CONSTRUCTION OF MINIMAL LINEAR CODES FROM
MULTI-VARIABLE FUNCTIONS

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(Communicated by Sihem Mesnager)

ABSTRACT. In this paper, we define a linear code by using multi-variable functions, and construct three classes of minimal linear codes with few-weight from multi-variable functions.

1. INTRODUCTION

Let $C$ be a linear code over the finite field. By a minimal codeword in $C$ we mean that its support does not contain the support of any other nonzero codeword in $C$. A linear code is minimal if every nonzero codeword is minimal.

In 1979, Blakley [3] and Shamir [19] introduced secret sharing schemes independently that the dealer distributes a secret divided by pieces (or, shares) to parties (or, participants), and only authorized subsets of participants (access structure of participants) should be able to recover the secret from using their respective shares. In 1981, McEliece and Sarwate [17] provided interconnection between the Shamir's secret sharing scheme and Reed-Solomon codes. Since then the construction of code-based secret sharing schemes has been actively researched as follows.

A construction method of a secret sharing scheme based on a linear code $C$ and its dual code was established by Massey [16]. From [21], there is a one-to-one correspondence between the set of minimal access sets of the secret sharing scheme

2010 Mathematics Subject Classification: Primary: 94B05; Secondary: 94A60.

Key words and phrases: Minimal linear code, multi-variable function, AB-condition, HDZ-condition.

The first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (NRF-2017R1A2B2004574). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2019R1I1A1A01060467). The third author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2018R1D1A1B07046315).

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based on $C$ and the set of minimal codewords of the dual code $C^\perp$ whose first coordinate is 1. In addition, to reduce the complexity of a decoding algorithm, the authors in [1, 15] took account of the set of minimal codewords in $C$. Thus it has been an interesting research topic in coding theory whether or not a codeword in $C$ is minimal. To find all minimal codewords of $C$ is however very hard for general linear codes such as binary Reed-Muller codes [1, 5, 6, 18].

On the other hand, a huge number of classes of minimal linear codes with few weights which satisfy the sufficient condition (called the AB-condition) [1] of Ashikhmin-Barg to be minimal have been reported in the literature (see, [9, 10, 12, 13, 14, 16, 20, 21], for example). The study of minimal linear codes which does not satisfy the AB-condition has been done recently on the fields of characteristic $p = 2, 3$ [7, 8, 11, 12, 22], and of odd characteristic $p$ [2].

The purpose of this paper is to construct new classes of minimal linear codes with few-weight. For this, we give a new method of defining a linear code by using multi-variable functions. Section 2 contains some definitions, and previous results related to our work. In Section 3, we define a linear code via multi-variable function, and provide some examples. In Sections 4 and 5, we construct three classes of minimal linear codes; Section 4 (Theorem 4.2) presents minimal linear codes which is not satisfying AB-condition, and Section 5 (Theorem 5.1 (i) and (ii)) gives minimal linear codes satisfying AB-condition.

2. Preliminaries

Let $p$ be a prime number. Let $\mathbb{F}_p^n$ be an $n$-dimensional vector space over $\mathbb{F}_p$ being the finite field of order $p$, and $\mathbb{F}_p^n\ast$ the set of non-zero vectors in $\mathbb{F}_p^n$.

A $p$-ary linear code of length $l$ is a subspace of $\mathbb{F}_p^l$. A linear code $C$ over $\mathbb{F}_p$ of length $l$, dimension $n$ and minimum weight $d$ is denoted by $[l,n,d]_p$. The support $\text{Supp}(u)$ of a vector $u \in \mathbb{F}_p^l$ is the set of non-zero coordinate positions. We denote by $w(v)$ the (Hamming) weight of any codeword $v$ in $C$, that is, $w(v)$ is the size of its support. We say that a linear code is $t$-weight if the number of its non-zero weights is $t$.

A codeword $c$ covers a codeword $c'$ if $\text{Supp}(c') \subseteq \text{Supp}(c)$. A codeword $c$ in a linear code $C$ is minimal if $c$ covers only the codeword $ac$ for all $a \in \mathbb{F}_p^n$, but no other codewords in $C$. A linear code is minimal if every codeword is minimal.

We present a sufficient condition for a linear code to be minimal introduced by Ashikhmin and Barg [1].

**Proposition 1.** (Ashikhmin-Barg) A $p$-ary linear code $C$ with minimum distance $d$ is minimal provided that

$$\frac{d}{d_{\text{max}}} > \frac{p-1}{p},$$

where $d_{\text{max}}$ is the largest weight of any codewords of $C$.

Recently, Heng-Ding-Zhou [12] provided a characterization of minimal linear codes over finite fields.

**Proposition 2.** (Heng-Ding-Zhou) Let $C$ be a linear code over $\mathbb{F}_p$. Then the code $C$ is minimal if and only if

$$\sum_{s \in \mathbb{F}_p^n} w(u + sv) \neq (p-1)w(u) - w(v)$$
Lemma 3.1. Let $w$ weight for non-zero vectors $u,v \in \mathbb{F}_p^n$ associated with $\mathcal{H}(x_1, x_2, \ldots, x_n)$ be a prime number and let $\mathcal{H}(x_1, x_2, \ldots, x_n)$ be an $n$-variable function defined by (3). If the dimension of linear code $C_{p_H}$ associated with $\mathcal{H}(x_1, x_2, \ldots, x_n)$ by

$$C_{p_H} = \{ c_{p_H}(u) = (u \cdot y)_{y \in P_H} : u \in \mathbb{F}_p^n \}$$

with the usual inner product. We call $P_H$ the defining set of $C_{p_H}$ associated with $\mathcal{H}(x_1, x_2, \ldots, x_n)$.

Lemma 3.1. Let $p$ be a prime number and let $\mathcal{H}(x_1, x_2, \ldots, x_n)$ be an $n$-variable function defined by (3). If the dimension of linear code $C_{p_H}$ equals $n$, then for $u, v \in \mathbb{F}_p^n$, two codewords $c_{p_H}(u)$ and $c_{p_H}(v)$ are $\mathbb{F}_p$-linearly dependent if and only if two vectors $u$ and $v$ are $\mathbb{F}_p$-linearly dependent.

