Strong Structural Controllability and Observability of Linear Time-Varying Systems

Gunther Reissig, Christoph Hartung, and Ferdinand Svaricek

Abstract—in this note we consider continuous-time systems \( \dot{x}(t) = A(t)x(t) + B(t)u(t) \), \( y(t) = C(t)x(t) + D(t)u(t) \) as well as discrete-time systems \( x(t+1) = A(t)x(t) + B(t)u(t) \), \( y(t) = C(t)x(t) + D(t)u(t) \) whose coefficient matrices \( A, B, C \) and \( D \) are not exactly known. More precisely, all that is known about the systems is their nonzero pattern, i.e., the locations of the nonzero entries in the coefficient matrices. We characterize the patterns that guarantee controllability and observability, respectively, for all choices of nonzero time functions at the matrix positions defined by the pattern, which extends a result by Mayeda and Yamada for time-invariant systems. As it turns out, the conditions on the patterns for time-invariant and for time-varying discrete-time systems coincide, provided that the underlying time interval is sufficiently long. In contrast, the conditions for time-varying continuous-time systems are more restrictive than in the time-invariant case.

Index Terms—MSC: Primary, 93B05; Secondary, 93B07, 93C05, 15A03, 05C50.

I. INTRODUCTION

In this note, we present novel results on controllability of the linear discrete-time control system

\[
x(t+1) = A(t)x(t) + B(t)u(t)
\]

and of the continuous-time control system

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t),
\]

each possibly extended by the output equation

\[
y(t) = C(t)x(t) + D(t)u(t).
\]

Here, the time \( t \) is integer valued and real valued, respectively, and the coefficient matrices \( A(t), B(t), C(t) \) and \( D(t) \) may be real or complex. If the latter matrices are all constant, the respective systems are time-invariant, and otherwise they are time-varying. We remark that the same notation is used for both discrete-time and continuous-time systems to keep the notation concise. In particular, the symbol \( t \) denotes time in both cases, and it will always be clear from context to which of the systems (1) and (2) we refer.

When such systems arise in applications, the coefficient matrices usually depend on physical parameters and other factors. Then the values of the entries are not known precisely, so that system properties can not, in general, be determined with complete certainty either. In contrast, the nonzero pattern of the system, i.e., the locations of the nonzero entries in its coefficient matrices, is usually completely defined by the modeling process. That fact can be exploited to determine structural properties or strong structural properties, two approaches which have found wide applications. See [1]–[7] and the references given there.

Here we follow the strong structural approach initiated in [1], which assumes that the nonzero pattern is all that is known about the system and seeks to characterize the patterns that guarantee certain system properties for all choices of nonzero values at the matrix positions defined by the pattern. In contrast, the structural approach would guarantee any property only for almost all choices [2]–[6].

Mayeda and Yamada have been the first to characterize the patterns that guarantee controllability of time-invariant systems [1]. Equivalent characterizations have been given subsequently, see [8] and the references therein, and related problems have been investigated in [7], [9], [10]. All these works consider only time-invariant systems, despite the fact that it is often most natural to assume that the physical parameters entering the coefficient matrices vary over time.

In this note, we characterize the patterns that guarantee controllability for all choices of nonzero time functions at the matrix positions defined by the pattern, which extends the results in [1] to time-varying systems. To this end we introduce basic notation and terminology in Section II and review controllability results for time-invariant systems in Section III-A, which includes an algorithm to verify the conditions in the result of [1]. Patterns guaranteeing controllability for time-varying discrete-time and continuous-time systems are characterized in Section III-B and III-C, respectively. In the former case, the conditions coincide with those for time-invariant systems if the underlying time interval is sufficiently long, whereas in the latter case, they turn out to be more restrictive. In Section IV we present our results on observability. The conditions in our characterizations can all be verified using the algorithm in Section III-A. Special cases of the results in the present paper have been announced in [11]–[13].

II. PRELIMINARIES

A. Basic Notation

\( \mathbb{C}, \mathbb{R} \) and \( \mathbb{Z} \) denote the sets of complex numbers, real numbers and integers, respectively. \( \mathbb{R}_+ \) and \( \mathbb{Z}_+ \) denote the subsets of non-negative elements of \( \mathbb{R} \) and \( \mathbb{Z} \), respectively, and \( \mathbb{N} = \mathbb{Z}_+ \setminus \{0\} \).

For \( a, b \in \mathbb{R} \cup \{\infty\} \) satisfying \( a \leq b \), the closed, open and half-open intervals with end points \( a \) and \( b \) are denoted \( [a, b], (a, b], [a, b) \) and \( [a, b] \), respectively. e.g. \( [0, \infty) = \mathbb{R}_+ \), and \( [a, b] = \emptyset \) if \( a > b \).

