Can steady-state Schrödinger cats survive in subharmonic generation with anharmonic nonlinearities?

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We discuss general properties of the equilibrium state in superconducting quantum circuits with anharmonic nonlinearities and detunings, in the nonlinear regime. By comparing moments of the steady state and those of a Schrödinger cat, we show that true Schrödinger cats cannot survive in the steady state if there is a single-photon loss. A delta-function ‘cat-like’ steady-state distribution can be formed, but this only exists in the limit of extremely large nonlinearity. In general, the steady state is a mixed state whose purity is reduced by driving.

The Schrödinger cat is a famous thought experiment of E. Schrödinger [1], where a cat is placed in a quantum superposition of two macroscopically distinct states, either alive or dead. It opens the fundamental question of whether quantum theory holds true in the macroscopic world [2–4]. A common strategy for Schrödinger cats [5] is via non-equilibrium subharmonic generation [6,7] leading to discrete time symmetry-breaking or time crystals [8]. Macroscopic superposition states have been proposed in quantum computation [9], quantum teleportation [10], quantum metrology [11] and quantum key distribution [12]. They have also been experimentally realized in atoms [13,14] and photons [15].

The steady state of above-threshold subharmonic generation is known for parametric down-conversion without anharmonicities or detuning [6,16]. In this case transient Schrödinger cats are possible [5,17,18]. Quantum subharmonic generation with anharmonic nonlinearities has been achieved in superconducting circuits [19]. Relatively large cat states were observed. In the experiment, the physics of the quantum steady state is different from previous studies [20]. This raises the question of the exact steady state limit as an example of how dissipation restores broken time symmetry, with applications to solving combinatorial optimization problems [21].

Quantum optical and quantum circuit physics are similar except that quantum circuits operate at microwave instead of optical frequencies. General driven quantum subharmonic generation with damping and weak nonlinearities was studied in a previous paper [20], where non-equilibrium quantum tunneling [22] occurs. Here we focus on the cat-like properties of the steady states in the case of strong combined parametric and anharmonic nonlinearities, as found in recent experiments.

We analytically calculate the exact steady state in subharmonic generation with strong parametric and anharmonic nonlinearities, and use the resulting correlation function to show that neither simple mixtures of coherent states nor Schrödinger cat states can occur in the steady state. We also expect this behavior in more complex time-crystal experiments. Although a steady-state mixture of coherent states [10] is achievable as a limiting case of extremely strong nonlinearities, it is still a mixed state. This is consistent with the superconducting experiment [19] where an approximate Schrödinger cat was observed in a transient regime. The steady-state in the zero loss case is not uniquely defined, due to conserved number parity.

Firstly, we summarize the system properties and theoretical techniques used previously [2,20,23], and then treat the detailed properties of the strongly coupled case. The annihilation and creation operators of the $k$th mode in two coupled resonant cavities are $a_k,a_k^\dagger$ at frequencies $\omega_k$. They have a non-interacting Hamiltonian in the rotating frame of $H_0 = \hbar \sum_\Delta_k a_k^\dagger a_k$, where $\Delta_k = \omega_k - k\omega_0 \ll \omega_0$ for an input laser frequency of $2\omega_0$. The interaction Hamiltonian is then given by

$$H_I = \hbar \frac{\chi}{2} a_1^{\dagger 2} a_1^2 + \left( i\hbar \frac{\kappa}{2} a_2 a_1^{\dagger 2} + i \hbar E_2 a_2^\dagger + h.c. \right).$$

Here $E_2$ is the envelope amplitude of the driving for the mode $a_2$, and $\kappa, \chi$ are the parametric and anharmonic nonlinearities [24] respectively. Anharmonic nonlinearities are only included for the mode $a_1$.

In addition, we include single-photon and two-photon losses in this open system. Defining $H = H_0 + H_I$, the master equation for the density matrix $\rho$ is

$$\dot{\rho} = -\frac{i}{\hbar} [H,\rho] + \sum_{k,j>0} \frac{\gamma_k}{j} E_k^{(j)} [\rho].$$

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Here $\gamma_k^{(j)}$ are the relaxation rates for $j$-photon losses in the $k$-th mode, with no two-photon losses in mode $k = 2$ for simplicity. The dissipative terms are

$$L_k^{(j)}[\rho] = 2i\rho\dot{O}^j - \rho\dot{O}^j - \dot{O}^j\rho,$$

where $\dot{O} = \dot{a}\hat{a}^j$. The corresponding thermal noises are set to zero. This allows us to study the steady-state properties in the low-temperature limit, in order to understand this exactly soluble case of maximal quantum coherence.

