TOROIDAL COLLECTIVE MOTIONS 
IN THE ATOMIC NUCLEUS

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Abstract

The work deals with one of the topics of collective motion. In the frame of Nuclear Fluid Dynamics, a model which portrays the nuclear matter as a quantum elastic body, the torus-like motions and their associated energies are computed using the thirteen moment approximation. Such excitations correspond to the Hill vortex known from classical Hydrodynamics. There are also calculated the nonvanishing contributions of transverse electric form factors and differential cross sections in the electroexcitation of these collective modes, which are purely toroidal. The spin-dependent collective excitations, with toroidal electromagnetic structure, are studied by means of the Generalized Goldhaber-Teller model, with emphasize on the $1^-$ spin-flip mode and its excitation in spherical nuclei by inelastic electron scattering. We discuss the importance of toroidal contributions in the inclusive electron scattering ($e,e'$) and exclusive coincidence electron scattering ($e,e'\gamma$). In order to extract the toroidal multipole, we use the backscattering angles in the first mentioned reaction, and the separation of the longitudinal/transverse interference in the second case. The introduction of a quantity which accounts for the deviations from the Siegert theorem, shows the importance of toroidal quadrupole transitions at high-momentum transfer. Another important result concerns the dependence of the intensity of toroidal effects on the nuclear vorticity.

1 Introduction

The toroidal multipole moments are a distinct family of electromagnetic moments, which occur in a special parametrization of charge and curent densities \cite{1, 2}.

Long time ago, Zeldovich showed that a 1/2 spin particle may interact with an electromagnetic field not only by means of a dipole electric interaction $d(\sigma \cdot E)$ that simultaneously violates the space and time inversion, but also by a different type of
interaction, \(a(\sigma \cdot J)\), which violates the parity but not the time reversal (Figure 1). This new electromagnetic characteristic was named anapole [3] and it is the first member of the class of toroidal moments, i.e. the static toroidal dipole moment. Inside nuclei, the anapole moment arises as a consequence of \(P\)-noninvariant [4] nuclear forces. In classical electrodynamics a very simple example of a toroidal dipole is given by a solenoid folded onto a torus. The radiation resistance of the corresponding antenna is proportional to \((R_T r_T^2/\lambda^3)^2\), where \(r_T\) and \(R_T\) are the torus small and large radii [5]. Therefore the toroidal antenna depends on the ratio between the geometrical size of the source \(d\) and the radiation wavelength \(\lambda\) like the charge octupole and magnetic quadrupole. Such solenoids have non-vanishing magnetic potential in those regions of the space where the toroidal dipole moment \(T \neq 0\) [6].

The work that we present below discusses the problem of those nuclear collective motions associated to purely or partially toroidal electromagnetic transitions. There will be also studied the properties of excitations associated to vortical nuclear currents of orbital isoscalar nature (\(dipole torus mode\) : DTM) and of isovector spin dependent nature (\(spin\)-\(isospin\) mode : s-is) along with the investigation of the possibility to detect them in electron inelastic scattering on nuclei.

The electroexcitation of collective rotational and vibrational motions with the account of toroidal quadrupole transitions will be also investigated.

2 The Dipole Torus Mode in Nuclear Fluid Dynamics

The Nuclear Fluid Dynamics (NFD) allows the description of isoscalar giant resonances as in phase harmonic vibrations of proton and neutron fluids, the restoring force being a consequence of the elasticity of nuclear matter. The nucleus is portrayed as a Fermi fluid which as a result of an external perturbation may undergo longitudinal oscillations (irrotational), as is the case of the liquid drop, but also transversal oscillations. Consequently an isoscalar giant resonance will be described by a small amplitude collective oscillation with multipolarity \(\lambda\), in which protons and neutrons are performing a divergenceless irrotational or vortical displacement [7].

Basically, the NFD equations can be deduced by taking the classical limit of time dependent Hartree-Fock equations (TDHF)

\[
\frac{i\hbar}{\partial t} \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \tag{2.1}
\]

which is nothing else than the collisionless Boltzmann equation from statistical physics of transport phenomena :

\[
\frac{\partial f}{\partial t} + \frac{1}{m} \mathbf{p} \cdot \nabla_r f - \nabla_r U(r) \cdot \nabla_p f = 0 \tag{2.2}
\]
and represents the equation of motion for the Wigner transform of the density matrix \( \hat{\rho} \)

\[
f(r, p, t) = \int ds \rho \left( r + \frac{s}{2}, r - \frac{s}{2}, t \right) e^{-i/h}p \cdot s \tag{2.3}
\]

In the argument of \( f \) the position vector \( r \) of a particle is determined with a precision larger than the wavelength of the perturbation in such a way that the Heisenberg principle is not violated.

In order to reduce the complexity of the non-linear partial derivatives of equation (2.2), one performs a transformation to coupled equations for the macroscopical variables. These physical quantities are introduced as \( p \)-order moments \( (= 0 \) for the density \( \rho, = 1 \) for the three components of the mean velocity \( u_i(r, t), = 2 \) for the nine components of the strain tensor \( P_{ij}(r, t) \) \) of the distribution function with respect to the momentum as follows :

\[
\rho (r, t) = m \int dp \ f (r, p, t) \tag{2.4}
\]

\[
\rho (r, t) u_i(r, t) = \int dp \ f (r, p, t) p_i \tag{2.5}
\]

\[
P_{ij}(r, t) = \frac{1}{m} \int dp \ f (r, p, t) (p_i - mu_i)(p_j - mu_j) \tag{2.6}
\]

By integrating the equation of Boltzmann over the momentum \( p \), with weight \( 1, p_j/m, p_i p_j/m^2 \), imposing the incompresibility condition for the nuclear matter, i.e. \( \rho = \rho_0 = \text{constant} \) and assuming for the strain tensor the ansatz

\[
P_{ij} = P_0 + p_{ij} \tag{2.7}
\]

one get the linearized NFD equations (the thirteen moments approximation)

\[
\frac{\partial u_k}{\partial x_k} = 0 \tag{2.8}
\]

\[
\rho \frac{\partial u_i}{\partial t} + \frac{\partial p_{ik}}{\partial x_k} = 0 \tag{2.9}
\]

\[
\frac{\partial p_{ij}}{\partial t} + P_0 \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 0 \tag{2.10}
\]

It is convenient to write the displacement \( dx_i \) of the nuclear matter at a certain point \( r \) inside the nucleus as

\[
dx_i = a_i^\lambda (r) \tag{2.11}
\]

or, alternatively by the mean velocity

\[
u_i = a_i^\lambda (r) \frac{d\alpha_\lambda}{dt} \tag{2.12}
\]

where \( a_i^\lambda (r) \) is the vector field of instantaneous displacements in the Fermi continuum, and \( \alpha_\lambda \) is the time-dependent amplitude of harmonic oscillations \( (\alpha_\lambda \sim \cdots) \).
sin ωλt) associated to the resonant phenomena that we study in this paper. If we differentiate (2.9) with respect to time, (2.10) with respect to position xk and we use equation (2.8), we obtain

$$\rho \frac{\partial^2 u_i}{\partial t^2} = P_0 \frac{\partial^2 u_i}{\partial x_k^2}$$  \hspace{1cm} (2.13)

