On auto and hetero Bäcklund transformations for the Hénon-Heiles systems

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Abstract

We consider a canonical transformation of parabolic coordinates on the plain and suppose that this transformation together with some additional relations may be considered as a counterpart of the auto and hetero Bäcklund transformations associated with the integrable Hénon-Heiles systems.

1 Introduction

According to classical definition by Darboux [4], a Bäcklund transformation between the two given PDEs

\[ E(u, x, t) = 0 \quad \text{and} \quad \tilde{E}(v, y, \tau) = 0 \]

is a pair of relations

\[ F_{1,2}(u, x, t, v, y, \tau) = 0 \quad (1.1) \]

and some additional relations between \((x, t)\) and \((y, \tau)\), which allow to get both equations \(E\) and \(\tilde{E}\). The BT is called an auto-BT or a hetero-BT depending whether the two PDEs are the same or not. The hetero-BTs describe a correspondence between equations rather than a one-to-one mapping between their solutions [1, 9]. In the modern theory of partial differential equation Bäcklund transformations are seen also as a powerful tool in the discretization of PDEs [2].

A counterpart of the auto Bäcklund transformations for finite dimensional integrable systems can be seen as the canonical transformation

\[(u, p_u) \rightarrow (v, p_v), \quad \{u_i, p_{u_j}\} = \{v_i, p_{v_j}\} = \delta_{ij}, \quad i, j = 1, \ldots, n, \quad (1.2)\]

preserving the algebraic form of the Hamilton-Jacobi equations

\[ H_i \left( u, \frac{\partial S}{\partial u} \right) = \alpha_i \quad \text{and} \quad H_i \left( v, \frac{\partial S}{\partial v} \right) = \alpha_i \]

associated with the Hamiltonians \(H_1, \ldots, H_n\) [11].

The counterpart of the discretization for finite dimensional systems is also currently accepted: by viewing the new \(v\)-variables as the old \(u\)-variables, but computed at the next time step; then the Bäcklund transformation \((1.2)\) defines an integrable symplectic map or discretization of the continuous model, see discussion in [6, 13].

The counterpart of the hetero Bäcklund transformations for finite dimensional integrable systems has to be a canonical transformation \((1.2)\), which has to relate two different systems of the Hamilton-Jacobi equations

\[ H_i \left( u, \frac{\partial \tilde{S}}{\partial u} \right) = \alpha_i \quad \text{and} \quad \tilde{H}_i \left( v, \frac{\partial \tilde{S}}{\partial v} \right) = \tilde{\alpha}_i \quad (1.3)\]
and has to satisfy some additional conditions. It is necessary to add these conditions to (1.2) and (1.3) in order to get a non-trivial, usable and efficient theory. The question of how to do it remains open.

One of the possible additional conditions may be found in the theory of superintegrable systems. For instance, let us consider integrals of motion for the two-dimensional harmonic oscillator

\[ H_1 = p_x^2 + p_y^2 + a(x^2 + y^2), \quad H_2 = p_x^2 - p_y^2 + a(x^2 - y^2), \]

which yield the Hamilton-Jacobi equations separable in Cartesian coordinates \( u = (x, y) \) on the plane. Another pair of Hamiltonians for the same harmonic oscillator

\[ \tilde{H}_1 = p_x^2 + p_y^2 + a(x^2 + y^2), \quad \tilde{H}_2 = xp_y - yp_x, \]

is separable in polar coordinates \( v = (r, \varphi) \) on the plane. Canonical transformation of variables

\[ (u, p_u) = (x, y, p_x, p_y) \rightarrow (v, p_v) = (r, \varphi, p_r, p_\varphi) \]  \hspace{1cm} (1.4)

defines a correspondence between the two different systems of the Hamilton-Jacobi equations

\[ H_{1,2}(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}) = \alpha_{1,2} \quad \text{and} \quad \tilde{H}_{1,2}(r, \varphi, \frac{\partial \tilde{S}}{\partial r}, \frac{\partial \tilde{S}}{\partial \varphi}) = \tilde{\alpha}_{1,2}. \]

This correspondence may be considered as a hetero-BT defined by the generating function

\[ F = p_x r \cos \varphi + p_y r \sin \varphi, \]

relations between \((x, y)\) and \((r, \varphi)\)

\[ x = r \cos \varphi, \quad y = r \sin \varphi, \]

and with an additional condition that Hamilton function \( H_1 = \tilde{H}_1 \) is simultaneously separable in \( u \) and \( v \) variables.

