HOMOTOPY TYPES OF SPACES OF FINITE PROPAGATION UNITARY OPERATORS ON \( \mathbb{Z} \)

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Abstract. In [1], certain structure of the set of finite propagation unitary operators has been studied from the viewpoint of quantum walks in mathematical physics. In particular they determined \( \pi_0 \) of the space. In this article we study homotopy theoretic aspects of the space. In particular we compute higher homotopy groups of the space. We also study some periodic versions of finite propagation unitary operators, and compute their homotopy groups.

1. Introduction

It is a remarkable fact that a wave operator on a complete Riemannian manifold satisfies the finite propagation speed (see [6]). Let \( X \) be a complete Riemannian manifold and \( D \) be the Dirac operator acting on spinor bundles \( S^\pm \). Then the wave operator \( e^{itD} \) acts on \( L^2(X; S^\pm) \) as a unitary operator. Moreover its kernel function \( k_i(x, y) \) has the support within \( |t| \) neighborhood of the diagonal. Such characteristics has been playing important roles in mathematical physics. More abstractly, it would be of particular interest to study the space of finite propagation unitary operators.

In [1], mathematical physicists, Gross, Nesme, Vogts and Werner studied finite propagation unitary operators from topology and combinatorics view points. They introduced the index theory on the set \( \mathcal{U}(\mathbb{C}) \) of finite propagation unitary operators on a separable Hilbert space, and discovered that each connected component of the set corresponds to the value of the index function. In particular the set of connected components is parametrized by \( \mathbb{Z} \), namely \( \pi_0(\mathcal{U}(\mathbb{C})) \equiv \mathbb{Z} \). This is a remarkable fact, since it is well known that the set of all unitary operators on a separable Hilbert space is contractible [2].

In this article we study homotopy types of the set of finite propagation unitary operators on a separable Hilbert space and their variants.

Our main result is the following (Corollary 3.7).

**Theorem 1.1.** For \( i \geq 0 \), the homotopy groups are computed as

\[
\pi_i(\mathcal{U}(\mathbb{C})) = \begin{cases} 
\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}} & \text{if } i \text{ is odd,} \\
\mathbb{Z} & \text{if } i \text{ is even.}
\end{cases}
\]

Here \( S : \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}, \mathbb{Z}) \) is the left shift operator by \( S(v)_i = (v_{i+1})_i \), and the coinvariant set is defined by

\[
\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}} := \ell^\infty(\mathbb{Z}, \mathbb{Z}) / \{ a - Sa \mid a \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) \}.
\]

There are variants of finite propagation unitary operators, which consist of subclasses of \( \mathcal{U}(\mathbb{C}) \). We also compute their homotopy groups.

Let \( \mathcal{U}(n, \mathbb{C}) \) be the set of finite propagation periodic unitary matrices. Then we have the next result (Corollary 4.8).

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**Theorem 1.2.** For $i \geq 0$,

$$\pi_i(\widehat{U}(\mathbb{C})) = \begin{cases} \mathbb{Q} & \text{if } i \text{ is odd}, \\ \mathbb{Z} & \text{if } i \text{ is even}. \end{cases}$$

Actually we have determined the homotopy type of the connected component $\widehat{U}^0(\mathbb{C})$ (Theorem 4.7 and Lemma 4.3).

**Theorem 1.3.** There is a homotopy equivalence

$$\widehat{U}^0(\mathbb{C}) \simeq B \times \prod_{n \geq 1} K(\mathbb{Q}, 2n - 1)$$

where $K(\mathbb{Q}, m)$ denotes the Eilenberg–MacLane space of type $(\mathbb{Q}, m)$ and $\prod_{n \geq 1} K(\mathbb{Q}, 2n-1) = \lim_{N \to \infty} \prod_{n=1}^{N} K(\mathbb{Q}, 2n-1)$.

Finally we consider finite propagation end-periodic unitary operator $T$ in the sense that $S^nT S^{-n} - T$ has only finitely many non-zero entries for some $n$. Let $\mathcal{U}(\mathbb{C}^d)$ be the set of finite propagation end-periodic unitary matrices. Then we obtain the following decomposition (Theorem 4.11).

**Theorem 1.4.** There is a weak homotopy equivalence

$$\pi_i(\mathcal{U}(\mathbb{C})) \simeq \pi_i(\widehat{U}(\mathbb{C})) \times U(\infty),$$

where $U(\infty) = \lim_{n \to \infty} U(n)$.

In particular we obtain the following result (Corollary 4.12).

**Corollary 1.5.** For $i \geq 0$

$$\pi_i(\mathcal{U}(\mathbb{C})) \simeq \begin{cases} \mathbb{Z} \times \mathbb{Q} & \text{if } i \text{ is odd}, \\ \mathbb{Z} & \text{if } i \text{ is even}. \end{cases}$$

We also discuss the relation between the results so far and $C^*$-algebras. Recall that the uniform Roe algebra $C^*_u(\mathbb{Z})$ of the metric space $\mathbb{Z}$ is the $C^*$-algebra defined to be the closure of the space of finite propagation operators in $\mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}))$ [6]. It is a natural to ask if the inclusion

$$\mathcal{U}(\mathbb{C}) \to U(C^*_u(\mathbb{Z}))$$

into the space of unitary elements in $C^*_u(\mathbb{Z})$ is a homotopy equivalence. We do not answer this question in the present work. Actually, even the homotopy groups of $U(C^*_u(\mathbb{Z}))$ seems unknown while its $K$-theory is easily computed. We will study how to relate the spaces $\mathcal{U}(\mathbb{C})$ and $\widehat{U}(\mathbb{C})$ with $K$-theory of $C^*$-algebras and Segal–Wilson’s restricted unitary group [7] having the homotopy type of the infinite disjoint copies of the Grassmannian.

2. **Index theory**

We recall definitions and results in [1]. Let $X$ be a separable and infinite-dimensional Hilbert space, and $\mathcal{B}(X)$ denote the $C^*$-algebra of bounded operators, where the operator norm of an element $T \in \mathcal{B}(X)$ is denoted by $\|T\|$.

Any two such Hilbert spaces are isometric. A standard representation will be $\ell^2(\mathbb{Z}) := \ell^2(\mathbb{Z}, \mathbb{C})$, where $\mathbb{Z}$ is the integer set. A slightly more general representation will be $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ for $d \geq 1$. An element can be written as $(v_i)_i \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$ with $v_i \in \mathbb{C}^d$ such that $\sum_{i \in \mathbb{Z}} |v_i|^2 < \infty$.

A bounded operator $T : \ell^2(\mathbb{Z}, \mathbb{C}^d) \to \ell^2(\mathbb{Z}, \mathbb{C}^d)$ can be expressed by the matrix form as

$$T = (T_{ij})_{ij},$$
More precisely $T_{ij} : \mathbb{C}^d \to \mathbb{C}^d$ is given by
\[
T_{ij} = (\text{projection onto the } i\text{-th component}) \circ T \circ (\text{inclusion into the } j\text{-th component}).
\]

Definition 2.1. We say that a bounded operator $T \in \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d))$ has finite propagation if
\[
\text{prop}(T) := \sup \{ |i - j| : T_{ij} \neq 0 \}
\]
is finite.

It is easy to check the following Lemma.

Lemma 2.2. For any finite propagation operators $S, T \in \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d))$, the following inequality holds:
\[
\text{prop}(ST) \leq \text{prop}(S) + \text{prop}(T).
\]

We consider the following space of finite propagation unitary operators.

Definition 2.3. Let
\[
\mathcal{U}_L(\mathbb{C}^d) \subset \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d)).
\]
the space of unitary operators on $l^2(\mathbb{Z}, \mathbb{C}^d)$ of propagation $\leq L$. Define the space $\mathcal{U}(\mathbb{C}^d)$ as the direct limit
\[
\mathcal{U}(\mathbb{C}^d) = \lim_{L \to \infty} \mathcal{U}_L(\mathbb{C}^d),
\]
which is the set of finite propagation unitary operators topologized with the directive limit topology, i.e. a subset $O \subset \mathcal{U}(\mathbb{C}^d)$ is open if and only if the intersection $O \cap \mathcal{U}_L(\mathbb{C}^d)$ is open in $\mathcal{U}_L(\mathbb{C}^d)$ with respect to the norm topology for any $L$.

Remark 2.4. The inclusion map $\mathcal{U}_L(\mathbb{C}^d) \to \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d))$ induces the injective continuous map $\mathcal{U}(\mathbb{C}^d) \to \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d))$. However it is not a homeomorphism onto the image.

Remark 2.5. By “regrouping” mentioned in [1 Section 3], we have a canonical homeomorphism between $\mathcal{U}(\mathbb{C})$ and $\mathcal{U}(\mathbb{C}^d)$. Later we consider the inclusion $\mathcal{U}(\mathbb{C}) \to \mathcal{U}(\mathbb{C}^d)$ corresponding to the inclusion $\mathbb{C} \to \mathbb{C}^d$ when we study stability of some homotopy groups. It is a different map from the former one.

We have another matrix expression for $T \in \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d))$ by
\[
T = \begin{pmatrix} T_{--} & T_{+-} \\ T_{+--} & T_{++} \end{pmatrix},
\]
where
\[
T_{--} : l^2(\mathbb{Z}_{<0}, \mathbb{C}^d) \to l^2(\mathbb{Z}_{<0}, \mathbb{C}^d), \quad T_{+-} : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^d) \to l^2(\mathbb{Z}_{<0}, \mathbb{C}^d),
\]
\[
T_{+--} : l^2(\mathbb{Z}_{<0}, \mathbb{C}^d) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^d), \quad T_{++} : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^d) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^d).
\]

Remark 2.6. We will use a different orientation from the standard finite-size matrix expression for indices $(x, y) \in \mathbb{Z}^2$. Actually our choice of the indices turns out to be natural when we use the infinite-size matrices.

Definition 2.7. The index of $U \in \mathcal{U}(\mathbb{C}^d)$ is given by
\[
\text{ind}(U) = \|U_{--}\|_{\text{HS}}^2 - \|U_{+-}\|_{\text{HS}}^2
\]
where $\| \cdot \|_{\text{HS}}$ denotes the Hilbert–Schmidt norm, i.e. its square is the sum of the squares of the norm values of all the entries.

