LOCALIZATION PHENOMENA IN THE RANDOM XXZ SPIN CHAIN

ALEXANDER ELGART AND ABEL KLEIN

Abstract. It is shown that the infinite random Heisenberg XXZ spin-1/2 chain exhibits, with probability one, spectral, eigenstate, and weak dynamical localization in an arbitrary (but fixed) energy interval in a non-trivial parameters range. The crucial step in the argument is a proof that if the Green functions for the associated finite systems Hamiltonians exhibit certain (volume-dependent) decay properties in a fixed energy interval, then the infinite volume Green function decays in the same interval as well. The pertinent finite systems decay properties for the random XXZ model had been previously verified by the authors.

Contents

1. Introduction 1
2. Main result and technical steps 5
3. Basic features of the XXZ spin chain 9
4. Proof of Theorem 2.4 12
5. Proof of Theorem 2.6 22
Appendix A. Localization types and nomenclature 22
Appendix B. Exponential sums 24
Appendix C. Quasi-locality of the filter function 27
References 29

1. Introduction

1.1. The model and (informal) main result. A quantum model is associated with a Hamiltonian (self-adjoint operator) on the state space (Hilbert space). Primary goals include studying its spectrum, eigenstates, and associated dynamics.

The simplest non-trivial quantum system is a single spin-1/2, characterized by a two-dimensional complex state space $\mathbb{C}^2$, spanned by two orthonormal vectors called qubits: the spin-up $\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the spin-down $\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ states. The self-adjoint operators on this space are real linear combinations of the identity $\mathbb{1}_{\mathbb{C}^2}$ and the three Pauli matrices,

$$
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Spin chains are arrays of spins indexed by subsets $\Lambda \subset \mathbb{Z}$. If $\Lambda$ is finite, the corresponding state space is the tensor product Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{i \in \Lambda} \mathcal{H}_i$, where each $\mathcal{H}_i$ is a copy of $\mathbb{C}^2$. For infinite $\Lambda$, we let $\mathcal{H}_{\Lambda,0}$ be the vector subspace of $\bigotimes_{i \in \mathbb{Z}} \mathcal{H}_i$ spanned by tensor products of the form $\bigotimes_{i \in \mathbb{Z}} \varphi_i$, $\varphi_i \in \{\uparrow, \downarrow\}$, with a finite number of spin-downs.
equipped with the tensor product inner product, and let $H_{\Lambda}$ be its Hilbert space completion. A single spin operator $\sigma^z$ acting on the $i$-th spin is lifted to $H_{\Lambda}$ by identifying it with $\sigma^z \otimes 1_{\Lambda\setminus\{i\}}$, where $1_{\Lambda\setminus\{i\}}$ denotes the identity operator on $H_{\Lambda\setminus\{i\}}$, i.e., it acts non-trivially only in the tensor product’s $i$-th component. To stress the $i$-th dependence, we will denote such single spin operator by $\sigma^z_i$. More generally, if $S \subset \Lambda$, and $A_S$ is an operator on $H_S$, we often identify it with its natural embedding on $H_{\Lambda}$, namely with $A_S \otimes 1_{\Lambda\setminus S}$.

The original motivation to study quantum spin systems goes back to the 1920s when their usefulness in explaining ferromagnetism was realized by Lenz, Ising, Dirac, and Heisenberg, among others. They are now playing a role in explaining various phenomena in a plethora of physics, computer science, chemistry, and biology topics. The rich structure associated with these systems is related to their complexity: While a single spin has a very simple state space, the dimensionality of $n$ spins grows exponentially fast with $n$. On the flip side, even for modestly sized spin systems where $n$ ranges in dozens, the computational cost of their numerical analysis is prohibitive. This problem is colloquially known in physics as the curse of dimensionality and is the main cause of our very limited theoretical understanding of such models, especially of their thermodynamic limit $\Lambda \to \mathbb{Z}$.

In this work, we study spectral and dynamical properties of the random XXZ quantum spin-$\frac{1}{2}$ chain. The random Hamiltonian $H^\Lambda = H_\omega^\Lambda$ on $H_{\Lambda}$ is given by\footnote{Our definition of $H^\Lambda$ incorporates a choice of boundary condition if $\Lambda \neq \mathbb{Z}$.}

$$H^\Lambda = -\frac{1}{2\pi} \sum_{\{i,i+1\} \subset \Lambda} (\sigma^+_i \sigma^-_{i+1} + \sigma^-_i \sigma^+_i) + \sum_{i \in \Lambda} N_i - \sum_{\{i,i+1\} \subset \Lambda} N_i N_{i+1} + \lambda \sum_{i \in \Lambda} \omega_i N_i,$$

where $\sigma^\pm = \frac{1}{2} (\sigma^x \pm i \sigma^y)$ are called the ladder operators and $N = \frac{1}{2} (1_{\mathbb{C}^2} - \sigma^z)$ is known as the number operator. The constant $\Delta$ is the anisotropy parameter; we assume $\Delta > 1$ (the Ising phase). The parameter $\lambda > 0$ determines the strength of a random transversal field $\sum_{i \in \Lambda} \omega_i N_i$, where $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ is a family of random variables. Throughout this work we assume that $\{\omega_i\}_{i \in \mathbb{Z}}$ are independent, identically distributed random variables, whose common probability distribution $\mu$ is absolutely continuous with a bounded density and satisfies $\{0,1\} \subset \text{supp} \mu \subset [0,1]$.

$H^\Lambda$ is a well defined positive bounded self-adjoint operator for finite sets $\Lambda$. For infinite sets $H^\Lambda$ is understood as an unbounded positive self-adjoint operator on $H_{\Lambda}$, Alternatively, one can exploit the fact that $H^\Lambda$ commutes with the total number of particles operator $N^\Lambda = \sum_{i \in \Lambda} N_i$ to represent it as a direct sum of bounded Hamiltonians of systems with a fixed number $N$ of particles. The corresponding $N$-particles Hamiltonian $H_N^\Lambda$ is a random Schrödinger operator on a certain subgraph of $\Lambda^N$.

The free ($\lambda = 0$) XXZ system is a special variant of the famous Heisenberg model. For $\Lambda = \mathbb{Z}$, its spectrum can be determined using the method, introduced by Bethe in 1931, known as the Bethe ansatz. Its ground state energy 0 is separated by a gap of size $1 - \frac{1}{\Delta}$ from the rest of the spectrum, which is expected to be absolutely continuous due to the translation invariance of the underlying Hamiltonian. This feature has been verified for some energy intervals $[11,10]$.

Starting from the first decade of the new millennium, the randomized version of this operator ($\lambda > 0$) has been proposed as a prototypical model for study of the many-body localization (MBL) phenomenon in solid state physics. The initial investigations (of numerical and heuristic nature) in the physics community seemed to indicate that for large values of $\lambda$ a completely different system’s behavior emerges: The spectrum becomes pure point almost surely and the system’s dynamics changes drastically, with thermalization not occurring even in the asymptotic limit of infinite system size and evolution time.
Such behavior is called the **MBL phase**. While localization phenomenon for single-particle systems is well understood, and indeed persists in infinite volume systems for all times, it is an open question in physics whether the MBL phase does occur. It is also not clear what is its precise characterizations, see [23] for the recent review (other physics reviews on this topic include [20, 5, 1]).

For single-particle systems, one usually distinguishes three types of localization: Spectral, eigenstate, and dynamical localization. We will now briefly describe these forms of localization to put our results for the random XXZ model in that context; a more detailed description can be found in Appendix A. We will use the same nomenclature for the many-body case as well, but we warn the reader that single-particle localization and MBL describe different phenomena.

**Spectral localization** for a random operator $H_\omega$ in a prescribed energy interval $I$ manifests itself as pure point spectrum in $I$, almost surely. This type of localization is informative only for infinite systems, but it is necessary for formulation of the subsequent types of localization in such models.

**Eigenstate localization** is described at the level of the eigenvectors for $H_\omega$, and reflects their spatial confinement. A strong form of eigenstate localization is exponential decay of the eigencorrelator, see (2.20) below for its definition.

**Dynamical localization** is the non-spreading of initially localized wave packets (in the Schrödinger picture) or of local observables (in the Heisenberg picture), in the course of the time evolution generated by $H_\omega$. Eigenstate localization is typically not sufficient to guarantee dynamical localization. The decay of the eigencorrelator can be seen as a weak (in the operator-topological sense) form of dynamical localization. Since local observables for a single-particle system are either compact (in the discrete case) or relatively compact (in the continuum case), weak dynamical localization implies dynamical localization for such systems. As the result, for single-particle models the decay of the eigencorrelator is essentially synonymous with dynamical localization. This is no longer the case for many-body systems, where local observables are full rank operators.

The existing mathematical results for few-particles systems (e.g., [8, 3, 18, 17]) show that for sufficiently large parameters $\Delta$ and $\lambda$ the infinite volume Hamiltonian $H_N$ obtained by restricting the full Hamiltonian to the $N$ particle sector is spectrally, eigenstate, and weakly dynamically localized for all energies, provided $N \leq N_0(\Delta, \lambda) < \infty$. These methods could not be significantly improved by considering energies in a fixed interval $[0, E_0]$.

Our main result, stated informally below, shows that the infinite volume random XXZ model is spectrally, eigenstate, and weakly dynamically localized in a fixed energy interval $[0, E_0]$, uniformly in $N$, as long as $\lambda \Delta^2$ is sufficiently large. This regime is sometimes referred in physics as **zero temperature MBL**. We sketch both few-particles and zero temperature localization regimes in Figure 1. The precise mathematical formulation is given in Theorem 2.1.

**Theorem 1.1** (Informal formulation). Let $H^Z$ be the the random XXZ Hamiltonian on $\mathcal{H}_Z$ with parameters $\Delta > 1$ and $\lambda > 0$. Fix the energy interval $I(E_0) = [0, E_0]$, where $E_0 > 0$. Then, if $\lambda \Delta^2$ is sufficiently large, we have:

(i) Spectral and eigenstate localization in the interval $I(E_0)$: The spectrum of $H^Z$ in $I(E_0)$ is almost surely pure point, and the corresponding eigenvectors in $I(E_0)$ decay exponentially fast away from their localization centers (in a suitable sense).

(ii) Weak dynamical localization in the interval $I(E_0)$: The expectation of the absolute value of the matrix elements of $\chi_{I(E_0)}(H^Z)e^{itH^Z}$ decays exponentially fast (in a suitable sense), uniformly in $t \in \mathbb{R}$. 
A localization cartoon for the infinite volume XXZ model in strong disorder/weak interaction regimes. The blue region $A$ is the few-particles localization $N \in [0, N_0]$, the green region $B$ is the zero temperature localization $E \in [0, E_0]$ (our result). The total region of localization can be extended to include the pink sector $C$ using existing methods. The white region $D$ is currently not understood.

1.2. Relation with existing research results and open problems. As we already mentioned, despite intensive efforts in the condensed physics community in the past two decades, even the very existence of the MBL phase remains a point of debate in the physics literature. On the mathematical level, limited progress in understanding this phenomenon has been made, mostly related to the random XXZ model. As far as we are aware, Theorem 1.1 is the first result establishing localization properties for not exactly solvable infinite spin systems in this generality. That being said, from the physics perspective results of this kind constitute a clear indication of the MBL phase only if the energy intervals are allowed to grow with the system size.

Up to a few years ago, rigorous MBL-related results were restricted to the class of exactly solvable models (see the review [2]). More recently, for the random XXZ model, spectral and dynamical localization in the special energy interval $[1 - \frac{1}{5}, 2 (1 - \frac{1}{5})]$, called the droplet spectrum, was established in [7, 15]. These results led to the validation of other important many-body features associated with MBL in the droplet spectrum, among them the exponential clustering properties for associated eigenfunctions and non-propagation of information [14], and the area law for the entanglement entropy [6].

The finite volumes bound obtained in [12] for a fixed energy interval provides a critical input for the current paper, and were used to obtain dynamical (rather than weak dynamical) localization type results for finite systems in the same regime [13]. However, the bounds in [13] depend on the system’s volume, precluding any conclusions for infinite system. If the volume dependence could be suppressed, one could use these results in establishing the stability of the logarithmic light cone against generic local perturbations [24]. It would be interesting to see whether the estimates developed in this paper could yield significantly stronger version of the result established in [13], but it is an open question if this volume dependence can be completely removed using these methods.

Most of the MBL attributes would be achieved if one can show the existence of a quasi-local unitary $U$ such that with large probability $U^* H U$ is a diagonal Hamiltonian,
in the sense that it commutes with every $\sigma_i^z$ operator. The implication is that $U^* H U$ can be represented as a sum of weighted products $\prod_{x \in S} \sigma_i^z$ over subsets $X \subseteq \Lambda$ which are referred to as local integrals of motions (LIOM) in the physics literature. The LIOM representation was first proposed in [22] as a possible mechanism explaining MBL, and became a popular physics tool for the heuristic derivation of the majority of MBL features. In particular, the existence of such $U$ implies that one can construct an eigenbasis for $H$ consisting of vectors of the form $U \psi$, where $\psi$ is a product state.

For the Anderson model in the strong disorder regime, in any finite volume one can construct a (semi-uniformly) quasi-local $U$ that diagonalizes the Hamiltonian. (This follows from the results in [11].) The construction relies on eigenstate localization for all energies and the possibility to label the underlying eigenfunctions according to the spatial position of their localization centers. Such labeling is attainable for the Anderson model due to the fact that one can show that with large probability the spectrum of $H$ is level spaced for sufficiently regular distribution of the random potential. The methods used in the present work are not sufficient to establish LIOM localization for two reasons: (a) We can only prove localization in a fixed energy interval and not on the whole spectrum of $H$; and (b) It is not known whether the spectrum of $H$ is level spaced with large probability. It is not clear whether it is reasonable (even from the physics perspective) to expect that such $U$ exists in the first place.

The recent preprint [9] considers a weak deterministic perturbation of a diagonal (and thus exactly solvable) random model, namely the Ising model in a random longitudinal field, on intervals $\Lambda$ of size $\gamma^{-c}$, where $\gamma$ is the perturbation strength and $c > 0$ is a small but nonzero exponent. One of the main results announced there is the construction of a uniformly quasi-local unitary $U$ such that $U^* H U$ is a diagonal operator. The authors exploit it to extract a useful information about this spin chain consistent with MBL, namely that the spin transport is anomalous in this system. It will be interesting to consider the analogue of this scaling type result in the XXZ setting. Namely, one would want to consider $\lambda$ fixed and $\Delta$ large, and investigate a possible existence of a quasi-local $U$ that diagonalizes $H$ on scales $|\Lambda| \sim \Delta^c$.

From the technical point of view, the proof of Theorem 1.1 uses the finite volume results developed in [12] as the input to obtain results for infinite systems, suppressing the volume dependence in these estimates. In the next section we introduce the necessary technical notation, state the precise formulation of Theorem 1.1 and present the technical results that establish it.

2. Main result and technical steps

We fix $\Delta > 1$ and $\lambda > 0$. Let $|S|$ stand for the cardinality of $S \subseteq \mathbb{Z}$. Given $\Lambda \subseteq \mathbb{Z}$, we let $\mathcal{P}_f(\Lambda) = \{x \subset \Lambda, |x| < \infty\}$ be the collection of finite subsets of $\Lambda$, and let $\mathcal{P}_N(\Lambda) = \{x \subset \Lambda, |x| = N\}$ for $N \in \mathbb{N}^0$ be the subset of $\mathcal{P}_f(\Lambda)$ consisting of sets with cardinality $N$. We will use notation $\mathcal{P}_+(\Lambda)$ for the set $\mathcal{P}_f(\Lambda) \setminus \{\emptyset\}$.

