REFLECTION LENGTH WITH TWO PARAMETERS IN THE ASYMPTOTIC REPRESENTATION THEORY OF TYPE B/C AND APPLICATIONS

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ABSTRACT. We introduce a two-parameter function \( \phi_{q^+, q^-} \) on the infinite hyperoctahedral group, which is a bivariate refinement of the reflection length. We show that this signed reflection function \( \phi_{q^+, q^-} \) is positive definite if and only if it is an extreme character of the infinite hyperoctahedral group and we classify the corresponding set of parameters \( q^+, q^- \). We construct the corresponding representations through a natural action of the hyperoctahedral group \( B(n) \) on the tensor product of \( n \) copies of a vector space, which gives a two-parameter analog of the classical construction of Schur–Weyl.

We apply our classification to construct a cyclic Fock space of type B generalizing the one-parameter construction in type A found previously by Bożejkko and Guta. We also construct a new Gaussian operator acting on the cyclic Fock space of type B and we relate its moments with the Askey–Wimp–Kerov distribution by using the notion of cycles on pair-partitions, which we introduce here. Finally, we explain how to solve the analogous problem for the Coxeter groups of type D by using our main result.

1. INTRODUCTION

Positive definite functions on a group \( G \) play a prominent role in harmonic analysis, operator theory, free probability and geometric group theory. When \( G \) has a particularly nice structure of combinatorial/geometric origin one can use it to construct positive definite functions which leads to many interesting properties widely applied in these fields. This phenomenon was first recognized in the pioneering work of Haagerup [Haa79], who proved that the function \( g \to q^{\ell_S(g)} \) is positive definite on the free group \( F_N \) with \( N \) generators for \(-1 \leq q \leq 1\), where \( \ell_S \) is the standard word length. Haagerup’s result was applied to show the completely bounded approximation property (CBAP) of the regular \( C^* \)-algebra of the free group \( F_N \) [DCH85] and it had an impact on free probability, non-commutative harmonic analysis and the operator algebras [HP93]. Bożejkko, Januszkiewicz and Spatzier, inspired by the work of Haagerup, were studying arbitrary Coxeter groups \((G, S)\) and they proved that the Coxeter function \( g \to q^{\ell_S(g)} \) is positive definite for every Coxeter group if and only if \(-1 \leq q \leq 1\) [BJS88]. Here \( \ell_S \) denotes the Coxeter length, which is the standard word length with respect to the set of Coxeter generators:

\[
\ell_S(g) := \min(k \colon s_1 \cdots s_k = g, s_1, \ldots, s_k \in S).
\]

They additionally showed that this implies that infinite Coxeter groups do not have Kazhdan’s property \((T)\). This result was further generalized to multi-parameters [BS03] and also other variants of the Coxeter function (colour-length) were studied [BS96, BGM18].

All the considered functions share two distinctive features:

(I) they are positive definite on the continuous set \(-1 \leq q \leq 1\),
Functions on a group $G$ which are invariant by conjugation are called central (also known as class functions). In his seminal work Thoma [Tho64] defined characters as positive definite, central functions on a group $G$ normalized to take value $1$ at the identity of the group. Extreme characters are extreme points in the set of all characters and they play a prominent role in asymptotic representation theory and specifically in the representation theory of infinite dimensional groups developed independently by Thoma [Tho64] in the case of the infinite symmetric group $S_{\infty}$ and Voiculescu [Voi74, Voi76] in the case of infinite dimensional Lie groups $U(\infty), SO(\infty), Sp(\infty)$. When $G$ is compact (e.g. finite) the extreme characters correspond to normalized traces of the irreducible representations, but when $G$ is infinite dimensional, the conventional representation theory of irreducible characters does not work well and the ideas of Thoma and Voiculescu have laid the foundations for the new and quickly developing field of asymptotic representation theory. This new area of research naturally linked representation theory with harmonic analysis, the theory of symmetric functions, probability theory, random matrix theory and mathematical physics (see [Ker03, BO17, M17] for the development of the asymptotic representation theory of $S_n$ and $S_\infty$ with their wide applications). The most natural way to modify the Coxeter function $g \to q^{S(g)}$ in order to obtain its analog which is central on $G$ is to replace the Coxeter length $\ell_S$ by the reflection length $\ell_R$, i.e. the minimal number of reflections that we need to use to decompose $g$ as their product

$$\ell_R(g) = \min(k: r_1 \cdots r_k = g, r_1, \ldots, r_k \in R := \{gs^{-1}: s \in S, g \in G\}).$$

The reflection length and factorizations into reflections in general are ubiquitous in the enumerative problems of finite Coxeter groups, or more generally (well-generated) complex reflection groups [Hur91, Loo74, Bes15]. In the case of the infinite symmetric group $S_{\infty}$, which is the inductive limit of the ascending tower of the symmetric groups $S_1 < S_2 < \cdots$ one can show that the reflection function $g \to q^{S(g)}$ is a character of $S_{\infty}$ for $q^{-1} \in \mathbb{Z}$. This is a straightforward consequence of the description of the Thoma simplex given by Vershik and Kerov [VK81] – in this case its restriction to the finite symmetric group $S_n$ gives the normalized trace of the natural action of the symmetric group on $n$ tensor copies of an $N$-dimensional vector space appearing in the classical construction of Schur–Weyl. Bożejko and Guta [BG02] used this positive definite function to construct a white noise functor and also to obtain certain “exclusion principle” of the operator counting the number of one-particle states. It is interesting to ask whether these are the only parameters $q$ for which the reflection function is a character. The answer for this question is affirmative\footnote{this is not discussed in the work of Bożejko and Guta [BG02], but it can be shown by the same methods as we are using in this paper to prove our main theorem - see Remark 3.5.} (after adding the point $q = 0$), which shows that the properties (I), (II) of the Coxeter function seem to be correlated. In particular positive definiteness of the reflection function is a different and complicated problem, strongly dependent on the choice of the underlying group.

In this paper we generalize the notion of the reflection function in the case of the Coxeter group of type $B$, that is the hyperoctahedral group $B(n)$, by introducing its two-parameter version. In contrast to type $A$, where all the reflections are conjugated, there are two conjugacy classes of reflections in Coxeter groups of type $B$ and we can refine the reflection function by introducing the signed reflection function which distinguishes short and long
reflections appearing in the factorization of a given element (see Definition 2.2 for the definitions):
\[ \sigma_{q_+,q_-}(g) := \frac{\ell_{R_+}(g)}{q_+} \frac{\ell_{R_-}(g)}{q_-}. \]

It is clear that this is a two-parameter refinement of the reflection function, which can be obtained by the substitution \( q_+ = q_- = q \). We prove in Section 2 that the signed reflection function is central on the hyperoctahedral group for any value of \( q_+, q_- \in \mathbb{C} \) and all orders \( n \geq 0 \), thus it gives rise to a central function on the infinite group \( B(\infty) \), which is the inductive limit of the ascending tower of the hyperoctahedral groups \( B(1) < B(2) < \cdots \).

Our main theorem, proved in Section 3, gives the complete characterization of the set of parameters \( q_+, q_- \) for which the signed reflection function is positive definite.

**Theorem 1.1.** Let \( q_+, q_- \in \mathbb{C} \). The following conditions are equivalent:

1. The signed reflection function \( \phi_{q_+,q_-} : B(\infty) \to \mathbb{C} \) is positive definite on \( B(\infty) \);
2. The signed reflection function \( \phi_{q_+,q_-} : B(\infty) \to \mathbb{C} \) is a character of \( B(\infty) \);
3. The signed reflection function \( \phi_{q_+,q_-} : B(\infty) \to \mathbb{C} \) is an extreme character of \( B(\infty) \);
4. \( q_+ = \frac{\epsilon}{M+N}, q_- = \frac{M-N}{M+N} \) for \( M, N \in \mathbb{N}, M+N \neq 0, \epsilon \in \{1, -1\} \) or \( q_+ = 0, -1 \leq q_- \leq 1 \).

Note that the set of parameters for which the signed reflection function is positive definite is a discrete set (except the degenerate case \( q_+ = 0 \)), which confirms that the behaviour of the reflection function is very different to its Coxeter counterpart and that the properties (I) and (II) are correlated. Another difference is visible in the strong correlation the parameters \( q_+ \) and \( q_- \), while the parameters in the multivariate versions of the Coxeter function studied in [BS03, BGM18] can take any value in the interval \([-1, 1]\) independently. The methods used in the previously mentioned works on the Coxeter functions are not applicable here and our approach is based on the connection between the representation theory of the hyperoctahedral group and symmetric functions. We prove that the Frobenius formula for the hyperoctahedral group expressed in terms of symmetric functions provides the full information on the expansion of the signed reflection function into normalized irreducible characters. Additionally, for the parameters \( N, M, \epsilon \) and the associated parameters \( q_+, q_- \) described in Theorem 1.1 we construct an explicit action of the hyperoctahedral group \( B(n) \) on the tensor product of \( N \) copies of the \( M+N \)-dimensional space \( V \), whose normalized character is given by the signed reflection function \( \phi_{q_+,q_-} \). This action gives a representation which is a two-parameter analog of the construction of Schur-Weyl.

In Section 4 we describe various applications of our main theorem in theory of operator algebras and probability theory: we start by constructing a cyclic Fock space of type B and proving in Theorem 4.5 that it admits a new cyclic commutation relation of type B:

\[ b_{q_+,q_-}(x \otimes y)b^*_{q_+,q_-}(\xi \otimes \eta) = \langle x, \xi \rangle \langle y, \eta \rangle \mathbb{1} + q_+ \langle x, \eta \rangle \langle y, \xi \rangle \mathbb{1} + \Gamma_{q_+}(\langle \xi \rangle \langle x | \otimes | \eta \rangle \langle y |), \]

where \( b_{q_+,q_-}(x \otimes y) \) and \( b^*_{q_+,q_-}(\xi \otimes \eta) \) are the annihilation and creation operators deformed by the use of the signed reflection function \( \phi_{q_+,q_-} \) and \( \Gamma_{q_+} \) is the differential second quantisation operator (see Section 4.1 for the precise definitions). In Section 4.1.1 we deduce the analog of the Pauli exclusion principle on this space. These results extend the previous work of Bożejko and Guta [BG02] in Coxeter groups of type A to type B and refine it by introducing two parameters \( q_+, q_- \). Note that the different constructions for type B were presented in [BEH15, BEH17], where the role of the signed reflection function was played by the Coxeter
function, which is not a character of $B(\infty)$. In Section 4.2 we introduce the notion of positive and negative cycles on the set of generalized symmetric pair-partitions, which share many similarities with the partitions of type B introduced by Reiner [Rei97]. We find the explicit formula (Theorem 4.15) for the moments of the generalized cyclic Gaussian operator of type B that we introduce in Section 4.3. This formula is an analog of the Wick-type formula for Coxeter groups of type B and is naturally expressed in terms of the combinatorial statistic on the set of positive and negative cycles, developed in Section 4.2. We show that the probability distribution $\mu_{q^+,q^-}$ of the cyclic Gaussian operator of type B with respect to the vacuum state is strictly related to the Askey–Wimp–Kerov distribution (Theorem 4.17) and the associated Hermite polynomials:

$$tH_n(t) = H_{n+1}(t) + (n+c)H_{n-1}(t), \quad n = 0, 1, 2, \ldots, \quad H_{-1}(t) = 0, H_0(t) = 1.$$ 

It is known [BBLS11] that the Askey–Wimp–Kerov distribution $\nu_c$ is freely infinitely divisible for $c \in [-1,0]$. Our work might shed new light on this phenomenon. We use the connection between the moments of the cyclic Gaussian operator of type B and the moments of the Askey–Wimp–Kerov distribution to show that the bivariate generating function of the symmetric pair-partitions (which distinguishes negative and positive cycles) specializes to the moments of the classical probability measures. In particular, this gives new combinatorial interpretations of the classical sequences $\frac{2^m}{m+1}\binom{2n}{n}$ and $\frac{(2n)!}{n!}$ and their deformations with respect to certain combinatorial statistics. Finally, we discuss the analogous problem for Coxeter groups of type D and we show that it can be obtained as a special case of our main result. In particular, we classify the positive-definite reflection functions for all the irreducible infinite Coxeter groups of Weyl type.

2. COXETER GROUPS OF TYPE B/C

2.1. Preliminaries on Coxeter groups of type B/C. General Coxeter system $(W, S)$ is defined as a group $W$ with chosen set $S$ of generators such that $W$ can be presented, with help of auxiliary symmetric function $m: S \times S \to \mathbb{N} \cup \{\infty\}$ satisfying $m_{st} = 1$ if and only if $s = t$, as

$$W = \langle S \rangle | (st)^{m_{st}} \text{ for } m_{st} < \infty \rangle.$$ 

Elements conjugated to the generators are called reflections. Two generators $s$ and $t$ are conjugated if and only if there exists a sequence of generators $s = s_1, \ldots, s_n = t$ such that all $m_{s_is_{i+1}}$ are odd.

The main example for this paper is the hyperoctahedral group $B(n)$, which is the group of symmetries of the $n$-dimensional hypercube. It is a finite real reflection group of type B/C and rank $n$ generated by reflections $s_0, \ldots, s_{n-1}$ where $s_0$ reflects in the plane $x_1 = 1$ and, for $i > 0$, $s_i$ reflects in the plane $x_i = x_{i+1}$.

In terms of Coxeter groups it is defined with help of the function

$$m_{s_is_j} = \begin{cases} 1 & \text{if } i = j \in \{0, \ldots, n-1\}, \\ 2 & \text{if } |i-j| > 1, \\ 4 & \text{if } \{i,j\} = \{0,1\}, \\ 3 & \text{otherwise}. \end{cases}$$

The Coxeter diagram for $B(n)$ is described in Fig. 1.
One can also define the group $B(n)$ as wreath product $\mathbb{Z}_2 \wr S_n$ with multiplication given by
\[
(g_1, \ldots, g_n; \sigma) \cdot (g'_1, \ldots, g'_n; \sigma') = (g_1g'_{\sigma^{-1}(1)}, \ldots, g_ng'_{\sigma^{-1}(n)}; \sigma\sigma'),
\]
where $g_i, g'_i \in \{1, -1\}, \sigma, \sigma' \in S_n$. We refer to this presentation as the signed model.

In this model $s_0$ corresponds to $((-1, 1, \ldots, 1), \text{id})$ and $s_i$ corresponds to $((1, \ldots, 1), (i \ i + 1))$.

Denote by $[\pm n]$ the set of integers $\{\overline{n}, \ldots, \overline{1}, 1, \ldots, n\}$. The hyperoctahedral group $B(n)$ can be also realised as the group of permutations $\sigma$ of the set $[\pm n]$ such that $\sigma(i) = \sigma(i)$ for any $i \in [\pm n]$ (with the convention that $\overline{i} = i$). These are precisely parmutations of $[\pm n]$ which commute with the involution $(\overline{\overline{n}} \ldots \overline{\overline{1}})$. We will refer to this realization as the permutation model.

