ON INTRINSIC CHARACTERIZATION OF REAL LOCALLY C*- AND LOCALLY JB-ALGEBRAS

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Dedicated to the memory of Professor George Bachman, Polytechnic University, Brooklyn, NY, USA.

Abstract. In the present paper we obtain an intrinsic characterization of real locally C*-algebras (projective limits of projective families of real C*-algebras) among complete real lmc*-algebras, and of locally JB-algebras (projective limits of projective families of JB-algebras) among complete fine Jordan locally multiplicatively-convex topological algebras.

1. Introduction

Banach associative regular *-algebras over \( \mathbb{C} \), so called \( C^* \)-algebras, were first introduced in 1940’s by Gelfand and Naimark in the paper \[7\]. Since then these algebras were studied extensively by various authors, and now, the theory of C*-algebras is a huge part of Functional Analysis which found applications in almost all branches of Modern Mathematics and Theoretical Physics. For the basics of the theory of C*-algebras, see for example Pedersen’s monograph [13].

The real analogues of complex C*-algebras, so called real C*-algebras, which are real Banach *-algebras with regular norms such that their complexifications are complex C*-algebras, were studied in parallel by many authors. For the current state of the basic theory of real C*-algebras, see Li’s monograph [11].

The real Jordan analogues of complex C*-algebras, so called JB-algebras, were first defined by Alfsen, Schultz and Størmer in [1] as the real Banach–Jordan algebras satisfying for all pairs of elements \( x \) and \( y \) the inequality of fineness

\[ \|x^2 + y^2\| \geq \|x\|^2, \]

and regularity identity

\[ \|x^2\| = \|x\|^2. \]

The basic theory of JB-algebras is fully treated in monograph of Hanche-Olsen and Størmer [8]. If \( A \) is a C*-algebra, or a real C*-algebra, then the self-adjoint part
$A_{sa}$ of $A$ is a JB-algebra under the Jordan product
\[ x \circ y = \frac{(xy + yx)}{2}. \]
Closed subalgebras of $A_{sa}$, for some C*-algebra or real C*-algebra $A$, become relevant examples of JB-algebras, and are called JC-algebras.

Complete locally multiplicatively-convex algebras or equivalently, due to Arens-Michael Theorem, projective limits of projective families of Banach algebras, were first studied by Arens in [3] and Michael in [12]. They were since studied by many authors under different names. In particular, projective limits of projective families of C*-algebras were studied by Inoue in [9], Apostol in [2], Schmüdgen in [15], Phillips in [14], Bhatt and Karia in [4], etc. We will follow Inoue [9] in the usage of the name locally C*-algebras for these topological algebras. The current state of the basic theory of locally C*-algebras is treated in the monograph of Fragoulopoulou [5].

Topological algebras which are projective limits of projective families of real C*-algebras under the name of real locally C*-algebras, and projective limits of projective families of JB-algebras under the name of locally JB-algebras were first introduced by Katz and Friedman in [10].

Bhatt and Karia in [4] studied the structure of locally C*-algebras. They obtain the following characterization of locally C*-algebras among complete lmC*-algebras (topological algebras which are projective limits of projective families of complex Banach *-algebras):

**Theorem 1** (Bhatt and Karia [4]). Let $\mathfrak{A}$ be a complex complete lmC*-algebra. Then $\mathfrak{A}$ is a locally C*-algebra iff $\mathfrak{A}$ contains a *-subalgebra $\mathfrak{B}$ such that:

1. $\mathfrak{B}$ is a C*-algebra with some norm $\| \cdot \|_\mathfrak{B}$; and
2. the inclusion $$(\mathfrak{B}, \| \cdot \|_\mathfrak{B}) \to \mathfrak{A},$$ is a continuous embedding with dense range.

