More Efficient Purifying scheme via Controlled- Controlled NOT Gate

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Abstract

A new modified version of the Oxford purification protocol is proposed. This version is based on the controlled-controlled NOT gate instead of controlled NOT in the original one. Comparisons between the results of the new version and the original and an earlier modification are given. It is found that the new version converges faster and consumes fewer initial qubit pairs of low fidelity per final qubit pair of high fidelity.

1 Introduction

It is well known that to perform most of the quantum information schemes efficiently, one requires maximally entangled states to be performed. However in reality, it is mandatory to consider the effect of the decoherence. In this situation, the maximally entangled states turn into partially entangled or product states. Therefore, these schemes my not be realized faithfully. Thus purifying these states is of a great importance in the context of quantum information. The entanglement purification, that is often required, distills a small number of strongly entangled pairs of qubits from a large number of weakly entangled pairs. This can be achieved by using local quantum operations, classical communication and measurements. Bennett et. al have proposed the first entanglement purification scheme which is called the IBM protocol $^?$. This protocol assumes that the initial states are of Werner type state. The initial density state for the IBM protocol is given by: $\rho = x|\psi^-\rangle\langle\psi^-| + \frac{1-x}{4}I$ which means that an $x$ fraction of the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ $^?$ and the rest is randomized. Another standard protocol has been proposed by Deutsch

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et al. and it is called the Oxford protocol [?]. This protocol is designed for quantum cryptography. The initial density operator for this protocol takes the form: 

$$\rho_{Ox} = A|\phi^+\rangle\langle\phi^+| + B|\psi^-\rangle\langle\psi^-| + C|\psi^+\rangle\langle\psi^+| + D|\phi^-\rangle\langle\phi^-|$$

where 

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

and 

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

are the Bell basis. Since then many theoretical [?] and experimental[?] schemes have been proposed.

The standard protocols [?,?] are based on the controlled NOT operations, CN and Bell state measurements. The IBM protocol has been improved by Feng et al [?], where they considered the controlled-controlled NOT operation (CCN) instead of CN. They showed that the new version is more efficient than the original one. Because in order to get a final state of certain fidelity from pairs of the same fidelity, the successful purification probability is larger and the amount of resources minimally consumed is less. Also the Oxford protocol has been improved by Metwally, where in his version the usual local unitary transformations are only performed when they are really helpful [?]. For some initial states this modification makes the new version converges faster, more efficient and needs fewer operations. For some other initial states the two versions are equivalent.

In this contribution, we study the Oxford protocol in the dynamical variables i.e., Pauli’s operators, $\sigma_i$ and $\tau_i$, where $i = 0, x, y, z$ for the first and the second qubit respectively. In this current version we consider the CCN operation instead of the CN. We see that the modification makes the suggested version more efficient. In section 2, we describe briefly the Oxford protocol $Ox_1$ and its first modification $Ox_2$[?]. Also we achieve the CN operation using the dynamical variables. Section 3 is devoted to a study of the second modified version, $Ox_3$. A comparison of the three protocols is discussed in section 4. Finally we summarize our results in section 5.

## 2 The Oxford protocol, $Ox_1$ and the first modified version, $Ox_2$

Assume that Alice and Bob are given an ensemble of the so called generalized Werner states or self transposed states [?], or simply Bell-diagonal states [?], and they are asked to use the Oxford protocol to purify them. The users pick two pairs of the form

$$\rho^{(1)} = \frac{1}{4}(1 + c_x\sigma_x^{(1)}\tau_x^{(1)} - c_y\sigma_y^{(1)}\tau_y^{(1)} + c_z\sigma_z^{(1)}\tau_z^{(1)})$$

where

$$1 \geq |c_x| \geq |c_y| \geq |c_z| \geq 0$$

(2)
the order being a matter of convention. These coefficients \( c_i, i = x, y, z \) are related to the coefficient of the original Oxford protocol by the following relations:

\[
\begin{align*}
  c_x &= A - B + C - D \\
  c_y &= A - B - C - D \\
  c_z &= A + B - C - D
\end{align*}
\]

(3)

The initial fidelities of states \((??)\) are given by:

\[
F_1 = \text{tr} \left\{ \rho^{(1)} \rho_{\phi^+}^{(1)} \right\} = \frac{1}{4} (1 + c_x + c_y + c_z)
\]

