A General Framework of Online Updating Variable Selection for Generalized Linear Models with Streaming Datasets

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Abstract

In the research field of big data, one of important issues is how to recover the sequentially changing sets of true features when the data sets arrive sequentially. The paper presents a general framework for online updating variable selection and parameter estimation in generalized linear models with streaming datasets. This is a type of online updating penalty likelihoods with differentiable or non-differentiable penalty function. The online updating coordinate descent algorithm is proposed to solve the online updating optimization problem. Moreover, a tuning parameter selection is suggested in an online updating way. The selection and estimation consistencies, and the oracle property are established, theoretically. Our methods are further examined and illustrated by various numerical examples from both simulation experiments and a real data analysis.

Keywords: Streaming data environment; Penalty likelihoods; Online updating BIC criterion; Oracle property.

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1 Introduction

A data set that arrives in streams and chunks is termed a streaming data set. In the era of big data, the streaming data sets from various areas such as bioinformatics, medical imaging and computer vision are rapidly increasing in volume and velocity. This brings us challenges to learn efficient statistical models and inferences. How to make statistical inferences without storage requirement for previous raw data is the key in the streaming data environment.

Assume we have the data set \( \{(x_{ti}, y_{ti}), i = 1, \ldots, n_t; t = 1, \ldots, b\} \) up to the \( b \)-th batch, where \( n_t \) is the the sample size at the \( t \)-batch and the total sample size is \( N_b = \sum_{t=1}^{b} n_t \). Denote by \( D_t = \{y_t, X_t\} \) the \( t \)-th batch data set, where \( y_t = (y_{t1}, \ldots, y_{tm})^T, X_t = (x_{t1}, \ldots, x_{tm})^T \). We suppose that \((y_{ti}, x_{ti})\) for all \( t \) and \( i \) are independent and identically distributed observations of \((y, X)\) with response \( y \in \mathbb{R} \) and covariate \( X \in \mathbb{R}^p \). To our knowledge, amounts of methodologies have been developed for related parameter estimation and statistical inference for the case of low-dimensional covariate \( X \), especially for the case when the dimension \( p \) is less than every batch sample sizes \( n_t, t \in \{1, \ldots, b\} \). Robbins & Monro (1951) proposed the well-known stochastic gradient descent (SGD) algorithm that has been extensively used in the filed of machine learning. It is obvious that the SGD algorithm is applied to the streaming data because of using a part of gradients in every steps. However, the SGD method is not robust to learning rate, and it may fail to converge if the learning rate is too large. As improvements, the implicit SGD (ISGD) algorithm and averaged implicit SGD (AISGD) algorithm were developed by Toulis et al. (2014), which could be used in the streaming data to estimate model parameters. Another type of recursive algorithm, the cumulative updating method has been well developed, including the online least squares estimator (OLSE) for linear model by Stengel (1994), the cumulative
estimating equation (CEE) estimator and the cumulatively updating estimating equation (CUEE) estimator of Schifano et al. (2016) for non-linear models. Lin & Zhang (2002) proposed an updating weighted sum for two samples. Recently, Luo & Song (2020) proposed a renewable estimation and inference for generalized linear model, and Lin et al. (2020) introduced a general framework of renewable weighted sums for various online updating estimation.

Although the methods aforementioned attain favourable estimation and achieve satisfactory theoretical conclusions, they need the condition that the dimension $p$ of covariate is smaller than every batch sample sizes $n_t$, $t \in \{1, \ldots, b\}$. For the case of high-dimensional covariate, an important issue is how to select important variables in an online updating framework. Several algorithms were developed for the variable selection in the models with streaming data sets. Xiao (2010) developed regularized dual averaging (RDA) method to solve the general online regularized problem, which can select active variables via appropriate penalty functions. Langford et al. (2009) proposed a variant of the truncated SGD. Fan et al. (2018) applied the truncated SGD to linear model and gave the statistical analysis of the truncated SGD. Sun et al. (2020) introduced a novel framework for variable selection in linear model, which combines the updated statistics and truncation techniques. These SGD algorithms and truncation techniques, however, are sensitive to learning rate or step size, and tend to select the set with larger cardinality for including all important variables. Moreover, the related works are applied mainly to linear models, without the well-developed methodology for general models.

In this paper, we suggest a renewable optimized objective function for parameter estimation and variable selection in generalized linear models (GLMs) with high-dimensional covariates. The new method is an online updating strategy for general penalty likelihoods,
including smooth penalty likelihoods and non-smooth penalty likelihoods. This optimized objective function with various penalty functions could select important variables efficiently. In order to realize numerical solution, we introduce efficient algorithms for this optimization problem. Our method enjoys a fast convergence rate and can efficiently recover the support of the true signals. Moreover, the proposed method can choose tuning parameters via a data-driven and online updating BIC criterion. Theoretically, the method achieves the estimation and selection consistencies and oracle property. It can be seen from the theoretical conclusions that the method is free of the constraint on the number of batches, which means that the new method is adaptive to the situation where streaming data sets arrive fast and perpetually.

The remainder of this paper is organized as follows. Section 2 presents our renewable optimized objective function and concludes the key large sample estimation and selection properties. Section 3 proposes incremental updating algorithm to compute renewable selection and estimation results. Section 4 reports main simulation studies and the real data analysis. Concluding remarks are provided in Section 5. Proofs of the theorems and all technical details are relegated to Supplementary Material.

2 Methodologies and Main Results

2.1 Setup

Suppose we have the streaming data sets \( \{(x_{ti}, y_{ti}), i = 1, \ldots, n_t; t = 1, \ldots, b\} \) satisfying the GLM: \( E(y_{ti}|x_{ti}) = g(x_{ti}^T\beta) \), where \( g(\cdot) \) is a known link function, \( \beta \in \mathbb{R}^p \) is the unknown parameter vector, and all the data \((x_{ti}, y_{ti})\) are independent and identically distributed from
the distribution

$$f(y; X, \beta_0, \phi_0) = \exp \left\{ \frac{yX^T \beta_0 - b(X^T \beta_0)}{a(\phi_0)} + c(y, \phi_0) \right\},$$

(2.1)

where $\beta_0 \in \mathbb{R}^p$ is the true value of the parameter of interest and $\phi_0$ is the true value of a nuisance parameter. We permit the dimension of covariate $X$ is large than the sample sizes $n_t$, and there is no a pre-decided size relationship between total sample size $N_b$ and the dimension $p$ in the streaming data sets. We suppose that only a part of components of covariate $X$ affect the response. Denote by $S = \{ j \in \{ 1, \cdots, p \} | \beta_j \neq 0 \}$ the index set of active covariates and by $s = |S|$ the cardinality of the set $S$. Our goal is then to select the active index set $S$ and estimate the corresponding parameters $\beta_j, j \in S$ under the environment of streaming data.

### 2.2 Renewable Optimized Objective Function

Denote by $D_b^* = \{ D_1, \cdots, D_b \}$ the cumulative data set up to the $b$-batch. For the GLM, we write the associated log-likelihood function as:

$$\ell (\beta, \phi; D_b^*) = \sum_{i=1}^{n_t} \sum_{i=1}^{n_t} \log f (y_{ti}; x_{ti}, \beta, \phi) = \sum_{t=1}^{b} \sum_{i=1}^{n_t} \frac{y_{ti}x_{ti}^T \beta - b(x_{ti}^T \beta)}{a(\phi)} + c(y_{ti}, \phi).$$

It is equivalent to $\sum_{i=1}^{n_t} \sum_{i=1}^{n_t} \{ y_{ti}x_{ti}^T \beta - b(x_{ti}^T \beta) \}$ when we maximize $\ell (\beta, \phi; D_b^*)$ with respect to $\beta$. For convenience, slightly abusing the notation, we use $\ell_t (\beta; D_t) := \{ y_t^T X_t \beta - 1^T b(X_t \beta) \}$ to represent the simplified log-likelihood function in the $t$-th batch for any $t \in \{ 1, \cdots, b \}$. Denote the score function as $U_t (\beta; D_t) := \nabla_{\beta} \ell_t (\beta; D_t)$, and the negative Hessian matrix as $J_t (\beta; D_t) := -\nabla_{\beta} U_t (\beta; D_t)$ for any $t \in \{ 1, \cdots, b \}$, where $\nabla_{\beta}$ stands for
the derivative of a function with respect to $\beta$.

Under high dimension and sparsity situation, by commonly used techniques, we establish a penalty likelihood by adding a penalty term into the likelihood function. That is to say, we estimate the parameters by solving the following penalty likelihood:

$$
\hat{\beta}_b = \arg \min_{\beta \in \mathbb{R}^p} \left\{ \sum_{t=1}^b \left\{ -\ell_t(\beta; D_t) + P(\beta; \lambda_t) \right\} \right\},
$$

where $P(\beta; \lambda_t)$ is a known penalty function. Here the penalty function $P(\beta; \lambda_t)$ can be chosen to be differentiable or non-differentiable with respect to $\beta$. We then denote by $\partial P(\beta; \lambda_t)$ the derivative of $P(\beta; \lambda_t)$ with respect to $\beta$ if $P(\beta; \lambda_t)$ is differentiable, and use the same notation $\partial P(\beta; \lambda_t)$ to denote the subgradient if $P(\beta; \lambda_t)$ is non-differentiable.

In the case of streaming data sets, our first task is to represent the above penalty likelihood by an online updating form. Assume we have got a consistent penalty estimator $\hat{\beta}_{b-1}$ until the $(b-1)$-th batch. Then, by the Taylor expansions of $\ell_t(\beta; D_t)$ and $P(\beta; \lambda_t)$ for $t = 1, \cdots, b-1$ around the accumulative estimator $\hat{\beta}_{b-1}$, we have

$$
\hat{\beta}_b = \arg \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{b-1} \left\{ -\ell_t(\hat{\beta}_{b-1}; D_t) + (\beta - \hat{\beta}_{b-1})^T U_t(\hat{\beta}_{b-1}; D_t) 
- \frac{1}{2}(\beta - \hat{\beta}_{b-1})^T J_t(\hat{\beta}_{b-1}; D_t)(\beta - \hat{\beta}_{b-1}) + o_p(\|\beta - \hat{\beta}_{b-1}\|) \right\} 
+ \left\{ P(\hat{\beta}_{b-1}; \lambda_t) + (\beta - \hat{\beta}_{b-1})^T \partial P(\hat{\beta}_{b-1}; \lambda_t) + o_p(\|\beta - \hat{\beta}_{b-1}\|) \right\} 
+ \left\{ -\ell_b(\beta; D_b) + P(\beta; \lambda_b) \right\} 
= \arg \min_{\beta \in \mathbb{R}^p} \sum_{t=1}^{b-1} \left\{ -\ell_t(\hat{\beta}_{b-1}; D_t) + P(\hat{\beta}_{b-1}; \lambda_t) \right\}
$$
\[ + (\beta - \hat{\beta}_{b-1})^T \sum_{t=1}^{b-1} \left\{ -U_t(\hat{\beta}_{b-1}; D_t) + \partial P(\hat{\beta}_{b-1}; \lambda_t) \right\} \]

\[ + \frac{1}{2} (\beta - \hat{\beta}_{b-1})^T \sum_{t=1}^{b-1} J_t(\hat{\beta}_{b-1}; D_t) (\beta - \hat{\beta}_{b-1}) \]

\[ - \ell_b(\beta; D_b) + P(\beta; \lambda_b) + o_p(\|\beta - \hat{\beta}_{b-1}\|) \].

