NUMERICAL EVALUATION OF SINE-GORDON CHAIN
ENERGY CONTROL VIA SUBDOMAINS STATE FEEDBACK
UNDER QUANTIZATION AND TIME SAMPLING

Boris Andrievsky
Inst. Problems Mechanical Eng.,
Russian Acad. of Sciences
Saint Petersburg State University
Saint Petersburg, Russia
boris.andrievsky@gmail.com

Yury Orlov
Mexican Scientific Research
Advanced Studies Center
of Ensenada,
Carretera Tijuana-Ensenada,
B.C. 22860, México,
yorlov@cicese.mx

Abstract
The paper is devoted to the numerical performance evaluation of the speed-gradient algorithms, recently developed in (Orlov et al., 2018; Orlov et al., 2019) for controlling the energy of the sine-Gordon spatially distributed systems with several in-domain actuators. The influence of the level quantization of the state feedback control signal (possibly coupled to the time sampling) on the steady-state energy error and the closed-loop system stability is investigated in the simulation study. The following types of quantization are taken into account: sampling-in-time control signal quantization, the level quantization for control, continuous in time; control signal quantization on level jointly with time sampling; control signal transmission over the binary communication channel with time-invariant first order coder; control signal transmission over the binary communication channel with first order coder and time-based zooming; control signal transmission over the binary communication channel with adaptive first-order coder. A resulting impact on the closed-loop performance in question is concluded for each type of the quantization involved.

Key words
energy control, sine-Gordon equation, Speed-gradient, quantization, data rate constraints

1 Introduction
Energy control is motivated by numerous applications in physics and engineering, see [Spong, 1995; Shiriaev et al., 1999; Fantoni et al., 2000; Xin and Kaneda, 2005; Acosta, 2010; Garofalo and Ott, 2017; Fradkov et al., 2018; Tang and Zuo, 2012; Rodriguez et al., 2001; Wang et al., 2003], for mentioning a few. Energy harvesting [Siang et al., 2018; Leong et al., 2018], deployment of tethered systems [Andrievsky and Guzenko, 2014; Nikpoorparizi et al., 2018], quantum control [Boussaid and Caponigro, 2013; Bonnard et al., 2011; Mantile, 2008] are among modern applications which are relevant to energy control as well.

During the last years the energy control problem of spatially distributed systems has been widely studied and proper solutions have been proposed and rigorously justified. In the series of papers the speed-gradient method of [Fradkov and Pogromsky, 1998; Fradkov, 2007; Fradkov et al., 1999] was used to design energy control algorithms for spatially distributed systems, using the boundary control [Dolgopolik et al., 2016; Dolgopolik et al., 2018] and distributed control [Orlov et al., 2017b; Orlov et al., 2017a]. The authors of [Orlov et al., 2017b] analyzed energy control problems for linear wave partial differential equation (PDE) and nonlinear sine-Gordon PDE where the distributed yet uniform over the space control was chosen. The speed-gradient method for energy control of Hamiltonian systems was developed and justified for the above partial differential equations (PDEs). An infinite dimensional version of the Krasovskii–LaSalle principle was validated for the resulting closed-loop systems. By applying this principle, the closed-loop trajectories were shown to either approach the desired energy level set or converge to a system equilibrium. In [Orlov et al., 2017a] a linear wave equation, governing 1-D (one-dimensional) string oscillations, was considered. A distributed control input, independently enforcing the
underlying string over its entire spatial location, was assumed to be available. The speed-gradient method was developed and justified for the considered PDE model. Capabilities of the proposed speed-gradient algorithm of reaching the energy goal were supported by numerical simulations. In [Orlov et al., 2018] an energy control of the sine-Gordon chain driven by several in-domain actuators was considered. The speed-gradient method was generalized to the in-domain actuation such as in [Christofides, 2001; Fridman and Blighovsky, 2012; Fridman and Am, 2013; Pisano and Orlov, 2017] for the purpose of pump/dissipate the energy of the model to a desired level. This result has been extended to control via output feedback by means of developing the Luenberger-type spatial observer, which got measurement data from the sensors, placed within small spatial plant subdomains in [Orlov et al., 2019].

