THE DIRAC SEA FOR THE NON-SEPARABLE HILBERT SPACES

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Abstract. We give a rigorous construction of the Dirac Sea for the fermionic quantization in the non-separable Hilbert spaces. These CAR-representations depend on the Axiom of Choice, hence are not unique, nevertheless they are unitarily equivalent to the classic Fock representation.

1. Introduction

The old hole theory of Dirac, where the vacuum is replaced by a sea filled of all the negative energy states, continues to arouse the interest of physicists and mathematicians despite the severe criticism by Weinberg. For instance the Dirac sea plays an important role in the theory of the fermionic projector of Finster [9], and recently it appears also in quantum cosmology [4]. Of the mathematical point of view, a rigorous definition of such a sea is an interesting issue. In the case of the usual framework of the separable Hilbert spaces, an elegant construction is provided by a semi-infinite wedge product in the work of Dimock [6] for the free Dirac equation, and for the external field problem by Deckert and co-authors [5]. At the first glance, this approach heavily depends of the countability of the Hilbert basis and its generalization to the non-separable Hilbert spaces is not obvious. The purpose of this paper consists in showing that this goal is achievable with a little work of set theory. This one consists in defining in a coherent way a suitable notion describing the “parity” of the size of all the subsets of an infinite set $X$. In short we prove the existence of a homomorphism $\pi$ from the Abelian group of the power set of $X$ endowed with the symmetric difference, to $\mathbb{Z}/2\mathbb{Z}$, such that $\pi(A) = 0$ (resp. 1) if $A$ is a finite part of $X$ with an even (resp. odd) cardinal. The proof that is based on the technics of the ultrapowers, is strongly inspired by the theory of the numerosities of Benci et alii [2], [3] (a shorter proof uses powerful arguments from the Boolean Algebras Theory). As regards the axiomatic framework, the price to pay to be able to treat the case of the non-separable Hilbert spaces, is a triple recourse to some consequences of the Axiom of Choice (see e.g [12]): 1) for the existence of a Hilbert basis $(e_x)_{x \in X}$ on a non-separable Hilbert space; 2) for the existence of a linear order on the set $X$ that indexes this basis; 3) in the proofs of the existence of $\pi$, we use one of the following three consequences of Zorn’s Lemma: i) the existence of a suitable ultrafilter on the set of the finite parts
of $X$, ii) Sikorski’s extension theorem, iii) the equivalence between “completeness” and “injectivity” for the Boolean Algebras. As a consequence, our construction leads to a lot of different quantizations. Fortunately, they are all unitarily equivalent to the classic Fock quantization. Concerning the role of the non-separable Hilbert spaces in Physics, the negative opinion of Streater and Wightman in *PCT, Spin and Statistics, and All That*, is well known. Nevertheless this issue is always matter to debate, see Earman [8] for a highlighting discussion. We also remark that the non-separable Hilbert spaces naturally arise in loop quantum gravity (see e.g. [1]).

We now introduce our strategy. We first fix the notations by recalling the well known basics of the fermionic quantization (see e.g. [7], [13], [15]). We consider a complex Hilbert space $(\mathfrak{h}, \langle ., . \rangle_\mathfrak{h})$ where $\langle ., . \rangle_\mathfrak{h}$ is the inner product linear with respect to the second argument, and we look for a Hilbert space $\mathfrak{H}$ and an antilinear map $\Psi$ from $\mathfrak{h}$ to the space of the linear maps on $\mathfrak{H}$ satisfying the canonical anticommutation relations (CAR): for any $u, v \in \mathfrak{h}$ we have

$$\{\Psi(u), [\Psi(v)]^*\} = \langle u; v \rangle_\mathfrak{h} \text{Id}_{\mathfrak{H}},$$

$$\{\Psi(u), \Psi(v)\} = 0,$$

where $\{A, B\} := AB + BA$ is the anticommutator of two operators $A$ and $B$. Taking the adjoint of (2) we also have

$$\{[\Psi(u)]^*, [\Psi(v)]^*\} = 0.$$

Another important consequence of the CAR is that $\Psi(u)$ belongs to the space $\mathcal{L}(\mathfrak{H})$ of the bounded linear maps on $\mathfrak{H}$ and we have

$$\|\Psi(u)\|_{\mathcal{L}(\mathfrak{H})} = \|\Psi(u)^*\|_{\mathcal{L}(\mathfrak{H})} = \|u\|_\mathfrak{h}.$$

The classic representation of the CAR on $\mathfrak{h}$ is given by choosing the antisymmetric Fock space

$$\mathfrak{H} = \mathcal{F}^\wedge(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathfrak{h}^\wedge n,$$

and $\Psi = a$, the usual annihilation operator that is the adjoint of the creation operator $a^*$, which is nicely expressed by using the wedge product:

$$a^*(u)(v_1 \wedge v_2 \wedge \ldots \wedge v_n) = u \wedge v_1 \wedge v_2 \wedge \ldots \wedge v_n.$$

Up to an unitary transform, the Fock quantization is the unique irreductible representation of the CAR on $\mathfrak{h}$.

We are now ready to introduce the issue of the Dirac sea in this framework. The idea is that, unlike the classic Dirac quantum field, which annihilates a particle but *creates* an anti-particle, the fermionic quantum field should just be an annihilation operator in a suitable sense: the creation of an antiparticle in the Fock quantization should be understood as the creation
of a hole in the Dirac sea, \( i.e. \) the annihilation of a state of negative energy, the Dirac sea being filled with all these states. This idea has been rigorously implemented by Dimock in [6] when \( \mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+ \) is \emph{separable}. In order to point out the role of the separability, we briefly describe his approach based on the semi-infinite wedge products. In the sequel, if \( X \) is a set of integers or ordinals, we put \( X^* := X \setminus \{0\} \).

Given a Hilbert basis \( (e_j)_{j \in \mathbb{N}^*} \) of \( \mathfrak{h}_\pm \), we consider the Hilbert completion \( H \) of the free vector space spanned by the formal symbols

\[
e_I := e_{i_1} \wedge e_{i_2} \wedge \ldots
\]

where \( I = \{i_s, \ s \in \mathbb{N}^*\} \) and \( (i_s)_{s \in \mathbb{N}^*} \in (\mathbb{Z}^*)^{\mathbb{N}^*} \) is strictly decreasing sequence (for the usual order on \( \mathbb{Z} \)) satisfying for \( s \) large enough

\[
i_{s+1} = i_s - 1.
\]

Then the Dirac sea is the vector

\[
\Omega_D := e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge \ldots
\]

and the quantum field is the antilinear map \( \Psi \) from \( \mathfrak{h} \) to \( \mathcal{L}(\mathfrak{h}) \) defined by:

\[
\Psi(e_j)e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots = \begin{cases} 
0 & \text{if } \forall s \in \mathbb{N}, \ j \neq i_s, \\
(-1)^{s+1}e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \ldots & \text{if } \exists s \in \mathbb{N}, \ j = i_s.
\end{cases}
\]

The key point in definition (10) is the factor \((-1)^s\) that is well defined thanks to the obvious fact that \( s = s(j, I) \) defined by

\[
j = i_s
\]

is a \emph{finite} ordinal that is just the cardinal of the subset

\[
X(j, I) := \{ i \in I, \ i \geq j\}.
\]

A crucial property to obtain the CAR is that

\[
k > j, \ k \in I \Rightarrow (-1)^{s(j, I)} = -(-1)^{s(j, I \setminus \{k\})},
\]

since

\[
k > j, \ k \in I \Rightarrow s(j, I) = s(j, I \setminus \{k\}) + 1.
\]

