Making the decoy-state measurement-device-independent quantum key distribution practically useful

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The relatively low key rate seems to be the major barrier to its practical use for the decoy state measurement device independent quantum key distribution (MDIQKD). We present a method for the decoy-state MDIQKD that hugely raises the key rate, especially in the case the total data size is not large. For example, it can raise the key rate of the recent Shanghai experimental set-up by more than 600 times. Calculation shows that our method can be immediately applied for practical secure private communication with fresh secure keys generated from MDI-QKD.

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I. INTRODUCTION

One of the most important expected advantage for Quantum key distribution (QKD)\(^1,2\) is to generate fresh secure key for instant use, so as to achieve a higher-level security in private communications. This demands a considerable final key generation in a time scale of seconds. The existing technologies can achieve such a goal if the decoy-state BB84 is applied \(^3\). This method can keep the unconditional security of QKD with an imperfect single-photon source \(^1,2\). However, to patch up the security loophole caused by the limited detection efficiency (including channel loss) \(^6\), one has to seek more methods such as the so called device independent QKD (DI-QKD) \(^7\) and the measurement-device independent QKD (MDI-QKD) which was based on the idea of entanglement swapping \(^8\,9\). By using the decoy-state method, Alice and Bob can use imperfect single-photon sources \(^10\,11\) securely in the MDI-QKD. The decoy-state MDI-QKD has the advantage of getting rid of all detector side-channel attacks with imperfect single-photon sources. The method has been studied extensively both experimentally \(^12\,13\) and theoretically \(^20\,27\,30\,39\).

However, the key rate of the decoy-state MDI-QKD is rather low, e.g., in the well known Shanghai experiment \(^24\), it is 0.018bps, with the set up running for 130 hours. The low key rate seems to be the only barrier to its final practical use. On the other hand, prior art results show that, if the statistical fluctuation is taken into consideration, we need a large data size so as to reach a considerable final key rate \(^27\,28\,30\,50\). In particular, the number of total pulses at each side \(N_t\) is assumed to be larger that \(10^{12}\) or even larger. It seemed to be a rather challenging task to reach a considerable key rate with a small data size, such as \(N_t \sim 10^{9} \sim 10^{10}\), which can be done in a time of fewer than 1 second or a few seconds, given a detector repetition rate of 1.25GHz. Note that none of the prior art works can generate any final key given a small data size such as \(N_t \sim 10^{9} \sim 10^{10}\).

Here we present a method that can produce a key rate much higher than any existing theoretical or experimental results and can generate considerable key rate in a very short time. Therefore our method can be applied immediately for fresh key generation with decoy-state MDI-QKD in practice.

In what follows, we shall first review the decoy-state MDI-QKD and our protocol in section II. We then we take improved analysis in section III. There we show a very tricky result that the lower bound of the averaged value of yield and the upper bound phase-flip error rate of all single-photon pairs in both bases can be estimated tightly with observed data in X basis only. Based on this fact, we show with formulas that instead of taking the worst-case estimations for the yield and phase-flip error rate of single-photon pairs separately, we can treat them jointly pointing directly to the worst-case result of the final key rate. We then present the numerical simulation results in section IV. The results there show huge advantages of this work in the key rates. The article is ended with a concluding remark.

II. PROTOCOL

Here we consider a protocol where sources \(x_A\) and \(y_A\) (\(y_A\) and \(y_B\)) only emit pulses in \(X\) basis while source \(z_A\) (\(z_B\)) only emits pulses in \(Z\) basis. And those sources only emit four different states \(\rho_{oA} = |0\rangle\langle 0|\), \(\rho_{xA}\), \(\rho_{yA}\), \(\rho_{yB}\), \(\rho_{zA}\) and \(\rho_{zB}\).
\( \rho_{zA} (\rho_{0B} = |0\rangle\langle 0|, \rho_{xA}, \rho_{yB}, \rho_{zB} ) \) respectively, in their respective basis.

