The weak Galerkin finite element method for Stokes interface problems with curved interface

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Abstract

In this paper, we develop a weak Galerkin (WG) finite element scheme for the Stokes interface problems with curved interface. The conventional numerical schemes rely on the use of straight segments to approximate the curved interface and the accuracy is limited by geometric errors. Hence in our method, we directly construct the weak Galerkin finite element space on the curved cells to avoid geometric errors. For the integral calculation on curved cells, we employ non-affine transformations to map curved cells onto the reference element. The optimal error estimates are obtained in both the energy norm and the $L^2$ norm. A series of numerical experiments are provided to validate the efficiency of the proposed WG method.

Keywords: Weak Galerkin finite element methods, Curved interface, Stokes equations, Weak divergence, Weak gradient.

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1. Introduction

In this paper, we focus on the Stokes interface problems. For simplicity, we adopt a specific model to describe the problems. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain, which is partitioned into two subdomains, $\Omega_1$ and $\Omega_2$. In each subdomain, the flow is governed by the incompressible Stokes equations, i.e.

\begin{align*}
-\nabla \cdot (A_i \nabla u_i) + \nabla p_i &= f_i, \quad \text{in } \Omega_i, \quad (1.1) \\
\nabla \cdot u_i &= 0, \quad \text{in } \Omega_i, \quad (1.2) \\
u_i &= g_i, \quad \text{on } \partial \Omega_i \setminus \Gamma, \quad (1.3)
\end{align*}

with the viscosity coefficient $A_i > 0$ defined in the $\Omega_i$. For simplicity of analysis, let $A_i$ be the piecewise constant matrix in this paper. And $\Gamma = \partial \Omega_1 \cap \partial \Omega_2$ denotes the interface between two subdomains and belongs to $C^2$ piecewise. The interface conditions on $\Gamma$ are described by the following equations:

\begin{equation}
\begin{split}
   u_1 - u_2 &= \phi, \quad \text{on } \Gamma,
\end{split}
\end{equation}

\footnotesize
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\[(A_1 \nabla u_1 - p_1 I) n_1 + (A_2 \nabla u_2 - p_2 I) n_2 = \psi, \quad \text{on} \quad \Gamma, \quad (1.5)\]

where \(n_1\) and \(n_2\) are the unit outward normal vectors on \(\Gamma\). \(n_1\) points from \(\Omega_1\) into \(\Omega_2\) and \(n_2 = -n_1\) (see Figure 1) and \(I\) represents the identity matrix. We shall drop the subscript \(i\) when velocity function \(u\) and pressure function \(p\) are defined in the whole domain \(\Omega\). The Stokes interface problems are classical fluid mechanics problems that describe the flow of viscous fluid through interfaces. The interface problems have been widely applied in different fields such as groundwater resource management, petroleum engineering, biomedicine, and more \[8, 12, 19, 23, 25, 30\].

In practical problems, the interface is usually a complex curved surface. In the two-dimensional problems, when dealing with such curved interface, a common approach involves approximating these curves with straight line segments. However, when employing high-degree polynomials to approximate the exact solution, the presence of geometric errors leads to a reduction in the orders of convergence \[27, 29, 36\]. Therefore, in domain with curved edges, various numerical methods have been proposed to solve the problems. For example, finite element method \[1, 2, 16, 26, 27\], discontinuous Galerkin finite element method \[4, 15, 17, 28, 37\], virtual element method \[10, 3\], weak Galerkin finite element method \[13, 18, 20\], etc.

In this work, we use the weak Galerkin (WG) finite element method to solve the Stokes interface problems with curved edges on fitted meshes. The WG method was first proposed in \[32\] for solving second order elliptic equations. Compared with other methods, the method uses separate polynomial functions on each cell and adds stabilizers to the cell boundaries to ensure weak continuity of the approximative functions. Since polynomial spaces are easy to construct, the WG method can be applied to polygonal or polyhedral meshes. At the same time, this method uses weakly defined differential operators to replace the classical differential operators. The WG method is used to solve various problems, such as Stokes equations \[34, 38\], Brinkman equations \[21, 35\], linear elasticity equations \[31\], parabolic equations \[39\], elliptic interface problems \[22\], Stokes-Darcy problems \[7, 24\], etc.

In this paper, we mainly treat curved cells. For the curved cells, instead of replacing curved edges with straight segments, we directly construct the weak Galerkin space on the curved cells. For the integral calculation on the curved cells, we use non-affine transformation \[40, 41\] to convert curved cell to the reference element. This treatment avoids geometric errors and makes the scheme easier to implement.

The outline of this paper is as follows: In Section 2, some notations used in this paper are presented. We also give the definitions of the weak Galerkin finite element space and weak differential operators. Based on these definitions, the weak Galerkin finite element scheme is established. Moreover, the proof of the existence and uniqueness is given. Section 3 is devoted to the proof of the stability of the solution of the WG scheme and presents some important inequalities. In Section 4 and 5, the error estimates in the energy norm and the \(L^2\) norm are proved separately. Finally, we give some numerical examples to verify our proposed theories in Section 6.

2. The Weak Galerkin Numerical Scheme

In this section, we first give the introduction of notations used in the paper. Then, we define the weak Galerkin finite element space and the corresponding weak differential operators. Next, the weak Galerkin finite element scheme is established. Based on the scheme, the proof of existence and uniqueness are given.
2.1. The notations

Assume that $\mathcal{T}^1_h$ and $\mathcal{T}^2_h$ are the polygon partitions of the domain $\Omega_1$ and $\Omega_2$ containing both straight and curved cells. Set $\mathcal{T}_h = \mathcal{T}^1_h \cup \mathcal{T}^2_h$. For $T \in \mathcal{T}_h$, when all edges of $T$ are straight, it is called a straight cell. The set of such straight cells is denoted by $\mathcal{T}^S_h$. When $T$ has a curved edge which intersects with the interface, it is called a curved cell. The set of the curved cells is denoted by $\mathcal{T}^C_h$. For $T \in \mathcal{T}_h$, $|T|$ and $h_T$ are the area and the diameter of the cell $T$, respectively. Denote that $h = \max_{T \in \mathcal{T}_h} h_T$ is the mesh size. Let $\mathcal{E}_h$ be the set of all edges on the partition $\mathcal{T}_h$ and $\mathcal{E}^S_h$ be the set of all interior edges including interface edges. Denote by $\mathcal{E}^S_h$ and $\mathcal{E}^I_h$ the set of all straight edges and the set of all curved edges, respectively. The cell $T \in \mathcal{T}_h$ should satisfy regularity conditions in [34]. In addition to these conditions, the following conditions also need to be satisfied.

(A1) There exists a positive number $C_1$ satisfying

$$|e| \leq C_1 h_T,$$

for all edges $e \in \partial T$ and $T \in \mathcal{T}_h$.

(A2) For every cell $T$, there exists a ball in the interior of $T$.

2.2. Weak Galerkin finite element space

For $T \in \mathcal{T}_h$, we define the weak function $v = \{v_0, v_b\}$ on the cell $T$, where $v_0$ represents the interior function in $T$ and $v_b$ represents the boundary function on $\partial T$. Note that $v_b$ has one unique value on the edge $e \in \mathcal{E}^S_h$ and there are two values $v_{1b}$ and $v_{2b}$ on the edge $e \in \mathcal{E}^I_h$. And there is no relationship between $v_0$ and $v_b$. Next, we give definitions of some projection operators. For a given integer $k \geq 1$, let $Q_0$ be the $L^2$ projection operator from $[L^2(T)]^2$ onto $[P_k(T)]^2$, $T \in \mathcal{T}_h$. $Q_h$ is defined as the $L^2$ projection operator from $[L^2(e)]^2$ onto $[P_{k-1}(e)]^2$ when $e \in \mathcal{E}^S_h$, and from $[L^2(e)]^2$ onto $[P_k(e)]^2$ when $e \in \mathcal{E}^I_h$. Finally, we define the weak Galerkin finite element space with respect to the velocity function $u$ and the pressure function $p$.

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in [P_k(T)]^2, T \in \mathcal{T}_h, v_b|_e \in [P_{k-1}(e)]^2, e \in \mathcal{E}^S_h, v_b|_e \in [P_k(e)]^2, e \in \mathcal{E}^I_h\},$$

$$V_h^0 = \{v = \{v_0, v_b\} \in V_h, v_b = 0 \text{ on } \partial \Omega, v_{1b} - v_{2b} = 0, e \in \mathcal{E}^I_h\},$$

$$W_h = \{q : q \in L^2_0(\Omega), q|_T \in P_{k-1}(T), T \in \mathcal{T}_h\}.$$
Firstly, we introduce some weak differential operators used in this paper.

**Definition 2.1.** [34] For \( v \in V_h \), its discrete weak gradient \( \nabla_w v \in [P_{k-1}(T)]^{2 \times 2} \) satisfies
\[
(\nabla_w v, \tau)_T = -(v_0, \nabla \cdot \tau)_T + \langle v_b, \tau \cdot n \rangle_{\partial T}, \quad \forall \tau \in [P_{k-1}(T)]^{2 \times 2},
\]
where \( n \) is the unit outward normal vector on \( \partial T \).

**Definition 2.2.** [34] For every \( v \in V_h \), its discrete weak divergence \( \nabla_w \cdot v \in P_{k-1}(T) \) satisfies
\[
(\nabla_w \cdot v, \varphi)_T = -(v_0, \nabla \varphi)_T + \langle \varphi n \rangle_{\partial T}, \quad \forall \varphi \in P_{k-1}(T),
\]
where \( n \) is the unit outward normal vector on \( \partial T \).

