TENSOR PRODUCTS OF SUBSPACE LATTICES AND RANK ONE DENSITY

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Abstract. We show that, if \( M \) is a subspace lattice with the property that the rank one subspace of its operator algebra is weak* dense, and \( L \) is a commutative subspace lattice, then \( L \otimes M \) possesses property (p) introduced in [14]. If \( M \) is moreover an atomic Boolean subspace lattice while \( L \) is any subspace lattice, we provide a concrete lattice theoretic description of \( L \otimes M \) in terms of projection valued functions defined on the set of atoms of \( M \). As a consequence, we show that the Lattice Tensor Product Formula holds for \( \text{Alg} M \) and any other reflexive operator algebra and give several further corollaries of these results.

1. Introduction

Let \( A \) and \( B \) be unital operator algebras acting on Hilbert space. The Lattice Tensor Product Formula (LTPF) problem asks if the invariant subspace lattice \( \text{Lat}(A \otimes B) \) of the (weak* spatial) tensor product of \( A \) and \( B \) is the tensor product of the invariant subspace lattices \( \text{Lat} A \) and \( \text{Lat} B \). The origins of this problem can be found in the Tomita Commutation Theorem, which asserts that the “dual” statement, namely the Algebra Tensor Product Formula, holds for the projection lattices of von Neumann algebras. The LTPF problem is related to the question of reflexivity for subspace lattices, which asks to decide whether a given lattice of projections on Hilbert space is the invariant subspace lattice of some operator algebra (see P. R. Halmos’ pivotal paper [5]). Although reflexivity questions have attracted considerable attention in the literature, little progress has been made on the LTPF problem since the initiation of its study in [8]. One of the reasons for this is the lack of useful descriptions of the tensor product of two subspace lattices which, in its own right, is due to the lack of compatibility between the lattice operations and the strong operator topology. It is known, however, that the LTPF problem has an affirmative answer if \( A \) and \( B \) are von Neumann algebras one of which is injective [14], if \( A \) is a completely distributive CSL algebra, while \( B \) is any other operator algebra [7], as well as when both \( A \) and \( B \) are CSL algebras [16]. Even the special case, where \( A \) consists of the scalar multiples of the identity operator, is in general open, although several partial results were obtained in [14].

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Properties related to the subspace generated by the rank one operators in a given operator algebra \( A \) (hereafter referred to as the rank one subspace of \( A \)) have been widely studied (see, e.g. [4, Chapter 23]). In this paper, we continue the study of the LTPF problem by considering the case where one of the algebras has the property that its rank one subspace is weak* dense. The class of operator algebras with this property is rather large; it includes as a special case the algebras of all operators leaving two fixed non-trivial subspaces invariant [9], [13], as well as the operator algebras of more general atomic Boolean subspace lattices [2].

The paper is organised as follows: in Section 3, we show that if \( M \) is a subspace lattice such that the rank one subspace of the algebra \( A = \text{Alg} M \) is weak* dense in \( A \), then the tensor product of \( M \) with the full projection lattice on an infinite dimensional separable Hilbert space is reflexive. This establishes the LTPF for the algebras \( A \) and \( \mathcal{C}_I \). The result is then extended to lattices of the form \( M \otimes \mathcal{L} \), where \( \mathcal{L} \) is a CSL, thus generalising a corresponding result proved earlier in [14]. In Section 4 we restrict our attention to the case where \( M \) is an atomic Boolean subspace lattice (ABSL), and achieve a convenient description of the tensor product \( M \otimes \mathcal{L} \), where \( \mathcal{L} \) is an arbitrary subspace lattice, showing that it is isomorphic to the lattice of \( \mathcal{L} \)-valued maps defined on the set of atoms of \( M \). We also show that the property of semistrong closedness of subspace lattices, introduced and studied in [14], is preserved under tensoring with \( M \) (see Proposition 4.9 for the complete statement). In Section 5 we show that if \( \mathcal{L} \) is any reflexive subspace lattice then the LTPF holds for the algebras \( A \) and \( \text{Alg} \mathcal{L} \). Some further consequences of the description of the tensor product from Section 4 are also included in Section 5. In the next section, we collect some preliminaries and fix notation.

2. Preliminaries

Let \( H \) be a Hilbert space and \( S_H \) be the set of all closed subspaces of \( H \). The set \( S_H \) is a complete lattice with respect to the operations of intersection \( \wedge \) and closed linear linear span \( \vee \). Using the bijective correspondence between \( S_H \) and the set \( \mathcal{P}_H \) of all orthogonal projections on \( H \), under which a closed subspace \( F \) corresponds to the projection with range \( F \), we transfer the lattice structure of \( S_H \) to \( \mathcal{P}_H \), and denote the lattice operations on \( \mathcal{P}_H \) obtained in this way again by \( \wedge \) and \( \vee \). A subspace lattice on \( H \) is a sublattice \( \mathcal{L} \) of \( \mathcal{P}_H \) containing 0 and \( I \) and closed in the strong operator topology.

Let \( \mathcal{B}(H) \) be the algebra of all bounded linear operators acting on \( H \). If \( A \subseteq \mathcal{B}(H) \), it is customary to denote by \( \text{Lat} A \) the set of all projections on \( H \) whose ranges are invariant under all operators in \( A \). It is easy to show that \( \text{Lat} A \) is a subspace lattice. Conversely, given any set of projections \( \mathcal{L} \subseteq \mathcal{P}_H \), let \( \text{Alg} \mathcal{L} \) be the set of all operators on \( H \) leaving invariant each element of \( \mathcal{L} \). It is easy to see that \( \text{Alg} \mathcal{L} \) is a unital subalgebra of \( \mathcal{B}(H) \) closed in the weak
operator topology. A subspace lattice \( \mathcal{L} \) is called reflexive if \( \mathcal{L} = \text{Lat}\, \text{Alg}\, \mathcal{L} \). Similarly, an operator algebra \( \mathcal{A} \) is called reflexive if \( \mathcal{A} = \text{Alg}\, \text{Lat}\, \mathcal{A} \). A subspace lattice \( \mathcal{L} \) is called a commutative subspace lattice (or CSL for short) if \( PQ = QP \) for all \( P, Q \in \mathcal{L} \).

If \( H_1 \) and \( H_2 \) are Hilbert spaces and \( \mathcal{L}_1 \subseteq \mathcal{P}_{H_1} \) and \( \mathcal{L}_2 \subseteq \mathcal{P}_{H_2} \) are subspace lattices, we denote by \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) the subspace lattice generated by the projections of the form \( L_1 \otimes L_2 \) acting on the Hilbert space tensor product \( H_1 \otimes H_2 \), where \( L_1 \in \mathcal{L}_1 \) and \( L_2 \in \mathcal{L}_2 \). Given operator algebras \( \mathcal{A}_1 \subseteq \mathcal{B}(H_1) \) and \( \mathcal{A}_2 \subseteq \mathcal{B}(H_2) \), we let \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) be the weak* closed operator subalgebra of \( \mathcal{B}(H_1 \otimes H_2) \) generated by the elementary tensors \( A_1 \otimes A_2 \), with \( A_1 \in \mathcal{A}_1 \) and \( A_2 \in \mathcal{A}_2 \). We denote by \( I \) the identity operator acting on a separable infinite dimensional Hilbert space, and set \( 1 \otimes \mathcal{A} = \mathbb{C}I \otimes \mathcal{A} \), where \( \mathbb{C}J = \{ \lambda I : \lambda \in \mathbb{C} \} \).

