A new generation of atomic sensors using ultra-narrow optical clock transitions and composite pulses are pushing quantum engineering control to a very high level of precision for applied and fundamental physics. Here, we propose a new version of Ramsey-Bordé interferometry introducing arbitrary composite laser pulses with tailored pulse duration, Rabi field, detuning and phase-steps. We explore quantum metrology below the $10^{-18}$ level of fractional accuracy by a fine tuning control of light excitation parameters protecting ultra-narrow optical clock transitions against residual light-shift coupled to laser-probe field fluctuation. We present, for the first time, new developments for robust hyper Ramsey-Bordé and Mach-Zehnder interferometers, where we protect wavepacket interferences against distortion on frequency or phase measurement related to residual Doppler effects and light-shifts coupled to a pulse area error. Quantum matter-wave sensors with composite pulses and ultra-cold sources will offer detection of inertial effects inducing phase-shifts with better accuracy, to generate hyper-robust optical clocks and improving tests of fundamental physics, to realize a new class of atomic interferometers tracking space-time gravitational waves with a very high sensitivity.

I. INTRODUCTION

Seventy years ago, Ramsey established the first quantum mechanical description of an interferometric resonance with the method of separated oscillating fields [1]. Ramsey spectroscopy with coherent radiation and phase manipulation became a very effective tool to investigate internal properties of nuclei, atoms and molecules [2–4] while opening a revolution in quantum metrology with atomic fountains as primary frequency standards reaching today a fractional frequency accuracy of $2 \times 10^{-16}$ [5].

By labeling internal states with momentum quantization, Bordé has extended the method of separated oscillating field to atomic interferometry with optical transitions realizing laser beam splitters and mirrors for matter-waves [6–8]. Pioneering works on interferences with atoms were also made by Chebotayev and Dubetsky based on separated optical fields with standing waves [9–10]. Ramsey-Bordé interferometers using cold and ultra-cold atom techniques have reached very impressive high sensitivity to rotation [11–12], acceleration [13], accurate determination of the fine structure constant [14–15] or optical clock realization with supersonic beams reaching a fractional frequency instability around $2 \times 10^{-16}$ [19]. Mach-Zehnder type quantum sensors have thus been developed for gravitational field measurements [20–22].

In parallel to laser spectroscopy and atom interferometry, composite pulses in Nuclear Magnetic Resonance (NMR) became a powerful tool to compensate for several imperfections due to radio-frequency (rf) pulses applied on large samples of nuclear spins [23–24]. Various signal distortions from rf field inhomogeneities, off-resonance effects and field amplitude error were reduced to a very low order of correction by means of complex sequences of pulses adapted to single or even dual compensation of these systematics. Composite pulses have also demonstrated to be useful for robust error compensation in high-fidelity qubit gates dedicated to quantum computation [25–27].

So far, understanding how to improve the robustness of precision measurements while reducing laser-probe-induced systematics still remains a critical goal for quantum sensing. But contrary to recent composite NMR-like pulse techniques applied in interferometers [28–30], composite laser pulses are required to reduce or eliminate residual phase-shifts leading to distortions of interferometric resonances. A major step in that direction was realized in 2010 with the introduction of the hyper-Ramsey scheme to experimentally reduce laser-probe-induced frequency shifts by several orders of magnitude in optical clocks requiring large probe intensities [31–32].
II. HYPER RAMSEY-BORDÉ BUILDING BLOCK

The formal derivation of the generalized Ramsey-Bordé amplitude probability $\Psi(t)$ is based on Cayley-Klein parametrization of rotation spinors as $^{35}$:

$$M(\vec{\vartheta}_l) = \begin{pmatrix} \cos \vartheta_l e^{i\varphi_l} & -ie^{-i\varphi_l} \sin \vartheta_l \\ -ie^{i\varphi_l} \sin \vartheta_l & \cos \vartheta_l e^{-i\varphi_l} \end{pmatrix},$$

with the action of a phase $\varphi_l$ on the Rabi complex frequency $\Omega_l$. Phase angles are introduced by the following definitions:

$$\vartheta_l = \arcsin \left( \frac{\Omega_l}{\omega_l} \sin \tilde{\vartheta}_l \right),$$

$$\varphi_l = \arctan \left( \frac{\delta_l}{\omega_l} \tan \tilde{\vartheta}_l \right).$$

The effective pulse area is $\tilde{\vartheta}_l = \vartheta_l/2 = \omega_l \tau_l/2$ with a generalized Rabi frequency denoted as $\omega_l = \sqrt{\delta_l^2 + \Omega_l^2}$. The effective detuning $\delta_l = \delta + \Delta_l$ is a free detuning including Doppler shift and quantized atomic recoil when required as in $^{39}$ with the residual uncompensated part of the light-shift $\Delta_l$. The Cayley-Klein parametrization is emphasizing the role of any residual light-shift correction as an additional phase factor acting on diagonal elements of the interaction matrix related to interference detection through atomic population transfer.

Our model is based on the exact description of full composite wave-function with spinors $^{38,40}$ incorporating independent control and fine tuning of coherent radiation parameters in the following form:

$$\Psi(t) = \left[ \prod_{l=1}^{p} M(\vec{\vartheta}_l) \right] \cdot M(\delta T) \cdot \left[ \prod_{l=1}^{q} M(\vec{\vartheta}_l) \right] \Psi(0).$$

Each arrow indicates the direction to develop the product of matrices around a single free evolution time $M(\delta T)$ where laser fields, thus the light-shifts, are switch-off.

Complex amplitudes of $\Psi(t)$ for a two-level spin system being initially prepared in $\Psi(t = 0)$, can be obtained by the application of successive $p$ pulses before and $q$ pulses after the free evolution time (See Fig.1) leading to a complex matrix given by $^{41}$:

$$\Psi(t) = M \cdot \Psi(0) \begin{pmatrix} C_g(t) \\ C_e(t) \end{pmatrix} = \begin{pmatrix} q_{pC_{gg}} q_{pC_{ge}} \\ p_{C_{cg}} q_{pC_{ce}} \end{pmatrix} \cdot \begin{pmatrix} C_g(0) \\ C_e(0) \end{pmatrix},$$

where $q_{M}$ is a special unitary operator and $q_{pC_{gg}} = q_{pC_{gg}}$, $q_{pC_{ge}} = q_{pC_{ge}}$, $q_{pC_{cg}}^2 = |q_{pC_{cg}}|^2 + |q_{pC_{ge}}|^2 = 1$.

The complex probability amplitude associated to $q_{pC_{gg}}$ and $q_{pC_{ge}}$ can be recast into a symmetric canonical form as following:

$$q_{pC_{gg}} = q_{pC_{gg}} \alpha_{gg} e^{i\delta T/2} \left[ 1 - |q_{pC_{gg}}|^2 e^{-i(\delta T + \delta)} \right],$$

$$q_{pC_{ge}} = q_{pC_{ge}} \alpha_{ge} e^{i\delta T/2} \left[ 1 + |q_{pC_{ge}}|^2 e^{-i(\delta T + \delta)} \right].$$

A sequence of two Ramsey pulses was used where an additional third one acts like a spin echo compensation of field amplitude error. This extra pulse can be inserted either before or after the free evolution time to strongly reduce the residual phase-shift $^{33}$. Moreover, new generalized hyper-Ramsey protocols have extended robustness of probing clock transitions against residual light-shifts coupled to decoherence $^{34}$.

