THE VERTICAL SLICE TRANSFORM IN SPHERICAL TOMOGRAPHY

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Abstract. The vertical slice transform takes a function on the unit sphere in $\mathbb{R}^{n+1}$, $n \geq 2$, to integrals of that function over spherical slices parallel to the last coordinate axis. In the case $n = 2$ these transforms arise in thermoacoustic tomography. We obtain new inversion formulas for the vertical slice transform and its singular value decomposition. The results can be applied to the inverse problem for the Euler-Poisson-Darboux equation associated to the corresponding spherical means.

1. Introduction

Let $\mathcal{T}$ be the set of all cross-sections of the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ by the hyperplanes parallel to the last coordinate axis. The vertical slice transform takes a function $f$ on $S^n$ to a function $Vf$ on $\mathcal{T}$ by integration over these cross-sections. Specifically,

$$ (Vf)(\tau) = \int_{\tau} f(x) \, d_{\tau}x, \quad \tau \in \mathcal{T}, $$

where $d_{\tau}x$ stands for the induced surface measure on $\tau$.

Transformations (1.1) with $n = 2$ arise in thermoacoustic tomography; see [10, 13, 33], where inversion of $Vf$ is reduced to the classical Radon transform for lines in the plane [9, 12]. In Section 2 we extend this approach to all $n \geq 2$, derive new inversion formulas and investigate the corresponding singular value decompositions.

An alternative way to study integrals (1.1) is to treat them as the spherical means over geodesic spheres centered on the equator of $S^n$. This situation resembles the Euclidean case having wide application in photoacoustic and thermoacoustic tomography, when the spherical means are evaluated over spheres centered on a boundary of a ball; see, e.g., [1, 2, 3, 4, 6, 7, 17, 18, 19, 20, 21, 24, 26, 32], to mention a few. In

2010 Mathematics Subject Classification. Primary 44A12; Secondary 35L05, 45Q05, 92C55.

Key words and phrases. Spherical tomography, inversion formulas, thermoacoustic tomography.
Section 3 we will show that the operator (1.1) can be explicitly inverted using the method of analytic continuation developed in our previous papers [29, 4]. This alternative approach might be of independent interest.

**Notation.**

We write \( x \in \mathbb{R}^{n+1} \) as \( (x', x_{n+1}) \), where \( x' \in \mathbb{R}^n \), \( n \geq 2 \). In the following, \( S^n \) is the unit sphere in \( \mathbb{R}^{n+1} \), \( S_n^+ = \{ x \in S^n : x_{n+1} \geq 0 \} \) is the “upper” hemisphere, \( B_n = \{ x' \in \mathbb{R}^n : |x'| \leq 1 \} \) is the unit ball in \( \mathbb{R}^n \); \( S^{n-1} \) is the boundary of \( B_n \); \( \sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2) \) is the area of \( S^{n-1} \). If \( x \in S^n \), then \( d\sigma(x) \) stands for the Riemannian surface measure on \( S^n \). The standard notation \( C(\Omega) \) and \( L^p(\Omega) \) is used for the corresponding spaces of continuous and \( L^p \) functions on the set \( \Omega \) under consideration.

Everywhere in the following, we assume that the function \( f \) in (1.1) is even in the last variable, because if \( f(x', \cdot) \) is odd, the integral (1.1) is zero.

**2. Connection with the Hyperplane Radon Transform**

The following lemma is a standard fact from Calculus.

**Lemma 2.1.** If \( f \in L^1(S_n^+) \), then

\[
\int_{S_n^+} f(x) \, d\sigma(x) = \int_{B_n} \varphi(x') \, dx', \quad \varphi(x') = \frac{f(x', \sqrt{1-|x'|^2})}{\sqrt{1-|x'|^2}}. \tag{2.1}
\]

**Proof.**

\[
\int_{S_n^+} f(x) \, d\sigma(x) = \int_{B_n} f(x', \sqrt{1-|x'|^2}) \sqrt{1+\left(\frac{\partial x_{n+1}}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial x_{n+1}}{\partial x_n}\right)^2} \, dx' = \int_{B_n} f(x', \sqrt{1-|x'|^2}) \frac{dx'}{\sqrt{1-|x'|^2}} = \int_{B_n} \varphi(x') \, dx'.
\]

Every vertical hyperplane section \( \tau \) of \( S^n \) can be parametrized by the pair \( (\theta, t) \), where \( \theta \in S^{n-1} \) and \( t \in (-1, 1) \) is the signed distance from the origin to the hyperplane containing \( \tau \). We denote

\[
\mathcal{C}_n = \{ (\theta, t) : \theta \in S^{n-1}, t \in (-1, 1) \}.
\]
Owing to the evenness of \( f(x', \cdot) \),
\[
(Vf)(\tau) \equiv (Vf)(\theta, t) = 2 \int_{S^n_\tau} f(x) \, d_\sigma(x),
\]
where \( d_\sigma(x) \) stands for the surface element on \( \tau \). Thus, without loss of generality, we can restrict ourselves to the hemispherical transform
\[
(V_+ f)(\tau) \equiv (V_+ f)(\theta, t) = \int_{S^n_\tau} f(x) \, d_\sigma(x)
\]
that might be of independent interest.

Let
\[
(R\varphi)(\theta, t) \equiv (R\varphi)(t) = \int_{x' \cdot \theta = t} \varphi(x') \, d_{\theta,t}x', \quad \theta \in S^{n-1}, \ t \in \mathbb{R},
\]
be the hyperplane Radon transform of the function \( \varphi \) from (2.1), where \( d_{\theta,t}x' \) is the volume element of the hyperplane \( x' \cdot \theta = t \). Clearly,
\[
(R\varphi)(\theta, t) = 0 \quad \text{whenever} \ |t| \geq 1.
\]

**Lemma 2.2.** If \( f \in L^1(S^n_+) \), then for almost all \( (\theta, t) \in \mathcal{E}_n \),
\[
(V_+ f)(\theta, t) = \sqrt{1-t^2} (R\varphi)(\theta, t). \tag{2.5}
\]

If \( f \in C(S^n_+) \), then (2.5) holds for all \( (\theta, t) \in \mathcal{E}_n \).

