On the proper push-forward of the characteristic cycle of a constructible sheaf

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Abstract

We study the compatibility with proper push-forward of the characteristic cycles of a constructible complex on a smooth variety over a perfect field.

The characteristic cycle of a constructible complex on a smooth scheme over a perfect field is defined as a cycle on the cotangent bundle supported on the singular support. It is characterized by the Milnor formula for the vanishing cycles defined for morphisms to curves.

We study the compatibility with proper push-forward. First, we formulate Conjecture on the compatibility with proper direct image. We prove it in some cases, for example, morphisms from surfaces to curves under a mild assumption in Theorem. We briefly sketch the idea of proofs, which use the global index formula computing the Euler-Poincaré characteristic. For the compatibility Theorem, it amounts to prove a conductor formula at each point of the curve. By choosing a point and killing ramification at the other points using Epp’s theorem, we deduce the conductor formula from the index formula. We give a characterization of characteristic cycle in terms of functorialities at the end of the article.

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For the definitions and basic properties of the singular support of a constructible complex on a smooth scheme over a perfect field, we refer to and.

Let $F$ be a constructible complex on a smooth scheme over a perfect field. The singular support $SS\mathcal{F}$ is defined in as a closed conical subset of the cotangent bundle $T^*X$. By Theorem 1.3 (ii)], every irreducible component $C_a$ of the singular support $SS\mathcal{F} = C = \bigcup_a C_a$ is of dimension $n$. The characteristic cycle $CC\mathcal{F} = \sum_a m_a C_a$ is defined as a linear combination with $\mathbb{Z}$-coefficients in Definition 5.10]. It is characterized by the Milnor formula

$$- \dim \text{tot}_u(\mathcal{F}, f) = (CC\mathcal{F}, df)_{T^*\mathcal{U}, u}$$
for morphisms \( f: U \to Y \) to smooth curves \( Y \) defined on an étale neighborhood \( U \) of an isolated characteristic point \( u \). For more detail on the notation, we refer to [4, Section 5.2].

We say that a constructible complex \( \mathcal{F} \) is \textit{locally constant} if every cohomology sheaf \( \mathcal{H}^q \mathcal{F} \) is locally constant. In this case, we have

\[
CC \mathcal{F} = (-1)^n \text{rank } \mathcal{F} \cdot [T^*_X X]
\]

where \( T^*_X X \) denotes the 0-section and \( n = \text{dim } X \) by [4, Lemma 5.11.1]. Assume \( \text{dim } X = 1 \) and let \( U \subset X \) be a dense open subset where \( \mathcal{F} \) is locally constant. For a closed point \( x \in X \), the Artin conductor \( a_x \mathcal{F} \) is defined by

\[
a_x \mathcal{F} = \text{rank } \mathcal{F}|_U - \text{rank } \mathcal{F}_x + \text{Sw}_x \mathcal{F}
\]

where \( \text{Sw}_x \mathcal{F} \) denotes the alternating sum of the Swan conductor at \( x \). Then, by [4, Lemma 5.11.3], we have

\[
CC \mathcal{F} = -\left( \text{rank } \mathcal{F} \cdot [T^*_X X] + \sum_{x \in X \cap U} a_x \mathcal{F} \cdot [T^*_x X] \right)
\]

where \( T^*_x X \) denotes the fiber.

To state the compatibility with push-forward, we fix some terminology and notations. We say that a morphism \( f: X \to Y \) of noetherian scheme is proper (resp. finite) on a closed subset \( Z \subset X \) if its restriction \( Z \to Y \) is proper (resp. finite) with respect to a closed subscheme structure of \( Z \subset X \).

Let \( h: W \to X \) and \( f: W \to Y \) be morphisms of smooth schemes over \( k \). Let \( C \subset X \) be a closed subset such that \( f \) is proper on \( h^{-1}(C) \) and let \( C' = f(h^{-1}(C)) \subset Y \) be the image of \( C \) by the algebraic correspondence \( X \leftarrow W \to Y \). If \( \text{dim } W = \text{dim } X - c \), the intersection theory defines the pull-back and push-forward morphisms

\[
CH_\bullet(C) \xrightarrow{h^!} CH_{\bullet-c}(h^{-1}(C)) \xrightarrow{f_*} CH_{\bullet-c}(C').
\]

We call the composition the morphism defined by the algebraic correspondence \( X \leftarrow W \to Y \). If every irreducible component of \( C \) is of dimension \( n \) and if every irreducible component of \( C' \) is of dimension \( m = n - c \), the morphism \((5)\) defines a morphism \( Z_n(C) \to Z_m(C') \) of free abelian groups of cycles.

