Weakened linearity for quantum fields

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Abstract. There are still no interacting models of the Wightman axioms, suggesting that the axioms are too tightly drawn. Here a weakening of linearity for quantum fields is proposed, with the algebra still linear but with the quantum fields no longer required to be tempered distributions, allowing explicit interacting quantum field models. Interacting quantum fields should be understood to be nonlinear quantum fields in this sense, because a set of effective field theories encodes a dependence on the energy scale of measurement — which is a nontrivial property of the test functions — so that correlation functions are implicitly nonlinear functions of test functions in the conventional formalism. In Local Quantum Physics terms, the algebraic models constructed here do not satisfy the additivity property. Finite nonlinear deformations of quantized electromagnetism are constructed as examples.

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1. Introduction

The free Klein-Gordon quantum field is an operator valued linear map from a suitable space of functions, \( \hat{\phi} : f \mapsto \hat{\phi}_f \). We will take \( f \) to be from a Schwartz space of functions\([1, \S II.1.2]\), so that \( f(x) \) is infinitely often differentiable and decreases as well as its derivatives faster than any power as \( x \) moves to infinity in any direction. For the free Klein-Gordon quantum field, \( \hat{\phi} \) is then a tempered distribution. This is the linearity we will weaken: we will allow the operator valued map \( \hat{\phi} : f \mapsto \hat{\phi}_f \) to be nonlinear, so that the linear operators \( \hat{\phi}_f, \hat{\phi}_g \) and \( \hat{\phi}_{f+g} \) will in general not satisfy the linear dependence \( \hat{\phi}_f + \hat{\phi}_g = \hat{\phi}_{f+g} \). With this weakening, we cannot take a quantum field to be an operator-valued distribution \( \hat{\phi}(x) \), we will be concerned only with operators \( \hat{\phi}_f \). Note, however, that allowing \( \hat{\phi} \) to be nonlinear does not weaken the linearity of the algebra generated by the operators \( \hat{\phi}_f \), and we will be able to construct a linear Hilbert space representation of the algebra of observables. The construction here is thus different from the nonlinear relativistic approach of Kibble, for example, who introduces a nonlinear Hamiltonian operator\([2]\).

The nonlinear quantum fields constructed here do not satisfy the additivity property of Local Quantum Physics \([1, \text{Axiom B, } \S III.1]\). This axiom requires that two algebras of observables, associated with regions \( O_1 \) and \( O_2 \) in space-time, together generate the algebra of observables associated with their union, \( \mathcal{A}(O_1 \cup O_2) = \mathcal{A}(O_1) \vee \mathcal{A}(O_2) \), but this is generally not possible if, for \( f \) and \( g \) with support in \( O_1 \) and \( O_2 \) respectively, \( \hat{\phi}_{f+g} \neq \hat{\phi}_f + \hat{\phi}_g \). The construction of this paper therefore casts some doubt on the necessity of the additivity property as an axiom of quantum field theory.

Locality and Lorentz covariance, however, will be preserved absolutely. The algebraic structure of a free linear quantum field is given by the hermitian inner product corresponding to the commutator, \([\hat{a}_g, \hat{a}_f^\dagger] = (f, g)\), with \( \hat{\phi}_f = \hat{a}_f + \hat{a}_f^\dagger \). A free linear quantum field is local just because \([\hat{\phi}_f, \hat{\phi}_g] = (g, f) - (f, g)\) is zero whenever the test functions \( f \) and \( g \) have space-like separated supports. Nonlinearity will be introduced in two ways, firstly by the simple expedient of taking the commutator to be a sum of a number of inner products such as, for example, without worrying here about constants,

\[
[\hat{a}_g, \hat{a}_f^\dagger] = \xi(f, g) = (f, g) + (f + f^2, g + g^2) + (f^2, g^2) + (f(f, f), g(g, g)) + (f + \partial_\mu f \partial^\mu f, g + \partial_\mu g \partial^\mu g) + \ldots,
\]

which will result in a local nonlinear quantum field just because invariant polynomials in the field and its derivatives such as \( f^n(x) = [f(x)]^n \) or \( \partial_\mu f \partial^\mu f \) have support contained in \( \text{Supp}(f) \). \([\hat{\phi}_f, \hat{\phi}_g] = \xi(g, f) - \xi(f, g)\) is zero, as for the free field, whenever the test functions \( f \) and \( g \) have space-like separated supports. The constraints of locality, positive semi-definiteness and Lorentz invariance on the form of \( \xi(f, g) \) are satisfied by many models, and it will turn out to be as easy to construct a vacuum state over this algebra as over the linear free field, allowing the GNS construction of a Hilbert space.

