VARIETIES OF CHARACTERS

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ABSTRACT. Let $G$ be a connected reductive affine algebraic group. In this short note we define the variety of $G$-characters of a finitely generated group $\Gamma$ and show that the quotient of the $G$-character variety of $\Gamma$ by the action of the trace preserving outer automorphisms of $G$ normalizes the variety of $G$-characters when $\Gamma$ is a free group, free abelian group, or a surface group.

1. Introduction

For a finitely generated group $\Gamma$ and a connected reductive affine algebraic group $G$ over the complex numbers, the $G$-character variety of $\Gamma$ is the GIT quotient $\mathcal{X}(\Gamma, G) := \text{Hom}(\Gamma, G)/\!/G$ of the $G$-representation space of $\Gamma$, $\text{Hom}(\Gamma, G)$, (considered as an algebraic set) by the adjoint action of $G$.

This algebraic set is known for its relationship to moduli spaces of geometric structures ([CG97, Gol88, GM88, JM87, KM98, Thu97]), 3 and 4 dimensional topology ([CCG+94, CS83, BZ98, PS00, Bul97, Cur01]), mathematical physics ([Ati90, AB83, BFM02, Hit87, KS00, Wit01, JW92, KW07]) and invariant theory ([BH95, Law08, Sik01]).

The term “$G$-character variety” is motivated by the fact that its closed points for $G = \text{SL}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}), \text{SO}(2n + 1, \mathbb{C})$ are classified by the $G$-characters of $\Gamma$. (Perhaps the first proof of this was given by Culler-Shalen in [CSS83] for the case $\text{SL}(2, \mathbb{C})$.) However, it is not the case for $\text{SO}(2n, \mathbb{C})$, see [FL11 Appendix A], [Sik15a, Sik15b]. For that reason, it is useful to consider the space of characters of $\Gamma$, which we show to have a natural structure of an algebraic set, leading to the notion of the variety of characters.

In this short note we first make clear the relationship between the variety of characters of $\Gamma$ and the character variety of $\Gamma$ by showing that although the variety of characters is not generally isomorphic to the character variety, for many groups (including free groups, free abelian groups, and surface groups), the quotient of the $G$-character variety by a finite group (the trace-preserving outer automorphisms of $G$) normalizes the variety of characters (see Corollary 8). As a consequence, we conclude that regular functions on the character varieties as above are rational functions in characters.

In the last section we compute the the trace-preserving outer automorphisms of $G$ for various simple complex Lie groups.

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2. Variety of Characters

Let $\Gamma$ be a finitely generated group. Fix a connected reductive subgroup $G$ of $\text{SL}(n, \mathbb{C})$. A $G$-character of $\Gamma$, denoted $\chi_{\rho}$, is the trace of a $G$-representation of $\Gamma$:

$$
\Gamma \xrightarrow{\rho} G \hookrightarrow \text{SL}(n, \mathbb{C}) \xrightarrow{\text{tr}} \mathbb{C}.
$$

Denote the set of $G$-characters of $\Gamma$ by $\text{Ch}(\Gamma, G)$.

Let $\text{Hom}(\Gamma, G)$ and $X(\Gamma, G) = \text{Hom}(\Gamma, G) / \! / G$ be the representation variety and the $G$-character variety, respectively (considered as affine varieties). Let $\mathcal{T}(\Gamma, G)$ be the trace $G$-algebra of $\Gamma$, that is, the subalgebra of $\mathbb{C}[X(\Gamma, G)]$ generated by trace functions $\tau_\gamma$ for $\gamma \in \Gamma$ defined by $\tau_\gamma(\rho) = \text{tr}(\rho(\gamma))$, see [CS83, FL11, Sik13] for more details.

It was proven in [FL11, Appendix A] and in [Sik13, Thm. 5] that $\mathcal{T}(\Gamma, G)$ coincides with $\mathbb{C}[X(\Gamma, G)]$ for any $\Gamma$ and $G$ equal to $\text{SL}(m, \mathbb{C})$, $\text{Sp}(2m, \mathbb{C})$, or $\text{SO}(2m + 1, \mathbb{C})$. For reader’s convenience, we enclose a concise proof of that result here.

