SUBPRODUCT SYSTEMS

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ABSTRACT. The notion of a subproduct system, a generalization of that of a product system, is introduced. We show that there is an essentially 1 to 1 correspondence between cp-semigroups and pairs \((X, T)\) where \(X\) is a subproduct system and \(T\) is an injective subproduct system representation. A similar statement holds for subproduct systems and units of subproduct systems. This correspondence is used as a framework for developing a dilation theory for cp-semigroups. Results we obtain: (i) a \(\ast\)-automorphic dilation to semigroups of \(\ast\)-endomorphisms over quite general semigroups; (ii) necessary and sufficient conditions for a semigroup of CP maps to have a \(\ast\)-endomorphic dilation; (iii) an analogue of Parrot’s example of three contractions with no isometric dilation, that is, an example of three commuting, contractive normal CP maps on \(B(H)\) that admit no \(\ast\)-endomorphic dilation (thereby solving an open problem raised by Bhat in 1998). Special attention is given to subproduct systems over the semigroup \(\mathbb{N}\), which are used as a framework for studying tuples of operators satisfying homogeneous polynomial relations, and the operator algebras they generate. As applications we obtain a noncommutative (projective) Nullstellensatz, a model for tuples of operators subject to homogeneous polynomial relations, a complete description of all representations of Matsumoto’s subshift \(C^\ast\)-algebra when the subshift is of finite type, and a classification of certain operator algebras – including an interesting non-selfadjoint generalization of the noncommutative tori.

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References

INTRODUCTION

Motivation: dilation theory of $CP_0$-semigroups. We begin by describing the problems that motivated this work.

Let $H$ be a separable Hilbert space, and let $M \subseteq B(H)$ be a von Neumann algebra. A CP map on $M$ is a contractive, normal and completely positive map. A CP$_0$-semigroup on $M$ is a family $\Theta = \{\Theta_t\}_{t \geq 0}$ of unital CP maps on $M$ satisfying the semigroup property

$$\Theta_{s+t}(a) = \Theta_s(\Theta_t(a)), \quad s, t \geq 0, a \in M,$$

and the continuity condition

$$\lim_{t \to t_0} \langle \Theta_t(a)h, g \rangle = \langle \Theta_{t_0}(a)h, g \rangle, \quad a \in M, h, g \in H.$$

A CP$_0$-semigroup is called an $E_0$-semigroup if each of its elements is a $*$-endomorphism.

Let $\Theta$ be a CP$_0$-semigroup acting on $M$, and let $\alpha$ be an $E_0$-semigroup acting on $R$, where $R$ is a von Neumann subalgebra of $B(K)$ and $K \supseteq H$. Denote the orthogonal projection of $K$ onto $H$ by $p$. We say that $\alpha$ is an $E_0$-dilation of $\Theta$ if for all $t \geq 0$ and $b \in R$

$$\Theta_{s+t}(pbp) = p\alpha_s(\beta_t(b))p$$

(0.1)

In the mid 1990’s Bhat proved the following result, known today as “Bhat’s Theorem” (see [9] for the case $M = B(H)$, and also [39, 14, 28, 6] for different proofs and for the general case):

Theorem 0.1. (Bhat). Every CP$_0$-semigroup has a unique minimal $E_0$-dilation.

A natural question is then this: given two commuting CP$_0$-semigroups, can one simultaneously dilate them to a pair of commuting $E_0$-semigroups? In [42] the following partial positive answer was obtained.$^1$

Theorem 0.2. [42] Theorem 6.6] Let $\{\phi_t\}_{t \geq 0}$ and $\{\theta_t\}_{t \geq 0}$ be two strongly commuting CP$_0$-semigroups on a von Neumann algebra $M \subseteq B(H)$, where $H$ is a separable Hilbert space. Then there is a separable Hilbert space $K$ containing $H$ and an orthogonal projection $p : K \to H$, a von Neumann algebra $R \subseteq B(K)$ such that $M = pRp$, and two commuting $E_0$-semigroups $\alpha$ and $\beta$ on $R$ such that

$$\phi_s \circ \theta_t(pbp) = p\alpha_s \circ \beta_t(b)p$$

for all $s, t \geq 0$ and all $b \in R$.

$^1$The same result was obtained in [41] for nonunital semigroups acting on $M = B(H)$.
In other words: every two-parameter CP$_0$-semigroup that satisfies an additional condition of strong commutativity has a two-parameter E$_0$-dilation. The condition of strong commutativity was introduced in [47]. A precise definition will not be given here. **The main tools in the proof of Theorem 0.2 and also in some of the proofs of Theorem 0.1 were product systems of W$^*$-correspondences and their representations.** In fact, the only place in the proof of Theorem 0.2 where the assumption of strong commutativity is used, is in the construction of a certain product system. More about that later.

In [10], Bhat showed that given a pair of commuting CP maps $\Theta$ and $\Phi$ on $B(H)$, there is a Hilbert space $K \supseteq H$ and a pair of commuting normal $*$-endomorphisms $\alpha$ and $\beta$ acting on $B(K)$ such that

$$\Theta^m \circ \Phi^n(pb)p = p\alpha^m \circ \beta^n(b)p, \; b \in B(K)$$

for all $m, n \in \mathbb{N}$ (here $p$ denotes the projection of $K$ onto $H$). Later on Solel, using a different method (using in fact product systems and their representations), proved this result for commuting CP maps on arbitrary von Neumann algebras [47]. Neither one of the above results requires strong commutativity.

In light of the above discussion, and inspired by classical dilation theory [46, 48], it is natural to conjecture that every two commuting (not necessarily strongly commuting) CP$_0$-semigroups have an E$_0$-dilation, and in fact that the same is true for any $k$ commuting CP$_0$-semigroups, for any positive integer $k$. However, the framework given by product systems seems to be too weak to prove this. Trying to bypass this stoppage, we arrived at the notion of a subproduct system.

**Background: from product systems to subproduct systems.** Product systems of Hilbert spaces over $\mathbb{R}^+$ were introduced by Arveson some 20 years ago in his study of E$_0$-semigroups [3]. In a few imprecise words, a product system of Hilbert spaces over $\mathbb{R}^+$ is a bundle $\{X(t)\}_{t \in \mathbb{R}^+}$ of Hilbert spaces such that

$$X(s+t) = X(s) \otimes X(t), \; s, t \in \mathbb{R}^+.$$ 

We emphasize immediately that Arveson’s definition of product systems required also that the bundle carry a certain Borel measurable structure, but we do not deal with these matters here. To every E$_0$-semigroup Arveson associated a product system, and it turns out that the product system associated to an E$_0$-semigroup is a complete cocycle conjugacy invariant of the E$_0$-semigroup.

Later, product systems of Hilbert C$^*$-correspondences over $\mathbb{R}^+$ appeared (see the survey [15] by Skeide). In [14], Bhat and Skeide associate with every semigroup of completely positive maps on a C$^*$-algebra $A$ a product system of Hilbert $A$-correspondences. This product system was then used in showing that every semigroup of completely positive maps can be “dilated” to a semigroup of $*$-endomorphisms. Muhly and Solel introduced a different construction [25]: to every CP$_0$-semigroup on a von Neumann algebra $\mathcal{M}$ they associated a product system of Hilbert W$^*$-correspondences over $\mathcal{M}'$, the commutant of $\mathcal{M}$. Again, this product system is then used in constructing an E$_0$-dilation for the original CP$_0$-semigroup.

Product systems of C$^*$-correspondences over semigroups other than $\mathbb{R}^+$ were first studied by Fowler [20], and they have been studied since then by many authors. In [17], product systems over $\mathbb{N}^2$ (and their representations) were studied, and the results were used to prove that every pair of commuting CP maps has a $*$-endomorphic dilation. Product systems over $\mathbb{R}^2_+$ were also central to the proof of
Theorem 0.2, where every pair of strongly commuting CP$_0$-semigroups is associated with a product system over $\mathbb{R}_+^2$. However, the construction of the product system is one of the hardest parts in that proof. Furthermore, that construction fails when one drops the assumption of strong commutativity, and it also fails when one tries to repeat it for $k$ strongly commuting semigroups.

On the other hand, there is another object that may be naturally associated with a semigroup of CP maps over any semigroup: this object is the subproduct system, which, when the CP maps act on $B(H)$, is the bundle of Arveson's "metric operator spaces" (introduced in [4]). Roughly, a subproduct system of correspondences over a semigroup $S$ is a bundle $\{X(s)\}_{s \in S}$ of correspondences such that

$$X(s + t) \subseteq X(s) \otimes X(t), \ s, t \in S.$$ 

See Definition 1.1 below. Of course, a difficult problem cannot be made easy just by introducing a new notion, and the problem of dilating $k$-parameter CP$_0$-semigroups remains unsolved. However, subproduct systems did already provide us with an efficient general framework for tackling various problems in operator algebras, and in particular it has led us to a progress toward the solution of the discrete analogue of the above unsolved problem.

This paper consists of two parts. In the first part we introduce subproduct systems over general semigroups, show the connection between subproduct systems and CP-semigroups, and use this connection to obtain three main results in dilation theory of CP-semigroups. The first result is that every $c_0$-semigroup over a (certain kind of) semigroup $S$ can be dilated to a semigroup of $\ast$-automorphisms on some type I factor. The second is some necessary conditions and sufficient conditions for a CP-semigroup to have a (minimal) $\ast$-endomorphic dilation. The third is an analogue of Parrot’s example of three contractions with no isometric dilation, that is, an example of three commuting, contractive normal CP maps on $B(H)$ that admit no $\ast$-endomorphic dilation. The CP maps in the stated example can be taken to have arbitrarily small norm, providing the first example of a theorem in the classical theory of isometric dilations that cannot be generalized to the theory of $e$-dilations of CP-semigroups.

Having convinced the reader that subproduct systems are an interesting and important object, we turn in the second part of the paper to take a closer look at the simplest examples of subproduct systems, that is, subproduct systems of Hilbert spaces over $\mathbb{N}$. We study certain tuples of operators and operator algebras that can be naturally associated with every subproduct system, and explore the relationship between these objects and the subproduct systems that give rise to them.

Some preliminaries. $\mathcal{M}$ and $\mathcal{N}$ will denote von Neumann subalgebras of $B(H)$, where $H$ is some Hilbert space.

In Sections 1 through 5, $\mathcal{S}$ will denote a sub-semigroup of $\mathbb{R}_+^k$. In fact, in large parts of the paper $\mathcal{S}$ can be taken to be any semigroup with unit, or at least any Ore semigroup (see [24] for a definition), but we prefer to avoid this distraction.

Definition 0.3. A cp-semigroup is a semigroup of CP maps, that is, a family $\Theta = \{\Theta_s\}_{s \in S}$ of completely positive, contractive and normal maps on $\mathcal{M}$ such that

$$\Theta_{s+t}(a) = \Theta_s(\Theta_t(a)) \ , \ s, t \in S, a \in \mathcal{M}.$$
and

\[ \Theta_0(a) = a , \ a \in M. \]

A cp0-semigroup is a semigroup of unital CP maps. An e-semigroup is a semigroup of \(*\)-endomorphisms. An cp0-semigroup is a semigroup of unital \(*\)-endomorphisms.

For concreteness, one should think of the case \( S = \mathbb{N}^k \), where a cp-semigroup is a \( k \)-tuple of commuting CP maps, or the case \( S = \mathbb{R}^k_+ \), where a cp-semigroup is a \( k \)-parameter semigroup of CP maps, or \( k \) mutually commuting one-parameter cp-semigroups.

**Definition 0.4.** Let \( \Theta = \{ \Theta_s \}_{s \in S} \) be a cp-semigroup acting on a von Neumann algebra \( M \subseteq B(H) \). An \( e \)-dilation of \( \Theta \) is a triple \(( \alpha, K, R )\) consisting of a Hilbert space \( K \supseteq H \) (with orthogonal projection \( P_H : K \to H \)), a von Neumann algebra \( R \subseteq B(K) \) that contains \( M \) as a corner \( M = P_H R P_H \), and an e-semigroup \( \alpha = \{ \alpha_s \}_{s \in S} \) on \( R \) such that for all \( T \in R \), \( s \in S \),

\[ \Theta_s(P_H TP_H) = P_H \alpha_s(T)P_H. \]

**Definition 0.5.** Let \( A \) be a \( C^* \)-algebra. A Hilbert \( C^* \)-correspondence over \( A \) is a (right) Hilbert \( A \)-module \( E \) which carries an adjointable, left action of \( A \).

**Definition 0.6.** Let \( M \) be a \( W^* \)-algebra. A Hilbert \( W^* \)-correspondence over \( M \) is a self-adjoint Hilbert \( C^* \)-correspondence \( E \) over \( M \), such that the map \( M \to \mathcal{L}(E) \) which gives rise to the left action is normal.

**Definition 0.7.** Let \( E \) be a \( C^* \)-correspondence over \( A \), and let \( H \) be a Hilbert space. A pair \(( \sigma, T \)\) is called a completely contractive covariant representation of \( E \) on \( H \) (or, for brevity, a c.c. representation) if

1. \( T : E \to B(H) \) is a completely contractive linear map;
2. \( \sigma : A \to B(H) \) is a nondegenerate \(*\)-homomorphism; and
3. \( T(xa) = T(x)\sigma(a) \) and \( T(a \cdot x) = \sigma(a)T(x) \) for all \( x \in E \) and all \( a \in A \).

If \( A \) is a \( W^* \)-algebra and \( E \) is \( W^* \)-correspondence then we also require that \( \sigma \) be normal.

Given a \( C^* \)-correspondence \( E \) and a c.c. representation \(( \sigma, T \)\) of \( E \) on \( H \), one can form the Hilbert space \( E \otimes_\sigma H \), which is defined as the Hausdorff completion of the algebraic tensor product with respect to the inner product

\[ \langle x \otimes h, y \otimes g \rangle = \langle h, \sigma(\langle x, y \rangle)g \rangle. \]

One then defines \( \tilde{T} : E \otimes_\sigma H \to H \) by

\[ \tilde{T}(x \otimes h) = T(x)h. \]

**Definition 0.8.** A c.c. representation \(( \sigma, T \)\) is called isometric if for all \( x, y \in E \),

\[ T(x)^*T(y) = \sigma(\langle x, y \rangle). \]

(This is the case if and only if \( \tilde{T} \) is an isometry). It is called fully coisometric if \( \tilde{T} \) is a coisometry.

Given two Hilbert \( C^* \)-correspondences \( E \) and \( F \) over \( A \), the balanced (or inner) tensor product \( E \otimes F \) is a Hilbert \( C^* \)-correspondence over \( A \) defined to be the Hausdorff completion of the algebraic tensor product with respect to the inner product

\[ \langle x \otimes y, w \otimes z \rangle = \langle y, \langle x, w \rangle \cdot z \rangle , \ x, w \in E, y, z \in F. \]
The left and right actions are defined as $a \cdot (x \otimes y) = (a \cdot x) \otimes y$ and $(x \otimes y)a = x \otimes (ya)$, respectively, for all $a \in A, x \in E, y \in F$. When working in the context of $W^*$-correspondences, that is, if $E$ and $F$ are $W^*$-correspondences and $A$ is a $W^*$-algebra, then $E \otimes F$ is understood to be the self-dual extension of the above construction.

Detailed overview of the paper. Subproduct systems, their representations, and their units, are defined in the next section. The following two sections, 2 and 3, can be viewed as a reorganization and sharpening of some known results, including several new observations.

Section 2 establishes the correspondence between $cp$-semigroups and subproduct systems. It is shown that given a subproduct system $X$ of $N$-correspondences and a subproduct system representation $R$ of $X$ on $H$, we may construct a $cp$-semigroup $\Theta$ acting on $N'$. We denote this assignment as $\Theta = \Sigma(X, R)$. Conversely, it is shown that given a $cp$-semigroup $\Theta$ acting on $N'$, there is a subproduct system $E$ (called the Arveson-Stinespring subproduct system of $\Theta$) of $M'$-correspondences and an injective representation $T$ of $E$ on $H$ such that $\Theta = \Sigma(E, T)$. Denoting this assignment as $(E, T) = \Xi(\Theta)$, we have that $\Sigma \circ \Xi$ is the identity. In Theorem 2.6 we show that $\Xi \circ \Sigma$ is also, after restricting to pairs $(X, R)$ with $R$ an injective representation (and up to some “isomorphism”), the identity. This allows us to deduce (Corollary 2.8) that a subproduct system that is not a product system has no isometric representations. We introduce the Fock spaces associated to a subproduct system and the canonical shift representations. These constructs allow us to show that every subproduct system is the Arveson-Stinespring subproduct system of some $cp$-semigroup.

In Section 3 we briefly sketch the picture that is dual to that of Section 2. It is shown that given a subproduct system and a unit of that subproduct system one may construct a $cp$-semigroup, and that every $cp$-semigroup arises this way.

In Section 4, we construct for every subproduct system $X$ and every fully coisometric subproduct system representation $T$ of $X$ on a Hilbert space, a semigroup $\hat{T}$ of contractions on a Hilbert space that captures “all the information” about $X$ and $T$. This construction is a modification of the construction introduced in [40] for the case where $X$ is a product system. It turns out that when $X$ is merely a subproduct system, it is hard to apply $\hat{T}$ to obtain new results about the representation $T$. However, when $X$ is a true product system $\hat{T}$ is very handy, and we use it to prove that every $e_0$-semigroup has a $*$-automorphic dilation (in a certain sense).

Section 5 begins with some general remarks regarding dilations and pieces of subproduct system representations, and then the connection between the dilation theories of $cp$-semigroups and of representations of subproduct systems is made. We define the notion of a subproduct subsystem and then we define dilations and pieces of subproduct system representations. These notions generalize the notions of commuting piece or $q$-commuting piece of [12] and [18], and also generalizes the definition of dilation of a product system representation of [28]. Proposition 5.8, Theorem 5.12 and Corollary 5.13 show that the 1-1 correspondences $\Sigma$ and $\Xi$ between $cp$-semigroups and subproduct systems with representations take isometric dilations of representations to $e$-dilations and vice-versa. This is used to obtain an example of three commuting, unital and contractive CP maps on $B(H)$ for which there exists no $e$-dilation acting on a $B(K)$, and no minimal dilation acting on any von Neummann algebra (Theorem 5.14).
In Section \[5\] we also present a reduction of both the problem of constructing an \(e_0\)-dilation to a \(cp_0\)-semigroup, and the problem of constructing an \(e\)-dilation to a \(k\)-tuple of commuting CP maps with small enough norm, to the problem of embedding a subproduct system in a larger product system. We show that not every subproduct system can be embedded in a product system (Proposition \[5.15\]), and we use this to construct an example of three commuting CP maps such that for any \(\lambda > 0\) the three-tuple \(\lambda \theta_1, \lambda \theta_2, \lambda \theta_3\) has no \(e\)-dilation (Theorem \[5.16\]). This unexpected phenomenon has no counterpart in the classical theory of isometric dilations, and provides the first example of a theorem in classical dilation theory that cannot be generalized to the theory of \(e\)-dilations of \(cp\)-semigroups.

The developments described in the first part of the paper indicate that subproduct systems are worthwhile objects of study, but to make progress we must look at plenty concrete examples. In the second part of the paper we begin studying subproduct systems of Hilbert spaces over the semigroup \(\mathbb{N}\). In Section \[6\] we show that every subproduct system (of \(W^\ast\)-correspondences) over \(\mathbb{N}\) is isomorphic to a standard subproduct system, that is, it is a subproduct subsystem of the full product system \(\{E \otimes \mathbb{N}\}_{n \in \mathbb{N}}\) for some \(W^\ast\)-correspondence \(E\). Using the results of the previous section, this gives a new proof to the discrete analogue of Bhat’s Theorem: every \(cp_0\)-semigroup over \(\mathbb{N}\) has an \(e_0\)-dilation. Given a subproduct system we define the standard \(X\)-shift, and we show that if \(X\) is a subproduct subsystem of \(Y\), then the standard \(X\)-shift is the maximal \(X\)-piece of the standard \(Y\)-shift, generalizing and unifying results from \[12, 13, 38\].

In Section \[7\] we explain why subproduct systems are convenient for studying noncommutative projective algebraic geometry. We show that every homogeneous ideal \(I\) in the algebra \(C(x_1, \ldots, x_d)\) of noncommutative polynomials corresponds to a unique subproduct system \(X_I\), and vice-versa. The representations of \(X_I\) on a Hilbert space \(H\) are precisely determined by the \(d\)-tuples in the zero set of \(I\),

\[
Z(I) = \{ \mathbf{T} = (T_1, \ldots, T_d) \in B(H)^d : \forall p \in I.p(\mathbf{T}) = 0 \}. 
\]

A noncommutative version of the Nullstellensatz is obtained, stating that

\[
\{ p \in C(x_1, \ldots, x_d) : \forall \mathbf{T} \in Z(I).p(\mathbf{T}) = 0 \} = I. 
\]

Section \[8\] starts with a review of a powerful tool, Gelu Popescu’s “Poisson Transform” \[37\]. Using this tool we derive some basic results (obtained previously by Popescu in \[38\]) which allow us to identify the operator algebra \(A_X\) generated by the \(X\)-shift as the universal unital operator algebra generated by a row contraction subject to homogeneous polynomial identities. We then prove that every completely bounded representation of a subproduct system \(X\) is a piece of a scaled inflation of the \(X\)-shift, and derive a related “von Neumann inequality”.

In Section \[9\] we discuss the relationship between a subproduct system \(X\) and \(A_X\), the (non-selfadjoint operator algebra generated by the \(X\)-shift). The main result in this section is Theorem \[9.7\] which says that \(X \cong Y\) if and only if \(A_X\) is completely isometrically isomorphic to \(A_Y\) by an isomorphism that preserves the vacuum state. This result is used in Section \[10\] where we study the universal norm closed unital operator algebra generated by a row contraction \((T_1, \ldots, T_d)\) satisfying the relations

\[
T_i.T_j = q_{ij}T_j.T_i, \quad 1 \leq i < j \leq d, 
\]

where \(q = (q_{ij})_{i,j=1}^d \in M_n(\mathbb{C})\) is a matrix such that \(q_{ii} = q_{ij}^{-1}\). These non-selfadjoint analogues of the noncommutative tori, are shown to be classified by
their subproduct systems when \( q_{i,j} \neq 1 \) for all \( i, j \). In particular, when \( d = 2 \), we obtain the universal algebra for the relation

\[
T_1 T_2 = q T_2 T_1,
\]

which we call \( A_q \). It is shown that \( A_q \) is isomorphically isomorphic to \( A_r \) if and only if \( q = r \) or \( q = r^{-1} \).

In Section 11 we describe all standard maximal subproduct systems \( X \) with \( \dim X(1) = 2 \) and \( \dim X(2) = 3 \), and classify their algebras up to isometric isomorphisms.

In the closing section of this paper, Section 12, we find that subproduct systems are also closely related to subshifts and to the subshift \( C^* \)-algebras introduced by K. Matsumoto [27]. We show how every subshift gives rise to a subproduct system, and characterize the subproduct systems that come from subshifts. We use this connection together with the results of Section 8 to describe all representations of subshift \( C^* \)-algebras that come from a subshift of finite type (Theorem 12.7).

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Part 1. Subproduct systems and \( C^\ast \)-semigroups

1. Subproduct systems of Hilbert \( W^\ast \)-correspondences

**Definition 1.1.** Let \( \mathcal{N} \) be a von Neumann algebra. A subproduct system of Hilbert \( W^\ast \)-correspondences over \( \mathcal{N} \) is a family \( X = \{X(s)\}_{s\in\mathcal{S}} \) of Hilbert \( W^\ast \)-correspondences over \( \mathcal{N} \) such that

1. \( X(0) = \mathcal{N} \),
2. For every \( s, t \in \mathcal{S} \) there is a coisometric mapping of \( \mathcal{N} \)-correspondences
   \[ U_{s,t} : X(s) \otimes X(t) \rightarrow X(s+t), \]
3. The maps \( U_{s,0} \) and \( U_{0,s} \) are given by the left and right actions of \( \mathcal{N} \) on \( X(s) \),
4. The maps \( U_{s,t} \) satisfy the following associativity condition:
   \[ U_{s,t+r} (U_{s,t} \otimes I_{X(r)}) = U_{s,t+r} (I_{X(s)} \otimes U_{t,r}) . \]

The difference between a subproduct system and a product system is that in a subproduct system the maps \( U_{s,t} \) are only required to be coisometric, while in a product system these maps are required to be unitaries. Thus, given the image \( U_{s,t}(x\otimes y) \) of \( x\otimes y \) in \( X(s+t) \), one cannot recover \( x \) and \( y \). Thus, subproduct systems may be thought of as irreversible product systems. The terminology is, admittedly, a bit awkward. It may be more sensible – however, impossible at present – to use the term product system for the objects described above and to use the term full product system for product system.

**Example 1.2.** The simplest example of a subproduct system \( F = F_E = \{F(n)\}_{n\in\mathbb{N}} \) is given by

\[ F(n) = E^\otimes n, \]

where \( E \) is some \( W^\ast \)-correspondence. \( F \) is actually a product system. We shall call this subproduct system the full product system (over \( E \)).

**Example 1.3.** Let \( E \) be a fixed Hilbert space. We define a subproduct system (of Hilbert spaces) \( SSP = SSP_E \) over \( \mathbb{N} \) using the familiar symmetric tensor products (one can obtain a subproduct system from the anti-symmetric tensor products as well). Define

\[ E^\otimes n = E \otimes \cdots \otimes E, \]

\( n \) times. Let \( p_n \) be the projection of \( E^\otimes n \) onto the symmetric subspace of \( E^\otimes n \), given by

\[ p_n k_1 \otimes \cdots \otimes k_n = \frac{1}{n!} \sum_{\sigma \in S_n} k_{\sigma^{-1}(1)} \otimes \cdots \otimes k_{\sigma^{-1}(n)}. \]

We define

\[ SSP(n) = E^\otimes n := p_n E^\otimes n, \]

the symmetric tensor product of \( E \) with itself \( n \) times \( (SSP(0) = \mathbb{C}) \). We define a map \( U_{m,n} : SSP(m) \otimes SSP(n) \rightarrow SSP(m+n) \) by

\[ U_{m,n}(x \otimes y) = p_{m+n}(x \otimes y). \]

The \( U \)'s are coisometric maps because every projection, when considered as a map from its domain onto its range, is coisometric. A straightforward calculation shows that (1.1) holds (see \[34, Corollary 17.2\]). In these notes we shall refer to \( SSP \) (or \( SSP_E \) to be precise) as the symmetric subproduct system (over \( E \)).
Definition 1.4. Let $X$ and $Y$ be two subproduct systems over the same semigroup $S$ (with families of coisometries $\{U_{s,t}^X\}_{s,t \in S}$ and $\{U_{s,t}^Y\}_{s,t \in S}$). A family $V = \{V_s\}_{s \in S}$ of maps $V_s : X(s) \to Y(s)$ is called a morphism of subproduct systems if $V_0$ is a unital $*$-isomorphism, if for all $s \in S \setminus \{0\}$ the map $V_s$ is a coisometric correspondence map, and if for all $s,t \in S$ the following identity holds:

\[(1.2) \quad V_{s+t} \circ U_{s,t}^X = U_{s,t}^Y \circ (V_s \otimes V_t).\]

$V$ is said to be an isomorphism if $V_s$ is a unitary for all $s \in S \setminus \{0\}$. $X$ is said to be isomorphic to $Y$ if there exists an isomorphism $V : X \to Y$.

The above notion of morphism is not optimized in any way. It is simply precisely what we need in order to develop dilation theory for $cp$-semigroups.

Definition 1.5. Let $\mathcal{N}$ be a von Neumann algebra, let $H$ be a Hilbert space, and let $X$ be a subproduct system of Hilbert $\mathcal{N}$-correspondences over the semigroup $S$. Assume that $T : X \to B(H)$, and write $T_s$ for the restriction of $T$ to $X(s)$, $s \in S$, and $\sigma$ for $T_0$. $T$ (or $(\sigma, T)$) is said to be a completely contractive covariant representation of $X$ if

1. For each $s \in S$, $(\sigma, T_s)$ is a c.c. representation of $X(s)$; and
2. $T_{s+t}(U_{s,t}(x \otimes y)) = T_s(x)T_t(y)$ for all $s, t \in S$ and all $x \in X(s), y \in X(t)$.

