BREAKDOWN OF $I$–LOVE–$Q$ UNIVERSALITY IN RAPIDLY ROTATING RELATIVISTIC STARS

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Received 2013 November 11; accepted 2013 December 5; published 2013 December 23

ABSTRACT

It was shown recently that normalized relations between the moment of inertia ($I$), the quadrupole moment ($Q$), and the tidal deformability (Love number) exist and for slowly rotating neutron stars they are almost independent of the equation of state (EOS). We extend the computation of the $I$–$Q$ relation to models rotating up to the mass-shedding limit and show that the universality of the relations is lost. With increasing rotation rate, the normalized $I$–$Q$ relation departs significantly from its slow-rotation limit, deviating up to 40% for neutron stars and up to 75% for strange stars. The deviation is also EOS dependent and for a broad set of hadronic and strange matter EOSs the spread due to rotation is comparable to the spread due to the EOS, if one considers sequences with fixed rotational frequency. Still, for a restricted sample of modern realistic EOSs one can parameterize the deviations from universality as a function of rotation only. The previously proposed $I$–Love–$Q$ relations should thus be used with care, because they lose their universality in astrophysical situations involving compact objects rotating faster than a few hundred Hz.

Key words: equation of state – gravitation – stars: neutron – stars: rotation

Online-only material: color figures

1. INTRODUCTION

Neutron stars are at the intersection of astrophysics, gravitational physics, and nuclear physics and are thus excellent laboratories for investigating strong gravitational fields and the behavior of matter at very high densities. Isolated neutron stars or binary systems can be a testbed for general relativity (GR) and other alternative theories of gravity in the strong field regime, while their oscillations and binary inspiral and merger phases are central to the current efforts in gravitational wave detection. Successful detections may allow tighter constraints on the equation of state (EOS) of nuclear matter.

The properties of neutron stars, their spacetime curvature and gravitational wave emission, are ultimately connected to their internal structure. However, at present there are still large uncertainties regarding the properties of matter at very high densities. Some information on the EOS can be indirectly extracted from the characteristics of events on the surface of neutron stars and their exterior spacetime using astrophysical observations. Steps in this direction are being taken (see, e.g., Özel 2013; Steiner et al. 2013 and references therein) and future observations of X-ray busters and the properties of accretion disks in low-mass X-ray binaries may allow the determination of the radius of neutron stars with an accuracy of about 10%. On the other hand, observations of millisecond pulsars in binary systems may allow the measurement of their moment of inertia with similar accuracy (see Lattimer & Schutz 2005; Kramer & Wex 2009). Furthermore, gravitational wave observations of the inspiral of binary neutron star systems will allow the measurement of their tidal deformability (see Flanagan & Hinderer 2008; Hinderer et al. 2010; Bini et al. 2012; Hotokezaka et al. 2013; Radice et al. 2014).

In most cases, such astrophysical measurements are EOS dependent. This prevents us from accurately determining neutron star properties such as mass, radius, and moment of inertia, or distinguishing between neutron stars and strange stars. The EOS uncertainty does not allow for accurate tests of alternative theories of gravity when the deviations from GR are of the same order as the deviations induced by different EOSs. The gravitational wave observations of binary inspirals also face certain problems because of the degeneracy between the spins and the quadrupole moments of the neutron stars. A welcome development in these fields has been the recent discovery of relations between (normalized expressions of) the moment of inertia $I$, the quadrupole moment $Q$ and the tidal Love number, by Yagi & Yunes (2013a, 2013b) (see also Urbanec et al. 2013; Yagi 2013), which are practically EOS independent. However, these relations were derived only for non-magnetized stars in the slow-rotation and small tidal deformation approximations, and the question of their general validity remained open. The small tidal deformation approximation was later relaxed by Maselli et al. (2013). They derived similar universal relations for the different phases of neutron star inspirals and concluded that these relations do not deviate significantly from the small tidal deformation approximation. On the other hand, it was shown recently by Haskell et al. (2013) that the universality of the $I$–Love–$Q$ relations is broken for strongly magnetized neutron stars with low rotational frequencies.

