OPINION DYNAMICS MODELS WITH MEMORY IN COOPETITIVE SOCIAL NETWORKS: ANALYSIS, APPLICATION AND SIMULATION

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Abstract. In some social networks, the opinion forming is based on its own and neighbors' (initial) opinions, whereas the evolution of the individual opinions is also influenced by the individual’s past opinions in the real world. Unlike existing social network models, in this paper, a novel model of opinion dynamics is proposed, which describes the evolution of the individuals’ opinions not only depends on its own and neighbors’ current opinions, but also depends on past opinions. Memory and memoryless communication rules are simultaneously established for the proposed opinion dynamics model. Sufficient and/or necessary conditions for the equal polarization, consensus and neutralizability of the opinions are respectively presented in terms of the network topological structure and the spectral analysis. We apply our model to simulate Kahneman’s seminal experiments on choices in risky and riskless contexts, which fits in with the experiment results. Simulation analysis shows that the memory capacity of the individuals is inversely proportional to the speeds of the ultimate opinions formational.

1. Introduction

Analysis and control of agent-based network systems have been an active research topic in the past decades, and the consensus problems have been extensively studied in the literature (see, [25, 21, 19, 40] and the references therein). The problem of multi-agent consensus from a graph signal processing perspective was considered in [37], where analytic solutions were provided for the optimal convergence rate as well as the corresponding control gains. In the theory of agent-based network systems, the social network is one of the important and interesting case study, the opinions of the social individuals usually reach disagreement in the social networks [28]. For example, in the cooperative social networks, the disagreement of the heterogeneous belief systems was investigated in [35], where it was revealed that the disagreement behaviors of opinion dynamics were affected by the logical interdependence structure [35], and for the opinion dynamics in the antagonistic social networks, the disagreement problem under the leader-follower hierarchical framework was studied in [23]. In the opinion dynamics model with biased assimilation, how individual biases influence social equilibria was reported in [8]. In the nonlinear...
opinion dynamics model, sufficient conditions guaranteeing asymptotic convergence of opinions were provided in [1].

In the past few years, there has been an increasing interest in the study of the opinion dynamics, and many results have been reported in the literature to analyze the individuals’ opinions on a topic evolve over time as they interact [27, 36]. For instance, the opinion dynamics model with bounded confidence was investigated in [13], which is called Hegselmann-Krause (H-K) model, the consensus and polarization problem were also addressed in [13], the extension opinion dynamics of the H-K model with decaying confidence was considered in [24], the evolution of opinions on a sequence of issues was studied in [14], and the quasi-consensus behavior of H-K opinion dynamics was analyzed in [31]. Some other well-known opinion dynamics model (for example, DeGroot model [9] and Friedkin-Johnsen (F-J) model [11]) were also studied in the literature, such as, in strongly connected networks, a nonlinear opinion dynamics model was analyzed in [33]. It was shown that the stability of certain equilibria subject to both the degree of bias and the neighbors’ topology [33]. The multidimensional F-J model was addressed by [28] and F-J model over issue sequences with bounded confidence was studied in [32]. To describe the stochastic evolution of opinion dynamics, a novel opinion dynamics model was proposed in [5].

All the references mentioned above mainly take into account the evolution of opinions in the cooperative networks. However, in some real world scenarios or social networks, it is reasonable to assume that some individuals cooperate with each other, the other individuals compete with each other, wherein the positive weights among individuals implies cooperation and the negative weights among individuals means competition, which can be described as signed graphs. Recently, the signed graph was firstly applied to the social networks by Altafini [2, 3], it was shown that each agent can be asymptotically converged to a value that equal size but opposite sign in the structurally balanced networks. Afterwards, social networks with competitive interactions have attracted extensive attentions, such as, by using the Perron-Frobenius theorem to predict the outcomes of the opinion forming process, which was applied to the PageRank with negative links [4]; in the time-varying network topology, the consensus and polarization of the continuous-time opinion dynamics with hostile camps was studied in [26], and the opinion-forming process on the coopetitive social networks was considered in [34], where the problem of how the mass media formulates and changes public opinions was solved in [34].
Notice that the literature [7] reports an interesting discovery on how human memory is encoded in social networks and the capacity of memory mentioned in [30]. Furthermore, [16] shows that people makes judgments and choices influenced by individual’s memory. In order to characterize the effect of human memory in the social networks, we propose a new model of opinion dynamics on coopetitive (cooperative and competitive) social networks, which describes the evolution of the individuals’ opinions not only depending on its own and neighbors’ current opinions, but also depending on its own and neighbors’ past opinions. Memory and memoryless communication rules are respectively established for the proposed opinion dynamics model in the coopetitive social networks, and moreover, sufficient and/or necessary conditions for the equal polarization, consensus and neutralizability of the opinions are proposed on the basis of the network topological structure and the spectral analysis. We apply the proposed model to Kahneman’s seminal experiments on choices in risky and riskless contexts, which are parts of Kahneman’s work in winning the Nobel Prize. We show that the model can reproduce the results of Kahneman experiments. To the best of our knowledge, few mathematical model regenerates Kahneman’s experiments. According to the simulation analysis, we find that the memory capacity of the individuals is inversely proportional to the speeds of the ultimate opinions formation.