Proof. For non-zero vectors $u, v \in \mathbb{F}_p^n$, we have

$$c_{p_H}(u) = s \cdot c_{p_H}(v) \text{ for some } s \in \mathbb{F}_p^* \iff c_{p_H}(u - sv) = 0 \iff \dim(C_{p_H}) = n \iff u = s \cdot v \text{ for some } s \in \mathbb{F}_p^*.$$

Lemma 3.1 is useful for the proof of our main results (Theorems 4.2 and 5.1).

Lemma 3.2. Let $C_{p_H}$ be a linear code defined in (4). Then the length of $C_{p_H}$ is $\mathcal{H}(1, 1, \ldots, 1)$, and the dimension of $C_{p_H}$ is at most $n$. The Hamming weight $w(c_{p_H}(u))$ of any codeword $c_{p_H}(u)$ is

$$w(c_{p_H}(u)) = |P_H| - \sum_{y \in P_H} \delta_0(u, y) \sum_{l \in \mathbb{F}_p^*} \mathcal{H}(\zeta_p^{l \cdot u_1}, \zeta_p^{l \cdot u_2}, \ldots, \zeta_p^{l \cdot u_n}),$$

where $\zeta_p = e^{2\pi \sqrt{-1}/p}$ is a primitive $p$-th root of unity.

Proof. The length and the dimension of $C_{p_H}$ are obtained obviously. The Hamming weight $w(c_{p_H}(u))$ is given by

$$w(c_{p_H}(u)) = |P_H| - \sum_{y \in P_H} \delta_0(u, y) \sum_{l \in \mathbb{F}_p^*} \mathcal{H}(\zeta_p^{l \cdot u_1}, \zeta_p^{l \cdot u_2}, \ldots, \zeta_p^{l \cdot u_n}),$$

where $\delta$ is the Kronecker delta function.
Remark 1. Let $D$ be a subset of $\mathbb{F}_p^n$. Then we can find the $n$-variable function $\mathcal{H}(x_1, x_2, \ldots, x_n)$ corresponding to $D$ as follows:

$$\mathcal{H}_D(x_1, x_2, \ldots, x_n) = \sum_{u \in \mathbb{F}_p^n} d_u x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n},$$

where $d_u = 1$ if $u \in D$, and $d_u$ is 0 otherwise. Thus the works on the constructions of minimal linear codes from the typical linear code $C_D = \{c_D(u) = (u \cdot y) : u \in \mathbb{F}_p^n\}$ can be employed with multivariable functions in view of Lemma 3.2.

On the other hand, the Hamming weight of a codeword $c_{P_n}(u)$ in the $C_{P_n}$, where $X^c$ is the complement of a set $X$, is as follows:

$$(5) \quad w(c_{P_n}(u)) = p^n - 1 - (1 - \delta_0, u) - w(c_{P_n}(u)),$$

where $\delta$ is the Kronecker delta function.

In the following example, we provide a class of linear codes not satisfying both AB-condition and HDZ-condition.

Example 1. Let us consider an $n$-variable function $\mathcal{H}(x_1, x_2, \ldots, x_n)$ defined by

$$\mathcal{H}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n (1 + x_i + x_i^2 + \cdots + x_i^{p-1}) - 1.$$

Then the length of $C_{P_n}$ is $\mathcal{H}(1, 1, \ldots, 1) = p^n - 1$. Let us compute the Hamming weight $w(c_{P_n}(u))$ of any codeword $c_{P_n}(u)$ in $C_{P_n}$. We have

$$\mathcal{H}(c_{P_n}^{(1)}, c_{P_n}^{(1)}, \ldots, c_{P_n}^{(1)}) = \prod_{i=1}^n (1 + c_{P_n}^{(1)} + c_{P_n}^{(1)} + \cdots + c_{P_n}^{(1)}) - 1 = \begin{cases} p^n - 1 & \text{if } u = 0, \\ -1 & \text{otherwise}, \end{cases}$$

so that $w(c_{P_n}(u)) = (p - 1)p^{n-1}$ for any $u \in \mathbb{F}_p^n$ by Lemma 3.2. Consequently, the $C_{P_n}$ is a constant weight linear code with parameters $[p^n - 1, n, (p - 1)p^{n-1}]_p$, which is equivalent to a simplex code by [4].

In the following example, we provide a class of linear codes not satisfying both AB-condition and HDZ-condition.

Example 2. Let

$$\mathcal{H}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \mathcal{H}_1(x_1, x_2, \ldots, x_n) \cdot \mathcal{H}_1(y_1, y_2, \ldots, y_m),$$

where

$$\mathcal{H}_1(x_1, x_2, \ldots, x_n) = (x_1 + x_1^2 + \cdots + x_1^{p-1}) \prod_{i=2}^n (1 + x_i + x_i^2 + \cdots + x_i^{p-1}).$$

Then the length of $C_{P_n}$ is $(p - 1)^2 p^{n+m-2}$. For $(u|v) = (u_1, \ldots, u_n, v_1, \ldots, v_m) \in \mathbb{F}_p^{n+m}$, the weight $w(c_{P_n}(u|v))$ of $C_{P_n}$ is given by

$$\begin{cases} 0 & \text{if } u = 0 \text{ and } v = 0, \\ (p - 1)^2 p^{n+m-2} & \text{if } u = 0 \text{ and } v \neq 0, v' = 0, \text{ or } u_1 \neq 0, u' = 0 \text{ and } v = 0, \\ (p - 1)(p - 2) p^{n+m-2} & \text{if } u_1, v_1 \neq 0 \text{ and } u', v' = 0, \\ (p - 1)^3 p^{n+m-3} & \text{if } u_1 = 0 \text{ and } u' \neq 0, \text{ or } v_1 = 0 \text{ and } v' \neq 0. \end{cases}$$
Now, we suppose $p \geq 3$. Then $C_{P_{H}}$ is given by

$$[(p-1)^2 p^{n+m-2}, n + m, (p-1)(p-2)p^{n+m-2}]_p.$$ 

Since $\frac{d}{d_{\text{max}}} = \frac{p-2}{p} < \frac{p-1}{p}$, it does not satisfy the AB-condition. Indeed, let $a = (0, \ldots, 0, v_1, 0, \ldots, 0)$ with $v_1 \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p^{n+m*}$ be a $\mathbb{F}_p$-linearly independent vector with $a$. We also show that it does not satisfy the HDZ-condition by using Proposition 2.

