Two-dimensional simulations of nonlinear beam-plasma interaction in isotropic and magnetized plasmas

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Nonlinear interaction of a low density electron beam with a uniform plasma is studied using two-dimensional particle-in-cell (PIC) simulations. We focus on formation of coherent phase space structures in the case, when a wide two-dimensional wave spectrum is driven unstable, and we also study how nonlinear evolution of these structures is affected by the external magnetic field. In the case of isotropic plasma, nonlinear buildup of filamentation modes due to the combined effects of two-stream and oblique instabilities is found to exist and growth mechanisms of secondary instabilities destroying the BGK-type nonlinear wave are identified. In the weak magnetic field, the energy of beam-excited plasma waves at the nonlinear stage of beam-plasma interaction goes predominantly to the short-wavelength upper-hybrid waves propagating parallel to the magnetic field, whereas in the strong magnetic field the spectral energy is transferred to the electrostatic whistlers with oblique propagation.

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I. INTRODUCTION

Investigations of various aspects of the beam-plasma instability are important for our understanding of physical processes occurring in space and laboratory plasmas. In recent years, regimes of collective beam-plasma interaction typical to the fast ignition scheme or space and ionospheric phenomena have attracted particular interest. That is why most of recent theoretical and numerical studies deals with strong instabilities excited by counterstreaming electron beams of comparable densities. Indeed, in the case of isotropic plasma, results of linear theory relevant for such regimes have been revisited and dominance of different unstable modes in parameter space have been determined [1, 2]. Nonlinear effects such as beam trapping and coupling of the Weibel-filamentation instability with the two-stream instability have been studied in the framework of the initial-condition problem using both two- and three-dimensional PIC simulations. As to the case of magnetized plasma, the particular attention has been focused on formation and stability of nonlinear phase space structures (electron holes and tubes) arising during saturation of the two-stream instability [9–14].

The goal of this paper is to investigate in details the nonlinear evolution scenario of a beam-plasma system in the case of low density beams \( n_b \ll n_p \) and to study how such a scenario is modified by the external magnetic field. Qualitatively, the overall picture of weak beam-plasma interaction resembles that for a strong beam, but, in our opinion, more attention should be given to the study of consecutive energy transfer from one plasma mode to another. Our interest to this problem is motivated by the fact that wave-wave interaction due to the beam nonlinearity that is responsible for the energy exchange between plasma modes in a uniform beam-plasma system does also play an important role in the realistic problem of beam injection through a plasma boundary. In one-dimensional simulations [15] we have found the regime of beam relaxation, which is characterized by the regular nonlinear beam dynamics in electric fields of resonant waves even in a quasistationary turbulent state. The beam nonlinearity in this regime results in growth of coherent wave packets instead of waves with chaotic phases and leads to saturation of the pumping power. The key argument against feasibility of such a regime is a one-dimensional character of our simulations. Thus, it is interesting to study whether nonlinear beam dynamics plays an essential role in multidimensional problems and whether multimode character of the beam-plasma instability is able to destroy correlation effects. For this purpose, we simulate two-dimensional beam-plasma interaction in the framework of the periodic boundary problem, in which different resonant modes interact with each other for a long time due to the beam nonlinearity.

In Sec. II we choose physical parameters of the beam-plasma system and numerical parameters of our PIC model. In Sec. III we study the linear stage of the beam-plasma instability and compare simulation results with theoretical predictions. Sec. IV presents simulation studies for the nonlinear stage of beam-plasma interaction. Here, we identify the main nonlinear processes responsible for the energy exchange between different plasma modes and study the magnetic field effects. In final Sec. V we discuss our main results.

