A CONSEQUENCE OF THE FACTORISATION THEOREM FOR POLYNOMIAL ORBITS ON NILMANIFOLDS

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Abstract. We discuss a consequence of Green and Tao’s factorisation theorem for polynomial orbits on nilmanifolds, adjusted to the requirements of certain arithmetic applications. More precisely, we prove a generalisation of Theorem 16.4, Acta Arith. 154 (2012), 235–306, by slightly rearranging its proof. The thus achieved strengthening of the result removes an oversight in the above-cited paper which resulted from the previously too weak conclusion. Since this type of result proved essential for further applications, we take the opportunity to discuss it in more detail.

1. Introduction

The quantitative factorisation theorem for polynomial orbits on nilmanifolds that was proved in Green–Tao [3] plays a fundamental role in applications of Green and Tao’s ‘nilpotent Hardy–Littlewood method’, the machinery behind their celebrated work [2]. The theorem allows one to factorise an arbitrary polynomial sequence \( g \) on a nilmanifold as a product \( \varepsilon g' \gamma \) of three polynomial sequences, where \( \varepsilon \) is slowly varying, \( g' \) is highly equidistributed and \( \gamma \) is periodic. It is usually the equidistribution properties of the sequence \( g' \) one seeks to exploit. In order to access these one splits the range of \( n \) into subprogressions on which \( \varepsilon \) is almost constant and on which \( \gamma \) is constant. For this approach to work it is crucial that \( g' \) is still equidistributed when restricted to these subprogressions.

The nilpotent Hardy–Littlewood method can be used to study correlations of arithmetic functions; in Green and Tao’s case the function studied is the von Mangoldt function. It is usually desirable that the relevant arithmetic function, \( f \) say, has the same average value on each of the new subprogressions as on the original (possibly \( W \)-tricked) range. We will refer to this property as the major arc property.

The purpose of this paper is to discuss a consequence of Green and Tao’s factorisation theorem that is suitable for specific arithmetic applications such as questions involving a function \( f \) that counts the number of representations of an integer by a form or a polynomial, c.f. [4] and [1]. In particular, we are interested in functions \( f \) whose average value in arithmetic progressions is determined by a product of local densities. The problem one faces in such applications is the following:

The product of local densities will typically remain constant when the \( p \)-adic densities of elements within two arithmetic progressions is the same. In other words, if \( \{n \equiv r_1 \pmod{q_1}\} \) and \( \{n \equiv r_2 \pmod{q_2}\} \) are two progressions such that \( p|q_1 \Leftrightarrow p|q_2 \) and such
that \( r_1 \equiv r_2 \pmod{\text{lcm}(q_1, q_2)} \), then the average value of \( f \) restricted to either progression is expected to be the same. This information allows us to employ a \( W \)-trick to obtain a major arc property as follows. Let \( W \) be a product of small prime powers. Then the average value of \( f \) is preserved when passing from a progression of the form \( \{Wn + A\} \), \( A \neq 0 \pmod{p^{v_p(W)}} \) for any \( p \mid W \), to a subprogression of the form \( \{W(qn + r) + A\} \) where \( q \) is entirely composed of primes dividing \( W \).

When applying the factorisation theorem it is necessary to ensure that the periodic sequence \( \gamma \) has a period that belongs to the set of integers \( q \) from above. The only way to do so in the setting of the original factorisation theorem is to ensure that the upper bound, \( Q \), for the period of \( \gamma \) does not exceed the largest prime factor dividing \( W \). This however produces at the same time an upper bound of the shape \( Q^A \), for some fixed \( A \geq 1 \) that we are free to choose, on the common difference of subprogressions on which \( g' \) is still equidistributed.

In an arithmetic application with polynomial structure one is however naturally lead to situations where one would like to know that for instance \( n \mapsto g'(aW^\ell n + b) \) for some fixed \( \ell \in \mathbb{N} \) is still equidistributed to some extent. Since \( A \) is fixed whereas \( W = W(N) \) usually grows with \( N \), this is difficult to achieve with the original factorisation theorem. More precisely, while \( q = aW^\ell \) would be a valid choice for the major arc condition provided \( q = a \) is, it is impossible to simultaneously guarantee that \( W^\ell < Q^A \) holds and that \( Q \leq P^r(W) \) is bounded above by the largest prime factor of \( W \).

