DENSITIES OF THE RANEY DISTRIBUTIONS

WOJCIECH MLOTKOWSKI, KAROL A. PENSON, KAROL ŻYCZKOWSKI

Abstract. We prove that if \( p \geq 1 \) and \( 0 < r \leq p \) then the sequence \( \binom{mp + r}{m} \frac{r}{mp + r} \), \( m = 0, 1, 2, \ldots \), is positive definite, more precisely, is the moment sequence of a probability measure \( \mu(p, r) \) with compact support contained in \([0, +\infty)\). This family of measures encompasses the multiplicative free powers of the Marchenko-Pastur distribution as well as the Wigner’s semicircle distribution centered at \( x = 2 \). We show that if \( p > 1 \) is a rational number, \( 0 < r \leq p \), then \( \mu(p, r) \) is absolutely continuous and its density \( W_{p, r}(x) \) can be expressed in terms of the Meijer and the generalized hypergeometric functions. In some cases, including the multiplicative free square and the multiplicative free square root of the Marchenko-Pastur measure, \( W_{p, r}(x) \) turns out to be an elementary function.

May 5, 2014

INTRODUCTION

For \( p, r \in \mathbb{R} \) we define the Raney numbers (or two-parameter Fuss-Catalan numbers) by

\[
A_m(p, r) := \frac{r}{m!} \prod_{i=1}^{m-1} (mp + r - i),
\]

\( A_0(p, r) := 1 \). For \( m = 0, 1, 2, \ldots \) we can also write

\[
A_m(p, r) = \binom{mp + r}{m} \frac{r}{mp + r},
\]

(provided \( mp + r \neq 0 \)) where the generalized binomial is defined by

\[
\binom{a}{m} := \frac{a(a-1) \ldots (a-m+1)}{m!}.
\]

Let \( B_p(z) \) denote the generating function of the sequence \( \{A_m(p, 1)\}_{m=0}^\infty \), the Fuss numbers of order \( p \):

\[
B_p(z) := \sum_{m=0}^\infty A_m(p, 1) z^m,
\]

convergent in some neighborhood of 0. For example

\[
B_2(z) = \frac{2}{1 + \sqrt{1 - 4z}}.
\]

2010 Mathematics Subject Classification. Primary 44A60; Secondary 33C20.

Key words and phrases. Mellin convolution, free convolution, Meijer function.

K. Ż. is supported by the Grant DEC-2011/02/A/ST1/00119 of Polish National Centre of Science. K. A. P. acknowledges support from PAN/CNRS under Project PICS No. 4339 and from Agence Nationale de la Recherche (Paris, France) under Program PHYSCOMB No. ANR-08-BLAN-0243-2.
Lambert showed that
\[ \mathcal{B}_p(z)^r = \sum_{m=0}^{\infty} A_m(p, r) z^m, \]
see [8]. These generating functions also satisfy
\[ \mathcal{B}_p(z) = 1 + z \mathcal{B}_p(z)^p, \]
which reflects the identity \( A_m(p, p) = A_{m+1}(p, 1) \), and
\[ \mathcal{B}_p(z) = \mathcal{B}_{p-r}(z \mathcal{B}_p(z)^r). \]

It was shown in [13] that if \( p \geq 1 \) and \( 0 \leq r \leq p \) then the sequence \( \{A_m(p, r)\}_{m=0}^{\infty} \) is positive definite, i.e. is the moment sequence of a probability measure \( \mu(p, r) \) on \( \mathbb{R} \). Moreover, \( \mu(p, r) \) has compact support (and therefore is unique) contained in the positive half-line \([0, \infty)\) (for example \( \mu(p, 0) = \delta_0 \)). The proof involved methods from the free probability theory (see [23, 15, 5]). In particular, for \( p \geq 1 \)
\[ \mu(p, 1) = \mu(2, 1)^{\mathbb{R}(p-1)}, \]
where \( \ominus \) denotes the multiplicative free power, and \( \mu(2, 1) \) is known as the Marchenko-Pastur (called also the free Poisson) distribution. It is given by
\[ \mu(2, 1) = \frac{1}{2\pi} \sqrt{\frac{4 - x}{x}} \, dx \quad \text{on } [0,4], \]
and plays an important role in the theory of random matrices, see [24, 9, 10, 2, 1, 4]. It was proved in [1] that the measure \( \mu(2, 1)^{\mathbb{R}(p-1)} \) is the limit of the distribution of squared singular values of the power \( G^n \) of a random matrix \( G \), when the size of the matrix \( G \) goes to infinity.

In this paper we are going to prove positive definiteness of \( \{A_m(p, r)\}_{m=0}^{\infty} \) using more classical methods. Namely, we show that if \( p > 1 \), \( 0 < r \leq p \) and if \( p \) is a rational number then \( \mu(p, r) \) is absolutely continuous and can be represented as Mellin convolution of modified beta measures. Next we provide a formula for the density \( W_{p,r}(x) \) of \( \mu(p, r) \) in terms of the Meijer function and consequently, of the generalized hypergeometric functions (cf. [25, 18], where \( p \) was assumed to be an integer). This allows us to draw graphs of these densities and, in some particular cases, to express \( W_{p,r}(x) \) as an elementary function. It is worth to point out that for \( r = 1 \) an alternative description of the densities \( W_{p,1}(x) \) has been recently given by Haagerup and Möller, see Corollary 3 in [11].

Finally let us also mention that the measures \( \mu(p, r) \) satisfy a peculiar relation:
\[ \mu(p, r) \triangleright \mu(p + s, s) = \mu(p + s, r + s), \]
for \( p \geq 1 \), \( 0 < r \leq p \) and \( s > 0 \), see [13], involving monotonic convolution “\( \triangleright \)”, an associative, noncommutative operation on probability measures on \( \mathbb{R} \), introduced by Muraki [14].

