Master equation approach for non-Hermitian quadratic Hamiltonians: Original and phase space formulations

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Abstract. We consider evolution of dynamical systems described by non-Hermitian Hamiltonians, using the density operator approach. The latter is formulated both at the level of the Hilbert space and the phase space, and adapted for applications to open quantum systems. We illustrate the formalism using a family of non-Hermitian system, which generators are quadratic with respect to both momentum and position. Despite the initial simplicity of a Hamiltonian, the structure of its solutions and spectral characteristics are nontrivial, and they can drastically change depending on parameters of the model and its symmetry in phase space. We present analytical solutions in $L^2(\mathbb{R})$ and in phase space, and an explicit form of the similarity transformation changing these generators into the corresponding normal operators.

1. Introduction

Non-Hermitian (NH) evolution generators (more commonly known as non-Hermitian Hamiltonians) appear in different contexts in description of quantum phenomena. Although there is no place for them in the formalism of standard quantum mechanics [1], they have been the focus of both theoretical and experimental studies in a theory of many-particle and open quantum systems [2, 3, 4]. When working within the frameworks of NH Hamiltonian approach to quantum dissipative systems, one assumes that the anti-Hermitian part arises in Hamiltonian as a result of the interaction of otherwise conservative system with its environment or reservoir [4, 5].

Despite the long history of the field, the core formalism of NH quantum dynamics is still a subject of active research from the viewpoint of a density-operator approach [6, 7, 8, 9, 10, 11, 12, 13] with specific applications to different phenomena [14, 15, 16, 17, 18], to mention just a few examples. From a mathematical point of view, relaxing the condition of hermiticity of Hamiltonian leads to serious difficulties, e.g., absence of the spectral representation, which results in the lack of properly defined functional calculus. From a physical point of view, difficulties in interpretation arise, related mostly to the lack of probability conservation and problems with calculation of average values.

The goal of this paper is to describe evolution of a system characterized by NH $\dot{\hat{G}} = \hat{H} - i\hat{\Gamma}$, where $\hat{H}$ and $\hat{\Gamma}$ are Hermitian operators themselves, in a Schrödinger picture and using Wigner phase space distribution. We aim to provide a complete description of such a NH case, analyzing the Schrödinger and Heisenberg equations for states in $L^2(\mathbb{R})$ and considering the corresponding
equations in phase space. In the main part of the paper, we use a NH evolution generator square in position and momentum to illustrate this general scheme. We will study \( \hat{G} \) of the form

\[
\hat{G} = \hat{H} - \hat{\Gamma}, \\
\hat{H} = \epsilon \left( \frac{1}{2m} \hat{p}^2 + \frac{m \omega^2}{2} \hat{x}^2 \right), \\
\hat{\Gamma} = \frac{A}{2m} \hat{p}^2 + \frac{Bm \omega^2}{2} \hat{x}^2 + \gamma \hat{1},
\]

where \( A, B, \gamma \in \mathbb{R} \); \( m > 0 \) denotes a mass, \( \omega > 0 \) is an (angular) frequency, and a dimensionless parameter \( \epsilon > 0 \) allows to trace parts of equations related to \( \hat{H} \). Square integrable solutions of the Schrödinger equation, valid for arbitrary parameters in (1), are derived in Section 3.1; the corresponding phase space solutions are presented in Section 3.2. In Section 2.1, a general description of evolution in \( L^2(\mathbb{R}) \) and in the Wigner representation is discussed. In Section 2.2, a phase space equivalent to the von Neumann equation is derived.

The important point to note is that, in this paper, Hermitian is a synonym of self-adjoint. Because we allow for a NH operator in the Schrödinger equation, its eigenvalues are, in general, complex-valued. By definition, in a NH case operators \( \hat{G} \) and \( \hat{G}^\dagger \) are not equal to each other and each has its own set of eigenvectors. Moreover, whenever \( \hat{G} \) is not a normal operator its eigenfunctions do not form an orthogonal basis.

2. General theoretical description

In this section, we present a general description of NH evolution on two levels: states in a Hilbert space, \( L^2(\mathbb{R}) \), and in phase space. We also show that, in the Heisenberg picture, eigenoperators are constructed from eigenfunctions of \( \hat{G}^\dagger \) when evolution is governed by \( \hat{G} \), and from eigenfunctions of operator \( \hat{G} \) when evolution is governed by \( \hat{G}^\dagger \).

