ABSTRACT

Effective potential for scalar $\lambda \phi^4$ theory is obtained using the exact renormalization group method which includes both the usual one-loop contribution as well as the dominant higher loop effects. Our numerical calculation indicates a breakdown of naive one-loop result for sufficiently large renormalized coupling constant.
Renormalization group (RG) has been a powerful non-perturbative method in probing how fundamental laws of physics are modified with varying observational length scale [1]. Starting from a bare action $S$ characterizing the system at a typical microscopical length scale $\Lambda^{-1}$, where $\Lambda$ is the momentum cut-off of the theory, if one wishes to examine the physics at a scale $\tilde{\Lambda} < \Lambda$, it is often desirable to consider a low energy effective action $\tilde{S}$ by a systematic elimination of the modes between $\tilde{\Lambda}$ and $\Lambda$. In other words, the cut-off is lowered to $\tilde{\Lambda}$ at the expense of having to use a more complicated action $\tilde{S}$ which in general also contains non-local interactions. How $\tilde{S}$ connects to the original $S$ is dictated by Wilson’s functional differential RG flow equation. Therefore, a complete solution of the flow equation would provide a complete knowledge of the theory at any length scale.

In the present work, we derive a RG flow equation based on the concept of blocking transformation [2], [3]. Consider for simplicity a scalar theory described by the bare lagrangian:

$$L = \frac{1}{2} \left( \partial_\mu \phi \right)^2 + V(\phi).$$  

Instead of using the original field variable $\phi(x)$, we introduce the coarse-grained “blocked variable”:

$$\phi_k(x) = \int d^4 y \, \rho_k(x-y) \phi(y),$$  

via a smearing function $\rho_k(x)$, with $k^{-1}$ being the characteristic linear dimension of the region over which the field averaging is performed. For simplicity, the smearing function shall be chosen as a sharp momentum regulator:

$$\rho_k(x) = \int_k^\Lambda \frac{d^4 p}{(2\pi)^4} \, e^{-ipx} \rightarrow \rho_k(p) = \Theta(k-p).$$  

By splitting $\phi(x)$ into the slowly-varying background $\chi(x)$, and the fast-fluctuating modes $\xi(x)$:

$$\phi(p) = \begin{cases} 
\chi(p), & 0 \leq p \leq k \\
\xi(p), & k < p < \Lambda,
\end{cases}$$  

such that $\phi_k(p) = \rho_k(p)\phi(p) = \chi(p)$, and by demanding that the field average $\Phi$ of a given block coincides with the slowly varying background, one then integrates out $\xi(x)$ using the loop expansion to obtain the blocked action $\tilde{S}_k[\Phi]$, which is the effective action at the energy scale $k$. In the one loop approximation, it takes on the familiar form:

$$\tilde{S}_k[\Phi] = S[\Phi] + \frac{1}{2} \text{Tr}' \ln \left[ -\partial^2 + V''(\Phi) \right],$$  

where the prime notation stands for derivatives taken with respect to $\phi$, and Tr’ implies that the trace in momentum space is to be carried out for $k \leq p \leq \Lambda$, i.e., the modes which are to be eliminated by blocking transformation.

The blocked action in its most general form, can however be written as

$$\tilde{S}_k[\Phi] = \int_x \left( \frac{Z_k(\Phi)}{2} (\partial_\mu \Phi)^2 + U_k(\Phi) + O(\partial^4) \right),$$  

2
where $Z_k$ is the wavefunction renormalization constant. The task of analyzing the RG flow pattern of the theory can be simplified if one concentrates on the blocked potential $U_k(\Phi)$ which is the derivative-independent sector of $\tilde{S}_k$. In this framework, the wavefunction renormalization is set to be unity and its small correction can be computed using the derivative expansion technique illustrated in [4], [5]. The lowering of the cut-off from $\Lambda$ to $\sim k$ can be achieved in a “smooth” manner by first dividing the volume of momentum integration into a large number of “thin shells” of width $\delta k$, each containing a small number of modes, followed by a systematic integration of each individual shell. This results in the following non-linear RG evolution equation at the one-loop level:

$$k \frac{\partial U_k(\Phi)}{\partial k} = -\frac{k^4}{16\pi^2} \ln \left( \frac{k^2 + U_k''(\Phi)}{k^2 + U_k''(0)} \right).$$