Further, if the unit ball of $\mathfrak{B}$,
\[ \mathfrak{B}_1 = \{ x \in \mathfrak{B} : \|x\|_\mathfrak{B} \leq 1 \}, \]
is closed in $\mathfrak{A}$ in projective topology of $\mathfrak{A}$, then
\[ \mathfrak{B} = \mathfrak{A}_b, \]
where by $\mathfrak{A}_b$ we mean the bounded part of $\mathfrak{A}$ (see below Section 2 for precise definitions). □

**Proof.** The “only if” part is due to Apostol (see [2]), and, by different methods, to Schmüdgen (see [15]) and Phillips (see [14]). The “if” part is based on numerical range theory in lmC*-algebras, developed by Giles and Koehler in [6]. □

The present paper is devoted to the presentation of analogues of Theorem 1 for real locally C*-algebras and locally JB-algebras. In particular, we give an intrinsic characterization of a real locally C*-algebra as if and only if it is a complete real lmC*-algebra (projective limit of a projective family of real Banach *-algebras) with a continuously embedded dense *-subalgebra which is a real C*-algebra under some norm, as well as an intrinsic characterization of a locally JB-algebra as if and only if it is a complete fine locally Banach-Jordan algebra (projective limit of a
projective family of fine Banach-Jordan algebras) with a continuously embedded dense Jordan subalgebra which is a JB-algebra under some norm.

2. Preliminaries

Let us briefly recall some of the basic material from the aforementioned sources one needs to comprehend what follows.

A Hausdorff topological vector space over the field of \( \mathbb{R} \) or \( \mathbb{C} \), in which any neighborhood of the zero element contains a convex neighborhood of the zero element; in other words, a topological vector space is a \textit{locally convex space} if and only if the topology of is a Hausdorff locally convex topology.

A number of general properties of locally convex spaces follows immediately from the corresponding properties of locally convex topologies; in particular, subspaces and Hausdorff quotient spaces of a locally convex space, and also products of families of locally convex spaces, are themselves locally convex spaces. Let \( \Lambda \) be an upward directed set of indices and a family

\[ \{ E_\alpha, \alpha \in \Lambda \}, \]

of locally convex spaces (over the same field) with topologies

\[ \{ \tau_\alpha, \alpha \in \Lambda \}. \]

Suppose that for any pair \( (\alpha, \beta) \),

\[ \alpha \leq \beta, \]

\[ \alpha, \beta \in \Lambda, \] there is defined a continuous linear mapping

\[ g^\beta_\alpha : E_\beta \to E_\alpha. \]

A family

\[ \{ E_\alpha, \alpha \in \Lambda \} \]

is called \textit{projective}, if for each triplet \( (\alpha, \beta, \gamma) \),

\[ \alpha \leq \beta \leq \gamma, \]

\[ \alpha, \beta, \gamma \in \Lambda, \]

\[ g^\gamma_\alpha = g^\gamma_\beta \circ g^\beta_\alpha, \]

and for each \( \alpha \in \Lambda \),

\[ g^\alpha_\alpha = Id. \]

Let \( E \) be the subspace of the product

\[ \prod_{\alpha \in \Lambda} E_\alpha, \]

whose elements

\[ x = (x_\alpha), \]

satisfy the relations

\[ x_\alpha = g^\beta_\alpha(x_\beta), \]

for all \( \alpha \leq \beta \). The space \( E \) is called the \textit{projective limit} of the projective family \( E_\alpha, \alpha \in \Lambda \), with respect to the family \( (g^\beta_\alpha) \), \( \alpha, \beta \in \Lambda \) and is denoted by

\[ \lim g^\beta_\alpha E_\beta, \]

or

\[ \lim E_\alpha. \]
The topology of $E$ is the projective topology with respect to the family

$$(E_\alpha, \tau_\alpha, \pi_\alpha),$$

$\alpha \in \Lambda$, where $\pi_\alpha$, $\alpha \in \Lambda$, is the restriction to the subspace $E$ of the projection

$$\hat{\pi}_\alpha : \prod_{\beta \in \Lambda} E_\beta \to E_\alpha,$$

and

$$\pi_\beta = g_\alpha^\beta \circ \pi_\alpha,$$

$\forall \alpha, \beta \in \Lambda$.

