A New Algebraic Inequality and Some Applications in Submanifold Theory

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Abstract: We give a simple proof of the Chen inequality involving the Chen invariant \( \delta(k) \) of submanifolds in Riemannian space forms. We derive Chen’s first inequality and the Chen–Ricci inequality. Additionally, we establish a corresponding inequality for statistical submanifolds.

Keywords: Riemannian space form; submanifold; Chen invariants; Chen inequalities; statistical manifold; statistical submanifold

MSC: 53C40; 53C05

1. Introduction

One of the most important topics of research in the geometry of submanifolds in Riemannian manifolds is to establish sharp relationships between extrinsic and intrinsic invariants of a submanifold.

The most used intrinsic invariants are sectional curvature, scalar curvature and Ricci curvature. The main extrinsic invariant is the squared mean curvature.

There are well-known relationships between the above extrinsic and intrinsic invariants for a submanifold in a Riemannian space form: (generalized) Euler inequality, Chen–Ricci inequality, Wintgen inequality, etc.

In [1,2], B.-Y. Chen introduced a sequence of Riemannian invariants, which are known as Chen invariants. They are different in nature from the classical Riemannian invariants. B.-Y. Chen established optimal relationships between the squared mean curvature and Chen invariants for submanifolds in Riemannian space forms, known as Chen inequalities (see [2]). The proofs of these inequalities use an algebraic inequality, discovered by B.-Y. Chen in [1].

In the present paper, we give simple proofs of some Chen inequalities by using a different algebraic inequality.

Other Chen inequalities were proved in [3] by applying another inequality.

2. Preliminaries

The theory of Chen invariants and Chen inequalities was initiated by B.-Y. Chen [1,2].

Let \( (M,g) \) be an n-dimensional (\( n \geq 2 \)) Riemannian manifold, \( \nabla \) its Levi-Civita connection and \( R \) the Riemannian curvature tensor field on \( M \). The sectional curvature \( K(\pi) \) of the plane section \( \pi \subset T_pM, p \in M \), is defined by

\[
K(\pi) = R(e_1,e_2,e_1,e_2) = g(R(e_1,e_2)e_2,e_1),
\]

where \( \{e_1,e_2\} \) is an orthonormal basis of \( \pi \).
Let \( \{e_1, ..., e_n\} \) be an orthonormal basis of \( T_p M \). The scalar curvature \( \tau \) at \( p \) is given by

\[
\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),
\]

where \( K(e_i \wedge e_j) \) is the sectional curvature of the plane section spanned by \( e_i \) and \( e_j \).

If \( X \) is a unit vector tangential to \( M \) at \( p \), consider the orthonormal basis \( \{e_1 = X, e_2, ..., e_n\} \) of \( T_p M \). The Ricci curvature is defined by

\[
\text{Ric}(X) = \sum_{j=2}^{n} K(X \wedge e_j).
\]

Let \( L \) be an \( r \)-dimensional subspace of \( T_p M \) and \( \{e_1, ..., e_r\} \) an orthonormal basis of \( L, \ 2 \leq r \leq n \). The scalar curvature \( \tau(L) \) of \( L \) is given by

\[
\tau(L) = \sum_{1 \leq k < \beta \leq r} K(e_k \wedge e_\beta).
\]

In particular, for \( r = 2 \), \( \tau(L) \) is the sectional curvature of \( L \) and for \( r = n \), \( \tau(T_p M) = \tau(p) \) is the scalar curvature of \( M \) at \( p \).

B.-Y. Chen introduced a sequence of Riemannian invariants \( \delta(n_1, ..., n_l) \), known as Chen invariants (see [2]). The Chen first invariant is \( \delta_M = \tau - \inf K \), where

\[
(\inf K)(p) = \inf \{K(\pi)|\pi \subset T_p M \text{ plane section}\}.
\]

Let \( l > 0 \) be an integer and \( n_1, ..., n_l \geq 2 \) integers such that \( n_1 < n \) and \( n_1 + ... + n_l \leq n \). The Chen invariant \( \delta(n_1, ..., n_l) \) is defined by

\[
\delta(n_1, ..., n_l)(p) = \tau(p) - \inf \{\tau(L_1) + ... + \tau(L_l)\},
\]

where \( L_1, ..., L_l \) are mutually orthogonal subspaces of \( T_p M \) with dim \( L_j = n_j, j = 1, ..., l \).