1. Preliminaries

For probability measures \( \mu_1, \mu_2 \) on the positive half-line \([0, \infty)\) the Mellin convolution is defined by
\[ (\mu_1 \circ \mu_2)(A) := \int_0^\infty \int_0^\infty 1_A(xy) d\mu_1(x) d\mu_2(y) \]
DENSITIES OF THE RANEY DISTRIBUTIONS

for every Borel set \( A \subseteq [0, \infty) \). This is the distribution of product \( X_1 \cdot X_2 \) of two independent nonnegative random variables with \( X_i \sim \mu_i \). In particular, if \( c > 0 \) then \( \mu \circ \delta_c \) is the dilation of \( \mu \):

\[
(\mu \circ \delta_c)(A) = D_c\mu(A) := \mu \left( \frac{1}{c} A \right).
\]

Note that if \( \mu \) has density \( f(x) \) then \( D_c\mu \) has density \( f(x/c)/c \).

If both the measures \( \mu_1, \mu_2 \) have all moments

\[
s_m(\mu_i) := \int_0^\infty x^m d\mu_i(x)
\]

finite then so has \( \mu_1 \circ \mu_2 \) and

\[
s_m(\mu_1 \circ \mu_2) = s_m(\mu_1) \cdot s_m(\mu_2)
\]

for all \( m \).

If \( \mu_1, \mu_2 \) are absolutely continuous, with densities \( f_1, f_2 \) respectively, then so is \( \mu_1 \circ \mu_2 \) and its density is given by the Mellin convolution:

\[
(f_1 \circ f_2)(x) := \int_0^\infty f_1(x/y) f_2(y) \frac{dy}{y}.
\]

We will need the following modified beta distributions:

**Lemma 1.1.** Let \( u, v, l > 0 \). Then

\[
\left\{ \frac{\Gamma(u + n/l)\Gamma(u + v)}{\Gamma(u + v + n/l)\Gamma(u)} \right\}_{n=0}^\infty
\]

is the moment sequence of the probability measure

\[
b(u + v, u, l) := \frac{l}{B(u, v)} x^{lu-1} (1 - x^l)^{-v-1} dx
\]

on \([0,1]\), where \( B \) is the Euler beta function.

**Proof.** Using the substitution \( t = x^l \) we obtain:

\[
\frac{\Gamma(u + n/l)\Gamma(u + v)}{\Gamma(u + v + n/l)\Gamma(u)} = \frac{B(u + n/l, v)}{B(u, v)} = \frac{1}{B(u, v)} \int_0^1 t^{u+n/l-1} (1 - t)^{-v-1} \, dt
\]

\[
= \frac{l}{B(u, v)} \int_0^1 x^{lu+n-1} (1 - x^l)^{-v-1} \, dx.
\]

Note that if \( X \) is a positive random variable whose distribution has density \( f(x) \) and if \( l > 0 \) then the distribution of \( X^{1/l} \) has density \( lx^{-1} f(x^l) \). In particular, if \( X \) has beta distribution \( b(u + v, u, 1) \) then \( X^{1/l} \) has distribution \( b(u + v, u, l) \).

For \( u, l > 0 \) we also define

\[
b(u, u, l) := \delta_1.
\]
2. Applying Mellin convolution

From now on we assume that $p > 1$ is a rational number, say $p = k/l$, with $1 \leq l < k$, and that $0 < r \leq p$. We will show, that then the sequence $A_m(p, r)$ is a moment sequence of a probability measure $\mu(p, r)$, which can be represented as Mellin convolution of modified beta distributions. In particular, $\mu(p, r)$ is absolutely continuous and we will denote its density by $W_{p,r}$. The case when $p$ is an integer was studied in [18, 25].

First we need to express the numbers $A_m(p, r)$ in a special form.

Lemma 2.1. If $p = k/l$, where $k, l$ are integers, $1 \leq l < k$ and $0 < r \leq p$ then

\begin{equation}
A_m(p, r) = \frac{r}{\sqrt{2\pi kl(k-l)}} \left( \frac{p}{p-1} \right)^r \prod_{j=1}^k \frac{\Gamma(\beta_j + m/l)}{\Gamma(\alpha_j + m/l)} c(p)^m,
\end{equation}

where $c(p) = p^p(p-1)^{1-p}$,

\begin{align}
\alpha_j &= \begin{cases} 
\frac{j}{l} & \text{if } 1 \leq j \leq l, \\
\frac{r+j-l}{k-l} & \text{if } l+1 \leq j \leq k,
\end{cases} \\
\beta_j &= \frac{r+j-1}{k}, \quad 1 \leq j \leq k.
\end{align}

Proof. First we write:

\begin{equation}
\left( \frac{mp+r}{m} \right) \frac{r}{mp+r} = \frac{r\Gamma(mp+r)}{\Gamma(m+1)\Gamma(mp-m+r+1)}.
\end{equation}

Now we apply the Gauss’s multiplication formula:

$$\Gamma(nz) = (2\pi)^{(1-n)/2}n^{nz-1/2}\Gamma(z)\Gamma \left( z + \frac{1}{n} \right) \Gamma \left( z + \frac{2}{n} \right) \ldots \Gamma \left( z + \frac{n-1}{n} \right)$$

to get:

$$\Gamma(mp+r) = \Gamma \left( k \left( \frac{m}{l} + \frac{r}{k} \right) \right) = (2\pi)^{(1-k)/2}k^{mk/l+r-1/2} \prod_{j=1}^k \Gamma \left( \frac{m}{l} + \frac{r+j-1}{k} \right),$$

$$\Gamma(m+1) = \Gamma \left( \frac{m+1}{l} \right) = (2\pi)^{(1-l)/2}l^{m+1/2} \prod_{j=1}^l \Gamma \left( \frac{m}{l} + \frac{j}{l} \right)$$

and

$$\Gamma(mp-m+r+1) = \Gamma \left( (k-l) \left( \frac{m}{l} + \frac{r+1}{k-l} \right) \right) = (2\pi)^{(1-k+l)/2}(k-l)^{m(k-l)/l+r+1/2} \prod_{j=l+1}^k \Gamma \left( \frac{m}{l} + \frac{r+j-l}{k-l} \right).$$