Some important issues have not yet been analytically studied due to nonlinearity and complexity of the systems under consideration. One of the challenging task is to evaluate the closed-loop system performance in the presence of external disturbances and measurement and data transmission errors. The main aim of the paper is obtaining an impression on the limitations of the energy control for a class of nonlinear distributed-parameter physical systems (the sine-Gordon chains) by means of the spatially-discretized control with time and level quantization. In the present work, a step is made for clarifying some of the mentioned issues, namely the effect of level quantization separately, or jointly with a time sampling, is investigated for the control signal applied to the nonlinear chain with the state feedback control, developed in [Orlov et al., 2018].

2 Plant Model and Problem Statement

2.1 Model of Controlled Plant

First, let us briefly recall the key points of [Orlov et al., 2018].

The system of interest is the 1-D chain, described by the following dissipation-free sine-Gordon PDE

\[ x_{tt} = \kappa x_{rr} - F_0 \sin x + u(r,t), \quad t \geq 0, \quad (1) \]

where \( t \) is the time instant, \( r \in [0, 1] \) is the scalar spatial variable, \( x = x(\cdot,t) \) is the instant state of the system, the parameter \( \kappa \) is the elasticity of the system, \( F_0 \) stands for the torque gain, \( u(r,t) \) is for the manipulable input. The PDE (1) is accompanied with the Dirichlet boundary conditions

\[ x(0,t) = 0, \quad x(1,t) = 0. \quad (2) \]

2.2 Sampled-in-space Actuators

The sampled-in-space actuation

\[ u(r,t) = \sum_{i=1}^{m} b_i(r) u_i(t) \quad (3) \]

is in play to control the sine-Gordon model (1) which is thus representable in the form

\[ x_{tt} = \kappa x_{rr} - F_0 \sin x + \sum_{i=1}^{m} b_i(r) u_i(t), \quad t \geq 0. \quad (4) \]

As in [Orlov et al., 2019], the spatial domain \([0, 1]\) is uniformly partitioned into \( m = 10 \) subdomains \([r_i, r_i + h_i] \) of lengths \( h_i = 0.1, \quad i = 1, \ldots, 10 \) so that \( r_i = 0.1(i - 1) \). Within each subdomain the corresponding actuator distribution \( b_i(r) \) is specified as

\[ b_i(r) = \begin{cases} 1, & \text{if} \quad r_i + 0.02 \leq r \leq r_i + 0.08, \\ 0, & \text{otherwise,} \end{cases} \quad (5) \]

i.e., the first and the last actuators are located in the distance 0.02 from the left and right boundaries, respectively, whereas the neighboring actuators possessed a slot of the length 0.04 between them.

2.3 Control Objective

The control objective is to pump or dissipate the energy

\[ E(x, x_t) = \frac{1}{2} \int_0^1 \left( \frac{1}{2} x_t^2 + k x^2 + 2F_0 (1-\cos x) \right) dr \quad (6) \]

of the sine-Gordon system (1)–(3) to a prespecified level \( E_s \geq 0 \) for guaranteeing the limiting relation

\[ \lim_{t \to \infty} E(x(\cdot,t), \dot{x}(\cdot,t)) = E_s \quad (7) \]

on the solutions \( x(r, t) \) of the closed-loop sine-Gordon model (1), (2).

3 Energy Control Synthesis Using State Feedback

Let us pick-up the goal functional as

\[ V(t) = \frac{1}{2} (E(t) - E_s)^2. \quad (8) \]

Following the speed-gradient design procedure [Fradkov, 2007; Fradkov et al., 1999], compute the time
derivative of $V(t)$ along the trajectories of system (1), (2), provisionally assuming that $u(r, t)$ is constant on $t$. Differentiating (8) in time, integrating then the resulting equality by part, and employing a consequence of the boundary condition (2) yield [Orlov et al., 2018]

$$ V = (E(t) - E_s) \int_0^1 u(r, t) \cdot x_t \, dr .$$

As the second step of the speed-gradient procedure, one should derive the gradient $\nabla_u V \in \mathbb{R}^m$ of the resulting expression of $V$ with respect to the control components $u_i(t), i = 1, \ldots, m$, thus arriving at

$$ \nabla_u V = (E - E_s) \left[ \frac{r_i + h_i}{r_i} b_i x_t \, dr \right] .$$

The third step of the speed-gradient procedure is to pick up a certain function $\eta(x, x_t)$ which forms an acute angle with $\nabla_u V$, i.e., satisfies an inequality $\eta^T \nabla_u V \geq 0$. In [Orlov et al., 2018], it is chosen in the form

$$ \eta(\cdot) = \text{sign}(E - E_s) \left[ \frac{r_i + h_i}{r_i} b_i x_t \, dr \right] .$$