**Example 1.** Let $\sigma = (124)(1\overline{2}4)(3\overline{5}5)(6)(\overline{6}) \in B(6)$ be an element of the hyperoctahedral group $B(6)$ considered as a permutation. Note that this element is uniquely determined by the word
\[
w(\sigma) := (\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5), \sigma(6)) = (2, 4, 5, 1, 3, 6).
\]
This word is uniquely determined by the permutation
\[
w_{+}(\sigma) := \left(\frac{1}{|\sigma(1)|} \frac{2}{|\sigma(2)|} \frac{3}{|\sigma(3)|} \frac{4}{|\sigma(4)|} \frac{5}{|\sigma(5)|} \frac{6}{|\sigma(6)|}\right) = \left(\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{1}\right) = (124)(35)(6)
\]
and by the sequence of signs $\text{sgn}(1)$ in $w(\sigma), \ldots, \text{sgn}(6)$ in $w(\sigma)) = (1, -1, 1, -1, -1, 1)$.
Here
\[
\text{sgn}(i) = \begin{cases} 
1 & \text{if } i \in \{1, 2, \ldots, n\}, \\
-1 & \text{if } i \in \{\overline{1}, \overline{2}, \ldots, \overline{n}\},
\end{cases}
\]
and $|\cdot| : [\pm n] \to [n]$ is the absolute value with the obvious definition. Therefore the associated element in $\mathbb{Z}_2 \wr S_n$ is given by $(1, -1, 1, -1, -1, 1; (124)(35)(6))$. This procedure adapted for arbitrary $n$ gives an isomorphism between the permutation model and the signed model.

### 2.2. Conjugacy classes

A partition $\rho$ of size $n$, where $n$ is a non-negative integer (also denoted $|\rho| = n$ or $\rho \vdash n$) is a non-increasing sequence $(\rho_1, \ldots, \rho_\ell)$ of integers which sum up to $n$:
\[
\sum_i \rho_i = n.
\]
The integer $\ell$ is called the length of the partition $\rho$ and it is denoted by $\ell(\rho)$. There is a unique partition $\rho = \emptyset$ of size 0 by convention.

The conjugacy classes of $B(n)$ are naturally identified with pairs of partitions $(\rho^+, \rho^-)$ of total size at most $n$, where the first partition $\rho^+$ has no parts equal to 1, i.e.
\[
|\rho^+| + |\rho^-| \leq n; \quad \rho_i^+ > 1 \text{ for } i = 1, \ldots, \ell(\rho^+).
\]

**Remark 2.1.** Note that these partitions are in a natural bijection with the set of pairs of partitions $(\rho^+, \rho^-)$ of total size equal to $n$. Indeed, if $\tilde{\rho}^+$ is obtained from $\rho^+$ by removing all its parts equal to 1, then the pair $(\tilde{\rho}^+, \rho^-)$ satisfies the conditions Eq. (2). This is clearly
invertible and for every pair of partitions \((\rho^+, \rho^-)\) which satisfies Eq. (2) there is a unique way of adding an appropriate number of parts equal to 1 to \(\rho^+\) to obtain a pair of partitions of total size equal to \(n\).

This identification is given by the following procedure. For each element \((g_1, \ldots, g_n; \sigma) \in B(n)\) we write \(\sigma = c_1 \cdots c_k\) as the product of disjoint cycles (fix-points of \(\sigma\) do not appear in this decomposition). For any cycle \(c = (a_1, \ldots, a_k)\) we define

\[
\text{sgn}(c) := g_{a_1} \cdots g_{a_k} = \pm 1.
\]

Then \(\rho^+\) is the partition given by the lengths of positive cycles
\[
\{c_i : i \in [\ell], \text{sgn}(c_i) = 1\}
\]
and \(\rho^-\) is the partition given by the lengths of negative cycles
\[
\{c_i : i \in [\ell], \text{sgn}(c_i) = -1\}.
\]

In the permutation model the pair of partitions \((\rho^+, \rho^-)\) can be understood as follows. Let \(\sigma \in B(n)\) be presented as a product of disjoint cycles \(\sigma = c_1 \cdots c_k\) (here, as before, fix-points do not appear as cycles in the decomposition). For any \(i \in [k]\) consider a cycle \(c_i := (a_{i_1}^i \cdots a_{i_k}^i)\), where \(c_i = (a_{i_1}^i \cdots a_{i_k}^i)\). Then either \(c_i\) is disjoint with \(c_i^\perp\) (and we call it a positive cycle) or \(c_i = c_i^\perp\) (and we call it a negative cycle). Let \(\tilde{\rho}^+\) denote the partition of lengths of positive cycles and \(\tilde{\rho}^-\) denote the partition of lengths of negative cycles. Note that \(\tilde{\rho}^+(\rho_+, \rho_-)\) is necessarily of the form \((\rho_1^+, \ldots, \rho_\ell^+; \rho_1^-, \ldots, \rho_\ell^-)\) and \(\tilde{\rho}^-\) is necessarily of the form \((2\rho_1^+, \ldots, 2\rho_\ell^-)\). Then the pair \((\rho_+, \rho_-)\) is given by \(\rho^+ = (\rho_1^+, \ldots, \rho_\ell^+)\) and \(\rho^- = (\rho_1^-, \ldots, \rho_\ell^-)\).

We denote by \(C_{\rho^+, \rho^-}\) the conjugacy class associated with the pair \((\rho_+, \rho_-)\).

**Example 2.** We continue with Example 1

\[
\sigma = (1, -1, 1, -1, 1; (124)(35)(6)) \in C_{\rho^+, \rho^-} \subset B(6),
\]
where \(\rho^+ = (3), \rho^- = (2)\). Using different presentation we have

\[
\sigma = (124)(124)(3535),
\]
which consists of one pair of positive cycles \((124)(124)\) of length 3 and one negative cycle \((3535)\) of length 4 = \(2 \cdot 2\).

2.3. **Central functions.** A function \(\phi : G \to \mathbb{C}\) on a group \(G\) is central if it is conjugacy invariant

\[
\phi(g) = \phi(hgh^{-1}) \text{ for any } g, h \in G.
\]

In particular \(\phi\) is fixed on the conjugacy classes.

For a Coxeter group \((G, S)\) and an element \(g \in G\) we denote by \(\ell_R(g)\) its reflection length, i.e. the minimal number of reflections that we need to use to decompose \(g\) as their product

\[
\ell_R(g) = \min(k : r_1 \cdots r_k = g, r_1, \ldots, r_k \in R := \{gs^{-1}g^{-1} : s \in S, g \in G\}).
\]

The set of reflections in the symmetric group \(\mathfrak{S}_n\) is given by all the transpositions, and for a permutation \(\sigma \in \mathfrak{S}_n\) its reflection length \(\ell_R(\sigma)\) is naturally expressed in terms of lengths of
cycles of $\sigma$. Indeed, let $\rho \vdash n$ be a partition whose parts are equal to lengths of cycles in $\sigma$. Then

$$\ell_{\mathcal{R}}(\sigma) = \sum_{i=1}^{\ell(\rho)} (p_i - 1) = |\rho| - \ell(\rho) =: \|\rho\|.$$  

A similar formula holds true for the hyperoctahedral group. The set of reflections $\mathcal{R}$ in $B(n)$ consists of two conjugacy classes $\mathcal{R}_+$ and $\mathcal{R}_-$ that we will call long reflections, and short reflections, respectively. These names reflects the fact that the corresponding roots have length $\sqrt{2}$ in the case of $\mathcal{R}_+$ and 1 in the case of $\mathcal{R}_-$. In the signed model these reflections are given by

$$\mathcal{R}_+ = \{(1, \ldots, 1, \epsilon, 1, \ldots, 1, \epsilon, 1, \ldots, 1; (i, j)) : \epsilon = \pm 1, 1 \leq i < j \leq n\},$$

$$\mathcal{R}_- = \{(1, \ldots, 1, -1, 1, \ldots, 1; id) : i \in [n]\},$$

and in the permutation model by

$$\mathcal{R}_+ = \{(ij)(ij) : i, j \in [\pm n], |i| \neq |j|\}$$

$$\mathcal{R}_- = \{(ii) : 1 \leq i \leq n\}.$$  

It is easy to see that we need at least $k - 1$ reflections $r_1, \ldots, r_{k-1}$ to express a positive cycle $c_+ \in B(n)$ of length $k$ (in the signed model) as their product $c_+ = r_1 \cdots r_{k-1}$. Similarly, we need at least $k$ reflections $r_1, \ldots, r_k$ to write a negative cycle $c_- \in B(n)$ of length $k$ as a product $c_- = r_1 \cdots r_k$. In particular, the reflection length of $\sigma$ is given by

$$\ell_{\mathcal{R}}(\sigma) = \sum_{i=1}^{\ell(\rho^+)} (p_i^+ - 1) + \sum_{i=1}^{\ell(\rho^-)} (p_i^- - 1) = \|\rho^+\| + |\rho^-|,$$

where $\rho^+$ and $\rho^-$ list the lengths of positive and negative cycles in $\sigma$, i.e. $\sigma \in C_{\rho^+, \rho^-}$.

In the case of the hyperoctahedral group $B(n)$ the notion of long and short reflections suggests the possibility of extending the univariate central function $\sigma \rightarrow q^{\ell_{\mathcal{R}}(\sigma)}$ to its bivariate refinement $\sigma \rightarrow q_{\mathcal{R}_+}^{\ell_{\mathcal{R}_+}(\sigma)} q_{\mathcal{R}_-}^{\ell_{\mathcal{R}_-}(\sigma)}$ defined as follows.

Suppose that $\sigma \in B(n)$ is expressed as a product of reflections, where the number of reflections is minimal:

$$\sigma = r_1 \cdots r_k, \quad r_i \in \mathcal{R},$$  

Then, we would like to set

$$\ell_{\mathcal{R}_+}(\sigma) = \text{The number of long reflections } r_i, 1 \leq i \leq k$$

appearing in the factorization Eq. (4),

$$\ell_{\mathcal{R}_-}(\sigma) = \text{The number of short reflections } r_i, 1 \leq i \leq k$$

appearing in the factorization Eq. (4).

The problem is that the functions $\ell_{\mathcal{R}_+}, \ell_{\mathcal{R}_-}$ are not well-defined, since the number of long/short reflections appearing in the minimal factorization Eq. (4) is not an invariant of
the factorization. The minimal example to see this can be already realized in $B(2)$, where the element $(-1, -1; id)$ (or the corresponding element $(1 \bar{1})(2 \bar{2})$ in the permutation model) can be expressed as the product of two negative reflections

$(-1, -1; id) = (1, -1; id) \cdot (-1, 1; id); \quad (1 \bar{1})(2 \bar{2}) = (1 \bar{1}) \cdot (2 \bar{2}),$

but also as the product of two long reflections

$(-1, -1; id) = (-1, 1; (12)) \cdot (1, 1; (12)); \quad (1 \bar{1})(2 \bar{2}) = (12)(\bar{1}2) \cdot (1\bar{2})(\bar{2}).$

In fact, it can be proved that the definition Eq. (5) does not depend on the choice of the factorization Eq. (4) if and only if $\sigma$ contains at most one negative cycle, which is the consequence of the work [BGRW17]. Nevertheless, there is a way to correct the definition of $\ell_{R_+}(\sigma), \ell_{R_-}(\sigma)$ which allows us to introduce a bivariate central function on $B(n)$ called the signed reflection function $\phi_{q_+,q_-} : B(n) \to \mathbb{C}$.

Consider the permutation model of $B(n)$. We say that a factorization Eq. (4) of $\sigma \in B(n)$ is minimal, non-mixing if:

(F1) the number $k$ of reflections is minimal,

(F2) for each $1 \leq i \leq k$ the support of $r_i$ belongs to a cycle of $\sigma$. In other terms the orbits of the action of $r_i$ on $[\pm n]$ form a sub-partition of the orbits of the action of $\sigma$ on $[\pm n]$ for each $1 \leq i \leq k$.

**Definition 2.2.** Let $q_+, q_- \in \mathbb{C}$ be parameters, and let $\sigma \in B(n)$. Let

\[\ell_{R_+}(\sigma) = \text{The number of long reflections } r_i, 1 \leq i \leq k\]

appearing in the minimal, non-mixed factorization Eq. (4),

\[\ell_{R_-}(\sigma) = \text{The number of short reflections } r_i, 1 \leq i \leq k\]

appearing in the minimal, non-mixed factorization Eq. (4).

We define the signed reflection function $\phi_{q_+, q_-} : B(n) \to \mathbb{C}$ by

\[\phi_{q_+, q_-}(\sigma) := q_+^{\ell_{R_+}(\sigma)} q_-^{\ell_{R_-}(\sigma)}.\]

It is clear (assuming that the minimal, non-mixing factorizations exist, which is proved in Proposition 2.3 and Lemma 2.4) that this function can be interpreted as a bivariate refinement of the reflection function. Indeed

\[\ell_{R_+}(\sigma) + \ell_{R_-}(\sigma) = \ell_R(\sigma)\]

so for $q_+ = q_- = q$ we recover the reflection length function $\phi_{q_+, q_-}(\sigma) = q^{\ell_R(\sigma)}$.

**Proposition 2.3.** The signed reflection function $\phi_{q_+, q_-}$ is central on $B(n)$ and for the element $\sigma \in C_{\rho^+, \rho^-}$ of the conjugacy class associated with the pair $(\rho^+, \rho^-)$ it is given by the explicit formula:

\[\phi_{q_+, q_-}(\sigma) = q^{|\rho^+| + |\rho^-|} \ell_R(\rho^-).\]

**Proof.** We need to prove that $\ell_{R_+}$ and $\ell_{R_-}$ do not depend on the choice of the minimal, non-mixed factorization (4). Strictly from the definition (non-mixing property (F2)), it is enough to prove the statement for the case when $\sigma$ is a positive/negative cycle. Indeed, each pair of reflections in the factorization (4) whose support belong to different orbits of $\sigma$ commutes, therefore the functions $\ell_{R_+}, \ell_{R_-}$ are additive with respect to the decomposition into disjoint cycles. Let $|\sigma|$ denote the minimal number of transpositions to write $\sigma$ as its product (so that
\[ \|\sigma\| = \|\rho\| \] for the partition \( \rho \) which lists the lengths of cycles in \( \sigma \), see Eq. (3)). For \( r \in \mathcal{R}_+ \) we have \( \|r\| = 2 \) and for \( r \in \mathcal{R}_- \) we have \( \|r\| = 1 \).

Suppose that \( \sigma \) is a positive cycle \( \sigma = c \cdot \tau \), where \( c \cap \tau = \emptyset \). Then \( \|\sigma\| = 2\|c\| \), so that \( \ell(\mathcal{R}(\sigma)) = \|c\| \) implies that in the minimal, non-mixed factorization of \( \sigma \) all the reflections are long and each such a factorization is of the form \( r_i = \tau_i \cdot \tau_i^\prime \), where \( \tau_1 \cdot \cdots \cdot \tau_k = c \) is a minimal factorization of \( c \) as a product of transpositions.