The situation drastically changes when \( \mathfrak{h} \) is a \emph{non-separable} Hilbert space. In this case we have to consider a Hilbert basis \( (e_j)_{j \in X} \) of \( \mathfrak{h} \), where \( X \) is a totally ordered set of uncountable cardinal \( |X| = \aleph > \aleph_0 \), and, instead of the \emph{countable} wedge products, the straight generalization of (7) and (8) would consist in considering the formal symbols

\[
e_I = \bigwedge_{j \in I} e_j
\]
where \( I = \{i_s, \ s \in \mathbb{N}^*\} \) is uncountable, and \((i_s)_{s \in \mathbb{N}^*} \in X^{\mathbb{N}^*} \) is a strictly decreasing generalized sequence satisfying for some \( a \in X \)
\[
(16) \quad x < a \Rightarrow x \in I.
\]
Now given \( I \) and \( j \in X \), \( s = s(j, I) \) solution of (11) is an ordinal that belongs to \( \mathbb{N} \) and we have to define \((-1)^s\) satisfying (13). Of course the difficulty arises when \( s \) is an infinite ordinal. We could try to use the Cantor’s normal form theorem that assures that \( s \) can be uniquely expressed as
\[
s = \lambda + N,
\]
with \( N \in \mathbb{N} \) and \( \lambda \) is a limit ordinal. Deciding classically that the limit ordinals are even, we could define
\[
(17) \quad (-1)^s := (-1)^N.
\]
Unfortunately this definition does not assure the key point (13) since (14) can be wrong due to the absorbing property of the ordinal addition
\[
1 + \lambda = \lambda.
\]
For an elementary example, we can take \( I = \{-n; \ n \in \mathbb{N}^*\} \cup \{-\omega\} \) where \(-\omega < -n \) for any integer \( n \). Then with \( j = -\omega \) and \( k = -n \) for some integer \( n \), we have \( s(-\omega, I) = \aleph_0 \) and also \( s(-\omega, I \setminus \{-n\}) = \aleph_0 \). Therefore with definition (17) we have
\[
(-1)^s(-\omega, I) = 1 = (-1)^s(-\omega, I \setminus \{-n\})
\]
that contradicts (13). We conclude that the ordinal calculus is not sufficient to associate 1 or \(-1\) to the set \( X(j, I) \) in such a way that (13) is satisfied even if \( X(j, I) \) is infinite. To overcome this difficulty, we simply remark that in the separable case for which \( X(j, I) \) is finite, \((-1)^{s(j, I)} = 1\) if the cardinal of \( X(j, I) \) is even and \((-1)^{s(j, I)} = -1\) if the cardinal of \( X(j, I) \) is odd. Finally we are led to ask a somewhat weird question: what is the parity of the size of an infinite set? Clearly, in the infinite case, the cardinal is not a tool sufficiently subtle to distinguish the size of \( X(j, I) \) from the size of \( X(j, I \setminus \{k\}) \). In fact a refined concept of size of an infinite set has been introduced by Benci and co-authors [2], [3]. It is a hypernatural \( s \) and we could define \((-1)^s\) in the framework of this theory of the numerosities, from the parity of \( s \), but we prefer to give a direct construction in the next section. We introduce the quantum fields in an abstract setting in the third part. We present the application to the Dirac theory in the last section.

2. Parity of an infinite set

We first introduce some notations. The cardinal of a set \( X \) is denoted \(|X|\), \( \mathcal{P}(X) \) is its power set, and \( \mathcal{P}_F(X) \) is the set of the finite parts of \( X \). The symmetric difference \( \Delta \) is defined by
\[
A, B \in \mathcal{P}(X), \quad A \Delta B := (A \cup B) \setminus (A \cap B).
\]
Given an infinite set $X$ we look for a homomorphism $\pi$ from $(\mathcal{P}(X), \Delta)$ to $(\mathbb{Z}/2\mathbb{Z}, +)$ such that

$$\forall a \in X, \quad \pi(\{a\}) = 1.$$ \hfill (18)

The requirement

$$\pi(A \Delta B) = \pi(A) + \pi(B)$$ \hfill (19)

is equivalent to the couple of properties

$$\forall A, B \in \mathcal{P}(X), \quad A \cap B = \emptyset \Rightarrow \pi(A \cup B) = \pi(A) + \pi(B) \text{ in } \mathbb{Z}/2\mathbb{Z},$$ \hfill (20)

$$\forall A, B \in \mathcal{P}(X), \quad B \subset A \Rightarrow \pi(A \setminus B) = \pi(A) - \pi(B) \text{ in } \mathbb{Z}/2\mathbb{Z}.$$ \hfill (21)

If $\pi$ exists, then (18) and (20) imply that for $A \in \mathcal{P}_F(X)$, $\pi(A)$ is the usual parity of the cardinal of $A$, defined as 0 if it is even and 1 if it is odd. The extension of $\pi$ from $\mathcal{P}_F(X)$ to the whole $\mathcal{P}(X)$ is not at all obvious: its existence depends on the Axiom of Choice, and then such a $\pi$ is not unique since we may impose $\pi(X) = 0$ or $\pi(X) = 1$ as well.

**Theorem 2.1** (Parity function). *We consider an infinite set $X$ and $p \in \mathbb{Z}/2\mathbb{Z}$. Then there exists a homomorphism $\pi : (\mathcal{P}(X), \Delta) \to (\mathbb{Z}/2\mathbb{Z}, +)$ satisfying (18) and

$$\pi(X) = p.$$ \hfill (22)*

We present two proofs of this result. The first one, rather long and pedestrian, uses a suitable Ultrafilter. The second one, short and elegant, has been suggested by the anonymous referee; it is based on a powerful tool of the Boolean Algebras Theory: either Sikorski’s extension theorem, or the injectivity of $\{0, 1\}$.

**First proof.** We identify the additive group $\mathbb{Z}/2\mathbb{Z}$ and $\{0, 1\}$ and we denote $\mathcal{I} := \mathcal{P}_F(X) \setminus \{\emptyset\}$. Given $p \in \{0, 1\}$, we define for any $i \in \mathcal{I}$

$$C_p(i) := \{j \in \mathcal{I}; \quad i \subset j, \quad | j | \in 2\mathbb{N} + p\} \in \mathcal{P}(\mathcal{I}) \setminus \{\emptyset\}. \hfill (23)$$

We have for any $i, j \in \mathcal{I}$

$$C_p(i) \cap C_p(j) = C_p(i \cup j), \hfill (24)$$

hence the family $\mathcal{B}_p := (C_p(i))_{i \in \mathcal{I}}$ is a filter basis on $\mathcal{I}$. Applying the Axiom of Choice, we consider an ultrafilter $\mathcal{U}_p$ containing $\mathcal{B}_p$. $\{0, 1\}^\mathcal{I}$ being the additive group of the maps from $\mathcal{I}$ to $\mathbb{Z}/2\mathbb{Z}$, we consider its ultrapower

$$\{0, 1\}^\mathcal{I} / \mathcal{U}_p := \{\dot{\varphi}, \quad \varphi \in \{0, 1\}^\mathcal{I}, \quad \dot{\varphi} := \{\psi \in \{0, 1\}^\mathcal{I}; \quad \{i; \quad \varphi(i) = \psi(i)\} \in \mathcal{U}_p\}. \hfill (25)$$

In fact this ultrapower is just a pair set: since $\mathcal{U}_p$ is an ultrafilter, either $\{i \in \mathcal{I}; \quad \varphi(i) = 0\}$ or $\{i \in \mathcal{I}; \quad \varphi(i) = 1\}$ belongs to $\mathcal{U}_p$. We denote $p_\varphi$ the
element of \(\{0, 1\}\) such that \(\{i \in \mathcal{I}; \ \varphi(i) = p_\varphi\} \in \mathcal{U}_p\). Now for \(\psi \in \varphi\) we have

\[
\{i \in \mathcal{I}; \ \psi(i) = p_\varphi\} \supset \{i \in \mathcal{I}; \ \psi(i) = \varphi(i)\} \cap \{i \in \mathcal{I}; \ \varphi(i) = p_\varphi\} \in \mathcal{U}_p,
\]

hence \(\{i \in \mathcal{I}; \ \psi(i) = p_\varphi\} \in \mathcal{U}_p\) and thus \(p_\varphi = p_\psi\), and we may introduce the map \(p\)

\[
p : \varphi \in \{0, 1\}^\mathcal{I} / \mathcal{U}_p \longrightarrow p(\varphi) := p_\varphi \in \{0, 1\}.
\]

\(p\) is a group isomorphism. It is obviously surjective, it is a homomorphism since

\[
\{i; \ \varphi(i) + \psi(i) = p_\varphi + p_\psi\} \supset \{i; \ \varphi(i) = p_\varphi\} \cap \{i; \ \psi(i) = p_\psi\} \in \mathcal{U}_p,
\]

hence \(\{i; \ \varphi(i) + \psi(i) = p_\varphi + p_\psi\} \in \mathcal{U}_p\) and thus \(p(\varphi + \psi) = p(\varphi) + p(\psi)\), and finally it is injective since

\[
p(\varphi) = 0 \iff p_\varphi = 0 \iff \{i; \ \varphi(i) = 0\} \in \mathcal{U}_p \iff \varphi = 0.
\]