The main motivation of this work is to bring optical composite pulses to matter-wave interferometry with efficient nonlinear compensation of pulse-defects induced phase-shifts while these methods are usually absent in modern atomic interferometry $^{35,37}$. We will revisit Ramsey-Bordé interferometry including internal state labeling, light-shift, Doppler-shift and atomic recoil with arbitrary sequences of composite pulses around a single free evolution time. A universal building-block, shown in Fig.1 is developed through section II offering an efficient computational algorithm to explore interferometric resonances and phase-shifts produced by composite pulses dragging matter-waves. We will review a few robust composite pulse schemes for hyper-clocks in section III. Then, in section IV, we will extend the application of composite pulses to hyper-interferometers which act against laser pulse errors induced by laser intensity variation during interrogation protocols. We will conclude about the potential impact of applying composite pulses with laser phase-step protocols in Ramsey-Bordé matter-wave interferometry within section V.
Remarkably, we have found that complex parameters $\alpha$ and $\beta$ driving the overall envelop and composite phase-shifts can be also separated in two independent contributions from $p$ pulses driven by pulse area $\vec{v}_1$ and $q$ pulses driven by $\vec{v}_1$ as following:

\begin{align}
q\alpha_{gg} &= \alpha_{p}^{gg} \cdot \alpha_{q}^{qq}, \\
q\beta_{gg} &= \beta_{p}^{gg} \cdot \beta_{q}^{qq}, \\
q\alpha_{ge} &= \alpha_{p}^{ge} \cdot \alpha_{q}^{gg}, \\
q\beta_{ge} &= \beta_{p}^{ge} \cdot \beta_{q}^{gg}.
\end{align}

Envelop terms $q\alpha_{gg}$ and $q\alpha_{ge}$ have been explicitly developed for arbitrary cases in the Appendix section S0 following [42]. From Eq. (5a) and Eq. (5b) it follows that interferometric phase-shifts affecting the central interference $q\Phi_{gg}$ or $q\Phi_{ge}$ are given by:

\begin{align}
q\Phi_{gg} &= \varphi_L + \phi_L - (\text{Arg}[\beta_{p}^{gg}(gg)] + \text{Arg}[\beta_{q}^{gg}(gg)]), \\
q\Phi_{ge} &= \varphi_L + \phi_L - (\text{Arg}[\beta_{p}^{ge}(ge)] + \text{Arg}[\beta_{q}^{gg}(gg)]),
\end{align}

with a remnant phase definition $\varphi_L = \varphi_1 - \varphi_1^{\prime}$ corrected by a light-shifted contribution $\phi_L = \phi_1^{\prime} + \phi_1$ from pulses forming the original two-pulse Ramsey configuration. Note that phase-factors are now including a contribution from an arbitrary number of optical composite pulses extending previous results with three pulses [38].

Let’s now derive the formal expression of complex factors $\beta_{p}^{gg}(gg)$ and $\beta_{p}^{ge}(gg)$ leading to a main distortion of matter-waves interferences. Composites phase-shifts $q\Phi_{gg}$ and $q\Phi_{ge}$ are driven by a truncated continued fraction expansion with $p, q$ pulses as following:

\begin{align}
\beta_{p}^{gg}(gg) &= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ... \\
&= \tan \vec{v}_1 + e^{-i\varepsilon_1} \tan \vec{v}_2 + e^{-i\varepsilon_2} \tan \vec{v}_3 + ...
\end{align}

where $\{ \}$* means complex conjugate. Phase-factor expressions are respectively $\Xi_{\ell+1} = \varphi_{\ell}^{\prime} - \varphi_{\ell}^{\prime} + \phi_{\ell}^{\prime} + \phi_{\ell+1}^{\prime}$ and $\Xi_{\ell+1} = \varphi_{\ell+1} - \varphi_{\ell} + \phi_{\ell} + \phi_{\ell+1}$.

Turning to applications, we demonstrate the capacity of composite pulses to reduce or eliminate laser-probe induced systematics on quantum interferences. We explore clock interrogation protocols limited in accuracy by laser-probe-intensity fluctuations and atomic interferometers limited in accuracy by residual Doppler-shift and light-shift induced by pulse area variation between sets of beam splitters from different Ramsey-Bordé building-blocks. Generalized transition probabilities related to laser pulse protocols are analytically derived in the Appendix (S1 for hyper-clock interrogation scheme with $p = q = 4$ pulses, S2 for a generalized hyper-Ramsey-Bordé (GHRB) interferometer and S3 for a hyper-Mach-Zehnder (HMZ) configuration).

### III. HYPER-CLOCKS

For a clock configuration, a single trapped-ion with a very narrow atomic transition is confined into a Lamb-Dicke regime where Doppler-shift and recoil are eliminated [43-45]. Dispersive error signals are produced to estimate precisely the center of the interferometric reso-
nance eliminating any shape distortion \cite{2,31,33}. They are generated by applying opposite phase-steps \( \pm \phi \) on the same transition probability \( _p \! P_e = 1 - _p \! P_g \) and computing the difference as:

\[
\Delta E = _p \! P_e(\varphi) - _p \! P_e(-\varphi),
\]

(9)

We give here the matrix components to evaluate all laser pulse protocols reported in Tab. I, and used in the main text. Components \( S_{gg}^{\tau}, S_{ge}^{\tau} \) are computed with the following elements:

\[
\alpha^2_1(gg) = \left( \prod_{p=2} \cos \theta^p e^{i\varphi^p} \right) \cdot (1 - S_{2,2}'), \quad (10a)
\]

\[
\alpha^2_1(gg) = \left( \prod_{q=2} \cos \theta^q e^{i\varphi^q} \right) \cdot (1 - S_{2,2}'), \quad (10b)
\]

\[
\alpha^2_1(ge) = -ie^{-i(\varphi_2' + \varphi_2 + \Xi_{12})} \left( \prod_{p=2} \cos \theta^p e^{i\varphi^p} \right) \cdot (S_{2,1}'), \quad (10c)
\]

We have respectively for \( S_{2,2}^{\prime}(gg), S_{2,1}^{\prime}(ge) \) and \( S_{2,2}^{\prime}(gg) \):

\[
S_{2,2} = e^{-i\Xi_{12}} \tan \vartheta_1 \tan \vartheta_2, \quad (11a)
\]

\[
S_{2,1} = \tan \vartheta_1' + e^{i\xi} \tan \vartheta_2, \quad (11b)
\]

\[
S_{2,2} = e^{-i\Xi_{12}} \tan \vartheta_1 \tan \vartheta_2, \quad (11c)
\]

The corresponding complex phase factor \( \beta^2_1(gg), \beta_1^2(gg) \) leading to a phase-shift correction are:

\[
\beta^2_1(gg) = \frac{\tan \vartheta_1' + e^{-i\Xi_{12}} \tan \vartheta_2'}{1 - e^{-i\Xi_{12}} \tan \vartheta_1' \tan \vartheta_2'}, \quad (12a)
\]

\[
\beta^2_1(ge) = \frac{\tan \vartheta_1 + e^{-i\Xi_{12}} \tan \vartheta_2}{1 - e^{-i\Xi_{12}} \tan \vartheta_1 \tan \vartheta_2}, \quad (12b)
\]

\[
\beta^2_1(ge) = \frac{1}{\{\beta^2_1(ge)\}^*}, \quad (12c)
\]

We give the decomposition of phase factor expressions as following:

\[
\Xi_1' = 0 \quad (13a)
\]

\[
\Xi_2' = \Xi_{12} \quad (13b)
\]

\[
\Xi_{12} = \varphi_1' - \varphi_2' + \varphi_1 + \varphi_2, \quad (14a)
\]

\[
\Xi_{12} = \varphi_2 - \varphi_1 + \varphi_1 + \varphi_2. \quad (14b)
\]

See also \cite{40} as another computational way to obtain directly \( S_{gg}^{\tau}, S_{ge}^{\tau} \) from \( S_{gg}^{\tau} \).