We say that the Lattice Tensor Product Formula (LTPF) holds for \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) if

\[
\text{Lat}(\mathcal{A}_1 \otimes \mathcal{A}_2) = \text{Lat}\, \mathcal{A}_1 \otimes \text{Lat}\, \mathcal{A}_2.
\]

Similarly, the Algebra Tensor Product Formula (ATPF) is said to hold for the subspace lattices \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) if

\[
\text{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \text{Alg}\, \mathcal{L}_1 \otimes \text{Alg}\, \mathcal{L}_2.
\]

The following notion will play an essential role in this paper.

**Definition 2.1** ([14]). A subspace lattice \( \mathcal{L} \) is said to possess property (p) if the lattice \( \mathcal{P}_{\mathcal{L}} \otimes \mathcal{L} \) is reflexive.

It follows from [14, Proposition 4.2] that \( \mathcal{L} \) possesses property (p) if and only if \( \mathcal{P}_{\mathcal{L}} \otimes \mathcal{L} = \text{Lat}(1 \otimes \text{Alg}\, \mathcal{L}) \).

If \( x, y \in H \), we denote by \( R_{x,y} \) the rank one operator on \( H \) given by \( R_{x,y}(z) = (z,y)x \), \( z \in H \). It was shown in [11] that the rank one operator \( R_{x,y} \) belongs to \( \text{Alg}\, \mathcal{L} \) if and only if there exists \( L \in \mathcal{L} \) such that \( x = Lx \) and \( L-y = 0 \), where

\[
L_+ = \vee\{ P \in \mathcal{L} : L \preceq P \}.
\]

We say that a subspace lattice \( \mathcal{L} \) possesses the rank one density property if the subspace of \( \text{Alg}\, \mathcal{L} \) generated by the rank one operators contained in \( \text{Alg}\, \mathcal{L} \) is weak* dense in \( \text{Alg}\, \mathcal{L} \). It was shown in [12] that if \( \mathcal{L} \) possesses the rank one density property then it is completely distributive.

An atomic Boolean subspace lattice (ABSL) is a distributive and complemented subspace lattice for which there exists a set \( \mathcal{E} = \{ E_j \}_{j \in J} \subseteq \mathcal{L} \) of minimal projections (called atoms) such that for every \( L \in \mathcal{L} \) there exists \( J_L \subseteq J \) with \( L = \vee_{j \in J_L} E_j \). A special case of interest arises when \( \mathcal{E} \) has two elements, see [9] and [13].

Along with the strong operator topology, we will also use the semi-strong convergence introduced in [9]. Namely, a sequence \( (P_n)_{n \in \mathbb{N}} \) of projections acting on a Hilbert space \( H \) is said to converge semistrongly to a projection \( P \) on \( H \), if (a) for every \( x \in PH \) there exists a sequence \( (x_n)_{n \in \mathbb{N}} \subseteq H \) with \( x_n \in P_n H, \ n \in \mathbb{N}, \) such that \( x_n \to_{n \to \infty} x \), and (b) if \( (x_k)_{k \in \mathbb{N}} \subseteq H \) is a convergent sequence of vectors such that \( x_k \in P_{n_k} H, \) for some increasing...
sequence \((n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}\), then \(\lim_{k \to \infty} x_k \in PH\). It was shown in [3] that \(P_n \to_{n \to \infty} P\) in the strong operator topology if and only if \(P_n \to_{n \to \infty} P^\perp\) semistrongly and \(P_n^\perp \to_{n \to \infty} P^\perp\) semistrongly, where, for a projection \(Q\), we let \(Q^\perp = I - Q\) be its orthogonal complement. The weak operator (resp. strong operator, weak*) topology will be denoted by \(w\) (resp. \(s\), \(w^*\)).

3. Property (p)

Let \(H\) and \(K\) be Hilbert spaces, with \(K\) infinite dimensional and separable, let \(\mathcal{P} = \mathcal{P}_K\) be the full projection lattice on \(K\) and let \(\mathcal{L} \subseteq \mathcal{P}\) be a subspace lattice. For any subset \(\mathcal{E} \subseteq \mathcal{P}_H\), we let \(m(\mathcal{E}, \mathcal{L})\) be the set of all maps from \(\mathcal{E}\) to \(\mathcal{L}\). If \(f, g \in m(\mathcal{E}, \mathcal{L})\), we define \(f \lor g\) and \(f \land g\) to be the elements of \(m(\mathcal{E}, \mathcal{L})\) given by

\[
(f \lor g)(E) = f(E) \lor g(E) \quad \text{and} \quad (f \land g)(E) = f(E) \land g(E), \quad E \in \mathcal{E}.
\]

It is clear that, under these operations, \(m(\mathcal{E}, \mathcal{L})\) is a complete lattice. Let \(\phi_{\mathcal{E}, \mathcal{L}} : \mathcal{P}_{K \otimes H} \to m(\mathcal{E}, \mathcal{L})\) be the map sending a projection \(Q\) on \(K \otimes H\) to the map \(f_Q\) given by

\[
(1) \quad f_Q(E) = \lor\{P \in \mathcal{L} : P \otimes E \leq Q\}, \quad E \in \mathcal{E}.
\]

We note that if \(E_1, E_2 \in \mathcal{E}\) are such that \(E_1 \land E_2 \in \mathcal{E}\), then

\[
(2) \quad f_Q(E_1) \lor f_Q(E_2) \leq f_Q(E_1 \land E_2).
\]

Dually, let \(\theta : m(\mathcal{E}, \mathcal{L}) \to \mathcal{P}_{K \otimes H}\) be the map given by

\[
\theta(f) = \lor\{f(E) \otimes E : E \in \mathcal{E}\}, \quad f \in m(\mathcal{E}, \mathcal{L}).
\]

For the rest of this section, fix a subspace lattice \(\mathcal{M} \subseteq \mathcal{P}_H\) and let \(\mathcal{A} = \text{Alg} \mathcal{M} \subseteq B(H)\). It is clear that the map \(\theta\) sends \(m(\mathcal{M}, \mathcal{L})\) into \(\mathcal{L} \otimes \mathcal{M}\), for every subspace lattice \(\mathcal{L} \subseteq \mathcal{P}\).

We first note that \(\theta\) is \(\lor\)-preserving; the proof is straightforward and we omit it.

**Proposition 3.1.** If \((f_\alpha)_{\alpha \in \mathcal{A}} \in m(\mathcal{M}, \mathcal{L})\) then \(\theta(\lor_{\alpha \in \mathcal{A}} f_\alpha) = \lor_{\alpha \in \mathcal{A}} \theta(f_\alpha)\).