2.1. Phase space formulation of time-independent Schrödinger equation

Let us consider operator \( \hat{G} = \hat{H} - \hat{\Gamma} \) with Hermitian \( \hat{H} \) and \( \hat{\Gamma} \). Taking the Wigner-Weyl transform of eigenequation \( (\hat{G} - \lambda \hat{1}) \Psi = 0 \), we obtain condition

\[
(\hat{G} - \lambda \hat{1})_W \star W_{\Psi} = 0,
\]

where \( W_{\Psi} \) is the Wigner function [19] corresponding to \( \Psi \),

\[
W_{\Psi}(x,p) = \frac{1}{2\pi \hbar} \int \Psi(x-\xi/2)\Psi(x+\xi/2)e^{\frac{i\xi p}{\hbar}} d\xi,
\]

a star product \( \star \) [20] is defined as

\[
f \star g = f \exp \left( -\frac{i\hbar}{2}\vec{P} \right) g, \quad \vec{P} := \hat{\partial}_p \hat{\partial}_x - \hat{\partial}_x \hat{\partial}_p,
\]

and a subscript \( ()_W \) denotes the Wigner-Weyl transform.

For \( \lambda = \lambda_re + i\lambda_im \), where \( \lambda_re \) and \( \lambda_im \) are the real and imaginary parts of \( \lambda \), respectively, (2) reads

\[
(H_w - \lambda_re - i(\Gamma_w + \lambda_im)) \exp \left( -\frac{i\hbar}{2}\vec{P} \right) W_{\Psi} =
\]

\[
= (H_w - \lambda_re - i(\Gamma_w + \lambda_im))(1 - \frac{i\hbar}{2}\vec{P} - \frac{\hbar^2}{8}\vec{P}^2 + \frac{i\hbar^3}{48}\vec{P}^3 + \frac{\hbar^4}{384}\vec{P}^4 - \ldots)W_{\Psi} = 0.
\]
The Wigner function is always real which allows us to easily divide the above equation into its real and imaginary parts. For the real part we have
\[
(H_w - \lambda_{re}) \cos \left(\frac{\hbar}{2} \vec{P}\right) W_\Psi - (\Gamma_w + \lambda_{im}) \sin \left(\frac{\hbar}{2} \vec{P}\right) W_\Psi = 0,
\]
and for the imaginary part we obtain
\[
(H_w - \lambda_{re}) \sin \left(\frac{\hbar}{2} \vec{P}\right) W_\Psi + (\Gamma_w + \lambda_{im}) \cos \left(\frac{\hbar}{2} \vec{P}\right) W_\Psi = 0.
\]
This set of two independent equations, (5) and (6), forms a phase space equivalent of the stationary Schrödinger equation.

