Geometric approach to stable homotopy groups of spheres II. The Kervaire invariant

P.M. Akhmet’ev

[translation by Peter Landweber of sections 1–5 of version sent 24 June 2009, including displays and references; there are minor corrections, and some requests for clarifications; further comments are given in brackets PSL]

Abstract

A solution to the Kervaire invariant problem is presented. We introduce the concepts of abelian structure on skew-framed immersions, bicyclic structure on $\mathbb{Z}/2^3$-framed immersions, and biquaternionic structure on $\mathbb{Z}/2^5$-framed immersions. Using these concepts, we prove that for sufficiently large $n$, $n = 2^l - 2$, an arbitrary skew-framed immersion in Euclidean $n$-space $\mathbb{R}^n$ has zero Kervaire invariant. The proof makes use of a compression theorem for skew-framed immersions in a given normal cobordism class. The proof of the compression theorem is also given in this paper.

1 Self-intersections of immersions and the Kervaire invariant

The Kervaire invariant one problem is an open problem in algebraic topology. For algebraic approaches to the problem see Snaith [S], Barratt-Jones-Mahowald [B-J-M] and Cohen-Jones-Mahowald [C-J-M]. We consider a geometric approach to the solution of the problem, which is based on results of P.J. Eccles [E1]. For a different geometric approach see Carter [C1], [C2].

Consider a smooth immersion $f : M^{n-1} \hookrightarrow \mathbb{R}^n$, $n = 2^l - 2$, $l > 1$ in general position and having codimension 1. We denote by $g : N^{n-2} \hookrightarrow \mathbb{R}^n$ the immersion of the manifold of self-intersections.
Definition 1. The Kervaire invariant of the immersion \( f \) is defined by the formula

\[
\Theta_{sf}(f) = \langle \eta_{N}^{n-2}, [N^{n-2}] \rangle,
\]

(1)

where \( \eta_{N} = w_{2}(N^{n-2}) \) denotes the second normal Stiefel-Whitney class of the manifold \( N^{n-2} \).

The Kervaire invariant \( \Theta_{sf} \) is an invariant of the regular cobordism class of the immersion \( f \). Moreover, the Kervaire invariant determines a homomorphism

\[
\Theta_{sf} : \text{Imm}^{sf}(n-1, 1) \rightarrow \mathbb{Z}/2.
\]

(2)

The normal bundle \( \nu_{g} \) of the immersion \( g : N^{n-2} \rightarrow \mathbb{R}^{n} \) is a 2-dimensional bundle over \( N^{n-2} \), which is naturally equipped with a \( D_{4} \)-framing, where \( D_{4} \) denotes the dihedral group of order 8. The classifying map of this bundle (and its corresponding characteristic class) are denoted by \( \eta_{N} : N^{n-2} \rightarrow K(D_{4}, 1) \). The pair \((g, \eta_{N})\) represents an element of the cobordism group \( \text{Imm}^{D_{4}}(n-2, 2) \). The passage from \( f \) to \((g, \eta_{N})\) gives rise to a well defined homomorphism

\[
\delta_{D_{4}} : \text{Imm}^{sf}(n-1, 1) \rightarrow \text{Imm}^{D_{4}}(n-2, 2).
\]

(3)

The cobordism group \( \text{Imm}^{sf}(n - k, k) \) generalizes the cobordism group \( \text{Imm}^{sf}(n - 1, 1) \). The new group is defined as the cobordism group of triples \((f, \Xi, \kappa_{M})\), where \( f : M^{n-k} \rightarrow \mathbb{R}^{n} \) is an immersion; moreover there is given a morphism of bundles (a bundle map) \( \Xi : \nu(f) \cong k \kappa_{M} \), which is invertible, i.e., which is a fiberwise isomorphism, and which is called a skew-framing, where \( \nu(f) \) denotes the normal bundle of the immersion \( f \) and \( \kappa_{M} \) is a given line bundle over \( M^{n-k} \), whose characteristic class is also denoted by \( \kappa_{M} \in H^{1}(M^{n-k}; \mathbb{Z}/2) \). The relation of cobordism on the set of triples is the standard one.

The group \( \text{Imm}^{D_{4}}(n-2, 2) \) is generalized in the following way. We shall define cobordism groups \( \text{Imm}^{D_{4}}(n - 2k, 2k) \). Each element of the group \( \text{Imm}^{D_{4}}(n - 2k, 2k) \) is represented by a triple \((g, \Psi, \eta_{N})\), where \( g : N^{n-2k} \rightarrow \mathbb{R}^{n} \) is an immersion, \( \Psi \) is a dihedral framing of codimension \( 2k \), i.e., a fixed isomorphism \( \Xi : \nu_{g} \cong k \eta_{N} \), and where \( \eta_{N} \) is a 2-dimensional bundle over \( N^{n-2k} \) with structure group \( D_{4} \). The characteristic mapping of this bundle, and also the corresponding characteristic class (respectively, the universal characteristic class) is denoted also by \( \eta_{N} : N^{n-2k} \rightarrow K(D_{4}, 1), \eta_{N} \in H^{2}(N^{n-2k}; \mathbb{Z}/2) \)
(respectively, \(\tau \in H^2(K(D_4, 1); \mathbb{Z}/2)\)). The mapping \(\eta_N\) is also called characteristic for the bundle \(\nu_g\), since \(\nu_g \cong k\eta_N\).

We define the Kervaire homomorphism (2) as the composition of the homomorphism (3) and a homomorphism

\[ \Theta_{D_4} : \text{Imm}^{D_4}(n - 2, 2) \to \mathbb{Z}/2. \]  

The homomorphism (4) is called the Kervaire invariant of a \(D_4\)-framed immersion.

The Kervaire homomorphism can be defined in more general situations by means of a direct generalization of the homomorphisms (2) and (4):

\[ \Theta_{sf}^k : \text{Imm}^{D_4}(n - k, k) \to \mathbb{Z}/2, \quad \Theta_{sf}^k := \Theta_{D_4}^k \circ \delta_{D_4}^k. \]  

\[ \Theta_{D_4}^k : \text{Imm}^{D_4}(n - 2k, 2k) \to \mathbb{Z}/2, \quad \Theta_{D_4}^k[(g, \Psi, \eta_N)] = \langle \eta_N^{n-2k}, [N^{n-2k}] \rangle. \]  

For \(k = 1\) the new homomorphism (6) coincides with the homomorphism (4) already defined, moreover the following diagram, in which the homomorphisms \(J_{sf}\) and \(J_{D_4}\) were defined in the first part of Akhmet’iev [A], is commutative:

\[
\begin{array}{ccc}
\text{Imm}^{sf}(n - 1, 1) & \xrightarrow{\delta_{D_4}^k} & \text{Imm}^{D_4}(n - 2, 2) \\
\downarrow J_{sf}^k & & \downarrow J_{D_4}^k \\
\text{Imm}^{sf}(n - k, k) & \xrightarrow{\delta_{D_4}^k} & \text{Imm}^{D_4}(n - 2k, 2k) \\
& \xrightarrow{\theta_{D_4}^k} & \mathbb{Z}/2 \\
& \xrightarrow{\theta_{D_4}^k} & \mathbb{Z}/2.
\end{array}
\]

We shall need to generalize formula (6) for immersions with framings of a more general form. Denote by \(\mathbb{Z}/2^{[d]}\) the wreath product of \(2^{d-1}\) copies of the cyclic group \(\mathbb{Z}/2\). This group is a subgroup of the orthogonal group \(O(2^{d-1})\), and can be defined in the following way:

- Transformations in \(\mathbb{Z}/2^{[d]}\) leave invariant the collection of \(d - 1\) sets \(\Upsilon_d, \Upsilon_{d-1}, \ldots, \Upsilon_2\) of coordinate subspaces. The set of subspaces \(\Upsilon_i, 2 \leq i \leq d\), consists of the \(2^i-1\) coordinate subspaces, spanned by the basis vectors

\[
((e_1, \ldots, e_{2^{d-i}}), \ldots, (e_{2^{d-i-1}} - 2^{d-i+1}, \ldots, e_{2^{d-i}})).
\]

[The blocks of basis vectors are disjoint and all of the same size. PSL]

In particular, in this new notation the dihedral group \(D_4\) will be denoted by \(\mathbb{Z}/2^{[2]}\). This group is defined as the subgroup of orthogonal transformations of the plane, carrying the set \(\Upsilon_2 = \{[e_1], [e_2]\}\) of lines into itself. In this paper
we shall make use of the groups $\mathbb{Z}/2^d$ for $2 \leq d \leq 6$. By definition, there is an inclusion $\mathbb{Z}/2^d \subset \mathbb{Z}/2 \wr \Sigma(2^{d-1})$, which coincides with the inclusion of a 2-Sylow subgroup of the symmetric group $\Sigma(2^d)$. [E.g., the dihedral group $D_4$ is a 2-Sylow subgroup of $\Sigma(4)$. PSL]

Consider an immersion $g : N^{n-k2^{d-1}} \rightarrow \mathbb{R}^n$ in general position and of codimension $k2^{d-1}$. We say that the immersion $g$ is $\mathbb{Z}/2^d$-framed (with multiplicity $k$), if an isomorphism $\Psi : \nu_g \cong k\eta_N$ is given between the normal bundle $\nu_g$ of the immersion $g$ and the Whitney sum of $k$ copies of a $2^{d-1}$-dimensional bundle $\eta_N$ with structure group $\mathbb{Z}/2^d$.

The bundle $\eta_N$ is classified by a mapping $\eta_N : N^{n-k2^{d-1}} \rightarrow K(\mathbb{Z}/2^d, 1)$. (The corresponding characteristic class is also denoted by $\eta_N$.) The characteristic class of the universal $2^{d-1}$-dimensional $\mathbb{Z}/2^d$-bundle over $K(\mathbb{Z}/2^d, 1)$ is denoted by $\tau[d]$. Therefore, $\eta_N^*(\tau[d]) = \eta_N$. The mapping $\eta_N$ is also called a characteristic map for the bundle $\nu_g$, since $\nu_g \cong k\eta_N$ via $\Psi$.

The set of all possible triples $(g, \Psi, \eta_N)$, as described above, generate the cobordism group $\text{Imm}^{\mathbb{Z}/2^d}(n - k2^{d-1}, k2^{d-1})$. In some considerations we use an additional index in the notation, connected with the structure group. For example, a representative of the group $\text{Imm}^{\mathbb{Z}/2^d}(n - 2k, 2k)$ will sometimes be denoted by $(g_{[2]}, \Psi_{[2]}, \eta_{N_{[2]}})$ and so on.

The manifold of self-intersections of an arbitrary $\mathbb{Z}/2^d$-framed immersion admits a natural $\mathbb{Z}/2^{d+1}$-framed immersion. Thus, the manifold of self-intersections yields a triple $(h, \Lambda, \zeta_L)$, where $h : L^{n-k2^d} \rightarrow \mathbb{R}^n$ is an immersion, $\Lambda : \nu_h \cong k\zeta_L$, and $\zeta_L : L^{n-k2^d} \rightarrow K(\mathbb{Z}/2^{d+1}, 1)$ is the classifying map of the $2^d$-dimensional bundle $\zeta_L$. We therefore obtain a homomorphism

$$\delta_{\mathbb{Z}/2^{d+1}}^k : \text{Imm}^{\mathbb{Z}/2^d}(n - k2^{d-1}, k2^{d-1}) \rightarrow \text{Imm}^{\mathbb{Z}/2^{d+1}}(n - k2^d, k2^d),$$

assigning to the normal cobordism class $[(g, \Psi, \eta_N)]$ the normal cobordism class $[(h, \Lambda, \zeta_L)]$.

A subgroup $i_{[d+1]} : \mathbb{Z}/2^d \subset \mathbb{Z}/2^{d+1}$ is defined as the subgroup of transformations of the subspace $\mathbb{R}^{2^d-1} \subset \mathbb{R}^{2^d}$ generated by the first $2^{d-1}$ basis vectors. [and acting as the identity on the remaining basis vectors, PSL]

A subgroup of index 2

$$\tilde{i}_{[d+1]} : \mathbb{Z}/2^d \times \mathbb{Z}/2^d \subset \mathbb{Z}/2^{d+1}$$

is defined as the subgroup of transformations leaving invariant each subspace in the set $Y_2$. 