This paper is organized as follows. Some preliminaries and the problem formulation are given in Section 2. Memory communication rules and memoryless communication rules are established respectively in Section 3 and Section 4, and sufficient and/or necessary conditions guaranteeing the equal polarization, consensus and neutralizability of the opinions are presented. In Section 5, Kahneman’s seminal experiments on choices in risky and riskless contexts is examined by using our proposed opinion dynamics model. In Section 6, two numerical examples are carried out to analyse the influences of the susceptibility coefficient and the memory capacity of the individuals. Section 7 concludes this paper.

2. Preliminaries and Problem Formulation

Before formulating the problem of the present paper, we first give some basic concepts and properties of signed graphs. Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted signed directed graph, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and $\mathcal{W} = [w_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix with elements $w_{ij}, i, j \in \mathcal{V}$. In addition, $w_{ij} > 0$ means that individual $j$ is cooperative to individual $i$, and $w_{ij} < 0$ means that individual $j$ is competitive to individual $i$. Similarly to [3], we assume that $w_{ii} = 0$ and $w_{ij}w_{ji} \geq 0$ for all $i, j \in \mathcal{V}$, which is called digon sign-symmetry. The Laplacian matrix
of $\mathcal{G}$ is defined as $L = [l_{ij}] \in \mathbb{R}^{N \times N}$, where $l_{ii} = \sum_{j=1}^{N} |w_{ij}|$, $l_{ij} = -w_{ij}, i \neq j$. Specially, the unsigned graph of the signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is expressed as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where the weighted adjacency matrix $\mathcal{W} = [w_{ij}] \in \mathbb{R}^{N \times N}$ with nonnegative elements $w_{ij} = |w_{ij}|, i, j \in \mathcal{V}$, and the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$, in which, $l_{ii} = \sum_{j=1}^{N} w_{ij}$, $l_{ij} = -w_{ij}, i \neq j$. The digraph is quasi-strongly connected if it has at least one root node, where root node has no parent and which has directed paths to all other nodes [18, 20]. A subgraph is in-isolated if there is no edge coming from outside to itself.

**Definition 2.1.** A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is structurally balanced if the node set can be split into two disjoint subsets $\mathcal{V}^+$ and $\mathcal{V}^-$ with the property that $\mathcal{V}^+ \cup \mathcal{V}^- = \mathcal{V}$ and $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$, $w_{ij} > 0$, if $i \in \mathcal{V}^+, j \in \mathcal{V}^+$ (or $i \in \mathcal{V}^-, j \in \mathcal{V}^-$) and $w_{ij} < 0$, if $i \in \mathcal{V}^+, j \in \mathcal{V}^-$ (or $i \in \mathcal{V}^-, j \in \mathcal{V}^+$).

**Lemma 2.2.** [3] The signed graph $\mathcal{G}$ is structurally balanced if and only if there exists a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_N)$, $d_i = \pm 1, i \in \mathcal{V}$, such that $\mathcal{W} = [w_{ij}] = D\mathcal{W}D$ is nonnegative matrix.

If the unsigned graph $\mathcal{G}$ is quasi-strongly connected (has oriented spanning tree), then there exists a nonsingular matrix $U \in \mathbb{R}^{N \times N}$, whose first column is $1_N \triangleq [1, 1, \ldots, 1]^T \in \mathbb{R}^N$, such that [29]

\[
(2.1) \quad U^{-1}\mathcal{L}U = \begin{bmatrix}
0 & \lambda_2 & \epsilon_2 & \cdots & \cdots & \lambda_{N-1} & \epsilon_{N-1} & \lambda_N
\end{bmatrix} \triangleq \begin{bmatrix}
0 & 0 \\
0 & J
\end{bmatrix} \triangleq J_L,
\]

where $\lambda_i, i \in I[2, N] \triangleq \{2, 3, \ldots, N\}$ are the eigenvalues of the Laplacian matrix $\mathcal{L}$, and $\epsilon_i \in \{0, 1\}, i \in I[2, N-1]$. Let $\epsilon_N = 0$. $\mathcal{G}$ is quasi-strongly connected implying that $\text{Re}\{\lambda_i\} > 0, i \in I[2, N]$.