To any \(A \in \mathcal{P}(X)\) we associate \(\varphi_A \in \{0, 1\}^\mathcal{I}\) by

\[
\forall i \in \mathcal{I}, \ \varphi_A(i) = 0 \iff A \cap i \in 2\mathbb{N}, \ \varphi_A(i) = 1 \iff A \cap i \in 2\mathbb{N} + 1,
\]

and we define

\[
(27) \quad \pi(A) := p(\hat{\varphi}_A).
\]

We remark that for any \(a \in X\), we have

\[
(28) \quad \{i \in \mathcal{I}; \ a \in i\} \in \mathcal{U}_p
\]

since

\[
\{i \in \mathcal{I}; \ a \in i\} \supset C_p(\{a\}) \in \mathcal{U}_p.
\]

We have

\[
\varphi_{\{a\}}(i) = 1 \iff i \in \{j \in \mathcal{I}; \ a \in j\},
\]

hence \(p_{\hat{\varphi}_{\{a\}}} = 1\) and we deduce that \((18)\) is satisfied. Now for \(A, B \in \mathcal{P}(X)\) with \(A \cap B = \emptyset\), we have

\[
\{i; \ \varphi_{A \cup B}(i) = p_{\hat{\varphi}_A} + p_{\hat{\varphi}_B}\} \supset \{i; \ \varphi(A)(i) = p_{\hat{\varphi}_A}\} \cap \{i; \ \varphi(B)(i) = p_{\hat{\varphi}_B}\} \in \mathcal{U}_p
\]

hence \(p_{\hat{\varphi}_{A \cup B}} = p_{\hat{\varphi}_A} + p_{\hat{\varphi}_B}\) and \((20)\) is established. Moreover \((21)\) is a consequence of \((18)\) and \((20)\) since if \(B \subset A\) we have \(\pi(A) = \pi(A \setminus B) + \pi(B)\). Finally we have

\[
\{i; \ \varphi_X(i) = p\} = \{i; \ |i| \in 2\mathbb{N} + \pi\} \supset C_p(\{a\}) \in \mathcal{U}_p
\]

hence \(\{i; \ \varphi_X(i) = p\} \in \mathcal{U}_p\), and thus \(\pi(X) = p\).

**Q.E.D.**

**Second proof.** We consider the collection \(\mathcal{A}\) of the subsets of \(X\) that are either finite or co-finite: \(\mathcal{A}\) is a Boolean algebra. We define a map \(\pi_0 : \mathcal{A} \rightarrow \{0, 1\}\) as follows: If \(A\) is a finite subset of \(X\), then \(\pi_0(A) = 0\) if \(|A|\) is even, otherwise \(\pi_0(A) = 1\). If \(A\) is a co-finite subset of \(X\), then \(\pi_0(A) = p\) if \(|X \setminus A|\) is even, otherwise \(\pi_0(A) = 1 - p\). \(\pi_0\) is clearly a Boolean homomorphism from \(\mathcal{A}\) to \(\{0, 1\}\) considered as a Boolean Algebra.
Since \{0, 1\} is trivially a complete algebra, \(\pi_0\) can be extended to a Boolean homomorphism \(\pi\) from \(\mathcal{P}(X)\) to \{0, 1\}. The existence of \(\pi\) is assured either by Sikorski’s extension theorem ([14], p.141, Theorem 33.1) or by the injectivity of \{0, 1\} due to the theorem on the equivalence between the completeness and the injectivity ([11], p.141, Theorem 19).

Q.E.D.

We now are ready to deduce the main tool for the quantization, the “\(\epsilon\)-functions” that replace \((-1)^s\) in the definition of the quantum fields.

**Corollary 2.2** ("\(\epsilon\)-function”). Given a partially ordered set \((X, \leq)\), there exists a map

\[
(29) \quad \epsilon : I \in \mathcal{P}(X) \mapsto \epsilon_I \in \{-1, +1\}^I
\]

satisfying

\[
(30) \quad \forall x \in X, \quad \epsilon_{\{x\}}(x) = 1,
\]

\[
(31) \quad j, k \in I, \quad j < k \Rightarrow \epsilon_I(k) = -\epsilon_{I \setminus \{j\}}(k),
\]

\[
(32) \quad j \in I, \quad k \in X, \quad j < k \Rightarrow \epsilon_I(j) = \epsilon_{I \cup \{k\}}(j).
\]

**Proof.** We choose a parity function \(\pi\) given by the previous theorem. For \(I \in \mathcal{P}(X), \ x \in X\), we put \(I_x := I \cap \{y \in X; \ y \leq x\}\). We obviously have: for any \(x \in X\),

\[
\{x\}_x = \{x\}, \quad \pi(\{x\}_x) = 1,
\]

for any \(j, k \in I\) with \(j < k\),

\[
I_k = (I \setminus \{j\})_k \cup \{j\}, \quad \pi(I_k) = \pi((I \setminus \{j\})_k) + 1,
\]

for any \(j \in I\) and \(k > j\)

\[
I_j = (I \cup \{k\})_j.
\]

Therefore it is sufficient to define for \(I \in \mathcal{P}(X), \ j \in I\)

\[
\epsilon_I(j) := (-1)^{\pi(I_j)+1}.
\]

Q.E.D.

To make the link with the countable case, we consider \(X = \mathbb{Z}^*\) and \(I = \{i_s, \ s \in \mathbb{N}^*\}\) where \((i_s)_{s \in \mathbb{N}^*} \in (\mathbb{Z}^*)^{\mathbb{N}^*}\) is a strictly decreasing sequence (for the usual order on \(\mathbb{Z}\)) satisfying (30) for \(s\) large enough. We now endow \(\mathbb{Z}^*\) with the reverse order of the usual order, then \(\epsilon_I(j) := (-1)^{s+1}\) where \(j = i_s\), satisfies (30), (31) and (32).
3. Abstract quantization

We now consider an infinite ordered set \((X, \leq)\) and a subset \(\mathcal{I} \subset \mathcal{P}(X)\) such that

\[
X \text{ is totally ordered},
\]

\[
\forall I \in \mathcal{I}, \, \forall A \in \mathcal{P}_F(X), \quad I \cup A \in \mathcal{I}, \quad I \setminus A \in \mathcal{I}.
\]

We also take an \(\epsilon\)-function given by Corollary 2.2.

For the quantization process, \(X\) enumerates a Hilbert basis \((e_x)_{x \in X}\) of the Hilbert space \(\mathfrak{h}\) of the classical fields, and \(\mathcal{I}\) indexes a Hilbert basis of the Hilbert space \(\mathfrak{h}\) on which the quantum fields act. The existence of a linear order on \(X\) is assured by the Ordering Principle which is strictly weaker than the Zermelo Axiom. The \(\epsilon\)-function will play the role of \((-1)^s\) in the definition of the quantum fields.

We introduce the equivalence relation \(\mathcal{R}\) on the abelian group \((\mathcal{P}(X), \Delta)\) associated to the subgroup \(\mathcal{P}_F(X)\). \(\mathcal{R}\) is defined by

\[
\forall A, B \in \mathcal{P}(X), \quad A \mathcal{R} B \iff A \Delta B \in \mathcal{P}_F(X).
\]

Since \(A = (A \cap B) \cup (A \cap (A \Delta B))\) we also have

\[
A \mathcal{R} B \iff \exists A', B' \in \mathcal{P}_F(X), \quad C \in \mathcal{P}(X), \quad A = A' \cup C, \quad B = B' \cup C.
\]

\(\mathcal{R}\) is an equivalence relation on \(\mathcal{P}(X)\) for which we denote \(\mathcal{P}(X)/\mathcal{P}_F(X)\) its quotient set and \([A]_X\) the equivalence class of \(A \in \mathcal{P}(X)\). The equivalence class \([A]_X\) is also described as

\[
I \in [A]_X \iff \exists I_F \in \mathcal{P}_F(X), \quad I = A \Delta I_F,
\]

and the map

\[
I_F \in \mathcal{P}_F(X) \mapsto I = A \Delta I_F \in [A]_X
\]

is a bijection from \(\mathcal{P}_F(X)\) onto \([A]_X\). We note that \([\emptyset]_X = \mathcal{P}_F(X)\) and \([X]_X\) is just the set of the co-finite subsets of \(X\). Putting \(I_- := I_F \cap A\) and \(I_+ := I_F \cap (X \setminus A)\), we also have

\[
\forall A \in \mathcal{P}(X), \quad \forall I \in [A]_X, \quad \exists I_- \in \mathcal{P}_F(A), \quad \exists I_+ \in \mathcal{P}_F(X \setminus A), \quad I = (A \setminus I_-) \cup I_+.
\]

We obviously have for any \(A \subset X\):

\[
| [A]_X | = |X|,
\]

and since

\[
\mathcal{P}(X) = \bigcup_{[A]_X \in \mathcal{P}(X)/\mathcal{P}_F(X)} [A]_X, \quad | \mathcal{P}(X) | = 2^{|X|},
\]

we have

\[
| \mathcal{P}(X)/\mathcal{P}_F(X) | = 2^{|X|}.
\]
Moreover \( \mathcal{I} \subset \mathcal{P}(X) \) satisfies (34) iff
\[
\mathcal{I} = \bigcup_{A \in \mathcal{I}} [A]_X.
\]

Now we introduce the free vector space \( V(\mathcal{I}) \) spanned by \( \mathcal{I} \), i.e. we consider the elements \( I \) of \( \mathcal{I} \) as vectors denoted \( e_I \), and \( V(\mathcal{I}) \) becomes a pre-Hilbert space if we decide \( \mathcal{I} \) is an orthonormal basis.

