A NONCOMMUTATIVE DE FINETTI THEOREM FOR BOOLEAN INDEPENDENCE

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Abstract. We introduce a family of quantum semigroups and its natural coactions on noncommutative polynomials. We define three invariance conditions for the joint distribution of sequences of selfadjoint noncommutative random variables associated with these coactions. For one of the invariance conditions, we show that the joint distribution of an infinite sequence of noncommutative random variables satisfies it is equivalent to the fact that the sequence of the random variables is identically distributed and boolean independent with respect to the conditional expectation onto its tail algebra. This is a boolean analogue of de Finetti theorem on exchangeable sequences. In the end of the paper, we also discuss the other two invariance conditions which lead to some trivial results.

1. Introduction

In classical probability, the study of random variables with probabilistic symmetries was started by the pioneering work of de Finetti on 2-point valued random variables. One of the most general versions of de Finetti’s work states that an infinite sequence of random variables, whose joint distribution is invariant under all finite permutations, is conditionally independent and identically distributed. One can see e.g. [13] for an exposition on the classical de Finetti theorem for more details. Also, see [11], Hewitt and Savage considered the probabilistic symmetries of random variables which are distributed on \( X = E \times E \times E \times \cdots \), where \( E \) is a compact Hausdorff space. Later, in [21], an early noncommutative version of de Finetti theorem was given by Størmer. His work focused on exchangeable states on the infinite reduced tensor product of \( C^\ast \)-algebras. Roughly speaking, in noncommutative probability, Størmer studied symmetric states on commuting noncommutative random variables. Recently, in [14], without the commuting relation, Köstler studied exchangeable sequences of noncommutative random variables in \( W^\ast \)-probability spaces with normal faithful states. In classical probability, if the second moment of a real valued random variable is 0, then the random variable is 0 a.e.. Faithfulness is a natural generalization of this property in noncommutative probability, readers are refered to [23]. Köstler showed that exchangeable sequences of random variables possess some kind of factorization property, but the exchangeability does not imply any kind of universal relation. In other words, we can not expect to determine mixed moments of an exchangeable sequence of random variables in Speicher’s universal sense [19]. By strengthening “exchangeability” to invariance under certain coactions of the free quantum permutations, in [15], Köstler and Speicher discovered that the de Finetti theorem has a natural analogue in Voiculescu’s free probability theory(see [23]). Here, free quantum permutations refer to Wang’s quantum groups \( \mathcal{A}_s(n) \) in [25].
Köstler and Speicher’s work starts a systematic study of the probabilistic symmetries on noncommutative probability theory. Most of the further projects are developed by Banica, Curran and Speicher, see [1], [6], [5]. They showed their de Finetti type theorems in both of the classical (commutative) probability theory and the noncommutative probability theory under the invariance conditions of easy groups and easy quantum groups, respectively. All these works in noncommutative case were proceeded under the assumption that the state of a probability space is faithful. This is a natural assumption in free probability theory, because in [7], Dykema showed that the free product of a family of $W^*$-probability spaces with normal faithful states is also a $W^*$-probability space with a normal faithful state. Thus the category of $W^*$-probability spaces with faithful states is closed under the free product construction. Since a normal state on $W^*$-probability space is not necessarily faithful, one may need to consider what happens to probability spaces with states which are not faithful. More specific, what are de Finetti type theorems for noncommutative probability spaces with normal states which are not necessarily faithful?

Recall that in the noncommutative realm, besides the freeness and the classical independence, there are many other kinds of independence relations, e.g. monotone independence [16], boolean independence [20], type B independence [3] and more recently two-face freeness for pairs of random variables [24]. All these types of independence are associated with certain products on probability spaces. Among these products, in [19], Speicher showed that there are only two universal products on the unital noncommutative probability spaces, namely the tensor product and the free product. The corresponding independent relations associated with these two universal products are the classical independence and the free independence. It was also shown in [19] that there is a unique universal product in the non-unital framework which is called boolean product. This non-unital universal product provides a way to construct probability spaces with non-faithful states from probability spaces with faithful states. By modifying the faithfulness, a more general noncommutative probability space is defined in section 6 which is a noncommutative probability space with a non-degenerated state. We would expect that boolean independence plays the same role in noncommutative probability spaces with non-degenerated states as the classical independence and the freeness play in commutative probability spaces and noncommutative probability spaces with faithful states, respectively. The main purpose of this work is to give certain probabilistic symmetries which can characterize conditionally boolean independence in de Finetti theorem’s form.

To proceed this work, we will construct a class of quantum semigroups $B_s(n)$’s and their sub quantum semigroups $B_s(n)$’s. Then, we can define a coaction of $B_s(n)$ on the set of noncommutative polynomials in $n$-variables. Unlike $B_s(n)$, there are two natural ways to define coactions of $B_s(n)$ on the set of noncommutative polynomials. The first way considers the set of noncommutative polynomials as a linear space, the coaction of $B_s(n)$ defined on this linear space will be called the linear coaction of $B_s(n)$ on the set of noncommutative polynomials. The second way defines the coaction of $B_s(n)$ by considering the set of noncommutative polynomials as an algebra, the coaction of $B_s(n)$ defined as a coaction on the algebra will be called the algebraic coaction of $B_s(n)$ on the set of noncommutative polynomials. With these three coactions of the quantum semigroups on the set of noncommutative polynomials in $n$ variables, we can describe three invariance conditions for the joint distribution of any sequence of $n$
random variables \((x_1, ..., x_n)\). We will show that the invariance conditions determined by the algebraic coaction of \(B_s(n)\) and the coaction of \(B_s(n)\) are so strong such that if the joint distribution of the sequence of \(n\) random variables \((x_1, ..., x_n)\) satisfies one of the invariance conditions, then \(x_1 = x_2 = \cdots = x_n\) or \(x_1 = x_2 = \cdots = x_n = 0\), respectively. In addition, we can see that Wang’s quantum permutation group \(A_s(n)\) is a quotient algebra of \(B_s(n)\) for each \(n\). Moreover, both of the invariance conditions associated with the linear coactions and the algebraic coactions of the quantum semigroups \(B_s(n)\)’s are stronger than the invariance condition associated with the quantum permutations \(A_s(n)\)’s. In this paper, we are mainly concerned with the invariance conditions which are determined by the linear coactions of the quantum semigroups \(B_s(n)\)’s.

To state a de Finetti type theorem, we need to define a conditional expectation and a tail algebra first. As K"ostler did in [14], we define our conditional expectation by taking the WOT limit of “shifts”. The situation for tail algebras is a little complicated. In a \(W^*\)-probability space with a non-degenerated state, we need to carefully choose a tail algebra so that the conditional expectation works. Basically, we need to consider two kinds of tail algebras, one contains the unit of the original algebra and the other may not. In section 6, we provide two examples to illustrate this phenomena: If the unit of a probability space is contained in the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\), then we should use the non-unital tail algebra, otherwise the conditional expectation may not be normal. If the unit of a probability space is not contained in the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\), then we should use the unital tail algebra, otherwise the conditional expectation is not \(\phi\)-preserving. The reason we need to consider two kinds of tail algebra is that the state on the probability space is not faithful.

Then, we prove the following theorem in the case that the tail algebra does not contain the unit of the original algebra:

**Theorem 1.1.** Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space and \((x_i)_{i \in \mathbb{N}}\) be an infinite sequence of selfadjoint random variables which generate \(\mathcal{A}\) as a von Neumann algebra and the unit of \(\mathcal{A}\) is contained in the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\). Then the following statements are equivalent:

- a) The joint distribution of \((x_i)_{i \in \mathbb{N}}\) is boolean exchangeable.
- b) The sequence \((x_i)_{i \in \mathbb{N}}\) is identically distributed and boolean independent with respect to the \(\phi\)-preserving conditional expectation \(E\) onto the non-unital tail algebra of \((x_i)_{i \in \mathbb{N}}\).

If the unit of \(\mathcal{A}\) is not contained in the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\), then we show that \(\mathcal{A}\) is the unitalization of the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\). Therefore, the tail algebra is the unitalization of the non-unital tail algebra of \((x_i)_{i \in \mathbb{N}}\). Then, we show that

**Theorem 1.2.** Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space and \((x_i)_{i \in \mathbb{N}}\) be a sequence of self-adjoint random variables. Suppose the unit \(I_\mathcal{A}\) of \(\mathcal{A}\) is not contained in the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\) and \(\phi\) is non-degenerated. Then the following statements are equivalent:

- a) The joint distribution of \((x_i)_{i \in \mathbb{N}}\) is boolean exchangeable.
b) The sequence \((x_i)_{i \in \mathbb{N}}\) is identically distributed and boolean independent with respect to a \(\phi\)-preserving normal conditional expectation \(\bar{E}\) onto the unital tail algebra \(A_{\text{tail}}\) of the \((x_i)_{i \in \mathbb{N}}\).

We see that the situation we need to consider the unital tail algebra is not “natural”, i.e. it is always the unitalization of a situation for which we use the non-unital tail algebra.

The paper is organized as follows: In Section 2, we recall basic definitions and notation from the noncommutative probability theory, Wang’s quantum groups and exchangeable sequences of random variables. In Section 3, we introduce our quantum semigroups \(B_s(n)\)’s and its sub quantum semigroups \(B_s(n)\)’s. Then, we introduce a linear coaction of the quantum semigroup \(B_s(n)\) on the set of the noncommutative polynomials. We will define an invariance condition associated with the linear coaction of \(B_s(n)\). In section 4, we have a brief discussion on a relation between freeness and boolean independence. We show that operator valued boolean independence implies operator valued freeness in some special cases. In section 5, we show that the joint distribution of a finite sequence of \(n\) boolean independent operator valued random variables is invariant under the linear coaction of \(B_s(n)\). In section 6, we recall some properties of the tail algebra of infinite exchangeable sequences of noncommutative variables and study the properties of the tail algebra of \(W^*\)-probability space with non-degenerated normal states. We show that a normal conditional expectation exists under the condition of boolean exchangeability. In section 7, we prove the main theorems. section 8, we define a coaction of \(B_s(n)\) and an algebraic coaction of \(B_s(n)\) on the set of noncommutative polynomials in \(n\) variables. Then, we define the invariance conditions associated with these coactions. We study the set of random variables \((x_1,\ldots,x_n)\) whose joint distribution satisfies one of these invariance conditions.

2. Preliminaries and notation

2.1. Noncommutative probability spaces. In this section, we recall some necessary definitions and notation in noncommutative probability. For further details, see texts [15], [17], [2], [23].

Definition 2.1. A non-commutative probability space \((\mathcal{A}, \phi)\) consists of a unital algebra \(\mathcal{A}\) and a linear functional \(\phi : \mathcal{A} \to \mathbb{C}\). \((\mathcal{A}, \phi)\) is called a \(*\)-probability space if \(\mathcal{A}\) is a \(*\)-algebra and \(\phi(xx^*) \geq 0\) for all \(x \in \mathcal{A}\). \((\mathcal{A}, \phi)\) is called a \(W^*\)-probability space if \(\mathcal{A}\) is a \(W^*\)-algebra and \(\phi\) is a normal state on it. We say \((\mathcal{A}, \phi)\) is tracial if

\[\phi(xy) = \phi(yx), \quad \forall x, y \in \mathcal{A}.\]

The elements of \(\mathcal{A}\) are called random variables. Let \(x \in \mathcal{A}\) be a random variable, the distribution of \(x\) is a linear functional \(\mu_x\) on \(\mathbb{C}[X]\) such that \(\mu_x(P) = \phi(P(x))\) for all \(P \in \mathbb{C}[x]\), where \(\mathbb{C}[x]\) is the set of complex polynomials in one variable.

Note that we do not require the state on \(W^*\)-probability space to be tracial. We will specify the probability spaces we concern in section 6 and section 8.

Definition 2.2. Let \(I\) be an index set. The algebra of noncommutative polynomials in \(|I|\) variables, \(\mathbb{C}\langle X_i | i \in I \rangle\), is the linear span of 1 and noncommutative monomials of the form \(X_{i_1}^{k_1}X_{i_2}^{k_2}\cdots X_{i_n}^{k_n}\) with \(i_1 \neq i_2 \neq \cdots \neq i_n \in I\) and all \(k_j\)’s are positive integers.
For convenience, we use \( \mathbb{C}(X_i| i \in I) \) to denote the set of noncommutative polynomials without a constant term.

Let \( (x_i)_{i \in I} \) be a family of random variables in a noncommutative probability space \((A, \phi)\). Their joint distribution is a linear functional \( \mu : \mathbb{C}(X_i| i \in I) \rightarrow \mathbb{C} \) defined by
\[
\mu(X_i^{k_1} X_i^{k_2} \cdots X_i^{k_n}) = \phi(x_i^{k_1} x_i^{k_2} \cdots x_i^{k_n}),
\]
and \( \mu(1) = 1. \)

**Remark 2.3.** In general, the joint distribution depends on the order of the random variables. For example, let \( I = \{1, 2\} \), then \( \mu_{x_1, x_2} \) may not equal \( \mu_{x_2, x_1} \). According to our notation, \( \mu_{x_1, x_2}(X_1 X_2) = \phi(x_1 x_2) \), but \( \mu_{x_2, x_1}(X_1 X_2) = \phi(x_2 x_1) \).

**Definition 2.4.** Let \((A, \phi)\) be a noncommutative probability space. A family of unital subalgebras \((A_i)_{i \in I}\) is said to be boolean independent if
\[
\phi(a_1 \cdots a_n) = 0,
\]
whenever \( a_k \in A_k, i_1 \neq i_2 \neq \cdots \neq i_n \) and \( \phi(a_k) = 0 \) for all \( k \). Let \((x_i)_{i \in I}\) be a family of random variables and \( A_i \)'s be the unital subalgebras generated by \( x_i \)'s, respectively. We say the family of random variables \((x_i)_{i \in I}\) is free if the family of unital subalgebras \((A_i)_{i \in I}\) is free.

**Definition 2.5.** Let \((A, \phi)\) be a noncommutative probability space, A family of (not necessarily unital) subalgebras \( \{A_i|i \in I\} \) of \( A \) is said to be boolean independent if
\[
\phi(x_{i_1} x_{i_2} \cdots x_{i_n}) = \phi(x_{i_1}) \phi(x_{i_2}) \cdots \phi(x_{i_n})
\]
whenever \( x_k \in A_{i_k} \) with \( i_1 \neq i_2 \neq \cdots \neq i_n \). A set of random variables \( \{x_i \in A|i \in I\} \) is said to be boolean independent if the family of non-unital subalgebras \( A_i \), which are generated by \( x_i \) respectively, is boolean independent.

One refers to [9] for more details of boolean product of random variables. Since the framework for boolean independence is a non-unital algebra in general, we will not require our operator valued probability spaces to be unital:

**Definition 2.6.** An operator valued probability space \((A, B, E : A \rightarrow B)\) consists of an algebra \( A \), a subalgebra \( B \) of \( A \) and a \( B - B \) bimodule linear map \( E : A \rightarrow B \) i.e.
\[
E[b_1 a b_2] = b_1 E[a] b_2, \quad E[b] = b
\]
for all \( b_1, b_2, b \in B \) and \( a \in A \). According to the definition in [22], we call \( E \) a conditional expectation from \( A \) to \( B \) if \( E \) is onto, i.e. \( E[A] = B \). The elements of \( A \) are called random variables.