We suppose the second harmonic mode is strongly excited, as in the recent Yale experiments [19]. Complex single-photon loss terms are defined as $\gamma_k = \gamma_k^{(1)} + i\Delta_k$. An adiabatic Hamiltonian is obtained for $a \equiv a_1$ as:

$$\frac{H_A}{\hbar} = \Delta_1 a\hat{a} + i \left[\frac{\epsilon}{2} a^2 + h.c.\right] + \frac{\chi_e}{2} a^4, \quad (4)$$

The effective driving field $\epsilon$ and nonlinearity $\chi_e$ are:

$$\epsilon = \frac{\kappa}{\gamma_2} \epsilon_2, \quad \chi_e = \chi - \frac{\Delta_2}{2} \left|\frac{\kappa}{\gamma_2}\right|^2. \quad (5)$$

The master equation of the reduced density matrix $\rho_1 = Tr_2(\rho)$ is then obtained as:

$$\frac{\partial}{\partial t} \rho_1 = \frac{1}{i\hbar} [H_A, \rho_1] + \gamma_1^{(1)} (2ap_1a^\dagger - a^\dagger ap_1 - p_1a^2)$$

$$+ \frac{\gamma_2}{2} (2\epsilon p_1a^2 - a^2\epsilon p_1 - \epsilon a^2p_1 - \epsilon a^2p_1^\dagger), \quad (6)$$

with an effective two-photon loss $\gamma_e^{(2)}$, where

$$\gamma_e^{(2)} = \gamma_1^{(2)} + \frac{\gamma_1^{(1)}}{2} - \frac{\kappa^2}{\gamma_2}. \quad (7)$$

We introduce a generalized $P$-representation [25] to obtain the exact solution to the steady-state. If we expand the reduced quantum density matrix in terms of coherent state projection operators and a complex P-distribution $P(\alpha, \alpha^*, t)$, one then obtains

$$\dot{\rho}_1 = \iint d\alpha d\alpha^* P(\alpha, \alpha^*) \left|\alpha\right\rangle \left\langle \alpha^+ \right| + \gamma_e^{(2)} \frac{\gamma_1^{(1)} + i\Delta_1}{\left|\gamma_e^{(2)} - i\chi_e\right|^2} - 1. \quad (12)$$

The steady-state solution of the Fokker-Planck equation [9] can then be derived with the potential equations [23, 24], $P(\alpha, \alpha^+) = N \exp \left[-\Phi(\alpha, \alpha^+)\right]$, where $N$ is a normalization constant and the steady-state potential solution is

$$\Phi(\alpha, \alpha^+) = -2\alpha^+\alpha - c\ln[\lambda_c - \alpha^2] - c^*\ln[\lambda_c^* - \alpha^+\alpha], \quad (10)$$

with dimensionless parameters $c = \gamma/g - 1$ and $\lambda_c = \epsilon/g$. Thus, the steady-state probability distribution is

$$P(\alpha, \alpha^+) = N(\lambda_c - \alpha^2)e^{(\lambda_c^* - \alpha^+\alpha)^+}e^{2\alpha^+\alpha}. \quad (11)$$

This is the exact zero-temperature steady-state solution for the density matrix. All the parameters here can have complex values, which is necessary when treating the situations in recent quantum circuit experiments [19].