$T$ is said to be an isometric (fully coisometric) representation if it is an isometric (fully coisometric) representation on every fiber $X(s)$.

Since we shall not be concerned with any other kind of representation, we shall call a completely contractive covariant representation of a subproduct system simply a representation.

Remark 1.6. Item 2 in the above definition of product system can be rewritten as follows:

\[\tilde{T}_{s+t}(U_{s,t} \otimes I_H) = \tilde{T}_s(I_{X(s)} \otimes \tilde{T}_t).\]

Here $\tilde{T}_s : X(s) \otimes_\sigma H \to H$ is the map given by

\[\tilde{T}_s(x \otimes h) = T_s(x)h.\]

Example 1.7. We now define a representation $T$ of the symmetric subproduct system $SSP$ from Example 1.3 on the symmetric Fock space. Denote by $\mathfrak{F}_+$ the symmetric Fock space

\[\mathfrak{F}_+ = \bigoplus_{n \in \mathbb{N}} E^{\otimes n}.\]

For every $n \in \mathbb{N}$, the map $T_n : SSP(n) = E^{\otimes n} \to B(\mathfrak{F}_+)$ is defined on the $m$-particle space $E^{\otimes m}$ by putting

\[T_n(x)y = p_{n+m}(x \otimes y)\]

for all $x \in X(n), y \in E^{\otimes m}$. Then $T$ extends to a representation of the subproduct system $SSP$ on $\mathfrak{F}_+$ (to see that item 2 of Definition 1.5 is satisfied one may use again [34 Corollary 17.2]).

Definition 1.8. Let $X = \{X(s)\}_{s \in S}$ be a subproduct system of $\mathcal{N}$-correspondences over $S$. A family $\xi = \{\xi_s\}_{s \in S}$ is called a unit for $X$ if

\[(1.3) \quad \xi_s \otimes \xi_t = U_{s,t}^* \xi_{s+t}.\]
A unit $\xi = \{\xi_s\}_{s \in S}$ is called unital if $\langle \xi_s, \xi_s \rangle = 1_N$ for all $s \in S$, it is called contractive if $\langle \xi_s, \xi_s \rangle \leq 1_N$ for all $s \in S$, and it is called generating if $X(s)$ is spanned by elements of the form

$$(1.4) \quad U_{s_1 + \cdots + s_{n-1}, s_n}(\cdots U_{s_1 + s_2, s_3}(U_{s_1, s_2}(a_1 \xi_{s_1} \otimes a_2 \xi_{s_2}) \otimes a_3 \xi_{s_3}) \otimes \cdots \otimes a_n \xi_{s_n}, a_{n+1}),$$

where $s = s_1 + s_2 + \cdots + s_n$.

From (1.3) follows the perhaps more natural looking

$$U_{s, t}(\xi_s \otimes \xi_t) = \xi_{s + t}.$$

**Example 1.9.** A unital unit for the symmetric subproduct system $SSP$ from Example [1.3] is given by defining $\xi_0 = 1$ and

$$\xi_n = v \otimes v \otimes \cdots \otimes v$$

for $n \geq 1$. This unit is generating only if $E$ is one dimensional.

## 2. Subproduct system representations and $cp$-semigroups

In this section, following Muhly and Solel’s constructions from [28], we show that subproduct systems and their representations provide a framework for dealing with $cp$-semigroups, and allow us to obtain a generalization of the classical result of Wigner that any strongly continuous one-parameter group of automorphisms of $B(H)$ is given by $X \mapsto U_t X U_t^*$ for some one-parameter unitary group $\{U_t\}_{t \in \mathbb{R}}$.

### 2.1. All $cp$-semigroups come from subproduct system representations.

**Proposition 2.1.** Let $N$ be a von Neumann algebra and let $X$ be a subproduct system of $N$-correspondences over $S$, and let $R$ be completely contractive covariant representation of $X$ on a Hilbert space $H$, such that $R_0$ is unital. Then the family of maps

$$(2.1) \quad \Theta_s : a \mapsto \tilde{R}_s(I_{X(s)} \otimes a)\tilde{R}_s^*, \quad a \in R_0(N)'$$

is a semigroup of $CP$ maps on $R_0(N)'$. Moreover, if $R$ is an isometric (a fully coisometric) representation, then $\Theta_s$ is a $*$-endomorphism (a unital map) for all $s \in S$.

**Proof.** By Proposition 2.21 in [28], $\{\Theta_s\}_{s \in S}$ is a family of contractive, normal, completely positive maps on $R_0(N)'$. Moreover, these maps are unital if $R$ is a fully coisometric representation, and they are $*$-endomorphisms if $R$ is an isometric representation. It remains is to check that $\Theta = \{\Theta_s\}_{s \in S}$ satisfies the semigroup condition $\Theta_s \circ \Theta_t = \Theta_{s + t}$. Fix $a \in R_0(N)'$. For all $s, t \in S$,

$$\Theta_s(\Theta_t(a)) = \tilde{R}_s \left( I_{X(s)} \otimes \left( \tilde{R}_t(I_{X(t)} \otimes a)\tilde{R}_t^* \right) \right) \tilde{R}_s^*$$

$$= \tilde{R}_s(I_{X(s)} \otimes \tilde{R}_t)(I_{X(s)} \otimes I_{X(t)} \otimes a)(I_{X(s)} \otimes \tilde{R}_t^*)\tilde{R}_s^*$$

$$= \tilde{R}_{s + t}(U_{s, t} \otimes I_G)(I_{X(s)} \otimes I_{X(t)} \otimes a)(U_{s, t}^* \otimes I_G)\tilde{R}_{s + t}^*$$

$$= \tilde{R}_{s + t}(I_{X(s + t)} \otimes a)\tilde{R}_{s + t}^*$$

$$= \Theta_{s + t}(a).$$
Using the fact that $R_0$ is unital, we have

$$\Theta_0(a)h = \widetilde{R_0}(I_N \otimes a)\widetilde{R_0}^* h = R_0(I_N \otimes a)(1_N \otimes h) = R_0(1_N)ah = ah,$$

thus $\Theta_0(a) = a$ for all $a \in N$. \qed

We will now show that every cp-semigroup is given by a subproduct representation as in (2.1) above. We recall some constructions from [28] (building on the foundations set in [4]).

Fix a CP map $\Theta$ on von Neumann algebra $M \subseteq B(H)$. We define $M \otimes_{\Theta} H$ to be the Hausdorff completion of the algebraic tensor product $M \otimes H$ with respect to the sesquilinear positive semidefinite form

$$\langle T_1 \otimes h_1, T_2 \otimes h_2 \rangle = \langle h_1, \Theta(T_1^* T_2)h_2 \rangle.$$

We define a representation $\pi_{\Theta}$ of $M$ on $M \otimes_{\Theta} H$ by

$$\pi_{\Theta}(S)(T \otimes h) = ST \otimes h,$$

and we define a (contractive) linear map $W_{\Theta} : H \to M \otimes H$ by

$$W_{\Theta}(h) = I \otimes h.$$

If $\Theta$ is unital then $W_{\Theta}$ is an isometry, and if $\Theta$ is an endomorphism then $W_{\Theta}$ is a coisometry. The adjoint of $W_{\Theta}$ is given by

$$W_{\Theta}^*(T \otimes h) = \Theta(T)h.$$

For a given CP semigroup $\Theta$ on $M$, Muhly and Solel defined in [28] a $W^*$-correspondence $E_{\Theta}$ over $M'$ and a c.c. representation $(\sigma, T_{\Theta})$ of $E_{\Theta}$ on $H$ such that for all $a \in M$

$$(2.2) \quad \Theta(a) = \widetilde{T_{\Theta}}(I_{E_{\Theta}} \otimes a) \widetilde{T_{\Theta}}^*.$$

The $W^*$-correspondence $E_{\Theta}$ is defined as the intertwining space

$$E_{\Theta} = \mathcal{L}_M(H, M \otimes_{\Theta} H),$$

where

$$\mathcal{L}_M(H, M \otimes_{\Theta} H) := \{X \in B(H, M \otimes_{\Theta} H) \mid \forall T \in M, XT = \pi_{\Theta}(T)X\}.$$  

The left and right actions of $M'$ are given by

$$S \cdot X = (I \otimes S)X, \quad X \cdot S = XS$$

for all $X \in E_{\Theta}$ and $S \in M'$. The $M'$-valued inner product on $E_{\Theta}$ is defined by $\langle X, Y \rangle = X^* Y$. $E_{\Theta}$ is called the Arveson-Stinespring correspondence (associated with $\Theta$).

The representation $(\sigma, T_{\Theta})$ is defined by letting $\sigma = \text{id}_{M'}$, the identity representation of $M'$ on $H$, and by defining

$$T_{\Theta}(X) = W_{\Theta}^* X.$$

$(\text{id}_{M'}, T_{\Theta})$ is called the identity representation (associated with $\Theta$). We remark that the paper [28] focused on unital CP maps, but the results we cite are true for nonunital CP maps, with the proofs unchanged.
The case where $\mathcal{M} = B(H)$ in the following theorem appears, in essence at least, in [11].

**Theorem 2.2.** Let $\Theta = \{\Theta_s\}_{s \in S}$ be a cp-semigroup on a von Neumann algebra $\mathcal{M} \subseteq B(H)$, and for all $s \in S$ let $E(s) := E_{\Theta_s}$ be the Arveson-Stinespring correspondence associated with $\Theta_s$, and let $T_s := T_{\Theta_s}$ denote the identity representation for $\Theta_s$. Then $E = \{E(s)\}_{s \in S}$ is a subproduct system of $\mathcal{M}'$-correspondences, and $(\text{id}_{\mathcal{M}'}, T)$ is a representation of $E$ on $H$ that satisfies

\begin{equation}
\Theta_s(a) = \tilde{T}_s (I_{E(s)} \otimes a) \tilde{T}_s^*
\end{equation}

for all $a \in \mathcal{M}$ and all $s \in S$. $T_s$ is injective for all $s \in S$. If $\Theta$ is an $e$-semigroup (cp$_0$-semigroup), then $(\text{id}_{\mathcal{M}'}, T)$ is isometric (fully coisometric).

**Proof.** We begin by defining the subproduct system maps $U_{s,t} : E(s) \otimes E(t) \to E(s + t)$. We use the constructions made in [28 Proposition 2.12] and the surrounding discussion. We define

$$U_{s,t} = V_{s,t}^* \Psi_{s,t},$$

where the map

$$\Psi_{s,t} : \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_{\Theta_s} H) \otimes \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_{\Theta_t} H) \to \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_{\Theta_s} \mathcal{M} \otimes_{\Theta_t} H)$$

is given by $\Psi_{s,t}(X \otimes Y) = (I \otimes X)Y$, and

$$V_{s,t} : \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_{\Theta_s} H) \otimes \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_{\Theta_t} H) \to \mathcal{L}_\mathcal{M}(H, \mathcal{M} \otimes_{\Theta_s} \mathcal{M} \otimes_{\Theta_t} H)$$

is given by $V_{s,t}(X) = \Gamma_{s,t} \circ X$, where $\Gamma_{s,t} : \mathcal{M} \otimes_{\Theta_{s+t}} H \to \mathcal{M} \otimes_{\Theta_t} \mathcal{M} \otimes_{\Theta_t} H$ is the isometry

$$\Gamma_{s,t} : S \otimes_{\Theta_{s+t}} h \mapsto S \otimes_{\Theta_t} I \otimes_{\Theta_t} h.$$

By [28 Proposition 2.12], $U_{s,t}$ is a coisometric bimodule map for all $s, t \in S$. To see that the $U$s compose associatively as in [11], take $s, t, u \in S$, $X \in E(s), Y \in E(t), Z \in E(u)$, and compute:

$$U_{s,t+u}(I_{E(s)} \otimes U_{t,u})(X \otimes Y \otimes Z) = U_{s,t+u}(X \otimes V_{t,u}^*(I \otimes Y)Z)$$

$$= V_{s,t+u}^* ((I \otimes X)V_{t,u}^*(I \otimes Y)Z)$$

$$= \Gamma_{s,t+u}^* (I \otimes X)\Gamma_{t,u}^*(I \otimes Y)Z$$

and

$$U_{s+t,u}(U_{s,t} \otimes I_{E(u)})(X \otimes Y \otimes Z) = U_{s+t,u}(V_{s,t}^*(I \otimes X)Y \otimes Z)$$

$$= V_{s+t,u}^* ((I \otimes V_{s,t}^*(I \otimes X)Y)Z)$$

$$= \Gamma_{s+t,u}^* (I \otimes X)\Gamma_{s,t}^*(I \otimes X)(I \otimes Y)Z.$$

So it suffices to show that

$$\Gamma_{s,t+u}^* (I \otimes X)\Gamma_{t,u}^* = \Gamma_{s+t,u}^* (I \otimes \Gamma_{s,t}^*)(I \otimes I \otimes X).$$

It is easier to show that their adjoints are equal. Let $a \otimes h$ be a typical element of $\mathcal{M} \otimes_{\Theta_{s+t+u}} h$.

$$\Gamma_{t,u}(I \otimes X^*)\Gamma_{s,t+u}(a \otimes_{\Theta_{s+t+u}} h) = \Gamma_{t,u}(I \otimes X^*)(a \otimes_{\Theta_{s+t+u}} I \otimes_{\Theta_s} h)$$

$$= \Gamma_{t,u}(a \otimes_{\Theta_{s+t+u}} X^*(I \otimes_{\Theta_s} h))$$

$$= a \otimes_{\Theta_s} I \otimes_{\Theta_s} X^*(I \otimes_{\Theta_s} h).$$
On the other hand

\[(I \otimes I \otimes X^*)(I \otimes \Gamma_{s,t})\Gamma_{s+t,u}(a \otimes \Theta_{s+u+t}) h = (I \otimes I \otimes X^*)(I \otimes \Gamma_{s,t})(a \otimes \Theta_{s+u+t}, h) = (I \otimes I \otimes X^*)(a \otimes \Theta_{u}, I \otimes \Theta_{s+t}, h) = a \otimes \Theta_{u} I \otimes \Theta_{s} X^*(I \otimes \Theta_{s}, h).\]

This shows that the maps \(\{U_{s,t}\}\) make \(E\) into a subproduct system.

Let us now verify that \(T\) is a representation of subproduct systems. That \((\text{id}_{\mathcal{M}^*}, T_s)\) is a c.c. representation of \(E(s)\) is explained in [28, page 878]. Let \(X \in E(s), Y \in E(t)\).

\[T_{s+t}(U_{s,t}(X \otimes Y)) = W_{\Theta_{s+t}}^* \Gamma_{s,t}^*(I \otimes X)Y,\]

while \(T_s(X)T_t(Y) = W_{\Theta_s}^* XW_{\Theta_t}^* Y\).

But for all \(h \in H\),

\[W_{\Theta_s}^* X W_{\Theta_t}^* h = W_{\Theta_s} X^*(I \otimes \Theta_s, h) = I \otimes \Theta_s X^*(I \otimes \Theta_s, h) = (I \otimes X^*)(I \otimes \Theta_s, I \otimes \Theta_s, h) = (I \otimes X^*) \Gamma_{s,t}^*(I \otimes \Theta_{s+t}, h) = (I \otimes X^*) \Gamma_{s,t} W_{\Theta_{s+t}} h,\]

which implies \(W_{\Theta_s}^* X W_{\Theta_t}^* Y = W_{\Theta_{s+t}}^* \Gamma_{s,t}^*(I \otimes X)Y\), so we have the desired equality

\[T_{s+t}(U_{s,t}(X \otimes Y)) = T_s(X)T_t(Y).\]

Equation (2.2) is a consequence of (2.1). The injectivity of the identity representation has already been noted by Solel in [17] (for all \(h, g \in H\) and \(a \in M\),

\[\langle W_{\Theta_s}^* X a^* h, g \rangle = \langle X a^* h, I \otimes g \rangle = \langle (I \otimes a^*) X h, I \otimes g \rangle = \langle X h, a \otimes g \rangle.\]

Finally, assertion of the theorem is trivial (if \(\Theta_s\) is a *-endomorphism, then \(W_{\Theta_s}\) is a coisometry).

\[\Box\]

**Definition 2.3.** Given a cp-semigroup \(\Theta\) on a \(W^*\) algebra \(\mathcal{M}\), the pair \((E, T)\) constructed in Theorem 2.2 is called the identity representation of \(\Theta\), and \(E\) is called the Arveson-Stinespring subproduct system associated with \(\Theta\).

**Remark 2.4.** If follows from [28, Proposition 2.14], if \(\Theta\) is an \(e\)-semigroup, then the identity representation subproduct system is, in fact, a (full) product system.

**Remark 2.5.** In [26], Daniel Markiewicz has studied the Arveson-Stinespring subproduct system of a \(\text{CP}_0\)-semigroup over \(\mathbb{R}_+\) acting on \(B(H)\), and has also shown that it carries a structure of a measurable Hilbert bundle.

2.2.** Essentially all injective subproduct system representations come from cp-semigroups.** The following generalizes and is motivated by [17, Proposition 5.7]. We shall also repeat some arguments from [31, Theorem 2.1].

By Theorem 2.2 with every cp-semigroup \(\Theta = \{\Theta_s\}_{s \in S}\) on \(\mathcal{M} \subseteq B(H)\) we can associate a pair \((E, T)\) - the identity representation of \(\Theta\) - consisting of a subproduct system \(E\) (of correspondences over \(\mathcal{M}'\)) and an injective subproduct system c.c. representation \(T\). Let us write \((E, T) = \Xi(\Theta)\). Conversely, given a pair \((X, R)\) consisting of a subproduct system \(X\) of correspondences over \(\mathcal{M}'\) and a c.c. representation \(R\) of \(X\) such that \(R_0 = \text{id}\), one may define by equation (2.1)
a cp-semigroup $\Theta$ acting on $\mathcal{M}$, which we denote as $\Theta = \Sigma(X, R)$. The meaning of equation (2.3) is that $\Sigma \circ \Xi$ is the identity map on the set of cp-semigroups of $\mathcal{M}$. We will show below that $\Xi \circ \Sigma$, when restricted to pairs $(X, R)$ such that $R$ is injective, is also, essentially, the identity. When $(X, R)$ is not injective, we will show that $\Xi \circ \Sigma(X, R)$ “sits inside” $(X, R)$.

**Theorem 2.6.** Let $\mathcal{N}$ be a $W^*$-algebra, let $X = \{X(s)\}_{s \in S}$ be a subproduct system of $\mathcal{N}$-correspondences, and let $R$ be a c.c. representation of $X$ on $H$, such that $\sigma := R_0$ is faithful and nondegenerate. Let $\mathcal{M} \subseteq B(H)$ be the commutant $\sigma(\mathcal{N})'$ of $\sigma(\mathcal{N})$. Let $\Theta = \Sigma(X, R)$, and let $(E, T) = \Xi(\Theta)$. Then there is a morphism of subproduct systems $W : X \to E$ such that

$$R_s = T_s \circ W_s, \ s \in S.$$ 

$W_s^* W_s = I_{X(s)} - q_s$, where $q_s$ is the orthogonal projection of $X(s)$ onto $\ker R_s$. In particular, $W$ is an isomorphism if and only if $R_s$ is injective for all $s \in S$.

**Remark 2.7.** The construction of the morphism $W$ below basically comes from [47, 31], and it remains only to show that it respects the subproduct system structure. The details are carried out for completeness.

**Proof.** We may construct a subproduct system $X'$ of $\mathcal{M}'$-correspondences (recall that $\mathcal{M}' = \sigma(\mathcal{N})$), and a representation $T'$ of $X'$ on $H$ such that $T'_s$ is the identity, in such a way that $(X, T)$ may be naturally identified with $(X', T')$. Indeed, put

$$X'(0) = \mathcal{M}' \quad X'(s) = X(s) \text{ for } s \neq 0,$$

where the inner product is defined by

$$\langle x, y \rangle_{X'} = \sigma(\langle x, y \rangle_X),$$

and the left and right actions are defined by

$$a \cdot x \cdot b := \sigma^{-1}(a)x\sigma^{-1}(b),$$

for $a, b \in \mathcal{M}'$ and $x, y \in X'(s), s \in S \setminus \{0\}$. Defining $T'_0 = id$ and $W_0 = \sigma$; and $T'_s = T_s$ for and $W_s = id$ for $s \in S \setminus \{0\}$, we have that $W$ is a morphism $X \to X'$ that sends $T$ to $T'$.

Assume, therefore, that $\mathcal{N} = \mathcal{M}'$ and that $\sigma$ is the identity representation.

We begin by defining for every $s \neq 0$

$$v_s : \mathcal{M} \otimes_{\Theta_s} H \to X(s) \otimes H$$

by

$$v_s(a \otimes h) = (I_{X(s)} \otimes a)\bar{R}_s^* h.$$ 

We claim that for all $s \in S$ the map $v_s$ is a well-defined isometry. To see that, let $a, b \in \mathcal{M}$ and $h, g \in H$, and compute:

$$\langle a \otimes_{\Theta_s} h, b \otimes_{\Theta_s} g \rangle = \langle h, \Theta_s(a^* b)g \rangle$$

$$= \langle h, \bar{R}_s(I_{X(s)} \otimes a^* b)\bar{R}_s^* g \rangle$$

$$= \langle (I_{X(s)} \otimes a)\bar{R}_s^* h, (I_{X(s)} \otimes b)\bar{R}_s^* g \rangle.$$ 

$$[I_{X(s)} \mathcal{M}\bar{R}_s^* H]$$ is invariant under $I_{X(s)} \mathcal{M}$, thus the projection onto the orthocomplement of this subspace is in $(I_{X(s)} \mathcal{M})' = \mathcal{L}(X(s)) \otimes I_H$, so it has
the form \( q_s \otimes I_H \) for some projection \( q_s \in \mathcal{L}(X(s)) \). In fact, \( q_s \) is the orthogonal projection of \( X(s) \) onto \( \text{Ker} R_s \); for all \( g, h \in H, \ a \in \mathcal{M}, \)
\[
\langle \xi \otimes h, (I \otimes a)\bar{R}_s^*g \rangle = \langle \bar{R}_s(\xi \otimes a^*h), g \rangle = \langle R_s(\xi)a^*h, g \rangle,
\]
thus, \( R_s(\xi) = 0 \) if and only if \( \xi \otimes h \in (\text{Im} v_s)^{-1} \) for all \( h \in H \), that is, \( \xi \in q_s X(s) \).

By the definition of \( v_s \) and by the covariance properties of \( T \), we have for all \( a \in \mathcal{M} \) and \( b \in \mathcal{M}' \),
\[
v_s(a \otimes I) = (I \otimes a)v_s, \ v_s(I \otimes b) = (b \otimes I)v_s.
\]

Fix \( s \in \mathcal{S} \) and \( x \in E(s) \). For all \( \xi \in X(s), h \in H \), write
\[
\psi(\xi)h = x^*v_s^*(\xi \otimes h).
\]

For \( a \in \mathcal{M} \) we have
\[
\psi(\xi)ah = x^*v_s^*(\xi \otimes ah) = x^*v_s^*(I \otimes a)(\xi \otimes h) = x^*(a \otimes I)v_s^*(\xi \otimes h) = ax^*v_s^*(\xi \otimes h) = a\psi(\xi)h.
\]

Thus the linear map \( \xi \mapsto \psi(\xi) \) maps from \( X(s) \) into \( \mathcal{M}' \) and it is apparent that this map is bounded. \( \psi \) is also a right module map: for all \( b \in \mathcal{M}' \), \( \psi(\xi)bh = x^*v_s^*(\xi \otimes bh) = x^*v_s^*(\xi \otimes bh) = \psi(\xi)bh \). From the self duality of \( X(s) \) it follows that there is a unique element in \( X(s) \), which we denote by \( V_s(x) \), such that for all \( \xi \in X(s), h \in H \),
\[
(V_s(x), \xi)h = x^*v_s^*(\xi \otimes h).
\]

For \( a, b \in \mathcal{M}', \ \xi \in X(s) \) and \( h \in H \), we have
\[
\langle V_s(axb), \xi \rangle h = \langle V_s((I \otimes a)xb), \xi \rangle h = b^*x^*(I \otimes a^*)v_s^*(\xi \otimes h) = b^*x^*v_s^*(a^*\xi \otimes h) = b^*\langle V_s(x), a^*\xi \rangle h = \langle (aV_s(x)b), \xi \rangle h.
\]

That is, \( V_s(axb) = aV_s(x)b \). In a similar way one proves that \( V_s : E(s) \to X(s) \) is linear.

Let us now show that \( V_s \) preserves inner products. For all \( \xi \in X(s) \), write \( L_\xi \) for the operator \( L_\xi : H \to X(s) \otimes H \) that maps \( h \) to \( \xi \otimes h \). One checks that
\[
L_\xi^*(\eta \otimes h) = (\xi, \eta)h, \text{ so equation (2.5) becomes}
\]
\[
L_{V_s(x)}^*L_\xi = x^*v_s^*L_\xi, \ \xi \in X(s),
\]
or \( L_{V_s(x)} = v_sx \), for all \( x \in E(s) \). But since \( v_s \) preserves inner products, we have for all \( x, y \in E(s) \):
\[
\langle x, y \rangle = x^*y = x^*v_s^*v_sy = L_{V_s(x)}^*L_{V_s(y)} = \langle V_s(x), V_s(y) \rangle.
\]

We now prove that \( V_sV_s^* = I_{X(s)} - q_s \), \( \xi \in \text{Im} V_s^+ \) if and only if \( L_\xi^*v_sE(s)H = 0 \). But by [28] Lemma 2.10, \( E(s)H = \mathcal{M} \otimes_{\Theta} H \), thus \( L_\xi^*v_sE(s)H = 0 \) if and only if \( \langle \xi, \eta \rangle = 0 \) for all \( \eta \in (I_{X(s)} - q_s)X(s) \), which is the same as \( \xi \in q_s X(s) \).
Fix \( h, k \in H \). For \( x \in E(s) \), we compute:

\[
\langle T_s(x)h, k \rangle = \langle W^*_s xh, k \rangle = \langle xh, I \otimes \Theta_s k \rangle = \langle v_s xh, v_s(I \otimes \Theta_s k) \rangle = \langle V_s(x) \otimes h, \tilde{R}^*_s k \rangle = \langle \tilde{R}_s(V_s(x))h, k \rangle,
\]

thus \( T_s = R_s \circ V_s \) for all \( s \in S \). Define \( W_s = V^*_s \). Then \( T_s = R_s \circ W^*_s \). Multiplying both sides by \( W_s \) we obtain \( T_s \circ W_s = R_s \circ W^*_s W_s \). But \( W^*_s W_s = I - q_s \) is the orthogonal projection onto \((\text{Ker} R_s)^\perp\), thus we obtain \( (2.1) \).

Finally, we need to show that \( W = \{ W_s \} \) respects the subproduct system structure: for all \( s, t \in S \), \( x \in X(s) \) and \( y \in X(t) \), we must show that

\[ W_{s+t}(U^X_{s,t}(x \otimes y)) = U^E_{s,t}(W_s(x) \otimes W_t(y)). \]

Since \( T_{s+t} \) is injective, it is enough to show that after applying \( T_{s+t} \) to both sides of the above equation we get the same thing. But \( T_{s+t} \) applied to the left hand side gives

\[ T_{s+t}W_{s+t}(U^X_{s,t}(x \otimes y)) = R_{s+t}(U^X_{s,t}(x \otimes y)) = R_s(x)R_t(y), \]

and \( T_{s+t} \) applied to the right hand side gives

\[ T_{s+t}(U^E_{s,t}(W_s(x) \otimes W_t(y))) = T_s(W_s(x))T_t(W_t(y)) = R_s(x)R_t(y). \]

\[ \square \]

**Corollary 2.8.** Let \( X \) be a subproduct system that has an isometric representation \( V \) such that \( V_0 \) is faithful and nondegenerate. Then \( X \) is a (full) product system.