Given $\Lambda \subseteq \mathbb{Z}$, let $H^\Lambda$ be the Hamiltonian given in (1.1), and consider the canonical orthonormal basis $\Phi_\Lambda = \{\phi_x\}_{x \in \mathcal{P}_f(\Lambda)}$ for $\mathcal{H}_\Lambda$, where

$$\phi_{\emptyset} = \phi_{\emptyset}^\Lambda = \otimes_{i \in \Lambda} \uparrow_i, \quad \phi_x = \phi_x^\Lambda = \left(\prod_{i \in x} \sigma_i^z\right) \phi_{\emptyset}^\Lambda \quad \text{for x } \in \mathcal{P}_+(\Lambda). \quad (2.1)$$

Note that $\phi_{\emptyset}$, the vacuum state, is an eigenvector for $H^\Lambda$ with the simple eigenvalue 0. (It is the ground state for $H^\Lambda$, as we shall see later.) Note also that $\phi_x = \phi_x^S \otimes \phi_{\emptyset}^{\Lambda \setminus S}$ for $x \subset S \subset \Lambda$. (We will suppress the $\Lambda$ dependence from $\phi_x$ for ease of notation when it is
clear from the context.) \(\Phi_\Lambda\) can be decomposed as the disjoint union

\[
\Phi_\Lambda = \bigcup_{N=0}^{\lfloor |\Lambda| \rfloor} \Phi_\Lambda^{(N)}, \quad \text{where} \quad \Phi_\Lambda^{(N)} = \{\phi_x, x \in \mathcal{P}_N(\Lambda)\}.
\]  

(2.2)

If \(x \in \mathcal{P}_N(\Lambda)\) with \(N \geq 1\), we identify \(x\) with \((x_1, \ldots, x_N) \in \Lambda^N\), where \(x_1 < x_2 < \ldots < x_N\), and set \(|x|_1 = \sum_{i=1}^{N} |x_i|\) and \(|x|_2 = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}\).

Given \(S \subset \mathbb{Z}\) finite, we set

\[
P_S^+ = \otimes_{i \in S} (I_i - N_i) \quad \text{and} \quad P_S^- = I_S - P_S^+, \quad \text{if} \ S \neq \emptyset,
\]

\[
P_+^{(\varnothing)} = I_S \quad \text{and} \quad P_-^{(\varnothing)} = I_S - P_+^{(\varnothing)} = 0.
\]

(2.3)

Note \(N_i = P_+^{(\{i\})}\) for all \(i \in \mathbb{Z}\).

For a unit vector \(\varphi \in \mathcal{H}_\Lambda\), we denote by \(\pi_\varphi\) the orthogonal projection onto \(\varphi\). If \(u \in \mathcal{P}_f(\Lambda)\), we write \(\pi_u = \pi_{\phi_u}\). We have

\[
\pi_u = P_+^{\Lambda_u} \Pi_{i \in \Lambda_u} N_i \quad \text{for all} \quad u \in \mathcal{P}_+(\Lambda),
\]

(2.4)

We denote by \(R_x^\Lambda = (H^\Lambda - z)^{-1}\) the resolvent operator, which is well defined for \(z \in \mathbb{C} \setminus \mathbb{R}\) (and for almost all \(z \in \mathbb{R}\) if \(\Lambda\) is finite), and use the Green function notation,

\[
G_x^\Lambda(x, y) = \langle \phi_x, R_x^\Lambda \phi_y \rangle \quad \text{for finite} \quad x, y \in \Lambda.
\]

(2.5)

More generally, for an operator \(T\) on \(\mathcal{H}_\Lambda\) we denote by \(T(x, y)\) its matrix elements with respect to the standard basis, i.e.,

\[
T(x, y) = \langle \phi_x, T \phi_y \rangle; \quad \text{note} \quad |T(x, y)| = \|\pi_x T \pi_y\|.
\]

(2.6)

In addition, we set

\[
\psi(x) = \langle \phi_x, \psi \rangle \quad \text{for all} \quad \psi \in \mathcal{H}_\Lambda \quad \text{and} \quad x \in \mathcal{P}_f(\Lambda).
\]

(2.7)

We let \(B(I)\) denote the collection of all bounded Borel measurable functions \(f\) supported on the interval \(I\), and set \(B_1(I) = \{f \in B(I), \sup_{t \in I} |f(t)| \leq 1\}\).

We equip \(\mathbb{Z}\) with the usual graph distance \(d_\varnothing(x, y) = |x - y|\) for \(x, y \in \mathbb{Z}\). We will consider \(\Lambda \subset \mathbb{Z}\) as a subgraph of \(\mathbb{Z}\), and denote by \(\dist(\cdot, \cdot)\) the graph distance in \(\Lambda\), which can be infinite if \(\Lambda\) is not a connected subset of \(\mathbb{Z}\). Given \(S \subset \Lambda \subset \mathbb{Z}\) and \(p \in \mathbb{N}^0 = \{0\} \cup \mathbb{N}\), we set

\[
[S]_p^\Lambda = \{x \in \Lambda : \dist(\Lambda, (x, S) \leq p)\},
\]

\[
c^\Lambda \varnothing S = \{x \in \Lambda : d_{\Lambda}(x, S) = 1\} = [S]_1^\Lambda \setminus S,
\]

(2.8)

\[
c^\Lambda \varnothing S = \{x \in \Lambda : \dist(\Lambda, (x, \Lambda \setminus S) = 1\},
\]

\[
c^\Lambda S = c^\Lambda \varnothing S \cup c^\Lambda \varnothing S.
\]

We also consider the Hausdorff distance between subsets of \(\Lambda\), given by

\[
d_H^I(U, V) = \max \left\{ \max_{u \in U} d_{\Lambda}(u, V), \max_{v \in V} d_{\Lambda}(v, U) \right\} \quad \text{for} \quad U, V \subset \Lambda,
\]

(2.9)

and observe that

\[
d_H^S(U, V) \geq d_H^I(U, V) \quad \text{if} \quad U, V \subset S \subset \Lambda.
\]

(2.10)

Due to the conservation of the total magnetization in the XXZ spin chain (see the next section), for any \(z \in \mathbb{C}\) and, more generally, for any bounded Borel measurable function \(f\), we have

\[
G_z^\Lambda(x, y) = f(H^\Lambda)(x, y) = 0 \quad \text{for all} \quad x, y \in \mathcal{P}_f(\Lambda) \text{ with } |x| \neq |y|.
\]

(2.11)
This justifies the introduction of a modified Hausdorff distance between finite subsets $x, y$ of $\Lambda$:

$$
\tilde{d}_H^2(x, y) = \begin{cases} 
    d_H^2(x, y) & \text{if } |x| = |y| \\
    \infty & \text{otherwise}
\end{cases}.
$$

(2.12)

We consider the following energy intervals, labeled by $t \in \mathbb{R}$, and defined by

$$
I_{\leq t} = (-\infty, (t + \frac{1}{\Delta}) (1 - \frac{1}{\lambda})) \cup \mathbb{R} \setminus \{z \in \mathbb{C} : \Re z \in I_{\leq t}\},
$$

$$
I_t = [(t - \frac{1}{\lambda}), (t + \frac{1}{\Delta}) (1 - \frac{1}{\Delta})].
$$

(2.13)

We will denote by $\mathbb{E}_\Lambda$ the expectation with respect to the random variables $\{\omega_i\}_{i \in \Lambda}$. In this paper we will use generic constants $C, c, \ldots$, whose values will be allowed to change from line to line, even in the same line in a displayed equation. These constants will, in general, depend on the fixed parameters of the model such as $\Delta, \lambda, \mu, \nu$, and on the fractional moment exponent $s$, but (critically) they will be volume-independent. We will not indicate the dependence on the fixed parameters and on $s$, but, when necessary, we will indicate the dependence of a constant on other parameters, say $q, N, \ldots$, explicitly by writing the constant as $C_q, C_{q, N}, \ldots$. If we write $C_q$, it does not depend on $N$. These constants can always be estimated from the arguments, but we will not track the changes to avoid complicating the arguments. We will use $C$ to indicate that the constant should be sufficiently large for a bound to hold, and $c$ to indicate that the constant should be sufficiently small, but still requiring $c > 0$. We generally use the same $C$ and $c$ for different constants in the same equation.

We are now ready to give the mathematically precise formulation of Theorem 1.1.

**Theorem 2.1.** Fix parameters $\Delta_0 > 1$ and $\lambda_0 > 0$. Given $q \in \frac{1}{2} \mathbb{N}_0$, there exists a constant $Y$ (which depends on $\Delta_0, \lambda_0, \mu,$ and $q$) such that, for all $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ satisfying $\Delta \lambda^2 \geq Y$ the following holds:

(i) $H^Z$ exhibits spectral and eigenstate localization in the interval $I_{\leq q}$, more precisely, there exists an event $E$, with $\mathbb{P}_Z(E) = 1$, such that for $\omega \in E$ the spectrum of $H^Z$ in $I_{\leq q}$ is pure point, and if $\psi = \psi_\omega$ is an eigenfunction of $H^Z$ with corresponding eigenvalue in $I_{\leq q}$, so $N^Z \psi = N_\psi \psi$, where $N_\psi \in \mathbb{N}_0$, it decays exponentially in the following sense:

$$
|\psi(y)| \leq C_{\omega, N_\psi} |x_\psi|^2 |x_\psi|^2 e^{-c_q \tilde{d}_H^2(y, x_\psi)} \quad \text{for all } y \in \mathcal{P}_+(\mathbb{Z}),
$$

(2.14)

where $x_\psi \in \mathcal{P}_{N_\psi}(\mathbb{Z})$ is a center of localization for $\psi$, that is, it satisfies

$$
|\psi(x_\psi)|^2 \geq \frac{(|x_\psi|^2 + 1)^{-1}-(N_\psi+1)}{\sum_{u \in \mathcal{P}_{N_\psi}(\mathbb{Z})} (|u|^2 + 1)^{-1}(N_\psi+1)}.
$$

(2.15)

(ii) $H^Z$ exhibits weak dynamical localization in the interval $I_{\leq q}$, more precisely,

$$
\mathbb{E}_Z \left\{ \sup_{f \in B_1(I_{\leq q})} |f(H^Z)(x, y)| \right\} \leq C_q e^{-c_q \tilde{d}_H^2(x, y)} \quad \text{for all } x, y \in \mathcal{P}_+(\mathbb{Z}).
$$

(2.16)

**Remark 2.2.** The result above is not vacuous as the spectrum $\sigma(H^Z) = \{0\} \cup [1 - \frac{1}{\Delta}, \infty)$ with probability one. (See, e.g., the discussion in [13]). Note also that $0$ is a simple eigenvalue.

The key input for proving Theorem 2.1 is an immediate corollary to [12, Theorem 2.4], which we now state.
Theorem 2.3. Fix parameters $\Delta_0 > 1$ and $\lambda_0 > 0$. Let $q \in \mathbb{Z} \setminus 0$ and $s \in (0, \frac{1}{3})$. Then there exists a constant $Y$ (which depends on $E_0$, $\lambda_0$, $\mu$, $q$, and $s$) such that, for all $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ satisfying $\lambda \Delta^2 > Y$, the following holds: For all finite $D \subset \mathbb{Z}$ we have

\[ \sup_{z \in \mathbb{H}_q} \mathbb{E}_D \{ |G^D_z(x, y)|^4 \} \leq C_q |D|^{C_q} e^{-c_q d^D_\alpha(x, y)} \text{ for all } x, y \in \mathcal{P}_+(D). \]  

(2.17)

Proof. We proved a slightly stronger result in [12, Theorem 2.4], where it is shown that under the hypotheses of the theorem there exists a constant $Y$ (which depends on $\Delta_0$, $\lambda_0$, $\mu$, $s$, and $q$) such that, for all $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ satisfying $\lambda \Delta^2 > Y$, for all $D \subset \mathbb{Z}$ finite we have

\[ \sup_{z \in \mathbb{H}_q} \mathbb{E}_D \{ \| P^A R^D_z P^B \|_s \} \leq C_q |D|^{C_q} e^{-c_q \text{dist}_D(A, \Lambda \cup B)}, \]  

(2.18)

for all $A \subset B \subset D$ with $A$ connected in $D$. ([12, Theorem 2.4] is stated and proved for real energies in the intervals $(-\infty, k + \frac{3}{4}]$, where $k \in \mathbb{N}$. The proof is also valid for complex energies $z$ with $\text{Re } z \leq (k + \frac{3}{4})(1 - \frac{1}{\lambda})$, with the same constants. The above result follows.)

Given $x, y \in \mathcal{P}_+(D)$ with $|x| = |y|$, and letting $r = d^D_H(x, y)$, then either $r = d^D_D(x, y)$ for some $x \in x$, or $r = d^D_D(y, x)$ for some $x \in y$. Both cases being similar, we introduce the former. In this case, using (2.6) and (2.4), we have

\[ |G^D_z(x, y)| = \| \pi_x R^D_z \pi_y \| \leq \| \mathcal{N} z R^D_z P^0_r \|_{\mathcal{P}_+(\Lambda \setminus x)} \leq \| d^D_D(x, y) \|. \]  

(2.19)

and hence (2.17) follows from (2.18) as $d^D_D(\{ x \}, \Lambda \setminus x \|_r) \geq r$. 

We now state our main technical result, Theorem 2.4 below. But first we need to introduce some additional notation and observations.

Let $S \subset \mathbb{Z}$. Given an energy interval $I$, we set $\sigma_I(H^S) = \sigma(H^S) \cap I$. If $\nu \in \mathbb{R}$, we set $\pi^S_\nu = \chi(\nu)(H^S)$, the spectral projection of $H^S$ on the set $\{ \nu \}$.

The eigencorrelator $Q_I^S$ for $H^S$ in the energy interval $I$ is given by

\[ Q_I^S(x, y) = \sum_{\nu \in \sigma_I(H^S)} |\pi^S_\nu(x, y)| \text{ for } x, y \in \mathcal{P}_I(S). \]  

(2.20)

If $S$ is finite, or, more generally, if $H^S$ has pure point spectrum in $I$, we have

\[ Q_I^S(x, y) = \sup_{f \in \mathcal{B}_I(I)} |f(H^S)(x, y)| \text{ for } x, y \in \mathcal{P}_I(S). \]  

(2.21)

We will write $\sigma_q(H^S) = \sigma_{I_q}(H^S)$ and $Q_q^S(u, v) = Q_{I_q}(u, v)$ for $q \in \frac{1}{2} \mathbb{N}^0$.

Theorem 2.4 (Finite volumes criterion). Fix $\Delta > 1$ and $\lambda > 0$. Let $s \in (0, \frac{1}{9})$ and $q \in \frac{1}{2} \mathbb{N}^0$. Suppose that for all finite $D \subset \mathbb{Z}$ we have

\[ \sup_{z \in \mathbb{H}_q} \mathbb{E}_D \{ |G^D_z(x, y)|^4 \} \leq C_q |D|^{C_q} e^{-c_q d^D_\alpha(x, y)} \text{ for all } x, y \in \mathcal{P}_+(D). \]  

(2.22)

Then for all $\Lambda \subset \mathbb{Z}$ we have

\[ \sup_{z \in \mathbb{H}_q} \mathbb{E}_\Lambda \{ |G^\Lambda_z(x, y)|^4 \} \leq C_q e^{-c_q d^\Lambda_\alpha(x, y)} \text{ for all } x, y \in \mathcal{P}_+(\Lambda). \]  

(2.23)

Furthermore, for all $D \subset \mathbb{Z}$ finite we have

\[ \mathbb{E}_D \{ Q^D_q(x, y) \} \leq C_q e^{-c_q d^D_\alpha(x, y)} \text{ for all } x, y \in \mathcal{P}_+(D). \]  

(2.24)

We only consider $x, y \in \mathcal{P}_+(\Lambda)$ because $G^2_z(\emptyset, \emptyset) = -\frac{1}{2}$ for $z \neq 0$ and and $G^\Lambda_z(\emptyset, x) = 0$ for $x \in \mathcal{P}_+(\Lambda)$. More generally, given a bounded Borel measurable function $f$, we have $f(H^\Lambda)(\emptyset, \emptyset) = f(0)$ and $f(H^\Lambda)(\emptyset, x) = 0$ for $x \in \mathcal{P}_+(\Lambda)$.
Remark 2.5. The input in the theorem, the estimate (2.22) (the finite volumes criterion), allows for volume dependence, whereas the output (2.23) completely suppresses this dependence. From the technical point of view, this is one of the delicate points in the analysis, and the suppression of the volume dependence is a crucial step in proving Theorem 2.4. In addition, the output of the theorem is also valid for infinite subsets $\Lambda$ of $\mathbb{Z}$.

