Existence of Nontrivial Solutions for $p$-Laplacian Equations in $\mathbb{R}^N$

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Abstract In this paper, we consider a $p$-Laplacian equation in $\mathbb{R}^N$ with sign-changing potential and subcritical $p$-superlinear nonlinearity. By using the cohomological linking method for cones developed by Degiovanni and Lancelotti in 2007, an existence result is obtained. We also give a result on the existence of periodic solutions for one-dimensional $p$-Laplacian equations which can be proved by the same method.

Key words $p$-Laplacian equation; sign-changing potential; cohomological link; Cerami condition; periodic solution

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1 Introduction and main results

We mainly consider the following $p$-Laplacian equation in the entire space

\begin{equation}
\begin{cases}
-\Delta_p u + U(x)|u|^{p-2}u = f(x, u), \\
u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}),
\end{cases}
\end{equation}

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator with $p > 1$.

For $p = 2$, (1.1) turns into a kind of Schrödinger equation of the form

\begin{equation}
-\Delta u + U(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N, \mathbb{R}),
\end{equation}

which has been studied extensively. In [1, 5, 6, 25], (1.2) with a constant sign potential $U(x)$ was considered. More precisely, the potential in these papers is of the form $a_0(x) + \lambda a(x)$,
$a_0(x)$ has positive lower bound, $a(x) \geq 0$ and $\lambda > 0$ large enough. And in \cite{[12],[13],[19]}, the authors considered \cite{[12]} with a potential $U(x)$ that may change sign.

For general $p > 1$, most of the work, as the authors of this paper known, deal with the problem \eqref{1.1} with a constant sign potential $U(x)$, see for example \cite{[14],[26],[23]} and the reference therein.

In this paper, we consider \eqref{1.1} with sign-changing potential and subcritical $p$-superlinear nonlinearity, moreover, periodic conditions on the potential and nonlinearity are not needed. Assume that $U(x)$ is of the form $b(x) - \lambda V(x)$, here $\lambda$ is a real number, $b(x)$, $V(x)$ and $f(x,t)$ satisfy the following conditions

(B) $b \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x \in \mathbb{R}^N} b(x) \geq b_0 > 0$, $\text{meas}(\{x \in \mathbb{R}^N : b(x) \leq M\}) < \infty$, $\forall M > 0$,

(V) $V \in L^\infty(\mathbb{R}^N, \mathbb{R})$,

(f1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\exists q \in (p,p^*)$ s.t. $|f(x,t)| \leq C(1 + |t|^{q-1})$, $f(x,t)t \geq 0$, $\forall t \in \mathbb{R}$,

(f2) $\lim_{|t| \to \infty} \frac{f(x,t)t}{|t|^p} = +\infty$ uniformly in $x \in \mathbb{R}^N$,

(f3) $f(x,t) = o(|t|^{p-1})$ as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$,

(f4) $\exists \theta \geq 1$ s.t. $\theta \mathcal{F}(x,t) \geq \mathcal{F}(x,st)$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$,

here we have set $F(x,t) = \int_0^t f(x,t)dt$, $\mathcal{F}(x,t) = f(x,t)t - pF(x,t)$, $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ if $p \geq N$ and $\text{meas}(\cdot)$ means the Lebesgue measure in $\mathbb{R}^N$.

Our main result reads as

**Theorem 1.1** If (B), (V) and (f1)-(f4) hold, the problem \eqref{1.1} possesses a nontrivial solution for every $\lambda \in \mathbb{R}$.

We remark that the condition $\inf_{x \in \mathbb{R}^N} b(x) \geq b_0 > 0$ is not essential, it can be replaced by the condition $\inf_{x \in \mathbb{R}^N} b(x) > -\infty$. First note that the case $\lambda = 0$ can be replaced by the case $V = 0$ with a nonzero $\lambda$, we can always assume that $\lambda \neq 0$. If $\inf_{x \in \mathbb{R}^N} b(x) > -c_0$ for some $c_0 > 0$, one can replace $b$ and $V$ by $b + c_0$ and $V + \frac{\alpha}{\lambda}$, then $b + c_0$ and $V + \frac{\alpha}{\lambda}$ satisfy conditions (B) and (V). For $p = 2$, the condition (f1) was introduced in \cite{[21]}, and for $p \neq 2$ it was introduced in \cite{[27]}. Condition (B) was first introduced in \cite{[5]}, and then was used by many authors, for example, \cite{[33]}.

When dealing with superlinear problem, one usually needs a growth condition together with the following classical condition which was introduced by Ambrosetti and Rabinowitz
There exists $\mu > 2$ such that for $u \neq 0$ and $x \in \mathbb{R}^N$, $0 < \mu F(x,u) \leq uf(x,u)$.  \hspace{1cm} (1.3)

Since then, many authors tried to weaken this condition, see [13, 18, 26, 27, 28, 30, 32]. In [30] the authors obtained a weak solution of (1.2) under the following conditions:

$$
(C_1) \quad U(x) \in C(\mathbb{R}^N, \mathbb{R}), \quad \inf_{x \in \mathbb{R}^N} U(x) \geq U_0 > 0, \; U(x) \text{ is 1-periodic in each of } x_i, \; i = 1, \cdots, N,
$$

$$
(C_2) \quad f(x,t) \in C^1 \text{ is 1-periodic in each of } x_i, \; i = 1, \cdots, N, \; f'(t) \text{ is a Caratheodory function and there exists } C > 0, \text{ such that } |f'(x,t)| \leq C(1 + |t|^{2p-2}), \lim_{|t| \to \infty} \frac{|f(x,t)|}{|t|^{2^* - 1}} = 0, \text{ uniformly in } x \in \mathbb{R}^N,
$$

$$
(C_3) \quad f(x,t) = o(|t|), \text{ as } |t| \to 0, \text{ uniformly in } x,
$$

$$
(C_4) \quad \lim_{|u| \to \infty} \frac{F(x,u)}{u^2} = \infty, \text{ uniformly in } x,
$$

$$
(C_5) \quad \frac{f(x,t)}{|t|} \text{ is strictly increasing in } t.
$$

And in [26] the author got a weak solution of (1.1) with the following assumptions:

$$
(D_1) \quad V \in C(\mathbb{R}^N), \; V \text{ is bounded below, } V^{-1}(0) \text{ has nonempty interior},
$$

$$
(D_2) \quad f \in C(\mathbb{R}^N \times \mathbb{R}) \text{ is 1-periodic in each of } x_i, \; i = 1, \cdots, N, \; \lim_{|t| \to \infty} \frac{f(x,t)}{|t|^{p^* - 1}} = 0,
$$

$$
(D_3) \quad \lim_{|t| \to \infty} \frac{F(x,t)}{|t|^p} = +\infty \text{ uniformly in } x \in \mathbb{R}^N,
$$

$$
(D_4) \quad f(x,t) = o(|t|^{p-2}) \text{ as } |t| \to 0, \text{ uniformly in } x \in \mathbb{R}^N,
$$

$$
(D_5) \quad \text{There exists } \theta \geq 1 \text{ such that } \theta F(x,t) \geq F(x,st) \text{ for } (x,t) \in \mathbb{R}^N \times \mathbb{R} \text{ and } s \in [0,1].
$$

