Coverage control for mobile sensing networks

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Abstract—This paper presents control and coordination algorithms for groups of vehicles. The focus is on autonomous vehicle networks performing distributed sensing tasks where each vehicle plays the role of a mobile tunable sensor. The paper proposes gradient descent algorithms for a class of utility functions which encode optimal coverage and sensing policies. The resulting closed-loop behavior is adaptive, distributed, asynchronous, and verifiably correct.

Keywords—Coverage control, distributed and asynchronous algorithms, centroidal Voronoi partitions

I. INTRODUCTION

Mobile sensing networks

The deployment of large groups of autonomous vehicles is rapidly becoming possible because of technological advances in networking and in miniaturization of electro-mechanical systems. In the near future large numbers of robots will coordinate their actions through ad-hoc communication networks and will perform challenging tasks including search and recovery operations, manipulation in hazardous environments, exploration, surveillance, and environmental monitoring for pollution detection and estimation. The potential advantages of employing teams of agents are numerous. For instance, certain tasks are difficult, if not impossible, when performed by a single vehicle agent. Further, a group of vehicles inherently provides robustness to failures of single agents or communication links.

Working prototypes of active sensing networks have already been developed; see [1]. In [3], lauchable miniature mobile robots communicate through a wireless network. The vehicles are equipped with sensors for vibrations, acoustic, magnetic, and IR signals as well as an active network. The vehicles are tuned via a pan-tilt unit. A second system is suggested in [4] under the name of Autonomous Oceanographic Sampling Network; see also [5]. In this case, underwater vehicles are envisioned measuring temperature, currents, and other distributed oceanographic signals. The vehicles communicate via an acoustic local area network and coordinate their motion in response to local sensing information and to evolving global data. This mobile sensing network is meant to provide the ability to sample the environment adaptively in space and time. By identifying evolving temperature and current gradients with higher accuracy and resolution than current static sensors, this technology could lead to the development and validation of improved oceanographic models.

Optimal sensor allocation and coverage problems

A fundamental prototype problem in this paper is that of characterizing and optimizing notions of quality-of-service provided by an adaptive sensor network in a dynamic environment. To this goal, we introduce a notion of sensor coverage that formalizes an optimal sensor placement problem. This spatial resource allocation problem is the subject of a discipline called locational optimization [6], [7], [8], [9].

Locational optimization problems pervade a broad spectrum of scientific disciplines. Biologists rely on locational optimization tools to study how animals share territory and to characterize the behavior of animal groups obeying the following interaction rule: each animal establishes a region of dominance and moves toward its center. Locational optimization problems are spatial resource allocation problems (where to place mailboxes in a city or cache servers on the Internet) and play a central role in quantization and information theory (the design of a minimum-distortion fixed-rate vector quantizer is a locational problem). Other technologies affected by locational optimization include mesh and grid optimization methods, clustering analysis, data compression, and statistical pattern recognition.

Because locational optimization problems are so widely studied, it is not surprising that methods are indeed available to tackle coverage problems; see [1], [2], [3], [4]. However, most currently-available algorithms are not applicable to mobile sensing networks because they inherently assume a centralized computation for a limited size problem in a known static environment. This is not the case in multi-vehicle networks which, instead, rely on a distributed communication and computation architecture. Although an ad-hoc wireless network provides the ability to share some information, no global omniscient leader might be present to coordinate the group. The inherent spatially-distributed nature and limited communication capabilities of a mobile network invalidate classic approaches to algorithm design.

Distributed asynchronous algorithms for coverage control

In this paper we design coordination algorithms implementable by a multi-vehicle network with limited sensing and communication capabilities. Our approach is related to the classic Lloyd algorithm from quantization theory; see [13] for a reprint of the original report and [14] for a historical overview. We present Lloyd descent algorithms
that take into careful consideration all constraints on the mobile sensing network. In particular, we design coverage algorithms that are adaptive, distributed, asynchronous, and verifiably asymptotically correct:

**Adaptive:** Our coverage algorithms provide the network with the ability to address changing environments, sensing task, and network topology (due to agents departures, arrivals, or failures).

**Distributed:** Our coverage algorithms are distributed in the sense that the behavior of each vehicle depends only on the location of its neighbors. Also, our algorithms do not require a fixed-topology communication graph, i.e., the neighborhood relationships do change as the network evolves. The advantages of distributed algorithms are scalability and robustness.

**Asynchronous:** Our coverage algorithms are amenable to asynchronous implementation. This means that the algorithms can be implemented in a network composed of agents evolving at different speeds, with different computation and communication capabilities. Furthermore, our algorithms do not require a global synchronization and convergence properties are preserved even if information about neighboring vehicles propagates with some delay. An advantage of asynchronism is a minimized communication overhead.

**Verifiable Asymptotically Correct:** Our algorithms guarantee monotonic descent of the cost function encoding the sensing task. Asymptotically the evolution of the mobile sensing network is guaranteed to converge to so-called centroidal Voronoi configurations that are critical points of the optimal sensor coverage problem.

Let us describe in some detail what are the contributions of this paper. Section IV reviews certain locational optimization problems and their solutions as centroidal Voronoi partitions. Section V provides a continuous-time version of the classic Lloyd algorithm from vector quantization and applies it to the setting of multi-vehicle networks. In discrete-time, we propose a family of Lloyd algorithms. We carefully characterize convergence properties for both continuous and discrete-time versions (Appendix VII collects some relevant facts on descent flows). We discuss a worst-case optimization problem, we investigate a simple uniform planar setting, and we present numerical results.

Section VI presents two asynchronous distributed implementations of Lloyd algorithm for ad-hoc networks with communication and sensing capabilities. Our treatment carefully accounts for the constraints imposed by the distributed nature of the vehicle network. We present two asynchronous implementations, one based on classic results on distributed gradient flows, the other based on the structure of the coverage problem.

Section V-A considers vehicle models with more realistic dynamics. We present two formal results on passive vehicle dynamics and on vehicles equipped with individual local controllers. We present numerical simulations of passive vehicle models and of unicycle mobile vehicles. Next, Section V-B describes density functions that lead the multi-vehicle network to predetermined geometric patterns.

**Review of distributed algorithms for cooperative control**

Recent years have witnessed a large research effort focused on motion planning and coordination problems for multi-vehicle systems. Issues include geometric patterns [13], [14], formation control [16], [18], [19], [20], [22], and conflict avoidance [22], [23]. Algorithms for robotic sensing tasks are presented for example in [24], [25]. It is only recently, however, that truly distributed coordination laws for dynamic networks are being proposed; e.g., see [26], [27] and the conference versions of this work [28], [29].

Heuristic approaches to the design of interaction rules and emerging behaviors have been thoroughly investigated within the literature on behavior-based robotics; see [30], [31], [32], [33], [34], [35], [36]. An example of coverage control is discussed in [37]. Along this line of research, algorithms have been designed for sophisticated cooperative tasks. However, no formal results are currently available on how to design reactive control laws, ensure their correctness, and guarantee their optimality with respect to an aggregate objective.

