Extrapolating the precision of the Hypergeometric Resummation
to Strong couplings with application to the $\mathcal{PT}$–Symmetric $i\phi^3$
Field Theory

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Abstract

In PRL 115, 143001 (2015), H. Mera et al. developed a new simple but precise Resummation
 technique. In this work, we suggest to obtain half of the parameters of the Hypergeometric function
from the strong coupling expansion of the physical quantity. Since these parameters are taking
now their exact values they can improve the precision of the technique for the whole range of
the coupling values. The modified algorithm here gives the exact form of the zero-dimensional
partition function of the $\phi^4$ theory from just the first order of perturbation series. The second
order of the algorithm is applied to resum the perturbation series of the ground state energy of the
$\mathcal{PT}$–symmetric $(i\phi^3)_{0+1}$ field theory. It gives accurate results compared to exact calculations from
the literature specially for very large coupling values. The $\mathcal{PT}$– symmetry breaking of the Yang-
Lee model has been investigated where third, fourth and fifth orders were able to get very accurate
results when compared to other resummation methods involving 150 orders. The algorithm can be
extended easily to accommodate any order of perturbation series in using the generalized $k+1F_k$
as it shares the same analytic properties of $2F_1$.

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In quantum field theories one is always confronted with perturbation series of zero radius of convergence \cite{1}. For such cases Resummation techniques have been applied and successful results have been obtained\cite{1,4}. Although these techniques can produce accurate results but they sometimes use calculations for large number of loops as an input besides most of the calculations go numerically. Recently, a precise as well as simple Resummation technique has been introduced \cite{5} where it uses only four orders of the perturbation series as well as it is of analytic form. Such algorithm is very suitable for quantum field cases as it can give reasonable results with only few orders out of the perturbation series. The authors of Ref. \cite{5} have extended the technique to employ higher orders of the perturbation series \cite{6} via use of the general definition of the Hypergeometric functions known in the literature as Meijer-G function \cite{7}. Regarding the original version in Ref.\cite{5} and its upgrades, the prediction of the whole parameters of the Hypergeometric function is obtained from just the first terms of the perturbation series. Since for original version, the first two parameters of the Hypergeometric function are related to its asymptotic form for large values of the argument \cite{7}, the prediction of these parameters from small coupling information might not lead to well known strong coupling behaviors of the physical quantity. The other two parameters can be in fact deduced from the large order behavior but in this case the Hypergeometric function will lose all the information about the small coupling behavior. So to employ all the information that might be known about the perturbation series (small coupling, large order and strong coupling behaviors) one needs to employ a generalized form of the Hypergeometric function. In this work, introduce an algorithm that can lower the number of input perturbation terms to half of those needed by the algorithms in Refs.\cite{5,6} as well as guarantee the accuracy for very large coupling values.

To motivate for this work, we mention that in Ref.\cite{2} the strong coupling behaviors have been stressed for both the $\mathcal{PT}$—symmetric and the real cubic anharmonic oscillator. In applying resummation techniques that involve 150 orders for the $\mathcal{PT}$—symmetric case the authors showed that:

$$\lim_{g \to \infty} \frac{E_0}{|g|^\frac{1}{5}} = 0.372545790452207098250601(1),$$

\(1\)
while for the cubic oscillator they obtained

$$\lim_{g \to -\infty} \frac{E_0}{|g|^\frac{3}{2}} = 0.3013958756586835717823(7)$$

$$+ 0.2189769214314493762936(0)i$$

(2)

It is easy to check that the prediction of the original Hypergeometric Resummation in [5] gives zero in both cases. This is because there is like 6% error in the prediction of the second parameter in the Hypergeometric function which affects the precision of the algorithm for large coupling values. Accordingly, one needs to extrapolate the predictions of the algorithm to give accurate results for the whole coupling space.

