Random matrix models with log-singular level
confinement: method of fictitious fermions *

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Abstract

Joint distribution function of $N$ eigenvalues of $U(N)$ invariant random-
matrix ensemble can be interpreted as a probability density to find $N$
 fictitious non-interacting fermions to be confined in a one-dimensional
 space. Within this picture a general formalism is developed to study the
eigenvalue correlations in non-Gaussian ensembles of large random ma-
trices possessing non-monotonic, log-singular level confinement. An effective
 one-particle Schrödinger equation for wave-functions of fictitious fermions
 is derived. It is shown that eigenvalue correlations are completely de-
termined by the Dyson’s density of states and by the parameter of the
 logarithmic singularity. Closed analytical expressions for the two-point
 kernel in the origin, bulk, and soft-edge scaling limits are deduced in a
 unified way, and novel universal correlations are predicted near the end
 point of the single spectrum support.

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1 Introduction and basic relations

Random matrices are the field-theoretical models which study the universal features of physical phenomena resulting from the symmetry constraints only. This is the reason why quite different physical problems get a unified mathematical description within the framework of the random-matrix theory [1]. In particular, the applicability of so-called invariant matrix model given by the joint distribution function

\[ \rho_N (\{\lambda\}) \, d\lambda = \frac{1}{Z_N} \prod_{k=1}^{N} d\lambda_k |\lambda_k|^{\alpha \beta} \exp \{-\beta v(\lambda_k)\} \prod_{i>j=1}^{N} |\lambda_i - \lambda_j|^{\beta} \]  

(1)

of \(N\) eigenvalues \(\{\lambda\}\) of a large \(N \times N\) random matrix \(H\) ranges from the problem of two-dimensional gravity [2], through the spectral properties of the Dirac operator in quantum chromodynamics [3], to the mesoscopic electron transport in normal and superconducting quantum dots [4, 5]. Here the eigenvalues \(\{\lambda\}\) belong to entire real axis, \(-\infty < \{\lambda\} < +\infty\), and the partition function \(Z_N\) is determined from the normalization condition \(\int \rho_N (\{\lambda\}) \, d\lambda = 1\). Parameter \(\beta\) in Eq. (1) accounts for the symmetry of the problem, \(\alpha\) is a free parameter associated with a logarithmic singularity, while \(v(\lambda)\) is a non-singular part of the confinement potential

\[ V(\lambda) = v(\lambda) - \alpha \log |\lambda|. \]  

(2)

It is implied that dimension \(N\) of the matrix \(H\) is large enough, \(N \gg 1\). In this thermodynamic limit the matrix model Eq. (1) becomes exactly solvable.

Different physics, that is behind the model introduced, deals with different regions of spectrum that can be explored in the corresponding scaling limits. Up to now, the most study received the random-matrix ensemble Eq. (1) with \(U(N)\) symmetry \((\beta = 2)\), where three types of universal correlations have been established in the origin [6, 7], bulk [8, 9, 10, 11], and soft-edge [12, 13] scaling limits. Corresponding eigenvalue correlations are described by the universal Bessel, sine, and G–multicritical kernels, respectively. Various scaling limits of the model Eq. (1) have been investigated by using different methods, so that a unified treatment of the problem of spectral correlations in \(U(N)\) invariant ensembles is still absent. The purpose of this paper is to present a unified approach allowing us to explore the spectral properties of the \(U(N)\) invariant matrix model Eq. (1) with effective log-singular level confinement in an arbitrary spectrum range.

The following representation [1] of the joint distribution function \(\rho_N (\{\lambda\})\) is well-known in the random-matrix theory:

\[ \rho_N (\{\lambda\}) = |\Psi_0 (\lambda_1, ..., \lambda_N)|^2, \]  

(3)
As far as \( \Psi_0 \) takes the form of the Slater determinant, \( \rho_N (\{\lambda\}) \) can be thought of as a probability density to find \( N \) non-interacting fictitious fermions in the quantum states \( \varphi_0, ..., \varphi_{N-1} \) at the “spatial” points \( \lambda_1, ..., \lambda_N \). The “wave-functions” of such fermions are uniquely determined by the set of polynomials \( P_n (\lambda) \) orthogonal on the entire real axis with respect to the measure \( d\mu (\lambda) = \exp \{-2V (\lambda)\} d\lambda \),

\[
\varphi_n (\lambda) = P_n (\lambda) \exp \{-V (\lambda)\}
\]