The proof of Theorem 2.4 given in Section 4 proceeds by induction over $q \in \frac{1}{2}\mathbb{N}^0$, with constants $C_q$ and $c_q$ in (2.22) that deteriorate with $q$, rendering the method unpractical beyond fixed energy intervals. Similarly to the situation with random Schrödinger operators in dimension higher than one, it is not clear whether this restriction is a technical shortcoming or a feature (i.e., there is a phase transition for high energies for this model). There is no consensus among physicists whether such phase transition occurs or not in the infinite volume systems.

Theorem 2.1 follows immediately from Theorems 2.3, 2.4, and Theorem 2.6 below.

Theorem 2.6. Let $q \in \frac{1}{2}\mathbb{N}^0$, and suppose that for all $D \subset \mathbb{Z}$ finite we have

\[ E_D \{ Q^p_q (x, y) \} \leq C_q e^{-c_q d_H(x, y)} \quad \text{for all } x, y \in \mathcal{P}_+ (D). \quad (2.25) \]

Then

\[ E_{\mathbb{Z}} \left\{ \sup_{f \in B_1 (I \leq q)} |f (H^Z) (x, y)| \right\} \leq C_q e^{-c_q d_H(x, y)} \quad \text{for all } x, y \in \mathcal{P}_+ (\mathbb{Z}). \quad (2.26) \]

Moreover, there exists an event $\mathcal{E}$, with $\mathbb{P}_Z (\mathcal{E}) = 1$, such that for $\omega \in \mathcal{E}$ the spectrum of $H^Z$ in $I_{\leq q}$ is pure point, and if $\psi_\omega$ is an eigenfunction of $H^Z$ with corresponding eigenvalue in $I_q$, so $\psi \in \mathcal{H}_Z^{N_\psi}$ for some $N_\psi \in \mathbb{N}$, it decays exponentially in the following sense:

\[ |\psi (y)| \leq C_{\omega, N_\psi} |x_\psi|^{N_\psi + 1} e^{-\frac{c_q}{q} d_H (y, x_\psi)}, \quad (2.27) \]

where $x_\psi \in \mathcal{H}_Z^{(N_\psi)}$ is a center of localization for $\psi$, that is,

\[ |\psi (x_\psi)|^2 \geq \frac{(|x_\psi| + 1)^{-N_\psi + 1} \sum_{u \in \mathcal{H}_Z^{(N_\psi)}} (|u| + 1)^{-N_\psi + 1}}{(N_\psi + 1)}. \quad (2.28) \]

Theorem 2.6 is proven in Section 5, where it is derived from [3, Theorem 4.1].

The rest of this paper is organized as follows: In Section 3 we introduce notation and collect some basic properties of the XXZ spin chain that are used in our arguments. We prove Theorems 2.4 and 2.6 in Sections 4 and 5 respectively. In Appendix A we provide a more detailed discussion of localization types for single-particle and many-body systems. Appendix B provides bounds on exponential sums that will be encountered throughout the paper. In Appendix C we discuss useful properties of the so-called filter function that appears in the proof of Theorem 2.4.

3. Basic features of the XXZ spin chain

When working with a fixed $\Lambda \subset \mathbb{Z}$, we write $K^c = \Lambda \setminus K$ for $K \subset \Lambda$.

We note that $N_i$ is the projection onto the spin-down state (also called the local number operator) at site $i$. Given $S \subset \Lambda$, $N^S = \sum_{i \in S} N_i$ is the total (spin-down) number operator in $S$. The total number operator $N^\Lambda$ has eigenvalues $0, 1, 2, \ldots, |\Lambda|$. We set $\mathcal{H}_\Lambda^{(N)} = \text{Ran} \chi_{\{|N|\}} (N^\Lambda)$, obtaining the Hilbert space decomposition $\mathcal{H}_\Lambda = \bigoplus_{N=0}^{|\Lambda|} \mathcal{H}_\Lambda^{(N)}$. 
The operator $H^\Lambda$ is the sum of three operators,

$$H^\Lambda = -\frac{1}{2\Delta} \Delta^\Lambda + \mathcal{W}^\Lambda + \lambda V^\Lambda,$$

where

$$\Delta^\Lambda = \sum_{\{i,i+1\}\subset \Lambda} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+), \quad \mathcal{W}^\Lambda = \mathcal{N}^\Lambda - \sum_{\{i,i+1\}\subset \Lambda} N_i N_{i+1}, \quad V^\Lambda = \sum_{i\in \Lambda} \omega_i N_i.$$  \hspace{1cm} (3.1)

We note that the operators $\{N_i\}$ (and thus $\mathcal{W}^\Lambda$ and $V^\Lambda$) are diagonal in the canonical basis: $N_i \phi_x = \delta_{i,x} \phi_x$ for $i \in \mathbf{x}$ and 0 otherwise. $\mathcal{W}^\Lambda$ is the number of clusters operator: $\mathcal{W}^\Lambda \phi_x = \mathcal{W}^\Lambda N_i \phi_x$ for $x \subset \Lambda$ finite, where $\mathcal{W}^\Lambda$ is the number of clusters (connected components) of $x$ as a subset of $\Lambda$, so $\sigma(\mathcal{W}^\Lambda) \subset \{0, 1, 2, \ldots, |\Lambda|\}$. $V^\Lambda$ is the random potential:

$$V^\Lambda \phi_x = V^\Lambda(x) \phi_x \text{ for } x \subset \Lambda \text{ finite, where } V^\Lambda(x) = \left( \sum_{i \in x} \omega_i \right).$$ \hspace{1cm} (3.2)

An important feature of the XXZ Hamiltonian $H^\Lambda$ is total particle number (or magnetization) preservation: the operators $\Delta^\Lambda$ and the total number of particles operator $\mathcal{N}^\Lambda = \sum_{i\in\Lambda} N_i$ commute (all bounded functions of these operators commute), and hence $H^\Lambda$ and $\mathcal{N}^\Lambda$ also commute. If $\Lambda$ is finite, this is equivalent to

$$[H^\Lambda, \mathcal{N}^\Lambda] = -\frac{1}{2\Delta} [\Delta^\Lambda, \mathcal{N}^\Lambda] = 0.$$ \hspace{1cm} (3.3)

It can be verified (e.g., [12]) that

$$-2\mathcal{W}^\Lambda \leq -\Delta^\Lambda \leq 2\mathcal{W}^\Lambda.$$ \hspace{1cm} (3.4)

Since $\lambda \geq 0$, and $V^\Lambda \geq 0$ by our assumption on the random variables, it follows that

$$H^\Lambda \geq \left(1 - \frac{1}{\Delta}\right) \mathcal{W}^\Lambda,$$ \hspace{1cm} (3.5)

and, as a consequence, the spectrum of $H^\Lambda$ is of the form

$$\sigma(H^\Lambda) = \{0\} \cup \left(\left[1 - \frac{1}{\Delta}, \infty\right) \cap \sigma(H^\Lambda)\right).$$ \hspace{1cm} (3.6)

We will denote $d_1(x, y)$ the $\ell^1$ distance between $x$ and $y$ in $\mathbb{Z}^N$, that is,

$$d_1(x, y) = |x - y|_1 = \sum_{i=1}^N |x_i - y_i|.$$ \hspace{1cm} (3.7)

In view of (2.11), we also introduce a modified $\ell^1$ distance between subsets of $\mathbb{Z}$:

$$\tilde{d}_1(x, y) = \begin{cases} d_1(x, y) & \text{if } |x| = |y| \\ \infty & \text{otherwise}. \end{cases}$$ \hspace{1cm} (3.8)

An important property of Green functions is the Combes-Thomas bound [15, Proposition 4.1],

$$\sup_{x\in \mathbb{Z}_m} |G^\Lambda_z(x, y)| \leq C_m e^{-c_m \tilde{d}_1(x, y)} \text{ for } m < \frac{3}{4} \text{ and } x, y \in \mathcal{P}_+(\Lambda).$$ \hspace{1cm} (3.9)

**Remark 3.1.** [15, Proposition 4.1] gives a Combes-Thomas bound for $H^\Lambda$ with constants independent of $N$ and $\Lambda$, so combined with the definition (3.8) it clearly implies (3.3).

As mentioned in Section 1, $H^\Lambda_N$ is a random Schrödinger operator on a certain subgraph of $\Lambda^N$, but the standard Combes-Thomas bounds will only yield (3.9) with $N$ dependent constants. Although [15, Proposition 4.1] is stated and proven on each $\mathcal{H}^\Lambda_N$, the proof uses the special structure of $H^\Lambda_N$ to obtain constants independent of $N$. 


Unfortunately, we cannot use the Combes-Thomas bound directly for energies lying in $H_m$ for $m \geq \frac{3}{4}$. The way around is to lift the spectrum of the operator $H^\Lambda$, and note that the Combes-Thomas bound holds for the lifted operator.

Given $m \in \mathbb{N}^0$, we set $Q_m^\Lambda = \chi_m(Q_m^\Lambda)$, the orthogonal projection onto configurations $x$ with exactly $m$ clusters, and let $Q_B^\Lambda = \chi_B(Q_B^\Lambda) = \sum_{m \in B} Q_m^\Lambda$ for $B \subset \mathbb{N}^0$. For $k \in \mathbb{N}$, we set

$$Q_{\leq k}^\Lambda = Q_{\{1,2,\ldots,k\}}^\Lambda = \sum_{m=1}^k Q_m^\Lambda$$

and recall that [12, Lemma 3.5]

$$\|Q_{\leq k}^\Lambda\|_{HS} \leq \sqrt{k}|\Lambda|^k,$$

$$\text{tr} \chi_\ell(k(H^\Lambda)) \leq k|\Lambda|^{2k} + 1.$$  

(3.11)  

(3.12)

Given $q \in \frac{1}{2}\mathbb{N}^0$, we set

$$\hat{H}_q^\Lambda = \begin{cases} H^\Lambda + (1 - \frac{1}{\Lambda}) Q_0^\Lambda & \text{if } q = 0, \frac{1}{2} \\ H^\Lambda + |q| (1 - \frac{1}{\Lambda}) Q_{\leq |q|}^\Lambda & \text{otherwise} \end{cases}$$

(3.13)

The basic feature of these operators is that we have

$$\hat{H}_q^\Lambda \geq (1 - \frac{1}{\Lambda})$$

for $q = 0, \frac{1}{2}$ and $\hat{H}_q^\Lambda \geq (|q| + 1) (1 - \frac{1}{\Lambda})$ otherwise,

$$\hat{H}_q^\Lambda - E \geq \frac{1}{4} (1 - \frac{1}{\Lambda})$$

for all $E \in \ell_q$. 

(3.14)

Given $z \notin \sigma(\hat{H}_q^\Lambda)$, we set

$$\hat{R}_{q,z}^\Lambda = (\hat{H}_q^\Lambda - z)^{-1}$$

and

$$\hat{G}_{q,z}^\Lambda(x,y) = \langle \phi_x, \hat{R}_{q,z}^\Lambda \phi_y \rangle.$$  

(3.15)

The Combes-Thomas bound of [15, Proposition 4.1] then holds for the modified Green functions: For all $q \in \frac{1}{2}\mathbb{N}^0$ we have

$$\sup_{z \in \mathbb{H}_q} \left| \hat{G}_{q,z}^\Lambda(x,y) \right| \leq C_q e^{-c_q \bar{d}(x,y)}$$

for all $x,y \in P_+(\Lambda)$. 

(3.16)

We observe that, given $S \subset \mathbb{Z}$, $q \in \frac{1}{2}\mathbb{N}^0$, and $\nu \in \sigma_q(H^S)$, it follows from (3.13) that

$$\pi_\nu^S = [q] \left(1 - \frac{1}{\Lambda}\right) \hat{R}_{q,\nu}^S \hat{S} \hat{S} \pi_\nu^S = [q]^2 \left(1 - \frac{1}{\Lambda}\right)^2 \hat{R}_{q,\nu}^S \hat{S} \pi_\nu^S \hat{R}_{q,\nu}^S \hat{S} \pi_\nu^S.$$  

(3.17)

Since our arguments rely on a certain decoupling idea, we need to introduce yet another variety of Green functions. Given $K \subset \Lambda$, we consider the operator $H^K = H^\Lambda \otimes 1_{H^K}$ acting on $H_\Lambda$. Then the decoupled Hamiltonian and resolvent on $H_\Lambda$ are given by

$$H^{K,K^c} = H^K + H^{K^c}, \quad R_{z}^{K,K^c} = (H^{K,K^c} - z)^{-1}, \quad \Gamma^K = H^\Lambda - H^{K,K^c}.$$  

(3.18)

The corresponding decoupled Green function is then

$$G_z^{K,K^c}(x,y) = \langle \phi_x, R_{z}^{K,K^c} \phi_y \rangle.$$  

(3.19)

The corresponding modified Hamiltonian is $\hat{H}_m^{K,K^c} = \hat{H}_m^K + \hat{H}_m^{K^c}$ acting on $H_\Lambda$, and the modified Green function,

$$\hat{G}_{m,z}^{K,K^c}(x,y) = \langle \phi_x, \hat{R}_{m,z}^{K,K^c} \phi_y \rangle,$$  

(3.20)

satisfies (3.16) as well. We note that $G_z^{K,K^c}(x,y)$ (and its modified analogue) vanishes unless $|x \cap K| = |y \cap K|$ and $|x \cap K^c| = |y \cap K^c|$.
The proof of Theorem 2.4 will be facilitated by the following a-priori estimate (see, e.g., [12, Lemma 3.4]):

\[
\mathbb{E}_{[i,j]} \left( \left\| T_i N_i R_i z T_j \right\|_2^s \right) \leq C_s \lambda^{-s'} \left\| T_i \right\|_2^{s'} \left\| T_j \right\|_2^{s'} \quad \text{for all } z \in \mathbb{C} \text{ and } s' \in (0,1),
\]

where \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm, which implies that

\[
\sup_{z \in \mathbb{C}} \mathbb{E}_A \left\{ |G_z^A(x, y)|^{s'} \right\} \leq C_s' \quad \text{for all } x, y \in \mathcal{P}_+(\Lambda) \text{ and } s' \in (0,1).
\]

We note that for real valued \( z \), \( G_z^A(x, y) \) is understood here and below as \( G_{z+|0|}^A(x, y) \). If \( x \subset \Lambda \) and \( S \subset \Lambda \), we write \( s_S = x \cap S \). If \( P \) is an orthogonal projection, we write \( \tilde{P} = 1 - P \).

4. PROOF OF THEOREM 2.4

In this Section we prove Theorem 2.4. We fix \( \Delta > 1 \), \( \lambda > 0 \), and \( s \in (0, \frac{1}{3}) \).

We will use the following lemma. For \( k \in \{1, 2, \ldots, N\} \) we let

\[
\mathcal{P}_{N,k}(\Lambda) = \{ x \in \mathcal{P}_N(\Lambda), 1 \leq W^A_x \leq k \} = \{ x \in \mathcal{P}_N(\Lambda), \phi_x \in \text{Ran} Q^A_{\leq k} \}.
\]

Lemma 4.1. Let \( q \in \frac{1}{2}\mathbb{N} \), \( 1 \leq q \), and \( N \in \mathbb{N} \). Fix \( \Lambda \subset \mathbb{Z} \), and and suppose (2.23) holds for all \( x, y \in \mathcal{P}_{N, |q|}(\Lambda) \). Then (2.23) holds for all \( x, y \in \mathcal{P}_N(\Lambda) \) (with different constants, independent of \( \Lambda \) and \( N \)).