**Proof.** Let \( \Theta = \Sigma(X, V) \). Then \( \Theta \) is an \( e \)-semigroup. Thus, if \( (E, T) = \Xi(\Theta) \) is the identity representation of \( \Theta \), then, by Remark 2.4, \( E \) is a (full) product system. But if \( V_0 \) is faithful and \( V \) is isometric then \( V \) is injective. By the above theorem, \( X \) is isomorphic to \( E \), so it is a product system. \[ \square \]

### 2.3. Subproduct systems arise from cp-semigroups. The shift representation.

A question rises: does every subproduct system arise as the Arveson-Stinespring subproduct system associated with a cp-semigroup? By Theorem 2.6 this is equivalent to the question does every subproduct system have an injective representation? We shall answer this question in the affirmative by constructing for every such subproduct system a canonical injective representation.

The following constructs will be of most interest when \( S \) is a countable semigroup, such as \( \mathbb{N}^k \).

**Definition 2.9.** Let \( X = \{X(s)\}_{s \in S} \) be a subproduct system. The \( X \)-Fock space \( \mathcal{F}_X \) is defined as

\[ \mathcal{F}_X = \bigoplus_{s \in S} X(s). \]

The vector \( \Omega := 1 \in N = X(0) \) is called the vacuum vector of \( \mathcal{F}_X \). The \( X \)-shift representation of \( X \) on \( \mathcal{F}_X \) is the representation

\[ S^X : X \rightarrow B(\mathcal{F}_X), \]

given by \( S^X(x)y = U^X_{s,t}(x \otimes y) \), for all \( x \in X(s), y \in X(t) \) and all \( s, t \in S \).
Strictly speaking, $S^X$ as defined above is not a representation because it represents $X$ on a C*-correspondence rather than on a Hilbert space. However, since for any C*-correspondence $E$, $L(E)$ is a C*-algebra, one can compose a faithful representation $\pi : L(E) \to B(H)$ with $S^X$ to obtain a representation on a Hilbert space.

A direct computation shows that $\tilde{S}^X : X(s) \otimes \mathcal{F}_X \to \mathcal{F}_X$ is a contraction, and also that $S^X(x)S^X(y) = S^X(U_{s,t}^X(x \otimes y))$ so $S^X$ is a completely contractive representation of $X$. $S^X$ is also injective because $S^X(x)\Omega = x$ for all $x \in X$. Thus,

**Corollary 2.10.** Every subproduct system is the Arveson-Stinespring subproduct system of a cp-semigroup.

### 3. Subproduct System Units and cp-semigroups

In this section, following Bhat and Skeide’s constructions from [14], we show that subproduct systems and their units may also serve as a tool for studying cp-semigroups.

**Proposition 3.1.** Let $N$ be a von Neumann algebra and let $X$ be a subproduct system of $N$-correspondences over $S$, and let $\xi = \{ \xi_s \}_{s \in S}$ be a contractive unit of $X$, such that $\xi_0 = 1_N$. Then the family of maps

$$\Theta_s : a \mapsto \langle \xi_s, a \xi_s \rangle,$$

is a semigroup of CP maps on $N$. Moreover, if $\xi$ is unital, then $\Theta_s$ is a unital map for all $s \in S$.

**Proof.** It is standard that $\Theta_s$ given by (3.1) is a contractive completely positive map on $N$, which is unital if and only if $\xi$ is unital. The fact that $\Theta_s$ is normal goes a little bit deeper, but is also known (one may use [28, Remark 2.4(i)]).

We show that $\{ \Theta_s \}_{s \in S}$ is a semigroup. It is clear that $\Theta_0(a) = a$ for all $a \in N$. For all $s, t \in S$,

$$\Theta_s(\Theta_t(a)) = \langle \xi_s, \langle \xi_t, a \xi_t \rangle \xi_s \rangle$$

$$= \langle \xi_t \otimes \xi_s, a \xi_t \otimes \xi_s \rangle$$

$$= \langle U_{t,s}^* \xi_{s+t}, aU_{t,s}^* \xi_{s+t} \rangle$$

$$= \langle \xi_{s+t}, a \xi_{s+t} \rangle$$

$$= \Theta_{s+t}(a).$$

\[ \square \]

We recall a central construction in Bhat and Skeide’s approach to dilation of cp-semigroup [14], that goes back to Paschke [35]. Let $\mathcal{M}$ be a $W^*$-algebra, and let $\Theta$ be a normal completely positive map on $\mathcal{M}$. The GNS representation of $\Theta$ is a pair $(F_\Theta, \xi_\Theta)$ consisting of a Hilbert $W^*$-correspondence $F_\Theta$ and a vector $\xi_\Theta \in F_\Theta$ such that

$$\Theta(a) = \langle \xi_\Theta, a \xi_\Theta \rangle \quad \text{for all } a \in \mathcal{M}.$$

\[2\text{The construction works also for completely positive maps on C*-algebras, but in Theorem 3.2 below we will need to work with normal maps on } W^*-\text{algebras.} \]
$F_\Theta$ is defined to be the correspondence $\mathcal{M} \otimes_{\Theta} \mathcal{M}$ - which is the self-dual extension of the Hausdorff completion of the algebraic tensor product $\mathcal{M} \otimes \mathcal{M}$ with respect to inner product

$$\langle a \otimes b, c \otimes d \rangle = b^* \Theta(a^* c)d.$$

$\xi_\Theta$ is defined to be $\xi_\Theta = 1 \otimes 1$. Note that $\xi_\Theta$ is a unit vector, that is -$\langle \xi_\Theta, \xi_\Theta \rangle = 1$, if and only if $\Theta$ is unital.

**Theorem 3.2.** Let $\Theta = \{\Theta_s\}_{s \in \mathcal{S}}$ be a cp-semigroup on a $W^*$-algebra $\mathcal{M}$. For every $s \in \mathcal{S}$ let $(F(s), \xi_s)$ be the GNS representation of $\Theta_s$. Then $F = \{F(s)\}_{s \in \mathcal{S}}$ is a subproduct system of $\mathcal{M}$-correspondences, and $\xi = \{\xi_s\}_{s \in \mathcal{S}}$ is a generating contractive unit for $F$ that gives back $\Theta$ by the formula

$$\Theta_s(a) = \langle \xi_s, a\xi_s \rangle \text{ for all } a \in \mathcal{M}. \tag{3.2}$$

$\Theta$ is a cp0-semigroup if and only if $\xi$ is a unital unit.

**Proof.** For all $s, t \in \mathcal{S}$ define a map $V_{s,t} : F(s + t) \to F(s) \otimes F(t)$ by sending $\xi_{s+t}$ to $\xi_s \otimes \xi_t$ and extending to a bimodule map. Because

$$\langle a\xi_s \otimes \xi_t b, c\xi_s \otimes \xi_t d \rangle = \langle \xi_t b, \langle a\xi_s, c\xi_s \rangle \xi_t d \rangle = \langle \xi_t b, \Theta_s(a^* c)\xi_t d \rangle = b^* \langle \xi_t, \Theta_s(a^* c)\xi_t \rangle d = b^* \Theta_t(a^* c)d = \langle a\xi_t + b, c\xi_t \rangle,$$

$V_{s,t}$ extends to a well defined isometric bimodule map from $F(s+t)$ into $F(s) \otimes F(t)$. We define the map $U_{s,t}$ to be the adjoint of $V_{s,t}$ (here it is important that we are working with $W^*$ algebras - in general, an isometry from one Hilbert $C^*$-module into another need not be adjointable, but bounded module maps between self-dual Hilbert modules are always adjointable, [35 Proposition 3.4]). The collection $\{U_{s,t}\}_{s,t \in \mathcal{S}}$ makes $F$ into a subproduct system. Indeed, these maps are coisometric by definition, and they compose in an associative manner. To see the latter, we check that $(I_{F(r)} \otimes V_{s,t})V_{r,s+t} = (I_{F(r)} \otimes I_{F(t)})V_{r+s,t}$ and take adjoints.

$$(I_{F(r)} \otimes V_{s,t})V_{r,s+t}(a\xi_{r+s+t}b) = (I_{F(r)} \otimes V_{s,t})(a\xi_r \otimes \xi_{s+t}b) = a\xi_r \otimes \xi_s \otimes \xi_t b.$$ 

Similarly, $(V_{r,s} \otimes I_{F(s)})V_{r+s,t}(a\xi_{r+s+t}b) = a\xi_r \otimes \xi_s \otimes \xi_t b$. Since $F(r + s + t)$ is spanned by linear combinations of elements of the form $a\xi_{r+s+t}b$, the $U$’s make $F$ into a subproduct system, and $\xi$ is certainly a unit for $F$. Equation (3.2) follows by definition of the GNS representation. Now,

$$\langle \xi_s, \xi_s \rangle = \Theta_s(1), \text{ } s \in \mathcal{S},$$

so $\xi$ is a contractive unit because $\Theta_s(1) \leq 1$, and $\xi$ it is unital if and only if $\Theta_s$ is unital for all $s$. $\xi$ is in fact more then just a generating unit, as $F(s)$ is spanned by elements with the form described in equation (1.4) with $(s_1, \ldots, s_n)$ a fixed $n$-tuple such that $s_1 + \cdots + s_n = s$. \hfill \square

**Definition 3.3.** Given a cp-semigroup $\Theta$ on a $W^*$ algebra $\mathcal{M}$, the subproduct system $F$ and the unit $\xi$ constructed in Theorem 3.2 are called, respectively, the GNS subproduct system and the GNS unit of $\Theta$. The pair $(F, \xi)$ is called the GNS representation of $\Theta$. 

Remark 3.4. There is a precise relationship between the identity representation (Definition 2.3) and the GNS representation of a cp-semigroup. The GNS representation of a CP map is the dual of the identity representation in a sense that is briefly described in [30]. This notion of duality has been used to move from the product-system-and-representation picture to the product-system-with-unit picture, and vice versa. See for example [43] and the references therein. It is more-or-less straightforward to use this duality to get Theorem 3.2 from Theorem 2.2 (or the other way around).

4. ∗-AUTOMORPHIC DILATION OF AN $e_0$-SEMIGROUP

We now apply some of the tools developed above to dilate an $e_0$-semigroup to a semigroup of ∗-automorphisms. We shall need the following proposition, which is a modification (suited for subproduct systems) of the method introduced in [40] for representing a product system representation as a semigroup of contractive operators on a Hilbert space.

Proposition 4.1. Let $\mathcal{N}$ be a von Neumann algebra and let $X$ be a subproduct system of unital $\mathcal{W}^*$-correspondences over $\mathcal{S}$. Let $(\sigma, T)$ be a fully coisometric covariant representation of $X$ on the Hilbert space $\mathcal{H}$, and assume that $\sigma$ is unital. Denote

$$H_s := (X(s) \otimes_{\sigma} H) / \text{Ker} \tilde{T}_s$$

and

$$\mathcal{H} = \bigoplus_{s \in \mathcal{S}} H_s.$$  

Then there exists a semigroup of coisometries $\tilde{T} = \{\tilde{T}_s\}_{s \in \mathcal{S}}$ on $\mathcal{H}$ such that for all $s \in \mathcal{S}$, $x \in X(s)$ and $h \in H$,

$$\tilde{T}_s (\delta_s \cdot x \otimes h) = T_s(x)h.$$  

$\tilde{T}$ also has the property that for all $s \in \mathcal{S}$ and all $t \geq s$

$$(4.1) \quad \tilde{T}_s^* \tilde{T}_t |_{H_s} = I_{H_t}, \quad (t \geq s).$$

Proof. First, we note that the assumptions on $\sigma$ and on the left action of $\mathcal{N}$ imply that $H_0 \cong H$ via the identification $a \otimes h \leftrightarrow \sigma(a)h$. This identification will be made repeatedly below.

Define $\mathcal{H}_0$ to be the space of all finitely supported functions $f$ on $\mathcal{S}$ such that for all $s \in \mathcal{S}$,

$$f(s) \in H_s.$$  

We equip $\mathcal{H}_0$ with the inner product

$$\langle \delta_s \cdot \xi, \delta_t \cdot \eta \rangle = \delta_{s,t} \langle \xi, \eta \rangle,$$

for all $s, t \in \mathcal{S}, \xi \in H_s, \eta \in H_t$. Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ with respect to this inner product. We have

$$\mathcal{H} \cong \bigoplus_{s \in \mathcal{S}} H_s.$$  

It will sometimes be convenient to identify the subspace $\delta_s \cdot H_s \subseteq \mathcal{H}$ with $H_s$, and for $s = 0$ this gives us an inclusion $H \subseteq \mathcal{H}$. We define a family $\tilde{T} = \{\tilde{T}_s\}_{s \in \mathcal{S}}$ of

---

3 An $\mathcal{N}$-correspondence is said to be unital if the left action of $\mathcal{N}$ is unital. The right action of every unital $C^*$-algebra on every Hilbert $C^*$-correspondence is unital.
operators on $\mathcal{H}_0$ as follows. First, we define $\hat{T}_0$ to be the identity. Now assume that $s > 0$. If $t \in S$ and $t \not\geq s$, then we define $\hat{T}_s(\delta_t \cdot \xi) = 0$ for all $\xi \in H_t$. If $t \geq s > 0$ we would like to define (as we did in [40])

$$\hat{T}_s(\delta_t \cdot (x_{t-s} \otimes x_s \otimes h)) = \delta_{t-s} \cdot \left( x_{t-s} \otimes \hat{T}_s(x_s \otimes h) \right),$$

but since $X$ is not a true product system, we cannot identify $X(t-s) \otimes X(s)$ with $X(t)$. For a fixed $t > 0$, we define for all $s \leq t$, $\xi \in X(t)$ and $h \in H$

$$\hat{T}_s(\delta_t \cdot (\xi \otimes h)) = \delta_{t-s} \cdot \left( (I_{X(t-s)} \otimes \hat{T}_s)(U^*_{t-s, s}\xi \otimes h) \right).$$

$\hat{T}_s$ can be extended to a well defined contraction from $X(t) \otimes H$ to $X(t-s) \otimes H$, for all $t \geq s$, and has an adjoint given by

$$\hat{T}_s^* \delta_{t-s} \cdot \eta \otimes h = \delta_t \cdot \left( (U_{t-s, s} \otimes I_H)(\eta \otimes \hat{T}_s^* h) \right).$$

We are going to obtain $\hat{T}_s$ as the map $H_t \rightarrow H_{t-s}$ induced by $\hat{T}_s$. Let $Y \in H_t$ satisfy $\tilde{T}_1(Y) = 0$. We shall show that $\tilde{T}_s \delta_t \cdot Y = 0$ in $\delta_{t-s} \cdot H_{t-s}$. But

$$\tilde{T}_s \delta_t \cdot Y = \delta_{t-s} \cdot \left( (I_{X(t-s)} \otimes \tilde{T}_s)(U^*_{t-s, s} \otimes I_H)Y \right),$$

and

$$\tilde{T}_{t-s} \left( (I_{X(t)} \otimes \tilde{T}_s)(U^*_{t-s, s} \otimes I_H)Y \right) \in (\tilde{T}_1(U_{t-s, s} \otimes I_H)(U^*_{t-s, s} \otimes I_H)Y \in \tilde{T}_1(Y) = 0,$n

where the equation marked by (*) follows from the fact that $T$ is a representation of subproduct systems, and the one marked by (**) follows from the fact that $U_{t-s, s}$ is a coisometry. Thus, for all $s, t \in S$,

$$\hat{T}_s \left( \delta_t \cdot \text{Ker}\tilde{T}_t \right) \subseteq \delta_{t-s} \cdot \text{Ker}\tilde{T}_{t-s},$$

thus $\hat{T}_s$ induces a well defined contraction $\hat{T}_s$ on $\mathcal{H}$ given by

$$\hat{T}_s(\delta_t \cdot (\xi \otimes h)) = \delta_{t-s} \cdot \left( (I_{X(t-s)} \otimes \tilde{T}_s)(U^*_{t-s, s}\xi \otimes h) \right),$$

where $\xi \otimes h$ and $(I_{X(t-s)} \otimes \tilde{T}_s)(U^*_{t-s, s}\xi \otimes h)$ stand for these elements’ equivalence classes in $(X(t) \otimes H)/\text{Ker}\tilde{T}_t$ and $(X(t-s) \otimes H)/\text{Ker}\tilde{T}_{t-s}$, respectively. It follows that we have the following, more precise, variant of (1.2):

$$\hat{T}_s(\delta_t \cdot (U_{t-s, s}(x_{t-s} \otimes x_s) \otimes h)) = \delta_{t-s} \cdot \left( x_{t-s} \otimes \hat{T}_s(x_s \otimes h) \right).$$

In particular,

$$\hat{T}_s(\delta_s \cdot x_s \otimes h) = T_s(x_s)h,$$

for all $s \in S, x_s \in X(s), h \in H$.

It will be very helpful to have a formula for $\hat{T}_s^*$ as well. Assume that $\sum_i \xi_i \otimes h_i \in \text{Ker}\tilde{T}_t$.

$$\hat{T}_s^* \left( \delta_t \cdot \sum_i \xi_i \otimes h_i \right) = \delta_{s+t} \cdot \left( (U_{t,s} \otimes I_H) \sum_i \xi_i \otimes \tilde{T}_s^* h_i \right),$$
and applying $T_{s+t}$ to the right hand side (without the $\delta$) we get
\[
T_{s+t} \left( (U_{t,s} \otimes I_H)(\sum_i \xi_i \otimes \tilde{T}_s^* h_i) \right) = T_t(I_{X(t)} \otimes \tilde{T}_s)(\sum_i \xi_i \otimes \tilde{T}_s^* h_i)
\]
\[
= T_t(\sum_i \xi_i \otimes \tilde{T}_s \tilde{T}_s^* h_i)
\]
\[
= T_t(\sum_i \xi_i \otimes h_i) = 0,
\]
because $T$ is a fully coisometric representation. So
\[
\hat{T}_s^* \left( \delta_t \cdot \text{Ker} \tilde{T}_t \right) \subseteq \delta_{s+t} \cdot \text{Ker} \tilde{T}_{s+t},
\]
and this means that $\hat{T}_s^*$ induces on $\mathcal{H}$ a well defined contraction which is equal to $\hat{T}_s^*$, and is given by the formula (1.3).

We now show that $\hat{T}$ is a semigroup. Let $s, t, u \in S$. If either $s = 0$ or $t = 0$ then it is clear that the semigroup property $\hat{T}_s \hat{T}_t = \hat{T}_{s+t}$ holds. Assume that $s, t > 0$. If $u \not\geq s + t$, then both $\hat{T}_s \hat{T}_t$ and $\hat{T}_{s+t}$ annihilate $\delta_u \cdot \xi$, for all $\xi \in H_u$. Assuming $u \geq s + t$, we shall show that $\hat{T}_s \hat{T}_t$ and $\hat{T}_{s+t}$ agree on elements of the form
\[
Z = \delta_u \cdot (U_{u-t,s} (x_{u-t-s} \otimes x_s) \otimes x_t) \otimes h,
\]
and since the set of all such elements is total in $H_u$, this will establish the semigroup property.
\[
\hat{T}_s \hat{T}_t Z = \hat{T}_s \left( \delta_{u-t} \left( x_{u-t-s} \otimes x_s \otimes \tilde{T}_t (x_t \otimes h) \right) \right)
\]
\[
= \delta_{u-s-t} \left( x_{u-s-t} \otimes \tilde{T}_s (x_s \otimes \tilde{T}_t (x_t \otimes h)) \right)
\]
\[
= \delta_{u-s-t} \left( x_{u-s-t} \otimes \tilde{T}_s (I \otimes \tilde{T}_t) (x_s \otimes x_t \otimes h) \right)
\]
\[
= \delta_{u-s-t} \left( x_{u-s-t} \otimes \tilde{T}_{s+t} (U_{s,t} (x_s \otimes x_t) \otimes h) \right)
\]
\[
= \hat{T}_{s+t} \delta_u \cdot (U_{u-t-s,t+s} (x_{u-s-t} \otimes U_{s,t} (x_s \otimes x_t)) \otimes h)
\]
\[
= \hat{T}_{s+t} Z.
\]
The final equality follows from the associativity condition (1.1).

To see that $\hat{T}$ is a semigroup of coisometries, we take $\xi \in X(t), h \in H$, and compute
\[
\hat{T}_t \left( \hat{T}_s \hat{T}_s^* \delta_t \cdot (\xi \otimes h) \right) = \hat{T}_t \left( (I_{X(t)} \otimes \tilde{T}_s) (U_{t,s} \otimes I_H) (I_{X(t)} \otimes \tilde{T}_s^*) (\xi \otimes h) \right)
\]
\[
= \hat{T}_{s+t} (U_{t,s} \otimes I_H) (I_{X(t)} \otimes \tilde{T}_s^*) (\xi \otimes h)
\]
\[
= \hat{T}_t (\xi \otimes \hat{T}_s \hat{T}_s^* h) = \hat{T}_t (\xi \otimes h),
\]
so $\hat{T}_s \hat{T}_s^*$ is the identity on $H_t$ for all $t \in S$, thus $\hat{T}_s \hat{T}_s^* = I_H$. Equation (1.1) follows by a similar computation, which is omitted.

We can now obtain a $*$-automorphic dilation for any $c_0$-semigroup over any subsemigroup of $\mathbb{R}_+$. The following result should be compared with similar-looking results of Arveson-Kishimoto \cite{8}, Laca \cite{24}, Skeide \cite{44}, and Arveson-Courtney \cite{7} (none of these cited results is strictly stronger or weaker than the result we obtain for the case of $c_0$-semigroups).
Theorem 4.2. Let \( \Theta \) be a \( c_0 \)-semigroup acting on a von Neumann algebra \( \mathcal{M} \). Then \( \Theta \) can be dilated to a semigroup of \( * \)-automorphisms in the following sense: there is a Hilbert space \( \mathcal{K} \), an orthogonal projection \( p \) of \( \mathcal{K} \) onto a subspace \( \mathcal{H} \) of \( \mathcal{K} \), a normal, faithful representation \( \varphi : \mathcal{M} \to B(\mathcal{K}) \) such that \( \varphi(1) = p \), and a semigroup \( \alpha = \{ \alpha_s \}_{s \in \mathcal{S}} \) of \( * \)-automorphisms on \( B(\mathcal{K}) \) such that for all \( a \in \mathcal{M} \) and all \( s \in \mathcal{S} \)

\[
\alpha_s(\varphi(a))|_{\mathcal{H}} = \varphi(\Theta_s(a)),
\]

so, in particular,

\[
p\alpha_s(\varphi(a))p = \varphi(\Theta_s(a)).
\]

The projection \( p \) is increasing for \( \alpha \), in the sense that for all \( s \in \mathcal{S} \),

\[
\alpha_s(p) \geq p.
\]

Remark 4.3. Another way of phrasing the above theorem is by using the terminology of “weak Markov flows”, as used in \([14]\). Denoting \( \varphi \) by \( j_0 \), and defining \( j_s := \alpha_s \circ j_0 \), we have that \( (B(\mathcal{K}), j) \) is a weak Markov flow for \( \Theta \) on \( \mathcal{K} \), which just means that for all \( t \leq s \in \mathcal{S} \) and all \( a \in \mathcal{M} \),

\[
j_t(1)j_s(a)j_t(1) = j_t(\Theta_{s-t}(a)).
\]

Equation \((4.8)\) for \( t = 0 \) is just \((4.9)\), and the case \( t \geq 0 \) follows from the case \( t = 0 \).

Remark 4.4. The assumption that \( \Theta \) is a unital semigroup is essential, since \((4.6)\) and \((4.7)\) imply that \( \Theta(1) = 1 \).

Remark 4.5. It is impossible, in the generality we are working in, to hope for a semigroup of automorphisms that extends \( \Theta \) in the sense that

\[
\alpha_s(\varphi(a)) = \varphi(\Theta_s(a)),
\]

because that would imply that \( \Theta \) is injective.

Proof. Let \((E, T)\) be the identity representation of \( \Theta \). Since \( \Theta \) preserves the unit, \( T \) is a fully coisometric representation. Let \( \hat{T} \) and \( \mathcal{H} \) be the semigroup and Hilbert space representing \( T \) as described in Proposition \((4.1)\) \( \{ \hat{T}_s \}_{s \in \mathcal{S}} \) is a commutative semigroup of isometries. By a theorem of Douglas \([19]\), \( \{ \hat{T}_s^* \}_{s \in \mathcal{S}} \) can be extended to a semigroup \( \{ \hat{V}_s \}_{s \in \mathcal{S}} \) of unitaries acting on a space \( \mathcal{K} \supseteq \mathcal{H} \). We obtain a semigroup of unitaries \( V = \{ \hat{V}_s \}_{s \in \mathcal{S}} \) that is a dilation of \( T \), that is

\[
P_T \hat{V}_s|_{\mathcal{H}} = \hat{T}_s, \ s \in \mathcal{S}.
\]

For any \( b \in B(\mathcal{K}) \), and any \( s \in \mathcal{S} \), we define

\[
\alpha_s(b) = \hat{V}_s b \hat{V}_s^*.
\]

Clearly, \( \alpha = \{ \alpha_s \}_{s \in \mathcal{S}} \) is a semigroup of \( * \)-automorphisms.

Put \( p = P_H \), the orthogonal projection of \( \mathcal{K} \) onto \( \mathcal{H} \). Define \( \varphi : \mathcal{M} \to B(\mathcal{K}) \) by \( \varphi(a) = p(I \otimes a)p \), where \( I \otimes a : \mathcal{H} \to \mathcal{H} \) is given by

\[
(I \otimes a)\delta_t \cdot x \otimes h = \delta_t \cdot x \otimes ah, \quad x \otimes h \in E(t) \otimes H.
\]

\( \varphi \) is well defined because \( T \) is an isometric representation (so \( \text{Ker}\hat{T}_t \) is always zero). We have that \( \varphi \) is a faithful, normal \( * \)-representation (the fact that \( T_0 \) is the identity representation ensures that \( \varphi \) is faithful). It is clear that \( \varphi(1) = p \).
To see (4.7), we note that since \( \tilde{V}_s^* \) is an extension of \( \hat{T}_s^* \), we have \( \hat{T}_s^* = \tilde{V}_s^* p = p \tilde{V}_s^* p \), thus

\[
po_s(p)p = p \tilde{V}_s \tilde{V}_s^* p \\
= p \tilde{V}_s \tilde{V}_s^* p \\
= p,
\]

that is, \( po_s(p)p = p \), which implies that \( \alpha_s(p) \geq p \).

We now prove (4.6). Let \( \delta \cdot x \otimes h \) be a typical element of \( \mathcal{H} \). We compute

\[
po_s(\varphi(a))p \delta \cdot x \otimes h = p \tilde{V}_s p (I \otimes a) \tilde{V}_s^* p \delta \cdot x \otimes h \\
= \hat{T}_s (I \otimes a) \hat{T}_s^* \delta \cdot x \otimes h \\
= \hat{T}_s (I \otimes a) \delta_{s+t} \cdot (U_{t,s} \otimes I_H) \left( x \otimes \tilde{T}_s^* h \right) \\
= \hat{T}_s \delta_{s+t} \cdot (U_{t,s} \otimes I_H) \left( x \otimes (I \otimes a) \tilde{T}_s^* h \right) \\
= \delta \cdot x \otimes (\Theta_s(a) h) \\
= \varphi(\Theta_s(a)) \delta \cdot x \otimes h.
\]

Since both \( po_s(\varphi(a))p \) and \( \varphi(\Theta_s(a)) \) annihilate \( K \otimes \mathcal{H} \), we have (4.6).

