A multiresolution algorithm to approximate the Hutchinson measure for IFS and GIFS

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Abstract

We introduce the discrete version of the Hutchinson–Barnsley theory providing algorithms to approximate the Hutchinson measure for iterated function systems (IFS) and generalized iterated function systems (GIFS) complementing the discrete version of the deterministic algorithm considered in our previous work [DOS] to generate attractors of both classical and fuzzy IFS and GIFS.

Key words and phrases: iterated function system (IFS), generalized iterated function system (GIFS), Markov operator, invariant measures, Hutchinson measures, attractor, discrete deterministic algorithm, discretization.

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1 Introduction

This work is a sequel of [DOS] where we approximate attractors for IFS and GIFS and its fuzzy versions by a discrete version of the deterministic algorithm.

Here we adapt the developed theory to find a discrete version of the Markov operator acting on probabilities. Once we prove that this operator satisfy Theorem 3.2 we guarantee the existence of a discrete measure approximating the Hutchinson Measure associated to the IFS or GIFS with probabilities. This is a very important problem addressed by several authors such as [O1], [CJ], et. al.

The iterative procedure can be coded in to a discrete deterministic algorithm for measures supported on the attractor.

The paper is organized as follows:

In Section 2 we recall some basic facts on the Hutchinson–Barnsley theory. After, in Section 3 and Section 4 we recall the theory developed in [DOS] to approximate fixed points by considering its projections on ε-nets and the discretization of sets.

In sections 5, 6 and 7 we recall IFS and the discrete theory to approximate fractal attractors, the Markov operator and its discretization introducing the concept of discrete Hutchinson measure of an IFS $S$ with resolution $\delta$. The main theorem of this part, which will be used to build an algorithm for IFS, is Theorem 7.3.
The analogous construction is made for GIFS in Section 8, Section 9, and Section 10, introducing the concept of discrete Hutchinson measure of a GIFS $S$ with resolution $\delta$. The main theorem of this part, which will be used to built algorithm for GIFS, is Theorem 10.2.

In Section 11 we introduce two algorithms for IFS and GIFS used to generate discrete Hutchinson measures and present some examples to illustrate their workings.

Finally in Section 12 we address the problem of estimating the integral of functions with respect to the Hutchinson measures from [CJ], the problem of the Projected Hutchinson measures from [EO] and the problem of IFS/GIFS with place dependent probabilities from [Mi] et. al.

2 Basics of the Hutchinson–Barnsley theory

Let $(X, d)$ be a metric space. We say that $f : X \to X$ is a Banach contraction, if the Lipschitz constant $\text{Lip}(f) < 1$. The classical Banach fixed point theorem states that each Banach contraction on a complete metric space has a unique fixed point $x_*$, and for every $x_0 \in X$, the sequence of iterates $(f^k(x_0))_{k=0}^\infty$ converges to $x_*$. 

**Definition 2.1.** An iterated function system (IFS in short) $S = (X, (\phi_j)_{j=1}^L)$ consists of a finite family $\phi_1, \ldots, \phi_L$ of continuous selfmaps of $X$. Each IFS $S$ generates the map $F_S : K^*(X) \to K^*(X)$ (where $K^*(X)$ denotes the family of all nonempty and compact subsets of $X$), called the Hutchinson operator, defined by

$$\forall K \in K^*(X) \quad F_S(K) := \bigcup_{j=1}^L \phi_j(K).$$

By the attractor of an IFS $S$ we mean the unique set $A_S \in K^*(X)$ which satisfies

$$A_S = F_S(A_S) = \bigcup_{j=1}^L \phi_j(A_S)$$

and such that for every $K \in K^*(X)$, the sequence of iterates $(F_S^k(K))_{k=0}^\infty$ converges to $A_S$ with respect to the Hausdorff metric $h$ on $K^*(X)$.

The classical Hutchinson–Barnsley theorem [Bar, Hut] states that each IFS $S$ consisting of Banach contractions on a complete metric space $X$ admits the attractor. This result can be proved with a help of the Banach fixed point theorem as it turns out that $F_S$ is a Banach contraction provided each $\phi_j$ is a Banach contraction.

**Lemma 2.2.** Let $(X, d)$ be a metric space and $S = (X, (\phi_j)_{j=1}^L)$ be an IFS consisting of Banach contractions. Then $F_S$ is a Banach contraction and $\text{Lip}(F_S) \leq \max\{\text{Lip}(\phi_j) : j = 1, \ldots, L\}$.

Given an IFS $S = (X, (\phi_j)_{j=1}^L)$ consisting of Banach contractions, we will denote

$$\alpha_S := \max\{\text{Lip}(\phi_1), \ldots, \text{Lip}(\phi_L)\}.$$

3 Discretization of fixed point theorems

**Definition 3.1.** A subset $\hat{X}$ of a metric space $(X, d)$ is called an $\varepsilon$-net of $X$, if for every $x \in X$, there is $y \in \hat{X}$ such that $d(x, y) \leq \varepsilon$.

A map $r : X \to \hat{X}$ such that $r(x) = x$ for $x \in \hat{X}$ and $d(x, r(x)) \leq \varepsilon$ for all $x \in X$ will be called an $\varepsilon$-projection of $X$ to $\hat{X}$.

For $f : X \to X$, by its $r$-discretization we will call the map $\hat{f} := (r \circ f)|_{\hat{X}}$

Clearly, for each $\varepsilon$-net $\hat{X}$ of $X$, an $\varepsilon$-projection exists, but it need not be unique. The following result can be considered as a discrete version of the Banach fixed point theorem. The proof can be founded in [DOS].

**Theorem 3.2.** Assume that $(X, d)$ is a complete metric space and $f : X \to X$ is a Banach contraction with the unique fixed point $x_*$ and a Lipschitz constant $\alpha$. Let $\varepsilon > 0$, $\hat{X}$ be an $\varepsilon$-net, $r : X \to \hat{X}$ be an
\(\varepsilon\)-projection and \(\hat{f}\) be an \(r\)-discretization of \(f\).

For every \(x \in \hat{X}\) and \(n \in \mathbb{N}\),

\[
\text{d}(\hat{f}^n(x), x^*) \leq \frac{5\varepsilon}{1 - \text{Lip}(f)} + \alpha^nd(x, x^*). \tag{1}
\]

In particular, there exists a point \(y \in X\) so that \(d(x^*, y) \leq \frac{6\varepsilon}{1 - \text{Lip}(f)}\) and which can be reached as an appropriate iteration of \(\hat{f}\) of an arbitrary point of \(\hat{X}\).

4 Discretization of sets

**Definition 4.1.** We say that an \(\varepsilon\)-net \(\hat{X}\) of a metric space \(X\) is proper, if for every bounded \(D \subset X\), the set \(D \cap \hat{X}\) is finite.

Note that proper \(\varepsilon\)-nets are discrete (as topological subspaces), but the converse need not be true. For example, there exists an infinite subset \(E\) of a unit sphere in an infinite dimensional normed space, so that \(|x - y| = 1\) for all \(x, y \in E, x \neq y\).

The existence of proper \(\varepsilon\)-nets for every \(\varepsilon > 0\) is guaranteed by the assumption that \(X\) has so-called Heine–Borel property, that is, the assumption that each closed and bounded set is compact. In particular, Euclidean spaces and compact spaces admit such nets.

Now assume that \((X, d)\) is a metric space and \(\hat{X}\) is a proper \(\varepsilon\)-net. Clearly, \(\mathcal{K}(\hat{X})\) consists of all finite subsets of \(\hat{X}\). Now if \(r : X \to \hat{X}\) is an \(\varepsilon\)-projection, then by the same letter \(r\) we will denote the map \(r : \mathcal{K}^*(X) \to \mathcal{K}(\hat{X})\) defined by \(r(K) = \{r(x) : x \in K\}\). Finally, let \(\hat{S} = (X, (\phi_j)_{j=1}^\delta)\) be an IFS.

It is straightforward to verify that:

**Lemma 4.2.** In the above frame,

(a) \(\mathcal{K}(\hat{X})\) is an \(\varepsilon\)-net of \(\mathcal{K}^*(X)\);
(b) \(r\) is an \(\varepsilon\)-projection of \(\mathcal{K}^*(X)\) to \(\mathcal{K}(\hat{X})\);
(c) \(F_{\hat{S}} = \hat{F}_S\), where \(\hat{F}_S = (r \circ F_S)_{|\mathcal{K}^*(X)}\) is the discretization of \(F_S\), and \(\hat{S} := (X, (\hat{\phi}_j)_{j=1}^\delta)\) and \(\hat{\phi}_j = (r \circ \phi_j)_{|\hat{X}}\) is the discretization of \(\phi_j\).

5 IFS

**Definition 5.1.** Given an IFS \(S\) on a metric space \(X\) with the attractor \(A_S\) and \(\delta > 0\), a set \(A_{\delta} \subset \mathcal{K}^*(X)\) will be called an attractor of \(S\) with resolution \(\delta\), if \(h(A_{\delta}, A_S) \leq \delta\).

**Lemma 2.2** Theorem 3.2 used for the Hutchinson operator and Lemma 4.2 imply the following “discrete” version of the Hutchinson–Barnsley theorem. The proof can be found in [DDS].