The cases \( a \in \mathbb{R}_+ \) and \( b \in \mathbb{R}_+ \) denote intervals for discrete intervals, e.g. \( [a, b] = [a, b] \cap \mathbb{Z} \). We often drop the subscript \( \mathbb{Z} \) when the type of interval to which we refer is obvious.

The set of \( n \times m \)-matrices over the field \( \mathbb{F} \) is denoted \( \mathbb{F}^{n \times m} \), where \( \mathbb{F} \in \{\mathbb{C}, \mathbb{R}\} \) throughout this note. \( X_{i,j} \) denotes the entry at position \( (i, j) \) of \( X \in \mathbb{F}^{n \times m} \). For any \( x \in \mathbb{F}^n \) and \( X \in \mathbb{F}^{n \times r} \), \( x^\top \) and \( X^\top \) denote the transpose of \( x \) and \( X \), respectively, if \( \mathbb{F} = \mathbb{R} \), and the conjugate transpose, if \( \mathbb{F} = \mathbb{C} \).

B. Systems, Solutions, Transition Matrices

The coefficient matrices in (1), (2), and (3), \( A(t), B(t), C(t) \) and \( D(t) \), are defined for all \( t \in \mathbb{Z} \) and all \( t \in \mathbb{R} \), respectively. We consider systems over \( \mathbb{R} \) and over \( \mathbb{C} \), so \( A(t) \in \mathbb{F}^{n \times n}, B(t) \in \mathbb{F}^{n \times r}, C(t) \in \mathbb{F}^{m \times n}, \) and \( D(t) \in \mathbb{F}^{m \times r} \) for each time \( t \), where \( n \in \mathbb{N} \) and \( r, m \in \mathbb{Z}_+ \). The cases \( r = 0 \) and \( m = 0 \), which stand for systems without inputs and outputs, respectively, are included here for the sake of notational simplicity.

Given \( u : \mathbb{Z} \to \mathbb{F}^r \), a map \( x : [t_0, \infty] \to \mathbb{F}^n \) is a solution of the system (1) (generated by the input signal \( u \)) if \( t_0 \in \mathbb{Z} \) and (1) holds for all \( t \in [t_0, \infty] \). Analogously, if \( u : \mathbb{R} \to \mathbb{F}^r \), a map \( x : [t_0, \infty] \to \mathbb{F}^n \) is a solution of the system (2) (generated by \( u \)) if \( t_0 \in \mathbb{R} \), \( x \) is absolutely continuous, and (2) holds for almost every (a.e.) \( t \in [t_0, \infty] \), i.e., for all \( t \in [t_0, \infty] \) with the possible exception of a set of (Lebesgue) measure zero. In the case of the system (2) we will always assume that the matrices \( A \) and \( B \) are locally integrable and that input signals \( u : \mathbb{R} \to \mathbb{F}^r \) are measurable and locally essentially bounded. This hypothesis implies both existence and uniqueness of solutions [14] and is satisfied, e.g. if \( A, B \) and \( u \) are piecewise continuous.

The authors are with the University of the Federal Armed Forces Munich, Dept. Aerospace Eng., Chair of Control Eng., D-85577 Neubiberg (Munich), Germany, http://www.reiszig.de/gunther/, http://www.unibw.de/lrt15/.

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The general solution of the system (1) and the system (2) is the map \( \varphi \) defined by the requirement that for all \( x_0 \in \mathbb{R}^n \), \( t_0 \) and \( u \), \( \varphi(\cdot, t_0, x_0, u) \) is the unique solution of (1) and (2), respectively, defined on \([t_0, \infty)\) and satisfying \( \varphi(t_0, t_0, x_0, u) = x_0 \). Of course, we do not need to specify \( u \) on the whole time axis, i.e., we define \( \varphi(t, t_0, x_0, u|_{[t_0, t)}) := \varphi(t, t_0, x_0, u) \), where \( t \geq t_0 \) and \( u|_{[t_0, t)} \) denotes the restriction of \( u \) to the (discrete or continuous) interval \([t_0, t)\]. The map \( \varphi(t, t_0, \cdot, 0) \), which is linear, is called the transition matrix at \((t, t_0)\) of the system and is usually denoted by \( \Phi(t, t_0) \).

\[
\varphi(t, t_0, x_0, u) = \Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau + 1)B(\tau)u(\tau), \quad (4)
\]

\[
\varphi(t, t_0, x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau, \quad (5)
\]

for the systems (1) and (2), respectively. We additionally have \( \Phi(t, t_0) = A(t - 1) \cdots A(t_0) \) for the system (1), and if \( n = 1 \) or \( A \) is constant, then \( \Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\tau)d\tau\right) \) for the system (2). See [14], [15].

