NEW RELATIONS ON ZETA AND $L$ FUNCTIONS

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Abstract. We prove new relations on zeta function at even arguments and Dirichlet $L$ function at odd. The key idea is to make use of Taylor series and partial fraction decomposition of cotangent and secant functions as we discuss in calculus and complex analysis.

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1. Introduction

Riemann zeta function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots, \quad \text{Re}(s) > 1$$

and the following type of Dirichlet $L$ function (also called Dirichlet beta function)

$$L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots, \quad \text{Re}(s) \geq 1$$

are of great importance in number theory.

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Table 1. signed Bernoulli numbers ($B_3 = B_5 = \cdots = 0$)

| $m$ | 0   | 1    | 2    | 4    | 6    | 8    |
|-----|-----|------|------|------|------|------|
| $B_m$ | 1   | $-\frac{1}{2}$ | $\frac{1}{6}$ | $-\frac{1}{30}$ | $\frac{1}{12}$ | $-\frac{1}{30}$ |

| $m$ | 10  | 12   | 14   | 16   | 18   | \cdots |
|-----|-----|------|------|------|------|--------|
| $B_m$ | $\frac{5}{66}$ | $-\frac{691}{2730}$ | $\frac{7}{6}$ | $-\frac{3617}{510}$ | $\frac{43867}{798}$ | \cdots |

Table 2. unsigned Euler numbers

| $m$ | 0   | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | \cdots |
|-----|-----|------|------|------|------|------|------|------|------|------|--------|
| $E_m$ | 1   | 0    | 1    | 0    | 5    | 0    | 61   | 0    | 1385 | 0    | \cdots |

Table 3. known values of $\zeta$ and $L$ functions

| $m$ | 0   | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |
|-----|-----|------|------|------|------|------|------|------|------|
| $\zeta(m)$ | $-\frac{1}{2}$ | $+\infty$ | $\frac{\pi^2}{6}$ | $\frac{\pi^4}{90}$ | $\frac{\pi^6}{945}$ | $\frac{\pi^8}{9450}$ |

| $\zeta(m)$ | $\frac{\pi}{4}$ | $\frac{\pi^3}{32}$ | $\frac{5\pi^5}{1536}$ | $\frac{61\pi^7}{184320}$ |

| $L(m)$ | | | | | | | | | | | |

Theorem 1.1 (Euler). Set $\zeta(0) = -\frac{1}{2}$. Then for each integer $m \geq 0$,

$$
\zeta(2m) = (-1)^{m+1} \left(\frac{2\pi}{2m}\right)^{2m}\frac{B_{2m}}{2(2m)!},
$$

$$
L(2m + 1) = \left(\frac{\pi}{2}\right)^{2m+1} \frac{1}{2(2m)!} E_{2m}
$$

where $B_{2m}, E_{2m}$ are Bernoulli and Euler numbers as in Tables 1, 2.

See Ayoub [1] for history of these series and numbers.

The aim of this article is to find new relations on $\{\zeta(2m)\}_{m=0}^{\infty}$ and $\{L(2m + 1)\}_{m=0}^{\infty}$. Often many researchers discover infinite series involving $\zeta(2m)$ as follows.
Table 4. Summary of this article

| ζ                      | L                       |
|------------------------|--------------------------|
| ζ(2m) = (-1)^{m+1}(2π)^{2m} B_{2m} | L(2m + 1) = \left(\frac{π}{2}\right)^{2m+1} \frac{1}{2(2m)!} E_{2m} |
| 1 < ζ(m) < 2 (m > 1)   | 0 < L(m) < 1 (m ≥ 1)     |
| ζ(m) → 1 (m → ∞)      | L(m) → 1 (m → ∞)         |
| ∞ \sum_{m=2} (ζ(m) - 1) = 1 | Theorem 2.3 |
| \frac{π z}{2} cot(πz) = \sum_{m=0}^{∞} ζ(2m)z^{2m} | Theorem 3.4 |
| ζ(2m) = \frac{2}{2m+1} \sum_{k=1}^{m-1} ζ(2k)ζ(2m-2k) | Theorem 4.2 |
| Theorem 5.1 (mixed formula) |