From above we can see (1.3) is weaken to (C_4) with the cost (C_5) and to (D_3) with the cost (D_5), respectively. And condition (D_5) is weaker than (C_5)(c.f. [27]). In our result, (f_2) takes place the condition (1.3) but we need (f_4).

We should also mention that there is another line to weaken (1.3). In [13] for $\lambda$ large enough the authors got a nontrivial solution of (1.2) with $U(x) = \lambda V(x)$ under the following conditions:

$$
(E_1) \quad V \in C(\mathbb{R}^N, \mathbb{R}), \; V \text{ is bounded below, } V^{-1}(0) \text{ has nonempty interior},
$$
(E₂) there exists $M > 0$ such that the set $\{x \in \mathbb{R}^N | V(x) < M\}$ is nonempty and has
finite measure,

(E₃) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $F(x, u) \geq 0$ for all $(x, u)$, $f(x, u) = o(u)$ uniformly in $x$ as $u \to 0$,

(E₄) $F(x, u)/u^2 \to \infty$ uniformly in $x$ as $|u| \to \infty$,

(E₅) $\frac{1}{2}f(x, u)u - F(x, u) > 0$ whenever $u \neq 0$,

(E₆) $|f(x, u)|^\tau \leq a_1 (\frac{1}{2}f(x, u)u - F(x, u)) |u|^\tau$ for some $a_1 > 0$, $\tau > \max\{1, N/2\}$ and all $(x, u)$ with $|u|$ large enough.

The authors of [13] also proved that conditions (E₄)(E₅)(E₆) are weaker than (1.3).

The main method used in the proof of Theorem 1.1 is the linking structure over cones which was developed in [9]. We will use the Cerami condition instead of (PS) condition. This method is also valid in finding periodic solutions for one-dimensional $p$-Laplacian equation. We will give a brief argument in section 5 for this topic.

The paper is organized as follows. In section 2, we give the variational settings, recall a critical point theorem and some important properties of cohomological index. In section 3, an eigenvalue problem is studied. We get a divergent sequence of eigenvalues for this eigenvalue problem by cohomological index theory. In section 4, we give a proof of Theorem 1.1. In section 5, we state an existence result for the periodic solutions of one-dimensional $p$-Laplacian equation.

2 Preliminaries

Let $\mathcal{W} := \{u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla u|^p + b(x)|u|^p)dx < \infty\}$ with $b(x)$ satisfying the condition (B). Then $\mathcal{W}$ is a reflexive, separable Banach space with norm $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^p + b(x)|u|^p)dx)^{\frac{1}{p}}$. From Gagliardo-Nirenberg inequality and Hölder inequality, we have $\mathcal{W} \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R})$ for $p \leq q \leq p^*$. Moreover, we have the following compactness result. It was proved in [34] in the case $p = 2$. For the general case, the proof is similar. We give it here for reader’s convenience.

Lemma 2.1 $\mathcal{W} \hookrightarrow \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R})$ for $p \leq s < p^*$.

Proof: Let $\{u_n\} \subset \mathcal{W}$ be a bounded sequence of $\mathcal{W}$ such that $u_n \rightharpoonup u$ weakly in $\mathcal{W}$. Then, by the Sobolev embedding theorem, $u_n \to u$ strongly in $L^s_{loc}(\mathbb{R}^N, \mathbb{R})$ for $p \leq s < p^*$. We first claim that

$$u_n \to u \text{ strongly in } L^p(\mathbb{R}^N, \mathbb{R}).$$  \hspace{1cm} (2.4)
In fact, by the uniformly convex properties of $L^p(\mathbb{R}^N, \mathbb{R})$, we only need to prove that $\alpha_n := \|u_n\|_p \to \|u\|_p$ (cf. p295 in [11]). Assume, up to subsequence, that $\alpha_n \to \alpha$.

Set

\[
B_R = \{x \in \mathbb{R}^N : |x| < R\},
\]

\[
A(R, M) = \{x \in \mathbb{R}^N \setminus B_R : b(x) \geq M\},
\]

\[
B(R, M) = \{x \in \mathbb{R}^N \setminus B_R : b(x) < M\}.
\]

Then

\[
\int_{A(R, M)} |u_n|^p \, dx \leq \int_{\mathbb{R}^N} \frac{b(x)}{M} |u_n|^p \, dx \leq \frac{\|u_n\|^p}{M}.
\]

Choose $t \in (1, \frac{p}{p'})$ and $t'$ such that $\frac{1}{t} + \frac{1}{t'} = 1$. Then

\[
\int_{B(R, M)} |u_n|^p \, dx \leq \left( \int_{B(R, M)} |u_n|^{pt'} \, dx \right)^{\frac{1}{t'}} \left( \text{meas}(B(R, M)) \right)^{\frac{1}{t}} \leq C\|u_n\|^p(\text{meas}(B(R, M)))^{\frac{1}{t}}.
\]

Since $\{\|u_n\|\}$ is bounded and condition (B) holds, we may choose $R, M$ large enough such that $\frac{\|u_n\|^p}{M}$ and $\text{meas}(B(R, M))$ are small enough. Hence, $\forall \varepsilon > 0$, we have

\[
\int_{\mathbb{R}^N \setminus B_R} |u_n|^p \, dx = \int_{A(R, M)} |u_n|^p \, dx + \int_{B(R, M)} |u_n|^p \, dx < \varepsilon.
\]

Thus,

\[
\|u\|^p_p = \|u\|^p_{L^p(B_R)} + \|u\|^p_{L^p(\mathbb{R}^N \setminus B_R)}
\]

\[
\geq \lim_{n \to \infty} \|u_n\|^p_{L^p(B_R)} = \lim_{n \to \infty} (\|u_n\|^p - \|u_n\|^p_{L^p(\mathbb{R}^N \setminus B_R)}) \geq \alpha^p - \varepsilon.
\]

On the other hand, let $\Omega$ be a arbitrary domain in $\mathbb{R}^N$, then

\[
\int_{\Omega} |u_n|^p \, dx \leq \int_{\mathbb{R}^N} |u_n|^p \, dx \to \alpha^p,
\]

hence $\|u\|_p \leq \alpha$. Thanks to the arbitrariness of $\varepsilon$, we have $\alpha = \|u\|_p$. So (2.4) is proved.

