Exact solution of the random bipartite matching model

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Abstract

In this paper we present the exact solution for the average minimum energy of the random bipartite matching model with an arbitrary finite number of elements where random paired interactions are described by independent exponential distribution. This solution confirms the Parisi conjecture proposed for this model earlier, as well as the result of the replica solution of this model in the thermodynamic limit.

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1 The Model

The model under consideration can be formulated as follows. We have a society consisting of \(N\) "men" (labeled by \(i = 1, 2, \ldots, N\)) and \(N\) "women" (labeled by \(j = 1, 2, \ldots, N\)) described by a given set of \(N^2\) random non-negative interactions \(\{J_{ij}\}\) between every man and every woman. The statistics of \(J_{ij}\)'s is defined by a probability distribution function \(P[J_{ij}]\).

Then we consider all possible "marriages" with the strict monogamy: every man can be connected with one and only one women, and vice versa. Thus, a particular marriage configuration in this society can be described by the \(N \times N\) permutation matrix \(S_{ij}\) with the elements taking values 0 or 1 ("0" for all non-coupled pairs of men and women, and "1" for married couples) constrained by two conditions:

\[
\sum_{i=1}^{N} S_{ij} = \sum_{j=1}^{N} S_{ij} = 1 \tag{1.1}
\]

which allow one and only one "1" in each row and in each column of the matrix \(\hat{S}\). The total number of all possible marriage configurations in this society is thus equal to \(N!\).

Now for every marriage configuration \(\hat{S}\) we introduce the total energy, or total weight (the Hamiltonian):

\[
H[\hat{S}; \hat{J}] = \sum_{i,j=1}^{N} S_{ij}J_{ij} \tag{1.2}
\]

For a given matrix \(\hat{S}\) this energy is equal to the sum of \(N\) particular \(J_{ij}\)'s (one from each line and each column) corresponding to the particular married couples. In this paper we consider the simplest possible model in which the interactions \(\{J_{ij}\}\) are assumed to be independent and described by the bounded exponential distribution:

\[
P[J_{ij}] = \prod_{i,j=1}^{N} \exp(-J_{ij}) ; \quad (0 \leq J_{ij} < +\infty) \tag{1.3}
\]

The problem studied below is formulated as follows: one has to find the value \(E_N\) of the average (over the distribution \(P[J_{ij}]\)) minimum (over all configurations of the permutation matrix \(S_{ij}\)) energy \(\langle 1.2 \rangle\):

\[
E_N = \left[ \prod_{i,j=1}^{N} \int_{0}^{\infty} dJ_{ij} \right] P[J_{ij}] \min_{S_{ij}} \left( \sum_{i,j=1}^{N} S_{ij}J_{ij} \right) \tag{1.4}
\]

Equivalently, in the language of statistical mechanics \(E_N\) can be obtained as the zero-temperature limit of the average free energy:

\[
E_N = -\lim_{\beta \to \infty} \frac{1}{\beta} \left[ \prod_{i,j=1}^{N} \int_{0}^{\infty} dJ_{ij} \exp(-J_{ij}) \right] \log \left( \sum_{S_{ij}} \exp \left( -\beta \sum_{i,j=1}^{N} S_{ij}J_{ij} \right) \right)
\]
\[
\equiv - \lim_{\beta \to \infty} \frac{1}{\beta} \left( \log \left\{ \sum_{S_{ij}} \exp\{ -\beta H[\hat{S}; \hat{J}] \} \right\} \right) \quad (1.5)
\]

Thus, we face the typical problem of statistical mechanics with quenched disorder: first, for given values of random parameters \( \{ J_{ij} \} \) one has to compute the partition function and the free energy, and only after that one has to carry out the averaging over \( J_{ij} \)'s.

In the thermodynamic limit \( (N \to \infty) \) this problem has been solved some years ago in the framework of the replica symmetric ansatz \( [1] \), yielding the result:

\[
E_{N \to \infty} = \zeta(2) = \frac{\pi^2}{6} \quad (1.6)
\]

In this paper we present the exact solution of this problem for an arbitrary (finite) value of \( N \).

