THE NEUMANN PROBLEM FOR A CLASS OF MIXED COMPLEX HESSIAN EQUATIONS

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Abstract. In this paper, we consider the Neumann problem of a class of mixed complex Hessian equations
\[ \sigma_k(\partial\bar{\partial} u) = \sum_{l=0}^{k-1} \alpha_l(z) \sigma_l(\partial\bar{\partial} u) \] with \( \alpha_l \) positive and \( 2 \leq k \leq n \), and establish the global \( C^1 \) estimates and reduce the global second derivative estimate to the estimate of double normal second derivatives on the boundary. In particular, we can prove the global \( C^2 \) estimates and the existence theorems when \( k = n \).

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1. Introduction

In this paper, we consider the Neumann problem for the following mixed complex Hessian equations
\[ \sigma_k(\partial\bar{\partial} u) = \sum_{l=0}^{k-1} \alpha_l(z) \sigma_l(\partial\bar{\partial} u), \quad z \in \Omega \subset \mathbb{C}^n, \] where \( k \geq 2 \), \( \Omega \subset \mathbb{C}^n \) is a bounded domain, \( \partial\bar{\partial} u = \{ \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \}_{1 \leq i, j \leq n} \) is the complex Hessian matrix of the real valued function \( u \), \( \alpha_l(z) > 0 \) in \( \Omega \) with \( l = 0, 1, \cdots, k - 1 \), are given real valued positive functions in \( \Omega \), and for any \( m = 1, \cdots, n \),
\[ \sigma_m(\partial\bar{\partial} u) = \sigma_m(\lambda(\partial\bar{\partial} u)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m}, \] with \( \lambda(\partial\bar{\partial} u) = (\lambda_1, \cdots, \lambda_n) \) being the eigenvalues of \( \partial\bar{\partial} u \). We also set \( \sigma_0 = 1 \). Recall that the Gårding’s cone is defined as
\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}. \]
If $\lambda(\bar{\partial}\bar{\partial}u) \in \Gamma_k$ for any $z \in \Omega$, we say $u$ is a $k$-admissible function.

The equation (1.1) is a general class of mixed Hessian equation. Specially, it is complex Monge-Ampère equation when $k = n$, $\alpha_0(z) > 0$ and $\alpha_1(z) = \cdots = \alpha_{n-1}(z) \equiv 0$, complex $k$-Hessian equation when $\alpha_0(z) > 0$ and $\alpha_1(z) = \cdots = \alpha_{k-1}(z) \equiv 0$, and complex Hessian quotient equation when $\alpha_m(z) > 0$ $(k - 1 \geq m > 0)$ and $\alpha_0(z) = \cdots = \alpha_{m-1}(z) = \alpha_{m+1}(z) = \cdots = \alpha_{k-1}(z) \equiv 0$. This kind of equations is motivated from the study of many important geometric problems. For example, the problem of prescribing convex combination of area measures was proposed in $[29]$, which leads to mixed Hessian equations of the form

$$\sigma_k(\nabla^2 u + uI_n) + \sum_{i=0}^{k-1} \alpha_i \sigma_i(\nabla^2 u + uI_n) = \phi(x), x \in \mathbb{S}^n.$$

The special Lagrangian equation introduced by Harvey-Lawson $[12]$ in the study of calibrated geometries is also a mixed type Hessian equation

$$\text{Im} \det(I_{2n} + \sqrt{-1}D^2 u) = \sum_{k=0}^{\left\lfloor (n-1)/2 \right\rfloor} (-1)^k \sigma_{2k+1}(D^2 u) = 0.$$

Another important example is Fu-Yau equation in $[8, 9]$ arising from the study of the Hull-Strominger system in theoretical physics, which is an equation that can be written as the linear combination of the first and the second elementary symmetric functions

$$\sigma_1(i\bar{\partial}\partial(e^u + \alpha'e^{-u})) + \alpha'\sigma_2(i\bar{\partial}\partial u) = 0.$$

For the Dirichlet problem of elliptic equations in $\mathbb{R}^n$, many results are well known. For example, the Dirichlet problem of the Laplace equation was studied in $[10]$. Caffarelli-Nirenberg-Spruck $[2]$ and Ivochkina $[15]$ solved the Dirichlet problem of the Monge-Ampère equation. Caffarelli-Nirenberg-Spruck $[3]$ solved the Dirichlet problem of the $k$-Hessian equation. For the general Hessian quotient equation, the Dirichlet problem was solved by Trudinger in $[32]$. Also, the Neumann or oblique derivative problem of partial differential equations has been widely studied. For a priori estimates and the existence theorem of Laplace equation with Neumann boundary condition, we refer to the book $[10]$. Also, we can see the recent book written by Lieberman $[23]$ for the Neumann or oblique derivative problem of linear and quasilinear elliptic equations. In 1986, Lions-Trudinger-Urbas solved the Neumann problem of the Monge-Ampère equation in the celebrated paper $[26]$. For related results on
the Neumann or oblique derivative problem for some class of fully nonlinear elliptic equations can be found in Urbas [33] and [34]. For the Neumann problem of $k$-Hessian equations, Trudinger [31] established the existence theorem when the domain is a ball, and Ma-Qiu [27] and Qiu-Xia [28] solved the strictly convex domain case. D.K. Zhang and the first author [6] solved the Neumann problem of general Hessian quotient equations. Jiang and Trudinger [16, 17, 18], studied the general oblique boundary problem for augmented Hessian equations with some regular conditions and concavity conditions.

Krylov in [19] considered the Dirichlet problem of real case of (1.1), and Guan-Zhang in [11] considered the $(k - 1)$-admissible solution without the sign of $\alpha_{k-1}$ and obtained the global $C^2$ estimates. Recently, the classical Neumann problem of real case of (1.1) is solved by the authors [4].

The recent development of the Neumann boundary problem for real equation is a motivation for us to study the corresponding complex equation.

For the complex Monge-Ampère equation, the Dirichlet problem is solved by Caffarelli-Kohn-Nirenberg-Spruck [1], and the Neumann problem is solved by S.Y. Li [20]. Recently, W. Wei and the first author obtained part results about the Neumann problem of complex Hessian quotient equations in [5].

Naturally, we want to know how about the Neumann problem of the mixed complex Hessian equation (1.1). In this paper, we obtain the existence theorem as follows.

