CHOW POINTS OF C-ORBITS

ANNETT PÜTTMANN

Abstract. We consider free algebraic actions of the additive group of complex numbers on a complex vector space \( X \) embedded in the complex projective space. We find an explicit formula for the map \( p \) that assigns to a generic point \( x \in X \) the Chow point of the closure of the orbit through \( x \). The properties Hausdorff quotient topology and proper action are equivalently characterized by the closure of the image of \( p \) in the closed Chow variety.

1. Introduction

In this article we consider free algebraic \( \mathbb{C} \)-actions, i.e., actions of the additive group of complex numbers, on \( \mathbb{C}^n = X \). Such an action is given by a locally nilpotent derivation \( \delta : \mathbb{C}[X] \to \mathbb{C}[X] \). We have \( t.f = \sum_{j=0}^{\infty} t^j \delta^j f \) for \( t \in \mathbb{C} \) and \( f \in \mathbb{C}[X] \), where \( \mathbb{C} \times \mathbb{C}[X] \to \mathbb{C}[X], (t, f) \mapsto t.f \), denotes the action of \( \mathbb{C} \) on \( \mathbb{C}[X] \).

Closures of \( \mathbb{C} \)-orbits in a compactification \( \mathbb{P}_n \mathbb{C} \) of \( X \) are projective curves in \( \mathbb{P}_n \mathbb{C} \).

In section 2, Theorem 1, we derive an explicit formula for the map \( p : U_0 \to C_{d,n} \) from the Zariski-open set \( U_0 \subset X \) of orbits of maximal degree \( d \) to the open Chow variety \( C_{d,n} \) of curves of degree \( d \) in \( \mathbb{P}_n \mathbb{C} \) that assigns to a point \( x \in X \) the Chow point of \( C.x \subset \mathbb{P}_n \mathbb{C} \).

Since the closed Chow variety \( \overline{C}_{d,n} \) is projective, there is a sequence of blow-ups \( \pi : \tilde{X} \to X \) and an regular map \( \tilde{p} : \tilde{X} \to \overline{C}_{d,n} \) that lifts \( p \).

We are interested in properties of the \( \mathbb{C} \)-action on \( \mathbb{C}^n \) that are relevant for geometric quotients. By definition, the quotient topology is Hausdorff if the orbits can be separated by continuous functions. Recall that the action of a Lie group \( G \) on a manifold \( X \) is called proper if the map \( G \times X \to X \times X, (g, x) \mapsto (x, g.x) \), is proper.

In section 3 we express properness and Hausdorff quotient topology using properties of the cycles in \( \tilde{p}(X) \). Corollary 3 states that the quotient topology is Hausdorff if and only if for all \( \tilde{x} \in \tilde{X} \) all irreducible components of \( \tilde{p}(\tilde{x}) \) except the closure of \( \mathbb{C}.\pi(\tilde{x}) \) are contained in infinity. In Corollary 4 we prove that the action is proper if and only if it has Hausdorff quotient topology and the multiplicity of the closure of \( \mathbb{C}.\pi(\tilde{x}) \) in \( \tilde{p}(\tilde{x}) \) is 1 for all \( \tilde{x} \in \tilde{X} \).

We exploit the fact that the analytic space associated the Chow scheme of curves in \( \mathbb{P}_n \mathbb{C} \) is the Barlet space of compact 1-cycles in \( \mathbb{P}_n \mathbb{C} \) [1]. This enables us to describe convergence of Chow points by the behavior of the corresponding cycles.

In section 4 we apply our results to free \( \mathbb{C} \)-actions of degree two that have been studied by other authors using different methods. So far our approach does not describe the geometric quotient in the case where the action is proper. But

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the map \( \tilde{p} \) gives complete information about non-separable orbits if the quotient topology is not Hausdorff.

The construction of the explicit form of the map \( p : U_0 \to C_{d,n} \) strongly depends on the fact that the acting group is \( \mathbb{C} \). Whereas it should be possible to generalize the results of section 3 to free actions of other groups on more general affine varieties than \( \mathbb{C}^n \).

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2. CHOW POINT OF A GENERIC ORBIT

Fixing a set of coordinates \( \{x_1, \ldots, x_n\} \subset \mathbb{C}[X] \) on \( X = \mathbb{C}^n \) defines an embedding \( \mathbb{C}^n \hookrightarrow \mathbb{P}_n \mathbb{C} \) by \( x \mapsto [1 : x_1(x) : \ldots : x_n(x)] \). Furthermore, every derivation \( \delta \) can be uniquely written as \( \delta = \sum_{j=1}^n \delta(x_j) \frac{\partial}{\partial x_j} \). The \( \mathbb{C} \)-action on \( X \) corresponding to a locally nilpotent derivation \( \delta \) on \( \mathbb{C}[X] \) is completely described by \( x_k(-t,x) = \sum_{j=0}^t \frac{\partial^j}{\partial t^j}(\delta^j x_k)(x) \).