**Theorem 5.2.** Let \((X, d)\) be a complete metric space and \(S = (X, (\phi_j)_{j=1}^\delta)\) be an IFS on \(X\) consisting of Banach contractions. Let \(\varepsilon > 0\), \(\hat{X}\) be a proper \(\varepsilon\)-net, \(r : X \to \hat{X}\) be an \(\varepsilon\)-projection on \(\hat{X}\) and \(\hat{S} := (\hat{X}, (\hat{\phi}_j)_{j=1}^\delta), \) where \(\hat{\phi}_j = (r \circ \phi_j)_{|\hat{X}}\) is the discretization of \(\phi_j\).

For any \(K \in \mathcal{K}(\hat{X})\) and \(n \in \mathbb{N}\),

\[
h(F^n_{\hat{S}}(K), A_{\delta}) \leq \frac{5\varepsilon}{1 - \alpha_{\delta}} + \alpha_{\delta}^n h(K, A_S), \tag{2}
\]

where \(A_{\delta}\) is the attractor of \(S\).

In particular, there is \(n_0 \in \mathbb{N}\) such that for every \(n \geq n_0\), \(F^n_{\hat{S}}(K)\) is an attractor of \(S\) with resolution \(\frac{6\varepsilon}{1 - \alpha_{\delta}}\).

Let us explain the thesis. Starting with an IFS \(S = (X, (\phi_j)_{j=1}^\delta)\) consisting of Banach contractions, we switch to \(\hat{S} := (\hat{X}, (\hat{\phi}_j)_{j=1}^\delta)\), which is the IFS consisting of discretizations of maps from \(S\) to \(\varepsilon\)-net \(\hat{X}\). Then, picking any \(K \in \mathcal{K}^*(\hat{X})\) (that is, any finite subset of \(\hat{X}\)), it turns out that the sequence of iterates \((F^n_{\hat{S}}(K))_{n=0}^\infty\) (of the Hutchinson operator \(F_{\hat{S}}\) adjusted to \(\hat{S}\)) gives an approximation to the attractor \(A_{\delta}\) of \(S\) with resolution \(\frac{6\varepsilon}{1 - \alpha_{\delta}}\).
6 Markov operator for IFSp

Let $\mathcal{S} = (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L)$ be a Lipschitz contractive IFS with probabilities, i.e. $\sum_{j=1}^L p_j = 1$.

Recall that $\mathcal{P}(X)$ is the set of all compactly supported measures of probability on $(X, d)$ with respect to the Borel sigma algebra.

We introduce the Markov operator $M_\mathcal{S} : \mathcal{P}(X) \to \mathcal{P}(X)$ given by

$$\nu(B) = (M_\mathcal{S}\mu)(B) = \sum_{j=1}^N p_j \mu(\phi_j^{-1}(B)), \quad \text{for all measurable sets } B \subseteq X.$$  \hfill (3)

for all measurable functions $f : X \to \mathbb{R}$. Alternatively, we can define $M_\mathcal{S}$ by duality as being the linear functional acting on continuous functions given by

$$\int_X f(x) d(M_\mathcal{S}\mu)(x) = \sum_{j=1}^N p_j \int_X f(\phi_j(x)) d\mu(x), \quad \text{for any continuous } f : X \to \mathbb{R}. \hfill (4)$$

7 Discretization of the Markov Operator for IFSs

Let $(X, d)$ be a complete metric space and $\mathcal{S} = (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L)$ be an IFS with probabilities on $X$ consisting of Banach contractions. Consider $\varepsilon > 0$, $\hat{X}$ a proper $\varepsilon$-net, $r : X \to \hat{X}$ a measurable $\varepsilon$-projection on $\hat{X}$ and $\hat{\mathcal{S}} := (\hat{X}, (\hat{\phi}_j)_{j=1}^L, (p_j)_{j=1}^L)$, where $\hat{\phi}_j = (r \circ \phi_j)|_{\hat{X}}$ is the discretization of $\phi_j$.

We recall that the push-forward map $\pi^*$ is a correspondence that associate to any measure $\nu \in \mathcal{P}(X)$ a new measure $\pi^*(\nu) \in \mathcal{P}(\hat{X})$ given by

$$\int_{\hat{X}} f(x) d\pi^*(\nu)(x) := \int_X f(r(y)) d\nu(y) \hfill (6)$$

for any continuous $f : \hat{X} \to \mathbb{R}$.

It is obvious that the support of $\pi^*(\nu)$ is finite because $\text{supp}(\nu)$ is compact and $\hat{X}$ is a proper $\varepsilon$-net. In particular,

$$\pi^*(\mathcal{P}(X)) = \mathcal{P}(\hat{X}).$$

Another characterization of $\pi^*$ is

$$\pi^*(\nu)(B) = \nu(B) = \nu(r^{-1}(B)), \quad \text{for all measurable sets } B \subseteq \hat{X}. \hfill (7)$$

We introduce the discrete Markov operator $M_{\hat{\mathcal{S}}} : \mathcal{P}(\hat{X}) \to \mathcal{P}(\hat{X})$ given by

$$(M_{\hat{\mathcal{S}}}\mu)(B) = \sum_{j=1}^N p_j \mu(\hat{\phi}_j^{-1}(B)), \quad \text{for all measurable sets } B \subseteq \hat{X}. \hfill (8)$$

Alternatively, we can define $M_{\hat{\mathcal{S}}}$ by duality as being the linear functional acting on continuous functions given by

$$\int_{\hat{X}} f(x) d(M_{\hat{\mathcal{S}}}\mu)(x) = \sum_{j=1}^N p_j \int_{\hat{X}} f(\hat{\phi}_j(x)) d\mu(x) \hfill (9)$$
for any continuous \( f : \hat{X} \to \mathbb{R} \).

For the next result we need to introduce a new hypothesis:

**Definition 7.1.** We said that an \( \varepsilon \)-net \( \hat{X} \) is regular with respect to a \( \varepsilon \)-projection \( r \) if the set

\[
\{ y \in \hat{X} \mid r^{-1}(y) \cap K \neq \varnothing \}
\]

is finite, for any compact \( K \subseteq X \).

From now on, we assume that our \( \varepsilon \)-net is always regular.

Consider the family \( \Omega \subseteq 2^X \) defined by

\[
\Omega := \{ r^{-1}(y) \mid y \in \hat{X} \}. \tag{10}
\]

This family is obviously a measurable partition of \( X \). Moreover, for any \( y \in \hat{X} \) we have \( r^{-1}(y) \subseteq B_\varepsilon(y) \) where \( B_\varepsilon(y) := \{ z \in X \mid d(z,y) < \varepsilon \} \) because \( z \in r^{-1}(y) \) implies \( r(z) = y \) and by definition \( d(z,y) \leq \varepsilon \).

**Lemma 7.2.** Considering the natural embedding of \( \mathcal{P}(\hat{X}) \subseteq \mathcal{P}(X) \), given by \( e^\varepsilon \), where \( e : \hat{X} \to X \) is the inclusion, we claim that

a) \( \mathcal{P}(\hat{X}) \) is a proper \( \varepsilon \)-net of \( \mathcal{P}(X) \);

b) \( r^\varepsilon \) is a \( \varepsilon \)-projection;

c) \( (M_\varepsilon)|_{\mathcal{P}(\hat{X})} = M_\varepsilon \circ e^\varepsilon \) and \( r^\varepsilon(M_\varepsilon \circ e^\varepsilon) = M_\varepsilon \) is the \( r^\varepsilon \)-discretization of \( M_\varepsilon \).

**Proof.** (a) We know that \( \mathcal{P}(\hat{X}) := \{ \sum_{i=1}^m a_i \delta_{y_i} \mid y_i \in \hat{X}, \sum_{i=1}^m a_i = 1, m < \infty \} \). From this, we conclude that

\[
\Omega_\mu := \{ r^{-1}(y) \cap K \mid y \in \hat{X} \},
\]

where \( K := \text{supp}(\mu) \) and \( \mu \in \mathcal{P}(X) \), is also a measurable partition of the support of \( \mu \).

Consider the set \( \hat{K} := \{ y \in \hat{X} \mid r^{-1}(y) \cap K \neq \varnothing \} \), we know that this set is finite, nominally \( \hat{K} = \{ y_1, ..., y_m \} \). Then we introduce the measure \( \nu \in \mathcal{P}(\hat{X}) \) by

\[
\nu := \sum_{i=1}^m \mu(r^{-1}(y_i) \cap K) \delta_{y_i}. \tag{11}
\]

We claim that \( d_H(\mu, \nu) \leq \varepsilon \). To see that consider \( f \in \text{Lip}_1(X, \mathbb{R}) \) then

\[
\left| \int_X f(x)d\mu(x) - \int_X f(x)d\nu(x) \right| \leq \left| \sum_{i=1}^m \int_{r^{-1}(y_i) \cap K} f(x)d\mu(x) - \sum_{i=1}^m \mu(r^{-1}(y_i) \cap K) f(y_i) \right| \leq \sum_{i=1}^m \int_{r^{-1}(y_i) \cap K} |f(x) - f(y_i)|d\mu(x) \leq \varepsilon
\]

because \( f \in \text{Lip}_1(X, \mathbb{R}) \) and \( r^{-1}(y) \subseteq B_\varepsilon(y) \).