The transition probability can be detected by measuring the atomic population fraction in the ground state or flipping to the excited state. Hyper-Ramsey (HR-\( \pi \)) and generalized hyper-Ramsey GHR(\( \pi/4 \)) protocols with 3 pulses are compared to a new hybrid hyper-Ramsey (HHR\(-\pi \)) protocol using 4 pulses from Tab. II. We employ the Beloy’s model to analyze laser-probe-intensity fluctuation between repetitive sequences of composite pulses for clock interrogation \cite{47}. Typical dispersive shapes for \( \Delta E \) are presented in Fig. 3 (a), (b) and (c). The fractional clock error, related to the specific electric octupole (E3) clock transition of a single trapped ion \( ^{171}Yb^{+} \), is reported in Fig. 3 (a) and (b). The robustness of the
shifts in optical lattices offered by the trap depth \[48\], we have investigated the pulse area control within the variation of the Rabi frequency through the q parameter changed from 0.9 to 1.1 in Fig. 4(a) and from 1.05 to 1.07 in Fig. 4(b). Several ”magic” values of the mean Rabi frequency \(\Omega\tau = q_m\pi/2\), if well controlled, can be used with HHR–π protocol to remove clock sensitivity to laser amplitude fluctuation. For example, from Fig. 4(b), we get \(q_m = 1.05636\) where a cancelation of the fractional clock error at the \(10^{-20}\) level (yellow region) seems possible for a frequency fluctuation at the \(1\%\) level. If our Rabi frequency fluctuation is pushed below 0.6%, the value \(q_m = 1.07\) seems a better choice. In any case, the GHR protocol remains very robust as long as the phase-step is precisely controlled as seen in Fig. 4(a) and (b) since the residual part of the compensated light-shift is removed at all order in the clock detuning \[49\]. Note that another hybrid error signal \(\Delta E_{\text{GHR}}\) can be generated by mixing GHR(\(\pi/4\)) and GHR(3\(\pi/4\)) schemes in the following way \[50\]:

\[
\Delta E_{\text{GHR}} = \frac{1}{2} (\Delta E_{\text{GHRB}(\pi/4)} - \Delta E_{\text{GHRB}(3\pi/4)}) .
\]

Combining error signals are still offering better robustness for example to the signal distortion from ion motion heating in a single ion rf trap. A potential fractional clock error around \(5 \times 10^{-20}\) was estimated following Eq. 15 in comparison to the original HR–π protocol \[50\]. It has been also recently demonstrated that GHR(\(\pi/4\)) and GHR(3\(\pi/4\)) schemes have a strong robustness against residual light-shift coupled to spontaneous emission in an optically dense medium of cold atoms \[51\].

Indeed, there is a large possibility of unexplored exotic composite pulse protocols for optical clocks due to the richness of the quantum Hilbert space engineering \[52\]. For instance, an additional technique is the fault-tolerant Hahn-Ramsey interferometry \[53\]. This other composite pulse scheme may be used with GHR(\(\pi/4\)) and GHR(3\(\pi/4\)) laser phase modulation protocols from Tab. I to suppress simultaneously residual light-shifts and small drifts in the clock detuning. Such a scheme is a modification of the Ramsey spectroscopy by adding a reversal pulse between two separated Ramsey pulses similarly to the Hahn’s spin-echo protocol \[54\] but using alternating detunings with opposite sign to restore phase oscillations with the clock detuning \[55\].

**IV. HYPER-INTERFEROMETERS**

We focus now on the field of atomic interferometry where the two-pulse Ramsey configuration is still a fundamental building-block of more elaborated multiple interrogation schemes including external degrees of freedom with Doppler shift, atomic recoil and a residual uncompensated light-shift in the detuning. Laser pulses initially based on single photon transition act as beam-splitters or mirrors to spatially separate or recombine...
atomic wave-packets \([6, 59]\). Stimulated Raman transitions in a microwave excitation by two-photon process were pioneered by Kasevich and Chu for internal and external state manipulation of alkali species \([12, 15, 18]\) while Bragg diffraction was preferred to eliminate potential action of one photon light-shift in atomic states but requiring narrow momentum distribution to be very efficient \([55, 57]\). Stimulated hyper-Raman-Ramsey optical transitions in bosonic alkaline-earth systems such as Yb, Hg, Sr and Mg, have been already proposed as ultra-robust two-photon clock transitions against detrimental light-shift and Zeeman effect \([58]\). This model can be upgraded inserting laser phase-step, Doppler-shift and atomic recoil state labeling for optical two-photon matter-wave interferometry by analogy with velocity-selective stimulated Raman transitions in alkali atoms \([59]\).

Degraded performances of atomic interferometers often rely on imperfect overlapping of wave-packets due to phase-shift accumulation during light-pulses \([60, 62]\). Revisiting Ramsey-Bordé interferometry is thus motivated by the use of composite phase-shifts from Eq. 7a and Eq. 7b in order to compensate simultaneously for residual Doppler-shift and light-shift when the pulse area is different between pairs of beam splitters from two successive Ramsey-Bordé building-blocks.

Several types of interferometers are existing from devices sensitive to clock detuning centered at the recoil frequency or devices measuring inertial effects like rotation or acceleration. Composite pulses with Bragg and Raman type transitions and butterfly geometry with four pulses have already been employed to improve current cold-atom gyroscopes in sensitivity and accuracy to rotations \([64, 65]\). Here, complex sequences of pulses with phase-steps are rather proposed to shield matter-wave interferences against pulse defects inside atomic interferometers.

In this context, we study an asymmetric Ramsey-Bordé configuration used to determine the fine structure constant for fundamental test in QED \([17, 39]\) and a Mach-Zehnder (MZ) interferometer for inertial measurement \([20, 61]\). We extend composite pulse protocols for hyper-clocks to realize a generalized hyper-Ramsey-Bordé (GHRB) interferometer reducing or eliminating residual corrections from light-shift and sensitivity to residual transverse Doppler-shifts. We also present an hyper-Mach-Zehnder (HMZ) to strongly reduce the sensitivity against asymmetrical pulse area variation of beam splitters investigated in \([61]\) and recently reported for a symmetric Ramsey-Bordé configuration \([62]\).

### A. HYPER-RAMSEY-BORDÉ

The original Ramsey-Bordé (RB) interferometer is based on four laser pulses separated by three Ramsey free evolution times as shown in Fig. 5. If Doppler is taking into account, the first Ramsey two-zone setup denoted as MI(\(\uparrow\)) is recovering a strongly dependence to the transverse first-order Doppler effect. While with microwaves or radio-frequencies, atomic wave-packets are still interfering after a Ramsey two pulse interrogation, fringes are destroyed by optical frequencies with large wave-vectors. Bordé proposed in the 1980’s a configuration with four traveling waves setup consisting of two separated Ramsey zones MI(\(\uparrow\)) and MII(\(\downarrow\)) with two counter-propagating sets of co-propagating laser pulses \([6]\). Within this geometry, opposite sets of wave-vectors cancel the Doppler-shift and interferences are retrieved \([69]\) (see Fig. 5).

Inserting composite pulses in atomic interferometry is motivated by eliminating potential uncompensated residual part of Doppler-shift sensitivity and light-shift on a recoil frequency determination when the pulse area changes over the whole pulse interrogation. Indeed, for an original RB interferometer, a uniform distribution of laser amplitude over the full sequence will be rejected by a differential measurement between the two sets of shifted wave-packets. These terms term drop out in any interferometric comparison between paths. However, if pulse area of pairs of beam splitters are modified between two Ramsey-Bordé interaction zones, a small parasitic shift may be recovered.

We start our analysis of a robust hyper-interferometer by splitting the sequence of pulses in two building-blocks as MI(\(\uparrow\)) and MII(\(\downarrow\)). The components required to compute each amplitude of probability associated to different path trajectories of the wave-packets are given in section S2 from the appendix.