For \( l = 1 \) in particular, one has \( \delta(2) = \delta_M \) and \( \delta(n-1) = \max \text{Ric} \), with

\[
\max \text{Ric}(p) = \max \{\text{Ric}(X)|X \in T_p M, g(X, X) = 1\}.
\]

We shall consider the Chen invariant \( \delta(k) \), which is given by

\[
\delta(k)(p) = \tau(p) - \inf \tau(L_k),
\]

where \( L_k \) is any \( k \)-dimensional subspace of \( T_p M \).

3. An Algebraic Inequality

In this section, we give an algebraic inequality and study its equality case. As an application, we get a simple proof of the Chen inequality for the invariant \( \delta(k) \).

**Lemma 1.** Let \( k, n \) be nonzero natural numbers, \( 2 \leq k \leq n - 1 \), and \( a_1, a_2, ..., a_n \in \mathbb{R} \). Then

\[
\sum_{1 \leq i < j \leq n} a_i a_j - \sum_{1 \leq \alpha < \beta \leq k} a_\alpha a_\beta \leq \frac{n - k}{2(n - k + 1)} \left( \sum_{i=1}^{n} a_i \right)^2.
\]

Moreover, the equality holds if and only if \( \sum_{\alpha = 1}^{k} a_\alpha = a_j \), for all \( j \in \{k + 1, ..., n\} \).

**Proof.** We prove this Lemma by using the Cauchy–Schwarz inequality. We have

\[
\left( \sum_{i=1}^{n} a_i \right)^2 \leq \left( \sum_{\alpha = 1}^{k} a_\alpha + a_{k+1} + ... + a_n \right)^2 \leq \frac{n - k}{2(n - k + 1)} \left( \sum_{i=1}^{n} a_i \right)^2.
\]
which implies the desired inequality.

The equality holds if and only if we have equality in the Cauchy–Schwarz inequality, i.e., \( \sum_{a=1}^{k} a_{a} = a_{j} \), for all \( j \in \{ k + 1, ..., n \} \). \( \square \)

4. Proof of the Chen Inequality for \( \delta(K) \)

We apply Lemma 1 for obtaining a simple proof of the Chen inequality corresponding to the Chen invariant \( \delta(k) \) for submanifolds in Riemannian space forms.

Let \( \bar{M}(c) \) be an \( m \)-dimensional Riemannian space form of constant sectional curvature \( c \). The Euclidean space \( \mathbb{E}^{m} \), the sphere \( S^{m} \) and the hyperbolic space \( H^{m} \) are the standard examples.

Consider \( M \) an \( n \)-dimensional submanifold of \( \bar{M}(c) \) and denote by \( h \) the second fundamental form of \( M \) in \( \bar{M}(c) \). The mean curvature vector \( H(p) \) at \( p \in M \) is defined by

\[
H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_{i}, e_{i}),
\]

where \( \{ e_{1}, ..., e_{n} \} \) is an orthonormal basis of \( T_{p}M \).

The submanifold \( M \) is called minimal if the mean curvature vector \( H(p) \) vanishes at any \( p \in M \).

We recall the Gauss equation (see [4]):

\[
R(X, Y, Z, W) = c + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),
\]

for all vector fields \( X, Y, Z, W \) tangential to \( M \).

Theorem 1. Let \( \bar{M}(c) \) be an \( m \)-dimensional Riemannian space form of constant sectional curvature \( c \) and \( M \) an \( n \)-dimensional submanifold of \( \bar{M}(c) \). Then, for any \( 2 \leq k \leq n - 1 \), one has the following Chen inequality:

\[
\delta(k) \leq \frac{n^{2}(n - k)}{2(n - k + 1)} ||H||^{2} + \frac{1}{2}[n(n - 1) - k(k - 1)]c.
\]

Moreover, the equality holds at a point \( p \in M \) if and only if there exist suitable orthonormal bases \( \{ e_{1}, ..., e_{n} \} \subset T_{p}M \) and \( \{ e_{n+1}, ..., e_{m} \} \subset T_{p}^{\perp}M \) such that the shape operators take the forms

\[
A_{e_{k+1}} = \begin{pmatrix}
  a_{1} & 0 & 0 & ... & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \ldots & a_{k} & 0 & 0 \\
  0 & \ldots & 0 & \mu & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \ldots & 0 & 0 & \mu
\end{pmatrix}, \quad \sum_{a=1}^{k} a_{a} = \mu,
\]

\[
A_{e_{r}} = \begin{pmatrix}
  A_{r} & 0 \\
  0 & O_{n-k}
\end{pmatrix}, \quad r = n + 2, ..., m,
\]
where $A_r$ is a symmetric $k \times k$ matrix with trace $A_r = 0$ and $O_{n-k}$ is the $(n-k) \times (n-k)$ null matrix.

