Non-Universal Fluctuations of the Empirical Measure for Isotropic Stationary Fields on $\mathbb{S}^2 \times \mathbb{R}$

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Abstract

In this paper, we consider isotropic and stationary real Gaussian random fields defined on $\mathbb{S}^2 \times \mathbb{R}$ and we investigate the asymptotic behavior, as $T \to +\infty$, of the empirical measure (excursion area) in $\mathbb{S}^2 \times [0, T]$ at any threshold, covering both cases when the field exhibits short and long memory, i.e. integrable and non-integrable temporal covariance. It turns out that the limiting distribution is not universal, depending both on the memory parameters and the threshold. In particular, in the long memory case a form of Berry’s cancellation phenomenon occurs at zero-level, inducing phase transitions for both variance rates and limiting laws.

- Keywords and Phrases: Sphere-cross-time random fields; Empirical measure; Berry’s cancellation; Central and non-Central Limit Theorems.
- AMS Classification: 60G60; 60F05, 60D05, 33C55.

1 Introduction

1.1 Background and motivations

In recent years, special interest has been devoted to the study of random fields $Z = \{ Z(x), \ x \in \mathbb{S}^2 \}$ defined on the two-dimensional unit sphere $\mathbb{S}^2$, finding applications in several areas such as medical imaging, atmospheric sciences, geophysics, solar physics and cosmology (see e.g. [10, 11, 20, 26]). In particular, considerable attention has been drawn by the investigation of geometric functionals of Gaussian excursion sets on manifolds (see e.g. [1, 2]). Indeed, aiming to study the geometry of a random field $Z$, it is natural to introduce the family of excursion sets

$$\{ x \in \mathbb{S}^2 : Z(x) \geq u \}$$

indexed by the threshold $u \in \mathbb{R}$; under Gaussianity and isotropy, the expected value of their Lipschitz-Killing curvatures (i.e. area, boundary length and Euler-Poincaré characteristic), is easily obtained as a special case of the celebrated Gaussian Kinematic Formula, see e.g. [1, Ch. 13]. However, what is more challenging is to investigate fluctuations around these expected values and for this purpose, asymptotic methods must be exploited, considering sequences of random fields. In particular, a number of recent papers has focussed on the asymptotic behavior of sequences of Gaussian Laplace eigenfunctions (random spherical harmonics), in the high-energy limit, i.e. as the eigenvalues diverge. Several results have been given concerning the asymptotic variance, the limiting distribution and the correlation for different values of the thresholding parameter $u \in \mathbb{R}$ of Lipschitz-Killing curvatures of their excursion sets, see e.g. [9, 8, 22, 23, 25, 36, 34] and the references therein; see also [7, 15, 21, 28, 30] for related results on the standard flat torus and on the Euclidean plane. Some of these results entail rather surprising issues, for instance the cancellation of the leading variance terms for specific threshold values and the possibility to express wide classes of functionals as simple polynomial integrals on $\mathbb{S}^2$ of the underlying fields, up to lower order terms.
The purpose of this paper is to begin the investigation of these same issues for a different class of fields, i.e., isotropic and stationary Gaussian fields on $S^2 \times \mathbb{R}$, which can be immediately interpreted as spherical random fields evolving over time (see e.g. [3, 6, 19] and the references therein). Although the present manuscript is mainly of theoretical nature, it is very easy to figure out several areas of applications where such random fields emerge most naturally, including the scientific research streams mentioned above. In the next subsection, we introduce our setting in more detail.

1.2 Sphere-cross-time random fields

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $S^2$ the two-dimensional unit sphere with the round metric. A space-time real-valued spherical random field

$$Z = \{Z(x, t), x \in S^2, t \in \mathbb{R}\} \quad (1)$$

is a collection, indexed by $S^2 \times \mathbb{R}$, of real random variables such that the map

$$Z : \Omega \times S^2 \times \mathbb{R} \to \mathbb{R}$$

is $\mathcal{F} \otimes \mathcal{B}(S^2 \times \mathbb{R})$-measurable, where $\mathcal{B}(S^2 \times \mathbb{R})$ stands for the Borel $\sigma$-field of $S^2 \times \mathbb{R}$. We say that $Z$ is Gaussian if for every $n \geq 1$, $x_1, \ldots, x_n \in S^2$, $t_1, \ldots t_n \in \mathbb{R}$, the random vector $(Z(x_1, t_1), \ldots, Z(x_n, t_n))$ is Gaussian.

**Condition 1.** The space-time real-valued spherical real random field $Z$ in (1) is Gaussian and

- zero-mean, i.e. $\mathbb{E}[Z(x, t)] = 0$ for every $x \in S^2, t \in \mathbb{R}$;
- stationary and isotropic, i.e.

  $$\mathbb{E}[Z(x, t)Z(y, s)] = \Gamma((x, y), t - s) \quad (2)$$

  for every $x, y \in S^2, t, s \in \mathbb{R}$, where $\Gamma : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ is a positive semidefinite function and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$;
- mean square continuous, i.e. $\Gamma$ is continuous.

The assumption of zero-mean is of course just a convenient normalization with no mathematical impact. The assumption of Gaussianity ensures that we need to make no distinction between so-called weak and strong stationarity, see e.g. [20, Definition 5.9], and it simplifies some of our proofs to follow; moreover, it is the common background with basically all the previous literature on the geometry of excursion sets (starting from [1]), likewise the assumption of mean square-continuity, see e.g. [3, 17] and the references therein.

From now on we assume that $Z$ in (1) satisfies Condition 1.

1.2.1 Karhunen-Loève expansions

It is well known (see e.g. [3, Theorem 3.3] or [19, Theorem 3]) that the following expansion for the covariance function $\Gamma$ in (2) holds:

$$\Gamma(\theta, \tau) = \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(\tau) P_\ell(\theta), \quad (\theta, \tau) \in [-1, 1] \times \mathbb{R}, \quad (3)$$

where $\{C_\ell, \ell \geq 0\}$ is a sequence of continuous positive semidefinite functions on $\mathbb{R}$, $P_\ell$ denotes the $\ell$-th Legendre polynomial [32, §4.7] and the series is uniformly convergent, which is equivalent to

$$\sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(0) < +\infty. \quad (4)$$
Obviously $C_\ell(0) \geq 0$ for every $\ell = 0, 1, 2, \ldots$. Let $T > 0$, it is straightforward (see e.g. [6]) to prove that the following Karhunen-Loève expansion for $Z$ holds in $L^2(\Omega \times S^2 \times [0, T])$:

$$Z(x, t) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m}(t)Y_{\ell,m}(x),$$  \hspace{1cm} (5)

where $\{Y_{\ell,m}, \ell \geq 0, m = -\ell, \ldots, \ell\}$ is the standard real orthonormal basis of spherical harmonics [20, §3.4] for $L^2(S^2)$, and

$$a_{\ell,m}(t) = \int_{S^2} Z(x, t)Y_{\ell,m}(x) \, dx,$$  \hspace{1cm} (6)

so that $\{a_{\ell,m}, \ell \geq 0, m = -\ell, \ldots, \ell\}$ is a family of independent, stationary, centered, Gaussian processes on $\mathbb{R}$ such that for every $t, s \in \mathbb{R}$

$$\mathbb{E}[a_{\ell,m}(t)a_{\ell,m}(s)] = C_\ell(t - s).$$

Now let

$$\tilde{N} := \{\ell \geq 0 : C_\ell(0) \neq 0\}.$$

From now on, we will consider only $\ell \in \tilde{N}$ unless otherwise specified. Let us define

$$Z_\ell(x, t) := \sum_{m=-\ell}^{\ell} a_{\ell,m}(t)Y_{\ell,m}(x), \quad (x, t) \in S^2 \times \mathbb{R}. \hspace{1cm} (7)$$

By construction, $\{Z_\ell, \ell \in \tilde{N}\}$ is a sequence of independent random fields and each $Z_\ell(\cdot, t)$ almost surely solves the Helmholtz equation

$$\Delta_{S^2} Z_\ell(\cdot, t) + \ell(\ell + 1)Z_\ell(\cdot, t) = 0,$$

where $\Delta_{S^2}$ denotes the spherical Laplacian. For notational convenience and without loss of generality we also assume that

$$\mathbb{E} \left[ Z^2(x, t) \right] = \sum_{\ell \in \tilde{N}} \sigma^2_\ell = 1, \quad \sigma^2_\ell := \mathbb{E}[Z^2_\ell(x, t)] = \frac{2\ell + 1}{4\pi} C_\ell(0). \hspace{1cm} (8)$$

1.2.2 Long and short range dependence

For $\ell \in \tilde{N}$, Bochner Theorem ensures that there exists a probability measure $\mu_\ell$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\frac{C_\ell(\tau)}{C_\ell(0)} = \int_{\mathbb{R}} e^{i\lambda \tau} \, d\mu_\ell(\lambda), \quad \tau \in \mathbb{R}. \hspace{1cm}$$

If $\mu_\ell$ is absolutely continuous with respect to the Lebesgue measure, then we may introduce the normalized spectral density as the function $f_\ell : \mathbb{R} \to \mathbb{R}^+$ such that

$$\frac{C_\ell(\tau)}{C_\ell(0)} = \int_{\mathbb{R}} e^{i\lambda \tau} f_\ell(\lambda) \, d\lambda, \quad \tau \in \mathbb{R}; \hspace{1cm}$$

we have of course

$$\int_{\mathbb{R}} f_\ell(\lambda) \, d\lambda = 1. \hspace{1cm}$$

If $C_\ell$ is integrable on $\mathbb{R}$, then clearly $f_\ell$ exists.

Let us now define the family of symmetric real-valued functions $\{g_\beta, \beta \in (0, 1]\}$ as follows:

$$g_\beta(\tau) = \begin{cases} (1 + |\tau|)^{-\beta} & \text{if } \beta \in (0, 1) \\ (1 + |\tau|)^{-\alpha} & \text{if } \beta = 1 \end{cases} \hspace{1cm} (10)$$

for some $\alpha \in [2, +\infty)$.  

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**Condition 2.** There exists a sequence \( \{\beta_\ell \in (0,1], \ell \in \tilde{N}\} \) such that

\[
C_\ell(\tau) = G_\ell(\tau) \cdot g_\beta_\ell(\tau), \quad \ell \in \tilde{N},
\]

where \( g_\beta_\ell \) is as in (10) and

\[
\sup_{\ell \in \tilde{N}} \left| \frac{G_\ell(\tau)}{C_\ell(0)} - 1 \right| = o(1), \quad \text{as } \tau \to +\infty.
\]

Moreover \( 0 \in \tilde{N} \) (that is, \( C_0(0) \neq 0 \)) and if \( \beta_0 = 1 \) then

\[
\int_{\mathbb{R}} C_0(\tau) d\tau > 0.
\]

From now on we assume that Condition 2 holds for the sequence \( \{C_\ell, \ell \in \tilde{N}\} \). Note that \( G_\ell(0) = C_\ell(0) \) for every \( \ell \in \tilde{N} \).

**Remark 1.1** (Abelian/Tauberian type results). Let \( \ell \in \tilde{N} \). The coefficient \( \beta_\ell \) in Condition 2 can be interpreted as a “memory” parameter; in particular, for \( \beta_\ell = 1 \) (resp. \( \beta_\ell \in (0,1) \)) the covariance function \( C_\ell \) is integrable on \( \mathbb{R} \) (resp. \( \int_{\mathbb{R}} |C_\ell(\tau)| d\tau = +\infty \)) and the corresponding process has so-called short (resp. long) memory behavior. Under some regularity assumptions, an equivalent characterization could be given in terms of the behavior at the origin of the spectral density \( f_\ell \) in (9): long-memory entailing divergence to infinity, whereas in the short-memory/integrable case \( f_\ell \) is immediately seen to be bounded in 0.

**Some conventions.** From now on, \( c \in (0, +\infty) \) will stand for a universal constant which may change from line to line. Let \( \{a_n, n \geq 0\}, \{b_n, n \geq 0\} \) be two sequences of positive numbers: we will write \( a_n \sim b_n \) if \( a_n/b_n \to 1 \) as \( n \to +\infty \), \( a_n \approx b_n \) whenever \( a_n/b_n \to c \), \( a_n = o(b_n) \) if \( a_n/b_n \to 0 \), and finally \( a_n = O(b_n) \) if eventually \( a_n/b_n \leq c \).

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### 2 Main results

Let \( u \in \mathbb{R} \). We consider the random process \( A_u \) on \( \mathbb{R} \) defined as

\[
A_u(t) := \text{area}\left( (Z(\cdot, t)^{-1}([u, \infty))) \right) = \int_{\mathbb{R}^2} 1_{Z(x,t) \geq u} \, dx, \quad t \in \mathbb{R}.
\]

(11)

In words, \( A_u(t) \) represents the empirical measure (i.e., the excursion area) of \( Z(\cdot, t) \) corresponding to the level \( u \); its expected value is immediately seen to be given by \( \mathbb{E}[A_u(t)] = 4\pi(1 - \Phi(u)) \), where

\[
\Phi(u) := \int_u^{+\infty} \phi(t) \, dt, \quad \phi(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2},
\]

\( \Phi \) (resp. \( \phi \)) denoting the tail distribution (resp. probability density) function of a standard Gaussian random variable.
We are interested in the fluctuations of \( A_u(t) \) around its expected value, and we hence introduce the following statistics: for \( T > 0 \)

\[
M_T(u) := \int_{[0,T]} \left( A_u(t) - \mathbb{E}[A_u(t)] \right) dt
\]

and its normalized version

\[
\tilde{M}_T(u) := \frac{M_T(u)}{\sqrt{\text{Var} M_T(u)}}.
\]

**Condition 3.** Let \( \{\beta_\ell, \ell \in \mathbb{N}\} \) be the sequence defined in Condition 2.

- The sequence \( \{\beta_\ell, \ell \in \mathbb{N}, \ell \geq 1\} \) admits minimum. Let us set
  
  \[
  \beta_{\ell^*} := \min\{\beta_\ell, \ell \in \mathbb{N}, \ell \geq 1\}, \quad \mathcal{I}^* := \{\ell \in \mathbb{N} : \beta_\ell = \beta_{\ell^*}\}. \quad (14)
  \]

- If \( \mathcal{I}^* \neq \mathbb{N} \), then the sequence \( \{\beta_\ell, \ell \in \mathbb{N} \setminus \mathcal{I}^*, \ell \geq 1\} \) admits minimum. Let us set
  
  \[
  \beta_{\ell^{**}} := \min\{\beta_\ell, \ell \in \mathbb{N} \setminus \mathcal{I}^*, \ell \geq 1\}. \quad (15)
  \]

Note that \( \beta_{\ell^*}, \beta_{\ell^{**}} \in (0,1] \) and for \( \ell \in \mathcal{I}^* \), obviously \( C_\ell(0) > 0 \). In words, \( \beta_{\ell^*} \) represents the smallest exponent corresponding to the largest memory, \( \mathcal{I}^* \) the set of multipoles where this minimum is achieved, and \( \beta_{\ell^{**}} \) the second smallest exponent \( \beta_\ell \) governing the time decay of the autocovariance \( C_\ell \) at some given multipole \( \ell \). Note that we are excluding the multipole \( \ell = 0 \) by the definition of \( \beta_{\ell^*} \) and \( \beta_{\ell^{**}} \) in (14) and (15), on the other hand \( \ell = 0 \) may belong to \( \mathcal{I}^* \). We assume that Condition 3 holds from now on.

As we shall see below, the asymptotic behavior of \( M_T(u) \) in (12), as \( T \to +\infty \), is governed by a subtle interplay between the value of the parameters \( \beta_{\ell^*}, \beta_{\ell^{**}} \) and the threshold level \( u \).

### 2.1 Long memory behavior

We start investigating the case of long-range dependence.

**Theorem 1.** If either \( u \neq 0 \) and \( \beta_0 < \min(2\beta_{\ell^*}, 1) \) or \( u = 0 \) and \( \beta_0 < \min(3\beta_{\ell^*}, 1) \), then

\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u))}{T^{2-\beta_0}} = \frac{2\phi^2(u)C_0(0)}{(1-\beta_0)(2-\beta_0)},
\]

and

\[
\tilde{M}_T(u) \xrightarrow{d} Z, \quad T \to +\infty
\]

where \( Z \sim \mathcal{N}(0,1) \) is a standard Gaussian random variable and \( \xrightarrow{d} \) denotes convergence in distribution.