(b) We need to show that \( d_H(\mu, r^\varepsilon(\mu)) \leq \varepsilon \) for any \( \mu \in \mathcal{P}(X) \). Recall that \( r^\varepsilon(\mu)(B) = \mu(r^{-1}(B)) \) for any measurable \( B \subseteq \hat{X} \). By a similar construction as we made in (a), we can prove that fact.

(c) Consider \( \nu \in \mathcal{P}(\hat{X}) \) and \( f \in C^0(\hat{X}, \mathbb{R}) \). Then,

\[
r^\varepsilon(M_\varepsilon \circ e^\varepsilon)(f) = \int_X f(r(x))d(M_\varepsilon \circ e^\varepsilon)(x) = \sum_{j=1}^L p_j \int_X f(r(\phi_j(x)))d(\varepsilon^\varepsilon)(x) =
\]

\[
= \sum_{j=1}^L p_j \int_{\hat{X}} f(r(\phi_j(e(y))))d\nu(y) = \sum_{j=1}^L p_j \int_{\hat{X}} f(\phi_j(e(y)))d\nu(y) = (M_\varepsilon(\nu))(f).
\]

A measure \( \nu \in \mathcal{P}(\hat{X}) \) is called a discrete Hutchinson measure with resolution \( \delta \) if \( d_H(\nu, \mu_\varepsilon) \leq \delta \).
**Theorem 7.3.** Let \((X, d)\) be a complete metric space and \(S = (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L)\) be an IFS with probabilities on \(X\) consisting of Banach contractions. Let \(\varepsilon > 0\), \(\hat{X}\) be a proper \(\varepsilon\)-net, \(r : X \to \hat{X}\) be a proper measurable \(\varepsilon\)-projection on \(\hat{X}\) and \(\hat{S} := (\hat{X}, (\hat{\phi}_j)_{j=1}^L, (\hat{p}_j)_{j=1}^L)\), where \(\hat{\phi}_j = (r \circ \phi_j)\vert_{\hat{X}}\) is the discretization of \(\phi_j\).

For any \(\nu \in \mathcal{P}(\hat{X})\) and \(n \in \mathbb{N}\),

\[
d_H(M^n_\hat{S}(\nu), \mu_\hat{S}) \leq \frac{5\varepsilon}{1 - \alpha_\hat{S}} + a_\hat{S}^n d_H(\nu, \mu_\hat{S}),
\]

where \(\mu_\hat{S}\) is the attractor of \(M_\hat{S}\) (i.e., the Hutchinson measure).

In particular, for every \(\delta > \frac{5\varepsilon}{1 - \alpha_\hat{S}}\), there is \(n_0 \in \mathbb{N}\) such that for every \(n \geq n_0\), \(M^n_\hat{S}(\nu)\) is a discrete Hutchinson measure of \(\hat{S}\) with resolution \(\delta\).

## 8 Generalized IFSs

We first recall some basics of a generalization of the classical IFS theory introduced by R. Miculescu and A. Mihail in 2008. For references, see \[M1\], [MM], [SS1] and references therein.

If \((X, d)\) is a metric space and \(m \in \mathbb{N}\), then by \(X^m\) we denote the Cartesian product of \(m\) copies of \(X\).

We consider it as a metric space with the maximum metric

\[
d_m((x_0, \ldots, x_{m-1}),(y_0, \ldots, y_{m-1})) := \max\{d(x_0, y_0), \ldots, d(x_{m-1}, y_{m-1})\}.
\]

A map \(f : X^m \to X\) is called a generalized Banach contraction, if \(\text{Lip}(f) < 1\).

It turns out that a counterpart of the Banach fixed point theorem holds. Namely, if \(f : X^m \to X\) is a generalized Banach contraction, then there is a unique point \(x_* \in X\) (called a generalized fixed point of \(f\)), such that \(f(x_*, \ldots, x_*) = x_*\). Moreover, for every \(x_0, \ldots, x_{m-1} \in X\), the sequence \((x_k)\) defined by

\[
x_{k+m} = f(x_k, \ldots, x_{k+m-1}), \quad k \geq 0,
\]

converges to \(x_*\).

This result can be used to prove a counterpart of the Hutchinson–Barnsley theorem.

**Definition 8.1.** A **generalized iterated function system of order** \(m\) (GIFS in short) \(S = (X, (\phi_j)_{j=1}^L)\) consists of a finite family \(\phi_1, \ldots, \phi_L\) of continuous maps from \(X^m\) to \(X\). Each GIFS \(S\) generates the map \(F_S : \mathcal{K}^*(X)^m \to \mathcal{K}^*(X)\), called the **generalized Hutchinson operator** (GH), defined by

\[
\forall K_0, \ldots, K_{m-1} \in \mathcal{K}^*(X) \quad F_S(K_0, \ldots, K_{m-1}) := \bigcup_{j=1}^L \phi_j(K_0 \times \ldots \times K_{m-1}).
\]

By the **attractor** of a GIFS \(S\) we mean the unique set \(A_S \in \mathcal{K}^*(X)\) which satisfies

\[
A_S = F_S(A_S, \ldots, A_S) = \bigcup_{j=1}^L \phi_j(A_S \times \ldots \times A_S)
\]

and such that for every \(K_0, \ldots, K_m \in \mathcal{K}^*(X)\), the sequence \((K_k)\) defined by

\[
K_{k+m} := F_S(K_k, \ldots, K_{k+m-1}), \quad k \geq 0,
\]

converges to \(A_S\).

The following lemma is known:

**Lemma 8.2.** Let \((X, d)\) be a metric space and \(S = (X, (\phi_j)_{j=1}^L)\) be a GIFS consisting of generalized Banach contractions. Then \(F_S\) is a generalized Banach contraction with \(\text{Lip}(F_S) \leq \max\{\text{Lip}(\phi_j) : j = 1, \ldots, L\}\).

From now on, given a GIFS \(S\) consisting of generalized Banach contractions, we denote \(\alpha_S := \max\{\text{Lip}(\phi_j) : j = 1, \ldots, L\}\).
From the perspective of the algorithms presented later, it is worth to consider also a slight different approach. Namely, given a GIFS \( S = (X, (\phi_j)_{j=1}^L) \), define the map \( \mathcal{F}_S : K^*(X) \rightarrow K^*(X) \) by

\[
\forall K \in K^*(X) \quad \mathcal{F}_S(K) := F_S(K, \ldots, K) = \bigcup_{j=1}^L \phi_j(K \times \ldots \times K).
\]

Lemma 8.3 implies the following:

**Lemma 8.3.** Let \( (X, d) \) be a metric space and \( S = (X, (\phi_j)_{j=1}^L) \) be a GIFS consisting of generalized Banach contractions. Then \( \mathcal{F}_S \) is a Banach contraction with \( \operatorname{Lip}(\mathcal{F}_S) \leq \alpha_S \).

Clearly, in the above frame, if additionally \( X \) is complete, then the fixed point of \( \mathcal{F}_S \) is the attractor of \( S \).

**Definition 8.4.** Given a GIFS \( S \) on a metric space \( X \) with the attractor \( A_S \) and \( \delta > 0 \), a set \( A_S \in K^*(X) \) will be called an attractor of \( S \) with resolution \( \delta \), if \( h(A_S, A_S) \leq \delta \).

It can be easily proved (similarly as Lemma 1.2(c)) that for a GIFS \( S = (X, (\phi_j)_{j=1}^L, (\rho_j)_{j=1}^L) \), an \( \varepsilon \)-net \( \hat{X} \) and an \( \varepsilon \)-projection \( r : X \rightarrow \hat{X} \), the discretization \( (r \circ \mathcal{F}_S)_{|\hat{X}} \) equals the operator \( \mathcal{F}_{\hat{S}} \) adjusted to the GIFS \( \hat{S} = (\hat{X}, (\hat{\phi}_j)_{j=1}^L, (\hat{\rho}_j)_{j=1}^L) \) consisting of discretizations \( \hat{\phi}_j := (r_m \circ \phi_j)_{|\hat{X}} \), where \( r_m(x_1, \ldots, x_m) = (r(x_1), \ldots, r(x_m)) \) is the natural projection of \( X^m \) to \( \hat{X}^m \). Hence Theorem 8.5 imply the following “discrete” version of the Hutchinson–Barnsley theorem for GIFSs. The proof may be found in [DOS].

**Theorem 8.5.** Let \( (X, d) \) be a complete metric space and \( S = (X, (\phi_j)_{j=1}^L) \) be a GIFS on \( X \) consisting of generalized Banach contractions. Let \( \varepsilon > 0 \), \( \hat{X} \) be a proper \( \varepsilon \)-net, \( r : X \rightarrow \hat{X} \) be an \( \varepsilon \)-projection on \( X \) and \( S = (X, (\phi_j)_{j=1}^L, (\rho_j)_{j=1}^L) \), where \( \hat{\phi}_j := (r_m \circ \phi_j)_{|\hat{X}} \) for \( j = 1, \ldots, L \).

Then for any \( K \in K^*(\hat{X}) \) and \( n \in \mathbb{N} \),

\[
d_\infty(\mathcal{F}_{\hat{S}}^n(K), A_S) \leq \frac{5\varepsilon}{1 - \alpha_S} + \alpha_S^nd_\infty(K, A_S),
\]

where \( A_S \) is the attractor of \( S \).