**Proof.** Let $p \in M$, $L \subset T_pM$ be a $k$-dimensional subspace and $\{e_1, ..., e_k\}$ be an orthonormal basis of $L$. We take $\{e_1, ..., e_k, e_{k+1}, ..., e_m\} \subset T_pM$ and $\{e_{n+1}, ..., e_m\} \subset T_p^\perp M$ as orthonormal bases, respectively.

Denote as usual by $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j = 1, ..., n$, $r \in \{n + 1, ..., m\}$, the components of the second fundamental form.

The Gauss equation implies

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_r) = \frac{n(n-1)}{2} c + \sum_{r=n+1}^m \sum_{1 \leq i < j \leq n} [h^r_{ij}h^r_{ij} - (h^r_{ij})^2].$$

Additionally, by the Gauss equation one has

$$\tau(L) = \frac{k(k-1)}{2} c + \sum_{r=n+1}^m \sum_{1 \leq a < b \leq k} [h^r_{ab}h^r_{ab} - (h^r_{ab})^2].$$

Then we get

$$\tau - \tau(L) = \frac{1}{2}[n(n-1) - k(k-1)]c + \sum_{r=n+1}^m \left( \sum_{1 \leq i < j \leq n} h^r_{ij}h^r_{ij} - \sum_{1 \leq a < b \leq k} h^r_{ab}h^r_{ab} \right) - \sum_{r=n+1}^m \sum_{1 \leq i < j \leq n, (i, j) \notin \{1, ..., k\}^2} (h^r_{ij})^2.$$

By using the algebraic inequality from the previous section, we obtain

$$\tau - \tau(L) \leq \frac{n-k}{2(n-k+1)} \sum_{r=n+1}^m \left( \sum_{i=1}^n h^r_{ii} \right)^2 + \frac{1}{2} [n(n-1) - k(k-1)]c = \frac{n^2(n-k)}{2(n-k+1)} ||H||^2 + \frac{1}{2} [n(n-1) - k(k-1)]c,$$

which implies the inequality to prove.

If the equality case holds at a point $p \in M$, then we have equalities in all the inequalities in the proof, i.e.,

$$\left\{ \begin{array}{l}
\sum_{a=1}^k h^r_{ab} = h^r_{ij}, \forall j \in \{k+1, ..., n\}, \\
h^r_{ij} = 0, \forall 1 \leq i < j \leq n, (i, j) \notin \{1, ..., k\}^2,
\end{array} \right.$$ 

for any $r \in \{n+1, ..., m\}$.

If we choose $e_{n+1}$ parallel to $H(p)$, then the shape operators take the above forms. \(\square\)

**Corollary 1.** Let $\tilde{M}(c)$ be an $m$-dimensional Riemannian space form of constant sectional curvature $c$ and $M$ an $n$-dimensional submanifold of $\tilde{M}(c)$. If there exists a point $p \in M$ such that $\delta(k)(p) > \frac{1}{2} [n(n-1) - k(k-1)]c$, then $M$ is not minimal.

If $k = 1$, we derive Chen’s first inequality:
Corollary 2. [1] Let $\tilde{M}(c)$ be an $m$-dimensional Riemannian space form of constant sectional curvature $c$ and $M$ an $n$-dimensional submanifold of $\tilde{M}(c)$. Then one has
\[
\inf K \geq \tau - \frac{n - 2}{2} \left[ \frac{n^2}{n - 1} ||H||^2 + (n + 1)c \right].
\]

Equality holds at a point $p \in M$ if and only if, with respect to suitable orthonormal bases $\{e_1, ..., e_n\} \subset T_p M$ and $\{e_{n+1}, ..., e_m\} \subset T_p M$, the shape operators take the following forms:
\[
A_{e_n} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & \mu - a & 0 & \ldots & 0 \\
0 & 0 & \mu & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \mu
\end{pmatrix}
\]
\[
A_{e_r} = \begin{pmatrix}
h_{r11} & h_{r12} & 0 & \ldots & 0 \\
h_{r12} & -h_{r11} & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}, \quad r = n + 2, ..., m.
\]