**Lemma 2.3.** (Jensen Inequality [12]) For $P \geq 0$, two scalars $\gamma_1$ and $\gamma_2$ with $\gamma_1 \leq \gamma_2$, and a vector valued function $\omega : [\gamma_1, \gamma_2] \to \mathbb{R}^n$ such that the integrals in the following are well defined, then

\[
\left(\int_{\gamma_1}^{\gamma_2} \omega^T(\beta) \, d\beta\right) P \left(\int_{\gamma_1}^{\gamma_2} \omega(\beta) \, d\beta\right) \leq (\gamma_2 - \gamma_1) \left(\int_{\gamma_1}^{\gamma_2} \omega^T(\beta) P \omega(\beta) \, d\beta\right).
\]
Motivated by the problem that how human memory is encoded and represented in social networks [7] and [16] shows that people makes judgments and choices influenced by individual’s memory (see Fig. 1 for brevity). In this paper, to characterize the effect of human memory in the social networks $G$, we propose a new opinion dynamics model as follows,

$$\dot{x}_i(t) = Ax_i(t) + \int_{-h}^{0} \sigma(\theta) (I_n - A) u_i(t + \theta) \, d\theta,$$

where $x_i(t) \in \mathbb{R}^n$, $i \in I[1, N]$, denotes the opinion of individual $i$ on some topics, $u_i(t) \in \mathbb{R}^n$ is the communication rule of individual $i$, $h > 0$ denotes the memory capacity of the individuals, and $\int_{-h}^{0} \sigma(\theta) \, d\theta = 1$. The matrix $A = [A_{ij}] \in \mathbb{R}^{n \times n}$ describes the ability of cognizance on their topic opinions, which is formed by some endogenous factors and exogenous conditions, for instance, personal intelligence, education level and the social experience. For example, $A_{12} = 1$ means that individual $i$ has very strong cognitive ability on topic 2, and $A_{13} = -1$ implies that individual $i$ has very weak cognitive ability on topic 3. In order to make the ideas in the present paper easy to follow, we consider the following simplified model of (2.2), $i \in I[1, N]$,

$$\dot{x}_i(t) = Ax_i(t) + \sigma \Lambda u_i(t) + (1 - \sigma) \Lambda u_i(t - h),$$

which is seen as the opinion update mechanism with memory, where $\Lambda = I_n - A$. We consider both types of rules, one with memory and the other without memory. The communication rule with memory is given by

$$u_i(t) = \mathcal{F} \sum_{j=1}^{N} |w_{ij}| \left( \text{sgn}(w_{ij}) x_j(t) - x_i(t) \right)$$

$$+ \mathcal{H}(v_i(t - \theta)), \theta \geq 0,$$

(2.4)

which is seen as the communication rule (process) involved the memory capacity, where $u_i(t)$ is defined later in (3.6) and the memoryless communication rule of individual $i$ is
given by

\[(2.5) \quad u_i(t) = \mathcal{H} \sum_{j=1}^{N} |w_{ij}| (\text{sgn}(w_{ij}) x_j(t) - x_i(t)),\]

in which, \(\mathcal{H} \in \mathbb{R}^{n \times n}\) is an ‘opinion adjustment matrix’, \(\mathcal{H}\) is a linear map and \(\text{sgn}(\cdot)\) denotes the signum function. From the opinion dynamics point of view model (2.3) can be explained as follows. The matrix \(\sigma \Lambda \in \mathbb{R}^{n \times n}\) denotes the susceptibility of individual \(i\) to the current interpersonal influence and the matrix \((1 - \sigma)\Lambda \in \mathbb{R}^{n \times n}\) denotes the susceptibility of individual \(i\) to the past interpersonal influence, and the susceptibility coefficient \(\sigma\) satisfies \(0 \leq \sigma \leq 1\). We mention that \(\Lambda = I_n - A\) is to be such that the opinion of individual \(i\) on topic \(p\) is in accord with his initial opinion on topic \(p\) before the individual communicates with his neighbors. In this paper, we let \(x_{ip}(t), p \in \mathcal{I}[1, n]\) denote the opinion on topic \(p\) of individual \(i\). Specially, if \(x_{ip} > 0\), we say that individual \(i\) support topic \(p\), if \(x_{ip} < 0\), we say that individual \(i\) reject topic \(p\), and we say that individual \(i\) remain neutral on topic \(p\) if \(x_{ip} = 0\). In this paper, we not only consider the opinion forming based on the update rule consisting of (2.3) and (2.4), but also consider the opinion forming based on the update rule consisting of (2.3) and (2.5).