Secondly, we can deform the simple relationship \( \hat{\phi}_f = \hat{a}_f + \hat{a}_f^\dagger \), setting \( \hat{\phi}_f \) to be an
arbitrary self-adjoint operator-valued function of $\hat{a}_f + \hat{a}_f^\dagger$, $\hat{a}_{P_1[f]} + \hat{a}_{P_1[f]}^\dagger$, $\hat{a}_{P_2[f]} + \hat{a}_{P_2[f]}^\dagger$, $X_1(f), X_2(f), \ldots$, ...

$$\hat{\phi}_f = \hat{F}(\hat{a}_f + \hat{a}_f^\dagger, \hat{a}_{P_1[f]} + \hat{a}_{P_1[f]}^\dagger, \hat{a}_{P_2[f]} + \hat{a}_{P_2[f]}^\dagger, X_1(f), X_2(f), \ldots),$$  

(2)

where $\text{Supp}(P_i[f]) \subseteq \text{Supp}(f)$, and $X_i(f)$ are arbitrary Poincaré invariant scalar functions of $f$ — microcausality is satisfied whatever such scalar functions are introduced. In the general case this deformation is quite nontrivial, more general than a nonlinear coordinate transformation.

The energy scale of an experiment is essentially a pragmatic matter that is obvious to an experimenter: an experiment deals with phonons on a lattice, with atomic energy levels, with nuclear energy levels, etc., without an exact explicit discussion being necessary, and we can choose the cutoff appropriately for a given experiment without too much detailed concern. From a quantum field perspective, however, the energy scale of an experiment is a very non-detailed measure of the structure of the test functions involved in its description: if a test function appropriate to a description of an experiment determines an effective real-space length scale, or if the Fourier transform of the same (or another) test function is concentrated at a particular energy scale, then such scales pragmatically determine what effective field model we use. Hence, there is a *prima facie* case that the correlation functions of a quantum field are nonlinearly determined by properties of the test functions that describe an experiment, because the test functions are involved in an explicit description of correlation functions not only by smearing, so that interacting quantum fields should be understood to be nonlinear quantum fields. This significantly reconceptualizes our understanding of interacting quantum fields.

We will not here concern ourselves with the Hamiltonian operators of the theories we discuss, because the Hamiltonian is a global (non)observable, so that any constraint on it is essentially theoretical. Additionally, the Hamiltonian is inessential to the algebraic constructions of quantum field theories given here. Instead, we will take $n$-measurement correlation functions to be the observables of the theory, with empirical adequacy achieved if a theory can accurately model experimental correlations.

Section 2 first discusses free quantum fields, then section 3 introduces a large class of models that weaken the linearity of the quantum field by the introduction of a nonlinear inner product, and section 4 discusses the introduction of a nonlinear map between $\hat{\phi}_f$ and creation and annihilation operators. Section 5 applies the methods of section 3 to an electromagnetic field, leaving the application of the methods of section 4 to the future.

2. Free field preliminaries

A simple way to construct the free Klein-Gordon quantum field is to project $\hat{\phi}_f$ into two parts, $\hat{\phi}_f = \hat{a}_f + \hat{a}_f^\dagger$, and specify the algebraic properties of $\hat{a}_f^\dagger$ and $\hat{a}_f$ by the commutation relations

$$[\hat{a}_g, \hat{a}_f^\dagger] = (f, g), \quad [\hat{a}_g, \hat{a}_f] = 0.$$  

(3)
The manifestly Poincaré invariant hermitian inner product \((f, g)\) is given by

\[
(f, g) = \hbar \int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^\mu k_\mu - m^2) \theta(k_0) \hat{f}^*(k) \hat{g}(k).
\]

(4)

This fixes the algebraic structure of the observables \(\hat{\phi}_f, [\hat{\phi}_f, \hat{\phi}_g] = i\omega(f, g)\), where \(\omega(f, g) = i((f, g) - (g, f)) = -\omega(g, f)\). Note that the self-adjoint operators \(\hat{\phi}_f' = i(\hat{a}_f - \hat{\alpha}_f)\) are taken not to be observable (if they were observable then we would be able to send messages faster than light because \([\hat{\phi}_f', \hat{\phi}_g] = i((f, f) + (f, g))\) is non-zero when \(f\) and \(g\) have space-like separated supports\(^\dagger\)). The vacuum expectation values are fixed by the trivial action of the operators \(\hat{a}_f\) on the vacuum state, \(\hat{a}_f |0\rangle = 0\), and the normalization \(\langle 0 | 0 \rangle = 1\). To compute any vacuum expectation value, apply the commutation relations above repeatedly, eliminating any terms in which \(\hat{a}_f |0\rangle\) or \(\langle 0 | \hat{a}^+_f\) appear, until we obtain a number by finally applying \(\langle 0 | 0 \rangle = 1\). For example, \(\langle 0 | \hat{\phi}_f\hat{\phi}_g |0\rangle = \langle 0 | \hat{a}_f \hat{a}^+_g |0\rangle = \langle 0 | ((f, g) + \hat{a}_g \hat{a}_f) |0\rangle = (g, f)\).

The commutator algebra and the specification of the vacuum state fix the Wightman functions of the theory at all times, which effectively encodes all dynamical information, so that a Hamiltonian and Lagrangian are superfluous in this approach to quantum fields. Since the algebra and the definition of the vacuum are the only structures in this approach, those are what we have to deform to create an interacting field theory.