According to the speed-gradient method, the control action $u(x, \dot{x}) = -\Gamma \eta(x, \dot{x}, t)$ is then constituted with the matrix design parameter $\Gamma = \text{diag} \{ \gamma_i, \ldots, \gamma_m \}$, composed of positive entries $\gamma_i, i = 1, \ldots, m$. Summarizing, the sampled-in-space actuation (3) is specified with

$$ u_i(t) = \gamma_i \text{sign} \Delta E(t) \int_{r_i}^{r_i + h_i} b_i x_t \, dr,$$

where the energy error $\Delta E(t) = E_s - E(t)$ is introduced, $i = 1, \ldots, m$.

The present research deals with an energy dissipation problem, where desired energy level $E_s$ is set to zero (in other words, the regulation problem is considered). In this particular case, $\Delta E(t) < 0$, therefore $\text{sign} \Delta E = -1$, and control law (11) takes the following form

$$ u_i(t) = -\gamma_i \int_{r_i}^{r_i + h_i} b_i x_t \, dr .$$

### 4 Numerical Evaluation Setup

#### 4.1 Computational algorithm

Following the previous works [Orlov et al., 2017b; Orlov et al., 2017a; Orlov et al., 2018], in the numerical study, the PDE (1) is discretized in the spatial variable $r \in \mathbb{R}$ by uniformly splitting the segment $[0, 1]$ into $N$ sub-intervals. The discretization step $\nu$ is introduced as $\nu = 1/N$. The resulting system of $N - 1$ ordinary differential equations (ODEs) of the second order are then numerically solved over a time interval $[0, T]$ by applying the medium order variable step Runge–Kutta Method [Dormand and Prince, 1980], performed with the standard MATLAB routine ode45.

At the discretization nodes $r_i = i \nu, i = 1, \ldots, N - 1$, the second-order spatial derivatives of $x(r, t)$ are approximately computed as

$$ x_r(r_i, t) = \frac{x(r_{i+1}, t) - x(r_{i-1}, t)}{2\nu}, \quad x_{rr}(r_i, t) = \frac{x(r_{i+1}, t) - 2x(r_i, t) + x(r_{i-1}, t)}{\nu^2} .$$

The boundary values $x(r_0, t) = x(r_1, t), x(r_N, t) = x(r_{N-1}, t)$ are specified according to boundary conditions (2), thereby yielding the following values $x(r_0, t) = x(r_N, t) = 0$ at the boundaries. The remaining discrete values $x(r_1, t), \ldots, x(r_{N-1}, t)$ are found by solving $(N - 1)$ ODEs numerically.

To numerically find the values of definite integrals in (12), the MATLAB standard routine trapz of the trapezoidal numerical integration are used.

#### 4.2 System Parameters and Initial Conditions

In the simulations of the closed-loop boundary-value problem (1)–(2), (12), the control gains were set to $\gamma_i = \gamma_0 / h_i, h_i = m^{-1}, i = 1, \ldots, m$ with $\gamma_0 = 5$ and the desired energy level was set to $E_s = 5$. The initial states were pre-specified in the form

$$ x(r, 0) = A(1 - \cos(2\pi r))^7, \quad x_t(r, 0) = 0$$

with a certain “magnitude” parameter $A$. In the present study, $A = 0.02$ is taken. Parameters of (1) were specified to $\kappa = 0.12, F_0 = 25$. For avoiding consideration of the various combinations of initial and desired energies, for certainty, the desired energy level $E_s = 0$
is taken, therefore actually, the stabilization problem is studied in the present work. The simulation results for various cases of the energy pumping/dissipation may be found in the above mentioned papers [Orlov et al., 2017b; Orlov et al., 2018; Orlov et al., 2019]. For given parameters $\kappa = 0.12$, $\mathcal{F}_0 = 25$ and initial conditions (13) with $A = 0.02$, initial system energy $E(0)$ is as $E(0) = 12.47$.

