How to improve the interpretability of kernel learning

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Abstract: In recent years, machine learning researchers have focused on methods to construct flexible and interpretable prediction models. However, the interpretability evaluation, the relationship between the generalization performance and the interpretability of the model and the method for improving the interpretability are very important factors to consider. In this paper, the quantitative index of the interpretability is proposed and its rationality is given, and the relationship between the interpretability and the generalization performance is analyzed. For traditional supervised kernel machine learning problem, a universal learning framework is put forward to solve the equilibrium problem between the two performances. The uniqueness of solution of the problem is proved and condition of unique solution is obtained. Probability upper bound of the sum of the two performances is analyzed.

Keywords: Interpretability, Generalization Performance, Kernel Learning, Supervised Learning, Error Estimation

1. Introduction

Safe, controllable and credible artificial intelligence has been the goal which the humanity has been pursuing. In the field of machine learning, in order to achieve this goal, it is necessary for learning algorithm to really interact with the humanity; It is necessary for the learning algorithm to have the ability to correct errors, so as to avoid a prediction model with serious errors caused by unnecessary deviation in training data; It needs to be able to check its own learning process or decision-making process based on unsuccessful prediction results, especially for complex learning tasks; It is necessary to establish a learning algorithm for capturing and learning causal relationships in the world around us, so that the prediction model could predict what will happen under certain conditions, even if these conditions are significantly different from those of the past; It needs the learning algorithm which can really take full control of generalization performance of the prediction model. As big data accelerates transformation of scientific research pattern, scientific research is translating from a hypothetical drive mode to a data-driven one, which needs learning algorithm to discover
new natural phenomena and laws through big data mining, statistic and analysis. However, recently, all of this is out of reach. The reason is that the prediction model and its training process are not yet understood by human beings, and are not covered by the knowledge base we currently have.

In practice, occurrence of random events always causes deviation of measurement data, which generates many intermittent and continuous noise data, so as to make the prediction model deviated from known relationship and real law between data. Even if there is no noise data, because the training data set is just a sample in sample space, if we can't scout out its distribution, as well as sample size is not large enough, even without the influence of the noise data, finally the prediction model can't accurately express the true relationship and law between data. Even if a lot of data were collected, if they do not conform to its real distribution, the final prediction model will be the same as the result from usual small sample. Even if distribution is known, but if the basis of the prediction model's space is not known in advance, traditional machine learning algorithms still can't guarantee the final prediction model can express exactly the true relationship and the real law between data and is difficult to ensure the model full compliance with professional knowledge. We believe this is because that in conjugate space of high dimensional feature space obtained by kernel method the linear functional set is nowhere dense in the square integrable function space. Even if mathematical form of the prediction model is known in advance, if the optimization problem is multi-peak complex objective function, recently there is no strong optimization mechanism to solve this optimization problem effectively for the optimal interpretable prediction model. In order to achieve this goal, the sample must be dense enough, its distribution must be accurately known, and interference of the noise data can be avoided easily, the kernel function should be reasonable or mathematical expression of the prediction model must be fully known, even prediction model posterior distribution is clear and a good optimization algorithm is also essential. However, in fact, all learning algorithms we face do not have such strict prerequisites.

How to make the prediction model and training process understood by us, and conform to human cognition, even to generate new cognitive for human, is in essence an optimization problem which can promote the interpretability of the prediction model and make the model more suitable to its causality or discover faults in the causality. As literature [1] points out, when we pay attention to scientific problems, it is a motivation of scientific research to trace their origins, or to pursue their causality. Professor Zoubin Ghahramani also pointed out that current machine learning theorists should consider how to construct more flexible and interpretable prediction models[2]. In ICML 2017, the theme of the best paper ”Understanding black-box Predictions via Influence Functions” is to use influence functions to understand black-box predictions and study how to explain source of prediction models[3]. Many
researchers will present their results on NIPS'17 Interpretable Machine Learning Symposium and on CVPR Tutorial 2018 Interpretable Machine Learning for Computer Vision. FICO, Google, UC Berkeley, Oxford, Imperial, MIT and UC Irvine jointly launched a competition to generate new researches for the interpretability of algorithms.

The interpretability of the prediction model is generally regarded as human simulatability. If humans can explain every calculation steps and finally make prediction at right time by using input data and model parameter, the prediction model will have this kind of imitative which is the interpretability (Lipton, 2016). For example, given a simulation model for a diagnosis, a doctor can easily check each step of the model with their professional knowledge and even infer fairness and system deviation of the diagnosis result. However, this is a strict definition. If based on this definition to improve the interpretability, the domain knowledge must be forced to every step of the training process of the prediction model. The optimal prediction model tends to lose its generalization performance, such as decision tree algorithm.

We posit that the interpretability should be a potential ability to help experts discover an essential reason of a prediction result and provide research clues and possibilities for researchers to further research. Specifically, that is, when the prediction model is the same as a theoretical model in the form of geometric shape or mathematical expression, despite different value and different scale, we can think that the model has good interpretability, that the model can be explained well by the theoretical model.

We posit that in machine learning, it is more realistic to apply this definition to solve an interpretability improvement problem of the prediction model. Its key problem is how to ensure the prediction model as much as possible consistent with its explanatory description, but not lost its generalization performance. In a training process of the prediction model, we not only should consider its generalization performance but also need to consider the deviation between the prediction model and the theoretical model, namely the interpretability.

Currently, there are two methods for improving the interpretability of the prediction model: analytical interpretability and statistical interpretability. In the analytical interpretability, Pang et al. [3], Wu et al. [4], and Zhou et al. [5-6] respectively discovered and analyzed the prediction model by influential function and visualization of internal feature data of neural networks. Craven et al.[7], and Baehrens et al.[8] proposed model-agnostic method by learning an interpretable model on the predictions of the black box model. Strumbelj et al.[9] and Krause et al. [10] proposed perturbing inputs and seeing how the black box model reacts. In statistical interpretability, James et al.[11] proposed to build a prediction model of automatic statistical learning by mining the functional relationship between input and output from training samples. Ribeiro et al.[12-13] proposed a measure of the interpretability complexity for obtaining an interpretable linear model and realizing a local interpretability
and proposed a submodular pick algorithm for a global interpretability. We in a literature [14] proposed use of prior knowledge in a hypothesis space (such as Sobolev space) to construct a compact subset, which can ensure the interpretability of the prediction model and correct prior knowledge in its training process.

However, how is the interpretability of the prediction model evaluated? How does promotion of its interpretability affect its generalization performance? If there is an equilibrium problem between the two, what is the probability upper bound for the sum of the two? What factors are involved in this upper bound? What are the relationship between the factors and the sum? What is the learning framework for solving this equilibrium problem? Is there a unique solution to this optimization problem? What are the conditions for producing the unique solution? These questions have not yet been answered. For this purpose, this paper first introduces the learning framework of traditional machine learning based on $L_2$ norm regular term. Then, the quantitative index of the interpretability is proposed and its rationality is given, and the relationship between the interpretability and the generalization performance is analyzed. For traditional supervised kernel machine learning problem, a universal learning framework is put forward to solve the equilibrium problem between the two performances. The uniqueness of solution of the problem is proved and condition of unique solution is obtained. Probability upper bound of the sum of the two performances is analyzed.

2. Learning framework of traditional machine learning

Suppose $X$ is a compact domain or a manifold in Euclidean space and $Y \in R^k$, $k = 1$, $\rho$ is a Borel probability measure of a space $Z = X \times Y$.

$f_\rho: X \rightarrow Y$ as $f_\rho(x) = \int_Y y d\rho(y|x)$ is defined. The function $f_\rho$ is a regression function of $\rho$.

In machine learning, $\rho$ and $f_\rho$ are unknown. At some conditions, an edge probability measure $\rho_X$ of $X$ is known.

The goal of the learning is to find the best approximation of $f_\rho$ in a functional space. Therefore, Tikhonov regularization learning framework[23-24] can be obtained

$$f_{x,Y} = \operatorname{argmax}_{f \in \mathcal{H}_K} \left\{ \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - y_i)^2 + \gamma \| f \|_K^2 \right\} \quad (1)$$
3. Learning framework for improving the interpretability of the prediction model

In traditional kernel machine learning framework\cite{15}, generalization error bound\cite{16} representing the generalization performance of the prediction model is related to sample size and feature space of kernel function. If trying to ensure that the prediction model can be explained as far as possible, but not lose its generalization performance, so in the learning process, not only the generalization error bound should be considered also the deviation boundary between the prediction model and a mathematical model describing prior knowledge, denoted by interpretability model, need to be considered. Literature \cite{2} proposed a uniform description of all optimization problems based on prior knowledge, assuming that the interpretability model and the prediction model are in the same Hilbert space, using the prior knowledge as strong constraints, and conducting the risk of consistency analysis and error analysis. However, the following two problems are not well solved: 1. How to quantify the interpretability of the prediction model? 2. When the interpretability model and learning function are not in the same Hilbert space, how to make use of prior knowledge constraint learning process for a well interpretability. This chapter will be the first to put forward a quantitative evaluation index of the interpretability, and then a learning framework for the well interpretability, and finally the uniqueness of the solution of the framework is proved.

3.1 Evaluation of the interpretability of the prediction model

The interpretability of the prediction model is not innate, and it requires the domain experts to provide based on professional terms or common sense, such as prior knowledge. This kind of professional explanation should be expressed in a form of a mathematical function, denoted by interpretation function or interpretation model such as linear models\cite{9, 17}, gradient vector\cite{18}, an additive model\cite{19}, decision trees\cite{20}, falling rule lists\cite{21-22}, attention-based networks. However, the prior knowledge is usually uncertain and incomplete, which leads to the uncertainty of the interpretation model. How to design interpretation models to express uncertain and incomplete prior knowledge? The representation method of the uncertain interpretation model and how to obtain complete knowledge from incomplete knowledge, in another article, has been introduced. So this article focuses on the second question. How to ensure that the prediction model is close to the uncertain interpretation model?

Inspired by induction and analysis coupling learning method, the differences between the prediction model $f(x)$ and the interpretation function $P(x)$ can be computed by the mean square error between the two. However, in practice, the mean square error is too strict in evaluating the difference between the two models, and there will be some problems caused by
different orders of magnitude and different function subspace. We posit that the correctness of the interpretability of the prediction model itself depends on correct expression of causal relationship between the output attribute and the input attributes. When the attributes in both models satisfy the same causal relationship, we can assume that both models express the same interpretation, even if their magnitude is different. This conclusion can be explained by Fig. 1.

In Fig.1, there are three functions, \( f(x) = ax \), \( f_1(x) = bx \) and \( f_2(x) = ax + c \). It is obvious when \( c \gg \frac{1}{\sqrt{3}} |a - b|(x_2^2 + x_1 x_2 + x_1^2)^{\frac{1}{2}} \), the mean square error between \( f_2(x) \) and \( f(x) \) is \( \int_{x_1}^{x_2} (f_2(x) - f(x))^2 dx = c^2(x_2 - x_1) \), which is greater than \( \int_{x_1}^{x_2} (f(x) - f_1(x))^2 dx = \frac{1}{3} |a - b|^2 (x_2^3 - x_1^3) \) which is the mean square error between \( f_1(x) \) and \( f(x) \). In order of magnitude, \( f_1(x) \) is similar to \( f(x) \), but the causal relationship between the input and output attributes in \( f_2(x) \) and \( f(x) \) is consistent. So the mean square error can't compare difference of the causality between the input attributes and the output attribute. In the example, the variance of the error between \( f_2(x) \) and \( f(x) \) is exactly 0. Thus it can be seen that the variance of the error is better.

![Fig.1 Two different evaluative methods of the interpretability](image)

From what has been discussed above, in the square integrable function space the variance \( \mathcal{E}^P(f) \) of the error between a model \( f(x) \) and an interpretation model \( P(x) \) is used to calculate the interpretability of \( f(x) \).

\[
\mathcal{E}^P(f) = \int_Z (f(x) - P(x) - \mu^P(f))^2
\] (2)
This formula calculates the distance between the prediction model and the interpretation function, also known as interpretation distance, where

$$\mu^p(f) = \int_k \left( f(x) - P(x) \right)$$

is a mean error between $f(x)$ and $P(x)$.