Here, we investigate the effect of rapid rotation on the universality of the $I$–Love–$Q$ relations, by computing the $I$–$Q$ relation for rapidly rotating relativistic stars (up to the mass-shedding limit) for a number of different EOSs. We show that the EOS universality breaks down at fast rotation rates, such as those encountered for the fastest known millisecond pulsars, and the $I$–$Q$ relation departs significantly from its slow rotation limit. We should note that other universal relations were derived very recently by Pappas & Apostolatos (2013) which are nearly EOS independent for all rotational rates of neutron stars within GR. The EOS independence is obtained when using a normalized angular momentum, instead of the spin frequency.
2. FORMALISM

The metric of a stationary and axisymmetric spacetime can be written in the following form:

$$ds^2 = -e^{2\nu}dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu}(d\phi - \omega dt)^2 + e^{2\xi(-\nu)}(dr^2 + r^2 d\theta^2).$$ (1)

The multipole moments of the spacetime are encoded in the asymptotic expansion (at large $r$) of the metric functions $\nu$, $B$, $\xi$, and $\omega$. To leading orders:

$$\nu = -\frac{M}{r} + \left(\frac{B_0 M}{3r} + \frac{v_2 P_2}{r^3}\right) \frac{1}{r^3} + O\left(\frac{1}{r^4}\right),$$ (2)

$$B = 1 + \frac{B_0}{r^2} + O\left(\frac{1}{r^4}\right),$$ (3)

$$\omega = \frac{2J}{r^3} + O\left(\frac{1}{r^4}\right),$$ (4)

where $M$ is the gravitational mass, $J$ is the angular momentum, and $P_2$ is a Legendre polynomial, while $v_2$ and $B_0$ are expansion coefficients. The moment of inertia for a uniformly rotating star

$$I = J/\Omega^2,$$

where $\Omega$ is the angular frequency.

The relativistic quadrupole moment is given by:

$$Q = -v_2 - \frac{4}{3} \left(\frac{1}{4} + \frac{b}{4}\right) M^3,$$ (5)

where $b = B_0/M^2$. A detailed derivation of the above formula is given in the book by Friedman & Stergioulas (2013) and also in the papers by Ryan (1995), Berti & Stergioulas (2004), and Pappas & Apostolatos (2012a, 2012b). Our numerical computation was checked against previous results by Berti & Stergioulas (2004) and Pappas & Apostolatos (2012a, 2012b).

We consider models of uniformly rotating stars, obtained with the {	extsc{vns}} code (Stergioulas & Friedman 1995; Nozawa et al. 1998). Two classes of EOSs are examined: hadronic EOSs, describing neutron stars; and the MIT bag model of self-bound strange quark matter, describing strange stars. We choose six representative neutron star EOSs, which are all in agreement with the observational constraint of a two solar mass constraint, respectively. Our representative set of hadronic EOSs are: Wiringa et al. (1988, WFF2); Akmal et al. (1998, APR); Goriely et al. (2010, GCP); and Hebeler et al. (2010, HLPS), where the Douchin & Haensel (2001) crust EOS is used. Furthermore, we include EOS L by Pandharipande et al. (1976), which is one of the stiffest proposed EOSs, and the zero-temperature limit of the EOS by Shen et al. (1998a, 1998b). The two strange star EOSs are taken from Gondek-Rosinska & Limousin (2008) (denoted by SQSB40 and SQSB60 therein).

3. NEUTRON STARS

In accordance with Yagi & Yunes (2013a, 2013b) we plot the normalized moment of inertia $I = I/M^3$ as a function of the normalized quadrupole moment $\bar{Q} = Q/(M^2 \chi^2)$ where $\chi = J/M^2$. Notice that we are using the gravitational mass $M$, which is the natural choice for fast rotating stars. Using the mass $M_\star$ of a corresponding non-rotating model, as was done by Yagi & Yunes (2013a, 2013b), might also be relevant for some astrophysical applications, because several empirical relations are available that connect the properties of fast rotating neutron stars (such as the spin frequency at the mass-shedding limit or the moment of inertia) to the masses and radii of a non-rotating star (see Friedman & Stergioulas 2013). But, we verified that the latter choice leads to considerably larger deviations from universality for different EOSs in the rapidly rotating case. Figure 1 summarizes our results for all hadronic EOSs and for different rotational frequencies $f$. Different colors correspond to different fixed rotational frequencies, while the universal numerical fit (valid at slow rotation) derived by Yagi & Yunes (2013a, 2013b) is shown as a solid black line. The deviations of the numerical results from a fourth order fit is shown in the lower panel for two representative rotational frequencies— in the slow rotation limit $f = 160$ Hz and for $f = 800$ Hz. (A color version of this figure is available in the online journal.)

Figure 1. Upper panel: the normalized moment of inertia as a function of the normalized quadrupole moment. Different colors correspond to different rotational frequencies, while different symbol shapes are used for the different EOSs. The EOS independent relation found by Yagi & Yunes (2013a, 2013b) in the slow rotation limit is shown as a black line. Lower panel: the deviations from fourth order fits to the data for two representative frequencies $f = 160$ Hz and $f = 800$ Hz.

