Introduction — Since the seminal observation of the quantum Hall effect in graphene, the zero modes of the massless Dirac fermion, which is essential for the characteristic quantum Hall effect in graphene, has been intensively discussed. Specifically, the zero mode is stable against ripples in graphene, which has been discussed in terms of the index theorem, or more explicitly for wave functions with the argument due to Aharonov and Chasher. For the stability of zero modes, a crucial ingredient is the chiral symmetry defined as the existence of an operator $\Gamma$ that anti-commutes with the effective Hamiltonian $H$, $\{\Gamma, H\} = 0$, with $\Gamma^2 = 1$, which is more relevant than the Dirac-cone dispersion per se, since the stability is also shown for a chiral symmetric bilayer graphene with a quadratic dispersion. In the case of graphene, the Dirac cone is vertical, so that the chiral operator has a simplest possible form of $\Gamma = \sigma_x$. Even when the system is disordered, the anomalous criticality at the $n = 0$ Landau level is retained as far as the randomness respects the chiral symmetry. The stability of zero modes has been observed experimentally as well for a mono-layer graphene.

On the other hand, we encounter Dirac cones in a wider class of materials, where the effective theory is more generally described by tilted Dirac cones, as is the case with an organic material $\alpha$-(BEDT-TTF)$_2$I$_3$. Since the conventional chiral symmetry is broken in tilted Dirac cones, it becomes an essential question to ask whether (i) the symmetry is entirely broken, and (ii) whether the anomaly in the systems with usual chiral symmetry is washed out in tilted Dirac cones. In the absence of disorder, the eigenvalues and eigenfunctions for tilted Dirac cones in magnetic fields have already been obtained in existing literatures, which indicate that the zero-energy modes themselves persist in clean tilted cones.

Effects of disorder on the zero modes is indeed an important issue, since, for graphene with the conventional chiral symmetry, it has been established that the disorder has an anomalous effect of retaining a sharp (delta-function-like) zero Landau level accompanied by a sharp (step-function-like) quantum Hall step, if the disorder respects the chiral symmetry. We can thus pose a question: what is the effect of disorder for the zero Landau level in tilted Dirac cones, where the conventional chiral symmetry is absent. This is exactly the motivation of the present work. In particular, we shall look at how the stability of the zero modes possessing an anomalous criticality at the $n = 0$ Landau level in the presence of disorder (e.g., random components in magnetic fields) is affected by the breakdown of the conventional chiral symmetry in tilted Dirac cones. Curiously, the study leads us to find a “generalized chiral symmetry”, which is then shown to give rise to an even wider stability of the zero modes persisting in tilted Dirac cones.

Thus we shall first generalize the conventional chiral symmetry so that the tilted and untitled Dirac dispersions can be captured in a unified manner. The generalized chiral symmetry is defined by the existence of an operator $\gamma$, which is not necessarily Hermitian, satisfying the relation $\gamma^4 H = -H$ and $\gamma^2 = 1$. This is consistent with the condition for the Dirac operator to be elliptic that is required for the index theorem. It enables us to show a topological stability of zero modes that is present in tilted Dirac cones. With this generalized chiral symmetry, we reformulate the eigenvalue problem so that the Aharonov-Casher argument for counting the number of zero modes is extended to tilted cones. This implies the zero Landau level is indeed delta-function-like. We then numerically confirm how these field theoretic treatments on the stability of zero modes appears in a lattice fermion model that has tilted Dirac cones where the titling is varied continuously.

Formalism — The effective Hamiltonian for a tilted Dirac cone can be generically expressed as

$$H = \sigma_0 (W \cdot \pi / \hbar) + (\sigma \cdot X) \pi_x / \hbar + (\sigma \cdot Y) \pi_y / \hbar,$$

where $(\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices while $\sigma_0$ is a $2 \times 2$ unit matrix, and $X, Y, W$ are real. The Dirac cone is...
tilted when \(W = \langle W_x, W_y \rangle\) is nonzero, while \(X = \langle X_x, X_y, X_z \rangle\) and \(Y = \langle Y_x, Y_y, Y_z \rangle\) characterize the anisotropy of the Dirac cone. Here \(\pi = p + eA\) \((e > 0)\) is the dynamical momentum with \(p = -i\hbar(\partial_x, \partial_y)\) and the vector potential \(A = (A_x, A_y)\) for the magnetic field \(B = \partial_x A_y - \partial_y A_x\) perpendicular to the \(x-y\) plane. Note that the dynamical momentum satisfies the commutation relation \([\pi_x, \pi_y] = -i\hbar B\). For the conventional Dirac cone \((W = 0)\), the Hamiltonian has the chiral symmetry associated with an operator defined (and generalized to anisotropic cases) by \(\Gamma = \sigma \cdot (X \times Y)/|X \times Y|\), which anti-commutes with the Hamiltonian \(H = (\sigma \cdot X)\pi_x/h + (\sigma \cdot Y)\pi_y/h\) with \(\Gamma^2 = \sigma_0\mathbb{I}\).