so that the orthogonality relation

\[
\int_{-\infty}^{+\infty} d\lambda \varphi_n (\lambda) \varphi_m (\lambda) = \delta_{nm}
\]

holds. It follows from Eq. (3) that the joint distribution function \( \rho_N (\{\lambda\}) \) can be represented as

\[
\rho_N (\{\lambda\}) = \frac{1}{N!} \det \|K_N (\lambda_i, \lambda_j)\|_{i,j=1...N},
\]

where \( K_N (\lambda, \lambda') \) (referred to as the “two-point kernel”)

\[
K_N (\lambda, \lambda') = \sum_{k=0}^{N-1} \varphi_k (\lambda) \varphi_k (\lambda').
\]

is completely determined by the wave-functions \( \varphi_n \). Due to an additional constraint on the wave-functions of three successive quantum states that results from the recurrence equation Eq. (10) below, only the highly excited states, \( \varphi_{N-1} \) and \( \varphi_N \), contribute to the two-point kernel in accordance with the Christoffel-Darboux theorem [14]:

\[
K_N (\lambda, \lambda') = c_N \frac{\varphi_N (\lambda') \varphi_{N-1} (\lambda) - \varphi_N (\lambda) \varphi_{N-1} (\lambda')}{\lambda' - \lambda}.
\]

This formula simplifies significantly the mathematical calculations in the thermodynamic limit \( N \gg 1 \). Effective Schrödinger equation for \( \varphi_N \), that is the cornerstone of our unified approach, will be derived in the next Section.

## 2 Effective Schrödinger equation

In the particular case of the Gaussian unitary ensemble (GUE) the wave-functions \( \varphi_n (\lambda) \) are well-known. They are eigenfunctions of a fermion confined by a parabolic potential [1]. For general non-Gaussian ensemble Eq. (1) the calculation of such effective wave-functions can be done by an extension of the
Shohat’s method \cite{15,16} that previously has been used by the authors \cite{13} to treat the problem of eigenvalue correlations in random-matrix ensembles with non-singular, strong level confinement. This method allows us to map a three-term recurrence equation

\[ \lambda P_{n-1} (\lambda) = c_n P_n (\lambda) + c_{n-1} P_{n-2} (\lambda) \]  

(10)

for polynomials \( P_n (\lambda) \) orthogonal on the entire real axis with respect to the measure \( d\mu (\lambda) = \exp \{-2V (\lambda)\} \, d\lambda, \)

\[ \int_{-\infty}^{+\infty} d\mu (\lambda) P_n (\lambda) P_m (\lambda) = \delta_{nm}, \]  

(11)

onto a second-order differential equation for corresponding fictitious wave-functions \( \varphi_n \). Coefficients \( c_n \) appearing in Eq. (10) are uniquely determined by the measure \( d\mu \).

In order to derive an effective Schrödinger equation, we note the following identity

\[ \frac{dP_n (\lambda)}{d\lambda} = A_n (\lambda) P_n (\lambda) - B_n (\lambda) P_n (\lambda), \]  

(12)

with functions \( A_n (\lambda) \) and \( B_n (\lambda) \) to be determined from the following consideration. Since \( dP_n (\lambda) /d\lambda \) is a polynomial of the degree \( n - 1 \), it can be represented \cite{14} through the Fourier expansion in the terms of the kernel \( Q_n (t, \lambda) = \sum_{k=0}^{n-1} P_k (\lambda) P_k (t) \) as:

\[ \frac{dP_n (\lambda)}{d\lambda} = \int_{-\infty}^{+\infty} d\mu (t) \frac{dP_n (t)}{dt} Q_n (t, \lambda). \]  

(13)

Integrating by parts in the last equation we get that

\[ \frac{dP_n (\lambda)}{d\lambda} = 2 \int_{-\infty}^{+\infty} d\mu (t) Q_n (t, \lambda) \left( \frac{dV}{dt} - \frac{dV}{d\lambda} \right) P_n (t). \]  

(14)

Now, making use of the Christoffel-Darboux theorem, we conclude that unknown functions \( A_n (\lambda) \) and \( B_n (\lambda) \) in Eq. (12) are

\[ A_n (\lambda) = 2c_n \int_{-\infty}^{+\infty} \frac{d\mu (t)}{t - \lambda} \left( \frac{dV}{dt} - \frac{dV}{d\lambda} \right) P_n^2 (t), \]  

