Derivations of continuous and discrete energy equations in wave and shallow-water equations

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Abstract
Symmetry-preserving (mimetic) discretization aims to preserve certain properties of a continuous differential operator in its discrete counterpart. For these discretizations, stability and (discrete) conservation of mass, momentum and energy are proven in the same way as for the original continuous model.

In our papers [1] and [2], we presented space discretization schemes for various models, which had exact conservation of mass, momentum and energy. Mass and momentum conservation followed from the left null spaces of the discrete operators used. The conservation of energy in the continuous and discrete models is more complicated, and the papers had little space for their complete derivation. This paper contains the derivation of the energy equations in more detail than was given in the papers [1] and [2].

Symmetry-preserving discretizations, Mimetic methods, Finite-difference methods, Mass, momentum and energy conservation, Curvilinear staggered grid

1 Introduction and motivation
All of the models presented in [1] and [2], except the scalar wave equation, consist of continuity, momentum and state equations. The energy equation is derived by combining these three equations. All the continuous energy equations express the change in the energy density $\varepsilon$ in terms of energy fluxes $\vec{f}_\varepsilon$, and therefore have the general form

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \vec{f}_\varepsilon = 0.$$  

(1)

The time derivative of the total energy $E$, which is the integral of the energy density over a domain $V$, has only boundary terms:

$$\frac{\partial E}{\partial t} + \oint_{\delta V} \vec{f}_\varepsilon \cdot dS = 0,$$

where $\delta V$ is the boundary of the domain $V$. In cases without boundary effects, such as periodic domains and domains with energy-conserving boundary conditions, the total energy remains constant.

In the discrete models, the energy is located on all the points in the staggered grid, and a local energy balance like (1) cannot be given. Instead, the conservation of total discrete energy $E$ is shown by deriving

$$\frac{\partial E}{\partial t} = 0,$$

(2)

using certain properties of the discrete operators used in the discretizations, like the discrete Laplacian LAPL, the discrete divergence DIV, the discrete gradient GRAD and others.

The subsequent four sections each present one of the models. In each case a continuous and a discrete model is presented, and the energy equations are derived.
2 Energy equation in scalar wave equations

The scalar wave equation describes the change in the pressure $p$ or its discrete approximation $p$, and is given by

\[
\begin{align*}
\text{Continuous} & \quad \frac{\partial^2 p}{\partial t^2} = \nabla^2 p, \\
\text{Discrete} & \quad \frac{d^2 p}{dt^2} = \text{LAPL}_p,
\end{align*}
\]

where \(\text{LAPL}\) is the discrete approximation of the Laplacian operator \(\nabla^2\), which is symmetrical:

\[
\text{LAPL}_p^* = \text{LAPL}_p.
\]

In the continuous model, the total energy \(E\) is the integral of the energy density \(e\). In the discrete model, the total energy \(E\) is presented using scalar products:

\[
\begin{align*}
\text{Continuous} & \quad e := \frac{1}{2} \left( \frac{\partial p}{\partial t} \right)^2 + \frac{1}{2} |\nabla p|^2, \\
\text{Discrete} & \quad E := \frac{1}{2} \left\langle \frac{dp}{dt}, \frac{dp}{dt} \right\rangle_c - \frac{1}{2} \left\langle p, \text{LAPL}_p \right\rangle_c. \tag{5}
\end{align*}
\]

The time derivative of the energy is

\[
\begin{align*}
\frac{\partial e}{\partial t} & = \frac{\partial p}{\partial t} \frac{\partial^2 p}{\partial t^2} + \nabla p \cdot \frac{\partial}{\partial t} \nabla p \\
& = \nabla \cdot \left( \frac{\partial p}{\partial t} \nabla p \right),
\end{align*}
\]

\[
\begin{align*}
\frac{dE}{dt} & = \left\langle \frac{dp}{dt}, \frac{d^2 p}{dt^2} \right\rangle_c \\
& = \left\langle \frac{dp}{dt}, \text{LAPL}_p \right\rangle_c - \frac{1}{2} \left\langle p, \text{LAPL}_p \frac{dp}{dt} \right\rangle_c \\
& = \left\langle \frac{dp}{dt}, \text{LAPL}_p \right\rangle_c - \frac{1}{2} \left\langle \frac{dp}{dt}, \left(\text{LAPL}_p + \text{LAPL}_p^*\right) p \right\rangle_c. \tag{7}
\end{align*}
\]