When the Dirac cone is tilted (with \(H\) containing \(\sigma_0\) for \(W \neq 0\)), the conventional chiral symmetry is broken. However, here we find that a generalized symmetry does exist, which can be seen if we introduce a new operator \(\gamma\) defined by

\[
\gamma = \sigma \cdot [(X \times Y) - i(W_y X - W_x Y)]/\Delta
\]

with \(\Delta^2 = |X \times Y|^2 - (W_y X - W_x Y)^2\). This operator \(\gamma\) is non-Hermitian for \(W \neq 0\), but its eigenvalues are \(\pm 1\) since \(\gamma^2 = (\gamma^2 \pm) = \sigma_0\). We can show that, if \(\Delta^2 > 0\), \(\gamma\) satisfies a relation with the Hamiltonian,

\[
\gamma^\dagger H \gamma = -H,
\]

which we call the generalized chiral symmetry. This symmetry reduces to the conventional chiral symmetry \((\gamma \rightarrow \Gamma)\) for \(W = 0\). It should be noted that the cross section of the tilted Dirac cone with a constant energy plane is an ellipse as long as \(\Delta^2 > 0\) (while a hyperbola when \(\Delta^2 < 0\)).

Can we say anything about the wave functions as a direct consequence of this generalized chiral symmetry? For this purpose it is instructive to choose the (right-) eigenvectors \(|\pm\rangle\) of the operator \(\gamma\) (with \(\gamma|\pm\rangle = \pm|\pm\rangle\)) in the spinor space as a basis of the eigenvalue problem for \(H\). If we express the (normalized) wave function as \(\psi = |+\rangle\psi_+ + |\mp\rangle\psi_-\), the Schrödinger equation \(H\psi = E\psi\) reduces to

\[
\begin{bmatrix}
|+\rangle H |+\rangle \\
|\mp\rangle H |\mp\rangle
\end{bmatrix} = E
\begin{bmatrix}
\psi_+ \\
\psi_-
\end{bmatrix} = E
\begin{bmatrix}
1 \\
|\mp\rangle
\end{bmatrix}
\begin{bmatrix}
|+\rangle \\
1
\end{bmatrix}
\begin{bmatrix}
\psi_+ \\
\psi_-
\end{bmatrix},
\]

Note that the operator \(\gamma\), being non-Hermitian, has eigenvectors that are in general not orthogonal with each other with \(\beta = \langle +\rangle \neq 0\). When the generalized chiral symmetry, \(\gamma^\dagger H \gamma = -H\), holds, we have \(\langle +\rangle |H|+\rangle = \langle -\rangle |H|\mp\rangle = 0\). The Schrödinger equation then becomes

\[
\begin{bmatrix}
0 & \alpha \cdot \pi \\
\alpha^* \cdot \pi & 0
\end{bmatrix}
\begin{bmatrix}
\psi_+ \\
\psi_-
\end{bmatrix} = E
\begin{bmatrix}
1 & \beta \\
\beta^* & 1
\end{bmatrix}
\begin{bmatrix}
\psi_+ \\
\psi_-
\end{bmatrix},
\]

where we have introduced a complex \(\alpha = (\alpha_X, \alpha_Y) = h^{-1}\langle +|W_x \sigma_0 + X \cdot \sigma|+\rangle, \langle +|W_y \sigma_0 + Y \cdot \sigma|\mp\rangle\). The Schrödinger equation for the \(E = 0\) states (zero modes) therefore amounts to that for the zero modes of the untitled (but can be anisotropic) Dirac cones (with zero diagonal elements). We see that the zero mode is then given by

\[
\alpha \cdot \pi \psi_- = 0 \quad \text{and} \quad \alpha^* \cdot \pi \psi_+ = 0.
\]

As we shall see in the following, the chirality \(\pm\) has to be assigned to (normalizable) wave functions \(\psi_\pm\). We can also note that the equations for zero modes apply to spatially varying magnetic fields \(B(x, y)\) as well, which can even be random.

Aharonov-Casher argument extended to the general chirality — Is the stability of zero modes inherited by the general chiral symmetry? For this we can look at the zero modes for the tilted Dirac cone in a spatially varying magnetic field \(B(x, y)\), which are therefore determined by eq. [2]. Here it is convenient to adopt “principal coordinates” by rotating \(\alpha\) with an orthogonal matrix \(T\) (with \(det T = 1\)), so that the complex numbers \((z_x, z_y) = (\alpha_X, \alpha_Y)T^{-1}\) become orthogonal with each other on the complex plane. It is indeed possible to do this with \(\pi_\pm = \pi_\mp\), and \(\psi_-\) becomes \((\lambda \Pi_X - i\lambda \chi^{-1} \Pi_Y)\psi_- = 0\), where the “ellipticity” \(\lambda = \sqrt{|z_x|^2/|z_y|^2} > 0\) is a positive real number. The equation for \(\psi_-\) then becomes \((\lambda \Pi_X - i\lambda \chi^{-1} \Pi_Y)\psi_- = 0\), This, along with a similar equation for \(\psi_+\), reduces to

\[
(\lambda \Pi_X \pm i\lambda \chi^{-1} \Pi_Y)\psi_\pm = 0.
\]