In operator valued free probability theory, \( A \) and \( B \) are unital and have the same unit

**Definition 2.7.** Given an algebra \( B \), we denote by \( B\langle X \rangle \) the algebra which is freely generated by \( B \) and the indeterminant \( X \). Let \( 1_X \) be the identity of \( C\langle X \rangle \), then \( B\langle X \rangle \) is set of linear combinations of the elements in \( B \) and the noncommutative monomials \( b_0 X b_1 X b_2 \cdots b_{n-1} X b_n \) where \( b_k \in B \cup \{C1_X\} \) and \( n \geq 0 \). The elements in \( B\langle X \rangle \) are called \( B \)-polynomials. In addition, \( B\langle X \rangle_0 \) denotes the subalgebra of \( B\langle X \rangle \) which does not contain a constant term i.e. the linear span of the noncommutative monomials \( b_0 X b_1 X b_2 \cdots b_{n-1} X b_n \) where \( b_k \in B \cup \{C1_X\} \) and \( n \geq 1 \).
Definition 2.8. Given an operator valued probability space $\langle A, B, E : A \to B \rangle$ such that $A$ and $B$ are unital. A family of unital subalgebras $\{A_i \supset B \}_{i \in I}$ is said to be freely independent with respect to $E$ if

$$E[a_1 \cdots a_n] = 0,$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $a_k \in A_{i_k}$ and $E[a_k] = 0$ for all $k$. A family of $(x_i)_{i \in I}$ is said to be freely independent over $B$, if the unital subalgebras $\{A_i\}_{i \in I}$ which are generated by $x_i$ and $B$ respectively are free, or equivalently

$$E[p_1(x_{i_1})p_2(x_{i_2}) \cdots p_n(x_{i_n})] = 0,$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $p_1, \ldots, p_n \in B \langle X \rangle$ and $E[p_k(x_{i_k})] = 0$ for all $k$.

Let $\{x_i\}_{i \in I}$ be a family of random variables in an operator valued probability space $\langle A, B, E : A \to B \rangle$. $A$, $B$ are not necessarily unital. $\{x_i\}_{i \in I}$ is said to be boolean independent over $B$ if for all $i_1, \ldots, i_n \in I$, with $i_1 \neq i_2 \neq \cdots \neq i_n$ and all $B$-valued polynomials $p_1, \ldots, p_n \in B \langle X \rangle_0$ we have

$$E[p_1(x_{i_1})p_2(x_{i_2}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})].$$

2.2. Wang’s quantum permutation groups. In [25], Wang introduced the following quantum groups $A_s(n)$’s.

Definition 2.9. $A_s(n)$ is defined as the universal unital $C^*$-algebra generated by elements $u_{i,j}$ ($i, j = 1, \ldots, n$) such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \ldots, n$.
- the elements in each row and column of $u = (u_{i,j})_{i,j=1,\ldots,n}$ form a partition of unit, i.e. are orthogonal and sum up to 1: for each $i = 1, \ldots, n$ and $k \neq l$ we have

$$u_{i,k}u_{i,l} = 0 \quad \text{and} \quad u_{k,i}u_{l,i} = 0;$$

and for each $i = 1, \ldots, n$ we have

$$\sum_{k=1}^{n} u_{i,k} = 1 = \sum_{k=1}^{n} u_{k,i}.$$

$A_s(n)$ is a compact quantum group in the sense of Woronowicz [26], with comultiplication, counit and antipode given by the formulas:

$$\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j},$$

$$\epsilon(u_{i,j}) = \delta_{i,j},$$

$$S(u_{i,j}) = u_{j,i}.$$

In [15], the right coaction of $A_s(n)$ on $\mathbb{C}\langle X_1, \ldots, X_n \rangle$ is a linear map $\alpha : \mathbb{C}\langle X_1, \ldots, X_n \rangle \to \mathbb{C}\langle X_1, \ldots, X_n \rangle \otimes A_s(n)$ given by:

$$\alpha(X_{i_1}X_{i_2} \cdots X_{i_m}) = \sum_{j_1,\ldots,j_m=1}^{n} X_{j_1}X_{j_2} \cdots X_{j_m} \otimes u_{j_1,i_1}u_{j_2,i_2} \cdots u_{j_m,i_m},$$

where $\otimes$ denotes the algebraic tensor product.

In the earlier papers, $\alpha$ is defined as an algebraic homomorphism. We put emphasis on the linearity here because we will define some coactions of our quantum semigroups.
Definition 3.1. By a quantum semigroup we mean a $\Delta$ additional structure described by a morphism $\phi$ of $A \otimes B$ into the minimal tensor product algebra homomorphisms acting from $A \otimes \phi A$ to $B$. For any $\phi$ in $[15]$, boolean quantum semigroups are stronger than the quantum exchangeability defined in [15].

In the end of this section, we will show that the invariance conditions associated with quantum semigroups and their coactions on the joint distribution of random variables.

Let $(x_i)_{i \in \mathbb{N}}$ be an infinite sequence of random variables in a noncommutative probability space $(\mathcal{A}, \phi)$. $(x_i)_{i \in \mathbb{N}}$ is said to be quantum exchangeable if their joint distribution is invariant under Wang’s quantum permutation groups, i.e. for all $n$, we have

$$\mu_{x_1, \ldots, x_n} = \mu_{x_{\sigma(1)}, \ldots, x_{\sigma(n)}}$$

where $\sigma$ is the permutation group on $\{1, \ldots, n\}$. Let $S_n$ be the permutation group on $\{1, \ldots, n\}$. The joint distribution of $(x_i)_{i \in \mathbb{N}}$ is said be exchangeable if for all $n, \sigma \in S_n$, we have

$$\mu_{x_1, \ldots, x_n} = \mu_{x_{\sigma(1)}, \ldots, x_{\sigma(n)}}$$

where $\mu_{x_1, \ldots, x_n}$ is the joint distribution of $x_1, \ldots, x_n$ with respect to $\phi$. It was shown, in [15], that quantum exchangeability implies classical exchangeability.

3. Quantum semigroups $\mathcal{B}_s(n)$

In this section, we will introduce quantum semigroups $\mathcal{B}_s(n)$’s. Our probabilistic symmetries are described by linear coactions of $\mathcal{B}_s(n)$’s. First, we recall the related definitions and notation of quantum semigroups. Then, we will introduce our boolean quantum semigroups and their coactions on the joint distribution of random variables. In the end of this section, we will show that the invariance conditions associated with boolean quantum semigroups are stronger than the quantum exchangeability defined in [15].

A quantum space is an object of the category dual to the category of $C^*$-algebras (27). For any $C^*$-algebras $A$ and $B$, the set of morphisms $\text{Mor}(A, B)$ consists of all $C^*$-algebra homomorphisms acting from $A$ to $M(B)$, where $M(B)$ is the multiplier algebra of $B$, such that $\phi(A)B$ is dense in $B$. If $A$ and $B$ are unital $C^*$-algebras, then all unital $C^*$-homomorphisms from $A$ to $B$ are in $\text{Mor}(A, B)$. In [18],

Definition 3.1. By a quantum semigroup we mean a $C^*$-algebra $\mathcal{A}$ endowed with an additional structure described by a morphism $\Delta \in \text{Mor}(\mathcal{A}, A \otimes A)$ such that

$$(\Delta \otimes id_{\mathcal{A}})\Delta = (id_{\mathcal{A}} \otimes \Delta)\Delta.$$
define a coassociative comultiplication on it. The definition of $B_s(n)$ is close to Wang’s quantum group $A_s(n)$:

**Quantum semigroup** $(B_s(n), \Delta)$: The algebra $B_s(n)$ is defined as the universal unital $C^*$-algebra generated by elements $u_{i,j}$ $(i, j = 1, \cdots n)$ and a projection $P$ such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \cdots, n$.
- $u_{i,k}u_{i,l} = 0$ and $u_{k,i}u_{l,i} = 0$ whenever $k \neq l$.
- For all $1 \leq i \leq n$, $P = \sum_{k=1}^{n} u_{i,k}P$.

The third universal condition is defined via taking the sum over the first index. If we fix the first index, we can get the same equality:

**Lemma 3.2.** For all $1 \leq i \leq n$, $P = \sum_{k=1}^{n} u_{i,k}P$.

**Proof.** According to the definition, we have

$$\sum_{i=1}^{n} \sum_{k=1}^{n} u_{i,k}P = nP$$

whose spectrum consists of 0 and $n$. Notice that

$$\| \sum_{k=1}^{n} u_{i,k}P \| \leq \| \sum_{k=1}^{n} u_{i,k} \| \| P \| = 1$$

for all $i = 1, \ldots, n$. Therefore,

$$P = \sum_{k=1}^{n} u_{i,k}P.$$ 

We will denote the unit of $B_s(n)$ by $I$, the projection $P$ is called the invariant projection of $B_s(n)$. On this unital $C^*$-algebra, we can define a unital $C^*$-homomorphism $\Delta : B_s(n) \rightarrow B_s(n) \otimes B_s(n)$ by the following formulas:

$$\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$$

and

$$\Delta P = P \otimes P, \quad \Delta I = I \otimes I.$$

We will see that $(B_s(n), \Delta)$ is a quantum semigroup. To show this we need to check that $\Delta$ defines a unital $C^*$-homomorphism from $B_s(n)$ to $B_s(n) \otimes B_s(n)$ and satisfies the coassociative condition:

Because $u_{i,k}, u_{k,j}$ are orthogonal projections and $u_{i,k}u_{i,l} = 0$ if $k \neq l$, $\{ u_{i,k} \otimes u_{k,j} \}_{k=1, \ldots, n}$ is a set of orthogonal projections which are orthogonal to each other. Therefore, $\Delta u_{i,j} =$
\[ \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j} \text{ is an orthogonal projection.} \]
\[ \Delta P = P \otimes P \text{ is a projection as well. Let} \]
\[ l \neq m, \text{ then} \]
\[ \Delta(u_{i,l})\Delta u_{i,m} = \left( \sum_{k=1}^{n} u_{i,k} \otimes u_{k,l} \right) \left( \sum_{j=1}^{n} u_{i,j} \otimes u_{j,m} \right) \]
\[ = \sum_{k,j=1}^{n} u_{i,k} u_{i,j} \otimes u_{k,l} u_{j,m} \]
\[ = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,l} u_{k,m} \]
\[ = 0. \]

The same, we have \( \Delta(u_{l,i})\Delta u_{m,i} = 0, \) for \( m \neq l. \) Moreover, we have
\[ \Delta \left( \sum_{l=1}^{n} u_{l,i} \right) \Delta P = \left( \sum_{l,k=1}^{n} u_{l,k} \otimes u_{k,i} \right) P \otimes P \]
\[ = \sum_{l,k=1}^{n} u_{l,k} P \otimes u_{k,i} P \]
\[ = \sum_{k=1}^{n} P \otimes u_{k,i} P \]
\[ = P \otimes P. \]

and \( \Delta \) sends the unit of \( B_s(n) \) to the unit of \( B_s(n) \otimes B_s(n). \) Therefore, \( \Delta \) defines a unital \( C^* \)-homomorphism on \( B_s(n) \) by the universality of \( B_s(n). \)

The coassociative condition holds, because on the generators we have:
\[ (\Delta \otimes id_A)\Delta u_{i,j} = \sum_{k,l=1}^{n} u_{ik} \otimes u_{k,l} \otimes u_{l,j} = (id_A \otimes \Delta)\Delta u_{i,j} \]
\[ (\Delta \otimes id_A)\Delta P = P \otimes P \otimes P = (id_A \otimes \Delta)\Delta P \]
\[ (\Delta \otimes id_A)\Delta I = I \otimes I \otimes I = (id_A \otimes \Delta)\Delta I. \]

Therefore, \( (B_s(n), \Delta) \) is a quantum semigroup.

**Remark 3.3.** If we let the invariant projection to be the identity, then we get Wang’s free quantum permutation group. Therefore, \( A_s(n) \) is a quotient \( C^* \)-algebra of \( B_s(n), \) i.e. there exists a unital \( C^* \)-homomorphism \( \beta : B_s(n) \to A_s(n) \) such that \( \beta \) is surjective.

The following two examples are nontrivial representations of \( B_s(n) \)’s:

**Example 1.** Let \( C^6 \) be the standard 6-dimensional complex Hilbert space with orthonormal basis \( v_1, \ldots, v_6. \) Let
\[
\begin{align*}
P_{11} &= P_{v_1+v_2}, & P_{12} &= P_{v_3+v_4}, & P_{13} &= P_{v_5+v_6}, \\
P_{21} &= P_{v_3+v_6}, & P_{22} &= P_{v_5+v_2}, & P_{23} &= P_{v_1+v_4}, \\
P_{31} &= P_{v_4+v_5}, & P_{32} &= P_{v_1+v_6}, & P_{33} &= P_{v_2+v_3}.
\end{align*}
\]

and \( P = P_{v_1+v_2+v_3+v_4+v_5+v_6}, \) where \( P_v \) denotes the one dimensional orthogonal projection onto the subspace spanned by \( v. \) Then the unital algebra generated by \( P_{i,j} \) and \( P \) gives a representation \( \pi \) of \( B_s(3) \) on \( C^6 \) by the following formulas on the generators of \( B_s(3): \)
\[
\pi(I) = I_{C^6}, \quad \pi(u_{i,j}) = P_{i,j}, \quad \pi(P) = P.
\]
\( \pi \) is well defined by the universality of \( B_s(3) \). In this example, we see that sums of columns are different, e.g. \( P_{11} + P_{21} + P_{31} \neq P_{12} + P_{22} + P_{32} \). Therefore, we can not construct \( B_s(3) \) from \( A_s(3) \) by simply adding a special projection \( P \). In general, \( B_s(n) \) is quit different from \( A_s(n) \) for all \( n > 1 \).

**Example 2.** Again, let \( \mathbb{C}^6 \) be the standard 6-dimensional complex Hilbert space with orthonormal basis \( v_1, \ldots, v_6 \). Let

\[
\begin{align*}
P_{11} &= P_{v_1 + v_2}, & P_{12} &= P_{v_4 + v_5}, & P_{13} &= P_{v_3 + v_6}, \\
P_{21} &= P_{v_3 + v_6}, & P_{22} &= P_{v_1 + v_2}, & P_{23} &= P_{v_4 + v_5}, \\
P_{31} &= P_{v_4 + v_5}, & P_{32} &= P_{v_3 + v_6}, & P_{33} &= P_{v_1 + v_2}.
\end{align*}
\]

and \( P = P_{v_1 + v_2 + v_3 + v_4 + v_5 + v_6} \), where \( P_v \) denotes the one dimensional orthogonal projection onto the subspace spaned by \( v \). Then the unital algebra generated by \( P_{i,j} \) and \( P \) gives a representation \( \pi \) of \( B_s(3) \) on \( \mathbb{C}^6 \) by the following formulas on the generators of \( B_s(3) \):

\[
\pi(I) = I_{\mathbb{C}^6}, \quad \pi(u_{i,j}) = P_{i,j}, \quad \pi(P) = P.
\]

\( \pi \) is well defined by the universality of \( B_s(3) \).

Moreover, the matrix form for \( P_{1,1} \) and \( P \) with respect to the basis are

\[
P_{11} = 1/2 \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\quad \text{and} \quad
P = 1/6 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},
\]

then we have

\[
PP_{11}P = 1/18 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1/3P.
\]

In general, we have

**Lemma 3.4.** Let \( v_1, \ldots, v_{2n} \) be an orthonormal basis of the standard \( 2n \)-dimensional Hilbert space \( \mathbb{C}^{2n} \), and let \( v_k = v_{k+2n} \) for all \( k \in \mathbb{Z} \), let

\[
P_{i,j} = P_{v_2(i-j)+1+v_2(j-i)+2},
\]

where \( P_v \) is the orthogonal projection the one dimensional subspace generated by the vector \( v \) and \( P = P_{v_1+v_2+ \cdots + v_{2n}} \), \( 1 \) is the identity of \( B(\mathbb{C}^{2n}) \). Then \( \{P_{i,j}\}_{i,j=1, \ldots, n} \), \( P \) and \( 1 \) satisfy the defining conditions of the algebra \( B_s(n) \). In addition, \( PP_{i,j}P = 1/nP \) for all \( i, j = 1, \ldots, n \).