To surpass this difficulty, a modified factorisation theorem was established in [4, §16]. While the chosen strategy of proof is sufficiently powerful, it has unfortunately been overlooked that in the form the result is stated not all requirements are met; specifically, [4, Theorem 16.4] does not guarantee that \( g' \) is equidistributed on the subprogressions on which \( \gamma \) is constant. By going through the proof slightly more carefully, we obtain in Theorem 2.3 below a generalisation of the result which resolves the issue.

2. A NEW FACTORISATION LEMMA FOR POLYNOMIAL NILSEQUENCES

To start with, we recall the statement of the factorisation theorem from Green and Tao [3, Thm 1.19]:

**Theorem 2.1.** Let \( m, d > 0 \), and let \( M_0, N > 1 \) and \( A > 0 \) be real numbers. Suppose that \( G/\Gamma \) is an \( m \)-dimensional nilmanifold together with a filtration \( G_\bullet \) of degree \( d \). Suppose that \( X \) is an \( M_0 \)-rational Mal’cev basis \( \mathcal{X} \) adapted to \( G_\bullet \) and that \( g \in \text{poly}(\mathbb{Z}, G_\bullet) \). Then there is an integer \( M \) with \( M_0 \leq M \ll M_0^{O(A,m,d)} \), a rational subgroup \( G' \subseteq G \), a Mal’cev basis \( \mathcal{X}' \) for \( G'/\Gamma' \) in which each element is an \( M \)-rational combination of the elements of \( \mathcal{X} \), and a decomposition \( g = \varepsilon g'\gamma \) into polynomial sequences \( \varepsilon, g', \gamma \in \text{poly}(\mathbb{Z}, G_\bullet) \) with the following properties:

(1) \( \varepsilon : \mathbb{Z} \to G \) is \((M, N)\)-smooth;
(2) \( g' : \mathbb{Z} \to G' \) takes values in \( G' \), and the finite sequence \((g(n)\Gamma')_{n \leq N}\) is totally \(1/M^A\)-equidistributed in \( G'/\Gamma' \), using the metric \( d_{\mathcal{X}'} \) on \( G'/\Gamma' \);
(3) \( \gamma : \mathbb{Z} \to G \) is \( M \)-rational, and \((\gamma(n)\Gamma)_{n \in \mathbb{Z}}\) is periodic with period at most \( M \).
We will employ this result in an iterative process. To guarantee the termination of this process, we will ensure that in each of our applications of the above result the rational subgroup \( G' \) will be of strictly lower dimension than that of the ambient group \( G \).

**Lemma 2.2.** Under the hypotheses of Theorem 2.1 let \( g \in \text{poly}(\mathbb{Z}, G_\bullet) \) and suppose that \( g(n) = \varepsilon(n)g'(n)\gamma(n) \) is a factorisation which satisfies the conditions (1)–(3). Then there is a positive constant \( C \) only depending on \( m \) and \( d \) such that whenever \( A \) is sufficiently large and \( G' = G \), then \( g \) is totally \( M^{-A/2C} \)-equidistributed.

**Proof.** Let \( C \geq 1 \) to be determined in the course of the proof and let \( P \subseteq \{1, \ldots, N\} \) be a progression of length at least \( M^{-A/2CN} \). Since \( \gamma \) is periodic with period bounded above by \( M \), we may split \( P \) into at most \( M \) subprogressions, each of length at least \( M^{-(A/2C+1)N} \), on which \( \gamma \) is constant. Next, we split each of these subprogressions into pieces of diameter between \( M^{-(A/2C+1)N} \) and \( 2M^{-(A/2C+1)N} \) and let \( \mathcal{P} \) denote the collection of all resulting bounded diameter pieces. For each progression \( Q \in \mathcal{P} \), let \( s_Q \) denote its smallest element. If \( F : G/\Gamma \to \mathbb{C} \) is a Lipschitz function, then the right-invariance of the metric \( d_{\mathcal{X}} \) (c.f. [3, Definition 2.2]) implies for any \( n, n' \) that belong to the same element \( Q \) of \( \mathcal{P} \) that:

\[
|F(\varepsilon(n)g'(n)\gamma(n)\Gamma) - F(\varepsilon(n')g'(n)\gamma(n)\Gamma)| \leq \|F\|_{\text{Lip}} d_{\mathcal{X}}(\varepsilon(n)g'(n)\gamma(n), \varepsilon(n')g'(n)\gamma(n)) = \|F\|_{\text{Lip}} |n - n'|M/N \leq 2\|F\|_{\text{Lip}} M^{-A/(2C)}.
\]

