Gradual partial-collapse theory for ideal nondemolition measurements of qubits in circuit QED

Wei Feng,† Cheng Zhang,‡ Zhong Wang,‡ Lupei Qin,* and Xin-Qi Li†,‡,†

†Center for Joint Quantum Studies and Department of Physics, Tianjin University, Tianjin 300072, China
‡Department of Physics, Beijing Normal University, Beijing 100875, China

(Dated: September 27, 2017)

The conventional method of qubit measurements in circuit QED is employing the dispersive regime of qubit-cavity coupling, which results in an approximated scheme of quantum nondemolition (QND) readout. However, this scheme breaks down owing to the Purcell effect in the case of strong coupling and/or strong measurement drive. To remove the drawbacks of the dispersive readout, a recent proposal by virtue of longitudinal coupling suggests a new scheme to realize fast, high-fidelity and ideal QND readout of qubit state. In the present work, in parallel to the dispersive readout, we carry out a study on the gradual partial-collapse theory for this new scheme, in terms of both the quantum trajectory equation and quantum Bayesian approach. In particular, we construct the qubit-plus-cavity entangled joint state under continuous measurement and present a comprehensive analysis for the quantum efficiency, qubit-state purity, and signal-to-noise ratio in the output currents. Insightfully, the combination of the joint state and the quantum Bayesian rule provides a route to improve the multiple partial-collapse measurements in the presence of qubit rotation. This is particularly relevant to the advantage that the longitudinal coupling promises a convenient method of cavity reset. In combination with the reset procedure, the partial-collapse weak measurement theory is expected to be useful for such as the measurement-based feedback control and other quantum applications.

PACS numbers: 03.67.-a,32.80.Qk,42.50.Lc,42.50.Pq

I. INTRODUCTION

The superconducting circuit quantum electrodynamics (cQED) architecture has been a fascinating platform for quantum information processing and for quantum measurement and control studies [1–13]. For quantum measurement of qubits, which is a central ingredient of many applications, the dispersive coupling regime is often exploited. In this regime, the qubit and microwave cavity are off resonance, and the dispersive coupling regime is often exploited. In this regime, the noisy back-action onto the measured state, the stochastic environment with stochastic measurement records. Despite drive to the cavity will result in qubit-state dependent coher- dependent shift of the cavity frequency. Then, a microwave drive to the cavity will result in qubit-state dependent coherent state of the cavity field and a homodyne detection of the field quadrature can reveal information of the qubit state.

Moreover, a particular interest is the type of continuous weak measurement, i.e., a continuous real-time monitoring of environment with stochastic measurement records. Despite the noisy back-action onto the measured state, the stochastic evolution of the measured state, i.e., the so-called quantum trajectory (QT), can be faithfully tracked [14–15]. Therefore, this type of partial-collapse weak measurements can be useful for quantum feedback [15–16], generating pre- and post-selected quantum ensembles to improve quantum state prepa- ration, smoothing and high-fidelity readout, and developing novel schemes of quantum metrology [16–18].

For continuous weak measurements, the most celebrated theory is the quantum trajectory equation (QTE), as broadly applied in quantum optics and quantum control studies [14–15]. However, in some cases, a larger-step state update scheme known as quantum Bayesian approach can be more ef-

cient, by using the accumulated output currents [13, 24–28]. The Bayesian approach also allows for efficient and analytical studies for interesting problems such as quantum weak values [14–18], quantum state smoothing and quantum metrology [19–23], etc.

For qubit measurements in cQED, as mentioned above, the typical scheme is based on an effective dispersive coupling between the qubit and cavity field. However, the dispersive scheme is approximate, being applicable only for weak drive and weak coupling. Increasing either the measurement drive amplitude \( \epsilon_m \) or the decay rate \( \kappa \) of the cavity photons, the qubit will suffer state flip, with a rate given by \( \gamma_p = (\epsilon_m/\Delta)^2 \kappa \), where \( \Delta \) is the detuning between the qubit energy and cavity frequency. This is the so-called Purcell effect extensively discussed in literature [29–34], which results in the dispersive scheme being not quantum nondemolition (QND). One may notice other schemes of qubit measurements in cQED, e.g., the Josephson-bifurcation-amplifier readout [35–38], and the more recent studies of quantum trajectories of superconducting qubit undergoing homodyne/heterodyne detection of fluorescence [39–42]. However, the former is also non-ideal QND while the latter is simply not.

In a recent work by Blais et al. [13], an ideal-QND readout scheme for qubits in circuit QED was proposed via parametric modulation of the longitudinal coupling between the qubit and cavity field [14–15]. This longitudinal-coupling scheme has more recently employed to study the quantum dynamics of simultaneously measured non-commuting observables of a superconducting qubit [43]. Differing from the (approximated) dispersive scheme, the longitudinal coupling in the new scheme acts as a qubit-state dependent drive on the cav- ity. Then, applying an external microwave drive, the cavity field will evolve also into qubit-state dependent coherent state and a homodyne detection of the field quadrature can reveal the qubit state information. The analysis carried out in Ref.

†Electronic address: wphv@tju.edu.cn
‡Electronic address: lijxmqi@bnu.edu.cn
showed that the new protocol can outperform the dispersive scheme to achieve, for instance, faster, high-fidelity and ideally QND readout. In present work we focus a study on the gradual partial-collapse theory associated with this new measurement scheme which, as for the dispersive readout, should be useful for feedback control and many other quantum applications. The work is organized as follows. In Sec. II we introduce the longitudinal coupling model and present the original QTE with the cavity-photon degrees of freedom included, conditioned on continuous homodyne detection of the cavity field quadrature. In Sec. III, following Ref. [53], we present the result of effective QTE which contains only the qubit degrees of freedom. Based on the result of the effective QTE and in particular the constructed qubit-plus-cavity entangled state, we present a comprehensive analysis for the quantum efficiency, qubit-state purity and signal-to-noise ratio in the output currents, by comparison with the dispersive readout. In Sec. IV we further construct the quantum Bayesian rule and present more discussions on the partial-collapse weak measurement theory. Finally, we summarize the work in Sec. V.