Using the symmetry property that \(\text{LAPL}_p^* = \text{LAPL}_p\), the following energy equation is found:

\[
\begin{align*}
\text{Continuous} & \quad \frac{\partial e}{\partial t} + \nabla \cdot \left( -\frac{\partial p}{\partial t} \nabla p \right) = 0, \\
\text{Discrete} & \quad \frac{\partial E}{\partial t} = 0. \tag{9}
\end{align*}
\]

3 Energy equation in linear-wave equations

The linear-wave equations describe the change in the flow velocity $\vec{v}$, the density $\rho$ and the pressure $p$, and their discrete approximations $v$, $\rho_0$ and $p$. The equations are given in the form of the continuity, momentum and state equations

\[
\begin{align*}
\text{Continuous} & \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho_0 \vec{v} = 0, \\
\frac{\partial \rho_0 \vec{v}}{\partial t} + \nabla p = 0, \\
\rho & = c^2 p, \tag{11}
\end{align*}
\]

\[
\begin{align*}
\text{Discrete} & \quad \frac{\partial \rho}{\partial t} + \rho_0 \text{DIV} v = 0, \\
\frac{\rho_0}{\partial t} + \text{GRAD} p = 0, \\
\rho_0 & = c^2 p. \tag{12}
\end{align*}
\]

where $c$ is the wave propagation speed, $\rho_0$ is a constant reference density, \(\text{DIV}\) is the discrete approximation of the divergence \(\nabla\), and \(\text{GRAD}\) of the gradient $\nabla$. The discrete divergence and gradient are each other’s negative adjoint:

\[
\text{GRAD}_p^* = -\text{DIV}_p.
\]
Time derivative of kinetic energy

The local kinetic energy $e_{\text{kin}}$ and the total kinetic energy $E_{\text{kin}}$ are given by

**Continuous**

$$e_{\text{kin}} := \frac{\rho_0}{2} |\vec{v}|^2. \quad (13)$$

**Discrete**

$$E_{\text{kin}} := \frac{\rho_0}{2} \langle \vec{v}, \vec{v} \rangle_v. \quad (14)$$

The time derivative of the kinetic energy is given by

**Continuous**

$$\frac{\partial e_{\text{kin}}}{\partial t} = \rho_0 \vec{v} \cdot \frac{\partial \vec{v}}{\partial t}. \quad (15)$$

**Discrete**

$$\frac{\partial E_{\text{kin}}}{\partial t} = \rho_0 \left\langle \vec{v}, \frac{d\vec{v}}{dt} \right\rangle_v. \quad (16)$$

The time derivatives in the right-hand sides of (15-16) are eliminated using the momentum equation and the following expression is found

**Continuous**

$$\frac{\partial e_{\text{kin}}}{\partial t} = -\vec{v} \cdot \nabla p. \quad (17)$$

**Discrete**

$$\frac{\partial E_{\text{kin}}}{\partial t} = -\langle \vec{v}, \text{GRAD} p \rangle_v. \quad (18)$$

Internal energy

The local internal energy $e_{\text{int}}$ and total internal energy $E_{\text{int}}$ are defined by

**Continuous**

$$e_{\text{int}} := \frac{c^2}{2\rho_0} \rho^2, \quad (19)$$

**Discrete**

$$E_{\text{int}} := \frac{c^2}{2\rho_0} \langle \rho, \rho \rangle_c. \quad (20)$$

Their time derivatives are given by

**Continuous**

$$\frac{\partial}{\partial t} e_{\text{int}} = \frac{c^2}{\rho_0} \frac{\partial \rho}{\partial t}. \quad (21)$$

**Discrete**

$$\frac{d}{dt} E_{\text{int}} := \frac{c^2}{\rho_0} \langle \rho, \frac{d\rho}{dt} \rangle_c. \quad (22)$$

The time derivatives in the right-hand sides are eliminated using the continuity equation:

**Continuous**

$$\frac{\partial}{\partial t} e_{\text{int}} = -c^2 \rho \nabla \cdot \vec{v} = -p \nabla \cdot \vec{v}, \quad (23)$$