**Theorem 1.1.** Suppose that $\Omega \subset \mathbb{C}^n$ is a $C^4$ strictly convex domain, $2 \leq k \leq n$, $\nu$ is the outer unit normal vector of $\partial \Omega$, $\alpha_l(z) \in C^2(\overline{\Omega})$ with $l = 0, 1, \cdots, k - 1$ are positive real valued functions and $\varphi \in C^3(\partial \Omega)$ is a real valued function. Moreover, if $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is the $k$-admissible solution of

\[
\left\{ \begin{array}{l}
\sigma_k(\partial \bar{\partial} u) = \sum_{l=0}^{k-1} \alpha_l(z) \sigma_l(\partial \bar{\partial} u), \quad \text{in} \quad \Omega, \\
D_\nu u = -\varepsilon u + \varphi(z), \quad \text{on} \quad \partial \Omega,
\end{array} \right.
\]

for small $\varepsilon > 0$. Then we have

\[
\sup_{\Omega} |u| \leq C_0, \quad \sup_{\Omega} |Du| \leq C_1,
\]

and

\[
\sup_{\Omega} |D^2 u| \leq C_2(1 + \max_{\partial \Omega} |D_{\nu\nu} u|),
\]
where $C_0$ depends on $n, k, \varepsilon$, $\text{diam}(\Omega)$, $\max |\varphi|$ and $\sum_{l=0}^{k-1} \sup_{\partial \Omega} \alpha_l$; $C_1$ depends on $n, k, \Omega$, $|\varphi|_{C^2}$, $\inf_{\Omega} \alpha_l$ and $|\alpha_l|_{C^1}$; and $C_2$ depends on $n, k, \Omega$, $|\varphi|_{C^3}$, $\inf_{\Omega} \alpha_l$ and $|\alpha_l|_{C^2}$.

Remark 1.2. In this paper, we always denote $z = (z_1, \ldots, z_n) \in \overline{\Omega}$, $z_j = t_j + \sqrt{-1}t_{n+j}$, $t = (t_1, \ldots, t_n, t_{n+1}, \ldots, t_{2n})$;

$$\partial_j u = \frac{\partial u}{\partial z_j} = u_{z_j}, \quad \partial_{\overline{\tau}} u = \frac{\partial u}{\partial z_j} = u_{\overline{\tau}}, \quad \partial u = (\partial_1 u, \ldots, \partial_n u).$$

$$D_k u = \frac{\partial u}{\partial t_k}, \quad D u = (D_1 u, \ldots, D_{2n} u),$$

where $\sqrt{-1}$ is the imaginary unit. It is easy to see

$$\partial_j u = \frac{1}{2} [D_j u - \sqrt{-1}D_{n+j} u], \quad |\partial u|^2 = \langle \partial u, \partial u \rangle = \sum_{j=1}^{n} \partial_j u \overline{\partial_j u} = \frac{1}{4} |Du|^2,$$

$$\partial_{\overline{\tau}} u = \frac{1}{4} [D_{jj} u + D_{(n+j)(n+j)} u].$$

In particular, for $k = n$, we can obtain the estimate of double normal second derivatives, and obtain the existence theorem as follows.

**Theorem 1.3.** Suppose that $\Omega \subset \mathbb{C}^n$ is a $C^4$ strictly convex domain, $k = n$, $\nu$ is the outer unit normal vector of $\partial \Omega$, $\alpha_l(z) \in C^2(\overline{\Omega})$ with $l = 0, 1, \ldots, n-1$ are positive real valued functions and $\varphi \in C^3(\partial \Omega)$ is a real valued function. Then there exists a unique plurisubharmonic solution $u \in C^{3,\alpha}(\Omega)$ for the Neumann problem of mixed complex Hessian equations (1.2) with $k = n$.

Also, we can obtain the existence theorem for the corresponding classical Neumann problem of mixed complex Hessian equation with $k = n$.

**Theorem 1.4.** Suppose that $\Omega \subset \mathbb{C}^n$ is a $C^4$ strictly convex domain, $k = n$, $\nu$ is the outer unit normal vector of $\partial \Omega$, $\alpha_l(z) \in C^2(\overline{\Omega})$ with $l = 0, 1, \ldots, n-1$ are positive real valued functions and $\varphi \in C^3(\partial \Omega)$ is a real valued function. Then there exists a unique constant $c$, such that the Neumann problem of mixed complex Hessian equations

$$\begin{cases}
\sigma_n(\partial \overline{\partial} u) = \sum_{l=0}^{n-1} \alpha_l(z) \sigma_l(\partial \overline{\partial} u), & \text{in } \Omega, \\
D_\nu u = c + \varphi(z), & \text{on } \partial \Omega,
\end{cases}$$

(1.3)

has plurisubharmonic solutions $u \in C^{3,\alpha}(\overline{\Omega})$, which are unique up to a constant.
The rest of this paper is organized as follows. In Section 2, collect some properties of the elementary symmetric function $\sigma_k$, and establish some key lemmas. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.3 and 1.4.

2. Preliminaries

In this section, we give some basic properties of elementary symmetric functions, which could be found in [22], and establish some key lemmas.

2.1. Basic properties of elementary symmetric functions. First, we denote by $\sigma_m(\lambda|i)$ the symmetric function with $\lambda_i = 0$ and $\sigma_m(\lambda|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

**Proposition 2.1.** Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $m = 1, \cdots, n$, then

$$
\sigma_m(\lambda) = \sigma_m(\lambda|i) + \lambda_i \sigma_{m-1}(\lambda|i), \quad \forall \ 1 \leq i \leq n,
$$
$$
\sum_i \lambda_i \sigma_{m-1}(\lambda|i) = m \sigma_m(\lambda),
$$
$$
\sum_i \sigma_m(\lambda|i) = (n - m) \sigma_m(\lambda).
$$

We also denote by $\sigma_m(W|i)$ the symmetric function with $W$ deleting the $i$-row and $i$-column and $\sigma_m(W|ij)$ the symmetric function with $W$ deleting the $i, j$-rows and $i, j$-columns. Then we have the following identities.

