Polynomial-Time Sum-of-Squares Can Robustly Estimate Mean and Covariance of Gaussians Optimally

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Abstract

In this work, we revisit the problem of estimating the mean and covariance of an unknown $d$-dimensional Gaussian distribution in the presence of an $\epsilon$-fraction of adversarial outliers. The work of [DKK+16] gave a polynomial time algorithm for this task with optimal $\tilde{O}(\epsilon)$ error using $n = \text{poly}(d, 1/\epsilon)$ samples.

On the other hand, [KS17b] introduced a general framework for robust moment estimation via a canonical sum-of-squares relaxation that succeeds for the more general class of certifiably subgaussian and certifiably hypercontractive [BK20] distributions. When specialized to Gaussians, this algorithm obtains the same $\tilde{O}(\epsilon)$ error guarantee as [DKK+16] but incurs a super-polynomial sample complexity ($n = d^{O(\log 1/\epsilon)}$) and running time ($n^{O(\log 1/\epsilon)}$). This cost appears inherent to their analysis as it relies only on sum-of-squares certificates of upper bounds on directional moments while the analysis in [DKK+16] relies on lower bounds on directional moments inferred from algebraic relationships between moments of Gaussian distributions.

We give a new, simple analysis of the same canonical sum-of-squares relaxation used in [KS17b, BK20] and show that for Gaussian distributions, their algorithm achieves the same error, sample complexity and running time guarantees as of the specialized algorithm in [DKK+16]. Our key innovation is a new argument that allows using moment lower bounds without having sum-of-squares certificates for them. We believe that our proof technique will likely be useful in designing new robust estimation algorithms.

Keywords: Robust estimation, sum-of-squares, mean estimation

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1 Introduction

Designing estimation algorithms for estimating basic parameters of probability distributions from samples is a foundational computational problem in machine learning. However, natural estimation algorithms, such as taking the sample mean for population mean, can be brittle – even a single outlier in the data can lead to an arbitrarily large estimation error. In the 1960s, Tukey and Huber began systematic efforts to build robust estimators that can tolerate minor deviations of the input from the chosen model, such as the injection of a small constant fraction of adversarially chosen outliers into the sample. While this effort has led to a burgeoning body of work called robust statistics, the algorithms from this line of work typically require exponential time in the underlying dimension to succeed and are thus inefficient in high-dimensional settings.

In 2016, two papers [DKK+16, LRV16] pioneered a systematic effort to build computationally efficient robust estimators. Since their work, the study of algorithmic robust statistics has evolved into an active area, that, in addition to yielding concrete solutions to basic robust estimation problems, has led to the synthesis of truly new algorithmic ideas (often improving even the classical, non-robust algorithms) that identify and clarify general principles for efficient robust estimation.

A key insight from this line of work has been a general blueprint for robust estimation using the sum-of-squares (SoS) method. A sequence of works in 2018 gave a canonical sum-of-squares relaxation and a rounding algorithm that gives the nearly statistically optimal outlier-robust estimation of moments [KS17b] and robust clustering [KS17a, HL18] of spherical mixtures of a broad class of probability distributions. Since then, this framework has been refined and expanded to obtain state-of-the-art robust estimation algorithms for problems such as outlier-robust regression [KKM18, BP21], clustering of non-spherical mixtures [BK20, BDH+20], heavy tailed estimation [Hop18, CHK+20], list-decodable regression and subspace recovery [KKK19, BK20, RY20b, RY20a] and robust learning of a mixture of arbitrary Gaussians [LM21, BDJ+20].

In addition, algorithms from the SoS-based robust estimation framework have the advantage of abstracting out natural analytic properties of the statistical model in question and yielding robust estimators for all distributions that satisfy such properties in a blackbox way. For example, the algorithms for robust estimation of moments [KS17b] and clustering [HL18, KS17a] apply to all certifiably subgaussian distributions, that, informally speaking, are distributions that admit “sum-of-squares certificates” of the property of having subgaussian low order moments. As another example, the covariance estimation algorithm of [BK20] applies to all distributions that admit sum-of-squares certificates of bounds on moments of degree-2 polynomials (certifiable hypercontractivity). Such properties are already known to hold for a broad class of distributions and verifying them for a new family immediately generalizes such results. In fact, one can interpret the analysis in the sum-of-squares framework as identifying structural properties (certificates of appropriate analytic properties) of the distribution families that can be exploited for the design of efficient robust estimation algorithms.\(^1\)

\(^1\)See this recent talk for this perspective and its applications to weakening distributional assumptions in robust estimation.
Robust Mean Estimation for Gaussians. While the SoS-based framework above typically achieves the best known recovery guarantees (among polynomial time algorithms), a striking exception so far has been the task of robustly estimating the mean and covariance of an unknown Gaussian distribution. In this problem, the algorithm is given input data $Y = \{y_1, y_2, \ldots, y_n\} \subseteq \mathbb{R}^d$ that is obtained by arbitrarily and adversarially corrupting $\varepsilon n$ points in an i.i.d. sample $X = \{x_1, x_2, \ldots, x_n\}$ from an unknown Gaussian distribution $\mathcal{N}(\mu, \Sigma)$. The algorithm of [DKK+16] obtains estimates $\hat{\mu}, \hat{\Sigma}$ so that the total variation distance $d_{TV}(\mathcal{N}(\mu, \Sigma), \mathcal{N}(\hat{\mu}, \hat{\Sigma})) \leq \tilde{O}(\varepsilon)$. This is optimal up to logarithmic factors in $\varepsilon$ in the bound (and there is evidence [DKS17] that such a loss might be necessary for polynomial time algorithms). Their algorithm requires $n = \text{poly}(d, 1/\varepsilon)$ samples and $\text{poly}(n)$ running time. On the other hand, the best known SoS-based algorithm for the problem is obtained by specializing the analysis in [BK20] for mean and covariance estimation of certifiably hypercontractive distributions to the case of Gaussians. This analysis yields the same error bound of $\tilde{O}(\varepsilon)$ on the total variation error but requires super-polynomially many $d^{O(\log 1/\varepsilon)}$ samples and $n^{O(\log 1/\varepsilon)}$ running time.

There is an important technical bottleneck in the analysis of the canonical SoS algorithm for obtaining the stronger guarantees in [DKK+16]. The analysis in [KS17b] (and extensions in [BK20]) only uses upper bounds on the higher moments of distributions. On the other hand, the stronger analysis in [DKK+16] implicitly relies on a non-trivial lower bound on moments of arbitrary subsets of the original sample of size $(1 - \varepsilon)n$. The best known sum-of-squares certificates for such a lower bound property appear to require an exponential cost in $O(\log 1/\varepsilon)$ in both sample complexity and running time. And, it is plausible (though, still unproven) that such a cost is necessary! This state of affairs leads us to the main motivating question of this work:

*Can the canonical SoS based algorithm give a robust estimate with $\tilde{O}(\varepsilon)$ total variation error for the mean and covariance of Gaussian distributions in polynomial time and samples? Or is the SoS framework for moment estimation weaker, when specialized to Gaussian distributions?*

In this work, we give a new analysis of the canonical sum-of-squares-based algorithm for robust mean estimation for Gaussians (that only has subgaussian upper bounds on 4th moment as constraints) that recovers the polynomial running time and sample complexity guarantees of [DKK+16] and same error up to poly log $1/\varepsilon$ factors. Our key innovation (that we explain later in this section) is a new argument that works around the issue of finding efficient sum-of-squares certificates for moment lower bounds and yet manages to prove the stronger guarantee. We believe that this new technique will likely find further applications in efficient robust estimation.

1.1 Our results

Formally, our algorithms work in the following strong contamination model for corrupted samples used in several prior works on robust estimation, beginning with [DKK+16] [LRV16].

**Definition 1.1** (Strong contamination model). Let $D$ be a distribution on $\mathbb{R}^d$ and let $X = \{x_1, x_2, \ldots, x_n\}$ be an i.i.d. sample from $D$. In the strong contamination model, an $\varepsilon$-corrupted sample is obtained by replacing any adversarially chosen $\varepsilon n$ points from $X$ with arbitrary outliers to obtain $Y = \{y_1, y_2, \ldots, y_n\}$. 

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Our main result is an analysis of the following canonical SoS relaxation for mean and covariance estimation along with a simple rounding (used in [KS17b, BK20]) for estimating the mean and covariance of an unknown distribution.

**Algorithm 1** (Mean and spectral norm covariance estimation).

**Input:** Parameter $\varepsilon \in (0, 1)$, and corrupted samples $y_1, \ldots, y_n \in \mathbb{R}^d$.

**Operation:** Find a degree-12 pseudo-expectation $\tilde{E}$ (solution to the SoS semidefinite programming relaxation) in variables $x'_1, \ldots, x'_n \in \mathbb{R}^d$, $w_1, \ldots, w_n \in \mathbb{R}$, $\mu' := \mathbb{E}_i x'_{i}$, $\Sigma' := \mathbb{E}_i (x'_i - \mu') (x'_i - \mu')^T$ satisfying the following set of constraints:

1. **Booleanity of intersection Variables:** $w_i^2 = w_i$ for every $i \in [n]$.
2. **Size of intersection:** $\sum_{i=1}^n w_i = (1 - \varepsilon)n$.
3. **Intersection constraints:** $w_i x'_i = w_i y_i$ for every $i \in [n]$.
4. **Certifiable Subgaussianity of 4th moments:**
   $$\frac{1}{n} \sum_{i=1}^n (\langle x'_i - \mu', v \rangle^2 - v^T \Sigma' v)^2 \leq (2 + \tilde{O}(\varepsilon))(v^T \Sigma' v)^2$$
   for every $v \in \mathbb{R}^d$.

**Output:** $\hat{\mu} = \tilde{E}[\mu']$, $\hat{\Sigma} = \tilde{E}[\Sigma']$.

The constraints of the program encode the task of finding a set of points $X'$ that intersects the input sample $Y$ in $(1 - \varepsilon)n$ points (encoded by the first 3 sets of constraints) such that the empirical 4-th moments of $X'$ are bounded above by at most $\sim 2$ times the square of the 2nd moments (the last set of constraints). The last set of constraints, though apparently infinitely many (one for every $v \in \mathbb{R}^d$) admit a succinct representation via techniques of [KS17b, HL18] (see Appendix B, or [FKP19], Chapter 4 for an exposition). The intended solution for this polynomial system is $X' = X$ – the unknown, true i.i.d. sample. (And then setting $w_i = 1(x_i = y_i)$, and $\mu', \Sigma'$ to be the empirical mean/covariance of $X$.) It is easy to check that $X$ satisfies the last set of constraints – the only property of i.i.d. Gaussian samples that we enforce.

We prove the following formal guarantees on **Algorithm 1**.

**Theorem 1** (Mean and spectral norm covariance estimation). **Algorithm 1** takes input an $\varepsilon$-corrupted sample of size $n$ from a Gaussian distribution with mean $\mu$ and covariance $\Sigma$ and in $\text{poly}(n)$-time, outputs estimates $\hat{\mu} \in \mathbb{R}^d$, $\hat{\Sigma} \in \mathbb{R}^{d \times d}$ with the following guarantee. If $\Sigma \geq 2^{-\text{poly}(d)} I$, and $n \geq \tilde{O}(d^2 \log^5(1/\delta)/\varepsilon^2)$, with probability at least $1 - \delta$ over the draw of the original uncorrupted sample $X$, the estimates $\hat{\mu}, \hat{\Sigma}$ satisfy:

1. **Mean estimation** $\|\Sigma^{-1/2}(\hat{\mu} - \mu)\|^2 \leq \tilde{O}(\varepsilon)$, and

2. **Covariance estimation in spectral norm** $(1 - \tilde{O}(\varepsilon))\Sigma \leq \hat{\Sigma} \leq (1 + \tilde{O}(\varepsilon))\Sigma$.

**Remark 1.2** (Computational Model and Numerical Issues). Our algorithm succeeds in the standard word RAM model of computation. In this model, the input sample $Y$ is given to the algorithm after “truncating” the real numbers to rational numbers with $\text{poly}(d)$ bits of precision. The running time of our algorithm is polynomial in the size of the bit representation of the input. The assumption on the smallest eigenvalue of $\Sigma$ in the statement above is entirely an artefact of the numerical issues.
as the truncation of \( Y \) to rational numbers, in general, does not allow recovering eigenvalues of \( \Sigma \) that are not representable in polynomially many bits of precision. Such an assumption is required (but sometimes not stated explicitly) by all prior works on robust estimation when implemented in the standard word RAM model of computation.

We note that it is possible (though, requires additional steps in the algorithm) to remove the assumption on the smallest eigenvalue of \( \Sigma \) if we instead assume that the unknown \( \Sigma \) has rational entries. Such an assumption is clearly necessary as algorithms in the word RAM model can only output \( \hat{\Sigma} \) with rational entries. We omit the description of such a method and instead choose to make an assumption that the smallest eigenvalue of \( \Sigma \) can be written down in \( \text{poly}(d) \) bits.

Theorem 1 shows that the algorithm of [KS17b], when analyzed for Gaussian distributions, achieves the information-theoretically optimal \( O(\epsilon) \) error guarantee using \( n = \text{poly}(d, 1/\epsilon) \) samples and \( \text{poly}(n) \) running time. This shows that the analysis of [KS17b], which is tight for the more general class of certifiable subgaussian and certifiable hypercontractive distributions, can be improved in the specific case of Gaussians.

The guarantees achieved by Theorem 1 are weaker than the guarantees of the algorithm in [DKK +16], whose estimate \( \hat{\Sigma} \) is additionally close to \( \Sigma \) in relative Frobenius error. We show that by analyzing the degree-12 SoS relaxation of the following program (that replaces the certifiable subgaussianity constraints by certifiable hypercontractivity constraints on degree-2 polynomials), we can upgrade the guarantees of Theorem 1 to achieve the stronger Frobenius norm guarantee of [DKK +16]. We note that this program (with additional higher-degree certifiable hypercontractivity constraints) was analyzed in [BK20] to obtain similar guarantees on the mean and covariance estimation of the more general class of all certifiably hypercontractive distributions, but needed \( n = d^{O(\log 1/\epsilon)} \) samples and \( n^{O(\log 1/\epsilon)} \) running time. Our contribution is obtaining a sharper analysis of the same program for the case of Gaussian distributions.