For a locally nilpotent derivation \( \delta \) on \( \mathbb{C}[X] \) we define its degree to be the smallest number \( d \) such that \( \delta^{d+1}(x_j) = 0 \) for all \( j \). Of course, this notion of degree depends on the chosen coordinates. Let \( C_x := \mathbb{C}[x] \subset \mathbb{P}_n \mathbb{C} \) be the closure in \( \mathbb{P}_n \mathbb{C} \) of the orbit through the point \( x \in X \). Although all orbits of a free \( \mathbb{C} \)-action are isomorphic to \( \mathbb{C} \), their closures in \( \mathbb{P}_n \mathbb{C} \) may lead to non-isomorphic projective curves distinguishable by their degree as projective varieties.

**Lemma 1.** If \( \delta \) defines a free algebraic \( \mathbb{C} \)-action on \( \mathbb{C}^n \) of degree \( d \), then \( \deg C_x \leq d \) for all \( x \in \mathbb{C}^n \). Furthermore, \( \deg C_x = d \) iff \( x \in \cup_{j=1}^n \{ \delta^j(x_j) \neq 0 \} \).

**Proof.** For a general hyperplane \( H = \{ \sum_{j=0}^n a_j X_j = 0 \} \) in \( \mathbb{P}_n \mathbb{C} \) the intersection \( H \cap C_x \) is given by the solutions of \( a_0 + \sum_{k=1}^n a_k \sum_{j=0}^d \frac{\partial^j}{\partial t^j}(\delta^j x_k)(x) = 0 \) which is a polynomial in \( t \) of degree at most \( d \).

The set \( U_0 := \cup_{j=1}^n \{ \delta^j(x_j) \neq 0 \} \) is Zariski-open and \( \mathbb{C} \)-invariant, because \( \delta^{d+1} x_j = 0 \) for all \( j \) and \( \delta^d x_j \neq 0 \) for at least one \( j \). An orbit in \( U_0 \) is called generic orbit. Hence, given a free algebraic \( \mathbb{C} \)-action on \( \mathbb{C}^n \) of degree \( d \) and a compactification \( \mathbb{C}^n \subset \mathbb{P}_n \mathbb{C} \) the closures of generic orbits are projective curves of degree \( d \) in \( \mathbb{P}_n \mathbb{C} \). Projective curves of degree \( d \) in \( \mathbb{P}_n \mathbb{C} \) are parametrized by the open Chow variety \( C_{d,n} \). We will give an explicit description of the map \( p : U_0 \to C_{d,n} \) that assigns to a point \( x \in X \) the Chow point of \( C_x \).

We refer to [2] for an introduction to the Chow variety. The Chow variety \( C_{d,n} \) is a quasiprojective variety in \( \mathbb{P}(\Gamma(\mathbb{P}_n^2(\mathbb{C}), H^d)) \). Here, \( \Gamma(\mathbb{P}_n^2(\mathbb{C}), H^d) \) is the vector space of bihomogeneous polynomials of bidegree \((d,d)\) with respect to the two sets of \( n+1 \) variables \( \alpha_0, \ldots, \alpha_n \) and \( \beta_0, \ldots, \beta_n \). The Chow point \( P_C \in C_{d,n} \) of a curve \( C \) of degree \( d \) in \( \mathbb{P}_n \mathbb{C} \) is the polynomial defining the hypersurface \( H_C := \cup_{r=1}^n (\{ (\alpha, \beta) | x \in H_{\alpha} \cap H_{\beta} \} \subset \mathbb{P}_n^2(\mathbb{C}) \). Here, the hyperplane \( H_{\alpha} \subset \mathbb{P}_n \mathbb{C} \) assigned to a point \( \alpha \in \mathbb{P}_n \mathbb{C} \) is \( H_{\alpha} = \{ \sum_{j=0}^n \alpha_j X_j = 0 \} \).

We denote by \( \tau : \mathbb{P}_n^2(\mathbb{C}) \to \mathbb{P}_n^2(\mathbb{C}), (\alpha, \beta) \mapsto (\beta, \alpha) \), the permutation of the components of \( \mathbb{P}_n^2(\mathbb{C}) \). Note that \( \gamma_{j_1 j_2} := \beta_{j_1} \alpha_{j_2} - \beta_{j_2} \alpha_{j_1} \) are \( \tau \)-invariant as elements of \( \mathbb{P}(\Gamma(\mathbb{P}_n^2(\mathbb{C}), H^1)) \) for \( j_1, j_2 = 0, \ldots, n \). For \( k, l = 0, \ldots, d \) we set

\[
(1) \quad f_{lk}(x) := \sum_{j=0}^n \beta_j (\delta^j x_j)(x) \sum_{j=0}^n \alpha_j (\delta^j x_j)(x) - \sum_{j=0}^n \alpha_j (\delta^j x_j)(x) \sum_{j=0}^n \beta_j (\delta^j x_j)(x)
\]
Note that $f_{kl}(x) = \sum_{j=0}^n \gamma_{j_1,j_2} \delta^l(x_{j_1})(x) \delta^k(x_{j_2})(x)$. Since $\gamma_{j_1,j_2} = \tau.\gamma_{j_1,j_2} = -\gamma_{j_2,j_1}$, we have $f_{lk} = -f_{kl}$. Let

