NEW ROBUST ESTIMATOR OF CHANGE POINT IN SEGMENTED REGRESSION MODEL FOR BED-LOAD OF RIVERS

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https://doi.org/10.26782/jmcms.2019.12.00027

Abstract

Segmentation has vital employment in regression analysis where data have some change point. Traditional estimation methods such as Hudson, D.J. (1966) and Muggeo, V. M., (2003) have been reviewed. But these methods do not take into account robustness in the presence of outliers values. However, third method was used as rank-based method, where the analysis will be devoted to the ranks of data instead of the data themselves. Our contribution in this paper is to use M-estimator methodology with three distinct weight functions (Huber, Tukey, and Hampel) which has been combined with Muggeo version approach to gain more robustness. Thus we get robust estimates from the change point and regression parameters simultaneously. We call our new estimator as robust Iterative Rewighted M-estimator: IRWm-method with respect to its own weight function. Our primary interest is to estimate the change point that joins the segments of regression curve, and our secondary interest is to estimate the parameters of segmented regression model. The real data set were used which concerned to bed-loaded transport as dependent variable (y) and discharge explanatory variable (x). The comparison has been conducted by using several criteria to select the most appropriate method for estimating the change point and the regression parameters. The superior results were marked for IRWm-estimator with respect to Tukey weight function.

Keywords: Segmented regression, change point, rank-based estimator, iterative reweighted least squares, M-estimator.

I. Introduction

Sometimes we need to use segmentation in regression model according to some "transition" or "change" in the studied phenomenon data. Segmented regression based on existence of some point at which the explanatory variable (x) changed and consequently, the functional form changed as well, this point is called: change point.
This type of regression is called: Broken line models [XVI], Joinpoint models [XIII], or Broken-stick models [IV]. There are two cases to deal with change point. The first type suppose change point to be known in advance, while the second type suppose change point to be unknown and need to be estimated which is practically, the most popular case among the recent researches.

Segmented regression is used to deal with the data represent partial linear relationship of dependent variable (y) that associated with explanatory variable (x) when the original phenomenon changes its behavior at some change point. This can be considered in many applications such as: medical, genetic, ecological, econometrics, financial or insurance studies.

Segmented (or piecewise) regression was initially introduced by Quandt, R.E. (1958) [XX], Blicscheke, (1961) [III], Robinson (1964) [XXI] to deal with two regimes only. While Hudson (1966) [VIII] emphasized this methodology to more than two regimes and described the types of change points. Küchenhoff H. (1996) [XI] was interested in unknown change point estimation for segmented generalized linear models. Furthermore, Muggeo, V.M.R. (2003, 2017) [XVI, XVII] can be considered as the most recent up-to-date contributor in his suggestion to deal with segmented regression model estimation. He presented new simple, alternative, and general method summarized by conducting confidence interval for change point estimation. Fasola, S. et. al. (2017) [V] introduced an iterative algorithm as approximation to maximum likelihood estimation to deal with the case of big data. Meanwhile, rank-based estimator was presented recently by Zhang F. and Li Q. (2017) [XXIX] to be applied on bent linear regression model with unknown change point with by successive replacing residual sum of squares in segmentation approach according to rank dispersion function and If data contain outliers it may cause inaccurate estimation of change point. The M-estimation method is most common general method of robust regression which introduced by Huber (1964) as a generalization to maximum likelihood estimation in context of location models, and it used to minimize cases where significant residues are due to the presence of outliers values[I].

This paper attempts to use new robust methods in estimate of change point and regression parameters where it combines robust M-estimator by three distinct weights (Huber, Tukey, and Hampel) withMuggeo approach at once inside algorithm, the yielding estimator can be called as robust iterative reweighted M-estimators (robust IRWm-estimators).

II. Methods and Materials

II.i. Segmented Linear Regression Model

Let \( \{(x_i, y_i); i = 1, \cdots, n\} \) is a random sample was drawn from unknown population such that: \( x_i \)'s are distinct and \( x_i < x_{i+1} \) for all \( i = 1, \cdots, n - 1 \). Define \( \{\tau_i\}_{i=1}^{n-1} \) as a sequence of join-points within sample range itself, i.e. \( x_i \leq \tau_i < x_{i+1} \). So, under these assumptions segmented regression model can be represented as simple linear regression has segmentation into two segments with one change point for each interval \((x_1, x_n)\) [VIII, XXII].
\[ f(x_i) = f_1(x_i; \beta_1) \quad x_1 \leq x_i \leq \tau = f_2(x_i; \beta_2) \tau \leq x_i \leq x_n \] (1)

Where \( x_1 \) and \( x_n \) are known boundaries of \( x \) values.