**Algorithm 2** (Frobenius norm covariance estimation).

**Input:** Parameter \( \epsilon \in (0, 1) \), and corrupted samples \( y_1, \ldots, y_n \in \mathbb{R}^d \).

**Operation:** Find a degree-12 pseudo-expectation \( \hat{\mathbb{E}} \) in the variables \( x'_1, \ldots, x'_n \in \mathbb{R}^d \), \( w_1, \ldots, w_n \in \mathbb{R} \), \( \mu' := \mathbb{E}_i x'_i, \Sigma' := \mathbb{E}_i (x'_i - \mu')(x'_i - \mu')^\top \) satisfying the following set of constraints:

1. \( w_i^2 = w_i \) for every \( i \in [n] \),
2. \( \sum_{i=1}^n w_i = (1 - \epsilon)n \),
3. \( w_i x'_i = w_i y_i \) for every \( i \in [n] \),
4. \( \mathbb{E}_i \langle (x'_i - \mu')(x'_i - \mu')^\top - \Sigma', P \rangle^2 \leq (2 + 0(\epsilon))\|P\|_F^2 \) for every symmetric \( P \in \mathbb{R}^{d \times d} \).

**Output:** \( \hat{\Sigma} := \hat{\mathbb{E}}[\Sigma'] \).

**Theorem 2** (Frobenius norm covariance estimation with \( \Sigma \approx I \)). Algorithm 2 takes input an \( \epsilon \)-corrupted sample of size \( n \) from a Gaussian distribution with mean \( \mu \) and covariance \( \Sigma \) with \( (1 - \tilde{O}(\epsilon))I \preceq \Sigma \preceq (1 + \tilde{O}(\epsilon))I \), and in \( \text{poly}(n) \)-time, outputs an estimate \( \hat{\Sigma} \in \mathbb{R}^{d \times d} \) with the following guarantee. If
We now give a high level sketch of the key idea used in our proof. First, let’s briefly recap and in the sum-of-squares method. The polynomial constraints in our program (Algorithm 1) encode analysis in these works utilizes the “proofs to algorithms” framework of algorithm design via the \( \Delta \). In particular, we can obtain \( \sim \) analysis proceeds by using the constraints to derive, via a \( \Delta \)-degree SoS proof, the error in \( \hat{\mu} = \tilde{E}[\mu'] \) also satisfies the inequality above giving us the required guarantee.

We thus obtain the final corollary:

**Corollary 1.3 (Mean and Frobenius norm covariance estimation).** There is an SoS-based algorithm that takes as input an \( \epsilon \)-corrupted sample of size \( n \) from a Gaussian distribution with mean \( \mu \) and covariance \( \Sigma \) and in \( \text{poly}(n) \)-time, outputs estimates \( \hat{\mu} \in \mathbb{R}^d, \hat{\Sigma} \in \mathbb{R}^{d \times d} \) with the following guarantee. If \( \Sigma \geq 2^{-\text{poly}(d)} I \), and \( n \geq \tilde{O}(d^2 \log^5(1/\delta)/\epsilon^2) \), with probability at least \( 1 - \delta \) over the draw of the original uncorrupted sample \( X \), the estimates \( \hat{\mu}, \hat{\Sigma} \) satisfy:

1. (Mean estimation) \( \|\Sigma^{-1/2}(\hat{\mu} - \mu)\|_2 \leq \tilde{O}(\epsilon) \), and
2. (Covariance estimation in Frobenius norm) \( \|\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I\|_F \leq \tilde{O}(\epsilon) \).

In particular, \( \Delta_{TV}(N(\mu, \Sigma), N(\hat{\mu}, \hat{\Sigma})) \leq \tilde{O}(\epsilon) \).

Thus, we obtain the same guarantee\(^2\) on the total variation distance as in [DKK +16].

1.2 A brief overview of our key idea

We now give a high level sketch of the key idea used in our proof. First, let’s briefly recap the style of analysis in [KS17b, BK20] by focusing on the guarantee for mean estimation. The analysis in these works utilizes the “proofs to algorithms” framework of algorithm design via the sum-of-squares method. The polynomial constraints in the our program (Algorithm 1) encodes finding a set \( X' \) of samples that intersects the input corrupted sample \( Y \) in \( (1 - \epsilon)n \) points and has 4th moments upper bounded in terms of the squared 2nd moments in every direction. The analysis proceeds by using the constraints to derive, via a \( O(1) \)-degree SoS proof that the error in the so called Mahalanobis norm, \( \|\Sigma^{-1/2}(\mu' - \mu)\|_2 \leq O(\epsilon^{3/4}) \). Such an inequality implies that any degree \( O(1) \)-pseudo-expectation \( \tilde{E} \) that satisfies the constraints of our program must also satisfy all consequences obtainable via \( O(1) \)-degree SoS proofs. As a result, the rounded estimate \( \hat{\mu} = \tilde{E}[\mu'] \) also satisfies the inequality above giving us the required guarantee.

The Mahalanobis error bound of \( \epsilon^{3/4} \) here comes from the upper bound on the 4th moments (in general, we can obtain \( \sim \sqrt[4]{\epsilon^{1-1/t}} \) error bounds by working with upper bounds on \( t \)-th moments) encoded in our constraints and is polynomially off from the optimal \( \tilde{O}(\epsilon) \) bound we intend to achieve when the unknown distribution is Gaussian. In fact, the analysis and the bounds obtained

\(^2\)Our formal guarantees are not explicit about \( \text{polylog}(1/\epsilon) \) factors in the error, and as a result, formally speaking our error bounds only match that of [DKK +16] up to \( \text{polylog}(1/\epsilon) \) factors. We believe that our argument in fact gets the same \( \text{polylog}(1/\epsilon) \) dependence, as we rely on the same concentration bounds as in [DKK +16], which is where the \( \text{polylog}(1/\epsilon) \) factors arise. But, our proofs currently do not explicitly show this.
by [KS17b, BK20] are information-theoretically optimal: for any $t$, there are two distributions with means that are $\sqrt{t}\varepsilon^{1-1/t}$ far, satisfy the $2t$-th moment upper bound condition, and are $\sim \varepsilon$-different in total variation distance. In particular, one can take such a pair of distributions and produce corrupted samples that are statistically indistinguishable!

At this point, one might wonder – how can one hope the same set of constraints to yield a tighter guarantee for Gaussians? Indeed, our analysis follows a substantially different path to use the Gaussianity of the underlying input distribution.

Enter Resilience. At a high-level, in retrospect, the key property of Gaussians that [DKK+16] exploit (which is not satisfied by distributions that constitute the “hard examples” above) is a certain mild anti-concentration property (that we call resilience, following [SCV17]) inferred from lower bounds on moments of subsamples. Specifically, if $X$ is a typical i.i.d. sample from a $d$-dimensional, 0-mean Gaussian distribution of size $n \gg d/\varepsilon$, and $S \subseteq X$ is any subset of size $(1-\varepsilon)n$, then the empirical covariance $\Sigma_S$ of points in $S$ satisfies $\Sigma_S \geq (1-\tilde{O}(\varepsilon))\Sigma_X$. Or, equivalently, that for $\tilde{S} = X \setminus S$, it holds that $\Sigma_{\tilde{S}} \preceq \text{polylog}(1/\varepsilon)\Sigma_X$. The analysis of [KS17b] can only infer (via Cauchy-Schwarz inequality) the exponentially worse bound of $\Sigma_{\tilde{S}} \leq O(1/\sqrt{\varepsilon})\Sigma_X$.

Crucially, resilience of the covariance cannot be inferred from 4th moment upper bounds, such as those encoded by our constraints. It can indeed be inferred from an argument that relies on boundedness of $O(\log 1/\varepsilon)$ moments, but if we wanted our sample $X$ to have all of its $\leq O(\log 1/\varepsilon)$ moments close to that of the true distribution, we would need $d^{O(\log 1/\varepsilon)}$ (in particular, superpolynomially many) samples.

A key insight of [DKK+16] is the observation that one can prove the resilience of covariance by a simple Hoeffding + union bound for a sample of size $\sim d/\varepsilon$. Notice that Hoeffding’s inequality itself relies on subgaussianity of all moments of the distribution but the relevant consequence of it – namely resilience – can be “seen” in typical samples of size $\sim d/\varepsilon$.

Resilience is likely not efficiently certifiable. While this is encouraging, using this property within the sum-of-squares framework poses a major issue. Notice that, a priori, verifying that a sample $X$ satisfies resilience requires an exponential search since we need to verify some property for every subset $S$ of size $(1-\varepsilon)n$. Indeed, given a sample of size $n$ – as far as we know, there is no known polynomial time algorithm to output a certificate (whether via sum-of-squares or otherwise) of such a property. On the other hand, since the analysis style in [KS17b] involves deriving a bound on the Mahalanobis distance between $\mu'$ and the true mean $\mu$, we would need a low-degree sum-of-squares certificate of resilience in order to plug it into the SoS framework. This is the key technical issue that prevented prior attempts to “SoSize” the argument of [DKK+16] for mean (and more generally, covariance estimation) for Gaussians.

Circumventing certificates by proving “only in pseudo-expectation”. Our main contribution is an argument that allows us to use resilience without requiring an SoS certificate. Notice that, though powerful and elegant, obtaining a low-degree sum-of-squares proof of a bound on the Mahalanobis distance $\|\Sigma^{-1/2}(\mu' - \mu)\|_2$ is overkill for our purpose! We only need the inequality after
taking pseudo-expectations. Our key idea, thus, is to directly prove a bound on \(\|\Sigma^{-1/2}(\mathbb{E}[\mu'] - \mu]\|_2\) without going through low-degree sum-of-squares proofs.

If, for a second, we pretend that pseudo-expectations are in fact actual probability distributions over solutions \(X'\), then this is akin to proving an inequality on the expectation of the solution \(X'\) without establishing (the considerably stronger claim) that it holds “pointwise” in the support of the distribution. Thus, our idea above can be summarized as attempting to prove a fact “in pseudo-expectation” without establishing it pointwise in the support of the “pseudo-distribution”.

We show that for the purpose of arguing “after taking pseudo-expectations”, we can in fact leverage the resilience bound discussed above. Our final argument thus derives some facts “within low-degree sum-of-squares proof system” – with some technical choices that make the composition with facts “after taking expectations” possible. Making this work and extending to covariance estimation in spectral and then Frobenius norms requires some more technical work which, for the purpose of this overview, we omit.

To the best of our knowledge, this is the first example in the SoS proofs to algorithms framework for robust statistics where the difference between facts “derived via low-degree SoS proofs” vs “proved only in pseudo-expectations” appears to make a significant material difference to the results so obtained. We believe that this style of analysis might come in handy in future applications of the SoS method to robust statistics and more generally, problems in statistical estimation.

2 Preliminaries

In this section, we give an overview of the sum-of-squares algorithm and state the concentration properties of Gaussians that we need for our results.

2.1 A crash course in sum-of-squares

We give a brief overview of the sum-of-squares (SoS) algorithm. For a more in-depth survey, see [FKP19].

The sum-of-squares algorithm works in the standard word RAM model of computation. We assume that all numerical inputs are rational numbers represented as a pair of integers describing the numerator and the denominator. In order to measure the running time of algorithms, we will need to account for the length of the numbers that arise during the run of the algorithm. The following definition captures the size of the representations of the rational numbers:

**Definition 2.1** (Bit complexity). The bit complexity of an integer \(p \in \mathbb{Z}\) is \(1 + \lceil \log_2 p \rceil\). The bit complexity of a rational number \(p/q\) where \(p, q \in \mathbb{Z}\) is the sum of the bit complexities of \(p\) and \(q\).

We now move to discussing the SoS algorithm. Consider a generic polynomial feasibility problem of the form

\[
\text{find } \ x \in \mathbb{R}^m \quad \text{s.t. } \ f_i(x) \geq 0 \ \forall i, \ g_j(x) = 0 \ \forall j
\]

where \(f_i\) and \(g_j\) are arbitrary polynomial functions of \(x\) with rational coefficients of bit complexity \(B\), and the total number of constraints is \(\text{poly}(m)\). Let \(P_{m,k}\) denote the set of polynomials \(p\) in \(m\)
variables with degree at most \( k \). A degree-\( k \) pseudo-expectation is an object that mimics a real expectation \( \mathbb{E} \) for low-degree polynomials, and is defined as follows.

**Definition 2.2** (Degree-\( k \) pseudo-expectation). A degree-\( k \) pseudo-expectation (\( k \) even) over \( m \) variables is a linear operator \( \tilde{\mathbb{E}} : \mathcal{P}_{m,k} \rightarrow \mathbb{R} \) satisfying:

1. (Normalization) \( \tilde{\mathbb{E}}[1] = 1 \), and
2. (PSDness) \( \tilde{\mathbb{E}}[p^2] \geq 0 \) for all \( p \in \mathcal{P}_{m,k/2} \).

We say that the PSDness condition is satisfied with error \( \tau \) if \( \tilde{\mathbb{E}}[p^2] \geq -\tau \|p\|^2_2 \) for each \( p \in \mathcal{P}_{m,k/2} \), where \( \|p\|_2 \) is the \( \ell_2 \)-norm of the vector of coefficients of \( p \).

We now define what it means for \( \tilde{\mathbb{E}} \) to (approximately) satisfy constraints.

**Definition 2.3** (Satisfying constraints). For a polynomial \( g \), we say that a degree-\( k \) \( \tilde{\mathbb{E}} \) satisfies the constraint \( \{ g = 0 \} \) exactly if for every polynomial \( p \) of degree \( \leq k - \deg(g) \), \( \tilde{\mathbb{E}}[pg_j] = 0 \) and \( \tau \)-approximately if \( |\tilde{\mathbb{E}}[pg_j]| \leq \tau \|p\|_2 \). We say that \( \tilde{\mathbb{E}} \) satisfies the constraint \( \{ g \geq 0 \} \) exactly if for every polynomial \( p \) of degree \( \leq k/2 - \deg(g)/2 \), it holds that \( \tilde{\mathbb{E}}[p^2g] \geq 0 \) and \( \tau \)-approximately if \( \tilde{\mathbb{E}}[p^2g] \geq -\tau \|p\|^2_2 \).

