The algebraic matroid of the funtf variety

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Abstract. The affine funtf variety is the Zariski closure of the set of finite unit norm tight frames. Determining the fiber of a projection of the funtf variety onto a set of coordinates is called the algebraic funtf completion problem. The algebraic matroid of an algebraic variety encodes the dimensions of fibers of coordinate projections. This work characterizes the bases of the algebraic matroid underlying the affine funtf variety of funtfs in $\mathbb{R}^3$, and partial results towards similar characterizations for funtfs in $\mathbb{R}^n$ with $n \geq 4$ are also given. We provide a method to bound the degree of the projections based on combinatorial data.

1. Introduction to algebraic frame theory

Frames generalize the notion of a basis of a vector space and have found use in numerous fields of science and engineering. In many applications, frames of (infinite dimensional) Hilbert spaces are of interest, but finite (dimensional) frames are frequently used when computing. Given a Hilbert space $H$, a frame is a collection of elements $\{f_k\}_{k \in \mathbb{N}} \subset H$ such that there exist real numbers $A, B$ such that $0 < A \leq B < \infty$ and for every $h \in H$

$$A\|h\|^2 \leq \sum_{k \in \mathbb{N}} |\langle h, f_k \rangle|^2 \leq B\|h\|^2.$$ 

These frame conditions are given by Duffin and Schaeffer in [8]. If $A = B$, then the frame is called tight. If $H$ is $n$-dimensional, then any frame has at least $n$ elements. A frame where each element has norm one is said to be a unit norm frame. In the literature unit norm frames are also known as normalized frames, uniform frames, and spherical frames.

Frames which are both tight and unit norm are called funtf and are the focus of much research as these frames minimize various measures of error in signal reconstruction [7, 12, 14, 16]. Algebraic frame theory uses the powerful tools of computational algebraic geometry to solve problems involving finite frame varieties. Such approaches have found success in [4, 9, 20, 24].

Given an $n \times r$ matrix where only a subset of the entries are observed, the funtf completion problem asks for values of the missing entries such that the resulting completed matrix is a funtf. The jumping off point for this work is the relaxation of this problem that allows for the missing entries to take on complex values. We call this relaxation the algebraic funtf completion problem. The Zariski closure (over $\mathbb{C}$) of the $n \times r$ matrices which are finite unit norm tight frames forms an algebraic variety in $\mathbb{C}^{n \times r}$ which we will denote $X_{n,r}$. Note that complex frames are also studied where a Hermitian inner product is used, but that is not the focus of this article. Studying the variety $X_{n,r}$ in place of the set of funtfs gives one access to tools from algebraic geometry, and results about $X_{n,r}$ can lead to insight about the set of funtfs - see e.g. [5]. Many works have studied the properties of various sets of frames considered as varieties. For example, dimensions of $(\mu, S)$-frame varieties, i.e. spaces...
of matrices $W = [w_1 \cdots w_r]$, real or complex, satisfying $WW^* = S$ for some Hermitian (symmetric) positive definite matrix $S$ such that $\|w_k\| = \mu_k$, were considered in [23]. Funfts are a special case of these $(\mu, S)$-frames where $\mu_i = 1$ and $S$ is a scalar multiple of the identity matrix. Along with the fundamental groups, the dimensions of funtf varieties were observed in [9].

In [24], nonsingular points of $(\mu, S)$-frames are characterized along with the tangent spaces at these nonsingular points on these varieties. The connectivity of the funtf variety along with its irreducibility are studied in [4]. In [15], the polytope of eigensteps of finite equal norm tight frames is studied. These eigensteps are sequences of interlacing spectra used by [3] to construct finite frames of prescribed norms and the dimension of funtf varieties is noted to be related to the dimension of these polytopes.

Our paper is structured as follows. In Section 2, we recall algebraic funtf completion and give a brief survey of work preceding this. In Section 3, we provide the minimum necessary background on algebraic matroids. In Section 4 we set up our notation and collect previous results we will need about $X_{n,r}$. In particular, we discuss how matroids arise in connection with algebraic frame completion problems. In Section 5, we provide our main results on the algebraic matroid underlying $X_{n,r}$. This includes a complete characterization of the bases of the algebraic matroid underlying $X_{3,r}$ (Theorem 5.2). Section 6 gives a recursive formula for computing the degree of a finite-to-one coordinate projection of $X_{n,r}$ (Theorem 6.2) which we then use to completely characterize the degrees of projection onto a basis of $X_{3,r}$.

2. ALGEBRAIC FUNTF COMPLETION

Due to their robustness to erasures and additive noise, unit norm tight frames play an important role in signal processing. Explicit constructions for unit norm tight frames are quite recent despite theoretical work regarding existence being quite classical. The Schur-Horn Theorem [17, 22] characterizes the pairs $(\lambda, \mu)$ such that there exists a frame whose frame operator has spectrum $\lambda$ and lengths $\mu$. However, explicit constructions for these frames have remained scarce. In [13] the authors give a constructive characterization for all unit norm tight frames in $\mathbb{R}^2$ and provide a construction technique known as harmonic frames for unit norm tight frames in $\mathbb{R}^n$. An alternative constructive technique called spectral tetris is given in [6]. An explicit construction of every unit norm tight frame was finally given by [3, 11].

The previous paragraph covers results on explicitly constructing frames with prescribed spectrum and whose vectors’ lengths are prescribed. However, what if you have specific vectors you want included in your frame? How do you complete this partial set of vectors into a tight frame? The work [10] answers how many vectors must be added to complete your set of vectors into a tight frame, and in the case when all vectors are unit norm, they also provide a lower bound (which is not sharp) for the number of vectors required to complete the set of vectors into a tight frame. The minimum number of vectors needed to add to your set of vectors to complete it to a frame when their norms are prescribed is provided by [18]. In both papers, it is assumed that you start with a set of vectors.

In this paper, we take a different approach than Feng, Wang, and Wang or Massey and Ruiz. Instead of starting with a set of vectors and asking how many more vectors are needed to have a tight frame, we have the following generalization of the problem.

Problem 2.1. Given some known entries of an $n \times r$ matrix, determine if the matrix can be completed such that the columns form a finite unit norm tight frame (funtf).
3. The basics of algebraic matroids

We now take a detour to introduce the minimum necessary background on algebraic matroids. Since the only matroids considered in this paper will be algebraic, we will not discuss or define abstract matroids. Moreover, our study will be limited to those that are algebraic over $\mathbb{R}$ or $\mathbb{C}$. The reader who is interested in learning about more general (algebraic) matroids is advised to consult the textbook [21].