The free field algebra determines that the probability density associated with an observable \(\hat{\phi}_f\) in the vacuum state is Gaussian. The characteristic function can be computed as \(\langle 0 | e^{i\lambda\hat{\phi}_f} |0\rangle = e^{-\frac{\lambda^2}{4}f^2(f, f)}\) by applying a Baker-Campbell-Hausdorff formula, leading to the probability density \(\frac{1}{\sqrt{2\pi i (f, f)}} \exp\left(-\frac{\pi^2}{\lambda^2 f^2 f, f}\right)\), which is well-defined if we take \(f\) to be a Schwartz space function, but not if we take \(f\) to be a point-like delta function. In a similar way, we can compute the joint quasiprobability density associated with two observables \(\hat{\phi}_f\) and \(\hat{\phi}_g\) in the vacuum state, which is also Gaussian. The characteristic function is \(\langle 0 | e^{i\lambda\hat{\phi}_f + i\mu\hat{\phi}_g} |0\rangle = e^{-\frac{1}{4}(\lambda f + i\mu g)^2 + \lambda \mu \text{Re}(f, g) + \mu^2 \text{Re}(g, g)}\), leading to the quasiprobability density

\[
\exp\left(-\frac{1}{2}\frac{e^{2(g, g) - 2x y \text{Re}(f, g) + y^2 (g, g)}}{(f, f)(g, g) - |\text{Re}(f, g)|^2}\right)
\]

(5)

Note that this quasiprobability is independent of the imaginary parts of \((f, g)\). Finally for the vacuum state, for a set of observables \(\{\hat{\phi}_f\}\) we obtain a characteristic function \(\langle 0 | e^{i\lambda j\hat{\phi}_f} |0\rangle = e^{-\frac{1}{4}(\lambda f + i\mu g)^2 + \lambda \mu \text{Re}(f, g) + \mu^2 \text{Re}(g, g)}\), where the matrix \(F_{ij} = \text{Re}(f_i, f_j)\) describes the relative geometry of the \(n\) joint measurements for the purposes of the free field theory, leading\(^\dagger\) We can eliminate the creation and annihilation operators (which are too prominent in many presentations of the free quantized Klein-Gordon field), by presenting the algebra directly as \([\hat{\phi}_f, \hat{\phi}_g] = i\omega(f, g)\) and presenting the vacuum state using the generating function \(\langle 0 | e^{i\lambda \hat{\phi}_f} |0\rangle = e^{-\frac{1}{4}(\lambda f, f)}\), with \(\omega(f, g)\) and \((f, g)\) defined as above. Together these are as sufficient to fix the Wightman functions of the theory as the construction in the main text is.
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to the $n$-measurement joint quasiprobability density

$$e^{-\frac{1}{2}x^TF^{-1}x} \sqrt{\text{det}(F)}.$$

\text{(6)}

The singular condition $\text{det}(F) = 0$ is fairly innocuous, since it is the expectation values that are significant rather than any characteristic functions that can be used to generate them.

For the non-vacuum state $|0\rangle / \sqrt{(g, g)}$ and a set of observables $\{\hat{\phi}_f\}$, we obtain a characteristic function $\langle 0 | \hat{a}_g e^{i \sum_j \lambda_j \hat{\phi}_j} \hat{a}_g^\dagger |0\rangle / (g, g) = (1 - |S|^2) e^{-\frac{1}{2}x^TFx}$, where $S_i = (f_i, g) / \sqrt{(g, g)}$ describes the relation between the state preparation and the chosen measurements. This leads to the $n$-measurement joint quasiprobability density

$$\left[|x^TF^{-1}S|^2 + (1 - S^\dagger F^{-1}S)\right] e^{-\frac{1}{2}x^TF^{-1}x} \sqrt{(2\pi)^n \text{det}(F)}. \quad \text{(7)}$$

The imaginary parts of $(f, g)$ contribute to equation (7), which consequently may be not positive semi-definite. It is straightforward, but progressively more time-consuming, to compute $n$-measurement joint quasiprobability densities for higher states, which introduce increasing deviations from a Gaussian distribution. We can in principle also compute probability densities straightforwardly for higher order observables such as $\hat{\phi}_{f_1} \hat{\phi}_{f_2} + \hat{\phi}_{f_2} \hat{\phi}_{f_1}$.

The intention of this rather lengthy elementary discussion of characteristic functions and quasiprobabilities is to give some sense of how we can compute empirically relevant results quite effectively by only considering the relations between explicit measurement and state descriptions without ever considering operator-valued distributions $\hat{\phi}(x)$. We have exclusively used inner products between the functions $f_i$ and $g$ that were used above to construct measurements and states. Using test functions universally has the useful effect of ensuring manifest Poincaré invariance of the resulting formalism very straightforwardly. Note that we have used the term “$n$-measurement” correlations instead of “$n$-point”, because we never measure anything at a point, and the idealization of point-like measurements will become impossible when we introduce nonlinearity. All calculations involve only Schwartz space functions, which are much easier to manipulate than distributions, in particular because Schwartz space is closed under multiplication. In a simple-minded way, it is arguable that the infinities profusely generated by the conventional perturbation of free quantum fields are caused by the introduction of higher than quadratic products of distributions.