4
The subgroup (9) induces a double covering \( \pi_{[d+1]} : K(\mathbb{Z}/2[d] \times \mathbb{Z}/2[d], 1) \to K(\mathbb{Z}/2[d+1], 1) \). The characteristic mapping \( \zeta_L : L^{n-k2^d} \to K(\mathbb{Z}/2[d+1], 1) \) induces a double covering \( \pi_{[d+1],L} : T^{n-k2^d} \to L^{n-k2^d} \) from the covering \( \pi_{[d+1]} \) over the classifying space. The double covering \( \pi_{[d+1],L} \) can be defined geometrically, namely it coincides with the canonical double covering of the manifold \( L^{n-k2^d} \) of points of self-intersection of the \( \mathbb{Z}/2[d] \)-framed immersion \( (g, \Psi, \eta_N) \) (see Akhmet’ev [A], section 1).

The projection \( p_{[d]} : \mathbb{Z}/2[d] \times \mathbb{Z}/2[d] \to \mathbb{Z}/2[d] \) onto the first factor induces a mapping \( p_{[d]} : K(\mathbb{Z}/2[d] \times \mathbb{Z}/2[d], 1) \to K(\mathbb{Z}/2[d], 1) \).

For the manifold of self-intersections \( (h, \Lambda, \zeta_L) \) of an arbitrary \( \mathbb{Z}/2[d] \)-framed immersion \( (g, \Psi, \eta_N) \) we consider the double covering \( \overline{\zeta}_L : \overline{T}^{n-k2^d} \to K(\mathbb{Z}/2[d] \times \mathbb{Z}/2[d], 1) \) over the classifying mapping \( \zeta_L \), which is induced from the covering \( \pi_{[d+1],L} \). This covering coincides with the canonical double covering over the classifying mapping \( \zeta_L : L^{n-2k^d} \to K(\mathbb{Z}/2[d+1], 1) \), which is defined by geometric considerations. The characteristic class \( (p_{[d]} \circ \overline{\zeta}_L)^*(\tau_{[d]}) \in H^{2d-1}(\overline{T}^{n-k2^d}; \mathbb{Z}/2), \tau_{[d]} \in H^{2d-1}(K(\mathbb{Z}/2[d], 1); \mathbb{Z}/2) \), is denoted by \( \overline{\zeta}_{[d],L} \).

We define the mapping \( i_{\text{tot}} = i_{[d]} \circ \cdots \circ i_{[3]} \) from the tower:

\[
K(D_4, 1) \xrightarrow{i_{[3]}} K(\mathbb{Z}/2[3], 1) \xrightarrow{i_{[4]}} \cdots \xrightarrow{i_{[d]}} K(\mathbb{Z}/2[d], 1) \xrightarrow{i_{[d+1]}} K(\mathbb{Z}/2[d+1], 1).
\]

There is defined a tower of canonical double coverings

\[
\overline{T}^{n-k2^d}_{[2]} \xrightarrow{\pi_{[3]}} \overline{T}^{n-k2^d}_{[3]} \xrightarrow{\pi_{[4]}} \cdots \xrightarrow{\pi_{[d]}} \overline{T}^{n-k2^d}_{[d]} \xrightarrow{\pi_{[d+1]}} L^{n-k2^d}.
\]

This tower of coverings is endowed with characteristic mappings to the diagram (10). We denote by

\[
\pi_{\text{tot}} = \pi_{[d]} \circ \cdots \circ \pi_{[3]} : K(D_4, 1) \to K(\mathbb{Z}/2[d], 1)
\]

the tower of coverings induced by compositions of the homomorphisms in the diagram (10). There is defined a sequence of characteristic classes [coefficients in \( \mathbb{Z}/2 \) are omitted from the notation, PSL]

\[
\overline{\zeta}_{[2],L} \in H^2(\overline{T}^{n-k2^d}_{[2]}), \ldots, \overline{\zeta}_{[d],L} \in H^{2d-1}(\overline{T}^{n-k2^d}_{[d]}), \zeta_{[d+1],L} \in H^{2d}(L^{n-k2^d}).
\]

Each element in this sequence is induced from the characteristic class of the corresponding universal space in (10). The tower of coverings (11) and
the sequence of characteristic classes (13) are defined not only for \( \mathbb{Z}/2^{[d+1]} \)-framed manifolds which occur as manifolds of self-intersection of suitable \( \mathbb{Z}/2^{[d]} \)-framed immersions, but also for arbitrary \( \mathbb{Z}/2^{[d+1]} \)-framed manifolds. [It is left to the reader to define such manifolds! PSL]

**Definition 2.** The Kervaire invariant \( \Theta_{\mathbb{Z}/2^{[d+1]}}^k \) of an arbitrary \( \mathbb{Z}/2^{[d+1]} \)-framed immersion \((h, \Lambda, \zeta_L)\) is defined by the following formula:

\[
\Theta_{\mathbb{Z}/2^{[d+1]}}^k(h, \Lambda, \zeta_L) = \langle \eta^{n-k2^d}, [N_{[2]}] \rangle,
\]

where \([N_{[2]}]\) denotes the fundamental class of the covering manifold in the sequence (13). [I'm confused, should the reference be to (11)? A clearer explanation is needed. PSL]

The invariant just constructed defines a homomorphism

\[
\Theta_{\mathbb{Z}/2^{[d]}}^k : \text{Imm}^{\mathbb{Z}/2^{[d]}}(n-k2^{d-1}, k2^{d-1}) \to \mathbb{Z}/2,
\]

which is included in the following commutative diagram:

\[
\begin{array}{ccc}
\text{Imm}^{\mathbb{Z}/2^{[d]}}(n-k2^{d-1}, k2^{d-1}) & \xrightarrow{\Theta_{\mathbb{Z}/2^{[d]}}^k} & \mathbb{Z}/2 \\
\downarrow{\delta_{\mathbb{Z}/2^{[d+1]}}^k} & & \downarrow{\mid} \\
\text{Imm}^{\mathbb{Z}/2^{[d+1]}}(n-k2^d, k2^d) & \xrightarrow{\Theta_{\mathbb{Z}/2^{[d+1]}}^k} & \mathbb{Z}/2.
\end{array}
\]

In section 2 we define the concept of an \( \textbf{I}_d \times \hat{\textbf{I}}_d \)-structure (abelian structure) on a skew-framed immersion representing an element of the cobordism group \( \text{Imm}^{\text{sf}}(n-k, k) \). It is proved in Theorem 6 that under an appropriate dimensional restriction and modulo elements of odd order, an arbitrary cobordism class of skew-framed immersions admits an \( \textbf{I}_d \times \hat{\textbf{I}}_d \)-structure. The hypothesis of this theorem presupposes the existence of a compression of characteristic classes of skew-framed immersions, see Definition 5. The compression theorem is proved in section 8.

In section 3 we formulate the concept of an \( \textbf{I}_a \times \hat{\textbf{I}}_a \)-structure (bicyclic structure) on a \( \mathbb{Z}/2^{[3]} \)-framed immersion. In Corollary 13 to Theorem 12, it is proved that, under the conditions of Theorem 6 (here and below, the natural number \( n_a \) defined in the hypothesis of this theorem can be taken to be 126 if an additional compression hypothesis is satisfied) an arbitrary
element of the group [probably want the image of a composition here, PSL]

\[ \text{Im}(\delta_{\mathbb{Z}/2[3]}^\mathfrak{n-n_s} : \text{Imm}^\mathfrak{s}(n - \frac{n - n_s}{32}, \frac{n - n_s}{32}) \to \text{Imm}^\mathbb{Z/2[3]}(n - \frac{n - n_s}{8}, \frac{n - n_s}{8})) \]  \hspace{1cm} (16)

is represented by a \( \mathbb{Z}/2[3] \)-framed immersion with bicyclic structure. For such an immersion the Kervaire invariant can be evaluated in terms of an \( \mathcal{I}_d \times \hat{\mathcal{I}}_a \)-characteristic class of the manifold of self-intersections.

In section 4 we formulate the concept of a \( \mathbb{Q}_a \times \hat{\mathbb{Q}}_a \)-structure (biquaternionic structure) on a \( \mathbb{Z}/2[5] \)-framed immersion. In Corollary 19 to Theorem 18, it is proved that, under the conditions of Theorem 6, an arbitrary element of the group [again probably want the image of a composition, PSL]

\[ \text{Im}(\delta_{\mathbb{Z}/2[5]}^\mathfrak{n-n_s} : \text{Imm}^\mathfrak{s}(n - \frac{n - n_s}{32}, \frac{n - n_s}{32}) \to \text{Imm}^\mathbb{Z/2[5]}(n - \frac{n - n_s}{2}, \frac{n - n_s}{2})) \]  \hspace{1cm} (17)

is represented by a \( \mathbb{Z}/2[5] \)-framed immersion with biquaternionic structure. For such an immersion the Kervaire invariant can be evaluated in terms of a \( \mathbb{Q}_a \times \hat{\mathbb{Q}}_a \)-characteristic class of the manifold of self-intersections. [I’ve changed [6] to [5] three times in this paragraph. PSL]

The following two diagrams explain the plan of the proof. In the following diagram various subgroups of the structure groups of cobordism groups of immersions are given, as well as the names of the structures on immersions corresponding to each subgroup:

\[
\begin{array}{cccc}
\mathbb{I}_d \times \hat{\mathbb{I}}_d & \subset & \mathbb{Z}/2[2] & \text{Abelian structure} \\
\downarrow & & \downarrow \hat{i}[3] \\
\mathbb{I}_a \times \hat{\mathbb{I}}_d & \subset & \mathbb{Z}/2[3] & \text{cyclic-Abelian structure} \\
\downarrow & & \downarrow \hat{i}[4] \\
\mathbb{I}_a \times \hat{\mathbb{I}}_a & \subset & \mathbb{Z}/2[4] & \text{bicyclic structure} \\
\downarrow & & \downarrow \hat{i}[5] \\
\mathbb{Q}_a \times \hat{\mathbb{Q}}_a & \subset & \mathbb{Z}/2[5] & \text{quaternionic-cyclic structure} \\
\downarrow & & \downarrow \hat{i}[6] \\
\mathbb{Q}_a \times \hat{\mathbb{Q}}_a & \subset & \mathbb{Z}/2[6] & \text{biquaternionic structure}.
\end{array}
\]

In the following diagram the natural homomorphisms of cobordism groups of immersions that will be used are shown, and the Kervaire invariants on each of these groups are indicated:
In view of the commutativity of this diagram, it suffices to show that the Kervaire invariant defined in the last row of the diagram is zero. This is proved with the help of the concept of biquaternionic structure.

The author thanks Prof. M. Mahowald (2005), Prof. R. Cohen (2007), and Prof. P. Landweber (2009) for discussions, Prof. A.A. Voronov for an invitation to lecture at the University of Minnesota (2005), and Prof. V. Chernov for an invitation to lecture at Dartmouth College (2009).

This paper was begun at the seminar of M.M. Postnikov in 1998. The paper is dedicated to the memory of Prof. Yu.P. Solov’ev. The compression theorem was proved in the seminar of A.S. Mishchenko.

## 2 Geometric control of the manifold of self-intersections of a skew-framed immersion

In this and the following sections, we shall make use of the cobordism groups $\text{Imm}^s(n-k,k)$ and $\text{Imm}^{Z/2^2}(n-2k,2k)$. It is well known that if the first argument in parentheses, denoting the dimension of the immersed manifold, is strictly positive, then the indicated group is a finite 2-group.

The dihedral group $\mathbb{Z}/2^2$ is defined by its presentation

$$\{a, b \mid a^4 = b^2 = e, [a, b] = a^2\}.$$
This group is a subgroup of the orthogonal group $O(2)$, namely, the group of orthogonal transformations of the standard Euclidean plane, preserving the pair of lines generated by a basis of orthogonal unit vectors $\{f_1, f_2\}$. The element $a$ is represented by a rotation of the plane through an angle $\frac{\pi}{2}$. The element $b$ is represented by a reflection of the plane with respect to the line generated by the vector $f_1 + f_2$.

Consider the subgroup $I_d \times \hat{I}_d \subset Z/2^2$ of the dihedral group, generated by the elements $\{a^2, b\}$. Notice that this is an elementary 2-group of rank 2. This is the subgroup of $O(2)$ consisting of transformations preserving individually each of the lines $l_1, l_2$ in the directions of the vectors $f_1 + f_2, f_1 - f_2$, respectively. The cohomology group $H^1(K(I_d \times \hat{I}_d, 1); Z/2)$ is also an elementary 2-group with two generators. We now describe these generators.