Let $\mathbb{K}$ be a field. Given a finite set $E$, we let $\mathbb{K}^E$ denote the vector space whose coordinates are indexed by the elements of $E$. Each subset $S \subseteq E$ of coordinates is associated with the linear projection $\pi_S : \mathbb{K}^E \to \mathbb{K}^S$ that sends each point $e \in E$ to $(e)_{e \in S}$. The ring of polynomials with coefficients in $\mathbb{K}$ and indeterminates indexed by $E$ will be denoted $\mathbb{K}[x_e : e \in E]$ and the corresponding field of rational functions will be denoted $\mathbb{K}(x_e : e \in E)$. The ideal in $\mathbb{K}[x_e : e \in E]$ generated by a finite set of polynomials $f_1, \ldots, f_k \in \mathbb{K}[x_e : e \in E]$ will be notated as $(f_1, \ldots, f_k)$. Given a set $X \subseteq \mathbb{K}^E$, we let $I(X)$ denote the ideal of all polynomial functions that vanish on $X$.

**Definition 3.1.** Let $E$ be a finite set, let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $X \subseteq \mathbb{K}^E$ be an irreducible variety. A subset of coordinates $S \subseteq E$ is

1. independent in $X$ if $I(\pi_S(X)) = 0$
2. spanning in $X$ if $\dim(\pi_S(X)) = \dim(X)$
3. a basis of $X$ if $S$ is both independent and spanning.

Any one of the three set systems consisting of the independent sets, the spanning sets, or the bases of an irreducible variety determines the other two. This combinatorial structure specified by any one of these set systems is called the algebraic matroid underlying $X$.

**Example 3.2.** Let $X \subset \mathbb{R}^4$ be the linear variety defined by the vanishing of the linear forms $x_1 - 5x_2 = 0$ and $x_3 + 2x_4 = 0$. The bases of $X$ are

$$\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}.$$ 

The independent sets of $X$ are the subsets of the bases, and the spanning sets are the supersets. Note that all the bases have cardinality 2, which is also the dimension of $X$. This is not a coincidence - see Proposition 3.3 below.

We now describe the intuition behind the algebraic matroid underlying an irreducible variety $X \subseteq \mathbb{K}^E$. When $S \subseteq E$ is independent, the coordinates $(x_e)_{e \in S}$ can be given arbitrary generic values, and the resulting vector can be completed to a point in $X$. When $S \subseteq E$ is spanning and $x \in X$ is generic, then the coordinates $(x_e)_{e \in S}$ can be determined by solving a zero-dimensional system of polynomials whose coefficients are polynomials in $(x_e)_{e \in S}$. In other words, the set $\pi_S^{-1}(\pi_S(x)) \cap X$ is generically finite.

Note that definition 3.1 requires that $X$ be an irreducible variety. This ensures that the algebraic matroid underlying $X$ is indeed a matroid (see e.g. [11 Proposition 1.2.9]). Proposition 3.3 below then follows from the fact that all bases of a matroid have the same size $[21]$ Chapter 1).

**Proposition 3.3.** Let $B \subseteq E$ be a basis of an irreducible variety $X \subseteq \mathbb{K}^E$. Then $|B| = \dim(X)$.

Given finite sets $A$ and $B$ and a field $\mathbb{K}$, we let $\mathbb{K}^{A \times B}$ denote the set of matrices with entries in $\mathbb{K}$ whose rows are indexed by elements of $A$ and whose columns are indexed by elements...
of $B$. Given polynomials $f_1, \ldots, f_k \in \mathbb{K}[x_e : e \in E]$ such that $(f_1, \ldots, f_k)$ is a prime ideal, the Jacobian matrix of $f_1, \ldots, f_k$ is the matrix $J(f_1, \ldots, f_k) \in (\mathbb{K}(x_e : e \in E))^{[k] \times E}$ whose $(i, e)$ entry is the partial derivative $\frac{\partial f_i}{\partial x_e}$. In Section 4 we will often work with submatrices of a Jacobian matrix. For this reason, we introduce the following notation.

**Notation 3.4.** Let $M$ denote a matrix whose columns are indexed by a set $E$. The submatrix of a given $M$ with columns corresponding to the elements of a subset $S$ of $E$ is denoted $M_S$.

The following proposition is useful for computing the bases of the algebraic matroid underlying a given irreducible variety. It is well known, and usually stated in terms of matroid duals. We state it here in more elementary terms for the purposes of keeping the necessary matroid theory background at a minimum.

**Proposition 3.5.** Let $E$ be a finite set, let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $X \subseteq \mathbb{K}^E$ be an irreducible variety of dimension $d$ such that $I(X) = (f_1, \ldots, f_k)$. A subset $S \subseteq E$ of size $d$ is a basis of $X$ if and only if the rank of $J(f_1, \ldots, f_k)_{E \setminus S}$ is $|E| - d$.

### 4. Algebraic Matroids to Algebraic Funtf Completion

The Zariski closure in $\mathbb{C}^{n \times r}$ of the set of $n \times r$ matrices $W$ such that the columns form a finite unit norm tight frame is denoted by $X_{n,r}$; in other words,

$$(4.1) \quad X_{n,r} = \{W \in \mathbb{C}^{n \times r} : WW^T = \frac{r}{n} \text{Id}_n, \text{diag}(W^T W) = \text{diag}(\text{Id}_r)\},$$

where $\text{diag}(M)$ denotes the diagonal entries of a matrix $M$. This paper studies the following algebraic relaxation of Problem 2.1

**Problem 4.1** (The algebraic frame completion problem). Given some known entries of an $n \times r$ matrix, determine if the matrix can be completed to an element of $X_{n,r}$.