In photon number space, suppose

\[
\rho_{xA} = \sum_k a_k |k\rangle\langle k|, \quad \rho_{xB} = \sum_k b_k |k\rangle\langle k|, \tag{1}
\]

\[
\rho_{yA} = \sum_k a'_k |k\rangle\langle k|, \quad \rho_{yB} = \sum_k b'_k |k\rangle\langle k|, \tag{2}
\]

\[
\rho_{zA} = \sum_k a''_k |k\rangle\langle k|, \quad \rho_{zB} = \sum_k b''_k |k\rangle\langle k|, \tag{3}
\]

We call \( x_A \) and \( x_B \) as well as \( y_A, y_B \) the decoy sources, and \( z_A, z_B \) the signal sources, and the intensities.

At each time, Alice will randomly choose source \( l_A \) with probability \( p_{lA} \) for \( l = o, x, y, z \). Similarly, Bob will randomly choose source \( l_B \) with probability \( p_{lB} \) for \( r = o, x, y, z \). The emitted pulse pairs (one pulse from Alice, one pulse from Bob) are sent to the un-trusted third party (UTP). After postprocessing, Alice and Bob evaluate the data sent in two bases separately. The \( Z \)-basis is used for key generations, while the \( X \)-basis is used for error test in the phase-flip error rate for single-photon pulse pairs in \( Z \)-basis. We shall use notation \( lr \) to indicate the two-pulse source when Alice use source \( l_A \) and Bob use source \( r_B \). For the notation simplicity, we omit the scripts of any \( l \) and \( r \) for a two-pulse source, e.g., source \( xy \) is the source that Alice uses source \( x_A \) and Bob uses source \( y_B \). Also, here in our protocol, the intensity for pulses in \( Z \) basis can be different from those of \( X \) basis, this makes more freedom in choosing the intensities and hence further raises the key rate.

### III. Improved Analysis for Final Key Rate

We need lower bound value for \( s_{11}^Z \). However, as discussed by[30, 36], since in actual applications, the number of pulse-pairs in \( Z \) is much larger than the number of pulse-pairs in \( X \) basis. Therefore, we can use the lower bound of the averaged values of the yield of single-photon pairs in all bases for the quantity in \( Z \) basis only. A very tricky point here is that we can tightly lower bound the yield of all single-photon pairs using the observed data in \( X \)-basis only.

#### A. Theorems for statistical fluctuation

We define the counting rate (yield) of pulses of a certain set \( C \)

\[
S_C = \frac{n_C}{N_C} \tag{4}
\]

where \( n_C \) is the number of valid counts due to pulses in \( C \), and \( N_C \) is the number of pulses in set \( C \). Actually, in MDI-QKD, we always use pulses pairs from Alice and Bob. So, more strictly speaking, “pulses” above are actually “pulse pairs”. Given this definition, we have the following theorem:

**Theorem 1**: Suppose set \( M = \{M_1, M_2, \ldots, M_i, \ldots, M_K\} \) with \( K \geq 1 \), and any subset \( M_i \) contains \( N_{M_i} \) pulse pairs. Set

\[
L_i = \{L_1, L_2, \ldots, L_i, \ldots, L_K\}.
\]

Define quantity

\[
\langle S_M \rangle_L = \sum_{i=1}^{K} a_i S_{L_i} \tag{5}
\]

with \( c_i = \frac{N_{M_i}}{N} \). The following inequality holds with a probability larger than \( 1 - \epsilon \):

\[
|S_M - \langle S_M \rangle_L| \leq \gamma \sqrt{\frac{SC}{N_C}}
\]

provided that the following conditions hold for any \( i \):

i) \( M_i \subset L_i \); \( M_i \cap M_j = \emptyset \), \( L_i \cap L_j = \emptyset \) for \( i \neq j \) where \( \phi \) is the empty set.

ii) States of pulse pairs in \( L_i \) are independent and identical.

iii) Pulse pairs in set \( L_i \) are randomly mixed.

Here the value of coefficient \( \gamma \) is dependent on \( \epsilon \), e.g., \( \gamma = 5.3 \) if \( \epsilon = 10^{-7} \). Though contents of this theorem have been studied and applied elsewhere[13, 30, 36], we believe that theorem presented here offers a clearer picture for study of statistical fluctuation in the decoy-state MDI-QKD. In particular, in considering the averaged value of yield and phase-flip error rate of the single-photon pairs, we don’t have to limit them to the average over a certain basis, as was so in Ref.[30, 36, 39]. With our theorem 1, we can use the average over pulses in different bases. As demonstrated later in this article, this theorem can help us do calculations more efficiently, e.g., our theorem 2 and theorem 3.