\( \tau \) : unknown value needed to be estimated as the change point which joining the two segments of regression.

\( \beta_j \) : unknown regression coefficients have to get their estimates.

\( f_j(x; \beta_j) \) : sub-regression model and linear in coefficients \( \beta_j \) and differentiable with respect to \( x \), and can be represented by:

\[
\begin{align*}
    f_1(x_i; \beta_1) &= \beta_0 + \beta_1 x_i \\
    f_2(x_i; \beta_2) &= \beta_{00} + \beta_2 x_i
\end{align*}
\] (2)

Equation (1) can be rewritten as [20, 26]:

\[
\begin{align*}
    f(x_i) &= \beta_0 + \beta_1 x_i \quad x_1 \leq x_i \leq \tau \\
    &= \beta_{00} + \beta_2 x_i \tau \leq x_i \leq x_n
\end{align*}
\] (3)

\( \beta_0 \) : intercept parameter.

\( \beta_1 \) : slope parameter.

Both of sub-models \( f_1(x_i; \beta_1), f_2(x_i; \beta_2) \) were associated with \( (x = \tau) \) and be equivalent at that point, i.e. [XXII, XXX]:

\[
    \beta_0 + \beta_1 \tau = \beta_{00} + \beta_2 \tau
\] (4)

Then it is obtained [XIV]:

\[
    \tau = \frac{\beta_{00} - \beta_0}{\beta_1 - \beta_2} = -\frac{\Delta_0}{\Delta_1}
\] (5)

**Alternative form of segmented regression model**

From equations (3) and (4) it can be got alternative form of segmented regression model as the following form [IV, VI, XXVII]:

\[
    f(x_i) = \beta_0 + \beta_1 x_i + \Delta_1 (x_i - \tau) +, \quad i = 1,2,\cdots, n
\] (6)

Where \( \tau \) is the change point.

\( (x_i - \tau)_+ = (x_i - \tau) \times I(x_i > \tau) \)

Where \( I(x_i > \tau) = 1 \) if \( x_i > \tau \) and 0 otherwise, i.e. [IV]:

\[
    (x_i - \tau)_+ = \begin{cases} x_i - \tau, & x_i > \tau \\ 0, & x_i \leq \tau \end{cases}
\]

where is:

\[
\begin{align*}
    \Delta_0 &= \beta_0 - \beta_{00}, \quad \Delta_1 = \beta_1 - \beta_2 \\
    \Delta_1 : \text{difference parameter between the slope of the first segment and the second segment.}
\end{align*}
\]

\( \beta_2 \) can be obtained by [VI, XXVII, XXVIII]:

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\[ \beta_2 = \beta_1 + \Delta_1 \]  
(7)
and \( \beta_{00} \) can be found from [VI, XXX]:
\[ \beta_{00} = \beta_0 + \beta_1 \tau - \beta_2 \tau \]  
(8)

II.ii. Hudson Method

Hudson, (1966) [VIII] introduced global least squares (OLS) estimation when the regression model consisting of two (or more) sub-models which has to be joined according to their closeness of each other. The join points or change points have to be estimated. Each sub-model has its own local least squares estimation.

Equations (3) and (4) can be used to get the estimations of regression parameters \( \beta \)'s and change point \( \tau \). Hudson's algorithm can be written as follows [VIII, XII, XXVIII]:

1. Setting \( x \) values in ascending order.
2. Set the local least square estimations for each segment of: \([x_i, x_{i+1}], [x_{i+1}, x_n]\) where is: \( 2 \leq i \leq n - 2 \) and

\[
Y_1 = \begin{pmatrix}
    y_1 \\
    \vdots \\
    y_i \\
    \vdots \\
    y_n
\end{pmatrix}, \quad Y_2 = \begin{pmatrix}
    y_{i+1} \\
    \vdots \\
    y_n
\end{pmatrix}, \quad X_1 = \begin{pmatrix}
    1 & \vdots & 1
\end{pmatrix}, \quad X_2 = \begin{pmatrix}
    1 & \vdots & 1
\end{pmatrix}
\]

Regardless of change point, weighted least squares can be obtained for each segment as follows:
\[
\hat{\beta}_1 = (X_j'W_jX_j)^{-1}X_j'W_jY_j, j = 1,2
\]
(9)
\[
\hat{\beta}_1 = (\beta_0, \beta_1), \quad \hat{\beta}_2 = (\beta_{00}, \beta_2)
\]
Assuming \( w=1 \), the local least squares (unconstrained) can be found.