II. MODEL AND FORMULATION

In solid-state circuit QED, the standard scheme of qubit measurements is the dispersive readout protocol |43]. Similar to its quantum optics counterpart (i.e. the atomic cavity-QED system), the most natural coupling between the superconducting qubit and the resonator cavity is the Jaynes-Cummings Hamiltonian, or, being a little bit more general (without making rotating-wave approximation), the so-called transversal coupling given by $H_x = g_x \sigma_z (a + a^\dagger)$. Here $a^\dagger (a)$ and $\sigma_z$ are respectively the creation (annihilation) operator of the cavity photon and the quasi-spin operator for the qubit. The so-called dispersive regime is defined by the criterion that the detuning between the cavity frequency ($\omega_r$) and qubit energy ($\omega_q$), $\Delta = \omega_r - \omega_q$, is much larger than the coupling strength $g_x$. Under this, the coupling Hamiltonian can be approximated by $H_x \simeq \chi \sigma_z a^\dagger a$, with $\chi = g_x^2 / \Delta$. Therefore, associated with this dispersive coupling Hamiltonian, it is clear that the qubit states $|e\rangle$ and $|g\rangle$ will change, respectively, the cavity frequency by $\pm \chi$. Under measurement drive, the cavity field will evolve into qubit-state-dependent coherent states and a dyne-type quadrature detection can provide the qubit state information.

In this work, following Ref. [53], we consider an alternative coupling between the qubit and cavity field, given by

$$H_z = g_z \sigma_z (a + a^\dagger) \quad (1)$$

In contrast to the transversal coupling which leads to an approximate dispersive coupling Hamiltonian as shortly mentioned above, this new type of coupling is termed as longitudinal coupling, with the subscript $z$ here to mark it. This interaction Hamiltonian has been employed to discuss the realization of multi-qubit gates |44–48]. In particular, in the superconducting circuits, e.g., a flux or transmon qubit coupled to a LC oscillator |44–45], the longitudinal coupling can emerge from the mutual inductance between the flux-tunable qubit and the oscillator. Other examples include the general approach developed in Ref. [47] based on a transmon qubit |49], and the realization based on a transmission-line resonator |43].

Let us now consider the parametric modulation of the longitudinal coupling strength at the resonator frequency $\omega_r$, i.e., $g_z(t) = g_z + \dot{g}_z \cos(\omega_r t)$. In the rotating frame with respect to the free Hamiltonian $H_0 = \omega_r a^\dagger a + \frac{\Delta}{2} \sigma_z$, the interaction Hamiltonian reads

$$H_z = \frac{1}{2} \dot{g}_z \sigma_z (a + a^\dagger) \quad (2)$$

In obtaining this result, all the terms of fast oscillations have been neglected, including the “$\dot{g}_z$” term. It is clear that this modulation of the longitudinal coupling strength plays a role of qubit-state dependent drive to the cavity. Instead of the dispersive readout where the qubit-state dependent shift of the cavity frequency is employed, this alternative method can as well result in qubit-state dependent coherent state of the cavity field. Via a homodyne quadrature detection of the cavity field, the qubit state can be inferred.

As analyzed in detail in Ref. [43], the longitudinal coupling scheme has remarkable advantages to improve the qubit readout. That is, it can lead to a faster, high-fidelity and ideally QND qubit readout with a simple reset mechanism. Even with a conservative modulation, the qubit-state dependent shift of the cavity field can be easily distinguished. Owing to the ideal QNDness of the coupling, i.e., being ideally immune to the measurement caused qubit-state transition, this scheme is free from the limitation of critical photon numbers.

The quadrature of the cavity field is detected as usual by a homodyne measurement. The measurement result can be expressed as [44–45]

$$I(t) = \sqrt{\kappa} (ae^{-i\varphi} + a^\dagger e^{i\varphi}) \varrho(t) + \xi(t), \quad (3)$$

where $\varphi$ is the phase of the local oscillator in the homodyne detection set-up, $\kappa$ is the leaky rate of the cavity photons, and $\xi(t)$ satisfies the ensemble-average properties of $E[\xi(t)\xi(t')] = 0$ and $E[\xi(t)\xi(t')] = \delta(t - t')$. The quantum average $\langle \cdots \rangle_{\varrho(t)}$ is defined by $\langle \cdots \rangle_{\varrho(t)} = Tr[\cdots \varrho(t)]$, with $\varrho(t)$ the qubit-cavity conditional state given by the quantum trajectory equation [44–45]

$$\dot{\varrho} = -i[H_z, \varrho] + \kappa \mathcal{D}[a] \varrho + \sqrt{\kappa} \mathcal{H}[ae^{-i\varphi}] \varrho(t), \quad (4)$$

where the Lindblad superoperator is defined as $\mathcal{D}[a] \varrho = a \varrho a^\dagger - \frac{1}{2} \{a^\dagger a, \varrho\}$, and the unraveling superoperator as $\mathcal{H}[a] \varrho = a \varrho a^\dagger - \frac{1}{2} \{a^\dagger a, \varrho\}$. In experiments, $\xi(t)$ in Eq. (3) is obtained from the output current by using Eq. (3), while calculating the average $\langle \cdots \rangle_{\varrho(t)}$ in Eq. (3) needs the knowledge of $\varrho(t)$ solved from Eq. (4). It is clear that this job is almost intractable if the cavity photon number is large. It would be thus desirable to establish an effective QTE that contains only the degrees of freedom of the qubit. This can be done by applying the so-called polaron transformation to eliminate the degrees of freedom of the cavity photons |51].
III. QUANTUM TRAJECTORY EQUATION

The basic idea of the polaron transformation is performing a qubit-state dependent displacement to the cavity field, shifting it to a new vacuum state. This allows us to simplify the procedures of tracing the cavity states and obtain compact results for the equation-of-motion of the (stochastic) reduced state of the qubit and the output current.