A reasonably high number $N = 2500$ was selected for the PDEs (1) to discretize the spatial variable $r$ and duration of the computation time $T$ was confined to 15.

5 Numerical Results

Below, the several sources of the errors, caused by the discretization and time sampling are numerically studied for the system under consideration. In all of them, discrepancy of the “ideal” control signals given by (12) and the actual ones, denoted by $\tilde{u}_i$, $i = 1, \ldots, 10$ is considered as an error source in the examined system.

5.1 Sampled in Time Control Signal

5.1.1 Time Sampling Description In the case of the control signal time sampling, controls $u_i(t)$ are calculated, accordingly to (12) at the uniformly distributed instants $t_k = kT_0$, where constant $T_0$ is for the given sampling interval, $k = 0, 1, \ldots$ denotes the step number (the discrete time). The resulting signals $u_i(t_k)$ are subjected to the first-order hold procedure which expands $u_i(t_k)$ fixing them over the sampling intervals $[t_k, t_{k+1})$. This leads to the following expression for actuator inputs $\tilde{u}_i$, where $i = 1, \ldots, 10$:

$$\tilde{u}_i(t) = u_i(kT_0), \quad \text{as} \quad kT_0 \leq t < (k+1)T_0. \quad (14)$$

5.1.2 Simulation Results for Time Sampling

The simulation results in the form of the energy error evolution $\Delta E(t)$, depending on the sampling interval $\delta \in \{0.01, 0.05, 0.10, 0.20\}$ are plotted on Fig. 1. The logarithmic scale is chosen along the $y$-axis (for the energy error values). The plots show that the energy tends in time to a certain steady-state value, depending on the corresponding $\delta$. In Tab. 2, the steady-state relative energy error $\delta E$ is given (in percents) for various quantization intervals $\delta$, where $t_{\text{fin}} = 25$ stands for the simulation final time.

The results show that the steady-state energy error is negligibly small at least for $T_0 \leq 0.10$, but if $T_0$ exceeds some threshold, loss of stability occurs.

Table 1. Steady-state relative energy error $\delta E$ (in percents) vs. sampling interval $T_0$.

| $T_0$  | 0.01 | 0.05 | 0.10 | 0.20 |
|--------|------|------|------|------|
| $\delta E,\%$ | $<3.4\times10^{-5}$ | $7.3\times10^{-4}$ | 0.005 | $\infty$ |

Table 2. Steady-state relative energy error $\delta E$ (in percents) vs quantization interval $\delta$.

| $\delta$ | $0.01$ | $0.20$ | $0.50$ |
|-----------|--------|--------|--------|
| $\delta E,\%$ | $5.61\times10^{-5}$ | $0.0017$ | $0.0080$ |

5.2 Level Quantization of Control Signal

5.2.1 Quantization Description In this case no time sampling is taken into account, and it is assumed that the control signals

$$\tilde{u}_i = \delta \cdot \left\lfloor \frac{u_i}{\delta} \right\rfloor, \quad i = 1, \ldots, 10 \quad (15)$$

are applied to the chain which are results of the uniform quantization of the signals $u_i$, $i = 1, \ldots, 10$ with a given quantization interval $\delta > 0$. Hereinafter, $\lfloor \cdot \rfloor$ denotes the round function, which rounds each element of the input signal to the nearest integer.

5.2.2 Simulation Results for Level Quantization

The simulation results in the form of the energy error evolution $\Delta E(t)$, depending on the quantization interval $\delta \in \{0.01, 0.20, 0.50, 1.0, 2.0, 5.0\}$ are plotted on Fig. 2. The logarithmic scale is chosen along the $y$-axis (for the energy error values). The plots show that the energy tends in time to a certain steady-state value, depending on the corresponding $\delta$. In Tab. 2, the steady-state relative energy error $\delta E$ is given (in percents) for various quantization intervals $\delta$, where $t_{\text{fin}} = 25$ stands for the simulation final time.

One may notice that even in the worst case, the steady-state energy error does not exceed 1%. Therefore, simulation results show that the energy is stabilized with an affordable accuracy for the all chosen quantization intervals.