(4)

where

\[
\rho_{\phi^+}^{(1)} = \frac{1}{4} \left[ 1 + \sigma_x^{(1)} \tau_x^{(1)} - \sigma_y^{(1)} \tau_y^{(1)} + \sigma_z^{(1)} \tau_z^{(1)} \right].
\]

(5)

Assume that Alice and Bob want to purify their pairs by using the first modified Oxford protocol Ox\(_2\). To achieve this aim, they perform the Bilateral CN operation, see table 1, on the pairs, followed by measuring the target qubit (second pair) in the computational basis. They measure the z components of the target spin \( \sigma_z^{(2)} \) and \( \tau_z^{(2)} \). They keep those first pairs for which they get the same results for the measurements and discard the others. After one step purification, they get

\[
\rho_{Ox_2} = \frac{1}{4} \left[ 1 + \frac{c_x c'_x + c_y c'_y}{1 + c_z c'_z} \sigma_x \tau_x - \frac{c_x c'_y + c_y c'_y}{1 + c_z c'_z} \sigma_y \tau_y + \frac{c_z c'_z + c_z c'_z}{1 + c_z c'_z} \sigma_z \tau_z \right]
\]

(6)

this is another new Bell state with fidelity,

\[
F_{Ox_2} = \frac{1}{4 P_1} \left[ (1 + c_z)^2 + (c_x + c_y)^2 \right]
\]

(7)

|   | \( I^{(2)} \) | \( \sigma_x^{(2)} \) | \( \sigma_y^{(2)} \) | \( \sigma_z^{(2)} \) |
|---|---|---|---|---|
| \( I^{(1)} \) | 1 | \( \sigma_x^{(1)} \) | \( \sigma_y^{(1)} \) | \( \sigma_z^{(1)} \) |
| \( \sigma_x^{(1)} \) | \( \sigma_x^{(2)} \) | \( \sigma_x^{(1)} \) | \( \sigma_y^{(1)} \) | \( \sigma_z^{(1)} \) |
| \( \sigma_y^{(1)} \) | \( \sigma_z^{(1)} \) | \( \sigma_y^{(1)} \) | \( \sigma_x^{(1)} \) | \( \sigma_z^{(1)} \) |
| \( \sigma_z^{(1)} \) | \( \sigma_z^{(2)} \) | \( -\sigma_y^{(1)} \) | \( \sigma_x^{(1)} \) | \( \sigma_y^{(1)} \) |

Table 1
Bilateral CN operation between the two qubits which is defined by \( \sigma^{(1)}_\mu \) and \( \sigma^{(2)}_\mu \). The same table is applied to the two qubits \( \tau^{(1)}_\mu \) and \( \tau^{(2)}_\mu \).
where $P_1 = \frac{1}{2}(1 + c_z^2)$, is the probability that Alice and Bob obtain coinciding outcomes in the measurement of target pair[?]. Now Alice and Bob have a new ensemble described by $(\text{??})$. If this ensemble does not obey the ordering required by $(\text{??})$, then Alice and Bob use local rotations to bring the state into the wanted form. These are rotations by $\pi/2$ about the $x, y$ or $z$ axis, namely

$$U_{12j} = e^{i\pi(\sigma_j - \tau_j)} \quad \text{and} \quad j = x, y, z. \quad (8)$$

In fact it is only necessary to ensure that $|c_z|$ is smaller than $|c_x|$ and $|c_y|$; the relative size of $|c_x|$ and $|c_y|$ does not matter. In the standard protocol, Ox1 [?], Alice and Bob perform the transformation $(\text{??})$ in $x-$ direction on all pairs. This operation changes the positions of $c_y$ and $c_z$. After applying the Bilateral CN operation and measuring the target qubits, Alice and Bob get a new ensemble with fidelity

$$F_{Ox1} = \frac{1}{4P_2} \left[ (1 + c_y)^2 + (c_x + c_z)^2 \right] \quad (9)$$

where $P_2 = \frac{1}{2}(1 + c_y^2)$ is the probability that Alice and Bob’s measurements are the same.

