Variational problems with splitting linear-superlinear growth conditions

Michael Bildhauer & Martin Fuchs

Abstract

Variational problems with mixed linear-superlinear growth conditions are considered. In the twodimensional case the minimizing problem is given by

$$J[w] = \int_{\Omega} \left[ f_1(\partial w) + f_2(\partial^2 w) \right] dx \to \min$$

w.r.t. a suitable class of comparison functions. Here $f_1$ is supposed to be a convex energy density with linear growth, $f_2$ is supposed to be of superlinear growth, for instance to be given by a $N$-function or just bounded from below by a $N$-function. One motivation for this kind of problem located between the well known splitting type problems of superlinear growth and the splitting type problems with linear growth (recently considered in [1]) is the link to mathematical problems in plasticity (compare [2]). Here we prove results on the appropriate way of relaxation including approximation procedures, duality, existence and uniqueness of solutions as well as apriori bounds.

1 Introduction

In the last decades the study of variational problems with nonstandard growth conditions developed to one of the main topics in the calculus of variations and related areas. We do not want to go into details and will not present the historical line of this development. The reader may find this background information, for instance, in the recent paper [3].

Let us just mention a few aspects, which serve as a motivation for the manuscript at hand.

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As one of the first main contributions Giaquinta considered the most common prototype of \((p,q)\)-growth in the sense of minimizing a splitting functional

\[
\int_{\Omega} f(\nabla u) \, dx = \int_{\Omega} \left[ f_1(\partial_1 u) + f_2(\partial_2 u) \right] \, dx \rightarrow \min
\]

in a suitable class of comparison functions where \(f_1\) and \(f_2\) are supposed to have different growth rates. In [4] he presented a famous counterexample which shows that in general and even in the scalar case we cannot expect the smoothness of solutions to this variational problem.

Of course (1.1) also serves as a motivation to study variational problems with non-uniform ellipticity conditions. As one variant we may consider energy densities of class \(C^2\) satisfying with different exponents \(1 < p < q\) (\(c_1, c_2 > 0, \xi, \eta \in \mathbb{R}^{nN}\))

\[
c_1 \left( 1 + |\xi|^2 \right)^{\frac{2}{p-2}} |\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2 \left( 1 + |\xi|^2 \right)^{\frac{2}{q-2}} |\eta|^2.
\]

Here a lot of important contributions in the scalar and also in the vectorial setting can be found, we just mention [5] as one central contribution in the long series of papers in this direction.

Related to (1.2), Frehse and Seregin (6) considered plastic materials with logarithmic hardening, i.e. the energy density \(f(\xi) = |\xi| \ln \left( 1 + |\xi| \right)\) of nearly linear growth. Due to [7] we have full regularity for this particular kind of model.

We finally pass to the case of linear growth conditions which have a uniform growth w.r.t. the energy density and nevertheless just satisfy a non-uniform ellipticity condition in the sense of (1.2). Of course the minimal surface case is the most prominent representative for this kind of elliptic problems satisfying with suitable constants \(a_1, b_1, c_1, c_2 > 0, a_2, b_2 \geq 0\), and for all \(\xi, \eta \in \mathbb{R}^{nN}\)

\[
a_1 |\xi| - a_2 \leq f(|\xi|) \leq b_1 |\xi| + b_2,
\]

\[
c_1 \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}} |\eta|^2 \leq D^2 f(\xi)(\eta, \eta) \leq c_2 \left( 1 + |\xi|^2 \right)^{-\frac{1}{2}} |\eta|^2.
\]

In the minimal surface case we have \(\mu = 3\) and we like to mention the pioneering work of Giaquinta, Modica and Souček [8], [9] in the list of outstanding contributions. In [8] and [9] a suitable relaxation is discussed together with a subsequent proof of apriori estimates. We note that the uniqueness of solutions in general is lost by passing to the relaxed problem.
In [10], condition (1.3) was introduced defining a class of $\mu$-elliptic energy densities. The regularity theory for minimizers was studied in a series of subsequent papers, e.g., [11]. We also like to mention the Lipschitz estimates of Marcellini and Papi [12], which cover a broad class of the functionals we discussed up to now.

Very recently, the authors [1] considered variational problems of splitting type as given in (1.1) but now with two energy parts being of linear growth. Here it turns out that the right-hand side of the ellipticity condition in (1.3) is no longer valid and we just have for all $\xi, \eta \in \mathbb{R}^{nN}$ with a postive constant $c$

$$D^2 f(\xi)(\eta, \eta) \leq c|\eta|^2.$$ 

Nevertheless, some natural assumptions still imply regularity and uniqueness properties of solutions to the relaxed problem.

In the manuscript at hand we follow this line of studying variational problems of splitting type by now considering variational problems with mixed linear- superlinear growth conditions.

A first step in this direction was already made in Chapter 6 of [13]. The results given there follow from suitable apriori estimates which are available if the non-uniform ellipticity is not too bad. This leads to the analysis of the set of cluster points of minimizing sequences and to some kind of local interpretation for the stress tensor, although the existence and the uniqueness of dual solutions were not established (compare Remark 6.15 of [13]).

We like to finish this introductory remarks by mentioning a prominent application of a mixed linear- superlinear growth problem, which in [2] is discussed as the Hencky plasticity model. This problem takes the form (compare (4.17) of [2])

$$\inf_{v \in C} \left\{ \int_{\Omega} (\text{div} \, v)^2 \, dx + \int_{\Omega} \psi(\varepsilon^D(v)) \, dx - L(v) \right\}$$

with a suitable class $C \subset W^{1,2}(\Omega; \mathbb{R}^3)$, some volume force $L$ and the deviatomic part $\varepsilon^D(u)$ of the symmetric gradient $\varepsilon(u)$. Here $\text{div} \, v$ enters with quadratic growth while the function $\psi$ is of linear growth w.r.t. the tensor $\varepsilon^D(u)$. Let us note that this plasticity model is based on the dual point of view, i.e. the so-called sur-potential $\psi$ is introduced through the conjugate function $\psi^*$. In general a more explicit expression cannot be given (see Remark 4.1, p. 75 of [2]). One particular example is of the form $\psi(\xi^D) = \Phi(|\xi^D|)$.
with (for some positive constants $k, \mu$)
\[
\Phi(s) = \begin{cases} 
\mu s^2 & \text{if } |s| \leq \frac{k}{\sqrt{2\mu}}, \\
\sqrt{2k}|s| - \frac{k^2}{2\mu} & \text{if } |s| \geq \frac{k}{\sqrt{2\mu}}.
\end{cases}
\]

However, at this stage our main difficulty in comparison to the known results for the Hencky model is quite hidden. We postpone a refined discussion to Remark 2.1.