\section*{C. Nonzero Patterns and Graphs}

We define the equivalence relation \( \sim \) on \( \mathbb{R}^{n \times m} \) by the requirement that \( X \sim Y \) iff the positions of the zeros in \( X \) and \( Y \) coincide, i.e., \( X \sim Y \) iff \( X_{i,j} = 0 \) implies \( Y_{i,j} = 0 \) and vice versa. The equivalence classes \([X]_\sim \in \mathbb{R}^{n \times m}/\sim\), which we call nonzero patterns, or just patterns, will be represented by matrices whose entries are asterisks and circles. Each asterisk stands for a nonzero, and each circle, for a zero. For example, if the coefficient matrices \( A \) and \( B \) in (2) are given by

\[
A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix},
\]

then \([A(t)]_\sim = A \) and \([B(t)]_\sim = B \) for every \( t \), where

\[
A = \begin{pmatrix} * & 0 \\ 0 & \circ \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} * \\ \circ \end{pmatrix}. \quad (7)
\]

In particular, \( A_{1,1} = * \) and \( A_{1,2} = 0 \).

\subsection*{II.1 Definition.}

Let \( A \in \mathbb{R}^{n \times n}/\sim \) and \( B \in \mathbb{R}^{n \times r}/\sim \). The system (1) is of pattern \((A, B)\) if

\[
A(t) \in A \quad \text{and} \quad B(t) \in B
\]

for all \( t \in \mathbb{R} \). Analogously, the system (2) is of pattern \((A, B)\) if \( A \) and \( B \) are locally integrable and (8) holds for almost every \( t \in \mathbb{R} \).

Some remarks are in order. First of all, we would like to emphasize that the coefficient matrices are required to satisfy the condition (8) (almost) everywhere on the whole time axis for the sake of simplicity only. In any of our subsequent results, these matrices actually need to be defined only on \([t_0, t_1]\). Still, there do exist systems that are not of any pattern, and their controllability can not be decided from the results in the present paper. On the other hand, our notion of ‘pattern’ is general enough to allow nonzero matrix entries to change their signs. The sets of zeros of such entries would be empty in the discrete-time case, but could be non-empty of measure zero in the continuous-time case, where the condition (8) is required to hold for almost everywhere \( t \) rather than for every \( t \) and Definition II.1 does not impose any continuity requirements. For example, a polynomial matrix entry \( \mathbb{R} \to \mathbb{R} \) is considered a nonzero entry iff it does not vanish identically on \( \mathbb{R} \).

The operations of addition, matrix composition and transposition for patterns are defined by

\[
[X]_\sim + [Y]_\sim = [[X] + [Y]]_\sim,
\]

\[
([X]_\sim, [Y]_\sim) = [(X, Y)]_\sim,
\]

\[
[X]_\sim^T = [X^*]_\sim,
\]

whenever the operations on the right hand sides are defined. Here, \( X \) and \( Y \) are matrices. \([X]_\sim \) denotes the matrix with entries \([X_{i,j}]_\sim \), and \((X, Y)\) is the matrix consisting of the columns of \( X \) and \( Y \).

Patterns of systems are conveniently represented by graphs [2–4]. Specifically, if \( A \in \mathbb{R}^{n \times n}/\sim \) and \( B \in \mathbb{R}^{n \times r}/\sim \), then the graph \( G(A, B) \) of \((A, B)\) has vertices \( 1, \ldots, n + r \), and there is a (directed) edge from the vertex \( v \) to the vertex \( w \) if \( 1 \leq w \leq n \) and \((A, B)w,v = * \). In this case, \( v \) is a predecessor of \( w \), and \( w \) is a successor of \( v \). For any set \( V \) of vertices, \( \text{Pre}(V) \) denotes the set of predecessors of \( V \), i.e., \( v \in \text{Pre}(V) \) if there exists an edge from \( v \) to some vertex in \( V \). Analogously, \( \text{Post}(V) \) denotes the set of successors of \( V \). The notations \( \text{Pre}(V) \) and \( \text{Post}(V) \) do not contain any reference to the graph \( G(A, B) \), which will always be clear from context.

\section*{III. CONTROLLABILITY}

In this section we present novel characterizations of controllability of time-varying systems in the strong structural sense. We rely on controllability notions from [16] throughout; for variants of controllability concepts, see e.g. [15], [17], [18].

\subsection*{III.1 Definition.}

Let \( \Sigma \) denote the discrete-time system (1) or the continuous-time system (2), assume \( t_0, t_1 \in \mathbb{Z} \) or \( t_0, t_1 \in \mathbb{R} \), respectively, and let \( \varphi \) denote the general solution of \( \Sigma \).

The pair \((t_0, x_0)\) can be controlled to the pair \((t_1, x_1)\) if \( x_0, x_1 \in \mathbb{F}^n \), \( t_0 \leq t_1 \), and there exists a control input \( u: [t_0, t_1] \to \mathbb{F}^r \) such that \( x_1 = \varphi(t_1, t_0, x_0, u) \). The system \( \Sigma \) is controllable on the interval \([t_0, t_1]\) if \((t_0, x_0)\) can be controlled to \((t_1, x_1)\) for all \( x_0, x_1 \in \mathbb{F}^n \), and \( \Sigma \) is controllable if for all \( x_0, x_1 \in \mathbb{F}^n \) there exist \( \tau_0 \) and \( \tau_1 \) such that \((\tau_0, x_0)\) can be controlled to \((\tau_1, x_1)\).

