Electric instability in superconductor-normal conductor ring

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Non-linear electrodynamics of a ring-shaped Andreev interferometer (superconductor-normal conductor-superconductor hybrid structure) inductively coupled to a circuit of the dissipative current is investigated. The current-voltage characteristics (CVC) is demonstrated to be a series of loops with several branches intersecting in the CVC origin. The sensitivity of the transport current to a change of the applied external magnetic flux can be comparable to the one of the conventional SQUID's. Spontaneous arising of coupled non-linear oscillations of the transport current, the Josephson current and the magnetic flux in Andreev interferometers are also predicted and investigated. The frequency of these oscillations can be varied in a wide range, while the maximal frequency can reach $\omega_{\text{max}} \sim 10^{12} \text{ sec}^{-1}$.

Recent years much attention has been paid to the charge transport in mesoscopic systems which combine normal conductor (N) and superconductor (S) elements (for review papers see, e.g., [2,3] and references there). In such a hybrid structure superconducting correlations penetrate into the normal conductor within mesoscopic distances changing its transport properties there. This quantum effect is most pronounced in S-N-S structures ("Andreev interferometers") in which the quantum interference gives rise to a high sensitivity of these superconducting correlations to the phase difference between the superconductors $\varphi$. It results, in particular, in oscillations of the Josephson current $J_s = J_s(\varphi)$ with a change of $\varphi$, the amplitude of the oscillations [3] being $J_c \sim N_\perp e v_F / L_N$ ($N_\perp = S_N / \lambda_F^2$, $\lambda_F = h / p_F$ - the Fermi wave length, $p_F$ - the electron Fermi momentum, $e$ - the electron charge, $v_F$ - the Fermi velocity, $S_N$ and $L_N$ are the cross-section area and the length of the normal conductor, respectively).

If the Andreev interferometer is coupled to reservoirs of normal electrons, the conductance $G(\varphi)$ and hence the dissipative current, $J_d = G(\varphi)V$, between the reservoirs ($V$ is the bias voltage) are also phase-dependent, the amplitude of the conductance oscillations being proportional to $N_\perp e^2 / h$ (in many experiments $N_\perp \sim 10^2 \pm 10^5$).

In a ring-shaped geometry of the S-N-S structure, the phase difference between the superconductors is controlled by the magnetic flux $\Phi = HS_r$ ($H$ is the magnetic field threading the ring, $S_r$ is the area of the ring): $\varphi = -2\pi \Phi / \Phi_0$ where the flux quantum $\Phi_0 = \pi \hbar c / e$, $c$ -the light velocity. This allows to control the Josephson and the dissipative currents with a change of the external magnetic field $H_{\text{ext}}$ that is favorable for various applications. On the other hand, the Josephson current (which depends on the magnetic flux in the ring) creates its proper magnetic field which modifies this flux. As a result, for a ring of a large enough self-inductance, the relation between the magnetic flux $\Phi$ and the external magnetic field $H_{\text{ext}}$ turned out to be highly non-linear with hysteresis loops in the dependence of $\Phi$ on $H_{\text{ext}}$ as shown in Fig.2 (see, e.g., [4]). This self-inductive property of the Josephson current has been used for creation of Superconducting Quantum Interference Devices (SQUID) which have found extensive applications for extremely precise measurements [4].

A dissipative current flow through the S-N-S ring creates a new situation in which the Josephson and dissipative currents are coupled together through their joint influence on the magnetic flux inside the S-N-S ring that, in its turn, affects the both currents themselves.

The aim of this paper is to show that the inductive interaction between the dissipative and Josephson currents results in a complicated loop-shaped form of the current-voltage characteristics (CVC) of S-N-S structures schematically presented in Fig.1. In the general case several branches of the CVC (which correspond to different values of the magnetic flux $\Phi$ inside the ring) intersect in the origin of CVC ($J_d = 0, V = 0$) as is shown in the insertion of Fig.(4). Stable non-linear periodic in time oscillations of the dissipative current $J_d$, the Josephson current $J_s$ and the magnetic flux $\Phi$ inside the ring (and hence the phase difference $\varphi$) are predicted and investi-
1. Loop-shaped current-voltage characteristics. The total flux of the magnetic field threading the ring, Φ, is created by the external magnetic field \( H_{ext} \), the proper magnetic fields of the Josephson current \( J_s \) and the dissipative current \( J_d \), and hence can be written as

\[
\Phi = \Phi_{ext} + \frac{L_r}{c} J_s + \frac{L_{12}}{c} J_d
\]

where \( \Phi_{ext} = H_{ext} S_r \) is the magnetic flux of the external magnetic field \( H_{ext} \), \( L_r \) is the self-inductance of the ring, \( L_{12} \) is the mutual inductance of the ring and the dissipative current circuit.

