVM loss on real data from the LIBSVM dataset (Chang and Lin 2011). Moreover, for non-convex problems, we compare \( \text{STP} \) and \( \text{STP}_{IS} \) on various continuous control environments on MuJoCo (Todorov, Erez, and Tassa 2012). We also compare against state-of-art solvers for the continuous control task.

**Ridge regression on synthetic data**

We compare \( \text{STP} \) against \( \text{STP}_{IS} \) on synthetic data on the regularized ridge regression problem: \( f(x) = \frac{1}{2m} \| Ax - y \|^2 + \frac{\lambda}{2} \| x \|^2 \), where \( A \in \mathbb{R}^{m \times n} \), \( y \in \mathbb{R}^m \) are the data and \( \lambda > 0 \) is the regularization parameter. The elements of \( A \) and \( y \) were sampled from the standard Gaussian distribution \( \mathcal{N}(0, 1) \). Note that for ridge regression, \( L_i = \frac{1}{m} \| A(:, i) \|^2 + \lambda \) where, following (Gower, Richtárik, and Bach 2018), we normalize data such that \( \| A(:, 1) \|_2 = 1 \) and \( \| A(:, i) \|_2 = \frac{1}{m}, i = 2, \ldots, m \) and set \( \lambda = \frac{1}{m^2} \). We compute a high accuracy solution \( x_\ast \) by solving the ridge regression problem exactly with a linear solver. Thereafter, the metric
used is the difference between the current objective value and the optimal one, i.e. \( f(x) - f(x_*) \). Since the objective is \( \lambda \)-strongly convex, we use the stepsize suggested by Theorem 4. In all experiments, we set \( t = 10^{-4} \). We perform experiments across difference choices of \( m \) and \( n \). In the first row of Figure 1, we compare both methods with a fixed \( n = 10 \) and a varying \( m \), i.e. \( m \in \{10^3, 10^4, 10^5\} \). The superior performance of \( \text{STP}_{\text{IS}} \) over \( \text{STP} \) is evident from Figure 1. Moreover, we conduct further experiments where \( m \) is fixed such that \( m = 100 \) but with a varying dimension, i.e. \( n \in \{10^1, 10^2, 10^3\} \). All experiments are conducted 10 times and we report the average, worst and best performances. A similar behaviour is also present as seen in the second row of Figure 1 where \( \text{STP}_{\text{IS}} \) is far more superior to \( \text{STP} \). In all experiments, the stopping criterion is set such that both \( \text{STP} \) and \( \text{STP}_{\text{IS}} \) run for exactly \( 5 \times 10^2 \) iterations for small problems, i.e. \( n = 10 \), while for problems of size \( n = 10^2 \) and \( n = 10^3 \), both methods are terminated at \( 5 \times 10^3 \) and \( 15 \times 10^3 \) iterations, respectively.

### Ridge regression and squared SVM on real data

We also conduct experiments on the regularized ridge regression problem on real datasets where \( A \) and \( y \) are from LIBSVM data. We follow the same protocol as the experiments on synthetic data. We compare both algorithms on 6 different datasets, namely, \text{australian}, \text{mushrooms}, \text{a9a}, \text{heart}, \text{cov1} and \text{ijcnn1}. In addition to ridge regression, we conduct experiments on the same real datasets on the regularized squared SVM loss: \[
f(x) = \frac{1}{2} \sum_{i=1}^{m} \max \left( 0, 1 - y_i a_i^\top x \right)^2 + \frac{1}{2} \|x\|^2 + \lambda \|A(:,i)\|^2_2,\]
where \( a_i \) is the \( i \)th row of \( A \). Note that \( L_i = \|A(:,i)\|^2_2 + \lambda \). Since the squared SVM problem does not exhibit a closed form solution, we compare both \( \text{STP} \) and \( \text{STP}_{\text{IS}} \) in terms of the objective value \( f(x) \). In Figure 2, we show the comparison between both \( \text{STP} \) and \( \text{STP}_{\text{IS}} \) on the ridge regression on all 6 datasets. It is clear that using the proposed importance sampling is far more superior to standard uniform sampling. The improvement is also consistently present on the squared SVM problem as seen in Figure 3.

### Continuous control experiments

Here, we address the problem of model-free control of a dynamical system. Model-free reinforcement learning algorithms (especially policy gradient methods), provide an off-the-shelf model-free approach to learn how to control a dynamical system. Such models have been typically benchmarked in a simulator. Thus, we adopt the MuJoCo (Todorov, Erez, and Tassa 2012) continuous control suite following its wide adaptation. We choose 5 problems with various difficulty \text{Swimmer-v1}, \text{Hopper-v1}, \text{HalfCheetah-v1}, \text{Ant-v1}, and \text{Humanoid-v1}. In all experiments, we use linear policies similar to (Mania, Guy,
and Recht 2018; Rajeswaran et al. 2017).