The case \( N = 1 \) is trivial:

\[
E_{N=1} = 1 \quad (1.7)
\]

The case \( N = 2 \) is only slightly more complicated, and it can also be easily calculated explicitly. Here the \( 2 \times 2 \) permutation matrix \( \hat{S} \) can have only two configurations:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \quad (1.8)
\]

and

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \quad (1.9)
\]

Thus, according to the definitions \( (1.4) \) or \( (1.5) \) we have:

\[
E_{N=2} = 2 \int_0^\infty dJ_{11} dJ_{12} dJ_{21} dJ_{22} (J_{11} + J_{22}) \exp\{ -J_{11} - J_{12} - J_{21} - J_{22} \} \theta(J_{12} + J_{21} - J_{11} - J_{22}) \quad (1.10)
\]

Here the \( \theta \)-function ensures that the state \( (1.8) \) has lower energy than the one \( (1.9) \) (due to obvious symmetry of the system the contribution from the opposite situation turns out to be the same, and this provides the factor 2 in the above equation). Simple integration yields:

\[
E_{N=2} = 1 + \frac{1}{4} \quad (1.11)
\]

Noting that the result \( (1.6) \) for \( N = \infty \) can be represented also in the form:

\[
E_{N \to \infty} = \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} \quad (1.12)
\]

and taking into account the results \( (1.7) \) and \( (1.11) \), G.Parisi has recently proposed very elegant conjecture that the solution of the problem for arbitrary value of \( N \) must be the following \( [2] \):
\[ E_N = \sum_{k=1}^{N} \frac{1}{k^2} \quad (1.13) \]

For the direct calculation of \( E_N \) (in the style of eq.(1.10)) with an arbitrary \( N \) one should perform the integration over the parameters \( \{J_{ij}\} \) in the constrained positive subspace \( J_{ij} \geq 0 \) of the \( N^2 \)-dimensional space. Since the total number of states of the \( N \times N \) permutation matrix is equal to \( N! \) this integration is also constrained by \((N! - 1)\) hyperplanes which guarantee that chosen one particular state has the minimum energy. One can easily verify that even in the case \( N = 3 \) such calculation turns out to be extremely difficult problem. Nevertheless, simple numerical tests for \( N = 3, 4, 5 \) proved to be compatible with the above conjecture with the precision \( \sim 10^{-5} \) \[2\]. Moreover, recent analytical studies have provided the exact solution of this problem up to \( N = 4 \) and the result of this solution confirms the conjecture \((1.13)\) \[3\]. Here we use the original idea (proposed by S.Bravyi) of this unpublished work to prove that the conjecture \((1.13)\) is indeed correct for arbitrary \( N \).

2 The Proof

To ease further presentation of the proof let us introduce the following notation. The operation of finding the average of minimum energy of the \( N \times N \) problem (defined in eqs.\((1.4)\) or \((1.5)\)) will be denoted by the symbol

\[
E \begin{pmatrix}
... & ... & ... \\
... & ... & ... \\
... & ... & ... \\
\end{pmatrix} = E_N \quad (2.1)
\]

It is assumed that "empty" boxes in the above matrix actually contain random elements \( \{J_{ij}\} \)

Let us consider the first line of the random matrix \( J_{ij} \), and among \( N \) its elements \( J_{ij} \) let us find the minimum one: \( J^{(1)} = \min_j (J_{1j}) \). Due to obvious symmetry of the problem with respect to permutations of the columns of the matrix \( J_{ij} \) we can always place this minimum element in the position \((1, 1)\). Now let us redefine the elements of the first line as follows:

\[ J_{ij} = J^{(1)} + \tilde{J}_{ij}, \quad (j \neq 1) \quad (2.2) \]

and leave all the other elements unchanged. According to \((1.3)\), the elements \( \tilde{J}_{ij} \) are described by the same exponential distribution: \( P[\tilde{J}_{ij}] = \exp(-\tilde{J}_{ij}), \quad (\tilde{J}_{ij} \geq 0) \), while for \( J^{(1)} \) the distribution is:

\[ P[J^{(1)}] = N \exp(-N J^{(1)}) \quad (2.3) \]

Due to the constrains \((1.3)\) the above redefinition produces only simple shift of the Hamiltonian \((1.2)\):
\[ H = J^{(1)} + \sum_{i,j=1}^{N} S_{ij} \tilde{J}_{ij} \]  

(2.4)

where, the random matrix \( \tilde{J}_{ij} \) contains "0" in the position (1,1), while the rest of its elements are described by the same distribution (1.3). Now using the definition of \( E_N \), eq.(1.3), we can easily integrate out \( J^{(1)} \) to get:

\[ E_N = \frac{1}{N} + E_N^{(1)} \]  

(2.5)

where

\[ E_N^{(1)} = \begin{pmatrix} 
0 & \ldots & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots \\
\end{pmatrix} \]  