### 2.3. The numerical scheme

We define bilinear forms as follows:
\[
s(v, w) = \sum_{T \in T_h} A h_T^{-1} (Q_b v_0 - v_b, Q_b w_0 - w_b)_{\partial T \setminus (\partial T \cap \Gamma)},
\]
\[
a(v, w) = \sum_{T \in T_h} (A \nabla_w v, \nabla_w w)_T,
\]
\[
a_s(v, w) = a(v, w) + s(v, w),
\]
\[
b(v, p) = -\sum_{T \in T_h} (\nabla_w \cdot v, p)_T,
\]
for all \( v, w \in V_h \) and \( p \in W_h \).

**Algorithm 1** Weak Galerkin Scheme

For the Stokes interface problems (1.1)-(1.5), the WG scheme is seeking \( \mathbf{u}_h \in V_h \) and \( p_h \in W_h \) to satisfy \( \mathbf{u}_h = Q_b \mathbf{g} \) on \( \partial \Omega \), \( \mathbf{v}_{1b} - \mathbf{v}_{2b} = Q_b \mathbf{\phi} \) on \( e \in E^I_h \), and
\[
a_s(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (f, v_0) + \langle \psi, v_h \rangle_{\Gamma}, \quad \forall \mathbf{v}_h \in V_h^0, \quad \forall p_h \in W_h.
\]
\[
b(\mathbf{u}_h, q_h) = 0, \quad \forall q_h \in W_h.
\]

We now define a semi-norm in the space \( V_h \) as follows:
\[
\|v\|^2 = a_s(v, v).
\]

Then we have the following properties.

**Lemma 2.1.** \( \| \cdot \| \) provides a norm in \( V_h^0 \).

**Proof.** Assume that \( \|v\| = 0 \) with \( v \in V_h^0 \), then we obtain
\[
0 = a_s(v, v) = \sum_{T \in T_h} (A \nabla_w v, \nabla_w v)_T + \sum_{T \in T_h} A h_T^{-1} (v_0 - v_b, v_0 - v_b)_{\partial T \cup \Gamma}
\]
\[
+ \sum_{T \in T_h} A h_T^{-1} (Q_b v_0 - v_b, Q_b v_0 - v_b)_{\partial T \cup (\partial T \cap \Gamma)},
\]
which leads to
\[ \nabla_w v = 0 \text{ in } T \in \mathcal{T}_h, \quad Q_b v_0 = v_b \text{ on } e \in \mathcal{E}_h^2, \quad v_0 = v_b \text{ on } e \in \mathcal{E}_h^I. \]

Therefore, for any \( \tau \in [P_{k-1}(T)]^{2 \times 2} \), we have the following results.

For \( T \in \mathcal{T}_h^S \), according to the definition \([2.2]\), integration by parts and the property of the \( L^2 \) projection operator, we obtain
\[ 0 = (\nabla_w v, \tau)_T \]
\[ = - (v_0, \nabla \cdot \tau)_T + (v_b, \tau n)_{\partial T} \]
\[ = (\nabla v_0, \tau)_T - (v_0, \tau n)_{\partial T} + (v_b, \tau n)_{\partial T} \]
\[ = (\nabla v_0, \tau)_T - (Q_b v_0 - v_b, \tau n)_{\partial T} - (v_0 - v_b, \tau n)_{\partial T \cap \Gamma} \]
\[ = (\nabla v_0, \tau)_T. \]

Similarly, for \( T \in \mathcal{T}_h^I \), we get
\[ 0 = (\nabla_w v, \tau)_T \]
\[ = - (v_0, \nabla \cdot \tau)_T + (v_b, \tau n)_{\partial T} \]
\[ = (\nabla v_0, \tau)_T - (v_0, \tau n)_{\partial T} + (v_b, \tau n)_{\partial T} \]
\[ = (\nabla v_0, \tau)_T - (Q_b v_0 - v_b, \tau n)_{\partial T \cap \Gamma} - (v_0 - v_b, \tau n)_{\partial T \cap \Gamma} \]
\[ = (\nabla v_0, \tau)_T. \]

Choosing \( \tau = \nabla v_0 \) in the above equations derives \( \nabla v_0 = 0 \) in \( T \in \mathcal{T}_h \). It follows that \( v_0 \) is constant in \( T \in \mathcal{T}_h \). Due to \( v_0 = Q_b v_0 = v_b \) on \( e \in \mathcal{E}_h \), we have \( v_0 = v_b = C \). According to the fact that \( v_0 = 0 \) on \( \partial \Omega \), we obtain that \( v_0 = 0 \) and \( v_b = 0 \).

**Lemma 2.2.** \([2.2]\) For any \( v, w \in V_h \), there holds
\[
|a_s(v, w)| \leq ||v|| \cdot ||w||.
\]

**Lemma 2.3.** The WG scheme \([2.4]-[2.5]\) has a unique solution.

**Proof.** Since \([2.4]-[2.5]\) are finite-dimensional square linear equations, existence and uniqueness are equivalent. Let \( \{u_h, p_h\} \) and \( \{u_h', p_h'\} \) be the solutions of the WG scheme \([2.4]-[2.5]\), respectively. For \( u_h \in V_h^0 \) and \( p_h \in W_h \), we have
\[
a_s(u_h, v_h) + b(v_h, p_h) = (f, v_0) + (\psi, v_b), \quad (2.7)
\]
\[
b(u_h, q_h) = 0, \quad (2.8)
\]
\[
a_s(u_h', v_h) + b(v_h, p_h') = (f, v_0) + (\psi, v_b), \quad (2.9)
\]
\[
b(u_h', q_h) = 0. \quad (2.10)
\]

Subtracting Eq. \((2.9)\) from Eq. \((2.7)\) and subtracting Eq. \((2.10)\) from Eq. \((2.8)\) derives that
\[
a_s(u_h - u_h', v_h) + b(v_h, p_h - p_h') = 0, \quad v_h \in V_h^0 \quad (2.11)
\]
\[
b(u_h - u_h', q_h) = 0, \quad q_h \in W_h. \quad (2.12)
\]

Taking \( v_h = u_h - u_h' \in V_h^0 \) in Eq. \((2.11)\) and \( q_h = p_h - p_h' \in W_h \) in Eq. \((2.12)\) leads to
\[
a_s(u_h - u_h', u_h - u_h') + b(u_h - u_h', p_h - p_h') = 0, \quad (2.13)
\]
\[ b(u_h - u'_h, p_h - p'_h) = 0. \] (2.14)

Since \( \| \cdot \| \) is a norm in \( V_h^0 \), we obtain \( u_h = u'_h \) and \( b(v_h, p_h - p'_h) = 0 \). Then taking \( v_h = \{ \nabla(p_h - p'_h), 0 \} \) gives

\[ 0 = \sum_{T \in T_h} \| \nabla(p_h - p'_h) \|^2_T, \]

which leads to \( p_h - p'_h \) is constant in \( T \in T_h \).

Next, setting \( v_0 = 0 \) in \( T \in T_h, v_h = [p_h - p'_h]n_e \) on \( e \in E_h^S \) and \( v_b = 0 \) on \( e \in E_h^I \) in Eq. (2.11) leads to

\[ \sum_{e \in E_h^S} \| [p_h - p'_h]n_e \|^2_e = 0. \]

The jump \([v]\) is defined as

\[ [v] = v \text{ if } e \subset \partial \Omega, \quad [v] = v|_{T_1} - v|_{T_2}, \text{ if } e \in E_h^0, \]

where \( e \) is the common edge of \( T_1 \) and \( T_2 \).

Moreover, when \( e \in E_h^I \), define the other two edges of the same cell as \( e_1 \) and \( e_2 \) and their unit outward normal vectors as \( n_{e_1} \) and \( n_{e_2} \). Choosing \( v_0 = 0 \) on \( T \in T_h, v_h = 0 \) on \( e \in E_h^S \) and \( v_b = [p_h - p'_h](n_{e_1} + n_{e_2}) \) on \( e \in E_h^I \) to derive that

\[ 0 = [p_h - p'_h]^2_e(n_{e_1} + n_{e_2}, n_e)_e. \]

Since \((n_{e_1} + n_{e_2}, n_e)_e < 0\), we get \([p_h - p'_h]_e = 0\). Noting that \( p_h \in L_0^2(\Omega) \), we have \( p_h - p'_h = 0 \) on \( \Omega \). The proof of the lemma is complete.

\[ \square \]

3. Stability

For \( T \in T_h \), let \( Q_h = \{ Q_0, Q_h \} \) be the \( L^2 \) projection operator onto \( V_h \). \( Q_h \) and \( Q_b \) are defined as the \( L^2 \) projection operators onto \( P_{k-1}(T) \) and \( [P_{k-1}(T)]^{2 \times 2} \), respectively.

**Lemma 3.1.** For \( u \in [H^1(\Omega)]^2 \) and \( \tau \in [P_{k-1}(T)]^{2 \times 2} \), on every cell \( T \in T_h \), we have the following properties for the discrete weak gradient operator.

1. For \( T \in T_h^S \), we have

\[ (\nabla_w(Q_hu), \tau)_T = (Q_h(\nabla u), \tau)_T. \] (3.1)

2. For \( T \in T_h^I \), we have

\[ (\nabla_w(Q_hu), \tau)_T = (Q_h(\nabla u), \tau)_T + (Q_hu - u, \tau n)_{\partial T \cap T}. \] (3.2)

**Proof.** For \( T \in T_h^S \) and \( \tau \in [P_{k-1}(T)]^{2 \times 2} \), according to Eq. (2.2), definitions of \( Q_0, Q_h \) and \( Q_h \) and integration by parts, we have

\[ (\nabla_w(Q_hu), \tau)_T = - (Q_0u, \nabla \cdot \tau)_T + (Q_hu, \tau \cdot n)_{\partial T} \]

\[ = - (u, \nabla \cdot \tau)_T + (u, \tau \cdot n)_{\partial T} \]

\[ = (\nabla u, \tau)_T. \]
Similarly, for \( T \in \mathcal{T}_k^i \) and \( \tau \in [P_{k-1}(T)]^{2 \times 2} \), we have

\[
(\nabla_w(Q_h u), \tau)_T = -(Q_0 u, \nabla \cdot \tau)_T + (Q_h u, \tau \cdot n)_{\partial T} \\
= -(u, \nabla \cdot \tau)_T + (u, \tau \cdot n)_{\partial T} + (Q_h u - u, \tau \cdot n)_{\partial T \cap \Gamma} \\
= (\nabla_w(Q_h u), \tau)_T + (Q_h u - u, \tau \cdot n)_{\partial T \cap \Gamma}.
\]

The proof of Lemma 3.1 is complete.