Borwein-Bradley-Crandall [3, p.255, 262-263]:

\[ \sum_{k=1}^{m-1} ζ(2k)ζ(2m-2k) = \left( m + \frac{1}{2} \right) ζ(2m), \]

\[ \sum_{m=1}^{∞} (ζ(2m) - 1) = \frac{3}{4}, \]

\[ \sum_{m=1}^{∞} \frac{ζ(2m) - 1}{m} = \log 2, \]

\[ \sum_{m=1}^{∞} \frac{ζ(2m) - 1}{2^{2m}} = \frac{1}{6}, \]

\[ \sum_{m=1}^{∞} \frac{ζ(2m) - 1}{4^{2m}} = \frac{13}{30} - \frac{π}{8}, \]

\[ \sum_{m=1}^{∞} \frac{ζ(2m) - 1}{8^{2m}} = \frac{61}{126} - \frac{π}{16} \frac{\sqrt{2} + 1}{\sqrt{2} - 1}. \]
\[
\sum_{m=1}^{\infty} (\zeta(4m) - 1) = \frac{7}{8} - \frac{\pi}{4} \coth(\pi).
\]

Choi [4, p.388]:
\[
\sum_{m=1}^{\infty} \frac{\zeta(2m) - 1}{m + 1} = \frac{3}{2} - \log \pi.
\]

Srivastava [7, p.133, 136]:
\[
\sum_{m=1}^{\infty} \frac{\zeta(2m)}{m(2m+1)} = \log(2\pi) - 1,
\]
\[
\sum_{m=1}^{\infty} \frac{\zeta(2m)}{m 2^{2m}} = \log \left( \frac{1}{2\pi} \right),
\]
\[
\sum_{m=1}^{\infty} \frac{\zeta(2m)}{(2m+1) 2^{2m}} = \frac{1}{2} - \log 2.
\]

We will see several new series in Example 3.3.

Presumably, we understand zeta and \(L\) functions together. Thus, we will extend known results on \(\zeta(2m)\) to the ones on \(L(2m+1)\) as Theorems 2.3, 3.4 and 4.2. Furthermore, we will prove the equality (which we call the mixed formula) involving both of \(\{\zeta(2m)\}\) and \(\{L(2m+1)\}\) as Theorem 5.1; Table 4 shows summary of this article.

2. Fractional parts of \(\zeta(2m)\) and \(L(2m+1)\)

Throughout \(j, k, l, m, n\) denote nonnegative integers and \(z\) a complex number. Since \(\zeta(m) = 1 + \frac{1}{2^m} + \cdots > 1\) and
\[
2 > \frac{\pi^2}{6} = \zeta(2) > \zeta(3) > \zeta(4) > \zeta(5) > \cdots,
\]
we have \(1 < \zeta(m) < 2\) for all \(m \geq 2\). That is, \(\zeta(m) - 1\) is the fractional part of \(\zeta(m)\). For example,
\[
\zeta(2) - 1 = 0.6449 \cdots,
\]
\[
\zeta(3) - 1 = 0.2020 \cdots,
\]
\[
\zeta(4) - 1 = 0.0823 \cdots,
\]
\[
\zeta(5) - 1 = 0.0369 \cdots.
\]

Shallit-Zikan gave the following surprising identity:

**Theorem 2.1** (Shallit-Zikan [6]).
\[
\sum_{m=2}^{\infty} (\zeta(m) - 1) = 1.
\]
Some other consequences are
\[ \sum_{m=1}^{\infty} (\zeta(2m+1) - 1) = \frac{1}{4}, \]
\[ \sum_{m=1}^{\infty} (\zeta(2m) - 1) = \frac{3}{4}, \]
\[ \sum_{m=2}^{\infty} (-1)^m (\zeta(m) - 1) = \frac{1}{2}. \]

See also Bibiloni-Paradis-Viader [2] and Choi-Quine-Srivastava [4] for Euler-Goldbach Theorem, the origin of such equalities.