(A color version of this figure is available in the online journal.)
consider sequences of fixed rotational frequency.\(^6\) Whereas the \(I \approx Q\) relation is practically independent of the EOS in the slow rotation limit (less than 160 Hz in Figure 1), for higher rotational frequencies the relation becomes more and more EOS dependent, and the EOS universality is completely lost at rapid rotation. The spread of the data due to different EOSs is comparable to the spread due to rotation and the deviation from a fourth order fit \(|I - I_{\text{fit}}|/I_{\text{fit}}\) can reach 15\% for millisecond pulsars (lower panel of Figure 1).

There is one additional qualitative feature of our results worth mentioning. As we can see in Figure 1, the spread of the dependences for different EOSs decreases for lower values of \(I\) and \(Q\). Also, the differences between the dependences for different rotation rates is smaller in that case. This is an expected effect, since smaller values of the normalized quantities \(I\) and \(Q\) correspond to larger masses and compactness, which loosely speaking means that we are approaching the black hole limit, where the \(I \approx Q\) relation is independent of the internal structure, as commented by Yagi & Yunes (2013a, 2013b).

If one restricts attention to the set of modern EOSs that produce 1.4–2.0 \(M_\odot\) models with intermediate values for radii [\(10.5 \text{ km} \cdots 12.5 \text{ km}\)] (WFF2, APR, GCP, and HLPS), as suggested by the observations (see Özel 2013; Lattimer 2012; Steiner et al. 2010) then at each fixed rotational frequency a \(I \approx Q\) relation that is roughly EOS independent still holds (with deviations from it increasing as the mass-shedding limit is approached).

For the above-mentioned restricted sample of EOSs, we can quantify how the \(I \approx Q\) relation changes with rotation in the following way: we calculated a series of models with different EOSs and central energy densities for several fixed rotational frequencies. For each rotational frequency, we fit the \(I \approx Q\) relation with a second order polynomial of the form

\[
\ln I = a_0 + a_1 \ln Q + a_2 (\ln Q)^2. \tag{6}
\]

Here, we use a second order polynomial, in contrast to the fourth order fit by Yagi & Yunes (2013a, 2013b) as we find that it is of sufficient accuracy, while it is easier to quantify the rotational dependence of the coefficients \(a_i\). A plot of the three fitting coefficients as a function of the rotational frequency is given in Figure 2. This dependence can be well approximated with a third order polynomial fit of the form\(^7\)

\[
a_i = c_0 + c_1 \left( \frac{f}{1 \text{ kHz}} \right) + c_2 \left( \frac{f}{1 \text{ kHz}} \right)^2 + c_3 \left( \frac{f}{1 \text{ kHz}} \right)^3 \tag{7}
\]

In Table 1 we give the results for the coefficients \(c_i\).

\(^6\) Yagi & Yunes (2013b) suggested that a universal relation should still exist at fixed rotational frequencies, estimating a maximum deviation from the slow-rotation limit of the order of 10\% (while our current results show a maximum deviation of 40\% for hadronic EOSs and 75\% for strange matter EOSs).

\(^7\) If we use instead a fourth order polynomial in Equation (6), as was done in Yagi & Yunes (2013a, 2013b), the error in the fitting of \(a_i(f)\) is considerably larger, especially for low rotational frequencies.

### Table 1

| \(a_i\) | \(c_0\) | \(c_1\) | \(c_2\) | \(c_3\) |
|-------|-------|-------|-------|-------|
| \(a_0\) | 1.406 | −0.051 | 0.154 | −0.131 |
| \(a_1\) | 0.489 | 0.183 | −0.562 | 0.471 |
| \(a_2\) | 0.098 | −0.136 | 0.463 | −0.273 |

The question is whether these fitting formulae for the coefficients \(a_i\) are sufficiently accurate. Figure 3 shows that this is indeed the case and the results in Table 1 can be used in practice to accurately obtain the \(I \approx Q\) dependence (6) for our restricted sample of EOSs at different rotation rates. The deviation from these approximate fits, defined as \(|I - I_{\text{fit}}|/I_{\text{fit}}\), reaches roughly 2\% for all frequencies below 500 Hz and it increases up to 4.5\% for the higher rotational rates. The deviations for the slowly rotating models are mostly due to the error in the approximate fitting formulae (6) and (7) we derive, while for rotational frequencies above a few hundred Hz it is caused mainly by the spread of the different EOSs. In general, the least massive models for each frequency lead to the largest deviations from the approximate fits. One should keep in mind that these deviations depend also on the choice of the set of restricted EOSs—if...
certain EOSs are included or excluded from the set, the percent-
geages might change.