In more abstract terms, for free fields the properties of the vacuum state define a state $\varphi_0 : \mathcal{A} \mapsto \langle 0 | A | 0 \rangle$ over the $\ast$-algebra $\mathcal{A}$ generated by a finite number of creation and annihilation operators, a linear map satisfying $\varphi_0(A^\dagger) = \overline{\varphi_0(A)}$, $\varphi_0(A^\dagger A) \geq 0$.

§ If instead of the inner product of equation (4), we use the real form $(f, g) + (g, f)$, we still obtain a quantum field theory, but it is classical in the sense that $[\hat{\phi}_f, \hat{\phi}_g] = 0$ whatever the space-time relationship between $f$ and $g$, and equation (7) is accordingly positive semi-definite. For a comparable perspective on the relationship between random fields and quantum fields see [3].
\( \varphi_0(1) = 1 \), which allows the Gelfand-Naimark-Segal construction of a pre-Hilbert space acted on by \( \mathcal{A} \), which can be closed in the norm to obtain a Hilbert space \( \mathcal{H}_{\varphi_0} \) (see Haag\[^{11}\] §III.2).

For free fields, \( \varphi_0(A) = \langle 0 \mid A \mid 0 \rangle \) satisfies \( \varphi_0(A^\dagger A) = \langle 0 \mid A^\dagger A \mid 0 \rangle \geq 0 \) because

\[
\langle 0 \mid \left[ \prod_{k=1}^{K} \hat{a}_{f_k} \right] \left[ \prod_{j=1}^{J} \hat{a}_{g_j}^\dagger \right] \mid 0 \rangle = \delta_{IJ} \text{per}[(g_j, f_k)],
\]

where \( \text{per}[(g_j, f_k)] \) is the permanent of the \( K \times K \) complex matrix \( (g_j, f_k) \). It is well-known\[^{4, 5}\] that

\[
\mathcal{S}^{\otimes K} \times \mathcal{S}^{\otimes K} \rightarrow \mathbb{C}; \quad (g_1 \otimes \ldots \otimes g_K, f_1 \otimes \ldots \otimes f_K) \mapsto \text{per}[(g_j, f_k)],
\]

is a complex hermitian positive semi-definite inner product on the symmetrized tensor product space \( \mathcal{S}^{\otimes K} \), so that equation (8) defines a complex hermitian positive semi-definite inner product on a direct sum of symmetrized tensor product spaces.

Any operator constructed as a multinomial in \( \hat{a}_f \) is not in the algebra \( \mathcal{B}(\mathcal{H}_{\varphi_0}) \) of bounded observables acting on \( \mathcal{H}_{\varphi_0} \), so we generally have to pay attention to the domain of \( A \in \mathcal{A} \). The insistence on at least a Banach \( * \)-algebra structure for the algebra of observables is useful for analysis (allowing, for example, the extension of the action of the algebra of observables to the Hilbert space \( \mathcal{H}_{\varphi_0} \)), but for constructive calculations of expectation values, characteristic functions, and probability distributions in particular states, as above, if \( \langle \psi \mid A \mid \psi \rangle \) is finite for a normalized vector \( |\psi\rangle \in \mathcal{H}_{\varphi_0} \) then we can interpret \( A \) as an observable for that state. This is a nontrivial extension of the pre-Hilbert space because, for example, the normalized vector \( e^{i\lambda \hat{a}_f} |0\rangle / \sqrt{\varepsilon(g, \varphi)} \) gives us a finite state over \( \mathcal{A} \). As well as extending the pre-Hilbert space, we have already implicitly extended the algebra \( \mathcal{A} \) by using \( \langle 0 \mid e^{i\lambda \hat{a}_f} |0\rangle \) above as a characteristic function, since \( e^{i\lambda \hat{a}_f} \) is not a polynomial in the field.

### 3. Weakened linearity I

Suppose now that we replace equation (9) by a commutation relation that depends nonlinearly on \( f \) and \( g \),

\[
[\hat{a}_g, \hat{a}_f^\dagger] = \xi(f, g), \quad [\hat{a}_g, \hat{a}_f] = 0,
\]

where \( \xi(f, g) \) must be complex hermitian positive semi-definite on Schwartz space (in the sense that the matrix \( \xi(f_i, f_j) \) is complex hermitian positive semi-definite for any \( \sigma \in S_K \).