When you take instead of $E_\alpha$, $\alpha \in \Lambda$, a projective family of algebras, *-algebras, Jordan algebras, etc., you naturally get a correspondent algebra, *-algebra or Jordan algebra structure in the projective limit algebra

$$E = \lim_{\leftarrow} E_\alpha.$$

Let $E$ be a vector space. A real function $p : E \to \mathbb{R}$ on $E$ is called a seminorm, if:

1. $p(x) \geq 0$, $\forall x \in E$;
2. $p(\lambda x) = |\lambda| p(x)$, $\forall \lambda \in \mathbb{R}$ or $\mathbb{C}$, and $x \in E$;
3. $p(x + y) \leq p(x) + p(y)$, $\forall x, y \in E$.

One can see that $p(0) = 0$.

If $p(x) = 0$, implies $x = 0$,

seminorm is called a norm and is usually denoted by $\|\|$.
If a space with a norm is complete, it is called a Banach space.

Let $(E, p)$ be a seminormed space, and

$$N_p = \ker(p) = p^{-1}\{0\}.$$

The quotient space $E/N_p$ is a linear space and the function

$$\|\|_p : E/N_p \to \mathbb{R}_+ :$$

$$x_p = x + N_p \to \|x_p\|_p = p(x),$$

is a well defined norm on $E/N_p$ induced by the seminorm $p$. The corresponding quotient normed space will be denoted by $E/N_p$, and the Banach space completion of $E/N_p$ by $E_p$. One can easily see that $E_p$ is the Hausdorff completion of the seminormed space $(E, p)$.

The algebras considered below will be without the loss of generality unital. If the algebra does not have an identity, it can be adjoint by a usual unitialization procedure.

A Jordan algebra is an algebra $E$ in which the identities

$$x \circ y = y \circ x,$$

$$x^2 \circ (y \circ x) = (x^2 \circ y) \circ x,$$

hold.
If $E$ is an algebra, the seminorm $p$ on $E$ compatible with the multiplication of $E$, in the sense that
\[ p(xy) \leq p(x)p(y), \]
\[ \forall x, y \in E, \] is called submultiplicative or $m$-seminorm.

For submultiplicative seminorm on a Jordan algebra $E$, the following inequality holds:
\[ p(x \circ y) \leq p(x)p(y), \]
\[ \forall x, y \in E. \] A seminorm on a Jordan algebra $E$ is called fine, if the following inequality holds:
\[ p(x^2 + y^2) \geq p(x^2), \]
\[ \forall x, y \in E. \] A Banach-Jordan algebra is Jordan algebra which is as well a Banach algebra.

Let $E$ be an algebra. A subset $U$ of $E$ is called multiplicative or idempotent, if
\[ UU \subseteq U, \]
in the sense that $\forall x, y \in U$, the product
\[ xy \in U. \]

If $p$ is an $m$-seminorm on $E$ the unit semiball $U_p(1)$ corresponding to $p$, that is
\[ U_p(1) = \{ x \in E : p(x) \leq 1 \}, \]
and one can see that this set is multiplicative. Moreover, $U_p(1)$ is an absolutely-convex (balanced and convex),absorbing subset of $E$. It is known that given an absorbing absolutely-convex subset
\[ U \subset E, \]
the function
\[ p_U : E \to \mathbb{R}_+ : \]
\[ x \to p_U(x) = \inf \{ \lambda > 0 : x \in \lambda U \}, \]
called gauge or Minkowski functional of $U$, is a seminorm. One can see that a real-valued function $p$ on the algebra $E$ is an $m$-seminorm iff
\[ p = p_U, \]
for some absorbing, absolutely-convex and multiplicative subset
\[ U \subset E. \]
In fact, one can take
\[ U = U_p(1). \]

By topological algebra we mean a topological vector space which is also an algebra, such that the ring multiplication is separately continuous. A topological algebra $E$ is often denoted by $(E, \tau)$, where $\tau$ is the topology of the underlying topological vector space of $E$. The topology $\tau$ is determined by a fundamental $0$-neighborhood system, say $\mathcal{B}$, consisting of absorbing, balanced sets with the property
\[ \forall V \in \mathcal{B} \exists U \in \mathcal{B}, \]
satisfying the condition $U + U \subseteq V$. Since translations by $y$ in $(E, \tau)$, i.e. the maps
\[ x \to x + y : \]
\[ (E, \tau) \to (E, \tau), \]
$y \in E$, are homomorphisms, an $x$-neighborhood in $(E, \tau)$ is of the form

$$x + V,$$

with $V \in \mathcal{B}$. A closed, absorbing and absolutely-convex subset of a topological algebra $(E, \tau)$ is called **barrel**. An $m$-**barrel** is a multiplicative barrel of $(E, \tau)$.