**Discrete**

$$\frac{d}{dt} E_{\text{int}} := -\langle p, \text{DIV} \vec{v} \rangle_c. \quad (24)$$

Energy equation

The local energy $e = e_{\text{kin}} + e_{\text{int}}$ is the sum of local kinetic and internal energies, and the total energy $E = E_{\text{kin}} + E_{\text{int}}$ is the sum of the total kinetic and internal energies, so their time derivatives are

**Continuous**

$$\frac{\partial e}{\partial t} = -\vec{v} \cdot \nabla p - p \nabla \cdot \vec{v} = -\nabla (p\vec{v}). \quad (25)$$

**Discrete**

$$\frac{dE}{dt} = -\langle \vec{v}, \text{GRAD} p \rangle_v - \langle p, \text{DIV} \vec{v} \rangle_c$$

$$= -\langle p, (\text{GRAD}^* + \text{DIV}) \vec{v} \rangle_c. \quad (26)$$

Using the symmetry property $\text{GRAD}^* = -\text{DIV}$, the following energy equation is found:

**Continuous**

$$\frac{\partial e}{\partial t} + \nabla \cdot p\vec{v} = 0. \quad (27)$$

**Discrete**

$$\frac{dE}{dt} = 0. \quad (28)$$
4 Energy equation in compressible-wave equations

Compressible-wave equations

The compressible-wave equations are given by the continuity, momentum and state equations

\[
\begin{align*}
\text{Continuous} & \\
\frac{\partial \rho}{\partial t} & + \nabla \cdot (\rho \vec{v}) = 0, \\
\frac{\partial \vec{v}}{\partial t} & + \nabla Q(p) = 0, \\
\rho & = R(p). \quad (29)
\end{align*}
\]

\[
\begin{align*}
\text{Discrete} & \\
\frac{\partial \rho}{\partial t} & + \text{DIV} \rho \vec{v} = 0, \\
\frac{\partial \vec{v}}{\partial t} & + \text{GRAD} Q(p) = 0, \\
\rho & = R(p). \quad (30)
\end{align*}
\]

where the function \(Q\) is given in terms of the density function \(R\) as

\[
Q(p) := \int p \frac{1}{R(p)} dq,
\]

so the momentum equation may also be written as

\[
\frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho} \nabla p = 0. \quad (31)
\]

Another form of the momentum equation uses the function \(S(p) := \int p \frac{1}{R(p)} dq\), and reads

\[
\begin{align*}
\text{Continuous} & \\
\frac{\partial \vec{v}}{\partial t} + \rho \nabla S(p) & = 0, \\
\frac{\partial \vec{v}}{\partial t} & + \vec{r} \text{GRAD} S(p) = 0. \\
\end{align*}
\]

\[
\text{Discrete} \quad \frac{\partial \vec{v}}{\partial t} + \vec{r} \text{GRAD} S(p) = 0. \quad (33)
\]

The operator \(\vec{r} \text{GRAD}\), which is the discrete approximation of the operator \(\rho \nabla\), is related to the discrete gradient \(\text{GRAD}\) in the discrete chain rule

\[
\vec{r} \text{GRAD} S(p) = \text{GRAD} Q(p),
\]

and the operator \(\text{DIV} \vec{r}\), the discrete approximation of the operator \(\nabla \cdot \rho\), is given by

\[
\text{DIV} \vec{r} = -\vec{r} \text{GRAD}^\ast.
\]

Time derivative of kinetic energy

The local kinetic energy \(e_{kin}\) and the total kinetic energy \(E_{kin}\) are given by

\[
\begin{align*}
\text{Continuous} & \\
e_{kin} & := \frac{\rho_0}{2} |\vec{v}|^2, \\
E_{kin} & := \frac{\rho_0}{2} \langle \vec{v}, \vec{v} \rangle_v. \quad (34)
\end{align*}
\]