(2.6)

To calculate \( E_N^{(1)} \) let us consider the second line of the above random matrix, and among \( N \) its elements \( J_{2j} \) let us find the minimum one: \( J^{(2)} \equiv \min_j (J_{2j}) \). Now, due to “0” in the position (1,1) the first column of this matrix is no more equivalent to the rest of \( (N - 1) \) columns (which remain to be equivalent among themselves). Therefore, with the probability \( 1/N \) the minimum element can be in the position (2,1), and with the probability \( (N - 1)/N \) it can be in the rest of the positions of the second line, and in this last case we can place it in the position (2,2). Then we shift the values of the elements of the second line: \( J_{2j} = J^{(2)} + \tilde{J}_{2j} \) (which leave the distribution of \( \{ \tilde{J}_{2j} \} \) unchanged). The integration over \( J^{(2)} \) gives one more factor \( 1/N \), and for \( E_N \) we get:

\[ E_N = \frac{2}{N} + \frac{(N - 1)}{N} E_N^{(2)} + \frac{1}{N} \tilde{E}_N^{(2)} \]  

(2.7)

where

\[ E_N^{(2)} = \begin{pmatrix} 
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots \\
\end{pmatrix} \]  

(2.8)

and
\[
\tilde{E}_N^{(2)} = \begin{pmatrix}
0 & \cdots & \\
0 & \cdots & \\
\cdots & \cdots & \\
\end{pmatrix}
\]

Eq.(2.7) can be represented in the form:

\[
E_N = \frac{2}{N} + E_N^{(2)} + \frac{1}{N} \delta E_N^{(2)}
\]

where

\[
\delta E_N^{(2)} = \begin{pmatrix}
0 & \cdots & \\
0 & \cdots & \\
\cdots & \cdots & \\
\end{pmatrix} - \begin{pmatrix}
0 & \cdots & \\
0 & \cdots & \\
\cdots & \cdots & \\
\end{pmatrix}
\]

To calculate the value \(E_N^{(2)}\) defined by the matrix:

\[
\begin{pmatrix}
0 & \cdots & \\
0 & \cdots & \\
\cdots & \cdots & \\
\end{pmatrix}
\]

let us consider its third line, and among \(N\) elements \(J_{3j}\) let us find the minimum one: \(J^{(3)} \equiv \min_j (J_{3j})\). Due to two "0" in the positions (1,1) and (2,2) the first and the second columns of this matrix are equivalent between themselves, but they are not equivalent to the rest of \((N-2)\) columns (which remain to be equivalent among themselves). Therefore, with the probability \(2/N\) the minimum element can be placed in the position (3,2), and with the probability \((N-2)/N\) it can be in the rest of the positions of the third line, and here we can place it in the position (3,3). Then we shit the values of the elements of the third line: \(J_{3j} = J^{(3)} + \tilde{J}_{3j}\) (which again leave the distribution of \(\{\tilde{J}_{3j}\}\) unchanged), and integrate over \(J^{(3)}\) which gives one more factor \(1/N\). In this way we get:

\[
E_N = \frac{3}{N} + \frac{(N-2)}{N} E_N^{(3)} + \frac{2}{N} \tilde{E}_N^{(3)} + \delta E_N^{(2)}
\]

where
Eq. (2.13) can be represented in the form:

\[
E_N = 3N + E_N^{(3)} + 2N \delta E_N^{(3)} + 1N \delta E_N^{(2)}
\]  

(2.16)

where

\[
\delta E_N^{(3)} = E - E_N^{(3)}
\]

(2.17)

Proceeding in this way up to the last line we eventually get:

\[
E_N = 1 + \sum_{k=2}^{N} \frac{k - 1}{N} \delta E_N^{(k)}
\]  

(2.18)

(note that \(E_N^{(N)} \equiv 0\) since it is given by the matrix with all zeros on the diagonal) where
Here the double lines mark the positions of the $k$-th column and the $k$-th line.