Finally, it is easy to prove that $u_n \to u$ in $L^s(\mathbb{R}^N, \mathbb{R})$ for $p \leq s < p^*$. In fact, if

$s \in (p, p^*)$, there is a number $\lambda \in (0, 1)$ such that $\frac{1}{s} = \frac{1}{p} + \frac{1 - \lambda}{p^*}$. Then by the Hölder inequality,

\[
\|u_n - u\|_s^s = \int_{\mathbb{R}^N} |u_n - u|^{\lambda s} |u_n - u|^{(1 - \lambda)s} \, dx \leq \|u_n - u\|_{L^p}^{\lambda s} \|u_n - u\|_{L^{p^*}}^{(1 - \lambda)s}.
\]

Since $u_n$ is bounded in $L^{p^*}(\mathbb{R}^N, \mathbb{R})$ and $\|u_n - u\|_p \to 0$, we have $u_n \to u$ in $L^s(\mathbb{R}^N, \mathbb{R})$.

In the following, we consider the $C^1$ functional $\Phi : \mathcal{W} \to \mathbb{R}$ defined by

\[
\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + b(x) |u|^p) \, dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} V(x) |u|^p \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx.
\]
It is clear that critical points of $\Phi$ are weak solutions of (1.1). In order to find a critical point of this functional, we will use the following critical point theorem. It was proved in [9], where the functional was supposed to satisfy the ($PS$) condition. Recently, in [8], the author extended it to more general case (the functional space is completely regular topological space or metric space). If the functional space is a real Banach space, according to the proof of Theorem 6.10 in [8], the Cerami condition is sufficient for the compactness of the set of critical points at a fixed level and the first deformation lemma to hold (see [31]). So this critical point theorem still hold under the Cerami condition.

**Lemma 2.2** ([9]) Let $W$ be a real Banach space and let $C_-, C_+$ be two symmetric cones in $W$ such that $C_+$ is closed in $W$, $C_- \cap C_+ = \{0\}$ and

$$i(C_- \setminus \{0\}) = i(W \setminus C_+) = m < \infty.$$

Define the following four sets by

$$D_- = \{ u \in C_- : \|u\| \leq r_- \},$$
$$S_+ = \{ u \in C_+ : \|u\| = r_+ \},$$
$$Q = \{ u + te : u \in C_-, t \geq 0, \|u + te\| \leq r_- \}, \quad e \in W \setminus C_-,$$
$$H = \{ u + te : u \in C_-, t \geq 0, \|u + te\| = r_- \}.$$

Then $(Q, D_- \cup H)$ links $S_+$ cohomologically in dimension $m + 1$ over $\mathbb{Z}_2$. Moreover, suppose $\Phi \in C^1(W, \mathbb{R})$ satisfying the Cerami condition, and $\sup_{x \in D_- \cup H} \Phi(x) < \inf_{x \in S_+} \Phi(x)$, $\sup_{x \in Q} \Phi(x) < \infty$. Then $\Phi$ has a critical value $d \geq \inf_{x \in S_+} \Phi(x)$.

For convenience, let us recall the definition and some properties of the cohomological index of Fadell-Rabinowitz for a $\mathbb{Z}_2$-set, see [16, 17, 31] for details. For simplicity, we only consider the usual $\mathbb{Z}_2$-action on a linear space, i.e., $\mathbb{Z}_2 = \{1, -1\}$ and the action is the usual multiplication. In this case, the $\mathbb{Z}_2$-set $A$ is a symmetric set with $-A = A$.

Let $E$ be a normed linear space. We denote by $S(E)$ the set of all symmetric subsets of $E$ which do not contain the origin of $E$. For $A \in S(E)$, denote $\tilde{A} = A/\mathbb{Z}_2$. Let $\rho : \tilde{A} \to \mathbb{R}P^\infty$ be the classifying map and $\rho^* : H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega] \to H^*(\tilde{A})$ the induced homomorphism of the cohomology rings. The cohomological index of $A$, denoted by $i(A)$, is defined by $\sup\{k \geq 1 : \rho^*(\omega^{k-1}) \neq 0\}$. We list some properties of the cohomological index here for further use in this paper. Let $A, B \in S(E)$, there hold

(i) **(monotonicity)** if $A \subseteq B$, then $i(A) \leq i(B)$,
(i2) (invariance) if $h : A \to B$ is an odd homeomorphism, then $i(A) = i(B)$,

(i3) (continuity) if $C$ is a closed symmetric subset of $A$, then there exists a closed symmetric neighborhood $N$ of $C$ in $A$, such that $i(N) = i(C)$, hence the interior of $N$ in $A$ is also a neighborhood of $C$ in $A$ and $i(\text{int}N) = i(C)$,

(i4) (neighborhood of zero) if $V$ is bounded closed symmetric neighborhood of the origin in $E$, then $i(\partial V) = \dim E$.