**Lemma 3.2.** For \( u \in [H^1(\Omega)]^2 \) and \( \varphi \in P_{k-1}(T) \), on every cell \( T \in \mathcal{T}_h \), we have the following properties for the discrete weak divergence operator.

1. For \( T \in \mathcal{T}_h^S \), we obtain

\[
(\nabla_w \cdot (Q_h u), \varphi)_T = (Q_h(\nabla \cdot u), \varphi)_T.
\]  

2. For \( T \in \mathcal{T}_k^i \), we obtain

\[
(\nabla_w \cdot (Q_h u), \varphi)_T = (Q_h(\nabla \cdot u), \varphi)_T + (Q_h u - u, \varphi n)_{\partial T \cap \Gamma}.
\]  

**Proof.** For \( T \in \mathcal{T}_h^S \) and \( \varphi \in P_{k-1}(T) \), according to Eq. (2.3), definitions of \( Q_0 \), \( Q_b \) and \( Q_h \) and integration by parts, we get

\[
(\nabla_w \cdot (Q_h u), \varphi)_T = -(Q_0 u, \nabla \varphi)_T + (Q_b u, \varphi n)_{\partial T} \\
= -(u, \nabla \varphi)_T + (u, \varphi n)_{\partial T} \\
= (\nabla \cdot u, \varphi)_T \\
= (Q_h(\nabla \cdot u), \varphi)_T.
\]

Similarly, for \( T \in \mathcal{T}_k^i \) and \( \varphi \in P_{k-1}(T) \), using Eq. (2.3) to derive

\[
(\nabla_w \cdot (Q_h u), \varphi)_T = -(Q_0 u, \nabla \varphi)_T + (Q_b u, \varphi n)_{\partial T} \\
= -(u, \nabla \varphi)_T + (u, \varphi n)_{\partial T} + (Q_b u - u, \varphi n)_{\partial T \cap \Gamma} \\
= (\nabla \cdot u, \varphi)_T + (Q_b u - u, \varphi n)_{\partial T \cap \Gamma} \\
= (Q_h(\nabla \cdot u), \varphi)_T + (Q_h u - u, \varphi n)_{\partial T \cap \Gamma}.
\]

The proof of the above lemma is complete.

Now, we give some important inequalities for the proof. The trace inequality and the inverse inequality are essential technique tools for the analysis. For the straight triangular cells, these inequalities have been proved in [33]. Next we extend two inequalities to the curved cells.

**Lemma 3.3.** (Inverse Inequality) For all \( T \in \mathcal{T}_h^i \), \( \varphi \) is the piecewise polynomial on \( T \), then we have

\[
\|\nabla \varphi\|_T \leq C h_T^{-1} \|\varphi\|_T. \tag{3.5}
\]
Proof. For any $T \in \mathcal{T}_h^I$, assume that $S(T)$ is the circumscribed simplex satisfying the shape regularity conditions. According to the standard inverse inequality \[33\] on $S(T)$, we have
\[
\|\nabla \varphi\|_T \leq \|\nabla \varphi\|_{S(T)} \leq Ch_{S(T)^{-1}}\|\varphi\|_{S(T)}.
\] (3.6)
Next, consider a ball, denoted as $S$, which is contained entirely within $T$ and has a diameter that is proportional to $h_T$. Then by the domain inverse inequality \[33\], we get
\[
\|\varphi\|_{S(T)} \leq \|\varphi\|_S \leq \|\varphi\|_T.
\] (3.7)
Substituting inequality (3.7) into the inequality (3.6) leads to
\[
\|\nabla \varphi\|_T \leq Ch_{T^{-1}}\|\varphi\|_T.
\]
The proof of the inverse inequality on the curved cells is complete. \(\square\)

Lemma 3.4. (Trace Inequality) For any $T \in \mathcal{T}_h^I$ and $\varphi \in H^1(T)$, we have
\[
\|\varphi\|_e^2 \leq C \left( h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla \varphi\|_T^2 \right), \quad e \subset \partial T.
\] (3.8)
Proof. For simplify, we consider curved triangular cells. Similarly, the trace inequality holds true for polygons. For $T \in \mathcal{T}_h^I$, assume that the non-affine transformation between $T$ and the reference element $\hat{T}$ is as follows
\[
\begin{align*}
x &= \tilde{x}(\xi, \eta), \quad (\xi, \eta) \in \hat{T} \\
y &= \tilde{y}(\xi, \eta), \quad (\xi, \eta) \in \hat{T},
\end{align*}
\]
and denote that $\tilde{\varphi}(\xi, \eta) = \varphi(\tilde{x}(\xi, \eta), \tilde{y}(\xi, \eta))$. $\tilde{J}$ is the corresponding Jacobian of the above mapping. Suppose that the edge $e$ possesses parametric representations given by $x = x(t)$ and $y = y(t)$, $0 \leq t \leq 1$. According to the regularity conditions and Theorem 1 in [40], every term of Eq. (3.8) has the following estimate,
\[
\begin{align*}
\|\varphi\|_e^2 &= \int_e \varphi^2(x, y)ds \\
&= \int_0^1 \varphi^2(x(t), y(t))|e|dt \\
&= \int_0^1 \tilde{\varphi}^2(\xi(t), \eta(t))|e|dt \\
&\leq Ch_T\|\tilde{\varphi}\|_e^2,
\end{align*}
\]
and
\[
\begin{align*}
\|\varphi\|_T^2 &= \int_T \varphi^2(x, y)^2dT \\
&= \int_{\tilde{T}} \tilde{\varphi}^2(\xi, \eta)|\tilde{J}|d\tilde{T} \\
&\geq Ch_T^2 \int_{\tilde{T}} \tilde{\varphi}^2(\xi, \eta)d\tilde{T} \\
&\geq Ch_T^2 h_T^{-1}\|\tilde{\varphi}\|_{\tilde{T}}^2,
\end{align*}
\]
and
\[
\left\|\nabla \varphi\right\|^2_T = \int_T \left( \frac{\partial \varphi}{\partial x}(x, y) \right)^2 + \left( \frac{\partial \varphi}{\partial y}(x, y) \right)^2 \, dT \\
= \int_T \left( \frac{\partial \varphi}{\partial x}(\tilde{x}(\xi, \eta), \tilde{y}(\xi, \eta)) \right)^2 + \left( \frac{\partial \varphi}{\partial y}(\tilde{x}(\xi, \eta), \tilde{y}(\xi, \eta)) \right)^2 \, |J| d\tilde{T} \\
\geq C \int_T \left( \frac{\partial \varphi}{\partial x} \frac{\partial \tilde{x}}{\partial \xi} + \frac{\partial \varphi}{\partial y} \frac{\partial \tilde{y}}{\partial \xi} \right)^2 + \left( \frac{\partial \varphi}{\partial x} \frac{\partial \tilde{x}}{\partial \eta} + \frac{\partial \varphi}{\partial y} \frac{\partial \tilde{y}}{\partial \eta} \right)^2 \, d\tilde{T} \\
= \int_T \left( \frac{\partial \varphi}{\partial \xi} \right)^2 + \left( \frac{\partial \varphi}{\partial \eta} \right)^2 \, d\tilde{T} \\
\geq C_h \left\| \nabla \tilde{\varphi} \right\|^2_{T, \xi}.
\]

Therefore, according to the trace inequality \[33\] on the reference element, we have
\[
\left\|\varphi\right\|^2_T \leq C h_T \left\| \varphi \right\|^2_T + C h_T \left\| \nabla \tilde{\varphi} \right\|^2_{T, \xi} \leq C h_T^{-1} \left\| \varphi \right\|^2_T + C h_T \left\| \nabla \varphi \right\|^2_{T, \xi}.
\]

The proof of the trace inequality is complete. \(\square\)

**Lemma 3.5.** For \(w \in [H^{l+1}(\Omega)]^2, \rho \in H^l(\Omega)\) with \(1 \leq l \leq k\) and \(0 \leq s \leq 1\), we have the following estimates
\[
\begin{align*}
\sum_{T \in \mathcal{T}_h} h_T^{2s} \| w - Q_h w \|^2_{T,s} & \leq C h^{2(l+1)} \| w \|^2_{l+1}, \quad (3.9) \\
\sum_{T \in \mathcal{T}_h} h_T^{2s} \| \nabla w - Q_h \nabla w \|^2_{T,s} & \leq C h^{2l} \| w \|^2_{l+1}, \quad (3.10) \\
\sum_{T \in \mathcal{T}_h} h_T^{2s} \| \rho - Q_h \rho \|^2_{T,s} & \leq C h^{2l} \| \rho \|^2_{l}. \quad (3.11)
\end{align*}
\]

Here \(C\) represents a positive constant that remains independent of both the mesh size \(h\) and the functions involved in the above estimates.