Notice that, a \(q\)-voter model with memory was studied in [15], where the social network represented by a complete graph and the model agents are described by a single binary variable. However, in this paper, the social network represented by a signed graph and the model agents are described by the multi-variable. A stochastic model and a hybrid model in opinion dynamics with collective memory were studied in [6] and [22], respectively. Nevertheless, the memory that appears in the models, being a memory of past relations (network connections) [6, 22], which is completely different from that the memory of past opinions in the present paper.

**Definition 2.4.** The opinions in the social networks are equal polarization if there exists \(x^*(t)\), such that \(\lim_{t \to \infty} x_i(t) = x^*(t), i \in \mathcal{V}^+\) and \(\lim_{t \to \infty} x_j(t) = -x^*(t), j \in \mathcal{V}^–\), and moreover, if \(\mathcal{V}^+ = \emptyset\) or \(\mathcal{V}^– = \emptyset\), then equal polarization reduces to consensus. Specially, the opinions are neutralizable if \(\lim_{t \to \infty} x_i(t) = 0, i \in \mathcal{V}\).

**Remark 2.5.** Notice that the opinion is nonstationary equal polarization/consensus in Definition 2.4. However, the opinion is stationary equal polarization/consensus when \(x^*(t) = c\), where \(c\) is a constant, which is a special case of Definition 2.4, and also is the case of Definition 1 in [3].
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Before closing this subsection, we give the following lemma to show the properties between $L$ and $L$, whose proof can be obtained with the help of [38].

**Lemma 2.6.** The eigenvalues of Laplacian matrix $L$ is the same as that of $L$, if the signed graph $G$ is structurally balanced. Moreover, the Jordan canonical form of Laplacian matrix $L$ is the same as (2.1) if the structurally balanced signed graph $G$ is quasi-strongly connected.

3. Communication Rules with Memory

The main purpose of this paper is to analyze the evolution of opinions in social networks by our proposed model and reproduce the results of Kahneman experiments. In order to clearly present the theoretical results and interpretations as social phenomena, all proofs are moved to the Appendix.

In what follows, we will establish the memory communication rule for the opinion dynamics model (2.3) and analyze the evolution of opinions in the social networks. Define a new state vector as

$$z_i(t) = x_i(t) + (1 - \sigma) \int_{t-h}^{t} e^{A(t-h-s)} \Lambda u_i(s) \, ds,$$

where $i \in I[1,N]$. It then follows from (2.3) and (3.1) that

$$\dot{z}_i(t) = Az_i(t) + Au_i(t), \; i \in I[1,N],$$

where

$$A = \sigma \Lambda + (1 - \sigma) e^{-Ah} \Lambda.$$

Then the opinion communication of each individual obeys the following rules,

$$u_i(t) = F \sum_{j \in N_i} |w_{ij}| (sgn(w_{ij}) z_j(t) - z_i(t))$$

where $i \in I[1,N]$ and $F$ is referred to as an ‘opinion adjustment matrix’. It yields from (3.3) that

$$u_i(t) = F \left( \chi_i(t) + (1 - \sigma) \int_{t-h}^{t} e^{A(t-h-s)} \Lambda v_i(s) \, ds \right),$$

where $i \in I[1,N]$,

$$\chi_i(t) = \sum_{j=1}^{N} |w_{ij}| (sgn(w_{ij}) x_j(t) - x_i(t)) = - \sum_{j=1}^{N} l_{ij} x_j(t),$$
and

$$(3.6) \quad v_i(t) = \sum_{j=1}^{N} |w_{ij}| \left( \text{sgn}(w_{ij}) u_j(t) - u_i(t) \right) = -\sum_{j=1}^{N} l_{ij} u_j(t).$$

For simplicity, we define the set of matrices as

$$S = \{ S : \lambda(S) \subseteq \mathbb{C}_{\leq 0}, s \text{ is semi-simple}, \forall s \in \lambda(S) \cap i\mathbb{R} \neq \emptyset \},$$

where $\lambda(S)$ denotes the spectrum of a square matrix $S$, $\mathbb{C}_{\leq 0}$ denotes the set of negative complex numbers and 0, a semi-simple eigenvalue possesses equal algebraic and geometric multiplicities, $i^2 = -1$ and $i\mathbb{R}$ denotes the imaginary axis. Now, we have the following new theorem.

**Theorem 3.1.** Consider the opinion dynamic model (2.3) in the coopetitive social network $G$. The individuals’ opinions are equal polarization under the communication rule (3.3) if the following conditions are satisfied: 1) $G$ is structurally balanced and quasi-strongly connected; 2) $A \in S$; 3) There exists an ‘opinion adjustment matrix’ $F$ such that $A - \lambda_iAF, i \in I[2, N]$ are Hurwitz.