**Lemma 3.2.** Let \(\mathcal{M} \subseteq \mathcal{P}_H\) be a subspace lattice with the rank one density property, \(\mathcal{A} = \text{Alg} \mathcal{M} \subseteq B(H)\). There exists \(f \in m(\mathcal{M}, \mathcal{P})\) such that the projection onto the cyclic subspace \((1 \otimes \mathcal{A})\xi\) coincides with \(\theta(f)\).

**Proof.** Let \(\xi = \sum_{j=1}^{\infty} e_j \otimes x_j\), where \((e_j)_{j \in \mathbb{N}}\) is an orthonormal basis of \(K\) and \((x_j)_{j \in \mathbb{N}}\) is a square-summable sequence in \(H\). Let \(f \in m(\mathcal{M}, \mathcal{P})\) be the mapping which sends the projection \(L \in \mathcal{M}\) to the projection \(f(L)\) onto the subspace

\[
\left\{ \sum_{j=1}^{\infty} (x_j, q)e_j : q \in H, Lq = 0 \right\}.
\]

Let \(\mathcal{R}\) be the rank one subspace of \(\mathcal{A}\) and \(F \in \mathcal{R}\). By [11], there exists \(m \in \mathbb{N}\), pairwise distinct projections \(L_i \in \mathcal{M}\), \(i = 1, \ldots, m\), and vectors
\( p_k^{(i)} = L_i p_k^{(i)} \) and \( q_k^{(i)} = (L_i) \perp q_k^{(i)} \), \( k = 1, \ldots, l_i \), \( l_i \in \mathbb{N} \), such that \( F = \sum_{i=1}^{m} (\sum_{k=1}^{l_i} R_{p_k^{(i)} q_k^{(i)}}) \). We have

\[
(I \otimes F)(\xi) = \sum_{i=1}^{m} \left( \sum_{k=1}^{l_i} \left( (I \otimes R_{p_k^{(i)} q_k^{(i)}}) \left( \sum_{j=1}^{\infty} e_j \otimes x_j \right) \right) \right) = \sum_{i=1}^{m} \left( \sum_{k=1}^{l_i} \left( \sum_{j=1}^{\infty} (x_j, q_k^{(i)}) e_j \right) \otimes p_k^{(i)} \right).
\]

It follows that \((I \otimes F)(\xi) \in \theta(f)(K \otimes H)\) and since \( F \) is an arbitrary element of \( \mathcal{R} \), we have that \((1 \otimes \mathcal{R})\xi \subseteq \theta(f)(K \otimes H)\). The property \( \mathcal{R}^w = \mathcal{A} \) easily implies that \( 1 \otimes \mathcal{R}^w = 1 \otimes \mathcal{A} \). A standard application of Hahn-Banach’s Theorem shows that \((1 \otimes \mathcal{A})\xi = (1 \otimes \mathcal{R})\xi \). Thus, \((1 \otimes \mathcal{A})\xi \subseteq \theta(f)(K \otimes H)\).

On the other hand, if \( L \in \mathcal{M}, p \in LH \) and \( q \in (L^\perp)^H \), then

\[
\left( \sum_{j=1}^{\infty} (x_j, q) e_j \right) \otimes p = (I \otimes R_{p,q})\xi \in (1 \otimes \mathcal{A})\xi.
\]

Hence, \((f(L) \otimes L)(K \otimes H) \subseteq (1 \otimes \mathcal{A})\xi \) and so we have that \( \theta(f)(K \otimes H) \subseteq (1 \otimes \mathcal{A})\xi \); thus, \((1 \otimes \mathcal{A})\xi = \theta(f)(K \otimes H)\) and the proof is complete. \( \square \)

**Theorem 3.3.** Let \( \mathcal{M} \) be a subspace lattice with the rank one density property. The restriction of the map \( \phi = \phi_{\mathcal{M},\mathcal{P}} \) to \( \text{Lat}(1 \otimes \mathcal{A}) \) is injective, \( \land \)-preserving and

\[
(3) \quad \theta \circ \phi|_{\text{Lat}(1 \otimes \mathcal{A})} = \text{id}|_{\text{Lat}(1 \otimes \mathcal{A})}.
\]

In particular, \( \mathcal{M} \) has property \((p)\) and every element of \( \mathcal{P} \otimes \mathcal{M} \) has the form \( \vee_{M \in \mathcal{M}} f(M) \otimes M \), for some map \( f : \mathcal{M} \to \mathcal{P} \).

**Proof.** Let \( Q \in \text{Lat}(1 \otimes \mathcal{A}) \) and \( P_L = f_Q(L), L \in \mathcal{M} \) (see [1] for the definition of the map \( f_Q \)). Obviously

\[
N \overset{\text{def}}{=} \theta(\phi(Q)) = \vee_{L \in \mathcal{M}} (P_L \otimes L) \leq Q.
\]

Assume, by way of contradiction, that there exists \( \xi \in Q(K \otimes H) \backslash N(K \otimes H) \). By Lemma [3.2] there exists \( f \in m(\mathcal{M}, \mathcal{P}) \) such that \((1 \otimes \mathcal{A})\xi = \theta(f)(K \otimes H)\).

There exists \( M \in \mathcal{M} \) such that \( f(M) \not\leq P_M \), for otherwise we would have that \( \xi \in N(K \otimes H) \). Thus,

\[
\vee_{L \in \mathcal{M}} ((f(L) \lor P_L) \otimes L) = (\vee_{L \in \mathcal{M}} (P_L \otimes L)) \lor (\vee_{L \in \mathcal{M}} (f(L) \otimes L)) \leq Q;
\]

in particular we have that \((f(M) \lor P_M) \otimes M \leq Q\), contradicting the maximality of \( P_M \). This proves that \( Q = N = \theta(\phi(Q)) \).

Since the range of \( \theta \) is contained in \( \mathcal{P} \otimes \mathcal{M} \), \([3]\) implies that \( \text{Lat}(1 \otimes \mathcal{A}) \subseteq \mathcal{P} \otimes \mathcal{M} \). Since the converse inclusion is trivial, we conclude that \( \text{Lat}(1 \otimes \mathcal{A}) = \mathcal{P} \otimes \mathcal{M} \), that is, that \( \mathcal{M} \) has property \((p)\).

We next observe that if \( E_1, E_2 \in \mathcal{P} \otimes \mathcal{M} \) then

\[
(4) \quad E_1 \leq E_2 \iff \phi(E_1) \leq \phi(E_2).
\]
Remarks (i) In Theorem 3.3, the assumption that $D$ density property is essential. Indeed, let
\[ H \]
and set $\theta = \theta(\phi(E_1)) \leq \theta(\phi(E_2)) = E_2$.

The converse direction follows directly from the definition of $\phi$, and (1) is proved.

It follows from (1) that $\phi(\mathcal{P} \otimes \mathcal{M})$ is injective. It remains to show that $\phi$ is $\wedge$-preserving. Let $\{E_j\}_{j \in J} \subseteq \mathcal{P} \otimes \mathcal{M}$, $f_j = \phi(E_j)$, $j \in J$, and $f = \phi(\wedge E_j)$. By (1) and the fact that $\wedge E_i \leq E_j$ for all $j \in J$, we have that $f \leq f_j$ for all $j \in J$. Thus,
\[ f \leq \wedge f_i. \]

Now let $g = \phi(\theta(\wedge f_i))$. By the definition of $\phi$, we have that $\wedge f_i \leq g$. On the other hand, for every $j \in J$, we have by (3) that
\[ \theta(\wedge f_i) \leq \theta(f_j) = \theta(\phi(E_j)) = E_j. \]

Hence, $\theta(\wedge f_i) \leq \wedge E_i$. By (1), $g \leq f$ and hence $\wedge f_i \leq f$; now (5) implies that $\wedge f_i = f$, showing that $\phi$ is $\wedge$-preserving.