It can be proved (see Appendix A) that the above value $\delta E_N^{(k)}$ is given by the rectangular $N \times k$ random matrix problem:

$$
\delta E_N^{(k)} = E \begin{pmatrix}
0 & \ldots & 0 & \ldots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots \\
\vdots & \ldots & \vdots & \vdots \\
\vdots & \ldots & \vdots & \vdots \\
\vdots & \ldots & \vdots & \vdots \\
\end{pmatrix} - E 
$$

(2.19)

defined by the Hamiltonian:

$$
H[\hat{S}; \hat{J}] = \sum_{i=1}^{N} \sum_{j=1}^{k} S_{ij} J_{ij} 
$$

(2.21)

where the random matrix $J_{ij}$ is shown in eq.(2.20) (with the same independent exponential distributions of non-zero elements). Here the ”truncated” $N \times k$ part of the original permutation matrix $\hat{S}$ again can have only one ”1” in each line, and besides it has $k$ columns each containing only one ”1” and $(N - k)$ columns each containing only ”0”.

It turns out that the above ”rectangular” problem, eq.(2.20) can be solved explicitly (the proof see in Appendix B):

$$
\delta E_N^{(k)} = E \begin{pmatrix}
0 & \ldots & 0 & \ldots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots \\
\vdots & \ldots & \vdots & \vdots \\
\vdots & \ldots & \vdots & \vdots \\
\vdots & \ldots & \vdots & \vdots \\
\end{pmatrix}
$$

(2.20)

Substituting this result into eq.(2.18) we find:

$$
E_N = 1 + \frac{1}{N} \sum_{k=2}^{N} \frac{1}{k} \sum_{l=1}^{k-1} \frac{l}{N - l} 
$$

(2.23)

After simple algebra one eventually finds:
\[ E_N - E_{N-1} = \frac{1}{N^2} \]  

(2.24)

which proves the result (1.13).

It should be noted in conclusion that obtained solution is valid only for the considered exponential type distribution, eq.(1.3). It is crucial for the above proof that the form of the distribution of a random element \( J_{ij} \) doesn’t change after its shift on a constant value. On the other hand, it is clear from the above proof that in the thermodynamic limit \( N \to \infty \) the leading (in \( 1/N \)) contribution to \( E_N \) is defined only by the very beginning of the distribution, \( P[J \to 0] \). Therefore, the result \( E_{N \to \infty} = \zeta(2) \) must be correct, also for the “rectangular” type distribution: \( P[0 \leq J \leq 1] = 1 \); \( P[J > 1] = 0 \) (it is actually the model with this type of distribution which was studied in the replica solution [3]). For the discussion of other types of the matching models see e.g. [4] and references therein.

3 Appendix A

In this Appendix we prove that value of \( \delta E_N^{(k)} \) defined in eq.(2.19) is given by the rectangular \( N \times k \) problem (2.20).

First, let us consider the most simple case \( k = 2 \):

\[
\delta E_N^{(2)} = E \begin{pmatrix}
0 & & & \\
0 & & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\vdots & & \ldots & \\
\end{pmatrix}
- E \begin{pmatrix}
0 & & & \\
0 & & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\vdots & & \ldots & \\
\end{pmatrix}
\equiv \tilde{E}_N^{(2)} - E_N^{(2)}
\]  

(3.1)

The above two problems, \( \tilde{E}_N^{(2)} \) and \( E_N^{(2)} \), differ only by the permutation of two elements: (2,1) and (2,2), while all the other matrix elements \( J_{ij} \) in both matrices are the same. Nevertheless, even this "tiny" permutation, in general, can make the ground state configurations of the matrix \( \hat{S} \) in the two problems to be quite different. Note that for the calculation of the above average energy difference \( \delta E_N^{(2)} \) we can make averaging over \( J_{ij} \) both simultaneously (keeping \( J_{ij} \) to be the same in both problems) as well as separately for \( \tilde{E}_N^{(2)} \) and for \( E_N^{(2)} \).

For further proof it is important to introduce the concept of equivalence among the columns (and among the lines). We call the two columns \( j_1 \) and \( j_2 \) (or the two lines \( i_1 \) and \( i_2 \)) equivalent if the probabilities of the positions \((i,j_1) \) and \((i,j_2) \) (or \((i_1,j) \) and \((i_2,j) \)) to be occupied in the ground state are equal.

Due to obvious symmetry properties of the systems under consideration, it is evident that in
each of the above problems, $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, all the columns on the right of the double vertical line, and all the lines below the double horizontal line are equivalent among themselves. On the other hand, the first two lines in each of the above problems are also equivalent between themselves, but they are not equivalent to the rest of the $(N - 2)$ lines. Besides, in the problem $E_N^{(2)}$ we have first two columns which are equivalent between themselves, but which are not equivalent to the rest of the $(N - 2)$ columns. Finally in the problem $\tilde{E}_N^{(2)}$ the first column is not equivalent to the rest of the $(N - 1)$ columns.

