THE COMPLEXITY OF POSSIBLE WINNERS ON PARTIAL CHAINS

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ABSTRACT

The POSSIBLE WINNER (PW) problem, a fundamental algorithmic problem in computational social choice, concerns elections where voters express only partial preferences between candidates. Via a sequence of investigations, a complete classification of the complexity of the PW problem was established for all pure positional scoring rules: the PW problem is in P for the plurality and veto rules, and NP-complete for all other such rules. More recently, the PW problem was studied on classes of restricted partial orders that arise in natural settings, such as partitioned partial orders and truncated partial orders; in particular, it was shown that there are rules for which the PW problem drops from NP-complete to P on such restricted partial orders. Here, we investigate the PW problem on partial chains, i.e., partial orders that are a total order on a subset of their domains. Such orders arise naturally in a variety of settings, including rankings of movies or restaurants. We classify the complexity of the PW problem on partial chains by establishing that, perhaps surprisingly, this restriction does not change the complexity of the problem, namely, the PW problem is NP-complete for all pure positional scoring rules other than the plurality and veto rules. As a byproduct, we obtain a new and more principled proof of the complexity of the PW problem on arbitrary partial orders.

1 Introduction

Determining the winners in an election under various voting rules has been a mainstream topic of research in computational social choice. Ideally, each voter has a clear ranking among the candidates, from the most preferred one to the least preferred one. In reality, however, a voter may have only limited information about the candidates, which translates to the voter providing only a partial order among the candidates that reflects the voter’s incomplete preferences (see the survey [1]). This state of affairs motivated [2] to introduce the notion of possible winners and necessary winners, where a candidate is a possible (necessary) winner if the candidate is a winner in at least one (respectively, in all) sets of linear orders that extend the set of partial orders provided by the voters.

There has been an extensive study of the complexity of the associated decision problems POSSIBLE WINNER (PW) and NECESSARY WINNER (NW) with respect to a variety of voting rules. Through a series of investigations [2, 3, 4, 5], the complexity of these problems has been classified for all pure positional scoring rules (see Section 2 for the precise definitions). Specifically, NW is in P w.r.t. every pure positional scoring rule (where P is the class of all decision problems solvable in polynomial time), while PW is in P w.r.t. the plurality rule and the veto rule, but it is NP-complete w.r.t. all other such rules.

More recently, the PW problem was studied on classes of restricted partial orders that arise in natural settings. For example, the preferences of the voters may be provided as top-truncated partial orders, that is, partial orders in which each voter linearly orders some top candidates, but expresses no preference on the rest. At the other end, we may have bottom-truncated partial orders, where each voter linearly orders some bottom candidates (i.e., “anybody but” candidates), but expresses no preference on the rest. We may also have doubly-truncated partial orders, where each voter linearly orders some top and bottom candidates, but expresses no preference for the ones in the middle. In [6, 7, 8], the complexity of the PW problem on such truncated partial orders was investigated. While no complete classification was obtained, it was shown that there are pure positional scoring rules, such as the 2-approval rule, for which the complexity of PW drops from NP-complete to P on doubly-truncated partial orders.
A partial order is *partitioned* if its elements can be partitioned into disjoint sets with a linear order between the disjoint sets, but no preference between elements in each set. In the machine learning community, such partial orders were shown to be common in many real-life datasets; furthermore, they have been used for learning statistical models on full and partial rankings [9][10][11]. Clearly, doubly-truncated partial orders are a special case of partitioned partial orders. In [12], the complexity of the PW problem on partitioned partial orders was investigated and a nearly complete classification was obtained for positional scoring rules. In particular, it was shown that, for all 2-valued rules (which contain 2-approval as a special case) and also for the rule with scoring vectors of the form \((2, 1, \ldots, 1, 0)\), the complexity of PW on partitioned partial orders drops from NP-complete to P.

**Summary of results** In this paper, we investigate the PW problem on *partial chains*, i.e., partial orders that are a total order on a subset of their domains. Such orders arise naturally in elections in which the number of candidates is large and, as a result, each voter can rank only a subset of the candidates. For example, consider the movies released in 2019. Most viewers have seen only a subset of these movies and so they can only rank the movies they have seen. A similar state of affairs holds for songs, books, restaurants, and so on. Partial chains are the most fitting model for this type of scenario. Indeed, it might be the case that a voter will like a movie they have not seen so far more than any of the the movies they have already seen (or less than any of the the movies they have already seen). This state of affairs can be modelled by partial chains, but not by partitioned, doubly-truncated, top-truncated, or bottom-truncated partial orders.

We obtain a complete classification of the complexity of the PW problem on partial chains by establishing that this restriction does not change the complexity of the problem, namely, PW is NP-complete for all pure positional scoring rules other than the plurality rule and the veto rule. This result should be contrasted with the aforementioned results about the drop in complexity of PW on doubly-truncated partial orders and on partitioned partial orders. Our result also yields a new, self-contained proof of the classification of the complexity of PW on arbitrary partial orders. Moreover, unlike the proof of the original classification theorem, our proof uses reductions from a single NP-complete problem, namely, the 3-DIMENSIONAL MATCHING problem.

Finally, we obtain new results about the complexity of the PW problem on doubly-truncated partial orders by establishing that this problem is NP-complete for a variety of pure positional scoring rules that were not covered by the earlier work on this problem. These rules include a broad group of both p-valued rules as well as unbounded rules.

## 2 Preliminaries and Earlier Work

### Voting profiles

A (strict) *partial order* on a set \(C\) is a binary relation \(\succ\) on \(C\) that is irreflexive (i.e., \(a \not\succ a\), for every \(a \in C\)) and transitive (i.e., \(a \succ b\) and \(b \succ c\) imply \(a \succ c\), for all \(a, b, c \in C\)). A *total order* on \(C\) is a partial order \(\succ\) on \(C\) such that for all \(a, b \in C\), we have \(a = b\) or \(a \succ b\) or \(b \succ a\).

Let \(C = \{c_1, \ldots, c_m\}\) be a set of *candidates* and let \(V = \{v_1, \ldots, v_n\}\) be a set of voters. A (complete) *voting profile* is a tuple \(T = (T_1, \ldots, T_n)\) of total orders on elements of \(C\), where each \(T_i\) represents the ranking (preference) of voter \(v_i\) on the candidates in \(C\). Similarly, a partial *voting profile* is a tuple \(P = (P_1, \ldots, P_n)\) of partial orders on \(C\), where each \(P_i\) represents the partial preferences of voter \(v_i\) on the candidates in \(C\). A *completion* of a partial voting profile \(P = (P_1, \ldots, P_n)\) is a complete voting profile \(T = (T_1, \ldots, T_n)\) such that each \(T_i\) is a completion of the partial order \(P_i\), i.e., \(T_i\) is a total order that extends \(P_i\). Note that a partial voting profile may have exponentially many completions.

### Voting rules

We focus on *positional scoring rules*, a widely studied class of voting rules. A positional scoring rule \(r\) on a set of \(m\) candidates is specified by a *scoring vector* \(s = (s_1, \ldots, s_m)\) of non-negative integers, called the *score values*, such that \(s_1 \geq s_2 \geq \ldots \geq s_m\) and \(s_1 > s_m\). Suppose that \(T = (T_1, \ldots, T_n)\) is a total voting profile. The score \(s(T, c)\) of a candidate \(c\) on \(T\) is the score value \(s_k\) where \(k\) is the position of candidate \(c\) in \(T\). The score of \(c\) under the positional scoring rule \(r\) on the total profile \(T\) is the sum \(\sum_{i=1}^n s_i(T_i, c)\). A candidate \(c\) is a *winner* if \(c\)'s score is greater than or equal to the scores of all other candidates; similarly, \(c\) is a *unique winner* if \(c\)'s score is greater than the scores of all other candidates. The set of all winners is denoted by \(W(r, T)\).

We consider positional scoring rules that are defined for every number \(m\) of candidates. Thus, a *positional scoring rule* is an infinite sequence \(s_1, s_2, \ldots, s_m, \ldots\) of scoring vectors such that each \(s_m\) is a scoring vector of length \(m\). Alternatively, a positional scoring rule is a function \(r\) that takes as argument a pair \((m, s)\) of positive integers with \(s \leq m\) and returns as value a non-negative integer \(r(m, s)\) such that \(r(m, 1) \geq r(m, 2) \geq \ldots \geq r(m, m)\). We assume that the function \(r\) is computable in time polynomial in \(m\), hence the winners can be computed in polynomial time. Such a rule is *pure* if the scoring vector \(s_{m+1}\) of length \((m + 1)\) is obtained from the scoring vector \(s_m\) of length \(m\) by inserting a score value in some position of \(s_m\), provided that the non-increasing order of score values is maintained. For every scoring rule \(s_m\), multiplying all score values by the same value, and adding the same constant to all score values does not change the winners; thus, we assume that the \(s_1, \ldots, s_m\) are co-prime and that there exists a \(k\) such that
Furthermore, the same classification holds for necessary unique winners and possible unique winners.

**Theorem 2.** Let $r$ be a voting rule and $P$ a partial voting profile. The following notions were introduced by Konczak and Lang [2].

- The set $\text{PW}(r, P)$ of the possible winners w.r.t. $r$ and $P$ is the union of the sets $W(r, T)$, where $T$ varies over all completions of $P$. Thus, a candidate $c$ is a possible winner w.r.t. $r$ and $P$, if $c$ is in the set $W(r, T)$ of winners, for at least one completion $T$ of $P$.

The **Possible Winner Problem (PW)** w.r.t. $r$ asks: given a set of candidates $C$, a partial profile $P$, and a distinguished candidate $c \in C$, is $c \in \text{PW}(r, P)$?

- The set $\text{NW}(r, P)$ of the necessary winners w.r.t. $r$ and $P$ is the intersection of the sets $W(r, T)$, where $T$ varies over all completions of $P$. Thus, a candidate $c$ is a necessary winner w.r.t. $r$ and $P$, if $c$ is in the set $W(r, T)$ of winners, for every completion $T$ of $P$.

The **Necessary Winner Problem (NW)** w.r.t. $r$ asks: given a set of candidates $C$, a partial profile $P$, and a distinguished candidate $c \in C$, is $c \in \text{NW}(r, P)$?

The notions of necessary unique winners and possible unique winners are defined in an analogous manner.

Through the initial investigation by [2] and subsequent investigations by [3], [4], and [5], the following classification of the complexity of the necessary and the possible winners for all pure positional scoring rules was established.

**Theorem 1.** [Classification Theorem] The following hold.

- For every pure positional scoring rule $r$, the necessary winner problem $\text{NW}$ w.r.t. $r$ is in P.

- The possible winner problem $\text{PW}$ w.r.t. the plurality rule and the veto rule is in P. For all other pure positional scoring rules $r$, this problem is $\text{NP}$-complete.

Furthermore, the same classification holds for necessary unique winners and possible unique winners.

The proof of the above classification is rather involved; also, it is not self-contained as it spans several papers. The proofs of NP-hardness for various positional scoring rules use reductions from several different known $\text{NP}$-complete problems, including $3$-Dimensional Matching, Exact $3$-Cover, Hitting Set, $3$-SAT, and Multicoloured Cliques.

### 3 Complexity of PW on Partial Chains

This section contains the main result of the paper. We begin by defining the concept of a partial chain.

**Definition 1.** A partial order on a set $C$ is a partial chain if it is a linear order on a non-empty subset $C'$ of $C$.

Let $C = \{a, b, c, d, e\}$ be a set of candidates. Clearly, every total order on $C$ is a partial chain. Two other examples of partial chains on $C$ are $a \succ d \succ c$ and $d \succ a \succ c \succ b$.

**Definition 2.** We write $\text{PW-PC}$ to denote the restriction of the PW problem to partial chains. More precisely, the PW-PC problem asks: given a set of candidates $C$, a partial profile $P$ in which every partial order $P_l$, $1 \leq l \leq n$, is a partial chain, and a distinguished candidate $c \in C$, is $c \in \text{PW}(r, P)$?

Since PW-PC is a special case of PW, Theorem 1 implies that if $r$ is the plurality rule or the veto rule, then the PW-PC problem with respect to $r$ is in P. The main result of this paper asserts that these are the only tractable cases.

**Theorem 2.** Let $r$ be a pure positional scoring rule other than the plurality and the veto rules. Then the PW-PC problem with respect to $r$ is $\text{NP}$-complete.

**Corollary 1.** [Classification Theorem for Partial Chains]

1. If $r$ is plurality rule or the veto rule, then the PW-PC problem with respect to $r$ is in P.

2. For all other pure positional scoring rules $r$, the PW-PC problem with respect to $r$ is $\text{NP}$-complete.
3.1 Proof outline of Theorem\textsuperscript{2}

NP-complete problem used As mentioned earlier, the NP-completeness of PW for rules other than plurality and veto in Theorem\textsuperscript{1} was established via reductions from a variety of well known NP-complete problems. Furthermore, none of these reductions used partial chains in the PW-instances constructed. Here, we will establish the NP-hardness of PW-PC for rules other than plurality and veto via reductions from a single well known NP-complete problem, namely, the 3-DIMENSIONAL MATCHING (3DM) Problem (Problem [SP1] in \cite{13}). This problem asks: given three disjoint sets $X = \{x_1, \ldots, x_q\}$, $Y = \{y_1, \ldots, y_h\}$, $Z = \{z_1, \ldots, z_q\}$ of the same size, and a set $S \subseteq X \times Y \times Z$, is there a subset $S' \subseteq S$ such that $|S'| = q$ and $S'$ does not contain two different triples that agree in at least one of their coordinates?