Considering the stochastic nature of the dynamical systems, i.e. \( f \) is stochastic, we take multiple \( (K) \) measurements for \( f(x_k), f(x_+), \) and \( f(x_-) \) and use their mean as the function values. Considering the varying dimensionality of the state space, we use different \( K \) for each problem, in particular, we set \( K = 2 \) for Swimmer-v1, \( K = 4 \) for Hopper-v1 and HalfCheetah-v1, \( K = 40 \) for Ant-v1 and \( K = 120 \) for Humanoid-v1. These values are decided using grid search over the set of \( K \in \{1, 2, 4, 8, 16\} \) for low dimensional problems and \( K \in \{20, 40, 120, 240\} \) for high dimensional Ant-v1 and Humanoid-v1 problems. Following our remark given by Equation 2, we use a square matrix \( B \) sampled from a standard Gaussian distribution \( N(0, 1) \). In our experiments, this coordinate transform resulted in a better performance. Since the Lipschitz constants are not available for continuous control, we learn an estimate of the function using a parametric family (specifically multi-layer perceptron) and use its Lipschitz constants as the estimates to decide importance sampling weights. This is similar to actor-critic methods (Sutton and Barto 1988) used in the policy gradient literature. Similar to us, actor-critic methods learn an estimate of the value function and use it to decide which point to evaluate. We defer the details of estimation procedure to the appendix. Following the common practice, we perform all experiments with 5 random initialization and measure the mean average reward at each iteration. We give detailed comparison of STP and our proposed importance sampling variant \( \text{STP}_{\text{IS}} \) in terms of reward vs. sample complexity in Figure 4 for both adaptive and fixed step size cases (see Theorems 1 and 2). Shaded regions in figures show standard deviations.

As seen from Figure 4, our proposed importance sampling version \( \text{STP}_{\text{IS}} \) significantly improves sample complexity when compared to STP. Moreover, the difference is significant for high dimensional problems like HalfCheetah, Ant and Humanoid. The results suggest that STP fails to scale to very high dimensional problems like Humanoid. Our method tackles this and improves the sample complexity of STP. Such results also suggest that it is feasible to estimate the coordinate-wise Lipschitz gradient constants, detailed in Assumption 1, of a complicated non-convex function using a data-driven approach. An interesting conclusion from Figure 4 is that adaptive step size performs better than fixed step size even after a large hyper-parameter search, particularly, for higher dimensional problems.

In order to compare our method with the existing state-of-the-art DFO and policy gradient methods, we also tabulate the sample complexity of our method and several existing baselines. Similar to (Mania, Guy, and Recht 2018), we compute the average number of episodes needed to reach a predefined threshold. Although there are many DFO and policy gradient methods in literature, we report ARS (Mania, Guy, and Recht 2018) as a representative DFO method since it outperforms other baselines. As for policy gradient approaches, we report TRPO (Schulman et al. 2015) as a representative policy gradient method since it is widely used in the community. Moreover, we use NG (Rajeswaran et al. 2017) as a

Figure 3: Shows the superiority of \( \text{STP}_{\text{IS}} \) over STP on real LIBSVM dataset on the squared SVM loss. The datasets used in the experiments are \texttt{australian}, \texttt{mushrooms} and \texttt{a9a} in the first row and \texttt{heart}, \texttt{cov1} and \texttt{ijcnn1} for the second row.
Table 2: For each MuJoCo task, we report the average number of episodes required to achieve a predefined reward threshold. Results for our method is averaged over five random seeds, the rest is copied from (Mania, Guy, and Recht 2018) (N/A means the method failed to reach the threshold. UNK means the results is unknown since they are not reported in the literature.)

| Task          | Threshold | Fixed Step Size | Adaptive Step Size | ARS(V1-t) | ARS(V2-t) | NG-lin | TRPO-nn |
|---------------|-----------|-----------------|--------------------|-----------|-----------|--------|---------|
| Swimmer-v1    | 325       | 320             | 110                | 200       | 90        | 100    | 427     | 1450    | N/A     |
| Hopper-v1     | 3120      | 3970            | 2400               | 3720      | 1870      | 51840  | 1973    | 13920   | 10000   |
| HalfCheetah-v1| 3430      | 13760           | 4420               | 5040      | 2710      | 8106   | 1707    | 11250   | 4250    |
| Ant-v1        | 3580      | 107220          | 43860              | 96980     | 26480     | 58133  | 20800   | 39240   | 73500   |
| Humanoid-v1   | 6000      | N/A             | 530200             | N/A       | 296800    | N/A    | 142600  | 130000  | UNK     |

As the results suggest, STP is competitive with existing solutions for low dimensional problems (Swimmer, Hopper and HalfCheetah) whereas it underperforms existing solutions for Ant and fails to solve the Humanoid problem. Our theoretically proposed importance sampling version STP$_{IS}$ significantly improves STP and results in a performance either competitive with or better than existing baselines in all problems except for Humanoid. Although our method successfully solves the Humanoid problem, it has worse sample complexity than other solutions. Hence, scaling STP to very large dimensional continuous control problems (e.g. Humanoid-v1 state space has more than 1000 dimensions) is still an open problem. Moreover, for lower dimensional problems like (Swimmer and Hopper), our method outperforms all existing methods.

An interesting question is whether we can use first order methods utilizing the estimated function. In order to estimate the Lipschitz smoothness constant of the function, we utilize a parametric family and one can argue that its gradients can be used as a surrogate gradient in first order optimization. We compare our method with a simple first order baseline using the surrogate gradient in SGD; it fails in all environments. Hence, we do not show the results in the paper.

Conclusion

We propose and analyze a DFO algorithm with importance sampling STP$_{IS}$ enjoying the best known complexity bounds known in DFO literature. Experiments on ridge regression and squared SVM objectives for both synthetic and LIBSVM datasets demonstrate the superiority of STP$_{IS}$ over its uniform version. We also conduct experiments on a collection of continuous control tasks on several MuJoCo environments. We are orders of magnitudes better than the uniform sampling and comparable or better than the state-of-art methods.