The permanent of a \( K \times K \) matrix \( M \) is a sum over the symmetric group, \( \text{per}(M) = \sum_{\sigma \in S_K} M_{\sigma(1)}M_{\sigma(2)}\ldots M_{\sigma(K)} \). This is the determinant without the sign of the permutation. The normalized permanent per\([(g_j, g_k)] / \prod_{i=1}^{K} (g_i, g_i) \) of a complex hermitian positive semi-definite matrix that is generated using inner products \( (g_j, g_k) \) measures how close the \( K \) functions \( g_i \) are to being parallel, independently of the relative lengths \( (g_i, g_i) \) of the functions, except in the singular case when \( \prod_{i=1}^{K} (g_i, g_i) = 0 \). If the functions are all parallel, the normalized permanent is \( K! \); if they are all orthogonal, the normalized permanent is 1. Comparably, the normalized determinant is zero if any subset of the functions is linearly dependent; if all the functions are orthogonal the normalized determinant is 1.
finite set of Schwartz space functions \( \{ f_i \} \). We will call \( \xi(f, g) \) a “nonlinear inner product”; the term “inner product” historically indicates a sesquilinear form, so we will always be explicit about nonlinearity. The operator valued map \( \hat{\phi} : f \mapsto \hat{\phi}_f \) cannot be linear if \( \xi(f, g) \) is nonlinear. The algebra \( \mathcal{A}_d \) generated by \( \hat{\phi}_f \) is still linear, but the linear dependence \( \hat{\phi}_f + \hat{\phi}_g = \hat{\phi}_{f+g} \) generally does not hold.

Essentially, for any set of vectors \( \{ g_i \} \) used to construct an operator in the deformed free field algebra, we obtain a complex hermitian positive semi-definite matrix \( \xi(g_i, g_j) \).

As a complex hermitian positive semi-definite matrix, it is a Gram matrix based on some other functions \( \{ f_i \} \) chosen so that \( (f_i, f_j) = \xi(g_i, g_j) \). The action of the vacuum state on an operator \( A^\dagger A \) in \( \mathcal{A}_d \) that is constructed using \( \{ \hat{a}^\dagger_{g_i} \} \) is positive semi-definite, therefore, just because the action of the vacuum state on an operator constructed in the same way in \( \mathcal{A} \) using \( \{ \hat{a}^\dagger_{f_i} \} \) is positive semi-definite.

To ensure locality,

\[
[\hat{\phi}_f, \hat{\phi}_g] = \xi(g, f) - \xi(f, g),
\]

must be zero when \( f \) and \( g \) have space-like separated supports. There is a wide range of possibilities for \( \xi(f, g) \): we can use the sum of any number of complex hermitian positive semi-definite inner products such as

\[
(f, g), (f + f^2, g + g^2), (f^2, g^2), \ldots, (f^n, g^n), \ldots,
\]

just because the sum of positive semi-definite matrices is positive semi-definite. All these terms satisfy locality because \( f^n \) has the same support as \( f \), so that, for example, \( \omega(f^n, g^n) \) is zero if \( f \) and \( g \) have space-like separated support. We can also introduce invariant polynomials in derivatives of the field, such as \( \partial_\mu f \partial^\mu f \), which again have the same support as \( f \). Furthermore, we need not restrict ourselves to one inner product \( (f, g) \), we can introduce different mass Poincaré invariant inner products for different invariant polynomials in the field and its derivatives. If the free quantum field is a 4-vector or other nontrivial representation space of the Lorentz group, “\( f^n \)”, perhaps contracted in some way, will usually require a different inner product than \( f \) (see section \[ \text{4} \] for a concrete example). In general, \( \xi(f, g) \) can be a sum

\[
\xi(f, g) = \sum_i (\mathcal{P}_i[f], \mathcal{P}_i[g]),
\]

for a list of local functionals \( \mathcal{P}_i \), satisfying \( \text{Supp}(\mathcal{P}_i[f]) \subseteq \text{Supp}(f) \), and a list of linear inner products \( (\cdot, \cdot)_i \).

That we cannot in general expect the linear dependencies \( \hat{\phi}_f + \hat{\phi}_g = \hat{\phi}_{f+g} \) and \( \hat{\phi}_{\lambda f} \neq \lambda \hat{\phi}_f \) to hold requires a fresh understanding of what we do when we describe a measurement using a function \( f + g \) or \( \lambda f \), which we must derive from the mathematical structure of the nonlinear inner product. In the linear case, we can imagine in folk terms that when we use the operator \( \hat{\phi}_f \) we are asking how much \( f \) “resonates” with the quantum state, insofar as the inner product of \( f \) with the functions \( g_i \) that are used to construct the state is a measure of similarity between the on-shell fourier components of the functions. There is of course a minimal “resonance” of \( f \) with vacuum state
fluctuations. In the nonlinear case, in the same folk terms, the nonlinear inner product is a measure of similarity between not only the on-shell components of $f$ and $g_i$, but also between the on-shell components of $f^2$ and $g_i^2$, $f + f^2$ and $g_i + g_i^2$, etc. We cannot, therefore, just add the results of measuring $\hat{\phi}_f$ and $\hat{\phi}_g$ to compute what we would have observed if we had measured $\hat{\phi}_{f+g}$, because the nonlinear resonances are not taken into account by simple addition of the operators.

