Motzkin numbers out of Random Domino Automaton

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Abstract

Motzkin numbers are derived from a special case of Random Domino Automaton - recently proposed toy model of earthquakes [1]. An exact solution of the set of equations describing stationary state of Random Domino Automaton in "inverse-power" case is presented. A link with Motzkin numbers allows to present explicit form of asymptotic behaviour of the automaton.

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It is known how to obtain Catalan numbers out of the bond directed percolation on a square lattice [2]. Here we derive Motzkin numbers [3–6] from the recently proposed Random Domino Automaton [1, 7, 8], which may be regarded as a toy model of earthquakes.

Random Domino Automaton comes from very simplified view of earthquakes. The space - one dimensional lattice - corresponds to boundaries of two tectonic plates moving with relative constant velocity. Due to irregularities of surfaces, relative motion can be locked at some places producing stress accumulation. Beyond some threshold of the stress, a relaxation took place. The size of relaxation depends on the nearby accumulated stress. Energy in the automaton is represented by balls added to the randomly chosen cell (each one is equally possible) with a constant rate - one ball in one time step. If the chosen cell is empty, it becomes occupied with probability $\nu$ or the ball is scattered with probability $(1 - \nu)$ leaving the state of the automaton unchanged. If the chosen place is already occupied, there are also two possibilities: the ball is scattered with probability $(1 - \mu)$ or with probability $\mu$ the incoming ball triggers a relaxation - balls from the chosen cell and all adjacent occupied cells are removed. An example of relaxation of size five is presented in the diagram below.

The stationary state of the system may be described by the distribution of clusters. The number of clusters of the length $i$, for $i = 1, 2, \ldots$, is denoted by $n_i$; the number of empty clusters of length 1 is denoted by $n_0$. Then the number of all clusters $n$ and and the density $\rho$ are

$$n = \sum_{i \geq 1} n_i, \quad \rho = \frac{1}{N} \sum_{i \geq 1} i n_i.$$  \hspace{1cm} (1)
The following set of equations is derived from the stationarity conditions (see [1])

\[ n_1 = \frac{1}{\mu_1 + 2} \left( (1 - \rho)N - 2n + n_1^0 \right), \]  
(2)

\[ n_2 = \frac{2}{2\mu_2 + 2} \left( 1 - \frac{n_1^0}{n} \right) n_1, \]  
(3)

\[ n_i = \frac{1}{\mu_i + 2} \times \]  
\[ \times \left( 2n_{i-1} \left( 1 - \frac{n_1^0}{n} \right) + n_1^0 \sum_{k=1}^{i-2} n_k n_{i-1-k} \right), \]  
(4)

for \( i \geq 3 \), where

\[ n_1^0 = \frac{2n}{\left( 3 + \frac{2\sum_{i \geq 1} \mu_i n_i}{\nu n} \right)}. \]

From the above set the balance equation for the total number of clusters \( n \) and for the density \( \rho \) can be derived

\[ (1 - \rho)N - 2n = \sum_{i \geq 1} \frac{\mu_i}{\nu} n_i, \]  
(5)

\[ \nu(1 - \rho) = \frac{1}{N} \left( \sum_{i \geq 1} \mu_i n_i^2 \right). \]  
(6)

In the case which refers to equal probability of triggering an avalanche for each cluster, the parameters are fixed as follows: \( \nu = \text{const} \) and \( \mu = \frac{\delta}{\nu} \), where \( \delta = \text{const} \). Equations (6) and (5) give \( \rho = (\theta + 1)^{-1} \), and \( n = N\theta[(\theta + 1)(\theta + 2)]^{-1} \), where \( \theta = \frac{\delta}{\nu} \in (0, \infty) \). Together with simple form of \( n_1^0 = 2n/(3 + 2\theta) \) it allows to reduce the set of equations (2)-(4) to the following recurrence:

\[ n_1 = \frac{N\theta}{(\theta + 1)(\theta + 2)^2} \left( \theta + \frac{2}{(2\theta + 3)} \right), \]  
(7)

\[ n_2 = \frac{2}{\theta + 2} \left( \frac{2\theta + 1}{2\theta + 3} \right) n_1, \]  
(8)

\[ n_{i+1} = \frac{2}{(\theta + 2)} \left( \frac{2\theta + 1}{2\theta + 3} \right) n_i + \]  
\[ + \frac{2}{N\theta} \left( \frac{\theta + 1}{2\theta + 3} \right) \sum_{k=1}^{i-1} n_k n_{i-k} \]  
(9)

for \( i \geq 2 \).