The study of distributed algorithms is concerned with providing mathematical models, devising precise specifications for their behavior, and formally proving their correctness and complexity. Via an automata-theoretic approach, the references [38], [39] treat distributed consensus, resource allocation, communication, and data consistency problems. From a numerical optimization viewpoint, the works in [40], [41], [42] discuss distributed asynchronous algorithms as networking algorithms, rate and flow control, and gradient descent flows. Typically, both these sets of references consider networks with fixed topology, and do not address algorithms over ad-hoc dynamically changing networks. Another common assumption is that, any time an agent communicates its location, it broadcasts it to every other agent in the network. In our setting, this would require a non-distributed communication set-up.

II. FROM LOCATION OPTIMIZATION TO CENTROIDAL VORONOI PARTITIONS

A. Locational optimization

In this section we describe a collection of known facts about a meaningful optimization problem. References include the theory and applications of centroidal Voronoi partitions, see [33], and the discipline of facility location, see [43]. Along the paper, we interchangeably refer to the elements of the network as sensors, agents, vehicles, or robots.

Let \( Q \) be a convex polytope in \( \mathbb{R}^N \) and let \( \| \cdot \| \) denote the Euclidean distance function. We call a map \( \phi: Q \rightarrow \mathbb{R}_+ \) a distribution density function if it represents a measure of information or probability that some event take place over \( Q \). In equivalent words, we can consider \( Q \) to be the bounded support of the function \( \phi \). Let \( P = (p_1, \ldots, p_n) \) be the location of \( n \) sensors, each moving in the space \( Q \). Because of noise and loss of resolution, the sensing performance at point \( q \) taken from \( i \)th sensor at the position \( p_i \) degrades with the distance \( \| q - p_i \| \) between \( q \) and \( p_i \);
we describe this degradation with a non-decreasing differentiable function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \). Accordingly, \( f(\|q - p_i\|) \) provides a quantitative assessment of how poor the sensing performance is.

**Remark II.1:** As an example, consider \( n \) mobile robots equipped with microphones attempting to detect, identify, and localize a sound-source. How should we plan to robots’ motion in order to maximize the detection probability? Assuming the source emits a known signal, the optimal detection algorithm is a matched filter (i.e., convolve the known waveform with the received signal and threshold). The source is detected depending on the signal-to-noise-ratio, which is inversely proportional to the distance between the microphone and the source. Various electromagnetic and sound sensors have signal-to-noise ratios inversely proportional to distance.

Within the context of this paper, a partition of \( Q \) is a collection of \( n \) polytopes \( W = \{W_1, \ldots, W_n\} \) with disjoint interiors whose union is \( Q \). We say that two partitions \( W \) and \( W' \) are equal if \( W_i \) and \( W'_i \) only differ by a set of \( \phi \)-measure zero, for all \( i \in \{1, \ldots, n\} \).

We consider the task of minimizing the locational optimization function

\[
\mathcal{H}(P, W) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|) d\phi(q),
\]

where we assume that the \( i \)th sensor is responsible for measurements over its “dominance region” \( W_i \). Note that the function \( \mathcal{H} \) is to be minimized with respect to both (1) the sensors location \( P \), and (2) the assignment of the dominance regions \( W \). This problem is referred to as a facility location problem and in particular as a continuous \( p \)-median problem in \([8]\).

**Remark II.2:** Note that if we interchange the positions of any two agents, along with their associated regions of dominance, the value of the locational optimization function \( \mathcal{H} \) is not affected. To eliminate this discrete redundancy, one could take the discrete group of permutations \( \Sigma_n \) with the natural action on \( Q^n \), and consider \( Q^n/\Sigma_n \) as the configuration space for the position \( P \) of the \( n \) vehicles.

### B. Voronoi partitions

One can easily see that, at fixed sensors location, the optimal partition of \( Q \) is the Voronoi partition \( \mathcal{V}(P) = \{V_1, \ldots, V_n\} \) generated by the points \( (p_1, \ldots, p_n) \):

\[
V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\}.
\]

We refer to \([13]\) for a comprehensive treatment on Voronoi diagrams, and briefly present some relevant concepts. The set of regions \( \{V_1, \ldots, V_n\} \) is called the Voronoi diagram for the generators \( (p_1, \ldots, p_n) \). When the two Voronoi regions \( V_i \) and \( V_j \) are adjacent, \( p_i \) is called a (Voronoi) neighbor of \( p_j \) (and vice-versa). The set of indexes of the Voronoi neighbors of \( p_i \) is denoted by \( \mathcal{N}(i) \). Clearly, \( j \in \mathcal{N}(i) \) if and only if \( i \in \mathcal{N}(j) \). We also define the \((i, j)\)-face as \( \Delta_{ij} = V_i \cap V_j \). Voronoi diagrams can be defined with respect to various distance functions, e.g., the 1-, 2-, \( s \)-, and \( \infty \)-norm over \( Q = \mathbb{R}^m \), and Voronoi diagrams can be defined over Riemannian manifolds; see \([43]\). Some useful facts about the Euclidean setting are the following: if \( Q \) is a convex polytope in a \( N \)-dimensional Euclidean space, the boundary of each \( V_i \) is the union of \((N - 1)\)-dimensional convex polytopes.

In what follows, we shall write

\[
\mathcal{H}_V(P) = \mathcal{H}(P, \mathcal{V}(P)).
\]

Note that

\[
\begin{align*}
\mathcal{H}_V(P) &= \int_{Q} \min_{i \in \{1, \ldots, n\}} f(\|q - p_i\|) d\phi(q), \\
&= E \left[ \min_{i \in \{1, \ldots, n\}} f(\|q - p_i\|) \right],
\end{align*}
\]

that is, the locational optimization function can be interpreted as an expected value composed with a min operation. This is the usual way in which the problem is presented in the facility location and operations research literature \([8], [9]\). Remarkably, one can show \([12]\) that

\[
\frac{\partial \mathcal{H}_V}{\partial p_i}(P) = \frac{\partial \mathcal{H}(P, \mathcal{V}(P))}{\partial p_i} = \int_{V_i} \frac{\partial}{\partial p_i} f(\|q - p_i\|) d\phi(q),
\]

and deduce some smoothness properties of \( \mathcal{H}_V \). Since the Voronoi partition \( \mathcal{V} \) depends at least continuously on \( P = (p_1, \ldots, p_n) \), the function \( \mathcal{H}_V \) is at least continuously differentiable.

### C. Centroidal Voronoi partitions

Let us recall some basic quantities associated to a region \( V \subset \mathbb{R}^N \) and a mass density function \( \rho \). The (generalized) mass, centroid (or center of mass), and polar moment of inertia are defined as

\[
M_V = \int_V \rho(q) \, dq, \quad C_V = \frac{1}{M_V} \int_V q \rho(q) \, dq,
\]

\[
J_{V,p} = \int_V \|q - p\|^2 \rho(q) \, dq.
\]

Additionally, by the parallel axis theorem, one can write,

\[
J_{V,p} = J_{V,C_V} + M_V \|p - C_V\|^2
\]

where \( J_{V,C_V} \in \mathbb{R}_+ \) is defined as the polar moment of inertia of the region \( V \) about its centroid.