(7)

However, since each loop integral is proportional to the volume of the thin shell, the higher loop contributions to the functional flow pattern are suppressed in the small $\delta k$ limit. Hence, (7) can be taken as an “exact” RG equation [3]. This can be contrasted with the blocked potential $\tilde{U}_k(\Phi)$ which is derived by eliminating each individual mode independently from one another:

$$k \frac{\partial \tilde{U}_k(\Phi)}{\partial k} = -\frac{k^4}{16\pi^2} \ln \left( \frac{k^2 + V_R''(\Phi)}{k^2 + V_R''(0)} \right),$$

(8)

where $V_R(\Phi)$ is the classical potential less the counterterm sector. Our RG equation generated by sharp cut-off regulator method can also be compared with that obtained in [6], where a smooth decrease of the cut-off from $\Lambda$ to $e^{-t}\Lambda$ for an arbitrarily small scale factor $t$ leads to:

$$\frac{\partial S}{\partial t} = \frac{1}{2t} \int \left\{ \ln \left( \frac{\partial^2 S}{\partial \phi(p) \partial \phi(-p)} \right) - \frac{\partial S}{\partial \phi(p)} \frac{\partial S}{\partial \phi(-p)} \left( \frac{\partial^2 S}{\partial \phi(p) \partial \phi(-p)} \right)^{-1} \right\} - \int p \phi(p) p_{\mu} \phi_{\mu} \frac{\partial S}{\partial \phi(p)} + dS + \left( 1 - \frac{d}{2} - \eta \right) \int p \phi(p) \frac{\partial S}{\partial \phi(p)} + \text{const.},$$

(9)

where the prime notations in the integration and the derivative indicate respectively, that $p$ lies in the range $e^{-t}\Lambda \leq p \leq \Lambda$, and that the derivative does not act on the $\delta$ function in $\partial S/\partial \phi(p)$. Further projecting $\phi(p)$ onto $\phi(0) = \Phi$ [7], (9) is reduced to a RG equation for the effective potential $U_t$:

$$\frac{\partial U_t(\Phi)}{\partial t} = \frac{p^4}{(4\pi)^{d/2} \Gamma(\frac{d}{2})} \ln \left( 1 + \frac{1}{p^2} \frac{\partial^2 U_t(\Phi)}{\partial \Phi^2} \right) + d \cdot U_t(\Phi) + \left( 1 - \frac{d}{2} - \eta \right) \Phi \frac{\partial U_t(\Phi)}{\partial \Phi},$$

(10)

which is reminiscent to what we have in (7) if the terms generated from rescalings are dropped. An alternative formulation of RG and proof of renormalizability can be found in [8].

The power of (7) is that any operator generated in the effective potential as the modes are being eliminated will be kept throughout [5]. If one is interested in the critical phenomena and the values of critical exponents, the complicated RG flow equation can be simplified in the vicinity of fixed points. For example, for the $\lambda \phi^4$ theory in $d = 4$ near the Gaussian infrared fixed point, it suffices to retain only the relevant $\Phi^2$ and $\Phi^4$ terms for determining the critical exponents to high accuracy; all other terms are irrelevant [9]. However, one must
remember that operators are always classified with respect to a particular fixed point. If a theory has, say two fixed points, one ultraviolet and one infrared, it is possible to generate in the effective theory an operator which becomes relevant in the infrared while being irrelevant in the ultraviolet [5]. As the RG trajectory reaches the crossover regime, a new classification of operators becomes necessary since the number of relevant operators corresponds to the number of unstable directions. Failure to provide an accurate classification would lead to substantial deformation of the RG trajectory. In lacking of a general classification scheme, it is desirable to retain as many new operators as possible in the effective lagrangian. Such feature indeed can easily be incorporated in our RG approach.