Now let us introduce the general framework of our considerations in a more precise way. For the sake of notational simplicity we restrict our considerations to the case that a linear growth condition is satisfied in only one coordinate direction.

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and denote let $f: \mathbb{R}^n \to \mathbb{R}$ be an energy density of class $C^2(\mathbb{R}^n)$ which is decomposed in the form
\[
f(\xi) = f_1(\xi_1) + f_2(\xi_2), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^{n-1}.
\]
(1.4)
Here we assume that $f_1: \mathbb{R} \to \mathbb{R}$ and $f_2: \mathbb{R}^{n-1} \to \mathbb{R}$ are convex functions of class $C^2(\mathbb{R})$ satisfying with $a_i, b_i \geq 0, a_1, a_3, b_1 > 0$ and with a $N$-function $A: \mathbb{R} \to \mathbb{R}$:
\[
a_1|\xi_1| - a_2 \leq f_1(\xi_1) \leq a_3|\xi_1| + a_4, \quad \xi_1 \in \mathbb{R},
\]
\[
b_1A(|\xi_2|) - b_2 \leq f_2(\xi_2), \quad \xi_2 \in \mathbb{R}^{n-1}.
\]
(1.5)
For the definition and the properties of $N$-functions and Orlicz-Sobolev space we refer to the monographs [14] or [15]. The basics needed here are summarized in [16] or [17]. We suppose that $A: [0, \infty) \to [0, \infty)$, satisfies

$(N1)$ $A$ is continuous, strictly increasing and convex.

$(N2)$ $\lim_{t \to 0} \frac{A(t)}{t} = 0$ and $\lim_{t \to \infty} \frac{A(t)}{t} = \infty$.

$(N3)$ There exist constants $k, t_0 > 0$ such that $A(2t) \leq kA(t)$ for all $t \geq t_0$, where $(N3)$ is called a $\Delta_2$-condition near infinity.

Concerning the function $f_1$ we suppose that there exist a real number $\mu > 1$ such that
\[
c_1(1 + |\xi_1|)^{-\mu} \leq f_1''(\xi_1) \leq c_1, \quad \xi_1 \in \mathbb{R},
\]
(1.6)
holds with constants $c_1, c_2 > 0$.

For $f_2$ we suppose that for some real number $\hat{\mu} < 2$
\begin{equation}
  c_2 (1 + |\xi|)^{-\hat{\mu}} |\eta|^2 \leq D^2 f_2(\xi)(\eta, \eta) \leq c_2 |\eta|^2, \quad t \in \mathbb{R},
\end{equation}
holds with constants $c_2, c_2 > 0$ and for all $\xi, \eta \in \mathbb{R}^{n-1}$. Moreover we suppose some kind of triangle inequality for $f_2$: there exists a real number $c_3 > 0$ such that for all $\xi_2, \hat{\xi}_2 \in \mathbb{R}^{n-1}$
\begin{equation}
  f_2(\xi_2 + \hat{\xi}_2) \leq c_3 \left[ f_2(\xi_2) + f_2(\hat{\xi}_2) \right].
\end{equation}
This condition, for instance, follows from the convexity of $f_2$ together with some $\Delta_2$-condition.

For the definition of the Sobolev spaces $W^k_p$ and their local variants we refer to the textbook of Adams ([18]), the notation in the case of functions of bounded variation can be found, e.g., in the monographs [19] and [20]. For the sake of completeness we recall the definition of the Orlicz-Sobolev space generated by a $N$-function $A$ (see [14], [15]).

For a bounded domain $\Omega$
\begin{equation}
  L_A(\Omega) := \left\{ u : \Omega \to \mathbb{R}, \text{u is a measurable function such that} \right. \end{equation}
\begin{equation}
  \left. \text{there exists $\lambda > 0$ with } \int_{\Omega} A(\lambda |u|) \, dx < +\infty \right\}
\end{equation}
is called Orlicz-space equipped with the Luxemburg norm
\begin{equation}
  \|u\|_{L_A(\Omega)} = \inf \left\{ l > 0 : \int_{\Omega} A\left( \frac{|u|}{l} \right) \, dx \leq 1 \right\}.
\end{equation}
The Orlicz-Sobolev space is given by
\begin{equation}
  W^1_A(\Omega) = \left\{ u : \Omega \to \mathbb{R}, u \text{ is a measurable function, } u, |\nabla u| \in L_A(\Omega) \right\},
\end{equation}
where we have the norm
\begin{equation}
  \|u\|_{W^1_A(\Omega)} = \|u\|_{L_A(\Omega)} + \|\nabla u\|_{L_A(\Omega)}.
\end{equation}
The closure in $W^1_A(\Omega)$ of $C^\infty_0(\Omega)$-functions w.r.t. this norm is according to Theorem 2.1 of [16] (recall that we suppose (N3))
\begin{equation}
  \overset{\circ}{W^1_A}(\Omega) = W^1_A(\Omega) \cap \overset{\circ}{W^1}(\Omega).
\end{equation}
We additionally use the notation
\[ W_{1,A}^1(\Omega) := \left\{ w \in W^{1,1}(\Omega) : \partial_2 w \in L_A(\Omega) \right\}, \]
and define the functional
\[ E[v] := \int_{\Omega} f_2(v) \, dx < \infty, \quad v \in L_A(\Omega). \]

Recalling (1.9), we define the class
\[ \overset{\circ}{W}_{1,A}^1(\Omega) = W_{1,A}^1(\Omega) \cap \overset{\circ}{W}_1^1(\Omega). \]

By \( \nu = (\nu_1, \nu_2) \) we denote the outward unit normal to \( \partial \Omega \). The main classes of functions under consideration are:

\begin{align*}
C_{\text{Sob}} & := u_0 + \overset{\circ}{W}_{1,A}^1(\Omega), \\
C_{\text{BV}} & := \left\{ w \in BV(\Omega) : \| \partial_2 w \|_{L_A(\Omega)} < \infty, (w - u_0)\nu_2 = 0 \text{ } H^1\text{-a.e. on } \partial \Omega \right\}.
\end{align*}

Note that in the definition of the space \( C_{\text{BV}} \) we require that the distributional derivative \( \partial_2 w \) is generated by a function from the space \( L_A(\Omega) \). Moreover, we consider the BV-trace of \( w \).