II. SIMULATION PARAMETERS

Let us consider relaxation of an electron beam with the density \( n_b/n_p = 0.002 \) and the temperature \( T_b = 10 \) eV propagating with the velocity \( u_b/c = 0.382 \) \((c\) is the speed of light) in a uniform plasma with the initial electron temperature \( T_e = 60 \) eV. This set of parameters is of a great importance for laboratory experiments on turbu-
lent plasma heating in mirror traps [16,17]. Since plasma in such traps is confined by the external magnetic field \( B \) that can be characterized by the ratio of the cyclotron frequency \( \omega_c = eB/(mc) \) to the plasma frequency \( \omega_p = (4\pi e^2n_p/m_e)^{1/2} \). In addition to beam and plasma electrons, we take into account the dynamics of ions with the mass \( m \). We choose \( \Omega = 0, \Omega = 0.5 \) and \( \Omega = 2 \) ( \( e \) and \( m \) are the charge and the mass of electrons).

To simulate nonlinear evolution of the beam-plasma system, we use the standard two-dimensional electromagnetic PIC code with periodic boundary conditions for particles and fields. Initially, the charge and current of the electron beam propagating along \( x \) are compensated by charges and currents of plasma electrons. We choose the simulation box \( L_x \times L_y = 360 \Delta x \times 384 \Delta y \) and grid sizes \( \Delta x = \Delta y = 0.02c/\omega_p \) in such a way as to fulfill the Cherenkov condition for the modes with wavenumbers \( k_x = 6\pi/L_x = \omega_p/v_b \) and resolve the Debye length. For the time step we get \( \tau = 0.01\omega_p^{-1} \) and for the reasonable level of noise we use 256 computation particles in each cell. In addition to beam and plasma electrons, we take into account the dynamics of ions with the mass \( m_i = 1836m_e \).

### III. LINEAR ANALYSIS

Let us find out what plasma modes should be driven unstable in the beam-plasma system according to the linear theory and how their growth-rates depend on the external magnetic field. Due to low beam and plasma

\[
|k_\alpha k_\beta - k^2\delta_{\alpha\beta} + \frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}| = 0, \tag{1}
\]

where \( \varepsilon_{\alpha\beta} \) is the dielectric tensor with the components:

\[
\begin{align*}
\varepsilon_{xx} &= 1 - \frac{\omega_p^2}{\omega^2} - \frac{n_b}{n_p \gamma^3} \frac{\omega_p^2}{(\omega - k_x v_b)^2}, \\
\varepsilon_{yy} &= 1 - \frac{\omega_p^2}{\omega^2}, \\
\varepsilon_{xy} &= -\frac{n_b}{n_p \gamma^2} \frac{\omega_p^2}{(\omega - k_x v_b)^2 - \omega_c^2/\gamma^2}, \\
\varepsilon_{xz} &= -\frac{i n_b}{n_p \gamma^2} \frac{\omega_c^2}{\omega (\omega - k_x v_b)^2 - \omega_c^2/\gamma^2}, \\
\varepsilon_{yz} &= -\frac{i n_b}{n_p \gamma^2} \frac{\omega_c^2}{\omega (\omega - k_x v_b)^2 - \omega_c^2/\gamma^2}. \\
\end{align*}
\]
Here, $\gamma$ is the relativistic factor of the beam. In our axis the beam velocity and the magnetic field are directed along $x$ and the wavevector has the components $\mathbf{k} = (k_x, k_y, 0)$.

In the spatially periodic system $L_x \times L_y$ possible wavenumbers are restricted by the following discrete sets:

$$k_x = \frac{2\pi n}{L_x} = \frac{\omega_p n}{3v_b}, \quad k_x = \frac{2\pi m}{L_y} = \frac{\omega_p m}{3v_b},$$

hence, the position of each plasma mode in the plane $(k_x, k_y)$ can be marked by two integers $n-m$. In our case, the unstable spectrum contains only modes with numbers $n = 0, 1, 2, 3$. Numerical solutions of the dispersion relation for the growth-rates of these modes as functions of the transverse wavenumber $k_y$ are shown in Fig. (a)-(c). It is seen that the transition from the case of isotropic plasma with dominant oblique instabilities to the case of strong magnetic field is accompanied by the essential reduction in the growth-rates of oblique modes and complete stabilization of the filamentation instability.