Afterwards, employing the ansatz (2.12) we arrive at the Helmholtz equation (see (2.1)) for stationary spherical waves

$$\frac{\partial^2 a^\lambda_i}{\partial x_i^2} + k^2 a^\lambda_i = 0$$  \hspace{1cm} (2.14)

where \( k = \sqrt{\rho \omega^2 / P_0} \) is the wave number. Equation (2.14) admits three independent solutions

$$a^\lambda_i = N^\lambda_i \nabla \times (k \hat{r}) Y_{\lambda\mu}(\theta, \phi)$$  \hspace{1cm} (2.15)

$$a^\lambda_t = N^\lambda_t \nabla \times r \nabla \times (k \hat{r}) Y_{\lambda\mu}(\theta, \phi)$$  \hspace{1cm} (2.16)

$$a^\lambda_p = N^\lambda_p \nabla \times \nabla \times (k \hat{r}) Y_{\lambda\mu}(\theta, \phi)$$  \hspace{1cm} (2.17)

in a frame with fixed axis. The longitudinal and poloidal solutions describe compressional and transversal oscillations of the elastic nuclear globe, being responsible for the electric-like resonances with parity \( \pi = (-)^\lambda \), whereas the torsional solution describes magnetic-like resonances with parity \( \pi = (-)^{\lambda+1} \). In the longwavelength limit \( kr \ll 1 \), the poloidal vector field becomes proportional to the longitudinal one

$$a^\lambda_p(r) = N^\lambda_p \nabla \times \nabla \times r \hat{r} Y_{\lambda0}(\theta, \phi) = N^\lambda_p (\lambda + 1) \nabla r^\lambda Y_{\lambda0}(\theta, \phi) = (\lambda + 1) a^\lambda_i(r)$$  \hspace{1cm} (2.18)

and the torsional field is merely

$$a^\lambda_t(r) = N^\lambda_t \nabla \times r r^\lambda Y_{\lambda0}(\theta, \phi)$$  \hspace{1cm} (2.19)

It is important to substantiate that in the above mentioned limit, the poloidal solution is simultaneously irrotational and divergenceless, i.e. \( \nabla \cdot a^\lambda_p = \nabla \times a^\lambda_p = 0 \), whereas the torsional solution is purely solenoidal, \( \nabla \cdot a^\lambda_t = 0 \) but \( \nabla \times a^\lambda_t \sim a^\lambda_p \neq 0 \).

In the dipole case \( \lambda = 1 \), the solution (2.18) corresponds to the displacement as a whole of the nucleus, without the change of the internal state. The equality (2.18) is obtained from (2.15) and (2.17) by keeping the first term in the asymptotic expansion of the spherical Bessel function

$$j_\lambda(x) \longrightarrow \frac{x^\lambda}{(2\lambda + 1)!!} \left( 1 - \frac{x^2}{2(2\lambda + 3)} + ... \right)$$  \hspace{1cm} (2.20)

Consequently, in order to investigate the dipole response of an incompressible elastic globe one need to go beyond the limit imposed by the longwavelength approximation and to introduce the high-order terms in the expansion (2.20). Then, the poloidal solution becomes

$$a^1_p = N^1_p \nabla \times \nabla \times r r^3 Y_{10}(\theta, \phi)$$  \hspace{1cm} (2.21)
In order to establish an explicit expression for the displacements field, corrected in such a way to take into account the center of mass motion we impose the condition

$$\delta R_{c.m.} = \frac{\int dr \rho a^1_p}{\int dr \rho} = 0$$  \hspace{1cm} (2.22)

This procedure leads to the following expression for the displacements field

$$a^1_p = N^1_p \nabla \times \nabla \times r(r^2 - R^2)Y_{10}(\theta, \phi)$$

$$= \frac{2}{\sqrt{3}} N^1_p \left[ \sqrt{2} r^2 Y^0_{12}(\theta, \phi) + (5r^2 - 3R^2)Y^0_{10}(\theta, \phi) \right]$$  \hspace{1cm} (2.23)

We can rewrite eq.(2.23) on spherical components

$$(a^1_p)_r = \sqrt{\frac{3}{\pi}} N^1_p (r^2 - R^2) \cos \theta \hspace{1cm} (2.24)$$

$$(a^1_p)_\theta = -\sqrt{\frac{3}{\pi}} N^1_p (2r^2 - R^2) \sin \theta \hspace{1cm} (2.25)$$

$$(a^1_p)_\phi = 0$$  \hspace{1cm} (2.26)

The dipole poloidal displacement field, or the Dipole Torus Mode (MDT) coincide with that for the Hill vortex known from Hydrodynamics [10]. The Stokes current function corresponding to this vortical flow is given by

$$\psi(r, \theta) = N^1_p (r^2 - R^2) r^2 \sin^2 \theta$$  \hspace{1cm} (2.27)

The contour lines given by eq.(2.27) are plotted in Fig.2. The critical or stagnation points are fixed by the conditions $a_r = 0$ and $a_\theta = 0$, i.e. $r_c = R/\sqrt{2}$ si $\theta_c = \pm \pi$. The geometric locus of these points is represented by a ring in the equatorial plane of the sphere. The vortical flow of the fluid takes place around this ring. The stream lines rotated around the globe axis generate tori. Such a vortical flow is known in Classical Hydrodynamics under the name of Hill ring vortex, contrary to the linear vortex where the critical points are located on the symmetry axis of the spheroid. We called the collective excitation corresponding to the Hill vortex of the nucleus dipole torus mode ( DTM ) [11]. A similar kind of collective motion have been studied in [11] and [12]. In the equatorial plane, i.e. the plane containing the vortical ring ($\theta = \frac{\pi}{2}$)

$$\zeta = (\nabla \times \mathbf{u}_p)_{\theta=\frac{\pi}{2}} = 5 \sqrt{\frac{3}{\pi}} N^1_p \dot{\alpha}(t) r e_\phi$$  \hspace{1cm} (2.28)

Thus, the vorticity depends on the radial coordinate. The energy is given by

$$E(1_{\text{tor}}) = \sqrt{\frac{21}{5}} \hbar \omega_F \approx 2\hbar \omega$$  \hspace{1cm} (2.29a)
Therefore, DTM may be interpreted as a dipole transversal isoscalar resonance of $2\hbar\omega$ type. Using realistic parameters for the Fermi distribution we get \[ E(1_{\text{tor}}) = 93.72A^{-1/3}\text{MeV} \] (2.29b)
i.e. the predicted mode is most probably located between the giant isovector resonance and the giant isoscalar octupole resonance.

