METHODS OF MINIMIZATION OF
CALCULATIONS IN HIGH ENERGY PHYSICS:
II. Minimization of Number of Vectors in Problem

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December 23, 2021

Abstract

A number of different ways of reducing the number of vectors describing the
condition of particles for high energy physics problems are presented. In particular
the decomposition of any vector with respect to the basis, consisting of any four
linearly independent vectors, including the orthonormal basis and the construction
of orthonormal bases from the vectors of a problem (these bases may be used as the
polarization ones for vector bosons) as well as the expression of one vector of the
problem through the other are considered.

1 Introduction

As is well known, the maximum number of linearly independent vectors is equal to four
in the Minkowski space. Therefore for any problem the number of vectors, describing the
condition of particles, may be reduced to four, by choosing four vectors as the basic ones
and by expressing all the other vectors through them.

In Section 2 we show, how to decompose any vector with respect to the basis, consisting
of any four linearly independent vectors, including the case of the orthonormal basis.

In Section 3 the construction of orthonormal bases from the vectors of a problem is
considered. In particular these bases may be used as the polarization ones for vector
bosons.

In Section 4 some possibilities of expression of some vectors of a problem through the
other are considered.
The methods described are especially effective in a combination with the covariant method for the calculating of the amplitudes (see [1], [5], [6], [8] – [14], [17], [19], [21]), but can also be used irrespective of it (see [7], [13], [16]).

2 The decomposition of any vector with respect to the basis in the Minkowski space

Let us consider the determinant of the fifth order in the Minkowski space

\[
\begin{vmatrix}
g_{\mu\nu} & g_{\mu\alpha} & g_{\mu\beta} & g_{\mu\lambda} & g_{\mu\rho} \\
g_{\sigma\nu} & g_{\sigma\alpha} & g_{\sigma\beta} & g_{\sigma\lambda} & g_{\sigma\rho} \\
g_{\tau\nu} & g_{\tau\alpha} & g_{\tau\beta} & g_{\tau\lambda} & g_{\tau\rho} \\
g_{\kappa\nu} & g_{\kappa\alpha} & g_{\kappa\beta} & g_{\kappa\lambda} & g_{\kappa\rho} \\
g_{\omega\nu} & g_{\omega\alpha} & g_{\omega\beta} & g_{\omega\lambda} & g_{\omega\rho}
\end{vmatrix} \equiv 0
\]  

(1)

where

\[
g_{\mu\nu} = \begin{cases} 
1 & \text{if } \mu = \nu = 0 \\
-1 & \text{if } \mu = \nu = 1, 2, 3 \\
0 & \text{if } \mu \neq \nu
\end{cases}
\]

The validity of the equation (1) in the Minkowski space follows from the properties of determinants and four-dimensionality of the Minkowski space (see [2]). Really, using the properties of determinants we notice, that the tensor in the left-hand side of the equation (1) is completely antisymmetric with respect to each of five indices: \( \nu, \alpha, \beta, \lambda, \rho \) (and \( \mu, \sigma, \tau, \kappa, \omega \) as well). In a four-dimensional space, every tensor which is antisymmetric with respect to more than four indices is equal to zero, since the values of at least two of them must be equal.

By the analogy with the Gram determinant (see [3]) we introduce the notation

\[
\begin{vmatrix}
g_{\mu\nu} & g_{\mu\alpha} & g_{\mu\beta} & g_{\mu\lambda} & g_{\mu\rho} \\
g_{\sigma\nu} & g_{\sigma\alpha} & g_{\sigma\beta} & g_{\sigma\lambda} & g_{\sigma\rho} \\
g_{\tau\nu} & g_{\tau\alpha} & g_{\tau\beta} & g_{\tau\lambda} & g_{\tau\rho} \\
g_{\kappa\nu} & g_{\kappa\alpha} & g_{\kappa\beta} & g_{\kappa\lambda} & g_{\kappa\rho} \\
g_{\omega\nu} & g_{\omega\alpha} & g_{\omega\beta} & g_{\omega\lambda} & g_{\omega\rho}
\end{vmatrix} = G\left( \begin{pmatrix} \mu & \sigma & \tau & \kappa & \omega \\ \nu & \alpha & \beta & \lambda & \rho \end{pmatrix} \right).
\]