Let us see how accurately theoretical predictions are reproduced in PIC simulations. Fig. (d) shows that the growth-rates of dominant Langmuir 3-1 modes agree well with theoretical predictions up to large $m$ thus confirming flow pattern of the observed instability, whereas instabilities of beam modes 1-m and 2-m are found to be suppressed. As to the pure transverse filamentary perturbations, it is shown, using the 0-3 mode as an example, that barely visible exponential growth of these modes with linear growth-rates switches rapidly to the nonlinear regime with effective rates exceeding the maximum growth-rate of the linear theory. Nonlinear nature of this regime is confirmed by the fact that by the moment $\omega_p t = 75$ fluctuations of beam density become comparable in magnitude with the unperturbed value.

The weak magnetic field $\Omega = 0.5$ reduces growth-rates of oblique upper-hybrid modes 3-m in a good agreement with the linear theory and keeps beam modes 1-m and 2-m stable. In the strong magnetic field the spectrum of unstable plasma waves narrows significantly. It results in the situation when, in addition to the 3-0 and 3-1 modes, we observe the instability of the 2-2 mode. In contrast to previous cases, this mode now falls in the $k_y$ range, inside which it becomes dominant.

IV. NONLINEAR STAGE

Before proceeding to the detailed study of nonlinear evolution scenarios of the beam-plasma system in three different cases $\Omega = 0$, $\Omega = 0.5$ and $\Omega = 2$, let us briefly describe the main stages of beam-plasma interaction by considering time evolution of dominant Fourier harmonics of the electric field $E_x$ presented in Fig. 2. In the case $\Omega = 0$, at the stage of beam trapping, the highest level of saturation is achieved by modes 3-0 and 3-1. The resulting nonlinear wave, however, turns out to be unstable against perturbations with shorter longitudinal wavelengths ($n = 4$ and $n = 5$). Among these primarily stable plasma oscillations, the main role is played by the Langmuir waves with small propagation angles ($m = 0, 1$). Fig. (a) shows that the time-averaged growth of these modes is found to be exponential and saturation levels of secondary instabilities appear to be so high that from

![Figure 2: (Color online) Time evolution of $E_x$ Fourier harmonics at the nonlinear stage of beam plasma interaction in different magnetic fields $\Omega = 0$ (a), $\Omega = 0.5$ (b) and $\Omega = 2$ (c).](image-url)
some time the spectrum of $E_x$ field is dominated by 4-0 and 4-1 modes. Further damping of these modes is accompanied by the modulation instability resulting in absorption of their wave energy by plasma electrons.

In the weak magnetic field $\Omega = 0.5$, the picture of nonlinear evolution does not change drastically. Upon saturation of primary unstable upper-hybrid modes 3-0 and 3-1, we observe the buildup of secondary instabilities with almost the same averaged growth-rates as in the previous case (Fig. 2(b)). At the saturation state, however, secondary upper-hybrid modes with $n = 4$ do not play the dominant role. Significant changes in the picture of spectral energy transfer are found to occur only in the strong magnetic field $\Omega = 2$. In this case, the buildup of initially stable oblique modes 3-2 with $m > 1$ becomes more efficient than the growth of longitudinal oscillations with $n = 4$ and $n = 5$ (Fig. 2(c)).

A. Isotropic plasma $\Omega = 0$

To study nonlinear beam-plasma interaction in more details, we present time evolution of the beam density map $n_b(x,y)$ as well as snapshots of beam phase spaces $(x, p_x)$ and $(x, p_y)$ (Fig. 3).

Beam trapping in oblique directions is the first nonlinear consequence of the prevailing growth of almost electrostatic oblique modes 3-2. Indeed, in the time interval $\omega_p t = 75.1 \div 100.5$, Fig. 3 demonstrates the process of beam density modulation under the most unstable oblique modes, which is followed by the formation of characteristic vortex structures in the phase space of beam electrons. As we have mentioned, at the beginning of beam modulation we observe that the filamentation instability switches to the nonlinear regime. Let us discuss the possible mechanism of nonlinear excitation of filamentary perturbations.