A **locally convex algebra** is a topological algebra in which the underlying topological vector space is a locally convex space. The topology $\tau$ of a locally convex algebra $(E, \tau)$ is defined by a fundamental 0-neighborhood system consisting of closed absolutely-convex sets. Equivalently, the same topology $\tau$ is determined by a family of nonzero seminorms. Such a family, say

$$\Gamma = \{p\},$$

or, for distinction purposes

$$\Gamma_E = \{p\},$$

is always assumed without a loss of generality **saturated**. That is, for any finite subset

$$F \subset \Gamma,$$

the seminorm

$$p_F(x) = \max_{p \in F} p(x),$$

$x \in E$, again belongs to $\Gamma$. Saying that

$$\Gamma = \{p\},$$

is a **defining family of seminorms** for a locally convex algebra $(E, \tau)$, we mean that $\Gamma$ is a saturated family of seminorms defining the topology $\tau$ on $E$. That is

$$\tau = \tau_\Gamma,$$

with $\tau_\Gamma$ completely determined by a fundamental 0-neighborhood system given by the $\varepsilon$-semiballs

$$U_p(\varepsilon) = \varepsilon U_p(\varepsilon) = \{x \in E : p(x) \leq \varepsilon\},$$

$\varepsilon > 0$, $p \in \Gamma$. More precisely, for each 0-neighborhood

$$V \subset (E, \tau),$$

there is an $\varepsilon$-semiball $U_p(\varepsilon)$, $\varepsilon > 0$, $p \in \Gamma$, such that

$$U_p(\varepsilon) \subseteq V.$$

The neighborhoods $U_p(\varepsilon)$, $\varepsilon > 0$, $p \in \Gamma$, are called **basic 0-neighborhoods**.

A locally C*-algebra (real locally C*-algebra, resp. locally JB-algebra) is a projective limit of projective family of C*-algebras (real C*-algebras, resp. JB-algebras). This is equivalent for locally C*- and real locally C*-algebras to the requirement that the family of defining continuous seminorms be regular:

$$p(x^*x) = p(x)^2.$$

In the case of locally JB-algebras this is equivalent to the requirement that the family of defining continuous seminorms be fine and regular:

$$p(x^2 + y^2) \geq p(x^2),$$

and

$$p(x^2) = p(x)^2,$$
\[ \forall p \in \Gamma, x, y \in E. \]

For a locally C*-algebra (real locally C*-algebra, resp. locally JB-algebra) \( E \), by the bounded part we mean the subalgebra

\[ E_b = \{ x \in E : \|x\|_\infty = \sup_{p \in \Gamma(E)} p(x) < \infty \}. \]

3. **Intrinsic characterization of real locally C*-algebras**

In the current section we present a real analogue of Theorem 1.

**Theorem 2.** Let \( A \) be a complex complete real lmc*-algebra. Then \( A \) is a real locally C*-algebra iff \( A \) contains a *-subalgebra \( B \) such that:

1. \( B \) is a real C*-algebra with some norm \( \|\cdot\|_B \);

and

2. the inclusion

\[ (B, \|\cdot\|_B) \rightarrow A, \]

is a continuous embedding with dense range.

Further, if the unit ball of \( B \),

\[ B_1 = \{ x \in B : \|x\|_B \leq 1 \}, \]

is closed in \( A \) in projective topology, then

\[ B = A_b, \]

where by \( A_b \) we mean the bounded part of \( A \).