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Preliminaries

We establish the key lemma which will be used to prove the theorems stated in the paper.

**Lemma 1.** If $f$ satisfies Assumption 1 and following the STP update, the following holds:

$$
\mathbb{E}[f(x_{k+1}) | x_k] \leq f(x_k) - \mathbb{E}[\alpha_i \langle \nabla f(x_k), e_i \rangle | x_k] + \frac{1}{2} \mathbb{E}[\alpha_i^2 L_{ik} | x_k].
$$

**Proof.** Since

$$f(x_{k+1}) \leq \min \{f(x_k + \alpha_i e_i), f(x_k - \alpha_i e_i)\} \leq f(x_k) - |\alpha_i \langle \nabla f(x_k), e_i \rangle| + \frac{\alpha_i^2 L_{ik}}{2}.
$$

Then the result follows by taking conditional expectation on $x_k$. \(\square\)

**Non convex case**

**Theorem 1.** Let Assumption 1 hold. Choose $\alpha_i = \frac{\alpha_0}{\sqrt{v_i} \sqrt{k+1}}$, where $\alpha_0 > 0$. If

$$K \geq \frac{2 \left( \frac{\sqrt{2}}{\alpha_0} \frac{f(x_0) - f_*}{\varepsilon} + \frac{\alpha_0 \sum_{i=1}^{n} \frac{p_i L_i}{v_i}}{2} \right)^2}{\left( \min_i \frac{p_i}{v_i} \right)^2 \varepsilon^2},$$

then

$$\min_{k=0,1,\ldots,K} \mathbb{E}[\|\nabla f(x_k)\|_1] \leq \varepsilon.$$

**Proof.** We have from Lemma 1

$$\mathbb{E}[f(x_{k+1}) | x_k] \leq f(x_k) - \mathbb{E}[\alpha_i \langle \nabla f(x_k), e_i \rangle | x_k] + \frac{1}{2} \mathbb{E}[\alpha_i^2 L_{ik} | x_k], \quad (12)$$

From ① we have

$$\mathbb{E}[\alpha_i \langle \nabla f(x_k), e_i \rangle | x_k] = \frac{\alpha_0}{\sqrt{k+1}} \sum_{i=1}^{n} \frac{p_i}{v_i} |\langle \nabla f(x_k), e_i \rangle| \geq \frac{\alpha_0}{\sqrt{k+1}} \min_i \frac{p_i}{v_i} \|\nabla f(x_k)\|_1.$$

From ② we have

$$\frac{1}{2} \mathbb{E}[\alpha_i^2 L_{ik} | x_k] = \frac{1}{2} \frac{\alpha_0^2}{k+1} \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2}.$$

By injecting ① and ② in (12) and taking the expectation we get

$$\theta_{k+1} \leq \theta_k - \frac{\alpha_0 \min_i \frac{p_i}{v_i}}{\sqrt{k+1}} g_k + \frac{\alpha_0^2 \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2}}{2(k+1)},$$

where $\theta_k = \mathbb{E}[f(x_k)]$ and $g_k = \mathbb{E}[\|\nabla f(x_k)\|_1]$. By re-arranging the terms we get:

$$g_k \leq \frac{1}{\min_i \frac{p_i}{v_i}} \left( \frac{\theta_k - \theta_{k+1}}{\sqrt{k+1}} \frac{\alpha_0}{\alpha_0} + \frac{\alpha_0 \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2}}{2 \sqrt{k+1}} \right).$$

We have that the sequence $\{f(x_k)\}_{k \geq 0}$ is monotonically decreasing and $f$ is bounded from below by $f(x_0)$, hence $f(x_0) \leq \theta_{k+1} \leq \theta_k \leq f(x_0)$ for all $k$. Letting $l = \lfloor K/2 \rfloor$, this implies that

$$\sum_{j=l}^{2l} \theta_j - \theta_{j+1} = \theta_l - \theta_{2l+1} \leq f(x_0) - f(x_0) \overset{\text{def}}{=} C,$$
from which we conclude that there must exist \( j \in \{l, \ldots, 2l\} \) such that \( \theta_j - \theta_{j+1} \leq C/(l+1) \). This implies that

\[
g_j \leq \frac{1}{\min_i \frac{p_i}{v_i}} \left( \frac{(\theta_j - \theta_{j+1})\sqrt{j+1}}{\alpha_0} + \frac{\alpha_0 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2\sqrt{j+1}} \right)
\]

\[
\leq \frac{1}{\min_i \frac{p_i}{v_i}} \left( \frac{C\sqrt{j+1}}{\alpha_0(l+1)} + \frac{\alpha_0 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2\sqrt{j+1}} \right)
\]

\[
\leq \frac{1}{\min_i \frac{p_i}{v_i}} \left( \frac{C\sqrt{2l+1}}{\alpha_0(l+1)} + \frac{\alpha_0 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2} \right)
\]

\[
\leq \frac{1}{\min_i \frac{p_i}{v_i}} \sqrt{\frac{2C}{\alpha_0}} + \frac{\alpha_0 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2}
\]

\[
\leq \frac{1}{\min_i \frac{p_i}{v_i}} \sqrt{\frac{K}{2}} + \frac{\alpha_0 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2}
\]

\[
\leq \frac{1}{\min_i \frac{p_i}{v_i}} \sqrt{\frac{K}{2}} + \frac{\alpha_0 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2}.
\]

\[
\leq \epsilon.
\]

\[\square\]

