SOME NEW RESULTS ABOUT $q$-TRINOMIAL COEFFICIENTS

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Abstract. In this paper, we present several new congruences on the $q$-trinomial coefficients introduced by Andrews and Baxter. A new congruence on sums of central $q$-binomial coefficients is also established.

1. Introduction

In 2019, Straub [18] gave the following $q$-supercongruence:

$$\begin{bmatrix} an \\ bn \end{bmatrix} \equiv \begin{bmatrix} a \\ b \end{bmatrix}_{q^{n^2}} - (a - b)b \begin{bmatrix} a \\ b \end{bmatrix} \frac{n^2 - 1}{24} (q^n - 1)^2 \pmod{\Phi_n(q)^3},$$

(1.1)

which is a $q$-analogue of the Wolstenholme–Ljunggren congruence: for any prime $p \geq 5$,

$$\begin{bmatrix} ap \\ bp \end{bmatrix} \equiv \begin{bmatrix} a \\ b \end{bmatrix} \pmod{p^3}.$$

Here and throughout the paper, $\Phi_n(q)$ stands for the $n$-th cyclotomic polynomial in $q$:

$$\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(n,k)=1} (q - \zeta^k),$$

where $\zeta$ is an $n$-th primitive root of unity. The $q$-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, & \text{if } 0 \leq k \leq n; \\
0, & \text{otherwise}, \end{cases}$$

where the $q$-shifted factorial is defined as $(a; q)_0 = 1$ and $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ with $n \in \mathbb{Z}^+$. For $n \in \mathbb{N}$ and integer $j$ with $-n \leq j \leq n$, the trinomial coefficient is the coefficient of $x^j$ in the expansion of $(1 + x + x^{-1})^n$. Namely,

$$\begin{bmatrix} n \\ j \end{bmatrix} = [x^j](1 + x + x^{-1})^n,$$
and it has a simple expression (see \[17\])

\[
\left(\binom{n}{j}\right) = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k+j}.
\]

Six different \(q\)-analogues of the trinomial coefficients, which play significant roles in hard hexagon model, were introduced by Andrews and Baxter \[1\]. One of these \(q\)-analogues is

\[
\left(\binom{n}{j}\right)_q = \sum_{k=0}^{n} q^{k(k+j)} \binom{n}{k} \binom{n-k}{k+j}.
\]

During the past few years, some experts have paid attention to \(q\)-analogues of supercongruences. We refer the reader to \[5, 6, 9, 14, 15\] for some of their work. Moreover, some congruences for \(q\)-binomial coefficients and \(q\)-trinomial coefficients can be found in \[2, 3, 7, 8, 13, 16, 21\].

Recently, Liu \[12\] established the following beautiful \(q\)-supercongruences: for any positive integer \(n\), modulo \(\Phi_n(q)^2\),

\[
\left(\binom{n}{0}\right)_q \equiv \begin{cases} 
(-1)^{m}(1 + q^{m})q^{m(3m-1)/2}, & \text{if } n = 3m; \\
(-1)^{m}q^{m(3m+1)/2}, & \text{if } n = 3m + 1; \\
(-1)^{m}q^{m(3m-1)/2}, & \text{if } n = 3m - 1,
\end{cases} \tag{1.3}
\]

\[
\left(\binom{2n}{0}\right)_q \equiv \begin{cases} 
2(-1)^{m}(1 + q^{m})q^{m(3m-1)/2} - 3m(1 - q^{3m}), & \text{if } n = 3m; \\
2(-1)^{m}q^{m(3m+1)/2} - (3m + 1)(1 - q^{3m+1}), & \text{if } n = 3m + 1; \\
2(-1)^{m}q^{m(3m-1)/2} - (3m - 1)(1 - q^{3m-1}), & \text{if } n = 3m - 1.
\end{cases} \tag{1.4}
\]

Motivated by the work just mentioned, we shall establish the following \(q\)-supercongruence similar to \(1.3\) and \(1.4\).