Grouping of Pure Positional Scoring Rules The NP-hardness of PW with respect to rules other than plurality and veto in Theorem\textsuperscript{1} was established by considering either groups of rules with similar characteristics \cite{3} or individual rules, e.g., the rule with scoring vectors of the form $(2, 1, \ldots, 1, 0, 0, 0)$ \cite{5}. Here, we will establish the NP-hardness of PW-PC with respect to rules other than plurality and veto by grouping the pure positional scoring rules into two different groups, namely, bounded rules and unbounded rules.

Definition 3. Let $r$ be a pure positional scoring rule.

- We say that $r$ is $p$-valued, where $p$ is a positive integer greater than 1, if there exists a positive integer $n_0$ such that for all $m \geq n_0$, the scoring vector $s_m$ of $r$ contains exactly $p$ distinct values.
- We say that $r$ is bounded if $r$ is $p$-valued, for some $p > 1$; otherwise, $r$ is unbounded.

Clearly, the plurality rule, the veto rule, and the $t$-approval rule with fixed $t \geq 2$, are 2-valued rules. For a different example of a 2-valued rule, consider the rule with scoring vectors $s_{2m} = (1, \ldots, 1, 0, \ldots, 0)$ and $s_{2m+1} = (1, \ldots, 1, 0, \ldots, 0)$, where $m \geq 1$. Furthermore, the rule with scoring vectors of the form $(2, 1, \ldots, 1, 0, 0)$ is 3-valued, while the Borda count $(m - 1, m - 2, \ldots, 0)$ is an unbounded rule. Note also that, unlike the Borda count, an unbounded scoring rule may have score values that are not decreasing at the same rate or may have arbitrarily long repeating score values.

Main Steps The technical cornerstones of the proof of Theorem\textsuperscript{2} are three polynomial-time reductions, each of which reduces the 3DM problem to the PW-PC problem with respect to the following types of pure positional scoring rules:

- 2-approval, which is then extended to all 2-valued rules other than plurality and veto.
- 3-valued rules, which is then extended to all $p$-valued rules with $p > 3$.
- unbounded scoring rules.

In each reduction, the partial profile we construct from an arbitrary 3DM instance consists of two parts. The first part is a set of partial chains (which are not total orders). These encode the given instance of the 3DM problem. It is worth pointing out that these partial chains have at most two candidates “missing”. The high-level idea of the construction is as follows. In order for candidate $c$ to win in some completion of the partial chains, some other candidates have to lose points. Suppose $c'$ is one such candidate. To lose points, $c'$ has to be in a higher position. Whenever $c'$ is in a higher position, a few other candidates are “pushed up” to lower positions, and they gain points. The score of these candidates are set in such a way that they can be “pushed up” only once. We set the specific scores for every candidate using the second part of the partial profile, which consists of a total profile. These votes, which fulfil certain properties, can be “pushed up” only once. We set the specific scores for every candidate using the second part of the partial profile, which consists of a total profile. These votes, which fulfil certain properties, can be constructed in time polynomial in the number of candidates due to a result similar to the one in \cite{14} Lemma 4.2. A variant of this result has been used in the literature \cite{15,16,17,18,12}. To make our work self-contained, we state and prove the following variant of the original result and use it in all our reductions from 3DM to PW-PC.

Lemma 1. Given a set $C = \{c_1, \ldots, c_m\}$ of candidates, a singleton $D = \{d\}$, a normalised scoring vector $s$ of length $m + 1$, and for every $c_i$ a list of integers $\eta_{i,1}, \ldots, \eta_{i,m}$ with $\sum_{j=1}^{m} |\eta_{i,j}| \leq O(m^4)$, one can construct, in time polynomial in $m$, a total voting profile $Q$ and a $\lambda_Q \in \mathbb{N}$ such that, for $1 \leq i \leq m$, the score $s(Q, c_i) = \lambda_Q + R_i$ where $R_i = \sum_{j=1}^{m} \eta_{i,j}(s_j - s_{j+1})$ and $s(Q, d) < \lambda_Q$. In particular, the number of votes in the profile $Q$ is polynomial in $m$. 

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Proof. Before proving the lemma, we introduce some notation which will be useful for the proof. Let, for $1 \leq j \leq m$, the value $\delta_j = s_j - s_{j+1}$. For $1 \leq j \leq m + 1$, let $\beta(d, j)$ be a block of $|C| = m$ votes where

- in each vote in the block, $d$ is in position $j$.
- in the $m$ votes, all the candidates besides $d$ are in each position exactly once.

For a given $j$, note that there are many ways to construct such a block $\beta(d, j)$. Given $j$, we fix a block $\beta(d, j)$. The following is an example of the block $\beta(d, 1)$.

\[
\begin{array}{cccccccc}
  d & > & c_1 & > & c_2 & > & \ldots & > & c_{m-1} & > & c_m \\
  d & > & c_2 & > & c_3 & > & \ldots & > & c_m & > & c_1 \\
  d & > & c_3 & > & c_4 & > & \ldots & > & c_1 & > & c_2 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  d & > & c_m & > & c_1 & > & \ldots & > & c_{m-2} & > & c_{m-1} \\
\end{array}
\]

When we talk about the score of a candidate in a block, we refer to the score of the candidate in a profile containing only the votes of that block. Observe that no matter how a block is constructed, the score of all the candidates is always the same. The score of the candidates in the block $\beta(d, j)$ are as follows.

- $s(\beta(d, j), d) = ms_j$
- For all $x \in C \setminus \{d\}$, we have $s(\beta(d, j), x) = -s_j + \sum_{k=1}^{m+1} s_k = \lambda_{\beta(d, j)}$

If candidate $d$ in position $j$ and candidate $c$ in position $j + 1$ are swapped, the score of $c$ increases by $\delta_j$, i.e., the score of $c$ is $\lambda_{\beta(d, j)} + \delta_j$. The score of $d$ decreases and the scores of all the candidates in $C \setminus \{d, c\}$ remain unchanged. We will use this idea construct the total profile $Q$.

Let $Q$ be an empty profile and $\lambda_Q = 0$. We will construct the total profile $Q$ incrementally. For each $c_i \in C$, for each $\eta_{i,j}$, where $1 \leq j \leq m$, and $\eta_{i,j} \neq 0$, we add votes to $Q$ in the following two steps.

1. This consists of two cases.

   **Case I.** $\eta_{i,j} > 0$
   We take the block $\beta(d, j)$.
   - $s(\beta(d, j), d) = ms_j$
   - For all $x \in C \setminus \{d\}$, we have $s(\beta(d, j), x) = -s_j + \sum_{k=1}^{m+1} s_k = \lambda_{\beta(d, j)}$

   Consider the vote where $c_i$ is in position $j + 1$. By construction, in every block, such a vote exists. Swap the positions of candidate $d$ and $c_i$. Let this block of votes be $\beta^+(d, j)$.
   - $s(\beta^+(d, j), d) = ms_j - s_j + s_{j+1} = ms_j - \delta_j$
   - $s(\beta^+(d, j), c_i) = \lambda_{\beta(d, j)} - s_{j+1} + s_j = \lambda_{\beta(d, j)} + \delta_j$
   - For all $x \in C \setminus \{c_i, d\}$, we have $s(\beta^+(d, j), x) = \lambda_{\beta(d, j)}$.

   We add $\eta_{i,j}$ copies of the votes in the block $\beta^+(d, j)$ to the profile $Q$. We add $\eta_{i,j} \lambda_{\beta(d, j)}$ to $\lambda_Q$.

   **Case II.** $\eta_{i,j} < 0$
   We take the block $\beta(d, j + 1)$.
   - $s(\beta(d, j + 1), d) = ms_{j+1}$
   - For all $x \in C \setminus \{d\}$, we have $s(\beta(d, j + 1), x) = -s_{j+1} + \sum_{k=1}^{m+1} s_k = \lambda_{\beta(d, j+1)}$

   Consider the vote where $c_i$ is in position $j$. By construction, in every block, such a vote exists. Swap the positions of candidate $d$ and $c_i$. Let this block of votes be $\beta^-(d, j)$.
   - $s(\beta^-(d, j + 1), d) = ms_{j+1} - s_j - s_{j+1} = ms_j + \delta_j$
   - $s(\beta^-(d, j + 1), c_i) = \lambda_{\beta(d,j+1)} + s_{j+1} - s_j = \lambda_{\beta(d,j+1)} - \delta_j$
   - For all $x \in C \setminus \{c_i, d\}$, we have $s(\beta^-(d, j + 1), x) = \lambda_{\beta(d,j+1)}$. 

   We add $\eta_{i,j}$ copies of the votes in the block $\beta^-(d, j)$ to the profile $Q$. We add $\eta_{i,j} \lambda_{\beta(d,j+1)}$ to $\lambda_Q$.
We first present the reduction of 3DM to PW-PC w.r.t. $S$. We will often need to define some arbitrary total order on a set of candidates with specific properties. For a set $m$ the total time required to construct these votes is also bounded above by a polynomial in $s$. The number of votes produced in this step is $\eta_{i,j} m$. The time to construct these votes is bounded by $O(\eta_{i,j} m)$.

2. Observe that the score of $d$ in the block of votes obtained from either of the above cases can be more than that of some $c_w \in C$. In particular, the difference between the scores of $d$ and $c_w$ is always strictly less than $\eta_{i,j} m s_1$. Consider the block $\beta(d, m + 1)$. The score of $d$ in this block is 0. The score of all the candidates in $C$, including $c_w$, in this block is $\lambda_{\beta(d, m + 1)} = \left( \sum_{k=1}^{m} s_k \right)$. To ensure that $c_w$ is never defeated by $d$, we add $\eta_{i,j} m$ copies of the votes in the block $\beta(d, m + 1)$ to the profile $Q$. We add $\eta_{i,j} m \lambda_{\beta(d, m + 1)}$ to $\lambda_Q$.

The number of votes produced in this step is $\eta_{i,j} m^2$. The time required to construct these votes is bounded by $O(\eta_{i,j} m^2)$.

Now we compute the upper-bound of the number of votes in $Q$ and time taken to construct the profile. For $1 \leq i \leq m$, to set the score of candidate $c_i$ we added $(m + m^2)$ $\sum_{j=1}^{m} \eta_{i,j}$ votes in $Q$, and thus a total of $(m + m^2) \sum_{i=1}^{m} \sum_{j=1}^{m} \eta_{i,j}$ votes.

Since for each $i$, we have $\sum_{j=1}^{m} \eta_{i,j} \leq O(m^2)$, the total number of votes in $Q$ is bounded above by a polynomial in $m$. The total time required to construct these votes is also bounded above by a polynomial in $m$.

Let $(X, Y, Z, \mathcal{S})$ be a 3DM instance where $\mathcal{S} = \{S_1, \ldots, S_t\} \subseteq X \times Y \times Z$ such that $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \leq i \leq t$. In all the reductions from 3DM to PW-PC, for each $c_i \in C$, the value $R_i$ will be of the form $R_i = \sum_{k=1}^{m} l_k \delta_k + \sum_{k=1}^{m+1} h_k s_k$ where $\sum_{k=1}^{m} l_k \leq O(m)$ and each $\sum_{k=1}^{m+1} h_k \leq t \leq O(m^3)$. Since, for $1 \leq k \leq m$, the score value $s_k = (\delta_k + \ldots + \delta_m)$, and $s_m = 0$, we have that $R_i = \sum_{k=1}^{m} l_k \delta_k + \sum_{k=1}^{m+1} h_k \left( \sum_{l=k}^{m} \delta_l \right)$. From this, it follows that $R_i = \sum_{j=1}^{m} \eta_{i,j} \delta_j$, where each $\eta_{i,j}$ is the sum of suitable $l_k$’s and $h_k$’s.

In the reductions, we call the candidates corresponding to the elements of the sets in 3DM, the element candidates. We will often need to define some arbitrary total order on a set of candidates with specific properties. For a set $S$, we denote an arbitrary total order on $S$ as $\vec{S}$. For $a, b \in S$, if we want $a$ to be in a higher position than $b$, i.e., $a$ has score less than or equal to $b$, in the total order, we simply state that $b \succ a$ in $\vec{S}$.

We start with the reduction from 3DM to the PW-PC problem with respect to 2-approval, and then show how to extend the reduction to the PW-PC problem with respect to an arbitrary 2-valued rule other than plurality and veto. This is an interesting case because, as mentioned in the Introduction, the PW problem with respect to 2-approval is in P, when restricted to partitioned partial orders [12] and to truncated partial orders [8].

3.2 Hardness of PW-PC w.r.t. 2-valued rules

We first present the reduction of 3DM to PW-PC w.r.t. 2-approval, and then prove its correctness.