We say $X_{n,r}$ is an affine funtf variety and call a matrix in $X_{n,r}$ a funtf matrix. The $\binom{n+1}{2} + r$ constraints of $X_{n,r}$ in (4.1) were found in [5]. We will express the polynomials defining the affine funtf variety in the ring $\mathbb{K}[x_{ij} : 1 \leq i \leq n, 1 \leq j \leq r]$ where $x_{ij}$ will represent the $ij$ entry of a matrix. Indeed, the column norm constraints on $W$ can be expressed as the following $r$ polynomials set to zero:

$$(4.2) \quad (g_1, \ldots, g_r) = \text{diag}(W^T W - \text{Id}_r),$$

while the orthogonal row constraints on $W$ can be expressed as the following $\binom{n+1}{2}$ polynomials $f_{ij}$, $i \leq j$:

$$(4.3) \quad \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix} = WW^T - \frac{r}{n} \text{Id}_n.$$

The problem of algebraic funtf completion can be cast as the problem of projecting an affine funtf variety to a subset of coordinates. Let $E \subseteq [n] \times [r]$ denote a subset of coordinates of $\mathbb{C}^{n \times r}$. We will think of $E$ as indexing “known” entries, and the algebraic funtf completion problem is to determine the remaining “unknown” entries so that the completed matrix is a funtf. Let $\pi_E$ denote the respective coordinate projection. The algebraic funtf completions of a given $M \in \mathbb{C}^E$ are the elements of the fiber $\pi_E^{-1}(M)$. It follows that $E$ is independent
in $X_{n,r}$ if and only if every generic $M \in \mathbb{C}^E$ has an algebraic funtf completion and that $E$ is spanning in $X_{n,r}$ if and only if each nonempty fiber $\pi^{-1}(\pi_E(M))$ is finite when $M$ is generic. Thus in the generic case, Problem 4.1 is equivalent to the following.

**Problem 4.2.** Find a combinatorial description of the algebraic matroid underlying $X_{n,r}$.

The first steps towards solving Problem 4.2 are determining the irreducibility and dimension of $X_{n,r}$. Fortunately, this was done in [9, Theorem 4.3(ii)], [24, Corollary 3.5],[4, Theorem 1.4] to give the following theorem.

**Theorem 4.3.** The dimension of the affine funtf variety $X_{n,r}$ is

\begin{equation}
\dim(X_{n,r}) = nr - \frac{(n+1)}{2} - r + 1 \quad \text{provided } r > n \geq 2.
\end{equation}

It is irreducible when $r \geq n + 2 > 4$.

In our work, we look to determine each basis (Definition 3.1, item 3) of $X_{n,r}$. We restrict our study to $r \geq n + 2 > 4$ so that $X_{n,r}$ is irreducible and thus gives a matroid. We seek a combinatorial description using bipartite graphs. Bipartite graphs provide a natural language for attacking Problem 4.2. Given finite sets $A$ and $B$ and a subset $S \subseteq A \times B$, we let $(A,B,S)$ denote the bipartite graph with partite vertex sets $A$ and $B$ and edge set $S$. We call two bipartite graphs $(A_1,B_1,S_1)$ and $(A_2,B_2,S_2)$ bipartite isomorphic if there exists a graph isomorphism $\phi: A_1 \cup B_1 \rightarrow A_2 \cup B_2$ such that $\phi(A_1) = A_2$ and $\phi(B_1) = B_2$.

Every subset $E$ of entries of an $n \times r$ matrix can be identified with the bipartite graph $([n],[r],[n] \times [r] \setminus E)$, which we denote by $G_E$. Note that the edges of $G_E$ are in bijection with the complement of $E$ and not $E$ itself. This stands in contrast to what is often done in the algebraic matrix completion literature, but will make our results much cleaner to state. Neither row-swapping nor column-swapping affects whether a given subset $E$ of entries of an $n \times r$ matrix is an independent set (or a basis, or spanning set) of $X_{n,r}$. Therefore, whether a given subset $E$ of entries is independent (or a basis or spanning) in $X_{n,r}$ only depends on the bipartite isomorphism equivalence class of $G_E$. The (non-bipartite) graph isomorphism class of $G_E$ may not be sufficient to determine whether $E$ is independent (or a basis or spanning) in $X_{n,r}$ because the transpose of a funtf matrix $W$ may not be funtf. So from now on, we will only consider bipartite graphs up to their bipartite isomorphism classes. We may now phrase Problem 4.2 more concretely as follows.

**Problem 4.4.** For which (bipartite isomorphism classes of) bipartite graphs $G_E$ is $E$ a basis of $X_{n,r}$?

We will sometimes find it useful to represent a subset $E \subseteq [n] \times [r]$ as the $\{0,1\}$-matrix whose $ij$ entry is 1 if $(i,j) \in E$ and 0 otherwise. Such a representation will be called a matrix entry representation.

5. **The algebraic matroid underlying the funtf variety**

In this section we give combinatorial criteria on the bases of $X_{n,r}$. First we show that if $E$ is a basis of $X_{n,r}$, then the graph $G_E$ is connected. Moreover, when $n = 3$ the converse is true as well. Second, we show that whether or not $E$ is a basis of $X_{n,r}$ only depends on the 2-core of $G_E$. This allows us to determine a combinatorial criterion for every $r$ after fixing $n$. 

Figure 1. Depicting $E := \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\} \subseteq [3] \times [5]$ as the bipartite graph $G_E$ and as a $\{0, 1\}$-matrix.

5.1. **Graph connectivity.** We begin with some graph theoretic definitions. Let $G = (A, B, S)$ be a bipartite graph. The *greater 2-core of $G$,* denoted $\overline{\text{core}}_2(G)$, is the graph obtained from $G$ by iteratively removing all edges that are incident to a vertex of degree one. The *2-core of $G$,* denoted $\text{core}_2(G)$, is the graph obtained by deleting the isolated vertices from $\overline{\text{core}}_2(G)$. Figure 2 shows a graph alongside its greater 2-core and its 2-core.

Figure 2. A bipartite graph $G$ alongside its greater 2-core and its 2-core.

Given a spanning forest $F$ of $G$ and an edge $e$ of $G$ not appearing in $F$, the graph $F \cup \{e\}$ has exactly one cycle which must contain $e$. This cycle is called *the fundamental circuit of $e$ with respect to $F$.* Given a subset $S' \subseteq S$ of the edge set of $G$, the *characteristic vector of $S'$* is the vector in $\{0, 1\}^S$ that has ones at entries corresponding to elements of $S'$ and zeros at all other entries. The *incidence matrix of $G$* is the matrix whose rows are indexed by the vertices of $G$, and the row corresponding to a vertex $v$ is the characteristic vector of the set of edges that are incident to $v$. Note that the columns of the incidence matrix of $G$ are naturally indexed by the edges of $G$.

**Example 5.1.** We use the notation $K_{a,b}$ to denote the complete bipartite graph on partite sets of size $a$ and $b$. The incidence matrix of $K_{3,5}$ is given by the $8 \times 15$ matrix below

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
$$

This matrix is naturally partitioned via the vertices in each partite of the graph $K_{3,5}$. 