Remarks (i) In Theorem 3.3, the assumption that $\mathcal{M}$ have the rank one density property is essential. Indeed, let $\mathcal{D}_0$ (resp. $\mathcal{D}$) be the multiplication masa of $L^\infty((0, 1))$ (resp. $L^\infty([0, 1)^2]$) acting on $L^2(0, 1)$ (resp. $L^2(0, 1) \otimes L^2(0, 1)$), and let $\mathcal{N}_0$ (resp. $\mathcal{N}$) be the projection lattice of $\mathcal{D}_0$ (resp. $\mathcal{D}$). We have that $\mathcal{N} \equiv \mathcal{N}_0 \otimes \mathcal{N}_0 \subseteq \mathcal{P} \otimes \mathcal{N}_0$. For a measurable subset $\gamma$ of $(0, 1)$ or of $(0, 1)^2$, we write $M_\gamma$ for the projection of multiplication by the characteristic function of $\gamma$.

Let $C$ be a non-null Cantor subset of $[0, 1)$ and equip $[0, 1)$ with the group operation of addition mod 1. Set
\[ F = \{(x, y) \in [0, 1) \times [0, 1) : x - y \in C\}. \]

The set $F$ is clearly non-null; we claim that it does not contain any non-trivial measurable rectangles. Indeed, suppose, by way of contradiction, that there exist non-null measurable subsets $\alpha$ and $\beta$ of $[0, 1)$, such that $\alpha \times \beta \subseteq F$. It follows by the definition of $F$ that $\alpha - \beta = \{a - b : a \in \alpha, b \in \beta\}$ is contained in $C$. By a well-known version of Steinhaus’ Theorem, we have that $\alpha - \beta$ contains an open interval. However, $\alpha - \beta \subseteq C$ and $C$ has empty interior, a contradiction. Thus, $F$ does not contain any non-trivial measurable rectangles and hence there exist no non-null subsets $\alpha$ and $\beta$ of $[0, 1)$ such that $M_\alpha \otimes M_\beta \leq M_F$.

We will prove that $\phi(M_F)(M_\beta) = 0$ for every measurable $\beta$. Fix such a $\beta$ and set $P = \phi(M_F)(M_\beta)$. Let $P_1$ be the projection onto $\mathcal{D}_0PK$; then $P_1 \in \mathcal{N}_0$ and $P \leq P_1$. Since $P_1 \otimes M_\beta \leq M_F$, we have that $P_1 = 0$, showing that $P = 0$. It follows that identity (3) from Theorem 3.3 does not hold in the case $\mathcal{M} = \mathcal{N}_0$.

(ii) Let $\mathcal{M}$ be an ABSL with the rank one density property, and $E_1$ and $E_2$ be atoms of $\mathcal{M}$. Also let $L_i \in \mathcal{P}$, $i = 1, 2$, be such that $L_1 \wedge L_2 \neq 0$ and
\[ M = (L_1 \otimes E_1) \lor (L_2 \otimes E_2). \]

Clearly
\[ M = (L_1 \otimes E_1) \lor (L_2 \otimes E_2) \lor ((L_1 \land L_2) \otimes (E_1 \lor E_2)) \]

and thus the representation in Theorem 3.3 is not unique.

The map \( \phi \mid_{P \otimes M} \) is not \( \lor \)-preserving and thus not a lattice homomorphism. Indeed, it is easy to check that
\[ \phi(L_i \otimes E_i)(E_i) = L_i, \quad i = 1, 2, \]

and
\[ \phi(L_i \otimes E_i)(E_1 \lor E_2) = 0. \]

Thus,
\[ \phi(M)(E_1 \lor E_2) \neq 0 = (\phi(L_1 \otimes E_1)(E_1 \lor E_2)) \lor (\phi(L_2 \otimes E_2)(E_1 \lor E_2)) \]

and hence \( \phi(M) \neq (\phi(L_1 \otimes E_1)) \lor (\phi(L_2 \otimes E_2)) \).

Theorem 3.3 can now be extended as follows.

**Theorem 3.4.** Let \( \mathcal{L} \) be a separably acting CSL and \( \mathcal{M} \) be a subspace lattice with the rank one density property. Then \( \mathcal{L} \otimes \mathcal{M} \) possesses property (p).

**Proof.** By Theorem 3.3, \( P \otimes \mathcal{M} \) is reflexive. If \( \mathcal{L} \) is a finite CSL then \( \mathcal{L} \) is totally atomic and [7, Theorem 12, Corollary 2] imply that \( \mathcal{L} \otimes \mathcal{P} \otimes \mathcal{M} \) is reflexive. Hence, \( \mathcal{L} \otimes \mathcal{M} \) has property (p).

Now let \( \mathcal{L} \) be an arbitrary separably acting CSL, \( \{L_i\}_{i \in \mathbb{N}} \) be a strongly dense subset of \( \mathcal{L} \), and \( \mathcal{L}_n \) be the subspace lattice generated by the set \( \{L_i\}_{i=1}^n \), \( n \in \mathbb{N} \); clearly, \( \mathcal{L}_n \) is finite for all \( n \in \mathbb{N} \). Since \( \mathcal{L} = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n \), we have that \( \mathcal{L} \otimes \mathcal{M} = \bigvee_{n \in \mathbb{N}} (\mathcal{L}_n \otimes \mathcal{M}) \). By the previous paragraph, \( \mathcal{L}_n \otimes \mathcal{M} \) has property (p) for all \( n \in \mathbb{N} \). By the strict approximativity of property (p) (see [15, Proposition 4.1]), \( \mathcal{L} \otimes \mathcal{M} \) has property (p). \( \square \)

### 4. Tensoring with atomic Boolean subspace lattices

In this section, we restrict our attention to the case where \( \mathcal{M} \) is an Atomic Boolean Subspace Lattice (ABSL) possessing the ultraweak rank one density property. Two atom ABSLs, namely lattices of the form \( \{0, P, Q, I\} \), where \( P \land Q = 0 \) and \( P \lor Q = I \), satisfy this property [13] and it is not difficult to show that the rank one density property is preserved under taking meshed product (see [2] for the definition and properties of this construction).

Our aim is to show that, if \( \mathcal{M} \) is an ABSL with the rank one density property, \( \mathcal{E} \) is the set of its atoms, and \( \mathcal{L} \) is an arbitrary subspace lattice, then the map \( \theta \) is an isomorphism from \( m(\mathcal{E}, \mathcal{L}) \) onto \( \mathcal{L} \otimes \mathcal{M} \). We first establish an important special case.

