A note on the topological insulator phase in non-Hermitian quantum systems

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Abstract

Examples of non-Hermitian quantum systems admitting a topological insulator phase are presented in one, two and three space dimensions. All of these non-Hermitian Hamiltonians have entirely real bulk eigenvalues and unitarity is maintained with the introduction of appropriate inner products in the corresponding Hilbert spaces. The topological invariant characterizing a particular phase is shown to be identical for a non-Hermitian Hamiltonian and its Hermitian counterpart, to which it is related through a non-unitary similarity transformation. A classification scheme for topological insulator phases in pseudo-Hermitian quantum systems is suggested.

1. Introduction

Topological ideas have resurfaced very often in analyzing physical problems, the recently discovered topological insulator [1–7] being one such example. The main characteristic of a topological insulator is the appearance of a boundary zero mode within the bulk gap, the stability of which is guaranteed due to the existence of some associated topological invariant. The classifications of topological insulators are based on the underlying discrete symmetries of the Hamiltonian admitting such phases, like the parity (P), time-reversal (T) and particle–hole symmetries [6, 7]. Perturbations and/or deformations preserving these symmetries cannot destabilize a phase characterized by a topological invariant, which is usually an integer or a $\mathbb{Z}_2$ quantity, as long as the bulk gap remains open. The decay of a particular phase within a given topological sector due to the time evolution of the states is also ruled out completely for Hermitian quantum systems.

The Hermiticity of an operator crucially depends on the metric of the Hilbert space on which it is defined. In the standard treatment of quantum mechanics, the metric is always taken to be an identity operator. An emerging view [8–13] in the context of the $\mathcal{PT}$ symmetric and/or pseudo-Hermitian quantum systems is that a consistent non-dissipative description of non-Hermitian quantum systems is admissible with a modified inner product in the Hilbert space. A few examples of non-dissipative non-Hermitian quantum systems with a complete and consistent description include the asymmetric XXZ spin chain, non-Hermitian transverse Ising model, non-Hermitian Dicke model, non-Hermitian quadratic form of bosonic (fermionic) operators and non-Hermitian many-particle rational Calogero model [13, 14]. Relativistic [11] and supersymmetric [12] non-Hermitian quantum systems have also been investigated within the same context. A proposal for optical realization of a non-Hermitian relativistic quantum system is described in [15]. Experimental results related to $\mathcal{PT}$ symmetric optical systems are also available [16].

The purpose of this paper is to present a general discussion on the construction of non-Hermitian Dirac Hamiltonians admitting a topological insulator phase, followed by a few explicit examples in one, two and three space dimensions. The method prescribed by the present author, in [11–13] and particularly in [11], is used to construct these models through suitable pseudo-Hermitian deformations of known Hermitian Hamiltonians admitting a topological insulator phase. All the model Hamiltonians presented in this paper have entirely real bulk eigenvalues and unitary time evolution. The topological invariant characterizing a particular topological insulator phase is identical for the non-Hermitian and the corresponding Hermitian Hamiltonian from which it is obtained by a pseudo-Hermitian deformation. The decay of a phase within a given topological sector is also ruled out, since the time evolution is unitary by construction.
A few attempts have been made in the recent past (see [17, 18]) to construct model non-Hermitian Hamiltonians admitting a topological insulator phase. The finding of [17] is that the topological insulator phase is absent in non-Hermitian $PT$ symmetric Hamiltonians. It is worth recalling in this context that the reality of the entire spectra as well as the unitarity of a non-Hermitian Hamiltonian can generally be understood in terms of an unbroken anti-linear symmetry, which is not necessarily the standard $PT$ symmetry [9, 11–14]. The anti-linear symmetry may be identified as the standard $PT$ symmetry for some specific quantum systems. The topological insulator phase in a non-Hermitian Dirac Hamiltonian having an anti-linear symmetry is described in this paper. The second reference [18] contains examples of non-Hermitian Hamiltonians admitting a topological insulator phase in which the bulk eigenvalues of the models are in general complex. Consequently, the time evolution is not unitary and the states describing the topological insulator phase are expected to decay within a given topological sector. The examples presented in this paper are free from these shortcomings; all of the bulk eigenvalues are real and the time evolution is unitary.

