Minimal random attractors

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It is well-known that random attractors of a random dynamical system are generally not unique. We show that for general pullback attractors and weak attractors, there is always a minimal (in the sense of smallest) random attractor which attracts a given family of (possibly random) sets. We provide an example which shows that this property need not hold for forward attractors. We point out that our concept of a random attractor is very general: The family of sets which are attracted is allowed to be completely arbitrary.

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1 Introduction

For deterministic dynamical systems on metric spaces the notion of an attractor is well established. The most common notion, mainly used for partial differential equations (PDEs) on suitable Hilbert or Banach spaces, is that of a global set attractor. It is characterized by being a compact set, being strictly invariant, and attracting every compact, or even every bounded set. Uniqueness of the global set attractor is immediate.

Another very common notion is that of a (global) point attractor, which is often used for systems on locally compact spaces (which are often Euclidean spaces or finite-dimensional manifolds). A global point attractor is again compact and strictly invariant, but it is only assumed to attract every point (or, equivalently, every finite set). The global point attractor is in general not unique, which can be seen from the simple dynamical system induced by the scalar differential equation $\dot{x} = x - x^3$. Here the unique global set attractor is the interval $[-1, 1]$, which, of course, is also a point attractor. But also

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[-1, 0] ∪ \{1\}, and \{-1\} ∪ [0, 1], are point attractors. Obviously, there is a minimal point attractor, namely \{-1, 0, 1\}.

In fact, for deterministic systems the following result is well known: Whenever \(\mathcal{B}\) is an arbitrary family of non-empty subsets of the state space then there exists an attractor for \(\mathcal{B}\) (i.e., a compact, strictly invariant set attracting every \(B \in \mathcal{B}\)) if and only if there exists a compact set such that every \(B \in \mathcal{B}\) is attracted by this set. Furthermore, there exists a unique minimal attractor for \(\mathcal{B}\), which is given by the closure of the union of the \(\omega\)-limit sets of all elements of \(\mathcal{B}\). This minimal attractor for \(\mathcal{B}\) is addressed as the \(\mathcal{B}\)-attractor. If a global set attractor exists then whenever a \(\mathcal{B}\)-attractor exists it is always a subset of the global set attractor.

For random dynamical systems an analogous statement has been established in [6]. However, this result had to assume a separability condition for \(\mathcal{B}\).

The aim of the present paper is to remove this separability condition, i.e. to establish that for any family \(\mathcal{B}\) of (possibly even random) sets for which there is a compact random set attracting every element of \(\mathcal{B}\), there exists a unique minimal random attractor for \(\mathcal{B}\). Furthermore, it is in general not true that this \(\mathcal{B}\)-attractor is given by the closed random set

\[
\bigcup_{B \in \mathcal{B}} \overline{\Omega_B(\omega)} \quad \text{almost surely,}
\]

where \(\Omega_B(\omega)\) denotes the (random) \(\Omega\)-limit set of \(B\). This is shown using an example of a random dynamical system induced by a stochastic differential equation on the unit circle \(S^1\). Here the global set attractor is the whole \(S^1\) (which is a strictly invariant compact set). If \(\mathcal{B}\) is taken to be the family of all deterministic points in \(S^1\) it is shown that (1) gives \(S^1\), the global set attractor, almost surely. However, the minimal point attractor is a one point set consisting of a random variable, which (pullback) attracts every solution starting in a deterministic point.

\section{Notation and Preliminaries}

Let \(E\) be a Polish space, i.e. a separable topological space whose topology is metrizable with a complete metric. Several assertions in the following are formulated in terms of a metric \(d\) on \(E\) which is referred to without further mentioning. This metric will always be assumed to generate the topology of \(E\) and to be complete, even if some of the assertions hold also if \(d\) is not complete. For \(x \in E\) and \(A \subset E\) we define \(d(x, A) = \inf\{d(x, a) : a \in A\}\) with the convention \(d(x, \emptyset) = \infty\). For non-empty subsets \(A\) and \(B\) of \(E\) we denote the Hausdorff semi-metric by \(d(B, A) := \sup\{d(b, A) : b \in B\}\) and define \(d(\emptyset, A) := 0\) and \(d(B, \emptyset) := \infty\) in case \(B \neq \emptyset\).

We note that with this convention both for the empty family \(\mathcal{B} = \emptyset\) as well as for \(\mathcal{B} = \{\emptyset\}\) the empty set \(A = \emptyset\) is an attractor, in fact the minimal one.
We denote the Borel $\sigma$-algebra on $E$ (i.e. the smallest $\sigma$-algebra on $E$ which contains every open set) by $\mathcal{E}$.

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathbb{T}_1 \in \{\mathbb{Z}, \mathbb{R}\}$, and
\[
\vartheta : \mathbb{T}_1 \times \Omega \to \Omega \\
(t, \omega) \mapsto \vartheta t \omega
\]
is a measurable map, such that $\vartheta_t : \Omega \to \Omega$ preserves $\mathbb{P}$, and such that $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $s, t \in \mathbb{T}_1$ and $\vartheta_0 = \text{id}$. Thus $(\vartheta_t)$ is a classical measurable dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 1.** Given $(\Omega, \mathcal{F}, \mathbb{P})$ and $\vartheta_t, t \in \mathbb{T}_1$ as above, $E$ a Polish space, and $\mathbb{T}_2$ either $\mathbb{R}, [0, \infty), \mathbb{Z},$ or $\mathbb{N}_0$ such that $\mathbb{T}_2 \subseteq \mathbb{T}_1$, a measurable map
\[
\varphi : \mathbb{T}_2 \times E \times \Omega \to E \\
(t, x, \omega) \mapsto \varphi(t, \omega)x
\]
(or the pair $(\varphi, \vartheta)$) is a *random dynamical system (RDS)* on $E$ if

(i) $\varphi(t, \omega) : E \to E$ is continuous for every $t \in \mathbb{T}_2$, $\mathbb{P}$-almost surely

(ii) $\varphi(t+s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}_2$, and $\varphi(0, \omega) = \text{id}$, for $\mathbb{P}$-almost all $\omega$.

**Remarks 2.** (i) Note that we do not assume continuity in $t$ here.

(ii) An RDS $(\varphi, \vartheta)$ is said to be *two-sided* if $\mathbb{T}_2$ is two-sided. For a two-sided RDS $\varphi$ the maps $\varphi(t, \omega)$ are invertible, and $\varphi(t, \omega)^{-1} = \varphi(-t, \vartheta_t \omega)$ a.s.

(iii) Proposition 25 shows that one can always change $\varphi$ on a set of measure 0 in such a way that properties (i) and (ii) in Definition 1 hold without exceptional sets. In the following we will tacitly assume that $\varphi$ satisfies these slightly stronger assumptions. The question whether exceptional sets of measure zero in (ii) of Definition 1 which may depend on $s$ and $t$ can be eliminated (without destroying possible (right-)continuity properties of $\varphi$ in the time variable) has been addressed in [1, 2, 17], and [15].

**Definition 3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $E$ a Polish space. A *random set* $C$ is a measurable subset of $E \times \Omega$ (with respect to the product $\sigma$-algebra $\mathcal{E} \otimes \mathcal{F}$).