Note that the components $U_{--}$ and $U_{+-}$ consist of finitely many non-zero entries. Hence the definition of index is justified.
Example 2.8. The (left) shift operator \( S \in \mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)) \) is defined by

\[
S(v_i)_i = (v_{i+1})_i \quad (v_i \in \mathbb{C}^d).
\]

The operator \( S \) is a unitary operator with \( \text{prop}(S) = 1 \) and \( \text{ind}(S) = d \). Note that the index of the shift \( S \) depends on \( d \). For this reason, we will sometimes give a proof only in the case when \( d = 1 \) for simplicity (see Remark 2.5).

We consider the subalgebra of block diagonal operators

(1) \( \mathcal{B}_k(\ell^2(\mathbb{Z}, \mathbb{C}^d)) = \{ T \in \mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)) \mid T_{ij} = 0 \text{ unless } k + nL \leq i, j < k + (n + 1)L \text{ for some } n \in \mathbb{Z} \} \)

for \( k \in \mathbb{Z}, L \geq 1 \). It is a \( \mathbb{C}^* \)-algebra. An element in \( \mathcal{B}_k(\ell^2(\mathbb{Z}, \mathbb{C}^d)) \) looks like:

\[
\begin{array}{ccc}
(k,k) & & 0 \\
\uparrow & L & \downarrow \\
(k+L,k+L) & & 0 \\
\end{array}
\]

Define \( \mathcal{U}(A) \) as the group of unitary elements in a \( \ast \)-algebra \( A \):

\[
\mathcal{U}(A) = \{ U \in A \mid U^*U =UU^* = 1 \}.
\]

Let us recall the basic properties of the index given by Gross–Nesme–Vogts–Werner.

Lemma 2.9. \([1, \text{Theorem 3 and its proof}]\)

(1) For any \( U \in \mathcal{U}(\mathbb{C}^d) \), \( \text{ind}(U) \) is an integer.

(2) For any \( U, V \in \mathcal{U}(\mathbb{C}^d) \), the additivity holds:

\[
\text{ind}(UV) = \text{ind}(U) + \text{ind}(V).
\]

(3) The index defines a continuous function \( \text{ind}: \mathcal{U}(\mathbb{C}^d) \to \mathbb{Z} \), which induces a bijection

\[
\pi_0(\mathcal{U}(\mathbb{C}^d)) \cong \mathbb{Z}
\]

between the set of path components and the integer set.

Moreover, the map \( n \mapsto S^n \) gives a section of the index function.

(4) If \( U \in \mathcal{U}(\mathbb{C}^d) \) satisfies \( \text{ind}(U) = 0 \) with \( \text{prop}(U) \leq L \), then it admits a decomposition \( U = VV' \) into the product for some \( V \in \mathcal{U}(\mathcal{B}_0(2L, \mathbb{C}^d)) \) and \( V' \in \mathcal{U}(\mathcal{B}_{-L}(2L, \mathbb{C}^d)) \).

To verify its variant for “periodic” operators we will consider later, we give a proof of the assertion (4) for \( d = 1 \) here. Let \( \{ e_i \}_{i \in \mathbb{Z}} \) denote the standard orthonormal basis of \( \ell^2(\mathbb{Z}, \mathbb{C}) \) and \( P: \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C}) \)
be the projection defined by
\[ P e_i = \begin{cases} 
  e_i & i \geq 0, \\
  0 & i < 0.
\end{cases} \]

**Lemma 2.10.** Let \( Q : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C}) \) be an orthogonal projection which satisfies the following conditions
\[ Q e_i = \begin{cases} 
  e_i & i \geq k + L, \\
  0 & i < k.
\end{cases} \]
Moreover assume \( \text{trace}(P - Q) = 0 \) where the trace is well-defined since \( P - Q \) has rank \( \leq L \).
Then there exists a unitary operator \( V : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C}) \) such that
- \( P = V^* Q V \)
- \( V e_i = e_i \) if \( i < k \) or \( i \geq k + L \).

**Proof.** Let \( W \subset \ell^2(\mathbb{Z}, \mathbb{C}) \) be the span of \( e_k, e_{k+1}, \ldots, e_{k+L-1} \). Then, \( W \) is a finite dimensional linear subspace. Note that the following properties hold.
- The inclusions \( P(W) \subset W \) and \( Q(W) \subset W \) both hold. Then, the inclusions \( P(W^\perp) \subset W^\perp \) and \( Q(W^\perp) \subset W^\perp \) also hold.
- \( k+L \geq 0 \) and \( k < 0 \) both hold (otherwise \( \text{trace}(P - Q) \) cannot vanish). Hence, the equality \( P = Q \) holds on \( W^\perp \).

Then, we obtain the equalities
\[ 0 = \text{trace}(P - Q) = \text{trace}(P - Q)_{|W} = \text{trace}(P|_W - Q|_W) = \text{trace}(P|_W) - \text{trace}(Q|_W). \]
This implies that \( \text{rank}(Q|_W) = \text{rank}(P|_W) \).
Choose an orthonormal basis \( v_k, v_{k+1}, \ldots, v_{k+L-1} \) of \( W \) so that the set \( \{v_0, v_1, \ldots, v_{k+L-1}\} \) consists of an orthonormal basis of \( Q(W) \). This implies that \( Q v_i = v_i \) for \( i \geq 0 \) and \( Q v_i = 0 \) for \( i < 0 \). Define the unitary operator \( V \) by
\[ V = (\ldots, e_{k-2}, e_{k-1}, v_k, v_{k+1}, \ldots, v_{k+L-1}, e_{k+L}, e_{k+L+1}, \ldots) \]
Thus we have \( P = V^* Q V \) and \( V e_i = e_i \) if \( i < k \) or \( i \geq k + L \). \( \Box \)

**Proof of (4) of Lemma 2.9** The composite \( U^* P U \) is an orthogonal projection. We have \( U^* P U e_i = e_i \) for \( i \geq L \) and \( U^* P U e_j = 0 \) for \( j < -L \). We also have
\[
P - U^* P U = \begin{pmatrix} 0 & 0 \\ 0 & \text{id} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} U_{++}^* & U_{+-}^* \\ U_{+-} & U_{--} \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \text{id} \end{pmatrix} \begin{pmatrix} U_{--} & U_{+-} \\ U_{+-} & U_{++} \end{pmatrix} = \begin{pmatrix} -U_{++}^* U_{++} & -U_{+-}^* U_{+-} \\ -U_{+-}^* U_{+-} & -U_{--} U_{++} \end{pmatrix}.
\]
Then we have the equalities
\[
\text{trace}(U^* P U - P) = ||U_{--}||_{\text{HS}}^2 - ||U_{+-}||_{\text{HS}}^2 = \text{ind}(U) = 0.
\]
See [1] subsection 6.2.
Then, by Lemma 2.10 there exists a unitary operator \( V_0 \) such that \( P = V_0^* U^* P U V_0 \) and \( V_0 e_i = e_i \) if \( i < -L \) or \( i \geq L \).
- The former condition implies that \( U V_0 \) commutes with the projection \( P \). Then it is straightforward to check that the \( ++ \) and \( -- \) parts of \( U V_0 \) are both zero.
- The latter condition implies that the non-trivial block component of \( V_0 \) is a \( 2L \) by \( 2L \) matrix \( V'_0 \in M_{2L}(\mathbb{C}) \).
Remark 2.11. It follows from the construction that if $U$ is periodic in the sense that $S^{2L}US^{-2L} = U$, then $V$ and $V'$ can be taken as $S^{2L}VS^{-2L} = V$ and $S^{2L}V'S^{-2L} = V'$. See Section 4.

3. Homotopy type of $\mathcal{U}(\mathbb{C}^d)$

3.1. The Banach manifold $\mathcal{W}_L(\mathbb{C}^d)$. We study the homotopy type of $\mathcal{U}(\mathbb{C}^d)$. For this purpose, we approximate $\mathcal{U}(\mathbb{C}^d)$ by a family of Banach manifolds. We note that the group of unitaries $U(A)$ of a unital $C^*$-algebra $A$ is a Banach Lie group whose Lie algebra is given by

$$u(A) = \{T \in A \mid T^* = -T\}.$$ 
Recall $B_k(L, \mathbb{C}^d)$ in (1) in the previous section.

**Proposition 3.1.** For any integer $L \geq 1$, we have a smooth principal fiber bundle of Banach manifolds

$$U(B_0(L, \mathbb{C}^d)) \longrightarrow U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d)) \longrightarrow \mathcal{W}_L(\mathbb{C}^d)$$

where $U(B_0(L, \mathbb{C}^d))$ acts on $U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d))$ by $(U, U')V = (UV, V^{-1}U')$ and $\mathcal{W}_L(\mathbb{C}^d)$ is its quotient.

Moreover, the multiplication map

$$\Phi_L : \quad \Phi_L : U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d)) \rightarrow U(\mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)))$$

by $\Phi_L(U, U') := UU'$ is smooth, and factors through the projection

$$U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d)) \longrightarrow \mathcal{W}_L(\mathbb{C}^d) \longrightarrow \mathcal{U}(\mathbb{C}^d)$$

where $\phi_L$ is a smooth embedding.

**Proof.** The image of the differential of $U(B_0(L, \mathbb{C}^d)) \rightarrow U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d))$ at each point is a closed direct summand. Then it follows from the implicit function theorem that the quotient $\mathcal{W}_L(\mathbb{C}^d)$ admits a natural smooth Banach manifold structure and that the projection $U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d)) \rightarrow \mathcal{W}_L(\mathbb{C}^d)$ is a smooth principal bundle.

Since $(UV)(V^{-1}U') = UU'$, the map $\phi_L : \mathcal{W}_L(\mathbb{C}^d) \rightarrow U(\mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)))$ is well-defined and smooth. If $UU' = WW'$ for some $U, W \in U(B_0(\mathbb{Z}, 2L, \mathbb{C}^d))$ and $U', W' \in U(B_{-L}(2L, \mathbb{C}^d))$, we have

$$W^{-1}U = W'(U')^{-1} \in U(B_0(2L, \mathbb{C}^d)) \cap U(B_{-L}(2L, \mathbb{C}^d)) = U(B_0(L, \mathbb{C}^d)).$$

This implies that the map $\phi_L$ is injective. The map $\Phi_L$ is expressed by the coordinate around $(U, U')$ as

$$u(B_0(2L, \mathbb{C}^d)) \times u(B_{-L}(2L, \mathbb{C}^d)) \ni (T, T') \mapsto Ue^T e^{T'} U' \in \mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)),$$

of which the kernel of the differential at $(0, 0)$ coincides with the tangent space of the orbit of the action by $U(B_0(L, \mathbb{C}^d))$. Thus $\phi_L$ is an embedding.  