**Reduction 1.** Let $(X, Y, Z, \mathcal{S})$ be a 3DM instance where $\mathcal{S} = \{S_1, \ldots, S_t\} \subseteq X \times Y \times Z$ such that $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \leq i \leq t$. We construct the partial profile $P$ as follows.

1. The set of candidates is $C = X \cup Y \cup Z \cup \{c, d_1, w\}$ where the sets $X, Y,$ and $Z$ comprise of candidates corresponding to the elements of the sets $X, Y, Z$ of the 3DM instance.

2. We construct the partial profile $P$ as follows.
   - For each $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C_i' = C \setminus \{x_{i_1}, y_{i_2}, z_{i_3}\} \cup \{d_1\}$ and $C_i''$ be such that $c \succ w$, i.e., $w$ is in a position higher than that of $c$.
     - $p'_i = x_{i_1} \succ y_{i_2} \succ z_{i_3} \succ d_1 \succ C_i''$
     - $p_i = x_{i_1} \succ y_{i_2} \succ C_i''$
   - $P = \bigcup_{i=1}^{t} p_i$ is a partial profile where each vote is a partial chain.
When we say that a candidate "gains" or "loses" points, it is in relation to the complete profile. Given a 3DM instance, we prove the following:

3. Consider $C = X \cup Y \cup Z \cup \{c, d_1\} \cup \{w\}$. Let $\{w\}$ be the set $D$ required in Lemma $\square$ and $R$ be as follows.

| $R_{xi}$ | $= 1 - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$. |
|---------|-------------------------------------------------|
| $R_{yi}$ | $= 1 - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$. |
| $R_{zi}$ | $= -1 - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$. |
| $R_{d_1}$ | $= -q - (s(P', d_1) - \lambda_{P'})$. |
| $R_c$   | $= 0$. |

4. By Lemma $\square$ there exist $\lambda_Q \in \mathbb{N}$ and a total profile $Q$ which can be constructed in time polynomial in $m$ such that the scores of the candidates in the profile $P' \cup Q$ are as follows. Let $\lambda_{P'} + \lambda_Q = \lambda$.

- For all $x \in X$, we have $s(P' \cup Q, x) = s(P', x) + s(Q, x)$
  $$= (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x) = \lambda + 1.$$ 

- For all $y \in Y$, we have $s(P' \cup Q, y) = s(P', y) + s(Q, y)$
  $$= (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y) = \lambda + 1.$$ 

- For all $z \in Z$, we have $s(P' \cup Q, z) = s(P', z) + s(Q, z)$
  $$= (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z) = \lambda - 1.$$ 

- $s(P' \cup Q, c) = s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda$.

- $s(P' \cup Q, d_1) = s(P', d_1) + s(Q, d_1)$
  $$= (\lambda_{P'} + s(P', d_1) - \lambda_{P'}) + (\lambda_Q + R_{d_1}) = \lambda - q.$$ 

- $s(P' \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda$.

5. We let $C$, the partial profile $V = P \cup Q$, and $c$ be the input to the PW-PC problem.

**Proposition 1.** Let $P$, $P'$, and $Q$ be the profiles as in the construction above. For all $P$ which extend $P$, we have $s(P \cup Q, c) = s(P', \cup Q, c) = \lambda$.

**Proof.** Recall that the scoring vector is $(1, 1, 0, \ldots, 0)$. By construction, in every completion of the partial chain $p_i$ in $P$, for $1 \leq i \leq m$, candidate $c$ is always in a position greater than two. Thus, for all total profiles $P$ which extend the partial profile $P$, the score of $c$ never changes. $\square$

**Lemma 2.** PW-PC w.r.t. 2-approval is NP-complete.

**Proof.** Given a 3DM instance $(X', Y', Z', \mathcal{A})$, we construct a PW-PC instance, $(C, V = P \cup Q, c)$, according to Reduction $\square$. Let $|C| = m$.

First, we prove the $\iff$ direction. Assume that the instance $(C, V = P \cup Q, c)$ of the PW-PC problem obtained from the reduction is a positive instance. Therefore, there exists a total profile $P^* = \bigcup_{i=1}^{q} p^*_i$ such that

- for all $1 \leq i \leq t$, we have $p^*_i$ extends $p_i$;
- $c$ is a possible winner and, by Proposition $\square$, has score $\lambda$.

When we say that a candidate "gains" or "loses" points, it is in relation to the complete profile $P'$ in the reduction.

1. For $1 \leq i \leq q$, each element candidate $x_i$ in $X'$, has to lose at least one point. Since a candidate can lose at most one point in any vote, let $p^*_i$ be the vote in which the element candidate $x_i$ loses a point, where $1 \leq i \leq q$. Let $K = \{k_i| 1 \leq i \leq q\}$.

2. Observe that in all the $q$ votes in $K$, both $d_1$ and an element candidate from $Z$ must be in the top two positions. Without loss of generality, assume that, in these $q$ votes, candidate $d_1$ is in the first position and the element candidate from $Z$ is in the second position.
3. Therefore, candidate \( d_1 \) gains a total of \( q \) points. Since each \( z \in Z \) can gain at most a point, the element candidate of \( Z \) in the second position in each of the above \( q \) votes must be distinct, i.e., no two votes in \( K \) have the same element candidate of \( Z \) in the second position.

4. By construction, candidates \( d_1 \) and \( z \) cannot gain any more points. Since \( c \) is a possible winner, it must be the case that each of the \( q \) element candidates in \( Y \) also lose at least a point each in the \( q \) votes in \( K \). Therefore, the element candidates of \( Y \) in the \( q \) votes in \( K \) must be distinct.

5. Therefore, the set \( \{S_i | i \in K \} \) must form a cover for \( \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z} \).

Now, we prove the other direction. Let \( (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{S}) \) be a positive instance of 3DM. Let \( \mathcal{S}' \subseteq \mathcal{S} \) be the cover. Recall that \( |\mathcal{S}'| = q \). We show that \( c \) is, indeed, a possible winner in the PW-PC instance constructed as above.

1. We extend each partial vote \( p_i \in P \) as follows.

\[
\begin{align*}
p_1^*: d_1 &\succ z_{i_3} \succ x_{i_1} \succ y_{i_2} \succ \overrightarrow{C_i} \text{ if } S_i \in \mathcal{S}' \\
p_2^*: x_{i_1} &\succ y_{i_2} \succ z_{i_3} \succ d_1 \succ \overrightarrow{C_i} \text{ if } S_i \notin \mathcal{S}'
\end{align*}
\]

Let \( P^* = \bigcup_{i=1}^q p_i^* \).

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
1 & 1 & 0 & 0 & \ldots & 0 \\
\hline
\hline
\end{tabular}
\caption{Completions for 2-approval.}
\end{table}

2. The following are the scores of the candidates in the profile \( P^* \cup Q \). Recall, that \( s(P^* \cup Q, c) = \lambda \).

- For all \( x \in X \), we have \( s(P^* \cup Q, x) = s(P^*, x) + s(Q, x) = s(P', x) - 1 + s(Q, x) \)
  \[= (\lambda_{P'} + s(P', x) - \lambda_{P'}) - 1 + (\lambda_Q + R_{x}) = \lambda.\]

- For all \( y \in Y \), we have \( s(P^* \cup Q, y) = s(P^*, y) + s(Q, y) = s(P', y) - 1 + s(Q, y) \)
  \[= (\lambda_{P'} + s(P', y) - \lambda_{P'}) - 1 + (\lambda_Q + R_{y}) = \lambda.\]

- For all \( z \in Z \), we have \( s(P^* \cup Q, z) = s(P^*, z) + s(Q, z) = s(P', z) + 1 + s(Q, z) \)
  \[= (\lambda_{P'} + s(P', z) - \lambda_{P'}) + 1 + (\lambda_Q + R_{z}) = \lambda.\]

\[s(P^* \cup Q, c) = s(P^*, c) + s(Q, c) = s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda.\]

\[s(P^* \cup Q, d_1) = s(P^*, d_1) + s(Q, d_1) = s(P', d_1) + q + s(Q, d_1)\]
\[= (\lambda_{P'} + s(P', d_1) - \lambda_{P'}) + q + (\lambda_Q + R_{d_1}) = \lambda.\]

\[s(P^* \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda.\]

Therefore, \( c \) is a possible winner.

Next, we generalise the construction to all 2-valued rules.

**Lemma 3.** If \( r \) is a 2-valued rule, then PW-PC w.r.t. \( r \) is NP-complete.

**Proof.** (Outline) The set \( C \) of candidates is the same as in the reduction for 2-approval. One can always construct the total profile \( P' = \bigcup_{i=1}^q p_i' \), such that in the vote \( p_i' \), corresponding to \( S_i \in \mathcal{S}' \), candidates \( x_{i_1} \) and \( y_{i_2} \) are in the two highest positions with score value one, and candidates \( z_{i_3} \) and \( d_1 \) are in the two lowest positions with score value zero. The score of each candidate in the profile \( P' \cup Q \) is set identical to that in the reduction for 2-approval. We drop \( d_1 \) and \( z_{i_3} \) in each \( p_i' \) to obtain the partial profile \( P \). More precisely, if the scoring vector is \((1, \ldots, 1, 0, 0, \ldots, 0)\), we do the following.
• For each $S_i$, we construct a total order $p_i'$ such that candidates $x_{i_1}$ and $y_{i_2}$ are in positions $k - 1$ and $k$ respectively, while candidates $z_{i_3}$ and $d$ are in positions $k + 1$ and $k + 2$.

$$p_i' = C_i^1 \succ x_{i_1} \succ y_{i_2} \succ z_{i_3} \succ d_1 \succ C_i^2,$$

where $C_i^1$ and $C_i^2$ are partitions of $C \setminus \{x_{i_1}, y_{i_2}, z_{i_3}, d_1\}$ such that $|C_i^1| = r - 2$ and $C_i^2 = C \setminus (C_i^1 \cup \{x_{i_1}, y_{i_2}, z_{i_3}, d_1\})$. Since $2 \leq r \leq (m - 2)$, the positions $(k - 1), (k), (k + 1), \text{ and } (k + 2)$ are always valid.

• Construct partial votes $p_i$ by dropping $d_1$ and $z_{i_3}$ from $p_i'$.

• For Proposition 1 to hold for all 2-valued scoring rules, one ensures that and $|C_i^1| = k - 2$ where $k$ is the number of times the largest score value is in the scoring vector.

• To apply Lemma 1, one ensures that in each $p_i'$ candidate $c$ is an position smaller than that of $w$.

• The relative scores of each candidate in the profile $P' \cup Q$ is set in a way similar to that in the reduction for 2-approval (Reduction 1), and thus the proof of $\iff$ direction is similar.

• For the $\implies$ direction, the partial chains in $P$ are completed as in Table 2. This makes the score of candidate $c$ greater than or equal to the score of all the other candidates.

\[
\begin{array}{cccccccc}
1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0 \\
\overrightarrow{C_i^1} & x_{i_1} & y_{i_2} & z_{i_3} & d_1 & \overrightarrow{C_i^2} & \text{if } S_i \in \mathcal{I}' \\
C_i^1 & d_1 & z_{i_3} & x_{i_1} & y_{i_2} & C_i^2 & \text{if } S_i \notin \mathcal{I}' \\
\end{array}
\]

Table 2: Completions for a 2-valued rules.

### 3.3 Hardness of PW-PC w.r.t. $p$-valued rules, where $p \geq 3$

In this section, we show NP-completeness of $p$-valued positional scoring rules, for $p \geq 3$. Consider a $p$-valued rule, where $p \geq 3$, which has a size $m$ scoring vector with the distinct values $a_1 > a_2 > \ldots > a_p$, we define, for $1 \leq j \leq p$, a function $\ell(m, j)$ which returns the number of times the score value $a_j$ repeats in the scoring vector. Schematically, a scoring vector of a $p$-valued rule, where $p \geq 3$, can be represented as follows.

$$\left( a_{1}, \ldots, a_{1}, a_{2}, \ldots, a_{2}, \ldots, a_{p}, \ldots, a_{p}, \ell(m,1), \ell(m,2), \ldots, \ell(m,p) \right)$$

The following proposition follows from the purity of the scoring rules considered in this paper.

**Proposition 2.** Let $r$ be a $p$-valued scoring rule. For all positive integers $\gamma$, there exists a length $m \leq \gamma p$ such that, in the scoring vector $s_m$, there exists $1 \leq u \leq p$ such that $\ell(m, u) = \gamma$.

We give the reduction first and then prove its correctness.

**Reduction 2.** Let $\mathcal{I} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{I})$ be a 3DM instance, with $|\mathcal{X}| = |\mathcal{Y}| = |\mathcal{Z}| = q$. Let $r$ be the $p$-valued scoring rule which has scoring vectors with blocks of repeating score values. More precisely, in the scoring vector of length $m$, the score value $a_j$ repeats $\ell(m, j)$ times, for $1 \leq j \leq p$. Let $\delta_j = a_j - a_{j+1}$, for $1 \leq j < p$. Let $\gamma = 3q$. By Proposition 2, there is a number $m \leq 3qp$ such that in the scoring vector $s_m$, there exists $1 \leq u \leq p$ such that the block of repeating score value $a_u$ has length $\ell(m', u) = 3q$. We consider the following three cases.

