A GEOMETRIC PROOF OF CONN’S LINEARIZATION THEOREM FOR ANALYTIC POISSON STRUCTURES

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Abstract. We give a geometric proof of Conn’s linearization theorem for analytic Poisson structures, without using the fast convergence method.

1. Introduction

In [2], Jack Conn proved the following theorem:

Theorem 1.1 (Conn). Let $\Pi = \Pi_1 + \ldots$ be an analytic Poisson structure in a neighborhood of $0$ in $K^n$ (where $K = \mathbb{R}$ or $\mathbb{C}$), which vanishes at 0 and whose linear part $\Pi_1$ corresponds to a semi-simple Lie algebra. Then $\Pi$ admits a local analytic linearization at 0.

This is probably the first major result about analytic local normal forms for Poisson structures. A similar result in the smooth case is also obtained by Conn in [3].

Conn’s proof is based on the powerful method of fast convergence. Using this same method, in [9, 4] we extended Conn’s results, proving the Levi decomposition for Poisson structures, and the linearization of Lie algebroids with a semisimple linear part.

The method of fast convergence has its drawbacks: besides the fact that it often requires heavy analytical estimations, it also hides the geometrical picture/structure of the problem. Because of this, Alan Weinstein [6, 7] and other people were asking for a more geometric proof of Theorem 1.1 and similar results.

The aim of this Note is to sketch such a geometric proof of Theorem 1.1. The techniques used here are first developed in [8] where we studied Birkhoff normal forms for Hamiltonian systems. Our proof consists of the following steps:

Step 1. Find $m$ Casimir functions $F_1,\ldots,F_m$ for the Poisson structure $\Pi$ in question, where $m$ is the rank of the semisimple Lie algebra $g$ associated to the linear part $\Pi_1$ of $\Pi$. In order to define these functions, we will use period integrals (of the symplectic form over 2-cycles in symplectic leaves) and arguments from [8].

Step 2. Use the above Casimir functions to show that the symplectic foliation (by symplectic leaves) of $\Pi$ is locally analytically diffeomorphic to the symplectic foliation of $\Pi_1$. So we may assume that $\Pi$ has the same foliation as $\Pi_1$. In particular, the symplectic foliation of $\Pi$, considered as a singular foliation (without the symplectic structure), is invariant under the coadjoint action of $g$. 

Date: First draft, July 2002.
1991 Mathematics Subject Classification. 53D17, 32S65.
Key words and phrases. Poisson structure, linearization, normal form, Casimir function, period integral.
Step 3. Use averaging and Moser’s path method to show that there is a non-autonomous vector field tangent to the symplectic foliation whose time-1 map moves $\Pi$ to a $\mathfrak{g}$-invariant Poisson structure. It implies that $\Pi$ admits a Hamiltonian $\mathfrak{g}$-action. The components of the corresponding equivariant moment map will then form a local linear system of coordinates for $\Pi$.

In the next three sections, we will carry out the above three steps, for holomorphic Poisson structures. Then in Section 5 we will indicate why things work the same in the real analytic case.

2. Step 1: Casimir functions

First let us look at Casimir functions for the linear Poisson structure $\Pi_1$ on $\mathfrak{g}^*$, where $\mathfrak{g}$ is the complex semi-simple Lie algebra of rank $m$ associated to $\Pi_1$. (We will consider $\Pi$ as a Poisson structure in a neighborhood of 0 in $\mathfrak{g}^*$). It is well known that the symplectic foliation of $\Pi_1$ (by coadjoint orbits) has codimension $m$, and there are $m$ independent homogeneous polynomial Casimir functions. We will refer to ??? for this and other informations concerning semi-simple Lie algebras used here.

What is important for us here is the fact that Casimir functions for $\Pi_1$ can be defined by period integrals as follows:

Let $P$ be a generic (semisimple regular) coadjoint orbit. Then the dimension of $H_2(P, \mathbb{R})$ is $m$. We can choose a basis $\Gamma_1, \ldots, \Gamma_m$ of $H_2(P, \mathbb{R})$, which may be represented by real 2-spheres on $P$ (these 2-spheres can be constructed explicitly by and fixing a Cartan algebra and a root system).

Define the following period integrals:

$$\rho_i^1 = \int_{\Gamma_i} \omega_1$$

where $\omega_1$ denotes the induced symplectic structure of $\Pi_1$ on $P$ (the so-called Kostant-Kirillov-Souriau symplectic structure).