**Proof.** For \(T \in \mathcal{T}_h^S\), the inequalities \[3.9\]–\[3.11\] are same as the inequalities in \[34\]. Therefore we only consider the curved cell \(T \in \mathcal{T}_h^C\). Assume that \(S(T)\) is the circumscribed simplex satisfying the shape regularity conditions. And let \(u\) smoothly extend onto \(S(T)\). Denote by \(\tilde{Q}_h\) the \(L^2\) projection operator onto \([P_k(S(T))]^2\). Then we have
\[
\sum_{T \in \mathcal{T}_h} \| u - Q_h u \|^2_T \leq C \sum_{T \in \mathcal{T}_h} \| u - \tilde{Q}_h u \|^2_T \\
\leq C \sum_{T \in \mathcal{T}_h} \| u - \tilde{Q}_h u \|^2_{S(T)} \\
\leq C h^{2(r+1)} \| u \|^2_{r+1, S(T)} \\
\leq C h^{2(r+1)} \| u \|^2_{r+1, S(T)}.
\]
From the regularity conditions, the number of overlaps of circumscribed simplex sets is fixed. Then we derive

$$\sum_{T \in \mathcal{T}_h} \|u - Q_h u\|_T^2 \leq C h^{2(r+1)} \|u\|_{r+1}^2.$$  

Similarly, we get

$$\sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla (u - Q_h u)\|_{T,s}^2 \leq C h^{2(r+1)} \|u\|_{r+1}^2.$$  

Therefore, the proof of Eq. (3.9) is complete. The proof of Eqs. (3.10)-(3.11) is quite similar to the Eq. (3.9) and so is omitted.

Next we give the stability analysis of the Stokes interface problems.

**Lemma 3.6.** (Inf-Sup Condition) There are two positive constants $C_1$ and $C_2$ to satisfy the following inequality

$$\sup_{v \in V_h^0} \frac{b(v, \rho)}{\|v\|} \geq C_1 \|\rho\| - C_2 h |\rho|_1,$$  

for all $\rho \in W_h$, where $C_1$ and $C_2$ are independent of the mesh size $h$.

**Proof.** For $\rho \in W_h$, according to [5, 6, 9, 11, 14, 34], we can find a function $v \in [H^1_0(\Omega)]^2$ to satisfy

$$\nabla \cdot v = -\rho,$$

and $\|v\|_1 \leq C \|\rho\|$. Here $C > 0$ is a constant depending on the domain $\Omega$. Let $\tilde{v} = Q_h v \in V_h^0$, then we need to verify the following inequality holds true:

$$\|\tilde{v}\| \leq C_0 \|v\|_1.$$  

(3.13)

First, for $T \in \mathcal{T}_h^S$, it follows from the definition of $\| \cdot \|$ and Eq. (3.1) that

$$A \|\nabla w \tilde{v}\|_T^2 = A \|\nabla (Q_h v)\|_T^2 = A \|Q_h (\nabla v)\|_T^2 \leq A \|\nabla v\|_T^2 \leq C \|v\|_{1,T}^2.$$  

(3.14)

Next, using the trace inequality, the definition of $Q_h$ and the estimate (3.9), we derive

$$s(\tilde{v}, \tilde{v})_T = A h_T^{-1} \|Q_h (Q_h v) - Q_h v\|_{\partial T}^2$$

$$\leq C (h_T^{-2} \|Q_h v - v\|_{1,T}^2 + \|Q_h (\nabla v - v)\|_{0,T}^2)$$

(3.15)

$$\leq C \|v\|_{1,T}^2.$$  

For $T \in \mathcal{T}_h^I$, by Eq. (3.2), the Cauchy-Schwarz inequality and the trace inequality, we have

$$\|\nabla w \tilde{v}\|_T^2 = (Q_h (\nabla v), \nabla w \tilde{v})_T + (Q_h v - v, (\nabla w \tilde{v})_{\partial T})$$

$$\leq \|Q_h (\nabla v)\|_T \|\nabla w \tilde{v}\|_T + \|Q_h v - v\|_{\partial T} \|\nabla w \tilde{v}\|_{\partial T}$$

$$\leq C \|v\|_{1,T} \|\nabla w \tilde{v}\|_T.$$  

(3.16)
Consequently, combining inequalities (3.14)-(3.18) yields inequality (3.13).

For stabilization on $T \in T_h^I$, according to the property of the $L^2$ projection operator, the trace inequality and the estimate (3.9), we get

\[
s(\tilde{v}, \tilde{v})_T = Ah^{-1}_T \|Q_b(Q_0v) - Q_0v\|_{\tilde{T}}^2 + Ah^{-1}_T \|Q_0v - Q_bv\|_{\tilde{T}}^2
\leq Ch^{-1}_T \|Q_0v - v\|_{\tilde{T}}^2 + Ch^{-1}_T \|Q_0v - v\|_{\tilde{T}}^2 + C h^{-1}_T \|v - Q_bv\|_{\tilde{T}}^2
\leq C\|v\|^2_{1,T}.
\]

Consequently, combining inequalities (3.14)-(3.18) yields inequality (3.13).

The next step in the proof is to estimate $b(v, \rho)$. For all $\rho \in W_h$, based on Lemma 3.2 and the definition of $Q_h$, it’s easy to see that

\[
\begin{align*}
- \sum_{T \in T_h} (\nabla \cdot (Q_bv), \rho)_T & = - \sum_{T \in T_h} (Q_b(\nabla \cdot v), \rho)_T - \sum_{e \in \Gamma} (Q_bv - v, \rho n)_e \\
& = - \sum_{T \in T_h} (\nabla \cdot v, \rho)_T - \sum_{e \in \Gamma} (Q_bv - v, \rho n)_e \\
& = \|\rho\|^2 - \sum_{e \in \Gamma} (Q_bv - v, \rho n - Q_b(\rho n))_e \\
& \geq \|\rho\|^2 - C h \|v\|_1 \|\rho\|_1,
\end{align*}
\]

where we have used the fact that $\sum_{e \in \Gamma} (Q_bv - v, Q_b(\rho n))_e = 0$. Combining inequality (3.13) with inequality (3.19), we have

\[
\frac{b(v, \rho)}{\|v\|} \geq \frac{\|\rho\|^2 - C h \|v\|_1 \|\rho\|_1}{C_0 \|v\|_1} \geq C_1 \|\rho\| - C_2 h \|\rho\|_1.
\]

This indicates that inequality (3.12) holds true.

\[\square\]

4. Error Equations

In this section, we present the error equations for $u$ and $p$. We use $(u_h, p_h)$ to represent the numerical solutions obtained from the WG scheme. At the same time, denote by $(u, p)$ the exact solutions of Eqs. (1.1)-(1.5). The errors associated with $u$ and $p$ are defined as follows:

\[
e_h = Q_hu - u_h, \quad \varepsilon_h = Q_hp - p_h.
\]

**Lemma 4.1.** For $(u_i, p_i) \in [H^1(\Omega_i)]^2 \times L^2(\Omega_i)$ with $i = 1, 2$ and $p \in L^2(\Omega)$ satisfying Eqs. (1.1)-(1.5), we derive the following equation:

\[
a(Q_hu, v) + b(v, Q_hp) = (f, v_0) + \sum_{e \in \Gamma} (\psi, v_0)_e + \ell_1(u, v) - \ell_2(p, v) + \ell_3(u, v),
\]

\[(4.2)\]
where

\begin{align}
\ell_1(u, v) &= \sum_{T \in T_h} \langle v_0 - v_b, A \nabla u \cdot n - A Q_h (\nabla u) \cdot n \rangle_{\partial T}, \\
\ell_2(p, v) &= \sum_{T \in T_h} \langle v_0 - v_b, p n - (Q_h p) n \rangle_e, \\
\ell_3(u, v) &= \sum_{i=1}^{2} \sum_{T \in T_h} \langle Q_b u_i - u_i, A_i \nabla_w v_i \cdot n_i \rangle_e. 
\end{align}

**Proof.** It follows from Lemma 3.1, the definition of the weak gradient operator and integration by parts that

\begin{align}
\langle \nabla_w (Q_h u), A \nabla_w v \rangle \\
= \sum_{i=1}^{2} \sum_{T \in T_h} \langle \nabla_w (Q_h u_i), A_i \nabla_w v_i \rangle_T \\
= \sum_{i=1}^{2} \sum_{T \in T_h} \langle Q_h (\nabla u_i), A_i \nabla_w v_i \rangle_T + \sum_{i=1}^{2} \sum_{T \in T_h} \langle Q_b u_i - u_i, A_i \nabla_w v_i \cdot n_i \rangle_{\partial T \cap \Gamma} \\
= \sum_{i=1}^{2} \sum_{T \in T_h} \langle (\nabla v_{i,0}, A_i \nabla u_i) - \langle v_{i,0} - v_{i,b}, A_i Q_h (\nabla u_i) \cdot n_i \rangle_{\partial T} \rangle_T \\
&+ \sum_{i=1}^{2} \sum_{T \in T_h} \langle Q_b u_i - u_i, A_i \nabla_w v_i \cdot n_i \rangle_{\partial T \cap \Gamma}.
\end{align}

Similarly, using the definition of the weak divergence operator and integration by parts, we have

\begin{align}
- \langle \nabla \cdot v, Q_h p \rangle \\
= - \sum_{i=1}^{2} \sum_{T \in T_h} \langle \nabla \cdot v_i, Q_h p_i \rangle_T \\
= \sum_{i=1}^{2} \sum_{T \in T_h} \langle v_{i,0}, \nabla (Q_h p_i) \rangle_T - \langle v_{i,b}, (Q_h p_i) n_i \rangle_{\partial T} \\
= \sum_{i=1}^{2} \sum_{T \in T_h} \langle \nabla v_{i,0}, Q_h p_i \rangle_T + \langle v_{i,0} - v_{i,b}, (Q_h p_i) n_i \rangle_{\partial T} \\
= \sum_{i=1}^{2} \sum_{T \in T_h} \langle \nabla v_{i,0}, p_i \rangle_T + \langle v_{i,0} - v_{i,b}, (Q_h p_i) n_i \rangle_{\partial T}.
\end{align}
Then integrate with \(v_0 \in v_0^0 \in V_h^0\) on two sides of Eq.(1.1), we obtain
\[
-(\nabla \cdot (A\nabla u), v_0) + (\nabla p, v_0) = (f, v_0).
\] (4.8)