Proposition 1. For $G$ equal to $\text{SL}(m, \mathbb{C})$, $\text{Sp}(2m, \mathbb{C})$, or $\text{SO}(2m + 1, \mathbb{C})$, and for all finitely generated $\Gamma$, $\mathcal{T}(\Gamma, G) = \mathbb{C}[X(\Gamma, G)]$.

Proof. If $\Gamma$ is a group on $r$ generators, then the epimorphism $F_r \to \Gamma$ mapping the free generators of $F_r$ to the generators of $\Gamma$ yields an embedding $X(\Gamma, G) \to X(F_r, G)$ and the corresponding dual epimorphism $\mathbb{C}[X(F_r, G)] \to \mathbb{C}[X(\Gamma, G)]$. Since that map sends trace functions to trace functions, it is enough to show that $\mathbb{C}[X(F_r, G)] = \mathbb{C}[G^r]^G$ is generated by trace functions.

By the assumptions of the theorem, $G$ is an algebraic subgroup of $\text{SL}(n, \mathbb{C})$ for some $n$ and, hence, of the space of the $n \times n$ complex matrices $\text{Mat}(n, \mathbb{C})$. The group $G$ acts on it by conjugation and the embedding $G \subset \text{SL}(n, \mathbb{C}) \subset \text{Mat}(n, \mathbb{C})$, induces a $G$-equivariant epimorphism $\mathbb{C}[\text{Mat}(n, \mathbb{C})^r] \to \mathbb{C}[G^r]$, which restricts to an epimorphism $\mathbb{C}[\text{Mat}(n, \mathbb{C})^r]^G \to \mathbb{C}[G^r]^G$ by the existence of Reynolds operators. Now the statement follows from the results of [Pro76] stating that $\mathbb{C}[\text{Mat}(n, \mathbb{C})^r]^G$ is generated by traces of monomials in matrices and their inverses for $G = \text{SL}(m, \mathbb{C}), \text{SO}(2m + 1, \mathbb{C})$, and $\text{Sp}(2m, \mathbb{C})$. □
The above statement does not hold for $G = \text{SO}(2m, \mathbb{C}), m \geq 1$. In that case $\mathbb{C}[[\mathcal{X}(\Gamma, G)]]$ is a $\mathcal{T}(\Gamma, G)$-algebra finitely generated by expressions involving Pfaffians, [Sik15a, Proposition 16].

In this paper we intend to elaborate on the relationship between coordinate rings of character varieties and their trace algebras.

Denote by $\varphi$ the natural projection map

$$\varphi : \text{Hom}(\Gamma, G) / \mathcal{N}(G) \to \mathcal{X}(\Gamma, \text{SL}(n, \mathbb{C})),$$

where $\mathcal{N}(G)$ is the normalizer of $G$ in $\text{SL}(n, \mathbb{C})$. Since a normalizer of a connected reductive group is reductive, the above GIT quotient is well defined. By [Vin96, Corollary 2 to Theorem 1], $\varphi$ is a finite morphism and, hence, its image is closed.

**Lemma 2.** There is an isomorphism

$$\psi : \mathcal{T}(\Gamma, G) \to \mathbb{C}[[\varphi(\text{Hom}(\Gamma, G) / \mathcal{N}(G))]]$$

sending regular functions $\tau_\gamma$ to regular functions $\tilde{\tau}_\gamma$ on $\varphi(\text{Hom}(\Gamma, G) / \mathcal{N}(G))$ defined by $\tilde{\tau}_\gamma([\rho]) = \text{tr}(\rho(\gamma))$. Since $\tau_\gamma$ generate $\mathcal{T}(\Gamma, G)$, this condition defines $\psi$ uniquely.

**Proof.** Since any polynomial identity in $\tau_\gamma$'s on $\mathcal{X}(\Gamma, G)$ holds if and only if it holds on $\varphi(\text{Hom}(\Gamma, G) / \mathcal{N}(G))$ as well, the above homomorphism is well defined and injective. Since $\mathbb{C}[[\mathcal{X}(\Gamma, \text{SL}(n, \mathbb{C}))]]$ is generated by trace functions, $\mathbb{C}[[\varphi(\text{Hom}(\Gamma, G) / \mathcal{N}(G))]]$ is generated by trace functions as well, and hence, $\psi$ is onto. \(\square\)

We have a map

$$\Psi : \text{Ch}(\Gamma, G) \to \text{Spec} \mathcal{T}(\Gamma, G)$$

sending $\chi_\rho$ to a algebra homomorphism $\mathcal{T}(\Gamma, G) \to \mathbb{C}$, determined by $\tau_\gamma \mapsto \chi_\rho(\gamma)$.