In inquiring the importance of the contributions from higher loops and the irrelevant operators in making up the effective potential, we shall compare $U_{k=0}(\Phi)$ generated from (7) with the standard one-loop Coleman-Weinberg result [10], $\tilde{U}_{k=0}(\Phi)$ which ignores the impact of these terms. For simplicity, we consider $\lambda \phi^4$ theory where

$$V(\Phi) = \frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4!} \Phi^4.$$  (11)

The main message here is to report the difference between $U_{k=0}(\Phi)$ and $\tilde{U}_{k=0}(\Phi)$ even for small renormalized coupling constant $\lambda_R = U_{k=0}^{(4)}(0)$, and show that the discrepancy grows with increasing $\lambda_R$. This is directly related to the fact that the latter utilizes independent-mode approximation, and hence neglects the continuous feedback from the modes which are being eliminated successively. As we shall see below, the most severe error, however, comes from its truncation at the one-loop order, thereby ignoring the higher loop terms which turn out to be large.

Our strategy consists of the following: Suppose we are given the bare mass parameter $\mu^2$, the bare coupling constant $\lambda$ and the cut-off $\Lambda$ as the input parameters. This allows us to determine the shape of the initial bare potential $V(\Phi)$. In our numerical integration, all dimensional parameters will be scaled with respect to $\Lambda \equiv 1$. A negative $\mu^2$ ensures the characteristic double-welled feature for $V(\Phi)$. The one-loop potential derived from (8) takes on the form:

$$\tilde{U}_k(\Phi) = \frac{\mu^2}{2} \Phi^2 \left[ 1 - \frac{\lambda}{64\pi^2} \left( 1 + \frac{k^2}{\mu_R^2} \right) \right] + \frac{\lambda}{4!} \Phi^4 \left( 1 - \frac{9\lambda}{64\pi^2} \right) + \frac{1}{64\pi^2} \left[ \left( \frac{\mu^2}{2} + \frac{\lambda}{2} \Phi^2 \right)^2 - k^4 \right] \ln \left( \frac{k^2 + \mu^2}{k^2 + \mu_R^2} \right),$$  (12)

which in the $k = 0$ limit, simplifies to [10]:

$$\tilde{U}_{k=0}(\Phi) = \frac{\mu^2}{2} \Phi^2 \left( 1 - \frac{\lambda}{64\pi^2} \right) + \frac{\lambda}{4!} \Phi^4 \left( 1 - \frac{9\lambda}{64\pi^2} \right) + \frac{1}{64\pi^2} \left( \mu^2 + \frac{\lambda}{2} \Phi^2 \right)^2 \ln \left( \frac{\mu^2}{\mu_R^2} + \frac{\lambda}{2} \Phi^2 / \mu^2 \right).$$  (13)

The above forms are deduced with the help of the one-loop renormalization conditions:

$$\begin{align*}
\mu^2 &= \mu_R^2 - \frac{\lambda_R}{32\pi^2} \left[ \lambda^2 + \mu_R^2 \ln \left( \frac{\mu_R^2}{\lambda^2} \right) \right] \\
\lambda &= \lambda_R + \frac{3\lambda_R}{32\pi^2} \left[ \ln \left( \frac{\lambda^2}{\mu_R^2} \right) - 1 \right].
\end{align*}$$  (14)
On the other hand, the RG improved potential $U_k(\Phi)$ is solved numerically. In Fig. 1, the gradual transition from a double-welled bare potential $U_k(\Phi) = V(\Phi)$ to $U_k = 0(\Phi)$ which has a unique minimum at $\Phi = 0$ is depicted. At large $k$, $U_k(\Phi)$ and $\tilde{U}_k(\Phi)$ are relatively close to one another. However, the deviation becomes more noticeable as $k$ is lowered, as can be seen in Fig. 2. For comparative purpose, one may simply examine the ratio of the renormalized mass parameters, $R = U''_{k=0}(0)/\tilde{U}''_{k=0}(0)$, where $\tilde{U}''_{k=0}(0) = \mu_R^2$. Since the resulting renormalized coupling constants from either approach do not differ appreciably: $U^{(4)}_{k=0}(0)/\tilde{U}^{(4)}_{k=0}(0) = 1.006$, we shall simply denote them as $\lambda_R$. We notice that even for $\lambda_R = 0.1$, the ratio of the mass parameters is $R = 4.93$. Such discrepancy can be explained by the following arguments: Our RG approach makes use of the “dressed”, effective vertex functions at each step of integration for deducing the next lower energy improved vertex functions. Therefore, one would naturally expect additional contributions from higher loops as well as higher order field operators. This method is analogous to a resummation over daisy and superdaisy diagrams in finite temperature theory [11]. Fig. 3 shows that the accumulation of higher order field operators only gives a small correction to $U_k(\Phi)$, thereby making it evident that the discrepancy is largely due to higher loop effects. One is then lead to the inevitable conclusion that the one-loop independent-mode approximation must break down. That is, it is insufficient to use the one-loop $\tilde{U}_{k=0}(\Phi)$ as the effective potential in the infrared regime.

