Approximate Hamiltonian for baryons in heavy-flavor QCD

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Building a method for describing gluons in hadrons in the Minkowski space-time, a pilot application of the renormalization group procedure for effective particles (RGPEP) to QCD of bottom and charm flavors is extended from quarkonia to baryons. Using the effective-particle basis in the Fock space, the bound-state eigenvalue problem for baryons is posed in terms of the Hamiltonian obtained by solving the RGPEP equations with accuracy to terms of second order in the expansion in powers of the coupling constant. The eigenvalue problem including the Fock components with effective gluons is reduced to the eigenvalue problem for the component of three effective quarks and no gluons. Namely, we use a hypothesis that all the components with gluons can be approximated by a component with one gluon that is massive, and we take this component into account using second-order perturbation theory. The effective three-quark Hamiltonian contains three quark-quark interaction terms, each of which consists of a Coulomb term with the Breit-Fermi spin couplings and a spin-independent harmonic oscillator term. As in quarkonia, the oscillator frequencies in baryons turn out to be not sensitive to the value of the gluon mass. The quark masses are adjusted at the corresponding scales to reproduce the masses of three lightest charm and bottom quarkonia of spin one. The dynamics in one-flavor baryons involves no free parameters and the resulting estimates for $bbb$ and $ccc$ baryon mass spectra match estimates obtained in quark models and lattice approach to QCD. Masses of $ccb$ and $bbc$ baryons are also estimated. All approximate baryon wave functions have simple oscillator forms. In $ccb$ baryons, charm quarks tend to form diquarks. In order to identify the dynamics of effective gluons beyond the assumption of a mass term, the RGPEP needs to be applied in higher order than second and beyond the perturbative expansion.

I. INTRODUCTION

Quark model represented baryons as bound states of three quarks, e.g. see [1,2]. In QCD, baryons are instead superpositions of states of quanta of quark and gluon fields. A priori, the number of quanta varies from three to infinity, across an infinite set of components. The range of momenta these quanta may have is also infinite. Their interactions diverge with their momenta. This article contributes to a development of a Hamiltonian approach to QCD that appears capable of filling the gap between the complex quantum-field picture and simple quark-model picture for hadrons [3]. We report results for heavy baryons that are obtained using our approach in its pilot version, which includes severe simplifications.

We limit the theory to quarks that have masses much greater than $\Lambda_{QCD}$ and we consider the weak coupling limit. Creation of quark-antiquark pairs is neglected. Components with more than one gluon are eliminated, by assuming that their dominant effect in the component with one gluon is that the gluon has a mass, allowed to be a function of the gluon kinematic relative momentum with respect to the quarks. We use second-order perturbation theory to derive the resulting effective Hamiltonian for baryons that only acts in the component with three quarks. We compare the quark interaction terms in this Hamiltonian to similar terms in the Hamiltonian that only acts in the quark-anti-quark component in quarkonia, previously derived using the same method [3]. Masses of heavy baryons are estimated by extrapolating the coupling constant at the quark-mass scale from a formally infinitesimal value of the weak-coupling limit to the value implied by the known coupling constant at the scale of $Z$-boson mass. Quark masses are adjusted to the known spectra of heavy quarkonia. Our estimates for baryon masses contain no new parameters. The scale parameter for hadrons built from different flavors is fixed by a linear interpolation between its one-flavor values.

The concept of effective-gluon mass that we use is explained in Sec. II followed by a brief outline of our method in Sec. III. The method is called the renormalization group procedure for effective particles (RGPEP). The second-order baryon eigenvalue problem is outlined in Sec. IV. Details of the effective quark-quark-interaction terms in $ccc$ and $bbb$ baryons, implied by the gluon mass, are described in Sec. V including a comparison with the case of heavy quarkonia. Sec. VI extends the calculation to the $ccb$ and $bbc$ baryons. The resulting estimates for baryon masses are described in Sec. VII. Comments concerning the RGPEP calculation of effective Hamiltonians in QCD in orders higher than second and beyond perturbation theory conclude the paper in Sec. VIII. Details of our fit to the spectra of quarkonia are described in App. A. Values of the RGPEP scale...
parameter we use are listed in App. [4] Appendix [C] discusses dependence of harmonic oscillator frequencies on the scale parameter. Appendix [D] provides a detailed description of the baryon wave functions that are used in our estimates and App. [E] presents explicit formulas for the associated heavy-baryon masses.

II. ASSUMPTION OF GLUON MASS

Theoretically, a baryon state in heavy-flavor QCD is a superposition of states of virtual, point-like quarks and gluons,

$$|\Psi\rangle = |3Q\rangle + |3Q \, G\rangle + |3Q \, 2G\rangle + \ldots . \tag{1}$$

Components that include quark-anti-quark pairs are considered very small because quarks are heavy. In contrast, components with gluons are included because in canonical QCD gluons are massless. However, using the massless gluons in the expansion and limiting their number in a computation one expects to obtain the spectrum of excited baryons that gets dense toward the free quark threshold. The same feature is expected to occur in such computations of spectrum of quarkonia. Physically, the threshold. The same feature is expected to occur in such excited baryons that gets dense toward the free quark in a computation one expects to obtain the spectrum of less gluons in the expansion and limiting their number considered very small because quarks are heavy. In contrast, Components that include quark-anti-quark pairs are considered very small because quarks are heavy. In contrast, components with gluons are included because in canonical QCD gluons are massless. However, using the massless gluons in the expansion and limiting their number in a computation one expects to obtain the spectrum of excited baryons that gets dense toward the free quark threshold. The same feature is expected to occur in such computations of spectrum of quarkonia. Physically, the threshold.

The mass assumption is falsifiable by extending the calculation to explicitly include more components and relegating the gluon mass ansatz to states with more gluons than one. The purpose would be to verify if the finite value of the RGPEP parameter \( t \) on the order of quark Compton wave-length is sufficient to prevent the spill of probability to states with many gluons, especially when the coupling constant is small. The latter situation is expected to occur for the quark masses that are much greater than \( \Lambda_{QCD} \). This is precisely the reason for us to study the dynamics of gluons using the RGPEP first in the context of heavy-flavor QCD. In order to simplify the problem and thus increase a chance of understanding the dynamics of effective gluons whose masses are likely to be much larger than \( \Lambda_{QCD} \), we exclude from the theory the quarks that have masses much smaller than \( \Lambda_{QCD} \). If the latter were included in the theory, they could appear in large numbers in the effective Fock-space basis and complicate the dynamics, as massless gluons do.

III. RGPEP FOR HADRONS

The RGPEP provides equations for calculating the renormalized Hamiltonian \( H_t \) from the canonical one that includes regularization and counterterms. It is also used to calculate the counterterms. Eigenstates of \( H_t \) define hadrons in terms of effective particle basis in the Fock space. We first consider QCD of only one flavor of heavy quarks, useful in discussing dynamics in baryons made of quarks of one flavor. The case of baryons made of two types of heavy quarks is discussed in Sec. IV. We calculate \( H_t \) using expansion in powers of a formally infinitesimal coupling constant, up to terms of second-order. Results of our second-order calculations are later compared with results obtained in quark models and in lattice approach to QCD.
A. Canonical Hamiltonian

The Lagrangian for one-flavor QCD is
\[ \mathcal{L} = \bar{\psi}(i\slashed{\partial} - m)\psi - \frac{1}{2} \text{tr} F^{\mu\nu} F_{\mu\nu} \, . \]  
(3)

We use the front form (FF) of Hamiltonian dynamics \[ \text{and employ canonical quantization to derive the corresponding Hamiltonian } \hat{H}^{\text{can}}_{\text{QCD}} \text{ in the gauge } A^+ = 0, \]
\[ \hat{H}^{\text{can}}_{\text{QCD}} = \hat{\mathcal{H}}^- = \int dx^+ d^2 x_- : \mathcal{H}_x = 0 : \]  
(4)

We adopt the FF notation of Refs. \[8, 9\]. The Hamiltonian operator density, \( \mathcal{H}_x = 0 \) : integrated over the front \( x^+ = 0 \), is expressed in terms of the quantum fields
\[ \hat{\psi} = \sum_{sc} \int [p] \left[ \chi_c \gamma_{\mu} \hat{b}_{\sigma c} e^{-i p x} + \chi_c \gamma_{\mu} \hat{d}_{\sigma c} e^{i p x} \right] x^+ = 0 , \]
(5)
\[ \hat{A}^\mu = \sum_{sc} \int [p] \left[ T_c \gamma_{\mu} \hat{d}_{\sigma c} e^{-i p x} + T_c \gamma_{\mu} \hat{b}_{\sigma c} e^{i p x} \right] x^+ = 0 , \]
(6)

where \( \int [p] = \int_0^\infty dp^+ \int d^2 p_- /[2p^+(2\pi)^3] \), \( u_{\sigma c} \) and \( v_{\sigma c} \) are the Dirac spinors, \( c^\mu_{\sigma c} \) is the transverse-gluon polarization vector, \( \chi_c \) and \( T_c \) denote three-component color vector for quarks and eight-component color matrix vector for gluons, while \( \sigma \) and \( c \) stand for their spins and colors, respectively. We omit the hats and normal ordering symbols in further formulas. In second-order calculation, only two interaction terms count, the quark-gluon interaction,
\[ \mathcal{H}_{\psi A} A = A^\mu A_{\mu}^{\text{quark}} , \]
(7)
and the instantaneous quark-quark interaction,
\[ \mathcal{H}_{(\psi \psi)^2} = \frac{1}{2} J^{a+} \frac{1}{(i\partial^+)^2} J^{a+} \text{quark} , \]
(8)
where
\[ J^{a+} = g_{\text{bare}} \bar{\psi} \gamma^\mu T^a \psi . \]
(9)

