On a class of submanifolds in a tangent bundle with a \( g \)-natural metric - normal lift

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Abstract

An isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( N \) gives rise in a natural way to variety of immersions into the tangent bundle \( T N \) with a non-degenerate \( g \)-natural metric \( G \). In the paper we introduce and study an immersion into \( T N \) defined by the immersion \( f : M \rightarrow N \) itself and the normal bundle.

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Key words: Riemannian manifold, tangent bundle, \( g \)-natural metric, submanifold, isometric immersion, totally geodesic distribution, non-degenerate metric.

1 Introduction

An isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( N \) gives rise in a natural way to variety of immersions into the tangent bundle \( T N \) with a non-degenerate \( g \)-natural metric \( G \). The isometric immersion defined by the tangent bundle of the submanifold was introduced by the author in [11], [12]. In the present paper we introduce and study an immersion \( \tilde{f} : LM \rightarrow T N \) defined by the immersion \( f : M \rightarrow N \) itself and the normal bundle.

In Preliminaries we recall basic facts on the decomposition of the tangent bundle and \( g \)-natural metrics. We also present basic notions on submanifolds and give short resumé on van der Waerden-Bortolotti covariant derivative. In Section 3 basic equations are presented. The main results are given in Section 4. We give the condition sufficient for \( LM \) being totally geodesic submanifold of \( T N \).

Throughout the paper all manifolds under consideration are Hausdorff and smooth. The metrics on the base manifolds are Riemannian and the metrics on tangent spaces are non-degenerate. We adopt the Einstein summation convention.
2 Preliminaries

2.1 Decomposition of the tangent space

Let \( \pi : T N \to N \) be the tangent bundle of a Riemannian manifold \( N \) with the Levi-Civita connection \( \nabla \) on \( N \), \( \pi \) being the projection. Then at each point \( (x, u) \in T N \) the tangent space \( T_{(x,u)}TN \) splits into direct sum of two isomorphic spaces \( V_{(x,u)}T N \) and \( H_{(x,u)}T N \), where

\[
V_{(x,u)}TN = \ker (d\pi|_{(x,u)}), \quad H_{(x,u)}TN = \ker (K|_{(x,u)}),
\]

and \( K \) is the connection map \([7], \text{see also } [15]\).

More precisely, if \( Z = \left( Z^r \frac{\partial}{\partial x^r} + \mathcal{Z} \frac{\partial}{\partial u^r} \right)|_{(x,u)} \in T_{(x,u)}TN, r = 1, \ldots, n \), then the vertical and horizontal projections of \( Z \) on \( T_xN \) are given by

\[
(d\pi)_{(x,u)}Z = Z^r \frac{\partial}{\partial x^r}|_x, \quad K_{(x,u)}(Z) = \left( \mathcal{Z}^r + u^s Z^t \Gamma_{rst}^r \right) \frac{\partial}{\partial x^r}|_x,
\]

where \( \Gamma_{rst}^r \) are components of the Levi-Civita connection on \( N \).

On the other hand, to each vector field \( X \) on \( N \) there correspond uniquely determined vector fields \( X^v \) and \( X^h \) on \( T N \) such that

\[
d\pi|_{(x,u)}(X^v) = 0, \quad K|_{(x,u)}(X^v) = X,
\]

\[
K|_{(x,u)}(X^h) = 0, \quad d\pi|_{(x,u)}(X^h) = X.
\]

\( X^v \) and \( X^h \) are called the vertical lift and the horizontal lift of a given \( X \) to \( T N \), respectively.

In local coordinates \( ((x^r), (u^r)), r = 1, \ldots, n \), on \( TN \), the horizontal and vertical lifts of a vector field \( X = X^r \frac{\partial}{\partial x^r} \) on \( N \) to \( TN \) are vector fields given respectively by

\[
X^h = X^r \frac{\partial}{\partial x^r} - u^s X^t \Gamma_{rst}^r \frac{\partial}{\partial u^s}, \quad X^v = X^r \frac{\partial}{\partial u^r}. \tag{1}
\]

Recall that for a given isometric immersion \( f : M \to N \), we have two tangent bundles \( \pi_N : TN \to N \) and \( \pi_M : TM \to M \), where the latter is the subbundle of the former. Let \( M, N \) be two Riemannian manifolds with metrics \( g_M \) and \( g_N \) and Levi-Civita connections \( \nabla_M \) and \( \nabla_N \) respectively. Then \( T_p TM \) and \( T_p TN \) have at a common point \( p \) their own decompositions into vertical and horizontal parts, i.e.

\[
T_p TN = V_p TN \oplus H_p TN = V_M \oplus H_M
\]

and

\[
T_p TM = V_p TM \oplus H_p TM = V_M \oplus H_M
\]

but neither \( V_M \subset V_N \) nor \( H_M \subset H_N \) need to hold along \( TM \). See also \([15]\).

Remark also that totally geodesic submanifolds of tangent bundle with \( g \)-natural metric are also studied in \([11]\) and \([10]\).
2. PRELIMINARIES

2.2 Preliminaries on $g$-natural metrics

In [13] the class of $g$-natural metrics was defined. We have

**Lemma 1** ([13], [3]) Let $(M, g)$ be a Riemannian manifold and $G$ be a $g$-natural metric on $TM$. There exist functions $a_j, b_j : <0, \infty \rightarrow R, j = 1, 2, 3$, such that for every $X, Y, u \in T_x M$

$$G_{(x,u)}(X^h, Y^h) = (a_1 + a_3)(r^2)g_x(X, Y) + (b_1 + b_3)(r^2)g_x(X, u)g_x(Y, u),$$
$$G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = a_2(r^2)g_x(X, Y) + b_2(r^2)g_x(X, u)g_x(Y, u),$$
$$G_{(x,u)}(X^v, Y^v) = a_3(r^2)g_x(X, Y) + b_1(r^2)g_x(X, u)g_x(Y, u),$$

where $r^2 = g_x(u, u)$. For dim $M = 1$ the same holds for $b_j = 0, j = 1, 2, 3$.