□
Through the inclusions
\[ U(B_0(L, \mathbb{C}^d)) \subset U(B_0(3L, \mathbb{C}^d)), \]
\[ U(B_0(2L, \mathbb{C}^d)) \subset U(B_0(6L, \mathbb{C}^d)), \]
\[ U(B_{-L}(2L, \mathbb{C}^d)) \subset U(B_{-3L}(6L, \mathbb{C}^d)), \]
we have the following inclusion of fiber bundles as

\[
\begin{array}{ccc}
U(B_0(L, \mathbb{C}^d)) & \longrightarrow & U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d)) \longrightarrow \mathcal{W}_L(\mathbb{C}^d) \\
\downarrow & & \downarrow \\
U(B_0(3L, \mathbb{C}^d)) & \longrightarrow & U(B_0(6L, \mathbb{C}^d)) \times U(B_{-3L}(6L, \mathbb{C}^d)) \longrightarrow \mathcal{W}_{3L}(\mathbb{C}^d)
\end{array}
\]

To study \( U(\mathbb{C}^d) \), it is sufficient to consider the direct limit \( \lim_{n \to \infty} \mathcal{W}_{3^nL}(\mathbb{C}^d) \) by the following lemma, where the choice of \( L \) does not matter.

Let \( U^0(\mathbb{C}^d) \) denote the identity component in \( U(\mathbb{C}^d) \). Actually it coincides with the subset of index 0 elements in \( U(\mathbb{C}^d) \) [1]. Let \( U^0_L(\mathbb{C}^d) \) denote the subset of index 0 elements in \( U_L(\mathbb{C}^d) \).

**Lemma 3.2.** The map
\[
\lim_{n \to \infty} \mathcal{W}_{3^nL}(\mathbb{C}^d) \rightarrow U^0(\mathbb{C}^d)
\]
is a homeomorphism, which is induced from the map \( \mathcal{W}_L(\mathbb{C}^d) \rightarrow U^0_L(\mathbb{C}^d) \) by \( U \mapsto \phi_L(U) \).

**Proof.** \( U^0_L(\mathbb{C}^d) \) is contained in \( \phi_L(\mathcal{W}_L(\mathbb{C}^d)) \) by (4) of Lemma 2.9.

Since \( \phi_L(\mathcal{W}_L(\mathbb{C}^d)) \) is homeomorphic to \( \mathcal{W}_L(\mathbb{C}^d) \) by Proposition 3.1, we obtain the continuous inverse
\[
U^0(\mathbb{C}^d) \rightarrow \lim_{n \to \infty} \mathcal{W}_{3^nL}(\mathbb{C}^d).
\]
This verifies the lemma. \( \square \)

**3.2. Homotopy groups of \( U(B_k(L, \mathbb{C}^d)) \).** Let
\[
\ell^\infty(\mathbb{Z}, \mathbb{Z}) = \{ \text{\( \mathbb{Z} \)-valued bounded sequences indexed by \( \mathbb{Z} \)} \}
\]
and \( X^{\infty \mathbb{Z}} \) be the countable infinite product of a space \( X \) indexed by \( \mathbb{Z} \). Recall

\[
\pi_i(U(dL)) = \begin{cases} 
0 & i : \text{even}, \\
\mathbb{Z} & i : \text{odd}
\end{cases}
\]

for \( 1 \leq i \leq 2dL - 1 \) (see [3]).

Note that the obvious map from \( U(B_k(L, \mathbb{C}^d)) \) to \( U(dL)^{\infty \mathbb{Z}} \) is one to one onto, however their topologies are different. Hence it is not a homeomorphism.

**Proposition 3.3.** The obvious map from \( U(B_k(L, \mathbb{C}^d)) \) to \( U(dL)^{\infty \mathbb{Z}} \) induces injection on their homotopy groups. Moreover, we compute the homotopy groups as follows:

\[
\pi_i(U(B_k(L, \mathbb{C}^d))) = \begin{cases} 
\ell^\infty(\mathbb{Z}, \mathbb{Z}) & i \leq 2dL - 1 \text{ is odd}, \\
0 & i \leq 2dL - 2 \text{ is even}, \\
\pi_i(U(dL))^{\infty \mathbb{Z}} & i \geq 2dL,
\end{cases}
\]

where the last isomorphism coincides with the inclusion \( \pi_i(U(B_k(L, \mathbb{C}^d))) \subset \pi_i(U(dL))^{\infty \mathbb{Z}} \).

The remaining of this subsection is devoted to the proof of Proposition 3.3. Take any based continuous map \( f: S^i \to U(B_k(L, \mathbb{C}^d)) \). Consider the projection

\[
\pi_j: B_k(L, \mathbb{C}^d) \to M_{dL}(\mathbb{C})
\]

onto the block component containing the \((k + jL, k + jL)\)-entry.

The following is the key to the proof. Recall that a family of maps \( \{f_i: X \to Y\}_{i \in \mathbb{Z}} \) between metric spaces is called uniformly equicontinuous, if for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( d_Y(f_n(x), f_n(x')) < \epsilon \) holds for any \( n \in \mathbb{Z} \) and \( x, x' \in X \) with \( d_X(x, x') < \delta \).

**Lemma 3.4.** The map \( f \) is homotopic to a map \( g: S^i \to U(B_k(L, \mathbb{C}^d)) \) such that there exists a finite set \( S_g \) of maps from \( S^i \) to \( U(dL) \) and each \( \pi_j \circ g \) is contained in \( S_g \).

**Proof.** Note that \( \|T\| = \sup_j \|\pi_j(T)\| \) for any \( T \in B_k(L, \mathbb{C}^d) \). Hence, the sequence of maps \( \{\pi_j \circ f\}_j \) is uniformly equicontinuous, because \( S^i \) is compact.

Moreover, since \( U(dL) \) is compact and hence is bounded in \( M_{dL}(\mathbb{C}) \cong \mathbb{C}^{dL^2} \), it follows from the Arzelà–Ascoli theorem that the sequence \( \{\pi_j \circ f\}_j \) is relatively compact in the space of maps \( S^i \to U(dL) \). Hence, we can find a finite family of maps \( h_1, \ldots, h_r: S^i \to U(dL) \) such that any \( \pi_j \circ f \) is sufficiently close to one of \( h_1, \ldots, h_r \). Thus we can find homotopies between them by using geodesics in \( U(dL) \). Hence, we have a homotopy from \( f \) to the resulting map \( g: S^i \to U(B_k(L, \mathbb{C}^d)) \) which is the desired map with \( S_g = \{h_1, \ldots, h_r\} \).

**Proof of Proposition 3.3** Any element in \( \pi_i(U(B_k(L, \mathbb{C}^d))) \) is represented by a continuous map

\[
g: S^i \to U(B_k(L, \mathbb{C}^d))
\]

with a set of maps \( S_g \) as in Lemma 3.4. Then, the homotopy class \( [g] \) is trivial if and only if all the maps in \( S_g \) are null-homotopic.

Note that we have the isomorphisms

\[
\pi_i(U(dL)^{\infty \mathbb{Z}}) \cong \pi_i(U(dL))^{\infty \mathbb{Z}}, \quad [h] \mapsto ([\pi_j \circ h])_j,
\]

\[
\pi_i(U(B_k(L, \mathbb{C}^d))) \cong \pi_i(U(dL)^{\infty \mathbb{Z}}) \quad \text{for } i \geq 2dL,
\]

where the latter is given by the obvious map from \( U(B_k(L, \mathbb{C}^d)) \) to \( U(dL)^{\infty \mathbb{Z}} \).

Then the case \( i \geq 2dL \) follows by these isomorphisms. For the case \( i < 2dL \) and even, this follows from the former isomorphism with (2).
Consider the remaining case $i < 2dL$ and odd. By Lemma 3.4, any element
$$\alpha = (\alpha_j) \in \pi_i(U(dL))^{\times Z} \cong \mathbb{Z}^{\times Z}$$
of its image admits a finite set $S_\alpha \subset \pi_i(U(dL)) \cong \mathbb{Z}$ such that $\alpha_j \in S_\alpha$ for all $j$. Hence, through the isomorphisms, $\alpha$ gives an element in $\ell^\infty(\mathbb{Z}, \mathbb{Z})$. Conversely, any element in $\ell^\infty(\mathbb{Z}, \mathbb{Z})$ comes from some $\alpha \in \pi_i(U(B_\ell(L, \mathbb{C}^d)))$. Therefore, we obtain the proposition. \hfill \Box

### 3.3. Homotopy groups of $\mathcal{W}_L^i(\mathbb{C}^d)$. We compute the homotopy groups of $\mathcal{W}_L^i(\mathbb{C}^d)$. Note that $\mathcal{W}_L(\mathbb{C}^d)$ is path-connected since so is $U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d))$. Consider the left shift operator
$$S : \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}, \mathbb{Z})$$
and the coinvariant set
$$\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}} := \ell^\infty(\mathbb{Z}, \mathbb{Z}) / \{ a - Sa | a \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) \},$$
which is a non-trivial group. For example, the element represented by $(\ldots, 1, 1, 1, \ldots)$ is not trivial in $\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}}$.