Recall that by assumption \( C_0(0) > 0 \) (see Condition 2), hence the limiting variance constant in Theorem 1 is strictly positive.

**Remark 2.1.** In words, Theorem 1 holds when the zero-th order multipole component \( \{a_{00}(t), t \in \mathbb{R}\} \) is long memory \( (\beta_0 < 1) \) and all the other multipoles have asymptotically smaller variance (a consequence of either \( \beta_0 < 2\beta_{\ell^*}, \) when \( u \neq 0 \), or \( \beta_0 < 3\beta_{\ell^*}, \) when \( u = 0 \), as we will show below). It should be recalled that, by (6),

\[
a_{00}(t) = \int_{\mathbb{R}^2} Z(x,t)Y_{00}(x) \, dx = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}^2} Z(x,t) \, dx,
\]

that is, \( a_{00}(t) \) corresponds to the sample mean of the random field \( Z(\cdot,t) \) at the instant \( t \in \mathbb{R} \).
The limiting distribution in Theorem 1 is universal; this is not the case for the theorems to follow. We need first to recall one more definition.

**Definition 2.1.** The random variable $X_\beta$ has the standard\(^1\) Rosenblatt distribution (see e.g. [12]) with parameter $\beta \in (0, \frac{1}{2})$ if it can be written as

$$X_\beta = a(\beta) \int_{\mathbb{R}^2} e^{i(\lambda_1 + \lambda_2)} \frac{1}{i(\lambda_1 + \lambda_2)|\lambda_1\lambda_2|^{(1-\beta)/2}} W(d\lambda_1)W(d\lambda_2),$$

(16)

where $W$ is a white noise Gaussian measure on $\mathbb{R}$, the stochastic integral is defined in the Itô’s sense (excluding the diagonals\(^2\)), and

$$a(\beta) := \frac{\sigma(\beta)}{2\Gamma(\beta) \sin((1-\beta)\pi/2)},$$

(17)

with

$$\sigma(\beta) := \sqrt{\frac{1}{2}(1-2\beta)(1-\beta)}.$$  

We say the random vector $V$ satisfies a composite Rosenblatt distribution of degree $N \in \mathbb{N}$ with parameters $c_1, \ldots, c_N \in \mathbb{R}$, if

$$V = V_N(c_1, \ldots, c_N; \beta) = \sum_{k=1}^{N} c_k X_{k; \beta},$$

(18)

where $\{X_{k; \beta}\}_{k=1,\ldots,N}$ is a collection of i.i.d. standard Rosenblatt random variables of parameter $\beta$.

**Remark 2.2.** The characteristic function $\Xi_V$ of $V = V_N(c_1, \ldots, c_N; \beta)$ in (18) is given by (see e.g. [35])

$$\Xi_V(\theta) = \prod_{k=1}^{N} \xi_\beta(c_k \theta), \quad \xi_\beta(\theta) = \exp \left( \frac{1}{2} \sum_{j=2}^{+\infty} (2i\theta \sigma(\beta))^{j} \frac{a_j}{j} \right),$$

where $\xi_\beta$ is the characteristic function of $X_\beta$ in (16), the series is only convergent near the origin and

$$a_j := \int_{[0,1]^j} |x_1 - x_2|^{-\beta} |x_2 - x_3|^{-\beta} \cdots |x_{j-1} - x_j|^{-\beta} |x_j - x_1|^{-\beta} dx_1 dx_2 \cdots dx_j.$$  

Note that when $\beta \rightarrow 0^+$ then $\xi_\beta$ approaches the characteristic function of $\frac{1}{\sqrt{2}}(Z^2 - 1)$, where $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable. As $\beta \rightarrow \frac{1}{2}^-$ the limit is the characteristic function of $Z$.

**Theorem 2.** Assume that $u \neq 0$. If $2\beta_{t^*} < \min(\beta_0, 1)$ we have

$$\lim_{T \rightarrow \infty} \frac{\text{Var} (\mathcal{M}_T(u))}{T^{2-2\beta_{t^*}}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_{t^*})(1-\beta_{t^*})} \sum_{\ell \in I^*} (2\ell + 1)C_{\ell}(0)^2.$$

If $\beta_0 = 1$ and $2\beta_{t^*} = 1$ we have

$$\lim_{T \rightarrow \infty} \frac{\text{Var} (\mathcal{M}_T(u))}{T \log T} = u^2 \phi(u)^2 \sum_{\ell \in I^*} (2\ell + 1) C_{\ell}(0)^2.$$  

\(^1\)Indeed $\mathbb{E}[X_\beta] = 0$ and $\text{Var}(X_\beta) = 1$

\(^2\)(\mathbb{R}^2)’ stands for the set $\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \neq \lambda_2 \}$
Assume in addition that \#\(\mathcal{I}^*\) is finite, then as \(T \to +\infty\)
\[
\tilde{M}_T(\mathbf{u}) \overset{d}{\to} \sum_{\ell \in \mathcal{I}^*} \frac{C_\ell(0)}{\sqrt{v^*}} V_{2\ell+1}(1, \ldots, 1; \beta_{\ell^*}),
\]  
(19)
where \(\{V_{2\ell+1}(1, \ldots, 1; \beta_{\ell^*}), \ell \in \mathcal{I}^*\}\) is a family of independent composite Rosenblatt random variables as in (18) and
\[
v^* = a(\beta_{\ell^*})^2 \sum_{\ell \in \mathcal{I}^*} \frac{2(2\ell + 1)C_\ell(0)^2}{(1 - \beta_{\ell^*})(1 - 2\beta_{\ell^*})},
\]
where \(a(\beta_{\ell^*})\) is as in (17).

Recall that for \(\ell \in \mathcal{I}^*\) we have \(C_\ell(0) > 0\) (see (14)) hence the limiting variance constants in Theorem 2 are strictly positive. For the limiting random variable in (19), note that
\[
\sum_{\ell \in \mathcal{I}^*} \frac{C_\ell(0)}{\sqrt{v^*}} V_{2\ell+1}(1, \ldots, 1; \beta_{\ell^*}) \overset{d}{=} V_{N^*}(c_1, \ldots, c_{N^*}; \beta_{\ell^*}),
\]
where \(N^* := \sum_{\ell \in \mathcal{I}^*} (2\ell + 1)\) and
\[
(c_1, \ldots, c_{N^*}) = \frac{1}{\sqrt{v^*}} \left( \frac{C_{\ell_1}(0)}{(2\ell_1+1)\text{ times}}, \ldots, \frac{C_{\ell_k}(0)}{(2\ell_k+1)\text{ times}} \right),
\]
with \(\mathcal{I}^* = \{\ell_1, \ldots, \ell_k\}\).

**Remark 2.3** (Normal approximation of Rosenblatt distributions). The distribution of the random variable in (18) is, of course, non-Gaussian. However, in some circumstances it can be closely approximated by a Normal law. Indeed, consider for simplicity the case where the minimum for \(\beta, \ell \in \mathbb{N}\) is attained in a single multipole that we call \(\ell^*\), i.e., \(\mathcal{I}^* = \{\ell^*\}\). Then the limiting distribution in (19) is
\[
C_\ell(0) V_{2\ell^*+1}(1, \ldots, 1; \beta_{\ell^*}) = \frac{1}{\sqrt{2\ell^* + 1}} V_{2\ell^*+1}(1, \ldots, 1; \beta_{\ell^*})
\]
and by an immediate application of the classical Berry-Esseen Theorem (see e.g. [14]) one has that
\[
d_{Kol} \left( \frac{1}{\sqrt{2\ell^* + 1}} V_{2\ell^*+1}(1, \ldots, 1; \beta_{\ell^*}), Z \right) \leq c \cdot \frac{\mathbb{E} \left[ |X_{\beta_{\ell^*}}|^{3/2} \right]}{\sqrt{2\ell^* + 1}},
\]
where \(Z \sim \mathcal{N}(0,1)\) and \(d_{Kol}\) denotes Kolmogorov distance, see e.g. [27, §C.2]. The value of \(\ell^*\) for a given random field is fixed, so no Central Limit Theorem occurs; however for \(\ell^*\) large enough the resulting composite Rosenblatt distribution can become arbitrary close to a standard Gaussian variable.

**Theorem 3.** Assume that \(u = 0\) and that there exists an even\(^3\) multipole \(\ell \in \mathcal{I}^*\). If \(3\beta_{\ell^*} < \min(1, \beta_0)\), then
\[
\lim_{T \to \infty} \frac{\text{Var} \left( \mathcal{M}_T(\mathbf{u}) \right)}{T^{2 - 3\beta_{\ell^*}}} = \frac{2}{3!(1 - 3\beta_{\ell^*})(2 - 3\beta_{\ell^*})} \sum_{\ell_1, \ell_2, \ell_3 \in \mathcal{I}^*} G^{000}_{\ell_1, \ell_2, \ell_3} \prod_{i=1}^3 \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0),
\]
\(^3\)The motivation for this assumption is described just after the statement of Theorem 3.
where
\[
G_{\ell_1\ell_2\ell_3}^{000} := \int_{S^2} Y_{\ell_1,0}(x)Y_{\ell_2,0}(x)Y_{\ell_3,0}(x) \, dx
\]  
(20)
is a so-called Gaunt integral (cf. (22). If \( \beta_0 = 1 \) and \( 3\beta_\ell = 1 \) then
\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u))}{T \log T} = \frac{8\pi}{3!\sqrt{2\pi}} \sum_{\ell_1,\ell_2,\ell_3 \in \mathcal{I}^*} G_{\ell_1\ell_2\ell_3}^{000} \prod_{i=1}^3 \sqrt{2\ell_i + 1} \left( \frac{4\pi}{C_0(0)} \right).
\]
Moreover we have, as \( T \to +\infty \),
\[
\tilde{M}_T(u) = -\frac{\text{Var}(M_T(u))^{-1/2}}{3!\sqrt{2\pi}} \int_{S^2 \times [0,T]} H_3(Z(x,t)) \, dx dt + o_T(1),
\]
(21)
where \( H_3(t) \) is the third Hermite polynomial (cf. (24)) and \( o_T(1) \) is a family of random variables converging to zero in probability.

Recall that for \( \ell \in \mathcal{I}^* \) we have \( C_\ell(0) > 0 \) (see (14)) and note that \( G_{\ell_1\ell_2\ell_3}^{000} \) in (20) is nonnegative (see e.g. [20, Remark 3.45]); moreover if \( \ell \) is even, then \( G_{\ell\ell\ell}^{000} > 0 \) [20, Proposition 3.43, (3.61)]. Hence under the assumptions of Theorem 3 the limiting variance constants are strictly positive.

**Remark 2.4.** Using the same steps as in the classical papers [12, 33], it seems possible to prove that the right hand side of (21) (and hence \( \tilde{M}_T(u) \)), under the setting of Theorem 3, converges in distribution to a weighted sum of higher order Rosenblatt random variables (more precisely, of order 3). However, because for the probability laws of the latter very little is known, and even less so for their linear combinations, we refrain from rigorously investigating this issue here.

**Remark 2.5.** For simplicity of presentation, we are ruling out some boundary cases (such as \( \beta_0 = 2\beta_\ell \)), which could be dealt with the same techniques as we shall exploit below: the limit distributions would just correspond to linear combinations of the asymptotic random variables that we obtained above.

### 2.2 Short memory behavior

Theorems 1, 2 and 3 have all considered cases where some form of long-memory behavior is present on the temporal side, meaning that \( \beta_\ell < 1 \) for at least one instance of the multipole \( \ell \). In this section we investigate the case where on all scales no form of long-range dependence occurs.

We first need to introduce some more notation: for \( q \geq 3 \), let \( \ell_1,\ldots,\ell_q \geq 0 \) and \( m_i \in \{-\ell_i,\ldots,\ell_i\} \) for \( i = 1,\ldots,q \). The *generalized Gaunt integral* [20, p. 82] of parameters \( q,\ell_1,\ldots,\ell_q,m_1,\ldots,m_q \) is defined as (cf. (20))
\[
G_{\ell_1,\ldots,\ell_q}^{m_1,\ldots,m_q} := \int_{S^2} Y_{\ell_1,m_1}(x) \cdots Y_{\ell_q,m_q}(x) \, dx,
\]
(22)
where \( \{Y_{\ell,m}, \ell \geq 0, m = -\ell,\ldots,\ell\} \) still denotes the family of spherical harmonics introduced in §1.2.1.

**Theorem 4.** Assume \( \beta_0 = 1 \). If either \( u \neq 0 \) and \( 2\beta_\ell > 1 \) or \( u = 0 \) and \( 3\beta_\ell > 1 \), we have
\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u))}{T} = \sum_{q=1}^{+\infty} \frac{s_q^2}{q^2},
\]
where
\[
s_1^2 := \phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau,
\]
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\[ s_2^2 := \frac{u^2 \phi(2u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \int_{\mathbb{R}} C_\ell(\tau)^2 d\tau, \]
\[ s_q^2 := \frac{4\pi H_{q-1}(u)^2 \phi(2u)^2}{q!} \sum_{\ell_1, \ldots, \ell_q=0}^{\infty} G_{\ell_1, \ldots, \ell_q}^0 \int_{\mathbb{R}} \prod_{i=1}^q \sqrt{2\ell_i + 1} \frac{2\ell_i + 1}{4\pi} C_{\ell_i}(\tau) d\tau, \quad q \geq 3. \]

Moreover, as \( T \to +\infty \),
\[ \tilde{M}_T(u) \xrightarrow{d} Z, \]
\[ Z \sim \mathcal{N}(0, 1) \] being a standard Gaussian random variable.

Remark 2.6 (On Berry’s cancellation). It is interesting to note that a phase transition occurs at \( u = 0 \). Indeed, for \( 2\beta^* < 1 \) one observes a form of Berry’s cancellation phenomenon (see e.g. \([4, 36]\)), in the sense that the variance diverges with a smaller order rate. More precisely, there are two possibilities:

- for \( 3\beta^* < 1 \) (resp. \( 3\beta^* = 1 \)), the rate of the variance changes from \( T^{2-2\beta^*} \) to \( T^{2-3\beta^*} \), (resp. \( T \log T \)) and the limiting distribution is nonGaussian (Theorem 3 and Remark 2.4);
- for \( 3\beta^* > 1 \), the rate of the variance changes from \( T^{2-2\beta^*} \) to \( T \), and the limiting distribution is Gaussian (Theorem 4).

3 Outline of the paper

The results in Theorems 1, 2, 3 and 4 fully characterize the behavior of the empirical measure for sphere-cross-time random fields. The resulting scheme is, in the end, rather simple and can be summarized as follows.

- **Short Memory Behavior**: this setting corresponds to integrable covariance functions and occurs either when \( \beta_0 = 1 \) and \( 2\beta^* > 1 \), for \( u \neq 0 \), or when \( \beta_0 = 1 \) and \( 3\beta^* > 1 \), for \( u = 0 \). In such circumstances, the limiting distribution is always Gaussian and the variance, as \( T \to +\infty \), is asymptotic to \( T \), for all values of the threshold parameter \( u \). Hence, no form of Berry’s cancellation, as in Remark 2.6, can occur.
- **Long Memory Behavior**: this setting corresponds to non-integrable temporal autocovariance and in this case the picture is more complicated:
  - for \( \beta_0 < \min(2\beta^*, 1) \), the variance grows as \( T^{2-\beta_0} \) and the limiting distribution is Gaussian, for all values of \( u \);
  - for \( 2\beta^* < \min(\beta_0, 1) \), the variance grows as \( T^{2-2\beta^*} \) and the limiting distribution is nonGaussian (we denote it as composite Rosenblatt), for \( u \neq 0 \); however, for \( u = 0 \), a form of Berry’s cancellation occurs, the variance is of order \( T^{\max(2-3\beta^*, 1)} \), the limiting distribution being nonGaussian for \( 2-3\beta^* > 1 \) and Gaussian for \( 2-3\beta^* < 1 \).

3.1 Overview of the proofs

The rationale behind these results can be more easily understood if we review the main ideas behind the proof.
3.1.1 Chaotic expansions

The main technical tool that we are going to exploit is the possibility to expand our area functional $\mathcal{M}_T(u)$ in (12) into so-called Wiener chaoses, by means of the Stroock-Varadhan decomposition, see [27, §2.2] as well as our §4. Briefly, the latter is based on the fact that the sequence of (normalized) Hermite polynomials \( \{ H_q / \sqrt{q!} \}_{q \geq 0} \)

\[ H_0 \equiv 1, \quad H_q(u) := (-1)^q \phi(u)^{-1} \frac{d^q}{du^q} \phi(u), \quad q \geq 1 \] (24)

(where $\phi$ still denotes the probability density function of a standard Gaussian random variable) is a complete orthonormal basis of the space of square integrable functions on the real line with respect to the Gaussian measure. The first three polynomials are $H_0(u) = 1$, $H_1(u) = u$, $H_2(u) = u^2 - 1$, $H_3(u) = u^3 - 3u$.