The time derivative of the kinetic energy is given by

\[
\begin{align*}
\text{Continuous} & \\
\frac{\partial e_{kin}}{\partial t} & = \rho_0 \vec{v} \cdot \frac{\partial \vec{v}}{\partial t}, \\
\frac{\partial E_{kin}}{\partial t} & = \rho_0 \langle \vec{v}, \frac{\partial \vec{v}}{\partial t} \rangle_v. \quad (36)
\end{align*}
\]

\[
\text{Discrete} \quad \frac{\partial E_{kin}}{\partial t} = \rho_0 \langle \vec{v}, \text{GRAD} Q(p) \rangle_v. \quad (37)
\]

Time derivative of kinetic energy converted to spatial derivatives

The time derivatives in the right-hand sides of (36-37) are replaced by the expression given in the momentum equation and the following expression is found

\[
\begin{align*}
\text{Continuous} & \\
\frac{\partial e_{kin}}{\partial t} & = -\rho_0 \vec{v} \cdot \nabla Q(p), \\
\frac{\partial E_{kin}}{\partial t} & = -\rho_0 \langle \vec{v}, \text{GRAD} Q(p) \rangle_v. \quad (38)
\end{align*}
\]

\[
\text{Discrete} \quad \frac{\partial E_{kin}}{\partial t} = -\rho_0 \langle \vec{v}, \text{GRAD} Q(p) \rangle_v. \quad (39)
\]

Internal energy

The local internal energy \(e_{int}\) is given by

\[
e_{int} := \rho_0 \int p \frac{R(p) - R(q)}{R^2(q)} dq = \rho_0 R(p) \int p \frac{1}{R^2(q)} dq - \rho_0 \int p \frac{1}{R(q)} dq. \quad (40)
\]
Its derivative with respect to the pressure is given by

\[
e_{\text{int}}'(p) = \rho_0 R'(p) \int_0^p \frac{1}{R^2(q)} dq + \rho_0 R(p) \frac{1}{R^2(p)} - \rho_0 \frac{1}{R(p)} = \rho_0 R'(p) \int_0^p \frac{1}{R^2(q)} dq = \rho_0 R'(p) S(p).
\] (41)

**Time derivative of the internal energy**

The chain rule is applied to find the following expression for the time derivative of the internal energy:

\[
\frac{\partial e_{\text{int}}}{\partial t} = e_{\text{int}}'(p) \frac{\partial p}{\partial t} = \rho_0 R'(p) S(p) \frac{\partial p}{\partial t} = \rho_0 S(p) \frac{\partial \rho}{\partial t}.
\] (42)

Using the continuity equation, the time derivative is eliminated

**Continuous**

\[
\frac{\partial e_{\text{int}}}{\partial t} = -\rho_0 S(p) \nabla \cdot \rho \nu
\] (43)

**Discrete**

\[
\frac{\partial e_{\text{int}}}{\partial t} = -\rho_0 \text{diag}(S(p)) \text{DIV} \nu.
\]

**Energy equation**

The time derivatives of local and total energies \(e\) and \(E\) are

**Continuous**

\[
\frac{\partial e}{\partial t} = -\rho \nu \cdot \nabla Q(p) - \rho_0 S(p) \nabla \cdot \rho \nu
\] (45)

**Discrete**

\[
\frac{\partial E}{\partial t} = -\rho_0 \langle \nu, \text{GRAD} Q(p) \rangle_v - \rho_0 \langle S(p), \text{DIV} \nu \rangle_v
\]

(46)

Now we use the symmetry property that \(\text{DIV} \nu = -\nu \nabla\text{GRAD} S(p)\) and the chain rules \(\nabla Q = \rho \nabla S, \text{GRAD} Q(p) = \nu \text{GRAD} S(p)\), to find

**Continuous**

\[
\frac{\partial e}{\partial t} = -\rho_0 \nu \cdot \nabla S(p) - \rho_0 S(p) \nabla \cdot \rho \nu = -\rho_0 \nabla \cdot (\rho \nu S(p))
\] (47)

**Discrete**

\[
\frac{\partial E}{\partial t} = -\rho_0 \langle \nu, \text{GRAD} Q(p) \rangle_v + \rho_0 \langle \nu \text{GRAD} S(p), \nu \rangle_v
\]

(48)

The energy equation is therefore

**Continuous**

\[
\frac{\partial e}{\partial t} + \rho_0 \nabla \cdot (\rho \nu S(p)) = 0.
\] (49)