**Proof.** By rotation invariance, it suffices to prove the lemma for \( \theta = e_n = (0, \ldots, 1, 0) \), when \( x' = te_n + y, \ y \in \mathbb{R}^{n-1}, \ |y| < r = \sqrt{1-t^2} \). As in the proof of Lemma 2.1,
\[
(V_+ f)(e_n, t) = \int_{|y|<r} f(te_n + y, \sqrt{r^2 - |y|^2}) \sqrt{1 + \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial y_j} \sqrt{r^2 - |y|^2} \right)^2} \, dy
\]
\[
= r \int_{|y|<r} f(te_n + y, \sqrt{r^2 - |y|^2}) \frac{dy}{\sqrt{r^2 - |y|^2}}.
\]
This gives (2.5). \[\square\]

**Remark 2.3.** By (2.5), inversion of \( V_+ \) reduces to the similar problem for the Radon transform \( R \). Most of the known inversion formulas for \( R \) are applicable in the entire space \( \mathbb{R}^n \). From now on, when dealing with \( (R\varphi)(\theta, t) \), we assume that \( \varphi \) extends by zero outside \( B_n \), so that
\[
(V_+ f)(\theta, t) = (R\varphi)(\theta, t) = 0 \quad \text{whenever} \ |t| > 1.
\]

Lemmas 2.1 and 2.2 imply the following result.
Theorem 2.4. A function $f \in L^1(S^n_+)$ can be recovered from its vertical slice transform $V_+f$ by the formula

$$f(x', x_{n+1}) = x_{n+1} (R^{-1}\Phi)(x'), \quad \Phi(\theta, t) = \frac{1}{\sqrt{1-t^2}} (V_+f)(\theta, t),$$

(2.6)

where $R^{-1}$ is the inverse Radon transform over hyperplanes in $\mathbb{R}^n$.

There is a big variety of explicit formulas for $R^{-1}$; see, e.g., [30] and references therein. For convenience of the reader, below we present some of them adapted to our case.

Let $h \equiv h(\theta, t) = \{x' \in \mathbb{R}^n : x' \cdot \theta = t\}$ be a hyperplane in $\mathbb{R}^n$ parametrized by $(\theta, t) \in S^{n-1} \times \mathbb{R}$. The dual Radon transform of the hyperplane function $\Phi(h) \equiv \Phi(\theta, t)$ is defined by the formula

$$(R^*\Phi)(x') = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} \Phi(\theta, x' \cdot \theta) d\sigma(\theta), \quad x' \in \mathbb{R}^n,$$

(2.7)

and averages $\Phi$ over the set of all hyperplanes passing through $x'$.

2.1. F. John’s Inversion Method. An advantage of this method in comparison with many others is that it can be easily adapted to functions on a ball. Below we present this reasoning following our exposition in [30, p. 195] that relies on the original work of F. John [14, Chapter I].

The starting point is the decomposition of $|x'|^{2-n}$ in spherical waves. Specifically,

$$|x'|^{2-n} = \frac{a_1}{c_1} \frac{(-\Delta)^{(n-1)/2}}{\pi^{n-1/2}} \int_{S^{n-1}} |x' \cdot \theta| d\theta, \quad n = 3, 5, \ldots,$$

(2.8)

and

$$|x'|^{2-n} = \frac{a_2}{c_2} \frac{(-\Delta)^{(n-2)/2}}{\pi^{n/2}} \int_{S^{n-1}} \log |x' \cdot \theta| d\theta, \quad n = 4, 6, \ldots.$$  

(2.9)

Here $\Delta$ is the Laplace operator in the $x'$-variable,

$$a_1 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}}, \quad c_1 = -2^{n-2} \pi^{-1/2} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n-2}{2}\right),$$

$$a_2 = \frac{\Gamma(n/2)}{2\pi^{n/2}}, \quad c_2 = -2^{n-3} \pi^{-1/2} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-2}{2}\right).$$

(2.10)

We say that a function $\varphi$ on a set $G \subset \mathbb{R}^n$ satisfies the Lipschitz condition with exponent $0 < \lambda \leq 1$ and write $\varphi \in \text{Lip}_\lambda(G)$ if there is a constant $A > 0$ such that

$$|\varphi(x') - \varphi(y')| \leq A |x' - y'|^\lambda \quad \forall x', y' \in G.$$  

(2.11)
**Theorem 2.5.** Let $\Phi = R\varphi$, where $\varphi \in \text{Lip}_\lambda(B_n)$ for some $0 < \lambda \leq 1$.

(i) If $n$ is odd, then

$$\varphi(x') = c_n(-\Delta)^{(n-1)/2}(R^*\Phi)(x'), \quad c_n = \frac{\pi^{1-n/2}}{2^{n-1} \Gamma(n/2)}. \quad (2.12)$$

(ii) If $n$ is even, then

$$\varphi(x') = \tilde{c}_n(-\Delta)^{n/2}(R^L\Phi)(x'), \quad (2.13)$$

$$\tilde{c}_n = \frac{\pi^{(1-n)/2}}{2^{n-1} \Gamma(n/2)}, \quad (L\Phi)(\theta, s) = \int_{-1}^{1} \log |s-t| \Phi(\theta, t) \, dt.$$ 