Now we consider the anti-code of $C_{P_{H}}$. Then the length of $C_{(P_{H})^*}$ is $p^{n+m} - |P_{H}| - 1$. For $(u|v) \in \mathbb{F}_p^{n+m}$, the weight $w(c_{(P_{H})^*}(u|v))$ of $C_{(P_{H})^*}$ is given by

$$\begin{cases} 
0 & \text{if } u = 0 \text{ and } v = 0, \\
(p-1)p^{n+m-2} & \text{if } u = 0 \text{ and } v \neq 0, \\
2(p-1)p^{n+m-2} & \text{if } u_1, v_1 \neq 0 \text{ and } u', v' = 0, \\
(p-1)(2p-1)p^{n+m-3} & \text{if } u_1 = 0 \text{ and } u' \neq 0, \\
& \text{or } v_1 = 0 \text{ and } v' \neq 0.
\end{cases}$$

Then the code $C_{P_{H}}$ is $[(2p-1)p^{n+m-2} - 1, n + m, (p-1)p^{n+m-2}]_p$-code. Since $\frac{d}{d_{\text{max}}} = \frac{1}{2}$, it does not satisfy the AB-condition. With $a = (u_1, 0, \ldots, 0, v_1, 0, \ldots, 0)$ and $b = (0, \ldots, 0, v_1', 0, \ldots, 0)$, we also show that it does not satisfy the HDZ-condition by using Proposition 2.

**Example 3.** Let $p$ be an odd prime number, and $n \geq 2$ be a fixed positive integer. Let

$$H(x_1, x_2, \ldots, x_n) = (1 + x_1 + \cdots + x_1^{p-2}) \prod_{i=2}^{n}(1 + x_i + \cdots + x_i^{p-1}) - 1.$$ 

Then $C_{P_{H}}$ is a two-weight linear code with parameters $[(p-1)p^{n-1} - 1, n, (p-2)p^{n-1}]_p$, and its weight distribution is given as follows.

| Weight | Number of distinct codewords |
|--------|-----------------------------|
| 0      | 1                           |
| $(p-2)p^{n-1}$ | $p-1$ |
| $(p-1)^2p^{n-2}$ | $p^n - p$ |

The code $C_{P_{H}}$ is a minimal linear code which satisfies AB-condition. Indeed, let $u = (u_1, u_2, \ldots, u_n) = (u_1|u')$ be an element of $\mathbb{F}_p^n$. We have that

$$\sum_{l \in \mathbb{F}_p^n} \mathcal{H}(\zeta_p^{l|u_1}, \zeta_p^{l|u_2}, \ldots, \zeta_p^{l|u_n})$$

$$\begin{cases} 
(p-1)(p^n - p^{n-1} - 1) & \text{if } u = 0, \\
(p^n - p^{n-1} - 1) & \text{if } u = (u_1|0) \text{ and } u_1 \neq 0, \\
-p + 1 & \text{otherwise.}
\end{cases}$$
By Lemma 3.2, we have

\[ w(c_{\mathcal{P}_n}(u)) = \begin{cases} 
0 & \text{if } u = 0, \\
(p - 2)p^{n-1} & \text{if } u = (u_1|0) \text{ and } u_1 \neq 0, \\
(p - 1)^2p^{n-2} & \text{otherwise.} 
\end{cases} \tag{6} \]

Moreover, the number of codewords can be obtained which is depending on the number of vectors \( u \); for example, the number of codewords \( c_{\mathcal{P}_n}(u) \) such that \( u = (u_1|0) \) and \( u_1 \neq 0 \) is equal to \( p - 1 \). Thus \( C_{\mathcal{P}_n} \) is a linear code with parameters \([(p - 1)p^{n-1} - 1, n, (p - 2)p^{n-1}]_p \); the dimension of the code \( C_{\mathcal{P}_n} \) is equal to \( n \) since the sum of all distinct codewords is \( p^n \). By (6), we have that

\[ \frac{d}{d_{\text{max}}} = \frac{(p - 2)p^{n-1}}{(p - 1)^2p^{n-2}} > \frac{p - 1}{p}. \]

It means that the AB-condition is satisfied.

**Remark 2.** Example 3 is interesting since \( C_{(\mathcal{P}_n)^*} \) is a two-weight linear code with the same parameters \([p^{n-1}, n, (p - 1)p^{n-2}]_p \) as primitive \( p \)-ary first order Reed-Muller codes, and it attains the Griesmer bound with equality.

### 4. Minimal linear codes not satisfying AB-condition

Let \( D_1 \) and \( D_2 \) be non-empty subsets of \( \mathbb{F}_p \) with \( 0 \notin D_1 \cap D_2 \). Let us define the equivalence relation \( \sim \) on \( D_1 \times D_2 \) by

\[ (i, j) \sim (i', j') \iff (i', j') = (ki, kj) \text{ for some } k \in \mathbb{F}_p^*. \]

The equivalence class of \((i, j) \in D_1 \times D_2 \) under \( \sim \) is denoted by

\[ L_{i,j} = \{(ki, kj) \in D_1 \times D_2 : k \in \mathbb{F}_p^* \}, \]

and let \( r \) be the number of equivalence classes. For \((i, j) \in D_1 \times D_2 \), we put

\[ H^*_{i,j} = \{(u_1, u_2) \in \mathbb{F}_p^* \times \mathbb{F}_p^* : u_1 \cdot i + u_2 \cdot j = 0 \}. \]

It is obvious that if \((i', j') \in L_{i,j} \), then \( H^*_{i,j} = H^*_{i', j'} \).

In this section, we define an \( n \)-variable function as follows:

\[ \mathcal{H}(x_1, x_2, \ldots, x_n) = \left( \sum_{i \in D_1} x_i \right) \left( \sum_{i \in D_2} x_i \right) \prod_{i=3}^n (1 + x_i + \cdots + x_i^{(p-1)}). \tag{7} \]

From now on, we denote by \([n]\) the set \( \{1, 2, \ldots, n\} \). By the following lemma, we can obtain the Hamming weight of any codeword for a linear code associated with an \( n \)-variable function (7).