Canonical transformation (1.4) can be considered as the semi hetero-BT relating different Hamilton-Jacobi equations, which are various faces of the same superintegrable system. We know that theory of such semi hetero-BTs is a profound and very useful theory, both in classical and quantum cases [7].

The main aim of this note is to discuss a correspondence between integrable Hénon-Heiles systems proposed in [10]. This correspondence between different Hamiltonians may be considered as a counterpart of the generic hetero-BTs relating different but simultaneously separable in \( v \)-variables Hamilton-Jacobi equations.

2 The Jacobi method

Let us consider some natural Hamilton function on \( T^* \mathbb{R}^n \)

\[ H = p_1^2 + \cdots + p_n^2 + V(q_1, \ldots, q_n). \]  \hspace{1cm} (2.5)

The corresponding Hamilton-Jacobi is said to be separable in a set of canonical coordinates \( u_i \) if it has the additively separated complete integral

\[ S(u_1, \ldots, u_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} S_i(u_i; \alpha_1, \ldots, \alpha_n), \]

where \( S_i \) are found by quadratures as solutions of ordinary differential equations. In order to express initial physical variables \((q, p)\) in terms of canonical variables of separation \((u, p_u)\) we have to obtain momenta \( p_{u_i} \) from the second Jacobi equations

\[ p_{u_i} = \frac{\partial S_i(u_i; \alpha_1, \ldots, \alpha_n)}{\partial u_i}, \quad i = 1, \ldots, n. \]  \hspace{1cm} (2.6)
Solving these equations with respect to \( \alpha_i \) one gets integrals of motion \( H_i = \alpha_i \) as functions on variables of separation \((u, p_u)\).

According to Jacobi we can use canonical transformation \((q, p) \rightarrow (u, p_u)\) in order to construct other integrable systems. For instance, if \( u_i \) are parabolic coordinates on \( \mathbb{R}^n \)

\[
\frac{\prod_{k=1}^{n}(\lambda - u_k)}{\prod_{j=1}^{n-1}(\lambda - a_j)} = \lambda - 2q_n - \sum_{i=1}^{n-1} \frac{q_i^2}{\lambda - a_i},
\]

the second Jacobi equations (2.6) may be rewritten in the following form

\[
p^2_{u_i} + U_i(u_i) = H + \sum_{k=1}^{n-1} \frac{H_k}{u_i - a_k}, \quad i = 1, \ldots, n,
\]

where \( U_i(u_i) \) are functions defined by the potential part \( V(q_1, \ldots, q_n) \) of the Hamiltonian (2.5).

Adding together all the separated relations (2.8) one gets another integrable Hamiltonian

\[
\tilde{H} = n^{-1} \sum_{i=1}^{n} \left( p^2_{u_i} + U_i(u_i) \right) = H + n^{-1} \sum_{k=1}^{n-1} \sum_{i=1}^{n-1} \frac{H_k}{u_i - a_k},
\]

which can be considered as an integrable perturbation of \( H \) (2.5). Of course, in generic case this perturbation has no physical meaning. Auto-BTs of the initial Hamilton-Jacobi equation

\[
(u_i, p_u) \rightarrow (v_i, p_v)
\]

preserve an algebraic form of the initial Hamiltonian \( H \) (2.5) and change the form of the second Hamiltonian

\[
\tilde{H} = H + n^{-1} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \frac{H_k}{v_i - a_k} = H - n^{-1} \sum_{k=1}^{n-1} \frac{H_k}{a_k} \left. \frac{d \ln \prod_{i=1}^{n}(\lambda - u_i)}{d\lambda} \right|_{\lambda = a_k}.
\]

We can try to pick out a special and maybe unique auto-BT, which gives physical meaning to the second Hamiltonian \( \tilde{H} \), as some of the possible counterparts of the hetero-BTs relating different but simultaneously separable Hamilton-Jacobi equations.