**Proof.** (i) We combine (2.8) with the classical equality $\varphi(x') = -\Delta I^2 \varphi(x')$, where

$$(I^2\varphi)(x') = \frac{1}{\gamma_n(2)} \int_{B_n} \frac{\varphi(y') \, dy}{|x'-y'|^{n-2}}, \quad \gamma_n(2) = \frac{4\pi^{n/2}}{\Gamma((n-2)/2)}, \quad (2.14)$$

is the Newtonian potential and $x'$ lies in the interior of $B_n$; see, e.g., [22, p. 231]. Let $\lambda_1 = a_1/c_1 \gamma_n(2)$. Then, using Remark 2.3, we have

$$\varphi(x') = \lambda_1 (-\Delta)^{(n+1)/2} \int_{B_n} \varphi(y') \, dy \int_{S^{n-1}} \frac{|(x'-y') \cdot \theta|}{\gamma_n(2)} \, d\theta$$

$$= \lambda_1 (-\Delta)^{(n+1)/2} \int_{S^{n-1}} \int_{-\infty}^{\infty} \frac{|t|}{\gamma_n(2)} \, d\theta \int_{-\infty}^{\infty} (R_{\theta}^\varphi)(x' \cdot \theta - t) \, dt$$

$$= \lambda_1 (-\Delta)^{(n+1)/2} \int_{S^{n-1}} \int_{-1}^{1} \frac{|x' \cdot \theta - t|}{\gamma_n(2)} \, d\theta \int_{-1}^{1} (R_{\theta}^\varphi)(t) \, dt.$$ 

Observing that

$$\Delta \int_{-1}^{1} |x' \cdot \theta - t| \, (R_{\theta}^\varphi)(t) \, dt$$

$$= \frac{d^2}{dz^2} \left[ \int_{-1}^{1} (z-t) \, (R_{\theta}^\varphi)(t) \, dt + \int_{-1}^{1} (t-z) \, (R_{\theta}^\varphi)(t) \, dt \right]_{z=x' \cdot \theta} = 2(R_{\theta}^\varphi)(x' \cdot \theta)$$

(here we use an obvious formula $\Delta_{x'} [g(x' \cdot \theta)] = g''(x' \cdot \theta)$), we get

$$\varphi(x') = -2\lambda_1 (-\Delta)^{(n-1)/2} \int_{S^{n-1}} (R_{\theta}^\varphi)(x' \cdot \theta) \, d\theta.$$
This coincides with (2.12).

(ii) For \( n \) even, we start with (2.9) if \( n > 2 \), and with the formula

\[
\log |x'| = \frac{1}{2\pi} \int_{S^1} \log |x' \cdot \theta| d\theta - \frac{1}{\pi^{1/2}} \frac{d}{d\alpha} \frac{\Gamma(\alpha/2)}{\Gamma((1 + \alpha)/2)} \bigg|_{\alpha=1}, \tag{2.15}
\]

if \( n = 2 \) (cf. [30, formula (4.8.70)]). Let

\[
\lambda_2 = \frac{a_2}{c_2 \gamma_n(2)} = \frac{1}{2^n \pi^{n/2}}.
\]

Then

\[
\varphi(x') = \lambda_2 (-\Delta)^{n/2} \int_{\mathbb{R}^n} \varphi(y') dy' \int_{S^{n-1}} \log |(x' - y') \cdot \theta| d\theta
\]

\[
= \lambda_2 (-\Delta)^{n/2} \int_{S^{n-1}} d\theta \int_{-\infty}^{\infty} \log |t| (R_{\theta} \varphi)(x' \cdot \theta - t) dt
\]

\[
= \lambda_2 (-\Delta)^{n/2} \int_{S^{n-1}} d\theta \int_{-1}^{1} \log |x' \cdot \theta - t| (R_{\theta} \varphi)(t) dt
\]

\[
= \frac{\pi^{(1-n)/2}}{2^{n-1} \Gamma(n/2)} (-\Delta)^{n/2} (R^* LR \varphi)(x').
\]

\[
\square
\]

Theorem 2.5 gives the following result for \( V_+ f \).

**Theorem 2.6.** Given an even function \( f \) on \( S^n \), suppose that

\[
\frac{f(x', \sqrt{1 - |x'|^2})}{\sqrt{1 - |x'|^2}} \in \text{Lip}_\lambda(B_n) \quad \text{for some } 0 < \lambda \leq 1. \tag{2.16}
\]

Then \( f \) can be recovered from its vertical slice transform \( V_+ f \) by the formula \( f(x', x_{n+1}) = |x_{n+1}| \varphi(x') \), where \( \varphi \) is defined by (2.12) and (2.13) with \( \Phi(\theta, t) = (1 - t^2)^{-1/2} (V_+ f)(\theta, t) \).

2.2. Hypersingular integrals. Hypersingular integrals have proved to be a powerful tool for explicit inversion of operators of the potential type [27, 31]. Here we follow [30, Theorem 4.87]. Let

\[
(\Delta^\ell y' \varphi)(x') = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \varphi(x' - ky')
\]
be the finite difference of \( \varphi \) of order \( \ell \) with step \( y' \) at the point \( x' \). We denote

\[
c_n = 2^{n-1} \pi^{n/2} \Gamma(n/2), \quad B_l(\alpha) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^\alpha,
\]

\[
d_{n,\ell}(\alpha) = \frac{\pi^{n/2}}{2^n \Gamma((n+\alpha)/2)} \begin{cases} 
\Gamma(-\alpha/2) B_l(\alpha), & \text{if } \alpha \neq 2, 4, 6, \ldots, \\
\frac{2(-1)^{n/2-1}}{(\alpha/2)!} \frac{d}{d\alpha} B_l(\alpha), & \text{if } \alpha = 2, 4, 6, \ldots.
\end{cases}
\]

The following result is a consequence of (2.6) and [30, Theorem 4.87].

**Theorem 2.7.** Let \( V_+ f \) be the vertical slice transform of a function \( f \in L^1(S^n_+) \) and let \( \Phi(\theta, t) = (1-t^2)^{-1/2}(V_+ f)(\theta, t) \), as in (2.6). Suppose that \( \ell > n-1 \) if \( n \) is odd, and \( \ell = n-1 \) if \( n \) is even. Then \( f \) can be reconstructed by the formula

\[
f(x', x_{n+1}) = c_{x_{n+1}} \int_{\mathbb{R}^n} \frac{(\Delta_{y'} R^* \Phi)(x')}{|y'|^{2n-1}} \, dy', \quad c = \frac{c_n}{d_{n,\ell}(n-1)}, \tag{2.18}
\]

where \( \int (...) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} (...) \) in the norm of the space \( L^1(\mathbb{R}^n) \) and in the a.e. sense. The constant \( d_{n,\ell}(n-1) \) is defined by (2.17) with \( \alpha = n-1 \). If \( f \) belongs to \( C(S^n_+) \) and is supported away from the boundary, the limit is uniform on \( \mathbb{R}^n \).