Under some mild regularity conditions, when \( n_{b-1} \) is large enough, the error term \( o_p(\|\beta - \hat{\beta}_{b-1}\|) \) can be ignored. Note that the accumulative estimator \( \hat{\beta}_{b-1} \) satisfies \( \sum_{t=1}^{b-1} \{ -U_t(\hat{\beta}_{b-1}; D_t) + \partial P(\hat{\beta}_{b-1}; \lambda_t) \} = 0 \). Ignoring these constant terms and error term \( o_p(\|\beta - \hat{\beta}_{b-1}\|) \), the above optimization problem can be expressed as following form:

\[ \hat{\beta}_b = \arg \min_{\beta \in \mathbb{R}^p} \frac{1}{2} (\beta - \hat{\beta}_{b-1})^T \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_{b-1}; D_t) \right\} (\beta - \hat{\beta}_{b-1}) - \ell_b(\beta; D_b) + P(\beta; \lambda_b). \]

If each sample size \( n_t \) is large enough, we have \( \hat{\beta}_{b-1} \approx \hat{\beta}_t \). Replacing \( \hat{\beta}_{b-1} \) with \( \hat{\beta}_t \) in \( J_t(\hat{\beta}_{b-1}; D_t) \) leads to the following incremental updating expression:

\[ \hat{\beta}_b = \arg \min_{\beta \in \mathbb{R}^p} \frac{1}{2} (\beta - \hat{\beta}_{b-1})^T \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right\} (\beta - \hat{\beta}_{b-1}) \]

\[ - \ell_b(\beta; D_b) + P(\beta; \lambda_b), \quad (2.2) \]

Obviously, the objective function in (2.2) has online updating structure, which only uses the summary statistics \( J_t(\hat{\beta}_t; D_t) \), \( \hat{\beta}_{b-1} \) and the raw data in the last batch. Thus, it can be used for variable selection and parameter estimation in the model with streaming data sets.
2.3 Theoretical Properties

In this subsection, we establish the estimation and selection consistencies, and the oracle properties for the proposed methods. Suppose that \( (y_{ti}, x_{ti}), i = 1, \ldots, n; t = 1, \ldots, b \) are i.i.d. samples from the exponential dispersion model (2.2). Write \( \beta_0 = (\beta_{01}, \ldots, \beta_{0p})^T \) with \( \beta_{01} \in \mathbb{R}^s \) and \( \beta_{02} \in \mathbb{R}^{p-s} \). Without loss of generality, assume that \( \beta_{02} = 0 \). Let \( I(\beta_0) \) be the true Fisher information matrix and \( I_1(\beta_{01}, 0) \) be the true Fisher information matrix knowing \( \beta_{02} = 0 \). Assume the penalty function has the additive form: \( P(\beta; \lambda_b) = \sum_{j=1}^{p} p(\beta_j; \lambda_b) \). This additivity condition is mild and most classical penalty functions have this property, see for example the penalty functions in LASSO (Tibshirani (1996)), SCAD (Fan & Li (2001)) and MCP(Zhang (2010)).

Throughout this section, we write \( p_{\lambda_{Nb}}(\cdot) \) for \( p(\cdot; \lambda_b) \) to emphasize \( \lambda_b \) depending on the sample size \( N_b \). Denote \( L_b(\beta) = \frac{1}{2}(\beta - \hat{\beta}_{b-1})^T \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right\} (\beta - \hat{\beta}_{b-1}) - \ell_b(\beta; D_b) \), \( Q_b(\beta) = -L_b(\beta) - N_b \sum_{j=1}^{p} p_{\lambda_{Nb}}(\beta_j) \), and \( \hat{\beta}_b \) is the local maximizer of \( Q_b(\beta) \).

We postulate the following regularity conditions.

**Condition 1** The Fisher information matrix \( I_{Nb}(\beta) = \frac{1}{N_b} \sum_{t=1}^{b} \sum_{i=1}^{n_t} b''(x_{ti}^T \beta)x_{ti}x_{ti}^T/a(\phi) \) is finite and positive definite for all \( \beta \) in some neighborhood of \( \beta_0 \).

**Condition 2** For all \( \beta \) in some neighborhood of \( \beta_0 \), there is \( |b^{(3)}(X^T \beta)| \leq M_0(X) \) for some functions \( M_0(X) \) satisfying \( E\{M_0(X)X_jX_kX_l\} < \infty \) for all \( j, k, l \).

**Condition 3** The log-likelihood function \( \ell(\beta, \phi, X; y) \) is twice continuously differentiable and \( I_{Nb}(\beta) \) is Lipschitz continuous in parameter space \( \Theta \).

**Remark 1** The condition 1 and condition 2 are the standard regularity conditions, see, for example, Lehmann (1983). The condition 3 is needed for renewable estimators, see Luo & Song (2020).
We then have the following estimation consistency.

**Theorem 1 (Estimation Consistency)** Under conditions 1-3, if \( \max \{ |p'_{\lambda N_b}(|\beta_j|)| : \beta_{0j} \neq 0 \} \to 0 \), then \( \hat{\beta}_b \) satisfies \( \| \hat{\beta}_b - \beta_0 \| = O_P(N_b^{-1/2} + a_{N_b}) \), where \( a_{N_b} = \max\{ p'_{\lambda N_b}(|\beta_j|) : \beta_{0j} \neq 0 \} \).

It is clear from Theorem 1 that by choosing a proper \( \lambda_{N_b} \), the online updating estimator is root-\( N_b \) consistent. The proof of Theorem 1 is given in Supplementary Material. We now show that this estimator possesses the sparsity property \( \hat{\beta}_{b2} = 0 \) and the oracle property. To this end, we need following lemma.

**Lemma 1** Under conditions 1-3, assume that

\[
\lim_{n \to \infty} \inf_{\theta \to 0^+} p'_{\lambda N_b}(\theta)/\lambda_{N_b} > 0, \tag{2.3}
\]

where \( \theta \) is the argument of the function. If \( \lambda_{N_b} \to 0 \) and \( \sqrt{N_b} \lambda_{N_b} \to \infty \) as \( N_b \to \infty \), then with probability tending to 1, for any given \( \beta_1 \) satisfying \( \| \beta_1 - \beta_{01} \| = O_P(N_b^{-1/2}) \) and any constant \( C \),

\[
Q_b \left\{ \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix} \right\} = \max_{\| \beta_2 \| \leq CN_b^{-1/2}} Q_b \left\{ \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\}.
\]

The proof of Lemma 1 is given in Supplementary Material. This lemma also implies the selection consistency.

**Theorem 2 (Sparsity and Oracle Property)** Under conditions 1-3, assume that the penalty function \( p_{\lambda N_b}(|\theta|) \) satisfies condition (2.3). If \( \lambda_{N_b} \to 0 \) and \( \sqrt{N_b} \lambda_{N_b} \to \infty \) as
\( N_b \to \infty \), then with probability tending to 1, the root-\( N_b \) consistent estimator \( \hat{\beta}_b = \begin{pmatrix} \hat{\beta}_{b1} \\ \hat{\beta}_{b2} \end{pmatrix} \) in Theorem 1 satisfies

(a) Sparsity: \( \hat{\beta}_{b2} = 0 \);

(b) Asymptotic normality:

\[
\sqrt{N_b} \left\{ \hat{\beta}_{b1} - \beta_{01} \right\} \to N \left\{ 0, I_1^{-1}(\beta_{01}) \right\}
\]

in distribution, where \( I_1(\beta_{01}) = I(\beta_{01}, 0) \), the Fisher information knowing \( \beta_{02} = 0 \).

The proof of Theorem 2 is given in Supplementary Material. From the theorem, it can be seen that the sparsity property \( \hat{\beta}_{b2} = 0 \) holds, the online updating estimators for nonzero components have the asymptotic normality and the standard convergence rate of order \( O_P(N_b^{-1/2}) \), and the asymptotic covariance is the same as that of the estimator obtained by supposing that zero components \( \beta_{02} \) were known beforehand. Moreover, these theoretical results always hold without any constraint on \( b \), the number of batches. It is an important property for the models with streaming data sets because it means that the method is adaptive to the situation where streaming data sets arrive fast and perpetually.
3 Online Updating Coordinate Descent Algorithm

3.1 Online Updating Coordinate Descent

For convenience, we denote the objective function in (2.2) as

$$L(\beta; D_b^*) = \frac{1}{2} (\beta - \hat{\beta}_{b-1})^T \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right\} (\beta - \hat{\beta}_{b-1}) - \ell_b(\beta; D_b) + P(\beta; \lambda_b).$$

The first derivative of $L(\beta; D_b^*)$ with respect to $\beta$ is

$$\frac{\partial L}{\partial \beta} = \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right\} (\beta - \hat{\beta}_{b-1}) - U_b(\beta; D_b) + \partial P(\beta; \lambda_b).$$

By taking the first-order Taylor expansion of $U_b(\beta; D_b)$ around the accumulative estimator $\hat{\beta}_{b-1}$, and ignoring higher terms, we have

$$\frac{\partial L}{\partial \beta} \approx \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right\} (\beta - \hat{\beta}_{b-1}) - \left\{ U_b(\hat{\beta}_{b-1}; D_b) - J_b(\hat{\beta}_{b-1}; D_b) (\beta - \hat{\beta}_{b-1}) \right\} + \partial P(\beta; \lambda_b)$$

$$= \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) + J_b(\hat{\beta}_{b-1}; D_b) \right\} (\beta - \hat{\beta}_{b-1}) - U_b(\hat{\beta}_{b-1}; D_b) + \partial P(\beta; \lambda_b)$$