**Proposition 3.5.** If $1 \leq i \leq 2dL$, then
$$\pi_i(\mathcal{W}_L(\mathbb{C}^d)) = \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}} & \text{if } i \text{ is odd}, \\ \mathbb{Z} & \text{if } i \text{ is even}. \end{cases}$$

**Proof.** Consider the homotopy exact sequence of the fiber bundle
$$U(0, L, \mathbb{C}^d) \to U(0, 2L, \mathbb{C}^d) \times U(B(2L, \mathbb{C}^d)) \to (U(B_0, \mathbb{C}^d))^L.$$Then, it follows from Proposition 3.3 that we have the exact sequence
$$0 \to \pi_2(\mathcal{W}_L(\mathbb{C}^d)) \to \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \pi_{2i-1}(\mathcal{W}_L(\mathbb{C}^d)) \to 0$$for $1 \leq i \leq dL$, where the middle map $\ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is given by
$$(a_j) \mapsto ((a_{2j} + a_{2j-1})_j, -(a_{2j} + a_{2j-1})_j).$$The kernel and the cokernel of this map can be computed as $\mathbb{Z}$ and $\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}}$, respectively. \hfill \Box

**Theorem 3.6.** If $i \geq 1$, then
$$\pi_i(\biglim_{n \to \infty} \mathcal{W}_{3^{n}L}(\mathbb{C}^d)) = \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}} & \text{if } i \text{ is odd}, \\ \mathbb{Z} & \text{if } i \text{ is even}. \end{cases}$$

**Proof.** Consider the map between fiber bundles
$$U(0, L, \mathbb{C}^d) \to U(0, 2L, \mathbb{C}^d) \times U(B(2L, \mathbb{C}^d)) \to (U(B_0, \mathbb{C}^d))^L.$$Then, we obtain the following diagram of the exact sequences for $1 \leq i \leq dL$:

$$\begin{array}{ccccccccc}
0 & \to & \pi_2(\mathcal{W}_L) & \to & \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \to & \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \to & \pi_{2i-1}(\mathcal{W}_L(\mathbb{C}^d)) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \pi_2(\mathcal{W}_{3L}) & \to & \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \to & \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \to & \pi_{2i-1}(\mathcal{W}_{3L}(\mathbb{C}^d)) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \pi_2(\mathcal{W}_{9L}) & \to & \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \to & \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \to & \pi_{2i-1}(\mathcal{W}_{9L}(\mathbb{C}^d)) & \to & 0
\end{array}$$
we obtain the natural isomorphism
\[\ell^\infty(Z, Z) \to \ell^\infty(Z, Z)\]
\[\ell^\infty(Z, Z) \times \ell^\infty(Z, Z) \to \ell^\infty(Z, Z) \times \ell^\infty(Z, Z)\]
where the vertical arrows are given as
\[(a)_j \mapsto (a_{3j} + a_{3j+1} + a_{3j+2})_j\]
\[(a)_j, (b)_j \mapsto ((a_{3j} + a_{3j+1} + a_{3j+2})_j, (b_{3j-1} + b_{3j} + b_{3j+1}))_j\].

Since the homomorphisms \(Z \cong \pi_2(W_L) \to \ell^\infty(Z, Z)\) and \(Z \cong \pi_2(W_{3L}) \to \ell^\infty(Z, Z)\) are given by \(a \mapsto (\ldots, a, -a, a, -a, \ldots)\), we can see that the homomorphism \(\pi_2(W_L) \to \pi_2(W_{3L})\) is an isomorphism. For the homomorphism \(\pi_2(W_L(C^d)) \to \pi_2(W_{3L}(C^d))\), since the homomorphisms \(\ell^\infty(Z, Z) \times \ell^\infty(Z, Z) \to \ell^\infty(Z, Z) \times \ell^\infty(Z, Z)\) and \(\ell^\infty(Z, Z) \to \ell^\infty(Z, Z)\) are surjective, the homomorphism \(\pi_2(W_L(C^d)) \to \pi_2(W_{3L}(C^d))\) in the diagram are surjective, the homomorphism \(\pi_{2i-1}(W_L(C^d)) \to \pi_{2i-1}(W_{3L}(C^d))\) is surjective. The injectivity can be seen as follows: any element in \(\pi_{2i-1}(W_L(C^d)) \cong \ell^\infty(Z, Z)\) can be represented as \([\ldots, a_{3j}, 0, a_{3j+3}, 0, a_{3j+6}, \ldots]\) by shift. Its image is written as \([\ldots, a_{3j}, a_{3j+3}, a_{3j+6}, \ldots]\). If it is zero, we can find \(b = (b_j)_j \in \ell^\infty(Z, Z)\) such that \((\ldots, a_{3j}, a_{3j+3}, a_{3j+6}, \ldots) = c - S b\) in \(\ell^\infty(Z, Z)\). Then we have \((\ldots, a_{3j}, 0, a_{3j+3}, 0, a_{3j+6}, \ldots) = c\) where \(c\) with \(c_j = (c_j)_j\). This implies that \([\ldots, a_{3j}, 0, a_{3j+3}, 0, a_{3j+6}, \ldots]\) is zero in \(\pi_{2i-1}(W_L(C^d))\). Thus the homomorphism \(\pi_{2i-1}(W_L(C^d)) \to \pi_{2i-1}(W_{3L}(C^d))\) is an isomorphism. By the standard argument using compactness in Hausdorff spaces, we obtain the natural isomorphism
\[\pi_i(\lim_{n \to \infty} W_{3^n L}(C^d)) \cong \lim_{n \to \infty} \pi_i(W_{3^n L}(C^d))\]
and hence the theorem. \(\square\)

**Corollary 3.7.** For \(i \geq 0\), the homotopy groups are computed as
\[\pi_i(U(C)) = \begin{cases} \ell^\infty(Z, Z)_{\text{shift}} & \text{if } i \text{ is odd,} \\ Z & \text{if } i \text{ is even.} \end{cases}\]

**Proof.** The conclusion follows from Lemma [3.2] for \(i \geq 1\).

The equality \(\pi_0(U(C)) = Z\) is verified in [11]. \(\square\)

We also obtain the following stability.

**Theorem 3.8.** The following canonical map is an isomorphism for any \(i \geq 0\) and \(d \geq 1\).
\[\pi_i(U(C)) \to \pi_i(U(C^d))\]

**Proof.** By Proposition [3.3] the homomorphism
\[\pi_i(U(B_0(L, C))) \to \pi_i(U(B_0(L, C^d)))\]
is an isomorphism for \(i \leq 2L - 1\). Then
\[\pi_i(W_L(C)) \to \pi_i(W_L(C^d))\]
is an isomorphism for \(1 \leq i \leq 2L\) by the five lemma. Thus the desired isomorphism for the direct limit follows from the commutative square
\[\begin{array}{ccc}
W_L(C) & \to & W_L(C^d) \\
\downarrow & & \downarrow \\
W_{3L}(C) & \to & W_{3L}(C^d).
\end{array}\] \(\square\)
4. Homotopy of periodic finite propagation operators

4.1. Periodic finite propagation operators. We say that an operator \( T \in \mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)) \) is periodic if \( S^n T S^{-n} = T \) holds for some \( n > 0 \). For the matrix expression \( T = (T_{ij})_{i,j} \), the condition above is equivalent to

\[
T_{i+n,j+n} = T_{ij}
\]

for all \( i, j \). For positive integers \( L \) and \( n \), let us set

\[
\tilde{U}_L(n, \mathbb{C}^d) := \{ U \in \mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^d)) \mid \text{prop}(U) \leq L, \ S^n US^{-n} = U \}.
\]

We define the ordering \( (L, n) \leq (L', n') \), if \( L \leq L' \) and \( n \) divides \( n' \). Note that the inclusion

\[
\tilde{U}_L(n, \mathbb{C}^d) \subset \tilde{U}_{L'}(n', \mathbb{C}^d)
\]

holds if \( (L, n) \leq (L', n') \).

Definition 4.1. The space of periodic finite propagation unitary operators

\[
\tilde{U}(\mathbb{C}^d) := \lim_{(L,n)} \tilde{U}_L(n, \mathbb{C}^d)
\]

is given by the direct limit, with respect to the inclusions \( \tilde{U}_L(n, \mathbb{C}^d) \subset \tilde{U}_{L'}(n', \mathbb{C}^d) \) for \( (L, n) \leq (L', n') \).

Here the topology is similar to \( U(\mathbb{C}^d) \) (see Definition 2.3). The aim of this section is to study the homotopy type of the space \( \tilde{U}(\mathbb{C}^d) \).

4.2. The manifold \( \tilde{W}_L(\mathbb{C}^d) \). Consider the action of \( U(dL) \times U(dL) \) on \( U(2dL) \times U(2dL) \) given by

\[
(U, U') \cdot (V, V') = (U(V \oplus V'), ((V')^{-1} \oplus V^{-1})U')
\]

for \( (U, U') \in U(2dL) \times U(2dL) \) and \( (V, V') \in U(dL) \times U(dL) \), and the orbit space \( \tilde{W}_L(\mathbb{C}^d) \).

Since this action is free, there is a principal bundle

\[
U(dL) \times U(dL) \to U(2dL) \times U(2dL) \to \tilde{W}_L(\mathbb{C}^d)
\]

where the fiber inclusion is given by

\[
(V, V') \mapsto (V \oplus V', (V')^{-1} \oplus V^{-1})
\]

for \( (V, V') \in U(dL) \times U(dL) \).

Let us consider the block diagonal matrix as

\[
\Delta_k(U) = \begin{pmatrix}
\ddots & U \\
U & U \\
\ddots & \ddots
\end{pmatrix}
\]

so that each building block consists of the same \( n \) by \( n \) unitary matrix on \( k + Nn \leq i, j < k + (N + 1)n \) for some \( N \in \mathbb{Z} \). Then, one can obtain the inclusion

\[
\Delta_k : U(dL) \to U(B_k(L, \mathbb{C}^d))
\]

whose image is a periodic

\[
S^n \Delta_k(U) S^{-n} = \Delta_k(U).
\]
Recall Proposition 3.1, where the action of $U(B_0(L, \mathbb{C}^d))$ on $U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d))$ was given, and it restricts to the case of finite unitary matrices through the inclusions

\[ \Delta_0: U(dL) \rightarrow U(B_0(L, \mathbb{C}^d)), \]
\[ \Delta_0 \times \Delta_{-L}: U(2dL) \times U(2dL) \rightarrow U(B_0(2L, \mathbb{C}^d)) \times U(B_{-L}(2L, \mathbb{C}^d)). \]

The building block of the restriction is actually given by (4) above.

The following proposition is proved verbatim as Proposition 3.1.

**Proposition 4.2.** The smooth map

\[ U(2dL) \times U(2dL) \rightarrow U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d))), \quad (U, U') \mapsto \Delta_0(U)\Delta_{-L}(U') \]

factors as the composition of maps

\[ U(2dL) \times U(2dL) \xrightarrow{\text{proj}} U(2dL) \xrightarrow{\phi_L} U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d))) \]

where the map $\phi_L$ is a smooth embedding.

If $L$ divides $L'$, then the diagonal map $U(2dL) \rightarrow U(2dL')$ induces a map $\widehat{W}_L(\mathbb{C}^d) \rightarrow \widehat{W}_{L'}(\mathbb{C}^d)$ satisfying a commutative diagram

\[ \begin{array}{ccc} \widehat{W}_L(\mathbb{C}^d) & \xrightarrow{\phi_L} & U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d))) \\ & \downarrow & \quad \downarrow \\ \widehat{W}_{L'}(\mathbb{C}^d) & \xrightarrow{\phi_{L'}} & U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d))). \end{array} \]

Thus we can define the direct limit

\[ \widehat{W}(\mathbb{C}^d) = \lim_{\leftarrow} \widehat{W}_L(\mathbb{C}^d) \]

and a map

\[ (5) \quad \phi: \widehat{W}(\mathbb{C}^d) \rightarrow U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d))). \]

Let $\widehat{U}^0(\mathbb{C}^d)$ denote the subspace of $\widehat{U}(\mathbb{C}^d)$ consisting of the index zero elements.