### 2.2. Phase space description of master equation

For $\hat{G} = \hat{H} - i\dot{\Gamma}$, the dynamics of the system described by density operator $\hat{\rho}$ is governed by [5, 21]
\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right] - \frac{1}{\hbar} \left\{ \hat{\Gamma}, \hat{\rho} \right\}.
\]
In the context of theory of open quantum systems, this is the evolution equation for a reduced density operator $\hat{\rho}$, which effectively describes the original subsystem (with Hamiltonian $\hat{H}$) together with the effect of environment (represented by $\hat{\Gamma}$). Upon taking the trace of both sides of (7), one obtains an evolution equation for the trace of $\hat{\rho}$, $\frac{\partial \text{Tr}\hat{\rho}}{\partial t} = -\frac{2}{\hbar} \text{Tr}(\hat{\Gamma} \hat{\rho})$, which shows that dynamics defined by (7) does not conserve the probability. Equation that rules evolution of a normalized density operator $\hat{\rho}' = \hat{\rho}/(\text{Tr} \hat{\rho})$, suggested in [6], reads
\[
\frac{\partial \hat{\rho}'}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho}' \right] - \frac{1}{\hbar} \left\{ \hat{\Gamma}, \hat{\rho}' \right\} + \frac{2}{\hbar} \hat{\rho}' \text{Tr}(\hat{\Gamma} \hat{\rho}').
\]
Using definitions of Lee and Jordan multiplications: $\hat{A} \cdot \hat{B} = \frac{1}{\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A})$ and $\hat{A} \circ \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$, respectively, we can rewrite (7) as
\[
\frac{\partial \hat{\rho}}{\partial t} = \hat{H} \cdot \hat{\rho} - \frac{2}{\hbar} \hat{\Gamma} \circ \hat{\rho}.
\]
It is known [22], that the following relations hold between the Wigner-Weyl transforms of operators $\hat{A}$ and $\hat{B}$: $(\hat{A} \cdot \hat{B})_W = -\frac{2}{\hbar} A_W \sin \left(\frac{\hbar}{2} \vec{P}\right) B_W$, $(\hat{A} \circ \hat{B})_W = A_W \cos \left(\frac{\hbar}{2} \vec{P}\right) B_W$. Combining these with (9) we find that, in phase space formulation, (7) is given by
\[
\frac{\partial}{\partial t} W_\rho = -\frac{2}{\hbar} H_W \sin \left(\frac{\hbar}{2} \vec{P}\right) W_\rho - \frac{2}{\hbar} \Gamma_w \cos \left(\frac{\hbar}{2} \vec{P}\right) W_\rho,
\]
where $W_\rho$ is the Wigner function corresponding to $\hat{\rho}$,
\[
W_\rho(x, p, t) = \frac{1}{2\pi \hbar} \int d\xi (x - \frac{\xi}{2}) \hat{\rho}(t)|x + \frac{\xi}{2}\rangle e^{i\xi p}.
\]
Obviously, for $\hat{\rho} = |\psi\rangle\langle\psi|$, definition (11) simplifies to (3). Let us note that, in a slightly different form, eq. (10) appeared previously in [23, 24, 25], in the context of pulse propagation in dispersive media. Yet another formulation, through the Wigner function flow, was given in [26].

Equation (10) is especially convenient to describe phase space evolution when interactions can be approximated by polynomials, i.e., when its RHS is represented by terms proportional to finite powers of $\hbar$. Unlike master equation, (10) acts on functions not on the operators, which makes it much easier to solve. Additionally, phase space description is very intuitive and connected to classical perception, which makes it even more interesting to study.
3. Example: evolution generator quadratic in position and momentum

In this Section, we apply the formalism of Section 2 to the evolution generator quadratic in position and momentum, i.e., \( \hat{G} \) defined by (1). We present analytical form of general, square-integrable solutions of the Schrödinger equation and use them to derive solutions of evolution equation in phase space. We also show how this particular \( \hat{G} \) can be transformed into a normal operator.

3.1. Schrödinger picture description

Let us consider the eigenproblem \( \hat{G}f_n = \lambda_nf_n \) with \( \hat{G} \) taken from (1). For \( \gamma = 0 \) and \( A, B > 0 \), the decay rate operator \( \hat{\Gamma} \) is reduced to a simple harmonic oscillator; for \( A > 0 \) and \( B < 0 \), it corresponds to a “chain falling from a table” Hamiltonian. The solutions presented below are not limited to these two special cases but are valid for arbitrary real \( A, B \), and \( \gamma \).

It can be shown, that square integrable solutions of equation

\[
-\frac{\hbar^2(\epsilon - iA)}{2m} \frac{d^2 f}{dx^2} + \frac{m\omega^2}{2}(\epsilon - iB)x^2 f - (\lambda + i\gamma)f = 0,
\]

are given by

\[
f_n(x) = \mathcal{N}_n \exp\left(-\frac{m\omega}{2\hbar} \sqrt{\epsilon - i\frac{B}{A}} x^2\right) H_n\left(\sqrt{\frac{\epsilon - i\frac{B}{A}}{\epsilon - i\frac{A}{B}}} \frac{m\omega}{\hbar} x\right), \tag{12}
\]

where \( n \in \mathbb{N} \), \( \mathcal{N}_n \) is a normalization factor [presented later in (23)], \( H_n \) denotes \( n \)th Hermite polynomial, and from square roots of \( \frac{iA}{\epsilon - iB} \) we choose the one with a positive real part\(^1\).