We would like to complete the analysis on toroidal dipole by presenting the calculation of the transverse electric form factor and of the current and transition vorticity densities using the Born’s plane wave approximation. The knowledge of these quantities is important because the form factor may be directly measured in electron inelastic scattering processes \[ [14] \].

In the case of the DTM, the current density associated to the transition is given in the fluid-dynamic representation by

\[ J_{\text{tor}} = n_e u_p = n_e a_p^1(r) \dot{\alpha}(t) \] (2.30)

where \( n_e = eZ/A n_0 \) and \( n_0 = 3A/4\pi R^3 \) is the particle density; \( \dot{\alpha}(t) = \alpha_0 \omega \cos \omega t \) and \( \alpha_0 = \left(\frac{\hbar}{2BC}\right)^{1/2} \left( B_1 = \frac{6}{7\pi} (N_p^1)^2 M R^4, C_1 = \frac{18}{5\pi} (N_p^1)^2 M v_F R^2 \right) \). We introduce the electric transverse form factor

\[ |F_{\lambda}^{\text{el}}(k)|^2 = \frac{4\pi}{3} \langle |\hat{T}^{\text{el}}_{\lambda 0}(k,t)|^2 \rangle_t \] (2.31)

where by \(< ... >_t \) we understand the time averaging. Normalizing to \( c^2 \) one obtains the dimensionless quantity

\[ \frac{1}{c^2} |F_{1}^{\text{el}}(k)|^2 = \frac{1}{(9\pi)^{1/3}} \sqrt{\frac{35}{6} \left( \frac{\omega}{kc} j_3(k R) \right)^2 \frac{Z^2}{A^{1/3}}} \] (2.32)

In Fig.3 we represented \( |F_{1}^{\text{el}}(k)|^2 \) for three spherical nuclei : \(^{40}\text{Ca}, ^{90}\text{Zr}\) and \(^{208}\text{Pb}\). This plot exquibite an enhancement of the DTM in heavy nuclei compared to light nuclei. Moreover, we notice that the first diffraction maxima of the form factor is shifted towards small momentum transfer when we pass to nuclei with \( Z \) and \( A \) large.

A simple calculation shows that the longitudinal and magnetic multipoles \[ [2] \] vanish for the current density (2.30). Thus, the transverse electric form factor has the only non-vanishing contribution in the excitation with electromagnetic probes (photons,electrons) of DTM! The Coulomb multipoles does not participate in the excitation due to the fact that the charge transition density \( \rho_1^{\text{tor}} \) vanishes as a consequence of the incompressibility condition (2.8) imposed to the Fermi globe. In other words, since the excited mode is purely rotational, the longitudinal form factor vanishes. Although the magnetic multipole should be associated to the excitations of rotational motions, these are of oposite parity to DTM, which has a natural parity like the electric excitations.

The electroexcitation differential cross-section of DTM looks as follows

\[ \left( \frac{d\sigma}{d\Omega} \right)_{1-\text{tor}} = 4\pi \sigma_{\text{Mott}} f_{\text{rec}}^{-1} \left( \frac{4\mu}{2q^2} + \tan^2 \frac{\theta}{2} \right) |\langle 1^- | \hat{T}^{\text{el}}_1(q) | 0^+ \rangle|^2 \] (2.33)
The dependence of (2.33) on the scattering angle is given in Fig.4. We should notice that since the longitudinal part does not contribute to the electroexcitation of DTM, it does not appear necessary to consider the backscattering case. However, in the electroexcitation process there will occur also modes which contribute to the longitudinal part and consequently in order to separate their influence it is better to chose the case with scattering angle $\theta = 180^\circ$.

An important characteristic in the study of collective oscillations by inelastic electron scattering is given by the transition current density, a quantity susceptible to experimental determination. Basically we are interested in the multipole component $I_{\lambda+1}(r)$ which can be expressed as inverse Fourier transform of the transverse electric form factor [15]:

$$J_{12}(r) = -\frac{1}{\sqrt{3\pi^3}} \int_0^\infty F_{11}^\text{el}(q) j_2(qr) q^2 dq$$

(2.34)

which integrated gives

$$J_{12}(r) = -\gamma \frac{r^2}{2\sqrt{3\pi} R}, \quad 0 < r < R$$

$$= -\gamma \frac{1}{4\sqrt{3\pi} R^2}, \quad r = R$$

$$= 0, \quad r > R$$

This radial function is plotted in fig.5.

Another quantity of interest in $(e,e')$ inelastic processes is the vorticity $\omega_{\lambda\lambda}$. It determines the nuclear current properties unconstrained by the charge-current conservation law. The vorticity transition density is [10]

$$\omega_{11}(r) = \sqrt{3} \left( \frac{d}{dr} + \frac{2}{r} \right) J_{12}(r)$$

(2.35)

This equation tells us that inside the nucleus the vorticity density varies linearly with the radius in the same manner as the vorticity vector (2.28). This radial function is represented in Fig.6.

Introducing the multipolar parametrization of Dubovik and Cheshkov [2] the transverse electric multipole corresponding to DTM reads

$$\hat{T}_{10}^\text{el}(q) = q^2 \hat{T}_{10}^\text{tor}$$

(2.36)

if we take into account that the Coulomb multipole $\hat{Q}_{10} = 0$. For the toroidal dipole moment associated to the transition $0^+ \rightarrow 1^-_{\text{tor}}$ the following proportionality relation is available

$$\langle T_1 \rangle \sim \alpha Z$$

(2.37)

A similar result was known for the electron transition $1s_{1/2} \rightarrow 2p_{1/2}$ in Hydrogen-like atoms [17]. This is the reason why the electromagnetic effects of toroidal nature are so weak for small $Z$. But whereas in the above mentioned atomic transition the toroidal moment enters as a small correction, in the DTM transition it is the principal electromagnetic characteristic of the nuclear response.
3 The spin-flip resonance

If we consider the excitation of nuclei with $0^+$ ground state, then the Wigner supermultiplet theory \[18\] leads to the classification of giant dipole resonances given in Table 1. In the frame of generalized Goldhaber-Teller mode for isobaric nuclei ($N = Z$), having ground state with $J = 0^+, T = 0$, the current density associated to the GT mode is

$$J(r) = \frac{1}{2} \hat{a}_n \rho_0(r) \quad (3.1)$$

and the spherical components of the magnetization density, for the s-is and sw modes, are given by

$$(\mu_\nu)_{s-is} = \frac{\hbar}{4mc} \frac{g_p - g_n}{2} \delta_{\nu \nu'} q_n \cdot \nabla \rho_0(r) \quad (3.2)$$

$$(\mu_\nu)_{sw} = \frac{\hbar}{4mc} \frac{g_p + g_n}{2} \delta_{\nu \nu'} q_n \cdot \nabla \rho_0(r) \quad (3.3)$$

where $\rho_0(r)$ is the ground state density, and $q_n$ is the relative coordinate. The contribution of the sw state (3.3) may be neglected due to the smallness of the factor $[(g_p + g_n)/(g_p - g_n)]^2$.