The lower panel in Figure 3 shows the gravitational mass as a function of the normalized quadrupole moment. For very large masses (above $2M_\odot$), the $\overline{I} - \overline{Q}$ relation becomes relatively insensitive to the rotation rate. However, at $M = 1.4M_\odot$ the relation depends strongly on rotation.

### 3.1. Strange Stars

The results for the two strange star EOSs we are considering are shown in Figure 4 for several values of $f$. In order to compare these with the corresponding relations for neutron stars, we also show the approximate polynomial fits for neutron stars derived in the previous section (given by Equations (6) and (7)), for the same values of $f$. The maximum value of $\chi$ for the sample of strange star models considered here is 1.05.

In the slow rotation limit, the relations for neutron stars and for both strange star EOSs are similar, in accordance with the results in Yagi & Yunes (2013a). However, for rapid rotation the deviations from the neutron star case become significant as Figure 4 shows. The two strange star EOSs lead to very different $\overline{I} - \overline{Q}$ relations (distinct data lines of the same color correspond to the two difference strange star EOSs) and the deviation of the $\overline{I} - \overline{Q}$ relation from its slow-rotation limit reaches approximately 75%. As a result, it is not possible to derive a single fit of the form given by Equation (6) (at a fixed rotation rate) that would be valid for all strange star EOSs.

Another qualitative difference between neutron stars and strange stars is the following: as we commented above, for neutron star masses near or above $2M_\odot$ the deviation from universality of the $\overline{I} - \overline{Q}$ relation decreases significantly, i.e., the relation becomes relatively insensitive to the rotational frequency and to the internal structure. But this is not the case for strange stars—even for masses above $2M_\odot$, the $\overline{I} - \overline{Q}$ relations are quite different for different values of $f$, even though they become more insensitive to the strange matter EOSs. One of the main reasons is that strange stars reach smaller values of compactness compared to neutron stars.

### 4. CONCLUSIONS

We have calculated the $\overline{I} - \overline{Q}$ relation for rapidly rotating neutron stars and strange stars and showed that it differs significantly from the EOS independent relation in the slow-
rotation limit found by Yagi & Yunes (2013a, 2013b). The deviation from universality can reach about 40% for hadronic EOSs and about 75% for strange matter EOSs, and for the fastest spinning pulsars currently known, the deviation is around 35% for the two classes of EOSs. The EOS independence of the $\overline{I}$-Love-$\overline{Q}$ relations is broken when rapid rotation is considered, because the deviations due to different EOSs are comparable to the deviations induced by rotation if one considers sequences with fixed rotational frequency. Only for a restricted set of EOSs, that do not include models with extremely small or large radii, were we still able to find relations that are roughly EOS independent at fixed rotational frequencies, which can be accurately represented by second order fits. In addition, we find that the complete rotational dependence of the $\overline{I} - \overline{Q}$ relation can be well captured if the coefficients in the second order fits are represented as a third order polynomial expansion in the rotational frequency $f$. One should keep in mind that these fits are valid for our chosen restricted set of EOSs and if one includes EOSs that reach more extreme values of the radii, then the relations would change.

For strange stars, we find that rapid rotation leads to a completely different $\overline{I} - \overline{Q}$ relation for different choices of the strange matter EOSs, thus prohibiting more general fits, such as those for neutron stars.

Our results do not affect the applicability of the $I$-Love-$\overline{Q}$ relations in cases where the slow-rotation approximation is sufficient, such as the inspiral phase of binary neutron star mergers or for the case of some relativistic binaries where the moment of inertia could be measured. In contrast, our results are important for astrophysical situations involving compact objects rotating faster than a few hundred Hz.

We would like to thank K. Yagi and N. Yunes for critical comments and suggestions on an early version of the manuscript. We are grateful to Andreas Bauswein for providing us with an updated collection of recent EOS tables. D.D. would like to thank the Alexander von Humboldt Foundation for support. K.K. and S.Y. would like to thank the Research Group Linkage Programme of the Alexander von Humboldt Foundation for support. The support from the Bulgarian National Science Fund under Grant DMU-03/6, by the Sofia University Research Fund under Grant 33/2013, by the German Science Foundation (DFG) via SFB/TR7, and the networking support by the COST Action MP1304 is gratefully acknowledged. N.S. is grateful for the hospitality of the Tübingen group, during an extended stay.
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