We note that in the above two definitions, the requirements on the degree of the polynomial is such that \( \tilde{\mathbb{E}} \) is well-defined, e.g., \( \tilde{\mathbb{E}}[pg_j] \) is only well-defined when \( \deg(pg_j) \leq k \).

For intuition, it is helpful to observe that a pseudo-expectation is a relaxation of the familiar notion of expectations: it may be useful to think of pseudo-expectation as satisfying \( \tilde{\mathbb{E}}[p] = \mathbb{E}_{z \sim D}[p(z)] \) for some distribution \( D \) over \( \mathbb{R}^m \). Clearly, if \( D \) is a distribution over feasible solutions of the constraints in \((2.1)\), then \( \tilde{\mathbb{E}} \) satisfies all constraints.

We are now ready to define the **sum-of-squares algorithm**.

**Fact 2.4** (Sum-of-Squares algorithm, [Sho87, Par00, Nes00, Las01]). There is an algorithm, the degree-\( k \) sum-of-squares algorithm, with the following properties: The algorithm takes as input \( B \in \mathbb{N}, \tau > 0, k \in \mathbb{N} \), and a problem of the form \((2.1)\) with \( \text{poly}(m) \) constraints, each with rational coefficients of bit complexity \( B \). If there is a degree-\( k \) pseudo-distribution satisfying \((2.1)\), then the algorithm outputs in \( \text{poly}(B, \log \frac{1}{\tau} \cdot m^{O(k)}) \) a degree-\( k \) pseudo-expectation \( \tilde{\mathbb{E}} \) that \( \tau \)-approximately satisfies all the constraints in \((2.1)\), and otherwise outputs “infeasible”.

For the purposes of this paper, we can set \( \tau = 2^{-m} \) and \( B = \text{poly}(m) \). The “total error” that we will incur will be \( O(\text{poly}(m, B)2^{-m}) = O(\text{poly}(m)2^{-m}) \) which is negligible.

We state some basic known facts about the pseudo-expectations that we use below.

**Proposition 2.5** (see for e.g., [BS16, FKP19]). For any degree-\( k \) \( \tilde{\mathbb{E}} \), the following Cauchy-Schwarz inequalities hold:

1. For any \( p, q \in \mathcal{P}_{m,k/2} \), we have \( \tilde{\mathbb{E}}[pq]^2 \leq \tilde{\mathbb{E}}[p^2] \tilde{\mathbb{E}}[q^2] \).
2. For any \( p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathcal{P}_{m,k/2} \) and distribution \( D \) over \( [n] \), \( \tilde{\mathbb{E}} \) satisfies the polynomial inequality \( \tilde{\mathbb{E}}_{z \sim D}[p_1^2q_1^2] \leq \tilde{\mathbb{E}}_{z \sim D}[p_1^2] \tilde{\mathbb{E}}_{z \sim D}[q_1^2] \). In particular, \( (p_1^2 + p_2^2 + p_3^2)^2 \leq 3(p_1^4 + p_2^4 + p_3^4) \).

**Definition 2.6** (SoS proofs of non-negativity). Let \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) be a polynomial. We say that \( h \) has a degree-\( \ell \) SoS proof of nonnegativity if \( h = \sum_{i=1}^r h_i^2 \) for some polynomials \( h_i \in \mathcal{P}_{d,\ell/2} \).
2.2 Resilience and certifiable subgaussianity of Gaussian moments

We now give a brief overview of the key properties of Gaussian moments that we use.

Our algorithm relies on concentration bounds of Gaussians from [DKK+16], which prove resilience of the first 4 moments of the Gaussian distribution. We state the bounds for the first two moments below.

**Lemma 2.7** (Resilience of first and second moments; Lemmas 4.4, 4.3 in [DKK+16]). Let \( x_1, \ldots, x_n \sim \mathcal{N}(0, I_{d \times d}) \), and \( n \geq O((d + \log(1/\delta))/\epsilon^2) \). Then with probability \( 1 - \delta \), for all \( v \in \mathbb{R}^d \) and vectors \( a \in [0, 1]^n \) such that \( \mathbb{E}_i a_i \geq 1 - \epsilon \), we have

\[
\left| \mathbb{E}_i a_i \langle x_i, v \rangle \right| \leq \tilde{O}(\epsilon) \|v\|_2 ,
\]

and

\[
\left| \mathbb{E}_i a_i \left[ \langle x_i, v \rangle^2 - \|v\|_2^2 \right] \right| \leq \tilde{O}(\epsilon) \|v\|_2^2 .
\]

To see the importance of Lemma 2.7, we note that, when combined with Proposition 2 in [SCV18], Lemma 2.7 immediately yields an exponential time algorithm to robustly estimate the mean \( \mu \) of a Gaussian \( \mathcal{N}(\mu, \Sigma) \) with known covariance \( \Sigma \), i.e., output \( \hat{\mu} \) satisfying (1) in Theorem 1.

The second resilience property we need is an upgrade of the resilience property of the second moment in Lemma 2.7, as well as the resilience of the fourth moment.

**Lemma 2.8** (Stronger resilience of second and resilience of fourth moments). Let \( x_1, \ldots, x_n \sim \mathcal{N}(0, I_{d \times d}) \), and \( n \geq \tilde{O}(d^2 \log^5(1/\delta)/\epsilon^2) \). Then with probability \( 1 - \delta \), for all symmetric \( P \in \mathbb{R}^{d \times d} \) and vectors \( a \in [0, 1]^n \) such that \( \mathbb{E}_i a_i \geq 1 - \epsilon \), we have

\[
\left| \mathbb{E}_i a_i \langle x_i x_i^T - I, P \rangle \right| \leq \tilde{O}(\epsilon) \cdot \|P\|_F ,
\]

and

\[
\left| \mathbb{E}_i a_i \left[ \langle x_i x_i^T - I, P \rangle^2 - 2\|P\|_F^2 \right] \right| \leq \tilde{O}(\epsilon) \cdot \|P\|_F^2 .
\]

Lemma 2.8 follows from Corollary 4.8, Lemma 5.17 and Lemma 5.21 of [DKK+16]; we include a short proof for completeness in Appendix C.

We will also need slightly different forms of the above resilience results. We now state the results in the form that we need, and postpone the proof to Appendix C. The proofs are a straightforward (but somewhat tedious) consequence of the above results from [DKK+16].

**Lemma 2.9.** Let \( x_1, \ldots, x_n \sim \mathcal{N}(\mu, \Sigma) \) for \( \mu \in \mathbb{R}^d \) and \( \Sigma \in \mathbb{R}^{d \times d} \) positive definite. Let \( n \) be as in Lemma 2.8, and \( \mu_0 = \mathbb{E}_i x_i \) and \( \Sigma_0 = \mathbb{E}_i (x_i - \mu_0)(x_i - \mu_0)^T \) be the sample mean and covariance respectively. Let \( X_{ij} = \frac{1}{n} (x_i - x_j)(x_i - x_j)^T \), and let \( a_{ij} \in [0, 1] \) for \( i, j \in [n] \) and \( a_i \) for \( i \in [n] \) be such that

1. \( a_{ij} = a_{ji} \) for all \( i, j \),
2. \( \mathbb{E}_{ij} a_{ij} \geq 1 - 4\epsilon \), and
3. \( \mathbb{E}_i a_{ij} \geq a_i (1 - 2\epsilon) \) for all \( i \), and \( a_{ij} \leq a_i \) for all \( i \) and \( j \).

Then, with probability \( 1 - O(\delta) \), for all \( v \in \mathbb{R}^d \) and symmetric \( P \in \mathbb{R}^{d \times d} \), we have

1. \( |\langle \mu - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \sqrt{v^T \Sigma v} \),
We now prove Theorem 1, restated below.

**Theorem (Restatement of Theorem 1).** Algorithm 1 takes input an \( \epsilon \)-corrupted sample of size \( n \) from a Gaussian distribution with mean \( \mu \) and covariance \( \Sigma \) and in \( \text{poly}(n) \)-time, outputs estimates \( \hat{\mu} \in \mathbb{R}^d, \hat{\Sigma} \in \mathbb{R}^{d \times d} \) with the following guarantee. If \( \Sigma \geq 2^{-\text{poly}(d)} I \), and \( n \geq \tilde{O}(d^2 \log^5(1/\delta) / \epsilon^2) \), with probability at least \( 1 - \delta \) over the draw of the original uncorrupted sample \( X \), the estimates \( \hat{\mu}, \hat{\Sigma} \) satisfy:

1. (Mean estimation) \( \| \Sigma^{-1/2}(\hat{\mu} - \mu) \|_2 \leq \tilde{O}(\epsilon) \), and
2. (Covariance estimation in spectral norm) \( (1 - \tilde{O}(\epsilon)) \Sigma \leq \hat{\Sigma} \leq (1 + \tilde{O}(\epsilon)) \Sigma \).

For convenience, we shall assume without loss of generality that \( \epsilon n \) is an integer; this can be done by changing \( \epsilon \) by at most a constant factor.

In this section, we prove Theorems 1 and 2. First, we prove Theorem 1. Then, we prove Theorem 2 in Section 3.2. We combine Theorems 1 and 2 to prove Corollary 1.3 in Appendix A.

### 3.1 Analyzing the canonical SoS program: proof of Theorem 1

The final property we will need of Gaussians is certifiable subgaussianity, which says that certain moment inequalities have low-degree SoS proofs.

**Lemma 2.10** (Certifiable fourth moments of Gaussian samples, Section 5 in [KS17b]). Let \( \epsilon, \delta > 0 \), and \( n \geq \tilde{O}((d \log(1/\delta)) / \epsilon^2) \). Let \( x_1, \ldots, x_n \sim \mathcal{N}(0, \Sigma) \) be samples from a \( d \)-dimensional Gaussian. Then with probability \( 1 - \delta \),

\[
h(x, v) := (3 + \epsilon) \langle v, \Sigma v \rangle^2 - \mathbb{E}_{i \sim [n]} \langle x_i, v \rangle^4
\]

has a degree-4 SoS proof of nonnegativity in \( v \) (Definition 2.6).

### 3. Mean and Covariance Estimation of Gaussians via SoS

In this section, we prove Theorems 1 and 2. First, we prove Theorem 1. Then, we prove Theorem 2 in Section 3.2. We combine Theorems 1 and 2 to prove Corollary 1.3 in Appendix A.
The canonical degree-12 SoS relaxation of Algorithm 1 outputs a degree-12 pseudo-expectation \( \hat{E} \) in the variables \( x_1', \ldots, x_n' \in \mathbb{R}^d, w_1, \ldots, w_n \in \mathbb{R} \), satisfying the constraints of Algorithm 1 if such a \( \hat{E} \) exists. The estimates produced by the algorithm are \( \hat{\mu} := \hat{E}[\mu] \) and \( \hat{\Sigma} := \hat{E}[\Sigma] \).

Let \( x_1, \ldots, x_n \sim \mathcal{N}(\mu, \Sigma) \). Let \( \mu_0 = E_i x_i \) be the sample mean, and let \( \Sigma_0 = E_i (x_i - \mu_0)(x_i - \mu_0)^T \) be the sample covariance. Fix \( \epsilon \in (0, 1) \), and let \( y_1, \ldots, y_n \) be an \( \epsilon \)-corruption of \( x_1, \ldots, x_n \).

By Lemma 2.9, with probability \( 1 - \delta \), the following inequalities hold for any \( a_1, \ldots, a_n \in [0, 1] \) with \( \sum_i a_i \geq (1 - 2\epsilon)n \) and \( v \in \mathbb{R}^d \):

1. \[ |\langle \mu - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \sqrt{v^T \Sigma v} , \] (3.1)
2. \[ |E_i a_i \langle x_i - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \cdot \sqrt{v^T \Sigma_0 v} , \] (3.2)
3. \[ |E_i a_i [\langle x_i - \mu_0, v \rangle^2 - v^T \Sigma_0 v]| \leq \tilde{O}(\epsilon) \cdot v^T \Sigma_0 v . \] (3.3)

Next, we let \( X_{ij} := \frac{1}{2}(x_i - x_j)(x_i - x_j)^T \), for any \( i, j \in [n] \). Let \( T \subseteq [0, 1]^{n^2} \) denote the set of \((a_{ij})_{i,j=[n]}\) such that:

1. \( a_{ij} = a_{ji} \) for all \( i, j \),
2. \( \sum_{i,j=1}^n a_{ij} \geq (1 - 4\epsilon)n \), and
3. there exist \( a_1, \ldots, a_n \in [0, 1] \) such that \( a_i \geq E_j a_{ij} \geq a_i(1 - 2\epsilon) \) for all \( i \in [n] \).

By Lemma 2.9 (setting \( P = vv^T \)), with probability \( 1 - \delta \), the following inequalities hold for any \((a_{ij})_{i,j=[n]} \in T\):

1. \[ |v^T (\Sigma_0 - \Sigma)v| \leq \tilde{O}(\epsilon) v^T \Sigma_0 v \] (3.4)
2. \[ |E_i [(\langle x_i - \mu_0, v \rangle^2 - v^T \Sigma_0 v)^2 - 2(v^T \Sigma_0 v)^2]| \leq \tilde{O}(\epsilon) \cdot (v^T \Sigma_0 v)^2 \] (3.5)
3. \[ |E_{ij} a_{ij} [v^T (X_{ij} - \Sigma_0)v]| \leq \tilde{O}(\epsilon) v^T \Sigma_0 v \] (3.6)
4. \[ |E_{ij} a_{ij} [(v^T (X_{ij} - \Sigma_0)v)^2 - 2(v^T \Sigma_0 v)^2]| \leq \tilde{O}(\epsilon) (v^T \Sigma_0 v)^2 . \] (3.7)

We proceed with the rest of the proof, assuming that the above resilience conditions hold. From this point on, we will no longer need to use the randomness of the \( x_i \)’s.