**Proposition 2.2.** Suppose $W = (W_{ij})$ is diagonal, and $m$ is a positive integer, then

$$
\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} 
\sigma_{m-1}(W|i), & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
$$

Recall that the Gårding’s cone is defined as

$$(2.1) \quad \Gamma_m = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq m \}.$$

**Proposition 2.3.** Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_m$ and $m \in \{1, 2, \cdots, n\}$. Suppose that

$$
\lambda_1 \geq \cdots \geq \lambda_m \geq \cdots \geq \lambda_n,
$$

Then...
then we have

\begin{align}
\sigma_{m-1}(\lambda|n) &\geq \sigma_{m-1}(\lambda|n - 1) \geq \cdots \geq \sigma_{m-1}(\lambda|m) \geq \cdots \geq \sigma_{m-1}(\lambda|1) > 0; \\
\lambda_1 &\geq \cdots \geq \lambda_m > 0, \quad \sigma_m(\lambda) \leq C_n^m \lambda_1 \cdots \lambda_m; \\
\lambda_1 &\sigma_{m-1}(\lambda|1) \geq m(n - m) \sigma_{m-1}(\lambda); \\
\sigma_{m-1}(\lambda|m) &\geq c(n, m) \sigma_{m-1}(\lambda);
\end{align}

where $C_n^m = \frac{n!}{m!(n-m)!}$.

Proof. All the properties are well known. For example, see [22] or [14] for a proof of (2.2), [21] for (2.3), [7] or [13] for (2.4), and [25] for (2.5). □

The generalized Newton-MacLaurin inequality is as follows, which will be used all the time.

**Proposition 2.4.** For $\lambda \in \Gamma_m$ and $m > l \geq 0$, $r > s \geq 0$, $m \geq r$, $l \geq s$, we have

\[
\left[ \frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l} \right]^{\frac{1}{r-l}} \leq \left[ \frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s} \right]^{\frac{1}{r-s}}.
\]

Proof. See [30]. □

**2.2. Key Lemmas.** In the establishment of a priori estimates, the following inequalities and properties play an important role.

For the convenience of notations, we will denote

\begin{align}
G_k(\partial \bar{\partial}u) := \frac{\sigma_k(\partial \bar{\partial}u)}{\sigma_{k-1}(\partial \bar{\partial}u)}, & \quad G_l(\partial \bar{\partial}u) := -\frac{\sigma_l(\partial \bar{\partial}u)}{\sigma_{k-1}(\partial \bar{\partial}u)}, \quad 0 \leq l \leq k - 2, \\
G(\partial \bar{\partial}u, z) &:= G_k(\partial \bar{\partial}u) + \sum_{l=0}^{k-2} \alpha_l(z) G_l(\partial \bar{\partial}u),
\end{align}

and

\begin{align}
G^{ij} := \frac{\partial G}{\partial u_{ij}}, & \quad 1 \leq i, j \leq n.
\end{align}

**Lemma 2.5.** If $u$ is a $C^2$ function with $\lambda(\partial \bar{\partial}u) \in \Gamma_k$, and $\alpha_l(z)$ ($0 \leq l \leq k - 2$) are positive, then the operator $G$ is elliptic and concave.

Proof. The proof is similar with the real case in [11]. □
Lemma 2.6. If $u$ is a $k$-admissible solution of (1.1), and $\alpha_l(z)$ $(0 \leq l \leq k - 1)$ are positive, then

\begin{align*}
0 < \frac{\sigma_l(\partial \bar{\partial} u)}{\sigma_{k-1}(\partial \bar{\partial} u)} & \leq C(n, k, \inf_{\Omega} \alpha_l), \; 0 \leq l \leq k - 2; \\
0 < \inf_{\Omega} \alpha_k & \leq \frac{\sigma_k(\partial \bar{\partial} u)}{\sigma_{k-1}(\partial \bar{\partial} u)} \leq C(n, k, \sum_{l=0}^{k-1} \sup_{\Omega} \alpha_l). \tag{2.9}
\end{align*}

Proof. The left hand sides of (2.9) and (2.10) are easy to prove. In the following, we prove the right hand sides.

Firstly, if $\frac{\sigma_k}{\sigma_{k-1}} \leq 1$, then we get from the equation (1.1)

$$\alpha_l \leq \frac{\sigma_k}{\sigma_{k-1}} \leq 1, \; 0 \leq l \leq k - 2.$$  

Secondly, if $\frac{\sigma_k}{\sigma_{k-1}} > 1$, i.e. $\frac{\sigma_{k-1}}{\sigma_k} < 1$. We can get for $0 \leq l \leq k - 2$ by the Newton-MacLaurin inequality,

$$\frac{\sigma_l}{\sigma_{k-1}} \leq \frac{(C_n^k)^{k-1-l}C_n^l}{(C_n^{k-1})^{k-l}} \leq C(n, k),$$

and

$$\frac{\sigma_k}{\sigma_{k-1}} = \sum_{l=0}^{k-1} \alpha_l \leq C(n, k) \sum_{l=0}^{k-1} \sup_{\Omega} \alpha_l.$$ 

\[\Box\]

Lemma 2.7. If $u$ is a $k$-admissible solution of (1.1), and $\alpha_l(z)$ $(0 \leq l \leq k - 1)$ are positive, then

\begin{align*}
\frac{n-k+1}{k} & \leq \sum G^{\bar{i}j} < n - k - 1; \tag{2.11} \\
\inf_{\Omega} \alpha_{k-1} & \leq \sum G^{\bar{i}j} u_{\bar{i}j} \leq C(n, k, \sum_{l=0}^{k-1} \sup_{\Omega} \alpha_l). \tag{2.12}
\end{align*}

Proof. By direct computations, we can get

$$\sum G^{\bar{i}j} \geq \sum \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_{k-1}}{\sigma_k} \right) = \sum \frac{\sigma_{k-1} \lambda | i \rangle \sigma_{k-1} - \sigma_k \sigma_{k-2} \lambda | i \rangle}{\sigma_{k-1}^2} \sigma_{k-1}^2

= \frac{(n-k+1)\sigma_{k-1}^2 - (n-k+2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2}