We evaluate first complex coefficients \(3^{C_{g, p}}(t)\) and

![FIG. 5. (color online). Generalized hyper-Ramsey-Bordé (GHRB) interferometer with multiple traveling waves designed by \(p\) and \(q\) composite pulses. Two counter-propagating sets of three co-propagating composite laser pulses are introduced by interaction zones MII(\(\uparrow\)) and MII(\(\downarrow\)) where \(\uparrow\downarrow\) arrows are corresponding to \(kv\) transverse Doppler wave-vector orientation. Laser pulse phases for each set are indicated respectively by \(\varphi_{I}^{\uparrow}(\varphi_{I}^{\downarrow})\) with \(\uparrow\) and \(\varphi_{II}^{\uparrow}(\varphi_{II}^{\downarrow})\) with \(\downarrow\).](image)
TABLE II. Composite pulses interrogation protocols for hyper-Ramsey-Bordé atom interferometry. Pulse area $\theta_i^j(\Omega_i)$ is given in degrees and phase-steps $\pm \varphi_i^j(\varphi_i)$ are indicated in subscript-brackets with radian unit. The standard Rabi frequency for all pulses is $\Omega = \pi/2\tau$ where $\tau$ is the pulse duration reference. Free evolution time regions are given by $\delta^\uparrow T$ and $\delta^\downarrow T$ where $\uparrow \downarrow$ denotes the transverse Doppler-shift orientation within each building-block MI($\uparrow$) and MI($\downarrow$) separated by the intermediate $\delta^\uparrow T'$ zone. Reverse protocols in time are denoted by $(\downarrow)$. 

| protocols | composite pulse building-blocks MI($\uparrow$), MI($\downarrow$) |
|-----------|--------------------------------------------------|
| RB $\varphi = \pi/4$ | $90^\uparrow_{t,\varphi} - \delta^\uparrow T + 90^\downarrow_{t,\varphi} - \delta^\downarrow T + 90^\uparrow_{t,\varphi}$ $(\downarrow) 90^\uparrow_{t,\varphi} - \delta^\uparrow T + 90^\downarrow_{t,\varphi} - \delta^\downarrow T + 90^\uparrow_{t,\varphi}$ |
| HRB $\varphi = \pi/4$ | $90^\uparrow_{t,\varphi} - \delta^\uparrow T + 180^\downarrow_{t,\varphi} 90^\downarrow_{t,\varphi} - \delta^\downarrow T + 90^\uparrow_{t,\varphi}$ $(\downarrow) 90^\uparrow_{t,\varphi} 180^\downarrow_{t,\varphi} - \delta^\uparrow T + 90^\downarrow_{t,\varphi} 180^\downarrow_{t,\varphi} - \delta^\downarrow T + 90^\uparrow_{t,\varphi}$ |
| GHRB $\varphi = \pi/8, 3\pi/8$ | $90^\uparrow_{t,\varphi} - \delta^\uparrow T + 180^\downarrow_{t,\varphi} 90^\downarrow_{t,\varphi} - \delta^\downarrow T + 90^\uparrow_{t,\varphi} 180^\downarrow_{t,\varphi} - \delta^\uparrow T + 90^\downarrow_{t,\varphi}$ $(\downarrow) 90^\uparrow_{t,\varphi} 180^\downarrow_{t,\varphi} - \delta^\uparrow T + 90^\downarrow_{t,\varphi} 180^\downarrow_{t,\varphi} - \delta^\downarrow T + 90^\uparrow_{t,\varphi}$ |

expression $\mathfrak{q}_p C_{e,\varphi} - h\vec{K}(t)$ within the first Ramsey zone MI($\uparrow$) with $p = 1$ and $q = 2$ pulses. Then we evaluate complex coefficients for the second interaction zone MI($\downarrow$) starting from previous solutions of interfering trajectories closing the interferometer. The first hyper-Ramsey-Bordé MI($\uparrow$) building-block is thus computed taking $C_{g,\varphi}(0) = 1, C_{e,\varphi} + h\vec{K}(0) = 0$ and gives:

$$\begin{align*}
\mathfrak{q}_p C_{g,\varphi}(t) &= C_{g,\varphi}, \quad (16a) \\
\mathfrak{q}_p C_{e,\varphi} + h\vec{K}(t) &= -\mathfrak{q}_p C_{g,\varphi}, \quad (16b)
\end{align*}$$

where $* \equiv$ complex conjugate. The common laser detuning $\delta_i$ for all spinor matrix component in MI($\uparrow$) zone is defined in section S2 from the Appendix. At the end of this first GHRB MI($\uparrow$) interaction zone, wave-packets are separated in several components in space during an intermediate $T'$ free evolution time. This is seen as a simple additional phase-factor of the form $e^{\pm i\delta T'/2}$. In asymmetric or symmetric Ramsey-Bordé interferometers, this delay allows for Bloch-oscillations to transfer large number of photon momenta to the wave-packets [17, 18].

The final complex GHRB matter-wave amplitudes $\mathfrak{q}_p C_{e,\varphi} - h\vec{K}(t)$ and $\mathfrak{q}_p C_{e,\varphi} + h\vec{K}(t)$ after successive interaction with MI($\uparrow$) and MI($\downarrow$) regions are four overlapping wave-packets centered at the recoil frequency $\pm \delta_r$ ($\pm h\vec{K}$) [39]. The final GHRB amplitude of the wavefunction after successive interaction is now given by:

$$\begin{align*}
\mathfrak{q}_p C_{e,\varphi} - h\vec{K}(t) &= -\mathfrak{q}_p C_{g,\varphi} \frac{\mathfrak{q}_p C_{g,\varphi}}{\mathfrak{q}_p C_{g,\varphi}}, \quad (17a) \\
\mathfrak{q}_p C_{e,\varphi} + h\vec{K}(t) &= -\mathfrak{q}_p C_{g,\varphi} \frac{\mathfrak{q}_p C_{g,\varphi}}{\mathfrak{q}_p C_{g,\varphi}}, \quad (17b)
\end{align*}$$

After algebraic manipulation, one up-shifted wave-packet
FIG. 6. (color online). Resolving the $^{40}$Ca recoil doublet with matter-wave composite pulse interferometry integrated over a narrow gaussian transverse velocity distribution around $T=250$ pK versus frequency detuning $\delta/2\pi$. (a) RB fringes. (b) HRB$-\pi$ fringes. (c) GHRB fringes. Laser pulse duration is $\tau=0.1ms$ and free evolution times around $T=30/\delta_r$ where we apply for the intermediate free evolution time $T'\rightarrow0$ and $\delta_r=\hbar k^2/2m$.

The computational algorithm allows us to derive the analytical formulae of an asymmetrical generalized hyper-Ramsey-Bordé interferometer generating a composite phase-shift on matter-wave interferences. Combination of phase-step protocols within two-successive building-blocks following Eq. 21a required to produce dispersive error signals, are using half values needed for an hyperclock interrogation scheme based on a single building-block. The overall effect of the residual uncorrected part of the light-shift remnant to the original Ramsey-Bordé scheme is finally encoded in Eq. 21b. This last result generalizes the description of atom interferometers neglecting potential light-shift distortion.

Note that there is also the RB interferometer configuration where the last set of optical traveling waves used to close the interferometer are not reversed. This symmetric Ramsey-Bordé interferometer is also exploited for the fine structure determination [18, 63]. Such a geometry is not sensitive to the net frequency dependence of the interference signal such that the relative phase-shift accumulation between arms of this symmetrical RB configuration is given by

$$\varphi_L = \varphi_I^L - \varphi_I^R = \varphi_I^1 - \varphi_I^1 + \varphi_{II}^I - \varphi_{II}^1,$$  \hspace{1cm} (22a)

$$\phi_L = \phi_I^L - \phi_I^R = \phi_I^1 + \phi_I^1 - \phi_{II}^I - \phi_{II}^1.$$  \hspace{1cm} (22b)

It is interesting to note that the sensitivity to residual Doppler-shifts and light-shifts are equivalently coming from Eq. 21b for an asymmetric interferometer and from

FIG. 7. (color online). (a) Numerical tracking of the error signal frequency position for HRB$-\pi$ and GHRB protocols around zero for various uncompensated part of the residual light-shift for $T\rightarrow0$ K. (b) Distortion effect of matter-waves due to a non zero temperature after integration over a narrow gaussian transverse velocity distribution around $T=250$ pK. All tracking points are generated with same parameters as in Fig. 6 except pulse area variation $\Omega\tau=q\pi/2$ which is driven by the parameter $q$ from 0.9 to 1.1. between the two interaction zones $M_I(\uparrow), M_{II}(\downarrow)$. 
Eq. 22b for a symmetric configuration. These terms are responsible for a velocity-dependent phase-shift leading to an imperfect overlapping of wave-packets originally derived in [22].