3 Eigenvalue problem

In this section, we consider the following eigenvalue problem
\[
\begin{cases}
-\Delta_p u + b(x)|u|^{p-2}u = \lambda V(x)|u|^{p-2}u, \\
u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}).
\end{cases}
\]

We assume $V$ satisfies condition (V) and further assume that $V^+(x) := \frac{V(x) + |V(x)|}{2} \neq 0$ on some positive measure subset of $\mathbb{R}^N$ in this section. Define on $\mathcal{W}$ the functionals
\[
H(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + b(x)|u|^p)dx,
\]
\[
I(u) = \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^pdx.
\]

Then
\[
H \in C^1(\mathcal{W}, \mathbb{R}), \quad \langle H'(u), v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + b(x)|u|^{p-2}uv)dx
\]

and
\[
I \in C^1(\mathcal{W}, \mathbb{R}), \quad \langle I'(u), v \rangle = \int_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx.
\]

Our aim is to solve the eigenvalue problem
\[
H'(u) = \lambda I'(u). \tag{3.6}
\]

Lemma 3.1 For any $u, v \in \mathcal{W}$, it holds that
\[
\langle H'(u) - H'(v), u - v \rangle \geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).
\]
Proof: We follow the idea of the proof of Lemma 2.3 in [24]. By direct computations, we have

\[ \langle H'(u) - H'(v), u - v \rangle = \int_{\mathbb{R}^N} |\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla v - |\nabla v|^{p-2}\nabla v \cdot \nabla u dx + \int_{\mathbb{R}^N} b(x)(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu) dx. \]

From the definition of the norm in \( \mathcal{W} \), we can get

\[ \langle H'(u) - H'(v), u - v \rangle = \|u\|^p + \|v\|^p - \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + b(x)|u|^{p-2}uv) dx \]

\[ - \int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v \cdot \nabla u + b(x)|v|^{p-2}vu) dx. \]

Applying Hölder inequality,

\[ \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + b(x)|u|^{p-2}uv) dx \]

\[ \leq \left( \int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} |\nabla v|^p dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^N} b(x)|u|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} b(x)|v|^p dx \right)^{\frac{1}{p}}. \]

Using the following inequality

\[(a + b)^\alpha (c + d)^{1-\alpha} \geq a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha}\]

which holds for any \( \alpha \in (0, 1) \) and for any \( a > 0, b > 0, c > 0, d > 0 \), set \( \alpha = \frac{p-1}{p} \) and

\[ a = \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad b = \int_{\mathbb{R}^N} b(x)|u|^p dx, \quad c = \int_{\mathbb{R}^N} |\nabla v|^p dx, \quad d = \int_{\mathbb{R}^N} b(x)|v|^p dx, \]

we can deduce that

\[ \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + b(x)|u|^{p-2}uv) dx \leq \|u\|^{p-1}\|v\|. \]

Similarly, we can obtain

\[ \int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v \cdot \nabla u + b(x)|v|^{p-2}vu) dx \leq \|v\|^{p-1}\|u\|. \]

Therefore, we have

\[ \langle H'(u) - H'(v), u - v \rangle \geq \|u\|^p + \|v\|^p - \|u\|^{p-1}\|v\| - \|v\|^{p-1}\|u\| \]

\[ = (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|). \]

Lemma 3.2 If \( u_n \to u \) and \( \langle H'(u_n), u_n - u \rangle \to 0 \), then \( u_n \to u \) in \( \mathcal{W} \).
Proof: Since $\mathcal{W}$ is a reflexive Banach space, it is isometrically isomorphic to a locally uniformly convex space, so as it was proved in [11], weak convergence and norm convergence imply strong convergence. Therefore we only need to show that $\|u_n\| \to \|u\|$.

Note that
\[
\lim_{n \to \infty} \langle H'(u_n) - H'(u), u_n - u \rangle = \lim_{n \to \infty} \left( \langle H'(u_n), u_n - u \rangle - \langle H'(u), u_n - u \rangle \right) = 0.
\]
By Lemma 3.1 we have
\[
\langle H'(u_n) - H'(u), u_n - u \rangle \geq (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \geq 0.
\]
Hence $\|u_n\| \to \|u\|$ as $n \to \infty$ and the assertion follows.

Lemma 3.3 $I'$ is weak-to-strong continuous, i.e. $u_n \rightharpoonup u$ in $\mathcal{W}$ implies $I'(u_n) \to I'(u)$.

Proof: This is a direct consequence of Theorem 1.22 in [34] and Lemma 2.1.

Lemma 3.4 If $u_n \rightharpoonup u$ in $\mathcal{W}$, then $I(u_n) \to I(u)$.

Proof:
\[
p|I(u_n) - I(u)| = |\langle I'(u_n), u_n \rangle - \langle I'(u), u \rangle| = |\langle I'(u_n) - I'(u), u_n + (I'(u), u_n - u)\rangle| \leq \|I'(u_n) - I'(u)\|\|u_n\| + o(1).
\]
Because $u_n \rightharpoonup u$, $u_n$ is bounded. From Lemma 3.3 we have $I(u_n) \to I(u)$.

Set $\mathcal{M} = \{u \in \mathcal{W} : I(u) = 1\}$. Clearly, $I(u) = \frac{1}{p}\langle I'(u), u \rangle$, so $1$ is a regular value of the functional $I$. Hence by the implicit theorem, $\mathcal{M}$ is a $C^1$-Finsler manifold. It is complete, symmetric, since $I$ is continuous and even. Moreover, $0$ is not contained in $\mathcal{M}$, so the trivial $\mathbb{Z}_2$-action on $\mathcal{M}$ is free. Set $\overline{H} = H|_{\mathcal{M}}$.

Lemma 3.5 If $u \in \mathcal{M}$ satisfies $\overline{H}(u) = \lambda$ and $\overline{H}'(u) = 0$, then $(\lambda, u)$ is a solution of (3.6).

Proof: By Proposition 3.14.9 in [31], the norm of $\overline{H}'(u) \in T^*_u\mathcal{M}$ is given by $\|\overline{H}'(u)\|_U^* = \min_{\mu \in \mathbb{R}} \|H'(u) - \mu I'(u)\|^*$ (here the norm $\|\cdot\|_U^*$ is the norm in the fibre $T^*_u\mathcal{M}$, and $\|\cdot\|^*$ is the operator norm, the minimal can be attained was proved in Lemma 3.14.10 in [31]). Hence there exists $\mu \in \mathbb{R}$ such that $H'(u) - \mu I'(u) = 0$, that is $(\mu, u)$ is a solution of (3.6) and
\[
\lambda = \overline{H}(u) = \frac{1}{p}\langle H'(u), u \rangle = \frac{1}{p}\langle \mu I'(u), u \rangle = \mu \frac{1}{p}\langle I'(u), u \rangle = \mu I(u) = \mu.
\]

Lemma 3.6 $\overline{H}$ satisfies the $(PS)$ condition, i.e. if $(u_n)$ is a sequence on $\mathcal{M}$ such that $\overline{H}(u_n) \to c$, and $\overline{H}'(u_n) \to 0$, then up to a subsequence $u_n \to u \in \mathcal{M}$ in $\mathcal{W}$. 9
Proof: First, from the definition of $H$, we can deduce that $(u_n)$ is bounded. Then, up to a subsequence, $u_n$ converges weakly to some $u$, by Lemma 3.4 we have $I(u) = 1$, so $u \in \mathcal{M}$.