To prove (4.5), it just remains to show that

\[
po_s(\varphi(a))\big|_{\mathcal{H}} = \alpha_s(\varphi(a))|_{\mathcal{H}},
\]

that is, \( \alpha_s(\varphi(a))\mathcal{H} \subseteq \mathcal{H} \). Now, \( \tilde{V}_s^* \) is an extension of \( \hat{T}_s^* \). Moreover (4.1) shows that if \( \xi \in H_u \) with \( u \geq s \), then \( \| \hat{T}_s(\xi) \| = \| \xi \| \). Thus

\[
\| \xi \| = \| \tilde{V}_s \xi \|^2 = \| P_H \tilde{V}_s \xi \|^2 + \| (I_K - P_H) \tilde{V}_s \xi \|^2 = \| \hat{T}_s(\xi) \|^2 + \| (I_K - P_H) \tilde{V}_s \xi \|^2.
\]

So \( \tilde{V}_s \xi = \hat{T}_s \xi \) for \( \xi \in H_u \) with \( u \geq s \). Now, for a typical element \( \delta \cdot x \otimes h \) in \( H_t \), \( t \in \mathcal{S} \), we have

\[
\alpha_s(\varphi(a)) \delta \cdot x \otimes h = \tilde{V}_s (I \otimes a) \tilde{V}_s^* \delta \cdot x \otimes h \\
= \tilde{V}_s (I \otimes a) \hat{T}_s^* \delta \cdot x \otimes h \\
= \tilde{V}_s \delta_{s+t} \cdot (U_{s,t} \otimes I_H) \left( x \otimes (I \otimes a) \tilde{T}_s^* h \right) \\
= \hat{T}_s \delta_{s+t} \cdot (U_{s,t} \otimes I_H) \left( x \otimes (I \otimes a) \tilde{T}_s^* h \right) \in \mathcal{H},
\]

because \( \delta_{s+t} \cdot x \otimes (I \otimes a) \tilde{T}_s^* h \in H_{s+t} \), and \( s + t \geq s \). \( \square \)

5. Dilations and pieces of subproduct system representations

5.1. Dilations and pieces of subproduct system representations.

**Definition 5.1.** Let \( X \) and \( Y \) be subproduct systems of \( \mathcal{M} \) correspondences (\( \mathcal{M} \) a W*-algebra) over the same semigroup \( \mathcal{S} \). Denote by \( U_{X,s}^Y \) and \( U_{s,t}^Y \) the coidmertic maps that make \( X \) and \( Y \), respectively, into subproduct systems. \( X \) is said to be a subproduct subsystem of \( Y \) (or simply a subsystem of \( Y \) for short) if for all
s ∈ S the space $X(s)$ is a closed subspace of $Y(s)$, and if the orthogonal projections $p_s : Y(s) → X(s)$ are bimodule maps that satisfy

\[(5.1)\quad p_{s+t} \circ U^Y_{s,t} = U^X_{s,t} \circ (p_s \otimes p_t), \quad s, t ∈ S.\]

One checks that if $X$ is a subproduct subsystem of $Y$ then

\[(5.2)\quad p_{s+t+u} \circ U^Y_{s,t+u} (I \otimes (p_{t+u} \circ U^Y_{t,u})) = p_{s+t+u} \circ U^Y_{s+t,u} (p_{s+t} \circ U^Y_{s,t}) \otimes I),\]

for all $s, t, u ∈ S$. Conversely, given a subproduct system $Y$ and a family of orthogonal projections $\{p_s\}_{s ∈ S}$ that are bimodule maps satisfying $(5.2)$, then by defining $X(s) = p_s Y(s)$ and $U^X_{s,t} = p_{s+t} \circ U^Y_{s,t}$ one obtains a subproduct subsystem $X$ of $Y$ (with $(5.1)$ satisfied).

The following proposition is a consequence of the definitions.

**Proposition 5.2.** There exists a morphism $X → Y$ if and only if $Y$ is isomorphic to a subproduct subsystem of $X$.

**Remark 5.3.** In the notation of Theorem 2.4 we may now say that given a subproduct system $X$ and a representation $R$ of $X$, then the Arveson-Stinespring subproduct system $E$ of $Θ = Σ(X, R)$ is isomorphic to a subproduct subsystem of $X$.

The following definitions are inspired by the work of Bhat, Bhattacharyya and Dey [12].

**Definition 5.4.** Let $X$ and $Y$ be subproduct systems of $W^*$-correspondences (over the same $W^*$-algebra $M$) over $S$, and let $T$ be a representation of $Y$ on a Hilbert space $K$. Let $H$ be some fixed Hilbert space, and let $S = \{S_s\}_{s ∈ S}$ be a family of maps $S_s : X(s) → B(H)$. $(Y, T, K)$ is called a dilation of $(X, S, H)$ if

1. $X$ is a subsystem of $Y$,
2. $H$ is a subspace of $K$, and
3. for all $s ∈ S$, $T^*_s H ⊆ X(s) ⊗ H$ and $T^*_s |_{H} = S^*_s$.

In this case we say that $S$ is an $X$-piece of $T$, or simply a piece of $T$. $T$ is said to be an isometric dilation of $S$ of $T$ is an isometric representation.

The third item can be replaced by the three conditions

1’. $T_0(\cdot)P_K = P_H T_0(\cdot)P_H = S_0(\cdot),$
2’. $P_H T^*_s |_{X(s) ⊗ H} = S_s$ for all $s ∈ S$, and
3’. $P_H T^*_s |_{Y(s) ⊗ K ⊗ X(s) ⊗ H} = 0$.

So our definition of dilation is identical to Muhly and Solel’s definition of dilation of representations when $X = Y$ is a product system [28 Theorem and Definition 3.7].

**Proposition 5.5.** Let $T$ be a representation of $Y$, let $X$ be a subproduct subsystem of $Y$, and let $S$ an $X$-piece of $T$. Then $S$ is a representation of $X$.

**Proof.** $S$ is a completely contractive linear map as the compression of a completely contractive linear map. Item 1’ above together with the coinvariance of $T$ imply that $S$ is covariant: if $a, b ∈ M$ and $x ∈ X(s)$, then

$$S_s(axb) = P_H T_s(axb)P_H = P_H T_0(a)T_s(x)T_0(b)P_H$$

$$= P_H T_0(a)P_H T_s(x)P_H T_0(b)P_H$$

$$= S_0(a)S_s(x)S_0(b).$$
Finally, (using Item 3’ above),
\[ S_{s+t}(U^X_{s,t}(x \otimes y))h = S_{s+t}(p_{s+t}U^Y_{s,t}(x \otimes y))h \]
\[ = \tilde{S}_{s+t}(p_{s+t}U^Y_{s,t}(x \otimes y) \otimes h) \]
\[ = P_H \tilde{T}_{s+t}(U^X_{s,t}(x \otimes y) \otimes h) \]
\[ = P_HT_s(x)T_t(y)h \]
\[ = P_HT_s(x)P_HT_t(y)h \]
\[ = S_s(x)S_t(y)h. \]

\[ \square \]

**Example 5.6.** Let \( E \) be a Hilbert space of dimension \( d \), and let \( X \) be the symmetric subproduct system constructed in Example 1.3. Fix an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( E \). There is a one-to-one correspondence between c.c. representations \( S \) of \( X \) (on some \( H \)) and commuting row contractions \( (S_1, \ldots, S_d) \) (of operators on \( H \)), given by

\[ S \leftrightarrow \mathcal{S} = (S(e_1), \ldots, S(e_d)). \]

If \( Y \) is the full product system over \( E \), then any dilation \( (Y, T, K) \) gives rise to a tuple \( \mathcal{T} = (T(e_1), \ldots, T(e_d)) \) that is a dilation of \( \mathcal{S} \) in the sense of [12], and vice versa. Moreover, \( \mathcal{S} \) is then a commuting piece of \( \mathcal{T} \) in the sense of [12].

Consider a subproduct system \( Y \) and a representation \( T \) of \( Y \) on \( K \). Let \( X \) be some subproduct subsystem of \( Y \). Define the following set of subspaces of \( K \):

\[ (5.3) \quad \mathcal{P}(X, T) = \{ H \subseteq K : \tilde{T}^*_s H \subseteq X(s) \otimes H \text{ for all } s \in S \}. \]

As in [12], we observe that \( \mathcal{P}(X, T) \) is closed under closed linear spans (and intersections), thus we may define

\[ K^X(T) = \bigvee_{H \in \mathcal{P}(X, T)} H. \]

\( K^X(T) \) is the maximal element of \( \mathcal{P}(X, T) \).

**Definition 5.7.** The representation \( T^X \) of \( X \) on \( K^X(T) \) given by

\[ T^X(x)h = P_{K^X(T)}T(x)h, \]

for \( x \in X(s) \) and \( h \in K^X(T) \), is called the maximal \( X \)-piece of \( T \).

By Proposition 5.8, \( T^X \) is indeed a representation of \( X \).

5.2. Consequences in dilation theory of \( cp \)-semigroups.

**Proposition 5.8.** Let \( X \) and \( Y \) be subproduct systems of \( W^* \)-correspondences (over the same \( W^* \)-algebra \( \mathcal{M} \) ) over \( S \), and let \( S \) and \( T \) be representations of \( X \) on \( H \) and of \( Y \) on \( K \), respectively. Assume that \( (Y, T, K) \) is a dilation of \( (X, S, H) \). Then the \( cp \)-semigroup \( \Theta \) acting on \( V_0(\mathcal{M})' \), given by

\[ \Theta_s(a) = \tilde{T}_s(I_{Y(s)} \otimes a)\tilde{T}^*_s, \quad a \in V_0(\mathcal{M})', \]

is a dilation of the \( cp \)-semigroup \( \Phi \) acting on \( T_0(\mathcal{M})' \) given by

\[ \Phi_s(a) = \tilde{S}_s(I_{X(s)} \otimes a)\tilde{S}^*_s, \quad a \in T_0(\mathcal{M})', \]
in the sense that for all $b \in V_0(M)'$ and all $s \in S$,

$$
\Phi_s(P_H b P_H) = P_H \Theta_s(b) P_H.
$$

**Proof.** This follows from the definitions.

Although the above proposition follows immediately from the definitions, we hope that it will prove to be important in the theory of dilations of $cp$-semigroups, because it points to a conceptually new way of constructing dilations of $cp$-semigroups, as the following proposition and corollary illustrate.

**Proposition 5.9.** Let $X = \{X(s)\}_{s \in S}$ be a subproduct system, and let $S$ be a fully coisometric representation of $X$ on $H$ such that $S_0$ is unital. If there exists a (full) product system $Y = \{Y(s)\}_{s \in S}$ such that $X$ is a subproduct subsystem of $Y$, then $S$ has an isometric and fully coisometric dilation.

**Proof.** Define a representation $T$ of $Y$ on $H$ by

$$
T_s = S_s \circ p_s,
$$

where, as above, $p_s$ is the orthogonal projection $Y(s) \to X(s)$. A straightforward verification shows that $T$ is indeed a fully coisometric representation of $Y$ on $H$. By [42, Theorem 5.2], $(Y, T, H)$ has a minimal isometric and fully coisometric dilation $(Y, V, K)$. $(Y, V, K)$ is also clearly a dilation of $(X, S, H)$.

**Corollary 5.10.** Let $\Theta = \{\Theta_s\}_{s \in S}$ be a $cp_0$-semigroup and let $(E, T) = \Xi(\Theta)$ be the Arveson-Stinespring representation of $\Theta$. If there is a (full) product system $Y$ such that $E$ is a subproduct subsystem of $Y$, then $\Theta$ has an $e_0$-dilation.

**Proof.** Combine Propositions 2.1, 5.8 and 5.9.

Thus, the problem of constructing $e_0$-dilations to $cp_0$-semigroups is reduced to the problem of embedding a subproduct system into a full product system. In the next subsection we give an example of a subproduct system that cannot be embedded into full product system. When this can be done in general is a challenging open question.

**Corollary 5.11.** Let $\Theta = \{\Theta_s\}_{s \in \mathbb{N}^k}$ be a $cp$-semigroup generated by $k$ commuting $CP$ maps $\theta_1, \ldots, \theta_k$, and let $(E, T) = \Xi(\Theta)$ be the Arveson representation of $\Theta$. Assume, in addition, that

$$
\sum_{i=1}^k \|\theta_i\| \leq 1.
$$

If there is a (full) product system $Y$ such that $E$ is a subproduct subsystem of $Y$, then $\Theta$ has an $e$-dilation.

**Proof.** As in [54], we may extend $T$ to a product system representation of $Y$ on $H$, which we also denote by $T$. Denote by $e_i$ the element of $\mathbb{N}^k$ with 1 in the $i$th element and zeros elsewhere. Then

$$
\sum_{i=1}^k \|\tilde{T}_{e_i} \tilde{T}_{e_i}^*\| = \sum_{i=1}^k \|\theta_i\| \leq 1.
$$

By the methods of [40], one may show that $S$ has a minimal (regular) isometric dilation. This isometric dilation provides the required $e$-dilation of $\Theta$. \[\square\]
Theorem 5.12. Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra, let $X$ be a subproduct system of $\mathcal{M}'$-correspondences, and let $R$ be an injective representation of $X$ on a Hilbert space $H$. Let $\Theta = \Sigma(X, R)$ be the cp-semigroup acting on $R_\alpha(\mathcal{M}')'$ given by \[2.7\]. Assume that $(\alpha, K, R)$ is an $e$-dilation of $\Theta$, and let $(Y, V) = \Xi(\alpha)$ be the Arveson-Stinespring subproduct system of $\alpha$ together with the identity representation. Assume, in addition, that the map $\mathcal{R}' \ni b \mapsto P_H b P_H$ is a $*$-isomorphism of $\mathcal{R}'$ onto $R_\alpha(\mathcal{M}')$. Then $(Y, V, K)$ is a dilation of $(X, R, H)$.

Proof. For every $s \in S$, define $W_s : Y(s) \to B(H)$ by $W_s(y) = P_H V_s(y) P_H$. We claim that $W = \{W_s\}_{s \in S}$ is a representation of $Y$ on $H$. First, note that $P_H \alpha_s (I - P_H)P_H = \Theta_s (P_H (I - P_H)P_H) = 0$, thus $P_H \tilde{V}_s (I \otimes (I - P_H)) \tilde{V}_s^* P_H = 0$, and consequently $P_H \tilde{V}_s (I \otimes P_H) = P_H \tilde{V}_s$. It follows that $W_s(y) = P_H V_s(y) P_H = P_H V_s(y)$. From this it follows that

$$W_s(y_1) W_t(y_2) = P_H V_s(y_1) P_H V_t(y_2) = P_H V_s(y_1) V_t(y_2) = P_H V_{s+t}(y_1 \otimes y_2) = W_{s+t}(y_1 \otimes y_2).$$

By Theorem 2.6, we may assume that $(X, R) = (E, T) = \Xi(\Theta)$ is the Arveson-Stinespring representation of $\Theta$. Because $\alpha$ is a dilation of $\Theta$, we have

$$\tilde{W}_s(I \otimes a) \tilde{W}_s^* = P_H \tilde{V}_s(I \otimes a) \tilde{V}_s^* P_H = \Theta_s(a).$$

That is, $\Theta = \Sigma(Y, W)$. Thus, by Theorem 2.6 and Remark 5.3 we may assume that $E$ is a subproduct subsystem of $Y$, and that $T_s \circ p_s = W_s, p_s$ being the projection of $Y(s)$ onto $E(s)$. In other words, for all $y \in Y$,

$$\tilde{T}_s(p_s \otimes I_H) = P_H \tilde{W}_s.$$

Therefore, $\tilde{W}_s^* H \subseteq E(s) \otimes H$, and $\tilde{W}_s^* |_H = \tilde{T}_s^*$. That is, $(Y, W, H)$ is a dilation of $(E, T, H)$. But $(Y, V, K)$ is a dilation of $(Y, W, H)$, so it is also a dilation of $(E, T, H)$.

The assumption that $\mathcal{R}' \ni b \mapsto P_H b P_H \in \mathcal{M}'$ is a $*$-isomorphism is satisfied when $\mathcal{M} = B(H)$ and $\mathcal{R} = B(K)$. More generally, it is satisfied whenever the central projection of $P_H$ in $\mathcal{R}$ is $I_K$ (see Propositions 5.5.5 and 5.6.6 in [21]).

Let $(\alpha, K, R)$ be an $e$-dilation of a semigroup $\Theta$ on $\mathcal{M} \subseteq B(H)$. $(\alpha, K, R)$ is called a minimal dilation if the central support of $P_H$ in $\mathcal{R}$ is $I_K$ and if

$$\mathcal{R} = W^* \left( \bigcup_{s \in S} \alpha_s(\mathcal{M}) \right).$$

Corollary 5.13. Let $\Theta$ be cp-semigroup on $\mathcal{M} \subseteq B(H)$, and let $(\alpha, K, R)$ be a minimal dilation of $\Theta$. Then $\Xi(\alpha)$ is an isometric dilation of $\Xi(\Theta)$.

5.3. cp-semigroups with no $e$-dilations. Obstructions of a new nature. By Parrot’s famous example [33], there exist 3 commuting contractions that do not have a commuting isometric dilation. In 1998 Bhat asked whether 3 commuting CP maps necessarily have a commuting $*$-endomorphic dilation [10]. Note that it is not obvious that the non-existence of an isometric dilation for three commuting contractions would imply the non-existence of a $*$-endomorphic dilation for 3 commuting CP maps. However, it turns out that this is the case.
Theorem 5.14. There exists a cp-semigroup $\Theta = \{\Theta_n\}_{n \in \mathbb{N}^3}$ acting on a $B(H)$ for which there is no $e$-dilation $(\alpha, K, B(K))$. In fact, $\Theta$ has no minimal $e$-dilation $(\alpha, K, \mathcal{R})$ on any von Neumann algebra $\mathcal{R}$.

Proof. Let $T_1, T_2, T_3 \in B(H)$ be three commuting contractions that have no isometric dilation and such that $T_1^{n_1}T_2^{n_2}T_3^{n_3} \neq 0$ for all $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ (one may take commuting contractions $R_1, R_2, R_3$ with no isometric dilation as in Parrot’s example [33], and define $T_i = R_i \pm 1$). For all $n = (n_1, n_2, n_3) \in \mathbb{N}^3$, define

$$\Theta_n(a) = T_1^{n_1}T_2^{n_2}T_3^{n_3}\times(T_3^{n_3})^*(T_2^{n_2})^*(T_1^{n_1})^*, \quad a \in B(H).$$

Note that $\Theta = \Sigma(X, R)$, where $X = \{X(n)\}_{n \in \mathbb{N}^3}$ is the subproduct system given by $X(n) = \mathbb{C}$ for all $n \in \mathbb{N}^3$, and $R$ is the (injective) representation that sends $1 \in X(n)$ to $T_1^{n_1}T_2^{n_2}T_3^{n_3}$ (the product in $X$ is simply multiplication of scalars).

Assume, for the sake of obtaining a contradiction, that $\Theta = \{\Theta_n\}_{n \in \mathbb{N}^3}$ has an $e$-dilation $(\alpha, K, B(K))$. Let $(Y, V) = \Xi(\alpha)$ be the Arveson-Stinespring subproduct system of $\alpha$ together with the identity representation. By Theorem 5.12 $(Y, V, K)$ is a dilation of $(X, R, H)$. It follows that $V_1, V_2, V_3$ are a commuting isometric dilation of $T_1, T_2, T_3$ where $V_1 \coloneqq V(1)$ with $1 \in X(1,0,0)$, $V_2 \coloneqq V(1)$ with $1 \in X(0,1,0)$, and $V_3 \coloneqq V(1)$ with $1 \in X(0,0,1)$. This is a contradiction.

Finally, a standard argument shows that if $(\alpha, K, \mathcal{R})$ is a minimal dilation of $\Theta$, then $\mathcal{R} = B(K)$. \qed

Until this point, all the results that we have seen in the dilation theory of cp-semigroups have been anticipated by the classical theory of isometric dilations. We shall now encounter a phenomena that has no counterpart in the classical theory.

By [18, Proposition 9.2], if $T_1, \ldots, T_k$ is a commuting $k$-tuple of contractions such that

\begin{equation}
\sum_{i=1}^k \|T_i\|^2 \leq 1,
\end{equation}

then $T_1, \ldots, T_k$ has a commuting regular unitary dilation (and, in particular, an isometric dilation). One is tempted to conjecture that if $\theta_1, \ldots, \theta_k$ is a commuting $k$-tuple of CP maps such that

\begin{equation}
\sum_{i=1}^k \|	heta_i\| \leq 1,
\end{equation}

then the tuple $\theta_1, \ldots, \theta_k$ has an $e$-dilation. Indeed, if $\theta_i(a) = T_i a T_i^*$, where $T_1, \ldots, T_k$ is a commuting $k$-tuple satisfying (5.5), then it is easy to construct an $e$-dilation of $\theta_1, \ldots, \theta_k$ from the isometric dilation of $T_1, \ldots, T_k$. However, it turns out that (5.6) is far from being sufficient for an $e$-dilation to exist. We need some preparations before exhibiting an example.

Proposition 5.15. There exists a subproduct system that is not a subsystem of any product system.

Proof. We construct a counter example over $\mathbb{N}^3$. Let $e_1, e_2, e_3$ be the standard basis of $\mathbb{N}^3$. We let $X(e_1) = X(e_2) = X(e_3) = \mathbb{C}^2$. Let $X(e_i + e_j) = \mathbb{C}^2 \otimes \mathbb{C}^2$ for all $i, j = 1, 2, 3$. Put $X(n) = \{0\}$ for all $n \in \mathbb{N}^3$ such that $|n| > 2$. To complete the construction of $X$ we need to define the product maps $U_{m,n}^X$. Let $U_{e_i,e_j}^X$ be the identity on $\mathbb{C}^2 \otimes \mathbb{C}^2$ for all $i, j$ except for $i = 3, j = 2$, and let $U_{e_3,e_2}^X$ be the flip.
Define the rest of the products to be zero maps (except the maps $U_{m,n}^X, U_{m,0}^X$ which are identities). This product is evidently coisometric, and it is also associative, because the product of any three nontrivial elements vanishes.

Let $Y$ be a product system “dilating” $X$. Then for all $k, m, n \in \mathbb{N}^k$ we have

$$U_{k+m,n}^Y(U_{k,m}^Y \otimes I) = U_{k,m+n}^Y(I \otimes U_{m,n}^Y),$$

or

$$U_{k+m,n}^Y = U_{k,m+n}^Y(I \otimes U_{m,n}^Y)(U_{k,m}^Y \otimes I)^*,$$

and

$$U_{k,m+n}^Y = U_{k+m,n}^Y(U_{k,m}^Y \otimes I)(I \otimes U_{m,n}^Y)^*.$$

Iterating these identities, we have, on the one hand,

$$U_{e_1,e_2} = U_{e_1,e_2}(U_{e_1}^Y \otimes I)(I \otimes U_{e_2}^Y)^* = U_{e_1+e_2}^Y((U_{e_1}^Y \otimes I)(I \otimes U_{e_2}^Y)^*) = U_{e_1+e_2}^Y(I \otimes U_{e_1}^Y)^*(U_{e_2}^Y \otimes I)(I \otimes U_{e_1+e_2}^Y)^*,$$

and on the other hand

$$U_{e_1+e_2} = U_{e_1+e_2}^Y(U_{e_1}^Y \otimes I)(I \otimes U_{e_2}^Y)^* = U_{e_1+e_2}^Y((U_{e_1}^Y \otimes I)(I \otimes U_{e_2}^Y)^*) = U_{e_1+e_2}^Y(I \otimes U_{e_1}^Y)^*(U_{e_2}^Y \otimes I)(I \otimes U_{e_1+e_2}^Y)^*.$$

Canceling $U_{e_1,e_2}^Y$, we must have

$$(U_{e_1,e_2}^Y \otimes I)(I \otimes U_{e_2}^Y)^*(I \otimes U_{e_2}^Y)(U_{e_1}^Y \otimes I)(I \otimes U_{e_1}^Y)^* = (U_{e_1}^Y \otimes I)(I \otimes U_{e_1}^Y)^*(I \otimes U_{e_2}^Y)(U_{e_2}^Y \otimes I)(I \otimes U_{e_2}^Y)^*.$$

Now, $U_{e_1,e_2}^X$ were unitary to begin with, so the above identity implies

$$(U_{e_1,e_2}^X \otimes I)(I \otimes U_{e_2}^X)^*(I \otimes U_{e_2}^X)(U_{e_1}^X \otimes I)(I \otimes U_{e_1}^X)^* = (U_{e_1}^X \otimes I)(I \otimes U_{e_1}^X)^*(I \otimes U_{e_2}^X)(U_{e_2}^X \otimes I)(I \otimes U_{e_2}^X)^*.$$

Recalling the definition of the product in $X$ (the product is usually the identity), this reduces to

$$I \otimes U_{e_2,e_2}^X = U_{e_2,e_2}^X \otimes I.$$

This is absurd. Thus, $X$ cannot be dilated to a product system.

We can now strengthen Theorem 5.14.

**Theorem 5.16.** There exists a $cp$-semigroup $\Theta = \{\Theta_n\}_{n \in \mathbb{N}^3}$ acting on $B(H)$, such that for all $\lambda > 0$, $\lambda \Theta$ has no $e$-dilation $(\alpha, K, B(K))$, and no minimal $e$-dilation $(\alpha, K, \mathcal{R})$ on any von Neumann algebra $\mathcal{R}$.

**Proof.** Let $X$ be as in Proposition 5.15. Let $\Theta$ be the $cp$-semigroup generated by the $X$-shift, as in Section 2.5 of the paper. Of course, $\Theta$, as a semigroup over $\mathbb{N}^3$, can be generated by three commuting CP maps $\Theta_1, \Theta_2, \Theta_3$. $X$ cannot be embedded into a full product system, so by Theorem 5.12 $\Theta$ has no minimal $e$-dilation, nor does it have an $e$-dilation acting on a $B(K)$. Note that if $\Theta$ is scaled its product system is left unchanged (this follows from Theorem 2.6 if you take $X$ and scale...
the representation $S^X$ you get a scaled version of $\Theta$). So no matter how small you take $\lambda > 0$, $\lambda \theta_1, \lambda \theta_2, \lambda \theta_3$ cannot be dilated to three commuting $*$-endomorphisms on $B(K)$, nor to a minimal three-tuple on any von Neumann algebra. \hfill \Box

Note that the obstruction here seems to be of a completely different nature from the one in the example given in Theorem 5.14. The subproduct system arising there is already a product system, and, indeed, the $cp$-semigroup arising there can be dilated once it is multiplied by a small enough scalar.
Part 2. Subproduct systems over \( \mathbb{N} \)

6. Subproduct systems of Hilbert spaces over \( \mathbb{N} \)

We now specialize to subproduct systems of Hilbert \( W^\ast \)-correspondences over the semigroup \( \mathbb{N} \), so from now on any subproduct system is to be understood as such (soon we will specialize even further to subproduct systems of Hilbert spaces).

6.1. Standard and maximal subproduct systems. If \( X \) is a subproduct system over \( \mathbb{N} \), then \( X(0) = M \) (some von Neumann algebra), \( X(1) \) equals some \( W^\ast \)-correspondence \( E \), and \( X(n) \) can be regarded as a subspace of \( E^\otimes n \). The following lemma allows us to consider \( X(m + n) \) as a subspace of \( X(m) \otimes X(n) \).