The polaron transformation is designed as [51]

\[ P(t) = \Pi_e D[\alpha_e(t)] + \Pi_g D[\alpha_g(t)]. \] (5)

are, respectively, the displacement operators of the cavity fields corresponding to the qubit states |e⟩ and |g⟩. The specific form reads \( D[\alpha] = e^{\alpha a^+ - \alpha^* a} \). The qubit-state dependence in the polaron transformation is characterized by the projection operators of the qubit states \( \Pi_j = |j⟩⟨j| (j = e, g) \). Under the parametric modulation of the longitudinal coupling and in the presence of leakage of the cavity photons (with rate \( \kappa \)), the qubit-state dependent evolution of the cavity field is governed by

\[ \dot{\alpha}_{e,g}(t) = \mp i\tilde{g}_z/2 - \kappa\alpha_{e,g}(t)/2. \] (6)

Starting to drive (modulate) the cavity from a vacuum, the solution of the cavity field simply reads

\[ \alpha_{e,g}(t) = \mp i\tilde{g}_z \left(1 - e^{-\kappa t/2}\right). \] (7)

It should be noted that the solutions of the cavity fields solved here for the longitudinal readout are purely imaginary and have different signs, whereas for the dispersive readout the complex amplitudes of the cavity states corresponding to |e⟩ and |g⟩ differ in the sign of real parts, but have the same imaginary parts [27]. As will be seen in the following, this difference will result in a different choice of the local oscillator’s phase in the homodyne measurement in order to optimize the information gain of |e⟩ and |g⟩ of the qubit. That is, for dispersive readout, \( \varphi = 0 \); while for the longitudinal readout, \( \varphi = \pi/2 \). It is also this difference of \( \alpha_e(t) \) and \( \alpha_g(t) \) that will result in different rates which fully characterize the effective QTE and the output current, after eliminating the degrees of freedom of the cavity photons.

With the canonical transformation introduced above, we transform the qubit-cavity joint state \( \rho^P(t) = P^\dagger \rho(t) P \), and as well the two sides of Eq. (4). We may expand the transformed state in the qubit and the photon-number (Fock) basis states, \( \rho^P(t) = \sum_{n,m=0}^{\infty} \sum_{i,j=e,g} \rho^P_{n,m,i,j}(t)|n,i⟩⟨m,j| \).

Owing to the cavity field shifted to a new vacuum, essentially, only the zero-photon component will survive in this expansion. However, non-trivial dynamics will result in excited states of the cavity appearing in the transformed QTE. Of course, our interest is the reduced state of the qubit, which is defined by tracing the cavity states from the un-transformed joint state, \( \rho(t) = \text{Tr}_R[\rho^P(t) P^\dagger] \). This establishes a connection between \( \rho_{i,j} \) and \( \rho^P_{n,m,i,j} \). Accordingly, we obtain the equation-of-motion of \( \rho(t) \), from the ones of \( \rho^P_{n,m,i,j}(t) \) by taking into account the fact that only the \( n, m = 0 \) components survive in the shifted state. Following the straightforward procedures outlined above, we obtain

\[ \dot{\rho} = \Gamma_d(t) \mathcal{D}[\sigma_z] \rho - \sqrt{\Gamma_\text{ci}(t)} \mathcal{H}[\sigma_z] \rho \xi(t) \]

\[-i\sqrt{\Gamma_{ba}(t)} [\sigma_z, \rho] \xi(t). \] (8)

Here the various rates read

\[ \Gamma_d(t) = \frac{\tilde{g}_z}{2} |\beta(t)|, \] (9a)

\[ \Gamma_{ci}(t) = \frac{\kappa}{4} |\beta(t)|^2 \cos^2(\varphi - \theta_\beta), \] (9b)

\[ \Gamma_{ba}(t) = \frac{\kappa}{4} |\beta(t)|^2 \sin^2(\varphi - \theta_\beta). \] (9c)

In these results we introduced \( \alpha_e(t) - \alpha_g(t) = \beta(t) = |\beta(t)| e^{i\theta_\beta} \). Similarly, applying the same technique of transformation to the calculation of the output current \( I(t) \), i.e., \( \text{Tr}[(\bullet) \xi(t)] = \text{Tr}[P(\bullet) \mathcal{P}^\dagger \rho^P(t)] \), we simply obtain

\[ I(t) = -2\sqrt{\Gamma_{ci}(t)} |\sigma_z⟩⟨\sigma_z| t + \xi(t). \] (10)

In deriving this result, the fact that the shifted cavity field is a vacuum has been used.

The above results for the longitudinal coupling are almost the same as those of the dispersive scheme, except for the overall decoherence rate \( \Gamma_m \) which reads, there, \( \Gamma_m = \chi \lambda |\alpha_e(t)|^2 |\alpha_g(t)|^2 \). Similar results are also obtained in Ref. [51]. However, the behaviors of the measurement rates appearing in the QTE and the resultant consequences differ in both schemes, owing to the different cavity fields. That is, in the dispersive readout scheme, the qubit-state dependent cavity fields are given [27] by \( \alpha_{e,g}(t) = \tilde{g} \left(1 - e^{-\kappa t/2}\right) \). In dispersive readout, the microwave driving amplitude \( \epsilon_m \) corresponds to the coupling modulation strength \( \tilde{g} \) in the longitudinal scheme; and moreover, there is an additional parameter \( \chi \), i.e., the dispersive coupling strength between the qubit and the cavity field.