From the four possible values of \( \sqrt{\frac{iA}{\epsilon - iB}} \), we, again, select the one with a positive real part, calculating it as a square root of the previously chosen square root of \( \frac{iA}{\epsilon - iB} \). The eigenvalues corresponding to (12) are given by

\[
\lambda_n = \hbar\omega(n + 1/2)(\epsilon - iA)\sqrt{\frac{\epsilon - iB}{\epsilon - iA}} - i\gamma
\]

\[
= \hbar\omega(n + 1/2)\sqrt{\epsilon^2 - AB - i\epsilon(A + B)} - i\gamma. \tag{13}
\]

Therefore, solutions of the time-dependent Schrödinger equation are of the form

\[
F_n(x,t) = f_n(x)e^{-\frac{i}{\hbar}\lambda_n t}, \tag{14}
\]

where \( f_n \) and \( \lambda_n \) are defined by (12) and (13), respectively. Note that, in general, \( \lambda_n \) has a nonzero imaginary part which changes the normalization of \( F_n(x,t) \) during evolution. Nonetheless, functions (14) are square integrable for any finite \( t \). If \( \text{Im}(\lambda_n) \neq 0 \), then, in the limit of \( t \to \pm\infty \), the \( L^2(\mathbb{R}) \) norm of \( F_n(x,t) \) may diverge. The same applies for all finite linear combinations of (14). On the other hand, infinite combinations of (14) may diverge for any \( t \neq 0 \) even if initial condition is square integrable.

From the form of (13) it is clear that, for a given \( n \in \mathbb{N} \) and set parameters \( m, \omega, \epsilon > 0, A \neq 0 \neq B, \) and \( B \neq -A \), one can always find such a \( \gamma \) that the corresponding \( F_n(x,t) \) is a stationary state, i.e., \( |F_n(x,t)| = |F_n(x,0)| \). If \( F_n \) is a stationary state then, during evolution

\(^1\) It guarantees that \( f_n(x) \) vanishes for \( x \to \pm\infty \). Note that, for \( \epsilon > 0 \), there always exists a square root of \( \sqrt{\frac{iA}{\epsilon - iB}} \) with a positive real part. Only for \( \epsilon = 0, A > 0, B < 0 \), i.e., the case corresponding to \( \hat{H} = 0 \) and a chain falling from a table type of \( \hat{\Gamma} \), the roots are purely imaginary. This case is not considered here, as in the beginning we assumed that \( \epsilon > 0 \).
specified by (1), all initial states $\Psi(x,0)$ that have $F_n$ as a lowest state in their series expansion, i.e.,

$$
\Psi(x,0) = \sum_{k=n}^{\infty} a_k F_k(x,0), \quad a_n \neq 0,
$$

are, in the limit of $t \to \infty$, reduced to $a_n F_n(x,t)$. It is interesting that, for $\gamma \neq 0$, the only two instances when it is not possible to obtain stationary state are the cases of $A = B = 0$ or $B = -A$. The first condition corresponds to a simple harmonic oscillator coupled to the environment by a damping factor $i\gamma$; the second matches the "chain falling from a table" type. If $B = -A$ and $\gamma = 0$, the eigenvalues take on purely real values.

### 3.2. Phase space description

In the case of (1), phase space equivalent of the time independent Schrödinger equation (5), (6) and a phase space evolution equation (10) can be easily calculated, because both $\hat{H}$ and $\hat{\Gamma}$ are simple polynomials in $\hat{p}$ and $\hat{x}$. In this case, the phase space equations corresponding to the time independent Schrödinger equation are

$$
\frac{2}{\hbar} \left[ \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 - \text{Re}(\lambda) \right] W_\Psi = \epsilon \left( \frac{A p}{m \partial x} - B m \omega^2 x \frac{\partial}{\partial p} \right) W_\Psi + \frac{\hbar}{4} \left( \frac{1}{m \partial x^2} + m \omega^2 \frac{\partial^2}{\partial p^2} \right) W_\Psi, \quad (15)
$$

$$
\frac{2}{\hbar} \left[ \frac{Ap^2}{2m} + \frac{B m \omega^2}{2} x^2 + \gamma + \text{Im}(\lambda) \right] W_\Psi = -\epsilon \left( \frac{p}{m} \frac{\partial}{\partial x} - m \omega^2 x \frac{\partial}{\partial p} \right) W_\Psi + \frac{\hbar}{4} \left( \frac{A}{m \partial x^2} + B m \omega^2 \frac{\partial^2}{\partial p^2} \right) W_\Psi. \quad (16)
$$