We denote by $p_d : I_d \times \hat{I}_d \rightarrow I_d$ the homomorphism, whose kernel consists of the reflection with respect to the bisector of the first coordinate angle and the identity. Define $\kappa_d = p_d^*(t_d)$, where $e \neq t_d \in H^1(K(I_d, 1); Z/2) \cong Z/2$. Denote by $p_{\hat{d}} : I_d \times \hat{I}_d \rightarrow \hat{I}_d$ the homomorphism, whose kernel consists of the reflection with respect to the bisector of the second coordinate angle and the identity, or equivalently whose kernel consists of the composition of the central symmetry and the symmetry with respect to the first coordinate angle and the identity. Define $\kappa_{\hat{d}} = p_{\hat{d}}^*(t_{\hat{d}})$, where $e \neq t_{\hat{d}} \in H^1(K(\hat{I}_d, 1); Z/2) \cong Z/2$.

We denote by $\tau_{d \times \hat{d}}$ the cohomology class $\kappa_d \kappa_{\hat{d}} \in H^2(K(I_d \times \hat{I}_d, 1); Z/2)$.

We consider the mapping $i_{d \times \hat{d}} : K(I_d \times \hat{I}_d, 1) \rightarrow K(Z/2^2, 1)$ and the pull-back $i_{d \times \hat{d}}^*(\tau_{[2]})$ of the Euler class $\tau_{[2]} \in H^2(K(Z/2^2, 1); Z/2)$ of the universal 2-plane bundle $\zeta_{[2]}$ over the Eilenberg-MacLane space $K(Z/2^2, 1)$. It is easy (as we now show) to verify that

$$
\tau_{d \times \hat{d}} = \kappa_d \kappa_{\hat{d}} = i_{d \times \hat{d}}^*(\tau_{[2]}). \tag{18}
$$

We consider the Euler class $e(\zeta_{[2]} \in H^2(K(Z/2^2, 1); Z/2)$ of the bundle $\zeta_{[2]}$, which coincides with $\tau_{[2]}$, and the restriction of this class to the group $H^2(K(I_d \times \hat{I}_d, 1); Z/2)$ by means of the homomorphism $i_{d \times \hat{d}}$. The restricted cohomology class coincides with the Euler class $\tau_{d \times \hat{d}} = e(\kappa_d \oplus \kappa_{\hat{d}})$ of the bundle $\kappa_d \oplus \kappa_{\hat{d}}$, in view of the naturality and multiplicativity of the Euler class and since $i_{d \times \hat{d}}^*(\zeta_{[2]} = \kappa_d \oplus \kappa_{\hat{d}}$. In this formula we denote by $\kappa_d$ and $\kappa_{\hat{d}}$ the line bundles over $K(I_d \times \hat{I}_d)$ corresponding to the one-dimensional cohomology classes with the same names.
Since $i^*_{d \times d}(\xi_{[2]})$ is the Whitney sum of the pair of corresponding line bundles, we obtain (18) for the Euler classes.

**Definition 3.** Let the skew-framed immersion $(f, \Xi, \kappa_M), f : M^{n-k} \looparrowright \mathbb{R}^n$, represent an element $x \in \text{Imm}^{sf}(n-k, k)$. Let the element $y = \delta^k_{\mathbb{Z}/2^2}(x) \in \text{Imm}^{\mathbb{Z}/2^2}_1(n-2k, 2k)$ be represented by a $\mathbb{Z}/2^2$-framed immersion $(g, \Psi, \eta_N), g : N^{n-2k} \looparrowright \mathbb{R}^n$. We say that the skew-framed immersion $(f, \Xi, \kappa_M), f : M^{n-k} \looparrowright \mathbb{R}^n$, is an $I_d \times \hat{I}_d$-immersion (abelian immersion) if the structure mapping $\eta_N : N^{n-2k} \to K(\mathbb{Z}/2^2, 1)$ can be represented as the composition of a mapping $\eta_{d \times d, N} : N^{n-2k} \to K(I_d \times \hat{I}_d, 1)$ and the mapping $i_{d \times d} : K(I_d \times \hat{I}_d, 1) \to K(\mathbb{Z}/2^2, 1)$.

**Definition 4.** Let the skew-framed immersion $(f, \Xi, \kappa_M), f : M^{n-k} \looparrowright \mathbb{R}^n$, represent an element $x \in \text{Imm}^{sf}(n-k, k), n > 32k$. Let the $\mathbb{Z}/2^2$-framed immersion $(g, \Psi, \eta_N), g : N^{n-2k} \looparrowright \mathbb{R}^n$, be the immersion of the manifold of self-intersections of the immersion $f$, so $g$ represents the element $y = \delta^k_{\mathbb{Z}/2^2}(x) \in \text{Imm}^{\mathbb{Z}/2^2}_1(n-2k, 2k)$. We say that the skew-framed immersion $(f, \Xi, \kappa_M)$ admits an abelian structure $(I_d \times \hat{I}_d$-structure) if there exists a mapping $\eta_{d \times d, N} : N^{n-2k} \to K(I_d \times \hat{I}_d, 1)$ satisfying the following equation:

$$
\Theta^k_{\mathbb{Z}/2^2}(y) = \langle \eta^*_N \eta^*_{d \times d, N}, [N] \rangle.
$$

(19)

In this equation, the characteristic class $\eta^*_{d \times d, N}(\tau_{d \times d}) \in H^2(N^{n-2k}; \mathbb{Z}/2)$ has been denoted by $\eta_{d \times d, N}$, $[N]$ is the fundamental class of the manifold $N^{n-2k}$, $\eta_N \in H^2(N^{n-2k}; \mathbb{Z}/2)$ is the characteristic class of the $\mathbb{Z}/2^2$-framing $\Psi$ of the manifold $N^{n-2k}$, and finally $\Theta^k_{sf}(x)$ is defined by formula (5).

**Example**

Let the skew-framed immersion $(f, \Xi, \kappa_M), f : M^{n-k} \looparrowright \mathbb{R}^n$, represent an element $x \in \text{Imm}^{sf}(n-k, k), n > 32k$ and be an $I_d \times \hat{I}_d$-immersion. Then the skew-framed immersion $(f, \Xi, \kappa_M)$ admits an abelian structure.

**Definition 5.** Let $(f, \Xi, \kappa_M) \in \text{Imm}^{sf}(N-k, k), f : M^{n-k} \looparrowright \mathbb{R}^n, \kappa_M \in H^1(M^{n-k}; \mathbb{Z}/2)$ be skew-framed by $\Xi$. We say that the pair $(M^{n-k}, \kappa_M)$ admits a compression of order $q$ if the mapping $\kappa_M : M^{n-k} \to \mathbb{RP}^\infty$ can be represented as a composition $\kappa_M = I \circ \kappa'_M : M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^\infty$, where $I$ denotes the inclusion. We say that the element $[(f, \Xi, \kappa_M)]$ admits a compression of order $q$ if this cobordism class contains a triple $(f', \Xi', \kappa_{M'}), f' : M^{n-k} \looparrowright \mathbb{R}^n$, so that the pair $(M^{n-k}, \kappa_{M'})$ admits a compression of order $q$. 

10
Theorem 6. Let $n_s, n > n_s$, be a natural number of the form $2^s - 2, s \geq 6$. Assume that the element $\alpha \in \text{Imm}^\sigma(n - \frac{n - n_s}{32}, \frac{n - n_s}{32})$ admits a compression of order $q = \frac{n_s}{2} + 1$. Then the element $\alpha$ admits an $I_d \times I_d$-structure.

We shall prove the following lemma.

Lemma 7. For an integer $k', k' \equiv 1 \pmod{2}, k' \geq 7$, there exists a PL-mapping $d : \mathbb{R}P^{n-k'} \to \mathbb{R}^n$ in general position, for which the manifold $N(d)$ with singularities and with boundary the points of self-intersection of the mapping $d$ admits a mapping $\kappa_{N(d)} : N(d) \to K(I_d, 1)$, for which the restriction to the boundary $\partial N(d)$ (the boundary of this manifold with singularities consists of the critical points of the mapping $d$) coincides with the composition $\partial N(d) \to \mathbb{R}P^{n-k'} \subset \mathbb{R}P^n = K(I_d, 1)$.

Construction of the mapping $d : \mathbb{R}P^{n-k'} \to \mathbb{R}^n$

Denote by $J_0$ the standard $(n - k')$-dimensional sphere of codimension $k'$ in $\mathbb{R}^n$, which is represented as the join of $\frac{n-k'+1}{2} = r$ copies of the circle $S^1$. We denote the standard embedding of $J_0$ into $\mathbb{R}^n$ by $i_{J_0} : J_0 \subset \mathbb{R}^n$.

A mapping $p' : S^{n-k'} \to J_0$ is obtained as a result of taking the join of $r$ copies of the standard double covering $S^1 \to \mathbb{R}P^1$. The standard antipodal action $\mathbb{Z}/2 \times S^{n-k'} \to S^{n-k'}$ commutes with the mapping $p'$. Hence, there results a mapping $p : \mathbb{R}P^{n-k'} \to J_0$.

We consider the composition $i_{J_0} \circ p : \mathbb{R}P^{n-k'} \to J_0 \subset \mathbb{R}^n$. The mapping $d$ is defined as a result of a small $\delta$-deformation to general position of this mapping. The deformation $i_{J_0} \mapsto d$ and its caliber are chosen in the process of the proof. [What is a better word than “caliber”? PSL]

Proof of Lemma 7

The proof is analogous to (and substantially simpler than) the proof of Proposition 22 of Akhmet’ev [A]. [I have not looked at this other proof. PSL]

Proof of Theorem 6

Put $k = \frac{n-n_s}{32}$. Let the element $\alpha$ be represented by a skew-framed immersion $(f, \Xi, \kappa_M), f : M^{n-k} \looparrowright \mathbb{R}^n$. By assumption there exists a mapping $\kappa'_M : M^{n-k} \to \mathbb{R}P^{n-k-q-1}$ such that the composition $M^{n-k} \to \mathbb{R}P^{n-k-q-1} \subset K(I_d, 1)$ coincides with the mapping $\kappa_M : M^{n-k} \to K(I_d, 1), q = \frac{n_s}{2} + 1$. Notice that $k + q + 1$ is odd; we denote this number by $k'$ and consider the mapping $d : \mathbb{R}P^{n-k'} \to \mathbb{R}^n$ constructed in Lemma 7.
Choose a positive number $\varepsilon$ less than the radius of the regular (immersed) neighborhood $U_{\text{reg}}$ of the regular points of the mapping $d$, and less than the radius of the regular (immersed) neighborhood $\partial U_{\text{reg}}$ of the critical points of this mapping. Consider an immersion $f_1 : M^{n-k} \to \mathbb{R}^n$ in the regular homotopy class of the immersion $f$ such that the inequality
\[
\text{dist}(d \circ \kappa', f_1)_{C^0} < \varepsilon
\]
holds. The immersion $f_1$ is skew-framed by means of $\Xi_1$, with the same characteristic class of skew-framing $\kappa_M$, and in addition $[(f_1, \Xi_1, \kappa_M)] = \alpha$; see Akhmet’ev [A], Corollary 23.

We now show that the skew-framed immersion $f_1$ admits an $I_d \times I_d$-structure. Let $N_1^{n-2k}$ denote the manifold of self-intersections of the immersion $f_1$. We shall define a mapping $\eta_{d \times d} : N_1^{n-2k} \to K(I_d \times I_d, 1)$. We represent the manifold $N_1^{n-2k}$ as a union of two manifolds with parts of their boundaries in common: $N_1^{n-2k} = N_{N(d)}^{n-2k} \cup _{\partial} N_{\text{reg}}^{n-2k}$. In this formula, $N_{N(d)}^{n-2k}$ is a manifold with boundary obtained from the regular (immersed) neighborhood $U_{N(d)}$ of the manifold with singularities with boundary $N(d)$ of points of self-intersection of the mapping $d$. The manifold with boundary $N_{\text{reg}}^{n-2k}$ is immersed in the immersed neighborhood $U_{\text{reg}}$. The common boundary of the manifolds $N_{N(d)}^{n-2k}$ and $N_{\text{reg}}^{n-2k}$ is a closed manifold of dimension $n - 2k - 1$; this manifold is immersed in the boundary of the immersed neighborhood $U_{\partial N(d)}$.