(15)

\[ B_n (\lambda) = 2c_n \int_{-\infty}^{+\infty} \frac{d\mu (t)}{t - \lambda} \left( \frac{dV}{dt} - \frac{dV}{d\lambda} \right) P_n (t) P_{n-1} (t). \]  

(16)

We also notice the identity that directly follows from Eqs. (13), (14), and from oddness of \( dV/d\lambda \):

\[ B_n (\lambda) + B_{n-1} (\lambda) = \frac{\lambda}{c_n} A_{n-1} (\lambda) = -2 \frac{dV}{d\lambda}. \]  

(17)
Differentiating Eq. (12), making use of the recurrence equation Eq. (10), and bearing in mind relation Eq. (5) between $P_n(\lambda)$ and $\varphi_n(\lambda)$, one can obtain an exact differential equation for the wave-functions of fictitious fermions, that is valid for arbitrary $n$:

\[
\frac{d^2 \varphi_n(\lambda)}{d\lambda^2} - \mathcal{F}_n(\lambda) \frac{d\varphi_n(\lambda)}{d\lambda} + \mathcal{G}_n(\lambda) \varphi_n(\lambda) = 0, \tag{18}
\]

where

\[
\mathcal{F}_n(\lambda) = \frac{1}{A_n} \frac{dA_n}{d\lambda}, \tag{19}
\]

\[
\mathcal{G}_n(\lambda) = \frac{dB_n}{d\lambda} + \frac{c_n}{c_{n-1}} A_n A_{n-1} - B_n \left( B_n + 2 \frac{dV}{d\lambda} + \frac{1}{A_n} \frac{dA_n}{d\lambda} \right) + \frac{d^2 V}{d\lambda^2} - \left( \frac{dV}{d\lambda} \right)^2 - \frac{1}{A_n} \frac{dA_n}{d\lambda} \frac{dV}{d\lambda}. \tag{20}
\]

Previously, equation of this type was known in the context of the random-matrix theory only for GUE, where $V(\lambda) = \lambda^2 / 2$. For such a confinement potential both functions $A_n$ and $B_n$ can easily be computed from Eqs. (15) and (16), and are given by $A_n(\lambda) = 2c_n$ and $B_n(\lambda) = 0$. Taking into account that for GUE $c_n = \sqrt{n}/2$ we end up with $\mathcal{F}_n(\lambda) = 0$ and $\mathcal{G}_n(\lambda) = 2n + 1 - \lambda^2$. This allows us to interpret $\varphi_n(\lambda)$ as a wave-function of the fermion confined by a parabolic potential:

\[
\frac{d^2 \varphi_n^{GUE}(\lambda)}{d\lambda^2} + \left( 2n + 1 - \lambda^2 \right) \varphi_n^{GUE}(\lambda) = 0. \tag{21}
\]

In principle, the effective Schrödinger equation Eq. (18) applies to general non-Gaussian random-matrix ensembles as well, although the explicit calculation of $\mathcal{F}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ in this situation may be a rather complicated task. However, significant simplifications arise in the thermodynamic limit $n = N \gg 1$.

To proceed with derivation of the asymptotic Schrödinger equation, we have to specify the form of confinement potential $V$ introduced by Eq. (4). Choosing the regular part $v(\lambda)$ to be an even function, we set

\[
V^{(\alpha)}(\lambda) = \sum_{k=1}^{p} \frac{d_k}{2k} \lambda^{2k} - \alpha \log |\lambda|, \tag{22}
\]

with $d_p > 0$. The signs of the rest $d_k$’s can be arbitrary, allowing for non-monotonic level confining, but they should lead to an eigenvalue density supported on a single connected interval $(-D_N, +D_N)$. Confinement potential $V^{(\alpha)}(\lambda)$ determines its own set of orthogonal polynomials $P_n^{(\alpha)}(\lambda)$, and functions $A_n^{(\alpha)}$ and $B_n^{(\alpha)}$ which are needed to construct an asymptotic second-order differential equation for the function $\varphi_N^{(\alpha)}(\lambda) = |\lambda|^\alpha P_N^{(\alpha)}(\lambda) \exp \{-v(\lambda)\}$. [Here
upper index $\alpha$ reflects the presence of the log-singular component in $V^{(\alpha)}(\lambda)$, and the restriction $\alpha > -\frac{1}{2}$ takes place due to normalization Eq. (11).