Proof. We use the following resolvent identity:

\[
R^A_x = \hat{R}^A_{q,z} + |q| (1 - \frac{1}{N}) \hat{R}^A_{q,z} \beta_{\leq |q|} \hat{R}^A_{q,z} = \hat{R}^A_{q,z} + |q| (1 - \frac{1}{N}) \hat{R}^A_{q,z} \beta_{\leq |q|} \hat{R}^A_{q,z}.
\]

Using \[
R^A_x = \hat{R}^A_{q,z} + |q| (1 - \frac{1}{N}) \hat{R}^A_{q,z} \beta_{\leq |q|} \hat{R}^A_{q,z} + |q|^2 (1 - \frac{1}{N})^2 \hat{R}^A_{q,z} \beta_{\leq |q|} R^A_{q,z} \beta_{\leq |q|} \hat{R}^A_{q,z},
\]

Suppose now that (2.23) holds for all \( u, v \in \mathcal{P}_{N, |q|}(\Lambda) \). Then, using also (4.3) and (3.10), we can bound

\[
\sup_{z \in \mathbb{H}_q} \sup_{\Lambda \subset \mathbb{Z}} \mathbb{E}_{\Lambda} \left\{ |G_z^A(x, y)|^{s'} \right\} \leq C_q e^{-c_q |x-y|_1} + C_q \sum_{u \in \mathcal{P}_{N, |q|}(\Lambda)} e^{-c_q |x-u|_1} e^{-c_q |u-y|_1}
\]

\[
+ C_q \sum_{u, v \in \mathcal{P}_{N, |q|}(\Lambda)} e^{-c_q |x-u|_1} e^{-c_q |u-v|_1} e^{-c_q |v-y|_1} \leq C_q e^{-c_q d_{\mathbb{H}}^q(x, y)},
\]

where in the last step we used properties of exponential sums, see [13.2] below. \( \square \)

Proof of Theorem 2.4. We take \( q \in \frac{1}{2}\mathbb{N}^0 \), and assume that (2.22) holds for all finite \( D \subset \mathbb{Z} \). Given \( \Lambda \subset \mathbb{Z} \), in view of (2.11) we only have to prove (2.23) for \( x, y \in \mathcal{P}_+(\Lambda) \) with \( |x| = |y| \).

The proof will proceed by induction on \( q \in \frac{1}{2}\mathbb{N}^0 \). For \( q = 0, \frac{1}{2} \), the theorem (i.e., (2.23)) follows from the Combes Thomas bound (3.9). Given \( q \in \frac{1}{2}\mathbb{N}^0 \), \( q \geq 1 \), we assume the theorem holds for \( q - \frac{1}{2} \), and will prove it then holds for \( q \).

The proof proceeds by a series of Lemmas. In view of Lemma 4.1, it suffices to prove (2.23) for all \( x, y \in \mathcal{P}_{N, |q|}(\Lambda) \).

Lemma 4.2. Let \( D \subset \mathbb{Z} \) be finite, let \( N \in \mathbb{N} \), and assume

\[
\sup_{z \in \mathbb{H}_q} \mathbb{E}_D \left\{ |G_z^D(x, y)|^{s'} \right\} \leq C_q e^{-c_q d_{\mathbb{H}}^D(x, y)} \quad \text{for all } x, y \in \mathcal{P}_{N, |q|}(D).
\]

Then

\[
\mathbb{E}_D \{ Q^D_q(x, y) \} \leq C_q e^{-c_q d_{\mathbb{H}}^D(x, y)} \quad \text{for all } x, y \in \mathcal{P}_N(D).
\]
Proof. Let \( D \subset \mathbb{Z} \) finite, \( N \in \mathbb{N} \), and \( x, y \in \mathcal{P}_N \). We assume that there is \( x \in x \) such that
\[
\text{dist}_D(x, y) = d_H^D(x, y),
\]
(4.7)
the other case being similar.

We first prove the lemma for \( x, y \in \mathcal{P}_{N, \{q\}}(D) \). This is done using the reduction to resolvents achieved by using the estimate \([4, \text{Eq. (7.44)}]\) and the spectral averaging as in \([3, \text{Theorem 4.5}]\). The final result can be re-formulated in our setting as:

Let \( r \in (0, 1), N \in \mathbb{N}, \) and let \( I \subset \mathbb{R} \) be an interval. Then for all finite \( D \subset \mathbb{Z} \) and \( x, y \in \mathcal{P}_N(D) \) we have
\[
\mathbb{E}_D \left\{ Q_D^P(x, y) \right\} \leq C_r \sum_{u \in \mathcal{P}_N(D) : x \in u} \int_I \mathbb{E}_D \left\{ \left| G_E^D(u, y) \right|^r \right\} dE \quad \text{for any } x \in x.
\]
(4.8)

Note that that \( d_H^P(u, y) \geq d_H^D(x, y) \) if \( x \in u \subset D \).

Given \( x, y \in \mathcal{P}_{N, \{q\}} \), we estimate \( \mathbb{E}_D \left\{ Q_D^P(x, y) \right\} \) by \([4.8]\), and estimate the term \( \mathbb{E}_D \left\{ \left| G_E^D(u, y) \right|^r \right\} \) inside the integral as in \([4.4]\), using \([4.5]\), getting
\[
\mathbb{E}_D \left\{ Q_D^P(x, y) \right\} \leq C_q \sum_{u \in \mathcal{P}_N(D) : x \in u} e^{-c_q|u-y|_1} + C_q \sum_{u \in \mathcal{P}_N(D) : x \in u} \sum_{v \in \mathcal{P}_{N, \{q\}}(D)} e^{-c_q|u-v|_1} e^{-c_q|v-y|_1}
\]
\[
+ C_q \sum_{v \in \mathcal{P}_{N, \{q\}}(D)} \sum_{v \in \mathcal{P}_{N, \{q\}}(D)} e^{-c_q|u-v|_1} e^{-c_qd_H^P(v, w)} e^{-c_q|w-y|_1}.
\]
(4.9)

To bound the first sum, we note that
\[
|u - y|_1 \geq d_H^D(u, y) \geq \text{dist}_D(x, y) = d_H^D(x, y),
\]
(4.10)
using \([4.7]\). Hence
\[
\sum_{u \in \mathcal{P}_N(D) : x \in u} e^{-c_q|u-y|_1} \leq e^{-c_qd_H^D(x, y)} \sum_{u \in \mathcal{P}_N(D)} e^{-c_q|u-y|_1} \leq C_q e^{-c_qd_H^D(x, y)},
\]
(4.11)
where in the last step we used \([B.3]\) and \( y \in \mathcal{P}_{N, \{q\}}(D) \).

To estimate the second sum in \([4.9]\), we use the triangle inequality to conclude that
\[
|u - v|_1 + |v - y|_1 \geq |u - y|_1 \geq d_H^D(x, y),
\]
\[
|u - v|_1 + |v - y|_1 \geq \frac{1}{2} (|u - y|_1 + |v - y|_1).
\]
(4.12)

Hence
\[
\sum_{u \in \mathcal{P}_N(D) : x \in u} \sum_{v \in \mathcal{P}_{N, \{q\}}(D)} e^{-c_q|u-v|_1} e^{-c_q|v-y|_1}
\]
\[
\leq e^{-c_qd_H^D(x, y)} \sum_{u \in \mathcal{P}_N(D)} \sum_{v \in \mathcal{P}_{N, \{q\}}(D)} e^{-c_q|u-v|_1} e^{-c_q|v-y|_1} \leq C_q e^{-c_qd_H^D(x, y)}
\]
(4.13)
using \( y \in \mathcal{P}_{N, \{q\}}(D) \) and \([B.3]\) twice in the last step.

Finally, to estimate the last sum in \([4.9]\), we use the triangle inequality and \([4.10]\) to conclude that
\[
|u - v|_1 + d_H^D(v, w) + |w - y|_1 \geq d_H^D(x, y),
\]
\[
|u - v|_1 + d_H^D(v, w) + |w - y|_1 \geq |u - v|_1 + \frac{1}{2} (d_H^D(v, y) + |w - y|_1).
\]
(4.14)
Thus, for any $N$ estimate (recall (3.2)) gives

$$\sum_{u \in \mathcal{P}_N(D): x \in u} \sum_{v, w \in \mathcal{P}_{N,[q]}(D)} e^{-c_q |u-v|_1} e^{-c_q d_H^2(v,w)} e^{-c_q |w-y|_1} \leq e^{-\frac{c_q}{4} d_H^2(x,y)} \sum_{u \in \mathcal{P}_N(D)} \sum_{v, w \in \mathcal{P}_{N,[q]}(D)} e^{-c_q |u-v|_1} e^{-\frac{c_q}{4} d_H^2(v,w)} e^{-\frac{c_q}{4} |w-y|_1},$$

(4.15)

where we used (4.22), and (4.15), we get

$$\mathbb{E}_D \{ Q_1^D(x, y) \} \leq C_q N^{2|q|} e^{-c_q d_H^2(x,y)}$$

for all $x, y \in \mathcal{P}_{N,[q]}(D)$.

To remove the $N$ dependence in (4.10), we will show

$$\mathbb{E}_D \{ Q_1^D(x, y) \} \leq C_q e^{-c_q N}$$

for all $x, y \in \mathcal{P}_{N,[q]}(D)$,

(4.17)

using a large deviation estimate.

Let $\bar{\mu} = \mathbb{E} \{ \omega_0 \}$, and assume $N \lambda \bar{\mu} > 2|q| \left( 1 - \frac{1}{\Delta} \right)$.

Then the standard large deviations estimate (recall (3.2)) gives

$$\mathbb{P} \{ \lambda \omega_0(u) < |q| \left( 1 - \frac{1}{\Delta} \right) \} \leq \mathbb{P} \{ \lambda \omega_0(u) < N \frac{\bar{\mu}}{2} \} \leq e^{-c_q N}$$

for all $u \in \mathcal{P}_{N,[q]}(Z)$. (4.18)

Thus, for any $N \in \mathbb{N}$, letting $S = [x]_N^D$, and defining the event

$$\mathcal{E}_N^S = \{ \lambda \omega_0(u) < |q| \left( 1 - \frac{1}{\Delta} \right) \}$$

we have

$$\mathbb{P} \{ \mathcal{E}_N^S \} \leq C_{\mu,q} \mathbb{P} \{ \mathcal{P}_{N,[q]}(S) \} e^{-c_q N} \leq C_{\mu,q}|q| \left( N(2N + 1) \right)^{2|q|} e^{-c_q N} \leq C_{\mu,q} e^{-c_{\mu,q} N},$$

(4.20)

where we used $\mathbb{P} \{ \mathcal{P}_{N,[q]}(S) \} = \text{tr} Q_{\omega_0}^S$, (3.11), and $|S| \leq N(2N + 1)$. Moreover, on the complimentary event $(\mathcal{E}_N^S)^c$ we have

$$\chi_S(N^S) H^S \geq \left( |q| + 1 \right) \left( 1 - \frac{1}{\Delta} \right) \chi_N(N^S),$$

(4.21)

so we can use (1.12) Proposition 4.1 to obtain the Combes-Thomas bound

$$\sup_{z \in \mathcal{H}^2} |G^S(z, y)| \leq C_q e^{-c_q |x-y|_1}$$

for all $x, y \in \mathcal{P}_{N,[q]}(S)$. (4.22)

To show (4.17), we start by observing that for $\nu \in \sigma_q(H^D)$ we have

$$\pi_{\nu} \phi_x = \pi_{\nu} \left( H^{S,S^c} - \nu \right) R_{\nu}^{S,S^c} \phi_x = \pi_{\nu} \Gamma^S R_{\nu}^{S,S^c} \phi_x.$$ 

(4.23)

By the construction of $S$ we have

$$R_{\nu}^{S,S^c} \phi_x = P_{\nu}^{S,S^c} R_{\nu}^{S} \phi_x.$$ 

(4.24)

It follows that on the complimentary event $(\mathcal{E}_N^S)^c$ we have

$$\sum_{\nu \in \sigma_q(H^D)} |\langle \phi_y, \pi_{\nu} \phi_x \rangle| \leq \sum_{\nu \in \sigma_q(H^D)} \sum_{u \in \mathcal{P}_N(D)} |\langle \phi_y, \pi_{\nu} \phi_u \rangle| |\langle \phi_u, \Gamma^S R_{\nu}^{S} \phi_x \rangle|$$

(4.25)

where we used (4.22), and

$$\mathcal{P}_{N}^{\hat{c}_S}(D) = \mathcal{P}_N(D, \hat{c}_S^x D) = \{ u \in \mathcal{P}_N(D) : u \cap \hat{c}_S^x D \neq \emptyset \}.$$ 

(4.26)
We next observe that by the Cauchy-Schwarz inequality,
\[
\left( \sum_{\nu \in \sigma_q(H^D)} |\langle \phi_y, \pi_\nu \phi_u \rangle| \right)^2 \leq \sum_{\nu \in \sigma_q(H^D)} \langle \phi_y, \pi_\nu \phi_y \rangle \sum_{\nu \in \sigma_q(H^D)} \langle \phi_u, \pi_\nu \phi_u \rangle
\]
\[
= \langle \phi_y, \chi_{I_q}(H^D) \phi_y \rangle \langle \phi_u, \chi_{I_q}(H^D) \phi_u \rangle \leq 1.
\]
Plugging this into (4.25), we see that
\[
\sum_{\nu \in \sigma_q(H^D)} |\langle \phi_y, \pi_\nu \phi_x \rangle| \leq C_q \sum_{u \in \Psi_N} e^{-c_q|x-u|_1}.
\]
(4.28)
Since for any \( u \in \mathcal{P}^R_N(D) \) we have \(|u - x|_1 \geq N \), we deduce from (4.28) and (3.3) (recall \( y \in \mathcal{P}_{N,q}(D) \)), that on \( E_N^y \) we have
\[
Q_q^D(x, y) = \sum_{\nu \in \sigma_q(H^D)} |\langle \phi_y, \pi_\nu \phi_x \rangle| \leq C_q e^{-c_q N}.
\]
(4.29)
Hence
\[
E_D \{ Q_q^D(x, y) \} = E_D \left\{ \chi_{E_N^y} Q_q^D(x, y) \right\} + E_D \left\{ \chi_{(E_N^y)^c} Q_q^D(x, y) \right\}
\]
\[
\leq \frac{1}{P(E_N^y)} (E_N^y) + C_q e^{-c_q N} \leq C_q e^{-c_q N},
\]
(4.30)
where we have used (4.20) and (4.30). The relation (4.17) follows.