Let us consider

1 We use the same metric as in the book [4]:

\( a^\mu = (a_0, \vec{a}) \), \( a_\mu = (a_0, -\vec{a}) \), \( ab = a_\mu b^\mu = a_0 b_0 - \vec{a} \cdot \vec{b} \), sign of the Levi-Civita tensor is determined as \( \varepsilon_{0123} = +1 \).
\[
G \left( \begin{array}{cccc}
\mu & \sigma & \tau & \kappa \\
\nu & \alpha & \beta & \lambda \\
\lambda \rho & \lambda \rho & \lambda \rho & \lambda \rho
\end{array} \right)
(l_0)^\mu (l_1)^\sigma (l_2)^\tau (l_3)^\kappa a^\omega (l_0)^\alpha (l_1)^\beta (l_2)^\gamma (l_3)^\delta
\]

\[
= G \left( \begin{array}{cccc}
l_0 & l_1 & l_2 & l_3 & a \\
l_0 & l_1 & l_2 & l_3 & \rho
\end{array} \right) = \left| \begin{array}{cccc}
l_0^2 & (l_0 l_1) & (l_0 l_2) & (l_0 l_3) & (l_0 \rho) \\
l_1^2 & (l_1 l_2) & (l_1 l_3) & (l_1 \rho) \\
l_2^2 & (l_2 l_3) & (l_2 \rho) \\
l_3^2 & (l_3 \rho)
\end{array} \right| = 0. \quad (2)
\]

From (2) we have

\[
a = \frac{1}{G \left( \begin{array}{cccc}
l_0 & l_1 & l_2 & l_3
\end{array} \right)} \left[ G \left( \begin{array}{cccc}
a & l_0 & l_1 & l_2 & l_3
\end{array} \right) l_0 + G \left( \begin{array}{cccc}
l_0 & a & l_1 & l_2 & l_3
\end{array} \right) l_1
\right.
\]

\[
+ G \left( \begin{array}{cccc}
l_0 & l_1 & a & l_3
\end{array} \right) l_2 + G \left( \begin{array}{cccc}
l_0 & l_1 & l_2 & a
\end{array} \right) l_3 \right]
\]

where \( G \) are the usual Gram determinants of the fourth order.

If the condition

\[
G \left( \begin{array}{cccc}
l_0 & l_1 & l_2 & l_3
\end{array} \right) \neq 0
\]

is carried out, that is if the vectors \( l_0, l_1, l_2, l_3 \) are linearly independent, then the equality (3) allows us to decompose any vector \( a \) with respect to the four vectors \( l_0, l_1, l_2, l_3 \).

If the vectors \( l_0, l_1, l_2, l_3 \) form the orthonormal basis, that is if

\[
(l_\mu l_\nu) = g_{\mu\nu}
\]

then (2) takes the form

\[
\left| \begin{array}{cccc}
1 & 0 & 0 & 0 & (l_0)_{\rho} \\
0 & -1 & 0 & 0 & (l_1)_{\rho} \\
0 & 0 & -1 & 0 & (l_2)_{\rho} \\
0 & 0 & 0 & -1 & (l_3)_{\rho}
\end{array} \right| = 0. \quad (4)
\]

And it follows that

\[
a = (a l_0) l_0 - (a l_1) l_1 - (a l_2) l_2 - (a l_3) l_3. \quad (5)
\]
3 The construction of orthonormal bases from the vectors of a problem

With the help of three vectors of a problem and completely antisymmetric Levi-Civita tensor we can always construct an orthonormal basis. Let us consider the three bases of this sort.

1. Let \( p \) be an arbitrary 4-momentum such that

\[ p^2 = m^2 \neq 0, \]

\( a \) and \( b \) are arbitrary vectors. Then the following four vectors \( l_0, l_1, l_2, l_3 \) form the orthonormal basis:

\[
l_0 = \frac{p}{m} ,
\]

\[
(l_1)_\rho = -\frac{G\left(\begin{array}{c} p \\ p \\ a \end{array}\right)}{m\left[-G\left(\begin{array}{c} p \\ p \\ a \end{array}\right)\right]^{1/2}} = \frac{(pa)p_\rho - m^2 a_\rho}{m\sqrt{(pa)^2 - m^2a^2}} ,
\]

\[
(l_2)_\rho = \frac{G\left(\begin{array}{c} p \\ p \\ a \\ b \end{array}\right)}{\left[-G\left(\begin{array}{c} p \\ p \\ a \\ b \end{array}\right)\right]^{1/2}} = \frac{[\{(pa)(ab) - a^2(pb)\} p_\rho + [(pa)(pb) - m^2(ab)] a_\rho + [m^2a^2 - (pa)^2] b_\rho]}{\sqrt{(pa)^2 - m^2a^2\sqrt{2(pa)(pb)(ab) + m^2a^2b^2 - m^2(ab)^2 - a^2(pb)^2 - b^2(pa)^2}} ,
\]

\[
(l_3)_\rho = \frac{\epsilon_{\rho\alpha\beta\lambda}p^\alpha a^\beta b^\lambda}{\left[G\left(\begin{array}{c} p \\ p \\ a \\ b \end{array}\right)\right]^{1/2}} = \frac{\epsilon_{\rho\alpha\beta\lambda}p^\alpha a^\beta b^\lambda}{\sqrt{2(pa)(pb)(ab) + m^2a^2b^2 - m^2(ab)^2 - a^2(pb)^2 - b^2(pa)^2}} .
\]

In particular, the authors of \( \text{[5], [6]} \) use the following set:

\[ p = p_i \ , \ a = p_f \ , \ b = r \]

where \( p_i \) and \( p_f \) are the 4-momenta of the initial and final particles respectively and \( r \) is an arbitrary 4-momentum.