Case 1. $u = 1$

Case 2. $u = p$

Case 3. $1 < u < p$
We construct the PW-PC instance as follows.

1. The set of candidates is $C = X \cup Y \cup Z \cup \{c, w\} \cup H$ where $c$ denotes the distinguished candidate, the sets $X, Y,$ and $Z$ comprise of candidates corresponding to the elements of the sets $X, Y$ and $Z$. The set $H$ consists of dummy candidates such that $|H| = m - 3q + 2$.

2. We construct the partial profile $P$ as follows.

- For each $S_i = (x_{i1}, y_{i2}, z_{i3})$, let $C_i' = C \setminus \{(x_{i1}, y_{i2}, z_{i3}) \cup H\}$. Let $C_i$ be such that candidate $c$ is ranked lower than $w$, i.e., we have $c \succ w$.

   **Case 1.** $u = 1$. Let $H_1 \subseteq H$ such that $|H_1| = \ell(m, 2) - 1$ and $H' = H \setminus H_1$.
   
   $p'_i = C_i' \succ x_{i1} \succ y_{i2} \succ H_1 \succ z_{i3} \succ H'$
   
   $p_i = C_i \succ y_{i2} \succ H_1 \succ z_{i3} \succ H'$

   **Case 2.** $u = p$. Let $H_1 \subseteq H$ such that $|H_1| = \ell(m, p - 1) - 1$ and $H' = H \setminus H_1$.
   
   $p'_i = H_1 \succ x_{i1} \succ y_{i2} \succ H_1 \succ z_{i3} \succ C_i'$
   
   $p_i = H_1 \succ y_{i2} \succ H_1 \succ z_{i3} \succ C_i'$

   **Case 3.** $1 < u < p$. Let $H_1 \subseteq H$ such that $|H_1| = \sum_{i=1}^{u-1} \ell(m, i) + \ell(m, u - 1) - 1$ and $H' = H \setminus H_1$.
   
   $p'_i = H_1 \succ x_{i1} \succ y_{i2} \succ H_1 \succ z_{i3} \succ C_i'$
   
   $p_i = H_1 \succ y_{i2} \succ H_1 \succ z_{i3} \succ C_i'$

- $P = \bigcup_{i=1}^u p_i$ is a partial profile where each vote is a partial chain.

- $P' = \bigcup_{i=1}^u p'_i$ is a total profile. Moreover, each $p'_i$ extends $p_i$. Let $s(P', c) = \lambda_{P'}$. Since $w$ is placed at a position greater $c$ in all the votes of $P'$, we have $s(P', w) < \lambda_{P'}$.

3. Consider $C = X \cup Y \cup Z \cup \{c\} \cup H \cup \{w\}$. Let $\{w\}$ be the set $D$ required in Lemma[1] and $R$ be as follows.

   **Case 1.** $u = 1$
   
   - $R_{x_i} = \delta_1 + \delta_2 - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_{y_i} = -\delta_1 - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_{z_i} = -\delta_2 - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_e = 0$.
   - $R_h = 0 - (s(P', h) - \lambda_{P'})$, for all $h \in H$.

   **Case 2.** $u = p$
   
   - $R_{x_i} = \delta_{p-2} + \delta_{p-1} - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_{y_i} = -\delta_{p-2} - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_{z_i} = -\delta_{p-1} - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_e = 0$.
   - $R_h = 0 - (s(P', h) - \lambda_{P'})$, for all $h \in H$.

   **Case 3.** $1 < u < p$
   
   - $R_{x_i} = \delta_{u-1} + \delta_u - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_{y_i} = -\delta_{u-1} - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_{z_i} = -\delta_u - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
   - $R_e = 0$.
   - $R_h = 0 - (s(P', h) - \lambda_{P'})$, for all $h \in H$.

4. By Lemma[1] there exist a $\lambda_Q \in \mathbb{N}$ and a total profile $Q$ which can be constructed in time polynomial in $m'$ such that the scores of the candidates in the profile $P' \cup Q$ are as follows. Let $\lambda_{P'} + \lambda_Q = \lambda$. 

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Case 1.  \( u = 1 \)
- For all \( x \in X \), we have \( s(P' \cup Q, x) = s(P', x) + s(Q, x) \)
  \[
  = (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x) = \lambda + \delta_2 + \delta_1.
  \]
- For all \( y \in Y \), we have \( s(P' \cup Q, y) = s(P', y) + s(Q, y) \)
  \[
  = (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y) = \lambda - \delta_1.
  \]
- For all \( z \in Z \), we have \( s(P' \cup Q, z) = s(P', z) + s(Q, z) \)
  \[
  = (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z) = \lambda - \delta_2.
  \]

Case 2.  \( u = p \)
- For all \( x \in X \), we have \( s(P' \cup Q, x) = s(P', x) + s(Q, x) \)
  \[
  = (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x) = \lambda + \delta_{p-2} + \delta_{p-1}.
  \]
- For all \( y \in Y \), we have \( s(P' \cup Q, y) = s(P', y) + s(Q, y) \)
  \[
  = (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y) = \lambda - \delta_{p-2}.
  \]
- For all \( z \in Z \), we have \( s(P' \cup Q, z) = s(P', z) + s(Q, z) \)
  \[
  = (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z) = \lambda - \delta_{p-1}.
  \]

Case 3.
- For all \( x \in X \), we have \( s(P' \cup Q, x) = s(P', x) + s(Q, x) \)
  \[
  = (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_x) = \lambda + \delta_u + \delta_{u-1}.
  \]
- For all \( y \in Y \), we have \( s(P' \cup Q, y) = s(P', y) + s(Q, y) \)
  \[
  = (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_y) = \lambda - \delta_u - 1.
  \]
- For all \( z \in Z \), we have \( s(P' \cup Q, z) = s(P', z) + s(Q, z) \)
  \[
  = (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_z) = \lambda - \delta_u.
  \]

For all the cases, the score of candidate \( c \), the dummy candidates in \( H \), and \( w \) in the profile \( P' \cup Q \) are the same.
- \( s(P' \cup Q, c) = s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda \)
- For all \( h \in H \), we have \( s(P' \cup Q, h) = s(P', h) + s(Q, h) = (\lambda_{P'} + s(P', h) - \lambda_{P'}) + (\lambda_Q + R_h) = \lambda \)
- \( s(P' \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda \)

5. We let \( C \), the profile \( V = P \cup Q \), and \( c \) be the input to the PW-PC problem.

**Lemma 4.** Let \( r \) be a \( p \)-valued scoring rule, where \( p \geq 3 \). Reducation 2 is a polynomial time reduction of 3DM to PW-PC w.r.t. \( r \).

**Proof.** We prove the “\( \implies \)” direction first. Let \( (X, Y, Z, S) \) be a positive instance of 3DM. Let \( S' \subseteq S \) be the cover. Recall that \( |S'| = q \). We construct a PW-PC instance as in Reducation 2 and show that \( c \) is, indeed, a possible winner.

1. We extend each partial vote \( p_i \in P \) as follows.

   **Case 1.**  \( u = 1 \)
   - \( p_i^* = C_t^i \triangleright y_{i_2} \triangleright H_1^i \triangleright z_{i_3} \triangleright x_{i_1} \triangleright H^i \) if \( S_i \in S' \)
   - \( p_i^* = C_t^i \triangleright x_{i_1} \triangleright y_{i_2} \triangleright H_1^i \triangleright z_{i_3} \triangleright H^i \) if \( S_i \notin S' \)
When we say that a candidate "gains" or "loses" points, it is in relation to the complete profile $P$

Given a 3DM instance $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$, for $i = 1, \ldots, n$,

1. For each $i$, we compute the scores of all the candidates in the completed profile to verify that candidate $c$ is, indeed, a possible winner. We show the detailed computation for Case 1. The other two cases are similar.

**Case 1.** $u = 1$

The following are the scores of the candidates in the profile $P^* \cup Q$. Recall, that $s(P^* \cup Q, c) = \lambda$.

- For all $x \in X$, we have $s(P^* \cup Q, x) = s(P^*, x) + s(Q, x) = s(P', x) - (\delta_2 + \delta_1) + s(Q, x)$
- $= (\lambda_{P'} + s(P', x) - \lambda_{P'}) - (\delta_2 + \delta_1) + (\lambda_Q + R_x) = \lambda$.

- For all $y \in Y$, we have $s(P^* \cup Q, y) = s(P^*, y) + s(Q, y) = s(P', y) + \delta_{p-2} + s(Q, y)$
- $= (\lambda_{P'} + s(P', y) - \lambda_{P'}) + \delta_1 + (\lambda_Q + R_y) = \lambda$.

- For all $z \in Z$, we have $s(P^* \cup Q, z) = s(P^*, z) + s(Q, z) = s(P', z) + \delta_{p-1} + s(Q, z)$
- $= (\lambda_{P'} + s(P', z) - \lambda_{P'}) + \delta_2 + (\lambda_Q + R_z) = \lambda$.

- $s(P^* \cup Q, c) = s(P^*, c) + s(Q, c) = s(P', c) + \lambda_{P'} = \lambda$.

- For all $c' \in H$, we have $s(P^* \cup Q, c') = s(P^*, c') + s(Q, c') = s(P', c') + 0 + s(Q, c')$.

- $= (\lambda_{P'} + s(P', c') - \lambda_{P'}) + 0 + (\lambda_Q + R_{c'}) = \lambda$.

- $s(P^* \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_{Q} < \lambda$.

Therefore, $c$ is a possible winner.

In the other direction, we prove the correctness of the reduction for Case 1 in full detail. The other two cases are similar. Given a 3DM instance $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{S})$, we construct a PW-PC instance $(C, V = P \cup Q, c)$, according to the above reduction. Assume that the PW-PC instance $(C, V = P \cup Q, c)$ is a positive one. Therefore, there exists a total profile $P^* = \bigcup_{i=1}^t p_i^*$ such that

- for all $1 \leq i \leq t$, the vote $p_i^*$ extends $p_i$;
- $c$ is a possible winner. Moreover, no matter how the partial orders are completed, the score of $c$ is $\lambda$.

When we say that a candidate "gains" or "loses" points, it is in relation to the complete profile $P'$ in the reduction.

1. For $1 \leq i \leq q$, each element candidate $x_i$ in $X$, has to lose at least $(\delta_2 + \delta_1)$ points. Therefore, it has to be in a position greater than $\ell(m, 1)$ in at least one vote. Assume, for now, that each $x_i$ loses at least $(\delta_2 + \delta_1)$ points in one vote, i.e., it is in a position greater than or equal to $\ell(m, 1) + \ell(m, 2)$. Let these $q$ votes be $p_{k_1}, \ldots, p_{k_q}$ where $1 \leq k_i \leq t$ and $K = \{k_i | 1 \leq i \leq q\}$.

2. For each $i \in K$, in the completion $p_i^*$, the candidate $x_i$ loses the points (and, therefore, is in position greater than $\ell(m, 1) + \ell(m, 2)$), candidates $z_{i_1}$ and $y_{i_2}$ gain $\delta_1$ and $\delta_2$ points respectively.

3. By construction, each element candidate of $Y$ can gain at most $\delta_1$ points, and each element candidate of $Z$ can gain at most $\delta_2$ points. Moreover, there are no votes where these element candidates can lose points. Therefore, the element candidates of $Y$ and the element candidates of $Z$, which gain points in the $q$ votes in $K$ must be distinct.
4. We had assumed that each element candidates of $X$ loses at least $(\delta_2 + \delta_1)$ points in one vote. Observe that whenever $x \in X$ is in a position greater than $\ell(m',1)$, an element candidate of $Y$ gains the maximum points it can without defeating $c$, i.e., $\delta_1$ points. Since there are $q$ element candidates in $X$ and $q$ element candidates in $Y$, every time an element candidate of $Y$ gains $\delta_1$ points, an element candidate of $X$ must lose at least $(\delta_1 + \delta_2)$ points.

5. The remaining partial votes in $P$ ($p_i$ for $1 \leq i \leq t$ and $i \not\in K$), must have the same completion as in $P'$.

6. Therefore, the set $\{S_i|i \in K\}$ must form a cover for $X \cup Y \cup Z$. □

3.4 Hardness of PW-PC w.r.t. unbounded rules

In this section, we focus on unbounded rules. The Borda count is an example of such rules. As noted earlier, unbounded scoring rules may have score values which repeat in blocks. Moreover, unlike Borda count, the score values can be non-uniformly decreasing. Recall, that for a scoring vector of length $m$, with $m'$ distinct score values, the function $\ell(m,j)$ returns the number of times the distinct score value $a_j$ repeats in a block, for $1 \leq j \leq m'$. Schematically, such a scoring vector can be represented as

$$\left(\begin{array}{c}
(a_1, \ldots, a_{\ell(m,1)}, a_2, \ldots, a_{\ell(m,2)}, \ldots, a_{m'}, \ldots, a_{m'})
\end{array}\right).$$

Now, we prove a fundamental property of scoring vectors of all unbounded rules.