The $\omega$-section of a set $C \subset E \times \Omega$ is defined by

\[
C(\omega) = \{ x : (x, \omega) \in C \}, \quad \omega \in \Omega.
\]

In the case that a set $C \subset E \times \Omega$ has closed or compact $\omega$-sections it is a random set as soon as the mapping $\omega \mapsto d(x, C(\omega))$ is measurable (from $\Omega$ to $[0, \infty)$) for every $x \in E$, see [7, Chapter 2]. Then $C$ will be said to be a *closed* or a *compact*, respectively, random set. For any set $C \subset E \times \Omega$, we define $\overline{C} := \{ (x, \omega) : x \in C(\omega) \}$.
Definition 5. If \( \varphi \) is an RDS then a set \( D \subset E \times \Omega \) is said to be forward invariant or strictly invariant, respectively, with respect to \( \varphi \), if \( \varphi(t, \omega)D(\omega) \subset D(\vartheta_t \omega) \) or \( \varphi(t, \omega)D(\omega) = D(\vartheta_t \omega) \) \( \mathbb{P}\)-a.s., respectively, for every \( t \geq 0 \).

Definition 6. For \( B \subset E \times \Omega \) we define the \( \Omega \)-limit set of \( B \) by

\[
\Omega_B(\omega) = \bigcap_{T \geq 0} \bigcup_{t \geq T} \varphi(t, \vartheta_{-t}\omega)(B(\vartheta_{-t}\omega)), \quad \omega \in \Omega,
\]
as in \([5, \text{Definition 3.4}]\).

Remark 7. It is easy to verify that an \( \Omega \)-limit set is always forward invariant and also that it is strictly invariant for an RDS with two-sided time.

The following Lemma, which generalizes \([6, \text{Theorem 3.4}]\), provides another sufficient condition for \( \Omega_B \) to be strictly invariant.

Lemma 8. Suppose that \( B \subset E \times \Omega \), \( K \subset E \times \Omega \), and \( \Omega_0 \subset \Omega \). Assume that for all \( \omega \in \Omega_0 \), the set \( K(\omega) \) is compact and

\[
\lim_{t \to \infty} d(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0. \tag{2}
\]

Then, for every \( t \geq 0 \) and \( \omega \in \Omega_0 \), \( \Omega_B(\vartheta_t \omega) \subset \varphi(t, \omega)\Omega_B(\omega) \) and \( \Omega_B(\omega) \subset K(\omega) \). If, moreover, \( \Omega_0 \in \mathcal{F} \) and \( \mathbb{P}(\Omega_0) = 1 \), then \( \Omega_B \) is strictly invariant.

Proof. Fix \( \omega \in \Omega_0 \) throughout the proof. Compactness of \( K(\omega) \) and (2) imply \( \Omega_B(\omega) \subset K(\omega) \). Suppose that \( y \in \Omega_B(\vartheta_t \omega) \) for some \( t \geq 0 \). Then \( y = \lim_{n \to \infty} \varphi(t_n, \vartheta_{-t_n}(\vartheta_t \omega))b_n \) for sequences \( t_n \to \infty \) and \( b_n \in B(\vartheta_{-t_n} \omega) \). Consider the sequence \( \varphi(t_n - t, \vartheta_{-(t_n - t)} \omega)b_n \), defined for \( n \) with \( t_n - t \geq 0 \). By (2) we have \( \lim_{n \to \infty} d(\varphi(t_n - t, \vartheta_{-(t_n - t)} \omega)b_n, K(\omega)) = 0 \) for \( n \to \infty \). Compactness of \( K(\omega) \) implies that this sequence has a convergent subsequence. Choose one and denote its limit by \( z(\omega) \), then \( z(\omega) \in \Omega_B(\omega) \). Using the same notation for the subsequence, continuity of \( \varphi(t, \omega) \) implies

\[
\varphi(t, \omega)z(\omega) = \lim_{n \to \infty} \varphi(t_n, \vartheta_{-(t_n - t)} \omega)b_n = y.
\]

Thus, for any \( y \in \Omega_B(\vartheta_t \omega) \) there exists \( z \in \Omega_B(\omega) \) with \( \varphi(t, \omega)z = y \), whence \( \Omega_B(\vartheta_t \omega) \subset \varphi(t, \omega)\Omega_B(\omega) \).

The final statement of Lemma 8 follows since \( \Omega_B \) is forward invariant. \( \square \)

Now let \( B \) be a non-empty family of sets \( B \subset E \times \Omega \). At this point we make no measurability assumptions on the sets in \( B \) and therefore say that a property depending on \( \omega \) holds almost surely if there is a measurable set of full measure on which the
property holds. Analogously, we will interpret statements like “$Y \to 0$ in probability” for a real-valued (possibly non-measurable) function $\omega \mapsto Y(\omega)$. As usual, we introduce the universal completion $\mathcal{F}^u$ of $\mathcal{F}$ as the intersection of all completions of $\mathcal{F}$ with respect to probability measures on $\mathcal{F}$. Note that one automatically has $\mathcal{F}^u = \mathcal{F}$ in the case of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We define the concept of a random pullback, forward and weak $\mathcal{B}$-attractor of an RDS $(\varphi, \theta)$ as usual (except that we do not impose any measurability assumptions on $\mathcal{B}$). Random pullback attractors were first introduced in [8] while the concept of a weak attractor is due to G. Ochs [10].

**Definition 9.** Suppose that $\varphi$ is an RDS on a Polish space $E$ and $\mathcal{B}$ is a non-empty family of subsets of $E \times \Omega$. Then a set $A \subset E \times \Omega$ is a random attractor for $\mathcal{B}$ if

(i) $A$ is a compact random set

(ii) $A$ is strictly $\varphi$-invariant, i.e.

$$\varphi(t, \omega)A(\omega) = A(\theta_t \omega)$$

$\mathbb{P}$-almost surely for every $t \in T_2$ with $t \geq 0$

(iii) $A$ attracts $\mathcal{B}$, i.e.

$$\lim_{t \to \infty} d\left(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)\right) = 0 \quad \mathbb{P}\text{-a.s.} \quad (3)$$

for every $B \in \mathcal{B}$; in this case $A$ is a random pullback attractor for $\mathcal{B}$,

or

$$\lim_{t \to \infty} d\left(\varphi(t, \omega)B(\omega), A(\theta_t \omega)\right) = 0 \quad \mathbb{P}\text{-a.s.},$$

for every $B \in \mathcal{B}$; in this case $A$ is a random forward attractor for $\mathcal{B}$,

or

$$\lim_{t \to \infty} d\left(\varphi(t, \omega)B(\omega), A(\theta_t \omega)\right) = 0 \quad \text{in probability} \quad (4)$$

for every $B \in \mathcal{B}$; in this case $A$ is a weak (random) attractor for $\mathcal{B}$.

**Remark 10.** Note that the property of being a pullback or weak attractor does not depend on the choice of the metric $d$ metrizing the topology of $E$. This is not true for forward attractors, not even when $E = \mathbb{R}$, see [13, Example 2.4] or the example in Section 5.