**Theorem 2.** Let Assumption 1 be satisfied. Choose \( \alpha_{ik} = \frac{\epsilon}{n v_i} \) where \( \sum_{i=1}^n \frac{p_i L_i}{v_i^2} < 2n \left( \min_i \frac{p_i}{v_i} \right) \). If

\[
K \geq \frac{2n (f(x_0) - f(x_*))}{\min_i \frac{p_i}{v_i} \left( 1 - \frac{\sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2n \min_i \frac{p_i}{v_i}} \right) \epsilon^2},
\]

then \( \min_{k=0, 1, \ldots, K} \mathbb{E} \|\nabla f(x_k)\|_1 \leq \epsilon. \)

**Proof.** From equation (12) we have

\[
\mathbb{E} [f(x_{k+1}) | x_k] \leq f(x_k) - \mathbb{E} [\alpha_{ik} \langle \nabla f(x_k), e_{ik} \rangle | x_k] + \frac{1}{2} \mathbb{E} [\alpha_{ik}^2 L_{ik}].
\]

From \( \odot \) we have

\[
\mathbb{E} [\alpha_{ik} \langle \nabla f(x_k), e_{ik} \rangle | x_k] = \frac{\epsilon}{n} \sum_{i=1}^n \frac{p_i}{v_i} |\langle \nabla f(x_k), e_{ik} \rangle |
\]

\[
\geq \frac{\epsilon \min_i \frac{p_i}{v_i}}{n} \|\nabla f(x_k)\|_1.
\]

From \( \oplus \) we have

\[
\frac{1}{2} \mathbb{E} [\alpha_{ik}^2 L_{ik}] = \frac{1}{2} \frac{\epsilon^2}{n^2} \sum_{i=1}^n \frac{p_i L_i}{v_i^2}
\]

By injecting \( \odot \) and \( \oplus \) in (16) and taking the expectation we get

\[
\theta_{k+1} \leq \theta_k - \frac{\epsilon}{n} \min_i \frac{p_i}{v_i} g_k + \frac{\epsilon^2}{2n^2} \sum_{i=1}^n \frac{p_i L_i}{v_i^2},
\]

where \( \theta_k = \mathbb{E} [f(x_k)] \) and \( g_k = \mathbb{E} [\|\nabla f(x_k)\|_1]. \) By re-arranging the terms we get:

\[
g_k \leq \left( \frac{(\theta_k - \theta_{k+1})n}{\epsilon \min_i \frac{p_i}{v_i}} + \frac{\epsilon^2 \sum_{i=1}^n \frac{p_i L_i}{v_i^2}}{2n \min_i \frac{p_i}{v_i}} \right).
\]
We have that the sequence \( \{f(x_k)\}_{k \geq 0} \) is monotonically decreasing and \( f(x) \) is bounded from below by \( f(x_\ast) \), hence \( f(x_\ast) \leq \theta_{k+1} \leq \theta_k \leq f(x_0) \) for all \( k \). Letting \( l = \lfloor K/2 \rfloor \), this implies that

\[
\sum_{j=l}^{2l} \theta_j - \theta_{j+1} = \theta_l - \theta_{2l+1} \leq f(x_0) - f(x_\ast) = C,
\]

from which we conclude that there must exist \( j \in \{l, \ldots, 2l\} \) such that \( \theta_j - \theta_{j+1} \leq C/(l+1) \). This implies that

\[
g_k \leq \left( \frac{Cn}{\epsilon \min_i \frac{p_i}{v_i} (l+1)} + \frac{\epsilon \sum_{i=1}^n p_i L_i}{2n \min_i \frac{p_i}{v_i}} \right)
\leq \left( \frac{Cn}{\epsilon \min_i \frac{p_i}{v_i} K/2} + \frac{\epsilon \sum_{i=1}^n p_i L_i}{2n \min_i \frac{p_i}{v_i}} \right)
\leq \epsilon. \tag{15}
\]

☐

**Convex case**

We state a lemma which will be useful latter on in the analysis

**Lemma 2.** Let assumption 1 be satisfied. Let \( \alpha_{i_k} = \frac{\alpha_0 (f(x_k) - f(x_\ast))}{v_{i_k}} \), then the following inequality holds:

\[
f(x_{k+1}) \leq f(x_k) - \frac{1}{t v_{i_k}} |\langle \nabla f(x_k), e_{i_k} \rangle|^2 + \frac{\epsilon}{2v_{i_k}} |\langle \nabla f(x_k), e_{i_k} \rangle| L_{i_k}
+ \frac{1}{2 v_{i_k}} |\langle \nabla f(x_k), e_{i_k} \rangle|^2 L_{i_k} + \frac{t}{2 v_{i_k}} |\langle \nabla f(x_k), e_{i_k} \rangle| L^2_{i_k} + \frac{t^2}{8 v^2_{i_k}} L^3_{i_k} \tag{19}
\]

**Proof.** It follows directly by noting that

\[
|\langle \nabla f(x_k), e_{i_k} \rangle| - \frac{1}{2} L_{i_k} \leq \frac{|f(x_k + \epsilon e_{i_k}) - f(x_k)|}{t v_{i_k}} \leq \frac{|\langle \nabla f(x_k), e_{i_k} \rangle|}{v_{i_k}} + \frac{1}{2} L_{i_k},
\]

which follows in a similar fashion to Theorem [13] in (Bergou, Gorbunov, and Richtárik 2019).