Setting $a_1 = 1, a_2 = a_3 = b_3 = 0$ we obtain the Sasaki metric, while setting $a_1 = b_1 = \frac{1}{1 + r^2}, a_2 = b_2 = 0 = 0, a_1 + a_3 = 1, b_1 + b_3 = 1$ we get the Cheeger-Gromoll one.

Following [2] we put

1. $a(t) = a_1(t) (a_1(t) + a_3(t)) - a_2^2(t)$,
2. $F_j(t) = a_j(t) + b_j(t)$,
3. $F(t) = F_1(t) [F_1(t) + F_3(t)] - F_2^2(t)$

for all $t \in <0, \infty \rangle$.

We shall often abbreviate: $A = a_1 + a_3, B = b_1 + b_3$.

**Lemma 2** ([2], Proposition 2.7) The necessary and sufficient conditions for a $g$-natural metric $G$ on the tangent bundle of a Riemannian manifold $(M, g)$ to be non-degenerate are

$$a(t) \neq 0, \quad F(t) \neq 0$$

for all $t \in <0, \infty \rangle$. If dim $M = 1$, this is equivalent to $a(t) \neq 0$ for all $t \in <0, \infty \rangle$.

Moreover, $(TM, G)$ is Riemannian one if and only if

$$a(t) > 0, \quad F(t) > 0, \quad a_1(t) > 0, \quad F_1(t) > 0$$

hold for all $t \in <0, \infty \rangle$.

We also have

**Proposition 3** ([3], [2]) Let $(N, g)$ be a Riemannian manifold, $\nabla$ its Levi-Civita connection and $R$ its Riemann curvature tensor. If $G$ is a non-degenerate $g$-natural metric on $TN$, then the Levi-Civita connection $\nabla$ of $(TN, G)$ is given at a point $(x, u) \in TN$ by

$$\left(\nabla_{X^h}Y^h\right)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h \{A(u, X_x, Y_x)\} + v \{B(u, X_x, Y_x)\},$$
\[
\begin{align*}
\left(\tilde{\nabla}_{X^h}Y^v\right)_{(x,u)} &= (\nabla_{X^h}Y^v)_{(x,u)} + h \{C(u, X_x, Y_x)\} + v \{D(u, X_x, Y_x)\}, \\
\left(\tilde{\nabla}_{X^h}Y^h\right)_{(x,u)} &= h \{C(u, X_x, X_x)\} + v \{D(u, Y_x, X_x)\}, \\
\left(\tilde{\nabla}_{X^h}Y^v\right)_{(x,u)} &= h \{C(u, Y_x, X_x)\} + v \{D(u, Y_x, Y_x)\},
\end{align*}
\]

where \(A, B, C, D, E, F\) are some \(F\)-tensors defined on the product \(TN \otimes TN \otimes TN\).

Remark 4 Expressions for \(A, B, C, D, E, F\) were presented for the first time in the original papers ([2], [3]). Unfortunately, they contain some misprints and omissions. Therefore, for the correct form, we refer the reader to ([4], [5]), see also ([8], [9]).

2.3 Submanifolds

Let \(M\) be a manifold isometrically immersed in a pseudo-Riemannian manifold \(N\) with metric \(g\). Denote by \(\tilde{\nabla}\) and \(\nabla\) the Levi-Civita connections of the metric \(g\) on \(N\) and that of the induced metric on \(M\) and by \(D^\perp\) the connection induced in the normal bundle \(T^\perp M\). Then the Gauss and Weingarten equations

\[
\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + H(X, Y), \\
\tilde{\nabla}_X \eta &= -A_\eta X + D^\perp_X \eta,
\end{align*}
\]

hold for all vectors fields \(X, Y\) tangent to \(M\) and all vector fields \(\eta\) normal to \(M\). Here \(H(X, Y)\) is the second fundamental form which is symmetric and takes values in \(T^\perp M\) while \(A_\eta X\) is the shape operator taking values in \(TM\). It is well known that \(A_\eta\) and \(H\) are related by

\[
g(A_\eta X, Y) = g(\eta, H(X, Y)).
\]

\(M\) is said to be totally geodesic if \(H(X, Y) = 0\) for all \(X, Y \in TM\).

For the local immersion \(x^r = x^r(y^a), r = 1, \ldots, n, a = 1, \ldots, m\), the components of the Levi-Civita connection \(\nabla\) of the induced metric \(g_{ab} = g_{rs}B^c_r B^b_s\), \(B^r_a = \frac{\partial y^a}{\partial x^r}\), are

\[
\Gamma^c_{ab} = \left[ B^c_{a,b} + \Gamma^r_{st} B^s_r B^b_t \right] B^c_t, \quad B^c_r = g^{cd} B^d_s g_{tr},
\]

where the dot denotes partial derivative with respect to \(y^b\).

Similarly, the components of the connection \(D^\perp\) are

\[
\Gamma^x_{ay} = \left[ N^r_{a,y} + \Gamma^r_{st} B^s_r N^t_y \right] N^x_r, \quad N^x_r = g^{xy} N^t_y g_{tr},
\]

where \(\eta_z = N^r_z \frac{\partial}{\partial x^r}, z = m + 1, \ldots, n\) are unit vector fields normal to \(M\).
2. PRELIMINARIES

2.3.1 Van der Waerden-Bortolotti covariant derivative

Van der Waerden-Bortolotti covariant derivative $\nabla$ is a covariant differentiation of tensor fields of mixed types defined along a submanifold $M$ immersed in a pseudo-Riemannian manifold $(N, g)$ and can be considered as a direct sum $\nabla \oplus \nabla \oplus \nabla^\perp$ of the Levi-Civita connections of the metric $g$ on $N$, the one induced on $M$ and of the metric induced in normal bundle $T^\perp M$.

