INDUCTION AND RESTRICTION IN FORMAL DEFORMATION OF COVERINGS

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Abstract: Let $X/S$ be a semistable curve with an action of a finite group $G$ and let $H$ be a normal subgroup of $G$. We present a new condition under which for any base change $T \to S$, $(X/G) \times_S T$ is isomorphic to $(X \times_S T)/G$. This allows us to define induction and restriction morphisms between the $G$-equivariant deformation functor of $X$ and the $G/H$-equivariant (resp. $H$-equivariant) deformation functor of $X/H$ (resp. $X$).

Résumé: Soit $X/S$ une courbe semi-stable munie de l’action d’un groupe fini $G$. Soit $H$ un sous-groupe normal de $G$. Nous proposons une nouvelle condition sous laquelle $(X/G) \times_S T$ est isomorphe à $(X \times_S T)/G$ quel que soit le changement de base $T \to S$. Ce résultat nous permet de définir des morphismes d’induction et de restriction entre le foncteur de déformations $G$-équivariantes de $X$ et le foncteur de déformations $G/H$-équivariantes (resp. $H$-équivariantes) de $X/H$ (resp. $X$).

1. Introduction

Let $S = \text{Spec } R$ be a base scheme, where $R$ is a commutative local noetherian ring. Let $X \to S$ be a semistable curve with a faithful action of a finite group $G$, and let $H \triangleleft G$ be a normal subgroup. There is an induced action of $G/H$ on the quotient $X/H$ which makes the quotient map $X \to X/H$ equivariant with respect to $G \to G/H$. We wish to compare the $G$-equivariant deformation functor $\text{Def}_G(X)$ of $X$ (in the sense of [BeMé]) with the corresponding $G/H$-equivariant deformation functor $\text{Def}_{G/H}(X/H)$ of $X/H$. This amounts to establish the existence of a morphism between both deformation functors. This will be realized provided that base change and passage to quotient are commuting operations. This means that for a base change $T \to S$ the natural morphism

$$(X/G) \times_S T \longrightarrow (X \times_S T)/G$$

is an isomorphism. After ([KaMa] Theorem 7.1.3), we know (1.1) universally is an isomorphism if one of the following two conditions is satisfied:

- $G$ acts freely
- the order of $G$ is invertible in the structural sheaf of $S$.

Our first result is a new condition under which (1.1) universally is an isomorphism:

**Theorem 1.1.** Let $X \to S$ be a semistable curve, and $G$ be a finite group of $S$-automorphisms of $X/S$. If the action of $G$ is free on an open dense set on any geometric fiber, then (1.1) is universally an isomorphism.

In addition we provide a local version (Proposition 3.7) of theorem 1.1 for $G$ a finite flat $S$-group scheme acting on a smooth affine $S$-curve. The reason for considering in the local case a finite flat group scheme is the following: under a stable
specialisation in characteristic $p > 0$ dividing the order of a Galois group $G$ of the covering, the ordinary group $G$ may specialize, at least locally, in an infinitesimal group scheme. This suggests that we must take into account in the general problem of deformation-specialisation of coverings, not only constant groups, but also infinitesimal groups.

What is important for application of theorem 1.1 to deformations, is that there is no condition on the residue characteristic of $S$, relatively on the order of $G$. Under theorem 1.1’s hypotheses, we construct induction and restriction morphisms

\[(1.2) \quad \text{Def}_H(X) \xleftarrow{\text{res}} \text{Def}_G(X) \xrightarrow{\text{ind}} \text{Def}_{G/H}(X/H)\]

The outline of this paper is as follows: first we present a version of the Kleiman-Lonsted algebra ([KLo]) in a slightly generalized form, i.e. to a finite flat $S$-group scheme admissibly acting on $X$ ([2]). By using the Kleiman-Lonsted algebra we prove a base change criterion ([3.1]) and a local version of 1.1 ([3.3]). After having worked out the double points case we prove theorem 1.1 ([§3.4]). At last we define the induction-restriction morphisms between the deformation functors and we compute the differentials of these morphisms for local deformations functors ([4]).

2. Kleiman-Lonsted’s algebra

In this section we briefly present the basic properties of the Kleiman-Lonsted algebra ([KLo]) associated to a finite flat group scheme action. This algebra is an approximation of the algebra of invariants ([§2.1]), but in the case of an infinitesimal group scheme acting on a singular curve, this algebra is more narrow. We prove some properties of reduction of this algebra ([§2.2]). Note that most of the proofs given below remain equally valid for an action of a finite flat groupoid ([DeGa]), but due to lack of applications, we shall limit ourselves in the statement, to group schemes actions.

2.1. Definition and elementary properties. Let $R$ be a commutative noetherian ring. Let $G$ be finite flat $R$-group scheme. We assume that the rank of $R[G]$ over $R$ is constant and equals the order $n = |G|$ of $G$. We consider an action of $G$ on a finite type $R$-algebra $A$. The coaction is denoted by $\mu^*: A \rightarrow R[G] \otimes_R A$. The norm mapping $N_G: A \rightarrow A$ is defined as usual ([Mu], [DeGa]) by $N(a) = Nm(\mu^*(a))$, where $Nm$ is the norm of the $A$-algebra $R[G] \otimes_R A$, the structural morphism being $a \rightarrow 1 \otimes a$. The map $N_G$ extends canonically to $N_G: A[T] \rightarrow A[T]$. The characteristic polynomial of $a \in A$ is $\chi_a(T) = N_G(T - a)$. The $G$-symmetric functions $\sigma_i^G = \sigma_i$, $1 \leq i \leq n$ are defined by

\[\chi_a(T) = T^n - \sigma_1(a)T^{n-1} + \cdots + (-1)^n \sigma_n(a)\]

After Cayley-Hamilton theorem $\chi_a(a) = 0$ and $N(a) = \sigma_n(a)$. Similar definitions hold for an action of a finite flat groupoid ([DeGa], [3]).