### 2.1 The Hénon-Heiles systems

There are three integrable Hénon-Heiles systems on the plane, which can be identified with appropriate finite-dimensional reductions of the integrable fifth order KdV, Kaup-Kupershmidt and Sawada-Kotera equations [5]. An explicit integration for all these cases is discussed in [3].

We can try to get a hetero-BT for the finite-dimensional Hénon-Heiles systems taking the hetero-BT between these integrable PDEs and then applying the Fordy finite-dimensional reduction [5]. We believe that the same information may be directly extracted from the well-known Lax representation for the Hénon-Heiles system separable in parabolic coordinates.

Let us take a Lax matrix for the first Hénon-Heiles system separable in parabolic coordinates

\[
L(\lambda) = \begin{pmatrix}
\frac{p_2}{2} + \frac{p_1 q_1}{2\lambda} & \lambda - 2q_2 - \frac{q_1^2}{\lambda} \\
\alpha \lambda^2 + 2aq_2 \lambda + a(q_1^2 + 4q_2^2) + \frac{p_1^2}{4\lambda} + \frac{p_2}{2} - \frac{p_1 q_1}{2\lambda}
\end{pmatrix}, \quad a \in \mathbb{R}.
\]

Characteristic polynomial of this matrix

\[
\det \left( L(\lambda) - \mu \right) = \mu^2 - a\lambda^3 - \frac{H_1}{4} + \frac{H_2}{\lambda}
\]
contains the Hamilton function of the first Hénon-Heiles system associated with a fifth order KdV

\[ H_1 = p_1^2 + p_2^2 - 16aq_2(q_1^2 + 2q_2^2) \quad (2.12) \]

and a second integral of motion

\[ H_2 = aq_1^2(q_1^2 + 4q_2^2) + \frac{p_1(q_2p_1 - q_1p_2)}{2}. \]

The auto-BTs preserve the algebraic form of the Hamiltonians [11]. Since the characteristic polynomial is the generating function of these integrals of motion, their invariance amounts to requiring the existence of a similarity transformation for the Lax matrix

\[ \hat{L} = V L V^{-1}, \]

associated with the given auto-BT. The matrix \( V \) needs not to be unique because a dynamical system can have different auto BTs [6, 13].

Let us consider a special, unique similarity transformation associated with matrix

\[ V = \begin{pmatrix} L_{12} & 0 \\ 4(L_{11} - \hat{L}_{11}(\lambda)) & 4L_{12} \end{pmatrix}, \quad (2.13) \]

where \( L_{ij} \) are entries of the Lax matrix (2.11) and

\[ \hat{L}_{11}(\lambda) = \frac{p_2}{2} + \frac{q_1p_1(\lambda - 2q_2)}{2q_1^2}. \]

The Lax matrix \( \hat{L}(\lambda) = V L V^{-1} \) is completely defined by the following conditions:

1. first off-diagonal element of the Lax matrix

\[ \hat{L}_{12}(\lambda) = \frac{(\lambda - u_1)(\lambda - u_2)}{4\lambda} \]

yields initial parabolic coordinates on the plane (2.7);

2. second off-diagonal element

\[ \hat{L}_{21} = 4a(\lambda - v_1)(\lambda - v_2) = 4a\lambda^2 + \frac{(8aq_2^2 - p_2^2)\lambda}{q_1^2} + 4a(q_1^2 + 4q_2^2) + \frac{2p_1(p_1q_2 - p_2q_1)}{q_1^2} \quad (2.14) \]

has only two commuting and functionally independent zeroes \( v_{1,2} \);

3. the conjugated momenta for \( u \) and \( v \) variables are the values of the diagonal element

\[ p_{ui} = \hat{L}_{11}(\lambda = u_i), \quad p_{vi} = \hat{L}_{11}(\lambda = v_i), \quad i = 1, 2. \]

In generic case such \( 2 \times 2 \) Lax matrices \( \hat{L}(\lambda) \) exist only if the genus of hyperelliptic curve defined by equation

\[ \det \left( \hat{L}(\lambda) - \mu \right) = 0 \]

is no more a number of degrees of freedom. The corresponding transformation of the classical \( r \)-matrix is discussed in [10].