In this work we apply the Hypergeometric Resummation algorithm to the \( \mathcal{PT} \)-symmetric Yang-Lee model but in guiding the Hypergeometric functions with parameters from the strong coupling behavior. In 0+1 space-time dimension (quantum mechanics), one can follow a scaling as well as gauge canonical transformations to obtain the strong coupling expansion of the theory [2]. In higher dimensions (Quantum field), there exists known tools to get the strong-coupling expansions of a physical quantity too [8, 9]. So feeding the resummation technique with two parameters (for the second order) from the perturbation series and the other two from the strong coupling expansion is possible for both quantum and quantum field problems. As we will see in this work this algorithm lowers the number of orders from the perturbation series to two instead of four needed for the original algorithm in Ref.[5]. Besides the prediction is then more accurate for large couplings. The extension of the algorithm to higher orders is direct and shall be applied here to investigate \( \mathcal{PT} \)-symmetry breaking of the Yang-Lee model with increasing precision when moved from \( {}_2F_1 \) to \( {}_3F_2 \) and very high precision obtained from \( {}_4F_3 \) and \( {}_5F_4 \) when compared to resummation results from Ref.[2] where methods involved 150 orders.

The Yang-Lee model or equivalently \( \mathcal{PT} \)-symmetric \( i\phi^3 \) field theory has been exposed to recent discussions because it has an imaginary potential but on the other hand has a real spectrum[11-16]. In fact the ground state energy has a zero radius of convergence and thus non-perturbative Resummation algorithms are in a need to get reliable results from perturbative calculations as an input. Another aspect for which non-perturbative approaches of the Yang-Lee model are essential is because it represents the Landau-Ginzberg approximation of the Ising model near the Yang-Lee edge singularity[17-21]. Pade, Borel and other algorithms applied to the model in Refs.[22-25]. While Pade approximation can
not account for the strong coupling behavior, most of Borel calculations are achieved via numerical calculations. The recent Hypergeometric Resummation technique introduced in [5] is characterized by being simple, closed form as well as employs only few number of terms from the perturbation series as an input. In $0+1$ space-time dimensions (quantum mechanics) the Yang-Lee model has been stressed by the authors of Ref. [5] but we realized that the precision of the results for strong coupling (even for the real potential case) is questionable. As we suggested above, a way to have better fitting with available exact results is to provide the Hypergeometric function with known results from the strong coupling behavior. In fact, strong coupling expansion can be obtained in many cases. For instance, Hermitian theories like the $\phi^4$ field theory has been extensively stressed in the literature and its strong coupling as well as large order behaviors are known [1]. Applying a simple and accurate Resummation algorithms to such cases might have a strong impact on the field of $\mathcal{PT}$– symmetric field theories where one can resum the series from the known results of just the first two terms in the perturbation series and its strong coupling behavior. The $\mathcal{PT}$–symmetric $(-\phi)^4$ model is assumed to be asymptotically free [26–29] but up to the best of our knowledge no non-perturbative calculation for the Beta function appeared yet. Another application that the Hypergeometric Resummation can play a vital role is in the very recently introduced $\mathcal{PT}$–symmetric Higgs Mechanism and such Resummation technique may offer a non-perturbative tool that saves the effort and time for the calculation in such cases where high order of loop calculations is time consuming.

For the Hypergeometric Resummation there is another precision realization concerning the small coupling predictions where it has been realized by the authors themselves in Ref. [10]. According to them, the series expansion of the Hypergeometric function does not have a zero radius of convergence while the aim is to sum a series of zero radius of convergence. To solve this problem, the authors set an algorithm that results in a Hypergeometric Resummation with zero radius of convergence [10]. In this work, we shall stress only the impact of employing the strong coupling behavior on the accuracy of the algorithm. In fact, this saves the effort of loop calculations at higher orders as this version of the algorithm shall need two orders only from the perturbation series.

The Hypergeometric function $\mathit{2F_1}$ can have a power law behavior near singular points [7] and thus in principle can account for the calculation of the critical exponents of the Yang-Lee model near the edge singularity but this will be out of the scope of this work.
Before we start the application of the algorithm to the $\mathcal{PT}$-symmetric $i\phi^3$ theory, we need to test its accuracy from examples where exact results are known. An example that always used for that goal is the the zero-dimensional partition of the $\phi^4$ which is given by:

$$Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi \exp \left( -\frac{1}{2} \phi^2 - \frac{g}{4!} \phi^4 \right).$$

(3)

The strong coupling expansion can be obtained as:

$$Z (g) = \frac{\sqrt{2\pi}}{\Gamma \left( \frac{3}{4} \right)} g^{-\frac{1}{4}} - \frac{\sqrt{3}}{2} \Gamma \left( \frac{3}{4} \right) g^{-\frac{3}{4}} + O \left( g^{-\frac{5}{4}} \right),$$