3 The second modified protocol, Ox3

Before performing this protocol, one needs to describe the controlled-controlled Not, CCN. It is a three qubit gate, two of them are used as a control qubit and the third is a target. Mathematically it takes the following form,

$$CCN = \frac{1}{4} \left[ (1 + \sigma_z^{(1)})(1 + \sigma_z^{(2)})\sigma_z^{(3)} + (1 + \sigma_z^{(1)})(1 - \sigma_z^{(2)}) 
+ (1 - \sigma_z^{(1)})(1 + \sigma_z^{(2)}) + (1 - \sigma_z^{(1)})(1 - \sigma_z^{(2)}) \right] \quad (10)$$

The target qubit pairs change only when the two control qubits are in the state 1. In a generic form one can write its effect as $CCN|abc\rangle = |ab, c \oplus a, b\rangle$. As an example it transforms the vector $|\phi_1^+ \phi_2^+ \phi_3^+\rangle$ to $\frac{1}{2}(|\phi_1^+ \phi_2^+ \phi_3^+\rangle + |\phi_1^+ \phi_2^+ \phi_3^-\rangle + |\phi_1^- \phi_2^- \phi_3^-\rangle - |\phi_1^- \phi_2^- \phi_3^+\rangle)$. It is possible to consider the effect on the dynamical variables, but its form is too complicated to be written in this text.

Assume that Alice and Bob are given an ensemble of identical pairs in the form of equation $(\text{??})$. To perform the second modified version of the Oxford protocol Ox3, they pick three pairs and both of them perform the transformation $(\text{??})$ on all pairs. After applying the CCN operations, they measure the
target qubits in the projection $|\phi^+\rangle_{A_1 A_2}(|\phi^+\rangle \otimes |\phi^+\rangle)_{B_1 B_2}$, $|\phi^-\rangle_{A_1 A_2}(|\phi^-\rangle \otimes |\phi^-\rangle)_{B_1 B_2}$, where $A_1 A_2$ and $B_1 B_2$ stands for the first and the second qubits for Alice and Bob respectively. They will keep those pairs for which the measurement results coincide. These pairs are used for the second round. After one step purification they get a new ensemble of states,

$$\rho_{Ox_3} = \frac{1}{4} \left( 1 + c^\text{new}_x \sigma_x \tau_x + c^\text{new}_y \sigma_y \tau_y + c^\text{new}_z \sigma_z \tau_z \right)$$

with a fidelity

$$F_{Ox_3} = \frac{1}{2P_3} \left[ (A^2 + D^2)A + (B^2 + C^2)C \right]$$

and,

$$C^\text{new}_x = A^\text{new} - B^\text{new} + C^\text{new} - D^\text{new}$$

$$C^\text{new}_y = A^\text{new} - B^\text{new} - C^\text{new} - D^\text{new}$$

$$C^\text{new}_z = A^\text{new} + B^\text{new} - C^\text{new} - D^\text{new}$$

where

$$A^\text{new} = F_{Ox_3}, \quad B^\text{new} = \frac{1}{N_3} [(A + C)DB]$$

$$C^\text{new} = \frac{1}{2P_3} \left[ A^2 D + (B^2 + C^2)A + D^2 C \right]$$

$$D^\text{new} = \frac{1}{P_3} \left[ AD^2 + CB^2 \right]$$

and $P_3$ the probability of success

$$P_3 = \frac{1}{2} \left[ A^3 + (3B^2 + D^2)C + (3A + D)AD + 2(A + C)BD + C^3 \right]$$

Now assume that the pairs in the initial ensemble are not identical i.e with different $c_i, i = 1, 2, 3$ and with different fidelities. Alice and Bob want to perform the Ox_3. To achieve this goal, they pick three pairs of the form

$$\rho = \frac{1}{4} \left( 1 + c_{xi} \sigma_x \tau_x - c_{yi} \sigma_y \tau_y + c_{zi} \sigma_z \tau_z \right), \quad i = 1, 2, 3$$

They perform all the required local operations as described above. After one iteration, they will get a new ensemble, with a new fidelity, to be used for the
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where

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

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are the Bell basis. Since then many theoretical [4] and experimental[5] schemes have been proposed.