| States                        | $L$ | $S$ | $J$          | $T$ |
|-------------------------------|-----|-----|-------------|-----|
| Goldhaber - Teller (GT)       | 1   | 0   | 1$^-$       | 1   |
| Spin - Isospin (s-is)         | 1   | 1   | 0$^-$, 1$^-$ | 1   |
| Spin Wave (us)                | 1   | 1   | 0$^-$, 1$^-$ | 0   |

The calculation of electromagnetic multipoles shows that whereas the GT mode is mainly longitudinal the s-is one is purely toroidal like DTM. It is worthwhile to make a comparison between the electroexcitation differential cross-sections of the isovector resonances 1$^-$ GT and 1$^-$ s-is.

$$\left( \frac{d\sigma}{d\Omega} \right)_{1^- \text{GT}} = \sigma_{\text{Mott}} b^2 \frac{F^2(q)}{2A} \left[ q^2 V_L(\theta) + 2 \left( \frac{\omega}{c} \right)^2 V_T(\theta) \right] \quad (3.4)$$

$$\left( \frac{d\sigma}{d\Omega} \right)_{1^- \text{s-is}} = \sigma_{\text{Mott}} b^2 \frac{F^2(q)}{2A} \left( \frac{q^2 \hbar c}{2mc^2} \right) \left( \frac{g_p - g_n}{2} \right)^2 V_T(\theta) \quad (3.5)$$

where $b = \sqrt{m\omega/\hbar}$ is the characteristic length of the harmonic oscillator with frequency $\omega$, and the charge density form factor in the ground state is

$$F(q) = \int dr e^{iqr}\rho_0(r) \quad (3.6)$$
In Fig. 7 we have plotted the differential cross-sections of the two resonances, GT and s-is, corresponding to the electroexcitation of the $^{12}$C and $^{16}$O nuclei. Notice that for small scattering angles, the differential cross-sections of the GT resonances are much larger than those for the s-is resonances, even at large momentum transfer. However, for backscattering angles, the differential cross-sections of the s-is modes become important and exceed those of GT resonances at momentum transfer $q > 0.5 \text{fm}^{-1}$.

4 Toroidal quadrupole transitions in the Riemann Rotational Model

The dynamic character of the nuclear rotational motion is one of the basic problems, unsolved yet, in nuclear structure theory. Many attempts have been made up to the present time to clarify whether the nuclear matter can be portrayed as a quantum fluid that could support only irotational flows (IF) or it is a quantum rotor which gives rise to a rigid rotation (RR) of the whole nucleus. In a naive image irotational flow may be viewed as a deformation which propagates on the surface of the nucleus, along with a corresponding motion of the intrinsic structure with small angular momentum (Fig. 8c). At the other extreme lays the rigid body rotation (Fig. 8a).

The calculation of inertia moments of these two different types of flow underestimates the predictions of IF and overestimates those of RR, in comparison with the experimental values. Thence one must consider a model which takes into account the existence of currents with intermediate values between IF and RR.

The Riemann rotational model [19] is a simple generalization of the Bohr-Mottelson model, with the current ranging between the limits mentioned above. Moreover the associated velocity field is supposed to depend linearly on position. A Riemann rotator is an ellipsoid whose principal axes have lengths $a_1, a_2, a_3$ and in stationary conditions are at rest with respect to a frame rotating with constant angular velocity $\omega$. In this rotating frame internal motions with vorticity $\zeta = \nabla \times u$ occur. These two vectors are parallel and oriented along one of the ellipsoid principal axes. Thence the velocity field measured by an observatory at rest with respect to the rotating frame is given by

\begin{align*}
    u_1 &= -\frac{a_1^2}{a_1^2 + a_2^2} \zeta_3 x_2 + \frac{a_1^2}{a_1^2 + a_3^2} \zeta_2 x_3 \\
    u_2 &= -\frac{a_2^2}{a_2^2 + a_3^2} \zeta_1 x_3 + \frac{a_2^2}{a_2^2 + a_1^2} \zeta_3 x_1 \\
    u_3 &= -\frac{a_3^2}{a_3^2 + a_1^2} \zeta_2 x_1 + \frac{a_3^2}{a_3^2 + a_2^2} \zeta_1 x_2
\end{align*}

(4.1)

whereas the velocity field defined in a space fixed inertial frame and projected on the rotating frame

\[ U = u + \omega \times x \]

(4.2)
A Riemann sequence is characterized by a parameter

$$ f = \frac{\zeta}{\omega} - 2 $$

(4.3)

When \( f = 0 \), the ellipsoid is rigidly rotating, and when \( f = -2 \) the flow is irrotational. Defining the rigidity parameter through the relation

$$ r = 1 + \frac{f}{2} $$

(4.4)

the RR case is reproduced for \( r = 1 \) and the IF case for \( r = 0 \). Considering that \( \zeta \) and \( \omega \) are parallel with the principal axis \( x_3 \), and that \( a_1 \geq a_2 \), the velocity field (4.2) reads

$$ U(r) = (1 - r) \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} \omega \nabla(x_1 x_2) + r \omega \times r $$

(4.5)

The above equation states that the velocity field \( U(r) \) is a convex combination of rigid \( U_{RR} = \omega \times r \) and irrotational \( U_{IF} = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} \omega \nabla(x_1 x_2) \) contributions. A similar relation is valid for the inertia moment \( \mathcal{I}_r \). A particular example of Riemann flow \( r \in (0, 1) \) is represented in Fig. 8(b).

In order to check the predictions given by the Riemann model a direct determination of the nuclear current is required. The electron-nucleus scattering is an useful tool which allows the measurement of the electromagnetic charge and current densities inside the ground-state band [20]. We showed in a previous section that Coulomb multipoles are associated to the charge distribution of the nucleus and thus, in order to obtain the quantities depending on the nuclear current we need to determine the transversal part of the cross-section, i.e. the electric and magnetic multipoles.

Since the task of this work is to substantiate the toroidal multipoles which are active in such processes we will focus on the study of electric transverse multipoles arising in the Dubovik and Cheshkov multipolar parametrization.