**Proof.** It is easy to see that the inner product

\[
\langle v_2(i-j)+1+v_2(j-i)+2, v_2(i-k)+1+v_2(k-i)+2 \rangle = 2\delta_{j,k},
\]
so $P_{ik} P_{ij} = 0$ if $j \neq k$. The same $P_{ki} P_{ji} = 0$ if $k \neq j$. Fix $i$, we see that $v_1 + v_2 + \cdots + v_{2n} \in \text{span}\{v_2(i-j)+1 + v_2(j-i)+2 \mid j = 1, \ldots, n\}$, so $\sum_{k=1}^{n} P_{ik} P = P$. By a direct computation, we have

$$\langle P P_{i,j} P \sum_{i=1}^{2n} v_i, \sum_{i=1}^{2n} v_i \rangle = 2$$

and

$$\| \sum_{i=1}^{2n} v_i \| = 2n.$$

Since $PP_{i,j} P$ is a selfadjoint operator with rank 1 and $\sum_{i=1}^{2n} v_i$ is in the range of $PP_{i,j} P$, we have

$$PP_{i,j} P = \frac{\langle PP_{i,j} P \sum_{i=1}^{2n} v_i, \sum_{i=1}^{2n} v_i \rangle}{\| \sum_{i=1}^{2n} v_i \|} P = \frac{1}{n} P.$$

The proof is complete. \qed

Therefore, by lemma 3.4, there exists a representation $\pi$ of $B_s(n)$ on $\mathbb{C}^{2n}$ which is defined by the following formulas:

$$\pi(1_{B_s(n)}) = 1, \quad \pi(P) = P$$

and

$$\pi(u_{i,j}) = P_{i,j},$$

for all $i, j = 1, \ldots, n$.

Now, we turn to introduce a sub quantum semigroup of $(B_s(n), \Delta)$. Since $P \neq I$ is a projection in $B_s(n)$, $B_s(n) = PB_s(n)P$ is a $C^*$-algebra with identity $P$ and generators

$$\{P u_{i_1,j_1} \cdots u_{i_k,j_k} P \mid i_1, j_1, \ldots, i_k, j_k \in \{1, \ldots, n\}, k \geq 0\}.$$

If we restrict the comultiplication $\Delta$ onto $B_s(n)$, then we have

$$\Delta(P u_{i_1,j_1} \cdots u_{i_k,j_k} P) = (P \otimes P)(\sum_{i_1, \ldots, i_k} u_{i_1,j_1} \cdots u_{i_k,j_k} \otimes u_{i_1,j_1} \cdots u_{i_k,j_k})(P \otimes P),$$

which is contained in $B_s(n) \otimes B_s(n)$. Therefore, $(B_s(n), \Delta)$ is also a quantum semigroup and $P$ is the identity of $B_s(n)$. We will call $B_s(n)$ the boolean permutation quantum semigroup of $n$.

**Remark 3.5.** If we require $P u_{i,j} = u_{i,j} P$ for all $i, j = 1, \ldots, n$, then the universal algebra we constructed in the above way is exactly Wang’s quantum permutation group. Therefore, $A_s(n)$ is also a quotient algebra of $B_s(n)$.

In the following definition, $\otimes$ denotes the tensor product for linear spaces:

**Definition 3.6.** Let $S = (A, \Delta)$ be a quantum semigroup and $V$ be a complex vector space, by a (right) coaction of the quantum group $S$ on $V$ we mean a linear map $L : V \to V \otimes A$ such that

$$(L \otimes id)L = (id \otimes \Delta)L.$$
We say a linear functional $\omega : \mathcal{V} \to \mathbb{C}$ is invariant under $L$ if

$$(\omega \otimes \text{id}) L(v) = \omega(v) I_A,$$

where $I_A$ is the identity of $A$.

Given a complex vector space $\mathcal{W}$, We say a linear map $T : \mathcal{V} \to \mathcal{W}$ is invariant under $L$ if

$$(T \otimes \text{id}) L(v) = T(v) \otimes I_A.$$

**Remark 3.7.** This definition is about coactions on linear spaces but not coactions on algebras.

Let $\mathbb{C}\langle X_1, \ldots, X_n \rangle$ be the set of noncommutative polynomials in $n$ variables, which is a linear space over $\mathbb{C}$ with basis $X_1 \cdots X_n$ for all integer $k \geq 0$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Now, we define a right coaction $L_n$ of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \ldots, X_n \rangle$ as follows:

$$L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1, \ldots, j_k=1}^{n} X_{j_1} \cdots X_{j_k} \otimes P u_{j_1,i_1} \cdots u_{j_n,i_n} P$$

and

$$L_n(1) = 1 \otimes P.$$

It is a well defined coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \ldots, X_n \rangle_0$, because:

$$(L_n \otimes \text{id}) L_n(X_{i_1} \cdots X_{i_k})$$

$$= (L_n \otimes \text{id}) \sum_{j_1, \ldots, j_k=1}^{n} X_{j_1} \cdots X_{j_k} \otimes P u_{j_1,i_1} \cdots u_{j_n,i_n} P$$

$$= \sum_{j_1, \ldots, j_k=1}^{n} \sum_{l_1, \ldots, l_k=1}^{n} X_{l_1} \cdots X_{l_k} \otimes P u_{l_1,j_1} \cdots u_{l_n,j_n} P \otimes P u_{j_1,i_1} \cdots u_{j_n,i_n} P$$

$$= \sum_{l_1, \ldots, l_k=1}^{n} X_{l_1} \cdots X_{l_k} \otimes (\sum_{j_1, \ldots, j_k=1}^{n} P u_{l_1,j_1} \cdots u_{l_n,j_n} P \otimes P u_{j_1,i_1} \cdots u_{j_n,i_n} P)$$

$$= \sum_{l_1, \ldots, l_k=1}^{n} X_{l_1} \cdots X_{l_k} \otimes (\Delta P u_{l_1,i_1} \cdots u_{l_n,i_n} P)$$

$$= (id \otimes \Delta) \sum_{j_1, \ldots, j_k=1}^{n} X_{l_1} \cdots X_{l_k} \otimes (P u_{l_1,i_1} \cdots u_{l_n,i_n} P)$$

$$= (id \otimes \Delta)L_n(X_{i_1} \cdots X_{i_k}).$$

We will call $L_n$ the linear coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \ldots, X_n \rangle$. The algebraic coaction will be defined in section 7.

**Lemma 3.8.** Let $L_n$ be the linear coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, \ldots, X_n \rangle$, $\{u_{i,j}\}_{i,j=1}^{n}$ and $P$ be the standard generators of $\mathcal{B}_s(n)$. Then,

$$L_n(p_1(X_{i_1}) \cdots p_k(X_{i_k})) = \sum_{j_1, \ldots, j_k=1}^{n} p_1(X_{j_1}) \cdots p_k(X_{j_k}) \otimes P u_{j_1,i_1} \cdots u_{j_k,i_k} P,$$

for all $i_1 \neq i_2 \neq \cdots \neq i_k$ and $p_1, \ldots, p_k \in \mathbb{C}\langle X \rangle_0$. 

Proof. Since the map is linear, it suffices to show that the equation holds by assuming
\( p_t(X) = X^{t_l} \) where \( t_l \geq 1 \) for all \( l = 1, \ldots, k \). Then, we have
\[
\mathcal{L}_n(x_{i_1} \cdots x_{i_{t_1}} \cdots x_{i_1} \cdots x_{i_{t_k}})
= \sum_{j_1, \ldots, j_{t_1}, \ldots, j_{t_k}} x_{j_1,1} \cdots x_{j_{t_1},1} \cdots x_{j_{t_k},1} \cdots x_{j_{t_1},t_1} \cdots x_{j_{t_k},t_k} \otimes p_{u_{j_1,1} \cdots u_{j_{t_1},t_1} \cdots u_{j_{t_k},t_k}} P.
\]
Notice that \( u_{j_m, s} u_{j_{m+1}, s} = \delta_{j_m, j_{m+1}} u_{j_m, s} \), the right hand side of the above equation becomes
\[
\sum_{j_1, \ldots, j_k=1}^{n} x_{j_1} \cdots x_{j_k} \otimes p_{u_{j_1,1} \cdots u_{j_k,1}} P.
\]
The proof is now completed.

We will be using the following invariance condition to characterize conditionally boolean independence.

Definition 3.9. Let \((A, \phi)\) be a noncommutative probability space and \((x_i)_{i \in \mathbb{N}}\) be an
infinite sequence of random variables in \(A\). We say that the joint distribution of \((x_i)_{i \in \mathbb{N}}\)
is boolean exchangeable if for all \(n\), we have
\[
\mu_{x_1, \ldots, x_n}(p) P = \mu_{x_1, \ldots, x_n} \otimes id_{\mathcal{B}_s(n)}(\mathcal{L}_n p)
\]
for all \(p \in \mathbb{C}(X_1, \ldots, X_n)\), where \(\mu_{x_1, \ldots, x_n}\) is the joint distribution of \(x_1, \ldots, x_n\).

Let \(\{\bar{u}_{ij}\}_{i, j=1, \ldots, n}\) be the standard generators of \(A_s(n)\), and \(\{u_{ij}\}_{i, j=1, \ldots, n} \cup \{P\}\) be the standard
generators of \(\mathcal{B}_s(n)\), then there exists a unital \(C^*\)-homomorphism \(\beta : \mathcal{B}_s(n) \to A_s(n)\) such that:
\[
\beta(u_{ij}) = \bar{u}_{ij}, \quad \beta(P) = 1_{A_s(n)}.
\]
The \(C^*\)-homomorphism is well defined because of the universality of \(\mathcal{B}_s(n)\). Let \(p = X_{i_1} \cdots X_{i_k} \in \mathbb{C}(X_1, \ldots, X_n)\), then
\[
\mu_{x_1, \ldots, x_n}(p) P = \mu_{x_1, \ldots, x_n} \otimes id_{\mathcal{B}_s(n)}(\mathcal{L}_n p)
\]
implies
\[
\mu_{x_1, \ldots, x_n}(X_{i_1} \cdots X_{i_k}) P = \sum_{j_1, \ldots, j_k=1}^{n} (\mu_{x_1, \ldots, x_n} \otimes id_{\mathcal{B}_s(n)})(X_{j_1} \cdots X_{j_k} \otimes \bar{P}_{u_{j_1,1} \cdots u_{j_k,1}} P).
\]
Now, apply \(\beta\) on both sides of the above equation, we get
\[
\mu_{x_1, \ldots, x_n}(X_{i_1} \cdots X_{i_k}) 1_{A_s(n)} = \sum_{j_1, \ldots, j_k=1}^{n} (\mu_{x_1, \ldots, x_n} \otimes id_{A_s(n)})(X_{j_1} \cdots X_{j_k} \otimes \bar{u}_{j_1,1} \cdots \bar{u}_{j_k,1}),
\]
which is the free quantum invariance condition. Since \(p\) is arbitrary, we have:

Proposition 3.10. Let \((A, \phi)\) be a noncommutative probability space and \((x_i)_{i=1, \ldots, n}\)
be a sequence of random variables in \(A\), the joint distribution of \((x_i)_{i=1, \ldots, n}\) is invariant
under the free quantum permutations \(A_s(n)\) if it satisfies the invariance condition associated with the linear coaction of the boolean quantum permutation semigroup \(\mathcal{B}_s(n)\).
4. Boolean independence and freeness

In this section, we will show that operator valued boolean independent variables are sometimes operator valued freely independent. Therefore, we should not be surprised that the joint distribution of any sequence of identically boolean independent random variables is invariant under the coaction of the free quantum permutations. The main idea we will use is the unitalization of $C^*$-algebras. Hence, we provide a brief review of the unitalization of $C^*$-algebras here:

To every $C^*$ algebra $\mathcal{A}$ one can associate a unital $C^*$ algebra $\bar{\mathcal{A}}$ which contains $\mathcal{A}$ as a two-sided ideal and with the property that the quotient $\mathcal{C}$-idea we will use is the unitalization of variables is invariant under the coaction of the free quantum permutations. The main distribution of any sequence of identically boolean independent random variables is sometimes operator valued freely independent. Therefore, we should not be surprised.

Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator-valued probability space where $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital. Let $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ be the unitalization defined above, then we can extend $E$ to $\bar{E}$ s.t $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$ is also an operator-valued probability space where $\bar{E}$ is a conditional expectation on $\bar{\mathcal{A}}$.

It is natural to define $\bar{E}$ as

$$\bar{E}[x, a] = (x, E[a]).$$

$E[(1, 0)] = (1, 0)$, so $\bar{E}$ is unital. The linear property is easy to check.

Take $(x_1, b_1), (x_2, b_2) \in \mathcal{B}$ and $(y, a) \in \bar{\mathcal{A}}$, we have

$$\bar{E}[(x_1, b_1)(y, a)(x_2, b_2)] = \bar{E}[x_1 y x_2, x_1 x_2 a + y x_2 b + x_2 b_1 a + x_1 ab + y b_1 b_2 + b_1 ab_2]$$

$$= (x_1 y x_2, E[x_1 x_2 a + y x_2 b + x_2 b_1 a + y b_1 b_2 + b_1 ab_2])$$

$$= (x_1 y x_2, x_1 x_2 E[a] + y x_2 b + x_2 b_1 E[a] + x_1 E[a] b_2 + y b_1 b_2 + b_1 E[a] b_2)$$

$$= (x_1, b_1)(y, E[a])(x_2, b_2)$$

$$= (x_1, b_1)\bar{E}[(y, a)](x_2, b_2).$$

It is obvious that $\bar{E}^2 = \bar{E}$. Hence, $\bar{E}$ is a $\mathcal{B}\mathcal{B}$ bimodule from the unital algebra $\bar{\mathcal{A}}$ to the unital subalgebra $\bar{\mathcal{B}}$, i.e. a conditional expectation.

**Proposition 4.1.** Let $(\mathcal{A}, \mathcal{B}, E) : A \rightarrow B$ be an operator valued probability space, $\{\mathcal{A}_i\}_{i \in I}$ be a $\mathcal{B}$-boolean independent family of sub-algebras and $\mathcal{B} \subset \mathcal{A}_i$ for all $i$. Then, in the unitalization operator probability space $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$, $\{\bar{\mathcal{A}}_i\}_{i \in I}$ is a $\mathcal{B}$-freely independent family of sub-algebras.

**Proof.** Let $(x, a) \in \bar{\mathcal{A}}$, where $a \in \mathcal{A}$ and $x$ is a complex number, then $\bar{E}[(x, a)] = (x, E[a])$, thus $\bar{E}[(x, a)] = 0$ if $x = 0$ and $E[a] = 0$.

Now, we can check the freeness directly. Let $(x_k, a_k) \in \bar{\mathcal{A}}_{i_k}$, i.e $a_k \in \mathcal{A}_{i_k}$ and $x_k$’s are complex numbers, for $k = 1, \cdots, n$ and $\bar{E}[x_k, a_k] = 0$ and $i_1 \neq i_2 \neq \cdots \neq i_n$, then we have $x_k = 0$ for all $k = 1, \cdots, n$ and

$$\bar{E}[(x_1, a_1)(x_2, a_2) \cdots (x_n, a_n)] = \bar{E}[(0, a_1)(0, a_2) \cdots (0, a_n)]$$

$$= \bar{E}[(0, a_1 a_2 \cdots a_n)]$$

$$= (0, E[a_1 a_2 \cdots a_n])$$

$$= (0, E[a_1 E[a_2] \cdots E[a_n]])$$

$$= (0, 0) = 0.$$

and $\bar{\mathcal{B}} \subset \bar{\mathcal{A}}_i$ for all $i$. □
By checking the conditions for operator valued freeness directly as we did in the above theorem, we also have

**Corollary 4.2.** Let \((A, B, E) : A \to B\) be an operator valued probability space, \(\{B \subset A_i\}_{i \in I}\) be a \(B\)-freely independent family of sub-algebras. Then, in their unitalization operator probability space \((A, B, E)\), \(\{A_i\}_{i \in I}\) is a \(B\)-freely independent family of sub-algebras.