Now we will consider the relevant parameters in the experiment [19]. In our notation, we obtain the parameters as $\gamma/2\pi = 3.98$ kHz, $g/2\pi = (7.96 + 4i)$ kHz and $\epsilon = (-19.2 - 0.07i)$ kHz. Thus we have $c = -0.6 - 0.2i$ and $\lambda_c = 1.93 + 0.96i$. Since the real part of $c$ is negative, there will be singularities occurring at $\alpha = \pm\sqrt{\lambda_c}$ or $\alpha^+ = \pm\sqrt{\lambda_c}$. From now on, we will treat the strong coupling regime, which corresponds to the parameter region of $Re(c) < 0$. Using the definitions of $c$ and $g$, we have

$$c = \left(\frac{\gamma_1^{(1)} + i\Delta_1}{\left|\gamma_e^{(2)} - i\chi_e\right|^2}\right).$$

Hence, $Re(c) < 0$ is equivalent to $\gamma_e^{(2)}(\gamma_1^{(1)} - \gamma_e^{(2)}) + \chi_e(\Delta_1 - \chi_e) < 0$. This is satisfied if there is either a weak single-photon damping $\gamma_1^{(1)}$ or strong nonlinear couplings $\chi_e$, $\gamma_e^{(2)}$. It is easily checked, provided there are no detunings, that $Re(c) \geq -1$ and the limit $c \rightarrow -1$ occurs if $\gamma_1^{(1)} \ll \gamma_e^{(2)}$ or $\gamma_1^{(1)} \ll \chi_e$. Considering nonlinear losses are always weak, the relation $\gamma_1^{(1)} \ll \gamma_e^{(2)}$ can occur with large $\kappa$ refer to Eq. (7). Thus the limit $c \rightarrow -1$ occurs either with large nonlinearities $\kappa$ or $\chi_e$. Due to the negative powers $c$, there are cuts on the complex integration manifold, and corresponding branch points that describe a topological structure, rather than local potential minima. This is why there is no quantum tunneling, although transient Schrödinger cats can be formed in this type of experiment [19]. This is a completely different manifold to that investigated in the previous work [20], where the real part of $c$ is positive and there is quantum tunneling between local potential minima on a finite, bounded manifold. To define the distribution, we must choose complex integration contours which are closed and continuous [7, 23, 24]. This is obtained from inserting a cut between the branch-points combined with complex Pochhammer contours, as used to represent the beta and hypergeometric functions [59-62].
We will use these exact analytic results to check the validity of approximate delta-function steady-state distributions which we introduce later [18] via the second-order correlation function, which is defined as
\[ g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2}, \]  
where the \( n \)-th moment can be calculated with P-representation integrals as
\[ I_{nn'} = \langle a^{n^\dagger} a^{n'} \rangle = \int \alpha^{n^\dagger} \alpha^{n'} P(\alpha, \alpha^+) d\alpha^+ d\alpha. \]  
it is well known that nonclassical effects like photon antibunching will occur if \( g^{(2)}(0) < 1 \) and classical bunching takes place if \( g^{(2)}(0) > 1 \). Thus, \( g^{(2)}(0) \) is often used to distinguish classical from non-classical behavior [33].

The exact solution for the moments [27] is obtained by expanding the term \( e^{2\alpha^+ \alpha} = \sum_m m^m e^{m\alpha^+ \alpha} / m! \) in Eq. (11). In this way, we obtain the form of moment after normalization and integration over the complex manifold, as:
\[ I_{n'n}^{\pi} = N' \sum_m \frac{(2\lambda)^m}{m!} (-\sqrt{\lambda_c})^{n'} (-\sqrt{\lambda_c})^n \times_2 F_1(-m-n', c+1, 2c+2, 2) \times_2 F_1(-m-n, c^*+1, 2c^*+2, 2). \]  
Here \( _2 F_1 \) is the hypergeometric function, and \( N' \) is the normalization factor,
\[ N'^{-1} = \sum_m \frac{(2\lambda)^m}{m!} _2 F_1(-m, c+1, 2c+2, 2) \times_2 F_1(-m, c^*+1, 2c^*+2, 2). \]  
The case of real \( c \) has been investigated in Ref. [16] [17], where there was no anharmonic nonlinearity, and a real manifold was used. It was suggested that the steady-state distribution approaches a set of \( \delta \) functions in strong coupling limits. The case without single-photon loss and anharmonic nonlinearity has also been studied in Ref. [22], where one always has \( c = -1 \). In this case, steady-state Schrödinger cats can be achieved with initial Fock states. Other work studying this potential in different parameter regimes was used to benchmark our numerical results, given below [23].