Now what can we say about the sum of fractional parts of \( \{L(2m+1)\} \)? As the proposition just below says, things are little different.

**Proposition 2.2.** For each \( m \geq 1 \), we have \( 0 \leq L(m) \leq 1 \).

**Proof.** For fixed \( m \geq 1 \), let
\[ S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(2n-1)^m} \]
be a partial sum of \( L(m) \). It follows that
\[ S_{2N+1} = \sum_{n=1}^{2N+1} \frac{(-1)^{n+1}}{(2n-1)^m} \]
\[ = 1 - \frac{1}{3^m} + \frac{1}{5^m} - \cdots - \frac{1}{(4N-1)^m} + \frac{1}{(4N+1)^m} \]
\[ < 1 \]
and
\[ S_{2N+2} = S_{2N+1} - \frac{1}{(4N+3)^m} < S_{2N+1} < 1. \]

Hence \( L(m) = \lim_{N \to \infty} S_N \leq 1 \). Moreover, there exists the following infinite product (known as Euler product)
\[ L(m) = \prod_{p \equiv 1 \pmod{4}} (1 - p^{-m})^{-1} \prod_{p \equiv 3 \pmod{4}} (1 + p^{-m})^{-1} \]
with \( p \) prime numbers and each factor \( (1 \pm p^{-m})^{-1} \geq 0 \). Thus, \( L(m) \geq 0 \) and this completes the proof. \( \Box \)
For this reason, let us consider $1 - L(m)$ instead of the fractional part of $L(m)$. For example,

$$1 - L(1) = 0.21460 \cdots,$$
$$1 - L(2) = 0.08403 \cdots,$$
$$1 - L(3) = 0.03105 \cdots,$$
$$1 - L(4) = 0.01105 \cdots.$$

The following is an analogy of Theorem 2.1.

**Theorem 2.3.**

1. $$\sum_{m=1}^{\infty} (1 - L(m)) = \frac{1}{2} \log 2 = 0.34657 \cdots,$$

2. $$\sum_{m=1}^{\infty} (1 - L(2m)) = \frac{1}{2} \log 2 - \frac{1}{4},$$

3. $$\sum_{m=1}^{\infty} (1 - L(2m - 1)) = \frac{1}{4},$$

4. $$\sum_{m=1}^{\infty} (-1)^{m-1} (1 - L(m)) = \frac{1}{2} (\log 2 - 1).$$

**Proof.** First, we remark that

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{(2n - 1)^m} \right| = \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{(2n - 1)^m} \leq \sum_{m=2}^{\infty} (\zeta(m) - 1) = 1$$

(absolute convergence) and then

$$\sum_{m=1}^{\infty} (1 - L(m)) = (1 - L(1)) + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n - 1)^m}$$

must be convergent; in particular, we can exchange summations (likewise below).
\[ \sum_{m=1}^{\infty} (1 - L(m)) = \sum_{m=1}^{\infty} \sum_{n=2}^{\infty} \frac{(−1)^n}{(2n - 1)^m} = \sum_{n=2}^{\infty} \frac{(−1)^n}{2n - 1} \sum_{m=0}^{\infty} \left( \frac{1}{2n - 1} \right)^m = \sum_{n=2}^{\infty} \frac{(−1)^n}{2n - 1} \left( \frac{1}{1 - \frac{1}{2n - 1}} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \frac{(−1)^n}{n - 1} = \frac{1}{2} \log 2. \]

\[ \sum_{m=1}^{\infty} (1 - L(2m)) = \sum_{m=1}^{\infty} \sum_{n=2}^{\infty} \frac{(−1)^n}{(2n - 1)^{2m}} = \sum_{n=2}^{\infty} \frac{(−1)^n}{(2n - 1)^2} \sum_{m=0}^{\infty} \left( \frac{1}{2n - 1} \right)^{2m} = \sum_{n=2}^{\infty} \frac{(−1)^n}{(2n - 1)^2} \left( \frac{1}{1 - \left( \frac{1}{2n - 1} \right)^2} \right) = \sum_{n=2}^{\infty} \frac{(−1)^n}{(2n - 2)2n} = \frac{1}{4} \left( \sum_{n=2}^{\infty} \frac{(−1)^n}{n - 1} - \sum_{n=2}^{\infty} \frac{(−1)^n}{n} \right) = \frac{1}{4} (\log 2 + (\log 2 - 1)) = \frac{1}{2} \log 2 - \frac{1}{4}. \]