Definition 2.1. The Kleiman-Lonsted algebra is defined as the $R$-algebra

\[\Sigma^G_R(A) = R[\sigma_1 \leq i \leq n \sigma_i(A)]\]

After [Mu] §12 Theorem 1, the norm maps $A$ into $A^G = \{a \in A, \mu^*(a) = 1 \otimes a\}$. The $\sigma_i(a)$’s also belong to $A^G$ and $\Sigma^G_R(A) \subset A^G$. We point out that as in the case of an action of a constant group ([KLo]), the algebras $\Sigma^G_R(A)$ and $A^G$ are closely related. We list below some basic facts about the Kleiman-Lonsted algebra. First,
the $R$-algebra $\Sigma^G_R(A)$ is of finite type over $R$. Second the $R$-algebras $A$ and $A^G$ are finite modules over $\Sigma^G_R(A)$. Then the formation of $A^G$ and $\Sigma^G_R(A)$ commutes with flat base change:

**Lemma 2.2.** Let $R'$ be a $R$-algebra. Denote $\phi : \Sigma^G_R(A) \otimes_R R' \to \Sigma^{G \otimes R'}_R(A \otimes_R R')$ and $\psi : A^G \otimes_R R' \to (A \otimes_R R')^{G \otimes R'}$. Assume $G$ is defined over an integral base ring.

i. The morphism $\phi$ is surjective.

ii. If $R'$ is a flat $R$-algebra, $\phi$ and $\psi$ are bijective maps.

iii. If $R'$ is a faithfully flat $R$-algebra, or if the order of $G$ is invertible, $\Sigma^G_R(A) = A^G$ if and only if $\Sigma^{G \otimes R'}_R(A \otimes_R R') = (A \otimes_R R')^{G \otimes R'}$.

**Proof.** The proof of assertions ii. and iii. in ([KLo] prop 4.4) works without any change provided i. holds.

First assume $G$ to be a constant group. The proof in this case follows from the following known fact (see Appendix). Let $n = |G|$ and $s_j(x)$, $1 \leq j \leq n$ denote the usual symmetric functions of order $j$ in $n$ variables $x = (x_1, \ldots, x_n)$:

$$s_j(x) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} x_{i_1} \cdots x_{i_j}$$

Let $x^{(\ell)}$, $1 \leq \ell \leq q$ be $q$ sets of $n$ variables. The quantity $s_j(x^{(1)} + \cdots + x^{(q)})$ may be expressed as a polynomial with integer coefficients in the symmetric functions $s_m$, $m \leq j$, where the arguments in the $s_m$'s are monomials in $x^{(\ell)}$, $1 \leq \ell \leq q$.

Denote by $(g_i)_{1 \leq i \leq n}$ the elements of $G$. Since $\sigma_k(a) = s_k(g_1a, \ldots, g_n a)$, $a \in A$, $1 \leq k \leq n$, then

$$\forall 1 \leq k \leq n, a \in A, r' \in R' \sigma_k(a \otimes r') = \sigma_k(a) \otimes r' \in \Sigma^G_R(A \otimes_R R')$$

hence $\forall a' = \sum_{i=1}^m a_i \otimes r'_i \in A \otimes_R R'$, $\sigma_k(a') \in \Sigma^G_R(A \otimes_R R')$ and $\phi$ is surjective.

Second assume that $G$ is etale over $R$. Perform an etale surjective (and thus faithfully flat) base change and use the compatibility of the norm map to deduce the assertion i. in an overing of $A$.

Then assume that $G$ is not etale and $R = k$ is a perfect field of characteristic $p > 0$. Denote by $\mu^* : A \to k[G] \otimes_k A$ the coaction. The $k$-group scheme $G$ is the semi-direct product of its etale part $G_{\text{red}}$ and its purely infinitesimal connected component $G_0$ ([DeGa]). Recall ([DeGa], III 6) that the scheme structure of $G$ is given by the local structure theorem of Cartier-Dieudonné

$$k[G] = k[G_0] \otimes_k k[G_{\text{red}}] = (k[X_1, \ldots, X_m]/(X_1^{x_1}, \ldots, X_m^{x_m})) \times k[G_{\text{red}}]$$

Express the matrix of the multiplication in $k[G]$ by $\mu^*(a)$ on the basis $X_1^{x_1} \cdots X_m^{x_m} \otimes e_j$, $(0 \leq i_\alpha < p^{e_\alpha})$, the $e_j$'s being a basis of $k[G_{\text{red}}]$. This matrix is a sum of upper triangular blocks. Hence the characteristic polynomial of $a \in A$ is given by:

$$\chi_{G,a}(T) = \chi_{G_{\text{red}},a}(T)^{p^{e_j}}$$

Assertion i. of lemma 2.2 follows from (2.1) in the same way that in the constant group case. Remark that this result is valid not only for $G$ over the base field $k$, but also after any base change $k \to R$, where $R$ is a $k$-algebra.

Consider at last the general case: assume that $G$ is a $k$-group scheme for $k$ any field (not perfect). Recall ([DeGa] II 4) that the connected component of
the identity of $G$ commutes with a base field extension. Write same universal expressions relating the coefficients of $\chi_{a}(T)$ with the coefficients of $\chi_{\sum_{i=1}^{m} a_{i}(T)}$ and also over $K$; we conclude as above to get i. when $G$ is defined over a field $k$ or any $k$-algebra.

**Remark 2.3.** We observe that the proof of lemma 2.2 does not rely on the finite flat group scheme structure: Let $H$ be a finite commutative free $R$-algebra. For any commutative $R$-algebra $A$, define as above the characteristic polynomial $\chi_{a}(T)$, and the $s_{k}(a)$’s for $a \in H \otimes_{R} A$. As in the proof of lemma 2.2, we want to express $s_{k}(x^{(1)} + \cdots + x^{(t)})$’s in terms of the $s_{j}(x^{(1)}r_{1} \cdots x^{(t)}r_{t})$’s by means of polynomial relations with integer coefficients. Assume $H$ to be defined over a domain $R_{0}$, i.e. $H = H_{0} \otimes_{R_{0}} R$, which depends only on $H_{0}$. Since we have $H \otimes_{R} A = H_{0} \otimes_{R_{0}} A$, we may restrict our discussion to $R$ a domain. Since the $s_{j}$’s commute with an arbitrary base change $A \to B$, we may assume that $A$ is a polynomial ring in a finite number of variables over $R_{0}$. It suffices now to get the polynomials relations over the fraction field $K$ of $R_{0}$. Then $H$ splits in a finite product of local algebras: $H = \prod_{\alpha=1}^{m} H_{\alpha}$. Performing a finite base field extension of $K$, we may assume that the residue field of each local factor $H_{\alpha}$ is equal to $K$. Now working with an explicit basis of $H$, the matrix of the multiplication by $a \in H \otimes_{K} A$ is a sum of blocks $M_{\alpha}$’s, each of the form $M_{\alpha} = a_{\alpha} Id + N_{\alpha}$, with $a_{\alpha} \in A$, and the matrix $N_{\alpha}$ being nilpotent. We get from this

$$\chi_{a}(T) = \prod_{\alpha} (T - a_{\alpha})^{n_{\alpha}}$$

In this way the conclusion of the lemma 2.2 extends to a finite flat groupoid action.