3.2 The existence proof of equilibrium problem

In order to build a learning framework for improving the interpretability of the prediction model, it is necessary to understand the relationship between the interpretability and the generalized performance. If $X$ is a compact metric space, $\nu$ is a Borel measure in $X$, such as Lebesgue measure or edge measures, $\mathcal{L}_2^2(X)$ is a square integrable function space on $X$, $K: X \times X \to \mathbb{R}$ is a continuous function, and then an integral transform $(L_K f)(x) = \int K(x, t)f(t)d\nu(t)$ are a linear mapping: $L_K: \mathcal{L}_2^2(X) \to C(X)$ and composite a linear operator $L_K: \mathcal{L}_2^2(X) \to \mathcal{L}_2^2(X)$ with a contain $C(X) \subset \mathcal{L}_2^2(X)$, denoted by $L_K$. The function $K$ is a kernel function of $L_K$. Let $K_\nu: X \to \mathbb{R}$ become $K_\nu(t) = K(x, t)$.

The Hilbert-Schmidt theorem shows that if all eigenvalues of the operator $L_K$ are not strictly nonzero and $\phi_k(x)$ is an eigenfunction of $L_K, k = 1, 2, 3, ...$, in the Hilbert space $\mathcal{L}_2^2(X)$ there is a function $f(x) = \sum_k c_k \phi_k(x) + \xi(x)$, and $\xi(x) \in \text{Ker}L_K$ which is $L_K\xi(x) = 0$. $\xi(x)$ usually can be approximated by a bias.

Because the interpretation function have no connection with the kernel function, it usually can’t be obtained by linear sum of the eigenfunctions of $L_K$, unless eigenvalues of $L_K$ are strictly non-zero. The optimal prediction model for dividing two types of sample is a linear functional while the optimal interpretation function is a nonlinear functional. In the square integrable function space $\mathcal{L}_2^2(X)$ of a separable Hilbert space, which is a conjugate space of the high dimensional feature space, the two models could not be equal. So let us prove that. First, lemma 1 is given.

**Lemma 1.** A continuous linear functional set of a separable Hilbert space $X$ is nowhere dense in an integrable function space $\mathcal{L}_2^2(X)$.

**Proof:** The limit of any nonlinear functional sequence is a simple function with finite values in the sense of average convergence, but it does not have features of linear function, such as, additive and homogeneous. If $M$ is a measurable set of the separable Hilbert space $X$, and $\nu(M) < \infty$, based on the conditions of lebesgue measure (In $R$ every open set and every closed set are measurable, and for all measurable set $M \subset R$ and arbitrary $\varepsilon > 0$ there is an
open set $G \supset M$ and $\nu(G \setminus M) < \varepsilon$ can be obtained) we can obtain that for any $\varepsilon > 0$ a closed set $F_M$ and an open set $G_M$ can have

$$F_M \subset M \subset G_M \quad \text{and} \quad \nu(G_M) - \nu(F_M) < \varepsilon \quad (4)$$

The function $\varphi_\varepsilon(x)$ is defined as the following:

$$\varphi_\varepsilon(x) = \frac{\rho(x, R \setminus G_M)}{\rho(x, R \setminus G_M) + \rho(x, F_M)} \quad (5)$$

where $\rho(x, D)$ is a distance between a point $x$ and a subspace $D$. $\varphi_\varepsilon(x)$ is zero when $x \in R \setminus G_M$, and one when $x \in F_M$. Because $\rho(x, F_M)$ and $\rho(x, R \setminus G_M)$ are continuous and the sum of the two functions keeps at nonzero, $\varphi_\varepsilon(x)$ is continuous. In $L_1^V(X)$, a linear functional sequence, $f_n(x) = (x, x_n), n = 1, 2, ...$, can be found. If $\lim_{n \to \infty} f_n(x_1) = f(x_1) = 1$ where $x_1 \in F_M$ is an element of a dense everywhere countable set in $F_M$, based on linearity of the linear functional $\alpha x_1$ can always be found and $\alpha$ is an arbitrary number, which makes $f(\alpha x_1) = \alpha$, namely $f(x) = \frac{x}{x_1}$. It is less difficult to know that

$$\int |f(x) - \varphi_\varepsilon(x)| \, dv = \int \left| x - \varphi_\varepsilon(x) \right| \, dv$$

has not upper bound. If $\lim_{n \to \infty} f_n(x_1) = f(x_1) = 0$ and $x_1 \in R \setminus G_M$, based on linearity of the linear functional $\alpha x_1$ can always be found, which makes $f(\alpha x_1) = 0$, namely $f(x) = 0$. Evidently though, $\int |f(x) - \varphi_\varepsilon(x)| \, dv = \int |0 - \varphi_\varepsilon(x)| \, dv = \nu(F_M)$.

So it's impossible to find a $N$, which makes $\int |f_n(x) - \varphi_\varepsilon(x)| \, dv < \varepsilon$ for any $n > N$.

That is to say, continuous linear functional set of the separable Hilbert space $X$ is nowhere dense in $L_1^V(X)$. Evidenced by the same token, the continuous linear functional set of the separable Hilbert space $X$ is nowhere dense in $L_2^V(X)$ too. Lemma 2 can be obtained.

**Lemma 2.** Continuous linear functional set of the separable Hilbert space $X$ is nowhere dense in $L_1^V(X)$.

It can be known from lemma 2 that continuous nonlinear functional set of the separable Hilbert space $X$ is everywhere dense in $L_2^V(X)$. Thus, Lemma 3 is true.

**Lemma 3.** Continuous nonlinear functional set of the separable Hilbert space $X$ is everywhere dense in $L_2^V(X)$.

From Lemma 3, it is less difficult to know that in $L_2^V(X)$ of a separable Hilbert space the optimal prediction model and the optimal interpretation function could not be equal. In other words, there is an equilibrium problem between the two models.
3.3 Learning framework for improving the interpretability of the prediction model

Based on Tikhonov regularized learning framework[23-24] and the evaluation formula of the interpretability, a learning framework for improving the interpretability of the prediction model can be obtained.

\[ f_{\mathcal{L}, \lambda} = \arg\min_{f \in \mathcal{H}} \left\{ \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - y_i)^2 + \lambda \| f \|_K^2 + \frac{1}{m} \sum_{i=1}^{m} \left( f(x_i) - P(x_i) - \frac{1}{m} \sum_{i=1}^{m} |f(x_i) - P(x_i)| \right)^2 \right\} \]  

(6)

3.4 The uniqueness of the optimal solution is proved

Now let us prove the solution of the optimal problem is unique.

In a close subspace \( \mathcal{H} \) of \( \mathcal{L}_2^p(X) \), \( f^p_{\mathcal{H}}(x) \) is defined as an optimal function which has the distance as small as possible with \( f_\rho(x) \), while has the smallest distance with the interpretation model \( P(x) \). We will prove that if the close subspace \( \mathcal{H} \) is convex, \( f^p_{\mathcal{H}}(x) \) must be unique.

**Lemma 4.** A compact space \( \mathcal{H} \) is a convex subset of \( \mathcal{L}_2^p(X) \), then Eq. (6) must have unique solution \( f^p_{\mathcal{H}}(x) \in \mathcal{H} \).

**Proof:** In \( \mathcal{H} \), suppose \( s = f^p_{\mathcal{H}} \) is a segment with two endpoints \( f^p_{\mathcal{H}} \) and \( f \). Because \( \mathcal{H} \) is a convex subset, and \( s \subset \mathcal{H} \), in \( \mathcal{L}_2^p(X) \) the sum of the distance between \( f^p_{\mathcal{H}}(x) \) and \( f_\rho(x) \) and the distance between \( f^p_{\mathcal{H}}(x) \) and \( P(x) \) is the smallest, all \( f \in s \),

\[ \int_Z \left( f^p_{\mathcal{H}}(x) - P(x) - \mu(f^p_{\mathcal{H}}, P) \right)^2 + \int_Z \left( f^p_{\mathcal{H}}(x) - f_\rho(x) \right)^2 \leq \int_Z \left( f(x) - P(x) - \mu(f, P) \right)^2 + \int_Z \left( f(x) - f_\rho(x) \right)^2. \]  

(7)
It can be known from the triangle cosine theorem that in $\triangle P_{f\mu}f$

$$
\int_Z \left[ (f(x) - P(x) - \mu(f,P))^2 - \left( f_{\mu}^P(x) - P(x) - \mu(f_{\mu}^P,P) \right)^2 \right] = \int_Z \left( f_{\mu}^P(x) - f(x) - \mu(f_{\mu}^P,f) \right)^2 - 2 \int_Z |f_{\mu}^P(x) - f(x) - \mu(f_{\mu}^P,f)||f_{\mu}^P(x) - P(x) - \mu(f_{\mu}^P,P)| \cos \angle P_{f\mu}f
$$

(8)

And in $\triangle f_{\mu}P_{f\mu}f$,

$$
\int_Z \left[ (f(x) - f_{\mu}(x))^2 - \left( f_{\mu}^P(x) - f_{\mu}(x) \right)^2 \right] = \int_Z \left( f_{\mu}^P(x) - f(x) \right)^2 - 2 |f_{\mu}^P(x) - f(x)||f_{\mu}^P(x) - f_{\mu}(x)| \cos \angle f_{\mu}P_{f\mu}f
$$

(9)

Combining the above two equations, we can get

$$
\epsilon_{f\mu}^P(f) = \int_Z \left( f_{\mu}^P(x) - f(x) - \mu(f_{\mu}^P,f) \right)^2 + \int_Z \left( f_{\mu}^P(x) - f(x) \right)^2 - 2 \int_Z |f_{\mu}^P(x) - f(x)||f_{\mu}^P(x) - f(x)| \cos \angle f_{\mu}P_{f\mu}f + \int_Z |f_{\mu}^P(x) - f(x)||f_{\mu}^P(x) - f_{\mu}(x)| \cos \angle f_{\mu}P_{f\mu}f
$$

(10)

Suppose $f(x) = f_{\mu}^P(x)$, and $f_{\mu}^P(x)$ also is the optimal one, then
\[
\int \left| f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^r(x) - \mu\left(f_{\mathcal{H}}^p, f_{\mathcal{H}}^r\right)\right| - \left| f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^r(x) - \mu\left(f_{\mathcal{H}}^p, f_{\mathcal{H}}^r\right)\right| - 2|f_{\mathcal{H}}^p(x) - p(x) - \mu(f_{\mathcal{H}}^p, p)|\cos\angle Pf_{\mathcal{H}}^p f_{\mathcal{H}}^r + \int \left| f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^r(x)\right|\left| f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^r(x) - \mu(f_{\mathcal{H}}^p, f_{\mathcal{H}}^r)\right| \\
= 0. \quad (11)
\]

Because \( \mathcal{H} \) is convex, from the above formula, it can be seen that there are a function \( f_{\mathcal{H}}^* \) between \( f_{\mathcal{H}}^p \) and \( f_{\mathcal{H}}^r \), which can guarantee:

\[
|f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^*(x) - \mu(f_{\mathcal{H}}^p, f_{\mathcal{H}}^*)| < |f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^r(x) - \mu(f_{\mathcal{H}}^p, f_{\mathcal{H}}^r)|
\]

and:

\[
|f_{\mathcal{H}}^r(x) - f_{\mathcal{H}}^*(x) - \mu(f_{\mathcal{H}}^p, f_{\mathcal{H}}^*)| < |f_{\mathcal{H}}^p(x) - f_{\mathcal{H}}^r(x) - \mu(f_{\mathcal{H}}^p, f_{\mathcal{H}}^r)|.
\]

\( \angle Pf_{\mathcal{H}}^p f_{\mathcal{H}}^r \) and \( \angle Pf_{\mathcal{H}}^r f_{\mathcal{H}}^* \). And \( \angle f_{\mathcal{H}}^p f_{\mathcal{H}}^r f_{\mathcal{H}}^*(x) \) < 0.

If so, we can always find a better function \( f_{\mathcal{H}}^*(x) \) than \( f_{\mathcal{H}}^p(x) \) and \( f_{\mathcal{H}}^r(x) \), as shown in the following figure. Therefore, the conclusion is a contradiction to the previous hypothesis.

So \( f_{\mathcal{H}}^p(x) \) is the unique solution that meets the condition in \( \mathcal{L}_{\mathcal{H}}^2(X) \).