From (12) we have the following orthogonal expansion

\[ \mathcal{M}_T(u) = \sum_{q=0}^{\infty} \mathcal{M}_T(u)[q], \] (25)

the series converging in $L^2(\Omega)$, where (see Lemma 4.1)

\[ \mathcal{M}_T(u)[q] = \frac{H_{q-1}(u) \phi(u)}{q!} \int_{[0,T]} \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt \] (26)

is the orthogonal projection of $\mathcal{M}_T(u)$ onto the so-called $q$-th Wiener chaos. Note that if $u = 0$, then $\mathcal{M}_T(u)[q] = 0$ whenever $q$ is even.

In particular, the zeroth projection is

\[ \mathcal{M}_T(u)[0] = \mathbb{E}[\mathcal{M}_T(u)] = 0; \] (27)

for the first one we have, recalling (5) and (7),

\[ \mathcal{M}_T(u)[1] = \phi(u) \int_{[0,T]} \int_{\mathbb{S}^2} Z(x,t) \, dx \, dt = \phi(u) \lim_{L \to \infty} \int_{[0,T]} \int_{\mathbb{S}^2} \sum_{\ell=0}^{L} Z_{\ell}(x,t) \, dx \, dt, \]

where the limit is in the $L^2(\Omega)$-sense. Hence

\[ \mathcal{M}_T(u)[1] = \phi(u) \int_{[0,T]} \frac{a_{00}(t)}{\sqrt{4\pi}} \, dt, \] (28)

the spherical harmonics of degree $\ell \geq 1$ having zero mean on the sphere. Furthermore

\[ \mathcal{M}_T(u)[2] = \frac{u \phi(u)}{2} \int_{[0,T]} \int_{\mathbb{S}^2} (Z^2(x,t) - 1) \, dx \, dt, \]

\[ \mathcal{M}_T(u)[3] = \frac{(u^2 - 1) \phi(u)}{2} \int_{[0,T]} \int_{\mathbb{S}^2} (Z^3(x,t) - 3Z(x,t)) \, dx \, dt. \]

3.1.2 Sharp asymptotics

The crucial step behind our arguments is to investigate the sharp asymptotic behavior, as $T \to +\infty$, of the variances for these chaotic projections. In order to simplify this discussion we assume here that $\beta_\ell \leq \beta_0$, see the next sections for a complete analysis. For every $u \in \mathbb{R}$,

\[ \text{Var}(\mathcal{M}_T(u)[1]) = c_1 \cdot T^{\max(2-\beta_0, 1)}(1 + o(1)). \] (29)
For $q \geq 2$ and either $u \neq 0$ or $u = 0$ and $q$ odd (recall that for $u = 0$ the projections onto even order chaoses vanish), we have

$$\text{Var} (\mathcal{M}_T(u)[q]) = c_q \cdot T^{\max(2 - q\beta_\ell^*, 1)} (1 + 1_{q\beta_\ell^* = 1} \cdot \log T) (1 + o(1)).$$

(30)

Here, for $q \geq 1$, $c_q = c_q(u, \beta_\ell^*, T^*)$ is a finite and positive constant depending in particular on $q$, the level $u$ and the coefficient $\beta_\ell^*$. Thanks to (25) and (27),

$$\text{Var} (\mathcal{M}_T(u)) = \sum_{q=1}^{\infty} \text{Var} (\mathcal{M}_T(u)[q])$$

and hence, up to controlling the sequence $\{c_q, q \geq 1\}$, from (29) and (30) we have that, as $T \to \infty$,

\[
\begin{align*}
\tilde{\mathcal{M}}_T(u) &= \frac{\mathcal{M}_T(u)[1]}{\sqrt{\text{Var} (\mathcal{M}_T(u)[1])}} + o_P(1), \quad \text{for } \beta_0 < \min(2\beta_\ell^*, 1), \ u \neq 0, \\
\tilde{\mathcal{M}}_T(u) &= \frac{\mathcal{M}_T(u)[1]}{\sqrt{\text{Var} (\mathcal{M}_T(u)[1])}} + o_P(1), \quad \text{for } \beta_0 < \min(3\beta_\ell^*, 1), \ u = 0, \\
\tilde{\mathcal{M}}_T(u) &= \frac{\mathcal{M}_T(u)[2]}{\sqrt{\text{Var} (\mathcal{M}_T(u)[2])}} + o_P(1), \quad \text{for } 2\beta_\ell^* < \min(\beta_0, 1), \ u \neq 0, \\
\tilde{\mathcal{M}}_T(u) &= \frac{\mathcal{M}_T(u)[3]}{\sqrt{\text{Var} (\mathcal{M}_T(u)[3])}} + o_P(1), \quad \text{for } 3\beta_\ell^* < \min(\beta_0, 1), \ u = 0,
\end{align*}
\]

where $o_P(1)$ denotes a sequence of random variables converging to zero in probability. The asymptotic distribution of (13) can then be derived in the cases considered just above by a careful analysis of these single components: the first chaotic term is Gaussian for every $T > 0$, the second one asymptotically follows a composite Rosenblatt distribution. On the other hand, in the remaining cases, (e.g. $u \neq 0$, $\beta_0 = 1$ and $2\beta_\ell^* > 1$ or $u = 0$, $\beta_0 = 1$ and $3\beta_\ell^* > 1$) it is not possible to identify a single dominating component; indeed, all the chaotic projections contribute with a variance of the same rate $T$, and the Gaussian limiting behaviour will follow from a Breuer-Major type argument [27, §5.3, §7].

### 3.2 Discussion

We can further summarize our results as follows:

| $u \neq 0$ | $u = 0$ | asymptotic distribution |
|------------|----------|-------------------------|
| first chaos dominates if | $\beta_0 < \min(2\beta_\ell^*, 1)$ (Var $\approx T^{2-\beta_0}$) | $\beta_0 < \min(3\beta_\ell^*, 1)$ (Var $\approx T^{2-\beta_0}$) | Gaussian |
| second chaos dominates if | $2\beta_\ell^* < \min(\beta_0, 1)$ (Var $\approx T^{2-2\beta_\ell^*}$) | never (it vanishes) | non-Gaussian (composite Rosenblatt 2) |
| third chaos dominates if | never | $3\beta_\ell^* < \min(\beta_0, 1)$ (Var $\approx T^{2-3\beta_\ell^*}$) | non-Gaussian |
| all chaoses dominate if | $\beta_0 = 1$, $2\beta_\ell^* > 1$ (Var $\approx T$) | $\beta_0 = 1$, $3\beta_\ell^* > 1$ (Var $\approx T$) | Gaussian |

These findings should be compared with a rapidly growing literature devoted to the investigation of geometric functionals over spherical random fields in a different regime: in particular, a number of papers (see e.g. [9, 22, 23, 25, 36, 34]) have considered the high-frequency behaviour (e.g., when the eigenvalues diverge) for spherical random eigenfunctions with no form of temporal dependence. The results we exhibited here have some analogies, but also important differences, with this stream of literature. In particular
for the excursion area of random spherical harmonics at \( u \neq 0 \) [25, 22] it is indeed the case that the high-energy behaviour is dominated by the second-order chaotic projection, whose asymptotic distribution is, however, Gaussian. The same asymptotic behaviour occurs for other geometric functionals, such as the boundary length of excursion sets and their Euler-Poincaré characteristic, see [31, 9];

- for \( u = 0 \), the limiting variance is always of smaller-order, and asymptotic Gaussianity holds [25, 23].

These differences can be explained as follows. Because in the case of high-frequency asymptotics one deals with sequences of eigenspaces of growing dimensions, the second chaotic components correspond to a sum of a growing number of i.i.d. coefficients, whence a standard Central Limit Theorem holds. In our case here, the dimension of the sum of the eigenspaces which correspond to the strongest memory does not diverge in general, and hence asymptotic Gaussianity need not to hold. Moreover, in the case of high-frequency asymptotics the linear projection term \( a_{00} \) is dropped by construction: on the contrary, for the random fields we investigate here this term can be dominant for instance when \( \beta_0 < \min(2\beta^{\ell^{\ast}}, 1) \), in which case Gaussianity follows trivially.

As far as Berry’s cancellation is concerned, this can occur in the present circumstances only when \( H_2(Z(\cdot, \cdot)) \) exhibits long memory behaviour, i.e., non-integrable temporal autocovariance: this is indeed the case for \( 2\beta^{\ell^{\ast}} < 1 \). If these condition is not met, all chaotic components have integrable temporal autocovariance, none of them dominates and a Central Limit Theorem is established by means of a Breuer-Major Theorem. Note that the presence of long memory behaviour in the field \( Z \) is a necessary, but not sufficient condition for the covariance of \( H_2(Z(\cdot, \cdot)) \) to be non-integrable.

As a final analogy, a remarkable feature of high-frequency asymptotics for random eigenfunctions is the fact that geometric functionals turn out to be asymptotically fully correlated over different levels, and even among themselves, see [8] and the references therein. It is then of interest to investigate if similar features appear in the present framework. We present here a small result that highlights this point.

**Proposition 3.1.** Assume that \( u \neq 0 \), \( 2\beta^{\ell^{\ast}} < \min(\beta_0, 1) \) and that there exists a unique

\[ \ell^{\ast} := \arg \min_{\ell \in \mathbb{N}, \ell \geq 1} \beta_\ell, \]

then, as \( T \to \infty \),

\[ \text{Corr} (\mathcal{M}_T(u), m_{T,\ell^{\ast}}(u)) \to 1, \]

where

\[ m_{T,\ell^{\ast}}(u) := \frac{u}{2\sigma_{\ell^{\ast}}} \phi \frac{u}{\sigma_{\ell^{\ast}}} \int_{\mathbb{R}^2} \int_0^T H_2 \left( \frac{Z_{\ell^{\ast}}(x,t)}{\sigma_{\ell^{\ast}}} \right) dx \, dt. \]

**Remark 3.1.** Note that, if we introduce the process

\[ \mathcal{M}_{T,\ell}(u) = \int_0^T (A_{u,\ell}(t) - \mathbb{E}[A_{u,\ell}(t)]) \, dt, \]

where

\[ A_{u,\ell}(t) := \int_{\mathbb{R}^2} 1_{Z_{\ell}(x,t) \geq u} \, dx, \]

(cf. (7) and (11)) then \( m_{T,\ell^{\ast}}(u) = \mathcal{M}_{T,\ell^{\ast}}(u)[2] \), the second order chaotic component of the functional of the monochromatic field \( Z_{\ell^{\ast}} \).
4 Stroock-Varadhan decompositions

The first tool that is needed in order to establish our asymptotic results is the derivation of the analytic form for the chaotic expansion (25) of the empirical measure. The result is very close to analogous findings given by [13, 25]. For a complete discussion on Wiener chaos and related topics see e.g. [27, §2.2].

Lemma 4.1. For every $T > 0$ we have that

$$M_T(u) = \sum_{q \geq 1} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{S^2} H_q(Z(x,t)) \, dx \, dt$$

(31)

where $J_q(u) = H_{q-1}(u)\phi(u)$, $\phi$ still being the density function of a standard Gaussian random variable and $H_q$ the Hermite polynomial (24) of order $q$. The convergence of the series in (31) is in the $L^2(\Omega)$-sense.

Proof. Let $Z \sim N(0,1)$, then

$$1_{Z \geq u} = \sum_{q=0}^{\infty} \frac{J_q(u)}{q!} H_q(Z),$$

where the right hand side converges in the $L^2(\Omega)$-sense and the coefficients $J_q(u)$ are given by

$$J_q(u) := \mathbb{E}[1_{Z \geq u} H_q(Z)] = \int_{\mathbb{R}} 1_{x \geq u} H_q(x) \phi(x) \, dx = (-1)^q \int_{\mathbb{R}} \frac{d^q}{dx^q} \phi(x) \, dx = H_{q-1}(u)\phi(u)$$

(note that for fixed $x \in S^2, t \in \mathbb{R}, Z(x,t)$ is standard Gaussian). Now consider the sequence of random variables

$$\left\{ \sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{S^2} H_q(Z(x,t)) \, dx \, dt, \ Q \geq 1 \right\};$$

let us prove that

$$\sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{S^2} H_q(Z(x,t)) \, dx \, dt \xrightarrow{Q \to +\infty} M_T(u)$$

(32)

in the $L^2(\Omega)$-sense. We have, thanks to Jensen inequality and Fubini-Tonelli Theorem,

$$\mathbb{E} \left[ \left( M_T(u) - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{S^2} H_q(Z(x,t)) \, dx \, dt \right)^2 \right]$$

$$= \mathbb{E} \left[ \left( \int_{[0,T]} \int_{S^2} \left( 1_{Z(x,t) \geq u} - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(Z(x,t)) \right) \, dx \, dt \right)^2 \right]$$

$$\leq 4\pi T \int_{[0,T]} \int_{S^2} \mathbb{E} \left[ \left( 1_{Z(x,t) \geq u} - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(Z(x,t)) \right)^2 \right] \, dx \, dt$$

$$= (4\pi)^2 T^2 \mathbb{E} \left[ \left( 1_{Z \geq u} - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(Z) \right)^2 \right] \xrightarrow{Q \to +\infty} 0,$$

hence (32) holds and the proof is concluded. \qed
Thanks to orthogonality of the chaotic components, from Lemma 4.1 we get

$$\text{Var} (\mathcal{M}_T(u)) = \sum_{q=1}^{\infty} \text{Var} (\mathcal{M}_T(u)[q])$$

$$= \sum_{q=1}^{\infty} \frac{J_q(u)^2}{q!} \int_{[0,T]^2} \int_{S^2 \times S^2} \Gamma((x,y),t-s)^q dx dy dt ds,$$

where $\Gamma$ is the covariance function in (2).

4.1 First order chaotic projections

In this subsection we investigate the variance behavior of the first chaotic component (28).