Coming back to the asymmetric RB interferometer, the generalized hyper-Ramsey Bordé transition probability is thus given by:

\[
\frac{q}{p} P_{e,\overline{e},\pm\hbar k} = \left| \frac{q}{p} C_{e,\overline{e},\pm\hbar k}(t) \right|^2 ,
\]

(23)

Similar to the generation of error signals based on Eq. 9 we also generate dispersive fringes as following:

\[
\Delta E = \frac{q}{p} P_{e,\overline{e},\pm\hbar k}(\varphi) = \frac{q}{p} P_{e,\overline{e},\pm\hbar k}(-\varphi) .
\]

(24)

where we can apply phase-step protocols reported in Tab. II. The extraction of the composite phase-shift from analytical expressions of error signal shapes is not always an easy task and sometimes requires a numerical tracking of the central dispersive feature. In the following figures that we have produced, we have numerically plotted the error signal and associated frequency-shifts for an accurate evaluation of interference distortion. We have also checked that nonlinear effects leading to these distortions are effectively related in some part to corrections resulting from Eq. 20a and Eq. 20b.

Dispersive errors signals based on hyper-Ramsey-Bordé (HRB−π) and generalized hyper-Ramsey-Bordé (GHRB) protocols from Tab. II are generated following Eq. 24. The last error signal \(\Delta E_{\text{GHRB}}\) reported in Fig. 6(c) is built by a combination of ±π/8 and ±3π/8 phase-steps:

\[
\Delta E_{\text{GHRB}} = \frac{1}{2} \left( \Delta E_{\text{GHRB}(\pi/8)} - \Delta E_{\text{GHRB}(3\pi/8)} \right) .
\]

(25)

Associating two error signals in a cooperative manner is a robust way to reduce again residual uncompensated Doppler-shifts and light-shifts coupled to pulse area variation between multiple building-blocks with better efficiency than a single error signal [59].

We report typical dispersive error signals integrated over a narrow transverse gaussian velocity distribution around \(T = 250\) pK using Ca atomic parameters as example in Fig. 8(a) for the RB interrogation scheme, (b) for the HRB−π protocol and in (c) for the GHRB protocol. Matter-waves are all split in two wave-packets that are centered on the atomic recoil term \(\pm \delta r/2\pi \sim \pm 23\) kHz for Ca.

We have also plotted the correction to the recoil due to residual light-shifts for two different transverse velocity distributions of the wave packet. In the ideal case \(T \to 0\) K presented in Fig. 7(a), the HRB−π interferometric scheme exhibits a highly nonlinear cubic dependence of the recoil correction under residual uncompensated parts of the light-shift \(\Delta /2\pi\). As expected, the GHRB scheme is still completely removing the dependence in the residual light-shift at all order in the detuning. However, the assumption that a sample of trapped atoms are in the \(T \to 0\) K regime is unrealistic.

FIG. 8. (color online). (a) Typical HGHRB error signal versus frequency detuning \(\delta /2\pi\) (Hz) centered around each recoil frequency and integrated over various narrow gaussian velocity distributions between \(0 \leq T \leq 1\) \(\mu\)K for different uncompensated part of residual light-shift \(\Delta /2\pi\). (b) Frequency correction to the recoil with the HGHRB protocol versus uncompensated part of the residual light-shift \(\Delta /2\pi\) over a wide distribution of ultra-cold temperatures between \(0 \leq T \leq 1\) \(\mu\)K. All Numerical tracking points are generated with same parameters as in Fig. except pulse area variation \(\Omega T = q/\pi /2\) which is driven by the parameter \(q\) from 0.9 to 1.1 between the two interaction zones MI(\(\uparrow\)), MII(\(\downarrow\)).

By integrating the interferometric error signal over a transverse gaussian distribution of velocities at \(T = 250\) K as shown in Fig. 7(b), the nonlinear compensation of the residual light-shift is lost and a small linear dependence of the recoil correction with \(\Delta /2\pi\) is restored for both protocols. A small asymmetry in pulse area between the two sets of Ramsey-Bordé interaction zones also generate a small sensitivity to potential light-shifts. However, let us remark that it is possible to cool atomic samples to ultra-cold temperatures relying on delta-kick techniques or sub-recoil cooling to reach nK to pK temperatures with very narrow momentum dispersion [67, 68]. Reaching lower temperature is thus an additional benefit for robust matter-wave interferometry with composite pulses.
In order to make an error signal more robust, even at relatively higher temperatures, to residual light-shifts and pulse area errors between interaction zones coupled to the transverse atomic motion, we present a new hybrid GHRB error signal (HGHRB) based on the following protocol:

$$\Delta E_{\text{HGHRB}} = \frac{1}{2}(\Delta E_{\text{GHRB}} + \Delta E_{\text{GHRB-\pi}}). \quad (26)$$

Such a combination of phase-shifted signals by $\pi$, reported in Tab. [I] is cooperatively adding interference fringes [66] with opposite phase-shifts to completely cancel any residual light-shifts and transverse Doppler-shifts. We check that symmetric or asymmetric residual light-shifts and transverse Doppler-shifts [54] are canceled when pulse area is changing by $\pm 10\%$. We now plot our new dispersive error signal in Fig. 8(a) versus the clock detuning $\delta/2\pi$ centered around each recoil frequency component for different residual light-shifts and large transverse temperatures $0 \leq T \leq 1 \mu K$. The robustness of the recoil correction to the residual light-shift coupled to pulse area variation by $\pm 10\%$ is presented in Fig. 8(b). The solid blue stars are the numerical tracking of the error signal near the recoil frequency for a GHRB scheme working at a transverse temperature around $= 1 \mu K$. The solid black line with dots is the error signal correction from the hybrid scheme HGHRB based on Eq. 26. The combination of GHRB and GHRB−$\pi$ sequences are acting like a Hahn spin-echo compensation against pulse area variation coupled to residual light-shifts and transverse Doppler-shifts [54]. Finally, note that a class of mirror-like interrogation schemes is reported Tab. [I] denoted as ($\dagger\dagger$) protocols. It is realized by applying a change in pulse order requiring also a sign flip in laser phase-step as $\varphi \rightarrow -\varphi$.

As association of GHRB and GHRB($\dagger$) schemes are motivated by a potential elimination of decoherence-induced residual frequency-shifts as proposed for hyper-clocks [69].

**B. HYPER-MACH-ZEHNDER**

We turn now to the atomic Mach-Zehnder (MZ) interferometer. Such a scheme is insensitive to constant clock detunings and Doppler-shifts all over the sequence of pulses making them accurate and sensitive inertial-force sensors [35, 55]. However, variations of the laser field amplitude during beam splitter action may restore a parasitic distortion related to a residual Doppler-shift [61]. So far, we consider wave-packets trajectories to be straight-lines to preserve the inherent symmetry of the MZ interferometer.

A composite pulse interferometer called hyper MZ (HMZ) is shown in Fig. 9. We propose to remove limits to the symmetry of the original MZ type under pulse area variation (see appendix, section S3 for analytical components needed to calculate the transition probability expression $P_{\text{HMZ}}$). The first Ramsey $90^\circ_{\psi}$ pulse is replaced by a composite sequence $90^\circ_{\psi}180^\circ_{\psi}90^\circ_{\psi}$ and the last Ramsey $90^\circ_{\psi}$ by the sequence $90^\circ_{\psi}180^\circ_{\psi}90^\circ_{\psi}$ with opposite phase-steps as reported in Tab. III.