### 3.1.1 Feasibility

Let us now argue that the constraints in Algorithm 1 are feasible. Set \( x_i' = x_i \) for each \( i \in [n] \), and let \( w_i = 1 \) if \( y_i = x_i \) and 0 otherwise. Constraints (1), (2), (3) of Algorithm 1 are clearly satisfied, so it remains to argue that constraint (4) is satisfied. By Eq. (3.5) (with \( a_i = 1 \) for all \( i \)) constraint (4) is satisfied. Hence, the constraints in Algorithm 1 are feasible. In particular, Algorithm 1 will output in \( \text{poly}(n) \) time a degree-12 pseudo-expectation \( \hat{E} \) in the variables \( x_1', \ldots, x_n', w_1, \ldots, w_n \), satisfying the constraints of Algorithm 1. From this point on, we shall think of the pseudo-expectation \( \hat{E} \) as fixed.

In light of the above, we summarize our notation in the box below.
Then, for every \( u \),

We now analyze the estimate \( v \).

Let \( v \), \( w \), the SoS variables for the mean/covariance

\[ w, \ldots, w_n, \text{ the SoS variables for the indicators } 1(x'_i = y_i) \]

\subsection{Guarantees for the mean}

We now finish the proof, assuming Lemma 3.1. We first observe that the hypotheses of Lemma 3.1 hold. Indeed, the two resilience conditions of Lemma 3.1 follow by Eqs. (3.2) and (3.3).

| Notation: |
|------------------|
| \( \mu, \Sigma \), the true mean/covariance of the Gaussian \( \mathcal{N}(\mu, \Sigma) \) |
| \( \mu_0, \Sigma_0 \), the sample mean/covariance of the true samples \( x_1, \ldots, x_n \) |
| \( \mu', \Sigma' \), the SoS variables for the mean/covariance |
| \( \hat{\mu} = \hat{E}[\mu'] \), \( \hat{\Sigma} = \hat{E}[\Sigma'] \), the estimates for the mean/covariance |
| \( y_1, \ldots, y_n \), the \( \epsilon \)-corruption of the true samples \( x_1, \ldots, x_n \) |
| \( x'_1, \ldots, x'_n \), the SoS variables for the samples |

\section{3.1.2 Guarantees for the mean}

We now analyze the estimate \( \hat{\mu} : = \hat{E}[\mu'] = \hat{E}[E_i x'_i] \) for the mean \( \mu \), where \( \hat{E} \) is a degree-12 pseudo-expectation satisfying the constraints in Algorithm 1. We need to show that \( \hat{\mu} \) satisfies

\[ |\langle \hat{\mu} - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \sqrt{v^\top \Sigma v}. \]

The key ingredient in the proof is the following lemma, which we prove in Section 4.

\textbf{Lemma 3.1.} Let \( x_1, \ldots, x_n \in \mathbb{R}^d \). Suppose that there is some \( \Sigma \in \mathbb{R}^{d \times d} \) such that for all \( v \in \mathbb{R}^d \) and \( a \in [0, 1]^d \) with \( \sum_{i=1}^n a_i \geq (1 - 2\epsilon)n \), we have

\[ |E a_i \langle x_i - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \sqrt{a_i v^\top \Sigma a_i v} \quad \text{and} \quad |E a_i [(x_i - \mu_0, v)^2 - v^\top \Sigma a_i v]| \leq \tilde{O}(\epsilon) v^\top \Sigma v. \]

Let \( y_1, \ldots, y_n \) be any \( \epsilon \)-corruption of \( x_1, \ldots, x_n \), let \( \hat{E} \) be a degree-6 pseudo-expectation in the variables \( x'_1, \ldots, x'_n \in \mathbb{R}^d \) and \( w_1, \ldots, w_n \in \mathbb{R} \). Let \( \mu' = E_i x'_i \). Suppose that

\begin{enumerate}
  \item \( \hat{E} \) satisfies \( w_i^2 = w_i \) for every \( i \in [n] \),
  \item \( \hat{E} \) satisfies \( \sum_{i=1}^n w_i = (1 - \epsilon)n \),
  \item \( \hat{E} \) satisfies \( w_i x'_i = w_i y_i \) for every \( i \in [n] \),
  \item \( \hat{E}[E_i (x'_i - \mu', v)^2] \leq v^\top \hat{\Sigma} v \) for every \( v \in \mathbb{R}^d \)
\end{enumerate}

Then, for every \( v \in \mathbb{R}^d \), it holds that:

\[ |\langle \hat{\mu} - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \sqrt{v^\top \Sigma_0 v} + \sqrt{O(\epsilon) \cdot v^\top (\hat{\Sigma} - \Sigma_0) v} + \tilde{O}(\epsilon^2) v^\top (\hat{\Sigma} + \Sigma_0) v. \]

We now finish the proof, assuming \textbf{Lemma 3.1} We first observe that the hypotheses of Lemma 3.1 hold. Indeed, the two resilience conditions of Lemma 3.1 follow by Eqs. (3.2) and (3.3). Second, \( \hat{E} \) is a degree-12 pseudo-expectation (and so is also degree-6) with the required properties:
We now analyze the estimate \( \hat{\Sigma} = \hat{E}[\Sigma'] \). As the hypotheses of Lemma 3.1 are satisfied, we thus conclude that
\[
|\langle \hat{\mu} - \mu_0, v \rangle| \leq \tilde{O}(\epsilon)\sqrt{v^\top \Sigma_0 v} + \sqrt{O(\epsilon) \cdot v^\top (\hat{\Sigma} - \Sigma_0) v + \tilde{O}(\epsilon^2) v^\top (\hat{\Sigma} + \Sigma_0) v} .
\tag{3.8}
\]

Suppose that the estimate for the covariance \( \hat{\Sigma} \) satisfies the desired conclusion, i.e., that \(|v^\top (\hat{\Sigma} - \Sigma) v| \leq \tilde{O}(\epsilon) v^\top \Sigma v\) for all \( v \in \mathbb{R}^d \) (we will prove this next). Then, Eqs. (3.4) and (3.8) imply that
\[
|\langle \hat{\mu} - \mu_0, v \rangle| \leq \tilde{O}(\epsilon)\sqrt{v^\top \Sigma v} .
\]

Finally, by Eq. (3.1) we conclude that
\[
|\langle \hat{\mu} - \mu, v \rangle| \leq \tilde{O}(\epsilon)\sqrt{v^\top \Sigma v} ,
\]
assuming that \( \hat{\Sigma} \) satisfies its desired property. By choosing \( v \) appropriately, this implies (1) in Theorem 1.

### 3.1.3 Spectral guarantees on the covariance

We now analyze the estimate \( \hat{\Sigma} := \hat{E}[\Sigma'] \) for the covariance, where \( \hat{E} \) is a degree-12 pseudo-expectation satisfying Algorithm 1. First, we observe that the polynomial \( \Sigma' := \mathbb{E}_i(x_i' - \mu) (x_i' - \mu)^\top \) is equal to \( \mathbb{E}_{ij} X_{ij} \), where \( X_{ij} := \frac{1}{2}(x_i' - x_j')(x_i' - x_j')^\top \), and similarly we also have \( \Sigma_0 = \mathbb{E}_{ij} X_{ij} \), where \( X_{ij} := \frac{1}{2}(x_i - x_j)(x_i - x_j)^\top \).

Let \( T \subseteq [0, 1]^{n^2} \) denote the set of \((a_{ij}), i, j \in [n]\) such that:

1. \( a_{ij} = a_{ji} \) for all \( i, j \),
2. \( \sum_{i,j=1}^n a_{ij} \geq (1 - 4\epsilon)n^2 \), and
3. there exist \( a_1, \ldots, a_n \in [0, 1] \) such that \( \mathbb{E}_j a_{ij} \geq a_i(1 - 2\epsilon) \) for all \( i, j \) and \( a_{ij} \leq a_i \) for all \( i \) and \( j \).

The key ingredient here is the following lemma, which is very similar to Lemma 3.1 that appeared in the case of mean estimation.

**Lemma 3.2.** Let \( X_1, \ldots, X_{n^2} \in \mathbb{R}^{d \times d} \) and let \( \Sigma_0 := \mathbb{E}_{ij} X_{ij} \). Let \( T \subseteq [0, 1]^{n^2} \). Suppose that, for all \( v \in \mathbb{R}^d \) and \( a \in T \) such that \( \sum_{i,j=1}^n a_{ij} \geq (1 - 4\epsilon)n^2 \), we have
\[
|\mathbb{E}_{ij} a_{ij} v^\top (X_{ij} - \Sigma_0) v| \leq \tilde{O}(\epsilon) \cdot v^\top \Sigma_0 v \quad \text{and} \quad |\mathbb{E}_{ij} a_{ij} [(v^\top (X_{ij} - \Sigma_0) v)^2 - 2(v^\top \Sigma_0 v)^2]| \leq \tilde{O}(\epsilon) (v^\top \Sigma_0 v)^2 .
\]

Let \( Y_1, \ldots, Y_{n^2} \) be any \((2\epsilon - \epsilon^2)\)-corruption of \( X_1, \ldots, X_{n^2} \), let \( \hat{E} \) be a degree-6 pseudo-expectation in the variables \( X_1', \ldots, X_{n^2}' \in \mathbb{R}^{d \times d} \) and \( w_1, \ldots, w_{n^2} \in \mathbb{R} \). Let \( \Sigma' = \mathbb{E}_{ij} X_{ij}' \). Suppose that

1. \( \hat{E} \) satisfies \( w_{ij}^2 = w_{ij} \) for every \( i, j \in [n] \),
2. \( \hat{E} \) satisfies \( \sum_{i,j=1}^n w_{ij} \geq (1 - \epsilon)^2 n^2 \),
(3) $\mathbf{E}$ satisfies $w_{ij}X'_{ij} = w_{ij}Y_{ij}$ for every $i, j \in [n]$.

(4) $\mathbf{E}[\mathbf{E}_{ij}(v^T(X'_{ij} - \Sigma)v)^2] \leq (2 + O(\epsilon)) \cdot \mathbf{E}[(v^T \Sigma'v)^2]$ for every $v \in \mathbb{R}^d$, and

(5) $a \in T$, where $a$ is the vector with $a_{ij} := \mathbf{E}[w_{ij}]\mathbf{1}(X_{ij} = Y_{ij})$ for each $i, j \in [n]$.

Then, for every $v \in \mathbb{R}^d$, the following hold:

$$\mathbf{E}(v^T(\Sigma' - \Sigma_0)v)^2 \leq O(\epsilon)(\mathbf{E}(v^T \Sigma'v)^2 + (v^T \Sigma_0v)^2) ,$$

$$|\langle \hat{\Sigma} - \Sigma_0, v \rangle| \leq \tilde{O}(\epsilon)v^T \Sigma_0v + \sqrt{\mathbf{E}[(1 - w'_{ij}) \cdot v^T(X'_{ij} - \Sigma_0)v]^2} ,$$

where $w'_{ij} := w_{ij}\mathbf{1}(X_{ij} = Y_{ij})$, $\hat{\Sigma} := \mathbf{E}[\Sigma']$, and

$$\mathbf{E}[\mathbf{E}_{ij}(1 - w'_{ij}) \cdot v^T(X'_{ij} - \Sigma_0)v]^2] \leq O(\epsilon) \cdot (\mathbf{E}(v^T \Sigma'v)^2 - (v^T \Sigma_0v)^2) + \tilde{O}(\epsilon) \cdot (\mathbf{E}(v^T \Sigma'v)^2 + (v^T \Sigma_0v)^2) .$$

As before, we postpone the proof of Lemma 3.2 to Section 4, and use it to finish the proof.

We apply Lemma 3.2 as follows. First, we note that $\Sigma_0$ defined in Lemma 3.2 is the same as the sample mean $\Sigma_0$. Next, let $T$ be the subset of vectors $(a_{ij})_{i, j \in [n]}$ defined earlier. We see that Eqs. (3.6) and (3.7) imply that the $X_{ij}$'s defined satisfy the resilience conditions in Lemma 3.2.

Now, we let $Y_{ij} = \frac{1}{2}(y_i - y_j)(y_i - y_j)^T$, and let $X'_{ij} = \frac{1}{2}(x'_i - x'_j)(x'_i - x'_j)^T$. We note that $\Sigma'$ as defined in Lemma 3.2 is the same polynomial as $\Sigma'$ defined earlier. We observe that the $Y_{ij}$'s must be a $(2\epsilon - \epsilon^2)$-corruption of the $X_{ij}$'s. Moreover, if we let $w_{ij} := w_i w_j$, then the pseudo-expectation defined by $\mathbf{E}$ on the polynomials $X'_{ij}$ and $w_{ij}$ is a degree-6 pseudo-expectation, and additionally satisfies properties (1) – (3). To see that (4) holds, we observe the following polynomial equality:

$$\mathbf{E}(v^T(X'_{ij} - \Sigma)v)^2 = \frac{1}{2}(\mathbf{E}(x'_i - \mu', v)^4 + (v^T \Sigma'v)^2) .$$

Combining with constraint (4) in Algorithm 1 and taking pseudo-expectations shows that property (4) holds.

Finally, property (5) in Lemma 3.2 holds, as $(a_{ij})_{i, j \in [n]} \in T$ because it satisfies the required properties with respect to the vector $a_i = \mathbf{E}[w_i]\mathbf{1}(x_i = y_i)$ for each $i$.