Let the first derivative $\frac{\partial L}{\partial \beta}$ equal zero, we attain the following equation:

$$\left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) + J_b(\hat{\beta}_{b-1}; D_b) \right\} (\beta - \hat{\beta}_{b-1}) - U_b(\hat{\beta}_{b-1}; D_b) + \partial P(\beta; \lambda_b) = 0. \quad (3.1)$$

This is an online updating estimating equation.
Under the high dimensional setting, however, the Newton-Raphson algorithm is not practical because it is difficult or impossible to get the inverse of the negative Hessian matrix. Motivated by Friedman et al. (2010), we solve the equation (3.1) via coordinate descent algorithm when the penalty function has the additive form: $P(\beta; \lambda_b) = \sum_{j=1}^{p} p(\beta_j; \lambda_b)$. As shown in the previous section, the additive penalty function is commonly used in the existing literature. In the following, we take the LASSO penalty $P(\beta; \lambda_b) = \lambda_b \sum_{j=1}^{p} |\beta_j|$, $\lambda_b > 0$ as an example to introduce the algorithm. We begin with the scenario where $D_b$ arrives after $D_{b-1}$. In this case, there are the summary statistics $\sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t), J_b(\hat{\beta}_{b-1}; D_b), U_b(\hat{\beta}_{b-1}; D_b)$ and the consistent estimator $\hat{\beta}_{b-1}$. Consider a coordinate descent step from the $(b-1)$-th batch to the $b$-th batch for the equation for $\hat{\beta}_b$:

$$\left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) + J_b(\hat{\beta}_{b-1}; D_b) \right\} \left( \hat{\beta}_b - \hat{\beta}_{b-1} \right) - U_b \left( \hat{\beta}_{b-1}; D_b \right) + \lambda_b \partial \left\{ \sum_{j=1}^{p} |\hat{\beta}_{b,j}| \right\} = 0. \quad (3.2)$$

Let estimators $\hat{\beta}_{b,l} = \hat{\beta}_{b-1,l}$ for $l \neq j$. Our goal is to solve the solution to $\hat{\beta}_{b,j}$. If $\hat{\beta}_{b,j} > 0$, then

$$\left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) + J_b(\hat{\beta}_{b-1}; D_b) \right\} \left[ j,j \right] \left( \hat{\beta}_{b,j} - \hat{\beta}_{b-1,j} \right) - U_b \left[ j \right] \left( \hat{\beta}_{b-1}; D_b \right) + \lambda_b \hat{\beta}_{b,j} = 0, \quad (3.3)$$

where the subscripts $[j,j]$ and $[j]$ represent the $j$-th row and $j$-th column element of the matrix and the $j$-th element of the vector, respectively. We have the similar expression when $\hat{\beta}_{b,j} < 0$, and $\hat{\beta}_{b,j} = 0$ is treated separately. Denote $Z_{b,j} = U_b \left[ j \right] \left( \hat{\beta}_{b-1}; D_b \right) + \hat{\beta}_{b-1,j} W_{b,j}$, and $W_{b,j} = \left\{ \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) + J_b(\hat{\beta}_{b-1}; D_b) \right\} \left[ j,j \right]$. Simple calculation shows
that the coordinate-wise updating for LASSO has the form
\[
\hat{\beta}_{LASSO}^{b,j} = \frac{\text{Soft}(Z_{b,j}, \lambda_b)}{W_{b,j}},
\]
where \(\text{Soft}(z, \gamma)\) is the soft-thresholding operator with value
\[
\text{sign}(z)(|z| - \gamma)_+ = \begin{cases} 
  z - \gamma & \text{if } z > 0 \text{ and } \gamma < |z|, \\
  z + \gamma & \text{if } z < 0 \text{ and } \gamma < |z|, \\
  0 & \text{if } \gamma \geq |z|.
\end{cases}
\]

We can get the similar expressions for SCAD and MCP penalties by the same way, which have the following forms respectively:

\[
\begin{align*}
W_{b,j} > \frac{1}{r - 1}, \hat{\beta}_{SCAD}^{b,j} &= \begin{cases} 
  \frac{\text{Soft}(Z_{b,j}, \lambda_b)}{W_{b,j}} & \text{if } |Z_{b,j}| \leq \lambda_b + \lambda_b W_{b,j} \\
  \frac{\text{Soft}(Z_{b,j}, \frac{r\lambda_b}{r - 1})}{(W_{b,j} - 1/r)} & \text{if } \lambda_b + \lambda_b W_{b,j} < |Z_{b,j}| < r\lambda_b W_{b,j} \\
  Z_{b,j}/W_{b,j} & \text{if } |Z_{b,j}| \geq r\lambda_b W_{b,j} 
\end{cases} \\
W_{b,j} < \frac{1}{r - 1}, \hat{\beta}_{SCAD}^{b,j} &= \begin{cases} 
  \frac{\text{Soft}(Z_{b,j}, \lambda_b)}{W_{b,j}} & \text{if } |Z_{b,j}| \leq r\lambda_b W_{b,j} \\
  \frac{\text{Soft}(Z_{b,j}, \frac{r\lambda_b}{r - 1})}{(W_{b,j} - 1/r)} & \text{if } r\lambda_b W_{b,j} < |Z_{b,j}| < \lambda_b + \lambda_b W_{b,j} \\
  Z_{b,j}/W_{b,j} & \text{if } |Z_{b,j}| \geq \lambda_b + \lambda_b W_{b,j} 
\end{cases} \\
W_{b,j} = \frac{1}{r - 1}, \hat{\beta}_{SCAD}^{b,j} &= \begin{cases} 
  (r - 1) \text{Soft}(Z_{b,j}, \lambda_b) & \text{if } |Z_{b,j}| \leq \frac{r\lambda_b}{r - 1} \\
  (r - 1)Z_{b,j} & \text{if } |Z_{b,j}| > \frac{r\lambda_b}{r - 1}.
\end{cases}
\]
\[
\begin{align*}
W_{b,j} > \frac{1}{r}, \hat{\beta}_{b,j}^{MCP} &= \left\{ \begin{array}{ll}
\text{Soft}(Z_{b,j}, \lambda_b) & \text{if } |Z_{b,j}| \leq r\lambda_b W_{b,j} \\
\frac{Z_{b,j}}{W_{b,j}} & \text{if } |Z_{b,j}| > r\lambda_b W_{b,j}
\end{array} \right. \\
W_{b,j} < \frac{1}{r}, \hat{\beta}_{b,j}^{MCP} &= \left\{ \begin{array}{ll}
0 & \text{if } |Z_{b,j}| \leq r\lambda_b W_{b,j} \\
\frac{(Z_{b,j} - \lambda_b)}{(W_{b,j} - 1/r)} & \text{if } r\lambda_b W_{b,j} < Z_{b,j} < \lambda_b \\
\frac{(Z_{b,j} + \lambda_b)}{(W_{b,j} - 1/r)} & \text{if } -\lambda_b < Z_{b,j} < -r\lambda_b W_{b,j} \\
\frac{Z_{b,j}}{W_{b,j}} & \text{if } |Z_{b,j}| \geq \lambda_b
\end{array} \right. \\
W_{b,j} = \frac{1}{r}, \hat{\beta}_{b,j}^{MCP} &= \left\{ \begin{array}{ll}
0 & \text{if } |Z_{b,j}| \leq \lambda_b \\
\frac{rZ_{b,j}}{r} & \text{if } |Z_{b,j}| > \lambda_b,
\end{array} \right.
\end{align*}
\]

(3.6)

in which \( r \) is a constant.

After the coordinate descent steps, we can complete variable selection and get the sparse estimator \( \hat{\beta}_b \). Like the common penalty estimators, however, the online updating estimator also has a relatively large bias because of the use of penalty. In order to get a bias-reduced estimator, after variable selection by the method proposed above, we estimate the nonzero components again by renewable MLE in Luo & Song (2020). Thus, in the each step above, we use the corresponding bias-reduced penalty estimator as our choice of parameter estimation; such a bias-reduced estimator is still denoted by \( \hat{\beta}_b \).

In the above process, only the diagonal elements of the negative Hessian matrix are considered. This is because maintaining a full Hessian matrix requires \( O(p^2) \) memory space and \( O(p^2) \) computational complexity, which is impractical for handling large-scale ultra-high dimensional data. We only use the diagonal elements of the negative Hessian matrix for reducing the computation and storage burdens.
3.2 Tuning and Solution Paths

The accuracy of selecting active variables is affected largely by the chosen value of the tuning parameter $\lambda_b$. Motivated by the BIC criterion provided by Schwarz (1978), in the online updating procedure of variable section and parameter estimation, we propose the following online updating BIC criterion at the $b$-th step to choose $\lambda_b$:

$$
\lambda_b = \arg \min_{\lambda \in \mathbb{R}} \left\{ \hat{s} \ln(N_b) + (\hat{\beta}_{b,\lambda} - \hat{\beta}_{b-1})^T \left( \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right) \left( \hat{\beta}_{b,\lambda} - \hat{\beta}_{b-1} \right) - 2\ell_b \left( \hat{\beta}_{b,\lambda}; D_b \right) \right\},
$$

where $\hat{\beta}_{b,\lambda}$ is an estimator given a $\lambda$ via online coordinate descent until the $b$-th batch. The reason for this choice is given in Supplementary Material. However, it is difficult to realize the above optimization procedure because of the infinite select interval $\lambda \in \mathbb{R}$. We see from (3.4), (3.5) and (3.6) that $\hat{\beta}_{b,j}$ will stay zero if $|Z_{b,j}| < \lambda_b$, implying $\lambda_{\text{max}} = \max_j |Z_{b,j}|$. Thus, to easily realize the above optimization procedure, the tuning parameter is restricted in $(0, \lambda_{\text{max}})$. Consequently, a realizable BIC criterion for choosing tuning parameter is given by

$$
\lambda_b = \arg \min_{\lambda \in (0, \lambda_{\text{max}})} \left\{ \hat{s} \ln(N_b) + (\hat{\beta}_{b,\lambda} - \hat{\beta}_{b-1})^T \left( \sum_{t=1}^{b-1} J_t(\hat{\beta}_t; D_t) \right) \left( \hat{\beta}_{b,\lambda} - \hat{\beta}_{b-1} \right) - 2\ell_b \left( \hat{\beta}_{b,\lambda}; D_b \right) \right\},
$$

(3.7)

Obviously, it has the online updating form. Amounts of simulations given in Section 4 will illustrate that the way of choosing $\lambda_b$ via (3.7) is efficient and computationally simple.
3.3 Summary of Algorithm

Finally, we summarize the general algorithm and give the pseudocode for key numerical calculations in following Algorithm.