}
Theorem 5.2. Assume \( r \geq n + 2 > 4 \) and let \( E \subseteq [n] \times [r] \) have cardinality \( nr - \binom{n+1}{2} - r + 1 \). If \( E \) is a basis of \( X_{n,r} \), then \( G_E \) is connected. When \( n = 3 \), the converse is true as well.

Proof. Let \( g_i \) and \( f_{ij} \) denote the polynomials as in (4.2) and (4.3); this set of polynomials generate the ideal of \( X_{n,r} \).

Let \( J \) be the \( (r + \binom{n+1}{2}) \times nr \) Jacobian matrix

\[
J := J(g_1, \ldots, g_r, f_{11}, f_{12} \ldots, f_{mn}).
\]

Proposition 3.5 implies that \( E \) is a basis of \( X_{n,r} \) if and only if the \( (r + \binom{n+1}{2}) \times (r + \binom{n+1}{2} - 1) \) matrix \( J_{[n] \times [r]} \) has full rank, or equivalently,

\[
\text{rank}(J_{[n] \times [r]} \setminus E) = r + \binom{n+1}{2} - 1.
\]

Let \( J' \) and \( J'' \) denote the following matrices of size \( (r + \binom{n+1}{2}) \times nr \) and \( (r + n) \times nr \), respectively

\[
J' := J \cdot \text{diag}(1/x_{11}, 1/x_{12} \ldots, 1/x_{nn}),
\]

\[
J'' := J(g_1, \ldots, g_r, f_{11}, f_{12} \ldots, f_{ii}, \ldots, f_{nn}) \cdot \text{diag}(1/x_{11}, 1/x_{12} \ldots, 1/x_{nn}),
\]

where \( \text{diag}(w) \) denotes the matrix with the vector \( w \) along its diagonal. The matrix \( J''_{[n] \times [r]} \setminus E \) is twice the incidence matrix of \( G_E \). Thus if \( G_E \) has \( c \) connected components, then

\[
\text{rank}(J''_{[n] \times [r]} \setminus E) = r + n - c.
\]

On the other hand, since \( J' \) can be obtained from \( J'' \) by including \( \binom{n}{2} \) additional rows, we have

\[
\text{rank}(J'_{[n] \times [r]} \setminus E) \leq \text{rank}(J''_{[n] \times [r]} \setminus E) + \binom{n}{2} = r + \binom{n+1}{2} - c.
\]

Since

\[
\text{rank}(J'_{[n] \times [r]} \setminus E) = \text{rank}(J''_{[n] \times [r]} \setminus E),
\]

we have that if \( G_E \) is disconnected, then \( J'_{[n] \times [r]} \setminus E \) is rank deficient and thus \( E \) is not a basis.

Now, having proved that \( E \) being a basis implies connectivity of \( G_E \), we assume that \( n = 3 \) and prove the converse. Further assume that \( G_E \) is connected with \( \binom{n+1}{2} + r - 1 = r + 2 \) edges. We will show that \( E \) is a basis of \( X_{n,r} \) by showing that \( J'_{[n] \times [r]} \setminus E \) has full rank. This is done by splitting \( J'_{[n] \times [r]} \setminus E \) into two row submatrices whose kernels intersect trivially.

Twice the incidence matrix of the complete bipartite graph \( K_{3,r} \) is a row-submatrix of \( J' \). Therefore, any linear relation among the columns of \( J' \) must lie in the linear space

\[
\mathbb{C}\{v_C : C \text{ is a circuit of } K_{3,r}\}
\]

where \( v_C \in \mathbb{C}^{[3] \times [r]} \) is the \( \{1, -1, 0\} \)-vector obtained from the characteristic vector of \( C \) by giving adjacent edges opposite signs. The row of \( J' \) corresponding to the constraint \( f_{ab} = 0 \) with \( a \neq b \) has \( x_{bi} / x_{ai} \) at the column corresponding to \( x_{ai} \) and \( x_{ai} / x_{bi} \) at the column corresponding to \( x_{bi} \). For ease of notation, we introduce the change of variables

\[
t_{1i} := x_{2i} / x_{1i} \quad t_{2i} := x_{3i} / x_{1i}.
\]
With this change of variables, the rows of $J'$ corresponding to the constraints $f_{12} = f_{13} = f_{23} = 0$ form the matrix $K$ shown below

$$K := \begin{pmatrix} x_{11} & \ldots & x_{1r} & x_{21} & \ldots & x_{2r} & x_{31} & \ldots & x_{3r} \\ f_{12} & t_{1l} & \ldots & t_{1r} & t_{1r}^{-1} & \ldots & t_{1r}^{-1} & 0 & \ldots & 0 \\ f_{13} & t_{21} & \ldots & t_{2r} & 0 & \ldots & 0 & t_{21}^{-1} & \ldots & t_{2r}^{-1} \\ f_{23} & 0 & \ldots & 0 & t_{2i}t_{r}^{-1} & \ldots & t_{2r}t_{r}^{-1} & t_{1i}t_{r}^{-1} & \ldots & t_{1r}t_{r}^{-1} \end{pmatrix}.$$  

Fix a spanning tree $T$ of $G_E$ and let $e_1, e_2, e_3$ denote the three edges of $G_E$ that are not contained in $T$. Let $C_i$ denote the fundamental circuit of $e_i$ with respect to $T$. The space of linear relations among the columns of $J'$ corresponding to the edges of $G_E$ lies within the three-dimensional subspace $\mathbb{C}\{v_{C_i} : i = 1, 2, 3\}$. We now show that no nonzero element of $\mathbb{C}\{v_{C_i} : i = 1, 2, 3\}$ lies in the kernel of $K$. It will then follow that the column-submatrix of $J'$ corresponding to the edges of $G_E$ has maximum rank.

The three fundamental circuits $C_1, C_2,$ and $C_3$ all lie in core$_2(G_E)$, which is a bipartite graph on partite sets of size $n' \leq 3$ and $r' \leq r$. Each vertex of core$_2(G_E)$ has degree at least 2, so $n', r' \geq 2$. Since $G_E$ is connected, core$_2(G_E)$ must also be connected. Hence since $C_1, C_2,$ and $C_3$ are in core$_2(G_E)$, core$_2(G_E)$ has exactly $n' + r' + 2$ edges. Since each vertex has degree at least 2, $2r' \leq n' + r' + 2$ and so $r' \leq n' + 2$. So thus far, we only need to consider $(n', r') = (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (3, 5).$ Among these, the only $(n', r')$-pairs such that there even exists such a bipartite graph with the correct number of edges are $(2, 4), (3, 3), (3, 4), (3, 5)$. For these values of $(n', r')$, we may compute all the connected bipartite graphs on partite sets of size $n'$ and $r'$ with minimum degree 2 and exactly $n' + r' + 2$ edges using the genbg command of Nauty and Traces [19]. There are seven such graphs and they are displayed in Figure 3 with vertices labeled according to which row or column they correspond to.