With respect to these classes we define the problem
\[ J[w] := \int_{\Omega} f(\nabla w) \, dx \rightarrow \min \text{ in the class } C_{\text{Sob}} \quad (1.10) \]
and its relaxed version
\[ K[w] := \int_{\Omega} f_1(\partial_a^s w) \, dx + \int_{\Omega} f_1^\infty \left( \frac{\partial_a^s w}{|\partial_a^s w|} \right) d|\partial_a^s w| + \int_{\partial \Omega} f_1^\infty ((u_0 - w)\nu_1) dH^1 + E[\partial_2 w] =: K_1[w] + E[\partial_2 w] \rightarrow \min \text{ in the class } C_{\text{BV}}. \quad (1.11) \]

Here \( \nabla^a w \) denotes the absolutely continuous part of \( \nabla w \) w.r.t. the Lebesgue measure, \( \nabla^s w \) represents the singular part.

As a matter of fact, problem (1.10) in general is not solvable and one has to pass to the relaxed version in order to have the existence of at least generalized minimizers. The approach to relaxation in the case of linear growth is
well known and outlined, e.g., in the monographs [19] or [20].

It will turn out in the sections 2 and 3 that the functional $K$ equipped with the class $\mathcal{C}_{BV}$ is the suitable choice in the setting at hand: in Section 3.1 we show that there exists a solution $\hat{u}$ of problem (1.11). Moreover, with the help of the geometric approximation procedure of Section 2, we show in Corollary 3.1 that the infima of (1.10) and (1.11) are equal. In the case that the superlinear part is given by a $N$-function, we obtain in addition a complete dual point of view.

In Section 4 the apriori higher integrability and regularity results of the recent paper [1] on splitting type variational problems with linear growth are carried over to the mixed linear- superlinear setting and suitable generalizations are presented.

In Section 5 we turn our attention to the question of uniqueness of solutions. Here we again have to distinguish two cases. In the $N$-function case we obtain the uniqueness of the dual solution $\sigma$ under very mild assumptions. If the ellipticity parameter from 1.6 satisfies $\mu < 2$, then the smoothness properties of $\sigma$ together with the uniqueness of $\sigma$ imply the uniqueness of generalized solutions.

If the superlinear part is not given by an $N$-function, then it is not obvious how to define a dual solution. Here we suppose $\mu < 2$ and in Chapter 6 of [13] the uniqueness of $L^1$-cluster points of $J$-minimizing sequences up to constants is shown. By our Corollary 3.1 and on account of the boundary data respecting the superlinear part we now have the uniqueness of generalized minimizers.

## 2 Approximation procedure

In this section we present an approximation procedure which is adapted to the particular linear- superlinear setting. Although the arguments seem to be quite technical, the principle idea is a geometric one.

We have to take care of various aspects:

- A retracting and smoothing procedure of the form $u_0 + \eta \epsilon * [(u - u_0)(x + \delta e_2)]$ is compatible with Lebesgue spaces. However it does not work w.r.t. the “BV”-direction $e_1$ which is due to the possible concentration of a measure on the boundary.

- In the linear growth situation the methods of local approximation (compare, e.g., [20], Theorem 1.17, p. 14) serve as a powerful tool together
with some extension by $u_0$ outside of $\Omega$ (see, e.g., [8], [9]). However, a partition of the unity $\{\varphi_i\}$ is involved in this kind of argument. This provides serious difficulties proving the convergence of $f_2(\partial_2 w_m)$ for the approximating sequence $w_m$ since the derivatives of $\varphi_i$ do not cancel inside the function $f_2$.

Combining and adjusting both methods and using the geometric structure of the problem we obtain a partition of the unity such that the derivatives w.r.t. the relevant direction vanishes.

We start with a generalization of Lemma B.1 of [13] including strong $L^p$-convergence of $\partial_s w_m$. The main feature is the way of constructing the sequence $\{w_m\}$ which is crucial for proving Lemma 2.2.

**Lemma 2.1.** Let $w \in \text{BV}(\Omega)$ such that $\partial_2 w \in L^p(\Omega)$ for some $1 \leq p < \infty$ and such that $(w - u_0)\nu_2 = 0$ a.e. on $\partial\Omega$. Then there exists a sequence $\{w_m\}$ in $u_0 + C^\infty(\Omega)$ satisfying

$$\lim_{m \to \infty} \int_{\Omega} |w_m - w| \, dx = 0,$$

$$\lim_{m \to \infty} \int_{\Omega} \sqrt{1 + |\nabla w_m|^2} \, dx = \int_{\Omega} \sqrt{1 + |\nabla w|^2},$$

$$\lim_{m \to \infty} \int_{\Omega} |\partial_2 w_m - \partial_2 w|^p \, dx = 0.$$

Moreover, the trace of each $w_m$ on $\partial\Omega$ coincides with the trace of $w$, in particular $(w_m - u_0)\nu_2 = 0$ a.e. on $\partial\Omega$ and for all $m \in \mathbb{N}$.

**Proof.** For the sake of simplicity we consider the case

$$\Omega = (-1, 1) \times (-1, 1). \quad (2.1)$$

We proceed in five steps.

**Step 1.** In the following we suppose that $u_0 = 0$. The general case is obtained by considering $w - u_0$ and adding $u_0$ at the end of the proof.

We then reduce the problem by choosing two smooth functions $\psi_1$, $\psi_2$: $[-1,1] \to [0,1]$ such that $\psi_1 + \psi_2 \equiv 1$, $\psi_1(t) = 0$ on $[-1, -1/2]$, $\psi_1(t) = 1$ on $[1/2, 1]$ and $\psi_2(t) = 1$ on $[-1, -1/2]$, $\psi_2(t) = 0$ on $[1/2, 1]$.