Using (4.16) for \( d_H^2(x, y) \geq N \) and (4.17) for \( d_H^2(x, y) < N \) we get
\[
E_D \{ Q_q^D(x, y) \} \leq C_q e^{-c_q d_H^2(x, y)} \text{ for all } x, y \in \mathcal{P}_{N,q}(D).
\]
(4.31)
We now consider the general case \( x, y \in \mathcal{P}_N(D) \). For any \( \nu \in \sigma_{q}[1](D) \), using (3.17), we have
\[
\pi_\nu^D(x, y) = C_q \sum_{u, v \in \mathcal{P}_{N,q}(D)} \hat{G}_{[q], \nu}(x, u) \pi_\nu^D(u, v) \hat{G}_{[q], \nu}(v, y)
\]
(4.32)
It follows that
\[
Q^D(x, y) \leq \sum_{u, v \in \mathcal{P}_{N,q}(D)} \left| \hat{G}_{[q], \nu}(x, u) \right| Q^D(u, v) \left| \hat{G}_{[q], \nu}(v, y) \right|
\]
\[
\leq C_q \sum_{u, v \in \mathcal{P}_{N,q}(D)} e^{-c_q|x-u|_1} Q^D(u, v) e^{-c_q|v-y|_1},
\]
(4.33)
where we used (3.16). Using (3.31), we conclude that
\[
E_D \{ Q^D(x, y) \} \leq C_q N^2 |q| \sum_{u, v \in \mathcal{P}_{N,q}(D)} e^{-c_q|x-u|_1} e^{-c_q d_H^2(u, v)} e^{-c_q|v-y|_1}
\]
\[
\leq C_q e^{-c_q d_H^2(x, y)} \sum_{u, v \in \mathcal{P}_{N,q}(D)} e^{-c_q|x-u|_1} e^{-c_q|v-y|_1},
\]
(4.34)
where we used (3.2) twice. The Lemma is proven. \( \square \)

**Lemma 4.3.** Let \( q \in \frac{1}{2} \mathbb{N}, 1 \leq q, \) and assume the induction hypothesis, that is, Theorem 2.4 is proven for \( q - \frac{1}{2} \). Let \( \Lambda \subset \mathbb{Z} \) and \( N \in \mathbb{N} \). Then for all \( x, y \in \mathcal{P}_{N,q} \) and any finite connected set \( D \subset \Lambda \) satisfying \( (x \cup y)_D \neq \emptyset \) and \( (x \cup y)_{D^c} \neq \emptyset \) we have
\[
\sup_{z \in \mathbb{H}_q} \mathbb{E}_\Lambda \left\{ \left| G_{z, D^c}^D(x, y) \right|^* \right\} \leq C_q |D|^{2|q|} e^{-c_q - \frac{1}{2} d_H^2(x, y)} \text{ for all } x, y \in \mathcal{P}_{N,q}(D).
\]
(4.35)
Proof. Let $D \subset \Lambda$ be finite and connected, and let $x, y \in \mathcal{P}_{N, q}(\Lambda)$, with $(x \cup y)_D \neq \emptyset$ and $(x \cup y)_{D^e} \neq \emptyset$. We only need to consider the case $1 \leq |x_D| = |y_D| \leq N - 1$, as otherwise $G^{D,D^e}_z(x, y) = 0$.

To do so, given $a \in \mathbb{R}$, let $F_a$ be the analytic function on $\mathbb{R}$ given by

$$
F_{\xi,a}(x) = \frac{1 - e^{-\xi x^2}}{x - ia} \quad \text{for } x \in \mathbb{R} \text{ if } a \neq 0,
$$

$$
F_{\xi,0}(x) = \frac{1 - e^{-\xi x^2}}{x} \quad \text{for } x \in \mathbb{R} \setminus \{0\} \quad \text{and } F_{\xi,0}(0) = 0.
$$

(4.36)

Given $z \in \mathbb{H}_q$, let $E = \text{Re } z$ and $a = \text{Im } z$. Setting $r = \frac{d^2_H}{2}(x, y) - 1$, and taking $F_a(x) = F_{\xi,a}(x)$ with $\xi = \frac{A^2}{200} r$, we have the following bound (recall (2.6)):

$$
\begin{align*}
|G^{D,D^e}_z(x, y)| &\leq \left| \left( R^{D,D^e}_{z} P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| + \left| \left( R^{D,D^e}_{z} P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| \\
&\leq \left| \left( R^{D,D^e}_{z} P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| + \left| \left( F_a(H^{D,D^e} - E) \bar{P}_{I_{\frac{q}{\xi} - \frac{1}{2}}} (H^D) \right) (x, y) \right| \\
&\quad + \left| \left( R^{D,D^e}_{z} \exp \left( -r(H^{D,D^e} - E)^2 \right) P_{I_{\frac{q}{\xi} + \frac{1}{2}}} (H^{D,D^e}) \bar{P}_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| \\
&\quad + \left| \left( R^{D,D^e}_{z} \exp \left( -r(H^{D,D^e} - E)^2 \right) P_{I_{\frac{q}{\xi} + \frac{1}{2}}} (H^{D,D^e}) \bar{P}_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right|.
\end{align*}
$$

(4.37)

Since $z \in \mathbb{H}_q$, we have

$$
\begin{align*}
&\left| \left( R^{D,D^e}_{z} \exp \left( -r(H^{D,D^e} - E)^2 \right) P_{I_{\frac{q}{\xi} + \frac{1}{2}}} (H^{D,D^e}) \bar{P}_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| \\
&\leq \left\| R^{D,D^e}_{z} \exp \left( -r(H^{D,D^e} - E)^2 \right) P_{I_{\frac{q}{\xi} + \frac{1}{2}}} (H^{D,D^e}) \right\| \leq 2e^{-\frac{r}{4}}.
\end{align*}
$$

(4.38)

Moreover, since $|x_{D^e}| = |y_{D^e}| \geq 1$, we have $P_{I_{\frac{q}{\xi} + \frac{1}{2}}} (H^{D,D^e}) \bar{P}_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) = 0$. Since

$$
\begin{align*}
&\left| \left( F_a(H^{D,D^e} - E) \bar{P}_{I_{\frac{q}{\xi} - \frac{1}{2}}} (H^D) \right) (x, y) \right| \\
&\leq \left| \left( R^{D,D^e}_{z} P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| + \left| \left( F_a(H^{D,D^e} - E) P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right|,
\end{align*}
$$

(4.39)

we obtain the estimate

$$
\begin{align*}
|G^{D,D^e}_z(x, y)| &\leq \left| \left( R^{D,D^e}_{z} P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| + \left| \left( F_a(H^{D,D^e} - E) \right) (x, y) \right| \\
&\quad + \left| \left( F_a(H^{D,D^e} - E) P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| + 2e^{-\frac{r}{4}} \\
\leq \left| \left( R^{D,D^e}_{z} P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| + \left| \left( F_a(H^{D,D^e} - E) P_{I_{\frac{q}{\xi} - \frac{1}{4}}} (H^D) \right) (x, y) \right| \\
&\quad + C \left( e^{-\frac{r}{4}} + e^{-\frac{r}{4}} \right),
\end{align*}
$$

(4.40)

where we used (C.3) to get the last inequality.
We first estimate the second term in the last line of (4.40). Given \( \nu \in \sigma_{q^{-\frac{1}{2}}}(H^D) \), we get, using (3.17),

\[
\left| \left( F_a(H^{D,Y^c} - E - \nu)_\pi^D \right)(x, y) \right| \\
= \left| q - \frac{1}{2} \right| \left( 1 - \frac{1}{3} \right) \left| \left( F_a(H^{D,Y^c} - E - \nu)\hat{R}_{q^{-\frac{1}{2}}} - \frac{1}{q}\hat{Q}_{\pi^D} \right)(x, y) \right| \\
\leq q \sum_{u_D \in \mathcal{P}_{N_D}(D)} \sum_{v_D \in \mathcal{P}_{N_D,|\nu|}(D)} \left| F_a(H^{D,Y^c} - E - \nu)(x, u_D \cup y_D) \right| \\
\times \left| \hat{G}_{q^{-\frac{1}{2}}}(u_D, v_D) \right| \left| \pi^D(v_D, y_D) \right| \\
\leq C_q \sum_{u_D \in \mathcal{P}_{N_D}(D)} \sum_{v_D \in \mathcal{P}_{N_D,|\nu|}(D)} e^{-cd_H^\Lambda(x, u_D \cup y_D^c)} e^{-c_{q^{-\frac{1}{2}}}|u_D - v_D|_1} \left| \pi^D(v_D, y_D) \right|, \tag{4.41}
\]

where we used (3.16) and (C.3) for the last inequality and \( N_D = |x_D| \).

It follows that

\[
\left| \left( F_a(H^{D,Y^c} - E)P_{\sigma_{q^{-\frac{1}{2}}}}(H^D) \right)(x, y) \right| \\
\leq C_q \sum_{u_D \in \mathcal{P}_{N_D}(D)} \sum_{v_D \in \mathcal{P}_{N_D,|\nu|}(D)} e^{-cd_H^\Lambda(x, u_D \cup y_D^c)} e^{-c_{q^{-\frac{1}{2}}}|u_D - v_D|_1} \\
\times \sum_{\nu \in \sigma_{q^{-\frac{1}{2}}}(H^D)} \left| \pi^D(v_D, y_D) \right|, \tag{4.42}
\]

\[
\leq C_q \sum_{u_D \in \mathcal{P}_{N_D}(D)} \sum_{v_D \in \mathcal{P}_{N_D,|\nu|}(D)} e^{-cd_H^\Lambda(x, u_D \cup y_D^c)} e^{-c_{q^{-\frac{1}{2}}}|u_D - v_D|_1} \hat{Q}_{q^{-\frac{1}{2}}}(v_D, y_D). \tag{4.43}
\]

We have

\[
d_H^\Lambda(x, u_D \cup y_D^c) + |u_D - v_D|_1 + d_H^D(v_D, y_D) \geq d_H^\Lambda(x, y), \tag{4.44}
\]

since

\[
d_H^D(u_D, y_D) = d_H^\Lambda(u_D \cup y_D^c, y_D \cup y_D^c) = d_H^\Lambda(u_D \cup y_D^c, y), \tag{4.45}
\]

and hence

\[
d_H^\Lambda(x, u_D \cup y_D^c) + |u_D - v_D|_1 + d_H^D(v_D \cup y_D) \geq d_H(x, u_D \cup y_D^c) + d_H^D(u_D, y_D) \geq d_H^\Lambda(x, y), \tag{4.46}
\]
Taking expectations in (4.42), using Lemma 4.2 for $q - \frac{1}{2}$ (the hypotheses of the Lemma are satisfied for $q - \frac{1}{2}$ by the induction hypothesis), and using (4.43), we obtain the bound

\[
\mathbb{E}_\Lambda \left\{ \left| \left( F_a(H^{D,e} - E)P_{\frac{1}{2}}(H^D) \right)(x,y) \right| \right\} \\
\leq C_q \sum_{v_D \in P_N(D)} \sum_{v_D \in \mathcal{P}_{N,D}(|1/2| - 1)} e^{-c_D \frac{1}{2} d_H(x,u_D \cup y_{D^c})} e^{-\frac{c_D}{2} |u_D - v_D|_1} e^{-c_D \frac{1}{2} d_H(v_D, y_D)} \\
\leq C_q e^{-c_D \frac{1}{2} d_H(x,y)} \sum_{v_D \in P_N(D)} \sum_{v_D \in \mathcal{P}_{N,D}(|1/2| - 1)} e^{-\frac{c_D}{2} |u_D - v_D|_1} e^{-c_D \frac{1}{2} d_H(v_D, y_D)} \\
\leq C_q e^{-c_D \frac{1}{2} d_H(x,y)} \sum_{v_D \in P_N(D)} e^{-c_D \frac{1}{2} d_H(v_D, y_D)} \\
\leq C_q N^2 e^{-c_D \frac{1}{2} d_H(x,y)} \\
\leq C_q |D|^{|2q|} e^{-c_D \frac{1}{2} d_H(x,y)},
\]

where in the last two steps we used (B.3) and (B.10). The use of the latter is justified as $y_D \in \mathcal{P}_{N,D,|q|}(D)$ since $y \in \mathcal{P}_{N,|q|}(\Lambda)$ and $D$ is connected.

It remains to estimate the first term in (4.46). We use the decomposition (recall that $D$ is assumed to be finite)

\[
R^{D,e}_z = \sum_{\nu \in \sigma(H^D)} R^{D,e}_z \otimes \pi^{D}_\nu \quad \text{on} \quad \mathcal{H}_\Lambda = \mathcal{H}_D^e \otimes \mathcal{H}_D,
\]

(4.47)

Since $1 \leq |x_D| = |y_D| \leq N - 1$, we have

\[
\left( R^{D,e}_z P_{\frac{1}{2}}(H^D) \right)(x,y) = \sum_{\nu \in \sigma_q \frac{1}{2}(H^D)} G^{D,e}_{z-\nu}(x_{D^c}, y_{D^c}) \pi^{D}_\nu (x_D, y_D),
\]

(4.48)

so

\[
\left| \left( R^{D,e}_z P_{\frac{1}{2}}(H^D) \right)(x,y) \right|^s \leq \sum_{\nu \in \sigma_q \frac{1}{2}(H^D)} |G^{D,e}_{z-\nu}(x_{D^c}, y_{D^c})|^s |\pi^{D}_\nu (x_D, y_D)|^s. \quad (4.49)
\]

If $z \in \mathcal{H}_q$, we have $z - \nu \in \mathcal{H}_{\frac{1}{2}}$ for $\nu \in \sigma_q \frac{1}{2}(H^D)$, so it follows from the induction hypothesis that

\[
\sup_{z \in \mathcal{H}_{\frac{1}{2}}} \mathbb{E}_{D^e} \left| G^{D,e}_{\zeta}(x_{D^c}, y_{D^c}) \right|^s \leq C_q \frac{1}{2} e^{-c_D \frac{1}{2} d_H(x_{D^c}, y_{D^c})}. \quad (4.50)
\]
Thus, for \( z \in \mathbb{H}_q \), using also Hölder’s inequality and the deterministic estimate \((3.12)\), we get

\[
\mathbb{E}_{\mathcal{D}^c} \left\{ \left| \left( R^{D,D^c}_z P_{\Lambda_{q-\frac{1}{2}}} (H^D) \right) (x,y) \right|^s \right\} \\
\leq C_{q-1} e^{-c_{q-1}d_H^{D^c} (x_D, y_D)} \sum_{\nu \in \sigma_q \frac{1}{2} (H^D)} |\pi^D_{\nu} (x_D, y_D)|^s
\]

\[
= C_{q-1} e^{-c_{q-1}d_H^{D^c} (x_D, y_D)} \left( \left| \pi^D_{\nu} (x_D, y_D) \right| \right)^s (\text{tr} \chi_{q-\frac{1}{2}} (H^D))^{1-s}
\]

\[
\leq C_{q-1} \left[ \left| q - \frac{1}{2} \right| |D|^2 |q - \frac{1}{2}| + 1 \right] e^{-c_{q-1}d_H^{D^c} (x_D, y_D)} \mathbb{E}_{\mathcal{D}^c} \left\{ Q_{\frac{1}{2}}^D (x_D, y_D)^s \right\}
\]

\[
\leq C_q |D|^2 e^{-c_q d_H^D (x,y)} Q_{\frac{1}{2}}^D (x_D, y_D)^s.
\]

It follows from Hölder’s inequality, the induction hypothesis, and Lemma 4.2 for \( q - \frac{1}{2} \) that

\[
\mathbb{E}_{\mathcal{D}} \left\{ \left( Q_{\frac{1}{2}}^D (x_D, y_D)^s \right) \right\} \leq \left( \mathbb{E}_{\mathcal{D}} \left\{ Q_{\frac{1}{2}}^D (x_D, y_D) \right\} \right)^s
\]

\[
\leq C_q e^{-c_q d_H^{D^c} (x_D, y_D)}.
\]

Combining (4.51) and (4.52) we get

\[
\mathbb{E}_{\mathcal{A}} \left\{ \left| R^{D,D^c}_z P_{\Lambda_{q-\frac{1}{2}}} (H^D) \right| (x,y) \right| \right]^s \leq C_q |D|^2 |q - \frac{1}{2}| e^{-c_q d_H^D (x,y)}
\]

\[
\leq C_q |D|^2 e^{-c_q d_H^D (x,y)},
\]

where we used

\[
d_H^D (x,y) \leq \max (d_H^D (x_D, y_D), d_{H^c}^D (x_D, y_D)).
\]

It now follows from (4.40), (4.46), and (4.53) that, incorporating \( s \) into the constants, that

\[
E_{\mathcal{A}} \left\{ \left| G^{D,D^c}_z (x,y) \right|^s \right\} \leq C_q |D|^2 |q - \frac{1}{2}| e^{-c_q d_H^D (x,y)},
\]

and the lemma is proved. \( \square \)

**Lemma 4.4.** Let \( q \in \frac{1}{2} \mathbb{N}, 1 \leq q, \) and assume the the induction hypothesis, that is, Theorem 2.4 is proven for \( q - \frac{1}{2} \). Let \( \Lambda \subset \mathbb{Z} \) and \( N \in \mathbb{N} \). Then

\[
\sup_{z \in \mathbb{H}_q} \mathbb{E}_{\mathcal{A}} \left\{ \left| G^{\Lambda}_z (x,y) \right|^s \right\} \leq C_q e^{-c_q d_H^{\Lambda} (x,y)} \text{ for } x,y \in \mathcal{P}_{N,|q|} (\Lambda),
\]

(4.56)

*Proof.* Fix \( z \in \mathbb{H}_q \) and \( x,y \in \mathcal{P}_{N,|q|} (\Lambda) \). We assume \( d_H^\Lambda (x,y) = d_{\Lambda} (x,y) \) for some \( x \in \mathbb{x} \), with the other case being similar.