From $\tilde{H}'(u_n) \to 0$, we have $H'(u_n) - \mu_n I'(u_n) \to 0$ for a sequence of real numbers $(\mu_n)$. So $\langle H'(u_n) - \mu_n I'(u_n), u_n \rangle \to 0$, thus we get $\mu_n \to c$. By Lemma 3.3, we have $H'(u_n) \to c I'(u)$. Hence

$$\langle H'(u_n), u_n - u \rangle = \langle H'(u_n) - c I'(u), u_n - u \rangle + \langle c I'(u), u_n - u \rangle \to 0.$$  

By Lemma 3.2 we obtain $u_n \to u$.

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, $\mathcal{F}_n = \{ M \in \mathcal{F} : i(M) \geq n \}$ and

$$\lambda_n = \inf_{M \in \mathcal{F}_n} \sup_{u \in M} \tilde{H}(u).$$ (3.7)

Since $\mathcal{F}_n \supset \mathcal{F}_{n+1}$, $\lambda_n \leq \lambda_{n+1}$.

**Lemma 3.7** For every $\mathcal{F}_n$, there is a symmetric compact set $M \in \mathcal{F}_n$.

Proof: We follow the idea of the proof of Theorem 3.2 in [20]. Since $\text{meas}\{ x \in \mathbb{R}^N : V(x) > 0 \} > 0$, it implies that $\forall n \in \mathbb{N}^*$, there exist $n$ open balls $(B_i)_{1 \leq i \leq n}$ in $\mathbb{R}^N$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\text{meas}\{ x \in \mathbb{R}^N : V(x) > 0 \} \cap B_i > 0$. Approximating the characteristic function $\chi_{\{ x \in \mathbb{R}^N : V(x) > 0 \} \cap B_i}$ by $C^\infty(\mathbb{R}^N, \mathbb{R})$ functions in $L^p(\mathbb{R}^N, \mathbb{R})$, we can infer that there exists a sequence $\{ u_i \}_{1 \leq i \leq n} \subseteq C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\int_{\mathbb{R}^N} V(x)|u_i|^pdx > 0$ for all $i = 1,...,n$ and $\text{supp} u_i \cap \text{supp} u_j = \emptyset$ when $i \neq j$. Normalizing $u_i$, we assume that $I(u_i) = 1$. Denote $U_n$ the space spanned by $\{ u_i \}_{1 \leq i \leq n}$. $\forall u \in U_n$, we have $u = \sum_{i=1}^{n} \alpha_i u_i$ and

$$I(u) = \sum_{i=1}^{n} |\alpha_i|^p.$$  

So $u \to \left( I(u) \right)^{\frac{1}{p}}$ defines a norm on $U_n$. Since $U_n$ is $n$ dimensional, this norm is equivalent to $\| \cdot \|$. Thus $\{ u \in U_n : I(u) = 1 \} \subseteq \mathcal{M}$ is compact with respect to the norm $\| \cdot \|$ and by (i4), $i(\{ u \in U_n : I(u) = 1 \}) = n$. So $\{ u \in U_n : I(u) = 1 \} \in \mathcal{F}_n$.

By Lemma 3.7, we have $\lambda_n < +\infty$, and by condition (B), there holds $\lambda_n \geq 0$. Furthermore, by Lemma 3.6 and Proposition 3.14.7 in [31], we see that $\lambda_n$ is sequence of critical values of $\tilde{H}$ and $\lambda_n \to +\infty$, as $n \to \infty$. By Lemma 3.6, we get a divergent sequence of eigenvalues for problem (3.6). So we have the following result.

**Theorem 3.8** Problem (3.6) has an increasing sequence eigenvalues $\lambda_n$ which are defined by (3.7) and $\lambda_n \to +\infty$, as $n \to \infty$.

**Lemma 3.9** Set

$$\mu_n = \inf_{K \in \mathcal{F}_n^c} \sup_{u \in K} H(u),$$ (3.8)

where $\mathcal{F}_n^c = \{ K \in \mathcal{F}_n : K \text{ is compact} \}$. Then we have $\lambda_n = \mu_n$.  

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Proof: From Lemma 3.7, \( \mathcal{F}_n^c \neq \emptyset \) and so \( \mu_n < +\infty \). It is obvious that \( \lambda_n \leq \mu_n \). If \( \lambda_n < \mu_n \), there is \( M \in \mathcal{F}_n \) such that \( \sup_{u \in M} H(u) < \mu_n \). The closure \( \overline{M} \) of \( M \) in \( \mathcal{M} \) is still in \( \mathcal{F}_n \), by continuity of \( H \), \( \sup_{u \in \overline{M}} H(u) < \mu_n \) holds. Applying the property (i3) of the cohomological index, we can find a small open neighborhood \( A \in \mathcal{F}_n \) of \( \overline{M} \) in \( \mathcal{M} \) such that \( \sup_{u \in A} H(u) < \mu_n \).

If \( \lambda_n < \lambda_{m+1} \), then \( i(A) = \sup \{ i(K) : K \text{ is compact and symmetric with } K \subseteq A \} \). So we can choose a symmetric compact subset \( K \subseteq A \) with \( i(K) \geq n \) and \( \sup_{u \in K} H(u) < \mu_n \). This contradicts to the definition of \( \mu_n \). Therefore we have \( \lambda_n = \mu_n \).

Motivated by Theorem 3.2 in [9], we have the following statement.

**Theorem 3.10** If \( \lambda_m < \lambda_{m+1} \) for some \( m \in \mathbb{N}^* \), then
\[
i(\{u \in W \setminus \{0\} : H(u) \leq \lambda_m I(u)\}) = i(\{u \in W : H(u) < \lambda_{m+1} I(u)\}) = m.\]

Proof: Suppose \( \lambda_m < \lambda_{m+1} \). If we set \( A = \{u \in \mathcal{M} : H(u) \leq \lambda_m\} \) and \( B = \{u \in \mathcal{M} : H(u) < \lambda_{m+1}\} \), by the definition (3.7), we have \( i(A) \leq m \). Assume that \( i(A) \leq m - 1 \). Thanks to (i3), there exists a symmetric neighborhood \( N \) of \( A \) in \( \mathcal{M} \) satisfying \( i(N) = i(A) \). By the equivariant deformation theorem (see [7]), there exists \( \delta > 0 \) and an odd continuous map \( \iota : \{u \in \mathcal{M} : H(u) \leq \lambda_m + \delta\} \to \{u \in \mathcal{M} : H(u) \leq \lambda_m - \delta\} \cup N = N \).