A cohomology class $\kappa_{d,N_1} \in H^1(N_1^{n-2k}; \mathbb{Z}/2)$ on the manifold $N_1^{n-2k}$ is defined by a mapping $\kappa_{d,N_1} : N_1^{n-2k} \to K(I_d, 1)$, which we now construct. A mapping $\kappa_{d,N_{\text{reg}}} : N_{\text{reg}}^{n-2k} \to K(I_d, 1)$ on the submanifold $N_{\text{reg}}^{n-2k}$ is defined as the composition of the projection $N_{\text{reg}}^{n-2k} \to \mathbb{RP}^{n-k'}$ and the inclusion $\mathbb{RP}^{n-k'} \subset \mathbb{RP}^\infty = K(I_d, 1)$. A mapping $\kappa_{N(d)} : N_{N(d)}^{n-2k} \to K(I_d, 1)$ is defined on the submanifold $N_{N(d)}^{n-2k}$ as the composition of the projection $N_{N(d)}^{n-2k} \to N(d)$ and the mapping $\kappa_d : N(d) \to K(I_d, 1)$ constructed in Lemma 7. The restrictions of the mappings $\kappa_{N(d)}$ and $\kappa_{d,N_{\text{reg}}}$ to the boundaries $\partial N_{N(d)}^{n-2k}$ and $\partial N_{\text{reg}}^{n-2k}$ are homotopic, since the mapping $\kappa_{N(d)} : N(d) \to K(I_d, 1)$ satisfies the boundary condition on $\partial N(d)$. As a result, the mapping $\kappa_{d,N_1} : N_1^{n-2k} \to K(I_d, 1)$ can be defined by gluing the mappings $\kappa_{N(d)}$ and $\kappa_{d,N_{\text{reg}}}$. The cohomology class $\kappa_{d,N_1} \in H^1(N_1^{n-2k}; \mathbb{Z}/2)$ is defined as the sum of the characteristic classes of the canonical double coverings $\overline{N}_1^{n-2k} \to N_1^{n-2k}$ with class $\kappa_{d,N_1}$. [This is confusing. PSL] The pair of cohomology classes $\kappa_{d,N_1}$, $\kappa_{d,N_1} \in H^1(N_1^{n-2k}; \mathbb{Z}/2)$ define the desired mapping $\eta_{d \times d} : N_1^{n-2k} \to K(I_d \times I_d, 1)$. This
mapping is uniquely characterized by the equations \( \eta^*_{d \times d}(\kappa_d) = \kappa_{d,N_1} \) and 
\( \eta^*_{d \times d}(\kappa_d) = \kappa_{d,N_1} \).

We now turn to the verification of equation (19). Consider the mapping 
\[ \kappa'_M : M^{n-k} \to \mathbb{R}P^{n-k-q-1}, \]
and the mapping 
\[ \kappa''_M : M^{n-16k}_2 \to \mathbb{R}P^{n-k-q-1} \]
which is defined as the restriction of \( \kappa'_M \) to the inverse image \( M^{n-16k}_2 = (\kappa'_M)^{-1}(\mathbb{R}P^{n-16k-q-1}-q-1) \) of the projective subspace \( \mathbb{R}P^{n-16k-q-1} \subset \mathbb{R}P^{n-k-q-1} \) of codimension 15k, under the assumption that the mapping \( \kappa'_M \) is transverse along this submanifold.

Consider the immersion \( f_2 : M^{n-16k} \to \mathbb{R}^n \) which is defined as the restriction of the immersion \( f_1 \) to the submanifold \( i_M : M^{n-16k}_2 \subset M^{n-k} \). This immersion has codimension 16 and is skew-framed. We denote the skew-framing of this immersion by \( \Xi_2 \), and its characteristic class by \( \kappa'_{M_2} \in H^1(M^{n-16k}_2, \mathbb{Z}/2) \). By construction, \( \kappa'_{M_2} = i^*_M(\kappa_M) \). The triple \( (f_2, \Xi_2, \kappa'_{M_2}) \) defines the element \( J^{16k}_n(\alpha) \in \text{Im} \xi^1(n-16k, 16k) \).

Consider the manifold of points of self-intersection of the immersion \( f_2 \), which we denote by \( N^{n-32k}_2 \). There is a natural inclusion

\[ i_{N_1} : N^{n-32k}_2 \subset N_1^{n-2k}. \tag{20} \]

The fundamental class of this submanifold represents the cycle \( (i_{N_1})_*([N_2]) \in H_{n-32k}(N^{n-2k}_1; \mathbb{Z}/2) \). This cycle is dual in the sense of Poincaré to the cohomology class \( \eta\xi_{N_1}^{15k} \in H^{30k}(N_1^{n-2k}; \mathbb{Z}/2) \).

We shall prove that the submanifold (20) is contained in the submanifold \( N^{n-2k}_{\text{reg}} \subset N^{n-2k}_1 \), so that we have the inclusions

\[ \text{Im}(i_{N_1}(N^{n-32k}_2)) \subset N^{n-2k}_{\text{reg}} \subset N^{n-2k}_1. \tag{21} \]

Recall the structure mapping \( \eta_{N(d)} : (N(d), \partial N(d)) \to (K(\mathbb{Z}/2^2, 1), K(\mathbb{L}_1, 1)) \), and as before denote by \( \eta_{N(d)} \in H^2(N(d); \mathbb{Z}/2) \) the characteristic class induced from the universal class \( \eta_2 \in H^2(K(\mathbb{Z}/2^2, 1); \mathbb{Z}/2) \) (from the Euler class of the universal \( \mathbb{Z}/2^2 \)-bundle) by the mapping \( \eta_{N(d)} \). The Poincaré dual homology class \( (\eta_{N_1}^{15k})^{\text{op}} \in H_{n-32k}(N_1^{n-2k}; \mathbb{Z}/2) \) represents a cycle, whose restriction to the submanifold with boundary \( N^{n-2k}_N \subset N^{n-2k}_1 \) coincides with the complete preimage of the relative cycle \( ((\eta_{N(d)})^{15k})^{\text{op}} \) under the projection \( N^{n-2k}_N \to N(d) \) onto the central submanifold with singularities with boundary in the regularly immersed neighborhood.

For a dimensional reason, since the dimension \( \dim(N(d)) = n-2k-2q = n-\frac{n-2k}{16}-n_s-2 \) is less than the codimension of the submanifold
(20), which is equal to $30k = \frac{15(a-n_s)}{16}$, the relative homology class dual to the cocycle $(\eta_{N(d)})^{15k}$, represents in $N(d)$ the empty manifold. This verifies the formula (21).

Now for the proof of (19) it suffices to remark that in view of formula (21), the cocycle $\eta_{d \times d,N_1} \in H^2(N_1^{n-2k};\mathbb{Z}/2)$, restricted to the submanifold (20), coincides with the restriction of the cocycle $\eta_{N_1} \in H^2(N_1^{n-2k};\mathbb{Z}/2)$ to the same manifold. Hence, the formula (19) and Theorem 6 are proved.

3 $I_a \times \hat{I}_a$-structure (bicyclic structure) on a $\mathbb{Z}/2^3$-framed immersion

We define the group $I_a$ as the cyclic subgroup of order 4 in the dihedral group $\mathbb{Z}/2^2$, see Akhmet’ev [A], section 2. We shall now define a subgroup

$$i_{a \times \hat{a}} : I_a \times \hat{I}_a \subset \mathbb{Z}/2^4.$$ (22)

Recall that the group $\mathbb{Z}/2^4$ was defined in terms of a basis $(e_1, \ldots, e_8)$ of the Euclidean space $\mathbb{R}^8$.

We denote the generators of the factors of the group $I_a \times \hat{I}_a$ by $a$ and $\hat{a}$ respectively. We shall describe transformations in $\mathbb{Z}/2^4$ corresponding to each generator. We introduce a new basis $\{f_1, \ldots, f_8\}$ of $\mathbb{R}^8$ by the formulas:

$$f_{2i-1} = e_{2i-1} + e_{2i},$$

$$f_{2i} = e_{2i-1} - e_{2i},$$

$i = 1, \ldots, 4$. The generator $a$ of order 4 is represented by a rotation through angle $\frac{\pi}{2}$ in each of the planes $(f_1, f_3), (f_5, f_7)$, and by the central symmetry in the plane $(f_2 - f_4, f_6 - f_8)$. [Here and below, I’m copying what I see, but think it needs correction. I was expecting to see four mutually orthogonal planes. I expect there are planes (which should be named) in which the generators act as the identity. PSL] The generator $\hat{a}$ is represented by a rotation through angle $\frac{\pi}{2}$ in the planes $(f_2 - f_4, f_6 - f_8), (f_2 + f_4, f_6 + f_8)$, and by the central symmetry in the plane $(f_1 - f_5, f_3 - f_7)$.

We shall show that the group of transformations $I_a \times \hat{I}_a$ has invariant $(2, 2, 2, 2)$-dimensional subspaces, which we denote by $\mathbb{R}^2_{a,+}, \mathbb{R}^2_{a,-}, \mathbb{R}^2_{\hat{a},+}, \mathbb{R}^2_{\hat{a},-}$.

The subspace $\mathbb{R}^2_{a,+}$ is generated by the pair of vectors $(f_1 + f_3, f_5 + f_7)$. The subspace $\mathbb{R}^2_{a,-}$ is generated by the pair of vectors $(f_1 - f_3, f_5 - f_7)$. The subspace $\mathbb{R}^2_{\hat{a},+}$ is generated by the pair of vectors $(f_2 + f_4, f_6 + f_8)$. The subspace $\mathbb{R}^2_{\hat{a},-}$ is generated by the pair of vectors $(f_2 - f_4, f_6 - f_8)$.

The generator $a$ acts as a rotation through angle $\frac{\pi}{2}$ in each of the planes $\mathbb{R}^2_{a,+}, \mathbb{R}^2_{a,-}$, and acts by central symmetry in the plane $\mathbb{R}^2_{\hat{a},-}$, and so obviously
commutes with the action of the generator \( \hat{a} \) in this plane. The generator \( \hat{a} \) acts as a rotation through angle \( \frac{\pi}{2} \) in each of the planes \( \mathbb{R}^2_{a,+}, \mathbb{R}^2_{a,-} \), and acts by central symmetry in the plane \( \mathbb{R}^2_{a,-} \), and so obviously commutes with the action of the generator \( a \) in this plane. So the subgroup (22) is well defined.

It is convenient to pass to a new basis

\[
(h_{1,+}, h_{2,+}, h_{1,-}, h_{2,-}, \hat{h}_{1,+}, \hat{h}_{2,+}, \hat{h}_{1,-}, \hat{h}_{2,-}).
\]

The pairs of vectors \((h_{1,+}, h_{2,+}), (h_{1,-}, h_{2,-})\) are bases for the subspaces \( \mathbb{R}^2_{a,+}, \mathbb{R}^2_{a,-} \) respectively. In addition, the pairs of vectors \((\hat{h}_{1,+}, \hat{h}_{2,+}), (\hat{h}_{1,-}, \hat{h}_{2,-})\) are bases for the subspaces \( \mathbb{R}^2_{a,+}, \mathbb{R}^2_{a,-} \) respectively.