\geq \frac{n-k+1}{k},$$ \tag{2.13}
and
\[ \sum G^\bar{a}_i = \sum \frac{\partial (\frac{\sigma_k}{\sigma_{k-1}})}{\partial \lambda_i} - \sum_{l=0}^{k-2} \alpha_l \sum_i \frac{\partial (\frac{\sigma_l}{\sigma_{k-1}})}{\partial \lambda_i} \]
\[ = \sum \frac{\sigma_{k-1}(\lambda|i)\sigma_{k-1} - \sigma_k \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2} - \sum_{l=0}^{k-2} \alpha_l \sum_i \frac{\sigma_{l-1}(\lambda|i)\sigma_{k-1} - \sigma_l \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2} \]
\[ = \frac{(n-k+1)\sigma_{k-1}^2 - (n-k+2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2} \]
\[ + \sum_{l=0}^{k-2} \frac{(n-k+2)\sigma_l \sigma_{k-2} - (n-l+1)\sigma_{l-1} \sigma_{k-1}}{\sigma_{k-1}^2} \]
\[ \leq (n-k+1) - \frac{(n-k+2)\sigma_{k-2}}{\sigma_{k-1}} \left( \frac{\sigma_k}{\sigma_{k-1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} \right) \]
(2.14) \[ < n-k+1, \]
hence (2.11) holds. Also, we can get
\[ \sum G^\bar{a}_i u_{ij} = \sum \frac{\partial (\frac{\sigma_k}{\sigma_{k-1}})}{\partial \lambda_i} \lambda_i - \sum_{l=0}^{k-2} \alpha_l \sum_i \frac{\partial (\frac{\sigma_l}{\sigma_{k-1}})}{\partial \lambda_i} \lambda_i \]
\[ = \frac{\sigma_k}{\sigma_{k-1}} + \sum_{l=0}^{k-2} (k-1-l)\alpha_l \frac{\sigma_l}{\sigma_{k-1}} \]
(2.15) \[ = \alpha_{k-1} + \sum_{l=0}^{k-2} (k-l)\alpha_l \frac{\sigma_l}{\sigma_{k-1}}, \]
hence (2.12) holds.

\[ \square \]

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1

3.1. $C^0$ estimate. The $C^0$ estimate is easy. For completeness, we produce a proof here following the idea of Lions-Trudinger-Urbas [26].

Theorem 3.1. Suppose $\Omega \subset \mathbb{C}^n$ is a $C^1$ bounded domain, $\alpha_l(z) \in C^0(\overline{\Omega})$ with $l = 0, 1, \cdots, k-1$ are positive functions and $\varphi \in C^0(\partial \Omega)$, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is the
\( k \)-admissible solution of the equation (1.2) with \( \varepsilon \in (0, 1) \), then we have

\[
\sup_{\Omega} |\varepsilon u| \leq M_0,
\]

where \( M_0 \) depends on \( n, k, \text{diam}(\Omega), \max_{\partial \Omega} |\varphi| \) and \( \sum_{l=0}^{k-1} \sup_{\Omega} \alpha_l \).

**Proof.** Firstly, since \( u \) is subharmonic, the maximum of \( u \) is attained at some boundary point \( z_0 \in \partial \Omega \). Then we can get

\[
0 \leq D_\nu u(z_0) = -\varepsilon u(z_0) + \varphi(z_0).
\]

Hence

\[
\max_{\Omega}(\varepsilon u) = \varepsilon u(z_0) \leq \varphi(z_0) \leq \max_{\partial \Omega} |\varphi|.
\]

For a fixed point \( z_1 \in \Omega \), and a positive constant \( A \) large enough, we have

\[
G(\partial \bar{\partial}(A|z - z_1|^2), z) = 2A \frac{C_k}{C_{n-k-1}} - \sum_{l=0}^{k-2} \alpha_l(2A)^{-(k-1-l)} \frac{C_l}{C_{n-k-1}} \geq \sup_{\Omega} \alpha_{k-1} \geq \alpha_{k-1}(z) = G(\partial \bar{\partial}u, z).
\]

By the comparison principle, we know \( u - A|z - z_1|^2 \) attains its minimum at some boundary point \( z_2 \in \partial \Omega \). Then

\[
0 \geq D_\nu(u - A|z - z_1|^2)|_{z=z_2} = D_\nu u(z_2) - AD_\nu(|z - z_1|^2)|_{z=z_2} \geq -\varepsilon u(z_2) - \max_{\partial \Omega} |\varphi| - 2A \text{diam}(\Omega).
\]

Hence

\[
\min_{\Omega}(\varepsilon u) \geq \varepsilon \min_{\Omega}(u - A|z - z_1|^2) \geq \varepsilon u(z_2) - A|z_2 - z_1|^2 \geq -\max_{\partial \Omega} |\varphi| - 2A \text{diam}(\Omega) - A \text{diam}(\Omega)^2.
\]

\[\square\]

### 3.2. Global Gradient estimate

In this subsection, we prove the global gradient estimate (independent of \( \varepsilon \)), using a similar argument of complex Monge-Ampère equation in Li [20].

**Theorem 3.2.** Suppose \( \Omega \subset \mathbb{C}^n \) is a \( C^3 \) strictly convex domain, \( \alpha_l(z) \in C^1(\overline{\Omega}) \) with \( l = 0, 1, \ldots, k - 1 \) are positive functions and \( \varphi \in C^2(\partial \Omega) \), and \( u \in C^3(\Omega) \cap C^2(\overline{\Omega}) \) is
the $k$-admissible solution of the equation (3.8) with $\varepsilon > 0$ sufficiently small, then we have

$$
(3.7) \quad \sup_{\Omega} |Du| \leq M_1,
$$

where $M_1$ depends on $n$, $k$, $\Omega$, $|\varphi|_{C^2}$, $\inf_{\Omega} \alpha_l$ and $|\alpha_l|_{C^1}$.

Proof. In order to prove (3.8), it suffices to prove the following.

$$
(3.9) \quad W(z, \xi) = D_\xi u(z) - \langle \nu, \xi \rangle (-\varepsilon u + \varphi(z)) + \varepsilon^2 u^2 + K|z|^2,
$$

where $K$ is a large constant to be determined later, and $\nu$ is a $C^2(\partial \Omega)$ extension of the outer unit normal vector field on $\partial \Omega$.

Assume $W$ achieves its maximum at $(z_0, \xi_0) \in \overline{\Omega} \times S^{2n-1}$. It is easy to see $D_{\xi_0} u(z_0) > 0$. We claim $z_0 \in \partial \Omega$. Otherwise, if $z_0 \in \Omega$, we shall get a contradiction in the following.