**Remarks 11.** (i) For a weak random attractor condition (4) is equivalent to (3) with almost sure convergence replaced by convergence in probability. See [10] for sufficient conditions for the existence of weak global set attractors. For monotone RDS, the concept of a weak attractor turns out to be more suitable than that of a pullback attractor, see [3] and [14].

(ii) While for non-autonomous systems it is very simple to find examples of attractors which are pullback but not forward, and vice versa (see, e.g., [9]), this is not so simple.
for random attractors. One of the authors [18] has constructed examples for this to happen as well as examples where only weak random attractors exist which are neither pullback nor forward attractors.

(iii) Clearly each random pullback attractor $A$ for $\mathcal{B}$ must satisfy $\Omega_{\mathcal{B}}(\omega) \subset A(\omega)$ almost surely for every $B \in \mathcal{B}$ (the exceptional sets may depend on $B$).

(iv) Instead of assuming an attractor $A$ to be a compact random set it suffices to assume $A$ to be a random set such that $A(\omega)$ is compact $\mathbb{P}$-almost surely, which is slightly weaker (see [7, Proposition 2.4] for details). Proposition 19 below, applied to the singleton $A$, implies existence of a compact random set satisfying the conditions of Definition 9.

Remark 12. For a general family $\mathcal{B}$ there may or may not exist a (pullback, forward or weak) random $\mathcal{B}$-attractor for an RDS $\varphi$ and if such an attractor exists then it need not be unique in general. However, as soon as $\mathcal{B}$ contains every compact deterministic set, then whenever a weak random attractor for $\mathcal{B}$ exists then it is unique, see [13, Lemma 1.3]. Since every pullback and every forward attractor is also a weak attractor, the same uniqueness statement holds for pullback and forward attractors (for pullback attractors this was already established in [5]).

Whenever a $\mathcal{B}$-attractor (in whatever sense) is not unique it is natural to ask whether there is a smallest (or minimal) $\mathcal{B}$-attractor. In [6, Theorem 3.4] a condition on $\mathcal{B}$ (formulated only for families of deterministic sets) for the existence of a minimal random pullback attractor has been given. It is one of the aims of this paper to show that a minimal random pullback (respectively weak) attractor exists for arbitrary $\mathcal{B}$, provided existence of at least one pullback (respectively weak) $\mathcal{B}$-attractor.

3 Existence of a minimal pullback attractor for $\mathcal{B}$

Given a random dynamical system $(\varphi, \vartheta)$ on the space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in the Polish space $(E, d)$ we consider a general family $\mathcal{B}$ of subsets of $E \times \Omega$ for which we assume existence of a pullback attractor or, equivalently, the existence of an attracting compact random set. Note that no measurability conditions whatsoever are imposed on (the elements of) $\mathcal{B}$. The major result of this section is the existence of a minimal random pullback attractor for $\mathcal{B}$. Of course this is immediate as soon as $\mathcal{B}$ contains every compact deterministic subset of $E$ (or, more precisely, every $C \times \Omega$ with $C \subset E$ compact); in this case the minimal attractor is the unique global set attractor. However, for instance for $\mathcal{B}$ consisting of all (deterministic) points (or, equivalently, of all finite sets), there may be several random pullback attractors.

Theorem 13. Let $\mathcal{B}$ be an arbitrary family of sets $B \subset E \times \Omega$ and let $(\varphi, \vartheta)$ be an RDS on $E$. 

(i) Assume that there exists a random set $K \subset E \times \Omega$ such that $K(\omega)$ is $\mathbb{P}$-a.s. compact and $K$ attracts $B$, i.e.

$$\lim_{t \to \infty} d(\varphi(t, \vartheta_{-t}\omega) B(\vartheta_{-t}\omega), K(\omega)) = 0 \quad \text{P-a.s.}$$

(and therefore $\Omega_B(\omega) \subset K(\omega)$ a.s.) for every $B \in \mathcal{B}$. Then $(\varphi, \vartheta)$ has at least one pullback attractor for $B$.

(ii) If $(\varphi, \vartheta)$ has at least one pullback attractor for $B$, then the RDS has a minimal pullback $\mathcal{B}$-attractor $A$. In addition, there is a countable sub-family $\mathcal{B}_0 \subset \mathcal{B}$ such that $A$ is also the minimal pullback $\mathcal{B}_0$-attractor.

(iii) Under the assumptions of (i) the minimal pullback $\mathcal{B}$-attractor $A$ is almost surely contained in $K(\omega)$.

Note that under the assumptions of (ii) any pullback attractor $K$ for $B$ will satisfy the assumptions of (i). Therefore, it suffices to show that under the assumptions of (i) there exists a minimal pullback attractor for $B$ and that it satisfies the conclusions of (ii) and (iii).

For the proof of the theorem we are going to prepare several results which will also be used in the next section, where we investigate the same question for weak attractors. For the first few results we just need an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Polish metric space $(E, d)$, but no RDS. We will denote the open ball with centre $x \in E$ and radius $r > 0$ by $B(x, r)$. By $D$ we always denote a fixed countable dense set in $E$. The following lemma just requires a separable metric (not necessarily Polish) space $(E, d)$.

**Lemma 14.** For each fixed $r > 0$ there exists a cover $R(x, r)$, $x \in D$, of $E$ such that $R(x, r)$ is a closed (possibly empty) subset of $B(x, r)$, and for each $y \in E$ there exists a neighbourhood of $y$ which intersects only finitely many of the sets $R(x, r)$, $x \in D$, and there exists $x \in D$ such that $y$ is in the interior of $R(x, r)$.

**Proof.** For given $r > 0$, the family $B(x, r)$, $x \in D$, is an open cover of $E$. Since every metric space is paracompact, Theorem VIII.4.2 in Dugundji [12] implies the existence of a cover $R(x, r)$, $x \in D$, as claimed in the lemma.

**Remark 15.** Note that for each $r > 0$ the closed cover $\bar{R}(x, r)$, $x \in D$, given in Lemma [14] has the property that for every (possibly infinite) subset $D_0 \subset D$ the union $\bigcup_{x \in D_0} R(x, r)$ is closed. This will be important in the following.

**Remark 16.** Note that for each $r > 0$, the family $\bar{R}(x, r)$, $x \in D$, is an open cover of $E$ thanks to the last assertion of Lemma [14] (here $\bar{S}$ denotes the interior of the set $S$).

In the following $R(x, r)$, $x \in D$, will always denote a closed cover as in Lemma [14] (which, of course, is not unique). The following result asserts that every subset of $E \times \Omega$ has a closed random hull.
Proposition 17. Let $K \subset E \times \Omega$. There exists a (unique) smallest closed random set $\hat{K}$ which contains $K$ in the following sense: $K \subset \hat{K}$ and for every random set $S$ for which $K(\omega) \subset S(\omega)$ for almost all $\omega \in \Omega$ and for which $S(\omega)$ is closed for almost all $\omega \in \Omega$, we have $\hat{K}(\omega) \subset S(\omega)$ for almost all $\omega \in \Omega$.