**Discrete**

\[
\frac{\partial E}{\partial t} = 0.
\] (50)
5 Energy equation in isentropic compressible Euler equations

Isentropic compressible Euler equations

The isentropic compressible Euler equations are given by the continuity, momentum and state equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \]
\[ \frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) + \nabla p = 0, \]
\[ \rho = R(p). \]  

(51)

where ADVEC is the discrete approximation of the advection operator \( \nabla \cdot \rho \vec{v} \otimes \vec{v} \), \( \nabla r \) of the operator \( \nabla \cdot \rho \), \( r_{GRAD} \) of \( \rho \nabla \), and where the discrete local momentum \( r \vec{v} \) is given by

\[ r \vec{v} := \text{diag}(v) \text{Interp}_{v+c} \rho, \]  

(53)

where \( \text{Interp}_{v+c} \) is an interpolation which uses the densities at the cell-centers of the staggered grid to calculate densities at the cell-faces.

The operators \( \nabla r \) and \( r_{GRAD} \) are each other’s negative adjoints:

\[ r_{GRAD}^* = -\nabla r, \]

and the operator \( r_{GRAD} \) is related to the discrete gradient \( \nabla r \) in the discrete chain rule

\[ \nabla r p = r_{GRAD} Q(p). \]

The advection operator has the following symmetry property:

\[ \text{ADVEC} + \text{ADVEC}^* = \text{diag}(\text{Interp}_{v+c} \nabla r). \]

Time derivative of kinetic energy

The local continuous kinetic energy \( e_{kin} \) and the total discrete kinetic energy \( E_{kin} \) are given by

\[ \begin{align*}
\text{Continuous} & : e_{kin} := \frac{\rho}{2} |\vec{v}|^2, \\
\text{Discrete} & : E_{kin} := \frac{1}{2} \langle \vec{v}, \vec{v} \rangle. 
\end{align*} \]  

(54)

(55)

Using the product rule for differentiation, the time derivative of the kinetic energy is given by

\[ \begin{align*}
\text{Continuous} & : \frac{\partial e_{kin}}{\partial t} = \vec{v} \cdot \frac{\partial (\rho \vec{v})}{\partial t} - \frac{1}{2} \frac{\partial \rho}{\partial t} |\vec{v}|^2, \\
\text{Discrete} & : \frac{\partial E_{kin}}{\partial t} = \langle \vec{v}, \frac{\partial r \vec{v}}{\partial t} \rangle - \frac{1}{2} \langle \text{diag}(\vec{v}) \vec{v}, \text{Interp}_{v+c} \frac{d}{dt} \rho \rangle. 
\end{align*} \]  

(56)

(57)

Time derivative of kinetic energy converted to spatial derivatives

The time derivatives in the right-hand sides of (56-57) are eliminated using the continuity and momentum equations and the following expression is found

\[ \begin{align*}
\text{Continuous} & : \frac{\partial e_{kin}}{\partial t} = -\vec{v} \cdot \nabla \cdot \rho \vec{v} \otimes \vec{v} - \vec{v} \cdot \nabla p + \frac{|\vec{v}|^2}{2} \nabla \cdot \rho \vec{v}, \\
\text{Discrete} & : \frac{\partial E_{kin}}{\partial t} = -\langle \vec{v}, \text{ADVEC} \vec{v} \rangle - \langle \vec{v}, \text{GRAD} p \rangle + \frac{1}{2} \langle \text{diag}(\vec{v}) \vec{v}, \text{Interp}_{v+c} \nabla r \vec{v} \rangle. 
\end{align*} \]  

(58)

(59)
To derive the local energy balance, we need the product rule for advection, given by

$$\nabla \cdot \rho |\vec{v}|^2 / 2 = \vec{v} \cdot \nabla \cdot \rho \vec{v} = \vec{v} \cdot \nabla \cdot (\rho \vec{v}) = \frac{1}{2} (\vec{v} \cdot \nabla |\vec{v}|^2 - |\vec{v}|^2 \nabla \cdot \rho \vec{v}).$$  \hspace{1cm} (60)