We will frequently need the well-known controllability criteria in Proposition III.2 below, which follow immediately from the formulas (4) and (5). Here and in the remainder of this note, \( \Phi \) denotes the transition matrix of the systems (1) and (2), in which it will always be clear from context to which of the two systems we refer.

\subsection*{III.2 Proposition.}

Let \( t_0, t_1 \in \mathbb{Z} \), \( t_0 < t_1 \). Then the system (1) is controllable on \([t_0, t_1]\) iff the condition

\[
p^*\Phi(t_1, \tau + 1)B(\tau) = 0 \quad \text{for every} \quad \tau \in [t_0, t_1]
\]

implies \( p = 0 \). Analogously, if \( t_0, t_1 \in \mathbb{R} \), \( t_0 < t_1 \), then the system (2) is controllable on \([t_0, t_1]\) iff the condition

\[
p^*\Phi(t_1, \tau)B(\tau) = 0 \quad \text{for a.e.} \quad \tau \in [t_0, t_1]
\]

implies \( p = 0 \).

\section*{A. Controllability of Time-Invariant Systems}

Next, we review results for time-invariant systems. If the systems (1) and (2) are time-invariant, the property of controllability of the system (1) (resp., the system (2)) on \([t_0, t_1]\) does not depend on the actual times \( t_0 \) and \( t_1 \), provided that \( t_0 + n \leq t_1 \) (resp., \( t_0 < t_1 \)). Moreover, the characterizations of controllability in terms of the pair \((A(0), B(0))\) of matrices in the discrete-time and the continuous-time case coincide. It is therefore justified to call the pair \((A(0), B(0))\) controllable if the time-invariant system (1), or, equivalently, the time-invariant system (2), is so. One of the well-known results
III.3 Theorem. Consider the conditions (G₀) and (G₁) below.

(G₀) For every non-empty subset \( V \subseteq \{1, \ldots, n\} \) of vertices of \( G(A, B) \) there exists a vertex \( v \in \{1, \ldots, n + r\} \) such that \( V \cap \text{Post}(\{v\}) \) is a singleton.

(G₁) For every non-empty subset \( V \subseteq \{1, \ldots, n\} \) of vertices of \( G(A, B) \) that satisfies \( V \subseteq \text{Pre}(V) \) there exists a vertex \( v \in \{1, \ldots, n + r\} \) such that \( V \cap \text{Pre}(\{v\}) \) is a singleton.

The condition (G₀) is equivalent to the requirement that (11) holds for \( \lambda = 0 \) and every pair \((A, B)\) of pattern \((A, B)\). Analogously, the condition (G₁) is equivalent to the requirement that (11) holds for every \( \lambda \in \mathbb{C} \setminus \{0\} \) and every pair \((A, B)\) of pattern \((A, B)\). Thus, every pair \((A, B)\) of pattern \((A, B)\) is controllable iff both \((G₀)\) and \((G₁)\) hold.

The conditions (G₀) and (G₁) can be verified using the algorithm in Fig. 1, whose correctness is immediate from the proof of the above Theorem given in [11, Section III]. We emphasize that the number of iterations performed by the algorithm does not exceed the state space dimension \( n \). Hence, despite the fact that the conditions \((G₀)\) and \((G₁)\) impose requirements on every non-empty subset \( V \subseteq \{1, \ldots, n\} \), only at most \( 2n \) such subsets are actually tested in the verification of \((G₀)\) and \((G₁)\). We remark in passing that the conditions \((G₀)\) and \((G₁)\) are equivalent to the possibility of transforming, through row and column permutations, the patterns \((A, B)\) and \(((id)_\sim + A, B)\) into special forms. See [8], [11].

Figure 1. Algorithm for the verification of the condition \((G₀)\) (if \( L = 0 \)) and the condition \((G₁)\) (if \( L = 1 \)) in Theorem III.3. It returns the empty set if the respective condition holds, or a nonempty set \( V \) for which the condition fails to hold.
that the system (1) is of nonzero pattern \((A, B)\). If \(n = 1\), application of Theorem III.3 yields \(B \neq 0\), and in particular, \(B(t_1 - 1) \neq 0\). Then the system (1) is controllable on \([t_0, t_1]\) by Proposition III.2 since \(\Phi(t_1, t_1) = 1\).

If \(n > 1\), we assume that the theorem holds for all systems with \((n - 1)\)-dimensional state space. We let \(p \in F^n\) satisfy (9) and show below that then \(p = 0\) necessarily, so that the system (1) is controllable on \([t_0, t_1]\) by Proposition III.2.