$$F_{ik}(x) := \sum_{r=0}^{\min d-l,k} \sum_{j=0}^{d-l-r-1} f_{l+r,k-r}(x) \prod_{j=0}^{r-1} (d-j) \prod_{j=0}^{k-j}(k-j),$$

and

$$P(x) := \det(F_{ik}(x))_{i\neq 0, k \neq d}.$$

**Lemma 2.** The polynomial $P$ is a $\tau$-invariant element of $\mathbb{P}(\Gamma(\mathbb{P}_n^d(\mathbb{C}), H^d))$.

**Proof.** By construction, $P$ is a homogeneous polynomial of degree $d$ in the terms $\gamma_{ik}$. It remains to show that $P$ is non-trivial, $P \neq 0$.

To see this, we examine the coefficients $c_{j0}$ of the monomials $\gamma_{d,j0}$. Among the polynomials $f_{ik}$ only $f_{0k}$ and $f_{0l}$ contribute to the coefficients $c_{j0}$, because $\delta^l(x_0) = 0$ for all $l > 0$. Thus, the coefficient of $\gamma_{d,j0}$ in $F_{lk}$ vanishes if $d-l < k$. Since the definition of $P$ involves a determinant, $c_{j0}$ equals the coefficient of $\gamma_{d,j0}$ in $F_{ik}$. By definition, $f^{(0)}_{dk} = (\sum_{j=1}^n \gamma_{d,j0} \delta^l x_j + \sum_{j=1}^n \gamma_{j1,j2} (\delta^l x_j) x_j)(x)$. Hence, $c_{j0} = (\delta^l x_j)^d(x)$. Since the action has degree $d$, we have $c_{j0} \neq 0$ for at least one $j$. □

**Lemma 3.** Let $f_{ik}$ be elements of a polynomial ring satisfying $f_{ik} = -f_{kl}$ for all $k, l = 0, \ldots, d$ and $t \in \mathbb{C}$. The equations $\sum_{k=0}^d f_{ik} \frac{x^k}{k!} = 0$ are satisfied for all $l = 0, \ldots, d$ if $\sum_{k=0}^{d-1} F_{ik} \frac{x^k}{k!} = 0$ for all $l = 1, \ldots, d$, where $F_{ik}$ is defined as in (2).

**Proof.** Consider the following recursion: $f_{ik}^{(0)} := f_{ik}$,

$$f_{ik}^{(s+1)} := \begin{cases} f_{ik}^{(s)}, & l \neq d-s-1 \\ f_{d-s,k-1}^{(s)} + f_{d-s-1,k} \prod_{j=0}^{s}(d-j), & l = d-s-1 \end{cases}$$

We will prove by induction on $s$ that $f_{ik}^{(s)} = 0$ for $l \geq d-s$ and

$$f_{ik}^{(s)} = \begin{cases} f_{ik}, & l < d-s \\ \sum_{r=0}^{\min d-l,k} f_{l+r,k-r} \prod_{j=0}^{d-l-r-1} (d-j) \prod_{j=0}^{r-1}(k-j), & l \geq d-s \end{cases}$$

If $s = 0$, then $f_{ik}^{(0)} = f_{dk} = \sum_{r=0}^{d-1} f_{d+0,k-0}$. In particular, $f_{dd}^{(0)} = 0$. Now,

$$f_{d-s-1,k}^{(s+1)} = k \sum_{r=0}^{\min s,k-1} f_{d-s+r,k-1-r} \prod_{j=0}^{r-1}(d-j) \prod_{j=0}^{s}(k-j) + f_{d-s-1,k} \prod_{j=0}^{s}(d-j)$$

$$= \sum_{r=0}^{\min s+1,k} f_{d-s-1+r,k-r} \prod_{j=0}^{s}(d-j) \prod_{j=0}^{r-1}(k-j) + f_{d-s-1,k} \prod_{j=0}^{s}(d-j)$$

Furthermore, $f_{d-s-1,d}^{(s+1)} = \sum_{r=0}^{s+1} f_{d-s-1+r,d-r} \prod_{j=0}^{s+1-r-1}(d-j) \prod_{j=0}^{r-1}(d-j) = 0$, since $f_{d-s-1,d-r} = -f_{d-r,d-(s+1-r)}$.