This estimate allows one to fix for any \( Q \in \mathcal{P} \) the contribution of \( \varepsilon \):

\[
\sum_{n \in Q} F(g(n)\Gamma) = \sum_{n \in Q} F(\varepsilon(s_Q)g'(n)\gamma(n)\Gamma) + O(#Q\|F\|_{\text{Lip}}M^{-A/(2C)}).
\]

Let \( H_Q : G/\Gamma \to \mathbb{C} \) denote the map \( H_Q(h) := F(\varepsilon(s_Q)h\Gamma) \) and observe that the approximate left-invariance of \( d_{\mathcal{X}} \) (c.f. [3, Lemma A.5]) implies that \( \|H_Q\|_{\text{Lip}} \leq M_0^{O(1)}\|F\|_{\text{Lip}} \). Furthermore we have \( \int_{G/\Gamma} F = \int_{G/\Gamma} H_Q \). The fact that \( (g'(n)\Gamma)_{n \in N} \) is totally \( M^{-A} \)-equidistributed in \( G/\Gamma \) allows us to deduce a similar property for each of the sequences \( (g'(n)\gamma(m)\Gamma)_{n \in \mathbb{N}} \) for fixed \( m \). It follows from [3, Proposition 14.3], which is a consequence of [3, Theorem 2.9], that there is a constant \( C' = BB' > 1 \), only depending on \( m \) and \( d \), such that \( (g'(n)\gamma(m)\Gamma)_{n \in N} \) is totally \( M^{-A/C} \)-equidistributed. We set \( C = C' \). Applying the above to any progression \( Q \in \mathcal{P} \), we obtain

\[
\sum_{n \in Q} F(\varepsilon(s_Q)g'(n)\gamma(n)\Gamma) = \sum_{n \in Q} F(\varepsilon(s_Q)g'(n)\gamma(n_Q)\Gamma) = \left( \int_{G/\Gamma} F + O\left(M_0^{O(1)} M^{-A/C} \|F\|_{\text{Lip}} \right) \right) #Q,
\]
and, hence,
\[
\sum_{n \in N} F(g(n)\Gamma) = N \left( \int_{G/\Gamma} F + \|F\|_{\text{lip}} O \left( M^{-A/(2C)} + M_0^{O(1)}M^{-A/C} \right) \right).
\]

This completes the proof. \(\square\)

**Theorem 2.3** (Factorisation lemma). Let \(N\) and \(T = T(N)\) be positive integer parameters that satisfy \(N^{1-\varepsilon} \ll T \ll N\) and let \(k : \mathbb{N} \to \mathbb{N}\) be a slowly growing function. Let \(m, d, B, E\) and \(Q_0 \geq 1\) be positive integers. Suppose that \(G/\Gamma\) is an \(m\)-dimensional nilmanifold together with a filtration \(G_\bullet\) of degree \(d\). Suppose that \(\mathcal{X}\) is a \(Q_0\)-rational Mal’cev basis adapted to \(G_\bullet\), and that \(g \in \text{poly}(\mathbb{Z}, G_\bullet)\). Suppose further that \(Q_0 \leq \log k(N)\). Let \(R = R(N)\) be a parameter that satisfies \(R \geq Q_0\) and \(R(N)^t \ll N\) for all \(t > 0\).

Then there is an integer \(Q\) with \(Q_0 \leq Q \ll Q_0^{O_{B,m,d}(1)}\) and a partition of \(\{1, \ldots, T\}\) into at most \(R_{O_{m,d,B,E}(1)}\) disjoint subprogressions \(P\), each of \(k(N)\)-smooth common difference \(q(P) \ll R_{O_{m,d,B,E}(1)}\) and each of length \(T/q(P) + O(1)\), such that the restriction \((g(n))_{n \in P}\) of \(g\) to any of the progressions \(P\) can be factorised as follows.

There is a rational subgroup \(G' \leq G\), depending on \(P\), and a Mal’cev basis \(\mathcal{X}'\) for \(G'T/\Gamma\) such that every element of \(\mathcal{X}'\) is a \(Q\)-rational combination of elements from \(\mathcal{X}\) (that is, each coefficient is rational of height bounded by \(Q\)). Suppose \(P = \{qn + r : 1 \leq n \leq T/q + O(1)\}\), where \(q = q(P)\), then we have a factorisation
\[
g(qn + r) = \varepsilon_P(n)g_P(n)\gamma_P(n),
\]
where \(\varepsilon_P, g_P, \gamma_P\) are polynomial sequences from \(\text{poly}(\mathbb{Z}, G_\bullet)\) with the properties
1. \(\varepsilon_P : \mathbb{Z} \to G\) is \((Q, T/q)\)-smooth;
2. \(\gamma_P : \mathbb{Z} \to G\) arises as the product of at most \(m\) \(Q\)-rational polynomial sequences and the sequence \((\gamma_P(n)\Gamma)_{n \in \mathbb{Z}}\) is periodic with a \(k(N)\)-smooth period \(q_{\gamma_P} \ll Q\);
3. \(g_P : \mathbb{Z} \to G\) takes values in \(G'\) and for each \(k(N)\)-smooth number \(\tilde{q} < (q_{\gamma_P}R)^E\) the finite sequence \((g_P(\tilde{q}n)\Gamma')_{n \in T/(\tilde{q}q)}\) is totally \(Q^{-B}\)-equidistributed in \(G'T/\Gamma\).