In accordance with Eqs. (15) and (22) it is convenient to represent $A^{(\alpha)}_N$ in the form

$$A^{(\alpha)}_N(\lambda) = A^{(N)}_{\text{reg}}(\lambda) + \alpha A^{(\alpha)}_{\text{sing}}(\lambda),$$

where

$$A^{(N)}_{\text{reg}}(\lambda) = 2c_N \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - \lambda} \left( P^{(\alpha)}_N(t) \right)^2 \left( \frac{dv}{dt} - \frac{dv}{d\lambda} \right),$$

$$A^{(\alpha)}_{\text{sing}}(\lambda) = 2c_N \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t} \left( P^{(\alpha)}_N(t) \right)^2.$$  

(23)

Analogously, Eq. (16) leads to the similar representation

$$B^{(\alpha)}_N(\lambda) = B^{(N)}_{\text{reg}}(\lambda) + \alpha B^{(\alpha)}_{\text{sing}}(\lambda),$$

with

$$B^{(N)}_{\text{reg}}(\lambda) = 2c_N \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - \lambda} P^{(\alpha)}_N(t) P^{(\alpha)}_{N-1}(t) \left( \frac{dv}{dt} - \frac{dv}{d\lambda} \right),$$

$$B^{(\alpha)}_{\text{sing}}(\lambda) = 2c_N \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t} P^{(\alpha)}_N(t) P^{(\alpha)}_{N-1}(t).$$

(24)

In the above formulas $A^{(N)}_{\text{reg}}$ and $B^{(N)}_{\text{reg}}$ result from the regular component of confinement potential, while $A^{(\alpha)}_{\text{sing}}$ and $B^{(\alpha)}_{\text{sing}}$ are caused by its log-singular part.

First, it is easy to see that $A^{(\alpha)}_{\text{sing}}(\lambda) \equiv 0$ due to evenness of the measure $d\mu$. Second, the exact expression for $B^{(\alpha)}_{\text{sing}}$ immediately follows from the recurrence equation Eq. (10), whence we get

$$B^{(\alpha)}_N(\lambda) = \frac{1 - (-1)^N}{\lambda}.$$  

(25)

Calculation of the regular parts $A^{(N)}_{\text{reg}}$ and $B^{(N)}_{\text{reg}}$ can be done along the lines presented in Ref. [13]. Then, we immediately obtain that $A^{(\alpha)}_N(\lambda)$ is expressed in terms of Dyson’s density $\nu_D(\lambda)$ as follows:

$$A^{(\alpha)}_N(\lambda) = \frac{\pi \nu_D(\lambda)}{\sqrt{1 - (\lambda/D_N)^2}},$$

$$\nu_D(\lambda) = \frac{2}{\pi^2} \mathcal{P} \int_0^{D_N} \frac{\xi d\xi}{\xi^2 - \lambda^2} \frac{dv}{d\xi} \sqrt{\frac{1 - (\lambda/D_N)^2}{1 - (\xi/D_N)^2}},$$

(30)

(31)

where $D_N = 2c_N$ should be identified with the soft edge of the spectrum. It is easy to see that a log-singular part of the confinement potential does not
contribute to the Dyson’s density, so that in the thermodynamic limit there are no changes in \( D_N \) due to logarithmic singularity of confinement potential. Expression for \( B_{\text{reg}}^{(N)} \) can be obtained by the use of the large-\( N \) version of the identity Eq. (17), that yields

\[
B_{\text{reg}}^{(N)}(\lambda) = \frac{\lambda}{D_N} A_N^{(\alpha)}(\lambda) - \frac{d\nu}{d\lambda},
\]

whence

\[
B_N^{(\alpha)}(\lambda) = \frac{\lambda}{D_N} \frac{\pi \nu_D(\lambda)}{\sqrt{1 - (\lambda/D_N)^2}} - \frac{d\nu}{d\lambda} + \frac{\alpha}{\lambda} \frac{1 - (-1)^N}{\lambda}.
\]

Now, having asymptotic representations for \( A_N^{(\alpha)} \) and \( B_N^{(\alpha)} \) given by Eqs. (30) and (33), and taking into account Eqs. (18), (19) and (20), it is straightforward to obtain the following remarkable effective asymptotic Schrödinger equation for the wave-functions \( \phi_N^{(\alpha)}(\lambda) = |\lambda|^\alpha P_N^{(\alpha)}(\lambda) \exp\{\nu(\lambda)\} \) of highly excited states (\( N \gg 1 \)) of fictitious fermions:

\[
\frac{d^2 \phi_N^{(\alpha)}}{d\lambda^2} - \left[ \frac{d}{d\lambda} \log \left( \frac{\pi \nu_D(\lambda)}{\sqrt{1 - (\lambda/D_N)^2}} \right) \right] \frac{d\phi_N^{(\alpha)}}{d\lambda} + \left[ \frac{\pi^2 \nu_D^2(\lambda)}{\lambda^2} + \frac{(-1)^N \alpha - \alpha^2}{\lambda^2} \right] \phi_N^{(\alpha)}(\lambda) = 0
\]