Can we reconcile the perturbative result (13) with that obtained through the “exact” RG flow equation? Fortunately, there is one parameter which can be tuned: the renormalized coupling constant $\lambda_R$. One sees that the higher loop effects included in the RG approach are all multiplied by some power of $\lambda_R$, and only by judicious choice of very small $\lambda_R$ can their effects be safely neglected. In Fig. 4, we see that as $\lambda_R$ is decreased, the agreement between $U_{k=0}(\Phi)$ and $\tilde{U}_{k=0}(\Phi)$ becomes better. At $\lambda_R = 0.01$, the two results differ only by 6%. Improvement of the one-loop result perhaps can best be seen from Fig. 5 in which $R \to 1$ as $\lambda_R$ becomes vanishingly small. One therefore concludes that the naive one-loop result $\tilde{U}_{k=0}(\Phi)$ that ignores the impacts of higher loops can be valid for very small $\lambda_R$. For $\lambda_R$ not too small, one must take into account their effects.

If one takes the cut-off $\Lambda$ seriously as part of the effective theory, then by choosing a large yet finite $\Lambda$, an interacting theory consistent with perturbation expansion may be defined without confronting the complication of “triviality”. Nevertheless, the value $\lambda_R$ takes should be checked by our improved RG method to ensure the reliability of one-loop perturbative result. For a given $\Lambda$, we shall denote by $\lambda_0$ the coupling constant which results in a 20% difference between our RG method and the standard one-loop integration, i.e., $R = 1.2$ for a given $\lambda_0$. A 20% difference, in our opinion, still lies within the limit of tolerance for perturbation. In Fig. 6, the relation between $\lambda_0$ and cut-off $\Lambda$ is illustrated in logarithmic scale. One easily sees that the larger the $\Lambda$, the smaller $\lambda_0$ must be used in order to trust the simple one-loop calculation.

We also comment on the sensitivity of $R$ to the choice of cut-off $\Lambda$. Eq. (14) shows that the finite renormalized mass parameter $\mu_R^2$ results from cancellation of two large numbers, namely, the $\Lambda^2$-dependent counterterm and the bare mass parameter $\mu_0^2$. Therefore, even a small adjustment of $\Lambda$ can lead to a substantial change in $\mu_R^2$. By integrating (7) with slight variation of initial choice of $\Lambda$, we find linear dependence of mass ratio $R$ on $\Lambda$. As illustrated in Fig. 7, the slope of the line can be approximated by $\frac{\lambda_0^2}{16\pi^2\mu_R}$, which agrees with (14). On the other hand, we also see from Fig. 7 that the mass parameter obtained using
the RG flow equation (7) with $\Lambda = 0.994$ is the same as the simple one-loop result with $\Lambda = 1$. Interpreting the result differently, we say that the higher loop contributions can be compensated by using a slightly higher cut-off. This observation has yet one other implication: If $\mu^2_R$, the mass of the scalar particle can be measured precisely, then it becomes imperative to know the cut-off of the theory to a very high accuracy so that when RG is applied to the microscopic lagrangian at the cut-off level, one eventually arrives at a macroscopic lagrangian describing a large-distance physics that agrees with experiments. It would be interesting to explore the dependence of the Higgs field mass in the Standard Model on the choice of $\Lambda$ [12]. Such peculiar sensitivity is only characteristic of the scalar theory, and should not be expected in gauge theories where only logarithmic divergences appear.