**Theorem 1.1.** For any positive integer \(n\), modulo \(\Phi_n(q)^2\),

\[
\left(\binom{2n}{0}\right)_q \equiv \begin{cases} 
2(-1)^{m}(1 + q^{m})q^{m(3m-1)/2} - 9m(1 - q^{3m}) + 1, & \text{if } n = 3m; \\
2(-1)^{m}q^{m(3m+1)/2} - 3(3m + 1)(1 - q^{3m+1}) + 1, & \text{if } n = 3m + 1; \\
2(-1)^{m}q^{m(3m-1)/2} - 3(3m - 1)(1 - q^{3m-1}) + 1, & \text{if } n = 3m - 1.
\end{cases} \tag{1.5}
\]

More generally, we shall give the following two new \(q\)-supercongruences.
Theorem 1.2. For any positive integers \( a \) and \( n \), modulo \( \Phi_n(q)^2 \),

\[
\left(\binom{an}{an-n}\right)_q \equiv \begin{cases} 
(-1)^m a(1 + q^m)q^{m(3m-1)/2} - 3m(1 - q^{3m})(a^3_2), & \text{if } n = 3m; \\
(-1)^m aq^{m(3m+1)/2} - (3m + 1)(1 - q^{3m+1})(a^2_2), & \text{if } n = 3m + 1; \\
(-1)^m aq^{m(3m-1)/2} - (3m - 1)(1 - q^{3m-1})(a^2_2), & \text{if } n = 3m - 1. 
\end{cases}
\]

Theorem 1.3. For any positive integers \( a \) and \( n \) with \( a \geq 2 \), modulo \( \Phi_n(q)^2 \),

\[
\left(\binom{an}{an-2n}\right)_q \equiv \begin{cases} 
(-1)^m 2^{(a)_2}(1 + q^m)q^{m(3m-1)/2} - 9m(1 - q^{3m})(a^{a+1}_3) + \frac{3a-a^2}{2}, & \text{if } n = 3m; \\
(-1)^m 2^{(a)_2}q^{m(3m+1)/2} - 3(3m + 1)(1 - q^{3m+1})(a^{a+1}_3) + \frac{3a-a^2}{2}, & \text{if } n = 3m + 1; \\
(-1)^m 2^{(a)_2}q^{m(3m-1)/2} - 3(3m - 1)(1 - q^{3m-1})(a^{a+1}_3) + \frac{3a-a^2}{2}, & \text{if } n = 3m - 1. 
\end{cases}
\]

It is easily seen that Liu's two supercongruences (1.3) and (1.4) are the special cases of Theorem 1.2 with \( a = 1 \) and \( a = 2 \), respectively, and Theorem 1.1 can be obtained by taking \( a = 2 \) in Theorem 1.3.

As serendipitous discoveries, we obtain the following \( q \)-congruences which are the contiguous forms of (1.6) and (1.7) modulo \( \Phi_n(q) \).

Theorem 1.4. For any integer \( a \geq 1 \), integer \( j \) and positive odd integer \( n \) with \( 0 \leq j \leq n-1 \), there holds

\[
\left(\binom{an-1}{an-n+j}\right)_q \equiv \left(\frac{n-j}{3}\right)q^{(n-j-1)(n-j-2)/6-(j^2+j)/2} \pmod{\Phi_n(q)}. \tag{1.8}
\]

Here and in what follows, \( \left(\frac{a}{p}\right) \) denotes the Legendre symbol.

Theorem 1.5. For any integer \( a \geq 2 \), integer \( j \) and positive odd integer \( n \) with \( 0 \leq j \leq n \), there holds, modulo \( \Phi_n(q) \),

\[
\left(\binom{an-1}{an-2n+j}\right)_q \equiv (a-1)\left(\frac{n-j}{3}\right)q^{(n-j-1)(n-j-2)/6-(j^2+j)/2} + \left(\frac{j}{3}\right)q^{1-j^2}. \tag{1.9}
\]

It should be mentioned that if we take \( j = 0 \) in Theorem 1.4 and \( j = n \) in Theorem 1.5, then the right-hand sides of (1.8) and (1.9) are congruent to each other modulo \( \Phi_n(q) \).