Also, due to Eq. (12), one can verify that the wave-functions of two successive quantum states are connected by the relationship

\[
\frac{d\phi_N^{(\alpha)}}{d\lambda} = \frac{\pi \nu_D(\lambda)}{\sqrt{1 - (\lambda/D_N)^2}} \left( \phi_{N-1}^{(\alpha)}(\lambda) - \frac{\lambda}{D_N} \phi_N^{(\alpha)}(\lambda) \right) + (-1)^N \frac{\alpha}{\lambda} \phi_N^{(\alpha)}(\lambda).
\]

Equations (34) and (35) provide a general basis for the study of eigenvalue correlations in non-Gaussian random-matrix ensembles in an arbitrary spectral range. In particular case of GUE, the Dyson’s density of states is the celebrated semicircle, \( \nu_D^{\text{GUE}}(\lambda) = \pi^{-1} \sqrt{D_N^2 - \lambda^2} \) with \( D_N = \sqrt{2N} \). The square-root law for \( \nu_D^{\text{GUE}}(\lambda) \) immediately removes the first derivative \( d\phi_N^{(\alpha)}/d\lambda \) in Eq. (34), providing us the possibility to interpret the fictitious fermions as those confined by a quadratic potential (\( \alpha = 0 \)). As far as the semicircle is a distinctive feature of density of states in GUE only, one will always obtain a first derivative in the effective Schrödinger equation for the non-Gaussian unitary ensembles of random matrices. Therefore, fictitious non-interacting fermions associated with non-Gaussian ensembles of random matrices live in a non-Hermitian quantum mechanics.
An interesting property of these equations is that they do not contain the regular part of confinement potential explicitly, but only involve the Dyson’s density $\nu_D$ (analytically continued on the entire real axis) and the spectrum end point $D_N$. In contrast, the logarithmic singularity (that does not affect the Dyson’s density) introduces additional singular terms into Eqs. (34) and (35), changing significantly the behavior of the wave-function $\varphi_N^{(\alpha)}$ near the origin $\lambda = 0$. The influence of the singularity decreases rather rapidly outward from the origin.

Structure of the effective Schrödinger equation leads us to the following fundamental statements: (i) *Eigenvalue correlations are stable with respect to non-singular deformations of the confinement potential.* (ii) In the random-matrix ensembles with well-behaved confinement potential the knowledge of Dyson’s density (that is rather crude one-point characteristics coinciding with the real density of states only in the spectrum bulk) is sufficient to determine the genuine density of states, as well as the $n$–point correlation function, everywhere. The latter conclusion is rather unexpected since it considerably reduces the knowledge required for computing $n$–point correlators.

### 3 Local eigenvalue correlations

Effective Schrödinger equation obtained in the preceding Section allows us to examine in a unified way the eigenvalue correlations in non-Gaussian ensembles with $U(N)$ symmetry in different scaling limits. As we show below, it inevitably leads to the universal Bessel correlations in the origin scaling limit [6, 7], to the universal sine correlations in the bulk scaling limit [8, 9, 10, 11], and to the universal G–correlations in the soft-edge scaling limit [13]. Corresponding two-point kernels are given by Eqs. (40), (42) and (52), respectively.

#### 3.1 Origin scaling limit and the universal Bessel law

Origin scaling limit deals with the region of spectrum close to $\lambda = 0$ where confinement potential displays the logarithmic singularity. In the vicinity of the origin the Dyson’s density can be taken as being approximately a constant, $\nu_D(0) = 1/\Delta_N(0)$, where $\Delta_N(0)$ is the mean level spacing at the origin in the absence of the logarithmic deformation of potential $v$. In the framework of this approximation, Eq. (34) reads

$$\frac{d^2\varphi_N^{(\alpha)}}{d\lambda^2} + \left( \frac{\pi^2}{\Delta_N^2(0)} + \frac{(-1)^N \alpha - \alpha^2}{\lambda^2} \right) \varphi_N^{(\alpha)}(\lambda) = 0. \tag{36}$$