It is not hard to see that $\Psi$ is well defined. Furthermore, we have:

**Proposition 3.** $\Psi$ is a bijection.

**Proof.** If $\Psi(\chi_\rho) = \Psi(\chi_\rho')$ then $\chi_\rho(\gamma) = \chi_\rho'(\gamma)$ for all $\gamma \in \Gamma$, implying $\chi_\rho = \chi_\rho'$. Thus, $\Psi$ is injective.

By Lemma 2 it remains to prove that

$$\Psi : \text{Ch}(\Gamma, G) \to \varphi(\text{Hom}(\Gamma, G) / \mathcal{N}(G))$$

is surjective. Indeed, for every $\rho \in \text{Hom}(\Gamma, G)$, $\Psi(\chi_\rho) = \varphi(\rho)$ and, hence, $\Psi$ is onto. \(\square\)

The above proposition provides for the structure of an algebraic set on $\text{Ch}(\Gamma, G)$ with its coordinate ring being $\mathcal{T}(\Gamma, G)$. By analogy with the $G$-character variety of $\Gamma$, we will call $\text{Ch}(\Gamma, G)$ the variety of $G$-characters of $\Gamma$ or simply the variety of characters when the context is clear.

By Lemma 2, $\varphi(\text{Hom}(\Gamma, G) / \mathcal{N}(G)) = \text{Ch}(\Gamma, G)$ and, hence, $\varphi$ can be written as

$$\varphi : \text{Hom}(\Gamma, G) / \mathcal{N}(G) \to \text{Ch}(\Gamma, G).$$
Denote the set of \textit{trace-preserving automorphisms} of $G$ by $\text{Aut}_T(G)$, that is, the automorphisms $\alpha \in \text{Aut}(G)$ such that for all $g \in G$, $\text{tr}(\alpha(g)) = \text{tr}(g)$. Note that $\text{Aut}_T(G)$ acts naturally on $\text{Hom}(\Gamma, G)$ by $(\alpha, \rho) \mapsto \alpha \circ \rho$ and that this action descends to an action of $\text{Aut}_T(G)/\text{Inn}(G)$ on $\mathcal{X}(\Gamma, G)$, where $\text{Inn}(G)$ is the inner automorphism group of $G$. We have a natural map
\[ \pi : N(G) \to \text{Aut}_T(G) \]
given by $h \mapsto C_h$ where $C_h(g) = hgh^{-1}$. Since $\text{Ker}(\pi)$ contains the center of $\text{SL}(n, \mathbb{C})$, $\pi$ is not one-to-one.

Recall that $\rho : \Gamma \to G$ is irreducible if its image is not properly contained in any parabolic subgroup of $G$, see for example [Sik12].

\textbf{Lemma 4.} $\pi$ is onto.

\textbf{Proof.} Let $G_c$ be a maximal compact subgroup of $G$. By [Gel08, Lemma 1.10], there exists a representation $\rho : F_2 \to G_c$ with a dense image. Then the image of $\rho$ in $G$ is Zariski dense and, in particular, $\rho : F_2 \to G_c \to G$ is irreducible. Let $\alpha \in \text{Aut}_T(G)$. Since $\mathbb{C}[\mathcal{X}(F_2, \text{SL}(n, \mathbb{C}))] = \mathcal{T}(F_2, \text{SL}(n, \mathbb{C}))$ and $\rho$ and $\alpha \rho$ have the same character, they coincide in $\mathcal{X}(F_2, \text{SL}(n, \mathbb{C}))$. Since $G$ is a reductive subgroup of $\text{SL}(n, \mathbb{C})$, $\rho$ and $\alpha \rho$ are completely reducible representations in $\text{SL}(n, \mathbb{C})$ and, therefore, their $\text{SL}(n, \mathbb{C})$-conjugation orbits are closed in $\text{Hom}(F_2, G)$, see [LMS85, Theorem 1.27]. Since there is a unique closed orbit in each equivalence class in $\mathcal{X}(F_2, \text{SL}(n, \mathbb{C}))$ (see for example [Del03, Corollary 6.1]), $\rho$ and $\alpha \rho$ are conjugate in $\text{SL}(n, \mathbb{C})$. Thus, since $\rho$ is irreducible, this implies that $\alpha : G \to G$ coincides with conjugation of $G$ by some element of $\text{SL}(n, \mathbb{C})$. \hfill \Box

We denote
\[ \text{Out}_T(G) := \text{Aut}_T(G)/\text{Inn}(G), \]
and call this group the \textit{trace-preserving outer automorphisms}.