Obviously, this theorem also holds for the error yields, and hence for the phase-flip error of single-photon pairs, as shall be studied latter.

Before any further investigation, we list the following simplified mathematical notations first.

1. \( C^{lr} \): the set for all pulse pairs from source \( lr \); Sometimes we simply use notation \( lr \) for \( C^{lr} \), if this does not cause any confusion.
2. \( C_{mn} \): the set of pulse pairs of state \( |m\rangle\langle m| \otimes |n\rangle\langle n| \) from set \( C \). For example, \( C_{mn}^{lr} \) and \( C_{mn} \) are sets for pulse pairs of state \( |m\rangle\langle m| \otimes |n\rangle\langle n| \) from sets \( C^{lr} \) and \( C \), respectively.
3. \( s_{mn}^{lr} \): yield of set \( C_{mn}^{lr} \), i.e., \( s_{mn}^{lr} = S_{C_{mn}^{lr}} \).
4. \( S_{mn}^{lr} \): yield of source \( lr \), i.e., \( S_{C_{mn}^{lr}} \).

In any real experimental set-up, the total pulses sent by both sides are finite. In order to extract the secure final key, the effect of statistical fluctuations caused by the finite-size key must be considered. In this case, in general

\[
s_{mn}^{lr} \neq s_{mn}^{lr'}
\]

if \( lr' \neq lr \). In this case, to obtain the lower bound value for \( s_{11} \) (yield of single-photon pairs) and upper bound value of \( e_{11}^{ph} \) (phase-flip rate of single-photon pairs), one can apply theorem 1 to a suitably chosen set of two pulse
sources, $D$. As an example we choose
\begin{equation}
D = \{oo, ox, xo, oy, yo, xy, yx\}. 
\end{equation}
To apply our theorem, we also need to choose set $\mathcal{L}$. First, we use
\begin{equation}
\mathcal{L} = D. 
\end{equation}
We can write the density matrix of any two-pulse source $lr \in D$ in the following form
\begin{equation}
\rho_{lr} = \sum_{m,n} c_{mn}^{lr}|m\rangle \otimes |n\rangle \langle m| \langle n|. 
\end{equation}
Relating this, we now define $\langle S_{lr} \rangle_{\mathcal{L}}$ as
\begin{equation}
\langle S_{lr} \rangle_{\mathcal{L}} = \sum_{m,n} c_{mn}^{lr} \mathcal{L}_{mn}. 
\end{equation}
According to this equation, we can list 7 equations for $lr \in D$. Using these, one can formulate the lower bound of $s_{11}^{lr}$ by
\begin{equation}
s_{11}^{lr} \geq s_{11}^{lr} = \frac{|a_1' b_2'(S_{xx})_{\mathcal{L}} + a_1 b_2(S_{yy})_{\mathcal{L}} - |a_1 b_2(S_{yy})_{\mathcal{L}}| + a_1 b_2 a_1' b_2'(S_{oo})_{\mathcal{L}} - a_1' b_2' \mathcal{H}|}{a_1 a_1' (b_1 b_2' - b_1' b_2)},
\end{equation}
and $\mathcal{H} = a_0(S_{ox})_{\mathcal{L}} + b_0(S_{xo})_{\mathcal{L}} - a_0 b_0(S_{oo})_{\mathcal{L}}$.

Since quantities $\langle S_{lr} \rangle_{\mathcal{L}}$ are not exactly determined, we can only find out the lower bound for $s_{11}^{lr}$ with constraints for fluctuations given by Eq. (3) in our theorem 1. We can rewrite $\mathcal{L}$ in $\mathcal{L} = \{\mathcal{L}_{mn}|m = 0,1,2,\cdots;n = 0,1,2,\cdots\}$ where the subset $\mathcal{L}_{mn}$ is for all pulse pairs in state $|m\rangle \otimes |n\rangle \langle m| \langle n|$ from set $\mathcal{L}$. We immediately find that for any source $lr \in D$, $\mathcal{L}_{mn} \in \mathcal{L}_{mn}$ and also $c_{mn}^{lr} \in \mathcal{L}_{mn}$. Regarding each $\mathcal{L}_{mn}$ as $\mathcal{M}$ and $\{\mathcal{L}_{mn}\}$ as $\{\mathcal{M}_l\}$ in our theorem, we find that condition 1 in theorem 1 holds. Moreover, one can easily find that all conditions in our theorem 1 hold for sets $\{\mathcal{L}_{mn}\}$ above. Therefore we can use Eq. (13) for constraints of fluctuations. We shall use the following constraints:

\begin{equation}
N_{is} S_{lr} + \sqrt{N_{is} S_{lr}} \geq N_{ir} (S_{lr})_{\mathcal{L}} \geq N_{is} S_{lr} - \sqrt{N_{is} S_{lr}} \quad \text{for any } lr \in D
\end{equation}

\begin{equation}
N_{xx}(S_{xx})_{\mathcal{L}} \geq S_{xx} - \gamma \sqrt{N_{xx} S_{xx}}
\end{equation}

\begin{equation}
N_{yy}(S_{yy})_{\mathcal{L}} \leq S_{yy} + \gamma \sqrt{N_{yy} S_{yy}}
\end{equation}

\begin{equation}
N_{yo}(S_{yo})_{\mathcal{L}} \geq N_{yo} S_{yo} + N_{yo} S_{oy} - \gamma \sqrt{N_{yo} S_{yo} + N_{yo} S_{oy}}
\end{equation}

\begin{equation}
N_{xx}(S_{xx})_{\mathcal{L}} + N_{yy}(S_{yy})_{\mathcal{L}} \geq N_{xx} S_{xx} + N_{yy} S_{yy} + N_{yo} S_{oy} - \gamma \sqrt{N_{xx} S_{xx} + N_{yy} S_{yy} + N_{yo} S_{oy}}
\end{equation}

\begin{equation}
N_{yy}(S_{yy})_{\mathcal{L}} + N_{oo}(S_{oo})_{\mathcal{L}} \leq N_{yy} S_{yy} + N_{oo} S_{oo} + \gamma \sqrt{N_{yy} S_{yy} + N_{oo} S_{oo}}
\end{equation}

The first line of Eq. (12) gives individual ranges of statistical fluctuations related to each separate sources, the other lines are joint constraints among different sources, as was studied in detail in Ref. [39].

Second, we use set
\begin{equation}
\mathcal{L}' = D \cup \mathcal{L}_{11}^{oo}.
\end{equation}
Note that $\mathcal{L}'$ is simply the set for pulse pairs from sources in $D$ and single-photon pairs from source $zz$. As shown in Ref. [20], the states in polarization space for single-photon pairs in $X$ basis or in $Z$ basis are identical. Both of them are $\frac{1}{2}I$. Similar to the study above for set $\mathcal{L}$, it is easy
to see that all conditions in our theorem 1 hold for sets \( \{C^r, L'\} \) above. Similar to Eq. (11), we have the lower bound of \( s_{11}^{L'} \) by

\[
s_{11}^{L'} \geq s_{11}^{L} = \frac{[a_1 b_2 (S_{xx})_{L'} + a_1 b_2 b_0 (S_{oo})_{L'} + a_1 b_2 b_0 (S_{yo})_{L'}] - [a_1 b_2 (S_{yy})_{L'} + a_1 b_2 b_0 (S_{oo})_{L'}] - a'_1 b'_1 \mathcal{H}'}{a_1 a'_1 (b_1 b'_2 - b'_1 b_2)},
\]

and \( \mathcal{H}' = a_0 (S_{oo})_{L'} + b_0 (S_{zo})_{L'} - a_0 b_0 (S_{oo})_{L'} \).