Let us consider again the locational optimization problem \((\mathcal{V})\), and suppose now we are strictly interested in the setting

\[
\mathcal{H}(P, W) = \sum_{i=1}^{n} \int_{W_i} \|q - p_i\|^2 d\phi(q),
\]

that is, we assume \( f(\|q - p_i\|) = \|q - p_i\|^2 \). Applying the parallel axis theorem leads to simplifications for both the
function $\mathcal{H}_V$ and its partial derivative:

$$\frac{\partial \mathcal{H}_V}{\partial p_i}(P) = 2M_V(p_i - C_V).$$

It is convenient to define $\mathcal{H}_V, 1 = \sum_{i=1}^{n} J_{V_i, C_{V_i}}$ and $\mathcal{H}_V, 2 = \sum_{i=1}^{n} M_V\|p_i - C_V\|^2$.

Therefore, the (not necessarily unique) local minimum points for the location optimization function $\mathcal{H}_V$ are centroidal Voronoi configurations. If this set is finite, then $\mathcal{H}_V, 1(0)$ consists of a finite collection of points, then $P(t)$ converges to one of them, see Corollary VII.2.

Proposition III.3 (Discrete-time Lloyd descent) Let $P_0 \in \mathbb{Q}^n$ denote the initial sensors location. Then, the sequence $\{T^m(P_0)\}_{m \geq 1}$ converges to the set of centroidal Voronoi configurations. If this set is finite, then $\{T^m(P_0)\}_{m \geq 1}$ converges to a centroidal Voronoi configuration.

Proof: Consider $\mathcal{H}_V : \mathbb{Q}^n \rightarrow \mathbb{R}$ as an objective function for the algorithm $T$. Note that

$$\mathcal{H}(P, V(P)) \leq \mathcal{H}(P, W),$$

with strict inequality if $W \neq V(P)$. Moreover, the parallel axis theorem guarantees

$$\mathcal{H}(P', W) \leq \mathcal{H}(P, W),$$

as long as $\|p_i' - C_W\| \leq \|p_i - C_W\|$ for all $i \in \{1, \ldots, n\}$, with strict inequality if for any $i$, $\|p_i' - C_W\| < \|p_i - C_W\|$.

Q. Assuming this set is finite, the sensors location converges to a centroidal Voronoi configuration.

Proof: Under the control law (6), we have

$$\frac{d}{dt} \mathcal{H}_V(P(t)) = \sum_{i=1}^{n} \frac{\partial \mathcal{H}_V}{\partial p_i}(p_i)\dot{p}_i = -2k_{prop} \sum_{i=1}^{n} M_V\|p_i - C_V\|^2 = -2k_{prop} \mathcal{H}_V, 2(P(t)).$$

By LaSalle’s principle, the sensors location converges to the largest invariant set contained in $\mathcal{H}_V, 1(0)$, which is precisely the set of centroidal Voronoi configurations. Since this set is clearly invariant for $H$, we get the stated result. If $\mathcal{H}_V, 1(0)$ consists of a finite collection of points, then $P(t)$ converges to one of them, see Corollary VII.2.

Remark III.2: If $\mathcal{H}_V, 1(0)$ is finite, and $P(t) \rightarrow C$, then a sufficient condition that guarantees exponential convergence is that the Hessian of $\mathcal{H}_V$ be positive definite at $C$. This property is known to be an open problem, see [12]. Note that this gradient descent is not guaranteed to find the global minimum. For example, in the vector quantization and signal processing literature [14], it is known that for bimodal distribution density functions, the solution to the gradient flow reaches local minima where the number of generators allocated to the two region of maxima are not optimally partitioned.
In particular, \( \mathcal{H}(C_W, W) \leq \mathcal{H}(P, W) \), with strict inequality if \( P \neq C_W \), where \( C_W \) denotes the set of centroids of the partition \( W \).

Now, we have

\[
\mathcal{H}_V(T(P)) = \mathcal{H}(T(P), V(T(P))) \leq \mathcal{H}(T(P), V(P)),
\]

because of (8). In addition, because of property (a) of \( T \), inequality (9) yields

\[
\mathcal{H}(T(P), V(P)) \leq \mathcal{H}(P, V(P)) = \mathcal{H}_V(P),
\]

and the inequality is strict if \( P \) is not centroidal by property (b) of \( T \). Hence, \( \mathcal{H}_V \) is a descent function for the algorithm \( T \). The result now follows from the global convergence Theorem VII.3 and Proposition VII.4.

**Remark III.4:** Lloyd algorithm in quantization theory (13), (14) is usually presented as follows: given the location of \( n \) agents, \( p_1, \ldots, p_n \), (i) construct the Voronoi partition corresponding to \( P = (p_1, \ldots, p_n) \); (ii) compute the mass centroids of the Voronoi regions found in step (i). Set the new location of the agents to these centroids; and return to step (i). Lloyd algorithm can also be seen as a fixed point iteration. Consider the mappings \( LL_i : Q^n \to Q \) for \( i \in \{1, \ldots, n\} \)

\[
LL_i(p_1, \ldots, p_n) = \left( \int_{V_i(P)} \phi(q) dq \right)^{-1} \int_{V_i(P)} \phi(q) dq.
\]

Let \( LL : Q^n \to Q^n \) be defined by \( LL = (LL_1, \ldots, LL_n) \). Clearly, \( LL \) is continuous (indeed, \( C^2 \)), and corresponds to Lloyd algorithm. Now, \( \|LL_i(P) - C_V\| = 0 \leq \|p_i - C_V\| \), for all \( i \in \{1, \ldots, n\} \). Moreover, if \( P \) is not centroidal, then the inequality is strict for all \( p_i \neq C_V \). Therefore, \( LL \) verifies properties (a) and (b).

**C. Generalized settings, worst-case design, and the p-center problem**

Different sensor performance functions \( f \) in equation (1) correspond to different optimization problems. Provided one uses the Euclidean distance in the definition of \( \mathcal{H}_V \), the standard Voronoi partition computed with respect to the Euclidean metric remains the optimal partition. For arbitrary \( f \), it is not possible anymore to decompose \( \mathcal{H}_V \) into the sum of terms similar to \( \mathcal{H}_{V,1} \) and \( \mathcal{H}_{V,2} \). Nevertheless, it is still possible to implement the gradient flow via the expression for the partial derivative (3).

**Proposition III.5:** Assume the sensors location obeys a first order dynamical behavior, \( \dot{p}_i = u_i \). Then, for the closed-loop system induced by the gradient law (3), \( u_i = -\partial \mathcal{H}_V / \partial p_i \), the sensors location \( P = (p_1, \ldots, p_n) \) converges asymptotically to the set of critical points of \( \mathcal{H}_V \). Assuming this set is finite, the sensors location converges to a critical point.