Let us consider an even-even nucleus whose surface oscillates harmonically and simultaneously undergo a rigid rotation around an axis perpendicularly on its symmetry axis. Expressing the velocities \( U_{RR} \) and \( U_{IF} \) as one and two-rank spherical tensors, we readily obtain the spherical component \( \mu \) of the total velocity.

$$ U_{1\mu}(r) = (1 - r) [V_2 \otimes r_1]_{1\mu} + r [V_1 \otimes r_1]_{1\mu} $$

(4.6)

where the spherical components of the tensors being coupled to \( r_\mu = \sqrt{\frac{4\pi}{3}} r Y_{1\mu} \) are given by

$$ V_{1\mu} = -i\sqrt{2} \omega_\mu, \quad V_{2\mu} = i \sqrt{\frac{10}{3} \frac{\mathcal{I}_{IF}}{\mathcal{I}_{RR}}} \mu \omega_\mu $$

(4.7)

with \( \mu = \pm 1, \omega_\mu = -\frac{\omega}{\sqrt{2}} \), and \( \mathcal{I}_{IF} \) and \( \mathcal{I}_{RR} \) are the inertia moments of IF and RR models.

The current density reads

$$ \hat{J}(r) = \rho^p(r) U(r) $$

(4.8)
where
\[ \rho^p = \sum_{L \geq 2} \rho^p_L(r) Y_{L0}(\theta, \phi) \]  
(4.9)

is the proton charge density expanded in even multipolar components \((L = 2, 4, \ldots)\) of an axially symmetric nucleus.

We shall express the electric transverse \((2.4)\) and longitudinal \((2.3)\) multipoles as follows \([21]\)

\[
\hat{T}^e_{\lambda\mu}(q) = \frac{i^{\lambda+1}}{\sqrt{2\lambda + 1}} \sum_{\lambda \chi \mu} (\sqrt{\lambda + 1} \delta_{\lambda \lambda - 1} - \sqrt{\lambda} \delta_{\lambda \lambda + 1}) \times 
\int_0^{\infty} r^3 dr \, j_{\lambda'}(qr) \rho^p_L(r) \sqrt{3 \lambda \lambda 0} \left( \begin{array}{ccc} \lambda' & L & k \\ 0 & 1 & \mu \\ 1 & L & -M \end{array} \right) (-)^{\mu+k} \left( \begin{array}{ccc} \lambda & 0 & L \\ -\mu & -M & -M \end{array} \right) V_{k\mu}
\]  
(4.10)

\[
\hat{L}_{\lambda\mu}(q) = \frac{i^{\lambda+1}}{\sqrt{2\lambda + 1}} \sum_{\lambda \chi \mu} (\sqrt{\lambda} \delta_{\lambda \lambda - 1} + \sqrt{\lambda + 1} \delta_{\lambda \lambda + 1}) \times 
\int_0^{\infty} r^3 dr \, j_{\lambda'}(qr) \rho^p_L(r) \sqrt{3 \lambda \lambda 0} \left( \begin{array}{ccc} \lambda' & L & k \\ 0 & 1 & \mu \\ 1 & L & -M \end{array} \right) (-)^{\mu+k} \left( \begin{array}{ccc} \lambda & 0 & L \\ -\mu & -M & -M \end{array} \right) V_{k\mu}
\]  
(4.11)

where \(\ldots\) and \(\ldots\) are 3\(j\) and 6\(j\) coefficients \([22]\). In the RR case we take \(k = 1\) and an axial symmetric quadrupole static deformation in \((4.9)\) : \(\beta = \beta_2 \neq 0, \gamma = 0\)

\[ \rho^p(r) = \frac{3eZ}{4\pi R_0^3} \Theta[R_0(1 + \beta Y_{20}) - r] \]  
(4.12)

Thus the charge quadrupole components of the density reads

\[ \rho^p_2(r) = \int d\Omega \, Y_{20}^\ast(\theta, \phi) \rho^p(r) = \frac{3eZ\beta}{4\pi R_0^3} \delta(R_0 - r) \]  
(4.13)

and we obtain in place of \((4.10)\) and \((4.11)\)

\[
\hat{T}^e_{2\mu}(q, \text{RR}) = -Ze\sqrt{3\pi} \frac{\sqrt{30} \, Q_0}{40 \, R_0} \left[ j_1(qR_0) - \frac{2}{3} j_3(qR_0) \right] \mu \omega_{\mu}
\]  
(4.14)

\[
\hat{L}_{2\mu}(q, \text{RR}) = -Ze\sqrt{3\pi} \frac{\sqrt{30} \, Q_0}{40 \, \sqrt{3 \, R_0}} [j_1(qR_0) + j_3(qR_0)] \mu \omega_{\mu}
\]  
(4.15)

For the irotational case \(k = 2\) we consider the deformed charge density distribution with monopole component

\[ \rho^p_0(r) = \frac{3eZ}{\sqrt{4\pi R_0^3}} \Theta(R_0 - r) \]  
(4.16)

such that the multipoles \((4.10)\) and \((4.11)\) may be rewritten

\[
\hat{T}^e_{2\mu}(q, \text{IF}) = -Ze\sqrt{3\pi} \frac{\sqrt{30} \, Q_0}{40 \, R_0} [j_1(qR_0) + j_3(qR_0)] \mu \omega_{\mu}
\]  
(4.17)
Using the convexity property of the velocity field (4.6) the electromagnetic multipoles for a certain value of the rigidity parameter are

\[
\hat{L}_{2\mu}(q, r) = -Ze\sqrt{2\pi} \sqrt{30} \frac{Q_0}{R_0} \left[ j_1(qR_0) - \left( 1 - \frac{5}{3}r \right) j_3(qR_0) \right] \mu\nu \lambda \mu (4.18)
\]

where \(Q_0 = \sqrt{\frac{3}{5\pi}} R_0^2 \beta\) is the static quadrupole moment, and \(R_0 = r_0 A^{1/3}\). The above formulas allow a first interesting remark: The longitudinal multipole does not depend on the rigidity parameter and is proportional to the transverse electric multipole in the IF limit when \(r = 0\)

\[
\hat{L}_{2\mu}(q, r) = \sqrt{\frac{2}{3}} \hat{T}_{2\mu}(q, r = 0)
\]

Therefore the longitudinal multipoles are insensitive to rotational components of the velocity field, their values being constant for any value of \(r\).

Next we will focus on the behaviour of electromagnetic multipoles at small momentum transfer.