(4)

while the weak coupling perturbation is given by:

$$Z (g) = 1 - \frac{g}{8} + \frac{35}{384} g^2 + O (g^3)$$

(5)

The Hypergeometric summation of the weak coupling series is suggested to be of the form:

$$Z (g) = {}_2F_1 (a, b, c, d g).$$

Since the strong coupling behavior of ${}_2F_1 (a, b, c, d g) \propto \alpha g^{-a} + \beta g^{-b}$ one can get from Eq.(4) the result $a = \frac{1}{4}$ and $b = \frac{3}{4}$. To get the values of the parameters $c$ and $d$ we compare the series expansion of ${}_2F_1 (a, b, c, d g)$ with the weak coupling perturbation series in Eq.(5) to get the result:

$$\frac{3d}{16c} = -\frac{1}{8}$$

$$\frac{105 d^2}{512 c (c + 1)} = \frac{35}{384}$$

In the second equation, insert the value of $d$ from the first to get the equation:

$$\frac{35}{384} c = \frac{35}{384} c + \frac{35}{384},$$

which has the solution $c = 0$. In other words, the suggested Hypergeometric resummation should be ${}_2F_0 (a, b, d g)$ instead of ${}_2F_1 (a, b, c, d g)$. The series expansion of ${}_2F_0 \left( \frac{1}{4}, \frac{3}{4}, d g \right)$ is

$${}_2F_0 \left( \frac{1}{4}, \frac{3}{4}, d g \right) = 1 + \frac{3}{16} d g + O (g^2),$$

Then from Eq.(3) we get:

$$\frac{3}{16} d = -\frac{1}{8} \Rightarrow d = -\frac{2}{3}.$$
Accordingly the Hypergeometric resummation of the perturbation series in Eq.\((3)\) is

$$Z(g) = _2 F_0 \left( \frac{1}{4}, \frac{3}{4}, \frac{-2g}{3} \right).$$

In using the identity

$$_2 F_0 (-n, n + 1, x) = \frac{1}{\sqrt{\pi}} \sqrt{-\frac{1}{x}} \exp \left( -\frac{1}{2x} \right) K_{n+\frac{1}{2}} \left( -\frac{1}{2x} \right),$$

we get the exact result reported in Ref.[6] but the authors used five orders while we used here only the first order as an input. The acceleration of the convergence of the algorithm is expected due to the employment of the exact strong coupling parameters.

For more tests of the algorithm before we go to the \(\mathcal{PT}\)–symmetric \(i\phi^3\) theory, the \(_2 F_1\) resummation has been tested to resum the ground state energy of the anharmonic oscillator where it has a perturbation series with zero radius of convergence [30]. We obtained the ground state function \(E_0 (g) = _2 F_1 \left( \frac{1}{3}, \frac{-1}{3}, c, -dg \right)\) and find \(E_0 (50) = 2.4484029106721046\). Although this result is better than Borel-Pade resummation in using 24\(^{th}\) order \(BP_{12}/12\) where it gives \(E_0 (50) = 2.3157388197\) [6], one can get better results in involving more terms from the perturbation series. When we use \(_4 F_3\) we get \(E_0 (50) = 2.4856072532925255\). Note that the exact result is 2.4997087726 and the Hypergeometric resummation in Ref.[6] gives 2.4997107287 but in involving 25 orders. So it seems that the algorithm we use accelerates the convergence to the exact prediction as the second order in our algorithm have accuracy that lies between the 24\(^{th}\) order of Borel-Bade resummation and the 25\(^{th}\) of the generalized Hypergeometric resummation in Ref.[6] while our fourth order results in a very close result to the exact one. We will stop at this order for the anharmonic oscillator example as our main problem is to investigate the \(\mathcal{PT}\)–symmetric \(i\phi^3\) theory and the closely related issue of \(\mathcal{PT}\)–symmetry breaking in the Yang-Lee model.

