Relations between Non-Commutative and Commutative Spacetime

Ken-Ichi Tezuka

Graduate School of Science and Technology Chiba University, Japan

Abstract

Spacetime non-commutativity appears in string theory. In this paper, the non-commutativity in string theory is reviewed. At first we review that a Dp-brane is equivalent to a configuration of infinitely many D\((p-2)\)-branes. If we consider the worldvolume as that of the Dp-brane, coordinates of the Dp-brane is commutative. On the other hand if we deal with the worldvolume as that of the D\((p-2)\)-branes, since coordinates of many D-branes are promoted to matrices the worldvolume theory is non-commutative one. Next we see that using a point splitting regularization gives a non-commutative D-brane, and a non-commutative gauge field can be rewritten in terms of an ordinary gauge field. The transformation is called the Seiberg-Witten map. And we introduce second class constraints as boundary conditions of an open string. Since Neumann and Dirichlet boundary conditions are mixed in the constraints when the open string is coupled to a NS B field, the end points of the open string is non-commutative.

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\footnote{e-mail: tezuka@physics.s.chiba-u.ac.jp}
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Chapter 1

Introduction

It goes without saying that the common and traditional language in particle physics is the quantum field theory\cite{1}. The theory represents the interaction, namely electroweak and strong, among elementary particles and matter (electrons and quarks etc.) by fields. Quantum field theory is used also in cosmology, solid state physics and so on. When we deal with phenomena with respect to elementary particles, gravitation can be ignored since this is weaker than other forces in almost cases. On the other hand, gravitation is the most conspicuous force in the macroscopic world. Gravitation is successfully described by general relativity\cite{2} which is also able to be viewed as the theory of spacetime. In the macroscopic world the quantum effect can be negligible. In other words the macroscopic world is classical which is a state with large quantum number. The universe should be described by the general relativity and it is adequate to be treated classically. Since the early universe, however, is microscopic, we also need to consider quantum effects to it. A relation between the metric of spacetime and matter is given by Einstein equation which is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}.$$  

The left hand side describes geometric properties of spacetime, and the right hand side describes matter fields. We can deal with the matter fields in the framework of quantum field theory. The equation means that quantum effects of the matter fields affects the spacetime. Hence we need quantum gravity. However it is difficult to deal with the quantum aspects of gravitational force in terms of quantum field theory.

At present there are mainly two types of standpoints for quantum gravity\cite{3}. One of which is that we do not change the classical theory of gravity, and we quantize it. This is called quantum geometry. Another one is that we should construct a new classical theory of gravity which is reduced to the general relativity in a limit and quantize it. The most typical one is superstring theory\cite{4, 5, 6, 7, 8}. Quantization of the general relativity is essentially equivalent to that of spacetime. Quantum gravity naively identified with non-commutative geometry. On non-commutative space, coordinates do not commute with each other; $[X^\mu, X^\nu] \neq 0$. Recently it is attracted much attention by a lot of authors that we can regard the spacetime where strings live as non-commutative space\cite{9, 10, 11, 12, 13, 14}. In this master thesis we would like to review the subject.
Here we would like to see necessities of introducing strings. It have been proven that
gauge theories except for the general relativity (which is a kind of gauge theory) are
renormalizable. The renormalization of quantum electrodynamics is formulated indepen-
dently by Schwinger, Feynman and Tomonaga. Tomonaga have however thought that
a renormalization is not a final resolution but a temporary method to the problem of
divergences. And in those days the theory of weak interaction is Fermi theory. This the-
ory is not renormalizable. For these reasons many people have made efforts to construct
a theory which does not have divergences. One of the efforts is non-local field theory.
But there is no one who successes to construct consistent non-local field theory. Up to
now elementary particles have been regarded as point like objects which is coincide with
experimental data. Since divergences come from the point like interactions in quantum
field theory, some physicists have thought that quantum theory of extended objects does
not give such divergences. This is one of reasons why we investigate string theory.

Next let us see the strong interaction. In scattering experiments unstable particles
appear as resonances. For resonances there is a relation between mass and spin;

$$m^2 = \frac{J}{\alpha'}$$  \hspace{1cm} (1.0.1)

where $\alpha' \sim 1(\text{GeV})^{-2}$. This is tested up to $J = \frac{11}{2}$. We can not describe this behavior by
field theory. Then we need a model to do this.

Venetiano has suggested a scattering amplitude of hadrons phenomenologically. It was
suggested by Nambu and Goto that the Venetiano amplitude is derived from the bosonic
string theory. Strings are one dimensionally extended objects. The strings are sorted into
open and closed strings. The open string has end points and the closed string does not.
The theory is defined perturbatively in the sense that there is no interaction term in the
string action and the interaction is included by vertex operators. The relation (1.0.1)
can be derived from this theory. The bosonic string theory has some difficulties. This
contains a tachyon which has mass $m^2 < 0$. The presence of the tachyon makes systems
unstable since the potential is not bounded from the below. In order to preserve the
Lorentz invariance at quantum level the spacetime dimension should be $1+25[1]$. However
our world is $1+3$ dimension. Although in the real world there are a lot of fermions, the
bosonic string theory does not contain fermion.

At the same period, it was suggested by Weinberg and Salam that the weak inter-
action is described by spontaneously broken $SU(2) \times U(1)$ gauge theory which is called
electroweak theory. The strong interaction is described by $SU(3)$ gauge theory which is
called quantum chromodynamics (QCD). It was proved by t’Hooft that these theories are
renormalizable. Because of the success of the gauge theories, studies of string theory as a
theory of strong interaction have disappeared.

In the spectrum of closed string theory, there is a spin 2 field. Scherk, Schwarz and
Yoneya have interpreted it as a graviton. In other words there is the possibility that

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1It was pointed out by Kato and Ogawa [15] that the critical dimension $D = 26$ is needed in order
to preserve the nil-potency of BRS charge. However it was suggested by Abe and Nakanishi [16, 17, 18]
that there is the possibility that the critical dimension is ill-defined. Their claim is that the fact "$Q^2 = 0$
only for $D = 26$" is true only if we use the conformal gauge which was used in [15]. We need further
investigation for the subject. We would like to discuss this in another publication.
the string theory is not the theory of hadron, but quantum theory of gravitation. It is believed that the theory has no UV divergence.

Neveu, Schwarz and independently Ramond have constructed the dual model which contains fermions [19, 20]. The dual model has worldsheet scalars and worldsheet fermions. The NSR string theory has the worldsheet supersymmetry. The supersymmetry is defined as a symmetry which exchanges bosons and fermions. We need to determine boundary conditions of strings. For fermionic variables, there are two types of conditions; Neveu-Schwarz (NS) and Ramond (R) boundary conditions. Because of the existing of two types of conditions, degrees of freedom increase. Hence we have to project out the extra degrees of freedom. This is called GSO (Gliozzi, Schwarz and Olive) projection [21]. After performing the GSO projection the NSR string theory has the spacetime supersymmetry, and does not contain a tachyon. In the theory the worldsheet supersymmetry is manifest but the spacetime supersymmetry is not.

Green and Schwarz [22, 23, 24] have constructed the manifestly spacetime supersymmetric string theory which is equivalent to the NSR string theory. The Green-Schwarz string does not have manifest worldsheet supersymmetry and it is difficult to quantize it with covariant gauge fixing conditions [25] because of the \( \kappa \) symmetry [26] which is needed in order to have equivalence with the NSR string.

The critical dimension of superstring theory is \( D=10 \) [27, 28, 29, 30, 31]. We should solve this problem. The most popular method is a compactification method. The word “compactification” means that some space directions are periodic and their radii are very small \( (R \rightarrow 0) \). The compactified directions are so small that we can not find motion of particles along to the compactified directions. Let us see a plain example. A stick is a three dimensionally extended object. However if the stick is seen from afar, this looks as if this is one dimensional object. Of course this is not the only possibility. Recently the brane world scenario is studied by a lot of authors [32].

We can construct five kinds of the perturbative string theories which are sorted by their symmetries. Type I theory has \( N=1 \) supersymmetry where \( N \) is the number of the supercharges. Type IIA (IIB) theory has \( N=2 \) supersymmetry whose chiralities are opposite (same). We can also construct string theories whose left mover is bosonic string and right mover is superstring. The only allowed gauge groups are \( E_8 \times E_8 \) and \( SO(32) \). These are called \( E_8 \times E_8 \) and \( SO(32) \) Heterotic string theories respectively.

It is expected that the string theory describes physics at the Planck scale. Hence we need to construct the low energy effective theory. However we do not know how to choose a true way to compactify the extra dimensions. There are infinitely many ways to construct the effective theories. This is the limit of applicability of the perturbative string theory. It is guessed that the true vacuum is determined by the construction of non-perturbative string theory.

When we see the dynamics of open strings we should determine boundary conditions of them. There are two types of conditions; Neumann and Dirichlet boundary conditions. For the Dirichlet, an end point of string can not move along to the direction. When the number of the directions to which open string satisfies the Dirichlet condition is \( p \), we can represent this situation by the open string which is attached to a \( p \)-dimensional object and freely moves on the brane. We call it as Dp-brane.

The ten dimensional supergravity is the effective theory of the superstring theory in the
sense that this contains only massless modes, and all of the massive modes are integrated out. The theory has solitons. We can understand non-perturbative effects through the solitons. The effective theory of string theory has a p-dimensional solitons. These solitons correspond to D-branes in string theory [33].

Various non-perturbative definitions of string theory are proposed which use lower dimensional D-branes as fundamental degrees of freedom. In the IIB matrix model [34] D-instantons are fundamental degrees of freedom. And in the BFSS matrix theory [35] it is conjectured that M-theory in the infinite momentum frame is equivalent to a large N limit of the theory of N D0-branes. It is guessed that the string theories correspond to a weak coupling limit of the M-theory. But the theory does not have any correct definition yet. In these matrix models coordinates are represented by matrices. Because of this the spacetime becomes naturally non-commutative. It was pointed out by Connes, Douglas and Schwarz [10] that M-theory with background constant three form tensor field compactified on a torus can be identified with matrix theory compactified on a non-commutative torus. Corresponding to this, the string theory with background NS B field is equivalent to the string theory on a non-commutative space [11].

The non-commutative spacetime is studied not only in the string theory, but also in field theory. Pauli has suggested that quantized spacetime can be used as a regulator in field theory. Ydri [36] has shown that all infinities in $\phi^4$ theory can be removed perturbatively by choosing appropriate non-commutative space.

Some authors have shown that non-commutativities come from differences of viewpoints. A Dp-brane with a constant gauge field is equivalent to $\infty$ D$(p-2)$-branes [37]. The worldvolume theory of a Dp-brane is ordinary gauge theory, and that of $\infty$ D$(p-2)$-branes is non-commutative gauge theory. Another way to give non-commutative coordinates is related to regularization methods. In this paper, we would like to see mainly relations between commutative and non-commutative field theories.

The subjects covered in this paper are organized as follows. In chapter 2 we see a relation between T-duality and D-brane. Non-commutativity in the string theory always appears at end points of strings at which D-branes are. The T-duality in closed string theory is defined as the exchange of Kaluza-Klein mode and winding mode, and simultaneously the exchange of a compactification radius $R$ and $\tilde{R} \equiv \frac{\alpha'}{R}$. Since in open string theory there is no winding mode, we define it as the exchange of Dirichlet condition and Neumann condition. We see the definition of boundary state which represents a boundary of a string (D-brane). In chapter 3 we construct the boundary state corresponding to infinite D$(p-2)$-branes. And we show equivalence between a Dp-brane and $\infty$ D$(p-2)$-branes [37]. An important point is that the theory of a Dp-brane is the ordinary DBI theory, and that of D$(p-2)$-branes is a gauge theory on non-commutative space. In chapter 4 another relation between commutative and non-commutative gauge theories is given. We see that products of functions on non-commutative space are Moyal products. If there is a constant background NS B field, coordinates of end points of open string become non-commutative. We observe this in terms of conformal field theories (CFT). In quantum field theories when we calculate an amplitude, its value is infinite. In order to make theory converge, we need to regularize the theory. At the quantum level symmetry depends on the regularization method. We find that the use of point splitting regularization gives non-commutative gauge theory. The Seiberg-Witten map which relates gauge theory on
ordinary space with non-commutative gauge theory is reviewed [12] in chapter 5. In the framework of the operator formalism, we can understand that the mixed type (coordinate and canonical momentum) boundary condition is a source of non-commutativity when we regard the boundary conditions as primary constraints [38, 40, 41, 42] which is reviewed in chapter 6. The final chapter contains summary and remarks.
Chapter 2

T-Duality and D-Brane

2.1 Open String and Boundary Conditions

In the conformal gauge, the bosonic string action is stationary where the coordinate $X^{\mu}$ satisfies the equation of motion:

$$\partial_{\alpha} \partial^{\alpha} X^{\mu} = 0,$$

and furthermore in the open string case the Neumann boundary condition

$$\partial_{\sigma} X^{\mu}|_{\sigma=0,\pi} = 0$$

or the Dirichlet condition

$$\delta X^{\mu}|_{\sigma=0,\pi} = 0.$$

These boundary conditions are described by a D-brane at which the end points of the open string are. Along longitudinal directions of the D-brane, the string satisfies the Neumann condition, on the other hand along transverse directions of the D-brane the string satisfies the Dirichlet condition. The end points of the open string can not leave the D-brane. There are four kinds of choices of the boundary conditions. We write down the solutions of the Eq. of motion for each cases below.

**N-N boundary condition**

If an open string satisfies the Neumann boundary condition at both end points, the solution of the Eq. of motion is

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2}\alpha' \sum_{n \neq 0} \left( \frac{\alpha^\mu_n}{n} e^{-in\tau} \cos n\sigma \right). \hspace{1cm} (2.1.1)$$

**D-D boundary condition**

The solution is

$$X^\mu = \frac{c^\mu (\pi - \sigma) + d^\mu \sigma}{\pi} - \sqrt{2}\alpha' \sum_{n \neq 0} \left( \frac{\alpha^\mu_n}{n} e^{-in\tau} \sin n\sigma \right).$$
2.2 T-Duality and Closed String

In this section we see about relations between T-duality and D-branes [43, 44]. At first consider bosonic closed strings with a condition 

\[ X^\mu(\sigma) = X^\mu(\sigma + \pi). \]

The solution of the equation of motion is given by

\[
X^\mu(\tau, \sigma) = x^\mu + \sqrt{2\alpha'} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left( \frac{\alpha^\mu}{r} e^{-i\tau r} \sin r\sigma \right)
\]

\[
+ i \sqrt{2\alpha'} \sum_{r \notin \mathbb{N}} \frac{1}{n} \left( \alpha^\mu e^{-2im(r-\sigma)} + \tilde{\alpha}^\mu e^{-2im(r+\sigma)} \right).
\]

In uncompactified directions, the term proportional to \( \sigma \) is not allowed. Then \( \alpha^\mu_0 - \tilde{\alpha}^\mu_0 = 0 \).