The fundamental example arising for the quantization in the separable case is given by:
\[
X = \mathbb{Z}^{\ast} = \mathbb{Z}^{\ast}_{-} \cup \mathbb{Z}^{\ast}_{+}, \quad \mathbb{Z}^{\pm} := \{ n \in \mathbb{Z}; \; \pm n \geq 1 \},
\]
and \(<\) is just the reverse of the usual strict order on \( \mathbb{Z} \). We choose \( \mathcal{I} \) to be the set of the “maya diagrams” (see e.g. [10]):
\[
\mathcal{I} = \{ I = (\mathbb{Z}^{\ast}_{-} \setminus I^{-}) \cup I^{+}; \; I^{\pm} \in \mathcal{P}_F(\mathbb{Z}^{\pm}) \} = [\mathbb{Z}^{\ast}_{-}]_{[\mathbb{Z}^{\ast}]}.
\]
The Dirac sea is the vector
\[
\Omega_D := e_{\mathbb{Z}^{\ast}_{-}} \in \mathcal{V}([\mathbb{Z}^{\ast}_{-}]_{\mathbb{Z}^{\ast}}).
\]

We easily generalize this example to take into account the non-separable Hilbert spaces for which \( X \) is uncountable. Given two infinite ordinals \( \Lambda^{\pm} \) we introduce
\[
X = \Lambda^{\ast}_{-} \cup \Lambda^{\ast}_{+}
\]
endowed with the total strict order \(<\) defined by
\[
\forall \lambda^{\pm}_{\pm}, \lambda^{\prime}_{\pm} \in \Lambda^{\ast}_{\pm}, \quad \lambda_{+} < \lambda_{-}, \quad \lambda_{-} < \lambda^{\prime}_{-}, \quad \lambda^{\prime}_{-} < \lambda^{\prime}_{+}, \quad \lambda_{+} < \lambda^{\prime}_{+} \iff \lambda^{\prime}_{+} < \lambda_{+},
\]
where \(<\) is the usual strict well order on the ordinals. To generalize (44) and (45), we can take
\[
\mathcal{I} = [\Lambda^{\ast}_{-}]_{\Lambda^{\ast}_{-} \cup \Lambda^{\ast}_{+}} = \{ I : (\Lambda^{\ast}_{-} \setminus I^{-}) \cup I^{+}; \; I^{\pm} \in \mathcal{P}_F(\Lambda^{\ast}_{\pm}) \},
\]
\[
\Omega_D := e_{\Lambda^{\ast}_{-}} \in \mathcal{V}([\Lambda^{\ast}_{-}]_{\Lambda^{\ast}_{-} \cup \Lambda^{\ast}_{+}}).
\]

We now choose the Hilbert framework. Given a set \( E \), we introduce the Hilbert space of the complex-valued square integrable functions on \( E \) with respect to the counting measure, that is also the Hilbert closure of the free vector space \( \mathcal{V}(E) \) spanned by \( E \),
\[
l^2(E) := \left\{ u \in \mathbb{C}^E; \; \| u \| := \left( \sum_{x \in E} | u(x) |^2 \right)^{\frac{1}{2}} < \infty \right\}.
\]
We use the canonical Hilbert basis
\[
\mathcal{B}(E) := \{ e_x \in \mathbb{C}^E; \; x \in E \}, \quad \forall x, y \in E, \quad e_x(y) = \delta_{x,y},
\]
For any infinite set $X$ and $u \in l^2(E)$ can be written as
\begin{equation}
 u = \sum_{x \in E} c_x e_x, \quad c_x \in \mathbb{C},
\end{equation}
where $\{x; c_x \neq 0\}$ is countable and
\begin{equation}
\|u\|^2 = \sum_{x \in E} |c_x|^2.
\end{equation}
For $F \subset E$ we denote $P_F$ the orthogonal projector on $l^2(F)$
\begin{equation}
P_F(u) = \sum_{x \in F} \langle \delta_x, u \rangle \delta_x.
\end{equation}
For any infinite set $X$ and $A \subset X$, we have
\begin{equation}
l^2(X) = l^2(A) \oplus l^2(X \setminus A), \quad l^2(\mathcal{P}(X)) = \bigoplus_{[A] \in \mathcal{P}(X)/\mathcal{P}_F(X)} l^2([A]_X),
\end{equation}
and since the elements $I$ of $[A]_X$ can be indexed by $I_F \in \mathcal{P}_F(X)$ or $(I_- , I_+) \in \mathcal{P}_F(A) \times \mathcal{P}_F(X \setminus A)$, we can identify the following spaces with natural isometries:
\begin{align*}
l^2([A]_X), \quad l^2(\mathcal{P}(X)), \quad l^2(\mathcal{P}_F(X) \times \mathcal{P}_F(X \setminus A)), \quad l^2(\mathcal{P}_F(A)) \otimes l^2(\mathcal{P}_F(X \setminus A)), \quad l^2([\emptyset]_A) \otimes l^2([\emptyset]_{X \setminus A}).
\end{align*}

We now construct two maps, $\psi$, the “annihilation operator”, and $\psi^*$, the “creation operator”, from $l^2(X)$ to the space $\mathcal{L}(l^2(\mathcal{P}(X)))$ of the bounded linear maps on $l^2(\mathcal{P}(X))$. Given $j \in X$ and $I \in \mathcal{P}(X)$, we put:
\begin{equation}
\psi(e_j)e_I := \begin{cases} 
0 & \text{if } j \notin I, \\
\epsilon_{I\setminus \{j\}}e_{I\setminus \{j\}} & \text{if } j \in I, 
\end{cases}
\end{equation}
\begin{equation}
\psi^*(e_j)e_I := \begin{cases} 
0 & \text{if } j \notin I, \\
\epsilon_{I\setminus \{j\}}e_{I\setminus \{j\}} & \text{if } j \in I. 
\end{cases}
\end{equation}
Since $\psi(e_j)$ and $\psi^*(e_j)$ map $\mathcal{B}(\mathcal{P}(X))$ into $\mathcal{B}(\mathcal{P}(X)) \cup \{0\}$, we can extend $\psi(e_j)$ and $\psi^*(e_j)$ as bounded linear operators on $l^2(\mathcal{P}(X))$ with
\begin{equation}
\|\psi(e_j)\|_{\mathcal{L}(l^2(I))} = 1, \quad \|\psi^*(e_j)\|_{\mathcal{L}(l^2(I))} = 1.
\end{equation}

We note that (51) implies that
\begin{equation}
\psi(e_j) e_{\{j\}} = e_{\emptyset}, \quad \psi^*(e_j) e_{\emptyset} = e_{\{j\}}.
\end{equation}
The physical interpretation of these operators is classic: $\psi(e_j)$ annihilates the state $e_{\{j\}}$ and $\psi^*(e_j)$ creates the state $e_{\{j\}}$. We also have with (50), (51) and (52)
\begin{equation}
j < k \Rightarrow \psi^*(e_k)\psi^*(e_j)e_{\emptyset} = -e_{\{j,k\}} \quad \psi^*(e_j)\psi^*(e_k)e_{\emptyset} = e_{\{j,k\}}
\end{equation}
More generally, we arrive to the fundamental algebraic properties: $\psi(e_j)$ and $\psi^*(e_j)$ are adjoint to each other and satisfy the canonical anticommutation relations (CAR).
Lemma 3.1. For any \( j, k \in X \), we have
\[ \psi^*(e_j) = [\psi(e_j)]^* , \]
\[ \{ \psi(e_j), \psi^*(e_k) \} = \delta_{jk} \text{Id}, \]
\[ \{ \psi(e_j), \psi(e_k) \} = 0, \]
\[ \{ \psi^*(e_j), \psi^*(e_k) \} = 0. \]

For any \( A \subset X \), \( \psi(e_j) \) and \( \psi^*(e_k) \) leave invariant \( l^2([A]_X) \).