Now, the \(\mathcal{PT}\)–symmetric \(i\phi^3\) theory has a Lagrangian density of the form:

$$\mathcal{L} [\phi] = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 (x) - \frac{i\sqrt{g}}{6} \phi^3 (x), \quad (6)$$

with a corresponding Hamiltonian density:

$$H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 (x) + \frac{i\sqrt{g}}{6} \phi^3 (x). \quad (7)$$

The Hamiltonian operator is \(\mathcal{PT}\)–symmetric and thus the spectrum is real. This Hamiltonian is closely related to the Hamiltonian

$$H_J = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{i}{6} \phi^3 (x) + iJ \phi, \quad (8)$$
where in 0+1 space-time dimensions, one can start from $H$ and apply scaling as well as gauge transformation to get $H_J$ \[2\] \[23\] \[31\]. Although $H_J$ is $\mathcal{PT}$-symmetric, the $\mathcal{PT}$-Symmetry is not exact for all real $J$ values and the $\mathcal{PT}$-Symmetry is broken for some critical $J$ value \[2\]. According to Yang and Lee, the partition function or equivalently the vacuum to vacuum amplitude can have a zero for negative $J$ values known as Yang-Lee edge singularity \[17\] \[20\]. This specific zero is associated with non-analyticity of the ground state energy and this is supposed to be associated with $\mathcal{PT}$-symmetry breaking. Near the edge singularity the theory is totally non-perturbative and one needs to apply non-perturbative techniques. The perturbation series for the ground state energy of the Hamiltonian $H$ also has a zero radius of convergence and thus Resummation is needed anyway. We will stress the resummation of ground state energy of both Hamiltonians $H$ and $H_J$ and show that our second to fourth orders are very competitive to the results with the 150th order of resummation methods in Ref.\[2\]. For $H_J$, the important $\mathcal{PT}$-symmetry breaking near the edge singularity will be shown in our results while the first 20th orders of weak coupling expansion cannot account for it.

For the model represented by the above Lagrangian density, up to first order in $g$, the ground state energy receives only contributions from the sunset and dumbbell diagrams shown in Fig.1 while Mercedes and other four diagrams contribute to the $g^2$ order which are shown in Fig.2 (these diagrams are all listed in Ref.\[11\] too). Accordingly, up to $g^2$ order, the vacuum energy is given by:

\[
E_0 = \frac{1}{2} + \frac{11g}{288} - \frac{930}{288^2}g^2,
\]  

(9)

where we assumed in Eq.(6) that $m = 1$ and the space-time dimension is $0 + 1$. This result can be checked in many articles although some of them obtained it in different basis \[2\] \[11\]. It is well known that this series is Borel summable as well as having a zero radius of convergence. Accordingly, non-perturbative techniques are needed in order to get reasonable results for the vacuum energy. In many articles different techniques applied ranging from Pade approximation \[12\] and Borel Resummation \[2\] to the recent Hypergeometric Resummation \[5\]. In fact, Pade approximation although can account for needed branch cuts of the divergent series, it fails to reproduce the strong coupling behavior of the physical quantity under consideration. The Hypergeometric algorithm in \[5\] needs information from at least four orders in perturbation series. According to the Hypergeometric Resummation of a di-
FIG. 1: The 2-vertices Feynman diagrams contributing to the first order in squared-coupling \( g \) of vacuum energy for the \( PT \)-symmetric \( (i\phi^3)_{0+1} \) field theory. The symmetry factor \( S \) is written for each diagram.

vergent series, the vacuum energy of the Hamiltonian \( H \) in \( 0+1 \) space-time dimensions is given by:

\[
E_0 = \frac{1}{2} \, _2F_1 \left( a, b, c, -\frac{g}{d} \right),
\]

(10)

where \( _2F_1 \) is the Hypergeometric function and the parameters \( a, b, c \) and \( d \) can be obtained from the series expansion of the Hypergeometric function and then matching them with the first four terms in the perturbation series. The series expansion of Eq.(10) can be obtained as

\[
E_0 = \frac{1}{2} - \frac{(ab)}{2(cd)} g + \frac{a(a+1)b(b+1)}{4(c+1)d^2} g^2 - \frac{(a(a+1)(a+2)b(b+1)(b+2))}{12 (c(c+1)(c+2)d^3)} g^3 + \frac{a(a+1)(a+2)(a+3)b(b+1)(b+2)(b+3)}{48c(c+1)(c+2)(c+3)d^4} g^4 + O(g^5)
\]

The order of the series in Eq.(9) does not have enough information to solve for the four unknown parameters. So one have to add contributions from vacuum diagrams up to eight vertices. Although this is possible but in applying the method to a more realistic field theory
FIG. 2: The vacuum diagrams contribution to the order $g^2$ of the ground state energy of the $\mathcal{P}\mathcal{T}$-symmetric $i\phi^3$ field theory.