In particular, there is \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \), \( \mathcal{F}_{\hat{S}}^n(K) \) is an attractor of \( S \) with the resolution \( \frac{6\varepsilon}{1 - \alpha_S} \).

### 9 Markov operator for GIFSs

A generalized iterated function system, of order \( m \), with probabilities \( (\text{GIFSp in short}) \vec{S} = (X, (\phi_j)_{j=1}^L, (\rho_j)_{j=1}^L) \) consists of a finite family \( \phi_1, \ldots, \phi_L \) of continuous maps from \( X^m \) to \( X \) and a system of probabilities, i.e. \( \sum_{j=1}^L \rho_j = 1 \).

Recall that \( \mathcal{P}(X) \) is the set of all compactly supported measures of probability on \( (X, d) \) with respect to the Borel sigma algebra.

We introduce the Markov operator \( M_S : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) given by duality as being the linear functional acting on continuous functions given by

\[
\int_X f(x)d(M_S(\mu, \nu))(x) = \sum_{j=1}^N \rho_j \int_X f(\phi_j(x, y))d\mu(x)d\nu(y),
\]

for any continuous \( f : X \rightarrow \mathbb{R} \).

We consider the distance in \( \mathcal{P}(X) \times \mathcal{P}(X) \) given by

\[
d_\infty((\mu, \nu), (\mu', \nu')) = \max\{d_H(\mu, \mu'), d_H(\nu, \nu')\}.
\]

Considering that \( M_S \) is a contraction in the space \( \mathcal{P}(X), d_H \) we have that, by the Banach contraction principle, there exists a unique \( \mu_S \) such that \( M_S(\mu_S, \mu_S) = \mu_S \). Moreover, for any \( \nu_0, \nu_1 \in \mathcal{P}(X) \) the sequence \( \nu_{i+2} := M_S(\nu_{i+1}, \nu_i) \) converges to \( \mu_S \) (meaning that \( d_H(\nu_i, \mu_S) \rightarrow 0 \) when \( i \rightarrow \infty \)).
Following [M2] we introduce a simpler operator $\overline{M}_S : \mathcal{P}(X) \to \mathcal{P}(X)$ given by
\[
\overline{M}_S(\nu) := M_S(\nu, \nu).
\]
(15)

This operator is also a contraction and $\mu_S$ is its only fixed point. Moreover, for any $\nu \in \mathcal{P}(X)$ the sequence $\nu_i := \overline{M}_S(\nu)$ converges to $\mu_S$ (meaning that $d_H(\nu_i, \mu_S) \to 0$ when $i \to \infty$).

10 Discretization of the Markov Operator for GIFSs

Let $(X, d)$ be a complete metric space and $\mathcal{S} = (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L)$ be a GIFS with probabilities on $X$ consisting of Banach contractions. Consider $\varepsilon > 0$, $\hat{X}$ a proper $\varepsilon$-net, $r : X \to \hat{X}$ a measurable $\varepsilon$-projection on $X$ and $\hat{\mathcal{S}} := (\hat{X}, (\hat{\phi}_j)_{j=1}^L, (\hat{p}_j)_{j=1}^L)$, where $\hat{\phi}_j = (r \circ \phi_j)_{|\hat{X}}$ is the discretization of $\phi_j$.

We recall that the push-forward map $r^\sharp$ is a correspondence that associate to any measure $\nu \in \mathcal{P}(X)$ a new measure $\hat{\nu} := r^\sharp(\nu) \in \mathcal{P}(\hat{X})$ given by
\[
\int_{\hat{X}} f(x) d\hat{\nu}(x) := \int_X f(r(y)) d\nu(y)
\]
(16)
for any continuous $f : \hat{X} \to \mathbb{R}$.

It is obvious that the support of $\hat{\nu}$ is finite because $\text{supp}(\nu)$ is compact and $\hat{X}$ is a proper $\varepsilon$-net. In particular,
\[
r^\sharp(\mathcal{P}(X)) = \mathcal{P}(\hat{X}).
\]

Another characterization of $r^\sharp$ is
\[
r^\sharp(\nu)(B) = \hat{\nu}(B) = \nu \left( r^{-1}(B) \right)
\]
(17)
for all measurable sets $B \subseteq \hat{X}$.

We introduce the discrete Markov operator $\overline{M}_\hat{\mathcal{S}} : \mathcal{P}(\hat{X}) \to \mathcal{P}(\hat{X})$ by duality as being the linear functional acting on continuous functions given by
\[
\int_{\hat{X}} f(z) d\left( \overline{M}_\hat{\mathcal{S}}(\mu) \right)(z) = \sum_{j=1}^N \hat{p}_j \int_{\hat{X}} f(\hat{\phi}_j(x,y)) d\mu(x) d\mu(y),
\]
(18)
for any continuous $f : \hat{X} \to \mathbb{R}$.

We recall that an $\varepsilon$-net $\hat{X}$ is regular with respect to a $\varepsilon$-projection $r$ if the set
\[
\{ y \in \hat{X} \mid r^{-1}(y) \cap K \neq \emptyset \}
\]
is finite, for any compact $K \subseteq X$. From now on, we assume that our $\varepsilon$-net is always regular.

Consider the family $\Omega \subseteq 2^X$ defined by $\Omega := \{ r^{-1}(y) \mid y \in \hat{X} \}$. This family is obviously a measurable partition of $X$. Moreover, for any $y \in \hat{X}$ we have $r^{-1}(y) \subseteq B_\varepsilon(y)$ where $B_\varepsilon(y) := \{ z \in X \mid d(z,y) < \varepsilon \}$ because $z \in r^{-1}(y)$ implies that $r(z) = y$ and by definition $d(z,y) \leq \varepsilon$.

**Lemma 10.1.** Considering the natural embedding of $\mathcal{P}(\hat{X}) \subseteq \mathcal{P}(X)$, given by $e^\sharp$, where $e : \hat{X} \to X$ is the inclusion, we claim that
\begin{enumerate}
  \item $\mathcal{P}(\hat{X})$ is a proper $\varepsilon$-net of $\mathcal{P}(X)$;
  \item $r^\sharp$ is a $\varepsilon$-projection;
  \item $\left( \overline{M}_\hat{\mathcal{S}} \right)|_{\mathcal{P}(\hat{X})} = \overline{M}_\mathcal{S} \circ e^\sharp$ and $r^\sharp(\overline{M}_\hat{\mathcal{S}} \circ e^\sharp) = \overline{M}_S$ is the $r^\sharp$-discretization of $\overline{M}_S$.
\end{enumerate}

**Proof.** (a) We know that $\mathcal{P}(\hat{X}) := \left\{ \sum_{i=1}^m a_i \delta_{y_i} \mid y_i \in \hat{X}, \sum_{i=1}^m a_i = 1, \ m < \infty \right\}$. From this, we conclude that
\[
\Omega_{\mu} := \{ r^{-1}(y) \cap K \mid y \in \hat{X} \},
\]
where $K := \text{supp}(\mu)$ and $\mu \in \mathcal{P}(X)$, is also a measurable partition of the support of $\mu$. 

Consider the set \( \hat{K} := \{ y \in \hat{X} \mid r^{-1}(y) \cap K \neq \emptyset \} \); we know that this set is finite, nominally \( \hat{K} = \{ y_1, \ldots, y_n \} \). Then we introduce the measure \( \nu \in \mathcal{P}(\hat{X}) \) by

\[
\nu := \sum_{i=1}^{m} \mu(r^{-1}(y_i) \cap K) \delta_{y_i}.
\]

We claim that \( d_H(\mu, \nu) \leq \varepsilon \). To see that consider \( f \in \text{Lip}_1(X, \mathbb{R}) \) then

\[
\left| \int_X f(x) d\mu(x) - \int_X f(x) d\nu(x) \right| \leq \sum_{i=1}^{m} \int_{r^{-1}(y_i) \cap K} f(x) d\mu(x) - \sum_{i=1}^{m} \mu(r^{-1}(y_i) \cap K) f(y_i) \leq \sum_{i=1}^{m} \int_{r^{-1}(y_i) \cap K} |f(x) - f(y_i)| d\mu(x) \leq \varepsilon
\]

because \( f \in \text{Lip}_1(X, \mathbb{R}) \) and \( r^{-1}(y) \subseteq B_{\varepsilon}(y) \).

(b) We need to show that \( d_H(\mu, \tau(\mu)) \leq \varepsilon \) for any \( \mu \in \mathcal{P}(X) \). Recall that \( \tau(\mu)(B) = \mu(r^{-1}(B)) \) any measurable \( B \subseteq \hat{X} \). By a similar construction as we made in (a) we can prove that.