We first assume \( d_H^\Lambda (x,y) > 6|q| N \). In this case, we claim there exists \( r < d_H^\Lambda (x,y) \), such that, setting \( D = [r]_\Lambda \), we have

\[
d_{\Lambda} (x, \partial^\Lambda D) \geq \frac{1}{|q|} d_H^\Lambda (x,y) - 1 \quad \text{and} \quad d_{\Lambda} (y, \partial^\Lambda D) \geq \frac{1}{|q|} d_H^\Lambda (x,y) - 1.
\]

(4.57)

Note that it follows that \( x_D \neq \emptyset \) and \( y_D = \emptyset \), which implies \( G^{D,D^c}_z (x,y) = 0 \).
The claim can be proven as follows. If \([q] = 1\), or if \(x\) consists of one cluster, simply take \(r = 3N\). If \([q] \geq 2\), and \(x\) consists of \(p\) clusters where \(2 \leq p \leq [q]\), we must have \(N \geq 2\). Let \(b = \left\lfloor \frac{1}{[q]}d_\Lambda (x, y) \right\rfloor > N - 1\), and set

\[
S_1 = [x]_{b}^A \quad \text{and} \quad S_j = [x]_{j_b}^A \setminus [x]_{(j-1)b}^A \quad \text{for} \quad j = 2, 3, \ldots, [q].
\]

Since \(x \in \mathcal{P}_{N,[q]}(\Lambda)\), \(x\) has at most \([q]\) clusters of length \(\leq N - 1\), so a cluster can intersect at most two of the \(S_j\)'s (as \(b > N - 1\)), hence \(x\) can intersect at most \(2[q]\) of the \(S_j\), \(j = 2, 3, \ldots, [q]\). It follows that there exists \(j_* \in \{2, 3, \ldots, [q] - 2\}\) such that

\[
x \cap (S_{j_*} \cup S_{j_*+1}) = \emptyset,
\]

(4.59)

Setting \(r = j_* b\), we get (4.57).

The resolvent identity and \(G_{z,D'}^A (x, y) = 0\) give

\[
G_{z}^A (x, y) = \left( R_{z}^{D,D'} \right)_{\Lambda} (x, y),
\]

(4.60)

so using (1.2) and inserting partitions of identity, we get

\[
\left| G_{z}^{A} (x, y) \right| \leq C \sum_{u \in \mathcal{P}_{N}^{D} (\Lambda)} \left| \hat{G}_{z}^{D,D'} (x, u) \right| \left| \left( \Gamma^{D} R_{z}^{A} \right) (u, y) \right| + C \sum_{u \in \mathcal{P}_{N}^{D} (\Lambda)} \sum_{v \in \mathcal{P}_{N,[q]} (\Lambda)} \left| G_{z}^{D,D'} (x, v) \right| \left| \hat{G}_{z}^{D,D'} (v, u) \right| \left| \left( \Gamma^{D} R_{z}^{A} \right) (u, y) \right|,
\]

(4.61)

where \(\mathcal{P}_{N}^{D} (\Lambda) = \{ u \in \mathcal{P}_{N} (\Lambda), u_{\Lambda D} = \emptyset \}\).

For all \(u', v' \in \mathcal{P}_{N} (\Lambda)\) we have

\[
\mathbb{E} \left\{ \left| \left( \Gamma^{D} R_{z}^{A} \right) (u', v') \right|^{s'} \right\} \leq C_{s'} \quad \text{for all} \quad s' \in (0, 1),
\]

(4.62)

where the first bound follows from (3.22) since \(\Gamma^{D} \phi_{uv'}\) can be decomposed into a linear combination of at most 4 canonical basis vectors, and the second is just (3.10).

We also have the inequality

\[
\mathbb{E}_{\Lambda} \left\{ \left| G_{z}^{D,D'} (x, v) \right|^{s} \right\} \leq C_{q} |D| C_{q} e^{-C_{q} d_{H}^{A} (x, v)} \quad \text{for all} \quad v \in \mathcal{P}_{N,[q]} (\Lambda).
\]

(4.63)

If \(x_{D'} = \emptyset\), this inequality follows from Lemma 4.3. On the other hand, if \(x_{D'} = \emptyset\), \(G_{z}^{D,D'} (x, v) = 0\) unless \(v_{D'} = \emptyset\), and in this case \(G_{z}^{D,D'} (x, v) = G_{z}^{D} (x, v)\) and \(d_{H}^{A} (x, v) = d_{H}^{A} (x, v)\), and hence (4.63) follows from the hypothesis (2.22). Moreover, since \(0 < s < 2s < \frac{2}{3}\), using the Riesz-Thorin Interpolation Theorem, it follows from (1.63) and (3.22) (with \(s' = \frac{2}{3}\)) that

\[
\mathbb{E}_{\Lambda} \left\{ \left| G_{z}^{D,D'} (x, v) \right|^{2s} \right\} \leq C_{q} |D| C_{q} e^{-C_{q} d_{H}^{A} (x, v)}.
\]

(4.64)

It then follows from (1.61), (1.62), (1.64), and \(|D| \leq 2r + 1\), using also Hölder’s inequality that

\[
\sup_{x \in H_{q}} \mathbb{E}_{\Lambda} \left\{ \left| G_{z}^{A} (x, y) \right|^{s} \right\} \leq C_{q} \sum_{u \in \mathcal{P}_{N}^{D}} e^{-C_{q} |x - u|_{1}}
\]

\[
+ C_{q} r C_{q} \sum_{u \in \mathcal{P}_{N}^{D} (\Lambda)} \sum_{v \in \mathcal{P}_{N,[q]} (\Lambda)} e^{-C_{q} d_{H}^{A} (x, v)} e^{-C_{q} |u - v|_{1}}.
\]

(4.65)

Since \(|x - u|_{1} \geq \frac{1}{[q]}d_{H}^{A} (x, y) - 1\) for any \(u \in \mathcal{P}_{N}^{D} (\Lambda)\) by (4.57), we can bound

\[
\sum_{u \in \mathcal{P}_{N}^{D}} e^{-C_{q} |x - u|_{1}} \leq C_{q} e^{-\frac{C_{q}}{[q]}d_{H}^{A} (x, y)} \sum_{u \in \mathcal{P}_{N}} e^{-\frac{C_{q}}{[q]} |x - u|_{1}} \leq C_{q} e^{-C_{q} d_{H}^{A} (x, y)},
\]

(4.66)
where in the last step we used (3.3). On the other hand, since it follows from (4.57) that
\[ d_H^\Lambda(x, v) + |u - v| \geq d_H^\Lambda(x, u) \geq \frac{1}{6[q]} d_H^\Lambda(x, y) - 1 \] (4.67)
for \( u \in \mathcal{P}_N^\Lambda(\Lambda) \), we can bound
\[
\sum_{u \in \mathcal{P}_N^\Lambda(\Lambda)} \sum_{v \in \mathcal{P}_N \setminus \{u\}} e^{-c q d_H^\Lambda(x, v)} e^{-c q |u - v|_1} \leq C_q e^{-c q d_H^\Lambda(x, y)} \sum_{u \in \mathcal{P}_N^\Lambda(\Lambda)} \sum_{v \in \mathcal{P}_N \setminus \{u\}} e^{-c q d_H^\Lambda(x, v)} e^{-c q |u - v|_1} \\
\leq C_q e^{-c q N^2 |q|} e^{-c q d_H^\Lambda(x, y)} \leq C_q e^{-c q d_H^\Lambda(x, y)} \leq C_q e^{-c q d_H^\Lambda(x, y)},
\]
(4.68)
using (3.10) and (3.3). Using these bounds in (4.65) yields (4.56) if \( d_H^\Lambda(x, y) > 6[q]N \).

It remains consider the case \( d_H^\Lambda(x, y) \leq 6[q]N \). To do so we will prove
\[
\sup_{z \in \mathcal{H}_q} \mathbb{E}_\Lambda \left\{ |G^\Lambda_z(x, y)|^s \right\} \leq C_q e^{-c q |x - y|_1} + C_q e^{-c q N} \quad \text{for all} \quad N \in \mathbb{N},
\]
(4.69)
which yields, for \( d_H^\Lambda(x, y) \leq 6[q]N \),
\[
\sup_{z \in \mathcal{H}_q} \mathbb{E}_\Lambda \left\{ |G^\Lambda_z(x, y)|^s \right\} \leq C_q e^{-c q |x - y|_1} + C_q e^{-c q d_H^\Lambda(x, y)} \leq C_q e^{-c q d_H^\Lambda(x, y)},
\]
(4.70)
which is (4.55).

To prove (4.69) we use a large deviation argument. For \( N \in \mathbb{N} \), letting \( S = [x]^N \), let \( \mathcal{E}_N^S \) be the event defined in (4.19), so we have (4.20), and (4.22) holds on the complimentary event \( (\mathcal{E}_N^S)^c \).

For \( z \in \mathcal{H}_q \), we also have, using Hölder’s inequality and the a-priori bound (3.22),
\[
\mathbb{E}_\Lambda \left\{ \chi_{\mathcal{E}_N^S} |G^\Lambda_z(x, y)|^s \right\} \leq \left( \mathbb{P} \left\{ \chi_{\mathcal{E}_N^S} \right\} \right)^{\frac{s}{2}} \mathbb{E}_\Lambda \left\{ |G^\Lambda_z(x, y)|^{2s} \right\} \leq C_q e^{-c q N}.
\]
(4.71)

On the complimentary event \( (\mathcal{E}_N^S)^c \) we use
\[
G^\Lambda_z(x, y) = G^{S, S^c}_z(x, y) - (R^\alpha \Gamma R^{S, S^c}_z)(x, y),
\]
(4.72)
where \( \Gamma = H^{\Lambda} - H^{S, S^c} \). Since \( x \subset S \), we have \( G^{S, S^c}_z(x, y) = 0 \) unless \( y \subset S \), in which case \( G^{S, S^c}_z(x, y) = G^{S}_z(x, y) \). Thus (4.22) implies that in this case we have
\[
\sup_{z \in \mathcal{H}_q} \left| G^{S, S^c}_z(x, y) \right|^s \leq C_q e^{-c q |x - y|_1}.
\]
(4.73)

On the other hand, setting \( \mathcal{P}_N(S) = \{ u \in \mathcal{P}_N(S) : u \cap \partial S \neq \emptyset \} \), we have, for \( \omega \notin \mathcal{E}_N^S \) and \( z \in \mathcal{H}_q \),
\[
\left| (R^\alpha \Gamma R^{S, S^c}_z)(x, y) \right|^s \leq \sum_{u \in \mathcal{P}_N^\Lambda(S)} \left| (R^\alpha \Gamma)(x, u) \right|^s \left| G^{S, S^c}(u, y) \right|^s \\
\leq C_q \sum_{u \in \mathcal{P}_N^\Lambda(S)} e^{-c q |u - y|_1} \left| (R^\alpha \Gamma)(x, u) \right|^s.
\]
(4.74)

It follows, using (4.62), that
\[
\sup_{z \in \mathcal{H}_q} \mathbb{E}_\Lambda \left\{ \chi_{(\mathcal{E}_N^S)^c} \left| (R^\alpha \Gamma R^{S, S^c}_z)(x, y) \right|^s \right\} \leq C_q \sum_{u \in \mathcal{P}_N^\Lambda(S)} e^{-c q |u - y|_1} e^{-c q d_H^\Lambda(x, u)}.
\]
(4.75)

Since \( u \in \mathcal{P}_N^\Lambda(S) \), we have \( d_H^\Lambda(x, u) \geq N \), and hence
\[
\sup_{z \in \mathcal{H}_q} \mathbb{E}_\Lambda \left\{ \chi_{(\mathcal{E}_N^S)^c} \left| (R^\alpha \Gamma R^{S, S^c}_z)(x, y) \right|^s \right\} \leq C_q e^{-c q N} \sum_{u \in \mathcal{P}_N^\Lambda(S)} e^{-c q |u - y|_1} \leq C_q e^{-c q N},
\]
(4.76)
where we used (3.3) (recall \( y \in \mathcal{P}_{N,q}(\Lambda) \)) to get the last inequality.
Combining (4.72), (4.73), and (4.76) we get
\[
\sup_{x \in \mathbb{R}^d} \left| \chi(e_N^d) \left| G_2^A(x, y) \right|^\ell \right| \leq C_q e^{-c_q |x-y|_1} + C_q e^{-c_q N}.
\] (4.77)

The estimate (4.69) now follows from (4.71) and (4.77). □

The first statement of the theorem (i.e., (2.23)) then follows from the first statement and Lemma 4.2.

The second statement (i.e., (2.24)) then follows from the first statement and Lemma 4.1. □

5. Proof of Theorem 2.7

Proof. We will show that the theorem can be derived from [3, Theorem 4.1]. We start by reviewing the representation of the XXZ quantum spin chain Hamiltonian by a direct sum of discrete Schrödinger-like operators.

As discussed in Section 3, given \( \Lambda \subset \mathbb{Z} \), we have the the Hilbert space decomposition
\[
\mathcal{H}_\Lambda = \bigoplus_{N=0}^{\left| \Lambda \right|} \mathcal{H}^{(N)}_\Lambda, \quad \text{where} \quad \mathcal{H}^{(N)}_\Lambda = \text{Ran} \chi_N(\Lambda^{(N)}).
\]
We define
\[
\mathcal{K}^{(N)} = \{ (x_1, x_2, \ldots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \ldots < x_N \} \quad \text{and} \quad \Lambda^{(N)} = \Lambda^{(N)} \cap \mathcal{K}^{(N)}.
\] (5.1)

Since \( \mathcal{H}^{(N)}_\Lambda \) has the orthonormal basis \( \Phi^{(N)}_\Lambda \), identifying \( x \in \mathcal{P}_N(\Lambda) \) with \( (x_1, x_2, \ldots, x_N) \in \Lambda^{(N)} \) yields the identification of \( \mathcal{H}^{(N)}_\Lambda \) with \( \ell^2(\Lambda^{(N)}) \).

Since \( \mathcal{H}^A, \Delta^A, W^A, V^A \) commute with the number of particles operator \( N^A \), they leaves each \( \mathcal{H}^{(N)}_\Lambda \) invariant. Let \( T^A_N \) be the restriction of \( T^A \) to \( \mathcal{H}^{(N)}_\Lambda = \ell^2(\Lambda^{(N)}) \), where \( T^A = \mathcal{H}^A, \Delta^A, W^A, V^A \). We still have the decomposition given in (3.3):
\[
H^A_N = -\frac{1}{2\Delta} \Delta_N^A + W^A_N + \lambda V^A_{N,\omega} \quad \text{acting on} \quad \ell^2(\Lambda^{(N)}),
\] (5.2)

where \( \Delta^A_N \) is the adjacency operator on the graph \( \Lambda^{(N)} \), \( W^A_N \) is a deterministic bounded potential, and \( V^A_{N,\omega} \) is a random potential. In other words, \( H^A_N \) is a random Schrödinger operator on \( \ell^2(\Lambda^{(N)}) \). For a fixed \( N \in \mathbb{N} \), \( H^A_N \) satisfies all the hypothesis of the operators studied on [3] except that it is a Schrödinger operator on \( \ell^2(\Lambda^{(N)}) \), not on \( \ell^2(\Lambda^{\Lambda}) \). This does not affect the analysis in [3], and all the results of [3] hold for \( H^A_N \) for a fixed \( N \).