Analogously to equations (6) and (7), we can construct the pseudoprobabilities

$$e^{-\frac{1}{2}x^T F^{-1} x} \sqrt{(2\pi)^n \det(F)},$$

$$\left[ x^T F^{-1} S |^2 + (1 - S^\dagger F^{-1} S) \right] \frac{e^{-\frac{1}{2}x^T F^{-1} x}}{\sqrt{(2\pi)^n \det(F)}},$$

$$F_{ij} = \text{Re} \left[ \xi(f_i, f_j) \right], \quad S_i = \frac{\xi(f_i, g_i)}{\sqrt{\xi(g, g)}}$$

in which the only change, predictably enough, is that we replace the inner product $(f, g)$ by the “nonlinear inner product” $\xi(f, g)$ wherever it occurs. The probability densities generated for the vacuum state are still Gaussian (which will be addressed by the method of the next section), but, for example, the fall-off of the 2-measurement correlation coefficient with increasing distance is controlled by $\xi(f, g)$, so the fall-off is in general nontrivially different from the fall-off for the free field. For scalar functions $f(x)$ and $f_a(x) = f(x+a)$ representing two measurements at separation $a^\mu$, and supposing the dynamics is described by the inner product (4) with masses $m_i$, then the 2-measurement correlation function is given by

$$\xi(f, f_a) = \hbar \sum_i \int \mathcal{P}_i[f] \mathcal{P}_i[f_a] 2\pi \delta(k^\mu k_\mu - m_i^2) \theta(k_0) \frac{d^4k}{(2\pi)^4},$$

$$= \hbar \sum_i \int |\mathcal{P}_i[f]|^2 e^{-ik_\mu a^\mu} 2\pi \delta(k^\mu k_\mu - m_i^2) \theta(k_0) \frac{d^4k}{(2\pi)^4},$$

so with a suitable choice of $\mathcal{P}_i$, we have considerable control over the change of the 2-measurement correlation with increasing separation and for different functions $f$.

4. Weakened linearity II

If we observe non-Gaussian probability densities, we can model them in linear quantum field theory by acting on the vacuum state with as many creation operators as necessary, spread over as large a region of space-time as necessary, or by constructing representations of the weakened linear commutation relations that are unitarily inequivalent to the vacuum sector. This section discusses the nonlinear alternative already introduced in the introduction, which maps the creation and annihilation operators nonlinearly to the quantum field $\hat{\phi}_f$,

$$\hat{\phi}_f = \check{F}(\hat{a}_f + \hat{a}_f^\dagger, \hat{a}_{\mathcal{P}_1[f]} + \hat{a}_{\mathcal{P}_1[f]}^\dagger, \hat{a}_{\mathcal{P}_2[f]} + \hat{a}_{\mathcal{P}_2[f]}^\dagger, X_1(f), X_2(f), ...),$$

(18)
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where \( \text{Supp}(\mathcal{P}_i[f]) \subseteq \text{Supp}(f) \), and \( X_i(f) \) are arbitrary Poincaré invariant scalar functions of \( f \). Microcausality is preserved, \([\hat{\phi}_f, \hat{\phi}_g] = 0 \) whenever \( f \) and \( g \) have spacelike separated supports, because \([\hat{a}_{\mathcal{P}_i[f]} + \hat{a}_{\mathcal{P}_i[f]}^\dagger, \hat{a}_{\mathcal{P}_j[g]} + \hat{a}_{\mathcal{P}_j[g]}^\dagger] = 0 \) \( \forall i, j \), but if \( \hat{F} \) includes a dependency on \((f, f)\), for example, there is a larger sense in which the algebra of observables is nonlocal. We take the set of observables to be the subalgebra of the algebra of operators generated by \( \hat{a}_f \) and \( \hat{a}_f^\dagger \) that is generated by \( \hat{\phi}_f \) (as noted above, the set of observables in the linear free field case is generated by \( \hat{\phi}_f \), not by the creation and annihilation operators).

In the simplest case, we can set \( G(\hat{\phi}_f) = \hat{a}_f + \hat{a}_f^\dagger \) for some invertible function \( G(x) \); with this deformation, the gaussian probability density \( Pr(\hat{a}_f + \hat{a}_f^\dagger = x) = \exp\left(\frac{-x^2}{2(f, f)}\right)/\sqrt{2\pi(f, f)} \) becomes

\[
Pr(\hat{\phi}_f = y) = \frac{1}{\sqrt{2\pi(f, f)}} \exp\left(\frac{-G(y)^2}{2(f, f)}\right)G'(y). \tag{19}
\]

This simplest case is of course more-or-less trivial, but in the most general case the nonlinear map \( F \) is not so easily dismissed. Whether trivial or not, even for \( G(x) = x - \tanh x \) we obtain a probability density with the double maximum characteristic of symmetry breaking,

\[
Pr(\hat{\phi}_f = y) = \frac{1}{\sqrt{2\pi(f, f)}} \exp\left(\frac{-(y - \tanh y)^2}{2(f, f)}\right)(1 - \text{sech}^2 y) \tag{20}
\]

(though this is not enough to claim that such a state corresponds to conventional symmetry breaking). Calculating \( n \)-measurement correlation functions in this superficially simple model for \( n \geq 2 \) is not straightforward.