Let \( f: X \to Y \) be a morphism of smooth schemes over a field \( k \). Assume that every irreducible component of \( X \) is of dimension \( n \) and that every irreducible component of \( Y \) is of dimension \( m \). Let \( C \subset T^*X \) be a closed conical subset. The intersection \( B = C \cap T^*_X X \) with the 0-section regarded as a closed subset of \( X \) is called the base of \( C \).

Assume that \( f: X \to Y \) is proper on the base \( B \). Then, the morphism \( X \times_Y T^*Y \to T^*Y \) induced by \( f \) is proper on the inverse image \( df^{-1}(C) \) by the canonical morphism \( df: X \times_Y T^*Y \to T^*X \). Let \( f_\circ C \subset T^*Y \) denote the image of \( df^{-1}(C) \) by \( X \times_Y T^*Y \to T^*Y \). Then \( f_\circ C \subset T^*Y \) is a closed conical subset.

By applying the construction of \((5)\) to the algebraic correspondence \( T^*X \leftarrow X \times_Y T^*Y \to T^*Y \) and \( C \subset T^*X \), we obtain a morphism

\[
f_*: CH_n(C) \to CH_m(f_\circ C).
\]
Further if every irreducible component of $C \subset T^*X$ is of dimension $n$ and if every irreducible component of $f_\circ C \subset T^*Y$ is of dimension $m$, we obtain a morphism

\[(7) \quad f_\circ: Z_n(C) \to Z_m(f_\circ C).\]

Now, let $f: X \to Y$ be a morphism of smooth schemes over $k$ and assume that every irreducible component of $Y$ is of dimension $m$. The base $B \subset X$ of the singular support $C = SSF \subset T^*X$ equals the support of $X$ by [2, Lemma 2.1(i)]. Then, the direct image

\[(8) \quad f_\circ CCF \in CH_m(f_\circ C)\]

of the characteristic cycle $CCF$ is defined by the algebraic correspondence $T^*Y \leftarrow X \times_Y T^*Y \to T^*X$. Further if every irreducible component of $f_\circ C \subset T^*Y$ is of dimension $m$, the direct image

\[(9) \quad f_\circ CCF \in Z_m(f_\circ C)\]

is defined as a linear combination of cycles.

**Conjecture 1.** Let $f: X \to Y$ be a morphism of smooth schemes over a perfect field $k$. Assume that every irreducible component of $X$ is of dimension $n$ and that every irreducible component of $Y$ is of dimension $m$. Let $F$ be a constructible complex on $X$ and $C = SSF$ be the singular support. Assume that $f$ is proper on the support of $F$.

1. We have

\[(10) \quad CCRf_\circ F = f_\circ CCF\]

in $CH_m(f_\circ C)$.

2. In particular, if every irreducible component of $f_\circ C \subset T^*Y$ is of dimension $m$, we have an equality (10) of cycles.

If $Y = \text{Spec } k$ is a point and $X$ is proper over $k$, the equality (10) is nothing but the index formula

\[(11) \quad \chi(X_k, F) = (CCF, T^*X)_{T^*X}.\]

This is proved in [4, Theorem 7.13] under the assumption that $X$ is projective. For a closed immersion $i: X \to P$ of smooth schemes over $k$, Conjecture 1 holds [4, Lemma 5.13.2]. Hence, for a proper morphism $g: P \to Y$ of smooth schemes over $k$, Conjecture 1 for $F$ and $f = g \circ i: X \to Y$ is equivalent to that for $i_*F$ and $g: P \to Y$. For the singular support, an inclusion $SSRF_\circ F \subset f_\circ SSF$ is proved in [2, Theorem 1.4 (ii)].