**Remark 2.4.** If the base change $R \to R'$ does not satisfy the condition ii of lemma 2.2, then the canonical morphism $A^{G} \otimes_{R} R' \to (A \otimes_{R} R')^{G \otimes R'}$ is in general neither injective nor surjective. With the Kleinman-Lonsted algebra, the situation is as seen above slightly better. The morphism $\phi : \Sigma^{G}_{R}(A) \otimes_{R} R' \to \Sigma^{G \times R'}(A \otimes_{R} R')$ is under conditions i of lemma 2.2 always surjective. When $G$ is infinitesimal, even if the action is free, $\Sigma^{G}_{R}(A)$ may be different from $A^{G}$. As an example, the field $k$ being of characteristic $p > 0$, take $A = k[\alpha_{p}] \otimes B$, where $B$ is a $k$ algebra of finite type, and the action is by left translation on $G = \alpha_{p}$, i.e. Spec$(A)$ is a trivial $G$-bundle. Then $A^{G} = B$, and $\Sigma^{G}_{k}(A) = k[BP]$.

It may be of interest to point out that the algebras $A^{G}$ and $\Sigma^{G}_{R}(A)$ are very near; first they have the same maximal spectrum (see Proposition 4.6 [KLM]). This leads to an extension of the Gabber lemma ([KaMa] A7.2.1) to the present situation, in fact to a finite flat groupoid.

**Lemma 2.5.** i. The morphism $\text{Spec}(A^{G}) \to \text{Spec}(\Sigma^{G}_{R}(A))$ is radical, i.e. universally injective.

ii. If $R$ is a $\mathbb{Z}_{p}$-algebra, and if the order of $G$ is $|G| = p^{m}$, with $(p, m) = 1$, then for any $a \in A^{G}$, $a^{p^{m}} \in \Sigma^{G}_{R}(A)$;

iii. Under the same hypothesis as in ii., for any base change $R \to R'$ the morphism $\phi : A^{G} \otimes_{R} R' \to (A \otimes_{R} R')^{G \times R'}$ is radical.

**Proof.** We first prove assertion iii. The argument of [KaMa] A7.2.1 shows that there is no loss of generality in assuming $R' = R/I$ to be the quotient of $R$ by an
ideal $I$. Let $x \in (A \otimes_R R')^G \times R'$. Lift $x$ to $a \in A$ such that $\mu^*(a) \equiv 1 \otimes a$ modulo $IR[G] \otimes_R A + R[G] \otimes_R IA$. We now show that $a^{p^r}$ lies in $\text{Im } \phi$. The characteristic polynomial of $a$ satisfies

$$\chi_a(T) \equiv (T - a)^{p^r m} \mod IA[T]$$

with $|G| = p^r m$. By identifying coefficients of $T^{p^r}$, we obtain

$$\sigma_{p^r m - p^r}(a) \equiv \binom{p^r m}{p^r} a^{p^r} \mod IA$$

Since $\binom{p^r m}{p^r} \equiv m \mod p$, this coefficient is a unit in the $\mathbb{Z}_p$-algebra $R$ and $a^{p^r} \in \text{Im } \phi$. We next show that the $p^r$-th power of any $x \in \ker \phi$ is equal to zero. Let $a \in A^G$ be a lift of $x$. Write $a = \sum \alpha_i a_i$ with $\alpha_i \in I$ and $a_i \in A$. On one hand, see that the coefficient of $T^j$ in the polynomial $\chi_a(T)$ is in $\mathcal{I}^j A^G$, so in $IA^G$ if $j > 0$. On the other hand, we have $\chi_a(T) = (T - a)^{p^r m}$; thus by identifying the coefficients of $T^{p^r}$ we get iii. This also proves that the morphism induced by $\phi$ on the spectra is radicial.

The proof of ii. is similar. For $a \in A^G$, one has

$$\chi_a(T) = (T - a)^{p^r m} = T^{p^r m} - \sigma_1(a) T^{p^r m - 1} + \cdots + (-1)^{p^r m} \sigma_{p^r m}(a)$$

By identifying the coefficients of $T^{p^r}$ in the r.h.s. and l.h.s. we get $\binom{p^r m}{p^r} a^{p^r} \in \Sigma^G_R(A)$ as required.

Consider now i. If $R$ is a $\mathbb{Z}_p$-algebra, then the result follows from ii. and iii. It suffices to prove that $\text{Spec } A^G \to \text{Spec } \Sigma^G_R(A)$ is injective. The argument of [DeGa] III,2,3 (used in the proof that $\text{Spec } A^G$ is the topological quotient of $\text{Spec } A$ by the $G$-action) shows without any change the injectivity as required.

**Remark 2.6.** If the action of the constant group $G$ is free, then we have $\Sigma^G_R(A) = A^G$, and the formation of $A^G$ commutes with an arbitrary base change. To see this, after a faithfully flat base change, we may assume that $A$ is a trivial $G$-torsor, that is $A = R[G] \otimes_R B$ and $B = A^G$. Since $\Sigma^G_R(A)$ is a surjective image of $\Sigma^G_R(R[G]) \otimes_R B = R \otimes_R B = B$, we have $\Sigma^G_R(A) = B$.

**Remark 2.7.** The group $G$ being constant, assume the action of $G$ is free on a dense open set. After Lemma 2.5 the morphism $\varphi : \text{Spec } (A^G) \to \text{Spec } (\Sigma^G_R(A))$ is radicial and is an isomorphism over a dense open set. If $A$ is a domain, $\varphi$ is birational.

### 2.2. Reductions.

In this paragraph, we show two properties of reduction of the Kleiman-Lonsted algebra: reduction to a quotient group scheme (Lemma 2.8) and to an induced action by a group scheme (Lemma 2.9). We assume the conditions of lemma 2.2 fulfilled, i.e. either $G$ is constant or $G$ is a $k$-group scheme and all algebras are over $k$.

