A high-order approach to elliptic multiscale problems

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ABSTRACT

Computational multiscale methods are popular tools to deal with microscopic features of partial differential equations (PDEs) that are typically encoded in an underlying material coefficient. It is well-known that standard finite element methods only achieve acceptable results if varying micro-features are resolved by the corresponding finite element mesh. Multiscale methods aim to overcome this problem and achieve good approximation properties already for coarse-level simulations at the cost of a moderate computational overhead. To obtain convergence rates beyond first order, however, suitable smoothness assumptions on the domain, the diffusion coefficient, and the exact solution need to hold in general.

Here, we analyze a high-order multiscale method for the elliptic model problem

$$\begin{align*}
-\text{div}(A \nabla u) &= f \quad \text{in } D, \\
u &= 0 \quad \text{on } \partial D
\end{align*}$$

on a Lipschitz domain $D \subseteq \mathbb{R}^d$ and with possibly highly varying diffusion coefficient $A$. The method achieves high-order convergence rates already for very general unstructured $L^\infty$-coefficients only from additional (piecewise) smoothness assumptions on the force term $f$. The construction is motivated by the localized orthogonal decomposition (LOD) method [MP14] and based on an appropriate two-scale gamblet construction [Owh17]. It computes an (ideal) coarse-scale approximation $u_{ms}$ of the exact solution $u$ for which we can prove an error estimate of the form

$$\|u - u_{ms}\|_{H^1(D)} \leq C(s) \left( \frac{H}{p} \right)^{s+1} |f|_{H^s(T_H)}, \quad s \leq p + 1$$

for a chosen polynomial degree $p$, where $T_H$ is an underlying mesh with coarse mesh size $H$ that not necessarily resolves oscillations in $A$. We also show that a similar result can be retained for a localized variant of the method. For more details, see [Mai20, Mai21].

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