Thus, we have two families of variables of separation for the first Hénon-Heiles system and can explicitly define the canonical transformation between them.
Proposition 1 The auto-BT for the first Hénon-Heiles system consists of canonical transformation of the parabolic coordinates

\[(u_1, u_2, p_{u_1}, p_{u_2}) \rightarrow (v_1, v_2, p_{v_1}, p_{v_2})\]

and separated relations

\[\Phi(\lambda, \mu) = \mu^2 - a\lambda^3 = \frac{H_1}{4} - \frac{H_2}{\lambda}, \quad \lambda = u_{1,2}, \quad \mu = p_{u_{1,2}}, \quad (2.15)\]

which allow to get two equivalent systems of the Hamilton-Jacobi equations

\[H_{1,2} \left( \lambda, \frac{\partial S}{\partial \lambda} \right) = \alpha_{1,2}, \quad \lambda = u, v.\]

This auto Bäcklund transformation changes coordinates on an algebraic invariant manifold defined by \(H_{1,2}\) without changing the manifold itself [6].

We can convert this special, unique auto-BT to some analogue of the hetero-BT by adding one more relation. Namely, substituting off-diagonal element \(\tilde{L}_{2,1} (2.14)\) into the definition \(2.10\) one gets the Hamilton function for the second integrable Hénon-Heiles system associated with the Kaup-Kupershmidt equation

\[\tilde{H}_1 = p_1^2 + p_2^2 - 2aq_2(3q_1^2 + 16q_2^2)\] (2.16)

up to rescaling \(p_1 \rightarrow \sqrt{2}p_1\) and \(q_1 \rightarrow q_1/\sqrt{2}\).

After canonical transformation

\[(q, p) \rightarrow (Q, P), \quad P_{1,2} = \frac{p_{v_1} \pm p_{v_2}}{\sqrt{2}}, \quad Q_{1,2} = \frac{v_1 \pm v_2}{\sqrt{2}} \quad (2.17)\]

the same Hamiltonian

\[\tilde{H}_1 = P_1^2 + P_2^2 - 2aQ_2(3Q_1^2 + Q_2^2)\] (2.18)

defines a third integrable Hénon-Heiles system associated with the the Sawada-Kotera equation. According [5] canonical transformation \(2.17\) is a counterpart of the gauge equivalence of the Sawada-Kotera and Kaup-Kupershmidt equations.

So, all the Hénon-Heiles systems on the plane are simultaneously separable in \(v\)-variables, and we suppose that this fact allows us to define some natural counterpart of the hetero-BT.

Proposition 2 For the Hénon-Heiles systems on the plane \((2.12)\) and \((2.16, 2.18)\) an analogue of the hetero-BT consists of the same canonical transformation of the parabolic coordinates

\[(u_1, u_2, p_{u_1}, p_{u_2}) \rightarrow (v_1, v_2, p_{v_1}, p_{v_2})\]

the same separated relations \(\Phi(\lambda, \mu) (2.17)\) and an additional rule

\[\tilde{H}_{1,2} = \Phi(v_1, p_{v_1}) \pm \Phi(v_2, p_{v_2}),\]

which allow to get two different systems of the Hamilton-Jacobi equations

\[H_{1,2} \left( u, \frac{\partial S}{\partial u} \right) = \alpha_{1,2} \quad \text{and} \quad \tilde{H}_{1,2} \left( v, \frac{\partial \tilde{S}}{\partial v} \right) = \tilde{\alpha}_{1,2}.\]

This analogue of the hetero-BT relates different algebraic invariant manifolds associated with Hamiltonians \(H_{1,2}\) and \(\tilde{H}_{1,2}\) similar to the well-studied relations between different invariant manifolds in the theory of superintegrable systems.