We have effectively constructed a class of quantum fields that is analogous to the class of integrable systems in classical field theory in that they are reducible to a free quantum field by nonlinear (and possibly microcausality preserving but otherwise nonlocal) maps. In other attempts to construct algebras of observables using the nonlinear operator-valued map \( \hat{\phi} : f \to \hat{\phi}_f \), using algebra deformations similar to those of Arik-Coons type\[6\] (which work nicely in the one-dimensional case), I have not so far found it possible to construct quantum field algebras that are both microcausal and associative, which I have taken to be essential requirements.

5. Deformation of electromagnetism

The electromagnetic potential and Dirac spinors are not observable fields, so we will here deform the quantized electromagnetic field. To avoid excessive complexity, we will use only the method of section\[3\]. The dynamics of the electromagnetic field in terms of a positive semi-definite inner product on test functions is given by Menikoff and Sharp\[7\] (equation (3.27)) (except for a missing factor of \((2\pi)^{-3}\) that is present in their equation (3.25)):

\[
(f_1, f_2)_{EM} = \hbar \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k_\alpha k^\alpha)\theta(k_0)k^\mu \tilde{f}_1^{*\mu\beta}(k)k^\nu \tilde{f}_2^{\beta\nu}(k). \tag{21}
\]
Note that $f_1$ and $f_2$ are not electromagnetic field tensors, they are classical test functions that contribute to a description of measurement and/or state preparation of the quantized electromagnetic field. The electromagnetic field in an interacting theory of the sort introduced here is not measurable at a point, so we always have to consider $\hat{\phi}_f$.

Supposing there is an observable 4-current field, and that $J_{1\mu}$ and $J_{2\nu}$ are test functions for it, we can introduce a massive free field inner product

$$\langle J_1, J_2 \rangle_V = \hbar \int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k_\alpha k^\alpha - m^2) \theta(k_0) \left( \sigma_T k_\mu k_\nu - \sigma_S m^2 \delta_\mu^\nu \right) \tilde{J}_{1\mu}^*(k) \tilde{J}_2^\nu(k),$$

(22)

where $\sigma_T \geq \sigma_S \geq 0$ determine the relative significance of time-like and space-like components (relative to $k_\mu$) of the 4-current. Note that any test function component for which $(f, f)$ is zero is in effect infinitely suppressed in the free theory $\dagger$, so $\sigma_S = \sigma_T$ makes only components orthogonal to $k_\mu$ significant and $\sigma_S = 0$ makes only the component parallel to $k_\mu$ significant. In terms of these free field inner products, we can introduce an interacting nonlinear inner product,

$$\langle (J_1, f_1), (J_2, f_2) \rangle_I = (f_1, f_2)_{EM} + \langle J_1, J_2 \rangle_V + \lambda_1 (J_1^\alpha + \kappa_1 J_{1\mu} f_1^{\mu\alpha}, J_2^\beta + \kappa_1 J_{2\nu} f_2^{\nu\beta})_V + \lambda_2 (J_1 f_1^{\mu\alpha}, J_2 f_2^{\nu\beta})_V + \lambda_3 (\epsilon^{\rho\sigma\alpha} J_{1\mu} f_1^{\rho\sigma}, \epsilon^{\tau\nu\beta} J_{2\nu} f_2^{\tau\nu})_V$$

(23)

with $\lambda_1$, $\lambda_2$, and $\lambda_3$ all $\geq 0$, and of course higher order terms are possible. Degrees of freedom that make no contribution to a noninteracting inner product may make a contribution after we introduce a new term to a nonlinear inner product. Fourier components of $J_1$ that are not on mass-shell, for example, so that they make no contribution to $\langle J_1, J_2 \rangle_V$, may contribute to the on mass-shell fourier components of $J_{1\mu} f_1^{\mu\alpha}$. Introducing nonlinearity in this way, therefore, effectively adds new degrees of freedom as well.

Polynomial invariants in derivatives of both $J$ and $f$ can also be added, such as $(J_1^\alpha + \kappa_2 \partial_{\mu} f_1^{\mu\alpha}, J_2^\beta + \kappa_2 \partial_{\nu} f_2^{\nu\beta})_V$ or $(\partial_{[\alpha} J_{1\beta]} + \kappa_3 f_{1\alpha\mu}, \partial_{[\beta} J_{2\nu]} + \kappa_3 f_{2\beta\nu})_{EM}$, again with higher orders as necessary.

All the nonlinear terms introduced above can result in correlations between the current and the electromagnetic field. In the noninteracting case, the inner product between test functions $(J_1, 0)$ and $(0, f_2)$ will always be zero, so there is no correlation in the vacuum state between 4-current observables and electromagnetic field observables, but with the introduction of the nonlinear terms above there will generally be correlations between 4-current observables and electromagnetic field observables in the vacuum state. Such interactions between the 4-current and the electromagnetic field through the action of nonlinearity in this approach are not immediately comparable to the description of correlations in conventional perturbation theory through the annihilation and creation of photon and charge lines in Feynman diagrams.