If $\tilde{X}$, $X$, and $\eta$ are vector fields, respectively, tangent to $N$, tangent to $M$, normal to $M$ and $\tilde{X}^*$, $X^*$, $\eta^*$ are respective 1-forms, then,

$$(\nabla_Y T) \left( \tilde{X}, X, \eta, \tilde{X}^*, X^*, \eta^* \right) = Y \left( T \left( \tilde{X}, X, \eta, \tilde{X}^*, X^*, \eta^* \right) \right) -$$

$$T \left( \tilde{\nabla}_Y \tilde{X}, X, \eta, \tilde{X}^*, X^*, \eta^* \right) - T \left( \tilde{X}, \tilde{\nabla}_Y X, \eta, \tilde{X}^*, X^*, \eta^* \right) -$$

$$T \left( \tilde{X}, X, \tilde{\nabla}_Y \eta, \tilde{X}^*, X^*, \eta^* \right) - T \left( \tilde{X}, X, \eta, \tilde{X}^*, \tilde{\nabla}_Y X^*, \eta^* \right) -$$

$$T \left( \tilde{X}, X, \eta, \tilde{X}^*, X^*, \tilde{\nabla}_Y \eta^* \right)$$

for any vector field $Y$ tangent to $M$ and tensor field $T$ of mixed type $(3,3)$.

Let $x^k = x^k(y^a)$ be the local expression of the immersion, $B^k_a = \frac{\partial x^k}{\partial y^a}$. Let $\eta_x = N^r_x \frac{\partial}{\partial x^r}$, $x = m + 1, ..., n$, be an orthonormal set of vectors normal to $M$.

For the local coordinate vector fields $\frac{\partial}{\partial x^r}$ tangent to $N$, $\frac{\partial}{\partial y^a}$ tangent to $M$, $\frac{\partial}{\partial v^s}$ normal to $M$ and the respective 1-forms $dx^k$, $dy^a$, $dv^s$, denote by $\Gamma^r_{hk}$, $\Gamma^r_{ab}$, $\Gamma^r_{xy}$ components of the connections $\nabla$, $\tilde{\nabla}$ and $\nabla^\perp$. If

$$T = T^{kby}_{hax} \frac{\partial}{\partial x^k} \otimes dx^h \otimes \frac{\partial}{\partial y^b} \otimes dy^a \otimes \frac{\partial}{\partial v^s} \otimes dv^s,$$

then

$$\nabla_c T^{kby}_{hax} = \partial_b T^{kby}_{haz} - \Gamma^r_{hr} B^r_{cz} T^{kby}_{taz} + \Gamma^r_{rs} B^r_{cz} T^{tby}_{haz} -$$

$$\Gamma^r_{cd} T^{kby}_{haz} + \Gamma^r_{cb} T^{kdy}_{haz} - \Gamma^r_{za} T^{kby}_{haz} - \Gamma^r_{ze} T^{kby}_{hax},$$

where $h, k, r, s = 1, ..., n$, $a, b, c, d = 1, ..., m$, $m < n$, and $x, y, z = m + 1, ..., n$.

In particular, $\nabla_a B^r_b$ and $\nabla_a N^r_z$ give rise to the components of the second fundamental form and the shape operator:

$$\nabla_b B^r_a \partial_r = (B^r_{a,b} + \Gamma^s_{st} B^r_s B^t_{b} - \Gamma^s_{ab} B^r_t) \partial_r = h^r_{ab} N^s_z \partial_r,$$

$$\nabla_a N^r_z \partial_r = (\partial_a N^r_z + \Gamma^t_{st} B^a_t N^r_z - \Gamma^t_{az} N^r_t) \partial_r.$$

(2)

In a free of coordinate notation we have respectively:

$$\nabla_X Y = \nabla_X Y - \nabla_X Y,$$

$$\nabla_X \eta = \nabla_X \eta - D^r_X \eta.$$
2.3.2 Isometric immersion defined by normal bundle

Let \( f : M \rightarrow N \) be an isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( N \). Suppose that the following diagram commute

\[
\begin{array}{ccc}
\pi_N & \to & \pi_M \\
| & & | \\
(\pi_N^{-1}(U), (x^r, u^r)) & \xrightarrow{\tilde{f}} & ((\pi_M^\perp)^{-1}(V), (y^a, v^a)) \\
\downarrow & & \downarrow \\
(U, (x^r)) & \xrightarrow{f} & (V, (y^a)) \\
\end{array}
\]

where \((U, (x^r))\) and \((V, (y^a))\) are coordinate neighbourhoods on \( N \) and \( M \) respectively, while the local expression for \( f \) is:

\[
f : x^r = x^r(y^a).
\]

Let

\[
\eta_x = N_x^r \frac{\partial}{\partial x^r}, \quad x = m + 1, \ldots, n
\]

be a set of orthonormal vectors normal to \( M \).

The coordinate vector fields tangent to \( \text{LM} = \tilde{f}(TM^\perp) \) are

\[
\frac{\partial}{\partial v^x} = N_x^r \frac{\partial}{\partial u^r} = (\eta_x)^r,
\]

where \((u^r)_{r=1,\ldots,n}\) are components of the vector \( u \) tangent to \( N \) at a point with coordinates \((x^r)\) and \((v^x)_{x=m+1,\ldots,n}\) are components of the vector normal to \( M \) at a point \( x^r = x^r(y^a) \).

If

\[
f : M \rightarrow N; \quad (y^a) \mapsto x^r(y^a)
\]

then

\[
\tilde{f} : x^r = x^r(y^a), \quad u^r = v^x N_x^r
\]

defines locally an immersion into \( TN \).