The plan for presenting the results is the following. A brief review of pseudo-Hermitian quantum systems as applied to Dirac Hamiltonians is presented at the beginning of section 2. Results regarding the allowed forms of topological invariants for pseudo-Hermitian quantum systems are also included in this section. Examples of non-Hermitian Dirac Hamiltonians admitting topological insulator phases in one, two and three space dimensions are presented in sections 3.1, 3.2 and 3.3, respectively. Finally, the paper ends with a summary of the results obtained and relevant discussions in section 4. Suggestions for a possible classification scheme for topological insulators in non-Hermitian quantum systems are included.

2. Pseudo-Hermiticity and topological invariants

The continuum description of topological insulators is given in terms of free particle Dirac Hamiltonians with translational invariance in $D + 1$ space–time dimensions. The main objective of this paper is to present Dirac Hamiltonians $H$, admitting a topological insulator phase, which are non-Hermitian in the Hilbert space $\mathcal{H}_D$ that is endowed with the standard inner product $\langle \cdot | \cdot \rangle$. Any operator that is related to its adjoint in $\mathcal{H}_D$ through a non-unitary similarity transformation is known as a pseudo-Hermitian operator [9], i.e.,

$$H^\dagger = \eta H \eta^{-1}. \quad (1)$$

A positive-definite similarity operator $\eta := \eta_+$, if it exists, can be identified as a metric operator. A Hilbert space that is endowed with the metric $\eta_+$ and the inner product $\langle \cdot | \cdot \rangle_{\eta_+} := \langle \cdot | \eta_+ \cdot \rangle$ is denoted as $\mathcal{H}_{\eta_+}$. The Hilbert spaces $\mathcal{H}_D$ and $\mathcal{H}_{\eta_+}$ are identical in the limit of $\eta_+$ being an identity operator.

In general, Hermitian operators in $\mathcal{H}_{\eta_+}$ are non-Hermitian in $\mathcal{H}_D$ and vice versa. A Hermitian Hamiltonian $H$ in $\mathcal{H}_{\eta_+}$ is related to a Hermitian Hamiltonian $h$ in $\mathcal{H}_D$ through a non-unitary similarity transformation [9, 10],

$$h = \rho H \rho^{-1}, \quad \rho := \sqrt{\eta_+}. \quad (2)$$

An operator $O_{\eta_+}$ in $\mathcal{H}_{\eta_+}$ may be introduced corresponding to each operator $O_D$ in $\mathcal{H}_D$ as $O_D = \rho O_{\eta_+} \rho^{-1}$ such that $\langle O_D \rangle = \langle (O_{\eta_+}) \rangle_{\eta_+}$. This relation allows us to find the symmetry generator of $H$ from that of $h$ or vice versa, and defines physical observables in $\mathcal{H}_{\eta_+}$. Both the operators $O_D$ and $O_{\eta_+}$ are Hermitian in $\mathcal{H}_D$ as well as in $\mathcal{H}_{\eta_+}$ for the special case for which these operators commute with the similarity operator $\rho$.