Modulation of the beam density under unstable modes 3-2 and 3-(-2) drives density perturbations with the following structure:

$$\delta n_b \propto e^{i\omega_p(x/v_b - t)} + \Gamma_3 - 2t \cos(2\pi y/L_y).$$

Here, we neglect the difference between the eigenfrequency of the mode 3-2 and the plasma frequency $\omega_p$. Since at the same time the mode 3-0 perturbs the parallel velocity of the beam

$$\delta v_b \propto e^{i\omega_p(x/v_b - t) + \Gamma_3 - 0 t},$$

the parallel beam current contains the nonlinear term

$$\delta j_x \propto \delta n_b \delta v_b \propto e^{(\Gamma_3 - 2 + \Gamma_3 - 0)t} \cos(2\pi y/L_y).$$
which generates the magnetic field $B_z$ with the same spatial structure. Thus, at the early stage, when density and velocity fluctuations are small enough, filamentary perturbations $0-m$ due to the combined effects of two-stream and oblique instabilities should grow with the approximate rate $2\Gamma_{3-0}$. In our simulations the corresponding growth-rate is slightly lower. Apparently, it is explained by the fact that growing beam perturbations at some time cease to be weakly nonlinear.

As it is seen from Fig. 3 after the trapping stage the beam breaks up into bunches localized in space in both parallel and perpendicular directions. One can also see that these bunches are arranged one behind the other thus forming current layers along the magnetic field. Since the beam acquires the large spread in transverse momentum under oblique modes, different current layers exchange beam particles and merge into one by the moment $\omega_p t = 252.5$. At the same time, in phase spaces $(x, p_x)$ and $(x, p_y)$ we observe the formation of nonlinear BGK-wave, the amplitude of which, in contrast to the one-dimensional problem, is localized in the transverse direction.

Let us investigate stability of the resulting nonlinear wave. As we have mentioned, at this stage, the beam-plasma system is unstable against short-wavelength al-

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most longitudinal Langmuir oscillations with $n = 4$ and

$n = 5$. The exponential growth of secondary instabilities

indicates that the initial stage of these instabilities can

be described by the theory linearized in the amplitudes

of unstable perturbations. In this case, however, we cannot

consider a uniform beam-plasma system even with

the increased momentum spread of the beam as an un-

perturbed state. Excitation of secondary plasma modes

differs markedly from Cherenkov mechanism. Fig. 2 and

3 show that the instability of the mode 5-0, for example,

starts to grow when there are no any particles in resonance

with this mode (the momentum corresponding to the phase

velocity of this mode is shown in the phase space $(x, p_x)$ as

the dash-dotted line). The stability analysis of a one-dimensional (uniform along $y$) non-

linear BGK-wave as an exact stationary solution of the

Vlasov-Poisson system seems to be a more adequate theo-

retical approach in our case. According to the theory pro-

posed in Ref. 18 [19], efficient energy exchange between

a plasma mode with the frequency $\omega$ and the wavenumber $k$ and a nonlinear BGK-wave with the phase velocity $v_0$ and the bounce-frequency $\omega_b$ occurs if the following resonance condition is fulfilled: $\omega = kv_0 + N\omega_b$. Let us compute growth-rates of sideband modes 4-0 and 5-0 driven unstable due to resonances $N = -1$ and $N = -3$, respectively.

For this purpose, we simplify the general theory of Ref. [19] using the following assumptions. Most of beam particles is concentrated near the bottom of the potential well with the parabolic profile $V(x) = -e\varphi_0(1 - k_1^2x^2/2)$ and their distribution function $f(W)$ is constant inside the energy range $-e\varphi_0 < W < -0.9e\varphi_0 = W_0$ ($\varphi_0$ denotes the amplitude of electrostatic potential in the BGK-wave).