**Lemma 4.3.** There is a homeomorphism

\[ \widehat{U}^0(\mathbb{C}^d) \cong \widehat{W}(\mathbb{C}^d). \]

**Proof.** We prove that the injective map $\phi: \widehat{W}(\mathbb{C}^d) \rightarrow U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d)))$ factors through a homeomorphism $\widehat{W}(\mathbb{C}^d) \xrightarrow{\cong} \widehat{U}^0(\mathbb{C}^d)$. Since $\widehat{W}_L(\mathbb{C}^d)$ is compact for each $L$ and $U(B(\ell^2(\mathbb{Z}, \mathbb{C}^d)))$ is Hausdorff, the map $\phi$ is closed. Then it is sufficient to show that the image of $\phi$ is $\widehat{U}^0(\mathbb{C}^d)$.

Applying (4) of Lemma 2.9 one can see that for each $U \in \widehat{U}_L^0(n, \mathbb{C}^d)$ and an integer $M$ divisible by $L$ and $n$, there are $V, W \in U(2dM)$ such that

\[ U = \Delta_0(V)\Delta_{-L}(W). \]

See Remark 2.11. Then it follows that $\widehat{U}_L^0(n, \mathbb{C}^d) \subset \phi_M(\widehat{W}_M(\mathbb{C}^d))$.

On the other hand, for any $U \in \widehat{W}_L(\mathbb{C}^d)$, the following properties

\[ \text{prop}(\phi_L(U)) \leq 4L, \quad S^{2L} \phi_L(U)S^{-2L} = \phi_L(U) \]

hold. Hence, $\phi_L(\widehat{W}_L(\mathbb{C}^d)) \subset \widehat{U}_L^0(2L, \mathbb{C}^d)$. Thus, by the naturality (5),

\[ \phi(\widehat{W}(\mathbb{C}^d)) = \widehat{U}^0(\mathbb{C}^d) \]

as desired. \qed
Proposition 4.5. The cohomology of the spectral sequences of the top and the bottom homotopy fibrations in the diagram. Then since $\rho$ is the transgression map in the Serre spectral sequence of the universal bundle $U(n) \to E U(n) \to B U(n)$. In particular, the map $\sum_{i+j=k} c_i \times c_j$, one gets $\tau(e_{2k-1}) = g^*(c_k) = \sum_{i+j=k} c_i \times c_j$ as claimed by naturality of transgression.

Lemma 4.4. In the Serre spectral sequence of $\tau(e_{2k-1})$, we have

$$\tau(e_{2k-1}) = \sum_{i+j=k} c_i \times c_j.$$ 

Proof. There is a homotopy commutative diagram

$$\begin{array}{ccc}
U(2n) & \longrightarrow & Gr_n(C^{2n}) \\
\downarrow & & \downarrow g \\
U(2n) & \longrightarrow & E U(2n) \\
\downarrow & & \downarrow g \\
B U(2n) & \longrightarrow & B U(2n)
\end{array}$$

where the map $g$ is induced from the inclusion $U(n) \times U(n) \to U(2n), (A, B) \mapsto A \oplus B$. Compare the Serre spectral sequences of the top and the bottom homotopy fibrations in the diagram. Then since $g^*(c_k) = \sum_{i+j=k} c_i \times c_j$, one gets $\tau(e_{2k-1}) = g^*(c_k) = \sum_{i+j=k} c_i \times c_j$ as claimed by naturality of transgression.

Proposition 4.5. The cohomology of $\tilde{W}_L(C^d)$ is given by

$$H^*(\tilde{W}_L(C^d)) \cong H^*(Gr_d(C^{2dL})) \otimes H^*(U(2dL)).$$

In particular, the map $\rho_1 : \tilde{W}_L(C^d) \to Gr_d(C^{2dL})$ is injective in cohomology.

Proof. By the definition of $\tilde{W}_L(C^d)$, there is a commutative diagram

$$\begin{array}{ccc}
U(2dL) & \longrightarrow & Gr_d(C^{2dL}) \\
\downarrow \pi_1 & & \downarrow \rho_1 \\
U(2dL) \times U(2dL) & \longrightarrow & \tilde{W}_L(C^d) \\
\downarrow \iota \circ \pi_2 & & \downarrow \rho_2 \\
U(2dL) & \longrightarrow & Gr_d(C^{2dL}) \\
\downarrow & & \downarrow \\
B U(dL) \times B U(dL) & \longrightarrow & B U(dL) \times B U(dL)
\end{array}$$

where $\pi_i : U(2dL) \times U(2dL) \to U(2dL)$ denotes the $i$-th projection for $i = 1, 2$ and $\iota : U(2dL) \to U(2dL)$ is the map $A \to A^{-1}$. Consider the Serre spectral sequence of the middle homotopy fibration. The $E_2$-term is given as

$$E_2^{*,*} = H^*(B U(dL) \times B U(dL)) \otimes H^*(U(2dL) \times U(2dL)).$$

By Lemma 4.4,

$$\tau(e_{2k-1} \times 1) = \sum_{i+j=k} c_i \times c_j, \quad \tau(1 \times e_{2k-1}) = - \sum_{i+j=k} c_i \times c_j.$$ 

Thus we get

$$E_\infty^{*,*} = E_\infty^{0,0} \otimes E_\infty^{0,*}.$$
where $E_{\infty}^{0} \cong H^{*}(\text{Gr}_{dL}(\mathbb{C}^{2dL}))$ and $E_{\infty}^{0,*} \cong H^{*}(U(2dL))$. Actually, one can see that $E_{\infty}^{0,0}$ coincides with the injective image of $\rho_1^*$ by a similar computation on the Serre spectral sequence of the top row in the diagram (7). Since $E_{\infty}^{0,*}$ is free as a graded commutative algebra, the canonical surjection $H^{*}(\hat{W}_{L}(\mathbb{C}^{d})) \to E_{\infty}^{0,*}$ admits a section as algebras. Then we obtain an algebra homomorphism

$$H^{*}(\text{Gr}_{dL}(\mathbb{C}^{2dL})) \otimes H^{*}(U(2dL)) \to H^{*}(\hat{W}_{L}(\mathbb{C}^{d})).$$

It is bijective by the previous computation of the $E_{\infty}$-term. \hfill $\Box$

4.4. Homotopy decomposition. We begin with a technical lemma.

**Lemma 4.6.** Let $\{F_{\lambda} \to E_{\lambda} \to B_{\lambda}\}_{\lambda \in \Lambda}$ be a direct system of homotopy fibrations such that all $F_{\lambda}, E_{\lambda}, B_{\lambda}$ are path-connected and all maps $F_{\lambda} \to F_{\mu}, E_{\lambda} \to E_{\mu}, B_{\lambda} \to B_{\mu}$ are cofibrations. Then

$$\lim_{\lambda \in \Lambda} F_{\lambda} \to \lim_{\lambda \in \Lambda} E_{\lambda} \to \lim_{\lambda \in \Lambda} B_{\lambda}$$

is a homotopy fibration.

**Proof.** Since direct limits commute with exact sequences, there is an exact sequence

$$\cdots \to \lim_{\lambda \in \Lambda} \pi_{*}(F_{\lambda}) \to \lim_{\lambda \in \Lambda} \pi_{*}(E_{\lambda}) \to \lim_{\lambda \in \Lambda} \pi_{*}(B_{\lambda}) \to \cdots.$$

Since all maps $F_{\lambda} \to F_{\mu}, E_{\lambda} \to E_{\mu}, B_{\lambda} \to B_{\mu}$ are cofibrations,

$$\lim_{\lambda \in \Lambda} \pi_{*}(F_{\lambda}) \cong \pi_{*}(\lim_{\lambda \in \Lambda} F_{\lambda}), \quad \lim_{\lambda \in \Lambda} \pi_{*}(E_{\lambda}) \cong \pi_{*}(\lim_{\lambda \in \Lambda} E_{\lambda}), \quad \lim_{\lambda \in \Lambda} \pi_{*}(B_{\lambda}) \cong \pi_{*}(\lim_{\lambda \in \Lambda} B_{\lambda}).$$

Thus the proof is done. \hfill $\Box$

Let $U(\infty) = \lim_{n \to \infty} U(n)$.

**Theorem 4.7.** There is a homotopy equivalence

$$\hat{W}_{L}(\mathbb{C}^{d}) \simeq B U(\infty) \times \prod_{n \geq 1} K(\mathbb{Q}, 2n-1)$$

where $K(\mathbb{Q}, m)$ denotes the Eilenberg–MacLane space of type $(\mathbb{Q}, m)$ and $\prod_{n \geq 1} K(\mathbb{Q}, 2n-1) = \lim_{N \to \infty} \prod_{n=1}^{N} K(\mathbb{Q}, 2n-1)$.

**Proof.** By definition, $\hat{W}_{L}(\mathbb{C}^{d})$ is the quotient of $U(2dL) \times U(2dL)$, and so the right $U(2dL)$ acts on $\hat{W}_{L}(\mathbb{C}^{d})$ such that the quotient is the Grassmannian $\text{Gr}_{dL}(\mathbb{C}^{2dL})$ of $(dL)$-dimensional subspaces in $\mathbb{C}^{2dL}$. Then there is a principal fibration

$$(8) \quad U(2dL) \to \hat{W}_{L}(\mathbb{C}^{d}) \xrightarrow{\rho_{1}} \text{Gr}_{dL}(\mathbb{C}^{2dL})$$

where $\rho_{1}$ is as in (7). Note that this fibration satisfies a commutative diagram

$$(9) \quad \begin{array}{ccc}
\text{U(2dL)} & \longrightarrow & \hat{W}_{L}(\mathbb{C}^{d}) \\
\downarrow & & \downarrow \\
\text{U(2dL)} & \longrightarrow & \text{Gr}_{dL}(\mathbb{C}^{2dL})
\end{array}$$

$$(9) \quad \begin{array}{ccc}
\text{U(2dL)} & \longrightarrow & \hat{W}_{L}(\mathbb{C}^{d}) \\
\downarrow & & \downarrow \\
\text{U(2dmL)} & \longrightarrow & \text{Gr}_{dmL}(\mathbb{C}^{2dmL})
\end{array}$$

where the left vertical map is the diagonal block inclusion and the right vertical map is the canonical inclusion. Clearly, the direct limit of the sequence of the natural inclusion

$$\cdots \to \text{Gr}_{n}(\mathbb{C}^{2n}) \to \text{Gr}_{n+1}(\mathbb{C}^{2n+2}) \to \cdots$$

is $BU$. Note that the direct limit of the sequence

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n+1} \mathbb{Z} \xrightarrow{n+2} \cdots$$

is \( \mathbb{Q} \). Then by looking at the homotopy groups in the stable range, one can see that the direct limit of the sequence of diagonal block inclusions

\[
U(n) \to U(kn) \to U(k(k + 1)n) \to \cdots
\]

is the rationalization of \( BU \). It is well known that the rationalization of \( BU \) is homotopy equivalent to \( \bigodot_{n \geq 1} K(\mathbb{Q}, 2n - 1) \), and so by Lemma 4.6 one gets a principal homotopy fibration

\[
\bigodot_{n \geq 1} K(\mathbb{Q}, 2n - 1) \to \hat{W}(\mathbb{C}^d) \to BU.
\]