Thus, by Lemma 3.2, we have

$$|\langle \hat{\Sigma} - \Sigma_0, v v^T \rangle| \leq \tilde{O}(\epsilon) \cdot v^T \Sigma_0v + \sqrt{R} ,$$

where

$$R := \mathbf{E}[\mathbf{E}_{ij}(1 - w'_{ij}) \cdot v^T(X_{ij} - \Sigma_0)v]^2] \leq O(\epsilon) \cdot (\mathbf{E}[(v^T \Sigma'v)^2 - (v^T \Sigma_0v)^2] + \tilde{O}(\epsilon^2) \cdot (\mathbf{E}[(v^T \Sigma'v)^2] + (v^T \Sigma_0v)^2) .$$

Write $\Sigma' = A + B$, where $B = \mathbf{E}_{ij}(1 - w'_{ij})X_{ij}$ and $A = \mathbf{E}_{ij}w'_{ij}X_{ij} = \mathbf{E}_{ij}w_i X_{ij}$, the latter equality holds because the following polynomial equalities are satisfied by $\mathbf{E}$:

$$w'_{ij}X_{ij} = w_i w_j \mathbf{1}(x_i = y_i) \mathbf{1}(x_j = y_j) \cdot \frac{1}{2}(x'_i - x'_j)(x'_i - x'_j)^T$$
Additionally, let \( A_v := v^T A v \) and \( B_v := v^T B v \), and \( \Sigma_v = v^T \Sigma_0 v \). We have that
\[
\bar{E}[A_v^2] = \bar{E}[(E \omega_{ij} v^T X_{ij} v)^2] = E \ E \bar{E}[w_{i_j} w_{i_j}'] v \cdot v^T X_{i_j} v \cdot v^T X_{i_j} v \\
\leq E \ E \sqrt{\bar{E}[w_{i_j}'] \bar{E}[w_{i_j}']} \cdot v^T X_{i_j} v \cdot v^T X_{i_j} v \quad \text{(as \( \bar{E}[w_{i_j}^2] = \bar{E}[w_{i_j}] \))} \\
= \bar{E}[E_i \sqrt{\bar{E}[w_{i_j}] v^T X_{i_j} v]^2} \leq (1 + \tilde{O}(\epsilon)) \Sigma_v^2 \quad \text{(by Eq. (3.6) applied to \( a_{i_j} = \sqrt{\bar{E}[w_{i_j}]} \))}
\]

We now bound \( R \). In this notation, we have
\[
R = \bar{E}[(B_v - E_i[1 - w_{i_j}']) \cdot \Sigma_v)^2] \leq O(\epsilon) \cdot (\bar{E}[(A_v + B_v)^2] - \Sigma_v^2) + \tilde{O}(\epsilon^2) \cdot (\bar{E}[(A_v + B_v)^2] + \Sigma_v^2) \quad \text{(3.9)}
\]

First, we have
\[
\bar{E}[(B_v - E_i[1 - w_{i_j}']) \cdot \Sigma_v)^2] = \bar{E}[B_v^2 + E_i[1 - w_{i_j}']^2 \cdot \Sigma_v^2 - 2B_v E_i[1 - w_{i_j}] \cdot \Sigma_v] \geq \bar{E}[B_v^2] - 4\epsilon \Sigma_v \bar{E}[B_v] \quad \text{(3.10)}
\]

as \( \Sigma_v \gg 0 \) and \( \bar{E} \) satisfies \( B_v \gg 0 \) because \( B_v \) is a sum-of-squares polynomial. As \( \bar{E}[A_v^2] \leq (1 + \tilde{O}(\epsilon)) \Sigma_v^2 \) and \( \bar{E}[A_v B_v]^2 \leq \bar{E}[A_v^2] \bar{E}[B_v^2] \) (by Proposition 2.5), it follows that
\[
\bar{E}[(A_v + B_v)^2] \leq \bar{E}[B_v^2] + 2 \sqrt{\bar{E}[A_v^2] \bar{E}[B_v^2]} + (1 + \tilde{O}(\epsilon)) \Sigma_v^2 \leq \bar{E}[B_v^2] + 2 \Sigma_v \sqrt{\bar{E}[B_v^2]} + (1 + \tilde{O}(\epsilon)) \Sigma_v^2 \quad \text{(3.11)}
\]

Combining Eqs. (3.9) to (3.11) thus yields
\[
\bar{E}[B_v^2] - 4\epsilon \Sigma_v \bar{E}[B_v] \leq R \leq O(\epsilon) \bar{E}[B_v^2] + 2 \Sigma_v \sqrt{\bar{E}[B_v^2]} + \tilde{O}(\epsilon) \Sigma_v^2 \quad \text{(3.11)}
\]

Rearranging, applying \( \bar{E}[B_v] \leq \sqrt{\bar{E}[B_v^2]} \), and solving for \( \bar{E}[B_v^2] \) yields
\[
\bar{E}[B_v^2] \leq \tilde{O}(\epsilon^2) \cdot \Sigma_v^2 \\
\implies R \leq O(\epsilon) \bar{E}[B_v^2] + 2 \Sigma_v \sqrt{\tilde{O}(\epsilon^2) \Sigma_v^2} + \tilde{O}(\epsilon) \Sigma_v^2 = \tilde{O}(\epsilon^2) \Sigma_v^2 \\
\implies |v^T (\hat{\Sigma} - \Sigma_0) v| \leq \tilde{O}(\epsilon) v^T \Sigma_0 v + \sqrt{R} = \tilde{O}(\epsilon) \cdot v^T \Sigma_0 v + \sqrt{R}.
\]

This is the desired spectral norm guarantee, only with \( \Sigma_0 \) in place of \( \Sigma \). Using Eq. (3.4) and the triangle inequality, we have \( |v^T (\hat{\Sigma} - \Sigma) v| \leq \tilde{O}(\epsilon) v^T \Sigma_0 v \), and so we thus have the desired spectral norm guarantee. This finishes the proof, as we have shown that \( \hat{\Sigma} \) satisfies its desired property, assuming that \( \hat{\Sigma} \) has this property.
3.2 Relative Frobenius guarantees on the covariance: proof of Theorem 2

We now prove Theorem 2, restated below.

**Theorem (Restatement of Theorem 2).** [Algorithm 2 takes input an $\varepsilon$-corrupted sample of size $n$ from a Gaussian distribution with mean $\mu$ and covariance $\Sigma$ with $\Sigma \leq (1 - \tilde{O}(\varepsilon))I$ and in poly(n)-time, outputs an estimate $\hat{\Sigma} \in \mathbb{R}^{d \times d}$ with the following guarantee. If $n \geq \tilde{O}(d^2 \log^3(1/\delta)/\varepsilon^2)$, then with probability at least $1 - \delta$ over the draw of the original uncorrupted sample $X$, the estimate $\hat{\Sigma}$ satisfies $\|\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I\|_F \leq \tilde{O}(\varepsilon)$.

Let $x_1, \ldots, x_n \sim \mathcal{N}(\mu, \Sigma)$, where $(1 - \tilde{O}(\varepsilon))I \preceq \Sigma \preceq (1 + \tilde{O}(\varepsilon))I$. Fix $\varepsilon \in (0, 1)$, and let $y_1, \ldots, y_n$ be an $\varepsilon$-corruption of $x_1, \ldots, x_n$. Let $\mu_0 = \mathbb{E}_i x_i$ be the sample mean, and let $\Sigma_0 = \mathbb{E}_i(x_i - \mu_0)(x_i - \mu_0)^T$ be the sample covariance.

We observe that for every symmetric $P \in \mathbb{R}^{d \times d}$, it holds that

$$\|P\|_F - \|\Sigma^{1/2} P \Sigma^{1/2}\|_F \preceq \tilde{O}(\varepsilon) \min\{\|P\|_F, \|\Sigma^{1/2} P \Sigma^{1/2}\|_F\},$$

as $(1 - \tilde{O}(\varepsilon))I \preceq \Sigma \preceq (1 + \tilde{O}(\varepsilon))I$, using the standard inequality $\|AB\|_F \leq \|A\|_2\|B\|_F$.

For each $i, j \in [n]$, let $X_{ij} := \frac{1}{2}(x_i - x_j)(x_i - x_j)^T$. Let $T \subseteq [0, 1]^{n^2}$ denote the set of $(a_{ij})_{i,j \in [n]}$ such that:

1. $a_{ij} = a_{ji}$ for all $i, j$,
2. $\sum_{i,j=1}^n a_{ij} \geq (1 - 4\varepsilon)n$, and
3. there exist $a_1, \ldots, a_n \in [0, 1]$ such that $\mathbb{E}_i a_{ij} \geq a_i(1 - 2\varepsilon)$ for all $i$, and $a_{ij} \leq a_i$ for all $i$ and $j$.

By Lemma 2.9, with probability $1 - \delta$ the following hold for any $(a_{ij}) \in T$ and symmetric $P \in \mathbb{R}^{d \times d}$:

$$\left|\langle \Sigma_0 - \Sigma, P \rangle\right| \preceq \tilde{O}(\varepsilon) \|\Sigma^{1/2} P \Sigma^{1/2}\|_F,$$

$$\left|\mathbb{E}_i \left[\langle x_i - \mu_0, (x_i - \mu_0)^T - \Sigma_0, P \rangle^2 - 2\|\Sigma^{1/2} P \Sigma^{1/2}\|_F^2\right]\right| \leq \tilde{O}(\varepsilon) \cdot \|\Sigma^{1/2} P \Sigma^{1/2}\|_F^2,$$

$$\left|\mathbb{E}_{ij} a_{ij} \left[\langle X_{ij}, P \rangle - \langle \Sigma_0, P \rangle\right]\right| \leq \tilde{O}(\varepsilon) \cdot \|\Sigma^{1/2} P \Sigma^{1/2}\|_F,$$

$$\left|\mathbb{E}_{ij} a_{ij} \left[\langle X_{ij} - \Sigma_0, P \rangle^2 - 2\|\Sigma^{1/2} P \Sigma^{1/2}\|_F^2\right]\right| \leq \tilde{O}(\varepsilon) \cdot \|\Sigma^{1/2} P \Sigma^{1/2}\|_F^2.$$

From the above, feasibility is simple: set $x'_i = x_i$ for all $i$, $w_i = \mathbb{I}(x_i = y_i)$, and observe that now $\mu' = \mu_0$, $\Sigma' = \Sigma_0$, and constraint (4) in Algorithm 2 is satisfied by Eq. (3.14) as $\|\Sigma^{1/2} P \Sigma^{1/2}\|_F \leq (1 + \tilde{O}(\varepsilon))\|P\|_F$. Thus, Algorithm 2 will output in poly(n) time a degree-12 pseudo-expectation $\hat{\Sigma}$ satisfying the constraints in Algorithm 2.

**Covariance estimation in Frobenius norm.** We now analyze the output $\hat{\Sigma} := \mathbb{E}[\Sigma']$ of the algorithm. We observe that $\mathbb{E}_{ij} X_{ij}$ is equal to the sample covariance $\Sigma_0$. Let $Y_{ij} := \frac{1}{2}(y_i - y_j)(y_i - y_j)^T$, and let $X'_{ij} := \frac{1}{2}(x'_i - x'_j)(x'_i - x'_j)^T$. Similarly, we have that the SoS variable $\Sigma' := \mathbb{E}_i(x'_i - \mu')(x'_i - \mu')^T$ is equal to $\mathbb{E}_{ij} X'_{ij}$.

The key ingredient in the proof is the following technical lemma, which we prove in Section 4. This lemma is similar to Lemmas 3.1 and 3.2.

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Lemma 3.3. Let \( X_1, \ldots, X_{n^2} \in \mathbb{R}^{d \times d} \), and let \( \Sigma_0 := \mathbf{E}_{ij} X_{ij} \). Let \( T \subseteq [0,1]^{n^2} \). Suppose that, for all symmetric \( P \in \mathbb{R}^{d \times d} \) and \( a \in T \), we have

\[
|\mathbf{E}a_{ij}(X_{ij} - \Sigma_0, P)| \leq \tilde{O}(\varepsilon) \cdot \|P\|_F \quad \text{and} \quad |\mathbf{E} a_{ij}[((X_{ij} - \Sigma_0, P)^2 - 2\|P\|_F^2]| \leq \tilde{O}(\varepsilon)\|P\|_F^2 .
\]

Let \( Y_1, \ldots, Y_{n^2} \) be any \((2\varepsilon^2)-\text{corruption of } X_1, \ldots, X_{n^2}\), and let \( \tilde{E} \) be a degree-6 pseudo-expectation in the variables \( X_1', \ldots, X_{n^2}' \in \mathbb{R}^{d \times d} \) and \( w_1, \ldots, w_{n^2} \in \mathbb{R} \). Let \( \Sigma' = \mathbf{E}_{ij} X_{ij}' \). Suppose that

1. \( \tilde{E} \) satisfies \( w_{ij}^2 = w_{ij} \) for every \( i, j \in [n] \),
2. \( \tilde{E} \) satisfies \( \sum_{i,j=1}^n w_{ij} = (1 - \varepsilon)^2 n^2 \),
3. \( \tilde{E} \) satisfies \( w_{ij} X_{ij}' = w_{ij} Y_{ij} \) for every \( i, j \in [n] \),
4. \( \tilde{E} ([\mathbf{E}_{ij}(X_{ij}' - \Sigma', v)^2] \leq (2 + \tilde{O}(\varepsilon))\|P\|_F^2 \) for every symmetric \( P \in \mathbb{R}^{d \times d} \), and
5. \( a \in T \), where \( a \) is the vector with \( a_{ij} := \tilde{E}[w_{ij}] 1(X_{ij} = Y_{ij}) \) for each \( i, j \in [n] \).

Then, for every symmetric \( P \in \mathbb{R}^{d \times d} \), it holds that

\[
|\langle \hat{\Sigma} - \Sigma_0, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F,
\]

where \( \hat{\Sigma} = \tilde{E}[\Sigma'] \).

We now apply Lemma 3.3. We observe that by Eqs. (3.15) and (3.16), and using Eq. (3.12), the resilience condition in Lemma 3.3 is satisfied by the \( X_{ij}'s \). We also observe that the pseudo-expectation \( \tilde{E} \), in the variables \( X_{ij}' \) and \( w_{ij} \) with \( w_{ij} := w_i w_j \), is a degree-6 pseudo-expectation, and trivially satisfies properties (1) – (3). Property (4) follows as \( \tilde{E} \) satisfies constraint (4) in Algorithm 2 and the following polynomial equality holds:

\[
\mathbf{E}_{ij}(X_{ij}' - \Sigma', v)^2 = \frac{1}{2} \mathbf{E}_i((x_i' - \mu')(x_i' - \mu')^T, P)^2 + \frac{1}{2} \langle \Sigma', v \rangle^2 .
\]

Property (5) follows by using the vector \( a \) with \( a_{i} = \tilde{E}[w_i] 1(x_i = y_i) \) to show membership of \((a_{ij})_{i,j \in [n]} \) in \( T \).