B. Regularization

\[ \hat{H}^{\text{can}}_{\text{QCD}} \text{ is regularized by inserting cutoff functions } r_{21,3} \text{ and } \Gamma_{121} = \mathcal{H} \text{ in the interaction vertices, as shown below and further explained in App. A of Ref. \[3\]. The terms that contribute to the baryon problem are } \]
\[ \hat{H}^{\text{can}}_{\text{QCD}} = \hat{H}^{\text{free}} + g_{\text{bare}} \hat{H}_{1}^{R} + \hat{g}_{\text{bare}} \hat{H}^{R}_{\text{inst}} \]. \]
(10)

The free, or kinetic term is
\[ \hat{H}^{\text{free}} = \sum_{sc} \int [p] E_q b^\dagger_{\sigma c} b_{\sigma c} + \sum_{sc} \int [p] E_g a^\dagger_{\sigma c} a_{\sigma c} \]  
(11)

where \( E_q \) and \( E_g \) are the FF quark and gluon energies, respectively. In the quark-gluon vertex,
\[ \hat{H}^{R}_{1} = \int_{123} r_{21,3} B_{21,3} b^\dagger_{1} a_{3} + h.c. , \]
(12)
the first term corresponds to emission and the second to absorption of a gluon by a quark. Numbers 1, 2, 3 stand for sets of quantum numbers of particles 1, 2 and 3, and \( \int_{123} \) includes integration over momenta and summation over spins and colors of particles 1, 2 and 3. In the factor
\[ B_{21,3} = \tilde{\delta}_{21,3} t_2^1 \tilde{u}_2 a_1^1 , \]
(13)
\[ r_{21,3} = \chi_{c_2}^+ T^{c_1} \chi_2 \]. The tilde over \( \delta \) indicates an implicit factor \( 2(2\pi)^3 \), multiplying the Dirac \( \delta \)-function of momentum conservation. The function \( r_{21,3} \) cuts off the large relative transverse momenta and small fractions of plus momenta for the particles involved in the vertex, cf. App. A in \[3\]. The instantaneous interaction of Eq. \[8\] yields
\[ \hat{H}^{R}_{\text{inst}} = \int_{121} \tilde{\delta}_{121} T_{121}^\mu \sqrt{x_1 x_2 x_1' x_2'} \delta_{\sigma_1 \sigma_2} \delta_{\sigma_3 \sigma_3'} \]
\[ \times \frac{\Gamma_{121} = \mathcal{H}}{(x_1 - x_1')^2} \tilde{u}_1^{b_1} \tilde{u}_2^{b_2} b_1^1 b_2^2 b_1' . \]
(14)

C. Renormalized Hamiltonian

We call the regularized canonical Hamiltonian for quanta of size \( s = 0 \) the initial Hamiltonian, since it provides an initial condition for solving the RGPEP equation. Its solution defines a family of renormalized Hamiltonians \( H_t \), which are written in terms of the operators \( q_t \) that create or annihilate particles of size \( s > 0, t = s^4 \). The latter operators are defined by means of a unitary transformation \( U_t \),
\[ q_t = U_t q_0 U_t^\dagger , \]
(15)

The idea is that nonzero size eliminates divergent integrals. Hence, the effective Hamiltonians cannot be sensitive to the cutoff parameters in the cutoff functions. This implies that the regularized canonical Hamiltonian needs to be supplemented with counter-terms, which ensures that the renormalized Hamiltonians do not depend on the regularization. The problem is to define \( U_t \) that generates Hamiltonians \( H_t = H_t(q_t) \) in a suitable operator basis. Instead of directly defining \( U_t \), we define \( \mathcal{H}_t = H_t(q_0) \), in which the products of operators from the bare theory have the coefficients from renormalized theory. By definition, it obeys
\[ \mathcal{H}_t = [\mathcal{G}_t, \mathcal{H}_t] , \]
(16)

where prime denotes derivative with respect to the parameter \( t \). \( \mathcal{G}_t \) is called a generator of the RGPEP. It is set to
\[ \mathcal{G}_t = [\mathcal{H}^{\text{free}}(t), \mathcal{H}_t] , \]
(17)
where $H_{\text{free}}$ is the free part of $H$, and is identical with $H_{\text{tree}}$. The tilde above $H$ means that coefficients in front of interaction terms are multiplied by the square of total + momentum entering the vertex. The RGPEP design guarantees that the interaction vertices that change invariant mass of interacting particles by more than 1/s are exponentially suppressed [6].

In the present work, we solve Eq. (18) using expansion in powers of renormalized coupling constant $g_t$, up to second order,

$$ H_i = H_{00} + g_t H_{11} + g_t^2 H_{22} + g_t^3 H_{33} . $$

The zero-order term, $H_{00}$, corresponds to terms without running coupling. The only difference between $H_{00}$ and the bare expression $H_{\text{free}}$ is the presence of effective creation and annihilation operators in place of bare ones,

$$ H_{00} = \sum_{\sigma \tau} \int \frac{d^4p}{(2\pi)^4} [E_q^\dagger a_{-\sigma \tau}^\dagger a_{\sigma \tau} + \sum_{\sigma \tau} \int [p] E_q a_{\sigma \tau} a_{-\sigma \tau} .$$

The solution for quark-gluon interaction term of order $g_t$, apart from substitution of bare operators by effective ones, differs from the bare theory term of Eq. (12) by the presence of form factors $f_{12,13}$. Namely,

$$ H_{11} = \int \frac{d^4x}{(2\pi)^4} \left[ \sum_{\sigma \tau} \int [p] E_q a_{\sigma \tau} a_{-\sigma \tau} + \sum_{\sigma \tau} \int [p] E_q^\dagger a_{\sigma \tau}^\dagger a_{-\sigma \tau} .$$

The differences between Eq. (24) and the corresponding equations for quarkonia [3], are: quark current $j_{22'}$ instead of anti-quark current $j_{22'}$, transposition of matrix $t'_{22}$, overall sign difference, color factor $2/3$ instead of 4/3 (when acting on a color singlet state) and symmetrization factor 1/2 that is absent in quarkonia.

Renormalized quark self-interaction mass correction is

$$ H_{12} \delta m = \int \frac{d^4x}{(2\pi)^4} \left[ \sum_{\sigma \tau} \int [p] E_q a_{\sigma \tau} a_{-\sigma \tau} + \sum_{\sigma \tau} \int [p] E_q^\dagger a_{\sigma \tau}^\dagger a_{-\sigma \tau} .$$

The other one that stems from exchange of transverse gluons,

$$ H_{12} \text{QQ} \text{exch} = \frac{1}{2} \int \frac{d^4x}{(2\pi)^4} \delta_{12,12'}\delta_{11',12'} \frac{d_{\mu\nu}(p_4)}{p_4^4} j_1^\mu j_{1'}^\nu \times \left[ \begin{array} {l} \theta_{22'} \right] + \theta_{22'} \right] \frac{d_{\mu\nu}(p_4)}{p_4^4} j_1^\mu j_{1'}^\nu \times \left[ \begin{array} {l} 

$$

where $j_1^\mu = \bar{u}_i \gamma^\mu u_j$, involves factors

$$ F_{\text{eff}}(\lambda; 12, 12') = \frac{1}{2} \int \frac{d^4x}{(2\pi)^4} \delta_{11',12'} \delta_{12,12'} \frac{d_{\mu\nu}(p_4)}{p_4^4} j_1^\mu j_{1'}^\nu \times \left[ \begin{array} {l} 

$$

D. Bound-state eigenvalue problem

Thanks to asymptotic freedom [10], $g_t = g$ is small in Hamiltonians with small $t$. This holds for $\lambda = 1/s$ much larger than the scale of $\Lambda_{\text{QCD}}$ in the RGPEP scheme. We formally consider

$$ m \gg \lambda = s^{-1} \gg \Lambda_{\text{QCD}} ,$$

which allows us to simplify the eigenvalue problem

$$ H_i |\Psi\rangle = E_i |\Psi\rangle .$$
In the baryon eigenstates represented using Eq. (2), we can neglect Fock sectors with more than three quarks, because of the first inequality in Eq. (32). In a matrix form, the eigenvalue problem reads

\[
\left\{ \begin{array}{c}
H_{t0} + g^2 H_{t2} \\
g H_{t1}
\end{array} \right\} - E \left[ \begin{array}{c}
| 3Q_t G_t \rangle \\
| 3Q_t \rangle
\end{array} \right] = 0 ,
\]

where \( H_{t2} = H_{t2}^{QQ} + H_{t2}^{\delta m} \) and dots stand for the Fock components with more than one effective gluon and for the Hamiltonian terms that involve those components.