It remains to apply them to (17). \hfill \Box

Now we need to modify enumeration of $\alpha$’s.
Lemma 2.2. For $1 \leq i \leq l + 1$ denote
\[ j_i := \left\lfloor \frac{(i - 1)k}{l} \right\rfloor + 1, \]
where $\lfloor \cdot \rfloor$ is the floor function, so that
\[ 1 = j_1 < j_2 < \ldots < j_l < k < k + 1 = j_{l+1}. \]
For $1 \leq j \leq k$ define
\[ \tilde{\alpha}_j = \begin{cases} 
\frac{i}{l} & \text{if } j = j_i, 1 \leq i \leq l, \\
\frac{r + j - i}{k - l} & \text{if } j_i < j < j_{i+1}.
\end{cases} \]

Then the sequence $\{\tilde{\alpha}_j\}_{j=1}^k$ is a rearrangement of $\{\alpha_j\}_{j=1}^k$. Moreover, if $0 < r \leq p = k/l$ then we have $\beta_j \leq \tilde{\alpha}_j$ for all $j \leq k$.

Proof. It is easy to verify the first statement.

Assume that $j = j_i$ for some $i \leq l$. Then we have to show that
\[ \frac{r + j_i - 1}{k} \leq \frac{i}{l}, \]
which is equivalent to
\[ lr + l \left\lfloor \frac{k(i - 1)}{l} \right\rfloor \leq ki, \]
and the latter is a consequence of the fact that $\lfloor x \rfloor \geq x$ and the assumption $r \leq p = k/l$.

Now assume that $j_i < j < j_{i+1}$. We ought to show that
\[ \frac{r + j - 1}{k} \leq \frac{r + j - i}{k - l}, \]
which is equivalent to
\[ lr + lj + k - l - ki \geq 0. \]
Using the inequality $\lfloor x \rfloor + 1 > x$ we obtain
\[ lj + k - l - ki \geq l(j_i + 1) + k - l - ki = lj_i + k - ki > k(i - 1) + k - ki = 0, \]
which completes the proof, as $r > 0$. \qed

Now we are ready to prove the main theorem of this section.

Theorem 2.3. Suppose that $p = k/l$, where $k, l$ are integers such that $1 \leq l < k$, and that $r$ is a real number, $0 < r \leq p$. Then there exists a unique probability measure $\mu(p, r)$ such that $\{\tilde{\alpha}_j\}_{j=1}^k$ is its moment sequence. Moreover $\mu(p, r)$ can be represented as the following Mellin convolution:
\[ \mu(p, r) = b(\tilde{\alpha}_1, \beta_1, l) \circ \ldots \circ b(\tilde{\alpha}_k, \beta_k, l) \circ \delta_{c(p)}, \]
where
\[ c(p) := \frac{p^p}{(p - 1)^{p-1}}. \]
Consequently, $\mu(p, r)$ is absolutely continuous and its support is $[0, c(p)]$. 

Note that the representation of densities in the form of Mellin convolution of modified beta distributions was used in different context in [7], see its Appendix A.

**Example.** For the Marchenko-Pastur measure we get the following decomposition:

\[(19) \quad \mu(2, 1) = b(1, 1/2, 1) \circ b(2, 1, 1) \circ \delta_4,\]

where \(b(1, 1/2, 1)\) has density \(1/(\pi \sqrt{x-x^2})\) on \([0, 1]\), the arcsine distribution with the moment sequence \((2^m)_m\) \(4^{-m}\), and \(b(2, 1, 1)\) is the Lebesgue measure on \([0, 1]\) with the moment sequence \(1/((m+1))\).

**Proof.** In view of Lemma 2.1 and Lemma 2.2 we can write

\[(20) \quad A_m(p, r) = D \prod_{j=1}^k \frac{\Gamma(\beta_j + m/l)\Gamma(\tilde{\alpha}_j)}{\Gamma(\tilde{\alpha}_j + m/l)\Gamma(\beta_j)} \cdot c(p)^m\]

for some constant \(D\). Taking \(m = 0\) we see that \(D = 1\). □

Note that a part of the theorem illustrates a result of Kargin [12], who proved that if \(\mu\) is a compactly supported probability measure on \([0, \infty)\), with expectation 1 and variance \(V\), and if \(L_n\) denotes the supremum of the support of the multiplicative free convolution power \(\mu \boxtimes n\), then

\[(21) \quad \lim_{n \to \infty} \frac{L_n}{n} = eV,\]

where \(e = 2.71\ldots\) is the Euler’s number. The Marchenko-Pastur measure \(\mu(2, 1)\) has expectation and variance equal to 1 and \(\mu(2, 1) \boxtimes n = \mu(n + 1, 1)\), so in this case \(L_n = (n + 1)^n/n^n\) (this was also proved in [24] and [10]) and (20) holds.

The density function for \(\mu(p, r)\) will be denoted by \(W_{p,r}(x)\). Since \(A_m(p, p) = A_{m+1}(p, 1)\), we have

\[(22) \quad W_{p,p}(x) = x \cdot W_{p,1}(x),\]

for example

\[(23) \quad W_{2,2}(x) = \frac{1}{2\pi} \sqrt{x(4-x)} \quad \text{on } [0, 4],\]

which is the famous semicircle Wigner distribution with radius 2, centered at \(x = 2\).