A comment is in order on the nature of the mapping described by equation (2). The non-Hermitian Hamiltonian $H$ may be mapped to a Hermitian Hamiltonian $\tilde{h} = \tilde{\rho} H \tilde{\rho}^{-1}$ in $\mathcal{H}_D$ that is different from $h$. Although the existence of such a similarity operator $\tilde{\rho}$ indicates non-uniqueness of the mapping (2), the Hermitian Hamiltonians thus obtained are known [9] to be unitarily equivalent to each other, i.e. $\tilde{h} = U^{-1} h U$. The unitary nature of $U$ follows [9] from the identities $\rho^{-1} \eta_+ \rho^{-1} = 1 = \tilde{\rho}^{-1} \eta_+ \tilde{\rho}^{-1}$. Without loss of generality, the symmetry generators of $H$ can thus be obtained by using the inverse similarity transformation of the corresponding generators of either $h$ or $\tilde{h}$. For example, a symmetry generator $T_D$ of $h$ is related to the corresponding generator $\tilde{T}_D$ of $\tilde{h}$ through the unitary transformation $T_D = U^{-1} \tilde{T}_D U$. Consequently, the symmetry operators $T_{\eta_+} := \rho^{-1} T_D \rho$ and $\tilde{T}_{\eta_+} := \tilde{\rho}^{-1} \tilde{T}_D \tilde{\rho}$ of $H$ are related to each other through a unitary transformation: $T_{\eta_+} = U^{-1} \tilde{T}_{\eta_+} U$ and $\langle T_{\eta_+} \rangle_{\eta_+} = \langle \tilde{T}_{\eta_+} \rangle_{\eta_+} = \langle T_D \rangle = \langle \tilde{T}_D \rangle$. This implies that the expectation values of the physical observables and/or associated topological invariants of $H$ can be computed either from that of $h$ or from that of $\tilde{h}$. The different choices of the similarity operator $\tilde{\rho}$ correspond to different quantum canonical transformations in $\mathcal{H}_D$. Thus, without any loss of generality, the discussions below involve the similarity operator $\rho$.

The Hamiltonians $H$ and $h$ are isospectral. The Bloch eigenstates $|\phi_\eta(p)\rangle$ of $H(p)$ at each $p$ with the eigenvalues $E_\eta(p)$ are related to the corresponding states $|\psi_h(p)\rangle$ of $h$ through the equation

$$|\phi_\eta(p)\rangle = \rho^{-1} |\psi_h(p)\rangle. \quad (3)$$

Further, the $|\psi_h(p)\rangle$ constitute a complete set of orthonormal states in $\mathcal{H}_D$, while the completeness and orthonormality of states $|\phi_\eta(p)\rangle$ can be shown only in $\mathcal{H}_{\eta_+}$. It is assumed throughout this paper that a bulk band gap exists for $h$ that is centered around some fixed energy (which may be chosen to be zero without loss of any generality) and the quantum ground state is obtained by filling all the $N$ states below this level. The isospectrality between $h$ and $H$ allows one to assume identical conditions for $H$. The Berry connection for the $N$ number of filled Bloch states for $h$ in the Hilbert space $\mathcal{H}_D$ is defined as [6, 7]

$$a^{\hat{b} \hat{b}}_\eta(p) dp_i := \langle \psi_\eta(p) | d\psi_\eta(p) \rangle, \quad \hat{a}, \hat{b} = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, D. \quad (4)$$
The Berry connection for the non-Hermitian Hamiltonian \( H \) is introduced as
\[
A_i^{ab}(p) \, dp_i := \langle (\phi_b(p)|d\phi_a(p)) \rangle_{\eta_{i+}},
\]
\[
= \partial_i^{ab}(p) \, dp_i.
\] (5)

Besides an implicit assumption that \( \rho \) is independent of the momentum \( p \), equation (3) and the modified inner product in \( \mathcal{H}_{\eta_+} \) have been used to obtain the expression in the second line of the above equation. The crucial observation at this point is that the gauge potentials \( A_i^{ab} \) in \( \mathcal{H}_{\eta_+} \) and \( A_i^{ab} \) in \( \mathcal{H}_D \) are identical, leading to the same Berry curvature. Consequently, the Chern numbers and Chern–Simons invariants will have identical values in \( \mathcal{H}_D \) as well as in \( \mathcal{H}_{\eta_+} \). The readers are referred to [7] for explicit expressions for different topological invariants in terms of Berry curvature.