3. Suppose \( \tau^{(i)} \) as a solution for: \( \beta_0 + \beta_1 \tau = \beta_{00} + \beta_2 \tau \)
   - If \( \tau^{(i)} \in [x_i, x_{i+1}] \), then: \( \tau^{(i)} \) will be at the right side of regression curve and the two sub-curves will be joined between the two observations \( x_i \) and \( x_{i+1} \), then:

\[
R(i) = \rho_1 + \rho_2
\]
(10)
where : \( \rho_1 \) and \( \rho_2 \) are (unconstrained) local residual sum of squares of the two segments respectively.

   - If \( \tau^{(i)} \notin [x_i, x_{i+1}] \), i.e. the two (unconstrained) sub-curves will be joined outside the interval \([x_i, x_{i+1}]\). Therefore, (constrained) least squares estimation has been used, i.e. \( \hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2) \) joined by change point belongs for both curves. Putting: \( \tau^{(i)} = x_i \) (change point constrain is joining the two segmented regression curves by \( \tau^{(i)} \)).

\[
\beta_0 + \beta_1 x_i - \beta_{00} - \beta_2 x_i = 0, \quad or(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_{00}, \hat{\beta}_2).q = 0,
\]
(11)
where: \( q' = (1, x, -1, -x) \)
\[ (\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_{00}, \tilde{\beta}_2). q = s \] (12)

Let:
\[ C^{-1} = \begin{pmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{pmatrix}, \quad C_1^{-1} = (X_1W_1'X_1)^{-1}, \]
\[ C_2^{-1} = (X_2W_2X_2)^{-1} \]
\[ q'C^{-1}q = m \] (13)

Using Lagrange Multipliers (constrained) least squares estimation will be:
\[ \hat{\beta} = \tilde{\beta} - \frac{s}{m} C^{-1}q \] (14)

The adjusted residual sum of squares will be:
\[ R(i) = (\rho_1 + \rho_2 + s^2/m) \] (15)

4. Repeat the previous steps for all \( i \) to select \( \tau^{(i)} \) that makes \( R(i) \) as minimum as possible, i.e.:
\[ \tau = \arg_{\tau^{(i)}} \min R(i) \] (16)

Hudson algorithm can be summarized as the following plan (1) below.
II.iii. Muggeo Method

Muggeo, (2003) [XVI] suggested an alternative technique to estimate segmented regression model or which he called it as (broken line model) to transform the problem into linear frame. Here, we rewrite the equation (6) into the following alternative form:

\[ y_i = \beta_0 + \beta_1 x_i + \Delta_i (x_i - \tau)_+ \quad i = 1, 2, \ldots, n \]  (17)

Linear reparametrization technique has to performed via the following [XVI, XXVII]:

\[ h(x; \tau) \approx h(x; \tau^{(0)}) + (\tau - \tau^{(0)}) h'(x; \tau^{(0)}) \]  (18)

\( \tau^{(0)} \): initial guess value of change point.

\( h'(\cdot; \tau^{(0)}) \): the first derivative of \( h(\cdot) \) at the point \( \tau^{(0)} \).

\( (x_i - \tau)_+ \) can be rewritten as :

\[ (x_i - \tau)_+ = (x_i - \tau^{(0)})_+ + (\tau - \tau^{(0)})(-1)I(x_i > \tau^{(0)}) \]  (19)
The equation (19) can be substituted in equation (17) to be:

\[ y_i = \beta_0 + \beta_1 x_i + \Delta_1[(x_i - \tau^{(0)})_+ + (\tau - \tau^{(0)})(-1)I(x_i > \tau^{(0)})] \] (20)

or written as:

\[ y = \beta_0 + \beta_1 x + \Delta_1 U + \gamma V \] (21)

where:

\[ \gamma = \Delta_1(\tau - \tau^{(0)}) \]

\[ U = (x_i - \tau^{(0)})_+ \text{ and } V = -I(x_i > \tau^{(0)}) \] (22)

Equation (21) is a regular linear regression model whose parameters and change point are estimable.