\textbf{Lemma 5.} $\text{Out}_T(G)$ is finite.

\textbf{Proof.} $\text{Out}_T(G)$ is a subgroup of $\text{Out}(G)$ and, consequently, of $\text{Out}(\mathfrak{g})$ which coincides with the automorphism group of the Dynkin diagram of $\mathfrak{g}$. Consequently, $\text{Out}_T(G)$ is finite for semisimple $G$.

For the general not semisimple case, note that $\text{Out}_T(G)$ is the epimorphic image of $\mathcal{N}(G)/G$ which is finite by [Vin96, Corollary 3b]. \hfill \Box

Clearly $\text{Out}_T(G)$ descends to an action on $\mathcal{X}(\Gamma, G)$ yielding an isomorphism
\[ \text{Hom}(\Gamma, G)/\mathcal{N}(G) \to \mathcal{X}(\Gamma, G)/\text{Out}_T(G). \]
Consequently, $\varphi$ becomes a map
\[ \varphi : \mathcal{X}(\Gamma, G)/\text{Out}_T(G) \to \mathcal{C}h(\Gamma, G). \]

By [Vin96, Corollary 2 of Theorem 1], the map $\mathcal{X}(\Gamma, G) \to \mathcal{C}h(\Gamma, G)$ is finite. Consequently, $\varphi$ is finite too.
Denote the set of characters of irreducible $G$-representations of $\Gamma$ by $\text{Ch}^i(\Gamma, G)$. It is a Zariski open subset of $\text{Ch}(\Gamma, G)$ by [Sik12 Proposition 27]. Let $\varphi^i$ be the restriction of $(\mathbb{I})$ to $\mathcal{X}^i(\Gamma, G)/\text{Out}_T(G) \subset \mathcal{X}(\Gamma, G)/\text{Out}_T(G)$.

**Theorem 6.** Let $G \subset \text{SL}(n, \mathbb{C})$ be a connected reductive group, and $\Gamma$ a finitely generated group. Then:

1. $\text{Out}_T(G)$ acts freely on $\mathcal{X}^i(\Gamma, G)$.
2. $\varphi^i : \mathcal{X}^i(\Gamma, G)/\text{Out}_T(G) \rightarrow \text{Ch}^i(\Gamma, G)$ is a finite, birational bijection.
3. If $\mathcal{X}^i(\Gamma, G)/\text{Out}_T(G)$ is normal, then $\varphi^i$ is a normalization map.

**Proof.** We first prove (1). Suppose for $[\alpha] \in \text{Out}_T(G)$ that $[\alpha] \cdot [\rho] = [\rho]$ for some $[\rho] \in \mathcal{X}^i(\Gamma, G)$. Any two irreducible $G$-representations equivalent in $\mathcal{X}(\Gamma, G)$ are conjugate by $G$. Thus, there exists $h \in G$ so for all $\gamma \in \Gamma$ we have $\alpha(\rho(\gamma)) = h\rho(\gamma)h^{-1}$. Since $[\rho]$ is irreducible, $\alpha(g) = hgh^{-1}$ for all $g \in G$. In particular, $\alpha \in \text{Inn}(G)$ and $[\alpha]$ is the identity in $\text{Out}_T(G)$.

Next we prove (2). Since $\varphi^i$ is a restriction of a finite map to an open set, it is also finite. It is injective by [AB94 Proposition 8.1], and $\varphi^i$ is onto by definition. Then, by [Vin96 Lemma 1], $\varphi^i$ is birational.

Lastly, (3) is an immediate consequence of (2). \qed

It is a general fact that bijective morphisms $f : X \rightarrow Y$ give equality $[X] = [Y]$ in the Grothendieck ring $K_0(\text{Var}/\mathbb{C})$, for example see [BBS13 Page 115]. Consequently, Theorem 6(2) implies the following corollary.