**Lemma 4.1.** Let \( \mathcal{M} \) be an ABSL acting on a Hilbert space \( H \) having the rank one density property and let \( \mathcal{E} = \{E_j : j \in J\} \) be the set of its atoms. Then \( \theta \mid_{m(\mathcal{E}, \mathcal{P})} \) is a complete lattice isomorphism of \( m(\mathcal{E}, \mathcal{P}) \) onto \( \mathcal{P} \otimes \mathcal{M} \) with inverse \( \phi_{\mathcal{E}, \mathcal{P}} \).
Proof. Let $M_j = \{ L \in M : E_j \leq L \}, j \in J$. Fix $M \in P \otimes M$ and let $f = \phi_{M, P}(M)$. By Theorem 3.3,

$$M = \bigvee_{L \in M} (f(L) \otimes L) = \bigvee_{L \in M} (f(L) \otimes (\bigvee_{E_j \leq L} E_j)) = \bigvee_{L \in M, E_j \leq L} (f(L) \otimes E_j) = \bigvee_{j \in J} f(E_j) \otimes E_j,$$

where the last identity follows from (2). Thus,

$$\theta \circ \phi_{E, P}(M) = M, \quad M \in P \otimes M,$$

Let $\phi = \phi_{E, P}|_{P \otimes M}$ for brevity. We next check that

$$\phi \circ \theta(f) = f, \quad f \in m(E, P).$$

Let $f \in m(E, P)$, $g = \phi \circ \theta(f)$, $M_j = f(E_j)$ and $P_j = g(E_j)$, $j \in J$. Set $M = \theta(f)$; by (3),

$$M = \theta(\phi(M)) = \theta(g).$$

By the definition of $\phi$, we have that $f \leq g$, that is, $M_j \leq P_j$ for all $j \in J$. Suppose that there exists $i \in J$ such that $M_i < P_i$. We have that

$$M = \bigvee_{j \neq i} (M_j \otimes E_j) \leq (M_i \otimes E_i) \vee \left( \bigvee_{j \neq i} (I \otimes E_j) \right)$$

and

$$= (M_i \otimes E_i) \vee \left( I \otimes \left( \bigvee_{j \neq i} E_j \right) \right)$$

where for last equality we have used the fact that $M_i \otimes I$ and $M_i \perp \left( \bigvee_{j \neq i} E_j \right)$ are orthogonal. Let now $0 \neq p \in (P_iK) \oplus (M_iK)$ and $0 \neq e \in E_iH$. Using (3), we have that $p \otimes e \in (P_i \otimes E_i)(K \otimes H) \subseteq M(K \otimes H)$ and that $(M_i \otimes I)(p \otimes e) = 0$. Hence

$$p \otimes e \in \left( M_i \perp \left( \bigvee_{j \neq i} E_j \right) \right)(K \otimes H).$$
and therefore
\[ 0 \neq e \in \left( \bigvee_{j \neq i} E_j \right) \wedge (E_i H) = \{0\}, \]
a contradiction. Hence \( f = g = \phi(\theta(f)) \) and (7) is proved.

By Proposition 3.1, \( \theta|_{m(\mathcal{E}, \mathcal{P})} \) is \( \vee \)-preserving. By Theorem 3.3, \( \phi \) is \( \wedge \)-preserving. Let \( (f_n)_{n \in \mathbb{N}} \subseteq m(\mathcal{E}, \mathcal{P}) \). Using (7), we have
\[ \phi(\theta(\wedge_{\alpha \in \mathcal{A}} f_{\alpha})) = \wedge_{\alpha \in \mathcal{A}} (f_{\alpha} \circ \theta) = \phi(\wedge_{\alpha \in \mathcal{A}} (f_{\alpha})). \]
By (6), \( \phi \) is injective and so \( \theta(\wedge_{\alpha \in \mathcal{A}} f_{\alpha}) = \wedge_{\alpha \in \mathcal{A}} \theta(f_{\alpha}). \) The proof is complete. \( \square \)

It will be helpful to isolate the following statement contained in Lemma 4.1 for future reference.

**Corollary 4.2.** Let \( \mathcal{M} \) be an ABSL acting on a Hilbert space \( H \) having the rank one density property and let \( \mathcal{E} = \{ E_j : j \in J \} \) be the set of its atoms. If \( M \in \mathcal{P} \otimes \mathcal{M} \), then there exists a unique family \( (P_j)_{j \in J} \subseteq \mathcal{P} \) such that \( M = \bigvee_{j \in J} (P_j \otimes E_j) \).

**Lemma 4.3.** Let \( H \) be a Hilbert space, \( \mathcal{M} \) be an ABSL on \( H \) with atoms \( E_j, j \in J \) having the rank one density property, and \( \{ f, f_n : n \in \mathbb{N} \} \subseteq m(\mathcal{E}, \mathcal{P}) \).

(i) If \( \theta(f_n) \rightarrow_{n \rightarrow \infty} \theta(f) \) semistrongly then \( f_n(E_j) \rightarrow_{n \rightarrow \infty} f(E_j) \) semistrongly for every \( j \in J \).

(ii) If \( f_n(E_j) \rightarrow_{n \rightarrow \infty} f(E_j) \) semistrongly for every \( j \in J \) then there exists a subsequence \( (\theta(f_{n_k}))_{k \in \mathbb{N}} \) of \( (\theta(f_n))_{n \in \mathbb{N}} \) such that \( \theta(f_{n_k}) \rightarrow_{k \rightarrow \infty} \theta(f) \) semistrongly.

**Proof.** Let \( L^j_n = f_n(E_j) \) and \( L_j = f(E_j), j \in J, n \in \mathbb{N} \).

(i) Fix \( k \in J \) and let \( (x_i)_{i \in \mathbb{N}} \) be a sequence such that \( x_i \in L^k_n K, i \in \mathbb{N} \), and \( x_i \rightarrow x \) (where the sequence \( (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N} \) is strictly increasing). Fix a non-zero vector \( p \in E_k H \). It follows that \( x_i \otimes p \rightarrow x \otimes p \). Clearly, \( x_i \otimes p \in \theta(f_{n_k})(K \otimes H) \) for all \( i \in \mathbb{N} \) and thus, by hypothesis, \( x \otimes p \in \theta(f)(K \otimes H) \).

Let \( \mathcal{W} = \{ y : y \otimes p \in \theta(f)(K \otimes H) \} \) for all \( p \in E_k H \).

Clearly, \( \mathcal{W} \) is a closed subspace such that \( L_k K \subseteq \mathcal{W} \) and \( x \in \mathcal{W} \). Also, \( \mathcal{W} \otimes E_k H \subseteq \theta(f)(K \otimes H) \). By Lemma 4.1,
\[ \mathcal{W} \otimes E_k H \subseteq \left( \bigvee_{j \in J} L_j \otimes E_j \right) (K \otimes H) \wedge (K \otimes E_k H) = L_k K \otimes E_k H. \]

It follows that \( \mathcal{W} \subseteq L_k K \) and so \( x \in L_k K \).