Using integration by parts, we get
\[
- (\nabla \cdot (A\nabla u), v_0) = \sum_{i=1}^2 \sum_{T \in T_h} (-\nabla \cdot (A_i \nabla u_i), v_{i,0})_T
\]
\[
= \sum_{i=1}^2 \sum_{T \in T_h} (A_i \nabla u_i \cdot \nabla v_{i,0})_T - \sum_{T \in T_h} (A_i \nabla u_i \cdot n_i, v_{i,0})_{\partial T}
\] (4.9)
\[
= \sum_{i=1}^2 \sum_{T \in T_h} ((A_i \nabla u_i, \nabla v_{i,0})_T - (v_{i,0} - v_{i,b}, A_i \nabla u_i \cdot n_i)_{\partial T})
\]
\[
- \sum_{\epsilon \in \Gamma} \langle v_{i,b}, A_i \nabla u_i \cdot n_1 + A_2 \nabla u_2 \cdot n_2 \rangle_e,
\]
where we have used the fact that \(\sum_{\epsilon \in E_h^\Gamma} \langle v_b, \nabla u \cdot n \rangle_e = 0\).

Then by integration by parts and the fact that \(\sum_{\epsilon \in E_h^\Gamma} \langle v_b, pn \rangle_e = 0\), we have
\[
(\nabla p, v_0) = \sum_{i=1}^2 \sum_{T \in T_h} (\nabla p_i, v_{i,0})_T
\]
\[
= \sum_{i=1}^2 \sum_{T \in T_h} (-\nabla \cdot (v_{i,0}, p_i) + \langle v_{i,0}, p_i n \rangle_{\partial T})
\] (4.10)
\[
= \sum_{i=1}^2 \sum_{T \in T_h} (-\nabla \cdot (v_{i,0}, p) + \langle v_{i,0}, p n \rangle_{\partial T}) + \sum_{\epsilon \in \Gamma} \langle v_{i,b}, p n_1 + p n_2 \rangle_e.
\]

Substituting Eq.(4.9) and Eq.(4.10) into Eq.(4.8) and using the interface condition (1.5), we have
\[
\sum_{T \in T_h} ((A\nabla u, \nabla v_0)_T - \langle v_0 - v_b, A\nabla u \cdot n \rangle_{\partial T} - (\nabla \cdot v_0, p)_T + \langle v_0 - v_b, pn \rangle_{\partial T})
\]
\[
= (f, v_0) + \sum_{\epsilon \in \Gamma} \langle v_b, \psi \rangle_e.
\] (4.11)

Adding Eq.(4.7) to Eq.(4.6), in light of Eq.(4.11), the proof of Eq.(4.12) is complete. \(\square\)

**Lemma 4.2.** For the \((u_i, p_i) \in [H^1(\Omega)]^2 \times L^2(\Omega)\) with \(i = 1, 2\) and \(p \in L_0^2(\Omega)\) satisfying Eqs.(1.1)-
(1.2), the errors \(e_h \) and \(\varepsilon_h \) satisfy the following equations:

\[
a_s(e_h, v) + b(v, \varepsilon_h) = \ell_4(u, v),
\]
\[
b(e_h, q) = -\ell_4(u, q),
\]

where
\[
\ell_4(u, q) = \sum_{i=1}^2 \sum_{\epsilon \in \Gamma} \langle Q_h u_i - u_i, q \rangle_e,
\]

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for any $v \in V_h^0$ and $q \in W_h$.

**Proof.** Since the exact solution $(u, p)$ satisfies the Eqs. (1.1)-(1.5), according to Lemma 4.1, we have

$$a(Q_h u, v) + b(v, Q_h p) = (f, v_0) + \sum_{e \in \Gamma} \langle \psi, v_b \rangle_e + \ell_1(u, v) - \ell_2(p, v) + \ell_3(u, v).$$

Adding $s(Q_h u, v)$ to both sides of the above equation and subtracting from Eq. (2.4) leads to Eq. (4.12). By incorporating with Lemma 3.2, we have

$$- (\nabla w \cdot (Q_h u), q) = \sum_{i=1}^2 \sum_{T \in T_h} - (\nabla w \cdot (Q_h u_i), q_i)_T$$

$$= \sum_{i=1}^2 \sum_{T \in T_h} (Q_h (\nabla \cdot u_i), q_i)_T - \sum_{i=1}^2 \sum_{e \in \Gamma} \langle Q_h u_i - u_i, q_i \rangle_e$$

$$= \sum_{i=1}^2 \sum_{e \in \Gamma} - \langle Q_h u_i - u_i, q_i \rangle_e.$$ (4.14)

Then, subtracting Eq. (4.14) from Eq. (2.5), we derive

$$b(e_h, q) = - \ell_4(u, q).$$ (4.15)

Hence the error equations are proved. $\square$

**5. Error Estimate in the Energy Norm**

In this section, we establish optimal order estimates for error $e_h$ of velocity function and error $\varepsilon_h$ of pressure function in the energy norm.

**Lemma 5.1.** \cite{[34]} For any $v = \{v_0, v_b\} \in V_h$ and $T \in T_h^S$, we obtain

$$\|\nabla v_0\|_T^2 \leq C \|v\|^2.$$ (5.1)

**Lemma 5.2.** Suppose that $u_i \in [H^{k+1}(\Omega_i)]^2$ and $p_i \in H^k(\Omega_i)$ with $i = 1, 2$, we have

$$|\ell_1(u, v)| \leq C h^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2})\|v\|,$$ (5.2)

$$|\ell_2(p, v)| \leq C h^k(\|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2})\|v\|,$$ (5.3)

$$|\ell_3(u, v)| \leq C h^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2})\|v\|,$$ (5.4)

$$|\ell_4(u, q)| \leq C h^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2})\|q\|,$$ (5.5)

$$|s(Q_h u, v)| \leq C h^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2})\|v\|.$$ (5.6)

for all $v \in V_h^0$ and $q \in W_h$. 14
Proof. As to the estimate (5.2), according to the Cauchy-Schwarz inequality, we have

\[ |\ell_1(u, v)| = \left| \sum_{T \in T_h} (v_0 - v_b, A(\nabla u) \cdot n - A_Qh(\nabla u) \cdot n)_{\partial T} \right| \]

\[ \leq A \sum_{T \in T_h} \|v_0 - v_b\|_{\partial T} \|\nabla u - Qh(\nabla u)\|_{\partial T} \]

\[ \leq A \left( \sum_{T \in T_h} \|v_0 - v_b\|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|\nabla u - Qh(\nabla u)\|^2_{\partial T} \right)^{\frac{1}{2}}. \]  (5.7)

It follows from the trace inequality and the estimate (3.10) that

\[ \sum_{T \in T_h} \|\nabla u - Qh(\nabla u)\|^2_{\partial T} \]

\[ \leq \sum_{T \in T_h} C \left( h^{-1}_T \|\nabla u - Qh(\nabla u)\|_{T}^2 + h_T \|\nabla(\nabla u - Qh(\nabla u))\|_{\partial T}^2 \right) \]  \[ \leq C h^{2k-1} (\|u_1\|^2_{k+1, \Omega_1} + \|u_2\|^2_{k+1, \Omega_2}). \]  (5.8)

Using the triangle inequality, the trace inequality, the Cauchy-Schwarz inequality and the estimate (3.9), as well as the estimate (5.1), we obtain

\[ \sum_{T \in T_h} \|v_0 - v_b\|^2_{\partial T} \]

\[ \leq \sum_{T \in T_h} 2 \left( \|v_0 - Qh v_0\|^2_{\partial T} + \|Qh v_0 - v_b\|^2_{\partial T} \right) + \sum_{T \in T_h} \|v_0 - v_b\|^2_{\partial T} \]

\[ \leq C \left( \sum_{T \in T_h} h_T \|\nabla v_0\|^2_{T} \right) + Ch \left( \sum_{T \in T_h} h^{-1}_T \|Qh v_0 - v_b\|^2_{\partial T} \right) + h \|v\|^2 \]

\[ \leq Ch \|v\|^2. \]  (5.9)

By the estimates (5.8)-(5.9), the proof of estimate (5.2) is complete.