5. **Operator valued boolean random variables are boolean exchangeable**

In this section, we prove that the joint distribution of \(n\) boolean independent operator valued random variables are invariant under the linear coactions of \(\bar{B}_s(n)\). The proof of the main theorem in this section involves combinatorial structure of the mixed moments of random variables. For boolean independence, the mixed moments of the random variables can be easily described by interval partitions. Therefore, we give an introduction below and we show some properties of this kind of partitions.

Given a set \(S\), a collection of disjoint nonempty sets \(P = \{V_i | i \in I\}\) is called a partition of \(S\) if \(\bigcup_{i \in I} V_i = S\). \(V_i \subset P\) is called a block of the partition \(P\). Let \(S\) be a finite ordered set, then all the partitions of \(S\) have finite blocks. A partition \(P = \{V_1, \cdots, V_r\}\) of \(S\) is interval if there are no two distinct blocks \(V_i\) and \(V_j\) and elements \(a, c \in V_i\) and \(b, d \in V_j\) s.t. \(a < b < c < d\) or \(b < c < d\). An interval partition \(P = \{W_s | 1 \leq s \leq r\}\) is ordered if \(a < b\) for all \(a \in W_s, b \in W_t\) and \(s < t\). We denote by \(P_I(S)\) the collection of ordered interval partitions of \(S\).

For convenience, we need to introduce an equivalence relation on indices sequences. Let \(I\) be an index set, \([k] = \{1, \cdots, k\}\) is an ordered set with the natural order. Let \(I^k = I \times I \times \cdots \times I\) be the \(k\)-fold Cartesian product of the index set \(I\). A sequence of indices \((i_m)_{m=1,\cdots,k} \in I^k\) is said to be compatible with an ordered interval partition \(P = \{W_1, \cdots, W_r\} \in P_I([k])\) if \(i_a = i_b\) whenever \(a, b\) are in the same block and \(i_a \neq i_b\) whenever \(a, b\) are in two consecutive blocks, i.e. \(W_s\) and \(W_{s+1}\) for some \(1 \leq s \leq r\). One should pay attention that \(i_a = i_b\) is allowed for \(a \in W_s, b \in W_t\) and \(s < t\).

Now, we define an equivalence relation \(\sim_P([k])\) on \(I^k\): two sequences of indices

\[ (i_m)_{m=1,\cdots,k} \sim_P([k]) (j_m)_{m=1,\cdots,k} \]

if the two sequences are both compatible with an ordered interval partition \(P \in P_I([k])\).

Given \(J = (i_m)_{m=1,\cdots,k}, J' = (j_m)_{m=1,\cdots,k} \in \{1, ..., n\}^k\), we denote \(P_{u_{i_1,j_1},u_{i_2,j_2}, \cdots, u_{i_k,j_k}}\) by \(U_J, J'\).

**Lemma 5.1.** Fix \(k \in \mathbb{N}\), let \(\bar{B}_s(n)\) be the boolean permutation quantum semigroup with standard generators \(\{u_{i,j}\}_{i,j=1,\cdots,n}\) and \(P\). Let \(J_1 = (i_1, \cdots, i_k), J_2 = (j_1, \cdots, j_k) \in [n]^k\) be two sequences of indices. Then, the product \(U_{J_1,J_2}\) is not vanishing if \(J_1 \sim_{P_I([k])} J_2\).

**Proof.** Suppose that each \(J_i\) is compatible with an ordered interval partition \(P_t\) for \(i = 1, 2\). Let \(P_1 = \{W_1, \cdots, W_{r_1}\}\) and \(P_2 = \{W'_1, \cdots, W'_{r_2}\}\), then \(P_1 \neq P_2\) implies that there exists a \(t\) such that \(W_t \neq W'_t\) for some \(1 \leq t \leq \min\{r_1, r_2\}\). Take the smallest \(t\), then \(W_s = W'_s\) whenever \(s < t\) and \(W_t \neq W'_t\). Then, these two intervals begin with the same number but end with different numbers. In other words, we have either \(W_t \subsetneq W'_t\)
or \( W'_t \subseteq W_t \). Without loss of generality, we assume \( W_t \subset W'_t \), then there is a number \( q \) s.t \( q \in W_t \) but \( q + 1 \notin W_t \) and \( q, q + 1 \in W'_t \). We have \( i_q \neq i_{q+1} \) and \( j_q = j_{q+1} \). Thus

\[
U_{\mathcal{J}_1, \mathcal{J}_2} = P_{u_{i_1,j_1}} \cdots u_{i_q,j_q} u_{i_{q+1},j_{q+1}} \cdots u_{i_k,j_k} P = 0.
\]

\[ \square \]

**Lemma 5.2.** Let \((A, B, E : A \to B)\) be an operator valued probability space. Let \((x_i)_{i=1, \ldots, n}\) be a sequence of \( n \) random variables which are identically distributed and boolean independent with respect to \( E \). Given two sequences of indices \( \mathcal{J} = (i_q)_{q=1, \ldots, k}, \mathcal{J}' = (j_q)_{q=1, \ldots, k} \in [n]^k \) and \( \mathcal{J} \sim_{P_t([n])} \mathcal{J}' \), then

\[
E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] = E[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}],
\]

where \( b_1, \ldots, b_{k-1} \in \mathcal{B} \cup \{I_A\} \).

**Proof.** Suppose that \( \mathcal{J} \) and \( \mathcal{J}' \) are compatible with an ordered interval partition \( P = \{W_1, \cdots, W_r\} \). Assume that \( W_1 = \{1, \cdots, k_1\} \), \( W_2 = \{k_1 + 1, \cdots, k_2\}, \cdots W_r = \{k_{r-1} + 1, \cdots, k\} \), then \( i_k \neq i_{k+1} \) and \( j_k \neq j_{k+1} \) for \( t = 1, \ldots, r \). For convenience, we let \( k_r = k \), \( k_0 = 0 \) and \( b_k = I_A \), we have

\[
\begin{align*}
E[x_{i_1} b_1 x_{i_2} b_2 \cdots b_{k-1} x_{i_k}] &= E[\prod_{s=1}^r (\prod_{t=n_{s-1}+1}^{n_s} x_{i_t} b_t)] \\
&= \prod_{s=1}^r E[\prod_{t=n_{s-1}+1}^{n_s} x_{i_t} b_t] \\
&= \prod_{s=1}^r E[\prod_{t=n_{s-1}+1}^{n_s} x_{j_t} b_t] \\
&= E[\prod_{s=1}^r (\prod_{t=n_{s-1}+1}^{n_s} x_{j_t} b_t)] \\
&= E[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}].
\end{align*}
\]

\[ \square \]

We denote \( \sim_{P_t([k])} \) by \( \sim_{P_t} \) when there is no confusion.

Let \( B_s(n) \) be the boolean permutation quantum semigroup with standard generators \( \{u_{i,j}\}_{i,j=1, \cdots, n} \) and \( P \). We have

**Lemma 5.3.** Fix \( k \) and \( 1 \leq i_1, \cdots, i_k \leq n \), then

\[
\sum_{j_1, \ldots, j_k=1}^n P_{u_{i_1,j_1}} \cdots u_{i_k,j_k} P = P
\]

**Proof.** The proof is straightforward:

\[
\begin{align*}
\sum_{j_1, \ldots, j_k=1}^n P_{u_{i_1,j_1}} \cdots u_{i_k,j_k} P &= \sum_{j_1, \ldots, j_{k-1}=1}^n P_{u_{i_1,j_1}} \cdots u_{i_{k-1},j_{k-1}} (\sum_{j_k=1}^n u_{i_k,j_k} P) \\
&= \sum_{j_1, \ldots, j_{k-1}=1}^n P_{u_{i_1,j_1}} \cdots u_{i_{k-1},j_{k-1}} P \\
&= \cdots = P.
\end{align*}
\]
According to the definition of $\mathcal{B}_s(n)$, it follows that the product $u_{i_1,j_1} \cdots u_{i_k,j_k}$ is not vanishing only if it satisfies that $i_t \neq i_{t+1}$ whenever $j_t \neq j_{t+1}$ for all $1 \leq t \leq k - 1$. Now, we can turn to prove the main theorem in this section.

**Theorem 5.4.** Let $(A, B, E : A \to B)$ be an operator valued probability space, $A$ be unital and $\{x_i\}_{i=1,...,n}$ be a sequence of $n$ random variables in $A$ which is identically distributed and boolean independent with respect to $E$. Let $\phi$ be a linear functional on $B$ and $\tilde{\phi}$ is a linear functional on $A$ where $\tilde{\phi}(\cdot) = \phi(E[\cdot])$. Then, the joint distribution of the sequence $\{x_i\}_{i=1,...,n}$ with respect to $\tilde{\phi}$ is invariant under the linear coaction of the boolean permutation quantum semigroup $\mathcal{B}_s(n)$.

**Proof.** Fix $k \in \mathbb{N}$, and indices $1 \leq i_1, \ldots, i_k \leq n$, and $b_1, \ldots, b_{k-1} \in B \cup \{I_A\}$, where $I_A$ is the unit of $A$, by the two lemmas above we have

$$
\sum_{j_1, j_2, \ldots, j_k = 1}^{n} E[x_{j_1}b_1x_{j_2}b_2 \cdots b_{k-1}x_{j_k}] \otimes P_{u_{i_1,j_1} \cdots u_{i_k,j_k}}
$$

$$
= \sum_{j_1, j_2, \ldots, j_k = 1}^{n} E[x_{j_1}b_1x_{j_2}b_2 \cdots b_{k-1}x_{j_k}] \otimes P_{u_{i_1,j_1} \cdots u_{i_k,j_k}}
$$

$$
= \sum_{j_1, j_2, \ldots, j_n = 1}^{n} E[x_{i_1}b_1x_{i_2}b_2 \cdots b_{k-1}x_{i_k}] \otimes P_{u_{i_1,j_1} \cdots u_{i_k,j_k}}
$$

$$
= \sum_{j_1, j_2, \ldots, j_n = 1}^{n} \tilde{\phi}(x_{j_1}x_{j_2} \cdots x_{j_n})P_{u_{i_1,j_1} \cdots u_{i_k,j_k}}.
$$

The last equality comes from Lemma 5.3. Let $b_1, \ldots, b_{k-1} = 1_A$ and let $\phi \otimes id_{\mathcal{B}_s(n)}$ act on the two sides of the above equation then we have

$$
\tilde{\phi}(x_{i_1}x_{i_2} \cdots x_{i_k})P
$$

$$
= \tilde{\phi}(x_{i_1}x_{i_2} \cdots x_{i_k})P
$$

$$
= \sum_{j_1, j_2, \ldots, j_n = 1}^{n} \tilde{\phi}(x_{j_1}x_{j_2} \cdots x_{j_n})P_{u_{i_1,j_1} \cdots u_{i_k,j_k}}.
$$

This is our desired conclusion. \hfill \Box

6. Tail algebra

In order to study boolean exchangeable sequences of random variables, we need to choose a suitable kind of noncommutative probability spaces. It is pointed out by Hasebe [10] that a $W^*$-probability with a faithful normal state does not contain a pair of boolean independent random variables with Bernoulli law. Moreover, in the end of the next section, we will show that, in a $W^*$-probability space with a faithful normal state, a boolean exchangeable sequence always contains identical elements. Therefore, in boolean situation, it is necessary to consider $W^*$-probability spaces with more general states rather than faithful states:

**Definition 6.1.** Let $A$ be a von Neumann algebra. A normal state $\phi$ on $A$ is said to be non-degenerated if $x = 0$ whenever $\phi(axb) = 0$ for all $a, b \in A$. 
Remark 6.2. By proposition 7.1.15 in [12], if $\phi$ is a non-degenerated normal state on $\mathcal{A}$ then the GNS representation associated to $\phi$ is faithful.

Let $(\mathcal{A}, \phi)$ be a $W^*$-probability space with a non-degenerated normal state $\phi$. Suppose that $\mathcal{A}$ is generated by an infinite sequence of exchangeable random variables $\{x_i\}_{i \in \mathbb{N}}$. In the usual way, the tail algebra $\mathcal{A}_{\text{tail}}$ of $\{x_i\}_{i \in \mathbb{N}}$ is defined by:

$$\mathcal{A}_{\text{tail}} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\},$$

where $vN\{x_k | k \geq n\}$ is the von Neumann algebra generated by $\{x_k | k \geq n\}$. We call $\mathcal{A}_{\text{tail}}$ the unital tail algebra of $\{x_i\}_{i \in \mathbb{N}}$ because it contains the unit of the original algebra. In a $W^*$-probability space with a non-degenerated normal state, we need to consider another kind of tail algebra $\mathcal{T}$ which is defined by the following formula:

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\},$$

where $W^*\{x_k | k \geq n\}$ is the WOT closure of the non-unital algebra generated by $\{x_k | k \geq n\}$. We call $\mathcal{T}$ the non-unital tail algebra of $\{x_i\}_{i \in \mathbb{N}}$. If the unit of $\mathcal{A}$ is contained in $\mathcal{T}$, then $\mathcal{T}$ is also the unital tail-algebra of $\{x_i\}_{i \in \mathbb{N}}$. Notice that the WOT closure of a non-unital algebra is different from the von Neumann algebra generated a non-unital algebra. The WOT closure of a non-unital algebra may not contain the unit of the original algebra. For example, let $P \in B(\mathbb{C}^2)$ be a one dimensional orthogonal projection. The weak operator closure of the algebra generated by $P$ is $CP$, but the vN-algebra generated by $P$ is $\mathbb{C}1_{B(\mathbb{C}^2)} + CP$.

In $W^*$-probability spaces with faithful states, the normal conditional expectation is constructed via the shift, i.e. a *-homomorphism which sends $x_i$ to $x_{i+1}$ for all $i \in \mathbb{N}$. In our situation, we should carefully choose the tail algebra so that we can construct a $\phi$-preserving normal conditional expectation via shifts. To illustrate this phenomena, we provide two examples here. For the details of the examples, see [4] and [8].

**Non-unital tail algebra case:** Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i \in \mathbb{N} \cup \{0\}}$. We define a sequence of operators $\{x_n\}_{n \in \mathbb{N}}$ as follows:

$$x_ne_0 = e_n, \text{ and } x_ne_i = \delta_{n,i}e_0 \text{ for } i \in \mathbb{N}.$$ 

Let $\mathcal{A}$ be the von Neumann algebra generated by $\{x_n\}_{n \in \mathbb{N}}$, then $e_0$ is cyclic for $\mathcal{A}$. Since $\mathcal{A}$ is WOT closed and contains all finite-rank operators, $\mathcal{A}$ is actually $B(\mathcal{H})$. On the other hand, if we denote by $P_i$, the orthogonal projection onto the one dimensional subspace generated by $e_i$ for $i \in \mathbb{N} \cup \{0\}$, then

$$x_1^2x_2^2 = P_{e_0}$$

and

$$\sum_{i=1}^{n} x_i^2 - (n-1)x_1^2x_2^2 = \sum_{i=0}^{n} P_{e_i}.$$ 

Since $\lim_{n \to \infty} \sum_{i=0}^{n} P_{e_i} = I_{B(\mathcal{H})}$ in WOT, $I_{B(\mathcal{H})}$ is contained in the WOT closure of the non-unital algebra generated by $\{x_i\}_{i \in \mathbb{N}}$. Let $\phi$ be the vector state $\phi(\cdot) = \langle \cdot e_0, e_0 \rangle$. We can see that the random variables $x_i$’s are identically distributed and boolean independent.
Since $e_0$ is cyclic for $B(\mathcal{H})$, the probability space $(\mathcal{A}, \phi)$ is non-degenerated. To construct a $\phi$-preserving conditional expectation, we need to use the non-unital tail algebra here. We have

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\} = CP_{e_0}.$$  

The tail algebra $\mathcal{T} = CP_{e_0}$ does not contain the unit of $B(\mathcal{H})$. The conditional expectation $E : \mathcal{A} \to \mathcal{T}$ is given by the following formula:

$$E[x] = P_{e_0} x P_{e_0},$$

for all $x \in \mathcal{A}$. One can check that the sequence $(x_i)_{i \in \mathbb{N}}$ is boolean independent in the operator valued probability space $(\mathcal{A}, \mathcal{T}, E : \mathcal{A} \to \mathcal{T})$.