In 1937, Siegert showed that using the charge-current conservation law in the longwavelength approximation it is possible to replace the charge density operator in the expression of the electric transverse operator with the charge density rate. Quantitatively this theorem may be expressed as follows

\[
\hat{T}_{\lambda}^{el}(q \rightarrow 0) = \sqrt{\frac{\lambda + 1}{\lambda}} \hat{L}_{\lambda}(q \rightarrow 0)
\]

Thus, independently of the model used for the nuclear current the transverse electric multipole is proportional to the longitudinal one in the longwavelength approximation. The approximate equation (4.21) is similar to the exact equation (4.22). In other words the Siegert theorem, quantitatively expressed by (4.22) is valid in any order of \(q\) for the irotational Riemann sequence \(r = 0\). This means that the reactions performed at low-\(q\) are not able to provide informations on the vorticity. The Riemann rotator behaves like an irotational liquid drop at low momentum transfer. From the view point of the multipolar parametrization adopted in this paper this fact may be justified invoking the following argument: in the low-\(q\) limit

\[
\langle I_f \parallel \hat{L}_{\lambda}(q \rightarrow 0) \parallel I_i \rangle = -\frac{iq}{15} \langle I_f \parallel \hat{Q}_{\lambda}(q \rightarrow 0) \parallel I_i \rangle
\]

where \(\hat{Q}_{2\mu} = \int d\mathbf{r} r^\lambda Y_{\lambda}(\theta, \phi) \hat{\rho}(\mathbf{r}, t)\) is the charge quadrupole moment. Using the charge-current conservation law, the time derivative of the charge quadrupole operator may be written as follows

\[
\dot{\hat{Q}}_{2\mu} = \int d\mathbf{r} r^\lambda Y_{\lambda\mu}(\theta, \phi) \nabla \cdot \hat{\mathbf{J}}(\mathbf{r}, t)
\]

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The presence of the gradient operator in the above equation ensures the cancellation of rotational (vortical) components of the nuclear current. Thus the time derivative of the charge quadrupole moment describes the curless quadrupole flows, i.e. \( \nabla \times \hat{J} = 0 \), like the well known \( \beta \) and \( \gamma \) vibrations. This is the reason why the nuclear response is vibrating-like, without shear components for small momentum transfer. Consequently, in order to get informations on the rotational currents inside the nucleus, one needs to investigate the electromagnetic structures neglected by applying the Siegert theorem. To go beyond the limitations of this theorem one needs to increase the energy transferred in the scattering reaction. The higher order terms in the \( q^2 \) expansion of the electric transverse multipole are free of the constraint dictated by the continuity law and therefore they are likely to provide data on the vortical components of the current \( \nabla \times \hat{J} \neq 0 \). Applying the Dubovik - Cheshkov parametrization \[2\] and defining a quantity which takes into account the deviation from Siegert’s theorem

\[ \eta_2(q) = q^2 \frac{\langle I_f \parallel \hat{T}_{2,\text{tor}}(q) \parallel I_i \rangle}{\langle I_f \parallel T_{2,\text{el}}(0) \parallel I_i \rangle} \] (4.25)

we are in the position to describe the importance of the toroidal multipoles. In the IF and RR cases, the function \( \eta_2 \) looks

\[ \eta_2(q, \text{IF}) = \frac{15}{(qR_0^2)^5} \left[ (3 - (qR_0)^2) \sin qR_0 - 3qR_0 \cos qR_0 \right] - 1 \] (4.26)

\[ \eta_2(q, \text{RR}) = \frac{15}{(qR_0^2)^6} \left[ ((qR_0)^2 - 2) \sin qR_0 + 2 - \frac{(qR_0)^2}{3} \right] qR_0 \cos qR_0 - 1 \] (4.27)

whereas in the intermediate case, this function will be expressed as a convex combination of IF and RR contributions

\[ \eta_2(q, r) = q^2 \frac{\langle I_f \parallel \hat{T}_{2,\text{tor}}(q, r) \parallel I_i \rangle}{\langle I_f \parallel T_{2,\text{el}}(0) \parallel I_i \rangle} = (1 - r)\eta_2(q, r = 0) + r\eta_2(q, r = 1) \] (4.28)

In Fig.9 we draw this quantity versus momentum transfer \( q \). Notice that the RR model presents a stronger deviation from the Siegert approximation than the IF model. In the hexadecupole case ( \( \lambda = 4 \) ) the deviation effect is smaller than in the quadrupole case ( \( \lambda = 2 \) ) for both models.

Another interesting quantity is the real electric transverse form factor which in the lowest order of \( q^2 \) is

\[ F^\lambda_{\text{el}}(q) = \frac{\langle I_f \parallel \hat{T}_{\lambda}^\text{el}(q) \parallel I_i \rangle}{\langle I_f \parallel T_{\lambda}^\text{el}(0) \parallel I_i \rangle} \approx 1 - \frac{q^2}{3} \frac{\mathcal{T}_2}{Q_2} \] (4.29)

where \( Q_2 = 3e^2 Z R_0^2 \beta/4\pi \) is the transition charge quadrupole moment, and \( \mathcal{T}_2 \) is the transition toroidal quadrupole moment. This last equation allows the measurement of the transition toroidal quadrupole moment \( \mathcal{T}_2 \) similar to that of the
charge (magnetic) mean square radius \( < r^2 >_{C(mag)} \) [24], which consists in the computation of \( F^2_{l}(q) \). For an arbitrary value of the rigidity parameter the toroidal quadrupole moment reads [25]

\[
\mathcal{T}_2(r) = \frac{3e^2 Z 3 + 2r}{56\pi L_r} R_0^4 \beta
\]

We plotted in Fig.10 \( \mathcal{T}_2 \) versus the rigidity for \(^{152}\text{Sm} \) and \(^{166}\text{Er} \). An important conclusion that we draw from this figure is that \( \mathcal{T}_2 \) increases sharply for values of \( r \) close to 1, being two orders of magnitude larger in the RR case than in the IF case. We conclude that in those nuclei where the vortical components of the nuclear current are stronger the toroidal quadrupole transitions are intensified with respect to the spherical vibrating nuclei.

Taking into account the definition of the toroidal multipoles, their measurement is equivalent with the measurement of electric transverse multipole at high momentum transfer followed by the removal of the Siegert limit.

In the excitations of ground state band of an even-even nucleus there will be involved the electric multipoles with \( \lambda = 2, 4, ... \) and and the magnetic multipoles with \( \lambda = 1, 3, ... \). Further we will not discuss the magnetic multipoles and we will focus on the longitudinal and electric transverse parts of the differential cross-sections. Since our purpose is to shed light onto the nature of toroidal transitions and their connection with nuclear vortical currents we will chose a convenient method to separate the dominant longitudinal components in the differential cross section and eventually to obtain the transverse multipoles. Separating the transverse multipoles by 180° scattering is a seducing method since at this angle these multipoles dominate the differential cross section. Another method of separation consisting in the scattering of polarized electrons will be discussed bellow.