Multiplying the equation $\sum_{k=0}^{d} f_{ik}^{(s)} \frac{x^k}{k!} = 0$ by $t \neq 0$ gives an equivalent equation $\sum_{k=0}^{d} f_{d-s,k}^{(s+1)} \frac{x^k}{k!} = 0$, since $f_{d-s,d} = 0$. Consequently, the systems $\sum_{k=0}^{d} f_{ik}^{(s)} \frac{x^k}{k!} = 0$ for all $l = 0, \ldots, d$ are equivalent for all $s$. 
Note that \( F_{lk} = f_{lk}^{(d)} \). Recall that \( F_{ld} = 0 \) for all \( l \). Since \( f_{r,k-r} = -f_{k-r,r} \), we have for all \( k \)

\[
F_{0k} = \sum_{r=0}^{k} f_{r,k-r} \prod_{j=0}^{d-r-1} (d-j) \prod_{j=0}^{r-1} (k-j) = \sum_{r=0}^{k} f_{r,k-r} \frac{d!}{r!} \frac{k^r}{(k-r)!} = 0.
\]

\( \square \)

**Theorem 1.** Let \( \delta \) be a locally nilpotent derivation of \( \mathbb{C}[x_1, \ldots, x_n] \) that defines a free \( \mathbb{C} \)-action on \( \mathbb{C}^n \) of degree \( d \) and \( x \in \mathbb{C}^n \). If \( \mathbb{C} \) a generic orbit, then the Chow point \( p(x) \) of \( C_x \) is \( P(x) \) as defined by equation (3).

**Proof.** Except for one point, the point at infinity, the curve \( C_x \) is parametrized by the \( \mathbb{C} \)-action. Therefore, \( H_{C_x} = \{(\alpha, \beta) \mid \exists t \in \mathbb{C} : t.x \in H_\alpha \cap H_\beta\} \). To find the defining polynomial of \( H_{C_x} \), we can assume that if \( (\alpha, \beta) \in H_{C_x} \) then there is \( t \in \mathbb{C} \) such that \( \sum_{j=0}^{n} \alpha_j x_j(t.x) = 0 \) and \( \sum_{j=0}^{n} \beta_j x_j(-t.x) = 0 \). Since for all \( j \) \( t.x_j = \sum_{k=0}^{d} \frac{k^j}{k!} x_j \), we obtain the equations

\[
\begin{align*}
\sum_{k=0}^{d} \frac{k^j}{k!} A_k &= 0, \quad A_k := \sum_{j=0}^{n} \alpha_j \delta^k x_j(x) \quad (4) \\
\sum_{k=0}^{d} \frac{k^j}{k!} B_k &= 0, \quad B_k := \sum_{j=0}^{n} \beta_j \delta^k x_j(x) \quad (5)
\end{align*}
\]

The terms \( A_k \) and \( B_k \) are polynomials of bidegree \((1, 0)\) and \((0, 1)\), respectively. The combination of the equations (4) and (5) leads to a system of \( d + 1 \) equations

\[
\sum_{k=0}^{d} \frac{t^k}{k!} (B_l A_k - A_l B_k) = 0, \quad l = 0, \ldots, d. \quad (6)
\]

Note that \( f_{lk} = B_l A_k - A_l B_k \). By Lemma 3, the system (6) is equivalent to

\[
\sum_{k=0}^{d-1} \frac{t^k}{k!} F_{lk} = 0, \quad l = 1, \ldots, d
\]

Since \( t^0 = 1 \), it follows that \( P = \det(F_{lk})_{l \neq 0, k \neq d} = 0 \) for all \( (\alpha, \beta) \in H_{C_x} \). But \( P \) is a homogeneous polynomial of bidegree \((d, d)\) and \( \tau \)-invariant as an element of \( \mathbb{P}(\mathbb{P}_d^2(\mathbb{C}), H^d) \).

A local slice of an action of a Lie group \( G \) on a manifold \( X \) through a point \( x \in X \) is a locally closed submanifold \( x \in S \subset X \) such that \( G.S \) is open in \( X \) and every orbit through \( S \) intersects \( S \) in exactly one point. For a free action a local slice establishes a local trivialization of the action, i.e., \( G \times S \to G.S, (g, s) \to g.s \), is a diffeomorphism. The properness of a Lie group action implies the existence of local slices. We include the proof of a special case of this general fact.

**Lemma 4.** A free action of a Lie group \( G \) on a manifold \( M \) is proper if and only if there are local slices through every point of \( M \). A proper action of a Lie group \( G \) on a manifold \( M \) has Hausdorff quotient topology.

**Proof.** The action of a Lie group \( G \) on a manifold \( M \) is proper iff for any convergent sequence \( \{x_n\} \subset M \) and any sequence \( \{g_n\} \subset G \) the convergence of the sequence \( \{g_n x_n\} \subset M \) implies the existence of a convergent subsequence of \( \{g_n\} \subset G \).
If there is a local slice through every point in $M$, then $M$ can be covered by local trivializations of the action. Then a local trivialization around $g := \lim g_n x_n$ shows that $\{g_n\}$ is convergent.