**Lemma 2.** Assume that $f$ is finite on the support of $F$. Then Conjecture 1 holds.

**Proof.** We may assume that $k$ is algebraically closed. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $f_\circ CCF$ satisfies the Milnor formula (1) for $Rf_\circ F$.

Let $Z \subset X$ denote the support of $F$. Let $V \to Y$ be an étale morphism and $g: V \to T$ be a morphism to a smooth curve $T$ with isolated characteristic point $v \in V$ with respect to $f_\circ C$. By replacing $Y$ by $V$, we may assume $V = Y$. 

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By [4] Lemma 3.9.3 (1)⇒(2)] and by the assumption that $Z$ is finite over $Y$, the composition $g \circ f : X \to T$ has isolated characteristic points at the inverse image $Z \times_Y v$. Hence, the composition $g \circ f : X \to T$ is locally acyclic relatively to $\mathcal{F}$ on a neighborhood of the fiber $X \times_Y v$ except at $Z \times_Y v$ and we have a canonical isomorphism
\[ \phi_v(Rf_*\mathcal{F}, g) \to \bigoplus_{u \in Z \times_Y v} \phi_u(\mathcal{F}, g \circ f). \]

Thus by the Milnor formula [1], we have
\[ -\dim \text{tot}\phi_v(Rf_*\mathcal{F}, g) = \sum_{u \in Z \times_Y v} -\dim \text{tot}\phi_u(\mathcal{F}, g \circ f) \]
\[ = \sum_{u \in Z \times_Y v} (\text{CCR}f_*(\mathcal{F}, d(g \circ f))_{T^*X,u} = (f_*\text{CCR}\mathcal{F}, dg)_{T^*Y,v} \]
and the assertion follows.

If $Y$ is a curve, Conjecture [2] may be rephrased as follows. Let $C = SS\mathcal{F}$ be the singular support and assume that on a dense open subscheme $V \subset Y$, the restriction $f_V : X_V = X \times_Y V \to V$ of $f$ is $C$-transversal. Then, $f_V : C \times_Y V$ is a subset of the $0$-section. Thus the condition that every irreducible component of $f_V C$ is of dimension $1$ is satisfied. Further $f_V$ is locally acyclic relatively to $\mathcal{F}$. Since $f$ is proper, $Rf_*\mathcal{F}$ is locally constant on $V$ by [1] Théorème 2.1 and we have
\[ CCRf_*\mathcal{F} = -\left( \text{rank } Rf_*\mathcal{F} : [T^*_Y]\right) + \sum_{y \in Y \setminus V} a_y Rf_*\mathcal{F} \cdot [T^*_yY]. \]

For a closed point $y \in Y$, the Artin conductor $a_y Rf_*\mathcal{F}$ is defined by
\[ a_y Rf_*\mathcal{F} = \chi(X_{\eta}, \mathcal{F}) - \chi(X_y, \mathcal{F}) + Sw_y H^*(X_{\eta}, \mathcal{F}). \]

In the right hand side, the first two terms denote the Euler-Poincaré characteristics of the geometric generic fiber and the geometric closed fiber respectively and the last term denotes the Swan conductor at $y$.

Let $df$ denote the section of $T^*X$ on a neighborhood of the inverse image $X_y$ defined by the pull-back of a basis $dt$ of the line bundle $T^*Y$ for a local coordinate $t$ on a neighborhood of $y \in Y$. Then, the intersection product $(\text{CCR} \mathcal{F}, df)_{T^*X,y}$ supported on the inverse image of $X_y$ is well-defined since $SS\mathcal{F}$ is a closed conical subset.

**Lemma 3.** Let $C = SS\mathcal{F}$ be the singular support and let $V \subset Y$ be a dense open subset such that $f_V : X_V \to V$ is projective, smooth and $C$-transversal.

1. The equality (10) is equivalent to the equality
\[ -a_y Rf_*\mathcal{F} = (\text{CCR} \mathcal{F}, df)_{T^*X,y} \]
at each point $y \in Y \setminus V$, where the right hand side denotes the intersection number supported on the inverse image of $y$.