**Proof.** Let \( A \) be a real locally C*-algebra. We show that the bounded part \( A_b \) of \( A \) is a real C*-algebra with required embedding. According to [10]

\[ \mathfrak{A} = A + iA, \]

is a complex locally C*-algebra, and there exists an involutive antiautomorphism

\[ \hat{\Psi} : \mathfrak{A} \rightarrow \mathfrak{A}, \]

of order 2 on \( \mathfrak{A} \), so that

\[ A = \{ x \in \mathfrak{A} : \hat{\Psi}(x) = \hat{\Psi}(x^*) \}. \]

Let \( \mathfrak{A}_b \) be the bounded part of \( \mathfrak{A} \). Then, according to [9] \( \mathfrak{A}_b \) is a complex C*-algebra with a norm

\[ \|\cdot\|_\infty \equiv \|\cdot\|_{\mathfrak{A}_b}. \]

Let

\[ \Psi : \mathfrak{A}_b \rightarrow \mathfrak{A}_b, \]

be a restriction of \( \hat{\Psi} \) to \( \mathfrak{A}_b \). Because \( \mathfrak{A}_b \) is a *-subalgebra of \( \mathfrak{A} \), \( \Psi \) is as well an involutive antiautomorphism of order 2 on \( \mathfrak{A}_b \), and, one can see that the set

\[ \{ x \in \mathfrak{A}_b : \Psi(x) = \Psi(x^*) \}, \]

is real isometrically *-isomorphic to the real C*-subalgebra of \( \mathfrak{A}_b \). It is a routine exercise to check that the bounded part \( A_b \) of the algebra \( A \) is exactly equal to it:

\[ A_b = \{ x \in \mathfrak{A}_b : \Psi(x) = \Psi(x^*) \}. \]

Conversely, let \( A \) be a real lmc*-algebra with dense real *-subalgebra \( B \) which is a real C*-algebra under a norm

\[ \|\cdot\|_B. \]
Let \( \mathfrak{A} = A + iA \),

and

\( \mathfrak{B} = B + iB \).

Similar to the arguments in [10] one can establish that \( \mathfrak{A} \) is a complex \( lmc \) \(*\)-algebra. From the definition of real \( C^* \)-algebras (see [11]) it follows that \( \mathfrak{B} \) is a complex \( C^* \)-algebra with the norm

\[ \| \cdot \|_B, \]

so that

\[ \| a + ib \|_B = \sqrt{ \| a \|_B^2 + \| b \|_B^2 }, \]

\( a, b \in B \). Also, from the construction of the complexification for the algebra \( A \) it follows that from the continuity of the embedding of \( B \) into \( A \) we can conclude a continuity of the embedding of \( \mathfrak{B} \) into \( \mathfrak{A} \). Thus, from Theorem 1 of Bhatt and Karia above (see as well [11]) it follows that \( \mathfrak{A} \) is a locally \( C^* \)-algebra. Because

\[ \mathfrak{A} = A + iA, \]

and

\[ A \cap iA = \{ 0 \}, \]

from [10] it follows that \( A \) is a real locally \( C^* \)-algebra.

Finally, because the cussedness of

\[ B_1 = \{ x \in B : \| x \|_B \leq 1 \}, \]

in the projective topology of \( A \) is, due to the aforementioned complexification arguments, equivalent to the cussedness of

\[ \mathfrak{B}_1 = \{ x \in \mathfrak{B} : \| x \|_{\mathfrak{B}} \leq 1 \}, \]

in the projective topology of \( \mathfrak{A} \), the proof is now completed. \( \square \)

4. Intrinsic characterization of locally JB-algebras

In the current section we present a Jordan-algebraic version of Theorem 1.

**Theorem 3.** Let \( M \) be a complete Jordan fine locally multiplicatively convex algebra. Then \( M \) is a locally JB-algebra iff \( M \) contains a Jordan subalgebra \( N \) such that:

1. \( N \) is a JB-algebra with some norm \( \| \cdot \|_N \);

and

2. the inclusion

\( (N, \| \cdot \|_N) \to M, \)

is a continuous embedding with dense range.