According to the Nöther method, it is easy to see the conserved momentum of the string is

\[
p^\mu = \frac{1}{\sqrt{2\alpha'}} (\alpha^\mu_0 + \tilde{\alpha}^\mu_0). \tag{2.2.1}
\]

We consider the situation that the 25th direction of spacetime is compactified into \( S^1 \). This statement is equivalent to the condition in which two points in the compactified direction are identified;

\[
x^{25} \sim x^{25} + 2\pi R, \tag{2.2.2}
\]

where \( R \) is the radius of \( S^1 \). The generator of the translation along the compactified direction is \( e^{ipx} \). From the identification (2.2.2), it can be understood that physical states are invariant under the translation from \( x = 0 \) to \( x = 2\pi R \), then the 25th direction of the momentum (2.2.1) must be

\[
p^{25} = \frac{n}{R} \quad n \in \mathbb{Z}. \tag{2.2.3}
\]

In order to satisfy the condition \( X^\mu(\sigma) = X^\mu(\sigma + \pi) \), the term proportional to \( \sigma \) must be

\[
\pi \sqrt{2\alpha'} (\alpha^{25}_0 - \tilde{\alpha}^{25}_0) = 2\pi w R
\]
where \( w \) is a winding number of the closed string about the compactified direction. From above equations, one find

\[
\alpha_{0}^{25} = \sqrt{\frac{\alpha'}{2}} \left[ \frac{n}{R} + \frac{wR}{\alpha'} \right]
\]

\[
\tilde{\alpha}_{0}^{25} = \sqrt{\frac{\alpha'}{2}} \left[ \frac{n}{R} - \frac{wR}{\alpha'} \right]
\]

With these situations, we would like to see the mass shell condition. To do this the zero mode of the Virasoro generators \( L_{n}, \tilde{L}_{n} \) are needed. These are written as the coefficients of the mode expansion of the energy momentum tensor

\[
T_{--} = 4\alpha' \sum_{n} L_{n} e^{-2in(\tau - \sigma)}
\]

\[
T_{++} = 4\alpha' \sum_{n} \tilde{L}_{n} e^{-2in(\tau + \sigma)}
\]

which can be rewritten:

\[
L_{n} = \frac{1}{4\pi\alpha'} \int_{0}^{\pi} d\sigma T_{--} e^{2in(\tau - \sigma)}
\]

\[
\tilde{L}_{n} = \frac{1}{4\pi\alpha'} \int_{0}^{\pi} d\sigma T_{++} e^{2in(\tau + \sigma)}.
\]

The energy momentum tensor is

\[
T_{++} = \partial_{+} X^{\mu} \partial_{+} X_{\mu}
\]

\[
T_{--} = \partial_{-} X^{\mu} \partial_{-} X_{\mu},
\]

which are obtained by a variation of the string action with respect to the world sheet metric. Performing the integrations in (2.2.4) and (2.2.5) with (2.2.6) and (2.2.7) gives \( L_{n} \) and \( \tilde{L}_{n} \) which are

\[
L_{m} = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_{n}
\]

\[
\tilde{L}_{m} = \frac{1}{2} \sum_{-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_{n}.
\]

The mass shell condition is

\[
M^{2} = \frac{2}{\alpha'} \sum_{n=1}^{\infty} \left( \alpha_{-n} \cdot \alpha_{n} + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_{n} \right) - \frac{4}{\alpha'} + \left( \frac{n}{R} \right)^{2} + \left( \frac{wR}{\alpha'} \right)^{2}
\]

These equations are invariant under the exchange of the Kaluza–Klein mode with the winding mode, and simultaneously \( R \) with \( \hat{R} \):

\[
w \leftrightarrow n, \quad R \leftrightarrow \hat{R} \equiv \frac{\alpha'}{R}.
\]
This is T-duality transformation. \( \hat{R} \) is a comapactification radius of the T-dual theory. In terms of zero modes of the string oscillator, this transformation is written as

\[
\alpha_0 \rightarrow \alpha_0 \quad , \quad \tilde{\alpha}_0 \rightarrow -\tilde{\alpha}_0
\]

(2.2.8)

Under this transformation, the physical space is changed. One can notice this fact by seeing the form of the Virasoro Operators. In \( \hat{L}_m \), a term with \( \tilde{\alpha}_0 \) is contained. Because we would like not to change the physical space under the T-duality transformation, we have to extend the definition of the T-duality transformation (2.2.8) to the non-zero modes:

\[
\alpha_n \rightarrow \alpha_n \quad , \quad \tilde{\alpha}_n \rightarrow -\tilde{\alpha}_n.
\]

(2.2.9)

It is convenient to deal with the T-duality in terms of string target space coordinates. The string target space coordinate \( X^\mu \) can be decomposed into the left and right moving modes:

\[
X^\mu(\tau, \sigma) = X^\mu_L(\tau - \sigma) + X^\mu_R(\tau + \sigma)
\]

(2.2.10)

where

\[
X^\mu_L(\tau - \sigma) = x^\mu + \sqrt{2\alpha'}(\tau - \sigma)\alpha_0^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)}
\]

\[
X^\mu_R(\tau + \sigma) = x^\mu + \sqrt{2\alpha'}(\tau + \sigma)\tilde{\alpha}_0^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}.
\]

We will extend the definition of the T-duality transformation to one including center of mass coordinates \( x^\mu \):

\[
X \rightarrow \hat{X} = X_L - X_R
\]

(2.2.11)

which is called T-dual coordinate. In the next section we will show that the T-dual symmetry is a key concept to understand D-branes in open string theory.

### 2.3 Open Strings and T-Duality

Open string theory does not have a winding mode. From this point of view, one may think that there is not the T-duality in open string theory. But this idea is inconsistent.

For example, an open string 1-loop worldsheet (cylinder) diagram can also be viewed as a closed string tree diagram. Therefore open string theory contains closed strings. Let us consider a \( R \rightarrow 0 \) limit. In this limit, if we see the worldsheet as the open string diagram, the K-K mode is infinitely massive. Hence the only allowed state which does not decouple from the theory is with \( n = 0 \) (i.e. zero momentum state). This suggests the open string can not have momentum along compactified directions, in other words, be living in D-k dimensional space (k is the number of the compactified dimensions). On the other hand, in the closed string channel, by virtue of the T-duality, the \( R \rightarrow 0 \) limit
correspond to the $\hat{R} \to \infty$ in the T-dual theory. This makes the closed string possible to have momentum along the compactified directions. This is not consistent.

Then let us define the T-duality in the open string theory so that it will solve this inconsistency. It is natural to define the T-dual coordinates also in open string theory as

$$X_L(\tau - \sigma) = \frac{1}{2} x + \sqrt{2\alpha'}(\tau - \sigma)\alpha_0 + i\sqrt{2\alpha'}\sum_{n \neq 0} \frac{\alpha_n}{n} e^{-in(\tau - \sigma)}$$

$$X_R(\tau + \sigma) = \frac{1}{2} x + \sqrt{2\alpha'}(\tau + \sigma)\alpha_0 + i\sqrt{2\alpha'}\sum_{n \neq 0} \frac{\alpha_n}{n} e^{-in(\tau + \sigma)}.$$ 

The T-dual coordinates satisfy the conditions

$$\partial_\tau X \to \partial_\sigma \hat{X} = -\partial_\sigma X$$

$$\partial_\sigma X \to \partial_\tau \hat{X} = -\partial_\tau X.$$  

(2.3.1)

These facts imply that the Neumann and Dirichlet boundary conditions are interchanged by T-duality transformation. In terms of the D-brane, the D-branes in the original theory and the T-dual theory are at angle $\frac{\pi}{2}$.

It is easy to see that an open string which satisfies these boundary conditions is interpreted to be attached to a hyper-surface which is called a Dp-brane, where $p$ is the number of uncompactified (non-T-dualized) directions. The distance between two T-dual coordinates of endpoints of the open string is

$$\hat{X}(\pi) - \hat{X}(0) = -2\pi n\hat{R}.$$  

(2.3.2)

This T-dual description means that the open string is winding around the compactified directions by $n$ times, whose endpoints are attached to same D-brane. This fact leads us to a conclusion that the bulk part of the open string freely moves in full D dimensional space, the boundary parts are however constrained to $p+1$ dimensional hyper-surface. This solved the problem for the bulk part of the string which can be viewed as both closed and open string world sheet.

As a final work in this section, we will show how string theory with Chan-Paton factors (which has U(N) gauge symmetry) can be realized in terms of D-branes.

Let $X$ be a compactified coordinate (here an index of the direction is omitted). We assume that along the direction there is a constant gauge field of the form

$$A_{ij} = \frac{1}{2\pi R} \text{diag}(\theta_1, \cdots, \theta_N).$$  

(2.3.3)

Here the string in which we are interested is an oriented one, whose charges at end points transform according to the adjoint representation $N \times \bar{N}$ of U(N). The action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X_\mu \partial^\alpha X^\mu + \int d\tau A_\mu \hat{X}^\mu|_{\sigma=\pi} - \int d\tau A_\mu \hat{X}^\mu|_{\sigma=0}$$  

(2.3.4)

where $A_\mu$ is the gauge field with $A_i = \frac{\theta_i}{2\pi R}$ coupled to the string endpoints. The assumption that the string is oriented is responsible for the difference between the signs in front of the terms with gauge field. The conserved current (momentum) is

$$P_\tau = \frac{1}{2\pi\alpha'} \partial_\tau X^\mu - \frac{1}{2\pi R} (\theta_j - \theta_i).$$
The conserved charge is

\[ p = \sqrt{\frac{2}{\alpha'} \alpha_0} - \frac{1}{2R} (\theta_j - \theta_i) \]

Similar procedure to (2.2.3) gives

\[ p = \frac{n}{R}. \]

The distance between two endpoints of the open string in T-dual theory is

\[ \hat{X}(\pi) - \hat{X}(0) = -(2\pi n + \theta_j - \theta_i) \hat{R}. \]

What we can read into this equation is the open string is attached to different D-branes whose coordinates are \( \theta_i \hat{R} \) and \( \theta_j \hat{R} \), respectively. (2.3.3) gives

\[ \theta_i \hat{R} = 2\pi \alpha' A_{ii}, \]

then i-th D-brane coordinate is written as

\[ X_i = -2\pi \alpha' A_{ii}. \]

The U(N) gauge field represents the D-brane’s coordinates. From this result, one can say that a string with Chan-Paton factors is equivalent to a string whose boundary points are constrained in D-branes.

### 2.4 Dirichlet Condition and Momentum Flow

It is well known that there is no momentum flowing out of open string’s end points for the Neumann boundary condition. The reason is the following.

By the Nöther method the current with respect to the translation on the worldsheet (i.e. momentum) is given by

\[ P^\mu_\alpha = T \partial_\alpha X^\mu, \]

where \( T \) is the string tension. The momentum flow across a line segment \( d\tau \) on the worldsheet is given by

\[ dP^\mu = P^\mu_\sigma d\tau. \quad (2.4.1) \]

The Neumann boundary condition implies that there is no momentum flowing out of the end of the string.

On the other hand, the open string with the Dirichlet boundary condition does not have the same property. This string satisfies the condition

\[ \partial_\tau X^\mu |_{\sigma=0,\pi} = 0. \]

Therefore the momentum flow (2.4.1) does not vanish. This flow into the D-brane to which the open string attaches. Hence the D-brane has the momentum, in other words, the D-brane is dynamical object.
2.5 Bosonic Boundary State

The interaction between two Dp-branes is described by the vacuum fluctuation of an open string which is between them. Its lowest order contribution is the open string 1-loop diagram which is illustrated by a cylinder. By exchanging the role of the world sheet coordinates $\tau$ and $\sigma$ this open string amplitude can also be viewed as a closed string tree amplitude which propagates between the Dp-branes. Here we would like to make use of this idea.

Our main purpose in this section is to construct a boundary state \[15\, 16\] which describes boundary conditions of a closed string which is attached to a Dp-brane. At a boundary $\tau = 0$ one defines the boundary state as

$$\partial \tau X^\alpha |_{\tau=0} |B\rangle = 0 \quad \alpha = 0, 1, \cdots, p$$ \hspace{1cm} (2.5.1)

$$X^i |_{\tau=0} |B\rangle = y^i |B\rangle \quad i = p + 1, \cdots, D - 1.$$ \hspace{1cm} (2.5.2)

Analogous conditions are hold at another endpoint of the string. In order to satisfy the boundary conditions (2.5.1) and (2.5.2) for arbitrary $\sigma$, the following expressions have to be hold;

$$ (\alpha^\alpha_n + \tilde{\alpha}^\alpha_{-n}) |B\rangle = 0$$ \hspace{1cm} (2.5.3)

$$ (\alpha^i_n - \tilde{\alpha}^i_{-n}) |B\rangle = 0$$ \hspace{1cm} (2.5.4)

$$ p^i |B\rangle = 0$$ \hspace{1cm} (2.5.5)

$$ (x^i - y^i) |B\rangle = 0.$$ \hspace{1cm} (2.5.6)

These expressions for the non-zero mode are also written as

$$ (\alpha^\mu_n + S^\mu_{\nu} \tilde{\alpha}^\nu_{-n}) |B\rangle = 0,$$

here we introduced the matrix

$$ S^\mu{}^\nu = (\eta^{\alpha\beta}, -\delta^i_j).$$

So far we have found the conditions which the boundary state for a Dp-brane must satisfy. The above expressions are adequate for investigation of the non-commutativity in string theory and we have no need to know the explicit form of the boundary state. It is easy problem, however, to write down the boundary state itself. The answer is

$$ |B\rangle = N_p \delta^{d-p-1}(x^i - y^i) \exp\left(-\frac{1}{n} \alpha_{-n} \cdot S \cdot \tilde{\alpha}_{-n}\right) |0\rangle$$ \hspace{1cm} (2.5.7)

where $N_p$ is a normalization factor, and $|0\rangle$ is a ground state with respect to the operators $\alpha, \tilde{\alpha}, p$.

For the zero-modes it is trivial for the boundary state to satisfy the conditions (2.5.3) and (2.5.6). Let’s check others (2.5.4) and (2.5.5). The commutation relation for the oscillators is given by

$$ [\alpha_m, \alpha_n] = m \delta_{m+n} \eta^\mu{}^\nu.$$

The counterpart for $\tilde{\alpha}$ is the same form. In the following the index of $\alpha_n$ are $n > 0$. It is easy to see that

$$ [\alpha^\mu_n, e^{-\frac{1}{n} \alpha_{-n} \cdot S \tilde{\alpha}_{-n}}] = -S^\mu_{\nu} \tilde{\alpha}_{-n} e^{-\frac{1}{n} \alpha_{-n} \cdot S \tilde{\alpha}_{-n}}.$$ \hspace{1cm} (2.5.8)

These are annihilation operators.
We can read into this equation that the boundary state (2.5.7) satisfies boundary conditions (2.5.3) and (2.5.4). Here we do not determine the normalization factor $N_p$, but this is very important when, for example, one calculates a beta function with respect to open string theory with background fields.

In this section we explained the basics of the boundary state in the case that there are not background fields. We will see the boundary states with constant background fields, when we review Ishibashi’s papers [37, 13] in the next chapter in which we see the equivalence between a Dp-brane and $\infty$ D(p-2)-branes.
Chapter 3

Dp-Brane from D(p − 2)-Branes

3.1 Classical Solution of the Theory of D-Branes

Lower dimensional D-branes such as D-particles and D-instantons are used to construct the matrix models as the fundamental degrees of freedom\[33, 34\]. N. Ishibashi et.al.\[34\] showed that a D-string can be expressed as a classical configuration of infinitely many D-instantons in the IIB matrix model. In the BFSS matrix theory\[35\] the conjecture is that M-theory in the infinite momentum frame is equivalent to the $N \to \infty$ limit of the theory of N D0-branes. P.K. Townsend suggested that a classical supermembrane configuration could be identified with D0-branes\[17\]. The purpose in this chapter is to generalize this to the relation between Dp-brane and D(p−2)-branes with $p > 1$ in the bosonic string theory. The consideration leads to the equivalence between non-commutative and commutative field theories\[37, 13\].

An open string which is attached to a Dp-brane has two kinds of massless modes; a gauge field on the D-brane $A_\alpha(\xi) (\alpha = 0,1, \cdots ,p)$ and corrective coordinates $M_i(\xi)$ $(i = p+1, \cdots ,D−1)$. $M_i(\xi)$ describes the position of fluctuating Dp-brane. The dynamics of the D-brane is represented by these fields.

If there are N parallel D-branes which are overlapped, the number of the ways the oriented open string attach to the D-branes is $N^2$. This is equivalent to the number of massless modes. As a result $A_\alpha(\xi)$ and $M_i(\xi)$ are promoted to $N \times N$ matrices which is first advocated by Witten\[48\].