For the estimate (5.3), the same techniques of proving the estimate (5.2) can be applied to derive

\[ |\ell_2(p, v)| = \left| \sum_{T \in T_h} (v_0 - v_b, p_n - Qh p_n)_{\partial T} \right| \]

\[ \leq \sum_{T \in T_h} \|v_0 - v_b\|_{\partial T} \|p - Qh p\|_{\partial T} \]

\[ \leq \left( \sum_{T \in T_h} h^{-1}_T \|v_0 - v_b\|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T \|p - Qh p\|^2_{\partial T} \right)^{\frac{1}{2}} \]

\[ \leq Ch^k (\|p_1\|_{\kappa, \Omega_1} + \|p_2\|_{\kappa, \Omega_2}) \|v\|. \]  (5.10)
Similarly, we obtain

\[
|s(Q_h u, v)| = \sum_{T \in T_h} Ah_T^{-1} \langle Q_0(Q_0 u - Q_h u, Q_0 v_0 - v_b)_{\partial T \setminus (\partial T \cap \Gamma)}
+ Ah_T^{-1} \langle Q_0 u - Q_h u, v_0 - v_b)_{\partial T \cap \Gamma} \rangle
\leq \sum_{T \in T_h} h_T^{-1} \|Q_0 u - u\|_{\partial T \setminus (\partial T \cap \Gamma)} \|Q_0 v_0 - v_b\|_{\partial T \setminus (\partial T \cap \Gamma)}
+ h_T^{-1} \|Q_0 u - Q_h u\|_{\partial T \cap \Gamma} \|v_0 - v_b\|_{\partial T \cap \Gamma}
\]

\[
\leq Ch^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2}) \|v\|.
\]

For \( \ell_3(u, v) \), by the definition of \( \|\cdot\| \), we have

\[
|\ell_3(u, v)| = \sum_{i=1}^2 \sum_{e \in \Gamma} \langle Q_h u_i - u_i, A_i \nabla_w v_i \cdot n_i \rangle_e
\leq C \sum_{i=1}^2 \sum_{e \in \Gamma} \|Q_h u_i - u_i\|_e \|A_i^{\frac{1}{2}} \nabla_w v_i\|_e
\leq \sum_{i=1}^2 \left( \sum_{e \in \Gamma} \|Q_h u_i - u_i\|_e^2 \right)^{\frac{1}{2}} \left( \sum_{e \in \Gamma} \|A_i^{\frac{1}{2}} \nabla_w v_i\|_e^2 \right)^{\frac{1}{2}}
\leq Ch^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2}) \|v\|.
\]

Similarly, we get

\[
|\ell_4(u, q)| = \sum_{i=1}^2 \sum_{e \in \Gamma} \langle Q_h u_i - u_i, q_i n_i \rangle_e
\leq C \sum_{i=1}^2 \sum_{e \in \Gamma} \|Q_h u_i - u_i\|_e \|q\|_e
\leq \sum_{i=1}^2 \left( \sum_{e \in \Gamma} \|Q_h u_i - u_i\|_e^2 \right)^{\frac{1}{2}} \left( \sum_{e \in \Gamma} \|q\|_e^2 \right)^{\frac{1}{2}}
\leq Ch^k(\|u_1\|_{k+1} + \|u_2\|_{k+1}) \|q\|.
\]

The proof of the lemma is complete.

Based on error equations \((4.12)-(4.13)\) and estimates \((5.2)-(5.6)\), we give the following error estimate.

**Theorem 5.1.** Assuming \((u_i, p_i) \in [H^{k+1}(\Omega_i)]^2 \times H^k(\Omega_i)\) with \(i = 1, 2\) are the exact solutions of the Eqs. \((1.4)-(1.5)\) and \((u_h, p_h) \in V_h \times W_h\) are numerical solutions obtained from the WG scheme, then we have

\[
\|Q_h u - u_h\| + \|Q_h p - p_h\| \leq Ch^k(\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2} + \|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2}).
\]
Proof. Choosing \( v = e_h \) in Eq. (4.12) and \( q = \varepsilon_h \) in Eq. (4.13) and adding the two equations, then we have

\[
 a_s(e_h, e_h) = \ell_1(u, e_h) - \ell_2(p, e_h) + \ell_3(u, e_h) + s(Q_h, u, e_h) + \ell_4(u, \varepsilon_h). \tag{5.15}
\]

To simplify, let \( \delta \) represent \( \|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2} + \|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2} \). According to Eqs. (5.2)–(5.6), we derive

\[
 \|e_h\|^2 \leq Ch^k \delta \|e_h\|^2 + Ch^k (\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2}) \|\varepsilon_h\|. \tag{5.16}
\]

According to Eq. (4.12) and the boundedness of \( a_s(\cdot, \cdot) \), we have

\[
 b(v, \varepsilon_h) = \ell_1(u, v) - \ell_2(p, v) - \ell_3(u, v) + s(Q_h, u, v) - a_s(e_h, v) \leq C \|e_h\| \|v\| + Ch^k \delta \|v\|. \tag{5.17}
\]

In particular, we take \( v = \{\nabla \varepsilon_h, 0\} \in V_h^0 \) to obtain

\[
 b(v, \varepsilon_h) = \sum_{T \in T_h} - \langle \nabla w \cdot v, \varepsilon_h \rangle_T = \sum_{T \in T_h} \left( \langle v_0, \nabla \varepsilon_h \rangle_T - \langle v_h, \varepsilon_h n \rangle_{\partial T} \right) \tag{5.18}
\]

\[
 = \| \nabla \varepsilon_h \|^2 - |\varepsilon_h|^2. \]

Moreover, for \( \|v\| \), it follows from the Cauchy-Schwarz inequality, the trace inequality and the inverse inequality that

\[
 \| \nabla w \|^2_T = - \langle v_0, \nabla \cdot (\nabla w) \rangle_T + \langle v_0, \nabla w \cdot n \rangle_{\partial T} \\
 = \langle \nabla v_0, \nabla w \rangle_T - \langle v_0, \nabla w \cdot n \rangle_{\partial T} \\
 = - \langle v_0, \nabla \cdot (\nabla v_0) \rangle_T - \langle v_0, \nabla w \cdot n \rangle_{\partial T} \\
 = \langle \nabla v_0, \nabla v_0 \rangle_T - \langle v_0, \nabla w \cdot n \rangle_{\partial T} \\
 \leq \| \nabla v_0 \|^2_T + \| v_0 \|^2_{\partial T} \| \nabla w \|^2_T \\
 \leq Ch^{-2} \| v_0 \|^2_T + Ch^{-1} \| v_0 \|^2_T \| \nabla v \|^2_T \\
 \leq Ch^{-2} \| v_0 \|^2_T + \frac{1}{2} \| \nabla w \|^2_T, \tag{5.19}
\]

i.e.

\[
 \sum_{T \in T_h} \| \nabla w \|^2_T \leq Ch^{-2} \sum_{T \in T_h} \| v_0 \|^2_T \leq Ch^{-2} | \varepsilon_h |^2. \tag{5.20}
\]

For the stabilizer term, we obtain

\[
 \sum_{T \in T_h} Ah^{-1} \| Q_h v_0 - v_b \|^2_{\partial T \cap \Omega} + A h^{-1} \| v_0 - v_b \|^2_{\partial T \cap \Gamma} \leq \sum_{T \in T_h} Ch^{-2} \| v_0 \|^2_{\partial T} \leq Ch^{-2} | \varepsilon_h |^2. \tag{5.21}
\]

Adding (5.20) to (5.21), we get

\[
 \| v \| \leq Ch^{-1} | \varepsilon_h |. \tag{5.22}
\]
Substituting Eq. (5.18) and Eq. (5.22) into Eq. (5.17), we have
\[ |\varepsilon_h| \leq C h^{-1} \| e_h \| + C h^{k-1} \delta. \]  
(5.23)

Next, according to the inf-sup condition (3.13), we derive
\[ C_1 \| e_h \| \leq C \| e_h \| + C h^k \delta. \]  
(5.24)

Substituting Eq. (5.24) into Eq. (5.16) gives rise to
\[ \| e_h \|^2 \leq C h^k \| e_h \| + C h^k (\| u_1 \|_{k+1,\Omega_1} + \| u_2 \|_{k+1,\Omega_2}) (C \| e_h \| + C h^k \delta) \leq C h^k \| e_h \| + C h^{2k} \delta^2 \leq C h^{2k} \delta^2 + \frac{1}{2} \| e_h \|^2, \]  
(5.25)

therefore we have
\[ \| e_h \| \leq C h^k \delta, \quad \| e_h \| \leq C h^k \delta. \]  
(5.26)

Therefore, the estimate (5.14) holds true.

\[ \square \]

6. Error Estimate in the $L^2$ Norm

In this section, we use the duality argument to derive an error estimate for $e_0 = Q_0 u - u_0$ in the $L^2$ norm. Now consider the following problem: seeking $(w, \theta)$ to satisfy
\[ -\nabla \cdot (A \nabla \omega) + \nabla \theta = e_0 \quad \text{in } \Omega, \]
\[ \nabla \cdot \omega = 0 \quad \text{in } \Omega, \]
\[ \omega = 0 \quad \text{on } \partial \Omega. \]
(6.1)

Assume the solution $(\omega, \theta)$ of dual problem (6.1)-(6.3) has $[H^2(\Omega)]^2 \times [H^1(\Omega)]$-regularity estimate, i.e.
\[ \| \omega \|_2 + \| \theta \|_1 \leq C \| e_0 \|. \]  
(6.4)

**Theorem 6.1.** Based on the assumptions in Theorem 5.1, we obtain the following error estimate:
\[ \| Q_0 u - u_0 \| \leq C h^{k+1} (\| u_1 \|_{k+1,\Omega_1} + \| u_2 \|_{k+1,\Omega_2} + \| p_1 \|_{k,\Omega_1} + \| p_2 \|_{k,\Omega_2}). \]  
(6.5)

**Proof.** Due to $(w, \theta)$ satisfies the Eq. (6.1) with $f = e_0 = Q_0 u - u_0$, then choosing $v = e_h$ in Eq. (4.12) and $q = Q_0 \theta$ in Eq. (4.13) leads to
\[ a_s(Q_h \omega, e_h) + b(e_h, Q_h \theta) = (e_0, e_h) + \ell_1(\omega, e_h) - \ell_2(\theta, e_h) + \ell_3(\omega, e_h) + s(Q_h \omega, e_h), \]
\[ b(e_h, Q_h \theta) = -\ell_4(u, Q_h \theta). \]
(6.6)

