Non–gaussian probability distribution functions in two dimensional Magnetohydrodynamic turbulence

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Abstract

Intermittency in MHD turbulence has been analyzed using high resolution 2D numerical simulations. We show that the Probability Distribution Functions (PDFs) of the fluctuations of the Elsässer fields, magnetic field and velocity field depend on the scale at hand, that is they are self–affine. The departure of the PDFs from a Gaussian function can be described through the scaling behavior of a single parameter $\lambda^2$ obtained by fitting the PDFs with a given curve stemming from the analysis of a multiplicative model by Castaing et al. [1]. The scaling behavior of the parameter $\lambda^2$ can be used to extract informations about the intermittency. A comparison of intermittency properties in different MHD turbulent flows is also performed.
One of the relevant features of turbulent systems is small–scale intermittency \cite{2}. Starting from the Kolmogorov 1962 theory \cite{3}, several models have been developed to take into account the intermittency effects on the statistics of the turbulent fields. Measurements show the presence of self–affine fields, which is the most remarkable of such effects, leading to the scaling departure from gaussianity of the Probability Distribution Functions (PDFs) of turbulent fluctuations (see \cite{2} and references therein). The usual statistical tool to check such departure is the analysis of the scaling exponents $\zeta_\psi$ of the longitudinal structure functions of an (as yet) unspecified vector field $\psi'$, i.e. the moments of the distributions of the fluctuations, namely: $S_r^{(p)}(\delta \psi'_r) = \langle (\delta \psi'_r)^p \rangle = \langle [(\psi'(x + r) - \psi'(x)) \cdot \frac{r}{r}]^p \rangle$; here $\psi'$ represents either the velocity field $v$, the normalized magnetic induction $b = B/\sqrt{4\pi\rho}$ ($\rho$ is the constant density), or the Elsässer variables $z^\pm = v \pm b$; $r = |r|$ is the scale we are examining and homogeneity is assumed. The structure functions are assumed to scale like $S_r^{(p)}(\delta \psi'_r) \sim r^{\zeta_\psi}$ in the inertial range. In the non–intermittent case the scaling exponents $\zeta_\psi$ should display a linear dependence on the order index of the moments; in the framework of the Kolmogorov theory, for example, this dependence is a $\zeta_v = p/3$ law. However, experimental observations have shown a nonlinear behavior of such scaling exponents, suggesting the multifractal nature of the energy transfer \cite{2,4}. This kind of behavior has been observed both in neutral fluids and in plasmas \cite{2,4}. A different approach to the study of intermittency consists in analysing the scaling behavior of the PDFs of fluctuations instead of the scaling of the moments \cite{2,5–7}. In the framework of the multiplicative cascade, the presence of very intense fluctuations of the fields becomes more probable as the scale decreases, because of the progressive strong localization of the active structures. As a result, the tails of the PDFs are higher than gaussian ones, and rare events become significant in the statistics. By fitting experimental data with a model distribution function, it is possible to describe intermittency through a small set of parameters. The characterization performed through the structure functions, on the contrary, requires, in principle, the determination of an infinite set of scaling exponents. Unfortunately, predictions and models concerning the scaling behavior of the PDFs are less common and less firmly established than those concerning
the scaling behavior of structure functions, a notable exception being the turbulence coming from Burgers equation \[8\]. In this paper, we follow one such model for PDFs \[1\], and we show that the informations thus obtained are in agreement with the previous results derived directly from moments.

The data we use stem from the turbulent fields obtained from two–dimensional magnetohydrodynamics incompressible simulations, with periodic boundary conditions at a resolution of \(1024^2\) grid points (see also \[9\]). The forcing consists in maintaining constant the amplitudes of Fourier modes with \(|k| = 1\), and the resulting energy is concentrated at large scales. There is on average an excess of magnetic energy \(E^M\) over its kinetic counterpart \(E^V\), as can be seen on the left of figure \[\]
showing the temporal evolution of the ratio \(E^M/E^V\); better equipartition is obtained in the small scales as diagnosed by the ratio of enstrophies (not shown). The data analyzed in the paper are averaged over approximatively 160 eddy turnover times in the statistically steady state, from \(t = 6.30\) to \(t = 12\). This lengthy computation is rendered necessary by the long–time fluctuations in the flow, as is visible on figure \[\]; these fluctuations are induced by the large–scale forcing. The data sets thus consist of about \(10^7\) points, and up to now, these are the largest data sets used to investigate intermittency in statistically steady MHD flows. The temporal window of analysis is chosen on the basis of the stationarity of the integral Reynolds number, shown on the right of figure \[\].