Theorem 1.6. For any integer \( a \geq 2 \), positive integer \( b \) satisfying \( a \leq b+2 \) and positive odd integer \( n \), there holds

\[
\left(\binom{an-1}{bn-1}\right)_q \equiv \left(\frac{a}{b}\right) + \left(\frac{a-1}{b}\right)\left(\frac{n+1}{3}\right)q^{n(n-1)/6} \pmod{\Phi_n(q)}. \tag{1.10}
\]
By taking \( j = n - 1 \) in Theorem \( 1.5 \) and \( b = a - 1 \) in Theorem \( 1.6 \), we can easily find that they have the same left-hand sides. As a result, the right-hand sides of \( (1.9) \) and \( (1.10) \) are congruent to each other modulo \( \Phi_n(q) \). In other words, for any integer \( a \geq 2 \) and positive odd integer \( n \), there holds

\[
a + \left( \frac{n + 1}{3} \right) q^{n(n-1)/6} \equiv (a - 1) q^{-n(n-1)/2} + \left( \frac{n - 1}{3} \right) q^{-n(n-2)/3} \pmod{\Phi_n(q)}.
\]

The rest of the paper is arranged as follows. In the next section, we shall give proofs of Theorems \( 1.2 \) and \( 1.3 \). The proofs of Theorems \( 1.4 \) and \( 1.5 \) will be presented in Sections \( 3 \) and \( 4 \), respectively. In the last section, a sketch of the proof of Theorem \( 1.6 \) will be provided.

2. Proofs of Theorems \( 1.2 \) and \( 1.3 \)

In order to prove Theorems \( 1.2 \) and \( 1.3 \), we need the following two lemmas which have been proved by Liu.

**Lemma 2.1 (Liu \[11\]).** For any non-negative integer \( n \), there holds

\[
(1 - q^n) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k q^{k(k-1)/2} \left\lfloor \frac{n-k}{k} \right\rfloor = \begin{cases} 
(-1)^m (1 + q^m) q^{m(3m-1)/2}, & \text{if } n = 3m; \\
(-1)^m q^{m(3m+1)/2}, & \text{if } n = 3m + 1; \\
(-1)^m q^{m(3m-1)/2}, & \text{if } n = 3m - 1,
\end{cases}
\]

where \( \left\lfloor x \right\rfloor \) is the integral part of real \( x \).

**Lemma 2.2 (Liu \[12\]).** For any positive integer \( n \), there holds

\[
\sum_{k=1}^{\left\lceil \frac{n}{2} \right\rceil} q^{-k(k-1)/2} \left\lfloor \frac{2k}{[2k]_q} \right\rfloor \equiv \frac{(1 - q)(1 - R_n(q))}{1 - q^n} \pmod{\Phi_n(q)},
\]

where \( R_n(q) \) denotes the right-hand side of \( (2.1) \).

It is not difficult to see that \( R_n(q) \equiv 1 \pmod{\Phi_n(q)} \).

**Proof of Theorem \( 1.2 \)** Firstly, we express the left-hand side of \( (1.6) \) by Andrews and Baxter’s expression \( (1.2) \). Note that \( \left\lfloor \frac{an}{k+an-n} \right\rfloor \) for any \( k > \left\lfloor \frac{a}{2} \right\rfloor \). Therefore, we get

\[
\left( \begin{array}{c} an \\ an - n \end{array} \right)_q = \sum_{k=0}^{\left\lfloor \frac{a}{2} \right\rfloor} q^{k(k+an-n)} \left[ \begin{array}{c} an \\ k \end{array} \right] \left[ \begin{array}{c} an - k \\ k + an - n \end{array} \right]
\]

\[
= \sum_{k=0}^{\left\lfloor \frac{a}{2} \right\rfloor} q^{k(k+an-n)} \left[ \begin{array}{c} an \\ k \end{array} \right] \left[ \begin{array}{c} an - k \\ k + an - n \end{array} \right]
\]
For 1 \leq k \leq \left[ \frac{n}{2} \right],
we have
\[
\begin{bmatrix}
an
\end{bmatrix}
\begin{bmatrix}
\frac{n}{k}
\end{bmatrix} + \sum_{k=1}^{n} q^{k(\frac{n}{k}-1)} \left[ \begin{array}{c}
an
\end{array} \right] \left[ \begin{array}{c}
\frac{n}{k} - k
\end{array} \right] + \sum_{k=1}^{n} q^{k\left(\frac{n}{k}+\frac{k}{n}-n\right)} \left[ \begin{array}{c}
an
\end{array} \right] \left[ \begin{array}{c}
\frac{n}{k} + \frac{k}{n} - n
\end{array} \right].
\] (2.2)