The topological invariants associated with chiral topological insulators are constructed in terms of the projectors onto the filled Bloch states. In this context, an idempotent projector \( P \) may be introduced in odd size dimensions that anti-commutes with the Dirac Hamiltonian \( h \) of a massive free particle:
\[
\Gamma_D^2 = 1, \quad \{h, \Gamma_D\} = 0. \quad (6)
\]

The existence of such an operator implies that corresponding to each eigenstate \( |\psi_\alpha(p)\rangle \) of \( h \) with the eigenvalue \( E_\alpha \), \( |\tilde{\psi}_\alpha(p)\rangle = \Gamma_D|\psi_\alpha(p)\rangle \) is an eigenstate of \( h \) with the eigenvalue \( -E_\alpha \). The operator that anti-commutes with \( H \) for a given \( \Gamma_D \) may be found as \( \Gamma_{\eta_+} = \rho^{-1}\Gamma_D\rho \). Consequently, \( \Gamma_{\eta_+} \) relates the eigenstates \( |\tilde{\phi}_\alpha(p)\rangle \) and \( |\phi_\alpha(p)\rangle \) of \( H \) corresponding to the eigenvalues \( -E_\alpha \) and \( E_\alpha \), respectively. In particular, \( |\tilde{\phi}_\alpha(p)\rangle = \Gamma_{\eta_+}|\phi_\alpha(p)\rangle \).

The projector \( P_{\eta_+}(p) \) onto the filled Bloch states of the Hamiltonian \( H \) at fixed \( p \) and the associated ‘\( Q_{\eta_+} \) matrix’ are defined as
\[
P_{\eta_+}(p) := \sum_\alpha |\tilde{\phi}_\alpha(p)\rangle \langle \tilde{\phi}_\alpha(p)|_{\eta_+},
\]
\[
Q_{\eta_+}(p) := 1 - 2P_{\eta_+}(p),
\] (7)
where \( \hat{a} \) indicates a summation over all the filled Bloch states. The operators \( P_{\eta_+}(p) \) and \( Q_{\eta_+}(p) \) are Hermitian in the Hilbert space \( \mathcal{H}_{\eta_+} \) and satisfy the standard relations:
\[
P_{\eta_+}^2(p) = P_{\eta_+}(p), \quad Q_{\eta_+}^2(p) = 1. \quad (8)
\]

The projector \( P_D(p) \) onto the filled Bloch states of the Hamiltonian \( h \) at fixed \( p \) and the associated ‘\( Q_D \) matrix’ are related to \( P_{\eta_+}(p) \) and \( Q_{\eta_+}(p) \) through a non-unitary similarity transformation:
\[
P_D(p) := \sum_\alpha |\phi_\alpha(p)\rangle \langle \phi_\alpha(p)| = \rho P_{\eta_+}(p) \rho^{-1}, \quad (9)
\]
\[
Q_D(p) := 1 - 2P_D(p) = \rho Q_{\eta_+}(p) \rho^{-1},
\]
where \( \hat{a} \) indicates a summation over all the filled Bloch states and \( P_D^2(p) = P_D(p), \quad Q_D^2(p) = 1 \). The topological invariants for class AIII topological insulators in Hermitian quantum systems are constructed by using the projector \( Q_D(p) \) [6, 7], and equation (9) implies that the topological invariants for \( H \) and \( h \) are identical.

A comment is in order before the end of this section. The existence of \( \Gamma_D \) implies that the pseudo-Hermitian \( H \) is also pseudo-anti-Hermitian [18] with respect to an operator \( \kappa \):
\[
H^\dagger = -\kappa H_{\kappa}^{-1}, \quad \kappa := \rho \Gamma_D \rho = \eta_{\kappa} \Gamma_{\eta_{\kappa}}. \quad (10)
\]