☐

**Theorem 3.** Let Assumptions 1 and 2 be satisfied. Choose \( \alpha_{i_k} = \frac{\alpha_0 (f(x_k) - f(x_\ast))}{v_{i_k}} \), where \( 0 < \alpha_0 < \frac{2 \min_i \frac{p_i}{v_i}}{R_0 \sum_{i=1}^n \frac{p_i}{v_i}} \). If

\[
k \geq \frac{1}{\alpha_0} \min_i \frac{p_i}{v_i} - \frac{\alpha_0^2}{2} \sum_{i=1}^n \frac{p_i L_i}{v_i} \left( \frac{1}{\epsilon} - \frac{1}{r_0} \right), \tag{20}
\]

then \( \mathbb{E}[f(x_k)] - f(x_\ast) \leq \epsilon \).

**Proof.** We have from Lemma 1

\[
\mathbb{E}[f(x_{k+1}) \mid x_k] \leq f(x_k) - \mathbb{E}[\alpha_{i_k} |\langle \nabla f(x_k), e_{i_k} \rangle| \mid x_k] + \frac{1}{2} \mathbb{E} \left[ \alpha_{i_k}^2 L_{i_k} \mid x_k \right]. \tag{21}
\]

From \( \circledast \) we have

\[
\mathbb{E}[\alpha_{i_k} |\langle \nabla f(x_k), e_{i_k} \rangle| \mid x_k] = \alpha_0 (f(x_k) - f(x_\ast)) \sum_{i=1}^n \frac{p_i}{v_i} |\langle \nabla f(x_k), e_i \rangle|
\geq \alpha_0 (f(x_k) - f(x_\ast)) \min_i \frac{p_i}{v_i} \|\nabla f(x_k)\|_1
\geq \frac{\alpha_0}{R_0} (f(x_k) - f(x_\ast))^2 \min_i \frac{p_i}{v_i}
\]
From \( \| \) we have
\[
\frac{1}{2} \mathbb{E} \left[ \alpha_k^2 L_{i_k} | x_k \right] = \frac{\alpha_0^2}{2} \left( f(x_k) - f(x) \right)^2 \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2}.
\]
Let \( r_k = \mathbb{E} [f(x_k)] - f(x_*) \). By substituting \( \| \) and \( \| \) in (21) and then taking the expectation we get
\[
r_{k+1} \leq r_k - \frac{\alpha_k^2}{2} \min_i \frac{p_i}{v_i} + \frac{\alpha_0^2}{2} \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2} = r_k - \alpha_k^2 r_k,
\]
where \( a = \frac{\alpha_0}{R_0} \min_i \frac{p_i}{v_i} - \frac{\alpha_0^2}{2} \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2} \). Therefore,
\[
1 + \frac{a}{r_k} \leq 1 + \frac{r_k - r_{k+1}}{r_k t_{k+1}} \geq 1 + \frac{r_k - r_k}{r_k} \geq a.
\]
Therefore, we have \( \frac{1}{r_k} \geq 1 + \frac{ka}{r_k} \) and hence \( r_k \leq \frac{1}{r_0 + ka} \). Therefore, if \( k \geq \frac{1}{a} \left( \frac{1}{r_0} - 1 \right) \), then \( r_k \leq \frac{1}{r_0 + ka} \leq \epsilon. \)

Note that the stepsizes in the previous theorem depend on the optimal value \( f(x_*) \). In practice, we cannot always use these stepsizes as we usually do not know \( f(x_*) \). Next theorem will suggest stepsizes that are independent from \( f(x_*) \) for which we get an optimized complexity as well.

**Theorem 4.** Let Assumptions 1 and 2 be satisfied. Choose \( \alpha_{ik} = \frac{f(x_{k+1}) - f(x_k)}{r_{ik}} \), then for a small \( t \) that satisfies
\[
t \leq \min \left\{ \frac{e}{2} \min_i \frac{p_i}{v_i}, \frac{e}{2} \left( \frac{\min_i \frac{p_i}{v_i}}{8 R_0 n \max_i \sqrt{\frac{1}{p_i}} \sqrt{2nL R_0}} \right) \right\},
\]
where \( r_0 = f(x_0) - f(x_*) \), and if
\[
k \geq 8 R_0^2 n \min_i \frac{e}{v_i} \left( \frac{1}{r_0} - 1 \right),
\]
then \( \mathbb{E} [f(x_k)] - f(x_*) \leq \epsilon. \)