**Proposition 3.** Let $r$ be a positional scoring rule and let $\gamma$ and $\beta$ be two positive integers greater than 1. Consider the scoring vector $s_m$ of $r$ with length $m = \gamma \beta$. Then either $s_m$ contains at least $\beta$ distinct values or there exists $1 \leq u \leq \gamma \beta$ such that $\ell(\gamma \beta, u) \geq \gamma$.

**Proof.** If the scoring vector of length $\gamma \beta$ contains at least $\beta$ distinct values we are done. Assume it contains fewer than $\beta$ distinct values. But, by the monotonicity of the rules, if two score values are the same, then they must be in the same block. So, we must have a block in which the same score value repeats more than $\gamma$ times, else the total length would be less than $\gamma \beta$, i.e., there exists $u \leq \gamma \beta$ such that $\ell(\gamma \beta, u) \geq \gamma$. □

**Reduction 3.** Let $(X, Y, Z, \mathcal{S})$ be a 3DM instance where and $\mathcal{S} = \{S_1, \ldots, S_t\} \subseteq X \times Y \times Z$ such that $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, for $1 \leq i \leq t$. Let $s_m$ be the scoring vector of length $m = (3q + 4)(3q)$. By Proposition 3 we need to consider the following two cases.

- Case 1. There exists a $u$ such that $\ell(m, u) = 3q$.
- Case 2. There are $m' = 3q + 4$ distinct values.

For Case 1, the reduction mimics Reduction 2 to create a PW-PC instance. For Case 2, the reduction proceeds as follows.

Let $a_1 > a_2 > \ldots > a_{m'}$ be the $m'$ distinct values. We define $\delta = (\delta_1, \ldots, \delta_{m'-1})$ where, $\delta_j = a_j - a_{j+1}$, for $1 \leq j < m'$. We construct the following instance of the PW-PC problem.

1. The set of candidates is $C = X \cup Y \cup Z \cup \{c, g, d, w\} \cup H$ where $X$, $Y$, and $Z$ contains candidates corresponding to the elements in $X$, $Y$, and $Z$ respectively. These candidates are called elements candidates. The set $H$ contains dummy candidates such that $|H| = m - m'$.

2. We construct the partial profile $P$ as follows.

- Let the set $H$ be partitioned into $H_1, \ldots, H_{m'}$, such that $|H_j| = \ell(m, j) - 1$, for $1 \leq j \leq m'$. For each $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C'_i = C \setminus \left\{x_{i_1}, y_{i_2}, z_{i_3}\right\} \cup \left\{g, d\right\} \cup \bigcup_{j=m'+4}^{m'} H_j$ and $C''_i$ be such that the dummy candidates in $H_j$ are in a position with score value $a_j$, for $1 \leq j \leq m' - 3$ and candidate $c$ is
ranked lower than candidate $w$.

\[ p'_i = C_i = C \setminus g \supset H_{m'-4} \supset d \supset H_{m'-3} \supset x_i \supset H_{m'-2} \supset y_i \supset H_{m'-1} \supset z_i \supset H_{m'} \]

\[ p_i = C_i = C \setminus g \supset H_{m'-4} \supset d \supset H_{m'-3} \supset x_i \supset H_{m'-2} \supset y_i \supset H_{m'-1} \supset z_i \supset H_{m'} \]

- $P = \bigcup_{i=1}^t p_i$ is partial profile where each vote is a partial chain where only one candidate ($g$) has been dropped.
- $P' = \bigcup_{i=1}^t p'_i$ is a total profile. Moreover, each $p'_i$ extends $p_i$. Let $s(P', c) = \lambda_{P'}$. Observe that $s(P', w) < \lambda_{P'}$ since $w$ is in a position greater than $c$ in all $C_i$, for $1 \leq i \leq t$.

3. Consider $C = X \cup Y \cup Z \cup \{c, g, d\} \cup \{w\}$. Let $\{w\}$ be the set $D$ required in Lemma 1 and $R$ be as follows

- $R_{z_i} = -\delta_{m'-3} - (s(P', x_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{y_i} = -\delta_{m'-2} - (s(P', y_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_{z_i} = -\delta_{m'-1} - (s(P', z_i) - \lambda_{P'})$, for $1 \leq i \leq q$.
- $R_c = 0$.
- $R_q = q \left( \sum_{j=1}^{4} \delta_{m'-j} \right) - (s(P', g) - \lambda_{P'})$.
- $R_d = -q(\delta_{m'-3}) - (s(P', d) - \lambda_{P'})$.
- $R_h = 0 - s(P', h) - \lambda_{P'}$, for all $h \in H$.

4. By Lemma 1, there exist a $\lambda_Q \in \mathbb{N}$ and a total profile $Q$ which can be constructed in time polynomial in $m'$ such that the scores of the candidates in the profile $P' \cup Q$ are as follows. Let $\lambda_{P'} + \lambda_Q = \lambda$.

- For all $x \in X$, we have $s(P' \cup Q, x) = s(P', x) + s(Q, x)$
  \[ = (\lambda_{P'} + s(P', x) - \lambda_{P'}) + (\lambda_Q + R_{z_i}) = \lambda - \delta_{m'-3}. \]
- For all $y \in Y$, we have $s(P' \cup Q, y) = s(P', y) + s(Q, y)$
  \[ = (\lambda_{P'} + s(P', y) - \lambda_{P'}) + (\lambda_Q + R_{y_i}) = \lambda - \delta_{m'-2}. \]
- For all $z \in Z$, we have $s(P' \cup Q, z) = s(P', z) + s(Q, z)$
  \[ = (\lambda_{P'} + s(P', z) - \lambda_{P'}) + (\lambda_Q + R_{z_i}) = \lambda - \delta_{m'-1}. \]
- $s(P' \cup Q, c) = s(P', c) + s(Q, c) = \lambda_{P'} + \lambda_Q = \lambda$.
- $s(P' \cup Q, g) = s(P', g) + s(Q, g) = (\lambda_{P'} + s(P', g) - \lambda_{P'}) + (\lambda_Q + R_g) = \lambda + q \left( \sum_{j=1}^{4} \delta_{m'-j} \right)$.
- $s(P' \cup Q, d) = s(P', d) + s(Q, d) = (\lambda_{P'} + s(P', d) - \lambda_{P'}) + (\lambda_Q + R_d) = \lambda - q(\delta_{m'-4})$.
- For all $h \in H$, we have
  - $s(P' \cup Q, h) = s(P', h) + s(Q, h) = (\lambda_{P'} + s(P', h) - \lambda_{P'}) + (\lambda_Q + R_{h_i}) = \lambda$.
  - $s(P' \cup Q, w) = s(P', w) + s(Q, w) < \lambda_{P'} + \lambda_Q < \lambda$.

5. We let $C$, the profile $V = P \cup Q$, and $c$ be the input to the PW-PC problem.

The following propositions follow quite naturally from the construction of the partial profile in the above reduction (Reduction 3).

Proposition 4. Let $P$, $P'$, and $Q$ be the profiles as constructed in Reduction 3. For all total profiles $P$ that extend $P$, we have $s(P \cup Q, c) \leq s(P' \cup Q, c) \leq \lambda$.

Proposition 5. Let $P$, $P'$, and $Q$ be the profiles as constructed in Reduction 3. For all total profiles $P$ that extend $P$, if $c$ is a possible winner in $P \cup Q$ then $s(P \cup Q, c) = s(P \cup Q, c) = \lambda$.

Note that the converse of the above proposition is not true.

Lemma 5. Let $r$ be an unbounded scoring rule. Reduction 3 is a polynomial time reduction of 3DM to PW-PC w.r.t. $r$.
Proof. Given a 3DM instance $I = (X, Y, Z, \mathcal{S})$, we construct a PW instance, $(C, V = P \cup Q, c)$, according to Reduction 3. Note that there are two cases in the reduction. In both the cases, the PW instance is polynomial in $|Z|$. For Case 1, correctness follows from Lemma 4. Here, we consider Case 2.

First, we prove the ‘$\Leftarrow$’ direction. Assume that the PW-PC instance $(C, V = P \cup Q, c)$ obtained above is a positive one. Therefore, there exists a total profile $P^* = \bigcup_{i=1}^t p_i^*$ such that

- for all $1 \leq i \leq t$, the vote $p_i^*$ extends $p_i$;
- $c$ is a possible winner and, by Proposition 5 has score $\lambda$.

In the following, when one says that a candidate "gains" or "loses" points, it is in relation to the complete profile $P'$ in the reduction.

1. Candidate $g$ must lose at least $q \sum_{j=1}^{m'} \delta_{m'-j}$ points for $c$ to be a possible winner. Therefore, it must be in a position greater than $\sum_{j=1}^{m'-4} \ell(m, j)$ at least $q$ times.

2. Whenever $g$ is in a position greater than $m - \sum_{j=m'-3}^{m'} \ell(m, j)$, candidate $d$ gains $\delta_{m'-4}$. Since $d$ cannot gain more than $q(\delta_{m'-4})$ points, there are at most $q$ votes where $g$ is in position greater than $m - \sum_{j=m'-3}^{m'} \ell(m, j)$.
Let these votes be $p_{i_1}^*, \ldots, p_{i_q}^*$ where each $1 \leq k_j \leq t$ and $K = \{k_j | 1 \leq j \leq q\}$.

3. Note that candidate $g$ has to lose at least $q(\sum_{j=1}^{m'} \delta_{m'-j})$ points in these $q$ votes. This is possible if and only if it is in position greater than $\sum_{j=1}^{m'-4} \ell(m, j)$. Furthermore, whenever $g$ is in position greater than $\sum_{j=1}^{m'-4} \ell(m, j)$ in a vote $p_i^*$, candidate $x_{i_1}$ gains $\delta_{m'-3}$ points, candidate $y_{i_2}$ gains $\delta_{m'-2}$ points, and candidate $z_{i_3}$ gains $\delta_{m'-1}$ points, for $i \in K$.

4. Since $|X| = |Y| = |Z| = q$, and each $x \in X$ can gain at most $\delta_{m'-3}$ points, each $y \in Y$ can gain at most $\delta_{m'-2}$ points, and each $z \in Z$ can gain at most $\delta_{m'-1}$ points, it must be the case that the element candidates of $Y$ and $Z$ which gained points in the $q$ votes in $K$ are distinct.

5. Since no other candidate can gain any more points, the remaining partial votes in $P$ ($p_i$ for $1 \leq i \leq t$ and $i \notin K$), must have the same completion as in $P'$.

6. Therefore, the set $\{S_i | i \in K\}$ must form a cover for $X \cup Y \cup Z$.

For the ‘$\Rightarrow$’ direction, let $(X, Y, Z, \mathcal{S})$, is a positive instance. We show that in the PW instance, $(C, V = P \cup Q, c)$, constructed in the reduction, $c$ is, indeed, a possible winner. By hypothesis, there is a $\mathcal{S}' \subseteq \mathcal{S}$ and $|\mathcal{S}'| = q$.

1. Complete each vote $p_i \in P$ to $p_i^*$.

- $p_i^* = C_i \succ \overline{H}_{m'-4} \succ d \succ \overline{H}_{m'-3} \succ x_{i_1} \succ \overline{H}_{m'-2} \succ y_{i_2} \succ \overline{H}_{m'-1} \succ z_{i_3} \succ g \succ \overline{H}_{m'}$ if $S_i \in \mathcal{S}'$
- $p_i^* = C_i \succ g \succ \overline{H}_{m'-4} \succ d \succ \overline{H}_{m'-3} \succ x_{i_1} \succ \overline{H}_{m'-2} \succ y_{i_2} \succ \overline{H}_{m'-1} \succ z_{i_3} \succ \overline{H}_{m'}$ if $S_i \notin \mathcal{S}'$

Let $P^* = \bigcup_{i=1}^t p_i^*$. Note that the score of candidate $c$ does not change in these votes and is, therefore, $\lambda$.

2. We compute the scores of each candidate in $P^* \cup Q$.

- For all $x \in X$, we have $s(P^* \cup Q, x) = s(P^*, x) + s(Q, x) = s(P', x) + \delta_{m'-3} + s(Q, x)$
  $= \lambda_{P'} + s(P', x) - s(P, x) = \lambda_{P'} + s(P', x) - \lambda_{P'} + \delta_{m'-2} + \lambda_{Q} + R_x = \lambda$.

- For all $y \in Y$, we have $s(P^* \cup Q, y) = s(P^*, y) + s(Q, y) = s(P', y) + \delta_{m'-2} + s(Q, y)$
  $= \lambda_{P'} + s(P', y) - \lambda_{P'} + \delta_{m'-2} + \lambda_{Q} + R_y = \lambda$.
Theorem 3. The result in [12, Theorem 5] along with [15, Lemma 6] provide the following (incomplete) classification of the PW-PP problem. Before summarising the main results of that paper, we introduce the concept of a differentiating rule and notation for a family of rules.

Partitioned preferences [12] and truncated preferences [8] are two restricted types of partial orders that have received attention in the literature.

Definition 4. Let $\succ$ be a partial order on a set $C$.

- We say that $\succ$ is a partitioned preference if $C$ can be partitioned into disjoint subsets $A_1, \ldots, A_q$ such that:
  1. For all $i < j \leq q$, if $c \in A_i$ and $c' \in A_j$ then $c \succ c'$;
  2. For each $i < q$, the elements in $A_i$ are incomparable under $\succ$ (i.e., $a \not\succ b$ and $b \not\succ a$, for every $a, b \in A_i$).