More generally, various distance notions can be used to define locational optimization functions. Different performance function gives rise to corresponding notions of “center of a region” (any notion of geometric center, mean, or average is an interesting candidate). These can then be adopted in designing coverage algorithms. We refer to (44) for a discussion on Voronoi partitions based on non-Euclidean distance functions and to (45), (46) for a discussion on the corresponding locational optimization problems.

Next, let us discuss an interesting variation of the original problem. In (45), minimizing the expected minimum distance function \( \mathcal{H}_V \) in equation (2) is referred to as the continuous p-median problem. It is instructive to consider the worst-case minimum distance function, corresponding to the scenario where no information is available on the distribution density function. In other words, the network seeks to minimize the largest possible distance from any point in \( Q \) to any of the sensor locations, i.e., to minimize the function

\[
\max_{q \in Q} \left[ \min_{i \in \{1, \ldots, n\}} \|q - p_i\| \right] = \max_{i \in \{1, \ldots, n\}} \left[ \max_{q \in V_i} \|q - p_i\| \right].
\]

This optimization is referred to as the p-center problem in (8), (85). One can design a strategy for the p-center problem analog to the Lloyd algorithm for the p-median problem: each vehicle moves, in continuous or discrete-time, toward the center of the minimum-radius sphere enclosing the polytope. To the best of our knowledge, no convergence proof is available in the literature for this algorithm; e.g., see (85). We refer to (47) for a convergence analysis of the continuous and discrete time algorithms.

In what follows, we shall restrict our attention to the p-median problem and to centroidal Voronoi partitions.

**D. Computations over polygons with uniform density**

In this section, we investigate closed-form expression for the control laws introduced above. Assume the Voronoi region \( V_i \) is a convex polygon (i.e., a polytope in \( \mathbb{R}^2 \)) with \( N_i \) vertices labeled \( \{(x_0, y_0), \ldots, (x_{N_i - 1}, y_{N_i - 1})\} \) such as in Figure 1. It is convenient to define \((x_i, y_i) = (x_0, y_0)\). Furthermore, we assume that the density function is \( \phi(q) = 1 \). By evaluating the corresponding integrals, one can ob-
tain the following closed-form expressions

\[ M_{V_i} = \frac{1}{2} \sum_{k=0}^{N_i-1} (x_k y_{k+1} - x_{k+1} y_k) \]
\[ C_{V_i,x} = \frac{1}{6M_{V_i}} \sum_{k=0}^{N_i-1} (x_k + x_{k+1})(x_k y_{k+1} - x_{k+1} y_k) \]
\[ C_{V_i,y} = \frac{1}{6M_{V_i}} \sum_{k=0}^{N_i-1} (y_k + y_{k+1})(x_k y_{k+1} - x_{k+1} y_k) . \]

To present a simple formula for the polar moment of inertia, let \( \bar{x}_k = x_k - C_{V_i,x} \) and \( \bar{y}_k = y_k - C_{V_i,y} \), for \( k \in \{0, \ldots, N_i - 1\} \). Then, the polar moment of inertia of a polygon about its centroid, \( J_{V_i,C} \) becomes

\[ J_{V_i,C} = \frac{1}{12} \sum_{k=0}^{N_i-1} (\bar{x}_k \bar{y}_{k+1} - \bar{x}_{k+1} \bar{y}_k) \cdot \]
\[ (\bar{x}_k^2 + \bar{x}_{k+1}^2 + \bar{y}_k^2 + \bar{y}_{k+1}^2 + \bar{y}_{k+1}^2) . \]

The proof of these formulas is based on decomposing the polygon into the union of disjoint triangles. We refer to [17] for analog expressions over \( \mathbb{R}^N \).

A second observation is that the Voronoi polygon’s vertices can be expressed as a function of the neighboring vehicles. The vertexes of the \( i \)th Voronoi polygon which lie in the interior of \( Q \) are the circumcenters of the triangles formed by \( p_i \) and any two neighbors adjacent to \( p_i \). The circumcenter of the triangle determined by \( p_i, p_j, \) and \( p_k \) is

\[ \frac{1}{4M} \left( \|\alpha_{kj}\|^2 (\alpha_{ji} \cdot \alpha_{ik})p_i + \|\alpha_{ik}\|^2 (\alpha_{kj} \cdot \alpha_{ji})p_j + \|\alpha_{ji}\|^2 (\alpha_{ik} \cdot \alpha_{kj})p_k \right), \]

where \( M \) is the area of the triangle, and \( \alpha_{ik} = p_k - p_i \).

Equation (3) for a polygon’s centroid and equation (4) for the Voronoi cell’s vertices lead to a closed-form algebraic expression for the control law in equation (3) as a function of the neighboring vehicles’ location.

### E. Numerical simulations

To illustrate the performance of the continuous-time Lloyd algorithm, we include some simulation results. The algorithm is implemented in Mathematica as a single centralized program. For the \( \mathbb{R}^2 \) setting, the code computes the bounded Voronoi diagram using the Mathematica package ComputationalGeometry, and computes mass, centroid, and polar moment of inertia via the numerical integration routine NIntegrate. Careful attention was paid to numerical accuracy issues in the computation of the Voronoi diagram and in the integration. We illustrate the performance of the closed-loop system in Figure 3.

### IV. Asynchronous distributed implementations

In this section we show how the Lloyd gradient algorithm can be implemented in an asynchronous distributed fashion. In Section IV-A we describe our model for a distributed asynchronous network of robotic agents. Next, we provide two distributed algorithms for the local computation and maintenance of the Voronoi cells. Finally, in Section IV-C we propose two distributed asynchronous implementations of Lloyd algorithm: the first one is based on the gradient optimization algorithms as described in [10] and the second one relies on the special structure of the coverage problem.

#### A. Modeling an asynchronous distributed network of mobile robotic agents

We start by modeling a robotic agent that performs sensing, communication, computation, and control actions. We are interested in the behavior of the asynchronous network resulting from the interaction of finitely many robotic agents. A theoretical framework to formalize the following concepts is that developed in the theory of distributed algorithms; see [18].

Let us here introduce the notion of **robotic agent with computation, communication, and control capabilities** as the \( i \)th element of a network. The \( i \)th agent has a processor with the ability of allocating continuous and discrete states and performing operations on them. Each vehicle has access to its unique identifier \( i \). The \( i \)th agent occupies a location \( p_i \in Q \subset \mathbb{R}^N \) and it is capable of moving in space, at any time \( t \in \mathbb{R}_+ \) for any period of time \( \delta t \in \mathbb{R}_+ \), according to a first order dynamics of the form:

\[ \dot{p}_i(s) = u_i , \quad \|u_i\| \leq 1 , \quad \forall s \in [t, t + \delta t] . \]

The processor has access to the agent’s location \( p_i \) and determines the control pair \( (\delta t, u_i) \). The processor of the \( i \)th agent has access to a local clock \( t_i \in \mathbb{R}_+ \cup \{0\} \), and a scheduling sequence, i.e., an increasing sequence of times \( \{T_{i,k} \in \mathbb{R}_+ \cup \{0\} \mid k \in \mathbb{N} \cup \{0\}\} \) such that \( T_{i,0} = 0 \) and \( t_{i,\min} < T_{i,k+1} - T_{i,k} < t_{i,\max} \). The processor of the \( i \)th agent is capable of transmitting information to any other agent within a closed disk of radius \( R_i \in \mathbb{R}_+ \). We assume the communication radius \( R_i \) to be a quantity controllable by the \( i \)th processor and the corresponding communication bandwidth to be limited.