This system is described by p+1 dimensional U(N) Yang-Mills theory with U(N) adjoint scalar fields $M_i$;

$$S = \int d^{p+1}\xi tr[−\frac{1}{4g_s l_s^{p−3}}F_{\alpha\beta}F^{\alpha\beta} − \frac{1}{2g_s l_s^{p+1}}D_\alpha M_i D^\alpha M^i + \frac{1}{4g_s l_s^{p+5}}[M_i, M_j]^2]$$

where

$$D_\alpha M_i = \partial_\alpha M_i − ig_s[A_\alpha , M_i],$$

$g_s$ is a string coupling constant and $l_s$ is a string length. Under the condition that the gauge field vanish and $M^i$ is static, the action becomes

$$S = \frac{1}{4g_s l_s^{p+5}} \int d^{p+1}\xi tr[M_i, M_j]^2.$$
This has an equation of motion;
\[ [M_i, [M_i, M_j]] = 0. \]
This has a trivial solution;
\[ [M_i, M_j] = 0 \]
which implies that the spacetime is an ordinary commutative one. However there is also a non-trivial solution;
\[ [M_i, M_j] = i\theta_{ij}, \quad (3.1.1) \]
where \( \theta_{ij} \) is a constant antisymmetric tensor times \( N \times N \) unit matrix. And this expression is valid only for the case that \( X^i \) is \( \infty \times \infty \) matrix. In other words, the open string has infinitely many Chan-Paton factors. One can understand this fact by operating a trace on the both sides of \((3.1.1)\). Hence the string theory contains automatically a non-commutative structure of a spacetime.

It is easy to naively understand that the theory of \( \infty \) \( Dp \)-branes with the background \((3.1.1)\) may be identical to the theory of \( D(p + 2) \)-brane. Transverse directions of \( Dp \)-brane is non-commutative. This causes the uncertainty \( \delta X^i \delta X^j \sim \theta^{ij} \). If one might represent the configuration of \( D5 \)-branes as \((1, 2, 3, 4, 5, \times, \times, \times, \times)\), we can’t detect the precise position of the \( D5 \)-brane along \( X^6, \cdots, X^9 \) directions by the effect of the uncertainty. Because of this, it can be viewed as if the \( Dp \)-brane is extended to these directions. Therefore one can consider the branes as a \( D(p + 2) \)-brane effectively. In section 3.3 and 3.4 we will see this more precisely in the framework of the boundary state formalism.

### 3.2 Boundary Condition of an Open String

The coupling term with gauge field in the bosonic string action takes the form
\[ S_A = \int d\tau A_\mu \dot{X}^\mu. \quad (3.2.1) \]
One can change the line integral to the surface integral form using Stokes theorem;
\[ \int_{\partial C} \omega = \int_C d\omega. \]
In our case, \( \partial C \) is a path on which an end point of an open string is, and \( \omega = A_\mu dX^\mu \). Hence \( S_A \) is rewritten as
\[ S_A = \frac{1}{2} \int d\tau d\sigma F_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu. \quad (3.2.2) \]
In this form of the action \( U(1) \) gauge invariance is manifest. We would like to know how the Neumann boundary condition is deformed by the effect of the gauge field. If \( F_{\mu\nu} \) is not constant, the equation of motion will be modified from the ordinary free field equation. The form of string’s boundary condition doesn’t depend on whether \( F_{\mu\nu} \) is constant or not. The boundary condition is
\[ \partial_\sigma X_\mu + 2\pi \alpha' \partial_\tau X^\nu F_{\mu\nu} = 0 \]
at boundaries. In the closed string picture \[ 1 \] the role of \( \tau \) and \( \sigma \) are exchanged.

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1. to which is referred as the tree channel in [49](#)
### 3.3 The Boundary State

A boundary state corresponding to a D$(p - 2)$-brane which is at $X^i = 0$ is defined by

$$|B\rangle_{p-2} = |X = 0\rangle \otimes |B\rangle_{gh}$$

where the state $|X = f\rangle$ satisfies

$$X^i(\sigma) |X = f\rangle = f^i(\sigma) |X = f\rangle,$$

and $|B\rangle_{gh}$ is a ghost part which is included so that the boundary state may be BRS invariant. This part is neglected in this section since this is not important in our purpose.

A boundary state which describes a system of $N$ D$(p - 2)$-branes with background (3.1.1) is written in terms of a Wilson line factor as

$$|B\rangle_{N(p-2)} = \text{tr} P \exp(-i \int_0^{2\pi} d\sigma P^i(\sigma) M_i) |B\rangle_{p-2}, \quad (3.3.1)$$

where $P$ denotes a path ordering with respect to the path $\sigma$. $P^i$ is the conjugate momentum with respect to the string coordinates ($P^i = -\frac{1}{2\pi\alpha} \partial_0 X^i$) and $M^i$ satisfies the commutation relation (3.1.1). In this chapter, we are interested in the boundary state $|B\rangle_{N(p-2)}$ with $N = \infty$.

Changing the Wilson line factor to the path integral representation is our next task. In this work it is the key aspect that the coordinates of the D-brane satisfy the commutation relation (3.1.1). It is helpful to deal with this system in analogy with the quantum mechanics. We deal with the $M^i$ as operators. The eigenvector of $M^i$ is denoted by $|y^i\rangle$;

$$M^i |y^i\rangle = y^i |y^i\rangle.$$  

In this equation repeated indices are unsummed. The $y^i$ representation of $M^j$ is

$$M^j = i\theta^{ij} \frac{\partial}{\partial y^i}.$$  

In the representation space (3.1.1) up to normalization $y^i$ representation of the eigenstate of $y^j$ is

$$\langle y^i |y^j\rangle \sim \exp[i\omega^{ij} y^i y^j]$$

where the $\omega$ is inverse of $\theta$. We will restrict ourselves to the case in which the value of the parameter $\theta$ is

$$\theta^{kl} = \theta \quad (k = p - 1, l = p)$$
$$\theta^{ij} = 0 \quad (i, j = \text{others}).$$

Let us divide the path $\sigma$ in the Wilson line factor in (3.3.1) into $N$ points with $\sigma_{i+1} - \sigma_i = \epsilon$ ($\epsilon$ is infinitesimal constant). The full Wilson line is from $\sigma = 0$ to $\sigma = 2\pi$. We
denote the Wilson lines which is from $\sigma_i$ to $\sigma_{i+1}$ as $W(i+1, i)$. The full Wilson line factor $W(2\pi, 0)$ is decomposed as

$$W(2\pi, 0) = \text{tr} P \exp(-i \int_0^{2\pi} P_i M_i^\prime d\sigma) = \text{tr} W(2\pi, N) W(N, N-1) \cdots W(1, 0)$$

with

$$W(i+1, i) = \exp[-i \epsilon P_i M^i].$$

Performing the trace gives the path integral representation of the Wilson line factor. Hence $\infty$ D$(p-2)$-branes boundary state is written as

$$|B\rangle_{N(p-2)} = \int \mathcal{D}y \exp[i\omega \int d\sigma y^p \partial_\sigma y^p - i \int d\sigma (P_{p-1} y^p + P_p y^p)] |B\rangle_{p-2}.$$  (3.3.2)

Next, we would like to see what conditions the $\infty$ D$(p-2)$-branes boundary state $|B\rangle_{N(p-2)}$ satisfy. We use the fact an integral of a total derivative vanish.

$$0 = \int \mathcal{D}y \frac{\delta}{\delta y^{p-1}} \exp[i\omega \int d\sigma y^p \partial_\sigma y^p - i \int d\sigma (P_{p-1} y^p + P_p y^p)] |B\rangle_{p-2} = [i\omega \partial_\sigma X^p - iP^{p-1}] |B\rangle_{N(p-2)}$$

This implies that the $\infty$ D$(p-2)$-branes boundary state satisfies the condition of a D$p$-brane boundary state with gauge field $F_p = \omega$. Hence one can say that the configuration of N D$(p-2)$-branes with $N = \infty$ is equivalent to that of a D$p$-brane with gauge field.

### 3.4 Worldvolume Theory

In the last section, we have explored the case in which the D$p$-branes are flat which correspond to the D-brane without scalar fields $\phi^i$. Here we would like to include effects of the collective coordinates.

#### 3.4.1 D$p$-Brane Picture

In the open string spectrum the collective coordinates of a D-brane correspond to scalar fields $\phi^i (i = p + 1, \cdots, D - 1)$. The boundary state with the scalar fields is written as

$$|B\rangle_{p, \phi} = \exp(-i \int_0^{2\pi} d\sigma P_i \phi^i) |B\rangle_p.$$  

Next we construct a boundary state with a gauge field background [51, 50]. At first, we introduce the coherent state $|x\rangle$ which satisfies

$$X^i(\sigma)|x\rangle = x^i(\sigma)|x\rangle$$
This is described as

$$|x\rangle = \exp(-i \int d\sigma P_i(\sigma)x^i(\sigma))|D\rangle$$

where the state $|D\rangle$ is defined by the condition $X^i(\sigma)|D\rangle = 0$. The boundary state with $U(1)$ gauge field is

$$|B\rangle_{pA} = \int \mathcal{D}y \exp(i \int A(y))|y\rangle.$$  

$$= \int \mathcal{D}y \exp(i \int A(y) - i \int d\sigma P_i(\sigma)y^i(\sigma))|D\rangle.$$  

In the Dp-brane picture, the open string mode is $A_\alpha$ and $\phi^i$. In these backgrounds the boundary state is guessed to be

$$|B\rangle_{A,\phi,p} = \int \mathcal{D}y \exp[i \int d\sigma A_\alpha \partial_\sigma y^\alpha - i \int d\sigma (P_{p-1}y^{p-1} + P_p y^p + \bar{P}^i \phi_i))]|B\rangle_{p-2} (3.4.1)$$

which is coincide with (3.3.2) when $F_{p-1,p} = \omega, \phi^i = 0$. Small variations $\delta A_\alpha, \delta \phi^i$ from the backgrounds $F_{p-1,p} = \omega, \phi^i = 0$ in (3.3.2) can be described by acting the following vertex operator on $|B\rangle_{N(p-2)}$

$$(1 + i \int d\sigma (\delta A_\alpha \partial_\sigma X^\alpha - \delta \phi^i P_i))|B\rangle_{N(p-2)} (3.4.2)$$

which is consistent with (3.4.1). It is easy to notice that because of the part $\exp(-i \int_{0}^{2\pi} d\sigma (P_{p-1}y^{p-1} + P_p y^p + \bar{P}^i \phi_i))$ the above boundary state corresponds to the one $|X^{p-1} = y^{p-1}, X^p = y^p, X^i = \phi^i\rangle$. This corresponds to the background

$$X^{p-1} = M^{p-1} \quad X^p = M^p \quad M^i = \phi^i(X^\alpha, M^{p-1}, M^p). \quad (3.4.3)$$

### 3.4.2 D(p − 2)-Brane Picture

We can parameterize the background more generally as

$$X^\mu = \phi^\mu(X^\alpha, M^{p-1}, M^p),$$

in which $X^\alpha, M^{p-1}$ and $M^p$ play a role of the coordinates on the worldvolume of Dp-brane. If we regard the worldvolume as that of D(p − 2)-brane, the worldvolume theory is non-commutative. The boundary state with such background is

$$|B\rangle_{p\phi} = \int \mathcal{D}y \exp[i\omega \int d\sigma y^{p-1}\partial_\sigma y^p - i \int_{0}^{2\pi} d\sigma P^\mu \phi_\mu]|B\rangle_{p-2}. \quad (3.4.4)$$

In the Dp-brane picture, on the worldvolume there are $A_\alpha$ $(\alpha = 0, \cdots, p)$ and $\phi^i$ $(i = p+1, \cdots, D-1)$. On the other hand, in the D(p − 2)-brane picture the worldvolume fields are $A_\alpha$ $(\alpha = 0, \cdots, p-2)$ and $\phi^i$ $(i = p-1, \cdots, D-1)$. In D(p − 2)-brane picture, there are not $A_\alpha$ $(\alpha = p-1, p)$. And in the Dp-brane picture, there are not $\phi^i$ $(i = p-1, p)$. 20
Here we would like to see the relation between these fields. Using (3.4.2) variations $\delta A$ and $\delta \phi^i$ in $|B\rangle_N(p{-}2)$ is written as

$$\delta |B\rangle_{N(p{-}2)} = i \int \mathcal{D}y \delta A\alpha \partial_\sigma y^\alpha \exp[i\omega \int d\sigma y^{p-1}\partial_\sigma y^p - i \int d\sigma (P_{p-1}y^{p-1} + P_p y^p)] |B\rangle_{p{-}2}$$

(3.4.5)

and

$$\delta |B\rangle_{N(p{-}2)} = -i \int \mathcal{D}y \delta \phi^i P_i \exp[i\omega \int d\sigma y^{p-1}\partial_\sigma y^p - i \int d\sigma (P_{p-1}y^{p-1} + P_p y^p)] |B\rangle_{p{-}2}$$

(3.4.6)

respectively. Using the identity

$$\delta |B\rangle = \omega \delta \phi$$

(3.4.7)

which implies that the reparametrization $\delta y$ is equivalent to the variation of the gauge field.

The relation between D$p$-brane picture and D$(p-2)$-brane picture can be viewed from another point of view. We consider the boundary state involving all fields $A^\alpha$ and $\phi^i$. Since there are extra fields there must be symmetry to reduce the extra degrees of freedom. This is reparametrization invariance. If we use the static gauge, the boundary state becomes that of D$p$-brane picture. On the other hand the gauge condition $F_{p-1}p = \omega$ correspond to the D$(p-2)$-brane picture.

We summarize the result. We observed that a D$p$-brane with gauge field $F = \omega$ can be viewed as a configuration of infinitely many D$(p-2)$-branes. If we consider the system as one D$p$-brane, the worldvolume theory is the gauge theory on ordinary commutative space. However in the D$(p-2)$-brane picture the worldvolume theory is non-commutative gauge theory.

Let us see about the symmetry. In the D$(p-2)$-brane picture the deformation of a configuration of D-branes along $X^{p-1}$ and $X^p$ directions is parameterized by the scalar fields $\phi^i$. The space of deformation would be

$$\frac{\text{space of } \phi^i}{\text{Diff}_F}$$

(3.4.8)

where $\text{Diff}_F$ is the group of diffeomorphism which preserve the field strength. The deformation of a D$p$-brane along $X^{p-1}$ and $X^p$ directions is also parameterized by $A^\alpha$. The counterpart of (3.4.8) is

$$\frac{\text{space of } A^\alpha}{G}$$

(3.4.9)

where $G$ is the gauge group. (3.4.8) and (3.4.9) are equivalent to each other by virtue of the relation $\delta A = \omega \delta \phi$. 

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### 3.5 Extension to multiple Dp-branes

In the last section we showed that a Dp-brane with a constant gauge field strength is equivalent to a configuration of infinitely many D(p − 2)-branes. Here we extend this result to the relation between n Dp-branes (n is finite) and ∞ D(p − 2)-branes. On the worldvolume of n Dp-branes the gauge field and the scalar field are in adjoint representation of U(n) group. In the Dp-brane picture the boundary state is written as

\[
|B\rangle_{A,\phi,p} = \int D\sigma A_{\alpha} \partial_{\alpha} y^\alpha - i \int d\sigma (P_{p-1} y^{p-1} + P_p y^p + P^{i} \phi_i) |B\rangle_{p-2} \quad (3.5.1)
\]

where \(A_{\alpha}\) and \(\phi^i\) are \(n \times n\) matrices. The difference between (3.5.1) and (3.4.1) is that there is \(\text{trP}\) in (3.5.1). As we did in section 3.3, we would like to make the factor \(\text{trP}\) easy to deal with. The technique to deal with \(\text{trP}\) is suggested by Samuel[54]. The technique is applied by many authors[53, 54, 55, 56, 57, 58, 59, 60].