The energy difference $\delta E_N^{(2)}$, eq(3.1), can be represented as follows:

$$\delta E_N^{(2)} = \left( \sum_{j=1}^{N} \left[ J_{\tilde{i}(j)j} - J_{i(j)j} \right] \right)$$  \hspace{1cm} (3.2)

where $J_{\tilde{i}(j)j}$ and $J_{i(j)j}$ represent the elements of the $j$-the column occupied in the ground states of the problems $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$ correspondingly.

In any ground state in each of these problems there are two columns in which the occupied elements belongs to one the first two lines. Therefore, in the summation over the columns in eq.(3.2) we can find two, three or four columns in which one of the elements (or both) of the difference $(J_{\tilde{i}(j)j} - J_{i(j)j})$ belongs to one of the first two lines. In the rest of the columns (their number can be $(N - 2)$, $(N - 3)$ or $(N - 4)$) both the element $(\tilde{i}(j), j)$ and the element $(i(j), j)$ belong to the rest of $(N - 2)$ equivalent lines. Now, if with some probability we have the ground states in which the elements $(\tilde{i}(j), j)$ is occupied in the problem $\tilde{E}_N^{(2)}$ and the element $(i(j), j)$ is occupied in the problem $E_N^{(2)}$, then with the same probability we must have the inverse situation: the element $(\tilde{i}(j), j)$ is occupied in the problem $E_N^{(2)}$, while the element $(i(j), j)$ is occupied in the problem $\tilde{E}_N^{(2)}$. Since both problems are defined by the same matrix $J_{ij}$ (except for the two elements (2,1) and (2,2)) in the averaging over $J_{ij}$ in eq.(3.2) the contributions from all the columns with no elements in one of the first two lines must cancel.

Thus, to compute the difference $\delta E_N^{(2)}$, we should take care only of those columns which in the ground states of the problems $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$ contain elements of the first two lines.

Due to the equivalence of the first two lines and due to equivalence of the $(N - 2)$ columns $(j = 3, ..., N)$ we can reduce all the relevant ground states of the problem $E_N^{(2)}$ to the following four non-equivalent basic configurations:

|          |          |          |          |
|----------|----------|----------|----------|
| 0        | 0        | $\odot$  | $\odot$  |
| 0        | $\bullet$| 0        | $\bullet$|
| (a)      | (b)      | (c)      | (d)      |

where "$\bullet$" represent the elements occupied in the ground state configuration of the matrix $\hat{S}$, and "$\odot$" denote occupied element "0". Note that each of the above configurations represents the whole set of equivalent configurations. For instance, (3.3(b)) represents all configurations with "$\bullet$" in any of $(N - 2)$ positions $(2, j)$, $(j = 3, ..., N)$, as well as all configurations with "$\bullet$" in the position (2,1) and another "$\bullet$" in any of $(N - 2)$ positions $(1, j)$, $(j = 3, ..., N)$. The diagram (3.3(c)) represents
all configurations in which one of the zeros is occupied. Note also that the configurations of the type:

\[
\begin{array}{cccc}
0 & \bullet & 0 \\
\bullet & 0 & \end{array}
\quad (3.4)
\]

must be excluded from the consideration since they can not be the ground state being always higher in energy than the states represented in \((3.3(d))\).

Due to equivalence of \((N−1)\) columns \((j = 2, ..., N)\) in the problem \(\tilde{E}_N^{(2)}\) here we have only two non-equivalent basic configurations:

\[
\begin{array}{cccc}
0 & * & 0 \\
0 & * & \end{array} \quad ; \quad \begin{array}{cccc}
\bigcirc & * & 0 \\
0 & * & \end{array}
\quad (3.5)
\]

Here for the occupied positions we use the notation ”*” instead of ”•” to distinguish them from the ones in the ground states of the problem \(E_N^{(2)}\).

Now to compute the contribution to the difference of the energies \(\delta E_N^{(2)}\), eq.\((3.2)\), we have to consider all possible combinations of the ground state configurations of the problem \(E_N^{(2)}\), eq.\((3.3)\), and of the ones of the problem \(\tilde{E}_N^{(2)}\), eq.\((3.5)\).