Now we can reprove the main result of [13].

**Theorem 2.4.** Suppose that \(p, r\) are real numbers such that \(p \geq 1\) and \(0 \leq r \leq p\). Then there exists a unique probability measure \(\mu(p, r)\), with support contained in \([0, c(p)]\), such that \(\{A_m(p, r)\}_{m=0}^\infty\) is its moment sequence.

**Proof.** It follows from the fact that the class of positive definite sequence is closed under pointwise limits. □

### 3. Applying Meijer G-function

The aim of this section is to describe the density function \(W_{p,r}(x)\) of \(\mu(p, r)\) in terms of the Meijer G-function (see [16] for example) and consequently, as a linear combination of generalized hypergeometric functions. We will see that in some particular cases \(W_{p,r}\) can be represented as an elementary function.
Theorem 3.1. Let $p = k/l > 1$, where $k, l$ are integers such that $1 \leq l < k$, and let $r$ be a positive real number, $r \leq p$. Then the density $W_{p,r}$ of the probability measure $\mu(p, r)$ can be expressed as

$$W_{p,r}(x) = \frac{rp^r}{x(p-1)^{r+1/2}2^{k+1}} G^{k,0}_{k,k}\left(\begin{array}{c} \alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k \end{array}\right| \frac{x^j}{c(p)^j}),$$

$x \in (0, c(p))$, where $c(p) = p^p(p-1)^{1-p}$ and the parameters $\alpha_j, \beta_j$ are given by (15) and (16).

Proof. Define

$$\phi_{p,r}(\sigma) = \frac{r\Gamma(\sigma p - p + r)}{\Gamma(\sigma)\Gamma(\sigma p - p + r + 2)}.$$

If $m$ is a natural number then

$$\phi_{p,r}(m+1) = \left(\frac{mp + r}{m}\right) \frac{r}{mp + r},$$

so $\phi_{p,r}$ is the Mellin transform of the density function $W_{p,r}$ of $\mu(p, r)$:

$$\phi_{p,r}(\sigma) = \int_0^\infty x^{\sigma-1} W_{p,r}(x) \, dx.$$

In order to reconstruct $W_{p,r}$ we apply the inverse Mellin transform:

$$W_{p,r}(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} x^{-\sigma} \phi_{p,r}(\sigma) \, d\sigma,$$

see [3, 16, 19] for details. Putting $m = \sigma - 1$ in (14) we get

$$\phi_{p,r}(\sigma) = \frac{r(p-1)^{p-1-r}}{p^{p-r}2^{k+1}} \prod_{j=1}^{k} \frac{\Gamma(\beta_j - 1/l + \sigma/l)}{\Gamma(\alpha_j - 1/l + \sigma/l)} c(p)^\sigma.$$

Writing the right hand side as $\Phi(\sigma/l)c(p)^\sigma$, using the substitution $\sigma = lu$ and the definition of the Meijer $G$-function (see [16] for example) we obtain

$$W_{p,r}(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Phi(u) \left(x/l c(p)\right)^{-u} \, du$$

$$= \frac{r(p-1)^{p-r-3/2}}{p^{p-r}2^{k+1}} G^{k,0}_{k,k}\left(\begin{array}{c} \alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k \end{array}\right| \frac{z}{\phi(c(p))}),$$

where $z = x/l c(p)$, $\alpha_j = \alpha_j - 1/l$, $\beta_j = \beta_j - 1/l$. Finally we use formula (16.19.2) in [16] and obtain

$$W_{p,r}(x) = \frac{r(p-1)^{p-r-3/2}}{z^1 p^{p-r}2^{k+1}} G^{k,0}_{k,k}\left(\begin{array}{c} \alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k \end{array}\right| \frac{z}{\phi(c(p))}),$$

which is equivalent to (23).

Now applying Slater’s theorem (see (16.17.2) in [16]) we can represent $W_{p,r}$ as a linear combination of hypergeometric functions.
Theorem 3.2. For for $p = k/l$, with $1 \leq l < k$, $r > 0$ and $x \in (0, c(p))$ we have

$$W_{p,r}(x) = \gamma(k, l, r) \sum_{h=1}^{k} c(h, k, l, r) kF_{k-1} \left( \begin{array}{c} a(h, k, l, r) \\ b(h, k, l, r) \end{array} \mid z \right) z^{r+h-1}/c(r^{2})$$

where $z = x^{l}/c(p)^{l}$,

$$\gamma(k, l, r) = \frac{(p-1)^{p-r-3/2}}{p^{r} \pi^{k}},$$

$$c(h, k, l, r) = \frac{\prod_{j=1}^{k} \Gamma \left( \frac{j-h}{k} \right) \prod_{j=h+1}^{k} \Gamma \left( \frac{j-h}{k} \right)}{\prod_{j=1}^{l} \Gamma \left( \frac{j-r h}{k} \right) \prod_{j=l+1}^{k} \Gamma \left( \frac{j-r h}{k} \right)}.$$

and the parameter vectors of the hypergeometric functions are

$$a(h, k, l, r) = \left( \left\{ \frac{r + h - 1}{k} \right\}_{j=1}^{l}, \left\{ \frac{r + h - 1}{l} \right\}_{j=1}^{k} \right),$$

$$b(h, k, l, r) = \left( \left\{ \frac{k + h - j}{k} \right\}_{j=1}^{h-1}, \left\{ \frac{k + h - j}{k} \right\}_{j=h+1}^{k} \right).$$

The most tractable case is $p = 2$.