Suppose that \( \sigma \) is a negative cycle \( \sigma = c = \overline{\tau} \). Then \( \|\sigma\| \) is odd, so that \( \ell(\mathcal{R}(\sigma)) \leq \frac{\|c\|+1}{2} \) and when the equality holds then either
\[ \ell_+ (\mathcal{R}(\sigma)) = \frac{\|c\|+1}{2} \text{ and } \ell_- (\mathcal{R}(\sigma)) = 0 \]
or
\[ \ell_+ (\mathcal{R}(\sigma)) = \frac{\|c\|+1}{2} \text{ and } \ell_- (\mathcal{R}(\sigma)) = 1. \]

We claim that for the non-mixed factorization the first case is impossible. Indeed, if \( \|c\| = 1 \), then \( c \) is a short reflection. Suppose that this hypothesis holds for all the negative cycles such that \( \|c\| < 2n + 1 \) and let \( \|c\| = 2n + 1 \). Let \( r \) be a long reflection whose support is a subset of the support of \( c \). These are the only reflections which appear in the minimal, non-mixed factorization of \( c \). Note that \( c \cdot r \) is the disjoint product of two positive cycles \( c_1, c_2 \) (possibly fix-points) and one negative cycle \( c' \) such that \( \|c\| - 2 = \|c_1\| + \|c_2\| + \|c'\| \). Then, by induction, \( \ell_+ (\mathcal{R}(c)) = 1 \) which finishes the proof.

Finally, for \( \sigma \in \mathcal{C}_{\rho^+, \rho^-} \) we will use the fact that the functions \( \ell_+ , \ell_- \) are additive with respect to the decomposition into disjoint cycles, so that
\[ \ell_+ (\mathcal{R}(\sigma)) = \|\rho_+\| + \|\rho_-\|, \quad \ell_- (\mathcal{R}(\sigma)) = \ell(\rho_-). \]
Therefore the formula Eq. (6) holds true and the signed reflection function is constant on the conjugacy classes, so it is central. \( \square \)

2.3.1. **Factorization lemma.** There is a natural embedding of \( B(n-1) \) into \( B(n) \), which implies that there exists a canonical choice of the minimal, non-mixing factorization Eq. (4). We will describe it using the permutation model.

**Lemma 2.4.** We have the following factorization of the signed reflection length:

\[ \sum_{\sigma \in B(n)} \phi_{q_+, q_-}(\sigma) \cdot \sigma = \prod_{i=1}^{n} (1 + q_+ \cdot J_i^+ + q_- \cdot J_i^-), \]  \hspace{1cm} \text{(7)}

where
\[ J_i^+ = \sum_{j \in [\pm (i-1)]} (ji)(ji'), \quad J_i^- = (ii'). \]

**Proof.** It is known that there exist unique left coset representatives \( \{w(j) : j \in [\pm n]\} \) for \( B(n) \backslash B(n-1) \) with minimal lengths given by
\[ w(j) = \begin{cases} \text{id} & \text{for } j = n, \\ (jn)(jn') & \text{for } j \in [\pm (n-1)], \\ (jn) & \text{for } j = \overline{n}. \end{cases} \]  \hspace{1cm} \text{(8)}

Let \( \sigma \in B(n-1) \) and consider three cases according to the Equation (8). Obviously, in the first situation one has \( \phi_{q_+, q_-}(\text{id} \cdot \sigma) = \phi_{q_+, q_-}(\sigma) \). Suppose that \( j \in [\pm (n-1)] \). If \( j \) belongs to a positive cycle \( c \) of \( \sigma \) or \( j \) is a fix-point of \( \sigma \) and \( c = \text{id} \) then \( (jn)(jn') \cdot c \) is a positive cycle.
of \((jn)\bar{n}\cdot \sigma\) of length increased by one. Similarly, if \(j\) belongs to a negative cycle \(c\) of \(\sigma\) then \((jn)\bar{n}\cdot c\) is a negative cycle of \((jn)\bar{n}\cdot \sigma\) of length increased by one. In both cases

\[\phi_{q_+,q_-}(jn)\bar{n}\cdot \sigma = q_+\phi_{q_+,q_-}(\sigma)\]

In the third situation we have \(\phi_{q_+,q_-}(\bar{n}\cdot \sigma) = q_-\phi_{q_+,q_-}(\sigma)\), which finishes the proof. \(\square\)

**Remark 2.5.** Note that this embedding gives rise to the ascending tower of groups:

\[B(1) < B(2) < \ldots,\]

which allows to define the infinite group \(B(\infty)\) as the inductive limit of this tower. It is clear that the conjugacy classes of \(B(\infty)\) are parametrized by pairs of partitions \((\rho^+,\rho^-)\) such that all parts of \(\rho^+\) are greater or equal to two. In particular the signed reflection length \(\phi_{q_+,q_-}\) can be naturally extended to the infinite hyperoctahedral group \(B(\infty)\) by using the same definition as in equation (6) for \(\sigma \in C_{\rho^+\rho^-} \subset B(\infty)\). Thus, \(\phi_{q_+,q_-}\) is a central function on \(B(\infty)\).

### 2.4. Another view on non-mixing presentations.

Note that, in case of a single positive cycle, any minimal presentation consists of long reflections and one short one. It is easy to see that a minimal presentation is non-mixing if and only if it contains the minimal number of long reflections (among all minimal presentations). It can be understood in the following way.

Consider a natural map \(\varphi : B(n) \to A(n-1)\) between the groups of signed and unsigned permutations (since we would like to present our results in a way that might be applied to other Coxeter groups we use standard notation \(A(n-1)\) for the Coxeter group of type \(A\) and rank \(n-1\) which is isomorphic with the permutation group \(\mathfrak{S}_n\). It shall not be confused with the alternating group i.e. the group of even permutations). It sends reflections from \(R^+_\) to the reflections \(R'_+\) of \(A(n-1)\) and reflections from \(R^-\) to the identity. Both groups carry their reflection lengths \(\ell_{R^+}\) and \(\ell_{R'}\) respectively. Then \(\ell_{R^+}(\sigma) = \ell_{R'}(\varphi(\sigma))\) and \(\ell_{R^-}(\sigma) = \ell_{R^+}(\sigma) - \ell_{R'}(\varphi(\sigma))\).

Let \((W,S)\) be an arbitrary Coxeter system and assume that \(S\) is partitioned as \(S = S' \cup S''\) such that \(m_{s's''}\) is even for all \(s' \in S'\) and \(s'' \in S''\) (that is no element of \(S'\) is conjugated to an element of \(S''\)). Then there is a natural map \(\varphi : W \to W'\) between the Coxeter groups \(W\) and \(W'\), where \(W'\) is the group generated by \(S'\). This map sends \(S'\) into itself and \(S''\) into the identity. Given \(w \in W\) we define \(\ell_{R^+}(w)\) define the length of \(w\) with respect to the reflections of \(W\) and \(\ell_{R'}(w)\) define the length of \(\varphi(w)\) with respect to the reflections of \(W'\) (which are the same as reflections in \(W\) that belong to \(W_{S'}\) [Gal05, Corollary 1.4]).

**Question 2.6.** Is it true that \(\ell_{R'}(w)\) equals the minimal number of reflections conjugated to elements of \(S'\) among all minimal presentations of \(w\) as a product of reflections?

### 3. The signed reflection function and the representation theory of \(B(\infty)\)

In this section we will present our main result which gives a complete characterization of the parameters \(q_+,q_- \in \mathbb{C}\) for which the signed reflection function is positive definite on \(B(\infty)\). Since \(\phi_{q_+,q_-}\) is central it means that we classify when the signed reflection function is a character of \(B(\infty)\). We also find the corresponding points in the Thoma simplex of \(B(\infty)\).

Finally, we provide an explicit construction of the representations which realize the character \(\phi_{q_+,q_-}\), generalizing the Schur-Weyl construction.
3.1. **Characters.** A function $\phi : G \to \mathbb{C}$ is positive definite if for any number $k \in \mathbb{Z}_{>0}$ and any $z_1, \ldots, z_k \in \mathbb{C}$, $g_1, \ldots, g_k \in G$ we have

$$\sum_{i,j=1}^{k} z_i \overline{z_j} \phi(g_j^{-1} g_i) \geq 0.$$ 

Note that the representation theory of a finite group $G$ can be described by central, positive definite functions. Lemma 3.1 below is well known (see for instance [BO17] for the proof), and it gives a correspondence between normalized central positive definite functions on $G$ and probability measures on the space $\hat{G}$ of its irreducible representations. This is an analogy to the classical Bochner theorem which gives a correspondence between normalized continuous positive definite functions on $\mathbb{R}$ and probability measures on $\mathbb{R}$.

**Lemma 3.1.** Let $G$ be a finite group and $f : G \to \mathbb{C}$ be a central, normalized function. The function $f$ is positive definite if and only if it is a convex combination of normalized characters of irreducible representations of $G$.

When $G$ is infinite its irreducible representations might be infinite dimensional therefore the definition of characters as traces do not apply here. Motivated by Lemma 3.1 it turns out that the definition of characters which plays the essential role in the representation theory of infinite groups should be modified as follows.

**Definition 3.2.** Let $G$ be an arbitrary group. A character $\phi : G \to \mathbb{C}$ is a central, positive-definite function which takes value 1 on the identity.

Our main theorem gives a classification of the values $q_+, q_-$ for which the signed reflection function $\phi_{q_+, q_-} : B(\infty) \to \mathbb{C}$ is a character of $B(\infty)$.

**Theorem 3.3.** Let $q_+, q_- \in \mathbb{C}$. The following conditions are equivalent:

(i) The signed reflection function $\phi_{q_+, q_-} : B(\infty) \to \mathbb{C}$ is positive definite on $B(\infty)$;
(ii) The signed reflection function $\phi_{q_+, q_-} : B(\infty) \to \mathbb{C}$ is a character of $B(\infty)$;
(iii) $q_+ = \frac{\epsilon}{M+N}, q_- = \frac{M-N}{M+N}$ for $M, N \in \mathbb{N}, M + N \neq 0$, $\epsilon \in \{1, -1\}$ or $q_+ = 0, -1 \leq q_- \leq 1$.

Moreover, when these conditions are satisfied and $q_+ \neq 0$, the signed reflection function restricted to $B(n)$ is a normalized character of the explicitly constructed unitary representation, generalizing the Schur-Weyl construction.

Before we prove our main theorem we quickly recall some information about the representation theory of the hyperoctahedral group $B(n)$; this information will be a necessary tool in our proof of Theorem 3.3.

3.2. **Representation theory of the hyperoctahedral group.** The irreducible representations of $B(n)$ are parametrized by pairs of partitions $(\lambda^+, \lambda^-)$ such that $|\lambda^+| + |\lambda^-| = n$. We denote by $\chi_{(\lambda^+, \lambda^-)}$ the trace of the irreducible representation $\rho_{(\lambda^+, \lambda^-)} \in \overline{B(n)}$ associated with the pair $(\lambda^+, \lambda^-)$. The irreducible characters $\chi_{(\lambda^+, \lambda^-)}$ are central on $B(n)$ and we denote by $\chi_{(\lambda^+, \lambda^-)}(\rho^+, \rho^-)$ the value of $\chi_{(\lambda^+, \lambda^-)}$ on the conjugacy class $C_{\rho^+, \rho^-} \subset B(n)$. Here, we use the convention that the conjugacy classes of $B(n)$ are parametrized by pairs of partitions of total size $n$, so that some parts of $\rho^+$ might be equal to 1 (see Remark 2.1).
Following [Poi98] we are going to relate the representation theory of $B(n)$ with the theory of symmetric functions. We denote by $x$ ($y$ respectively) the infinite alphabet $x_1, x_2, \ldots$ ($y_1, y_2, \ldots$). Let $\epsilon \in \{+, -\}$ and let

$$p_\epsilon^\rho(x, y) := \sum_{i \geq 1} x_i^\epsilon + \epsilon \sum_{i \geq 1} y_i^\epsilon.$$ 

Finally, for a partition $\rho$ we define

$$p_\rho^\epsilon(x, y) := \prod_{i=1}^{\ell(\rho)} p_\epsilon^\rho(x, y)$$

and for a pair of partitions $(\rho^+, \rho^-)$ we set

$$p_{(\rho^+, \rho^-)}(x, y) := p_{\rho^+}(x, y)p_{\rho^-}(y, x).$$

Let $s_\lambda(x)$ denote the Schur symmetric function in the infinite alphabet $x = (x_1, x_2, \ldots)$. We use the following Frobenius formula for the direct calculation of $p_{(\rho^+, \rho^-)}(x, y)$ in terms of irreducible characters $\chi_{(\lambda^+, \lambda^-)}$.

**Lemma 3.4** (Frobenius formula, [Poi98]). For any pair of partitions $(\rho^+, \rho^-)$ of total size equal to $n$ we have the following equality:

$$p_{(\rho^+, \rho^-)}(x, y) = \sum_{(\lambda^+, \lambda^-) \in B(n)} \chi_{(\lambda^+, \lambda^-)}(\rho^+, \rho^-)s_{\lambda^+}(x)s_{\lambda^-}(y).$$

**3.3. Proof of the main result.** We are ready to prove Theorem 3.3.

**Proof of Theorem 3.3.** The signed reflection function is normalized $\phi_{q_+, q_-}(id) = 1$, therefore it is clear from Definition 3.2 and from Remark 2.5 that the conditions (i) and (ii) are equivalent.
We will now prove that (i) $\Leftrightarrow$ (iii). The signed reflection length $\phi_{q_+,q_-}$ is positive definite on $B(\infty)$ if and only if it is positive definite on $B(n)$ for each $n$. Fix two positive integers $N,M \in \mathbb{Z}_{>0}$, set

$$x_i = \begin{cases} 1 & \text{if } i \leq M, \\ 0 & \text{otherwise}; \end{cases} \quad y_i = \begin{cases} 1 & \text{if } i \leq N, \\ 0 & \text{otherwise}; \end{cases}$$

and plug these families $(x,y)$ into (9). We obtain the following equality

$$(N + M)^{n-\|\rho^+\|\|\rho^-\|}(M - N)^{\ell(\rho^-)} = \sum_{(\lambda^+,\lambda^-) \in \mathcal{B}_n} \chi_{(\lambda^+,\lambda^-)}(\rho^+,\rho^-) \left( s_{\lambda^+}(1, \ldots, 1, 0 \ldots)_{M} s_{\lambda^-}(1, \ldots, 1, 0 \ldots)_{N} \right).$$

We use the classical content-formula for Schur polynomials (see for instance [Sta99])

$$s_{\lambda}(1^N) = \prod_{\square \in \lambda} (N + c(\square)), \tag{10}$$

where $c(\square) := x - y$ is the content of the box $\square = (x,y)$. Plugging it into the previous identity we obtain

$$(N + M)^{n-\|\rho^+\|\|\rho^-\|}(M - N)^{\ell(\rho^-)} = \sum_{(\lambda^+,\lambda^-) \in \mathcal{B}_n} \chi_{(\lambda^+,\lambda^-)}(\rho^+,\rho^-) \prod_{\square \in \lambda^+} (M + c(\square)) \prod_{\square \in \lambda^-} (N + c(\square)). \tag{11}$$

Note that the left and the right hand sides of (10) are both polynomials in $N, M$ which are equal on the grid $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. Therefore equality (10) holds true for any real parameters $N, M \in \mathbb{R}$. Using the following change of variables $q^{-1} = N + M, q_+^{-1}q_- = M - N$ we obtain

$$\phi_{q_+,q_-} = \sum_{(\lambda^+,\lambda^-) \in \mathcal{B}_n} \chi_{(\lambda^+,\lambda^-)} \prod_{\square \in \lambda^+} \left( q_+ c(\square) + \frac{1 + q_-}{2} \right) \prod_{\square \in \lambda^-} \left( q_+ c(\square) + \frac{1 - q_-}{2} \right). \tag{11}$$

This in conjunction with Lemma 3.1 gives a sufficient and necessary condition for $\phi_{q_+,q_-}$ to be positive definite:

$$\prod_{\square \in \lambda^+} \left( q_+ c(\square) + \frac{1 + q_-}{2} \right) \prod_{\square \in \lambda^-} \left( q_+ c(\square) + \frac{1 - q_-}{2} \right) \geq 0$$

for any partitions $\mu, \lambda$. This condition is satisfied only for $q_+, q_-^{}$ as in our hypothesis, which finishes the proof of the equivalence (i) $\Leftrightarrow$ (iii).