According to the definition of the weak divergence operator and Lemma 3.2 we have
\[ b(Q_h \omega, \varepsilon_h) = - \sum_{i=1}^2 \sum_{T \in T_h} (\nabla w \cdot Q_h \omega_i, \varepsilon_h)_T \]
\[ = - \sum_{i=1}^2 \sum_{T \in T_h} (Q_h \nabla \omega_i, \varepsilon_h)_T - \sum_{i=1}^2 \sum_{e \in T} (Q_h \omega_i - \omega_i, \varepsilon_h n_e)_e \]
\[ = - \ell_4(\omega, \varepsilon_h). \]  
(6.8)
Therefore, we get
\[
a_s(Q_h\omega, e_h) + b(Q_h\omega, \varepsilon_h) = \|e_0\|^2 + \ell_1(\omega, e_h) - \ell_2(\theta, e_h) + \ell_3(\omega, e_h) + s(Q_h\omega, e_h) - \ell_4(\omega, \varepsilon_h) + \ell_4(u, Q_h\theta). \tag{6.9}
\]

From the above equations and Eq.(4.12), we obtain
\[
\|e_0\|^2 = \ell_1(u, Q_h\omega) - \ell_2(p, Q_h\omega) + \ell_3(u, Q_h\omega) + s(Q_hu, Q_h\omega) - \ell_4(u, Q_h\theta)
- \ell_1(\omega, e_h) + \ell_2(\theta, e_h) - \ell_3(\omega, e_h) - s(Q_h\omega, e_h) - \ell_4(\omega, \varepsilon_h).
\]

According to Lemma 5.2, we have
\[
| - \ell_1(\omega, e_h) + \ell_2(\theta, e_h) - \ell_3(\omega, e_h) - s(Q_h\omega, e_h - \ell_4(\omega, \varepsilon_h)| 
\leq Ch(\|\omega\|_2 + \|\theta\|_1)(\|e_h\| + \|\varepsilon_h\|)
\leq Ch\|e_h\|\|\varepsilon_0\|.
\]

Each of the remaining terms is handled as follows.

(1) For \(\ell_1(u, Q_h\omega)\), we use the Cauchy-Schwarz inequality, the trace inequality and the estimate (3.9) to derive
\[
\sum_{T \in T_h} \langle Q_0\omega - \omega, A\nabla u \cdot n - A\nabla_h(\nabla u) \cdot n \rangle_{\partial T}
\leq C \left( \sum_{T \in T_h} \|Q_0\omega - \omega\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|\nabla u - Q_h(\nabla u)\|_{\partial T}^2 \right)^{\frac{1}{2}}
\leq \left( C \sum_{T \in T_h} (h_T^{-1}\|Q_0\omega - \omega\|_T^2 + h_T\|\nabla(Q_0\omega - \omega)\|_T^2) \right)^{\frac{1}{2}}
\leq \left( C \sum_{T \in T_h} (h_T^{-1}\|\nabla u - Q_h(\nabla u)\|_T^2 + h_T\|\nabla(\nabla u - Q_h(\nabla u))\|_T^2) \right)^{\frac{1}{2}}
\leq \left( \sum_{T \in T_h} Ch_T^3\|\omega\|_T^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} Ch_T^{2k-1}\|u\|_{k+1}^2 \right)^{\frac{1}{2}}
\leq Ch^{k+1}\|\omega\|_{2}||\nabla u||_{k+1,\Omega_2} ||\nabla u||_{k+1,\Omega_2}.
\]

Next, we use same techniques and the fact that \(\sum_{c \in E_h^o} \langle \omega - Q_h\omega, A(\nabla u) \cdot n - A\nabla_h(\nabla u) \cdot n \rangle_c = 0\) to obtain
\[
\bigg| \sum_{T \in T_h} \langle \omega - Q_h\omega, A\nabla u \cdot n - A\nabla_h(\nabla u) \cdot n \rangle_{\partial T} \bigg|
= \sum_{c \in E_h^o} \langle \omega - Q_h\omega, A(\nabla u) \cdot n - A\nabla_h(\nabla u) \cdot n \rangle_c
\leq \sum_{c \in E_h^o} \|\omega - Q_h\omega\|_c \|A\nabla u - A\nabla_h(\nabla u)\|_c
\]
\[
\leq C \left( \sum_{e \in E^I_h} \|\omega - Q_h \omega\|^2_e \right)^{\frac{1}{2}} \left( \sum_{e \in E^I_h} \|\nabla u - Q_h (\nabla u)\|^2_e \right)^{\frac{1}{2}} \\
\leq C \left( h^{k - \frac{1}{2}} \|u\|_{k+1} \right) \left( \sum_{e \in E^I_h} \|\omega - Q_h \omega\|^2_e \right)^{\frac{1}{2}},
\]

where
\[
\|\omega - Q_h \omega\|^2_e \leq \|\omega - Q_0 \omega\|^2_e \leq \|\omega - Q_0 \omega\|^2_{\partial T} \leq Ch^3 \|\omega\|_2^2.
\]

Using the above inequality, we have
\[
\sum_{T \in T_h} \langle \omega - Q_h \omega, A\nabla u \cdot n - AQ_h (\nabla u) \cdot n \rangle_{\partial T} \leq Ch^{k+1} \|\omega\|_2 (\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2}).
\]

Therefore,
\[
|\ell_1(u, Q_h \omega)| = \left| \sum_{T \in T_h} \langle Q_0 \omega - Q_h \omega, A\nabla u \cdot n - AQ_h (\nabla u) \cdot n \rangle_{\partial T} \right|
\leq \left| \sum_{T \in T_h} \langle Q_0 \omega - \omega, A\nabla u \cdot n - AQ_h (\nabla u) \cdot n \rangle_{\partial T} \right| \\
+ \left| \sum_{T \in T_h} \langle Q_0 \omega - Q_h \omega, A\nabla u \cdot n - AQ_h (\nabla u) \cdot n \rangle_{\partial T} \right|
\leq Ch^{k+1} \|\omega\|_2 (\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2}).

(2) For \( \ell_2(p, Q_h \omega) \), we use the same method as the proof of \( \ell_1(u, Q_h \omega) \) to get
\[
\sum_{T \in T_h} \langle Q_0 \omega - \omega, (p - Q_h p) \rangle_{\partial T}
\leq \left( \sum_{T \in T_h} \|Q_0 \omega - \omega\|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|p - Q_h p\|^2_{\partial T} \right)^{\frac{1}{2}}
\leq C \left( \sum_{T \in T_h} (h^{-1} \|Q_0 \omega - \omega\|^2_T + hT \|\nabla (Q_0 \omega - \omega)\|^2_T) \right)^{\frac{1}{2}}
\leq \left( \sum_{T \in T_h} Ch^3 \|\omega\|^2_T \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} Ch^{2k-1} \|p\|^2_T \right)^{\frac{1}{2}}
\leq Ch^{k+1} \|\omega\|_2 \|p\|_k.
\]

Similarly, we have
\[
\sum_{T \in T_h} \langle \omega - Q_h \omega, (p - Q_h p) \rangle_{\partial T} \leq Ch^{k+1} \|\omega\|_2 (\|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2}).
\]
Therefore, for $\ell_2(p, Qh\omega)$, we derive the following estimate

$$|\ell_2(p, Qh\omega)| \leq CH^{k+4}\|\omega\|_2\|p\|_k. \tag{6.11}$$

(3) For $s(Qh u, Qh\omega)$, we get

$$|s(Qh u, Qh\omega)| \leq \left| \sum_{T \in T_h} Ah^{-1} T \langle Q_h (Q_0 u) - Q_0 u, Q_h (Q_0 \omega) - Q_0 \omega \rangle_{\partial T \setminus (\partial T \cap \Gamma)} \right|$$

$$+ \left| \sum_{T \in T_h} Ah^{-1} T \langle Q_0 u - Q_h u, Q_0 \omega - Q_h \omega \rangle_{\partial T \cap \Gamma} \right|$$

$$\leq C \left( \sum_{T \in T_h} h^{-1}_T \|Q_0 u - u\|_{\partial T \setminus (\partial T \cap \Gamma)}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h^{-1}_T \|Q_0 \omega - \omega\|_{\partial T \setminus (\partial T \cap \Gamma)}^2 \right)^{\frac{1}{2}}$$

$$+ \left( \sum_{T \in T_h} h^{-1}_T \|Q_0 u - u\|_{\partial T \cap \Gamma} + \|Q_h u - u\|_{\partial T \cap \Gamma} \right)$$

$$\left( \sum_{T \in T_h} h^{-1}_T \|Q_0 \omega - \omega\|_{\partial T \cap \Gamma}^2 + \|Q_h \omega - \omega\|_{\partial T \cap \Gamma}^2 \right)^{\frac{1}{2}}$$

$$\leq C h^{k+4} \left( \|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2} \right) \|\omega\|_2. \tag{6.12}$$

(4) For $\ell_3(u, Qh\omega)$, using the fact that $\sum_{i=1}^{2} \sum_{c \in \Gamma} \langle Q_h u_i - u_i, Q_h(\nabla \omega_i \cdot n_i) \rangle_c = 0$, the Cauchy-Schwarz inequality, the trace inequality and the inverse inequality, we obtain

$$|\ell_3(u, Qh\omega)|$$

$$= \left| \sum_{i=1}^{2} \left| \sum_{c \in \Gamma} \langle Q_h u_i - u_i, A_i \nabla_w(Q_h \omega_i) \cdot n_i \rangle_c \right| \right|$$

$$\leq C \sum_{i=1}^{2} \sum_{c \in \Gamma} \|Q_h u_i - u_i\|_c \|\nabla_w(Q_h \omega_i) \cdot n_i - Q_h(\nabla \omega_i) \cdot n_i\|_c$$

$$+ C \sum_{i=1}^{2} \sum_{c \in \Gamma} \|Q_h u_i - u_i\|_c \|Q_h(\nabla \omega_i \cdot n_i) - (\nabla \omega_i \cdot n_i)\|_c$$

$$\leq \left( \sum_{i=1}^{2} \sum_{c \in \Gamma} \|Q_h u_i - u_i\|_c^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{2} \sum_{T \in T_h^c} h^{-1}_T \|\nabla_w(Q_h \omega_i) - Q_h(\nabla \omega_i)\|_T^2 \right)^{\frac{1}{2}}$$

$$+ \left( \sum_{i=1}^{2} \sum_{c \in \Gamma} \|Q_h u_i - u_i\|_c^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{2} \sum_{c \in \Gamma} \|Q_h(\nabla \omega_i) - \nabla \omega_i\|_c^2 \right)^{\frac{1}{2}}$$