Muggeo method can be introduced as the following iterative algorithm [XVI, XXVII]:

1. Fixing \( \tau^{(s)} \) initial value which is a guess value belonging to \( x \) observations themselves.
2. Calculate \( V^{(s)} \) and \( U^{(s)} \) according to equation (22).
3. Fitting the model with the additional variables \( V^{(s)} \) and \( U^{(s)} \) as equation (21) to get all estimations of parameters including \( \Delta \) and \( \gamma \).
4. Improving and updating change point by the following formulae:

\[ \hat{\tau} = \frac{\hat{\gamma}}{\Delta_1} + \tau^{(s)} \] (23)

5. Repeat the previous steps (2 to 4) until converging has been achieved, i.e. \( \hat{\gamma} \approx 0 \).

Muggeo algorithm can be summarized as the following plan (2) below.
II.iv. Iterative rank-based Method

The origin idea was initialized by Jureckova [X] and Jaeckel [IX] to minimize the objective function \( \|e\|_2^2 = \sum_{i=1}^{n} e_i e_i \) through the criterion:

\[
\|e\|_\phi = \sum_{i=1}^{n} \phi \left( \frac{R_i}{n+1} \right) e_i,
\]

(24)

Where \(e_1, \cdots, e_n\) are the residuals

\(\|e\|_\phi\): is called "dispersion function"

\(R_i\): is the rank of \(i^{th}\) residual \((e_i)\)

\[
\text{Start}
\]

Input the Data X, Y

Input the Initial value \(\tau^{(0)}\)

Calculate \(U^{(0)}, V^{(0)}\) According to the formula (22)

Fit the model with Additional variables \(U^{(0)}, V^{(0)}\) According to the formula (21)

Calculate the Maximum likelihood Estimator (MLE) for the formula (21)

Improve the change point estimate by the formula (23)

\[
\text{If The Parameter } \gamma \text{ in the formula (21) } \approx 0
\]

\[
\text{yes} \quad \text{No}
\]

Print (SSE, MLE, \(\tau^{(0)} = \hat{\tau}\))

END

Plan (2) Muggeo Algorithm
∅(·): is squared objective function integralable non-decreasing on the period (0 , 1) such as: \( \int \varphi(u)\,du = 0 \) and \( \int \varphi(u)^2\,du = 1 \) [XV]

Rank-based estimator is obtained by minimizing the dispersion function \( \|e\|_\varphi \) rather than \( \|e\|_\varphi^2 \) [XXV, XXIX].

Choosing the objective function is often subjective to the probability distribution of the error (see; Hettmansperger and McKean, 2011) [VII]. Wilcoxon is one of the common objective functions which is represented by:

\[
\varphi(t) = \sqrt{\frac{12}{7}} (t - 0.5)
\]

[XV].

The residual sum of squares decomposition in Muggeo, (2003) is replaced by rank dispersion which consists Wicoxon objective function [XXV, XXIX] through the following algorithm to minimize dispersion function:

1. Preparing initial values of parameters \( \hat{\beta}^{(s)} = (\hat{\beta}_1^{(s)}, \hat{\Delta}_1^{(s)}), \hat{\beta}_0^{(s)}, \hat{\tau}^{(s)} \) and setting a small initial value for \( \hat{\gamma}^{(s)} \).

2. Estimating the parameters \( \hat{\theta}^{(s+1)} = (\hat{\beta}_1^{(s+1)}, \hat{\Delta}_1^{(s+1)}) \) and \( \hat{\varphi}^{(s+1)} \) by using rank-based regression estimation.

\[
y_i = \beta_0 + \beta_1 x_i + \Delta_1 (x_i - \tau^{(s)}) + \gamma [-I(x_i > \tau^{(s)})] + e_i
\]

(25)

where is:

\[
(\hat{\theta}^{(s+1)}, \hat{\varphi}^{(s+1)}) = \arg \min \sum_{i=1}^{n} \sqrt{12} \left( \frac{R_i^{(s)}}{n + 1} - 0.5 \right) e_i^{(s)}
\]

(26)

and \( R_i^{(s)} \) is the rank of \( (i^{th}) \) residual \( e_i^{(s)} \).

\[
e_i^{(s)} = y_i - \beta_0 - \beta_1 x_i - \Delta_1 (x_i - \tau^{(s)}) + \gamma [-I(x_i > \tau^{(s)})]
\]