Solution to this equation that remains finite at $\lambda = 0$ can be expressed by means of Bessel functions:
\[ \varphi_{2N}^{(\alpha)}(\lambda) = a \sqrt{\lambda} J_{\alpha - \frac{1}{2}} \left( \frac{\pi \lambda}{\Delta(0)} \right), \quad (37) \]

\[ \varphi_{2N+1}^{(\alpha)}(\lambda) = b \sqrt{\lambda} J_{\alpha + \frac{1}{2}} \left( \frac{\pi \lambda}{\Delta(0)} \right), \quad (38) \]

where \( a \) and \( b \) are constants to be determined later, and \( \Delta(0) = \Delta_{2N}(0) \approx \Delta_{2N+1}(0) \). In accordance with Eq. (3), the two-point kernel can be written down as

\[ K^{(\alpha)}_{2N}(\lambda, \lambda') = c \sqrt{\lambda \lambda'} \left[ J_{\alpha + \frac{1}{2}} \left( \frac{\pi \lambda}{\Delta(0)} \right) J_{\alpha - \frac{1}{2}} \left( \frac{\pi \lambda'}{\Delta(0)} \right) - J_{\alpha - \frac{1}{2}} \left( \frac{\pi \lambda'}{\Delta(0)} \right) J_{\alpha + \frac{1}{2}} \left( \frac{\pi \lambda}{\Delta(0)} \right) \right], \quad (39) \]

where the unknown factor \( c \) can be found from the requirement \( K^{(\alpha=0)}_{2N}(\lambda, \lambda) = 1/\Delta(0) \). This immediately yields us \( c = -\pi/\Delta(0) \). Defining now the scaled variable \( s = \lambda_s/\Delta(0) \), we obtain that in the origin scaling limit the two-point kernel \( K_{\text{orig}}(s, s') = \lim_{N \to \infty} \left[ K^{(\alpha)}_{N}(\lambda_s, \lambda_s') d\lambda_s/ds \right] \) takes the universal Bessel law

\[ K_{\text{orig}}(s, s') = \pi \sqrt{ss'} J_{\alpha + \frac{1}{2}} \left( \pi s \right) J_{\alpha - \frac{1}{2}} \left( \pi s' \right) - J_{\alpha - \frac{1}{2}} \left( \pi s \right) J_{\alpha + \frac{1}{2}} \left( \pi s' \right) s - s'. \quad (40) \]

Formula (40) is valid for arbitrary \( \alpha > -\frac{1}{2} \). Note, that a recent proof of universality of the Bessel kernel given in Ref. [7] was based on the Christoffel theorem [14], that imposed an artificial restriction on parameter \( \alpha \) to be only positive integer.

### 3.2 Bulk scaling limit and the universal sine law

Bulk scaling limit has been explored in a number of works [8, 9, 10, 11]. It is associated with a spectrum range where the confinement potential is well behaved (that is far from the logarithmic singularity \( \lambda = 0 \)), and where the density of states can be taken as being approximately a constant on the scale of a few levels. In accordance with this definition one has

\[ K_{\text{bulk}}(s, s') = \lim_{s, s' \to \infty} K_{\text{orig}}(s, s'), \quad (41) \]

where \( s \) and \( s' \) should remain far enough from the end point \( D_N \) of the spectrum support.

Taking this limit in Eq. (40), we arrive at the universal sine law

\[ K_{\text{bulk}}(s, s') = \frac{\sin \left[ \pi (s - s') \right]}{\pi (s - s')}, \quad (42) \]
3.3 Soft-edge scaling limit and the universal \( G \)-multicritical law

Soft-edge scaling limit is relevant to the tail of eigenvalue support where crossover occurs from a non-zero density of states to a vanishing one. It is known [17] that by tuning coefficients \( d_k \) which enter the regular part \( v \) of confinement potential [see Eq. (22)], one can obtain a bulk (Dyson's) density of states which possesses a singularity of the type

\[
\nu_D (\lambda) = \left( 1 - \frac{\lambda^2}{D_N^2} \right)^{m+1/2} \mathcal{R}_N \left( \frac{\lambda}{D_N} \right)
\]

with the multicritical index \( m = 0, 2, 4, \) etc., and \( \mathcal{R}_N \) being a well-behaved function with \( \mathcal{R}_N (\pm 1) \neq 0 \). [Odd indices \( m \) are inconsistent with our choice that the leading coefficient \( d_p \), entering the regular component \( v (\lambda) \) of confinement potential, be positive in order to keep a convergence of integral for partition function \( Z_N \) in Eq. (3)]. Such an \( m \)-th multicriticality can be achieved by many means, and the corresponding plethora of multicritical potentials \( V^{(m)} \) is given by the equation

\[
\frac{dV^{(m)} (\lambda)}{d\lambda} = \mathcal{P} \int_{-D_N}^{+D_N} \frac{dt}{\lambda - t} \left( 1 - \frac{t^2}{D_N^2} \right)^{m+1/2} \mathcal{R}_N \left( \frac{t}{D_N} \right).
\]