Corollary 3.3. For $p = 2$, $0 < r \leq 2$, the density function is

$$W_{2,r}(x) = \frac{\sin \left( r \cdot \arccos \sqrt{x/4} \right)}{\pi x^{1-r/2}},$$

$x \in (0, 4)$. In particular for $r = 1/2$ we have

$$W_{2,1/2}(x) = \frac{\sqrt{2 - \sqrt{x}}}{2\pi x^{3/4}},$$

and for $r = 3/2$

$$W_{2,3/2}(x) = \frac{(\sqrt{x} + 1) \sqrt{2 - \sqrt{x}}}{2\pi x^{1/4}}.$$

Note that if $r > 2$ then $W_{2,r}(x) < 0$ for some values of $x \in (0, 4)$.

Proof. We take $k = 2$, $l = 1$ so that $c(2) = 4$, $z = x/4$ and $\gamma(2, 1, r) = r^{2p}/(8\sqrt{\pi})$.

Using the Euler’s reflection formula and the identity $\Gamma(1 + r/2) = \Gamma(r/2)r/2$ we get

$$c(1, 2, 1, r) = \frac{\Gamma(1/2)}{\Gamma(1 - r/2)\Gamma(1 + r/2)} = \frac{2 \sin(p\pi/2)}{r\sqrt{\pi}},$$

$$c(2, 2, 1, r) = \frac{\Gamma(-1/2)}{\Gamma((1 - r)/2)\Gamma((1 + r)/2)} = \frac{-2 \cos(p\pi/2)}{r\sqrt{\pi}}.$$

We also need formulas for two hypergeometric functions, namely

$$\binom{r}{2} \binom{-r}{2} \binom{1/2}{z} = \cos(r \arcsin z),$$

$$\binom{1 + r}{2} \binom{1 - r}{2} \binom{3/2}{z} = \sin(r \arcsin z)/r\sqrt{z},$$

$$\binom{1}{2} \binom{-1}{2} \binom{1/2}{z} = \sin(r \arcsin z)/r\sqrt{z}.$$
see 15.4.12 and 15.4.16 in [16]. Now we can write
\[ W_{2,r}(x) = \frac{\sin(\pi r/2) \cos \left( r \arcsin \sqrt{x/4} \right) - \cos(\pi r/2) \sin \left( r \arcsin \sqrt{x/4} \right)}{\pi x^{1-r/2}} \]
\[ = \frac{\sin \left( \pi r/2 - r \arcsin \sqrt{x/4} \right)}{\pi x^{1-r/2}} = \frac{\sin \left( r \arccos \sqrt{x/4} \right)}{\pi x^{1-r/2}}, \]
which concludes the proof. \( \square \)

**Remark.** Note that
\[ \frac{W_{2,1}(\sqrt{x})}{2\sqrt{x}} = \frac{1}{4} W_{2,1/2} \left( \frac{x}{4} \right). \]
It means that if \( X, Y \) are random variables such that \( X \sim \mu(2,1) \) and \( Y \sim \mu(2,1/2) \) then \( X^2 \sim 4Y \). This can be also derived from the relation \( A_m(2,1/2)4m = A_{2m}(2,1) \).

### 4. Some particular cases

In this part we will see that for \( k = 3 \) some densities still can be represented as elementary functions. We will need two families of formulas (cf. 15.4.17 in [16]).

**Lemma 4.1.** For \( c \neq 0, -1, -2, \ldots \) we have
\[ \begin{align*}
\text{34} & \quad _2F_1 \left( \frac{c}{2}, \frac{c-1}{2}; c \mid z \right) = 2^{c-1} (1 + \sqrt{1-z})^{1-c}, \\
\text{35} & \quad _2F_1 \left( \frac{c+1}{2}, \frac{c-2}{2}; c \mid z \right) = \frac{2^{c-1}}{c} (1 + \sqrt{1-z})^{1-c} (c - 1 + \sqrt{1-z}).
\end{align*} \]

**Proof.** We know that \( _2F_1(a, b; c \mid z) \) is the unique function \( f \) which is analytic at \( z = 0 \), with \( f(0) = 1 \), and satisfies the hypergeometric equation:
\[ z(1-z)f''(z) + [c - (a + b + 1)z] f'(z) - abf(z) = 0 \]
(see [3]). Now one can check that this equation is satisfied by the right hand sides of (34) and (35) for given parameters \( a, b, c \). \( \square \)

Now consider \( p = 3/2 \).

**Theorem 4.2.** Assume that \( p = 3/2 \). Then for \( r = 1/2 \) we have
\[ W_{3/2,1/2}(x) = \frac{\left( 1 + \sqrt{1 - 4x^2/27} \right)^{2/3} - \left( 1 - \sqrt{1 - 4x^2/27} \right)^{2/3}}{2^{5/3} \pi x^{1/3}}, \]
for \( r = 1/2 \):
\[ W_{3/2,1}(x) = 3^{1/2} \frac{\left( 1 + \sqrt{1 - 4x^2/27} \right)^{1/3} - \left( 1 - \sqrt{1 - 4x^2/27} \right)^{1/3}}{2^{4/3} \pi x^{1/3}} + 3^{1/2}x^{1/3} \frac{\left( 1 + \sqrt{1 - 4x^2/27} \right)^{2/3} - \left( 1 - \sqrt{1 - 4x^2/27} \right)^{2/3}}{2^{5/3} \pi} \]
and, finally, \( W_{3/2,3/2}(x) = x \cdot W_{3/2,1}(x) \), with \( x \in (0, 3\sqrt{3}/2) \).
Proof. For arbitrary \( r \) we have
\[
W_{3/2,r}(x) = \frac{2^{1-2r} \sin(2\pi r/3)}{3^{3/2-r} \pi} \binom{3 + 2r, r, 2r, 1}{6, 3, 3, 3} 3F_2 \left( \frac{3 + 2r}{6}, \frac{r}{3}, \frac{2r}{3}; 3, 3 \right) z^{r-3/2}
\]
\[
+ \frac{2^{(4-2r)/3} r \sin((1-2r)\pi/3)}{3^{3/2-r} \pi} \binom{5 + 2r, 1 + r, 1 - 2r}{6, 3, 3} 3F_2 \left( \frac{5 + 2r}{6}, \frac{1 + r}{3}, \frac{1 - 2r}{3}; 3, 3 \right) z^{(r+1)/3 - 1/2}
\]
\[
+ \frac{r(1 + 2r) \sin((1 + 2r)\pi/3)}{2^{1+2r/3} 3^{3/2-r} \pi} \binom{7 + 2r, 2 + r, 2 - 2r}{6, 3, 3} 3F_2 \left( \frac{7 + 2r}{6}, \frac{2 + r}{3}, \frac{2 - 2r}{3}; 3, 3 \right) z^{(r+2)/3 - 1/2},
\]
where \( z = 4x^2/27 \). If \( r = 1/2 \) or \( r = 1 \) then one term vanishes and in the two others the hypergeometric functions reduce to \( 2F_1 \).