Eq.(1) together with the conventional relation \( J_d = G(\varphi)V \) define a parametric form of the CVC:

\[
J_d = J_d(\varphi) \equiv \frac{1}{L_{12}} \left( \frac{e\Phi_0}{2\pi} (\varphi_{ext} - \varphi) - L_r J_s(\varphi) \right)
\]

\[
V = \frac{J_d(\varphi)}{G(\varphi)}; \quad \varphi_{ext} \equiv -2\pi\Phi_{ext}/\Phi_0
\]

One of the distinguishing features of the CVC of the system under consideration is its "many-fold degeneracy", that is several branches of the CVC (which correspond to different values of the magnetic flux threading the ring \( \Phi \)) can intersect in its origin \( (V = 0, J_d = 0) \). As is seen from Eq.(2) at \( J_d = 0 \) and Fig.2, the "degeneracy factor" is equal to the number of intersections of the vertical line \( \Phi_{ext} = \text{const} \) and the curve \( \Phi = \Phi(\Phi_{ext}) \).

Other key features of the CVC can be found if one consider the differential resistance \( dV/dJ_d \) which is readily obtained from Eq.(2):

\[
\frac{dV}{dJ_d} = \frac{1}{G(\varphi)} \left( 1 + \frac{2\pi L_{12} J_s}{e\Phi_0} G'(\varphi) \right) \quad \varphi = \varphi(J_d)
\]

Here and below the sign "prime" means the derivative with respect to \( \varphi \); the function \( A(\varphi) \) [5] is

\[
A(\varphi) = 1 + \frac{2\pi L_r}{e\Phi_0} J_s'(\varphi)
\]

All quantities in Eq.(3) are taken at \( \varphi = \varphi(J_d) \) which is a solution of the first equation in Eq.(2). Using Eq.(2) one finds

\[
d\varphi/dJ_d = -\frac{2\pi L_{12}}{e\Phi_0} \frac{1}{A(\varphi)}
\]

If \( L_r > L^{(1)}_{cr} = e\Phi_0/(2\pi \max(|J_s'|)) \) there are values of \( \varphi \) at which \( A(\varphi) = 0 \). Therefore, in this case, as follows from Eq.(3) the CVC \( J_d(V) \) inevitably has points at which the differential resistance \( dV/dJ_d \) goes to infinity changing its sign there. On the other hand, the derivative \( G' \) also changes its sign with a change of \( \varphi \) that can provide points at which \( dV/dJ_d \) goes to zero changing its sign at them. The consecutive order of this peculiarities with an increase of \( J_d \) (that is the form of the CVC) depends on the relative positions of the maxima and minima of \( G(\varphi) \) and \( J_s(\varphi) \).

For the case of a low transparency of the potential barrier \( t_r \ll 1 \) the conductance \( G(\varphi) \) has maxima at odd numbers of \( \pi \) [2] while the Josephson current \( J_s \equiv 0 \) at them and has a maximum and a minimum in their vicinities [3] as schematically shown in Fig.3. This general information together with Eq.(3 - 5) would suffice to find the CVC to be loop-shaped if \( L_r > L^{(1)}_{cr} \). In order to see it let us start from the external magnetic field which corresponds to \( \varphi = 0 \) and hence \( G' = 0 \), \( J' > 0 \) and therefore \( dV/dJ_d > 0 \) according to Eq.(3). As at this point \( d\varphi/dJ_d < 0 \) the phase difference \( \varphi \) decreases with an increase of current \( J_d \) (see Eq.(5)). Therefore, with an increase of the current \( G' \) becomes negative and after passing the minimum of \( J_s(\varphi) \), \( A(\varphi) \) starts to decrease because \( J' < 0 \) now (see Fig.3), and when \( A \) is small enough (but positive yet), the differential resistance becomes \( dV/dJ_d = 0 \). With a further increase of the current \( A(\varphi) \to +0 \) while \( G' \) is yet negative, and hence \( dV/dJ_d \to -\infty \). When \( A(\varphi) = 0 \) the differential resistance \( dV/dJ_d = +\infty \). In order to follow this second branch of CVC one should decrease the current because \( d\varphi/dJ_d > 0 \) (see Eq.(5)). Pursuing such a reasoning...
one easily finds the current-voltage characteristic to be a series of loops which touch the lines \( J_d = G_{\text{min}} V \) and \( J_d = G_{\text{max}} V \) (in our case \( G_{\text{min}} = G(0), \ G_{\text{max}} = G(\pi) \)); the number of loops intersecting in the origin of the CVC increases with an increase of the ring inductance. Such a loop-shaped CVC takes place for the both cases of a diffusive or a ballistic normal conductor of the S–N–S ring-shaped interferometer.