3. Updating change point estimation \( \hat{\tau}^{(s+1)} \) according to the formulae:

\[
\hat{\tau}^{(s+1)} = \hat{\tau}^{(s)} + \frac{\hat{\Delta}_1^{(s+1)}}{\hat{\Delta}_1^{(s+1)}}
\]

(27)

4. Repeat steps (2) & (3) until converging achievement, i.e.

\[
\|\hat{\theta}^{(s+1)} - \hat{\theta}^{(s)}\|_\infty < 10^{-5}
\]

where is \( \|v\|_\infty = \max |v_j| \) for each \( v \in \mathbb{R}^d \).

Furthermore, when the above algorithm converges according to (Muggeo) also as a stopping rule \( (\hat{\varphi} \approx 0) \) then, the estimates \( (\hat{\beta}, \hat{\varphi}) \) will be consistence and followed an asymptotic normal distribution (Hettmansperger and McKean, 2011) [VII].

Rank-based regression procedures can be done and calculated by "Rfit" routine in R package.

Rank-based Regression algorithm can be summarized as the following plan (3) below.
II.v. Proposed Robust IRWm-Method

M-estimator methodology is considered as one of the robust regression estimations which reduces large residuals in case of existence of extreme values. M-estimator uses maximum likelihood form via obtaining the optimal weight for a certain data set. M-

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version can be employed for Muggeo method in iterative reweighting manner to estimate segmented regression which yields robustness subjected to minimize loss function $P(\cdot)$ instead of least squares [XXVI], i.e. $\sum P(e_i) = \sum e_i^2$

We can get robust estimation of the parameters through minimizing $\sum P(e_i)$. Consequently, we can call it robust IRWm-estimator which can be obtained by the following steps:

1. Initializing parameters with $\hat{\theta}^{(s)} = (\hat{\beta}_0^{(s)}, \hat{\beta}_1^{(s)}, \hat{\Delta}_1^{(s)}, \hat{\tau}^{(s)})$ and assigning $\hat{\tau}^{(s)}$ with tiny initial value (say; 0.01).

2. Starting to estimate parameters $\hat{\theta}^{(s+1)} = (\hat{\beta}_0^{(s+1)}, \hat{\beta}_1^{(s+1)}, \hat{\Delta}_1^{(s+1)}, \hat{\tau}^{(s+1)})$ through employing M-estimator and apply it on equation (21) of Muggeo method after using Linear reparametrization technique as follows:
   a. Determine the initial estimator $\hat{\theta}^{(r)} = (\beta_0, \beta_1, \Delta_1, \gamma)$ such as ordinary least squares [I, XVIII] of Muggeo equation (21) which is rewritten as:

   $$y_i = \beta_0 + \beta_1 x_i + \Delta_1 U_i^{(r)} + \gamma V_i^{(r)} + e_i$$  

   (28)

   b. Compute the residuals:

   $$e_i^{(r)} = y_i - (\beta_0 + \beta_1 x_i + \Delta_1 U_i^{(r)} + \gamma V_i^{(r)})$$

   c. Estimate the parameters $\hat{\theta}^{(r+1)} = (\hat{\beta}_0^{(r+1)}, \hat{\beta}_1^{(r+1)}, \hat{\Delta}_1^{(r+1)}, \hat{\gamma}^{(r+1)})$ as robust M-estimator according to the formula [I]:

   $$D(\hat{\theta}^{(r+1)}) = \min \sum_n p(u_i^{(r)})$$  

   (29)

   Where:

   $$u_i^{(r)} = e_i^{(r)} / \sigma_{\text{MAD}}$$

   and:

   $$\sigma_{\text{MAD}} = \text{median}[|e_i^{(r)} - \text{median}(e_i^{(r)})|] / 0.6745$$

   By differentiation equation (29) with respect to $\theta$ and then setting to zero, we get [XXVI]:

   $$\sum_{i=1}^{n} Xw(u_i^{(r)}) = 0$$  

   (30)

   Where is: $X = (x, U^{(r)}, V^{(r)})$ and $w = \psi$

   $$\psi(u_i^{(r)}) = \frac{\psi(u_i^{(r)})}{u_i^{(r)}}$$

   Hence, equation (30) can be rewritten as follows:

   $$\sum_{i=1}^{n} Xw(u_i^{(r)}) (y_i - (\beta_0 + \beta_1 x_i + \Delta_1 U_i^{(r)} + \gamma V_i^{(r)})) = 0$$  