For \( r = 1/2 \) we apply (34) to obtain:
\[
W_{3/2,1/2}(x) = \frac{z^{-1/3}}{2^{1/3} 3^{1/2} \pi} 2F_1 \left( \frac{1}{6}, -\frac{1}{3}, \frac{1}{3}; z \right) = \frac{z^{1/3}}{2^{5/3} 3^{1/2} \pi} 2F_1 \left( \frac{5}{6}, \frac{1}{3}, \frac{5}{3}; z \right)
\]
\[
= \frac{z^{-1/3}}{2^{1/3} 3^{1/2} \pi} \left( 1 + \sqrt{1 - z} \right)^{2/3} \left( 1 + \sqrt{1 - z} \right)^{-2/3}
\]
and this yields (36).

For \( r = 1 \) we use (35):
\[
W_{3/2,1}(x) = \frac{z^{-1/6}}{2^{2/3} \pi} 2F_1 \left( \frac{5}{6}, -\frac{2}{3}, \frac{2}{3}; z \right) + \frac{z^{1/6}}{2^{1/3} \pi} 2F_1 \left( \frac{7}{6}, -\frac{1}{3}, \frac{4}{3}; z \right)
\]
\[
= \frac{z^{-1/6}}{4\pi} \left( 1 + \sqrt{1 - z} \right)^{1/3} \left( 3\sqrt{1 - z} - 1 \right) + \frac{z^{1/6}}{4\pi} \left( 1 + \sqrt{1 - z} \right)^{-1/3} \left( 3\sqrt{1 - z} + 1 \right)
\]
\[
= \frac{z^{-1/6}}{4\pi} \left( 1 + \sqrt{1 - z} \right)^{1/3} \left( 3\sqrt{1 - z} - 1 \right) + \frac{z^{1/6}}{4\pi} \left( 1 - \sqrt{1 - z} \right)^{1/3} \left( 3\sqrt{1 - z} + 1 \right).
\]
Now we have
\[
\left( 1 + \sqrt{1 - z} \right)^{1/3} \left( 3\sqrt{1 - z} - 1 \right) = -\left( 1 + \sqrt{1 - z} \right)^{1/3} \left( 3 - 3\sqrt{1 - z} - 2 \right)
\]
\[
= -3z^{1/3} \left( 1 - \sqrt{1 - z} \right)^{2/3} + 2 \left( 1 + \sqrt{1 - z} \right)^{1/3}
\]
and similarly
\[
\left( 1 - \sqrt{1 - z} \right)^{1/3} \left( 3\sqrt{1 - z} + 1 \right) = 3z^{1/3} \left( 1 + \sqrt{1 - z} \right)^{2/3} - 2 \left( 1 - \sqrt{1 - z} \right)^{1/3}.
\]
Therefore
\[
W_{3/2,1}(x) = \frac{z^{-1/6}}{2\pi} \left( (1 + \sqrt{1 - z})^{1/3} - (1 - \sqrt{1 - z})^{1/3} \right)
\]
\[
+ \frac{3z^{1/6}}{4\pi} \left( (1 + \sqrt{1 - z})^{2/3} - (1 - \sqrt{1 - z})^{2/3} \right),
\]
which entails (37). The last statement is a consequence of (24). \( \square \)
The dilation $D_2\mu(3/2, 1/2)$, with the density $W_{3,21/2}(x/2)/2$, is known as the \textit{Bures distribution}, see (4.4) in [21]. Moreover, the integer sequence $4^n A_n(3/2, 1/2)$, moments of $D_4\mu(3/2, 1/2)$, appear as A078531 in [20] and counts the number of symmetric noncrossing connected graphs on $2n + 1$ equidistant nodes on a circle. The axis of symmetry is a diameter of a circle passing through a given node, see [6].

The measure $\mu(3/2, 1)$ is equal to $\mu(2, 1)^{21/2}$, the multiplicative free square root of the Marchenko-Pastur distribution.

For the sake of completeness we also include the cases $p = 3, r = 1$ and $p = 3, r = 2$, which have already appeared in [17] [18].

\textbf{Theorem 4.3.} Assume that $p = 3$. Then for $r = 1$ we have

$$W_{3,1}(x) = 3 \left(1 + \sqrt{1 - 4x/27}\right)^{2/3} - 2^{2/3} x^{1/3}$$

(38)

for $r = 2$:

$$W_{3,2}(x) = 9 \left(1 + \sqrt{1 - 4x/27}\right)^{4/3} - 2^{4/3} x^{2/3}$$

(39)

and, finally, $W_{3,3}(x) = x \cdot W_{3,1}(x)$, with $x \in (0, 27/4)$.