The kinds of operators \( \Gamma_D \) and \( \rho \) that will be considered in this paper satisfy the identity
\[
\rho \Gamma_D \rho = \Gamma_D. \quad (11)
\]
The Hermitian Hamiltonian \( \hat{H} := H + H^\dagger \) in \( \mathcal{H}_D \) anti-commutes with the operator \( \Gamma_D \), when the validity of equation (11) is assumed, implying that \( \hat{H} \) can be transformed into a block off-diagonal form in a representation in which \( \Gamma_D \) is diagonal. The projector onto the filled Bloch states of the Hamiltonian \( \hat{H} \) can be used to construct topological invariants corresponding to \( H \) provided \( H \) is a normal operator, i.e. \([H, H^\dagger] = 0\), so \( \hat{H} \) and \( H \) are simultaneously diagonalizable. The non-Hermitian Hamiltonians considered in this paper are not normal operators and, hence, this scheme of constructing topological invariants is not applicable. On the other hand, the projector \( Q := 1 - [P_{\eta_+}(p) + P_D(p)] \) onto the filled Bloch states of \( H \) and \( H^\dagger \) is not an idempotent operator. Consequently, a block in the block off-diagonal form of \( Q \) cannot necessarily be identified as an element of \( U(N) \). Thus, the use of \( Q \) to construct a topological invariant corresponding to non-Hermitian \( H \) seems problematic.

3. Examples

A few examples of non-Hermitian Hamiltonians admitting topological insulator phases are presented in this section. All the non-Hermitian Hamiltonians considered in this paper can be mapped to Hermitian Hamiltonians, through non-unitary similarity transformations, which are known to admit a topological insulator phase. It follows from the discussions in section 2 that the bulk eigenvalues and the topological invariants are identical for both the non-Hermitian Hamiltonian and its similarity transformed Hermitian version.

The bulk eigenstates, zero modes and symmetry generators of the non-Hermitian Hamiltonian can be obtained from the corresponding quantities of the Hermitian Hamiltonian through the inverse similarity transformation. Thus, in general, only non-Hermitian Hamiltonians and associated non-unitary similarity transformations are mentioned explicitly in this note to avoid any repetition of known results.

The 1 + 1-dimensional Dirac equation is treated in some detail in section 3.1 to give a general outline of the technique involved, that can be applied in similar situations.

3.1. The Dirac Hamiltonian in 1 + 1 dimensions

The first example considered in this paper is the 1 + 1-dimensional non-Hermitian Dirac Hamiltonian
\[
H^{(1)} = \sigma^2 p_x + m(x) \cosh \phi \sigma^3 - im(x) \sinh \phi \sigma^1, \quad \phi \in \mathbb{R},
\]
(12)
where \( p_x \) is the linear momentum and \( m(x) \) is an arbitrary real function of its argument. The Hamiltonian \( H^{(1)} \) was
first introduced in [11] in the context of a non-associative non-Hermitian relativistic quantum system in $\mathcal{H}_D$. It was shown [11] that $H^{(1)}$ is Hermitian in the Hilbert space $\mathcal{H}_{\eta^+}$ that is endowed with the metric $\eta_{++}^{(1)}$:

$$\eta_{++}^{(1)} := e^{-\phi^{\gamma}}, \quad \rho^{(1)} := e^{-\frac{2}{5}\sigma^2}. \quad (13)$$

The non-unitary similarity operator $\rho^{(1)}$ maps $H^{(1)}$ to a Hermitian Hamiltonian $H^{(1)}$ in $\mathcal{H}_D$ [11]:

$$H^{(1)} := \rho^{(1)} H^{(1)} (\rho^{(1)})^{-1} = \sigma^2 p_x + m(\chi) \sigma^3, \quad (14)$$

implying that $H^{(1)}$ and $H^{(1)}$ are isospectral. The Hamiltonian $H^{(1)}$ anti-commutes with $\sigma^1$. Correspondingly, the Hamiltonian $H^{(1)}$ anti-commutes with the operator $(\rho^{(1)})^{-1} \sigma^1 \rho^{(1)} = \sigma^1 \eta_{++}^{(1)}$.