It is evident that if in the problem \(E_N^{(2)}\) we have one of the configurations of the type \((3.3(a))\) or \((3.3(b))\) and in the problem \(\tilde{E}_N^{(2)}\) we have one of the configurations of the type \((3.5(a))\) (all those in which no one ”0” is occupied), then (since the two problems contain the same set of \(J_{ij}\)’s) the positions of ”•” and ”*” must coincide. Therefore, these two cases give no contribution to \(\delta E_N^{(2)}\), eq.\((3.2)\).

It is also evident that the combination of one of the ground states of the type \((3.3(a))\) or \((3.3(b))\) with \((3.5(b))\) is impossible. For example, let us suppose that the ground state of the problem \(E_N^{(2)}\) is the configuration \((3.3(a))\), and the one of the problem \(\tilde{E}_N^{(2)}\) is the configuration \((3.3(b))\). Then, according to the definition of the ground state, the energy of \((3.3(a))\) must be smaller than the one of the configuration \((3.3(d))\), which in turn (since the problem \(\tilde{E}_N^{(2)}\) contain the same set of \(J_{ij}\)’s) must be smaller than the energy of the configuration \((3.3(b))\). On the other hand, the energy of the configuration \((3.3(a))\) is equal to the one

\[
\begin{array}{cccc}
0 & * & 0 \\
0 & * & \end{array}
\quad (3.6)
\]

of the problem \(\tilde{E}_N^{(2)}\). Thus, the energy of \((3.6)\) is smaller than the one of \((3.5(b))\), and therefore \((3.3(b))\) can not be the ground state.

Similar arguments shows that the combinations of \((3.3(c))\) with \((3.5(a))\), as well as \((3.5(d))\) with \((3.3(a))\) are also impossible.
The combination of (3.3(c)) and (3.5(b)) is allowed, but in this case, according to the definition of the ground state, the position of "•" in (3.3(c)) of the problem $E_N^{(2)}$ must coincide with the position of "∗" in (3.5(b)) of the problem $\tilde{E}_N^{(2)}$, and therefore this combination also gives no contribution to $\delta E_N^{(2)}$.

Finally we are left with the combination of the ground state configurations of the types (3.3(d)) and (3.5(b)) which indeed give finite contribution to $\delta E_N^{(2)}$, eq.(3.2). According to the discussion below eq.(3.2), here we get no contribution from all the elements $J_{ij}$ of all $(N-2)$ lines $(i = 3, \ldots, N)$, and thus all these elements fall out of the computation. The difference of energies between (3.5(b)) and (3.3(d)) which does enter into the computation is equal to one of $2(N-1)$ elements of the first two lines of the problem $\tilde{E}_N^{(2)}$ (note again, that the diagram (3.5(b)) represents one of $2(N-1)$ configurations with one occupied "0"). Therefore, the average difference $\delta E_N^{(2)}$ can be represented in the form of the "truncated" problem:

$$\delta E_N^{(2)} = E \begin{pmatrix} 0 & & \cdots & \cdots \\ 0 & & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$  \hspace{1cm} (3.7)

where the ground state is just one the configurations of the type (3.5(b)).

The generalization of the proof for arbitrary $k$ is straightforward. First of all, it is evident that to get the proof for a general value of $k$ it is actually sufficient to consider the problem with $k = 3$:

$$\delta E_N^{(3)} = E \begin{pmatrix} 0 & & \cdots & \cdots \\ 0 & & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} - E \begin{pmatrix} 0 & & \cdots & \cdots \\ 0 & & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \tilde{E}_N^{(3)} - E_N^{(3)}$$  \hspace{1cm} (3.8)

because here the first line and the first columns can "represent" all the other equivalent lines and columns of the problems with $k > 3$.

Then we should make quite similar to eqs.(3.3) and (3.5) classification (which turns out to be only slightly more cumbersome) of all non-equivalent ground state configurations of the problems $E_N^{(3)}$ and $\tilde{E}_N^{(3)}$ according to the positions of the occupied elements of the first three lines. Simple analysis shows that here again the only relevant (for $\delta E_N^{(3)}$) configurations of the problem $E_N^{(3)}$ are the ones with all three zeros occupied, while in the problem $\tilde{E}_N^{(3)}$ these are the configurations with one or two of the zeros occupied. On the other hand, due to equivalence of the rest of $(N-3)$ lines $(i = 4, \ldots, N)$ one finds that all the elements $J_{ij}$ of these lines fall out of the computation. In this way one obtains that energy difference $\delta E_N^{(3)}$ is given by two or one of the elements of the three first lines of the problem $\tilde{E}_N^{(3)}$, which is just defined in terms of the "truncated" problem:
\[ \delta E^{(3)}_N = E \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \] (3.9)

The calculation of the actual values of \( \delta E^{(k)}_N \) is presented in the next Appendix.