By relabeling vertices, we may assume that core$_2(G_E)$ is supported on partite vertex sets $1, \ldots, n'$ and $1, \ldots, r'$. Let $A_{\text{core}_2(G_E)}$ denote the incidence matrix of core$_2(G_E)$ and let $M_{\text{core}_2(G_E)}$ denote the matrix whose columns are a basis of the kernel of $A_{\text{core}_2(G_E)}$. Then $\mathbb{C}\{v_{C_i} : i = 1, 2, 3\}$ is the span of $M_{\text{core}_2(G_E)}$. Letting $K'$ be the submatrix of $K$ with columns corresponding to the edge set of core$_2(G_E)$, we are done if we show that $K'M_{\text{core}_2(G_E)}$ has rank 3 for the seven values of core$_2(G_E)$ above. This is verified in a Mathematica script available at the following url.

https://dibernstein.github.io/Supplementary_materials/funtf.html

$\square$

The following proposition is useful to construct examples showing that the converse of Theorem 5.2 is not true for $n \geq 4$.

**Proposition 5.3.** Assume $r \geq n + 2 > 4$ and let $E \subseteq [n] \times [r]$. If $E$ is spanning in $X_{n,r}$, then at most two vertices of $G_E$ corresponding to columns can have degree $n$.

**Proof.** Assume $G_E$ has $k$ column vertices of degree $n$. Without loss of generality, assume they correspond to the first $k$ columns so that $(a, i) \notin E$ for all $1 \leq a \leq n$, $1 \leq i \leq k$. Define $E' := \{(a, i) : 1 \leq a \leq n, k + 1 \leq i \leq r\}$. Note that $E \subseteq E'$. We show that the dimension of $\pi^{-1}_E(\pi_E(M))$ is positive for generic $M$. It follows that $E'$, and therefore $E$, is not spanning.

Let $M \in X_{n,r}$ be a generic funtf. The $(i, a)$ entry of $M$ will be denoted $m_{ia}$. Let $g_a, f_{ij}$ denote the polynomials obtained from $g_a$ and $f_{ij}$ by plugging in $m_{ia}$ for $x_{ia}$ when $(i, a) \in E'$.
The algebraic matroid of the Funtf variety

The Zariski closure of $\pi_{E'}^{-1}(\pi_E(M))$ can be identified with the variety in $\mathbb{C}^{n \times k}$ defined by the polynomials $\tilde{g}_1, \ldots, \tilde{g}_k$ and $\tilde{f}_{ij}$ for $1 \leq i \leq j \leq n$. Note that

$$\tilde{f}_{11} + \tilde{f}_{22} + \cdots + \tilde{f}_{nn} = \tilde{g}_1 + \cdots + \tilde{g}_k$$

and so $\pi_{E'}^{-1}(\pi_E(M))$ is in fact the vanishing locus of $\tilde{g}_1, \ldots, \tilde{g}_k$ and $\tilde{f}_{ij}$ for $1 \leq i, j \leq n$. Moreover, the polynomials

$$\tilde{f}_{ij} + \delta_{ij} \frac{r}{n} - \sum_{a=k+1}^{r} m_{ia} m_{ja} \quad \text{where} \quad \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

parameterize the variety of $n \times n$ symmetric matrices of rank at most $k$, which has dimension $nk - \binom{k}{2}$ (see e.g. [2, Lemma 6.2]). Thus the $\binom{n+1}{2}$ polynomials $\tilde{f}_{ij}, 1 \leq i \leq j \leq n$ together contribute at most $nk - \binom{k}{2}$ to the codimension of $\pi_{E'}^{-1}(\pi_E(M))$. Hence the codimension of $\pi_{E'}^{-1}(\pi_E(M))$ is at most $nk - \binom{k}{2} + k - 1$, which is strictly less than $nk$ for $k \geq 3$.

**Example 5.4.** Let $n \geq 4$ and $r \geq n + 2$ and let $E' := \{(i,a) : 2 \leq i \leq n, 4 \leq a \leq r\}$. Let $E$ be obtained from $E'$ by removing any $\binom{n-2}{2} - 1$ elements. Then $E$ has $nr - \binom{n+1}{2} - r + 1$ elements and $G_E$ is connected. However, Proposition 5.3 implies that $E$ cannot be a basis of $X_{n,r}$. Figure 4 shows examples of this construction for $n = 4$ and $n = 5$.

5.2. Combinatorial criteria with fixed row size. The goal of this section is to fix $n$ and find a combinatorial criteria to determine if $E$ is a basis of $X_{n,r}$ for any $r$. This is made precise in Remark 5.6.
The following theorem tells us that whether or not a given $E \subseteq [n] \times [r]$ of cardinality $nr - \binom{n+1}{2} - r + 1$ is a basis in $X_{n,r}$ depends only on $\text{core}_2(G_E)$.

**Theorem 5.5.** Assume $r \geq n+2 > 4$ and let $E \subseteq [n] \times [r]$ such that $|E| = nr - \binom{n+1}{2} - r + 1$. Then $E$ is a basis in the algebraic matroid underlying $X_{n,r}$ if and only if the set $E' \supseteq E$ satisfying $G_{E'} = \text{core}_2(G_E)$ is spanning in $X_{n,r}$. Moreover, for a fixed $n$, there are only finitely many possible graphs appearing as $\text{core}_2(G_E)$ as $E$ ranges over all bases of $X_{n,r}$.