Substituting the results (2.3) and (2.4) into the right-hand side of (2.2), we arrive at
\[
\text{following}
\]
where the second relation is due to the fact that 1 - q^{an} \equiv a(1 - q^n) \pmod{\Phi_n(q)}. Because the reduced form of \left[ \begin{array}{c}
an
\end{array} \right] contains the factor \Phi_n(q), it is enough to consider the following q-binomial coefficient modulo \Phi_n(q). For 1 \leq k \leq \left[ \frac{n}{2} \right], we have
\[
\begin{bmatrix}
an
\end{bmatrix}
\begin{bmatrix}
\frac{n}{k}
\end{bmatrix} = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n+1-2k})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}
\equiv a(1 - q^n)(1 - q^{-1}) \cdots (1 - q^{1-k}) \pmod{\Phi_n(q)}.
\] (2.3)

Because the reduced form of \left[ \begin{array}{c}
an
\end{array} \right] contains the factor \Phi_n(q), it is enough to consider the following q-binomial coefficient modulo \Phi_n(q). For 1 \leq k \leq \left[ \frac{n}{2} \right], we have
\[
\begin{bmatrix}
an
\end{bmatrix}
\begin{bmatrix}
\frac{n}{k}
\end{bmatrix} = \frac{(1 - q^n)(1 - q^{-1}) \cdots (1 - q^{-2k})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}
\equiv a(1 - q^n)(1 - q^{-1}) \cdots (1 - q^{1-k}) \pmod{\Phi_n(q)}.
\] (2.4)

Substituting the results (2.3) and (2.4) into the right-hand side of (2.2), we arrive at
\[
\left( \begin{array}{c}
an
\end{array} \right) \equiv \left( \begin{array}{c}
an
\end{array} \right) - (1 - q^n) \left( \begin{array}{c}
an
\end{array} \right) \left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) \sum_{k=1}^{1} \frac{q^{-k(1-1)}}{1 - q^k} \left[ \begin{array}{c}
2k - 1
\end{array} \right] \pmod{\Phi_n(q)^2}.
\] (2.5)

Then by Lemma 2.2, we obtain
\[
\sum_{k=1}^{1} \frac{q^{-k(1-1)}}{1 - q^k} \left[ \begin{array}{c}
2k - 1
\end{array} \right] = \frac{1}{1 - q} \sum_{k=1}^{1} \frac{q^{-k(1-1)}}{[2k]q} \left[ \begin{array}{c}
2k
\end{array} \right] \equiv a - \frac{1}{1 - q^n} \pmod{\Phi_n(q)}. (2.6)
\]

On the other hand, by the weaker version of (1.1), we have
\[
\begin{bmatrix}
an
\end{bmatrix} \equiv \left( \begin{array}{c}
a
\end{array} \right) \equiv 1 + q^n + q^{2n} + \cdots + q^{(a-1)n^2}
\equiv a - ((1 - q^n) + (1 - q^{2n}) + \cdots + (1 - q^{(a-1)n^2}))
\equiv a - n(1 - q^n) \frac{a(a - 1)}{2} \pmod{\Phi_n(q)^2},
\] (2.7)
where we have utilized the fact $q^n \equiv 1 \pmod{\Phi_n(q)}$ and the identity
$$1 - q^{kn^2} = (1 - q^n)(1 + q^n + q^{2n} + \cdots + q^{(kn-1)n}).$$
Finally, substituting (2.6) and (2.7) into the right-hand side of (2.5), we are led to
$$\left(\frac{an}{an-n}\right)_q \equiv aR_n(q) - n(1 - q^n)\left(\frac{a}{2}\right) \pmod{\Phi_n(q^2)}$$
as desired. \qed

**Proof of Theorem 1.3.** The proof is quite similar to that of Theorem 1.2. In order to apply Lemma 2.2 we express $(\binom{an}{an-2n})_q$ as follows:

$$\left(\frac{an}{an-2n}\right)_q = \left[\frac{an}{an-2n}\right] + q^{(a-1)n^2} \left[\frac{an}{n}\right] + \sum_{k=1}^{n-1} q^{k(an-2n)} \left[\frac{an}{k}\right] \left[\frac{an-k}{k+an-2n}\right].$$