Moreover, (3) is (1) − (2) and (4) is (2) − (3). \(\square\)

3. Generating function

To study sequences \{\zeta(2m)\} and \{L(2m + 1)\}, it is helpful to find their generating functions. In the sequel, we will often use the following fact in complex analysis implicitly.

**Fact 3.1.** If \(F(z) = \sum_{m=0}^{\infty} \alpha_m z^m \ (\alpha_m \in \mathbb{C})\) is a convergent power series with the radius of convergence \(R\), then so is \(F'(z)\) and moreover it is given by the power series \(\sum_{m=0}^{\infty} m \alpha_m z^{m-1}\).
Proposition 3.2. For $|z| < 1$,

$$-\frac{\pi z}{2} \cot(\pi z) = \sum_{m=0}^{\infty} \zeta(2m) z^{2m}.$$ 

Proof. Recall from complex analysis that

$$\cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$ 

Now let $|z| < 1$. Then, we have $|\frac{z}{n}|^2 < 1$ for all $n \geq 1$ so that

$$-\frac{\pi z}{2} \cot(\pi z) = -\frac{1}{2} - \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2}$$

$$= -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{z^2}{n^2} \left( 1 - \left(\frac{z}{n}\right)^2 \right)$$

$$= -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{z^2}{n^2} \left( \sum_{m=0}^{\infty} \left(\frac{z}{n}\right)^{2m} \right)$$

$$= -\frac{1}{2} + \sum_{m=0}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2m+2}} \right) z^{2m+2}$$

$$= -\frac{1}{2} + \sum_{m=0}^{\infty} \zeta(2m+2) z^{2m+2}$$

$$= \sum_{m=0}^{\infty} \zeta(2m) z^{2m}.$$ 

□
Example 3.3. Set \( f(z) = -\frac{\pi z}{2} \cot(\pi z) \) as above. Observe that
\[
\sum_{m=0}^{\infty} \frac{\zeta(2m)}{2^{2m}} = 0,
\sum_{m=0}^{\infty} \frac{\zeta(2m)}{3^{2m}} = -\frac{\pi}{6\sqrt{3}},
\sum_{m=0}^{\infty} \frac{\zeta(2m)}{4^{2m}} = -\frac{\pi}{8},
\sum_{m=0}^{\infty} \frac{\zeta(2m)}{5^{2m}} = -\frac{\sqrt{5}}{10\sqrt{5} - \sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right),
\sum_{m=0}^{\infty} \frac{\zeta(2m)}{6^{2m}} = -\frac{\sqrt{3}}{12}\pi,
\]
are \( f(\frac{1}{2}), f(\frac{1}{4}), f(\frac{1}{8}), f(\frac{1}{6}) \), respectively (recall \( \cot \frac{\pi}{5} = \frac{1 + \sqrt{5}}{\sqrt{5} - \sqrt{5}} \)). There are some consequences. We see that
\[
z f'(z) = \sum_{m=0}^{\infty} 2m \zeta(2m) z^{2m},
\]
that is,
\[
\sum_{m=0}^{\infty} 2m \zeta(2m) z^{2m} = -\frac{\pi z}{2} \left( \cot \pi z + \pi z (\cot^2(\pi z) - 1) \right).
\]
Then observe also that
\[
\sum_{m=0}^{\infty} \frac{2m}{2^{2m}} \zeta(2m) = \frac{1}{2} f'' \left( \frac{1}{2} \right) = \frac{\pi^2}{8},
\sum_{m=0}^{\infty} \frac{2m}{3^{2m}} \zeta(2m) = \frac{1}{3} f'' \left( \frac{1}{3} \right) = -\frac{\pi}{6} \left( \frac{1}{\sqrt{3}} - \frac{4}{9} \pi \right)
\]
and so on.