**Lemma 2.8.** Assume the subgroup $H \leq G$ acts trivially on $X = \text{Spec } A$.

i. There is an induced action of quotient group scheme $G/H$ on $X$ such that $X/G = X/(G/H)$, i.e. $A^G = A(G/H)$.

ii. If the order $m = |H|$ is invertible in $A$, then we also have $\Sigma^G_R(A) = \Sigma^G_R/(H)(A)$.
Proof. Assertion i. simply is the translation of the universal property of quotient scheme ([DeGa] III, 2, 6). For assertion ii., we first establish the following relation between the characteristic polynomials of \( a \in A \):

\[
\chi_a^G(T) = \chi_a^{G/H}(T)^m
\]

(2.2)

Since the action of \( G \) factorizes through an action of \( G/H \), the coaction morphism \( \mu^* : A \rightarrow R[G] \otimes_R A \) factorizes through the subring \( R[G/H] \otimes_R A \):

\[
\mu^* : A \rightarrow R[G/H] \otimes_R A \subset R[G] \otimes_R A
\]

By definition \( R[G] \) is free of rank \( m \) over \( R[G/H] \). Then taking the norm of \( \mu^*(a) \), we get the equality \( N_G(a) = N_{G/H}(a)^m \). We apply this to the action of \( G \) on \( \text{Spec}(A[T]) \), then we get the expected identity (2.2).

Expanding the two terms of (2.2), we get the inclusion \( \Sigma_R^G(A) \subset \Sigma_R^{G/H}(A) \).

To establish the opposite inclusion, denote \( n = |G|, \ell = n/m = |G/H| \) and

\[
\chi_a^G(T) = \sum_{j=0}^{n} \alpha_j^G(a)T^j, \quad \chi_a^{G/H}(T) = \sum_{j=0}^{\ell} \alpha_j^{G/H}(a)T^j
\]

After (2.2),

\[
\forall \ 0 \leq j \leq n, \quad \alpha_j^G(a) = \sum_{i_1 + \cdots + i_m = j \atop 0 \leq i_1, \ldots, i_m \leq \ell} \alpha_{i_1}^{G/H}(a) \cdots \alpha_{i_m}^{G/H}(a)
\]

In order to prove \( \Sigma_R^{G/H}(A) \subset \Sigma_R^G(A) \), it suffices to show

\[
\forall \ 0 \leq u \leq \ell, \quad \alpha_u^{G/H}(a) \in \Sigma_R^G(A)
\]

For this, we make a decreasing induction on \( u \). First \( \alpha_0^{G/H}(a) = 1 \in \Sigma_R^G(A) \).

Assume \( \forall \ u < i \leq \ell, \quad \alpha_i^{G/H}(a) \in \Sigma_R^G(A) \). Denote \( j = \ell(m-1) + u \) and consider the element of \( \Sigma_R^G(A) \)

\[
\alpha_j^G(a) = \sum_{i_1 + \cdots + i_m = j \atop 0 \leq i_1, \ldots, i_m \leq \ell} \alpha_{i_1}^{G/H}(a) \cdots \alpha_{i_m}^{G/H}(a)
\]

Since \( j = i_1 + \cdots + i_m = \ell(m-1) + u, \ 0 \leq i_1, \ldots, i_m \leq \ell \) and \( 0 \leq u \leq \ell \), the indices \( i_q, \ 1 \leq q \leq m \) of the previous sum are at least \( u \). If one of them is equal to \( u \), then all others are equal to \( \ell \):

\[
\alpha_j^G(a) = ma_u^{G/H}(a)(\alpha_{\ell}^{G/H}(a))^{m-1} + \sum_{i_1 + \cdots + i_m = j \atop u < i_q \leq \ell} \alpha_{i_1}^{G/H}(a) \cdots \alpha_{i_m}^{G/H}(a)
\]

By induction, \( i \in [u, \ell], \quad \alpha_i^{G/H}(a) \in \Sigma_R^G(A) \). Recall moreover that \( \alpha_{\ell}^{G/H}(a) = 1 \) and that \( m \) is invertible in \( A \). Hence \( \forall u \leq i \leq \ell, \quad \alpha_i^{G/H}(a) \in \Sigma_R^G(A) \). At last \( \Sigma_R^G(A) = \Sigma_R^{G/H}(A) \). \( \square \)

Assume that \( C \) is a finite type \( R \)-algebra with an action of a flat \( R \)-subgroup scheme \( H \) of \( G \). The induced \( R \)-algebra of \( C \) from \( H \) to \( G \) is by definition ([Ja] §2.12):

\[
A = \text{Ind}^G_H(C) = (R[G] \otimes C)^H
\]
where the left action of $H$ is diagonal: $h(g, x) = (gh^{-1}, hx)$ at the points level. In other words, Spec $A$ is the total space of the associated $H$-bundle $G \times_H \text{Spec } C \to G/H$. Note that the fiber over the origin $H \in G/H$ is Spec $C$, meaning there is a surjection $\phi: A \to C$, $\phi = e_G \otimes 1_C$, with $e_G$ the evaluation map.

**Lemma 2.9.** i. The morphism $\phi$ induces an isomorphism $A^G \cong C^H$.

ii. If the homogeneous space $G/H$ is étale over $R$, then $\Sigma_R(A) \xrightarrow{\phi} \Sigma_R^H(C)$.

**Proof.** i. The first assertion comes from the bundle interpretation of the induced $R$-algebra $\phi: A \to C$, $\phi = e_G \otimes 1_C$, with $e_G$ the evaluation map.

on the functor of points. The actions of $G$ and $H$ commute in $G \times H$ and $G$ acts on $G \times \text{Spec } C/H$. Taking the quotient by $G$ we obtain

$$(G \times_H \text{Spec } C)/G \simeq (G \times \text{Spec } C)/(G \times H) \simeq (G \times \text{Spec } C/G)/H \simeq \text{Spec } C/H$$

Hence $A^G \simeq C^H$.

ii. By hypothesis $G/H$ is étale over $R$. After an étale base change, we may assume that $G/H$ splits over $R$. The $R$-algebra $A$ splits into $[G/H]$ factors as an induced module usually does, $A = \prod_{\alpha \in G/H} A_\alpha$. We can see $A_\alpha \cong C$ as the ring of functions with support in the $\alpha$-coset in $G$, and $A_{\text{Spec } H} = C$ the fiber over the origin $H \in G/H$.

From this picture, it is clear that if $f \in C = A_{\text{Spec } H}$ the characteristic polynomials satisfy

$$(2.3) \quad \chi_G(f) = \prod_{\alpha} \chi_H(f)$$

Let us denote by $\sigma^H_k$ (resp $\sigma^G_k$) the corresponding coefficients. After $(2.3)$, $\sigma^G_k(f) = \sigma^H_k(f)$ for all $k < m = |H|$ and $f \in C = A_{\text{Spec } H}$. This gives the inclusion $\Sigma_R^H(C) \subset \Sigma_R^G(A)$. To get the equality, we now compute the symmetric functions $\sigma_k(f)$ for any $k$ and any $f \in A$. Write $f$ as a sum $f = \sum_{\alpha \in G/H} f_{\alpha} \in \prod_{\alpha \in G/H} A_\alpha$. As seen in the proof of Lemma 2.2, such a symmetric expression of $f$ is a polynomial expression of symmetric functions of “pure” elements (belonging to a factor $A_\alpha$). For a pure element, the formula $(2.3)$ gives the answer. Hence we have the required equality.