Given \( S \subset \mathbb{Z} \), we define the eigen correlator \( Q^{S}_{N,\omega}(x,y) \) for \( H^A_N \) similarly as we did for \( H^S \) in Section 2. The hypothesis of the theorem can then be rewritten as:

Let \( q \in \frac{1}{2} \mathbb{N}^0 \), and suppose that for all \( D \subset \mathbb{Z} \) finite and all \( N \in \mathbb{N} \) we have
\[
\mathbb{E}_D \left\{ Q^{D}_{N,\omega}(x,y) \right\} \leq c_q e^{-c_q D^2(x-y)} \quad \text{for all} \quad x, y \in \ell^2(D^{(N)}).
\] (5.3)

where the constants \( C_q \) and \( c_q \) are independent of \( N \).

We can then apply [3, Theorem 4.1] to \( H^A_N \) for all \( N \in \mathbb{N} \). We obtain the conclusions of the theorem for \( H^A_N \) for all \( N \in \mathbb{N} \), with the constants independent of \( N \) unless explicitly stated. It follows that the theorem holds as stated. □

### Appendix A. Localization Types and Nomenclature

Localization is a very rich phenomenon that manifests itself in variety of ways. As discussed in Section 3 for a single-particle systems one usually distinguishes between three types of localization: Spectral, eigenstate, and dynamical localization. In this Appendix we further describe these types, associated nomenclature, and the relationship between them.

(i) **Spectral localization**: The spectrum of a random operator \( H_\omega \) in a prescribed energy interval \( I \) is pure point, almost surely. When \( I = \mathbb{R} \), we say that \( H_\omega \) is completely spectrally localized.
(ii) Eigenstate localization: Consider the Hilbert space $\ell^2(\Lambda)$ where $\Lambda$ is a subset of $\mathbb{Z}^d$, the infinite system corresponding to $\Lambda = \mathbb{Z}^d$, and finite systems corresponding to bounded subsets $\Lambda$. In the mathematical literature, a frequently used formulation is the semi-uniformly localized eigenvectors (SULE) form of eigenstate localization: For a given energy interval $I$ and almost any random configuration $\omega$, one can construct an orthonormal basis $\psi_{n,\omega}$ for the range of the spectral projection $P_I$ of $H_\omega$ onto $I$, such that, for each $n$ we can find a site $k_n \in \Lambda$ so

$$|\psi_{n,\omega}(x)| \leq C_\omega \langle k_n \rangle e^{-m|x-k_n|}, \quad x \in \Lambda, \quad \langle k_n \rangle = |k_n| + 1, \quad (A.1)$$

with parameters $p, m > 0$ that do not depend on the choice of the configuration. That is, the normalized eigenfunction $\psi_{n,\omega}$ is exponentially confined near its localization center $k_n$, but the control over the confinement is only semi-uniform (it gets worse for localization centers further away from the origin). Unfortunately, for the Anderson model, the primary model for studying single-particle localization phenomena, SULE localization cannot be upgraded to ULE (uniformly localized eigenvectors). ULE does not occur for this model (and indeed for a broad class of random Schrödinger operators) [10].

A related form of the eigenstate localization is exponential decay of the eigen-correlator (in expectation), already discussed in Section 2. Roughly speaking, eigencorrelator decay implies SULE localization almost surely (see [3]).

In physics, a popular metric for a measure of localization is the inverse participation ratio (IPR): If $\psi$ is a normalized eigenvector for $H_\omega$, the IPR for $\psi$ is given by $\sum_{x \in \Lambda} |\psi(x)|^4$. Using the Cauchy-Schwarz inequality, it is easy to see that $1 \geq \text{IPR}(\psi) \geq \frac{1}{|\Lambda|}$, where the maximum is achieved when $\psi$ is a standard basis vector (that is maximally localized), and the minimum is achieved when $\psi$ is uniformly spread on $\Lambda$, i.e., $\psi(x) = \frac{1}{\sqrt{|\Lambda|}}$ for every $x \in \Lambda$ (maximally delocalized state). We note that SULE implies that $\text{IPR}(\psi_{n,\omega}) \geq C > 0$, where the constant $C$ is volume-independent and only depends (logarithmically) on the position of localization center for a given random configuration $\omega$.

If $H_\omega$ is completely spectrally localized, one can construct an orthonormal eigenbasis for $\ell^2(\Lambda)$, that is, there exists a unitary operator $U_\omega$ that diagonalizes $H_\omega$, i.e., $U_\omega^* H_\omega U_\omega$ is a diagonal matrix in the standard basis for $\ell^2(\Lambda)$. It turns out that for the Anderson model in the strong disorder regime, for any finite volume $\Lambda \subset \mathbb{Z}^d$, with high probability one can construct $U_\omega$, which is semi-uniformly quasi-local, meaning that the matrix elements $U_\omega(x, y)$ of $U_\omega$ satisfy the bound $|U_\omega(x, y)| \leq C_\omega \langle x \rangle e^{-m|x-y|}$ for some $p, m > 0$ that do not depend on the choice of the configuration. (This follows from the results in [11].) The existence of such $U_\omega$ not only yields a SULE basis, but also allows to label these eigenfunctions according to the spatial position of their localization centers. Such labeling is possible for the Anderson model due to the fact that one can show that with large probability the spectrum of $H_\omega$ is level spaced for sufficiently regular distribution of the random potential. Motivated by the concept of LIOM in the many-body context introduced in Section 1.2, we will refer to this form of the eigenstate localization as semi-uniform LIOM localization, As we already indicated, ULE does not occur for the Anderson model, so one can never upgrade a semi-uniform LIOM localization to a uniform LIOM localization in this context.
Lemma B.1. Let $k, N \in \mathbb{N}, k \leq N, \alpha > 0$, and let
\[ C_\alpha = (1 - e^{-\alpha})^{-1} \left( \prod_{n=1}^{\infty} (1 - e^{-\alpha n})^{-1} \right)^2. \]
Then
\[ \sup_{y \in \mathcal{P}_N(\mathbb{Z})} \sum_{x \in \mathcal{P}_{N,k}(\mathbb{Z})} e^{-\alpha |x-y|_1} \leq C_{\alpha}^k, \]  
\hfill (B.2) 
and
\[ \sup_{x \in \mathcal{P}_{N,k}(\mathbb{Z})} \sum_{y \in \mathcal{P}_N(\mathbb{Z})} e^{-\alpha |x-y|_1} \leq C_{\alpha}^k, \]  
\hfill (B.3)

**Proof.** The lemma is proven by adapting the argument of [1, Lemma B.2], who estimate the case \( k = 1 \) of (B.2).

Let \( x \in \mathcal{P}_{N,k}(\mathbb{Z}) \), and suppose that \( x \) has \( m = m_x \) clusters (where \( m \in \{1, \ldots, k\} \)). Then \( x = (x_1, \ldots, x_N) \), where \( x_1 < \ldots < x_N \), and let \( x_{j_1} < \ldots < x_{j_{2m}} \), where \( j_1 = 1, j_{2m} = N \), are the end points for its \( m \) clusters. (Note that \( m \) and \( j_2, \ldots, j_{2m-1} \) are \( x \)-dependent.) Given \( y \in \mathcal{P}_N(\mathbb{Z}) \) with \( y = (y_1, y_N) \), where \( y_1 < \ldots < y_N \), we set \( t_i = y_i - x_i, i = 1, \ldots, N \). Then the finite sequences \( \tau_q = (t_{j_{2q-1}}, \ldots, t_{j_{2q}}) \) are monotone non-decreasing for each \( q = 1, \ldots, m \). Let \( \mathcal{T}_q \) denote the collection of such monotone non-decreasing finite sequences \( \tau_q \). Let \( \mathcal{T}^{(N)} \) denote the collection of all monotone non-decreasing finite sequences \( \tau^{(N)} = (t_1, \ldots, t_N) \).

To prove (B.2), fix \( y \in \mathcal{P}_N(\mathbb{Z}) \). Then each \( x \in \mathcal{P}_{N,k}(\mathbb{Z}) \) is determined uniquely by the corresponding \( \{t_i\}_{i=1}^N \), so we have
\[
\sum_{x \in \mathcal{P}_{N,k}(\mathbb{Z})} e^{-\alpha |x-y|_1} = \sum_{x \in \mathcal{P}_{N,k}(\mathbb{Z})} \prod_{q=1}^{m_x} e^{-\alpha \sum_{j=2q-1}^{2q} |x_j - y_j|} \leq \sum_{m=1}^{k} \sum_{\tau_1, \tau_2, \ldots, \tau_m \in \mathcal{T}^{(N)}} \prod_{q=1}^{m} e^{-\alpha \sum_{i \leq q} (\tau_i)} = \sum_{m=1}^{k} \left( \sum_{\tau \in \mathcal{T}^{(N)}} e^{-\alpha \sum_{i \leq N} (\tau_i)} \right)^m. \]  
\hfill (B.4)

Given \( \tau^{(N)} \in \mathcal{T}^{(N)} \), since \( \tau^{(N)} \) is monotone non-decreasing there exists an index \( 0 \leq p \leq N + 1 \), such that such that \( t_j < 0 \) for \( 1 \leq j \leq p \) and \( t_j \geq 0 \) for \( p + 1 \leq j \leq N \). (Note that sets are allowed to be empty). Thus,
\[
\sum_{\tau \in \mathcal{T}^{(N)}} e^{-\alpha \sum_{i=1}^{N} (\tau_i)} = \sum_{p=0}^{N+1} \left( \sum_{t_{t_1} \leq \ldots \leq t_{t_p} \leq -1} e^{\alpha (t_1 + t_2 + \ldots + t_p)} \right) \left( \sum_{0 \leq t_{p+1} \leq \ldots \leq t_N} e^{-\alpha (t_{p+1} + t_{p+2} + \ldots + t_N)} \right) 
= \sum_{p=0}^{N+1} e^{-\alpha p} \left( \sum_{0 \leq t_{p+1} \leq \ldots \leq t_N} e^{-\alpha (t_{p+1} + t_{p+2} + \ldots + t_N)} \right) 
\leq \left( \sum_{p=0}^{\infty} e^{-\alpha p} \right) \left( \sum_{n=0}^{\infty} P(n) e^{-\alpha n} \right)^2 = (1 - e^{-\alpha})^{-1} \left( \prod_{n=1}^{\infty} (1 - e^{-\alpha})^{-1} \right)^2 = C_{\alpha}, \]  
\hfill (B.5)

where \( P(n) \) is the number of integer partitions of \( n \), and we used the formula for the generating function for \( P(n) \).

It follows from (B.4) and (B.5) that
\[
\sum_{x \in \mathcal{P}_{N,k}(\mathbb{Z})} e^{-\alpha |x-y|_1} \leq \sum_{m=1}^{k} C_{\alpha}^m = \frac{C_{\alpha}^{k+1} - C_{\alpha}}{e_{\alpha} - 1} \leq C_{\alpha}^{k+1}, \]  
\hfill (B.6)

which yields (B.2).
To establish (B.3), we modify the above argument. We fix \( x \in \mathcal{P}_{N,k}(\mathbb{Z}) \), and note that every \( y \in \mathcal{P}_N(\mathbb{Z}) \) is determined uniquely by the corresponding \( \{t_i\}_{i=1}^m \), so we have

\[
\sum_{y \in \mathcal{P}_N(\mathbb{Z})} e^{-\alpha|x-y|_1} = \sum_{y \in \mathcal{P}_N(\mathbb{Z})} \prod_{q=1}^m e^{-\alpha \sum_{j=2q-1}^{2q-1} |x_j-y_j|} \leq \prod_{q=1}^m \sum_\tau e^{-\alpha \sum_{j=1}^{t_{2q-1}} |t_j|}.
\]  

(B.7)

where \( n_q = n_q(x) = j_{2q} - j_{2q-1} \) for \( q = 1, 2, \ldots, m \).

Let \( n \in \mathbb{N} \), then

\[
\sum_{\tau \in \mathcal{T}^{(n)}} e^{-\alpha \sum_{j=1}^{m} |t_j|} = \sum_{p=0}^{N+1} \left( \sum_{0 \leq t_{p+1} \leq t_{p+2} \leq \ldots \leq t_N} e^{\alpha(t_1+t_2+\ldots+t_p)} \right) \left( \sum_{0 \leq t_{p+1} \leq t_{p+2} \leq \ldots \leq t_N} e^{-\alpha(t_{p+1}+t_{p+2}+\ldots+t_N)} \right)
\]

\[
= \sum_{p=0}^{N+1} e^{-ap} \left( \sum_{0 \leq t_{p+1} \leq t_{p+2} \leq \ldots \leq t_N} e^{-\alpha(t_1+t_2+\ldots+t_p)} \right) \left( \sum_{0 \leq t_{p+1} \leq t_{p+2} \leq \ldots \leq t_N} e^{-\alpha(t_{p+1}+t_{p+2}+\ldots+t_N)} \right)
\]

\[
\leq \left( \sum_{p=0}^{\infty} e^{-ap} \right) \left( \sum_{n=0}^{\infty} P(n)e^{-an} \right)^2 = C_\alpha,
\]  

(B.8)

as in (B.5).

It follows from (B.7) and (B.8) that

\[
\sum_{y \in \mathcal{P}_N(\mathbb{Z})} e^{-\alpha|x-y|_1} \leq C_\alpha^m,
\]  

(B.9)

which yields (B.3).

\[\square\]

**Lemma B.2.** Let \( N \in \mathbb{N} \), \( k \in \mathbb{N} \), \( k \leq N \), and \( \alpha > 0 \). Then

\[
\sup_{x \in \mathcal{P}_{N,k}(\mathbb{Z})} \sum_{y \in \mathcal{P}_{N,k}(\mathbb{Z})} e^{-\alpha d_H(x,y)} \leq C_{\alpha,k} N^{2k}.
\]  

(B.10)

**Proof.** Fix \( N \in \mathbb{N} \) and \( k \in \mathbb{N} \), \( k \leq N \). For \( m \in \mathbb{N} \) let \( \mathcal{P}_{N}^{(m)}(\mathbb{Z}) = \{ x \in \mathcal{P}_N(\mathbb{Z}), W_x^\mathbb{Z} = m \} \).

In addition, for \( x \in \mathcal{P}_N(\mathbb{Z}) \), and \( r \in \mathbb{N} \) let

\[
\mathcal{S}_{x,r} = \{ y \in \mathcal{P}_N(\mathbb{Z}), d_H(x,y) = r \}, \quad \mathcal{S}_{x,r}^{(m)} = \{ y \in \mathcal{P}_{N}^{(m)}(\mathbb{Z}), d_H(x,y) = r \}
\]

\[
\mathcal{S}_{x,r,k} = \bigcup_{m=1}^{k} \mathcal{S}_{x,r}^{(m)}.
\]  

(B.11)

Let now \( x \in \mathcal{P}_{N,k}(\mathbb{Z}) \). We note that \( y \in \mathcal{S}_{x,r} \) implies \( y \subset [x]_r^\mathbb{Z} \). Since \( |[x]_r| \leq N + 2kr \), we deduce that

\[
|\mathcal{S}_{x,r}| \leq \left( \frac{N + 2kr}{N} \right) \leq (N + 2kr)^N.
\]  

(B.12)
As a consequence, for \( x \in \mathcal{P}_{N,k}(\mathbb{Z}) \) and \( \alpha > 0 \), we obtain the estimate
\[
\sum_{y \in \mathcal{P}_{N}(\mathbb{Z})} e^{-\alpha d_H(x,y)} = 1 + \sum_{r=1}^{\infty} |S_{x,r}| e^{-ar}
\]
\[\leq 1 + \sum_{r=1}^{[N/2k]} (2N)^r e^{-ar} + \sum_{r=[N/2k]+1}^{\infty} (4kr + 1)^N e^{-ar} \leq C_{\alpha,k}^N N^{N+1}.
\]
\[\text{(B.13)}\]

We also have the following bounds:
\[
|S_{x,r}^{(m)}| \leq (N + 2kr)^{2m}, \quad |S_{x,r,k}| \leq k(N + 2kr)^{2k}.
\]
\[\text{(B.14)}\]

Clearly, the second bound follows immediately from the first one by summing over \( m \). To obtain the first bound, we note that \( y \in S_{x,r}^{(m)} \) implies \( y \in [x]^r \), and hence \( y \) is completely determined by the \( 2m \) points in \([x]^r\) that are the end points for its \( m \) clusters. Since \( |[x]^r| \leq N + 2kr \), we deduce that
\[
|S_{x,r}^{(m)}| \leq \left( \frac{N + 2kr}{2m} \right)^m \leq (N + 2kr)^{2m}.
\]
\[\text{(B.15)}\]

As a consequence, for \( x \in \mathcal{P}_{N,k}(\mathbb{Z}) \) and \( \alpha > 0 \), we obtain the estimate
\[
\sum_{y \in \mathcal{P}_{N,k}(\mathbb{Z})} e^{-\alpha d_H(x,y)} = 1 + \sum_{r=1}^{\infty} |S_{x,r}| e^{-ar}
\]
\[\leq 1 + k \sum_{r=1}^{[N/2k]} (2N)^r e^{-ar} + k \sum_{r=[N/2k]+1}^{\infty} (4kr + 1)^r e^{-ar} \leq C_{\alpha,k} N^{2k}.
\]
\[\text{(B.16)}\]

**Appendix C. Quasi-locality of the Filter Function**

We fix \( \Lambda \subset \mathbb{Z} \) and consider the Hilbert space \( \mathcal{H}_\Lambda \). We consider disjoint subsets \( K_1, K_2 \) of \( \Lambda \), and let \( H = H_{K_1} + H_{K_2} \) acting on \( \mathcal{H}_\Lambda \). We observe that the following holds:

(i) For all \( K \subset \Lambda \) and \( K' = [K]^K \) we have \( \{P^-_K, H\} P^+_K = 0 \).