The exact hyper Mach-Zehnder (HMZ) transition probability $P_{\text{HMZ}}$ is established. Some materials from section S3 in the appendix are used to evaluate a composite set of $p = 3$ pulses for the left side of the HMZ (building-block $M(\dagger)$) and a composite set of $q = 4$ pulses for the right side of the HMZ (building-block $M(\dagger)$). In such a configuration starting with $C_\alpha(0) = 1, C_\beta(0) = 0$, the clock frequency dependence is removed in the Hahn-echo like scheme [53, 54]. So we take $\delta T \rightarrow 0$ for the calculation while the intermediate $180^\circ_{\psi}$ reverse pulse is played by index $j = 1$ in the second set of $q$ pulses ($\text{MIII}(\dagger)$). The matter-wave interferometric signal can be thus directly computed leading to the following complex expression:

$$P_{\text{HMZ}} = \left| 4 \alpha_{gg} \left[ 1 - |3\beta_{gg}| e^{-i2\Phi_{gg}} \right] \right|^2 \quad (27)$$

where respective expressions for $\alpha_{gg}$, $\alpha_{gg} \rightarrow \alpha_{gg}$ and $\beta_{gg} \rightarrow \beta_{gg}$ (see Appendix section S3 for laser index modification due to non overlap-
The HMZ composite phase-shift is:

$$\Phi^{gg} - \phi_1' + 2\varphi_1 - \varphi_2 + \phi_1' - \phi_2$$

The HMZ composite phase-shift is:

$$\frac{4}{3}\Phi^{gg} = -\varphi_1' + 2\varphi_1 - \varphi_2 + \phi_1' - \phi_2$$

However, a direct use of the HMZ phase-shift expression does not correspond to the correct evaluation of the true central fringe phase-shift. This is due to a simultaneous action of laser phases within envelop terms $\alpha_1^{gg}(gg), \alpha_2^{gg}(gg)$ and complex terms $\beta_1^g(gg), \beta_2^g(gg)$. For this reason, we cannot identify the expression of the composite phase-shift as Eq. 33, requiring to plot numerically the phase-shift when fringes extremum are recorded by sweeping pulse laser phases.

As a validation of previous analytical calculations, we re-derive the original three-pulse MZ configuration with the intermediate $180^\circ$ pulse still played with index $l = 1$ in the second set of $q$ pulses. We obtain, after straightforward simplification on envelops $\alpha_1^{gg}(gg), \alpha_2^g(gg)$ and complex terms $\beta_1^g(gg), \beta_2^g(gg)$, the MZ transition probability expressed as:

$$P_{MZ} = \left| -\cos\bar{\vartheta}_1 \sin\bar{\vartheta}_1 \sin\bar{\vartheta}_2 \cdot \left[ 1 + \tan\bar{\vartheta}_2 e^{-i\Phi^{gg}} \right] \right|$$

where, this time, the MZ phase-shift is easily identified to be:

$$\Phi^{gg} = -\varphi_1' + 2\varphi_1 - \varphi_2 + \phi_1' - \phi_2$$

consistent with Eq. 33 when $\varphi_1' = \phi_2$. An additional MZ phase sensitivity to residual Doppler shifts is retrieved when $\varphi_1' \neq \varphi_2$ due to the imbalance of Rabi fields between the first and the last beam splitter pulse as expected.

We point out that a MZ interferometer is a particular case of the symmetric RB configuration we have shortly discussed in the previous subsection. By fixing $\varphi_1' = \phi_1$, $\varphi_2' = \varphi_1' = \phi_2$ in Eq. 22a and $\varphi_1' = \phi_1$, $\varphi_2' = \phi_2$ in Eq. 22a, we retrieve the MZ phase-shift given by Eq. 33.

We have reported in Fig. 10 matter-wave interferences versus the laser phase recorded at the output of the interferometer with three pulses (MZ with solid red and dashed lines) and using sets of composite pulses (HMZ with solid blue and dashed lines). The laser phase is scanned only during the first (or the last) beam splitter of the three-pulse interferometer while all laser phases are simultaneously changed over the two sets of composite pulse beam splitters in the HMZ configuration. This results in a different inter-fringe between interferometers as shown in Fig. 10.
Several numerical plots of the center of fringes versus a residual Doppler shift $\delta_D/2\pi$ ($\delta_D \equiv kv_z$) is reported in Fig. 11(a) and (b) under a 10% pulse area error between the first beam splitter pulse (or the first set of composite pulses) and the last one (or the last set of composite pulses). The composite phase-shift generated by the HMZ scheme is highly nonlinear as it can be observed from Fig. 11(a). We remark that the phase-shift presents a divergence near Doppler-shift $\delta_D/2\pi \approx \pm 20$ kHz. It is related to a sudden change from a minimum to a maximum of the fringe interference transition probability. The plots with red star from Fig. 11(a) corresponds to the MZ type interferometer exhibiting a dispersive phase-shift about 24 mrad for a Doppler-shift of 10 kHz in accordance with [61]. The plots with blue dots from Fig. 11(b) corresponds to the HMZ configuration with a nonlinear reduction of the phase-shift around 0.3 mrad for the same Doppler-shift. Following error signal generation based on Eq. 9, a new set of plots with open circles in Fig. 11(b) are produced by an hybrid combination of HMZ interferometric signals (HHMZ) with opposite phase-steps as (see the last protocol from Tab. III):

$$P_{\text{HHMZ}} = \frac{P_{\text{HMZ}}(\varphi) + P_{\text{HMZ}}(-\varphi)}{2}$$

(34)

Such a combination of signals is again cooperatively adding interference fringes with opposite phase-shifts to completely cancel the residual correction [62]. The HMZ interferometer is demonstrated to be more robust against velocity-distribution asymmetries activated by pulse area error that may lead to parasitic shifts compromising the accuracy. We have also verified that an additional contribution from any uncompensated asymmetric light-shifts of a few % between sets of composite pulses are still strongly reduced to the same level of correction [62].

V. CONCLUSIONS AND OUTLOOK

We have explored new applications of composite pulses from optical clocks to Ramsey-Bordé and Mach-Zehnder atom interferometers, and demonstrate that they are more robust against pulse defects related to laser field amplitude variation coupled to residual Doppler-shifts and light-shifts between multiple interaction zones. Combining atomic sensors with ultra-cold sources [67, 68] and recent quantum technologies through entanglement and spin squeezing [70, 72] while offering arbitrarily Hilbert space engineering process with composite laser pulses in phase, duration and frequency-steps will probably lead to a next generation of ultra-robust devices. Composite pulse interferometry will offer an efficient correction of residual light-shift and driving-field frequency drifts [55, 56]. It will improve high-precision laser spectroscopy and metrology below a relative level of $10^{-19}$.
in accuracy, opening new applications to track gravity
induced phase-shifts in Ramsey interferometry \cite{76} or to
detect gravitational waves in a frequency band between
the LISA and LIGO detectors \cite{77,78}. A new class
of hyper-Ramsey-Bordé interferometers using weakly al-
lowed or forbidden optical clock transitions inducing
large momentum transfer \cite{58,78,79} will benefit from
composite pulses for bringing quantum sensors to robust
real-world application \cite{80} while searching for new funda-
mental physics behind the standard model with a better
accuracy \cite{81}.

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APPENDIX

S0: SYMMETRIC FUNCTIONS $S_{p,k}^r$ AND $S_{q,k}^r$

We turn to look for an exact expression of the transi-
tion probability when initials conditions are arbitrary
with $p = q = 2$ and $p = q = 4$ pulses around a single
free evolution time. We use our algorithm to establish
an exact expression of the transition probability extend-
ing previous works by \cite{1 33,38,49}. The full composite
wave-function is:

$$\begin{align*}
\alpha_l^{p}(gg) &= \left( \prod_{l=1}^{p} \cos \vartheta_l e^{i \phi_l} \right) \cdot \left( \sum_{\text{even } k \geq 0} (-1)^{\frac{k}{2}} S_{p,k}(gg) \right) \\
\alpha_l^{q}(gg) &= \left( \prod_{l=1}^{q} \cos \vartheta_l e^{i \phi_l} \right) \cdot \left( \sum_{\text{even } k \geq 0} (-1)^{\frac{k}{2}} S_{q,k}(gg) \right) \\
\alpha_l^{p}(ge) &= \alpha' \left( \prod_{l=1}^{p} \cos \vartheta_l e^{i \phi_l} \right) \cdot \left( \sum_{\text{odd } k \geq 1} (-1)^{\frac{k-1}{2}} S_{p,k}(ge) \right)
\end{align*}$$

with $\alpha' = -ie^{-i(\phi_1'+\phi_2'+\sum_{r=3}^{p}(\phi_r'))}$ in Eq. \ref{36c}. The convention
is $S_{p,0}^r = S_{q,0}^r = 1$ in Eq. \ref{36a} and Eq. \ref{36b}. Symmetric
functions $S_{p,k}^r$ and $S_{q,k}^r$ having respectively $\frac{p!}{k!(p-k)!}$ and
$\frac{q!}{k!(q-k)!}$ elements are:

\begin{align*}
S_{p,k}^r(gg) &= \sum_{\Lambda \subseteq \{1, 2, \ldots, p\}, |\Lambda| = k} e^{-i\Xi_\Lambda} \prod_{l \in \Lambda} \tan \vartheta_l \tag{37a} \\
S_{q,k}^r(gg) &= \sum_{\Lambda \subseteq \{1, 2, \ldots, q\}, |\Lambda| = k} e^{-i\Xi_\Lambda} \prod_{l \in \Lambda} \tan \vartheta_l \tag{37b} \\
S_{p,k}^r(ge) &= \sum_{\Lambda \subseteq \{1, 2, \ldots, p\}, |\Lambda| = k} e^{i\Xi_\Lambda} \prod_{l \in \Lambda} \tan \vartheta_l \tag{37c}
\end{align*}