Suppose that the parameters $q_+, q_-^{}$ are given by (iii). We will construct a representation $\omega_n$ of $B(n)$ whose normalized character

$$\frac{\text{Tr} \omega_n(\cdot)}{\text{Tr} \omega_n(id)} : B(n) \to \mathbb{C}$$

coincides with the signed reflection function

$$\phi_{q_+,q_-} : B(n) \to \mathbb{C}. $$
Our construction generalizes the action of the symmetric group on the tensor product of a fixed vector space. Consider

$$V = \text{Span}\{e_1^+, \ldots, e_M^+, e_1^-, \ldots, e_N^-\}.$$  

The hyperoctahedral group $B(n)$ acts on $W := \bigotimes_{\ell=1}^n V$ as follows

$$(g_1, \ldots, g_n; \sigma) \cdot f_1 \otimes \cdots \otimes f_n := e^{\|\rho^+\|+\|\rho^-\|} c_1(f_{\sigma^{-1}(1)}) \otimes \cdots \otimes c_n(f_{\sigma^{-1}(n)}),$$

where $c_i$ is an operator $c_i : V \mapsto V$ defined as:

$$c_i(f) = \begin{cases} f & \text{if } f = e_j^+, \\ g_i \cdot f & \text{if } f = e_j^-, \end{cases}$$

and $(g_1, \ldots, g_n; \sigma) \in C_{\rho^+, \rho^-}$. It is straightforward to check that this action extended by multilinearity defines the representation of $B(n)$ on $W$. Let us compute the character of this representation. Pick an element $g = (g_1, \ldots, g_n; \sigma) \in C_{\rho^+, \rho^-} \subset B(n)$ and note that the only elementary tensors from $W$ contributing to the trace of $\omega_n(g)$ have necessarily the same vectors $e$ in all the coordinates corresponding to the points in the same cycle of $\sigma$. Let us choose this vector equal to $e_j^\pm$ for a fixed cycle $c$ of length $\ell(c)$. Then, the eigenvalue contributing to this cycle is given by

$$\begin{cases} e^{\ell(c)-1} & \text{when } c \text{ is a positive cycle,} \\ \pm \cdot e^{\ell(c)-1} & \text{when } c \text{ is a negative cycle.} \end{cases}$$

This means that

$$\text{Tr} \omega_n(g) = e^{\|\rho^+\|+\|\rho^-\|} (N + M)^n - \ell(\rho^-) (M - N)^{\ell(\rho^-)}.$$  

Moreover the dimension of $W$ is equal to $(N + M)^n$, therefore we have the following formula

$$\frac{\text{Tr} \omega_n(g)}{\text{Tr} \omega_n(\text{id})} = \left(\frac{e}{N + M}\right)^{\|\rho^+\|+\|\rho^-\|} \cdot \left(\frac{M - N}{N + M}\right)^{\ell(\rho^-)} = \phi_{q^+, q^-}(g).$$

\[\square\]

**Remark 3.5.** Note that the classical Frobenius formula relates the irreducible characters $\chi_\lambda$ of the symmetric group $\mathfrak{S}_n$ with the Schur symmetric function $s_\lambda(x)$:

$$p_\mu(x) = \sum_{\lambda \vdash n} \chi_\lambda(\mu) s_\lambda(x).$$

Using the same arguments as in the proof of Theorem 3.3 one can apply the Frobenius formula to show that the reflection function $g \mapsto q^{s(g)}$ on the infinite symmetric group $\mathfrak{S}_\infty$ is positive definite if and only if $q = 0$ or $q^{-1} \in \mathbb{Z}$.

### 3.4. The signed reflection length and extreme characters of $B(\infty)$.

The set of characters of the group $B(\infty)$ forms an infinite dimensional simplex. A character $\phi : B(\infty) \to \mathbb{C}$ is called *extreme* if it belongs to the extreme points of the simplex. Since any simplex is uniquely determined by its extreme points, the classification problem of the characters of $B(\infty)$ reduces to the problem of characterizing the extreme characters. The classification of all the extreme characters of the group $B(\infty)$, which is a type $B$ analog of the classical
Thoma’s theorem, was found by Hirai and Hirai [HH02]. They proved that the set of extreme characters of $B(\infty)$ is parametrized by the following sequences:

$$\alpha_1 \geq \alpha_2 \geq \ldots \in \mathbb{R}^\infty_{\geq 0},$$
$$\beta_1 \geq \beta_2 \geq \ldots \in \mathbb{R}^\infty_{\geq 0},$$
$$\gamma_1 \geq \gamma_2 \geq \ldots \in \mathbb{R}^\infty_{\geq 0},$$
$$\delta_1 \geq \delta_2 \geq \ldots \in \mathbb{R}^\infty_{\geq 0},$$
$$\kappa \in \mathbb{R}$$

such that

$$\|\alpha\| + \|\beta\| + \|\gamma\| + \|\delta\| + |\kappa| = \sum_{i \geq 0} (\alpha_i + \beta_i + \gamma_i + \delta_i) + |\kappa| \leq 1.$$ 

For any such sequences $\alpha, \beta, \gamma, \delta, \kappa$ the value of the associated extreme character $\psi_{\alpha, \beta, \gamma, \delta, \kappa}$ on the conjugacy class $C(\rho^+, \rho^-)$ is given by the following formula:

$$\psi_{\alpha, \beta, \gamma, \delta, \kappa}(\rho^+, \rho^-) = (\|\alpha\| + \|\beta\| - \|\gamma\| - \|\delta\| + \kappa) m_1(\rho^-) \times \prod_{\varepsilon \in \{+,-\}} \times \prod_{j=2}^\infty \left( \sum_{i=1}^\infty \alpha_i^j + (-1)^{j-1} \sum_{i=1}^\infty \beta_i^j + \varepsilon \sum_{i=1}^\infty \gamma_i^j + \varepsilon (-1)^{j-1} \sum_{i=1}^\infty \delta_i^j \right) m_j(\rho^\varepsilon).$$

It is straightforward to check that the signed reflection function $\phi_{q^+, q^-}$ with parameters $q^+, q^-$ given by (iii) coincides with the extreme character of $B(\infty)$ corresponding to the sequences

$$\alpha = \left( \frac{1}{M+N}, \ldots, \frac{1}{M+N}, 0 \ldots \right), \quad \beta = 0,$$
$$\gamma = \left( \frac{1}{M+N}, \ldots, \frac{1}{M+N}, 0 \ldots \right), \quad \delta = 0, \quad \kappa = 0$$

when $q^+ = \frac{1}{M+N}, q^- = \frac{M-N}{N+M},$

$$\alpha = 0, \quad \beta = \left( \frac{1}{M+N}, \ldots, \frac{1}{M+N}, 0 \ldots \right),$$
$$\gamma = 0, \quad \delta = \left( \frac{1}{M+N}, \ldots, \frac{1}{M+N}, 0 \ldots \right), \quad \kappa = 0$$

when $q^+ = \frac{-1}{M+N}, q^- = \frac{M-N}{N+M},$ and

$$\alpha = \beta = \gamma = \delta = 0, \quad \kappa = q_-$$

when $q^+ = 0, |q^-| \leq 1.$

These considerations in conjunction with Theorem 3.3 give directly Theorem 1.1.

4. APPLICATIONS

In this section, we assume that the parameters $q^+$ and $q^-$ are as in Theorem 1.1.
4.1. **Cyclic Fock space of type B.** Let $H_\mathbb{R}$ be a separable real Hilbert space and let $H$ be its complexification with the inner product $\langle \cdot, \cdot \rangle$ linear on the right component and anti-linear on the left. The Hilbert space $K := H \otimes H$ is the complexification of its real subspace $K_\mathbb{R} := H_\mathbb{R} \otimes H_\mathbb{R}$, with the inner product
\[ \langle x \otimes y, \xi \otimes \eta \rangle_K = \langle x, \xi \rangle \langle y, \eta \rangle. \]
We define $K_n := H^\otimes n \otimes H^\otimes n = H^\otimes 2n$ and instead of indexing its simple tensors by \{1, \ldots, 2n\} we will index them by $[\pm n]$:
\[ K_n \ni x_\pi \otimes x_n = x_\pi \otimes \cdots \otimes x_\pi \otimes x_1 \otimes \cdots \otimes x_n = x_\pi \otimes \cdots \otimes x_\pi. \]
We use this convention for indexing the elements of $K_n$ to define a natural action of the hyperoctahedral group $B(n)$ on $K_n$ by setting:
\[ \sigma : K_n \rightarrow K_n, \]
\[ x_\pi \otimes \cdots \otimes x_\pi \otimes x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(\pi)} \otimes \cdots \otimes x_{\sigma(\pi)} \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}. \]
for any $\sigma \in B(n)$. Let $F(K)$ be the (algebraic) full Fock space over $K$
\[ F(K) := \bigoplus_{n=0}^\infty K_n = \bigoplus_{n=0}^\infty H^\otimes 2n \]
with the convention that $K^{\otimes 0} = H^{\otimes 0} \otimes H^{\otimes 0} = \mathbb{C} \Omega \otimes \Omega$ is the one-dimensional normed space along the unit vector $\Omega \otimes \Omega$. Note that elements of $F(K)$ are finite linear combinations of the elements from $H^{\otimes 2n}, n \in \mathbb{N} \cup \{0\}$ and we do not take the completion. We equip $F(K)$ with the inner product
\[ \langle x_\pi \otimes \cdots \otimes x_\pi \otimes x_1 \otimes \cdots \otimes x_n, y_\pi \otimes \cdots \otimes y_\pi \otimes y_1 \otimes \cdots \otimes y_m \rangle_{0,0} := \delta_{m,n} \prod_{i=n}^n \langle x_i, y_i \rangle. \]
For the parameters $q_+$ and $q_-$ described in Theorem 1.1 (iv) we define the symmetrization operators
\[ P^{(n)}_{q_+, q_-} := \sum_{\sigma \in B(n)} \phi_{q_+, q_-}(\sigma) \sigma, \quad n \geq 1, \]
\[ P^{(0)}_{q_+, q_-} := \text{id}_{H^{\otimes 0} \otimes H^{\otimes 0}}. \]
Moreover let
\[ P_{q_+, q_-} := \bigoplus_{n=0}^\infty P^{(n)}_{q_+, q_-} \]
be the **cyclic type B symmetrization operator** acting on the algebraic full Fock space. For $x_\pi \otimes x_n \in K_n$ and $y_\pi \otimes y_m \in K_m$ we deform the inner product $\langle \cdot, \cdot \rangle_{0,0}$ by using the cyclic type B symmetrization operator:
\[ \langle x_\pi \otimes x_n, y_\pi \otimes y_m \rangle_{q_+, q_-} := \delta_{n,m} \langle x_\pi \otimes x_n, P^{(m)}_{q_+, q_-} y_\pi \otimes y_m \rangle_{0,0} \]
which by Theorem 1.1 is a semi-inner product from the positivity of $P_{q_+, q_-}$. For $x \in H$ let $l(x)$ and $r(x)$ be the free left and right annihilator operators on $H^{\otimes n}$, respectively, defined by
the equations
\[ l^*(x)(x_1 \otimes \cdots \otimes x_n) := x_1 \otimes x_1 \otimes \cdots \otimes x_n, \]
\[ l(x)(x_1 \otimes \cdots \otimes x_n) := \langle x, x_1 \rangle x_2 \otimes \cdots \otimes x_n, \]
\[ r^*(x)(x_1 \otimes \cdots \otimes x_n) := x_1 \otimes \cdots \otimes x_n \otimes x, \]
\[ r(x)(x_1 \otimes \cdots \otimes x_n) := \langle x, x_n \rangle x_1 \otimes \cdots \otimes x_{n-1}, \]
where the adjoint is taken with respect to the free inner product. The left-right creation and annihilation operators \( b^*(x \otimes y), b(x \otimes y) \) on \( \mathcal{F}(\mathcal{K}) \) are defined by
\[ b^*(x \otimes y)(x_{\pi} \otimes x_n) := l^*(x)x_{\pi} \otimes r^*(y)x_n, \quad b^*(x \otimes y)\Omega \otimes \Omega := x \otimes y, \]
\[ b(x \otimes y)(x_{\pi} \otimes x_n) := l(x)x_{\pi} \otimes r(y)x_n, \quad b(x \otimes y)\Omega \otimes \Omega := 0, \]
where \( x_{\pi} \otimes x_n \in \mathcal{K}_n, n \geq 1 \). Then it holds that \( b^*(x \otimes y)^* = b(x \otimes y) \) where the adjoint is taken with respect to \( \langle \cdot, \cdot \rangle_{0,0} \) and \( b^* : \mathcal{K} \to B(\mathcal{F}(\mathcal{K})) \) is linear, but \( b : \mathcal{K} \to B(\mathcal{F}(\mathcal{K})) \) is anti-linear (here and throughout the paper we will use the notation \( B(X) \) for the space of bounded operators on \( X \)).

**Definition 4.1.** Let \( q_+ \) and \( q_- \) be as in Theorem 1.1 (iv). The algebraic full Fock space \( \mathcal{F}(\mathcal{K}) \) equipped with the inner product \( \langle \cdot, \cdot \rangle_{q_+,q_-} \) is called the cyclic Fock space of type B and it is denoted by \( \mathcal{F}_{q_+,q_-}(\mathcal{K}) \). For \( x \otimes y \in \mathcal{K} \) we define \( b^*_{q_+,q_-}(x \otimes y) := b^*(x \otimes y) \) and we consider its adjoint operator \( b_{q_+,q_-}(x \otimes y) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{q_+,q_-} \) acting on the Hilbert space \( \mathcal{F}_{q_+,q_-}(\mathcal{K}) \) (note that a priori \( b^*_{q_+,q_-}(x \otimes y) \) might not be bounded). The operators \( b^*_{q_+,q_-}(x \otimes y) \) and \( b_{q_+,q_-}(x \otimes y) \) are called cyclic creation and cyclic annihilation operators of type B.

The following proposition can be derived directly from Lemma 2.4.

**Proposition 4.2.** We have the decomposition
\[ P_{q_+,q_-}^{(n)} = (\text{id} \otimes P_{q_+,q_-}^{(n-1)} \otimes \text{id}) R_{q_+,q_-}^{(n)} \text{ on } \mathcal{K}_n, \quad n \geq 1, \]
where
\[ R_{q_+,q_-}^{(n)} = \text{id} + q_- J_n^- + q_+ J_n^+. \]

Now, we can compute the annihilation operator in terms of \( R_{q_+,q_-}^{(n)} \).