Let \(H = iA_{\alpha} \partial_{\alpha} y^\alpha - i(P_{p-1} y^{p-1} + P_p y^p + P^{i} \phi_i)\) and \(\sigma_{j+1} - \sigma_j = \epsilon (j = 0, 1, \ldots, 2\pi)\). We introduce a set of anti-commuting variables \(\eta_l(\sigma_j) = \eta_l(j)\) and \(\eta^*_l(\sigma_j) = \eta^*_l(j)\) \((l = 1, \ldots, n)\) which satisfies anti-commutators

\[
[\eta_l(j), \eta_k(j)]^{(+)} = 0, \quad [\eta^*_l(j), \eta_k(j)]^{(+)} = 0 \quad \text{and} \quad [\eta_l^*(j), \eta_k^*(j)]^{(+)} = 0.
\]

Consider

\[
T_{lk} \equiv \int d\eta (2\pi) d\eta^* (2\pi) \int d\eta (N) d\eta^* (N) \cdots \int d\eta (0) d\eta^* (0) \exp(\sum_{j=0}^N C_j) \eta^*_k(0) \quad (3.5.2)
\]

where \(\int d\eta (i) d\eta^* (i) = \prod_k \int d\eta (i) d\eta^* (i)\) and

\[
C_j \equiv C(\sigma_j) \equiv \sum_k [\eta^*_k(j+1) - \eta^*_k(j)] \eta_k(j) + \sum_{il} \eta^*_l(j+1) H_{il}(x_j) \eta_l(j) \epsilon \quad (3.5.3)
\]

We would like to show

\[
\text{trP} \exp[\int d\sigma H] = \text{tr} T_{lk}
\]

In the exponential in (3.5.2) there is a factor;

\[
\exp[\sum_k \eta_k(j) \eta^*_k(j)] = 1 + \sum_k \eta_k(j) \eta^*_k(j) + \cdots + \frac{1}{n!} (\sum_k \eta_k(j) \eta^*_k(j))^n. \quad (3.5.4)
\]

Integrations of (3.5.4) multiplied with

\[
\begin{cases}
\frac{1}{n!} \eta_k(j) \eta^*_k(j) \\
\eta_{k_1}(j) \eta_{k_1}(j) \eta_{k_2}(j) \eta_{k_2}(j) \quad (k_1 \neq k_2) \\
\vdots \\
\eta_1(j) \eta^*_1(j) \cdots \eta_{n-1}(j) \eta^*_{n-1}(j) \eta_{n-1}(j) \eta^*_{n-1}(j) \eta_{n-1}(j) \cdots \eta_{n}(j) \eta^*_n(j) \eta_n(j)
\end{cases}
\quad (3.5.5)
\]

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respectively with respect to $\eta(i), \eta^*(i)$ are equivalent to one. All other integrations are zero. Another exponential factor is
\[
\exp\{\sum_k \eta_k(j + 1)\eta_k(j) + \sum_{il} \eta_i^*(j + 1)H_{il}(x_j)\eta_l(j)\epsilon\}.
\]
Expand this and perform $\eta(0), \eta^*(0)$ integrals in (3.5.2):
\[
\int d\eta(0)d\eta^*(0) \exp(\sum \eta(0)\eta^*(0)) \sum_n \frac{1}{n!}\left[\sum (\eta^*_k(1)\eta_k(0) + \eta^*_i(1)H_{il}(x_j)\eta_l(0)\epsilon)\right]^n\eta^*_k(0)
\]
\[
= \sum_k \eta^*_k(1)[I + \epsilon H(1)]_{kl}
\]
where we have used (3.5.5) and $I$ is an unit matrix. Repetition of these integrations in (3.5.2) gives
\[
T_{ij} = \{[I + \epsilon H(N)] \cdots [I + \epsilon H(1)]\}_{ij}
\]
Using this result the boundary state (3.5.1) is rewritten as
\[
\langle B|_{A,\phi,p} = \int D\eta D\eta^* \times \exp\left\{\int d\sigma \eta^*\partial_\sigma \eta + i \int d\sigma \eta^*[A_\alpha \partial_\sigma y^\alpha - i \int d\sigma (P_{p-1}y^{p-1} + P_py^p + P^i\phi_i)]\eta\right\}|B\rangle_{p-2}.
\]
In short performing $\eta, \eta^*$ integrals in (3.5.6) gives (3.5.2).

### 3.6 Diff Invariance and Non-Commutative Gauge Symmetry

In this section we analyze a flat $D_p$-brane with a constant NS B field in the Type II theory and show that the symmetry of the worldvolume coordinate can lead to a non-commutative gauge symmetry on the D-brane\[61\]. The $D_p$-brane is extended in the spacetime directions $X^\alpha (\alpha = 0, \cdots , p)$. The worldvolume coordinates are parameterized by $x^\alpha$, which are related to the spacetime coordinates as
\[
X^\alpha = x^\alpha.
\]
We assume that the NS B field has maximal rank $r = p + 1$. For simplicity we restrict ourselves to the type IIB theory. In this case the space dimensionality $p$ of a $D_p$-brane is odd and the matrix $B_{\alpha\beta}$ is invertible. We denote the inverse matrix of B as $\theta$. The effective theory of D-brane in the approximation that the derivative of the field strength $f_{\alpha\beta}$ of U(1) gauge field is negligible is the Dirac-Born-Infeld action. The DBI action has two types of gauge symmetry. One of which is ordinary one;
\[
A_\alpha \rightarrow A_\alpha + \partial_\alpha \lambda.
\]
And another is

\[ \begin{align*}
B_{\alpha\beta} & \rightarrow B_{\alpha\beta} + \partial_\alpha \Lambda_\beta - \partial_\beta \Lambda_\alpha \\
A_\alpha & \rightarrow A_\alpha + \Lambda_\alpha.
\end{align*} \]

In the action the gauge invariant combination of the U(1) gauge field and NS B field is

\[ F_{\alpha\beta}(x) \equiv B_{\alpha\beta} + f_{\alpha\beta}(x) \]

where

\[ f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \]

In the DBI action, we can fix the \(\Lambda\) gauge symmetry as \(F_{\alpha\beta} = B_{\alpha\beta}\). We transform the worldvolume coordinate \(x\) to \(\sigma\). This transformation causes a change of the total field strength;

\[ \tilde{F}_{\alpha\beta}(\sigma) = \frac{\partial x^\delta}{\partial \sigma^\alpha} \frac{\partial x^\gamma}{\partial \sigma^\beta} F_{\delta\gamma}(x). \]

Let us consider an infinitesimal transformation;

\[ x^\alpha = \sigma^\alpha + d^\alpha. \quad (3.6.2) \]

The total field strength becomes

\[ \tilde{F}_{\alpha\beta} = B_{\alpha\beta} + B_{\rho} \partial_\alpha d^\rho + B_{\alpha\rho} \partial_\beta d^\rho \]

The coordinate system \(\sigma^\alpha\) in which the total field strength is equivalent to the NS B field is not unique. Coordinate translation from the \(\sigma^\alpha\) to \(\sigma^\alpha + V^\alpha(\sigma), V^\alpha\) is infinitesimal, with \(V^\alpha\) satisfying

\[ B_{\delta\beta} \partial_\alpha V^\delta + B_{\alpha\delta} \partial_\beta V^\delta = 0, \]

the NS B field (in other words, total field strength) is invariant. \(V^\alpha = \theta^{\alpha\beta} \partial_\beta \rho\) for arbitrary scalar \(\rho\) satisfies this equation. The consequence in this paragraph is that the worldvolume with the total field strength \(\tilde{F} = B\) has symmetry;

\[ x^\alpha \rightarrow x^\alpha + i\{\rho, x^\alpha\}. \quad (3.6.3) \]

Here we deal with the worldvolume of the D-brane as a symplectic manifold with a Poisson bracket

\[ \{A, B\} \equiv i\theta^{\alpha\beta} \partial_\alpha A \partial_\beta B \]

where the \(A\) and \(B\) are functions depending on canonical variables. Using the definition of the Poisson bracket gives

\[ \{\sigma^\alpha, \sigma^\beta\} = i\theta^{\alpha\beta}. \]
We see the meaning of this equation. Define a field
\[ \hat{A}_\alpha = B_{\alpha\beta} d^\beta(\sigma). \]

In terms of the gauge field \( A \), the symmetry (3.6.3) is written as
\[ \hat{A}_\alpha \rightarrow \hat{A}_\alpha + \partial_\alpha \rho + i\{\rho, \hat{A}_\alpha\}. \] (3.6.4)

where (3.6.2) and (3.6.3) were used. Let us see a relation between the Poisson bracket and a canonical commutator on a non-commutative space. If the inverse of the NS B field \( \theta \) is small, one can expand a Moyal product factor. From this consideration, it is found that the commutator is equivalent to the Poisson bracket up to first order in \( \theta \).

\[ [A, B]_\theta = \{A, B\} + \mathcal{O}(\theta^2) \]

Hence the symmetry (3.6.4) can be interpreted as the gauge symmetry on the non-commutative space which has non-commutative parameter \( \theta \);
\[ [\sigma^\alpha, \sigma^\beta]_\theta = i\theta^{\alpha\beta}, \]
explicitly
\[ \hat{A}_\alpha \rightarrow \hat{A}_\alpha + \partial_\alpha \rho + i[\rho, \hat{A}_\alpha]_\theta. \]

The result is essentially equivalent to the result (3.4.7) in section 3.4.
Chapter 4

Moyal Product and
Non-Commutative Field Theory

4.1 Non-Commutative Geometry

In string theory the space-time non-commutativity is related to the NS B field which is investigated by many authors. In this section we would like to show that a product of fields on a non-commutative space is represented by a Moyal product \cite{10, 62, 63}. We are not interested in a non-commutative space itself, but functions like fields and geometrical objects on a non-commutative space.

The concepts of line, surface and space in geometry can be understood in terms of $C^*$ algebra. The definition of the $C^*$ algebra is as follows\cite{9}.

* algebra

A complex space with multiplication which is associative and distributive, and with an involution $a \rightarrow a^*$ is called * algebra. An involution in an algebra $B$ is a mapping $a \rightarrow a^*$ such that

\[
\begin{align*}
(T + S)^* &= T^* + S^* \\
(\alpha T)^* &= \bar{\alpha} T^* \\
(ST)^* &= T^* S^* \\
T^{**} &= T \\
||T^*|| &= ||T||
\end{align*}
\]

with $T, S \in B, \alpha$ is a complex number. If $ST = TS$ this is called as commutative * algebra.

$C^*$ algebra

$C^*$ algebra is a * algebra with a norm which satisfies the condition

\[
||ab|| \leq ||a|| ||b||, \quad ||a^* a|| = ||a||^2.
\]
A set $\mathrm{SpA}$ of homomorphism $\chi$ of $A$ into $\mathbb{C}$ such that $\chi(1) = 1$ is compact; the compact space $\mathrm{SpA}$ is called the spectrum of $A$. There is an important theorem with respect to the commutative $C^*$ algebra.

**Theorem 1** Let $A$ be a commutative $C^*$ algebra with unit and let $X = \mathrm{SpA}$ be its spectrum. The Gelfand transformation

$$x \in A \rightarrow \chi \in X$$

(4.1.1)

is an isomorphism of $A$ onto the $C^*$ algebra $C(X)$ of continuous complex function on $X$

This implies that a commutative $C^*$ algebra can be described in terms of functions. In the next section we will see it’s non-commutative version and specific identification of a non-commutative $C^*$ algebra with functions on a non-commutative space.

### 4.2 Moyal Product

An operator $\hat{O}$ which is an element of a non-commutative $C^*$ algebra can be mapped to an element of a function algebra. The Fourier decomposition of the operator $\hat{O}$ is

$$\hat{O} = \int \frac{d^Dk}{(2\pi)^D} e^{ik_\mu \hat{x}^\mu} \tilde{f}(k)$$

where the spacetime coordinates $\hat{x}^\mu$ satisfy the commutator

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta^\mu\nu.$$ 

$\theta^\mu\nu$ is a constant real anti-symmetric tensor and characterize this non-commutative space. A counterpart of (4.1.1) in non-commutative $C^*$ algebra is represented by a Fourier transformation of $\tilde{f}(k)$;

$$f(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k).$$

Fourier transformation of $\tilde{f}(k)$ gives a function $f(x)$ on the non-commutative space. Summarize the process of this.

\[\hat{O} \quad \downarrow \quad \tilde{f}(k) \quad \downarrow \quad f(x)\]

: an element of $C^*$ algebra

: Fourier transformation

: a function on the non–commutative space

Secondary let us see how the product of two operators transform.

$$\hat{O}_f \hat{O}_g \rightarrow \tilde{f}\tilde{g}(k) \rightarrow (f * g)(x)$$
\[ \hat{O}_f \hat{O}_g = \int \frac{d^Dk'}{(2\pi)^D} e^{ik'_\mu \tilde{x}^\mu} \tilde{f}(k') \int \frac{d^Dk''}{(2\pi)^D} e^{ik''_\mu \tilde{x}^\mu} \tilde{g}(k'') \]

The product of two operators transformed as
\[
(f * g)(x) = \int \frac{d^Dk}{(2\pi)^D} e^{ikx} \int \frac{d^Dk'}{(2\pi)^D} e^{-\frac{i}{2} \theta^{\mu\nu} k'_\mu (k - k')_\nu} \tilde{f}(k') \tilde{g}(k - k')
\]
\[\equiv e^{i\theta^{\mu\nu} \partial_\mu \partial_\nu} f(x + \alpha) g(x + \beta)|_{\alpha = \beta = 0} \]

This product is called “Moyal product” or “star product”. When \( \theta = 0 \), this is reduced to an ordinary product.

As a consistency check, let us try the case
\[ f(x) = x^1 \quad g(x) = x^2 \]
on a 2 dimensional space (\( \theta^{12} \equiv \theta \)). One can readily calculate the commutator of these coordinates whose product is the Moyal product.
\[
(f * g)(x) = (1 + i\theta^{12} \partial \partial x^1 \partial x^2)x^1 x^2
\]
\[= x^1 x^2 + i\theta \]

This gives
\[ [x^1, x^2]_\star = i\theta^{12}. \]

Hence the map preserves the non-commutativity of the space. One of the origins of difficulties of the non-commutative field theory is the Moyal product which makes a theory non-local.

For the Moyal product there are simple formulae. The first one is
\[ \int dxf \star g = \int dxg \star f. \]  \hspace{1cm} (4.2.1)

We can easily check this equation by seeing order by order in \( \theta \).

\( \theta^0 \) order
\[ \int fg = \int gf \]
(Notice that the product is an ordinary one in the equation.)

\( \theta^1 \) order
\[ \int \frac{i}{2} \theta^{\mu\nu} \partial_\mu f \partial_\nu g = - \int \frac{i}{2} \theta^{\mu\nu} f \partial_\mu \partial_\nu g \]
\[= 0 \]
where in the first line the integration by parts is performed. Hence
\[ \int dxf \star g = \int dxfg, \]
and (4.2.1) is satisfied. And the second one is
\[ \delta \theta^{ij} \frac{\partial}{\partial \theta^{ij}} (f \star g) = \frac{i}{2} \delta \theta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}. \]  \hspace{1cm} (4.2.2)
4.3 Non-Commutative Field Theory

Here we will show some actions of field theory on a non-commutative space. We are aware that the only difference with an ordinary theory (on commutative space) is products of fields are the Moyal products which is the result of the last section.