We shall alternatively consider networks of **robotic agents with computation, sensing, and control capabilities**. In this case, the processor of the \( i \)th agent has the same computation and control capabilities as before. Furthermore, we assume the processor can detect any other agent within a closed disk of radius \( R_i \in \mathbb{R}_+ \). We assume the sensing radius \( R_i \) to be a quantity controllable by the processor.

#### B. Voronoi cell computation and maintenance

A key requirement of the Lloyd algorithms presented in Section III is that each agent must be able to compute its own Voronoi cell. To do so, each agent needs to know the relative location (distance and bearing) of each Voronoi neighbor. The ability of locating neighbors plays a central role in numerous algorithms for localization, media access, routing, and power control in ad-hoc wireless communication networks; e.g., see [18], [19], [54], [51] and references.
provided twice as large as the maximum distance between \( p_i \) and the vertexes of \( W(p_i, R_i) \), all Voronoi neighbors of \( p_i \) are within distance \( R_i \) from \( p_i \) and the equality \( V_i = W(p_i, R_i) \) holds. The minimum adequate sensing radius is therefore \( R_{i, \min} = 2 \max_{q \in W(p_i, R_{i, \min})} \| p_i - q \| \). We are now ready to state the following algorithm.

\[
W(p_i, R_i) = B(p_i, R_i) \cap (\cap_j \{ p_j \mid \| p_i - p_j \| \leq R_i \}, S_{ij}),
\]

(11)

where \( B(p_i, R_i) = \{ q \in Q \mid \| q - p_i \| \leq R_i \} \) and the half planes \( S_{ij} \) are

\[
\{ q \in \mathbb{R}^N \mid 2q \cdot (p_i - p_j) \geq (p_i + p_j) \cdot (p_i - p_j) \}.
\]

Provided \( R_i \) is twice as large as the maximum distance between \( p_i \) and the vertexes of \( W(p_i, R_i) \), all Voronoi neighbors of \( p_i \) are within distance \( R_i \) from \( p_i \) and the equality \( V_i = W(p_i, R_i) \) holds. The minimum adequate sensing radius is therefore \( R_{i, \min} = 2 \max_{q \in W(p_i, R_{i, \min})} \| p_i - q \| \). We are now ready to state the following algorithm.

**Name:** ADJUST SENSING RADIUS ALGORITHM

**Goal:** distributed Voronoi cell

**Requires:** sensor with radius \( R_i \)

Local agent \( i \) performs:

1: initialize \( R_i \), detect vehicles \( p_j \) within radius \( R_i \)
2: update \( P^i(t_i) \), compute \( W(p_i(t_i), R_i) \)
3: while \( R_i < 2 \max_{q \in W(p_i(t_i), R_i)} \| p_i(t_i) - q \| \) do
4: \( R_i := 2 \max_{q \in W(p_i(t_i), R_i)} \| p_i(t_i) - q \| \)
5: detect vehicles \( p_j \) within radius \( R_i \)
6: update \( P^i(t_i) \)
7: compute \( W(p_i(t_i), R_i) \)
8: end while
9: set \( R_i := 2 \max_{q \in W(p_i(t_i), R_i)} \| p_i(t_i) - q \| \)
10: set \( V_i := W(p_i(t_i), R_i) \)

A similar algorithm can be designed for a robotic agent with communication capabilities. The specifications go as in the previous algorithm, except for the fact that steps 2 and 7 are substituted by

send ("request to reply", \( p_i(t_i) \)) within radius \( R_i \)
receive ("response", \( p_j \)) from all agents within radius \( R_i \)

Further, we have to require each agent to perform the following event-driven task: if the \( i \)th agent receives at any time \( t_i \) a “request to reply” message from the \( j \)th agent located at position \( p_j \), it executes

send ("response", \( p_i(t_i) \)) within radius \( \| p_i(t) - p_j \| \)

We call this algorithm ADJUST COMMUNICATION RADIUS ALGORITHM.

Next, we present an algorithm whose objective is to maintain the information about the Voronoi cell of the \( i \)th agent, and detect the presence of certain events. We consider only robotic agents with sensing capabilities. We call an agent active if it is moving and we assume that the \( i \)th agent can determine if any agent within radius \( R_i \) is active or not. Two events are of interest: (i) a Voronoi neighbor of
the \(i\)th agent becomes active and (ii) a new active agent becomes a Voronoi neighbor of the \(i\)th agent. In both cases, we require a trigger message “request recomputation” to an appropriate control algorithm that we shall present in the next section. Before presenting the algorithm, let us introduce the map weight that assigns to the state vector \(P^i \in \mathbb{R}^{N \times n}\) a tuple \((w_1, \ldots, w_n) \in \mathbb{N}^n\) according to

\[
w_j = \begin{cases} 
3 & \text{if } j \in \mathcal{N}(i) \text{ and } j \text{ is active} \\
1 & \text{if } j \in \mathcal{N}(i) \text{ and } j \text{ is not active} \\
0 & \text{if } j \notin \mathcal{N}(i) 
\end{cases}.
\]

The algorithm is designed to run for times \(t_i \in [t_0, t_0 + \delta t]\).

| Name:         | Monitoring algorithm                      |
|---------------|-------------------------------------------|
| Goal:         | Cell maintenance & event detection        |
| Requires:     | (i) sensor with radius \(R_i\)            |
|               | (ii) positive reals \(t_0, \delta t\)     |
|               | (iii) Adjust sensing radius algorithm     |

Local agent \(i\) performs for \(t_i \in [t_0, t_0 + \delta t]\):

1. initialize \(P^i(t_0)\) and \(V_i(t_0)\), set \(w = \text{weight}(P^i(t_0))\)
2. while \(t_i \leq t_0 + \delta t\) do
3. run Adjust sensing radius algorithm
4. if weight, \(P^i(t_i)) \geq w_j + 2\) then
5. send (“request recomputation”)
6. set \(w = \text{weight}(P^i(t_i))\)
7. end if
8. end while

C. Asynchronous distributed implementations of coverage control

Let us now present two versions of Lloyd algorithm for the solution of the optimization problem \([1]\) that can be implemented by an asynchronous distributed network of robotic agents. For simplicity, we assume that at time 0 all clocks are synchronized (although they later can run at different speeds) and that each agent knows at 0 the exact location of every other agent. The first algorithm is designed for robotic agents with communication capabilities, and requires the Adjust communication radius algorithm (while it does not require the Monitoring algorithm).