It remains to show that this homotopy fibration is trivial, which is equivalent to that the classifying map \( BU \to \bigodot_{n \geq 1} K(\mathbb{Q}, 2n) \) is trivial. To see this, it is sufficient to observe that the induced homomorphism \( H^{2n}(K(\mathbb{Q}, 2n); \mathbb{Q}) \to H^{2n}(BU; \mathbb{Q}) \) is trivial. By Proposition 4.5, the homomorphism \( \rho_n^* : H^*(Gr_{IL}(\mathbb{C}^d L); \mathbb{Q}) \to H^*(\hat{W}_L(\mathbb{C}^d L); \mathbb{Q}) \) is injective. Thus the induced map in rational cohomology must be trivial since the composite of the maps

\[
\hat{W}(\mathbb{C}^d) \to BU \to \bigodot_{n \geq 1} K(\mathbb{Q}, 2n),
\]

is null-homotopic. This completes the proof. \( \square \)

There is an immediate corollary of Theorem 4.7. Note that \( \hat{U} \) is homeomorphic to \( \hat{W} \times \mathbb{Z} \) by Lemma 4.3.

**Corollary 4.8.** For \( i \geq 0 \),

\[
\pi_i(\hat{U}(\mathbb{C}^d)) = \begin{cases}
\mathbb{Q} & \text{if } i \text{ is odd}, \\
\mathbb{Z} & \text{if } i \text{ is even}.
\end{cases}
\]

The stability for this case follows from a similar proof to Theorem 3.8.

**Theorem 4.9.** The following canonical map is an isomorphism for any \( i \geq 0 \) and \( d \geq 1 \).

\[
\pi_i(\hat{U}(\mathbb{C})) \to \pi_i(\hat{U}(\mathbb{C}^d))
\]

### 4.5. End-periodic finite propagation operators.

We say that a finite propagation operator \( T \in \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}^d)) \) is *end-periodic* if \( S^nTS^{-n} - T \) has only finitely many non-zero entries for some \( n \).

For positive integers \( L, m \) and \( n \), we consider the set

\[
\tilde{U}_L(n, m, \mathbb{C}^d)
\]

consisting of unitary operators \( U \) of propagation \( \leq L \) such that \( S^nUS^{-n}U^{-1} \) coincides with the identity except \( 2m \times 2m \) matrix operator on the entries between the \( -m \)-th and the \( (m - 1) \)-th ones. Note that \( S^nUS^{-n}U^{-1} \) is the identity operator, if \( U \) happens to be periodic of period \( n \).

The inclusion

\[
\tilde{U}_L(n, m, \mathbb{C}^d) \subset \tilde{U}_{L'}(n', m', \mathbb{C}^d)
\]

holds if \( L \leq L' \), \( n \) divides \( n' \) and \( m \leq m' \). As in the case of periodic unitary operators, we study the homotopy type of the space of finite propagation end-periodic unitary operators with respect to the topology given by the direct limit

\[
\tilde{U}(\mathbb{C}^d) = \lim_{(L, m)} \tilde{U}_L(n, m, \mathbb{C}^d).
\]
Consider the subgroup $U^\text{fin}(m, \mathbb{C}^d) \subset U(B(L^2(\mathbb{Z}, \mathbb{C}^d)))$ consisting of $U$ which coincides with the identity except $2m \times 2m$ matrix operator on the entries between the $-m$-th and the $(m-1)$-th ones. Then, $U^\text{fin}(m, \mathbb{C}^d)$ is isomorphic to $U(2m)$ and the inductive limit
\[
U^\text{fin}(\mathbb{C}^d) = \lim_{m \to \infty} U^\text{fin}(m, \mathbb{C}^d)
\]
is homotopy equivalent to $U(\infty) = \lim_{L \to \infty} U(L)$.

Recall $\hat{U}_L(n, \mathbb{C}^d)$ in Subsection 4.1 $\hat{U}_L(n, m, \mathbb{C}^d)$ decomposes into a product as follows.

**Lemma 4.10.** The subset $\hat{U}_L(n, \mathbb{C}^d) \subset \hat{U}_L(n, m, \mathbb{C}^d)$ admits a retraction
\[
r : \hat{U}_L(n, m, \mathbb{C}^d) \to \hat{U}_L(n, \mathbb{C}^d)
\]
and the map
\[
\hat{U}_L(n, \mathbb{C}^d) \times U^\text{fin}(m, \mathbb{C}^d) \to \hat{U}_L(n, m, \mathbb{C}^d), \quad (U, V) \mapsto UV
\]
is a homeomorphism.

**Proof.** For any $U \in \hat{U}_L(n, m, \mathbb{C}^d)$, there is a unique periodic operator $r(U) \in \hat{U}_L(n, \mathbb{C}^d)$ such that $r(U)^{-1}U$ coincides with the identity except $2m \times 2m$ matrix operator on the entries between the $-m$-th and the $(m-1)$-th ones. The correspondence $r$ is obviously continuous with respect to the norm topology in $\hat{U}_L(n, m, \mathbb{C}^d)$. Moreover, the correspondence
\[
\hat{U}_L(n, \mathbb{C}^d) \to \hat{U}_L(n, m, \mathbb{C}^d) \times U^\text{fin}(m, \mathbb{C}^d), \quad U \mapsto (r(U), r(U)^{-1}U)
\]
is the inverse to the map in the lemma. \hfill \Box

**Theorem 4.11.** There is a continuous bijection
\[
\hat{U}(\mathbb{C}^d) \to \hat{U}(\mathbb{C}^d) \times U^\text{fin}(\mathbb{C}^d).
\]
which induces isomorphisms on homotopy groups.

We do not say this map is a homeomorphism since product and direct limit do not commute in general. We do not go further to investigate this problem.

**Proof.** By Lemma 4.10 we have the isomorphism
\[
\pi_i(\hat{U}_L(n, m, \mathbb{C}^d)) \cong \pi_i(\hat{U}_L(n, \mathbb{C}^d) \times \pi_i(U^\text{fin}(m, \mathbb{C}^d)).
\]
Taking the direct limit of both sides as groups, we obtain the isomorphism
\[
\pi_i(\hat{U}(\mathbb{C}^d)) \cong \pi_i(\hat{U}(\mathbb{C}^d) \times \pi_i(U^\text{fin}(\mathbb{C}^d)).
\]
It coincides with the product of the induced homomorphisms of the direct limit maps $\hat{U}(\mathbb{C}^d) \to \hat{U}(\mathbb{C}^d)$ and $\hat{U}(\mathbb{C}^d) \to U^\text{fin}(\mathbb{C}^d)$.

**Corollary 4.12.** For $i \geq 0$
\[
\pi_i(\hat{U}(\mathbb{C}^d)) \cong \begin{cases} \mathbb{Z} \times \mathbb{Q} & \text{i is odd}, \\ \mathbb{Z} & \text{i is even}. \end{cases}
\]
**Proof.** This immediately follows from Corollary 4.8, Theorem 4.11 and the following well-known fact (see [3]):
\[
\pi_i(U^\text{fin}(\mathbb{C}^d)) \cong \begin{cases} \mathbb{Z} & \text{i is odd}, \\ 0 & \text{i is even}. \end{cases}
\]
The stability also holds. Compare Theorems 3.8 and 4.9.
**Theorem 4.13.** The following canonical map is an isomorphism for any \( i \geq 1 \) and \( d \geq 1 \).

\[
\pi_i(U(C)) \rightarrow \pi_i(U(C^d))
\]

Recall \( U(C) \) in Definition 2.3. We can describe the homomorphism induced from the obvious inclusion \( U^{\text{fin}}(C) \subset U(C) \) as follows.

**Proposition 4.14.** The induced homomorphism

\[
\pi_{2i-1}(U^{\text{fin}}(C)(\equiv \mathbb{Z}) \rightarrow \pi_{2i-1}(U(C)(\equiv \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}})
\]

is given by \( a \mapsto [\ldots, 0, 0, a, 0, 0, \ldots] \).

**Proof.** The following left commutative square induces the right commutative square of \((2i-1)\)-st homotopy groups:

\[
\begin{array}{ccc}
U^{\text{fin}}(L, C) & \rightarrow & U(B_{-L}(2L, C)) \\
\downarrow & & \downarrow \\
U^{\text{fin}}(C) & \rightarrow & U(C)
\end{array}
\quad
\begin{array}{ccc}
\mathbb{Z} & \rightarrow & \ell^\infty(\mathbb{Z}, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \rightarrow & \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}}
\end{array}
\]

where in the right square, the top arrow is given by \( a \mapsto (\ldots, 0, 0, a, 0, 0, \ldots) \) and the right vertical arrow is the canonical projection. \( \square \)

## 5. Comparison with Grassmannian

We recall the **restricted unitary group** as a Banach Lie group, introduced by Segal and Wilson in [7]. It has the homotopy type of the union of countable infinite copies of the infinite Grassmannian \( B U(\infty) \).

**Definition 5.1.** We define the \( C^* \)-algebra \( B^{\text{SW}} \) by

\[
B^{\text{SW}} := \{ T \in B(\ell^2(\mathbb{Z}, \mathbb{C})) \mid T_{-+}, T_{+-} \text{ are compact} \}.
\]

The symbol “SW” stands for Segal–Wilson. Later in Subsection 6.1 we recall the uniform Roe algebra \( C_u(\mathbb{Z}) \), and it is contained in \( B^{\text{SW}} \). The Banach Lie group \( U(B^{\text{SW}}) \) is called the Segal–Wilson restricted unitary group. We note that the Lie algebra \( u(B^{\text{SW}}) \) of \( U(B^{\text{SW}}) \) is described as

\[
u(B^{SW}) = \{ T \in B(\ell^2(\mathbb{Z}, \mathbb{C})) \mid T_{-+} \text{ is compact, } T^* = -T \}.
\]

The local coordinate around \( U \in U(B^{SW}) \) is given by the exponential map \( u(B^{SW}) \ni T \mapsto U \exp T \). By the condition \( UU^* = U^*U = \text{id} \), we can observe that the component \( U_{++} \) of \( U \in U(B^{SW}) \) is a Fredholm operator.