We define new variables \( c_i \) for \( i = 0, 1, \ldots \), by

\[ c_i = \frac{\beta}{\alpha^{i+1}} n_{i+1}, \]  
(10)
where

\[
\alpha = \frac{2}{(\theta + 2)} \left( \frac{2\theta + 1}{2\theta + 3} \right),
\]

\[
\beta = \frac{(\theta + 1)(\theta + 2)}{N\theta(2\theta + 1)}.
\]

Then, the equation (9) can be rewritten in the form

\[
c_{m+2} = c_{m+1} + \sum_{k=0}^{m} c_k c_{m-k},
\]

which is valid for \( m \geq 0 \) (\( m = i - 2 \)). Initial data \( c_0 \) and \( c_1 \) are easily obtained from equations (7)-(8), when it is transformed according the rule of equation (10), namely

\[
c_0 = c_1 = \frac{1 + \frac{3}{2}\theta + \theta^2}{1 + 4\theta + 4\theta^2}.
\]

The above equation (13) has the form of Motzkin numbers recurrence [3–6]. It is similar to the ubiquitous Catalan numbers recurrence (however the order of (13) is not 1 but is 2) and can be solved using generating functions technique [3]. For the limit case, when \( \theta = 0 \), we start with \( c_0 = c_1 = 1 \), and recurrence (13) produces Motzkin numbers: 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, . . . , etc. Thus, it is explicitly shown, how to obtain them from the Random Domino Automaton.

Below we present solution of the set (7)-(9). The generating function

\[
C(z) = \sum_{m \geq 0} c_m z^m
\]

for the recurrence (13) is equal to

\[
\frac{(1 - z) - \sqrt{1 - 2z + (1 - 4c_0)z^2 + 4(c_0 - c_1)z^3}}{2z^2}.
\]

For \( c_0 = c_1 \) it reduces to

\[
G(z) = \frac{(1 - z) - \sqrt{1 - 2z(1 + kz)}}{2z^2}
\]

where \( k = (2c_0 - \frac{1}{2}) \). From expansion of the formula (16) one can read explicitly the form of coefficients \( c_m \) being solutions of the recurrence (13). Using

\[
(1 + z)^a = \sum_{n \geq 0} \binom{a}{n} z^n,
\]

where \( a \) is a complex number. This formula is a generalization of the binomial theorem for negative exponents. The binomial theorem states that for any real or complex number \( a \) and any non-negative integer \( n \),

\[
\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}.
\]

Using this formula, we can express \( (1 + z)^a \) as a sum of terms involving \( z^n \), where \( n \) is a non-negative integer. This is particularly useful in combinatorics and probability theory, where generating functions are used to solve problems involving sequences of numbers.

The generating function for the sequence of Motzkin numbers is given by

\[
G(z) = \frac{(1 - z) - \sqrt{1 - 2z + (1 - 4c_0)z^2 + 4(c_0 - c_1)z^3}}{2z^2}.
\]

For \( c_0 = c_1 = 1 \), this reduces to

\[
G(z) = \frac{(1 - z) - \sqrt{1 - 2z}}{2z^2}.
\]

This generating function allows us to find the Motzkin numbers explicitly. The Motzkin numbers are a sequence of numbers that count the number ofMotzkin paths from \((0,0)\) to \((n,0)\) that do not cross below the \(x\)-axis. A Motzkin path is a sequence of steps \((1,1)\) and \((1,-1)\) in the \((x,y)\)-plane. The Motzkin numbers are given by

\[
M_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k}.
\]

where \( M_n \) is the \(n\)th Motzkin number. The Motzkin numbers have applications in various areas of mathematics, including combinatorics, computer science, and physics.
where
\[
\binom{a}{n} = \frac{a(a-1)(a-2)\ldots(a-n+1)}{n!},
\]
for \( m \geq 0 \) we have
\[
[z^m]g(f) = [z^{m+2}] \left( -\frac{1}{2} \sqrt{1 - 2z(1 + kz)} \right).
\]
Then it follows
\[
\sqrt{1 - 2z(1 + kz)} = \sum n \geq 0 \binom{n}{2}(\frac{1}{n})(-2)^n z^n(1 + kz)^n =
1 - z - kz^2 + \sum n \geq 2 \frac{-1}{n2^{n-2}} \binom{2n-3}{n-1} z^n(1 + kz)^n.
\]
For any \( m \geq 0 \) it gives
\[
c_m = [z^{m+2}] \left( \frac{kz^2}{2} + \frac{1}{2} \sum n \geq 2 \frac{1}{n2^{n-2}} \binom{2n-3}{n-1} z^n \sum j=0 \binom{n}{j} k^j z^j \right).
\]
Changing indices \( n + j = m + 2 \) for \( m \geq 1 \) we have
\[
c_m = \frac{1}{2} \sum j=0 \frac{(2c - 1/2)^j}{(m-j+2)2^{m-j}} \binom{2(m - j) + 1}{m-j+1} \binom{m-j+2}{j} \left( \frac{1}{3} \right)^i c_{i-1},
\]
Thus formula (10) gives explicit solution of equations (2)-(4) for the distribution \( n_i \)'s for any value of \( \theta \).

Using known asymptotics for Motzkin numbers (after Benoit Cloitre, see [5])
\[
c(i) \sim \sqrt{3/4/\pi} \frac{3^{i+1}}{i^{3/2}}
\]
we can present explicit formula for asymptotic behaviour for the \( n_i \) distribution. In the limit case \( \theta \rightarrow 0 \), while \( N\theta = const \),
\[
n_i = \frac{\alpha}{\beta} c_{i-1} \longrightarrow (N\theta)(\frac{1}{3})^i c_{i-1},
\]
and using (23) one obtains
\[
n_{i+1} \sim \frac{1}{i^{3/2}}.
\]
Thus we found one more example of inverse-power distribution with the power equal to \( \frac{3}{2} \).

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