To understand the physics more clearly, we note that in the limit of \( c \rightarrow -1 \), the exact solution is a product of simple poles with opposite contour integration directions. These can be integrated using Cauchy’s theorem, and correspond to a delta-function solution, so the ratio of the probabilities at the singularities is
\[ \frac{P_{ss}(\alpha = \pm \sqrt{\lambda_c}, \alpha^+ = \pm \sqrt{\lambda_c^*})}{P_{ss}(\alpha = \pm \sqrt{\lambda_c}, \alpha^+ = \mp \sqrt{\lambda_c^*})} = e^{4\lambda}, \]  
with \( \lambda = |\lambda_c| \). If we assume this is also true approximately for \( c \neq -1 \), we obtain a real distribution [16] in the form of
\[ P_{ss}(\alpha, \alpha^+) = \frac{1}{2(1 + e^{-2\lambda})} \left[ \delta(\alpha - \sqrt{\lambda_c}) \delta(\alpha^+ - \sqrt{\lambda_c^*}) + \delta(\alpha + \sqrt{\lambda_c}) \delta(\alpha^+ + \sqrt{\lambda_c^*}) \right] \]
\[ + \frac{1}{2(1 + e^{2\lambda})} \left[ \delta(\alpha - \sqrt{\lambda_c}) \delta(\alpha^+ + \sqrt{\lambda_c^*}) + \delta(\alpha + \sqrt{\lambda_c}) \delta(\alpha^+ - \sqrt{\lambda_c^*}) \right]. \]  
We now contrast this with an idealized, even cat state \( |\psi\rangle_{cat} \propto (|\sqrt{\lambda_c} \rangle + |-\sqrt{\lambda_c} \rangle) \), where the P-representation takes the form after normalization
\[ P_{cat}(\alpha, \alpha^+) = \frac{1}{2(1 + e^{-2\lambda})} \left[ \delta(\alpha - \sqrt{\lambda_c}) \delta(\alpha^+ - \sqrt{\lambda_c^*}) + \delta(\alpha + \sqrt{\lambda_c}) \delta(\alpha^+ + \sqrt{\lambda_c^*}) \right] \]
\[ + \frac{1}{2(1 + e^{2\lambda})} \left[ \delta(\alpha - \sqrt{\lambda_c}) \delta(\alpha^+ + \sqrt{\lambda_c^*}) + \delta(\alpha + \sqrt{\lambda_c}) \delta(\alpha^+ - \sqrt{\lambda_c^*}) \right]. \]  
The factor is \( e^{-2\lambda} \), rather than \( e^{-4\lambda} \) in Eq. (18), so even if the steady state does evolve to a delta-function distribution (18), it will be a mixed state instead of a true cat state. In this case, the density matrix can be derived to have the following form,
\[ \rho_{ss} = p |\psi\rangle_{cat} \langle \psi | + (1 - p) \rho_{mix}. \]  
Here \( p = (1 + e^{2\lambda})/(1 + e^{4\lambda}) \) and \( \rho_{mix} = \frac{1}{2} [ |\sqrt{\lambda_c} \rangle \langle \sqrt{\lambda_c} | + |-\sqrt{\lambda_c} \rangle \langle -\sqrt{\lambda_c} | \] is a mixed state. The purity can then be obtained as
\[ \mu = Tr[\rho_{ss}^2] = \frac{e^{8\lambda} + 6e^{4\lambda} + 1}{2(e^{4\lambda} + 1)^2}, \]  
which is a monotonic decreasing function of \( \lambda \) since
\[ \frac{d\mu}{d\lambda} = \frac{8e^{4\lambda} + 1}{(e^{4\lambda} + 1)^3} < 0, \]  
for \( \lambda > 0 \). Thus, the driving will weaken the purity of the steady state since \( \lambda \) is proportional to the driving \( E_2 \).

It is obvious that we will have \( p \rightarrow 1 \) in the limit of \( \lambda \rightarrow 0 \). Thus the delta-function distribution tends to be a true Schrödinger cat state in this limit. However, since \( |\lambda_c| = \lambda \rightarrow 0 \), the steady state will actually reduce to a vacuum state. This is natural that a non-driven system can be expressed as a vacuum state. In the opposite limit of \( \lambda \rightarrow \infty \), the delta-function steady-state distribution [18] will reduce to the mixed state \( \rho_{mix} \) since \( p \rightarrow 0 \). Therefore, a pure Schrödinger cat state is unreachable in the steady state of the system, even using an approximate delta-function solution.