Fig. 3 The unique solution of the new learning problem (II)
4. Error estimate of Hypothesis Space

Suppose the optimal solution \( f^p_H(x) \) of the optimal problem (6) can be found in the convex subset \( \mathcal{H} \) of \( L^2_P(X) \). The deviation between \( f \in \mathcal{H} \) and \( f^p_H(x) \) is defined as an error \( \mathbb{E}_H(f) = \mathbb{E}(f) - \mathbb{E}(f^p_H) + \mathbb{E}^p(f) - \mathbb{E}^p(f^p_H) \), where \( \mathbb{E}(f) \) is an error between \( f(x) \) and the real output \( y \). If \( f : X \rightarrow Y, \mathbb{E}(f) = \mathbb{E}_\rho(f) = \int_x (f(x) - y)^2 \).

For any function \( f \in \mathcal{H}, \mathbb{E}_H(f) \geq 0 \) and \( \mathbb{E}_H(f^p_H) = 0 \). Let us focus on that

\[
\mathbb{E}(f_{\mu x}) + \mathbb{E}^p(f_{\mu x}) = \mathbb{E}_H(f_{\mu x}) + \mathbb{E}(f^p_H) + \mathbb{E}^p(f^p_H) \tag{12}
\]

where \( \mathbb{E}_H(f_{\mu x}) \) is a distance between \( f_{\mu x}(x) \) and \( f^p_H(x) \), denoted by sample error. \( \mathbb{E}(f^p_H) \) is a distance between \( f^p_H(x) \) and \( y \), and \( \mathbb{E}^p(f^p_H) \) is a distance between \( f^p_H(x) \) and \( P(x) \), the sum of the two distances is approximate error.

4.1 Sample error estimation

From the above formula (12), it can be seen that

\[
\mathbb{E}_H(f) = \mathbb{E}(f) - \mathbb{E}(f^p_H) + \mathbb{E}^p(f) - \mathbb{E}^p(f^p_H) \tag{13}
\]

The formula can be divided into two parts: \( \mathbb{E}(f) - \mathbb{E}(f^p_H) \) and \( \mathbb{E}^p(f) - \mathbb{E}^p(f^p_H) \).

Probability bound of the former, \( \mathbb{E}(f) - \mathbb{E}(f^p_H) \), can be deduced by Theorem B and theorem C of reference [1].

**Theorem 1.** Suppose \( \mathcal{H} \) is a compact subset of \( L^2_P(X) \), and for all \( f \in \mathcal{H} \), \(|f(x) - y| \leq M \) is true almost everywhere. If

\[
\sigma^2 = \sigma^2(\mathcal{H}) = \sup_{f \in \mathcal{H}} \sigma^2(f^2_I)
\]

where \( \sigma^2(f^2_I) \) is a variance of \( f^2_I = (f(x) - y)^2 \), for all \( \epsilon > 0 \),

\[
prob_{x \in Z^m}[|\mathbb{E}(f_I) - \mathbb{E}(f^p_H)| \leq \epsilon] \geq 1 - \mathcal{N}\left(\mathcal{H}, \frac{\epsilon}{16M}\right) 2e^{-\frac{m\epsilon^2}{8(4\sigma^2+\frac{M^2\epsilon}{3})}} \tag{14}
\]

where \( \mathcal{N}\left(\mathcal{H}, \frac{\epsilon}{16M}\right) \) is a covering number on \( \mathcal{H} \) in the radius \( \frac{\epsilon}{16M} \).

According to lemma 5 in literature [1], it is easy to deduce Theorem 2 in the convex hypothesis space \( \mathcal{H} \).

**Theorem 2.** Suppose \( \mathcal{H} \) is a compact convex subset of \( L^2_P(X) \) which can sure that the interpretation distance between \( f^p_H \) and \( P(x) \) is as small as possible, and for all \( f \in \mathcal{H} \), \(|f(x) - y| \leq M \) is true almost everywhere. For all \( \epsilon > 0 \),
\[
\text{prob}_{z \in \mathcal{Z}^m}[|\mathcal{E}(f) - \mathcal{E}(f^p_z)\| \leq \varepsilon] \geq 1 - \mathcal{N}\left(\mathcal{H}, \frac{\varepsilon}{2M} \right) 2e^{-\frac{m\varepsilon^2}{8M}}
\] (15)

Probability bound of the latter, \(\mathcal{E}^p(f) - \mathcal{E}^p(f^p_z)\), can be obtained by the following process.

Suppose  
\[\mu^p_z(f) = \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - P(x_i))\] and  
\[\varepsilon^p_z(f) = \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - P(x_i) - \mu^p_z(f))^2.\]

Because  
\[\mathcal{E}^p(f) - \mathcal{E}^p_z(f) = \int_Z (f(x) - P(x) - \mu^p_z(f))^2 - \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - P(x_i) - \mu^p_z(f))^2\]
where \(\mu^p_z(f)\) and \(\mu^p_z(f)\) are changeless, based on theorem A of reference [1] Theorem 3 can be obtained.

**Theorem 3.** If \(f : X \rightarrow Y\) and \(P(x)\) is an interpretation function, when \(|f(x) - P(x) - \mu^p_z(f)| \leq M_p\) is true almost everywhere for \(M_p > 0\), for all \(\varepsilon_p > 0\), inequality

\[
\text{prob}_{z \in \mathcal{Z}^m}[|\mathcal{E}^p(f) - \mathcal{E}^p_z(f)| \leq \varepsilon_p] \geq 1 - 2e^{-\frac{m\varepsilon_p^2}{2(\sigma_p^2 + M_p^2)}}
\] (16)

holds, where \(\sigma_p^2\) is a variance of \((f(x) - P(x) - \mu^p_z(f))^2\).

Theorem 4 can be derived from theorem 3.

**Theorem 4.** Suppose \(\mathcal{H}\) is a compact subset of \(\mathcal{L}_p^2(X)\), if \(P(x)\) is the interpretation function and for all \(f \in \mathcal{H}\), \(|f(x) - P(x) - \mu^p_z(f)| \leq M_p\) is true almost everywhere, for all \(\varepsilon_p > 0\), inequality

\[
\text{prob}_{z \in \mathcal{Z}^m}\{\sup_{f \in \mathcal{H}}|\mathcal{E}^p(f) - \mathcal{E}^p_z(f)| \leq \varepsilon_p\} \geq 1 - 2\mathcal{N}\left(\mathcal{H}, \frac{\varepsilon_p}{16M_p} \right) e^{-\frac{m\varepsilon_p^2}{4(2\sigma^2_p + 3M_p^2)}}
\] (17)

holds, where \(\sigma_p^2\) is a maximum variance of \((f(x) - P(x) - \mu^p_z(f))^2\),

\[\sigma^2_p = \sigma^2_p(\mathcal{H}) = \sup_{f \in \mathcal{H}} \sigma^2_p \left((f(x) - P(x) - \mu^p_z(f))^2\right)\]

Lemma 5 or Lemma 5* give out a linearly dependent bound with \(\varepsilon\) when \(\mathcal{H}\) is a compact convex subset of \(\mathcal{L}_p^2(X)\). Then, according to Lemma 5 or Lemma 5*, the probability bound of sample error is given.
Lemma 5. Suppose \( \mathcal{H} \) is a compact subset of \( L_p^2(\mathcal{X}) \), if \( \varepsilon, \varepsilon_p > 0 \), \( 0 < \delta < 1 \),
\[
\text{prob}_{x \in \mathcal{Z}} \{ \sup_{f \in \mathcal{H}} |L_x(f)| \leq \varepsilon \} \geq 1 - \delta, \text{ and } \text{prob}_{x \in \mathcal{Z}} \{ \sup_{f \in \mathcal{H}} |L_x^p(f)| \leq \varepsilon_p \} \geq 1 - \delta,
\]
holds.

Lemma 5*. Suppose \( \mathcal{H} \) is a compact subset of \( L_p^2(\mathcal{X}) \), if \( \varepsilon, \varepsilon_p > 0 \), \( 0 < \delta, \delta_p < 1 \),
\[
\text{prob}_{x \in \mathcal{Z}} \{ \sup_{f \in \mathcal{H}} |L_x(f)| \leq \varepsilon \} \geq 1 - \delta, \text{ and } \text{prob}_{x \in \mathcal{Z}} \{ \sup_{f \in \mathcal{H}} |L_x^p(f)| \leq \varepsilon_p \} \geq 1 - \delta_p,
\]
holds.

In Lemma 5, if \( \varepsilon \) is replaced by \( \varepsilon/4 \), based on Theorem 1 and 4, we can obtain the following conclusion.

Theorem 5. Suppose \( \mathcal{H} \) is a compact subset of \( L_p^2(\mathcal{X}) \), for all \( f \in \mathcal{H} \) \( P(x) \) is interpretation function, \( |f(x) - P(x) - \mu^p(f)| \leq M_p \) and \( |f(x) - y| \leq M \) is true almost everywhere. If \( \sigma^2_p = \sigma^2_p(\mathcal{H}) = \sup_{f \in \mathcal{H}} \sigma^2_p((f(x) - P(x) - \mu^p(f))^2) \) and \( \sigma^2 = \sigma^2(\mathcal{H}) = \sup_{f \in \mathcal{H}} \sigma^2(f^2_x), \text{ where } \sigma^2_p((f(x) - P(x) - \mu^p(f))^2) \) is a variance of \( (f(x_i) - P(x_i) - \mu^p(f))^2 \) and \( \sigma^2(f^2_x) \) is a variance of \( f^2_x \), for all \( \varepsilon > 0 \),
\[
\text{prob}_{x \in \mathcal{Z}} \{ |\mathcal{E}_{f_x}^p - \mathcal{E}_{f^p_x}| + |\mathcal{E}(f_x) - \mathcal{E}(f^p_x)| \leq \varepsilon \} \geq \left[ 1 - 2N \left( \mathcal{H}, \frac{\varepsilon}{\delta 4M_p} \right) e^{-\frac{m_x^2}{16(\delta^2 + 1/4M_p^2)}} \right] \left[ 1 - N \left( \mathcal{H}, \frac{\varepsilon}{4M} \right) 2e^{-\frac{m_x^2}{32(16\delta^2 + 1/4M_p^2)}} \right]
\]
(20)

Under the condition of no noise, for all \( f \in L_p^2(\mathcal{X}) \), we have \( \sigma^2(f^2_x) = 0 \). \[1\]
So, \( \sigma^2 = 0 \). Similarly, \( \sigma^2_p = 0 \). Exponents in Theorem 5 turn into \( \frac{3m \varepsilon}{32M^2} \) and \( \frac{3m \varepsilon}{16M^2} \). Theorem 5 give out a linearly dependent bound with \( \varepsilon \). Meanwhile, Theorem 5* can give out the linearly dependent bound with \( \varepsilon \), but need not suppose \( \sigma^2_p = \sigma^2 = 0 \).

Theorem 5*. Suppose \( \mathcal{H} \) is a compact convex subset of \( L_p^2(\mathcal{X}) \). If to all \( f \in \mathcal{H} \), \( P(x) \) is an interpretation function, \( |f(x_i) - P(x_i) - \mu^p(f)| \leq M_p \), and \( |f(x) - y| \leq M \) is true almost everywhere, for all \( \varepsilon_p > 0 \), inequality
\[
\text{prob}_{x \in \mathcal{Z}} \{ |\mathcal{E}_{f_x}^p - \mathcal{E}_{f^p_x}| + |\mathcal{E}(f_x) - \mathcal{E}(f^p_x)| \leq \varepsilon \} \geq \left[ 1 - N \left( \mathcal{H}, \frac{\varepsilon}{8(3M+2M_p)} \right) e^{-\frac{m_x}{32(M^2+M_p)(3M+2M_p)^2}} \right]^2
\]
(21)
4.2 Approximation error estimate

Based on the Hilbert-Schmidt theorem, we can get the theorem 6.

**Theorem 6.** Suppose $\mathcal{H}$ is a Hilbert space, $A$ is a strict positive definite self-adjoint compact operator.

1. If $0 < r \leq s, r \in \mathbb{R}$, for all $a \in \mathcal{H}$, let $\mathcal{L} = \mathbb{I} - \mathcal{G}$, $\Gamma(b - p) = \int (b - p) \, dp$, then
   \[
   \min_{b \in \mathcal{H}} \|b - a\|^2 + \tau \|b - p - \int (b - p) \, dp\|^2 + \gamma \|A^{-s} b\|^2 \leq \|\mathcal{L}^2 + \gamma A^{-2s}\|^{-1} \left[\|\mathcal{L}^2 p - (\mathcal{L}^2 + \gamma A^{-2s}) a\|^2 + \tau \|\mathcal{L}(\mathcal{L}^2 + \gamma A^{-2s})^{-1} [a - (1 + \gamma A^{-2s}) p]\|^2 \right] \tag{22}
   \]
   where $\gamma \leq (r + s)^{\frac{r+s}{s-r}} R^{\frac{2s}{s-r}} (s - r)^{-\frac{r+s}{s-r}} (1 + \mathcal{L}^2)^{-\frac{r+s}{s-r}} \|A^{-r} (a + \gamma A^{-2s} p)\|^{\frac{2s}{s-r}}$.