A. Quantum Efficiency and Purity

Briefly speaking, \( \Gamma_{ci} \) is the rate of coherent information gain, i.e., inferring qubit state |e⟩ or |g⟩. \( \Gamma_{ba} \) characterizes the backaction of the measurement not associated with information gain of the qubit state, but on qubit-level fluctuations. \( \Gamma_d \) is the overall decoherence rate, after ensemble average over large number of quantum trajectories. The sum of the former two rates, \( \Gamma_m = \Gamma_{ci} + \Gamma_{ba} \), is the total measurement rate. If \( \Gamma_m = \Gamma_d \), the measurement is ideally quantum limited, with quantum efficiency \( \eta = \Gamma_m/\Gamma_d = 1 \). Otherwise, if \( \Gamma_m < \Gamma_d \), the measurement is not ideal, implying some information loss.

In Fig. 1(a) we plot the transient quantum efficiency, defined as \( \eta = \Gamma_m(t)/\Gamma_d(t) \), of the longitudinal readout against its counterpart of the dispersive scheme, by setting \( \epsilon_m = \tilde{g}_z \) and taking several different \( \chi \). We find that, as the cavity field approaches to the steady state, \( \eta \rightarrow 1 \) for both
readout schemes, which seemingly implies quantum-limited measurements. However, this is only true in the sense of *transient differential gain* of qubit state information. Starting the measurement over \((0, t)\) with a pure state of the qubit, the resultant state of the qubit is given by Eq. (3), which is no longer a pure state. Actually, conditioned on the outcomes of the homodyne detection, the qubit-plus-cavity joint state can be expressed as [27]

\[
\Psi(t) = c_1(t)|e\rangle|\alpha_c(t)\rangle + c_2(t)e^{i\phi(t)}|g\rangle|\alpha_g(t)\rangle,
\]

(11)

where the evolution of \(c_1(t)\) and \(c_2(t)\) corresponds to information gain of the qubit states, and the (random) phase \(\phi(t)\) stems from qubit energy fluctuations caused by measurement backaction. The qubit state given by Eq. (3) corresponds to the one obtained by tracing the cavity degrees of freedom from this entangled state. This leads to a decoherence factor for the reduced state of qubit given by [27]

\[
D(t) = |\langle \alpha_c(t) | \alpha_g(t) \rangle| = e^{-2 \int_0^t d\tau [\Gamma_a(\tau) - \Gamma_m(\tau)]},
\]

(12)

In general, this factor is smaller than unity, implying a degradation of purity of the qubit state.

In Fig. 1(a) we also find that, at the early stage of partial-collapse measurement, the quantum efficiency \(\eta\) of the longitudinal readout is better than the dispersive scheme. However, this does not necessarily imply a higher purity for the qubit state. In contrast, as shown in Fig. 1(b), the corresponding purity of the qubit state, characterized by \(D(t)\), is lower than the dispersive readout. The reason for this behavior is that the faster readout of the longitudinal scheme is associated with larger measurement rates \(\Gamma_a(t)\) and \(\Gamma_m(t)\). Then, from Eq. (12), we have a smaller purity factor \(D(t)\), owing to the larger difference of \(\Gamma_a(t) - \Gamma_m(t)\). However, this is not a serious problem to the longitudinal scheme. Via introducing a fast reset procedure to the cavity, the qubit can be disentangled with the cavity from Eq. (11), resulting thus in an ideal pure state for the qubit. We will discuss this point in more detail near the end of next section.

In some surprise, we notice that in Fig. 1(a), in the intermediate stage of dispersive readout, it is possible to have \(\eta > 1\), i.e., the measurement rate \(\Gamma_m\) is larger than the associated decoherence rate \(\Gamma_d\). This is in general impossible. From a general consideration of quantum system coupled to environment (with continuum of energy spectrum), the continuous monitoring/measurement of the environment has a consequence of unraveling the usual master equation. However, in this case, the unraveling rate \(\Gamma_m\) must be smaller than or at most equal to the decoherence rate \(\Gamma_d\). Otherwise, consider starting with a pure state: if \(\Gamma_m > \Gamma_d\), the resultant state (inferred from the measurement outcome) will be ‘super-pure’, being unphysical with the off-diagonal elements of the density matrix violating the basic inequality of a physical state. In the circuit QED system, however, this ‘anomaly’ of the transient differential rates does not violate the requirement of a physical state. Based on the expression Eq. (11) and the expression Eq. (12), the transient \(\Gamma_m(t) > \Gamma_d(t)\) simply means that the purity (coherence) of the qubit state has been recovered some amount. This is interesting but possible. Indeed, in Fig. 1(b), the plot of \(D(t)\) supports this understanding.

**FIG. 1:** Comparison between the longitudinal coupling (LC) readout and the conventional dispersive readout scheme (with several different coupling strengths, defined from \(\chi\sigma_xa^\dagger a\)). In (a) and (b) we plot, respectively, the transient quantum efficiency \(\eta = \Gamma_m/\Gamma_d\), and the purity degradation factor \(D(t) = |\langle \alpha_c(t) | \alpha_g(t) \rangle|\), as a result of tracing out the degrees of freedom of the cavity photons. In (c) and (d), the same data of the signal-to-noise ratio (SNR) are plotted, for different visualization purposes.