Figure 1. Energy error $\Delta E(t)$: time histories for various values of sampling interval $T_0$. Figure 2. The logarithmic scale is chosen along the $y$-axis (for the energy error values).
Table 3. Steady-state relative control error $\delta E$ (in percents) vs. quantization interval $\delta$ for $T_0 = 0.03$.

| $\delta$ | 0.20 | 0.50 | 2.0 |
|----------|------|------|-----|
| $\delta E$, % | 0.00706 | 0.015 | 0.127 |

$\delta$, $\delta E$, %

| $\delta$ | 5.0 | 10 | -- |
|----------|-----|----|-----|
| $\delta E$, % | 0.633 | 1.92 | -- |

5.3 Control Signal Quantization on Level and Time Sampling

5.3.1 Quantization and Time Sampling Description

Consider now the case when both level quantization and the time sampling of the control signal exist in the actuation link. The control signals

$$\tilde{u}_i(t) = \delta \left\langle u_i(kT_0) \right\rangle$$

as $(k-1)T_0 < t \leq kT_0$,

where $i = 1, \ldots, 10$, applied to the chain, are then obtained by zero-order holding of the uniformly quantized signals $u_i$ where constant $T_0$ denotes the sampling interval, $k = 0, 1, \ldots$ is for the step number (the discrete time).

5.3.2 Simulation Results for Level Quantization and Time Sampling

Time histories of the energy error $\Delta E(t)$ for $T_0 = 0.03$ and varying values of the quantization interval $\delta \in \{0.20, 0.50, 2.0, 5.0, 10\}$ are plotted in Fig. 3 in the logarithmic scale for the energy error values. As above, the closed-loop system demonstrate stable performance and the steady-state errors, depending on the quantization interval, occur. The steady-state relative energy errors $\delta E = \frac{\Delta E(t_{\infty})}{\Delta E(0)}$ are given in Tab. 3 for various quantization intervals $\delta$ and $T_0 = 0.03$. Table 3 shows that the steady-state error is less than 2 for the worst case with $\delta = 10$.

Let us now consider the dependence of the system accuracy on sampling interval $T_0$ for fixed value of $\delta$. Time histories of the energy error $\Delta E(t)$ are depicted on Fig. 4 for various values of sampling interval $T_0 \in \{0.01, 0.02, 0.03, 0.04, 0.05, 0.06\}$ as $\delta = 1$.

The plots of Fig. 4 show that there exists a critical sampling time which should not be exceed to preserve the closed-loop system stability. In the presented example, the transients of the system have approximately the same shape and asymptotically tend to certain steady-state values with the exception of the case $T_0 = 0.06$, where the stability is lost and the process diverges. Table 4 exhibits the dependence of the steady state error on sampling time $T_0$ for $\delta = 1$. As seen, the steady-state error does not depend practically of $T_0$ and remains negligibly small before the stability is lost.

| $T_0$ | 0.01 | 0.02 | 0.03 |
|-------|------|------|------|
| $\delta E$, % | 0.0298 | 0.0319 | 0.0385 |

| $T_0$ | 0.04 | 0.05 | 0.06 |
|-------|------|------|------|
| $\delta E$, % | 0.0478 | 0.0503 | $\infty$ |

### Figure 2
Energy error $\Delta E(t)$: time histories for various values of quantization interval $\delta$.

### Figure 3
Energy error $\Delta E(t)$: time histories with $T_0 = 0.03$ and varying values of quantization level $\delta$.

### Figure 4
Energy error $\Delta E(t)$: time histories for various values of sampling time $T_0$ and $\delta = 1$. 

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**Table 4. Steady-state relative control energy error $\delta E$ (in percents) vs. sampling time $T_0$ for $\delta = 1$.**
5.4 Control Signal Transmission Over Binary Communication Channel with Time-invariant First Order Coder

5.4.1 Method Description In this Section, the control signal transmission is considered over the digital communication channel subject to a limiting data transmission rate. It should be noted that performance limitations under constraints, imposed by a finite capacity of communication channel, are well-recognized from the control literature, see [Nair et al., 2007; Andrievsky et al., 2010b; Matveev and Savkin, 2009; Matveev and Pogromsky, 2016] and the references therein.