So-called minimal multicritical potentials which correspond to \( \mathcal{R}_N = \text{const} \) can be found in Refs. [17, 18].

Below we demonstrate that as long as multicriticality of order \( m \) is reached, the eigenvalue correlations in the vicinity of the soft edge become universal, and are independent of the particular potential chosen. The order \( m \) of the multicriticality is the only parameter which governs spectral correlations in the soft-edge scaling limit.

Let us move the spectrum origin to its endpoint \( D_N \), making the replacement

\[
\lambda_s = D_N \left[ 1 + s \cdot \frac{1}{2} \left( \frac{2}{\pi D_N \mathcal{R}_N (1)} \right)^{1/\nu^*} \right],
\]

that defines the \( m \)-th soft-edge scaling limit provided \( s \ll (D_N \mathcal{R}_N (1))^{1/\nu^*} \propto N^{1/\nu^*} \), with

\[
\nu^* = m + \frac{3}{2}.
\]

It is straightforward to show from Eqs. (34) and (35) that in the vicinity of the end point \( D_N \) the function \( \tilde{\varphi}_N (s) = \varphi_N^{(s)} (\lambda - D_N) \) obeys the universal differential equation

\[
\tilde{\varphi}_N'' (s) - \frac{(\nu^* - \frac{3}{2})}{s} \tilde{\varphi}_N' (s) - s^{2(\nu^* - 1)} \tilde{\varphi}_N (s) = 0,
\]

(47)
and that the following relation takes place:

\[
\hat{\phi}_{N-1} (s) = \hat{\phi}_N (s) + (-1)^{\nu^* - \frac{d}{2}} \left( \frac{2}{\pi D N R_N (1)} \right)^{\frac{1}{2}} s^{\frac{d}{2} - \nu^*} \hat{\phi}'_N (s) .
\]  

(48)

Solution to Eq. (47), which decreases at \( s \to +\infty \) (that is at far tails of the density of states) can be represented through the function

\[
G (s | \nu^*) = \frac{1}{2 \sqrt{\nu^*}} \left[ \sin \left( \frac{\pi}{4 \nu^*} \right) + (-1)^{\nu^* - \frac{d}{2}} \right]^{-1/2}
\]

\[
\times \begin{cases} 
  s^{\frac{d}{2}(\nu^* - \frac{1}{2})} \left[ I_{\pm \frac{d}{2}(1 - \frac{1}{2\nu^*})} \left( \frac{|s|^\nu^*}{\nu^*} \right) - I_{\pm \frac{d}{2}(1 - \frac{1}{2\nu^*})} \left( \frac{|s|^{\nu^*}}{\nu^*} \right) \right], & s > 0, \\
  |s|^{\frac{d}{2}(\nu^* - \frac{1}{2})} \left[ J_{\pm \frac{d}{2}(1 - \frac{1}{2\nu^*})} \left( \frac{|s|^{\nu^*}}{\nu^*} \right) + (-1)^{\nu^* - \frac{d}{2}} J_{\pm \frac{d}{2}(1 - \frac{1}{2\nu^*})} \left( \frac{|s|^\nu^*}{\nu^*} \right) \right], & s < 0,
\end{cases}
\]

where \( J_{\pm \frac{d}{2}(1 - \frac{1}{2\nu^*})} \) and \( I_{\pm \frac{d}{2}(1 - \frac{1}{2\nu^*})} \) are the Bessel functions] as follows:

\[
\hat{\phi}_N (s) = a G (s | \nu^*) ,
\]  

(50)

where \( a \) is an unknown constant. Making use of Eq. (48), we obtain that in the vicinity of the soft edge the two-point kernel is

\[
K_N (\lambda_s, \lambda_{s'}) = \frac{b G (s | \nu^*) G' (s' | \nu^*) \cdot s^{\frac{d}{2} - \nu^*} - G (s' | \nu^*) G' (s | \nu^*) \cdot (s')^{\frac{d}{2} - \nu^*}}{s - s'},
\]