We consider the subgroup \( i_{a \times d, a \times \hat{a}} : I_a \times \hat{I}_d \subset I_a \times \hat{I}_d \), which is defined as the direct product of the group \( I_a \) with the elementary subgroup \( \hat{I}_d \) of the second factor. There is an inclusion \( i_{a \times d} : I_a \times \hat{I}_d \subset \mathbb{Z}/2[3] \), which is compatible with the inclusion (22). Moreover, the following diagram is commutative:

\[
\begin{array}{c}
\begin{array}{ccc}
I_d \times \hat{I}_d & \Rightarrow & \mathbb{Z}/2[2] \\
i_{d \times d, a \times \hat{a}} & & i_{[3]} \\
i_{a \times d} & \Rightarrow & \mathbb{Z}/2[3] \\
i_{[4]} & & \\
I_a \times \hat{I}_d & \Rightarrow & \mathbb{Z}/2[4].
\end{array}
\end{array}
\]

An inclusion \( i_{d \times d, a \times d} = i_{d \times d, a} \times p_{d \times d, d} : I_d \times \hat{I}_d \subset I_d \times \hat{I}_d \) is also defined. The homomorphism \( i_{d \times d, a} : I_d \times \hat{I}_d \rightarrow I_a \) is defined as the composition of the projection homomorphism \( I_d \times \hat{I}_d \rightarrow I_d \) and the inclusion \( I_d \subset I_a \). I’ve made a few changes. PSL] Moreover, the following diagram is commutative:

\[
\begin{array}{c}
\begin{array}{ccc}
I_d \times \hat{I}_d & \Rightarrow & \mathbb{Z}/2[2] \\
i_{d \times d, a \times \hat{d}} & \Rightarrow & \mathbb{Z}/2[2] \times \mathbb{Z}/2[2] \\
i_{[3]} & \cap & \\
i_{a \times d} & \Rightarrow & \mathbb{Z}/2[3].
\end{array}
\end{array}
\]

**Definition 8.** Let a \( \mathbb{Z}/2[3] \)-framed (\( \mathbb{Z}/2[4] \)-framed) immersion \((h, \Lambda, \zeta_L), h : L^{n-4k} \leftrightarrow \mathbb{R}^n (h : L^{n-8k} \leftrightarrow \mathbb{R}^n)\) represent an element \( z \in \Imm \mathbb{Z}/2[3] (n - 4k, 4k) (z \in \Imm \mathbb{Z}/2[4] (n - 8k, 8k))\). We say that this \( \mathbb{Z}/2[3] \)-framed (\( \mathbb{Z}/2[4] \)-framed) immersion is an \( I_a \times \hat{I}_d \)-framed \( (I_a \times \hat{I}_a \text{-framed}) \) immersion if the
structure mapping \( \zeta_L : L^{n-4k} \to K(\mathbb{Z}/2[3], 1) \) \((\zeta_L : L^{n-8k} \to K(\mathbb{Z}/2[4], 1)) \) can be represented as a composition of a mapping \( \zeta_{a \times d,L} : L^{n-4k} \to K(I_a \times I_d, 1) \) \((\zeta_{a \times \hat{d},L} : L^{n-8k} \to K(I_a \times \hat{I}_a, 1)) \) and the mapping \( i_{a \times d} : K(I_a \times \hat{I}_d, 1) \to K(\mathbb{Z}/2[3], 1) \) \((i_{a \times \hat{a}} : K(I_a \times \hat{I}_a, 1) \to K(\mathbb{Z}/2[4], 1)) \) below.

We now consider the analog of the relation (18) for the groups of \( I_a \times \hat{I}_d \) and \( I_a \times \hat{I}_a \)-framed immersions, respectively.

The cohomology group \( H^4(K(I_a \times \hat{I}_d, 1); \mathbb{Z}/2) \) \( (H^8(K(I_a \times \hat{I}_a); \mathbb{Z}/2) \) contains an element \( \tau_{a \times d} (\tau_{a \times \hat{a}}) \), which will be defined in the equation (25) \((26) \) below.

Consider the mapping \( i_{a \times d} : K(I_a \times \hat{I}_d, 1) \to K(\mathbb{Z}/2[3], 1) \) \((i_{a \times \hat{a}} : K(I_a \times \hat{I}_a, 1) \to K(\mathbb{Z}/2[4], 1)) \) and the pull-back \( i^{*}_{a \times d}(\tau_{[3]}) \) \((i^{*}_{a \times \hat{a}}(\tau_{[4]}) \) of the characteristic Euler class \( \tau_{[3]} \in H^4(K(\mathbb{Z}/2[3], 1); \mathbb{Z}/2) \) \((\tau_{[4]} \in H^8(K(\mathbb{Z}/2[4], 1); \mathbb{Z}/2) \) of the universal bundle. We define

\[
i^{*}_{a \times d}(\tau_{[3]}) = \tau_{a \times d} \in H^4(K(I_a \times \hat{I}_d, 1); \mathbb{Z}/2),
\]

\[
i^{*}_{a \times \hat{a}}(\tau_{[4]}) = \tau_{a \times \hat{a}} \in H^8(K(I_a \times \hat{I}_a, 1); \mathbb{Z}/2).
\]

In section 1, for a \( \mathbb{Z}/2^{[d+1]} \)-framed immersion \((h, \Lambda, \zeta_L) \), together with a \( 2^d \)-dimensional characteristic class \( \zeta_L \in H^2(L^{n-k2^d}; \mathbb{Z}/2) \), we also considered a \( 2 \)-dimensional characteristic class \( \zeta[2]_L \in H^2(L^{n-k2^d}; \mathbb{Z}/2) \). For a mapping \( \zeta_{a \times d} : L^{n-4k} \to K(I_a \times \hat{I}_d, 1) \) \((\zeta_{a \times \hat{d},L} : L^{n-8k} \to K(I_a \times \hat{I}_a, 1)) \) as an analog of the characteristic class \( \zeta[2]_L \) there serves the characteristic class \( \tilde{\zeta}_{d \times d,L} \in H^2(T^{n-k2^d}; \mathbb{Z}/2) \) for \( d = 3 \) \((\tilde{\zeta}_{d \times \hat{d},L} \in H^2(T^{n-8k}; \mathbb{Z}/2) \) for \( d = 4 \). We now define this \( 2 \)-dimensional characteristic class. [Beware two uses of \( d \). PSL]

The characteristic class \( \tilde{\zeta}_{d \times d,L} \) is induced from the universal class \( \tau_{d \times d} \in H^2(K(I_d \times \hat{I}_d, 1); \mathbb{Z}/2) \) by the mapping \( \tilde{\zeta}_{d \times d,L} : T^{n-4k} \to K(I_d \times \hat{I}_d, 1) \) \((\tilde{\zeta}_{d \times \hat{d},L} : \overline{T}_d \to K(I_d \times \hat{I}_d, 1)) \). The mapping \( \tilde{\zeta}_{d \times \hat{d},L} \) is defined as the \( 2 \)-sheeted covering of the mapping \( \zeta_N \) with respect to the subgroup \( i_{d \times d,a \times \hat{d}} : I_d \times \hat{I}_d \subset I_a \times \hat{I}_a \) (as the \( 4 \)-sheeted covering with respect to the subgroup \( i_{d \times d,a \times \hat{a}} : I_d \times \hat{I}_d \subset I_a \times \hat{I}_a \)). This \( 2 \)-sheeted (\( 4 \)-sheeted) covering over the manifold \( L^{n-4k} \) \((L^{n-8k}) \) with respect to the subgroup \( i_{d \times d,a \times \hat{d}} \) \((i_{d \times d,a \times \hat{a}}) \) is denoted by \( \pi_{d \times d,a \times \hat{d}} : \overline{T} \to L^{n-4k} \) \((\pi_{d \times \hat{d},a \times \hat{a}} : \overline{T} \to L^{n-8k}) \).
Definition 9. Let the \( \mathbb{Z}/2^2 \)-framed immersion \((g, \Psi, \eta_N), g : N^{n-2k} \hookrightarrow \mathbb{R}^n\), represent an element \( y \in \text{Imm}^{\mathbb{Z}/2^2}(n - 2k, 2k) \), with \( n > 32k \). Let the \( \mathbb{Z}/2^3 \)-framed immersion \((h, \Lambda, \zeta_L), h : L^{n-8k} \hookrightarrow \mathbb{R}^n\), be the immersion of the points of self-intersection of the immersion \( g \), representing the element \( z = \delta \mathbb{Z}/2^2,k(y) \in \text{Imm}^{\mathbb{Z}/2^2}(n - 4k, 4k) \). [Note the different notation for this homomorphism. PSL] We say that the \( \mathbb{Z}/2^2 \)-framed immersion \((g, \Psi, \eta_N)\) admits an \( I_a \times I_d \)-structure if there exists a mapping \( \zeta_{a \times d, L} : L^{n-4k} \rightarrow K(I_a \times I_d, 1) \), satisfying the equation:

\[
\Theta^k_{\mathbb{Z}/2^2}(y) = \langle \pi^*_d \times d, a \times d, L, \langle \zeta^7_L \rangle \zeta^{n-32k}_{d \times d, L}, [L_{d \times d}] \rangle,
\]

where \([L_{d \times d}]\) is the fundamental class of the manifold \( L_{d \times d} \) and the characteristic number \( \Theta^k_{\mathbb{Z}/2^2} \) is defined by formula (6).

Definition 10. Let the \( \mathbb{Z}/2^3 \)-framed immersion \((g, \Psi, \eta_N), g : N^{n-4k} \hookrightarrow \mathbb{R}^n\), represent an element \( y \in \text{Imm}^{\mathbb{Z}/2^3}(n - 4k, 4k) \), with \( n > 32k \). Let the \( \mathbb{Z}/2^4 \)-framed immersion \((h, \Lambda, \zeta_L), h : L^{n-8k} \hookrightarrow \mathbb{R}^n\), be the immersion of the points of self-intersection of the immersion \( g \), representing the element \( z = \delta \mathbb{Z}/2^2,k(y) \in \text{Imm}^{\mathbb{Z}/2^4}(n - 8k, 8k) \). We say that the \( \mathbb{Z}/2^3 \)-framed immersion \((g, \Psi, \eta_N)\) admits a bicyclic structure (an \( I_a \times I_d \)-structure) if there exists a mapping \( \zeta_{a \times d, L} : L^{n-8k} \rightarrow K(I_a \times I_d, 1) \), satisfying the equation:

\[
\Theta^k_{\mathbb{Z}/2^3}(y) = \langle \pi^*_d \times d, a \times d, L, \langle \zeta^3_L \rangle \zeta^{n-32k}_{d \times d, L}, [L_{d \times d}] \rangle,
\]

where \([L_{d \times d}]\) is the fundamental class of the manifold \( L_{d \times d} \) and the characteristic number \( \Theta^k_{\mathbb{Z}/2^3} \) is defined by formula (14).

Example

Let the \( \mathbb{Z}/2^2 \)-framed (\( \mathbb{Z}/2^3 \)-framed) immersion \((g, \Psi, \eta_N), g : N^{n-2k} \hookrightarrow \mathbb{R}^n\) \((g : N^{n-4k} \hookrightarrow \mathbb{R}^n)\) represent an element \( y \in \text{Imm}^{\mathbb{Z}/2^2}(n - 2k, 2k) \) \((y \in \text{Imm}^{\mathbb{Z}/2^3}(n - 4k, 4k))\) and be an \( I_d \times I_d \)-framed (\( I_a \times I_d \)-framed) immersion, where \( n > 32k \). Let the \( \mathbb{Z}/2^3 \)-framed (\( \mathbb{Z}/2^4 \)-framed) immersion \((h, \Lambda, \zeta_L), h : L^{n-8k} \hookrightarrow \mathbb{R}^n\) \((h : L^{n-8k} \hookrightarrow \mathbb{R}^n)\) represent the element \( z = \delta \mathbb{Z}/2^3,k(y) \in \text{Imm}^{\mathbb{Z}/2^3}(n - 4k, 4k) \) \((z = \delta \mathbb{Z}/2^4,k(y) \in \text{Imm}^{\mathbb{Z}/2^4}(n - 8k, 8k))\) and be an \( I_a \times I_d \)-framed (\( I_a \times I_d \)-framed) immersion. Then the \( \mathbb{Z}/2^2 \)-framed (\( \mathbb{Z}/2^3 \)-framed) immersion \((g, \Psi, \eta_N)\) admits an \( I_a \times I_d \)-structure (\( I_a \times I_d \)-structure), given by the reduction \( \zeta_{a \times d, L} (\zeta_{a \times d, L}) \) of the structure mapping of \( \zeta_L \).
Justification of the example

We restrict attention to the case of an \( I_d \times \hat{I}_d \)-framed immersion. Consider the cohomology class \( \zeta_{a \times a, L} = \zeta^*_{a \times a, L} \in H^8(L^{n-8k}; \mathbb{Z}/2) \). Notice that this cohomology class coincides with the class \( \zeta_L = \zeta^*_L \), since both classes are defined as Euler classes of the same bundle. Now formula (28) is obvious.

The following theorems are analogs of Theorem 6.

**Theorem 11.** Assume that the \( \mathbb{Z}/2^3 \)-framed immersion \( (g, \Psi, \eta_N) \) represents an element \( y \in \text{Imm}^{\mathbb{Z}/2^3} (n - \frac{n-n_s}{16}, \frac{n-n_s}{8}), n_s = 2^s - 2, n > n_s, s \geq 6 \). Assume given a mapping \( \eta_{d \times d, N} : N^{n-\frac{n-n_s}{8}} \rightarrow K(I_d \times \hat{I}_d, 1) \), satisfying the following equation:

\[
\Theta^{k}_{\mathbb{Z}/2^3}(y) = \langle \eta_{N}^{\frac{15(n-n_s)}{8}} \rangle_{d \times d, N}, \frac{n_s}{2}, [N],
\]

where \([N]\) is the fundamental class of the manifold \( N^{n-\frac{n-n_s}{8}} \), and \( \Theta^{k}_{\mathbb{Z}/2^3} \) is the characteristic number defined by the formula (6). Then the element \( J^{\mathbb{Z}/2^3, \frac{n-n_s}{8}}(y) \) in the group \( \text{Imm}^{\mathbb{Z}/2^3} (n - \frac{n-n_s}{16}, \frac{n-n_s}{8}) \) is represented by a \( \mathbb{Z}/2^3 \)-framed immersion \( (h, \Lambda, \zeta_L) \), which admits an \( I_d \times \hat{I}_d \)-structure. [A different notation is being used for the homomorphism \( J \).]