If we use unital tail algebra in this situation, then we have

$$\mathcal{A}_{\text{tail}} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} \supset \{I_{B(\mathcal{H})}, P_{e_0}\}.$$  

Notice that

$$P_{e_0} = E\left[\sum_{i=1}^{n} x_i^2 - (n - 1)x_1^2x_2^2\right]$$

for all $n$. We have

$$w^* - \lim_{n \to \infty} E\left[\sum_{i=0}^{n} P_{e_i}\right] = P_{e_0} \neq I_{B(\mathcal{H})},$$

but $\lim_{n \to \infty} \sum_{i=0}^{n} P_{e_i} = I_{B(\mathcal{H})}$ in WOT. In conclusion, if the conditional expectation is normal, then it may not be unital. In other words, the normal map $E$ is not a conditional expectation in this case.

**Unital tail algebra case:** Let $\mathcal{H}_1 = \mathcal{H} \oplus Ce_{-1}$ be the direct sum of the Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_i\}_{i \in \mathbb{N} \cup \{0\}}$ and $Ce_{-1}$. As we constructed in the previous example, we define a sequence of operators $\{x_n\}_{n \in \mathbb{N}}$ as follows:

$$x_n e_0 = e_n, \text{ and } x_n e_i = \delta_{n,i}e_0 \text{ for } i \in \mathbb{N}, \ x_n e_{-1} = 0.$$  

Let $\mathcal{A}$ be the von Neumann algebra generated by $\{x_n\}_{n \in \mathbb{N}}$, then $\mathcal{A} = B(\mathcal{H}) \oplus CP_{e_{-1}}$. Therefore, the WOT-closure of the non-unital algebra generated by $\{x_n\}_{n \in \mathbb{N}}$ is $B(\mathcal{H}) \oplus 0$ does not contain the unit of $\mathcal{A}$. Then $\phi$ be the vector state $\phi(\cdot) = \frac{1}{2}\langle (e_0 + e_{-1}), e_0 + e_{-1}\rangle$, then the random variables $x_i$’s are identically distributed and boolean independent. In this case, we need to use the unital tail algebra here. The unital tail algebra is

$$\mathcal{A}_{\text{tail}} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} = CI_{\mathcal{H}_1} + CP_{e_0}.$$  

According to our construction, the conditional expectation $E$ is given by the following formula:

$$E[x] = P_{e_0} x P_{e_0} + \langle xe_{-1}, e_{-1}\rangle(I_{\mathcal{H}_1} - P_{e_0}),$$

for all $x \in \mathcal{A}$.

We see that the the non-unital tail algebra here is

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\} = CP_{e_0}.$$
If $E$ is a conditional expectation from $\mathcal{A}$ to $\mathcal{T}$, then it sends the unit of $\mathcal{A}$ to the unit of $\mathcal{T}$. We have

$$E[I_{\mathcal{H}}] = P_{e_0}.$$ 

But,

$$\phi(I_{\mathcal{H}}) \neq \frac{1}{2} = \phi(P_{e_0}).$$

Therefore, there is no $\phi$-preserving conditional expectation $E : \mathcal{A} \to \mathcal{T}$ in this situation.

Now, we turn to define the conditional expectation for our de Finetti type theorem. In the rest of this section, we suppose that the joint distribution of $\{x_i\}_{i \in \mathbb{N}}$ is boolean exchangeable. Let $\mathcal{A}_0$ be the non-unital algebra generated by $\{x_i\}_{i \in \mathbb{N}}$. In addition, we assume that the unit $1_{\mathcal{A}}$ of $\mathcal{A}$ is contained in the WOT closure of $\mathcal{A}_0$. We will denote the GNS construction associated to $\phi$ by $(\mathcal{H}, \xi, \pi)$, then there is a linear map $\hat{\alpha} : \mathcal{A}_0 \to \mathcal{H}$ such that $\hat{a} = \pi(a)\xi$ for all $a \in \mathcal{A}_0$. For convenience, we denote by $\mathcal{A}_n$ the non-unital algebra generated by $\{x_k | k > n\}$. Now, we turn to define our $\mathcal{T}$-linear map. Recall that in [14], the normal conditional expectation is defined as the WOT limit of shifts. The construction works because the shift of an exchangeable sequence is automatically a normal isomorphism. This fact relies on the property that a faithful normal state on $\mathcal{A}$ is faithful on all its subalgebras. When we consider $W^*$-probability spaces with non-degenerated normal states, we can see that a non-degenerated normal state on $\mathcal{A}$ is not necessarily non-degenerated on $\mathcal{A}$’s subalgebras. Therefore, in our situation, the shift of exchangeable sequence is not automatically a normal isomorphism. In consequence, the conditional expectation defined by taking WOT limit of shifts could not be well defined. Indeed, we are not sure that if we can construct a normal conditional expectation under the assumption that the random variables are only exchangeable. But in our situation, this is not a problem, since boolean exchangeability is much stronger than classical exchangeability. For this reason, we check the details of our conditional expectation in the rest of this section:

**Lemma 6.3.** Let $\mathcal{A}$ be a von Neumann algebra generated by an infinite sequence of selfadjoint random variables $(x_i)_{i \in \mathbb{N}}$, $\phi$ be a non-degenerated normal state on $\mathcal{A}$. If the sequence $(x_i)_{i \in \mathbb{N}}$ is exchangeable in $(\mathcal{A}, \phi)$, then there is a $C^*$-isomorphism $\alpha : \mathcal{A}_n^{\|\cdot\|} \to \mathcal{A}_i^{\|\cdot\|}$ such that,

$$\alpha(x_i) = x_{i+1},$$

for all $i \in \mathbb{N}$, where $\mathcal{A}_i^{\|\cdot\|}$ is the $C^*$-algebra generated by $\mathcal{A}_i$.

**Proof.** Let $(\mathcal{H}, \xi, \pi)$ be the GNS construction associated to $\phi$, it follows that $\{\hat{a} | a \in \mathcal{A}_0\}$ is dense in $\mathcal{H}$. For each $n \in \mathbb{N}$, denote by $\mathcal{A}[n]$ the non-unital algebra generated by $\{x_i | i \leq n\}$. Then $\bigcup_{n=1}^{\infty} \{\pi(a)\xi | a \in \mathcal{A}[n]\}$ is dense in $\mathcal{H}$. Given $y \in \bigcup_{n=1}^{\infty} \mathcal{A}[n]$, there exists $N \in \mathbb{N}$ such that $y \in \mathcal{A}[N]$. We can assume $y = p(x_1, ..., x_N)$ for some $p \in \mathbb{C}[\langle X_1, ..., X_N \rangle]_0$, then we have

$$\|\pi(p(x_1, ..., x_N))\xi\|^2 = \phi(\pi(p(x_1, ..., x_N)^*(p(x_1, ..., x_N)))$$

$$= \phi(p(x_2, ..., x_{N+1})^*(p(x_2, ..., x_{N+1}))$$

$$= \|\pi(p(x_2, ..., x_{N+1}))\xi\|^2$$
We can define an isometry $U$ from $\mathcal{H}$ to its subspace $\mathcal{H}_1$ which is generated by $\{\hat{a}|a \in A_1\}$ by the following formula:

$$U\pi(x_{i_1} \cdots x_{i_k})\xi = \pi(x_{i_1+1} \cdots x_{i_k+1})\xi,$$

for all $i_1, \ldots, i_k \in \mathbb{N}$.

Since $\phi$ gives a faithful representation to $A$, it gives a faithful representation to $A_0 = A^\|$. For all $y \in A_1$, according to the faithfulness, we have

$$\|y\|^2 = \sup\left\{ \frac{(y^*\hat{a})}{\langle \hat{a}, \hat{a} \rangle} | a \in A_0, \hat{a} \neq 0 \right\} = \sup\left\{ \frac{\phi(a^*y^*ya)}{\phi(a^*a)} | a \in A_0, \phi(a^*a) \neq 0 \right\}.$$

Denote by $(\mathcal{H}', \xi', \pi')$ the GNS representation of $A_1$ associated to $\phi$. Indeed, $\mathcal{H}'$ can treated as $\mathcal{H}_1$. Because the identity of $A$ is contained in the weak*–closure of the non-unital algebra generated by $(x_i)_{i \in \mathbb{N}}$, by the Kaplansky density theorem, there exists a bounded sequence $\{y_i\|y_i\| \leq 1\} \in \bigcup_{n=1}^\infty A_{[n]}$ such that $y_i$ converges to $1_A$ in WOT. Therefore, $\pi(y_i)\xi$ converges to $\xi$ in norm. Again, by the exchangeability of $(x_i)_{i \in \mathbb{N}}$ and $U\pi(y_i)\xi \in \{b|b \in A_1\}$ for all $i$, we have

$$\|U\pi(y_i)\xi\| = \|\pi(y_i)\xi\| \leq 1$$

and

$$\langle U\pi(y_i)\xi, \xi \rangle = \langle \pi(y_i)\xi, \xi \rangle \to 1.$$

Therefore, $U\pi(y_i)\xi$ converges to $\xi$ in norm, namely, $\xi \in \mathcal{H}_1$.

Let $x \in A_1$, then $x = p(x_2, \ldots x_{N+1})$ for some $N$ and $p \in \mathbb{C}\langle X_1, \ldots, X_N \rangle$. For every $y \in A_0$ there exists an $M$, such that $y = p'(x_1, \ldots, x_M)$ for some $p' \in \mathbb{C}\langle X_1, \ldots, X_M \rangle$. By the exchangeability, we send $x_1$ to $x_{N+M}$. Then

$$\|\pi(x)\hat{y}\|^2_{\mathcal{H'}} = \phi(p'(x_1, \ldots, x_M)^*p(x_2, \ldots x_{N+1})^*p(x_2, \ldots x_{N+1})p'(x_1, \ldots, x_M))$$

$$= \phi(p'(x_{M+N}, \ldots, x_M)^*p(x_2, \ldots x_{N+1})^*p(x_2, \ldots x_{N+1})p'(x_{M+N}, x_2, \ldots, x_M))$$

$$= \|\pi'(x)p'(x_{M+N}, x_2, \ldots, x_M)\|^2_{\mathcal{H'}}$$

and

$$\|p'(x_{M+N}, x_2, \ldots, x_M)\|_{\mathcal{H}} = \|p'(x_{M+N}, x_2, \ldots, x_M)\|_{\mathcal{H'}}.$$

Therefore, we get

$$\{\frac{\|\pi(x)\hat{a}\|_{\mathcal{H}}}{\|\hat{a}\|_{\mathcal{H}}} | a \in A_0, \hat{a} \neq 0\} \subseteq \{\frac{\|\pi'(x)\hat{a}\|_{\mathcal{H'}}}{\|\hat{a}\|_{\mathcal{H'}}} | a \in A_1, \hat{a} \neq 0\},$$

which implies

$$\|x\| = \|\pi(x)\| = \sup\left\{ \frac{\|\pi x\hat{a}\|_{\mathcal{H}}}{\|\hat{a}\|_{\mathcal{H}}} | a \in A_0, \hat{a} \neq 0 \right\} \leq \sup\left\{ \frac{\|\pi'(x)\hat{a}\|_{\mathcal{H'}}}{\|\hat{a}\|_{\mathcal{H'}}} | a \in A_1, \hat{a} \neq 0 \right\} = \|\pi'(x)\|.$$

It follows that $\|x\| = \|\pi'(x)\|$ for all $x \in A_1$. By taking the norm limit, we have $\|x\| = \|\pi'(x)\|$ for all $x \in A_1^\|$, so the GNS representation of $A_1^\|$ associated to $\phi$ is faithful.

Now, we turn to define our $C^*$-isomorphism $\alpha$:

Since $U$ is an isometric isomorphism from $\mathcal{H}$ to $\mathcal{H}'$, we define a homomorphism $\alpha' : \pi(A_0) \to B(\mathcal{H}')$ by the following formula

$$\alpha'(y) = UyU^*,$$
for $y \in \pi(A_0)$. Let $y \in \pi(A_{[n]})$, then $y = \pi(p(x_1, \ldots, x_n))$ for some $p \in C(X_1, \ldots, X_n)$. For all $v \in \bigcup_{n=2}^{\infty} \{\pi(a)\xi | a \in A_{[n]} \subset H'\}$, there exists $N \in \mathbb{N}$ and $p_1 \in C(X_1, \ldots, X_N)$ such that $v = \pi(p_1(x_2, \ldots, x_{N+1}))\xi$. We have

$$\alpha'(y)v = U\pi(p(x_1, \ldots, x_n))U^*\pi(p_1(x_2, \ldots, x_{N+1}))\xi$$

$$= U\pi(p(x_1, \ldots, x_n)\pi(p_1(x_1, \ldots, x_N))\xi$$

$$= U\pi(p(x_1, \ldots, x_n)p_1(x_1, \ldots, x_N))\xi$$

$$= \pi(p(x_2, \ldots, x_{N+1})p_1(x_1, \ldots, x_{N+1}))\xi$$

Since $\bigcup_{n=2}^{\infty} \{\pi(a)\xi | a \in A_{[n]}\}$ is dense in $H_1$, we get $\alpha'(\pi(p(x_1, \ldots, x_n))) = \pi(p(x_2, \ldots, x_{N+1}))$.

Because $(H, \xi, \pi)$ and $(H', \xi', \pi')$ are faithful GNS representations for $A_0$ and $A_1$ respectively, there is a well defined norm preserving homomorphism $\alpha : A_0 \to A_1$, such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. Therefore, $\alpha$ extends to a $C^*$-isomorphism from $A_0^{[n]}$ to $A_1^{[n]}$.

Since $W^*\{x_k | k \geq n\}$'s are WOT closed, their intersection is a WOT closed subset of $A$. Following the proof of proposition 4.2 in [15], we have

**Lemma 6.4.** For each $a \in A_0$, $\{\alpha^n(a)\}_{n \in \mathbb{N}}$ is a bounded WOT convergent sequence. Therefore, there exists a well defined $\phi$-preserving linear map $E : A_0 \to T$ by the following formula:

$$E[a] = w^* - \lim_{n \to \infty} \alpha^n(a)$$

for $a \in A_0$

**Proof.** By lemma 6.3, there is a norm preserving endomorphism $\alpha$ of $A_0$ such that

$$\phi \circ \alpha = \phi \quad \text{and} \quad \alpha(x_i) = x_{i+1}.$$  

For $I \subset \mathbb{N}$, denote by $A_I$ the non-unital algebra generated by $\{x_i | i \in I\}$. Suppose $a, b, c \in \bigcup_{|I|<\infty} A_I$, we can assume $a \in A_I, b \in A_J$ and $c \in A_K$ for some finite sets $I, J, K \subset \mathbb{N}$. Because $I, J, K$ are finite, there exists an $N$ such that $(I \cup K) \cap (J + n) = \emptyset$, for all $n > N$. We infer from the exchangeability that $\phi(aa^n(b)c) = \phi(aa^{n+1}(b)c)$ for all $n > N$. This establishes the limit

$$\lim_{n \to \infty} \phi(aa^n(b)c)$$

on the weak*-dense algebra $\bigcup_{|I|<\infty} A_I$. We conclude from this and $\{\alpha^n(b)\}_{n \in \mathbb{N}}$ is bounded that the pointwise limit of the sequence $\alpha$ defines a linear map $E : A_0 \to A$ such that $E(A_0) \subset T$.

To extend $E$ to the $W^*$-algebra $A$, we need to make use of the boolean invariance conditions.