4 Appendix B

In this Appendix we prove that

\[ \delta E^{(k)}_N = \frac{1}{k(k-1)} \sum_{l=1}^{k-1} \frac{l}{N-l} \] (4.1)

The solution of the case \( k = 2 \), eq.(3.7), is trivial. Here the ground state configuration is of the type (3.5(b)), where the position of "*' must be at the smallest element out of 2(\( N-1 \)) non-zero \( J_{ij} \)'s. According to the distribution (1.3), for the average value of this element we get:

\[ \delta E^{(2)}_N = \frac{1}{2(N-1)} \] (4.2)

Now let us consider slightly more complicated case \( k = 3 \), eq.(3.9). Simple analysis of the the structures of possible ground state configurations shows that all of them can be taken into account in terms of only 3 × 3 matrix:

\[
\begin{array}{ccc}
0 & \otimes & z \\
\otimes & 0 & \otimes \\
y & 0 & x \\
\end{array}
\] (4.3)

Here \( x \) is the smallest element out of 2(\( N-2 \)) equivalent elements \( J_{2j} \) (\( j = 3, \ldots, N \)) and \( J_{3j} \) (\( j = 3, \ldots, N \)) of the second and the third lines; \( y \) is the smallest element out of two equivalent elements \( J_{21} \) and \( J_{31} \); \( z \) is the smallest element out of (\( N-2 \)) equivalent elements \( J_{1j} \) (\( j = 3, \ldots, N \)) of the first line; the symbol "\( \otimes \)" denotes the elements which do not enter into any ground state configuration. One can easily check that the matrix in eq.(3.9) can have only two ground state energies equal to \( x \) or equal to \( (y+z) \). Note that if we consider this problem in terms of the 3 × 3 matrix (4.3), the element \( x \) could as well be placed in the position (3,2) (instead of "\( \otimes \)" which then should be placed at the position (3,3)), as well as \( y \) could be interchanged with "\( \otimes \)" in the positions (2,1) and (3,1).

Now one can easily note that original 3 × 3 problem (4.3) is actually equivalent to the 2 × 2 problem:

\[ \delta E^{(3)}_N = E \begin{pmatrix} 0 & z \\ y & x \end{pmatrix} \] (4.4)
where, according to the definitions of the random parameters \( x, y \) and \( z \) their statistical distributions are:

\[
P(x) = 2(N - 2) \exp[-2(N - 2)x] \tag{4.5}
\]

\[
P(y) = 2 \exp(-2y) \tag{4.6}
\]

\[
P(z) = (N - 2) \exp[-(N - 2)z] \tag{4.7}
\]

Keeping in mind further generalization of the solution for an arbitrary \( k \), we solve the problem (4.4) in the following way. Similarly to the procedure described in the beginning of Section 2, we can ”shift” the elements of the second line (\( x \) and \( y \)) by the value of the smallest of them, and then integrate it out:

\[
\delta E_N^{(3)} = \frac{1}{2(N - 1)} + \frac{1}{(N - 1)} \left( \begin{array}{c}
0 \\
0
\end{array} \right) \left( \begin{array}{c}
z \\
x
\end{array} \right)
\]

The factor \( 1/(N - 1) \) in the second term of the above equation is the probability that \( y \) is smaller than \( x \) (if the smallest element is \( x \), then the remaining problem will have all zeros at the diagonal, and the minimum energy of this problem is identically equal to zero). The solution of the remaining \( 2 \times 2 \) problem is trivial, and eventually we get the following result:

\[
\delta E_N^{(3)} = \frac{1}{2(N - 1)} + \frac{1}{3(N - 1)(N - 2)} = \frac{1}{3} \cdot \frac{2}{N - 1} + \frac{2}{N - 2} \tag{4.9}
\]

Now the generalization of the above procedure for an arbitrary value of \( k \) becomes evident. First we note that all possible ground state configurations of the \( N \times k \) problem \( \delta E_N^{(k)} \), eq.(2.20), can be taken into account in terms of the \( k \times k \) matrix:

\[
\begin{array}{cccc}
0 & \ldots & \otimes & z_1 \\
0 & \ldots & \otimes & z_2 \\
\ldots & \ldots & \ldots & \ldots \\
\otimes & \otimes & \ldots & 0 \\
y_1 & y_2 & \ldots & y_{(k-2)} \\
& & & 0 \otimes \\