Let \( q \) be a non-zero vector in \( H \) such that \( \left( \bigvee_{j \neq k} E_j \right) q = (E_k)_{-} \) = 0. Write \( q = p_0 + p' \) where \( p_0 = E_k p_0 \) and \( E_k p'_0 = 0 \). Since \( E_k^\perp \cap \left( \bigvee_{j \neq k} E_j \right) = (E_k \cap \left( \bigvee_{j \neq k} E_j \right))^\perp = 0 \), it follows that \( p_0 \neq 0 \) and thus \( (p_0, q) \neq 0 \). Let \( p = \frac{p_0}{(p_0, q)} \); we have that \( R_{p,q} \in \text{Alg } \mathcal{M} \). Clearly \( R_{p,q} p = p \) and \( R_{p,q} \) annihilates \( \bigvee_{j \neq k} E_j \).
Fix \( x \in L_K K \). By hypothesis, there exist a sequence \( (\xi_n)_{n \in \mathbb{N}} \) such that \( \xi_n = \theta(f_n)\xi_n, \ n \in \mathbb{N}, \) and \( \xi_n \to_{n \to \infty} x \otimes p. \) Thus,
\[
x \otimes p = (I \otimes R_{p,q})(x \otimes p) = \lim_{n \to \infty} (I \otimes R_{p,q})\xi_n.
\]

By the definition of \( R_{p,q} \), we have that \((I \otimes R_{p,q})\xi_n \in (L^k_n \otimes E_k)(K \otimes H)\). Let \( \psi : K \otimes H \to K \) be the bounded linear operator such that
\[
\psi(x_1 \otimes x_2) = \frac{(x_2, p)}{\|p\|^2} \cdot x_1, \ x_1 \in K, \ x_2 \in H.
\]
Clearly, \( \psi(I \otimes R_{p,q})\xi_n \in L^k_n K \) for all \( n \in \mathbb{N}, \) and \( \psi((I \otimes R_{p,q})\xi_n) \to \psi(x \otimes p) = x \). This shows that \( L^k_n \to L_k \) semistrongly.

(ii) Suppose that \( f_n(E_j) \to f(E_j) \) semistrongly for all \( j \in J \). By the weak compactness of the unit ball of \( B(H) \) (see, e.g. [3, Proposition 5.5]), there exists a subsequence \( (\theta(f_{n_k}))_{k \in \mathbb{N}} \) of \( (\theta(f_n))_{n \in \mathbb{N}} \) and a positive contraction \( W \) on \( H \) such that \( \theta(f_{n_k}) \to_{k \to \infty} W \) in the weak operator topology. By [6], \( (\theta(f_{n_k}))_{k \in \mathbb{N}} \) converges semistrongly to the orthogonal projection \( Q \) onto \( \ker(I - W) \). By Theorem 3.3, \( \mathcal{P} \otimes \mathcal{M} \) is reflexive and, by [14, Proposition 3.1], it is semistrongly closed. Thus, \( Q \in \mathcal{P} \otimes \mathcal{M} \) and, by Lemma 4.1, \( Q = \theta(g) \) for some \( g \in m(\mathcal{E}, \mathcal{P}) \). By (i), \( f_{n_k}(E_j) \to_{k \to \infty} g(E_j) \) semistrongly. By the uniqueness of the semistrong limit, \( f(E_j) = g(E_j) \) for all \( j \in J, \) that is, \( f = g \) and so \( \theta(f_{n_k}) \to_{k \to \infty} \theta(f) \) semistrongly.

The next proposition is certainly well-known; since we were not able to find a corresponding reference, we include its short proof for the convenience of the reader.

**Proposition 4.4.** Let \( \mathcal{M} \) be an ABSL, \( \mathcal{E} = \{E_j\}_{j \in J} \) be the set of its atoms, and let \( D_j = \wedge_{i \neq j} (E_i^\perp), \ j \in J. \) Then \( \mathcal{M}^\perp \overset{def}{=} \{L^\perp : L \in \mathcal{M}\} \) is an ABSL whose set of atoms is \( \mathcal{D} = \{D_j\}_{j \in J}. \)

**Proof.** It is a direct consequence of the de Morgan laws that \( \mathcal{M}^\perp \) is distributive and that if \( L \in \mathcal{M} \) and \( L' \in \mathcal{M} \) is the complement of \( L \) in \( \mathcal{M}, \) then \( L' \) is a complement of \( L^\perp \) in \( \mathcal{M}^\perp \).

Let \( L \in \mathcal{M}^\perp. \) If \( 0 \leq L < D_j \) for some \( j \in J, \) then \( \vee_{i \neq j} E_i = D_j^\perp < L^\perp \in \mathcal{M}. \) Since \( L^\perp \) is equal to the closed linear span of the atoms that it majorises, it must contain \( E_i \) and hence \( L^\perp = I, \) that is, \( L = 0. \) Thus, \( D_j \) is an atom of \( \mathcal{M}^\perp, \) for each \( j \in J. \)

If \( L \in \mathcal{M}^\perp, \) then there exists \( S \subseteq J \) such that \( L^\perp = \vee_{j \in S} E_j. \) By distributivity,
\[
L = \vee_{j \notin S}(\wedge_{i \neq j} E_i^\perp) = \vee_{j \notin S} D_j.
\]
We thus showed that \( \mathcal{M}^\perp \) is an ABSL with atoms \( \{D_j : j \in J\}. \)

In the rest of the section, we adopt the notation from Proposition 4.4. If \( f \in m(\mathcal{E}, \mathcal{P}), \) let \( f^\perp \in m(\mathcal{D}, \mathcal{P}) \) be the map given by \( f^\perp(D_j) = f(E_j)^\perp, \ j \in J. \)
Lemma 4.5. Let $\mathcal{M}$ be an ABSL with the rank one density property and $\mathcal{E}$ be the set of its atoms. If $f \in m(\mathcal{E}, \mathcal{P})$ then $\theta(f) = \theta(f^\perp)$.

Proof. Since $\mathcal{M}$ has the rank one density property, the identity $\text{Alg}(\mathcal{M})^\perp = (\text{Alg}(\mathcal{M}))^\perp$ implies that $\mathcal{M}^\perp$ has the rank one density property as well. Let $f \in m(\mathcal{E}, \mathcal{P})$ and $L_j = f(E_j), j \in J$. Then

$$\theta(f) = (\cup_{j \in J} (L_j \otimes E_j))^\perp = \cap_{j \in J} (L_j \otimes E_j)^\perp$$

$$= \cap_{j \in J} ((L_j^\perp \otimes I) \cup (L_j \otimes E_j^\perp))$$

$$= \cap_{j \in J} ((L_j^\perp \otimes D_j) \cup (L_j \otimes E_j^\perp))$$

$$= \cap_{j \in J} ((L_j^\perp \otimes I) \cup (I \otimes E_j^\perp))$$

$$= \cap_{j \in J} ((L_j^\perp \otimes D_j) \cup (I \otimes (\cup_{i \neq j} D_i)))$$

$$= \cap_{j \in J} ((L_j^\perp \otimes D_j) \cup (\cup_{i \neq j} (I \otimes D_i)) = \cup_{j \in J} (L_j^\perp \otimes D_j) = \theta(f),$$

where at the second last equality we used Lemma 4.1. □

The main result of this section is the following.