2.3.3 Vectors tangent to \( \text{LM} \)

The coordinate vector fields tangent to \( \text{LM} = \tilde{f}(TM^\perp) \) are

\[
\frac{\partial}{\partial v^x} = N_x^r \frac{\partial}{\partial u^r} = (\eta_x)^r,
\]

6
3. Basic equations

In this section we derive, using the equations of Gauss and Weingarten and the formulas for the Levi-Civita connection on \((\mathbb{T}N, G)\) with a non-degenerate \(g\)-natural metric \(G\), the basic equations for the immersion given by (3) to be used throughout the paper.

\(H^h + V^v\) is a unique decomposition of a vector field normal to \(LM\) into its horizontal and vertical parts, where \(H\) and \(V\) are vector fields along \(M\), not necessary tangent to \(M\). \(\nabla\) and \(\tilde{\nabla}\) denote the Levi-Civita connections of the metric \(g\) and \(g\)-natural non-degenerate metric \(G\), respectively. \(\tilde{H}\) is the second fundamental form of the immersion \((3)\). Finally, \(R\) stands for the Riemann curvature tensor of \(g\). The computations in this section were performed and checked with Mathematica software. In virtue of Proposition 5, the pairs of equations in each subsection must satisfy

\[
G(\tilde{H}(\partial_x, \partial_y), H^h + V^v) - G(A_{H^h + V^v} \partial_x, \partial_y) = 0. \tag{4}
\]

It also can be used to verify the correctness of computations.

**Equation 1**

\[
G(\nabla_{\partial_x} \partial_y, H^h + V^v) = G(\tilde{H}(\partial_x, \partial_y), H^h + V^v) = \\
G(\nabla_{\partial_x} \eta_y^v, H^h + V^v) = \\
G(h\{E(u, \eta_x, \eta_y)\} + v\{F(u, \eta_x, \eta_y)\}, H^h + V^v) = \\
\]
In virtue of the equality (4) the above two equations yield

\[
\begin{align*}
\frac{b_2}{2}g(\eta_x, \eta_y)g(u, H) + (b_1 - a'_1)g(\eta_x, \eta_y)g(u, V) + \\
\frac{a'_1}{2}g(\eta_x, V)g(u, \eta_y) + a'_1g(\eta_y, V)g(u, \eta_x) + \\
\left(a'_2 + \frac{b_2}{2}\right) \left(g(\eta_x, H)g(u, \eta_y) + g(\eta_y, H)g(u, \eta_x)\right) + \\
g(u, \eta_x)g(u, \eta_y)g(u, b'_1V + 2b'_2H).
\end{align*}
\]

(5)

In virtue of the equality (4) the above two equations yield

\[g(u, \eta_x)g(\eta_y, T) - g(u, \eta_y)g(\eta_x, T) = 0,\]

where \(T = (b'_1 - 2a'_1)V + (b'_2 - 2a'_2)H\).

**Equation 2**

\[
\begin{align*}
G(\bar{\nabla}_{\partial_x} H^h + V^v) &= G(\overline{H}, \partial_x, \partial_a), H^h + V^v = \\
&= G(\bar{\nabla}_{\partial_a} (\delta^h_a + M^w_a + N^w_a), H^h + V^v) = \\
&= G(h\{C(u, H, \eta_x)\} + v\{D(u, H, \eta_x)\} + h\{E(u, \eta_x, V)\} + v\{F(u, \eta_x, V)\}, \eta^v_y) = \\
&= (b_1 - a'_1)g(\eta_x, V)g(u, \eta_y) + a'_1g(\eta_y, V)g(u, \eta_x) + a'_1g(\eta_x, \eta_y)g(u, V) + \\
&= \left(a'_2 - \frac{b_2}{2}\right) (g(\eta_y, H)g(u, \eta_x) - g(\eta_x, H)g(u, \eta_y)) + \\
&= b'_1g(u, \eta_x)g(u, \eta_y)g(u, V).
\end{align*}
\]

In virtue of the equality (4) the above two equations yield

\[
\begin{align*}
G(\bar{\nabla}_{\partial_a} H^h + V^v) &= G(-\overline{A}_h + V, \partial_x, \partial_a) = \\
&= G(\bar{\nabla}_{\partial_a} (H^h + V^v), \delta^h_a + M^w_a + N^w_a) = \\
&= G(h\{C(u, H, \eta_x)\} + v\{D(u, H, \eta_x)\}, \delta^h_a + M^w_a + N^w_a) + \\
&= G(h\{E(u, \eta_x, V)\} + v\{F(u, \eta_x, V)\}, \delta^h_a + M^w_a + N^w_a) = \\
&= a'_1g(\eta_x, \eta_y)g(u, \eta_x)g(u, \eta_y) + g(u, \eta_x)g(u, \eta_y)g(u, V) + \\
&= g(u, b'_1V + 2b'_2H)g(u, \eta_x)g(u, \eta_y)g(u, \eta_x).
\end{align*}
\]

(6)
\begin{align*}
\frac{1}{2}a_1 R(H, \delta_a, u, \eta_x) + (b_1 - a'_1) g(u, N_a) g(V, \eta_x) + a'_1 g(u, \eta_x) g(V, M_a + N_a) + \\
A'(H, \delta_a) g(u, \eta_x) + \left( a'_2 + \frac{b_2}{2} \right) g(u, \eta_x) g(V, \delta_a) + \\
\left( a'_2 - \frac{b_2}{2} \right) \left[ g(H, M_a + N_a) g(u, \eta_x) - g(H, \eta_x) g(u, N_a) \right] + \\
b'_1 g(u, V) g(u, N_a) g(u, \eta_x). 
\end{align*}