**Lemma 4.1.** Let \( \mathcal{H}(x_1, x_2, \ldots, x_n) \) be an \( n \)-variable function defined as (7). For non-zero vector \( u \), we obtain the value

\[ \sum_{i \in [n]} \mathcal{H}_{\mathcal{L}_i^{L_1 u_1}, \ldots, \mathcal{L}_i^{L_1 u_n}} = \begin{cases} 
0 & \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
(p|L_{i_k,j_k}| - |D_1||D_2|)p^{n-2} & \text{if } (u_1, u_2) \in \bigcup_{k \in [n]} H^*_{i_k,j_k}, \\
-|D_1||D_2|p^{n-2} & \text{otherwise.} 
\end{cases} \]
Proof. We now compute the sum \( \sum_{l \in \mathbb{F}_p} \mathcal{H}(\zeta_p^{l u_1}, \ldots, \zeta_p^{l u_n}) \). One can readily check that for \((u_1, u_2 | u') \in \mathbb{F}_p^n\) with \(u_1, u_2 \neq 0\), we obtain that

\[
\sum_{l \in \mathbb{F}_p} \mathcal{H}(\zeta_p^{l u_1}, \ldots, \zeta_p^{l u_n}) = \begin{cases} 
0 & \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
-|D_1||D_2|p^{n-2} & \text{if } u = (u_1, 0|0) \text{ or } u = (0, u_2|0).
\end{cases}
\]

For \(u = (u_1, u_2|0)\) with \(u_1, u_2 \neq 0\), the sum \( \sum_{l \in \mathbb{F}_p} \mathcal{H}(\zeta_p^{l u_1}, \ldots, \zeta_p^{l u_n}) \) becomes

\[
\sum_{l \in \mathbb{F}_p} \sum_{i \in D_1} \sum_{j \in D_2} \zeta_p^{l(u_1+i+u_2+j)} p^{n-2} = \sum_{l \in \mathbb{F}_p} \sum_{k=1}^r \sum_{i \in D_k} \zeta_p^{l(u_1-i_1+u_2-j_k)} p^{n-2}
\]

\[
= \left\{ \begin{array}{ll}
(p-1)|L_i|L_{i',j_k}|p^{n-2} - \sum_{1 \leq k \leq r} |L_{i_k,j_k}|p^{n-2} & \text{if } (u_1, u_2) \in \cup_{k' \in [r]} H^*_{i_k,j_{k'}}, \\
\sum_{k=1}^r |L_{i_k,j_k}|(-1)p^{n-2} & \text{otherwise}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
(p-1)|L_i|L_{i',j_k}|p^{n-2} - (|D_1||D_2| - |L_{i',j_{k'}}|)p^{n-2} & \text{if } (u_1, u_2) \in \cup_{k \in [r]} H^*_{i_k,j_{k'}}, \\
- (|D_1||D_2|)p^{n-2} & \text{otherwise}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
(p)|L_{i',j_{k'}}| - |D_1||D_2|p^{n-2} & \text{if } (u_1, u_2) \in \cup_{k \in [r]} H^*_{i_k,j_{k'}}, \\
- |D_1||D_2|p^{n-2} & \text{otherwise}.
\end{array} \right.
\]

Hence we obtain the result. \( \square \)

The following code \( C_{(p_n^0)} \) is a minimal linear code which does not satisfy AB-condition.

**Theorem 4.2.** Let \( p \geq 5 \) be an odd prime number, and \( n \geq 2 \) be a fixed positive integer. Let \( D := D_1 \subseteq \mathbb{F}_p^n \) and \( D_2 = \mathbb{F}_p \), such that \( |D| \geq \frac{p^2-1}{2p-1} \) and \( |D| \neq p-1 \). Then \( C_{(p_n^0)} \) is a two-weight minimal linear code which does not satisfy AB-condition. Its parameters are \([p^n - |D|p^{n-1} - 1, n, (p-1 - |D|)|p^{n-1}]_p\), and weight distribution is given as follows.

| Weight                           | Number of distinct codewords |
|----------------------------------|------------------------------|
| \(0\)                            | \(1\)                        |
| \((p-1-|D|)p^{n-1}\)            | \(p-1\)                      |
| \((p-1)(p-|D|)p^{n-2}\)        | \(p^n-p\)                    |

Proof. First, we investigate the Hamming weights for all codewords of \( C_{(p_n^0)} \). Let \( u = (u_1, u_2, \ldots, u_n) = (u_1 | u') \) be an element of \( \mathbb{F}_p^n \). By Lemma 4.1, we have that

\[
\sum_{l \in \mathbb{F}_p} \mathcal{H}(\zeta_p^{l u_1}, \ldots, \zeta_p^{l u_n}) = \sum_{l \in \mathbb{F}_p} \left( \sum_{i \in D} \zeta_p^{l u_1} \prod_{i=2}^n (1 + \zeta_p^{l u_1} + \cdots + \zeta_p^{l u_1(p-1)}) \right)
\]
\[
\begin{cases}
(p - 1) \cdot |D| \cdot p^{n-1} & \text{if } u = 0, \\
-|D| \cdot p^{n-1} & \text{if } u = (u_1|0) \text{ and } u_1 \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

By Lemma 3.2, we find that
\[
w(c_{p_n}(u)) = \begin{cases}
0 & \text{if } u = 0, \\
|D| \cdot p^{n-1} & \text{if } u = (u_1|0) \text{ and } u_1 \neq 0, \\
(p - 1) \cdot |D| \cdot p^{n-2} & \text{otherwise}.
\end{cases}
\]

From (5), we find that
\[
(8) \quad w(c_{(p_n^t)})(u) = \begin{cases}
0 & \text{if } u = 0, \\
(p - 1 - |D|)p^{n-1} & \text{if } u = (u_1|0) \text{ and } u_1 \neq 0, \\
(p - 1)(p - |D|)p^{n-2} & \text{otherwise}.
\end{cases}
\]

Moreover, we clearly obtain the number of distinct codewords which have the Hamming weights given as above; the number of distinct codewords which have Hamming weight \((p - 1 - |D|)p^{n-1}\) (resp. \((p - 1)(p - |D|)p^{n-2}\)) is equal to \(p - 1\) (resp. \(p^n - p\)). The sum of all frequencies is equal to \(p^n\), that is, the dimension of code \(C_{(p_n)}\) is equal to \(n\). Thus the parameters of \(C_{(p_n^t)}\) is \([p^n - |D|p^{n-1} - 1, n, (p - 1 - |D|)p^{n-1}]_p\). We easily check that this code does not satisfy AB-condition since \(|D| \geq \frac{p^2 - p}{2p - 1}\).