Proof. For $G \in \mathcal{E}$ define

$$M^G := \{\omega : K(\omega) \cap G \neq \emptyset\}.$$  

This set will not be measurable in general. Let $\beta_G := \mathbb{P}^\#(M^G) := \inf\{\mathbb{P}(M) : M \in \mathcal{F}, M \supseteq M^G\}$ and let $M_1 \supseteq M_2 \supseteq \ldots$ be sets in $\mathcal{F}$ with $M^G \subset M_i, i \in \mathbb{N}$, such that $\lim_{i \to \infty} \mathbb{P}(M_i) = \beta_G$. Define $\hat{M}^G := \bigcap_{i=1}^\infty M_i$ and

$$\hat{K} := \bigcap_{n=1}^\infty \bigcup_{x \in D} \left(R(x, \frac{1}{n}) \times \hat{M}^{R(x, \frac{1}{n})}\right).$$ (5)

$\hat{K}$ is a random set. Using Lemma 14 and Remark 15, we see that $\hat{K}(\omega)$ is closed for all $\omega \in \Omega$. Since $\omega \mapsto d(y, (R(x, \frac{1}{n}) \times \hat{M}^{R(x, \frac{1}{n})})(\omega))$ is measurable for each $x, y$ and $n$, the same is true for the map $\omega \mapsto d(y, \hat{K}(\omega))$, so that $\hat{K}$ is a closed random set. By construction, we have $K(\omega) \subset \hat{K}(\omega)$ for all $\omega \in \Omega$. Indeed,

$$\hat{K} = \bigcap_{n=1}^\infty \bigcup_{x \in D} \left(R(x, \frac{1}{n}) \times M^{R(x, \frac{1}{n})}\right).$$

It remains to show the minimality property of $\hat{K}$.

Let $S$ be a set as in the statement of the proposition and let $G \in \mathcal{E}$. Then, almost surely,

$$K(\omega) \cap G \subset S(\omega) \cap G$$

and therefore $S(\omega) \cap G \neq \emptyset$ for almost every $\omega \in M^G$. Since $E$ is Polish, the projection theorem (see, e.g., Dellacherie and Meyer [11, III.44]) implies that the set $\{\omega \in \Omega : S(\omega) \cap G \neq \emptyset\}$ is in $\mathcal{F}^u$ and therefore $S(\omega) \cap G \neq \emptyset$ for almost all $\omega \in M^G$. Let $\Omega_0 \in \mathcal{F}$ be a set of full measure such that $S(\omega)$ is closed for all $\omega \in \Omega_0$ and $S(\omega) \cap G \neq \emptyset$ for all $\omega \in M^G \cap \Omega_0$ and all $G = R(x, \frac{1}{n}), x \in D, n \in \mathbb{N}$. Let $\omega \in \Omega_0$ and $y \in \hat{K}(\omega)$. By definition of $\hat{K}$ this means that for every $n \in \mathbb{N}$ there exists some $x \in D$ such that $y \in R(x, \frac{1}{n})$ and $\omega \in \hat{M}^{R(x, \frac{1}{n})}$, so $d(y, S(\omega)) \leq 2/n$. Since $S(\omega)$ is closed this implies $y \in S(\omega)$, so the proof of the proposition is complete. \qed

Remark 18. The assertion of Proposition 17 is wrong without the word closed (at both places where it appears). As an example consider the deterministic case in which $\Omega$ is a singleton and let $K$ be a set which is not in $\mathcal{E}$. There exists no smallest measurable subset of $E$ which contains $K$. 

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Proposition 19. Assume that $K_\alpha$, $\alpha \in I$, is a family of sets in $E \otimes F$. Then there exists a closed random set $A$ such that for every $\alpha \in I$ we have $K_\alpha(\omega) \subset A(\omega)$ almost surely and $A$ is the minimal set with this property: $A(\omega) \subset S(\omega)$ almost surely for every $S \in E \times \Omega$ for which $S(\omega)$ is almost surely closed and for which $K_\alpha(\omega) \subset S(\omega)$ almost surely for every $\alpha \in I$.

Furthermore, there exists a countable subset $I_0 \subset I$ such that

$$A(\omega) = \bigcup_{\alpha \in I_0} K_\alpha(\omega)$$

(6)

up to a set of measure zero.

Proof. We can and will assume without loss of generality that the family $K_\alpha$, $\alpha \in I$, is closed under finite unions. For an arbitrary set $G \in E$ and $\alpha \in I$ we define

$$M^G_\alpha := \{\omega \in \Omega : K_\alpha(\omega) \cap G \neq \emptyset\}.$$ 

Since $E$ is Polish, $M^G_\alpha \in \mathcal{F}$ by the projection theorem [11, III.44]. We can extend the probability measure $P$ to $\mathcal{F}$ in a unique way (we will use the same notation for the extension). Note that on the set $M^G_\alpha$ each $S$ as in the lemma necessarily satisfies $S(\omega) \cap G \neq \emptyset$ almost surely and therefore the same must be true for $A$ (which we yet need to define). Let $\beta_{\alpha,G} := P(M^G_\alpha)$. Then, there exists a sequence $\alpha_j = \alpha_j(G), j = 1, 2, \ldots$ in $I$ such that $P(M^G_{\alpha_j}) \nearrow \beta_G := \sup_{\alpha \in I} \beta_{\alpha,G}$. Define

$$M^G := \bigcup_{j=1}^\infty M^G_{\alpha_j}.$$ 

The set $M^G$ depends on the choice of the sequence $(\alpha_j)$, but different choices give sets which only differ by a set of measure 0. Changing the sets $M^G_{\alpha}$ on a set of $P$-measure 0, we can and will in fact assume that all these sets (and also the sets $M^G$) are even in $\mathcal{F}$. Note that for each $\alpha \in I$ we have $M^G_\alpha \subset M^G$ up to a set of measure 0. Recall that $D$ is a countable and dense subset of $E$. Defining

$$C_r := \bigcup_{x \in D} (\overline{B(x,r)} \times M^{B(x,r)}), \quad C := \bigcap_{n \in \mathbb{N}} C_{1/n},$$

one may hope to be able to show that $A := C$ has the required properties. While the measurability property clearly holds it is not clear that $A(\omega)$ is closed. One might therefore take the closure of the right hand side in the definition of $C_r$, but then measurability of $A$ is not clear. We will therefore change the definition of $C_r$ in such a way that it becomes a closed random set for each $r > 0$. Then $C$ will be a closed random set as well.

Define $R(x,r)$ as in Lemma 15 and put

$$A_r := \bigcup_{x \in D} (R(x,r) \times M^{R(x,r)}), \quad A := \bigcap_{n \in \mathbb{N}} A_{1/n},$$

9
Clearly, $A_{r} \in \mathcal{E} \otimes \mathcal{F}$ and $A_{r}(\omega)$ is closed for each $\omega \in \Omega$ and each $r > 0$. As in the proof of Proposition 17 it follows that $A_{r}$ is even a closed random set and hence the same is true for $A$.