In this case, all beam particles participate in bounce osc-

illations with the same frequency $\omega_b = k_0(e\varphi_0/m)^{1/2}$ and their maximum deviation length from the center of the potential well is determined by

$$a = \frac{1}{k_0} \sqrt{2 \left( \frac{W_0}{e\varphi_0} + 1 \right)}.$$  

Goldman theory 19 results in the following dispersion relation

$$\varepsilon(k, \omega) + \chi_{0,0}(\varepsilon(k - 2k_0, \omega - 2\omega_0) + \chi_{-2,-2}) = \chi_{0,-2}^2,$$

where

$$\chi_{t,s} = -\omega_p^2 n_T^2 \sum_{n=1}^{\infty} \frac{J_2(z_n^n)J_N(z_n)}{4N^2 J_N^2(z_n) \sum_n^{\infty} z_n},$$

$\varepsilon(k, \omega)$ is the linear plasma permittivity, $J_N$ are the

Bessel functions of arguments $z_n = (k + jk_0)a$ and $n_T$ is the space-averaged density of trapped particles. In order to calculate the growth-rate of the mode 4-0 as the function of the bounce frequency of BGK-wave, we should take into account the contribution of only one resonance $N = 1$. Numerical solution of such a dispersion relation is presented in Fig. 4. From this figure we notice that

for the observed amplitude of the BGK-wave the best agreement between theoretical and simulation results is achieved, if the effective density of trapped particles is slightly lower than the beam density: $n_T = 0.9n_0$. If for the same amplitude of nonlinear wave we take into account the contribution of the resonance $N = 3$ in the dispersion equation, we are also able to predict the growth-rate of the mode 5-0, which is found to reach the value $\Gamma/\omega_p \approx 0.015$. Besides longitudinal modes 4-0 and 5-

0, which can be predicted theoretically, we also observe instabilities of oblique modes 4- and 5- with small propagation angles.

As to the nonlinear saturation of sideband instabilities, in the time interval $\omega_p t = 395.6 \div 659.4$ (Fig. 3) this

Figure 4: Dependence of the growth-rate of sideband instability for the mode 4-0 on the bounce frequency of trapped electrons in a one-dimensional BGK-wave.
process is accompanied by transformation of the primary BGK-wave with \( n = 3 \) to the other nonlinear wave with \( n = 4 \). From the motion of beam particles in the phase space \((x, p_x)\) we conclude that the wave energy comes to the secondary mode 4-0 not only from the kinetic energy of the beam that is decelerated by this slower wave, but also from the primary unstable mode 3-0 that loses most of its energy during nonlinear wave-wave interactions. In the later stage, plasma nonlinearities come into force and the BGK-wave with \( n = 4 \) is dissipated due to the local (in \( y \)) buildup of the modulation instability.

\[ t = 252.5p \]

\[ t = 115.6p \]

\[ t = 422p \]

\[ t = 688.3p \]

\[ \omega_p t = 115.6 \] and \( \omega_p t = 252.5 \) beam particles mix inside the trapping phase-space region of the 3-0 mode without a visible increase in their transverse momentum.

Due to further damping of the primary mode 3-1, we observe the formation of almost one-dimensional BGK-wave. Since the amplitude of this nonlinear wave is comparable with that in the previous case, secondary instabilities of almost longitudinal sideband modes with \( n = 4 \) and \( n = 5 \) grow with the same growth-rates as in the isotropic plasma [Fig. 5(b)]. Nonlinear saturation of these modes, however, is not accompanied by the complete wave energy transfer from primary unstable modes to sideband modes. As one can see from Fig. 5 demonstrating snapshots of the phase space \((x, p_x)\) at the moments \( \omega_p t = 422 \) and \( \omega_p t = 688.3 \), beam particles move simultaneously in fields of both primary \( (n = 3) \) and secondary modes \( (n = 4) \). The beam spread in transverse momentum, increasing due to excitation of oblique sideband modes 4-1 and 4-2, is found to be much smaller than in the case of isotropic plasma.

Thus, in the weakly magnetized plasma the nonlinear stage of the beam-plasma instability starts with almost

B. Weak magnetic field \( \Omega = 0.5 \)

In the weak magnetic field, due to decreasing in growth-rates of oblique instabilities, the main role at the nonlinear stage of beam-plasma interaction is played by modes 3-0 and 3-1. The exponential growth of these modes is saturated by beam trapping. In the chosen local region, this nonlinear process can be considered as one-dimensional. Indeed, Fig. 5 shows that at the moments \(\omega_p t = 115.6\) and \(\omega_p t = 252.5\) beam particles mix inside the trapping phase-space region of the 3-0 mode without a visible increase in their transverse momentum.