**Corollary 7.** Under the assumptions of Theorem 6, $[\mathcal{X}^1(\Gamma, G)/\text{Out}_T(G)] = [\text{Ch}^1(\Gamma, G)]$ in the Grothendieck ring $K_0(\text{Var}/\mathbb{C})$.

**Corollary 8.** Under the assumptions of Theorem 6, we have:

1. If $\mathcal{X}(\Gamma, G)/\text{Out}_T(G)$ is irreducible and contains an irreducible representation, then $\varphi$ is a birational morphism.
2. If $\mathcal{X}(\Gamma, G)/\text{Out}_T(G)$ is normal and contains an irreducible representation, then $\varphi$ is a normalization map.
3. $\varphi$ is a normalization map for $\Gamma$ a free group of rank $r \geq 2$ (compare [Vin96]).
4. $\varphi$ is a normalization map for $\Gamma$ a genus $g \geq 2$ surface group and $G = \text{SL}(m, \mathbb{C})$ or $\text{GL}(m, \mathbb{C})$, $m \geq 1$.
5. $\varphi$ is a normalization map for $\Gamma$ a free abelian group of rank $r \geq 1$ and $G = \text{SL}(m, \mathbb{C})$ or $\text{Sp}(2m, \mathbb{C})$, $m \geq 1$.

**Proof.** The irreducible representations form an open subset of $\text{Hom}(\Gamma, G)$. Being non-empty, $\mathcal{X}^i(\Gamma, G)/\text{Out}_T(G)$ is dense in $\mathcal{X}(\Gamma, G)/\text{Out}_T(G)$. Now (1) follows from Theorem 6 and [Vin96 Lemma 1]. Then (2) is an immediate consequence of (1) and the finiteness of $\varphi$.

Since $\text{Hom}(F_r, G)$ is smooth, and $\mathcal{X}(\Gamma, G)/\text{Out}_T(G)$ is its GIT quotient, both spaces are normal. Since there is an irreducible $G$-representation of $F_r$ for any $r \geq 2$ by [Sik12 Proposition 29], (3) follows from (2).
By [Sim94a, Sim94b], \( \mathcal{X}(\Gamma, G) \) is normal for surface groups when \( G \) is \( \text{SL}(n, \mathbb{C}) \) or \( \text{GL}(n, \mathbb{C}) \). Since there is an irreducible representation \( \Gamma \to F_g \to \text{SL}(n, \mathbb{C}) \) for every \( g \geq 2 \) again by [Sik12, Proposition 29], (4) also follows from (2).

Lastly, by [Sik14] \( \mathcal{X}(\mathbb{Z}^r, G) \) is irreducible and normal for \( G \) equal to \( \text{SL}(m, \mathbb{C}) \) or \( \text{Sp}(2m, \mathbb{C}) \). In these cases \( \text{Out}(G) \) is trivial (see Section 3), and by Proposition 1 \( \mathcal{X}(\mathbb{Z}^r, G) = \mathcal{C}h(\mathbb{Z}^r, G) \). Hence, (5) follows.

In particular, for \( \Gamma \) and \( G \) from (2) in Corollary 8, regular functions on \( \mathcal{X}(\Gamma, G)/\text{Out}(G) \) can be expressed as ratios of polynomials in trace functions.

As observed above, the map \( \varphi : \mathcal{X}(F_r, G)/\text{Out}(G) \to \mathcal{C}h(F_r, G) \) is injective on the non-empty Zariski open subset of irreducible representations. However, it is not always injective on \( \mathcal{X}(F_r, G)/\text{Out}(G) \), see examples in [Vin96].

On the other hand, Proposition 1 shows \( \varphi \) is an isomorphism for \( \Gamma = F_r \) and the minimal dimensional algebraic representations of \( G \) equal to \( \text{SL}(m, \mathbb{C}), \text{Sp}(2m, \mathbb{C}) \), or \( \text{SO}(2m + 1, \mathbb{C}) \). It is also an isomorphism for \( G = \text{SO}(2m, \mathbb{C}) \) by [Sik15a, Proposition 16].

**Question 9.** Is \( \varphi : \mathcal{X}(F_r, G)/\text{Out}(G) \to \mathcal{C}h(F_r, G) \) an isomorphism for any quasi-simple \( G \) and any minimal dimensional algebraic representation of \( G \)?