Let \(V = \{1, \ldots, n\}\) and observe that by Theorem III.3, the conditions \((G_0)\) and \((G_1)\) hold. In particular, there exists some vertex \(v \in \{1, \ldots, n + r\}\) such that \(V \cap \text{Post}(\{v\})\) is a singleton.

Assume first that \(v \notin V\). Then among the columns of \(B\) there exists one with exactly one nonzero component. So, without loss of generality, \(A\) and \(B\) can be partitioned according to \(F^n = F^{n-1} \times F\),

\[
A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}, \quad B = \begin{pmatrix} B_{1,1} \\ B_{2,1} \end{pmatrix},
\]

where \(B_{2,2} = \star \in F/\sim\), \(B_{1,2} = [0]_{\sim} \in F^{n-1}/\sim\), and \(\Phi\) as well as the coefficient matrices \(A\) and \(B\) are partitioned analogously. Moreover, since \(B_{2,1} = [0]_{\sim}\), the conditions \((G_0)\) and \((G_1)\) still hold when \(n\), \(A\) and \(B\) is replaced with \(n - 1\), \(A_{1,1}\) and \((B_{1,1}, A_{1,2})\), respectively. Consequently, by our induction hypothesis, the following system is controllable on \([t_0 + 1, t_1]\):

\[
x(t + 1) = A_{1,1}(t)x(t) + B_{1,1}(t)u(t) + A_{1,2}(t)u_2(t). \tag{13}
\]

Let \(p\) take the form \(p = (q, \alpha) \in F^{n-1} \times F\) and consider the last column of \(p^*\Phi(t)_{(1, s + 1)}(1 + B(s))\). That column equals \(q^*B_{1,1}(s) + \alpha^*B_{2,2}(s)\) if \(s = t_1 - 1\), and hence, the condition (9) implies \(\alpha = 0\) since \(B_{2,1} = 0\) and \(B_{2,2}(t_1 - 1) \neq 0\). It follows that

\[
q^*\Phi_{1,1}(t_1, s + 1) = 0 \quad \text{for all } s \in [t_0, t_1]. \tag{14}
\]

Next define \(z(s) = q^*\Phi_{1,1}(t_1, s) - \Phi(t_1, s)\), where \(\Phi\) is the transition matrix of the system (13). Consider the adjoint equation

\[
\Phi(t, s) = \Phi(t, s + 1)A(s) \tag{15}
\]

of the (1), which holds for all \(t, s \in \mathbb{Z}\) for which \(s < t\), to see that (14), (15) and the adjoint equation of the system (13) imply that \(z(s) = z(s + 1)A_{1,1}(s)\) for all \(s \in [t_0, t_1]\). From \(z(t_1) = 0\) it follows that \(z(s) = 0\), hence

\[
q^*\Phi_{1,1}(t_1, s + 1) = q^*\Phi(t_1, s + 1) \tag{16}
\]

Moreover, application of (14) and (16) to the difference equation for \(\Phi_{1,1}(t_1, \cdot)\) that is part of the adjoint equation (15) yields

\[
q^*\Phi(t, s + 1)A_{1,1}(s) = 0 \quad \text{for all } s \in [t_0 + 1, t_1], \tag{17}
\]

and condition (9) for \(p = (q, 0)\), (14) and (16) additionally show that

\[
q^*\Phi(t_1, s + 1)B_{1,1}(s) = 0 \quad \text{for all } s \in [t_0, t_1]. \tag{18}
\]

As the system (13) is controllable on \([t_0 + 1, t_1]\), it follows from Prop. III.2 and the identities (17) and (18) that \(q = 0\), hence \(p = 0\).

It remains to consider the case that \(v\) cannot be chosen from the complement of \(V\). Then the condition \((G_1)\) implies \(V \not\subseteq \text{Pre}(V)\), so one of the columns of \(A\) vanishes. Hence, without loss of generality, \(A\) and \(B\) can be partitioned according to \(F^n = F^{n-1} \times F\),

\[
A = \begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix}, \quad \Phi = \begin{pmatrix} A_{1,2} \\ A_{2,2} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_{1,1} \\ B_{2,1} \end{pmatrix},
\]

where \(A_{2,2} = [0]_{\sim} \in F/\sim\), \(A_{1,2} = [0]_{\sim} \in F^{n-1}/\sim\), and \(A\), \(B\) and \(\Phi\) are partitioned analogously. Moreover, since \(A_{1,2} = [0]_{\sim}\), the conditions \((G_0)\) and \((G_1)\) still hold when \(n\), \(A\) and \(B\) is replaced with

\[
\begin{pmatrix} \tilde{A}_{1,1} \\ \tilde{A}_{2,1} \end{pmatrix}, \quad \tilde{B}_{1,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{B}_{2,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

for discrete-time time-varying systems (1) on any interval \([t_0, t_1]\).