**Lemma 6.5.** Let $\{x_i\}_{i \in \mathbb{N}} \subset A$ be an infinite sequence of random variables whose joint distribution is invariant under the linear coactions of the quantum semigroups $B_s(k)$’s, then

$$\phi(x_{i_1}^{k_1}x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, and $k_1, \ldots, k_n \in \mathbb{N}$.
Proof. If \( i_l \neq i_m \) for all \( l \neq m \), then the statement holds by the exchangeability of the sequence. Suppose the number \( i_l \) appears \( m \) times in the sequence, which are \( \{i_{l_j}\}_{j=1,...,m} \) such that \( i_{l_j} = i_l \) and \( l_1 < l_2 < \cdots < l_m \). Without loss of generality, we can assume that \( i_1, ..., i_n \leq N + 1 \) and \( i_l = N + 1 \) for some \( N \) by the exchangeability.

For each \( M \in \mathbb{N} \), by lemma 3.3, we have the following representation \( \pi_M \) of the quantum semigroup \( \mathcal{B}_s(M + N) \):

\[
\pi_M(u_{i,j}) = \begin{cases} 
    P_{i-N,j-N}, & \text{if } \min\{i,j\} > N \\
    \delta_{ij}P, & \text{if } \min\{i,j\} \leq N
\end{cases},
\]

and \( \pi(P) = P \), where \( p_{i,j} \) and \( p \) are projections in \( B(\mathbb{C}^{2M}) \) given by lemma 3.3. Then we have

\[
PP_{i,j}P = \frac{1}{M}P,
\]

for \( 1 \leq i, j \leq N \).

According to the boolean invariance condition, we have:

\[
\phi(x_{i_1}^{k_1}x_{i_2}^{k_2}\ldots x_{i_n}^{k_n})P = \sum_{j_1,j_2,\ldots,j_n=1}^{M+N} \phi(x_{j_1}^{k_1}x_{j_2}^{k_2}\ldots x_{j_n}^{k_n})Pu_{j_1,i_1} \cdots u_{j_n,i_n}P
\]

\[
= \sum_{j_1,j_2,\ldots,j_n=1}^{M+N} \phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{i_2}^{k_2}\ldots x_{j_2}^{k_2}\ldots x_{i_n}^{k_n})P
\]

\[
= \sum_{j_1,j_2,\ldots,j_n=1}^{M+N} \phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{j_n}^{k_n})P
\]

\[
= \frac{1}{M^{m}}[\sum_{j_1, j_2, \ldots, j_m=1}^{N} \phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{j_n}^{k_n})P + \sum_{j_s=j_t}^{N} \phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{j_n}^{k_n})P]
\]

In the first part of the sum, by the exchangeability, it follows that

\[
\phi(x_{i_1}^{k_1}x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1}\ldots x_{N+1}^{k_1}\ldots x_{N+2}^{k_1}\ldots x_{i_n}^{k_n}),
\]

where we sent \( j_s \) to \( N + s \). Then, we have

\[
\frac{1}{M^{m}} \sum_{j_s=j_t}^{N} \phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{j_n}^{k_n})P = \frac{\prod_{s=0}^{m-1} (M - s)}{M^{m}} \phi(x_{i_1}^{k_1}\ldots x_{N+1}^{k_1}\ldots x_{N+2}^{k_1}\ldots x_{i_n}^{k_n})P,
\]

which converges to \( \phi(x_{i_1}^{k_1}\ldots x_{N+1}^{k_1}\ldots x_{N+2}^{k_1}\ldots x_{i_n}^{k_n})P \) as \( M \) goes to \( \infty \).

To the second part of the sum, we have

\[
\phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{i_n}^{k_n}) \leq \|x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{i_n}^{k_n}\| \leq \|x_{1}^{k_1+\cdots+k_n}\|,
\]

which is bounded, therefore,

\[
\left| \frac{1}{M^{m}} \sum_{j_s=j_t}^{N} \phi(x_{i_1}^{k_1}\ldots x_{j_1}^{k_1}\ldots x_{j_2}^{k_2}\ldots x_{j_n}^{k_n}) \right| \leq (1 - \frac{\prod_{s=0}^{m-1} (M - s)}{M^{m}}) \|x_{1}^{k_1+\cdots+k_n}\|
\]
goes to 0 as $M$ goes to $\infty$. By now, we have showed that if there are indices $i_s = i_t$ for $s \neq t$ in the the sequence, we can, without changing the value of the mixed moments, change them to two different large numbers $j_s, j_t$ such that $j_s, j_t$ differ the other indices. After finite steps, we will have

$$\phi(x_{i_1}^{k_1}x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_{j_1}^{k_1}x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}),$$

such that all the $j_i$’ are not equal to any of the other indices. By the exchangeability, the proof is complete. \[\Box\]

**Corollary 6.6.** Let $\{x_i\}_{i \in \mathbb{N}} \subset (\mathcal{A}, \phi)$ be an infinite sequence of random variables whose joint distribution is invariant under the linear coactions of the quantum semigroups $B_s(k)$’s, then

$$\phi(x_{i_1}^{k_1}x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_{j_1}^{k_1}x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $j_1 \neq j_2 \neq \cdots \neq j_n$, $i_1, \ldots, i_n, j_1, \ldots, j_n \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{N}$. Moreover, we have

$$\phi(a x_{i_1}^{k_1}x_{i_2}^{k_2} \cdots x_{i_n}^{k_n} b) = \phi(a x_{j_1}^{k_1}x_{j_2}^{k_2} \cdots x_{j_n}^{k_n} b),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $j_1 \neq j_2 \neq \cdots \neq j_n$, $i_1, \ldots, i_n, j_1, \ldots, j_n > M$, $k_1, \ldots, k_n \in \mathbb{N}$ and $a, b \in \mathcal{A}_{[M]}$ for some $M \in \mathbb{N}$.

**Lemma 6.7.** For all $a, b, y \in \mathcal{A}$, we have

$$\langle E(y) \hat{a}, \hat{b} \rangle = \langle yE(\hat{a}), E(\hat{b}) \rangle.$$

**Proof.** Because an element in $\mathcal{A}$ is a finite linear combination of the noncommutative monomials, it suffices to show the property in the case: $b^* = x_{i_1}^{r_1} \cdots x_{i_l}^{r_l}$, $y = x_{j_1}^{s_1} \cdots x_{j_m}^{s_m}$, $a = x_{k_1}^{t_1} \cdots x_{k_n}^{t_n}$, where $i_1 \neq i_2 \neq \cdots \neq i_t$, $j_1 \neq \cdots \neq j_m$, $k_1 \neq \cdots \neq k_n$ and all the power indices are positive integers. Let $N = \max\{i_1, \ldots, i_l, j_1, \ldots, j_m, k_1, \ldots, k_n\}$, for all $L > N$, we have $i_t \neq j_1 + L$ and $j_m + L \neq k_1$. Therefore, we have

$$\langle E(y) \hat{a}, \hat{b} \rangle = \lim_{M \to \infty} \langle \alpha^M(y) \hat{a}, \hat{b} \rangle$$

$$= \langle \alpha^L(y) \hat{a}, \hat{b} \rangle$$

$$= \phi(x_{i_1}^{r_1} \cdots x_{i_l}^{r_l} x_{j_1}^{s_1} \cdots x_{j_m}^{s_m} x_{k_1}^{t_1} \cdots x_{k_n}^{t_n}),$$

by corollary 6.6,

$$= \phi(x_{i_1}^{r_1} \cdots x_{i_l}^{r_l} x_{j_1}^{s_1} \cdots x_{j_m}^{s_m} x_{k_1}^{t_1} \cdots x_{k_n}^{t_n} x_{L+M}^{r_1} \cdots x_{L+M+1}^{r_l} x_{L+M}^{s_1} \cdots x_{L+M+1}^{s_m} x_{L+M}^{t_1} \cdots x_{L+M+1}^{t_n})$$

$$= \lim_{M \to \infty} \phi(\alpha^N(b^*) y \alpha^{2L+M}(a))$$

$$= \phi(\alpha^L(b^*) y E[a]).$$

Notice that $\{\alpha^L(b)| L \leq N\}$ is a bounded sequence of random variables which converges to $E[b^*]$ in WOT and $\phi(y E[a])$ is a normal linear functional on $\mathcal{A}$, we have

$$\phi(\alpha^L(b^*) y E[a]) = \lim_{M \to \infty} \phi(\alpha^M(b^*) y E[a])$$

$$= \phi(E[b^*] y E[a])$$

$$= \langle y E[a], E[b] \rangle.$$
Lemma 6.8. Let \( \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{A}_0 \) be a bounded sequence of random variables such that \( w^* - \lim_{n \to \infty} y_n = 0 \), then \( w^* - \lim E[y_n] = 0 \).

Proof. For all \( a, b \in \mathcal{A}_0 \), we have
\[
\lim_{n} \langle E[y_n]\hat{a}, E[b]\rangle = \lim_{n} \langle y_n E[\hat{a}], E[b]\rangle = 0.
\]
Since \( \{\hat{a}\mid a \in \mathcal{A}_0\} \) is dense in \( \mathcal{H}_\xi \), we get our desired conclusion. \( \square \)

Let \( y \in \mathcal{A} \) and \( \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{A}_0 \) be a bounded sequence such that \( y_n \) converges to \( y \) in WOT. For all \( a, b \in \mathcal{A}_0 \), we have
\[
\lim_{n} \langle E[y_n]\hat{a}, \hat{b}\rangle = \lim_{n} \langle y_n E[\hat{a}], \hat{b}\rangle = \langle y E[\hat{a}], \hat{b}\rangle.
\]
Therefore, \( \{E[y_n]\}_{n \in \mathbb{N}} \) converges to an element \( y' \) in pointwise weak topology, by the lemma above, we see that \( y' \) is independent of the choice of \( \{y_n\}_{n \in \mathbb{N}} \). Since \( \{E[y_n]\}_{n \in \mathbb{N}} \subset \mathcal{T} \), we have \( y' \in \mathcal{T} \). By now, we have defined a linear map \( E : \mathcal{A} \to \mathcal{T} \) and we have

Lemma 6.9. \( E \) is normal.

Proof. Let \( \{y_n\}_{n \in \mathbb{N}} \subset \mathcal{A} \) be a bounded WOT convergent sequence of random variables such that \( w^* - \lim_{n \to \infty} y_n = y \). Then, we have
\[
\lim_{n \to \infty} \langle E[y_n]\hat{a}, \hat{b}\rangle = \lim_{n \to \infty} \langle y_n E[\hat{a}], \hat{b}\rangle = \langle y E[\hat{a}], \hat{b}\rangle = \langle E[y]\hat{a}, \hat{b}\rangle,
\]
for all \( a, b \in \mathcal{A}_0 \). Therefore, \( E \) is normal. \( \square \)

Now, we can turn to show that \( E \) is a conditional expectation from \( \mathcal{A} \) to \( \phi \):

Lemma 6.10. \( E[a] = a \) for all \( a \in \mathcal{T} \).

Proof. Let \( a \in \mathcal{T}, b, c \in \mathcal{A}_0 \), then there exists an \( N \in \mathbb{N} \) such that \( a \in \overline{\mathcal{A}_{N+1}w^*} \) and \( b, c \in \mathcal{A}_{[N]} \). We can approximate \( a \) in WOT by a bounded sequence \( \{a_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_{N+1} \) in WOT. According to the definition of \( E \) and the exchangeability, we have

\[
\langle E[a]\hat{c}, \hat{b}\rangle = \phi(b^* E[a]c) = \lim_k \phi(b^* E[a_k]c) = \lim_k \lim_{n} \phi(b^* \alpha^n(a_k)c) = \lim_k \phi(b^* a_k c) = \phi(b^* ac) = \langle a\hat{c}, \hat{b}\rangle.
\]

The equation is true for all \( b, c \in \mathcal{A}_0 \), so \( E[a] = a \). \( \square \)

To check the bimodule property of \( E \), we need to show that the quality of 6.7 holds for all \( x \in \mathcal{A} \):

Lemma 6.11. For all \( a, b, x \in \mathcal{A} \), we have
\[
\phi(aE[x]b) = \phi(E[a]xE[b]).
\]
Proof. By the Kaplansky’s density theorem, there exist two bounded sequences \( \{a_n \in \mathcal{A}_0 \| \|a_n\| \leq \|a\|, n \in \mathbb{N}\} \) and \( \{b_n \in \mathcal{A}_0 \| \|b_n\| \leq \|b\|, n \in \mathbb{N}\} \) which converge to \( a \) and \( b \) in WOT, respectively. Since \( \phi \) and \( E \) are normal, we have

\[
\phi(aE[x]b) = \lim_{n} \phi(a_nE[x]b) \\
= \lim_{n} \lim_{m} \phi(a_nE[x]b_m) \\
= \lim_{n} \lim_{m} \phi(E[a_n]xE[b_m]) \\
= \lim_{n} \phi(E[a_n]xE[b]) \\
= \phi(E[a]xE[b]).
\]

\[\square\]

Lemma 6.12. \( E[ax] = aE[x] \) for all \( a \in \mathcal{T} \) and \( x \in \mathcal{A} \).

Proof. For all \( b, c \in \mathcal{A}_0 \), by lemma 6.11 and Lemma 6.10, we have

\[
\langle E[ax] \hat{b}, \hat{c} \rangle = \phi(c^*E[ax]b) \\
= \phi(E[c^*]xE[b]) \\
= \phi((E[c^*]a)xE[b]).
\]

since \( E[c^*]a \in \mathcal{T} \), \( E[E[c^*]a] = E[c^*]a \), then

\[
\phi((E[c^*]a)xE[b]) = \phi(E[E[c^*]a]xE[b]) \\
= \phi(E[c^*]aE[x]b) \\
= \phi(E[c^*]E[aE[x]]b) \\
= \phi(E[c^*]((aE[x])E[b]) \\
= \phi(E[c^*](aE[x])E[b]) \\
= \phi(c^*E[aE[x]]b) \\
= \phi(c^*aE[x]b) \\
= \langle aE[x] \hat{b}, \hat{c} \rangle.
\]

Since \( b, c \) are arbitrary, we get our desired conclusion

\[\square\]

Lemma 6.13.

\[
E[x_{i_{t_1}}^{k_1} \ldots x_{i_{t_s}}^{k_s} \ldots x_{i_{t_t}}^{k_t} \ldots x_{i_{t_n}}^{k_n} ] = E[x_{i_{t_1}}^{k_1} \ldots a^N (x_{i_{s_1}}^{k_1} \ldots x_{i_{s_t}}^{k_t} \ldots x_{i_{s_n}}^{k_n})]
\]

whenever \( i_1 \neq i_2 \neq \cdots \neq i_n \), \( N \geq \max\{i_1, \ldots, i_n\} \), \( k_j \)’s are positive integers.

Proof. Given \( a, b \in \mathcal{A}_0 \), then there exists an \( M \) such that \( a, b \in \mathcal{A}_{[M]} \). Then, we have

\[
\langle E[x_{i_{t_1}}^{k_1} \ldots x_{i_{t_s}}^{k_s} \ldots x_{i_{t_t}}^{k_t} \ldots x_{i_{t_n}}^{k_n}] \hat{a}, \hat{b} \rangle \\
= \lim_{l \to \infty} \langle \alpha^l(x_{i_{t_1}}^{k_1} \ldots x_{i_{t_s}}^{k_s} \ldots x_{i_{t_t}}^{k_t} \ldots x_{i_{t_n}}^{k_n}) \hat{a}, \hat{b} \rangle \\
= \langle \alpha^M (x_{i_{t_1}}^{k_1} \ldots x_{i_{t_s}}^{k_s} \ldots x_{i_{t_t}}^{k_t} \ldots x_{i_{t_n}}^{k_n}) \hat{a}, \hat{b} \rangle \\
= \langle x_{i_{t_1}}^{k_1} \ldots x_{i_{t_s}}^{k_s} \ldots x_{i_{t_t}}^{k_t} \ldots x_{i_{t_n}}^{k_n} + a_{i_{t_1} + M} \ldots x_{i_{t_s} + M} \ldots x_{i_{t_t} + M} \ldots x_{i_{t_n} + M} \hat{a}, \hat{b} \rangle,
\]
by lemma 6.6 and \(i_1 + M \neq \cdots \neq i_{s+1} + M \neq i_s + M + N \neq i_{s+1} + M + N \neq \cdots \neq i_t + M + N \neq i_{t+1} + M \neq \cdots \neq i_n + M\),

\[
\begin{align*}
&\langle x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n} \rangle \\
&= \langle x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n} \rangle + M \hat{a}, \hat{b} \\
&= \langle x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n} \rangle + M \hat{a}, \hat{b} \\
&= \alpha^M (x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}) \hat{a}, \hat{b} \\
&= \lim_{l \to \infty} \alpha^l (x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}) \hat{a}, \hat{b} \\
&= \langle E[x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}] \rangle \hat{a}, \hat{b}.
\end{align*}
\]

Because \(\{\hat{a} | a \in A_0\}\) is dense in \(H\), the proof is complete.

\[\square\]

**Corollary 6.14.**

\[E[x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}] = E[x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}],\]

whenever \(i_1 \neq i_2 \neq \cdots \neq i_n\).