**Example 2.8.** If \( n = 2 \), then

\[
f(x', x_{n+1}) = \frac{x_{n+1}}{4\pi} \int_{\mathbb{R}^2} \frac{(R^* \Phi)(x') - (R^* \Phi)(y')}{|x' - y'|^3} \, dy'. \tag{2.19}
\]

2.3. **Singular Value Decomposition.** By making use of the connection (2.5), known results about singular value decomposition of the Radon transform on the unit ball can be adapted for the vertical slice transform. Our reasoning relies on [30, Subsection 4.4.4], where the corresponding results for the Radon transform are presented in detail. Some information for the non-weighted case \( \varphi \in L^2(B_n) \) can be found in [23, pp. 17, 95].
Given a nonnegative measurable function $W$ on a measure space $X$, the corresponding weighted space $L^2(X; W)$ is defined by

$$L^2(X; W) = \left\{ f : ||f||_{L^2(X; W)} = \left( \int_X |f(x)|^2 W(x) \, dx \right)^{1/2} < \infty \right\}.$$ 

We will be dealing with different weighted spaces of this kind. Let

$$W(x') = (1 - |x'|^2)^{n/2-\lambda}, \quad \tilde{W}(x) = x_{n+1}^{n-2\lambda-1}; \quad x' \in B_n, \quad x \in S^n;$$

$$w(t) = (1 - t^2)^{-1/2-\lambda}, \quad \tilde{w}(t) = (1 - t^2)^{-1/2-\lambda}, \quad t \in (-1, 1);$$

$$\mathcal{C}_n = \{(\theta, t) : \theta \in S^{n-1}, t \in (-1, 1)\};$$

$$L^2_\mathcal{C}(\mathcal{C}_n; w) = \{ \Phi \in L^2(\mathcal{C}_n; w) : \Phi(-\theta, -t) = \Phi(\theta, \tau) \};$$

$$L^2_\mathcal{C}(\mathcal{C}_n; \tilde{w}) = \{ F \in L^2(\mathcal{C}_n; \tilde{w}) : F(-\theta, -t) = F(\theta, \tau) \}.$$ 

Everywhere in the following we assume

$$\lambda > n/2 - 1.$$ 

This assumption is motivated by the following lemmas.

**Lemma 2.9.** [30, Example 4.36] If $\lambda > n/2 - 1$, then the Radon transform $R$ is a linear bounded operator from $L^2(B_n; W)$ to $L^2_\mathcal{C}(\mathcal{C}_n; w)$ with the norm

$$||R|| = \left( \frac{\pi^{(n-1)/2} \Gamma(n/2 + 1)}{\Gamma(n/2 + 1/2)} \right)^{1/2}. \quad (2.20)$$

Note that the assumption $\lambda > n/2 - 1$ agrees with the argument of the gamma function $\Gamma(n/2 + 1)$.

A similar result for $V_+ f$ follows by simple calculation.

**Lemma 2.10.** If $\lambda > n/2 - 1$, then the vertical slice transform $V_+$ is a linear bounded operator from $L^2(S^n_+; \tilde{W})$ to $L^2_\mathcal{C}(\mathcal{C}_n; \tilde{w})$ and $||V_+|| = ||R||$, as in (2.20).

**Proof.** Let $\varphi(x') = (1 - |x'|^2)^{-1/2} f(x', (1 - |x'|^2)^{1/2})$, as in (2.1). Then (cf. the proof of Lemma 2.1),

$$||\varphi||^2_{L^2(B_n; W)} = \int_{B_n} |\varphi(x')|^2 (1 - |x'|^2)^{n/2-\lambda} \, dx'$$

$$= \int_{S^n_+} |f(x)|^2 x_{n+1}^{n-2\lambda-1} \, d\sigma(x) = ||f||^2_{L^2(S^n_+; \tilde{W})}. \quad (2.21)$$
Further, by (2.5), \( (R\varphi)(\theta, t) = (V_+ f)(\theta, t)(1 - t^2)^{-1/2} \). Therefore,

\[
||R\varphi||^2_{L^2(\mathcal{E}; n; w)} = \int_{\mathcal{E}_n} |(R\varphi)(\theta, t)|^2 (1 - t^2)^{1/2 - \lambda} d\theta dt \\
\int_{\mathcal{E}_n} |(V_+ f)(\theta, t)|^2 (1 - t^2)^{-1/2 - \lambda} d\theta dt = ||V_+ f||^2_{L^2(\mathcal{E}; n; w)}.
\](2.22)

Because \( ||R\varphi||^2_{L^2(\mathcal{E}; n; w)} \leq ||R|| ||\varphi||^2_{L^2(B_n; W)} \), the result follows. \(\Box\)

The equalities (2.21) and (2.22) yield the following statements.

**Lemma 2.11.**

(i) The maps

\[
\alpha : L^2(B_n; W) \to L^2(S^n; \tilde{W}), \quad (\alpha \varphi)(x) = x_{n+1}\varphi(x'),
\]

and

\[
\beta : L^2(\mathcal{E}; n; w) \to L^2(\mathcal{E}; n; \tilde{w}), \quad (\beta \Phi)(\theta, t) = (1 - t^2)^{1/2} \Phi(\theta, t),
\]

are isometric isomorphisms, so that

\[
V_+ f = \beta R \alpha^{-1} f. \quad (2.23)
\]

(ii) A system of functions \( \{\eta_{\nu}\}_{\nu \in \mathcal{N}} \) is an orthonormal basis of \( L^2(B_n; W) \) if and only if a system \( \{\tilde{\eta}_{\nu}\}_{\nu \in \mathcal{N}} \) with \( \tilde{\eta}_{\nu}(x) = x_{n+1}\eta_{\nu}(x') \) is an orthonormal basis of \( L^2(S^n; \tilde{W}) \).