$\dagger$ The variance associated with the observable $\hat{\phi}_f$ in the vacuum state is $(f, f)$; if this is zero, then the observed value of $\hat{\phi}_f$ is always zero in the vacuum state (and indeed in every state).
If there is also an observable axial 4-vector, and $S_{1\mu}$ and $S_{2\mu}$ are test functions for it, quite a few more terms become possible in a nonlinear inner product, even without introducing derivatives,

$$ ( (J_1, S_1, f_1), (J_2, S_2, f_2)_I = (f_1, f_2)_{EM} + (J_1, J_2)_V + (S_1, S_2)_V + \lambda_1(J_1^\mu + \kappa_1 J_\mu f_1^{\alpha}, J_2^\beta + \kappa_1 J_\nu f_2^{\beta})_V + \lambda_2(J_1^{\mu} f_1^{\alpha}, J_2^{\nu} f_2^{\beta})_V + \lambda_3(\epsilon^{\mu\nu\sigma\rho} J_\mu f_1^{\sigma}, \epsilon^{\tau\nu\beta\rho} J_\nu f_2^{\beta})_V + \lambda_4(S_{1\mu} f_1^{\mu}, S_{2\nu} f_2^{\nu})_V + \lambda_5(S_{1\mu} f_1^{\mu}, S_{2\nu} f_2^{\nu} + \kappa_2 \epsilon^{\mu\nu\sigma\rho} J_\mu f_1^{\sigma}, S_{2\nu} f_2^{\nu} + \kappa_2 \epsilon^{\rho\tau\nu\beta} J_\nu f_2^{\beta})_V + \lambda_6(S_{1\mu} J_{1\alpha} + \kappa_3 \epsilon_{\mu\alpha} f_1^{\rho}, S_{2\nu} J_{2\beta} + \kappa_3 \epsilon_{\nu\beta} f_2^{\tau})_V + \lambda_7(S_{1\mu} J_{1\alpha}, S_{2\nu} J_{2\beta})_{EM}$$

To these might also be added parity violating terms, and, with the introduction of a scalar inner product, terms involving $(J_{1\mu} J_{1\nu}, J_{2\nu} J_{2\mu})_s, (S_{1\mu} S_{2\nu} S_{1\nu} S_{2\mu})_s, (J_{1\mu} J_{1\nu}, J_{2\nu} S_{1\nu} S_{2\mu})_s, (f_{1\mu\alpha} f_{1\nu\beta}, f_{2\nu\beta} f_{2\nu\beta})_s$. Furthermore, every occurrence of an inner product could be modified to make each term have a unique mass (and a different contribution for the time-like and space-like components of each 4-current and axial 4-vector term).

In view of the number of parameters that are apparently possible in this approach, even in the case of electromagnetism, in contrast to the relatively tight constraints imposed by renormalizability, equation (24) presumably has to be regarded as only (potentially) phenomenologically descriptive, not as a fundamental theory, unless a theoretically natural constraint on admissible terms emerges. Note that this approach or some extension or modification of it might be empirically useful, for example if it can describe electromagnetic fields in nonlinear materials effectively, without it being at all equivalent to QED.

6. Conclusion

With all computations being entirely finite, it may be possible to use these nonlinear quantum field models more easily and with less conceptual uncertainty than using conventional perturbation theory. The universal use of Schwartz space test functions to describe measurement and state preparation ensures that there are none of the infinities that usually emerge in perturbative quantum field theory. Correlation functions for measurements in a given state are straightforwardly computed in terms of the nonlinear inner products between all the functions used to generate a state and to describe measurements.

The mathematics allows a reasonable understanding of the nonlinearity that has been introduced, and there seems to be no a priori reason to exclude this kind of nonlinearity, in which the linearity of the algebra is preserved. Indeed, on the classical precedent, nonlinearity ought to be expected. The apparent introduction of nonlinearity by renormalization through the implicit nonlinear use of test functions gives a stronger impetus to consider how empirically effective the nonlinear models introduced here — and perhaps more general models — can be.

The infinite range of possibilities is at present a little uncontrolled, and the mathematical analysis of the empirical consequences of particular terms in models of
the theory appear to be quite nontrivial — to my knowledge it is a novel mathematical
problem. Quantum theory has largely moved to supersymmetry and string theory
because of the apparent impossibility of putting interacting quantum field theory on a
sound mathematical footing, but the form of interacting quantum field theories presented
here is a mathematically reasonable alternative.

It will be interesting to see what range of physical situations can be modelled with
these nonlinear quantum fields. Free fields are already useful as a first approximation
in quantum optics, so it’s possible that the methods of this paper might make a useful
second approximation as a way to construct phenomenological models for nonlinear
materials. These nonlinear quantum fields, however, are conceptually significantly
different from the interacting quantum fields of conventional perturbation theory, and
are manifestly different from conventional constructive and axiomatic quantum fields.

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