**Proof.** Let $M \in X_{n,r}$ be a generic funt whose $(i,a)$ entry is $m_{ia}$. Let $\tilde{g}_a$ and $\tilde{f}_{ij}$ denote the polynomials obtained from $g_a$ and $f_{ij}$ by setting $x_{ia} = m_{ia}$ when $(i,a) \in E$. Then $\pi_E^{-1}(\pi_E(M))$ can be viewed as the zero-dimensional variety in $\mathbb{C}^{[n] \times [r] \setminus E}$ defined by the vanishing of the polynomials $\tilde{g}_a$, $1 \leq a \leq r$ and $\tilde{f}_{ij}$, $1 \leq i \leq j \leq n$. Since $G_E$ is connected, the edges of $G_E$ that are not in $\text{core}_2(G_E)$ can be ordered $(i_1,a_1), \ldots, (i_k,a_k)$ such that for each $j$, in either $\tilde{g}_{a_j}$ or $\tilde{f}_{ij}$, every variable other than $x_{i_1a_j}$ that appears is of the form $x_{i_ka_{j-1}}$ for some $l < j$. It follows that given $\pi_E(M)$, one can solve a series of quadratic equations in order to recover, up to finite ambiguity, the entries of $M$ at positions corresponding to edges of $G_E$ that are not in $\text{core}_2(G_E)$. One can then solve for the remaining entries of $M$ precisely when $\text{core}_2(G_E)$ is spanning in $X_{n,r}$. The “moreover” clause follows by Proposition 5.7 below. $\square$

**Remark 5.6.** Given a set $E \subseteq [n] \times [r]$ that is spanning in $X_{n,r}$, the set $E \cup \{ (1,r+1), \ldots, (n,r+1) \}$ is spanning in $X_{n,r+1}$. Thus Theorem 5.5 tells us that if we fix $n$ but allow $r$ to vary, then the problem of determining whether or not $E \subseteq [n] \times [r]$ is a basis of $X_{n,r}$ is equivalent to determining whether or not $\text{core}_2(G_E)$ appears on a certain finite list. Proposition 5.7 below gives us the finiteness statement in Theorem 5.5 as well as bounds on the size of $\text{core}_2(G)$.

**Proposition 5.7.** Let $r \geq n+2 > 4$ and let $E \subseteq [n] \times [r]$ such that $G_E$ is connected. Let $\alpha$ and $\beta$ be the number of row- and column-vertices (respectively) in $\text{core}_2(G_E)$. If $E$ is a basis of $X_{n,r}$, then

1. $\alpha = n - 1$ or $\alpha = n$
2. $\alpha \leq \beta \leq \binom{n}{2} + \alpha - 1$.

**Proof.** Let $M \in X_{n,r}$ be a generic funt whose $(i,a)$ entry is $m_{ia}$. Let $\tilde{g}_a$ and $\tilde{f}_{ij}$ denote the polynomials obtained from $g_a$ and $f_{ij}$ by setting $x_{ia} = m_{ia}$ when $(i,a) \in E$. Then $\pi_E^{-1}(\pi_E(M))$ can be viewed as the zero-dimensional variety in $\mathbb{C}^{[n] \times [r] \setminus E}$ defined by the vanishing of the polynomials $\tilde{g}_a$, $1 \leq a \leq r$ and $\tilde{f}_{ij}$, $1 \leq i \leq j \leq n$. 

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]
First we show $\alpha = n - 1$ or $\alpha = n$. Without loss of generality, assume that $\text{core}_2(G_E)$ has row-vertices $1, \ldots, \alpha$ and column vertices $1, \ldots, \beta$. Let $F := \{(i, a) : 1 \leq i \leq \alpha, 1 \leq a \leq \beta\} \setminus E$ be the edge set of $\overline{\text{core}}_2(G_E)$ and note that $|F| = \alpha + \beta + \binom{n}{2} - 1$. The elements of $F$ index the entries in the upper-left submatrix of $M$ that are, in principle, allowed to vary over the fiber $\pi_E^{-1}(\pi_E(M))$. After dropping one of the redundant $\tilde{g}_a$’s via (5.1), there are exactly $\binom{\alpha+1}{2} + \beta - 1 + \alpha(n - \alpha)$ equations among the $\tilde{g}_a$’s and $\tilde{f}_{ij}$’s that involve entries in the upper-left $\alpha \times \beta$ block of $M$. Since $\pi_E^{-1}(\pi_E(M))$ is zero-dimensional, we must have

$$\alpha + \beta + \binom{n}{2} - 1 \leq \left(\frac{\alpha + 1}{2}\right) + \beta - 1 + \alpha(n - \alpha)$$

which simplifies to

$$- \frac{1}{2} \alpha^2 + \left(n - \frac{1}{2}\right) \alpha - \binom{n}{2} \geq 0.$$  

(5.3)

Let us now consider the left-hand side of the inequality (5.3) as a polynomial $h$ in $\alpha$, treating $n$ as a constant. The only roots of $h$ are $n - 1$ and $n$, and $h(\alpha)$ is nonnegative if and only if $n - 1 \leq \alpha \leq n$. Therefore, we must have $\alpha \in \{n - 1, n\}$.

Now we show $\alpha \leq \beta$. As noted in the proof of Proposition 5.3, when $(i, a)$ is an edge in $G_E$ but not $\overline{\text{core}}_2(G_E)$, we may solve a zero-dimensional quadratic system for $x_{ia}$ given $\{m_{ia} : (i, a) \in E\}$. Thus we may now assume that $G_E = \overline{\text{core}}_2(G_E)$ and allow $E$ to be spanning in $X_{n, r}$ (as opposed to a basis of $X_{n, r}$).

Assume for the sake of contradiction that $\alpha > \beta$. Then, the $\binom{\alpha+1}{2}$ constraints $\tilde{f}_{ij} = 0$ where $1 \leq i \leq j \leq \alpha$ together can contribute at most $\alpha \beta - \binom{\beta}{2}$ to codimension. This is because for $1 \leq i \leq j \leq \alpha$, the polynomials from (5.2) with $k = \beta$ give the entries of an $\alpha \times \alpha$ symmetric matrix with rank at most $\beta$, and the dimension of the variety of $\alpha \times \alpha$ symmetric matrices of rank at most $\beta$ is $\alpha \beta - \binom{\beta}{2}$ (see e.g. [2] Lemma 6.2). Also, as before, at least one of the constraints $\tilde{g}_a = 0$, $1 \leq a \leq \beta$ is redundant. Since $\pi_E^{-1}(\pi_E(M))$ is zero-dimensional, we must have $|F| \leq \alpha \beta - \binom{\beta}{2} + \beta - 1$ and therefore

$$\alpha(\beta - 1) - \binom{\beta}{2} - \binom{n}{2} \geq 0.$$  

(5.4)

After plugging in $n - 1$ for $\alpha$, (5.4) becomes $(\beta - n)^2 + n + \beta \leq 2$, which is a contradiction because $n \geq 3$. Plugging in $n$ for $\alpha = n$ in (5.4), we get the inequality $-n - \beta \geq (\beta - n)^2$, which is a contradiction because the left hand side is strictly negative, and the right hand side is nonnegative. Hence, we have $\alpha \leq \beta$.