Hence \( i(u \in \mathcal{M} : H(u) \leq \lambda_m + \delta) \leq m - 1 \). By (3.7), there exists \( M \in \mathcal{F}_m \) such that \( \sup_{u \in M} H(u) < \lambda_m + \delta \). So \( M \subseteq \{u \in \mathcal{M} : H(u) \leq \lambda_m + \delta\} \) and thus \( i(M) \leq m - 1 \). This contradicts to the fact that \( M \in \mathcal{F}_m \). Thus we have \( i(A) = m \). By the invariance of the cohomological index under odd homeomorphism and the functionals \( H, I \) are \( p \)-homogeneous, we have \( i(\{u \in W \setminus \{0\} : H(u) \leq \lambda_m I(u)\}) = m \).

Since \( A \subseteq B \) and \( i(A) = m \), we have \( i(B) \geq m \). Assume that \( i(B) \geq m + 1 \). As in the proof of Lemma 3.9, there exists a symmetric, compact subset \( K \) of \( B \) with \( i(K) \geq m + 1 \). Since \( \max_{u \in K} H(u) < \lambda_{m+1} = \mu_{m+1} \), this contradicts to definition (3.8). By the invariance of the cohomological index under odd homeomorphism and the functionals \( H, I \) are \( p \)-homogeneous, we have \( i(\{u \in W : H(u) < \lambda_{m+1} I(u)\}) = m \).

4 Proof of the main theorem

Set \( J(u) = \int_{\mathbb{R}^N} F(x, u) \, dx \), by the definition of \( H \) and \( I \) in section 3, we can write the functional \( \Phi \) defined in section 2 as
\[
\Phi(u) = H(u) - \lambda I(u) - J(u), \quad u \in W.
\]
It follows from Lemma 1.22 in [34] that $J'$ is compact.

Replacing $(\lambda, V)$ with $(-\lambda, -V)$, we can assume that $\lambda \geq 0$.

First, we consider the case $V^+(x) \neq 0$ on some positive measure subset of $\mathbb{R}^N$ and there exist $m \geq 1$ such that $\lambda_m \leq \lambda < \lambda_{m+1}$. Set

$$C_- = \{ u \in W : H(u) \leq \lambda_m I(u) \}, \quad (4.9)$$

$$C_+ = \{ u \in W : H(u) \geq \lambda_{m+1} I(u) \}. \quad (4.10)$$

It is easy to see that $C_-, C_+$ are two symmetric closed cones in $W$ and $C_- \cap C_+ = \{ 0 \}$. By Theorem 3.10 we have

$$i(C_- \setminus \{0\}) = i(W \setminus C_+) = m. \quad (4.11)$$

**Theorem 4.1** There exist $r_+ > 0$ and $\alpha > 0$ such that $\Phi(u) > \alpha$ for $u \in C_+$ and $\|u\| = r_+$.

**Proof**: Let $\varepsilon > 0$ be small enough, from (f$_1$) and (f$_3$), we have $|F(x,t)| \leq \varepsilon|t|^p + C_\varepsilon|t|^q$, by the Sobolev embedding inequality, for $u \in C_+$, we can get

$$\Phi(u) = H(u) - \lambda I(u) - J(u)$$

$$= H(u) - \frac{\lambda}{\lambda_{m+1}} \lambda_{m+1} I(u) - J(u)$$

$$\geq H(u) - \frac{\lambda}{\lambda_{m+1}} H(u) - \varepsilon \int_{\mathbb{R}^N} |u|^p dx - C_\varepsilon \int_{\mathbb{R}^N} |u|^q dx$$

$$\geq H(u) - \frac{\lambda}{\lambda_{m+1}} H(u) - \varepsilon_0 \int_{\mathbb{R}^N} b(x)|u|^p dx - C_\varepsilon \int_{\mathbb{R}^N} |u|^q dx$$

$$\geq (1 - \frac{\lambda}{\lambda_{m+1}} - \frac{\varepsilon_0}{C_\varepsilon}) H(u) - C_\varepsilon \int_{\mathbb{R}^N} |u|^q dx$$

$$\geq \frac{(1 - \frac{\lambda}{\lambda_{m+1}} - \frac{\varepsilon_0}{C_\varepsilon})}{p} \|u\|^p - C\|u\|^q. \quad (4.12)$$

We remind that in the second inequality of (4.12), the condition (B) has been applied. Since $p < q$, the assertion follows.

Since $\lambda \geq \lambda_m$, by (f$_1$) it holds that

$$\Phi(u) \leq 0, \quad \forall u \in C_- \quad (4.13)$$

Set $\mathbb{R}^+ = [0, +\infty)$. Following the idea of the proof of Theorem 4.1 in [9], we have

**Theorem 4.2** Let $e \in \mathcal{W} \setminus C_-$, there exists $r_- > r_+$ such that $\Phi(u) \leq 0$ for $u \in C_- + \mathbb{R}^+ e$ and $\|u\| \geq r_-$. 