Theorem 3.4. For \( |z| < 1 \),
\[
\frac{\pi z}{4} \sec \frac{\pi z}{2} = \sum_{m=0}^{\infty} L(2m + 1) z^{2m+1}.
\]

Proof. Recall from complex analysis that
\[
\sec z = \sum_{n=1}^{\infty} \frac{(-1)^n(2n - 1)\pi}{z^2 - (\frac{2n-1}{2} \pi)^2}.
\]
Now let $|z| < 1$. Then, for all $n \geq 1$, we have $|\frac{z}{2n-1}|^2 < 1$ so that

\[
\frac{\pi z}{4} \sec \frac{\pi z}{2} = \frac{\pi z}{4} \sum_{n=1}^{\infty} \frac{(-1)^n(2n - 1)\pi}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{2n-1}{2}\pi\right)^2}
\]

\[
= z \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n - 1)^2} \left(1 - \frac{\pi^2}{(2n-1)^2}\right)
\]

\[
= z \sum_{n=1}^{\infty} \frac{(-1)^n}{2n - 1} \sum_{m=0}^{\infty} \left(\frac{z}{2n-1}\right)^{2m}
\]

\[
= \sum_{m=0}^{\infty} L(2m + 1) z^{2m+1}.
\]

\[
\square
\]

**Corollary 3.5.**

\[
\sum_{m=0}^{\infty} \frac{L(2m + 1)}{2^{2m+1}} = \frac{\sqrt{2}}{8} \pi,
\]

\[
\sum_{m=0}^{\infty} \frac{L(2m + 1)}{3^{2m+1}} = \frac{\pi}{6\sqrt{3}},
\]

\[
\sum_{m=0}^{\infty} \frac{L(2m + 1)}{4^{2m+1}} = \frac{\pi}{8\sqrt{2} + \sqrt{2}},
\]

\[
\sum_{m=0}^{\infty} \frac{L(2m + 1)}{5^{2m+1}} = \frac{\sqrt{2}}{10\sqrt{5} + \sqrt{5}} \pi,
\]

\[
\sum_{m=0}^{\infty} \frac{L(2m + 1)}{6^{2m+1}} = \frac{\sqrt{6} - \sqrt{2}}{24} \pi.
\]

**Proof.** Set $g(z) = \frac{\pi z}{4} \sec \frac{\pi z}{2}$ as above. These identities are $g \left(\frac{1}{2}\right)$, $g \left(\frac{1}{3}\right)$, $g \left(\frac{1}{4}\right)$, $g \left(\frac{1}{5}\right)$, $g \left(\frac{1}{6}\right)$ with

\[
\sec \frac{\pi}{8} = \frac{2}{\sqrt{2} + \sqrt{2}} \quad \sec \frac{\pi}{10} = \frac{2\sqrt{2}}{\sqrt{5} + \sqrt{5}} \quad \sec \frac{\pi}{12} = \sqrt{6} - \sqrt{2}.
\]

Moreover,

\[
zg'(z) = \frac{\pi z}{4} \sec \frac{\pi z}{2} \left(1 + \frac{\pi z}{2} \tan \frac{\pi z}{2}\right)
\]
leads us to
\[
\sum_{m=0}^{\infty} \frac{2m+1}{2^{2m+1}} L(2m+1) = \frac{1}{2} g'(\frac{1}{2}) = \frac{\sqrt{2}}{8} \pi \left(1 + \frac{\pi}{4}\right),
\]
\[
\sum_{m=0}^{\infty} \frac{2m+1}{3^{2m+1}} L(2m+1) = \frac{1}{3} g'(\frac{1}{3}) = \frac{1}{6\sqrt{3}} \pi \left(1 + \frac{\pi}{18\sqrt{3}}\right)
\]
and so on. \[ \square \]

4. Convolution

It is well-known that \(\{\zeta(2m)\}\) satisfies the following relation.