For $S(\omega)$ as in the proposition and $\alpha \in I$ we have to show that $K_{\alpha}(\omega) \subseteq A(\omega) \subseteq S(\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

Fix $\alpha \in I$. For each $r > 0$ we have

$$K_{\alpha}(\omega) \subseteq \bigcup_{x \in D} \left( R(x, r) \times M_{\alpha}^{R(x, r)} \right)(\omega) \subseteq \bigcup_{x \in D} \left( R(x, r) \times \hat{M}^{R(x, r)} \right)(\omega) = A_{r}(\omega)$$

up to a set of measure 0 (which may depend on $\alpha$). Hence, up to a set of measure 0,

$$K_{\alpha}(\omega) \subseteq A(\omega). \tag{7}$$

To finish the proof it suffices to show that

$$A(\omega) = \bigcup_{x \in D, n \in \mathbb{N}, j \in \mathbb{N}} K_{\alpha_{j}(R(x, 1/n))}(\omega)$$

up to a null set (observe that the right hand side is in $S(\omega)$ almost surely). Note that the inclusion “$\supset$” follows from (7) and the fact that $A(\omega)$ is closed, so it remains to show the inclusion “$\subseteq$”.

For every $n \in \mathbb{N}$ and almost every $\omega \in \Omega$ we have

$$A(\omega) \subseteq \bigcup_{x \in D, R^{(x, 1/n)}(\omega)} \hat{R}(x, \frac{1}{n}) \subseteq \bigcup_{x \in D, n \in \mathbb{N}, j \in \mathbb{N}} K_{\alpha_{j}(R(x, 1/m))}^{2/n}(\omega),$$

where the upper index $2/n$ denotes the closed $2/n$-neighbourhood of a set. Taking intersections over all $n \in \mathbb{N}$, (8) follows and the proof of the proposition is complete.

**Remark 20.** The assertion of Proposition 19 becomes wrong if the word closed is deleted. As an example take $\Omega$ a singleton, $E = \mathbb{R}$, $I$ a non-measurable subset of $\mathbb{R}$, and $K_{\alpha} = \{\alpha\}$ for $\alpha \in I$. There is no minimal measurable subset of $\mathbb{R}$ which contains $I$.

Our next goal is to clarify whether forward and strict invariance, respectively, of a set $K$ are inherited by $\hat{K}$ given by (the proof of) Proposition 17.

**Lemma 21.** (i) If $K \subseteq E \times \Omega$ is forward invariant then so is $\hat{K}$.

(ii) If $K \subseteq E \times \Omega$ is strictly invariant, and if $K(\omega)$ is compact for almost every $\omega \in \Omega$, then $\hat{K}$ is strictly invariant.

**Proof.** Let $K$ be forward invariant, so for fixed $t \geq 0$ we have $\varphi(t, \omega) K(\omega) \subseteq K(\varphi_{t}\omega)$. Here and in the following equalities and inclusions are meant up to sets of measure 0. Hence

$$K(\omega) \subseteq (\varphi(t, \omega))^{-1}(K(\varphi_{t}\omega)) \subseteq (\varphi(t, \omega))^{-1}(\hat{K}(\varphi_{t}\omega)).$$
The right hand side is a random subset with closed \( \omega \)-sections since \( \varphi(t, \omega) \) is continuous and therefore contains \( \hat{K} \), so \( \hat{K} \) is invariant.

Next suppose that \( K \) is strictly invariant. Changing \( \hat{K} \) on a set of measure 0 if necessary we can and will assume that \( \hat{K} \) is a compact random set. For \( t \geq 0 \) fixed we have

\[
\varphi(t, \omega)\hat{K}(\omega) \supset \varphi(t, \omega)K(\omega) \supset K(\vartheta, \omega).
\]

If the left hand side is a random set with closed \( \omega \)-sections, then it must contain \( \hat{K}(\vartheta, \omega) \) and strict invariance of \( \hat{K} \) follows. Using the representation [5] of the set \( \hat{K} \), we see that if \( \varphi(t, \cdot)(S) \), where \( S = C \times F \) with \( C \) a deterministic compact subset of \( E \) and \( F \in \mathcal{F} \), is a random set, then the same is true for \( \varphi(t, \cdot)(\hat{K}) \). Observe that the map \( y \mapsto d(y, \varphi(t, \omega)(S(\omega))) \) is measurable for each \( y \in E \) since \( d(y, \varphi(t, \omega)(S(\omega))) = \infty \) for \( \omega \notin F \), \( d(y, \varphi(t, \omega)(S(\omega))) = d(y, \varphi(t, \omega)C) \) for \( \omega \in F \) and \( d(y, \varphi(t, \omega)C) < r \) iff \( d(y, \varphi(t, \omega)x_i) < r \) for all \( i \) where \( (x_i) \) is a countable dense set in \( C \) and \( r > 0 \). This shows that \( \varphi(t, \omega)\hat{K}(\omega) \) is a random set. To see that it has closed \( \omega \)-sections we use the fact that \( \hat{K}(\omega) \) is compact and \( \varphi(t, \omega) \) is continuous. This completes the proof of the lemma.

**Proof** of Theorem 13. We apply Proposition 19 to \( B := \hat{\Omega}_B \), \( B \in \mathcal{B} \), where \( \hat{\Omega}_B \) is constructed from \( \Omega_B \) as in Proposition 17, and obtain a smallest closed random set \( A \) containing \( \Omega_B \) almost surely for every \( B \in \mathcal{B} \). Clearly \( A \) is also the minimal closed random set \( A \) containing \( \Omega_B \) almost surely for each \( B \in \mathcal{B} \). We claim that (a slight modification of) \( A \) is the minimal pullback \( \mathcal{B} \)-attractor. By assumption, the set \( \mathcal{K}(\omega) \) from the assertion of Theorem 13 contains \( \Omega_B(\omega) \) and, by Proposition 17, also \( \Omega_B(\omega) \) almost surely for every \( B \in \mathcal{B} \). By minimality, we have \( A(\omega) \subset \mathcal{K}(\omega) \) almost surely. Since \( \mathcal{K}(\omega) \) is almost surely compact so is \( A(\omega) \). Changing \( A \) on a set \( N \) of measure zero containing those \( \omega \) for which \( A(\omega) \) is not compact (e.g. by redefining \( A(\omega) = \{e\} \) for \( \omega \in N \) with some fixed \( e \in E \)) we can assume that \( A \) is a compact random set. Proposition 17 further implies that \( A \) can be represented as the closure of a countable union of sets \( \Omega_B \), \( B \in \mathcal{B} \), almost surely. Since \( \Omega_B \) is strictly invariant by Lemma 8 and \( \hat{\Omega}_B(\omega) \) is almost surely compact, \( \hat{\Omega}_B \) is strictly invariant by Lemma 21, and so is \( A \) and the proof of Theorem 13 is complete.