| Name: Coverage behavior algorithm I |
| Goal: distributed optimal agent location |
| Requires: (i) Voronoi cell computation |
|          (ii) centroid and mass computation |
|          (iii) positive real \(\delta_0\) |
|          (iv) Adjust communication radius algorithm |

For \(i \in \{1, \ldots, n\}\), \(i\)th agent performs at time \(t_i = T_{i,0} = 0\):

0. \(P^i(T_{i,0}) := (p^i_1(T_{i,0}), \ldots, p^i_n(T_{i,0}))\)
0. compute Voronoi region \(V_i(T_{i,0})\)
0. set \(V_i = V_i(T_{i,0}) \text{ and } R_i = 2 \max_{q \in V_i} \|p_i - q\|\)

For \(i \in \{1, \ldots, n\}\), the \(i\)th agent performs at time \(t_i = T_{i,k}\) either one of the following threads or both. For some \(B_i \in \mathbb{N}\), we require that after \(B_i\) steps of the scheduling sequence, each of the threads has been executed at least once.

| Information thread |
| 1: run Adjust communication radius algorithm |
| Control thread |
| 1: compute centroid \(C_{V_i}\) and mass \(M_{V_i}\) of \(V_i\) |
| 2: apply control pair \((\delta_0, M_{V_i}(C_{V_i} - p_i(T_{i,k})))\) |

As a consequence of Theorem 3.1 and Corollary 3.1 in [40], we have the following result.

**Proposition IV.1:** Let \(P_0 \in Q^n\) denote the initial sensors location. Let \(\{T_k\}\) be the sequence in increased order of all the scheduling sequences of the agents of the network. Assume \(\inf_k\{T_k - T_{k-1}\} > 0\). Then, there exists a sufficiently small \(\delta_* > 0\) such that if \(0 < \delta_0 \leq \delta_*\), the Coverage behavior algorithm I converges to the set of critical points of \(H_V\), that is, the set of centroidal Voronoi configurations.

Next, we focus on distributed asynchronous implementations of Lloyd algorithm that take advantage of the special structure of the coverage problem. The following algorithm is designed for robotic agents with sensing capabilities, it requires the Monitoring and the Adjust sensing radius algorithms. Two advantages of this algorithm over the previous one are that there is no need for each agent to exactly go toward the centroid of its Voronoi cell nor to take a small step at each stage.
CORTÉS, MARTÍNEZ, KARATAS AND BULLO: COVERAGE CONTROL FOR MOBILE SENSING NETWORKS

| Name: Coverage behavior algorithm II |
| Goal: distributed optimal agent location |
| Requires: (i) Voronoi cell computation |
| (ii) centroid computation |
| (iii) Monitoring algorithm |

For \( i \in \{1, \ldots, n\} \), \( i \)th agent performs at \( t_i = T_{i,0} = 0 \):

1. \( P_i(T_{i,0}) := \{p_1(T_{i,0}), \ldots, p_n(T_{i,0})\} \)
2. compute Voronoi region \( V_i(T_{i,0}) \)
3. set \( V_i = V_i(T_{i,0}) \) and \( R_i = 2 \max_{q \in V_i} \|p_i - q\| \)

For \( i \in \{1, \ldots, n\} \), \( i \)th agent performs at \( t_i = T_{i,k} \):

1. choose \( 0 < \delta t_i < t_i, \min \)
2. set \( s = T_{i,k}, \) compute centroid \( C_{V_i}(s) \)
3. choose \( u_i \), with \( u_i \cdot (C_{V_i} - p_i(s)) \geq 0 \), with strict inequality if \( p_i(s) \neq C_{V_i} \)
4. while \( t_i \leq T_{i,k} + \delta t_i \) do
5. run Monitoring algorithm for \( (T_{i,k}, \delta t_i) \)
6. while no warning do
7. \( \dot{p}_i = u_i \)
8. end while
9. set \( s = t_i, \) compute centroid \( C_{V_i}(s) \)
10. choose \( u_i \), with \( u_i \cdot (C_{V_i} - p_i(s)) \geq 0 \), with strict inequality if \( p_i(s) \neq C_{V_i} \)
11. end while

Proposition IV.3: For passive systems, the control law (12) achieves asymptotic convergence of the sensors location to the set of centroidal Voronoi configurations. If this set is finite, then the sensors location converges to a centroidal Voronoi configuration.

Proof: Consider the evolution of the function \( \mathcal{E} \),

\[
\frac{d}{dt} \mathcal{E} = \frac{1}{2} k_{prop} \mathcal{H}_V + \sum_{i=1}^{n} S_i \\
\leq k_{prop} M V_i (p_i - C_{V_i}) + \dot{p}_i u_i = -k_{deriv} \sum_{i=1}^{n} \dot{p}_i^2 \leq 0 .
\]

By LaSalle’s principle, the sensors location converges to the largest invariant set contained in \( \{\dot{p}_i = 0\} \). Given the assumption on the zero dynamics, we conclude that \( p_i = C_{V_i} \) for \( i \in \{1, \ldots, n\} \), i.e., the largest invariant set corresponds to the set of centroidal Voronoi configurations. If this set is finite, LaSalle’s principle also guarantees convergence to a specific centroidal Voronoi configuration.

In Figure 3 we illustrate the performance of the control law (12) for vehicles with second-order dynamics \( \dot{p}_i = u_i \).

Fig. 3

Coverage control for 32 vehicles with second order dynamics. The environment and Gaussian density function are as in Figure 2. The control gains are \( k_{prop} = 6 \) and \( k_{deriv} = 1 \).

A. Variations on vehicle dynamics

Here, we consider vehicles systems described by more general linear and nonlinear dynamical models.

Coordination of vehicles with passive dynamics. We start by considering the extension of the control design to nonlinear control systems whose dynamics is passive. Relevant examples include networks of vehicles and robots with general Lagrangian dynamics, as well as spatially invariant passive linear systems. Specifically, assume that for each \( i \in \{1, \ldots, n\} \), the \( i \)th vehicle state includes the spatial variable \( p_i \), and that the vehicle’s dynamics is passive with input \( u_i \), output \( \dot{p}_i \) and storage function \( S_i : Q \rightarrow \mathbb{R}_+ \). Furthermore, assume that the input preserving the zero dynamics manifold \( \{\dot{p}_i = 0\} \) is \( u_i = 0 \).

For such systems, we devise a proportional derivative (PD) control via,

\[
u_i = -k_{prop} M V_i (p_i - C_{V_i}) - k_{deriv} \dot{p}_i , \tag{12}\]

where \( k_{prop} \) and \( k_{deriv} \) are scalar positive gains. The closed-loop system induced by this control law can be analyzed with the Lyapunov function

\[
\mathcal{E} = \frac{1}{2} k_{prop} \mathcal{H}_V + \sum_{i=1}^{n} S_i ,
\]

yielding the following result.