### E. Gluon mass ansatz in the effective eigenvalue problem

Similarly to the case of heavy quarkonia [3], we remove the Fock components and Hamiltonian matrix elements that involve more than one gluon. The price for this removal is a gluon-mass ansatz for the component \( | 3Q_t G_t \rangle \). Indeed, a gluon mass term appears when one uses Gaussian elimination to express the component \( | 3Q_t 2G_t \rangle \) in terms of component \( | 3Q_t G_t \rangle \). Our working hypothesis is that other terms can also be dropped once a mass ansatz is introduced. The reduced eigenvalue problem with the ansatz mass term \( \mu^2 \) is

\[
\left\{ \begin{array}{c}
H_{t0} + \mu^2 g H_{t1} \\
g H_{t1}
\end{array} \right\} - E \left[ \begin{array}{c}
| 3Q_t G_t \rangle \\
| 3Q_t \rangle
\end{array} \right] = 0 .
\]

This two-component problem is reduced to an equation for the component \( | 3Q_t \rangle \), using a transformation \( R \) succinctly described in [11]. Up to terms order \( g^2 \) and using notation \( r \) and \( l \) for right and left states in the matrix elements, one obtains

\[
\langle l | H_{\text{eff}} | r \rangle = \langle l | H_{t0} + g^2 H_{t2} + \frac{1}{2} g H_{t1} \times \left( \frac{1}{E_l - H_{t0} - \mu^2} + \frac{1}{E_r - H_{t0} - \mu^2} \right) g H_{t1} | r \rangle .
\]

### IV. 3Q EIGENVALUE PROBLEM

The three-quark component of a baryon satisfies the FF eigenvalue equation

\[
H_{\text{eff}} | 3Q_t \rangle = \frac{M^2 + P^2}{P^+} | 3Q_t \rangle ,
\]

in which the state \( | 3Q_t \rangle \) is defined by

\[
| 3Q_t \rangle = \int_{123} P^+ \tilde{\delta}_{P,123} \psi_t(123) \frac{\epsilon^{123}}{\sqrt{6}} b^\dagger_{1 \perp} b^\dagger_{1 \parallel} b^\dagger_{1 3} | 0 \rangle .
\]

The spin-momentum wave function, \( \psi_t(123) \), is multiplied by the color factor \( \frac{\epsilon^{123}}{\sqrt{6}} \). The eigenvalue equation for the spin-momentum wave function reads

\[
\left( \frac{M^2_t + (p_{1 \perp}^+)^2}{p_{1 \perp}^+} + \frac{M^2_t + (p_{1 \parallel}^+)^2}{p_{1 \parallel}^+} + \frac{M^2_t + (p_{1 3}^+)^2}{p_{1 3}^+} \right) \times \psi_t(123) + g^2 \sum_{\sigma_1 \sigma_2} \int [1' 2'] \tilde{\delta}_{12.1'2'} U_{t \text{eff}}(12; 1'2') \psi_t(1'2'3) + g^2 \sum_{\sigma_1 \sigma_2} \int [3' 1] \tilde{\delta}_{31.3'1} U_{t \text{eff}}(31; 3'1') \psi_t(1'23') + g^2 \sum_{\sigma_1 \sigma_2} \int [2' 3'] \tilde{\delta}_{23.2'3'} U_{t \text{eff}}(23; 2'3') \psi_t(12'3') = \frac{M^2 + (P^+)^2}{P^+} \psi_t(123) .
\]
The interaction terms illustrated in Fig. 1 with

$$
M_{2i}^2 = m^2 + \frac{4}{3} g^2 \int [x_{4i}/\xi_4] r_{34, i}^2 x_{4i} \cdot S_{45} x_{4i} \cdot x_{4i} / (x_{4i} - 2 + x_{4i})
$$

(40)

$$
M_{2i}^2 = m^2 + \frac{4}{3} g^2 \int \frac{d^3k}{(2\pi)^3} r_{34, i}^2 e^{-2(M^2 - m^2)^2} x_{4i} \cdot S_{45} x_{4i} \cdot x_{4i} / (x_{4i} - 2 + x_{4i})
$$

(41)

where $x = x_{4i}$, $M^2 = M_{2i}^2$, $S_{45, \mu_i} = S_{45} + \mu_i^2 x_{4i}^{-1}$. The mass ansatz $\mu^2$, which is allowed to be a function of the gluon relativistic momentum with respect to the three quarks, yields $\mu_1^2 = \mu_2^2 (p_4, p_5, p_1, p_2)$, so that $\mu_2^2 = \mu_2^2 (p_1, p_5, p_2, p_3)$, $\mu_3^2 = \mu_3^2 (p_1, p_4, p_5, p_3)$ and $\mu_4^2 = \mu_4^2 (p_1, p_4, p_5, p_1, p_2)$, see Fig. 1.

The interactions include instantaneous and exchange terms $U_{\text{eff}} = H_{\text{inst}} + H_{\text{exch}}$.

$$
H_{\text{inst}}(12; 1'2') = \left[ -\frac{2}{3} r_{12, 1'2} f_{12, 1'2}^{++} \frac{j_{11', 122}^{++}}{(p_4^2)^2} \right]
$$

(42)

$$
H_{\text{exch}}(12; 1'2') = \left[ -\frac{2}{3} \frac{d_{\mu
u}(p_4)}{p_4} j^{\mu\nu}_{11', 122} \right]
$$

(43)

where $S$ denotes symmetrization $1 \leftrightarrow 2$, or, symbolically, $(12 + 21)/2$, see Fig. 1. One obtains

$$
\mathcal{F}_{12}^2 = f_{12, 1'2} F_{12}(12; 1'2') + f_{1'4, 12} f_{24, 1'2} R_{12}(12; 1'2')
$$

(44)

$$
\mathcal{F}_{12}^2 = f_{12, 1'2} F_{12}(12; 1'2') + f_{2'4, 1'2} R_{12}(12; 1'2')
$$

(45)

$$
R_{12}(12; 1'2') = \frac{-p_{1'}^2/2}{S_{1'4} + \mu_4^2 x_{1'2} x_{1'2}/4} + \frac{-p_{1'}^2/2}{S_{2'4} + \mu_4^2 x_{1'2} x_{1'2}/4}
$$

(46)

$$
R_{12}(12; 1'2') = \frac{-p_{1'}^2/2}{S_{1'4} + \mu_4^2 x_{1'2} x_{1'2}/4} + \frac{-p_{1'}^2/2}{S_{2'4} + \mu_4^2 x_{1'2} x_{1'2}/4}
$$

(47)

The other two interaction terms, $U_{\text{eff}}(31; 3'1')$ and $U_{\text{eff}}(23; 2'3')$, are obtained by cyclic permutations of 1, 2, 3 and 1', 2', 3' in the formulas for $U_{\text{eff}}(12; 1'2')$.

### A. Small-x dynamics

The interaction kernel $U_{\text{eff}}(12; 1'2')$ in Eq. (39) can be written in terms of relative momentum variables $x_{1/12}, \kappa_{1/12}^2$ and $x_{1'/12}, \kappa_{1'/12}^2$ (that is momenta relative to pair 12). The resulting expression has the same structure as in our quarkonium analysis [3], with the color factor 2/3 instead of 4/3. The gluon mass ansatz in baryons can be different from the one in quarkonia, since it is a function of gluon momentum relative to the three-quark subsystem instead of quark-antiquark subsystem. The small-x singular factors in the interaction do not produce divergences for the same reason as in quarkonia. The gluon-exchange integral is finite because we assume that the gluon mass ansatz vanishes when $x_5 \to 0$. An example of required behavior in quarkonia is $\mu^2 \sim x_5^{3/2} \kappa_5^2$. In the notation used for baryons, the same behavior is described by $\mu^2 \sim x_{4/12}^{3/2} \kappa_{4/12}^2$ when $x_{4/12} \to 0$. Let us assume that $\mu^2 \sim x_{4/12}^{3/2} \kappa_{4/12}^2$ in the limit $x_4 \to 0$. When $x_4$ goes to zero, so does $x_{4/12} = x_4/(x_1 + x_2)$, and $\kappa_{4/12}^2 \approx \kappa_4^2$. Therefore, $\mu^2 \sim x_{4/12}^{3/2} \kappa_{4/12}^2$ implies $\mu^2 \sim x_{4/12}^{3/2} \kappa_{4/12}^2$. The same reasoning applies to every pair of the quarks that exchange a gluon. Similarly, in the quark self-interaction terms, the integration variables are $x_{4/1} = x_{4/1} \equiv x_{4/1}$, $\kappa_{4/1} = \kappa_{4/1}$, $\mu_{4/1} = \mu_{4/1}$, and $x_{4/1} \approx x_4 \to 0$. Therefore, mass terms are finite when the gluon mass ansatz vanishes properly when $x_4 \to 0$.

### V. HARMONIC OSCILLATOR CORRECTION TO COULOMB TERMS

Given Eq. (32), the effective Hamiltonian $H_{\text{eff}}$ can be approximated by its non-relativistic (NR) limit. To define this limit, we introduce a set of convenient mo-
and these equations hold only for equal masses. The non-quark 1 with respect to 2 and $\vec{Q}$ momentum of quark 3 with respect to the pair of quarks 1 and 2 and interaction vertices of effective particles. In the leading $\lambda$ sigificantly exceed the RGPEP scale $\lambda$. We in-

\begin{align}
Q^1_3 &= \sqrt{\beta_3(1-\beta_3)} \frac{x_3(1-x_3)}{x_3(1-x_3)} \kappa^1_3 = \sqrt{\frac{2/9}{x_3(1-x_3)} \kappa^1_3}, \\
Q^2_3 &= \sqrt{\beta_3(1-\beta_3)} \frac{x_3(1-x_3)}{x_3(1-x_3)} \kappa^2_3 = \sqrt{\frac{2m(x_3-1/3)}{x_3(1-x_3)}}, \\
K^1_{12} &= \sqrt{\beta_3 \beta_2(1-x_3)} \frac{x_1 x_2 (1-\beta_3)}{x_1 x_2 (1-\beta_3)} \kappa^{1/12}_1 = \sqrt{\frac{1-x_3}{6 x_1 x_2} \kappa^{1/12}_1}, \\
K^2_{12} &= \sqrt{\beta_3 \beta_2(1-x_3)} \frac{x_1 x_2 m_2 - x_2 m_1}{x_1 x_2 (1-\beta_3)} (1-x_3) = \sqrt{\frac{1-x_3}{6 x_1 x_2} (x_1 - x_2) m},
\end{align}

where $\beta_3 = m_3/(m_1 + m_2 + m_3)$. The second equality in these equations holds only for equal masses. The non-relativistic limit is defined as $\vec{K}/m \to 0$, $\vec{Q}/m \to 0$. It is valid because the relative momentum regions that significantly exceed the RGPEP scale $\lambda \ll m$ are suppressed by the exponentially-fast vanishing form factors in the interaction vertices of effective particles. In the leading NR approximation, the momenta $K_{12}$ and $Q_{3}$ are related to the Jacobi momenta: $K_{12}$ is the relative momentum of quark 1 with respect to 2 and $Q_{3}$ is the relative momentum of quark 3 with respect to the pair of quarks 1 and 2, see Ref. [12] for more details. Generically, we denote by $\vec{K}$ the relative momentum of a quark with respect to another quark with which it is involved in an interaction term, and we denote by $\vec{Q}$ the relative momentum of a spectator with respect to the pair in interaction. We introduce three sets of such relative momentum variables, arranged using the cyclic permutation of indices 123: $\vec{K}_{ijk}$ and $\vec{Q}_{ij}$. In the NR limit,

\begin{align}
K_{31}^1 &= -\frac{1}{2} K_{12}^1 + \frac{3}{4} Q_3^1, \\
K_{23}^1 &= -\frac{1}{2} K_{12}^2 - \frac{3}{4} Q_3^2, \\
Q_1^1 &= K_{12}^2 - \frac{1}{2} Q_3^2.
\end{align}