(c) Consider \( \nu \in \mathcal{P}(\hat{X}) \) and \( f \in C^0(\hat{X}, \mathbb{R}) \).

\[
\tau(\mathcal{M}_\delta \circ \varepsilon(\nu))(f) = \int_X f(r(z)) d(\mathcal{M}_\delta \circ \varepsilon(\nu))(z) = \\
= \sum_{j=1}^{L} p_j \int_X \int_X f(r(\phi_j(x, y))) d(\varepsilon(\nu))(x) d(\varepsilon(\nu))(y) = \sum_{j=1}^{L} p_j \int_X \int_X f(r(\phi_j(e(x, y)))) d\nu(x) d\nu(y) = \\
= \sum_{j=1}^{L} p_j \int_X \int_X f(\phi_j(e(x, y))) d\nu(x) d\nu(y) = (\mathcal{M}_\delta(\nu))(f).
\]

A measure \( \nu \in \mathcal{P}(\hat{X}) \) is called a discrete Hutchinson measure with resolution \( \delta \) if \( d_H(\nu, \mu_S) \leq \delta \).

**Theorem 10.2.** Let \( (X, d) \) be a complete metric space and \( S = (X, (\phi_j)_{j=1}^{L}, (p_j)_{j=1}^{L}) \) be a GIFS with probabilities on \( X \) consisting of Banach contractions. Let \( \varepsilon > 0, \hat{X} \) be a proper \( \varepsilon \)-net, \( \tau : X \to \hat{X} \) be a proper measurable \( \varepsilon \)-projection on \( \hat{X} \) and \( S := (\hat{X}, (\hat{\phi}_j)_{j=1}^{L}, (\hat{p}_j)_{j=1}^{L}) \), where \( \hat{\phi}_j = (\tau \circ \phi_j)_{\hat{X}} \) is the discretization of \( \phi_j \).

For any \( \nu \in \mathcal{P}(\hat{X}) \) and \( n \in \mathbb{N} \),

\[
d_H(\mathcal{M}_\delta^{n}(\nu), \mu_S) \leq \frac{5\varepsilon}{1 - \alpha_S} + \alpha_S^2 d_H(\nu, \mu_S),
\]

where \( \mu_S \) is the attractor of \( M_S \) (i.e. the Hutchinson measure).

In particular, for every \( \delta > \frac{5\varepsilon}{1 - \alpha_S} \), there is \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \), \( \mathcal{M}_\delta(x) \) is a discrete Hutchinson measure of \( S \) with resolution \( \delta \).

### 11 Discrete Deterministic Algorithms for Hutchinson Measures

The central idea is that a discrete Hutchinson measure of \( S \) with resolution \( \delta \), is a measure \( \nu \) with finite support on \( X \), let us say

\[
\nu := \sum_{k=1}^{m} a_k \delta_{y_k}
\]

where \( y_k \in \hat{X} \) and \( \sum_{k=1}^{m} a_k = 1 \). Applying the discrete Markov operator \( M_S \) on this measure we get

\[
\int_X f(x) d(M_S(\nu))(x) = \sum_{j=1}^{L} p_j \int_X f(\hat{\phi}_j(x)) d\left( \sum_{k=1}^{m} a_k \delta_{y_k} \right)(x) = \\
= \sum_{k=1}^{m} a_k \int_X f(\hat{\phi}_j(x)) \delta_{y_k}(x) = \sum_{k=1}^{m} a_k f(y_k).
\]
for any continuous \( f : \mathbb{X} \to \mathbb{R} \), where \( z_q \in \{ z \mid z := \hat{\phi}_j(y_k), 1 \leq j \leq L, 1 \leq k \leq m \} \) and the \( a'_q \) are the sum of all the coefficients correspondent to a same \( z_q \). Therefore

\[
M_{\tilde{\mathcal{S}}} \left( \sum_{k=1}^{m} a_k \delta_{y_k} \right) := \sum_{q=1}^{m'} a'_q \delta_{z_q}. \tag{22}
\]

For practical purposes, we compute the set \( F_{\tilde{\mathcal{S}}}(\text{supp}(\nu)) \) where \( \text{supp}(\nu) := \{ y_k, 1 \leq k \leq m \} \). Then, we enumerate \( F_{\tilde{\mathcal{S}}}(\text{supp}(\nu)) := \{ z_q, 1 \leq q \leq m' \} \) and, for each \( q \) we define

\[
a'_q := \sum_{\hat{\phi}_j(y_k) = z_q} p_j a_k. \tag{23}
\]

The remarkable fact is that the successive iterations \( F_{\tilde{\mathcal{S}}}(\text{supp}(\nu)) \) will make the discrete Hutchinson measure \( \nu_{\tilde{\mathcal{S}}} \) of \( \mathcal{S} \) with resolution \( \delta \) be supported on an actual attractor \( A_{\delta} \) of \( \mathcal{S} \), with resolution \( \delta' \)!

Thus, the output of our algorithm is a bitmap image with the equal shape of \( A_{\delta} \) but each pixel represents the measure \( \nu_{\tilde{\mathcal{S}}} \) of the atom \( \{ y \} \), that is, a grey scale histogram. More than that, the value \( \nu_{\tilde{\mathcal{S}}}(|\{ y \}|) \) represents an approximation of the value

\[
\nu_{\tilde{\mathcal{S}}}(|\{ y \}|) \simeq \mu_{\delta}(r^{-1}(\{ y \})). \tag{24}
\]

Analogously, for GIFS, applying the discrete Markov operator \( \overline{M}_{\tilde{\mathcal{S}}} \) on the discrete measure \( \nu \) we get

\[
\int_X f(x) d \left( \overline{M}_{\tilde{\mathcal{S}}} (\nu) \right) (x) = \sum_{j=1}^{N} p_j \int_X f(\hat{\phi}_j(x, y)) d \left( \sum_{k=1}^{m} a_k \delta_{y_k} \right) (x) d \left( \sum_{k=1}^{m} a_k \delta_{y_k} \right) (y) =
\]

\[
= \sum_{j=1}^{N} p_j \left( \sum_{k=1, \ell=1}^{m} a_{k \ell} f(\hat{\phi}_j(y_k, y_\ell)) \right) = \sum_{j=1}^{N} \sum_{k=1, \ell=1}^{m} (p_j a_k a_\ell) f(\hat{\phi}_j(y_k)) = \sum_{q=1}^{m'} a'_q f(z_q),
\]

for any continuous \( f : \mathbb{X} \to \mathbb{R} \), where \( z_q \in \{ z \mid z := \hat{\phi}_j(y_k), 1 \leq j \leq L, 1 \leq k \leq m \} \) and the \( a'_q \) are the sum of all the coefficients correspondent to a same \( z_q \). Therefore

\[
\overline{M}_{\tilde{\mathcal{S}}} \left( \sum_{k=1}^{m} a_k \delta_{y_k} \right) := \sum_{q=1}^{m'} a'_q \delta_{z_q}. \tag{25}
\]

For practical purposes, we compute the set \( \overline{F}_{\tilde{\mathcal{S}}}(\text{supp}(\nu)) \) where \( \text{supp}(\nu) := \{ y_k, 1 \leq k \leq m \} \). Then, we enumerate \( \overline{F}_{\tilde{\mathcal{S}}}(\text{supp}(\nu)) := \{ z_q, 1 \leq q \leq m' \} \) and, for each \( q \) we define

\[
a'_q := \sum_{\hat{\phi}_j(y_k, y_\ell) = z_q} p_j a_k a_\ell. \tag{26}
\]

The remarkable fact is that the successive iterations \( \overline{F}_{\tilde{\mathcal{S}}}(\text{supp}(\nu)) \) will make the discrete Hutchinson measure \( \nu_{\tilde{\mathcal{S}}} \) of \( \mathcal{S} \) with resolution \( \delta \) be supported on an actual attractor \( A_{\delta} \) of \( \mathcal{S} \), with resolution \( \delta' \)!

Thus, the output of our algorithm is a bitmap image with the equal shape of \( A_{\delta} \) but each pixel represents the measure \( \nu_{\tilde{\mathcal{S}}} \) of the atom \( \{ y \} \), that is, a grey scale histogram. More than that, the value \( \nu_{\tilde{\mathcal{S}}}(|\{ y \}|) \) represents an approximation of the value

\[
\nu_{\tilde{\mathcal{S}}}(|\{ y \}|) \simeq \mu_{\delta}(r^{-1}(\{ y \})). \tag{27}
\]

### 11.1 Uniform \( \varepsilon \)-nets

In order to build an algorithm we are going to fix some notation and consider a special type of \( \varepsilon \)-net on Euclidean spaces such as \( X = \mathbb{R}^2 \).
Given $\varepsilon > 0$ we consider the sequence $x[i] \in \mathbb{R}, i \in \mathbb{Z}$ such that $x[i] < x[i+1]$ and $x[i+1] - x[i] = \varepsilon$, for all $i \in \mathbb{Z}$. Then the set $\hat{X} \subseteq \mathbb{R}^2$ given by
\[
\hat{X} := \{(x[i], x[j]) \mid i, j \in \mathbb{Z}\}
\] is a proper $\varepsilon$-net for $\mathbb{R}^2$ with respect to the Euclidean distance.

We need also to define an $\varepsilon$-projection on $\hat{X}$. For practical use, we consider a rectangle $\hat{X}' = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ large enough to contain the fractal attractor $A_S$, which is the attractor of $\mathcal{S} := \hat{X}, (\hat{\phi}_j)_{j=1}^L$.

Of course, $\hat{X}$ is also a $\varepsilon$-net for $\hat{X}'$ as well.