like QCD, it will be time consuming. It would be better to seek a way to lower the number of orders in the perturbation series needed to find the different parameters. Rather than this, the parameters when all are obtained from just the first few orders in perturbation series, the resumed function does not reproduce well known strong coupling limits of the Yang-Lee model shown in Eqs[1] & [2]. So it is very necessary to feed the Hypergeometric function with parameters obtained from strong coupling behavior. In fact, when $a - b$ is not an integer and for large values of $|g|$, the Hypergeometric function has the following asymptotic form[7]:

$$ _2F_1 (a, b, c, g) \sim \lambda_1 g^{-a} + \lambda_1 g^{-b}, |g| \gg 1. $$

In many cases the strong coupling behavior of a physical quantity is known. In such cases, the parameters $a$ and $b$ are known exactly and only a second order of the perturbation series is sufficient to predict the other two parameters. In $0 + 1$, the Hamiltonian takes the form

$$ H = \frac{1}{2} \pi^2 + \frac{1}{2} m^2 \phi^2 (x) + \frac{i \sqrt{g}}{6} \phi^3 (x). \quad (11) $$
A symmetry transformation of the form

\[ x \rightarrow \exp(-wp)x\exp(wp) = x - w[p,x] \]

\[ = x + iw, \quad w = \frac{m^2}{\sqrt{g}} \]

leads to:

\[ H = \frac{1}{2}\pi^2 + \frac{1}{2} \left( \nabla \phi \right)^2 + \left( \frac{1}{6}i\sqrt{g} \right) \phi^3 + \frac{im^4}{2\sqrt{g}}\phi - \frac{m^6}{3g} \]  

(12)

Note that this transformation changes the metric operator but keeping the spectrum invariant [32, 33]. If we follow this by a scaling transformation of the form \( \exp(i\ln\beta) \) it scales \( \phi \) by a factor \( \beta \) [32]. Taking \( \beta = g^{-\frac{1}{10}} \) leads to the result:

\[ H = \sqrt{g} \left( \frac{\pi^2}{2} + \frac{i\phi^3}{6} + \frac{1}{2} \frac{im^4}{g^2}\phi \right) - \frac{m^6}{3g}. \]  

(13)

This form suggests an expansion for the energy in the form

\[ E_0 = -\frac{m^6}{3g} + g^{\frac{1}{5}} \sum_{l=0}^{\infty} c_l g^{-\frac{4l}{5}}. \]

This result has been shown in Ref.[2] and it suggests that \( a = 1 \) while \( b \) equals \( -\frac{1}{5} \). Accordingly, the Hypergeometric Resummation of the perturbation series in Eq.(9) takes the form

\[ E_0 = \frac{1}{2} \binom{1}{-\frac{1}{5}, c, -\frac{g}{d}} \]  

(14)

The parameters \( c \) and \( d \) can be found from matching the coefficients of the first two terms from series expansion of this equation with those in Eq.(9) and then we get:

\[ E_0 = \frac{1}{2} \binom{1}{-\frac{1}{5}, \frac{465}{19}, \frac{8525}{912}, 912 g} \]  

The resummed form in Eq.(14) is fed with information from small coupling (parameters \( c \& d \)) and strong coupling (parameters \( a \& b \)) behaviors. Accordingly, one expect to give accurate results for the whole range of the coupling space. To test that expectation, let us check the accuracy of the Resummation formula in Eq.(14). For \( g = \frac{1}{2} \), we get \( E_0 = 0.516915 \) compared to the best of the resummation algorithms at 150th in Ref.[2] which gives \( E_0 = 0.516892 \). Also, for \( g = 1 \), we have \( E_0 = 0.530886 \) compared to \( E_0 = 0.5307818 \) form Ref.[2]. Now we compare with some larger values of \( g \). For \( g = \frac{288}{49} \), one gets \( E_0 = 0.614319 \) while the result in Ref.[2] gives \( E_0 = 0.612738 \). Also, for \( g = 4 \times 288 \), we get \( E_0 = 1.55851 \) while the exact value (reported in the last row in table III in Ref.[12]) is \( E_0 = 1.53078 \). So it seems that
the simple method of Hypergeometric Resummation gives precise results though it has been fed with information of the first two terms in the perturbation series. However, the original version introduced in Ref.[5] gives precise results for a wide range of coupling values but not for very large coupling values. For instance when $g = 4 \times 288$ it gives $E_0 = 1.48104$ which is not as accurate as our prediction when both are compared with the exact value above. For more tests of our results and also the original version introduced in Ref.[5] one needs to check for the limit at $g \to \pm \infty$. For $g \to -\infty$ our prediction is