Lemma 4.2. We have, as $T \to +\infty$,

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T^{2 - \beta_0}} = \frac{2\phi(u)^2C_0(0)}{(1 - \beta_0)(2 - \beta_0)}, \quad \text{if } \beta_0 \in (0, 1)$$

and

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} = \phi(u)^2 \int_{\mathbb{R}} C_0(\tau) d\tau, \quad \text{if } \beta_0 = 1.$$

Recall that Condition 2 ensures that $C_0(0) > 0$ and that for $\beta_0 = 1$

$$\int_{\mathbb{R}} C_0(\tau) d\tau \in (0, +\infty),$$

hence Lemma 4.2 gives the exact rate for the variance, the limiting constants being strictly positive. From (28) we can write

$$\text{Var}(\mathcal{M}_T(u)[1]) = \phi(u)^2 \int_{[0,T]^2} C_0(t-s) dt ds.$$  \hspace{1cm} (34)

Remark 4.1. We will often make use of the following standard computation, that we report in this remark and that are taken for granted in the rest of the article. Making the change of variable $\tau = t - s$ for the double integral on the right hand side of (34), one has

$$\int_{[0,T]^2} C_0(t-s) dt ds = \int_0^T ds \int_{s}^{T-s} C_0(\tau) d\tau$$

$$= \int_0^T ds \int_{-T}^{T} 1_{[-s,T-s]}(\tau) C_0(\tau) d\tau = \int_{-T}^{T} C_0(\tau) d\tau \int_0^T 1_{[-\tau,T-\tau]}(s) ds$$

$$= \int_{-T}^{0} C_0(\tau) d\tau \int_0^T 1_{[-\tau,T-\tau]}(s) ds + \int_0^T C_0(\tau) d\tau \int_{0}^{T} 1_{[0,T-\tau]}(s) ds$$

$$= \int_{-T}^{0} (T + \tau) C_0(\tau) d\tau + \int_0^T (T - \tau) C_0(\tau) d\tau$$

$$= 2T \int_0^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) d\tau = T \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) C_0(\tau) d\tau.$$

It is now easy to investigate the asymptotic behavior, as $T \to +\infty$, of the variance of the first order chaotic component.
Proof of Lemma 4.2. From Remark 4.1 and (34) we can write
\[
\text{Var}(\mathcal{M}_T(u)[1]) = 2T\phi(u)^2 \int_0^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau .
\]

(35)

If \(\beta_0 = 1\), recalling from Condition 2 that in this case the covariance \(C_0\) is integrable on \(\mathbb{R}\), we immediately have (thanks to Dominated convergence Theorem) the exact asymptotic behavior of the variance
\[
\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} = 2\phi(u)^2 \int_0^{+\infty} C_0(\tau) \, d\tau = \phi(u)^2 \int_\mathbb{R} C_0(\tau) \, d\tau .
\]

Now assume \(\beta_0 < 1\). Let \(\varepsilon > 0\), thanks to Condition 2, there exists \(M > 0\) such that, for \(\tau > M\),
\[
\sup_{\ell \in \mathbb{N}} \left| G_\ell(\tau) - 1 \right| < \varepsilon ,
\]

(36)

and we can write (from (35))
\[
\text{Var}(\mathcal{M}_T(u)[1]) = 2T\phi(u)^2 \int_0^M \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau + 2T\phi(u)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau 
\]

\[
= O(1) + 2T\phi(u)^2 \int_0^M C_0(\tau) \, d\tau + 2T\phi(u)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau .
\]

(37)

Consider the last integral on the right hand side of (37) and write
\[
\frac{1}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau = \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-\beta_0} \, d\tau 
\]

\[
+ \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left( \frac{G_0(\tau)}{C_0(0)} - 1 \right) (1 + \tau)^{-\beta_0} \, d\tau .
\]

We have
\[
\lim_{T \to \infty} \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-\beta_0} \, d\tau = \frac{C_0(0)}{(1-\beta_0)(2-\beta_0)}
\]

(38)

and
\[
\lim_{T \to \infty} \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left( \frac{G_0(\tau)}{C_0(0)} - 1 \right) (1 + \tau)^{-\beta_0} \, d\tau = 0 .
\]

(39)

The proof of (38) is straightforward; recall that by assumption \(C_0(0) > 0\). It remains to prove (39). For \(T > M\)
\[
\frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left| \frac{G_0(\tau)}{C_0(0)} - 1 \right| (1 + \tau)^{-\beta_0} \, d\tau \leq \varepsilon \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T (1 + \tau)^{-\beta_0} \, d\tau
\]

\[
= \varepsilon \frac{C_0(0)}{1-\beta_0} \left( 1 + \frac{1}{T} \right)^{1-\beta_0} - \left( \frac{M+1}{T} \right)^{1-\beta_0} \leq \varepsilon \frac{C_0(0)}{1-\beta_0} \left( 1 + \frac{1}{T} \right)^{1-\beta_0} .
\]

Hence
\[
\limsup_{T \to +\infty} \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left( \frac{G_0(\tau)}{C_0(0)} - 1 \right) (1 + \tau)^{-\beta_0} \, d\tau \leq \varepsilon \frac{C_0(0)}{1-\beta_0}
\]

and the result follows, \(\varepsilon\) being arbitrary. Plugging (38) and (39) into (37) we find
\[
\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T^{2-\beta_0}} = \frac{2\phi(u)^2 C_0(0)}{(1-\beta_0)(2-\beta_0)} , \quad \beta_0 \in (0,1)
\]

that concludes the proof.
4.2 Second order chaotic projections

Our next step is a careful analysis for the variance of the second order chaotic component $\mathcal{M}_T(u)[2]$ (25) and (31), which will play a dominating role in most long memory scenarios (see §2.1). For $q = 2$ we have

$$\text{Var} (\mathcal{M}_T(u)[2]) = \frac{u^2 \phi(u)^2}{2} \int_{[0,T]^2} \int_{S^2 \times S^2} \Gamma((x, y), t - s)^2 \, dx \, dy \, ds .$$

Now, thanks to (3) and (4),

$$\int_{[0,T]^2} \int_{S^2 \times S^2} \Gamma((x, y), t - s)^2 \, dx \, dy \, ds = \int_{[0,T]^2} \int_{S^2 \times S^2} \left( \sum_{\ell_1 = 0}^\infty C_{\ell_1} (t - s) \left(\frac{2\ell_1 + 1}{4\pi}\right) P_{\ell_1}(\langle x, y \rangle) \right)^2 \, dx \, dy \, ds \times \int_{S^2 \times S^2} P_{\ell_1}(\langle x, y \rangle) P_{\ell_2}(\langle x, y \rangle) \, dx \, dy \, ds$$

$$= \int_{[0,T]^2} \int_{S^2 \times S^2} \left( \sum_{\ell_1 = 0}^\infty C_{\ell_1} (t - s) C_{\ell_2} (t - s) \left(\frac{2\ell_1 + 1}{4\pi}\right) \left(\frac{2\ell_2 + 1}{4\pi}\right) \times \right.$$

$$\left. \int_{S^2 \times S^2} P_{\ell_1}(\langle x, y \rangle) P_{\ell_2}(\langle x, y \rangle) \, dx \, dy \, ds \right) (2\ell_1 + 1)(2\ell_2 + 1) \frac{(4\pi)^2}{2\ell_1 + 1} 1_{\ell_1 = \ell_2} \, dt \, ds$$

$$= \int_{[0,T]^2} \int_{S^2 \times S^2} (2\ell_1 + 1) C_{\ell_1} (t - s)^2 \, dt \, ds .$$

Hence

$$\text{Var} (\mathcal{M}_T(u)[2]) = \frac{u^2 \phi(u)^2}{2} \sum_{\ell_1 = 0}^\infty (2\ell_1 + 1) \int_{[0,T]^2} C_{\ell_1} (t - s)^2 \, dt \, ds .$$

The next result is of fundamental importance for the study of the asymptotic behavior of $\text{Var} (\mathcal{M}_T(u)[2])$, its proof will be given in §4.2.1.

**Lemma 4.3.** Fix $\ell \in \tilde{N}$. If $2\beta_\ell < 1$, then

$$\lim_{T \to \infty} \frac{1}{T^{2 - 2\beta_\ell}} \int_{[0,T]^2} C_{\ell}^2 (t - s) \, dt \, ds = \frac{C_{\ell}(0)^2}{(1 - \beta_\ell)(1 - 2\beta_\ell)} .$$

If $2\beta_\ell = 1$, then

$$\lim_{T \to \infty} \frac{1}{T \log T} \int_{[0,T]^2} C_{\ell}^2 (t - s) \, dt \, ds = 2C_{\ell}(0)^2 .$$

If $2\beta_\ell > 1$, then

$$\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]^2} C_{\ell}^2 (t - s) \, dt \, ds = \int_{\mathbb{R}} C_{\ell}(\tau)^2 \, d\tau .$$

Let us write

$$\text{Var} (\mathcal{M}_T(u)[2]) = \frac{u^2 \phi(u)^2}{2} \int_{[0,T]^2} C_{0}(t - s)^2 \, dt \, ds + \frac{u^2 \phi(u)^2}{2} \sum_{\ell = 1}^\infty (2\ell + 1) \int_{[0,T]^2} C_{\ell}(t - s)^2 \, dt \, ds .$$

(40)

Now recall the definition of $\beta_\ell\star$ in (14).
Proposition 4.1. Assume \( u \neq 0 \). For \( 2\beta_\star < 1 \) and \( \beta_\star \leq \beta_0 \), we have that
\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T^{2-2\beta_\star}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_\star)(1-\beta_0)} \sum_{\ell \in I^*} (2\ell + 1) C_0(0)^2 ;
\]
for \( 2\beta_\star < 1 \) and \( \beta_0 < \beta_\star \),
\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T^{2-2\beta_\star}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_0)(1-\beta_0)} C_0(0)^2 ;
\]
for \( 2\beta_\star = 1 \) and \( \beta_\star \leq \beta_0 \),
\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T \log T} = \frac{u^2 \phi(u)^2}{2(1-2\beta_0)(1-\beta_0)} C_0(0)^2 ;
\]
for \( 2\beta_\star > 1 \) and \( 2\beta_0 > 1 \),
\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T} = \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \int_{\mathbb{R}} C_\ell(\tau)^2 d\tau ;
\]
for \( 2\beta_\star > 1 \) and \( 2\beta_0 = 1 \),
\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T \log T} = u^2 \phi(u)^2 \int_{\mathbb{R}} C_0(\tau)^2 ;
\]
finally, for \( 2\beta_\star > 1 \) and \( 2\beta_0 < 1 \),
\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T^{2-2\beta_0}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_0)(1-\beta_0)} C_0(0)^2 .
\]

Recall that by assumption \( C_0(0) > 0 \), and for \( \ell \in I^* \) we have \( C_\ell(0) > 0 \) (see (14)); as a consequence, Proposition 4.1 gives the exact rate for the variance, the limiting constants being strictly positive.

Remark 4.2. In words, for \( \beta_\star \leq \beta_0 \), when \( 2\beta_\star < 1 \) (resp. \( 2\beta_\star = 1 \)), we have a form of long-range dependence and the second order chaotic component of the functional \( \mathcal{M}_T(u) \) is dominated by a subset of the multipoles; the variance scales as order \( T^{2-2\beta_\star} \) (resp. \( T \log T \)). On the contrary, when \( 2\beta_\star > 1 \), a form of short-range dependence holds and all frequencies contribute with variance terms of order \( T \).

In order to prove Proposition 4.1 we will also need the following technical results. The proofs of Lemma 4.4 and 4.5 are given in the Appendix §A, the proofs of Lemma 4.6 and Lemma 4.7 are indeed similar and we omit the details for brevity.

Lemma 4.4. Let \( \varepsilon, M > 0 \) be as in (36). For \( \ell \) such that \( \beta_\ell = 1 \) and \( T > \max(1, M) \)
\[
\frac{1}{T} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq 2C_\ell(0)^2 \left( M + 2 \frac{(\varepsilon + 1)^2}{\alpha - 1} \right) ,
\]
where \( \alpha \geq 2 \) comes from the definition in (10).
Lemma 4.5. Let \( \varepsilon, M > 0 \) be as in (36) and \( 2\beta_{t^*} < 1 \).

- For \( \ell \in I^* \), \( \beta_{t^*} < 1 \) and \( T > \max(1, M) \)
  \[
  \frac{1}{T^{2-2\beta_{t^*}}} \int_{[0,T]^2} C_{t^*}(t - s)^2 \, dtds \leq 2C_{t^*}(0)^2 \left( M + \frac{(\varepsilon + 1)^2}{-2\beta_{t^*} + 1} \left( 1 + \frac{1}{M} \right)^{1-2\beta_{t^*}} \right). \tag{45}
  \]

- Let \( m(\beta_{t^*}) := \max_{x > 0} \frac{\log(1+x)}{x^{1-2\beta_{t^*}}} \) and \( T_m \) the corresponding arg max. For \( \ell \notin I^* \), \( \ell \geq 1 \), \( \beta_{t^*} < 1 \) and \( T > \max(1, M, T_m) \) we have
  \[
  \frac{1}{T^{2-2\beta_{t^*}}} \int_{[0,T]^2} C_{t^*}(t - s)^2 \, dtds \leq 2C_{t^*}(0)^2 \left( M + 2m(\beta_{t^*})(\varepsilon + 1)^2 \mathbf{1}_{2\beta_{t^*} = 1} \right. \\
  \quad \left. + (\varepsilon + 1)^2 \frac{1}{-2\beta_{t^*} + 1} \left( 1 + \frac{1}{M} \right)^{-2\beta_{t^*} + 1} \mathbf{1}_{2\beta_{t^*} < 1} \right) \\
  \quad \left. + (\varepsilon + 1)^2 \frac{1}{2\beta_{t^*} - 1} \left( 1 + \frac{1}{M} \right)^{2\beta_{t^*} - 1} \mathbf{1}_{2\beta_{t^*} > 1} \right). \tag{46}
  \]

Lemma 4.6. Let \( \varepsilon, M > 0 \) be as in (36) and \( 2\beta_{t^*} = 1 \).

- For \( \ell \in I^* \), \( \beta_{t^*} < 1 \) and \( T > \max(1, M, e) \)
  \[
  \frac{1}{T \log T} \int_{[0,T]^2} C_{t^*}(t - s)^2 \, dtds \leq 2C_{t^*}(0)^2 \left( M + \log(e + 1) \right). 
  \]

- For \( \ell \notin I^* \), \( \ell \geq 1 \), \( \beta_{t^*} < 1 \) and \( T > \max(1, M) \)
  \[
  \frac{1}{T \log T} \int_{[0,T]^2} C_{t^*}(t - s)^2 \, dtds \leq 2C_{t^*}(0)^2 \left( M + \int_{\mathbb{R}} (1 + |\tau|)^{-2\beta_{t^*}} \, d\tau \right). 
  \]

Lemma 4.7. Let \( \varepsilon, M > 0 \) be as in (36). If \( 2\beta_{t^*} > 1 \), for \( \ell \geq 1 \) and \( \beta_{t^*} < 1 \) we have
  \[
  \frac{1}{T} \int_{[0,T]^2} C_{t^*}(t - s)^2 \, dtds \leq 2C_{t^*}(0) \left( M + (\varepsilon + 1)^2 \int_{\mathbb{R}} (1 + |\tau|)^{-2\beta_{t^*}} \, d\tau \right)
  \]
  whenever \( T > \max(1, M) \).

We are now in the position to prove Proposition 4.1.

Proof of Proposition 4.1. Assume first that \( 2\beta_{t^*} < 1 \) and \( \beta_{t^*} \leq \beta_0 \). For the asymptotic behavior of the first term on the right hand side of (40) we refer to Lemma 4.3:
  \[
  \lim_{T \to \infty} \frac{\int_{[0,T]^2} C_{0}(t - s)^2 \, dtds}{T^{2-2\beta_{t^*}}} = \begin{cases} 
  0 & \text{if } \beta_{t^*} < \beta_0, \\
  \frac{C_{0}(0)^2}{(1-\beta_{t^*})(1-2\beta_{t^*})} & \text{if } \beta_{t^*} = \beta_0.
  \end{cases} \tag{47}
  \]
  
  Now, since from (4) we have
  \[
  \sum_{\ell=0}^{\infty} (2\ell + 1)C_{t^*}(0)^2 < +\infty, \tag{48}
  \]
  thanks to Lemma 4.4 and Lemma 4.5 we can apply Dominance Convergence Theorem and then Lemma 4.3 to get
  \[
  \lim_{T \to \infty} \sum_{\ell \in I^*} \frac{(2\ell + 1)}{T^{2-2\beta_{t^*}}} \int_{[0,T]^2} C_{t^*}(t - s)^2 \, dtds
  \]
Thanks again to Lemma 4.4 and Lemma 4.5, since (48) holds, we can apply Dominated Convergence Theorem to obtain (50)), more precisely we have to distinguish between the possible cases: from Lemma 4.3

\[
\lim_{T \to \infty} \sum_{\ell \geq 1; \beta_{\ell} < \beta_{\ell} < \beta_{\ell}} (2\ell + 1) \int_{[0, T]^2} C_{\ell}(t - s)^2 \, dt \, ds = 0;
\]

moreover

\[
\lim_{T \to \infty} \sum_{\ell \geq 1; \beta_{\ell} = 1} (2\ell + 1) \int_{[0, T]^2} C_{\ell}(t - s)^2 \, dt \, ds = 0;
\]

\[
\lim_{T \to \infty} \sum_{\ell \geq 1; \beta_{\ell} > \frac{1}{2}} (2\ell + 1) \int_{[0, T]^2} C_{\ell}(t - s)^2 \, dt \, ds = 0.
\]

Putting together (47), (49) and (50) we immediately get (41) in Proposition 4.1.