2. Further, if $f$ has at most isolated characteristic points, then Conjecture [2] holds.

In particular, if $f : X \to Y$ is a finite flat generically étale morphism of smooth curves, then Conjecture [2] holds.

3. Let $\delta_y$ denote the difference of (14). If $X$ and $Y$ are projective, we have $\sum_{y \in Y \setminus V} \delta_y \cdot \deg y = 0$. 

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Proof. 1. Let \( V' \subset V \) be the complement of the images of irreducible components \( C_a \) of the singular support \( C = \bigcup_a C_a \) such that the image of \( C_a \) is a closed point of \( Y \). Then, for every closed point \( w \in V' \), the immersion \( i_w : X_w \to X \) is properly \( C \)-transversal and we have \( CCi_w^*F = i_w^! CC \) by [4, Theorem 7.6]. Further, we have

\[
(15) \quad f_*CCF = - \left( i_w^! CC, T_{X_w}^*X_w \right) \cdot [T_Y Y] - \bigcup_{y \in Y - V} (CC, df)_{T^*X, y} \cdot [T_Y Y].
\]

If we assume that \( f_V : X \times_Y V \to V \) is projective, the index formula (11) implies

\[
\operatorname{rank} Rf_*F = \chi(X, i_w^*F) = (CCi_w^*F, T_{X_w}^*X)_{T^*X}. \]

Thus, it suffices to compare (15) and (12).

2. If \( f \) has at most isolated characteristic points, (14) is an immediate consequence of the Milnor formula (1).

3. We have

\[
(CCRf_*F, T_Y^*Y)_{T^*Y} = \chi(Y, Rf_*F) = \chi(X, F) = (CC, T_X^*X)_{T^*X} = (f_*CCF, T_Y^*Y)_{T^*Y}
\]

by the index formula (11) and the projection formula. Thus it follows from 1.

We prove some cases of Conjecture 1.2 assuming that \( X \) is a surface.

Let \( X \) be a normal noetherian scheme and \( U \subset X \) be a dense open subscheme. Let \( G \) be a finite group and \( V \to U \) be a \( G \)-torsor. The normalization \( Y \to X \) in \( V \) carries a natural action of \( G \). For a geometric point \( \bar{x} \) of \( X \), the stabilizer \( I \subset G \) of a geometric point \( \bar{y} \) of \( Y \) above \( \bar{x} \) is called an inertia subgroup at \( \bar{x} \).

Lemma 4. Let \( G \) be a finite group and

\[
W \xleftarrow{j'} V \xrightarrow{r} X \xleftarrow{j} U
\]

be a cartesian diagram of smooth schemes over a field \( k \) where the horizontal arrows are dense open immersions, the right vertical arrow \( V \to U \) is a \( G \)-torsor and the left vertical arrow \( r : W \to X \) is proper. Assume that for every geometric point \( x \) of \( X \), the order of the inertia group \( I_x \subset G \) is prime to \( \ell \).

Let \( F \) be a locally constant sheaf on \( U \) such that the pull-back \( r^*F \) is a constant sheaf. Then, for the intermediate extension \( j_*F = j_*(F[\operatorname{dim} U])[- \operatorname{dim} U] \) on \( X \), we have an inclusion

\[
SS j_*F \subset r_!(T^*W).
\]

Proof. Since the assertion is étale local on \( X \), we may assume that \( G = I_x \) is of order prime to \( \ell \). Then, the canonical morphism \( F = (r'_i r^*F)G \to r'_i r^*F \) is a splitting injection and induces a splitting injection \( j_*F \to j_*r'_! r^*F \). Hence, we have

\[
SS(j_*F) \subset SS(j_*r'_! r^*F).
\]
Since every irreducible subquotient of the shifted perverse sheaf $j_!r^s_!H^r_!F$ is isomorphic to an irreducible subquotient of a shifted perverse sheaf $\mathcal{H}^0_!(Rr^s_!H^0_!F)[\dim U][−\dim U]$ extending $r^s_!H^r_!F$, we have

$$SS(j_!r^s_!H^r_!F) \subset SS(\mathcal{H}^0_!(Rr^s_!H^0_!F)) \subset SS(Rr^s_!H^r_!F)$$

by [2] Theorem 1.4 (ii). Since $j^*_sH^rF$ is a constant sheaf on $W$ and $r$ is proper, we have

$$SS(Rr^s_!H^r_!F) \subset r_0SS(j^*_sH^rF) \subset r_0(T^*_W\mathcal{W})$$

by [2] Lemma 2.2 (ii), Lemma 2.1 (iii). Thus the assertion follows.