**Lemma 5.2.** The index on \( U(C) \) continuously extends to the map

\[
\text{ind} : U(B^{SW}) \xrightarrow{U_{++}\text{ind}(U_{++})} \mathbb{Z}
\]

by taking the Fredholm index of \( U_{++} \).

In particular the inclusion \( U(C) \rightarrow U(B^{SW}) \) induces an isomorphism on \( \pi_0 \).

**Proof.** It follows from Proposition 6.4 that the index map on \( U(C) \) coincides with taking the Fredholm index of \( U_{++} \). Hence, the composite

\[
\begin{array}{ccc}
U(C) & \xrightarrow{\text{inclusion}} & U(B^{SW}) \\
& & \xrightarrow{U_{++}\text{ind}(U_{++})} \mathbb{Z},
\end{array}
\]

coincides with the index map in Section 2. Since the path components of \( U(B^{SW}) \) are classified by \( \text{ind}(U_{++}) \in \mathbb{Z} \) [7], the inclusion \( U(C) \rightarrow U(B^{SW}) \) induces an isomorphism on \( \pi_0 \) by Lemma 2.9. \( \square \)
Let $\mathcal{K}(X)$ be the space of compact operators on a separable Hilbert space $X$ of infinite dimension. The unitization

$$\mathcal{K}^+(X) := \mathbb{C} \oplus \mathcal{K}(X) \subset \mathcal{B}(X)$$

is a $C^*$-algebra equipped with the augmentation $\mathcal{K}^+(\ell^2(\mathbb{Z}, \mathbb{C})) \to \mathbb{C}$. Let $\mathcal{U}^{pl}(\mathbb{Z}, \mathbb{C})$ be the kernel of

$$U(\mathcal{K}^+(\ell^2(\mathbb{Z}, \mathbb{C}))) \to U(\mathbb{C}) = S^1$$

and consider similar spaces $\mathcal{U}^{pl}(\mathbb{Z}_{\geq 0}, \mathbb{C})$ and $\mathcal{U}^{pl}(\mathbb{Z}_{< 0}, \mathbb{C})$ of unitary operators on $\ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C})$ and $\ell^2(\mathbb{Z}_{< 0}, \mathbb{C})$ respectively. It is well-known that all of them have the homotopy type of $U(\infty)$.

We have the smooth principal bundle of Banach manifolds

$$\mathcal{U}^{pl}(\mathbb{Z}_{\geq 0}, \mathbb{C}) \times \mathcal{U}^{pl}(\mathbb{Z}_{< 0}, \mathbb{C}) \to U(\mathcal{B}(\ell^2(\mathbb{Z}_{< 0}, \mathbb{C}))) \times U(\mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}))) \times \mathcal{U}^{pl}(\mathbb{Z}, \mathbb{C}) \to U(\mathcal{B}^{SW})$$

by a similar argument to the proof of Proposition 3.1, where the left and the right hand side maps are given by

$$(U_-, U_+) \mapsto (U_-, U_+; U^{-1}_- \oplus U^{-1}_+),
\quad (U_-, U_+; U_0) \mapsto (U_- \oplus U_+)U_0.$$ 

**Theorem 5.3.** The inclusions $\tilde{U}(\mathbb{C}) \to U(\mathbb{C}) \to U(B^{SW})$ induce isomorphisms

$$\pi_2(\tilde{U}(\mathbb{C})) \cong \pi_2(U(\mathbb{C})) \cong \pi_2(U(B^{SW})) \cong \mathbb{Z}$$

for each $i \geq 0$.

**Proof.** The case $i = 0$ follows from Corollary 4.12 and Lemma 5.2. Hence, we may concentrate on the path-components of index 0. Consider the maps

$$\alpha : U(2L) \times U(2L) \to U(B_0(2L, \mathbb{C})) \times U(B_{-L}(2L, \mathbb{C})),
\alpha(U, U') = (\Delta_0(U), \Delta_{-L}(U')),$n
$$\beta : U(2L) \times U(2L) \to U(\mathcal{B}(\ell^2(\mathbb{Z}_{< 0}, \mathbb{C}))) \times U(\mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}))) \times \mathcal{U}^{pl}(\mathbb{Z}, \mathbb{C}),
\beta(U, U') = (\Delta_0(U)\Delta_{-L}(U')\gamma(U')^{-1}, \gamma(U')),$$

where $\gamma(U')$ is the block diagonal matrix

$$\begin{pmatrix}
\cdots \\
1 \\
U' \\
1 \\
\cdots
\end{pmatrix} \in U(B_{-L}(\mathbb{Z}, 2L, \mathbb{C}))$$

of which the entry $U'$ is the 0-th entry, and $\Delta_0(U)\Delta_{-L}(U')\gamma(U')^{-1} \in U(\mathcal{B}(\ell^2(\mathbb{Z}_{< 0}, \mathbb{C}))) \times U(\mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C})))$. Then we obtain the following commutative diagram of fiber bundles as

$$
\begin{array}{ccccccc}
U(B_0(L, \mathbb{C})) & \longrightarrow & U(B_0(2L, \mathbb{C})) \times U(B_{-L}(2L, \mathbb{C})) & \longrightarrow & W_1(L, \mathbb{C}) \\
\downarrow & & \downarrow \alpha & & \downarrow & \\
U(L) \times U(L) & \longrightarrow & U(2L) \times U(2L) & \longrightarrow & \tilde{W}_1(L, \mathbb{C}) \\
\downarrow & & \downarrow \beta & & \downarrow & \\
\mathcal{U}^{pl}(\mathbb{Z}_{\geq 0}, \mathbb{C}) \times \mathcal{U}^{pl}(\mathbb{Z}_{< 0}, \mathbb{C}) & \longrightarrow & U(\mathcal{B}(\ell^2(\mathbb{Z}_{< 0}, \mathbb{C}))) \times U(\mathcal{B}(\ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}))) \times \mathcal{U}^{pl}(\mathbb{Z}, \mathbb{C}) & \longrightarrow & U(B^{SW})
\end{array}$$

Thus we have the following diagram of homotopy groups

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi_{2i}(W_L(\mathbb{C})) & \rightarrow & \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \rightarrow & \ell^\infty(\mathbb{Z}, \mathbb{Z}) \oplus \ell^\infty(\mathbb{Z}, \mathbb{Z}) & \rightarrow & \pi_{2i-1}(W_L(\mathbb{C})) \\
\downarrow & & \alpha & & & & \downarrow \\
0 & \rightarrow & \pi_{2i}(W_L(\mathbb{C})) & \rightarrow & \mathbb{Z} \oplus \mathbb{Z} & \rightarrow & \mathbb{Z} \oplus \mathbb{Z} & \rightarrow & \pi_{2i-1}(W_L(\mathbb{C})) \\
\downarrow & & \beta & & & & \downarrow \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} \oplus \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & 0
\end{array}
\]

for \(1 \leq i \leq L\), where the rows are exact. By computing this, we obtain that the left vertical arrows are isomorphisms.

Since the composite \(\widehat{W}_L(\mathbb{C}) \rightarrow W_L(\mathbb{C}) \rightarrow U(B^{SW})\) coincides with \(\widehat{\phi}_L\), the homomorphism \(\pi_{2i}(W_L(\mathbb{C})) \rightarrow \pi_{2i}(U(B^{SW}))\) is an isomorphism for \(1 \leq i \leq L\). Therefore, we obtain the theorem. \(\square\)

**Remark 5.4.** The homomorphism

\[
Q \cong \pi_{2i-1}(\widehat{U}(\mathbb{C})) \rightarrow \pi_{2i-1}(U(\mathbb{C})) \cong \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}}
\]

is injective. This can be seen by computing the composite

\[
\mathbb{Z} \cong \pi_{2i-1}(U(2L)) \xrightarrow{(\Delta_0)} \pi_{2i-1}(\widehat{U}(\mathbb{C})) \rightarrow \pi_{2i-1}(U(\mathbb{C})) \cong \pi_{2i-1}(W_L(\mathbb{C})) \cong \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}}
\]

for sufficiently large \(L\) as \(a \mapsto [\ldots, a, a, a, \ldots] \neq 0\) in \(\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\text{shift}}\).

**Remark 5.5.** By Proposition\(^4.14\) Theorem\(^5.3\) and the previous remark, we can also compute the homomorphism \(\pi_i(U(\mathbb{C})) \equiv \pi_i(U(\mathbb{C})) \times \pi_i(U^{\text{fin}}(\mathbb{C})) \rightarrow \pi_i(U(\mathbb{C}))\) for each \(i\).

### 6. Uniform Roe algebra and its variants

We have studied various spaces of unitary operators defined by direct limits so far. In this section, we investigate their relation with corresponding completed spaces.

Let \(A\) be a \(C^*\)-algebra. Let \(\text{GL}_d(A)\) and \(U_d(A)\) denote the groups of invertible and unitary elements of the \(d\times d\)-matrix algebra \(M_d(A)\), respectively. Then it is well-known that the stabilization of the homotopy groups

\[
\lim_{d \to \infty} \pi_i(\text{GL}_d(A))
\]

admits Bott periodicity. The \(K\)-theory \(K_i(A)\) of \(A\) is given as

\[
K_0(A) = \lim_{d \to \infty} \pi_{2i-1}(\text{GL}_d(A)), \quad K_1(A) = \lim_{d \to \infty} \pi_{2i}(\text{GL}_d(A))
\]

for \(i \geq 1\), where the direct limits are taken over the usual inclusions \(\text{GL}_d(A) \subset \text{GL}_{d+1}(A)\).

In this section, we compute the \(K\)-groups of the uniform Roe algebra and its variant. The proofs can go rather formally, and very different from the method of our computations of the homotopy groups.