$$\lim_{g \to -\infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.30738 + 0.223325i$$

while the prediction of the original form in Ref.[5] is zero and the methods in Ref.[2] gives $0.30139588 + 0.2189769214i$. This is expected because any tiny difference in the parameters $a$ and $b$ will ruin the strong coupling behavior of the resummed function. In fact, the imaginary part of the vacuum energy for a real potential can be obtained by non-perturbative techniques only and thus it is always a good test for any Resummation tool. On the other hand, for the $\mathcal{PT}$-symmetric case ($g \to \infty$), we can find the result:

$$\lim_{g \to \infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.379943,$$

while the result from Ref.[2] is

$$\lim_{g \to \infty} \frac{E_0}{|g|^{\frac{1}{5}}} = 0.3723,$$

but as expected the original Hypergeometric Resummation algorithm gives zero again. These results show clearly that feeding the Hypergeometric Resummation with parameters from the strong coupling behavior is necessary to extrapolate the prediction to the large coupling behavior of the resummed function. The accuracy of our results for large coupling values over the predictions from the original algorithm in Ref.[5] is clear from Fig.3. In this figure one can realize that both of our formula and original one give reasonable results compared to exact results for not so large values of the coupling. For very large values however, one can realize that our formula fits well with exact results but the original formula deviates from the exact results. This is expected as the parameters in Ref.[5] are all predicted from the first four terms in the perturbation series and thus expected to loose memory for strong coupling predictions. Our calculations though uses only two terms from the perturbation series as an input gives more precise results because it was guided by exact parameters from
FIG. 3: Comparison between our Resummation formula $2F_1$ for the ground state energy of the Hamiltonian in Eq.(7) (solid), the original form in Ref.[5] (dashed) and exact results from Ref.[12] (dots). Note that the coupling in our work is rescaled from that in Ref.[12] where $g$ in our work is equivalent to $288\lambda^2$ in that reference.

the strong coupling behavior. Although the idea here applied for $0+1$ space-time dimensions only, it can be applied to the more important higher dimensional cases i.e Quantum field theory as it does exist well known methods to get the strong coupling expansion of physical quantities [8, 9].

The extension of the method to higher orders is direct as one suggests the resummation function as $pF_q(a_1,a_2,....a_p;b_1,b_2,....b_q;−\sigma z)$. When $p = q + 1$, the set of functions $pF_q(a_1,a_2,....a_p;b_1,b_2,....b_q;−\sigma z)$ are all sharing the same analytic properties. In our algorithm, the $a_i$ parameters are determined exactly from the strong coupling expansion of the theory under consideration while $b_i$ and $\sigma$ parameters are determined from $q + 1$ set of algebraic equations obtained by comparing the series expansion of $pF_q(a_1,a_2,....a_p;b_1,b_2,....b_q;−\sigma z)$ with the perturbation series of the physical quantity. In fact, this algorithm reduces the non linearity of the parameters equations to half. One can even get an equivalent set of equations which are all linear in $\sigma$ and all the equations consider only powers of one in each parameters. This strategy avoids troubles faced in the generalized Hypergeometric resummation technique in the literature in solving the set of equations of the $N$ parameters. For the Hamiltonian $H$ in Eq.(7) we resummed the ground
state energy using $2F_1$, $3F_2$, $4F_3$ and $5F_4$ and the results are listed in table I compared to exact results and the 150$^{th}$ resummation techniques from Ref.[2]. It is clear that our fourth order resummation ($4F_3$) gives accurate results and the accuracy is increasing systematically when moving to higher orders.