**Fact 4.1 ([3, p.255]).** For \(m \geq 2\),

\[
\zeta(2m) = \frac{2}{2m+1} \sum_{j,k>0}^{j+k=m} \zeta(2j)\zeta(2k).
\]

The following is an analogous result of this.

**Theorem 4.2.** For \(m \geq 1\), we have

\[
L(2m+1) = \frac{1}{(2m-1)2m} \left(8 \sum_{j,k,l \geq 0}^{j+k+l=m-1} L(2j+1)L(2k+1)L(2l+1) - \frac{\pi^2}{4} L(2m-1) \right).
\]

**Proof.** Note that

\[
\left(\sec \frac{\pi z}{2}\right)'' = \frac{\pi^2}{4} \left(2 \sec^3 \frac{\pi z}{2} - \sec \frac{\pi z}{2}\right).
\]

Let us now express each term in both sides as power series:

\[
\left(\sec \frac{\pi z}{2}\right)'' = \frac{4}{\pi} \sum_{m=0}^{\infty} (2m)(2m-1)L(2m+1)z^{2m-2},
\]
\[
\frac{\pi^2}{4} \left(2 \sec^3 \frac{\pi z}{2}\right) = \frac{\pi^2}{2} \left(\frac{4}{\pi} \sum_{m=0}^{\infty} L(2m+1)z^{2m}\right)^3
\]
\[
= \frac{32}{\pi} \sum_{m=0}^{\infty} \left(\sum_{j,k,l \geq 0}^{j+k+l=m} L(2j+1)L(2k+1)L(2l+1) \right) z^{2m},
\]
\[
-\frac{\pi^2}{4} \sec \frac{\pi z}{2} = -\pi \sum_{m=0}^{\infty} L(2m+1)z^{2m}.
\]
Equate the coefficients of $z^{2(m-1)}$ ($m \geq 1$) in (5):

$$
\frac{4}{\pi}(2m)(2m-1)L(2m+1) = \frac{32}{\pi} \left( \sum_{j,k,l \geq 0} L(2j+1)L(2k+1)L(2l+1) \right) - \pi L(2m-1).
$$

Conclude that

$$
L(2m+1) = \frac{1}{(2m-1)2m} \left( 8 \sum_{j,k,l \geq 0} L(2j+1)L(2k+1)L(2l+1) - \frac{\pi^2}{4} L(2m-1) \right).
$$

For example,

$$
L(5) = \frac{1}{3 \cdot 4} \left( 8 \cdot 3L(1)^2L(3) - \frac{\pi^2}{4} L(3) \right) = \frac{1}{12} \left( 24 \left( \frac{\pi}{4} \right)^2 - \frac{\pi^2}{4} \right) \frac{1}{32} \pi^3 = \frac{5}{1536} \pi^5.
$$

5. Mixed formula

We have seen many results for each of zeta function and $L$ function. Here, we prove the identity involving both of $\{\zeta(2m)\}$ and $\{L(2m+1)\}$.

**Theorem 5.1** (mixed formula). For $m \geq 1$,

$$
L(2m+1) = \frac{1}{4m} \left( \sum_{j,k \geq 0} \frac{\zeta(2j)}{2^{2j}} 8kL(2k+1) + \frac{\pi^2}{2} L(2m-1) \right).
$$

**Proof.** Note that

$$
\left( \sec \frac{\pi z}{2} \right)' = \frac{\pi}{2} \tan \frac{\pi z}{2} \sec \frac{\pi z}{2}
$$

and so

$$
\left( -\frac{\pi z}{4} \cot \frac{\pi z}{2} \right) \left( \sec \frac{\pi z}{2} \right)' = -\frac{\pi^2 z}{8} \sec \frac{\pi z}{2},
$$

that is,

$$
\left( \sum_{j=0}^{\infty} \zeta(2j) \left( \frac{z}{2} \right)^{2j} \right) \left( \frac{4}{\pi} \sum_{k=0}^{\infty} 2kL(2k+1)z^{2k-1} \right) = -\frac{\pi}{2} \left( \sum_{m=0}^{\infty} L(2m+1)z^{2m+1} \right).
$$