We thus have by Lemma 3.3 that \(|\langle \hat{\Sigma} - \Sigma_0, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F \) for all symmetric \( P \in \mathbb{R}^{d \times d} \). Using Eq. (3.13), it follows that \(|\langle \hat{\Sigma} - \Sigma, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F \). Hence,

\[
|\langle \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I, P \rangle| = |\langle \hat{\Sigma} - \Sigma, \Sigma^{-1/2} P \Sigma^{-1/2} \rangle| \leq \tilde{O}(\varepsilon)\|\Sigma^{-1/2} P \Sigma^{-1/2}\|_F \leq \tilde{O}(\varepsilon)\|P\|_F .
\]

Setting \( P = \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I \), we conclude that

\[
\|\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I\|_F \leq \tilde{O}(\varepsilon),
\]

as required.
4 A Generic Estimation Lemma

Lemmas 3.1 to 3.3 are special cases of a generic technical result, which we now state and prove.

**Lemma 4.1.** Let $x_1, \ldots, x_n \in \mathbb{R}^d$, and let $\mu_0 := \mathbb{E}_i x_i$. Let $V(\mu_0, v)$ for $v \in \mathbb{R}^d$ be a degree-2 polynomial in $\mu_0$, and let $S \subseteq \mathbb{R}^d$ be a set such that $V(\mu_0, v) \geq 0$ for all $v \in S$ and $\mu_0 \in \mathbb{R}^d$.

Let $T \subseteq [0,1]^n$. Suppose that, for all $v \in \mathbb{R}^d$ and $a \in T$ such that $\sum_i a_i \geq (1 - \epsilon)n$, we have

$$|\mathbb{E}_i a_i (x_i - \mu_0, v)| \leq \tilde{O}(\epsilon) \cdot \sqrt{V(\mu_0, v)} \quad \text{and} \quad |\mathbb{E}_i a_i [(x_i - \mu_0, v)^2 - V(\mu_0, v)]| \leq \tilde{O}(\epsilon) V(\mu_0, v) \quad (4.1)$$

Let $y_1, \ldots, y_n$ be any $\epsilon$-corruption of $x_1, \ldots, x_n$, let $\tilde{\mathbb{E}}$ be a degree-6 pseudo-expectation in the variables $x'_1, \ldots, x'_n \in \mathbb{R}^d$ and $w_1, \ldots, w_n \in \mathbb{R}$. Let $\mu' = \mathbb{E}_i x'_i$. Suppose that

1. $\tilde{\mathbb{E}}$ satisfies $w_i^2 = w_i$ for every $i \in [n]$,
2. $\tilde{\mathbb{E}}$ satisfies $\sum_{i=1}^n w_i = (1 - \epsilon)n$,
3. $\tilde{\mathbb{E}}$ satisfies $w_i x'_i = w_i y_i$ for every $i \in [n]$,
4. $\tilde{\mathbb{E}}[\mathbb{E}_i (x'_i - \mu', v)^2] \leq (1 + \tilde{O}(\epsilon)) \tilde{\mathbb{E}}[V(\mu', v)]$ for every $v \in S$, and
5. $a \in T$, where $a$ is the vector with $a_i := \tilde{\mathbb{E}}[w_i] 1(x_i = y_i)$ for each $i \in [n]$.

Then, for every $v \in S$, the following hold:

$$\tilde{\mathbb{E}}(\mu' - \mu_0, v)^2 \leq O(\epsilon)(\tilde{\mathbb{E}} V(\mu', v) + V(\mu_0, v)) \quad ,$$

$$|\langle \hat{\mu} - \mu_0, v \rangle| \leq \tilde{O}(\epsilon) \sqrt{V(\mu_0, v)} + \sqrt{\tilde{\mathbb{E}}[\mathbb{E}_i ((1 - w'_i)(x'_i - \mu_0, v))^2]} \quad ,$$

where $\hat{\mu} := \tilde{\mathbb{E}}[\mu']$, and

$$\tilde{\mathbb{E}}[\mathbb{E}_i ((1 - w'_i)(x'_i - \mu_0, v))^2] \leq O(\epsilon)(\tilde{\mathbb{E}} V(\mu', v) - V(\mu_0, v)) + \tilde{O}(\epsilon^2) \cdot (\tilde{\mathbb{E}} V(\mu', v) + V(\mu_0, v)) \quad .$$

One should think of $V$ as the variance of the distribution from which the $x_i$'s are drawn, along the direction $v \in \mathbb{R}^d$. We now turn to the proof of Lemma 4.1.

**Proof of Lemma 4.1.** For each $i \in [n]$, let $w'_i = w_i \cdot 1(x_i = y_i)$. One should think of $w_i$ as indicating that the algorithm “thinks” that $x_i = y_i$; the variable $w'_i$ then indicates that the algorithm correctly “thinks” that $x_i = y_i$.

We now notice that the constraints $w_i^2 = w'_i, w'_i x'_i = w'_i x_i$, and $\sum_i w'_i \geq (1 - 2\epsilon)n$ are all satisfied by $\tilde{\mathbb{E}}$. Indeed, e.g., the fact that $w'_i x'_i = w'_i x_i$ is satisfied is consistent with the logic that if the algorithm thinks that $x_i = y_i$, then it chooses $x'_i = y_i = x_i$, and therefore $x'_i = x_i$.

We then have

$$|\langle \hat{\mu} - \mu_0, v \rangle| = |\tilde{\mathbb{E}} \mathbb{E}_i (x'_i - x_i, v)|$$

$$= |\tilde{\mathbb{E}} \mathbb{E}_i w'_i (x'_i - x_i, v) + \tilde{\mathbb{E}} \mathbb{E}_i (1 - w'_i)(x'_i - x_i, v)|$$
we have that

\[ \bar{\text{E}} \mathbf{E}(1 - w_j')\langle x'_i - \mu_0 + \mu_0 - x_i, v \rangle \] (because \( \bar{\text{E}} \) satisfies \( w'_i x'_i = w'_i x_i \))

\[ \leq |\bar{\text{E}} \mathbf{E}(1 - w'_j)\langle x'_i - \mu_0, v \rangle| + |\mathbf{E}_i \bar{\text{E}}[1 - w'_j](x_i - \mu_0, v)| . \]

One should notice that in the calculation above, we split the estimation error into the term when the algorithm “thinks” correctly and the error term, and then we “center” the error term about the sample mean \( \mu_0 \).

We now apply the robustness assumption (4.1) to the second error term. Let \( a_i = \bar{\text{E}}[w'_j] \) for each \( i \). We have that \( \sum_{i=1}^n a_i \geq (1 - 2\varepsilon)n \) and \( a_i \in [0, 1] \) because \( \bar{\text{E}}[w'^2] = \bar{\text{E}}[w'_j] \), and \( a \in T \) by assumption. Hence, again by assumption, we have that

\[ |\mathbf{E}_i \bar{\text{E}}[w'_j](x_i - \mu_0, v)| \leq \bar{O}(\varepsilon) \sqrt{V(\mu_0, v)} \]

and

\[ |\mathbf{E}_i \bar{\text{E}}[w'_j](x_i - \mu_0, v)| \leq \bar{O}(\varepsilon) \sqrt{V(\mu_0, v)} . \]

Thus, \( |\mathbf{E}_i \bar{\text{E}}[1 - w'_j](x_i - \mu_0, v)| \leq \bar{O}(\varepsilon) \sqrt{V(\mu_0, v)} \). We note here that the robustness assumption we apply is an inequality that holds “outside” the pseudo-expectation \( \bar{\text{E}} \).

It thus remains to bound the first error term: \( |\bar{\text{E}} \mathbf{E}_i(1 - w'_j)(x'_i - \mu_0, v)| \). We do this by using constraint (4) to control its second moments.

First, by applying the Cauchy-Schwarz inequality, we have

\[ |\bar{\text{E}} \mathbf{E}_i(1 - w'_j)(x'_i - \mu_0, v)| \leq \sqrt{\bar{\text{E}}[\mathbf{E}_i((1 - w'_j)(x'_i - \mu_0, v))^2]} , \]

and that

\[ \bar{\text{E}}[\mathbf{E}_i((1 - w'_j)(x'_i - \mu_0, v))^2] \]

\[ \leq \mathbf{E}[\mathbf{E}_i((1 - w'_j)^2 \cdot (1 - w'_j)(x'_i - \mu_0, v)^2)] \quad \text{(by Item (2) in Proposition 2.5)} \]

\[ = \mathbf{E}[\mathbf{E}_i[1 - w'_j] \cdot (1 - w'_j)(x'_i - \mu_0, v)^2]] \quad \text{(as \( \bar{\text{E}} \) satisfies \( w'^2 = w'_j \))} \]

\[ \leq 2\varepsilon \cdot \bar{\text{E}} \mathbf{E}_i((1 - w'_j)(x'_i - \mu_0, v)^2] \quad \text{(as \( \bar{\text{E}} \) satisfies \( i[1 - w'_j] \leq 2\varepsilon \))} \]

Note that here we crucially need that \( \bar{\text{E}} \) is a degree-6 pseudo-expectation, as \( \mathbf{E}_i[(1 - w'_j)^2 \cdot \mathbf{E}_i[(1 - w'_j)(x'_i - \mu_0, v)^2] \) is a degree-6 polynomial in the SoS variables \( x'_i, \ldots, x'_n \) and \( w_1, \ldots, w_n \).

We thus need to control the second moments \( \mathbf{E}_i[(1 - w'_j)(x'_i - \mu_0, v)^2] \). Using constraint (4), we have that

\[ \bar{\text{E}} \mathbf{E}_i[(1 - w'_j)(x'_i - \mu_0, v)^2] \]

\[ = \bar{\text{E}} \mathbf{E}_i(x'_i - \mu_0, v)^2 - \bar{\text{E}} \mathbf{E}_i w'_i(x'_i - \mu_0, v)^2 \]

\[ = \bar{\text{E}} \mathbf{E}_i(x'_i - \mu_0, v)^2 - \bar{\text{E}} \mathbf{E}_i w'_i(x_i - \mu_0, v)^2 \quad \text{(as \( \bar{\text{E}} \) satisfies \( w'_i x'_i = w'_i x_i \))} \]

\[ \leq \bar{\text{E}} \mathbf{E}_i(x'_i - \mu + \mu - \mu_0, v)^2 - (1 - \bar{O}(\varepsilon))V(\mu_0, v) \] (by Eq. (4.1), setting \( a_i = \bar{\text{E}}[w'_i] \))

\[ = \bar{\text{E}} \mathbf{E}_i(x'_i - \mu)^2 + \bar{\text{E}}(\mu - \mu_0, v)^2 + \bar{\text{E}} \mathbf{E}_i(x'_i - \mu, v)\langle \mu' - \mu_0, v \rangle - (1 - \bar{O}(\varepsilon))V(\mu_0, v) \]
\[
\begin{align*}
\mathbb{E}[(1 + \hat{O}(\varepsilon))V(\mu', v)] + \mathbb{E}(\mu' - \mu_0, v)^2 + O(1 - \hat{O}(\varepsilon))V(\mu_0, v) & \quad \text{(by constraint (4))} \\
= \mathbb{E}V(\mu', v) - V(\mu_0, v) + \hat{O}(\varepsilon)(\mathbb{E}V(\mu', v) + V(\mu_0, v)) + \mathbb{E}(\mu' - \mu_0, v)^2 .
\end{align*}
\]

Again, we remark that Eq. (4.1) used above to lower bound the second moment, is an inequality that holds “outside” \( \hat{\mathbb{E}} \).

Finally, we upper bound \( \mathbb{E}(\mu' - \mu_0, v)^2 \). We compute

\[
\begin{align*}
\mathbb{E}(\mu' - \mu_0, v)^2 & \\
= \mathbb{E}\left[\mathbb{E}_i[(w'_i + (1 - w'_i))(x'_i - x_i, v)]\right] \\
= \mathbb{E}\left[\mathbb{E}_i[(1 - w'_i)(x'_i - x_i, v)]\right] & \quad \text{(as \( \hat{\mathbb{E}} \) satisfies \( w'_i x'_i = w'_i x_i \))} \\
\leq \mathbb{E}\left[\mathbb{E}_i[(1 - w'_i)^2] \cdot \mathbb{E}_i[(x'_i - x_i, v)]\right] & \leq 2\mathbb{E}_i \mathbb{E}(x'_i - x_i, v)^2 \\
= 2\mathbb{E}_i \mathbb{E}(x'_i - \mu') + (\mu' - \mu_0) + (\mu_0 - x_i, v)^2 \\
\leq 6\mathbb{E}_i \mathbb{E}(x'_i - \mu', v)^2 + 6\mathbb{E}_i (x_i - \mu_0, v)^2 + 6\mathbb{E}_i (\mu' - \mu_0, v)^2 & \quad \text{(by Proposition 2.5)} \\
\leq 6\mathbb{E}_i (1 + \hat{O}(\varepsilon))(\mathbb{E} V(\mu', v) + V(\mu_0, v)) + 6\mathbb{E}_i (\mu' - \mu_0, v)^2 ,
\end{align*}
\]

and so it follows that \( \mathbb{E}(\mu' - \mu_0, v)^2 \leq O(\varepsilon)(\mathbb{E} V(\mu', v) + V(\mu_0, v)) \).