   In both cases, $b$ is uniquely exists and finite and in the first part, the optimal $b$ is
   \[
   \hat{b} = \left(\mathcal{L}^2 + \gamma A^{-2s}\right)^{-1} (a + \gamma A^{-2s} p).
   \]

   Now, in a Hilbert space, let us introduce a general setting. Suppose $\nu$ is a Borel measure in $X$ and $A: \mathcal{L}^2_X(X) \rightarrow \mathcal{L}^2_X(X)$ is a strict positive definite self-adjoint compact operator, and $\mathbb{E} = \{g \in \mathcal{L}_2^2(X) | \|A^{-s} g\|_\nu < \infty\}$ where $\mathcal{L}^2_\mu(X)$ is a squared integrable function space with Lebesgue measure $\mu$ induced by $\mathbb{R}^n$ quotient space $X$. In $\mathbb{E}$, an inner product is defined as $\langle g, h \rangle_\mathbb{E} = \langle A^{-s} g, A^{-s} h \rangle_\nu$. $\mathbb{E}$ is a Hilbert space. So, $A^{-s}: \mathcal{L}^2_\mathbb{E}(X) \rightarrow \mathbb{E}$ is a Hilbert isomorphism. For the general setting, some supposes should be given. $\mathbb{E} \rightarrow \mathcal{L}^2_\mathbb{E}(X)$ can be decomposed into $\int_{\mathbb{E}}: \mathbb{E} \rightarrow \mathbb{C}(X)$ and $\mathbb{C}(X) \subset \mathcal{L}^2_\mathbb{E}(X)$. Suppose $\mathcal{H} = \mathcal{H}_{E,R}$ is $\int_{\mathbb{E}}(B_R)$, where $B_R$ is a sphere with radius $R$. If $\mathcal{D}_\nu$ is a norm of operator $J: \mathcal{L}^2_\mathbb{E}(X) \rightarrow \mathcal{L}^2_\mathbb{E}(X)$, we can obtain Theorem 7.

**Theorem 7.** In the general setting of a Hilbert space, for $0 < r \leq s, r \in \mathbb{R}$, the approximation error
\[ E(f^k_p) + E^p(f^k_p) = \min_{g(x) \in B_R} \left( \|f^k_p(x) - g(x)\|_p^2 + \tau \|g(x) - P(x) - \mu^p(g)\|_p^2 \right) + \sigma_p^2 \leq \]
\[ D_{\nu \rho}^2 \| (I \rho + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1} \left[ \tau \mathcal{L}^2 P(x) - (\tau \mathcal{L}^2 + \gamma A^{-2s}) f^k_p(x) \right] \|_\nu^2 + \tau D_{\nu \rho}^2 \| \mathcal{L}(I \rho + \tau \mathcal{L}^2 + \\
\gamma A^{-2s})^{-1} \left[ f^k_p(x) - (1 + \gamma A^{-2s}) P(x) \right] \|_\nu^2 + \sigma_p^2 \]

where
\[ \gamma \leq (r + s)^{r+s} R^{-\frac{2s}{s-r}} (s - r)^{-r+s} \left( 1 + \tau \mathcal{L}^2 \right)^{-r+s} \frac{\tau}{s-r} D_{\nu \rho}^2 \left\| \mathcal{A}^{-\tau} \left( f^k_p(x) + \tau \mathcal{L}^2 P(x) \right) \right\|_\nu^{\frac{2s}{s-r}}. \]

**Proof:** We apply Theorem 6 (2). And if \( \mathcal{H} = \mathcal{L}^2(\nu) \), \( a = f^k_p(x) \), and \( p = P(x) \), we get
\[ E(f^k_p) + E^p(f^k_p) = \min_{g(x) \in B_R} \left( \|f^k_p(x) - g(x)\|_p^2 + \tau \|g(x) - P(x) - \mu^p(g)\|_p^2 \right) + \sigma_p^2 \leq \]
\[ D_{\nu \rho}^2 \min_{g(x) \in B_R} \left( \|f^k_p(x) - g(x)\|_\nu^2 + \tau \|g(x) - P(x) - \mu^p(g)\|_\nu^2 \right) + \sigma_p^2 \leq \]
\[ D_{\nu \rho}^2 \| (I \rho + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1} \left[ \tau \mathcal{L}^2 P(x) - (\tau \mathcal{L}^2 + \gamma A^{-2s}) f^k_p(x) \right] \|_\nu^2 + \tau D_{\nu \rho}^2 \| \mathcal{L}(I \rho + \tau \mathcal{L}^2 + \\
\gamma A^{-2s})^{-1} \left[ f^k_p(x) - (1 + \gamma A^{-2s}) P(x) \right] \|_\nu^2 + \sigma_p^2 \]

where
\[ \gamma \leq (r + s)^{r+s} R^{-\frac{2s}{s-r}} (s - r)^{-r+s} \left( 1 + \tau \mathcal{L}^2 \right)^{-r+s} \frac{\tau}{s-r} D_{\nu \rho}^2 \left\| \mathcal{A}^{-\tau} \left( f^k_p(x) + \tau \mathcal{L}^2 P(x) \right) \right\|_\nu^{\frac{2s}{s-r}}. \]

\[ \square \]

If \( \nu = \rho \), \( D_{\nu \rho} = 1 \).

### 4.3 Approximation error estimate in Sobolev space and RKHS

In the section, suppose \( X \subset \mathbb{R}^n \) is a compact region with smooth boundary.

**Theorem 8.** If \( 0 < r < s \), \( B_R \) is a sphere with radius \( R \) in a conjugate space \( H(X) \) on \( X \), and \( \mathcal{H} = f_{H(X)}(B_R) \), the approximation error is
\[ (f^k_p) + E^p(f^k_p) \leq \mathcal{D}_{\nu \rho}^2 \| (I \rho + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1} \left[ \tau \mathcal{L}^2 P(x) - (\tau \mathcal{L}^2 + \gamma A^{-2s}) f^k_p(x) \right] \|_\nu^2 + \\
\tau \mathcal{D}_{\nu \rho}^2 \| \mathcal{L}(I \rho + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1} \left[ f^k_p(x) - (1 + \gamma A^{-2s}) P(x) \right] \|_\nu^2 + \sigma_p^2 \]

where \( \gamma \leq (r + s)^{\frac{r+s}{s-r}} (r - s)^{-r+s} \left( 1 + \tau \mathcal{L}^2 \right)^{-r+s} \frac{\tau}{s-r} \mathcal{D}_{\nu \rho}^2 \left\| \mathcal{A}^{-\tau} \left( f^k_p(x) + \tau \mathcal{L}^2 P(x) \right) \right\|_\nu^{\frac{2s}{s-r}} \), and \( \mathcal{C} \) is a constant only depends on \( r, X \).
Proof: Suppose $\Delta : H^2(X) \to L^2_2(X)$ is a Laplacian operator, and $A = (-\Delta + Id)^{-1/2}$. For all $\eta \geq 0$, $A^\eta : L^2_2(X) \to H^0(X)$ is a compact linear mapping with bounded inverse. If $C_0, C_1 > 0$, for all $g \in H^0(X)$,

$$C_0 \left\| g \right\|_v \leq \left\| A^{-\eta} g \right\|_v \leq C_1 \left\| g \right\|_v$$  \hspace{1cm} (26)

Because $H^1(X) \subset L^2_2(X)$, we can think $A : L^2_2(X) \to L^2_2(X)$. Then the general setting of a Hilbert space is considered.

If $E$ is a space in the general setting, a sphere $B_{RC_0}(E)$ with radius $RC_0$ in $E$ is contained in a sphere $B_{\rho}(H(X))$ in $H(X)$. Then

$$\mathcal{E}(f_{\rho}^p) + \mathcal{E}(f_{\rho}^p) = \min_{g(x) \in B_{\rho}(H(X))} \left( \| f_\rho(x) - g(x) \|_\rho^2 + \tau \| g(x) - P(x) - \mu^p(g) \|_\rho^2 \right) + \sigma_\rho^2 = \min_{g(x) \in B_{R C_0}(E)} \left( \| f_\rho(x) - g(x) \|_\rho^2 + \tau \| g(x) - P(x) - \mu^p(g) \|_\rho^2 \right) + \sigma_\rho^2$$  \hspace{1cm} (27)

From Theorem 7, we can obtain

$$\min_{g(x) \in B_{R C_0}(E)} \left( \| f_\rho(x) - g(x) \|_\rho^2 + \tau \| g(x) - P(x) - \mu^p(g) \|_\rho^2 \right) + \sigma_\rho^2 \leq \mathcal{D}^2_{\nu \rho} \left\| (I \tau + \gamma A^{-2s})^{-1} \left[ \tau L^2 P(x) - \gamma \gamma A^{-2s} f_\rho(x) \right] \right\|_v^2 + \tau D^2_{\nu \rho} \left\| L(I \tau + \gamma A^{-2s})^{-1} f_\rho(x) - (1 + \gamma A^{-2s}) P(x) \right\|_v^2 + \sigma_\rho^2$$  \hspace{1cm} (28)

where $\gamma \leq (r + s)^{2s - r} (s - r)^{-r - s - i} \left( 1 + \tau L^2 \right)^{r + s} \left( D^2_{\nu \rho} \right)^{2s - r - i} \tau^2 A^{-r} \left( f_\rho(x) + \tau L^2 P(x) \right) \|_v^{2s - r - i}$.

Finally, Eq.(26) is applied, and if $\eta = r$, we can get

$$\left\| A^{-r} \left( f_\rho(x) + \tau L^2 P(x) \right) \right\|_v \leq C_1 \left\| f_\rho(x) + \tau L^2 P(x) \right\|_v \hspace{1cm} (29)$$

If $C = C_0^{-2} C_1^2$, $\gamma \leq (r + s)^{2s - r} (R C)^{-s - r} \left( 1 + \tau L^2 \right)^{r + s} \left( D^2_{\nu \rho} \right)^{2s - r - i} \tau^2 A^{-r} \left( f_\rho(x) + \tau L^2 P(x) \right) \|_v^{2s - r - i}$ holds. \hfill \box

In a reproducing kernel Hilbert space (RKHS), we can get Theorem 9.

**Theorem 9.** Suppose $K$ is a Mercer kernel function, $\nu$ is a Borel measure on $X$, $R > 0$, $\mathcal{H} = L_K(B_R)$. For $0 < r < s$, the approximation error is

$$\mathcal{E}(f_{\rho}^p) + \mathcal{E}(f_{\rho}^p) \leq \mathcal{D}_{\nu \rho} \left\| (I \tau + \gamma A^{-2s})^{-1} \left[ \tau L^2 P(x) - \gamma A^{-2s} f_\rho(x) \right] \right\|_v^2 + \tau D^2_{\nu \rho} \left\| L(I \tau + \gamma A^{-2s})^{-1} f_\rho(x) - (1 + \gamma A^{-2s}) P(x) \right\|_v^2 + \sigma_\rho^2$$  \hspace{1cm} (30)
where $\gamma \leq (r + s)^{rt+2s}R^{-r+2s}(s - r)^{-r+2s}(1 + \tau\ell^2)^{\frac{r+2s}{s-r}}D_{\nu\rho}^2 \left\| L^{-r/2}_K \left( f_\nu(x) + \tau\ell^2 P(x) \right) \right\|_{\nu}^{\frac{2s}{s-r}}$.