From now on, we consider the HDZ-condition in order to prove minimality of the code \(C_{(p_n^t)}\) by using (8).

**Case 1.1.** Suppose that \(a = (u_1|0)\) and \(b = (v_1|v_2, \ldots, v_n) = (v_1|v')\) where \(u_1 \in F^*_p, v_1 \in F_p\) and \(v' \in F^{n-1*}_p\). Then \(a\) and \(b\) are linearly independent over \(F_p\), hence the codewords \(c_{(p_n^t)}(a)\) and \(c_{(p_n^t)}(b)\) are linearly independent over \(F_p\) by Lemma 3.1. It follows from the right hand side of (2) is \((p^2 - 2 + |D|)p + |D|(p - 1)p^{n-2}\), and the left hand side of (2) is \((p^2 - 1 + |D|)p + |D|(p - 1)p^{n-2}\). So it satisfies HDZ-condition.

**Case 1.2.** Suppose that \(a = (u_1|u_2, \ldots, u_n) = (u_1|u')\) and \(b = (v_1|0)\) where \(u_1 \in F_p, v_1 \in F^*_p\) and \(u', v' \in F^{n-1*}_p\). The codewords \(c_{(p_n^t)}(a)\) and \(c_{(p_n^t)}(b)\) are linearly independent over \(F_p\) by Lemma 3.1 since \(a\) and \(b\) are linearly independent over \(F_p\). It follows that the right hand side of (2) is \((p - 1)^2(p - |D|) - (p - 1 - |D|)p^{n-2}\), and the left hand side of (2) is \((p - 1)^2(p - |D|)p^{n-2}\). Since \(|D| \neq p - 1\), it satisfies HDZ-condition.

**Case 1.3.** Suppose that \(a = (u_1|u')\) and \(b = (0|v')\) where \(u_1 \in F^*_p\) and \(u', v' \in F^{n-1*}_p\). Clearly, the codewords \(c_{(p_n^t)}(a)\) and \(c_{(p_n^t)}(b)\) are linearly independent over \(F_p\) by Lemma 3.1 since \(a\) and \(b\) are linearly independent over \(F_p\). It follows that the right hand side of (2) is \((p - 2)(p - 1)(p - |D|)p^{n-2}\), and the left hand side of (2) is
\[
\begin{cases}
((p - 2)(p - 1)(p - |D|) + (p - 1 - |D|)p) p^{n-2} & \text{if } u' + sv' = 0 \text{ for some } s \in F^*_p, \\
(p - 1)^2(p - |D|)p^{n-2} & \text{otherwise}.
\end{cases}
\]

Since \(|D| \neq p - 1\), it satisfies HDZ-condition.
Case 1.4. If \( a = (0|u') \) and \( b = (v_1|v') \) where \( v_1 \in \mathbb{F}_p^* \) and \( u',v' \in \mathbb{F}_{p^n-1}^* \), then the code \( C_{(P_k^*)} \) satisfies HDZ-condition by the similar reason as Case 1.3.

Case 1.5. Suppose that \( a = (0|u') \) and \( b = (0|v') \), where \( a \in \mathbb{F}_{p^n-1}^* \) and \( b \in \mathbb{F}_{p^n}^* \) are linearly independent over \( \mathbb{F}_p \). Then the codewords \( c_{(P_k^*)}^*(a) \) and \( c_{(P_k^*)}^*(b) \) are linearly independent by Lemma 3.1. The right hand side of (2) is \((p-2)(p-1)(p-|D|)p^{n-2}\), and the left hand side of (2) is \((p-1)^2(p-|D|)p^{n-2}\). Hence it satisfies HDZ-condition.

Case 1.6. Suppose that \( a = (u_1|u') \) and \( b = (v_1|v') \), where \( u_1,v_1 \in \mathbb{F}_p^*, \ u',v' \in \mathbb{F}_{p^n-1}^* \), \( a \) and \( b \) are linearly independent over \( \mathbb{F}_p \). Then the codewords \( c_{(P_k^*)}^*(a) \) and \( c_{(P_k^*)}^*(b) \) are linearly independent by Lemma 3.1. The right hand side of (2) is \((p-2)(p-1)(p-|D|)p^{n-2}\), and the left hand side of (2) is

\[
\begin{cases}
(p-2)(p-1)(p-|D|)p^{n-2} + (p-1-|D|)p^{n-1} & \text{if } u'+sv' = 0 \text{ for some } s \in \mathbb{F}_p^*, \\
(p-1)^2(p-|D|)p^{n-2} & \text{otherwise}.
\end{cases}
\]

Since \(|D| \neq p-1\), it satisfies the HDZ-condition.

By Case 1.1 through Case 1.6, \( C_{(P_k^*)} \) is a minimal code violating AB-condition.

We illustrate Theorem 4.2 with the following example.

Example 4. Let \( p = 7, n = 3 \) and \( D = \mathbb{F}_5^* \). Let

\[
\mathcal{H}(x_1, x_2, x_3) = \left( \sum_{i \in D} x_i^1 \right) \prod_{i=2}^3 (1 + x_i + \cdots + x_i^7).
\]

Then we have the set \( (P_k^*) \) as follows:

\[
(P_k^*) = \\
\{(5,0,1), (5,0,0), (0,2,5), (0,2,4), (0,2,6), (0,2,1), (0,2,0), (0,2,3), (0,2,2), (6,4,1),
(6,4,0), (6,4,3), (6,4,2), (6,4,5), (6,4,4), (0,3,5), (0,3,4), (6,4,6), (0,3,6), (6,0,5),
(0,3,1), (6,0,4), (0,3,0), (0,3,3), (0,6,6), (0,3,2), (6,0,1), (6,0,0), (6,0,3), (6,0,2),
(0,5,3), (0,5,2), (0,5,1), (0,5,0), (6,3,6), (6,3,5), (0,5,6), (6,3,4), (0,5,5), (6,3,3),
(0,5,4), (6,3,2), (6,3,1), (6,3,0), (6,2,3), (6,2,2), (6,2,1), (6,2,0), (6,2,6), (6,2,5),
(6,2,4), (5,3,1), (5,3,0), (5,3,3), (0,0,3), (5,4,3), (0,0,2), (5,3,2), (5,4,2), (0,0,1),
(5,3,5), (5,4,1), (5,3,4), (5,4,0), (5,3,6), (0,0,6), (5,4,6), (0,0,5), (5,4,5), (0,0,4),
(5,4,4), (0,4,6), (0,4,5), (0,4,4), (0,4,3), (0,4,2), (0,4,1), (0,4,0), (6,1,5), (6,1,4),
(6,1,6), (6,1,1), (6,1,0), (6,1,3), (6,1,2), (6,6,6), (6,6,5), (6,6,4), (6,6,3), (6,6,2),
(6,6,1), (6,6,0), (0,6,1), (0,6,0), (0,6,3), (0,6,2), (0,6,5), (0,6,4), (0,6,6), (5,6,5),
(5,1,3), (5,6,4), (5,1,2), (5,1,1), (5,6,6), (5,1,0), (5,6,1), (5,6,0), (5,1,6), (5,6,3),
(5,1,5), (5,6,2), (5,1,4), (6,5,1), (6,5,0), (6,5,3), (6,5,2), (0,1,6), (6,5,5), (0,1,5),
(6,5,4), (0,1,4), (0,1,3), (6,5,6), (0,1,2), (0,1,1), (0,1,0), (5,2,1), (5,2,0), (5,5,6),
(5,2,3), (5,5,5), (5,2,2), (5,5,4), (5,2,5), (5,0,6), (5,5,3), (5,2,4), (5,0,5), (5,5,2),
(5,0,4), (5,5,1), (5,2,6), (5,0,3), (5,5,0), (5,0,2)\}.
\]

From this set \( (P_k^*) \), we obtain a two-weight minimal code \( C_{(P_k^*)} \) with parameters \([146,3,98]_7\). The code \( C_{(P_k^*)} \) has the weight distribution as below.

| Weight | 0   | 98 | 126 |
|--------|-----|----|-----|
| Number of distinct codewords | 1   | 6  | 336 |

5. MINIMAL LINEAR CODES SATISFYING AB-CONDITION

We find some minimal codes which satisfy AB-condition. We use the same notations as Section 4.
Theorem 5.1. Let \( p \) be an odd prime number, and \( n \geq 3 \) be a fixed positive integer. The following codes \( C(p_n^*) \) are minimal linear codes which satisfy AB-condition.

(i) Let \( D_1 = S \) be the set of all squares of \( p \), and \( D_2 = NS \) be the set of all non-squares of \( p \). Then \( C(p_n^*) \) is a three-weight minimal linear code with parameters \( \left[ \frac{3(p-1)(p+1)}{4} p^{n-2} - 1, n, \frac{(p-1)(3p+1)}{4} p^{n-2} \right]_p \), and its weight distribution is given as follows.

| Weight | Number of distinct codewords |
|--------|-------------------------------|
| 0      | 1                             |
| \( \frac{3(p-1)(p+1)}{4} p^{n-2} \) | \( p^n - p^2 \) |
| \( \frac{(p-1)}{2} \) | \( (p-1)^2 \) |
| \( \frac{3(p-1)(p+1)}{4} p^{n-2} \) | \( \frac{p^n - p^2}{2} \) |

(ii) Let \( D := D_1 = D_2 \) be a proper subgroup of \( \mathbb{F}_p^* \). Then \( C(p_n^*) \) is a three-weight minimal linear code with parameters \( \left[ p^{n-2}(p^2 - |D|^2) - 1, n, p^{n-2}(p^2 - p - |D|^2) \right]_p \), and its weight distribution is given as follows.

| Weight | Number of distinct codewords |
|--------|-------------------------------|
| 0      | 1                             |
| \( p^n - 3(p-1)(p^2 - |D|^2) \) | \( p^n - p^2 \) |
| \( p^n - 2(p^2 - p - |D|(|D| - 1)) \) | \( (p-1)|D| \) |
| \( p^{n-2}(p^2 - p - |D|^2) \) | \( (p-1)(p - |D| + 1) \) |

Proof. We recall an \( n \)-variable function

\[
\mathcal{H}(x_1, x_2, \ldots, x_n) = \left( \sum_{i \in D_1} x_i \right) \left( \sum_{i \in D_2} x_i^2 \right) \prod_{i=3}^n (1 + x_i + \cdots + x_i^{p-1}).
\]

(i) We note that \( |S| = |NS| = \frac{p^n-1}{2} \), \( |L_{i,k,j_k}| = \frac{p^n-1}{2} \) and \( |H_{i,k,j_k}^*| = p - 1 \) for all \( k \in \left[ \frac{p-1}{2} \right] \). For \( (u_1, u_2) \in \mathbb{F}_p^n \), by Lemma 4.1, we have

\[
\sum_{c \in \mathbb{F}_p^n} \mathcal{H}(c_{i} u_1, \ldots, c_{i} u_n) = \begin{cases} 
0 & \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
\frac{(p-1)(p+1)}{4} p^{n-2} & \text{if } (u_1, u_2) \in \bigcup_{k \in \left[ \frac{p-1}{2} \right]} H_{i,k,j_k}^*, \\
\frac{(p-1)^2}{4} p^{n-2} & \text{otherwise}.
\end{cases}
\]

By Lemma 3.2, we have

\[
w(c_{p_n}(u)) = \begin{cases} 
0 & \text{if } u = 0, \\
\frac{(p-1)^3}{4} p^{n-3} & \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
\frac{(p-1)(p-1)^2}{4} p^{n-2} & \text{if } (u_1, u_2) \in \bigcup_{k \in \left[ \frac{p-1}{2} \right]} H_{i,k,j_k}^*, \\
\frac{(p-1)^2}{4} p^{n-2} & \text{otherwise}.
\end{cases}
\]
By Lemma 3.2, we have

\begin{align*}
\sum_{i=0}^{n-1} H_i^* u_i & = 0 \\
\text{if } u = 0, \\
\sum_{i=0}^{n-1} H_i^* u_i & = 3p-1 \quad \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
\sum_{i=0}^{n-1} H_i^* u_i & = 3p-1 \quad \text{if } (u_1, u_2) \in \bigcup_{k \in [\frac{p-1}{2}]} H_{k,j,k}^*.
\end{align*}

We note that the number of codewords can be obtained which is depending on the number of vectors \( u \); for example, the number of codewords \( c(P_{n,k})^* \) such that \( u_i \neq 0 \) for some \( i = 3, 4, \ldots, n \) is equal to \((p^n - 1)^2 - p^2 = p^n - p^2\). Thus \( C(P_{n,k}^*) \) is a linear code with parameters \([3p-1](3p+1)p^{-2}-1, n, (p-1)(3p+1)p^{-2}]\). It is minimal code because

\[
\frac{d}{d_{\text{max}}} = \frac{3p+1}{3p+3} > \frac{p-1}{p}
\]

by (9).