**Proof.** Taking the expectation of (19) on \( x_k \) with the choice of \( \alpha_{ik} \) we have:
\[
\mathbb{E} [f(x_{k+1}) | x_k] \leq f(x_k) + \mathbb{E} [\langle f(x_k), \epsilon_{ik} \rangle] | x_k] - \frac{1}{2} \mathbb{E} \left[ \left( \sum_{i=1}^{n} \frac{p_i}{v_i} \right) | \nabla f(x_k), \epsilon_{ik} \rangle | x_k] \right)^2 + \frac{t^2}{2} \sum_{i=1}^{n} \frac{p_i L_i^2}{v_i^2}.
\]
As for \( \| \), taking the total expectation we have
\[
\mathbb{E} \left[ \mathbb{E} [\langle \nabla f(x_k), \epsilon_{ik} \rangle] | x_k] \right] = \mathbb{E} [\| \nabla f(x_k), \epsilon_{ik} \rangle] = \mathbb{E} \left[ \left( \sum_{i=1}^{n} \frac{p_i}{v_i} | \nabla f(x_k) | \right)^2 \right]
\]
\[
\leq \max_i \sqrt{p_i} \mathbb{E} [\| \nabla f(x_k) \|_2]
\]
\[
\leq \max_i \sqrt{p_i} \mathbb{E} [\| \nabla f(x_k) \|_2]
\]
\[
\leq \max_i \sqrt{p_i} \mathbb{E} [\| f(x_k) \| - f(x_*)] \leq \max \sqrt{p_i} \mathbb{E} [\| f(x_k) \| - f(x_*)].
\]
where \( r_0 = \mathbb{E} [f(x_0)] - f(x_*) \). Note that the first inequality follows by Jensen’s inequality. The last inequality follows from the fact that \( f \) has Lipschitz gradient, with Lipschitz constant \( n L \) (this is a direct property from Assumption 1).

As for \( \| \), taking total expectation, we have
\[
\mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{v_i} | \nabla f(x_k), \epsilon_{ik} \rangle | x_k] \right] \right] = \mathbb{E} \left[ \frac{1}{v_i} | \nabla f(x_k), \epsilon_{ik} \rangle | x_k] \right]
\]
\[
\geq \mathbb{E} \left[ \frac{1}{v_i} | \nabla f(x_k) \|_2^2 | x_k] \right]
\]
\[
\geq \mathbb{E} \left[ \frac{1}{v_i} \min_i \frac{p_i}{v_i} | \nabla f(x_k) \|_2 | x_k] \right]
\]
\[
\geq \frac{1}{n} \min_i \frac{p_i}{v_i} L_0^2 \mathbb{E} \left[ (f(x_k) - f(x_*))^2 \right] = \frac{r_k^2}{R_0^2 n} \min_i \frac{p_i}{v_i}. \]

(8)
where \( r_k = \mathbb{E} [f(x_k) - f(x_*)] \) Note that the second inequality follows since for \( h \in \mathbb{R}^n \) the following holds \( \|h\|_2 \geq \frac{1}{\sqrt{n}} \|h\|_1 \). Lastly, by subtracting \( f(x_*) \) and taking expectation of (24), we have

\[
r_{k+1} \leq r_k - \frac{1}{2} \min \left( \frac{\beta_i}{\tau_i} \right) + t \max \left( \frac{\sqrt{p_i} \sqrt{2nLr_0}}{c_1} \right) + \frac{t^2}{8} \sum_{i=1}^{n} \frac{p_i L_i^3}{v_i^2}.
\]

Thus, we have that

\[
\frac{1}{r_{k+1}} - \frac{1}{r_k} \geq \frac{1}{r_k (1 - c_1 r_k + c_2 + c_3)} - \frac{1}{r_k} = \frac{r_k - r_k (1 - c_1 r_k - c_2 - c_3)}{r_k [r_k (1 - c_1 r_k + c_2 + c_3)]} \tag{26}
\]

Note that by setting \( t \) to be the the first two upper bounds in (22), the numerator of (26) is lower bounded as

\[
c_1 r_k^2 - t \left( \max_i \sqrt{p_i} \sqrt{2nLr_0} + \frac{t}{8} \sum_{i=1}^{n} \frac{p_i L_i^3}{v_i^2} \right) \geq c_1 r_k^2 - \frac{\epsilon^2 \min_t \frac{p_t}{v_t}}{4R_0^2 n} \geq \frac{\min_t \frac{p_t^2}{v_t^2} r_k^2}{4R_0^2 n}.
\]

Moreover, by setting \( t \) to the last two upper bounds in (22), the denominator of (26) is lower bounded as

\[
r_k \left[ r_k (1 - c_1 r_k + c_2 + c_3) \right] \leq r_k^2 + (c_2 + c_3)r_k \leq r_k^2 + \left( \frac{\epsilon}{2} + \frac{\epsilon}{2} \right)r_k \leq 2r_k^2.
\]

Then \( \frac{1}{r_{k+1}} - \frac{1}{r_k} \geq \frac{\min \frac{p_t}{v_t}}{8R_0^2 n} \) and setting \( \frac{1}{r_k} \geq \frac{1}{r_0} + ka. \) Hence if

\[
k \geq \frac{1}{a} \left( \frac{1}{\epsilon} - \frac{1}{r_0} \right) = \frac{8R_0^2 n}{\min \frac{p_t}{v_t}} \left( \frac{1}{\epsilon} - \frac{1}{r_0} \right)
\]

we have \( \mathbb{E} [f(x_k) - f(x_*)] \leq \epsilon. \)

**Strongly Convex Case**

**Theorem 5.** Let Assumptions 1 and 3 be satisfied. Choose \( \alpha_i = \frac{\alpha_0}{\sqrt{v_i h_i}} \sqrt{2 \mu (f(x_k) - f(x_*))} \), where \( 0 < \alpha_0 < \frac{2}{\sum_{i=1}^{n} \frac{p_i L_i}{v_i} \epsilon} \). If

\[
k \geq \frac{\max_i \frac{v_i^2}{p_i^2}}{\lambda} \frac{1}{2 \alpha_0 - \alpha_0 \sum_{i=1}^{n} \frac{p_i L_i}{v_i}} \log \left( \frac{f(x_0) - f(x_*)}{\epsilon} \right),
\]

then \( \mathbb{E} [f(x_k) - f(x_*)] \leq \epsilon. \)