For $1 \leq k \leq n - 1$, we have

$$\left[\frac{an-k}{k+an-2n}\right] = \left[\frac{an}{2n}\right] \frac{1 - q^{2n} \cdots (1 - q^{2n+1-2k})}{(1 - q^{an}) \cdots (1 - q^{an-k+1})(1 - q^{(a-2)n+1}) \cdots (1 - q^{(a-2)n+k})}
\equiv \left[\frac{an}{2n}\right] \frac{1 - q^{2n} \cdots (1 - q^{1-2k})}{(1 - q^{an}) (1 - q^{1-2k}) \cdots (1 - q^{1}) \cdots (1 - q^{k})}
\equiv \left[\frac{an}{2n}\right] \frac{2}{a} (-1)^k q^{-(3k-1)k/2} \left[\frac{2k-1}{k}\right] \pmod{\Phi_n(q)}. \quad (2.8)$$

Applying (2.3), (2.6) and (2.8), we obtain

$$\left(\frac{an}{an-2n}\right)_q = \left[\frac{an}{an-2n}\right] + q^{(a-1)n^2} \left[\frac{an}{n}\right] - 2(1 - q^n) \left[\frac{an}{2n}\right] \sum_{k=1}^{n-1} q^{-k(k-1)} \left[\frac{2k-1}{k}\right]
\equiv \left[\frac{an}{2n}\right] + q^{(a-1)n^2} \left[\frac{an}{n}\right] - 2(1 - q^n) \left[\frac{an}{2n}\right] \frac{1 - R_n(q)}{1 - q^n} \pmod{\Phi_n(q^2)}. \quad (2.9)$$

The third step holds since $\left[\frac{2k-1}{k}\right] \equiv 0 \pmod{\Phi_n(q)}$ for $\left[\frac{n}{2}\right] + 1 \leq k \leq n - 1$.

By utilizing the method used in (2.7), we get

$$\left[\frac{an}{2n}\right] \equiv \frac{a}{2} \pmod{\Phi_n(q^2)}. \quad (2.10)$$

Substituting (2.7) and (2.10) into the right-hand side of (2.9), we arrive at

$$\left(\frac{an}{an-2n}\right)_q \equiv - \frac{(a+1)a(a-1)}{2} n(1 - q^n) + \frac{3a - a^2}{2} + a(a-1)R_n(q) \pmod{\Phi_n(q^2)}.$$
This completes the proof. □

3. Proof of Theorem 1.4

For the sake of proving Theorem 1.4, we need the following congruence on the sum of central \( q \)-binomial coefficients.

**Theorem 3.1.** For any positive odd integer \( n \) and integer \( j \) with \( 0 \leq j \leq n - 1 \), there holds
\[
\sum_{k=0}^{n-j-1} q^{-k(k+1+j)} \left[ \begin{array}{c} 2k + j \\ k \end{array} \right] \equiv (-1)^j \left( \frac{n-j}{3} \right) q^{(n-j-1)(n-j-2)/6} \pmod{\Phi_n(q)}.
\]

Obviously, the special case of Theorem 3.1 with \( j = 0 \) is
\[
\sum_{k=0}^{n-1} q^{-k(k+1)} \left[ \begin{array}{c} 2k \\ k \end{array} \right] \equiv \left( \frac{n}{3} \right) q^{(n-1)(n-2)/6} \pmod{\Phi_n(q)},
\]
which is a \( q \)-analogue of a congruence by Sun and Tauraso [19] (the modulo \( p \) version): for any prime \( p \geq 5 \),
\[
\sum_{k=0}^{p-1} \left( \begin{array}{c} 2k \\ k \end{array} \right) \equiv \left( \frac{p}{3} \right) \pmod{p^2}.
\]

In order to prove Theorem 3.1, we recall the following result as the lemma, which firstly appeared in Ekhad and Zeilberger [4] and Krattenthaler’s note [10] and latter been proved by Warnaar [20] as a special case of a cubic summation formula.

**Lemma 3.2.** For any integer \( n \geq 0 \), there holds
\[
\sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \left[ \begin{array}{c} n-k \\ k \end{array} \right] = (-1)^n \left( \frac{n+1}{3} \right) q^n(n-1)/6.
\]