And we shall use the following constraints from Eq. (17) in theorem 1:

\[
\begin{align*}
N_{tr} S_{tr} + \sqrt{N_{tr} S_{tr}} &\geq N_{tr} \langle S_{tr} \rangle_{L'} \geq N_{tr} S_{tr} - \sqrt{N_{tr} S_{tr}} ; \text{ for any } l r \in D \\
N_{xx} \langle S_{xx} \rangle_{L'} &\geq S_{xx} - \gamma \sqrt{N_{xx} S_{xx}} \\
N_{yy} \langle S_{yy} \rangle_{L'} &\leq S_{yy} + \gamma \sqrt{N_{yy} S_{yy}} \\
N_{yo} \langle S_{yo} \rangle_{L'} + N_{yo} \langle S_{yo} \rangle_{L'} &\geq N_{yo} S_{yo} + N_{yo} S_{yo} - \gamma \sqrt{N_{yo} S_{yo} + N_{yo} S_{yo}} \\
N_{xx} \langle S_{xx} \rangle_{L'} + N_{yy} \langle S_{yy} \rangle_{L'} + N_{yo} \langle S_{yo} \rangle_{L'} &\geq N_{xx} S_{xx} + N_{yy} S_{yy} + N_{yo} S_{yo} - \gamma \sqrt{N_{xx} S_{xx} + N_{yo} S_{yo} + N_{yo} S_{yo}} \\
N_{yy} \langle S_{yy} \rangle_{L'} + N_{ox} \langle S_{ox} \rangle_{L'} &\leq N_{yy} S_{yy} + N_{ox} S_{ox} + \gamma \sqrt{N_{yy} S_{yy} + N_{ox} S_{ox}} \\
N_{ox} S_{ox} + N_{zo} S_{zo} + \gamma \sqrt{N_{ox} S_{ox} + N_{zo} S_{zo}} &\geq N_{zo} \langle S_{zo} \rangle_{L'} + N_{ox} \langle S_{ox} \rangle_{L'} \geq N_{ox} S_{ox} + N_{zo} S_{zo} - \gamma \sqrt{N_{ox} S_{ox} + N_{zo} S_{zo}}
\end{align*}
\]

Therefore we arrive at our theorem 2 below:

**Theorem 2:** In the non-asymptotic case, the yield of single-photon pairs in \( X \) basis is lower bounded by Eq. (11), and the yield of all single-photon pairs in both \( X \) basis and \( Z \) basis is lower bounded Eq. (16).

To a good approximation, we can regard \( s_{11}^{L'} \) as the lower bound of the yield of single-photon pairs in \( Z \) basis, since in our protocol most of (actually, almost all) single-photon pairs which can cause effective events are produced in \( Z \) basis.

Obviously, according to the definition of \( L \) and \( L' \), we immediately find that

\[
\mathcal{H} = \mathcal{H}'.
\]

We shall simply use the notation \( \mathcal{H} \) for both of them. Given theorem 2, we can actually deduce the lower bound of yield of single-photon pairs in \( Z \) basis through using the observed data in \( X \) basis only, even for the non-asymptotic calculation. This makes it possible to treat the yield of single-photon pairs and the phase-flip error of single-photon jointly because both of them are dependent on the same quantity \( \mathcal{H} \). We shall take a detailed study on this very important point below.

**B. Worst-case results for \( s_{11}^{L} \) and \( s_{11}^{L'} \) and phase flip error, key rate**

We denote \( \mathcal{S}_{11} (H) \) as the lower bound of \( s_{11}^{L} \) with a given value \( H \). The value \( \mathcal{S}_{11} (H) \) can be calculated rather tightly if we use constraints in Eq. (12). The value \( \mathcal{S}_{11} (H) \) calculated by these constraints can also lower bound \( s_{11}^{L'} \) because there is a same set of constraints for quantities \( \langle S_{tr} \rangle_{L'} \), as given by Eq. (17). Therefore we shall simply use notation \( \mathcal{S}_{11} (H) \) for lower bound of both \( s_{11}^{L} \) and \( s_{11}^{L'} \), given \( H \). Actually, using the method in Ref. [39], the functional \( \mathcal{S}_{11} (H) \) can be analytically formulated.

Second, we can also formulate the phase-flip error rate of single-photon pairs for all single-photon pairs as a functional of \( H \). The traditional method [30, 36, 39] did the calculation for the upper bound of averaged bit error rate of single-photon pairs in \( X \) basis \( s_{11}^{X} \). However, as discussed earlier [30, 36, 39], there are also statistical fluctuation between phase flip error rate in \( Z \) basis and the bit flip rate in \( X \) basis. When this fluctuation is straightly added, the key rate and the distance is decreased.