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Proof: Define another norm on \( \mathcal{W} \) by \( \|u\|_{v} := (\int_{\mathbb{R}^N} (|V(x)| + 1)|u|^pdx)^{1/p} \). Then the same reason as the proof of Theorem 4.1 in [9], there exists some constant \( b > 0 \) such that \( \|u + te\| \leq b\|u + te\|_{v} \) for every \( u \in C_{-}, t \geq 0 \) and some \( b > 0 \). That is

\[
\int_{\mathbb{R}^N} (|\nabla(u + te)|^p + b(x)|u + te|^p)dx \leq b^p \int_{\mathbb{R}^N} (|V(x)| + 1)|u + te|^pdx. \quad (4.14)
\]

Let \( \{u_k\} \) be a sequence such that \( \|u_k\| \to +\infty \) and \( u_k \in C_{-} + \mathbb{R}^+e \). Set \( v_k = \frac{u_k}{\|u_k\|} \), then, up to a subsequence, \( \{v_k\} \) converges to some \( v \) weakly in \( \mathcal{W} \) and a.e.in \( \mathbb{R}^N \). Note that Lemma [3.4] is also true for functional \( \int_{\mathbb{R}^N} (|V(x)| + 1)|u|^pdx, \ u \in \mathcal{W} \), it follows from (4.14) that \( \int_{\mathbb{R}^N} (|V(x)| + 1)|v|^pdx \geq \frac{1}{bp} \). So \( |v| \neq 0 \) on a positive measure set \( \Omega_0 \). Since

\[
\lim_{|t| \to \infty} \frac{f(x, t)}{|t|^p} = +\infty \implies \lim_{|t| \to \infty} \frac{F(x, t)}{|t|^p} = +\infty, \quad \text{from (f2)}
\]

we have

\[
\lim_{k \to \infty} \frac{F(x, u_k(x))}{\|u_k\|^p} = \lim_{k \to \infty} \frac{F(x, \|u\|v_k(x))}{\|u\|^p|v_k(x)|^p}v_k(x)|^p = +\infty, \quad x \in \Omega_0.
\]

By (f1) and Fatou’s lemma we can get

\[
\frac{\int_{\mathbb{R}^N} F(x, u_k)dx}{\|u_k\|^p} \to +\infty, \quad \text{as} \quad k \to \infty.
\]

By the arbitrariness of the sequence \( \{u_k\} \), we have \( \frac{\int_{\mathbb{R}^N} F(x, u)dx}{\|u\|^p} \to +\infty \) as \( \|u\| \to +\infty \) and \( u \in C_{-} + \mathbb{R}^+e \). Noting that

\[
\frac{\Phi(u)}{|u|^p} = \frac{1}{p} - \frac{\lambda I(u)}{|u|^p} - \frac{\int_{\mathbb{R}^N} F(x, u)dx}{\|u\|^p}
\]

and by conditions (B) and (V),

\[
\frac{|I(u)|}{\|u\|^p} \leq C \frac{\int_{\mathbb{R}^N} |u|^pdx}{\|u\|^p} \leq C \frac{\int_{\mathbb{R}^N} b(x)|u|^pdx}{\|u\|^p} \leq C,
\]

the assertion follows.

**Theorem 4.3** \( \Phi \) satisfies the Cerami condition. i.e., for any sequence \( \{u_k\} \) in \( \mathcal{W} \) satisfying \((1 + \|u_k\|)|\Phi'(u_k)| \to 0 \) and \( \Phi(u_k) \to c \) possesses a convergent subsequence.

Proof: Let \( \{u_k\} \) be a sequence in \( \mathcal{W} \) satisfying \((1 + \|u_k\|)|\Phi'(u_k)| \to 0 \) and \( \Phi(u_k) \to c \). We claim that \( \{u_k\} \) is bounded in \( \mathcal{W} \). Otherwise, if \( \|u_k\| \to \infty \), we consider \( w_k := \frac{u_k}{\|u_k\|} \). Then, up to subsequence, we get \( w_k \to w \) in \( \mathcal{W} \), \( w_k \to w \) in \( L^s(\mathbb{R}^N) \) for \( p \leq s < p^* \) and \( w_k(x) \to w(x) \) a.e. \( x \in \mathbb{R}^N \) as \( k \to \infty \). If \( w \neq 0 \) in \( \mathcal{W} \), since \( \Phi'(u_k)u_k \to 0 \), that is to say

\[
\int_{\mathbb{R}^N} (|\nabla u_k|^p + b(x)|u_k|^p)dx - \lambda \int_{\mathbb{R}^N} V(x)|u_k|^pdx - \int_{\mathbb{R}^N} f(x, u_k)u_kdx \to 0, \quad (4.15)
\]
from condition (V), we have \( \left| \int_{\mathbb{R}^N} V(x) |u_k|^p dx \right| \leq C \), so by dividing the left hand side of (4.15) with \( \|u_k\|^p \) there holds
\[
\left| \int_{\mathbb{R}^N} \frac{f(x, u_k)u_k}{\|u_k\|^p} dx \right| \leq C. \tag{4.16}
\]

On the other hand, by Fatou’s lemma and condition (f_2) we have
\[
\int_{\mathbb{R}^N} \frac{f(x, u_k)u_k}{\|u_k\|^p} dx = \int_{\{u_k \neq 0\}} |u_k|^p \frac{f(x, u_k)u_k}{|u_k|^p} dx \to \infty,
\]
this contradicts to (4.16).

If \( w = 0 \) in \( W \), inspired by [21], we choose \( t_k \in [0, 1] \) such that \( \Phi(t_k u_k) := \max_{t \in [0, 1]} \Phi(t u_k) \).

For any \( \beta > 0 \) and \( \tilde{w}_k := (2p\beta)^{1/p} w_k \), by Lemma 3.4 and the compactness of \( J' \) we have that
\[
\Phi(t_k u_k) \geq \Phi(\tilde{w}_k) = 2\beta - \lambda \int_{\mathbb{R}^N} V(x) |\tilde{w}_k|^p dx - \int_{\mathbb{R}^N} F(x, \tilde{w}_k) dx \geq \beta,
\]
when \( k \) is large enough, this implies that
\[
\lim_{k \to \infty} \Phi(t_k u_k) = \infty. \tag{4.17}
\]

Since \( \Phi(0) = 0 \), \( \Phi(u_k) \to c \), we have \( t_k \in (0, 1) \). By the definition of \( t_k \),
\[
\langle \Phi'(t_k u_k), t_k u_k \rangle = 0. \tag{4.18}
\]

From (4.17), (4.18), we have
\[
\Phi(t_k u_k) - \frac{1}{p} \langle \Phi'(t_k u_k), t_k u_k \rangle = \int_{\mathbb{R}^N} \left( \frac{1}{p} f(x, t_k u_k) t_k u_k - F(x, t_k u_k) \right) dx \to \infty.
\]

By (f_1), there exists \( \theta \geq 1 \) such that
\[
\int_{\mathbb{R}^N} \left( \frac{1}{p} f(x, u_k) u_k - F(x, u_k) \right) dx \geq \frac{1}{\theta} \int_{\mathbb{R}^N} \left( \frac{1}{p} f(x, t_k u_k) t_k u_k - F(x, t_k u_k) \right) dx \to \infty. \tag{4.19}
\]

On the other hand,
\[
\int_{\mathbb{R}^N} \left( \frac{1}{p} f(x, u_k) u_k - F(x, u_k) \right) dx = \Phi(u_k) - \frac{1}{p} \langle \Phi'(u_k), u_k \rangle \to c_0. \tag{4.20}
\]

(4.19) and (4.20) are contradiction. Hence \( \{u_k\} \) is bounded in \( W \). So up to a subsequence, we can assume that \( u_k \to u \) for some \( W \).