**Proof:** In Theorem 7, suppose $A = L_{K}^{1/2}$. Then, from Mercer theorem, for all $f \in L_{\nu}^{2}(\mathcal{X})$, according the general setting, kernel norm $\|f\|_K = \|A^{-1/2}f\|_\nu$. Theorem 7 is applied. Then if $s = 1$,

$$
\mathcal{E}(f_\nu^P) + \mathcal{E}_P(f_\nu^P) \leq D_{\nu\rho}^2 \left\| (I + \tau\ell^2 + \gamma L_{K}^{-1})^{-1} (\tau\ell^2 P(x) - (\tau\ell^2 + \gamma L_{K}^{-1})f_\nu(x)) \right\|_\nu^2 + \\
\tau D_{\nu\rho}^2 \left\| (I + \tau\ell^2 + \gamma L_{K}^{-1})^{-1} [f_\nu(x) - (1 + \gamma L_{K}^{-1})P(x)] \right\|_\nu^2 + \sigma_\rho^2
$$

(31)

where $\gamma \leq (r + s)^{rt+2s}R^{-r+2s}(s - r)^{-r+2s}(1 + \tau\ell^2)^{\frac{r+2s}{s-r}}D_{\nu\rho}^2 \left\| L^{-r/2}_K \left( f_\nu(x) + \tau\ell^2 P(x) \right) \right\|_{\nu}^{\frac{2s}{s-r}}$.

\[ \square \]

5. How to solve this new learning problem

According the general setting in section 4.2, suppose sample size is $m$ and the confidence is $1 - \delta, 0 < \delta < 1$. For every $R > 0$, hypothesis space $\mathcal{H} = \mathcal{H}_{E,R}$. We consider $f_\nu^P$ and $f_\nu, z \in Z^m$. In the general setting, the optimal solution of the new learning problem (6) is how to find an optimal $R, M$ and $M_P$, where $|f(x) - P(x) - \mu^P(f)| \leq M_P$, and $|f(x) - y| \leq M$ are true almost everywhere.

**Theorem 10.** For all $m \in \mathbb{N}, \delta \in \mathbb{R}, 0 < \delta < 1$, and $r \in \mathbb{R}, 0 < r < s$, in the general setting, the optimal $R^*, M^*$ and $M^*_P$, can be found in the learning framework for improving the interpretability of a predication model.

**Proof:** we know that $\mathcal{E}(f_\nu) + \mathcal{E}_P(f_\nu) = \mathcal{E}_\nu(f_\nu) + \mathcal{E}(f_\nu^P) + \mathcal{E}_P(f_\nu^P)$. For $0 < r < s$, Theorem 7 provides probability bound of the approximation error,

$$
\mathcal{E}(f_\nu^P) + \mathcal{E}_P(f_\nu^P) \leq D_{\nu\rho}^2 \left\| (I + \tau\ell^2 + \gamma A^{-2s})^{-1} [\tau\ell^2 P(x) - (\tau\ell^2 + \gamma A^{-2s})f_\nu(x)] \right\|_\nu^2 + \\
\tau D_{\nu\rho}^2 \left\| (I + \tau\ell^2 + \gamma A^{-2s})^{-1} [f_\nu(x) - (1 + \gamma A^{-2s})P(x)] \right\|_\nu^2 + \sigma_\rho^2
$$

(32)

where $\gamma \leq (r + s)^{rt+2s}R^{-r+2s}(s - r)^{-r+2s}(1 + \tau\ell^2)^{\frac{r+2s}{s-r}}D_{\nu\rho}^2 \left\| A^{-r} \left( f_\nu(x) + \tau\ell^2 P(x) \right) \right\|_{\nu}^{\frac{2s}{s-r}}$.

For obtaining the probability bound of the sample error, given $M = M(R) = \|f_\nu\|_R + M_P + \|f_\nu\|_\infty$. 


\[ |f(x) - y| \leq |f(x)| + |y| \leq |f(x)| + |y - f_\mu(x)| + |f_\mu(x)| \leq \|f\|_E R + M_\mu + \|f_\mu\|_\infty \]  \hspace{1cm} (33)

Therefore, \(|f(x) - y| \leq M\) is true almost everywhere. So similarly, \(|f(x) - P(x) - \mu^\prime(f)| \leq M_\mu\) is true almost everywhere too. From Theorem 5* and Birman and Solomyak’s work[25], the sample error \(\varepsilon > 0\) in confidence \(1 - \delta\) meets

\[ \mathcal{N}(\mathcal{H}_{\frac{E}{2(M+2M_\mu)^2}}) e^{-\frac{m E}{32(M+2M_\mu)^2(M^2+M_\mu^2)}} \geq \delta \]  \hspace{1cm} (34)

And according to Proposition 6 of section 1.6 in reference [1] we get

\[ \frac{m E M^2}{32(M+2M_\mu)^2(M^2+M_\mu^2)} + \ln \frac{1}{\delta} - \left( \frac{B(3M+2M_\mu)R C E}{E M^2 \|f\|_E} \right)^{1/\ell_E} \leq 0 \]  \hspace{1cm} (35)

Suppose \(R \|f\|_E \leq \frac{(3M+2M_\mu)(M^2+M_\mu^2)}{M^2}\), we can obtain

\[ \frac{m E M^2}{32(M+2M_\mu)^2(M^2+M_\mu^2)} + \ln \frac{1}{\delta} - \left( \frac{B(3M+2M_\mu)^2(M^2+M_\mu^2) C E}{E M^2 \|f\|_E} \right)^{1/\ell_E} \leq 0 \]  \hspace{1cm} (36)

If \(\theta = \frac{E M^2}{(3M+2M_\mu)^2(M^2+M_\mu^2)}\), then

\[ \frac{m}{32} \theta + \ln \frac{1}{\delta} - \left( \frac{B C E}{\|f\|_E} \right)^{1/\ell_E} \leq 0 \]  \hspace{1cm} (37)

where \(\ell_E > \frac{1}{2}\) which is related to \(n\) and \(s\) of \(H^s(X)\), and \(s > n/2, \ell_E = \frac{s}{n} > \frac{1}{2}\).

If \(c_0 = \frac{m}{32}, c_1 = \ln \frac{1}{\delta}, c_2 = \left( \frac{B C E}{\|f\|_E} \right)^{1/\ell_E}, d = 1/\ell_E\), then we can obtain

\[ c_0 \theta + c_1 - c_2 \theta^{-d} \leq 0 \]  \hspace{1cm} (38)

The first derivative of the left-hand side of the above formula is \(c_0 + dc_2 \theta^{-d-1}\). The second derivative is \(-d(d-1)c_2 \theta^{-d-2}\). Therefore, given \(d \geq 1(\ell_E \leq 1)\), \(c_0 \theta + c_1 - c_2 \theta^{-d}\) is a monotonic increase convex function; If \(d < 1(\ell_E > 1)\), \(c_0 \theta + c_1 - c_2 \theta^{-d}\) is a monotonic increase concave function. In a word, if \(c_0 + dc_2 \theta^{-d-1} = 0\), a solution \(\theta^*(m, \delta)\) about \(\theta\) can be found. And

\[ \varepsilon(R, M, M_\mu) = (3M + 2M_\mu)^2 \left(1 + \frac{M_\mu^2}{M^2}\right) \theta^*(m, \delta) \]  \hspace{1cm} (39)

is the optimal bound of sample error from Theorem 5*.

From Theorem 7, we can obtain
\[
\alpha(R, M, M_p) \leq D_{\nu p}^2 \left\| (I + \tau L^2 + \gamma A^{-2s})^{-1} \left[ \tau L^2 P(x) - (\tau L^2 + \gamma A^{-2s}) f_\rho(x) \right] \right\|_v^2 + \tau D_{\nu p}^2 \left\| L(I + \tau L^2 + \gamma A^{-2s})^{-1} \left[ f_\rho(x) - (1 + \gamma A^{-2s}) P(x) \right] \right\|_v^2 + \sigma^2_p
\]

(40)

where \( \gamma \leq (r + s) \frac{r + s}{s - r} \) \( s - r \) \( r + s \) \( s - r \) \( r + s \) \( s - r \) \( r + s \) \( s - r \) \( r + s \) \( s - r \).

It can be seen from the proof of Theorem 6 that

\[
\left\| (I + \tau L^2 + \gamma A^{-2s})^{-1} \left[ \tau L^2 P(x) - (\tau L^2 + \gamma A^{-2s}) f_\rho(x) \right] \right\|_v^2 \leq M^2
\]

(41)

\[
\left\| L(I + \tau L^2 + \gamma A^{-2s})^{-1} \left[ f_\rho(x) - (1 + \gamma A^{-2s}) P(x) \right] \right\|_v^2 \leq M_p^2
\]

(42)

We can get \( \alpha(R, M, M_p) \leq D_{\nu p}^2 M^2 + \tau D_{\nu p}^2 M_p^2 + \sigma^2_p \). And

\[
\alpha(R, M, M_p) + \varepsilon(R, M, M_p) \leq D_{\nu p}^2 M^2 + \tau D_{\nu p}^2 M_p^2 + (3M + 2M_p) \left( 1 + \frac{M_p^2}{M^2} \right) \theta^*(m, \delta) + \sigma^2_p
\]

(43)

And \( R \left\| f_E \right\| \leq \frac{(3M + 2M_p)(M^2 + M_p^2)}{M^2} \). Because \( \left\| f_E \right\|_E \left( R + M_p + \left\| f_\rho \right\|_\infty \right) = M \), then

\[
M^2 \left( M - M_p - \left\| f_\rho \right\|_\infty \right) \leq (3M + 2M_p)(M^2 + M_p^2)
\]

(44)

The right-hand side of the above formula shrinks as

\[ 2M^2 + \left( M_p + \left\| f_\rho \right\|_\infty \right) M + 3M_p^2 \leq 0 \]

(45)

The left-hand side of the above formula is a quadratic function of \( M \). We always find an optimal equality relationship between \( M \) and \( M_p \), which is always going to make the equality hold up.

\[
M = \frac{-\left( M_p + \left\| f_\rho \right\|_\infty \right) + \sqrt{\left( M_p + \left\| f_\rho \right\|_\infty \right)^2 - 24M_p^2}}{4}
\]

(46)

It can be know from the above equation that \( M \) and \( M_p \) cannot always go down at the same time. Under the condition, \( M = \frac{-\left( M_p + \left\| f_\rho \right\|_\infty \right) + \sqrt{\left( M_p + \left\| f_\rho \right\|_\infty \right)^2 - 24M_p^2}}{4} \), if \( M \) and \( M_p \) go down simultaneous, the sample error and the approximation error will also be reduced. But there are contradiction between \( M \) and \( M_p \). One of them goes down, which is going to cause another increase.
The optimal $R^*$, $M^*$ and $M^*_p$, can be found in the learning framework for improving the interpretability of the predication model. And $\|E\|R^* + M_p + \|f_p\|_\infty = M^*$ and $M^* = \frac{-(M_p+\|f_p\|_\infty) + \sqrt{(M_p+\|f_p\|_\infty)^2 - 24M^*_p^2}}{4}$ will minimize $a(R, M, M_p) + \varepsilon(R, M, M_p)$. \hfill $\Box$

6. The Proof of Main results

In section, we will prove Theorem 4 , Theorem 6, Lemma 5 and Lemma 5*. Firstly, we give the following Lemma 6.

Suppose $f_1, f_2 \in L^2_p(Z)$, firstly, for all $z \in Z^m$, $|L^p_z(f_1) - L^p_z(f_2)|$ is estimated, where $L^p_z(f) = \mathcal{E}^p(f) - \mathcal{E}^p_z(f)$ is Lipshitz estimation. Then Lemma 6 can be obtained.

**Lemma 6.** In a completely measurable set $U \subset Z$, if $|f_j(x) - P(x) - \mu^p_z(f_j)| \leq M_p, j = 1, 2$, for $z \in U^m$,

$$|L^p_z(f_1) - L^p_z(f_2)| \leq 8M_p \|f_1 - f_2\|_\infty$$

**Proof:** Because

$$\left(f_1(x) - P(x) - \mu^p(f_1)\right)^2 - \left(f_2(x) - P(x) - \mu^p(f_2)\right)^2 = |f_1(x) - f_2(x) - \mu^p(f_1) + \mu^p(f_2)||f_1(x) - P(x) - \mu^p(f_1) + (f_2(x) - P(x) - \mu^p(f_2))|,$$

we get

$$|\mathcal{E}^p(f_1) - \mathcal{E}^p(f_2)| \leq 2\|f_1 - f_2\|_\infty \int_Z |f_1(x) - P(x) - \mu^p(f_1) + (f_2(x) - P(x) - \mu^p(f_2))| \leq 4\|f_1 - f_2\|_\infty M_p.$$  \hfill (49)

So,

$$\left|\mathcal{E}^p_z(f_1) - \mathcal{E}^p_z(f_2)\right| = \left|\frac{1}{m} \sum_{i=1}^m (f_1(x_i) - P(x_i) - \mu^p(f_1))\right|^2 - \left|\frac{1}{m} \sum_{i=1}^m (f_2(x_i) - P(x_i) - \mu^p(f_2))\right|^2 \leq 4\|f_1 - f_2\|_\infty M_p.$$  \hfill (50)

And

$$|L^p_z(f_1) - L^p_z(f_2)| = \left|\mathcal{E}^p(f_1) - \mathcal{E}^p_z(f_1) - \mathcal{E}^p(f_2) + \mathcal{E}^p_z(f_2)\right| \leq 8M_p \|f_1 - f_2\|_\infty.$$ \hfill $\Box$
If $\mathcal{H} \subseteq \mathcal{L}_p^2(X)$, and for all $f \in \mathcal{H}$, $|f(x) - P(x) - \mu^p(f)| \leq M_p$ is true almost everywhere, the inequalities $|E^p(f_1) - E^p(f_2)| \leq 4M_p\|f_1 - f_2\|_\infty$ and $|E^p(f_1) - E^p(f_2)| \leq 4M_p\|f_1 - f_2\|_\infty$ indicate that $E^p, E^p: \mathcal{H} \to \mathbb{R}$ is continuous.