**Proof.** Let \(N = \max\{i_1, \ldots, i_n\}\). Since \(E[x_{i_1} \cdots x_{i_t}] = w^* - \lim_{l \to \infty} \alpha^l (x_{i_1} \cdots x_{i_t})\), we have

\[E[x_{k_1} \cdots x_{k_t}] = w^* - \lim_{l \to \infty} \frac{1}{l} \sum_{s=1}^{l} \alpha^{N+l} (x_{k_1} \cdots x_{k_t}).\]

Then, by lemma 6.13,

\[
\begin{align*}
E[x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}] &= \frac{1}{l} \sum_{s=1}^{l} E[x_{k_1} \cdots x_{k_s} \cdots x_{k_t} \cdots x_{k_n}] \\
&= E[x_{k_1} \cdots [w^* - \lim_{l \to \infty} \frac{1}{l} \sum_{s=1}^{l} \alpha^{N+l} (x_{k_1} \cdots x_{k_t})] \cdots x_{k_n}] \\
&= E[x_{k_1} \cdots E[x_{k_s} \cdots x_{k_t}] \cdots x_{k_n}].
\end{align*}
\]

The last two equations follow the normality of \(E\) and

\[
x_{k_1} \cdots \left[\frac{1}{l} \sum_{s=1}^{l} \alpha^{N+l} (x_{k_1} \cdots x_{k_t})\right] \cdots x_{k_n} \to x_{k_1} \cdots E[x_{k_s} \cdots x_{k_t}] \cdots x_{k_n}
\]

in WOT.

\[\square\]

**Lemma 6.15.**

\[E[b_1 x_{k_1} x_{k_2} \cdots b_s x_{k_s} \cdots b_t x_{k_t} \cdots b_n x_{k_n}] = E[b_1 x_{k_1} x_{k_2} \cdots b_s x_{k_s} \cdots b_t x_{k_t} \cdots b_n x_{k_n}],\]

whenever \(i_1 \neq i_2 \neq \cdots \neq i_n, k_1, \ldots, k_n\) are positive integers, \(b_1, \ldots, b_n \in A_{N+1}\) where \(N = \max\{i_1, \ldots, i_n\}\).

**Proof.** By the linearity of \(E\), we can assume that \(b_j\’s\) are “monomials”, i.e. \(b_j = x_{i_{j,1}} \cdots x_{i_{j,r_j}}\) where \(i_{j,1}\)’s are greater than \(N\). Then,

\[
b_1 x_{k_1} x_{k_2} \cdots b_s x_{k_s} \cdots b_t x_{k_t} \cdots b_n x_{k_n} = b_1 x_{k_1} x_{k_2} \cdots b_s x_{i_{s,1}} \cdots x_{i_{s,r_s}} x_{k_s} \cdots x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{k_t} \cdots b_n x_{k_n},
\]

\(i_{s,1} \geq N + 1 > i_{s-1}\) and \(r_s + r_t \geq N + 1 > i_{s+1}\). Therefore, by lemma 6.14,

\[
\begin{align*}
E[b_1 x_{k_1} x_{k_2} \cdots x_{i_{s,1}} \cdots x_{i_{s,r_s}} x_{i_{s,1}} x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{k_t} \cdots b_n x_{k_n}] &= E[b_1 x_{k_1} x_{k_2} \cdots E[x_{i_{s,1}} \cdots x_{i_{s,r_s}}] x_{i_{s,1}} x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{k_t} \cdots b_n x_{k_n}] \\
&= E[b_1 x_{k_1} x_{k_2} \cdots E[x_{i_{s,1}} \cdots x_{i_{s,r_s}}] x_{i_{s,1}} x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{k_t} \cdots b_n x_{k_n}].
\end{align*}
\]
Proposition 6.16. Let \((\mathcal{A}, \phi)\) be a W*-probability space and \((x_i)_{i \in \mathbb{N}}\) be a sequence of selfadjoint random variables in \(\mathcal{A}\) whose joint distribution is invariant of under the boolean permutations. Let \(E\) be the conditional expectation onto the non-unital tail algebra \(\mathcal{T}\) of the sequence. Then, \(E\) has the following factorization property: for all \(n, k \in \mathbb{N}\), polynomials \(p_1, ..., p_n \in \mathcal{T}(X_1, ..., X_k)\) and \(i_1, ..., i_n \in \{1, ..., k\}\) such that \(i_1 \neq i_2 \neq \cdots \neq i_n\), we have
\[
E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1}) \cdots E[p_l(x_{i_m})] \cdots p_n(x_{i_n})].
\]

Proof. It suffices to prove the statement in the case: \(p_1, ..., p_n\) are \(\mathcal{T}\)-monomials but none of them is an element of \(\mathcal{T}\). Assume that
\[
p_i(X) = b_{i,0}X^{i_1,1}b_{i,1}X^{i_2,2}b_{i,2} \cdots X^{i_k,ki},
\]
where \(b_{i,j} \in \mathcal{T}\) and \(t_{i,j}'s\) are positive integers. Let \(N = \max\{i_1, ..., i_n\}\), then \(b_{i,j} \in \mathcal{T} \subset \overline{\mathcal{N}_{N+1}}\). By the Kaplansky theorem, for every \(b_{i,j}\), there exists a bounded sequence \(\{b_{i,j,n}\}_{n \in \mathbb{N}}\) such that \(b_{i,j,n}\) converges to \(b_{i,j}\) in strong operator topology (SOT). Let \(p_{n,i}(X) = b_{n,i,0}X^{i_1,1}b_{n,i,1}X^{i_2,2}b_{n,i,2} \cdots X^{i_k,ki}\), then \(p_{n,i}(x_{i_k})\) converges to \(p_k(x_{i_k})\) in SOT. By the normality of \(E\), we have
\[
E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})] = w^* - \lim_{l \to \infty} E[p_{l,1}(x_{i_1}) \cdots p_{l,m}(x_{i_m}) \cdots p_{l,n}(x_{i_n})].
\]
By lemma 6.15, we have
\[
E[p_{l,1}(x_{i_1}) \cdots p_{l,m}(x_{i_m}) \cdots p_{l,n}(x_{i_n})] = E[p_{l,1}(x_{i_1}) \cdots E[p_{l,m}(x_{i_m})] \cdots p_{l,n}(x_{i_n})].
\]
It follows that \(E[p_{l,m}(x_{i_m})]\) converges to \(E[p_m(x_{i_m})]\) in WOT. Therefore,
\[
p_{l,1}(x_{i_1}) \cdots E[p_{l,m}(x_{i_m})] \cdots p_{l,n}(x_{i_n})
\]
converges to \(p_1(x_{i_1}) \cdots E[p_m(x_{i_m})] \cdots p_n(x_{i_n})\) in WOT. Now, we have
\[
E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})]
\]
\[
= w^* - \lim_{l \to \infty} E[p_{l,1}(x_{i_1}) \cdots p_{l,m}(x_{i_m}) \cdots p_{l,n}(x_{i_n})]
\]
\[
= w^* - \lim_{l \to \infty} E[p_{l,1}(x_{i_1}) \cdots E[p_{l,m}(x_{i_m})] \cdots p_{l,n}(x_{i_n})]
\]
\[
= E[p_1(x_{i_1}) \cdots E[p_m(x_{i_m})] \cdots p_n(x_{i_n})],
\]
the last equality follows \(E\)'s WOT continuity. \(\square\)

7. Main theorem and examples

In this section, we will complete the proof of the main theorems. In the first subsection, we assume that the unit of the original algebra is contained in the WOT-closure of the non-unital algebra generated by random variables. The proof in this case is a summary of the results in section 6 and section 5. In the second subsection, we assume that the unit of the original algebra is not contained in the WOT-closure of the non-unital algebra generated by random variables. We will show that a probability space in this situation is the unitalization of a case in subsection 7.1. Then, we prove the theorem by using results in section 5. In the last subsection, we show that boolean exchangeable sequences in a probability space with a faithful state is trivial, i.e. all the random variables are equal to each other.
7.1. Non-unital tail algebra case.

**Theorem 7.1.** Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space and \((x_i)_{i \in \mathbb{N}}\) be an infinite sequence of selfadjoint random variables. Suppose \(\mathcal{A}\) is the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\) and \(\phi\) is non-degenerated. Then the following statements are equivalent:

a) The joint distribution of \((x_i)_{i \in \mathbb{N}}\) satisfies the invariance conditions associated with the linear coactions of the quantum semigroups \(\mathcal{B}_s(n)\)'s.

b) The sequence \((x_i)_{i \in \mathbb{N}}\) is identically distributed and boolean independent with respect to a \(\phi\)-preserving normal conditional expectation \(E\) onto the non-unital tail algebra \(\mathcal{T}\) of the sequence \((x_i)_{i \in \mathbb{N}}\).

**Proof.**

a) \(\Rightarrow\) b): By choosing \(m = 1\) in proposition 6.16, we have

\[
E[p_1(x_{i_1}) \cdots p_2(x_{i_2}) \cdots p_n(x_{i_n})] = E[E[p_1(x_{i_1})]p_2(x_{i_2}) \cdots p_n(x_{i_n})]
\]

\[
= E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})],
\]

whenever \(i_1 \neq i_2 \neq \cdots \neq i_n\), \(p_1, \ldots, p_n \in \mathcal{T}(X)_0\).

b) \(\Rightarrow\) a) is a special case of theorem 5.4 \(\square\)

The non-unital tail algebra case example in the previous section is a special case of this theorem. The range of the conditional expectation in example is a one dimensional algebra.

7.2. Unital tail algebra case. Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space with a non-degenerated normal state \(\phi\) and \((x_i)_{i \in \mathbb{N}}\) be a sequence of selfadjoint random variables. Suppose \(\mathcal{A}\) is the WOT closure of the unital algebra generated \((x_i)_{i \in \mathbb{N}}\) and \(\phi\) is non-degenerated. Again, we denote by \(\mathcal{A}_0\) the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\). Let \(I_{\mathcal{A}}\) be the unit of \(\mathcal{A}\), we assume that \(I_{\mathcal{A}}\) is not contained in \(\overline{\mathcal{A}_0}^{w^*}\) and denote by \(I_1\) the unit of \(\overline{\mathcal{A}_0}^{w^*}\). Then,

\[
I_2 = I_{\mathcal{A}} - I_1 \neq 0
\]

and

\[
\mathcal{A} = \mathbb{C}I_2 \oplus \overline{\mathcal{A}_0}^{w^*}.
\]

For all \(x \in \overline{\mathcal{A}_0}^{w^*}\), we have

\[
I_2 x = (I_{\mathcal{A}} - I_1) x = 0.
\]

Let \(a \in \overline{\mathcal{A}_0}^{w^*}\) such that \(\phi(xay) = 0\) for all \(x, y \in \overline{\mathcal{A}_0}^{w^*}\). For \(\bar{x}, \bar{y} \in \mathcal{A}\), there exist two constants \(c_1, c_2 \in \mathbb{C}\) and \(x, y \in \overline{\mathcal{A}_0}^{w^*}\) such that \(x = c_1 I_2 + x\) and \(y = c_2 I_2 + y\), then

\[
\phi(\bar{x}a\bar{y}) = \phi(xab) = 0,
\]

Since our \(\bar{x}, \bar{y}\) are chosen arbitrarily, we have \(a = 0\). Therefore, \((\overline{\mathcal{A}_0}^{w^*}, \frac{1}{\phi(I_1)}\phi)\) is a \(W^*\)-probability space with a non-degenerated normal state. Let \(\mathcal{A}_{\text{tail}}\) be the unital
tail algebra of \((x_i)_{i \in \mathbb{N}}\) in \((\mathcal{A}, \phi)\) and \(\mathcal{T}\) be the non-unital tail algebra of \((x_i)_{i \in \mathbb{N}}\) in \((\mathcal{A}^{w^*}, \frac{1}{\phi(I_1)} \phi)\). Then, we have

\[
\mathcal{A}_{\text{tail}} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} = \bigcap_{n=1}^{\infty} (W^*\{x_k | k \geq n\} + \mathbb{C}I_A) = \mathcal{T} + \mathbb{C}I_A.
\]

Since \(\mathcal{A}^{w^*}_0\) is a two-sided ideal of \(\mathcal{A}\), for all \(\bar{x} \in \mathcal{A}_{\text{tail}}\) we have \(\bar{x} = aI_A + x\) for some \(x \in \mathcal{A}^{w^*}_0\) and \(a \in \mathbb{C}\). By theorem 7.1, there is a \(\phi\)-preserving normal conditional expectation \(E\) from \(\mathcal{A}^{w^*}_0\) onto \(\mathcal{T}\). As we proceeded in section 4, we can extend this conditional expectation \(E\) to an conditional expectation \(\bar{E}\) which is from the unitalization of \(\mathcal{A}^{w^*}_0\) to the unitalization of \(\mathcal{T}\). The unitalizations of the two algebras are isomorphic to \(\mathcal{A}\) and \(\mathcal{A}_{\text{tail}}\), respectively. We have

**Lemma 7.2.** The conditional expectation \(\bar{E}\) is \(\phi\)-preserving and normal.

**Proof.** The normality is obvious, we just check the \(\phi\)-preserving condition here. Let \(\bar{x} = aI_A + x\) for some \(x \in \mathcal{A}^{w^*}_0\) and \(a \in \mathbb{C}\), we have

\[
\phi(E[\bar{x}] = \phi(E[aI_A + x]) = \phi(aI_A + E[x]) = a + \phi(E[x]) = a + \phi(x).
\]

The last equality holds because \(E\) is a \(\frac{1}{\phi(I_1)} \phi\)-preserving conditional expectation on \((\mathcal{A}^{w^*}_0, \frac{1}{\phi(I_1)} \phi)\). \(\square\)

Together with proposition 5.4, we have the following theorem:

**Theorem 7.3.** Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space and \((x_i)_{i \in \mathbb{N}}\) be a sequence of self-adjoint random variables. Suppose the unit \(I_A\) of \(\mathcal{A}\) is not contained in the WOT closure of the non-unital algebra generated by \((x_i)_{i \in \mathbb{N}}\) and \(\phi\) is non-degenerated. Then the following statements are equivalent:

a) The joint distribution of \((x_i)_{i \in \mathbb{N}}\) is boolean exchangeable.

b) The sequence \((x_i)_{i \in \mathbb{N}}\) is identically distributed and boolean independent with respect to a \(\phi\)-preserving normal conditional expectation \(\bar{E}\) onto the unital tail algebra \(\mathcal{A}_{\text{tail}}\) of the \((x_i)_{i \in \mathbb{N}}\).