Regarding the conditions in Theorem 3.1, we give some discussions and social interpretations as follows. In condition 1), the structurally balanced property means that the social community divides into two hostile camps, the positive influence $w_{ij} > 0$ implies that the individuals come from the same camp and cooperate with each other in the social ties, whereas the negative influence $w_{ij} < 0$ denotes the individuals compete with each other in the social ties. In addition, quasi-strongly connectivity implies that there is at least one individual who can transmit directly or indirectly his/her opinions to the remaining individuals. Condition 2) not only emphasizes that the individuals’ dynamical properties are important, but also eliminates the case that opinions are neutralizable and unbounded. Finally, condition 3) guarantees that the individual is open to interpersonal influence.

The social interpretations of Theorem 3.1 can be summarized as below. In the social community with two hostile camps, if there is at least one direct/indirect transmission line from the one individuals to the remaining individuals and the opinions of individuals influence each other. Then some individuals support a topic and some one reject a topic under the memory communication rule. According to Theorem 3.1, we have the following new corollary whose proof is similar to that of Theorem 3.1, thus is omitted.
Corollary 3.2. For opinion dynamics model (2.3) in the cooperative social network $\mathcal{G}$, the individuals’ opinions achieve consensus under the communication rule (3.3) if $\mathcal{G}$ is quasi-strongly connected, $A \in \mathcal{S}$ and there exists an ‘opinion adjustment matrix’ $F$ such that $A - \lambda_i A F, i \in [2, N]$ are Hurwitz.

Corollary 3.2 means that, under the memory communication rules, all individuals support or reject a topic in the cooperative social community if there is at least one direct/indirect transmission line from the one individuals to the remaining individuals and the opinions of individuals are influenced by others. Necessary and sufficient conditions guaranteeing the neutralizability of the opinions are given by the following new theorem.

Theorem 3.3. For opinion dynamic model (2.3) in cooperative social network $\mathcal{G}$. The individuals’ opinions are neutralizable under the communication rule (3.3) if and only if $\mathcal{G}$ does not involve an in-isolated structurally balanced subgraph and $(A, A)$ is controllable.

The meaning of Theorem 3.3 is that, under the memory communication rule, the individuals’ opinions are neutralizable in the both cooperative and competitive if the opinions of individuals influence each other and the social community in the absence of two hostile camps.

4. Memoryless Communication Rules

In order to establish the memoryless communication rules, we first let the opinion adjustment matrix $F$ be parameterized as $F = F(\gamma) : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n}$ [39] and such that

$$\lim_{\gamma \downarrow 0} \frac{1}{\gamma} \|F(\gamma)\| < \infty. \quad (4.1)$$

Hence, the communication rules $u_i(t)$ is “of order 1” with respect to $\gamma$. As a result, we can get that the second term $F \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-h-s)} A u_j(s) \, ds$ in (3.4) are at least “of order 2” with respect to $\gamma$, which means that the term $F \sum_{j=1}^{N} l_{ij} \int_{t-h}^{t} e^{A(t-h-s)} A u_j(s) \, ds$ is dominated by the term $F \chi_i(t)$ in (3.4), and thus might be safely neglected in $u_i(t)$ when $\gamma$ is sufficiently small. Hence, the memoryless communication rules can be represented as

$$u_i(t) = F(\gamma) \chi_i(t). \quad (4.2)$$

The following assumption guarantees that there exists a opinion adjustment matrix $F(\gamma)$ satisfying (4.1).

Assumption 1. The matrix pair $(A, A)$ is controllable, and all the eigenvalues of $A$ are on the imaginary axis.
If Assumption 1 is satisfied, the following parametric algebraic Riccati equation (ARE) (4.3)
\[ A^T P + PA - PA A^T P = -\gamma P, \]
exists a unique positive definite solution \( P(\gamma) \), \( \forall \gamma > 0 \), and \( F = -A^T P(\gamma) \) satisfy (4.1).

**Theorem 4.1.** Consider opinion dynamics model (2.3) with Assumption 1 in the competitive social networks \( \mathcal{G} \). Let \( P(\gamma) \) be the unique solution of (4.3) and define \( F = -A^T P(\gamma) \). Then, for any \( h > 0 \) and \( \varrho \geq \max_{i \in [2,N]} \{1/\text{Re}\{\lambda_i}\} \), there exists a number \( \gamma^* = \gamma^*(\varrho, h, \{\lambda_i\}_{i=2}^N) \) such that the individuals’ opinions are equal polarization by the memoryless communication rules (4.4)
\[ u_i(t) = \varrho F(\gamma) \chi_i(t), \gamma \in (0, \gamma^*), \]
if the signed graph \( \mathcal{G} \) is structurally balanced and is quasi-strongly connected.

This new theorem implies that, there exist possibilities of opinion adjustment such that some individuals support a topic and others reject a topic under memoryless communication rules if the social community can be divided into two hostile camps. With Theorem 4.1, we have a new corollary as follows.