For equal quark masses, we write the baryon mass as $M = 3m + B$, divide Eq. (39) by $6m$, take the NR limit and obtain

\begin{align}
\frac{K_{12}^2}{2 \mu_{12}} + \frac{Q_3^2}{2 \mu_{31(2)}} - B + 3 \frac{\delta m^2_i}{2m} \psi_t(123) + \sum_{\sigma_i' \sigma_{i'}} \int \frac{d^3 \vec{K}'}{(2\pi)^3} \left[ f_{i12,1'2'} V_{C,BF}^{i12} + W_{i12}^{12'} \right] \psi_t(1'2'3), \\
\sum_{\sigma_{i'} \sigma_{i''}} \int \frac{d^3 \vec{K}_{23}'}{(2\pi)^3} \left[ f_{i23,2'3'} V_{C,BF}^{231} + W_{23}^{23'} \right] \psi_t(12'3'), \\
\sum_{\sigma_{i'} \sigma_{i''}} \int \frac{d^3 \vec{K}_{31}'}{(2\pi)^3} \left[ f_{i31,3'1'} V_{C,BF}^{31} + W_{31}^{31} \right] \psi_t(1'2'3) = 0,
\end{align}

where $V_{C,BF}^{ij} = V_{C,BF}(\vec{K}_{ij}, \vec{K}_{ij}')$ and $W_{ij} = W(\vec{K}_{ij} - \vec{K}_{ij}')$ are, respectively, the Coulomb term with Breit-Fermi (BF) corrections and the additional interaction resulting from the gluon mass ansatz. $\mu_{12} = m/2$, $\mu_{31(2)} = 2m/3$ are the reduced masses. Both $V$ and $W$ are similar to the ones in the quarkonium case [3].

\begin{align}
V_{C,BF}(\vec{K}, \vec{K}') &= \frac{2}{3} \frac{g^2}{\Delta K^2} (1 + BF), \\
W(\Delta \vec{K}) &= \frac{2}{3} \frac{g^2}{\mu^2 + \Delta K^2} \left[ \frac{1}{(\Delta K^2)^2} - \frac{1}{\Delta K^2} \right] \frac{n_{\mu}}{\Delta K^2} \exp \left[ -2 t m^2 \frac{\Delta K^4}{(\Delta K^2)^2} \right],
\end{align}

where $\Delta \vec{K} = \vec{K} - \vec{K}'$ and the RGPEP form factor is

\begin{equation}
\int \frac{d^3 \vec{K}_{12}'}{(2\pi)^3} W_{12}^{12'} [\psi_t(1'2'3) - \psi_t(123)] = -w^n \frac{\partial^2 \psi_t(123)}{\partial (K_{12}^n)^2}.
\end{equation}

As in quarkonia, we assume that the ansatz $\mu^2$ dominates $\Delta K^2$ in the relevant integration range. In this case, in
Eq. (56), \( \mu^2/(\mu^2 + \Delta K^2) \approx 1 \), which further leads to the conclusion that \( w^n \) for \( n = 1, 2, 3 \), corresponding to different directions in space, are the same. Thus, the effective oscillator interaction respects rotational symmetry in the Jacobi variables. However, there are only two independent relative momenta for three quarks. We distinguish one pair of quarks, e.g., 12, and rewrite the oscillators in terms of \( K_{12} \) and \( Q_3 \),

\[
\left( \frac{\partial}{\partial K_{12}} \right)^2 + \left( \frac{\partial}{\partial K_{23}} \right)^2 + \left( \frac{\partial}{\partial K_{31}} \right)^2 = \frac{3}{2} \left( \frac{\partial}{\partial K_{12}} \right)^2 + 2 \left( \frac{\partial}{\partial Q_3} \right)^2 . \tag{60}
\]

Thus, we obtain an oscillator force between quarks 1 and 2 and an oscillator force between quark 3 and the pair 12. Their strengths are in ratio 3/2 : 2. Since the ratio of corresponding reduced masses \( \mu_{12} \) and \( \mu_{12} \) is 4/3, the frequencies of these oscillators are the same and equal

\[
\omega_{\text{baryon}} = \sqrt{\frac{3}{2}} \frac{\alpha}{18\sqrt{2\pi}} \frac{\lambda^3}{m^2} . \tag{61}
\]

This expression differs from the result for quarkonia by a factor \( \sqrt{3}/2 \), rendering \( \omega_{\text{baryon}}^2/\omega_{\text{meson}}^2 = 3/4 \), assuming that \( m, \lambda \) and \( \alpha \) are the same for mesons and baryons built from one flavor of heavy quarks. This result is very close to the ratio 5/8 suggested by models that employ the concept of gluon condensate in vacuum [14] or only inside hadrons [12].

The oscillator interaction may appear to be in contradiction with the linear confinement picture in QCD. However, the eigenvalues of the FF Hamiltonian are the baryon masses squared, in distinction from the instant form (IF) Hamiltonian eigenvalues that are the baryon energies, reducing to the baryon masses only for bound states at rest. At large distances between quarks, the quadratic potential in the FF corresponds to the linear potential in the IF of Hamiltonian dynamics [13].

**VI. TWO FLAVORS OF HEAVY QUARKS**

Several new elements appear when one of the three quarks, say quark 3, is of different flavor than the other two. Besides smaller particle-exchange symmetry and the fact that \( \beta_1 \neq \beta_3 \neq 1/3 \), a new feature emerges that the NR effective quark masses are modified. The reason for NR mass modification is that, when we deal with two different flavors of quarks, the constant term that cancels completely in Eq. (59) for the same flavor no longer does so for different flavors. A finite function of \( x \) and \( \kappa \) is left and it multiplies \( \psi(1, 2, 3) \). This effect is small, but in principle ought to be considered. The correction shifts the minimal invariant mass squared value around which the NR approximation is obtained. Namely, the optimal values of \( \beta_i \) around which one expands are slightly altered, cf. Eqs. (48) to (51).

The shifts spoil the rotational symmetry of the second-order Coulomb and harmonic oscillator potentials. For \( b \) and \( c \) quarks, the deviation from spherical symmetry appears to be on the order of a few percent. It depends on the gluon mass.

This effect is certainly going to change in calculations of higher order than second, because it depends on the gluon mass ansatz and the ansatz will be replaced by theory. Since this effect is relatively small, we neglect it in what follows. Apart from the neglected effect, the Coulomb interactions between quarks are not altered when flavors differ.

The harmonic oscillator forces between pairs of quarks depend on the quark masses. Instead of Eq. (60), in which a common coefficient \( w \) is omitted, one obtains

\[
w_{12} \left( \frac{\partial}{\partial K_{12}} \right)^2 + w_{23} \left( \frac{\partial}{\partial K_{23}} \right)^2 + w_{31} \left( \frac{\partial}{\partial K_{31}} \right)^2 = \left( w_{12} + \frac{1}{2} w_{23} \right) \left( \frac{\partial}{\partial K_{12}} \right)^2 + 2w_{23} \left( \frac{\partial}{\partial Q_3} \right)^2 , \tag{62}
\]

where

\[
w_{ij} = \frac{\alpha \lambda^3}{18 \sqrt{\pi}} \left( \frac{\lambda^2}{m_i^2 + m_j^2} \right)^{3/2} . \tag{63}
\]

and \( w_{31} = w_{23} \). Using the reduced masses \( \mu_{12} \) and \( \mu_{3(12)} \), one can write the frequencies squared for the Jacobi oscillation modes 12 and 3(12) as

\[
\omega_{12}^2 = \frac{1}{m_1} \frac{\alpha \lambda^3}{18 \sqrt{\pi}} \left[ \left( \frac{\lambda^2}{2m_1^2} \right)^{3/2} + 1 \left( \frac{\lambda^2}{m_1^2 + m_3^2} \right)^{3/2} \right] , \tag{64}
\]

\[
\omega_{3(12)}^2 = \frac{2m_1 + m_3}{2m_1 m_3} \frac{\alpha \lambda^3}{18 \sqrt{\pi}} \left( \frac{\lambda^2}{m_1^2 + m_3^2} \right)^{3/2} . \tag{65}
\]

The frequencies strongly depend on the quark masses and the RGEP scale parameter \( \lambda \). The latter needs to be adjusted to justify the assumption that the component \( |3Q_1\rangle \) in the baryon eigenstate is dominant. This is unlikely in the case when \( \lambda \) is much greater than the heavy quark mass and additional components are significantly involved in the dynamics. One can solve our effective eigenvalue problem for the component \( |3Q_1\rangle \) and thus obtain approximate baryon masses and wave functions when \( \lambda \) is optimized by demanding agreement with data for quarkonia.

The effective eigenvalue equation for heavy baryons in QCD with two heavy flavors, implied by our gluon mass
ties we use data for heavy quarkonia. Our premise that
is satisfied in the limit of infinitesimal strong coupling constant
momentum and \( \alpha \) where

\[ \alpha = \frac{\beta_0 \log(\lambda^2/\Lambda_{\text{QCD}}^2)}{\Lambda_{\text{QCD}}} \]      \hspace{1cm} (71)\]

where \( \beta_0 = (33-2n_f)/(12\pi) \) and \( n_f = 2 \), valid in QCD of
two heavy flavors \( b \) and \( c \), ignoring \( u, d, s \) and \( t \). Demand that \( \alpha = 0.1181 \) for \( \lambda = M_Z = 91.1876 \text{ GeV} \),
would enforce the RGPEP value of \( \Lambda_{\text{QCD}} = 371 \text{ MeV} \) and we use
this value. The resulting spectra do not change significantly when we change \( n_f \) in the range from 2 to 5.