**Algorithm 1** Online Variable Selection and Parameter Estimation in GLM with Streaming Datasets

1: **Input:** model $f(y|X; \beta_0)$, streaming data sets $D_1, \ldots, D_b, \ldots$
2: Initialize $\hat{\beta}_1$ in $D_1$ via offline LASSO/SCAD/MCP, and compute the components of $J_1(\hat{\beta}_1, D_1)$, where the corresponding indexes are in the selected active index set $\hat{S}_1$;
3: **for** $b = 2, 3, \cdots$ **do**
4: Choose $\hat{\lambda}_b$ via (3.7);
5: Compute $W_{b,j} = W_{b-1,j} + J_b(\hat{\beta}_{b-1}, D_b)[j,j]$, $Z_{b,j} = U_b(\hat{\beta}_{b-1}; D_b)_{[j]} + \hat{\beta}_{b-1,j}W_{b,j}$, for $j \in \{1, \cdots, p\}$;
6: Choose active index set via LASSO(3.4)/SCAD(3.5)/MCP(3.6), $\hat{S}_b = \{j \in \{1, \cdots, p\} | \hat{\beta}_{b,j} \neq 0\}$;
7: Estimate the parameters with indexes in $\hat{S}_b$ via renewable MLE in Luo & Song (2020);
8: Compute the components of $J_1(\hat{\beta}_b, D_b)$, where the corresponding indexes are in the selected active index set $\hat{S}_b$; and update the cumulative negative Hessian matrix;
9: **end for**
10: **Output** $\hat{\beta}_b$.

4 Simulation Studies and Real Data Analysis

In this section, we conduct simulation experiments to assess the performance of the proposed method in the settings of linear and logistic models. For a thorough evaluation, we compare our method with the following competitors:

(a) The total data offline LASSO proposed by Tibshirani (1996), denoted by Total_data_LASSO. This method processes the entire data in a procedure of variable section
and parameter estimation. Thus, it is not an online updating method. It is employed only as an object of reference;

(b) The total data offline SCAD proposed by Fan & Li (2001), denoted by Total_data_SCAD. Like the Total_data_LASSO, this method is also employed as an object of reference;

(c) The total data offline MCP proposed by Zhang (2010), denoted by Total_data_MCP. Also this is an object of reference;

(d) The truncated SGD proposed by Fan et al. (2018), denoted by Fan_LASSO, Fan_SCAD and Fan_MCP respectively according to the methods in the burn-in step;

(e) Our online updating coordinate descent algorithms, denoted by Renew_LASSO, Renew_SCAD and Renew_MCP respectively according to the types of the penalty function.

In the procedure of simulation, the parameter estimation is assessed by the average squared $\ell_2$ error, and the variable selection is evaluated by the following five criteria:

(a) the average number of the variables being selected, denoted by NV;

(b) the percentage of occasion on which all active variables are included in the selected model, denoted by IN;

(c) the percentage of occasion on which correct components are selected, denoted by CS;

(d) the type I error, irrelevant predictor inclusion rate, denoted by I;

(e) the type II error, active predictor exclusion rate, denoted by II.

4.1 Example 1 (Gaussian linear model)

In this example, we generate the full data set $D^*_B$ and $N_B$ observations independently from the mean model $E(y_{ti}|x_{ti}) = g(x^T_{ti}\beta), i = 1, \cdots , n_t; t = 1, \cdots , B$, where $g(\cdot)$ is the identical transformation, and $B$ is a terminal point. For different dimensions of covariates,
$p = 10, 100 \text{ and } 1000$, we always set the active index set as $S = \{1, 2, 3, 4, 5\}$, the true parameter vector as $\beta_0 = (0.5, -0.5, 0.5, -0.5, 0.5, 0, 0, \ldots, 0)^T$, the intercept $x_{ti, [1]} \equiv 1$, and $x_{ti, [2:p]} \sim \mathcal{N}(0_{p-1}, V_{p-1})$, where $V_{p-1}$ is a $(p-1) \times (p-1)$ compound symmetry covariance matrix with correlation $\rho = 0.5$. As suggested in Fan et al. (2018), for all renewable methods, we use the 1000 samples to give the starting values. In fact, it is always satisfied in the streaming data sets. For the methods of Fan LASSO, Fan SCAD and Fan MCP provided by Fan et al. (2018), we choose the stepwise as $\eta = \log(N_B)/N_B$.

For SCAD and MCP, we choose the $r = 3.7$ and $r = 3$ as defaults respectively, which are suggested in Fan & Li (2001) and Zhang (2010). Assume that in every batches the numbers of observation data are the same as $n_t = n = N_B/B$. All the results are obtained via over 100 rounds of simulations. We summarize the simulation results in Tables 1-6, and Figure 1.

We first report the variable selection results as follows. From the Tables 1, 3, 5, we have the following findings.

(1) When we fix the total sample size $N_B$ and vary the number of batches $B$, all the methods can choose the active variable sets such that $S \subseteq \hat{S}$ for different dimensions of covariates, and our methods have smaller I, NV and larger CS. These illustrate that our methods could select all active variables and less irrelevant variables than the other methods, and our results are competitive with the total data offline methods.

(2) The Fan LASSO, Fan SCAD and Fan MCP all have larger I, NV and smaller CS than ours.

From the Tables 2, 4, 6, we have the following findings.

(3) When we fix the sample size $n$ of every batches and vary the number of batch $B$, all the methods can select all the important variables.
Table 1: Linear model p=10 Fixed N=10^6.

| Size  | Method       | NV  | IN  | CS  | I   | H   |
|-------|--------------|-----|-----|-----|-----|-----|
| n=100 | Fan_LASSO    | 7.85| 1.00| 0.07| 0.285| 0.000|
|       | Renew_LASSO  | 5.47| 1.00| 0.80| 0.047| 0.000|
|       | Fan_SCAD     | 5.24| 1.00| 0.84| 0.024| 0.000|
|       | Renew_SCAD   | 5.51| 1.00| 0.77| 0.051| 0.000|
|       | Fan_MCP      | 5.27| 1.00| 0.80| 0.027| 0.000|
|       | Renew_MCP    | 6.25| 1.00| 0.29| 0.125| 0.000|
| n=500 | Fan_LASSO    | 7.79| 1.00| 0.06| 0.279| 0.000|
|       | Renew_LASSO  | 5.45| 1.00| 0.80| 0.045| 0.000|
|       | Fan_SCAD     | 5.23| 1.00| 0.84| 0.023| 0.000|
|       | Renew_SCAD   | 5.42| 1.00| 0.79| 0.042| 0.000|
|       | Fan_MCP      | 5.21| 1.00| 0.84| 0.021| 0.000|
|       | Renew_MCP    | 6.24| 1.00| 0.27| 0.124| 0.000|
| n=1000| Fan_LASSO    | 7.67| 1.00| 0.07| 0.267| 0.000|
|       | Renew_LASSO  | 5.48| 1.00| 0.79| 0.048| 0.000|
|       | Fan_SCAD     | 5.22| 1.00| 0.83| 0.022| 0.000|
|       | Renew_SCAD   | 5.44| 1.00| 0.79| 0.044| 0.000|
|       | Fan_MCP      | 5.21| 1.00| 0.84| 0.021| 0.000|
|       | Renew_MCP    | 6.08| 1.00| 0.34| 0.108| 0.000|
| n=2000| Fan_LASSO    | 7.96| 1.00| 0.03| 0.296| 0.000|
|       | Renew_LASSO  | 5.46| 1.00| 0.81| 0.046| 0.000|
|       | Fan_SCAD     | 5.17| 1.00| 0.88| 0.017| 0.000|
|       | Renew_SCAD   | 5.46| 1.00| 0.80| 0.046| 0.000|
|       | Fan_MCP      | 5.14| 1.00| 0.89| 0.014| 0.000|
|       | Renew_MCP    | 5.91| 1.00| 0.36| 0.091| 0.000|
| n=10000| Fan_LASSO   | 7.79| 1.00| 0.06| 0.279| 0.000|
|       | Renew_LASSO  | 5.35| 1.00| 0.85| 0.035| 0.000|
|       | Fan_SCAD     | 5.01| 1.00| 0.99| 0.001| 0.000|
|       | Renew_SCAD   | 5.35| 1.00| 0.84| 0.035| 0.000|
|       | Fan_MCP      | 5.02| 1.00| 0.98| 0.002| 0.000|
|       | Renew_MCP    | 5.10| 1.00| 0.02| 0.310| 0.000|
|       | Total_data_LASSO | 5.50| 1.00| 0.79| 0.050| 0.000|
|       | Total_data_SCAD | 5.60| 1.00| 0.72| 0.060| 0.000|
Table 2: Linear model p=10 Fixed n=100.

| Size   | Method       | NV  | IN  | CS  | I   | H   |
|--------|--------------|-----|-----|-----|-----|-----|
| B=10   |Fan LASSO     | 7.94| 1.00| 0.03| 0.294| 0.000|
| N=1000 |Renew LASSO  | 6.21| 1.00| 0.29| 0.121| 0.000|
|        |Total data LASSO | 8.04| 1.00| 0.03| 0.304| 0.000|
|        |Fan SCAD     | 5.38| 1.00| 0.83| 0.038| 0.000|
|        |Renew SCAD  | 5.28| 1.00| 0.84| 0.028| 0.000|
|        |Total data SCAD | 5.36| 1.00| 0.83| 0.036| 0.000|
|        |Fan MCP      | 5.34| 1.00| 0.83| 0.034| 0.000|
|        |Renew MCP  | 5.25| 1.00| 0.85| 0.025| 0.000|
|        |Total data MCP | 5.36| 1.00| 0.80| 0.036| 0.000|

| B=100  |Fan LASSO     | 7.83| 1.00| 0.05| 0.283| 0.000|
| N=10000|Renew LASSO  | 6.33| 1.00| 0.30| 0.133| 0.000|
|        |Total data LASSO | 8.10| 1.00| 0.04| 0.310| 0.000|
|        |Fan SCAD     | 5.41| 1.00| 0.84| 0.041| 0.000|
|        |Renew SCAD  | 5.27| 1.00| 0.88| 0.027| 0.000|
|        |Total data SCAD | 5.76| 1.00| 0.69| 0.076| 0.000|
|        |Fan MCP      | 5.36| 1.00| 0.83| 0.036| 0.000|
|        |Renew MCP  | 5.22| 1.00| 0.89| 0.022| 0.000|
|        |Total data MCP | 5.67| 1.00| 0.68| 0.067| 0.000|