**Corollary 4.2.** Consider opinion dynamics model (2.3) with Assumption 1 in the cooperative social network \( \mathcal{G} \). Then, for any \( h > 0 \) and \( \varrho \geq \max_{i \in [2,N]} \{1/\text{Re}\{\lambda_i}\} \), there exists a number \( \gamma^* = \gamma^*(\varrho, h, \{\lambda_i\}_{i=2}^N) \), such that the individuals’ opinions achieve consensus for all \( \gamma \in (0, \gamma^*) \) by the memoryless communication rule (4.4) if \( \mathcal{G} \) is quasi-strongly connected.

Corollary 4.2 means that, in the cooperative social community, there exist possibilities of opinion adjustment such that all individuals support or reject a topic under the memoryless communication rules. By Theorem 4.1, necessary and sufficient conditions guaranteeing the neutralizability of the opinions for the proposed opinion dynamics model (2.3) are given as follows.

**Theorem 4.3.** Consider opinion dynamics model (2.3) with Assumption 1 in competitive social networks \( \mathcal{G} \). Let \( P(\gamma) \) be unique solution of (4.3). Then, for any \( h > 0 \) and \( \varrho \geq \max_{i \in [2,N]} \{1/\text{Re}\{\lambda_i}\} \), there exists a number \( \gamma^* = \gamma^*(\varrho, h, \{\lambda_i\}_{i=2}^N) \), such that the individuals’ opinions are neutralizable for all \( \gamma \in (0, \gamma^*) \) by the memoryless communication rule (4.4) if and only if \( \mathcal{G} \) does not involve an in-isolated structurally balanced subgraph.
Theorem 4.3 means that there exist possibilities of opinion adjustment so that individual opinions are neutralizable under the memoryless communication rules if the social community in the absence of two hostile camps.

Remark 4.4. Compared with [18], this section possesses some differences and novelty. On the one hand, we would like to emphasize that the opinion dynamics model (2.3) involves memory \( h \), and the model in this paper is more complex. However, the opinion dynamics model in absence of the memory in [18]. On the other hand, in this paper, the opinion adjustment matrix \( F(\gamma) \) of the memoryless communication rule (process) is a variable with respect to \( \gamma \), which is easily adjusted in the real world.

Remark 4.5. We will give a method to estimate the upper bound of \( \gamma^* \) as follows. According to the proof of Theorem 4.1, we have

\[
\dot{\eta}(t) = A\eta(t) + \sigma \Lambda F \eta(t) + (1 - \sigma) \Lambda F \eta(t-h),
\]

is asymptotically stable, where \( F = -gA^T P(\gamma) \). The stability of the above equation is completely determined by the rightmost roots of its characteristic equation

\[
c(s,\gamma) = \det\left(sI_n - A - \sigma \Lambda F - (1 - \sigma) \Lambda F e^{-sh}\right).
\]

Let

\[
\lambda_{\text{max}}(\gamma) = \max\{\text{Re}\{s\} : c(s,\gamma) = 0\},
\]

which can be efficiently computed by the software package DDE-BIFTOOL [10]. Then the (35) is asymptotically stable if and only if \( \lambda_{\text{max}}(\gamma) < 0 \). The maximal \( \gamma \) such that \( \lambda_{\text{max}}(\gamma) = 0 \) is defined as \( \gamma^* \).

5. Application to Kahneman’s Experiments

In this section, we will use our proposed model to revisit Kahneman’s seminal experiments on choices in risky and riskless contexts [17]. To totally appreciate and understand the results, we present a brief overview of the famous experiments. Assume that a city is preparing for an outbreak of an unusual disease (such as, COVID-19), and COVID-19 is expected to kill 600 people. Two alternative programs to combat COVID-19 have been proposed, wherein the “lives saved version” (LSV) of the scientific estimates of the consequences for the proposed programs are given as follows.

- **A1**: If Program 1 is adopted, 200 people will be saved.
- **B1**: If Program 2 is adopted, the probability of 600 people being saved is \( \frac{1}{3} \), and the probability of no one being saved is \( \frac{2}{3} \).
Figure 2. Left: Structurally balanced community. Right: Structurally unbalanced community.

Figure 3. Equal polarization of opinion dynamics model consisting of (2.3) and (3.3) in the coopetitive community.

The “lives lost version” (LLV) of the scientific estimates of the consequences for the proposed programs are stated as below, which is undistinguishable in real terms from the “LSV” [17].

- A2: If Program 1 is adopted, 400 people will die.
- B2: If Program 2 is adopted, the probability that no one will die is 1/3, and the probability that 600 people will die is 2/3.