**Lemma 7.** If $\mathcal{H} = S_1 \cup \cdots \cup S_\ell$ and $\epsilon > 0$, then

$$\text{prob}_{z \in \mathbb{Z}^m}\{\sup_{f \in \mathcal{H}}|L^p_z(f)| \leq \epsilon_p\} \leq \sum_{j=1}^\ell \text{prob}_{z \in \mathbb{Z}^m}\{\sup_{f \in \mathcal{S}_j}|L^p_z(f)| \leq \epsilon_p\}$$

(51)

**Proof:** Because of the equivalence property

$$\sup_{f \in \mathcal{H}}|L^p_z(f)| \geq \epsilon_p \Leftrightarrow \exists j \leq \ell \text{ s.t.} \sup_{f \in \mathcal{S}_j}|L^p_z(f)| \geq \epsilon_p$$

(52)

and a fact, that the union probability of some events is bounded by the sum of the probabilities of these events, the result of the lemma can be obtained. $\square$

**Proof of Theorem 4.** If $\ell_p = N_p\left(\mathcal{H}, \frac{\epsilon_p}{8M_p}\right)$, there some disks $D_j, j = 1,2, \ldots, \ell_p$, to cover $\mathcal{H}$, which makes $f_j$ as the center and $\frac{\epsilon_p}{8M_p}$ as its radius. In $U$ which is a completely measurable set, we have $|f(x_i) - P(x_i) - \mu^p(f)| \leq M_p$. From Lemma 6, for all $z \in U^m$ and $f \in D_j$, we have

$$|L^p_z(f_1) - L^p_z(f_2)| \leq 8M_p\|f_1 - f_2\|_\infty \leq 8M_p\frac{\epsilon_p}{8M_p} = \epsilon_p.$$ 

(53)

Then,

$$\sup_{f \in D_j}|L^p_z(f)| \geq 2\epsilon_p \Rightarrow |L^p_z(f_j)| \geq \epsilon_p.$$ 

(54)

Therefore, for $j = 1, \ldots, \ell_p$, from Theorem 3 we can obtain

$$\text{Prob}_{x \in \mathbb{Z}^m}\left\{\sup_{f \in D_j}|E^p(f) - E^p_z(f)| \geq 2\epsilon_p\right\} \leq \text{prob}_{x \in \mathbb{Z}^m}\{|L^p_z(f_j)| \geq \epsilon_p\} \leq 2e^{-\frac{m\epsilon^2}{8M_p}}.$$ 

(55)

Now, we replace $\epsilon_p$ with $\epsilon_p/2$, from Lemma 7 we obtain the following conclusion.

$$\text{Prob}_{x \in \mathbb{Z}^m}\left\{\sup_{f \in \mathcal{H}}|E^p(f) - E^p_z(f)| \leq \epsilon_p\right\} \geq 1 - \mathcal{N}\left(\mathcal{H}, \frac{\epsilon_p}{16M_p}\right) 2e^{-\frac{m\epsilon^2}{32M_p}}.$$ 

$\square$

**Proof of Lemma 5.** In accordance with the probability of at least $(1-\delta)^2$, we have

$$|E(f_2) - E_z(f_2)| \leq \delta \Rightarrow |E^p(f_2) - E^p_z(f_2)| \leq \epsilon_p,$$

(56)
Therefore,
\[ \mathcal{E}(f_z) + \mathcal{E}^p(f_z) \leq \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) + \epsilon_p + \epsilon \]  \hspace{1cm} (57)

In the same way, we have
\[ \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) \leq \mathcal{E}(f_z) + \mathcal{E}^p(f_z) + \epsilon_p + \epsilon \]  \hspace{1cm} (58)

Moreover, since in \( \mathcal{H} \) \( f_z \) minimizes \( \mathcal{E}_z \), we have
\[ \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) \leq \mathcal{E}_z(f_{\mathcal{H}}^p) + \mathcal{E}_z^p(f_{\mathcal{H}}^p) \]  \hspace{1cm} (59)

So, in accordance with the probability of at least \((1 - \delta)^2\), we get
\[ \mathcal{E}(f_z) + \mathcal{E}^p(f_z) \leq \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) + \epsilon_p + \epsilon \leq \mathcal{E}_z(f_{\mathcal{H}}^p) + \mathcal{E}_z^p(f_{\mathcal{H}}^p) + \epsilon_p + \epsilon \leq \mathcal{E}(f_{\mathcal{H}}^p) + \mathcal{E}^p(f_{\mathcal{H}}^p) + 2\epsilon_p + 2\epsilon \]  \hspace{1cm} (60)

Therefore, \( \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) \leq 2(\epsilon_p + \epsilon) \)

\textbf{Proof of Lemma 5*}. In accordance with the probability of at least \((1 - \delta)(1 - \delta^p)\), we have
\[ |\mathcal{E}(f_z) - \mathcal{E}_z(f_z)| \leq \epsilon \] \hspace{0.5cm} and \hspace{0.5cm} \[ |\mathcal{E}^p(f_z) - \mathcal{E}_z^p(f_z)| \leq \epsilon \],  \hspace{1cm} (61)

Therefore
\[ \mathcal{E}(f_z) + \mathcal{E}^p(f_z) \leq \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) + 2\epsilon. \]  \hspace{1cm} (62)

In the same way, we have
\[ \mathcal{E}_z(f_{\mathcal{H}}^p) + \mathcal{E}_z^p(f_{\mathcal{H}}^p) \leq \mathcal{E}(f_{\mathcal{H}}^p) + \mathcal{E}^p(f_{\mathcal{H}}^p) + 2\epsilon. \]  \hspace{1cm} (63)

Moreover, since in \( \mathcal{H} \) \( f_z \) minimizes \( \mathcal{E}_z \), we have
\[ \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) \leq \mathcal{E}_z(f_{\mathcal{H}}^p) + \mathcal{E}_z^p(f_{\mathcal{H}}^p). \]  \hspace{1cm} (64)

So, in accordance with the probability of at least \((1 - \delta)(1 - \delta^p)\), we get
\[ \mathcal{E}(f_z) + \mathcal{E}^p(f_z) \leq \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) + 2\epsilon \leq \mathcal{E}_z(f_{\mathcal{H}}^p) + \mathcal{E}_z^p(f_{\mathcal{H}}^p) + 2\epsilon \leq \mathcal{E}(f_{\mathcal{H}}^p) + \mathcal{E}^p(f_{\mathcal{H}}^p) + 4\epsilon. \]  \hspace{1cm} (65)

Therefore, \( \mathcal{E}_z(f_z) + \mathcal{E}_z^p(f_z) \leq 4 \epsilon \)

\textbf{Proof of Theorem 5*}. To prove Theorem 5*, we have to prove Lemma 8.
**Lemma 8.** Given $\mathcal{H}$ is a compact convex subset of $\mathcal{L}_\rho^2(\mathcal{X})$ which can ensure that the interpretation distance between $f_{\mathcal{H}}^p$ and $P(x)$ is as small as possible, then for all $f \in \mathcal{H}$,

$$\int_Z \left( |f_{\mathcal{H}}^p(x) - f(x) - \mu(f, f_{\mathcal{H}}^p)| + |f_{\mathcal{H}}^p(x) - f(x)| \right)^2 \leq \varepsilon_{\mathcal{H}}(f)$$

where $\varepsilon_{\mathcal{H}}(f) = \varepsilon(f) - \varepsilon(f_{\mathcal{H}}^p) + \varepsilon^p(f) - \varepsilon^p(f_{\mathcal{H}}^p)$.

**Proof:** Firstly, we consider the first case.

\[ \varepsilon^p(f) - \varepsilon^p(f_{\mathcal{H}}^p) = \int_z (f_{\mathcal{H}}^p(x) - f(x) - \mu(f, f_{\mathcal{H}}^p))^2 - 2 \int_z f_{\mathcal{H}}^p(x) - f(x) - \mu(f, f_{\mathcal{H}}^p) \left\| f_{\mathcal{H}}^p(x) - P(x) - \mu^p(f_{\mathcal{H}}^p) \right\| \cos \angle P f_{\mathcal{H}}^p f \] (67)

And

\[ \varepsilon(f) - \varepsilon(f_{\mathcal{H}}^p) = \int_z (f_{\mathcal{H}}^p(x) - f(x))^2 - 2 \int_z f_{\mathcal{H}}^p(x) - f(x) \left\| f_{\mathcal{H}}^p(x) - f_{\mu}(x) \right\| \cos \angle f_{\mu} f_{\mathcal{H}} f \] (68)

From Eq.(67) and Eq.(68), we get
\[ \mathcal{E}_H(f) = \mathcal{E}(f) - \mathcal{E}(f_{\mu_1}) + \mathcal{E}(f_{\mu_2}) - \mathcal{E}(f_{\mu_1}) \]
\[ = \int_Z (f_{\mu_1}(x) - f(x))^2 + \]
\[ 2 \int_Z \left( |f_{\mu_1}(x) - f(x)| - |\hat{f}(x) - f_{\mu_1}(x) - \mu(f_{\mu_1})| \right) \]
\[ + \int_Z (f_{\mu_2}(x) - f(x) - \mu(f, f_{\mu_1}))^2 \]

(69)

Suppose \( C_1 = |f_{\mu_1}(x) - P(x) - \mu(f_{\mu_1})| \) and \( C_2 = |f_{\mu_1}(x) - f(x)| \), we have
\[
\mathcal{E}_H(f) = \int_Z \left( |f_{\mu_1}(x) - f(x)| + |f_{\mu_1}(x) - f(x) - \mu(f_{\mu_1})| \right)^2 + \]
\[ 2 \int_Z \left( |f_{\mu_1}(x) - f(x)| C_2 - |f_{\mu_1}(x) - f(x) - \mu(f_{\mu_1})| C_1 \right) - \]
\[ 2 \int_Z |f_{\mu_1}(x) - f(x)| f_{\mu_1}(x) - f(x) - \mu(f, f_{\mu_1}) | \]

(70)

Since \( \mathcal{E}_H(f) \) is greater than zero, when \( f \) infinite close to \( f_{\mu_1} \), we have
\[ 2 \int_Z \left( |f_{\mu_1}(x) - f(x)| C_2 - |f_{\mu_1}(x) - f(x) - \mu(f, f_{\mu_1})| C_1 - |f_{\mu_1}(x) - f(x) - \mu(f_{\mu_1})| \right) \geq 0 \]

Therefore,
\[ \mathcal{E}_H(f) \geq \int_Z \left( |f_{\mu_1}(x) - f(x) - \mu(f, f_{\mu_1})| + |f_{\mu_1}(x) - f(x)| \right)^2 \]

(71)

For the second case,
We can easily obtain

\[
\mathcal{E}_\mathcal{H}(f) \geq \int_Z \left( |f_{\mathcal{H}}^p(x) - f(x) - \mu(f, f_{\mathcal{H}}^p)| + |f_{\mathcal{H}}(x) - f(x)| \right)^2 \quad (72)
\]

The proof of Lemma 8 is completed. \qed

Let us focus on the function \( \ell(f) : \mathbb{Z} \to \mathbb{Y} \), \( \ell(f) = (f(x) - y)^2 - (f_{\mathcal{H}}^p(x) - y)^2 + (f(x) - P(x) - \mu^p(f))^2 - (f_{\mathcal{H}}^p(x) - P(x) - \mu^p(f_{\mathcal{H}}^p))^2 \). It can be abbreviated to

\[
\ell(f) = (f - y)^2 - (f_{\mathcal{H}}^p - y)^2 + (f - P - \mu^p(f))^2 - (f_{\mathcal{H}}^p - P - \mu^p(f_{\mathcal{H}}^p))^2 \quad (73)
\]

Then, \( \mathbb{E}\ell(f) = \mathcal{E}_\mathcal{H}(f) = \mathcal{E}(f) - \mathcal{E}(f_{\mathcal{H}}^p) + \mathcal{E}^p(f) - \mathcal{E}^p(f_{\mathcal{H}}^p) \), and for \( z \in \mathbb{Z}^m \), \( \mathbb{E}_z \ell(f) = \mathcal{E}_{\mathcal{H}, z}(f) = \mathcal{E}_z(f) - \mathcal{E}_z(f_{\mathcal{H}}^p) + \mathcal{E}^p_z(f) - \mathcal{E}^p_z(f_{\mathcal{H}}^p) \). Moreover, we know that for all \( f \in \mathcal{H} \), \( |\ell(f)(x, y)| \leq M^2 + M_P^2 \) is true almost everywhere.