   (31)

   Equation (31) is known as weighted least squares and its solution yielding $\theta^{r+1}$ estimation which can be given as follows [I, XXVI]:

   

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\[ \hat{\theta}^{r+1} = (XWX)^{-1}X'WY \]  
(32)

Where: W is a diagonal matrix with order \( n \times n \) which its elements are \( w(u_i^{(r)}) \).

d. Repeat steps from (a) to (c) until satisfying the stopping rule by convergence to the following value:
\[ \| \hat{\theta}^{r+1} - \hat{\theta}^r \|_\infty < 10^{-5} \]

where \( \|v\|_\infty = \max |v_j| \) for each \( v \in R^n \)

After being converged we getting:
\[ \hat{\theta}^{s+1} = \hat{\theta}^{r+1} \]

3. Update change point \( \tau \) according to equation (23) in Muggeo method which rewritten as:
\[ \hat{\tau}^{(s+1)} = \hat{\tau}^{(s)} + \frac{\rho^{(s+1)}}{\Delta \hat{\tau}^{(s+1)}} \]  
(33)

4. Repeat the previous steps after updating change point \( \tau \) until convergence achievement according to Muggeo criterion \( (\hat{\tau} \approx 0) \)

It is worthy to mention that there are many other patterns can be used as \( P(\cdot) \) as objective functions. In table (1) below listing of Objective function and weight function for M-estimator.

Table (1): Objective function and weight function for M-estimator

| Functions      | Objective Function | Weightfunction (w) |
|----------------|-------------------|-------------------|
| Huber [I,XIX] | \( p(u) = \begin{cases} \frac{1}{2}u_i^2, & \text{if } |u| < c \\ 2u_i|c - u_i|c^2, & \text{if } |u| \geq c \end{cases} \) with \( c = 1.345 \) | \( W(u) = \begin{cases} 1, & \text{if } |e| < c \\ \frac{c}{|e|}, & \text{if } |e| \geq c \end{cases} \) |
| Tukeybisquare [XIX,XXIV] | \( p(u) = \begin{cases} \frac{u_i^2}{2} - \frac{u_i^4}{2c^2} + \frac{u_i^6}{6c^2}, & \text{if } |u| \leq c \\ \frac{u_i^2}{c^2}, & \text{if } |u| > c \end{cases} \) with \( c = 4.685 \) | \( W(u) = \begin{cases} \left(1 - \left(\frac{u_i}{c}\right)^r\right)^2, & \text{if } |u| \leq c \\ 0, & \text{if } |u| > c \end{cases} \) |
| Hampel[XXIV] | \( p(u) = \begin{cases} \frac{1}{2}u_i^2, & \text{if } |u| \leq a \\ a|u| - \frac{1}{2}a^2, & \text{if } a < |u| \leq b \\ c|u| - \frac{1}{2}c^2, & \text{if } b < |u| \leq c \\ \frac{u_i - c}{c - b}, & \text{if } |u| > c \end{cases} \) with \( a = 1.5, b = 3, c = 6 \) | \( W(u) = \begin{cases} 1, & \text{if } |u| \leq a \\ a|u| - \frac{1}{2}a^2, & \text{if } a < |u| \leq b \\ \frac{c - |u|}{(c - b)|u|}, & \text{if } b < |u| \leq c \\ 0, & \text{if } |u| > c \end{cases} \) |

IRWm-estimator can be summarized as algorithm described in Plan (4) below.
II.vi. The standard error and Confidence intervals for the change point

Confidence Intervals estimation is one of the major alternatives to represent the statistical uncertainty \[ \sigma \] of \( \hat{\tau} \), which based on the distribution of \( \hat{\tau} \) itself. If it is imposed that \( y \) has Gaussian distribution, then the distribution of \( \Delta_0 \) and \( \Delta_1 \) in equation (5) will be Gaussian distribution too. Although of that, but \( \hat{\tau} \) in equation (5) does not distributed as Gaussian. Alternatively, \( \hat{\tau} \) is approximately normally distributed. Under these conditions, confidence intervals of \( \hat{\tau} \) can be calculated.