Assume that $G$ acts properly on $M$. Let $x \in M$ be a point. Choose any locally closed submanifold $S'$ through $x$ that is transversal to the $G$-orbit $Gx$. Such a $S'$ can be shrunk to $S$ such that $G \times S \to G.S$, $(g, s) \mapsto g.s$, is a diffeomorphism. Otherwise two convergent sequences $\{s_n\} \to x$, $\{s'_n = g_n s_n\} \to x$ would exist. The properness would imply that $\{g_n\}$ is convergent to the identity $\epsilon$ contradicting the fact that $G \times S \to G.S$ is locally diffeomorphic at $(e, x)$.

If $\{x_n\}$ and $\{g_n x_n\}$ are convergent sequences in $M$, then the properness of the action implies that $g_n$ tends to an element $g \in G$. Then $g. \lim_{n \to \infty} x_n = \lim_{n \to \infty} g_n x_n$. \hfill $\Box$

**Lemma 5.** There are local slices through every $x \in U_0$.

**Proof.** There are local trivializations $\{\delta^d x_j \neq 0\} \to \mathbb{C} \times \{\delta^d x_j \neq 0, \delta^d-1 x_j = 0\}$ given by $x \mapsto (t(x), -t(x), x)$ with $t(x) = (\delta^d-1 x_j)(x)/(\delta^d x_j)(x)$, because

$$\delta^d-1 x_j(-t(x), x) = \sum_{k=0}^d \frac{(-t(x))^k}{k!} \delta^k \delta^d-1 x_j(x) = \delta^d-1 x_j(x) - t(x) \delta^d x_j(x) = 0.$$

\hfill $\Box$

**Corollary 1.** The restriction of the $\mathbb{C}$-action on $U_0$ is a proper action.

Note that $U_0 = X$ if $d = 1$. Hence, a free algebraic $\mathbb{C}$-action of degree one on $\mathbb{C}^n$ is proper.

### 3. Limit Cycles

The regular map $p : U_0 \to \mathcal{C}_{d,n}$, $x \mapsto P(x)$ defines a rational map $p : X \to \mathcal{C}_{d,n}$. Since the open Chow variety $\mathcal{C}_{d,n}$ is quasiprojective, there is a finite sequence of blow-ups $\pi : \tilde{X} = X_m \to X_{m-1} \to \ldots \to X_0 = X$ along $Y_j \subset X_j$ and a regular morphism $\tilde{p} : \tilde{X} \to \mathcal{C}_{d,n}$ into the projective Chow variety $\mathcal{C}_{d,n}$ such that $\tilde{p} = p \circ \pi$. A point in $\mathcal{C}_{d,n}$ is a cycle $C = \sum_{j=1}^n n_j C_j$ which consists of irreducible curves $C_j \subset \mathbb{P}_n \mathbb{C}$ with multiplicities $n_j \in \mathbb{N}$ such that $d = \sum_{j=1}^n n_j \deg C_j$.

For $\tilde{x} \in \tilde{X} \setminus \pi^{-1}(U_0)$ we call $\tilde{p}(\tilde{x}) \in \mathcal{C}_{d,n}$ a limit cycle. A cycle $\tilde{p}(\tilde{x})$ can be uniquely written as $\tilde{p}(\tilde{x}) = n_{\tilde{x}} C_{\tilde{x}} + Z_{\tilde{x}}$ with $n_{\tilde{x}} \in \mathbb{N}$, $C_{\tilde{x}} = \tilde{x} \pi(\tilde{x})$, and a cycle $Z_{\tilde{x}} \subset \mathbb{P}_n \mathbb{C}$ of degree $d - n_{\tilde{x}} \deg C_{\tilde{x}}$ that does not contain $C_{\tilde{x}}$. Note that $n_{\tilde{x}} = 1$ and $Z_{\tilde{x}} = 0$ if $\pi(\tilde{x}) \in U_0$.

In the sequel we use the fact that the analytic space associated to the closed Chow variety is contained in the Barlet space.

**Theorem 2.** If $Y$ is a projective variety, then the Barlet space of $Y$ coincides with the analytic space associated with the Chow scheme of $Y$.

**Proof.** See [1] and [4]. \hfill $\Box$

This allows us to discuss convergence of Chow points $\tilde{p}(x_n)$ in terms of the corresponding projective curves $C_{x_n}$.

**Lemma 6.** Let $\tilde{x} \in \tilde{X}$. For every open neighborhood $W \subset \mathbb{P}_n \mathbb{C}$ of $\tilde{p}(\tilde{x})$, there is an open neighborhood $U \subset \tilde{X}$ of $\tilde{x}$ such that $\tilde{p}(y) \in W$ for all $y \in U$. 

Proof. The family of compact 1-cycles parametrized by the points in the Barlet space is an analytic family of compact 1-cycles, see [1] and [4]. □

Corollary 2. For all \( \hat{x} \in \hat{X} \) the curve \( C_{\hat{x}} \) is a component of the limit cycle \( \hat{p}(\hat{x}) \), i.e., \( n_{\hat{x}} \geq 1 \).