- We say that $\succ$ is a doubly-truncated ballot if there is a permutation $\pi$ over $\{1, \ldots, |C|\}$ and natural numbers $l, b$ such that $\succ$ is of the form $c_{\pi(1)} \succ \ldots \succ c_{\pi(l)} \succ \{c_{\pi(t+1)}, \ldots, c_{\pi(m-b)}\} \succ c_{\pi(m-b+1)} \succ \ldots \succ c_{\pi(m)}$.

A doubly-truncated ballot is called top-truncated if $b = 0$; it is called bottom-truncated if $t = 0$.

Note that doubly-truncated ballots are a special case of partitioned preferences; thus, so are top-truncated ballots and bottom-truncated ballots.

We write PW-PP to denote the only restriction of the PW problem to partial profiles consisting of partitioned preferences. Similarly, we write PW-DTB, PW-TTB, and PW-BTB for the restriction of the PW problem to partial profiles consisting of, respectively, doubly-truncated, top-truncated, and bottom-truncated preferences.

The PW-PP problem has been studied in [12]. Before summarising the main results of that paper, we introduce the concept of a differentiating rule and notation for a family of rules.

Definition 5. A scoring rule $r$ is differentiating if there exists a $n_0 \in \mathbb{N}_0$ such that for all $m > n_0$, the scoring vector $s_m$ contains two positions $i$ and $j$, where $1 \leq i < j < m$, such that $(s_i - s_{i+1}) > (s_j - s_{j+1})$. We say $r$ is non-differentiating if it is not differentiating.

Let $f$ and $l$ be two positive integers ($f$ for “first” and $l$ for “last”). We write $R(f, l)$ to denote the 3-valued rule with scoring vectors $s_m = (2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$. Note that $R(1, 1)$ is the rule $(2, 1, \ldots, 1, 0)$ encountered earlier.

The result in [12] Theorem 5] along with [15] Lemma 6] provide the following (incomplete) classification of the PW-PP problem.

Theorem 3. [12] [15] Let $r$ be a pure positional scoring rule. Then the following statements hold.

- If $r$ is a 2-valued or if $r$ is the rule $R(1, 1)$, then the PW-PP problem is in $P$.
- If $r$ is a differentiating rule, then the PW-PP problem is in NP-complete.
- If $r$ is a non-differentiating $p$-valued rule, where $p \geq 3$, other than $R(f, l)$ with $f + l > 2$, then the PW-PP problem w.r.t. $r$ is NP-complete.
We need the notion of maximum partial score where the distinguished candidate is fixed. Before presenting the notion formally, we present an example. Consider a partial profile constructed in the NP-hardness proof for all differentiating rules in [15, Lemma 6] and for p-valued rules with \( p \geq 4 \) in [12, Lemma 13] has only doubly-truncated ballots. Therefore, the PW-DTB problem w.r.t. unbounded rules with scoring vectors containing at least four distinct score values is obtained as a corollary to [12, Lemma 14]. The proof of this lemma implicitly uses doubly-truncated profiles. This is a generalisation of an earlier result in [6, 7], which established that the PW-DTB problem is NP-complete for Borda count. Thus, the existing results for the PW-PP problem can be summarised as follows.

**Theorem 4.** [6, 7, 8, 15, 12] The following are true.

- If \( r \) is a 2-valued rule or \( r \) is the rule \( R(1, 1) \), then the PW-DTB problem w.r.t. \( r \) is in P.
- If \( r \) is a 3-valued rule other than \( R(f, l) \) with \( f + l > 2 \), or \( r \) is a \( p \)-valued rule, where \( p \geq 4 \), or \( r \) is an unbounded rule with scoring vectors containing at least four distinct score values, then the PW-DTB problem w.r.t. \( r \) is NP-complete.

Therefore, the complexity of the PW-DTB problem w.r.t. unbounded rules with scoring vector containing three distinct score values remains open. Since PW-TTB and PW-BTB are special cases of PW-DTB, the P results in Theorem 3 and in Theorem 4 also hold for PW-TTB and PW-BTB problems. The complexity of these two problems w.r.t. \( p \)-valued rules, where \( p \geq 3 \), other than the rule \( R(1, 1) \), and unbounded rules remain open. We settle the complexity of PW-TTB and PW-DTB w.r.t. a broad group of unbounded rules (such that all scoring vectors containing at least three distinct values and satisfies some additional properties) and a restricted group of 3 valued rules. This implies the NP-completeness of PW-DTB problem w.r.t. the same group of unbounded rules.

### 4.1 Maximum partial score

Before presenting the reductions, we introduce a few general notions which will help us reason about partial preferences beyond partial chains, namely doubly-truncated and partitioned preferences.

**Definition 6.** Let \( C \) be a set of candidates and \( P \) be a partial profile.

- We say that a candidate \( c' \in C \) is fixed in a partial vote \( p \in P \) if \( c \) has the same position in all extensions \( p^* \) of \( p_i \).
- We say that a candidate \( c' \in C \) is fixed in the partial profile \( P \) if \( c' \) is fixed in every vote in \( P \), i.e., for every \( p_i \in P \), there is an integer \( b_i \) such that \( c \) has position \( b_i \) in every extension \( p_i^* \) of \( p_i \). (Note that, in general, \( b_i \) depends on \( p_i \).)
- We say that a position \( b \) in a partial vote is available if there is no fixed candidate in \( b \), i.e., there exists no candidate \( c' \in C \) such that \( c' \) is in position \( b \) in all extensions \( p^* \) of \( p \).
- Let \( fixed_P(c') \) be the total score made by \( c' \in C \) from those votes in \( P \) where \( c' \) is fixed.

Let \( P \) be a doubly-truncated profile. A candidate is fixed in \( P \) if for every \( p_i \in P \), candidate \( c \) is in the top or in the bottom.

We need the notion of maximum partial score, introduced in [4], to reason about the completions of doubly-truncated profiles. Since we consider only doubly-truncated profiles in this section, our discussion will be focused on these kinds of votes. However, we note that the notion of maximum partial score can be used for any partial profile where the distinguished candidate is fixed. Before presenting the notion formally, we present an example. Consider a doubly-truncated profile which contains the following partial votes.

\[
\begin{align*}
  p_1 &: e_3 \succ c \succ e_1 \succ \{e_2, e_4, e_7\} \succ e_6 \\
  p_2 &: e_1 \succ \{e_2, e_4, e_7, e_3\} \succ e_6 \succ c.
\end{align*}
\]
The candidate $c$ is fixed in both $p_1$ and $p_2$. Therefore, $c$ is fixed in the profile. Candidate $e_3$ is fixed in $p_1$ but not in $p_2$. Let $P$ be a partial profile where candidate $c$ is fixed. Let $Q$ be a total profile. Let $P'$ be an extension of $P$. The score of $c$ in $P' \cup Q$, namely, $s(P' \cup Q, c) = \lambda$. For a candidate $c' \neq c$, the maximum partial score of $c'$, denoted $s^{\text{max}}(c', P, Q)$, is defined as $s^{\text{max}}(c', P, Q) = \lambda - s(c', Q)$. Intuitively, there exists no completion $P'$ of $P$ such that $c$ is a possible winner in $P' \cup Q$ and $c'$ has a score more than $s^{\text{max}}(c', P, Q)$ in $P'$.

Since, in a doubly-truncated vote, all the candidates which are not in the top or in the bottom are not ordered, the concept of maximum partial score makes it convenient to reason about the completions of such votes. Furthermore, it helps us define the tightness property [4], which will be extremely helpful in proving the correctness of our reductions.

**Definition 7.** Let $C$ be a set of candidates, $c \in C$ be a distinguished candidate, $P$ be a partial profile such that $c$ is fixed in $P$, and $Q$ be a total profile. We say that $P \cup Q$ has the tightness property if the sum of the score values of all the available positions in all the partial votes in $P$ is equal to the quantity $\sum_{c' \in C \setminus \{c\}} s^{\text{max}}(c', P, Q) - \text{fixed}(P')$.

The following proposition is quite obvious.

**Proposition 6.** Let $C$ be a set of candidates, $c \in C$ be a distinguished candidate, $P$ be a partial profile in which $c$ is fixed, and $Q$ be a total profile. Let $P'$ be an extension of $P$ such that $c$ is a winner in $P' \cup Q$. If $P \cup Q$ has the tightness property, then $s(P' \cup Q, c') = s^{\text{max}}(c', P, Q)$ for every candidate $c' \in C \setminus \{c\}$.

**Proof.** Otherwise, there exists another candidate $c'' \in C \setminus \{c\}$ which makes more than $s^{\text{max}}(c'', P, Q)$ and, thus, defeats $c$. \qed

The reductions in this section require construction of partial profiles where candidates have certain pre-specified maximum partial scores. There is a polynomial time algorithm [4] Lemma 1 to construct votes to realise the maximum partial scores of the candidates. We restate this fact as follows.

**Lemma 6.** [4] Given a scoring rule $r$ with the scoring vector $(s_1, \ldots, s_m)$, a set $C$ of $m$ candidates with distinguished candidate $c \in C$, a value $\mu(c')$, for all $c' \in C \setminus \{c\}$, and a partial profile $P$ where the following properties hold.

1. Candidate $c$ is fixed in $P$.
2. For every $c' \in C \setminus \{c\}$, the value $\mu(c')$ can be written as a sum of at most $|P|$ integers from $s_1, \ldots, s_m$.
   Formally, $\mu(c') = \sum_{j=1}^{m} n_j s_j$ where $n_j \in \mathbb{N}_0$ denotes how often the score value $s_j$ is added. Moreover, $\sum_{j=1}^{m} n_j \leq |P|$.
3. There is a dummy candidate $w$, such that $w$ cannot beat the distinguished $c$ in any extension.

Then, a set $Q$ of total votes can be constructed in time polynomial in $|P|$ and $m$, such that for all $c' \in C \setminus \{c\}$, we have $s^{\text{max}}(c', P, Q) = \mu(c')$.

We conclude with a proposition on a linear combination of two distinct numbers which will help us prove correctness of the reductions.

**Proposition 7.** Let $a_1$ and $a_2$ be two distinct numbers. Let $n_1$ and $n_2$ be two natural numbers and $S = n_1 a_1 + n_2 a_2$. There exists no $n_1 \neq n_3$ and $n_2 \neq n_4$, such that $n_1 + n_2 = n_3 + n_4$ and $n_3a_1 + n_4a_2 = S$.

**Proof.** Suppose there exists $n_3 \neq n_1$ and $n_4 \neq n_2$ such that $n_1 + n_2 = n_3 + n_4$ and $n_3a_1 + n_4a_2 = S$.

By hypothesis, $n_1 = n_3 + n_4 - n_2$. Therefore,

$$n_1 a_1 + n_2 a_2 = n_3 a_1 + n_4 a_2$$

$$\implies (n_3 + n_4 - n_2)a_1 + n_2 a_2 = n_3 a_1 + n_4 a_2$$

$$\implies n_2 a_2 - n_2 a_1 = n_4 a_2 - n_4 a_1$$

$$\implies n_2(a_2 - a_1) = n_4(a_2 - a_1)$$

Since $a_1 \neq a_2$, it must be the case that $n_2 = n_4$ which is a contradiction. \qed
4.2 Hardness results for PW-TTB

Recall, that for a scoring vector of length \( m \), having \( m' \) distinct score values, the function \( \ell(m, j) \) returns the number of times the distinct score value \( a_j \), for \( 1 \leq j \leq m' \), repeats in a block. Schematically, such a scoring vector can be represented as

\[
\begin{pmatrix}
(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_{m'}, \ldots, a_{m'})
\end{pmatrix}
\]

\( \ell(m, 1) \), \( \ell(m, 2) \), \( \ell(m, m') \)

In the reduction below, we consider an unbounded rule \( r \) with the following properties. There exists a polynomial \( g(u) \) with the property that for all \( u \), every scoring vector \( s_m \) of \( r \) with length \( m = g(u) \) has \( m' \geq 3 \) distinct score values, and if the three smallest score values are \( a_{m' - 2} > a_{m' - 1} > a_{m'} \), it holds that \( m = \ell(m, m' - 1) - \ell(m, m') \geq 3u \). Borda count is an obvious example. The lexicographic scoring rule given by \((2^m, 2^{m-1}, \ldots, 1)\) for \( m \) candidates is another example of the scoring rules considered in this section.

**Reduction 4.** Let \( (X, Y, Z, S) \) be a 3DM instance where and \( S = \{S_1, \ldots, S_t\} \subseteq X \times Y \times Z \) such that \( S_i = (x_{i1}, y_{i1}, z_{i1}) \), for \( 1 \leq i \leq t \) and \( [Z] \). Let \( s_m \) be the scoring vector of length \( m = g(q) \). The scoring vector has at least three distinct values, i.e., \( m' \geq 3 \) there are values \( a_{m' - 2}, a_{m' - 1}, \) and \( a_{m'} \) such that each of the lengths \( \ell(m, m' - 1) \), and \( \ell(m, m') \) are fixed and at least one. We construct an instance of the PW-TTB problem as follows.