The standard protocols [1,3] are based on the controlled NOT operations, CN and Bell state measurements. The IBM protocol has been improved by Feng et al [6], where they considered the controlled-controlled NOT operation (CCN) instead of CN. They showed that the new version is more efficient than the original one. Because in order to get a final state of certain fidelity from pairs of the same fidelity, the successful purification probability is larger and the amount of resources minimally consumed is less. Also the Oxford protocol has been improved by Metwally, where in his version the usual local unitary transformations are only performed when they are really helpful [7]. For some initial states this modification makes the new version converges faster, more efficient and needs fewer operations. For some other initial states the two versions are equivalent.

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where

$$1 \geq |c_x| \geq |c_y| \geq |c_z| \geq 0 \ \ (2)$$
the order being a matter of convention. These coefficients $c_i, i = x, y, z$ are related to the coefficient of the original Oxford protocol by the following relations:

$$
\begin{align*}
    c_x &= A - B + C - D \\
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$$

The initial fidelities of states (1) are given by:

$$
F_1 = \text{tr} \left\{ \rho_1^{(1)} \rho_\phi^{(1)} \right\} = \frac{1}{4}(1 + c_x + c_y + c_z)
$$

where

$$
\rho_\phi^{(1)} = \frac{1}{4}(1 + \sigma_x^{(1)} \tau_x^{(1)} - \sigma_y^{(1)} \tau_y^{(1)} + \sigma_z^{(1)} \tau_z^{(1)}).
$$

Assume that Alice and Bob want to purify their pairs by using the first modified Oxford protocol $Ox_2$. To achieve this aim, they perform the Bilateral CN operation, see table 1, on the pairs, followed by measuring the target qubit(second pair) in the computational basis. They measure the $z$ components of the target spin $\sigma_z^{(2)}$ and $\tau_z^{(2)}$. They keep those first pairs for which they get the same results for the measurements and discard the others. After one step purification, they get

$$
\rho_{Ox_2} = \frac{1}{4} \left[ 1 + \frac{c_x c'_x + c_y c'_y}{1 + c_z c'_z} \sigma_x \tau_x - \frac{c_x c'_y + c_y c'_x}{1 + c_z c'_z} \sigma_y \tau_y \frac{c_z + c_z'}{1 + c_z c'_z} \sigma_z \tau_z \right]
$$

this is another new Bell state with fidelity,

$$
F_{Ox_2} = \frac{1}{4} \left[ (1 + c_z)^2 + (c_x + c_y)^2 \right]
$$

|   | $I^{(2)}$ | $\sigma_x^{(2)}$ | $\sigma_y^{(2)}$ | $\sigma_z^{(2)}$ |
|---|----------|------------------|------------------|------------------|
| $I^{(1)}$ | 1        | $\sigma_x^{(1)} \sigma_x^{(2)}$ | $\sigma_y^{(1)} \sigma_y^{(2)}$ | $\sigma_z^{(1)} \sigma_z^{(2)}$ |
| $\sigma_x^{(1)}$ | $\sigma_x^{(2)}$ | $\sigma_x^{(1)} \sigma_x^{(2)}$ | $\sigma_y^{(1)} \sigma_y^{(2)}$ | $\sigma_z^{(1)} \sigma_z^{(2)}$ |
| $\sigma_y^{(1)}$ | $\sigma_x^{(2)} \sigma_y^{(2)}$ | $\sigma_y^{(1)} \sigma_y^{(2)}$ | $- \sigma_x^{(1)} \sigma_x^{(2)}$ | $\sigma_y^{(2)}$ |
| $\sigma_z^{(1)}$ | $\sigma_x^{(2)} \sigma_z^{(2)}$ | $- \sigma_y^{(1)} \sigma_y^{(2)}$ | $\sigma_x^{(1)} \sigma_x^{(2)}$ | $\sigma_y^{(2)}$ |

Table 1
Bilateral CN operation between the two qubits which is defined by $\sigma^{(1)}_\mu$ and $\sigma^{(2)}_\mu$. The same table is applied to the two qubits $\tau^{(1)}_\mu$ and $\tau^{(2)}_\mu$. 