The quark masses are assumed to be independent of \( \lambda \) because their dependence is not known yet in the RG-
PEP. Confinement poses a conceptual difficulty concerning
the definition of quark mass \( [4] \). Quantitative estimates of quark masses would need the RGPEP calculation
to at least fourth order while we consider only second. Formulas for running masses of quarks in other
approaches, like in Eq. (9.6) in \([4]\), do not concern mass
terms in the FF Hamiltonian \( H_f \). At the current level of crude approximation and not knowing the masses precisely, we assume that \( m_b \) and \( m_c \) can be treated as constants in the range of values of \( \lambda \) that we use in fitting data, see below.

If our calculations of \( H_f \) and its eigenvalues were exact, observables we obtain would be independent of \( \lambda \), which hence could be chosen arbitrarily. Since we solve the
RGPEP equation only up to order \( \alpha \) and we introduce a
ghon mass ansatz to reduce the eigenvalue problem
to the hadron dominant Fock component, the effective
dynamics we obtain may provide a reasonable approximation
only in a certain window of values of \( \lambda \) \([15]\).\n
Note that keeping \( \lambda \) proportional to the square root of \( \alpha \) secures proportionality of the resulting
hadron binding energies to \( \alpha^2 \) \([16]\), which resembles analogous scaling in QED. This scaling is maintained with our oscillator
terms because their frequencies emerge proportional to \( \alpha^2 \). Knowing that the observed quarkonium mass
spectra can be characterized as intermediate between the Coulomb and oscillator spectra \([5]\), we expect that the
harmonic oscillator frequencies obtained from QCD may be comparable in size with the strong-interaction
Rydberg-like constant \( R = \mu(4\alpha/3)^2/2 \).

In the case of quarkonia, we set

\[ \lambda_{QQ} = \sqrt{\alpha} (a_{QQ} \bar{m}_{QQ} + b_{QQ}) \] \hspace{1cm} (72)\]
where \( m_{Q\bar{Q}} \) is the average mass of quark and antiquark that form a heavy meson, such as \( J/\psi, \Upsilon \) or \( B_c^+ \). The quark masses and unknown values of \( a_{QQ} \) and \( b_{QQ} \) are fitted to the spectra of heavy quarkonia. Separate fits for a set of \( bb \) states and a set of \( cc \) states give us most suitable \( \lambda_{\bar{b}b} \) and \( \lambda_{cc} \), and quark masses \( m_b \) and \( m_c \). This set of numbers allows us to fix values of \( a_{QQ} \) and \( b_{QQ} \) in the linear formula of Eq. (72). With \( a_{QQ} \) and \( b_{QQ} \) fixed, we test Eq. (72) by comparing our theoretical spectrum of \( B_c \) particles, computed for \( \lambda_{bc} \) given by Eq. (72), with experimental data. The agreement is satisfactory, cf. Sec. VII B. The adjustment of constants \( a_{QQ} \) and \( b_{QQ} \) reflects the current lack of knowledge of the values of \( \lambda \) at which one can most accurately approximate different hadron eigenvalue problems using merely their lowest Fock components and gluon mass ansatz. Details of our fits of two quark masses, \( m_c \) and \( m_b \) at most suitable values of \( \lambda_{\bar{b}b} \) and \( \lambda_{cc} \), are described in App. A.

In the case of baryons, we set
\[
\lambda_{Q\bar{Q}} = \sqrt{a} (a_{QQ} m_{3Q} + b_{QQ}) \; ,
\] (73)
where \( m_{3Q} \) is the average mass of the three quarks that form a lowest Fock component of a baryon at scale \( \lambda_{Q\bar{Q}} \). We set \( a_{QQ} = a_{QQ} \), \( b_{QQ} = b_{QQ} \) because it is the simplest possible choice that secures \( \lambda_{\bar{b}b} = \lambda_{\bar{b}b} \) and \( \lambda_{cc} = \lambda_{cc} \), which is a natural choice. It turns out to yield estimates for \( \bar{b} \bar{b} \) and \( cc \) spectra that resemble results of other approaches, see below.

Our estimates are quite crude. We ask two questions. One is if the oscillator terms that follow from the assumption of gluon mass are capable of providing a reasonable first approximation to heavy hadrons. Provided that in the case of heavy quarkonia the answer is yes, the other question is what character of the heavy baryons spectrum one expects using the assumption that effective gluons develop a mass. To address these qualitative questions, we ignore the BF spin-dependent terms and we estimate strong-Coulomb effects by evaluating expectation values of the corresponding interaction terms in the oscillator eigenstates. Details of our baryon wave functions are described in App. D. Comparison with other approaches, including lattice estimates, suggests that our extremely simple oscillator picture and thus possibly also the gluon mass hypothesis, appear reasonable. Reliable estimates of better accuracy require fourth-order solution to the RGPEP Eq. (10).

**B. Masses of quarkonia**

Details of fits of quark masses and scale parameter to quarkonium data are described in App. A. Most accurate fit to masses of \( \Upsilon(1S), \Upsilon(2S) \) and \( \chi_{b1}(1P) \), is obtained for
\[
\begin{align*}
    m_b &= 4698 \text{ MeV} \quad \text{and} \quad \lambda_{\bar{b}b} = 4258 \text{ MeV} \; ,
\end{align*}
\] (74)
These values are associated with \( \alpha(\lambda_{\bar{b}b}) = 0.2664 \) and \( \omega_{\bar{b}b} = 268.8 \) MeV. The resulting bottomonium masses are shown in the left panel of Fig. 3. To most accurately describe masses of \( J/\psi, \psi(2S) \) and \( \chi_{c1}(1P) \), one needs
\[
\begin{align*}
    m_c &= 1460 \text{ MeV} \quad \text{and} \quad \lambda_{c\bar{c}} = 1944 \text{ MeV} \; ,
\end{align*}
\] (75)
and these values are associated with \( \alpha(\lambda_{c\bar{c}}) = 0.3926 \) and \( \omega_{c\bar{c}} = 321.6 \) MeV. The resulting charmonium masses are illustrated in the right panel of Fig. 3. Values of \( \lambda_{bb}, \lambda_{cc}, \) and \( m_b, m_c \) allow us to fix
\[
\begin{align*}
    a_{Q\bar{Q}} &= 1.589 \; ,
    b_{Q\bar{Q}} &= 783 \text{ MeV} \; .
\end{align*}
\] (76)
(77)
These coefficients imply, according to Eq. (72),
\[
\lambda_{bc} = 3134 \text{ MeV} \; .
\] (78)
The middle panel of Fig. 3 shows the comparison of experimental masses of \( B_c \) and \( B_c^{(2S)} \) and an average of different predictions for a mass of \( B_c^+ \) with our theoretical levels. Note that because we fit the masses of spin-one quarkonia while we neglect spin-dependent interactions, we present our mass estimates as for \( ^1S \). Because there are no experimental data to compare for spin-one \( B_c^+ \), we provide an average of various theoretical predictions [17]. The agreement is satisfactory, given that we do not expect our estimates to be precise. Equation (78) gives
\[
\begin{align*}
    \alpha(\lambda_{bc}) &= 0.3047 \; ,
    \omega_{bc} &= 261.1 \text{ MeV} \; .
\end{align*}
\] (79)
(80)

**C. Estimates of masses of heavy baryons**

The fit to quarkonia described in Sec. VII B establishes optimal values of \( \lambda \) for all baryons. Values of the coupling constant are obtained from Eq. (71). The optimal values we obtain for these parameters are listed in App. B. The resulting masses of heavy baryons are shown in Fig. 3. Labels of states describe internal orbital motion of quarks, where the first part of a label corresponds to the motion of quark 1 with respect to quark 2 and the second part corresponds to the motion of quark 3 with respect to the pair of quarks 1 and 2. For example, in the state \( 1P1S \), the pair 12 is in a \( p \)-wave without radial excitation, while the quark 3 in its motion with respect to the pair 12 is in an \( s \)-wave state without radial excitation. In \( 1S2S \), both 1 with respect to 2 and 3 with respect to 12 are in an \( s \)-wave state but the latter is radially excited. States \( A, B, C \) and \( D \) in \( cc \) and \( bb \) correspond to the second excitation of harmonic oscillator with excitation energy \( 2\omega \) above the ground state. These states have spin-momentum wave functions that are symmetrized in a way due for fermions in colorless states. Details of the harmonic oscillator basis wave functions are described in App. D. Analytical formulas for masses of baryons are given in App. E.

The values of masses we obtain for \( bb \) and \( cc \) baryons agree well with model calculations [18–24] including quark-diquark [25] and hypercentral approxima-
sitions \[26, 27\], bag models \[28, 30\], Regge phenomenology \[31, 32\], sum rules \[33, 36\], pNRQCD \[37\], Dyson-Schwinger approach \[48, 49\], and lattice studies \[40–46\], where comparison is available. As an example of comparison, we note that the ground state of \(ccc\) is assigned masses from 4733 MeV to 4796 MeV, by different lattice calculations, with an average of 4768 MeV. Our result is 4797 MeV, differing by 29 MeV, or 0.6% from the average. For \(b\bar{b}\), the average of two lattice results we have identified is 14360 MeV, and our result is 14346 MeV, which is 23 MeV difference, or 0.2%. These comparisons refer to Table I in Ref. \[27\] that summarizes results of calculations of masses of \(\Omega_{ccc}\) and \(\Omega_{b\bar{b}}\) reported in twenty different articles. Ground state of \(b\bar{b}\)c is also very close to the lattice result \[40\]. In contrast, our \(c\bar{c}\) differs by about 300 MeV from the lattice. We comment on this feature below. Comparison with lattice calculations reported in Ref. \[41\] shows that our splittings in \(b\bar{b}\) differ only by about 10%. In case of \(ccc\) \[43\], the difference of splittings does not exceed 20%. This degree of agreement is surprising in view of the complexity of lattice calculations in comparison with the simplicity of our effective Hamiltonian calculation.