So to each generic $P$ we can associate $m$ numbers $\rho_1^1, \ldots, \rho_m^1$. By changing $P$, they become Casimir functions near each generic leaf. But they are not single-valued (holomorphic) Casimir functions for $\Pi_1$ due to the monodromy problem: By moving round circle $P$ around a singularity, the 2-cycles on $P$ change after the move. In other words, the locally flat fiber bundle whose fibers are $H_2(P, \mathbb{R})$ has a nontrivial monodromy. The corresponding monodromy group is nothing by the Weyl group (Borel’s theorem ??). Thus, if we denote by $S(\rho_1^1, \ldots, \rho_m^1)$ the symmetric algebra of $\rho_1^1, \ldots, \rho_m^1$, then they Weyl group acts on it by monodromy. In fact, the set of polynomial Casimirs functions for $\Pi_1$ coincides with the set $S(\rho_1^1, \ldots, \rho_m^1)^W$

of elements in $S(\rho_1^1, \ldots, \rho_m^1)$ which are invariant under the Weyl group action. We may choose a basis $F_1^1 = G_1^1(\rho_1^1, \ldots, \rho_m^1), \ldots, F_1^m = G_1^m(\rho_1^1, \ldots, \rho_m^1)$ of homogeneous polynomial Casimir functions of $\Pi_1$.

For examples, when $\mathfrak{g} = sl(m + 1, \mathbb{C})$, then $\rho_1^1, \ldots, \rho_m^1$ can be chosen to be the first $m$ eigenvalues of $(m + 1) \times (m + 1)$ matrices, while $F_1^1, \ldots, F_1^m$ are nonlinear symmetric functions of the eigenvalues.
Now look at $\Pi$. An important observation is that, since $\Pi$ is formally equivalent to $\Pi_1$ (Weinstein’s theorem [5]), most symplectic leaves of $\Pi$ in a sufficiently small neighborhood of 0 have $m$ independent 2-cycles inherited from $\Pi_1$. More precisely, for any large natural number $N$ we have $\Pi = \Pi_1^{(N)} + o(N)$, where $o(N)$ means terms of order greater than $N$, and $\Pi_1^{(N)}$ is a Poisson structure which is locally analytically equivalent to $\Pi_1$. Then most symplectic leaves of $\Pi_1^{(N)}$ in an sufficiently small neighborhood of 0 are “nearly tangent” to symplectic leaves of $\Pi$. Therefore, due to Reeb’s stability, 2-spheres that represent 2-cycles on most symplectic leaves of $\Pi_1^{(N)}$ can be projected (in a unique way homotopically) to 2-cycles on symplectic leaves of $\Pi$ (This is well defined outside a horn-shaped neighborhood of the singular set of $\Pi$, see [8] for details). Denote these cycles on symplectic leaves of $\Pi$ again by $\Gamma_1, \ldots, \gamma_m$, and define
\[ \rho_i = \int_{\Gamma_i} \omega \]
where $\omega$ is the symplectic form induced from $\Pi$, and
\[ F^i = G^i(\rho_1, \ldots, \rho_m) \]
Then $F^1, \ldots, F^m$ are single-valued Casimir functions for $\Pi$ outside a horn-shaped neighborhood of the singular set of $\Pi$. Now make $N$ tend to $\infty$ and use arguments from [8] to conclude that $F^1, \ldots, F^m$ are holomorphic Casimir functions for $\Pi$ in a neighborhood of 0.

3. Step 2: Symplectic foliation

Since $\Pi$ is formally equivalent to $\Pi_1$, the $m$-tuple of Casimir functions $F^1, \ldots, F^m$ for $\Pi$ is formally equivalent to the $m$-tuple of Casimir functions $F_1^1, \ldots, F_1^m$ for $\Pi_1$. In other words, if we denote by $y = y(x)$ a formal diffeomorphism which moves $\Pi$ to $\Pi_1$, then we have
\[ F^i(x) = F_1^i(y(x)) \]
Now applying Artin’s theorem [4] to the system of analytic equations
\[ F_1^i(y) - F_1^1(x) = 0, \ldots, F_1^m(y) - F_1^m(x) = 0 \]
and the formal solution $y = y(x)$, we find a local analytic diffeomorphism $z = z(x)$ (which is tangent to the formal solution $y = y(x)$ up to any desired order) such that
\[ F^i(x) = F_1^i(z(x)) \]
By applying this local diffeomorphism, we may assume that
\[ F^i(x) = F_1^i(x), \]
i.e. the Casimir functions for $\Pi$ are the same as the Casimir functions for $\Pi_1$. It implies that the symplectic foliation for $\Pi$ is locally the same as the symplectic foliation for $\Pi_1$, i.e. it is given by coadjoint orbits on $\mathfrak{g}^*$ (at least in a dense regular part, but then everywhere in a neighborhood of 0 by continuation). We may also assume that $\Pi - \Pi_1 = o(N)$ for some natural number $N$ high enough, i.e. $\Pi$ is tangent to $\Pi_1$ up to order $N$. 
Though we will not use it in the next section, let us mention the following fact: because the values of $F^1, \ldots, F^m$ on each symplectic leaf determines the cohomological class of the symplectic form on the leaf (via the period integrals), the cohomological class of the symplectic form induced by $\Pi$ coincides with the cohomological class of the Kirillov-Kostant-Souriau symplectic form on each coadjoint orbit. (One may be tempted to use this fact in order to apply Moser’s path method directly to $\Pi$ and $\Pi_1$).