Proposition V.1: For passive systems, the control law (12) achieves asymptotic convergence of the sensors location to the set of centroidal Voronoi configurations. If this set is finite, then the sensors location converges to a centroidal Voronoi configuration.

Proof: Consider the evolution of the function \( \mathcal{E} \),

\[
\frac{d}{dt} \mathcal{E} = \frac{1}{2} k_{prop} \frac{d}{dt} \mathcal{H}_V + \sum_{i=1}^{n} \dot{S}_i \\
\leq k_{prop} M V_i (p_i - C_{V_i}) + \dot{p}_i u_i = -k_{deriv} \sum_{i=1}^{n} \dot{p}_i^2 \leq 0 .
\]

By LaSalle’s principle, the sensors location converges to the largest invariant set contained in \( \{\dot{p}_i = 0\} \). Given the assumption on the zero dynamics, we conclude that \( p_i = C_{V_i} \) for \( i \in \{1, \ldots, n\} \), i.e., the largest invariant set corresponds to the set of centroidal Voronoi configurations. If this set is finite, LaSalle’s principle also guarantees convergence to a specific centroidal Voronoi configuration.

In Figure 3 we illustrate the performance of the control law (12) for vehicles with second-order dynamics \( \dot{p}_i = u_i \).
dynamics and is endowed with a local feedback and feed-forward controller. The controller is capable of strictly decreasing the distance to any specified position in \( Q \) in a specified period of time \( \delta \).

Assume the dynamics of the \( i \)th vehicle is described by
\[
\dot{x}_i = f_i(t, x_i, u), \quad x_i \in \mathbb{R}^m
\]
denotes its state, and
\[
\pi_i : \mathbb{R}^m \to Q
\]
is such that \( \pi_i(x_i) = p_i \). Assume also that for any \( p_{\text{target}} \in Q \) and any \( x_0 \in \mathbb{R}^m \), there exists \( \pi_i(t, x(t), p_{\text{target}}) \) such that the solution \( t_0 \) of
\[
\pi_i(t, x(t), p_{\text{target}}) = x_0,
\]
verifies \( \| \pi_i(t_0 + \delta) - p_{\text{target}} \| < \| \pi_i(t_0) - p_{\text{target}} \| \).

Proposition V.2: Consider the following coordination algorithm. At time \( t_k = k\delta \), \( k \in \mathbb{N} \), each vehicle computes \( V_i(t_k) \) and \( C_{V_i}(t_k) \); then, for time \( t \in [t_k, t_{k+1}] \), the vehicle executes \( t(x(t), C_{V_i}(t_k)) \). For this closed-loop system, the sensors location converges to the set of centroidal Voronoi configurations. If this set is finite, then the sensors location converges to a centroidal Voronoi configuration.

The proof of this result readily follows from Proposition II.3, since the algorithm verifies properties (a) and (b) of Section II.B.

As an example, we consider a classic model of mobile wheeled dynamics, the unicycle model. Assume the \( i \)th vehicle has configuration \( (\theta_i, x_i, y_i) \in \mathbb{SE}(2) \) evolving according to
\[
\dot{\theta}_i = \omega_i, \quad \dot{x}_i = v_i \cos \theta_i, \quad \dot{y}_i = v_i \sin \theta_i,
\]
where \((\omega_i, v_i)\) are the control inputs for vehicle \( i \). Note that the definition of \((\theta_i, v_i)\) is unique up to the discrete action \((\theta_i, v_i) \rightarrow (\theta_i + \pi, -v_i)\). Given a target point \( p_{\text{target}} \), we use this symmetry to require the equality \((\cos \theta_i, \sin \theta_i) \cdot (p_i - p_{\text{target}}) = 0\) for all time \( t \). Should the equality be violated at some time \( t = t_0 \), we shall redefine \( \theta_i(t_0^+) = \theta_i(t_0^-) + \pi \) and \( v_i \) as \(-v_i\) from time \( t = t_0 \) onwards.

Following the approach in [12], consider the control law
\[
\omega_i = 2k_{\text{prop}} \arctan \left( \frac{-\sin \theta_i, \cos \theta_i}{\cos \theta_i, \sin \theta_i} \right) \cdot (p_i - p_{\text{target}}),
\]
\[
v_i = -k_{\text{prop}} \cos \theta_i, \sin \theta_i \cdot (p_i - p_{\text{target}}),
\]
where \( k_{\text{prop}} \) is a positive gain. This feedback law differs from the original stabilizing strategy in [12] only in the fact that no final angular position is preferred. One can prove that \( p_i = (x_i, y_i) \) is guaranteed to monotonically approach the target position \( p_{\text{target}} \) when run over an infinite time horizon. We illustrate the performance of the proposed algorithm in Figure 4.

B. Geometric patterns and formation control

Here we suggest the use of decentralized coverage algorithms as formation control algorithms, and we present various density functions that lead the multi-vehicle network to predetermined geometric patterns. In particular, we present simple density functions that lead to segments, ellipses, polygons, or uniform distributions inside convex environments.

Consider a planar environment, let \( k \) be a large positive gain, and denote \( q = (x, y) \in Q \subseteq \mathbb{R}^2 \). Let \( a, b, c \) be real numbers, consider the line \( ax + by + c = 0 \), and define the density function
\[
\phi_{\text{line}}(q) = \exp(-k(ax + by + c)^2).
\]

Similarly, let \((x_c, y_c)\) be a reference point in \( \mathbb{R}^2 \), let \( a, b, r \) be positive scalars, consider the ellipse \( a(x - x_c)^2 + b(y - y_c)^2 = r^2 \), and define the density function
\[
\phi_{\text{ellipse}}(q) = \exp(-k(a(x - x_c)^2 + b(y - y_c)^2 - r^2)).
\]

We illustrate this density function in Figure 5. During the simulations, we observed that the convergence to the desired pattern was rather slow.

Finally, define the smooth ramp function \( \text{SR}_\ell(x) = x(\arctan(x/\pi + (1/2))) \), and the density function
\[
\phi_{\text{disk}}(q) = \exp(-k \text{SR}_\ell(a(x - x_c)^2 + b(y - y_c)^2 - r^2)).
\]

This density function leads the multi-vehicle network to obtain a uniform distribution inside the ellipsoidal disk \( a(x - x_c)^2 + b(y - y_c)^2 \leq r^2 \). We illustrate this density function in Figure 6.