Consider the auxiliary function $q : \mathbb{R} \to \{x[i] \in \mathbb{R}, i \in \mathbb{Z}\}$ given by
\[
q_1(t) := \begin{cases} 
    x[m], & \text{if } t \leq x[m] \\
    x[i], & \text{if } x[i] \leq t < \frac{x[i]+x[i+1]}{2}, i \geq m \\
    x[i+1], & \text{if } \frac{x[i]+x[i+1]}{2} \leq t \leq x[i+1], i \leq m \\
    x[M], & \text{if } t \geq x[M]
\end{cases}
\] (29)
where $x[m]$ is the smaller point such that $a \leq x[i]$ and $x[M]$ is the bigger point such that $b \geq x[i]$.
The function $q_2$ refers to the second coordinate and is defined by the same formula replacing $[a, b]$ by $[c, d]$.

Then, the actual $\varepsilon$-projection on $\hat{X}$ is given by
\[
r(x, y) := (q_1(x), q_2(y)).
\] (30)
We notice that $r$ is clearly proper and measurable because $r^{-1}(v_1, v_2)$ is a semi-open interval for any $(v_1, v_2) \in \hat{X}$.

Strictly speaking, $r$ is an $\varepsilon$-projection only on $\hat{X} \cap ([a, b] \times [c, d])$; however, the fractal $A_S$ is contained in this set and it contains the image of the discrete fractal operator, meaning that,
\[
F^*_\varepsilon(\hat{X}) \subseteq \hat{X} \cap ([a, b] \times [c, d])
\]
where $\hat{S} := (\hat{X}, (\hat{\phi}_j)_{j=1}^L)$, where $\hat{\phi}_j = (r \circ \phi_j)_{|\hat{X}}$ is the discretization of $\phi_j$.

This allow us to iterate the discrete fractal operator freely obtaining the right approximation on the target set $[a, b] \times [c, d]$. To establish an optimal target set we may execute a few iterations of the discrete deterministic algorithm from [DOS] or even the classic deterministic one [JLS] and then locate the boundaries of the actual fractal.

Finally, for a measure $\nu$ with finite support on $\hat{X}$, let us adopt the notation
\[
\nu := \sum_{k=1}^m \nu_{k,k} \delta(x[k], y[j_k]),
\] (31)
where $(x[k], y[j_k]) \in \hat{X}$ and $\sum_{k=1}^m \nu_{k,k} = 1$.

### 11.2 The IFS and GIFS algorithms

Consider $\mathcal{S} = (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L)$ be an IFS with probabilities on $X$ consisting of Banach contractions. Let $\varepsilon > 0$, $\hat{X}$ be a proper $\varepsilon$-net, $r : X \to \hat{X}$ be a measurable $\varepsilon$-projection on $\hat{X}$ and $\hat{S} := (\hat{X}, (\hat{\phi}_j)_{j=1}^L, (\hat{p}_j)_{j=1}^L)$, where $\hat{\phi}_j = (r \circ \phi_j)_{|\hat{X}}$ is the discretization of $\phi_j$.

**IFSMMeasureDraw**(S)

**input:**
- $\delta > 0$, the resolution.
- $K \subseteq \hat{X}$, any finite and not empty subset (a list of points in $\hat{X}$).
- $\nu$, any probability such that $\text{supp}(\nu) = K$.
- $\text{The diameter } D$ of a ball in $(X, d)$ containing $A_S$.

**output:**
- A bitmap representing a discrete attractor with resolution at most $\delta$.
- A bitmap image representing a discrete Hutchinson Measure with resolution at most $\delta$.

**Compute:**
- $\alpha_S := \text{Lip}(\mathcal{S})$
\( \varepsilon > 0 \) and \( N \in \mathbb{N} \) such that \( \frac{5\varepsilon}{1-\alpha_S} + \alpha_S^2 D < \delta \)

Initialize \( \mu := 0 \) and \( W := \emptyset \)

for \( n \) from 1 to \( N \) do

for \( \ell \) from 1 to \( \text{Card}(K) \) do

for \( j \) from 1 to \( L \) do

\((x[i_0], y[j_0]) := K[\ell] \) and \((x[i'], y[j']) := \phi_j(x[i_0], y[j_0]) \)

If \((x[i'], y[j']) \notin W \) then \( W := W \cup (x[i'], y[j']) \)

\( \mu_{i',j'} := \mu_{i',j'} + \nu_j \nu_{i_0,j_0} \)

end do

end do

\( K := W \) and \( W := \emptyset \)

\( \nu := \mu \) and \( \mu := 0 \)

end do

return: Print \( K \) and \( \nu \)

Remark 11.1. By construction, the measure \( \mu_N := M^N_S(\nu) \) has support on the finite set \( K_N := F^N_S(K) \) where \( K = \text{supp}(\nu) \). This shows that the discrete Hutchinson Measure \( \nu_S \), with resolution \( \delta \), is actually a measure of probability with finite support on a discrete fractal \( A_S \) with resolution \( \delta \). This is the discrete version of the classical result where the actual Hutchinson measure of a IFS has support on its attractor.

Now consider \( S = (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L) \) be a GIFS with probabilities on \( X \) consisting of Banach contractions. Let \( \varepsilon > 0 \), \( X \) be a proper \( \varepsilon \)-net, \( r : X \to X \) be a measurable \( \varepsilon \)-projection on \( X \) and \( S := (X, (\phi_j)_{j=1}^L, (p_j)_{j=1}^L) \), where \( \phi_j = (r \circ \phi_j)_1 \mathbb{Z}^2 \) is the discretization of \( \phi_j \).

GIFSMeasureDraw(S)

input:
\( \delta > 0 \), the resolution.
\( K \subseteq X \), any finite and not empty subset (a list of points in \( X \)).
\( \nu \), any probability such that \( \text{supp}(\nu) = K \).
The diameter \( D \) of a ball in \((X, d)\) containing \( A_S \).

output:
A bitmap representing a discrete attractor with resolution at most \( \delta \).
A bitmap image representing a discrete Hutchinson Measure with resolution at most \( \delta \).

Compute:
\( \alpha_S := \text{Lip}(S) \)
\( \varepsilon > 0 \) and \( N \in \mathbb{N} \) such that \( \frac{5\varepsilon}{1-\alpha_S} + \alpha_S^2 D < \delta \)

Initialize \( \mu := 0 \) and \( W := \emptyset \)

for \( n \) from 1 to \( N \) do

for \( \ell_1 \) from 1 to \( \text{Card}(K) \) do

for \( \ell_2 \) from 1 to \( \text{Card}(K) \) do

for \( j \) from 1 to \( L \) do

\((x[i_0], y[j_0]) := K[\ell_1] \) and \((x[i_1], y[j_1]) := K[\ell_2] \)

\((x[i'], y[j']) := \phi_j(x[i_0], y[j_0], x[i_1], y[j_1]) \)

If \((x[i'], y[j']) \notin W \) then \( W := W \cup (x[i'], y[j']) \)

\( \mu_{i',j'} := \mu_{i',j'} + \nu_j \nu_{i_0,j_0} \nu_{i_1,j_1} \)

end do

end do

\( K := W \) and \( W := \emptyset \)

\( \nu := \mu \) and \( \mu := 0 \)

end do

return: Print \( K \) and \( \nu \)

Remark 11.2. By construction, the measure \( \mu_N := \overline{M}^N_S(\nu) \) has support on the finite set \( K_N := F^N_S(K) \) where \( K = \text{supp}(\nu) \). This shows that the discrete Hutchinson Measure \( \nu_S \), with resolution \( \delta \), is actually a
measure of probability with finite support on a discrete fractal $A_d$ with resolution $\delta$. This is the discrete version of the classical result where the actual Hutchinson measure of a IFS has support on its attractor.

11.3 IFS examples

Example 11.3. Using our algorithms we can recover the results of [Bar] Chapter IX], for IFS with probabilities. In page 331, Table IX.1, that author considers the IFS $p$ given by

$$S: \begin{align*}
\phi_1(x, y) &= (0.5x, 0.5y) \\
\phi_2(x, y) &= (0.5x + 0.5, 0.5y) \\
\phi_3(x, y) &= (0.5x, 0.5y + 0.5) \\
\phi_4(x, y) &= (0.5x + 0.5, 0.5y + 0.5)
\end{align*}$$

with the probabilities $(p_1 = 0.1, p_2 = 0.2, p_3 = 0.3, p_4 = 0.4)$. The comparison is made in Figure 1.

![Figure 1](image.png)

Figure 1: From the left to the right the output of the algorithm $\text{IFSMedureDraw}(S)$ after 8 iterations, with $512 \times 512$ pixels and the picture IX.247 obtained in [Bar] through a random process with 100,000 iterations.

Example 11.4. This example is a classic geometric fractal, the Maple Leaf. The approximation of the attractor by the algorithm $\text{IFSDraw}(S)$ from [DOS] is presented in Figure 2 and the approximation of the discrete Hutchinson Measure, through $\text{IFSMedureDraw}(S)$, is presented in the same figure. Consider $(\mathbb{R}^2, d)$ a metric space and the IFS $\phi_1, ..., \phi_4 : X \rightarrow X$ with probabilities $(p_j)_{j=1}^{L=4}$ where

$$S: \begin{align*}
\phi_1(x, y) &= (0.8x + 0.1, 0.8y + 0.04) \\
\phi_2(x, y) &= (0.5x + 0.25, 0.5y + 0.4) \\
\phi_3(x, y) &= (0.355x - 0.355y + 0.266, 0.355x + 0.355y + 0.078) \\
\phi_4(x, y) &= (0.355x + 0.355y + 0.378, -0.355x + 0.355y + 0.434)
\end{align*}$$

As we can see, when the initial probabilities are small on the index of a map responsible for a part of the fractal attractor, the Hutchinson measure is very little concentrated on that part. For example, if we choose equal probability we will have a much more equal distribution as in Figure 3.