In \( \text{[7]} \) for the construction of the basis the following vectors are used:

\[ p = p_i \ , \ a = q_i \ , \ b = q_f \]
where \(q_i\) and \(q_f\) are the 4-momenta of the initial and final particles of the other line of the diagram respectively.

In [8] the construction of the basis uses the following vectors

\[
p = q_i, \quad a = q_f, \quad b = p_i
\]

(in this paper only the last three vectors of the basis are used).

In [9] – [13] a special form of the basis (6) – (9) with

\[
a^\mu = (m, 0, 0, 0), \quad b^\mu = (0, 0, 0, 1)
\]

is used. (In [11] – [13] only the last three vectors of the basis are used, in [13] there is the restriction \(p_y = 0\).)

2. Let \(p_1, p_2\) are the 4-momenta of a problem such that

\[
p_1^2 = m_1^2 \neq 0, \quad p_2^2 = m_2^2 \neq 0.
\]

Let us consider a special form of the basis (6) – (9) at

\[
p = \frac{m_2 p_1 + m_1 p_2}{\sqrt{m_1 m_2}}, \quad m = \sqrt{2 [(p_1 p_2) + m_1 m_2]}, \quad a = p_2:
\]

\[
l_0 = \frac{m_2 p_1 + m_1 p_2}{\sqrt{2 m_1 m_2 [(p_1 p_2) + m_1 m_2]}},
\]

\[
l_1 = \frac{m_2 p_1 - m_1 p_2}{\sqrt{2 m_1 m_2 [(p_1 p_2) - m_1 m_2]}},
\]

\[
(l_2)_\rho = \frac{G \left( \begin{array}{ccc} p_1 & p_2 & \rho \\ p_1 & p_2 & b \end{array} \right)}{\left[ -G \left( \begin{array}{ccc} p_1 & p_2 \\ p_1 & p_2 \end{array} \right) G \left( \begin{array}{ccc} p_1 & p_2 \\ p_1 & p_2 \end{array} \right) \right]^{1/2}}
\]

\[
= \frac{\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}{\sqrt{2(p_1 p_2)(p_1 b)(p_2 b) + m_1^2 m_2^2 b^2 - m_1^2 (p_2 b)^2 - m_2^2 (p_1 b)^2 - b^2 (p_1 p_2)^2}}.
\]

\[
(l_3)_\rho = \frac{\epsilon_{\rho \alpha \beta \lambda} \alpha \beta \lambda}{\sqrt{2(p_1 p_2)(p_1 b)(p_2 b) + m_1^2 m_2^2 b^2 - m_1^2 (p_2 b)^2 - m_2^2 (p_1 b)^2 - b^2 (p_1 p_2)^2}}.
\]
In [6] for the construction of the basis the vectors

\[ p_1 = p_i, \ p_2 = p_f, \ b = r \]

are used, where \( p_i \) and \( p_f \) are the 4-momenta of the initial and final particles respectively and \( r \) is an arbitrary 4-momenta.

In [14] for the construction of the basis the same vectors are used under the additional condition

\[ (p_i r) = (p_f r) = 0 \, . \]

In [15] the basis is constructed with the help of

\[ p_1 = p_f, \ p_2 = p_i, \ b = q_i \]

or

\[ p_1 = p_i, \ p_2 = p_f, \ b = q_i \]

(in this paper only vectors \( l_2 \) and \( l_3 \) are used).

3. Let \( p_1, \ p_2 \) be the 4-momenta of a problem such that \( p_1^2 = p_2^2 = 0 \). Let us consider a special form of the basis (1) – (3) at

\[ p = p_1 + p_2, \ m = \sqrt{2(p_1 p_2)}, \ a = p_2 : \]

\[ l_0 = \frac{p_1 + p_2}{\sqrt{2(p_1 p_2)}}, \quad (14) \]

\[ l_1 = \frac{p_1 - p_2}{\sqrt{2(p_1 p_2)}}, \quad (15) \]

\[ (l_2)_\rho = \frac{(p_2 b)(p_1)_\rho + (p_1 b)(p_2)_\rho - (p_1 p_2)_\rho b_\rho}{\sqrt{2(p_1 p_2)(p_1 b)(p_2 b) - b^2(p_1 p_2)^2}}, \quad (16) \]

\[ (l_3)_\rho = \frac{\varepsilon_\rho_{\alpha\beta\lambda} p_1^\alpha p_2^\beta b^\lambda}{\sqrt{2(p_1 p_2)(p_1 b)(p_2 b) - b^2(p_1 p_2)^2}}. \quad (17) \]

In [16] the basis is constructed with the help of the vectors

\[ p_1 = k_1, \ p_2 = k_2, \ b = p_i - p_f \]

where \( k_1 \) and \( k_2 \) are the 4-momenta of photons.