In summary, we have introduced in this paper an improved RG flow equation whose non-perturbative nature takes into consideration the additional dominant loop effects. It is concluded that one-loop result is inadequate unless $\lambda_R$ is set to be small or higher loop effects are included. Typically one chooses $\lambda_R < 0.02$ for $\Lambda^2/\mu^2_R = 10^6$ to safely ignore higher loops. Our conceptually simple yet powerful non-perturbative method is now being implemented to systems at finite temperature [13]. With our approach, daisy, superdaisy and higher order effects are automatically included. In addition, upon employing a suitable choice of smearing function in the proper-time formalism [14], the RG evolution equations for gauge theories can also be generated while preserving gauge symmetry.

ACKNOWLEDGEMENT

We thank Professor B. Müller critical for reading of the manuscript. S.-B. L is grateful to Professor J. Polonyi for stimulating discussions.

REFERENCES

[1] K. Wilson and J. Kogut, Phys. Rep. 12C (1975) 75.
[2] S.-B. Liao and J. Polonyi, Ann. Phys. 222 (1993) 122.
[3] C. Wetterich, Nucl. Phys. B352 (1991) 529.
[4] C. M. Fraser, Z. Phys. C28 (1985) 101.
[5] S.-B. Liao and J. Polonyi, Duke-TH-94-64, LPT 94-3.
[6] F.J. Wegner and A. Houghton, Phys. Rev A8 (1972) 401.
[7] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B270 (1986) 687.
[8] J. Polchinsky, Nucl. Phys B231 (1984) 269.
[9] P. E. Haagensen, Y. Kubysin, J. I. Latorre and E. Moreno, UB-ECM-PF#93-20; J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford University Press (1989).
[10] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888.
[11] L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3320.
[12] P. Hasenfratz and J. Nager, *Z. Phys.* C37 (1988) 477.

[13] S.-B. Liao and J. Polonyi,MIT-CTP-2143.

[14] M. Oleszczuk, “A symmetries Preserving Cut-Off Regularization”; S.-B. Liao and J. Polonyi, Duke-TH-94-65.

FIGURE CAPTIONS

Fig. 1 Potentials as function of $\Phi$ for $\Lambda = 10$ and $\mu_R = 10^{-2}$. The solid, dotted, and square dotted lines represent, respectively, $V(\Phi)$, $\tilde{U}_{k=0}(\Phi)$ and $U_{k=0}(\Phi)$.

Fig. 2 Comparisons of $\tilde{U}_{k}(\Phi)$ and $U_{k}(\Phi)$ for various values of $k\Lambda$, with $\Lambda = 10$ and $\mu_R = 10^{-2}$.

Fig. 3 Contribution from higher order field operators (solid line). Dotted line represents $U_{k=0}(\Phi)$.

Fig. 4 Comparisons of $\tilde{U}_{k=0}(\Phi)$ and $U_{k=0}(\Phi)$ for various values of $\lambda R$, with $\Lambda = 10$ and $\mu_R = 10^{-2}$.

Fig. 5 Ratio between the two mass curvatures as function of renormalized coupling constant $\lambda R$. Note that $R \to 1$ for $\lambda R \to 0$.

Fig. 6 Cut-off dependence of $\lambda_0$, the coupling constant which yields $R = 1.2$.

Fig. 7 Dependence of $R$ on the cut-off $\Lambda$. 

This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9404086v2
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9404086v2
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9404086v2
This figure "fig1-4.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9404086v2
This figure "fig1-5.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9404086v2