Now assume \(2\beta_0 > 1\) and \(2\beta_{\ell} > 1\). Then obviously \(2\beta_{\ell} > 1\) for each \(\ell \in \mathbb{N}\) and hence, using Lemma 4.4 and 4.7 and Lemma 4.3 as before, we have

\[
\lim_{T \to \infty} \frac{\text{Var} (\mathcal{M}_T(u)[2])}{T} = \frac{u^2 \phi(u)}{2} \sum_{\ell = 0}^{\infty} (2\ell + 1) \int_{[0, T]^2} C_{\ell}(t - s)^2 \, dt \, ds
\]

\[
= \frac{u^2 \phi(u)}{2} \sum_{\ell = 0}^{\infty} (2\ell + 1) \int_{0}^{T} \frac{1}{T^2 - 2\beta_0} \int_{[0, T]^2} C_{\ell}(t - s)^2 \, dt \, ds = 0.
\]

which is (43). Note that we automatically get

\[
\sum_{\ell = 0}^{\infty} (2\ell + 1) \int_{-\infty}^{+\infty} C_{\ell}(\tau)^2 \, d\tau < +\infty,
\]

and the proof is concluded, the remaining cases requiring analogous proofs.
4.2.1 Proof of Lemma 4.3

Proof. If \( \beta \in (\frac{1}{2}, 1] \) then from Remark 4.1, thanks to Dominated Convergence Theorem,

\[
\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]^2} C^2 \tau (t-s) dt ds = \lim_{T \to \infty} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) C^2 \tau (\tau)^2 d\tau = \int_{\mathbb{R}} C^2 \tau (\tau)^2 d\tau.
\]

Now assume that \( 2\beta < 1 \) and recall Condition 2, then as in Remark 4.1 and the proof of Lemma 4.2 we fix \( \varepsilon > 0 \) and we know there exists \( M > 0 \) such that, for \( \tau > M \),

\[
\sup_{\tau} \left| \frac{G^2 \tau (\tau)}{C^2 \tau (0)} - 1 \right| < \varepsilon
\]

(as in (36)), so that

\[
\int_{[0,T]^2} C^2 \tau (t-s) dt ds = 2T \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) C^2 \tau (\tau) d\tau
\]

\[
= 2T \int_{0}^{M} \left(1 - \frac{\tau}{T}\right) C^2 \tau (\tau) d\tau + 2T \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) C^2 \tau (\tau) d\tau.
\]

\[
= O(1) + 2T \int_{0}^{M} C^2 \tau (\tau) d\tau + 2TC^2 \tau (0)^2 \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) \left(\frac{G^2 \tau (\tau)}{C^2 \tau (0)} - 1\right) (1 + \tau)^{-2\beta} d\tau
\]

\[
+ 4TC^2 \tau (0)^2 \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-2\beta} d\tau.
\]

(51)

For the second and the last summands of (51) it is straightforward to check that

\[
\lim_{T \to \infty} \frac{2T}{T^2 - 2\beta} \int_{0}^{M} C^2 \tau (\tau) d\tau = 0,
\]

\[
\lim_{T \to \infty} \frac{2TC^2 \tau (0)^2}{T^2 - 2\beta} \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-2\beta} d\tau = \frac{C^2 \tau (0)^2}{(1 - \beta)(1 - 2\beta)}.
\]

On the other hand, for the third and the fourth summands

\[
\lim_{T \to \infty} \frac{2TC^2 \tau (0)}{T^2 - 2\beta} \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) \left(\frac{G^2 \tau (\tau)}{C^2 \tau (0)} - 1\right) (1 + \tau)^{-2\beta} d\tau
\]

\[
= \lim_{T \to \infty} \frac{4TC^2 \tau (0)}{T^2 - 2\beta} \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) \left(\frac{G^2 \tau (\tau)}{C^2 \tau (0)} - 1\right) (1 + \tau)^{-2\beta} d\tau = 0.
\]

(52)

Let us prove (52): we have that

\[
\limsup_{T \to +\infty} \frac{2TC^2 \tau (0)}{T^2 - 2\beta} \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) \left(\frac{G^2 \tau (\tau)}{C^2 \tau (0)} - 1\right) (1 + \tau)^{-2\beta} d\tau
\]

\[
\leq \varepsilon^2 \limsup_{T \to +\infty} \frac{2C^2 \tau (0)}{T^1 - 2\beta} \int_{M}^{T} (1 + \tau)^{-2\beta} d\tau \leq \varepsilon^2 \frac{4C^2 \tau (0)}{1 - 2\beta},
\]

and analogously that

\[
\limsup_{T \to +\infty} \frac{4TC^2 \tau (0)}{T^2 - 2\beta} \int_{M}^{T} \left(1 - \frac{\tau}{T}\right) \left| \frac{G^2 \tau (\tau)}{C^2 \tau (0)} - 1\right| (1 + \tau)^{-2\beta} d\tau \leq \varepsilon \frac{2C^2 \tau (0)}{1 - 2\beta},
\]

and (52) follows, \( \varepsilon \) being arbitrary. When \( 2\beta = 1 \), then one can prove using the same arguments that

\[
\lim_{T \to \infty} \frac{1}{T \log T} \int_{[0,T]^2} C^2 \tau (t-s) dt ds = 2C^2 \tau (0)^2
\]

and the proof of the lemma is concluded. \( \square \)
4.3 Higher order chaotic projections

In this subsection we want to investigate the behavior of higher order chaotic components. Let $q \geq 3$, from (31) we can write

$$ \text{Var}(\mathcal{M}_T(u)[q]) = \frac{\phi(u)^2 H_{q-1}(u)^2}{q!} \int_{[0,T]^2} \int_{S^2 \times S^2} \Gamma(\langle x, y \rangle, t - s)^q \, dx \, dy \, dt \, ds. \quad (53) $$

Thanks to (3) we have that

$$ \int_{[0,T]^2} \int_{S^2 \times S^2} \Gamma(\langle x, y \rangle, t - s)^q \, dx \, dy \, dt \, ds $$

$$ = \int_{[0,T]^2} \int_{S^2 \times S^2} \left( \sum_{\ell=0}^{\infty} C_\ell(t - s) \frac{(2\ell + 1)}{4\pi} P_\ell(\langle x, y \rangle) \right)^q \, dx \, dy \, dt \, ds $$

$$ = \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\infty} \int_{[0,T]^2} \int_{S^2 \times S^2} C_{\ell_1}(t - s) C_{\ell_2}(t - s) \cdots C_{\ell_q}(t - s) $$

$$ \times \frac{2\ell_1 + 1}{4\pi} P_{\ell_1}(\langle x, y \rangle) \frac{2\ell_2 + 1}{4\pi} P_{\ell_2}(\langle x, y \rangle) \cdots \frac{2\ell_q + 1}{4\pi} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy \, dt \, ds. $$

Recall the addition formula for spherical harmonics [20, (3.42)]

$$ \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(x) Y_{\ell,m}(y) = P_{\ell}(\langle x, y \rangle), \quad x, y \in S^2, $$

and the definition of generalized Gaunt integral in (22) to write

$$ \int_{S^2 \times S^2} \frac{2\ell_1 + 1}{4\pi} P_{\ell_1}(\langle x, y \rangle) \frac{2\ell_2 + 1}{4\pi} P_{\ell_2}(\langle x, y \rangle) \cdots \frac{2\ell_q + 1}{4\pi} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy $$

$$ = \sum_{m_1=-\ell_1}^{\ell_1} \cdots \sum_{m_q=-\ell_q}^{\ell_q} \left( G_{\ell_1 \ldots \ell_q}^{m_1 \ldots m_q} \right)^2. \quad (54) $$

Equivalently,

$$ \int_{S^2 \times S^2} \frac{2\ell_1 + 1}{4\pi} P_{\ell_1}(\langle x, y \rangle) \frac{2\ell_2 + 1}{4\pi} P_{\ell_2}(\langle x, y \rangle) \cdots \frac{2\ell_q + 1}{4\pi} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy $$

$$ = 4\pi \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) G_{\ell_1 \ldots \ell_q}^{0 \ldots 0}. \quad (55) $$

In particular $G_{\ell_1 \ldots \ell_q}^{0 \ldots 0} \geq 0$. In order to check (55) recall that

$$ Y_{\ell,0}(\theta, \varphi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos \theta), $$

where $(\theta, \varphi)$ are the angular coordinates of the point $x \in S^2$; then, letting $o$ be the north pole of the sphere, we have

$$ \int_{S^2 \times S^2} \sqrt{\frac{2\ell_1 + 1}{4\pi}} P_{\ell_1}(\langle x, y \rangle) \cdots \sqrt{\frac{2\ell_q + 1}{4\pi}} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy $$

$$ = 4\pi \int_{S^2} \sqrt{\frac{2\ell_1 + 1}{4\pi}} P_{\ell_1}(\langle x, o \rangle) \cdots \sqrt{\frac{2\ell_q + 1}{4\pi}} P_{\ell_q}(\langle x, o \rangle) \, dx $$

$$ = 1. $$

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As a consequence, from (53) we can write
\[
\text{Var}(M_T(u)[q]) = \frac{4\pi H_{q-1}(u)^2\phi(u)^2}{q!} \sum_{\ell_1,\ldots,\ell_q=0}^{\infty} k_{\ell_1,\ldots,\ell_q}(T) \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) g_{\ell_1,\ldots,\ell_q}^{0,0},
\]
where
\[
k_{\ell_1,\ldots,\ell_q}(T) := \int_{[0,T]^{2}} C_{\ell_1}(t-s) C_{\ell_2}(t-s) \cdots C_{\ell_q}(t-s) dtds.
\]
Note that
\[
k_{\ell_1,\ldots,\ell_q}(T) = E \left[ \left( \int_{[0,T]} a_{\ell_1,0}(t) \cdots a_{\ell_q,0}(t) dt \right)^2 \right].
\]
In order to study the asymptotic behavior, as $T \to +\infty$, of (56) we will need the following result whose proof is given in §4.3.1.

**Lemma 4.8.** Let $\ell_1,\ldots,\ell_q$ be such that $\beta_{\ell_1} + \cdots + \beta_{\ell_q} < 1$, then
\[
\lim_{T \to \infty} \frac{k_{\ell_1,\ldots,\ell_q}(T)}{T^{2 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q})}} = \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{(1 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))(2 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))}.
\]
and if $\beta_{\ell_1} + \cdots + \beta_{\ell_q} = 1$
\[
\lim_{T \to \infty} \frac{k_{\ell_1,\ldots,\ell_q}(T)}{T \log T} = 2 C_{\ell_1}(0) \cdots C_{\ell_q}(0) .
\]
On the contrary, let $\ell_1,\ldots,\ell_q$ be such that $\beta_{\ell_1} + \cdots + \beta_{\ell_q} > 1$, then
\[
\lim_{T \to \infty} \frac{k_{\ell_1,\ldots,\ell_q}(T)}{T} = \int_{-\infty}^{+\infty} C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) d\tau.
\]
Recall (56).

**Proposition 4.2.** Let $q \geq 3$. If $q\beta_{\ell_t} < 1$ and $\beta_{\ell_t} \leq \beta_0$ then
\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T^{2 - q\beta_{\ell_t}}} = \frac{4\pi H_{q-1}(u)^2\phi(u)^2}{q!(1 - q\beta_{\ell_t})(2 - q\beta_{\ell_t})} \sum_{\ell_1,\ell_2,\ldots,\ell_q \in Z^*} \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) g_{\ell_1,\ldots,\ell_q}^{0,0};
\]
on the other hand, If $q\beta_{\ell_t} < 1$ and $\beta_{\ell_t} > \beta_0$ then
\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T \log T} = \frac{8\pi H_{q-1}(u)^2\phi(u)^2}{q!(1 - q\beta_0)(2 - q\beta_0)} \sum_{\ell_1,\ell_2,\ldots,\ell_q \in Z^*} \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) g_{\ell_1,\ldots,\ell_q}^{0,0};
\]
If $q\beta_{\ell_t} = 1$ and $\beta_{\ell_t} \leq \beta_0$ then
\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T^{2 - q\beta_0}} = \frac{8\pi H_{q-1}(u)^2\phi(u)^2}{q!(1 - q\beta_0)(2 - q\beta_0)} \sum_{\ell_1,\ell_2,\ldots,\ell_q \in Z^*} \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) g_{\ell_1,\ldots,\ell_q}^{0,0};
\]
if $q\beta_{\ell_t} = 1$ and $\beta_{\ell_t} > \beta_0$ then
\[
\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T^{2 - q\beta_0}} = \frac{8\pi H_{q-1}(u)^2\phi(u)^2}{q!(1 - q\beta_0)(2 - q\beta_0)} \sum_{\ell_1,\ell_2,\ldots,\ell_q \in Z^*} \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) g_{\ell_1,\ldots,\ell_q}^{0,0};
\]
On the other hand, if $q\beta_* > 1$ and $q\beta_0 > 1$, then

$$\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T} = s_q^2,$$

where

$$s_q^2 := \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\ell} \mathbb{E}^{q_0 \ldots q_\ell} \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi C_{\ell_i}}(\tau)} \right) d\tau,$$

moreover if $q\beta_* > 1$ and $q\beta_0 = 1$, then

$$\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T \log T} = \frac{H_{q-1}(u)^2 \phi(u)^2}{(4\pi)^{q-2}q!} \cdot 2C_0(q);$$

finally if $q\beta_* > 1$ and $q\beta_0 < 1$, then

$$\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T^{2-q\beta_0}} = \frac{H_{q-1}(u)^2 \phi(u)^2}{(4\pi)^{q-2}q!} \cdot \frac{C_0(0)^q}{1 - q\beta_0}(2 - q\beta_0).$$

In order to prove Proposition 4.2 we will also need the following technical results, the proofs of Lemma 4.9 and Lemma 4.10 are postponed to the Appendix §A, the proofs of the remaining lemmas are very similar and we omit the details.

**Lemma 4.9.** Let $\varepsilon, M > 0$ be as in (36). If there is at least one index $j \in \{1, \ldots, q\}$ such that $\beta_{\ell_j} = 1$ we have for $T > \max(1, M)$,

$$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T} \leq 2C_{\ell_1}(0) \cdots C_{\ell_q}(0) \left( M + \frac{1}{q \min(\beta_0, \beta_*)} \left( \frac{\varepsilon + 1}{1 + M^{\min(\beta_0, \beta_*)}} \right)^q \right).$$

**Lemma 4.10.** Let $\varepsilon, M > 0$ be as in (36) and $q\beta_* < 1$.

- For $\ell_1, \ldots, \ell_q \in \mathcal{I}^*$, $\beta_{\ell_j} < 1$ for every $j$ and $T > \max(1, M)$ we have

  $$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T^{2-q\beta_*}} \leq 2C_{\ell_1}(0) \cdots C_{\ell_q}(0) \left( M + \frac{1}{1 - q\beta_*} \left( 1 + \frac{M}{1 - q\beta_*} \right)^{1-q\beta_*} \right).$$

- For $(\ell_1, \ldots, \ell_q) \notin (\mathcal{I}^*)^q$, $\beta_{\ell_j} \geq 1$, $\beta_{\ell_j} < 1$ for every $j$ and $T > \max(1, M, T_m)

  $$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T^{2-q\beta_*}} \leq 2C_{\ell_1}(0) \cdots C_{\ell_q}(0) \left( M + \frac{(\varepsilon + 1)^q}{1 - q\beta_*} \left( 1 + \frac{1}{M} \right)^{1-q\beta_*} \right).$$

**Lemma 4.11.** Let $\varepsilon, M > 0$ be as in (36) and $q\beta_* = 1$.

- For $\ell_1, \ldots, \ell_q \in \mathcal{I}^*$, $\beta_{\ell_j} < 1$ for every $j$ and $T > \max(1, M, e)$

  $$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T \log T} \leq 2C_{\ell_1}(0) \cdots C_{\ell_q}(0)(M + \log(e + 1)).$$

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For \((\ell_1, \ldots, \ell_q) \notin (I^*)^q\), \(\ell_j \geq 1\), \(\beta_{\ell_j} < 1\) for every \(j\) and \(T > \max(1, M, \varepsilon)\)
\[
\frac{k_{\ell_1, \ldots, \ell_q}(T)}{T \log T} \leq 2C \ell_1(0) \cdots C \ell_q(0) \left( M + (\varepsilon + 1)^q \int_{\mathbb{R}} (1 + |\tau|)^{-(\beta_{\ell_1} + \cdots + (q-1)\beta_{\ell_q})} d\tau \right).
\]

**Lemma 4.12.** Let \(\varepsilon, M > 0\) be as in (36) and \(q\beta_{\ell_i} > 1\). Then for \(T > \max(1, M)\)
\[
\frac{k_{\ell_1, \ell_2, \ldots, \ell_q}(T)}{T} \leq 2C \ell_1(0) \cdots C \ell_q(0) \left( M + \frac{(1 + M)}{q\beta_{\ell_i} - 1} \left( \frac{1 + \varepsilon}{(1 + M)^{\beta_{\ell_i}}} \right)^q \right)
\]
for any \(\ell_1, \ldots, \ell_q\) such that \(\beta_{\ell_j} < 1\), \(\ell_j \geq 1\) for every \(j\).