Proof. Let \( \hat{x} \in \hat{X} \). We take a sequence \( \{\hat{x}_n\} \subset \pi^{-1}(U_0) \) that tends to \( \hat{x} \) in \( \hat{X} \). Now, \( \hat{p}(\hat{x}_n) \to \hat{p}(\hat{x}) \), since \( \hat{p} \) is continuous. But \( \hat{p}(\hat{x}_n) = p(\pi(\hat{x}_n)) \) and the sequence \( \{\pi(\hat{x}_n)\} \) tends to \( \pi(\hat{x}) \). By lemma 6, \( C_{\hat{x}} \) is a component of \( \hat{p}(\hat{x}) \), because \( t.\pi(\hat{x}_n) \to t.\pi(\hat{x}) \) for all \( t \in \CC \).

\( \square \)

Corollary 3. The \( \CC \)-action on \( X \) has Hausdorff quotient topology if and only if \( Z_{\hat{x}} \cap X = \emptyset \) for all \( \hat{x} \in \hat{X} \).

Proof. Two distinct orbits \( \CC.x \) and \( \CC.x' \) in \( X \) can not be separated by \( \CC \)-invariant open subsets if and only if for any pair of open neighborhoods \( U \) of \( x \) and \( U' \) of \( x' \) there exists a \( t \in \CC \) such that \( t.U \cap U' \neq \emptyset \). This means that there are convergent sequences \( \{\hat{x}_n\} \to \hat{x}, \{\hat{x}_n'\} \to \hat{x}' \) in \( \hat{X} \) and a sequence \( \{t_n\} \subset \CC \) such that \( \pi(\hat{x}_n') = t_n.\pi(\hat{x}_n), \pi(\hat{x}_n) \) tends to \( x \), and \( \pi(\hat{x}_n') \) tends to \( x' \). By lemma 6 both limit cycles, \( \hat{p}(\hat{x}) \) and \( \hat{p}(\hat{x}') \), contain the irreducible components \( C_{\hat{x}} \) and \( C_{\hat{x}'} \).

The condition \( Z_{\hat{x}} \cap X = \emptyset \) means that \( Z_{\hat{x}} \) contains only points at infinity, i.e., \( Z_{\hat{x}} \) is contained in the hyperplane \( \{X_0 = 0\} \subset \PP_n.\CC \). Hence, \( Z_{\hat{x}} \cap X = \emptyset \) iff \( \hat{p}(\hat{x}) = p(\pi(\hat{x}))^{n+r} \) where \( r \) is a polynomial in the set of variables \( \{\gamma_{lk} : l, k \neq 0\} \) with coefficients in \( \CC[X] \).

Lemma 7. Let \( \hat{x} \in \hat{X} \) and \( U \subset X \) be a neighborhood of \( \pi(\hat{x}) \) with coordinates \( \{\xi_1, \ldots, \xi_n\} \) satisfying \( U \cap C_{\hat{x}} = \{\xi_2 = \ldots = \xi_n = 0\} \) and \( \xi_j(\pi(\hat{x})) = 0 \) \( \forall j \).

If \( Z_{\hat{x}} \cap X = \emptyset \), then there exists a neighborhood \( U' \subset X \) of \( \hat{x} \) such that the restriction of \( \xi_1 \) to \( \hat{p}(\hat{y}) \cap U \) is an unramified covering of degree \( n_{\hat{x}} \) for all \( y \in U \).

Proof. The local coordinates define a scale, that is adapted to \( \hat{p}(\hat{x}) \), since \( U \subset X \) and \( X \cap \hat{p}(\hat{x}) = n_{\hat{x}}C_{\hat{x}} \) ([1], [4]). The resulting covering is unramified, because distinct \( \CC \)-orbits do not intersect. □

Corollary 4. The \( \CC \)-action on \( X \) is proper if and only if \( Z_{\hat{x}} \cap X = \emptyset \) and \( n_{\hat{x}} = 1 \) for all \( \hat{x} \in \hat{X} \).

Proof. If the \( \CC \)-action on \( X \) is proper, \( Z_{\hat{x}} \cap X = \emptyset \), because the quotient topology of the action is Hausdorff. Furthermore, there is a local slice \( S \) through \( \pi(\hat{x}) \). By construction we can assume that \( S \) is a locally closed complex submanifold of dimension \( n - 1 \). Choosing local coordinates \( \{\xi_2, \ldots, \xi_n\} \) on \( S \) around \( \pi(\hat{x}) \) and a coordinate \( \xi_1 \) on \( \CC \) the biholomorphic map \( \Phi : \CC \times S \to \CC . S, (t, s) \mapsto (t.s) \), gives local coordinates on a small neighborhood \( U \) of \( \pi(\hat{x}) \) with the properties required in Lemma 7. Since \( \Phi \) is a local trivialization of the action, it then follows that \( n_{\hat{x}} = 1 \).