![Diagram](image)

**FIG. 3.** Typical dependences of the Josephson current and the conductance on the superconductor phase difference \( \varphi \) for the case of a low transparency \( (t_r \ll 1) \) of the potential barrier between the normal sections of the sample and the lead \( (J_s = N_1 e v_F / L_N \) and \( G_0 = N_1 e^2 / (\pi \hbar) \)).

An example of such a current-voltage characteristics is presented in Fig.4 for the case of a ballistic normal section of the Andreev interferometer in the presence of potential barriers at the N/S interfaces. In this case the dependences of the conductance \( G(\varphi) \) [7] and the Josephson current \( J_s(\varphi) \) can be written as follows:

\[
G(\varphi) = \frac{t_F^2 G_0}{\sqrt{1 + \left(1 + \frac{r_N^{(2)}}{r_A^{(2)}} \cos \varphi + \frac{t_f^2}{2} - \frac{r_N^{(1)}}{r_A^{(1)}} \right)^2}}
\]

\[
J_s(\varphi) = N_1 \frac{2 e v_F}{\pi L_N} \sin \varphi \times \int_0^{2\pi} \frac{d\phi}{2 \pi \sin \varphi} \tan \left( \frac{e^{-2t_r} \sin \varphi - \phi}{1 - e^{-2t_r} \cos \varphi - \phi} \right)
\]

Here \( G_0 = N_1 e^2 / (\pi \hbar) \), \( v_F = N_1^{-1} \sum \bar{n} v_{\bar{n}} \sim v_F \) and \( v_{\bar{n}} \) is the electron velocity in the \( \bar{n} \)-th transverse mode; \( r_N^{(1,2)} \) are the normal reflection amplitudes at N/S interfaces 1 and 2 while \( r_A^{(1,2)} = 1 \), (see [8]); \( \cos \varphi - \phi = \left[ r_N^{(1)} r_A^{(2)} \cos \phi - r_N^{(1)} r_A^{(1)} \right] \cos \phi \) and \( \sin \varphi - \phi = \sqrt{1 - \cos^2 \varphi - \phi} \). Expanding Eq.(7) in exp\{\varphi - \phi\} and taking the limit \( t_r \to 0, t_N^{(1,2)} \to 0 \) one reduces it to the well-known expression for the Josephson current in a 3D SNS junction [9].

Inserting Eq.(6,7) in Eq.(2) and performing numerical calculations one obtains the current-voltage characteristics shown in Fig.4.

![Current-voltage characteristics](image)

**FIG. 4.** Current - voltage characteristics of an SNS ring for the external flux \( \Phi_{\text{ext}} / \Phi_0 = (2l + 1) \pi, \ l = 0, \pm 1, \pm 2, \ldots; \ t_r = 0.1, \ t_N^{(1)} = 0.2, \ t_N^{(2)} = 0.25, \) and \( L_r = 2.5 e \Phi_0 / J_c; \ J_0 = (L_r / L_{12}) J_c, \ V_0 = 2 e v_F / (\pi c L_N) \). The insertion shows a zoomed vicinity of the CVC origin.

2. **Electromagnetic self-oscillations.** A change of the magnetic flux threading a superconductor-normal conductor ring results in the following current in the ring:

\[
J_r = J_s(\varphi) - \frac{1}{c R} \frac{d\Phi}{dt}
\]

where \( R \) is the resistance of the normal section of the S–N–S ring [10]. Using \( \varphi = -2\pi \Phi / \Phi_0 \) one sees Eq.(8) to be the equation of the ac Josephson effect.

Taking into account the self-inductances of the ring \( L_r \) and the transport current circuit \( L_{11} \) together with their mutual inductances \( L_{12} \), after using Eq.(1) (in which \( J_s \) should be changed to \( J_r \)) and Eq.(8) one gets a set of equations that describes the time evolution of the transport current and the superconductor phase difference as follows:

\[
\frac{\Delta L_{11}}{L_{11}^2} \frac{dJ_d}{dt} + \frac{\Delta L_{12}}{2\pi c R} \frac{d^2\varphi}{dt^2} + \frac{L_{12}}{L_{11}} \frac{dJ_s}{dt} = \frac{J_d}{G(\varphi)} = V;
\]

\[
\frac{\Phi_0 L_r}{2\pi c R} \frac{d\varphi}{dt} + c \Phi_0 + L_r J_s(\varphi) + L_{12} J_d = -e \Phi_{\text{ext}}
\]

The static solutions of Eq.(9) give the CVC (Eq.(2)).