(51)

where \( b \) is an unknown constant again. It can be found by fitting \[12\] the density of states \( K_N (\lambda_s, \lambda_s) \), Eq. (51), to the Dyson’s density of states \( \nu_D (\lambda_s) \), Eq. (13), near the soft edge provided \( 1 \ll s \ll N^{1/\nu^*} \). This yields us the value \( b = c_N (\pi c_N R_N (1))^{1/\nu^*} \). Thus, we obtain that in the \( m \)-th soft-edge scaling limit, Eq. (15), the two-point kernel \( K_{\text{soft}}^{(m)} (s, s') = \lim_{N \to -\infty} [K_N (\lambda_s, \lambda_{s'}) d\lambda_s / ds] \) satisfies the universal law

\[
K_{\text{soft}}^{(m)} (s, s') = \frac{G (s | \nu^*) G' (s' | \nu^*) \cdot s^{\frac{d}{2} - \nu^*} - G (s' | \nu^*) G' (s | \nu^*) \cdot (s')^{\frac{d}{2} - \nu^*}}{s - s'}.
\]

(52)

These \( G \)-multicritical correlations are universal in the sense that they do not depend on the details of confinement potential, but only involve such an “integral” characteristic of level confinement as the index \( m \) of the multicriticality. In particular case of \( m = 0 \), that is inherent in random-matrix ensembles with monotonic confinement potential, the function \( G \) coincides with the Airy function, \( G (s \frac{d}{2}) = \text{Ai} (s) \), and the previously supposed universal Airy correlations \[12\]

\[
K_{\text{soft}}^{(0)} (s, s') = \frac{\text{Ai} (s) \text{Ai}' (s') - \text{Ai} (s') \text{Ai}' (s)}{s - s'}
\]

(53)

are recovered.
It follows from Eq. (52) that the density of states in the same scaling limit
\[ \nu_{\text{soft}}^{(m)}(s) = \left( \frac{d}{ds} G(s|\nu^*) \right)^2 s^{-\frac{1}{2}\nu^* - \frac{1}{2}} - [G(s|\nu^*)]^2 s^{\nu^* - \frac{1}{2}} \]
is also universal. The large-\(|s|\) behavior of \(\nu_{\text{soft}}^{(m)}\) can be deduced from the known asymptotic expansions of the Bessel functions:
\[ \nu_{\text{soft}}^{(m)}(s) = \begin{cases} \frac{|s|^{\nu^* - 1}}{\pi} + \frac{|s|^\nu}{4\pi|s|} \cos \left( \frac{2|s|^\nu}{\nu^*} \right), & s \to -\infty, \\ \exp \left( -\frac{2|s|^\nu}{\nu^*} \right) \frac{\cos^2 \left( \frac{\pi}{2\nu^*} \right)}{\sin \left( \frac{\pi}{2\nu^*} \right) + (-1)^{\nu^* - 1}}, & s \to +\infty. \end{cases} \] (55)
Note that the leading order behavior as \(s \to -\infty\) is consistent with the \(|s|^{\nu^* - 1}\) singularity of the bulk density of states, Eq. (43).

4 Concluding remarks

We have presented a general formalism for a treatment of the problem of eigenvalue correlations in spectra of \(U(N)\) invariant ensembles of large random matrices with log-singular level confinement. An important ingredient of our analysis is an effective one-particle Schrödinger equation [see Eqs. (18) and (34)] for fictitious non-interacting fermions naturally appearing in the determinantal representation of the joint distribution function of \(N\) eigenvalues of large \(N \times N\) Hermitian random matrix. The structure of the asymptotic equation Eq. (34) allowed us to conclude that: (i) Eigenvalue correlations are stable with respect to non-singular deformations of confinement potential. (ii) In the random-matrix ensembles with well-behaved confinement potential the knowledge of Dyson’s density (that is rather crude one-point characteristics coinciding with the real density of states only in the spectrum bulk) is sufficient to determine the genuine density of states, as well as the \(n\)-point correlation function, everywhere. We have also demonstrated that effective Schrödinger equation contains all the information about eigenvalue correlations in arbitrary spectrum range: the universal Bessel kernel Eq. (40) was found to describe eigenvalue correlations in the origin scaling limit; the universal sine kernel Eq. (42) was revealed in the bulk scaling limit; finally, we have shown that the soft-edge scaling limit is described by the novel universal \(G\)-multicritical kernel Eq. (52).

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