$\phi^3$ theory

The ordinary $\phi^3$ theory is given by

$$S = \int d^D x \left\{ \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{\lambda}{3!} \phi^3(x) \right\}.$$ 

Its counterpart in non-commutative theory is

$$S = \int d^D x \left\{ \frac{1}{2} \partial_\mu \phi(x) \ast \partial^\mu \phi(x) + \frac{\lambda}{3!} \phi \ast \phi \ast \phi \right\} \equiv \int d^D x \left\{ \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{\lambda}{3!} \phi^3(x) \right\}_\ast.$$ 

$U(1)$ gauge theory

The ordinary $U(1)$ gauge theory is described by

$$S = - \int d^D x F^{\mu\nu} F_{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$ 

This is invariant under

$$\delta A_\mu = \partial_\mu \lambda.$$ 

On a non-commutative space, the field strength is

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu].$$ 

The action is

$$S = - \int d^D x \hat{F}^{\mu\nu} \ast \hat{F}_{\mu\nu}.$$ 

The field strength is defined by a commutator of covariant derivatives. The last term in (4.3.1) comes from the fact that the gauge fields are non-commutative. This theory is $U(1)$ gauge theory, but non-Abelian. Let us show that the action is invariant under the transformation

$$\hat{\delta} \hat{A}_\mu = \partial_\mu \hat{\lambda} + i[\hat{\lambda}, \hat{A}_\mu].$$ 

(4.3.2)
The gauge transformation changes the field strength (4.3.1) as
\[ \delta \hat{F}_{ij} = i \hat{\lambda} \ast \hat{F}_{ij} - i \hat{F}_{ij} \ast \hat{\lambda}. \]
Under the transformation, although the Lagrangian is not invariant:
\[ \delta (\hat{F}_{ij} \ast \hat{F}^{ij}) = i \hat{\lambda} \ast \hat{F}_{ij} \ast \hat{F}^{ij} - i \hat{F}_{ij} \ast \hat{F}^{ij} \ast \hat{\lambda}, \]
the action is invariant by virtue of (4.2.1).

In subsequent sections we will show that the low energy effective theory of the string with NS B field has non-commutative field theoretical representations.

### 4.4 Conformal Field Theory

The string theory has a conformal invariance on a string worldsheet. In this section our purpose is to see the symmetry and the operator product expansion. We use an Euclidean signature with respect to the string worldsheet. Defining complex variables;
\[ \rho = \tau + i\sigma \]
\[ \bar{\rho} = \tau - i\sigma, \]
the conformal symmetry is defined in terms of \( \rho \) and \( \bar{\rho} \) as
\[ \rho \rightarrow f(\rho) \]
\[ \bar{\rho} \rightarrow \bar{f}(\bar{\rho}) \]
where \( f(\rho) \) and \( \bar{f}(\bar{\rho}) \) are holomorphic functions. The left and right movers are not mixed under the transformation.

Define \( z \) plane as \( z = e^\rho \) which is mainly used below. Define a correlation function as
\[ \langle X(z)Y(w) \cdots \rangle \equiv \langle \Omega | T[X(z)Y(w) \cdots] | \Omega \rangle \quad (4.4.1) \]
where \( X \) and \( Y \) are operators on the \( z \) plane \( | \Omega \rangle \) is a conformal (SL(2,C)) vacuum whose definition is given here. The conformal vacuum differ from an oscillator vacuum which is usually used. A field \( \phi(z) \) with a conformal dimension \( d \) transforms with respect to a coordinate transformation \( z \rightarrow z' \) as
\[ \phi'(z') = \left( \frac{dz}{dz'} \right)^d \phi(z). \]

On the \( \rho \) plane oscillator modes \( \phi_n \) are defined by
\[ \phi(\rho) = \sum_{n=-\infty}^{\infty} \phi_n e^{-n\rho}. \]
On the \( z \) plane using \( \frac{d\rho}{dz} = \frac{1}{z} \) gives
\[ \phi(z) = \left( \frac{dz}{d\rho} \right)^d \phi(\rho) = \sum_n \phi_n z^{-n-d}. \]
φ_n is given by

\[ \phi_n = \oint C \frac{dz}{2\pi i} z^{n+d-1} \phi(z). \]

Here we demand that \( \phi(z) \) is regular at \( z = 0, \infty \). We consider the case that one of the operators in (4.4.1) is \( \phi_n \);

\[ \langle X(z_1)Y(z_2)\cdots \oint C \frac{dz}{2\pi i} z^{n+d-1} \phi(z) \rangle \equiv \langle \Omega|T[X(z_1)Y(z_2)\cdots \phi_n]|\Omega \rangle \]

We use a small contour C so that \( z_1, \cdots \) may not be in the contour. Because of the time (radial) ordering, \( \phi_n \) is at the right end. Since the integral on the left hand side vanishes for \( n \geq 1 - d \), we have conditions for the conformal ket vacuum;

\[ \phi_n|\Omega \rangle = 0 \quad \text{for } n \geq 1 - d. \] (4.4.2)

Similarly we obtain conditions for the bra vacuum. We have

\[ \langle \Omega|\phi_n = 0 \quad \text{for } n \leq d - 1. \] (4.4.3)

(4.4.2) and (4.4.3) are definitions of the conformal vacuum. \( \phi_n \) with \( n \geq 1 - d \) are interpreted as annihilation operators, and others are creation operators.

In a conformal field theory operators are usually normal ordered without any mention. The normal ordering of operators is defined so that modes with \( n \geq 1 - d \) is on the right of modes with \( n \leq 1 - d \). For example the normal ordered product \( : \phi_n \phi_l : \) with \( n \geq 1 - d \) and \( l \leq 1 - d \) is

\[ : \phi_n \phi_l := \phi_l \phi_n. \]

Singularity of products of operators are picked up by using of the operator product expansion (OPE) which can be generated from

\[ : X :: Y := XY + \sum \text{(cross – contractions)} \]

for arbitrary operators \( X \) and \( Y \). Then if we know a propagator, we can perform the calculation. A practical calculation is given in the next section.

### 4.5 Open String and Constant NS B Field

Coordinates of an open string with a background NS B field become non-commutative\[50\]. In this section we examine this in the framework of CFT.

The string theory has a Diff invariance and the Λ gauge invariance as discussed above. Here we use the conformal and \( F^{\alpha\beta} = 0 \) gauge fixing conditions. The string action with these gauge conditions is

\[ S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (g_{\alpha\beta} \partial_a X^\alpha \partial^a X^\beta - 2\pi\alpha' i B_{\alpha\beta} \epsilon^{ab} \partial_a X^\alpha \partial_b X^\beta) \]
where \( g_{\alpha \beta} \) and \( B_{\alpha \beta} \) are a spacetime metric and a NS B field which are constants. Boundary conditions are

\[
g_{\alpha \beta} \partial_\sigma X^\beta - 2\pi \alpha' i B_{\alpha \beta} \partial_\tau X^\beta = 0
\]
at boundaries. Here we consider the case that the string worldsheet \( \Sigma \) is a disc (in the open string picture this is a tree diagram). The worldsheet can be mapped to the upper half plane of the \( z \) plane on which the open string boundary condition is

\[
g_{\alpha \beta}(\partial - \bar{\partial})X^\beta + 2\pi \alpha' B_{\alpha \beta}(\partial + \bar{\partial})X^\beta \bigg|_{z = \bar{z}} = 0.
\]
The boundaries of the worldsheet are identified with the real axis \( (z = \bar{z}) \). We would like to obtain a Green function \( G(z, z') \) which satisfies

\[
\frac{1}{2\pi \alpha'} \partial \bar{\partial} G(z, z') = -\delta(z - z').
\]  
(4.5.1)

We use the method of images. The green function \( \langle X^\alpha(z) X^\beta(z') \rangle \) satisfies the boundary condition;

\[
(g_{\alpha \beta} + 2\pi \alpha' B_{\alpha \beta}) \partial \langle X^\beta(z) X^\gamma(z') \rangle = (g_{\alpha \beta} - 2\pi \alpha' B_{\alpha \beta}) \bar{\partial} \langle X^\beta(z) X^\gamma(z') \rangle
\]
at \( z = \bar{z} \). This is

\[
\langle X^\alpha(z) X^\beta(z') \rangle = -\alpha' [g^{\alpha \beta} \ln|z - z'| + \frac{1}{2} \left( \frac{g - 2\pi \alpha' B}{g + 2\pi \alpha' B} \right)^{\alpha \beta} \ln(z - z') + \frac{1}{2} \left( \frac{g + 2\pi \alpha' B}{g - 2\pi \alpha' B} \right)^{\alpha \beta} \ln(\bar{z} - z')] .
\]  
(4.5.2)

We rewrite this in terms of \( G \) and \( \theta \) which are defined by

\[
G^{\alpha \beta} = \left( \frac{1}{g + 2\pi \alpha' B} \frac{1}{g - 2\pi \alpha' B} \right)^{\alpha \beta}, \quad \theta^{\alpha \beta} = -2\pi \alpha' \left( \frac{1}{g + 2\pi \alpha' B} \frac{1}{g - 2\pi \alpha' B} \right)^{\alpha \beta} .
\]

What we would like to do here is to divide these terms into two parts. One of which is single-valued, and others are not. Therefore the Green function is

\[
\langle X^\alpha(z) X^\beta(z') \rangle = -\alpha [g^{\alpha \beta} \ln|z - z'| - g^{\alpha \beta} \ln|z - \bar{z}'| + G^{\alpha \beta} \ln|z - z'|^2
\]

\[
+ \frac{\theta^{\alpha \beta}}{2\pi \alpha'} \ln \frac{z - z'}{z - \bar{z}'} + D^{\alpha \beta}] .
\]

where \( D^{\alpha \beta} \) is a constant. In this chapter we only need the Green function at the boundary \( (\text{Im} Z = 0) \). The green function except for the fourth term is single-valued. We define a sign function as

\[
\epsilon(\tau) \equiv 2\theta(\tau) - 1 .
\]
Let \( D^{\alpha\beta} \) have a convenient value, at boundary \( \langle X^\alpha(z)X^\beta(z') \rangle \) is
\[
\langle X^\alpha(\tau)X^\beta(\tau') \rangle = -\alpha' G^{\alpha\beta} \ln(\tau - \tau')^2 + \frac{i}{2} \theta^{\alpha\beta} \epsilon(\tau - \tau'). \tag{4.5.3}
\]
It is easy to see that \( \theta^{\alpha\beta} \) is interpreted as a non-commutative parameter. Set \( \tau^\pm = \tau \pm \epsilon \) \((\epsilon > 0)\), the commutator of coordinates of the open string is written by using the time ordering operator \( T \);
\[
[X^\alpha(\tau), X^\beta(\tau)] = T(X^\alpha(\tau)X^\beta(\tau^-) - X^\alpha(\tau)X^\beta(\tau^+)).
\]
Using the Wick theorem gives
\[
[X^\alpha(\tau), X^\beta(\tau)] = i\theta^{\alpha\beta}
\]
which implies that the coordinates on the D-brane is non-commutative. If there is no background NS B field, \( \theta = 0 \) which means that the spacetime is commutative. As we have seen in section 4.2, on non-commutative space, a product of functions is the star product. Let us see a product of vertex operators of tachyons. Its OPE with \( \tau > \tau' \) is
\[
e^{ipX(\tau)}e^{iqX(\tau')} \sim \langle e^{ipX(\tau)}e^{iqX(\tau')} \rangle = (\tau - \tau')^{2\alpha'}G^{\alpha\beta}p_\alpha q_\beta e^{\frac{i}{2}ip\alpha q_\beta e^{i(p+q)X}}.
\]
If the factor \((\tau - \tau')^{2\alpha'}G^{\alpha\beta}p_\alpha q_\beta\) can be ignored, this is the star product;
\[
e^{ipX(\tau)}e^{iqX(\tau')} \sim e^{ipX} * e^{iqX}(\tau). \tag{4.5.4}
\]
For this reason we consider \( \alpha' \to 0 \) limit in which only massless fields are dominant. In the Seiberg-Witten limit
\[
\alpha' \sim \epsilon \rightarrow 0, \quad \alpha' \sim \epsilon \rightarrow 0
\]
\( G \) and \( \theta \) are
\[
G^{\alpha\beta} \to -\frac{1}{(2\pi\alpha')^2} \left( \frac{1}{B}g \frac{1}{B} \right)^{\alpha\beta},
\]
\[
G_{\alpha\beta} \to -(2\pi\alpha')^2(Bg^{-1}B)_{\alpha\beta},
\]
\[
\theta^{\alpha\beta} \to \left( \frac{1}{B} \right)^{\alpha\beta}. \tag{4.5.5}
\]
(4.5.3) is consistent with the result in section 3.7. And the propagator at boundary (4.5.3) becomes
\[
\langle X^\alpha(\tau)X^\beta(0) \rangle = \frac{i}{2} \theta^{\alpha\beta} \epsilon(\tau).
\]
We can generalize the equation (4.5.4) to a product of any functions. By performing formal power series in the limit we have
\[
: f(x(\tau)) : g(x(0)) : \sim: (f * g)(x(0)) : .
\]
And it is important to notice that at boundary on the worldsheet in the zero slope limit, the OPE of a derivative of $x$ and $x$ does not equivalent to the star product,

$$: \partial_\tau X^\alpha(\tau) :: X^\beta(0) : \sim 0 \quad (\text{for } \tau > 0)$$

since the propagator is proportional to the sign function $\epsilon(\tau)$. More generally we can say that the OPE of any polynomial in derivatives of $x$ does not equivalent to the star product.

Next, we would like to see that the low energy effective action of a D-brane with a point splitting regularization used is described by a non-commutative gauge theory\[12\]. We consider a background gauge field with a coupling term;

$$-i \int d\tau A_\alpha(X) \partial_\tau X^\alpha.$$

At a classical level this term is invariant under the gauge transformation\[3.6.1]\;

$$\delta \int d\tau A_\alpha(X) \partial_\tau X^\alpha = \int d\tau \partial_\tau \lambda$$

which is a total derivative. However at a quantum level the coupling term is in the argument of an exponential in the partition function, and we have to perform a regularization procedure. As well known the Pauli-Villars regularization preserves the $U(1)$ gauge invariance, and the low energy effective action has the $U(1)$ gauge invariance \[64, 65, 66\]. Let us consider a point splitting regularization. A relevant factor

$$\exp[-i \int A] \quad (4.5.6)$$

in the partition function is transformed by the gauge transformation. Expanding the exponential in powers of $A$ and $\lambda$, the first order term in $A$ is

$$- \int d\tau A_\alpha(X) \partial_\tau X^\alpha \int d\tau' \partial_\tau' \lambda.$$

The point splitting regularization is a procedure that we cut out the region $|\tau - \tau'| < \delta$ ($\delta$ is infinitesimal) in the integral.

$$- \int d\tau A_\alpha(X) \partial_\tau X^\alpha(\int_{\tau+\delta}^{\tau} + \int_{\tau-\delta}^{-\infty}) d\tau \partial_\tau' \lambda.$$

We explicitly write the normal ordering symbol which is omitted in the above

$$- \int d\tau : A_\alpha(X) \partial_\tau X^\alpha :: [\lambda(X(\tau^-)) - \lambda(X(\tau^+))] : \sim \int d\tau (A_\alpha * \lambda - \lambda * A_\alpha) \partial_\tau X^\alpha.$$