We consider $\psi_1(x_2)w$ and $\psi_2(x_2)w$ separately, hence w.l.o.g. $w \equiv 0$ in a neighborhood of $[x_2 = -1]$. 

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**Step 2.** Fix some $\varepsilon_0 > 0$ and let (w.r.t. the $x_2$-direction)

$$w_{\varepsilon_0}(x) = w(x + \varepsilon_0 e_2),$$

(2.2)

where $w$ is extended by 0 on $(-1, 1) \times [1, \infty)$. At the end of our proof we pass to the limit $\varepsilon_0 \to 0$.

Thus we may suppose w.l.o.g. that

$$w \equiv 0 \quad \text{on} \quad \left[(-1, 1) \times (1 - \varepsilon_0, 1)\right] \cap \left[(-1, 1) \times (-1, -1 + \varepsilon_0)\right].$$

(2.3)

**Step 3.** We now take [20], proof of Theorem 1.17, as a reference (compare [13]), Lemma B.1), fix $\varepsilon > 0$, $w \in BV(\Omega)$ (recalling Step 1 and Step 2) and for $l \in \mathbb{N}$ we let

$$\Omega_k = \Omega_k^l := \left\{ x \in \Omega : -1 + \frac{1}{l+k} < x_1 < 1 - \frac{1}{l+k} \right\}, \quad k \in \mathbb{N}_0,$$

where $l$ is chosen sufficiently large such that

$$\int_{\Omega_{-\Omega_0}} |\nabla w| < \varepsilon.$$

(2.4)

With this notation we define $A_1 := \Omega_2$ and

$$A_i = \Omega_{i+1} - \overline{\Omega_{i-1}} := \left\{ x \in \Omega : -1 + \frac{1}{l+i+1} < x_1 < -1 + \frac{1}{l+i-1}, \right.$$

$$\text{and} \quad 1 - \frac{1}{l+i-1} < x_1 < 1 - \frac{1}{l+i+1} \right\}$$

$$= \left\{ x \in \Omega : x_1 \in I_i^- \cup I_i^+ \right\}.$$

A partition $\{\varphi_i\}$ of the unity is defined w.r.t. these sets by

$$\varphi_i \in C^\infty(A_i), \quad 0 \leq \varphi_i \leq 1, \quad \sum_{i=1}^\infty \varphi_i = 1.$$

For proving Lemma 2.2 below it will be crucial to observe that the functions $\varphi_i$ may be chosen respecting the structure of the stripes, i.e. for all $i \in \mathbb{N}$

$$\varphi_i(x_1, x_2) = \tilde{\varphi}_i(x_1), \quad \tilde{\varphi}_i \in C^\infty_0(I_i^- \cup I_i^+).$$

(2.5)
Step 4. Now we proceed essentially as described in Lemma B.1 of [13]: let \( \Omega_{-1} = \emptyset \), denote by \( \eta \) a smoothing kernel and choose \( \varepsilon_i \) sufficiently small such that
\[
\text{spt} \eta_{\varepsilon_i} * (\varphi_i w) \subset \Omega_{i+2} - \overline{\Omega}_{i-2},
\]
\[
\int_{\Omega} |\eta_{\varepsilon_i} * (\varphi_i w) - \varphi_i w| \, dx < 2^{-i} \varepsilon,
\]
\[
\int_{\Omega} |\eta_{\varepsilon_i} * (w \nabla \varphi_i) - w \nabla \varphi_i| < 2^{-i} \varepsilon,
\]
\[
\int_{\Omega} |\eta_{\varepsilon_i} * \partial_2 (\varphi_i w) - \partial_2 (\varphi_i w)|^p \, dx < 2^{-i} \varepsilon. \tag{2.6}
\]
On account of (2.3) we select \( \varepsilon_i \) small enough such that the smoothing procedure is well defined. Moreover, the analogue to (2.3) holds for \( w_m \) with some \( \tilde{\varepsilon}_0 < \varepsilon_0 \). Here with the choice \( \varepsilon = 1/m \) we have set
\[
w_m = \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (\varphi_i w).
\]
By the above remarks we suppose with a slight abuse of notation (relabelling \( \varepsilon_0 \)) that we have in addition to (2.3) for all \( m \in \mathbb{N} \)
\[
w_m \equiv 0 \quad \text{on} \quad \left[ (-1, 1) \times (1 - \varepsilon_0, 1) \right] \cap \left[ (-1, 1) \times (-1, -1 + \varepsilon_0) \right]. \tag{2.7}
\]
Given (2.7) we follow exactly the proof of Lemma B.1, where in particular the notion of a convex function \( g \) of a measure (see [21]) is exploited via the representation
\[
\int_U g(\nabla w) := \sup_{\kappa \in C_0^\infty(U; \mathbb{R}^n), \| \kappa \| \leq 1} \left\{ -\int_U \text{div} \kappa \, dx - \int_U g^*(\kappa) \, dx \right\}
\]
and where \( g \) is of linear growth and \( g^* \) denotes the conjugate function (see the definition given in Section 3.2).

Step 5. With \( \varepsilon \ll \varepsilon_0 \) we pass to the limit \( \varepsilon_0 \to 0 \), which finally proves the lemma.

Following the lines of Lemma 2.1 we obtain the convergence of the superlinear part of the energy under consideration.

Lemma 2.2. Suppose that we have the notation and assumptions of Lemma 2.1 together with (1.8). Then the sequence \( \{w_m\} \) of Lemma 2.1 satisfies
\[
\int_{\Omega} f_2(\partial_2 w_m) \, dx \rightarrow \int_{\Omega} f_2(\partial w) \, dx \quad \text{as} \quad m \to \infty.
\]
Proof. We start with the first three steps of the proof of Lemma 2.1 in particular we have (2.3), (2.7) and (2.5).