**Lemma 4.13.** Let \(\varepsilon, M > 0\) be as in (36), and set \(U := U(\ell_1, \ldots, \ell_q) = \{ j \in \{1, \ldots, q\} : \ell_j = 0 \}\). If \(\beta_{\ell_i} \leq \beta_0\)
\[
k_{\ell_1, \ldots, \ell_q - \# U = 0}(T) \leq 2T C \ell_1(0) \cdots C_{\ell_q \# U} \text{C}(0) C_0(0)^{\# U} \times \left( M + (\varepsilon + 1)^q \int_{[M,T]} (1 + |\tau|)^{-(q)\beta_{\ell_i} + \beta_0} d\tau \right),
\]
otherwise if \(\beta_{\ell_i} > \beta_0\)
\[
k_{\ell_1, \ldots, \ell_q - \# U = 0}(T) \leq 2T C \ell_1(0) \cdots C_{\ell_q \# U} \text{C}(0) C_0(0)^{\# U} \left( M + (\varepsilon + 1)^q \int_{[M,T]} (1 + |\tau|)^{-q\beta_0} d\tau \right).
\]

We are now in the position to prove Proposition 4.2.

**Proof of Proposition 4.2.** Note first that
\[
\sum_{\ell_1, \ldots, \ell_q = 0}^{\infty} \left( \frac{2^q + 1}{4\pi} \right)^{\ell_1(0)} C_0(0) G_{\ell_1, \ldots, \ell_q}^{0, \ldots, 0} < +\infty,
\]
(60)
since the following estimate (see [24, § 4.2.1])
\[
G_{\ell_1, \ldots, \ell_q}^{0, \ldots, 0} \leq \sqrt{\frac{(2^q + 1)(2^q + 2) \cdots (2^q q - 1 + 1)}{(4\pi)^2 (2^q + 1)}}
\]
and (4) hold. Now, assume \(q\beta_{\ell_i} < 1\) and \(\beta_{\ell_i} \leq \beta_0\) and recall equation (56). Let \(J := \{(\ell_1, \ldots, \ell_q)\}: there is at least one index \(j \in \{1, \ldots, q\}\) such that \(\ell_j = 0\). Then, thanks to Lemma 4.13, we can apply Dominated Convergence Theorem and then Lemma 4.8 to get
\[
\lim_{T \to \infty} \sum_{(\ell_1, \ldots, \ell_q) \in J} \frac{k_{\ell_1, \ell_2, \ldots, \ell_q}(T)}{T^{2 - q\beta_{\ell_i}}} \left( \prod_{i=1}^{q} \sqrt{\frac{2^i + 1}{4\pi}} \right) G_{\ell_1, \ell_2, \ldots, \ell_q}^{0, \ldots, 0} = \begin{cases} 0 & \text{if } \beta_{\ell_i} < \beta_0, \\ \sum_{(\ell_1, \ldots, \ell_q) \in (I^*)^q \cap J} \frac{C_{\ell_1}(0) - C_{\ell_q}(0)}{(1 - q\beta_{\ell_i})(2 - q\beta_{\ell_i})} \left( \prod_{i=1}^{q} \sqrt{\frac{2^i + 1}{4\pi}} \right) G_{\ell_1, \ldots, \ell_q}^{0, \ldots, 0} & \text{if } \beta_{\ell_i} = \beta_0. \end{cases}
\]
(62)
Now, thanks to Lemma 4.9, Lemma 4.10, Lemma 4.13 and (60) together with Lemma 4.8 we have
\[
\lim_{T \to \infty} \sum_{(\ell_1, \ldots, \ell_q) \in (I^*)^q} \frac{k_{\ell_1, \ell_2, \ldots, \ell_q}(T)}{T^{2 - q\beta_{\ell_i}}} \left( \prod_{i=1}^{q} \sqrt{\frac{2^i + 1}{4\pi}} \right) G_{\ell_1, \ell_2, \ldots, \ell_q}^{0, \ldots, 0}
\]
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4.12 and Lemma 4.13, so that (64) is proved.

On the other hand, if we assume \( q \beta_0 > 1 \) and \( q \beta_{\ell^*} > 1 \), then obviously \( q \beta_{\ell} > 1 \) for all \( \ell \in \mathbb{N} \), and \( \beta_1 + \cdots + \beta_{q^*} > 1 \) for all \( \ell_1, \ldots, \ell_q \in \mathbb{N} \). Then, thanks to Lemma 4.8, Lemma 4.9, Lemma 4.12 and Lemma 4.13,

\[
\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T} = 4\pi \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\infty} \lim_{T \to \infty} \frac{k_{\ell_1 \ell_2 \ldots \ell_q}(T)}{T} \left( \prod_{i=1}^{q} \frac{2\ell_i + 1}{4\pi} \right) G_{\ell_1 \ell_2 \ldots \ell_q}^0 = 0,
\]

which concludes the proof. In particular, we have proved that the series on the right hand side of the previous formula converges. The remaining cases can be treated analogously.

\[
\square
\]
4.3.1 Proof of Lemma 4.8

Proof. This proof is similar to the one of Lemma 4.3. Consider $\varepsilon, M > 0$ as in (36). Then, using Remark 4.1, we have

$$k_{\ell_1, \ell_2, \ldots, \ell_q}(T) = \int_{[0,T]^2} C_{\ell_1}(t-s) C_{\ell_2}(t-s) \cdots C_{\ell_q}(t-s)dt ds$$

$$= 2T \int_0^M \left(1 - \frac{\tau}{T}\right) C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) d\tau + 2T \int_M^T \left(1 - \frac{\tau}{T}\right) C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) d\tau.$$  \hfill (65)

Now assume that $\beta_{\ell_1} + \cdots + \beta_{\ell_q} < 1$. For the first summand on the right hand side of (65) we have

$$\lim_{T \to \infty} \frac{2}{T^{1-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})}} \int_0^M \left(1 - \frac{\tau}{T}\right) |C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau)| d\tau \leq \lim_{T \to \infty} \frac{2}{T^{1-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})}} M = 0.$$  \hfill (66)

For the second summand on the right hand side of (65) we write

$$\int_M^T \left(1 - \frac{\tau}{T}\right) C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) d\tau$$

$$= C_{\ell_1}(0) \cdots C_{\ell_q}(0) \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})} \times$$

$$\times \sum_{k=1}^q \sum_{k_1, k_2, \ldots, k_q \in \{0,1\}} \left(\frac{G_{\ell_1}(\tau)}{C_{\ell_1}(0)} - 1\right)^{k_1} \cdots \left(\frac{G_{\ell_q}(\tau)}{C_{\ell_q}(0)} - 1\right)^{k_q} d\tau$$

$$+ C_{\ell_1}(0) \cdots C_{\ell_q}(0) \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})} d\tau.$$  \hfill (66)

For the first term on the right hand side of the previous equality it holds that

$$\lim_{T \to \infty} \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{T^{1-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})} \times$$

$$\times \sum_{k=1}^q \sum_{k_1, k_2, \ldots, k_q \in \{0,1\}} \left(\frac{G_{\ell_1}(\tau)}{C_{\ell_1}(0)} - 1\right)^{k_1} \cdots \left(\frac{G_{\ell_q}(\tau)}{C_{\ell_q}(0)} - 1\right)^{k_q} d\tau = 0.$$  \hfill (67)

Let us prove (67). Actually, for $\tau > M$ we have

$$\left| \sum_{k=1}^q \sum_{k_1, k_2, \ldots, k_q \in \{0,1\}} \left(\frac{G_{\ell_1}(\tau)}{C_{\ell_1}(0)} - 1\right)^{k_1} \cdots \left(\frac{G_{\ell_q}(\tau)}{C_{\ell_q}(0)} - 1\right)^{k_q} \right| \leq \sum_{k=1}^q \binom{q}{k} \varepsilon^k,$$

and (67) follows, $\varepsilon$ being arbitrary. On the other hand, for the second summand on the right hand side of (66),

$$\lim_{T \to \infty} \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{T^{1-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-(\beta_{\ell_1} + \cdots + \beta_{\ell_q})} d\tau$$

$$= \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{(1 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))(2 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))}.$$
Analogously, if $\beta_{\ell_1} + \cdots + \beta_{\ell_q} = 1$ 

$$\lim_{T \to \infty} \frac{k_{\ell_1, \ldots, \ell_q}(T)}{T \log T} = 2 C_{\ell_1}(0) \cdots C_{\ell_q}(0).$$

Otherwise, if $\beta_{\ell_1} + \cdots + \beta_{\ell_q} > 1$, it immediately follows from equation (65) that, as $T \to +\infty$, 

$$k_{\ell_1, \ldots, \ell_q}(T) = 2T \int_0^{+\infty} C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau + O(1).$$

Note that the limiting constant 

$$\int_{\mathbb{R}} C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau$$

in (59) is finite (see the proof of Proposition 4.2). 

\[ \square \]

### 5 Proofs of the main results

#### 5.1 Proof of Theorem 1

**Proof.** Recall (33). Assume first that $u \neq 0$ and $\beta_0 < \min(2\beta_{\ell_*}, 1)$. For the first chaotic projection, since $\beta_0 < 1$, from Lemma 4.2 we have 

$$\lim_{T \to \infty} \frac{\text{Var}(M_T(u)[1])}{T^{2-\beta_0}} = \frac{2\phi(u)^2 C_0(0)}{(1-\beta_0)(2-\beta_0)}. \quad (68)$$

Let $Q \in \{2, 3, \ldots\}$ be such that 

$$Q\beta_{\ell_*} > 1.$$ 

For $q \in \{2, 3, \ldots, Q-1\}$ we have, from Proposition 4.1 and Proposition 4.2, since $\beta_0 < 2\beta_{\ell_*}$, 

$$\lim_{T \to \infty} \sum_{q=2}^{Q-1} \frac{\text{Var}(M_T(u)[q])}{T^{2-\beta_0}} = \sum_{q=2}^{Q-1} \lim_{T \to \infty} \frac{\text{Var}(M_T(u)[q])}{T^{2-\beta_0}} = 0. \quad (69)$$

Let us now prove that 

$$\lim_{T \to \infty} \sum_{q=Q}^{+\infty} \frac{\text{Var}(M_T(u)[q])}{T^{2-\beta_0}} = 0. \quad (70)$$

Recall (56); thanks to (61) we can write for any $\ell_1, \ldots, \ell_q \geq 0$ 

$$\frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \left( \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) g^{\ell_1 \cdots \ell_q} \leq \frac{(4\pi)^2 H_{q-1}(u)^2 \phi(u)^2}{q!} \left( \prod_{i=1}^{q} \frac{2\ell_i + 1}{4\pi} \right) =: b_q(\ell_1, \ldots, \ell_q; u).$$

For $q \geq Q$ we have of course $q\beta_{\ell_*} > 1$. Let $\varepsilon, M > 0$ be as in (36). From Lemma 4.12 we have for $T > \max(1, M)$

$$\sum_{\ell_1, \ell_2, \ldots, \ell_q \geq 1, \atop \beta_{\ell_1}, \ldots, \beta_{\ell_q} < 1} b_q(\ell_1, \ldots, \ell_q; u) \frac{k_{\ell_1, \ldots, \ell_q}(T)}{T^{2-\beta_0}} \leq \sum_{\ell_1, \ell_2, \ldots, \ell_q \geq 1, \atop \beta_{\ell_1}, \ldots, \beta_{\ell_q} < 1} b_q(\ell_1, \ldots, \ell_q; u) \frac{k_{\ell_1, \ldots, \ell_q}(T)}{T}$$

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Gaussian process with mean expanded as follows

We will need the following well known result.

Theorem 5 ([12, 33])

5.2 Proof of Theorem 2

Consequently, since $M$ is a slowly varying function. Let $u ∈ ℝ$ be a real measurable mean-square continuous stationary Gaussian process with mean $𝔼 [ξ(t)]$ and covariance function $ρ(t−s) = ρ(|t−s|) = Cov(ξ(t), ξ(s))$. Moreover, assume that

$$ρ(t−s) = \frac{L(|t−s|)}{|t−s|^β}, \quad \text{with} \quad 0 < β < 1,$$

where $L$ is a slowly varying function. Let $F : ℝ → ℝ$ be a Borel function such that $𝔼 [F(N)^2] < +∞$, where $N$ is a standard Gaussian random variable. Then it is a well known fact that can be expanded as follows

$$F(ξ) = \sum_{k=0}^{∞} \frac{b_k}{k!} H_k(ξ), \quad \text{where} \quad b_k = \int_{ℝ} F(ξ) H_k(ξ) φ(ξ) dξ.$$
Assume there exists an integer \( r \), the so-called Hermitian rank, such that \( b_0 = b_1 = \cdots = b_{r-1} = 0 \) and \( b_r \neq 0 \). Then, if \( \beta \in (0, 1/r) \), we have that the finite-dimensional distributions of the random process

\[
X_T(s) = \frac{1}{T^{1-\beta r/2}L(T)^{\beta r/2}} \int_0^T [F(\xi(t)) - b_0] \, dt , \quad 0 \leq s \leq 1 ,
\]

converge weakly, as \( T \to \infty \), to the ones of the Rosenblatt process of order \( r \), that is

\[
X_\beta(s) := \frac{b_r}{r!} \int_{(\mathbb{R}^r)^{\gamma}} e^{(\lambda_1 + \cdots + \lambda_r)s} - 1 W(d\lambda_1) \cdots W(d\lambda_r) \left| \lambda_1 \cdots \lambda_r \right|^{(1-\beta)/2} \, \text{d}\lambda , \quad 0 \leq s \leq 1 ,
\]

where \( W \) is a complex Gaussian white noise.

**Proof of Theorem 2.** Recall that \( u \neq 0 \) and \( 2\beta_r < \min(\beta_0, 1) \). From Lemma 4.2 we have

\[
\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T^{2-2\beta_r}} = 0 .
\]

Moreover, thanks to Proposition 4.2, as for the proof of Theorem 1 (in particular (70)), we have, as \( T \to +\infty \),

\[
\lim_{T \to \infty} \frac{\sum_{q \geq 3} \text{Var}(\mathcal{M}_T(u)[q])}{T^{2-2\beta_r}} = 0 ,
\]

so that, recalling also Proposition 4.1,

\[
\mathcal{M}_T(u)[2] = \frac{J_2(u)}{2} \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \int_0^T H_2(a_{\ell m}(t)) \, dt ,
\]

it holds that

\[
\mathcal{M}_T(u)[2] \frac{T^{1-\beta_r}}{2} = \frac{J_2(u)}{2} \sum_{\ell \in \mathcal{I}} \sum_{m=-\ell}^\ell \int_0^T H_2(a_{\ell m}(t)) \, dt + o_P(1) .
\]

Moreover, since in \( L^2(\Omega) \) we have the following equality

\[
\mathcal{M}_T(u)[2] = \frac{J_2(u)}{2} \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \int_0^T H_2(a_{\ell m}(t)) \, dt ,
\]

it holds that

\[
\mathcal{M}_T(u)[2] \frac{T^{1-\beta_r}}{2} = \frac{J_2(u)}{2} \sum_{\ell \in \mathcal{I}} \sum_{m=-\ell}^\ell \int_0^T H_2(a_{\ell m}(t)) \, dt + o_P(1) .
\]

Indeed, reasoning exactly as in the Proposition 4.1, we have that

\[
\lim_{T \to \infty} E \left[ \left( \mathcal{M}_T(u)[2] \frac{T^{1-\beta_r}}{2} - \frac{1}{T^{1-\beta_r}} \sum_{\ell \in \mathcal{I}} \sum_{m=-\ell}^\ell \frac{J_2(u)}{2} \int_0^T H_2(a_{\ell m}(t)) \, dt \right)^2 \right] = \lim_{T \to \infty} \frac{J_2(u)^2}{2T^{2-2\beta_r}} \sum_{\ell \in \mathcal{I}} (2\ell + 1) \int_0^T \int_0^T C_\ell(t-s)^2 \, dt \, ds = 0 .
\]

From (74) and (76), in order to understand the asymptotic distribution of \( \mathcal{M}_T(u) \), it suffices to investigate the leading term on the right hand side of (76). Recall Condition 2, for \( \ell \in \mathcal{I}^* \) we have that

\[
C_\ell(\tau) = \frac{G_\ell(\tau)}{(1 + |\tau|)^{2\beta_r}} ,
\]

where in particular \( G_\ell \) is a slowly varying function. Hence, setting \( \xi(t) = a_{\ell m}(t) \), we automatically have that \( \rho = \rho_\ell = C_\ell, L = L_\ell = G_\ell \) and, as a consequence, that

\[
X_T^{\ell m} \asymp 1 \frac{1}{C_\ell(0)} T^{1-\beta_r} \int_0^T \frac{J_2(u)}{2} H_2(a_{\ell m}(t)) \, dt \overset{d}{\to} \frac{J_2(u)}{2\beta_r} X_m^{\beta_r} , \quad \text{as } T \to \infty ,
\]

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for all \( m = -\ell, \ldots, \ell \), where for each \( m \) \( X_{m; \beta^*} \) is a standard Rosenblatt random variable (16) of parameter \( \beta^* \). Moreover, since the \( X_{\ell,m} \) are all independent for each \( T \) we have that

\[
\tilde{M}_T(u) = \sqrt{T^{1-\beta^*}} \frac{T^2}{\text{Var}(M_T(u)[2])} \sum_{\ell \in \mathcal{I}^*} C_\ell(0) \sum_{m=-\ell}^{\ell} \frac{\int_0^T J_2(u) H_2(a_{\ell,m}(t)) \, dt}{C_\ell(0) T^{1-\beta^*}} + o_P(1)
\]

\[
\xrightarrow{d} \left( \frac{J_2(u)^2}{2} \sum_{\ell \in \mathcal{I}^*} \frac{(2\ell + 1) C_\ell(0)^2}{(1 - \beta^*)(1 - 2\beta^*)} \right)^{-1/2} \sum_{\ell \in \mathcal{I}^*} C_\ell(0) \sum_{m=-\ell}^{\ell} \frac{J_2(u)}{2a(\beta^*)} X_{m; \beta^*},
\]

where

\[
v^* = a(\beta^*)^2 \sum_{\ell \in \mathcal{I}^*} \frac{2(2\ell + 1) C_\ell(0)^2}{(1 - \beta^*)(1 - 2\beta^*)},
\]

and the proof is concluded.