**Proposition 4.3.** For \( n \geq 1 \), we have
\[ b_{q_+,q_-}(x \otimes y) = b(x \otimes y)R_{q_+,q_-}^{(n)} \text{ on } \mathcal{K}_n. \]

**Proof.** Let \( f \in \mathcal{K}_{n-1}, g \in \mathcal{K}_n \). Then
\[ \langle f, b_{q_+,q_-}(x \otimes y)g \rangle_{q_+,q_-} = \langle b^*_{q_+,q_-}(x \otimes y)f, g \rangle_{q_+,q_-} = \langle b^*_{q_+,q_-}(x \otimes y)f, P_{q_+,q_-}^{(n)}g \rangle_{0,0} = \langle b^*_{q_+,q_-}(x \otimes y)f, (\text{id} \otimes P_{q_+,q_-}^{(n-1)} \otimes \text{id}) R_{q_+,q_-}^{(n)}g \rangle_{0,0} = \langle f, b(x \otimes y)(\text{id} \otimes P_{q_+,q_-}^{(n-1)} \otimes \text{id}) R_{q_+,q_-}^{(n)}g \rangle_{0,0}. \]
Observe that \( b(x \otimes y) (\text{id} \otimes P_{q_+q_-}^{(n-1)} \otimes \text{id}) h = P_{q_+q_-}^{(n-1)} b(x \otimes y) h \) for \( h \in \mathcal{K}_n \) and so we get
\[
\langle f, b(x \otimes y) (\text{id} \otimes P_{q_+q_-}^{(n-1)} \otimes \text{id}) P_{q_+q_-}^{(n)} g \rangle_{0,0} = \langle f, P_{q_+q_-}^{(n-1)} b(x \otimes y) P_{q_+q_-}^{(n)} g \rangle_{0,0} = \langle f, b(x \otimes y) P_{q_+q_-}^{(n)} g \rangle_{q_+q_-}.
\]
(17)

In order to simplify the notation we define the operators
\[
\mathcal{J}_i(x \otimes y) : \mathcal{K}_n \to \mathcal{K}_{n-1} \text{ for } i \in [\pm n]
\]
by
\[
x_{\pi} \otimes x_n \mapsto \begin{cases} 
\langle x, x_\pi \rangle \langle y, x_n \rangle C(x_\pi \otimes x_n) & i = n \\
\langle x, x_\gamma \rangle \langle y, x_\iota \rangle C((i,n) \pi \otimes x_n) & i \in [\pm(n-1)] \\
\langle x, x_n \rangle \langle y, x_\pi \rangle C(x_\pi \otimes x_n) & i = \pi 
\end{cases}
\]
where \( C(x_\pi \otimes \cdots \otimes x_n) = x_{n-1} \otimes \cdots \otimes x_{n-1} \) and \( C(x_\pi \otimes x_1) = \Omega \otimes \Omega \).

**Remark 4.4.** By using the above notation we can decompose \( b_{q_+,q_-}(x \otimes y) \) as follows
\[
b_{q_+,q_-}(x \otimes y) = \alpha_0 (x \otimes y) + \beta_{q_+}(x \otimes y) + \gamma_{q_-}(x \otimes y), \quad x \otimes y \in \mathcal{K},
\]
where
\[
\alpha_0 (x \otimes y) = \mathcal{J}_n
\]
\[
\beta_{q_+}(x \otimes y) = q_+ \sum_{i \in [\pm(n-1)]} \mathcal{J}_i
\]
\[
\gamma_{q_-}(x \otimes y) = q_- \mathcal{J}_\pi.
\]
The second quantisation differential operator \( \Gamma_{q_+}(A \otimes B) \) is defined by the equation
\[
\Gamma_{q_+}(A \otimes B)x_{\pi} \otimes \cdots \otimes x_n = q_+ \sum_{1 \leq i \leq n} x_{\pi} \otimes \cdots \otimes Ax_\iota \otimes \cdots \otimes Bx_n + q_+ \sum_{1 \leq i \leq n} x_{\pi} \otimes \cdots \otimes Bx_i \otimes \cdots \otimes Ax_\iota \otimes \cdots \otimes x_n
\]
where \( A \otimes B \in B(\mathcal{K}) \) and \( \Gamma_{q_+} \Omega \otimes \Omega = 0 \). The above considerations provide a new commutation relation.

**Theorem 4.5.** For \( x \otimes y, \xi \otimes \eta \in \mathcal{K} \) we have the cyclic commutation relation of type B
\[
b_{q_+,q_-}(x \otimes y)b^*_{q_+,q_-}(\xi \otimes \eta) = \langle x, \xi \rangle \langle y, \eta \rangle \text{id} + q_- \langle x, \eta \rangle \langle y, \xi \rangle \text{id} + \Gamma_{q_+} (|\xi\rangle \langle x| \otimes |\eta\rangle \langle y|),
\]
where \( |\xi\rangle \langle x| : = \langle x, \cdot \rangle \xi \). We note that \( |x\rangle \langle x| \) is the projection on the one dimensional space spanned by \( x \) and \( \Gamma_{q_+} (|\xi\rangle \langle x| \otimes |\eta\rangle \langle y|) = \beta_{q_+}(x \otimes y)b^*_{q_+}(\xi \otimes \eta) \).

4.1.1. **Exclusion principle.** The Pauli exclusion principle is the quantum mechanical principle which states that two or more identical fermions cannot occupy the same quantum state within a quantum system simultaneously. We will explain that for \( q_+ < 0 \) an exclusion principle is found allowing at most \( M \) identical particles on the same state \( x \otimes x \), with \( \|x\| = 1 \). This might have some interest also from the physics point of view. First observe that \( b_{q_+,q_-}(x \otimes x)b^*_{q_+,q_-}(x \otimes x) \geq 0 \). Thus, Theorem 4.5 entails that
\[
(1 + q_-) \text{id} + \Gamma_{q_+} (|x\rangle \langle x| \otimes |x\rangle \langle x|) \geq 0.
\]
Computing $\langle (1 + q_-) \id + \Gamma_{q_+}(|x\rangle \otimes |x\rangle \langle x|) \rangle x^{\otimes 2n}, x^{\otimes 2n} \rangle_{q_+, q_-}$ we obtain that

$$1 + q_- \geq -q_+ 2n.$$ 

We recall that $q_+ = \frac{1}{M+N}$ and $q_- = \frac{M-N}{M+N}$ which gives the inequality

$$M \geq n.$$ 

We can also explain this phenomenon by using (13), namely

$$0 \leq \langle x^{\otimes 2n}, x^{\otimes 2n} \rangle_{q_+, q_-}$$

$$= \langle x^{\otimes 2n}, \Gamma_{q_+} \Gamma_{q_-} x^{\otimes 2n} \rangle_{0,0}$$

$$= (1 + 2(n - 1)q_+ + q_-) \langle x^{\otimes 2(n-1)}, \Gamma_{q_+} \Gamma_{q_-} x^{\otimes 2(n-1)} \rangle_{0,0}$$

$$= (1 + 2(n - 1)q_+ + q_-) \langle x^{\otimes 2(n-1)}, x^{\otimes 2(n-1)} \rangle_{q_+, q_-}.$$ 

This gives us the inequality $n < M + 1$, and consequently, we obtain $n \leq M$.

4.2. Cycles on pair partitions of type B. Let $S \subseteq \mathbb{N}$ be a finite subset. For an ordered set $S$, let $\mathcal{P}(S)$ denote the lattice of set partitions of that set. For a partition $\pi \in \mathcal{P}(S)$, we write $B \in \pi$ if $B$ is a class of $\pi$ and we say that $B$ is a block of $\pi$. Any partition $\pi$ defines an equivalence relation on $S$, denoted by $\sim_{\pi}$, such that the equivalence classes are the blocks of $\pi$. That is, $i \sim_{\pi} j$ if $i$ and $j$ belong to the same block of $\pi$. A block of $\pi$ is called a singleton if it consists of one element. Similarly, a block of $\pi$ is called a pair if it consists of two elements. $\mathcal{P}(n)$ is a lattice under the refinement order, where the relation $\pi \leq \rho$ holds if every block of $\pi$ is contained in a block of $\rho$.

A partition $\pi$ is called noncrossing if different blocks do not interlace, i.e., there is no quadruple of elements $i < j < k < l$ such that $i \sim_{\pi} k$ and $j \sim_{\pi} l$ but $i \not\sim_{\pi} j$. The set of non-crossing partitions of $S$ is denoted by $\mathcal{NC}(S)$. The subclass of noncrossing partitions whose every block is either a pair or a singleton (i.e. noncrossing matchings) is denoted by $\mathcal{NC}_{1,2}(S)$.

**Definition 4.6.** We denote by $\mathcal{P}_{1,2}^{sym}(n)$ the subset of partitions $\pi \in \mathcal{P}([\pm n])$ whose every block is either a pair or a singleton and such that they are symmetric $\overline{\pi} = \pi$, but every pair $B \in \pi$ is different than its symmetrization $\overline{B}$, i.e., $B \neq \overline{B}$. We will order elements in pairs $\{a, b\}$ of $\pi \in \mathcal{P}_{1,2}^{sym}(n)$ by writing $(a, b)$ which means that $a < b$ and we call $a$ (b respectively) the left (right, respectively) leg of $(a, b)$. A pair $(a, b)$ is called positive if $b > a$; otherwise it is called negative. The subset of pair partitions of $\mathcal{P}_{1,2}^{sym}(n)$ is denoted by $\mathcal{P}_{2}^{sym}(n)$.

**Proposition 4.7.** Let $\pi \in \mathcal{P}_{1,2}^{sym}(n)$. There exists a unique non-crossing partition $\hat{\pi} \in \mathcal{NC}_{1,2}([\pm n]) \cap \mathcal{P}_{1,2}(n)$, such that

(a) the set of right legs of the positive pairs of $\pi$ and $\hat{\pi}$ coincide;

(b) the set of left legs of the negative pairs of $\pi$ and $\hat{\pi}$ coincide;

(c) pairs of $\hat{\pi}$ do not cover singletons.

**Proof.** Notice that if $\sigma \in \mathcal{NC}_{1,2}([\pm n]) \cap \mathcal{P}_{1,2}(n)$ then each block of $\sigma$ is either contained in $\{1, \ldots, n\}$ or in $\{\overline{n}, \ldots, \overline{1}\}$. In particular $\sigma$ is completely determined by its restriction to $\{\overline{n}, \ldots, \overline{1}\}$. Take a non-crossing partition $\pi$ on $\{\overline{n}, \ldots, \overline{1}\}$ whose blocks are either pairs or singletons and associate it with the Motzkin path, whose left legs correspond to up steps, right legs correspond to down steps and singletons correspond to horizontal steps. We recall that a Motzkin path of length $n$ is a path starting at $(0, 0)$ and finishing in $(n, 0)$ which never goes
below the horizontal and consists of three steps: the up step \((1, 1)\), the down step \((1, -1)\) and the horizontal step \((1, 0)\). Notice that pairs do not cover singletons in this partition if and only if all the horizontal steps in the associated Motzkin path lie on the horizontal axis. Pick a partition \(\pi \in \mathcal{P}^{sym}_{1,2}(n)\) and associate with it a path with \(n\) steps by considering the set \(\{\pi, \ldots, \bar{1}\}\) and placing an up step in the place of left legs of the negative pairs of \(\pi\) and down steps in the other points of \(\{\pi, \ldots, \bar{1}\}\). Notice that there is a unique way for replacing some of the down steps by horizontal steps so that the resulting path is a Motzkin path with all the horizontal steps placed on the horizontal axis. Indeed, we successively change a down step into a horizontal step whenever we go below the horizontal axis. We consider the associated non-crossing partition of \(\{\pi, \ldots, \bar{1}\}\), and by using the symmetry it uniquely determines a partition \(\hat{\pi} \in \mathcal{NC}_{1,2}([\pm n]) \cap \mathcal{P}_{1,2}(n)\) which has all the desired properties. This finishes the proof. □

**Definition 4.8.** For the pairs \(A, B\) of \(\pi\), we say that \(A\) is connected with \(B\) if there exists a pair \(C \in \hat{\pi}\) such that \(A \cap C \neq \emptyset\) and \(B \cap C \neq \emptyset\).

We denote this equivalence relation by writing \(A \sim B\) see Fig. 3.

**Remark 4.9.** In the graphical presentation of the Definition 4.8 we will usually draw arcs of \(\pi\) above the points and \(\hat{\pi}\) below the points. From this definition, it also follows that \(A \sim B \iff \bar{A} \sim \bar{B}\), which shortens the notation in the next definition.

\[(2, 4) \sim (1, 3)\] by \(C = (2, 3)\)

\[(3, 2) \sim (\bar{2}, 3)\] by \(C = (2, 3)\)

Figure 3. Two examples of connected pairs. The bottom non-crossing partitions are the associated \(\hat{\pi}\).

A cycle in \(\pi \in \mathcal{P}^{sym}_{1,2}(n)\) is a sequence of pairwise distinct pairs of \(\pi\) cyclically connected by \(\hat{\pi}\), i.e.:

\[(l_1, r_1) \sim (l_2, r_2) \sim \cdots \sim (l_{m-1}, r_{m-1}) \sim (l_m, r_m) \sim (l_1, r_1)\]

Note that due to the symmetric nature of the considered partitions we distinguish two fundamentally different kinds of cycles: positive and negative, which resemble the description of the cycles in the hyperoctahedral group.

**Definition 4.10.** Let \(\pi\) and \(\hat{\pi}\) be as in Proposition 4.7. Suppose that

\[(\ast)\quad (l_1, r_1) \sim (l_2, r_2) \sim \cdots \sim (l_{m-1}, r_{m-1}) \sim (l_m, r_m) \sim (l_1, r_1)\]

is a cycle. There are two possibilities: either

\[\{l_1, r_1, \ldots, l_m, r_m\} \cap \{l_1, r_1, \ldots, l_m, r_m\} = \emptyset,\]
or
\[ \{l_1, r_1, \ldots, l_m, r_m\} = \{l_1, r_1, \ldots, l_m, r_m\}. \]

In the first case, there exists the associated cycle
\[ (l_m, r_m) \sim (l_{m-1}, r_{m-1}) \sim \cdots \sim (l_1, r_1) \sim (l_m, r_m) \]
and we call the pair of cycles
\[ \sigma = (l_1, r_1, \ldots, l_m, r_m)(l_m, r_m, \ldots, l_1, r_1) \]
a positive cycle in \( \pi \) of length \( m \). We will use the notation \( \sigma = (l_1, r_1, \ldots, l_m, r_m)^+ \) (or \( \sigma = (l_1, r_1, \ldots, l_m, r_m)^{\pm} \)) to indicate positivity. In the second case the parameter \( m = 2m' \) is necessarily even and due to symmetry this cycle has the following form
\[ (**) \quad (l_1, r_1) \sim \cdots \sim (l_{m'}, r_{m'}) \sim (l_1, r_1) \sim \cdots \sim (l_{m'}, r_{m'}) \sim (l_1, r_1). \]

We call it a negative cycle in \( \pi \) of length \( m' \) and we denote it \( \sigma = (l_1, r_1, \ldots, l_{m'}, r_{m'})^- \).

For a cycle \( \sigma \) of \( \pi \) (positive or negative) we denote its length by \( |\sigma| \) and by \( \text{Cyc}(\pi) \) the set of cycles of \( \pi \).

\[ l_c(\pi) = \sum_{\sigma \in \text{Cyc}(\pi)} (|\sigma| - 1) \]
is the total length of cycles of \( \pi \) reduced by 1. Let \( c_-(\pi) \) be the number of negative cycles of \( \pi \), see Fig. 4.