For $G$ a constant group, Lemma 2.9 is well known (see Raynaud ([Ra2], X)).

### 3. Base change theorem

**3.1. Base Change.** We first recall an elementary version of the exchange lemma (Proposition 7.7.10 [Gr]).

**Lemma 3.1.** Let $R \to A$ be a noetherian local ring homomorphism and $k$ be the residue field of $R$. Let $L \xrightarrow{\alpha} P$ be a homomorphism of finite type $R$ flat $A$-modules. If the morphism

$$\ker \alpha \otimes_R k \xrightarrow{\phi} \ker(\alpha \otimes 1_k)$$

is surjective then it is bijective and the formation of $\ker \alpha$ is compatible with the base change: for any $R$-algebra $R'$, $\ker \alpha \otimes_R R' \xrightarrow{\sim} \ker(\alpha \otimes 1_{R'})$

The exchange lemma allows us to prove:
Proposition 3.2. Let $\pi : X \to S$ be a finite type flat morphism between locally noetherian schemes. Let $G$ be a finite flat $S$-group scheme admissibly acting on $X$. Let $\{U_i = \text{Spec } A_i\}_{i \in I}$ be a $G$-invariant affine cover of $X$ such that $\pi(U_i)$ is contained in an affine open $S$-subscheme $V_i = \text{Spec } R_i$ for any $i$. Assume that for any morphism $R_i \to \Omega$ ($\Omega$ an algebraically closed field) the morphism

$$A_i^G \otimes_R \Omega \to (A_i \otimes_R \Omega)^{G \times \Omega}$$

is surjective. Under those conditions the morphism $X \to X/G$ commutes with any base change.

Proof. Let us restrict to a $G$-invariant open affine $U = \text{Spec } A$ with $\pi(U) \subset \text{Spec } R$. The ring $R$ is noetherian. Since $\pi : X \to S$ is a finite type morphism, $A$ is a finite type $R$-algebra.

Assume $R$ to be local. Let $k$ denote its residue field and $\Omega$ an algebraic closure of $k$. Since $A^G \otimes_R \Omega \to (A \otimes_R \Omega)^{G \times \Omega}$ is surjective, $A^G \otimes_R k \to (A \otimes_R k)^{G \times k}$ is surjective. Consider the finite type homomorphism of $A$-flat module

$$\alpha : A \to A \otimes_R R[G], \ a \mapsto (\mu^*(a) - a \otimes 1)$$

By definition $\ker \alpha = A^G$. The hypotheses of the exchange lemma 3.1 are satisfied. Hence $A^G \otimes_R R' \simeq (A \otimes_R R')^{G \times R'}$ for any $R$-algebra $R'$. If the noetherian ring $R$ is not local, the same proof at any localization of $R$ allows us to prove proposition 3.2.

Remark 3.3. In proposition 3.2, it is difficult to ensure the surjectivity of morphisms $A_i^G \otimes_R \Omega \to (A_i \otimes_R \Omega)^{G \times \Omega}$ (see remark 3.4).

The quotient $X/G$ is said to be co-generated by the $G$-symmetric functions on $S$ (K.L.La) if there exists a $G$-invariant open affine cover $\{U_i = \text{Spec } A_i\}_{i \in I}$ of $X$ such that for any $i \in I$, $A_i^G = \Sigma_{R_i}^{G}(A_i)$ and $\pi(U_i)$ is contained in an open affine subscheme $V_i = \text{Spec } R_i$ of $S$. It is not difficult to extend this definition and proposition 3.4 to a finite flat groupoid.

Proposition 3.4. Let $X \to S$ be a flat morphism and let $G$ be a finite flat group scheme admissibly acting on $X$. Assume that for any geometric point $s$ of $S$, the quotient $X_s/G_s$ of the geometric fiber $X_s$ is cogenerated by the $G_s$-symmetric functions. Then $X/G$ is cogenerated by the $G$-symmetric functions on $S$ and the quotient map $X \to X/G$ commutes with any base change.

Proof. In order to obtain proposition 3.4, it suffices to remark that the hypotheses of the proposition 3.2 are satisfied: let $s$ be a geometric point of $S$ corresponding to the algebraically closed field $\Omega : s : k(s) \to \Omega$. By hypothesis, the fiber $X_s$ is cogenerated by the $G_s$-symmetric functions. Let $\{U_i = \text{Spec } A_i\}_{i \in I}$ be a $G$-invariant open affine cover of $X$ such that for any $i \in I$, $\pi(U_i)$ is contained in an open affine subscheme $V_i = \text{Spec } R_i$ of $S$ and such that $(A_i \otimes_R \Omega)^{G_s} = \Sigma_{R_i}^{G_s}(A_i)$. The surjectivity of the base change morphism for the invariant algebra

$$A_i^G \otimes_R \Omega \to (A_i \otimes_R \Omega)^{G_s}$$

comes from the surjectivity of the base change morphism for the Kleiman-Lonsted algebra $\Sigma_{R_i}^{G_s}(A_i)$ (Lemma 2.2) and from the obvious inclusion $\Sigma_{R_i}^{G_s}(A_i) \subset A_i^G$. Proposition 3.2 then proves the announced result.
After proposition 3.4, to ensure that the quotient map commutes with an arbitrary base change, it is sufficient to test the cogenority hypothesis along the geometric fibers of $X \to S$. We establish two versions of the base change theorem: first, for $X$ an affine smooth curve with an action of a finite flat group scheme (§3.2, local version) and second, for $X$ a semistable $S$-curve with an action of a finite group (i.e. constant) of $S$-automorphisms (§3.3, semistable version).

### 3.2. Local version of base change theorem

Let us begin with a local result about the action of finite group schemes on discrete valuation rings:

**Lemma 3.5.** Let $G$ be a finite group scheme over the perfect field $k$ acting on $A = k[[t]]$, the action being free at the generic point. Then $A^G = k[[N_G(t)]]$.