With these equations, it is straightforward to generalize the analytic Aharonov-Casher argument for the zero modes to the present case. We use the “Coulomb gauge”, \(\lambda^2 \partial_X A_X + \lambda^{-2} \partial_Y A_Y = 0\), which is automatically satisfied if we introduce a “scalar potential” \(\phi\) with \(\langle A_X, A_Y\rangle = \langle -\lambda^{-2} \partial_Y \phi, \lambda^2 \partial_X \phi\rangle\). Then Eq. [2] simplifies to

\[
-i\hbar \left\{ \mathcal{D}_\pm \equiv \frac{2\pi}{\phi_0} \left( \mathcal{D}_\pm \varphi \right) \right\} \psi_\pm = 0
\]

with \(\mathcal{D}_\pm \equiv (\lambda \partial_X \pm i\lambda \chi^{-1} \partial_Y)\). Namely, the function \(\varphi_\pm\) is an entire function of \(Z_\pm = \hat{X} \pm i\hat{Y}\) over the whole complex plane, namely a polynomial in \(Z_\pm\). The function \(\varphi\) is determined by \(B_x = \partial_x A_Y - \partial_y A_X = (\lambda^2 \partial_X^2 + \lambda^{-4} \partial_Y^2) \varphi = (\partial_X^2 + \partial_Y^2) \varphi\), which implies that \(\varphi(\hat{R}) = \int d\hat{R} G(\hat{R} - \hat{R}') B_x(\hat{R}')\) with \(G(\hat{R}) = (1/2\pi)\log(r/r_0)\).
and $r^2 = \vec{R}^2$. When the magnetic field is nonzero only in a finite region, we have an asymptotic behavior, $\varphi \to (\Phi/2\pi) \log(r/r_0)$, for $r \to \infty$, where $\Phi = \int d\vec{R} B_z = \int d\vec{R} B_x$ is the total flux. Then we obtain

$$\psi_\pm \to \tilde{\psi}_\pm(r/r_0)^{\pm \chi(\Phi/\phi_0)}$$

for $r \to \infty$, in which $\pm \chi \Phi < 0$ is necessary for $\psi_\pm$ to be normalizable. The normalizability of $|\psi_\pm|^2$ indicates that the degeneracy of zero modes is $\Phi/\phi_0$. This is exactly equal to the total number of energy levels in the Landau level, which implies a remarkable property that no broadening occurs for the Landau level, which holds only as a low-energy effective model. In order to confirm whether the anomalously sharp zero-Landau level discussed above appears as well in a lattice model or not, we have introduced here as a random component, $\delta \phi(r)$, in the magnetic flux $\phi(r) = \phi + \delta \phi(r)$ piercing each plaquette, where $\phi$ is the uniform part. The band dispersion $\delta \phi(r)$ is assumed to have a gaussian distribution with a variance $\sigma$ and a spatial correlation length $\eta$ with $\langle \delta \phi(r) \delta \phi(r') \rangle = (\delta \phi^2) \exp(-|r-r'|^2/4\eta^2)$. We have chosen this disorder since it restores, for large enough $\eta$, the generalized chiral symmetry of the effective Hamiltonian at tilted Dirac cones.

![FIG. 1](image) (Color Online) (a) A lattice model possessing tilted Dirac cones. Hopping energies are $t$: solid thick lines, $-t$: solid thin lines, and $t'$: dotted lines. A unit cell is indicated by the primitive vectors $e_1$ and $e_2$. Energy dispersions $E(k)/t$ for (b) $t' = 0$, (c) $t'/t = 0.2$ and (d) $t'/t = 0.4$.

We apply a magnetic field to this model to examine the stability of zero modes against the disorder in the magnetic field. The magnetic field is taken into account by the Peierls substitution $t \to te^{-2\pi i \delta \phi(r)}$, $t' \to t' e^{-2\pi i \delta \phi(r')}$. So, the effective Hamiltonian $\mathcal{H}$ is obtained in a usual fashion. This anomaly, appearing only for the $n = 0$ Landau level, suggests that the $n = 0$ Landau...
states are degenerated at $E = 0$, endorses the stability of zero modes for tilted Dirac cones.

In summary, we have found that the conventional chiral symmetry can be extended to a generalized chiral symmetry that encompasses the models having tilted Dirac cone, so that the untitled and tilted Dirac cones can be treated in a unified way. The stability of zero modes can be proved under this generalized chiral symmetry with an Aharonov-Casher argument. We have further shown, numerically for a lattice model, that topologically protected zero modes of tilted Dirac fermions survive even in random magnetic fields correlated over a few lattice constants. These results suggest that the anomaly at $n = 0$ Landau level can be observed generally in systems with tilted Dirac dispersions.

As a significance of this, we can finally note that the existence of the generalized chiral symmetry ($\Delta^2 > 0$) is equivalent to the ellipticity of the Hamiltonian $H$ as a differential operator, under which the index theorem can be applied. It is an interesting future problem to extend the notion of the generalized chiral symmetry to a broader class of Dirac-cone systems.

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