**Remark 22.** It is natural to ask if Theorem 13 remains true if the set \( \mathcal{K} \) is not required to be \( \mathcal{E} \otimes \mathcal{F} \)-measurable (but all other assumptions hold). This is not the case in general. As an example, take \( \Omega = [0, 1] \) equipped with Lebesgue measure \( \mathbb{P} \) on the Borel sets, take \( \vartheta = \text{id} \), \( E = \mathbb{R} \), and \( \varphi = \text{id} \) (with discrete or continuous time). Let \( f : \Omega \to \mathbb{R} \) be a function whose graph \( B \subset E \times \Omega \) is non-measurable and let \( B := \{B\} \). Then \( \Omega_B(\omega) = B(\omega) = \{f(\omega)\} \) for all \( \omega \), hence \( \Omega_B = B \) and the assumptions of Theorem 13 hold with \( \mathcal{K} = B \) except that \( \mathcal{K} \) is not a random set. If a pullback \( \mathcal{B} \)-attractor exists then it cannot possibly be contained in \( \mathcal{K} \), so (iii) of Theorem 13 does not hold. One might hope that Theorem 13 remains true if one replaces \( \mathcal{K} \) by \( \mathcal{K} \) in part (iii). However, also this fails to hold true since \( \mathcal{K} \subset E \times \Omega \) with \( \mathcal{K}(\omega) \) compact for almost all \( \omega \) does not guarantee that \( \mathcal{K}(\omega) \) is almost surely compact and if we choose \( B = \mathcal{K} \) then no \( \mathcal{B} \)-pullback attractor exists.
If we impose additional measurability assumptions on $B$ then measurability of $K$ is not required for Theorem 13 to hold. Indeed, if $\Omega_B$ is a random set for every $B \in \mathcal{B}$ (or even just $\Omega_B \in \mathcal{E} \otimes \mathcal{F}^u$), then define $A$ as in Proposition 19. Due to formula (6) we see that $A(\omega) \subset K(\omega)$ almost surely, so all assumptions of Theorem 13 hold with $K$ replaced by $A$.

4 Existence of a minimal weak attractor for $\mathcal{B}$

Now we discuss the corresponding question for weak attractors. We continue to use covers $R(x, r)$, $x \in D$, as in the previous section.

**Theorem 23.** Let $\mathcal{B}$ be an arbitrary family of sets $B \subset E \times \Omega$ and let $(\varphi, \vartheta)$ be an RDS on $E$.

(i) Assume that there exists a random set $K \subset E \times \Omega$ such that $K(\omega)$ is $\mathbb{P}$-a.s. compact and $K$ attracts $\mathcal{B}$, i.e.

$$\lim_{t \to \infty} d\left(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)\right) = 0 \text{ in probability,}$$

for every $B \in \mathcal{B}$. Then $(\varphi, \vartheta)$ admits at least one weak attractor for $\mathcal{B}$.

(ii) If the RDS $(\varphi, \vartheta)$ has a weak attractor for $\mathcal{B}$, then it has a minimal weak $\mathcal{B}$-attractor $A$. In addition, there is a countable sub-family $\mathcal{B}_1 \subset \mathcal{B}$ such that $A$ is also the minimal weak $\mathcal{B}_1$-attractor.

(iii) Under the assumptions of (i) the minimal weak $\mathcal{B}$-attractor $A$ satisfies $A(\omega) \subset K(\omega)$ almost surely.

**Proof.** For $G \in \mathcal{E}$ and $B \in \mathcal{B}$ define

$$\mathcal{V}^G_B := \left\{ V \in \mathcal{F} : \lim_{t \to \infty} \mathbb{P}^*\left( V \cap \{ \omega : \varphi(t, \vartheta_{-t}\omega)(B(\vartheta_{-t}\omega)) \cap G \neq \emptyset \} \right) = 0 \right\}$$

and put $\beta := \sup\{ \mathbb{P}(V) : V \in \mathcal{V}^G_B \}$. Choose an increasing sequence $V_i \in \mathcal{V}^G_B$ such that $\lim_{i \to \infty} \mathbb{P}(V_i) = \beta$. Then $V := \bigcup_{i=1}^{\infty} V_i \in \mathcal{V}^G_B$ (since outer measures are monotone and subadditive). Put $M^G_B := \Omega \setminus V$ and define

$$K_B := \bigcap_{n=1}^{\infty} \bigcup_{x \in D} \left( R(x, \frac{1}{n}) \times M^R_B \right).$$

Fix $B \in \mathcal{B}$. Then $K_B$ is a closed random set. We first show that $K_B$ is a minimal weak $\{B\}$-attractor and that $K_B$ is contained in $K$.

**Step 1:** $K_B(\omega) \subseteq K(\omega)$ almost surely
Let $B \in \mathcal{B}$, $G \in \mathcal{E}$ and denote $S_t := \{\omega : \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \cap G \neq \emptyset\}$ for $t \geq 0$ and $\mathcal{M}_\delta := \{\omega : \mathcal{K}^\delta(\omega) \cap G \neq \emptyset\}$ for $\delta > 0$. Then $\lim_{t \to \infty} \mathbb{P}(S_t \setminus \mathcal{M}_\delta) = 0$ by assumption, whence $\mathcal{M}_\delta \in \mathcal{Y}_B^G$ and therefore $M_B^G \subseteq \mathcal{M}_\delta$, so $\mathcal{K}^\delta \cap G \neq \emptyset$ on $M_B^G$ almost surely for each $\delta \in \mathbb{Q} \cap (0, \infty)$. This implies

$$\mathcal{K} \cap G \neq \emptyset \quad \text{on } M_B^G \text{ a.s.,}$$

provided $G$ is closed (using the fact that $\mathcal{K}(\omega)$ is almost surely compact). In particular, we have

$$R(x, 1/n) \subseteq \mathcal{K}^{3/n}(\omega) \quad \text{on } M_B^{R(x, \frac{1}{n})} \text{ a.s.}$$

and therefore

$$K_B(\omega) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{x \in D} \left( \mathcal{K}^{3/n}(\omega) \times M_B^{R(x, \frac{1}{n})} \right)(\omega) \subseteq \bigcap_{n=1}^{\infty} \mathcal{K}^{3/n}(\omega) = \mathcal{K}(\omega) \quad \text{a.s.,}$$

proving Step 1.

Our next goal is to show that $K_B$ attracts $B$ in probability. Generally, we say (in this proof) that a random set $S \subset E \times \Omega$ attracts $B$ if

$$\lim_{t \to \infty} d\left(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), S(\omega)\right) = 0 \quad \text{in probability.}$$

**Step 2:** The following properties are easy to verify:

a) If $S_1$ and $S_2$ attract $B$, then so does $S_1 \cap S_2$.

b) If $S_n$, $n \in \mathbb{N}$, are random sets such that $S_n(\omega)$ is compact for almost all $\omega$ and such that $S_n$ attract $B$, $n \in \mathbb{N}$, then so does $\bigcap_{n=1}^\infty S_n$.

**Step 3:** We show that $E_n := \bigcup_{x \in D} \left( R(x, \frac{1}{n}) \times M_B^{R(x, \frac{1}{n})} \right)$ attracts $B$ for each $n \in \mathbb{N}$.