In order to get PDFs with zero average and unit standard deviation, we use, as is customary, the normalized fluctuations of the turbulent fields:

\[
\delta \psi_r = \frac{\delta \psi'_r - \langle \delta \psi'_r \rangle}{\sqrt{(\delta \psi'_r - \langle \delta \psi'_r \rangle)^2}}
\]

where \(\delta \psi'_r\) is any one of the relevant fields at scale \(r\). We built up the PDFs of the normalized fluctuations by dividing the range between \(-3.5\) and \(+3.5\) (in standard deviation units) in bins of length \(A_{\text{bins}}\). We then used the histogram of the fluctuations \(N_r(\delta \psi_r)\) to compute the probability density:

\[
P(\delta \psi_r) = \frac{N_r(\delta \psi_r)}{A_{\text{bins}}N_{\text{tot}}(r)},
\]

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where $N_{\text{tot}}(r)$ is the actual number of fluctuations at scale $r$ in the interval $(-3.5, 3.5)$. The PDFs have been computed for several values of the scale $r$, ranging from $r/L = 10^{-3}$ up to $r/L = 0.5$, where $L = 2\pi$ is the length of the simulation box. Samples of the so obtained PDFs are shown in figure 2 for the normalized fluctuations of the $z^+$ field; similar PDFs hold for the $z^-$, magnetic and velocity fields. As can be easily seen, the PDFs are gaussian at large scales, but become more and more stretched as the scale decreases. This scaling behavior is present for all fields and is an indication that these fields are self-affine. Such an intermittent behavior, in the framework of the Kolmogorov’s refined similarity hypothesis [2], can be attributed to the fluctuations of the energy transfer rate, and the dependence on scales of the PDFs can be eliminated by looking at the PDFs conditioned to a given value of the energy transfer rate at the scale $r$ (see [10,11] for the fluid case). In MHD, we can introduce the local energy transfer rates for both Elsässer variables, defined as:

$$ r\varepsilon^\pm_r = (\delta z^\pm_r)^2 \delta z^\mp_r $$

(see e.g. [9]). The PDFs $P(\delta z^+|r\varepsilon^+_r)$, conditioned to two different values of $\varepsilon^+_r$, are shown in figure 3. The same gaussian shape is observed at all scales, which means that intermittency has been eliminated. We found the same behavior for the PDFs $P(\delta z^-|r\varepsilon^-_r)$. As reported in the caption of figure 3, for a fixed scale $r$, the width of the PDFs can be different when conditioned by different values of the energy transfer rate. The departure from gaussianity observed in the unconditioned PDFs is then the imprint of the multifractal model of intermittency. In fact for each scale $r$, the intermittency can be described as the superposition of different PDFs of fluctuations belonging to subsets with different $\varepsilon^\pm_r$ transfer rates. Using the multifractal framework [2,4,12], we can give a quantitative analysis of the continuous scaling departure of PDFs from a gaussian. In fact, as stated by Castaing et al. [1] in order to describe the PDFs at a given scale $r$, two ingredients are needed: the parent distribution at the scale $L$ and the distribution of the energy transfer rate. The first information can be extracted directly from the experimental observations, just looking at the large scale PDFs of the fields fluctuations which is a Gaussian curve. The second item needs some a priori hypothesis about the shape of the PDF of the $\varepsilon^\pm_r$ transfer rates. In other words, the resulting PDFs of fluctuations at scale $r$ can be described as a
convolution of the parent gaussian distribution with a chosen distribution for $\varepsilon_r^\pm$. Following [1], this can be done in a continuous way by introducing a distribution $L_{\lambda_r}(\sigma_r)$ for the width $\sigma_r$ of the gaussians (which is directly proportional to $\varepsilon_r^\pm$), and by then computing the convolution:

$$P(\delta\psi_r) = \int_0^\infty L_{\lambda_r}(\sigma_r) \exp\left(-\frac{\delta\psi_r^2}{2\sigma_r^2}\right) \frac{1}{\sqrt{2\pi\sigma_r}} \frac{d\sigma_r}{\sigma_r}.$$  \hspace{1cm} (1)