### B. Signal-to-Noise Ratio

Based on the key results of Eqs. (8)-(11), it is also convenient to carry out another type of assessment, i.e., the signal-to-noise ratio (SNR) of the output currents. From Eq. (10), we can obtain the accumulated current over \((0, \tau)\) as \(\bar{Q} = \int_0^\tau dt I(t)\). Especially, corresponding to qubit states \(|e\rangle\) and \(|g\rangle\), the accumulated currents are, respectively, \(Q_e\) and \(Q_g\). Owing to involving the Gaussian noise \(\xi(t)\), \(Q_e\) and \(Q_g\) are random variables, satisfying Gaussian distributions with the distribution centers at \(\bar{Q}_{e(g)} = \mp 2 \int_0^\tau \sqrt{\Gamma_a(t)} dt\), and with a variance (half-width) of \(D_e = D_g = \sqrt{\tau}\). Based on this picture, the SNR of the measurement can be defined as \(\text{SNR} = |Q_e - Q_g|/(D_e + D_g)\). Choosing \(\varphi = \theta_0\), the results...
presented in Ref. [13] are simply recovered as
\[
\text{SNR}_{(L)} = \tilde{g}_z \sqrt{\frac{8\tau}{\kappa}} \left[ 1 - \frac{2}{\kappa \tau} (1 - e^{-\frac{\kappa \tau}{2}}) \right],
\]
for the longitudinal readout, and
\[
\text{SNR}_{(D)} = \epsilon_m \sqrt{\frac{8\tau}{\kappa}} \left[ 1 - \frac{2}{\kappa \tau} (1 - \cos(\frac{\kappa \tau}{2})e^{-\frac{\kappa \tau}{2}}) \right],
\]
for the dispersive readout. Here, the optimal condition $\chi = \kappa/2$ has been assumed for the dispersive readout, which ensures a maximal SNR in steady state of the cavity field under given $\epsilon_m$ and $\kappa$.

As a preliminary comparison, i.e., neglecting the Purcell effect in the dispersive readout, we set $\epsilon_m = \tilde{g}_z$. From the above analytic results, we see that both SNR are identical at steady state. We see also from Fig. 1(c) and (d) that, indeed, the condition of $\chi = \kappa/2$ can optimize the SNR at steady state in the dispersive readout. In the result of Eq. (13), the cosine term may have effect in the transient process. Indeed, we see from Fig. 1(d) that, for the choice of $\chi = 0.8 \kappa$ as an example, the SNR of the dispersive readout can exceed that of the longitudinal readout in certain intermediate stage of measurement process.

However, under equivalent parameter conditions, in the short time regime, the SNR of the longitudinal readout is better than the dispersive readout, just like the quantum efficiency shown in Fig. 1(a). This feature can benefit the short-time partial collapse measurement (i.e. weak measurement), which is of importance to such as the technique of measurement-based feedback control.

The really essential point is that, the longitudinal scheme is a genuine QND measurement, allowing for strong drive (with large $\tilde{g}_z$) but not suffering any qubit-state flip when the cavity field exceeds some critical photon numbers $n_{\text{crit}}$. So we can safely increase the modulation amplitude $\tilde{g}_z$ to improve the SNR. In a sharp contrast, in the dispersive readout, increasing either the drive amplitude $\epsilon_m$ or the decay rate of the cavity photons will induce the non-QNDness caused by Purcell effect, with a flip rate of the qubit state given by $\gamma_p = (\epsilon_m/\Delta)^2 \kappa$. Moreover, as will discuss later in connection with the partial-collapse weak measurement, the longitudinal coupling scheme allows also for a convenient way to reset the cavity field to vacuum.

\section*{IV. QUANTUM BAYESIAN RULE}

The effective QTE obtained above is much more efficient than the original Eq. (4) which contains the degrees of freedom of the cavity photons. However, as illustrated in recent cQED experiments [12–13], the quantum Bayesian approach is an alternative convenient method to update the quantum state based on accumulated output currents over relatively larger time steps. The quantum Bayesian rule allows also more efficient and analytical studies for some interesting problems such as quantum weak values, quantum smoothing, and quantum estimate [14–15].

Originally, the construction of the quantum Bayesian approach was largely based on the classical Bayes formula. For the diagonal elements of the qubit, the Bayes formula works perfectly; however, it does not work for the off-diagonal elements. One proceeds then by a purity consideration [24], together with some additional physical insights [25]. Especially, in a recent publication [24], a pedagogical scheme was proposed to generalize the quantum Bayesian approach constructed in Ref. [13] for cQED measurements, beyond the limits of bad-cavity and weak response. This scheme depends on a smart use of historical tail in terms of optical coherent states for the leaked cavity photons. Further approximating the photon-number distribution in the (leaked) coherent state to a Gaussian, and assuming successive measurements of the photon-number Fock states, the quantum Bayesian rule established in Refs. [27, 28] beyond the bad-cavity and weak response limits, was obtained. The methods employed in Refs. [23, 28] were based on the real measurements of homodyne detection of the cavity field quadrature, i.e., heavily based on the effective QTE after eliminating the cavity-photon degrees of freedom. In the following, we develop similar quantum Bayesian rule for the longitudinal readout, using the same method as in Ref. [24].

In the representation of the qubit basis states $|e\rangle$ and $|g\rangle$, Eq. (8) is rewritten as
\[
\hat{\rho}_{ee} = -4 \sqrt{\Gamma_{ci}} \hat{\rho}_{ee} \rho_{gg} z \xi,
\]
\[
\hat{\rho}_{eg} = -2 \sqrt{\Gamma_{ci}} \hat{\rho}_{eg} - \sqrt{\Gamma_{ci} (\sigma_z)} \rho_{gg} z \xi + i \sqrt{\Gamma_{ba}} \rho_{eg} z \xi.
\]
Following Ref. [28], we establish the Bayesian rule by straightforwardly integrating the stochastic differential equations. However, special care is required. While Eq. (8), or the above equations, can work well for numerical simulations based on the present Itô calculus form, analytic solution should be obtained by integrating its counterpart of the converted Stratonovich form.