![Diagram](image)

\[ \text{Cyc}(\pi) = \{(1, 3, 2, 4)^+, (7, 9, 8, 10)^+, (5, 6)^-\} \]

Figure 4. An example of a partition \( \pi \in \mathcal{P}^{sym}_2(10) \) with one negative and two positive cycles.

**Remark 4.11.**

1. In the case of a positive cycle \( \sigma = (l_1, r_1)(l_1, r_1) \) it should be understood that the pairs \( (l_1, r_1), (l_1, r_1) \) lie in both partitions \( \pi \) and \( \hat{\pi} \).

2. Our definition of cycles in \( \mathcal{P}^{sym}_2(n) \) is quite similar to the definition of partitions of type \( B \) [Rei97, Section 2], but in our situation a zero block (which is invariant under the bar operation) does not necessarily exist.

If some pairs are not connected cyclically, we call them *semi-cycles.*
Definition 4.12. Let $\pi$ and $\hat{\pi}$ be as in Definition 4.8. Consider the set of pairs in $\pi$, which are not connected cyclically. This set is partitioned into chains of connected pairs:

$$(l_1, r_1) \sim \cdots \sim (l_m, r_m) \not\sim (l_1, r_1)$$

where $(l_1, r_1)$ is a negative pair, and by symmetry

$$\overline{l_1} \sim \cdots \sim \overline{l_m} \not\sim (\overline{l_1}, l_1).$$

We say that

$$\sigma = \sigma^- \sigma^+ = (\overline{r_m}, \overline{r_m}, \overline{l_m}, \ldots, \overline{l_1}, l_1) \sim (l_1, r_1, \ldots, l_m, r_m, c_m)^+$$

is a semi-cycle of length $m + 1$ in $\pi$, where $(r_m, c_m), (\overline{r_m}, \overline{r_m}) \in \hat{\pi}$ (they exist by Proposition 4.7). Additionally, there are also semi-cycles $\sigma = (\overline{K})^{-}(k)^{+}$ in $\pi$ of length 1 formed by singletons of $\pi$ (with the convention $k \in \{1, \ldots, n\}$). Similarly as for the cycles we denote the length of a semi-cycle $\sigma$ by $|\sigma|$, the set of semi-cycles by $\text{SemiCyc}(\pi)$ and we introduce the total length $l_{\text{nc}}(\pi)$:

$$l_{\text{nc}}(\pi) = \sum_{\sigma \in \text{SemiCyc}(\pi)} (|\sigma| - 1).$$

See Fig. 5 for an example of semi-cycles. The points $c_m$, $l_1$ (resp. $\overline{r_m}$, $\overline{l_1}$) are called the edges of the positive (and the negative, respectively) parts of a semi-cycle $\sigma$ and they are denoted by

$$l(\sigma^f) = \begin{cases} c_m & \text{if } f = + \\ \overline{r_m} & \text{if } f = - \end{cases}, \quad r(\sigma^f) = \begin{cases} l_1 & \text{if } f = + \\ \overline{l_1} & \text{if } f = - \end{cases}.$$  

$\pi$

$\hat{\pi}$

$\text{SemiCyc}(\pi) = \{(5, 9, 3, 4, 1)^{-}, \overline{1}, 4, 3, 9, 8)^{+}, (\overline{5}, \overline{6}, \overline{2})^{-}, (2, 6, 5)^{+}, (\overline{7})^{-}(7)^{+}\}$

Figure 5. The partition $\pi$ has one semi-cycle $\sigma_1$ of length 3 with the edges of the negative part $l(\sigma_1^-) = \overline{5}$, $r(\sigma_1^-) = 1$, one semi-cycle $\sigma_2$ of length 2 with the edges of the negative part $l(\sigma_2^-) = \overline{5}$, $r(\sigma_1^-) = \overline{5}$, and one semi-cycle $\sigma_3$ of length 1 with only one edge of the negative part $r(\sigma_3^-) = \overline{7}$.

4.3. Gaussian operator. We construct generalized cyclic Gaussian operators of type B. They are given by the creation and annihilation operators on a cyclic Fock space of type B. We show that the distribution of these operators with respect to the vacuum expectation is a generalized Gaussian distribution, in the sense that all the moments can be calculated from the second moment by the explicit combinatorial formula. The operator

$$G(x) = b_{q+, q-}(x \otimes x) + b_{q+, q-}^{*}(x \otimes x), \quad x \in \mathcal{K}_{\mathbb{R}},$$
is called the **cyclic Gaussian operator of type B**. Given \( \epsilon = (\epsilon(1), \ldots, \epsilon(n)) \in \{1, *\}^n \), let \( \mathcal{P}_{*}^{\text{sym}}(n) \) be the set of partitions \( \pi \in \mathcal{P}_{1,2}^{\text{sym}}(n) \) such that

- if \((a, b)\) is a negative pair in \( \pi \) then \( \epsilon(|b|) = * \), \( \epsilon(|a|) = 1 \).
- if \( \{c\} \) is a singleton in \( \pi \) then \( \epsilon(|c|) = * \).

**Theorem 4.13.** Let \( x_7 \otimes x_i \in K_{\mathbb{R}} \) for \( 1 \leq i \leq n \). Then

\[
(22) \quad b_{q+,q-}^{\pi}(x_7 \otimes x_n) \cdots b_{q+,q-}^{\pi}(x_7 \otimes x_1) \otimes \Omega = \sum_{\pi \in \mathcal{P}_{1,2}^{\text{sym}}(n)} q_{-}^{\pi}(\pi) q_{+}^{\pi}(\pi) + b_{-}^{\pi}(\pi) \prod_{\{i,j\} \in \text{Pair}(\pi)} \langle x_i, x_j \rangle \\
\bigotimes_{\sigma \in \text{SemiCyc}(\pi)} \left\{ x_{l(\sigma^\pm)} \right\}_{r(\sigma^\pm)} = x_7 \otimes x_7 \otimes x_8 \otimes x_8 \otimes x_5 \otimes x_7.
\]

**Remark 4.14.** 1) In the above formula we use the following bracket notation \( \{ \circ \} \). It should be understood that the position of \( \circ \) (in the tensor product) is ordered with respect to the \( \ast \). For example, for the partition as in Fig. 5, we have

\[
\bigotimes_{\sigma \in \text{SemiCyc}(\pi)} \left\{ x_{l(\sigma^\pm)} \right\}_{r(\sigma^\pm)} = x_7 \otimes x_7 \otimes x_8 \otimes x_8 \otimes x_5 \otimes x_7.
\]

2) We denote by \((A, B)\) the concatenation of the semi-cycles \( A \) and \( B \).

**Proof.** The proof is by induction. When \( n = 1 \), \( b_{q+,q-}^{\pi}(x_7 \otimes x_1) \otimes \Omega = 0 \) and \( b_{q+,q-}^{\pi}(x_7 \otimes x_1) \otimes \Omega = x_7 \otimes x_1 \). Suppose that the formula (22) is true for \( n = k - 1 \) and for any \( \epsilon \in \{1, *\}^{k-1} \). We will show that the action of \( b_{q+,q-}^{\pi}(x_7 \otimes x_k) \) corresponds to the inductive pictorial description of set partitions, cycles and semi-cycles.

From now on, we fix a partition \( \pi \in \mathcal{P}_{1,2}^{\text{sym}}(k - 1) \) (then the associated non-crossing \( \hat{\pi} \) is automatically defined) and suppose that \( \pi \) has the semi-cycles \( \sigma_1^- \sigma_1^+, \ldots, \sigma_{i^*}^- \sigma_{i^*}^+, \ldots, \sigma_{p^*}^- \sigma_{p^*}^+ \), with lengths \( d_1, \ldots, d_p \) and we assume that \( \sigma_{p^*}^- \sigma_{p^*}^+ \) is the most left-right semi-cycle i.e. \( |r(\sigma_i^+)| = |r(\sigma_i^-)| < |r(\sigma_{p^*}^-)| = |r(\sigma_{p^*}^+)| \) for all \( 1 \leq i \leq p - 1 \). In this situation, the semi-cycles contribute to the tensor product

\[
(23) \quad \left\{ x_{l(\sigma_p^-)} \right\}_{r(\sigma_p^-)} \otimes \cdots \otimes \left\{ x_{l(\sigma_i^+)} \right\}_{r(\sigma_i^+)} \otimes \cdots \otimes \left\{ x_{l(\sigma_{p^*}^-)} \right\}_{r(\sigma_{p^*}^-)} \quad \text{or}
\]

\[
(24) \quad \left\{ x_{l(\sigma_{p^*}^+)} \right\}_{r(\sigma_{p^*}^+)} \otimes \cdots \otimes \left\{ x_{l(\sigma_i^+)} \right\}_{r(\sigma_i^+)} \otimes \cdots \otimes \left\{ x_{l(\sigma_{p^*}^-)} \right\}_{r(\sigma_{p^*}^-)}
\]

where \( f = \pm \) and \( g = -f \). For the reader it is convenient to visualize equations (23) and (24) as the following diagrams

![Diagram](image)

We understand that \( p = 0 \) when there is no semi-cycle. We will show that the action of \( b_{q+,q-}^{\pi}(x_7 \otimes x_k) \) corresponds to the inductive graphic description of set partitions and corresponding cycles semi-cycles. During this step, we create the new partition \( \tilde{\pi} \in \mathcal{P}_{1,2}^{\text{sym}}(k) \) and \( \tilde{\pi} \in \mathcal{NC}_{1,2}^{\ast}([\pm k]) \). Note that when there are no semi-cycles, the arguments below can be modified easily.
Case 1. If $\epsilon(k) = \ast$, then the operator $b^*(x_\pi \otimes x_k)$ acts on the tensor product (23) or (24), putting $x_k$ on the right and $x_\pi$ on the left. This operation pictorially corresponds to adding the semi-cycle $(\tilde{k})^-(k)^+ \to \pi \in \mathcal{P}_{1,2c}^{sym} (k-1)$, with length one. This map $\pi \mapsto \tilde{\pi}$ does not change the numbers $c_-, l_c$ or $l_{sc}$, which is compatible with the fact that the action of $b^*(x_\pi \otimes x_k)$ does not change the coefficient.

Case 2. Now we apply the operator $b(x_\pi \otimes x_k)$ on (23) and (24) with $f = \pm$. In the figures below we will illustrate our proof with $f = -$ because the case $f = +$ does not introduce any new difficulty and is fully analogous.

Consider the action of $\alpha_0(x_\pi \otimes x_k)$ and $\gamma_q^-(x_\pi \otimes x_k)$. There are four cases during which we create cycles.

Case 2a. Suppose that $\alpha_0(x_\pi \otimes x_k)$ acts on the semi-cycle appearing at the edges vectors of the tensor product (23) by using (18), which gives us the inner product $\langle x_k, x_{\pi(l(s_p^+))} \rangle \langle x_\pi, x_{\pi(l(s_p^-))} \rangle$. Pictorially this corresponds to getting a set partition $\tilde{\pi} \in \mathcal{P}_{1,2c}^{sym} (k)$ by adding $\tilde{k}$ and $k$ to $\pi$ and adding the pairs

$$(l(s_p^+), k), (\tilde{k}, l(s_p^-)) \to \pi.$$ 

This results in adding the pairs

$$(r(s_p^+), k), (\tilde{k}, r(s_p^-)) \to \tilde{\pi}$$

and creating the positive cycle $(s_p^+, k)^+$. Note that the above pairs create a closed cycle – see Fig. 6 (a). We also remove the semi-cycle $s^-_p s^+_p$. This new cycle has length $d_p$ and so increases the $l_c(\tilde{\pi})$ by $d_p - 1$ and decreases the length of semi-cycle by $d_p$ because originally $s^-_p s^+_p$ was the semi-cycle of $\pi$. Altogether we have: $l_c(\tilde{\pi}) = l_c(\pi) + d_p - 1$, $l_{sc}(\tilde{\pi}) = l_{sc}(\pi) - d_p + 1$ and $c_-(\tilde{\pi}) = c_-(\pi)$. So the exponent of $q_+$ and $q_-$ do not change, which is compatible with the action (18) on the tensor (23).

![Diagram](image_url)
Case 2b). In the next step, we have the analogous situation, with the case 2a) because then \( \alpha_0(x_\mathcal{T} \otimes x_k) \) acts on the tensor product (24), which gives us the inner product \( \langle x_k, x_{l(\sigma_p^+)}, x_k \rangle \) and we proceed as shown in the Fig. 6 (b) by creating the positive cycle \((k, \sigma_p^+)\).

Case 2c). Suppose that \( \gamma_{q_+} (x_\mathcal{T} \otimes x_k) \) acts on the tensor product (23) by using (20), which gives us the inner product \( \langle x_k, x_{l(\sigma_p^+)}, x_k \rangle \) with coefficient \( q_+ \). We create the negative cycle \((\overline{k}, \sigma_p^-)\) by adding the pairs

\[
\pi \quad \pi
\]

\[
(1(\sigma_p^-), k), (\overline{k}, 1(\sigma_p^+)) \text{ to } \pi \quad \text{and} \quad (r(\sigma_p^+), k), (\overline{k}, r(\sigma_p^-)) \text{ to } \overline{\pi},
\]

which create a closed cycle – see Fig. 7 (a). Next, we count the change of the exponent of \( q_+ \) similarly as in Case 2a, and get \( l_c(\overline{\pi}) = l_c(\pi) + d_p - 1 \), \( l_{sc}(\overline{\pi}) = l_{sc}(\pi) - d_p + 1 \). Moreover, the new negative cycle appears so that \( c_-(\overline{\pi}) = c_-(\pi) + 1 \). Thus the exponent of \( q_+ \) increases by 1 which agrees with the action (20).

Case 2d). Suppose that \( \gamma_{q_-} (x_\mathcal{T} \otimes x_k) \) acts on the tensor product (24), which gives us \( \langle x_k, x_{l(\sigma_p^-)}, x_k \rangle \) with coefficient \( q_- \). This situation is fully analogous to the Case 2c) and we proceed as shown in Fig. 7 (b) by creating the negative cycle \((\overline{k}, \sigma_p^-)\).

Case 3. Finally we consider the action of \( \beta_{q_+} (x_\mathcal{T} \otimes x_k) \), which creates semi-cycles. In this case we have two situations but similarly as before, they are very similar to each other and we describe in details only the first one.
Case 3a). If $\beta_{q^+}(x_k \otimes x_k)$ acts on the tensor product (23) then there are $2(p - 1)$ new terms which appear in (19). The action on the $i^{th}$ term in the tensor product (23), for $1 \leq i \leq p - 1$ contributes to the

(a) inner product $\langle x_k, x_{l(\sigma^+_i)} \rangle \langle x_{\overline{\sigma}^+_i}, x_{l(\sigma^-_i)} \rangle$ with coefficient $q_+$ if we apply the action of $q_+ \mathcal{F}_i$;

(b) inner product $\langle x_k, x_{l(\sigma^-_i)} \rangle \langle x_{\overline{\sigma}^-_i}, x_{l(\sigma^+_i)} \rangle$ with coefficient $q_+$ if we apply the action of $q_+ \mathcal{F}_i$.