Since \( \Phi'(u_k) = H'(u_k) - \lambda I'(u_k) - J'(u_k) \to 0 \) and \( I', J' \) are compact, we have that
\[
H'(u_k) \to \lambda I'(u) + J'(u) \text{ in } W^*. \text{ So }
\]
\[
\langle H'(u_k), u_k - u \rangle = \langle H'(u_k) - (\lambda I'(u) + J'(u)), u_k - u \rangle + \langle \lambda I'(u) + J'(u), u_k - u \rangle \to 0.
\]
By Lemma 4.2 \( w_k \to u \) in \( W \).

**Proof of Theorem 1.1** Define \( D_-, S_+, Q, H \) as lemma 2.2, then from Theorem 4.1
\[
\Phi(u) \geq \alpha > 0 \text{ for every } u \in S_+, \text{ from Theorem 4.2 } \Phi(u) \leq 0 \text{ for every } u \in D_- \cup H \text{ and } \Phi \\
\text{is bounded on } Q. \text{ Applying Theorem 4.3, it follows that } \Phi \text{ has a critical value } d \geq \alpha > 0.
\]
Hence \( u \) is a nontrivial weak solution of (1.1).

For the cases \( 0 \leq \lambda < \lambda_1 \) or \( \lambda_1 < \lambda \), set \( C_- = \{0\} \) and \( C_+ = W \), it is easy to see that the arguments in this section are also valid. So we get a nontrivial solution and the proof of Theorem 1.1 is complete.

**5 Periodic problem for one-dimensional \( p \)-Laplacian equation**

In this section, we state a result which can be proved by the same methods as in the proof of Theorem 1.1. We only outline the main points. Our result reads as

**Theorem 5.1** If \( p > 1, V \in L^\infty(\mathbb{R}, \mathbb{R}), f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) satisfies \( (f_1)-(f_3) \), both \( V(t) \) and \( f(t,u) \) are 1-periodic in \( t \), then

\[
\begin{aligned}
-\Delta_p u + |u|^{p-2}u &= \lambda V(t)|u|^{p-2}u + f(t,u), \\
 u(0) &= u(1), u'(0) = u'(1).
\end{aligned}
\]

(5.21)

has a nontrivial solution for every \( \lambda \in \mathbb{R} \).

The periodic solution of \( p \)-Laplacian equation has been considered in many papers, for example, [2, 3, 29]. Up to the author’s knowledge, Theorem 5.1 is new.

Let \( W := W^{1,p}(S^1, \mathbb{R}) \) with the norm \( \|u\| = (\int_{S^1} |\nabla u|^p + |u|^p dt)^{\frac{1}{p}} \), here \( S^1 = \mathbb{R}/\mathbb{Z} \). Then \( W \) is a reflexive, separable Banach space. And \( W \) can be embedded into \( L^q(\mathbb{R}, \mathbb{R}) \) for any \( p \leq q < \infty \). As in section 3, we consider the eigenvalue problem

\[
\begin{aligned}
-\Delta_p u + |u|^{p-2}u &= \lambda V(t)|u|^{p-2}u, \\
u(0) &= u(1), u'(0) = u'(1).
\end{aligned}
\]

We can get a divergent sequence of eigenvalues defined by \( \lambda_n = \inf_{M \in \mathcal{F}_n} \sup_{u \in \mathcal{M}} \int_{S^1} |\nabla u|^p + |u|^p dt \) if \( V^+(t) \neq 0 \) on a positive measure subset of \( S^1 \), here \( \mathcal{F}_n \) is the class of symmetrical subsets with Fadell-Rabinowitz index greater than \( n \) of \( \mathcal{M} := \{u \in W : \int_{S^1} |u|^p dt = 1\} \). And if \( \lambda_m < \lambda_{m+1} \) for some \( m \in \mathbb{N}^* \), then \( i(\{u \in W \setminus \{0\} : \int_{S^1} |\nabla u|^p + |u|^p dt \leq \lambda_m \int_{S^1} |u|^p dt\}) = i(\{u \in W : \int_{S^1} |\nabla u|^p + |u|^p dt < \lambda_{m+1} \int_{S^1} |u|^p dt\}) = m \). Then arguing as in section 4,
consider the functional on $\mathcal{W}$

$$
\Phi(u) = \frac{1}{p} \int_{S^1} (|\nabla u|^p + |u|^p)dt - \frac{\lambda}{p} \int_{S^1} V(t)|u|^p dt - \int_{S^1} F(t, u)dt,
$$

assume $\lambda \geq 0$, $V^+(t) \neq 0$ on a positive measure subset of $S^1$ and there exists $m \in \mathbb{N}^*$ such that $\lambda_m \leq \lambda < \lambda_{m+1}$, Set

$$
C_- = \{u \in \mathcal{W} : \int_{S^1} |\nabla u|^p + |u|^p dt \leq \lambda_m \int_{S^1} |u|^p dt\},
$$

$$
C_+ = \{u \in \mathcal{W} : \int_{S^1} |\nabla u|^p + |u|^p dt \geq \lambda_{m+1} \int_{S^1} |u|^p dt\},
$$

then we have

1. There exist $r_+ > 0$ and $\alpha > 0$ such that $\Phi(u) > \alpha$ for $u \in C_+$ and $\|u\| = r_+$,

2. Let $e \in \mathcal{W} \setminus C_-$, there exists $r_- > r_+$ such that $\Phi(u) \leq 0$ for $u \in C_- + \mathbb{R}^+e$ and $\|u\| \geq r_-$,

3. $\Phi$ satisfies the Cerami condition.

Then from Lemma 2.2 we can get a nontrivial solution for (5.21). The cases for $0 \leq \lambda < \lambda_1$ or $V^+(x) \equiv 0$ are similar as in the proof of Theorem 1.1.

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