The percentage who chose each option is given in Table 1 after each program [17]. It is interesting to see from Table 1 that the consequences of the “LSV” is completely different from the consequences of the “LLV” even A1 and B1 are undistinguishable from A2 and B2, respectively.
In the following, Kahneman’s seminal experiments are simulated by the proposed opinion dynamics model with the memory communication rules. Let the structurally balanced social community network be given in Fig. 2 (Left), and Jordan, Kevin, Alisa, Amy, James, Anni, David, Peter are assigned labels 1, 2, 3, 4, 5, 6, 7 and 8, respectively. In this particular case, cooperative link means that the individuals come from government officials or well-known doctors, whereas competitive link means that one comes from government officials and another one comes from well-known doctors, and memory information means that the past choices (opinions) of the government officials and well-known doctors in risky and riskless contexts. Let $h = 1$, $\sigma = 0.6$, and $A$ is given by

$$A = \begin{bmatrix}
-0.5 & 0.25 \\
0 & 0
\end{bmatrix}.$$ 

For simplicity, without loss of generality, we explain the meaning of matrix $A$ by A1 and B1 of “LSV” as follows. $A_{11} = -0.5$ means that individual $i$ lacks cognitive ability on A1, $A_{12} = 0.25$ means that individual $i$ possesses cognitive ability on B1 before the individual communicates with his neighbors. The corresponding matrices $A$ and $F$ are respectively computed as

$$A = \begin{bmatrix}
2.4731 & -0.7365 \\
0 & 1
\end{bmatrix}, F = \begin{bmatrix}
-0.0615 & -0.4059 \\
0 & -1.1564
\end{bmatrix},$$

where $F$ is such that $\lambda(A - 1.6434F) = \{-0.75, -1.9\}$.

Let the initial opinions for the six individuals be $x_1(0) = [-0.5, 0.5]^T$, $x_2(0) = [-0.3, 0.3]^T$, $x_3(0) = [-0.4, 0]^T$, $x_4(0) = [-0.8, 0.2]^T$, $x_5(0) = [-0.4, 0.4]^T$, $x_6(0) = [-0.3, 0.6]^T$, $x_7(0) = [0.4, -0.3]^T$ and $x_8(0) = [0.5, 0.6]^T$. The simulation results are recorded in Fig. 3, where $x_{i1}$ denotes the opinion of individual $i$ on A1 and $x_{i2}$ denotes the opinion of individual $i$ on B2.

Clearly, according to the left of Fig. 3, six individuals support A1, where the percentage who chose A1 is $\frac{6}{8} = 75\%$, which almost fits with the percentage 72% who chose A1 in Table 1. In addition, two individuals reject A1, namely, two individuals support B1.

### Table 1: The results of the experiments for different versions

| Different Versions | LSV | LLV |
|--------------------|-----|-----|
| Different Programs | A1  | B1  |
| Percentages       | 72% | 28% |
|                   | A2  | B2  |
| Percentages       | 22% | 78% |

In the following, Kahneman’s seminal experiments are simulated by the proposed opinion dynamics model with the memory communication rules. Let the structurally balanced social community network be given in Fig. 2 (Left), and Jordan, Kevin, Alisa, Amy, James, Anni, David, Peter are assigned labels 1, 2, 3, 4, 5, 6, 7 and 8, respectively. In this particular case, cooperative link means that the individuals come from government officials or well-known doctors, whereas competitive link means that one comes from government officials and another one comes from well-known doctors, and memory information means that the past choices (opinions) of the government officials and well-known doctors in risky and riskless contexts. Let $h = 1$, $\sigma = 0.6$, and $A$ is given by

$$A = \begin{bmatrix}
-0.5 & 0.25 \\
0 & 0
\end{bmatrix}.$$ 

For simplicity, without loss of generality, we explain the meaning of matrix $A$ by A1 and B1 of “LSV” as follows. $A_{11} = -0.5$ means that individual $i$ lacks cognitive ability on A1, $A_{12} = 0.25$ means that individual $i$ possesses cognitive ability on B1 before the individual communicates with his neighbors. The corresponding matrices $A$ and $F$ are respectively computed as

$$A = \begin{bmatrix}
2.4731 & -0.7365 \\
0 & 1
\end{bmatrix}, F = \begin{bmatrix}
-0.0615 & -0.4059 \\
0 & -1.1564
\end{bmatrix},$$

where $F$ is such that $\lambda(A - 1.6434F) = \{-0.75, -1.9\}$.

Let the initial opinions for the six individuals be $x_1(0) = [-0.5, 0.5]^T$, $x_2(0) = [-0.3, 0.3]^T$, $x_3(0) = [-0.4, 0]^T$, $x_4(0) = [-0.8, 0.2]^T$, $x_5(0) = [-0.4, 0.4]^T$, $x_6(0) = [-0.3, 0.6]^T$, $x_7(0) = [0.4, -0.3]^T$ and $x_8(0) = [0.5, 0.6]^T$. The simulation results are recorded in Fig. 3, where $x_{i1}$ denotes the opinion of individual $i$ on A1 and $x_{i2}$ denotes the opinion of individual $i$ on B2.