Let $\varepsilon > 0$ and let $K \subset E$ be compact such that $\mathbb{P}(\mathcal{K}(\omega) \subseteq K) \geq 1 - \varepsilon$. Let $D_0 \subset D$ be a finite set such that $K \subset \bigcup_{x \in D_0} R(x, \frac{1}{n})$ and let $\delta := \inf\{d(y, K) : y \in \bigcup_{x \in D_0} (R(x, \frac{1}{n}))^c\}$. Note that $\delta > 0$. We have

$$\mathbb{P}^\delta \left( d\left(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), E_n(\omega)\right) \geq \varepsilon \right)$$

$$\leq \mathbb{P}^\delta \left( \left( \bigcup_{x \in D} \left( \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \cap R(x, \frac{1}{n}) \right) \right) \cup \left( \bigcup_{x \in D_0} \left( R(x, \frac{1}{n}) \times M_B^{R(x, \frac{1}{n})} \right) \right) \geq \varepsilon \right)$$

$$\leq \mathbb{P}^\delta \left( \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \in K^\delta \right)$$

$$+ \sum_{x \in D_0} \mathbb{P}^\delta \left( d\left(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \cap R(x, \frac{1}{n})\right), \left( R(x, \frac{1}{n}) \times M_B^{R(x, \frac{1}{n})} \right) \geq \varepsilon \right).$$
By definition of $M_{B}^{R(x,t)}$ the sum converges to 0 as $t \to \infty$. Further, by the definition of $K$ and $\delta$,
\[
\limsup_{t \to \infty} \mathbb{P}^\ast(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \ni K^\delta) < \varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary it follows that $E_n$ attracts $B$.

**Step 4:** Define $S_n := E_n \cap K$. Then Steps 1, 2 and 3 and the definition of $K_B$ show that $K_B$ attracts $B$.

Note that the statements in Step 1 and Step 4 imply that $K_B$ is a minimal closed random weak $B$-attracting set. Further we know that $K_B(\omega)$ is compact for almost all $\omega \in \Omega$. Changing $K_B(\omega)$ on a set of measure 0 we can ensure that $K_B$ is a compact random set. In order to complete the proof that $K_B$ is a minimal weak $\{B\}$-attractor it remains to show strict invariance of $K_B$.

**Step 5:** $K_B$ is strictly invariant.

The proof of this step is similar to that of Lemma 21. Fix $t > 0$. Since $K_B$ attracts $B$ in probability, we have
\[
\lim_{s \to \infty} d(\varphi(t + s, \theta_{-s}\omega)B(\theta_{-s}\omega), K_B(\theta_{-t}\omega)) = 0 \quad \text{in probability} \quad \text{(9)}
\]
and
\[
\lim_{s \to \infty} d(\varphi(s, \theta_{-s}\omega)B(\theta_{-s}\omega), K_B(\omega)) = 0 \quad \text{in probability}. \quad \text{(10)}
\]
Using the cocycle property and continuity of $\varphi$ together with compactness of $K_B$, (10) implies
\[
\lim_{s \to \infty} d(\varphi(t + s, \theta_{-s}\omega)B(\theta_{-s}\omega), \varphi(t, \omega)K_B(\omega)) = 0 \quad \text{in probability}. \quad \text{(11)}
\]
Using the fact that $\varphi(t, \omega)K_B$ is a random set (as in the proof of Lemma 21), and using minimality of $K_B$, we see from (9) and (11) that $K_B(\theta_{-t}\omega) \subset \varphi(t, \omega)K_B(\omega)$ almost surely. Since (11) implies
\[
\lim_{s \to \infty} d(\varphi(s, \theta_{-s}\omega)B(\theta_{-s}\omega), (\varphi(t, \omega))^{-1}K_B(\theta_{-t}\omega)) = 0 \quad \text{in probability},
\]
equation (10) and minimality of $K_B$ imply the converse inclusion.

The rest of the proof of the theorem is identical to that in the pullback case: just replace the set $\hat{\Omega}_B$ by $K_B$. \qed
5 Non-existence of a minimal forward attractor

In this section we provide an example of an RDS which has a forward attractor, but which fails to have a smallest forward attractor. Consider a stationary Ornstein-Uhlenbeck process $Z$, i.e. a real-valued centered Gaussian process defined on $\mathbb{R}$ with covariance $\mathbb{E}(Z(t)Z(s)) = \frac{1}{2}\exp(-|t-s|)$. We define $Z$ on the canonical space $C(\mathbb{R}, \mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ together with the usual shift and equipped with the law of $Z$. Then $Z(t, \omega) = Z(0, \theta_{t}\omega)$, where $\theta_{t}$ is the shift operator. Let $g : [0,1] \to [-1,0]$ be continuous, non-decreasing such that $g(1) = 0$, $g(x) < 0$ for all $x \in [0,1)$ and $g(x) = -1$ for all $x \in [0,1/2]$, and let $h(t,y)$ be the unique solution of the ordinary differential equation

$$\dot{h}(t) = \begin{cases} g(h(t)) & \text{if } h(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

with initial condition $h(0) = y \in [0,1]$.

Next we define, for $x \in \mathbb{R}$, $y \in [0,1]$, and $t \geq 0$,

$$\varphi(t, \omega)(x,y) := \begin{cases} (x + Z(t, \omega) - Z(0, \omega), h(t,y)) & \text{if } t \leq \tau(y) \\ (e^\tau(x - Z(0, \omega)) + Z(t, \omega), 0) & \text{if } t \geq \tau(y), \end{cases}$$

where $\tau(y) := \min\{s \geq 0 : h(s, y) = 0\}$. Note that the motion in the $y$-direction is deterministic and all points with $y > 0$ move in parallel in the $x$-direction. If $y < 1$, then after the finite (deterministic) time $\tau(y)$, the $y$-component arrives at 0 and stays there, while the first coordinate is attracted by the process $Z$ exponentially fast. It is straightforward to check that $\varphi$ defines a continuous RDS on the Polish space $E := \mathbb{R} \times [0,1]$. If we consider $E$ equipped with the Euclidean metric then the singleton $\{(Z(0, \omega), 0)\}$ is a forward attractor for the family $\mathcal{B} := \{C \times \{0\} : C \subset \mathbb{R} \text{ compact}\}$. This is no longer true if we change the metric on $E$ in the following way (without changing the topology of $E$):

$$d((x,y), (\tilde{x}, \tilde{y})) := |\tilde{y} - y| + |\Gamma(\tilde{x}) - \Gamma(x)|,$$

where $\Gamma$ is strictly increasing, odd, continuous such that $\Gamma(x) = \exp\{\exp(\exp(x))\}$ for large $x$ (the fact that this metric works can be checked by using the fact that the running maximum of a stationary Ornstein Uhlenbeck process up to time $t$ is of the order $\sqrt{\log t}$). There are, however, many forward $\mathcal{B}$-attractors with respect to the metric $d$, for example

$$A_{\gamma}(\omega) := ([Z(0, \omega) - \gamma, Z(0, \omega) + \gamma] \times \{0\}) \cup \left( ([Z(0, \omega) - \gamma] \times [0,1]) \cup ([Z(0, \omega) + \gamma] \times [0,1]) \right)$$

for an arbitrary $\gamma > 0$ (note that this set is strictly invariant!). Now if there would be a smallest forward $\mathcal{B}$-attractor $A(\omega)$ then $A(\omega)$ would have to be contained in the
intersection $A_1(\omega) \cap A_2(\omega)$, which is a subset of $\mathbb{R} \times \{0\}$. It is clear, however, that the set $\{Z(0, \omega), 0\}$ is the only strictly invariant compact subset of $\mathbb{R} \times \{0\}$, and we already have noted that this is not a forward attractor. This contradicts the assumption that there is a smallest forward $B$-attractor. Consequently, this RDS does not have a smallest $B$-attractor.