The precision of change point estimation and its corresponding confidence intervals can be obtained for the model represented in equation (3) by considering of random error propagation law [II], which yields the change point variance as the following formulae [XIV]:

\[ \text{Plan (4) ProposedRobustIRWm-estimatoralgorithm} \]

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\[ \text{Var}(\hat{\tau}) = \frac{\text{Var}(\hat{\Delta}_0) + \hat{\tau}^2 \text{Var}(\hat{\Delta}_1) + 2\hat{\tau}\text{Cov}(\hat{\Delta}_0, \hat{\Delta}_1)}{\hat{\Delta}_1^2} \quad (34) \]

Where \( \text{Var}(\cdot) \) and \( (\cdot,\cdot) \) are the variance and covariance respectively.

\( (\hat{\Delta}_0, \hat{\Delta}_1) \) are the differences between the first and second segment from the segmented regression as the equation (5), i.e.

\[ \hat{\Delta}_0 = \hat{\beta}_0 - \hat{\beta}_{00}, \quad \hat{\Delta}_1 = \hat{\beta}_1 - \hat{\beta}_2 \]

and

\[ \text{Var}(\hat{\Delta}_0) = \text{Var}[\hat{\beta}_0 - \hat{\beta}_{00}] = \text{Var}(\hat{\beta}_0) + \text{Var}(\hat{\beta}_{00}) \quad (35) \]

\[ \text{Var}(\hat{\Delta}_1) = \text{Var}[\hat{\beta}_1 - \hat{\beta}_2] = \text{Var}(\hat{\beta}_1) + \text{Var}(\hat{\beta}_2) \quad (36) \]

\[ \text{Cov}(\hat{\Delta}_0, \hat{\Delta}_1) = \mu_{1,(x;\leq\tau)} \text{Var}(\hat{\beta}_1) - \mu_{2,(x;>\tau)} \text{Var}(\hat{\beta}_2) \quad (37) \]

\( \mu_{1,(x;\leq\tau)} \): the mean of the first segment before the change point at model described in equation (3).

\( \mu_{2,(x;>\tau)} \): the mean of the second segment after the change point at model described in equation (3).

Whereas the change point variance of the model in equation (21) is represented as [XXIX]:

\[ \text{Var}(\hat{\tau}) = \frac{\text{Var}(\hat{\gamma}) + \text{Var}(\hat{\Delta}_1) \left( \frac{\hat{\gamma}}{\hat{\Delta}_1} \right)^2 + 2 \left( \frac{\hat{\gamma}}{\hat{\Delta}_1} \right) \text{Cov}(\hat{\gamma}, \hat{\Delta}_1)}{\hat{\Delta}_1^2} \quad (38) \]

Consequently, the standard error of change point \( \hat{\tau} \) will be gained from (Wald statistics) according to segmented regression as the following:

\[ \text{SE}(\hat{\tau}) = \sqrt{\text{Var}(\hat{\tau})} \quad (39) \]

When the algorithm converges then it is expected that \( \hat{\gamma} \) approaches to zero then the standard error for change point will be simply as:

\[ \text{SE}(\hat{\tau}) = \text{SE}(\hat{\gamma})/|\hat{\Delta}_1| \quad (40) \]

while the confidence intervals of (1-\( \alpha \))% Wald-based as the formulae:

\[ [\hat{\tau} \mp z_{\alpha/2}\text{SE}(\hat{\tau})] \quad (41) \]

Where \( z_{\alpha/2} \) the tabulated value of standard normal distribution with significant level (1 - \( \alpha \)/2).

**I. Application Data-Set**

The real data set was based on published data by Ryan, S. E., Porth, L.S.,(2007) [XXIII]. The data collected to examine the bed-load transport kg/s as \( y \) depended on discharge sediments m\(^3\)/s as \( x \) at Colorado and Wyoming, USA. The sample size was 123 observations (n=123). Figure (1) shows the Scatter plot for the data.
Formulas and equations in sections: (II.ii, II.iii, II.iv, and II.v) were applied in order to get segmented regression parameters and change point estimators by using Matlab and R programs. Furthermore, mean of square error: MSE and relative efficiency: RE criteria were computed for Hudson method on a hand, and other estimation methods on the other hand. While standard error and confidence intervals of change point which corresponding estimation method, as well as coefficient of determination: $R^2$, mean of absolute error: MAE criteria. All summarized in table (2) below. Figures (2, 3, 4, and 5) represent segmented regression pattern and change point location. Equations (7) and (8) were applied also to obtain $\beta_0, \beta_2$ parameter estimation for Muggo, Iterative rank-based, and IRWM-estimators respectively.