For the \( 0^+ \rightarrow \lambda^+ \) excitation we have the following expression for the differential cross-section:

\[
\left( \frac{d\sigma}{d\Omega} \right)_{(e,e')} = \frac{4\pi \sigma_{\text{Mott}}}{f_{\text{rec}}} \left\{ \frac{q_\mu^4}{q^4} |\langle \lambda^+ || \hat{M}_\lambda(q) || 0^+ \rangle|^2 + \left( \frac{q_\mu^2}{2q^2} + \tan^2 \frac{\theta}{2} \right) |\langle \lambda || \hat{T}^{\text{ele}}_\lambda(q) || 0^+ \rangle|^2 \right\}
\]

Using the Dubovik - Cheshkov decomposition in the above formula and neglecting the Coulomb multipoles and the low-\( q \) limit of electric transverse multipole one get

\[
\left( \frac{d\sigma}{d\Omega} \right)_{(e,e')} = \frac{4\pi \sigma_{\text{Mott}}}{f_{\text{rec}}} q^2 \left( \frac{q_\mu^2}{2q^2} + \tan^2 \frac{\theta}{2} \right) |\langle \lambda || \hat{T}^{\text{tor}}_\lambda(q) || 0^+ \rangle|^2
\]

This approximation is equivalent to the neglecting of transition charge moments \( Q_\lambda \) mean 2\( n \)-power of charge distribution radii \( r^2_\lambda \).

The differential cross sections (4.31) and (4.32) are ploted in both cases, for RR and IF, in order to compare the exact formula and the approximate one in backscattering processes. The cross section of the quadrupole transition induced
by the scattered electron is represented in Fig. 11 for the $^{152}$Sm nucleus. The hexadecupolar transition is considered in Fig.12. From the study of these two graphics it becomes obvious that the above mentioned approximation is acceptable in the RR model for the whole range of momentum transfer for both considered transitions. The main difference consists in the location of diffraction minima. These differences are not important because in a phase-shift diffraction analysis the curve will be smoothed in the neighborhood of minima.

In the IF case, the exact differential cross section (4.31) matches quite well the approximate one for momentum transfer $q < 400$ MeV/c when $\lambda = 2$ and $q < 250$ MeV/c when $\lambda = 4$ for the case $^{166}$Er. The reason of these discrepancies in the IF model between the exact and approximate curves is that at high momentum transfer the mean $2n$-power charge distribution radius is drastically enhanced.

## 5 Separation of toroidal multipoles in electron coincidence processes

As we saw in the preceding sections of this work a method to separate the electric transverse multipoles is given by the Rosenbluth decomposition of the $(e, e')$ differential cross section at $180^\circ$ scattering angle. A more recent method which allows this separation is based on the $(e, e'\gamma)$ longitudinal/transversal interferences [26]. The differential cross section of the process is given by

$$\left( \frac{d^2\sigma}{d\Omega_e d\Omega_\gamma} \right)^{h,\sigma} = \frac{1}{2} \Sigma_0^{\lambda^+0^+} \left( \frac{\Gamma_\gamma^{\lambda^+\rightarrow(\lambda-2)^+}}{\Gamma_{\text{total}}^{\lambda^+}} \right) (W_{\Sigma}(\theta_\gamma, \phi_\gamma) + h\sigma W_{\Delta}(\theta_\gamma, \phi_\gamma))$$

where $\Sigma_0^{\lambda^+0^+}$ is the differential cross section of the corresponding $(e, e')$ process, given by (4.31), $\Gamma_\gamma^{\lambda^+\rightarrow(\lambda-2)^+}$ is the photodisintegration width of the transition $\lambda^+ \rightarrow (\lambda-2)^+$ and $\Gamma_{\text{total}}^{\lambda^+}$ is the total decay width for the state $\lambda^+$. The ratio of these two widths is close to unity. The differential cross section contains as labels the incident electron helicity, $h = \pm 1$, and the circular polarization of the photon detected in coincidence, $\sigma = \pm 1$. The angular distribution functions are normalized to unity ($\int d\Omega_\gamma W_{\Sigma} = 1$), and satisfy the integral condition ($\int d\Omega_\gamma W_{\Delta} = 0$). Their explicit expressions are

$$W_{\Sigma}(\theta_\gamma, \phi_\gamma) = \frac{1}{4\pi} (1 + A_{\Sigma}(\theta_\gamma, \phi_\gamma))$$

$$W_{\Delta}(\theta_\gamma, \phi_\gamma) = \frac{1}{4\pi} A_{\Delta}(\theta_\gamma, \phi_\gamma)$$

In the electroexcitation of the rotational g.s. band of a deformed even-even nucleus (0+, 2+, 4+, ...), the longitudinal multipoles (C2, C4, ... ) are larger than the electric transverse multipoles (E2, E4, ... ). The even-even nuclei are not good candidates to become polarized targets, because they have spin and parity 0+ in the ground state. However they are good candidates for the $(e, e'\gamma)$ study where the L/T interferences can be isolated and the matrix elements of the
electric transverse multipole computed. Introducing the ratio between the electric transverse multipole and the Coulomb one

$$\xi_\lambda = \frac{\langle I_f \parallel \hat{T}_{el}^\lambda \parallel I_i \rangle}{\langle I_f \parallel \hat{M}_\lambda \parallel I_i \rangle}$$

and considering that $\xi_\lambda \ll 1$, there will be considered only the linear terms in the angular distribution function. The transversal-longitudinal interference term, which is linear in $\xi_\lambda$ is the most interesting. It can be isolated performing measurements at $\phi_\gamma = 0^o$ and $\phi_\gamma = 180^o$ with unpolarized electrons

$$d^2\sigma_{nepol} \bigg|_{\phi_\gamma = 0^o} - d^2\sigma_{nepol} \bigg|_{\phi_\gamma = 180^o} \left( \frac{\Gamma_\lambda^{\lambda+\rightarrow(\lambda-2)^+}}{\Gamma_{total}} \right)^4 \pi \sigma_{Mott} f_{rec}^{-1} = -\sqrt{2} \frac{q^2}{q^2 + \tan^2 \frac{\theta}{2}} G_{TL} \xi_\lambda \left( \langle I_f \parallel \hat{M}_\lambda(q) \parallel I_i \rangle \right)^2$$

where

$$G_{TL} = \frac{\sqrt{2}}{7(2\lambda - 1)} \sqrt{\frac{\lambda + 1}{\lambda}} \left\{ 5P_2^1(\cos \theta_\gamma) - 3 \left( \frac{\lambda + 2}{2\lambda - 3} \right) P_4^1(\cos \theta_\gamma) \right\}$$

Therefore, choosing convenient values which maximize (5.5), it is possible to determine $\langle I_f \parallel \hat{M}_\lambda(q) \parallel I_i \rangle \langle I_f \parallel \hat{T}_{el}^\lambda(q) \parallel I_i \rangle$ for different values of $q$.