(iii) A system of functions \( \{\zeta_{\nu}\}_{\nu \in \mathcal{N}} \) is an orthonormal basis of \( L^2(\mathcal{E}; n; w) \) if and only if a system \( \{\tilde{\zeta}_{\nu}\}_{\nu \in \mathcal{N}} \) with \( \tilde{\zeta}_{\nu}(\theta, t) = (1 - t^2)^{1/2}\zeta_{\nu}(\theta, t) \) is an orthonormal basis of \( L^2(\mathcal{E}; n; \tilde{w}) \).

We remind explicit formulas for \( \eta_{\nu} \) and \( \zeta_{\nu} \) following [30, Subsection 4.12.3]. Let \( \{Y_{m,\mu}(\theta)\} \) be a real-valued orthonormal basis of spherical harmonics in \( L^2(S^{n-1}) \). Here \( m = 0, 1, 2, \ldots \) and \( \mu = 1, 2, \ldots d_n(m) \), where

\[
d_n(m) = (n + 2m - 2) \frac{(n + m - 3)!}{m!(n - 2)!} \quad (2.24)
\]
is the dimension of the subspace of spherical harmonics of degree \( m \).

We denote

\[
p_{2k}(x') = P_{k}^{(\lambda-n/2,m+n/2-1)}(2|x'|^2 - 1), \quad x' \in B_n, \quad (2.25)
\]

where \( P_{k}^{(\lambda-n/2,m+n/2-1)}(t) \) is the Jacobi polynomial [5], and introduce the index set

\[
\mathcal{N} = \{\nu = (m, \mu, k) : m, k = 0, 1, 2, \ldots ; \mu = 1, 2, \ldots d_n(m)\}. \quad (2.26)
\]
The notation \( C_{m}^{\lambda}(t) \) is commonly used for Gegenbauer polynomials [5].
Lemma 2.12. [30, pp. 237, 239] Let $\nu = (m, \mu, k) \in \mathcal{N}$. The functions
\[
\eta_{\nu}(x') = c_{\nu} (x')^{m} W^{-1}(x') p_{2k}(x') Y_{m, \mu}(x' / |x'|), \\
\zeta_{\nu}(\theta, t) = d_{\nu} w^{-1}(t) C_{m+2k}^{\lambda}(t) Y_{m, \mu}(\theta),
\]
with
\[
c_{\nu} = \left( \frac{2k! (2k + \lambda + m) \Gamma(k + m + \lambda)}{\Gamma(k + \lambda - n/2 + 1) \Gamma(k + m + n/2)} \right)^{1/2}, \\
d_{\nu} = 2^{\lambda - 1/2} \Gamma(\lambda) \left( \frac{(m + 2k)! (m + 2k + \lambda)}{\pi \Gamma(m + 2k + 2\lambda)} \right)^{1/2},
\]
form orthonormal bases of $L^{2}(B_{n}; W)$ and $L_{c}^{2}(\mathcal{E}_{n}; w)$, respectively.

Corollary 2.13. The functions
\[
\tilde{\eta}_{\nu}(x) = x_{n+1} \eta_{\nu}(x'), \\
\tilde{\zeta}_{\nu}(\theta, t) = (1 - t^{2})^{1/2} \zeta_{\nu}(\theta, t),
\]
where $\eta_{\nu}$ and $\zeta_{\nu}$ are defined by (2.27) and (2.28), form orthonormal bases of $L^{2}(S_{a}^{n}; \tilde{W})$ and $L_{c}^{2}(\mathcal{E}_{n}; \tilde{w})$, respectively.

Corollary 2.14. The number
\[
s_{\nu} = 2^{\lambda} \pi^{(n-1)/2} \left[ \frac{(m + 2k)! \Gamma(k + m + \lambda) \Gamma(k + 1 + \lambda - n/2)}{k! \Gamma(m + 2k + 2\lambda) \Gamma(k + m + n/2)} \right]^{1/2},
\]
is the singular value of the vertical slice transform $V_{+}$ with respect to the orthonormal systems $\{\tilde{\eta}_{\nu}\}$ and $\{\tilde{\zeta}_{\nu}\}$, that is, $V_{+} \tilde{\eta}_{\nu} = s_{\nu} \tilde{\zeta}_{\nu}$.

Proof. By Lemma 4.124 from [30], $R \eta_{\nu} = s_{\nu} \zeta_{\nu}$. Hence, by (2.23) and (2.31), $\beta R a^{-1} \tilde{\eta}_{\nu} = s_{\nu} \beta \tilde{\zeta}_{\nu} = s_{\nu} \zeta_{\nu}$, that is, $V_{+} \tilde{\eta}_{\nu} = s_{\nu} \tilde{\zeta}_{\nu}$. $\square$

The singular value decompositions of $V_{+}$ and its inverse are consequences of Theorem 4.125 from [30], Lemma 2.11, and Corollary 2.14. We set $F(\theta, t) = V_{+} f(\theta, t)$ and use the following notation for the corresponding Fourier coefficients:
\[
f_{\nu} = \int_{S_{a}^{n}} f(x) \tilde{\eta}_{\nu}(x) \tilde{W}(x) d\sigma(x), \\
F_{\nu} = \int_{\mathcal{E}_{n}} F(\theta, t) \tilde{\zeta}_{\nu}(\theta, t) \tilde{w}(t) d\theta dt.
\]

Theorem 2.15. Let $f \in L^{2}(S_{a}^{n}; \tilde{W})$.
(i) The singular value decomposition of the vertical slice transform $V_{+} f$ has the form
\[
(V_{+} f)(\theta, t) = \sum_{\nu} s_{\nu} f_{\nu} \tilde{\zeta}_{\nu}(\theta, t),
\]
where $s_{\nu}$ has the form (2.32) and $\tilde{\zeta}_{\nu}$ is defined by (2.31) and (2.28).
(iii) The function \( f \) can be reconstructed from \( F = V_+ f \) by the formula

\[
f(x) = \sum_{\nu} s_\nu^{-1} F_\nu \tilde{\eta}_\nu(x),
\]
where \( \tilde{\eta}_\nu \) is defined by (2.31) and (2.27).