Note that phase factors $\Xi_\Lambda$ as well as $\Xi_\Lambda$ that are
affected to the product in Eq. \ref{37a} and Eq. \ref{37b} are
determined by all possible k-combination of elements in A
ensemble with k even. The decomposition rules of
$\Xi_\Lambda$ ($\Xi_A$) are presented: for $|\Lambda| = 2$, phase-factors are
given by $\Xi_\Lambda = \Xi_\Lambda^{12}, \Xi_\Lambda^{13}, \Xi_\Lambda^{14}, \Xi_\Lambda^{23}, \Xi_\Lambda^{24}$, \ldots where the de-
composition is $\Xi_\Lambda^{12} = \Xi_\Lambda^{12} + \Xi_\Lambda^{13}, \Xi_\Lambda^{14}, \Xi_\Lambda^{23}, \Xi_\Lambda^{24}$, \ldots $|\Lambda| = 4, \Xi_\Lambda^{123} = \Xi_\Lambda^{123} + \Xi_\Lambda^{124}, \Xi_\Lambda^{134}, \Xi_\Lambda^{234}, \Xi_\Lambda^{1234}$, \ldots and so on.

Phase factors $\Xi_\Lambda$ affected to the product in Eq. \ref{37c}
are determined by all possible k-combination of elements in A
ensemble with k odd. For $|\Lambda| = 1$, phase-factors are
given by $\Xi_\Lambda^{1} = \Xi_\Lambda^{1}, \Xi_\Lambda^{2}, \Xi_\Lambda^{3}, \Xi_\Lambda^{4}$. Here, by convention $\Xi_1 = 0$
and $\Xi_2 = \Xi_2^{12}, \Xi_1^{12}, \Xi_2^{12}, \Xi_2^{13}, \Xi_2^{14}, \Xi_2^{123}, \Xi_2^{124}, \Xi_2^{134}, \Xi_2^{1234}$, \ldots where the decomposition is $\Xi_2^{12} = \Xi_2^{12} + \Xi_2^{13}, \Xi_2^{14}, \Xi_2^{123}, \Xi_2^{124}, \Xi_2^{134}, \Xi_2^{1234}$, \ldots $|\Lambda| = 3$, we get $\Xi_3^{123} = \Xi_3^{123} + \Xi_3^{124}, \Xi_3^{124}, \Xi_3^{134}, \Xi_3^{1234}$, \ldots $|\Lambda| = 4$, \Xi_4^{1234} = \Xi_4^{1234} + \Xi_4^{124}, \Xi_4^{124}, \Xi_4^{134}, \Xi_4^{1234}, \Xi_4^{1234}$, \ldots and so on. The basic structure of all phase-factors is

$$\Xi_{l+1} = \varphi_l + \varphi_{l+1}$$

for $\Xi_\Lambda$ and $\Xi_{l+1} = \varphi_l + \varphi_{l+1}$ for $\Xi_A$.

S1: SPINOR COMPONENTS WITH $p = q = 4$

We present an example of our computational algorithm
with $p = q = 4$ pulses to evaluate spinor components
expressions related to an extended scheme with 8 pulses. The components we need to compute $\frac{4}{3}C_{gg}, \frac{4}{3}C_{ge}$ are:

$$\alpha_1^4(gg) = \left( \prod_{i=1}^{\beta=4} \cos \vec{p}_i e^{i\phi_i} \right) \cdot (1 - S_{4,2} + S_{4,4})$$  \hspace{1cm} (38a)$$

$$\alpha_1^4(ge) = -ie^{-i(\phi_1 + \phi_2 + \Xi_1)}$$ \times \left( \prod_{i=1}^{\beta=4} \cos \vec{p}_i e^{i\phi_i} \right) \cdot (S_{4,1} - S_{4,3}) \hspace{1cm} (38b)$$

We have for $S_{4,k}^4(gg)$:

$$S_{4,0}^4 = 1$$

$$S_{4,2}^4 = e^{-i\Xi_2} \tan \vec{p}_1 \tan \vec{p}_2 + e^{-i\Xi_2} \tan \vec{p}_2 \tan \vec{p}_3$$

$$+ e^{-i\Xi_3} \tan \vec{p}_3 \tan \vec{p}_4 + e^{-i\Xi_3} \tan \vec{p}_4 \tan \vec{p}_1$$

$$S_{4,4}^4 = e^{-i\Xi_2} \tan \vec{p}_1 \tan \vec{p}_2 \tan \vec{p}_3 \tan \vec{p}_4 \hspace{1cm} (39)$$

and for $S_{4,k}^4(ge)$:

$$S_{4,1}^4 = \tan \vec{p}_1 + e^{i\Xi_1} \tan \vec{p}_2$$

$$+ e^{i\Xi_1} \tan \vec{p}_3 + e^{i\Xi_1} \tan \vec{p}_4$$

$$S_{4,3}^4 = e^{i\Xi_3} \tan \vec{p}_1 \tan \vec{p}_2 \tan \vec{p}_3$$

$$+ e^{-i\Xi_3} \tan \vec{p}_1 \tan \vec{p}_2 \tan \vec{p}_4$$

$$+ e^{i\Xi_3} \tan \vec{p}_1 \tan \vec{p}_2 \tan \vec{p}_3 \tan \vec{p}_4 \hspace{1cm} (40)$$

We also have for $S_{4,k}(gg)$ elements:

$$S_{4,0}^4 = 1$$

$$S_{4,2}^4 = e^{-i\Xi_2} \tan \vec{p}_1 \tan \vec{p}_2 + e^{-i\Xi_2} \tan \vec{p}_2 \tan \vec{p}_3$$

$$+ e^{-i\Xi_3} \tan \vec{p}_3 \tan \vec{p}_4 + e^{-i\Xi_3} \tan \vec{p}_4 \tan \vec{p}_1$$

$$S_{4,4}^4 = e^{-i\Xi_2} \tan \vec{p}_1 \tan \vec{p}_2 \tan \vec{p}_3 \tan \vec{p}_4 \hspace{1cm} (41)$$