3
where $P_1 = \frac{1}{2}(1 + c_z^2)$, is the probability that Alice and Bob obtain coinciding outcomes in the measurement of target pair[7]. Now Alice and Bob have a new ensemble described by (6). If this ensemble does not obey the ordering required by (2), then Alice and Bob use local rotations to bring the state into the wanted form. These are rotations by $\pi/2$ about the $x, y$ or $z$ axis, namely

$$U_{12j} = e^{i\pi(\sigma_j - \tau_j)} \quad \text{and} \quad j = x, y, z. \quad (8)$$

In fact it is only necessary to ensure that $|c_z|$ is smaller than $|c_x|$ and $|c_y|$; the relative size of $|c_x|$ and $|c_y|$ does not matter. In the standard protocol, $Ox_1[1]$, Alice and Bob perform the transformation (8) in $x$- direction on all pairs. This operation changes the positions of $c_y$ and $c_z$. After applying the Bilateral CN operation and measuring the target qubits, Alice and Bob get a new ensemble with fidelity

$$F_{Ox_1} = \frac{1}{4}P_2 \left[ (1 + c_y)^2 + (c_x + c_z)^2 \right] \quad (9)$$

where $P_2 = \frac{1}{2}(1 + c_y^2)$ is the probability that Alice and Bob’s measurements are the same.

### 3 The second modified protocol, Ox3

Before performing this protocol, one needs to describe the controlled-controlled Not, CCN. It is a three qubit gate, two of them are used as a control qubit and the third is a target. Mathematically it takes the following form,

$$CCN = \frac{1}{4} \left[ (1 + \sigma_z^{(1)}) (1 + \sigma_z^{(2)}) \sigma_z^{(3)} + (1 + \sigma_z^{(1)}) (1 - \sigma_z^{(2)}) + (1 - \sigma_z^{(1)}) (1 + \sigma_z^{(2)}) + (1 - \sigma_z^{(1)}) (1 - \sigma_z^{(2)}) \right] \quad (10)$$

The target qubit pairs change only when the two control qubits are in the state 1. In a generic form one can write its effect as $CCN|abc\rangle = |ab, c \oplus a.b\rangle$. As an example it transforms the vector $|\phi_1^+ \phi_2^+ \phi_3^\pm\rangle$ to $\frac{1}{2}[|\phi_1^+ \phi_2^+ \phi_3^\pm\rangle + |\phi_1^+ \phi_2^+ \phi_3^\pm\rangle + |\phi_1^- \phi_2^- \phi_3^\pm\rangle - |\phi_1^- \phi_2^- \phi_3^\pm\rangle]$. It is possible to consider the effect on the dynamical variables, but its form is too complicated to be written in this text.

Assume that Alice and Bob are given an ensemble of identical pairs in the form of equation (1). To perform the second modified version of the Oxford protocol Ox3, they pick three pairs and both of them perform the transformation (8) on all pairs. After applying the CCN operations, they measure the target qubits.
in the projection $|\phi^+\rangle_{A_1A_2}(|\phi^+\rangle \otimes |\phi^+\rangle)_{B_1B_2} + |\phi^-\rangle_{A_1A_2}(|\phi^-\rangle \otimes |\phi^-\rangle)_{B_1B_2}$, where $A_1A_2$ and $B_1B_2$ stands for the first and the second qubits for Alice and Bob respectively. They will keep those pairs for which the measurement results coincide. These pairs are used for the second round. After one step purification they get a new ensemble of states,

$$\rho_{O_{X3}} = \frac{1}{4}(1 + c_{x}^{{\text{new}}} \sigma_x \tau_x + c_{y}^{{\text{new}}} \sigma_y \tau_y + c_{z}^{{\text{new}}} \sigma_z \tau_z)$$

with a fidelity

$$F_{O_{X3}} = \frac{1}{2P_3} [(A^2 + D^2)A + (B^2 + C^2)C]$$

and,

$$C_x^{{\text{new}}} = A_{\text{new}} - B_{\text{new}} + C_{\text{new}} - D_{\text{new}}$$
$$C_y^{{\text{new}}} = A_{\text{new}} - B_{\text{new}} - C_{\text{new}} - D_{\text{new}}$$
$$C_z^{{\text{new}}} = A_{\text{new}} + B_{\text{new}} - C_{\text{new}} - D_{\text{new}}$$