In order to investigate time-evolution of the system it is convenient to eliminate \( J_d \) from the set of equations Eq.(9) and get the following closed equation for \( \varphi(t) \):

\[
\frac{\Delta L_{12} e f}{R \pi^2 c R^2} \frac{d^2\varphi}{dt^2} + \gamma(\varphi) \frac{d\varphi}{dt} + F(\varphi) = 0
\]

where \( L_{12} e f = (L_r L_{11} - L_{12}^2) / L_{11} \). Eq.(10) is the equation for a non-linear oscillator under a "friction"
\[ \gamma(\varphi) = 1 + \frac{2\pi L_{\text{eff}}}{c\Phi_0} J'_s(\varphi) + \frac{L_r}{L_{11}} \frac{1}{RG(\varphi)} \]  \hspace{1cm} (11) 

and a "force"

\[ F(\varphi) = \frac{c^2}{L_{11}G(\varphi)} \times 
\left( \varphi - \varphi_{\text{ext}} + \frac{2\pi L_{\text{eff}}}{c\Phi_0} J_s(\varphi) + \frac{2\pi L_{12}}{c\Phi_0} G(\varphi)V \right) \]  \hspace{1cm} (12)

For low values of \( L_{11} \) the "friction" \( \gamma > 0 \) at any value of \( \varphi \) and hence, starting from any initial state, the system approaches one of its static states determined by Eq.(1) (in which \( J_d = G(\varphi)V \)). With an increase of \( L_{11} \), the "friction" \( \gamma(\varphi) \) becomes negative in a certain range of \( \varphi \) and the static state can become unstable. Investigations of the stability of the static solutions of Eq.(10) \( \varphi = \varphi_{\text{st}} \) show that the critical value of \( L_{11} \) is determined by the condition \( \gamma(\varphi_{\text{st}}) = 0 \), that is

\[ L_{cr} = \frac{L_r}{BRG(\varphi_{\text{st}})}; \]

\[ B = - \left( 1 + \frac{2\pi L_{\text{eff}}}{c\Phi_0} J'_s(\varphi_{\text{st}}) \right) > 0 \]  \hspace{1cm} (13)

The Poincare method [11] shows that in the plane \((\varphi, \dot{\varphi})\) a stable limit cycle arises if

\[ L_{11} > L_{cr}; \quad b = 1 + \frac{2\pi L_r}{c\Phi_0} J'_s(\varphi_{\text{st}}) + \frac{2\pi L_{12}}{c\Phi_0} G'(\varphi_{\text{st}}) > 0 \]

These inequalities (together with the one in Eq.(13)) can be satisfied if only \( J'_s(\varphi_{\text{st}}) < 0 \) and \( G'(\varphi_{\text{st}}) > 0 \). From here and Eq.(3,4) it follows that the stable limit cycle (that is non-linear periodic time-oscillations of \( \varphi(t) \) and \( \dot{\varphi}(t) \)) can only be on those branches of the CVC with the negative differential resistance \( dV/dJ_d < 0 \) that are close to the \( J_d \)-axis in Fig.2 (and on the loop shown in the insertion of the figure). If \( 0 < L_{11} - L_{cr} \ll L_{cr} \) the frequency of these oscillations is

\[ \omega_0 = \sqrt{\frac{bc^4R}{(L_{11}L_r - L_{12}^2)G(\varphi_{\text{st}})}} \]

This frequency can be varied in a wide range, and estimations show the maximal frequency \( \omega_{\text{max}} \sim 10^{12} \text{ sec}^{-1} \).

In conclusion, for both the diffusive and ballistic cases I have shown that the current-voltage characteristics of the Andreev ring-shaped interferometer \( J_d(V) \) is a series of loops with several branches intersecting in its origin, and hence it has sections with a negative differential resistance \( dV/dJ_d \) if the ring inductance is large enough. These properties of the CVC are robust as the negative differential resistance and the intersection of the several branches in the CVC origin appear at any value of the mutual inductance, \( L_{12} \), between the ring and the dissipative transport circuit (which can be as small as is wished) as soon as the self-inductance of the ring \( L_r \) is large enough to provide \( A < 0 \) (see Eq.(3,4); of course, such a CVC can not be observed experimentally if the mutual inductance \( L_{12} \) is unreasonably small. This inequality \( (A < 0) \) is the same as the one needed for the conventional SQUID functioning. What’s is more, manipulations with the ring self-inductance \( L_r \) and the external magnetic flux \( \Phi_{\text{ext}} \) allow to get CVC loops (and therefore the hysteresis loops of the CVC) as small as wanted, especially it concerns the smallest loop around the CVC origin (see Insertion in Fig.4), and one can obtain the sensitivity of the transport current \( J_d \) (or the applied voltage \( V \)) to a change of the external magnetic flux comparable to the conventional SQUID sensitivity.

Spontaneous arising of coupled non-linear oscillations of the transport current \( J_d(t) \), the Josephson current \( J_s(t) \) and the flux \( \Phi(t) \) in Andreev interferometers have been also predicted and investigated. The frequency of the oscillations \( \omega \) can be varied in a wide range, and the maximal frequency can reaches \( \omega_{\text{max}} \sim 10^{12} \text{ sec}^{-1} \).

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