Proof. To prove (61) it is sufficient to establish for any \( I, J \in \mathcal{I} \) that:
\[ \langle \psi(e_j)e_I; e_J \rangle = \langle e_I; \psi^*(e_j)e_J \rangle. \]
We have
\[ \langle \psi(e_j)e_I; e_J \rangle = \begin{cases} 0 & \text{if } j \notin I, \\ \epsilon_I(j)\delta_{I \setminus \{j\}, J} & \text{if } j \in I, \end{cases} \]
\[ \langle e_I; \psi^*(e_j)e_J \rangle = \begin{cases} 0 & \text{if } j \in J, \\ \epsilon_{J \cup \{j\}}(j)\delta_{J \cup \{j\}, I} & \text{if } j \notin J. \end{cases} \]

Therefore \( \langle \psi(e_j)e_I; e_J \rangle \) and \( \langle e_I; \psi^*(e_j)e_J \rangle \) are non zero iff \( j \notin J \) and \( I = J \cup \{j\} \). Moreover in this case they are equal to \( \epsilon_I(j) \). That proves (65).

Now we consider \( j \in I \) and we evaluate
\[ \psi(e_j)\psi^*(e_j)e_I = 0, \quad \psi^*(e_j)\psi(e_j)e_I = \epsilon_I(j)\psi^*(e_j)e_{I \setminus \{j\}} = \epsilon_I(j)\epsilon_I(j)e_I = e_I \]

hence we have
\[ j \in I \Rightarrow \{ \psi(e_j), \psi^*(e_j) \}e_I = e_I. \]

Then we consider \( j \notin I \) and we evaluate
\[ \psi^*(e_j)\psi(e_j)e_I = 0, \quad \psi(e_j)\psi^*(e_j)e_I = \epsilon_{I \cup \{j\}}(j)\psi(e_j)e_{I \cup \{j\}} = \epsilon_{I \cup \{j\}}(j)\epsilon_{I \cup \{j\}}(j)e_I = e_I \]

that implies
\[ j \notin I \Rightarrow \{ \psi(e_j), \psi^*(e_j) \}e_I = e_I. \]

Now we take \( j \neq k \). 1) We first assume \( j \in I \) and \( k \notin I \). We compute:
\[ \psi(e_j)\psi^*(e_k)e_I = \epsilon_{I \cup \{k\}}(j)e_{I \cup \{k\}}(k)e_{I \setminus \{j\}}(k) \]
\[ \psi^*(e_k)\psi(e_j)e_I = \epsilon_I(j)e_{I \cup \{j\}}(k)e_{I \setminus \{j\}}(k). \]

If \( j < k \), then \( \epsilon_{I \cup \{k\}}(j) = \epsilon_I(j) \) by (32), \( \epsilon_{I \setminus \{j\}}(k) = -\epsilon_{I \cup \{k\}}(k) \) by (31) and thus
\[ \{ \psi(e_j), \psi^*(e_k) \}e_I = 0. \]

If \( k < j \), then \( \epsilon_{I \cup \{k\}}(j) = -\epsilon_I(j) \) by (31), \( \epsilon_{I \setminus \{j\}}(k) = \epsilon_{I \cup \{k\}}(k) \) by (32) and thus (72) is satisfied again. 2) We assume \( j \notin I \) and \( k \in I \). Then we have:
\[ \psi^*(e_k)e_I = 0 = \psi(e_j)e_I, \]
hence (72) is trivially satisfied. 3) We assume \( j \in I \) and \( k \in I \). Then \( k \in I \setminus \{j\} \) and so

\[
(74) \quad \{\psi(e_j), \psi^*(e_k)\} e_I = \psi^*(e_k) \psi(e_j) e_I = \epsilon_I(j) \psi^*(e_k) e_{I \setminus \{j\}} = 0.
\]

4) Finally we assume \( j \notin I \) and \( k \notin I \). Then \( \psi(e_j) e_I = 0 \) and \( j \notin I \cup \{k\} \), hence

\[
(75) \quad \{\psi(e_j), \psi^*(e_k)\} e_I = \psi(e_j) \psi^*(e_k) e_I = \epsilon_{I \cup \{k\}}(k) \psi(e_j) e_{I \cup \{k\}} = 0,
\]

and the proof of (62) is complete.

To prove (63), we distinguish the different cases again. 1) If \( j \notin I \), \( k \notin I \) then \( \psi(e_j) e_I = \psi(e_k) e_I = 0 \) and obviously we have

\[
(76) \quad \{\psi(e_j), \psi(e_k)\} e_I = 0.
\]

2) If \( j \notin I \), \( k \in I \), we have \( \psi(e_j) e_I = 0 \) and

\[
(77) \quad \{\psi(e_j), \psi(e_k)\} e_I = \epsilon_I(k) \psi(e_j) e_{I \setminus \{k\}} = 0
\]

since \( j \notin I \setminus \{k\} \). 3) The case \( j \in I \), \( k \notin I \) is analogous. 4) Now if \( j \in I \), \( k \in I \), \( j \neq k \), we compute

\[
(78) \quad \{\psi(e_j), \psi(e_k)\} e_I = (\epsilon_I(k) \epsilon_{I \setminus \{k\}}(j) + \epsilon_I(j) \epsilon_{I \setminus \{j\}}(k)) e_{I \setminus \{j,k\}}.
\]

If \( j < k \), then (62) implies that \( \epsilon_I(j) = \epsilon_{I \setminus \{j\}}(j) \) and (31) assures that \( \epsilon_I(k) = -\epsilon_{I \setminus \{j\}}(k) \), and we conclude that (76) follows from (78). 5) Finally if \( j = k \in I \), \( \psi(e_j) \psi(e_j) e_I = \epsilon_I(j) \psi(e_j) e_{I \setminus \{j\}} = 0 \). The proof of (63) is achieved. Moreover (64) follows from (63) by taking the adjoint.

To end, since the quantum fields map \( \delta_I \) to zero, or \( \pm \delta_{I \setminus \{j\}} \), or \( \pm \delta_{I \cup \{j\}} \), they leave invariant \( L^2([A]_X) \).

Q.E.D.

We now define \( \psi \) and \( \psi^* \) on the whole space \( L^2(X) \) by the usual way used in the separable case (6). If \( u \in L^2(X) \) is expressed as

\[
(79) \quad u = \sum_{k \in \mathbb{N}} c_k e_{jk}, \quad c_k \in \mathbb{C}, \quad \sum_{k \in \mathbb{N}} |c_k|^2 < \infty,
\]

we introduce for any \( N \in \mathbb{N} \)

\[
(80) \quad \psi_N(u) := \sum_{k \leq N} c_k^* \psi(e_{jk}), \quad \psi_N^*(u) := \sum_{k \leq N} c_k \psi^*(e_{jk}) = [\psi_N(u)]^*.
\]

Given two integers \( N, M \) with \( M \geq N + 1 \), we deduce from (62) that for any \( V \in L^2(\mathcal{P}(X)) \) we have

\[
<V; \{\psi_M(u) - \psi_N(u), \psi^*_M(u) - \psi^*_N(u)\} V > = \left( \sum_{k=N+1}^{M} |c_k|^2 \right) \|V\|^2.
\]
Therefore
\[ \| \psi_M(u) - \psi_N(u) \|_{L^2(\mathcal{P}(X))}^2 + \| \psi_M^*(u) - \psi_N^*(u) \|_{L^2(\mathcal{P}(X))}^2 \leq \sum_{k=N+1}^{M} | c_k |^2. \]

We conclude that we may define the quantum fields
\[ \psi(u), \psi^*(u) \] for any \( u, v \in l^2(X) \).