**Proof.** Since \( f \) is \( \lambda \)-strongly convex, then:

\[
f(x_{k+1}) \geq f(x_k) + \left\langle \nabla f(x_k), h \right\rangle + \frac{\lambda}{2} \|h\|_2^2
\]

\[
\mu \geq f(x_k) + \left\langle \nabla f(x_k), h \right\rangle + \frac{\mu}{2} \|h\|_2^2
\]

where \( \|h\|_2^{p-2} \geq \sum_{i=1}^{n} \frac{v_i^2}{p_i^2} h_i^2. \) Minimizing the right hand side in \( h \) and substituting again, we have the inequality:

\[
\|\nabla f(x_k)\|_{p=2}^{2} \geq \sqrt{2 \mu (f(x_k) - f(x_*))}
\]

(29)

By subtracting \( f(x_*) \) from both sides of Lemma 1 we get

\[
\mathbb{E} [f(x_{k+1}) - f(x_*)] \leq \mathbb{E} [f(x_k) - f(x_*)] - \mathbb{E} [\alpha_i \left\langle \nabla f(x_k), e_i \right\rangle] \|x_k\| + \mathbb{E} \left[ \alpha_i \frac{L_i}{2} |x_k| \right]
\]

(30)

From (30), we have

\[
\mathbb{E} [\alpha_i \left\langle \nabla f(x_k), e_i \right\rangle] = \alpha_0 \sqrt{2 \mu (f(x_k) - f(x_*))} \sum_{i=1}^{n} \frac{p_i}{v_i} |\nabla_i f(x_k)|.
\]

(31)
Since the following holds:
\[
\sum_{i=1}^{n} \frac{p_i}{v_i} |\nabla_i f(x_k)| = \sqrt{\sum_{i=1}^{n} \frac{p_i}{v_i} |\nabla_i f(x_k)|^2} \geq \sqrt{\sum_{i=1}^{n} \frac{L_i^2}{v_i^2} |\nabla_i f(x_k)|^2} = \|\nabla f(x_k)\|_{p^{2}\alpha^{-2}},
\]
Thus, combining the previous result with (31), we have
\[
\mathbb{E} [\alpha_k |\langle \nabla f(x_k), e_i \rangle\|x_k] \geq \alpha_0 \sqrt{2\mu (f(x_k) - f(x_\star))}\|\nabla f(x_k)\|_{p^{2}\alpha^{-2}} \geq 2\mu \alpha_0 (f(x_k) - f(x_\star)) \tag{29}
\]
From \(\Box\), we have
\[
\mathbb{E} \left[ \frac{\alpha_k^2}{2} L_{ik} | x_k \right] = \mu \alpha_0^2 (f(x_k) - f(x_\star)) \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2} \tag{32}
\]
Lastly, substituting the results of \(\Box\) and \(\Box\) in (30) and taking the expectation with respect to \(x_k\) where \(r_k = \mathbb{E}[f(x_k) - f(x_\star)]\), we get
\[
r_{k+1} \leq r_k - 2\alpha_0 \mu r_k + \mu \alpha_0^2 r_k \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2} = \left( 1 - \frac{\lambda}{\max_i \frac{v_i}{p_i}} \left( 2\alpha_0 - \alpha_0^2 \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2} \right) \right) r_k.
\tag{33}\]
\[
K \geq \frac{\max_i \frac{v_i^2}{p_i^2}}{\lambda} \frac{1}{2\alpha_0 - \alpha_0^2 \sum_{i=1}^{n} \frac{p_i L_i}{v_i^2}} \log \left( \frac{f(x_0) - f(x_\star)}{\epsilon} \right),
\tag{34}\]
then \(\mathbb{E} [f(x_k) - f(x_\star)] \leq \epsilon\).
\(\Box\)

**Theorem 6.** Let Assumptions 1 and 3 be satisfied. Choose \(\alpha_{ik} = \frac{\langle f(x_k + t\epsilon i) - f(x_k) \rangle}{t\epsilon_i} \) for some small \(t\), \(\mu = \frac{\lambda}{\max_i \frac{v_i}{p_i}}\). If
\[
k \geq \frac{\max_i \frac{v_i^2}{p_i^2}}{\lambda} \log \left( \frac{2(f(x_0) - f(x_\star))}{\epsilon} \right) \tag{35}\]
then \(\mathbb{E} [f(x_k) - f(x_\star)] \leq \epsilon\).