**Theorem 12.** Assume that the \( \mathbb{Z}/2^3 \)-framed immersion \( (g, \Psi, \eta_N) \) represents an element \( y \in \text{Imm}^{\mathbb{Z}/2^3} (n - \frac{n-n_s}{8}, \frac{n-n_s}{8}), n_s = 2^s - 2, n > n_s, s \geq 6 \). Assume given a mapping \( \eta_{a \times d, N} : N^{n-\frac{n-n_s}{8}} \rightarrow K(I_a \times \hat{I}_d, 1) \), satisfying the following equation:

\[
\Theta^{k}_{\mathbb{Z}/2^3}(y) = \langle (\pi_{d \times d, a \times d, N})^{\frac{15(n-n_s)}{8}} \rangle_{d \times d, N}, \frac{n_s}{2}, [\tilde{N}_{d \times d}],
\]

where the covering manifold \( N^{\frac{n-n_s}{8}}_{d \times d} \) and the mapping \( \eta_{d \times d, N} : N^{\frac{n-n_s}{8}}_{d \times d} \rightarrow K(I_d \times \hat{I}_d, 1) \) over the mapping \( \eta_{a \times d, N} \) are defined completely analogously to (27) and where \([\tilde{N}_{d \times d}]\) denotes the fundamental class of the manifold \( N^{\frac{n-n_s}{8}}_{d \times d} \).

Then the element \( J^{\mathbb{Z}/2^3, \frac{n-n_s}{32}}(y) \) in the group \( \text{Imm}^{\mathbb{Z}/2^3} (n - \frac{n-n_s}{32}, \frac{n-n_s}{32}) \) is represented by a \( \mathbb{Z}/2^3 \)-framed immersion \( (h, \Lambda, \zeta_L) \) which admits an \( I_a \times \hat{I}_d \)-structure.

**Corollary 13.** Assume that the hypotheses of Theorem 6 hold (i.e., \( n_s \) is an integer of the form \( 2^s - 2, n > n_s, s \geq 6 \) and that the element \( x \in \text{Imm}^f (n - \frac{n-n_s}{32}, \frac{n-n_s}{32}) \) admits a compression of order \( q = \frac{n_s}{2} \)). Then the
element \( \delta^k_{\mathbb{Z}/2^3} \circ \delta^k_{\mathbb{Z}/2^3} (x) \), defined by the composition of homomorphisms (8), with \( k = \frac{n-n_s}{32} \), is represented by a \( \mathbb{Z}/2^3 \)-framed immersion \((h, \Lambda, \zeta_L)\) which admits a bicyclic structure.

4 \( \mathbb{Q}_a \times \dot{\mathbb{Q}}_a \)-structure (biquaternionic structure) on a \( \mathbb{Z}/2^5 \)-framed immersion

[Section title slightly changed, from [6] to [5]. PSL] We shall recall the definition of the quaternion subgroup \( \mathbb{Q}_a \subset \mathbb{Z}/2^3 \), which contains the subgroup \( \mathbb{I}_a \subset \mathbb{Q}_a \), see Akhmet’ev [A], section 2.

We shall then define a subgroup

\[
i_{\mathbb{Q}_a \times \dot{\mathbb{Q}}_a} : \mathbb{Q}_a \times \dot{\mathbb{Q}}_a \subset \mathbb{Z}/2^6.
\]

Consider the basis \((h_{1,+}, h_{2,+}, h_{1,-}, h_{2,-}, \dot{h}_{1,+}, \dot{h}_{2,+}, \dot{h}_{1,-}, \dot{h}_{2,-})\) which was defined for the construction of bicyclic structures.

We now describe a basis of \( \mathbb{R}^{32} \). This basis consists of 32 vectors, divided into two subsets of 16 vectors

\[
h_{1,*,**}, h_{2,*,**}, h_{3,*,**}, h_{4,*,**}, \quad \dot{h}_{1,*,**}, \dot{h}_{2,*,**}, \dot{h}_{3,*,**}, \dot{h}_{4,*,**},
\]

where the symbols \(*, **\) independently take the values \(+, -\). In each of the subspaces, generated by the 4 vectors (32), for which the symbols \(*, **\) take the same values, the representation of the group \( \mathbb{Q}_a \times \dot{\mathbb{Q}}_a \) is trivial on the second factor \( \dot{\mathbb{Q}}_a \); moreover the generators \( i, j, k \) of the factor \( \mathbb{Q}_a \) act by the transformations defined by the matrices:

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]
The generator $i$ ($j$) of the first factor $Q_a$ acts by a central symmetry on each 4-dimensional subspace (33), generated by vectors for which the index $^*$ ($^*$) takes the value $-$, while the index $^{**}$ ($^*$) takes any fixed value, and on the remaining pair of subspaces generated by the vectors (33) it acts as the identity.

In each of the subspaces generated by the 4 vectors (33), for which the symbols $^*$, $^{**}$ take the same value, the representation of the group $Q_a \times \dot{Q}_a$ is trivial on the first factor $Q_a$, while the generators $i, j, k$ of the factor $Q_a$ are represented by the matrices (34), (35), and (36).

The generator $i$ ($j$) of the second factor $\dot{Q}_a$ acts by a central symmetry in each 4-dimensional subspace (32), generated by vectors for which the index $^*$ ($^{**}$) takes the value $-$, while the index $^{**}$ ($^*$) takes any fixed value, and on the remaining pair of subspaces generated by the vectors (32) it acts as the identity. Since the transformation of central symmetry lies in the center of the group $Q_a \times \dot{Q}_a$, the transformation of each element of $\dot{Q}_a$ ($Q_a$) in an irreducible subspace of the representation of $Q_a$ ($\dot{Q}_a$) commutes with the elements of the indicated representation. Hence, the subgroup (31) is well defined.

Consider the subgroup $i_{Q_a \times \dot{Q}_a}$ : $Q_a \times \dot{Q}_a \subset Q_a \times \dot{Q}_a$, which is defined as the product of the group $Q_a$ with the inclusion of the subgroup $\dot{I}_a$ of the second factor. An inclusion $i_{Q_a \times \dot{I}_a}$ : $Q_a \times \dot{I}_a \subset \mathbb{Z}/2^5$ is induced by the embedding (31). Moreover, we obtain a commutative diagram:

$$
\begin{array}{cccc}
\mathbb{I}_d \times \dot{I}_d & \overset{i_{d \times d}}{\longrightarrow} & \mathbb{Z}/2^2 \\
\downarrow i_{d \times d, a \times d} & & \downarrow i_{[3]} \\
\mathbb{I}_a \times \dot{I}_d & \overset{i_{a \times d}}{\longrightarrow} & \mathbb{Z}/2^3 \\
\downarrow i_{a \times d, a \times \dot{a}} & & \downarrow i_{[4]} \\
\mathbb{I}_a \times \dot{I}_a & \overset{i_{a \times \dot{a}}}{\longrightarrow} & \mathbb{Z}/2^4 \\
\downarrow i_{I_a \times I_a, Q_a \times \dot{Q}_a} & & \downarrow i_{[5]} \\
Q_a \times \dot{I}_a & \overset{i_{Q_a \times I_a}}{\longrightarrow} & \mathbb{Z}/2^5 \\
\downarrow i_{Q_a \times I_a, Q_a \times Q_a} & & \downarrow i_{[6]} \\
Q_a \times Q_a & \overset{i_{Q_a \times Q_a}}{\longrightarrow} & \mathbb{Z}/2^6,
\end{array}
$$

(37)
which contains the diagram (23) as a subdiagram.

The following definition is analogous to Definition 8.

**Definition 14.** Let a $\mathbb{Z}/2[5]$-framed ($\mathbb{Z}/2[6]$-framed) immersion $(h, \Lambda, \zeta_L)$, $h : L^{n-16k} \to \mathbb{R}^n$ ($h : L^{n-32k} \to \mathbb{R}^n$) represent an element $z \in \text{Imm}^{\mathbb{Z}/2[5]}(n - 16k, 16k)$ ($z \in \text{Imm}^{\mathbb{Z}/2[6]}(n - 32k, 32k)$). We say that this $\mathbb{Z}/2[5]$-framed ($\mathbb{Z}/2[6]$-framed) immersion is a $Q_a \times \hat{I}_a$-framed ($Q_a \times \hat{Q}_a$-framed) immersion if the structure mapping $\zeta_L : L^{n-16k} \to K(\mathbb{Z}/2[5], 1)$ ($\zeta_L : L^{n-32k} \to K(\mathbb{Z}/2[6], 1)$) can be represented as a composition of a mapping $\zeta_{Q_a \times I_a,L} : L^{n-16k} \to K(Q_a \times \hat{I}_a, 1)$ ($\zeta_{Q_a \times Q_a,L} : L^{n-32k} \to K(Q_a \times \hat{Q}_a, 1)$) and the mapping $i_{Q_a \times I_a} : K(Q_a \times \hat{I}_a, 1) \to K(\mathbb{Z}/2[5], 1)$ ($i_{Q_a \times Q_a} : K(Q_a \times \hat{Q}_a, 1) \to K(\mathbb{Z}/2[6], 1)$).

We now consider the analogs of the formulas (18), (25), (26) for the groups $Q_a \times \hat{I}_a$- and $Q_a \times \hat{Q}_a$-framed immersions, respectively.

The cohomology group $H^{16}(K(Q_a \times \hat{I}_a, 1); \mathbb{Z}/2) (H^{32}(K(Q_a \times \hat{Q}_a, 1); \mathbb{Z}/2))$ contains an element $\tau_{Q_a \times I_a} (\tau_{Q_a \times Q_a})$, which is defined by the equations \( (38) \) \((39)) given below.

Let us consider the mapping $i_{Q_a \times I_a} : K(Q_a \times \hat{I}_a, 1) \to K(\mathbb{Z}/2[5], 1)$ ($i_{Q_a \times Q_a} : K(Q_a \times \hat{Q}_a, 1) \to K(\mathbb{Z}/2[6], 1)$) and consider the pull-back $i^*_{Q_a \times I_a} (\tau_{[5]}$ ($i^*_a \tau_{[6]}$) of the characteristic Euler class $\tau \in H^{16}(K(\mathbb{Z}/2[5], 1); \mathbb{Z}/2)$ ($\tau \in H^{32}(K(\mathbb{Z}/2[6], 1); \mathbb{Z}/2)$) of the universal bundle. Define
\[
i^*_{Q_a \times I_a} (\tau_{[5]} = \tau_{Q_a \times I_a} \in H^{16}(K(Q_a \times \hat{I}_a, 1); \mathbb{Z}/2),
\]
\[
i^*_{Q_a \times Q_a} (\tau_{[6]} = \tau_{Q_a \times Q_a} \in H^{32}(K(Q_a \times \hat{Q}_a, 1); \mathbb{Z}/2).
\]

For the mapping $\zeta_{Q_a \times I_a} : L^{n-16k} \to K(Q_a \times \hat{I}_a, 1)$ ($\zeta_{Q_a \times Q_a} : L^{n-32k} \to K(Q_a \times \hat{Q}_a, 1)$), as an analog of the characteristic class $\zeta_{[2],L}$, serves the characteristic class $\tau_{d \times d,L} \in H^2(T_{d \times d}; \mathbb{Z}/2)$ ($\tau_{d \times d,L} \in H^2(T_{d \times d}; \mathbb{Z}/2)$). We now define this 2-dimensional characteristic class.