4. Step 3: $\mathfrak{g}$-action and isotopy

Denote by $G$ the compact group whose Lie algebra is the compact form of our semi-simple algebra $\mathfrak{g}$. Then $G$ acts on $\mathfrak{g}^*$ by coadjoint action. For each $g \in G$ and coadjoint orbit $P$ we denote by $\omega^P = g^*\omega^P_0$ the image of $\omega^P$ under the action of $g$ on $P$, where $\omega^P$ denotes the induced symplectic form of $\Pi$ on $P$.

Define

$$\omega^P_1 = \int_{g \in G} \omega^P_g d\mu$$

where $\mu$ is the Haar measure on $G$. Since $\Pi$ is tangent to $\Pi_1$ up to order $N$ and $\Pi_1$ is $G$-invariant, it implies that $\omega^P_1$ is nondegenerate.

Denote by $\Lambda_1$ the Poisson structure whose symplectic leaves are coadjoint orbits with symplectic form $\omega^P_1$. It is a holomorphic Poisson structure (because it is holomorphic at least in the regular part of $\Pi$, and the singular part is of codimension greater than 1, so we can use Hartogs’ extension theorem).

For each coadjoint orbit $P$, put $\omega^P_s = (1-s)\omega^P + s\omega^P_1$. Then we have an analytic 1-dimensional family of Poisson structure $\Lambda_s$ with induced symplectic structures $\omega^P_s$, which connects $\Pi = \Lambda_0$ with $\Lambda_1$. One checks that $\Pi_s$ is well-defined (again by Hartogs’ theorem).

Now we will find an analytic flow whose type-$s$ map moves $\Pi$ to $\Lambda_s$ by Moser’s path method. The corresponding time-dependent vector field is given by the following equation:

$$i_{X_s}\omega^P_s = \alpha$$

where $\alpha$ is an 1-form on each symplectic leaf such that $d\alpha = \omega^P_1 - \omega^P$.

The main difficulty here is how to define $\alpha$. For general singular foliations/fibrations this would be a highly nontrivial problem. (And if we chose $\Pi_1$ instead of $\Lambda_1$ in the isotopy, it would be more difficult to define $\alpha$). However, our situation here is a little bit special because we have a compact group action and can define $\alpha$ directly by the following formulas:

For each $g \in G$ denote by $\xi(g) \in \mathfrak{g}$ the “smallest” element of $\mathfrak{g}$ such that $g = \exp(\xi(g))$ (the set where $\xi(g)$ is not well defined (i.e. not unique) is of measure 0 so it will not matter), and denote by $X(g)$ the vector field on $\mathfrak{g}^*$ generated by $\xi(g)$.

Then we have

$$g^*\omega^P - \omega^P = \int_0^1 \mathcal{L}_{X(g)}\exp(tX(g))^*\omega^P dt = d\int_0^1 i_{X(g)}\exp(tX(g))^*\omega^P dt$$

and

$$\omega^P_1 - \omega^P = d\int_G \int_0^1 i_{X(g)}\exp(tX(g))^*\omega^P dt d\mu$$
So we can put
\[ \alpha = \int_G \int_0^1 i_{X(g)} \exp(tX(g))^* \omega^P \, dt \, d\mu \]

This last formula assures the analyticity of the time-dependent vector field \( X \) whose time-1 map moves \( \Pi \) to \( \Lambda_1 \).

Applying this analytic time-1 map, we may assume that \( \Pi \) is \( G \)-invariant, which is the same thing as \( g \)-invariant. Then since \( g \) is semisimple, the action of \( g \) is then Hamiltonian with respect to \( \Pi \) and is given by a (unique) equivariant moment map. Use the components of this moment map as local coordinates for a neighborhood of 0. Then \( \Pi \) becomes linear with respect to these coordinates, and we are done.

5. **The real analytic case**

In the real case, we can still proceed as above. Due to the complex conjugation, Casimir functions can be chosen to be real. In Step 3, we can still use the compact group \( G \) (which acts in the complex space, not the real one). But due to the complex conjugation, \( \Lambda_1 \) and \( \alpha \) are real ...

Question: does the method presented in this note work in the smooth case? I don’t know, but probably the most difficult part is to show that Casimir functions defined by period integrals in Step 1 are smooth. For this we probably have to control the singularity set of \( \Pi \) first. Step 2 and Step 3 probably still work, with a few modifications.

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