The evolution equation for the Wigner function obtained from (10) is given by

$$
-\frac{\partial W_\rho}{\partial t} = \frac{2}{\hbar} \left( \frac{Ap^2}{2m} + \frac{B m \omega^2}{2} x^2 + \gamma \right) W_\rho + \epsilon \left( \frac{p}{m} \frac{\partial}{\partial x} - m \omega^2 x \frac{\partial}{\partial p} \right) W_\rho + \frac{\hbar}{4} \left( \frac{A}{m \partial x^2} + B m \omega^2 \frac{\partial^2}{\partial p^2} \right) W_\rho, \quad (17)
$$

Because solutions of time dependent Schrödinger equation, i.e., functions $F_k$ from (14) are known, the square integrable solutions of (15)-(17) can be calculated as

$$
W_{kl}(x,p,t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} F_l(x-\xi/2,t) F_k(x+\xi/2,t) e^{i\xi p} d\xi. \quad (18)
$$

For $l = k$, formula (18) reduces to the Wigner function of $F_k$ and corresponds to solutions of analogues to Schrödinger equation. In general, (18) defines a wider class of nondiagonal Wigner functions [20, 27] that together with the standard Wigner function correspond to the solutions of von Neumann equation. This class does not include all possible solutions of (17) but any square integrable initial condition can be expanded as a series of functions (18) at $t = 0$. To calculate (18) it is convenient to introduce new real parameters $u$ and $v$ such that

$$
\sqrt{\frac{\epsilon - iB}{\epsilon + iA}} = u + iv. \quad (19)
$$
It transforms formula (14) into
\[
F_n(x, t) = \mathcal{N}_n \exp\left(-\frac{m\omega}{2\hbar}(u + iv)^2 x^2\right) H_n\left((u + iv)\sqrt{\frac{m\omega}{\hbar}} x\right) \exp\left(-\frac{i}{\hbar} \lambda_n t\right).
\] (20)

From four possible solutions of (19) we choose the one that fulfills conditions: \(u^2 - v^2 > 0\) and \(u > 0\). The explicit dependence of \(A\) and \(B\) on parameters \(u, v, \) and \(\epsilon\) is the following
\[
A = \frac{\epsilon(u^4 - 6uv^2 + v^4 - 1)}{4uvv^2 - u^2},
\]
\[
B = \frac{\epsilon((u^2 + v^2)^2 + 4u^2v^2 - (u^2 - v^2)^2)}{4uv(v^2 - u^2)}.
\]

To shorten notation, we introduce \(\sigma = \frac{u + iv}{\sqrt{u^2 - v^2}}\). Combining (18) with (20), we obtain after integration
\[
W_{kl}(x, p, t) = \frac{\mathcal{N}_k}{\sqrt{u^2 - v^2}} \frac{\mathcal{N}_l}{\sqrt{\pi\hbar\omega}} \exp\left[-\frac{m\omega(u^2 - v^2)}{\hbar} x^2 - (p + 2m\omega ux)^2\right] \times \exp\left[\frac{i}{\hbar} \left(\lambda_l - \lambda_k\right) t\right] \sum_{k=0}^{l} \frac{l!}{j!} \frac{1}{(2j)!} \sigma^j \bar{\sigma}^{l-j} H_{k-j} \left[(u + iv)\sqrt{\frac{m\omega}{\hbar}} x\right] \times H_{l-j} \left[(u + iv)\sqrt{\frac{m\omega}{\hbar}} x\right] H_{j+l} \left[\frac{p + 2m\omega ux}{\sqrt{\hbar\omega(u^2 - v^2)}}\right].
\] (21)