Pictorially this corresponds to getting a set partition $\overline{\pi} \in \mathcal{P}^{sym}_{1,2e}(k)$ by adding $\overline{k}$ and $k$ to $\pi$ and creating the pairs

(a) $(l(\sigma^+_i), k)$ and $(\overline{l}(k), l(\sigma^-_i))$ in $\overline{\pi}$; see Fig. 8 (a);

(b) $(l(\sigma^-_i), k)$ and $(\overline{l}(k), l(\sigma^+_i))$ in $\overline{\pi}$; see Fig. 8 (b).

In the above situations we also add the pairs $(r(\sigma^+_p), k)$ and $(\overline{k}, r(\sigma^-_p))$ to $\hat{\pi}$. These new pairs create a semi-cycle by connecting $\sigma^+_p$ and $\sigma^-_p$ through $k$ or $\overline{k}$ as in Fig. 8 (a), (b). More precisely the new semi-cycle is $\overline{\sigma}^- \overline{\sigma}^+$, where

(a) $\overline{\sigma}^- = (\sigma^-_p, \overline{k}, \sigma^-_i)$ and $\overline{\sigma}^+ = (\sigma^+_i, k, \sigma^+_p)$ with $r(\overline{\sigma}^+) = r(\sigma^+_i)$ and $l(\overline{\sigma}^+) = l(\sigma^+_p)$;

(b) $\overline{\sigma}^- = (\sigma^-_p, \overline{k}, \sigma^+_i)$ and $\overline{\sigma}^+ = (\sigma^-_i, k, \sigma^-_p)$ with $r(\overline{\sigma}^+) = r(\sigma^-_i)$ and $l(\overline{\sigma}^+) = l(\sigma^-_p)$.

We can analyze the change in the statistic generated by the new semi-cycle analogously to the previous cases and we see that the length of the new semi-cycle is $d_i = d_i + d_p$, which gives $l_{sc}(\overline{\pi}) = l_{sc}(\pi) + 1$. This is because we remove semi-cycles $\sigma^+_i \sigma^-_i$ and $\sigma^+_p \sigma^-_p$ from $\pi$, which have a contribution $d_i + d_p - 2$ to $l_{sc}(\pi)$, and replace them by one semi-cycle which has contribution $d_i + d_p - 1$ to $l_{sc}(\overline{\pi})$. The other statistics remain unchanged $l_c(\overline{\pi}) = l_c(\pi)$, $c_-(\overline{\pi}) = c_-(\pi)$, so the total change of the coefficient appears only in increasing the exponent of $q_+$ by 1 (see Fig. 8 (a) and (b)). This agrees with the action (19).

Case 3b). In the last situation we apply equation (19) to (24) and we proceed as in Fig. 9 (a), (b).
Figure 9. The visualization of the action of $\beta_{q_+}(x_k)$ on (24)

We emphasize that in all of the above steps $r(\sigma_+^i)$ is the most right or left point representing vector in the tensor product (23) and (24). By using this point we create the partition $\tilde{\pi}$. This explains why $\tilde{\pi}$ is non-crossing and don’t cover singletons.

Note that as $\pi$ runs over $P_{1,2,\epsilon}^{sym}(k-1)$, every set partition $\tilde{\pi} \in P_{1,2,\epsilon}^{sym}(k)$ appears exactly once in one of the cases 2 or 3. Therefore in Case 2 and Case 3, the pictorial inductive step and the actual action of $b_{q_+,q_-}(x_k \otimes x_k)$ both create the same terms with the same coefficients, and hence the formula (22) is true when $n = k$ and $\epsilon(k) = 1$. Therefore the formula (22) holds for all $n \in \mathbb{N}$ by induction, which finishes the proof.

When equipped with the vacuum expectation state $\varphi(\cdot) = \langle \Omega, \cdot \rangle_{q_+,q_-}$ the moments of the Gaussian operator can be expressed as cycles of type B.

**Theorem 4.15.** Suppose that $x_1, \ldots, x_{2n} \in H_\mathbb{R}$, then

$$
\varphi(G(x_{2n}) \dotsc G(x_1)) = \sum_{\pi \in \mathcal{P}_\text{sym}^*(2n)} q_{-c^-}(\pi) q_{-l}(\pi) \prod_{\{i,j\} \in \text{Pair}(\pi)} \langle x_i, x_j \rangle = \sum_{\pi \in \mathcal{P}_\text{sym}^*(2n)} q_{-c^-}(\pi) q_{+l}(\pi) \prod_{\{i,j\} \in \text{Pair}(\pi)} \langle x_i, x_j \rangle,
$$

where $c(\pi)$ is the number of cycles of $\pi$.

**Proof.** The first equation is a direct consequence of (22) by taking the sum over all $\epsilon \in \{1, \ast\}^{2n}$ and $\text{SemiCyc}(\pi) = \emptyset$. The second one follows from the observation that $l_c(\pi) = |\pi| - c(\pi)$. Indeed, the length of each cycle can be represented as the number of pairs which create it, and so

$$
l_c(\pi) = \sum_{\sigma \in \text{Cyc}(\pi)} (|\sigma| - 1) = \sum_{\sigma \in \text{Cyc}(\pi)} |\sigma| - c(\pi) = n - c(\pi).
$$

4.4. Orthogonal polynomials and bivariate generating functions.
4.4.1. Orthogonal polynomials. For a probability measure \( \mu \) with finite moments of all orders, let us orthogonalize the sequence \((1, x, x^2, x^3, \ldots)\) in the Hilbert space \( L^2(\mathbb{R}, \mu) \), following the Gram-Schmidt method. This procedure yields orthogonal polynomials \((P_n(t))_{n=0}^{\infty}\) with \( \deg P_n(x) = n \). Multiplying by constants, we take \( P_n(x) \) to be monic, i.e., the coefficient of \( x^n \) is 1. It is known that they satisfy a recurrence relation

\[
xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x), \quad n = 0, 1, 2, \ldots
\]

with the convention that \( P_{-1}(x) = 0 \). The coefficients \( \beta_n \) and \( \gamma_n \) are called Jacobi parameters and they satisfy \( \beta_n \in \mathbb{R} \) and \( \gamma_n \geq 0 \). It is known that

\[
\gamma_0 \cdots \gamma_n = \int_{\mathbb{R}} |P_{n+1}(x)|^2 \mu(dx), \quad n \geq 0.
\]

Moreover, the measure \( \mu \) has a finite support of cardinality \( N \) if and only if \( \gamma_n = 0 \) for \( n \geq N - 1 \) and \( \gamma_n > 0 \) for \( n = 0, \ldots, N - 2 \) — see [Chi78].

Let us recall here that for any \( c \in (-1, \infty) \) the Askey-Wimp-Kerov distribution \( \nu_c \) (see e.g. [AW84] or [Ker98, Section 8.4]) is the measure on \( \mathbb{R} \), with Lebesgue density

\[
\frac{1}{\sqrt{2\pi\Gamma(c+1)}} |D_{-c}(ix)|^{-2} \quad x \in \mathbb{R},
\]

where \( D_{-c}(z) \) is the solution to the differential Weber equation:

\[
\frac{d^2y}{dz^2} + \left( \frac{1}{2} - c - \frac{z^2}{4} \right) y = 0,
\]

satisfying the initial conditions:

\[
D_{-c}(0) = \frac{\Gamma \left( \frac{1}{2} \right) 2^{-c/2}}{\Gamma \left( \frac{1+c}{2} \right)} \quad \text{and} \quad D'_{-c}(0) = \frac{\Gamma \left( -\frac{1}{2} \right) 2^{-(c+1)/2}}{\Gamma \left( \frac{c+1}{2} \right)}.
\]

When \( c > 0 \), the solution \( D_{-c} \) has the integral representation

\[
D_{-c}(z) = \frac{e^{-z^2/4}}{\Gamma(c)} \int_0^\infty e^{-xz} x^{c-1} e^{-x^2/2} dx.
\]

It was proved in [AW84] that for any \( c \in (-1, \infty) \) the measure \( \nu_c \) is a probability measure. The family \( (\nu_c)_{c \in (-1, \infty)} \) can be extended continuously at \( c = -1 \) by defining \( \nu_{-1} \) to be the Dirac point mass \( \delta_0 \) at 0. The orthogonal polynomials \((H_n(t))_{n=0}^{\infty}\), with respect to \( \nu_c \) are called the associated Hermite polynomials and given by the recurrence relation:

\[
tH_n(t) = H_{n+1}(t) + (n + c) H_{n-1}(t), \quad n = 0, 1, 2, \ldots
\]

with \( H_{-1}(t) = 0, H_0(t) = 1 \).

**Remark 4.16.** For any \( c \in [-1, 0] \) the Askey–Wimp–Kerov distribution \( \nu_c \) is freely infinitely divisible and freely selfdecomposable — see [BBLS11, Theorem 3.1] and [HSTr19, Theorem 3.3].

Let \((Q_n(t))_{n=0}^{\infty}\) be the family of orthogonal polynomials with the recursion relation \( Q_{-1}(t) = 0, Q_0(t) = 1 \) and

\[
tQ_n(t) = Q_{n+1}(t) + \lambda(n) Q_{n-1}(t), \quad n = 0, 1, 2, \ldots
\]
where
\[ \lambda(n) = 1 + 2(n - 1)q_+ + q_- = 2q_+(n + c), \quad c = \frac{1 + q_-}{2q_+} - 1. \]

Let \( \mu_{q_+,q_-} \) be a probability measure associated with the orthogonal polynomials \( Q_n(t) \). If \( q_+ > 0 \), then \( c = M - 1 \geq -1 \), since \( q_+ = \frac{1}{2q_+ + M} \) and \( q_- = \frac{M - N}{M + N} \). The above information leads to the conclusion that the measure \( \mu_{q_+,q_-} \) is the dilatation by \( \sqrt{2q_+} \) (for \( q_+ > 0 \)) of the Askey-Wimp-Kerov distribution \( \nu^{1+q_+}_{2q_+} \), with Lebesgue density
\[
\frac{1}{2\sqrt{q_+} \pi^1(\frac{1+q_-}{2q_+})} |D_{-1+q_+}^{1+q_+} \frac{1}{\sqrt{2q_+}}|^2, \quad x \in \mathbb{R}.
\]

In the case \( q_- < 0 \), then by Section 4.1.1 we have \( \lambda(n) = 1 + 2(n - 1)q_+ + q_- > 0 \) for \( n \leq M \) and \( \lambda(n) = 0 \) for \( n > M \), thus the measure is discrete.

**Theorem 4.17.** Let \( x \otimes x \in \mathcal{K}_R \) and \( \|x \otimes x\| = 1 \). Let \( \kappa_{q_+,q_-} \) be the probability distribution of \( G(x) \), with respect to the vacuum state. Then \( \kappa_{q_+,q_-} \) is equal to \( \mu_{q_+,q_-} \).

**Proof.** Let \( \gamma_{n-1} = 1 + 2q_+(n - 1) + q_- \), \( n = 1, 2, \ldots \), then
\[
\|x^{\otimes n} \otimes x^{\otimes n}\|_{q_+,q_-}^2 = \langle x^{\otimes n} \otimes x^{\otimes n}, P_q^{(n)}(x^{\otimes n} \otimes x^{\otimes n}) \rangle_{0,0} \\
= \langle x^{\otimes n} \otimes x^{\otimes n}, (I \otimes P_q^{(n-1)} \otimes I) P_q^{(n)}(x^{\otimes n} \otimes x^{\otimes n}) \rangle_{0,0}.
\]

Using the identity \( P_q^{(n)}(x^{\otimes n} \otimes x^{\otimes n}) = (1 + 2q_+(n - 1) + q_-)x^{\otimes n} \otimes x^{\otimes n} \) we have that
\[
\|x^{\otimes n} \otimes x^{\otimes n}\|_{q_+,q_-}^2 = \gamma_{n-1}\|x^{\otimes (n-1)} \otimes x^{\otimes (n-1)}\|_{q_+,q_-}^2 = \gamma_0 \gamma_1 \cdots \gamma_{n-1},
\]
and hence by (26) we obtain that
\[
\|x^{\otimes n} \otimes x^{\otimes n}\|_{q_+,q_-} = \|Q_n\|_{L^2}, \quad n \in \mathbb{N} \cup \{0\}.
\]

Therefore, the map \( \Phi: (\text{span}\{x^{\otimes n} \otimes x^{\otimes n} \mid n \geq 0\}, \|\cdot\|_{q_+,q_-}) \to L^2(\mathbb{R}, \mu_{q_+,q_-}) \) defined by \( \Phi(x^{\otimes n} \otimes x^{\otimes n}) = Q_n(t) \) is an isometry. Note that
\[
G(x) x^{\otimes n} \otimes x^{\otimes n} = b_{q_+,q_-}(x \otimes x) x^{\otimes n} \otimes x^{\otimes n} + b_{q_+,q_-}(x \otimes x) x^{\otimes n} \otimes x^{\otimes n} \\
= x^{\otimes (n+1)} \otimes x^{\otimes n+1} + \gamma_{n-1} x^{\otimes (n-1)} \otimes x^{\otimes (n-1)}.
\]

Hence, by induction we can compute \( G^n(x) \Omega \otimes \Omega \) and show that \( \Phi(G^n(x) \Omega \otimes \Omega) = x^n \). Since \( \Phi \) is an isometry we get \( \langle \Omega \otimes \Omega, G^n(x) \Omega \otimes \Omega \rangle_{q_+,q_-} = m_n(\mu_{q_+,q_-}) \) for \( n \in \mathbb{N} \). Since the moment problem is determined (see Remark 4.18), the probability measure \( \mu_{q_+,q_-} \) giving the moment sequence \( m_n(\mu_{q_+,q_-}) \) is uniquely determined and hence \( \mu_{q_+,q_-} = \kappa_{q_+,q_-} \). □

**Remark 4.18.** In the above proof, the condition \( q_- < 0 \) implies that \( n \leq M \); see Section 4.1.1. In this situation we consider the finitely many vectors i.e. \( x^{\otimes n} \otimes x^{\otimes n} \) for \( n \leq M \).
4.4.2. Moments and combinatorial statistics. The moment problem for the probability measure $\mu_{q_+,q_-}$ is determined. Indeed, when $q_+ \geq 0$, then the best known criterion (in this case) of Hamburger moment problem is due to Carleman \cite{Car26,Chi89}. Carleman’s theorem states that the moment problem is determined if

$$\infty = \sum_{n \geq 1} \frac{1}{\lambda(n)} = \sum_{n \geq 1} \frac{1}{1 + 2(n-1)q_+ + q_-}, \quad q_+ \geq 0.$$  

The odd moments $m_n(\mu_{q_+,q_-})$ are equal to zero and even moments can be computed for instance using Viennot’s combinatorial theory of the orthogonal polynomials \cite{Vie85} (see also \cite{AB98} and \cite[H007, Section 1.6]).

A Dyck path $D$ of length $n$ is a finite sequence of steps $w_1, \ldots, w_{2n}$, where $w_i \in \{-1,1\}$, for $1 \leq i \leq 2n$ and

- $\sum_{i=1}^{\ell} w_i \geq 0$ for all $1 \leq \ell < 2n$,
- $\sum_{i=1}^{2n} w_i = 0$.