Here the OPE has been performed. Hence the point splitting regularization violates the gauge invariance\[3.6.1\]. We see that\[4.5.6\] is invariant under the non-commutative gauge transformation\[4.3.2\]. n-th order terms in $A$ of

$$\exp[-i \int (A + \partial \lambda + i[\lambda, A])_n]$$
are

\[ \frac{i^{n+1}}{n!} \int A(X(t_1)) \cdots A(X(t_n)) \partial_t \lambda(X(t)) \]
\[ + \frac{i^{n+1}}{(n-1)!} \int A(X(t_1)) \cdots A(X(t_{n-1})) (\lambda(X(t_n)) \ast A - A \ast \lambda(X(t_n))) , \]

after performing the point splitting regularization and the OPE the first term becomes

\[ \frac{i^{n+1}}{n!} \int : A(X(t_1)) \cdots A(X(t_n)) : \partial_t \lambda(X(t)) : \]
\[ \sim \frac{i^{n+1}}{n!} \int \sum_j A(X(t_1)) \cdots A(X(t_{j-1})) A(X(t_{j+1})) \cdots A(X(t_n)) \]
\[ \times (A \ast \lambda(X(t_j)) - \lambda \ast A(X(t_j))) \]

each term in the last line are identical;

\[ = \frac{i^{n+1}}{(n-1)!} A(X(t_1)) \cdots A(X(t_{n-1}))(A \ast \lambda(X(t_n)) - \lambda \ast A(X(t_n))). \]

Therefore the first and second terms are canceled out with each other. The theory has the non-commutative gauge invariance instead of the ordinary gauge symmetry. The extension of this consequence to the NSR superstring is also discussed in [12].
Chapter 5

The Seiberg-Witten Map

5.1 The Seiberg-Witten Map

In chapter 3 we have seen that a Dp-brane with a constant gauge field strength is equivalent to a configuration of $\infty$ D(p-2)-branes. The gauge field on the Dp-brane is an ordinary U(1) gauge field. On the other hand if we consider the worldvolume as that of D(p-2)-branes the worldvolume theory is the non-commutative gauge theory. In the last chapter it was observed the symmetry of the theory is dependent on the regularization. If we use the Pauli-Villars regularization ordinary gauge invariance is preserved. In contrast to this the point splitting regularization violates the gauge invariance and we have the non-commutative gauge theory. Usually two S-matrix which are obtained by using two different regularizations are related by a transformation. From this observation, there must be a map from ordinary to non-commutative gauge fields\[^{[12, 67]}\], which is called as the Seiberg-Witten map. We construct the transformation so that the map may preserve the gauge equivalence\[^{[12]}\]. An ordinary gauge field $A$ is gauge equivalent to $A + \delta_\lambda A$, and an non-commutative gauge field $\hat{A}$ is gauge equivalent to $\hat{A} + \delta_\hat{\lambda} \hat{A}$ where $\delta_\lambda$ and $\delta_\hat{\lambda}$ are ordinary and non-commutative gauge transformations with parameters $\lambda$ and $\hat{\lambda}$.

\[
\begin{align*}
A & \rightarrow A + \delta_\lambda A \\
\downarrow & \quad \downarrow \\
\hat{A}(A) & \rightarrow \hat{A}(A) + \delta_\hat{\lambda} \hat{A}(A) = \hat{A}(A + \delta_\lambda A)
\end{align*}
\]

From the requirement that the Seiberg-Witten map preserves the gauge equivalence we have

\[
\hat{A}(A) + \delta_\hat{\lambda} \hat{A}(A) = \hat{A}(A + \delta_\lambda A). \quad (5.1.1)
\]

We would like solutions of this equation up to first order in $\theta$. We set as

\[
\begin{align*}
\hat{A} &= A + A'(A) \\
\hat{\lambda} &= \lambda + \lambda'(\lambda, A)
\end{align*}
\]

\[^{1}\]The gauge theory is of arbitrary rank. We will deal with a simple case, namely, U(1) gauge theory later.
where $A'(A)$ and $\lambda'(\lambda, A)$ are of first order in $\theta$ and vanish when $\theta = 0$. (5.1.1) is written as

$$A'_i(A + \delta \lambda A) - A'_i - \partial_i \lambda' - i[\lambda', A_i] - i[\lambda, A'_i] = -\frac{1}{2} \theta^{kl}(\partial_k \lambda \partial_l A_i + \partial_i A_k \partial_k \lambda).$$

(5.1.2)

It is noticed that

$$\hat{A}_i = A_i - \frac{1}{4} \theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + \mathcal{O}(\theta^2)$$

(5.1.3)

$$\hat{\lambda} = \lambda + \frac{1}{4} \theta_{ij}\{\partial_i \lambda, A_j\} + \mathcal{O}(\theta^2)$$

are solutions of (5.1.2). Here $\{,\}$ is an anti-commutator. However it is easily noticed that the solutions have an ambiguity since the two functions $\hat{A}_i$ and $\hat{\lambda}$ are derived form one equation (5.1.2) [67]. There are other sources of ambiguities [67], but we do not discuss this in this paper. Although the calculation is tedious, the relation between ordinary and non-commutative field strength can be obtained by using (5.1.3) as

$$\hat{F}_{ij} = F_{ij} + \frac{1}{4} \theta^{kl}(2\{F_{ik}, F_{jl}\} - \{A_k, D_l F_{ij}\}) + \mathcal{O}(\theta^2).$$

Similarly we can construct a transformation from a non-commutative gauge field on a space with parameter $\theta$ to a non-commutative gauge field with $\theta + \delta \theta$. This case with $\theta = 0$ corresponds to the above case. Demanding that the map preserves the gauge equivalence relation

$$\hat{A}(\theta) \quad \rightarrow \quad \hat{A} + \delta \hat{\lambda} \hat{A}(\theta)$$

$$\downarrow$$

$$\hat{A}(\theta + \delta \theta) \quad \rightarrow \quad \hat{A}(\theta + \delta \theta) + \delta \hat{\lambda} \hat{A}(\theta + \delta \theta) = \hat{A}(\hat{A}(\theta) + \delta \hat{\lambda} \hat{A}(\theta))$$

gives

$$\hat{A}(\theta + \delta \theta) + \delta \hat{\lambda} \hat{A}(\theta + \delta \theta) = \hat{A}(\hat{A}(\theta) + \delta \hat{\lambda} \hat{A}(\theta)).$$

(5.1.4)

The Moyal product on non-commutative space with non-commutativity parameter $\theta + \delta \theta$ is

$$\exp\left[i\frac{\theta^{ij} + \delta \theta^{ij}}{2} \partial_i \partial_j^3 f(x + \alpha)g(x + \beta)\right]_{\alpha = \beta = 0} = \frac{i}{2} \delta \theta^{ij} \frac{\partial f}{\partial x_i} * \frac{\partial g}{\partial x_j} = \delta \theta^{ij} \frac{\partial}{\partial \theta^{ij}} (f * g)$$

where (4.2.2) is used. Let

$$\hat{A}(\theta + \delta \theta) \equiv \hat{A}(\theta) + \delta \hat{A}(\theta)$$

$$\hat{\lambda}(\theta + \delta \theta) \equiv \hat{\lambda}(\theta) + \delta \hat{\lambda}(\theta).$$

Using this,

$$\delta \hat{A}_i(\hat{A}(\theta) + \delta \hat{\lambda} \hat{A}(\theta)) - \delta \hat{A}_i(\theta) - \partial_i \hat{\lambda}(\theta) + i[\delta \hat{\lambda}(\theta), \hat{A}_i(\theta)]_* + i[\hat{\lambda}(\theta), \delta \hat{A}_i(\theta)]_*$$

$$= i \delta \theta^{ij} \frac{\partial}{\partial \theta^{ij}} [\hat{\lambda}(\theta), \hat{A}_i(\theta)]_*$$
whose solutions are given by
\[
\delta \hat{A}_i(\theta) = -\frac{1}{4} \delta \theta^{kl} [\hat{A}_k \ast (\partial_l \hat{A}_i + \hat{F}_{li}) + (\partial_l \hat{A}_i + \hat{F}_{li}) \ast \hat{A}_k]
\]
\[
\delta \hat{\lambda}(\theta) = \frac{1}{4} \delta \theta^{kl} (\partial_k \hat{\lambda} \ast \hat{A}_l \ast \hat{A}_l \ast \partial_k \hat{\lambda})
\]
\[
\delta \hat{F}_{ij} = -\frac{1}{4} \delta \theta^{kl} [2 \hat{F}_{ik} \ast \hat{F}_{jl} + \hat{F}_{jl} \ast \hat{F}_{ik} - \hat{A}_k \ast (\hat{D}_l \hat{F}_{ij} \ast \partial_l \hat{F}_{ij})]
\]
\[-(\hat{D}_l \hat{F}_{ij} \ast \partial_l \hat{F}_{ij}) \ast \hat{A}_k].
\]
(5.1.5)

If we would like a map from \( \theta = 0 \) to a finite \( \theta \), we need to integrate the above. For simplicity we deal with a U(1) gauge theory with a constant field strength. For this case (5.1.5) is reduced to
\[
\delta \hat{F}_{ij} = -\delta \theta^{kl} \hat{F}_{ik} \hat{F}_{lj},
\]
we rewrite this in the Lorentz indices omitted form as
\[
\delta \hat{F} = -\hat{F} \delta \theta \hat{F}.
\]
(5.1.6)

The solution of the differential equation with condition \( \hat{F}(\theta = 0) = F \) is
\[
\int_{\hat{F}} d\hat{F} \frac{1}{\hat{F}^2} = -\int_0^\theta d\theta \\
\hat{F} = \frac{1}{1 + F_0 F}.
\]
(5.1.7)

As a check a variation of this result is (5.1.6). The ordinary field strength in terms of the non-commutative field strength is written as
\[
F = \hat{F} \frac{1}{1 - \theta \hat{F}}.
\]

In \( \alpha' \to 0 \) limit \( \theta = B^{-1} \). When \( F + B = 0 \), it is noticed from (5.1.7) that we can not use the non-commutative description of the gauge theory. The criterion of whether we can use the non-commutative gauge theory or not is gauge invariant.

The effective theory of a D-brane for slowly varying fields is the Dirac-Born-Infeld Lagrangian \[66\];
\[
\mathcal{L}(F) = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi \alpha'(F + B))}.
\]

If we use the point splitting regularization, a field strength is \( \hat{F} \) and products of fields are the star products. Then the action \[70\] is
\[
\hat{\mathcal{L}}(\hat{F}) = \frac{1}{G_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(G + 2\pi \alpha' \hat{F})}.\]
From (5.1.7) \( F = 0 \) corresponds to \( \hat{F} = 0 \). Then the constant part of the two Lagrangian have to be equivalent. Then we have

\[
G_s = g_s \left( \frac{\det G}{\det(g + 2\pi\alpha'B)} \right)^{\frac{1}{2}}
\]

which becomes in the zero slope limit

\[
G_s = g_s \det(2\pi\alpha'Bg)^{-\frac{1}{2}}.
\] (5.1.8)

In the above we have seen that two different regularization scheme give commutative and non-commutative gauge theories. The point splitting regularization of the two dimensional theory on the string worldsheet with boundary gives the non-commutative gauge theory with field strength \( \hat{F} \) and the Moyal product. The Lagrangian is a function of \( \hat{F} \) and NS B field appear implicitly in the Moyal product and the open string metric \( G_{ij} \). (In the zero slope limit the non-commutativity parameter \( \theta \) is equivalent to \( B^{-1} \).) On the other hand if we use the Pauli-Villars regularization, in the DBI action the gauge field and NS B field appear in the form \( F + B \) and the product is ordinary one. It is natural to guess there are other regularizations which correspond to other non-commutative descriptions of the gauge theory. Seiberg and Witten\(^{[12]}\) suggested that the B dependence in the DBI action appear in \( \theta \) and some field \( \Phi \) which is in the form of \( \hat{F} + \Phi \). We see their argument below. However they did not give a proof for this. In order to prove this conjecture we have to look for another regularization which gives a non-commutative theory. Further investigation is needed\(^2\).

From the definition of the metric \( G \) and non-commutativity \( \theta \), they guessed relations between \( \Phi \) and other variables are

\[
\frac{1}{G + 2\pi\alpha'\Phi} = -\frac{\theta}{2\pi\alpha'} + \frac{1}{g + 2\pi\alpha'B},
\]

\[
G_s = g_s \left( \frac{\det(G + 2\pi\alpha'\Phi)}{\det(g + 2\pi\alpha'B)} \right)^{\frac{1}{2}}.
\] (5.1.9)

(5.1.10)

The second equation is from the suggestion for the \( \Phi \) dependence in the DBI action. We have some consistency checks. When \( \theta = 0 \) (5.1.9) and (5.1.10) are consistent with \( G = g, G_s = g_s \) and \( \Phi = B \). Next when \( \Phi = 0 \) (5.1.9) and (5.1.10) are reduced to the representation of the theory which has been observed in the last chapter which is derived from the point splitting regularization. We consider \( \alpha' \to 0 \) limit where

\[
g = \epsilon g^{(0)} + O(\epsilon^2)
\]

\[
B = B^{(0)} + \epsilon B^{(1)} + O(\epsilon^2)
\]

\[
\alpha' = \epsilon^{\frac{1}{2}}
\]

so that \( G \) and \( \Phi \) may be 0 th order in \( \epsilon \). In the limit from (5.1.9)

\[
\theta = \frac{1}{B^{(0)}}
\] (5.1.11)

\(^2\)The \( \Phi \) dependence is discussed in \[71, 72\] which are informed by O. Andreev
\[
G = - \frac{(2\pi\alpha')^2}{\epsilon} B^{(0)} \frac{1}{g^{(0)}} B^{(0)}
\]

(5.1.12)

\[
\Phi = -B^{(0)} + \frac{(2\pi\alpha')^2}{\epsilon} B^{(0)} \frac{1}{g^{(0)}} B^{(1)} \frac{1}{g^{(0)}} B^{(0)}.
\]

(5.1.13)

\[
G_s = g_s \text{det} \left( \frac{2\pi\alpha'}{\epsilon} B^{(0)} \frac{1}{g^{(0)}} \right).
\]

(5.1.14)

(5.1.11), (5.1.12) and (5.1.14) are coincide with the zero slope limit of the results of the point splitting regularization.

As a final check we see that there is no inconsistency for the combination \( \hat{F} + \Phi \) in the DBI Lagrangian. In the zero slope limit the Lagrangian is \( \text{tr}(\hat{F} + \Phi)^2 \). It is easily understood that the existence of the \( \Phi \) does not affect to a physical observable. The Lagrangian is

\[
\text{tr}(\hat{F} + \Phi)^2 = \text{tr}(\hat{F}^2 + 2\hat{F}\Phi + \Phi^2).
\]

The second and third terms are a total derivative and a constant term respectively which do not contribute to S-matrix. Then we can neglect the \( \Phi \) dependent terms.

So far we have seen regularization dependence of the theory with the NS B field fixed. Here we would like to vary the B field with the point splitting regularization used. In the commutative description there is a symmetry with respect to the B field, namely

\[
B_{ij} \rightarrow B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i.
\]

In the non-commutative description is there a counterpart of this symmetry?

For simplicity we deal with \( \theta \) whose rank is equivalent to the dimensionality of that of the D-brane so that \( \theta \) may be invertible. The covariant derivative of the gauge theory is defined by

\[
D_i = \partial_i - iA_i
\]

on the non-commutative space \([x^i, x^j] = \theta^{ij}\). We rewrite it as

\[
D_i = \partial_i' - iC_i.
\]

(5.1.15)

where

\[
\partial_i' = \partial_i + iB_{ij}x^j \\
C_i = A_i + B_{ij}x^j.
\]

\( \partial' \) commute with \( x^i; [\partial', x_j] = 0 \). For the sake of the unusual definition (5.1.13), the field strength takes the simple form

\[
\hat{F}_{ij} = B_{ij} - i[C_i, C_j].
\]

Introduce vierbeins \( e_i^a \) and \( E_i^a \) with respect to the metric \( g \) and \( G \) respectively \( (g_{ij} = \sum_a e_i^a e_j^a, \ G_{ij} = \sum_a E_i^a E_j^a) \). \( a \) is an index of the local Lorentz frame. We vary \( \theta \) with \( g_{ij} \)
(in other words $\epsilon^a_i$) and $C_a$ fixed ($C_a$ is defined by $C_i = E^a_i C_a$). Using the form of the metric $G$ in $\alpha \to 0$ limit

$$G^{il} = -(2\pi \alpha')^{-2} \theta_{ij} g_{jk} \theta^{kl}$$

gives that the vierbein for the metric $G$ is $E^a_i = 2\pi \alpha' B_{ij} e^j_a$. We would like to show that the Lagrangian

$$G^{ik} G^{jl} \text{tr}(\hat{F}_{ij} - \theta^{-1}_{ij}) \ast (\hat{F}_{kl} - \theta^{-1}_{kl})$$

(5.1.16)

is background independent. The problem is reduced to the work to prove

$$Q^{il} = -i \theta^{ij} [C_j, C_k] \theta^{kl}$$

is background independent which is from $\theta^{ij} C_i$ is independent; $\theta^{ij} C_i = -(2\pi \alpha')C_a e^j_a$. Hence the Lagrangian is background independent.