Suppose \( \sigma^2 = \sigma^2(\ell(f)) \) is a variance of \( \ell(f) \). We can obtain Lemma 9.

**Lemma 9.** For all \( f \in \mathcal{H} \), \( \sigma^2 \leq 4 \max (M^2, M_P^2) \mathcal{E}_\mathcal{H}(f) \).

**Proof:** Since
\[ \sigma^2 \leq \text{Var}(f) = \text{E}\left[(f - y)^2 - (f - \mu(f))^2\right] \leq \text{E}\left[2M(f - f_\mathcal{H}^p) + 2M_p(f - f_\mathcal{H}^p - \mu(f) + \mu'(f_\mathcal{H}^p))\right] \leq 4\max(M^2, M_p^2)\text{E}\left[(f - f_\mathcal{H}^p) + (f - f_\mathcal{H}^p - \mu(f) + \mu'(f_\mathcal{H}^p))\right]^2 \] (74)

And because \( \text{E}\left[(f - f_\mathcal{H}^p) + (f - f_\mathcal{H}^p - \mu(f) + \mu'(f_\mathcal{H}^p))\right]^2 \leq \mathcal{E}_\mathcal{H}(f) \), Lemma 9 is true. □

**Lemma 10.** If \( f \in \mathcal{H} \), for all \( \varepsilon > 0, 0 < \alpha \leq 1 \),

\[
\text{Prob}_{z \in \mathbb{Z}^m}\left\{ \frac{\mathcal{E}_\mathcal{H}(f)-\mathcal{E}_\mathcal{H}(f)}{\mathcal{E}_\mathcal{H}(f)+\varepsilon} \geq \alpha \right\} \leq e^{-\frac{\alpha^2 M \varepsilon}{8(M^2 + M_p^2)}}
\] (75)

**Proof:** If \( \mu = \mathcal{E}_\mathcal{H}(f) \), the unilateral Bernstein's inequality is applied to \( \ell(f) \), and \( |\ell(f)(z)| \leq M^2 + M_p^2 \) is true almost everywhere. We have

\[
\text{Prob}_{z \in \mathbb{Z}^m}\left\{ \frac{\mathcal{E}_\mathcal{H}(f)-\mathcal{E}_\mathcal{H}(f)}{\mathcal{E}_\mathcal{H}(f)+\varepsilon} \geq \alpha \right\} \leq e^{-\frac{\alpha^2 M \varepsilon}{8(M^2 + M_p^2)}}
\] (76)

We just have to prove the inequality.

\[
\frac{\varepsilon}{8(M^2 + M_p^2)} \leq \frac{(\mu + \varepsilon)^2}{2\left(\sigma^2 + \frac{\varepsilon}{3}(M^2 + M_p^2)\alpha(\mu + \varepsilon)\right)}
\] (77)

\[
\Leftrightarrow \frac{\varepsilon}{4(M^2 + M_p^2)} \left(\sigma^2 + \frac{1}{3}(M^2 + M_p^2)\alpha(\mu + \varepsilon)\right) \leq (\mu + \varepsilon)^2
\] (78)

\[
\Leftrightarrow \frac{\varepsilon}{4(M^2 + M_p^2)} + \frac{\varepsilon \alpha}{12} + \frac{\varepsilon^2}{12} \leq (\mu + \varepsilon)^2
\] (79)

Since \( 0 < \alpha \leq 1 \), bounds of the second term and the third term on the left side of the above inequality are \( \varepsilon \mu \) and \( \varepsilon^2 \) respectively. From Lemma 6, \( 4\max(M^2, M_p^2)\mu \) is the bound of \( \sigma^2 \). So, the first term is less than \( \varepsilon \mu \), and \( 2\varepsilon \mu + \varepsilon^2 \leq (\mu + \varepsilon)^2 \). □

**Lemma 11.** Given \( 0 < \alpha < 1, \varepsilon > 0, f \in \mathcal{H} \), we have

\[
\frac{\mathcal{E}_\mathcal{H}(f)-\mathcal{E}_\mathcal{H}(f)}{\mathcal{E}_\mathcal{H}(f)+\varepsilon} < \alpha
\] (80)

For all \( g \in \mathcal{H} \), \( \|f - g\|_\infty \leq \frac{\alpha \varepsilon}{4M} \), we have

\[
\frac{\mathcal{E}_\mathcal{H}(g)-\mathcal{E}_\mathcal{H}(g)}{\mathcal{E}_\mathcal{H}(g)+\varepsilon} < \left(\frac{3M + 2M_p}{M}\right)\alpha
\] (81)
Proof: Firstly, we have

\[
\frac{\mathcal{E}_\eta(g) - \mathcal{E}_{\eta,x}(g)}{\mathcal{E}_\eta(g) + \epsilon} = \mathcal{E}(g) - \mathcal{E}(f_\eta^p) + \mathcal{E}(g) - \mathcal{E}(f_\eta^p) + \mathcal{E}(g) - \mathcal{E}(f_\eta^p) + \mathcal{E}(g) + \mathcal{E}(f_\eta^p) = L_\epsilon(g) - L_\epsilon(f_\eta^p) - L_\epsilon(f_\eta^p) - L_\epsilon(f_\eta^p)
\]

(82)

The following inequality can be obtained from proposition 3 in literature [1] and Lemma 6.

\[
\frac{\mathcal{E}_\eta(g) - \mathcal{E}_{\eta,x}(g)}{\mathcal{E}_\eta(g) + \epsilon} \leq \frac{L_\epsilon(g) - L_\epsilon(f_\eta^p) + L_\epsilon(f_\eta^p) - L_\epsilon(f_\eta^p)}{\mathcal{E}_\eta(g) + \epsilon} + \frac{L_\epsilon(g) - L_\epsilon(f_\eta^p) + L_\epsilon(f_\eta^p) - L_\epsilon(f_\eta^p)}{\mathcal{E}_\eta(g) + \epsilon} + \frac{L_\epsilon(g) - L_\epsilon(f_\eta^p) + L_\epsilon(f_\eta^p) - L_\epsilon(f_\eta^p)}{\mathcal{E}_\eta(g) + \epsilon}
\]

(83)

Using the first part of the proof process of Lemma 6 and \(\alpha < 1\), we have

\[
\left|\mathcal{E}(f) - \mathcal{E}(g)\right| \leq 4\|f - g\|_e M_p < \frac{4\epsilon M_p}{M} \leq \frac{\epsilon M_p}{M}
\]

(84)

This implies

\[
\mathcal{E}_{\eta}(f) - \mathcal{E}_{\eta}(g) = \mathcal{E}(f) - \mathcal{E}(g) \leq \frac{\epsilon M_p}{M} \leq \frac{\epsilon M_p}{M} + \frac{\epsilon M_p}{M}
\]

(85)

And because \(\mathcal{E}(f) - \mathcal{E}(f_\eta^p) - \mathcal{E}(g) + \mathcal{E}(f_\eta^p) \leq \epsilon \leq \mathcal{E}(g) - \mathcal{E}(f_\eta^p) + \epsilon\).

If the both sides of this inequality of the both inequalities add, we have

\[
\mathcal{E}(f) - \mathcal{E}(f_\eta^p) - \mathcal{E}(g) + \mathcal{E}(f_\eta^p) + \mathcal{E}(f_\eta^p) - \mathcal{E}(f_\eta^p) - \mathcal{E}(f_\eta^p) \leq \mathcal{E}(g) - \mathcal{E}(f_\eta^p) + \epsilon + \mathcal{E}(f_\eta^p) + \frac{\epsilon M_p}{M}
\]

(86)

It is \(\mathcal{E}_{\eta}(f) - \mathcal{E}_{\eta}(g) \leq \mathcal{E}_{\eta}(g) + \frac{\epsilon (M + M_p)}{M}\).

This is equivalent to

\[
\frac{\mathcal{E}(f)}{\mathcal{E}(g)} + \frac{M + M_p}{M} \leq 2.
\]

(87)
So,
\[
\frac{\mathcal{E}_\mathcal{H}(g) - \mathcal{E}_{\mathcal{H}, z}(g)}{\mathcal{E}_\mathcal{H}(g) + \varepsilon} \leq \alpha + 2 \frac{M_P \alpha}{M} + 2 \alpha \leq \left( \frac{3M + 2M_P}{M} \right) \alpha
\]

From Lemma 11, we can obtain Lemma 12.

**Lemma 12.** For all \( \varepsilon > 0 \) and \( 0 < \alpha < 1 \),

\[
\text{prob}_{z \in \mathbb{Z}^m} \left\{ \sup_{f \in \mathcal{H}} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \left( \frac{3M + 2M_P}{M} \right) \alpha \right\} \leq \mathcal{N} \left( \mathcal{H}, \frac{\alpha \varepsilon}{4M} \right) e^{-\frac{\alpha^2 \text{me}}{\|M^2 + M_P^2\|}} (88)
\]

**Proof:** If \( \mathcal{H} = \mathcal{N} \left( \mathcal{H}, \frac{\alpha \varepsilon}{4M} \right) \), there some disks \( D_j, j = 1, 2, \ldots, \ell_p \), to cover \( \mathcal{H} \), which makes \( f_j \) as the center and \( \frac{\alpha \varepsilon}{4M} \) as its’ radius. In \( U \) which is a completely measurable set, we have \( |f(x) - y| \leq M \). From Lemmas 11 and 6, it can be seen that for all \( z \in U^m \) and \( f \in D_j \), we have

\[
\sup_{f \in D_j} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \left( \frac{3M + 2M_P}{M} \right) \alpha \Rightarrow \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \alpha \quad (89)
\]

Therefore, for \( j = 1, \ldots, \ell \),

\[
\text{prob}_{z \in \mathbb{Z}^m} \left\{ \sup_{f \in D_j} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \left( \frac{3M + 2M_P}{M} \right) \alpha \right\} \leq \text{prob}_{z \in \mathbb{Z}^m} \left\{ \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \alpha \right\} \leq e^{-\frac{\alpha^2 \text{me}}{\|M^2 + M_P^2\|}} (90)
\]

Likewise, if \( \mathcal{H} = D_1 \cup \cdots \cup D_\ell \) and \( \varepsilon > 0 \), we have

\[
\text{prob}_{z \in \mathbb{Z}^m} \left\{ \sup_{f \in \mathcal{H}} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \left( \frac{3M + 2M_P}{M} \right) \alpha \right\} \leq \sum_{j=1}^\ell \text{prob}_{z \in \mathbb{Z}^m} \left\{ \sup_{f \in D_j} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \left( \frac{3M + 2M_P}{M} \right) \alpha \right\} \leq e^{-\frac{\alpha^2 \text{me}}{\|M^2 + M_P^2\|}} (91)
\]

From Lemma 10, it can be seen

\[
\text{prob}_{z \in \mathbb{Z}^m} \left\{ \sup_{f \in \mathcal{H}} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H}, z}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} \geq \left( \frac{3M + 2M_P}{M} \right) \alpha \right\} \leq \mathcal{N} \left( \mathcal{H}, \frac{\alpha \varepsilon}{4M} \right) e^{-\frac{\alpha^2 \text{me}}{\|M^2 + M_P^2\|}} \quad \square
\]

From Lemma 12 Theorem 5* can be proved.