Hence, in virtue of (4), we get

\begin{align*}
g(u, b_1 V + b_2 H) g(N_a, \eta_x) + 2g(u, \eta_x) g(A'H + a'_2 V, \delta_a) + \\
2g(u, \eta_x) g(a'_1 V + a'_2 H, M_a + N_a) + \\
g(u, N_a) [g(b_1 V + b_2 H, \eta_x) + 2g(u, \eta_x) g(b'_1 V + b'_2 H, u)] = 0. 
\end{align*}

Equation 3

\begin{align*}
G(\nabla_{\partial_a} \eta_x, H^h + V^v) = G(\bar{H}(\partial_a, \partial_x), H^h + V^v) = \\
G(\nabla_{\delta_x + M_a^h + N_a^v} (\eta_x)^v, H^h + V^v) = \\
G((\nabla_{\partial_a} \eta_x)^v + h\{C(u, \delta_a, \eta_x)\} + v\{D(u, \delta_a, \eta_x)\}, H^h + V^v) + \\
G(h\{E(u, \eta_x, M_a + N_a)\} + v\{F(u, \eta_x, M_a + N_a)\}, H^h + V^v).
\end{align*}

Since \(\bar{H}(\partial_a, \partial_x)\) is symmetric, comparing the last equation with (8), we obtain

\begin{align*}
G \left( (\nabla_{\delta_a} \eta_x)^v, H^h + V^v \right) = 0. 
\end{align*}

\begin{align*}
G(\nabla_{\delta_a} (H^h + V^v), \partial_x) = G(-A^h + V, \partial_a, \partial_x) = \\
G(\nabla_{\delta_a + M_a^h + N_a^v} (H^h + V^v), \eta_x) = \\
G((\nabla_{\partial_a} H)^h + h\{A(u, \delta_a, H)\} + v\{B(u, \delta_a, H)\}, \eta_x^v) + \\
G((\nabla_{\partial_a} V)^v + h\{C(u, \delta_a, V)\} + v\{D(u, \delta_a, V)\}, \eta_x^v) + \\
G(h\{C(u, H, M_a + N_a)\} + v\{D(u, H, M_a + N_a)\}, \eta_x^v) + \\
G(h\{E(u, M_a + N_a, V)\} + v\{F(u, M_a + N_a, V)\}, \eta_x^v) =
\end{align*}

\begin{align*}
\frac{1}{2}a_1 R(H, \delta_a, u, \eta_x) + a_1 g(\eta_x, \nabla_{\delta_a} V) + b_1 g(u, \eta_x) g(u, \nabla_{\delta_a} V) + \\
a_2 g(\eta_x, \nabla_{\delta_a} H) + b_2 g(u, \eta_x) g(u, \nabla_{\delta_a} H) - A'(u, \eta_x) g(\delta_a) + \\
(b_1 - a'_1) g(u, \eta_x) g(V, M_a + N_a) + a'_1 g(u, N_a) g(V, \eta_x) + a'_1 g(u, V) g(N_a, \eta_x) + \\
\left( a'_2 - \frac{b_2}{2} \right) \left[ g(H, \eta_x) g(u, N_a) - g(H, M_a + N_a) g(u, \eta_x) - g(u, \eta_x) g(V, \delta_a) \right] + \\
b'_1 g(u, V) g(u, N_a) g(u, \eta_x). 
\end{align*}
3. BASIC EQUATIONS

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Equation 4

\[
G(\nabla_{\delta_b} \partial_b, H^h + V^v) = G(\tilde{H}(\partial_a, \partial_b), H^h + V^v) = \\
G(\nabla_{(\delta^h_b + M^h_b + N^h_b)}(\delta^h_b + M^h_b + N^h_b), H^h + V^v) = \\
G(\nabla_{\delta_b}) + h \{A(u, \delta_a, \delta_b)\} + v \{B(u, \delta_a, \delta_b)\} + \\
(\nabla_{\delta_a}(M_b + N_b))^2 + h \{C(u, \delta_a, M_b + N_b)\} + v \{D(u, \delta_a, M_b + N_b)\} + \\
h \{E(u, M_a + N_a, M_b + N_b)\} + v \{F(u, M_a + N_a, M_b + N_b)\}, H^h + V^v) = \\
- a_2 R(H, \delta_b, u, \delta_a) - \\
\frac{1}{2}a_1 R(H, \delta_a, u, M_b + N_b) - \frac{1}{2}a_1 R(H, \delta_b, u, M_a + N_a) - \frac{1}{2}a_1 R(u, V, \delta_a, \delta_b) + \\
g(AH + a_2 V, \nabla_{\delta_a} \delta_b) + g(BH + b_2 V, u)g(u, \nabla_{\delta_b} \delta_b) + \\
\frac{1}{2}Bg(H, u) (g(M_a, \delta_b) + g(M_b, \delta_a)) + \\
g(a_1 V + a_2 H, \nabla_{\delta_a}(M_b + N_b)) + g(b_2 H + b_1 V, u)g(u, \nabla_{\delta_b}(M_b + N_b)) + \\
+ g(b_2 H + (b_1 - a'_1) V, u)g(M_a, M_b + N_a) + g(N_a, N_b) + \\
A' [g(H, \delta_a)g(u, N_b) + g(H, \delta_b)g(u, N_a) - g(V, u)g(\delta_a, \delta_b)] + \\
a'_1 g(u, N_a)g(V, M_b + N_b) + a'_1 g(u, N_b)g(V, M_a + N_a) + \\
\left(\frac{a'_2 - \frac{b_2}{2}}{2}\right) \{g(u, N_a)g(V, \delta_a) + g(u, N_a)g(V, \delta_b) - g(u, V) [g(M_b, \delta_a) + g(M_a, \delta_b)]\} + \\
\left(\frac{a'_2 + \frac{b_2}{2}}{2}\right) \{g(H, M_a + N_a)g(u, N_b) + g(H, M_b + N_b)g(u, N_a) + \\
g(u, b'_1 V + 2b'_2 H)g(u, N_a)g(u, N_b)\}. \quad (10)
\]