The prominent feature visible in Fig. 4 is the extraordinary magnitude of splittings in the \(c\bar{c}\) baryons. It is a consequence of large oscillator frequency in \(cc\) subsystem in Eq. (64), due to large ratio of \(\lambda_{c\bar{c}}/m_c\), in which the scale parameter \(\lambda_{c\bar{c}}\) is large in comparison with \(m_c\) due to \(m_b\). This separation of scales may make precise calculations of masses of excited \(c\bar{c}\) baryons difficult. Since the harmonic excitation is so high, it is likely that components with gluons of mass on the order of 1 GeV have to be included in a nonperturbative way.

The surprising feature that the very crude, first approximation based on the RGPEP, with no free parameters left after adjusting quark masses and scale to \(b\bar{b}\) and \(c\bar{c}\) data, produces in an elementary analytic way similar splittings to the ones resulting from advanced calculations, is further illustrated in Fig. 3. It presents splittings in a second band of harmonic oscillator caused by Coulomb interactions. Splittings \(m_D - m_C\), \(m_C - m_B\), \(m_B - m_A\) are in relation 2:1:5, which is the general result in the first order of perturbation theory for harmonic oscillator perturbed by any potential \[47\].

Interestingly, analogous lattice QCD splittings with spin dependent interactions turned off \[41\], also appear in the ratios 2:1:5. These results suggest that the RGPEP constituent picture with a gluon mass ansatz may be grasping the physics of lowest-mass heavy baryons. Since experimentally triply heavy baryons are difficult to produce and detect \[48\], their theoretical understanding using standard techniques is weakly motivated and hence also limited \[49, 50\]. Therefore, the ease with which our method yields results for heavy baryons in agreement with complex approaches suggests that application of the RGPEP in fourth order and including components with one or more effective gluons in the eigenvalue problem
beyond perturbation theory, are worth attempting.

The effective Hamiltonians we finesse for heavy quarkonia and baryons from QCD of charm and beauty quarks using our gluon mass ansatz, lead to the baryon mass spectra in the ball park of expectations from other approaches to physics of $ccc$ and $bbb$ systems. In addition, the Hamiltonians suggest that quarks in $ccb$ baryons display extraordinarily large mass excitation for states $1P_{1S}$, $2S_{1S}$ and $1D_{1S}$ in $ccb$. Such high excitations are associated with formation of $cc$-diquarks, bound by a harmonic force that is strong because the charmed quarks are much lighter than the bottom quarks. Much less pronounced splittings appear in the $bbc$ baryons. See the text for further discussion.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Qualitative picture of triply heavy baryon mass spectrum implied by the second-order RGPEP in heavy-flavor QCD and our gluon mass ansatz. The figure shows excitations above the ground states $1S_{1S}$, whose absolute masses are written at the bottom of each column. The $ccb$ spectrum displays extraordinarily large mass excitation for states $1P_{1S}$, $2S_{1S}$ and $1D_{1S}$ in $ccb$. Such high excitations are associated with formation of $cc$-diquarks, bound by a harmonic force that is strong because the charmed quarks are much lighter than the bottom quarks. Much less pronounced splittings appear in the $bbc$ baryons. See the text for further discussion.}
\end{figure}

\section{VIII. CONCLUSION}

The effective Hamiltonians we finesse for heavy quarkonia and baryons from QCD of charm and beauty quarks using our gluon mass ansatz, lead to the baryon mass spectra in the ball park of expectations from other approaches to physics of $ccc$ and $bbb$ systems. In addition, the Hamiltonians suggest that quarks form tight diquarks in $ccb$ baryons. Diquarks are less likely in $bbc$ baryons. Other approaches do not foresee tight diquarks in $ccb$ baryons. This feature may thus distinguish a physically proper approach in future. However, such tight diquarks are hard to excite and mass splittings due their excitation are comparable or even exceed values of the quark mass one may expect in theory. In that case, the highly excited baryon component with a heavy gluon may be large and our approximation to the three-quark component as dominant may be invalid. Calculations that treat the highly excited baryons as having significant components with one heavy effective gluon may yield smaller masses than our approximation based on the dominance of the three-quark component. If it were the case, the RGPEP approach would still apply, but in the domain of hadron physics in which gluons appear as constituents in competition with quarks for probability of appearance.

Taking into account that the method of RGPEP that we use is invariant under boosts and that it is a priori capable of providing a relativistic theory of hadrons in terms of a limited number of their effective constituents with suitably adjusted size, an extension of the RGPEP calculation to fourth order appears worth undertaking. It is certainly needed for verifying if the gluon mass ansatz we introduced provides an adequate representation of dynamics of gluons in the presence of heavy color sources. Fourth-order Hamiltonian is also needed for control on the spin splittings and rotational symmetry.

The ratio $\sqrt{8/6}$ of harmonic oscillator frequencies in heavy quarkonia and triply heavy baryons is close to the ratio $\sqrt{8/5}$ obtained for $u$ and $d$ constituent quarks in models using the concept of gluon condensate. If this is not accidental, one may hope that the RGPEP formalism shall apply also to light hadrons as built from constituent quarks and massive gluons, the latter nearly decoupled after generating effective interactions for quarks on the way down in $\lambda$ toward $1/\text{fm}$ [13]. But even for heavy...
baryons alone, the effective oscillator picture provides simple wave functions that can be used in description of relativistic processes that involve heavy hadrons.

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Appendix A: Fits to masses of well-established quarkonia

Once the coupling constant $\alpha$ as a function of $\lambda$ is set, the eigenvalues of Eq. (69) estimated by evaluating expectation values of the Coulomb terms in known eigenstates of the oscillator part of the effective Hamiltonian,

$$E_{1S} = \frac{3}{2} \omega - \frac{4}{3} \alpha \sqrt{\frac{2}{\pi \nu}} ,$$  \hspace{1cm} (A1)

$$E_{2S} = \frac{7}{2} \omega - \frac{10}{9} \alpha \sqrt{\frac{2}{\pi \nu}} ,$$  \hspace{1cm} (A2)

$$E_{1P} = \frac{5}{2} \omega - \frac{8}{9} \alpha \sqrt{\frac{2}{\pi \nu}} ,$$  \hspace{1cm} (A3)

with

$$\omega = \sqrt{\frac{\alpha(\lambda_{QQ})}{18 \sqrt{2 \pi}} \frac{\lambda_{QQ}^3}{m^2}} , \hspace{1cm} \nu = \frac{1}{m \omega} ,$$  \hspace{1cm} (A4)

and $m = m_c$ or $m = m_b$, are used to evaluate corresponding masses from the formula

$$M = 2m \sqrt{1 + \frac{E}{m}} .$$  \hspace{1cm} (A5)

These are compared with data [3]. Thus, $\Upsilon(1S)$, $\Upsilon(2S)$ and $\chi_{b1}(1P)$ are used to find best values $m_b$ and $\lambda_{bb}$, using $\lambda^2$. We obtain,

$$m_b(\lambda_{bb}) = 4698 \text{ MeV} , \hspace{1cm} \lambda_{bb} = 4258 \text{ MeV} .$$  \hspace{1cm} (A6)

Using $\alpha(\lambda_{bb}) = 0.2664$, we find that $\omega_{bb} = 268.8$ MeV. For charmonia, $J/\psi$, $\psi(2S)$ and $\chi_{c1}(1P)$ masses are used in the same way to find best values of $m_c$ and $\lambda_{c\bar{c}}$, which turn out to be

$$m_c = 1460 \text{ MeV} , \hspace{1cm} \lambda_{c\bar{c}} = 1944 \text{ MeV} .$$  \hspace{1cm} (A7)

Using $\alpha(\lambda_{c\bar{c}}) = 0.3926$, we obtain $\omega_{c\bar{c}} = 321.6$ MeV.

Two heavy mesons made of quarks $b$ and $c$ were observed, $B_c$ and $B_c(2S)$ [4]. Fits to quarkonia fix masses of quarks, hence, the only free parameter is $\lambda_{cc}$, which we fix by assuming Eq. (72). Without any freedom left we plot in Fig. 3 the spectrum of $B_c$ using the same Eqs. [A1] and [A2] but with

$$\omega^2 = \frac{\alpha(\lambda_{bc})}{18 \sqrt{2 \pi \mu_{bc}}} \left( \frac{\lambda_{bc}^3}{m_b^2 + m_c^2} \right)^3 ,$$  \hspace{1cm} (A8)

$$\nu = \frac{1}{2 \mu_{bc} \omega} ,$$  \hspace{1cm} (A9)

$$\mu_{bc} = \frac{m_b m_c}{m_b + m_c} .$$  \hspace{1cm} (A10)

The physical quarkonia masses are read from

$$M = (m_b + m_c) \sqrt{1 + \frac{2E}{m_b + m_c}} .$$  \hspace{1cm} (A11)

Appendix B: Parameters for heavy baryons

We choose parameter $\lambda$ for a baryon system by assuming Eq. (73) with coefficients $a_{3Q} = a_{QQ}$ and $b_{3Q} = b_{QQ}$ where $a_{QQ}$ and $b_{QQ}$ are given in Eqs. (76) and (77). Values of $\lambda$ are solutions to the following equations

$$\lambda_{bbb} = \sqrt{\alpha(\lambda_{QQ})} \left( a_{3Q} m_b + b_{3Q} \right) ,$$  \hspace{1cm} (B1)

$$\lambda_{bbc} = \sqrt{\alpha(\lambda_{bbc})} \left( a_{3Q} m_b + \frac{2m_b + m_c}{3} + b_{3Q} \right) ,$$  \hspace{1cm} (B2)

$$\lambda_{ccb} = \sqrt{\alpha(\lambda_{ccb})} \left( a_{3Q} m_c + \frac{m_c + 2m_b}{3} + b_{3Q} \right) ,$$  \hspace{1cm} (B3)

$$\lambda_{ccc} = \sqrt{\alpha(\lambda_{ccc})} \left( a_{3Q} m_c + b_{3Q} \right) .$$  \hspace{1cm} (B4)

We obtain:

$$\lambda_{bbb} = 4258 \text{ MeV} , \hspace{1cm} \alpha(\lambda_{bbb}) = 0.2664 ,$$  \hspace{1cm} (B5)

$$\lambda_{bbc} = 3514 \text{ MeV} , \hspace{1cm} \alpha(\lambda_{bbc}) = 0.2892 ,$$  \hspace{1cm} (B6)

$$\lambda_{ccb} = 2746 \text{ MeV} , \hspace{1cm} \alpha(\lambda_{ccb}) = 0.3248 ,$$  \hspace{1cm} (B7)

$$\lambda_{ccc} = 1944 \text{ MeV} , \hspace{1cm} \alpha(\lambda_{ccc}) = 0.3926 .$$  \hspace{1cm} (B8)

For readers’ convenience, we also listed above the associated values of coupling constant.