For the first Hénon-Heiles system on the plane \((2.12)\) we can consider parabolic variables \((u_{1,2}, p_{u_{1,2}})\) as coordinates on the Jacobian variety defined by equations \((2.15)\). In order to get integrals of motion for the second or third Hénon-Heiles systems \((2.16, 2.18)\) we have to take linear combinations of these equations and to make simultaneously the special shift of the coordinates \((u, p_u) \rightarrow (v, p_v)\) on the Jacobian variety.
2.2 Integrable Hamiltonian with velocity dependent potential

It is well-known that Hamilton-Jacobi equation is separable in parabolic coordinates $u_{1,2}$ if the Hamilton function has the form

$$\dot{H} = p_1^2 + p_2^2 + V_N(q_1, q_2), \quad V_N = 4a \sum_{k=0}^{[N/2]} 2^{1-2k} \binom{N-k}{k} q_1^{2k} q_2^{N-2k},$$

where the positive integer $N$ enumerates the members of the hierarchy.

At $N = 3$ one gets the Hénon-Heiles system (2.12), at $N = 4$ the next member of hierarchy is a "(1:12:16)" system with the following Hamiltonian

$$\dot{H} = p_1^2 + p_2^2 - 4a (q_1^4 + 12q_1^2 q_2^2 + 16q_2^4). \quad (2.19)$$

The corresponding Lax matrix is equal to

$$L(\lambda) = \left( \begin{array}{cc} \frac{p_2}{2} + \frac{p_1 q_1}{2\lambda} & \lambda - 2q_2 - \frac{q_1^2}{\lambda} \\ a\lambda^3 + 2aq_2\lambda^2 + a(q_1^2 + 4q_2^2)\lambda + 4aq_2(q_1^2 + 2q_2^2) + \frac{p_1^2}{4\lambda} - \frac{p_2}{2} - \frac{p_1 q_1}{2\lambda} \end{array} \right). \quad (2.20)$$

After similarity transformation of $L(\lambda)$ with matrix $V (2.13)$, where

$$\hat{L}_{11}(\lambda) = \sqrt{a}\lambda^2 - \frac{4\sqrt{a}q_2q_1 - p_1}{2q_1}\lambda - \frac{2\sqrt{a}q_1^3 + 2p_1q_2 - p_2q_1}{2q_1},$$

one gets the Lax matrix with two off-diagonal elements $\hat{L}_{12}(\lambda)$ and $\hat{L}_{21}(\lambda)$, which yield two families of variables of separation.

As above first coordinates are parabolic coordinates $u_{1,2}$, whereas second coordinates $v_{1,2}$ are zeroes of the polynomial

$$\hat{L}_{21} = \frac{4(4aq_1q_2 - \sqrt{a}p_1)}{q_1} \lambda^2 + \frac{8aq_1^2(q_1^2 + 2q_2^2) + 4\sqrt{a}q_1(2p_1q_2 - p_2q_1) - p_1^2}{q_1^2} \lambda + \frac{16aq_1^2q_2(q_1^2 + 2q_2^2) + 2p_1(p_1q_2 - p_2q_1)}{q_1^2} = \frac{4(4aq_1q_2 - \sqrt{a}p_1)}{q_1}(\lambda - v_1)(\lambda - v_2).$$

Substituting this polynomial into the definition (2.10) one gets integrable Hamiltonian with velocity dependent potential

$$\hat{H} = \frac{p_1^2}{2} + p_2^2 + 4\sqrt{a}p_1q_1q_2 - 2\sqrt{a}p_2q_1^2 - 8aq_2^2(5q_1^2 + 8q_2^2).$$