**Proof of Theorem 3.1.** Let \( \omega = e^{2\pi mi/n} \) with \( \gcd(m, n) = 1 \). Then we have that \( \omega^n = 1 \) and \( \omega^k \neq 1 \) for \( 1 \leq k \leq n - 1 \). Observing that
\[
\omega^{-k(k+j+1)} \left[ \begin{array}{c} 2k + j \\ k \end{array} \right] = \omega^{-k(k+j+1)} \prod_{l=1}^{k} \frac{1 - \omega^{2k+j+1-l}}{1 - \omega^l} = (-1)^k \omega^{k(k-1)/2} \left[ \begin{array}{c} n-k-j-1 \\ k \end{array} \right] \omega,
\]
we immediately get
\[
\sum_{k=0}^{n-j-1} q^{-k(k+j+1)} \left[ \begin{array}{c} 2k + j \\ k \end{array} \right] \equiv \sum_{k=0}^{n-j-1} (-1)^k q^{k(k-1)/2} \left[ \begin{array}{c} n-k-j-1 \\ k \end{array} \right] 
\equiv (-1)^j \left( \frac{n-j}{3} \right) q^{(n-j-2)(n-j-1)/6} \pmod{\Phi_n(q)},
\]
where we have utilized Lemma 3.2. □

Now we begin to prove Theorem 1.4.
Proof of Theorem 1.4. For $1 \leq k \leq n - 1$, we have

$$\begin{align*}
\left[ \frac{an - 1}{k} \right] &= \frac{(1 - q^{an-1}) \cdots (1 - q^{an-k})}{(1 - q) \cdots (1 - q^k)} \equiv (-1)^k q^{-k(k+1)/2} \pmod{\Phi_n(q)}. \tag{3.2}
\end{align*}$$

We can also get the following congruence with $1 \leq k \leq \lfloor (n - 1 - j)/2 \rfloor$,

$$\begin{align*}
\left[ \frac{an - k - 1}{an - n + k + j} \right] &= \left[ \frac{an}{n} \right] \frac{(1 - q^n) \cdots (1 - q^{n-2k-j})}{(1 - q^{an-k})(1 - q^{an-k+1}) \cdots (1 - q^{an-k + n+k+j})}
\equiv (-1)^{k+j} q^{-(3k+j+1)(k+j)/2} \left[ \frac{2k + j}{k} \right] \pmod{\Phi_n(q)}. \tag{3.3}
\end{align*}$$

By combining the congruence (3.2) with (3.3), we arrive at the following result

$$\begin{align*}
\left( \frac{an - 1}{an - n + j} \right)_q &= \sum_{k=0}^{\lfloor \frac{n-j-1}{2} \rfloor} q^{k(\lfloor an - n + j \rfloor)} \left[ \frac{an - 1}{k} \right] \left[ \frac{an - k - 1}{k + an - n + j} \right]
\equiv \left[ \frac{an - 1}{an - n + j} \right] - (-1)^j q^{-(j^2+j)/2} \sum_{k=1}^{\lfloor \frac{n-j-1}{2} \rfloor} q^{-k^2-kj-k} \left[ \frac{2k+j}{k} \right]
\equiv \left( \frac{n-j}{3} \right) q^{(n-j-1)(n-j-2)/2-(j^2+j)/2} \pmod{\Phi_n(q)},
\end{align*}$$

where we have applied Theorem 3.1 and $\left[ \frac{2k+j}{k} \right] \equiv 0 \pmod{\Phi_n(q)}$ for $\lfloor \frac{n-j-1}{2} \rfloor + 1 \leq k \leq n - j - 1$ in the last step.

4. Proof of Theorem 1.5

Proof of Theorem 1.5. Because of the different range of $k$, the proof of Theorem 1.5 is a little different from that of Theorem 1.4. We first split the summation of the following $q$-trinomial coefficient into three parts as

$$\begin{align*}
\left( \frac{an - 1}{an - 2n + j} \right)_q &= \left[ \frac{an - 1}{an - 2n + j} \right] + \sum_{k=1}^{n-j-1} q^{k(\lfloor an - 2n + j \rfloor)} \left[ \frac{an - 1}{k} \right] \left[ \frac{an - k - 1}{k + an - 2n + j} \right]
\end{align*}$$

$$\begin{align*}
+ \sum_{k=n-j}^{\lfloor \frac{n-j}{2} \rfloor} q^{k(\lfloor an - 2n + j \rfloor)} \left[ \frac{an - 1}{k} \right] \left[ \frac{an - k - 1}{k + an - 2n + j} \right]. \tag{4.1}
\end{align*}$$