The mapping $\zeta_{d \times d,L}$ is defined as the 8-sheeted covering over the mapping $\zeta_{[5],L}$ relative to the subgroup $i_{d \times d, Q_a \times I_a} : I_d \times \hat{I}_d \subset Q_a \times \hat{I}_a$ (as the 16-sheeted covering over the mapping $\zeta_{[6],L}$ relative to the subgroup $i_{d \times d, Q_a \times Q_a} : I_d \times \hat{I}_d \subset Q_a \times \hat{Q}_a$). Over the manifold $L^{n-16k} (L^{n-32k})$ the resulting covering is denoted by $\pi_{d \times d, Q_a \times I_a,L}$ ($\pi_{d \times d, Q_a \times Q_a,L}$).
Definition 15. Let the $\mathbb{Z}/2^{[4]}$-framed immersion $(g, \Psi, \eta_N), g : N^{n-8k} \ni \mathbb{R}^n$ represent an element $y \in \text{Imm}^Z/2^{[4]}(n-8k, 8k)$, where $n > 32k$. Let the $\mathbb{Z}/2^{[5]}$-framed immersion $(h, \Lambda, \zeta_L), h : L^{n-16k} \ni \mathbb{R}^n$ be the immersion of the points of self-intersection of the immersion $g$, which represents the element $\delta_{Z/2^{[5]}}^k(y) \in \text{Imm}^Z/2^{[5]}(n-16k, 16k)$. We say that the $\mathbb{Z}/2^{[4]}$-framed immersion $(g, \Psi, \eta_N)$ admits a $Q_a \times \hat{L}_a$-structure if there exists a mapping $\zeta_{Q_a \times \hat{L}_a} : L^{n-16k} \rightarrow K(Q_a \times \hat{L}_a, 1)$, satisfying the equation:

$$\Theta_{Z/2^{[4]}}^k(y) = \langle \pi_{d \times Q_a \times \hat{L}_a}^* \zeta^k \dot{\zeta}^{n-32k}_{d \times d}, [\mathcal{T}_{d \times d}] \rangle,$$

where $[\mathcal{T}_{d \times d}]$ is the fundamental class of the manifold $\mathcal{T}_{d \times d}$, and the characteristic number $\Theta_{Z/2^{[4]}}^k(y)$ is defined by formula (14) for $d = 4$.

Definition 16. Let the $\mathbb{Z}/2^{[5]}$-framed immersion $(g, \Psi, \eta_N), g : N^{n-16k} \ni \mathbb{R}^n$ represent an element $y \in \text{Imm}^Z/2^{[6]}(n-16k, 16k)$, where $n > 32k$. Let the $\mathbb{Z}/2^{[6]}$-framed immersion $(h, \Lambda, \zeta_L), h : L^{n-32k} \ni \mathbb{R}^n$ be the immersion of the points of self-intersection of the immersion $g$, which represents the element $\delta_{Z/2^{[6]}}^k(y) \in \text{Imm}^Z/2^{[6]}(n-32k, 32k)$. We say that the $\mathbb{Z}/2^{[5]}$-framed immersion $(g, \Psi, \eta_N)$ admits a biquaternionic structure $(Q_a \times \hat{Q}_a)$-structure if there exists a mapping $\zeta_{Q_a \times \hat{Q}_a} : L^{n-32k} \rightarrow K(Q_a \times \hat{Q}_a, 1)$, satisfying the equation:

$$\Theta_{Z/2^{[5]}}^k(y) = \langle \zeta^{n-32k}_{d \times d}, [\mathcal{T}_{d \times d}] \rangle,$$

where $[\mathcal{T}_{d \times d}]$ is the fundamental class of the manifold $\mathcal{T}_{d \times d}$, and the characteristic number $\Theta_{Z/2^{[5]}}^k(y)$ is defined by formula (14) for $d = 5$.

Example

Let the $\mathbb{Z}/2^{[4]}$-framed ($\mathbb{Z}/2^{[5]}$-framed) immersion $(g, \Psi, \eta_N), g : N^{n-8k} \ni \mathbb{R}^n$, represent an element $y \in \text{Imm}^Z/2^{[4]}(n-8k, 8k) (y \in \text{Imm}^Z/2^{[5]}(n-16k, 16k))$ and be a $Q_a \times \hat{L}_a$-framed ($Q_a \times \hat{Q}_a$-framed) immersion, where $n > 32k$. Let the $\mathbb{Z}/2^{[5]}$-framed ($\mathbb{Z}/2^{[6]}$-framed) immersion $(h, \Lambda, \zeta_L), h : L^{n-16k} \ni \mathbb{R}^n$, and the element $\delta_{Z/2^{[6]}}^k(y) \in \text{Imm}^Z/2^{[6]}(n-32k, 32k))$ and be a $Q_a \times \hat{L}_a$-framed ($Q_a \times \hat{Q}_a$-framed) immersion. Then the $\mathbb{Z}/2^{[4]}$-framed ($\mathbb{Z}/2^{[5]}$-framed) immersion $(g, \Psi, \eta_N)$ admits a $Q_a \times \hat{L}_a$-structure ($Q_a \times \hat{Q}_a$-structure), given by the reduction $\zeta_{Q_a \times \hat{L}_a} : (\zeta_{Q_a \times \hat{Q}_a})$ of the structure mapping $\zeta_{[4], L} (\zeta_{[5], L})$. 

22
Justification of the example

We restrict attention to the case of a \( Q_a \times \hat{Q}_a \)-framed immersion. Consider the cohomology class \( \zeta_{Q_a \times Q_a} = \zeta^*_{Q_a \times Q_a} \langle \tau_{Q_a \times Q_a} \rangle \in H^{32}(\mathbb{Z}^{n-32k}; \mathbb{Z}/2) \). Notice that this cohomology class coincides with the class \( \zeta_\mathcal{L} = \zeta_{\mathcal{L}}(\tau_{\mathcal{L}}) \), since both classes are defined as the Euler class of the same bundle. Now the formula (41) is obvious.

The following theorems are analogous to Theorems 6, 11, and 12.

**Theorem 17.** Assume that a \( \mathbb{Z}/2^4 \)-framed immersion \((g, \Psi, \eta_N)\) represents an element \( y \in \text{Imm}^{\mathbb{Z}/2^4}(n - \frac{n-n_s}{4}, \frac{n-n_s}{4}) \), \( n_s = 2^s - 2, n > 5n_s + 168, s \geq 6 \). Assume given a mapping \( \eta_{a \times \hat{a}, N} : N^{n-\frac{n-n_s}{4}} \rightarrow K(I_a \times \hat{I}_a, 1) \), for which the following equation holds:

\[
\Theta^*_\mathbb{Z}/2^4(y) = \langle \pi^*_{\mathbb{Z}/2^4} \eta_{\mathbb{Z}/2^4}(\tau_{\mathbb{Z}/2^4}) \rangle_{\mathbb{Z}/2^4}, j(N_{\mathbb{Z}/2^4}), [N_{\mathbb{Z}/2^4}],
\]

where the covering manifold \( N_{\mathbb{Z}/2^4}^{n-\frac{n-n_s}{4}} \) and the mapping \( \eta_{d \times d, N} : N_{\mathbb{Z}/2^4}^{\frac{n-n_s}{4}} \rightarrow K(I_d \times \hat{I}_d, 1) \) over the mapping \( \eta_{[4], N} \) are defined in the same way as (40). Then the element \( J_{\mathbb{Z}/2^4}(y) \) in the group \( \text{Imm}^{\mathbb{Z}/2^4}(n - \frac{n-n_s}{2}, \frac{n-n_s}{2}) \) is represented by a \( \mathbb{Z}/2^5 \)-framed immersion \((h, \Lambda, \zeta_\mathcal{L})\) which admits a \( Q_a \times \hat{Q}_a \)-structure.

**Theorem 18.** Assume that a \( \mathbb{Z}/2^5 \)-framed immersion \((g, \Psi, \eta_N)\) represents an element \( y \in \text{Imm}^{\mathbb{Z}/2^5}(n - \frac{n-n_s}{2}, \frac{n-n_s}{2}) \), \( n_s = 2^s - 2, n > \frac{5n_s + 84}{3}, s \geq 6 \). Assume given a mapping \( \eta_{a \times \hat{a}, N} : N^{n-\frac{n-n_s}{2}} \rightarrow K(Q_a \times \hat{Q}_a, 1) \) for which the following equation holds:

\[
\Theta^*_\mathbb{Z}/2^5(y) = \langle \pi^*_{\mathbb{Z}/2^5} \eta_{\mathbb{Z}/2^5}(\tau_{\mathbb{Z}/2^5}) \rangle_{\mathbb{Z}/2^5}, j(N_{\mathbb{Z}/2^5}), [N_{\mathbb{Z}/2^5}],
\]

where the covering manifold \( N_{\mathbb{Z}/2^5}^{n-\frac{n-n_s}{2}} \) and the mapping \( \eta_{d \times d, N} : N_{\mathbb{Z}/2^5}^{\frac{n-n_s}{2}} \rightarrow K(I_d \times \hat{I}_d, 1) \) over the mapping \( \eta_{[4], N} \) are defined in the same way as (40), (42). Then the element \( J_{\mathbb{Z}/2^5}(y) \) in the group \( \text{Imm}^{\mathbb{Z}/2^6}(n_s, n - n_s) \) is represented by a \( \mathbb{Z}/2^6 \)-framed immersion \((h, \Lambda, \zeta_\mathcal{L})\) which admits a biquaternionic structure.
Corollary 19. Assume that the hypotheses of Theorem 6 hold (i.e., $n_s$ is an integer of the form $2^s - 2$, $n > n_s$, $s \geq 6$ and that the element $x \in \text{Imm}^s(n - \frac{n-n_s}{32}, \frac{n-n_s}{32})$, admits a compression of order $q = \frac{n_s}{2}$). Then the element

$$\delta_{Z/2[5]}^k \circ \delta_{Z/2[4]}^k \circ \delta_{Z/2[3]}^k \circ \delta_{Z/2[2]}(x),$$

defined by the composition of homomorphisms (8), $k = \frac{n-n_s}{32}$, is represented by a $Z/2[5]$-framed immersion $(g, \Psi, \eta_N)$ which admits a biquaternionic structure.

5 Solution of the Kervaire invariant problem

In this section we prove the following result.

Main Theorem

There exists a natural number $l_0$ such that for an arbitrary integer $l \geq l_0$ and $n = 2^l - 2$, the Kervaire invariant defined by formula (1) is trivial.

Remark

Under the hypothesis that an element $x \in \text{Imm}^s(n - 1, 1)$ admits a compression of order $2^7 - 1 = 127$, it follows that $\Theta(x) = 0$ for $l \geq 9$, i.e., for $n \geq 512$.

Proof of the Main Theorem

Put $n_s = 2^s - 2$ and denote $\frac{n-n_s}{32}, n > n_s$ by $k$. Clearly $k \equiv 0 \pmod{8}$, provided $l \geq 8$. By Theorem 29 (the Compression Theorem) there exists a natural number $l_0$ such that for each integer $l \geq l_0$ an arbitrary element $x \in \text{Imm}^s(n - k, k)$ admits a compression of order $\frac{n_s}{2} = 2^7 - 1$. Without loss of generality we shall assume that $l_0 \geq 9$. Because of the dimensional conditions formulated in Theorems 6, 17, 18 and Corollary 19 it follows that in the cobordism class (44) there exists a $Z/2[5]$-framed immersion $(g, \Psi, \eta_N)$, admitting a biquaternionic structure.

Consider the manifold of self-intersections $L^{n_s}$ of the immersion $g$, $\dim(L^{n_s}) = n_s$. The manifold $L^{n_s}$ is equipped with a mapping

$$\zeta_{Q_a \times \hat{Q}_a} = \zeta_{Q_a \times \hat{Q}_a}, L : L^{n_s} \to K(Q_a \times \hat{Q}_a, 1).$$

(45)
This mapping can be represented as a product of two mappings
\[ \zeta_{Q_a \times Q_a} = \zeta_{Q_a} \times \zeta_{Q_a} : L^{n_a} \to K(Q_a, 1) \times K(\hat{Q}_a, 1). \] (46)

We shall need the following lemma. Consider the generating cohomology class \( \tau_{Q_a \times Q_a} \in H^8(K(Q_a \times \hat{Q}_a, 1); \mathbb{Z}/2) \). Denote \( \zeta^*_{Q_a \times Q_a}(\rho_{Q_a \times \hat{Q}_a}) \in H^8(L^{n_a}; \mathbb{Z}/2) \) by \( \rho_{Q_a \times \hat{Q}_a} \). Consider the 16-dimensional covering \( \pi_{\text{tot},L} : \overline{T}^{n_a}_{[2]} \to L^{n_a} \). The class \( \rho_{Q_a \times \hat{Q}_a} \) pulls back to the class \( \pi_{\text{tot},L}^*(\rho_{Q_a \times \hat{Q}_a}) \in H^8(L^{n_a}_{[2]}; \mathbb{Z}/2) \).