\[
G(\nabla_{\delta_b}(H^h + V^v), \partial_b) = G(-\tilde{A}_{H^h + V^v} \partial_a, \partial_b) = \\
G(\nabla_{(\delta^h_b + M^h_b + N^h_b)}(H^h + V^v), \eta^v_b) = \\
\]

\[
a_2 R(H, \delta_b, u, \delta_a) + \\
\frac{1}{2}a_1 R(H, \delta_a, u, M_b + N_b) + \frac{1}{2}a_1 R(H, \delta_b, u, M_a + N_a) + \frac{1}{2}a_1 R(u, V, \delta_a, \delta_b) + \\
Ag(\delta_b, \nabla_{\delta_a} H) + a_2 g(\delta_b, \nabla_{\delta_a} V) + a_2 g(M_b + N_b, \nabla_{\delta_a} H) + a_2 g(M_b + N_b, \nabla_{\delta_a} V) + \\
\frac{1}{2}Bg(H, u) (g(M_a, \delta_b) - g(M_b, \delta_a)) + \\
g(u, N_b) (b_2 g(u, \nabla_{\delta_a} H) + b_1 g(u, \nabla_{\delta_a} V)) + \\
A' (g(H, \delta_a)g(u, N_b) + g(H, \delta_b)g(u, N_a) + g(V, u)g(\delta_a, \delta_b)) + \\
(b_1 - a'_1) g(u, N_a)g(V, M_a + N_a) + a'_1 g(u, N_a)g(V, M_b + N_b) + \\
\]

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April 24, 2018

Let \( \bar{f} \) be the immersion given by (3) defined by the isometric immersion \( f : M \to (N, g) \) into a Riemannian manifold. Suppose, moreover, that \( TN \) is endowed with non-degenerate \( g \)-natural metric \( G \). Then in the notation as above the following identities are satisfied.

1. 
\[
g(\eta_x, S) = 0, \tag{11}
\]
where \( S = a_2 H + a_1 V + g(u, b_2 H + b_1 V)u \).

2. 
\[
g(u, S) = g(N_a, S) = 0. \tag{12}
\]

3. 
\[
g(\nabla_{\delta_a} \eta_x, S) = 0. \tag{13}
\]
4. \[ g(\eta_x, \nabla_{\delta_a} S) = 0. \] (14)

5. \[ g(u, \nabla_{\delta_a} S) = g(N_b, \nabla_{\delta_a} S) = 0. \] (15)

6. \[ g(\nabla_{\delta_a} u, S) = g(\nabla_{\delta_a} N_b, S) = 0. \] (16)

7. \[ g(\delta_a, AH + a_2 V) = g(M_a, a_2 H + a_1 V) = 0. \] (17)

8. \[ g(\delta_a, A' H + a'_2 V) = g(M_a, a'_2 H + a'_1 V) = 0. \] (18)

9. \[ g(M_b, M_a) + g(u, \nabla_{\delta_a} M_b) = 0. \] (19)

10. \[ g(M_b, \delta_a) + g(u, \nabla_{\delta_a} \delta_a) = 0. \] (20)

11. \[ g(\nabla_{\delta_a} \eta_x, \delta_b) + g(\eta_x, \nabla_{\delta_a} \delta_b) = 0. \] (21)

Moreover, if $M$ is not a hypersurface of $N$, then

12. \[ X_u = g(u, b_2 H + b_1 V) = 0. \] (22)

13. \[ X_{\eta_x} = g(\eta_x, b_2 H + b_1 V) = 0. \] (23)

14. \[ Y_{\eta_x} = g(\eta_x, b'_2 H + b'_1 V) = 0. \] (24)

15. \[ Y_u = g(u, b'_2 H + b'_1 V) = 0. \] (25)

16. \[ Z_{\eta_x} = g(\eta_x, a'_2 H + a'_1 V) = 0, \quad Z_u = g(u, a'_2 H + a'_1 V) = 0. \] (26)

17. \[ g(\eta_x, a_2 H + a_1 V) = 0. \] (27)

Finally

18. \[ S = a_2 H + a_1 V. \]
Proof. (11) results from
\[ G(\partial_x, H^h + V^v) = G(\eta_x, H^h + V^v) = 0. \]
Then (12) is obvious since \( u = v^y N_y^r \partial_r = v^h \eta_y \) and \( N_a = N_a^y \eta_y \) are normal to \( M \). Now (13) is a consequence of (8), whence, by (11), (14) results.

Once again, by orthogonality of \( u \) and \( N_a \) with respect to \( M \), we have (15). Consequently, in virtue of (12), we obtain (16).

Observe that the identity
\[ (7) - (10) + g(\eta_x, \nabla_{\delta_a} S) - \sum_y \frac{\partial}{\partial \eta_y} g(\eta_x, S) N_a^y = 0 \] gives
\[ g(\delta_a, A^t H + a_2^t V) + g(M_a, a_2^t H + a_1^t V) = 0. \]
On the other hand, relations
\[ G(\partial_a, H^h + V^v) = G(\delta_a^t + M_a^u + N_a^v, H^h + V^v) = 0 \]
and (12) yield
\[ g(\delta_a, AH + a_2 V) + g(M_a, a_2 H + a_1 V) = 0. \]
Differentiating (29) with respect to \( v^x \) and using (28) we find
\[ g(M_a x, a_2 H + a_1 V) = 0, \]
where \( M_a x = \nabla_{\delta_a} N^r \partial_r \). Consequently, (29) yields (17). Hence, by differentiating with respect to \( v^x \), (18) results.