In Lemma 12, \( \alpha = \frac{1}{2} \left( \frac{M}{3M + 2M_P} \right) \), it can be seen that in accordance with the probability of at least
\[ 1 - \mathcal{N} \left( \mathcal{H}, \frac{\alpha \varepsilon}{4M} e^{-\frac{\alpha^2 m \varepsilon}{8(M^2 + M^2)})} \right) = 1 - \mathcal{N} \left( \mathcal{H}, \frac{\varepsilon}{4M} \left( \frac{M}{3M + 2M_2} \right) e^{-\frac{m \varepsilon}{8(M^2 + M^2)})^2 \left( \frac{M}{3M + 2M_2} \right)^2} = 1 - \mathcal{N} \left( \mathcal{H}, \frac{\varepsilon}{32(M^2 + M^2)}^{\frac{1}{2}} \left( \frac{M}{3M + 2M_2} \right)^2, \right) \]

we have

\[ \sup_{f \in \mathcal{H}} \frac{\mathcal{E}_\mathcal{H}(f) - \mathcal{E}_{\mathcal{H},x}(f)}{\mathcal{E}_\mathcal{H}(f) + \varepsilon} < \frac{1}{2} \]  \hspace{1cm} (93)

So, for all \( f \in \mathcal{H} \), we have \( \frac{1}{2} \mathcal{E}_\mathcal{H}(f) < \mathcal{E}_{\mathcal{H},x}(f) + \frac{1}{2} \varepsilon \). If \( f = f_2 \), both sides of the above inequality are multiplied by 2, and we get

\[ \mathcal{E}_\mathcal{H}(f) < 2\mathcal{E}_{\mathcal{H},x}(f) + \varepsilon \]  \hspace{1cm} (94)

In accordance with the definition of \( f_2 \), we have \( \mathcal{E}_{\mathcal{H},x}(f_2) \leq 0 \). Therefore, \( \mathcal{E}_\mathcal{H}(f_2) < \varepsilon \). The proof of Theorem 5* is completed.

**Proof of Theorem 6.** Firstly part (1) is proved. Since

\[ \varphi(b) = \|b - a\|^2 + \tau \|b - p - \int (b - p)d\rho\|^2 + \gamma \|A^{-s}b\|^2, \]  \hspace{1cm} (95)

if a function \( \vec{b} \) minimizes \( \varphi \), it must be a zero point on a derivative \( D\varphi \).

Suppose an operator \( \Gamma(b - p) = \int (b - p)d\rho \), then we have

\[ \vec{b} = [Id + \tau(Id - \Gamma)^2 + \gamma A^{-2s}]^{-1}[a + \tau(Id - \Gamma)^2p] = (Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}(a + \tau \mathcal{L}^2p) \]  \hspace{1cm} (96)

Because the operator \( Id + \tau(Id - \Gamma)^2 + \gamma A^{-2s} \) is a sum of an identity operator and a positive definite operator, it is invertible. If \( \lambda_1 \geq \lambda_2 \geq \cdots > 0 \) are the eigenvalues of \( A \), and function \( a = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} r_k \varphi_k \), where \( a_k = r_k \varphi_k \),

\[ \varphi(\vec{b}) = \|\vec{b} - a\|^2 + \tau \|(Id - \Gamma)(\vec{b} - p)\|^2 + \gamma \|A^{-s}\vec{b}\|^2 \]  \hspace{1cm} (97)

From Eq.(96), we have

\[ \|\vec{b} - a\|^2 = \|(Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}[\tau \mathcal{L}^2p - (\tau \mathcal{L}^2 + \gamma A^{-2s})a]\|^2 \]  \hspace{1cm} (98)

And we get

\[ \|(Id - \Gamma)(\vec{b} - p)\|^2 = \|\mathcal{L}(Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}(a - p - \gamma A^{-2s}p)\|^2 \]  \hspace{1cm} (99)
\[ \|A^{-s}\hat{b}\|^2 = \|A^{-s}(Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}(a + \tau \mathcal{L}^2 p)\|^2 \]
\[ = \|(Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}(A^{-s}a + \tau A^{-s} \mathcal{L}^2 p)\|^2 \]
\[ = \sum_{k=1}^{\infty} \frac{1}{(1 + \tau \mathcal{L}^2 + \gamma \lambda_k^{-2s})^{-2}} (\lambda_k^{-2s}a_k + \tau \lambda_k^{-2s} \mathcal{L}^2 p_k)^2 \]
\[ = \sum_{k=1}^{\infty} \lambda_k^{2r-2s} \frac{1}{(1 + \tau \mathcal{L}^2 + \gamma t^{-2s})^{-2}} (\lambda_k^{2r}a_k + \tau \mathcal{L}^2 p_k)^2 \]
\[ \leq \sup_{t \in \mathbb{R}} t^{2r-2s} (1 + \tau \mathcal{L}^2 + \gamma t^{-2s})^{-2} \sum_{k=1}^{\infty} \lambda_k^{2r}(a_k + \tau \mathcal{L}^2 p_k)^2 \]

Then we have
\[ (2r - 2s)\hat{t}^{2r-2s-1} (Id + \tau \mathcal{L}^2 + \gamma \hat{t}^{-2s})^{-2} + 4s \gamma \hat{t}^{2r-4s-1} (Id + \tau \mathcal{L}^2 + \gamma t^{-2s})^{-3} = 0 \quad (100) \]

Finally, \( \hat{t} = (r + s)^{\frac{1}{2s}}(s - r)^{-\frac{1}{2s}}(1 + \tau \mathcal{L}^2)^{-\frac{1}{2s}}. \)

Then \( \psi(\hat{t}) = \hat{t}^{2r-2s}(1 + \tau \mathcal{L}^2 + \gamma t^{-2s})^{-2} \leq (r + s)^{\frac{r+s}{s}} \gamma^{\frac{r-s}{s}} (s - r)^{-\frac{r-s}{s}} (1 + \tau \mathcal{L}^2)^{-\frac{r+s}{s}} \).

\[ \|A^{-s}\hat{b}\|^2 \leq \psi(\hat{t})\|A^{-r}(a + \tau \mathcal{L}^2 p)\|^2. \quad (101) \]

\[ \varphi(\hat{b}) \leq \|(Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}[\tau \mathcal{L}^2 p - (\tau \mathcal{L}^2 + \gamma A^{-2s})a]\|^2 + \tau\|\mathcal{L}(Id + \tau \mathcal{L}^2 + \gamma A^{-2s})^{-1}(a - p - \gamma A^{-2s} p)\|^2 + (r + s)^{\frac{r+s}{s}} \gamma^{\frac{r-s}{s}} (s - r)^{-\frac{r-s}{s}} (1 + \tau \mathcal{L}^2)^{-\frac{r+s}{s}} \|A^{-r}(a + \tau \mathcal{L}^2 p)\|^2. \quad (102) \]

For part (2), if \( \|A^{-s}a\| \leq R, \|A^{-s}p\| \leq R \) and \( a = p \), the minimum value of the expression is zero. Then the theorem is obviously true. Suppose the case is not true, in the subspace \( \|A^{-s}\hat{b}\| \leq R \) of \( \mathcal{H} \), the optimal \( \hat{b} \) is on the boundary of the subspace, which is \( \|A^{-s}\hat{b}\| = R. \)

\[ \gamma^{\frac{s-r}{s}} \leq (r + s)^{\frac{r+s}{s}} \gamma^{\frac{r-s}{s}} (s - r)^{-\frac{r-s}{s}} (1 + \tau \mathcal{L}^2)^{-\frac{r+s}{s}} \|A^{-r}(a + \tau \mathcal{L}^2 p)\|^2 \quad (103) \]

Then, we get
\[ \gamma \leq (r + s)^{\frac{r+s}{s}} \gamma^{\frac{2s}{s-r}} (s - r)^{-\frac{r+s}{s-r}} (1 + \tau \mathcal{L}^2)^{-\frac{r+s}{s-r}} \|A^{-r}(a + \tau \mathcal{L}^2 p)\|^2 \]

\[ \square \]
7. Conclusion

In this paper, we proposed a quantitative index of the interpretability, and analyzed the relationship between the interpretability and the generalization performance of the prediction model in machine learning. The equilibrium problem between the two performances was proven to exist. For traditional supervised kernel machine learning problem, we studied a universal learning framework for improving the interpretability of the prediction model and solving the equilibrium problem. Next, the uniqueness of solution of the problem was proved and condition of unique solution was found. Probability upper bound of the sum of the two performances is analyzed. The solving method was proposed for the equilibrium problem.

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Reference

[1] Cucker F, Smale S. On the mathematical foundations of learning[J]. Bulletin of the American Mathematical Society, 2002, 39(1):1-49.
[2] Sun Z, Zhang Z, Wang H. Consistency and error analysis of Prior-Knowledge-Based Kernel Regression[M]. Elsevier Science Publishers B. V. 2011.
[3] Koh P W, Liang P. Understanding Black-box Predictions via Influence Functions[J]. 2017.
[4] Wu J, Zhou B, Peck D, et al. DeepMiner: Discovering Interpretable Representations for Mammogram Classification and Explanation[J]. 2018.
[5] Zhou B. Interpretable Representation Learning for Visual Intelligence[D], MIT EECS, May 17, 2018.
[6] Bolei Zhou, Yiyou Sun, David Bau, Antonio Torralba. Interpretable Basis Decomposition for Visual Explanation. European Conference on Computer Vision (ECCV), 2018.
[7] Craven, Mark W and Shavlik, Jude W. Extracting treestructured representations of trained networks. Advances in neural information processing systems, pp. 24–30, 1996.
[8] Baehrens, David, Schroeter, Timon, Harmeling, Stefan, Kawanabe, Motoaki, Hansen, Katja, and M’uller, Klaus-Robert. How to explain individual classification decisions. Journal of Machine Learning Research, 11, 2010.
[9] Strumbelj, Erik and Kononenko, Igor. An efficient explanation of individual classifications using game theory. Journal of Machine Learning Research, 11, 2010.
[10] Krause J, Perer A, Ng K. Interacting with Predictions: Visual Inspection of Black-box Machine Learning Models[C]// CHI Conference on Human Factors in Computing Systems. ACM, 2016:5686-5697.

[11] R.L. James, D. David, G. Roger, B.T. Joshua, G. Zoubin, Automatic Construction and Natural-Language Description of Nonparametric Regression Models, In: Association for the Advancement of Artificial Intelligence (AAAI), July 2014.

[12] Ribeiro M T, Singh S, Guestrin C. "Why Should I Trust You?": Explaining the Predictions of Any Classifier[C]// ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. ACM, 2016:1135-1144.

[13] Ribeiro M T, Singh S, Guestrin C. Model-Agnostic Interpretability of Machine Learning[J]. 2016.

[14] Zhao J, Hei X, Shi Z, et al. Regression Learning Based on Incomplete Relationships between Attributes[J]. Information Sciences, 2017, 422.

[15] Hofmann T, Schölkopf B, Smola A J. Kernel methods in machine learning[J]. Annals of Statistics, 2008, 36(3):1171-1220.

[16] Vapnik, V. (1998). Statistical learning theory. 1998. Wiley, New York.

[17] Ustun, Berk and Rudin, Cynthia. Supersparse linear integer models for optimized medical scoring systems. Machine Learning, 2015.

[18] Baehrens, David et al. How to Explain Individual Classification Decisions. Journal of Machine Learning Research 11 (2010): 1803-1831.

[19] Caruana, Rich, Lou Y, Gehrke J, Koch, Paul,Sturm, Marc, and Elhadad, Noemie. Intelligible Models for HealthCare: Predicting Pneumonia Risk and Hospital 30-day Readmission[C]// ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. ACM, 2015:1721-1730.

[20] Karimi K, Hamilton H J. Generation and Interpretation of Temporal Decision Rules[J]. 2012, 3(1):314-323.

[21] Letham, Benjamin, Rudin, Cynthia, McCormick, Tyler H., and Madigan, David. Interpretable classifiers using rules and bayesian analysis: Building a better stroke prediction model. Annals of Applied Statistics, 2015.

[22] F. Wang and C. Rudin. Falling rule lists. Eprint Arxiv. 2014 :1013-1022

[23] Tihonov A N. On the regularization of ill-posed problems.[J]. Dok.akad.nau.ssr, 1963, 153(6):1111-1146(36).

[24] Tikhonov A N. Resolution of ill-posed problems and the regularization method (in Russian)[J]. Doklady Akademii Nauk Ssr, 1963, 151:501-504.

[25] Birman M S, Solomjak A. Piecewise-polynomial approximations of functions of the classes W?p[J]. English translation in Math. USSR Sb. 73 (1967), 331-355.