| B=1000 |Fan LASSO     | 7.96| 1.00| 0.05| 0.296| 0.000|
| N=100000|Renew LASSO  | 6.39| 1.00| 0.19| 0.139| 0.000|
|        |Total data LASSO | 8.01| 1.00| 0.07| 0.301| 0.000|
|        |Fan SCAD     | 5.65| 1.00| 0.75| 0.065| 0.000|
|        |Renew SCAD  | 5.31| 1.00| 0.78| 0.031| 0.000|
|        |Total data SCAD | 5.53| 1.00| 0.81| 0.053| 0.000|
|        |Fan MCP      | 5.52| 1.00| 0.78| 0.052| 0.000|
|        |Renew MCP  | 5.29| 1.00| 0.80| 0.029| 0.000|
|        |Total data MCP | 5.32| 1.00| 0.87| 0.032| 0.000|

| B=2000 |Fan LASSO     | 7.81| 1.00| 0.04| 0.281| 0.000|
| N=200000|Renew LASSO  | 6.31| 1.00| 0.19| 0.131| 0.000|
|        |Total data LASSO | 7.77| 1.00| 0.05| 0.277| 0.000|
|        |Fan SCAD     | 5.35| 1.00| 0.82| 0.035| 0.000|
|        |Renew SCAD  | 5.24| 1.00| 0.84| 0.024| 0.000|
|        |Total data SCAD | 5.40| 1.00| 0.82| 0.040| 0.000|
|        |Fan MCP      | 5.53| 1.00| 0.75| 0.053| 0.000|
|        |Renew MCP  | 5.25| 1.00| 0.80| 0.025| 0.000|
|        |Total data MCP | 5.27| 1.00| 0.84| 0.027| 0.000|

| B=10000|Fan LASSO     | 7.85| 1.00| 0.07| 0.285| 0.000|
| N=1000000|Renew LASSO  | 6.25| 1.00| 0.29| 0.125| 0.000|
|        |Total data LASSO | 8.10| 1.00| 0.02| 0.310| 0.000|
|        |Fan SCAD     | 5.47| 1.00| 0.89| 0.047| 0.000|
|        |Renew SCAD  | 5.24| 1.00| 0.84| 0.024| 0.000|
|        |Total data SCAD | 5.50| 1.00| 0.79| 0.050| 0.000|
|        |Fan MCP      | 5.51| 1.00| 0.77| 0.051| 0.000|
|        |Renew MCP  | 5.27| 1.00| 0.80| 0.027| 0.000|
|        |Total data MCP | 5.60| 1.00| 0.72| 0.060| 0.000|
Table 3: Linear model p=100 Fixed N=10^5.

| Size | Method     | NV  | IN  | CS  | I   | H   |
|------|------------|-----|-----|-----|-----|-----|
| n=100| Fan_LASSO  | 17.51 | 1.00 | 0.00 | 0.1251 | 0.0000 |
| B=1000| Renew_LASSO | 7.22  | 1.00 | 0.25 | 0.0222 | 0.0000 |
|      | Fan_SCAD   | 5.88  | 1.00 | 0.69 | 0.0088 | 0.0000 |
|      | Renew_SCAD | 5.43  | 1.00 | 0.73 | 0.0043 | 0.0000 |
|      | Fan_MCP    | 5.58  | 1.00 | 0.76 | 0.0058 | 0.0000 |
|      | Renew_MCP  | 5.30  | 1.00 | 0.78 | 0.0030 | 0.0000 |
| n=500| Fan_LASSO  | 17.82 | 1.00 | 0.00 | 0.1282 | 0.0000 |
| B=200| Renew_LASSO| 6.95  | 1.00 | 0.36 | 0.0195 | 0.0000 |
|      | Fan_SCAD   | 6.51  | 1.00 | 0.56 | 0.0151 | 0.0000 |
|      | Renew_SCAD | 5.52  | 1.00 | 0.65 | 0.0052 | 0.0000 |
|      | Fan_MCP    | 5.73  | 1.00 | 0.76 | 0.0073 | 0.0000 |
|      | Renew_MCP  | 5.25  | 1.00 | 0.81 | 0.0025 | 0.0000 |
| n=1000| Fan_LASSO | 17.65 | 1.00 | 0.00 | 0.1265 | 0.0000 |
| B=100| Renew_LASSO| 6.45  | 1.00 | 0.39 | 0.0145 | 0.0000 |
|      | Fan_SCAD   | 6.55  | 1.00 | 0.62 | 0.0155 | 0.0000 |
|      | Renew_SCAD | 5.43  | 1.00 | 0.75 | 0.0043 | 0.0000 |
|      | Fan_MCP    | 5.43  | 1.00 | 0.81 | 0.0043 | 0.0000 |
|      | Renew_MCP  | 5.22  | 1.00 | 0.85 | 0.0022 | 0.0000 |
| n=2000| Fan_LASSO | 17.33 | 1.00 | 0.01 | 0.1233 | 0.0000 |
| B=50| Renew_LASSO| 6.50  | 1.00 | 0.51 | 0.0150 | 0.0000 |
|      | Fan_SCAD   | 5.97  | 1.00 | 0.76 | 0.0097 | 0.0000 |
|      | Renew_SCAD | 5.27  | 1.00 | 0.88 | 0.0027 | 0.0000 |
|      | Fan_MCP    | 5.34  | 1.00 | 0.85 | 0.0034 | 0.0000 |
|      | Renew_MCP  | 5.18  | 1.00 | 0.88 | 0.0018 | 0.0000 |
| n=10000| Fan_LASSO| 19.12 | 1.00 | 0.01 | 0.1412 | 0.0000 |
| B=10| Renew_LASSO| 7.35  | 1.00 | 0.24 | 0.0235 | 0.0000 |
|      | Fan_SCAD   | 6.66  | 1.00 | 0.62 | 0.0166 | 0.0000 |
|      | Renew_SCAD | 5.19  | 1.00 | 0.91 | 0.0019 | 0.0000 |
|      | Fan_MCP    | 5.59  | 1.00 | 0.74 | 0.0059 | 0.0000 |
|      | Renew_MCP  | 5.00  | 1.00 | 1.00 | 0.0000 | 0.0000 |
| N=100000| Total_data_LASSO | 18.66 | 1.00 | 0.00 | 0.1366 | 0.0000 |
|      | Total_data_SCAD | 6.57  | 1.00 | 0.66 | 0.0157 | 0.0000 |
|      | Total_data_MCP | 5.60  | 1.00 | 0.78 | 0.0060 | 0.0000 |
Table 4: Linear model $p=100$ Fixed $n=100$.

| Size  | Method     | NV  | IN  | CS  | I    | H    |  
|-------|------------|-----|-----|-----|------|------|  
| B=10  | Fan_LASSO  | 16.98 | 1.00 | 0.00 | 0.1198 | 0.0000 |  
| N=1000| Renew_LASSO| 7.51  | 1.00 | 0.29 | 0.0251 | 0.0000 |  
|       | Total_LASSO| 17.64 | 1.00 | 0.00 | 0.1264 | 0.0000 |  
|       | Fan_SCAD   | 5.86  | 1.00 | 0.76 | 0.0086 | 0.0000 |  
|       | Renew_SCAD | 5.56  | 1.00 | 0.78 | 0.0056 | 0.0000 |  
|       | Total_SCAD | 6.10  | 1.00 | 0.72 | 0.00110| 0.0000 |  
|       | Fan_MCP    | 5.33  | 1.00 | 0.85 | 0.0033 | 0.0000 |  
|       | Renew_MCP  | 5.30  | 1.00 | 0.86 | 0.0030 | 0.0000 |  
|       | Total_MCP  | 5.31  | 1.00 | 0.84 | 0.0031 | 0.0000 |  
| B=50  | Fan_LASSO  | 18.36 | 1.00 | 0.00 | 0.1336 | 0.0000 |  
| N=5000| Renew_LASSO| 8.55  | 1.00 | 0.24 | 0.0353 | 0.0000 |  
|       | Total_LASSO| 17.91 | 1.00 | 0.00 | 0.1291 | 0.0000 |  
|       | Fan_SCAD   | 6.15  | 1.00 | 0.69 | 0.0115 | 0.0000 |  
|       | Renew_SCAD | 5.53  | 1.00 | 0.76 | 0.0054 | 0.0000 |  
|       | Total_SCAD | 5.85  | 1.00 | 0.78 | 0.0085 | 0.0000 |  
|       | Fan_MCP    | 5.41  | 1.00 | 0.82 | 0.0041 | 0.0000 |  
|       | Renew_MCP  | 5.21  | 1.00 | 0.86 | 0.0021 | 0.0000 |  
|       | Total_MCP  | 5.33  | 1.00 | 0.87 | 0.0033 | 0.0000 |  
| B=100 | Fan_LASSO  | 17.64 | 1.00 | 0.00 | 0.1251 | 0.0000 |  
| N=10000| Renew_LASSO| 8.15  | 1.00 | 0.21 | 0.0315 | 0.0000 |  
|       | Total_LASSO| 17.44 | 1.00 | 0.00 | 0.1244 | 0.0000 |  
|       | Fan_SCAD   | 5.87  | 1.00 | 0.77 | 0.0087 | 0.0000 |  
|       | Renew_SCAD | 5.39  | 1.00 | 0.86 | 0.0039 | 0.0000 |  
|       | Total_SCAD | 5.61  | 1.00 | 0.73 | 0.0061 | 0.0000 |  
|       | Fan_MCP    | 5.24  | 1.00 | 0.86 | 0.0024 | 0.0000 |  
|       | Renew_MCP  | 5.30  | 1.00 | 0.86 | 0.0024 | 0.0000 |  
|       | Total_MCP  | 5.33  | 1.00 | 0.87 | 0.0033 | 0.0000 |  
| B=200 | Fan_LASSO  | 18.92 | 1.00 | 0.00 | 0.1392 | 0.0000 |  
| N=20000| Renew_LASSO| 7.36  | 1.00 | 0.24 | 0.0236 | 0.0000 |  
|       | Total_LASSO| 18.31 | 1.00 | 0.01 | 0.1331 | 0.0000 |  
|       | Fan_SCAD   | 6.49  | 1.00 | 0.64 | 0.0149 | 0.0000 |  
|       | Renew_SCAD | 5.54  | 1.00 | 0.74 | 0.0054 | 0.0000 |  
|       | Total_SCAD | 6.08  | 1.00 | 0.72 | 0.0118 | 0.0000 |  
|       | Fan_MCP    | 5.45  | 1.00 | 0.79 | 0.0045 | 0.0000 |  
|       | Renew_MCP  | 5.18  | 1.00 | 0.87 | 0.0018 | 0.0000 |  
|       | Total_MCP  | 5.33  | 1.00 | 0.85 | 0.0033 | 0.0000 |  
| B=1000| Fan_LASSO  | 17.51 | 1.00 | 0.00 | 0.1251 | 0.0000 |  
| N=100000| Renew_LASSO| 7.22  | 1.00 | 0.25 | 0.0222 | 0.0000 |  
|       | Total_LASSO| 18.66 | 1.00 | 0.00 | 0.1266 | 0.0000 |  
|       | Fan_SCAD   | 5.88  | 1.00 | 0.69 | 0.0088 | 0.0000 |  
|       | Renew_SCAD | 5.43  | 1.00 | 0.73 | 0.0043 | 0.0000 |  
|       | Total_SCAD | 6.07  | 1.00 | 0.76 | 0.0157 | 0.0000 |  
|       | Fan_MCP    | 5.58  | 1.00 | 0.76 | 0.0058 | 0.0000 |  
|       | Renew_MCP  | 5.30  | 1.00 | 0.78 | 0.0030 | 0.0000 |  
|       | Total_MCP  | 5.60  | 1.00 | 0.78 | 0.0060 | 0.0000 |  