Recall that $\delta(n - 1) = \max \text{Ric}$. Then, from Theorem 1 we deduce the Chen–Ricci inequality:

Corollary 3. [5] Let $\tilde{M}(c)$ be an $m$-dimensional Riemannian space form of constant sectional curvature $c$ and $M$ an $n$-dimensional submanifold of $\tilde{M}(c)$. Then, for any $p \in M$ and any unit vector $X$ tangential to $M$, one has
\[
\text{Ric}(X) \leq \frac{n^2}{4} ||H||^2 + (n - 1)c.
\]

We present the following examples:

Example 1. Let $k, n$ be integers such that $k \geq 2$ and $n \geq 2k - 1$. Consider the hypercylinder $M = S^k \times \mathbb{E}^{n-k} \subset \mathbb{E}^{n+1}$.

Clearly $\delta(k) = \tau = \frac{1}{2}k(k-1)$. Then the equality case of Theorem 1 holds identically if and only if $n = 2k - 1$, i.e., $M = S^k \times \mathbb{E}^{k-1}$.

Moreover, $\max \text{Ric} = \frac{n^2}{4} ||H||^2$ if and only if $k = 2$ and $n = 3$, i.e., $M = S^2 \times \mathbb{E}$.

Example 2. The generalized Clifford torus.

Let $T = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}) \subset S^{n+1} \subset \mathbb{E}^{n+2}$, $n > k \geq 1$.

It is known that $T$ is a minimal hypersurface of $S^{n+1}$, but a non-minimal submanifold of $\mathbb{E}^{n+2}$.

Obviously $\max \text{Ric} = \max\{(k - 1)\frac{\sqrt{k}}{n}, (n - k - 1)\frac{\sqrt{n-k}}{n-k}\}$.

Then $T \subset S^{n+1}$ does not satisfy the equality case of Theorem 1 for $\delta(n - 1)$, $\forall n \geq 2$.

If we consider $T \subset \mathbb{E}^{n+2}$, then it does not satisfy the equality case of Theorem 1 for $\delta(n - 1)$, $\forall n \geq 2$. 
5. A Chen Inequality for Statistical Submanifolds

A statistical manifold is an $m$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$ endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, which satisfy

$$Z\tilde{g}(X,Y) = \tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(X, \tilde{\nabla}^*_Z Y),$$

for any $X,Y,Z \in \Gamma(T\tilde{M})$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called dual connections (see [6,7]), and it is easily seen that $(\tilde{\nabla}^*)^* = \tilde{\nabla}$. The pairing $(\tilde{\nabla}, \tilde{g})$ is said to be a statistical structure. If $(\tilde{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}$, then $(\tilde{\nabla}^*, \tilde{g})$ is a statistical structure too [6,8].

Any torsion-free affine connection $\tilde{\nabla}$ on $\tilde{M}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0,$$

where $\tilde{\nabla}^0$ is the Levi–Civita connection on $\tilde{M}$.

The dual connections are called conjugate connections in affine differential geometry (see [9]).

Denote by $\hat{R}$ and $\hat{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. They satisfy

$$\tilde{g}(\hat{R}^*(X,Y)Z, W) = -\tilde{g}(Z, \hat{R}(X,Y)W).$$

A statistical structure $(\tilde{\nabla}, \tilde{g})$ is said to be of constant curvature $\varepsilon \in \mathbb{R}$ if

$$\hat{R}(X,Y)Z = \varepsilon \{g(Y, Z)X - g(X, Z)Y\}. $$

A statistical structure $(\tilde{\nabla}, \tilde{g})$ of constant curvature 0 is called a Hessian structure.

The Equation (2) implies that if $(\tilde{\nabla}, \tilde{g})$ is a statistical structure of constant curvature $\varepsilon$, then $(\tilde{\nabla}^*, \tilde{g})$ is also a statistical structure of constant curvature $\varepsilon$ (obviously, if $(\tilde{\nabla}, \tilde{g})$ is Hessian, $(\tilde{\nabla}^*, \tilde{g})$ is also Hessian).

The dual connections are not metric, then we cannot define a sectional curvature in the standard way. A sectional curvature on a statistical manifold was defined by B. Opozda [10].