**Lemma 6.1.** Let \( X = \{X(n)\}_{n \in \mathbb{N}} \) be a subproduct system. \( X \) is isomorphic to a subproduct system \( Y = \{Y(n)\}_{n \in \mathbb{N}} \) with coisometries \( \{U_{m,n}^Y\}_{m,n \in \mathbb{N}} \) that satisfies

\[
Y(1) = X(1)
\]

and

\[
Y(m + n) \subseteq Y(m) \otimes Y(n).
\]

Moreover, if \( p_{m+n} \) is the orthogonal projection of \( Y(1)^{\otimes (m+n)} \) onto \( Y(m+n) \), then

\[
U_{m,n}^Y = p_{m+n} Y(m) \otimes Y(n)
\]

and the projections \( \{p_n\}_{n \in \mathbb{N}} \) satisfy

\[
p_{k+m+n} = p_{k+m+n} (I_{E^\otimes k} \otimes p_{m+n}) = p_{k+m+n} (p_{k+m} \otimes I_{E^\otimes n}).
\]

**Proof.** Denote by \( U_{m,n}^X \) the subproduct system maps \( X(s) \otimes X(t) \to X(s + t) \). Denote \( E = X(1) \). We first note that for every \( n \) there is a well defined coisometry \( U_n : E^\otimes n \to X(n) \) given by composing in any way a sequence of maps \( U_{k,m}^X \) (for example, one can take \( U_3 = U_{2,1}^X U_{1,1}^X \otimes I_E \) and so on). We define \( Y(n) = \ker(U_n)^\perp \), and we let \( p_n \) be the orthogonal projection from \( E^\otimes n \) onto \( Y(n) \), \( p_n = U_n^* U_n \), so in particular, \( p_n \) is a bimodule map. For all \( m, n \in \mathbb{N} \) we have that

\[
E^{\otimes (m)} \otimes \ker(U_n) \subseteq \ker(U_{m+n}).
\]

Thus \( E^{\otimes (m)} \otimes \ker(U_n)^\perp \supseteq \ker(U_{m+n})^\perp \), so \( p_{m+n} \leq I_{E^\otimes m} \otimes p_n \). This means that \( \text{(6.3)} \) holds. In addition, defining \( U_{m,n}^Y \) to be \( p_{m+n} \) restricted to \( Y(m) \otimes Y(n) \subseteq E^{\otimes (m+n)} \) gives \( Y \) the associative multiplication of a subproduct system.

It remains to show that \( X \) is isomorphic to \( Y \). For all \( n \), \( X(n) \) is spanned by elements of the form \( U_n(x_1 \otimes \cdots \otimes x_n) \), with \( x_1, \ldots, x_n \in E \). We define a map \( V_n : X(n) \to Y(n) \) by

\[
V_n(U_n(x_1 \otimes \cdots \otimes x_n)) = p_n(x_1 \otimes \cdots \otimes x_n).
\]

It is immediate that \( V_n \) preserves inner products (thus it is well defined) and that it maps \( X(n) \) onto \( Y(n) \). Finally, for all \( m, n \in \mathbb{N} \) and \( x \in E^{\otimes m}, y \in E^{\otimes n}, \)

\[
V_{m+n}(U_{m,n}^X(U_m(x) \otimes U_n(y))) = V_{m+n}(U_{m+n}(x \otimes y)) = p_{m+n}(x \otimes y) = p_{m+n}(p_m x \otimes p_n y) = p_{m+n}((V_m U_m(x)) \otimes (V_n U_n(y))) = U_{m+n}^Y((V_m U_m(x)) \otimes (V_n U_n(y))),
\]
and (1.2) holds. □

**Definition 6.2.** A subproduct system $Y$ satisfying (6.1), (6.2) and (6.3) above will be called a standard subproduct system.

Note that a standard subproduct system is a subproduct subsystem of the full product system $\{E^\otimes n\}_{n \in \mathbb{N}}$.

**Corollary 6.3.** Every cp-semigroup over $\mathbb{N}$ has an $e$-dilation.

**Proof.** The unital case follows from Corollary 5.10 together with the above lemma. The nonunital case follows from a similar construction (where the dilation of a nonfully-coisometric representation is obtained by adapting [40, Theorem 4.2] instead of [42, Theorem 5.2]). □

Let $k \in \mathbb{N}$, and let $E = X(1), X(2), \ldots, X(k)$ be subspaces of $E, E^\otimes 2, \ldots, E^\otimes k$, respectively, such that the orthogonal projections $p_n : E^\otimes n \to X(n)$ satisfy

$$p_n \leq I^\otimes n \otimes p_j$$

and

$$p_n \leq p_i \otimes I^\otimes j$$

for all $i, j, n \in \mathbb{N}_+$ satisfying $i + j = n \leq k$. In this case one can define a maximal standard subproduct system $X$ with the prescribed fibers $X(1), \ldots, X(k)$ by defining inductively for $n > k$

$$X(n) = \left( \bigcap_{i+j=n} E^\otimes i \otimes X(j) \right) \bigcap \left( \bigcap_{i+j=n} X(i) \otimes E^\otimes j \right).$$

It is easy to see that

$$X(n) = \bigcap_{n_1 + \ldots + n_m = n} X(n_1) \otimes \ldots \otimes X(n_m) = \bigcap_{i+j=n} X(i) \otimes X(j).$$

We then have obvious formulas for the projections $\{p_n\}_{n \in \mathbb{N}}$ as well, for example

$$p_n = \bigcap_{i+j=n} p_i \otimes p_j, \quad (n > k).$$

### 6.2. Examples

**Example 6.4.** In the case $k = 1$, the maximal standard subproduct system with prescribed fiber $X(1) = E$, with $E$ a Hilbert space, is the full product system $F_E$ of Example 1.2. If $\dim E = d$, we think of this subproduct system as the product system representing a (row-contractive) $d$-tuple $(T_1, \ldots, T_d)$ of non commuting operators, that is, $d$ operators that are not assumed to satisfy any relations (the idea behind this last remark must be rather vague at this point, but it shall become clearer as we proceed). In the case $k = 2$, if $X(2)$ is the symmetric tensor product $E$ with itself then the maximal standard subproduct system with prescribed fibers $X(1), X(2)$ is the symmetric subproduct system $SSP_E$ of Example 1.3. We think of $SSP$ as the subproduct system representing a commuting $d$-tuple.

**Example 6.5.** Let $E$ be a two dimensional Hilbert space with basis $\{e_1, e_2\}$. Let $X(2)$ be the space spanned by $e_1 \otimes e_1, e_1 \otimes e_2$, and $e_2 \otimes e_1$. In other words, $X(2)$ is what remains of $E^\otimes 2$ after we declare that $e_2 \otimes e_2 = 0$. We think of the maximal standard subproduct system $X$ with prescribed fibers $X(1) = E, X(2)$
as the subproduct system representing pairs \((T_1, T_2)\) of operators subject only to
the condition \(T_i^2 = 0\). \(E^\otimes n\) has a basis consisting of all vectors of the form
\(e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}\), where \(\alpha = \alpha_1 \cdots \alpha_n\) is a word of length \(n\) in \(\text{“1” and \text{“2”}}\). \(X(n)\) then
has a basis consisting of all vectors \(e_\alpha\) where \(\alpha\) is a word of length \(n\) not containing
\(\text{“22” as a subword. Let us compute} \dim X(n)\), that is, the number of such words.

Let \(A_n\) denote the number of words not containing \(\text{“22”}\) that have leftmost letter
\(\text{“1”}\), and let \(B_n\) denote the number of words not containing \(\text{“22”}\) that have leftmost
letter \(\text{“2”}\). Then we have the recursive relation \(A_n = A_{n-1} + B_{n-1}\) and \(B_n = A_n - 1\).
The solution of this recursion gives
\[
\dim X(n) = A_n + B_n \approx \left(\frac{1 + \sqrt{5}}{2}\right)^n.
\]
As one might expect, the dimension of \(X(n)\) grows exponentially fast.

**Example 6.6.** Suppose that we want a “subproduct system that will represent a
pair of operators \((T_1, T_2)\) such that \(T_iT_j = 0\) for \(i = 1, 2\).” Although we have not yet
made clear what we mean by this, let us proceed heuristically along the lines of the
preceding examples. We let \(E\) be as above, but now we declare \(e_1 \otimes e_2 = e_2 \otimes e_2 = 0\).

In other words, we define \(X(2) = \{e_1 \otimes e_2, e_2 \otimes e_2\}^\perp\). One checks that the maximal
standard subproduct system \(X\) with prescribed fibers \(X(1) = E, X(2)\) is given by
\(X(n) = \text{span}\{e_1 \otimes e_1 \cdots e_1, e_2 \otimes e_1 \cdots e_1\}\). This is an example of a subproduct
system with two dimensional fibers.

At this point two natural questions might come to mind. First, *is every standard
subproduct system \(X\) the maximal subproduct system with prescribed fibers
\(X(1), \ldots, X(k)\) for some \(k \in \mathbb{N}\) ? Second, does \(\dim X(n)\) grow exponentially fast
(or remain a constant) for every subproduct system \(X\)? The next example answers
both questions negatively.

**Example 6.7.** Let \(E\) be as in the preceding examples, and let \(X(n)\) be a subspace of \(E^\otimes n\) having basis the set
\[
\{e_\alpha : |\alpha| = n, \alpha \text{ does not contain the words 22, 212, 2112, 21112, \ldots}\}.
\]
Then \(X = \{X(n)\}_{n \in \mathbb{N}}\) is a standard subproduct system, but it is smaller than the
maximal subproduct system defined by any initial \(k\) fibers. Also, \(X(n)\) is the span
of \(e_\alpha\) with \(\alpha = 11 \cdots 11, 21 \cdots 11, 121 \cdots 11, \ldots, 11 \cdots 12\), thus
\[
\dim X(n) = n + 1,
\]
so this is an example of a subproduct system with fibers that have a linearly growing
dimension.

Of course, one did not have to go far to find an example of a subproduct system
with linearly growing dimension: indeed, the dimension of the fibers of the
symmetric subproduct system \(SSP_{\mathbb{C}^d}\) is known to be
\[
\dim SSP_{\mathbb{C}^d}(n) = \binom{n + d - 1}{n}.
\]
Taking \(d = 2\) we get the same dimension as in Example 6.7. However, \(SSP := SSP_{\mathbb{C}^2}\) and the subproduct system \(X\) of Example 6.7 are not isomorphic: for any nonzero \(x \in SSP(1)\), the “square” \(U_{1,1}^{SSP}(x \otimes x) \in SSP(2)\) is never zero, while
\(U_{1,1}^{X}(e_2 \otimes e_2) = 0\).
Here is an interesting question that we do not know the answer to: given a solution $f : \mathbb{N} \to \mathbb{N}$ to the functional inequality
\[ f(m + n) \leq f(m)f(n), \quad m, n \in \mathbb{N}, \]
does there exists a subproduct system $X$ such that $\dim X(n) = f(n)$ for all $n \in \mathbb{N}$?

**Remark 6.8.** One can cook up simple examples of subproduct systems that are not standard. We will not write these examples down, as we already know that such a subproduct system is isomorphic to a standard one.

### 6.3. Representations of subproduct systems

Fix a $W^*$-correspondence $E$. Every completely contractive linear map $T_1 : E \to B(H)$ gives rise to a c.c. representation $T^n$ of the full product system $F_E = \{ E^{\otimes n} \}_{n \in \mathbb{N}}$ by defining for all $x \in E^{\otimes n}$ and $h \in H$
\[
T^n(x)h = \tilde{T}_1(I_E \otimes \tilde{T}_1) \cdots (I_{E^{\otimes (n-1)}} \otimes \tilde{T}_1)(x \otimes h),
\]
where $\tilde{T}_1 : E \otimes H \to H$ is given by $\tilde{T}_1(e \otimes h) = T_1(e)h$. We will denote the operator acting on $x \otimes h$ in the right hand side of (6.4) as $T^n$, so as not to confuse with $\tilde{T}_n$, which sometimes has a different meaning (namely: if $T$ denotes a c.c. representation of a subproduct system $X$ then
\[
\tilde{T}_n : X(\cdot) \otimes H \to H
\]
is given by
\[
\tilde{T}_n(x \otimes h) = T(x)h
\]
for all $x \in X(n), h \in H$. Of course, when $X = F_E$, $T$ is a representation of $F_E$ and $T_1$ is the restriction of $T$ to $E$, then $\tilde{T}^n = \tilde{T}_n$ for all $n$). If $X$ is a standard subproduct system and $X(1) = E$, we obtain a completely contractive representation of $X(n)$ by restricting $T^n$ to $X(n)$. Let us denote this restriction by $\tilde{T}_n$, and denote the family $\{T_n\}_{n \in \mathbb{N}}$ by $T$.

**Proposition 6.9.** Let $X$ be a standard subproduct system with projections $\{p_n\}_{n \in \mathbb{N}}$, and let $T_1 : E \to B(H)$ be a completely contractive map. Construct the family of maps $T = \{T_n\}_{n \in \mathbb{N}}$, with $T_n : X(n) \to B(H)$ as in the preceding paragraph. Then the following are equivalent:

1. $T$ is a representation of $X$.
2. For all $m, n \in \mathbb{N}$,
\[
\tilde{T}_m(I_{X(m)} \otimes \tilde{T}_n)(p_m \otimes p_n \otimes I_H)(p_{m+n}^\perp \otimes I_H) = 0.
\]
3. For all $n \in \mathbb{N}$,
\[
\tilde{T}^n(p_n^\perp \otimes I_H) = 0.
\]

**Proof.** If $T$ is a representation, then
\[
\tilde{T}_m(I_{X(m)} \otimes \tilde{T}_n)(p_m \otimes p_n \otimes I_H)(p_{m+n}^\perp \otimes I_H) = \tilde{T}_{m+n}(p_m \otimes I_H)(p_{m+n}^\perp \otimes I_H) = 0,
\]
The necessity of (6.7) follows from Proposition 6.9. By the same proposition, Proposition 6.10. Let $X$ be the maximal standard subproduct system with prescribed fibers $X(1), \ldots, X(k)$, and let $T_1 : E \to B(H)$ be a completely contractive map. Construct $T$ as in Proposition 6.9. Then $T$ is a representation of $X$ if and only if
\begin{equation}
\tilde{T}^n(p_n^\perp \otimes I_H) = 0 \quad \text{for all} \quad n = 1, 2, \ldots, k.
\end{equation}

Proof. The necessity of (6.7) follows from Proposition 6.9. By the same proposition, to show that the condition is sufficient it is enough to show that (6.7) holds for all $n \in \mathbb{N}$. Given $m \in \mathbb{N}$, we have $p_m = \bigwedge q$, where $q$ runs over all projections of the form $q = I_{X(j)} \otimes p_i$ or $q = p_i \otimes I_{X(j)}$, with $i, j \in \mathbb{N}_+$ and $i + j = m$. But then $p_m^\perp = \bigvee q^\perp$, so if (6.7) holds for all $n < m$ then it also holds for $n = m$. □

6.4. Fock spaces and standard shifts.

Definition 6.11. Let $X$ be a subproduct system of Hilbert spaces. Fix an orthonormal basis $\{e_i\}_{i \in \mathcal{I}}$ of $E = X(1)$. $X(n)$, when considered as a subspace of $\mathfrak{S}_X$, is called the $n$ particle space. The standard $X$-shift (related to $\{e_i\}_{i \in \mathcal{I}}$) on $\mathfrak{S}_X$ is the tuple of operators $S_X = (S^X_i)_{i \in \mathcal{I}}$ in $B(\mathfrak{S}_X)$ given by
\begin{equation}
S^X_i(x) = U_{1,n}(e_i \otimes x),
\end{equation}
for all $i \in \mathcal{I}$, $n \in \mathbb{N}$ and $x \in X(n)$.

It is clear that $S^X_i = S^X(e_i)$, where $S^X$ is the shift representation given by Definition 2.9.

If $F$ denotes the usual full product system (Example 1.2) then $\mathfrak{S}_F$ is the usual Fock space and the tuple $(S^F_i)_{i \in \mathcal{I}}$ is the standard shift (the $\mathcal{I}$ orthogonal shift of $\mathfrak{S}_F$). We shall denote $\mathfrak{S}_F$ as $\mathfrak{S}$ and $(S^F_i)_{i \in \mathcal{I}}$ as $(S_i)_{i \in \mathcal{I}}$. It is then obvious that the tuple $(S^X_i)_{i \in \mathcal{I}}$ is a row contraction, as it is the compression of the row contraction
(S_i)_{i \in I}. Indeed, assuming (as we may, thanks to Lemma 6.1) that U_{m,n} is an orthogonal projection \( p_{m+n} : X(m) \otimes X(n) \to X(m+n) \), and denoting \( p = \oplus p_n \), we have for all \( i \) that \( S_i^X = p S_i \mathfrak{F}_X \).

**Example 6.12.** The \( q \)-commuting Fock space of [18] also fits into this framework. Indeed, let (as in [18]) \( \Gamma(C^d) \) be the full Fock space, let \( \Gamma_q(C^d) \) denote the \( q \)-commuting Fock space, and let \( Y(n) \) be the "\( n \) particle \( q \)-commuting space" with orthogonal projection \( p_n : (C^d)^n \to Y(n) \). Then a straightforward calculation shows that the projections \( \{ p_n \}_{n \in \mathbb{N}} \) satisfy equation (6.3) of Lemma 6.1, thus \( Y = \{ Y(n) \}_{n \in \mathbb{N}} \) is a subproduct system (satisfying (6.1) and (6.2)). With our notation from above we have that \( \mathfrak{F}_Y = \Gamma_q(C^d) \) and that the tuple \( (S_1^X, \ldots, S_{d}^X) \) is the standard \( q \)-commuting shift.

\( S^F \), the standard shift of the full product system on the full Fock space, will be denoted by \( S \), and will be called simply the standard shift.

By the notation introduced in Definition 5.7, the symbol \( S^X \) is also used to denote the maximal \( X \)-piece of the standard shift \( S \). The following proposition – which is a generalization of [12, Proposition 6], [18, Proposition 11] and [38, Proposition 2.9] – shows that this is consistent.

**Proposition 6.13.** Let \( X \) be a subproduct subsystem of a subproduct system \( Y \). Then the maximal \( X \)-piece of the standard \( Y \)-shift is the standard \( X \)-shift.

**Proof.** Let \( E = Y(1) \), and let \( F = F_E \) be the full product system. Viewing \( F(n) \otimes \mathfrak{F}_F \) as direct sum of \( |I|^n \) copies of \( \mathfrak{F}_F \), \( (S) \) is just the row isometry \( (S_i \cdot \cdots \cdot S_{i_n})_{i_1, \ldots, i_n \in I} \) from the space of columns \( \mathfrak{F}_F \oplus \mathfrak{F}_F \oplus \cdots \) into \( \mathfrak{F}_F \). In other words, for \( h \in \mathfrak{F}_F \) and \( i_1, \ldots, i_n \in I \),

\[
(S)\left((e_{i_1} \cdot \cdots \cdot e_{i_n}) \otimes h\right) = S_{i_1} \cdot \cdots \cdot S_{i_n} h = (e_{i_1} \cdot \cdots \cdot e_{i_n}) \otimes h.
\]

This is an isometry, and the adjoint works by sending \((e_{i_1} \cdot \cdots \cdot e_{i_n}) \otimes h \in \mathfrak{F}_F\) back to \((e_{i_1} \cdot \cdots \cdot e_{i_n}) \otimes h \in F(n) \otimes \mathfrak{F}_F\), and by sending the 0, 1, \ldots, \( n-1 \) particle spaces to 0.

Now, if \( Z \) is any standard subproduct subsystem of \( F \), then

\[
\left( S^Z \right)_n = P_{\mathfrak{F}_Z} \left( S \right)_n |_{Z(n) \otimes \mathfrak{F}_Z},
\]

thus

\[
(\tilde{S}^Z)_n = P_{Z(n) \otimes \mathfrak{F}_Z} \tilde{S}_n |_{\mathfrak{F}_Z}.
\]

Now if \( h \) is in the \( k \) particle space of \( \mathfrak{F}_F \) with \( k < n \), then \( (\tilde{S}^Z)^* h = 0 \). If \( k \geq n \), then since \( Z(k) \subseteq Z(n) \otimes Z(k-n) \) we may write \( h = \sum \xi_i \otimes \eta_i \), where \( \xi_i \in Z(n) \) and \( \eta_i \in Z(k-n) \). Thus by (6.8) we find that

\[
(\tilde{S}^Z)^*_n \left( \sum \xi_i \otimes \eta_i \right) = \sum p_n^Z \xi_i \otimes p_{k-n}^Z \eta_i = \sum \xi_i \otimes \eta_i.
\]

From these considerations it follows that the standard \( X \)-shift is in fact an \( X \)-piece of the standard \( Y \)-shift, as \((S^Y)^*_n |_{\mathfrak{F}_X} = (S^X)^*_n \). It remains to show that the \( X \)-shift is maximal.

Assume that there is a Hilbert space \( H \), \( \mathfrak{F}_X \subseteq H \subseteq \mathfrak{F}_Y \), such that the compression of \( S^Y \) to \( H \) is an \( X \)-piece of \( Y \), that is, \( H \in \mathcal{P}(X, S^Y) \) (see equation (5.3)). Let \( h \in H \otimes \mathfrak{F}_X \). We shall prove that \( h = 0 \). Being orthogonal to all of \( \mathfrak{F}_X \), \( p_n^Y h \)}
must be orthogonal to \(X(n)\) for all \(n\). Thus, we may assume that \(h \in Y(n) \cap X(n)\)
for some \(n\). But then by (6.3)
\[
(SY)_n^* h = h \otimes \Omega.
\]
But since \(H \in \mathcal{P}(X, SY)\), we must have \(h \otimes \Omega \in X(n) \otimes H\), and this, together with
\(h \in Y(n) \cap X(n)\), forces \(h = 0\).

\[\square\]

7. Zeros of homogeneous polynomials in noncommutative variables

In the next section we will describe a model theory for representations of subproduct systems. But before that we dedicate this section to build a precise connection between subproduct systems together with their representations and tuples of operators that are the zeros of homogeneous polynomials.

Remark 7.1. The notions that we are developing give a framework for studying tuples of operators satisfying relations given by homogeneous polynomials. One can go much further by considering subspaces of Fock spaces and “representations”, i.e., maps of the Fock space into \(B(H)\), that give a framework for studying tuples of operators satisfying arbitrary (not-necessarily homogeneous) polynomial and even analytic identities. Gelu Popescu [38] has already begun developing such a theory.

We begin by setting up the usual notation. Let \(\mathcal{I}\) be a fixed set of indices, and let \(\mathbb{C}(\{x_i\}_{i \in \mathcal{I}})\) be the algebra of complex polynomials in the noncommuting variables \((x_i)_{i \in \mathcal{I}}\). We denote \(x = (x_i)_{i \in \mathcal{I}}\), and we consider \(x\) as a “tuple variable”. We shall sometimes write \(\mathbb{C}(x)\) for \(\mathbb{C}(\{x_i\}_{i \in \mathcal{I}})\). The set of all words in \(\mathcal{I}\) is denoted by \(\mathbb{F}_+^\mathcal{I}\). For a word \(\alpha \in \mathbb{F}_+^\mathcal{I}\), let \(|\alpha|\) denote the length of \(\alpha\), i.e., the number of letters in \(\alpha\).

For every word \(\alpha = \alpha_1 \cdots \alpha_k\) in \(\mathcal{I}\) denote \(x^\alpha = x_{\alpha_1} \cdots x_{\alpha_k}\). If \(\alpha = 0\) is the empty word, then this is to be understood as 1. \(k\) is also referred to in this context as the degree of the monomial \(x^\alpha\). \(\mathbb{C}(x)\) is by definition the linear span over \(\mathbb{C}\) of all such monomials, and every element in \(\mathbb{C}(x)\) is called a polynomial. A polynomial is called homogeneous if it is the sum of monomials of equal degree. A homogeneous ideal is a two-sided ideal that is generated by homogeneous polynomials.

If \(\mathcal{T} = (T_i)_{i \in \mathcal{I}}\) is a tuple of operators on a Hilbert space \(H\) and \(\alpha = \alpha_1 \cdots \alpha_k\) is a word with letters in \(\mathcal{I}\), we define
\[
\mathcal{T}^\alpha = T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_k}.
\]
We consider the empty word 0 as a legitimate word, and define \(\mathcal{T}^0 = I_H\). If \(p(x) = \sum \alpha c_\alpha x^\alpha \in \mathbb{C}(x)\), we define \(p(\mathcal{T}) = \sum \alpha c_\alpha \mathcal{T}^\alpha\).

If \(E\) is a Hilbert space with orthonormal basis \(\{e_i\}_{i \in \mathcal{I}}\), An element \(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \in E^{\otimes k}\) will be written in short form as \(e_\alpha\), where \(\alpha = \alpha_1 \cdots \alpha_k\). If \(p(x) = \sum \alpha c_\alpha x^\alpha \in \mathbb{C}(x)\), we define \(p(e) = \sum \alpha c_\alpha e_\alpha\). Here \(e_0\) (0 the empty word) is understood as the vacuum vector \(\Omega\).

Proposition 7.2. Let \(E\) be a Hilbert space with orthonormal basis \(\{e_i\}_{i \in \mathcal{I}}\). There is an inclusion reversing correspondence between proper homogeneous ideals \(I\triangleleft \mathbb{C}(x)\) and standard subproduct systems \(X = \{X(n)\}_{n \in \mathbb{N}}\) with \(X(1) \subseteq E\). When \(|\mathcal{I}| < \infty\) this correspondence is bijective.

Proof. Let \(X\) be such a subproduct system. We define an ideal
\[
(I^X) := \text{span}\{p \in \mathbb{C}(x) : \exists n > 0, p(e) \in E^{\otimes n} \otimes X(n)\}.
\]
Once it is established that $I^X$ is a two-sided ideal the fact that it is homogeneous will follow from the definition. Let $p \in \mathbb{C}(\mathfrak{g})$ be such that $p(e) \in E_{\otimes n} \ominus X(n)$ for some $n > 0$. It suffices to show that for every monomial $x^\alpha$ we have that $x^\alpha p(x) \in I^X$, that is,

$$e_\alpha \otimes p(e) \in E_{\otimes |\alpha| + n} \ominus X(|\alpha| + n).$$

But since $X$ is standard, $X(|\alpha| + n) \subseteq X(|\alpha|) \otimes X(n)$, thus

$$E_{\otimes |\alpha|} \otimes (E_{\otimes n} \ominus X(n)) \subseteq E_{\otimes |\alpha| + n} \ominus X(|\alpha| + n).$$

It follows that $I^X$ is a homogeneous ideal.

Conversely, let $I$ be a homogeneous ideal. We construct a subproduct system $X_I$ as follows. Let $I^{(n)}$ be the set of all homogeneous polynomials of degree $n$ in $I$. Define

$$(7.2) \quad X_I(n) = E_{\otimes n} \ominus \{p(e) : p \in I^{(n)}\}.$$

Denote by $p_n$ the orthogonal projection of $E_{\otimes n}$ onto $X_I(n)$. To show that $X_I$ is a subproduct system it is enough (by symmetry) to prove that for all $m, n \in \mathbb{N}$

$$p_{m+n} \leq I_{E_{\otimes m}} \otimes p_n,$$

or, in other words, that

$$(7.3) \quad X_I(m+n) \subseteq E_{\otimes m} \otimes X_I(n).$$

Let $x \in X_I(m+n)$, let $\alpha \in I^m$, and let $q \in I^{(n)}$. Since $I$ is an ideal, $x^\alpha q(x)$ is in $I^{(m+n)}$, thus $\langle x, e_\alpha \otimes q(e) \rangle = 0$. This proves (7.3).

Assume now that $|I| < \infty$. We will show that the maps $X \mapsto I^X$ and $I \mapsto X_I$ are inverses of each other. Let $J$ be a homogeneous ideal in $\mathbb{C}(\mathfrak{g})$. Then

$$I^{X_J} = \text{span}\{p \in \mathbb{C}(\mathfrak{g}) : \exists n > 0, p(e) \in E_{\otimes n} \ominus X_J(n)\}$$

$$(*) = \text{span}\{p \in \mathbb{C}(\mathfrak{g}) : \exists n > 0, p(e) \in \{q(e) : q \in J^{(n)}\}\}$$

$$\quad = \text{span}\{p : \exists n > 0, p \in J^{(n)}\}$$

$$(** = J,$$

where (*) follows from the definition of $X_J$, and (**) from the fact that $J$ is a homogeneous ideal.