In this paper, we use the log–normal ansatz, as in Castaing et al. [1]:

$$L_{\lambda_r}(\sigma_r) = \frac{1}{\sqrt{2\pi\lambda_r}} \exp\left(-\frac{\ln^2 \sigma_r/\sigma_{0,r}}{2\lambda_r^2}\right).$$  \hspace{1cm} (2)

The PDF of a field is thus seen, in this model, as a superposition of gaussian curves whose standard deviations are distributed according to a log–normal law. The parameter $\sigma_{0,r}$ is the most probable value of the $\sigma_r$ deviations for a given scale $r$, while the parameter $\lambda_r$ represents the width of the log–normal distribution $L_{\lambda_r}(\sigma_r)$. In so doing, the scaling properties of the PDFs are concentrated into the single scaling behavior of the parameter $\lambda_r^2$. For $\lambda_r^2 = 0$, the distribution $L_{\lambda_r}(\sigma_r)$ is a delta function, and the convolution (1) is a single gaussian of width $\sigma_{0,r}$. As $\lambda_r^2$ increases, the spectrum of the values of $\sigma_r$, involved in the convolution, is wider and the tails of the PDFs consequently become stronger.

The results of the fit of the measured PDFs with the model (1) are shown in figure 2. It can be seen that the model reproduces quite well the scaling properties of the PDFs $P(\delta\psi_r)$ of the various fields, although with a lesser agreement at the largest scales of the flow. In order to get informations about intermittency, we compute $\lambda_r^2$ from the PDFs using (1)-(2) and we then look at the scaling of this parameter with the separation length $r/L$ (figure 4). When considering the scaling of the structure functions $S^{(p)}(\delta z^\pm)$, it was shown in [3] that the inertial scales for the Elsässer variables, fundamental in this problem [13], are in the domain $0.01 \leq r/L \leq 0.1$. This can be our main guide for scaling here. We however perform a careful examination of the data to check the scaling ranges of $\lambda_r^2$; in particular, for the velocity field, this will lead us to shorten this range at large scales (see table II). A typical nontrivial behavior can be distinguished as a power–law scaling:
\[
\lambda^2_r(r) \sim (r/L)^{-\beta},
\]

in the previously defined inertial range: a saturation of \(\lambda^2_r\) is reached at the onset of the dissipative range, while the gaussian regime is found on scales larger than \(r/L \simeq 0.1\).

This method leads to a characterisation of the turbulent system using only two parameters: the scaling exponent \(\beta\) and \(\lambda_{\text{max}}^2\), the maximum value of \(\lambda^2_r\) over \(r\). The parameter \(\beta = -d \log \lambda^2_r / d \log (r/L)\), the logarithmic derivative of \(\lambda^2_r\), represents how “fast” the generation of intermittency occurs throughout the inertial range. In fact, a greater value of \(\beta\) is related to an energy cascade mechanism which produces intermittency in a more efficient way (the wings of the distribution of \(\delta \psi_r\) increase more quickly). The parameter \(\lambda_{\text{max}}^2\), being related to both the maximum number of gaussians needed in the convolution (1) and to the weight of the widest gaussians in that convolution, tells us how “strong” is the intermittency, i.e. how deeply the intermittent cascade is active and generates the strongest events \(\delta \psi_r\).