More precisely, if one considers $p \in \tilde{M}$, $\pi$ a plane section in $T_p\tilde{M}$ and an orthonormal basis $\{X, Y\}$ of $\pi$, then a sectional curvature is defined by

$$\hat{\kappa}(\pi) = \frac{1}{2} \tilde{g}(\hat{R}(X, Y)Y + \hat{R}^*(X, Y)Y, X),$$

which is independent of the choice of the orthonormal basis.

Next, we consider a statistical manifold $(\tilde{M}, \tilde{g})$ and a submanifold $M$ of dimension $n$ of $\tilde{M}$. Then $(M, g|_M)$ is also a statistical manifold with the connection induced by $\tilde{\nabla}$ and induced metric $g$.

In Riemannian geometry, the fundamental equations are the Gauss and Weingarten formulae and the equations of Gauss, Codazzi and Ricci.

As usual, we denote by $\Gamma(T^\perp M)$ the set of the sections of the bundle normal to $M$.

In our case, for any $X,Y \in \Gamma(TM)$, according to [8], the corresponding Gauss formulae are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\nabla^*_X Y = \nabla^*_X Y + h^*(X,Y),$$

where $h, h^* : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(T^\perp M)$ are symmetric and bilinear, called the imbedding curvature tensor (see [6,8]) of $M$ in $\tilde{M}$ for $\tilde{\nabla}$ and the imbedding curvature tensor of $M$ in $\tilde{M}$ for $\tilde{\nabla}^*$, respectively.

In [8], it was also proven that $(\tilde{\nabla}, \tilde{g})$ and $(\tilde{\nabla}^*, \tilde{g})$ are dual statistical structures on $M$.

Since $h$ and $h^*$ are bilinear, there are linear transformations $A_{\tilde{g}}$ and $A^*_{\tilde{g}}$ on $TM$ defined by

$$\tilde{g}(A_{\tilde{g}} X, Y) = \tilde{g}(h(X, Y), \tilde{c}).$$
for any \( \xi \in \Gamma(T^\perp M) \) and \( X, Y \in \Gamma(TM) \).

Further (see [8]), the corresponding Weingarten formulae are

\[
\nabla_X \xi = -A^\perp_X \xi + \nabla_X^\perp \xi ,
\]

\[
\nabla^\perp_X \xi = -A^\perp_X \xi + \nabla_X^{\perp} \xi ,
\]

for any \( \xi \in \Gamma(T^\perp M) \) and \( X \in \Gamma(TM) \). The connections \( \nabla \) and \( \nabla^{\perp} \) are Riemannian dual connections with respect to the induced metric on \( \Gamma(T^\perp M) \).

Let \( \{e_1, \ldots, e_n\} \) and \( \{e_{n+1}, \ldots, e_m\} \) be orthonormal tangential and normal frames, respectively, on \( M \). Then the mean curvature vector fields are defined by

\[
H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} h^i_j \right) e_i, \quad h^i_i = g(h(e_i, e_i), e_i),
\]

and

\[
H^{\ast} = \frac{1}{n} \sum_{i=1}^{n} h^\ast(e_i, e_i) = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} h^\ast_{i j} \right) e_i, \quad h^\ast_{i i} = g(h^\ast(e_i, e_i), e_i),
\]

for \( 1 \leq i, j \leq n \) and \( n + 1 \leq \alpha \leq m \).

The Gauss equations for the dual connections \( \nabla \) and \( \nabla^{\ast} \), respectively, are given by (see [8])

\[
g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(h(X, Z), h^{\ast}(Y, W)) - g(h^{\ast}(X, W), h(Y, Z)),
\]

\[
g(R^{\ast}(X, Y)Z, W) = g(R^{\ast}(X, Y)Z, W) + g(h^{\ast}(X, Z), h(Y, W)) - g(h(X, W), h^{\ast}(Y, Z)),
\]

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [11].

In this section we prove the Chen inequality corresponding to the Chen invariant \( \delta(k) \) for statistical submanifolds in statistical manifolds of constant curvature.