11.4 GIFS examples

Example 11.5. The approximation of the attractor by the algorithm $\text{GIFSDraw}(S)$ from [DOS] is presented in Figure 4 and the approximation of the discrete Hutchinson Measure, through $\text{GIFSMedureDraw}(S)$, is presented in the same figure. Consider $(\mathbb{R}^2, d)$ a metric space and the GIFS $\phi_1, ..., \phi_3 : X^2 \rightarrow X$ with probabilities $(p_j)_{j=1}^{L=3}$ where

$$S: \begin{align*}
\phi_1(x_1, y_1, x_2, y_2) &= (0.25x_1 + 0.2y_2, 0.25y_1 + 0.2y_2) \\
\phi_2(x_1, y_1, x_2, y_2) &= (0.25x_1 + 0.2x_2, 0.25y_1 + 0.1y_2 + 0.5) \\
\phi_3(x_1, y_1, x_2, y_2) &= (0.25x_1 + 0.1x_2 + 0.5, 0.25y_1 + 0.2y_2)
\end{align*}$$

Once again, we can see that when the initial probabilities are small on the index of a map responsible for a part of the fractal attractor, the Hutchinson measure is very little concentrated on that part.
Figure 2: From the left to the right the output of the algorithm $\text{IFSMeasureDraw}(S)$ after 5 iterations, 512 \times 512 pixels, with the set of probabilities $(p_1 = 0.3, p_2 = 0.2, p_3 = 0.05, p_4 = 0.45)$ and $(p_1 = 0.05, p_2 = 0.2, p_3 = 0.3, p_4 = 0.45)$ respectively, for the rightmost, algorithm $\text{IFSDraw}(S)$ after 12 iterations.

Figure 3: The output of the algorithm $\text{IFSMeasureDraw}(S)$ after 12 iterations, $\varepsilon = 0.004$ (512 points in a uniform net) with the set of probabilities $(p_1 = 0.25, p_2 = 0.25, p_3 = 0.25, p_4 = 0.25)$. In this case the measure $\nu$ is a Hutchinson measure with resolution inferior to $\delta = 0.2441701363$.

Figure 4: On the top, from the left to the right the output of the algorithm $\text{GIFSMeasureDraw}(S)$ after 12 iterations, with the set of probabilities $(p_1 = 0.10, p_2 = 0.45, p_3 = 0.45)$, $(p_1 = 0.45, p_2 = 0.10, p_3 = 0.45)$ and $(p_1 = 0.45, p_2 = 0.45, p_3 = 0.10)$ respectively. On the bottom, algorithm $\text{GIFSDraw}(S)$ after 12 iterations. In all the cases we have, 512 \times 512 pixels.
12 Further applications

12.1 Approximating integrals with respect to stationary Probability measures

In a recent preprint [CJ] the authors describe a method to approximate integral of functions with respect to stationary probability measures (measures that are fixed points for the Markov operator associated to an IFS with probabilities, $M_S(\mu_S) = \mu_S$) which are the Hutchinson measures for those IFS. The setting is the interval $[0, 1]$ and the IFS is required to fulfill some additional regularity properties such as holomorphic extension, control of derivatives of the maps in the IFS and on the set of functions that one may integrate.

Since our algorithm $\text{IFSMeasureDraw}(S)$ for dimension 1 provides a discrete $\delta$-approximation of such measures $\mu_S$ in the form

\[ \nu := \sum_{k=1}^{m} \nu_k \delta_{x[i_k]} \]

where the points $x[i_k]$ are in the correspondent $\delta$-approximation of the actual attractor, we can approximate the integral of a Lipschitz function $g : [0, 1] \rightarrow \mathbb{R}$ by

\[ \int_{[0,1]} gd\mu_S \simeq \sum_{k=1}^{m} \nu_k g(x[i_k]) \]

with precision $\delta$.

As a demonstration of this we next describe the measures and the integrals for three examples found in [CJ] using our algorithms.

**Example 12.1.** In this first example we consider the Hausdorff moment of a Hutchinson measure $\mu_S$ which is given by

\[ \gamma_n = \gamma_n(\mu_S) := \int_{-\infty}^{\infty} x^n d\mu_S(x), \quad n = 0, 1, \ldots \]

\[ S : \begin{cases} 
\phi_1(x) &= \frac{1}{3}x \\
\phi_2(x) &= \frac{1}{3}x + \frac{2}{3}
\end{cases} \]

with probabilities $p_1 := 1/3$ and $p_2 := 2/3$.

For purpose of comparison we use the algorithm $\text{IFSMeasureDraw}(S)$ to approximate $\mu_S$ (see Figure 5) and then compute $\gamma_n, n = 0, 1, \ldots, 10$: The first 10 moments are displayed in Table 1.

![Figure 5: Histogram of $\mu_S$ produced by algorithm $\text{IFSMeasureDraw}(S)$ with resolution $\delta = 0.0001$ after 10 iterations, having a high definition 50,000 pixels taking 1.9 seconds.](image)

**Example 12.2.** In this second example we consider the Wasserstein distance between two Hutchinson measures $\mu_{S_1}$ and $\mu_{S_2}$ (see Figure 6) associated to different probabilities for a same IFS

\[ S : \begin{cases} 
\phi_1(x) &= \frac{\sin(\pi x/4)}{\sin(\pi/4)} + \frac{1}{3} \\
\phi_2(x) &= \frac{\sin(\pi x/4)}{\sin(\pi/4)} + \frac{4}{3}
\end{cases} \]

with probabilities $p_1 := 1/7, p_2 := 6/7$ and $q_1 := 1/2, q_2 := 1/2$ respectively.
It is easy to see that the IFS is $\frac{7}{12}$-Lipschitz then our algorithm $\text{IFSMeasureDraw}(S)$ can be used to approximate the integrals and, in particular, the Wasserstein distance

$$W_1(\mu_S, \mu_{S_2}) = \left| \int x d\mu_S - \int x d\mu_{S_2} \right| \simeq 0.22104594557263850324...$$

**Example 12.3.** For the last, we consider the problem of compute the Lyapunov exponent of the Hutchinson measure $\mu_S$ (see Figure 7) of a IFS given by

$$\chi_{\mu_S} := -\sum_{i=1}^{2} p_i \log |\phi_i'| \, d\mu_S(x) \simeq 1.7367208099326368...$$

for

$$S : \left\{ \begin{array}{c}
\phi_1(x) = \frac{\sin(\pi x/4)}{3} + \frac{1}{3} \\
\phi_2(x) = \frac{\sin(\pi x/4)}{3} + \frac{1}{3}
\end{array} \right.)$$

with probabilities $p_1 := 1/3, p_2 := 2/3$.

Figure 7: The histogram of $\mu_S$ produced by algorithm $\text{IFSMeasureDraw}(S)$ with resolution $\delta = 2.11 \times 10^{-6}$ after 20 iterations, having a high definition 3,200,000 pixels taking 0.5690 seconds.
12.2 Projected Hutchinson measures

One can easily adapt algorithm \texttt{IFSMeasureDraw}(S) to also compute the integral of a function with respect to the Hutchinson measure and compare this result against the typical averages, as predicted by Elton’s ergodic theorem, see \cite{Elt}.

For example, given the GIFS \( S \) from \cite[Example 11]{EO},

\[
S: \begin{cases} 
\phi_1(x, y) = \frac{1}{3}x + \frac{1}{4}y \\
\phi_2(x, y) = \frac{1}{3}x - \frac{1}{4}y + \frac{1}{2} 
\end{cases}
\]

we consider its extension

\[
\mathcal{R}: \begin{cases} 
\psi_1(x, y) = (y, \frac{1}{3}x + \frac{1}{4}y) \\
\psi_2(x, y) = (y, \frac{1}{3}x - \frac{1}{4}y + \frac{1}{2}) 
\end{cases}
\]

which is an eventually contractive IFS on \([0, 1]^2\) (the second power of \( \mathcal{R} \) is 0.5435541904-Lipschitz), so our theory works. In both cases we consider probabilities \( p_1 := 0.65 \) and \( p_2 := 0.35 \). As pointed out in \cite{EO}, if \( \nu \in \mathcal{P}([0, 1]) \) is the Hutchinson measure for the GIFS \( \mathcal{S} = (X, (\phi_j)_{j=1}^2, (p_j)_{j=1}^2) \) and \( \mu_{\mathcal{R}} \in \mathcal{P}([0, 1]^2) \) is the Hutchinson measure for the IFS \( \mathcal{R} = (X, (\psi_j)_{j=1}^2, (p_j)_{j=1}^2) \) (see Figure 8) then \( \nu \neq \nu' \), where \( \nu' \) is the projected Hutchinson measure \( \Pi^\#_1(\mu_{\mathcal{R}}) \).