\textbf{Proof.} For arbitrary $r$ we have

$$W_{3,r}(x) = \frac{2^{6-2r} \sin \left(\pi r/3\right)}{3^{3-r} \pi} 3F_2 \left(\frac{r}{3}, \frac{3-r}{6}, \frac{-r}{6}; \frac{2}{3}, \frac{1}{3}; z\right) z^{(r-3)/3} - \frac{2^{4-2r} \sin \left((1-2r)\pi/3\right)}{3^{3-r} \pi} 3F_2 \left(\frac{1+r}{3}, \frac{5-r}{6}, \frac{2-r}{6}; \frac{4}{3}, \frac{2}{3}; z\right) z^{(r-2)/3} + \frac{r(r-1) \sin \left((1-r)\pi/3\right)}{2^{1+2r} \pi^2 z^{3-r} \pi} 3F_2 \left(\frac{2+r}{3}, \frac{7-r}{6}, \frac{4-r}{6}; \frac{5}{3}, \frac{4}{3}; z\right) z^{(r-1)/3},$$

where $z = 4x/27$. For $r = 1$ and $r = 2$ we have similar reduction as in the previous proof. Here we will be using only (34).

Taking $r = 1$ we get

$$W_{3,1}(x) = \frac{2^{1/3} z^{-2/3}}{3^{3/2} \pi} 2F_1 \left(\frac{1}{3}, \frac{-1}{6}; \frac{2}{3}; z\right) - \frac{z^{-1/3}}{2^{1/3} 3^{3/2} \pi} 2F_1 \left(\frac{2}{3}, \frac{1}{6}; \frac{4}{3}; z\right)$$

$$= \frac{z^{-2/3}}{3^{3/2} \pi} (1 + \sqrt{1 - z})^{1/3} - \frac{z^{-1/3}}{3^{3/2} \pi} (1 + \sqrt{1 - z})^{-1/3}$$

$$= \frac{(1 + \sqrt{1 - z})^{2/3} - z^{1/3}}{3^{3/2} \pi z^{2/3} (1 + \sqrt{1 - z})^{1/3}},$$

which implies (38).

Now we take $r = 2$:

$$W_{3,2}(x) = \frac{z^{-1/3}}{2^{1/3} 3^{3/2} \pi} 2F_1 \left(\frac{1}{6}, \frac{-1}{3}; \frac{1}{3}; z\right) - \frac{z^{1/3}}{2^{5/3} 3^{3/2} \pi} 2F_1 \left(\frac{5}{6}, \frac{1}{3}; \frac{5}{3}; z\right)$$

(38).
we represent $W \leq r$ graphs of the functions 1–9. These graphs which are partly negative are drawn as dashed curves. In Fig. 1 the $p > r$ any rational function of the Wigner’s semicircle distribution of the extremities of their supports. They can be therefore considered as generalizations of the Marchenko-Pastur distribution. Note that the measure $\mu(3, 1)$ is equal to $\mu(2, 1)^{\otimes 2}$, the multiplicative free square of the Marchenko-Pastur distribution.

5. Graphical representation of selected cases

The explicit form of $W_{p,r}(x)$ given in Theorem 3.2 permits a graphical visualization for any rational $p > 0$ and arbitrary $r > 0$. We shall represent some selected cases in Figures 1–9. These graphs which are partly negative are drawn as dashed curves. In Fig. 1 the graphs of the functions $W_{3/2,r}(x)$ for values of $r$ ranging from 0.9 to 2.3 are given. For $r \leq 3/2$ these functions are positive, otherwise they develop a negative part. In Fig. 2 we represent $W_{5/2,r}(x)$ for $r$ ranging from 2 to 3.4. In Fig. 3 we display the densities $W_{p,p}(x)$ for $p = 6/5, 5/4, 4/3$ and $3/2$. All these densities are unimodal and vanish at the extremities of their supports. They can be therefore considered as generalizations of the Wigner’s semicircle distribution $W_{2,2}(x)$, see equation (22). In Fig. 4 we depict the functions $W_{1/3,r}(x)$, for values $r$ ranging from 0.8 to 2.4. Here for $r \geq 1.4$ negative contributions clearly appear. In Fig. 5 and 6 we present six densities expressible through elementary functions, namely $W_{3/2,r}(x)$ for $r = 1/2, 1, 3/2$, see Theorem 4.2 and $W_{3,r}(x)$ for $r = 1, 2, 3$, see Theorem 4.3. In Fig. 7 the set of densities $W_{p,1}(x)$ for five fractional values of $p$ is presented. The appearance of maximum near $x = 1$ corresponds to the fact that $\mu(p, 1)$ weakly converges to $\delta_1$ as $p \to 1^+$. In Fig. 8 the fine details of densities $W_{p,1}(x)$ for $p = 5/2, 7/3, 9/4, 11/5$, on a narrower range $2 \leq x \leq 4.5$ are presented. In Fig. 9 we display the densities $W_{p,1}(x)$ for $p = 2, 5/2, 3, 7/2, 4$, near the upper edge of their respective supports, for $3 \leq x \leq 9.5$.

The Fig. 10 summarizes our results in the $p > 0, r > 0$ quadrant of the $(p, r)$ plane, describing the Raney numbers (c.f. Fig. 5.1 in [13] and Fig. 7 in [18]). The shaded region $\Sigma$ indicates the probability measures $\mu(p, r)$ (i.e. where $W_{p,r}(x)$ is a nonegative function). The vertical line $p = 2$ and the stars indicate the pairs $(p, r)$ for which $W_{p,r}(x)$ is an elementary function, see Corollary 3.3, Theorem 4.2 and Theorem 4.3. The points $(3/2, 1)$ and $(3, 1)$ correspond to the multiplicative free powers $\text{MP}^{\otimes 1/2}$ and $\text{MP}^{\otimes 2}$ of the Marchenko-Pastur distribution MP. Symbol B at $(3/2, 1/2)$ indicates the Bures distribution and SC at $(2, 2)$ denotes the semicircle law centered at $x = 2$, with radius 2.