We consider an \( m \)-dimensional statistical manifold \( M(\varepsilon) \) of constant curvature \( \varepsilon \) and an \( n \)-dimensional statistical submanifold \( M \). Let \( p \in M \) and \( L \) be a \( k \)-dimensional subspace of \( T_pM \). Denote by \( \{e_1, \ldots, e_k\} \) an orthonormal basis of \( L \), \( \{e_{k+1}, \ldots, e_{n} \} \) an orthonormal basis of \( T_pM \) and \( \{e_{n+1}, \ldots, e_{m}\} \) an orthonormal basis of \( T^pM \), respectively.
The Gauss equation implies
\[
\tau = \frac{1}{2} \sum_{1 \leq i < j \leq n} [g(R(e_i, e_j)e_i, e_i) + g(R^* (e_i, e_j)e_i, e_i)] = \\
= \frac{n(n-1)}{2} \epsilon + \frac{1}{2} \sum_{1 \leq i < j \leq n} [g(h^*(e_i, e_i), h(e_j, e_j)) + g(h(e_j, e_i), h^*(e_i, e_j)) - 2g(h(e_i, e_j), h^*(e_i, e_j))] = \\
= \frac{n(n-1)}{2} \epsilon + \frac{1}{2} \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} (h^*_{ii}h^*_{jj} + h^*_{ij}h^*_{ij} - 2h^*_{ij}h^*_{ij}) = \\
= \frac{n(n-1)}{2} \epsilon + \frac{1}{2} \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} \left( (h^*_{ii} + h^*_{ij})(h^*_{ij} + h^*_{ij}) - h^*_{ii}h^*_{ij} - h^*_{ij}h^*_{ii} - (h^*_{ij})^2 \right) = \\
= \frac{n(n-1)}{2} \epsilon + \frac{1}{2} \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} \left\{ 2[h^*_{ii}h^*_{ij} - (h^*_{ij})^2] - \frac{1}{2} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2] - \frac{1}{2} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2] \right\} = \\
= \frac{n(n-1)}{2} \epsilon + \frac{1}{2} \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} \left\{ 2[h^*_{ii}h^*_{ij} - (h^*_{ij})^2] - \frac{1}{2} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2] - \frac{1}{2} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2] \right\}.
\]
where \( h^0 \) is the second fundamental form of the Riemannian submanifold \( M \).

We denote by \( \tau_0 \) the scalar curvature with respect to the Levi–Civita connection and by \( \tilde{\tau}_0 = \sum_{1 \leq i < j \leq n} \tilde{K}_0(e_i \wedge e_j) \).

The Gauss equation with respect to the Levi–Civita connection gives
\[
\tau_0 = \tau_0 + \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2].
\]
(5)

By substituting Equation (5) into (4), we get
\[
\tau = 2(\tau_0 - \tilde{\tau}_0) + \frac{n(n-1)}{2} \epsilon - \frac{1}{2} \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2] - \frac{1}{2} \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} [h^*_{ii}h^*_{ij} - (h^*_{ij})^2].
\]
(6)
By using Gauss equation, we have

\[ \tau(L) = \frac{1}{2} \sum_{1 \leq \alpha < \beta \leq k} [g(R(e_{\alpha}, e_{\beta})e_{\beta}, e_{\alpha}) + g(R^*(e_{\alpha}, e_{\beta})e_{\beta}, e_{\alpha})] = \]

\[ = \frac{k(k - 1)}{2} \varepsilon + \frac{1}{2} \sum_{1 \leq \alpha < \beta \leq k} \left[ g(h^*(e_{\alpha}, e_{\alpha}), h(e_{\beta}, e_{\beta})) + g(h(e_{\alpha}, e_{\alpha}), h^*(e_{\beta}, e_{\beta})) - 2g(h(e_{\alpha}, e_{\beta}), h^*(e_{\alpha}, e_{\beta})) \right] = \]

\[ = \frac{k(k - 1)}{2} \varepsilon + \frac{1}{2} \sum_{r = n + 1}^{m} \sum_{1 \leq \alpha < \beta \leq k} \left( h_{\alpha \alpha}^r h_{r \beta}^r + h_{\alpha \alpha}^r h_{r \beta}^r - 2h_{\alpha \beta}^r h_{r \beta}^r \right) = \]

\[ = \frac{k(k - 1)}{2} \varepsilon + \frac{1}{2} \sum_{r = n + 1}^{m} \sum_{1 \leq \alpha < \beta \leq k} \left( (h_{\alpha \alpha}^r + h_{\alpha \alpha}^r)(h_{r \beta}^r + h_{r \beta}^r) - h_{r \alpha}^r h_{r \beta}^r - h_{r \alpha}^r h_{r \beta}^r - (h_{r \beta}^r)^2 + (h_{r \beta}^r)^2 \right) = \]

\[ = \frac{k(k - 1)}{2} \varepsilon + \sum_{r = n + 1}^{m} \left( \sum_{1 \leq \alpha < \beta \leq k} \left( 2h_{\alpha \alpha}^r h_{r \beta}^r - (h_{r \beta}^r)^2 \right) - \frac{1}{2} \left( h_{r \alpha}^r h_{r \beta}^r - (h_{r \beta}^r)^2 \right) \right) = \]

\[ = 2\tau_0(L) - 2\tau_0(L) + \frac{k(k - 1)}{2} \varepsilon - \frac{1}{2} \sum_{r = n + 1}^{m} \sum_{1 \leq \alpha < \beta \leq k} \left\{ h_{r \alpha}^r h_{r \beta}^r - (h_{r \beta}^r)^2 \right\} \].