**Proposition 5.** Let $X$ be a normal scheme of finite type over a perfect field $k$, $Y$ be a smooth curve over $k$ and $f: X \to Y$ be a flat morphism over $k$. Let $V \subset Y$ be a dense open subscheme such that $f_V: X_V = X \times_Y V \to V$ is smooth.

1. There exist a finite flat surjective morphism $g: Y' \to Y$ of smooth curves over $k$ and a dense open subscheme $X'' \subset X'$ of the normalization $X'$ of $X \times_Y Y'$ satisfying the following condition:

   (1) We have inclusions $X'_V = X' \times_Y V \subset X'' \subset X'$ and $X'' \subset X'$ is dense in every fiber of $X' \to Y'$. The morphism $X'' \to Y'$ is smooth.

   2. Let $\mathcal{F}$ be a perverse sheaf on $X_V$ and let $C \subset T^*X_V$ be a closed conical subset on which $\mathcal{F}$ is micro-supported. Assume that $f_V: X_V \to V$ is $C$-transversal. Then, there exist $g: Y' \to Y$ and $X'' \subset X' \to X \times_Y Y'$ as in 1. satisfying the condition (1) above and the following condition:

   (2) Let $j': X'_V \to X''$ denote the open immersion and $C' = SSj'_s\mathcal{F}$ be the singular support of the intermediate extension $j'_s\mathcal{F}'$ of the pull-back $\mathcal{F}'$ of $\mathcal{F}$ to $X'_V$. Then, the morphism $X'' \to Y'$ is $C'$-transversal.

**Proof.** By deissage and approximation, we may assume that the complement $Y \dashv V$ consists of a single closed point $y$ and that the closed fiber $X_y$ is irreducible. The assertion is local on a neighborhood in $X$ of the generic point $\xi$ of $X_y$.

1. It follows from [3].

2. Since $f_V: X_V \to V$ is smooth, the $C$-transversality of $f_V$ and the condition that $\mathcal{F}$ is micro-supported on $C$ are preserved after base change by [4] Lemma 3.9.2, Lemma 4.2.4. After replacing $Y$ by $Y'$ and $X$ by $X''$ as in 1., we may assume that $X \to Y$ is smooth. Shrink $X$ and $Y$ further if necessary, we may assume that $\mathcal{F}$ is locally constant.

Let $W_V \to X_V$ be a $G$-torsor for a finite group $G$ such that the pull-back of $\mathcal{F}$ on $W_V$ is a constant sheaf. Let $r: W \to X$ be the normalization of $X$ in $W_V$. Applying 1 to $W \to Y$ and shrinking $X$ if necessary, we may assume that there exists a finite flat surjective morphism of smooth curves $Y' \to Y$ such that the normalization $W'$ of $W \times_Y Y'$ is smooth over $Y'$.

Let $r': W' \to X'$ be the canonical morphism. Since the ramification index at the generic point $\xi'$ of an irreducible component of the fiber $X'_y$ is 1, the inertia group at $\xi'$ is of order a power of $p$. Hence, after shrinking $X$ if necessary, we may assume that for every geometric point $w'$ of $W'$, the order of the inertia group is a power of $p$. Hence, by Lemma [4] we have $C' = SS(j'_s\mathcal{F}) \subset r'_0(T^*_W\mathcal{W}')$. Since $W' \to Y'$ is smooth, the morphism $j': X' \to Y'$ is $C'$-transversal by [4] Lemma 3.9.3 (2) ⇒ (1)].