### 7.3. On \(W^*\)-probability spaces with faithful states

Now, we turn to consider boolean exchangeable sequences of random variables in a \(W^*\)-probability space with a faithful state:

**Theorem 7.4.** Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space and \((x_i)_{i \in \mathbb{N}}\) be a sequence of self-adjoint random variables such that \(\mathcal{A}\) is generated by \((x_i)_{i \in \mathbb{N}}\) and \(\phi\) is faithful. Then the following statements are equivalent:

a) The joint distribution of \((x_i)_{i \in \mathbb{N}}\) is boolean exchangeable.

b) \(x_i = x_j\) for all \(i, j \in \mathbb{N}\)
Proof. b) ⇒ a): If \( x_i = x_j \) for all \( i, j \in \mathbb{N} \), given a monomial \( p = X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1, \ldots, X_n \rangle \), then

\[
\mu_{x_1, \ldots, x_n}(X_{i_1} \cdots X_{i_k})P = \phi(x_{i_1} \cdots x_{i_k})P = \phi(x_1^k)P = \sum_{j_1, \ldots, j_k=1}^{n} \phi(x_1^k)\pi(Pu_{j_1,i_1} \cdots u_{j_k,i_k}P) = \sum_{j_1, \ldots, j_k=1}^{n} \phi(x_{j_1} \cdots x_{j_k})Pu_{j_1,i_1} \cdots j_k, i_kP = \mu_{x_1, \ldots, x_n} \otimes id_{B_s(n)}(lp).
\]

b) ⇒ a): It is sufficient to show that \( x_1 = x_2 \). By theorem 7.1 and 7.3, there exists a \( \phi \)-preserving conditional expectation \( E \) maps \( A \) to its unital or non-unital tail algebra such that \( (x_i)_{i \in \mathbb{N}} \) is identically distributed and boolean independent with respect to \( E \). For \( k \in \mathbb{N} \) and \( k > 2 \), we have

\[
\phi((x_1 - x_2)x_k((x_1 - x_2)x_k)^\ast) = \phi((x_1 - x_2)x_k^2(x_1 - x_2)) = \phi(E[(x_1 - x_2)x_k^2(x_1 - x_2)]) = \phi(E[x_1 - x_2]E[x_k^2]E[x_1 - x_2]) = 0.
\]

Since \( \phi \) is faithful, we get

\[
(x_1 - x_2)x_k = 0
\]

for all \( k > 2 \). Let \( A_k \) be the WOT closure of the non-unital algebra generated by \( \{x_n|n > k\} \), then we have

\[
(x_1 - x_2)x = 0
\]

for all \( x \in A_k \). Notice that \( (x_i)_{i \in \mathbb{N}} \) is exchangeable, by the construction of proposition 4.2 in [15], there exists a normal \( \phi \)-preserving homomorphism \( \alpha : A_n \to A_{n+1} \) such that \( \alpha(x_i) = x_{i+1} \). For each \( n \in \mathbb{N} \), we denote by \( I_n \) the unit of \( A_n \). Then, \( \alpha(I_n) = I_{n+1} \) and \( I_nI_{n+1} = I_{n+1}I_n \), since \( I_{n+1} \) is a projection in \( A_n \). Then, we have

\[
\phi((I_n - I_{n+1})^2) = \phi(I_n - I_{n+1}) = \phi(I_n) - \phi(\alpha(I_n)) = 0,
\]

which implies that \( I_n = I_{n+1} \). It follows that

\[
I_0 = I_1 = I_2.
\]

Therefore,

\[
0 = (x_1 - x_2)I_2 = (x_1 - x_2)I_0 = x_1 - x_2.
\]

□

Especially, we have the following:

**Corollary 7.5.** Let \( (A, \phi) \) be a \( W^* \)-probability space and \( (x_i)_{i \in \mathbb{N}} \) be a sequence of boolean independent and identically distributed random variables. Then, there exists a real number \( a \) and a projection \( P \in A \), such that \( x_i = aP \) for all \( i \in \mathbb{N} \).
Proof. The joint distribution of \((x_i)_{i \in \mathbb{N}}\) is boolean exchangeable, since \((x_i)_{i \in \mathbb{N}}\) are boolean exchangeable and identically distributed. According to Theorem 7.4, all these random variables \((x_i)_{i \in \mathbb{N}}\) are equal to each other. Let \(x = x_1\), then

\[
\phi(x^n) = \phi(\prod_{i=1}^{n} x_i) = \prod_{i=1}^{n} \phi(x) = \phi(x)^n.
\]

Therefore, \(\phi(2x^2 - \phi(x)x^2) = 0\). It implies that \(x^2 = \phi(x)\). If \(\phi(x) = 0\), then \(x^2 = 0\). In this case, \(a = 0\), \(P\) can be any projection in \(\mathcal{A}\). If \(\phi(x) \neq 0\), then \(\frac{1}{\phi(x)}x\) is a projection. In this case, \(a = \phi(x), P = \frac{1}{\phi(x)}x\). \(\square\)

8. TWO MORE KINDS OF PROBABILISTIC SYMMETRIES

In this section, we study two more kinds of probabilistic symmetries. Since \(\mathbb{C}(X_1, \ldots, X_n)\) is an algebra, it would be natural to define coactions of the quantum semigroups \(\mathcal{B}_s(n)\) on \(\mathbb{C}(X_1, \ldots, X_n)\) as algebraic homomorphisms but not only linear maps. In the following, we will define algebraic coactions of the quantum semigroups \(\mathcal{B}_s(n)'s\) and \(\mathcal{B}_s(n)'s\) on \(\mathbb{C}(X_1, \ldots, X_n)\). The invariance conditions for the joint distribution will be defined as we did in previous sections.

In our situation, the algebraic coaction \(L'_n : \mathbb{C}(X_1, \ldots, X_n) \to \mathbb{C}(X_1, \ldots, X_n) \otimes \mathcal{B}_s(n)\) is given by the following formulas:

\[
L'_n(1) = 1 \otimes 1, \quad L'_n(X_i) = \sum_{k=1}^{n} X_k \otimes P_{u_{k,i}}P.
\]

Since \(\mathbb{C}(X_1, \ldots, X_n)\) is freely generated by \(n\) variables \(X_1, \ldots, X_n\), the homomorphism is well defined. Then, we would have

\[
L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1, \ldots, j_k=1}^{n} X_{j_1} \cdots X_{j_k} \otimes P_{u_{j_1,i_1}}P \cdots P_{u_{j_n,i_n}}P
\]

and

\[
(L'_n \otimes id_{\mathcal{B}_s(n)})L'_n = (id_{\mathbb{C}_n} \otimes \Delta)L'_n.
\]

We call \(L'_n\) the algebraic coaction of \(\mathcal{B}_s(n)\) on \(\mathbb{C}(X_1, \ldots, X_n)\). The invariance condition associated with \(L'_n\) would be the defined as following:

**Definition 8.1.** In a probability space \((\mathcal{A}, \phi)\), let \((x_i)_{i=1,\ldots,n}\) be random variables in \(\mathcal{A}\). We say that the joint distribution of \((x_i)_{i=1,\ldots,n}\) is invariant under the algebraic coaction \(L'_n\) of \(\mathcal{B}_s(n)\) if

\[
\mu_{x_1,\ldots,x_n}(p)P = \mu_{x_1,\ldots,x_n} \otimes id_{\mathcal{B}_s(n)}(L'_n(p)),
\]

for all \(p \in \mathbb{C}(X_1, \ldots, X_n)\), where \(\mu_{x_1,\ldots,x_n}\) is the joint distribution of \((x_i)_{i=1,\ldots,n}\) with respect to \(\phi\).

We will see that this invariance condition is so strong that we can study it in finitely generated probability spaces.

**Proposition 8.2.** Let \((\mathcal{A}, \phi)\) be a \(W^*\)-probability space with a non-degenerated state \(\phi\). Suppose \(\mathcal{A}\) is the WOT closure of the unital algebra generated by selfadjoint random variables \((x_i)_{i=1,\ldots,n}\). Then, the joint distribution of \((x_i)_{i=1,\ldots,n}\) is invariant under the algebraic coaction \(L'_n\) of \(\mathcal{B}_s(n)\) is equivalent to \(x_1 = x_2 = \cdots = x_n\).
Proof. Suppose \( x_1 = x_2 = \cdots = x_n \). Let \( p = X_{i_1} \cdots X_{i_m} \in \mathbb{C}(X_1, \ldots, X_n) \), then we have
\[
\mu_{x_1, \ldots, x_n} \otimes id_{B_s(n)}(L_n'(X_{i_1} \cdots X_{i_m})) = \mu_{x_1, \ldots, x_n} \otimes id_{B_s(n)} \left( \sum_{j_1, \ldots, j_m = 1}^{n} X_{j_1} \cdots X_{j_m} \otimes P u_{j_1, i_1} P u_{j_2, i_2} P \cdots P u_{j_m, i_m} P \right)
\]
\[
= \sum_{j_1, \ldots, j_m = 1}^{n} \phi(x_{j_1} \cdots x_{j_m}) P u_{j_1, i_1} P u_{j_2, i_2} P \cdots P u_{j_m, i_m} P
\]
\[
= \phi(x_{i_1}^m) P u_{i_1, i_1} P u_{i_2, i_2} P \cdots P u_{i_m, i_m} P
\]
\[
= \mu_{x_1, \ldots, x_n}(X_{i_1} \cdots X_{i_m}) P.
\]
Since \( p \) is arbitrary, we proved \( \Leftrightarrow \).

Suppose the joint distribution of \( (x_i)_{i=1, \ldots, n} \) is invariant under the algebraic coaction \( L_n' \). Let \( \{v_1, \ldots, v_{2n}\} \) be orthonormal basis of the standard \( 2n \)-dimensional Hilbert space \( \mathbb{C}^{2n} \) and denote \( v_k = v_{k+2n} \) for all \( k \in \mathbb{Z} \). Let
\[
P_{i,j} = P_{v_1(v_{i}v_{j})+1v_{(j+1)+2}}
\]
and
\[
P = P_{v_1+v_2+\cdots+v_{2n}},
\]
where \( P_v \) is the orthogonal projection onto the one dimensional subspace generated by the vector \( v \). By lemma 3.4, we have a representation \( \pi \) of \( B_s(n) \) on \( \mathbb{C}^{2n} \) defined by the following formulas:
\[
\pi(P) = P, \quad \pi(P u_{i_1, j_1} \cdots u_{i_k, j_k} P) = P P_{i_1, j_1} \cdots P_{i_k, j_k} P
\]
for all \( i_1, j_1, \ldots, i_k, j_k \in \{1, \ldots, n\} \) and \( k \in \mathbb{N} \). In particular, we have
\[
\pi(P u_{i,j} P) = P P_{i,j} P = \frac{1}{n} P.
\]
Let \( \pi \) act on the invariance condition, we get
\[
\phi(x_{i_1} \cdots x_{i_k}) P = \pi(\mu_{x_1, \ldots, x_n}(X_{i_1} \cdots X_{i_k}) P)
\]
\[
= \pi(\sum_{j_1, \ldots, j_k = 1}^{n} \mu_{x_1, \ldots, x_n} \otimes id_{B_s(n)}(X_{j_1} \cdots X_{j_k} \otimes P u_{j_1, i_1} P \cdots P u_{j_k, i_k} P))
\]
\[
= \sum_{j_1, \ldots, j_k = 1}^{n} \phi(x_{j_1} \cdots x_{j_k}) \pi(P u_{j_1, i_1} P \cdots P u_{j_k, i_k} P)
\]
\[
= \sum_{j_1, \ldots, j_k = 1}^{n} \phi(x_{j_1} \cdots x_{j_k}) \frac{1}{n^k} P,
\]
for all \( i_1, \ldots, i_k \in \{1, \ldots, n\} \). It implies that
\[
\phi(x_{i_1} \cdots x_{i_k}) = \frac{1}{n^k} \sum_{j_1, \ldots, j_k = 1}^{n} \phi(x_{j_1} \cdots x_{j_k}).
\]

Therefore, two mixed moments are the same if their degrees are the same. Given two monomials \( a = x_{s_1} \cdots x_{s_k} \) and \( b = x_{t_1} \cdots x_{t_{k_2}} \), then
\[
\phi(a(x_i-x_j)(x_i-x_j)^*b) = \phi(a(x_i-x_j)^2b) = \phi(ax_i x_b) - \phi(ax_i x_j b) - \phi(ax_i x_j b) + \phi(ax_i x_j b) = 0.
\]
The last equation holds because all those monomials have the same degree. By the linearity of \( \phi \), we have
\[
\phi(a(x_i - x_j)(x_i - x_j)^*b) = 0,
\]
for all \(a, b \in \mathcal{A}_{[n]}\), where \(\mathcal{A}_{[n]}\) is the unital algebra generated by \(x_1, \ldots, x_n\). Therefore,
\[
x_i = x_j,
\]
for all \(i \neq j\). \(\square\)

In the end of this section, we define an algebraic coaction
\[
L_n : \mathbb{C} \langle X_1, \ldots, X_n \rangle \rightarrow \mathbb{C} \langle X_1, \ldots, X_n \rangle \otimes B_s(n)
\]
of \(B_s(n)\) on \(\mathbb{C} \langle X_1, \ldots, X_n \rangle\) via the following formulas:
\[
L_n(1) = 1 \otimes I, \quad L_n(X_i) = \sum_{k=1}^{n} X_k \otimes u_{k,i},
\]
where 1 is the identity of \(\mathbb{C} \langle X_1, \ldots, X_n \rangle\) and \(I\) is the identity of \(B_s(n)\). According to the definition of \(L_n\), we have
\[
L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1, \ldots, j_k=1}^{n} X_{j_1} \cdots X_{j_k} \otimes u_{j_1,i_1} \cdots u_{j_k,i_k}.
\]

One can easily check
\[
(L_n \otimes id_{B_s(n)})L_n = (id_{\mathbb{C}^n} \otimes \Delta)L_n,
\]
where \(id_{B_s(n)}\) and \(id_{\mathbb{C}^n}\) are identity maps on \(B_s(n)\) and \(\mathbb{C} \langle X_1, \ldots, X_n \rangle\) respectively. The invariance condition associated with \(L_n\) is

**Definition 8.3.** Let \((\mathcal{A}, \phi)\) be a probability space and \((x_i)_{i=1}^{n}\) be random variables in \(\mathcal{A}\). The joint distribution of \((x_i)_{i=1}^{n}\) is invariant under the coaction \(L_n\) if for all \(p \in \mathbb{C} \langle X_1, \ldots, X_n \rangle\), we have
\[
\mu_{x_1, \ldots, x_n}(p)I = \mu_{x_1, \ldots, x_n} \otimes id(L_n(p)).
\]

Then, we have

**Proposition 8.4.** Let \((\mathcal{A}, \phi)\) be a probability space and \((x_i)_{i=1}^{n}\) be random variables in \(\mathcal{A}\). If the joint distribution of \((x_i)_{i=1}^{n}\) is invariant under the coaction \(L_n\), then \(\phi(x_{i_1} x_{i_2} \cdots x_{i_k}) = 0\) for all \(i_1, \ldots, i_k \in \{1, \ldots, n\}\) and \(k \in \mathbb{N}\).

**Proof.** Take a trivial representation \(\pi\) of \(B_s(n)\) on a 1-dimensional space \(V\) defined by the following formulas:
\[
\pi(I) = 1, \quad \pi(u_{i,j}) = \pi(P) = 0,
\]
where 1 is the identity of \(V\). By the universality of \(B_s(n)\), \(\pi\) is well defined. According to the invariance condition, we have
\[
\pi(\mu_{x_1, \ldots, x_n}(p)I) = \mu_{x_1, \ldots, x_n} \otimes \pi(L_n(p)),
\]
for all \(p \in \mathbb{C} \langle X_1, \ldots, X_n \rangle\). Let \(p = X_{i_1} \cdots X_{i_k}\), we get
\[
\phi(x_{i_1} x_{i_2} \cdots x_{i_k})1 = \sum_{j_1, \ldots, j_k=1}^{n} \phi(x_{j_1} \cdots x_{j_k})\pi(u_{j_1,i_1} \cdots u_{j_k,i_k}) = 0,
\]
which completes the proof. \(\square\)
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