Theorem 4.6. Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $K$ and $\mathcal{M}$ be an ABSL with the rank one density property. Let $\mathcal{E} = \{E_j : j \in J\}$ be the set of atoms of $\mathcal{M}$. Then $\theta|_{m(\mathcal{E}, \mathcal{L})}$ is a complete lattice isomorphism of $m(\mathcal{E}, \mathcal{L})$ onto $\mathcal{L} \otimes \mathcal{M}$ with inverse $\phi_{\mathcal{E}, \mathcal{P}}|_{\mathcal{L} \otimes \mathcal{M}}$.

Proof. Let

$$\mathcal{F} = \theta(m(\mathcal{E}, \mathcal{L})) = \{ \cup_{j \in J} (L_j \otimes E_j) : L_j \in \mathcal{L}, j \in J\}.$$ 

By Lemma 4.1, $\mathcal{F}$ is a projection lattice. We will show that $\mathcal{F}$ is strongly closed. Let $f_n \in m(\mathcal{E}, \mathcal{L}), n \in \mathbb{N}$, and let $Q$ be a projection with $\theta(f_n) \to Q$ in the strong operator topology. Since $\theta(f_n) \in \mathcal{P} \otimes \mathcal{M}$ for all $n$, we have that $Q \in \mathcal{P} \otimes \mathcal{M}$. By Lemma 4.1, $Q = \theta(f)$ for some $f \in m(\mathcal{E}, \mathcal{P})$. Since $\theta(f_n) \to_{n \to \infty} \theta(f)$ semistrongly [6], Lemma 4.3 (i) implies that $f_n(E_j) \to_{n \to \infty} f(E_j)$ semistrongly, for all $j \in J$.

Since $\mathcal{M}$ has the rank one density property, $\mathcal{M}^\perp$ does so as well. By [6], $\theta(f_n)^\perp \to_{n \to \infty} \theta(f)^\perp$ semistrongly and by Lemma 4.3 (i) $\theta(f_n^\perp) \to_{n \to \infty} \theta(f^\perp)$ semistrongly. By Lemma 4.3 (i), $f_n^\perp(D_j) \to_{n \to \infty} f^\perp(D_j)$ semistrongly for all $j \in J$, that is, $f_n(E_j)^\perp \to_{n \to \infty} f(E_j)^\perp$ semistrongly for all $j \in J$. By [6], $f_n(E_j) \to_{n \to \infty} f(E_j)$ in the strong operator topology and, since $\mathcal{L}$ is strongly closed, we conclude that $f(E_j) \in \mathcal{L}$ for all $j \in J$. Thus, $\mathcal{F}$ is strongly closed. It follows that $\mathcal{F} = \mathcal{L} \otimes \mathcal{M}$.

If $Q \in \mathcal{L} \otimes \mathcal{M}$ then by Corollary 4.2 there exists a unique $f \in m(\mathcal{E}, \mathcal{P})$ such that $\theta(f) = Q$. Since $\mathcal{L} \otimes \mathcal{M} = \theta(m(\mathcal{E}, \mathcal{L}))$, we have that $\theta(f) \in m(\mathcal{E}, \mathcal{L})$. Thus, $\phi_{\mathcal{E}, \mathcal{P}}(Q) \in m(\mathcal{E}, \mathcal{L})$, and the rest of the statements follow from Lemma 4.1. □
We include some immediate corollaries of Theorem 4.6.

Corollary 4.7. Let \( \mathcal{M} \) be an ABL acting on a Hilbert space \( H \) having the rank one density property, \( \mathcal{E} = \{ E_j : j \in J \} \) be the set of its atoms and \( \mathcal{L} \) be any subspace lattice. If \( M \in \mathcal{L} \otimes \mathcal{M} \), then there exists a unique family \( (P_j)_{j \in J} \subseteq \mathcal{L} \otimes \mathcal{M} \) such that
\[
M = \vee_{j \in J} (P_j \otimes E_j).
\]
Moreover, \( P_j \in \mathcal{L} \) for each \( j \in J \).

Corollary 4.8. Let \( \mathcal{L} \) be a subspace lattice and \( \mathcal{M} \) be an ABL with the rank one density property. If \( \mathcal{L} \) is distributive then so is \( \mathcal{L} \otimes \mathcal{M} \).

We finish this section with a stability result about semistrong closedness. We refer the reader to [14], where semistrongly closed subspace lattices were studied in detail.

Proposition 4.9. Let \( \mathcal{L} \) be a subspace lattice acting on a Hilbert space \( K \) and \( \mathcal{M} \) be an ABL acting on a Hilbert space \( H \), having the rank one density property. The lattice \( \mathcal{L} \) is semistrongly closed if and only if the lattice \( \mathcal{L} \otimes \mathcal{M} \) is semistrongly closed.

Proof. Suppose that \( \mathcal{L} \) is semistrongly closed and assume that \( \{ Q_n : n \in \mathbb{N} \} \subseteq \mathcal{L} \otimes \mathcal{M} \) with \( Q_n \to Q \) semistrongly for some projection \( Q \) on \( K \otimes H \). By Theorem 4.3, \( \mathcal{P} \otimes \mathcal{M} \) is reflexive, and by [14], it is semistrongly closed; hence, \( Q \in \mathcal{P} \otimes \mathcal{M} \). Thus, by Lemma 4.1, \( Q = \theta(f) \) for some \( f \in m(\mathcal{E}, \mathcal{P}) \), where \( \mathcal{E} \) is the set of atoms of \( \mathcal{M} \). By Theorem 4.6, there exist \( f_n \in m(\mathcal{E}, \mathcal{L}) \) such that \( Q_n = \theta(f_n), n \in \mathbb{N} \). By Lemma 4.3 (i), \( f_n(E_j) \to f(E_j) \) semistrongly and since \( \mathcal{L} \) is semistrongly closed, \( f(E_j) \in \mathcal{L} \); therefore, \( Q \in \mathcal{L} \otimes \mathcal{M} \).

Conversely, suppose that \( \mathcal{L} \otimes \mathcal{M} \) is semistrongly closed. Fix an atom \( E \) of \( \mathcal{M} \). Suppose that \( (L_n)_{n \in \mathbb{N}} \subseteq \mathcal{L} \) and that \( L_n \to_{n \to \infty} L \) semistrongly for some projection \( L \in \mathcal{P} \). By Lemma 4.3 (ii), there exists a subsequence \( (n_k)_{k \in \mathbb{N}} \) with \( L_{n_k} \otimes E \to_{k \to \infty} L \otimes E \) semistrongly. Since \( \mathcal{L} \otimes \mathcal{M} \) is semistrongly closed, \( L \otimes E \in \mathcal{L} \otimes \mathcal{M} \) and, by Corollary 4.7, \( L \in \mathcal{L} \). \hfill \( \square \)

5. LTPF and other consequences

The next theorem, along with Corollary 5.2, are the main results of this section. We also give some more consequences of the results from the previous sections.