**Theorem 6.** Let the notation be as in Conjecture [1] and let $C = SS\mathcal{F}$ be the singular support. Assume that $\dim X = 2$, $\dim Y = 1$ and that there exists a dense open subscheme $V \subset Y$ such that $f_V: X_V \to V$ is smooth and $C$-transversal. Then, Conjecture [1]2 holds.
Proof. We may assume $\mathcal{F}$ is a perverse sheaf by [2, Theorem 1.4 (ii)]. Since the resolution of singularity is known for curves and surfaces, we may assume $Y$ is projective. Since a proper smooth surface over a field is projective, the surface $X$ is projective. Let $y \in Y \subset V$ be a point. It suffices to show the equality (14).

By Proposition 3 and approximation, there exists a finite flat surjective morphism $Y' \to Y$ of proper smooth curves étale at $y$ and satisfying the conditions in Proposition 3 on the complement $Y - \{y\}$. Since the normalization $X'$ of $X \times_Y Y'$ is projective, we may take a projective smooth scheme $P$ and decompose $f': X' \to Y$ as a composition $X' \to P \to Y$ of a closed immersion $i: X' \to Y$ and $g: P \to Y$.

Let $U = V \cup \{y\}$. Let $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ to $X'_U = X' \times_Y U$ and let $j_{s*}\mathcal{F}'$ be the intermediate extension with respect to the open immersion $j': X'_U \to X'$. It suffices to show Conjecture 12 holds for $P \to Y$ and $\mathcal{G} = i_*j_{s*}\mathcal{F}'$. Outside the inverse image of $y$, the morphism $g: P \to Y$ has at most isolated characteristic points with respect to the singular support $SS\mathcal{G}$ by the condition (2) in Proposition 3 and Lemma 3.9.3 (2) applied to the restriction of the immersion $X' \to P$ on the complement of a finite closed subset of $X'$. Thus, we have $\delta_y = 0$ for any closed point $y' \in Y'$ not on $y$ by Lemma 3.2. This implies $[Y': Y] \cdot \delta_y = 0$ by Lemma 3.3. Thus the assertion follows.

Theorem 7. Let the notation be as in Conjecture 1 and let $C = SS\mathcal{F}$ be the singular support. Assume that $\dim X = \dim Y = 2$, that $f: X \to Y$ is proper surjective and that every irreducible component of $f_\ast C$ is of dimension 2. Then, Conjecture 12 holds.

Proof. By Lemma 2 the assertion holds except possibly for the coefficients of the fibers $T^*_yY$ of finitely many closed points $y \in Y$ where $X \to Y$ is not finite. Let $v \in Y$ be a closed point and we show that the coefficients of the fibers $T^*_vY$ are equal.

Since the resolution of singularity is known for surfaces and since a proper smooth surface over a field is projective, we may assume that $Y$ and hence $X$ are projective. By replacing $X$ by the Stein factorization of $X \to Y$ except on a neighborhood of $v$, we define $X \to X' \to Y$ such that $f': X' \to Y$ is finite on the complement of $v$ and $r: X \to X'$ is an isomorphism on the inverse image of a neighborhood of $v$.

Since $X'$ is projective, we may take a projective smooth scheme $P$ and decompose $f': X' \to Y$ as a composition $X' \to P \to Y$ of a closed immersion $i: X' \to P$ and $g: P \to Y$. Conjecture 12 holds for $\mathcal{G} = i_*Rr_\ast\mathcal{F}$ and $g: P \to Y$ except possibly for the coefficients of the fiber $T^*_vY$ by Lemma 2. Namely, we have $g_\ast CCR\mathcal{G} = CCRg_\ast\mathcal{G} = CCRf_\ast\mathcal{F}$ except possibly for the coefficients of the fiber $T^*_vY$. By the index formula (11), we have

$$(CCRf_\ast\mathcal{F}, T^*_vY)_{T^*_vY} = \chi(Y, Rf_\ast\mathcal{F}) = \chi(P, \mathcal{G}) = (CCR\mathcal{G}, T^*_vP)_{T^*_vP} = (g_\ast CCR\mathcal{G}, T^*_vY)_{T^*_vY}.$$  

Thus, we have an equality also for the coefficients of the fiber $T^*_vY$. 