Using our algorithms we are capable to display a histogram representation of such distributions on \([0, 1]\). In each case the height of each vertical bar represents the approximate measure of a cell in the \( \varepsilon \)-net with 350 points in \( X = [0, 1] \), and resolution \( 2 \times 10^{-2} \) (see Figure 9).

![Figure 8: The output of algorithm \texttt{IFSMeasureDraw}(\mathcal{R}) after 25 iterations, having a fairly high definition, 512 x 512 pixels. On the left the attractor \( A_{\mathcal{R}} \) on the right the histogram of \( \mu_{\mathcal{R}} \).](image)

![Figure 9: The output of algorithm \texttt{GIFSMeasureDraw}(\mathcal{S}) after 25 iterations, having a fairly high definition 350 x 350 pixels. On the left the attractor \( \nu \) on the right the histogram of \( \nu' = \Pi^\#_1(\mu_{\mathcal{R}}) \) which is the projection of the Figure 8.](image)

The ergodic theorem for projected Hutchinson measures from \cite{EO} claims that

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n(a)) = \int_X f(x) d\nu'(x)
\]

where \( x_n(a) \) is the projection on the first coordinate and \( a \) is a sequence of symbols chosen with probability one in the sequences of \( \{1, 2\}^N \) according to the probabilities \( p_1 := 0.65 \) and \( p_2 := 0.35 \).

Consider \( f(x) = x^2 \): using the measure obtained by \texttt{IFSMeasureDraw}(\mathcal{R}) for estimate \( \int_0^1 x^2 d\nu'(x) \), the projected measure, we obtain

\[
\int_0^1 x^2 d\nu'(x) \simeq 0.12177521930...
\]
On the other hand, using the ergodic theorem where each \( a_i \) in the sequence \( a = (a_0, a_1, a_2, \ldots) \) is picked from a random i.i.d. variable \( I \in \{1, 2\} \) with distribution \( p_1 := 0.65 \) and \( p_2 := 0.35 \), \( x_0 = 0.5 \) and \( N = 10,000 \) we get

\[
\frac{1}{10000} \sum_{n=0}^{10000-1} (x_n(a))^2 \simeq 0.1228183842857 \ldots
\]

with an absolute error of \( 5 \times 10^{-2} \). For a fractal computed on a resolution of \( 500 \times 500 \) pixels the error is \( 3 \times 10^{-4} \).

### 12.3 IFS and GIFS with place dependent probabilities

We consider variable probabilities, that is, each \( p_i \) is as function of \( x \), such as in \[Hu\], \[BDEC\] Theorem 2.1, assuming average-contractiveness, \[St\], \[Ob\] and more recently \[GMM\] for IFS and in \[Mi\] Section 3 for GIFS.

We notice that Lemma 7.2 and Lemma 10.1 are still valid under the variable probability hypothesis. Therefore, we just need to ensure that the respective Markov operator is \( \alpha \)-Lipschitz with \( \alpha < 1 \) to use Theorem 5.2. We are not going to remake several straightforward computations to update the algorithms. We only update the necessary computation of the probabilities in each case (both in the bidimensional version):

- In the algorithm \( \text{IFSMeasureDraw}(S) \) we replace \( \mu_{\nu^j} := \mu_{\nu^j} + p_j \nu_{\alpha j0} \) by
  \[
  \mu_{\nu^j'} := \mu_{\nu^j'} + p_j (x[j_0], y[j_0]) \nu_{\alpha j0};
  \]
- In the algorithm \( \text{GIFSMeasureDraw}(S) \) we replace \( \mu_{\nu^j} := \mu_{\nu^j} + p_j \nu_{\alpha j0} \nu_{1i1j} \) by
  \[
  \mu_{\nu^j'} := \mu_{\nu^j'} + p_j (x[j_0], y[j_0], x[i_1], y[j_1]) \nu_{\alpha j0} \nu_{1i1j}.
  \]

**Example 12.4.** In \[Mi\] p. 146, the author considers the one dimensional case \( X = [0, 1] \) and a GIFS of degree 2, \( S = (X, (\phi_j)_{j=1}^2) \) where

\[
\begin{align*}
\phi_1(x, y) &= \frac{x}{3} + \frac{2}{3}, \\
\phi_2(x, y) &= \frac{x}{3} + \frac{1}{3} + \frac{y}{3}.
\end{align*}
\]

Given a function

\[
\alpha(t) = \begin{cases} 1, & \text{if } t \in \left[0, \frac{1}{3}\right], \\
2 - 4t, & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
0, & \text{if } t \in \left[\frac{2}{3}, 1\right].
\end{cases}
\]

he considers the probabilities \( p_1(x, y) = \frac{1}{3} \alpha(x) \alpha(y) \),

\( p_2(x, y) = 1 - p_1(x, y) = 1 - \frac{1}{3} \alpha(x) \alpha(y) \),

and the GIFSpdp \( S = (X, (\phi_j)_{j=1}^2, (p_j)_{j=1}^2) \).

Under this hypothesis, he verifies that \( M_S \) is a contraction and \( \mu_S \) is its only fixed point. More than that, he verifies that the attractor is \( A_S = [0, 1] \) and \( \text{supp} \mu_S \subseteq \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \).

From the previous discussion it is easy to see that our algorithm \( \text{GIFSMeasureDraw}(S) \) converges to \( \mu_S \) as we can see in figure 17.

![Figure 10](image_url)

The output suggests that \( \mu_S = \delta_1 \). We can verify that by direct examination as follows.

The output suggests that \( \mu_S = \delta_1 \). We can verify that by direct examination as follows.
Consider a continuous function \( f : [0, 1] \to \mathbb{R} \) then,
\[
\int f(z) dM_S(\delta_1 \times \delta_1) = \int_x \int_y p_1(x, y) f(\phi_1(x, y)) + p_2(x, y) f(\phi_1(x, y)) d\delta_1(x) d\delta_1(y) = \\
= p_1(1, 1) f(\phi_1(1, 1)) + p_2(1, 1) f(\phi_1(1, 1)) = p_1(1, 1) f \left( \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right) + p_2(1, 1) f \left( \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right) = \\
= f(1) = \int f(z) d\delta_1(z),
\]
because \( p_1(1, 1) = 0 \). Therefore, \( M_S(\delta_1 \times \delta_1) = \delta_1 \) meaning that \( \mu_S = \delta_1 \).

Example 12.5. In this case we consider a negative case from \([CR]\). In this case the author considers the doubling map \( T(x) = 2x \mod 1 \) on the interval and studies the solutions of the equation
\[
P_u f(x) = u \left( \frac{x}{2} \right) f \left( \frac{x}{2} \right) + u \left( \frac{x}{2} + \frac{1}{2} \right) f \left( \frac{x}{2} + \frac{1}{2} \right)
\]
for a given potential \( u : [0, 1] \to \mathbb{R} \) and the \( P_u \)-harmonic functions, i.e. \( P_u(f) = f \). After he characterizes the dual solutions \( P^*_u(\mu) = \mu \), which he calls invariant measures.

It follows that this problem is the same as the problem of finding the Hutchinson measures for the \( \text{IFSdpd} \) \( S = (X, (\phi_j)_{j=1}^2, (p_j)_{j=1}^2) \) where
\[
\begin{align*}
\phi_1(x) &= \frac{x}{2} \\
\phi_2(x) &= \frac{x}{2} + \frac{1}{2}.
\end{align*}
\]

Given a function, say \( u(t) = \cos^2(3\pi t) \), he considers the probabilities \( p_1(x) = u(\phi_1(x)) = u(x/2) \) and \( p_2(x) = u(\phi_2(x)) = u(x/2 + 1/2) \).

In example 3 b, the author shows that the invariant measures (Hutchinson measures for the \( \text{IFSdpd} \)) are the probabilities supported on the \( T \)-periodic cycles \( \{0\} \), \( \{1\} \) and \( \{\frac{1}{2}, \frac{3}{2}\} \). In that case our Markov operator can not be a contraction because there is more than one Hutchinson measure. In any case we can still use \( \text{IFSMeasureDraw}(S) \) to see if it converges to any measure. We show in Figure 11 and Figure 12 a selection phenomena: the iteration seems to converge to some combination \( \mu := a\delta_0 + b\delta_{\frac{1}{2}} + c\delta_{\frac{3}{2}} + d\delta_1 \) depending on the initial measure we choose.

![Figure 11: Output of algorithm IFSMeasureDraw(S) after 1 and 100 iterations.](image)

Figure 11: The output of algorithm \( \text{IFSMeasureDraw}(S) \) after 1 and 100 iterations, having a fairly high definition of 500 pixels. In this case, after 100 iterations the discrete measure is \( \mu := 0.1030321845 \delta_0 + 0.8199775187 \delta_{\frac{1}{2}} + 0.0620948524 \delta_{\frac{3}{2}} + 0.01471437940 \delta_1 \).

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Figure 12: The output of algorithm \texttt{IFSMeasureDraw}(S) after 1 and 100 iterations, having a fairly high definition of 500 pixels. In this case, after 100 iterations the discrete measure is 
\[ \mu := 0.05792616463 \delta_0 + 0.4568641232 \delta_1 + 0.4317603436 \delta_2 + 0.05323785154 \delta_3. \]

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