We summarize the main results as follows. First, for the diagonal elements, we have
\[
\rho_{ii}(\tau) = \rho_{ii}(0) P_i(\tau) / N(\tau),
\]
where the index $i$ denotes $e$ and $g$, and the normalization factor reads $N(\tau) = \rho_{ee}(0) P_e(\tau) + \rho_{gg}(0) P_g(\tau)$. Importantly, in contrast to the usual simple Gaussian function, the prior distributions of the output currents corresponding to qubit $|e\rangle$ and $|g\rangle$ are, instead, Gaussian functionals given by
\[
P_i(\tau) = \exp \left\{ -\langle [I(t) - \bar{I}_i(t)]^2 \rangle / (2V) \right\}.
\]
Here we introduced the notations $\langle \bullet \rangle_t = \tau^{-1} \int_0^\tau dt \langle \bullet \rangle$, and $\bar{I}_{cg}(t) = \mp 2 \sqrt{\Gamma_{ci}(t)}$. The distribution variance $V$ simply reads $V = 1/\tau$.

Second, the off-diagonal element is obtained as
\[
\rho_{eg}(\tau) = \rho_{eg}(0) \left[ \sqrt{P_e(\tau) P_g(\tau) / N(\tau)} \right] \times D(\tau) \exp \left\{ -i \Phi(\tau) \right\}.
\]
The factor in the square brackets is associated with the updated change of the diagonal elements of the qubit state. Importantly, involved in this result for the off-diagonal element, there are two more factors:

\[
D(\tau) = \exp \left\{ -2 \int_0^{\tau} dt \left[ \Gamma_a(t) - \Gamma_m(t) \right] \right\},
\]

(20)

\[
\Phi(\tau) = 2 \int_0^{\tau} dt \sqrt{\Gamma_m(t)} I(t).
\]

(21)

Being closely related with the discussion in Sec. III A, the first factor, \(D(\tau)\), solved here from the quantum trajectory Eq. (8), accounts for the purity degradation of the qubit state after tracing the cavity degrees of freedom from the entangled qubit-plus-cavity state, Eq. (11). Actually, it can be proved \([27]\) that \(D(\tau) = \langle |\alpha_e(\tau)| \rangle \langle |\alpha_g(\tau)| \rangle\), which reveals, very directly, the physical meaning of this factor.

The second one, \(\Phi(\tau)\), is a random phase factor, as also appeared in the entangled state of Eq. (11). In contrast to the back-action owing to information gain of the qubit state \(|e\rangle\) or \(|g\rangle\), which alters the diagonal elements or \(c_1(t)\) and \(c_2(t)\) in Eq. (11), this phase factor characterizes the back-action of the measurement process (via the cavity photons) on the energy-level fluctuation of the qubit. Importantly, and quite interestingly, this qubit-level fluctuation also has consequence on the output currents. Therefore, one should take into account this effect as well in order to track the stochastic evolution of the qubit state. It should be noted that, in contrast to the usual reasoning of information loss (noting that this aspect of the output current does not carry any information of the qubit state \(|e\rangle\) or \(|g\rangle\)), this random fluctuation of the qubit energy is purity-preserving in the single realizations of continuous measurement, but not causing decoherence.

### A. More Discussions

As briefly discussed in Sec. III A, the factor \(D(t)\) implies that the partially collapsed qubit state is not quantum-mechanically pure. Actually, from Eq. (11), we know that the qubit is entangled with the cavity states \(|\alpha_e(t)\rangle\) and \(|\alpha_g(t)\rangle\). This is a typical feature for some mesoscopic detectors, where partial degrees of freedom of the detector are remained in the state description. A well-known example is the single-electron-transistor used as a charge-state detector \([52, 53]\), where the degrees of freedom of the central dot/island cannot be eliminated in general.

However, in the present cQED set-up, the cavity fields are well structured. After a partial-collapse measurement over \((0, t)\), let us consider to perform a fast (shorter than \(\kappa^{-1}\)) inverse-displacement at the moment \(t\) to the cavity field, based on the same coupling Hamiltonian of Eq. (3) but inverting the phase of the modulation, i.e., the action of \(U_{\text{cav}}(t + \delta t, t)\)\(|0\rangle\rightarrow e^{-i\hat{G}_t \delta t(|\alpha_e(a + a^1)\rangle}\) on the cavity state. Here we denote the coupling amplitude by \(\hat{G}_t\) (rather than \(\tilde{g}_t\)), implying a strongly pulsed drive. We may re-denote the result of Eq. (3) as \(\alpha_{e,g}(t) = \mp ia\). Now, if we make \(\hat{G}_t \delta t = -a\) (noting the particular sign opposite to the measurement drive \(\tilde{g}_t\)), we actually disentangle the qubit from the qubit-plus-cavity entangled state, as follows

\[
\Psi(t) = c_1(t)|e\rangle|\alpha_e(t)\rangle + c_2(t)e^{i\Phi(t)}|g\rangle|\alpha_g(t)\rangle
\]

\[
\rightarrow \Psi(t + \delta t) = [c_1(t)|e\rangle + c_2(t)e^{i\Phi(t)}|g\rangle]|0\rangle, \tag{22}
\]

where \(|0\rangle\) denotes the cavity vacuum. Note that this is essentially the same procedure of cavity reset discussed in Ref. [43], but a generalization from the projectively collapsed state \(|\alpha_e\rangle\) or \(|\alpha_g\rangle\), to the partially collapsed state of a superposition of \(|\alpha_e(t)\rangle\) and \(|\alpha_g(t)\rangle\). Therefore, after this qubit-state dependent displacement, we completely restore the desired qubit state, which is quantum mechanically pure, with now no degradation of purity owing to disentangling with the cavity states.