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Table 5: Linear model $p=1000$ Fixed $N=10^4$.

| Size   | Method    | NV   | IN   | CS   | I       | II       |
|--------|-----------|------|------|------|---------|----------|
| n=100  | Fan_LASSO | 29.51| 1.00 | 0.00 | 0.02451 | 0.00000  |
| B=100  | Renew_LASSO | 10.79| 1.00 | 0.24 | 0.00579 | 0.00000  |
|        | Fan_SCAD  | 7.77 | 1.00 | 0.66 | 0.00277 | 0.00000  |
|        | Renew_SCAD | 6.30 | 1.00 | 0.70 | 0.00130 | 0.00000  |
|        | Fan_MCP   | 5.68 | 1.00 | 0.79 | 0.00068 | 0.00000  |
|        | Renew_MCP | 5.34 | 1.00 | 0.82 | 0.00034 | 0.00000  |
| n=200  | Fan_LASSO | 27.67| 1.00 | 0.00 | 0.02267 | 0.00000  |
| B=50   | Renew_LASSO | 9.35 | 1.00 | 0.21 | 0.00435 | 0.00000  |
|        | Fan_SCAD  | 7.48 | 1.00 | 0.67 | 0.00248 | 0.00000  |
|        | Renew_SCAD | 6.24 | 1.00 | 0.71 | 0.00124 | 0.00000  |
|        | Fan_MCP   | 5.45 | 1.00 | 0.85 | 0.00045 | 0.00000  |
|        | Renew_MCP | 5.25 | 1.00 | 0.86 | 0.00025 | 0.00000  |
| n=400  | Fan_LASSO | 30.61| 1.00 | 0.00 | 0.02561 | 0.00000  |
| B=25   | Renew_LASSO | 11.23| 1.00 | 0.17 | 0.00623 | 0.00000  |
|        | Fan_SCAD  | 7.07 | 1.00 | 0.66 | 0.00207 | 0.00000  |
|        | Renew_SCAD | 6.06 | 1.00 | 0.67 | 0.00106 | 0.00000  |
|        | Fan_MCP   | 5.74 | 1.00 | 0.76 | 0.00074 | 0.00000  |
|        | Renew_MCP | 5.52 | 1.00 | 0.77 | 0.00052 | 0.00000  |
| n=500  | Fan_LASSO | 26.06| 1.00 | 0.01 | 0.02106 | 0.00000  |
| B=20   | Renew_LASSO | 11.46| 1.00 | 0.13 | 0.00646 | 0.00000  |
|        | Fan_SCAD  | 7.35 | 1.00 | 0.73 | 0.00235 | 0.00000  |
|        | Renew_SCAD | 5.84 | 1.00 | 0.74 | 0.00084 | 0.00000  |
|        | Fan_MCP   | 5.40 | 1.00 | 0.85 | 0.00040 | 0.00000  |
|        | Renew_MCP | 5.31 | 1.00 | 0.86 | 0.00031 | 0.00000  |
| n=1000 | Fan_LASSO | 27.01| 1.00 | 0.00 | 0.02201 | 0.00000  |
| B=10   | Renew_LASSO | 8.74 | 1.00 | 0.12 | 0.00374 | 0.00000  |
|        | Fan_SCAD  | 7.89 | 1.00 | 0.69 | 0.00289 | 0.00000  |
|        | Renew_SCAD | 5.97 | 1.00 | 0.78 | 0.00097 | 0.00000  |
|        | Fan_MCP   | 5.56 | 1.00 | 0.83 | 0.00056 | 0.00000  |
|        | Renew_MCP | 5.19 | 1.00 | 0.91 | 0.00019 | 0.00000  |
| N=10000| Total_data_LASSO | 26.33| 1.00 | 0.00 | 0.02133 | 0.00000  |
|        | Total_data_SCAD | 7.23 | 1.00 | 0.71 | 0.00223 | 0.00000  |
|        | Total_data_MCP | 5.58 | 1.00 | 0.76 | 0.00058 | 0.00000  |


Table 6: Linear model p=1000 Fixed n=100.

| Size  | Method    | NV  | IN  | CS  | I    | II   |
|-------|-----------|-----|-----|-----|------|------|
| B=10  | Fan_LASSO | 28.54 | 1.00 | 0.00 | 0.02354 | 0.00000 |
|       | Renew_LASSO | 10.27 | 1.00 | 0.21 | 0.00527 | 0.00000 |
|       | Total_data_LASSO | 27.33 | 1.00 | 0.00 | 0.02233 | 0.00000 |
|       | Fan_SCAD | 7.73 | 1.00 | 0.58 | 0.00273 | 0.00000 |
|       | Renew_SCAD | 7.39 | 1.00 | 0.58 | 0.00239 | 0.00000 |
|       | Total_data_SCAD | 6.71 | 1.00 | 0.69 | 0.00171 | 0.00000 |
|       | Fan_MCP | 5.81 | 1.00 | 0.76 | 0.00081 | 0.00000 |
|       | Renew_MCP | 5.68 | 1.00 | 0.77 | 0.00068 | 0.00000 |
|       | Total_data_MCP | 5.30 | 1.00 | 0.82 | 0.00030 | 0.00000 |
| B=20  | Fan_LASSO | 31.12 | 1.00 | 0.00 | 0.02612 | 0.00000 |
|       | Renew_LASSO | 9.56 | 1.00 | 0.28 | 0.00456 | 0.00000 |
|       | Total_data_LASSO | 27.33 | 1.00 | 0.00 | 0.02233 | 0.00000 |
|       | Fan_SCAD | 8.08 | 1.00 | 0.62 | 0.00308 | 0.00000 |
|       | Renew_SCAD | 7.32 | 1.00 | 0.64 | 0.00232 | 0.00000 |
|       | Total_data_SCAD | 6.76 | 1.00 | 0.65 | 0.00267 | 0.00000 |
|       | Fan_MCP | 5.55 | 1.00 | 0.74 | 0.00054 | 0.00000 |
|       | Renew_MCP | 5.54 | 1.00 | 0.74 | 0.00054 | 0.00000 |
|       | Total_data_MCP | 6.60 | 1.00 | 0.77 | 0.00060 | 0.00000 |
| B=40  | Fan_LASSO | 28.16 | 1.00 | 0.01 | 0.02316 | 0.00000 |
|       | Renew_LASSO | 10.72 | 1.00 | 0.19 | 0.00472 | 0.00000 |
|       | Total_data_LASSO | 30.57 | 1.00 | 0.00 | 0.02557 | 0.00000 |
|       | Fan_SCAD | 6.26 | 1.00 | 0.70 | 0.00126 | 0.00000 |
|       | Renew_SCAD | 7.34 | 1.00 | 0.69 | 0.00234 | 0.00000 |
|       | Total_data_SCAD | 6.58 | 1.00 | 0.72 | 0.00074 | 0.00000 |
|       | Fan_MCP | 5.54 | 1.00 | 0.72 | 0.00054 | 0.00000 |
|       | Renew_MCP | 5.53 | 1.00 | 0.81 | 0.00054 | 0.00000 |
|       | Total_data_MCP | 5.56 | 1.00 | 0.81 | 0.00054 | 0.00000 |
| B=50  | Fan_LASSO | 28.83 | 1.00 | 0.00 | 0.02383 | 0.00000 |
|       | Renew_LASSO | 10.86 | 1.00 | 0.26 | 0.00586 | 0.00000 |
|       | Total_data_LASSO | 28.89 | 1.00 | 0.00 | 0.02389 | 0.00000 |
|       | Fan_SCAD | 7.21 | 1.00 | 0.61 | 0.00221 | 0.00000 |
|       | Renew_SCAD | 5.93 | 1.00 | 0.67 | 0.00093 | 0.00000 |
|       | Total_data_SCAD | 6.63 | 1.00 | 0.67 | 0.00163 | 0.00000 |
|       | Fan_MCP | 5.59 | 1.00 | 0.79 | 0.00059 | 0.00000 |
|       | Renew_MCP | 5.39 | 1.00 | 0.82 | 0.00039 | 0.00000 |
|       | Total_data_MCP | 5.56 | 1.00 | 0.76 | 0.00056 | 0.00000 |
| B=100 | Fan_LASSO | 29.51 | 1.00 | 0.00 | 0.02451 | 0.00000 |
|       | Renew_LASSO | 10.76 | 1.00 | 0.24 | 0.00579 | 0.00000 |
|       | Total_data_LASSO | 26.33 | 1.00 | 0.00 | 0.02153 | 0.00000 |
|       | Fan_SCAD | 7.77 | 1.00 | 0.66 | 0.00277 | 0.00000 |
|       | Renew_SCAD | 6.30 | 1.00 | 0.70 | 0.00130 | 0.00000 |
|       | Total_data_SCAD | 7.23 | 1.00 | 0.71 | 0.00223 | 0.00000 |
|       | Fan_MCP | 5.68 | 1.00 | 0.79 | 0.00068 | 0.00000 |
|       | Renew_MCP | 5.34 | 1.00 | 0.82 | 0.00034 | 0.00000 |
|       | Total_data_MCP | 5.58 | 1.00 | 0.76 | 0.00058 | 0.00000 |
Figure 1: Parameter estimation results in the linear model.
(4) With larger total sample size $N_B$, our methods have the almost same selection results as the offline Total_data_LASSO, Total_data_SCAD, Total_data_MCP. In this case, the Fan_LASSO, Fan_SCAD and Fan_MCP still include more irrelevant covariates.