Firstly, we rotate the coordinates such that $\partial \bar{\varphi}(z_0)$ is diagonal. It is easy to see $\{G^i_j\}$ is diagonal. For fixed $\xi = \xi_0$, $W(z, \xi_0)$ achieves its maximum at the same point $z_0 \in \Omega$ and we can easily get at $z_0$,

$$
0 \geq G^i_j \partial_i W = G^i_j \left[ \partial_i D_{\xi_0} u - \langle \nu, \xi_0 \rangle \xi_i - \langle \nu, \xi_0 \rangle (-\varepsilon u + \varphi_i) - \langle \nu, \xi_0 \rangle (-\varepsilon u_i + \varphi_i) + 2\varepsilon^2 u_i u_i + 2\varepsilon^2 u \xi_i + K \right]
$$

$$
= D_{\xi_0} \alpha_{k-1} + \sum_{l=0}^{k-2} D_{\xi_0} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} + G^i_j \left[ 2\varepsilon^2 u_i u_i + \langle \nu, \xi_0 \rangle \sigma_i \varepsilon u_i + \langle \nu, \xi_0 \rangle \varepsilon u_i \right]
$$

$$
+ G^i_j \left[ \varepsilon \langle \nu, \xi_0 \rangle + 2\varepsilon^2 u \right]
$$

$$
+ G^i_j \left[ K - \langle \nu, \xi_0 \rangle \xi_i (-\varepsilon u + \varphi) - \langle \nu, \xi_0 \rangle \xi_i \varphi - \langle \nu, \xi_0 \rangle \xi_i \varphi_i - \langle \nu, \xi_0 \rangle \xi_i \varphi_i \right]
$$

$$
\geq - |D\alpha_{k-1}| - \sum_{l=0}^{k-2} |D\alpha_l| C(n, k, \inf \alpha_l) - (n - k + 1)|D\langle \nu, \xi_0 \rangle|^2
$$

$$
- C(n, k, \sum \sup \alpha_l)[1 + 2M_0]
$$

$$
+ \frac{n - k + 1}{k} [K - |D^2\langle \nu, \xi_0 \rangle|(M_0 + |\varphi|) - |D^2 \varphi| - 2|D\langle \nu, \xi_0 \rangle||D\varphi|] > 0,
$$

$$
(3.10)
$$
where $K$ is large enough, depending only on $n$, $k$, $\Omega$, $M_0$, $|\varphi|_{C^2}$ and $\alpha$. This is a contradiction.

So $z_0 \in \partial \Omega$. Then we continue our proof into the following three cases.

(a) If $\xi_0$ is normal at $z_0 \in \partial \Omega$, then

$$W(z_0, \xi_0) = \varepsilon^2 u^2 + K|z_0|^2 \leq C.$$ 

Then we can easily obtain (3.8).

(b) If $\xi_0$ is non-tangential at $z_0 \in \partial \Omega$, then we can write $\xi_0 = \alpha \tau + \beta \nu$, where $\tau \in S^{2n-1}$ is tangential at $x_0$, that is $\langle \tau, \nu \rangle = 0$, $\alpha = \langle \xi_0, \tau \rangle > 0$, $\beta = \langle \xi_0, \nu \rangle < 1$, and $\alpha^2 + \beta^2 = 1$. Then we have

$$W(z_0, \xi_0) = \alpha D_\tau u + \varepsilon^2 u^2 + K|z_0|^2 \leq \alpha W(z_0, \xi_0) + (1 - \alpha)(\varepsilon^2 u^2 + K|z_0|^2),$$

so

$$W(z_0, \xi_0) \leq \varepsilon^2 u^2 + K|z_0|^2 \leq C.$$ 

Then we can easily get (3.8).

(c) If $\xi_0$ is tangential at $x_0 \in \partial \Omega$, we may assume that the outer normal direction of $\Omega$ at $z_0$ is $(0, \ldots, 0, 1)$. By a rotation, we assume that $\xi_0 = (1, \ldots, 0) = e_1$. Then we have

$$0 \leq D_\nu W(z_0, \xi_0)$$

$$= D_\nu D_1 u - D_\nu \langle \nu, \xi_0 \rangle (-\varepsilon u + \varphi) + 2u \cdot D_\nu u + KD_\nu|z_0|^2$$

$$\leq D_\nu D_1 u + C_1$$

$$= D_1 D_\nu u - D_1 \nu_k D_k u + C_1.$$ 

By the boundary condition, we know

$$D_1 D_\nu u = D_1 (-\varepsilon u + \varphi) \leq D_1 \varphi.$$ 

Following the argument of [20], we can get

$$-D_1 \nu_k D_k u \leq -\kappa_{\min} W(z_0, \xi_0) + C_2,$$

where $\kappa_{\min}$ is the minimum principal curvature of $\partial \Omega$. So

$$W(z_0, \xi_0) \leq \frac{C_1 + |D\varphi| + C_2}{\kappa_{\min}}.$$ 

Then we can conclude (3.8).
3.3. Reduce global second derivatives to double normal second derivatives on the boundary. In this subsection, we reduce global second derivatives to double normal second derivatives on the boundary, following the ideas of Lions-Trudinger-Urbas [26] and Li [20].

**Theorem 3.3.** Suppose $\Omega \subset \mathbb{C}^n$ is a $C^4$ strictly convex domain, $\alpha_l(z) \in C^2(\overline{\Omega})$ with $l = 0, 1, \cdots, k - 1$ are positive functions and $\varphi \in C^3(\partial \Omega)$, and $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is the $k$-admissible solution of the equation (1.2) with $\varepsilon > 0$ sufficiently small, then we have

$$\sup_{\Omega} |D^2u| \leq C(1 + \max_{\partial \Omega} |D_{\nu\nu}u|),$$  
where $C$ depends on $n, k, \Omega, |\varphi|_{C^3}, \inf_{\Omega} \alpha_l$ and $|\alpha_l|_{C^2}$.

**Proof.** Since $u$ is subharmonic, by the argument in [20], we know that we only need to prove that

$$D_{\zeta\zeta}u(z) \leq C(1 + \max_{\partial \Omega} |D_{\nu\nu}u|), \quad \forall (z, \zeta) \in \Omega \times \mathbb{S}^{2n-1}.$$  

As the real case in [26], we use the auxiliary function

$$Q(z, \zeta) = D_{\zeta\zeta}u - v(z, \zeta) + |Du|^2 + K|z|^2,$$

where $v(z, \zeta) = 2(\zeta, \nu)\langle \zeta', D\varphi - \varepsilon Du - D_muD\nu^m \rangle = a_mD_mu + b, \nu = (\nu^1, \nu^2, \cdots, \nu^{2n}) \in \mathbb{S}^{2n-1}$ is a $C^3(\overline{\Omega})$ extension of the outer unit normal vector field on $\partial \Omega$, $\zeta' = \zeta - (\zeta, \nu)\nu, \pm a = -2(\zeta, \nu)\langle \zeta', D\nu^m \rangle - 2\varepsilon(\zeta, \nu)(\zeta')^m, b = 2(\zeta, \nu)\langle \zeta', D\varphi \rangle$, and $K > 0$ is to be determined later.
For any \( z \in \Omega \), we rotate the coordinates such that \( \partial \bar{\partial} u(z) \) is diagonal, and then \( \{ G^{ij} \} \) is diagonal. For any fixed \( \zeta \in S^{2n-1} \), we have