The Dirac Hamiltonian $H^{(1)}$ with $m(\chi) = m \in R$ provides an example of a chiral topological insulator in class AIII [7]. The relation equation (14) implies that the non-Hermitian Hamiltonian $H^{(1)}$ also admits a topological insulator phase. The bulk spectrum of $H^{(1)}$ contains a mass gap, $E_{\pm} = \pm \sqrt{p_x^2 + m^2} \equiv \pm \lambda$, and the Bloch wavefunction corresponding to the negative eigenvalue state reads

$$|\phi^-(p_x)\rangle = \left(\frac{(\rho^{(1)})^{-1}}{2\sqrt{p_x^2 + m^2}} \sum_{\pm} (ip_x - m + \lambda, \pm (ip_x - m - \lambda)\right). \quad (15)$$

The Berry connection $A(p_x)$ and the associated Chern–Simons invariant $CS_1$ for the state $|\phi^-(p_x)\rangle$ may be computed as

$$A(p_x) = \frac{im}{2\lambda^2} dp_x, \quad CS_1 = \frac{m}{4|m|}, \quad (16)$$

which are identical with the corresponding expressions [7] for $H^{(1)}$ in $\mathcal{H}_D$. The projector $Q_{\eta^+}$ can be expressed as follows:

$$Q_{\eta^+} = (U \rho^{(1)})^{-1} \left(\begin{array}{cc} 0 & q \\ q^* & 0 \end{array}\right) (U \rho^{(1)}),$$

$$q := \frac{-ip_x + m}{\lambda}, \quad U := e^{-\frac{2}{5}\sigma^2}, \quad (17)$$

where $q^*$ denotes the complex conjugate of $q$. It may be noted that the unitary operator $U$ and the similarity operator $\rho^{(1)}$ are independent of $p_x$ and $m$. The winding number is calculated as $n = \frac{2\pi}{2\pi} \int_0^\infty dq' q^{-1} dq' = \frac{m^2}{2|m|}$, which is twice the value of the Chern–Simons invariant $CS_1$.

The operator $H^{(1)}$ can be identified as the real supercharge of an $N = 1$ supersymmetric quantum system. The ground state of the supersymmetric Hamiltonian $h_5 := [H^{(1)}]^2$ in the supersymmetry-preserving phase is a zero mode of $H^{(1)}$.

The zero mode of $H$ can be obtained from the zero mode of $H^{(1)}$ by using equation (3). For example, assuming that $\text{Lt}_{x \rightarrow \pm \infty} m(x) = \pm m, m \in R^3$, the zero mode of $H$ reads

$$\phi_0 = \frac{1}{\sqrt{2}} (\rho^{(1)})^{-1} e^{-\frac{2}{5}\sigma^2} (m(x) \otimes 1). \quad (18)$$

The zero mode of $H$ for different shapes of $m(x)$ may be obtained in a similar way from the corresponding well-behaved zero-energy state of $H^{(1)}$.

3.2. The Dirac Hamiltonian in $2 + 1$ dimensions

The second example is a four-component non-Hermitian Dirac equation in $2 + 1$ dimensions. The relevant gamma matrices may be constructed in terms of the elements of the Clifford algebra:

$$[\xi^p, \xi^q] = \delta^{pq}, \quad p, q = 1, 2, \ldots, 5. \quad (19)$$

The three generators of the group $O(3)$ are expressed as

$$J^n := \frac{i}{8} e^{abc} [\xi^b, \xi^c], \quad a, b, c = 1, 2, 3. \quad (20)$$

The Hilbert space $\mathcal{H}_{\eta^+}$ is endowed with the metric $\eta_{++}^{(2)}$:

$$\eta_{++}^{(2)} := e^{-\phi^{\gamma}}, \quad \rho^{(2)} := e^{-\frac{2}{5}\sigma^2}, \quad \phi \in R, \quad n \cdot \bar{n} = 1, \quad (21)$$

The Hamiltonian $H^{(2)}$ given by

$$H^{(2)} = \xi^4 p_x + \xi^5 p_y + m \sum_{b=1}^3 R^{3b} \xi^b, \quad (22)$$

is non-Hermitian in $\mathcal{H}_D$ and Hermitian in $\mathcal{H}_{\eta^+}$. The non-Hermiticity of $H^{(2)}$ arises due to the fact that the complex conjugate of $R^{ab}$ is not equal to itself.