\end{array}
\]

(4.10)

Here \( x \) is the smallest element out of \( 2(N - k + 1) \) equivalent elements \( J_{(k-1)j} \) \((j = k, \ldots, N)\) and \( J_{kj} \) \((j = k, \ldots, N)\) of the last two lines; \( y_j \) \((j = 1, \ldots, (k - 2))\) is the smallest element out of two equivalent elements \( J_{(k-1)j} \) and \( J_{kj} \); \( z_i \) \((i = 1, \ldots, (k - 2))\) is the smallest element out of \( (N - k + 1) \) equivalent elements \( J_{ij} \) \((j = k, \ldots, N)\) of the \( i \)-th line; and again the symbol ”\( \otimes \)” denotes the elements which do not enter into any ground state configuration.

According to the above definitions of the random parameters \( x, \{y_j\} \) and \( \{z_i\} \) their probability distribution functions are:
In this way we can reduce the calculation of $\delta E^{(k)}_N$ to the $(k-1) \times (k-1)$ matrix problem:

$$
\delta E^{(k)}_N = E \left( \begin{array}{ccc}
0 & \ldots & z_1 \\
0 & \ldots & z_2 \\
\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
y_1 & y_2 & \ldots & y_{(k-3)} & x
\end{array} \right) 
$$

Taking into account the equivalence of the first $(k-2)$ columns here we can integrate out the smallest element of the last line to get:

$$
\delta E^{(k)}_N = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} E \left( \begin{array}{ccc}
0 & \ldots & z_1 \\
0 & \ldots & z_2 \\
\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
\otimes & \otimes & \cdots & \otimes & 0
\end{array} \right) 
$$

Now one can easily see that all possible ground state configurations in the remaining $(k-1) \times (k-1)$ problem can be taken into account in the same way as in the previous $k \times k$ one, eq.(4.10). Here we can reduce the number of relevant elements by choosing the smallest one between $x$ and $z_{(k-2)}$, as well as between each $y_j$ of the last line and $J_{(k-2)j}$ ($j = 1, \ldots, (k-3)$) of the previous line. In this way we get:

$$
\delta E^{(k)}_N = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} E \left( \begin{array}{ccc}
0 & \ldots & \otimes z_1 \\
0 & \ldots & \otimes z_2 \\
\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
\otimes & \otimes & \cdots & \otimes & 0
\end{array} \right) 
$$

where the random elements $x$ and $\{y_j\}$ according to their definitions are now described by the following distribution functions:

$$
P(x) = 3(N - k + 1) \exp \left[ -3(N - k + 1)x \right] \quad (4.17)
$$

$$
P(y_j) = 3 \exp (-3y_j) \quad (4.18)
$$
while the distribution functions of $z_i$'s remain unchanged, eq. (4.13). In this way we can reduce the calculation of $\delta E_N^{(k)}$ to the $(k-2) \times (k-2)$ matrix problem:

$$
\delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} E \left( \begin{array}{ccc}
0 & \ldots & z_1 \\
0 & \ldots & z_2 \\
\ldots & \ldots & \ldots \\
\ldots & 0 & z_{(k-3)} \\
y_1 & y_2 & \ldots & y_{(k-3)} & x
\end{array} \right)
$$

(4.19)

Here again we can integrate out the smallest element of the last line to get:

$$
\delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} + \frac{1}{3(N-2)} + \frac{k-3}{(N-2)} E \left( \begin{array}{ccc}
0 & \ldots & z_1 \\
0 & \ldots & z_2 \\
\ldots & \ldots & \ldots \\
\ldots & 0 & z_{(k-4)} \\
y_1 & y_2 & \ldots & y_{(k-4)} & 0 & x
\end{array} \right)
$$

(4.20)

Continuing these iterations till the last trivial $2 \times 2$ problem we eventually obtain the following result:

$$
\delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} + \frac{1}{3(N-2)} + \frac{k-3}{(N-2)} + \frac{1}{4(N-3)} + \frac{k-4}{(N-3)} + \ldots + \frac{1}{k(N-k+1)}
$$

(4.21)

After simple algebra the above expression can be easily reduced to the following form:

$$
\delta E_N^{(k)} = \frac{1}{k(k-1)} \left[ \frac{1}{N-1} + \frac{2}{N-2} + \ldots + \frac{k-1}{N-k+1} \right]
$$

(4.22)

which proves eq. (4.1).

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