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Thus, in the weakly magnetized plasma the nonlinear stage of the beam-plasma instability starts with almost
one-dimensional beam trapping. The resulting BGK-wave propagating parallel to the magnetic field drives sideband instabilities of various upper-hybrid modes. In contrast to the previous case, nonlinear interactions between primary and secondary modes do not result in the substantial energy transfer and dominant modes 3-0 and 4-0 have comparable amplitudes in the quasi-stationary state.

C. Strong magnetic field $\Omega = 2$

In the strong magnetic field the unstable spectrum narrows significantly. Beam trapping continues to be the main nonlinear process resulting in saturation of dominant unstable modes 3-0 and 3-1. The contribution of oblique mode becomes herewith more pronounced. In Fig. 6 demonstrating snapshots of the electric field map $E_x(x, y)$, the beam density map $n_b(x, y)$ and the beam phase space $(x, p_x)$, one can see that trapping of beam particles in different regions along $y$ starts at different moments of time. By the moment $\omega_p t = 187.1$, the nonlinear BGK-wave has not yet been formed, but primarily stable modes 3-2, 3-3 and 4-0 have already grown up to large amplitudes. All of these modes get into the Cherenkov resonance with beam particles, but their rapid intermittent growth does not allow to clarify unambiguously what excitation mechanism (Cherenkov or nonlinear) plays the main role.

Fig. 2(c) shows that, upon saturation of secondary unstable modes, the wave spectrum is dominated by 3-$m$ modes with the prevailing role played by the oblique mode 3-2. As to determine the branch of plasma oscillations, to which secondary modes with $m \geq 2$ belong, we should find real parts of their frequencies. This can be done by using the relationship between different field components $E_x, E_y$ and $B_z$:

$$\omega(k) = \frac{c}{B_z(k)} (k_x E_y(k) - k_y E_x(k)).$$

Substituting simulation results into this formula, we identify secondary modes as electrostatic whistler waves.

Dominance of plasma modes with $n = 3$ is also visible in the phase space of the beam. In Fig. 6 at the moment $\omega_p t = 252.5$, typical vortex structures arising from trapping of beam particles by different 3-$m$ modes are
observed clearly. Due to the difference in parallel phase velocity, trapping regions of these modes move relative to each other, but when bottoms of potential wells have the same $x$-coordinates, electric fields of excited modes are summed coherently and form the large potential well resulting in phase space mixing of beam particles from different trapping regions. From spatial structures of the electric field and the beam density at the moments $\omega_p t = 252.5$ and $\omega_p t = 715.9$ we see that the spatial region of coherent wave-wave interaction is localized in the transverse direction and moves along $y$. Intense phase-space mixing of beam wave particles inside this region makes the nonlinear interaction between plasma modes with the same parallel wavenumbers $k_{||}$ to be more efficient than interaction between modes with different $n$.

Thus, in strongly magnetized plasmas, nonlinear wave-wave interaction via shared trapped particles is marked by the prevailing energy transfer from linearly unstable modes to initially stable oblique whistlers with the same parallel wavenumber. In spite of excitation of oblique plasma waves, the beam spread in transverse momentum remains unchanged for the whole period of beam-plasma interaction.

V. DISCUSSION OF SIMULATION RESULTS

Detailed two-dimensional PIC simulations show that collective interaction of a cold low-density electron beam with an isotropic plasma evolves according to the following scenario. Prevailing growth of oblique linear instabilities is followed by trapping of beam particles in oblique directions and substantial spreading of the beam in transverse momentum. The linear growth-rate of the filamentation instability in our case is much smaller than growth-rates of oblique and two-stream instabilities, but from the certain time filamentary perturbations demonstrate rapid growth due to nonlinear interaction of primary unstable modes. Simultaneous buildup of two-stream, oblique and filamentation instabilities results in the situation when the beam appears to be divided into separate bunches localized spatially in both longitudinal and transverse directions. Current layers constructed from these bunches merge into one thus forming the nonlinear BGK-wave localized in $y$. This nonlinear equilibrium is found to be unstable against oscillations with frequencies, which in the reference frame of BGK-wave equal to the harmonics of the bounce frequency of trapped electrons. Growth-rates of secondary sideband modes propagating parallel to the beam direction are well predicted by the one-dimensional theory. Saturation of sideband instabilities is characterized by almost complete energy transfer from primary unstable modes to almost longitudinal Langmuir modes with shorter wavelengths. It results in the formation of the BGK-wave with $k_{||} > \omega_p / v_b$, which is followed by dissipation of its energy due to the modulation instability.