We give a characterization of characteristic cycle using functoriality. For the definition of $C$-transversal morphisms $h: W \to X$ of smooth schemes and the pull-back $Z_n(C) \to Z_m(h\circ C)$, we refer to [1, Definition 7.1]. For a constructible complex $\mathcal{F}$ on a projective space $P = P^n$, let $RF = Rp^*_\ast p^\ast \mathcal{F}[n-1]$ denote the Radon transform on the dual projective space $P^\vee$ where $p: Q \to P$ and $p^\vee: Q \to P^\vee$ denote the projections on the universal family of hyperplanes $Q = \{(x, H) \in P \times P^\vee \mid x \in H\}$. For a linear combination $A = \sum a_mC_a$ of irreducible closed conical subset $C_a \subset T^*P$ of dimension $n$, let $LA = (-1)^{n-m}p^\vee_\ast p^\ast A$ denote the Legendre transform (cf. [1, Corollary 7.5]).
Proposition 8. Let \( k \) be a perfect field and \( \Lambda \) be a finite field of characteristic \( \ell \) invertible in \( k \). Then, there exists a unique way to attach a linear combination \( A(\mathcal{F}) = \sum_{a} m_a C_a \) satisfying the conditions (1)–(5) below of irreducible components of the singular support \( SS F = C = \bigcup_{a} C_a \subset T^* X \) to each smooth scheme \( X \) over \( k \) and each constructible complex \( \mathcal{F} \) of \( \Lambda \)-modules on \( X \):

1. For every étale morphism \( j: U \to X \), we have \( A(j^* \mathcal{F}) = j^* A(\mathcal{F}) \).

2. For every properly \( C \)-transversal closed immersion \( i: W \to X \) of smooth schemes, we have \( A(i^* \mathcal{F}) = i^! A(\mathcal{F}) \).

3. For every closed immersion \( i: X \to P \) of smooth schemes, we have \( A(i_* \mathcal{F}) = i_* A(\mathcal{F}) \).

4. For the Radon transform, we have \( A(RF) = LA(\mathcal{F}) \).

5. For \( X = \text{Spec} \ k \), we have \( A(\mathcal{F}) = \text{rank} \, \mathcal{F} \cdot T_X X \).

If \( A(\mathcal{F}) \) satisfies the conditions (1)–(5), then we have

\[
A(\mathcal{F}) = CCF. 
\]

As the proof below shows, it suffices to assume the condition (2) in the case where \( i: x \to X \) is the closed immersion of a closed point or \( i: L \to P \) is the closed immersions of lines in projective spaces. By the definition of the naive Radon transform and the naive Legendre transform, the equality in condition (4) can be decomposed as \( A(Rp^y p^* \mathcal{F}) = p^y_* A(p^* \mathcal{F}) = p^y_* p^! A(\mathcal{F}) \). The first (resp. second) equality corresponds to a special case of Conjecture [4, Lemma 3.11] (resp. to a special case of [4, Proposition 5.17]).

Proof. The characteristic cycles satisfy the conditions (1)–(5) by [4, Lemma 5.11.2], [4, Theorem 6.6], [4, Lemma 3.11.2], [4, Corollary 7.12], [4, Lemma 5.11.1] respectively. Thus the existence is proved.

We show the uniqueness. It suffices to show the equality (17). By the condition (2) applied to the closed immersion \( x \to X \) of a closed point in a dense open subscheme \( U \subset X \) where \( \mathcal{F} \) is locally constant and by the conditions (1) and (5), the coefficient of the 0-section \( T_X X \) in \( A(\mathcal{F}) \) equals the rank of the restriction \( \mathcal{F}|_U \).

We show that this property and the conditions (3)–(5) imply the index formula for projective smooth scheme \( X \). By (3), we may assume that \( X \) is a projective space \( P = P^n \) for \( n \geq 2 \). Let \( R^y G = Rp_! p^* G(n-1)[n-1] \) and \( L^y B = (-1)^{n-1} p_* p^! B \) denote the inverse Radon transform and the inverse Legendre transform. Then, \( R^y RF \) is isomorphic to \( \mathcal{F} \) up to locally constant complex of rank \( (n-2) \cdot \chi(P_{\bar{k}}, \mathcal{F}) \). Hence the coefficient of the 0-section \( T_0 P \) in \( A(R^y RF) \) equals that in \( A(\mathcal{F}) \) plus \( (n-1) \cdot \chi(P_{\bar{k}}, \mathcal{F}) \). Similarly, the coefficient of \( T_0 P \) in \( L^y LA(\mathcal{F}) \) equals that in \( A(\mathcal{F}) \) plus \( (n-1) \cdot (A(\mathcal{F}), T_0 P)_{T_0 P} \). Thus, we obtain the index formula \( \chi(P_{\bar{k}}, \mathcal{F}) = (A(\mathcal{F}), T_0 P)_{T_0 P} \) for projective \( X \).