We denote the set of Dyck paths of length $n$ by $\text{Dyck}_n$ and its cardinality is given by the $n$-th Catalan number $C_n := \frac{1}{n+1} \binom{2n}{n}$. The combinatorial theory of the orthogonal polynomials asserts that the $2n$-th moment $m_n(\mu_{q_+,q_-})$ is given by the following weighted generating function of Dyck paths:

$$m_{2n}(\mu_{q_+,q_-}) = \sum_{(w_1,\ldots,w_{2n}) \in \text{Dyck}_n} \prod_{\ell,w_\ell = -1} \left(1 + q_- + q_+ \cdot 2 \sum_{i=1}^{\ell} w_i\right).$$

This in conjunction with Theorem 4.15 gives a curious identity between two bivariate generating functions of different combinatorial objects:

$$\sum_{\pi \in P_2^{sym}(2n)} q_-^{c(\pi)} q_+^{l(\pi)} = \sum_{(w_1,\ldots,w_{2n}) \in \text{Dyck}_n} \prod_{\ell,w_\ell = -1} \left(1 + q_- + q_+ \cdot 2 \sum_{i=1}^{\ell} w_i\right).$$

Indeed, note that both sides of the above identity are polynomials in $q_+, q_-$ which agree on a certain infinite set and by the same argument as used in the proof of Theorem 3.3 the equality between polynomials also holds true. Note that only the LHS is a bivariate generating function sensu stricto since the weights associated to Dyck paths appearing in the RHS are not monomials in $q_+, q_-$. Nevertheless it is convenient to rewrite the LHS in terms of another “bivariate generating function”, which is not a monomial in $q_+, q_-$ but it is a monomial in $x, y$ after the following change of variables $x = 1 + q_- , y = 2q_+$.

**Proposition 4.19.** Let $x = 1 + q_-, y = 2q_+$. Then

$$m_{2n}(\mu_{q_+,q_-}) = \sum_{\pi \in P_2^{sym}(2n)} q_-^{c(\pi)} q_+^{l(\pi)} = \sum_{\pi \in P_2(2n)} x^{c(\pi)} y^{n-c(\pi)},$$

where $P_2(2n)$ is the set of matchings (pair-partitions) on the set $[2n]$ and $c(\pi)$ denotes the number of cycles created by a concatenation of $\pi$ with the unique non-crossing matching $\hat{\pi} \in P_2(2n)$ whose left legs coincides with the leg legs of $\pi$.

**Proof.** Define the natural projection $P : P_2^{sym}(2n) \to P_2(2n)$ which is sending a pair $(i,j)$ of $\pi \in P_2^{sym}(2n)$ into a pair $(|i|,|j|)$ of $P(\pi)$. We claim that for every pair $\pi \in P_2(2n)$ one
has
\[
\sum_{\sigma \in P^{-1}(\pi)} q_{-}(\sigma)^{i_{-}(\sigma)} q_{+}(\sigma)^{i_{+}(\sigma)} = (1 + q_{-})c(\pi)(2q_{+})^{n-c(\pi)}.
\]

Then the proof follows immediately from the claim.

In order to prove the claim it is enough to notice the following properties of \( P \). First of all for any \( \pi \in \mathcal{P}_{2}(2n) \) there are precisely \( 2^{n} \) preimages in \( \mathcal{P}_{2}^{\text{sym}}(2n) \): for every pair \( \{ i, j \} \pi \) the corresponding pairs in the partition from \( P^{-1}(\pi) \) are either \( \{ i, j \} \) and \( \{ i', j \} \) or \( \{ i, j \} \) and \( \{ i', j' \} \). Exchanging these two pairs by each other in elements from \( P^{-1}(\pi) \) which has a property that it fixes the number of cycles and it changes the sign of the cycle, which contains the pairs associated with \( \{ i, j \} \). The group generated by \( \{ f_{B} : B \in \pi \} \) acts transitively on \( P^{-1}(\pi) \). In particular the number of cycles is an invariant of \( P^{-1}(\pi) \) so it is clearly equal to \( c(\pi) \), which proves the formula
\[
\sum_{\sigma \in P^{-1}(\pi)} q_{-}(\sigma)^{i_{-}(\sigma)} q_{+}(\sigma)^{i_{+}(\sigma)} = \sum_{\sigma \in P^{-1}(\pi)} q_{-}(\sigma)^{n-c(\sigma)} = (1 + q_{-})c(\pi)(2q_{+})^{n-c(\pi)}.
\]

\( \square \)

Notice that substituting \( x = c, y = 1 \) in (30) we obtain moments of the Askey–Wimp–Kerov distribution \( \nu_{c-1} \) with the shifted parameter \( c-1 \). These moments were interpreted by Drake [Dra09] as a generating function of pair-partitions with respect to two statistics.

**Theorem 4.20** ([Dra09]). The (even) moments \( m_{2n}(\nu_{c-1}) \) are the generating series of pair-partitions with respect to the following statistics:

\[
m_{2n}(\nu_{c-1}) = \sum_{\pi \in \mathcal{P}_{2}(2n)} c^{|\text{non-nested pairs in } \pi|},
\]

\[
m_{2n}(\nu_{c-1}) = \sum_{\pi \in \mathcal{P}_{2}(2n)} c^{|\text{pairs in } \pi \text{ with no right crossing}|}.
\]

We recall that a pair \( \{ i, j \} \in \pi \) is nested if there exists a pair \( \{ i', j' \} \in \pi \) such that \( i' < i < j < j' \), and a pair \( \{ i, j \} \in \pi \) has a right crossing if there exists a pair \( \{ i', j' \} \in \pi \) such that \( i < i' < j < j' \). As an immediate corollary from (30) we have an interpretation of the moments \( m_{2n}(\nu_{c-1}) \) as the generating series of pair-partitions with respect to a third, different statistic:

\[
m_{2n}(\nu_{c-1}) = \sum_{\pi \in \mathcal{P}_{2}(2n)} c^{|\text{cycles in } \pi|}.
\]

**Remark 4.21.** We would like to mention potential interpretations of the moments as bivariate generating functions of maps via unknown statistics. A (orientable) map is an embedding of a graph into a (orientable) compact, connected surface without a boundary such that the complement of the image of the graph is a disjoint union of simply connected pieces. A map is rooted if it is distinguished with an oriented corner (i.e. a small neighbourhood around a vertex, called the root, delimited by two consecutive half-edges). Drake, based on the result of showed that the moment \( m_{2n}(\nu_{c}) \) is the generating function of rooted orientable maps with \( n \) edges, where the exponent of \( c \) gives the number of vertices different from the root. This corresponds to the substitution \( y = 1, c = q_{+} \) in (30) and suggests that the bivariate
generating function
\[ \sum_{\pi \in \mathcal{P}_2(2n)} (1 + x)^{c(\pi)} y^{n - c(\pi)} \]
can be interpreted as the generating function of rooted orientable maps with \( n \) edges, where the exponent of \( x \) gives the number of vertices different from the root and the exponent of \( y \) is a statistic on maps yet to be found. Another interesting specialization is given by \( q_+ = 0 \), which gives the following univariate generating function of the symmetric-pair partitions with all cycles of size 1:
\[ \sum_{\pi \in \mathcal{P}_2^\text{sym}(2n)} q_{-}^{c(\pi)} = \sum_{\pi \in \mathcal{P}_2^\text{sym,orientable}(2n)} (1 + q_{-})^n = C_n (1 + q_{-})^n. \]
This is the weighted generating function for objects enumerated by \( 2^n C_n \), which is given by A151374 (see also A052701). Among others, it counts pointed, rooted, bipartite planar maps with \( n \) edges. This suggest that there might be a natural injection of pointed, rooted, bipartite planar maps with \( n \) edges into the set of rooted, orientable maps with \( n \) edges. Finally, the set \( \mathcal{P}_2^\text{sym}(2n) \) of the symmetric pair-partitions is in a natural bijection with the set of rooted (orientable and non-orientable) maps with only one-face (i.e. the complement of the image of the graph embedded into a surface is simply connected) and it is natural to ask for two statistics which expresses these moments as a bivariate generating function of rooted, one-face maps. Note that the set of rooted, one-face maps plays a crucial role in studying the structure of maps via bijective methods (see [Sch97, BDG04, CMS09, CFF13, CD17, Lep19, DL20] among others) and we believe that the statistics in question might be found through the aforementioned bijections. We leave these questions open, as they are out of scope of this paper.

**Appendix A. How to Derive the Picture of Type D from Type B**

In this section we explain that the analogous problem in the case of reflection groups of type D turned out to be similar to the case of type A and it follows from our main result (in particular the possible applications coincides with the special choice of parameters \( M, N \) in Theorem 1.1). This will exhaust the classification problem for the inductive limits of all three infinite series of Coxeter groups of Weyl type (A,B/C and D).

We recall that the Coxeter group \( D_n \) of rank \( n \) and type \( D \) is a normal subgroup of \( B_n \) of index two:
\[ D_n := \{(g_1, \ldots, g_n; \sigma) \in B_n : g_1 \cdots g_n = \text{id}\}. \]
In other terms, it is isomorphic to the kernel of the one-dimensional representation
\[ B_n \ni g \mapsto (-1)^{\text{number of negative cycles in } g}. \]

Therefore the conjugacy classes of \( D_n \) are parametrized by pairs of partitions \((\rho^+, \rho^-)\) of total size \( n \), where the second partition has an even number of parts (because the number of negative cycles is even). They coincide with the conjugacy classes \( C_{\rho^+, \rho^-} \), except of the class given by \( C_{\rho,0} \) for \( \rho \) of the form
\[ \rho = (\rho_1, \ldots, \rho_\ell) = 2 \cdot \mu = (2\mu_1, \ldots, 2\mu_\ell); \quad \mu \vdash n/2. \]
This conjugacy class of \( B_n \) splits into two conjugacy classes of \( D_n \) that we denote by \( C_{\rho,0}^+ \) and \( C_{\rho,0}^- \) (note that these classes exist only if \( 2 \mid n \)). The set of reflections in \( D_n \) coincides with the set \( R_+ \) of long reflections in \( B_n \). In particular all the reflections are conjugated to each
other and the reflection function is given by restricting the signed reflection function \( \psi_{q_+,q_-} \) to \( D_n \) and substituting \( q_+ = q_- = q \).

Representation theory of \( D_n \) can be derived from the representation theory of the hyper-octahedral group \( B_n \) through the Clifford theory (see for instance [Ker75]). The irreducible representations of \( D_n \) are indexed by non-ordered pairs of partitions \( (\lambda^+,\lambda^-) \) of total sum \( n \) such that \( \lambda^+ \neq \lambda^- \) (in other terms the irreducible representations \( \rho_{\lambda^+} \) and \( \rho_{\lambda^-} \) are the same) and there are two additional irreducible representations denoted \( \rho_{\lambda^+,\lambda^-} \) and \( \rho_{\lambda^-} \). The relation between these representations and irreducible representations of \( B_n \) can be described by the restriction. Let \( \rho_{\lambda^+,\lambda^-}^{B_n} \) be the irreducible representation of \( B_n \) indexed by a pair of partitions \( (\lambda^+,\lambda^-) \) and let \( \rho_{\lambda^+,\lambda^-}^{D_n} \) denote the restriction of \( \rho_{\lambda^+,\lambda^-}^{B_n} \) to \( D_n \). Then

\[
\rho_{\lambda^+,\lambda^-}^{D_n} = \begin{cases} 
\rho_{\lambda^+} + \rho_{\lambda^-} & \text{if } \lambda^+ \neq \lambda^-,
\rho_{\lambda^+,\lambda^-} + \rho_{\lambda^-} & \text{if } \lambda^+ = \lambda^- = \lambda.
\end{cases}
\]

Therefore, if we restrict (11) to \( D_n \) we obtain that the reflection function of \( D_n \) is given by

\[
\sum_{(\lambda^+,\lambda^-) \in \mathcal{P}_n} \left( \delta_{\lambda^+ \neq \lambda^-} \chi_{\lambda^+} \chi_{\lambda^-} + \delta_{\lambda^+ = \lambda^-} \left( \chi_{\lambda^+} \chi_{\lambda^-} \right) \right) \\
\cdot \frac{1}{2} \left( \prod_{\square \in \lambda^+} (qc(\square) + 1 + g) \prod_{\square \in \lambda^-} (qc(\square) + 1) \right)
\]

\[
+ \prod_{\square \in \lambda^-} \left( qc(\square) + 1 + g \right) \prod_{\square \in \lambda^+} \left( qc(\square) + 1 \right)
\]

In particular it is positive definite on \( D(\infty) \) if and only if

\[
\prod_{\square \in \lambda} \left( qc(\square) + 1 + g \right) \prod_{\square \in \mu} \left( qc(\square) + 1 \right)
\]

\[
+ \prod_{\square \in \mu} \left( qc(\square) + 1 + g \right) \prod_{\square \in \lambda} \left( qc(\square) + 1 \right) \geq 0
\]

for all partitions \( \lambda, \mu \). It is easy to check that this condition holds true if and only if

\[
q \in \left\{ \frac{1}{2N+1} : N \in \mathbb{Z} \right\} \cup \{0\}.
\]

Note that due to Theorem 1.1 this set of parameters corresponds precisely to the case when \( \phi_{q_+,q_-} \) is positive definite on \( B(\infty) \) and \( q_+ = q_- \) (which can happen for \( \epsilon = \pm 1, N \in \mathbb{Z}_{\geq 0} \) and \( M = N + 1 \)). Therefore we obtained the following theorem.

**Theorem A.1.** Let \( q \in \mathbb{C} \). The following conditions are equivalent:

(i) The reflection function \( D(\infty) \ni \sigma \rightarrow q^{f_\infty(\sigma)} \) is positive definite on \( D(\infty) \);

(ii) The reflection function \( D(\infty) \ni \sigma \rightarrow q^{f_\infty(\sigma)} \) is a character of \( D(\infty) \);

(iii) The reflection function \( D(\infty) \ni \sigma \rightarrow q^{f_\infty(\sigma)} \) is an extreme character of \( D(\infty) \);

(iv) The reflection function \( D(\infty) \ni \sigma \rightarrow q^{f_\infty(\sigma)} \) is the restriction of the positive definite reflection function \( B(\infty) \ni \sigma \rightarrow q^{f_\infty(\sigma)} \) on \( B(\infty) \);

(v) \( q \in \left\{ \frac{1}{2N+1} : N \in \mathbb{Z} \right\} \cup \{0\} \).
A.1. **Concluding remarks.** We conclude by making a remark that the problem of classifying positive definite reflection functions (or their multivariate refinements) seems to be approachable for a wide class of groups $\mathfrak{S}_\infty(T)$ (studied by Hirai and Hirai [HH05]) by using a method presented in this paper. These groups are of the following form: for a finite group $T$ we define $\mathfrak{S}_n(T)$ as the wreath product $T \wr \mathfrak{S}_n$, and we set $\mathfrak{S}_\infty(T)$ as the inductive limit of the ascending tower of groups $\mathfrak{S}_1(T) < \mathfrak{S}_2(T) < \cdots$. This class contains for instance the infinite analogue of the Sheppard–Todd groups $G(r, 1, \infty)$, which corresponds to the choice $T = \mathbb{Z}_r$. It turns out that the representation theory of the group $\mathfrak{S}_n(T)$ can be explicitly described in terms of the representation theory of $\mathfrak{S}_n$ and $T$. This makes it possible to use the same methods as we used in this paper to tackle the case of the hyperoctahedral group (which is given by the choice $T = \mathbb{Z}_2$). We did not study this more general context because of the applications that we presented in the second part of the paper and that are specific to type $B$. Therefore we leave this comment as an invitation for further investigations.

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