Table (2): Results of change point and regression parameters estimations of the methods with corresponding comparison measures

| Variables                   | Parameters of the segmented regression | Methods                                      | Robust IRWm-estimator |
|-----------------------------|----------------------------------------|----------------------------------------------|------------------------|
|                             |                                        | Hudson | Muggeo | Iterative rank-based estimator | Tukey Bisquare | Huber | Hampel |
| Change point                | $\hat{\tau}$                           | 4.83413 | 7.58400 | 7.53422 | 7.08386 | 7.45169 | 7.0990 |
| Constant offirst segment    | $\hat{\beta}_0$                        | -0.0022 | -0.07047 | -0.01299 | -0.0026 | -0.02085 | -0.00365 |
| $x_i \leq \tau$             | $\hat{\beta}_1$                        | 0.00355 | 0.03266 | 0.00889 | 0.00312 | 0.01187 | 0.00378 |
| Constant of second segment  | $\hat{\beta}_{00}$                     | -0.5137 | -1.12117 | -1.69180 | -1.50019 | -1.61797 | -1.50528 |
| $x_i > \tau$                | $\hat{\beta}_2$                        | 0.10936 | 0.1712 | 0.2317 | 0.21452 | 0.2262 | 0.21531 |

95% Confidence intervals of $\hat{\tau}$

|                           | L=3.859 | U=5.810 | L=7.399 | U=7.669 | L=7.035 | U=7.133 | L=7.174 | U=7.729 | L=7.036 | U=7.162 |
|---------------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Standard error for $\hat{\tau}$ | 0.49775 | 0.56547 | 0.06875 | 0.02489 | 0.14156 | 0.03218 |
| MSE                       | 0.02422 | 0.02460 | 0.02792 | 0.00003 | 0.00285 | 0.00006 |
| $\bar{\sigma}$            | 0.1556 | 0.1568 | 0.1671 | 0.0055 | 0.0534 | 0.0077 |
| $R^2$                     | 0.63785 | 0.63213 | 0.8102 | 0.99651 | 0.86683 | 0.99349 |
| MAE                       | 0.0764 | 0.0901 | 0.0731 | 0.0730 | 0.0743 | 0.0728 |
| Relative Efficiency       | 100     | 101.569 | 115.3014 | 0.1239 | 0.2477 |

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The results showed that minimum MSE value was assigned by Tukey function when used in Robust IRWm-estimator. After that Hampel function came in the second grade and finally, Huber function. While maximum $R^2$ value was assigned by Tukey function and then Hampel and Huber functions respectively. The priority in minimum MAE values were assigned for Hampel, Tukey, and Iterative rank-based methods respectively. The minimum values associated to standard error for change point estimator were assigned first for Tukey, Hampel, and then for Iterative rank-based method. It can be noticed that Tukey method had the best RE with respect to OLS (Hudson) estimator and then followed by Hampel and Huber respectively.

Therefore, it can be said that: Robust IRWm-estimator according to Tukey function was the best robust method for segmented regression model, and its parameter and change point estimators can be relied to find the predicted bed-load values, as described below.

\[
\hat{Y} = -0.0026 + 0.00312x_i x_i \leq 7.08386
\]
\[
= -1.50019 + 0.21452x_i x_i > 7.08386
\]
Alternatively, it can be written as follows:

\[ \hat{Y} = -0.0026 + 0.00312 x_i + (0.21452 - 0.00312)(x_i - 7.08386) \]

III. Conclusions

Based on the numerical results of real data set application, one can conclude the following main conclusions:

- Our new robust estimators of Robust IRWm-estimator can be relied as breakdown of regular conditions of traditional linear regression model.
- The best method was the proposed Robust IRWm-estimator which based on Tukey-bisquare function with respect to MSE, \( R^2 \), RE, and the precision of change point estimate (according to the standard error) respectively. While, Hample function came in the second grade.

Furthermore, (Robust IRWm-estimator) can be used due to another weighting functions which different from that used in this paper for future work.

IV. Acknowledgements

Our thanks go to Sandra E. Ryan and Laurie S. Porth for the benefit of their published data set. We would like to thank the Department of Statistics, College of Administration and Economics, University of Baghdad, for supporting this work. Additional thanks to the editor and anonymous reviewers of (JMCMS) their useful comments and notices.

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