The parity \( \hat{\mathcal{P}} = (-1)^{a^+a} \) can also be studied directly with the complex P-distribution [18]. In the P-representation, the parity operator is equivalent to the average of \( \mathcal{P} = \exp(-2\alpha^+a) \). In the steady state of the
delta-function approximation, we have $\mathcal{P}_{ss} = \text{sech}(2\lambda)$. This means that $\mathcal{P}_{ss} = 1$ in the case of $\lambda = 0$, and $\mathcal{P}_{ss} = 0$ in the limit of $\lambda \to \infty$. It is consistent with the density matrix \[20\] which is a vacuum state when $\lambda = 0$ and a mixed state when $\lambda \to \infty$. The parity is non-conserved because of the nonzero single-photon loss.

The steady-state distributions \[11\] with different parameters are shown in Fig. 1 plotted on a finite manifold. We see that delta-function distribution will be obtained approximately with large $|\text{Re}(c)|$ and small $|\text{Im}(c)|$, and reduced to classical mixture of coherent states with large $\lambda$. However, these graphs also demonstrate that the probability does not vanish at the boundaries, which means that with $\text{Re}(c) < 0$ on this bounded manifold, the potential solution is no longer a solution to the original master equation, since boundary terms from integration by parts are non-vanishing. An inspection of Fig. 1 shows that when assuming a real, bounded manifold, the distribution is neither a true delta function, nor does it vanish at the boundaries, which is the reason why the exact complex contour solution is preferable.

As a result, the true steady states are clearly different to either mixtures of delta functions or Schrödinger cats. This difference can be quantified by using the steady-state distribution \[18\], to compare moments. The approximate $n$-th moment is obtained directly with the definition \[14\] as,

$$
I_{n,n}^\delta = \frac{(\sqrt{\lambda_c})^{n'}}{(\sqrt{\lambda_c})^n} + \frac{(\sqrt{\lambda_c})^{n'}(-\sqrt{\lambda_c})^n}{2(1 + e^{-2\lambda})} + \frac{(\sqrt{\lambda_c})^{n'}(-\sqrt{\lambda_c})^n}{2(1 + e^{2\lambda})}.
$$

Similarly, the moment can be written down directly with the cat state \[19\] as:

$$
I_{n,n}^\delta = \frac{(\sqrt{\lambda_c})^{n'}}{(\sqrt{\lambda_c})^n} + \frac{(\sqrt{\lambda_c})^{n'}(-\sqrt{\lambda_c})^n}{2(1 + e^{-2\lambda})} + \frac{(\sqrt{\lambda_c})^{n'}(-\sqrt{\lambda_c})^n}{2(1 + e^{2\lambda})}.
$$

We have compared the average steady-state photon number $\langle a^\dagger a \rangle$ and the second order correlation function $g^{(2)}(0)$ changing with $c$ in Fig. 2. The results of Fig. 2 show that the delta-function distribution \[18\] is only attainable when $c \to -1$, which is valid when $\gamma_1^{(1)} \ll \gamma_2^{(1)}$ or $\gamma_2^{(1)} \ll \chi_c$, if there are no detunings. Mathematically, it is obtained by reaching the steady state first and then taking the limit $\gamma_1^{(1)} \to 0$, which is different from the magenta circles where we take $\gamma_1^{(1)} = 0$ exactly and then get the steady states assuming some particular parity \[23\]. Number parity is conserved only if $\gamma_1^{(1)} = 0$, and non-conserved if $\gamma_1^{(1)} \neq 0$. Thus the ordering of the limit is
important, which leads to the gap between the red line with \( c \rightarrow -1 \) (a mixed state) and the magenta circles (a pure cat state) in Fig. 2. In addition, the delta-function distribution can also be obtained in the region of extremely strong nonlinearity as the limit \( c \rightarrow -1 \) suggests, which is more practical than the case \( \gamma_1^{(1)} = 0 \).