Further, if the unit ball of \( N \),

\[ N_1 = \{ x \in N : \| x \|_N \leq 1 \}, \]

is closed in \( M \) in projective topology, then

\[ N = M_b, \]

where by \( M_b \) we mean the bounded part of \( M \).
Proof. Let $M$ be a locally JB-algebra. Using functional calculus, one can see that
\[ x \circ (1 + x^2)^{-1} \in M_b, \]
and
\[ \|x \circ (1 + x^2)^{-1}\| \leq 1. \]
Then, for each $n \in \mathbb{N}$, let us set
\[ x_n = x \circ \left(1 + \frac{x^2}{n}\right)^{-1}. \]
It is easy to see that $x_n \in M_b$, for $\forall n \in \mathbb{N}$. Now, for every $p$ be a continuous fine submultiplicative regular seminorm on $M$,
\[ p(x - x_n) \leq \frac{1}{\sqrt{n}} p(x) p\left(\frac{x}{\sqrt{n}} \circ (1 + \frac{x^2}{n})^{-1}\right) \leq \frac{1}{\sqrt{n}} p(x) \to 0, \]
as $n \to \infty$, which shows that $(M_b, \|\|_\infty)$ is continuously embedded in $M$ with sequentially dense range.

Conversely, let $M$ be a complete Jordan fine locally multiplicatively convex algebra. Let
\[ P = \{p_\alpha\}, \]
$\alpha \in \Lambda$ be a saturated separating directed family of continuous fine submultiplicative seminorms on $M$, and
\[ M = \lim_{\leftarrow} M_\alpha, \]
where
\[ M_\alpha = M/J_\alpha, \]
with
\[ J_\alpha = \{x \in M : p_\alpha(x) = 0\}, \]
$\alpha \in \Lambda$.

To begin with, let us assume that $M_b$ is a JB-algebra with a norm
\[ \|x\|_\infty = \sup_{\alpha \in \Lambda} p_\alpha(x) < \infty, \]
dense in $M$ in projective topology of $M$. Let us replace the family $P$ with another saturated separating directed family
\[ P' = \{p'_\alpha\}, \]
$\alpha \in \Lambda$ of continuous fine submultiplicative seminorms on $M$, with
\[ \sup_{\alpha \in \Lambda} p_\alpha(x) = \sup_{\alpha \in \Lambda} p'_\alpha(x), \]
\[ M = \lim_{\leftarrow} M'_\alpha, \]
where
\[ M'_\alpha = M/J'_\alpha, \]
with
\[ J'_\alpha = \{x \in M : p'_\alpha(x) = 0\}, \]
the following way:
\[ p'_\alpha(x) = \begin{cases} \|x\|_\infty, & \forall x \in M_b; \\ p_\alpha(x), & \forall x \notin M_b; \end{cases} \]
\[ \alpha \in \Lambda. \text{ One can easily see that with the norm} \]
\[ \|x_\alpha\|_\alpha = p'_\alpha(x), \]
\[ M'_\alpha \text{ becomes a fine Banach-Jordan algebra, where} \]
\[ x_\alpha = x + J'_\alpha. \]

For a given \( \alpha \in \Lambda \), let
\[ \pi_\alpha : M \to M'_\alpha : x \mapsto x_\alpha = x + J'_\alpha, \]
be a continuous projection from \( M \) onto \( M'_\alpha \). Because \( p'_\alpha(x) \) is regular on \( M_b \), \( \pi_\alpha(M_b) \) will be, due to projective topological density of \( M_b \) in \( M \), a dense in \( \|\|_\alpha \) norm Jordan subalgebra of \( M'_\alpha \), which is a pre-JB-algebra in the norm \( \|\|_\alpha \). It is remained to show that the norm \( \|\|_\alpha \) is regular on \( M'_\alpha \), i.e. that
\[ \forall y_\alpha \in M'_\alpha, \|y^2_\alpha\|_\alpha = \|y_\alpha\|^{2}_\alpha. \]

On the elements of \( \pi_\alpha(M_b) \),
\[ \|x^2_\alpha\|_\alpha = \|x_\alpha\|^{2}_\alpha, \]
and if
\[ y_\alpha \notin \pi_\alpha(M_b), \quad y_\alpha \in M'_\alpha, \]
from submultiplicativity of \( \|\|_\alpha \) it already follows that
\[ \|y^2_\alpha\|_\alpha \leq \|y_\alpha\|^{2}_\alpha. \]