So far, we have not specified the normalization factor \(\mathcal{N}_n\). To do so, we calculate the scalar product of functions \(f_m(x)\) and \(f_n(x)\) from (12), as
\[
\int \overline{f_m(x)} f_n(x) dx = \begin{cases} 0, & \text{for } n + m \in (2\mathbb{N} + 1), \\ (-1)^n (-2)^{m+n} \mathcal{N}_m \mathcal{N}_n \sqrt{\pi} \sigma \bar{\sigma}^{m+n} \sum_{k=0}^{[m/2]} \sum_{l=0}^{[n/2]} C_{kl}^{mn} \frac{\hbar^2 k! \bar{\sigma}^{2l}}{2^{2k+2l}}, & \text{for } n + m \in (2\mathbb{N}), \end{cases}
\] (22)

where \(C_{kl}^{mn} = \frac{\hbar^2}{2^{2k+2l}} \gamma^{m+n} \left[\begin{array}{c} 2k+1 \gamma^{2l+1} \\ 2k+1 \gamma^{m+n-2(k+l)+1} \end{array}\right]\). It leads to the following condition for the normalization constant:
\[
1/\mathcal{N}_n^2 = 2^n \sqrt{\pi} \sigma \bar{\sigma}^{2n} \sum_{k=0}^{[m/2]} \sum_{l=0}^{[n/2]} C_{kl}^{mn} \frac{\hbar^2 k! \bar{\sigma}^{2l}}{2^{2k+2l}}.
\] (23)

Note, that (22) can be also calculated as an integral of the product of the corresponding Wigner functions over all possible positions and momenta.

3.3. Transformation to a Normal Operator
We have considered evolution generator
\[
\hat{G} = \frac{\epsilon - iA}{2m} \hat{p}^2 + \frac{m\omega^2}{2}(\epsilon - iB)\hat{x}^2 - i\gamma \hat{\gamma},
\] (24)

where \(\epsilon > 0, A, B, \gamma \in \mathbb{R}\). In this Section, we will show that for any set of these parameters there exists a transformation that changes (24) into a normal operator. As a first step, let us note that (24) can be rewritten as
\[
\hat{G} = e^{\alpha} [\frac{\epsilon - iA}{2m}] \hat{p}^2 + \frac{e^{ib} m\omega^2}{2} [\epsilon - iB] \hat{x}^2 - i\gamma \hat{\gamma},
\] (25)
where parameters \(a\) and \(b\), such that \(0 \leq a, b < 2\pi\), are defined by values of \(\epsilon\), \(A\), and \(B\):

\[
|\epsilon - iA|e^{ia} := \epsilon - iA \quad \text{and} \quad |\epsilon - iB|e^{ib} := \epsilon - iB.
\]

For a complex number \(w\), operator \(\hat{Q}(w) := \exp\left(\frac{\hbar}{2}\hat{p}\hat{x} + \frac{\hbar}{2}\hat{x}\hat{p}\right)\) is positive. Besides,

\[
\hat{G}\hat{Q}\hat{G}^{-1} = e^{2i\text{Re}(w)\frac{h}{2m}}e^{-2i\text{Re}(w)\frac{m\omega^2}{2}(\epsilon - iB)\hat{x}^2 - i\gamma\hat{I}}. \quad (26)
\]

If we now assume that \(\exp(i2\text{Re}(w)) = \exp(i\frac{(b-a)}{2} + i\pi r)\), for \(r \in \{0, 1\}\), and insert this assumption into (26), we obtain

\[
\hat{G}_Q := \hat{Q}\hat{G}\hat{Q}^{-1} = \pm e^{i\frac{(b-a)}{2}|\epsilon - iA|}\left(\frac{\hbar^2}{2m} + \frac{m\omega^2}{2}|\epsilon - iB|\hat{x}^2\right) - i\gamma\hat{I}. \quad (27)
\]

Above, a plus sign corresponds to \(r = 0\) and a minus to \(r = 1\). Defining \(\hat{\omega} := \omega\sqrt{|\frac{\epsilon - iB}{|\epsilon - iA|}|}\), we get the following expression for \(\hat{G}_Q\)

\[
\hat{G}_Q = \pm e^{i\frac{(b-a)}{4}|\epsilon - iA|}\left(\frac{\hbar^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2\right) - i\gamma\hat{I}. \quad (28)
\]

Note, that \(\hat{G}_Q\) is a normal operator, i.e., \([\hat{G}_Q, (\hat{G}_Q)^\dagger] = 0\). It is yet another way to obtain already known eigenvalues:

\[
\lambda_n = \pm e^{i\frac{(b-a)}{2}|\epsilon - iA|}\sqrt{|\frac{\epsilon - iB}{|\epsilon - iA|}|}\hbar\omega(n + \frac{1}{2}) - i\gamma \quad (29)
\]

(compare with (13)). One has to ensure that the real part of \(\lambda_n\) is positive, determining the choice of a plus or minus sign, accordingly.