Appendix C: Frequency diagram

Harmonic oscillator frequencies, Eqs. (64) and (65), depend on quark masses and on the scale $\lambda$. Figure 6 shows the dependence of $\omega_{12}$ and $\omega_{3(12)}$ on $\lambda$ for four different choices of three quark masses that correspond to the systems $bbb$, $bcc$, $c\bar{c}$ and $ccc$. For $ccc$ and $bbb$ we have $\omega_{12} = \omega_{3(12)} = \omega$. Blue vertical lines indicate the values of $\lambda$s for baryons from Eqs. (B5) to (B8). They end on a higher of two blue dots. The dots show the values of $\omega_{12}(\lambda)$ and $\omega_{3(12)}(\lambda)$ for a given system and for $\lambda$ adjusted to that system. The frequencies are:

$$\omega_{bbb} = 232.8 \text{ MeV} ,$$  \hspace{1cm} (C1)

$$\omega_{12, bcc} = 166.2 \text{ MeV} , \hspace{1cm} \omega_{3(12), bcc} = 336.7 \text{ MeV} ,$$  \hspace{1cm} (C2)

$$\omega_{12, c\bar{c}} = 593.5 \text{ MeV} , \hspace{1cm} \omega_{3(12), c\bar{c}} = 142.7 \text{ MeV} ,$$  \hspace{1cm} (C3)

$$\omega_{ccc} = 278.5 \text{ MeV} .$$  \hspace{1cm} (C4)
On Fig. 6 there are also two green triangles at the bottom of the plot. They indicate the values of \( m_b \) and \( m_c \). Furthermore, Figure 8 presents also \( \alpha(\lambda) \) given in Eq. (71) with a decreasing black line. Red vertical line shows the asymptote of \( \alpha(\lambda) \) curve at \( \lambda = \Lambda_{QCD} \).

**FIG. 6.** Dependence of the harmonic-oscillator frequencies on the RGPEP scale \( \lambda \) and quark masses (left axis), and the dependence of \( \alpha \) on \( \lambda \) (right axis). For detailed description of this figure content, see App. C.

### Appendix D: Wave functions for baryons

Harmonic oscillator basis for baryons is constructed from products of wave functions of two harmonic oscillators associated with relative motion of particles 1 and 2 (with momentum \( \vec{k}_{12} \)), and relative motion of particle 3 of momentum \( Q_3 \) with respect to the pair 12.

Eigenfunctions for relative motion of two particles with reduced mass \( \mu \) interacting with harmonic oscillator force characterized by frequency \( \omega \) are

\[
    \psi_{klm}(p, \theta, \phi) = N_{kl} \ e^{-\nu p^2} P_{l}^{(l+1/2)}(2r\nu^2) \ Y_{lm}(\theta, \phi) ,
\]

where \( \nu = 1/(2\mu \omega) \), \( k \) is the radial excitation number, \( l \) is the orbital angular momentum number of the state, \( m \) is the projection of angular momentum on the z axis, \( Y_{lm} \) are spherical harmonics [4] and \( P_{l}^{(l+1/2)}(x) \) are generalized Laguerre polynomials. The first two polynomials are \( P_{0}^{(0)}(x) = 1 \), \( P_{1}^{(1/2)}(x) = 1 + a - x \). The normalization factors are

\[
    N_{kl} = \sqrt{\frac{2r^3 \ e^{2l+3k+1}}{\pi (2k+2l+1)!}} (2\pi)^{3/2} ,
\]

so that \( \int \frac{d^3p}{(2\pi)^3} \psi_{klm}^* \psi_{k'l'm'} = \delta_{kk'} \delta_{ll'} \delta_{mm'} \). Finally, the energies are

\[
    E = \omega (2k + l + \frac{3}{2}) .
\]

We use a convenient notation for products of wave functions of two harmonic oscillators,

\[
    |(k_{12} + 1)(l_{12})_{m_{12}}(k_{3(12)} + 1)(l_{3(12)})_{m_{3(12)}}, \alpha_\lambda(\lambda)\rangle = |\psi_{k_{12}l_{12}m_{12}}\rangle|\psi_{k_{3(12)}l_{3(12)}m_{3(12)}}, \alpha_\lambda(\lambda)\rangle ,
\]

where index 12 corresponds to harmonic oscillator between 1 and 2, with \( \nu_{12} = 1/(2\mu_{12}\omega_{12}) \), and index 3(12) corresponds to harmonic oscillator between 3 and 12, with \( \nu_{3(12)} = 1/(2\mu_{3(12)}\omega_{3(12)}) \). For example, the ground state is \( |1S_01S_0, 1⟩ \equiv |1S_1S⟩ \), while \( |1P_2S_0, 1⟩ \equiv |1P_2S⟩ \) is the state with harmonic oscillator between 1 and 2 excited to the first orbital excitation with angular momentum projection on z-axis equal 1 and the harmonic oscillator between 3 and 12 being radially excited. The quantum numbers \( m_{12} \) and \( m_{3(12)} \) are omitted below, unless they are relevant.

We consider states whose excitation energies are at most \( 2\omega_{12} \) or \( 2\omega_{3(12)} \) or \( \omega_{12} + \omega_{3(12)} \). That is, we consider states \( |1S_1S, 1P_1P⟩ \), \( |1S_1S, 1S_1P⟩ \), \( |1S_1P, 2S_1S⟩ \), \( |1S_2S, 1D_1S⟩ \), \( |1S_1D, 1P_1P⟩ \). The total angular momentum of a baryon is conserved, therefore, each state of the basis should have definite orbital angular momentum \( L \). Since states \( |1P_{m_{12}}, 1P_{m_{3(12)}}⟩ \) do not have definite angular momentum, we introduce instead the following states with angular momenta \( L = 2, 1 \) and 0, respectively,

\[
    |2, +2⟩ = |1P_11P_1⟩ , \quad |2, +1⟩ = \ldots \quad \text{(D5)}
\]

\[
    |1, +1⟩ = \frac{1}{\sqrt{2}} (|1P_11P_0⟩ - |1P_01P_1⟩) , \quad |1, 0⟩ = \ldots \quad \text{(D6)}
\]

\[
    |0, 0⟩ = \frac{1}{\sqrt{3}} \left( |1P_11P_{-1}⟩ - |1P_01P_0⟩ + |1P_{-1}1P_1⟩ \right) ,
\]

where only the highest \( L \) state is written explicitly. The construction of these states is done in accordance with the rules of adding angular momenta in quantum mechanics and we use convention defined in [4] in the tables of Clebsch-Gordan coefficients.

Because quarks have spin, we also need to construct the spin wave functions. We define spin-3/2 quadruplet, which is fully symmetric with respect to exchange of any pair of quarks,

\[
    \left| \frac{3}{2} \right\rangle = |↑↑↑⟩ , \quad \left| \frac{1}{2} \right⟩ = \ldots \quad \text{(D8)}
\]
spin-1/2 doublet, which we call \((1/2)_s\) and which is 12-symmetric,

\[
\begin{align*}
|+\frac{1}{2}S\rangle &= \sqrt{\frac{2}{3}} \left( |\uparrow\uparrow\downarrow\rangle - \frac{1}{2} |\uparrow\downarrow\uparrow\rangle - \frac{1}{2} |\downarrow\uparrow\uparrow\rangle \right), \\
|\frac{1}{2}S\rangle &= \sqrt{\frac{1}{3}} \left( |\uparrow\downarrow\downarrow\rangle + \frac{1}{2} |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle \right);
\end{align*}
\] (D9)

and another spin-1/2 doublet, which we call \((1/2)_A\) and which is 12-antisymmetric,

\[
\begin{align*}
|+\frac{1}{2}A\rangle &= \sqrt{\frac{1}{2}} \left( |\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle \right), \\
|\frac{1}{2}A\rangle &= \sqrt{\frac{1}{2}} \left( |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle \right). \\
\end{align*}
\] (D10)

For completeness, we describe our construction of states of definite total angular momentum. First consider ccb and bbc systems, where quarks 1 and 2 are identical and 3 is different. Because quarks 1 and 2 are identical, the total spin-momentum wave function has to be 12-symmetric; color-singlet wave function is antisymmetric. Therefore, one must add orbital angular momentum and spin respecting the Pauli exclusion principle for fermions. For example, the states \(|1P1S\rangle\) are 12-antisymmetric and to obtain 12-symmetric spin-momentum wave function we can combine them only with spin \((1/2)_A\), which is also 12-antisymmetric. States \(|1S1P\rangle\) are 12-symmetric and we can combine them only with spin 3/2 and \((1/2)_s\). The list of possible states is summarized in Table I. To obtain the explicit formulas for the wave functions, we use Clebsch-Gordan tables, as in Eqs. (D5) to (D7).