For the other direction, let $Y$ be a standard subproduct subsystem of $F_E = \{E_{\otimes n}\}_{n \in \mathbb{N}}$. Clearly, $(I_Y)^{(n)} = \{p \in \mathbb{C}(\mathfrak{g}) : p(e) \in E_{\otimes n} \ominus Y(n)\}$. Thus

$$X_{IV}(n) = E_{\otimes n} \ominus \{p(e) : p \in (I_Y)^{(n)}\}$$

$$(*) = E_{\otimes n} \ominus \{p(e) : p \in \{q(e) : q \in C(\mathfrak{g}) : q(e) \in E_{\otimes n} \ominus Y(n)\}\}$$

$$\quad = E_{\otimes n} \ominus (E_{\otimes n} \ominus Y(n))$$

$$\quad = Y(n).$$

We record the definitions of $I^X$ and $X_I$ from the above theorem for later use:

**Definition 7.3.** Let $E$ be a Hilbert space with orthonormal basis $\{e_i\}_{i \in I}$ (|I| is not assumed finite). Given a homogeneous ideal $I \subseteq \mathbb{C}(\mathfrak{g})$, the subproduct system $X_I$ defined by (7.2) will be called the subproduct system associated with $I$. If $X$ is a given subproduct subsystem of $F_E$, then the ideal $I^X$ of $\mathbb{C}(\mathfrak{g})$ defined by (7.1) will be called the ideal associated with $X$. 

□
We note that $X_I$ depends on the choice of the space $E$ and basis $\{e_i\}_{i \in \mathcal{I}}$, but different choices will give rise to isomorphic subproduct systems.

**Proposition 7.4.** Let $X$ and $Y$ be standard subproduct systems with $\dim X(1) = \dim Y(1) = d < \infty$. Then $X$ is isomorphic to $Y$ if and only if there is a unitary linear change of variables in $\mathbb{C}[x_1, \ldots, x_d]$ that sends $I^X$ onto $I^Y$.

Fix some infinite dimensional separable Hilbert space $H$. As in classical algebraic geometry, given a homogeneous ideal $I \triangleleft \mathbb{C}\langle \mathcal{X} \rangle$, it is natural to introduce and to study the zero set of $I$

$$Z(I) := \{T = (T_i)_{i \in \mathcal{I}} \in B(H)^\mathcal{I}: \forall p \in I.p(T) = 0\}.$$  

Also, given a set $Z \subseteq B(H)^\mathcal{I}$, one may form the following two-sided ideal in $\mathbb{C}\langle \mathcal{X} \rangle$

$$I(Z) := \{p \in \mathbb{C}\langle \mathcal{X} \rangle: \forall T \in Z.p(T) = 0\}.$$  

In the following theorem we shall use the notation of $\text{[6.9]}$. This simple result is the justification for viewing subproduct systems as a framework for studying tuples of operators satisfying certain homogeneous polynomial relations.

**Theorem 7.5.** Let $E$ be a Hilbert space with orthonormal basis $\{e_i\}_{i \in \mathcal{I}}$ (not necessarily with $|\mathcal{I}| < \infty$), and let $I$ be a proper homogeneous ideal in $\mathbb{C}\langle (x_i)_{i \in \mathcal{I}} \rangle$. Let $X_I$ be the associated subproduct system. Let $T_1 : E \to B(H)$ be a given representation of $E$. Define a tuple $T = (T(e_i))_{i \in \mathcal{I}}$. Construct the family of maps $T = \{T_n\}_{n \in \mathbb{N}}$, with $T_n : X(n) \to B(H)$ as in the paragraphs before Proposition $\text{[6.9]}$. Then $T$ is a representation of $X$ if and only if $T \in Z(I)$.

**Proof.** On the one hand, $E^\otimes n \otimes X_I(n) = \overline{\text{span}}\{p(e) : p \in I^{(n)}\}$. On the other hand, for every $p \in I^{(n)}$ and every $h \in H$,

$$\overline{T^n(p(e) \otimes h)} = p(T)h.$$  

Hence, the Theorem follows from Proposition $\text{[6.9]}$.

**Lemma 7.6.** Let $J \triangleleft \mathbb{C}\langle (x_i)_{i \in \mathcal{I}} \rangle$, $|\mathcal{I}| < \infty$, be a proper homogeneous ideal. Let $S^{X_J}$ be the $X_J$-shift representation, and define $T = (T_i)_{i \in \mathcal{I}}$ by $T_i = S^{X_J}(e_i), i \in \mathcal{I}$. If $p \in \mathbb{C}\langle \mathcal{X} \rangle$ is a homogeneous polynomial, then $p(T) = 0$ if and only if $p \in J$.

**Proof.** The “if” part follows from Theorem $\text{[7.5]}$. For the “only if” part, let $p \notin J$ be a homogeneous polynomial of degree $n$. Applying $p(T)$ to the vacuum vector $\Omega$, we have

$$p(T)\Omega = Pp(e),$$  

where $P$ is the orthogonal projection of $E^\otimes n$ onto $X_J(n)$. But as $p \notin J$, $p(e)$ is not in $E^\otimes n \otimes X_J(n) = \ker P$, thus $Pp(e) \neq 0$. In particular, $p(T) \neq 0$.

We have the following noncommutative projective Nullstellensatz.

**Theorem 7.7.** Let $H$ be a fixed infinite dimensional separable Hilbert space. Let $J$ be a homogeneous ideal in $\mathbb{C}\langle (x_i)_{i \in \mathcal{I}} \rangle$, with $|\mathcal{I}| < \infty$. Then

$$I(Z(J)) = J.$$  

In particular, $Z(J) = \{\mathcal{U} = (0, 0, \ldots)\}$ if and only if $J$ is the ideal generated by all the $x_i, i \in \mathcal{I}$.
Proof. \( I(Z(J)) \supseteq J \) is immediate. To see the converse, first note that equality is obvious when \( J = \mathbb{C}(x) \), so we may assume that \( J \) is proper. Also note that since \( J \) is homogeneous \( Z(J) \) is scale invariant. From this it follows that \( I(Z(J)) \) is also a homogeneous ideal. Indeed, if \( h, g \in \mathcal{Z} \), then for every tuple \( T = (T_i)_{i \in \mathcal{I}} \in Z(I) \),

\[
0 = \langle p(\lambda T) h, g \rangle = \sum_k \left( \sum_{|\alpha| = k} c_{\alpha} \langle T^\alpha h, g \rangle \right) \lambda^k,
\]

and since a nonzero univariate polynomial has only finitely many zeros, it follows the homogeneous components of \( p \) are all in \( I(Z(J)) \).

Assume now that \( p \) is a homogeneous polynomial not in \( J \). Let \( S^{X_J} \) be the \( X_J \)-shift representation, and define \( T = (T_i)_{i \in \mathcal{I}} \) by \( T_i = S^{X_J}(e_i) \), \( i \in \mathcal{I} \). It is clear that \( B(H)^J \) contains some unitarily equivalent copy of \( T \), which we also denote by \( T \). By Theorem \( 7.16 \) \( T \in Z(J) \). But by Lemma \( 7.16 \) \( p(T) \neq 0 \), so \( p \notin I(Z(J)) \). This completes the proof. \( \square \)

8. Universality of the shift: universal algebras and models

In [5], Arveson established a model for commuting, row-contraction tuples. Using an idea from that paper that appeared also in [12] and [13] – an idea that rests upon Popescu’s “Poisson Transform” introduced in [37] (and pushed forward in [32] and [38]) – we construct below a model for representations of subproduct systems. Roughly speaking, we will show that every representation of a subproduct system \( X \) is a piece of a scaled inflation of the shift. Our model should be compared with a similar model obtained by Popescu in [38]. We will also see below that the operator algebra generated by the shift \( S^X \) is the universal operator algebra generated by a representation of \( X \).

8.1. Notation for this section. We continue to use the notation set in the previous section. Let \( X \) be a standard subproduct system of Hilbert spaces over \( \mathbb{N} \), to be fixed throughout this section. Let \( p_n : E^{\otimes n} \to X(n) \) be the projections. Denote \( E = X(1) \). Let \( \{ e_i \}_{i \in \mathcal{I}} \) be an orthonormal basis for \( E \), fixed once and for all.

We denote the standard \( X \)-shift tuple by \( S^X = (S^X_i)_{i \in \mathcal{I}} \), and we denote the standard \( X \)-shift representation of \( X \) on \( \mathcal{F}_X \) by \( S^X \). We consider \( \mathcal{F}_X \) to be a subspace of the full Fock space \( \mathcal{F} \), we denote the full shift by \( S = (S_i)_{i \in \mathcal{I}} \), and we denote the full shift representation of \( F \) on \( \mathcal{F}_X \) to be \( \mathcal{F} \). For a representation \( T : X \to B(H) \), we will write \( T = (T_i)_{i \in \mathcal{I}} \) for the tuple \( (T(e_i))_{i \in \mathcal{I}} \).

We denote by \( A_X \) the unital algebra

\[
A_X := \overline{\text{span}} \{ S^X_\alpha : \alpha \in \mathbb{F}^+_I \}.
\]

We denote by \( E_X \) the operator system

\[
E_X := \overline{\text{span}} A_X A_X^*,
\]

and by \( T_X = C^*(S^X) \) the \( C^* \)-algebra generated by \( S^X_i \), \( i \in \mathcal{I} \) and \( I_{\mathcal{F}_X} \). We denote by \( K(\mathcal{F}_X) \) the algebra of compact operators on \( \mathcal{F}_X \).

If \( T \) and \( U \) are two representations of \( X \) on Hilbert spaces \( H \) and \( K \), respectively, then we define

\[
T \oplus U
\]
to be the representation of $X$ on $H \oplus K$ given by $(T \oplus U)(x) = T(x) \oplus U(x)$. We also define

$$T \otimes I_K$$

to be the representation of $X$ on $H \otimes K$ given by $(T \otimes I_K)(x) = T(x) \otimes I_K$.

8.2. Popescu’s “Poisson Transform”. After obtaining the results of this section, we discovered that they were obtained earlier by Popescu [38]. We are presenting them here since they are important for the rest of this paper.

**Proposition 8.1.** $\mathcal{K}(\mathfrak{g}_X) \subseteq \mathcal{E}_X$.

**Proof.** By the definition of representation, we have that $S(e_\alpha) = \mathfrak{g}_X^\alpha$. By Definition 5.4 and the remarks following it, we have that $S^\alpha \mid_{\mathfrak{g}_X} = (\mathfrak{g}_X)^{\alpha^*}$ for all $\alpha$. Let $x \in X(n)$. Let $|\alpha| = k$. If $n < k$ then $(\mathfrak{g}_X)^{\alpha^*} x = 0$. If $n \geq k$, then since $X(n) \subseteq X(k) \otimes X(n-k)$, we may write $x = \sum_i x_k^i \otimes x_m^i$, where $x_k^i \in X(k)$, $x_m^i \in X(m)$, and $m = n - k$. We have then

$$\sum_{|\alpha| = k} S^\alpha \sum_i (e_\alpha, x_k^i) x_m^i = \sum_i (e_\alpha, x_k^i) x_m^i \in X(m).$$

We then have for $x \in X(n)$:

$$\left( I - \sum_{|\alpha| = k} S^\alpha \right) x = \begin{cases} x, & n < k \\ x - \sum_{|\alpha| = k} S^\alpha \sum_i (e_\alpha, x_k^i) x_m^i, & n \geq k. \end{cases}$$

But

$$\sum_{|\alpha| = k} \sum_i (e_\alpha, x_k^i) x_m^i = \sum_{|\alpha| = k} p_n \left( \sum_i (e_\alpha, x_k^i) e_\alpha \otimes x_m^i \right)$$

$$= p_n \left( \sum_{|\alpha| = k} \sum_i (e_\alpha, x_k^i) e_\alpha \otimes x_m^i \right)$$

$$= p_n \left( \sum_i x_k^i \otimes x_m^i \right)$$

$$= x.$$
Given a representation $T$ of $X$ on a Hilbert space $H$ and given an integer $m \in \mathbb{N}$, we denote by $m \cdot T$ the representation

$$m \cdot T : X \to B(H \oplus H \oplus \cdots \oplus H)$$

given by $m \cdot T(x) = T(x) \oplus T(x) \oplus \cdots \oplus T(x)$, $T$ is a row contraction (i.e., $\sum_{i \in \mathcal{I}} T_i T_i^* \leq I_H$) if and only if $T$ is completely contractive. When $T$ is a row contraction the defect operator $\Delta(T)$ is defined as

$$\Delta(T) = I - \sum_{i \in \mathcal{I}} T_i T_i^*,$$

and the Poisson Kernel [37] associated with $T$ is the family of isometries $\{K_r(T)\}_{0 \leq r < 1}$ defined by

$$K_r(T) : H \to \mathcal{F} \otimes H,$$

given by

$$K_r(T) h = \sum_{\alpha \in \mathbb{F}^+_I} e_\alpha \otimes \left(r^{|\alpha|} \Delta(rT)_{1/2} T_\alpha^* h \right).$$

(See the beginning of [37, Section 8] for the remark that $T$ has “property (P)”, and [37, Lemma 3.2] for the fact that these are isometries). When it makes sense, we also define $K_1(T)$ by the same formula with $r = 1$. The Poisson transform is then defined as a map

$$\Phi : \mathcal{E}_X \to B(H)$$

$$\Phi(a) = \Phi_T(a) = \lim_{r \to 1} K_r(T)^* (a \otimes I) K_r(T).$$

By [37, Theorem 3.8], $\Phi$ is a unital, completely positive, completely contractive, satisfies

$$\Phi(\sum \alpha \sum \beta^* \mathcal{S}) = \sum \alpha \sum \beta^* \mathcal{T},$$

and is multiplicative on $\text{Alg}(\mathcal{S}, I_{\mathcal{F}})$, the algebra generated by $\mathcal{S}$ and $I_{\mathcal{F}}$ ($\Phi$ is in fact an $\text{Alg}(\mathcal{S}, I_{\mathcal{F}})$-morphism).

Theorem 8.2. Let $T$ be a c.c. representation of $X$ on $H$. There exists a unital, completely positive, completely contractive map

$$\Psi : \mathcal{E}_X \to B(H)$$

that satisfies

$$\Psi(\sum \alpha \sum \beta^* \mathcal{S}) = \sum \alpha \sum \beta^* \mathcal{T},$$

and

$$(8.3) \quad \Psi(ab) = \Psi(a)\Psi(b), \quad a \in \mathcal{A}_X, b \in \mathcal{E}_X.$$

Proof. By the lemma below, the range of $K_r(T)$ is contained in $\mathcal{F}_X \otimes H$ for all $0 \leq r < 1$, thus

$$(P_{\mathcal{F}_X} \otimes I_H) K_r(T) = K_r(T).$$

We may then define

$$\Psi(T)((\sum \alpha \sum \beta^* \mathcal{S})) = \lim_{r \to 1} K_r(T)^* (\sum \alpha \sum \beta^* \mathcal{S}) \otimes I) K_r(T),$$

$$(*) = \lim_{r \to 1} K_r(T)^* \left(\sum \alpha \sum \beta^* \mathcal{S} \otimes I \right) K_r(T)$$

$$= \sum \alpha \sum \beta^* \mathcal{T}.$$
where in (*) we have made use of the coinvariance of $\mathfrak{g}_X$ under $\mathfrak{S}$. This obviously extends to the desired map on $E_X$. □

Lemma 8.3. $K_r(T)H \subseteq \mathfrak{g}_X \otimes H$.

Proof. Let $h \in H$. It suffices to show that for all $n \in \mathbb{N}$, the element

$$\sum_{|\alpha|=n} e_\alpha \otimes r^n \Delta(rT^{1/2}) T^{\alpha^*} h$$

is in $X(n) \otimes H$. However, $X(n) \otimes H$ (considered as a subspace of $E^{\otimes n} \otimes H$) is reduced by $(I \otimes r^n \Delta(rT^{1/2}))$, so it is enough to show that

$$\xi := \sum_{|\alpha|=n} e_\alpha \otimes (T^{\alpha^*} h) \in X(n) \otimes H.$$ 

Let $x \in E^{\otimes n} \otimes X(n)$ and $g \in H$. The proof will be completed by showing that $\langle \xi, x \otimes g \rangle = 0$.

$$\langle \xi, x \otimes g \rangle = \sum_{|\alpha|=n} \langle e_\alpha \otimes T(e_\alpha)^* h, x \otimes g \rangle$$

$$= \sum_{|\alpha|=n} \langle e_\alpha, x \rangle \langle h, T(e_\alpha) g \rangle$$

$$= \langle h, T \left( \sum_{|\alpha|=n} \langle e_\alpha, x \rangle e_\alpha \right) g \rangle$$

$$= \langle h, \tilde{T}^n(x \otimes g) \rangle,$$

and by Proposition 6.9, the last expression in this chain of equalities is 0. □

8.3. The universal algebra generated by a tuple subject to homogeneous polynomial identities.

Theorem 8.4. $J \triangleleft \mathfrak{C}((x_i)_{i \in I})$, be a homogeneous ideal. Then $A_{X,J}$ is the universal unital operator algebra generated by a row contraction in $Z(J)$, that is: $A_{X,J}$ is a norm closed unital operator algebra generated by a tuple in $Z(J)$, (namely, $(S_X^N)_{i \in I}$), and if $B \subseteq B(H)$ is another norm closed unital operator algebra generated by a row contraction $(T_i)_{i \in I} \in Z(J)$, then there is a unique unital and completely contractive homomorphism $\varphi$ of $A_{X,J}$ onto $B$, such that $\varphi(S_X^N) = T_i$ for all $i \in I$.

Proof. This follows from Theorems 7.5 and 8.2. □

8.4. A model for representations: every completely bounded representation of $X$ is a piece of an inflation of $S_X$. We will now construct a model for representations of subproduct systems. In [38 Section 2], a similar but different model – that includes also a fully coisometric part and not only the shift – has been obtained.

Theorem 8.5. Let $T$ be a completely bounded representation of the subproduct system $X$ on a separable Hilbert space $H$, and let $K$ be an infinite dimensional, separable Hilbert space. Then for all $r > \|T\|_{cb}$, $T$ is unitarily equivalent to a piece of

$$S_X \otimes rI_K.$$ (8.4)
Moreover, \( \|T\|_{cb} \) is equal the infimum of \( r \) such that \( T \) is a piece of an operator as in \([8,3]\).

Proof. It is known that \( \|T\|_{cb} = \|T_i\|_{cb} \), where \( T_i = T(e_i) \). Thus if \( r > r_0 = \|T\|_{cb}, \) then \( \sum_{i \in I} T_i \leq r_0 I < r^2 I \). Put \( W_i = r^{-1} T_i \), so \( \sum_{i \in I} W_i W_i^* \leq r_0^2 / r^2 I \). Then \( K_1(W) \) is an isometry (it is equal to \( K_{r_0/r}(r/r_0 W) \), and \( r/r_0 W \) is a row contraction). Thus we may define a map (as in the proof of Theorem \([S,2]\))

\[
\Psi : B(\mathcal{F}X) \to B(H)
\]

by

\[
\Psi(a) = K_1(W) a (a \otimes I) K_1(W).
\]

\( \Psi \) is a normal completely positive unital map that satisfies

\[
\Psi \left((S^X)^\alpha (S^X)^\beta\right) = W^\alpha W^\beta, \quad \alpha, \beta \in \mathbb{F}_r^+.
\]

Since \( \Psi \) is normal it has a normal minimal Stinespring dilation \( \Psi(a) = V^* \pi(a)V \), with \( \pi : B(\mathcal{F}X) \to B(L) \) a normal \(*\)-homomorphism and \( V : H \to L \) is an isometry. It is well known that \( \pi \) is equivalent to a multiple of the identity representation. Thus we obtain, up to unitary equivalence and after identifying \( H \) with \( V H \), that \( r^{-1} T_i = P_H \pi(S_i^X) P_H = P_H(S_i^X \otimes I_G) P_H \), for some Hilbert space \( G \). To see that \( T \) is a piece of \( S^X \otimes I_G \) we need to show that \( (S_i^X \otimes I_G)^*|_H = T_i^* \) for all \( i \in I \). In other words, we need to show that \( P_H \pi(S_i^X) = P_H \pi(S_i^X) P_H \). But, for all \( b \in \mathcal{F}X \),

\[
P_H \pi(S_i^X) \pi(b) P_H = P_H \pi(S_i^X b) P_H
\]

\[
= \Psi(S_i^X b)
\]

\[
(*) = \Psi(S_i^X) \Psi(b)
\]

\[
= P_H \pi(S_i^X) P_H \pi(b) P_H,
\]

where \( (*) \) follows from \([8,3]\). By Proposition \([S,1]\) the strong operator closure of \( \mathcal{E}_X \) is \( B(\mathcal{F}X) \). \( P_H \pi(S_i^X) = P_H \pi(S_i^X) P_H \) now follows from the minimality and normality of the dilation.

It is clear that \( r^{-1} T \) is a also piece of \( S^X \otimes I_K \) for every \( K \) with \( \dim K \geq \dim G \), so we may choose \( K \) to be infinite dimensional.

We want to show that necessarily \( \dim K \geq \dim H \). Since \( S^X \otimes I_K \) is a dilation of \( r^{-1} T \), \( I_L = \sum_{i \in I} S_i^X(S_i^X)^* \otimes I_K \) is a dilation of \( I_H = \sum_{i \in I} r^{-2} T_i^* T_i^* \). But the latter operator is invertible so it has rank \( \dim H \). Thus the rank of \( P_C \otimes I_K = I_L - \sum_{i \in I} S_i^X(S_i^X)^* \otimes I_K \), which is \( \dim K \), must be greater.

Now the final assertion is clear. \( \square \)

We can now obtain a general von Neumann inequality.

**Theorem 8.6.** Let \( X \) be a subproduct system, and let \( T \) be a c.c. representation of \( X \) on a Hilbert space \( H \). Let \( \{e_1, \ldots, e_d\} \) be an orthonormal set in \( X(1) \), and define \( T_i = T(e_i) \) and \( S_i^X = S^X(e_i) \) for \( i = 1, \ldots, d \). Then for every polynomials \( p \) and \( q \) in \( d \) non commuting variables,

\[
\|p(T_1, \ldots, T_d) q(T_1, \ldots, T_d)^*\| \leq \|p(S_1^X, \ldots, S_d^X) q(S_1^X, \ldots, S_d^X)^*\|.
\]

**Proof.** Since \( T \) is a piece of \( S^X \otimes r I_K \) for all \( r > 1 \), we have

\[
p(T_1, \ldots, T_d) q(T_1, \ldots, T_d)^* = P \left( p(r S_1, \ldots, r S_d) q(r S_1, \ldots, r S_d)^* \otimes I_K \right) P
\]

for some projection \( P \), and the result follows by taking \( r \to 1 \). \( \square \)
9. The operator algebra associated to a subproduct system

9.1. Let $X$ be a subproduct system. Recall the definitions of $\mathcal{A}_X$ and $\mathcal{E}_X$ from Section 8.1. If $\{e_i\}_{i \in I}$ is an orthonormal basis for $X(1)$, then $\mathcal{A}_X$ is the unital operator algebra generated by $\{S_X^i\}_{i \in I}$ with $S_X^1 = S^X(e_1)$. If $\{f_i\}_{i \in I}$ is another orthonormal basis then the tuple $(S^X(f_i))_{i \in I}$ is not necessarily unitarily equivalent to $(S_X^i)_{i \in I}$. For instance (with the above notation), if $X$ and $\{e_1, e_2\}$ are as in Example 6.7 and

$$f_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad f_2 = \frac{1}{\sqrt{2}}(e_1 - e_2),$$

then $S_X^1, S_X^2$ are partial isometries, whereas $T_1 = S^X(f_1)$ and $T_2 = S^X(f_2)$ are not. Thus, the unitary equivalence of the row $(S_X^i)$ does not determine the isomorphism class of the subproduct system $X$.

Proposition 9.1. Let $X$ and $Y$ be two subproduct systems with $X(1) = E$ and $Y(1) = F$. Assume that $\{e_i\}_{i \in I}$ is an orthonormal basis for $E$ and that $\{f_i\}_{i \in I}$ is an orthonormal basis for $F$. Then the shifts $(S_X^i)_{i \in I}$ and $(S_Y^i)_{i \in I}$ are unitarily equivalent as rows (i.e., there exists a unitary $V : H_X \to H_Y$ such that $VS_X^i = S_Y^iV$ for all $i \in I$), if and only if there is an isomorphism of subproduct systems $W : X \to Y$ such that $W e_i = f_i$ for all $i \in I$.

Proof. If $X$ and $Y$ are isomorphic with the isomorphism $W$ sending $e_i$ to $f_i$, then define a unitary $V : H_X \to H_Y$ by

$$V = \bigoplus_{n \in \mathbb{N}} W|_{X(n)}.$$  

$VS_X^i = S_Y^i V$ follows from the properties of $W$. Conversely, a unitary $V$ intertwining $S_X^i$ and $S_Y^i$ must send $\Omega_X$ to $\Omega_Y$. Indeed, such a unitary must send $\{\Omega_X\}^\perp$ (which is equal to $\vee \text{Im } S_X^i$) onto a subspace of $\{\Omega_Y\}^\perp$ that has codimension 1 in $H_Y$, thus it must send $\{\Omega_X\}^\perp$ onto $\{\Omega_Y\}^\perp$. It follows that $V \Omega_X = \Omega_Y$. Thus, given a unitary $V$ intertwining $S_X^i$ and $S_Y^i$, we may define $W|_{X(n)} : X(n) \to Y(n)$ by

$$W S_X^i \Omega = V S_Y^i \Omega = S_Y^i \Omega,$$

for all $|\alpha| = n$, and it is easy to see that the maps $W|_{X(n)}$ assemble to form an isomorphism of subproduct systems. \hfill $\square$

In the example preceding the proposition, we saw how the shift “tuple” $(S_X^1, S_X^2)$ depends essentially on the choice of basis in $E$. However, the closed unital algebra generated by $(S_X^1, S_X^2)$ is isomorphic to the one generated by $(T_1, T_2)$. Similar remarks hold for $\mathcal{E}_X$ and $\mathcal{T}_X$.

Example 9.2. Let $X$ be the subproduct system given by $X(0) = \mathbb{C}, X(1) = \mathbb{C}^2$ and $X(n) = 0$ for all $n \geq 2$. Let $Y$ be the subproduct system given by $Y(0) = Y(1) = Y(2) = \mathbb{C}$ and $Y(n) = 0$ for all $n \geq 3$. Then since $\mathcal{E}_X$ and $\mathcal{E}_Y$ contain the compact operators on $\mathcal{F}_X$ and $\mathcal{F}_Y$ (the Fock spaces), we have $\mathcal{E}_X = \mathcal{F}_X \cong M_2(\mathbb{C}) \cong \mathcal{F}_Y = \mathcal{E}_Y$.

On the other hand, let $\{e_1, e_2\}$ be an orthonormal basis for $X(1)$. Then if $\Omega$ is the vacuum vector, then $\mathcal{A}_X$ is generated by $S^X(\Omega) = I, S^X(e_1), S^X(e_2)$. In the base $\{\Omega, e_1, e_2\}$ for $\mathcal{F}_X$, these operators have the form

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
$$
Thus,

\[ \mathcal{A}_X \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}. \]

On the other hand, \( \mathcal{A}_Y \) is generated by

\[ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S^Y(f_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, (S^Y(f_1))^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]

where \( \{f_1\} \) is an orthonormal basis for \( Y(1) \). Thus

\[ \mathcal{A}_Y \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & b & a \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}. \]

So \( \mathcal{A}_X \not\cong \mathcal{A}_Y \) (in \( \mathcal{A}_X \) the solutions of \( T^2 = 0 \) form a two dimensional subspace, and in \( \mathcal{A}_Y \) they form a one dimensional subspace).