Theorem 5.1. Let \( \mathcal{L} \) be a subspace lattice acting on a Hilbert space \( K \) and \( \mathcal{M} \) be an ABL acting on a Hilbert space \( H \) and having the rank one density property. Let \( \mathcal{E} = \{ E_j : j \in J \} \) be the set of atoms of \( \mathcal{M} \). Then
\[
\text{Lat Alg}(\mathcal{L} \otimes \mathcal{M}) = (\text{Lat Alg} \mathcal{L}) \otimes \mathcal{M}
\]
\[
= \{ \bigvee_{j \in J} (f(E_j) \otimes E_j) : f \in m(\mathcal{E}, \text{Lat Alg} \mathcal{L}) \}.
\]

Proof. The second equality follows from Corollary 4.7. By hypothesis, the subalgebra of \( \mathcal{A} = \text{Alg} \mathcal{M} \) generated by the rank one operators in \( \mathcal{A} \) is dense
Then the LTPF holds for \( \mathcal{A} \) and \( \mathcal{M} \).

Proof. The ATPF holds for \( \mathcal{L} \) and \( \mathcal{M} \) because \( \mathcal{M} \) has the ultraweak rank one density property (see [10, Theorem 2.1 and Proposition 1.1]). Let \( \mathcal{A} = \text{Alg}(\mathcal{L} \otimes \mathcal{M}) \) and \( \mathcal{B} = \text{Alg}(\mathcal{L}) \). Using Theorem 5.1 we have

\[
\text{Lat}(\mathcal{B} \otimes \mathcal{A}) = \text{Lat Alg}(\mathcal{L} \otimes \mathcal{M}) = (\text{Lat Alg}(\mathcal{L}) \otimes \mathcal{M}) = (\text{Lat Alg}(\mathcal{L}) \otimes \text{Lat Alg}(\mathcal{M})).
\]

\( \square \)

**Corollary 5.3.** Let \( \mathcal{M} \) be an ABSL having the rank one density property. A subspace lattice \( \mathcal{L} \) is reflexive if and only if \( \mathcal{L} \otimes \mathcal{M} \) is reflexive.

Proof. If \( \mathcal{L} \) is reflexive then \( \mathcal{L} \otimes \mathcal{M} \) is reflexive by Theorem 5.1. Conversely, suppose that \( \mathcal{L} \otimes \mathcal{M} \) is reflexive. Let \( L \in \text{Lat Alg}(\mathcal{L}) \) and \( E \in \mathcal{M} \) be an atom. By Theorem 5.1 \( L \otimes E \in \mathcal{L} \otimes \mathcal{M} \) and, by Corollary 4.7, \( L \in \mathcal{L} \). \( \square \)
Corollary 5.4. If $\mathcal{L}$ is a subspace lattice having property (p) and $\mathcal{M}$ is an ABSL having the rank one density property, then $\mathcal{L} \otimes \mathcal{M}$ has property (p).

Proof. By hypothesis, we have that $\mathcal{P} \otimes \mathcal{L}$ is reflexive. It follows from Corollary 5.2 that $\mathcal{P} \otimes \mathcal{L} \otimes \mathcal{M}$ is reflexive, that is, $\mathcal{L} \otimes \mathcal{M}$ has property (p). \hfill \Box

Corollary 5.5. Let $H$ be a Hilbert space and $P$ and $Q$ be projections acting on $H$ such that $P \land Q = 0$ and $P \lor Q = I$. If $\mathcal{M} = \{0, P, Q, I\}$ and $\mathcal{L}$ is a subspace lattice acting on a Hilbert space $K$, then

$$\text{Lat Alg}(\mathcal{L} \otimes \mathcal{M}) = \{(L_1 \otimes P) \lor (L_2 \otimes Q) : L_1, L_2 \in \text{Lat Alg} \mathcal{L}\}.$$  
Furthermore, if $\mathcal{L}$ is reflexive, then the LTPF holds for $\text{Alg} \mathcal{L}$ and $\text{Alg} \mathcal{M}$, and the lattice $\mathcal{L} \otimes \mathcal{P}$ is reflexive.

Proof. The statement is immediate from Theorem 5.1 Corollary 5.2 and the fact that two atom ABSLs satisfy the rank one density property \cite{13, Theorem 2.1}. \hfill \Box

We finish this section with the following additional consequence of the above results.

Theorem 5.6. Let $\mathcal{L}$ and $\mathcal{M}$ be ABSLs with sets of atoms $\{D_i : i \in I\}$ and $\{E_j : j \in J\}$, respectively. If either $\mathcal{L}$ or $\mathcal{M}$ has the rank one density property, then $\mathcal{L} \otimes \mathcal{M}$ is an ABSL whose set of atoms is $\{D_i \otimes E_j : (i, j) \in I \times J\}$.

Proof. Without loss of generality, we assume that $\mathcal{M}$ has the rank one density property. By Corollary 5.2 and the fact that every ABSL is reflexive \cite{5}, we have that

$$\mathcal{L} \otimes \mathcal{M} = \{ \lor_{j \in J} (P_j \otimes E_j) : P_j \in \mathcal{L}, j \in J\}.$$  
On the other hand, for $j \in J$, we have that

$$P_j \otimes E_j = (\lor_{D_i \subseteq P_j} D_i) \otimes E_j = \lor_{D_i \subseteq P_j} D_i \otimes E_j.$$  
Thus, every element in $\mathcal{L} \otimes \mathcal{M}$ is the span of elements of the set $\{D_i \otimes E_j : (i, j) \in I \times J\}$.

Suppose that $L = \lor_{j \in J} (P_j \otimes E_j) \subseteq (D_{i_0} \otimes E_{j_0})$ where $(i_0, j_0) \in I \times J$. Since $\mathcal{L}$ is an ABSL, we have that either $P_{j_0} \land D_{i_0} = 0$, or $D_i \subseteq P_{j_0}$. If $D_{i_0} \subseteq P_{j_0}$, then $D_{i_0} \otimes E_{j_0} \subseteq L$ and thus $D_{i_0} \otimes E_{j_0} = L$. By hypothesis, $D_{i_0} \otimes E_{j_0} \neq L$, hence $P_{j_0} \land D_{i_0} = 0$. By Theorem 4.6

$$L = L \land (D_{i_0} \otimes E_{j_0}) = (P_{j_0} \land D_{i_0}) \otimes E_{j_0} = 0.$$  
Thus, $D_i \otimes E_j$ is an atom of $\mathcal{L} \otimes \mathcal{M}$ for all $i$ and $j$.

It remains to prove that $\mathcal{L} \otimes \mathcal{M}$ is complemented and distributive. Let $L = \lor_{j \in J} (P_j \otimes E_j)$, where $P_j \in \mathcal{L}$, $j \in J$, and let $P_j'$ be the complement of $P_j$ in $\mathcal{L}$, for all $j \in J$. If $L' = \lor_{j \in J} (P_j' \otimes E_j)$, then

$$L \lor L' = \lor_{j \in J} ((P_j \lor P_j') \otimes E_j) = \lor_{j \in J} (I \otimes E_j) = I.$$
and, by Theorem 4.6
\[ L \land L' = \lor_{j \in J} ((P_j \land P'_j) \otimes E_j) = 0. \]
Hence $L'$ is a complement for $L$. Finally, the distributivity of $\mathcal{L} \otimes \mathcal{M}$ follows from Corollary 4.8. □

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