This reasoning shows that the similarity transformation, \(\hat{G} \to \hat{Q}\hat{G}\hat{Q}^{-1} = \hat{G}_Q\), transforms initial \(\hat{G}\) into a normal operator \(\hat{G}_Q\). One can show that under some natural physical assumptions (\(\hat{G}\) is dense-defined, closed and its eigenvectors span a dense-defined subspace of \(L^2(\mathbb{R})\)) there always exists similarity transformation that changes generator \(\hat{G}\) into a normal operator. It is an important observation because change to orthogonal basis formed by eigenvectors of such normal operator radically reduces challenges of calculations, especially, when one thinks about numerical approximations.

4. Conclusion

We have described evolution of the system governed by NH generator using the Schrödinger picture and Wigner distribution function, and derived the phase space equations that dictate system’s behavior in a general NH case. All this points were illustrated by the example of NH generator square in position and momentum, (24), which we studied in detail. Form of the operator that changes (24) into a normal operator was given explicitly.

Acknowledgments

The research of K.Z. was supported by the National Research Foundation of South Africa under Grants Nos. 95965 and 98892.
References

[1] Teschl G 1999 Mathematical Methods in Quantum Mechanics: with Applications to Schrödinger Operators (Providence: American Mathematical Soc.).

[2] Suura H 1954 Prog. Theor. Phys. 12 49-71.

[3] Feshbach F 1958 Ann. Phys. 5 357; 1962 Ann. Phys. 19 287.

[4] Breuer H P and Petruccione F 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press).

[5] Faisal F H M 1987 Theory of Multiphoton Processes (New York: Plenum Press).

[6] Sergi A and Zloshchastiev K G 2013 Int. J. Mod. Phys. B 27 1350163.

[7] Zloshchastiev K G and Sergi A 2014 J. Mod. Opt. 61 1298-1308.

[8] Sergi A and Zloshchastiev K G 2015 Phys. Rev. A 91 062108.

[9] Sergi A and Zloshchastiev K G 2016 J. Stat. Mech. 2016 033102.

[10] Zloshchastiev K G 2015 Eur. Phys. J. D 69 253.

[11] Sergi A and Giaquinta P V 2016 Entropy 18 451.

[12] Zhang S Y, Fang M F and Xu L 2017 Quant. Inf. Process. 16 234.

[13] Znojil M, Růžička F and Zloshchastiev K G 2017 Symmetry 9 165.

[14] Karakaya E, Altintas F, Güven K and Müstecaplıoğlu Ö E 2014 Eur. Phys. Lett. 105 40001.

[15] Shakeria S, Zandi M H and Bahrampour A 2016 J. Mod. Optics 64 507-514.

[16] Zloshchastiev K G 2016 Phys. Rev. B 94 115136.

[17] Zloshchastiev K G 2017 Ann. Phys. 529 1600185.

[18] Longhi S 2017 Phys. Rev. A 95 062122.

[19] Wigner E 1932 Phys. Rev. 40 749.

[20] Groenewold H J 1946 Physica 12 405-460.

[21] Graef E M and Schubert R 2011 Phys. Rev. A 83 060101.

[22] Tarasov V E 2008 Quantum Mechanics of non-Hamiltonian and Dissipative Systems (Amsterdam: Elsevier).

[23] Ben-Benjamin J S and Cohen L 2013 SPIE 8744 874413.

[24] Ben-Benjamin J S and Cohen L 2013 J. Mod. Opt. 61, 36.

[25] Ben-Benjamin J S, Cohen L and Loughlin P 2015 J. Acoust. Soc. Amer. 138 1122.

[26] Praxmeyer L, Yang P and Lee R-K 2016 Phys. Rev. A 93 042122.

[27] Moyal J 1949 Proc. Camb. Phil. Soc. 45 99-124.

[28] Praxmeyer L and Zloshchastiev K G 2018 Int. J. Mod. Phys. B 32 1850276.