9.2. \( \mathcal{A}_X \) as a graded algebra. For every subproduct system \( X \) there exists a unique completely contractive multiplicative linear functional \( \rho_0 : \mathcal{E}_X \to \mathbb{C} \) that sends \( \lambda I \) to \( \lambda \) and \( S^X_{\alpha} \) to 0 when \( |\alpha| > 0 \). The existence of \( \rho_0 \) follows from Theorem 8.2 (using the Poisson Transform), but it is also clear that \( \rho_0 \) is just the vector state associated with the vacuum vector \( \Omega_X \):

\[ \rho_0(T) = \langle T\Omega_X, \Omega_X \rangle, \ T \in \mathcal{A}_X. \]

\( \rho_0 \) can be considered also as a conditional expectation \( \rho_0 : \mathcal{A}_X \to \mathbb{C} \cdot \Omega_X \), inducing a direct sum

\[ (9.1) \quad \mathcal{A}_X = \rho_0 \mathcal{A}_X \oplus \ker \rho_0 = \mathbb{C} \cdot I \oplus \sum_i S^X_i \mathcal{A}_X. \]

\( \mathcal{A}_X \) contains a dense graded subalgebra, with the homogeneous elements of degree \( n \) being \( S^X(\xi) \), where \( \xi \in X(n) \). To be precise, we have the following proposition.

**Proposition 9.3.** Every \( T \in \mathcal{A}_X \) can be written in a unique way as

\[ T = \sum_{n=0}^{\infty} T_n, \]

where \( T_n \in \text{span}\{S^X(\xi) : \xi \in X(n)\} \) and the sum is Cesaro convergent in the norm topology.

**Proof.** The proof uses a familiar gadget in operator algebra theory, the gauge action of the torus. For every \( t \in [-\pi, \pi] \), let \( W_t : X \to X \) be the subproduct system automorphism given by

\[ X(n) \ni \xi \mapsto e^{int} \xi \in X(n). \]

The gauge action on \( \mathcal{A}_X \) is given by

\[ \gamma_t(T) = W_t T W_t^*, \ T \in \mathcal{A}_X. \]

Note that if \( \alpha \in \mathbb{T}^n \), then

\[ \gamma_t(S^X_\alpha) = e^{int} S^X_\alpha, \]

and it follows also that for all \( T \in \text{span}\{S^X(\xi) : \xi \in X(n)\} \),

\[ \gamma_t(T) = e^{int} T. \]
Moreover, for all \( T \in \mathcal{A}_X \), the path \( t \mapsto \gamma_t(T) \) is strong operator continuous. Given \( T \in \mathcal{A}_X \), we define
\[
\Phi_n(T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_t(T)e^{-int}dt,
\]
where this is interpreted in the strong operator sense. In more detail, \( \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_t(T)e^{-int}dt \) is the operator that for all \( h \in H \) acts as
\[
\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_t(T)e^{-int}dt \right) h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_t(T)e^{-int}hdt,
\]
where the integral on the right is usual vector valued integration. It follows that if \( T \in \overline{\text{span}}\{S^X(\xi) : \xi \in X(m)\} \) then
\[
\Phi_n(T) = \begin{cases} T, & \text{if } m = n \\ 0, & \text{else} \end{cases}
\]
\( \Phi_n \) is a completely contractive linear map. For a finite sum \( \sum_{m=0}^{N} T_m \) with \( T_m \in \overline{\text{span}}\{S^X(\xi) : \xi \in X(m)\} \) and \( N \geq n \),
\[
\Phi_n \left( \sum_{m=0}^{N} T_m \right) = T_n.
\]
As such finite sums are dense in \( \mathcal{A}_X \), it follows that the range of \( \Phi_n \) is equal to \( \overline{\text{span}}\{S^X(\xi) : \xi \in X(n)\} \).

Define linear maps on \( \mathcal{A}_X \) by
\[
\Psi_N(T) = \sum_{n=0}^{N} \left( 1 - \frac{n}{N} \right) \Phi_n(T).
\]
Our goal is to prove that \( \sum_n \Phi_n(T) \) is Cesaro convergent to \( T \) in the norm topology, that is, to show that for all \( T \in \mathcal{A}_X \),
\[
\| \Psi_N(T) - T \| \xrightarrow{N \to \infty} 0.
\]
But
\[
\sum_{n=0}^{N} \left( 1 - \frac{n}{N} \right) \Phi_n(T)h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{N} \left( 1 - \frac{n}{N} \right) e^{-int}\gamma_t(T)hdt
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N}^{N} \left( 1 - \frac{n}{N} \right) e^{-int}\gamma_t(T)hdt
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(t)\gamma_t(T)hdt,
\]
where \( F_N(t) \) is the Fejér Kernel (see page 12 in [23])
\[
F_N(t) = \frac{1}{N+1} \left( \frac{\sin \frac{(N+1)t}{2}}{\sin \frac{t}{2}} \right)^2.
\]
From the above integral representation and the fact that \( \int |F_N(t)|dt = 2\pi \), it follows that \( \Psi_N \) is a contraction for all \( N \).
Let \( T \in A_X \) and fix \( \epsilon > 0 \). Then there is a \( T_\epsilon = \sum_{m=0}^N T_m \) such that \( \| T - T_\epsilon \| < \epsilon \). Therefore \( \sum_n \Phi_n(T_\epsilon) \) converges to \( T_\epsilon \), so \( \Phi_n(T_\epsilon) \) converges to \( T_\epsilon \). There is some \( M \) such that for \( n > M \), \( \| T_\epsilon - \Phi_n(T_\epsilon) \| < \epsilon \), so for \( n > M \)

\[
\| T - \Phi_n(T) \| \leq \| T - T_\epsilon \| + \| T_\epsilon - \Phi_n(T_\epsilon) \| + \| \Phi_n(T_\epsilon) - T \| < \epsilon + \epsilon + \epsilon.
\]

That shows that \( \sum_n \Phi_n(T) \) is Cesaro convergent to \( T \), and it remains to prove the uniqueness assertion. Assume that \( T = \sum_n T_n \), where the sum is Cesaro convergent to \( T \), and \( T_n \in \overline{\text{span}}\{S^X(\xi) : \xi \in X(n)\} \). Then for all \( N > n \),

\[
\Phi_n \left( \sum_{m=0}^N \left( 1 - \frac{m}{N} \right) T_m \right) = \left( 1 - \frac{n}{N} \right) T_n \xrightarrow{N \to \infty} T_n.
\]

On the other hand,

\[
\Phi_n \left( \sum_{m=0}^N \left( 1 - \frac{m}{N} \right) T_m \right) \xrightarrow{N \to \infty} \Phi_n(T),
\]

whence \( T_n = \Phi_n(T) \). \( \square \)

### 9.3. Vacuum state preserving isometric isomorphisms of \( A_X \)

**Lemma 9.4.** Let \( \varphi : A_X \to A_Y \) be an isometric isomorphism. Then \( \varphi \) is unital.

**Proof.** A theorem of Arazy and Solel [1] implies that an isometric map between \( A_X \) and \( A_Y \) must send \( I \in A_X \) to an isometry in \( A_X \cap A_X^\ast \). It follows that \( \varphi(I) = cI \), \( |c| = 1 \). But since \( \varphi \) is a homomorphism, then \( c = 1 \). \( \square \)

**Lemma 9.5.** For all \( n \in \mathbb{N}, \xi \in X(n) \)

\[
\| S^X(\xi) \| = \| S^X(\xi) \Omega_X \| = \| \xi \|.
\]

**Proof.** Because \( S^X(\xi) \) maps the orthogonal summands \( X(k) \) of \( \mathfrak{A}_X \) into the orthogonal summands \( X(k+n) \), it suffices to show that for all \( \eta \in X(k), \| S^X(\xi)\eta \| \leq \| \xi \| \| \eta \| \) (because \( S^X(\xi)\Omega_X = \xi \)). Now, \( S^X(\xi)\eta = p^X_{n+k}(\xi \otimes \eta) \), thus

\[
\| S^X(\xi)\eta \|^2 \leq \| \xi \otimes \eta \|^2 = \| \xi \|^2 \| \eta \|^2.
\]

\( \square \)

**Lemma 9.6.** Let \( \varphi : A_X \to A_Y \) be an isometric isomorphism that preserves the direct sum decomposition \( \{Y^i(\eta) : \eta \in Y(i) \} \). Then \( \varphi \) preserves the grading: if \( \xi \in X(n) \) then \( \varphi(S^X(\xi)) \) is in the norm closure of \( \text{span}\{S^Y(\eta) : \eta \in Y(n) \} \).

**Proof.** Since \( \varphi \) is a homomorphism, it suffices to show, say, that \( \varphi(S^X_\xi) \) has “degree one”, that is, it is in the norm closure of \( \text{span}\{S^Y(\eta) : \eta \in Y(1) \} \). By assumption, we may write \( \varphi(S^X_\xi) = \sum_i a_i S^Y_i + T \), with \( T \) in the closure of \( \text{span}\{S^Y(\eta) : \eta \in Y(n), n \geq 2 \} \). But \( \varphi^{-1}(\sum_i a_i S^Y_i + T) = S^X_\xi \), and \( \varphi^{-1}(T) \) is in the norm closure of \( \text{span}\{S^X(\xi) : \xi \in X(n), n \geq 2 \} \), so \( \varphi^{-1}(\sum_i a_i S^Y_i) = S^X_\xi + B \), with \( B = -\varphi^{-1}(T) \) (note that \( \varphi^{-1} \) also preserves the direct sum decomposition \( \{Y^i(\eta) : \eta \in Y(1) \} \)).

If \( T = 0 \) then we are done, so assume \( T \neq 0 \). Then \( B \neq 0 \), also. But

\[
1 = \| S^X_1 \| = \| S^X_1 \Omega_X \| < \| (S^X_1 + B) \Omega_X \| \leq \| S^X_1 + B \| = \| \sum_i a_i S^Y_i \|,
\]
Theorem 9.7. \( X \cong Y \) if and only if \( \mathcal{A}_X \) and \( \mathcal{A}_Y \) are isometrically isomorphic with an isomorphism that preserves the direct sum decomposition \((9.1)\), and this \( \phi \) preserves isomorphism. In fact, if \( \|\cdot\| \) and at the same time \( m,n \in X \)
isomorphism then there is an isomorphism \( T \sim \) from \( \mathcal{V} \). This completes the proof. \( \square \)

Proof. \( X \cong Y \) implies \( \mathcal{A}_X \cong \mathcal{A}_Y \) because these algebras are then generated by unitarily equivalent tuples.

For the converse, we will assume that \( X \) and \( Y \) are standard subproduct systems. The isomorphism \( V : X \to Y \) is defined on the fiber \( X(n) \) by

\[
V(\xi) = V(S^X(\xi)\Omega_X) = \varphi(S^X(\xi))\Omega_Y, \quad \xi \in X(n).
\]

If it is well defined, then it is onto. Lemma 9.5 shows that \( V \) is an isometry on the fibers:

\[
\|S^X(\xi)\Omega_X\| = \|S^X(\xi)\| = \|\varphi(S^X(\xi))\| = \|\varphi(S^X(\xi))\Omega_Y\|.
\]

Lemma 9.6 implies that \( V(\xi) \) sits in \( Y(n) \). \( V \) respects the subproduct structure: if \( m,n \in \mathbb{N}, \xi \in X(n), \eta \in X(m) \), then

\[
Vp_{m,n}^X(\xi \otimes \eta) = VS^X(p_{m,n}^X(\xi \otimes \eta))\Omega_X
= \varphi(S^X(p_{m,n}^X(\xi \otimes \eta)))\Omega_Y
= \varphi(S^X(\xi))S^X(\eta)\Omega_Y
= \varphi(S^X(\xi))\varphi(S^X(\eta))\Omega_Y
= p_{m,n}^Y(V(\xi) \otimes V(\eta)).
\]

\((*)\) follows from the facts \( S^Y(y)\Omega_Y = y \) and \( S^Y(y_1)S^Y(y_2)\Omega_Y = S^Y(p_{m,n}^X(y_1 \otimes y_2))\Omega_Y = p_{m,n}^X(S^Y(y_1))\Omega_Y \otimes S^Y(y_2)\Omega_Y \).

Finally, let us show that for all \( T \in \mathcal{A}_X \), \( \varphi(T) = VTV^* \). What we mean by this is that for all \( \xi \in X \), \( \varphi(S^X(\xi)) = VS^X(\xi)V^* \). Let \( \varphi(S^X(\eta))\Omega_Y = V(\eta) \) be a typical element in \( \mathcal{Y} \).

\[
VS^X(\xi)V^*\varphi(S^X(\eta))\Omega_Y = VS^X(\xi)\eta
= Vp^X(\xi \otimes \eta)
= \varphi(S^X(p^X(\xi \otimes \eta)))\Omega_Y
= \varphi(S^X(\xi))S^X(\eta)\Omega_Y
= \varphi(S^X(\xi))\varphi(S^X(\eta))\Omega_Y,
\]

This completes the proof. \( \square \)
10. Classification of the Universal Algebras of $q$-commuting tuples

**Definition 10.1.** A matrix $q$ is called admissible if $q_{ii} = 0$ and $0 \neq q_{ij} = q_{ji}^{-1}$ for all $i \neq j$.

10.1. The $q$-commuting algebras $A_q$ and their universality. Let \{e_1, \ldots, e_d\} be an orthonormal basis for $E := \mathbb{C}^d$, to be fixed (together with $d$) throughout this section. Let $q \in M_d(\mathbb{C})$ be an admissible matrix, and let $X_q$ be the maximal standard subproduct system with fibers

$$X_q(1) = E, \quad X_q(2) = E \otimes E \oplus \text{span}\{e_i \otimes e_j - q_{ij} e_j \otimes e_i : 1 \leq i, j \leq d, i \neq j\}.$$ 

When $q_{ij} = 1$ for all $i < j$, then $X_q$ is the symmetric subproduct system SSP. The Fock spaces $S_X$ have been studied in [18].

For brevity, we shall write $S_i^q$ instead of $S_i^Z$. We denote by $A_q$ the algebra $A_{X_q}$. By Theorem 8.4, the algebra $A_q$ is the universal norm closed unital operator algebra generated by a row contraction $(T_1, \ldots, T_d)$ satisfying the relations

$$T_i T_j = q_{ij} T_j T_i, \quad 1 \leq i < j \leq d.$$

10.2. The character space of $A_q$. Let $M_q$ be the space of all (contractive) multiplicative and unital linear functionals on $A_q$, endowed with the weak-* topology. We shall call $M_q$ the character space of $A_q$. Every $\rho \in M_q$ is uniquely determined by the $d$-tuple of complex numbers $(x_1, \ldots, x_d)$, where $x_i = \rho(S_i^q)$ for $i = 1, \ldots, d$.

Since a contractive linear functional is completely contractive, $(x_1, \ldots, x_d)$ must be a row contraction, that is, $|x_1|^2 + \ldots + |x_d|^2 \leq 1$. In other words, $(x_1, \ldots, x_d)$ is in the unit ball $B_d$ of $\mathbb{C}^d$. The multiplicity of $\rho$ implies that $(x_1, \ldots, x_d)$ must lie inside the set

$$Z_q := \{(z_1, \ldots, z_d) \in B_d : (1 - q_{ij}) z_i z_j = 0, 1 \leq i < j \leq d\}.$$

Conversely, Theorem 8.4 implies that every $(x_1, \ldots, x_d) \in Z_q$ gives rise to a character $\rho \in M_q$ that sends $S_i^q$ to $x_i$. Thus the map

$$M_q \ni \rho \mapsto (\rho(S_1^q), \ldots, \rho(S_d^q)) \in Z_q$$

is injective and surjective. It is also obviously continuous (with respect to the weak-* and standard topologies). Since $M_q$ is compact, we have the homeomorphism

$$(10.1) \quad M_q \cong Z_q.$$

Note that the vacuum state $\rho_0$ corresponds to the point $0 \in Z_q \subset \mathbb{C}^d$.

When $q_{ij} = 1$, the condition $(1 - q_{ij}) z_i z_j = 0$ is trivially satisfied, so when $q_{i,j} = 1$ for all $i, j$, then $Z_q$ is the unit ball $B_d$. When $q_{ij} \neq 1$, the condition is that either $z_i = 0$ or $z_j = 0$. Thus, if for all $i, j$, $q_{ij} \neq 1$, then $Z_q$ is the union of $d$ discs glued together at their origins.

10.3. Classification of the $A_q$, $q_{ij} \neq 1$. Given a permutation $\sigma$ (on a set with $d$ elements), let $U_\sigma$ be the matrix that induces the same permutation on the standard basis of $\mathbb{C}^d$.

**Proposition 10.2.** Let $q$ and $r$ be two admissible $d \times d$ matrices. Assume that there is a permutation $\sigma \in S_d$ such that $r = U_\sigma q U_\sigma^{-1}$, and let $\lambda_1, \ldots, \lambda_d$ be any complex numbers on the unit circle. Then the map

$$(10.2) \quad e_i \mapsto \lambda_i e_{\sigma(i)}$$
extends to an isomorphism of $X_q$ onto $X_r$, and thus the map
\[ S^q_l \mapsto \lambda_l S^r_{\sigma(i)} \]
extends to a completely isometric isomorphism between $\mathcal{A}_q$ and $\mathcal{A}_r$.

**Proof.** For all $n$, the map (10.2) extends to a unitary $V_n$ of $E^{\otimes n}$. For $n = 2$, this unitary sends $e_i \otimes e_j - q_{ij} e_j \otimes e_i$ to $\lambda_l \lambda_j e_{\sigma(i)} \otimes e_{\sigma(j)} - \lambda_l \lambda_j q_{ij} e_{\sigma(j)} \otimes e_{\sigma(i)}$. But $r = U_{\sigma} q U_{\sigma}^{-1}$ implies $r_{\sigma(i)\sigma(j)} = q_{ij}$, thus
\[ V_2 : e_i \otimes e_j - q_{ij} e_j \otimes e_i \mapsto \lambda_l \lambda_j e_{\sigma(i)} \otimes e_{\sigma(j)} - \lambda_l \lambda_j q_{ij} e_{\sigma(j)} \otimes e_{\sigma(i)}, \]
so $V_2$ is a unitary between $X_q(2)$ and $X_r(2)$ that respects the product. By induction, it follows that $V = \{ V_n \mid X_q(n) \}_{n}$ is an isomorphism of subproduct systems. The final assertion follows from Proposition 9.1. \(\square\)

**Theorem 10.3.** Let $q$ and $r$ be two admissible $d \times d$ matrices such that $q_{ij}, r_{ij} \neq 1$ for all $i, j$. Then $X_q$ is isomorphic to $X_r$ if and only if there is a permutation $\sigma \in S_q$ such that $r = U_{\sigma} q U_{\sigma}^{-1}$. In this case the isomorphisms are precisely those of the form
\[ e_i \mapsto \lambda_i e_{\sigma(i)}, \]
where $\lambda_1, \ldots, \lambda_d$ are any complex numbers on the unit circle, and $\sigma$ is such that $r = U_{\sigma} q U_{\sigma}^{-1}$.

**Proof.** One direction is Proposition 10.2 so assume that there is an isomorphism of subproduct systems $V : X_q \to X_r$. Let $f_i := V^{-1} e_i$. There is a $d \times d$ unitary matrix $U = (u_{ij})$ such that $f_i = \sum_j u_{ij} e_j$. As $V$ is an isomorphism of subproduct systems, we have for all $i \neq j$
\[ V p_2^{\chi}(f_i \otimes f_j - r_{ij} f_j \otimes f_i) = p_2^{\chi}(e_i \otimes e_j - r_{ij} e_j \otimes e_i) = 0, \]
thus
\[ (\sum_k u_{ik} e_k) \otimes (\sum_l u_{jl} e_l) - r_{ij} (\sum_k u_{jk} e_k) \otimes (\sum_l u_{il} e_l) \in \text{span}\{e_m \otimes e_n - q_{mn} e_n \otimes e_m : m \neq n\}, \]
or
\[ (10.3) \sum_{k, l} (u_{ik} u_{jl} - r_{ij} u_{jk} u_{il}) e_k \otimes e_l \in \text{span}\{e_m \otimes e_n - q_{mn} e_n \otimes e_m : m \neq n\}. \]
The coefficients of the vectors $e_k \otimes e_k$ in the sum above must vanish, thus $u_{ik} u_{jk} - r_{ij} u_{jk} u_{ik} = 0$ for all $i \neq j$. Since $r_{ij} \neq 1$, we must have $u_{jk} u_{ik} = 0$ for all $k$ and all $i \neq j$. Thus the unitary matrix $U$ has precisely one nonzero element in each column, and it therefore must be of the form $U_{\sigma}^{-1} D$, where $D$ is a diagonal unitary matrix.

Equation (10.3) becomes
\[ u_{i\sigma(i)} u_{j\sigma(j)} e_{\sigma(i)} \otimes e_{\sigma(j)} - r_{ij} u_{j\sigma(j)} u_{i\sigma(i)} e_{\sigma(j)} \otimes e_{\sigma(i)} \in \text{span}\{e_m \otimes e_n - q_{mn} e_n \otimes e_m : m \neq n\}, \]
but this can only happen if
\[ u_{i\sigma(i)} u_{j\sigma(j)} e_{\sigma(i)} \otimes e_{\sigma(j)} - r_{ij} u_{j\sigma(j)} u_{i\sigma(i)} e_{\sigma(j)} \otimes e_{\sigma(i)} \]
is proportional to
\[ e_{\sigma(i)} \otimes e_{\sigma(j)} - q_{\sigma(i)\sigma(j)} r_{\sigma(j)} \otimes e_{\sigma(i)}, \]
that is $u_{i\sigma(i)} u_{j\sigma(j)} q_{\sigma(i)\sigma(j)} = u_{j\sigma(j)} u_{i\sigma(i)} r_{ij}$, or $r_{ij} = q_{\sigma(i)\sigma(j)}$. Replacing $\sigma$ with $\sigma^{-1}$, the proof is complete. \(\square\)
Corollary 10.4. Let $q$ be an admissible $d \times d$ matrix such that there is no permutation $\sigma \in S_d$ such that $q = U_\sigma q U_\sigma^{-1}$. Assume that $q_{ij} \neq 1$ for all $i, j$. Then the only automorphisms of $X_q$ are unitary scalings of the basis $\{e_1, \ldots, e_d\}$.

Theorem 10.5. Let $q$ and $r$ be two admissible $d \times d$ matrices such that $q_{ij}, r_{ij} \neq 1$ for all $i, j$. Then $A_q$ is isometrically isomorphic to $A_r$ if and only if there is a permutation $\sigma \in S_d$ such that $r = U_\sigma q U_\sigma^{-1}$. In this case the isometric isomorphisms between $A_q$ and $A_r$ are precisely those of the form

$$S^q_i \mapsto \lambda_i S^r_{\sigma(i)},$$

where $\lambda_1, \ldots, \lambda_d$ are any complex numbers on the unit circle.

Proof. If $r = U_\sigma q U_\sigma^{-1}$, then by Proposition 10.2 and Theorem 9.7, $A_q$ and $A_r$ are isomorphic (with an isomorphism that preserves the direct sum decomposition (9.1)).

Conversely, assume that $\varphi : A_q \to A_r$ is a completely isometric isomorphism. Then $\varphi$ induces a homeomorphism between $\mathcal{M}_r$ and $\mathcal{M}_q$ by $\rho \mapsto \rho \circ \varphi$. Recall that $\mathcal{M}_r$ and $\mathcal{M}_q$ are both homeomorphic to $d$ discs glued together at the origin. Thus the homeomorphism $\rho \mapsto \rho \circ \varphi$ must take $\rho_0$ of $X_r$ to $\rho_0$ of $X_q$, because these are the unique points in $\mathcal{M}_r$ and $\mathcal{M}_q$, respectively, that when removed from $\mathcal{M}_r$ and $\mathcal{M}_q$ leave $d$ disconnected punctured discs. Thus $\varphi$ sends the vacuum state of $A_r$ to the vacuum state of $A_q$, and must therefore preserve the direct sum decomposition (9.1).

By Theorem 9.7 there is an isomorphism of subproduct systems $V : X_q \to X_r$ such that $\varphi(\bullet) = V \bullet V^*$. By Theorem 10.3 we conclude that there is a permutation $\sigma \in S_d$ such that $r = U_\sigma q U_\sigma^{-1}$. It also follows that $\varphi(S^q_i) = \lambda_i S^r_{\sigma(i)}$. \qed

Corollary 10.6. Let $q$ be an admissible $d \times d$ matrix such that there is no permutation $\sigma \in S_d$ such that $q = U_\sigma q U_\sigma^{-1}$. Then the only isometric automorphisms of $A_q$ are unitary scalings of the shift $\{S^q_1, \ldots, S^q_d\}$.

As a corollary of the above discussion we have:

Corollary 10.7. Let $q$ and $r$ be two admissible $d \times d$ matrices such that $q_{ij}, r_{ij} \neq 1$ for all $i, j$. Then $A_q$ is isometrically isomorphic to $A_r$ if and only if $X_q \cong X_r$.

10.4. $X_q$ and $A_q$, $d = 2$. In the particular case $d = 2$, we let a complex number $q$ parameterize the spaces $X_q$ (we may allow also $q = 0$) defined to be the maximal standard subproduct system with fibers

$$X_q(1) = \mathbb{C}^2, \quad X_q(2) = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{span}\{e_1 \otimes e_2 - q e_2 \otimes e_1\}.$$ 

Since $\mathcal{M}_1 \cong B_2$, $A_1$ is not isomorphic to any $A_q$ with $q \neq 1$ (recall that when $q \neq 1$, $\mathcal{M}_q$ is homeomorphic to two discs glued together at the origin). Thus Theorem 10.5 gives:

Corollary 10.8. Assume that $d = 2$. Then $X_q \cong X_r$ if and only if $A_q$ is isometrically isomorphic to $A_r$, and either one of these happens if and only if either $r = q$ or $r = q^{-1}$.

Elias Katsoulis has pointed out to us that the above corollary also follows from the techniques of [17].

The above result is reminiscent to the fact that two rotation algebras $A_q$ and $A_{q'}$ are isomorphic if and only if either $e^{2\pi i \theta} = e^{2\pi i \theta'}$ or $(e^{2\pi i \theta})^{-1} = e^{2\pi i \theta'}$. One cannot help but wonder whether one can draw a deeper connection between these results.
then the superficial one, in particular, can the classification of rotation algebras be deduced from the classification of the algebras $A_q$?

By Corollaries 10.4 and 10.6 we have the following.

**Corollary 10.9.** Let $d = 2$ and let $q \neq 1$. Then subproduct system $X_q$ has no automorphisms aside from the unitary scalings of the basis. The algebra $A_q$ has no isometric automorphisms other than unitary scalings of the generators.

On the other hand, a direct calculation shows that every unitary on $\mathbb{C}^2$ extends to an automorphism of $X_1$, and thus induces a non-obvious automorphism of $A_1$.

11. Standard maximal subproduct systems with $\dim X(1) = 2$ and $\dim X(2) = 3$

Again, let $\{e_1, \ldots, e_d\}$ be an orthonormal basis for $E := \mathbb{C}^d$. We will soon turn attention to the case $d = 2$. For a matrix $A \in M_d(\mathbb{C})$, we define the symmetric part of $A$ to be $A_s := (A + A^t)/2$ and the antisymmetric part of $A$ to be $A_a := (A - A^t)/2$. Denote by $X_A$ the maximal standard subproduct system with fibers

$$X_A(1) = E, \quad X_A(2) = E \otimes E \oplus \operatorname{span} \left\{ \sum_{i,j=1}^d a_{ij} e_i \otimes e_j \right\}.$$ 

We will write $S^X$ for the shift $S^{X_A}$. We will also write $A_A$ for $A_{X_A}$.

**Proposition 11.1.** Let $A, B \in M_d(\mathbb{C})$. Then there is an isomorphism $V : X_A \to X_B$ if and only if there exists $\lambda \in \mathbb{C}$ and a unitary $d \times d$ matrix $U$ such that $B = \lambda U^t A U$. In this case, $U$ extends to the isomorphism $V$ between $X_A$ and $X_B$ by $V_1 = U$.