To begin with, consider the following memoryless (static) binary quantizer for a certain signal $y$:

$$q(y, M) = M \text{sign}(y),$$  (16)

where $\text{sign}(\cdot)$ is the signum function. Notice that given binary quantizer, each codeword symbol contains one bit of information. This quantizer is a part of the coder with a memory [Nair and Evans, 2003; Liberzon, 2003; Tatikonda and Mitter, 2004; Nair et al., 2007].

Let signal $u(t)$ should be transmitted over the limited capacity communication channel to the actuator. For the first-order coder the predicted value $\hat{u}[k + 1] \equiv \hat{u}((k + 1)T_0)$ is taken equal to $\hat{u}[k] \equiv \hat{u}(kT_0)$, see [Tatikonda and Mitter, 2004; Fradkov et al., 2006]. In the coders with a memory, the sequence of centroids $c[k]$, $k = 0, 1, \ldots$ with initial condition $c[0] = 0$ is employed. At step $k$ the coder compares the current measured output $u[k] \equiv u(kT_0)$ with value $c[k]$, forming the deviation signal $\partial u[k] = u[k] - c[k]$. Then this signal is discretized with a given quantizer range $M$ according to (16). Binary signal

$$s[k] = \text{sign} \partial u[k]$$  (17)

is transmitted at step $k$ over the communication channel to the decoder. Then centroid $c[k + 1]$ is renewed by the following update algorithm:

$$c[k + 1] = c[k] + q(\partial u[k], M), \quad c[0] = 0.$$  (18)

Equations (16)–(18) describe the coder algorithm. The same algorithm is realized by the decoder. Namely, the decoder calculates variable $\hat{c}[k]$ based on received codeword flow similarly to $s[k]$ and on the value of $M$ which is known on the receiver side.

As obtained in [Andrievsky et al., 2018], in order to avoid the failure of the received signal $\hat{u}(t)$ to track the transmitted signal $u(t)$, the following inequality

$$M = \alpha L_u T_0$$  (19)

should be valid for some $\alpha > 1$, where $L_u = \sup_{t \geq 0} |\dot{u}(t)|$ is the growth rate of the transmitted signal $u(t)$. In this case, the upper bound of the transmission error $\sup_{t \geq 0} |u(t) - \hat{u}(t)|$ does not exceed $(1 + \alpha)L_u T_0$.

5.4.2 Simulation Results for Signal Transmission with Time-invariant First Order Coder The closed-loop system is simulated for various values sampling period $T_0$. It is worth mentioning that for a binary quantizer (16) the communication channel bit-rate $R$ (in bits per a time unit) is defined as $R = 1/T_0$. This relation makes it possible to calculate the demanded communication channel capacity.

For the present study $L_u = 30$ has been experimentally found by the simulation of the “ideal” control system (1), (2), (6), (12) with the given initial conditions (13). Parameter $\alpha$ in (19) is set to $\alpha = 1.05$.

The simulation results are shown in Fig. 5, where time histories of the energy error $\Delta E(t)$ are depicted for the binary time-invariant coder with various sampling time $T_0$.

Since the transients do not tend to the steady-state values, the averaged limiting values are used to evaluate the system accuracy. The results are summarized in Tab. 5, where the dependence of the steady-state averaged relative energy error $\delta E$ on the time sampling interval $T_0$ and data transmission rate $R = 1/T_0$ (in bits per time unit) is given. These results allow one to conclude that under a data rate less than $2.0$ bits per time unit for every control channel $i = 1, \ldots, m$ lead to practically acceptable energy error.

The data transmission scheme described in this section relies upon the Lipschitz constant $L_u$ of $u(t)$. Since the average rate of change for $u(t)$ may vary on time (for example, it differs during the transients and in the steady-state mode, as demonstrated in [Orlov et al., 2017b; Orlov et al., 2017a; Orlov et al., 2018; Orlov et al., 2019]), calculation of $M$ through this constant by means of (19) may be too conservative, reducing
the data transmission accuracy. Therefore, when the transient is over, the coder saturation state and consequent failure of the signal transmission may be avoided for less values of $M$, than are given by (19). Potentially, reducing $M$ on $t$ makes it possible to ensure the asymptotical stabilization of the system. This approach is usually realized in the so-called *zooming* strategy [Brockett and Liberzon, 2000; Liberzon, 2003; Tatikonda and Mitter, 2004; Fradkov *et al.*, 2006; Nair *et al.*, 2007]. The values of $M[k]$ may be precomputed (the *time-based zooming*), or, alternatively, current quantized measurements may be used at each step to update $M[k]$ (the *event-based zooming*). Time-based zooming is described and evaluated in Sec. 5.5. The event-based zooming, in the form of the *adaptive coding* is considered in Sec. 5.6.