The series (2.33) and (2.34) converge in the \( L^2(\mathcal{C}_n; \tilde{w}) \)-norm and in the \( L^2(S_{n+1}^n; \tilde{W}) \)-norm, respectively.

3. Method of Analytic Continuation

Given \( x \in S^n \) and \( t \in (-1, 1) \), let

\[
(Mf)(x, t) = \frac{1 - t^2}{} \int \left[ f(y) \frac{1 - t^2}{\sigma_{n-1}} \right] dy \sigma(y)
\]
be the mean value of \( f \) over the planar section \( \{ y \in S^n : x \cdot y = t \} \).

This operator is commonly used in analysis on the sphere; see, e.g., [30] and references therein. If \( x \) lies on the equator \( S^{n-1} \), then (3.1) is exactly our vertical slice transform, so that

\[
(Vf)(\theta, t) = \sigma_{n-1} (Mf)(\theta, t) \frac{1 - t^2}{(n-1)/2}.
\]

Thus reconstruction of \( f \) from \((Vf)(\theta, t)\) is equivalent to reconstruction of \( f \) from the spherical mean \( Mf \) over geodesic spheres centered on the equator. It means that we can invoke the method which was suggested in our previous paper [4], of course, with suitable modifications.

Unlike the methods of the previous section, we can apply the method of analytic continuation only to infinitely differentiable functions which are even in the \( x_{n+1} \) variable and vanish identically in some neighborhood of the equator \( x_{n+1} = 0 \). The space of all such functions will be denoted by \( \tilde{C}_e^\infty(S^n) \). The subspace of integrable functions on \( S^n \) which are even in the \( x_{n+1} \)-variable will be denoted by \( L^1_e(S^n) \). The abbreviation \( a.c. \) means analytic continuation.

It is convenient to treat the cases \( n > 2 \) and \( n = 2 \) separately.

3.1. The case \( n > 2 \). We introduce an analytic family of operators

\[
(N^\alpha f)(\theta, t) = \int_{S^n} \left[ \frac{1 - t^2}{\Gamma(\alpha/2)} \right] f(y) dy \sigma(y), \quad (\theta, t) \in \mathcal{C}_n, \quad Re \alpha > 0,
\]
and a backprojection operator \( P \) that sends functions on \( \mathcal{C}_n \) to functions on \( S^n \) by the formula

\[
(PF)(x) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} F(\theta, \theta \cdot x) d\sigma(\theta), \quad x \in S^n.
\]
For \( x \) and \( y \) in \( S^n \) we keep the previous notation \( x = (x', x_{n+1}) \), \( y = (y', y_{n+1}) \), where \( x' \) and \( y' \) are points in \( B_n \).

**Lemma 3.1.** If \( f \in L^1_e(S^n) \),

\[
\varphi(y') = (1 - |y'|^2)^{-1/2} f(y', (1 - |y'|^2)^{1/2}),
\]

then

\[
a.c._{\alpha=3-n} (PN^\alpha f)(x) = \frac{2\Gamma(n/2)}{\pi^{1/2}} \int_{B_n} \frac{\varphi(y') dy'}{|x' - y'|^{n-2}}.
\]  

**Proof.** For \( Re \alpha > 0 \), changing the order of integration, we have

\[
(PN^\alpha f)(x) = \int_{S^n} f(y) k_\alpha(x, y) d\sigma(y),
\]

where

\[
k_\alpha(x, y) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} |\theta \cdot (x-y)|^{\alpha-1} d\sigma(\theta) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} |\theta \cdot (x'-y')|^{\alpha-1} d\sigma(\theta)
\]

\[
= \frac{|x' - y'|^{\alpha-1}}{\sigma_{n-1}} \int_{S^{n-1}} |\theta \cdot \omega|^{\alpha-1} d\sigma(\theta), \quad \omega = \frac{x' - y'}{|x' - y'|} \in S^{n-1}.
\]

The last integral can be evaluated by the formula

\[
\int_{S^{n-1}} |\theta \cdot \eta|^{\alpha-1} d\theta = \frac{2\pi^{(n-1)/2}\Gamma(\alpha/2)}{\Gamma((n + \alpha - 1)/2)};
\]

see, e.g., [30, formula (1.12.14)]. This gives

\[
k_\alpha(x, y) = c_{n, \alpha} |x' - y'|^{\alpha-1}, \quad c_{n, \alpha} = \frac{\Gamma(n/2) \pi^{-1/2}}{\Gamma((n + \alpha - 1)/2)},
\]

and therefore, by Lemma 2.1,

\[
(PN^\alpha f)(x) = c_{n, \alpha} \int_{S^n} f(y) |x' - y'|^{\alpha-1} d\sigma(y) = 2c_{n, \alpha} \int_{B_n} \varphi(y') |x' - y'|^{\alpha-1} dy',
\]

where \( \varphi \in L^1(B_n) \) has the form (3.5). The last integral is an analytic function of \( \alpha \) in the domain \( Re \alpha > 1 - n \) because one can differentiate in \( \alpha \) under the sign of integration. Taking analytic continuation at \( \alpha = 3 - n \), we complete the proof.