For the parameter estimation, from the Figure 1 we have the following observations.

(1) When we fix the total sample size $N_B$, and vary the numbers of batches and the dimensions of covariates, our methods have smaller average squared $\ell_2$ errors than the Fan_LASSO, Fan_SCAD and Fan_MCP, and have almost same results as the offline Total_data_LASSO, Total_data_SCAD, Total_data_MCP.

(2) When we fix the every batch sample size $n$, vary the dimensions of covariates, and increase numbers of batches, our methods still have almost same results as the offline Total_data_LASSO, Total_data_SCAD, Total_data_MCP. In this case, the Fan_LASSO, Fan_SCAD and Fan_MCP have larger the average squared $\ell_2$ errors than ours and total data offline methods.

In summary, our methods obtain better variable selection and parameter estimation results than the Fan_LASSO, Fan_SCAD and Fan_MCP, and almost the same results as the total data offline methods.

4.2 Example 2 (Logistic model)

In this example, the mean model is chosen as $E(y_{ti}|x_{ti}) = g(x_{ti}^T\beta), i = 1, \cdots, n_t; t = 1, \cdots, B$, where $g(\cdot) = \exp(\cdot)/\{1 + \exp(\cdot)\}$. We choose $\beta_0 = (1, -1, 1, -1, 1, 0, 0, \ldots, 0)^T$ and set the others as the same as in Example 1. Note that the Fan_LASSO, Fan_SCAD and Fan_MCP are only suitable for linear models. Thus, in the above logistic regression, we only compare our results with the offline LASSO, SCAD and MCP.

We fix the total data size $N_B$ and every batch sample size $n$ respectively. The selection
results are summarized in Tables 7-12, and the parameter estimation results are reported in Figure 2.

We report the variable selection results firstly. From the Tables 7, 9, 11, we have the following findings.

(1) All methods could achieve $S \subseteq \hat{S}$ when we fix the total sample size $N_B$ and vary the number of batches $B$. Our Renew_SCAD and Renew_MCP almost have the same NV, CS and I as those of the offline Total_data_SCAD and Total_data_MCP.

(2) The method Renew_LASSO has smaller NV and I and larger CS than the offline Total_data_LASSO.

From the Tables 8, 10, 12, we have the following observations.

(3) When we fix the every batch sample sizes $n$, vary the numbers of batches and dimensions of covariates, all the methods can select all the active variables.

(4) It is seen that the Renew_LASSO could select less irrelevant variables than the offline Total_data_LASSO.
Table 8: Logistic model p=10 Fixed n=100.

| Size | Method    | NV  | IN  | CS  | I   | H   |
|------|-----------|-----|-----|-----|-----|-----|
| B=10 | Renew_LASSO | 5.50 | 1.00 | 0.62 | 0.050 | 0.000 |
| N=1000 | Total data_LASSO | 8.37 | 1.00 | 0.02 | 0.337 | 0.000 |
|       | Renew_SCAD | 5.37 | 1.00 | 0.80 | 0.037 | 0.000 |
|       | Total data_SCAD | 5.43 | 1.00 | 0.84 | 0.043 | 0.000 |
|       | Renew_MCP | 5.25 | 1.00 | 0.84 | 0.025 | 0.000 |
|       | Total data_MCP | 5.38 | 1.00 | 0.86 | 0.038 | 0.000 |

| B=100 | Renew_LASSO | 5.81 | 1.00 | 0.42 | 0.081 | 0.000 |
| N=10000 | Total data_LASSO | 8.26 | 1.00 | 0.01 | 0.326 | 0.000 |
|       | Renew_SCAD | 5.13 | 1.00 | 0.99 | 0.013 | 0.000 |
|       | Total data_SCAD | 5.45 | 1.00 | 0.77 | 0.045 | 0.000 |
|       | Renew_MCP | 5.17 | 1.00 | 0.89 | 0.017 | 0.000 |
|       | Total data_MCP | 5.20 | 1.00 | 0.86 | 0.020 | 0.000 |

| B=1000 | Renew_LASSO | 6.00 | 1.00 | 0.31 | 0.100 | 0.000 |
| N=10000 | Total data_LASSO | 8.60 | 1.00 | 0.02 | 0.360 | 0.000 |
|       | Renew_SCAD | 5.20 | 1.00 | 0.83 | 0.020 | 0.000 |
|       | Total data_SCAD | 5.38 | 1.00 | 0.84 | 0.038 | 0.000 |
|       | Renew_MCP | 5.31 | 1.00 | 0.89 | 0.031 | 0.000 |
|       | Total data_MCP | 5.34 | 1.00 | 0.87 | 0.034 | 0.000 |

| B=2000 | Renew_LASSO | 6.01 | 1.00 | 0.29 | 0.101 | 0.000 |
| N=20000 | Total data_LASSO | 8.39 | 1.00 | 0.02 | 0.339 | 0.000 |
|       | Renew_SCAD | 5.23 | 1.00 | 0.82 | 0.023 | 0.000 |
|       | Total data_SCAD | 5.59 | 1.00 | 0.76 | 0.059 | 0.000 |
|       | Renew_MCP | 5.16 | 1.00 | 0.86 | 0.046 | 0.000 |
|       | Total data_MCP | 5.54 | 1.00 | 0.74 | 0.054 | 0.000 |

| B=10000 | Renew_LASSO | 6.03 | 1.00 | 0.28 | 0.103 | 0.000 |
| N=100000 | Total data_LASSO | 8.52 | 1.00 | 0.07 | 0.352 | 0.000 |
|       | Renew_SCAD | 5.25 | 1.00 | 0.81 | 0.025 | 0.000 |
|       | Total data_SCAD | 5.41 | 1.00 | 0.83 | 0.041 | 0.000 |
|       | Renew_MCP | 5.14 | 1.00 | 0.87 | 0.014 | 0.000 |
|       | Total data_MCP | 5.30 | 1.00 | 0.86 | 0.030 | 0.000 |

Table 9: Logistic model p=100 Fixed N=10^5.

| Size | Method    | NV  | IN  | CS  | I   | H   |
|------|-----------|-----|-----|-----|-----|-----|
| n=100 | Renew_LASSO | 5.82 | 1.00 | 0.58 | 0.0082 | 0.0000 |
| B=1000 | Renew_SCAD | 5.33 | 1.00 | 0.76 | 0.0033 | 0.0000 |
|       | Renew_MCP | 5.16 | 1.00 | 0.88 | 0.0016 | 0.0000 |

| n=500 | Renew_LASSO | 6.19 | 1.00 | 0.55 | 0.0119 | 0.0000 |
| B=200 | Renew_SCAD | 5.18 | 1.00 | 0.87 | 0.0018 | 0.0000 |
|       | Renew_MCP | 5.11 | 1.00 | 0.92 | 0.0011 | 0.0000 |

| n=1000 | Renew_LASSO | 6.21 | 1.00 | 0.57 | 0.0121 | 0.0000 |
| B=100 | Renew_SCAD | 5.15 | 1.00 | 0.89 | 0.0015 | 0.0000 |
|       | Renew_MCP | 5.15 | 1.00 | 0.92 | 0.0015 | 0.0000 |

| n=2000 | Renew_LASSO | 5.59 | 1.00 | 0.68 | 0.0059 | 0.0000 |
| B=500 | Renew_SCAD | 5.30 | 1.00 | 0.84 | 0.0030 | 0.0000 |
|       | Renew_MCP | 5.13 | 1.00 | 0.89 | 0.0013 | 0.0000 |

| n=10000 | Renew_LASSO | 7.18 | 1.00 | 0.24 | 0.0218 | 0.0000 |
| B=1000 | Renew_SCAD | 5.05 | 1.00 | 0.98 | 0.0005 | 0.0000 |
|       | Renew_MCP | 5.00 | 1.00 | 1.00 | 0.0000 | 0.0000 |

| N=100000 | Total data_LASSO | 21.56 | 1.00 | 0.00 | 0.1656 | 0.0000 |
| Total data_SCAD | 5.98 | 1.00 | 0.74 | 0.0098 | 0.0000 |
| Total data_MCP | 5.24 | 1.00 | 0.86 | 0.0024 | 0.0000 |
### Table 10: Logistic model p=100 Fixed n=100.

| Size | Method      | NV  | IN  | CS  | I   | H   |
|------|-------------|-----|-----|-----|-----|-----|
| B=10 | Renew_LASSO | 6.05 | 1.00 | 0.56 | 0.0105 | 0.0000 |
| N=1000 | Total_LASSO | 2.06 | 1.00 | 0.00 | 0.1586 | 0.0000 |
|       | Renew_SCAD  | 5.64 | 1.00 | 0.77 | 0.0064 | 0.0000 |
|       | Total_SCAD  | 6.12 | 1.00 | 0.73 | 0.0112 | 0.0000 |
|       | Renew_MCP   | 5.27 | 1.00 | 0.83 | 0.0027 | 0.0000 |
|       | Total_MCP   | 5.61 | 1.00 | 0.78 | 0.0061 | 0.0000 |
| B=50 | Renew_LASSO | 6.40 | 1.00 | 0.58 | 0.0140 | 0.0000 |
| N=10000 | Total_LASSO | 21.60 | 1.00 | 0.00 | 0.1660 | 0.0000 |
|       | Renew_SCAD  | 5.38 | 1.00 | 0.81 | 0.0038 | 0.0000 |
|       | Total_SCAD  | 6.68 | 1.00 | 0.67 | 0.0168 | 0.0000 |
|       | Renew_MCP   | 5.17 | 1.00 | 0.89 | 0.0017 | 0.0000 |
|       | Total_MCP   | 5.47 | 1.00 | 0.79 | 0.0047 | 0.0000 |