5.3 Proof of Theorem 3

First of all assume that \( 3\beta^* < \min(1, \beta_0) \). Since we are in the case where \( u = 0 \), we have that all even chaotic projections vanish and hence that

\[
\text{Var}(M_T) = \text{Var}(M_T[1]) + \text{Var}(M_T[3]) + \sum_{q \geq 2} \text{Var}(M_T[2q + 1]).
\]

where we used the notation \( M_T(0) := M_T \). As a consequence, as in the proof of Theorem 1, we have

\[
\lim_{T \to \infty} \frac{\text{Var}(M_T)}{T^{2-3\beta^*}} = \lim_{T \to \infty} \frac{\text{Var}(M_T[1])}{T^{2-3\beta^*}} + \lim_{T \to \infty} \frac{\text{Var}(M_T[3])}{T^{2-3\beta^*}} + \sum_{q \geq 2} \lim_{T \to \infty} \frac{\text{Var}(M_T[2q + 1])}{T^{2-3\beta^*}}.
\]

Now,

\[
\lim_{T \to \infty} \frac{\text{Var}(M_T[1])}{T^{2-3\beta^*}} = 0;
\]

while from Proposition 4.2 we know that

\[
\lim_{T \to \infty} \frac{\text{Var}(M_T[3])}{T^{2-3\beta^*}} = \frac{2}{3!(1 - 3\beta^*)(2 - 3\beta^*)} \sum_{\ell_1, \ell_2, \ell_3 \in \mathcal{I}^*} \left( \prod_{i=1}^3 \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) G^{000}_{\ell_1, \ell_2, \ell_3} =: K_3.
\]

Moreover

\[
\lim_{T \to \infty} \sum_{q \geq 2} \frac{\text{Var}(M_T[2q + 1])}{T^{2-3\beta^*}} = 0,
\]

which of course implies that

\[
\tilde{M}_T(u) = \frac{M_T(u)[3]}{\sqrt{K_3 T^{1-\frac{3}{2}\beta^*}}} + o_P(1),
\]

as claimed.
5.4 Proof of Theorem 4

The result below is just [27, Theorem 6.3.1] restated for our framework as a lemma. Recall the definition of cumulants for a random variable [20, §4.3].

**Lemma 5.1.** Assume that the functional \( \widehat{M}_T(u) \) in (13) satisfies the following conditions:

(a) For each \( q \geq 1 \), \( \text{Var}(\widehat{M}_T(u)[q]) \to \sigma_q^2 \), as \( T \to \infty \) and for some \( \sigma_q^2 \geq 0 \); 

(b) \( \sigma^2 := \sum_{q=1}^{\infty} \sigma_q^2 < +\infty; \)

(c) For each \( q \geq 2 \), \( \text{Cum}_4(\widehat{M}_T(u)[q]) \to 0 \), as \( T \to \infty \);

(d) \( \lim_{Q \to \infty} \sup_{T > 0} \sum_{q=Q+1}^{\infty} \text{Var}(\widehat{M}_T(u)[q]) = 0. \)

Then \( \widehat{M}_T(u) \overset{d}{\to} Z \), as \( T \to \infty \), where \( Z \sim \mathcal{N}(0, \sigma^2) \).

We will use Lemma 5.1 to prove Theorem 4. Let us first focus on Condition (c).

**Proposition 5.1.** Assume \( \beta_0 = 1 \). If either \( u \neq 0 \) and \( 2\beta_* > 1 \) or \( u = 0 \) and \( 3\beta_* > 1 \) we have

\[
\widehat{M}_T(u)[q] \overset{d}{\to} Z, \quad \text{as} \quad T \to \infty,
\]

where \( Z \sim \mathcal{N}(0, \sigma_q^2) \) is a standard Gaussian random variable whose variance is given by

\[
\sigma_q^2 := \frac{s_q^2}{\sum_{k=1}^{+\infty} s_k^2} \in [0, +\infty),
\]

where the sequence \( \{s_k^2, k \geq 1\} \) is defined in Theorem 4.

**Remark 5.1.** Note that some of the chaoses might converge to a degenerate Gaussian (that is, with zero expected value and variance).

**Proof of Proposition 5.1.** It suffices to check [27, Theorem 5.2.7] that the fourth cumulant goes to zero as \( T \to +\infty \), i.e.

\[
\lim_{T \to \infty} \text{Cum}_4 \left( \widehat{M}_T(u)[q] \right) = 0.
\]

Recall (26). For any \( 1 \leq \alpha \leq q - 1 \), we have (see e.g. [20, §4.3], in particular §4.3.1)

\[
\text{Cum}_4 \left( \int_0^T \int_{\mathbb{S}^2} H_q(Z(x,t)) dx dt \right) = \int_{[0,T]^4} \int_{(\mathbb{S}^2)^4} dx_1 dx_2 dx_3 dx_4 dt_1 dt_2 dt_3 dt_4 
\]

\[
\times \text{Cum} \left( H_q(Z(x_1, t_1)), H_q(Z(x_2, t_2)) H_q(Z(x_3, t_3)) H_q(Z(x_4, t_4)) \right) 
\]

\[
\leq c \int_{[0,T]^4} \int_{(\mathbb{S}^2)^4} |\mathbb{E}[Z(x_1, t_1)Z(x_2, t_2)]|^\alpha |\mathbb{E}[Z(x_2, t_2)Z(x_3, t_3)]|^\alpha 
\times |\mathbb{E}[Z(x_3, t_3)Z(x_4, t_4)]|^\alpha |\mathbb{E}[Z(x_4, t_4)Z(x_1, t_1)]|^\alpha dx_1 dx_2 dx_3 dx_4 dt_1 dt_2 dt_3 dt_4.
\]

For \( x, y \) positive numbers, it holds that

\[
x^\alpha y^\beta \leq x^{\alpha+\beta} + y^{\alpha+\beta},
\]

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as a consequence,

\[
\text{Cum}_4 \left( \int_0^T \int_{\mathbb{R}^2} H_q(Z(x,t)) dx dt \right)
\]

\[
\leq c \int_{[0,T]^4} \left| \mathbb{E} [Z(x_1,t_1)Z(x_2,t_2)] \right|^{q-\alpha} \left| \mathbb{E} [Z(x_2,t_2)Z(x_3,t_3)] \right|^{\alpha}
\times \left| \mathbb{E} [Z(x_3,t_3)Z(x_4,t_4)] \right|^q \ dx_1 dx_2 dx_3 dx_4 dt_1 \ldots dt_4
\]

\[
= c \int_{[0,T]^4} \left| \Gamma(\langle x_1, x_2 \rangle, t_2 - t_1) \right|^{q-\alpha} \left| \Gamma(\langle x_2, x_3 \rangle, t_3 - t_2) \right|^{\alpha}
\times \left| \Gamma(\langle x_3, x_4 \rangle, t_4 - t_3) \right|^q \ dx_1 dx_2 dx_3 dx_4 dt_1 \ldots dt_4
\]

\[
\leq c T \int_{[0,T]^4} \int_{[-T,T]} \left| \Gamma(\langle x_1, x_2 \rangle, s_1) \right|^{q-\alpha} ds_1 \int_{[-T,T]} \left| \Gamma(\langle x_2, x_3 \rangle, s_2) \right|^{\alpha} ds_2
\times \int_{[-T,T]} \left| \Gamma(\langle x_3, x_4 \rangle, s_3) \right|^q \ ds_3 \ dx_1 dx_2 dx_3 dx_4
\]

where for the last inequality we used Jensen inequality, recalling (8). For \( k = 1, \ldots, q - 1 \) we have that, as \( T \to +\infty \),

\[
\int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(0) \left| \frac{C_\ell(s_1)}{C_\ell(0)} \right|^{q-\alpha} ds_1 \int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(0) \left| \frac{C_\ell(s_2)}{C_\ell(0)} \right|^{\alpha} ds_2
\times \int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(0) \left| \frac{C_\ell(s_3)}{C_\ell(0)} \right|^q ds_3,
\]

whereas for \( k = q \) (since \( q\beta_* > 1 \))

\[
\int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(0) \left| \frac{C_\ell(\tau)}{C_\ell(0)} \right|^k \ d\tau = O \left( T^{1-k\beta_*} (1 + 1_{k\beta_* = 1} \log T) \right)
\]

Hence, as \( T \to +\infty \),

\[
\text{Cum}_4 \left( \int_0^T \int_{\mathbb{R}^2} H_q(Z(x,t)) dx dt \right) = O \left( T^{3-q\beta_*} (1 + 1_{(q-\alpha)\beta_* = 1} \log T)(1 + 1_{\alpha\beta_* = 1} \log T) \right).
\]

From Proposition 4.2 we know that \( \text{Var}(M_T(u)) \sim T \sum_{k=1}^{+\infty} s_k^2 \) thus as \( T \to +\infty \)

\[
\text{Cum}_4 \left( \frac{M_T(u)[q]}{\sqrt{\text{Var}(M_T(u))}} \right) = O \left( T^{1-q\beta_*} (1 + 1_{(q-\alpha)\beta_* = 1} \log T)(1 + 1_{\alpha\beta_* = 1} \log T) \right)
\]

so that

\[
\lim_{T \to \infty} \text{Cum}_4 \left( \frac{M_T(u)[q]}{\sqrt{\text{Var}(M_T(u))}} \right) = 0
\]

and the proof is concluded. \( \square \)

We are now in the position to prove Theorem 4.
Proof of Theorem 4. Here we have $\beta_0 = 1$. From Lemma 4.2 and Condition 2 we know that

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} = \phi(u)^2 \int_{-\infty}^{+\infty} C_0(\tau) \, d\tau > 0.$$  

Assume first that $u \neq 0$ and $2\beta_\epsilon > 1$, then, using Propositions 4.1 and 4.2 we have that

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[2])}{T} = \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) F_\ell,$$

and for $q \geq 3$, since of course $q\beta_\epsilon > 1$,

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T} = s_q^2,$$

where we recall that

$$s_q^2 = \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\infty} g_{\ell_1, \ell_2, \ldots, \ell_q} \prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \int_{-\infty}^{+\infty} C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau.$$

As in the proof of (70), thanks to Dominated Convergence Theorem, we can write

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u))}{T} = \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} + \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[2])}{T} + \sum_{q \geq 3} \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T}$$

$$= \phi(u)^2 \int_{-\infty}^{+\infty} C_0(\tau) \, d\tau + \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) F_\ell + \sum_{q \geq 3} s_q^2.$$

Now assume that $u = 0$ and $3\beta_\epsilon > 1$, then analogously

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u))}{T} = \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} + \sum_{q = 1}^{\infty} \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[2q+1])}{T}$$

$$= \phi(u)^2 \int_{-\infty}^{+\infty} C_0(\tau) \, d\tau + \sum_{q = 1}^{\infty} s_{2q+1}^2.$$

In order to prove Convergence in distribution to a Gaussian random variable, we are going to check the four conditions of Lemma 5.1. Conditions (a) and (b) are verified thanks to Theorem 4 and Proposition 4.2. Condition (c) of Lemma 5.1 is immediately verified by Proposition 5.1, thanks to the Fourth Moment Theorem [29, Theorem 1]. Let us now check Condition (d). Recall that

$$\phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau > 0.$$

Fix $\varepsilon > 0$ such that

$$\phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau - \varepsilon > 0.$$

Then we have for some $T_\varepsilon > 0$

$$\frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} \geq \phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau - \varepsilon,$$

for every $T > T_\varepsilon$. Hence

$$\sup_{T > T_\varepsilon} \sum_{q=Q}^{\infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{\text{Var}(\mathcal{M}_T(u))} \leq \sup_{T > T_\varepsilon} \sum_{q=Q}^{\infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{\phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau - \varepsilon} \to 0,$$

as $Q \to \infty$. As a consequence, Condition (d) of Lemma 5.1 is satisfied and the proof is concluded. □
5.5 Proof of Proposition 3.1

Proof. From Theorem 2 we have

\[
\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u))}{T^{2-2\beta_t'^*}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_t'^*)(1-\beta_t'^*)} (2\ell'^* + 1) C_{\ell'^*}(0)^2. \tag{77}
\]

Let us study the variance of \(m_{T,t'^*}(u)\).

\[
\text{Var}(m_{T,t'^*}(u)) = \frac{u^2 \phi(u/\sigma_{t'^*})^2}{2\sigma_{t'^*}^2} \int_{[0,T]^2} \frac{C_{\ell'^*}(t-s)^2}{C_{\ell'^*}(0)^2} P_{\ell'^*}(\langle x,y \rangle)^2 \, dx \, dy \, dt \, ds
\]

\[
= \frac{u^2 \phi(u/\sigma_{t'^*})^2}{2\sigma_{t'^*}^2} \int_{[0,T]^2} \frac{(4\pi)^2}{2\ell'^* + 1} C_{\ell'^*}(0)^2 \, dt \, ds
\]

\[
= \frac{u^2 \phi(u/\sigma_{t'^*})^2}{2\sigma_{t'^*}^2} 2T \int_{[0,T]} \frac{(4\pi)^2}{2\ell'^* + 1} \left(1 - \frac{\tau}{T}\right) C_{\ell'^*}(0)^2 \, d\tau.
\]

From Proposition 4.3

\[
\lim_{T \to \infty} \frac{2T}{T^{2-2\beta_t'^*}} \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) C_{\ell'^*}(\tau) \, d\tau = \frac{C_{\ell'^*}(0)^2}{(1-\beta_t'^*)(1-2\beta_t'^*)}
\]

so that

\[
\lim_{T \to \infty} \frac{\text{Var}(m_{T,t'^*}(u))}{T^{2-2\beta_t'^*}} = \frac{u^2 \phi(u/\sigma_{t'^*})^2}{2\sigma_{t'^*}^2} \frac{(4\pi)^2}{2\ell'^* + 1} \frac{1}{(1-\beta_t'^*)(1-2\beta_t'^*)}.
\tag{78}
\]