\[
G^{ii} \partial_{ii} Q = G^{ii} \left[ \partial_{ii} D_{\zeta \zeta} u - \partial_{ii} a_m D_m u - a_m \partial_{ii} D_m u - \partial_{ii} a_m \partial_l D_m u - \partial \bar{\partial} D_{\zeta \zeta} u + 2 \partial_i D_m u \partial_l D_m u + 2 D_m u \partial_{ii} D_m u + K \right] \\
= D_{\zeta \zeta} \alpha_{k-1} - 2 \sum_{l=0}^{k-2} D_{\zeta l} \alpha_l G_l^{ii} D_{\zeta} \partial_{ii} u - \sum_{l=0}^{k-2} D_{\zeta \zeta} \alpha_l G_l - G^{ij,rs} D_{\zeta} \partial_{ij} u D_{\zeta} \partial_{rs} u \\
+ G^{ii} [K - \partial_{ii} a_m D_m u - \partial_{ii} b] + G^{ii} [2 \partial_i D_m u \partial_l D_m u - \partial_{ii} a_m \partial_l D_m u - \partial_{ii} a_m \partial_l D_m u] \\
+ (-a_m + 2 D_m u) [D_{\zeta \zeta} \alpha_{k-1} - \sum_{l=0}^{k-2} D_m \alpha_l G_l] \\
\geq G^{ii} [K - C_2] - C_1 - 2 \sum_{l=0}^{k-2} D_{\zeta l} \alpha_l G_l^{ii} D_{\zeta} \partial_{ii} u - G^{ij,rs} D_{\zeta} \partial_{ij} u D_{\zeta} \partial_{rs} u \\
(3.20) \\
\geq \frac{n - k + 1}{k} [K - C_2] - C_1 - C_3 > 0,
\]

where \( K \) is large enough, and we used the fact

\[
-2 \sum_{l=0}^{k-2} D_{\zeta l} \alpha_l G_l^{ii} D_{\zeta} \partial_{ii} u - G^{ij,rs} D_{\zeta} \partial_{ij} u D_{\zeta} \partial_{rs} u \\
\geq -2 \sum_{l=0}^{k-2} D_{\zeta l} \alpha_l G_l^{ii} D_{\zeta} \partial_{ii} u - \sum_{l=0}^{k-2} \alpha_l G_l^{ij,rs} D_{\zeta} \partial_{ij} u D_{\zeta} \partial_{rs} u \\
= -2 \sum_{l=0}^{k-2} D_{\zeta l} \alpha_l G_l^{ii} D_{\zeta} \partial_{ii} u \\
- \sum_{l=0}^{k-2} \alpha_l \left[ \frac{k - l - 1}{\sigma_{k-1} \sigma_l} \sigma_{k-1} \sigma_l \frac{1}{\sigma_{k-1} \sigma_l} \right] D_{\zeta} \partial_{ij} u D_{\zeta} \partial_{rs} u \\
\geq \sum_{l=0}^{k-2} \frac{k - l}{k - l} \frac{(D_{\zeta \alpha_l})^2}{\alpha_l} G_l \\
\geq - C_3.
\]

So \( \max_{\Omega} Q(z, \zeta) \) attains its maximum on \( \partial \Omega \). Hence \( \max_{\Omega \times S^{2n-1}} Q(z, \zeta) \) attains its maximum at some point \( z_0 \in \partial \Omega \) and some direction \( \zeta_0 \in S^{2n-1} \).
Then we continue our proof in the following two cases following the idea of [20].

(a) If $\zeta_0$ is non-tangential at $z_0 \in \partial \Omega$.

Then we can write $\zeta_0 = \alpha \tau + \beta \nu$, where $\tau \in S^{2n-1}$ is tangential at $z_0$, that is $\langle \tau, \nu \rangle = 0$, $\alpha = \langle \zeta_0, \tau \rangle$, $\beta = \langle \zeta_0, \nu \rangle \neq 0$, and $\alpha^2 + \beta^2 = 1$. Then we have

$$D_{\zeta_0 \zeta_0} u(z_0) = \alpha^2 D_{\tau \tau} u(z_0) + \beta^2 D_{\nu \nu} u(z_0) + 2\alpha \beta D_{\tau \nu} u(z_0)$$

$$= \alpha^2 D_{\tau \tau} u(z_0) + \beta^2 D_{\nu \nu} u(z_0) + 2(\xi_0 \cdot \nu)[\zeta_0 - (\xi_0 \cdot \nu)\nu][D\varphi - \varepsilon Du - D_m u D^m u],$$

hence

$$(3.21) \quad Q(z_0, \zeta_0) = \alpha^2 Q(z_0, \tau) + \beta^2 Q(z_0, \nu).$$

From the definition of $Q(z_0, \zeta_0)$, we know

$$(3.22) \quad Q(z_0, \zeta_0) \leq Q(z_0, \nu) \leq C(1 + \max_{\partial \Omega} |D_{\nu \nu} u|),$$

and we can prove (3.18).

(b) If $\zeta_0$ is tangential at $z_0 \in \partial \Omega$.

Then we have by Hopf Lemma

$$0 \leq D_\nu Q(z_0, \zeta_0) = D_\nu D_{\zeta_0 \zeta_0} u - D_\nu a_m D_m u - a_m D_\nu D_m u$$

$$- D_\nu b + 2D_m u D_\nu D_m u + K D_\nu |\zeta|^2$$

$$(3.23) \quad \leq D_\nu D_{\zeta_0 \zeta_0} u + [2D_m u - a_m] D_\nu D_m u + C_3.$$

By the boundary condition, we know

$$D_\nu D_{\zeta_0 \zeta_0} u = D_{\zeta_0 \zeta_0} D_\nu u - (D_{\zeta_0 \zeta_0} \nu^m) D_m u - 2(D_{\zeta_0 \zeta_0} \nu^m) D_{\zeta_0} D_m u$$

$$= D_{\zeta_0 \zeta_0} (-\varepsilon \nu + \varphi) - (D_{\zeta_0 \zeta_0} \nu^m) D_m u - 2(D_{\zeta_0 \zeta_0} \nu^m) D_{\zeta_0} D_m u$$

$$(3.24) \quad \leq -\varepsilon Q(z_0, \zeta_0) + C_4 - 2(D_{\zeta_0} \nu^m) D_{\zeta_0} D_m u.$$

Following the argument of [20], we can get

$$|D_\nu D_m u| \leq C_5(1 + \max_{\partial \Omega} |D_{\nu \nu} u|),$$

$$- 2(D_{\zeta_0} \nu^m) D_{\zeta_0} D_m u \leq -2\kappa_{\min} Q(z_0, \zeta_0) + C_6(1 + \max_{\partial \Omega} |D_{\nu \nu} u|).$$

So

$$(3.25) \quad Q(z_0, \zeta_0) \leq C_3 + C_4 + (2|D u| + |a_m|)C_5 + C_6 \frac{1 + \max_{\partial \Omega} |D_{\nu \nu} u|}{2\kappa_{\min} + \varepsilon}.$$

Then we can easily get (3.18).