The non-unitary similarity operator $\rho^{(2)}$ maps $H^{(2)}$ to a Hermitian Hamiltonian $h^{(2)}$ in $\mathcal{H}_D$:

$$h^{(2)} := \rho^{(2)} H^{(2)} (\rho^{(2)})^{-1} = \xi^4 p_x + \xi^5 p_y + m k^3. \quad (23)$$

The elements of the Clifford algebra are realized in terms of the following matrices:

$$\xi_1 := \tau^1 \otimes I, \quad \xi_2 := \tau^3 \otimes I, \quad \xi_3 := \tau^1 \otimes \sigma^3, \quad \xi_4 := \tau^1 \otimes \sigma^1, \quad (24)$$

$$\xi_5 := \tau^1 \otimes \sigma^2, \quad (25)$$

where $\tau^a, \sigma^a$ with $a = 1, 2, 3$ are two sets of Pauli matrices corresponding to two different sublattices and $I$ is a $2 \times 2$ identity matrix. The Hamiltonian $h^{(2)}$ is known to admit a topological insulator phase [7]. In fact, $h^{(2)}$ describes two copies of the two-dimensional two-component Dirac Hamiltonian having a topological insulator phase. Following the general discussions in section 2, the conclusion is that $H^{(2)}$ admits a topological insulator phase with identically the same bulk eigenvalues and topological invariant as Hamiltonian $H^{(2)}$. The bulk eigenstates as well as the zero-mode state of $H^{(2)}$ can be obtained from the corresponding states of $h^{(2)}$ by using the relation (3) and making the identification $\rho := \rho^{(2)}$. 

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1. The Dirac Hamiltonian $H_{\text{ID}}$ in equation 6 of [11] is reduced to $H^{(1)}$ for $M(x) = V(x) = 0$ and $P(x) = -m(x)$. 

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3.3. The Dirac Hamiltonian in 3 + 1 dimensions

A Hamiltonian in 3 + 1 dimensions may be introduced as follows:

\[ H^{(3)} = \tilde{a} \cdot \tilde{p} + m e^{i \phi} \beta, \quad \phi \in \mathbb{R} \]
\[ a^a := \tau^1 \otimes \sigma^a, \quad \beta := \tau^5 \otimes I, \]
\[ \gamma^5 := \tau^1 \otimes I, \quad a = 1, 2, 3, \]

where the Dirac representation of the \( \gamma \) matrices has been used and \( \tau^a, \sigma^a \) correspond to two independent sets of Pauli matrices. The Hamiltonian is non-Hermitian in the Hilbert space \( \mathcal{H}_D \) for \( \phi \neq 0 \). The generator of the time-reversal transformation is defined as \( T := (I \otimes i \sigma^3)K \), where \( K \) is the complex conjugation operator. The Hamiltonian \( H^{(3)} \) is invariant under the time-reversal transformation. The last term spoils the invariance of the Hamiltonian under the parity transformation with the standard form of the generator \( P := \beta \). However, a non-standard generator of the parity transformation may be introduced as \( \tilde{P} := e^{i\phi} \beta \), which keeps \( H^{(3)} \) invariant.