Using canonical transformation we can rewrite this Hamiltonian in a more symmetric form

$$\hat{H} = p_1^2 + p_2^2 - 3\sqrt{a}p_2q_1^2 + 2a(q_1^4 - 12q_1^2q_2^2 - 32q_2^4). \quad (2.21)$$

The corresponding second integral of motion is fourth order polynomial in momenta

$$\hat{H}_2 = p_1^4 + 4q_1^4(q_1^2 - 8q_1^2q_2^2 - 112q_2^4) a^2 + 4q_1^3(64p_1q_2^3 - p_2q_1^3 - 12p_2q_1q_2^2) a^{3/2} + q_1^2(4p_1^2q_2^2 - 48p_1^2q_2^2 + 32p_1p_2q_1q_2 + p_2^2q_1^2) a - 6a^{1/2}p_1^2p_2q_1^2,$$

which also can be obtained from the Lax matrix $\hat{L}(\lambda)$.

Canonical transformation (2.17) allows us to identify a Hamilton function with velocity dependent potential (2.21) and Hamilton function

$$\hat{H} = P_1^2 + P_2^2 - a(Q_1^2 + 6Q_1^2Q_2^2 + Q_2^4)$$
similar to the relation between second and third Hénon-Heiles systems.

Canonical transformation $(u, p_u) \rightarrow (v, p_v)$ is the special auto-BT for the "(1:12:16)" system, which can be considered as a hetero-BT relating two different Hamilton-Jacobi equations associated with Hamiltonians $H$ (2.19) and $\tilde{H}$ (2.21), respectively.

Of course, integrable system on the plane (2.21) could be obtained in the framework of different theories, see [5, 8, 12] and references within. However, using auto-BT and $2 \times 2$ Lax matrices we obtain not only integrals of motion, but also variables of separation, separated relations etc. In similar manner we can construct various simultaneously separable integrable systems associated with other curvilinear coordinates, see examples in [10].

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References

[1] Ablowitz, M. J. and Segur, H., Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] Bobenko, A.I. and Suris, Yu.B., Discrete differential geometry: integrable structure, Graduate Studies in Mathematics, v. 98, AMS, Providence, 2008.
[3] Conte, R., Musette, M. and Verhoeven, C., Explicit integration of the Hénon-Heiles Hamiltonians, J. Nonlinear Math. Phys., v.12, Suppl. 1, pp.212-227, 2005.
[4] Darboux, G., Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, vol. III (Gauthier-Villars, Paris, 1894).
[5] Fordy A. P., The Hénon-Heiles system revisited, Phys. D, v.52, no.2-3, pp.204-210, 1991.
[6] Kuznetsov, V. and Vanhaecke, P., Bäcklund transformations for finite-dimensional integrable systems: a geometric approach, J. Geom. Phys., v.44, n.1, pp.1-40, 2002.
[7] Miller W., Post S. and Winternitz P., Classical and quantum superintegrability with applications, J. Phys. A: Math. and Theor., v.46, n.42, 423001, 2013.
[8] Pucacco G., On integrable Hamiltonians with velocity dependent potentials, Celestial Mech. and Dyn. Astronomy, v.90, n.1-2, pp.109-123, 2004.
[9] Rogers, C. and Schief, W.K., Bäcklund and Darboux transformations: geometry and modern applications in soliton theory, vol. 30, Cambridge University Press, 2002.
[10] Tsiganov, A.V., Simultaneous separation for the Neumann and Chaplygin systems, Reg. Chaotic Dyn., v. 20, n.1, pp.74-93, 2015.
[11] Wojciechowski S., The analogue of the Bäcklund transformation for integrable many-body systems, J. Phys. A: Math. Gen., v.15, pp.653-657, 1982.
[12] Yehia H.M. and Hussein A.M., New families of integrable two-dimensional systems with quartic second integrals, arXiv:1308.1442, 2013.
[13] Zullo F., Bäcklund transformations and Hamiltonian flows, J. Phys. A: Math. and Theor., v.46, n.14, 145203, 2013.