1. The set of candidates is \( C = X \cup Y \cup Z \cup \{c, w\} \cup H \) where \( X, Y, \) and \( Z \) contains candidates corresponding to the elements in \( X, Y, \) and \( Z \) respectively. These candidates are called *element candidates*. The set \( H \) contains dummy candidates such that \( |H| = m - 3q - 2 \).

2. We construct the partial profile \( P \) as follows.

   - Let the set \( H \) be partitioned into \( H', H_{m-1}', \) and \( H_m' \) such that \( |H_j| = \ell(m, j) - 1 \), for \( j = \{m' - 1, m'\} \).
   - \( H' = H \setminus (H_{m-1}' \cup H_m') \).
   - For each \( S_i = (x_{i1}, y_{i1}, z_{i1}) \), let \( C'_i = C \setminus \{\{x_{i1}, y_{i1}, z_{i1}\} \cup H_{m-1}' \cup H_m'\} \) and \( \overline{C}'_i \) be such that \( c \succ w \), i.e., candidate \( c \) is always ranked lower than \( w \).
   - \( p_i = \overline{C}'_i \succ \{x_{i1}, y_{i1}, z_{i1}\} \cup H_{m-1}' \cup H_m' \)

   - \( P = \bigcup_{i=1}^{t} p_i \) is profile where each vote is top-truncated.

   In the vote \( p_i \), note that all the candidates except \( \{x_{i1}, y_{i1}, z_{i1}\} \cup H_{m-1}' \cup H_m' \) are fixed. In other words, positions \( |\overline{C}'_i| + 1 \) though \( m \) are available.

   \( P' = \bigcup_{i=1}^{t} p'_i \) is a total profile. Moreover, each \( p'_i \) extends \( p_i \). Let \( s(P', c) = \lambda_{P'} \). Observe that \( s(P', w) < \lambda_{P'} \) since \( w \) is in a position greater \( c \) in all \( \overline{C}'_i \), for \( 1 \leq i \leq t \).

3. For an element candidate \( e \in X \cup Y \cup Z \), let \( f_e \) denote the number of triples in \( S \) containing the element of the 3DM instance corresponding to candidate \( e \). By construction of the partial profile, candidate \( e \) is not fixed in \( f_e \) votes. Let \( \text{fixed}_P(e') \) be the total score made by \( e' \in C \) from those votes in \( P \) where \( e' \) is fixed.

   Therefore, for any candidate \( e \in X \cup Y \cup Z \), we have \( \text{fixed}_P(e) = \sum_{i=1}^{t-f_e} s_{k_i} \), where \( 1 \leq k_i \leq m \) is the position of \( e \) in a vote where it is fixed.

4. Consider the following.

   - For all \( x \in X \), we have \( \mu(x) = a_{m'} + (f_x - 1)a_{m' - 2} + \text{fixed}_P(x) \).
   - For all \( y \in Y \), we have \( \mu(y) = a_{m' - 2} + (f_y - 1)a_{m' - 1} + \text{fixed}_P(y) \).
   - For all \( z \in Z \), we have \( \mu(z) = a_{m' - 1} + (f_z - 1)a_{m'} + \text{fixed}_P(z) \).
   - For all \( h \in H_j \), we have \( \mu(h) = t(a_j) \) where \( j = \{m' - 1, m'\} \).
   - For all \( h' \in H' \), we have \( \mu(h') = \text{fixed}_P(h') \).
   - \( \mu(w) \geq ta_1 \).
5. We verify that the profile $P$, and, for all $c' \in C \setminus \{c\}$, the number $\mu(c')$, as specified above, satisfy the properties required by Lemma [6]

- Property 1: By the construction of the votes in the reduction, this property is satisfied.
- Property 2: For all $c \in X \cup Y \cup Z$, the number $\mu(c)$ is the sum of $(t - f_c) + (f_c - 1) + 1 = t$ score values. For all $h \in H$, property 2 is satisfied trivially.
- Property 3: Candidate $w$ is fixed in $P$, and in every vote, has a position greater than that of $c$, and therefore, can never defeat $c$ in any extension.

Therefore, by the lemma, there is a total profile $Q$, which can be constructed in time polynomial in $|P|$ and $m$, such that $s^{\max}(c', P, Q) = \mu(c')$, for all $c' \in C \setminus \{c\}$.

6. We let $C$, the profile $V = P \cup Q$, and $c$ be the input to the PW-TTB problem.

**Proposition 8.** The profile $P \cup Q$ in Reduction 4 has the tightness property.

**Proof.** Recall that $|X| = |Y| = |Z| = q$. Therefore,

$$\sum_{x \in X} f_x = t \implies \sum_{x \in X} (f_x - 1) = \sum_{x \in X} f_x - \sum_{x \in X} 1 = t - q.$$ 

Similarly, $\sum_{y \in Y} f_y = \sum_{z \in Z} f_z = t$ and $\sum_{y \in Y} (f_y - 1) = \sum_{z \in Z} (f_z - 1) = t - q$.

We focus only on the available positions in the votes of $P$ and the scores the candidates can make in these positions. Thus, for a candidate $c \in X, Y,$ and $Z$ we consider the score $s^{\max}(c, P, Q) - \text{fixed}(c)$. All candidates $h' \in H'$ and the candidate $w$ are fixed in $P$ and therefore, contributes nothing to the sum. Whereas, all candidates $h \in H_{m'-1} \cup H_{m'}$ are not fixed in any vote in $P$, i.e. $\text{fixed}(P(h)) = 0$. Therefore, the sum of the maximum scores which the candidates can make in the available positions is

$$\sum_{c' \in (X \cup Y \cup Z \cup H_{m'-1} \cup H_{m'})} (s^{\max}(c', P, Q) - \text{fixed}(c'))$$

$$= \sum_{x \in X} (s^{\max}(x, P, Q) - \text{fixed}(x)) + \sum_{y \in Y} (s^{\max}(y, P, Q) - \text{fixed}(y))$$

$$+ \sum_{z \in Z} (s^{\max}(z, P, Q) - \text{fixed}(z)) + \sum_{h \in H_{m'-1}} s^{\max}(h, P, Q) + \sum_{h \in H_m} s^{\max}(h, P, Q)$$

$$= \sum_{x \in X} (a_{m'} + (f_x - 1)a_{m'-2}) + \sum_{y \in Y} ((a_{m'} + (f_y - 1)a_{m'-1}) + \sum_{z \in Z} ((a_{m'} + (f_z - 1)a_{m'}))$$

$$+ \sum_{h \in H_{m'-1}} t(a_{m'}) + \sum_{h \in H_m} t(a_{m'})$$

$$= q(a_{m'}) + (t - q)(a_{m'-2}) + q(a_{m'-2}) + (t - q)(a_{m'-1})$$

$$+ q(a_{m'-1}) + (t - q)(a_{m'}) + (t(m' - 1) - 1)t(a_{m'-1}) + (t(m' - 1) - 1)t(a_{m'})$$

$$= t(a_{m'-2} + (t(m' - 1)a_{m'-1} + t(m')a_{m'})$$

Recall, that there are $t$ votes in $P$, one corresponding to every triple in $\mathcal{F}$. Therefore, the sum of the score values of the available positions in the $t$ votes, namely

- position $m - \ell(m, m' - 1) - \ell(m, m')$, with score value $a_{m'-2}$,
- positions $m - \ell(m, m' - 1) - \ell(m, m') + 1$ through $m - \ell(m, m')$, each with score value $a_{m'-1}$, and
- positions $m - \ell(m, m') + 1$ through $m$, with score value $a_{m'}$

is $t(a_{m'-2} + (t(m' - 1)a_{m'-1} + t(m')a_{m'})$ which is the same as in [1].
Proposition 9. In Reduction 4, for all completions $P$ of $P$, if $c$ is a possible winner in $P \cup Q$ then the candidate which is in position $m - \ell(m, m') - (m, m')$, with score value $a_{m' - 2}$ is an element candidate of $X \cup Y$.

Proof. By the construction of the partial profile in Reduction 4, there are $t$ votes in $P$. By Proposition 8, the profile $P \cup Q$ has the tightness property. Therefore, the sum of the maximum scores which the elements in $X \cup Y$ can make from the available positions in all the votes in $P$ is

$$\sum_{x \in X} (s_{\max}(x, P, Q) - \text{fixed}_P(x)) + \sum_{y \in Y} (s_{\max}(y, P, Q) - \text{fixed}_P(y))$$

$$= q(a_{m'}) + (t - q)(a_{m' - 2}) + q(a_{m' - 2}) + (t - q)(a_{m' - 1})$$

$$= q(a_{m'}) + \ell(a_{m' - 2}) + (t - q)(a_{m' - 1}).$$

(2)

Let $P$ be a completion of $P$. In each vote $\pi \in P$, an element of $X \cup Y$ can gain $a_{m' - 2}$, $a_{m' - 1}$, or $a_{m'}$ points. If there exists a vote in $P$ such that an element candidate of $X \cup Y$ is not in position $m - \ell(m, m' - 1) - \ell(m, m')$ the total score will be strictly less than that in $P$ and therefore in violation of the tightness property.

Lemma 7. Let $r$ be an unbounded rule such that there exists a polynomial $q(u)$ with the property that for all $u$, every scoring vector $s_u$ of $r$, with length $m = q(u)$, has $m > 3$ distinct score values, and if the three smallest score values are $a_{m - 2} > a_{m - 1} > a_m$, it holds that $m - \ell(m, m' - 1) - \ell(m, m') \geq 3u$. Then reduction 4 is a polynomial-time reduction of 3DM to PW-TTB w.r.t. $r$.

Proof. Given a 3DM instance $\mathcal{I} = (X, Y, Z, \mathcal{P})$, we construct a PW-TTB instance, $(C, V = P \cup Q, c)$, according to the above reduction. We let $u = |\mathcal{I}| = q$.

First, we prove the "$\leftarrow$" direction. Assume that the PW-PC instance $(C, V = P \cup Q, c)$ obtained above is a positive one. Therefore, there exists a total profile $P^* = \bigcup_{i=1}^{t} p_i^*$ such that

- for all $1 \leq i \leq t$, $p_i$ extends $p_i$,
- $c$ is a possible winner and its score remains the same in all extensions (since by construction, $c$ is a fixed candidate in $P$).

By Proposition 8, $P$ has the tightness property. In the following, use tightness and other properties of $P$ to reason about $P^*$. Recall the schematic representation of the scoring vector is

$$\left(\ldots, a_{m' - 2}, a_{m' - 1}, \ldots, a_{m' - 1}, a_{m'}, \ldots, a_{m'}\right)$$

$$\geq 3q \begin{pmatrix} \ell(m, m' - 1) \\ \ell(m, m') \end{pmatrix}$$

1. By construction and the maximum partial scores set in the reduction, all candidates $h \in H_{m'}$ must be in a position with score value $a_{m'}$. If any candidate $h \in H_{m'}$ is in a position with score value greater than $a_{m'}$, it will defeat $c$. Without loss of generality, assume that they are in positions positions $m - \ell(m, m') + 2$ through $m$.

2. By Proposition 9, in all the votes, the candidate in the position $m - \ell(m, m') - \ell(m, m' - 1)$, which has a score value of $a_{m' - 2}$, is an element candidate of $X \cup Y$.

3. Observe that no candidate $h$ in $H_{m' - 1}$ can be in position $m - \ell(m, m') + 1$.

For, if there exists such an $h$, then, by tightness, it would have to be in position $m - \ell(m, m')$ at least once. By 2, this is not possible.

4. Observe that, by Proposition 7, for $1 \leq i \leq t$, in any completion of a vote $p_i$, for $S_i = (x_{i1}, y_{i2}, z_{i3})$, the element candidate corresponding to $y_{i2}$ has to be in a position with score value $a_{m' - 2}$ or $a_{m' - 1}$. Otherwise, since the total number of votes is fixed, it would violate tightness.

(a) Moreover, by the maximum partial scores set in the construction, for $1 \leq i \leq q$, every $y_i$ scores $a_{m' - 2}$ exactly once. More than once, $y_i$ defeats $c$ and less than once violates tightness. Let these votes be $p_{k_1}^*, \ldots, p_{k_q}^*$, where $1 \leq k_i \leq t$ and $K = \{k_i | 1 \leq i \leq q\}$.

(b) For $1 \leq i \leq t$ and $i \notin K$, in $p_i^*$, candidate $y_{i2}$ is in position $m - \ell(m, m') - \ell(m, m' - 1) + 1$.

5. Thus, for $1 \leq i \leq t$ and $i \notin K$, in the vote $p_i^*$,
The following lemma is a direct consequence of the Lemma 7.

(a) by [4a] and [4b] candidate \( x_i \) is in position \( m - \ell(m, m') - \ell(m, m' - 1) \).

(b) \( z_{i_3} \) must be in \( m - \ell(m, m') + 1 \).

6. By the tightness property, for \( i \in K \), in \( p_i^* \), candidate \( z_{i_3} \) must be in position \( m - \ell(m, m') - \ell(m, m' - 1) + 1 \). By the maximum partial scores set for \( z \in Z \), the element candidates of \( Z \) in these \( q \) votes must be distinct.