In the weak magnetic field, oblique instabilities cease to dominate in the linearly unstable spectrum and the nonlinear stage of beam-plasma interaction starts with almost one-dimensional beam trapping followed by the formation of one-dimensional BGK-wave. In contrast to the case of isotropic plasma, the beam does not acquire significant spread in transverse momentum at this stage. Although sideband instabilities of short-wavelength modes $4-m$ and $5-m$ grow with the same rates as in the isotropic plasma, saturation levels of these secondary upper-hybrid modes appear to be lower. In this case, the prevailing energy transfer at the saturation stage remains to be directed along $k_{||}$, but nonlinear interaction between primary and secondary modes becomes less efficient.

The strong magnetic field does not affect essentially the initial stage of beam trapping by linearly unstable modes with small propagation angles, but modifies substantially the spectrum of secondary modes arising due to nonlinear wave-wave interaction via the beam nonlinearity. In this case, the wave energy of primary unstable modes is transferred to oblique modes with the same parallel wavenumber. It means that the energy flow is directed predominantly along $k_{||}$.

![Figure 7](https://via.placeholder.com/150)

**Figure 7**: (Color online) Phase velocities of linearly stable (blue) and linearly unstable (red) modes in the beam-plasma system in terms of momentum of resonant electrons. Bold arrows indicate the prevailing direction of energy flow.

In order to understand why the direction of spectral
energy transfer at the nonlinear stage of beam-plasma interaction changes drastically with the increase of external magnetic field, let us consider some dispersion characteristics of plasma modes that get into the Cherenkov resonance with the trapped electron beam. Let us express phase velocities of these modes in terms of momentum of resonant electrons

\[ \frac{p_{\text{w}}}{m_e c} = \left( \frac{k_{\parallel} c}{\omega_k} \right)^2 - 1 \right)^{-1/2}. \]

Fig. 7 presents the dependence of this momentum on the transverse wavenumber for different branches of plasma waves and different magnetic fields. Squares show positions of modes that can be resolved in our periodic system and colors indicate amplitudes of these modes at the end of simulation run. The main difference between cases \( \Omega = 0.5 \) and \( \Omega = 2 \) is that in the weak magnetic field the unstable beam branch couples to the upper-hybrid branch whereas in the strong magnetic field it couples to the whistler branch. In the former case the phase velocity of unstable plasma modes with the same parallel wavenumber increases with the increase of \( k_{\perp} \), whereas in the latter case it decreases even after the transition from unstable beam modes to stable whistlers. Thus, the observed distribution of wave energy between different plasma modes can be explained, if nonlinear wave-wave interactions due to the beam nonlinearity are governed by the following rules: (i) interaction between modes is more efficient, if these modes have the same \( k_{\parallel} \); and (ii) wave energy is transferred to modes with lower phase velocities. It explains why, in the weak magnetic field, oblique unstable modes with large growth-rates, are saturated at rather low levels and why the mode 3-1 loses its energy at the later nonlinear stage. In the absence of slower modes with \( k_{\parallel} > \omega_p/v_b \), wave energy here is transferred to the modes with \( k_{\parallel} > \omega_p/v_b \) arising due to sideband instabilities. In the strong magnetic field, the efficient energy transfer from primary unstable modes 3-0 and 3-1 to other modes with the same \( k_{\parallel} \) and lower phase velocities becomes possible. That is why the spectrum of plasma oscillations in the relaxed beam-plasma system in this case is dominated by oblique whistler modes.

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