We will also need an alternative representation of \( a.c._{\alpha=3-n} (PN^\alpha f)(x) \) in terms of the spherical means (3.1).
Lemma 3.2. Let \( f \in \tilde{C}_c^\infty(S^n) \),
\[
\delta_n = \frac{(-1)^{[n/2-1]} \Gamma((n-1)/2)}{(n-3)!}.
\] (3.8)

(i) If \( n = 3, 5, \ldots \), then
\[
a.c. \ (PN^\alpha f)(x) = \delta_n \int_{S^{n-1}} (d/dt)^{n-3}[(Mf)(\theta, t) (1 - t^2)^{n/2-1}] \bigg|_{t=\theta \cdot x} d\theta,
\]

(ii) If \( n = 4, 6, \ldots \), then
\[
a.c. \ (PN^\alpha f)(x) = -\frac{\delta_n}{\pi} \int_{S^{n-1}} d\sigma(\theta)
\times \int_{-1}^{1} (d/dt)^{n-2}[(Mf)(\theta, t) (1 - t^2)^{n/2-1}] \log|t - \theta \cdot x| dt.
\]

Proof. For \( \Re \alpha > 0 \), by making use of the formula
\[
\int_{S^n} f(y) a(\theta \cdot y) dy = \sigma_{n-1} \int_{-1}^{1} a(s)(Mf)(\theta, s) (1 - s^2)^{n/2-1} ds
\] (3.9)
(see, e.g., [30, formula (A.11.18)], we have
\[
(N^\alpha f)(\theta, t) = \frac{\sigma_{n-1}}{\Gamma(\alpha/2)} \int_{-1}^{1} (Mf)(\theta, s) |s - t|^{\alpha-1} (1 - s^2)^{n/2-1} ds
\]
\[
= \int_{-\infty}^{\infty} \frac{|s|^{\alpha-1}}{\Gamma(\alpha/2)} h_\theta(s + t) dst, \quad h_\theta(t) = \sigma_{n-1} (Mf)(\theta, t) (1 - t^2)^{n/2-1}.\]

Here \((1 - t^2)^{n/2-1}\) denotes extension of \((1 - t^2)^{n/2-1}\) by zero outside \((-1, 1)\). Because \( f \) is smooth and its support is separated from the equator, \( h_\theta \) belongs to \( C^\infty(\mathbb{R}) \) uniformly in \( \theta \) and vanishes identically in the respective neighborhoods of \( t = \pm 1 \). Thus, we can invoke the standard procedure of analytic continuation (see, e.g., [8], [4, Lemma 2.1]), and obtain the following equalities.
For \( n = 3, 5, \ldots \):
\[
a.c. \ (N^\alpha f)(\theta, t) = \delta_n h_\theta^{(\alpha-3)}(t).
\]
For \( n = 4, 6, \ldots \):

\[
a.c. \quad (N^\alpha f)(\theta, t) = -\frac{\delta_n}{n} \int_{-1}^{1} h^{(n-2)}_\theta (s) \log |s-t| \, ds,
\]

\( \delta_n \) being defined by (3.8).

Combining these formulas with the backprojection \( P \) and noting that operations \( a.c. \) and \( P \) commute, we obtain

\[
(PN^\alpha f)(x) = \delta_n \sigma_{n-1}^{n-1} \int_{S^{n-1}} \left. (d/dt)^{n-3}(V f)(\theta, t) \right|_{t=\theta \cdot x'} \log |t-\theta \cdot x| \, dt,
\]

if \( n = 3, 5, \ldots \), and

\[
(PN^\alpha f)(x) = -\frac{\delta_n}{\pi \sigma_{n-1}^{n-1}} \int_{S^{n-1}} \left. (d/dt)^{n-2}(V f)(\theta, t) \right|_{t=\theta \cdot x'} \log |t-\theta \cdot x| \, dt,
\]

if \( n = 4, 6, \ldots \). This gives the result.

Now we compare different expressions of \( a.c. \quad PN^\alpha f \) in Lemmas 3.1 and 3.2. The right-hand side of (3.6) is \( c_n I^2 \varphi \), where \( I^2 \varphi \) is the Newtonian potential (2.14) and \( c_n = 4\pi^{(n-1)/2}(n-2) \). This gives the following corollary.

**Corollary 3.3.** Let \( f \in \tilde{C}_e^\infty (S^n) \), \( n > 2 \).

(i) If \( n = 3, 5, \ldots \), then

\[
(I^2 \varphi)(x') = \lambda_n \int_{S^{n-1}} (d/dt)^{n-3}(V f)(\theta, t) (1-t^2)^{-1/2} \bigg|_{t=\theta \cdot x'} \, d\theta,
\]

\[
\lambda_n = \frac{\delta_n}{c_n \sigma_{n-1}^{n-1}} = \frac{(-1)^{[n/2]} \Gamma((n-1)/2) \Gamma(n/2)}{8\pi^{n-1/2} (n-2)!}.
\]

(ii) If \( n = 4, 6, \ldots \), then

\[
(I^2 \varphi)(x') = -\frac{\lambda_n}{\pi} \int_{S^{n-1}} (d/dt)^{n-2}(V f)(\theta, t) (1-t^2)^{-1/2} \log |t-\theta \cdot x'| \, dt.
\]

Now, inverting \( I^2 \varphi \) by making use of the Laplace operator \( \Delta = \partial_1^2 + \ldots + \partial_n^2 \), we can reconstruct \( \varphi \), and therefore \( f \).

**Theorem 3.4.** If \( n > 2 \), then a function \( f \in \tilde{C}_e^\infty (S^n) \) can be reconstructed from the vertical slice transform \( V f \) as follows.
(i) If \( n = 3, 5, \ldots \), then
\[
  f(x) = \lambda_n |x_{n+1}| \left( -\Delta \right) \int_{S^{n-1}} (d/dt)^{n-3} [(V f)(\theta, t) (1 - t^2)^{-1/2}]_{t=\theta} \, d\theta,
\]
where \( \lambda_n \) is the constant (3.10).