Here we thoroughly solve the issue using our theorem 1. As its original definition in the Pauli channel...
model \[40, 41\], a phase-flip error is a \(\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) error that takes a phase shift in Z basis. We adopt this definition for all qubits in both X-basis and Z-basis. The \(\sigma_z\) errors on qubits in Z-basis are not physically detectable, but they do exist. More clearly, imagine a virtual protocol that replaces all single-photon pulses by entangled pairs, (one photon is sent out for and another photon is stored in the local lab). Before any measurement, the users determine which subset will be used for testing the \(\sigma_z\) error (which can be measured in X-basis) and which subset will be measured in Z basis for key distillation. In this case, one can properly define the \(\sigma_z\) error for qubits in both bases. Nothing changes if one measures one half of the entangled pairs before sending out the other half. This will be just the real protocol. This means that \(\sigma_z\) error can be defined for qubits in all bases.

Denote the phase-flip error \((\sigma_z, e)\) yield of any set \(C\) by \(T_C\), we have \(T_C = \frac{\bar{n}_C}{N_C}\), and \(\bar{n}_C\) is the number or error bits due to set \(C\). Denoting \(T_{xx} = T_{C,xx}\) we have

\[
T_{xx} = \sum_{m,n} E_{mn} T_{mn}^{xx}
\]

where \(T_{mn}^{xx}\) is error yield of set \(C_{mn}^{xx}\). Taking the same definition of set \(C'\) as used earlier, we define quantity \((T_{xx})_{C'}\) as

\[
(T_{xx})_{C'} = \sum_{m,n} E_{mn} T_{mn}^{xx}. \tag{21}
\]

We can now apply our theorem 1 to make a non-trivial treatment for error yield and bound the phase-flip error all single-photon pairs in both bases.

\[
e_{11}^{C'} \leq e_{11}^{ph} = \frac{\xi}{a_1 b_1 S_{11}}, \tag{22}
\]

and \(\xi = (T_{xx})_{C'} - a_0 (T_{ox})_{C'} + b_0 (T_{ox})_{C'} + a_0 b_0 (T_{oo})_{C'}\). This gives the phase flip error for all single-photon pairs by using observed data of source \(xx\) only. To a good approximation, this is also the phase-flip error rate of single-photon pairs in Z-basis only, because almost all single-photon pairs are from \(zz\).

**Theorem 3** Applying our theorem 1, together with Eq. (22) one can obtain the upper bound of phase-flip error rate for all single-photon pairs by using observed data of source \(xx\) only.

Given the obvious fact that the bit flip error rate must be 50% if the bit is caused by source state of \(|0\rangle\langle 0| \otimes \rho\) or \(\rho \otimes |0\rangle\langle 0|\), we have

\[
\xi_{xx} = (T_{xx})_{C'} - \frac{1}{2} H. \tag{23}
\]

We can therefore regard the upper bound of \(e_{11}^{ph}\) as functional of \(H\) as

\[
\tilde{E}_{11}(H) = \frac{T_{xx} + \gamma \sqrt{\frac{T_{xx}}{S_{11}}} - H/2}{a_1 b_1 S_{11}} \geq e_{11}^{ph}. \tag{24}
\]

Therefore, we have the following key rate formula as a functional of \(H\)

\[
R(H) = \frac{P_{zx} P_{zy}}{2} \cdot \{ e_{11}^{ph} \mu S_{11}(H)[1 - \xi H + f S_{zz}H(E_{zz})]\}, \tag{25}
\]

where \(f\) is the error correction inefficiency and \(E_{zz}\) is the observed bit error rate for source \(zz\). The final key is simply the worst-case result of \(R(H)\) over all possible values for \(H\), i.e.

\[
R = \min_{H \in \mathbb{Z}} R(H). \tag{26}
\]

According to Eq. (13), we have \(I = [h - \delta, h + \delta]\) and

\[
\begin{align*}
\delta &= a_0 \sqrt{S_{ox} + S_{oo}} \frac{N_{ox}}{N_{oo}} - a_0^2 \gamma \sqrt{S_{oo}} \\
E_{oo} &= \mu S_{11}(H)[1 - \xi H + f S_{zz}H(E_{zz})]
\end{align*}
\]

given the symmetric set-up that satisfies \(a_0 = b_0, N_{ox} = N_{ox}\). We shall use such a symmetric case in our numerical simulation.

