Asymptotic Curvature of Moduli Spaces for Calabi–Yau Threefolds

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Abstract Motivated by the classical statements of Mirror Symmetry, we study certain Kähler metrics on the complexified Kähler cone of a Calabi–Yau threefold, conjecturally corresponding to approximations to the Weil–Petersson metric near large complex structure limit for the mirror. In particular, the naturally defined Riemannian metric (defined via cup-product) on a level set of the Kähler cone is seen to be analogous to a slice of the Weil–Petersson metric near large complex structure limit. This enables us to give counterexamples to a conjecture of Ooguri and Vafa that the Weil–Petersson metric has non-positive scalar curvature in some neighborhood of the large complex structure limit point.

Keywords Mirror symmetry · Weil–Petersson metric · Large complex structure limit points · Large radius limit points · Curvature

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1 Introduction

In this paper, we aim to understand the asymptotic behavior of the Weil–Petersson metric near large complex structure limit points (defined in terms of maximally unipotent monodromy) on the complex moduli space of Calabi–Yau threefolds, by using
a classical form of mirror symmetry and calculating the curvature of certain Kähler metrics on the complexified Kähler cone of the mirror.

This is intimately connected with the theory developed in [28]. If $V$ is a Calabi–Yau threefold with $h^{2,0} = 0$ and $h^{1,1} = r$, we shall denote the cup-product cubic form on $H^2(V, \mathbb{R})$ by $f(y_1, \ldots, y_r)$. Let $\mathcal{K}(V) \subset H^2(V, \mathbb{R})$ denote the Kähler cone, and $\mathcal{K}_1 \subset \mathcal{K}(V)$ the level set given by $f = 1$. It is a consequence of the Hodge index theorem that the restriction of $-\frac{1}{6}(\partial^2 f/\partial y_i \partial y_j)$ to $\mathcal{K}_1$ defines a natural Riemannian metric on $\mathcal{K}_1$. In [28], it was argued in Sect. 2 that this Riemannian manifold reflects the asymptotic Weil–Petersson geometry on the moduli space of the mirror near large complex structure limit. This claim is given a more precise justification in this paper; in particular, see Remark 3.7 below.

The history of expectations concerning the curvature of the Weil–Petersson metric on the moduli space of Calabi–Yau threefolds is marked by unfulfilled hopes, probably over-influenced by the Weil–Petersson metric on the moduli space of curves, which was known to have negative curvature. In the Calabi–Yau threefold case, it was claimed in [23] that the holomorphic sectional curvatures were negative, and in [24] that all the sectional curvatures were negative. Both these statements were disproved by the calculations of Candelas et al. [5] for the mirror quintic, where the 1-dimensional moduli space was shown to have Weil–Petersson curvature tending to $+\infty$ as one approaches the orbifold point.

The mirror quintic moduli space, however, does have negative curvature as one approaches the large complex structure limit point in moduli. This fact was generalized in [26], where it was shown that, whenever the complex moduli space of a Calabi–Yau threefold is 1-dimensional, the Weil–Petersson metric is asymptotic to the Poincaré metric near a large complex structure limit point, and in particular, has negative curvature.

There was then a folklore expectation that this asymptotic negativity of the Weil–Petersson curvature near large complex structure limit should continue to hold for the complex moduli space having arbitrary dimension. This expectation motivated some of the work carried out in [28]. A weaker version of this expectation was articulated in Conjecture 3 of [20], where it was conjectured that at least the scalar curvature of the Weil–Petersson metric should be non-positive near the points at infinity in moduli (see also the evidence quoted in Example (v) in Sect. 3 of that paper). The second author pointed out in Sect. 2 of [29] that these expectations of asymptotic negativity were likely to be false, with conjectural counterexamples provided by the mirrors to the smooth Calabi–Yau Weierstrass fibrations over the Hirzebruch rational surfaces $F_0 = P^1 \times P^1$, $F_1$ and $F_2$. In Sect. 4 below, we study in detail the case of the Weierstrass fibration over $F_2$; this may also be conveniently described as the desingularization of a general hypersurface of degree 24 in $P(1, 1, 2, 8, 12)$, and is a threefold known in the Physics literature as the STU-model. In Theorem 4.3, we see that the mirror to this Calabi–Yau threefold does indeed provide a counterexample to the conjecture in [20], since the Weil–Petersson scalar curvature is unbounded above in any neighborhood of the large complex structure limit point. Moreover, we observe that the same thing happens for the mirrors to certain other toric hypersurface Calabi–Yau threefolds, and in Theorem 4.7 we extend this yet further to include a different type of Calabi–Yau threefold.