Let us now compute the covariance between \(\mathcal{M}_T(u)\) and \(m_{T,t'^*}(u)\): by orthogonality of Wiener chaoses

\[
\text{Cov}(\mathcal{M}_T(u), m_{T,t'^*}(u)) = \text{Cov}(\mathcal{M}_T(u)[2], m_{T,t'^*}(u))
\]

\[
= \frac{J_2(u) J_2 \left( \frac{u}{\sigma_{t'^*}} \right)}{4} \int_{[0,T]^2} \int_{S^2 \times S^2} \mathbb{E} \left[ H_2(Z(x,t)) H_2 \left( \frac{Z_{\ell'^*}(y,s)}{\sigma_{t'^*}} \right) \right] \, dx \, dy \, dt \, ds
\]

\[
= \frac{J_2(u) J_2 \left( \frac{u}{\sigma_{t'^*}} \right)}{2} \int_{[0,T]^2} \int_{S^2 \times S^2} \mathbb{E} \left[ Z(x,t) \frac{Z_{\ell'^*}(y,s)}{\sigma_{t'^*}} \right]^2 \, dx \, dy \, dt \, ds
\]

\[
= \frac{J_2(u) J_2 \left( \frac{u}{\sigma_{t'^*}} \right)}{2 \sigma_{t'^*}^2} \int_{[0,T]^2} \int_{S^2 \times S^2} C_{\ell'^*}(t-s)^2 \left( 2\ell'^* + 1 \right)^2 \frac{1}{4\pi} \, dx \, dy \, dt \, ds
\]

\[
= (2\ell'^* + 1) \frac{J_2(u) J_2 \left( \frac{u}{\sigma_{t'^*}} \right)}{2 \sigma_{t'^*}^2} \int_{[0,T]^2} C_{\ell'^*}(t-s)^2 \, dt \, ds.
\]

As before

\[
\lim_{T \to \infty} \frac{\text{Cov}(\mathcal{M}_T(u), m_{T,t'^*}(u))}{T^{2-2\beta_t'^*}} = (2\ell'^* + 1) \frac{J_2(u) J_2 \left( \frac{u}{\sigma_{t'^*}} \right)}{2 \sigma_{t'^*}^2} \frac{C_{\ell'^*}(0)^2}{(1-\beta_t'^*)(1-2\beta_t'^*)}.
\tag{79}
\]

Plugging (77) and (78) into (79) we get

\[
\lim_{T \to \infty} \frac{\text{Cov}(\mathcal{M}_T(u), m_{T,t'^*}(u))}{\sqrt{\text{Var}(\mathcal{M}_T(u)) \text{Var}(m_{T,t'^*}(u))}} = 1,
\]

that concludes the proof.
A Proofs of technical Lemmas

From Remark 4.1

\[
\int_{[0,T]^2} C(t-s)^2 \, dt \, ds = 2T \int_{[0,T]} \left( 1 - \frac{\tau}{T} \right) C(\tau)^2 \, d\tau
\]

\[
\leq 2T \left( \int_{[0,M]} C(\tau)^2 \, d\tau + \int_{[M,T]} C(\tau)^2 \, d\tau \right)
\]

\[
\leq 2TC(0)^2 \left( M + (\varepsilon + 1)^2 \int_{[M,T]} g_\beta(\tau)^2 \, d\tau \right).
\]

Proof of Lemma 4.4. For $\beta = 1$ from (80) we have, for $T > \max(1, M),

\[
\frac{1}{T} \int_{[0,T]^2} C(t-s)^2 \, dt \, ds
\]

\[
\leq 2C(0)^2 \left( M + (\varepsilon + 1)^2 \int_{[M,T]} g_\beta(\tau) \, d\tau \right)
\]

\[
\leq 2C(0)^2 \left( M + (\varepsilon + 1)^2 \int_{R} |g_\beta(\tau)| \, d\tau \right) = 2C(0)^2 \left( M + \frac{(\varepsilon + 1)^2}{\alpha - 1} \right),
\]

where we used the fact that $|g_\beta(\tau)| \leq 1$ for every $\tau \in \mathbb{R}$, see (10). \qed

Proof of Lemma 4.5. Let us start with (45). For $\ell \in \mathcal{I}^*$, $\beta < 1$ we have

\[
\frac{1}{T^{2-2\beta_\ell}} \int_{[0,T]^2} C(t-s)^2 \, dt \, ds \leq \frac{2C(0)^2}{T^{1-2\beta_\ell}} \left( M + (\varepsilon + 1)^2 \int_{[M,T]} (1 + \tau)^{-2\beta_\ell} \, d\tau \right)
\]

\[
= \frac{2C(0)^2}{T^{1-2\beta_\ell}} \left( M + \frac{(\varepsilon + 1)^2}{-2\beta_\ell + 1} \left( (1 + T)^{-2\beta_\ell + 1} - (1 + M)^{-2\beta_\ell + 1} \right) \right)
\]

\[
\leq \frac{2C(0)^2}{T^{1-2\beta_\ell}} \left( M + \frac{(\varepsilon + 1)^2}{-2\beta_\ell + 1} (1 + T)^{-2\beta_\ell + 1} \right)
\]

\[
\leq 2C(0)^2 \left( M + \frac{(\varepsilon + 1)^2}{-2\beta_\ell + 1} \left( 1 + \frac{1}{M} \right)^{-2\beta_\ell + 1} \right),
\]

where for the last inequality we recall that $T > \max(1, M)$.

Let us now prove (46). From (80), for $\ell \notin \mathcal{I}^*$, $\ell \geq 1$ and $\beta < 1,

\[
\frac{1}{T^{2-2\beta_\ell}} \int_{[0,T]^2} C(t-s)^2 \, dt \, ds \leq \frac{2C(0)^2}{T^{1-2\beta_\ell}} \left( M + (\varepsilon + 1)^2 \int_{[M,T]} (1 + \tau)^{-2\beta_\ell} \, d\tau \right).
\]

Now for $2\beta_\ell = 1$ we have

\[
\frac{1}{T^{1-2\beta_\ell}} \int_{[M,T]} (1 + \tau)^{-2\beta_\ell} \, d\tau \leq 2m(\beta_\ell),
\]

while for $2\beta_\ell < 1$ we have

\[
\frac{1}{T^{1-2\beta_\ell}} \int_{[M,T]} (1 + \tau)^{-2\beta_\ell} \, d\tau \leq \frac{1}{-2\beta_\ell + 1} \left( 1 + \frac{1}{M} \right)^{-2\beta_\ell + 1},
\]

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We have, from (81), for \( T > \), which concludes the proof.

Finally for \((\ell_1, \ldots, \ell_q) \notin \mathcal{I}^*, \ell_j \geq 1, \beta_{\ell_j} < 1\) for every \( j \) and \( T > \max(1, M) \), from (81) (note that \( \beta_{\ell_1} + \ldots + \beta_{\ell_q} \geq \beta_{r^*} + (q - 1) \beta_{r^*} > q \beta_{r^*} \)), we have

\[
\frac{k_{\ell_1, \ldots, \ell_q}(T)}{T^{2-q\beta_{r^*}}} \leq \frac{2Cq_{\ell_1}(0) \ldots Cq_{\ell_q}(0)}{T^{1-q\beta_{r^*}}} \left( M + (\varepsilon + 1)^q \int_{[M,T]} (1 + \tau)^{-q\beta_{r^*} - 1} d\tau \right),
\]

which concludes the proof.

**Proof of Lemma 4.9.** The proof is similar to the proof of Lemma 4.4. Assume that there is at least one index \( j \in \{1, \ldots, q\} \) such that \( \beta_{\ell_j} = 1 \). Let \( U := U(\ell_1, \ldots, \ell_q) = \{ j \in \{1, \ldots, q\} : \beta_{\ell_j} = 1 \} \). We have, from (81), for \( T > \max(1, M) \),

\[
\frac{k_{\ell_1, \ldots, \ell_q}(T)}{T} \leq \frac{2Cq_{\ell_1}(0) \ldots Cq_{\ell_q}(0)}{T} \left( M + (\varepsilon + 1)^q \int_{[M,T]} (1 + \tau)^{-q\beta_{r^*} - 1} d\tau \right)
\]

which concludes the proof.

**Proof of Lemma 4.10.** The proof is similar to the proof of Lemma 4.5. For \( \ell_1, \ldots, \ell_q \in \mathcal{I}^* \), \( \beta_{\ell_j} < 1 \) for every \( j \) and \( T > \max(1, M) \), from (81), we have

\[
\frac{k_{\ell_1, \ldots, \ell_q}(T)}{T^{2-q\beta_{r^*}}} \leq \frac{2Cq_{\ell_1}(0) \ldots Cq_{\ell_q}(0)}{T^{1-q\beta_{r^*}}} \left( M + (\varepsilon + 1)^q \int_{[M,T]} (1 + \tau)^{-q\beta_{r^*} - 1} d\tau \right)
\]

Finally for \((\ell_1, \ldots, \ell_q) \notin \mathcal{I}^* \), \( \ell_j \geq 1, \beta_{\ell_j} < 1 \) for every \( j \) and \( T > \max(1, M) \), from (81) (note that \( \beta_{\ell_1} + \ldots + \beta_{\ell_q} \geq \beta_{r^*} + (q - 1) \beta_{r^*} > q \beta_{r^*} \)), we have

\[
\frac{k_{\ell_1, \ldots, \ell_q}(T)}{T^{2-q\beta_{r^*}}} \leq \frac{2Cq_{\ell_1}(0) \ldots Cq_{\ell_q}(0)}{T^{1-q\beta_{r^*}}} \left( M + (\varepsilon + 1)^q \int_{[M,T]} (1 + \tau)^{-q(\beta_{r^*} + (q - 1) \beta_{r^*}) - 1} d\tau \right)
\]
\[ \leq 2C_{\ell_1}(0) \ldots C_{\ell_q}(0) \left( M + (\varepsilon + 1)q \frac{1}{T^{1-q\beta_{\ell^*}}} \int_{[M,T]} (1 + \tau)^{-\left(\beta_{\ell^*} + (q-1)\beta_{\ell^*}\right)} \, d\tau \right). \]

Now if \( \beta_{\ell^*} + (q-1)\beta_{\ell^*} < 1 \) then
\[ \frac{1}{T^{1-q\beta_{\ell^*}}} \int_{[M,T]} (1 + \tau)^{-\left(\beta_{\ell^*} + (q-1)\beta_{\ell^*}\right)} \, d\tau \leq \frac{(1 + \frac{1}{M})^{-\left(\beta_{\ell^*} + (q-1)\beta_{\ell^*}\right)+1}}{-(\beta_{\ell^*} + (q-1)\beta_{\ell^*}) + 1}; \]

if \( \beta_{\ell^*} + (q-1)\beta_{\ell^*} = 1 \), then
\[ \frac{1}{T^{1-q\beta_{\ell^*}}} \int_{[M,T]} (1 + \tau)^{-1} \, d\tau \leq \frac{1}{T^{1-2\beta_{\ell^*}}} \int_{[M,T]} (1 + \tau)^{-1} \, d\tau \leq 2m(\beta_{\ell^*}), \]

where \( m(\beta_{\ell^*}) \) is a constant defined in Lemma 4.4. Finally if \( \beta_{\ell^*} + (q-1)\beta_{\ell^*} > 1 \) then
\[ \frac{1}{T^{1-q\beta_{\ell^*}}} \int_{[M,T]} (1 + \tau)^{-\left(\beta_{\ell^*} + (q-1)\beta_{\ell^*}\right)} \, d\tau \leq \frac{(M + 1)^{-\left(\beta_{\ell^*} + (q-1)\beta_{\ell^*} - 1\right)}}{\beta_{\ell^*} + (q-1)\beta_{\ell^*} - 1} \]

and the proof is concluded. \( \square \)

References

[1] R. J. Adler, J. E. Taylor (2007) Random Fields and Geometry. Springer Monographs in Mathematics, Springer.
[2] J.-M. Azéïs, Wschebor, M. (2009) Level Sets and Extrema of Random Processes and Fields. Wiley.
[3] C. Berg, E. Porcu (2017) From Schoenberg coefficients to Schoenberg functions. Constructive Approximation, 45, 2, 217–241.
[4] M. V. Berry (2002) Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. Journal of Physics A 35(13), 3025–3038.
[5] N. H. Bingham (1972) A Tauberian Theorem for Integral Transforms of Hankel Type, Journal of the London Mathematical Society, 2-5, 3, 493–503.
[6] N. H. Bingham, T. L. Symons (2019) Gaussian random fields on the sphere and sphere cross line. Stochastic Processes and their Applications.
[7] V. Cammarota (2019) Nodal area distribution for arithmetic random waves. Transactions of the American Mathematical Society 372, 5, 3539–3564.
[8] V. Cammarota, D. Marinucci (2019) On the Correlation of Critical Points and Angular Trispectrum for Random Spherical Harmonics. Preprint arXiv:1907.05810.
[9] V. Cammarota, D. Marinucci (2018) A Quantitative Central Limit Theorem for the Euler-Poincaré Characteristic of Random Spherical Eigenfunctions, Annals of Probability, 46, 6, 3188–3288.
[10] G. Chiles, Delfinier (2012) Geostatistics. Wiley Series in Probability and Statistics.
[11] G. Christakos (2005) Random Field Models in Earth Sciences. Academic Press.
[12] R. L. Dobrushin, P. Major (1979) Non-Central Limit Theorems for Non-Linear Functionals of Gaussian Fields. Z. Wahrscheinlichkeitstheorie verw. Gebiete 50, 27–52.
[13] H. Dehling and M. Taqqu (1989) The empirical process of some long-range dependent sequences with an application to U-statistics, Annals of Statistics 17, 4,1767–1783.
[14] C. G. Esseen (1942) On the Liapunoff limit of error in the theory of probability. Arkiv för Matematik, Astronomi och Fysik A28, 1–19.
[15] M. Krishnapur, P. Kurlberg, I. Wigman (2013) Nodal length fluctuations for arithmetic random waves. Annals of Mathematics 177, 2, 699–737.
[16] P. Imkeller, V. Pérez-Abreu, J. Vives (1995) Chaos expansions of double intersection local time of Brownian motion in $\mathbb{R}^q$ and renormalization. *Stochastic Processes and their Applications*, 56(1):1–34.

[17] N. Leonenko, A. Olenko (2013) Tauberian and Abelian Theorems for Long-range Dependent Random Fields. *Methodology and Computing in Applied Probability*, 15, 715–742.

[18] N. N. Leonenko, M. D. Ruiz-Medina (2018) Increasing domain asymptotics for the first Minkowski functional of spherical random fields. *Theory of Probability and Mathematical Statistics* 97, 127–149.

[19] C. Ma, A. Malyarenko (2020) Time-Varying Isotropic Vector Random Fields on Compact Two-Point Homogeneous Spaces *Journal of Theoretical Probability* 33, 319–339.

[20] D. Marinucci, G. Peccati (2011) *Random Fields on the Sphere: Representations, Limit Theorems and Cosmological Applications*, Cambridge University Press.

[21] D. Marinucci, G. Peccati, M. Rossi, I. Wigman (2016) Non-universality of nodal length distribution for arithmetic random waves. *Geometric and Functional Analysis* 26, 3, 926–960.

[22] D. Marinucci, M. Rossi (2015) Stein-Malliavin approximations for nonlinear functionals of random eigenfunctions on $S^d$. *Journal of Functional Analysis*, 268, 8, 2379–2420.

[23] D. Marinucci, M. Rossi, I. Wigman (2020) The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 56, 1, 374–390.

[24] D. Marinucci, S. Vadlamani (2016) High-frequency asymptotics of Lipschitz-Killing curvatures of excursion sets on the sphere. *The Annals of Applied Probability* 26, 1, 462–506.

[27] I. Nourdin, G. Peccati (2012) *Normal Approximations Using Malliavin Calculus: from Stein’s Method to Universality*, Cambridge University Press.

[28] I. Nourdin, G. Peccati, M. Rossi (2019) Nodal statistics of planar random waves. *Communications in Mathematical Physics* 369, 1, 99–151.

[29] D. Nualart and G. Peccati (2005) Central limit theorems for sequences of multiple stochastic integrals. *Annals of Probability* 33, 1, 177-193.

[30] G. Peccati, A. Vidotto (2020) Gaussian Random Measures Generated by Berry’s Nodal Sets. *Journal of Statistical Physics* 178, 4, 996–1027.

[31] M. Rossi (2019) Random nodal lengths and Wiener chaos, *Probabilistic Methods in Geometry, Topology and Spectral Theory*, Contemporary Mathematics Series, 739, 155–169.

[32] G. Szegő (1975) *Orthogonal polynomials*, vol. XXIII, 4th edn. American Mathematical Society, Providence, RI.

[33] M. Taqqu (1979) Convergence of Integrated Processes of Arbitrary Hermite Rank, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 50, 53–83.

[34] A. P. Todino (2019) A Quantitative Central Limit Theorem for the Excursion Area of Random Spherical Harmonics over Subdomains of $S^2$, *Journal of Mathematical Physics* 60, 023505.

[35] M. K. Veillette and M. S. Taqqu (2013) Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli* 19, 3, 982–1005.

[36] I. Wigman (2010) Fluctuations of the nodal length of random spherical harmonics, *Communications in Mathematical Physics*, 298, 3, 787–831.
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