The strong \( L^p \)-convergence of the sequence \( \{ \partial_2 w_m \} \) yields
\[
\partial_2 w_m \to \partial_2 w \quad \text{a.e. in } \Omega. \tag{2.8}
\]

The first ingredient of the proof follows from our assumption (1.8) and Jensen’s inequality, where we recall that in fact only finite sums are considered: for all \( x \in \Omega \) and for all \( m \in \mathbb{N} \) we have
\[
f_2(\partial_2 w_m) = f_2\left( \partial_2 \sum_{i=1}^{\infty} \eta_{\varepsilon_i} \ast (\varphi_i w) \right) = f_2\left( \sum_{i=1}^{\infty} \eta_{\varepsilon_i} \ast \partial_2(\varphi_i w) \right) \leq c \sum_{i=1}^{\infty} f_2 \left( \eta_{\varepsilon_i} \ast \partial_2(\varphi_i w) \right) \leq c \sum_{i=1}^{\infty} \eta_{\varepsilon_i} \ast f_2 \left( \partial_2(\varphi_i w) \right). \tag{2.9}
\]

We recall (2.5) which means \( \partial_2 \varphi_i = 0 \). In conclusion, (2.9) shows
\[
f_2(\partial_2 w_m) \leq c \sum_{i=1}^{\infty} \eta_{\varepsilon_i} \ast f_2 \left( \varphi_i \partial_2 w \right). \tag{2.10}
\]

Now we benefit from (2.8) and Egoroff’s theorem: for any \( \bar{\varepsilon} > 0 \) and for any \( i \in \mathbb{N} \) there exists a measurable set \( A_{i,\bar{\varepsilon}} \) such that
\[
|A_i - A_{i,\bar{\varepsilon}}| < \varepsilon_i \ll \bar{\varepsilon} \quad \text{and} \quad \partial_2 w_m \Rightarrow \partial_2 w \quad \text{on } A_{i,\bar{\varepsilon}}. \tag{2.11}
\]

With the help of (2.10) one obtains for fixed \( i \in \mathbb{N} \) (note that by the first condition of (2.6), there exist at most three different numbers \( k \in \mathbb{N} \) such
that the function $\eta \ast f_2(\varphi_k \partial_2 w) \neq 0$ on $A_i$)
\[
\int_{A_i - A_i, \bar{\varepsilon}} f_2(\partial_2 w_m) \, dx
\]
\[\leq c \sum_{k=1}^{\infty} \int_{A_i - A_i, \bar{\varepsilon}} \eta \ast f_2(\varphi_k \partial_2 w)
\]
\[= c \sum_{k=1}^{\infty} \int_{A_i - A_i, \bar{\varepsilon}} \left[ \int_{B_1} \eta(z) f_2\left((\varphi_k \partial_2 w)(x - \varepsilon_k z)\right) \, dz \right] \, dx
\]
\[= c \sum_{k=1}^{\infty} \int_{B_1} \eta(z) \left[ \int_{T_{i, \varepsilon}^k} f_2\left((\partial_2 w)(y)\right) \, dy \right] \, dz,
\]
(2.12)
where it is abbreviated ($|U|$ denoting the Lebesgue measure of $U \subset \Omega$)
\[T_{i, \varepsilon}^k := \{ y = x - \varepsilon_k z : x \in A_i - A_i, \bar{\varepsilon} \}, \text{ in particular } |T_{i, \varepsilon}^k| = |A_i - A_i, \bar{\varepsilon}|.
\]
Now, since for fixed $i$ the sum is just taken over three indices, we may choose $\varepsilon_i$ sufficiently small and finally obtain from (2.12) (recalling $\int_{\Omega} f_2(\partial_2 w) \, dx < \infty$)
\[
\int_{A_i - A_i, \varepsilon} f_2(\partial_2 w_m) \, dx \leq 2^{-i} \varepsilon.
\]
(2.13)
Decreasing $\varepsilon_i$, if necessary, it may also be assumed that
\[
\int_{A_i - A_i, \varepsilon} f_2(\partial_2 w) \, dx \leq 2^{-i} \varepsilon.
\]
(2.14)
By (2.3) and (2.7) we note that once more only finite sums have to be con-
sidered and recalling (2.11), (2.13) and (2.14) we obtain
\[
\left| \int_{\Omega} f_2(\partial^2 w_m) \, dx - \int_{\Omega} f_2(\partial^2 w) \, dx \right|
\]
\[
\leq \sum_{i=1}^{N_0} \int_{A_i} \left| f_2(\partial^2 w_m) - f_2(\partial^2 w) \right| \, dx
\]
\[
\leq \sum_{i=0}^{N_0} \int_{A_{i,\varepsilon}} \left| f_2(\partial^2 w_m) - f_2(\partial^2 w) \right| \, dx + 2\varepsilon
\]
\[
\leq \sum_{i=0}^{N_0} \sup_{A_{i,\varepsilon}} \left| f_2(\partial^2 w_m) - f_2(\partial^2 w) \right| |A_i| + 2\varepsilon \leq 3\varepsilon
\]
provided that \( m > m_0 \) with \( m_0 \) sufficiently large. This finishes the proof of the lemma. \qed

**Remark 2.1.** Now we can shortly discuss one main difference to the model of Hencky plasticity investigated in [2]. There an approximation lemma is formulated as Theorem 5.3. The convergence of the deviatoric part in terms of \( f \) corresponds to the convergence of the square root in Lemma 2.1 which follows from the linear growth of \( f \) and the notion of a convex function of a measure.

Our main difficulty is proving the convergence w.r.t. the \( f_2 \)-energy. In the case of the Hencky plasticity the analogue is just a consequence of considering the intermediate topology (defined in formula (3.37) of [2]) which respects the linear operator \( \text{div} \, v \), see also Theorem 3.4 and (5.53) of [2].

Now we define
\[
\hat{\Omega} := (-2, 2) \times (-1, 1)
\]
and for \( w \in BV(\Omega) \) we let
\[
\hat{w} := \begin{cases} 
  w & \text{on } \Omega, \\
  u_0 & \text{on } \hat{\Omega} - \Omega.
\end{cases}
\]

We then have the validity of an approximation result corresponding to Lemma B.2 of [13]. It can be seen as a kind of generalization of Lemma 2.1 and Lemma 2.2 where now \( C^\infty_0 \) is replaced by \( C^\infty \).