Next, we show (17) assuming \( \dim X = 1 \). Let \( U \subset X \) be a dense open subscheme where \( \mathcal{F} \) is locally constant. Then, we have

\[
A(\mathcal{F}) = - \left( \text{rank} \, \mathcal{F} \cdot T_X^* X + \sum_{x \in X - U} m_x \cdot T_x^* X \right).
\]

We show \( m_x \) equals the Artin conductor \( a_x \mathcal{F} \). If \( \mathcal{F}|_U \) is unramified at \( x \) and if \( \mathcal{F}_x = 0 \), the same argument as above shows \( m_x = \text{rank} \, \mathcal{F}|_U = a_x \mathcal{F} \). We show the general case. By (1), we may assume \( U = X - \{ x \} \). Further shrinking \( X \), we may assume that there exists a finite étale surjective morphism \( \pi: X' \to X \) such that the pull-back \( \pi^* \mathcal{F} \) is unramified.
at every point $X' \cong X'$ of the boundary of a smooth compactification $j': X' \to \bar{X}'$. Then, we have

$$A(j'_* \pi^* \mathcal{F}) = - \left( \text{rank } \mathcal{F}|_U \cdot T_{X'}^* \bar{X}' + \sum_{x' \in \pi^{-1}(x)} m_x \cdot T_{x'}^* \bar{X}' + \sum_{x' \in X' \setminus X'} a_x \mathcal{F} \cdot T_{x'}^* \bar{X}' \right).$$

Thus, the index formula implies $m_x = a_x \mathcal{F}$.

We show the general case. By (1), we may assume $X$ is affine. We consider an immersion $X \to \mathbb{A}^n \subset \mathbb{P}^n$. Then, by (1) and (3), we may assume $X$ is projective. Set $SS \mathcal{F} = C = \bigcup_a C_a$ and take a projective embedding $X \to \mathbb{P}$ and a pencil $L \subset \mathbb{P}^r$ satisfying the following properties as in [5, Lemma 2.3]: The axis $A_L$ of the pencil meets $X$ transversely, that the blow-up $\pi_L: X_L \to X$ is $C$-transversal, that the morphism $p_L: X_L \to L$ defined by the pencil has at most isolated characteristic points, that the isolated characteristic points are not contained in the inverse image of $V$ and are unique in the fibers of $p_L$, and that for each irreducible component $C_a$ there exists an isolated characteristic point $u$ where a section $dp_L$ of $T^*X$ meets $C_a$.

Then, for $v = p_L(u)$, the coefficient $m_v$ of $T^*_v L$ in $A(Rp_L!* \pi^* \mathcal{F})$ equals the Artin conductor $- \dim \text{tot } \phi_u(\mathcal{F}, p_L) = -a_\mathcal{F} R p_L!* \pi^* \mathcal{F}$. Let $i_L: L \to \mathbb{P}^r$ denote the immersion. Then, by the proper base change theorem and by the conditions (2) and (3), we have $A(Rp_L!* \pi^* \mathcal{F}) = (-1)^{n-1} i_L^! A(R\mathcal{F}) = i_L^! L A(\mathcal{F})$. Thus, if $A(\mathcal{F}) = \sum_a m_a C_a$, the coefficient $m_v$ equals $m_a \cdot (C_a, dp_L)_{T^*X,u} \neq 0$ and we obtain

$$- \dim \text{tot } \phi_u(\mathcal{F}, p_L) = m_a \cdot (C_a, dp_L)_{T^*X,u} \neq 0.$$  

This means that the coefficient $m_a$ is characterized by the same condition as the Milnor formula (11) and we have (17). \hfill \square

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