Since \( M_a, \delta_a \) are tangent to \( M \) and \( u, \eta_x \) are normal, by covariant differentiation of \( g(u, M_a) = 0, g(u, \delta_a) = 0, g(\eta_x, \delta_a) = 0 \) we get (19) - (21).

Differentiating (11) with respect to \( v^v \) we get
\[ g(\eta_y, \eta_y) X_u + 2g(\eta_x, u) g(\eta_y, u) Y_u + g(\eta_x, u) X_{\eta_y} + 2g(\eta_x, \eta_y) Z_{\eta_y} = 0, \]
where \( X_u = g(u, b_2 H + b_1 V), X_{\eta_y} = g(\eta_x, b_2 H + b_1 V), Y_u = g(u, b_2 H + b_1 V), \)
\( Z_u = g(\eta_x, a_2^t H + a_1^t V) \) and \( Z_{\eta_y} = g(\eta_x, a_2^t H + a_1^t V). \)

Transvecting in turn with \( v^x, v^y, v^x v^y \) and, finally, contracting wit \( g^{uv} \) we get for each \( x = m + 1, ..., n \) a system of four equations:
\[
\begin{align*}
v_x X_u + 2r^2 v_x Y_u + r^2 X_{\eta_x} + 2v_x Z_u &= 0, \\
v_x X_u + 2r^2 v_x Y_u + v_x X_u + 2r^2 Z_{\eta_x} &= 0, \\
v^2 (X_u + r^2 Y_u + Z_u) &= 0, \\
(n - m + 1)X_u + 2r^2 Y_u + 2Z_u &= 0,
\end{align*}
\]
where \( v_x = g(u, \eta_x) \). Solving it with respect to \( X, X_{\eta_x}, Y, Z_{\eta_x} \) we obtain
\[ X_u = X_{\eta_x} = 0, \quad Y_u = -\frac{Z_u}{r^2}, \quad Z_{\eta_x} = -\frac{v_x Z_u}{r^2} \] (31)
for any \( u = v^x \eta_x \neq 0 \). By continuity, \( X_u = X_{v^x} = 0 \) hold for any \( u \). Then \( \frac{\partial}{\partial v^x} X_{\eta_x} = Y_{\eta_x} g(\eta_z, u) = 0 \) for all \( u \neq 0 \), whence, in virtue of continuity, \( Y_{\eta_x} = 0 \) for any \( u \). Consequently, we have \( Y_{\eta_x} = 0 \). Now, \( Z_u = 0 \) follows from (30) and \( Z_{\eta_x} = 0 \) results from (31). Finally, (27) is a consequence of (11) and (22). Thus the lemma is proved.  

**Theorem 6** Let \((x, u)\) be a point of \(LM\) immersed in \(TN\). If \(\text{codim } M > 1\) and 
\[ a_2 b_1 - a_1 b_2 \neq 0 \] 
(32)

at \( t = g(u, u) \), then the normal space at \((x, u) \in LM\) is spanned by lifts of vectors tangent to \(M\).

**Proof.** It results from the identities (27) and (23). Note that other conditions, similar to that of (32) can be deduced in the same way from (27), (26), (23) and (24).  

We shall prove that the conditions \(\text{codim } M > 1\) and (32) are essential in that sense that there exist immersion \(f : M \to N\), \(M\) being a hypersurface of \(N\), and a metric \(G\) on \(TN\) satisfying \(a_2 b_1 - a_1 b_2 = 0\) such that the normal component of at least one of the vectors \(H, V\) does not vanish.

**Example 7** Let \(f : S^1 \to (R^2, \text{Euclid metric})\) be the immersion given by \(f(t) = [\cos t, \sin t]\). The vector tangent to \(S^1\) is \(s = [0, 0, -\sin t, \cos t]\), and the normal one is \(n = [\cos t, \sin t, 0, 0]\). The vectors tangent to \(L(S^1)\) are 
\[ \frac{\partial}{\partial t} = s^h + v s^v, \quad \frac{\partial}{\partial v} = n^v. \]

Suppose 
\[ H^h + V^v = \alpha s^h + \beta n^h + \gamma s^v + \delta n^v \]
and consider a non-degenerate \(g\)-natural metric on \(T R^2\) such that \(B = b_1 = b_2 = 0\).

Then 
\[ G(\frac{\partial}{\partial v}, H^h + V^v) = G(n^v, H^h + V^v) = a_2 g(n, H) + a_1 g(n, V) = a_2 \beta + a_1 \delta \]
and 
\[ G(\frac{\partial}{\partial t}, H^h + V^v) = G(s^h, H^h + V^v) = A g(s, H) + a_2 g(s, V) = A \alpha + a_2 \gamma. \]

We put 
\[ \alpha = \frac{a_2 + \gamma}{A + \gamma a_2} \neq 0, \quad \beta = -\frac{a_1}{a_2} \delta \neq 0. \]
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The restriction on codimension can be omitted as the next proposition shows.

**Proposition 8** If \( a_2 = b_2 = 0 \) for all \( t \in < 0, \infty \), then the normal bundle of \( LM \) at a point \( (x, u) \), \( x \in M \), \( u \in T_x M \), is spanned by the vectors \( \eta_y \) and \( \delta_a - \frac{a_1}{a_2} (\nabla \delta_a u)^h \).