The above question is open for groups \( G \) other than those mentioned above. In particular, it is open for exceptional groups. (The example of \( G = \text{SL}(2, \mathbb{C}) \times \text{SO}(3, \mathbb{C}) \) of [Vin96] shows the assumption of \( G \) being quasi-simple is essential.)

Let \( \mathcal{X}^{\text{sm}}(\Gamma, G) \) and \( \mathcal{C}h^{\text{sm}}(\Gamma, G) \) denote the smooth loci of \( \mathcal{X}(\Gamma, G) \) and \( \mathcal{C}h(\Gamma, G) \), respectively.

**Corollary 10.** Let \( F_r \) be a free group of rank \( r \geq 2 \). If \( G \) is a connected reductive subgroup of \( \text{SL}(n, \mathbb{C}) \) such that all simple factors of the Lie algebra of the derived subgroup \( [G, G] \) have rank 2 or more, then \( \mathcal{X}^{\text{sm}}(F_r, G) \) is étale equivalent to \( \mathcal{C}h^{\text{sm}}(F_r, G) \).

**Proof.** Under the assumptions of the theorem, by [FLR16] Theorem 7.4 \( \mathcal{X}(F_r, G)^{\text{sm}} \subset \mathcal{X}^{i}(F_r, G) \). Since \( \text{Out}(G) \) is a finite group acting freely on \( \mathcal{X}^{i}(F_r, G) \), we conclude that \( \mathcal{X}^{\text{sm}}(F_r, G) \to \mathcal{X}^{\text{sm}}(F_r, G)/\text{Out}(G) \) is étale. Since \( \phi^i \) is a finite, bijective birational map, restricting it to the smooth locus gives a local analytic isomorphism. Hence \( \mathcal{X}^{\text{sm}}(F_r, G)/\text{Out}(G) \to \mathcal{C}h^{\text{sm}}(F_r, G) \) is also étale. Since the composition of étale morphisms is étale, the result follows.

\[ \square \]
More generally, denote the set of singular points of $\varphi^i$ by $Z$. Then

$$\mathcal{X}^s(\Gamma, G) = \varphi^{-1}(\text{Ch}^i(\Gamma, G) - Z)$$

is a Zariski open subset $\mathcal{X}^i(\Gamma, G)$. Since $\varphi^i$ is finite, its restriction to the subspace $\mathcal{X}^s(\Gamma, G)/\text{Out}_T(G)$ is étale and, hence, the composite map

$$\mathcal{X}^s(\Gamma, G) \rightarrow \mathcal{X}^s(\Gamma, G)/\text{Out}_T(G) \rightarrow \text{Ch}^i(\Gamma, G)$$

is étale as well. In particular, if $\text{Out}_T(G)$ is trivial then the above map restricts to a diffeomorphism of the smooth locus of $\mathcal{X}^s(\Gamma, G)$.

We next analyze the groups $\text{Out}_T(G)$ for quasi-simple $G$ and show that they are often trivial.

3. Trace preserving outer automorphisms

$\text{Out}(G)$ is a subgroup of the symmetries of the Dynkin diagram of the Lie algebra of $G$ (see [FH91, Appendix D]) and, hence, $\text{Out}(G)$ is finite for $G$ semisimple. By definition, $\text{Out}_T(G) \subset \text{Out}(G)$ and, thus, $\text{Out}_T(G)$ is trivial whenever $\text{Out}(G)$ is trivial.

**Example 11.** $\text{Out}_T(G)$ is trivial for every matrix realization of odd orthogonal groups $\text{SO}(2m + 1, \mathbb{C})$, symplectic groups $\text{Sp}(2m, \mathbb{C})$, and the complex groups $G_2, F_4, E_7, E_8$.

**Proof.** This follows from the fact that $B_n, C_n, G_2, F_4, E_7, E_8$ have no symmetries of their Dynkin diagrams, see for example [Sam90]. □

For $m \geq 3$, $\text{Out}(\text{SL}(m, \mathbb{C})) = \langle \sigma \mid \sigma^2 \rangle$, where $\sigma$ is the Cartan involution $\sigma(A) = (A^{-1})^T$. Thus, the group $\text{Out}_T(\text{SL}(m, \mathbb{C}))$ is either trivial or equal to $\text{Out}(\text{SL}(m, \mathbb{C}))$ for $m > 2$ depending on whether $\sigma$ is conjugation by an element of the normalizer of $\text{SL}(m, \mathbb{C})$ in the ambient group $\text{SL}(n, \mathbb{C})$.