The first and the second parts of the right-hand side of (4.1) can be handled just as what we have done in the proof of Theorem 1.4. Thus, we can get the following result with no difficulty,

$$\begin{align*}
\left[ \frac{an - 1}{an - 2n + j} \right] + \sum_{k=1}^{n-j-1} q^{k(\lfloor an - 2n + j \rfloor)} \left[ \frac{an - 1}{k} \right] \left[ \frac{an - k - 1}{k + an - 2n + j} \right]
\end{align*}$$
Now we begin to calculate the third part of the right-hand side of (4.1). Since the third part will disappear if $j = 0$, we only need to consider the case where $j$ is in the range $1 \leq j \leq n$. Noticing that $\left[\frac{an-k}{k+an-2n+j}\right]$ can be simplified through the method used in (3.3), and recalling the result (3.2), we have

\[
\sum_{k=n-j}^{2n-j-1} q^{k+an-2n+j} \left[\frac{an-1}{k}\right] \left[\frac{an-k-1}{k+an-2n+j}\right] \equiv (-1)^j q^{-(j^2+j)/2} \sum_{k=n-j}^{2n-j-1} q^{-kj-k} \left[\frac{2k+j}{k}\right] \pmod{\Phi_n(q)}. \tag{4.3}
\]

Performing $k \to n-j+k$ in the summation of the right-hand side of (4.3), we have

\[
\sum_{k=n-j}^{2n-j-1} q^{-kj-k} \left[\frac{2k+j}{k}\right] = \sum_{k=0}^{j-1} q^{-(k+an-j)^2-(k+an-j)(j-n-j)} \left[\frac{2(k+n-j)+j}{k+n-j}\right]
\]

\[
\equiv \sum_{k=0}^{j-1} q^{-kj-k} \left[\frac{n+2k-j}{k}\right]
\]

\[
\equiv (-1)^{n-j}\left(\frac{j}{3}\right) q^{(j-1)(j-2)/6} \pmod{\Phi_n(q)}, \tag{4.4}
\]

where we have utilized Theorem 3.1 with $j \to n-j$ in the last step.

Finally, by combining the formulas from (4.1) to (4.4) together, we obtain

\[
\left(\left[\frac{an-1}{an-2n+j}\right]\right)_q \equiv (a-1) \left(\frac{n-j}{3}\right) q^{(n-j-1)(n-j-2)/6} - \frac{j^2+j}{2} + \left(\frac{j}{3}\right) q^{-\frac{j^2}{2}} \pmod{\Phi_n(q)}. \tag{4.5}
\]

The right-hand side of (4.4) equals 0 if $j = 0$, and so (4.5) also holds when $j = 0$. Thus we finish the proof of Theorem 1.5. \hfill \Box

5. The Proof of Theorem 1.6

Sketch of Proof. For $1 \leq k \leq \left[\frac{(a-b)n}{2}\right]$, we have

\[
\left[\frac{an-k-1}{k+bn-1}\right] = \left[\frac{an}{bn}\right] \frac{(1-q^{(a-b)n}) \cdots (1-q^{(a-b)n-2k+1})}{(1-q^{bn+1}) \cdots (1-q^{bn+k-1})(1-q^{an-k}) \cdots (1-q^{an})}
\]

\[
\equiv (-1)^{k-1} \left(\frac{a-1}{b}\right) q^{-3k(k-1)/2} \left[\frac{2k-1}{k}\right] \pmod{\Phi_n(q)}. \]
Similarly to the proof of Theorem 1.4, we get
\[
\binom{an - 1}{bn - 1}_q = \sum_{k=0}^{\lfloor \frac{(a-b)n}{2} \rfloor} q^{k+bn-1} \binom{an - 1}{k} \binom{an - k - 1}{k + bn - 1}
\end{equation}
\begin{equation}
\equiv \binom{an - 1}{bn - 1} - \sum_{k=1}^{\lfloor \frac{(a-b)n}{2} \rfloor} \frac{q^{-k^2}}{k^2} \binom{2k - 1}{k}
\end{equation}
\begin{equation}
\equiv \binom{an - 1}{bn - 1} - \sum_{k=1}^{\lfloor \frac{(a-b)n}{2} \rfloor} (-1)^k q^{k(k-1)/2} \binom{n - k}{k}
\end{equation}
\begin{equation}
\equiv \binom{a}{b} + \binom{a - 1}{b} \left( \frac{n + 1}{3} \right) q^{n(n-1)/6} \pmod{\Phi_n(q)},
\end{equation}
where we have applied Lemma 3.2 in the last step. □

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