(ii) If \( n = 4, 6, \ldots \), then
\[
  f(x) = \frac{\lambda_n}{\pi} |x_{n+1}| \Delta \int_{S^{n-1}} d\sigma(\theta) \times \int_{-1}^{1} (d/dt)^{n-2} [(V f)(\theta, t) (1 - t^2)^{-1/2}] \log |t - \theta \cdot x'| \, dt.
\]

\[\text{(3.11)}\]

3.2. The case \( n = 2 \). In this case a substitute of the Newtonian potential \( I^2 \varphi \) is the logarithmic potential
\[
  (I^2_* \varphi)(x') = \frac{1}{2\pi} \int_{B_2} \varphi(y') \log |x' - y'| \, dy', \quad x' \in B_2.
\]

\[\text{(3.12)}\]

**Lemma 3.5.** If \( f \in \tilde{C}_c^\infty(S^2) \), \( \varphi(x') = (1 - |x'|^2)^{-1/2} f(x', (1 - |x'|^2)^{1/2}) \), then
\[
  (I^2_* \varphi)(x') = \frac{1}{4\pi} \int_{S^1} d\sigma(\theta) \int_{-1}^{1} (Mf)(\theta, s) \log |s - \theta \cdot x'| \, ds + \frac{\log 2}{4\pi} \int_{S^2} f(y) \, d\sigma(y).
\]

\[\text{(3.13)}\]

**Proof.** Let
\[
  (N_* f)(\theta, t) = \int_{S^2} f(y) \log |\theta \cdot y - t| \, d\sigma(y), \quad (\theta, t) \in S^1 \times (-1, 1),
\]
\[
  (P_* F)(x) = \frac{1}{2\pi} \int_{S^1} F(\theta, \theta \cdot x) \, d\sigma(\theta), \quad x \in S^2.
\]

\[\text{(3.14)}\]

Changing the order of integration, we obtain
\[
  (P_* N_* f)(x) = \int_{S^2} f(y) k_*(x, y) \, d\sigma(y),
\]
where
\[ k_*(x, y) = \frac{1}{2\pi} \int_{S^1} \log |\theta \cdot (x - y)| \, d\sigma(\theta) \]
\[ = \log |x' - y'| + \frac{1}{2\pi} \int_{S^1} \log |\theta \cdot \omega| \, d\sigma(\theta), \quad \omega = \frac{x' - y'}{|x' - y'|} \in S^1. \]

The second term can be easily evaluated:
\[ \frac{1}{2\pi} \int_{S^1} \log |\theta \cdot \omega| \, d\sigma(\theta) = \frac{1}{\pi} \int_{-1}^{1} \frac{\log|t|}{\sqrt{1 - t^2}} \, dt = -\log 2; \]
see, e.g., [11, formula 4.241 (7)]. Thus
\[ k_*(x, y) = \log |x' - y'| - \log 2, \]
and we have
\[ (P_*N_*f)(x) = \int_{S^2} f(y) \log |x' - y'| \, d\sigma(y) - c_f, \quad c_f = \log 2 \int_{S^2} f(y) \, d\sigma(y). \]

By Lemma 2.1 it follows that
\[ (P_*N_*f)(x) = 2 \int_{B_2} \varphi(y') \log |x' - y'| \, dy' - c_f = 4\pi (I^2 \varphi)(x') - c_f. \quad (3.15) \]

On the other hand, by (3.9),
\[ (N_*f)(\theta, t) = 2\pi \int_{-1}^{1} (Mf)(\theta, s) \log |s - t| \, ds \]
and
\[ (P_*N_*f)(x) = \int_{S^1} d\sigma(\theta) \int_{-1}^{1} (Mf)(\theta, s) \log |s - \theta \cdot x| \, ds. \quad (3.16) \]

Comparing (3.16) with (3.15), we obtain (3.13).

Lemma 3.5 gives the following inversion result.

**Theorem 3.6.** A function \( f \in \tilde{C}_c^\infty(S^2) \) can be recovered from its vertical slice transform \( Vf \) by the formula
\[ f(x) = \frac{|x_3|}{8\pi^2} \Delta \int_{S^1} d\sigma(\theta) \int_{-1}^{1} (Vf)(\theta, t) \frac{\log|t - \theta \cdot x'|}{\sqrt{1 - t^2}} \, dt. \quad (3.17) \]
Proof. By (3.13),

\[(I_2^2 \varphi)(x') = \frac{1}{4\pi} \int_{S^1} d\sigma(\theta) \int_{-1}^{1} (Mf)(\theta, s) \log |s - \theta \cdot x'| ds \]

\[+ \frac{\log 2}{4\pi} \int_{S^2} f(y) d\sigma(y).\]

Hence, owing to (3.2) and the equality \(\Delta I_2^2 \varphi = \varphi\), we obtain

\[\varphi(x') = (1 - |x'|^2)^{-1/2} f(x', (1 - |x'|^2)^{1/2})\]

\[= \frac{1}{8\pi^2} \Delta \int_{S^1} d\sigma(\theta) \int_{-1}^{1} (Vf)(\theta, t) \frac{\log |t - \theta \cdot x'|}{\sqrt{1 - t^2}} dt.\]

The latter is equivalent to (3.17). \(\Box\)

Note that (3.17) can be formally obtained from (3.11) if we set \(n = 2\).

Remark 3.7. Results of this section can be applied to explicit solution of the inverse problem for the Euler-Poisson-Darboux equation

\[\tilde{\Box}_\alpha u \equiv \Delta_S u - u_{\omega \omega} - (n - 1 + 2\alpha) \cot \omega u_{\omega} + \alpha(n - 1 + \alpha)u = 0, \quad (3.18)\]

where \(x \in S^n\) is the space variable, \(\omega \in (0, \pi)\) is the time variable, and \(\Delta_S\) is the Beltrami-Laplace operator on \(S^n\) acting in the \(x\)-variable. The particular case \(\alpha = (1 - n)/2\) gives the wave equation in spherical tomography. The Cauchy problem

\[\tilde{\Box}_\alpha u = 0, \quad u(x, 0) = f(x), \quad u_{\omega}(x, 0) = 0,\]

is well-known; see, e.g., [25], [28, p. 179], and references therein.

The corresponding inverse problem is to find the initial function \(f\) and reconstruct \(u(x, \omega)\) for all \((x, \omega) \in S^n \times (0, \pi)\) when the solution \(u(x, \omega)\) is known only for \(x\) restricted to the equator \(S^{n-1}\). The interested reader is referred to [4, Section 6], where this problem is solved in a more general setting for geodesic spheres of arbitrary fixed radius \(0 < \theta \leq \pi/2\).

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