In Fig. 3, the results of the delta-function distributions never agree with those of the cat states. This is consistent with the discussion above that the steady state of the system is always a mixed state \( |\psi\rangle \) instead of a pure cat state. Although there are crosses for the exact results of the steady state and those of the pure cat state, they are always at different \( c \) for \( \langle a^\dagger a \rangle \) and \( g^{(2)}(0) \). The exact steady state is therefore different from both the cat state and a mixture of delta-functions. Hence we can’t generate a pure steady-state cat state, unless the system has no single-photon losses.

We have stated that in the limit of small \( \lambda \), the delta-function distribution \( |\psi\rangle \) tends to an approximate Schrödinger cat. Now we show how \( \langle a^\dagger a \rangle \) and \( g^{(2)}(0) \) change with \( \lambda \) in Fig. 3. It is natural that the average photon number \( \langle a^\dagger a \rangle \) increases with large driving \( \mathcal{E}_2 \propto \lambda \) as shown in Fig. 3(a). It also shows that in the region of small \( \lambda \), their photon numbers agree with each other, but \( g^{(2)}(0) \) has a different behavior. This means that even with \( \lambda \rightarrow 0 \), the delta-function steady-state distribution \( |\psi\rangle \) is still different from the distribution of a Schrödinger cat. We also show in Fig. 3 that in the limit of \( c \rightarrow -1 \), the exact steady state will approach the delta-function steady-state distribution, although as before, this is not a cat state.

It is directly checked with Eqs. (23) and (24) that the second-order correlation functions are

\[
g_{\text{cat}}^{(2)}(0) = \left( \frac{e^{4\lambda} + 1}{e^{4\lambda} - 1} \right)^2, \quad g_{\text{cat}}^{(2)}(0) = \left( \frac{e^{2\lambda} + 1}{e^{2\lambda} - 1} \right)^2.
\]

Thus in the limit of \( \lambda \rightarrow 0 \), we have \( g_{\text{cat}}^{(2)}(0) \rightarrow 4 \). This is consistent with the Fig. 3. In addition, we will also have \( g_{\text{cat}}^{(2)}(0) > g_{\text{cat}}^{(2)}(0) > 1 \) in all range of \( \lambda \). This means that their probability distributions are both super-Poissonian. From all the discussions above, we demonstrate that the delta-function steady-state distribution \( |\psi\rangle \) is different from the Schrödinger cat state, even if \( \lambda \rightarrow 0 \).

Pure steady-state cats can occur in systems without single-photon loss and anharmonic nonlinearity. If we neglect the single-photon loss in our system from the beginning, the steady-state solution is obtained from solving \( \partial \rho_1 / \partial t = 0 \) in Eq. (6). We expand the density operator in the coherent state basis as \( \rho_1(t = \infty) = \int c_{\alpha,\alpha'} |\alpha\rangle \langle \alpha'| d^2 \alpha d^2 \alpha' \). Substitute it into Eq. (6) with \( \gamma_1^{(1)} = 0 \), then for arbitrary \( c_{\alpha,\alpha'} \) we have

\[
\alpha = \pm \sqrt{\lambda_c}, \quad \alpha' = \pm \sqrt{\lambda_c}.
\]

Thus the steady-state density matrix takes the form,

\[
\rho_1(\infty) = c_{++} |\sqrt{\lambda_c}\rangle \langle \sqrt{\lambda_c}| + c_{--} | - \sqrt{\lambda_c}\rangle \langle - \sqrt{\lambda_c}|
\]

+ \( c_{-+} | - \sqrt{\lambda_c}\rangle \langle \sqrt{\lambda_c}| + c_{+-} | \sqrt{\lambda_c}\rangle \langle - \sqrt{\lambda_c}| \),

where the coefficients \( c_{\alpha,\alpha'} \) are determined by the initial states. This is consistent with the earlier work.

In summary, we have studied the steady states of quantum subharmonic generation with strong nonlinearity, which has been experimental achieved. By comparing the correlation functions, we conclude that true Schrödinger cats cannot survive in the steady state unless there is no single-photon loss. With single-photon loss included, the steady state of the subharmonic generation will reduce to a delta-function steady-state distribution only if there is an extremely strong nonlinearity. More generally, neither type of delta-function solution is obtainable. However, the correct integration manifold is a Poisson contour which samples both sheets of a double Riemann sheet contour, intriguingly reflecting some of the character of the known transient macroscopic superposition.

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