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where by the trace inequality and the estimate (3.9), we have
\[ \sum_{i=1}^{2} \sum_{c \in \Gamma} \| Q_{b} u_{i} - u_{i} \|_{c}^{2} \leq \sum_{i=1}^{2} \sum_{c \in \Gamma} \| Q_{b} (\nabla \omega_{i} \cdot \mathbf{n}) - (\nabla \omega_{i} \cdot \mathbf{n}) \|_{c}^{2} \]

Similarly, we get the following estimates:
\[ \sum_{i=1}^{2} \sum_{c \in \Gamma} \| Q_{b} (\nabla \omega_{i} - \nabla \omega_{i}) \|_{c}^{2} \leq Ch \| \omega \|_{2}^{2} \]
\[ \sum_{i=1}^{2} \sum_{c \in \Gamma} \| Q_{b} (\nabla \omega_{i} \cdot \mathbf{n}) - (\nabla \omega_{i} \cdot \mathbf{n}) \|_{c}^{2} \leq Ch \| \omega \|_{2}^{2} \]

Next according to Eq. (3.2), we take \( \tau = \nabla_{w} (Q_{h} \omega) - Q_{h} (\nabla \omega) \) to lead to
\[ \| \nabla_{w} (Q_{h} \omega) - Q_{h} (\nabla \omega) \|_{T} \leq C h^{-1} \| Q_{h} \omega_{i} - \omega_{i} \|_{T} \leq Ch^{k+1} (\| u_{1} \|_{k+1, \Omega_{1}} + \| u_{2} \|_{k+1, \Omega_{2}}) \]

we have
\[ \| \nabla_{w} (Q_{h} \omega) - Q_{h} (\nabla \omega) \|_{T} \leq C h^{-1} \| Q_{h} \omega_{i} - \omega_{i} \|_{T} \leq Ch^{k+1} \| \omega \|_{2}^{2} \]

Combining the above four estimates, we get
\[ |\ell_{3}(\mathbf{u}, Q_{h} \theta) | \leq C h^{k+1} \| \omega \|_{2}^{2} (\| u_{1} \|_{k+1, \Omega_{1}} + \| u_{2} \|_{k+1, \Omega_{2}}) \]

(5) For \( \ell_{4}(\mathbf{u}, Q_{h} \theta) \), by the fact that \( \sum_{i=1}^{2} \sum_{c \in \Gamma} (Q_{b} u_{i} - u_{i}, Q_{b} \theta \mathbf{n})_{c} = 0 \), we have
\[ |\ell_{4}(\mathbf{u}, Q_{h} \theta) | = \left| \sum_{i=1}^{2} \sum_{c \in \Gamma} Q_{b} (u_{i} - u_{i}, (Q_{b} \theta \mathbf{n})_{c} \right| \leq \left| \sum_{i=1}^{2} \sum_{c \in \Gamma} (Q_{b} u_{i} - u_{i}, (Q_{b} \theta \mathbf{n})_{c} \right| + \left| \sum_{i=1}^{2} \sum_{c \in \Gamma} (Q_{b} u_{i} - u_{i}, Q_{b} (\theta \mathbf{n}) - \mathbf{n})_{c} \right| \leq Ch^{k+1} (\| u_{1} \|_{k+1, \Omega_{1}} + \| u_{2} \|_{k+1, \Omega_{2}}) \| \theta \|_{1} \]
Combining the five estimates (6.10)-(6.15), we obtain
\[
\|e_0\|^2 \leq Ch^{k+1} (\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2} + \|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2}) (\|w\|_2 + \|\theta\|_{1, 1}) \\
+ Ch\|e_h\|\|e_0\| \\
\leq Ch^{k+1} (\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2} + \|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2}) \|e_0\| \\
+ Ch\|e_h\|\|e_0\|.
\]

Finally, it follows from Theorem 5.1 that
\[
\|e_0\| \leq Ch^{k+1} (\|u_1\|_{k+1, \Omega_1} + \|u_2\|_{k+1, \Omega_2} + \|p_1\|_{k, \Omega_1} + \|p_2\|_{k, \Omega_2}) + Ch\|e_h\|\|e_0\|.
\]

The proof of theorem is complete.

7. Numerical Results

In this section, we give some numerical examples to validate the efficiency of the proposed WG method. We solve the interface problems in the domain \(\Omega = [-1, 1] \times [-1, 1]\) with different interfaces.

Example 7.1. In this example, we solve the interface problems with discontinuous velocity function and pressure function. The viscosity coefficient \(A\) is continuous in \(\Omega\). And the interface is described as
\[
x^2 + y^2 = \frac{1}{4}.
\]
The exact solutions are
\[
\begin{align*}
    u_1 &= \begin{pmatrix} 2 \sin y \cos y \cos x \\ (\sin^2 y - 2) \sin x \end{pmatrix}, \\
    u_2 &= \begin{pmatrix} -\cos(\pi x) \sin(\pi y) \\ \sin(\pi x) \cos(\pi y) \end{pmatrix}, \\
    p &= \begin{cases} 1 & \text{in } \Omega_1, \\
                        \frac{\pi}{16-\pi} & \text{in } \Omega_2 \\
    \end{cases}, \\
    A &= \begin{cases} 1 & \text{in } \Omega_1, \\
                        1 & \text{in } \Omega_2. \\
    \end{cases}
\end{align*}
\]

In Figures 2-4, we compare the numerical results on the straight triangular meshes and curved triangular meshes. On the straight triangular meshes, we use the straight segments to replace the curved interface. As we can see, the optimal order convergence is obtained by the \(P_1\) WG element in two cases. When using \(P_2\) and \(P_3\) WG elements to solve the problems, the orders of convergence are less than the optimal orders on the straight triangular meshes. However, all numerical solutions converge at the optimal rates on the curved triangular meshes. This comparison shows the advantages of our proposed WG scheme. The numerical solutions on the curved triangular meshes are plotted in Figure 5.

Example 7.2. In the example, the interface between two subdomains is described as:
\[
r = \frac{1}{2} + \frac{\sin(2\theta)}{4}.
\]
Figure 2: The numerical results for Example 7.1 on the curved triangular meshes (left) and straight triangular meshes (right) with $k = 1$.

Figure 3: The numerical results for Example 7.1 on the curved triangular meshes (left) and straight triangular meshes (right) with $k = 2$.

Figure 4: The numerical results for Example 7.1 on the curved triangular meshes (left) and straight triangular meshes (right) with $k = 3$. 

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The exact solutions are

\[
\begin{align*}
\mathbf{u}_1 & = \begin{pmatrix} 2 \sin y \cos y \cos x \\ (\sin^2 y - 2) \sin x \end{pmatrix}, \\
\mathbf{u}_2 & = \begin{pmatrix} -\cos(\pi x) \sin(\pi y) \\ \sin(\pi x) \cos(\pi y) \end{pmatrix}, \\
p & = \begin{cases} 
\cos(\pi x) \cos(\pi y) & \text{in } \Omega_1, \\
\cos(\pi x) \cos(\pi y) & \text{in } \Omega_2,
\end{cases}, \quad A = \begin{cases} 
1 & \text{in } \Omega_1, \\
10 & \text{in } \Omega_2.
\end{cases}
\end{align*}
\]

Figure 5: The numerical solutions by \( P_3 \) WG element on curved triangular meshes level 3 in Example 7.1. Left: The first component of \( \mathbf{u}_3 \). Middle: The second component of \( \mathbf{u}_3 \). Right: Pressure function \( p_0 \).

Figure 6: The numerical results for Example 7.2 on curved triangular meshes with \( k = 1 \) (left), 2 (middle), 3 (right).

Figure 7: The numerical results for Example 7.2 on curved quadrilateral meshes with \( k = 1 \) (left), 2 (middle), 3 (right).

In this example, we consider the interface problems with discontinuous velocity function \( \mathbf{u} \) and the viscosity coefficient \( A \). The pressure function \( p \) is continuous in the domain \( \Omega \). The velocity
Figure 8: The numerical solutions by $P_2$ WG element on triangular meshes level 3 in Example 7.2. Left: The first component of $u_h$. Middle: The second component of $u_h$. Right: Pressure function $p_h$.

The $u_h$ function and pressure function $p_h$ are plotted in Figure 8 respectively. The numerical results are plotted in Figures 6 - 7 by $P_1$ to $P_3$ WG elements on curved triangular meshes and curved quadrilateral meshes. The $P_k$ WG elements show the convergence orders $O(h^k)$ and $O(h^{k+1})$ for velocity functions in the energy norm and $L^2$ norm, respectively. For the pressure function, the $P_k$ WG elements achieve the convergence orders $O(h^k)$ in the $L^2$ norm. The orders of convergence are optimal in every case.

8. Conclusion

In this paper, we use the weak Galerkin finite element method to deal with Stokes interface problems with curved interface. We present a weak Galerkin finite element numerical scheme with two values at the interface. Based on the WG scheme, we prove that numerical solutions converge to the exact solutions at the optimal rates. Additionally, the numerical results from our examples show the optimal convergence orders are obtained in both the energy norm and the $L^2$ norm on the triangular meshes and quadrilateral meshes. These results align with the theoretical analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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