The computed values of \(\lambda^2_r\) provide \(\lambda_{\text{max}}^2\) and \(\beta\) which are given in table I, together with their variations obtained by a chi–square test, as well as the range of scales where (3) is verified. Note that the analysis of each temporal data set separately led us to eliminate some samples: \(\lambda^2_r\) was ill–defined on average on one third of the individual temporal data sets. The magnetic field appears more intermittent than the velocity, in agreement with previous numerical results \([9,14]\), and with the results in solar wind plasmas \([15]\) (see also \([16]\) for a 3D computation in a decaying helical MHD flow using hyper viscosity). This result resembles what happens in fluid turbulence where passive scalars are more intermittent than the velocity field \([17]\). In MHD the stronger intermittency of the magnetic field is perhaps due to the fact that, when non-linearly coupled with a velocity field, the magnetic field behaves like a passive vector, at least in the kinematic phase of the dynamo. The Elsässer variables (dominated by the magnetic field, see figure I) display also a strong intermittency; the values of the parameters found for the \(z^-\) field are comparable to those for the magnetic field, while the parameters found for \(z^+\) are intermediate between those obtained for the magnetic field and the velocity. The slight difference between the \(\pm \beta\)–parameter could appear significant:
although the global correlation coefficient between the velocity and the magnetic field is low when averaged over the whole flow, it presents regions with strong pointwise values of either signs (not shown) and this may reflect in the \( \pm \) discrepancy observed here. A similar result is found in the solar wind [18].

As a comparison with some previous experimental analysis, the values of \( \beta \) obtained here are larger than those found for fluid flows \( (\beta \simeq 0.3) \) [1], for solar wind turbulence \( (\beta \simeq 0.2) \) [15] and for magnetic turbulence in a Reverse Field Pinch laboratory plasma \( (\beta \simeq 0.4) \) [19]. This fact can be interpreted as an indication that 2D MHD turbulence in the present numerical simulations is strong, with a low degree of anisotropies and inhomogeneities which could exist, on the contrary, in geophysical or laboratory plasma turbulence. Again, the values of \( \lambda_{\text{max}}^2 \) found here are greater than those found in the solar wind (slightly so for the magnetic field, more clearly for the velocity) [18], and the magnetic field is strongly more intermittent than in the laboratory plasma mentioned above.

In conclusion, this paper shows that the analysis of the PDFs is a useful tool to quantify intermittency. In the present approach, only two parameters, \( \beta \) and \( \lambda_{\text{max}}^2 \), are used to characterize the intermittency in a way which is found consistent with previous analysis. An improvement of the method using different distributions for the energy transfer rates is presently in progress.

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FIGURES

FIG. 1. On the left, temporal evolution of the ratio of magnetic to kinetic energy $E^M / E^V(t)$ and, on the right, temporal evolution of the integral Reynolds number.

FIG. 2. PDFs of the fluctuations of the normalized Elsässer variable $z^+$ for three different scales (see insert). The full line represents the fit made with the convolution function $\square$. 
FIG. 3. Conditioned PDFs of the normalized fluctuations of the Elsässer variables $z^+$ for three different scales, namely $r/L = 0.002$, $r/L = 0.03$ and $r/L = 0.25$. The PDFs $P(\delta z^+_r | r\varepsilon^+_r)$ are conditioned by a given level of the energy dissipation $r\varepsilon^+_r = (\delta z^+_r)^2\delta z^-_r$ between $-0.1$ and $0.1$ (left panel) and between $0.9$ and $1$ (right panel). Gaussian fits are reported for comparison (dashed lines). The standard deviations of such curves, computed from the fit, are $\sigma_r = 0.4$ and $\sigma_r = 0.9$, respectively. The insets show the unconditioned PDFs for the same values of $r/L$. 

\[ P(\delta Z^+ | r\varepsilon^+), \]
FIG. 4. Scaling behavior of the exponent $\lambda_2^r$ for the magnetic field, the velocity and the Elsässer variables (see insert). Straight lines represent the fit with power-laws as reported in Table I.
TABLES

TABLE I. For the different fields, values of the two parameters $\beta$ and $\lambda_{max}^2$, together with their statistical error bars, determined from the indicated ranges of fit (see text).

|   | $\beta$  | $\lambda_{max}^2$ | range of fit    |
|---|----------|-------------------|-----------------|
| $v$ | 0.50 ± 0.15 | 0.8 ± 0.2         | 0.008 ≤ $r/L$ ≤ 0.05 |
| $b$ | 0.84 ± 0.13 | 1.1 ± 0.3         | 0.01 ≤ $r/L$ ≤ 0.1  |
| $z^+$ | 0.74 ± 0.14 | 0.9 ± 0.2         | 0.01 ≤ $r/L$ ≤ 0.1  |
| $z^-$ | 0.98 ± 0.19 | 1.0 ± 0.3         | 0.01 ≤ $r/L$ ≤ 0.1  |