It is our pleasure to thank M. Bożejko, Z. Burda, K. Górska, I. Nechita and M. A. Nowak for fruitful interactions.

References

[1] N. Alexeev, F. Götzte, A. Tikhomirov, Asymptotic distribution of singular values of powers of random matrices, Lith. Math. J. 50 (2010), 121-132.

[2] G. W. Anderson, A. Guionnet, O. Zeitouni, An introduction to random matrices, Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010.

[3] G. E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge 1999.
[4] T. Banica, S. Belinschi, M. Capitaine, B. Collins, *Free Bessel laws*, Canad. J. Math. 63 (2011), 3–37.

[5] P. Biane, *Free probability and combinatorics*, Proceedings of the International Congress of Mathematics, Higher Ed. Press, Beijing 2002, Vol. II (2002), 765–774.

[6] P. Flajolet and M. Noy, *Analytic combinatorics of non-crossing configurations*, Discrete Math., 204 (1999) 203–229.

[7] K. Górski, K. A. Penson, A. Horzela, G. H. E. Duchamp, P. Blasiak, A. I. Solomon, *Quasiclassical asymptotics and coherent states for bounded discrete spectra*, J. Math. Phys. 51 (2010), 122102, 12 pp.

[8] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics. A Foundation for Computer Science*, Addison-Wesley, New York 1994.

[9] U. Haagerup, F. Larsen, *Brown’s spectral distribution measure for R-diagonal elements in finite von Neumann algebras*, J. Funct. Anal. 176 (2000), 331-367.

[10] U. Haagerup, S. Thorbjørnsen, *A new application of random matrices: Ext(C∗red(F2)) is not a group*, Ann. of Math. 162 (2005), 711-775.

[11] U. Haagerup, S. Möller, *The law of large numbers for the free multiplicative convolution*, arXiv:1211.4357.

[12] V. Kargin, *On asymptotic growth of the support of free multiplicative convolutions*, Elec. Comm. Prob. 13 (2008), 415–421.

[13] W. Młotkowski, *Fuss-Catalan numbers in noncommutative probability*, Documenta Math. 15 (2010) 939–955.

[14] N. Muraki, *Monotonic independence, monotonic central limit theorem and monotonic law of small numbers*, Inf. Dim. Anal. Quantum Probab. Rel. Topics 4 (2001) 39–58.

[15] A. Nica, R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge University Press, 2006.

[16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge 2010.

[17] K. A. Penson, A. I. Solomon, *Coherent states from combinatorial sequences*, Quantum theory and symmetries, Kraków 2001, World Sci. Publ., River Edge, NJ, 2002, 527–530.

[18] K. A. Penson, K. Życzkowski, *Product of Ginibre matrices: Fuss-Catalan and Raney distributions*, Phys. Rev. E 83 (2011) 061118, 9 pp.

[19] A. D. Polygin, A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, Boca Raton, 1998.

[20] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, (2012), published electronically at: http://oeis.org/.

[21] H.-J. Sommers, K. Życzkowski, *Statistical properties of random density matrices*, J. Phys. A: Math. Gen. 37 (2004) 8457–8466.

[22] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge 1999.

[23] D. V. Voiculescu, K. J. Dykema, A. Nica, *Free random variables*, CRM, Montréal, 1992.

[24] R. Wegmann, *The asymptotic eigenvalue-distribution for a certain class of random matrices*, J. Math. Anal. Appl. 56 (1976) 113-132.

[25] K. Życzkowski, K. A. Penson, I. Nechita, B. Collins, *Generating random density matrices*, J. Math. Phys. 52 (2011) 062201, 20 pp.
Figure 1. Raney distributions $W_{3/2,r}(x)$ with values of the parameter $r$ labeling each curve. For $r > p$ solutions drawn with dashed lines are not positive.

Figure 2. As in Fig. 1 for Raney distributions $W_{5/2,r}(x)$. 
Figure 3. Diagonal Raney distributions $W_{p,p}(x)$ with values of the parameter $p$ labeling each curve.

Figure 4. The functions $W_{4/3,r}(x)$ for $r$ ranging from 0.8 to 2.4.
**Figure 5.** Raney distributions $W_{3/2,r}(x)$ with values of the parameter $r$ labeling each curve. The case $W_{3/2,1}(x)$ represents the multiplicative free square root of the Marchenko Pastur distribution.

**Figure 6.** Raney distributions $W_{3,r}(x)$ with values of the parameter $r$ labeling each curve. The case $W_{3,1}(x)$ represents the multiplicative free square of the Marchenko Pastur distribution.
Figure 7. Raney distributions $W_{p,1}(x)$ with values of the parameter $p$ labeling each curve. The case $W_{3/2,1}(x)$ represents the multiplicative free square root of the Marchenko–Pastur distribution, $MP^{\frac{1}{2}}$.

Figure 8. Tails of the Raney distributions $W_{p,1}(x)$ with values of the parameter $p$ labeling each curve.
Figure 9. As in Fig. 8 for larger values of the parameter $p$.

Figure 10. Parameter plane $(p, r)$ describing the Raney numbers. The shaded set $\Sigma$ corresponds to nonnegative probability measures $\mu(p, r)$. The vertical line $p = 2$ and the stars represent values of parameters for which $W_{p, r}(x)$ is an elementary function. Here MP denotes the Marchenko–Pastur distribution, $\text{MP}^{\otimes s}$ its $s$-th free mutiplicative power, B-the Bures distribution while SC denotes the semicircle law. For $p > 1$ the points $(p, p)$ on the upper edge of $\Sigma$ represent the generalizations of the Wigner semicircle law; see Fig. 3.