(ii) Let \( \ell \) be the cardinality \( |D| \) of the set \( D \). For \((u_1, u_2, u') \in F_p^n\), by Lemma 4.1, we have

\[
\sum_{i=0}^{n-1} H_i^* (u_i, u_i') = \begin{cases} 
0 & \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
\ell p^n - 2(p - \ell) & \text{if } (u_1, u_2, u') \in \bigcup_{k \in [\ell]} H_{k,j,k}^*, \\
-\ell^2 p^n - 2 & \text{otherwise}.
\end{cases}
\]

By Lemma 3.2, we have

\[
w(c_{P_{n,k}}^*) (u) = \begin{cases} 
0 & \text{if } u = 0, \\
p^n - 2 & \text{if } u_i \neq 0 \text{ for some } i = 3, 4, \ldots, n, \\
(p-2) \ell (\ell - 1) & \text{if } (u_1, u_2) \in \bigcup_{k \in [\ell]} H_{k,j,k}^*, \\
p^n - 2 & \text{otherwise}.
\end{cases}
\]

We note that the length of \( C(P_{n,k}^*) \) is \( p^n - 1 - \ell^2 p^n - 2 = p^n - 2(p^2 - \ell^2) - 1 \). The number of codewords can be obtained which is depending on the number of vectors \( u \); for example, the number of codewords \( c(P_{n,k}^*)^* \) such that \( (u_1, u_2) \in H_{k,j,k}^* \) for some \( k \in [\ell] \) is equal to \((p-1)\ell\). Thus \( C(P_{n,k}^*) \) is a linear code with parameters \([p^n - 2(p^2 - \ell^2) - 1, n, p^n - 2(p^2 - \ell^2)]\). Moreover, it is minimal code which satisfies the AB-condition because

\[
\frac{d}{d_{\text{max}}} = \frac{p^2 - p - \ell^2}{p^2 - p - \ell^2 + \ell} > \frac{p-1}{p}
\]

by (10).
We point out that if $D_1 = D_2 = S$ or $D_1 = D_2 = NS$, then we also have $|L_{i_k,j_k}| = \frac{p-1}{2}$ and $|H_{i_k,j_k}^+| = p - 1$ for all $k \in \left[ \frac{p-1}{2} \right]$. With these settings, we obtain the same results as Theorem 5.1(i).

The following example is corresponding to Theorem 5.1.

**Example 5.** (i) Let $p = 7$ and $n = 3$. Let $D_1 = \{1,2,4\}$ be the set of all squares of $p$, and $D_2 = \{3,5,6\}$ be the set of all non-squares of $p$. We give

$$\mathcal{H}(x_1, x_2, x_3) = \left( \sum_{i \in D_1} x_1^i \right) \left( \sum_{i \in D_2} x_2^i \right) (1 + x_3 + \cdots + x_3^6).$$

Then we have the set $(P^c_{\mathcal{H}})^*$ as follows:

$$(P^c_{\mathcal{H}})^* = \{(5,4,3),(5,4,2),(5,6,5),(5,4,1),(5,6,4),(5,4,0),(5,6,6),(5,4,6),(5,6,1),(6,5,1),$$

$$(5,4,5),(5,6,0),(5,4,4),(5,6,3),(5,6,5),(5,6,2),(6,5,2),(6,5,5),(6,5,4),$$

$$(6,5,6),(0,2,5),(0,2,4),(0,2,6),(0,2,1),(0,2,0),(0,2,3),(0,2,2),(6,2,3),(6,2,2),$$

$$(6,2,1),(6,2,0),(6,2,6),(6,2,5),(6,2,4),(1,0,3),(1,0,2),(1,0,1),(1,0,0),(1,0,6),$$

$$(0,0,3),(1,0,5),(0,0,2),(1,0,4),(0,0,1),(0,0,6),(5,3,1),(0,0,5),(5,3,0),(0,0,4),$$

$$(5,3,3),(5,3,2),(5,3,5),(5,3,4),(5,3,6),(5,3,5),(6,0,5),(5,5,5),(6,0,4),(5,5,4),$$

$$(5,5,3),(6,0,6),(5,5,2),(6,0,1),(5,5,1),(3,4,1),(5,5,0),(6,0,0),(3,4,0),(6,0,3),$$

$$(3,4,3),(6,0,2),(3,4,2),(3,4,5),(5,1,3),(3,4,4),(5,1,2),(5,1,1),(6,1,1),(3,4,6),$$

$$(5,1,0),(6,4,0),(6,4,3),(5,1,6),(6,4,2),(5,1,5),(6,4,5),(5,1,4),(6,4,4),(6,4,6),$$

$$(1,4,6),(1,4,5),(1,4,4),(1,4,3),(1,4,2),(1,4,1),(1,4,0),(3,2,3),(3,2,2),(3,2,1),$$

$$(3,2,0),(3,2,6),(3,3,6),(3,2,5),(3,3,5),(3,2,4),(3,3,4),(3,3,3),(3,3,2),(3,3,1),$$

$$(3,3,0),(2,0,1),(2,0,0),(2,0,3),(2,0,2),(2,0,5),(2,0,4),(2,0,6),(3,5,1),(3,5,0),$$

$$(3,5,3),(3,5,2),(3,5,5),(3,5,4),(3,5,6),(1,2,5),(1,2,4),(1,2,6),(1,2,1),(1,2,0),$$

$$(1,2,3),(2,4,5),(1,2,2),(2,4,4),(2,4,6),(3,1,5),(2,4,1),(3,1,4),(2,4,0),(2,4,3),$$

$$(3,1,6),(2,4,2),(3,1,1),(3,1,0),(3,1,3),(3,1,2),(0,3,5),(0,3,4),(0,3,6),(0,3,1),$$