where

$$A_{\text{new}} = F_{O_{X3}}, \quad B_{\text{new}} = \frac{1}{N_3} [(A + C)DB]$$
$$C_{\text{new}} = \frac{1}{2P_3} [A^2D + (B^2 + C^2)A + D^2C]$$
$$D_{\text{new}} = \frac{1}{P_3} [AD^2 + CB^2]$$

and $P_3$ the probability of success

$$P_3 = \frac{1}{2} [A^3 + (3B^2 + D^2)C + (3A + D)AD + 2(A + C)BD + C^3]$$

Now assume that the pairs in the initial ensemble are not identical i.e with different $c_i, i = 1, 2, 3$ and with different fidelities. Alice and Bob want to perform the $O_{X3}$. To achieve this goal, they pick three pairs of the form

$$\rho = \frac{1}{4}(1 + c_{xi} \sigma_x \tau_x - c_{yi} \sigma_y \tau_y + c_{zi} \sigma_z \tau_z), \quad i = 1, 2, 3$$

They perform all the required local operations as described above. After one iteration, they will get a new ensemble, with a new fidelity, to be used for the
second round. In this case the fidelity is given by

\[
F = \frac{1}{2P_d} \left[ (A_1A_2 + D_1D_2)A_3 + (B_1B_2 + C_1C_2)C_3 \right]
\]

(17)

where \(P_d\) is the probability of success,

\[
P_d = \frac{1}{2} \left[ (A_1A_2(A_3 + D_3) + (B_1B_2 + C_1C_2 + D_1D_2)(A_3 + C_3)
+ 2(A_1D_2 + D_1A_2 + B_1C_2 + C_1B_2)(B_3 + D_3) \right]
\]

(18)

4 Discussion

To study the efficiency of the protocols: the original one \(Ox_1\), the first modified one, \(Ox_2\) and the second modified one \(Ox_3\), we consider an ensemble of identical states given by (1). Let us assume that this ensemble has an initially fidelity, \(F = 0.52\). In Fig.(1), we plot the fidelity of the surviving states after each step as a function of the number of iteration. As a first remark, the fidelity of the purified state using the \(Ox_2\) and \(Ox_3\) after one step is larger than that obtained using the standard protocol \(Ox_1\). It decreases for the \(Ox_1\) and then increases, but for the modified ones it always increases. Concerning the fidelity
after one step obtained by using Ox₃ it is much larger that obtained from Ox₂. Since the fidelity in each next step depends on the previous one, the final state in the modified protocols converges faster than in the standard one. Also for the Ox₃ it converges much faster than Ox₂. The number of steps needed in the modified protocols are fewer than those in the standard one. In table (2) there is a comparison of the three protocols. It is clear that, in order to get a state with a fidelity \( \sim 0.8 \), one needs 9 steps for the Ox₁ and 8 steps for the Ox₂ and 3 steps for the Ox₃. The probability of success in each step, for the original protocol is larger than the modified ones. To complete comparing

| protocol | Fidelity | number of iterations | consumed pairs |
|----------|----------|----------------------|----------------|
| Ox₁      | 0.853    | 9                    | 256            |
| Ox₂      | 0.805    | 8                    | 128            |
| Ox₃      | 0.843    | 3                    | 9              |

Table 2
Comparison between the Ox₁ and its modified versions Ox₂ and Ox₃.

the efficiency between the three protocols, we have to examine the consumed pairs in each protocol. We know that the users in each step needs two pairs for the Ox₁ and Ox₂ protocol while three pairs for the Ox₃ are needed. In each success step, they consume one pair for the Ox₁ and Ox₂ but two pairs for the Ox₃. Table (2) shows the number of the consumed pairs in each protocol to get a certain fidelity. From this table it is clear that the resource consumed in the modified versions are fewer than the original one.

5 Summary

The new proposed version Ox₃ is more efficient than Ox₁ and Ox₂ for the following reasons: (i) The final state converges faster than the two other protocols. (ii) The steps needed to get a state with a certain fidelity are fewer than those needed if the other protocols are used. (iii) The consumed sources are much fewer. (iv) The new version, Ox₃ is efficient for any set of initial states, while the Ox₂ and Ox₁ are the same for the set of Werner states. However, we would point out to the only disadvantage of this protocol namely: the successful purification probability is smaller than the original one, however the Ox₃ is much better than Ox₂.

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