**Proof.** Taking the expectation of (19) conditioned on \(x_k\), we have:
\[
\mathbb{E} [f(x_{k+1}) | x_k] \leq f(x_k) + t \mathbb{E} [\langle \nabla f(x_k), e_i \rangle | x_k] - \frac{1}{2} \mathbb{E} \left[ \frac{1}{v_i} |\langle \nabla f(x_k), e_i \rangle\|x_k |\right] + \frac{r_0^2}{8} \sum_{i=1}^{n} \frac{p_i L_i^3}{v_i^2} \tag{36}\]
Taking total expectation in \(\Box\) we have
\[
\mathbb{E} [\mathbb{E} [\langle \nabla f(x_k), e_i \rangle | x_k]] = \mathbb{E} [\langle \nabla f(x_k), e_i \rangle] = \mathbb{E} \left[ \sum_{i=1}^{n} \frac{p_i}{v_i} |\nabla_i f(x_k)|^2 \right] \leq \max_i \sqrt{p_i} \mathbb{E} [\|\nabla f(x_k)\|_2] \leq \max_i \sqrt{p_i} \sqrt{\mathbb{E} [\|\nabla f(x_k)\|_2^2]} \leq \max_i \sqrt{p_i} \sqrt{2nL (\mathbb{E} [f(x_k)] - f(x_\star))} \leq \max_i \sqrt{p_i} \sqrt{2nLr_0},
\]
where \(r_0 = \mathbb{E} [f(x_0)] - f(x_\star)\). Note that the first inequality follows by Jensen’s inequality. The last inequality follows from the fact that \(f\) has Lipschitz gradient, with Lipschitz constant \(nL\). As for \(\Box\), note that since \(\|h\|_{p^{2}\alpha^{-2}} \leq \max_i \frac{v_i}{p_i} \|h\|_{2}\); thus, by strong convexity we have that
\[
f(x_{k+1}) \geq f(x_k) + \langle \nabla f(x_k), h \rangle + \frac{\mu}{2} \|h\|_{p^{2}\alpha^{-2}} \geq f(x_k) - \frac{1}{2\mu} \|\nabla f(x_k)\|_{p^{2}\alpha^{-2}}^2,
\tag{37}\]
where \( \|h\|^2_{p^{-1} \text{ov}} = \sum_{i=1}^{n} \frac{v_i}{p_i} h_i^2 \) and \( \|\nabla f(x_k)\|_{p^{ov-1}} \) def \( \sqrt{\sum_{i=1}^{n} \frac{v_i}{p_i} \nabla_i f(x_k)^2} \). Thus, it follows that \( \|\nabla f(x_k)\|^2_{p^{ov-1}} \geq 2\mu(f(x_k) - f(x_0)) \). Therefore, taking the expectation of \( \mathbb{E}[\nabla f(x_k)] \) and with the use of the total expectation rule, \( \mathbb{E}[\nabla f(x_k)] \) can be lower bounded as follows

\[
\mathbb{E} \left[ \left( \frac{\|\nabla f(x_k), e_{ik}\|}{\sqrt{v_{ik}}} \right)^2 \right] = \left( \sum_{i=1}^{n} \frac{p_i}{v_i} \nabla_i f(x_k)^2 \right) = \|\nabla f(x_k)\|^2_{p^{ov-1}}
\]

(38)

By subtracting \( f(x_0) \) in (36) and taking expectation, we get

\[
r_k \leq r_{k-1} + t \max_i \sqrt{p_i} \sqrt{2nL_0} - \mu r_{k-1} + \frac{t^2}{8} \sum_{i=1}^{n} \frac{p_i L_i^3}{v_i^2}
\]

\[
\leq (1 - \mu)^k r_0 + \left( t \max_i \sqrt{p_i} \sqrt{2nL_0} + \frac{t^2}{8} \sum_{i=1}^{n} \frac{p_i L_i^3}{v_i^2} \right) \sum_{i=1}^{k-1} (1 - \mu)^i
\]

(39)

Thus, by choosing \( t \) such that \( \left( t \max_i \sqrt{p_i} \sqrt{2nL_0} + \frac{t^2}{8} \sum_{i=1}^{n} \frac{p_i L_i^3}{v_i^2} \right) ^{1/\mu} \leq \frac{\mu}{2} \) and if

\[
k \geq \frac{\max_i \frac{p_i v_i}{p}}{\lambda} \log \left( \frac{2(f(x_0) - f(x_0))}{\epsilon} \right),
\]

we have \( \mathbb{E}[f(x_k)] - f(x_0) \leq \epsilon \).

**Estimating the Lipschitz Smoothness Constant**

As we explain in the Section, we do not have an access to the Lipschitz smoothness constants of the function for the continuous control case. Instead, we estimate them using the points which have been evaluated. Our experiments suggest that, estimating the function directly using a parametric family works better than estimating smoothness constants directly. In other words, we estimate the function to be optimized directly using a parametric family \( \tilde{f}(\cdot, \theta) \) and use its Lipschitz smoothness constants as an estimate.

Consider the DFO step \( n \); using the queried sampled \( \{x_i, f(x_i)\}_{i \in [n-1]} \), we can estimate the function of interest by solving empirical risk minimization problem as:

\[
\theta_n = \arg \min_{\theta} \sum_{i} (f(x_i) - \tilde{f}(x_i, \theta))^2
\]

(40)

We use the resulting \( \tilde{f}(\cdot, \theta_n) \) to compute Lipschitz smoothness constants and importance sampling weights. The parametric family we consider is a multi-layer perceptron with single hidden layer and tanh non-linearity. Input dimension is the policy size, output dimension is 1 and hidden layer dimension is chosen as 16 for Swimmer-v1 and Hopper-v1, 64 for HalfCheetah-v1 and Ant-v1, and 256 for Humanoid-v1. Learning has been performed using ADAM (Kingma and Ba 2015) optimizer at each iteration. Step size (learning rate) has been chosen as \( 10^{-3} \) for all experiments. At each iteration we choose a 32 random samples in uniform from \( x_0, \ldots, x_{n-1} \) and use it as a batch.