Using this rule and the chain rules that $\nabla p = \rho \nabla Q$ and $\text{GRAD} \ p = r \text{GRAD} \ Q(p)$, it is found that

$$\frac{\partial e_{\text{kin}}}{\partial t} = -\nabla \cdot \rho \frac{1}{2} |\vec{v}|^2 \vec{v} - \rho \vec{v} \cdot \nabla Q(p) + \frac{1}{2} \nabla \cdot \rho \vec{v},$$

$$\frac{\partial E_{\text{kin}}}{\partial t} = -\langle v, (\text{ADVEC} + \text{ADVEC}^*) v \rangle_v + \frac{1}{2} \langle \text{diag}(v) v, \text{Interp} \vec{v} \cdot \text{DIV} \vec{v} \rangle_v.$$  \hspace{1cm} (61)

The second and last terms in the continuous equation cancel each other. Also, the first and last terms in the discrete equation cancel, because of the symmetry property

$$\text{ADVEC} + \text{ADVEC}^* = \text{diag}(|\text{Interp} \vec{v} \cdot \text{DIV} \vec{v}|).$$

This leads to the shorter equations

$$\frac{\partial e_{\text{kin}}}{\partial t} = -\nabla \cdot \rho \frac{1}{2} |\vec{v}|^2 \vec{v} - \rho \vec{v} \cdot \nabla Q(p),$$

$$\frac{\partial E_{\text{kin}}}{\partial t} = -\langle v, \text{rGRAD} \ Q(p) \rangle_v.$$  \hspace{1cm} (62)

**Time derivative of internal energy**

The local internal energy $e_{\text{int}}$ is given by

$$e_{\text{int}} = \int R(p) - R(q) dq = R(p) \int \frac{1}{R(q)} dq - p.$$  \hspace{1cm} (65)

Its derivative with respect to the pressure is given by

$$e'_{\text{int}}(p) = R'(p) \int \frac{1}{R(q)} dq + R(p) \int \frac{1}{R(p)} dq - 1 = R'(p) \int dq = R'(p) Q(p).$$  \hspace{1cm} (66)

The time derivative of the internal energy follows from the chain rule:

$$\frac{\partial e_{\text{int}}}{\partial t} = e'_{\text{int}}(p) \frac{\partial p}{\partial t} = R'(p) Q(p) \frac{\partial p}{\partial t} = Q(p) \frac{\partial p}{\partial t}.$$  \hspace{1cm} (67)

**Time derivative of internal energy converted to spatial coordinates**

Eliminating the time derivative using the continuity equation and using the symmetry property $\text{DIV} r^* = -r \text{GRAD}$, we find

$$\frac{\partial e_{\text{int}}}{\partial t} = -Q(p) \nabla \cdot \rho \vec{v},$$

$$\frac{\partial E_{\text{int}}}{\partial t} = -\langle v, \text{rGRAD} \ S(p) \rangle_v.$$  \hspace{1cm} (68)

**Energy equation**

The time derivatives of local and total energies $e$ and $E$ are

$$\frac{\partial e}{\partial t} = -\nabla \cdot \rho \frac{1}{2} |\vec{v}|^2 \vec{v} - \rho \vec{v} \cdot \nabla Q(p) - Q(p) \nabla \cdot \rho \vec{v},$$

$$\frac{\partial E}{\partial t} = -\langle v, \text{rGRAD} \ Q(p) \rangle_v + \langle r \text{GRAD} \ S(p), v \rangle_v.$$  \hspace{1cm} (70)

$$\frac{\partial E_{\text{int}}}{\partial t} = -\langle v, \text{rGRAD} \ Q(p) \rangle_v + \langle r \text{GRAD} \ S(p), v \rangle_v = 0.$$  \hspace{1cm} (71)
so the energy equation is

\[ \frac{\partial e}{\partial t} + \nabla \cdot (\frac{1}{2} |\vec{v}|^2 + \rho Q(p)) \rho \vec{v} = 0. \quad (72) \]

\[ \frac{\partial E}{\partial t} = 0. \quad (73) \]

References

[1] B. van ’t Hof and M. Vuik. Symmetry-preserving discretizations of arbitrary order on structured curvilinear grids, 2017. arXiv:1710.07149 [math.NA].

[2] B. van ’t Hof and M. Vuik. Symmetry-preserving finite-difference discretizations of arbitrary order on structured curvilinear staggered grids, 2019. arXiv:1901.02264 [math.NA].