Opportunity of representation of the nonlinear wave equations through a variable action-angle

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The approach allowing is considered to represent the solutions such as stationary lonely waves of various nonlinear wave the equations as system of the ordinary differential equations in variable action - angle.

1. Introduction

In the nonlinear theory of oscillations the essentially important meaning has an opportunity transition in researched system from initial variable to variable action - angle. If oscillatory system with two degrees of freedom is integrable hamiltonian system, it is possible transformed to a kind

\[
\begin{align*}
\frac{dI}{dt} &= 0, \quad \frac{d\Theta}{dt} = \omega_0(I), \quad I = \frac{1}{2\pi} \oint p\,dq \\
\end{align*}
\]

where \( I = \text{const} \) - action, \((q(t), p(t))\) - canonical variable of hamiltonian system, \(\omega_0(I)\) - frequency of oscillations.

If to consider only oscillatory systems with two degree of freedom (on a phase plane), in a case hamiltonian systems, the action is the area limited on a phase plane to a trajectory of system [1]. Action and frequency of oscillations are determined, how in parameters of system, and initial conditions. If to consider nonhamiltonian (dissipative) system, stationary periodic (on an infinite interval of time) oscillations in them, correspond one or more limiting cycles, and, accordingly, as much of pairs of meanings variable action - angle. In a case, when a limiting cycle on plane, the action is equal to the area limited on a phase plane by a limiting cycle [2].

From nonlinear oscillations this formalism can be distributed to some kinds of nonlinear waves. In work [3] is shown, that basic nonlinear wave models having solutions as solitons (Korteweg-de Vriz equation, sin-Gordon equation and some other) are complete integrable systems and them it is possible to transform to the equations to a variable action - angle.

Opportunity of representation in a variable action - corner will be considered below the solutions such as stationary lonely waves for some kinds nonlinear the wave equations, that is actually in an implicit kind is carried out transformation from difficult nonlinear partial differential equation to the elementary system the ordinary differential equations.

2. The Korteweg-de Vriz equation

\[
\frac{\partial u}{\partial t} + Au \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{2}
\]
Through substitution such as a running wave \( u(x,t) = u(z) = u(x - vt), \) \( v = \text{const} \) - speed of a wave, it is possible to transform to the ordinary differential equation

\[
\frac{d^2 u}{dz^2} - vu + 0.5u^2 = 0, \tag{3}
\]

which is Hamiltonian system with canonical variable \((u, du/dz)\) and Hamilton function

\[
H(u, du/dz) = 0.5((du/dz)^2 - vu^2 + Au^3/3).\]

To the solution as a lonely wave of the Korteweg-de Vriz equation (2) corresponds the solution as the closed loop of separatrix (fig. 1) equation (3):

\[
u(z) = 3vA^{-1} \text{sech}^2(0.5\sqrt{vz}). \tag{4}\]

Proceeding from equality (1), action on each closed trajectory equation (3) is equal:

\[
I = \frac{1}{2\pi} \int pdq = \frac{1}{2\pi} \int (du/dz)du = \frac{1}{2\pi} \int (du/dz)^2dz = \text{const.}
\]

Separatrix also is the closed trajectory, as \( u(z \to \pm \infty) \to 0. \) On separatrix

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} (du/dz)^2dz = I_1 = \text{const}, \quad \frac{dI}{dz} = 0,
\]

where \( u(z) \) is defined by equality (4).

Let’s analyse sense a variable angle for the description stationary lonely waves. In case of stationary periodic oscillations variable angle describes phase of oscillations, that is complete angle on which the system an initial situation has deviated for an interval time from the moment of a beginning of oscillations. By analogy it is possible to assume, that in case of stationary lonely waves the variable angle will be to describe distance, on which the wave from has left initial situation and complete derivative of a angle is speed of a wave:

\[
\frac{d\Theta}{dz} = v
\]

Thus in case of dynamics such as a lonely stationary wave the Korteweg-de Vriz equation can be transformed to system of the equations in a variable action - angle:

\[
\frac{dI}{dz} = 0, \quad \frac{d\Theta}{dz} = v, \quad I = \frac{1}{2\pi} \int_{-\infty}^{\infty} (du/dz)^2dz,
\]

where \( z = x - vt, \quad u(z \to \pm \infty) \to 0. \) The speed \( v \) is any real number also is determined by the initial conditions and parameters of the equation (2).

3. The sin-Gordon equation

\[
\frac{\partial^2 u}{\partial t^2} + \sin u = \frac{\partial^2 u}{\partial x^2}. \tag{5}\]

Through substitution such as a running wave \( u(x,t) = u(z) = u(x - vt), v = \text{const} \) it can be transformed to the ordinary differential equation

\[
\frac{d^2 u}{dz^2} + (v^2 - 1)^{-1} \sin u = 0, \tag{6}\]
which is hamiltonian system with canonical variable \((u, du/dt)\). To the solution as lonely waves of the sin-Gordon equation corresponds solution as top or bottom halfloop of separatrix (fig. 2) of the equations (6):

\[
\text{(top halfloop)}: u(z) = 4arctg \exp[z(1-v^2)^{-1/2}],
\]

\[
u(z \to -\infty) \to 0, u(z \to +\infty) \to 2\pi.
\]

Value of action is equal

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} (du/dz)^2 dz = I_2 = \text{const},
\]

and the equation (5) will be transformed to a kind

\[
\frac{dI}{dz} = 0, \quad \frac{d\Theta}{dz} = v,
\]

where \(u(z \to -\infty) \to 0, u(z \to +\infty) \to 2\pi, z = x - vt\). The speed \(v\) is determined by the initial conditions and parameters of equation (5).

4. The Kolmogorov-Petrovskiy-Piskunov equation.

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + ku(1-u). \tag{7}
\]

By substitution such as a running wave \(u(x,t) = u(z) = u(x - vt), v = \text{const}\) it can be transformed to the ordinary differential equation

\[
D \frac{d^2 u}{dz^2} + v \frac{du}{dz} + ku(1-u) = 0, \tag{8}
\]

To the solution such as a running wave of the equation (7) there corresponds the solution of a type halfloop of separatrix (fig. 3) equation (8) [4]:

\[
u(z) = (1 + \exp(az))^{-2}, a = \pm \sqrt{k/(6D)}, v = 5aD
\]

with the boundary conditions \(u(z \to -\infty) \to 0, u(z \to +\infty) \to 1\). Value of action is equal

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} (du/dz)^2 dz = I_3 = \text{const},
\]

and equation (7) will be transformed to a kind

\[
\frac{dI}{dz} = 0, \quad \frac{d\Theta}{dz} = v = \pm \sqrt{\frac{25kD}{6}}.
\]

5. The Burgers equatian

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2}. \tag{9}
\]
By substitution such as a running wave \( u(x, t) = u(z) = u(x - vt) \), \( v = \text{const} \) it can be transformed to the ordinary differential equation, which under boundary conditions \( u(z \to \infty) \to u_1, u(z \to -\infty) \to u_2, u_1 < u < u_2 \) has the solution [2]:

\[
u(x, t) = u_1 + \frac{u_2 - u_1}{1 + \exp(0.5(u_2 - u_1)(x - vt)/D)}, v = \frac{u_1 + u_2}{2}.
\]

Value of action is equal

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} (du/dz)^2 dz = I_4 = \text{const},
\]

and equation (9) will be transformed to a kind

\[
\frac{dI}{dz} = 0, \quad \frac{d\Theta}{dz} = v = \frac{u_1 + u_2}{2}.
\]

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