Now, let us assume \( Z_{\hat{x}} \cap X = \emptyset \) and \( n_{\hat{x}} = 1 \). Choosing coordinates \( \{\xi_1, \ldots, \xi_n\} \) as described in Lemma 7 the locally compact complex submanifold \( \pi(U) \cap \{\xi_1 = 0\} \) is a local slice through \( \pi(\hat{x}) \).

\( \square \)

4. Examples

In this section we apply the explicit formula for the map \( p : \to \CC_{d,n} \) and the conditions derived in Corollary 3 and Corollary 4 to free \( \CC \)-actions of degree two.
These examples have been examined by different authors using other methods. An expansion of the determinant that defines $P$ in equation (3) using $f_{ik} = -f_{ki}$ yields $P = 2f_{10}f_{21} - f_{20}^2$ if $d = 2$.

**Example 1.** We consider the free $\mathbb{C}$-action on $\mathbb{C}^3$ that is defined by the derivation \( \delta = -x_1^2 \frac{\partial}{\partial x_2} + (1 - x_1 x_2) \frac{\partial}{\partial x_3} \). It is of degree two and $U_0 = \{ x_1 \neq 0 \}$, since $\delta^2 = x_1^3 \frac{\partial}{\partial x_3}$. We know that this action, which is induced by a free affine $\mathbb{C}^2$-action on $\mathbb{C}^4$, is proper and admits a global algebraic slice [3], [5]. Applying formula (1) we obtain $f_{10} = \sum_{j=0}^{3} ((1 - x_1 x_2) \gamma_{3j} - \gamma_{2j} x_1^2) x_j$, $f_{20} = \sum_{j=0}^{2} x_1^3 x_j \gamma_{3j}$ and $f_{21} = -x_1^5 \gamma_{32}$. Then by equation (3)

\[
p(x) = -x_1^5 (2 \gamma_{32} \sum_{j=0}^{3} ((1 - x_1 x_2) \gamma_{3j} - \gamma_{2j} x_1^2) x_j + x_1 (\sum_{j=0}^{2} x_j \gamma_{3j}))^2.
\]

The map $p$ extends to a regular map on $X$ with $p(\{ x_1 = 0 \}) = \gamma_{32} (\gamma_{30} x_0 + \gamma_{32} x_2)$. For $\tilde{x} \in X \setminus U_0$ the first factor corresponds the component $Z_{\tilde{x}}$, which lies at infinity, and the second factor represents the component $C_{\tilde{x}}$, i.e., $n_{\tilde{x}} = 1$.

**Example 2.** The derivation $\delta = (x_1 - x_2 x_3) \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_4} + (x_1 + 1) \frac{\partial}{\partial x_5}$ defines a free $\mathbb{C}$-action on $\mathbb{C}^5$. Since $\delta^2 = -x_2 (x_1 + 1) \frac{\partial}{\partial x_5}$, we have $U_0 = \{ x_2 x_1 + 1 \neq 0 \}$ and $d = 2$. We know that this action is proper and admits Zariski-open local trivializations [6]. Using the equations (3) and (1) we get

\[
p(x) = 2 ((x_1 + 1) \gamma_{35} + x_2 \gamma_{34}) \sum_{j=0}^{5} ((x_1 - x_2 x_3) \gamma_{3j} + x_2 \gamma_{4j} + (x_1 + 1) \gamma_{5j}) x_j + x_2(x_1 + 1) r
\]

with $r = (\sum_{j=0}^{5} \gamma_{3j} x_j)^2$. The map $p : U_0 \to \overline{C}_{2,5}$ extends to a regular map on $\{ x_2 \neq 0 \} \cup \{ x_1 + 1 \neq 0 \}$ and

\[
p|_{\{ x_1 + 1 = 0 \}} = -x_2 \gamma_{34} (\sum_{j=0}^{5} ((1 + x_2 x_5) \gamma_{3j} - x_2 \gamma_{4j}) x_j)
\]

\[
p|_{\{ x_2 = 0 \}} = (x_1 + 1) \gamma_{35} (\sum_{j=0}^{5} (x_1 \gamma_{3j} + (x_1 + 1) \gamma_{5j}) x_j)
\]

Hence, $Z_{\tilde{x}} \cap X = \emptyset$ and $n_{\tilde{x}} = 1$ for all $x \in \{ x_2 \neq 0 \} \cup \{ x_1 + 1 \neq 0 \}$. To find $\tilde{p}$ we have to construct the blow-up $\pi : \tilde{X} \to X$ of $X$ along $Y := \{ x_2 = x_1 + 1 = 0 \}$. There is a covering of $\tilde{X}$ by $U_1 = \{ \xi_2 \neq 0, x_1 + 1 = x_2 \gamma_{31} \}$ and $U_2 = \{ \xi_1 \neq 0, x_2 = (x_1 + 1) \gamma_{31} \}$. A direct calculation shows that the rational map $p : X \to \overline{C}_{2,5}$ lifts to a regular map $\tilde{p} : \tilde{X} \to \overline{C}_{2,5}$ given by