**Proof.** Let us denote the co-action of $G$ by $\mu^*: k[[t]] \to k[G] \otimes k[[t]]$. The group scheme $G$ is the semi-direct product of its étale part and its local part of order $p^r$ (\[\mathbb{G}_a\]). The action of $G$ being free at the generic point of $\text{Spec}(k[[t]])$, we know the fraction field of the ring $A^G$ has index $n = |G|$ in the field $k((t))$ (\[\text{Mu AV}\]). The computation made in lemma 2.2, tells us that the norm map is

$$N_G(f) = N_{G_{red}}(f)^{p^r}$$

From this we see that the norm of $\mu^*(t)$ has valuation exactly equal to $n$. Thus $k[[t]]$ has rank $n$ over $k[[N_G(t)]]$. This yields lemma 3.5. \[\square\]

**Remark 3.6.** It should be note that the conclusion of lemma 3.5 is false in general if we have more than one variable. In fact for the action of the group scheme $G = \alpha_p$ on $k[[x,y]]$ defined by the $p$-nilpotent vector field $\partial/\partial x + \partial/\partial y$, we get $A^G = k[[x-y,x^p,y^p]]$ and $\Sigma_k^G(A) = k[[x^p,y^p]]$.

**Proposition 3.7.** Let $k$ be a perfect field, and let $S = \text{Spec } k$. Let $X = \text{Spec } A$ be an affine smooth $S$-curve and $G$ be a finite flat group scheme effectively acting on $X$, the action being free at the generic point of $X$. Then $X \to X/G$ commutes with any base change.

**Proof.** We already know that $A$ and $A^G$ are finite module over $\Sigma_R^G(A)$. Since $\Sigma_R^G(A) = \Sigma_{\Sigma_R^G(A)}(A)$, we may assume without loss of generality that $R = \Sigma_R(A)$. Then $A$ is now finite over $R$ and $R = \Sigma_R(A) \subset A^G \subset A$. We must prove $R = A^G$. The latter equality is a local property on $R$, so after localization at a given maximal ideal of $R$, we may assume $R$ to be local (see lemma 2.4). Then $A$ is a semi local ring. The hypothesis about the residue fields of $A$ implies that the residue field of $R$ is perfect. The next reduction we wish to perform is to pass to the completion $\hat{R}$ of $R$, which leads to a faithfully flat base change. The completion $\hat{A} = A \otimes_R \hat{R}$ of $A$ satisfies

$$R = \Sigma_R^G(A) = A^G \text{ if and only if } \hat{R} = \Sigma_R^G(A \times_R \hat{R}) = (A \times_R \hat{R})^G$$

One can now assume $R$ to be a complete local ring. The local ring $A^G$ is complete and $A$ decomposes as the product of its local factors indexed by the maximal ideals of the semi local ring $A$. Since $G$ acts transitively on these factors, we have $A = \text{Ind}_H^G(A_0)$ where $A_0$ is one of these local rings and $H$ the stabilizer. We may finally confine our analysis to the case where $A$ is a complete discrete valuation ring, with perfect residue field (Lemma 2.4) and $H$-action. There exists an étale finite local extension $R \to R'$ such that the residue fields of $A$ and $R'$ coincide. We
Lemma 3.9. Let \( R \) be a noetherian ring such that \( \text{ker}(\phi) = 0 \) and \( \Delta \) be the projection of \( \Gamma = \text{Aut}_{A^G} \) (resp. on \( \text{Aut}_{\Sigma} \)).

Proof. The hypotheses of lemma 2.8 are satisfied for the normal subgroup \( \Delta \) being the product of complete discrete valuation rings. We may now suppose that \( R \) and \( A \) have the same residue fields. Now we have \( A = k[[t]] \); applying lemma 3.7 yields proposition 3.7.

Remark 3.8. The proof of proposition 3.7 also shows that \( \Sigma \) is finite and flat of rank \( |G| \) over \( A^G \) and \( A^G \) is a co-generated Dedekind ring: \( \Sigma_R^G(A) = A^G \) (i.e. a smooth curve).

3.3. Quotient of semistable curve. Let us begin by describing the passage to the algebra of invariants for the local ring at a double point (see \([\text{Appendix}]\) for the local ring at a double point. In other words, \( A \) is isomorphic to

\[
A \simeq \{(f(x), g(y)) \in k[[x]] \times k[[y]], \ f(0) = g(0)\}
\]

The group \( \Gamma = \text{Aut}_k \), \( A \)-automorphisms is the semi-direct product of \( \Gamma_0 \) by \((1, \tau_x, y)\) with \( \tau_0 = \{(\sigma, \sigma') \in \text{Aut}_k[[x]] \times \text{Aut}_k[[y]]\} \) and \( \tau_{x,y} \) the \( A \)-automorphism exchanging the two branches: \( \tau_{x,y}(x) = y \) and \( \tau_{x,y}(y) = x \).

Let \( R \) be a noetherian ring such that \( A \) is a finite type \( R \)-algebra. Let \( G \) be a finite subgroup of \( \text{Aut}_{A^G} \). Denote \( G_0 \) the maximal subgroup \( G \) fixing the branches of \( A \).

Let \( pr_x \) (resp. \( pr_y \)) be the projection of \( A \simeq \{(f(x), g(y)) \in k[[x]] \times k[[y]], \ f(0) = g(0)\} \) on \( k[[x]] \) (resp. on \( k[[y]] \)). Denote by \( \Delta_x \) the kernel of the projection \( H \) of \( G_0 \) on \( \text{Aut}_k[[x]] \) and by \( \Delta_y \) the kernel of the projection \( K \) of \( G_0 \) on \( \text{Aut}_k[[y]] \).

Lemma 3.9. If the order \( |\Delta_x| \) and \( |\Delta_y| \) are invertible in \( A \), then \( A^G \) is co-generated by the \( G \)-symmetric functions, \( A^G = \Sigma_R^G(A) \).

Proof. By definition \( A^{G_0} \simeq \{(f(x), g(y)) \in k[[x]]^H \times k[[y]]^K, \ f(0) = g(0)\} \). Hence

\[
A^{G_0} = \frac{k[[u,v]]}{(uv)} \quad \text{with} \quad u = \text{Nm}_H x, \ v = \text{Nm}_K y
\]

The hypotheses of lemma 2.8 are satisfied for the normal subgroups \( \Delta_x \) and \( \Delta_y \) of \( G \) respectively acting on \( pr_x(A) \) and \( pr_y(A) \). Hence \( \Sigma_R^G(pr_x(A)) \subseteq \Sigma_R^{G_0}(pr_x(A)) \) and \( \Sigma_R^G(pr_y(A)) \subseteq \Sigma_R^{G_0}(pr_y(A)) \). In particular \( u = \text{Nm}_H x \in \Sigma_R^G(A) \) and \( v = \text{Nm}_K y \in \Sigma_R^G(A) \). Then \( A^{G_0} = \Sigma_R^{G_0}(A) \).