**Lemma 2.3.** Given this notation suppose that \( w \in BV(\Omega) \), \( \|\partial^2 w\|_{L^A(\Omega)} < \infty \) and that we have (1.4), (1.5) and (1.8). Then there exists a sequence \( \{w_m\} \).
in $u_0 + C_0^\infty(\Omega)$ such that passing to the limit $m \to \infty$ we have

\begin{enumerate}
  \item $w_m \to \hat{w}$ in $L^1(\hat{\Omega})$,
  \item $\int_\Omega \sqrt{1 + |\nabla w_m|^2} \, dx \to \int_\Omega \sqrt{1 + |\nabla w|^2}$,
  \item $\int_\Omega f_2(\partial_2 w_m) \, dx \to \int_\Omega f_2(\partial_2 w) \, dx$.
\end{enumerate}

3 Existence of solutions

There are two approaches towards the existence of generalized solutions to problem (1.10). The first one follows the direct method and leads to the existence of solutions to problem (1.11). This works under quite weak assumptions.

The second approach yields the stress tensor as the unique solution of the dual problem and by the stress-strain relation a complete picture of the situation is drawn. However, following the duality approach we have to suppose that $f_2$ is a suitable $N$-function.

3.1 Generalized solutions

**Theorem 3.1.** Suppose again that $w \in BV(\Omega), \|\partial_2 w\|_{L^A(\Omega)} < \infty$ and that we have (1.4), (1.5) and (1.8). Then the relaxed problem (1.11) is solvable.

**Proof of Theorem 3.1.** Consider a $K$-minimizing sequence $\{u^{(n)}\}$ in the admissible class $C_{BV}$ of comparison functions. After passing to a subsequence we may assume that there exits a function $\hat{u} \in BV(\Omega)$ and a function $v \in L^p(\Omega)$ such that as $n \to \infty$

\begin{equation}
  u^{(n)} \to \hat{u} \text{ in } L^1(\Omega), \quad \partial_2 u^{(n)} \to v \text{ in } L^p(\Omega).
\end{equation}

We have for any $\varphi \in C_0^\infty(\Omega)$

\[
  \int_\Omega u^{(n)} \partial_2 \varphi \, dx = - \int_\Omega \partial_2 u^{(n)} \varphi \, dx,
\]

hence $v = \partial_2 \hat{u}$ and since we have for any $\psi \in C^\infty(\Omega)$

\[
  \int_\Omega u^{(n)} \partial_2 \psi \, dx = - \int_\Omega \partial_2 u^{(n)} \psi \, dx + \int_{\partial \Omega} u_0 \psi v_2 \, d\mathcal{H}^1,
\]

\[
  \int_\Omega \hat{u} \partial_2 \psi \, dx = - \int_\Omega \partial_2 \hat{u} \psi \, dx + \int_{\partial \Omega} \hat{u} \psi v_2 \, d\mathcal{H}^1,
\]
the convergences stated in (3.1) prove \( \hat{u} \in C_{BV} \).

We note that

\[
\liminf_{n \to \infty} K[u^{(n)}] \geq \liminf_{n \to \infty} K_1[u^{(n)}] + \liminf_{n \to \infty} E[\partial_2 u^{(n)}].
\]

By [22], see also [19], Theorem 5.47, p. 304, we have the lower semicontinuity

\[
K_1[\hat{u}] \leq \liminf_{n \to \infty} K_1[u^{(n)}].
\]

Diskussing \( J_2 \) we cite Theorem 2.3, p. 18, of [23], hence

\[
E[\partial_2 \hat{u}] \leq \liminf_{n \to \infty} E[\partial_2 u^{(n)}].
\]

Since \( \{u^{(n)}\} \) was chosen as a \( K \)-minimizing sequence, the proof of Theorem 3.1 is complete.

Now, on account of our approximation Lemma 2.3, we have

**Corollary 3.1.** With the notation of Theorem 3.1 we have

\[
\inf_{w \in C_{Sob}} J_w = \inf_{v \in C_{BV}} K_v = K[\hat{u}].
\]

### 3.2 The dual solution

Another approach leading to an analogue of the stress tensor is to consider the dual problem. As the main references on convex analysis we mention [24] and [25].

Let us assume that we have (1.5) with \( A(|\xi_2|) = f_2(\xi_2) \) for all \( \xi \in \mathbb{R}^{n-1} \) and with \( A \) being of class \( C^1([0, \infty)) \). In this case we suppose for notational simplicity that \( f: \mathbb{R}^2 \to \mathbb{R} \),

\[
f(\xi) = f_1(\xi_1) + A(|\xi_2|), \quad \xi \in \mathbb{R}^2.
\]

As usual we define the conjugate function \( A^*: [0, \infty) \to [0, \infty) \) by

\[
A^*(s) := \max_{t \geq 0} \{st - A(t)\}.
\]

and note that we have for all \( t \in [0, \infty) \)

\[
A(t) + A^*(A'(t)) = sA'(t).
\]