### 5.5 Control Signal Transmission Over the Binary Communication Channel with First Order Coder and Time-based Zooming

In *time-varying quantizers* [Brockett and Liberzon, 2000; Liberzon, 2003; Tatikonda and Mitter, 2004; Fradkov *et al.*, 2006; Nair *et al.*, 2007] the range $M$ is updated with time and different values of $M$ are used at each step, $M = M[k]$. Using such a “zooming” strategy it is possible to increase coder accuracy in the steady-state mode and at the same time, to prevent coder saturation at the beginning of the process. To this end, the following recursive algorithm for the quantizer range $M[k]$ may be employed

$$M[k] = (M_0 - M_{\infty}) \rho ^k + M_{\infty}, \; k = 0, 1, \ldots , (20)$$

where $0 < \rho \leq 1$ is the decay parameter, $M_{\infty}$ stands for the limit value of $M[k]$, $M_0$ is the initial value of $M$. Equations (16), (17), (18), (20) describe the coder algorithm. The similar algorithm is realized by the decoder.

#### 5.5.1 Simulation Results for Signal Transmission with First Order Coder and Time-based Zooming

As above, the closed-loop system is simulated for various values sampling period $T_0$. The coder parameters are taken as follows $M_0 = \alpha L_0 T_0$, $M_{\infty} = 0.05M_0$.

#### Figure 6. Control error $\Delta E(t)$; time histories for the binary coder with time-based zooming with various sampling time $T_0$; $s = 0.6$.

#### Figure 7. Energy error $\Delta E(t)$; time histories for the binary coder with time-based zooming with various sampling time $T_0$; $s = 0.15, M_{\infty} = 0$.

$\rho = \exp(-sT_0)$, where $s = 0.6$. The energy error $\Delta E(t)$ time histories for the binary coder with time-based zooming with various sampling time $T_0$ are depicted in Fig. 6. One may notice that for the considered system an asymptotic stabilization has not been achieved due to non-zero value of $M_{\infty}$. The case of $M_{\infty} = 0$ and $s = 0.15$ is depicted in Fig. 7. The simulation results in the form of dependence of the steady-state averaged relative control error $\delta E$ on time sampling interval $T_0$ and data transmission rate $R = 1/T_0$ (in bits per unit time) are summarized in Tabs. 5, 6.

### 5.6 Control Signal Transmission Over the Binary Communication Channel with Adaptive First-order Coder

The following *adaptive coding* procedure may be employed. It assumes adaptively changing quantizer range $M[k]$ depending on the current measurements, cf. [Goodman and Gersho, 1974; Andrievsky *et al.*, 2010a; Gomez-Estern *et al.*, 2011; Goodwin *et al.*, 2012; Andrievsky, 2013]. Range $M[k]$ in (16)–(18) is updated adaptively by means of the following proce-

| $T_0$ | $R$ | $\delta E, \%$ |
|------|-----|-------------|
| 0.01 | 100 | 0.0026 0.0577 2.57 |
| 0.20 | 5.0 | 0.802 4.17 30.5 |

Table 5. Steady-state averaged relative energy error $\delta E$ vs. time sampling interval $T_0$ and data transmission rate $R$. 