The proof of the factorisation lemma makes use of the fact that a polynomial sequence that fails to be totally equidistributed also fails to be equidistributed when allowing polynomial changes in the equidistribution parameter. This is made precise in [3, Lemma 6.2], which we restate here for simplicity:

**Lemma 2.4.** Let \(N\) and \(A\) be positive integers and let \(\delta : [0, 1] \to \mathbb{R}\) be a function that satisfies \(\delta(x)^{-t} \ll x\) for all \(t > 0\). Suppose that \(G\) has \(\frac{1}{\delta(N)}\)-rational Mal’cev basis adapted to the filtration \(G_\bullet\). Suppose that \(g \in \text{poly}(\mathbb{Z}, G_\bullet)\) is a polynomial sequence such that \((g(n)\Gamma)_{n \leq N}\) is \(\delta(N)^A\)-equidistributed. Then there is \(1 \leq B \ll d_G \cdot m_G\) such that \((g(n)\Gamma)_{n \leq N}\) is totally \(\delta(N)^{A/B}\)-equidistributed, provided \(A/B > 1\) and provided \(N\) is sufficiently large.

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1. The notion of smoothness was defined in [3, Def. 1.18]. A sequence \((\varepsilon(n))_{n \in \mathbb{Z}}\) is said to be \((M, N)\)-smooth if both \(d_x(\varepsilon(n), \text{id}_G) \leq N\) and \(d_x(\varepsilon(n), \varepsilon(n - 1)) \leq M/N\) hold for all \(1 \leq n \leq N\).
2. A sequence \(\gamma : \mathbb{Z} \to G\) is said to be \(Q\)-rational if for each \(n\) there is \(0 < r_n \leq Q\) such that \((\gamma(n))^{r_n} \in \Gamma\). See [3, Def. 1.17].
Proof of Lemma 2.3 assuming Lemma 2.4. We may suppose that \( g \) does not satisfy (3) with \( Q \) replaced by \( Q_0 \). That is, there is some \( k(N) \)-smooth integer \( q_1 \leq R \) such that \( (g(q_1 n) \Gamma)^{n < T/q_1} \) fails to be totally \( Q_0^{-B} \)-equidistributed. By Lemma 2.4, this sequence also fails to be \( Q_0^{-BC} \)-equidistributed for some \( C = O_{m,d}(1) \). Writing \( z_1 := (q_1)^d \), we deduce from \cite{4} Lemma 16.3] that each of the sequences \( (g(z_1 n + r_1) \Gamma)^{n < T/z_1} \) with \( 0 \leq r_1 < z_1 \) fails to be \( Q_0^{-BC} \)-equidistributed in \( G/\Gamma \) for some \( C' = O_{m,d}(1) \). Now, we run through all \( 0 \leq r_1 < z_1 \) in turn.

Applying Theorem 2.1 and Lemma 2.2 to any of these sequences yields some \( Q_0 \leq Q_1 \leq Q_0^{O(B,m,d)} \), a \( Q_1 \)-rational subgroup \( G_1 < G \) of dimension strictly smaller than that of \( G \), and a factorisation

\[
g(z_1 n + r_1) = \varepsilon_{r_1}(n)g'_1(n)\gamma_{r_1}(n),
\]

where the finite sequence \( (g'_1(n) \Gamma)^{n < T/z_1} \) is totally \( Q_1^{-B} \)-equidistributed in

\[
G_1/\Gamma_1 := G_1/(\Gamma \cap G_1),
\]

where \( (\varepsilon_{r_1}(n) \Gamma)^{n \in \mathbb{Z}} \) is \((Q_1, T/z_1)\)-smooth, and where \( (\gamma_{r_1}(n) \Gamma)^{n \in \mathbb{Z}} \) is periodic with period at most \( Q_1 \).