The previous lemma directly implies the main properties of the quantum fields:

**Theorem 3.2.** \( \psi \) is an anti-linear map from \( l^2(X) \) to \( L^2(\mathcal{P}(X)) \) and we have for any \( u, v \in l^2(X) \)

\begin{align*}
(82) & \quad [\psi(u)]^* = \psi^*(u), \\
(83) & \quad \| \psi(u) \|_{L^2(\mathcal{P}(X))} = \| u \|_{l^2(X)}, \\
(84) & \quad \{ \psi(u), \psi^*(v) \} = \langle u, v \rangle > I d_{l^2(\mathcal{P}(X))}, \\
(85) & \quad \{ \psi^*(u), \psi(v) \} = 0, \\
(86) & \quad \{ \psi^*(u), \psi^*(v) \} = 0,
\end{align*}

\begin{align*}
(87) & \quad \forall A \subset X, \forall u \in l^2(X), \quad \psi(u)l^2([A]X) \subset l^2([A]X), \\
(88) & \quad \forall A \subset X, \forall u \in l^2(X), \quad P_A(u) = 0 \Rightarrow \psi(u)e_A = 0, \\
(89) & \quad \forall A \subset X, \forall u \in l^2(X), \quad P_{X \setminus A}(u) = 0 \Rightarrow \psi^*(u)e_A = 0.
\end{align*}

The quantum field provided by the previous theorem is not unique since it depends on the choices of the linear order on \( X \), and the function \( \epsilon \). Nevertheless, the representation of the CAR on \( l^2(X) \) given by \( (\psi, l^2([A]X)) \) is actually unique up to an unitary transform and it depends only on \( A \) and \( X \setminus A \). We denote
\[ N_- := | A |, \quad N_+ := | X \setminus A |. \]

We take a bijection \( \theta_- \) from \( A \) onto \( N_- \) and a bijection \( \theta_+ \) from \( X \setminus A \) onto \( N_+ \). We denote \( \theta \) the bijection from \( X \) onto \( N_- \sqcup N_+ \) defined by
\[ \theta : j \in A \Rightarrow \theta(j) := \theta_-(j), \quad j \in X \setminus A \Rightarrow \theta(j) := \theta_+(j). \]

Therefore the map
\[ u \in l^2(X) \mapsto u \circ \theta^{-1} \in l^2(N_- \sqcup N_+) = l^2(N_-) \oplus l^2(N_+) \]
is an isometry.

We consider an anti-unitary operator \( C \) on \( l^2(N_-) \oplus l^2(N_+) \) and we introduce the fermionic Fock space
\[ \mathcal{F}_F := \mathcal{F}^\wedge(C l^2(N_-)) \otimes \mathcal{F}^\wedge(l^2(N_+)). \]
The Fock quantization \( (\mathcal{H}_F, \Psi_F) \) on \( l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+) \) is defined by choosing the antilinear map, \( \Psi_F \), from \( l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+) \) to \( \mathcal{L}(\mathcal{H}_F) \) defined by
\[
\Psi_F(u_- \oplus u_+) := a_-^*(C u_-) \otimes (-1)^{N_+} + Id \otimes a_+(u_+), \quad u_\pm \in l^2(\mathbb{R}_\pm),
\]
where \( a_-^* \) (resp. \( a_+ \)) is the creation (resp. annihilation) operator on \( \mathcal{F}^\wedge (C l^2(\mathbb{R}_-)) \) (resp. \( \mathcal{F}^\wedge (l^2(\mathbb{R}_+)) \)), and \( N_+ \) is the number operator on \( \mathcal{F}^\wedge (l^2(\mathbb{R}_+)) \). The Fock vacuum is the vector
\[
\Omega_F := (1, 0, 0, \ldots) \in [C l^2(\mathbb{R}_-)]^{\wedge 0} \otimes [l^2(\mathbb{R}_-)]^{\wedge 0}
\]
that obviously satisfies
\[
\forall u_\pm \in l^2(\mathbb{R}_\pm), \quad \Psi_F(u_+) \Omega_F = 0, \quad [\Psi_F(u_-)]^* \Omega_F = 0.
\]
We know that this representation is irreducible.

**Theorem 3.3.** For any \( A \subset X \), the quantization \( (\psi, l^2([A]_X)) \) is an irreducible representation of the CAR on \( l^2(X) \), moreover there exists a unitary transformation \( \mathbb{U}_A \) from \( \mathcal{H}_F \) onto \( l^2([A]_X) \) such that for any \( u \in l^2(X) \) we have
\[
\psi(u) = \mathbb{U}_A \Psi_F (u \circ \theta^{-1}) \mathbb{U}_A^{-1} \text{ on } l^2([A]_X).
\]

**Proof.** The irreducibility of the representation \( (\psi, l^2([A]_X)) \) follows from \( (97) \) since the Fock representation is irreducible. We express any subset \( I_F \in \mathcal{P}_F(X) \) as a finite decreasing sequence
\[
I_F = \{ j_n \mid 1 \leq n \leq N, \quad j_N < j_{N-1} < \ldots < j_1 \},
\]
and if \( \varphi = \psi \), \( \psi^* \) we denote
\[
\prod_{j \in I_F} \varphi(e_j) := \varphi(e_{j_N})\varphi(e_{j_{N-1}})\ldots\varphi(e_{j_1}),
\]
and for \( \varphi = \Psi_F \), \( \Psi_F^* \) we denote
\[
\prod_{j \in I_F} \varphi(e_{\theta(j)}) := \varphi(e_{\theta(j_N)})\varphi(e_{\theta(j_{N-1})})\ldots\varphi(e_{\theta(j_1)}).
\]
We note that \( (39) \) assures that any \( I \in [A]_X \) can be uniquely written as \( I = (A \setminus I_-) \sqcup I_+ \), and we have
\[
\prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A = \varepsilon e_I, \quad \varepsilon \in \{-1, 1\}.
\]
We conclude that the set
\[
(98) \quad \left\{ \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A; \quad I_- \in \mathcal{P}_F(A), \quad I_+ \in \mathcal{P}_F(X \setminus A) \right\},
\]
is a Hilbert basis of \( l^2([A]_X) \).
Since \( \{e_{\theta_{-}(j)}; \ j \in A\} \) is a Hilbert basis of \( l^2(\mathbb{N}_-) \) and \( \{e_{\theta_{+}(j)}; \ j \in X \setminus A\} \) is a Hilbert basis of \( l^2(\mathbb{N}_+) \), we know that

\[
\prod_{j \in I_+} \Psi_{F}^*(e_{\theta_{+}(j)}) \prod_{k \in I_-} \Psi_{F}(e_{\theta_{-}(k)}) \Omega_F; \ I_- \in \mathcal{P}_F(A), \ I_+ \in \mathcal{P}_F(X \setminus A)
\]

is a Hilbert basis of \( \mathfrak{H}_F \). We define the unitary transformation \( U_A \) from \( \mathfrak{H}_F \) onto \( l^2([A|_X]) \) by putting

\[
U_A \left( \prod_{j \in I_+} \Psi_{F}^*(e_{\theta_{+}(j)}) \prod_{k \in I_-} \Psi_{F}(e_{\theta_{-}(k)}) \Omega_F \right) = \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A.
\]

Now to prove (99) it is sufficient to show that for any \( l \in X, \ I_- \in \mathcal{P}_F(A), \ I_+ \in \mathcal{P}_F(X \setminus A) \) we have

\[
\psi(e_l) \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A = U_A \Psi_F(e_{\theta_{l}(l)}) \prod_{j \in I_+} \Psi_{F}^*(e_{\theta_{+}(j)}) \prod_{k \in I_-} \Psi_{F}(e_{\theta_{-}(k)}) \Omega_F.
\]

The CAR assure that:

1) If \( l \notin I_+ \) there exists \( \varepsilon \in \{-1, +1\} \) such that

\[
\psi(e_l) \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A = \left\{ \begin{array}{ll} 
0 & \text{if } l \notin A, \\
\varepsilon \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A & \text{if } l \in A,
\end{array} \right.
\]

and we conclude that (101) is satisfied;

2) If \( l \in I_+ \) there exists \( \varepsilon \in \{-1, +1\} \) such that

\[
\psi(e_l) \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A = \varepsilon \prod_{j \in I_+ \setminus \{l\}} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A,
\]

hence (101) is satisfied again. The proof is complete.

\[Q.E.D.\]

We deduce a form of the Shale-Stinespring criterion of implementation of unitary operators, adapted to our quantization.

**Corollary 3.4.** Given \( A \subset X \), let \( U \) be an unitary operator on \( l^2(X) = l^2(A) \oplus l^2(X \setminus A) \). Then there exists a unitary operator \( \mathbb{U} \) on \( l^2([A|_X]) \) satisfying for any \( u \in l^2(X) \)

\[
\psi(Uu) = \mathbb{U}\psi(u)\mathbb{U}^{-1}
\]
if and only if $P_A U P_{X \setminus A}$ and $P_{X \setminus A} U P_A$ are Hilbert-Schmidt operators.