The final inequality $\beta \leq \binom{n}{2} + \alpha - 1$ follows from the fact that $\text{core}_2(G_E)$ has $\binom{n}{2} + \alpha + \beta - 1$ edges and each of the $\beta$ non-isolated column vertices has degree at least 2. \qed

6. Degree of projection and algebraic identifiability

Now that we have a handle on which subsets $E \subseteq [n] \times [r]$ yield projections $\pi_E : X_{n, r} \to \mathbb{C}^E$ that are generically finite-to-one, we can ask about the cardinality of a generically finite fiber. In other words, we want to solve the following problem.

**Problem 6.1** (Algebraic identifiability complexity). Develop a combinatorial method for computing the degree of the map $\pi_E : X_{n, r} \to \mathbb{C}^E$ from $G_E$ when $E$ is a basis of $X_{n, r}$.

The following theorem gets us part of the way towards a solution to Problem 6.1.
Theorem 6.2. Let $r \geq n + 2 > 4$ and let $E \subseteq [n] \times [r]$. Define $F \subseteq [n] \times [r]$ such that $\text{core}_2(G_E) = G_F$, and let $k$ denote the number of vertices that are isolated in $\text{core}_2(G_E)$ but not in $G_E$. If $E$ is a spanning set of $X_{n,r}$, then

$$\deg \pi_E = 2^k \cdot \deg \pi_F.$$ 

Proof. Note that $E \subseteq F$ and so we have a projection map $h : \pi_F(X_{n,r}) \to \mathbb{C}^E$ that omits all the coordinates corresponding to elements of $F \setminus E$. Then, $\pi_E = h \circ \pi_F$ and

$$\deg \pi_E = \deg \pi_F \cdot \deg h,$$

which can be seen as follows. The maps $h : \pi_F(X_{n,r}) \to \mathbb{C}^E$ and $\pi_E : X_{n,r} \to \mathbb{C}^E$ are each branched covers of $\mathbb{C}^E$. In other words, there exist dense Zariski open subsets $U_1$ and $U_2$ of $\mathbb{C}^E$ such that $h$ restricted to $h^{-1}(U_1)$ and $\pi_E$ restricted to $\pi_E^{-1}(U_2)$ are covering spaces. Moreover, $h$ and $\pi_E$ restricted to $U_1 \cap U_2$ are also covering spaces. Since $\pi_E = h \circ \pi_F$, we have $\pi_F$ is a homomorphism of covering spaces and thus

the topological degree of the restricted maps with image $U_1 \cap U_2$ satisfy $\deg \pi_E = \deg \pi_F \cdot \deg h$. As $U_1$ and $U_2$ are dense Zariski open subsets of $\mathbb{C}^E$, we have $U_1 \cap U_2$ is also a dense Zariski open subset of $\mathbb{C}^E$. Therefore the equality above follows.

It now suffices to show $\deg h = 2^k$. Let $M \in X_{n,r}$ be generic and define $\tilde{g}_a$ and $\tilde{f}_{ij}$ as in the proofs of Theorem 5.5 and Proposition 5.7. We can order the elements of $F \setminus E$ as $(i_1, a_1), \ldots, (i_k, a_k)$ such that for each $j \in \{1, \ldots, k\}$, the only non-$x_{i_ja_j}$ variables in at least one of $\tilde{g}_a$ or $\tilde{f}_{ij}$ will be of the form $x_{i_la_l}$ with $l < j$. Let $\mathcal{F}$ denote the system of all such polynomials, and note that the non-constant coordinates of $h^{-1}(\pi_E(M))$ are given by the variety defined by the vanishing of $\mathcal{F}$. By solving $\mathcal{F}$ via “back-substitution” in the order $x_{a_1i_1}, \ldots, x_{a_ki_k}$, we see that this variety has exactly $2^k$ points. Thus, $|h^{-1}(f(M))| = 2^k$ and $\deg h = 2^k$. 

Let $E \subseteq [n] \times [r]$ with $E$ a basis of $X_{n,r}$ and define $F \subseteq [n] \times [r]$ so that $G_F = \overline{\text{core}_2(G_E)}$. Let $\beta$ denote the number of column vertices in $\text{core}_2(G_E)$ and let $M \in X_{n,r}$ be generic whose $(i, a)$ entry is $m_{ia}$. If $r \geq n + 2 > 4$ and $r \geq \beta + 1$, then the degree of the projection map $\pi_F : X_{n,r} \to \mathbb{C}^F$ only depends on $\text{core}_2(G_E)$ and not on $r$. This follows from the fact that if $r \geq \beta + 1$, then the set of non-constant polynomials $\tilde{f}_{ij}$ and $\tilde{g}_a$ obtainable by substituting $x_{ia} = m_{ia}$ for $(i, a) \in F$ does not depend on $r$. So for a graph $H$ such that $H = \text{core}_2(G_E)$ for some basis $E$ of $X_{n,r}$, let $\deg(H)$ denote the degree of $\pi_F$ when $r \geq \beta + 1$. If $r = \beta$ (note that $r < \beta$ is not possible), then $\deg(\pi_F) \leq \deg(\text{core}_2(G_E))$. Thus Theorem 6.2 gives us the bound $\deg(\pi_F) \leq 2^k \deg(\text{core}_2(G_E))$.

Theorem 5.5 tells us, that for fixed $n$, there are only finitely many $\text{core}_2(G_E)$. Thus one can compute all values of $\deg(\text{core}_2(G_E))$ for a fixed $n$ and use this to produce an algorithm that bounds the size of a finite fiber $|\pi_E^{-1}(\pi_E(M))|$ by computing the 2-core of $G_E$. This is done in Algorithm 1 and Example 6.3 illustrates this for the case $n = 3$. 


Algorithm 1 For fixed $n \geq 3$, bounds the size of a generic fiber $\pi_{E}^{-1}(\pi_{E}(M))$ when $E$ is a basis of $X_{n,r}$. Assumes that all possible values of $\deg(\text{core}_2(G_{E}))$ have been precomputed.