Let us consider as an example the study of $(e,e'\gamma)$ processes when the excited states are vibrational collective modes. In the incompressible liquid drop model the density operator is given by

$$\hat{\rho}_N(r) = \frac{3eZ}{4\pi R_0^3} \Theta \left[ R_0(1 + \sum_{l,m} \alpha_{lm} Y_{lm}^* - r) \right]$$

and the current density operator by

$$\hat{J}_N(r) = \frac{3eZ}{4\pi R_0^3} \sum_{l,m} \frac{1}{l} \alpha_{lm} \left[ \nabla \left( \frac{r}{R_0} \right)^l Y_{lm}(\theta, \phi) \right] \Theta(R_0 - r)$$

For the transition to the one-surfon $2^+$ state, the Coulomb multipole is

$$\langle 2^+ \parallel \hat{M}_2(q) \parallel 0^+ \rangle = \frac{3eZ}{4\pi} \sqrt{\frac{5}{2(B_2C_2)^{1/2}}} j_2(qR_0)$$

where $B_2$ and $C_2$ are the inertia and rigid parameters. Since the liquid drop velocity field is postulated to be irrotational, the contribution of the transverse multipole reads

$$\langle 2^+ \parallel \hat{T}_{el}^2(q) \parallel 0^+ \rangle = -\frac{\omega_{2^+}}{q} \sqrt{\frac{3}{2}} \langle 2^+ \parallel \hat{M}_2(q) \parallel 0^+ \rangle$$
with $\omega_2 \approx 36A^{-1/2}\text{MeV}$. In order to take into account the magnetization components of the current in eq.(5.8), a crude model for the magnetization density is used \cite{14}

$$\hat{\mu}_N(r) = \frac{\mu}{2mZ} \hat{\rho}_N(r) L \quad (5.11)$$

and the electric transverse multipole becomes

$$\langle 2^+ || \hat{T}_2^{\text{el}}(q) || 0^+ \rangle = -\frac{\omega_2}{q} \sqrt{3 \over 2} \left( 1 - \frac{\mu}{Z^2 2m\omega_2} \right) \langle 2^+ || \hat{M}_2(q) || 0^+ \rangle \quad (5.12)$$

Introducing the deviation from the Siegert theorem (4.25) it is then possible to establish a connection with the interference factor (5.4) as follows \cite{23}

$$\eta_2(q) = 1 + \sqrt{2 q \xi_2(q)} \frac{\langle 2^+ || \hat{M}_2(q) || 0^+ \rangle}{\langle 2^+ || \hat{M}_0(q) || 0^+ \rangle} \quad (5.13)$$

The dependence of $\eta_2$ on momentum transfer $q$ is given in Fig.13, for $^{16}\text{O}$ and $^{90}\text{Zr}$ in both cases: irotational flow and with nonzero magnetization ($\mu = 0.5$). Notice that for $^{16}\text{O}$ the magnetization contributions have a sensitive effect: the deviation from the Siegert theorem is enhanced. For the nucleus $^{90}\text{Zr}$, the magnetization contributions are less important.

As we saw earlier the deviation from the Siegert theorem increases when we pass from the IF model to the RR model. Although the microscopic nature of the magnetization current is different from that of the RR current from geometrical point of view they have the same rotational structure leading to the same effect: enhancement of the toroidal transitions. It can be also noticed that the deviation from the Siegert theorem is stronger in heavy nuclei than in light nuclei. This last conclusion is in agreement with the statement that we have made on the electric transverse form factor of DTM, i.e. the toroidal effect is intensified in heavy nuclei.

6 Conclusions

As we mentioned in the beginning the scope of this work was the study of collective motions with a toroidal electromagnetic structure or to extract the toroidal contribution in nuclear transitions with a mixed rotational-vibrational spectrum.

Considering the giant isoscalar resonance $1^- \text{DTM}$, we calculated for a group of spherical nuclei ($^{40}\text{Ca}$, $^{90}\text{Zr}$ and $^{208}\text{Pb}$) the transverse form factors and we underlined the shift of the toroidal effect toward small momentum transfers when we pass to nuclei with large $Z$ and $A$. Simultaneously we testify an enhancement of the dipole toroidal response in heavy nuclei. The probability to excite the DTM with photon probes is small according to the calculation we have made. However the possibility to use electron inelastic scattering offers the promise to detect DTM because in such reactions it is possible to vary independently the momentum transfer and the excitation energy along with the scattering angle $\theta$ dependence of the differential cross section. Since the DTM is a transversal flow
associated to the Hill ring vortex the electroexcitation at $180^\circ$ angles seems to be favourable since other modes which could occur, being especially of longitudinal nature, are suppressed. We have also extracted the longwavelength limit of the electric transverse form factor and we emphasized that it is proportional with the transition toroidal dipole moment. This dynamic characteristic associated to the DTM for small momentum transfers depends linearly on $\alpha Z$, which explains the smallness of toroidal effects for nuclei with small numbers of protons.

Another type of resonances that we studied are the spin dependent (spin - flip) modes. We emphasized their purely toroidal electromagnetic structure and their possible investigation with leptonic probes at $180^\circ$ scattering angles and large momentum transfer. These conclusions stems on the calculation of $(e, e')$ differential cross sections of Goldhaber - Teller resonances and $1^-\text{ electric spin - flip modes (s-is)}$ in $^{12}\text{C}, ^{16}\text{O}, ^{40}\text{Ca}$ and $^{208}\text{Pb}$.

In the second part of this work we focussed on the study of toroidal contributions in the excitation of g.s. band of even-even nuclei from the rare-earth region ($^{152}\text{Sm, } ^{166}\text{Er}$ ). We introduced a quantity which describes the deviations from the Siegert theorem or in other terms the contribution of higher order terms in the momentum transfer of the electric transverse multipole, i.e. the toroidal multipole moments and their mean square radii. For the RR which is a submodel of the Riemann rotator the deviation from the Siegert theorem has a slope larger than that of the IF model. Based on this observation we showed that defining the real electric transverse form factor and taking the first order approximation in the momentum transfer we can extract the transition toroidal quadrupole moment in the same manner in which we determine the charge or magnetic mean square radii : calculating the slope of the real Coulomb (magnetic) form factor.

We computed the transition toroidal quadrupole moment and we determined that it depends smoothly on the vorticity This observation strengths our opinion that the toroidal moments of arbitrary multipolarity give a measure of the intensity of vortical electromagnetic currents in the same manner the charge multipole moments are associated to charge distribution or irotational electromagnetic currents.

The electroexcitation differential cross sections of $2^+$ and $4^+$ levels from the g.s. band of the RR model may be approximated taking into account taking into account the toroidal form factors only, for a wide range of momentum transfer. This result emphasize the importance of toroidal multipoles relative to the Coulomb multipoles at $180^\circ$ angles, regardless of the momentum transfer.

We showed that in coincidence $(e, e'\gamma)$ reactions, by separating the longitudinal/transversal interference term we can measure the deviation from the Siegert theorem in the IF model and with non-zero magnetization components of the nuclear current. For light nuclei the deviation from the Siegert theorem is enhanced by the existence of magnetic currents.
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