In the commutative description only $\Lambda$ gauge invariant function is $B + F$. This is coincide with the above result. In the case of U(1) gauge theory with a constant field strength, using (5.1.7) gives $Q$ in terms of the ordinary field strength;

$$Q = -\frac{1}{B + F}$$

where we have used;

$$\hat{F} - B = -B \frac{1}{B + F} B.$$

In the both descriptions of the DBI theory $Q$ is a background independent value. We can define the transformation rule of a coupling constant so that a measure of the action will be background independent.

### 5.2 Dirac-Born-Infeld Action

We have three different descriptions of the gauge theory. In this section we would like to see that these descriptions are equivalent to each other. The low energy effective theory of a D-brane in a slowly varying field approximation is the Dirac-Born-Infeld Lagrangian. As we have seen in the last section we can describe a D-brane by the Lagrangian

$$\hat{\mathcal{L}} = \frac{1}{G_s(2\pi)^p(\alpha')^{p+1}} \sqrt{\det(G + 2\pi \alpha' (\hat{F} + \Phi))}$$

which is derived by using the point splitting regularization. In the Lagrangian products of functions are the Moyal products. Here what we would like to do is to prove the non-commutative DBI Lagrangian is equivalent to the ordinary DBI action up to total derivative and derivative terms of field strength of a gauge field. The DBI Lagrangian is valid for the case in which derivatives of the field strength is negligible. Then we replace the Moyal product to ordinary one. For simplicity we set $2\pi \alpha' = 1$. The process of the
proof is only to vary $\theta$ in the non-commutative DBI Lagrangian with $g, B$ and $g_s$ fixed. We will see its result is total derivative terms and $O(\theta^2)$. We take a variation of \(5.1.9\) with $g, B$ and $g_s$ fixed:

$$\delta G + \delta \Phi = (G + \Phi)\delta \theta(G + \Phi).$$

We take a variation of \(5.1.10\):

$$\delta G_s = \frac{G_s}{2} \text{tr}(G + \Phi)\delta \theta = \frac{G_s}{2} \text{tr}(\Phi \delta \theta),$$

where the fact that $G$ is a symmetric matrix, $\Phi$ and $\theta$ are anti-symmetric matrices is used. In the present situation \(5.1.5\) becomes

$$\delta \hat{F}_{ij}(\theta) = \delta \theta^{kl} \left( \hat{F}_{ik} \hat{F}_{jl} - \frac{1}{2} \hat{A}_k (\partial_l \hat{F}_{ij} + \hat{D}_l \hat{F}_{ij}) \right) + O(\hat{F}).$$

We take a variation of the Lagrangian with respect to $\theta$

$$\delta \left[ \frac{1}{G_s} \sqrt{\det(G + \hat{F} + \Phi)} \right] = \frac{1}{G_s} \sqrt{\det(G + \hat{F} + \Phi)} \left[ -\delta G_s + \frac{1}{2} \text{tr} \left( \frac{1}{G + \hat{F} + \Phi} (\delta G + \delta \hat{F} + \delta \Phi) \right) \right]
$$

$$= \frac{1}{2 G_s} \sqrt{\det(G + \hat{F} + \Phi)} \left[ -\text{tr} \delta \theta(G + \Phi) + \text{tr} \frac{1}{G + \hat{F} + \Phi} (G + \Phi) \delta \theta(G + \Phi) \right]
$$

$$+ \left( \frac{1}{G + \hat{F} + \Phi} \right) \delta \theta^{kl} \left( \hat{F}_{ik} \hat{F}_{jl} - \frac{1}{2} \hat{A}_k (\partial_l \hat{F}_{ij} + \hat{D}_l \hat{F}_{ij}) \right) + O(\hat{F}). \quad (5.2.1)$$

Here we need to write some calculations explicitly.

$$\text{tr} \frac{1}{G + \hat{F} + \Phi} (G + \Phi) \delta \theta(G + \Phi) = \text{tr} \delta \theta(G + \Phi) - \text{tr} \frac{1}{G + \hat{F} + \Phi} \hat{F} \delta \theta(G + \Phi)$$

$$\partial_i \text{det}(G + \hat{F} + \Phi) \frac{1}{2} = \frac{1}{2} \text{det}(G + \hat{F} + \Phi) \frac{1}{G + \hat{F} + \Phi} \partial_i \hat{F}_{ij}$$

$$\hat{D}_l \text{det}(G + \hat{F} + \Phi) \frac{1}{2} = \frac{1}{2} \text{det}(G + \hat{F} + \Phi) \frac{1}{G + \hat{F} + \Phi} \hat{D}_l \hat{F}_{ij}.$$

The last term in \(5.2.1\) is rewritten as

$$\frac{1}{2 G_s} \sqrt{\det(G + \hat{F} + \Phi)} \left( \frac{1}{G + \hat{F} + \Phi} \right) \delta \theta^{kl} \left( \hat{F}_{ik} \hat{F}_{jl} - \frac{1}{2} \hat{A}_k (\partial_l \hat{F}_{ij} + \hat{D}_l \hat{F}_{ij}) \right)
$$

$$= \frac{1}{2 G_s} \delta \theta^{kl} \hat{F}_{ik} \sqrt{\det(G + \hat{F} + \Phi)} + (\text{total derivative})$$
Hence (5.2.1) becomes

\[
\frac{1}{2 G_s} \det(G + \hat{F} + \Phi) \hat{\delta} \left[ -\text{tr} \frac{1}{G + \hat{F} + \Phi} \hat{F} \delta \theta (G + \Phi) - \text{tr} \frac{1}{G + \hat{F} + \Phi} \hat{F} \delta \theta \hat{F} + \text{tr} \delta \theta \hat{F} \right]
\]

\[+ O(\partial \hat{F}) + \text{total derivative}\]

\[= O(\partial \hat{F}) + \text{total derivative}.\]

Then we have the conclusion that two action whose non-commutativity parameters are different by \(\delta \theta\) (\(\delta \theta\) is infinitesimal) will give a same physical S-matrix. Furthermore infinite chain of the variation would give an equivalence of DBI Lagrangians with non-commutativity parameters whose difference is finite. Hence the effective theory of a D-brane has infinitely many equivalent descriptions. It will be important to see whether properties of D-branes (T-duality etc.), which are satisfied in the ordinary commutative description, are depend on descriptions or not. This is a future problem.

In constructing the effective theory of a D-brane derivative terms of fields were omitted\[64]\.

We need to find the derivative corrections and a systematic way to construct these terms. It was pointed out by Okawa and Terashima\[73, 74]\ that derivative corrections to the DBI Lagrangian can be constructed by the equivalence of ordinary gauge theory and non-commutative gauge theory.
Chapter 6

Canonical Quantization of an Open String with NS B Field

6.1 Dirac Formalism

In the canonical formalism a canonical momentum for a given Lagrangian \( L(q, \dot{q}) \) is defined by

\[
p_i = \frac{\partial L}{\partial \dot{q}^i}.
\]

The Hessian matrix is given by

\[
W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}.
\]

If \( \det W_{ij} = 0 \), the Lagrangian is called singular. For a singular Lagrangian we cannot write the \( \dot{q}^i \) in terms of the canonical momentum which is need if we construct a Hamiltonian from the Lagrangian. For example, in Yang-Mills gauge theory, the time component of a momentum \( p^0 \) is zero. Hence this system is singular. Dirac developed the method how to deal with the singular system \([75, 76]\). In this section we will review this in a formal way.

In the Lagrangian and Hamiltonian formalism, independent variables are \((q, \dot{q})\) and \((q, p)\) respectively. Because of the singularity, the Jacobian \( \frac{\partial p_i}{\partial \dot{q}^j} \) of the Legendre transformation is zero. This means that \( p_i \) are not independent of \( q^j \) and \( \dot{q}^i \) which are related by primary constraints

\[
\Phi_a^{(0)}(q, p) = 0.
\]

The total Hamiltonian is

\[
H_T = H + \lambda_a \Phi_a
\]

where \( \lambda_a \) are Lagrange multipliers. The equation of motion for a variable \( g(q, p) \) is

\[
\dot{g}(q, p) = \{g(q, p), H_T\}_p.
\]
The primary constraints should be invariant under the time evolution;

\[ \dot{\Phi}_{\alpha}(q,p) = \{\Phi_{\alpha}^{(0)}, H_T\}_p = \{\Phi_{\alpha}^{(0)}, H\}_p + \lambda_b \{\Phi_{\alpha}^{(0)}, \Phi_{\beta}^{(0)}\} = 0. \]  

(6.1.1)

In this step there are some possibilities.

- (6.1.1) holds identically
- \(\{\Phi_{\alpha}^{(0)}, \Phi_{\beta}^{(0)}\}\) vanishes and this gives new constraints
  \[ \Phi_{\alpha}^{(1)} \equiv \{\Phi_{\alpha}^{(0)}, H\}_p. \]
- \(\{\Phi_{\alpha}^{(0)}, \Phi_{\beta}^{(0)}\}\) does not vanish and we obtain \(\lambda_a\) for which (6.1.1) is satisfied.

Next similarly we should do the same procedure for \(\Phi^{(1)}\). And repeat this procedure until it does not give new constraints. The set of constraints \(\Phi^{(1)}, \Phi^{(2)}, \ldots\) are called secondary constraints. In quantum field theory, there is the possibility that this process does not have end, in other words, there are infinite constraints. However for almost cases, the number of secondary constraints is finite. In the next section we will consider the system of an open string coupled to the NS B field. We will regard boundary conditions of the open string as primary constraints. The constraints will give infinitely many secondary constraints.

There is another classification of constraints. If Poisson brackets of a constraint with all other constraints (both primary and secondary) vanish, the constraint is called first class. Other constraints are second class.

In a non-singular system the process of the quantization is equivalent to the replacement of the Poisson bracket with the commutator

\[ i\{q,p\}_p \rightarrow [q,p]. \]

Dirac suggested that when all constraints in theory is second class, the quantization is given by the replacement

\[ i\{q,p\}_D \rightarrow [q,p] \]

where

\[ \{A, B\}_D \equiv \{A, B\}_p - \{A, \Phi_M\}(C^{-1})^{MN}\{\Phi_N, B\}_p \]

is Dirac bracket \((C^{MN} \equiv \{\Phi^M, \Phi^N\}_p)\).

If there are also first class constraints the matrix \(C^{MN}\) does not have an inverse matrix. The first class constrains correspond to generators of transformations. For instance gauge theory includes first class constraints. In this case we have to fix the symmetries which correspond to the first class constraints. We choose the gauge fixing condition so that all of the constraints will become second class.
6.2 Dirac Quantization of an Open String

The action of an open string with the gauge and NS B fields is given by

\[ S = -\frac{1}{4\pi\alpha'} \int d\sigma^2 \left[ \partial_\alpha X^\alpha \partial_\beta X^\beta + \mathcal{F}_{\alpha\beta} \epsilon^{ab} \partial_\alpha X^\alpha \partial_\beta X^\beta \right] \]

where \( \mathcal{F} = F - B \) and Minkowski signature is used. We fix the \( \Lambda \) gauge symmetry as \( \mathcal{F} = -B \). The boundary condition is given by

\[ \partial_\sigma X_\alpha + B_{\alpha\beta} \partial_\tau X^\beta = 0 \quad \text{at } \sigma = 0, \pi \]

and canonical momentum is

\[ P^\alpha = \frac{1}{2\pi\alpha'} (\partial_\tau X^\alpha + B^{\alpha\beta} \partial_\sigma X^\beta). \]

This action is seen not to have primary constraints. Then as usual we would like to give equal \( \tau \) canonical commutators as

\[
\begin{align*}
[X^\alpha(\sigma), P^\beta(\sigma')] &= i\delta^{\alpha\beta} \delta(\sigma - \sigma') \\
[X^\alpha(\sigma), X^\beta(\sigma')] &= 0 \quad \text{(6.2.1)} \\
[P^\alpha(\sigma), P^\beta(\sigma')] &= 0.
\end{align*}
\]

However this does not coincide with the boundary condition. We can easily understand this. In terms of the canonical momentum the boundary conditions are

\[ 0 = M_{\alpha\beta} \partial_\sigma X^\beta + 2\pi\alpha' B_{\alpha\beta} P^\beta \]

where \( M \equiv \eta - B^2 \). From this we have

\[
\begin{align*}
-2\pi\alpha' B_{\alpha\beta} \left[ P^\beta, P^\gamma \right] &= M_{\alpha\beta} \partial_\sigma \left[ X^\beta, P^\gamma \right] \\
-2\pi\alpha' B_{\alpha\beta} \left[ P^\beta, X^\gamma \right] &= M_{\alpha\beta} \partial_\sigma \left[ X^\beta, X^\gamma \right].
\end{align*}
\]

If (6.2.1) is satisfied the commutator \( [X^\alpha(\sigma), P^\beta(\sigma')] \) should vanish. Hence the ordinary commutators are not true in this system. Then we need a way to determine the commutators.

In the case with no background fields, usually we use the symplectic form

\[ \Omega = \int d\sigma dP_\alpha dX^\alpha. \]

Using the mode expansion (2.1.1), we can find the symplectic form for the modes which is \( \tau \) independent. On the other hand if there is the NS B field the symplectic form for modes is \( \tau \) dependent. Then it is suggested in [38] that instead of \( \Omega \), we should use its time average

\[ < \Omega > = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \Omega d\tau \]
as the symplectic form which is \(\tau\) independent. This gives the commutator which is consistent with the boundary condition.

There is another method which will be reviewed in this section. We regard the boundary conditions as primary constraints \([41, 42, 78, 79, 80, 81]\). It will be found that the constraints are second class. The Dirac bracket of secondary constraints with a canonical variable identically vanish. Then the commutator which is constructed by this method is manifestly consistent with the boundary conditions.