7. By the tightness property, each of the \( q \) candidates in \( X \) have to score \( a_{m'} \) points exactly once. Therefore it must be true that the element candidates of \( X \) in position \( m - \ell(m, m') + 1 \) in the votes \( p_i^* \), for \( i \in K \), are distinct.

8. Therefore, the set \( \{ S_i | i \in K \} \) must form a cover for \( X \cup Y \cup Z \).

For the ‘ \( \Rightarrow \) ’ direction, let \((X, Y, Z, T)\) be a positive instance. We show that in the PW-DTB instance, \((C, V = P \cup Q, c)\), constructed in the reduction, \( c \) is, indeed, a possible winner. By hypothesis, there is a \( T' \subseteq T \) and \( |T'| = q \).

1. Complete each vote \( p_i \in P \) to \( p_i^* \) as follows.
   \begin{itemize}
   \item \( p_i^* = C_i \succ y_{i_2} \succ z_{i_3} \succ H_{m' - 1} \succ x_{i_1} \succ H_{m'} \) if \( S_i \in T' \)
   \item \( p_i^* = C_i \succ x_{i_1} \succ y_{i_2} \succ H_{m' - 1} \succ z_{i_3} \succ H_{m'} \) if \( S_i \notin T' \)
   \end{itemize}

Let \( P^* = \bigcup_{i=1}^{q} p_i^* \).

2. We verify that the candidates have, indeed, scored no more than the respective maximum partial scores in the completion \( P^* \) of \( P \). Recall that the position of \( c \) is fixed in both \( Q \) and \( P \).
   \begin{itemize}
   \item For all \( x \in X \), we have \( s(P^*, x) = a_{m'} + (f_x - 1)a_{m' - 2} + \text{fixed}_P(x) \).
   \item For all \( y \in Y \), we have \( s(P^*, y) = a_{m' - 2} + (f_y - 1)a_{m' - 1} + \text{fixed}_P(y) \).
   \item For all \( z \in Z \), we have \( s(P^*, z) = a_{m' - 1} + (f_z - 1)a_{m'} + \text{fixed}_P(z) \).
   \item For all \( h \in H_{m' - 1} \), we have \( s(P^*, h) = t(a_{m' - 1}) \).
   \item For all \( h \in H_{m'} \), we have \( s(P^*, h) = t(a_{m'} \) )
   \item The positions of all the candidates in \( H' \), as constructed in the reduction, are fixed and therefore their scores do not change, i.e., for all \( h' \in H' \), we have \( s(P^*, h') = \text{fixed}_P(h') \)
   \item By construction, we have \( s(P^*, w) < s(P^*, c) \).
   \end{itemize}

Therefore, \( c \) is a possible winner.

The following lemma is a direct consequence of the Lemma 7.

**Lemma 8.** Let \( r \) be a \( p \)-valued positional scoring rule, where \( p \geq 3 \), there exists a polynomial \( g(u) \) with the property that for all \( u \), every scoring vector \( s_m \) of \( r \), with length \( m = g(u) \), has \( m' = p \) distinct score values, and if the three smallest score values are \( a_{m' - 2} > a_{m' - 1} > a_{m'} \), it holds that \( m - \ell(m, m' - 1) - \ell(m, m') \geq 3u \). Then PW-TTB problem w.r.t \( r \) is NP-complete.

**Proof.** Observe that \( p \)-valued positional scoring rule, where \( p \geq 3 \), is sufficient for the construction in Reduction 4. In particular, to specify the value \( \mu(c') \) for each candidate \( c' \in C \setminus \{c\} \) we need at least three distinct score values which we always have.

### 4.3 Hardness results for PW-BTB

In this section, we prove NP-completeness for the PW-BTB problem w.r.t a restricted group of unbounded rules. We consider an unbounded rule \( r \) with the following properties. There exists a polynomial \( g(u) \) with the property that for all \( u \), every scoring vector \( s_m \) of \( r \) with length \( m = g(u) \) has at least three distinct score values, and if the largest three score values are \( a_1 > a_2 > a_3 \), it holds that \( m - \ell(m, 1) - \ell(m, 2) \geq 3u \). Besides the Borda count, examples include broad families of scoring rules with scoring vectors in which the first \( k \) score values are distinct. Schematically, such a scoring vector can be represented as follows

\[
\begin{pmatrix}
a_1, \ldots, a_1, a_2, \ldots, a_2, a_3, \ldots
\end{pmatrix}
\begin{pmatrix}
\ell(m, 1) \\
\ell(m, 2) \\
\geq 3u
\end{pmatrix}
\]
Reduction 5. The reduction is similar to Reduction\textsuperscript{5} We only provide the key steps below.

- The set of candidates is $C = X \cup Y \cup Z \cup \{c, w\} \cup H$ where $X$, $Y$, and $Z$ contains candidates corresponding to the elements in $X', Y'$, and $Z$ respectively. These candidates are called element candidates. The set $H$ contains dummy candidates such that $|H| = m - 3q - 2$.
- We construct the partial profile $P$ as follows.
  - Let the set $H$ be partitioned into $H_1, H_2,$ and $H'$ such that $|H_j| = \ell(m, j) - 1$ for $j \in \{1, 2\}$. $H' = H \setminus (H_1 \cup H_2)$. For each $S_i = (x_{i_1}, y_{i_2}, z_{i_3})$, let $C'_i = C \setminus (\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup H_1 \cup H_2)$ and $C_i^*$ be such that $c \succ w$, i.e., candidate $c$ is always ranked lower than $w$.
    \[
    p'_i = \overline{H_1} \succ x_{i_1} \succ y_{i_2} \succ \overline{H_2} \succ z_{i_3} \succ C_i^* \\
    p_i = (H_1 \cup \{x_{i_1}, y_{i_2}, z_{i_3}\} \cup H_2) \succ C_i^* \\
    \]
  - $P = \bigcup_{i=1}^{t} p_i$ is a partial profile where each vote is bottom-truncated.

In the vote $p_i$, note that all the candidates except $\{x_{i_1}, y_{i_2}, z_{i_3}\} \cup H_1 \cup H_2$ are fixed. In other words, positions one though $\ell(m, 1) + \ell(m, 2) + 1$ are available.

\[P' = \bigcup_{i=1}^{t} p'_i\] is a total profile. Moreover, each $p'_i$ extends $p_i$. Let $s(P', c) = \lambda_{P'}$. Observe that $s(P', w) < \lambda_{P'}$ since $w$ is in a position greater $c$ in all $C_i^*$, for $1 \leq i \leq t$.
- Now we specify the value $\mu(c')$ for each $c' \in C \setminus \{c\}$.
  - For all $x \in X$, we have $\mu(x) = a_3 + (f_x - 1)a_1 + \text{fixed}_P(x)$.
  - For all $y \in Y$, we have $\mu(y) = a_1 + (f_y - 1)a_2 + \text{fixed}_P(y)$.
  - For all $z \in Z$, we have $\mu(z) = a_2 + (f_z - 1)a_3 + \text{fixed}_P(z)$.
  - For all $h \in H_j$, we have $\mu(h) = t(a_j)$ where $1 \leq j \leq m'$.
  - $\mu(w) \geq t a_1$.

The remaining steps are identical to Reduction\textsuperscript{4}.

Proposition 10. The profile $P \cup Q$ in Reduction\textsuperscript{5} has the tightness property.

Proof. The proof of tightness property for the above construction is similar to the proof of Proposition\textsuperscript{8}.

Lemma 9. Let $r$ be an unbounded rule such that there exists a polynomial $g(u)$ with the property that for all $u$, every scoring vector $s_m$ of $r$ with length $m = g(u)$ has at least three distinct score values, and if the largest three score values are $a_1 > a_2 > a_3$, it holds that $m - \ell(m, 1) - \ell(m, 2) \geq 3u$. Then reduction\textsuperscript{5} is a polynomial-time reduction of 3DM to PW-TTB w.r.t. $r$.

Proof. The proof is similar to the proof of Lemma\textsuperscript{7}.

The following lemma is a direct consequence of Lemma\textsuperscript{9}.

Lemma 10. Let $r$ be a $p$-valued positional scoring rule, where $p \geq 3$, such that there exists a polynomial $g(u)$ with the property that for all $u$, every scoring vector $s_m$ of $r$ with length $m = g(u)$ has $p$ distinct score values, and if the largest three score values are $a_1 > a_2 > a_3$, it holds that $m - \ell(m, 1) - \ell(m, 2) \geq 3u$. Then the PW-BTB problem w.r.t. $r$ is NP-complete.

Proof. Observe that $p$-valued positional scoring rule, where $p \geq 3$, is sufficient for the construction in Reduction. In particular, to specify the value $\mu(c')$ for each candidate $c' \in C \setminus \{c\}$ we need at least three distinct score values which we always have.
4.4 Hardness of PW-DTB w.r.t. unbounded rules

We conclude with the following result for PW-DTB w.r.t. to a broad group of unbounded rules. This generalises the existing results in Theorem 4. Putting together the hardness results in Lemma 7 and Lemma 9, we can state the following theorem.

**Theorem 5.** Let $r$ be an unbounded rule that satisfies one of the following conditions:

1. There exists a polynomial $g(u)$ such that for every $u$, the scoring vector $s_m$ of $r$ with $m = g(u)$ has $m' \geq 3$ distinct score values, and if the three smallest score values are $a_{m' - 2} > a_{m' - 1} > a_{m'}$, it holds that $m - \ell(m, m') - 1 - \ell(m, m') \geq 3u$.

2. There exists a polynomial $g(u)$ such that for every $u$, the scoring vector $s_m$ of $r$ with $m = g(u)$ has at least three distinct score values, and if the largest three score values are $a_1 > a_2 > a_3$, it holds that $m - \ell(m, 1) - \ell(m, 2) \geq 3u$.

Then the PW-DTB problem w.r.t. $r$ is NP-complete.

**Proof.** Since PW-TTB and PW-BTB are special cases of PW-DTB, the NP-hardness results in Lemma 7 and Lemma 9 also hold for PW-DTB.

Therefore, PW-DTB w.r.t. unbounded rules with scoring vectors containing three values such that the length of the block containing the second score value is unbounded remains open. The complexity of PW-DTB w.r.t. to all $p$-valued rules except $R(f, l)$, such that $f + 1 > 2$, remains to be established.

5 Concluding Remarks

| Scoring Rule          | PW | PW-PC | PW-PP | PW-DTB | PW-TTB | PW-BTB |
|-----------------------|----|-------|-------|--------|--------|--------|
| Plurality & Veto      | P  | P     | P     | P      | P      | P      |
| 2-valued              | NP-c | NP-c | P | P | P | P |
| $R(1, 1)$            | NP-c | NP-c | P | P | P | P |
| $R(f, l), f + l > 2$ | NP-c | NP-c | ? | ? | ? | ? |
| All other 3-valued    | NP-c | NP-c | NP-c | NP-c | NP-c | NP-c | NP-c | [Lem. 8] | NP-c | [Lem. 10] |
| $p$-valued, $p \geq 4$ | NP-c | NP-c | NP-c | NP-c | NP-c | NP-c | NP-c | [Lem. 8] | NP-c | [Lem. 10] |
| Unbounded rules      | NP-c | NP-c | NP-c | NP-c | NP-c | NP-c | NP-c | [Thm. 5] | NP-c | [Lem. 7] | NP-c | [Lem. 9] |

Table 3: Classification of the PW problem and its various restrictions. The results in boldface have been established in this paper.

*See the respective results for restrictions.

The contributions in this paper can be summarised as follows.

- We obtained a complete classification of the complexity of the PW problem on partial chains w.r.t. to all pure positional scoring rules. Since the classification we obtained is the same as that of the PW problem on arbitrary partial orders, we gave a new, self-contained (and, in our view, more principled) proof of the original classification theorem for PW.
- We established new NP-completeness results for the PW problem on top-truncated, and bottom-truncated partial orders. These results also hold for the PW problem on doubly-truncated partial orders.
- Our results, together with their comparison to earlier related results in the literature, are depicted in Table 3.

In terms of future work, it remains an open problem to pinpoint the complexity of the PW problem w.r.t. rules $R(f, l)$ with $f + l > 2$ on doubly-truncated partial orders and on partitioned partial orders. In a different direction, there is a rich body of work on algorithmic problems about manipulation in voting (PW is a special case of one of these problems), where computational hardness is regarded as a feature because it provides an obstacle to such manipulation (see [19] for a survey). More recent work in this area includes the study of manipulation in voting when only incomplete preferences,
expressed as partial orders, are available [20]. Furthermore, top-truncated partial orders have been studied in this setting [21]. It would be natural to investigate manipulation in voting with partial chains as incomplete preferences.

As a broader agenda, we note that a framework aiming to create bridges between computational social choice and relational databases was introduced in [22] and studied further in [23]. In that framework, the main concepts are the necessary answers and the possible answers to queries about winners in elections together with relational context about candidates, voters, and candidates’ positions on issues. It should be pointed out that the necessary answers to natural database queries may be intractable (coNP-complete), even w.r.t. the plurality rule. Thus, our work motivates the investigation of the complexity of the necessary answers and the possible answers to queries on partial chains and other restricted classes of partial orders.

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