The proof is finished. \qed
4. Proof of Theorem 1.3 and Theorem 1.4

In this section, we prove Theorem 1.3 and Theorem 1.4.

4.1. Estimate of double normal second derivatives on boundary for \( k = n \).

**Theorem 4.1.** Suppose \( \Omega \subset \mathbb{C}^n \) is a \( C^4 \) strictly convex domain, \( k = n \), \( \alpha_l(z) \in C^2(\Omega) \) with \( l = 0, 1, \cdots, n-1 \) are positive functions and \( \varphi \in C^3(\partial \Omega) \), and \( u \in C^4(\Omega) \cap C^3(\overline{\Omega}) \) is the plurisubharmonic solution of the equation (1.2) with \( \varepsilon > 0 \) sufficiently small, then we have

\[
\max_{\partial \Omega} |D_{\nu\nu}u| \leq M_2, 
\]

where \( M_2 \) depends on \( n, \Omega, |\varphi|_{C^3}, \inf_{\Omega} \alpha_l \) and \( |\alpha_l|_{C^2} \).

**Proof.** Since \( \Omega \) is a \( C^4 \) strictly convex domain, there is a strictly plurisubharmonic defining function \( r \in C^4(\Omega) \) such that

\[
|Dr| = 1, \quad \text{on} \ \partial \Omega, 
\]

\[
\partial \overline{\partial} r \geq k_0 I_n, \quad \text{in} \ \overline{\Omega}; 
\]

where \( k_0 \) is a positive constant depending only on \( \Omega \), and \( I_n \) is the \( n \times n \) identity matrix.

Let \( z_0 \in \partial \Omega \) be an arbitrary point. By a shift and a rotation of the coordinates \( \{z_1, \cdots, z_n\} \), we can assume that \( z_0 = 0 \), \( \partial z_i r(0) = 0 \) for \( i < n \), and \( D_{t_n} r(0) = -1 \), \( D_{z_n} r(0) = 0 \). In \( \overline{B}(0, \delta) \bigcap \overline{\Omega} \), a sufficiently small neighborhood of \( z_0 \), we can get by the Taylor expansion of \( r \) up to second order

\[
r(z) = -Re(z_n - \sum_{i,j=1}^{n} a_{ij} z_i z_j) + \sum_{i,j=1}^{n} b_{ij} z_i \overline{z_j} + O(|z|^3),
\]

where \( \{b_{ij}\} = \partial \overline{\partial} r(0) \) is positive definite. We now introduce new coordinates \( z' = \psi(z) \) of the form

\[
z'_i = z_i, \quad \text{for} \ i < n; \quad z'_n = z_n - \sum_{i,j=1}^{n} a_{ij} z_i z_j.
\]

In \( \psi(\overline{B}(0, \delta) \bigcap \overline{\Omega}) \), we have

\[
r(z)|_{z = \psi^{-1}(z')} = -Re z'_n + \sum_{i,j=1}^{n} b_{ij} z'_i \overline{z'_j} + O(|z'|^3).
\]
Denote

\[ r_0(z') = -Re z'_n + \sum_{ij=1}^{n} b_{ij} \overline{z'_i z'_j}; \]

\[ B_j(z') = \left( \sum_{i=1}^{n} [a_{ij} + a_{ji}] z_i \right)_{z'=\psi^{-1}(z')}, \quad j = 1, \cdots, n; \]

\[ A_m(z') = B_m(1 - B_n)^{-1} \frac{\partial r_0(z')}{\partial z'_n}, \quad m = 1, \cdots, n - 1. \]

It is easy to know \(|B_j| = O(|z'|)\) for \(j = 1, \cdots, n\), and \(A_j\) is holomorphic in \(z' \in \psi(B(0, \delta) \cap \Omega)\). Following the calculations in [20], we know the Neumann boundary condition in \(z'\) coordinates

\[ 4Re \left( < \partial_{z'} u, \partial_{z'} r_0 > - \sum_{m=1}^{n-1} A_m \partial_{\overline{z'_m}} u \right) = \phi(z', u) + O(|z'|^2) \] (4.7)

where \(\phi(z', u) = |1 - B_n(z')|^{-2}(-\varepsilon u + \varphi(z))|_{z=\psi^{-1}(z')}\).

Following the idea of [20], we choose the auxiliary function

\[ h(z') = 4Re[< \partial_{z'} u, \partial_{z'} r_0 > - \sum_{m=1}^{n-1} A_m \partial_{\overline{z'_m}} u] - \phi(z', u) + K r(z)|_{z=\psi^{-1}(z')} - K_1 Re(z'_n), \] (4.8)

where \(K_1 > 0\) is sufficiently large such that

\[ h < 0, \quad \text{on} \quad \psi(\partial(B(0, \delta) \cap \Omega) \setminus \partial \Omega), \] (4.9)

and

\[ h = -K_1 Re(z'_n) + O(|z'|^2) \leq 0 \quad \text{on} \quad \psi(\partial(B(0, \delta) \cap \Omega) \cap \partial \Omega). \] (4.10)

Let

\[ G^{ij} = \frac{\partial G}{\partial u_{z'_i \overline{z'_j}}}, \quad F^{ij} = \frac{\partial G}{\partial u_{\overline{z'_i} z'_j}}. \]