(ii) For all \( K \subset K_1 \), connected in \( K_1 \), we have \( \|\{P^-_K, H\}\| \leq \gamma = \Delta^{-1} \).

We also observe that \( H \) commutes with the total particle number operator, and its restriction \( \mathcal{H}^N \) to the \( N \)-particle sector \( \mathcal{H}^{(N)}_\Lambda \) is a well defined bounded operator for each \( N \in \mathbb{N} \).

We use the following adaptation of [12, Lemma B.1] which does not require \( \Lambda \) to be finite.

**Lemma C.1.** For all \( A \subseteq B \subset \Lambda \) with \( A, B \) finite, \( A \subset K_1 \) connected in \( K_1 \), we have
\[
\|P^-_A e^{itH} P^+_B\| \leq \Delta^{-r} \frac{|t|^r}{r!} \text{ for all } t \in \mathbb{R}, \quad \text{where} \quad r = d_\Lambda(A, B^c).
\]
\[\text{(C.1)}\]

**Proof.** \( H \) satisfies the input conditions (i) and (ii) of [12, Lemma B.1], so its output (i.e., (C.1)) is valid as well.

**Theorem C.2.** Given \( t \in \mathbb{R}_+ \) and \( a \in \mathbb{R} \), let \( F_{t,a} \) be the \( C^\infty \) function on \( \mathbb{R} \) given in (4.36). Let \( S \subseteq T \subset \Lambda \), \( S, T \) finite, where \( S \subset K_1 \) is is connected in \( K_1 \), and let \( t = d_\mathbb{Z}(S, T^c) - 1 \). Then, taking \( \beta = \frac{\Lambda}{a} \), for all \( t > 1, a \in \mathbb{R} \), and \( E \in \mathbb{R} \) we have
\[
\|P^S F_{t,a}(H - E) P^T_a\| \leq C \left( e^{-\frac{4}{t}} + \sqrt{t} e^{-\frac{2\beta}{t}} \right),
\]
\[\text{(C.2)}\]
where all constants are $a$-independent.
In particular, taking $t = \frac{\beta^2}{8} \ell$, we have ($\ell \in \mathbb{N}^0$)
\[ \left\| P^8 F_{\hat{a},\ell_0}(H - E) P^*_+ \right\| \leq C e^{-\frac{1}{2}\ell}. \] (C.3)

We will refer to $F_{t,a}$ as a filter function.

**Proof.** We introduce a introduce a function $F_{t,a,\epsilon} \in \mathcal{S}(\mathbb{R})$, where $0 < \epsilon$, given by
\[ F_{t,a,\epsilon}(x) = \frac{e^{-ix^2} - e^{-ix^2}}{x - ia} \quad \text{for} \quad x \in \mathbb{R} \quad \text{if} \quad a \neq 0, \] (C.4)
\[ F_{t,0,\epsilon}(x) = \frac{e^{-ix^2} - e^{-ix^2}}{x} \quad \text{for} \quad x \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad F_{t,0}(0) = 0. \]

Let $\hat{f}$ denote the Fourier transform of the function $f$. We note that for $a > 0$, the Fourier transform of $f_a(x) = \frac{1}{x - ia}$ exists as an $L^2$ function, and is given by $\hat{f}_a(\xi) = 2i\pi e^{a\xi} \chi_{(-\infty,0)}(\xi)$, whereas for $a = 0$ it exists in a distributional sense, $\hat{f}_0(\xi) = -i\pi \text{sgn}(\xi)$.

We will only consider the more delicate case $a = 0$, the argument for $a \neq 0$ is very similar.

A standard calculation gives
\[ \hat{F}_{t,0,\epsilon}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-i \sqrt{\frac{\pi}{2}} \text{sgn}(\xi - s) \right) \left( \frac{1}{\sqrt{2\pi}} e^{-s^2/4\epsilon} - \frac{1}{\sqrt{2\pi}} e^{-s^2/4t} \right) ds \]
\[ = \frac{1}{\sqrt{2\pi}} \left( \int_{-\xi}^{\infty} - \int_{-\infty}^{-\xi} \right) \left( \frac{1}{\sqrt{\epsilon}} e^{-s^2/4\epsilon} - \frac{1}{\sqrt{t}} e^{-s^2/4t} \right) ds. \] (C.5)

Since $F_{t,0,\epsilon} \in L^1$, by the Riemann–Lebesgue Lemma we have
\[ 0 = \lim_{\xi \to \infty} \hat{F}_{t,0,\epsilon}(\xi) = -\frac{i}{2\sqrt{\epsilon}} \int_{-\infty}^{\infty} e^{i\lambda s} \left( \frac{1}{\sqrt{\epsilon}} e^{-s^2/4\epsilon} - \frac{1}{\sqrt{t}} e^{-s^2/4t} \right) ds, \] (C.6)
so it follows that
\[ \hat{F}_{t,0,\epsilon}(\xi) = \frac{i}{\sqrt{2}} \int_{-\xi}^{\xi} \left( \frac{1}{\sqrt{\epsilon}} e^{-s^2/4\epsilon} - \frac{1}{\sqrt{t}} e^{-s^2/4t} \right) ds \]
\[ = -\frac{i}{\sqrt{2}} \int_{-\infty}^{-\xi} \left( \frac{1}{\sqrt{\epsilon}} e^{-s^2/4\epsilon} - \frac{1}{\sqrt{t}} e^{-s^2/4t} \right) ds. \]

If $\xi > 0$, we estimate
\[ \left| \hat{F}_{t,0,\epsilon}(\xi) \right| \leq \frac{1}{\sqrt{2}} \int_{-\xi}^{\infty} \frac{1}{\sqrt{\epsilon}} e^{-s^2/4\epsilon} ds + \frac{1}{\sqrt{2}} \int_{-\xi}^{\infty} \frac{1}{\sqrt{t}} e^{-s^2/4t} ds. \] (C.7)

Using the Gaussian estimate
\[ \int_{-\infty}^{\infty} e^{-x^2/2} dx \leq \sqrt{2\pi} \quad \text{for} \quad x > 0, \] (C.8)
and recalling $\int_{-\infty}^{\infty} e^{-y^2/2} dy = \sqrt{2\pi}$, we conclude that
\[ \int_{-\infty}^{\infty} e^{-x^2/2} dx \leq \sqrt{\frac{2\pi}{2}} e^{-x^2/4} \quad \text{for all} \quad x > 0. \] (C.9)
(Note that $\int_{-\infty}^{\infty} e^{-x^2/2} dx \leq e^{-x^2/4}$ for $x \geq 1$ and $e^{-x^2/2} \sqrt{\epsilon} \geq 1$ for $x \in [0, 1]$.) Thus
\[ \int_{-\infty}^{\infty} e^{-s^2/4b} ds = \sqrt{2b} \int_{-\infty}^{\infty} e^{-s^2/2} ds \leq \sqrt{\frac{\pi}{2}} \sqrt{2b} e^{-\xi^2/4b} \]
\[ \leq \sqrt{\pi} e b \sqrt{e^{-\xi^2/4b}} \leq 3 \sqrt{b} e^{-\xi^2/4b} \quad \text{for} \quad \xi > 0, \quad b > 0. \] (C.10)
It follows from (C.7) and (C.10) that
\[
\left| \hat{F}_{t,0,\varepsilon}(\xi) \right| \leq \frac{3}{\sqrt{2}} e^{-\xi^2/4t} + \frac{3}{\sqrt{2}} e^{-\xi^2/2t} \leq 5e^{-\xi^2/4t},
\]
for all \( \xi > 0 \), and since the same estimate can be established for \( \xi < 0 \), for all \( \xi \in \mathbb{R} \).
Moreover, the same upper bound also holds for an arbitrary value of \( a \).
We can bound
\[
\| P^A f(H) P^B_+ \| \leq \int_{\mathcal{R}} \| P^A e^{itH} P^B_+ \| \left| \hat{f}(t) \right| dt + \int_{\mathcal{R}^c} \left| \hat{f}(t) \right| dt,
\]
where \( \mathcal{R} = [-R, R] \). Using (C.12) with \( R = ct \) and Lemma C.1, we have
\[
\| P^S f(H - E) P^T_+ \| \leq C \left\| \hat{f} \right\|_\infty \frac{|\Delta - 1| c \xi}{\ell} + \int_{|t| > ct} \left| \hat{f}(t) \right| dt.
\]
Hence for \( 0 < \varepsilon < t \) and appropriately chosen value for \( c \), say \( c = \frac{3}{5} \), (C.11) implies, via Stirling’s approximation, that
\[
\| P^S F_{t,0,\varepsilon}(H - E) P^T_+ \| \leq Ce^{-\frac{3}{5}t} + C \int_{|\xi| > \frac{3}{5}t} e^{-\xi^2/4t} d\xi \leq Ce^{-\frac{3}{5}t} + C\sqrt{t} e^{-\frac{3t^2}{500}}.
\]
Using \( |(F_{t,0} - F_{t,0,\varepsilon})(x)| \leq \varepsilon |x| \), restricting to the N-particle sector \( \mathcal{H}^{(N)}_A \), and recalling that \( H_N \) is a bounded operator, we get
\[
\| P^S F_{t,0}(H_N - E) P^T_+ \| \leq \| P^S F_{t,0,\varepsilon}(H_N - E) P^T_+ \| + \| (F_{t,0} - F_{t,0,\varepsilon})(H_N - E) \| \leq C \left( e^{-\frac{3}{5}t} + \sqrt{t} e^{-\frac{3t^2}{500}} \right) + \varepsilon \| H_N - E \|,
\]
where \( C \) is \( N \)-independent. Letting \( \varepsilon \to 0 \) we get
\[
\| P^S F_{t,a}(H_N - E) P^T_+ \| \leq C \left( e^{-\frac{3}{5}t} + \sqrt{t} e^{-\frac{3t^2}{500}} \right) \quad \text{for all} \quad N \in \mathbb{N}.
\]
The desired estimate (C.2) follows. \( \square \)

References

[1] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn. “Colloquium: Many-body localization, thermalization, and entanglement”. Rev. Mod. Phys. 91 (2019), 021001.
[2] H. Abdul-Rahman, B. Nachtergaele, R. Sims, and G. Stolz. “Localization properties of the disordered XY spin chain: A review of mathematical results with an eye toward many-body localization”. Ann. Phys. 529 (2017), 1600280.
[3] M. Aizenman and S. Warzel. “Localization bounds for multiparticle systems”. Comm. Math. Phys. 290 (2009), 903–934.
[4] M. Aizenman and S. Warzel. Random operators. 168. American Mathematical Soc., 2015.
[5] F. Alet and N. Laflorencie. “Many-body localization: An introduction and selected topics”. C. R. Phys. 19 (2018), 498–525.
[6] V. Beaud and S. Warzel. “Bounds on the entanglement entropy of droplet states in the XXZ spin chain”. J. Math. Phys 59 (2018).
[7] V. Beaud and S. Warzel. “Low-energy Fock-space localization for attractive hard-core particles in disorder”. Ann. Henri Poincaré 18 (2017), 3143–3166.
[8] V. Chulaevsky and Y. Suhov. “Multi-particle Anderson localisation: induction on the number of particles”. Math. Phys. Anal. Geom. 12 (2009), 117–139.
[9] W. De Roeck, L. Giacomini, F. Huveneers, and O. Prosnik. “Absence of normal heat conduction in strongly disordered interacting quantum chains”. arXiv preprint arXiv:2408.04338 (2024).
[10] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. “Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization”. J. d’Analyse Math. 69 (1996), 153–200.

[11] A. Elgart and A. Klein. “An eigensystem approach to Anderson localization”. J. Funct. Anal. 271 (2016), 3465–3512.

[12] A. Elgart and A. Klein. “Localization in the random XXZ quantum spin chain”. Forum Math., Sigma 12 (2024), e129. DOI: 10.1017/fms.2024.119

[13] A. Elgart and A. Klein. “Slow propagation of information on the random XXZ quantum spin chain”. Comm. Math. Phys. 405 (2024), article number 239. DOI: 10.1007/s00220-024-05127-y.

[14] A. Elgart, A. Klein, and G. Stolz. “Manifestations of dynamical localization in the disordered XXZ spin chain”. Comm. Math. Phys. 361 (2018), 1083–1113.

[15] A. Elgart, A. Klein, and G. Stolz. “Many-body localization in the droplet spectrum of the random XXZ quantum spin chain”. J. Funct. Anal. 275 (2018), 211–258.

[16] C. Fischbacher and G. Stolz. “The infinite XXZ quantum spin chain revisited: structure of low lying spectral bands and gaps”. Math. Model. Nat. Phenom. 9 (2014), 44–72.

[17] A. Klein and S. T. Nguyen. “Bootstrap multiscale analysis and localization for multi-particle continuous Anderson Hamiltonians”. J. Spectr. Theory 5 (2015), 399–444.

[18] A. Klein and S. T. Nguyen. “The bootstrap multiscale analysis for the multi-particle Anderson model”. J. Stat. Phys. 151 (2013), 938–973.

[19] B. Nachtergaele, W. Spitzer, and S. Starr. “Droplet excitations for the spin-1/2 XXZ chain with kink boundary condition”. Ann. Henri Poincaré (2007), 165–201.

[20] R. Nandkishore and D. A. Huse. “Many-Body Localization and Thermalization in Quantum Statistical Mechanics”. Annual Review of Condensed Matter Physics 6 (2015), 15–38.

[21] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. “What is localization?” Phys. Rev. Lett. 75 (1995), 117.

[22] M. Serbyn, Z. Papić, and D. A. Abanin. “Local conservation laws and the structure of the many-body localized states”. Phys. Rev. Lett. 111 (2013), 127201.

[23] P. Sierant, M. Lewenstein, A. Scardicchio, L. Vidmar, and J. Zakrzewski. “Many-body localization in the age of classical computing”. arXiv preprint arXiv:2403.07111 (2024).

[24] D. Toniolo and S. Bose. “Stability of slow Hamiltonian dynamics from Lieb-Robinson bounds”. arXiv preprint arXiv:2405.05958 (2024).

(A. Elgart) Department of Mathematics; Virginia Tech; Blacksburg, VA, 24061-1026, USA
Email address: aelgart@vt.edu

(A. Klein) University of California, Irvine; Department of Mathematics; Irvine, CA 92697-3875, USA
Email address: aklein@uci.edu