Theorem III.5 gives a complete characterization of nonzero patterns that guarantee controllability on \([t_0, t_1]\). As we have observed in the proof above, the condition \((G_2)\) is equivalent to the condition \((G_0)\) with \(G(A, \tilde{B})\) replaced by \(\tilde{G}([0]_{\sim}, K)\), which can be verified using the algorithm in Fig. 1. By the second claim of the Theorem, it suffices to verify both \((G_0)\) and \((G_1)\) instead if \(t_0 + n \leq t_1\), which is more efficient. The latter assumption can not be dropped as shown by the following example. In fact, the case \(t_1 - t_0 < n\) remains open for time-invariant systems (1).

**III.6 Example.** Let \(A\) and \(B\) be given by (12) and assume that the system (1) is of pattern \((A, B)\). By Example III.4, the system is controllable if it is time-invariant, and by Theorem III.5, the system is controllable in any case, on any interval \([t_0, t_1]\) satisfying \(t_0 + 6 \leq t_1\).

In the time-invariant case, let the \(6 \times 6\) matrix \(M(t)\) consist of the columns of \(\Phi(t, s + 1)B(s)\) for \(s \in [t - 3, t]\). Then the determinant \(\det(M(t))\) of \(M(t)\) equals \(\det(A_{1,2}^2 A_{2,1}^2 A_{1,4} A_{5,0} B_{1,2} B_{1,1}^2 B_{1,1} B_{2,1}^2 B_{2,1}^2)\), so the system is controllable on \([t_0, t_1]\) as soon as \(t_0 + 3 \leq t_1\). See Proposition III.2. On the other hand, if the non-zeros in \(A\) are all equal to 1 identically and the coefficient \(B\) is given by \(B(t)^* = \begin{pmatrix} -1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}\), then \(\det(M(t)) = 0\) for all \(t\), so the system is not
controllable on any interval of the form \([t_0, t_0 + 3]\). This is consistent with the fact that the condition \((G_2)\) is not satisfied if \(t_1 = t_0 + 3\). See Fig. 3.

C. Controllability of Continuous-Time Time-Varying Systems

We have just demonstrated that in the discrete-time case, controllability of all time-invariant systems of a given nonzero pattern implies the controllability of all time-varying systems of that pattern. It turns out that the continuous-time case is quite different in that respect.

III.7 Example. Consider the patterns \(A\) and \(B\) given by

\[
A = \begin{pmatrix}
  \circ & \star & \circ \\
  \circ & \circ & \star \\
  \circ & \circ & \circ
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
  \star
\end{pmatrix}.
\]

The fact that every time-invariant system \((2)\) of pattern \((A, B)\) is controllable follows from Theorem III.3, or, alternatively, from the Hautus criterion. Now consider the time-varying system \((2)\) of pattern \((A, B)\) in which the nonzero entries in \(A\) are all equal to 1 identically and the coefficient \(B\) is given by \(B(t) = (t^2 + 1, 0, -2)\). That system is not controllable since the choice \(p = (2, -2t_1, t_1^2 + 1)\) satisfies the condition \((10)\) whenever \(t_0 < t_1\).

III.8 Example. Consider the patterns \(A\) and \(B\) given in \((7)\). As before, if the system \((2)\) is time-invariant and of pattern \((A, B)\), it is controllable. What is different here is that the same conclusion holds if we merely assume the nonzero entries in \(A\) and \(B\) to be polynomials rather than constants. This fact is straightforward to verify. Surprisingly, however, controllability is lost if we assume the coefficients to be merely analytic. Indeed, if \(A\) and \(B\) are given by \((6)\), then the system \((2)\) is of pattern \((A, B)\), yet it is not controllable as the choice \(p = (1, -e^{-t})\) satisfies condition \((10)\).

As the examples demonstrate, we do not only need to distinguish between time-invariant and time-varying systems, but we also have to be precise about any regularity conditions imposed on the time-varying coefficients of the system \((2)\). In this respect, one rather restrictive class of time-varying systems, which is used in the formulation of Theorem III.9 below, is that of exponentially scaled systems, by which we mean systems \((2)\) over \(F\) that can be transformed into a time-invariant system over \(F\) by means of a time-varying change of coordinates of the form \((t, x) \rightarrow e^{\langle A(t)\rangle}x\) with \(A(t)\) being diagonal. In other words, we require that there exist a diagonal matrix \(A \in F^{n \times n}\) and matrices \(A_0 \in F^{n \times n}\) and \(B_0 \in F^{n \times r}\) such that

\[
A(t) = e^{-A t} A_0 e^{A t} - A \quad \text{and} \quad B(t) = e^{-A t} B_0 \tag{21}
\]

hold for every \(t \in \mathbb{R}\). It follows from \((21)\) that diagonal entries of \(A\) may be added or removed at will, without affecting any controllability properties of the system \((2)\). It is this fact that distinguishes the discrete-time from the continuous-time case and leads to more restrictive controllability conditions in the time-varying continuous-time case.