If we do not perform the reset procedure, and especially in the presence of qubit transition (by, e.g., unitary operation), the problem becomes much more difficult, as follows:

\[
\Psi(t) = c_1(t)|e\rangle|\alpha_e(t)\rangle + c_2(t)e^{i\Phi(t)}|g\rangle|\alpha_g(t)\rangle
\]

\[
\rightarrow [c_1(t)|e\langle\alpha_e(t)| + c_2(t)e^{i\Phi(t)}|g\langle\alpha_g(t)|]|c\rangle
\]

\[
+ [c_1(t)|b\langle\alpha_e(t)| + c_2(t)e^{i\Phi(t)}|a\langle\alpha_g(t)|]|g\rangle. \tag{23}
\]

Here we assumed a fast qubit-state rotation: \(|e\rangle \rightarrow a|e\rangle + b|g\rangle\); and \(|g\rangle \rightarrow a|g\rangle + b|e\rangle\). After this rotation, the subsequent measurement-caused evolution will be somehow complicated. That is, the evolution of the cavity field dragged by the qubit states \(|e\rangle\) and \(|g\rangle\) is started with a superposition of the coherent states (i.e. with a cat state), but not with the vacuum. This will invalidate the expressions of the measurement rates in the effective QTE or, equivalently, in the Bayesian approach. This will also affect the output currents. The complexities caused by qubit rotation have been discussed in the context of quantum trajectory equation by Gambetta et al. \([9\)] and of quantum Bayesian approach by Korotkov \([26\)], associated with the setup of dispersive readout.

It is clear that, from Eq. (23), if we consider a ‘bad’-cavity limit, the problem can be greatly simplified. That is, the cavity states (in the square brackets in Eq. (23)) associated with \(|e\rangle\) and \(|g\rangle\) will rapidly relax to \(|\alpha_e\rangle\) and \(|\alpha_g\rangle\). This makes the transient dynamics of the cavity field have negligible effect. As a result, each new step evolution affected by the measurement is updated by using the Bayesian inference, based on measurement rates determined by the stationary states of the cavity field. Beyond the ‘bad’-cavity limit, for the longitudinal readout scheme, the difficulty implied by Eq. (23) may be partly resolved by a fast reset procedure (before the qubit rotation) by means of the particular benefit of the longitudinal coupling, or developing more complicated theory based on insight into the state structure of Eq. (23). This will be an interesting problem for further investigations.

### V. SUMMARY

We carried out a study on the gradual partial-collapse weak measurement theory for a recently proposed genuinely QND scheme by virtual of longitudinal coupling in circuit QED.
This new coupling scheme has the great advantage of suffering no Purcell effect, i.e., no qubit-state-flip owing to measurement backaction mediated by the cavity photons. The theory was constructed in terms of both the quantum-trajectory-equation and quantum Bayesian approaches, which are expected to be useful for such as measurement-based feedback control and many other quantum applications associated with weak measurements. Based on the measurement rates appearing in the theory, we also presented comparative studies for the transient quantum efficiency, the qubit state purity, and the signal-to-noise ratio of the measurement, between the longitudinal coupling and the conventional dispersive readout schemes. In this context, we have explained a couple of unusual features.

With the insight gained from the theory, we also constructed the qubit-cavity joint state, from which a reset procedure to obtain the partially-collapsed desirable pure state for the qubit was proposed. This entangled joint state also shines light on possible future work to construct a complete theory (under generic parameter conditions) in the presence of qubit rotations, which remains so far an open problem within the framework of simplified theory after eliminating the cavity degrees of freedom.

Acknowledgements— This work was supported by the National Key Research and Development Program of China (No 2017YFA0303304), and the NNSF of China (No 11675016).