**TABLE I:** The Hypergeometric resummation $2F_1$, $3F_2$ and $4F_3$ for the ground state function in Eq.(7) compared to the 150$^{th}$ order of resummation methods in Ref.[2] and exact results. It is very clear that the second order $2F_1$ gives accurate results and we get higher precision in going to higher orders where our 4$^{th}$ order resummation $4F_3$ gives results competitive to the the 150$^{th}$ order of resummation methods in Ref.[2].

| $g$ | $2F_1$   | $3F_2$   | $4F_3$   | $E_{150}$ | $E_{exact}$ |
|-----|--------|--------|--------|--------|--------|
| 0.5 | 0.516915482 | 0.5168903301 | 0.516891566 | 0.516891764 | — |
| 1   | 0.530885535 | 0.5307696951 | 0.530779963 | 0.5307817593 | 0.53078176 |
| 288/27 | 0.614318594 | 0.6119805725 | 0.612782366 | 0.61273810639 | 0.612738106 |

For the investigation of $\mathcal{PT}$-symmetry breaking in the Yang-Lee model represented by the Hamiltonian $H$ in Eq.(8), the ground state energy up to second order in $J$ is given by [2];

$$E_J = 0.3725457904522070982506011 + 0.3675358055441936035304J + 0.1437877004150665158339 J^2 + O (J^3)$$

The resummation for this perturbation series is then obtained as:

$$E_J = 2F_1(a,b;c;x)(-3/2, -1/4, b_1; -\sigma z).$$

We compared our results with the 20$^{th}$ order of the perturbation series from Ref.[2] in Fig.4. The resumed series gives reasonable agreement with this relatively high order of perturbation series but deviate from each other near the critical region where the resumed formula starts to be complex ($\mathcal{PT}$-symmetry breaking) and also separated for a relatively strong coupling. A note to be mentioned that the resumed formula using four terms from the perturbation series as in Ref.[5], agrees well with our result but they deviates at very strong couplings as expected. To get more accurate results to be compared with 150$^{th}$ order of resummation methods in Ref.[2], we obtained also the resummation functions $2F_1$, $3F_2$, $4F_3$ and $5F_4$ where we listed them in table I. It is very clear that our resummation formula are giving precise
FIG. 4: Comparison between our Resummation formula $2F_1$ for $E_J$ (solid) and the 20th order of the perturbation series (dashed) from Ref.\[2\]. While the agreement is good for a range of the coupling $J$, the perturbation series fails (as expected) to produce the $\mathcal{P}\mathcal{T}$-symmetry breaking expected as well as fails to fit with strong coupling behavior.

results although we used only low orders of calculations compared to the the 150th order of resummation methods in Ref.\[2\]. Note that in this table for $J = -1$, $2F_1$ results in a tiny imaginary part to the ground state energy and this is because it predicts a smaller critical coupling than $3F_2$, $4F_3$ and $5F_4$ and the methods in Ref.\[2\]. In fact this is acceptable because near the critical point the theory is highly non-perturbative and thus higher orders like $3F_2$, $4F_3$ and $5F_4$ are expected to give more accurate result for the critical coupling.