By subtracting the last equation from (4), we obtain

\[ (\tau - \tau(L)) - 2(\tau_0 - \tau_0(L)) \geq 2(\tilde{\tau}_0(L) - \tilde{\tau}_0) + \frac{1}{2} |n(n - 1) - k(k - 1)| \varepsilon - \]

\[ - \frac{1}{2} \sum_{r = n + 1}^{m} \left( \sum_{1 \leq \alpha < \beta \leq n} h_{r \alpha}^r h_{r \beta}^r - \sum_{1 \leq \alpha < \beta \leq k} h_{r \alpha}^r h_{r \beta}^r \right) - \]

\[ - \frac{1}{2} \sum_{r = n + 1}^{m} \left( \sum_{1 \leq \alpha < \beta \leq n} h_{r \alpha}^r h_{r \beta}^r - \sum_{1 \leq \alpha < \beta \leq k} h_{r \alpha}^r h_{r \beta}^r \right). \]

We denote by \( \max \hat{K}_0(p) \) the maximum of the Riemannian sectional curvature function of \( \hat{M}(\varepsilon) \) restricted to 2-plane sections of the tangent space \( T_p M, p \in M \). Obviously

\[ \tau_0 - \tilde{\tau}_0(L) \leq \frac{1}{2} |n(n - 1) - k(k - 1)| \max \hat{K}_0(p). \]

On the other hand, by using Lemma 1, one has

\[ \sum_{1 \leq i < j \leq n} h_{ij}^r h_{ij}^r - \sum_{1 \leq \alpha < \beta \leq k} h_{r \alpha}^r h_{r \beta}^r \leq \frac{n - k}{2(n - k + 1)} \left( \sum_{i = 1}^{n} h_{ii}^r \right)^2, \]

\[ \sum_{1 \leq i < j \leq n} h_{ij}^r h_{ij}^r - \sum_{1 \leq \alpha < \beta \leq k} h_{r \alpha}^r h_{r \beta}^r \leq \frac{n - k}{2(n - k + 1)} \left( \sum_{i = 1}^{n} h_{ii}^r \right)^2. \]

It follows that

\[ \tau - \tau(L) \geq 2(\tau_0 - \tau_0(L)) + \frac{1}{2} |n(n - 1) - k(k - 1)| (\varepsilon - 2 \max \hat{K}_0(p)) - \]
We state the following result.

**Theorem 2.** Let $M$ be an $n$-dimensional statistical submanifold of an $m$-dimensional statistical manifold $\tilde{M}(\varepsilon)$ of constant curvature. Then, for any $p \in M$ and any $k$-plane section $L$ of $T_pM$, we have:

\[
\tau_0 - \tau_0(L) \leq \frac{1}{2}(\tau - \tau(L)) + \frac{n^2(n-k)}{4(n-k+1)} \left(||H||^2 + ||H^*||^2\right) + \\
\frac{1}{2} \left[n(n-1) - k(k-1)\right](\max \tilde{K}_0(p) - \varepsilon^2).
\]

Moreover, the equality holds at a point $p \in M$ if and only if there exist orthonormal bases $\{e_1, \ldots, e_n\}$ of $T_pM$ and $\{e_{n+1}, \ldots, e_m\}$ of $T^\perp_pM$ such that

\[
\begin{cases}
\sum_{k=1}^{k} h^r_{nn} = h^r_{jj}, \forall j \in \{k+1, \ldots, n\}, \\
\sum_{k=1}^{k} h^r_{NN} = h^r_{jj}, \forall j \in \{k+1, \ldots, n\}, \\
h^r_{ij} = h^r_{ij} = 0, \forall 1 \leq i < j \leq n, (i,j) \notin \{1, \ldots, k\}^2,
\end{cases}
\]

for any $r \in \{n+1, \ldots, m\}$.

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