### Table 11: Logistic model p=1000 Fixed N=10^4.

| Size   | Method      | NV  | IN  | CS  | I   | H   |
|--------|-------------|-----|-----|-----|-----|-----|
| n=100  | Renew_LASSO | 7.88 | 1.00 | 0.53 | 0.00288 | 0.00000 |
| B=100  | Renew_SCAD  | 5.74 | 1.00 | 0.76 | 0.00074 | 0.00000 |
|        | Renew_MCP   | 5.25 | 1.00 | 0.87 | 0.00025 | 0.00000 |
| n=200  | Renew_LASSO | 7.33 | 1.00 | 0.60 | 0.00233 | 0.00000 |
| B=50   | Renew_SCAD  | 5.99 | 1.00 | 0.69 | 0.00099 | 0.00000 |
|        | Renew_MCP   | 5.34 | 1.00 | 0.82 | 0.00034 | 0.00000 |
| n=400  | Renew_LASSO | 7.41 | 1.00 | 0.37 | 0.00241 | 0.00000 |
| B=25   | Renew_SCAD  | 6.38 | 1.00 | 0.55 | 0.00138 | 0.00000 |
|        | Renew_MCP   | 5.47 | 1.00 | 0.78 | 0.00047 | 0.00000 |
| n=500  | Renew_LASSO | 8.33 | 1.00 | 0.33 | 0.00333 | 0.00000 |
| B=20   | Renew_SCAD  | 6.40 | 1.00 | 0.63 | 0.00140 | 0.00000 |
|        | Renew_MCP   | 5.34 | 1.00 | 0.84 | 0.00034 | 0.00000 |
| n=1000 | Renew_LASSO | 7.45 | 1.00 | 0.23 | 0.00245 | 0.00000 |
| B=10   | Renew_SCAD  | 6.04 | 1.00 | 0.69 | 0.00104 | 0.00000 |
|        | Renew_MCP   | 5.26 | 1.00 | 0.89 | 0.00026 | 0.00000 |
| N=10000 | Total_LASSO | 36.46 | 1.00 | 0.00 | 0.03146 | 0.00000 |
|        | Total_SCAD  | 7.10 | 1.00 | 0.71 | 0.00210 | 0.00000 |
|        | Total_MCP   | 5.45 | 1.00 | 0.81 | 0.00045 | 0.00000 |
Table 12: Logistic model $p=1000$ Fixed $n=100$.

| Size | Method | NV   | IN   | CS   | I     | H     |
|------|--------|------|------|------|-------|-------|
|      | Renew LASSO | 5.58 | 1.00 | 0.71 | 0.00058 | 0.00000 |
| N=1000 | Total data LASSO | 34.95 | 1.00 | 0.00 | 0.02995 | 0.00000 |
|      | Renew SCAD | 5.75 | 1.00 | 0.66 | 0.00075 | 0.00000 |
|      | Total data SCAD | 7.78 | 1.00 | 0.62 | 0.00278 | 0.00000 |
|      | Renew MCP | 5.44 | 1.00 | 0.76 | 0.00044 | 0.00000 |
|      | Total data MCP | 5.55 | 1.00 | 0.80 | 0.00055 | 0.00000 |
|      | Renew LASSO | 6.16 | 1.00 | 0.61 | 0.00116 | 0.00000 |
| N=2000 | Total data LASSO | 32.52 | 1.00 | 0.00 | 0.02752 | 0.00000 |
|      | Renew SCAD | 6.16 | 1.00 | 0.59 | 0.00116 | 0.00000 |
|      | Total data SCAD | 7.76 | 1.00 | 0.59 | 0.00276 | 0.00000 |
|      | Renew MCP | 5.49 | 1.00 | 0.78 | 0.00049 | 0.00000 |
|      | Total data MCP | 5.75 | 1.00 | 0.74 | 0.00075 | 0.00000 |
|      | Renew LASSO | 6.34 | 1.00 | 0.58 | 0.00134 | 0.00000 |
| N=4000 | Total data LASSO | 39.35 | 1.00 | 0.00 | 0.03435 | 0.00000 |
|      | Renew SCAD | 5.90 | 1.00 | 0.68 | 0.00090 | 0.00000 |
|      | Total data SCAD | 8.20 | 1.00 | 0.61 | 0.00320 | 0.00000 |
|      | Renew MCP | 5.49 | 1.00 | 0.73 | 0.00049 | 0.00000 |
|      | Total data MCP | 5.69 | 1.00 | 0.79 | 0.00069 | 0.00000 |
|      | Renew LASSO | 6.66 | 1.00 | 0.62 | 0.00166 | 0.00000 |
| N=5000 | Total data LASSO | 36.27 | 1.00 | 0.00 | 0.03127 | 0.00000 |
|      | Renew SCAD | 6.06 | 1.00 | 0.67 | 0.00106 | 0.00000 |
|      | Total data SCAD | 8.04 | 1.00 | 0.64 | 0.00304 | 0.00000 |
|      | Renew MCP | 5.37 | 1.00 | 0.78 | 0.00037 | 0.00000 |
|      | Total data MCP | 5.75 | 1.00 | 0.77 | 0.00075 | 0.00000 |
|      | Renew LASSO | 7.88 | 1.00 | 0.53 | 0.00288 | 0.00000 |
| N=10000 | Total data LASSO | 36.46 | 1.00 | 0.00 | 0.03146 | 0.00000 |
|      | Renew SCAD | 7.10 | 1.00 | 0.71 | 0.00210 | 0.00000 |
|      | Total data SCAD | 7.10 | 1.00 | 0.71 | 0.00210 | 0.00000 |
|      | Renew MCP | 5.25 | 1.00 | 0.87 | 0.00025 | 0.00000 |
|      | Total data MCP | 5.45 | 1.00 | 0.81 | 0.00045 | 0.00000 |
Figure 2: Parameter estimation results in the logistic model.
For parameter estimation, from the Figure 2, we have the following observations.

(1) When we fix the total sample size \( N_B \), vary the dimensions of covariates and increase the numbers of batches, the results of our methods are almost accordance with the results of offline Total_data_LASSO, Total_data_SCAD and Total_data_MCP.

(2) When we fix the every batch sample size \( n \), vary dimensions of covariates, and increase numbers of batches, our methods still almost have the same performances as the offline methods.

In summary, our methods almost have the same behaviors as the total data offline methods. Especially, the Renew_LASSO performances are better than the offline Total_data_LASSO. In the following subsection of real data analysis, we will further illustrate the properties via the corresponding numerical results.

### 4.3 Real Data Analysis

In the real data analysis, we apply our renewable variable selection methods, such as renewable LASSO, renewable SCAD, renewable MCP, to the following real dataset. It is the YearPredictionMSD dataset from the UCI Machine Learning Repository Dua & Graff (2019). This dataset, with 90 features, 463715 training samples and 51630 testing samples, is about the prediction of the release year of a song from audio features. For all online updating methods, we give the starting values by \( m = 3715 \) training samples. We use the \( R^2 \) values of estimators from various methods to confirm the advantages of the new methods, where

\[
R^2 = 1 - \frac{\sum_{j=1}^{n_{test}} (y_j - \hat{y}_j)^2}{\sum_{i=1}^{n_{test}} (y_j - \overline{y}_j)^2},
\]

\( n_{test} \) is the total sample size in testing dataset, \( \hat{y}_j \) is the predicted value of \( y_j \), and \( \overline{y}_j \) is the sample mean of \( y_j \)'s. We summarize the \( R^2 \) values in Table 13, in which the Total_data_LS is the least square estimator using the total data.
Table 13: Results for YearPredictionMSD data set.

| Size   | Method  | $R^2$    | Size   | Method  | $R^2$    |
|--------|---------|----------|--------|---------|----------|
| n=230  | Fan LASSO | 0.17104  | n=4600 | Fan LASSO | 0.17104  |
| B=2000 | Renew LASSO | 0.23167  | B=100  | Renew LASSO | 0.23170  |
|        | Fan SCAD  | 0.17117  |        | Fan SCAD  | 0.17061  |
|        | Renew SCAD | 0.23169  |        | Renew SCAD | 0.23130  |
|        | Fan MCP   | 0.17210  |        | Fan MCP   | 0.17210  |
|        | Renew MCP | 0.23180  |        | Renew MCP | 0.23142  |
| n=460  | Fan LASSO | 0.17104  | n=9200 | Fan LASSO | 0.17224  |
| B=1000 | Renew LASSO | 0.23168  | B=50   | Renew LASSO | 0.23175  |
|        | Fan SCAD  | 0.17117  |        | Fan SCAD  | 0.17117  |
|        | Renew SCAD | 0.231723 |        | Renew SCAD | 0.22920  |
|        | Fan MCP   | 0.17147  |        | Fan MCP   | 0.17210  |
|        | Renew MCP | 0.23169  |        | Renew MCP | 0.23013  |
| n=920  | Fan LASSO | 0.17224  | N=463715 | Total data LS | 0.23169  |
| B=500  | Renew LASSO | 0.23166  |        | Renew LASSO | 0.23166  |
|        | Fan SCAD  | 0.17061  |        | Fan SCAD  | 0.17061  |
|        | Renew SCAD | 0.23163  |        | Renew SCAD | 0.23163  |
|        | Fan MCP   | 0.17210  |        | Fan MCP   | 0.17210  |
|        | Renew MCP | 0.23167  |        | Renew MCP | 0.23167  |

The results of the table clearly show that the $R^2$ values of the our renewable selection and estimators are larger than the other online updating estimators in general, and ours almost have the same results as the dotal data least squared estimators. Thus, our methods are the best choices for fitting the real data.

5 Conclusion

In the previous sections of this paper, for high-dimensional GLMs, we proposed a novel online updating objective function and related algorithm to achieve variable selection and parameter estimation. This is a unified framework because it can be applied to various penalty likelihoods with differentiable or non-differentiable penalty functions. Unlike the truncated method, ours choose the tuning parameter via data-driven criteria, such as BIC, and the proposed BIC criterion has an online updating form. Moreover, the proposed strategy can achieve the estimation and selection consistencies, and the oracle property, and these theoretical properties always holds, without any constraint on the number of the
data batches. Through various simulation studies, it is demonstrated that our proposed methods can recover the support of the true features, efficiently and accurately, and are much better the competitors and are competitive with the objects of reference by total data methods.

However, still there are some important issues needed to be investigated in the future. For example, in the environments of streaming data, usually the distributions of data will change or have heterogeneity in different data periods. How to design the objective function and the algorithm to adapt to the situation is a very significant issue in the streaming data. Another example is how to extend our methods into nonlinear models and nonparametric models. These are interesting issues and are worth further study in the future.

**SUPPLEMENTARY MATERIAL**

**Supplementary Material:** Supplementary Material includes the details of online updating BIC criterion and some proofs of a lemma and theorems. (.pdf file)

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