\begin{table}
\caption{The Hypergeometric resummation $2F_1$, $3F_2$, $4F_3$ and $5F_4$ for the ground state function of the Hamiltonian in Eq.(8) compared to the 150th order of resummation methods in Ref.\[2\] and the 20th order of the perturbation series ($E_{\text{per}}$) from Ref.\[2\]. Our resummation formulae all show up $\mathcal{P}\mathcal{T}$-symmetry Breaking and precision is improved using higher orders. Our third ($3F_2$), fourth ($4F_3$) and fifth ($5F_4$) orders are showing results with competitive precision to the 15th order of resummation methods in Ref.\[2\].}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textit{J} & $2F_1$ & $3F_2$ & $4F_3$ & $5F_4$ & $E_{150}$ & $E_{\text{per}}$
\hline
$-2^+$ & 0.289111-0.345157i & 0.388417-0.328504i & 0.394688-0.3560448i & 0.388902-0.358021i & 0.389 8-0.3644i & 2.52955
\hline
$-1$ & 0.229176 - 0.029543i & 0.197199 & 0.19580355 & 0.195741 & 0.1957508 & 0.19574
\hline
$-5^+$ & 0.282836 & 0.282700 & 0.282699 & 0.28269926 & 0.282699 & 0.282699
\hline
$-21.6^+$ & 0.342161 & 0.342158 & 0.342158 & 0.342158 & 0.342158 & 0.342158
\hline
\end{tabular}
\end{table}
To conclude, we stressed the recently introduced Hypergeometric Resummation algorithm \cite{5}. We realized that when applying the algorithm to the $\mathcal{PT}$-symmetric $i\phi^3$ field theory it gives accurate results for a range of the coupling values but for very large coupling values the results are deviated from expected ones either from exact calculations or from strong coupling limits where both are known from the literature. We expected that the reason behind this is that the four parameters of the Hypergeometric function are all predicted from the first four perturbative terms in the divergent series of the ground state energy and thus the resummed function has no guidance for strong coupling values. In fact, there exist well known techniques to obtain the strong coupling expansion of a physical quantity either for the quantum mechanical case or the more important quantum field case \cite{2,8,9}. Accordingly, we suggested to feed the Hypergeometric function with two parameters that can be predicted from the strong coupling behavior and the other two parameters from the first two terms in the perturbation series. In that way we obtained a Hypergeometric function that bears information from weak coupling as well as strong coupling behaviors and thus expected to give accurate results for the whole range of the coupling space. We tested the algorithm for resumming the perturbation series of the zero-dimensional partition function of the scalar $\phi^4$ theory and get the exact result from only input of the first order in the perturbation series while the last upgrade of the Hypergeometric resummation technique in Ref.\cite{6} the same result has been obtained in the fifth order. This shows that the effect of involving exact parameters from strong coupling expansion shall accelerate the convergence of the resummation function toward exact results. We showed that the extension of the algorithm is direct and one can use as a resummation function the generalized Hypergeometric function $\,_pF_q(a_1,a_2,...,a_p;b_1,b_2,....b_q;-\sigma z)$ where the parameters $a_i$ are all obtained exactly from the strong coupling expansion of the physical quantity under consideration. In fact, this algorithm reduces the noon-linearity issue which one faces when trying to find all the parameters from just perturbation series as it lowers the number of equations in the parameters by a factor of half. Besides, one can get an equivalent set of equations where are all linear in the parameter $\sigma$ as well as having only the power one in each parameter. Also, the algorithm grantees reliable results even for very large coupling values and feeding the resummation function with strong coupling exact parameters accelerates the convergence and thus is expected to have high precision even for low orders of calculations. We tested the idea for the anharmonic oscillator for $g = 50$ and found that our second order ($\,_2F_1$)
result is better than the 24th order for Borel- Pade \( BP_{12-12} \) but is not of same precision as the 25th order of the generalized Hypergeometric algorithm in Ref.[6]. However, our fourth order calculation \( _{4}F_{3}(a_1,a_2,....a_4;b_1,b_2,......b_3;−\sigma z) \) shows good results in comparison with exact ones.

Since our main problem is to resum the ground state function of the \( \mathcal{PT} \)-symmetric \( i\phi^3 \), we tested the prediction of the modified algorithm and found that the second order calculation \( _{2}F_{1}(a_1,a_2;b_1;−\sigma z) \) gives accurate results compared to exact results for a wide range of the coupling. Of course as the coupling increases our predictions go better than those from the original version of the algorithm as explained above. The modified algorithm introduced here is successful to reproduce the well known limit of the ground state energy as \( g \longrightarrow \pm \infty \). Also, the precession is improved when we use higher order where the fourth order predictions give competitive results compared to the 150th order of the resummation algorithms used in Ref.[2] (Table [I]).

The \( \mathcal{PT} \)-symmetry breaking of the Yang-Lee model has been tested where the perturbation series can not account for it at any order. The second order shows good agreement with 20th order of the perturbation series of the ground state energy but far from the critical region as well as large values of the coupling, the perturbation series fails to give reliable results. While the second order of our calculation gives reasonable results and accounts for \( \mathcal{PT} \)-symmetry breaking at a negative coupling as it proved by the work in Ref.[2], it predicts smaller (in value) critical coupling. The accuracy is highly increased in using higher orders where the third, fourth and fifth orders showed great results compared to the to the 150th resummation algorithm used in Ref.[2] (Table [II]).

The algorithm introduced here gives accurate results with less effort as one is not in a need to obtain more than the third order of the strong coupling expansion of a physical quantity and can then conclude all the strong coupling parameters in the series since coefficients in the strong coupling expansion do not matter here. We think that this might be the most simple, accurate and time saving resummation algorithm as it uses the already summed huge set of functions \( _{p}F_{q}(a_1,a_2,....a_p;b_1,b_2,......b_q;−\sigma z) \) where the variety of parameters can fit with
huge number of problems in physics.

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