Our main result for the continuous-time case, presented below, characterizes the nonzero patterns that ensure controllability of all time-varying systems \((2)\), or, what turns out to be equivalent, of all exponentially scaled systems \((2)\). Obviously then, the result also characterizes the patterns that ensure controllability of the systems in any class in between the two extremes. Examples of such classes include the ones defined by the requirement that certain entries of the coefficient matrices \(A\) and \(B\) must be continuous, smooth, or of constant sign.

III.9 Theorem. Let \(t_0, t_1 \in \mathbb{R}\), \(t_0 < t_1\), and consider the following condition.

\((G_2)\) For every non-empty subset \(V \subseteq \{1, \ldots, n\}\) of vertices of \(G(A, B)\) there exists some vertex \(v \in \{1, \ldots, n + r\} \setminus V\) such that \(V\) contains exactly one successor of \(v\) in \(G(A, B)\).

The condition \((G_2)\) is equivalent to each of the following three statements.

(i) Every system \((2)\) of pattern \((A, B)\) is controllable on \([t_0, t_1]\).

(ii) Every exponentially scaled system \((2)\) of pattern \((A, B)\) is controllable on \([t_0, t_1]\).

(iii) Every time-invariant system \((2)\) of pattern \(\langle [id]_+ + A, B \rangle\) is controllable.

Proof of Theorem III.9. In the chain \((i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (G_2) \Rightarrow (i)\), the first implication is obvious, and the third follows from Theorem III.3 and the remark preceding this proof.

In order to prove that \((iii)\) follows from \((ii)\), we let \(A_0 \in [id]_+ + A\) and \(B_0 \in B\) and define the diagonal matrix \(A \in F^{n \times n}\) by the requirement

\[
A(t) = (A_0)_{i,i} + \begin{cases} 0, & \text{if } A_{i,i} = 0, \\ 1, & \text{otherwise} \end{cases}
\]

for all \(i \in \{1, \ldots, n\}\). If the coefficients \(A\) and \(B\) of the system \((2)\) satisfy \((21)\) for all \(t \in \mathbb{R}\), that system is exponentially scaled and of nonzero pattern \((A, B)\), and hence, it is controllable on \([t_0, t_1]\) by \((ii)\). Then, as controllability is invariant with respect to the change of coordinates given by \((21)\), the pair \((A_0, B_0)\) is controllable, which proves \((iii)\).

In order to show that \((G_3)\) implies \((i)\), assume that the system \((2)\) is of pattern \((A, B)\). If \(n = 1\), application of \((G_3)\) yields \(B \neq 0\), so \(B(s) \neq 0\) for a.e. \(s \in [t_0, t_1]\). Then, since \(\Phi(t, s) \neq 0\) for all \(s \in [t_0, t_1]\), the system \((2)\) is controllable on \([t_0, t_1]\) by Prop. III.2.

The proof in the case that \(n > 1\) is analogous to that part of the proof of Theorem III.5 where it is assumed that \(v \notin V\), with only three differences. Firstly, the continuous-time system \(\dot{x}(t) = A_{1,1}(t)x(t) + B_{1,1}(t)u_1(t) + A_{1,2}(t)u_2(t)\), which is controllable on \([t_0, t_1]\), is used at the place of the system \((13)\). Secondly, from the fact that \(p = (q, \alpha) \in F^{n-1} \times F\) satisfies \((10)\) we conclude that \(\alpha = 0\) by the following argument. The identity \((10)\) yields \((q' \Phi_{1,2}(t, s) + \alpha' \Phi_{2,2}(t, s))B_{2,2}(s) = 0\) for a.e. \(s \in [t_0, t_1]\), so \(\alpha = 0\) as \(\Phi(t_1, t_1) = \text{id}, \Phi\) is continuous, and \(B_{2,2} \neq 0\) a.e.. Finally, the continuous-time adjoint equation \(D_2 \Phi(t, s) = -\Phi(t, s)A(s)\), which holds for all \(t \in \mathbb{R}\) and a.e. \(s \in R\) \([14]\), is used at the place of the discrete-time variant \((15)\). Here, \(D_2\) denotes the partial derivative with respect to the second argument. Then \(z = 0\) as \(z(s) = -z(s)A_{1,1}(s)\) holds for a.e. \(s \in [t_0, t_1]\), and we arrive at the identities \(q' \Psi(t_1, s)A_{1,2}(s) = 0\) and \(q' \Psi(t_1, s)B_{1,1}(s) = 0\) for a.e. \(s \in [t_0, t_1]\), at the place of \((17)\) and \((18)\), to conclude that \(p = 0\).

IV. Observability

In this section we present our results on strong structural observability for both discrete-time and continuous-time systems. At the end of the section, we point out a subtlety regarding the duality between observability and controllability in the discrete-time case.