In [17] for the construction of the basis the vectors

\[ p_1 = k, \ p_2 = p_i, \ b = p_f \]
or

\[ p_1 = k, \ p_2 = p_f, \ b = p_i \]

are used, where \( k \) is the 4-momentum of photon (in this work only the vectors \( l_2, \ l_3 \) are used).

An appropriate choice of the form of the basis and the vectors of a problem for its construction can essentially simplify the calculations.

Note that in [22] it was proposed the decomposition of any 4-vector \( q \)

\[
q^\mu = \frac{1}{\varepsilon(p_1, p_2, p_3, p_4)} [(qp_1)v_1^\mu + (qp_2)v_2^\mu + (qp_3)v_3^\mu + (qp_4)v_4^\mu]
\]

with respect to the basis

\[
v_1^\mu = \varepsilon^{\mu\nu\rho\lambda}(p_2)_\nu(p_3)_\rho(p_4)_\lambda, \ v_2^\mu = \varepsilon^{\nu\rho\lambda}(p_1)_\nu(p_3)_\rho(p_4)_\lambda, \\
v_3^\mu = \varepsilon^{\nu\rho\lambda}(p_1)_\nu(p_2)_\rho(p_4)_\lambda, \ v_4^\mu = \varepsilon^{\nu\lambda\rho}(p_1)_\nu(p_2)_\rho(p_3)_\lambda,
\]

where \( \varepsilon(p_1, p_2, p_3, p_4) = \varepsilon^{\mu\nu\rho\lambda}(p_1)_\mu(p_2)_\nu(p_3)_\rho(p_4)_\lambda \); \( p_1, \ p_2, \ p_3, \ p_4 \) are arbitrary vectors of a problem.

However the basis (18) is not orthonormal. Besides it is necessary to use four vectors of a problem for its construction.

4 The expression of vectors of a problem through the other

1. It is possible to express the vector \( n \) which determine the axis of the spin projections of a fermion through the 4-momentum of this fermion and any other vector. Really, let us consider the vector

\[
n(p, a) = \frac{(pa)p - m^2a}{m\sqrt{(pa)^2 - m^2a^2}}. \tag{19}
\]

where \( a \) is an arbitrary vector. One can easily see that the vector \( n(p, a) \) satisfies the standard conditions

\[
n^2 = -1, \ (pn) = 0.
\]

Choosing \( a^\mu = (1, 0, 0, 0) \), we have

\[
n^\mu = \frac{1}{m}(\frac{p_0}{|\vec{p}|}, \frac{p_0}{|\vec{p}|} \vec{p})
\]

that is in this case the state of polarization of a particle is the helicity (see e.g. [18]).
In [5] – [6] it was proposed to choose
\[ n_f = \frac{(p_ip_f)p_f - m_f^2p_i}{m_f\sqrt{(p_ip_f)^2 - m_i^2m_f^2}} \]
that is in this case \( a = p_i \).
In [7] \( a = q_i \), where \( q_i \) is the 4-momentum of the initial particle of the other line of the diagram.
In [19] \( a = q \), where \( q^2 = 0 \) and for the numerical calculations there is taken the vector \( q^\mu = (1, 1, 0, 0) \).
In [20] \( a^\mu = [(\vec{p}\vec{o}), (p_0 + m)\vec{o}] \), where \( \vec{o}^2 = 1 \).
A method of this sort suitable when we are not interested in the polarization state of the particle and therefore have to sum or average over its polarizations.

2. In the paper [21] for an arbitrary vector \( S \) such that
\[ S^2 \neq 0 \]
there was offered the decomposition
\[ S = \left[ S - \frac{S^2}{2(qS)}q \right] + \frac{S^2}{2(qS)}q = s + \frac{S^2}{2(qS)}q, \]  
(20)
where \( q \) is an arbitrary vector such that
\[ q^2 = 0. \]
As this takes place, the vector
\[ s = S - \frac{S^2}{2(qS)}q \]
also has the property
\[ s^2 = 0. \]
This method is especially effective for the calculations for the processes involving massless particles.
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