Hence if \( G \) fixes the branches of \( A \), proposition 3.8 is proved for \( G = G_0 \).

Assume now that \( G \) does not fix the branches of \( A \). Let \( \psi \in G - G_0 \). The image \( \tilde{\psi} \) of \( \psi \) in \( G/G_0 \) is an involution of \( A^{G_0} \) which exchanges the branches; then \( \tilde{\psi} \) can be written as \( \tilde{\psi} = \tau_{u,v} \circ (p(u) + p^{-1}(v)) \) with \( p \in \text{Aut}_k[[u]] \). An element \( f(u) + g(v) \in A^{G_0} \) is \( \tilde{\psi} \)-invariant if and only if \( g(v) = pf(v) = \bar{\psi}(f(u)) \). Then

\[
\frac{k[[u]]}{(u)} \overset{\tilde{\psi}}{\rightarrow} A^G, \quad u \mapsto u + \psi(u)
\]

In order to prove \( A^G = \Sigma_R^G(A) \), it suffices to show \( u + \psi(u) \in \Sigma_R^G(A) \).

Let \( \ell = |G_0| = n/2 \) and \( G = \{g_1, \ldots, g_n\} \). The \( \ell \)-th \( G \)-symmetric function at \( x \) is

\[
\sigma(x) = \prod_{1 \leq i_1 < \cdots < i_\ell \leq n} g_{i_1}(x) \cdots g_{i_\ell}(x)
\]

Since \( G = G_0 \prod \psi G_0 \),

\[
\sigma(x) = \prod_{g \in G_0} g(x) + \psi\left(\prod_{g \in G_0} g(x)\right) + B(x)
\]
The sum $B_\ell(x)$ of monic polynomial of degree $\ell$ contains terms of the form $g(x)$ and $\psi g'(x)$ with $g, g' \in G^0$. Since $g(x)$ is a serie in $x$ without any constant term and $\psi g'(x)$ is a serie in $y$ without any constant term, $g(x)\psi g'(x) = 0$ in $A$. This leads us to

$$\sigma_k(x) = NmC_n(x) + \psi(NmC_n(x)) = u + \psi(u) \in \Sigma_R^G(A)$$

and $A^G = \Sigma_R^G(A)$. \hfill $\Box$

Let $S$ be a locally noetherian scheme. A $S$-curve is a flat, separated finite type relative dimension 1 $S$-scheme. A $S$-curve $X \rightarrow S$ is said to be semistable if

- $X$ is proper,
- its geometric fibers are reduced connected curves where the singular points are ordinary double points,

In this section $G$ is a finite group of $S$-automorphisms on $X$. The quotient by $G$ of the semistable curve $X$ is semistable (Ra1 Appendix). In lemma 3.3 we have described the local action of $G$. We now prove our main result, the base change theorem 3.10.

**Theorem 3.10.** Assume $X$ to be a semistable $S$-curve and $G$ a finite group of $S$-automorphisms of $X$. For any geometric fiber $X_s = X \times_{k(s)} \Omega$ and for any point $P$ of $X_s$, we assume that the order of the kernels $\Delta_x$ and $\Delta_y$ (if $P$ is a double point) or $\Delta$ (if $P$ is regular) are prime to the characteristic of $\Omega$. Then the morphism $(X/G) \times_S T \rightarrow (X \times_S T)/G$ is an isomorphism for any base change $T \rightarrow S$.

**Proof.** Let $s$ be a geometric $S$-point corresponding to the algebraic closed field $\Omega$. Let $\{U_i = \text{Spec } A_i\}_{i \in I}$ be a open $G$-invariant affine cover of $X$ such that $\pi(U_i) \subset V_i = \text{Spec } R_i$. The $A_i$’s are finite type $R_i$-algebra with perfect residue field at the maximal ideals. We have to show that $(A_i \otimes_{R_i} \Omega)^G = \Sigma_{R_i}^G(A_i)$. Fix $i \in I$ and let us write $A = A_i \otimes_{R_i} \Omega$. We prove $A^G = \Sigma_{R_i}^G(A)$.

Performing the same reduction as in proposition 3.7, this equality is equivalent to $A^{G'} = \Sigma_{R_i}^G(A')$ with $A'$ a local complete discrete valuation ring finite on the local complete ring $R' = \Sigma_{R_i}^G(A')$ with algebraically closed residue field. The algebra $A'$ comes from $A$ after a faithfully flat base change. The group $G'$ is the stabilizer of $A'$. Since $A = A_i \otimes_{R_i} \Omega$ and $\text{Spec } A_i$ is an affine open set of the semistable curve $X$, the algebra $A'$ is

$$A' = \Omega[[x]] \text{ or } \Omega[[x,y]]/(xy)$$

If $A' = \Omega[[x]]$ then $A'^G = \Omega[[Nm_{G'/\Delta}(T)]] = \Sigma_{R_i}^{G'/\Delta}(A')$. After lemma 2.8 for the normal subgroup $\Delta$ of $G'$, $\Sigma_{R_i}^{G'/\Delta}(A') = \Sigma_{R_i}^G(A')$. Hence $A'^G = \Sigma_{R_i}^G(A')$.

If $A' = \Omega[[x,y]]/(xy)$, $A'^G = \Sigma_{R_i}^G(A')$ after lemma 3.3.

Hence $(A_i \otimes_{R_i} \Omega)^G = \Sigma_{R_i}^G(A_i)$ for any $i \in I$ and the quotient $X_s/G$ of the fiber $X_s$ is co-generated by the $G$-symmetric functions. Theorem 3.10 then proceeds from proposition 3.7. \hfill $\Box$

Theorem 3.11 is an easy corollary of theorem 3.10.

**Corollary 3.11.** Let $X \rightarrow S$ be a semistable curve and let $G$ be a finite group of $S$-automorphisms acting on $X$. If the action of $X$ is free on an open dense set on
any geometric fiber, the morphism \((X/G) \times_S T \to (X \times_S T)/G\) is an isomorphism for any base change \(T \to S\).

4. Induction and restriction for deformation functors

From here on \(k\) is an algebraically closed field and \(S = \text{Spec } k\). Let \(X\) be a semistable \(S\)-curve and \(G\) be a finite group of \(S\)-automorphisms freely acting on a dense open set of \(X\).