Let us assume on the contrary that
\[ \|y^2_\alpha\|_\alpha - \|y_\alpha\|^{2}_\alpha = \varepsilon > 0. \]
Then, because \( \pi_\alpha(M_b) \) is dense in \( M'_\alpha \) in the norm \( \|\|_\alpha \), we can find
\[ x_\alpha \in \pi_\alpha(M_b), \]
to be such that
\[ \|y^2_\alpha\|_\alpha - \|x^2_\alpha\|_\alpha < \frac{\varepsilon}{2}, \]
and
\[ \|y_\alpha\|^{2}_\alpha - \|x_\alpha\|^{2}_\alpha < \frac{\varepsilon}{2}. \]
Thus, we have
\[ \varepsilon = \|y^2_\alpha\|_\alpha - \|y_\alpha\|^{2}_\alpha = \|y^2_\alpha\|_\alpha - \|x^2_\alpha\|_\alpha + \|x_\alpha\|^{2}_\alpha - \|y_\alpha\|^{2}_\alpha = \]
\[ = |\|y^2_\alpha\|_\alpha - \|x^2_\alpha\|_\alpha| + |\|x_\alpha\|^{2}_\alpha - \|y_\alpha\|^{2}_\alpha| < \varepsilon. \]
Contradiction.

Now, let us assume that \( N \) is a JB-algebra with some norm \( \|\|_N \), and the inclusion
\[ (N, \|\|_N) \to M, \]
is a continuous embedding with dense in the projective topology \( \tau \) of \( M \) range.
Let \( B_M(\tau) \) be the collection of all absolutely-convex, closed, bounded, idempotent subsets of \( M \), containing identity \( 1 \) of \( M \).

Let
\[ M(S) = \{ \lambda x : \lambda \in \mathbb{R}, x \in S \}, \]
and the Minkowski functional of $S$ in $M(S)$ is
\[ |x|_S = \inf \{ \lambda > 0 : x \in M(S) \}. \]

If the set $S$ is absorbing in $M(S)$, one can see that
\[ M(S) = M, \]
and $|.|_S$ is defined on the whole of $M$. Because $(M, \tau)$ is complete, then for
\[ B \in B_M(\tau), \]
$(M(B), |x|_B)$ is a Banach-Jordan algebra. One can see that given a saturated separating directed family of continuous fine submultiplicative seminorms on $M$:
\[ P = \{ p_\alpha \}, \]
$\alpha \in \Lambda$, \[ U_P = \{ x \in M : p_\alpha(x) \leq 1, \forall \alpha \in \Lambda \} \in B_M(\tau), \]
and conversely, given a subset \[ B \in B_M(\tau), \]
there exists such a saturated separating directed family of continuous fine submultiplicative seminorms on $M$:
\[ P = \{ p_\alpha \}, \]
$\alpha \in \Lambda$, so that
\[ B \subset U_P. \]

Now, it is clear that the closure $\overline{N_1}$ of $N_1$ in $(M, \tau)$ belongs to $B_M(\tau)$, therefore there exists a saturated separating directed family of continuous fine submultiplicative seminorms on $M$:
\[ P = \{ p_\alpha \}, \]
$\alpha \in \Lambda$, such that
\[ N_1 \subset U_P, \]
so,
\[ (M(U_P), |x|_{U_P}), \]
is a Banach-Jordan algebra, and
\[ N \subset (M(U_P), |x|_{U_P}), \]
thus \[ |x|_{U_P} \leq \|x\|_N. \]
Moreover, because $(N, \| \cdot \|_N)$ is a JB-algebra, using spectral radii of an element in $N$ and $M(U_P)$, one can conclude that
\[ \|x\|_N = |x|_{U_P}, \]
for all $x \in N$.

Due to the fact that $(N, \| \cdot \|_N)$ is dense in $(M, \tau)$, the same way as above we now can establish that $M$ is a locally JB-algebra.

Finally, if $N_1$ is closed in $(M, \tau)$, then
\[ (M(U_P), |x|_{U_P}) = (N, \| \cdot \|_N). \]
Indeed, if \[ x \in M(U_P) \subset M, \]
then
\[ x_n = x \circ \left(1 + \frac{x^2}{n}\right)^{-1} \in N \subset M(U_P), \]
and
\[ |x - x_n|_{U_P} \leq \frac{1}{\sqrt{n}} |x|_{U_P} \to 0, \]
as \( n \to \infty \), therefore \( x \in N \). Thus
\[ M(U_P) = N = M_b. \]
The proof is now complete. \( \square \)

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