Proof. With the previous notations we have

$$
\psi(U u) = U A \Psi_F((U u) \circ \theta^{-1}) U_A^{-1} = U A \Psi_F(U(u \circ \theta^{-1})) U_A^{-1}
$$

where $U$ is the unitary operator on $l^2(\mathbb{R}_+)$ defined by

$$
v \in l^2(\mathbb{R}_+) \mapsto U v := \left[U(v \circ \theta)\right] \circ \theta^{-1}.
$$

Therefore $U$ exists and satisfies (102) iff $U$ is implementable in the Fock representation of $l^2(\mathbb{R}_+) \oplus l^2(\mathbb{R}_+)$. The famous theorem of Shale and Stinespring states that a necessary and sufficient condition is that $P_{\mathbb{R}_+} U P_{\mathbb{R}_+}$ and $P_{\mathbb{R}_+} U P_{\mathbb{R}_-}$ are Hilbert-Schmidt operators. We have

$$
P_{\mathbb{R}_+} U P_{\mathbb{R}_+} v = [P_A U P_{X \setminus A}(v \circ \theta)] \circ \theta^{-1}
$$

hence we deduce that

$$
\sum_{y,y' \in \mathbb{R}_- \sqcup \mathbb{R}_+} | < P_{\mathbb{R}_+} U P_{\mathbb{R}_+} e_y ; e_{y'} > |^2 = \sum_{y-y' \in \mathbb{R}_- \sqcup \mathbb{R}_+} \sum_{y-y' \in \mathbb{R}_+} | < U e_y ; e_{y_{-}} > |^2

= \sum_{j-j' \in \mathbb{R}_+ \sqcup \mathbb{R}_- \setminus A} \sum_{j-j' \in X_1 \setminus A} | < U e_j ; e_{j_{-}} > |^2

= \sum_{j-j' \in X} | < P_A U P_{X_1 \setminus A} e_y ; e_{y'} > |^2.

$$

(103)

We conclude that $P_{\mathbb{R}_+} U P_{\mathbb{R}_+}$ is Hilbert-Schmidt on $l^2(\mathbb{R}_- \sqcup \mathbb{R}_+)$ iff $P_A U P_{X \setminus A}$ is Hilbert-Schmidt on $l^2(X)$. And the same holds for $P_{\mathbb{R}_+} U P_{\mathbb{R}_-}$ and $P_{X \setminus A} U P_A$.

The proof is complete.

Q.E.D.

Finally we prove that, up to a unitary transform, our irreducible representations depend only on $| A |$ and $| X \setminus A |$. We consider two totally ordered infinite sets $X_i$, $i = 1, 2$ and two epsilon functions $\epsilon_i$ given by Corollary 4.2 and the quantum fields $\psi_i$ defined on $l^2(X_i)$ as previously. Let $A_i$ be in $\mathcal{P}(X_i)$ such that

$$
| A_1 | = | A_2 |, \quad | X_1 \setminus A_1 | = | X_2 \setminus A_2 |
$$

and we choose a bijection $\Theta$ from $X_1$ onto $X_2$ satisfying

$$
\Theta(A_1) = A_2.
$$

Theorem 3.5. There exists a unitary transform $\mathbb{U}$ from $l^2([A_2]_{X_2})$ onto $l^2([A_1]_{X_1})$ such that for any $u_2 \in l^2(X_2)$ we have

$$
\psi_2(u_2) = \mathbb{U}^{-1} \psi_1(u_2 \circ \Theta) \mathbb{U} \quad \text{on} \quad l^2([A_2]_{X_2}).
$$

Proof. We denote $\mathbb{R}_- := | A_1 |, \mathbb{R}_+ := | X_i \setminus A_i |$. We introduce a bijection $\theta_i$ from $X_i$ onto $\mathbb{R}_- \sqcup \mathbb{R}_+$ such that $\theta_i(A_i) = \mathbb{R}_-, \theta_i(X_i \setminus A_i) = \mathbb{R}_+$. The previous
Theorem assures there exists a unitary transform $U_i$ from $\mathcal{F}^\wedge(C{l^2}(\mathbb{R}_-)) \otimes \mathcal{F}^\wedge(l^2(\mathbb{R}_+))$ onto $l^2([A_i], X_i)$ such that for any $u_i \in l^2(X_i)$ we have
\begin{equation}
\psi_i(u_i) = U_i \Psi_F (u_i \circ \theta_{i}^{-1}) U_i^{-1}.
\end{equation}

Now we introduce an unitary operator $T$ on $l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+)$ by putting
\begin{equation}
T : v \in l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+) \mapsto v \circ \theta_2 \circ \Theta \circ \theta_1^{-1} \in l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+).
\end{equation}

Since $T$ leaves invariant $l^2(\mathbb{R}_\pm)$ it is implementable in the Fock representation: there exists a unitary operator $T$ on $\mathcal{F}^\wedge(C{l^2}(\mathbb{R}_-)) \otimes \mathcal{F}^\wedge(l^2(\mathbb{R}_+))$ such that for any $v \in l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+)$ we have
\begin{equation}
\Psi_F(T v) = T \Psi_F(v) T^{-1}.
\end{equation}

Since the previous Theorem assures that there exist unitary tranformations $U_{A_i}$ from $\mathcal{F}_F$ onto $l^2([A_i], X_i)$ such that for any $u_i \in l^2(X_i)$
\begin{equation}
\psi_i(u_i) = U_{A_i} \Psi_F (u_i \circ \theta_{i}^{-1}) U_{A_i}^{-1}
\end{equation}
on $l^2([A_i], X_i)$,
we compute
\begin{equation}
\begin{aligned}
\psi_1(u_2 \circ \Theta) &= U_1 \Psi_F (u_2 \circ \theta_2^{-1} \circ (\theta_2 \circ \Theta \circ \theta_1^{-1})) U_1^{-1} \\
&= U_1 T \Psi_F (u_2 \circ \theta_2^{-1}) T^{-1} U_1^{-1} \\
&= U_1 T U_2^{-1} \psi_2(u_2) U_2 T U_1^{-1}.
\end{aligned}
\end{equation}

Now it is sufficient to choose
\begin{equation}
\mathcal{U} := U_1 T U_2^{-1}.
\end{equation}

\textit{Q.E.D.}

Taking $X_1 = X_2$, $A_1 = A_2$, $\Theta = Id$, we immediately deduce the following

\textbf{Corollary 3.6.} \textit{Given an infinite set $X$, let $\psi_1$, $\psi_2$ be the quantizations associated to two linear orders on $X$ and two $\epsilon$-functions. Then for any $A \subset X$ there exists a unitary operator $\mathcal{U}$ on $l^2([A], X)$ such that for any $u \in l^2(X)$ we have}
\begin{equation}
\psi_2(u) = \mathcal{U}^{-1} \psi_1(u) \mathcal{U} \text{ on } l^2([A], X).
\end{equation}

\section{4. Application to the Dirac Theory}

We consider two Hilbert spaces $\mathfrak{h}_{\pm}$, possibly non-separable, and we define
\begin{equation}
\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+,
\end{equation}
and we denote $P_{\pm}$ the orthogonal projector on $\mathfrak{h}_{\pm}$. Invoking the Axiom of Choice, we take two Hilbert basis $(e_j)_{j \in X_{\pm}}$ of $\mathfrak{h}_{\pm}$. We put
\begin{equation}
X := X_- \sqcup X_+,
\end{equation}
then $(e_j)_{j \in X}$ is a Hilbert basis of $\mathfrak{h}$, and using the Axiom of Choice again, we endow $X$ with a total order. Finally, we use a last time this Axiom to get an ultrafilter on $\mathcal{P}_F(X) \setminus \{\emptyset\}$ to obtain an $\epsilon$-function given by Corollary 2.2.
ψ being the quantum field on $l^2(X)$ given by Theorem 3.2, ($\psi, l^2([X_-]_X)$ is an irreducible representation of the CAR on $l^2(X)$. Now we introduce the canonical isometry $J$ from $\mathfrak{h}$ onto $l^2(X)$,

$$(113) \quad u \in \mathfrak{h} \mapsto J(u) = (\langle e_j; u \rangle)_{j \in X} \in l^2(X),$$

and we define a quantum field on $\mathfrak{h}$ by putting

$$(114) \quad \Psi_D(u) := \psi(J(u)) \in \mathcal{L}(S_D), \quad S_D := l^2([X_-]_X).$$