[1] A. Blais, R. S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A 69, 062320 (2004).
[2] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R. S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature (London) 431, 162 (2004).
[3] I. Chiorescu, P. Bertet, K. Semba, Y. Nakamura, C. J. P. M. Harmans, and J. E. Mooij, Nature (London) 431, 159 (2004).
[4] R. J. Schoelkopf, and S. M. Girvin, Nature (London) 451, 664 (2008).
[5] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. 6, 442 (2010).
[6] J. P. Groen, D. Risté, L. Tornberg, J. Cramer, P. C. de Groot, T. Picot, G. Johansson, and L. DiCarlo, Phys. Rev. Lett. 111, 090506 (2013).
[7] M. Mariantoni, H. Wang, R. C. Bialczak, M. Lenander, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, M. Weides, J. Wenner, T. Yamamoto, Y. Yin, J. Zhao, J. M. Martinis, and A. N. Cleland, Nat. Phys. 7, 287 (2011).
[8] P. Campagne-Ibarcq, E. Flurin, N. Roch, D. Darson, P. Morfin, M. Mirrahimi, M. H. Devoret, F. Mallet, and B. Huard, Phys. Rev. X 3, 021008 (2013).
[9] D. Risté, J. G. van Leeuwen, H. S. Ku, K. W. Lehnert, and L. DiCarlo, Phys. Rev. Lett. 109, 050507 (2012).
[10] M. Hatridge, S. Shankar, M. Mirrahimi, F. Schackert, K. Gerelings, T. Brecht, K. M. Sliwa, B. Abdo, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Science 339, 178 (2013).
[11] K. W. Murch, S. J. Weber, C. Macklin, and I. Siddiqi, Nature 502, 211 (2013).
[12] D. Tan, S. J. Weber, I. Siddiqi, K. Molmer, and K. W. Murch, Phys. Rev. Lett. 114, 090403 (2015).
[13] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, Nature (London) 490, 77 (2012).
[14] Wiseman, H. M. & Milburn, G. J. Quantum Measurement and Control (Cambridge Univ. Press, Cambridge, 2009).
[15] Jacobs, K. Quantum Measurement Theory and Its Applications (Cambridge Univ. Press, Cambridge, 2014).
[16] N. S. Williams and A. N. Jordan, Phys. Rev. Lett. 100, 026804 (2008).
[17] L. Qin, P. Liang, and X. Q. Li, Phys. Rev. A 92, 012119 (2015).
[18] L. Qin, L. Xu, W. Feng, and X. Q. Li, New J. Phys. 19, 033036 (2017).
[19] S. Gammelmark, B. Jølsgaard, and K. Molmer, Phys. Rev. Lett. 111, 160401 (2013).
[20] D. Tan, S. J. Weber, I. Siddiqi, K. Molmer, and K. W. Murch, Phys. Rev. Lett. 114, 090403 (2015).
[21] T. Rybarczyk et al., arXiv:1409.0958.
[22] M. Tsang, Phys. Rev. A 80, 033840 (2009).
[23] I. Guevara, and H. Wiseman, Phys. Rev. Lett. 115, 180407 (2015).
[24] A. N. Korotkov, Phys. Rev. B 60, 5737 (1999).
[25] A. N. Korotkov, Quantum Bayesian approach to circuit QED measurement, arXiv:1111.4016; see also chapter 17 in Les Houches 2011 session XCVI on Quantum Machines.
[26] A. N. Korotkov, Phys. Rev. A 94, 042326 (2016).
[27] P. Y. Wang, L. P. Qin, and X.-Q. Li, New J. Phys. 16, 123047 (2014); ibid. 17, 059501 (2015).
[28] W. Feng, P. F. Liang, L. P. Qin, and X.-Q. Li, Sci. Rep. 6, 20492 (2016).
[29] F. Beaudoin, J. M. Gambetta, and A. Blais, Phys. Rev. A 84, 043832 (2011).
[30] E. A. Sete, J. M. Gambetta, and A. N. Korotkov, Phys. Rev. B 89, 104516 (2014).
[31] L. C. G. Govia and F. K. Wilhelm, Phys. Rev. A 93, 012316 (2016).
[32] M. Boissonneault, J. M. Gambetta, and A. Blais, Phys. Rev. A 79, 013819 (2009).
[33] D. H. Slichter, R. Vijay, S. J. Weber, S. Boutin, M. Boissonneault, J. M. Gambetta, A. Blais, and I. Siddiqi, Phys. Rev. Lett. 109, 153601 (2012).
[34] M. D. Reed, B. R. Johnson, A. A. Houck, L. Dicarlo, J. M. Chow, D. I. Schuster, L. Frunzio, and R. J. Schoelkopf, Appl. Phys. Lett. 96, 203110 (2010).
[35] I. Siddiqi, R. Vijay, F. Pierre, C. M. Wilson, M. Metcalfe, C. Rigetti, L. Frunzio, and M. H. Devoret, Phys. Rev. Lett. 93, 207002 (2004); I. Siddiqi et al., ibid. 94, 027005 (2005).
[36] I. Siddiqi, R. Vijay, M. Metcalfe, E. Boaknin, L. Frunzio, R. J. Schoelkopf, and M. H. Devoret, Phys. Rev. B 73, 054510 (2006).
[37] V. E. Manucharyan, E. Boaknin, M. Metcalfe, R. Vijay, I. Siddiqi, and M. Devoret, Phys. Rev. B 76, 014524 (2007).
[38] A. Lupascu, S. Saito, T. Picot, P. C. de Groot, C. J. P. M. Harmans, and J. E. Mooij, Nat. Phys. 3, 119 (2007).
[39] P. Campagne-Ibarcq, L. Bretheau, E. Flurin, A. Auffèves, F. Mallet, and B. Huard, Phys. Rev. Lett. 112, 180402 (2014).
[40] P. Campagne-Ibarcq, P. Six, L. Bretheau, A. Sarlette, M. Mirrahimi, P. Rouchon, B. Huard, Phys. Rev. X 6, 011002 (2016).
[41] A. N. Jordan, A. Chantasri, P. Rouchon, and B. Huard, Quantum Stud. Math. Found. 3, 237 (2016).
[42] M. Naghiloo, D. Tan, P. M. Harrington, P. Lewalle, A. N. Jordan, and K. W. Murch, arXiv: 1612.03189v2 (2017).
[43] N. Didier, J. Bourassa, and A. Blais, Phys. Rev. Lett. 115, 203601 (2015).
[44] A. J. Kerman, New J. Phys. 15, 123011 (2013).
[45] P. M. Billangeon, J. S. Tsai, and Y. Nakamura, Phys. Rev. B 91, 094517 (2015).
[46] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Phys. Rev. A 75, 032329 (2007).
[47] J. Bourassa, J. M. Gambetta, A. A. Abdumalikov, O. Astafiev, Y. Nakamura, and A. Blais, Phys. Rev. A 80, 032109 (2009).
[48] S. Richer and D. Divincenzo, Phys. Rev. B 93, 134501 (2016).
[49] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A 75, 042329 (2007).
[50] S. Hacohen-Gourgy, L. Martin, E. Flurin, V. V. Ramasesh, K. B. Whaley, and I. Siddiqi, Nature (London) 538, 491 (2016)
[51] J. Gambetta, A. Blais, M. Boissoneault, A. A. Houck, D. I. Schuster, and S. M. Girvin, Phys. Rev. A 77, 012112 (2008).
[52] M. H. Devoret and R. J. Schoelkopf, Nature (London) 406, 1039 (2000).
[53] Yu. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).