Now equate the coefficients of $z^{2m+1}$ ($m \geq 0$) in both sides to obtain

$$
\sum_{j,k \geq 0 \atop j+k=m+1} \frac{\zeta(2j)}{2^{2j}} \frac{4}{\pi} 2kL(2k+1) = -\frac{\pi}{2} L(2m+1).
$$
In the sum on the left hand side, the term for \((j, k) = (0, m + 1)\) is 
\[
\zeta(0) \frac{\pi}{2} 2(m + 1) L(2(m + 1) + 1) = -\frac{4(m + 1)}{\pi} L(2m + 3)
\]
and for \((j, k) = (m + 1, 0)\) is 0. Therefore, we have
\[
-\frac{4(m + 1)}{\pi} L(2m + 3) + \sum_{j,k>0 \atop j+k=m+1} \frac{\zeta(2j)}{2^{2j}} \frac{4}{\pi} 2k L(2k + 1) = -\frac{\pi}{2} L(2m + 1)
\]
and so
\[
L(2m + 3) = \frac{\pi}{4(m + 1)} \left( \sum_{j,k>0 \atop j+k=m+1} \frac{\zeta(2j)}{2^{2j}} \frac{8k}{\pi} L(2k + 1) + \frac{\pi}{2} L(2m + 1) \right).
\]
With \(m \mapsto m - 1\), we conclude that
\[
L(2m + 1) = \frac{1}{4m} \left( \sum_{j,k>0 \atop j+k=m} \frac{\zeta(2j)}{2^{2j}} 8k L(2k + 1) + \frac{\pi^2}{2} L(2m - 1) \right).
\]

For example, let \(m = 3\).
\[
L(7) = \frac{1}{12} \left( \frac{\zeta(2)}{2^2} 16 L(5) + \frac{\zeta(4)}{2^4} 8 L(3) + \frac{\pi^2}{2} L(5) \right)
\]
\[
= \frac{1}{12} \left( \frac{\pi^2}{6} \frac{5\pi^5}{1536} + \frac{1}{290} \frac{\pi^4}{32} + \frac{\pi^2}{2} \frac{5\pi^5}{1536} \right) = \frac{61}{184320} \pi^7.
\]

6. Conclusion

In this note, we proved many identities on \(\{\zeta(2m)\}_{m=0}^{\infty}\) and \(\{L(2m + 1)\}_{m=0}^{\infty}\). The key idea is to make use of Taylor series and partial fraction decomposition of cotangent and secant functions as we sometimes discuss in calculus and complex analysis. Indeed, this method is powerful enough to find more formulas. For example, we can derive other relations from
\[
2 \cot(z) \cot(2z) = \cot^2 z - 1,
\]
\[
1 + \cot^2 z = \cot^2 z \sec^2 z,
\]
\[
-\frac{\pi z}{4} (\cot(\pi z) + \coth(\pi z)) = \sum_{m=0}^{\infty} \zeta(4m) z^{4m},
\]
\[
-\frac{\pi z}{4} (\cot(\pi z) - \coth(\pi z)) = \sum_{m=0}^{\infty} \zeta(4m + 2) z^{4m+2},
\]

\[
-\frac{\pi z}{4} (\cot(\pi z) + \coth(\pi z)) = \sum_{m=0}^{\infty} \zeta(4m) z^{4m},
\]
\[
-\frac{\pi z}{4} (\cot(\pi z) - \coth(\pi z)) = \sum_{m=0}^{\infty} \zeta(4m + 2) z^{4m+2},
\]

\[
\frac{\pi z}{8} \left( \sec \frac{\pi z}{2} + \text{sech} \frac{\pi z}{2} \right) = \sum_{m=0}^{\infty} L(4m + 1)z^{4m+1},
\]

\[
\frac{\pi z}{8} \left( \sec \frac{\pi z}{2} - \text{sech} \frac{\pi z}{2} \right) = \sum_{m=0}^{\infty} L(4m + 3)z^{4m+3}.
\]

It is also easy to translate our results into relations for Bernoulli and Euler numbers as Dilcher discussed [5].

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