IV.1 Definition. Let \(\Sigma\) denote the discrete-time system \((1), (3)\) or the continuous-time system \((2), (3)\), assume \(t_0, t_1 \in \mathbb{Z}\) or \(t_0, t_1 \in \mathbb{R}\), respectively, \(t_0 \leq t_1\), and let \(\varphi\) denote the general solution of \(\Sigma\).

The events \((t_0, x_0)\) and \((t_0, x_1)\) are indistinguishable on the interval \([t_0, t_1]\) if \(x_0, x_1 \in F^n\) and for every control input \(u: [t_0, t_1] \rightarrow F^n\).
Moreover, the latter is equivalent to each of the following two statements:

(i) Every time-invariant system \((1), (3)\) of output pattern \((A, C)\) is observable.

(ii) The conditions \((G_0)\) and \((G_1)\) in Theorem III.3 hold with \(A^*, C^*\) and \(m\) at the place of \(A, B\) and \(r\), respectively.

Proof. We assume without loss of generality that \(B = 0\) and \(D = 0\) in \((1)\) and \((3)\). Then the requirement that \(\tilde{A}(t_1 - s) = A(t_0 + s)^* + B(t_1 - s) = C(t_0 + s)^*\) for all \(s \in \mathbb{Z}\) defines the adjoint system

\[
x(t + 1) = \tilde{A}(t)x(t) + \tilde{B}(t)u(t)
\]

for any given system \((1), (3)\), and vice versa, and the system \((1), (3)\) is of output pattern \((A, C)\) iff the condition \((G_2)\) holds with \(A^*, C^*\) (resp., time-invariant) iff the system \((23)\) is of pattern \((A^*, C^*)\) (resp., time-invariant). In addition, the condition \((9)\) for the system \((23)\) is equivalent to the condition that \(C(s)\Phi(s, t_0)p = 0\) for every \(s \in [t_0, t_1]\), where \(p\) is the transition matrix of the system \((1)\). The latter condition implies \(p = 0\) iff the system \((1), (3)\) is observable on \([t_0, t_1]\) [15, Th. 25.9]. Hence, on the interval \([t_0, t_1]\), the system \((1), (3)\) is observable iff the adjoint system \((23)\) is controllable, and an application of Theorem III.5 completes the proof.

In order to be able to present our continuous-time observability result we extend the notion of exponential scalability to systems with outputs in the obvious way: The system \((2), (3)\) is exponentially scalable if there exist a diagonal matrix \(\Lambda \in \mathbb{F}^{n \times n}\) and matrices \(A_0 \in \mathbb{F}^{n \times n}\), \(B_0 \in \mathbb{F}^{n \times r}\) and \(C_0 \in \mathbb{F}^{m \times n}\) such that both \((21)\) and \(C(t) = C_0 \exp(\Lambda t)\) hold for every \(t \in \mathbb{R}\).

IV.3 Corollary. Let \(t_0, t_1 \in \mathbb{R}\), \(t_0 < t_1\). Then the following four statements are equivalent:

(i) Every system \((2), (3)\) of output pattern \((A, C)\) is observable on \([t_0, t_1]\).

(ii) Every exponentially scaled system \((2), (3)\) of output pattern \((A, C)\) is observable on \([t_0, t_1]\).

(iii) Every time-invariant system \((2), (3)\) of output pattern \([id]_\sim + (A, C)\) is observable.

(iv) Condition \((G_3)\) holds with \((A^*, C^*)\) and \(m\) at the place of \((A, B)\) and \(r\), respectively.

The proof of Corollary IV.3 is omitted as it is analogous to that of Corollary IV.2. In particular, the adjoint system \(\tilde{x}(t) = -A(t)^*x(t) + C(t)^*u(t)\), which is controllable on \([t_0, t_1]\) iff the system \((2), (3)\) is observable on \([t_0, t_1]\), see [16], is used at the place of the system \((23)\).

Given the duality between our controllability and our observability results, we warn against a subtlety with the discrete-time case. While it follows from Theorem III.5 and Corollary IV.2 that every system \((1), (3)\) of output pattern \((A, C)\) is observable on \([t_0, t_1]\) iff every system

\[
x(t + 1) = A(t)^*x(t) + C(t)^*u(t)
\]

of pattern \((A^*, C^*)\) is controllable on \([t_0, t_1]\), the respective properties of the two systems are not, in general, equivalent.

IV.4 Example. The system \((1), (3)\) is not observable if

\[
A(t) = \left(\begin{array}{cc}
-2 & e^{1-t} \\
3e^t & -2
\end{array}\right) \quad \text{and} \quad C(t) = (e^t, -e^t)
\]

for all \(t \in \mathbb{Z}\), whereas the system \((24)\) is controllable on \([t_0, t_1]\) whenever \(t_0 + 2 \leq t_1\). Note also that \(A(t)\) is invertible for all \(t\).

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