Let us first adapt the definition of \([\text{BeM}]\) of the functor of the \(G\)-equivariant deformations of \((X,G)\). Let \(W(k)\) denote the Witt vectors ring of \(k\). Let \(C\) be the category of local artinian \(W(k)\)-algebras; the morphisms are \(W(k)\)-morphisms of local rings. A deformation of \((X,G)\) to \(A\) an object of \(C\) is a \(G\)-equivariant isomorphism class of Galois covers \(C \to C/G\) which further induces the identity on \(X\), with:

- \(C\) a semistable \(\text{Spec } A\)-curve such that the fiber over the closed point of \(\text{Spec } A\) is \(X \simeq C \otimes_A k\);
- the action of \(G\) on \(X\) lifts to \(C\) such that the isomorphism \(X \simeq C \otimes_A k\) is \(G\)-equivariant.

This defines a covariant functor:

\[ \text{Def}_G(X) : C \to \text{Set}, \quad A \mapsto \{\text{deformations of } (X,G) \text{ to } A\} \]

Let \(H\) be a normal subgroup of \(G\). The restriction of the action of \(G\) on \(X\) to \(H\) defines the canonical morphism

\[ \text{res} : \text{Def}_G(X) \to \text{Def}_H(X) \]

After theorem 3.10 the induction morphism is well-defined

\[ \text{ind} : \text{Def}_G(X) \to \text{Def}_{G/H}(X/H) \]

To summarize

**Theorem 4.1.** Let \(k\) be an algebraically closed field and \(S = \text{Spec } k\). Let \(X\) be a semistable \(S\)-curve and \(G\) be a finite group of \(S\)-automorphisms freely acting on a dense open set of \(X\). There exist induction and restriction morphisms between the deformation functors:

\[ \text{Def}_H(X) \xleftarrow{\text{res}} \text{Def}_G(X) \xrightarrow{\text{ind}} \text{Def}_{G/H}(X/H) \]

A local-global principle reduces the study of the deformation functor \(\text{Def}_G(X)\) to the study of local deformation functor at singular points and wildly ramified points (\([\text{BeM}2]\)).

**Remark 4.2.** If \(X\) is a smooth curve, we are able to determine the differential of the local restriction and induction morphisms. In the local case, we could study the functor \(D_G\) (resp. \(D_H\), resp. \(D_{G/H}\)) of deformations of an injective morphism \(G \to \text{Aut } k[[T]]\) (resp. \(H \to \text{Aut } k[[T]]\), resp. \(G/H \to \text{Aut } k[[T]]^G\)). The tangent spaces of these local deformation functors are isomorphic to cohomology groups (\([\text{BeM}2]\) Th. 2.2). Let \(k[\varepsilon]\) be the ring of dual numbers and \(k[[Y]] \simeq k[[T]]^G\). We have

\[ D_G(k[\varepsilon]) \simeq H^1(G, \Theta_T) \quad \text{with} \quad \Theta_T = \{h(T) \frac{d}{dT}, h(t) \in k[[T]]\} \]

\[ D_H(k[\varepsilon]) \simeq H^1(H, \Theta_T), \quad D_{G/H}(k[\varepsilon]) \simeq H^1(G/H, \Theta_Y) \]
The differential $\phi_{\text{res}}$ of the restriction morphism canonically coincides with the restriction map between cohomology groups:

$$\phi_{\text{res}} : H^1(G, \Theta_T) \xrightarrow{\text{res}} H^1(H, \Theta_T)$$

For the induction case, denote the inflation map by $\inf : H^1(G/H, \Theta^H_T) \rightarrow H^1(G, \Theta_T)$. We may show that the differential $\phi_{\text{ind}}$ of the induction morphism satisfies that:

$$\phi_{\text{ind}} \circ \inf : H^1(G/H, \Theta^H_T) \rightarrow H^1(G/H, \Theta_Y)$$

is the natural morphism arising from the inclusion $\Theta^H_T \subset \Theta_Y$. We are also able to make an analysis of the obstruction space for the $G$-deformation functor in the spirit of the Hochschild-Serre spectral sequence (paper in preparation). Beside the obstructions associated to $H$ and $G/H$, there are in general mixed obstructions (see an example in ([CoKa]).

5. Appendix

We want to explain briefly for convenience of the reader how to get lemma 5.1 used in the proof of lemma 2.2. Note that in ([Kil]) the relations are left as an exercise. Let $q, n \in \mathbb{N}^*$ and let $x^{(i)}$, $1 \leq i \leq q$ be $q$ sets of $n$ variables. Define the partial polarization of the elementary symmetric functions

$$(5.1) \quad s_{\alpha_1, \ldots, \alpha_q}(x^{(1)}, \ldots, x^{(q)}) = \sum(x^{(1)}_{i_1} \cdots x^{(1)}_{i_{\alpha_1}}) \cdots (x^{(q)}_{i_1} \cdots x^{(q)}_{i_{\alpha_q}})$$

where the sum is over all disjoints sequences $i_1^{(1)} < \cdots < i_1^{(1)}, \ldots, i_1^{(q)} < \cdots < i_{\alpha_q}$, with $\alpha_1 + \cdots + \alpha_q \leq n$.

**Lemma 5.1.** The partial polarization of the symmetric functions $s_{\alpha_1, \ldots, \alpha_q}(x^{(1)}, \ldots, x^{(q)})$ may be expressed as a polynomial with integer coefficients in the arguments

$$s_j((x^{(1)})^{r_1} \cdots (x^{(q)})^{r_q})$$

where the notation $((x^{(1)})^{r_1} \cdots (x^{(q)})^{r_q})$ means $(x_1^{(1)})^{r_1} \cdots (x_1^{(q)})^{r_q}, \ldots, (x_n^{(1)})^{r_1} \cdots (x_n^{(q)})^{r_q})$.

**Proof.** We work by induction on $|\alpha| = \sum_{i=1}^q \alpha_i$. Develop the product of elementary symmetric functions

$$\prod_{j} s_{\alpha_j}(x^{(j)}) = s_{\alpha_1, \ldots, \alpha_q}(x^{(1)}, \ldots, x^{(q)}) + \sum_{\beta_1, \ldots, \beta_q} s_{\beta_1, \ldots, \beta_q}(m_1, \ldots, m_q)$$

where the sum is over index $\beta$ with $|\beta| < |\alpha|$, and the $m_i$’s are suitable monomials in the formal variables $x^{(1)}, \ldots, x^{(q)}$. Inverting these triangular relations, we get the expected result.

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