6 Another example

We construct an example of an RDS for which a minimal pullback point attractor $\omega \mapsto A(\omega)$ exists which, however, does not coincide with $\bigcup_{x \in E} \Omega_x(\omega)$ (writing $\Omega_x(\omega)$ instead of $\Omega_{\{x\}}(\omega)$ for brevity). This shows that Theorem 4 in Crauel and Kloeden [9] is not entirely correct. In the following example, $A(\omega)$ consists of a single point while $\bigcup_{x \in E} \Omega_x(\omega)$ coincides with the whole space $E$ almost surely. Even though we have $\Omega_x(\omega) \subset A(\omega)$ for almost all $\omega \in \Omega$ ([6, Theorem 3.4] or Remark 11(iii)), the (uncountable) union of all $\Omega_x(\omega)$ will turn out to be considerably larger than $A(\omega)$.

Example 24. Let $E := S^1$ be the unit circle which we identify with the interval $[0, 2\pi)$ equipped with the usual metric $d(x, y) := |x - y| \land (2\pi - |x - y|)$. Consider the SDE

$$dX(t) = \cos(X(t))\,dW_1(t) + \sin(X(t))\,dW_2(t)$$

on $E$, where $W_1$ and $W_2$ are independent standard Brownian motions. Then there exists a ‘stable point’ $\omega \mapsto S(\omega)$, measurable with respect to “the past” $\sigma\{W(t) : t \leq 0\}$, $W = (W_1, W_2)$, which is the support of a random invariant measure, and whose Lyapunov exponent is negative (see Baxendale [3]). The random one point set $\{S(\omega)\}$ is a (minimal) weak point attractor (even a forward point attractor) of the RDS $\varphi$ which is generated by the SDE. Recall that the system is reversible. Reverting time and using the same argument for the time inverted system gives existence of an ‘unstable point’ $\omega \mapsto U(\omega)$, measurable with respect to $\{W(t) : t \geq 0\}$ and therefore independent of $S$, which is a weak point repeller. The domain of attraction of $\{S(\omega)\}$ is $E \setminus \{U(\omega)\}$ and that of $\{U(\omega)\}$ for the time-reverted flow is $E \setminus \{S(\omega)\}$.

When considering the system with continuous time, $\omega \mapsto \{S(\omega)\}$ is not a pullback point attractor of $\varphi$, though. In fact we even have $\Omega_x(\omega) = E$ almost surely for each fixed $x \in E$ since for each fixed $y \in E$, the process $t \mapsto \varphi(-t, \omega)y$ is a Brownian motion on $E$ and therefore hits $x$ for (some) arbitrarily large values of $t$ showing that $y \in \Omega_x(\omega)$ for almost all $\omega \in \Omega$. In particular, the unique pullback point attractor of $\varphi$ is the whole space $E$.

To obtain the required example we therefore evaluate the RDS $\varphi$ at integer times only, i.e. we define

$$\psi_n(\omega, x) := \varphi_n(\omega, x), \quad x \in E, \quad n \in \mathbb{N}_0,$$

and we work with $T = \mathbb{Z}$ instead of $T = \mathbb{R}$. We denote the restriction of $\varphi$ to $T = \mathbb{Z}$ with the same symbol. We claim that now $\{S(\omega)\}$ is a minimal pullback point attractor,
but that the closure of the union over all $\Omega$-limit sets of the points in $E$ equals $E$ almost surely.

To see the first claim, consider the neighbourhood $I = (S(\omega) - \varepsilon, S(\omega) + \varepsilon) \mod 2\pi$ of $S(\omega)$ for some $\varepsilon \in (0, 1)$. The claim follows once we show that for each $x \in E$ we have $\Omega_x(\omega) \subset I$ almost surely. Since the Lyapunov exponent of $\psi$ is negative, the normalized Lebesgue measure of the set $\psi_n^{-1}(\vartheta_n, \cdot)(I^c)$ converges to 0 geometrically fast as $n \to \infty$ almost surely. The first Borel-Cantelli lemma now implies that the Lebesgue measure of the set $\psi_n^{-1}(\vartheta_n, \cdot)(I^c)$ converges to 0 almost surely. The first Borel-Cantelli lemma now implies that the Lebesgue measure of the set $C(\omega)$ of all $x \in E$ which are contained in the set $\psi_n^{-1}(\vartheta_n, \cdot)(I^c)$ for infinitely many $n \in \mathbb{N}$ is zero. Since the distribution of $C(\omega)$ is invariant under rotations of $E$ this implies that for each fixed $x \in E$ we have $\mathbb{P}(x \in C(\omega)) = 0$. This being true for every $\varepsilon$ of the form $1/m$ we obtain $\Omega_x(\omega) = \{S(\omega)\}$ almost surely.

To see the second claim, take any non-empty (deterministic) compact interval $J \subset E$. Then the set $\psi_n^{-1}(\vartheta_n, \cdot)(J)$ is a non-trivial interval for each $n$. Note that for the centre point $x \in J$, the process $n \mapsto \psi_n^{-1}(\vartheta_n, \cdot)(x)$ performs a random walk on $E$ (Brownian motion evaluated at discrete time steps) and therefore, the set $\bigcup_n \psi_n^{-1}(\vartheta_n, \cdot)(x)$ is almost surely dense in $E$. Moreover, for any $y \in E$ and any $\delta > 0$, we almost surely find a sequence of integer random times $(\tau_n)$ such that $[y-\delta, y+\delta] \cap \bigcap_n \psi_n^{-1}(\vartheta_n, \cdot)(J) \neq \emptyset$. Therefore, for any $z$ in that set, we have

$$\Omega_z(\omega) \cap J \neq \emptyset \quad \text{almost surely},$$

showing the second claim. Note that we have actually proved more than we claimed insofar that for every non-empty open subset $I$ of $E$ we have

$$\bigcup_{z \in I} \Omega_z(\omega) = E \quad \text{for $\mathbb{P}$-almost every $\omega$}.$$

### 7 Perfection

**Proposition 25.** Let $\varphi, \vartheta$ be an RDS as in Definition 1. Then there exists an RDS $\psi$ on the same measurable dynamical system and a set $\Omega_1 \in \mathcal{F}$ of measure 1 such that $\psi$ agrees with $\varphi$ on $\Omega_1$ and $\psi$ satisfies (i) and (ii) of Definition 2 without exceptional sets.

**Proof.** By assumption, there exists a set $N \in \mathcal{F}$ such that for all $\omega \in N$ the following hold: $x \mapsto \varphi(t, \omega)x$ is continuous for every $t \in \mathbb{T}_2$, $\varphi(0, \omega) = \text{id}$, and $\varphi(t + s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}_2$. Define

$$\Omega_1 := \{\omega \in \Omega : \vartheta_s \omega \in N \text{ for almost all } s \in \mathbb{T}_1\}$$

(here, “almost all” refers to Lebesgue measure in the continuous case and counting measure in the discrete case). Clearly, $\Omega_1$ has full measure and is invariant under $\vartheta_t$ for every $t \in \mathbb{T}_1$. Then $\psi(t, \omega)x := \varphi(t, \omega)x$ in case $\omega \in \Omega_1$ and $\psi(t, \omega) = \text{id}$ in case $\omega \notin \Omega_1$ satisfies the claims in the proposition. \qed
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