1: procedure BoundFiber$(r, E)$ \Comment{$r \geq n + 2$ and $E$ is a basis of $X_{n,r}$}
2: $H \leftarrow \text{core}_2(G_{E})$
3: $k \leftarrow \text{number of vertices in } G_{E} \text{ but not } H$
4: $d \leftarrow \deg(H)$ \Comment{obtain by looking up in precomputed table}
5: Return: $d \cdot 2^k$
6: end procedure

Example 6.3. When $E$ is a basis of $X_{3,r}$, $\text{core}_2(G_{E})$ is one of seven graphs, displayed in Figure 3. For each possible $\text{core}_2(G_{E})$, we compute the cardinality of a projection of an $X_{3,r}$ onto $\text{core}_2(G_{E})$ using probability-one methods in bertini. When $\text{core}_2(G_{E})$ has five or more column vertices, we took $r = 5$ and $r = 6$ and observed that in both cases, the degree of projection was the same. When $\text{core}_2(G_{E})$ has fewer than five column vertices, we take $r = 5$. These degrees are given in Table 1. Via the above discussion, this characterizes the possible degrees of a projection of $X_{3,r}$ onto a basis. For example, if $E \subseteq [3] \times [5]$ where where as a zero-one matrix,

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

then the degree of a generic fiber of this projection is $128 = 2^2 \cdot 32$. This can be read off from Table 1 by noting that $F$ such that $G_{F} = \text{core}_2(G_{E})$ is given in the top row of the table and the degree of the corresponding fiber is 32.

Acknowledgments

This work was partially supported by the grants NSF CCF-1708884 and NSA H98230-18-1-0016. Daniel Irving Bernstein was supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship (DMS-1802902). He also completed some of this work while employed by the NSF-supported (DMS-1439786) Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Fall 2018 semester program on nonlinear algebra. Cameron Farnsworth’s research was supported in part by the National Research Foundation of Korea (Grant Number 2015R1X300350). Jose Israel Rodriguez was partially supported by the College of Letters and Science, UW-Madison.

References

[1] D. I. Bernstein. Matroids in Algebraic Statistics. PhD thesis, North Carolina State University, 2018. https://repository.lib.ncsu.edu/handle/1840.20/35009
[2] D. I. Bernstein, G. Blekherman, and R. Sinn. Typical and generic ranks in matrix completion. arXiv preprint arXiv:1802.09513, 2018.
[3] J. Cahill, M. Fickus, D. G. Mixon, M. J. Poteet, and N. Strawn. Constructing finite frames of a given spectrum and set of lengths. Appl. Comput. Harmon. Anal., 35(1):52–73, 2013.
[4] J. Cahill, D. G. Mixon, and N. Strawn. Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames. SIAM J. Appl. Algebra Geom., 1(1):38–72, 2017.
[5] J. Cahill and N. Strawn. Algebraic geometry and finite frames. In Finite frames, Appl. Numer. Harmon. Anal., pages 141–170. Birkhäuser/Springer, New York, 2013.
$F$ such that $G_F = \text{core}_2(G_E)$ | $\deg(\text{core}_2(G_E))$ |
---|---|
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} & 32 |
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} & 24 |
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} & 96 |
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix} & 128 |
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} & 288 |
\begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} & 576 |
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix} & 384 |

Table 1. Each possible value of $F \subseteq [3] \times [5]$ such that $G_F = \text{core}_2(G_E)$ when $E$ is a basis of $X_{3,r}$, alongside the degree of the corresponding coordinate projection map.

[6] P. G. Casazza, M. Fickus, D. G. Mixon, Y. Wang, and Z. Zhou. Constructing tight fusion frames. *Appl. Comput. Harmon. Anal.*, 30(2):175–187, 2011.

[7] P. G. Casazza and J. Kovačević. Equal-norm tight frames with erasures. *Adv. Comput. Math.*, 18(2-4):387–430, 2003. Frames.

[8] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[9] K. Dykema and N. Strawn. Manifold structure of spaces of spherical tight frames. *Int. J. Pure Appl. Math.*, 28(2):217–256, 2006.

[10] D.-J. Feng, L. Wang, and Y. Wang. Generation of finite tight frames by Householder transformations. *Adv. Comput. Math.*, 24(1-4):297–309, 2006.

[11] M. Fickus, D. G. Mixon, M. J. Poteet, and N. Strawn. Constructing all self-adjoint matrices with prescribed spectrum and diagonal. *Adv. Comput. Math.*, 39(3-4):585–609, 2013.

[12] V. K. Goyal. *Beyond traditional transform coding*. PhD thesis, University of California, Berkeley, 1998.

[13] V. K. Goyal, J. Kovačević, and J. A. Kelner. Quantized frame expansions with erasures. *Appl. Comput. Harmon. Anal.*, 10(3):203–233, 2001.

[14] V. K. Goyal, M. Vetterli, and N. T. Thao. Quantized overcomplete expansions in $\mathbb{R}^N$: analysis, synthesis, and algorithms. *IEEE Trans. Inform. Theory*, 44(1):16–31, 1998.

[15] T. Haga and C. Pegel. Polytopes of eigensteps of finite equal norm tight frames. *Discrete Comput. Geom.*, 56(3):727–742, 2016.

[16] R. B. Holmes and V. I. Paulsen. Optimal frames for erasures. *Linear Algebra Appl.*, 377:31–51, 2004.

[17] A. Horn. Doubly stochastic matrices and the diagonal of a rotation matrix. *American Journal of Mathematics*, 76:620–630, 1954.
[18] P. G. Massey and M. A. Ruiz. Tight frame completions with prescribed norms. *Sampl. Theory Signal Image Process.*, 7(1):1–13, 2008.

[19] B. D. McKay and A. Piperno. Practical graph isomorphism, {II}. *Journal of Symbolic Computation*, 60(0):94 – 112, 2014.

[20] L. Oeding, E. Robeva, and B. Sturmfels. Decomposing tensors into frames. *Adv. in Appl. Math.*, 73:125–153, 2016.

[21] J. G. Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[22] I. Schur. Über eine klasse von mittelbildungen mit anwendungen auf die determinantentheorie. *Sitzungsber. Berl. Math. Ges*, 22:9–20, 1923.

[23] N. Strawn. Geometry and constructions of finite frames. Master’s thesis, Texas A&M University, College Station, TX USA, 2007.

[24] N. Strawn. Finite frame varieties: nonsingular points, tangent spaces, and explicit local parameterizations. *J. Fourier Anal. Appl.*, 17(5):821–853, 2011.

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