For the canonical matrix realization of $\text{SL}(m, \mathbb{C})$ it is easy to see that $\sigma$ is not trace-preserving for any $m > 2$. On the other hand, consider the adjoint representation of $\text{SL}(m, \mathbb{C})$ which acts irreducibly on

$$M_0 := \{M \in \text{Mat}(m, \mathbb{C}) \mid \text{tr}(M) = 0\}$$

by conjugation. It is a representation of $\text{SL}(m, \mathbb{C})$ into $\text{GL}(m^2-1, \mathbb{C})$, which we denote by $\text{Ad}$.

**Example 12.** For the standard matrix realization of $\text{SL}(m, \mathbb{C})$, the group $\text{Out}_T(\text{SL}(m, \mathbb{C}))$ is trivial. For $m \geq 3$,

$$\text{Out}_T(\text{Ad}(\text{SL}(m, \mathbb{C}))) \cong \mathbb{Z}/2\mathbb{Z}.$$ 

**Proof.** Since $\text{Out}(\text{SL}(m, \mathbb{C}))$ is trivial for $m = 2$ and $\text{tr}(\sigma(A)) \neq \text{tr}(A)$ in general for $A \in \text{SL}(m, \mathbb{C})$ for $m \geq 3$, the first part follows.

For the second part, it is enough to prove that there is $P \in \text{GL}(M_0)$ such that $\sigma$ coincides with conjugation by $P$ via $\text{Ad}$, that is,

$$P\text{Ad}(\sigma)(A)P^{-1} = \text{Ad}(A)$$
for every $A \in M_0$. That means
\[ P(\sigma(A)(P^{-1}X)\sigma(A)^{-1}) = AXA^{-1} \]
for every $X \in M_0$. Note now that $P(M) = M^T$, obviously an invertible linear map, satisfies the above equation. Thus, $Out_T(\text{Ad}(\text{SL}(m, \mathbb{C}))) = \langle \sigma \mid \sigma^2 \rangle$, as required.

**Example 13.** For the standard matrix realization of $\text{SO}(2m, \mathbb{C})$, the group $Out_T(\text{SO}(2m, \mathbb{C}))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

**Proof.** Let $\sigma$ be an automorphism on $\text{SO}(2m, \mathbb{C})$ given by conjugation by an orthogonal matrix of determinant $-1$ as in [Sik15a]. Then $\sigma$ considered as an element of $Out(\text{SO}(2m, \mathbb{C}))$ does not depend on the choice of such matrix, [Sik15a]. Furthermore, $\sigma^2 = \text{Id}$. It is the only non-trivial outer automorphism of $\text{SO}(2m, \mathbb{C})$ seen by considering the Dynkin diagram (for $m \neq 4$). Note that for $m = 4$ the Dynkin diagram has order three symmetry (trality) and so $Out(\text{Spin}(8, \mathbb{C}))$ is the symmetric group on 3 letters, but descending to $\text{SO}(8, \mathbb{C})$ reduces the outer automorphisms back down to $\mathbb{Z}/2\mathbb{Z}$ again. So assuming the defining matrix realization of $\text{SO}(2m, \mathbb{C})$, $\sigma$ preserves the trace and, hence, $Out_T(\text{SO}(2m, \mathbb{C})) = \langle \sigma \mid \sigma^2 \rangle$. \hfill \Box

We note that as shown in [Sam90], for the complex algebraic form of $E_6$, $Out(E_6)$ is generated by an order 2 element as well.

**Example 14.** There exists a matrix realization of $E_6$ so that $Out_T(E_6) \cong \mathbb{Z}/2\mathbb{Z}$.

**Proof.** The minimal dimensional irreducible representation of $E_6$ gives an embedding of $E_6$ into $\text{GL}(27, \mathbb{C})$. Then the generator of $Out(E_6)$ is, as above, the Cartan involution $\sigma(A) = (A^{-1})^T$ [Yok09, page 68]. As above, we can then consider the adjoint representation into $\text{GL}(78, \mathbb{C})$ which will allow the Cartan involution to be conjugation by an element of the normalizer of $E_6$. \hfill \Box

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