Putting everything together, we conclude that:

\[
\mathbb{E}_i[(1 - w'_i)(x'_i - \mu_0, v)] \leq O(\varepsilon) \cdot (\mathbb{E}_i V(\mu', v) - V(\mu_0, v)) + \hat{O}(\varepsilon^2) \cdot (\mathbb{E}_i V(\mu', v) + V(\mu_0, v))
\]

and that

\[
|\langle \hat{\mu} - \mu_0, v \rangle| \leq \hat{O}(\varepsilon)\sqrt{V(\mu_0, v)} + \sqrt{\mathbb{E}_i[(1 - w'_i)(x'_i - \mu_0, v)]} ,
\]

for every \( v \) in \( S \). □

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We prove Corollary 1.3. Let $x_1, \ldots, x_{2n}$ be drawn from $\mathcal{N}(\mu, \Sigma)$, and let $y_1, \ldots, y_{2n}$ be an $\varepsilon$-corruption of $x_1, \ldots, x_{2n}$. Note that $y_1, \ldots, y_n$ is an $\varepsilon$-corruption of $x_1, \ldots, x_n$, and $y_{n+1}, \ldots, y_{2n}$ is an $\varepsilon$-corruption of $x_{n+1}, \ldots, x_{2n}$.

First, we run the algorithm in Theorem 1 on $y_1, \ldots, y_n$: this yields an estimate $\hat{\mu}$ satisfying Item (1) of Corollary 1.3 and an estimate $\hat{\Sigma}_1$ of $\Sigma$ satisfying $|v^\top (\Sigma_1 - \Sigma)v| \leq \tilde{O}(\varepsilon)\|v\|^2$ for all $v \in \mathbb{R}^d$.

Next, we run the algorithm in Theorem 2 on the transformed samples $\Sigma_1^{-1/2}y_{n+1}, \ldots, \Sigma_1^{-1/2}y_{2n}$. We observe that these samples are an $\varepsilon$-corruption of $\Sigma_1^{-1/2}x_{n+1}, \ldots, \Sigma_1^{-1/2}x_{2n}$, which are drawn from $\mathcal{N}(\mu, \Sigma_2)$, where $\Sigma_2 = \Sigma^{-1/2}\Sigma\Sigma^{-1/2}$. By our guarantee on $\Sigma_1$, we must have $|1 - \tilde{O}(\varepsilon)|I \leq \Sigma_2 \leq (1 + \tilde{O}(\varepsilon))I$. Hence, the output of the algorithm in Theorem 2 is $\hat{\Sigma}_3$ where $\hat{\Sigma}_3$ satisfies $\|\Sigma_2^{-1/2}\Sigma_3\Sigma_2^{-1/2} - I\|_F \leq \tilde{O}(\varepsilon)$.

Our final estimate for $\Sigma$ is $\hat{\Sigma} := \Sigma_1^{1/2}\Sigma_3\Sigma_1^{-1/2}$. We have that

$$\|\Sigma_2^{-1/2}\Sigma_3\Sigma_2^{-1/2} - I\|_F = \|\Sigma_2^{-1/2}(\Sigma_1^{1/2}\Sigma_3\Sigma_1^{-1/2})\Sigma_2^{-1/2} - I\|_F = \|\Sigma_2^{-1/2}\Sigma_3\Sigma_2^{-1/2} - I\|_F \leq \tilde{O}(\varepsilon),$$

where we use the following proposition. This finishes the proof of Corollary 1.3, as by Corollary 2.14 in [DKK+16] we have the desired bound on the total variation distance.

**Proposition A.1.** Let $A, B \in \mathbb{R}^{d \times d}$ be symmetric, PSD matrices, with $B$ invertible. Then for any invertible $C \in \mathbb{R}^{d \times d}$, it holds that

$$\|B^{-1/2}AB^{-1/2} - I\|_F = \|(BC)^{-1/2}CA(CB)^{-1/2} - I\|_F$$
Proof. Recall that for a symmetric matrix $M \in \mathbb{R}^{d \times d}$ with eigenvalues $\lambda_1, \ldots, \lambda_d$, $\|M\|_F = \sum_{i=1}^{d} \lambda_i^2$. It thus suffices to show that $B^{-1/2}AB^{-1/2}$ and $(CBC^T)^{-1/2}CAC^T(CBC^T)^{-1/2}$ are equivalent up to an orthogonal change of basis.

Let $D = (CBC^T)^{-1/2}CB^{1/2}$. We then have that $DB^{-1/2}AB^{-1/2}D^T = (CBC^T)^{-1/2}CAC^T(CBC^T)^{-1/2}$, so it remains to show that $D$ is orthogonal, i.e., $DD^T = D^TD = I$. We have

$$
DD^T = (CBC^T)^{-1/2}CB^{1/2}B^{1/2}C^T(CBC^T)^{-1/2} = (CBC^T)^{-1/2}
$$

$$
(CBC^T)^{1/2} = (CBC^T)^{-1/2} - I
$$

$$
D^TD = B^{1/2}C^T(CBC^T)^{-1/2}CB^{1/2} = B^{1/2}C^T(C^{-1})^TB^{-1}C^{-1}CB^{1/2} = I,
$$

which finishes the proof. \qed

## B Quantifier Elimination in Sum-of-Squares

In this section we will justify why the SoS relaxations of Algorithms 1 and 2, which are written as a family of infinitely many constraints, can be solved efficiently. The programs have the form

$$
\text{find } \quad x \in \mathbb{R}^m
$$

$$
\text{s.t. } \quad f_i(x) \geq 0 \quad \forall i
$$

$$
\text{and } \quad g_j(x) = 0 \quad \forall j
$$

$$
\text{and } \quad h(x, \nu) \geq 0 \quad \forall \nu \in \mathbb{R}^d.
$$

with poly$(m)$ constraints $f_i, g_j$. As such, we need a way to express constraints of the form “$h(x, \nu) \geq 0$ for all $\nu \in \mathbb{R}^d$” within degree-$k$ SoS. This will follow from the following result:

**Lemma B.1** (Quantifier elimination in SoS, e.g., Section 4.3.4 in [FKP19]). Suppose that there exists some $x^* \in \mathbb{R}^m$ such that $f_i(x^*) \geq 0$ for all $i$, $g_j(x^*) = 0$ for all $j$, and $h(x^*, \cdot)$ has a degree-$k$ SoS proof of nonnegativity. Then a degree-$k$ pseudoexpectation satisfying all constraints in (B.1) can be found by solving a semidefinite program of size $m^{O(k)}$.

Intuitively, this is true because “$h(x^*, \cdot)$ has a degree-$k$ SoS proof of nonnegativity” is equivalent to a particular moment matrix of size $m^{O(k)}$ being PSD, which can be expressed within SoS. We will use this result in two forms.

**Lemma B.2** (Lemma 4.27 in [FKP19]). If $h$ is a quadratic form, then $h$ has a degree-2 SoS proof of nonnegativity if and only if $h(\nu) \geq 0$ for every $\nu$.

In Algorithms 1 and 2, the constraint (4) is indeed a quadratic form in $\nu$ (or $P$), so we are done. For constraint (5) in Algorithm 1 we require the certifiable hypercontractivity of Gaussians (Lemma 2.10), which we restate here:

**Lemma** (Restatement of Lemma 2.10). Let $\epsilon, \delta > 0$, and $n \geq \tilde{O}(d \log(1/\delta)/\epsilon^2)$. Let $x_1, \ldots, x_n \sim \mathcal{N}(0, \Sigma)$ be samples from a $d$-dimensional Gaussian. Then with probability $1 - \delta$,

$$
h(x, \nu) := (3 + \epsilon)\langle \nu, \Sigma \nu \rangle^2 - \mathbb{E}_{i \sim [n]} \langle x_i, \nu \rangle^4
$$

has a degree-4 SoS proof of nonnegativity in $\nu$ (Definition 2.6).
By rearranging, the constraint (5) in Algorithm 1 is exactly the condition that \( h(x', v) \geq 0 \) for all \( v \) in the above lemma, so the lemma states that a degree-4 SoS proof exists with high probability for the true samples \( x'_i = x_i \). Thus, the conditions of Lemma B.1 are satisfied, and we are done.

C Deferred Proofs from Section 2.2

C.1 Proof of Lemma 2.8

Proof. The first statement is

\[
\left\| \mathbb{E}_i a_i [x_i x_i^T - \mathbb{I}] \right\|_F \leq \tilde{O}(\varepsilon)
\]

which is Corollary 4.8 in [DKK+16]. The second statement is

\[
\left| \mathbb{E}_i a_i [\langle x_i x_i^T - \mathbb{I}, P \rangle^2 - 2\|P\|_F^2] \right| \leq \tilde{O}(\varepsilon)\|P\|_F^2.
\]

By convexity, we may assume that \( a_i \in \{0, 1\} \) for all \( i \). Let \( S \) be the set of indices \( i \) for which \( a_i > 0 \). We have:

\[
\left| \mathbb{E}_i a_i [\langle x_i x_i^T - \mathbb{I}, P \rangle^2 - 2\|P\|_F^2] \right| \leq \mathbb{E}_i [\langle x_i x_i^T - \mathbb{I}, P \rangle^2 - 2\|P\|_F^2] + \varepsilon \left| \mathbb{E}_i \langle x_i x_i^T - \mathbb{I}, P \rangle^2 \right|
\]

We bound the two terms separately. Condition on the “good event” in Lemma 5.17 of [DKK+16]. Then,

\[
\left| \mathbb{E}_i \langle x_i x_i^T - \mathbb{I}, P \rangle^2 - 2\|P\|_F^2 \right| \leq O(\varepsilon)\|P\|_F^2
\]

follows from Item 3 of Definition 5.15 in [DKK+16] with \( p(x) = \langle xx^T - \mathbb{I}, P \rangle / (\sqrt{2}\|P\|_F) \). The fact that

\[
\varepsilon \left| \mathbb{E}_i \langle x_i x_i^T - \mathbb{I}, P \rangle^2 \right| \leq \tilde{O}(\varepsilon)
\]

follows from Lemma 5.21 of [DKK+16] with the same choice of \( p \). Combining these bounds completes the proof. \( \square \)

C.2 Proof of Lemma 2.9

The statements in Lemma 2.9 are similar to those in Lemma 2.7 and Lemma 2.8, so it should be reasonable to believe that they should hold. The proofs are tedious but ultimately mostly brute force.

All the statements are invariant to linear transformations, so assume WLOG that \( \mu = 0 \) and \( \Sigma = \mathbb{I} \). Condition on the conclusions of Lemmas 2.7 and 2.8 which hold with high probability for the chosen \( n \). Let \( z_i = x_i - \mu_0 \) for notational simplicity.

In the proof, instead of the stated conditions on \( a \), we will use instead normalized vectors, namely, we will assume that \( \mathbb{E}_{ij} a_{ij} = \mathbb{E}_{ij} a_{i} = 1 \), \( \mathbb{E}_{ij} a_{ij} = a_{i} < 1 + O(\varepsilon) \), and \( a_{ij} \leq a_{i}(1 + O(\varepsilon)) \). Since this amounts to nothing but scaling the coefficients by \( 1 + O(\varepsilon) \), the conclusions of Lemmas 2.7 and 2.8 hold verbatim.
(1) $|\langle \mu_0, v \rangle| \leq \tilde{O}(\varepsilon)\|v\|_2$

Proof. Set $a_i = 1$ for all $i$ in Lemma 2.7.

(2) $|\mathbb{E}_i a_i(x_i - \mu_0, v)| \leq \tilde{O}(\varepsilon) \cdot \sqrt{\sigma_0 \|v\|}$

Proof. By (1) above and Lemma 2.7, we have

$|\mathbb{E}_i a_i(x_i - \mu_0, v)| \leq |\mathbb{E}_i a_i(x_i, v)| + |\mathbb{E}_i a_i(\mu_0, v)| \leq \tilde{O}(\varepsilon)\|v\|_2$

But $\|v\|_2 = (1 \pm \tilde{O}(\varepsilon))\sqrt{\sigma_0 \|v\|}$ by (4) (with $P = vv^T$), so we are done.

The following intermediate results will be useful in the remaining proofs.

Lemma C.1. $\|P \mu_0\|_2 \leq \tilde{O}(\varepsilon)\|P\|_F$, $|\mathbb{E}_i a_i(x_i \mu_0^T, P)| \leq \tilde{O}(\varepsilon^2)\|P\|_F$, and $|\mu_0^T P \mu_0| \leq \tilde{O}(\varepsilon^2)\|P\|_F$.

Proof. The first inequality is $\|P \mu_0\|_2 \leq \|P\|_2\|\mu_0\|_2 \leq \tilde{O}(\varepsilon)\|P\|_F$ by Lemma 2.7 and $\|P\|_2 \leq \|P\|_F$.

The second is $|\mathbb{E}_i a_i(x_i \mu_0^T, P)| \leq \tilde{O}(\varepsilon)\|P \mu_0\|_2 \leq \tilde{O}(\varepsilon^2)\|P\|_F$, by Lemma 2.7 the first inequality. The third is the second when $a_i = 1$ for all $i$.

(3) $|\mathbb{E}_i a_i(z_i z_i^T - I, P)| \leq \tilde{O}(\varepsilon)\|P\|_F$

(The statement in the lemma follows from setting $P = vv^T$ and applying (4) below, but we will need this more generic statement later, so this is the one we prove.)

Proof. We have

$\mathbb{E}_i a_i(\langle x_i - \mu_0, (x_i - \mu_0)^T, P \rangle = \mathbb{E}_i a_i(\langle x_i x_i^T, P \rangle + \langle \mu_0^T, P \rangle - 2 \mathbb{E}_i a_i(\langle x_i \mu_0^T, P \rangle$)

The first term is $\mathbb{E}_i a_i(\langle x_i x_i^T, P \rangle = \langle I, P \rangle = \tilde{O}(\varepsilon)\|P\|_F$ by Lemma 2.8, and the other terms are $\pm \tilde{O}(\varepsilon)\|P\|_F$ by Lemma C.1.

(4) $|\langle \Sigma_0 - I, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F$

Proof. Set $a_i = 1$ for all $i$ in (3).