In order to obtain a well posed dual problem we suppose

\[
A^*(A'(t)) \leq c \left[ A(t) + 1 \right] \quad \text{for all} \; t \in \mathbb{R}.
\]
Since
\[ f_1^*(s) := \sup_{\xi_1 \in \mathbb{R}} \left\{ s \xi_1 - f_1(\xi_1) \right\}, \]
we obtain from the decomposition of \( f \)
\[ f^*(\xi) = f_1^*(\xi_1) + A^*(|\xi_2|) \quad (3.6) \]
as formula for the conjugate function \( f^*: \mathbb{R}^2 \to \mathbb{R} \). The conjugate function \( f^* \) satisfies in correspondence to (3.4)
\[ f(\xi) + f^*(Df(\xi)) = \xi \cdot Df(\xi), \quad \xi \in \mathbb{R}^2. \quad (3.7) \]
Given these preliminaries we define the Lagrangian
\[ l(v, \tau) := \int_{\Omega} \tau \cdot \nabla v \, dx - \int_{\Omega} f_1^*(\tau_1) \, dx - \int_{\Omega} A^*(|\tau_2|) \, dx, \]
with \( v \in u_0 + \overset{\circ}{W}^{1,1}_1(\Omega), \quad \tau \in L^{\infty,A^*}(\Omega; \mathbb{R}^2) \). (3.8)
In (3.8) we have set
\[ L^{\infty,A^*}(\Omega; \mathbb{R}^2) := \left\{ \kappa = (\kappa_1, \kappa_2) \in L^{\infty}(\Omega) \times L^{A^*}(\Omega) \right\}. \]
With the help of the formula for the conjugate function given in (3.6) we have the representation for the energy \( J \),
\[ J[w] = \sup_{\kappa \in L^{\infty,A^*}(\Omega; \mathbb{R}^2)} \left\{ \int_{\Omega} \kappa \cdot \nabla w \, dx - \int_{\Omega} f_1^*(\kappa_1) \, dx \right. \]
\[ \left. - \int_{\Omega} A^*(|\kappa_2|) \, dx \right\}, \]
\[ = \sup_{\kappa \in L^{\infty,A^*}(\Omega; \mathbb{R}^2)} l(w, \kappa), \quad w \in u_0 + \overset{\circ}{W}^{1,1}_1(\Omega). \quad (3.9) \]
The dual functional finally is defined via
\[ R[\tau] := \inf_{w \in u_0 + \overset{\circ}{W}^{1,1}_1(\Omega)} l(w, \tau), \quad \tau \in L^{\infty,A^*}(\Omega; \mathbb{R}^2). \quad (3.10) \]
This functional leads to the dual problem as the maximizing problem
\[ R[\tau] \to \text{max} \quad \text{in} \quad \tau \in L^{\infty,A^*}(\Omega; \mathbb{R}^2). \quad (3.11) \]
Then we have recalling Theorem 3.1
Theorem 3.2. Suppose that we have our general assumptions (1.4), (1.5) and (1.8). Moreover, suppose that $f$ is given in (3.2) with $A$ satisfying (3.5). Let $\hat{u}$ denote a generalized solution of the problem (1.11). Then the “stress tensor” defined by

$$\sigma(x) := Df(\nabla^a \hat{u}) = \left( f'_1(\partial^*_1 \hat{u}), A'(|\partial^*_2 \hat{u}|) \right)$$  \hspace{1cm} (3.12)

is of class $L^{\infty,A^*}(\Omega;\mathbb{R}^2)$ and maximizes the dual variational problem (3.11) with $R$ given in (3.10).

Proof. We first note that the boundedness of $|f'_1|$ and condition (3.5) imply $\sigma \in L^{\infty,A^*}(\Omega;\mathbb{R}^2)$.

We then follow an Ansatz similar to Lemma 5.1 of [26]. For any $v \in u_0 + \dot{W}_1^1(\Omega)$ we have recalling (3.8) and using (3.7)

$$l(v, \sigma) = \int_{\Omega} \nabla v \cdot Df(\nabla^a \hat{u}) \, dx - \int_{\Omega} f^*(Df(\nabla^a \hat{u})) \, dx$$

$$= \int_{\Omega} Df(\nabla^a \hat{u}) \cdot (\nabla v - \nabla^a \hat{u}) \, dx + \int_{\Omega} f(\nabla^a \hat{u}) \, dx. \quad (3.13)$$

Now given $|t| \ll 1$ let $\hat{u}_t := \hat{u} + t(v - \hat{u}) \in u_0 + \dot{W}_1^1(\Omega)$. The K-minimality of $\hat{u}$ obviously gives

$$\frac{d}{dt}|_{t=0} K[\hat{u}_t] = 0,$$

hence by $\nabla^s v = 0$

$$0 = \int_{\Omega} Df(\nabla^a \hat{u}) \cdot (\nabla v - \nabla^a \hat{u}) \, dx + \frac{d}{dt}|_{t=0} \int_{\Omega} f_1^\infty\left(\frac{\partial^*_1 \hat{u}_t}{|\partial^*_1 \hat{u}_t|}\right) d|\partial^*_1 \hat{u}_t|$$

$$+ \int_{\partial \Omega} f_1^\infty((u_0 - \hat{u}_t)\nu_1) d\mathcal{H}^1, \quad (3.14)$$

where $\partial^*_1 \hat{u}_t = (1 - t)\partial^*_1 \hat{u}$. Now we note that

$$\frac{d}{dt}|_{t=0} \int_{\Omega} f_1^\infty\left(\frac{\partial^*_1 \hat{u}_t}{|\partial^*_1 \hat{u}_t|}\right) d\left((1 - t)|\partial^*_1 \hat{u}|\right) = - \int_{\Omega} f_1^\infty\left(\frac{\partial^*_1 \hat{u}}{|\partial^*_1 \hat{u}|}\right) d|\partial^*_1 \hat{u}|,$$

and since $v$ takes the boundary data $u_0$ on $\partial \Omega$ we have

$$\frac{d}{dt}|_{t=0} \int_{\partial \Omega} f_1^\infty((u_0 - \hat{u}_t)\nu_1) d\mathcal{H}^1 = - \int_{\partial \Omega} f_1^\infty((u_0 - \hat{u})\nu_1) d\mathcal{H}^1.$$
Hence, inserting (3.13) in (3.14) we have shown
\[ l(v, \sigma) = K[\hat{u}] \quad \text{for any} \quad v \in u_0 + W^{1,1}_{1,A}(\Omega) \]
and taking the infimum w.r.t. the comparison function \( v \) we have
\[ R[\sigma] \geq K[\hat{u}] \tag{3.15} \]
We already know from Section 2 that \( \inf J = K[\hat{u}] \) and the representation (3.9) finally yields
\[ J[w] = \sup_{\chi \in L^{\infty,A}(\Omega;\mathbb{R}^2)} l(w, \chi) \]
\[ \geq \sup_{\chi \in L^{\infty,A}(\Omega;\mathbb{R}^2)} \left\{ \inf_{v \in u_0 + \hat{W}^{1,1}_{1,A}(\Omega)} l(v, \chi) \right\} \]
\[ = \sup_{\chi \in L^{\infty,A}(\Omega;\mathbb{R}^2)} R[\chi], \quad \text{i.e.} \]
\[ \inf_{w \in u_0 + \hat{W}^{1,1}_{1,A}(\Omega)} J[w] \geq \sup_{\chi \in L^{\infty,A}(\Omega;\mathbb{R}^2)} R[\chi]. \]
This together with (3.15) and Corollary 3.1 proves the theorem. \( \square \)

4 Apriori estimates

Going through the proofs of Theorem 1.1 and Corollary 1.1 of [1] we can prove the following theorem.