**Proof.** Direct calculation. Property (20) is applied. ■

Applying Proposition 5 to (5), (6) and (10) we obtain

Applying Proposition \([3] \) to \([5] \), \([6] \) and \([10] \) we obtain

**Theorem 9** Let \( M \), \( \text{codim} M > 1 \), be a submanifold isometrically immersed in a manifold \( N \). Then along \( LM \) we have:

1. 
   \[
   \tilde{G} \left[ H^h + V^v, \nabla_{\partial_x} \partial_y \right] = \tilde{G} \left[ H^h + V^v, \tilde{H}(\partial_x, \partial_y) \right] = \\
   g \left[ H, \frac{b_2}{2} g(u, \eta_x) \eta_y + \frac{b_2}{2} g(u, \eta_y) \eta_x + a_2' g(\eta_x, \eta_y) u + b_2' g(u, \eta_x) g(u, \eta_y) u \right],
   \]

2. 
   \[
   \tilde{G} \left[ H^h + V^v, \nabla_{\partial_\alpha} \partial_\beta \right] = \tilde{G} \left[ H^h + V^v, \tilde{H}(\partial_\alpha, \partial_\beta) \right] = \\
   g \left[ H, \frac{b_2}{2} g(u, \eta_x) \eta_y + \frac{b_2}{2} g(u, \eta_y) (M_a + N_a) + a_2' g(\eta_x, \eta_y) u + b_2' g(u, \eta_x) g(u, \eta_y) u \right] - \\
   g \left[ V, \frac{b_2}{2} g(u, \eta_x) \eta_y \right],
   \]

3. 
   \[
   \tilde{G} \left[ H^h + V^v, \nabla_{\partial_\alpha} \partial_\beta \right] = \tilde{G} \left[ H^h + V^v, \tilde{H}(\partial_\alpha, \partial_\beta) \right] = \\
   g[AH + a_2 V, \nabla_{\delta_\alpha} \delta_\beta] - a_2 R(H, \delta_\alpha, u, \delta_\beta) - \\
   \frac{a_1}{2} R(H, \delta_\alpha, u, M_b + N_b) + R(H, \delta_\alpha, u, M_a + N_a) + R(u, V, \delta_\alpha, \delta_\beta) + \\
   g[a_2 H + a_1 V, \nabla_{\delta_\alpha} (M_b + N_b)] - \\
   g(u, V) [A' g(\delta_\alpha, \delta_\beta) + a_1' (g(M_a, M_b) + g(N_a, N_b)) + 2a_2 g(M_a, \delta_\beta)] - \\
   \frac{1}{2} b_2 [g(u, N_a) g(V, \delta_\beta) + g(u, N_b) g(V, \delta_\alpha)] + \\
   g \left[ H, \frac{b_2}{2} g(u, \eta_x) (M_b + N_b) + \frac{b_2}{2} g(u, \eta_x) (M_a + N_a) \right] + \\
   b_2 g(u, H) g(u, N_a) g(u, N_b)
   \]

for a nondegenerate \( g \)-natural metric \( \tilde{G} \).
Proof. Straightforward computation. 

Remark 10 The third equation of the last Proposition can be written as

\[
\tilde{G} \left[ H^h + V^v, \tilde{\nabla}_{\partial_x} \partial_y \right] = \tilde{G} \left[ H^h + V^v, \tilde{H}(\partial_x, \partial_y) \right] = \\
G[H^h + V^v, (\nabla_{\delta_a} \delta_b)^h + (\nabla_{\delta_a} \nabla_{\delta_b} u)^v] - g(BH + b_2 V, u)g(u, \nabla_{\delta_a} \nabla_{\delta_b} u) - \\
a_2 R(H, \delta_a, u, \delta_a) - \frac{a_1}{2} [R(H, \delta_a, u, M_b + N_b) + R(H, \delta_b, u, M_a + N_a) + R(u, V, \delta_a, \delta_b) + \\
g(u, V) [A' g(\delta_a, \delta_b) + a_1' (g(M_a, M_b) + g(N_a, N_b)) + 2a_2 g(M_a, \delta_b)] - \\
\frac{1}{2} b_2 [g(u, M_a)g(V, \delta_a) + g(u, N_a)g(V, \delta_a)] + \\
g \left[ H, \frac{b_2}{2} g(u, N_a) (M_b + N_b) + \frac{b_2}{2} g(u, N_b) (M_a + N_a) \right] + \\
b_2 g(u, H)g(u, N_a)g(u, N_b).
\]

Definition 11 A distribution \( D \) on a manifold \( M \) is said to be totally geodesic if it is invariant with respect to covariant differentiation, i.e. \( \nabla_X Y \in D \) for all \( X, Y \in D \).

Theorem 12 Let \( M, \text{codim} \, M > 1 \), be a submanifold isometrically immersed in a manifold \((\mathbb{N}, g)\). Suppose that \( LM \) is a submanifold isometrically immersed by \( \mathbb{F} \) in \( TN \) with non-degenerate \( g \)-natural metric \( G \).

1. If either the normal bundle of \( LM \) is spanned by vectors of the form \( H^h + V^v \), where \( H \) and \( V \) are tangent to \( M \) or \( b_2 = a_2' = 0 \) along \( M \), then vector fields \( \{\partial_x\} \), \( x = m + 1, ..., n \) define on \( LM \) the totally geodesic distribution that is involutive.

2. If

(a) \( a_1 = 0, b_2 = 0, a_2 = \text{const} \neq 0 \) along \( M \) or

(b) \( N \) is a space of constant curvature and \( a_2' = 0, b_2 = 0 \) along \( M \),

then \( LM \) is mixed totally geodesic.

Here, along \( M \),

\[
\nabla_{\delta_a} u = \nabla_{\delta_a} (u^y \eta_y) = \nabla_{\delta_a} \left( u^y N^r_y \frac{\partial}{\partial x^r} \right) = M_a + N_a.
\]

Proof. In virtue of the assumptions, the first equation of Theorem \( \mathbb{F} \) yields \( \tilde{G} \left[ H^h + V^v, \tilde{H}(\partial_x, \partial_y) \right] = 0 \). Hence \( \nabla_{\delta_a} \partial_y \) is tangent to \( LM \). Since \( \partial_x = \eta^r_y \) are vertical vector fields, the distribution is involutive. This proves the first point. The proof of the second one is obvious. 

\[ \square \]
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