(5) $|\mathbb{E}_i(z_i z_i^T - \Sigma_0, P)^2 - 2\|P\|^2_F| \leq \tilde{O}(\varepsilon)\cdot\|P\|^2_F$

Proof. We have:

$\mathbb{E}_i a_i(\langle z_i z_i^T - \Sigma_0, P \rangle^2 = \mathbb{E}_i a_i(\langle z_i z_i^T - I, P \rangle^2 + \langle \Sigma_0 - I, P \rangle^2 - 2 \mathbb{E}_i a_i(\langle z_i z_i^T - I, P \rangle\langle \Sigma_0 - I, P \rangle$

For the second term, we have $\langle \Sigma_0 - I, P \rangle^2 \leq \tilde{O}(\varepsilon^2)\|P\|^2_F$ by (4). For the third term, we have

$|\mathbb{E}_i a_i(\langle z_i z_i^T - I, P \rangle\langle \Sigma_0 - I, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F\mathbb{E}_i a_i(\langle z_i z_i^T - I, P \rangle| \leq \tilde{O}(\varepsilon^2)\|P\|^2_F$
We analyze each term separately. The first term is by (4) and then (3). That only leaves the first term. We have:

\[
\mathbb{E} a_i \langle (x_i - \mu_0)(x_i - \mu_0)^\top - \mathbb{I}, P \rangle^2 = \mathbb{E} a_i \langle (x_i x_i^\top - \mathbb{I}) + \mu_0\mu_0^\top - 2x_i\mu_0^\top, P \rangle^2 \\
= \mathbb{E} a_i \langle x_i x_i^\top - \mathbb{I}, P \rangle^2 + \langle \mu_0\mu_0^\top, P \rangle \mathbb{E} a_i \langle x_i x_i^\top - \mathbb{I}, P \rangle - 2\langle \mu_0\mu_0^\top, P \rangle \mathbb{E} a_i \langle x_i x_i^\top, P \rangle \\
- 2 \mathbb{E} a_i \langle x_i x_i^\top - \mathbb{I}, P \rangle \langle x_i x_i^\top, P \rangle
\]

We analyze each term separately. The first term is \((2 \pm \tilde{O}(\varepsilon))\|P\|_F^2\) by Lemma 2.8, so it suffices to show that all remaining terms are small. The second term is \(\tilde{O}(\varepsilon^2)\|P\|_F^2\) by Lemma 2.1. The third term is \(|\mathbb{E} a_i \langle x_i, P \mu_0 \rangle^2| = (1 \pm \tilde{O}(\varepsilon))\|P\|_F^2 \leq \tilde{O}(\varepsilon^2)\|P\|_F^2\) by Lemmas 2.7 and C.1. The fourth term is \(\tilde{O}(\varepsilon^3)\|P\|_F^2\) by Lemmas 2.8 and C.1. The fifth term is \(\tilde{O}(\varepsilon^4)\|P\|_F^2\) by Lemma C.1. For the final term, we have

\[
\mathbb{E} a_i \langle (x_i x_i^\top - \mathbb{I}, P) \langle x_i x_i^\top, P \rangle \rangle \leq \sqrt{\left(\mathbb{E} a_i \langle x_i x_i^\top - \mathbb{I}, P \rangle^2 \right) \left(\mathbb{E} a_i \langle x_i x_i^\top, P \rangle^2 \right)}
\]

by Cauchy-Schwarz. The first term is \(\mathbb{E} a_i \langle x_i x_i^\top - \mathbb{I}, P \rangle^2 = (2 \pm \tilde{O}(\varepsilon))\|P\|_F^2\), and the second term is \(\tilde{O}(\varepsilon^2)\|P\|_F^2\) as argued above. Combining these yields \(|\mathbb{E} a_i \langle x_i x_i^\top - \mathbb{I}, P \rangle \langle x_i x_i^\top, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F^2\), as needed.

The proofs of the remaining two bounds will make repeated use of the following generic technique, roughly speaking. Suppose that resilience (Lemmas 2.7 and 2.8) gives us \(|\mathbb{E} a_i z_i| = \tilde{O}(\varepsilon)B\), and we want to argue that \(|\mathbb{E} a_{ij} z_i z_j| = \tilde{O}(\varepsilon)B^2\). This is not immediate, because \(a_{ij} \neq a_i a_j\) in general. Instead, we write \(\mathbb{E}_{ij} a_{ij} z_i z_j = \mathbb{E}_i z_i \mathbb{E}_j a_{ij} z_j\), and apply resilience to the inner expectation (noting that the vector whose \(j\)th entry is \(a_{ij}/a_i\) is, by construction, a valid resilience vector) to find \(|\mathbb{E}_j a_{ij} z_j| \leq a_i \tilde{O}(\varepsilon)B\) for each \(i\), so that \(|\mathbb{E}_{ij} a_{ij} z_i z_j| \leq B|\mathbb{E}_i z_i \tilde{O}(\varepsilon)|\). In this expression, the \(\tilde{O}(\varepsilon)\) may depend on \(i\). Let \(b_i\) be the term hidden by the \(\tilde{O}(\varepsilon)\) for each \(i\), and let \(a'_i := 1 - b_i + \mathbb{E}_i b_i\). Note that \(\mathbb{E}_i a'_i = 1\) and \(a'_i = 1 \pm \tilde{O}(\varepsilon)\) for all \(i\), so \(a'\) is a valid input to the resilience condition. Thus, we have

\[
|\mathbb{E}_i b_i z_i| \leq (1 + \mathbb{E}_i b_i)|\mathbb{E}_i z_i| + |\mathbb{E}_i a_i z_i|
\]

Now applying resilience to each of the two terms separately gives \(|\mathbb{E}_i b_i z_i| \leq \tilde{O}(\varepsilon)B\), so \(|\mathbb{E}_{ij} a_{ij} z_i z_j| \leq \tilde{O}(\varepsilon)B^2\), as desired.

The following intermediate result will also be useful:

**Lemma C.2.** \(\mathbb{E}_{ij} a_{ij} \langle X_{ij}, P \rangle = \mathbb{E}_i a_i \langle x_i x_i^\top, P \rangle + \tilde{O}(\varepsilon)\|P\|_F\)

**Proof.** We have

\[
\mathbb{E}_{ij} a_{ij} \langle X_{ij}, P \rangle = \frac{1}{2} \mathbb{E}_{ij} a_{ij} \langle (x_i - x_j)(x_i - x_j)^\top, P \rangle \\
= \mathbb{E}_i a_i \langle x_i x_i^\top, P \rangle - \mathbb{E}_{ij} a_{ij} \langle x_i x_j^\top, P \rangle
\]
where we use the symmetry of $P$ and the $a_{ij}$s. It thus remains to bound the last term. Write $P = \sum_k \lambda_k v_k v_k^\top$ for orthonormal vectors $v_k$. Note that $\|\sum_k \lambda_k v_k\|_2 = \sqrt{\sum_k \lambda_k^2} = \|P\|_F$ by Pythagorean theorem. Then:

$$E_a ij \langle x_i x_i^\top, P \rangle = E_i \sum_k \lambda_k \langle v_k, x_i \rangle E_a ij \langle v_k, x_i \rangle = E_i a_i (\pm \tilde{O}(\varepsilon)) \left( \sum_k \lambda_k v_k, x_i \right) = \tilde{O}(\varepsilon)\|P\|_F$$

by applying Lemma 2.7 twice using the generic technique.

We now prove the last two results in Lemma 2.9.

(6) $\left| E_a ij \langle X_{ij} - \Sigma_0, P \rangle \right| \leq \tilde{O}(\varepsilon) \cdot \|P\|_F$

Proof. By Lemma C.2, we have:

$$E_a ij \langle X_{ij} - \Sigma_0, P \rangle = E_i a_i \langle x_i x_i^\top - \Sigma_0, P \rangle \pm \tilde{O}(\varepsilon^2)\|P\|_F$$

It thus only remains to bound the first term. We have

$$\left| E_i a_i \langle x_i x_i^\top - \Sigma_0, P \rangle \right| \leq \left| E_i a_i \langle x_i x_i^\top - I, P \rangle \right| + \left| E_i a_i \langle \Sigma_0 - I, P \rangle \right| \leq \tilde{O}(\varepsilon)\|P\|_F$$

by applying Lemma 2.8 and (4).

(7) $\left| E_a ij \langle X_{ij} - \Sigma_0, P \rangle^2 - 2\|P\|_F^2 \right| \leq \tilde{O}(\varepsilon) \cdot \|P\|_F^2$

Proof. We follow the same structure as the proof of (5) above. We have:

$$E_a ij \langle X_{ij} - \Sigma_0, P \rangle^2 = E_a ij \langle X_{ij} - I, P \rangle^2 + \langle \Sigma_0 - I, P \rangle^2 - 2 E_a ij \langle X_{ij} - I, P \rangle \langle \Sigma_0 - I, P \rangle$$

For the second term, we have $\langle \Sigma_0 - I, P \rangle^2 \leq \tilde{O}(\varepsilon^2)\|P\|_F^2$ by (4). For the third term, we have

$$\left| E_a ij \langle X_{ij} - I, P \rangle \langle \Sigma_0 - I, P \rangle \right| \leq \tilde{O}(\varepsilon)\|P\|_F |E_a ij \langle X_{ij} - I, P \rangle|$$

$$= \tilde{O}(\varepsilon)\|P\|_F |E_a ij \langle x_i x_i^\top - I, P \rangle|$$

$$\leq \tilde{O}(\varepsilon^2)\|P\|_F^2$$

by Lemmas 2.8 and C.2. That only leaves the first term. We have

$$E_a ij \langle x_i x_i^\top - I, P \rangle^2 = E_a ij \left( \frac{1}{2} \langle x_i x_i^\top - I, P \rangle^2 + \frac{1}{2} \langle x_i x_i^\top - I, P \rangle - x_i x_i^\top, P \right)^2$$

$$= \frac{1}{2} E_a ij \langle x_i x_i^\top - I, P \rangle^2 + E_a ij \langle x_i x_i^\top, P \rangle^2 + \frac{1}{2} E_a ij \langle x_i x_i^\top - I, P \rangle \langle x_i x_i^\top - I, P \rangle$$

$$- 2 E_a ij \langle x_i x_i^\top - I, P \rangle \langle x_i x_i^\top, P \rangle$$
The first term is \((1 \pm \tilde{O}(\varepsilon))\|P\|_F^2\) by Lemma 2.8. For the second term, we have

\[
\mathbb{E} a_{ij} \langle x_i x_j^\top, P \rangle^2 = \mathbb{E} _i \mathbb{E} _j a_{ij} x_i^\top P x_j x_j^\top P x_i
\]

\[
= \mathbb{E} a_{ij} \langle x_i x_j^\top, P x_i x_j^\top P \rangle
\]

\[
= \mathbb{E} a_{i} \left[ \langle \mathbb{I}, P x_i x_i^\top P \rangle \pm \tilde{O}(\varepsilon) \|P x_i x_i^\top P\|_F \right]
\]

\[
= \mathbb{E} a_{i} (1 \pm \tilde{O}(\varepsilon)) \langle x_i x_i^\top, P^2 \rangle
\]

\[
= \langle \mathbb{I}, P^2 \rangle \pm \tilde{O}(\varepsilon) \|P^2\|_F
\]

\[
= (1 \pm \tilde{O}(\varepsilon))\|P\|_F^2
\]

where we use Lemma 2.8 twice (the second time exploiting the fact that \(a_i (1 \pm \tilde{O}(\varepsilon))\) is still a valid resilience vector), and the last line uses \(\langle \mathbb{I}, P^2 \rangle = \|P\|_F^2\) and \(\|P^2\|_F \leq \|P\|_F^2\).

Thus, it only remains to show that the other two terms are small. For the third term, we have

\[
|\mathbb{E} a_{ij} \langle x_i x_i^\top - \mathbb{I}, P \rangle \langle x_j x_j^\top - \mathbb{I}, P \rangle| = |\mathbb{E} _i \mathbb{E} _j a_{ij} \langle x_i x_i^\top - \mathbb{I}, P \rangle \mathbb{E} _j a_{ij} \langle x_j x_j^\top - \mathbb{I}, P \rangle|
\]

\[
\leq \|P\|_F |\mathbb{E} a_{i} (\pm \tilde{O}(\varepsilon)) \langle x_i x_i^\top - \mathbb{I}, P \rangle|
\]

\[
\leq \tilde{O}(\varepsilon)\|P\|_F^2,
\]

by the generic technique. For the final term, we have

\[
|\mathbb{E} a_{ij} \langle x_i x_i^\top - \mathbb{I}, P \rangle \langle x_j x_j^\top, P \rangle| = |\mathbb{E} a_{ij} \langle x_i x_i^\top - \mathbb{I}, P \rangle \mathbb{E} a_{ij} \langle x_j, P x_i \rangle|
\]

\[
= |\mathbb{E} \mathbb{E} a_{i} (\pm \tilde{O}(\varepsilon)) \langle x_i x_i^\top - \mathbb{I}, P \rangle \|P x_i\|_2|
\]

\[
\leq \sqrt{(\mathbb{E} a_{i} \langle x_i x_i^\top - \mathbb{I}, P^2 \rangle) (\mathbb{E} a_{i} \tilde{O}(\varepsilon^2) \|P x_i\|_2^2)}
\]

The first term is \(\mathbb{E} a_{i} \langle x_i x_i^\top - \mathbb{I}, P \rangle^2 = (2 \pm \tilde{O}(\varepsilon))\|P\|_F^2\). For the second term, we have

\[
|\mathbb{E} a_{i} \tilde{O}(\varepsilon^2) \|P x_i\|_2^2| = \tilde{O}(\varepsilon^2) |\mathbb{E} a_{i} \langle x_i x_i^\top, P^2 \rangle|
\]

\[
\leq \tilde{O}(\varepsilon^2) |\mathbb{E} a_{i} \langle x_i x_i^\top - \mathbb{I}, P^2 \rangle| + \tilde{O}(\varepsilon^2) \|\mathbb{I}, P^2\|
\]

\[
\leq \tilde{O}(\varepsilon^2) \|P\|_F^2
\]

where the last line applies Lemma 2.8 to the first term (noting again that \(\langle \mathbb{I}, P^2 \rangle = \|P\|_F^2\)), and the inequality \(\|P^2\|_F \leq \|P\|_F^2\) to the second. Combining these yields \(|\mathbb{E} a_{ij} \langle x_i x_i^\top - \mathbb{I}, P \rangle \langle x_j x_j^\top, P \rangle| \leq \tilde{O}(\varepsilon)\|P\|_F^2\), which is what we needed.

\(\square\)