We introduce primary constraints:

\[
\Phi^\alpha(0) = \Phi^\alpha(\pi) = 0
\]

where

\[
\Phi^\alpha(\sigma) \equiv 2\pi \alpha^\prime B^\alpha_\beta P^\beta + \partial_\sigma X^\beta M^\alpha_\beta \quad M^\alpha_\beta = (\eta - B^2)^\alpha_\beta.
\]

The procedure which we have seen in the previous section gives secondary constraints

\[
\phi^{(1an)} \equiv \partial^{2n}_\sigma \Phi^\alpha(\sigma) = 0, \quad \phi^{(2an)} \equiv \partial^{2n+1}_\sigma P^\alpha(\sigma) = 0 \quad (n = 0, 1, \cdots)
\]

at \(\sigma = 0, \pi\). We have infinite constraints. Here we would like to see that these constraints are second class. We set as

\[
C^{(i\alpha n)(j\beta m)} = \{\phi^{(i\alpha n)}, \phi^{(j\beta m)}\}_p
\]

where \(i = 1, 2\). Components of the \(C^{(i\alpha 0)(j\beta 0)}\) are

\[
\begin{align*}
\{\Phi^\alpha(\sigma), \Phi^\beta(\sigma')\}_p &= -2\pi \alpha^\prime (BM)^{\alpha\beta}[\partial_\sigma \delta(\sigma - \sigma') + \partial_{\sigma'} \delta(\sigma - \sigma')] \\
\{\Phi^\alpha(\sigma), \partial_\sigma P^\beta(\sigma')\}_p &= M^{\alpha\beta} \partial_\sigma \partial_{\sigma'} \delta(\sigma - \sigma') \\
\{\partial_\sigma P^\alpha(\sigma), \partial_{\sigma'} P^\beta(\sigma')\}_p &= 0.
\end{align*}
\]

For \(n, m \neq 0\), we have

\[
C^{(i\alpha n)(j\beta m)} = \partial^{2n}_\sigma \partial^{2m}_{\sigma'} C^{(i\alpha 0)(j\beta 0)}.
\]

The constraints are defined only at \(\sigma = 0, \pi\). Then we treat indices \(\sigma, \sigma', \cdots\) as discrete variables below. Its inverse matrix is formally written as

\[
(C^{-1})^{(i\alpha n)(j\beta m)}(\sigma'', \sigma''') = \begin{pmatrix} 0 & -(M^{-1})_{\alpha\beta} R_{nm}(\sigma'', \sigma''') \\ (M^{-1})_{\alpha\beta} R_{mn}(\sigma'', \sigma''') & 2\pi \alpha^\prime (BM^{-1})_{\alpha\beta} S_{nm}(\sigma'', \sigma''') \end{pmatrix}
\]

where matrices \(R\) and \(S\) satisfy

\[
\sum_{m\sigma''} \partial^{2n+1}_\sigma \partial^{2m+1}_{\sigma''} \delta(\sigma - \sigma'') R_{mp}(\sigma'', \sigma''') = \delta^n_\sigma \delta_{\sigma'''}
\]

\[(6.2.2)\]

\[
\sum_{m\sigma''} \partial^{2n}_\sigma \partial^{2m}_{\sigma''} [\partial_\delta \delta(\sigma - \sigma'') + \partial_{\sigma''} \delta(\sigma - \sigma'')] R_{mp}(\sigma'', \sigma''') = \sum_{m\sigma''} \partial^{2n+1}_\sigma \partial^{2m+1}_{\sigma''} \delta(\sigma - \sigma'') S_{mp}(\sigma'', \sigma''')
\]

\[(6.2.3)\]

---

1However in reality the symplectic form \(\Omega\) does not have \(\tau\) dependence even if there is the NS B field background. The unnecessary procedure in \([38]\) has been corrected in \([39]\). This is informed by C.-S. Chu.
We would like to find Dirac brackets. It is trivial that
\[ \{ P^\alpha(\sigma), P^\beta(\sigma') \}_D = 0. \]

Next we would like to see \( \{ X^\alpha(\sigma), X^\beta(\sigma') \}_D \). From the definition of the Dirac bracket,
\[
\{ X^\alpha(\sigma), X^\beta(\sigma') \}_D = -2\pi\alpha' \sum_{nm\sigma'\sigma''} [B^\gamma(\sigma'' - \sigma'') R_{nm}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma'' - \sigma')
- \delta^\gamma \partial^2_{\sigma'''} \delta(\sigma - \sigma'') R_{nm}(\sigma''', \sigma''') \partial^2_{\sigma'''} \delta(\sigma''' - \sigma') B^\delta
+ \delta^\gamma \partial^2_{\sigma'''} \delta(\sigma - \sigma'') (BM^{-1})_\gamma S_{nm}(\sigma''', \sigma''') \partial^2_{\sigma'''} \delta(\sigma''' - \sigma') \delta^\gamma] \quad (6.2.4)
\]

If \( \sigma, \sigma' \neq 0, \pi, \{ X^\alpha(\sigma), X^\beta(\sigma') \}_D = 0 \). We would like to see the value of the Dirac bracket for the case \( \sigma, \sigma' = 0 \). (6.2.3) for \( n = 0 \) and multiplying \( \sum_p \sum_{\sigma''} \partial^2_{\sigma''} \delta(\sigma' - \sigma''') \) and integrate over \( \sigma \) we have
\[
\int d\sigma \sum_{mp\sigma''\sigma'''} \partial^2_{\sigma''} [\partial_\sigma \delta(\sigma - \sigma'') + \partial_{\sigma'} \delta(\sigma - \sigma'')] R_{mp}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma'' - \sigma''')
= \int d\sigma \sum_{mp\sigma''\sigma'''} \partial_\sigma \partial^2_{\sigma''} \delta(\sigma - \sigma'') S_{mp}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma'' - \sigma''').
\]

Using
\[
\int d\sigma \partial_{\sigma''} \delta(\sigma'' - \sigma) = \partial_{\sigma''} \int d\sigma \delta(\sigma'' - \sigma) = 0
\]
gives
\[
\sum_{p\sigma''\sigma'''} \partial^2_{\sigma''} \delta(\sigma - \sigma'') R_{mp}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma'' - \sigma''') \bigg|_{\sigma = 0}^{\sigma = \pi}
= \sum_{p\sigma''\sigma'''} \partial^2_{\sigma''} \delta(\sigma - \sigma'') S_{mp}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma'' - \sigma''') \bigg|_{\sigma = 0}^{\sigma = \pi}
\]

By virtue of the formula (6.2.5), the Dirac bracket is simplified in the following combination of the brackets;
\[
\{ X^\alpha(\pi) - X^\alpha(0), X^\beta(\sigma') \}_D
= \{ X^\alpha(\pi), X^\beta(\sigma') \}_D - \{ X^\alpha(0), X^\beta(\sigma') \}_D
= -2\pi\alpha' (BM^{-1})_\alpha^\beta \sum_{\sigma''\sigma'''} \partial^2_{\sigma''} \delta(\sigma - \sigma'') R_{nm}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma''' - \sigma') \bigg|_{\sigma = 0}^{\sigma = \pi}
\]

(6.2.6)

where (6.2.4) has been used. Using (6.2.2) for \( n = 0 \) multiplying it with \( \sum_p \sum_{\sigma''} \partial^2_{\sigma''} \delta(\sigma' - \sigma''') \) and integrate over \( \sigma \) we have
\[
\int d\sigma \sum_{mp\sigma''\sigma'''} \partial_\sigma \partial^2_{\sigma''} \delta(\sigma - \sigma'') R_{mp}(\sigma'', \sigma''') \partial^2_{\sigma'''} \delta(\sigma'' - \sigma''')
\]
\[
\int d\sigma \sum_{p_{\alpha\beta}} \delta^0_\rho \delta_{\sigma,\sigma''} \partial^2_{\sigma''} \delta(\sigma' - \sigma''') \\
= \sum_{\sigma''''} \delta(\sigma' - \sigma''') \\
= \delta(\sigma') + \delta(\sigma' - \pi).
\]

(6.2.7)

At a same time there are not both \(X(0)\) and \(X(\pi)\) in a constraint in this theory. Then we have

\[\{X^\alpha(0), X^\beta(\pi)\}_D = 0.\]

From (6.2.6) and (6.2.7) we have

\[\{X^\alpha(0), X^\beta(0)\}_D = 2\pi \alpha'(M^{-1}B)^{\alpha\beta} \delta(0)\]

\[\{X^\alpha(\pi), X^\beta(\pi)\}_D = -2\pi \alpha'(M^{-1}B)^{\alpha\beta} \delta(0).\]

In this method of calculation we can not determine the explicit form of \(\{X^\alpha(\sigma), P^\beta(\sigma')\}_D\) at boundaries. Of course \(\{X^\alpha(\sigma), P^\beta(\sigma')\}_D\) takes usual form at \(\sigma, \sigma' \neq 0, \pi\).

### 6.3 Generalization to the NSR Superstring

So far we have seen that the effect of the NS B field to the commutation relations of string coordinates. In this section we extend this to the fermionic variables in the NSR formalism [41, 79]. In the conformal gauge we have to add the terms;

\[
S = \frac{i}{2\pi} \int d^2\sigma \bar{\psi}^\mu \rho^a \partial_a \psi^\mu - \frac{i}{4\pi \alpha'} \int d^2\sigma B_{\alpha\beta} \bar{\psi}^\sigma \epsilon^{ab} \rho_a \partial_b \psi^\beta
\]

(6.3.1)

where

\[
\psi = \begin{pmatrix}
\psi \\
\bar{\psi}
\end{pmatrix}
\]

is a majorana spinor and

\[
\rho^0 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \quad \rho^1 = \begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix}
\]

are 2 dimensional gamma matrices. Here we would like to find the boundary conditions for the fermionic variables in a manifestly supersymmetric way [82]. Using a superfield \(\Phi\) the NSR superstring action is given by

\[
S = \frac{1}{2\pi} \int dzd\bar{z}d\theta d\bar{\theta}(g_{\mu\nu} + 2\pi \alpha' B_{\mu\nu}) \bar{D} \Phi^\mu(z, \bar{z}) D \Phi_\mu(z, \bar{z})
\]

(6.3.2)

where \(z = (z, \theta)\) and \(\bar{z} = (\bar{z}, \bar{\theta})\) are coordinates of the worldsheet superspace and

\[
D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}
\]

\[
\bar{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \frac{\partial}{\partial \bar{z}}
\]
are the super covariant derivatives. The superfield is written in terms of the component fields as
\[ \Phi^\mu(z, \bar{z}) = \sqrt{\frac{2}{\alpha'}}X^\mu(z, \bar{z}) + i\theta\psi^\mu(z, \bar{z}) + i\bar{\theta}\bar{\psi}^\mu(z, \bar{z}) + i\theta\bar{\theta}F^\mu(z, \bar{z}) \]

We can eliminate the auxiliary field \( F^\mu \) by the equation of motion. Then the action is rewritten as
\[ S = \frac{1}{2\pi} \int dzd\bar{z}d\bar{\theta}d\theta(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})(\frac{2}{\alpha'}\bar{\partial}X^\mu \partial X^\nu - \bar{\partial}\psi^\mu \psi^\nu + \bar{\psi}^\mu \bar{\partial}\bar{\psi}^\nu) \]

which is same with (6.3.1). We set \( \tilde{g}_{\mu\nu} = g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu} \) and take a variation of the action (6.3.2);
\[ \delta S = \frac{1}{2\pi} \int dzd\bar{z}d\bar{\theta}d\theta \tilde{g}_{\mu\nu}(\bar{\theta}(\delta \Phi^\mu D\Phi^\nu) + \bar{D}\Phi^\mu \delta \Phi^\nu) = 0. \]

Along longitudinal directions of a D-brane
\[ g_{\alpha\beta}(\partial - \bar{\partial})X^\beta + 2\pi\alpha' B_{\alpha\beta}(\partial + \bar{\partial})X^\beta \big|_{\sigma=0,\pi} = 0, \]
for NS fermion
\[ g_{\alpha\beta}(\psi^\beta - \bar{\psi}^\beta) + 2\pi\alpha' B_{\alpha\beta}(\psi^\beta + \bar{\psi}^\beta) \big|_{\sigma=0,\pi} = 0, \]
and for R fermion
\[ g_{\alpha\beta}(\psi^\beta - \bar{\psi}^\beta) + 2\pi\alpha' B_{\alpha\beta}(\psi^\beta + \bar{\psi}^\beta) \big|_{\sigma=0} = 0, \]
\[ g_{\alpha\beta}(\psi^\beta + \bar{\psi}^\beta) + 2\pi\alpha' B_{\alpha\beta}(\psi^\beta - \bar{\psi}^\beta) \big|_{\sigma=\pi} = 0. \]

Although we can get the commutator for the fermionic variables by repeat the process in the previous section, instead we find the commutators by using the supersymmetry on the worldsheet explicitly. The symmetry is defined by
\[ \delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu, \]
\[ \delta_\epsilon \psi^\mu = -i\rho^\beta \partial_\alpha X^\mu \epsilon. \]

Since \( X \) and \( \psi \) are not mixed in the boundary conditions,
\[ \{\psi^\mu, X^\nu\}_D = 0 \]
is easily understood. The equation must be invariant under the supersymmetry transformation
\[ 0 = \delta \{\psi^\mu, X^\nu\}_D \]
\[ = \left( \{ -\bar{\epsilon}\partial_0 X^\mu + \bar{\epsilon}\partial_1 X^\mu, X^\nu\}_D \right) + \left( \{ \bar{\psi}^\mu(\sigma), i\bar{\epsilon}\psi^\nu - i\epsilon\bar{\psi}^\nu\}_D \right) + \left( \{ \psi^\mu(\sigma), i\bar{\epsilon}\psi^\nu - i\epsilon\bar{\psi}^\nu\}_D \right). \quad (6.3.3) \]
Because of the Dirichlet conditions the parameters are related by

\[ \epsilon = \lambda \tilde{\epsilon} \quad \lambda = \pm 1 \]

for which the Dirichlet conditions are preserved. We are aware the Dirac brackets of \( X \), then by using \( (6.3.3) \) we have the Dirac brackets of \( \psi \). For example for the NS fermion at boundaries the commutators are given by

\[
\{ \psi^\alpha, \psi^\beta \}_D = \pi \alpha' \eta^{\alpha \beta} \tilde{\delta}(\sigma - \sigma')
\]
\[
\{ \bar{\psi}^\alpha, \bar{\psi}^\beta \}_D = \pi \alpha' \eta^{\alpha \beta} \tilde{\delta}(\sigma - \sigma')
\]

where

\[ \tilde{\delta}(\sigma - \sigma') = \delta(\sigma - \sigma') - \sum_{\sigma'', \sigma'''} \partial^{2m+1}_{\sigma''} \delta(\sigma - \sigma'') R_{mk}(\sigma'', \sigma''') \partial^{2k+1}_{\sigma'''} \delta(\sigma''' - \sigma'). \]

For \( \sigma, \sigma' \neq 0, \pi \) these are reduced to the ordinary commutators. Although we can not obtain explicit form of the Dirac bracket (the \( R_{mk}(\sigma'', \sigma''') \) is an unknown function) these seems to be \( B \) independent. If this is true, the \( B \) independence of commutators of fermions is common in NSR and GS superstring theories \[83\].
In this paper we have seen relations between commutative and non-commutative space-time. In chapter 3 we have reviewed in the framework of the boundary state formalism that a Dp-brane with a constant field strength of gauge field is equivalent to \( \infty \) D\((p-2)\)-branes. The worldvolume theories of the Dp-brane and \( \infty \) D\((p-2)\)-branes are DBI theory on ordinary and non-commutative space, respectively. From the result, we have found that we can represent coordinates on the D-brane by commutative and also non-commutative one. In chapter 4 it has been reviewed that using point splitting regularization violates \( U(1) \) gauge symmetry, and make the theory non-commutative \( U(1) \) gauge invariant. In chapter 5 we could construct a transformation which connects an ordinary gauge field and a non-commutative gauge field which was first advocated by Seiberg and Witten. It is need to see whether the properties (T-duality etc.) which hold in commutative representation hold in non-commutative representation or not.

In chapter 6 we have found commutation relations which coincide with boundary conditions in the framework of the operator formalism. In the case that there is a constant background NS B field, we treat the boundary conditions (which is mixed type) as primary constraints, and we have non-commutative coordinates at endpoints of an open string. Because there are infinite secondary constraints, it is difficult to find commutators for a non-constant B field. We would like a method to deal with the case.

We obtained the commutators in the bosonic and NSR string theories. The case of Green-Schwarz string with supergravity background \([5,4]\) is studied in \([3]\).

Last of all, we would like to comment on a relation between results in these chapters. It is common in chapter 4 and 6 that the source of the non-commutativity of coordinates is the NS B field. The difference is the methods. The CFT and the operator formalism gives the same result. The connection between non-commutativities from dimension of D-brane and regularization is discussed in \([4]\) in the case of compactified space.
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