![Fig. 5](image1)

Coverage control for 32 vehicles with \( \phi_{\text{ellipse}} \). The parameter values are: \( k = 500, a = 1.4, b = .6, x_c = y_c = 0, r^2 = .3, \) and \( k_{\text{prop}} = 1. \)

![Fig. 6](image2)

Coverage control for 32 vehicles to an ellipsoidal disk. The density function parameters are the same as in Figure 5 and \( \ell = 10, k_{\text{prop}} = 1. \)

It appears straightforward to generalize these types of density functions to the setting of arbitrary curves or shapes. The proposed algorithms are to be contrasted with the classic approach to formation control based on rigidly encoding the desired geometric pattern. One disadvantage of the proposed approach is the requirement for a careful numerical computation of Voronoi diagrams and centroids. We refer to [13] for previous work on algorithms for geometric patterns, and to [14, 15] for formation control algorithms.
VI. Conclusions

We have presented a novel approach to coordination algorithms for multi-vehicle networks. The scheme can be thought of as an interaction law between agents and as such it is implementable in a distributed asynchronous fashion. Numerous extensions appear worth pursuing. We plan to investigate the setting of non-convex environments and non-isotropic sensors. We are currently implementing these algorithms on a network of all-terrain vehicles. Furthermore, we plan to extend the algorithms to provide collision avoidance guarantees and to vehicle dynamics which are not locally controllable.

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In this section we collect some relevant facts on descent flows both in the continuous and in the discrete-time settings. We do this following [53] and [54], respectively. We include Proposition VII.1 as we are unable to locate it in the linear and nonlinear programming literature.

**Continuous-time descent flows**

Consider the differential equation \( \dot{x} = X(x) \), where \( X : D \subset \mathbb{R}^N \rightarrow \mathbb{R}^N \) is locally Lipschitz and \( D \) is an open connected set. A set \( M \) is said to be (positively) invariant with respect to \( X \) if \( x(0) \in M \) implies \( x(t) \in M \), for all \( t \in \mathbb{R} \) (resp. \( t \geq 0 \)). A descent function for \( X \) on \( \Omega \), \( V : D \rightarrow \mathbb{R} \), is a continuously differentiable function such that \( L_X V \leq 0 \) on \( \Omega \). We denote by \( E \) the set of points in \( \Omega \) where \( L_X V = 0 \) and by \( M \) be the largest invariant set contained in \( E \). Finally, the distance from a point \( x \) to a set \( M \) is defined as \( d(x, M) = \inf_{p \in M} \| x - p \| \).

**Lemma VII.1** (LaSalle’s principle) Let \( \Omega \subset D \) be a compact set that is positively invariant with respect to \( X \). Let \( x(0) \in M \) and \( x_t \) be an accumulation point of \( x(t) \). Then \( x_t \in M \) and \( d(x(t), M) \rightarrow 0 \) as \( t \rightarrow \infty \).

The following corollary is Exercise 3.22 in [54].

**Corollary VII.2:** If the set \( M \) is a finite collection of points, then the limit of \( x(t) \) exists and equals one of them.

**Discrete-time descent flows**

Let \( X \) be a subset of \( \mathbb{R}^N \). An algorithm \( T \) is a continuous mapping from \( X \) to \( X \). A point \( x_0 \) is said to be a fixed point of \( T \) if \( T(x_0) = x_0 \). We denote the set of fixed points of \( T \) by \( \Gamma \). A descent function for \( T \) on \( \mathbb{R}_+ \), \( Z : X \rightarrow \mathbb{R}_+ \), is any nonnegative real-valued continuous function satisfying \( Z(T(x)) \leq Z(x) \) for \( x \in C \), where the inequality is strict if \( x \not\in \Gamma \). Typically, \( Z \) is the objective function to be minimized, and \( T \) reflects this goal by yielding a point that reduces (or at least does not increase) \( Z \).

**Lemma VII.3** (Global convergence theorem) Let \( C \subset X \) be a compact set and it is positively invariant with respect to \( T \). Let \( x_m \in C \) and denote \( x_m = T(x_{m-1}) \), \( m \geq 1 \). Let \( x_m \) be an accumulation point of the sequence \( \{ x_m \}_{m \geq 1} \).
Then $x_\ast \in \Gamma$, dist$(x_m, \Gamma) \to 0$ and $Z(x_m) \to Z(x_\ast)$ as $m \to \infty$.

Proposition VII.4: If the set $\Gamma$ is a finite collection of points, then $\{x_m\}$ converges and equals one of them.

Proof: Let $x_\ast$ be an accumulation point of $\{x_m\}$ and assume the whole sequence does not converge to it. Then, there exists an $\epsilon > 0$ such that for all $m_0$, there is a $m' > m_0$ such that $\|x_{m'} - x_\ast\| > \epsilon$. Let $d$ be the minimum of all the distances between the points in $\Gamma$. Fix $\epsilon' = \min\{d/2, \epsilon\}$. Since $T$ is continuous and $\Gamma$ is finite, there exists $\delta > 0$ such that $\|T(x) - z\| < \epsilon'$ (that is, for each $z \in \Gamma$, there exists such $\delta(z)$, and we take the minimum over $\Gamma$).

Now, since dist$(x_m, \Gamma) \to 0$, there exists $m_1$ such that for all $m \geq m_1$, dist$(x_m, \Gamma) < \delta$. Also, we know that there is a subsequence of $\{x_{m_k}\}$ which converges to $x_\ast$, let us denote it by $\{x_{m_{k_1}}\}_{k_1 \geq 1}$. For $\delta$, there exists $m_{k_1}$ such that for all $k_1 \geq m_{k_1}$, we have $\|x_{m_{k_1}} - x_\ast\| < \delta$.

Let $m_0 = \max\{m_1, m_{k_1}\}$. Take $k$ such that $m_k \geq m_0$. Then,

\[
\|x_{m_{k+1}} - x_\ast\| = \|T(x_{m_k}) - x_\ast\| < \epsilon'.
\] (13)

Now we are going to prove that $\|x_{m_{k+1}} - x_\ast\| < \delta$. If $d/2 \leq \delta$, then this claim is straightforward, since $\epsilon' \leq d/2$. If $d/2 > \delta$, suppose that $\|x_{m_{k+1}} - x_\ast\| > \delta$. Since $m_{k+1} \geq m_0 \geq m_1$, then dist$(x_{m_{k+1}}, \Gamma) < \delta$. Therefore, there exists $y \in \Gamma$ such that $\|x_{m_{k+1}} - y\| < \delta$. Necessarily, $y \neq x_\ast$.

Now, by the triangle inequality, $\|x - y\| \leq \|x - x_{m_{k+1}}\| + \|x_{m_{k+1}} - y\|$. Then,

$\|x_{m_{k+1}} - x_\ast\| \geq \|x - y\| - \|x_{m_{k+1}} - y\| \geq d - \delta > d/2$,

which contradicts (13). Therefore, $\|x_{m_{k+1}} - x_\ast\| < \delta$. This argument can be iterated to prove that for all $m \geq m_0$, we have $\|x_m - x_\ast\| < \delta$. Let us take now $m' > m_0$ such that $\|x_{m'} - x_\ast\| > \epsilon$. Since $m'-1 \geq m_0$, we have $\|x_{m'-1} - x_\ast\| < \delta$, and therefore

\[
\|x_{m'} - x_\ast\| = \|T(x_{m'-1}) - x_\ast\| < \epsilon' \leq \epsilon,
\]

which is a contradiction. Therefore, $\{x_m\}$ converges to $x_\ast$. \blacksquare