**Proof.** Let $V : X_A \to X_B$ be an isomorphism of subproduct systems. There is a $d \times d$ unitary matrix $U = (u_{ij})$ such that

$$f_i := V_1(e_i) = \sum_{j=1}^d u_{ij} e_j.$$ 

Then

$$0 = V_1(p_2^X (\sum_{i,j} a_{ij} e_i \otimes e_j)) = p_2^Y (\sum_{i,j} a_{ij} f_i \otimes f_j),$$

so $\sum_{i,j} a_{ij} f_i \otimes f_j$ must be a spanning vector of $\operatorname{span} \left\{ \sum_{i,j} b_{ij} e_i \otimes e_j \right\}$. Writing out fully what this means,

$$\lambda \sum_{i,j} a_{ij} \sum_{k,l} u_{ik} u_{jl} e_k \otimes e_l = \sum_{k,l} b_{kl} e_k \otimes e_l$$

for some $\lambda \in \mathbb{C}$, so

$$b_{kl} = \lambda \sum_{i,j} a_{ij} u_{ik} u_{jl}.$$ 

But the right hand side is precisely the $kl$-th element of $\lambda U^t A U$.

Conversely, assuming $B = \lambda U^t A U$, one can read the above argument from finish to start to obtain an isomorphism $V : X_A \to X_B$. \qed
We see that for $X_A$ and $X_B$ to be isomorphic the ranks of $A$ and $B$ must be the same, as well as the ranks of their symmetric and anti-symmetric parts. For example, if $A$ is symmetric and $B$ is not then $X_A \not\cong X_B$, a result which may not seem obvious at first glance.

**Theorem 11.2.** Assume that $d = 2$. Let $A, B \in M_2(\mathbb{C})$ be any two matrices. Then $\mathcal{A}_A$ is isometrically isomorphic to $\mathcal{A}_B$ if and only if $X_A \cong X_B$, and this happens if and only if there exists $\lambda \in \mathbb{C}$ and a unitary $2 \times 2$ matrix $U$ such that $B = \lambda U^*AU$.

The proof of Theorem 11.2 will occupy the rest of this section. Denote by $\mathcal{M}_A$ the character space of $\mathcal{A}_A$, that is, the topological space of contractive multiplicative and unital linear functionals on $\mathcal{A}_A$, endowed with the weak-* topology.

**Lemma 11.3.** The topology of $\mathcal{M}_A$ depends on the rank $r(A^s)$ of the symmetric part $A^s$ of $A$:

1. If $r(A^s) = 0$ then $\mathcal{M}_A \cong B_2$, the unit ball in $\mathbb{C}^2$.
2. If $r(A^s) = 1$ then $\mathcal{M}_A \cong D$, the unit disc in $\mathbb{C}$.
3. If $r(A^s) = 2$ then $\mathcal{M}_A$ is homeomorphic to two discs pasted together at the origin.

**Proof.** We proceed similarly to the lines of 10.2. Every character $\rho \in \mathcal{M}_A$ is uniquely determined by $\lambda_1 = \rho(S_1^A)$ and $\lambda_2 = \rho(S_2^A)$, which lie in $B_2$. Conversely, every $(\lambda_1, \lambda_2) \in B_2$ that satisfies

$$\sum_{i,j} a_{ij}\lambda_i\lambda_j = 0$$

gives rise to a character $\rho$ by defining $\lambda_1 = \rho(S_1^A)$ and $\lambda_2 = \rho(S_2^A)$. Thus,

$$\mathcal{M}_A \cong V_A := \left\{ (\lambda_i, \lambda_j) \in B_2 : \sum_{i,j} a_{ij}\lambda_i\lambda_j = 0 \right\}.$$

Clearly, $V_A = V_{A^*}$. However, every symmetric $2 \times 2$ matrix is complex congruent to one of the following:

$$D_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., there exists a nonsingular matrix $T$ such that $A^s = T^*D_iT$, for $i = r(A^s)$. But then $V_{A^s} = T^{-1}V_{D_i} \cong V_{D_i}$, so it remains to verify that $V_{D_i}$ is homeomorphic to the spaces listed in the statement of the lemma. \qed

**Corollary 11.4.** If $r(A^s) \neq r(B^s)$ then $\mathcal{A}_A \not\cong \mathcal{A}_B$.

We can use this corollary to break down the classification of the algebras $\mathcal{A}_A$ to the classification of the algebras $\mathcal{A}_A$ with fixed $r(A^s)$. The easiest case is $r(A^s) = 0$, because then $A$ is either the zero matrix or a multiple of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and these two matrices give rise to non isomorphic algebras (these are the algebras generated by the full and symmetric shifts, respectively).

The next easiest case is $r(A^s) = 2$.

**Lemma 11.5.** If $A, B \in M_2(\mathbb{C})$ and $r(A^s) = r(B^s) = 2$, then $\mathcal{A}_A$ is isometrically isomorphic to $\mathcal{A}_B$ if and only if $X_A \cong X_B$, and this happens if and only if there exists $\lambda \in \mathbb{C}$ and a unitary $2 \times 2$ matrix $U$ such that $B = \lambda U^*AU$. Any isometric
isomorphism between $\mathcal{A}_A$ and $\mathcal{A}_B$ arises as conjugation by the subproduct system isomorphism arising from $U$.

Proof. In light of Theorem 9.7 and Proposition 11.1, it suffices to show that any isometric isomorphism $\varphi : \mathcal{A}_A \to \mathcal{A}_B$ sends the vacuum state to the vacuum state. But the vacuum state in $\mathcal{M}_A$ and in $\mathcal{M}_B$ corresponds to the point where the two discs are glued together. Since $\varphi$ induces a homeomorphism between $\mathcal{M}_B$ and $\mathcal{M}_A$, it must send the vacuum state to the vacuum state. □

Remark 11.6. In the previous section we have seen already that there is a continuum of non-(completely isometrically)-isomorphic algebras $\mathcal{A}_A$ and subproduct systems $X_A$ with $r(A^*) = 2$, namely the algebras $\mathcal{A}_q$. One can see that these algebras $\mathcal{A}_A$ are not exhausted by the algebras $\mathcal{A}_q$ of the previous section. For example, all the algebras $\mathcal{A}_A$ with $A = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$, with $q > 0$, are non-isomorphic, and only for $q = 1$ is this algebra isomorphic to an $\mathcal{A}_q$ (in this case $q = -1$).

We now come to the trickiest case, $r(A^*) = 1$.

Lemma 11.7. If $A, B \in M_2(\mathbb{C})$ are two symmetric matrices of rank 1, then there exists $\lambda \in \mathbb{C}$ and a unitary $2 \times 2$ matrix $U$ such that $B = \lambda U^t A U$, and consequently $X_A \cong X_B$ and $\mathcal{A}_A$ is isometrically isomorphic to $\mathcal{A}_B$.

Proof. We only have to prove the first assertion, and we may assume that $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We may also assume that there is a unit vector $v = (v_1, v_2)^t$ such that $A = vv^t$. Now let

$$U = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}.$$ 

Then

$$U^t AU = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}^t \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

Below we will also need the following lemma.

Lemma 11.8. Let $A$ be a $2 \times 2$ matrix for which $r(A^*) = 1$. Then there exists one and only one $q \geq 0$ for which there is a $\lambda \in \mathbb{C}$ and a unitary $U$ such that

$$\begin{pmatrix} 1 & q \\ -q & 0 \end{pmatrix} = \lambda U^t AU.$$ 

Furthermore, if $A$ is non-symmetric then $A$ is congruent to the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Proof. Direct verification, using Lemma 11.7 and the fact that congruences preserves, up to a scalar, the anti-symmetric part. □

Let us write $A_q$ for the matrix

$$A_q = \begin{pmatrix} 1 & q \\ -q & 0 \end{pmatrix}.$$ 

By the above lemma, we may restrict attention only to the algebras $\mathcal{A}_{A_q}$ with $q \geq 0$. 


Recall that the character space $\mathcal{M}_{A_q}$ of $A_{A_q}$ is identified with the closed unit disc $\overline{B}$ by

$$\mathcal{M}_{A_q} \ni \rho \mapsto \rho(S^A_i) \in \overline{B}.$$ 

We write $\rho_z$ for the character that sends $S^A_i$ to $z \in \overline{B}$. This identifies the vacuum vector $\rho_0$ with the point 0. Recall also that if $\varphi : A_{A_q} \to A_{A_r}$ is an isometric isomorphism, then it induces a homeomorphism $\varphi_* : \mathcal{M}_{A_r} \to \mathcal{M}_{A_q}$ given by $\varphi_* \rho = \rho \circ \varphi$. We write $F_\varphi$ for the homeomorphism $\overline{D} \to \overline{D}$ induced by $\varphi$, that is, $F_\varphi$ is the unique self map of $\overline{D}$ that satisfies

$$\varphi_* \rho_z = \rho_{F_\varphi(z)}, \quad z \in \overline{B}.$$ 

Let us introduce the notation

$$\mathcal{O}(0; q, r) = \{ F_\varphi(0) | \varphi : A_{A_q} \to A_{A_r} \text{ is an isometric isomorphism} \},$$

and

$$\mathcal{O}(0; q) = \mathcal{O}(0; q, q).$$

**Lemma 11.9.** Let $q, r \geq 0$. If $q \neq r$ then 0 does not lie in $\mathcal{O}(0; q, r)$.

**Proof.** Assume that $0 \in \mathcal{O}(0; q, r)$. Then there is some isometric isomorphism $\varphi : A_{A_q} \to A_{A_r}$ that preserves the character $\rho_0$. It follows from Theorem 9.7 and Proposition 11.1 that, for some unitary $2 \times 2$ matrix $U$ and some $\lambda \in \mathbb{C}$, $A_q = \lambda U^* A_r U$. But, as noted in Lemma 11.8 this is impossible if $r \neq q$. □

**Lemma 11.10.** The sets $\mathcal{O}(0; q, r)$ are invariant under rotations around 0.

**Proof.** For $\lambda$ with $|\lambda| = 1$, write $\varphi_\lambda$ for the isometric isomorphism mapping $S^A_i$ to $\lambda S^A_i$ ($i = 1, 2$). For $b = F_\varphi(0) \in \mathcal{O}(0; q, r)$, consider $\varphi \circ \varphi_\lambda$. We have $\rho_0((\varphi \circ \varphi_\lambda)(S^A_i)) = \rho_0(\varphi(\lambda S^A_i)) = \lambda \rho_0(\varphi(S^A_i)) = \lambda b$. Thus $\lambda b \in \mathcal{O}(0; q, r)$. □

**Lemma 11.11.** Let $q, r \geq 0$. If $q \neq r$ then $A_{A_q}$ is not isometrically isomorphic to $A_{A_r}$.

**Proof.** Assume that $\varphi : A_{A_q} \to A_{A_r}$ is an isometric isomorphism. We have $\rho_0 \circ \varphi = \rho_b$, with $b = F_\varphi(0)$, and $F_\varphi$ is a homeomorphism of $\overline{D}$ onto itself.

By definition, $b \in \mathcal{O}(0; q, r)$. By Lemma 11.9 $b \neq 0$. Denote $C := \{ z | |z| = |b| \}$. By Lemma 11.10 $C \subseteq \mathcal{O}(0; q, r)$. Consider $C' := F_{\varphi}^{-1}(C)$. We have that $C' \subseteq \mathcal{O}(0; r)$. $C'$ is a simply connected closed path in $\overline{D}$ that goes through the origin. By Lemma 11.10 the interior of $C'$, $\text{int}(C')$, is in $\mathcal{O}(0; r)$. But then $F_{\varphi}(\text{int}(C'))$ is the interior of $C$, and it is in $\mathcal{O}(0; q, r)$. But then 0 $\in \mathcal{O}(0; q, r)$, contradicting Lemma 11.8. □

That concludes the proof of Theorem 11.2.

12. THE REPRESENTATION THEORY OF MATSUMOTO’S SUBSHIFT $C^*$-ALGEBRAS

In [27] Kengo Matsumoto introduced a class of $C^*$-algebras that arise from symbolic dynamical systems called “subshifts” (we note that in the later paper [16] Carlsen and Matsumoto suggest another way of associating a $C^*$-algebra with a subshift). Here we are discussing only the algebras originally introduced in [27]. These subshift algebras, as we shall call them, are strict generalizations of Cuntz-Krieger algebras and have been extensively studied by Matsumoto, T. M. Carlsen and others. For example, the following have been studied: criteria for simplicity
and pure-infiniteness; conditions on the underlying dynamical systems for subshift algebras to be isomorphic; the automorphisms of the subshift algebras; K-theory of the subshift algebras; and much more. In this section we will use the framework constructed in the previous sections to give a complete description of all representations of a subshift algebra when the subshift is of finite type.

12.1. Subshifts and the corresponding subproduct systems and C*-algebras.

Our references for subshifts are [27] and [15, Chapter 3].

Let \( I = \{1, 2, \ldots, d\} \) be a fixed finite set. \( I^\mathbb{Z} \) is the space of all two-sided infinite sequences, endowed with the product topology. The left shift (or simply the shift) on \( I^\mathbb{Z} \) is the homeomorphism \( \sigma : I^\mathbb{Z} \to I^\mathbb{Z} \) given by \( (\sigma(x))_k = x_{k+1} \). Let \( \Lambda \) be a shift invariant closed subset of \( I^\mathbb{Z} \). By this we mean \( \sigma(\Lambda) = \Lambda \). The topological dynamical system \((\Lambda, \sigma|_\Lambda)\) is called a subshift. Sometimes \( \Lambda \) is also referred to as the subshift.

If \( W \) is a set of words in \( 1, 2, \ldots, d \), one can define a subshift by forbidding the words in \( W \) as follows:

\[
\Lambda_W = \{ x \in I^\mathbb{Z} : \text{no word in } W \text{ occurs as a block in } x \}.
\]

Conversely, every subshift arises this way: i.e., for every subshift \( \Lambda \) there exists a collection of words \( W \), called the set of forbidden words, such that \( \Lambda = \Lambda_W \). In this context, if \( W \) can be chosen finite then \( \Lambda = \Lambda_W \) is called a subshift of finite type, or SFT for short. By replacing \( I \) if needed, we may always assume that \( W \) has no words of length one. If \( W \) can be chosen such that the longest word in \( W \) has length \( k+1 \) then \( \Lambda \) is called a \( k \)-step SFT. A 1-step SFT is also called a topological Markov chain. A basic result is that every SFT is isomorphic to a topological Markov chain ([15, Proposition 3.2.1]).

For a fixed subshift \((\Lambda, \sigma|_\Lambda)\), we set \( \Lambda^k = \{ \alpha : \alpha \text{ is a word with length } k \text{ occurring in some } x \in \Lambda \} \), and \( \Lambda_t = \bigcup_{k=0}^t \Lambda^k \), \( \Lambda_* = \bigcup_{k=0}^\infty \Lambda^k \). With the subshift \((\Lambda, \sigma|_\Lambda)\) we associate a subproduct system \( X_\Lambda \) as follows. Let \( \{e_i\}_{i \in I} \) be an orthonormal basis of a Hilbert space \( E \). We define

\[
X_\Lambda(0) = \mathbb{C},
\]

and for \( n \geq 1 \) we define

\[
X_\Lambda(n) = \text{span}\{e_\alpha : \alpha \in \Lambda^n\}.
\]

We define a product \( U_{m,n} : X_\Lambda(m) \otimes X_\Lambda(n) \to X_\Lambda(m+n) \) by

\[
U_{m,n}(e_\alpha \otimes e_\beta) = \begin{cases} e_{\alpha \beta}, & \text{if } \alpha \beta \in \Lambda^{m+n} \\ 0, & \text{else.} \end{cases}
\]

Since \( \Lambda^{m+n} \subseteq \Lambda^m \cdot \Lambda^n \), \( X_\Lambda \) is a standard subproduct system.

**Definition 12.1.** The C*-algebra associated with a subshift \((\Lambda, \sigma|_\Lambda)\) is defined as the quotient algebra

\[
\mathcal{O}_\Lambda := \mathcal{O}_{X_\Lambda} = T_{X_\Lambda}/\mathcal{K}(\mathfrak{H}_{X_\Lambda}).
\]

**Remark 12.2.** Just to prevent confusion: In [27], \( \mathcal{O}_\Lambda \) was defined as the quotient by the compacts of the C*-algebra generated by the “creation operators” (that is, the \( X \)-shift) on \( \mathfrak{H}_X \), without using the language of subproduct systems.
12.2. Subproduct systems that come from subshifts.

**Proposition 12.3.** Let $X$ be a standard subproduct system such that there is an orthonormal basis $\{e_i\}_{i \in I}$ of $X(1)$, with $I$ finite, such that

1. Every $X(n)$, $n \geq 1$, is spanned by vectors of the form $e_\alpha$ with $|\alpha| = n$.
2. For all $m, n \in \mathbb{N}$, $|\alpha| = n$ and $e_\alpha \in X(n)$, implies that there is some $\beta, \gamma \in I^m$ such that $e_\beta \otimes e_\alpha$ and $e_\alpha \otimes e_\gamma$ are in $X(m + n)$.

Then there is a shift invariant closed subset $\Lambda$ of $I^\mathbb{Z}$ such that $X = X_\Lambda$. $X$ is the maximal standard subproduct system with prescribed fibers $X(1), X(2), \ldots, X(k+1)$ if and only if $\Lambda$ is $k$-step SFT.

**Proof.** For all $k \in \mathbb{N}$, define

$$\Lambda^{(k)} = \{ \alpha \in I^k : e_\alpha \in X(k) \}.$$

For all $m \in \mathbb{Z}, k \in \mathbb{N}$, define the closed sets

$$A_{m,k} = \{ x \in I^\mathbb{Z} : (x_m, x_{m+1}, \ldots, x_{m+k-1}) \in \Lambda^{(k)} \}.$$

Condition (2) implies that $X(k)$ always contains a nonzero vector of the form $e_\alpha$, $|\alpha| = k$. That implies that the family $\{A_{m,k}\}_{m,k}$ has the finite intersection property. Indeed,

$$A_{m_1,k_1} \cap A_{m_2,k_2} \supseteq A_{M,K} \neq \emptyset,$$

where $M = \min\{m_1, m_2\}, \quad K = \max\{m_2 + k_2, m_1 + k_1\} - M$. By compactness of $I^\mathbb{Z}$ we conclude that the closed set

$$\Lambda := \bigcap_{m,k} A_{m,k}$$

is non-empty. $\Lambda$ is invariant under the left and the right shifts, so $\sigma(\Lambda) = \Lambda$, so $(\Lambda, \sigma|_{\Lambda})$ is a subshift. By condition (2), $\Lambda^k = \Lambda^{(k)}$. Condition (1) together with the definition of $X_\Lambda$ now imply that $X = X_\Lambda$.

The final assertion follows from the following facts, together with $X = X_\Lambda$. Fact number one:

$$E \otimes^n \cap X_\Lambda(n) = \operatorname{span}\{e_\alpha : \alpha \text{ is a forbidden word of length } n\}.$$  

Fact number two: $X$ is the maximal standard subproduct system with prescribed fibers $X(1), \ldots, X(k+1)$ if and only if for every $n > k + 1$,

$$X(n) = \bigcap_{i+j=n} X(i) \otimes X(j),$$

or in other words, if and only if

$$E \otimes^n \cap X(n) = \bigvee_{i+j=n} (E \otimes (X(i) \otimes X(j))).$$

Fact number three: $\Lambda$ is a $k$-step SFT if and only if for every $n > k + 1$,

$$\{\text{forbidden words of length } n\} = \bigcup_{i+j=n} (I^i \cdot \{\text{forbidden words of length } j\} \cup \{\text{forbidden words of length } i\} \cdot I^j).$$

These facts assemble together to complete the proof. □
Not every subproduct system is isomorphic to one that comes from a subshift. Indeed, in the symmetric subproduct system SSP (see Example 1.3) for any basis \( \{e_i\}_{i \in \mathcal{I}} \) of \( X(1) \), the product \( e_i \otimes e_j \) for \( i \neq j \) is never in \( X(2) \), and thus the images \( f_i \) and \( f_j \) of \( e_i \) and \( e_j \) in any isomorphic subproduct system \( X \) can never be such that \( f_i \otimes f_j \) is mapped isometrically to \( U_{1,1}^X(f_i \otimes f_j) \). Thus if SSP is isomorphic to \( X_\Lambda \) for some subshift \( \Lambda \), then \( \Lambda \) must be the subshift containing only constant sequences. But such \( X_\Lambda \) is clearly not isomorphic to SSP.

As another example, the subproduct system \( X(0) = \mathbb{C}, X(1) = \mathbb{C}^2 \), and \( X(n) = 0 \) for \( n > 1 \), cannot be of the form \( X_\Lambda \) for any \( \Lambda \subseteq \mathcal{T} \).

12.3. The representation theory of the C*-algebra associated with a subshift of finite type. Let \( \Lambda \) be a fixed subshift in \( \mathcal{T}^2 \) (with \( \mathcal{T} = \{1,2,\ldots,d\} \)), and let \( X = X_\Lambda \) be the associated subproduct system. We will denote the \( X \)-shift by \( S \) (instead of \( S^X \)) to make some formulas more readable. Let \( Z_i \) be the image of \( S_i \) in the quotient \( \mathcal{O}_\Lambda \). We define for \( i \in \mathcal{T}, k \in \mathbb{N} \) the sets

\[
E^k_i = \{ \alpha \in \Lambda^k : i\alpha \in \Lambda^* \}.
\]

**Lemma 12.4.** If \( \Lambda \) is a \( k \)-step SFT, then for all \( i \in \mathcal{T} \),

\[
\{ \gamma \in \Lambda^* : |\gamma| \geq k, i\gamma \in \Lambda^* \} = \{ \alpha \beta \in \Lambda^* : \alpha \in E^k_i, \beta \in \Lambda^* \}.
\]

**Proof.** Assume that \( \gamma \in \Lambda^* \) is such that \( |\gamma| \geq k \) and \( i\gamma \in \Lambda^* \). Defining \( \alpha = \gamma_1 \cdots \gamma_k \) and \( \beta = \gamma_{k+1} \cdots \gamma_{k+i} \), we have that \( \gamma = \alpha \beta \) where \( \alpha \in E^k_i \) and \( \beta \in \Lambda^* \).

Conversely, if \( \gamma = \alpha \beta \in \Lambda^* \) where \( \alpha \in E^k_i \) and \( \beta \in \Lambda^* \), then \( i\gamma \) must be in \( \Lambda^* \). Indeed, if not, then \( i\gamma \) must contain a forbidden word. But \( \gamma \in \Lambda^* \), thus the forbidden word must be in \( i\alpha \) (since \( \Lambda \) is a \( k \)-step SFT). But that is impossible because \( \alpha \in E^k_i \).

**Lemma 12.5.** If \( \Lambda \) is a \( k \)-step SFT then for all \( i,j \in \mathcal{T}, i \neq j \),

\[
S^*_i S_j = 0,
\]

and

\[
(12.1) \quad S^*_i S_i = \sum_{\alpha \in E^k_i} \mathbf{Z}^* \mathbf{Z}^\alpha \mod \mathcal{K}_X.
\]

Consequently, \( \mathcal{E}_X = \mathcal{T}_X \).

**Proof.** Since the \( S_i \) are partial isometries with orthogonal ranges, we have \( S^*_i S_j = 0 \) for all \( i \neq j \). Since \( \mathcal{K}_X \subseteq \mathcal{E}_X \subseteq \mathcal{T}_X \) (Proposition 8.11), \( \mathcal{E}_X = \mathcal{T}_X \) will be established once we prove (12.1).

\( S^*_i S_i \) is the projection onto the initial space of \( S_i \). Call this space \( G \). We have

\[ G = \text{span}\{e_\alpha : \alpha \in \Lambda^* \text{ such that } i\alpha \in \Lambda^* \}. \]

The space

\[ G' = \text{span}\{e_\alpha : \alpha \in \Lambda^* \text{ such that } i\alpha \in \Lambda^* \text{ and } |\alpha| \geq k \} \]

has finite codimension in \( G \). But by Lemma 12.4,

\[ G' = \{e_{\alpha \beta} : \alpha \beta \in \Lambda^*, \alpha \in E^k_i \}, \]

that is, \( G' \) is spanned by \( e_\gamma \) where \( \gamma \) runs through all legal words beginning with some \( \alpha \in E^k_i \). Thus, \( G' \) is the range of the projection \( \sum_{\alpha \in E^k_i} \mathbf{Z}^\alpha \mathbf{Z}^\alpha \). Since \( G' \) has finite codimension in \( G \), we have (12.1).

\[ \square \]
Proposition 12.6. For every subshift $\Lambda$, the $d$-tuple $Z = (Z_1, \ldots, Z_d)$ satisfies the following relations:

\begin{align}
(12.2) & \quad p(Z) = 0, \text{ for all } p \in I^X, \\
(12.3) & \quad Z_i^* Z_j = 0, \text{ for all } i, j \in I, i \neq j, \\
\text{and} & \quad \sum_{i=1}^d Z_i Z_i^* = 1.
\end{align}

In particular, $Z_i$ is a partial isometry for all $i \in I$. If $\Lambda$ is a $k$-step SFT, the $Z$ also satisfies

\begin{equation}
Z_i^* Z_i = \sum_{\alpha \in E^*_k} Z_\alpha Z_\alpha^*, \text{ for all } i \in I.
\end{equation}

Proof. The quotient map $T_X \to O_\Lambda$ is a $\ast$-homomorphism, so (12.2) follows from Theorem 7.5. (12.3) and (12.5) follow from the previous lemma, and (12.4) follows from equation (8.2). \qed

Theorem 12.7. Let $\Lambda$ be a $k$-step SFT. Every unital representation $\pi : O_\Lambda \to B(H)$ is determined by a row-contraction $T = (T_1, \ldots, T_d)$ satisfying relations (12.2)-(12.7) such that $\pi(Z_i) = T_i$ for all $i \in I$. Conversely, every row contraction in $B(H)^d$ satisfying the relations (12.2)-(12.5) gives rise to a unital representation $\pi : O_\Lambda \to B(H)$ such $\pi(Z_i) = T_i$ for all $i \in I$.

Proof. It is the second assertion that is non-trivial, and we will try to convince that it is true. By Theorem 8.2, there is unital completely positive map $\Psi : E_X \to B(H)$ sending $\sum_{i=1}^d S_i S_i^*$ to $T_i^\alpha T_i^\beta$. Since enough of the rank one operators on $\mathcal{F}_X$ arise as $S_i^\alpha (I - \sum_{i=1}^d S_i S_i^*) S_i^\beta$ (see equation (8.2)), and because $T$ satisfies (12.4), we must have that $\Psi(K) = 0$ for every $K \in \mathcal{K}(\mathcal{F}_X)$. By Lemma 12.5 $E_X = T_X$, and it follows that $\Psi$ induces a positive and unital (hence contractive) mapping $\pi : O_\Lambda \to B(H)$

\[ \Psi : E_X \to B(H) \]

that sends $Z_\alpha Z_\beta^*$ to $T_\alpha T_\beta$. Roughly speaking: $\pi$ must be multiplicative because $Z$ and $T$ satisfy the same relations. In more detail: every product $(Z_\alpha Z_\beta^*) (Z_\alpha' Z_\beta'^*)$ may be written, using the relations (12.2), (12.5) as some sum $\sum_{\gamma, \delta} Z_\gamma Z_\delta$. The mapping $\pi$ then takes this sum to $\sum_{\gamma, \delta} \pi(Z_\gamma) \pi(Z_\delta)$, and this can be rewritten (using the same relations) as

\[ (T_\gamma T_\delta^*) (T_\gamma' T_\delta'^*) = \pi(Z_\gamma Z_\delta^*) \pi(Z_\gamma' Z_\delta'^*). \]

This shows that

\[ \pi \left( (Z_\alpha Z_\beta^*) (Z_\alpha' Z_\beta'^*) \right) = \pi(Z_\alpha Z_\beta^*) \pi(Z_\alpha' Z_\beta'^*), \]

and since the elements of the form $Z_\alpha Z_\beta^*$ span $O_\Lambda$, and since $\pi$ is a positive linear map, it follows that $\pi$ is in fact a $\ast$-representation. \qed
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