**Theorem 4.1.** For the sake of simplicity suppose again that \( n = 2 \) and that we have (1.4), (1.5). Suppose in addition that we have (1.6) and (1.7).

i) Then there exists a generalized minimizer \( u \in \mathcal{M} \) such that
\[ \partial^2 u \in L^\chi_{\text{loc}}(\Omega) \quad \text{for any finite} \ \chi. \]

ii) Suppose that we have in addition
\[ \mu < 2. \]
Then the relaxed problem (1.11) admits a solution \( u \in C^{1,\alpha}(\Omega), \ 0 < \alpha < 1. \)
With obvious changes in notation these results easily extend to the case of arbitrary dimensions \( n \geq 3 \).

We also announce a generalized version of Theorem 4.1 without the restriction that \( f \) is at most of quadratic growth.

## 5 Uniqueness of solutions

Here we have to distinguish two subcases.

### 5.1 Superlinear parts in terms of \( N \)-functions

In this case the uniqueness of solutions to the generalized problems can be established without using the results of the previous section.

The reason is the existence of an unique dual solution which, by the duality relation, can be carried over to generalized minimizers.

So let us suppose that we have the situation as described in the hypotheses of Theorem 3.2 in particular

\[
\sigma := \nabla f \left( f'(\partial_1 \hat{u}), A'(|\partial_2 \hat{u}|) \right)
\]

is a solution of the dual variational problem (3.11) whenever \( \hat{u} \) is a generalized minimizer in the sense of (1.11).

Now, if we have a closer look at the arguments from measure theory leading to Theorem 7 of [27], then we may adapt the proof to the situation at hand and obtian:

**Theorem 5.1.** Suppose that the hypotheses stated in Theorem 3.2 are valid. Then the dual problem (3.11) is uniquely solvable.

We emphasize that Theorem 5.1 is valid without any restriction on the exponent \( \mu \) in (1.6). This is just needed for the following corollary.

**Corollary 5.1.** Suppose that in addition to the assumptions of Theorem 5.1 we now have \( \mu < 2 \). Then we also have the uniqueness of generalized solutions of problem (1.11).

**Sketch of the proof of Corollary 5.1.** The main idea proving Corollary 5.1 is to use the regularity of an particular minimizer to get some sufficient regularity...
of the dual solution. Then, as outlined in the proof of Theorem A.9 in [13], a suitable comparison argument w.r.t. $\sigma$ is admissible to obtain for any generalized minimizer $\hat{u} \in C_{BV}$

$$\nabla \hat{u} = \nabla f^*(\sigma). \quad (5.1)$$

By the uniqueness of $\sigma$ the uniqueness of generalized minimizers up to an additive constant is established. Finally, on account of $\hat{u} \in C_{BV}$ we have $(\hat{u} - u_0)\nu_2 = 0$ $H^1$ a.e. on $\partial \Omega$ and in conclusion the uniqueness of generalized minimizers. \hfill $\Box$

### 5.2 The general superlinear case

Here we even do not know, whether the dual problem in general is solvable. The main reason is: if $f_2$ is not given by a $N$-function and if we do not impose in addition (3.5), then we cannot follow the arguments proving Theorem 3.2, in particular we do not have enough information on the global integrability if $\sigma$.

Here the idea presented in Chapter 6 of [13] is to suppose $\mu < 2$ in (1.6) and to proceed as follows.

- Define the set $\mathcal{M}$ of $L^1$-cluster points of $J$-minimizing sequences.
- Construct the quadratic $\delta$-regularization $\{u_\delta\}$ although is it is not evident, that we have found a $J$-minimizing sequence.
- Let formally $\sigma_\delta := Df_\delta(\nabla u_\delta)$ and study the regularity properties of $\sigma_\delta$.
- Establish the stress-strain relation for the limits of these particular sequences $\{u_\delta\}$ an $\{\sigma_\delta\}$.
- Benefit from $\text{div} \sigma_\delta = 0$ in order to obtain (recall (3.5))

$$\int_{\Omega} f_2^*(\partial_2 u_\delta) \, dx \leq c,$$

which is a main tool to show $\inf_w J[w] \leq R[\sigma]$ for $\sigma$ as above. This actually shows that $\{u_\delta\}$ is a $J$-minimizing sequence.

- Prove that $\sigma$ is of class $C^{0,\alpha}$. In conclusion a comparison w.r.t. $\sigma$ gives a minimax inequality for $\sigma$ and any generalized minimizer defined as a cluster point of a $J$-minimizing sequence. This leads to $\sigma = \nabla f^*(\nabla u)$ and we have the counterpart of (5.1) without knowing that we have solutions of dual problems (see Remark 6.15 of [13]).
As in Section 5.1 the regularity of $\sigma$ gives the uniqueness of generalized minimizers (as cluster points) up to a constant.

Altogether we have the following theorem (Theorem 6.5 of [13]) which

**Theorem 5.2.** Suppose that we are given the assumptions (1.4), (1.5), (1.6) with $\mu < 2$ and (1.7). Let

$$u^* \in \mathcal{M} := \left\{ u \in BV(\Omega) : u \text{ is the } L^1\text{-limit of a } J\text{-minimizing sequence from } u_0 + W^{1,1}_1(\Omega) \right\}.$$

Then $u^*$ is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$. Moreover, the elements of $\mathcal{M}$ are uniquely determined up to constants.

With our approximation result 3.1 we identify the elements of $\mathcal{M}$ with $K$-minimizers and observe that the class $C_{BV}$ is defined respecting the condition

$$(w - u_0)\nu_2 = 0 \, H^1 \text{ a.e. on } \partial\Omega.$$

**Corollary 5.2.** Given the assumptions of Theorem 5.2, the problem (1.11) is uniquely solvable.

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Michael Bildhauer  bibi@math.uni-sb.de
Martin Fuchs  fuchs@math.uni-sb.de

Department of Mathematics
Saarland University
P.O. Box 15 11 50
66041 Saarbrücken
Germany