The Dirac sea describing the state fulfilled by $\mathfrak{h}_-$ is the vector

$$(115) \quad \Omega_D := e_{X_-} \in S_D.$$

These mathematical objects have the usual meaning of the Dirac theory: $\mathfrak{h}_+(-)$ is the space of the classical solutions of positive (negative) energy. The quantum field $\Psi_D$ is an operator of annihilation: $\Psi_D(e_j)$ annihilates the state $e_j$, therefore it annihilates a particle in the state $e_j$ when $j \in X_+$, and it creates a hole in the Dirac sea when $j \in X_-$. We shall compare with the classic Fock quantization defined by the Hilbert space

$$(116) \quad S_F := F^\wedge (\mathcal{C}h_-) \otimes F^\wedge (\mathfrak{h}_+),$$

the Fock vacuum vector

$$(117) \quad \Omega_F := (1, 0, 0, 0...) \in [\mathcal{C}h_-]^0 \otimes [\mathfrak{h}_+]^0,$$

and the quantum field

$$(118) \quad \Psi_F(u) := a^*_\wedge (\mathcal{C}P_- u) \otimes (-1)^{N_+} + Id \otimes a_+ (P_+ u) \in \mathcal{L}(S_F),$$

where $a^*_\wedge$ (resp. $a_+$) is the creation (resp. annihilation) operator on $F^\wedge (\mathcal{C}h_-)$ (resp. $F^\wedge (\mathfrak{h}_+)$), $N_+$ is the number operator on $F^\wedge (\mathfrak{h}_+)$ and $\mathcal{C}$ is an anti-unitary operator on $\mathfrak{h}$. Now $\mathfrak{h}_+$ is the one-particle space, $\mathcal{C}h_-\mathfrak{h}_+$ is the one-antiparticle space, and $\Psi_F(e_j)$ annihilates a particle if $j \in X_+$ and creates an antiparticle if $j \in X_-$. 

**Theorem 4.1.** $(\Psi_D, S_D)$ defined by (114) is an irreducible representation of the CAR on $\mathfrak{h}$. All the representations obtained by changing the total order on $X$, or the $\epsilon$-function, or the Hilbert basis, are unitarily equivalent. Moreover there exists a unitary transform $U$ from $S_F$ onto $S_D$ such that

$$(119) \quad \Omega_D = U\Omega_F,$$

$$(120) \quad \forall u \in \mathfrak{h}, \quad \Psi_D(u) = U\Psi_F(u)U^{-1}.$$

**Proof.** If we change the total order on $X$ or the $\epsilon$-function, Corollary 3.6 shows that we obtain a new representation that is unitarily equivalent to the previous one. More generally, if we take another basis $(e'_j)_{j \in X_-}$ of $\mathfrak{h}_\pm$, Theorem 3.5 assures that the quantizations built from these two choices of basis are unitarily equivalent. Moreover, if we take an anti-unitary transform $\mathcal{C}$ on $\mathfrak{h}$, Theorem 3.3 implies that these quantizations are unitarily equivalent.
to the Fock quantization $\Psi_F$ on $l^2(\mathbb{R}_-) \oplus l^2(\mathbb{R}_+)$ where $X_\pm = \mathbb{N}_\pm = \dim(\mathfrak{h}_\pm)$ and $C = J\mathcal{J}^{-1}$, which is obviously unitarily equivalent to the classic Fock quantization on $\mathfrak{h}_- \oplus \mathfrak{h}_+$.

Q.E.D.

We introduce a general setting for the abstract Dirac equations that have the form

\begin{equation}
  i\frac{d\psi}{dt} = H\psi,
\end{equation}

with the following assumptions:

1) $H$ is a densely defined selfadjoint operator on $\mathfrak{h}$ with domain $\mathcal{D}(H)$;

2) there exist two densely defined selfadjoint operators $H_\pm$ on $\mathfrak{h}_\pm$ with domain $\mathcal{D}(H_\pm)$ such that

\begin{equation}
  0 \leq H_+,
\end{equation}

\begin{equation}
  H = H_- \oplus H_+, \quad \mathcal{D}(H) = \mathcal{D}(H_-) \oplus \mathcal{D}(H_+);
\end{equation}

3) There exists an anti-unitary operator $C$ on $\mathfrak{h}$ such that,

\begin{equation}
  C\mathfrak{h}_\pm = \mathfrak{h}_\mp, \quad CP_\pm = P_\mp C,
\end{equation}

\begin{equation}
  C\mathcal{D}(H_\pm) = \mathcal{D}(H_\mp), \quad CH = -HC.
\end{equation}

In fact these assumptions imply that

\begin{equation}
  H_- \leq 0
\end{equation}

since we have:

\begin{align*}
  0 \leq <P_+ u; H_+ P_+ u> &= <C H_+ P_+ u; CP_+ u> \\
  &= <CP_+ H u; P_- C u> = -<P_- H C u; P_- C u> = -<P_- H_- C u; P_- C u>
\end{align*}

The time evolution of the classical Dirac fields is given by the unitary group $e^{-itH} = e^{-itH_-} \oplus e^{-itH_+}$ on $\mathfrak{h}$, that leaves invariant $\mathfrak{h}_\pm$. The time evolution of the quantum field is defined for $t \in \mathbb{R}$, $u \in \mathfrak{h}$ by putting:

\begin{equation}
  \Psi_D(t,u) := \Psi_D(e^{itH}u).
\end{equation}

**Theorem 4.2.** For any $t \in \mathbb{R}$, $(\Psi_D(t,\cdot),\mathfrak{H}_D)$ is an irreductible representation of the CAR on $\mathfrak{h}$ and for any $u \in \mathcal{D}(H)$ the map $t \mapsto \Psi_D(t,u) \in \mathcal{L}(\mathfrak{H}_D)$ is a strongly differentiable function satisfying

\begin{equation}
  i\frac{d}{dt}\Psi_D(t,u) = \Psi_D(t,Hu), \quad \Psi_D(0,u) = \Psi_D(u).
\end{equation}

Moreover there exists a densely defined self-adjoint operator $\mathbb{H}_D$ on $\mathfrak{H}_D$ such that

\begin{equation}
  0 \leq \mathbb{H}_D,
\end{equation}

\begin{equation}
  \forall t \in \mathbb{R}, \forall u \in \mathfrak{h}, \quad \Psi_D(t,u) = e^{it\mathbb{H}_D}\Psi_D(u)e^{-it\mathbb{H}_D}.
\end{equation}
Proof. The first assertion is obvious since $e^{itH}$ is unitary. Moreover if $u \in \mathcal{D}(H)$, the map $t \mapsto e^{itH}u$ belongs to $C^1(\mathbb{R}; \mathfrak{h}) \cap C^0(\mathbb{R}; \mathcal{D}(H))$, therefore, since $u \mapsto \Psi_D(u)$ is an antilinear bounded map, we can differentiate $\Psi_D(t, e^{itH}u)$ and we obtain (128). Now since $\mathfrak{c}_{\mathfrak{h}-} = \mathfrak{h}_+$, we consider $\mathfrak{h}_F = \mathcal{F}^\varphi(\mathfrak{h}_+) \otimes \mathcal{F}^\varphi(\mathfrak{h}_+)$, and the positive densely defined self-adjoint operator on $\mathfrak{h}_F$

$\mathbb{H}_D := \mathfrak{d} \mathcal{I}(H_+) \otimes Id + Id \otimes d \mathcal{I}(H_+)$

where $\mathcal{I}$ is the usual second-quantization functor. Therefore, using unitary operator $U$ of Theorem 4.1, the operator

$e^{it\mathbb{H}_D} = U e^{itH} \Psi_F(u) e^{itH} U^{-1}$

We achieve the proof by writting

$\Psi_D(e^{itH}u) = U \Psi_F(e^{itH}u) U^{-1}$

$= U \left( a^+_+(CP_- e^{itH} u) \otimes (-1)^{N_+} + Id \otimes a_+(P_+ e^{itH} u) \right) U^{-1}$

$= U e^{it\mathbb{H}_F} \Psi_F(u) e^{-it\mathbb{H}_F} U^{-1}$

$= e^{it\mathbb{H}_D} \Psi_D(u) e^{-it\mathbb{H}_D}$.

Q.E.D.

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