Note that in general, the homotopy group \(\pi_*(\text{GL}_1(A))\) and the \(K\)-theory \(K_{*+1}(A)\) are different. For example, \(\pi_{2i-1}(\text{GL}_d(\ell^\infty(\mathbb{Z}, \mathbb{C})))\) is isomorphic to the countably infinite product of the finite group \(\pi_{2i-1}(\text{GL}_d(\mathbb{C})))\) for \(i > d\) and is not isomorphic to \(K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) = \ell^\infty(\mathbb{Z}, \mathbb{Z})\) (see Proposition\(^3.3\)).

**Remark 6.1.** In fact, we can find an isomorphism \(C^*_d(\mathbb{Z}) \rightarrow M_d(C^*_u(\mathbb{Z}))\) of \(C^*\)-algebras. But, in general, the existence of an isomorphism \(A \rightarrow M_d(A)\) does not imply the stability of the homotopy groups \(\pi_*(\text{GL}_d(A))\).
6.1. The uniform Roe algebra $C_u^*(\mathbb{Z})$. Let $C_u(\mathbb{Z}) \subset \mathcal{B}(l^2(\mathbb{Z}, \mathbb{C}))$ denote the $*$-subalgebra consisting of finite propagation operators. The uniform Roe algebra $C_u^*(\mathbb{Z})$ of $\mathbb{Z}$ is the $C^*$-algebra defined as the norm closure of $C_u(\mathbb{Z})$.

Let $U_d(A)$ denote the set of $d$ by $d$ unitary matrices with entries in a $*$-algebra $A$. Then, we have a canonical inclusion

$$U(C^d) \to U_d(C_u^*(\mathbb{Z}))$$

This inclusion coincides with the inclusions with respect to the dimension $d$. More precisely, the following square commutes.

$$
\begin{array}{ccc}
U(C^d) & \to & U_d(C_u^*(\mathbb{Z})) \\
\downarrow & & \downarrow \\
U(C^{d+1}) & \to & U_{d+1}(C_u^*(\mathbb{Z}))
\end{array}
$$

We would like to propose the following:

**Conjecture 6.2.** The canonical inclusion

$$U(C^d) \to U_d(C_u^*(\mathbb{Z}))$$

is a homotopy equivalence for $d \geq 1$.

This is analogous to the fact that the canonical inclusion $U^{\text{fin}}(C) \to U^{\text{cpt}}(\mathbb{Z}, C)$ is a homotopy equivalence. Once this conjecture is verified, it follows that the inclusion

$$U_d(C_u^*(\mathbb{Z})) \to U_{d+1}(C_u^*(\mathbb{Z}))$$

is also a homotopy equivalence since $U(C^d) \to U(C^{d+1})$ is a homotopy equivalence by Theorem 3.8.

On the other hand, we can compute it without Conjecture 6.2 as follows. It supports our conjecture.

**Proposition 6.3.** The following isomorphism holds:

$$K_i(C_u^*(\mathbb{Z})) \cong \begin{cases} 
\ell^\infty(\mathbb{Z}, \mathbb{Z}) & i = 0, \\
0 & i = 1.
\end{cases}$$

**Proof.** Let $\alpha : \ell^\infty(\mathbb{Z}, \mathbb{C}) \to \ell^\infty(\mathbb{Z}, \mathbb{C})$ be the automorphism defined by the left shift $S$. Then $C_u^*(\mathbb{Z})$ is presented as the crossed product

$$C_u^*(\mathbb{Z}) = \ell^\infty(\mathbb{Z}, \mathbb{C}) \rtimes_\alpha \mathbb{Z}$$

Applying the Pimsner–Voiculescu exact sequence [5], we get the six-term cyclic exact sequence:

$$
\begin{array}{cccccc}
K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \xrightarrow{1-S} & K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \to & K_0(C_u^*(\mathbb{Z})) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(C_u^*(\mathbb{Z})) & \xrightarrow{1-S} & K_1(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \to & K_1(\ell^\infty(\mathbb{Z}, \mathbb{C}))
\end{array}
$$

As is well-known (or by Proposition 3.3), we have

$$K_i(\ell^\infty(\mathbb{Z}, \mathbb{C})) \cong \begin{cases} 
\ell^\infty(\mathbb{Z}, \mathbb{Z}) & i = 0, \\
0 & i = 1,
\end{cases}$$

where the induced homomorphism $S : \ell^\infty(\mathbb{Z}, \mathbb{C}) \to \ell^\infty(\mathbb{Z}, \mathbb{C})$ is the left shift as well. Thus we can compute $K_i(C_u^*(\mathbb{Z}))$ by the previous exact sequence. \qed
For $\pi_0$, we know that the index homomorphism $\pi_0(U(\mathbb{C})) \to \mathbb{Z}$ is an isomorphism. Let us reformulate the index map.

**Proposition 6.4.** For any finite propagation unitary operator $U$, the component

$$U_{++} : \ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^d) \to \ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^d)$$

is a Fredholm operator. Moreover, the index $\text{ind}(U)$ coincides with the index of the Fredholm operator $U_{++}$ as

$$\text{ind}(U) = \text{ind}(U_{++})$$

where $\text{ind}(U_{++}) := \dim \ker U_{++} - \dim \text{coker} U_{++}$.

**Proof.** As in Remark 2.5 it is sufficient to prove in the case when $d = 1$. Since $(U_-, U_{++}) : \ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C})$ is injective and $U_-$ has only finitely many non-zero entries, the kernel of $U_{++}$ is finite dimensional. By applying the same argument to the adjoint $U^*$, we can see that the cokernel of $U_{++}$ is also finite dimensional. Hence, $U_{++}$ is a Fredholm operator.

If $U$ belongs to the path component containing the iterated shift $S^n (n \in \mathbb{Z})$, it follows from a property of the index of Fredholm operators that $U_{++}$ is a Fredholm operator of index $n$. Thus, by Lemma 2.9 (3), the index of the Fredholm operator $U_{++}$ coincides with $\text{ind}(U_{++})$. $\square$

**Corollary 6.5.** The index map on $U(\mathbb{C})$ is continuously extended over $U(\mathcal{C}_u^*(\mathbb{Z}))$ as

$$\text{ind} : U(\mathcal{C}_u^*(\mathbb{Z})) \to \mathbb{Z}.$$  

**Proof.** This follows since the component $U_{++}$ of any $U \in U(\mathcal{C}_u^*(\mathbb{Z}))$ is Fredholm. $\square$

It follows from the proof of Proposition 6.3 that the class $[S] \in K_1(\mathcal{C}_u^*(\mathbb{Z}))$ represented by the shift $S$ generates $K_1(\mathcal{C}_u^*(\mathbb{Z})) \equiv \mathbb{Z}$. Then, from the fact $\text{ind}(S) = 1$ with Conjecture 6.2 we could conclude that $\text{ind}_* : \pi_0(U(\mathcal{C}_u^*(\mathbb{Z}))) \to \mathbb{Z}$ is also an isomorphism. At least we know it is surjective.

### 6.2. The $C^*$-algebra $\mathcal{P}_u^*(\mathbb{Z})$

Let $\mathcal{P}_u(\mathbb{Z}) \subset \mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}))$ denote the $*$-subalgebra consisting of finite propagation periodic operators. We define the $C^*$-algebra $\mathcal{P}_u^*(\mathbb{Z})$ as the $C^*$-algebra given by the norm closure of $\mathcal{P}_u(\mathbb{Z})$.

We would propose a parallel conjecture for the canonical inclusion $\widehat{U}(\mathbb{C}^d) \to U_d(\mathcal{P}_u^*(\mathbb{Z}))$.

**Conjecture 6.6.** The canonical inclusion

$$\widehat{U}(\mathbb{C}^d) \to U_d(\mathcal{P}_u^*(\mathbb{Z}))$$

is a homotopy equivalence for $d \geq 1$.

Similarly, the following proposition supports this conjecture.

**Proposition 6.7.** The following isomorphism holds:

$$K_i(\mathcal{P}_u^*(\mathbb{Z})) \cong \begin{cases} \mathbb{Q} & i = 0, \\ \mathbb{Z} & i = 1. \end{cases}$$

To verify this, let us introduce a $C^*$-algebra $\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}) \subset \ell^\infty(\mathbb{Z}, \mathbb{C})$ as the closure of the space of all the periodic sequences. Let $\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ denote the space of $\mathbb{Z}$-valued periodic sequences.
Lemma 6.8. The following isomorphism holds:
\[
\pi_i(U_d(\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}))) = \begin{cases} 
\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{Z}) & \text{if } i \leq 2d - 1 \text{ is odd}, \\
0 & \text{if } i \leq 2d - 2 \text{ is even}.
\end{cases}
\]
Moreover, the inclusion \( U_d(\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C})) \to U_{d+1}(\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C})) \) induces the isomorphism on the homotopy groups of degree \( \leq 2d - 1 \).

Proof. Let \( A_n \subset \ell^\infty(\mathbb{Z}, \mathbb{C}) \) be the subalgebra of periodic sequences of period \( n \). Then, \( A_n \) has dimension \( n \) and the union \( \bigcup_{n \geq 1} A_n \) is dense in \( \ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}) \), where \( A_n \subset A_{n'} \) if and only if \( n \) divides \( n' \). Note that we can easily extend the result [4, Theorem 12] to general sequential colimit by the same proof. Applying this to the invertible \( d \times d \)-matrices \( \text{GL}_d(A_n) \) and the unitaries \( U_d(A_n) \), we can compute the homotopy groups \( \pi_i(\text{GL}_d(\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}))) \cong \pi_i(U_d(\ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}))) \) as in the lemma. \qed

Proof of Proposition 6.7. Together with Lemma 6.8, the same proof as Proposition 6.3 works for the crossed product \( P^*_\alpha(\mathbb{Z}) = \ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}) \rtimes_\alpha \mathbb{Z} \), where \( \alpha: \ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}) \to \ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{C}) \) is the automorphism defined by the left shift. Note that the shift coinvariant of \( \ell^\infty_{\text{per}}(\mathbb{Z}, \mathbb{Z}) \) is isomorphic to \( \mathbb{Q} \). \qed

6.3. The closure of end-periodic finite propagation operators. It is obvious that the closure of the subspace of all the end-periodic finite propagation operators in \( B(\ell^2(\mathbb{Z}, \mathbb{C})) \) coincides with the direct sum \( P^*_\alpha(\mathbb{Z}) \oplus K(\ell^2(\mathbb{Z}, \mathbb{C})) \).

Thus, we can also compute the homotopy groups of its unitaries by Conjecture 6.6 and the homotopy equivalence \( \mathcal{U}^{\text{fin}}(\mathbb{C}) \to \mathcal{U}^{\text{cpt}}(\mathbb{Z}, \mathbb{C}) \). This coincides with Theorem 4.11.

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