The corresponding complex phase factor $\beta_1^4(\Xi_1), \beta_1^4(\Xi_2)$ leading to a phase-shift correction are now:

$$\beta_1^4(gg) = \frac{\tan \vec{p}_1 + e^{-i\Xi_1} \tan \vec{p}_2 + e^{-i\Xi_2} \tan \vec{p}_3 + e^{-i\Xi_3} \tan \vec{p}_4}{1 - e^{-i\Xi_1} \tan \vec{p}_1 + e^{-i\Xi_2} \tan \vec{p}_2 + e^{-i\Xi_3} \tan \vec{p}_3 + e^{-i\Xi_4} \tan \vec{p}_4} \hspace{1cm} (42a)$$

$$\beta_1^4(ge) = \frac{1 - e^{-i\Xi_2} \tan \vec{p}_1 + e^{-i\Xi_2} \tan \vec{p}_2 + e^{-i\Xi_3} \tan \vec{p}_3 + e^{-i\Xi_4} \tan \vec{p}_4}{1 - e^{-i\Xi_2} \tan \vec{p}_1 + e^{-i\Xi_2} \tan \vec{p}_2 + e^{-i\Xi_3} \tan \vec{p}_3 + e^{-i\Xi_4} \tan \vec{p}_4} \hspace{1cm} (42b)$$

$$\beta_1^4(ge) = \frac{1}{\beta_1^4(\Xi_1)} \hspace{1cm} (42c)$$

We give the decomposition of phase factor expressions as following:

$$\Xi_1 = 0$$

$$\Xi_2 = \Xi_2$$

$$\Xi_3 = \Xi_3$$

$$\Xi_4 = \Xi_4$$

$$\Xi_{123} = \Xi_1 + \Xi_2$$

$$\Xi_{124} = \Xi_1 + \Xi_3$$

$$\Xi_{134} = \Xi_2 + \Xi_4$$

$$\Xi_{1234} = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4$$

$$\Xi_{123} = \Xi_4$$

$$\Xi_{124} = \Xi_3$$

$$\Xi_{134} = \Xi_2$$

$$\Xi_{1234} = \Xi_1$$

and:

$$\Xi_{12} = \varphi_2 - \varphi_1 + \varphi_2$$

$$\Xi_{13} = \varphi_2$$

$$\Xi_{14} = \varphi_1 + \varphi_2 + \varphi_3$$

$$\Xi_{23} = \varphi_1$$

$$\Xi_{24} = \varphi_2$$

$$\Xi_{34} = \varphi_3$$

$$\Xi_{123} = \Xi_{12} + \Xi_{13}$$

$$\Xi_{124} = \Xi_{14}$$

$$\Xi_{134} = \Xi_{12} + \Xi_{34}$$

$$\Xi_{1234} = \Xi_{12}$$

See also [40] as another way to obtain $\frac{4}{3}C_{ge}$ from $\frac{4}{3}C_{gg}$.
S2: GHRB COMPONENTS WITH $p = 1$, $q = 2$

For a general purpose, we explicitly give matrix components for MI$(\uparrow, \downarrow)$ and MII$(\downarrow, \uparrow)$ where two successive building-blocks are including pairs of traveling waves with opposite orientation to close the interferometer (MI$(\uparrow)$, MI$(\downarrow)$ and MII$(\downarrow)$, MII$(\uparrow)$). We have for the first MI$(\uparrow, \downarrow)$ Ramsey zone, the following components:

$$
\begin{align*}
\tilde{\gamma}^I_{gg} &= \alpha_{11}^I (gg) \cdot \alpha_{12}^I (gg) \\
\tilde{\gamma}^I_{ge} &= \alpha_{11}^I (ge) \cdot \alpha_{12}^I (gg) \\
\beta_{11}^I (gg) &= \beta_{11}^I (gg) \cdot \beta_{12}^I (gg) \\
\beta_{12}^I (gg) &= \beta_{11}^I (ge) \cdot \beta_{12}^I (gg)
\end{align*}
$$

where we have:

$$
\begin{align*}
\alpha_{11}^I (gg) &= \cos \tilde{\vartheta}_1 e^{i\varphi_1} \\
\alpha_{12}^I (gg) &= \cos \tilde{\vartheta}_1 \cos \bar{\vartheta}_2 e^{i(\varphi_1 + \varphi_2)} \cdot (1 - S_{2,2}) \\
\beta_{11}^I (gg) &= \tan \tilde{\vartheta}_1 \\
\beta_{12}^I (gg) &= \frac{\tan \tilde{\vartheta}_1 + e^{-i\Xi_{12}} \tan \bar{\vartheta}_2}{1 - e^{-i\Xi_{12}} \tan \tilde{\vartheta}_1 \tan \bar{\vartheta}_2}
\end{align*}
$$

and

$$
\begin{align*}
\alpha_{11}^I (ge) &= -i \sin \tilde{\vartheta}_1 e^{-i(\varphi'_1 + \varphi'_2)} e^{i\varphi_1} \\
\beta_{11}^I (ge) &= \frac{1}{\tan \tilde{\vartheta}_1}
\end{align*}
$$

with a laser detuning definition given by $\delta_{I}^{\uparrow\downarrow} = \delta \mp kv_z - \delta_r + \Delta_I$ for wave-vector orientation.

We have for the second MII$(\downarrow, \uparrow)$ Ramsey zone with opposite wave-vectors the following components:

$$
\begin{align*}
\tilde{\gamma}^I_{gg} &= \alpha_{11}^I (gg) \cdot \alpha_{12}^I (gg) \\
\tilde{\gamma}^I_{ge} &= \alpha_{11}^I (ge) \cdot \alpha_{12}^I (gg) \\
\beta_{11}^I (gg) &= \beta_{11}^I (gg) \cdot \beta_{12}^I (gg) \\
\beta_{12}^I (gg) &= \beta_{11}^I (ge) \cdot \beta_{12}^I (gg)
\end{align*}
$$

where laser detunings are given by $\delta_{I}^{\uparrow\downarrow} = \delta \pm kv_z + 3\delta_r + \Delta_{II}$ for the down-shifted frequency component and $\delta_{I}^{\downarrow\uparrow} = \delta \pm kv_z - \delta_r + \Delta_{II}$ for the up-shifted component by the atomic recoil.

We have used definitions from \[39\,\[40\] for arbitrary transverse Doppler-shift orientation and momentum quantization along each path of the interferometer.

S3: HMZ COMPONENTS WITH $p = 3$, $q = 4$

Some materials from section S1 are used again to evaluate a composite set of $p = 3$ pulses for the left side of the HMZ (building-block MI$(\uparrow)$) and a composite set of $q = 4$ pulses for the right side of the HMZ (building-block MII$(\uparrow)$). The matter-wave interferometric signal can be thus directly computed leading to the following complex expression:

$$
P_{\text{HMZ}} = \left| \frac{4}{3} \alpha_{gg} \left[ 1 - \frac{1}{3} \alpha_{gg} e^{-i\varphi_{gg}} \right] \right|^2
$$

where

$$
\begin{align*}
\alpha_{gg} &= \alpha_{11}^I (gg) \cdot \alpha_{12}^I (gg) \\
\beta_{gg} &= \beta_{11}^I (gg) \cdot \beta_{12}^I (gg)
\end{align*}
$$

with

$$
\begin{align*}
\alpha_{11}^I (gg) &= \prod_{1}^{3} \cos \tilde{\vartheta}_1 e^{i\varphi_1} \cdot (1 - S_{3,2}) \\
\alpha_{12}^I (gg) &= \prod_{1}^{q=4} \cos \tilde{\vartheta}_1 e^{i\varphi_1} \cdot (1 - S_{4,2} + S_{4,4})
\end{align*}
$$

and

$$
\begin{align*}
\beta_{11}^I (gg) &= \frac{\tan \tilde{\vartheta}_1 + e^{-i\Xi_{12}} \tan \bar{\vartheta}_2 e^{-i\Xi_{12}} \tan \tilde{\vartheta}_1 \tan \bar{\vartheta}_2}{1 - e^{-i\Xi_{12}} \tan \tilde{\vartheta}_1 \tan \bar{\vartheta}_2}
\end{align*}
$$

We now apply $\cos \tilde{\vartheta}_1 e^{i\varphi_1} \rightarrow 0$ with $\varphi_1 = 0$ keeping $\sin \tilde{\vartheta}_1$ fixed in Eq.\[52b\] and Eq.\[53b\]. It takes into account non-overlapping matter-waves during the intermediate 180° action. We finally get modified expressions for $\alpha_{11}^I (gg) \rightarrow \alpha_{11}^I (gg) \rightarrow \beta_{12}^I (gg)$ within section IV.B of the main text.

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By applying a direct transformation to the last pulse $\vec{\vartheta}_p$ of $\rho C_{q_ge}$ as following:

$$
\cos \vec{\vartheta}_p \mapsto -i \sin \vec{\vartheta}_p e^{-i(\varphi'_p + \varphi'_p)}
$$

$$
\sin \vec{\vartheta}_p \mapsto i \cos \vec{\vartheta}_p e^{-i(\varphi'_p + \varphi'_p)}
$$

(54)

we directly get the complex component $\rho C_{q_ge}$ of the composite wave-function.

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