**IV. NUMERICAL SIMULATION**

In this section, we present some numerical simulations in comparison with the best known prior art results Refs. [36, 39]. Without we focus on the symmetric case where the two channel transmissions from Alice to UTP and from Bob to UTP are equal. We also assume that the UTP’s detectors are identical, i.e., they have the same dark count rates and detection efficiencies, and their detection efficiencies do not depend on the incoming signals. Also, we assume

\[
a_k = b_k, \quad a'_k = b'_k \tag{28}
\]

for all \(k\) and \(N_{ox} = N_{ox}, N_{yo} = N_{oo}\).

We shall estimate what values would be probably observed for the yields and error yields in the normal cases by the linear models as in \([3, 10, 23, 32, 36]\). We shall assume 2 types of detectors. Experimental conditions and the first type of detectors \([38]\) properties are listed in Table 1. For the second type of detector, we assume 40% detection and \(10^{-4}\) dark count rate, in Fig. [2][4][14]. In Fig. [4][14], we assumed the alignment error probability to be \(e_d = 1\%\). With these, the yields and error rate can be simulated in \([27, 32]\). We assume a coherent state for all sources. The density matrix of the coherent state with intensity \(\mu\) can be written into \(\rho = \sum_k \frac{\mu^k}{k!} |k\rangle\langle k|\). We calculate the key rate using Eq. (25)[20].

To make a fair comparison, we need to find out the full parameter optimizations for different methods \([36]\). The earlier methods \([31, 38]\) actually used the quantity \(e_{11}^{ph}\) which is the upper bound for averaged error rate for
Table I: List of experimental parameters used in numerical simulations of Fig. (1). $p_d$: the dark count rate. $\eta_d$: the detection efficiency of all detectors. $f_e$: the error correction inefficiency. $\epsilon$: the security bound considered in the finite-date analysis.

| $c_0$ | $c_d$ | $p_d$ | $\eta_d$ | $f_e$ | $\epsilon$ |
|-------|-------|-------|---------|------|----------|
| 0.5   | 1.5%  | $6.02 \times 10^{-6}$ | 14.5%   | 1.16 | $1.0 \times 10^{-7}$ |

A. Numerical results

Fig. (1, 2, 3, 4) make a clear view that our method improves the final key rate and transmission distance drastically, and the advantage is much more outstanding when the data size is smaller.

In these figures, the black dotted curve is the key rate obtained from the method in Ref. [36], where fluctuations of each sources are treated separately. The blue dashed curve is the improved key rate using the method of our previous work in Ref. [39], and the red solid curve is the result of this work.

B. The Shanghai set-up

With high-tech detectors such as the one used by Calgary group [21], it is possible to reach a distance of 300 kms for the MDI-QKD, as was predicted very recently [42]. However, the finite-size effect was not taken into account in making the prediction [42]. Now we consider the achievable performance of a real set-up, the Shanghai set-up that has run for the decoy-state MDI-QKD over 200 kms [26] with the statistical fluctuation being taken into account. We have calculated the key rate.
rate for set-up of Shanghai experiment at distance of 200 kms, where we assumed the detection efficiency of a single detector 40%, the dark count rate of a single detector 10^-7, η_d = 40%. Other parameters for the set-up are given by Table I.

In our calculation, we have chosen a special of sources for D in Eq.(7). As was pointed out already, there are other choices, e.g., \{oo, ox, xo, oy, yo, xx, xy, yx\}. The method here can also be applied to the traditional decoy-state BB84 protocol, say Alice has vacuum source, source x, y in X basis and source z in Z basis as the signal source. They use vacuum source and sources x, y to calculate the single photon yield and phase-flip error rate and use source z for key distillation. And also one can treat the single photon yield and phase-flip error rate jointly, taking the optimization directly pointing to the final key rate. This will be reported elsewhere.

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\[ D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
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