\[
\tilde{p}|_{U_1} = 2 (\xi_{35} + \xi_{34}) \sum_{j=0}^{5} ((x_1 - x_2 x_3) \gamma_{3j} + x_2 \gamma_{4j} + (x_1 + 1) \gamma_{5j}) x_j + x_2 r
\]

\[
\tilde{p}|_{U_2} = 2 (\gamma_{35} + \xi_{34}) \sum_{j=0}^{5} ((x_1 - x_2 x_3) \gamma_{3j} + x_2 \gamma_{4j} + (x_1 + 1) \gamma_{5j}) x_j + (x_1 + 1) r
\]

with $r = (\sum_{j=0}^{5} \gamma_{3j} x_j)^2$. Now,

\[
p|_{U_1 \cap \pi^{-1}(Y)} = (\xi_{35} + \gamma_{34}) \sum_{j \neq 2,3} \gamma_{3j} x_j, p|_{U_2 \cap \pi^{-1}(Y)} = (\xi_{34} + \gamma_{35}) \sum_{j \neq 2,3} \gamma_{3j} x_j
\]
where the second factors correspond to $C_2$ and the first factors to $Z_2$.

**Example 3.** The derivation $\delta = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + (x_3^2 - 2x_1x_3 - 1) \frac{\partial}{\partial x_4}$ defines a free $\mathbb{C}$-action on $\mathbb{C}^4$ of degree two with $\delta^2 = x_1 \frac{\partial}{\partial x_2}$ and $U_0 = \{x_1 \neq 0\}$. We know that this action has non-Hausdorff quotient topology [6]. Evaluating the map $\tilde{p}$ we can determine all inseparable orbits. As in the examples above we find the explicit form of $p : U_0 \to C_{2,4}$ using the equations (3) and (1):

$$p = 2(x_1 \gamma_{32} + (x_2^2 - 2x_1x_3 - 1)\gamma_{34})(\sum_{j=0}^{4}(x_2 \gamma_{3j} + (x_2^2 - 2x_1x_3 - 1)\gamma_{4j})x_j) - x_1r$$

with $r = (\sum_{j=0}^{4} \gamma_{3j} x_j)^2$. This map is well defined outside of $Y := \{x_1 = x_2^2 - 1 = 0\}$ and $p|_{\{x_1 = 0\}} = (x_2^2 - 1)\gamma_{34}(\sum_{j=0}^{4}(x_2 \gamma_{3j} + (x_2^2 - 1)\gamma_{4j})x_j)$

which means that $n_x = 1$ and $Z_x = (x_2^2 - 1)\gamma_{34}$ for $x \in \{x_1 = 0, x_2^2 \neq 1\}$. The blow-up $\pi : \tilde{X} \to X$ of $Y$ is covered by the charts $U_1^+ = \{\xi_1 \neq 0, x_2 + 1 = x_1\xi_2\}, U_2^+ = \{\xi_2 \neq 0, x_1 = (x_2 + 1)\xi_3\}, U_1^- = \{\xi_1 \neq 0, x_2 - 1 = x_1\xi_3\}$, and $U_2^- = \{\xi_2 \neq 0, x_1 = (x_2 - 1)\xi_3\}$. It can be checked by an explicit calculation that the rational map $p : X \to C_{2,4}$ lifts to regular map $\tilde{p} : \tilde{X} \to C_{2,4}$ and

$$\tilde{p}|_{U_1^+ \cap \pi^{-1}(Y)} = (\gamma_{30} - \gamma_{32} + x_4\gamma_{34})(\gamma_{30} + \gamma_{32} + (x_4 - 4x_3 - 4\xi)\gamma_{34}),$$

$$\tilde{p}|_{U_2^+ \cap \pi^{-1}(Y)} = (\gamma_{30} - \gamma_{32} + x_4\gamma_{34})(\gamma_{30} + \gamma_{32} + (\xi x_4 - 4)\gamma_{34}),$$

$$\tilde{p}|_{U_1^- \cap \pi^{-1}(Y)} = (\gamma_{30} + \gamma_{32} + x_4\gamma_{34})(\gamma_{30} - \gamma_{32} + (x_4 + 4x_3 - 4\xi)\gamma_{34}),$$

$$\tilde{p}|_{U_2^- \cap \pi^{-1}(Y)} = (\gamma_{30} + \gamma_{32} + x_4\gamma_{34})(\gamma_{30} - \gamma_{32} + (\xi x_4 + 4x_3 - 4)\gamma_{34}).$$

If $\tilde{x} \in \pi^{-1}(Y)$ and $\xi(\tilde{x}) \neq 0$, then $\tilde{p}(\tilde{x})$ contains two distinct $\mathbb{C}$-orbits. In particular, orbits in $\{x_1 = x_2 - 1 = 0\}$ can not be separated from orbits in $\{x_1 = x_2 + 1 = 0\}$.

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Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum

E-mail address: annett.puettmann@rub.de