It is easy to see

\[ F^{ij} = G^{pq}(\frac{\partial z'_i}{\partial z'_p})(\frac{\partial z'_j}{\partial z'_q}). \]
For any $z' \in \psi(B(0, \delta) \cap \Omega)$, we can get

$$F^{\bar{i}} \partial_{z'_j} h = 2F^{\bar{i}} \partial_{z'_j} \left( \partial_{z_m} u \overline{z_m} r_0 + \partial_{\overline{z_m}} u \partial_{z_m} r_0 \right) - 2F^{\bar{i}} \partial_{z'_j} \left( \sum_{m=1}^{n-1} A_m \partial_{z_m} u + \overline{A_m} \partial_{\overline{z_m}} u \right)$$

$$- F^{\bar{i}} \partial_{z'_j} \phi + K F^{\bar{i}} \partial_{z'_j} r$$

$$G^{\bar{p}} \partial_{z_p z_q} h = 2G^{\bar{p}} \left( \partial_{z_m} u \overline{z_m} \partial_{z_p z_q} r_0 + \partial_{\overline{z_m}} u \partial_{z_p z_q} \partial_{z_m} r_0 + u \overline{z_m} \partial_{\overline{z_m}} \partial_{z_p z_q} r_0 + u \partial_{\overline{z_m}} \partial_{z_p z_q} \partial_{z_m} r_0 \right)$$

$$- 2 \sum_{m=1}^{n-1} G^{\bar{p}} \left( A_m \partial_{z_m} u \overline{z_m} + \overline{A_m} \partial_{\overline{z_m}} u + \partial_{z_m} A_m u \overline{z_m} + \overline{\partial_{z_m} A_m u \overline{z_m}} \right)$$

$$+ \partial_{z_p} A_m \partial_{z_m} u \overline{z_m} + \overline{\partial_{z_p} A_m u \overline{z_m}}$$

$$- C_7 + K k_0 \sum_{p=1}^{n} G^{\bar{p}}$$

$$\geq K k_0 \frac{n - k + 1}{k} - C_8,$$

(4.11)

where $G^{\bar{p}} u_{z_p z_q} = G^{\bar{p}} u_{z_m = z_q} \partial_{z_m}$ and $G^{\bar{p}} u_{z_p z_q} = G^{\bar{p}} u_{z_p z_q} \overline{\partial_{z_m}}$ are bounded by rotating the coordinates $\{z_1, \cdots, z_n\}$ such that $\partial_{z_j} \overline{\partial_{z_j}} u$ is diagonal.

Taking $K$ large enough, we can have $G^{\bar{p}} \partial_{z'_j} h \geq 0$ in $\psi(B(0, \delta) \cap \Omega)$. By the maximum principle, we know $h(z')$ achieves its maximum at $\psi(B(0, \delta) \cap \Omega)$, and by (4.9) and (4.10) the maximum is attained at $z' = 0$. Hence $h(z')|_{z' = \psi(z)}$ achieves its maximum at $z_0 = 0$. Thus

(4.12) $0 \leq D_\nu h(0) \leq D_{\nu\nu} u(z_0) + C_9.$

So we have $D_{\nu\nu} u(z_0) \geq -C_9$.

The same argument for

$$\tilde{h}(z') = 4Re[< \partial_{z'} u, \partial_{z'} r_0 > - \sum_{m=1}^{n-1} A_m \partial_{z_m} u] - \phi(z', u) - K r(z)|_{z = \psi^{-1}(z')} + K_1 \Re(z'_n)$$

can give

(4.13) $D_{\nu\nu} u(z_0) \leq C_{10}.$

This completes the estimates of the double normal derivative on the boundary. □
4.2. **Proof of Theorem 1.3.** For $k = n$ and $\varepsilon > 0$ sufficiently small, we have established the global $C^2$ estimates for the plurisubharmonic solution of the Neumann problem of mixed complex Hessian equation (1.2) in Section 3 and Subsection 4.1. By the global $C^2$ a priori estimates, we obtain that the equation (1.2) are uniformly elliptic in $\Omega$. Due to the concavity of the operator $G$, we can get the global Hölder estimates of second derivative following the discussions in [24], that is, we can get

$$\|u\|_{C^2,\alpha(\Omega)} \leq C,$$

where $C$ and $\alpha$ depend on $n$, $\Omega$, $\varepsilon$, $\inf \alpha_l$, $|\alpha_l|_{C^2}$ and $|\varphi|_{C^3}$. Then the $C^{3,\alpha}(\Omega)$ estimates hold by differentiating the equation (1.2) and applying the Schauder theory for linear, uniformly elliptic equations.

Applying the method of continuity (see [10], Theorem 17.28), we can show the existence of the plurisubharmonic solution, and the solution is unique by Hopf lemma. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the high order regularity.

4.3. **Proof of Theorem 1.4.** By the argument in Subsection 4.2, we know there exists a unique plurisubharmonic solution $u^\varepsilon \in C^{3,\alpha}(\Omega)$ to (1.2) for any small $\varepsilon > 0$. Let $v^\varepsilon = u^\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} u^\varepsilon$, and it is easy to know $v^\varepsilon$ satisfies

$$\sigma_n(\bar{\partial}\bar{\partial} v^\varepsilon) = \sum_{l=0}^{n-1} \alpha_l \sigma_l(\bar{\partial}\bar{\partial} v^\varepsilon), \quad \text{in } \Omega,$$

$$D_\nu v^\varepsilon = -\varepsilon v^\varepsilon + \frac{1}{|\Omega|} \int_{\Omega} \varepsilon u^\varepsilon + \varphi(x), \quad \text{on } \partial \Omega.$$  

By the global gradient estimate (3.7), it is easy to know $\varepsilon \sup |Du^\varepsilon| \to 0$. Hence there is a constant $c$ and a function $v \in C^2(\Omega)$, such that $-\varepsilon v^\varepsilon \to c$, $-\varepsilon v^\varepsilon \to 0$, $-\frac{1}{|\Omega|} \int_{\Omega} \varepsilon u^\varepsilon \to c$ and $v^\varepsilon \to v$ uniformly in $C^2(\Omega)$ as $\varepsilon \to 0$. It is easy to verify that $v$ is a plurisubharmonic solution of

$$\sigma_n(\bar{\partial}\bar{\partial} v) = \sum_{l=0}^{n-1} \alpha_l \sigma_l(\bar{\partial}\bar{\partial} v), \quad \text{in } \Omega,$$

$$D_\nu v = c + \varphi(x), \quad \text{on } \partial \Omega.$$  

If there is another plurisubharmonic function $v_1 \in C^2(\Omega)$ and another constant $c_1$ such that

$$\sigma_n(\bar{\partial}\bar{\partial} v_1) = \sum_{l=0}^{n-1} \alpha_l \sigma_l(\bar{\partial}\bar{\partial} v_1), \quad \text{in } \Omega,$$

$$D_\nu v_1 = c_1 + \varphi(x), \quad \text{on } \partial \Omega.$$
Applying the maximum principle and Hopf Lemma, we can know \( c = c_1 \) and \( v - v_1 \) is a constant. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the high order regularity.

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