**Lemma 20.** In the group \( H^8(\overline{T}^{n_a}_{[2]}; \mathbb{Z}/2) \) the following equality holds:
\[ \pi_{\text{tot},L}^*(\rho_{Q_a \times \hat{Q}_a}) = \zeta_{d \times d,L}^4. \] (47)

**Proof of Lemma 20**
Consider the Euler class \( e(\zeta_{d \times d \times d}) \) of the bundle \( \zeta_{d \times d \times d} \). The bundle \( \zeta_{d \times d \times d} \) is isomorphic to the Whitney sum of 8 copies of a line bundle, namely 4 copies of the bundle \( \kappa_d \) and 4 copies of the bundle \( \kappa_d \). The lemma follows, since the cohomology class \( \zeta_{d \times d,L} \in H^2(L_{[2]}^{n_a}; \mathbb{Z}/2) \) is the Euler class of the bundle \( \kappa_d \oplus \kappa_d \).

Consider a submanifold \( i_K : K^{14} \subset L^{n_a} \), dual to the cohomology class \( \rho_{Q_a \times \hat{Q}_a}^{n_a-1} \in H^{n_a-14}(L^{n_a}; \mathbb{Z}/2) \). The manifold \( K^{14} \) is equipped with a mapping \( \zeta_{Q_a \times Q_a,K} : K^{14} \to K(Q_a \times \hat{Q}_a, 1) \), which is defined as the restriction of the mapping \( \zeta_{Q_a \times \hat{Q}_a} \) to the submanifold \( K^{14} \subset L^{n_a} \). There is defined a 16-sheeted covering \( \pi_{\text{tot},K} : \overline{K}^{14}_{d \times d} \to K^{14} \). There is defined an induced characteristic class \( \overline{\zeta}_{d \times d,K} \in H^2(\overline{K}_{d \times d}; \mathbb{Z}/2) \) which is equal to \( \pi_{\text{tot},K}^*(\overline{\zeta}_{d \times d,L}) \).

The formula (41) takes the form:
\[ \Theta^k_{sl}(y) = \langle \zeta_{d \times d,L}^n, [L_{[2]}] \rangle. \] (48)

By Lemma 20 the characteristic number (48) is equal to the characteristic number [L changed to K twice, PSL]
\[ \langle \overline{\zeta}_{d \times d,K}^n, [\overline{K}_{d \times d}] \rangle. \] (49)

We shall show that the characteristic number (49) vanishes. Over the space \( K(Q_a, 1) \) (\( K(\hat{Q}_a, 1) \)) there is defined a 4-dimensional bundle \( \rho_{Q_a} (\rho_{\hat{Q}_a}) \).
with structure group \(Q_a (\hat{Q}_a)\), see Akhmet’ev [A], section 2. Hence, over
the space \(K(Q_a \times \hat{Q}_a, 1) = K(Q_a, 1) \times K(\hat{Q}_a, 1)\) there are also defined 4-
dimensional bundles \(\chi_{Q_a}\) and \(\chi_{\hat{Q}_a}\) by the formulas
\[
\chi_{Q_a} = p^*_{Q_a \times Q_a, Q_a}(\rho_{Q_a}),
\]
\[
\chi_{\hat{Q}_a} = p^*_{\hat{Q}_a \times \hat{Q}_a, \hat{Q}_a}(\rho_{\hat{Q}_a}).
\]

The normal bundle to the manifold \(K^{14}\) is isomorphic to a Whitney sum
\[
\nu_K \cong \left(\frac{n_s - 14}{8}\right)(\zeta_{Q_a,K} \oplus \zeta_{\hat{Q}_a,K}) \oplus \Omega_K,
\] (50)
where \(\zeta_{Q_a,K} = \zeta^*_{Q_a \times Q_a, K}(\chi_{Q_a}), \zeta_{\hat{Q}_a,K} = \zeta^*_{\hat{Q}_a \times \hat{Q}_a, K}(\chi_{\hat{Q}_a}),\) and \(\Omega_K\) is the restriction
of the normal bundle of the manifold \(L^{n_s}\) to the submanifold \(K^{14} \subset L^{n_s}\).

The bundle \(\Omega_K\) is isomorphic to the sum of \(k\) copies of a 32-dimensional
bundle \(\zeta_{[6]}\) with structure group \(\mathbb{Z}/2^6\). Since \(k \equiv 0 \pmod{8}\), the reduction
\(p'_1(\Omega_K) \in H^4(K; \mathbb{Z}/2)\) of the Pontryagin class \(p_1(\Omega_K) \in H^4(K; \mathbb{Z})\) to the
coefficient group \(\mathbb{Z}/8\) is trivial.

Consider the manifold \(-K^{14}\), which is obtained from \(K^{14}\) by reversing
orientation. Consider the mapping \(F = \text{id} \cup -\text{id} : K^{14} \cup -K^{14} \to K^{14}\). We shall compute the characteristic cycles \([p_1(\nu_K)]^\text{op}\), \([p_1(\nu_{-K})]^\text{op}\) of the normal bundles to the manifolds \(K^{14}, -K^{14}\), where “\text{op}\" denotes Poincaré duality, the normal bundle to the manifold \(K^{14}\) is defined by formula (50), and the normal bundle to the manifold \(-K^{14}\) has the opposite orientation (see an
analogous calculation in Proposition 41 of Akhmet’ev [A]).

We shall show that if the characteristic number (49) does not vanish, then
the cycle \([\omega \text{ changed to } \zeta, \text{ PSL}]\)
\[
([\zeta_{Q_a \times Q_a,K} \circ F]_1([p_1(\nu_T)]^\text{op} + [p_1(\nu_{-T})]^\text{op})) \in H_{10}(K(Q_a \times \hat{Q}_a, 1); \mathbb{Z}),
\] (51)
for the expansion in a basis contains the monomial
\[
4u_3 \otimes v_7 \in H_3(K(Q_a, 1); \mathbb{Z}) \otimes H_7(K(Q_a, 1); \mathbb{Z}) \subset H_{10}(K(Q_a \times \hat{Q}_a, 1); \mathbb{Z}).
\]

In this formula \(u_7 \in H_7(K(Q_a, 1); \mathbb{Z}) = \mathbb{Z}/8\) and \(v_7 \in H_7(K(\hat{Q}_a, 1); \mathbb{Z}) = \mathbb{Z}/8\) are generators, and \(H_7(K(Q_a, 1); \mathbb{Z}) \otimes H_7(K(\hat{Q}_a, 1); \mathbb{Z}) \subset H_{14}(K(Q_a \times \hat{Q}_a, 1); \mathbb{Z})\) is the natural inclusion. [What is \(u_3\)? PSL]

Without loss of generality we can assume that \((\zeta_{Q_a \times Q_a,K})_*([K]) = u_7 \otimes v_7 + xu_3 \otimes v_{11} + \cdots\), where \(x\) is an integer. (For a dimensional reason, for
all other elements labeled by the symbol \(\cdots\), the characteristic class \(K\) does
not contain the element \(u_3 \otimes v_7\).) We obtain:

26
\[ F_s([p_1(\nu_K)]^{\text{op}}) = 4u_3 \otimes v_7 + 4xu_3 \otimes v_7 + \cdots \in H_3(K(Q_a, 1); \mathbb{Z}) \otimes H_7(K(\tilde{Q}_a, 1); \mathbb{Z}) \subset H_{10}(K(Q_a \times \tilde{Q}_a, 1); \mathbb{Z}), \]

\[ F_s([p_1(\nu_{-K})]^{\text{op}}) = 4xu_3 \otimes v_7 + \cdots. \]

We therefore conclude that:

\[ 0 \neq 4u_3 \otimes v_7 + \cdots = (\zeta_{Q_a \times Q_a, K} \circ F)_s([p_1(\nu_T)]^{\text{op}} + [p_1(\nu_{-T})]^{\text{op}}). \]

In particular, the mapping \( F \) is not cobordant to zero. But on the other hand, by construction the mapping \( F \) is cobordant to zero. This contradiction shows that the characteristic number (49) vanishes.

We have shown that on the framed cobordism class \( x \) the characteristic number (6) is zero. The Kervaire invariant problem is solved. [I have made slight changes, but have essentially copied this argument. There is much that confuses me; a clean argument with full explanations would be welcome. PSL]
6 Proof of Theorems 11 and 12

7 Proof of Theorems 17 and 18

8 The compression theorem

We prove the following compression theorem, which was used in the formulation of Theorem 6.

**Theorem 29. (Compression Theorem)** For an arbitrary natural number \( d \) there exists a natural number \( l = l(d) \) such that an arbitrary element of the cobordism group \( \text{Imm}^{sf}(2l' - 3, 1) \) with \( l' \geq l \) admits a compression of order \( d - 1 \) (see Definition 5).

**Remark**

The necessary and sufficient condition for the existence of a compression is reformulated below in Corollary 34. This allows us to reformulate Theorem 29 without use of the concept of compression, as follows. For an arbitrary natural number \( d \) there exists a natural number \( l = l(d) \) such that for \( l' \geq l \) the homomorphism \( J^d_{sf} : \text{Imm}^{sf}(2l' - 3, 1) \to \text{Imm}^{sf}(d, 2l' - 2 - d) \) is zero.

[The rest of the section, 23 pages, is devoted to the proof of the compression theorem. PSL]

**References**

[A] P.M. Akhmet’ev, *Geometric approach towards stable homotopy groups of spheres. The Adams-Hopf invariants*, a talk at the M.M. Postnikov Memorial Conference (2007) “Algebraic Topology: Old and New” and at the Yu.P. Soloviev Memorial Conference (2005) “Topology, analysis and applications to mathematical physics”. The complete version (in Russian) arXiv:0710.5779; arXiv:0710.5853.

[A-E] P.M. Akhmet’ev and P.J. Eccles, *The relationship between framed bordism and skew-framed bordism*, Bull. London Math. Soc., 39 (2007), 473–481.
[B-J-M] M.G. Barratt, J.D.S. Jones and M.E. Mahowald, *The Kervaire invariant one problem*, Contemporary Mathematics, vol. 19, (1983), 9–22.

[B-R-S] S. Buoncristiano, C.P. Rourke, B.J. Sanderson *A Geometric Approach to Homology Theory*, London Math. Soc. LNS, vol. 18, Cambridge Univ. Press, 1976.

[1] J.S. Carter, *Surgery on codimension one immersions in \( \mathbb{R}^{n+1} \): removing \( n \)-tuple points*, Trans. Amer. Math. Soc., 298 (1986), 83–101.

[2] J.S. Carter, *On generalizing Boy’s surface: constructing a generator of the third stable stem*, Trans. Amer. Math. Soc., 298 (1986), 103–122.

[C] R.L. Cohen *The Immersion Conjecture for differentiable manifolds*, Ann. Math., 122 (1985), 237–328.

[C-J-M] R.L. Cohen, J.D.S. Jones and M.E. Mahowald *The Kervaire invariant of immersions*, Invent. math., 79 (1985), 95–123.

[E1] P.J. Eccles, *Codimension one immersions and the Kervaire invariant one problem*, Math. Proc. Cambridge Phil. Soc., 90 (1981), 483–493.

[G-M] L. Guillou and A. Marin (eds.), *A la recherche de la topologie perdue* (French), (Prog. Math., 62) Birkhauser, Boston-Basel-Stuttgart (1986); Russian transl., *In search of lost topology*, Mir, Moscow, 1986.

[K1] U. Koschorke, *Multiple points of immersions and the Kahn-Priddy theorem*, Math. Z., 169 (1979), 223–236.

[K2] U. Koschorke, *Nonstable and stable monomorphisms of vector bundles*, Topology and Applications, 75 (1987), 261–286.

[P] L.S. Pontryagin, *Smooth manifolds and their applications to homotopy theory*, 2-nd ed., Nauka, Moscow (1976), English transl.: L.S.Pontryagin, Selected Works, Vol. 3, Algebraic and Differential topology, pp. 115–252, Gordon and Breach Scientific Publishers (1986).

[S] V.P. Snaith, *Stable Homotopy Theory around the Arf-Kervaire invariant*, preprint (2008), published 2009.

[T] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. Math. Studies 49, Princeton Univ. Press, 1962; Mir, Moscow, 1982.