If \( g'_1 \) is totally \( Q_1^{-B} \)-equidistributed on every subprogression \( \{n \equiv 0 \pmod{q_2} \} \) of \( k(N) \)-smooth common difference \( q_2 < (z_1 Q_1 R)^E \), then we stop (and turn to the next choice of \( r_1 \)). Otherwise, invoking Lemma 2.3 and \cite{4} Lemma 16.3, there are positive integers \( C, C' = O_{d,m}(1) \) and a \( k(N) \)-smooth integer \( q_2 \) as above such that, with \( z_2 := q_2^2 \), the finite sequence \( (g_{r_1,r_2}(n))^{n < T/(z_1 z_2)} \) defined by \( g_{r_1,r_2}(n) := g'_1(z_2 n + r_2) \) fails to be \( Q_1^{-BC} \)-equidistributed for every \( 0 \leq r_2 < z_2 \). We proceed as before.

This process yields a tree of operations which has height at most \( m = \dim G \), since each time the factorisation theorem is applied a new sequence \( g'_{r_1,\ldots,r_t} \) is found that takes values in some strictly lower dimensional submanifold \( G_i = G_i(r_1,\ldots,r_i) \) of \( G_{i-1}(r_1,\ldots,r_{i-1}) \). Thus, we can apply the factorisation theorem at most \( m \) times in a row before the manifold involved has dimension 0.

The tree we run through starts with \( g \), which has \( z_1 \) neighbours \( g_{r_1} \), one for each \( 0 \leq r_1 < z_1 \). For each \( r_1 \), the vertex \( g_{r_1} \) has \( z_2 = z_2(r_1) \) neighbours \( g_{r_1,r_2} \), one for each \( 0 \leq r_2 < z_2(r_1) \), etc. As a result, we obtain a decomposition of the range \( \{1,\ldots,T\} \) into at most \( O_{m,d,B,E}(1) \) subprogressions of the form

\[
P = \{z_1(z_2(z_3(\ldots(z_t n + r_t)\ldots) + r_3) + r_2) + r_1 : n \leq T/(z_1 z_2 \ldots z_t)\}
\]

for \( t \leq m \), some \( r_t \), and where each \( z_t \) depends on \( r_1,\ldots,r_{t-1} \). The common difference of such a progression \( P \) is \( k(N) \)-smooth and bounded by \( O_{m,d,B,E}(1) \). The iteration process furthermore yields a factorisation of \( g_{r_1,\ldots,r_t} \), which is the restriction of \( g \) to \( P \):

\[
g_{r_1,\ldots,r_t}(m) = g(z_1 z_2 \ldots z_t m + r) = \tilde{\varepsilon}_{r_1,\ldots,r_t}(m)g'_t(m)\gamma_{r_1,\ldots,r_t}(m),
\]

where

\[
\tilde{\varepsilon}_{r_1,\ldots,r_t}(m) = \varepsilon_{r_1}(z_2 \ldots z_t m + \tilde{r}_2) \ldots \varepsilon_{r_1,\ldots,r_{t-1}}(z_t m + \tilde{r}_t)\varepsilon_{r_1,\ldots,r_t}(m)
\]
for certain integers $\tilde{r}_2, \tilde{r}_3, \ldots, \tilde{r}_t$, and

$$\tilde{\gamma}_{r_1, \ldots, r_t}(m) = \gamma_{r_1, \ldots, r_t}(m) \gamma_{r_1, \ldots, r_{t-1}}(z_t m + \tilde{r}_t) \ldots \gamma_{r_1}(z_2 \ldots z_t m + \tilde{r}_2).$$

The factor $\tilde{\gamma}_{r_1, \ldots, r_t}(m)$ is a $(Q_{0}^{O_{B,d,m}(1)}, T/(z_1 \ldots z_t))$-smooth sequence. This follows from the triangle inequality, the right-invariance and the approximate left-invariance of $d_{X}$; we refer to the discussion following Definition 16.1 in [4] for details and to [3, App. A] for the properties of $d_{X}$.

Since each $\gamma_{r_1, \ldots, r_i}$ with $i \leq t$ is periodic with period at most $Q_{0}^{O_{m,d,B}(1)}$ and since $t \leq m$, we deduce that $\tilde{\gamma}_{r_1, \ldots, r_t}$ is periodic with period at most $Q_{0}^{O_{m,d,B}(1)}$. The bound $Q_{0} \leq \log k(N)$ implies that this period is $k(N)$-smooth provided $N$ is sufficiently large.

Finally, $g_x'$ satisfies property (3) by construction. \hfill \Box

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