1. Symmetric wave functions

In the case of three identical quarks, we need to use fully symmetric wave functions. One can symmetrize the wave functions given above. The ground state \(1S1S\) wave function is fully symmetric in momentum, and we can combine it only with spin 3/2. By the way, symmetrization of \(1S1S\) with \((1/2)_s\) gives zero. In this case the wave function is the same as in the case of only two quarks being identical,

\[
|0\omega, \frac{3}{2}, 1/2\rangle = |1S1S\rangle|J_z\rangle,
\] (D13)

where \(J_z = +3/2, +1/2, -1/2, -3/2\) is the projection of baryon spin on \(z\)-axis.

After symmetrization of the oscillator once-excited states \(1P1S\) and \(1S1P\), one is left with only two linearly independent multiplets of states, whose wave functions are (we write only the highest \(J_z\) state in each multiplet, more information is available in Table II)

\[
|1\omega, \frac{3}{2}, +\frac{3}{2}\rangle = \frac{1}{\sqrt{2}} \left( |1P1S\rangle |+\frac{1}{2}A\rangle - \frac{1}{\sqrt{2}} |1S1P\rangle |+\frac{1}{2}S\rangle \right),
|1\omega, \frac{3}{2}, +\frac{1}{2}\rangle = \frac{1}{\sqrt{2}} \left( |1P1S\rangle |+\frac{1}{2}A\rangle + \frac{1}{\sqrt{2}} |1S1P\rangle |+\frac{1}{2}S\rangle \right).
\] (D14)

The symmetrization of a band of twice-excited oscillator states, \(2S1S, 1S2S\), \(1D1S, 1S1D\) and \(1P1P\) reduces the number of linearly independent multiplets from 21 to 8 (compare tables I and II). Highest \(J_z\) states in each multiplet are

\[
|A_{\frac{3}{2}}^+, +\frac{3}{2}\rangle = |2S1S\rangle |+\frac{3}{2}\rangle, \\
|B_{\frac{3}{2}}^+, +\frac{1}{2}\rangle = \frac{1}{\sqrt{2}} |2S1S\rangle |+\frac{1}{2}S\rangle - \frac{1}{\sqrt{2}} |0, 0\rangle |+\frac{1}{2}\rangle, \\
|C_{\frac{7}{2}}^+, +\frac{7}{2}\rangle = |1D2S\rangle |+\frac{3}{2}\rangle, \\
|C_{\frac{5}{2}}^+, +\frac{5}{2}\rangle = -\sqrt{\frac{3}{7}} |1D1S\rangle |+\frac{3}{2}\rangle + \sqrt{\frac{4}{7}} |1D2S\rangle |+\frac{1}{2}\rangle,
\] (D16) (D17) (D18) (D19)
\[ |C_{\frac{3}{2}^+, \frac{3}{2}}\rangle = \sqrt{\frac{1}{5}} |1D_0 1S\rangle_+ \left| +\frac{3}{2} \right\rangle - \sqrt{\frac{2}{5}} |1D_1 1S\rangle_+ \left| +\frac{1}{2} \right\rangle + \sqrt{\frac{2}{5}} |1D_2 1S\rangle_+ \left| -\frac{1}{2} \right\rangle, \]
\[ |C_{\frac{3}{2}^+, \frac{1}{2}}\rangle = -\sqrt{\frac{1}{10}} |1D_{-1} 1S\rangle_+ \left| +\frac{3}{2} \right\rangle + \sqrt{\frac{2}{5}} |1D_0 1S\rangle_+ \left| +\frac{1}{2} \right\rangle - \sqrt{\frac{3}{10}} |1D_1 1S\rangle_+ \left| -\frac{1}{2} \right\rangle + \sqrt{\frac{2}{5}} |1D_2 1S\rangle_+ \left| -\frac{3}{2} \right\rangle, \]
\[ |D_{\frac{5}{2}^+, \frac{5}{2}}\rangle = \frac{1}{\sqrt{2}} |1D_2 1S\rangle_+ \left| +\frac{1}{2} S \right\rangle - \frac{1}{\sqrt{2}} |2, +2 \rangle \left| +\frac{1}{2} A \right\rangle, \]
\[ |D_{\frac{5}{2}^+, \frac{3}{2}}\rangle = -\sqrt{\frac{6}{5}} |1D_1 1S\rangle_+ \left| +\frac{1}{2} S \right\rangle + \sqrt{\frac{2}{5}} |1D_2 1S\rangle_+ \left| -\frac{1}{2} S \right\rangle + \sqrt{\frac{1}{10}} |2, +1 \rangle \left| +\frac{1}{2} A \right\rangle - \sqrt{\frac{3}{10}} |2, +2 \rangle \left| -\frac{1}{2} A \right\rangle. \]

where

\[ |2S1S\rangle_\pm \equiv \frac{[2S1S]\pm [1S2S]}{\sqrt{2}}, \]
\[ |1D_m 1S\rangle_\pm \equiv \frac{[1D_m 1S]\pm [1S1D_m]}{\sqrt{2}}. \]

States with \( J_z \) different than the highest \( J_z \) available in the multiplet can be constructed according to Table II.

**Table II. Summary of states for systems ccc and bbb.** For example, \( 1D1S_+ \otimes (\frac{1}{2})_S \) means that we use \( |1D_m 1S\rangle_\pm \) states and \((1/2)_S\), spin states to obtain one of \( J = 5/2 \) or \( J = 3/2 \) states according to the rules of adding angular momenta, i.e., using the Clebsch-Gordan coefficients. \( 1P1P_{L=2} \otimes (\frac{1}{2})_A \) means that we take \( L = 2 \) states, given in Eq. (D8), and \((1/2)_A\), spin states to obtain a state with the same quantum numbers. We then subtract the latter from the former, as indicated in the table, and normalize the result to obtain the final expression, such as in Eqs. (D22) or (D23), where the states with \( J = 5/2, J_z = +5/2 \) and \( J = 3/2, J_z = +3/2 \) are written explicitly. Our prescription differs in sign from the prescriptions known in the literature [17], because our momentum \( \hat{Q}_3 \) is a momentum of quark 3 with respect to pair 12, instead of pair 12 with respect to quark 3.

**Appendix E: Baryon masses**

1. **States ccb and bbc**

Baryon masses are given by Eq. (65), where

\[ E = \omega_{12} \left( 2k_{12} + l_{12} + \frac{3}{2} \right) + \omega_{3(12)} \left( 2k_{3(12)} + l_{3(12)} + \frac{3}{2} \right) + V, \]

and \( V = \langle \hat{V}_C \rangle \) is the expectation value of Coulomb interaction in the harmonic oscillator eigenstates. For example,

\[ E_{1P1P}^{L=0} = \frac{5}{2} \omega_{12} + \frac{5}{2} \omega_{3(12)} + V_{1P1P}^{L=0}, \]

where \( V_{1P1P}^{L=0} = \langle 0, 0 | \hat{V}_C | 0, 0 \rangle \). We define,

\[ V = -\frac{2}{3} \alpha \sqrt{\frac{2}{\pi \mu_{12}}} \hat{V}, \]

and

\[ x = \frac{4 \mu_{3(12)}}{\nu_{12}}. \]

We list the Coulomb interaction expectation values for ccb and bbc states in Fig. 4.

\[ \hat{V}_{1S1S} = 1 + \frac{4}{\sqrt{1 + x}}, \]
\[ \hat{V}_{1P1S} = \frac{2}{3} + \frac{4 (3x + 2)}{3 (1 + x)^{3/2}}, \]
\[ \hat{V}_{1S1P} = 1 + \frac{4 (2x + 3)}{3 (1 + x)^{3/2}}, \]
\[ \hat{V}_{1P1P}^{L=0} = \frac{2}{3} + \frac{4 (2x^2 + 7x + 2)}{3 (1 + x)^{5/2}}, \]
\[ \hat{V}_{1P1P}^{L=1} = \frac{2}{3} + \frac{8}{3 \sqrt{x + 1}}, \]
\[ \hat{V}_{1P1P}^{L=2} = \frac{2}{3} + \frac{8 (5x^2 + 13x + 5)}{15(x + 1)^{5/2}}. \]
\[ \tilde{V}_{1DIS} = \frac{8}{15} + \frac{4(15x^2 + 20x + 8)}{15(1 + x)^{5/2}}, \quad (E11) \]
\[ \tilde{V}_{1SID} = 1 + \frac{4(8x^2 + 20x + 15)}{15(1 + x)^{5/2}}, \quad (E12) \]
\[ \tilde{V}_{2S1S} = \frac{5}{6} + \frac{2(6x^2 + 8x + 5)}{3(1 + x)^{5/2}}, \quad (E13) \]
\[ \tilde{V}_{1S2S} = 1 + \frac{2(5x^2 + 8x + 6)}{3(1 + x)^{5/2}}. \quad (E14) \]

2. States ccc and bbb

For baryons ccc and bbb, we also make use of Eqs. [E1], [E2], and [E3]. Formulas for ground states and one-orbitally excited states do not change. For identical quarks \( x = 3 \), and
\[ \tilde{V}_{1S1S} = 3, \quad (E15) \]
\[ \tilde{V}_\omega = \frac{5}{2}. \quad (E16) \]

Energies for states \( A, B, C \) and \( D \) need to be evaluated separately. We have
\[ \tilde{V}_A = \frac{11}{4}, \quad (E17) \]
\[ \tilde{V}_B = \frac{19}{8}, \quad (E18) \]
\[ \tilde{V}_C = \frac{23}{10}, \quad (E19) \]
\[ \tilde{V}_D = \frac{43}{20}. \quad (E20) \]

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