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**Note:** The table content and the text are extracted from the given document. The specific values and calculations are not repeated here for brevity.
Table 6. Steady-state averaged relative energy error $\delta E$ vs. time sampling interval $T_0$ and data transmission rate $R$ for the coder with a time-based zooming; $s = 0.6, M_\infty = M_0/20$.

| $T_0$ | 0.01 | 0.05 | 0.10 |
|-------|------|------|------|
| $R$   | 100  | 20.0 | 10.0 |
| $\delta E, \%$ | 0.0273 | 0.2149 | 3.85 |

Table 7. Dependence of the steady-state averaged relative control error $\delta E$ on time sampling interval $T_0$ and data transmission rate $R$ for the coder with a time-based zooming; $s = 0.6, M_\infty = 0$.

| $T_0$ | 0.01 | 0.05 | 0.10 |
|-------|------|------|------|
| $R$   | 100  | 20.0 | 10.0 |
| $\delta E, \%$ | $< 4.57 \cdot 10^{-4}$ | 0.021 | 0.906 |

Figure 8. Energy error $\Delta E(t)$: time histories for the binary adaptive coder with $M_\infty = 0$ and various sampling time $T_0$; $s = 0.6$.

The closed-loop system is simulated for various values of the sampling period $T_0$. The coder parameters are taken as follows $M_0 = \alpha L_u T_0$, $M_\infty = 0$, $\rho = \exp(-s T_0)$, where $s = 0.6$. The time histories of the energy error $\Delta E(t)$ are depicted in Fig. 8. An impression of $M[k]$ adjustment along with the control action time history for $T_0 = 0.1$ may be obtained from the plots, shown in Fig. 9. The simulation results show that the adaptive coding procedure ensures the smooth transients with a relatively high steady-state accuracy.

5.6.1 Simulation Results for Signal Transmission with First Order Adaptive Coder

The closed-loop system is simulated for various values of the sampling period $T_0$. The coder parameters are taken as follows $M_0 = \alpha L_u T_0$, $M_\infty = 0$, $\rho = \exp(-s T_0)$, where $s = 0.6$. The time histories of the energy error $\Delta E(t)$ are depicted in Fig. 8. An impression of $M[k]$ adjustment along with the control action time history for $T_0 = 0.1$ may be obtained from the plots, shown in Fig. 9. The simulation results show that the adaptive coding procedure ensures the smooth transients with a relatively high steady-state accuracy.

In Tab. 8, the dependence of the steady-state averaged relative control error $\delta E$ on the time sampling interval $T_0$ and data transmission rate $R$ for the binary adaptive coder with $M_\infty = 0$ and various sampling time $T_0$; $s = 0.15$.

| $T_0$ | 0.01 | 0.05 | 0.10 |
|-------|------|------|------|
| $R$   | 100  | 20.0 | 10.0 |
| $\delta E, \%$ | $< 1.44 \cdot 10^{-4}$ | 0.0343 | 0.257 |

In Tab. 8, the dependence of the steady-state averaged relative control error $\delta E$ on the time sampling interval $T_0$ and data transmission rate $R = 1/T_0$ (in bits per time unit) is presented for the adaptive coder. The time histories of $M[k]$ and control action $u(t)$ are
shown in Fig. 11 for the system with the adaptive coder and $s = 0.15, T_0 = 0.1$.

6 Conclusions

In the paper, an impression on the limitations of the energy control for the sine-Gordon chains by means of the spatially-discretized control with time and level quantization is obtained. The speed-gradient-based state feedback energy control, developed in [Orlov et al., 2018; Orlov et al., 2019], is numerically evaluated for a sine-Gordon spatially distributed system. The closed-loop system robustness is investigated under various types of disturbances, caused by: sampling-in-time control signal quantization, the level quantization/sampling parameters is numerically evaluated on level jointly with time sampling; control signal transmission over the binary communication channel with time-invariant first order coder; control signal transmission over the binary communication channel with first order coder and time-based zooming; control signal transmission over the binary communication channel with adaptive first-order coder. The results obtained demonstrate robustness of the examined regulation algorithm with respect to the considered disturbances. The steady-state error dependence on the quantization/sampling parameters is numerically evaluated by the simulations. Particularly, it is shown that the most critical role is played by the time sampling, which may lead to loss of the system stability once a certain threshold is overcome.

In the future it is planned to expand this study on the output-feedback energy control systems of [Orlov et al., 2019] where both the control and measurement signals may be affected by the quantization, and to evaluation of non-discontinuous type speed gradient control algorithms. Numerical study of robustness in the presence of quantization disturbances and extension to non-collocated actuation and sensing of other nonlinear distributed parameter models, e.g., Klein-Gordon PDE and chains [Dolgopolik et al., 2018; Kovaleva, 2016] are among challenges to be tackled within the adopted framework.

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