The Hamiltonian \( H^{(3)} \) becomes Hermitian in \( \mathcal{H}_D \) in the limit \( \phi \to 0 \) and is Hermitian for any arbitrary \( \phi \) in the Hilbert space \( \mathcal{H}_H \), that is endowed with a positive-definite metric \( \eta^{(3)}_+ \):

\[ \eta^{(3)}_+ := e^{-i\phi} \gamma_5, \quad \rho^{(3)} := e^{-\frac{i}{2} \gamma_5}. \]

In fact, the non-unitary similarity operator \( \rho^{(3)} \) can be used to map \( H^{(3)} \) to a Hermitian Hamiltonian \( h^{(3)} \):

\[ H^{(3)} = \rho^{(3)} H^{(3)} (\rho^{(3)})^{-1} = \tilde{a} \cdot \tilde{p} + m \beta, \]

that is Hermitian in \( \mathcal{H}_D \). Thus, \( H^{(3)} \) and \( h^{(3)} \) are isospectral. It is known that the Hamiltonian \( h^{(3)} \) admits a topological insulator phase [7]. Thus, the Hamiltonian \( H^{(3)} \) also admits a topological insulator phase with identically the same bulk eigenvalues and topological invariant as Hamiltonian \( h^{(3)} \). Other relevant quantities for \( H^{(3)} \) may be obtained from \( h^{(3)} \) by using equation (3) with the replacement of \( \rho \) by \( \rho^{(3)} \).

4. Summary and discussions

Examples of non-Hermitian Dirac Hamiltonians admitting a topological insulator phase have been presented in one, two and three spatial dimensions. All of these Hamiltonians are Hermitian in a Hilbert space that is endowed with a positive-definite metric and a modified inner product. All of the bulk eigenvalues of a given non-Hermitian Hamiltonian are thus real. It has also been shown that any non-Hermitian Hamiltonian belonging to this class can be mapped to a Hermitian Hamiltonian through a non-unitary similarity transformation, implying that these two quantum systems are isospectral to each other. Further, the time evolution of the bulk states as well as the zero modes of the non-Hermitian Hamiltonians are unitary in the Hilbert space \( \mathcal{H}_+ \). It appears that the topological insulator phase in non-Hermitian Dirac Hamiltonians admitting entirely real spectra and unitary time evolution have been discussed in this paper for the first time in the literature.

The topological insulator phases in Hermitian quantum systems have been classified previously [6, 7] according to certain discrete symmetries of Dirac Hamiltonians. Perturbations preserving the symmetries of the original Hamiltonian cannot destabilize the topological insulator phase as long as the gap is not closed. A direct application of this classification scheme to the non-Hermitian Hamiltonians presented in this paper may give conflicting results, since the non-Hermitian terms may or may not preserve the symmetry of the Hermitian part of the same Hamiltonian. However, the same classification scheme can be used for a non-Hermitian Hamiltonian provided that the identification of the topological class is based on the analysis of the Hermitian Hamiltonian to which it is related through a non-unitary similarity transformation. It may be worth emphasizing here that the topological index characterizing a particular type of insulator, as presented in this paper, is identical for the non-Hermitian and the corresponding Hermitian Hamiltonians. The non-Hermitian Hamiltonian and its Hermitian counterpart thus belong to the same topological class by construction. It may be emphasized here that the unitary equivalence between different Hermitian Hamiltonians \( h, \tilde{h} \) in \( \mathcal{H}_D \), which are obtained from the same non-Hermitian Hamiltonian \( H \), allows a unique characterization of the topological class of \( H \) admitting topological insulator phases.

The approach taken in this paper in classifying topological insulator phases in non-Hermitian quantum systems may equally be extended to any generic non-Hermitian Hamiltonian that can be mapped to a Hermitian Hamiltonian. In general, finding the non-unitary similarity operator that maps the non-Hermitian Hamiltonian to a Hermitian one is a highly non-trivial task. However, the continuum descriptions of topological insulators are in terms of Dirac Hamiltonians which involve gamma matrices. The method described in [12] for a pseudo-Hermitian realization of the gamma matrices of arbitrary dimensions may be useful for finding the Hermitian Hamiltonian corresponding to a given non-Hermitian Hamiltonian. It is desirable that the method presented in this paper and in previous works [11–13] is utilized fully to construct at least one physically realizable non-Hermitian Dirac Hamiltonian admitting a topological insulator phase.

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