$$(0,3,0),(0,3,3),(4,0,6),(0,3,2),(4,0,5),(4,0,4),(4,0,3),(4,0,2),(4,0,1),(4,0,0),$$

$$(4,0,4),(4,0,5),(4,0,4),(4,0,3),(4,0,2),(4,0,1),(6,1,5),(0,4,0),(6,1,4),(6,1,6),$$

$$(6,1,1),(2,2,6),(6,1,0),(2,2,5),(6,1,3),(2,2,4),(6,1,2),(2,2,3),(2,2,2),(2,2,1),$$

$$(5,0,6),(2,2,0),(5,0,5),(5,0,4),(5,0,3),(5,0,2),(5,0,1),(5,0,0),(4,1,3),(4,1,2),$$

$$(4,1,1),(4,1,0),(4,1,6),(4,1,5),(4,1,4),(4,1,3),(4,1,2),(4,1,1),(4,1,0),(4,1,6),$$

$$(3,6,6),(1,0,6),(3,6,5),(0,6,0),(3,6,4),(0,6,3),(3,6,3),(0,6,2),(3,6,2),$$

$$(3,6,2),(0,6,5),(3,6,1),(0,6,4),(3,6,0),(0,6,6),(3,0,5),(3,0,4),(3,0,6),(3,0,1),$$

$$(3,0,0),(3,0,3),(3,0,2),(6,3,6),(6,3,5),(6,3,4),(6,3,3),(6,3,2),(6,3,1),(6,3,0),$$

$$(0,5,3),(0,5,2),(0,5,1),(0,5,0),(0,5,6),(0,5,5),(0,5,4),(4,4,3),(4,4,2),(4,4,1),$$

$$(4,4,0),(4,4,6),(4,4,5),(4,4,4),(4,2,1),(5,2,1),(4,2,0),(5,2,0),(4,2,2),(5,2,3),$$

$$(4,2,2),(5,2,2),(4,2,5),(5,2,5),(4,2,4),(5,2,4),(4,2,6),(5,2,6),(0,1,6),(0,1,5),$$

$$(0,1,4),(0,1,3),(0,1,2),(0,1,1),(0,1,0),(2,1,1),(2,1,0),(2,1,3),(2,1,2),(2,1,5),$$

$$(2,1,4),(2,1,6),(6,6,2),(6,6,3),(6,6,2),(6,6,1),(6,6,0))$$

From this set $(P^c_{\mathcal{H}})^*$, we obtain a three-weight minimal linear code $C(P^c_{\mathcal{H}})^*$ with parameters $[279,3,231]_7$. The code $C(P^c_{\mathcal{H}})^*$ has the weight distribution as below.

| Weight | 0 | 231 | 240 | 252 |
|--------|---|-----|-----|-----|
| Number of distinct codewords | 1 | 30  | 294 | 18  |

(ii) Let $p = 7$, $n = 4$ and $D := D_1 = D_2 = \{2\} = \{1,2,4\}$. Let

$$\mathcal{H}(x_1, x_2, x_3, x_4) = \left( \sum_{i \in D} x_1^i \right) \left( \sum_{i \in D} x_2^i \right) \prod_{i = 3}^{4} (1 + x_i + \cdots + x_i^6).$$
Then we have the set $P_H$ as follows:

$$P_H = \{ (4, 4, 6, 6), (4, 4, 6, 5), (4, 4, 6, 4), (4, 4, 6, 3), (4, 4, 6, 2), (4, 4, 6, 1), (4, 4, 6, 0), (4, 4, 5, 5), (4, 4, 5, 4), (4, 4, 5, 3), (4, 4, 5, 2), (4, 4, 5, 1), (4, 4, 5, 0), (4, 4, 4, 6), (4, 4, 4, 5), (4, 4, 4, 4), (4, 4, 4, 3), (4, 4, 4, 2), (4, 4, 4, 1), (4, 4, 4, 0), (4, 4, 3, 6), (4, 4, 3, 5), (4, 4, 3, 4), (4, 4, 3, 3), (4, 4, 3, 2), (4, 4, 3, 1), (4, 4, 3, 0), (4, 4, 2, 6), (4, 4, 2, 5), (4, 4, 2, 4), (4, 4, 2, 3), (4, 4, 2, 2), (4, 4, 2, 1), (4, 4, 2, 0), (4, 4, 1, 6), (4, 4, 1, 5), (4, 4, 1, 4), (4, 4, 1, 3), (4, 4, 1, 2), (4, 4, 1, 1), (4, 4, 1, 0), (4, 4, 0, 6), (4, 4, 0, 5), (4, 4, 0, 4), (4, 4, 0, 3), (4, 4, 0, 2), (4, 4, 0, 1), (4, 4, 0, 0), (4, 3, 6), (4, 3, 5), (4, 3, 4), (4, 3, 3), (4, 3, 2), (4, 3, 1), (4, 3, 0), (4, 2, 6), (4, 2, 5), (4, 2, 4), (4, 2, 3), (4, 2, 2), (4, 2, 1), (4, 2, 0), (4, 1, 6), (4, 1, 5), (4, 1, 4), (4, 1, 3), (4, 1, 2), (4, 1, 1), (4, 1, 0), (4, 0, 6), (4, 0, 5), (4, 0, 4), (4, 0, 3), (4, 0, 2), (4, 0, 1), (4, 0, 0), (3, 6), (3, 5), (3, 4), (3, 3), (3, 2), (3, 1), (3, 0), (2, 6), (2, 5), (2, 4), (2, 3), (2, 2), (2, 1), (2, 0), (1, 6), (1, 5), (1, 4), (1, 3), (1, 2), (1, 1), (1, 0), (0, 6), (0, 5), (0, 4), (0, 3), (0, 2), (0, 1), (0, 0) \) .

So the cardinality of the set $(P_H)^*$ is equal to 1959. By using the set $(P_H)^*$, we obtain a three-weight minimal linear code $C(P_H)^*$ with parameters $[1959, 4, 1617]_7$. The code $C(P_H)^*$ has the weight distribution as below.

| Weight | 0 | 1617 | 1680 | 1764 |
|--------|---|------|------|------|
| Number of distinct codewords | 1 | 30 | 2352 | 18 |

**Acknowledgments**

We would like to thank the anonymous referees for their helpful comments and suggestions that have improved our paper.
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Received July 2019; revised October 2019.

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