Clearly, according to the left of Fig. 3, six individuals support A1, where the percentage who chose A1 is $\frac{6}{8} = 75\%$, which almost fits with the percentage 72% who chose A1 in Table 1. In addition, two individuals reject A1, namely, two individuals support B1.
Figure 4. Equal polarization of opinion dynamics model consisting of (2.3) and (4.4) in the coopetitive community.

Figure 5. Consensus of opinion dynamics model consisting of (2.3) and (4.4) in the cooperative community.

where the percentage who chose B1 is $\frac{2}{8} = 25\%$, which almost fits with the percentage 28% who chose B1 in Table 1.

Similarly, in terms of the right of Fig. 3, six individuals support B2, where the percentage who chose B2 is $\frac{6}{8} = 75\%$, which almost fits with the percentage 78% who chose B2 in Table 1. In addition, two individuals reject B2, namely, two individuals support A2, where the percentage who chose A2 is $\frac{2}{8} = 25\%$, which almost fits with the percentage 22% who chose A2 in Table 1. Therefore, the simulation results in Fig. 3 nearly fit in Kahneman experiment.

6. Simulation Analysis

In this section, two numerical examples are worked out to support the obtained theoretical results and analyse the influences of the memory capacity. Consider a paradigmatic social community consisting of eight individuals, which is shown in Fig. 2.
Example 1: This example demonstrates the evolution of the individual opinions on different topics under the proposed memoryless communication rule (4.4). Matrix $A$ is given by

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}.$$  

Let $h = 1$ and $\sigma = 0.6$. Then, we have

$$\Lambda = \begin{bmatrix} 1 & -1 \\ 0.5 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7204 & -1.2716 \\ 0.6358 & 0.7204 \end{bmatrix}.$$  

We choose $\varrho = 1/0.2 > \max_{i \in [2,N]} \{1/\Re\{\lambda_i\}\}$, by solving the parametric ARE (4.3), and with the help of (4.4), we have

$$F = (1/0.2) \begin{bmatrix} -0.0688 & -0.1336 \\ 0.1224 & -0.1525 \end{bmatrix},$$

where $\gamma = 0.2$, in which $\gamma^*$ can be estimated by Remark 4.5.

Example 2: This example studies the evolution of opinions affected by the memory capacity $h$ under the proposed memory communication rule (3.3). For convenience of description, we only consider the influence of the memory capacity on topic 1. Let $h = 15$ (half a month) and $h = 30$ (a month). The corresponding matrices $A$ are respectively computed as

$$A = \begin{bmatrix} 1.0857 & -0.5424 \\ 0 & 0.0010 \end{bmatrix} \times 10^3,$$

and

$$A = \begin{bmatrix} 1.9614 & -0.9807 \\ 0 & 0 \end{bmatrix} \times 10^6.$$
Figure 7. Opinions formation processes are affected by $h$ (Equal polarization).

Figure 8. Consensus formation processes are affected by $h$.

Figure 9. Opinions formation processes are affected by $h$ (Neutralization).

Let the initial opinions be the same as that in Section 5. The process of opinion evolution are shown in Fig. 7, Fig. 8 and Fig. 9, by which, it is clear to see that the memory capacity $h$ of the individuals is inversely proportional to the speeds of the ultimate opinions formation. Moreover, we can see that, if the memory capacity $h$ is contained in both the
opinion update mechanism and the communication rule (process), then the opinions is stationary. If the memory capacity $h$ is contained only in the opinion update mechanism, then the opinions is nonstationary (see Fig. 4, Fig. 5 and Fig. 6 for details).

In addition, although the coopetitive social network $G$ is structurally balanced and quasi-strongly connected. Under the memory communication rules, it is interesting to see from Fig. 10 that the individuals’ opinions are neutralizable if $x_8(0) = [0.5, 0.6]^T$ is replaced by $x_8(0) = [0.5, -0.6]^T$, where $h = 1$.

7. Conclusion

This paper proposed a novel model of opinion dynamics on coopetitive social networks to describe the evolution of the individuals’ opinions depending on its own and neighbors’ current opinions and past opinions, wherein the cooperative and competitive interactions coexist. Memory and memoryless communication rules have been given to analysis the evolution of opinions in the coopetitive social networks, respectively. Sufficient and/or necessary conditions guaranteeing the equal polarization, consensus and neutralizability of the opinions were obtained on the basis of the network topological structure and the spectral analysis. We use the proposed model to revisit Kahneman’s seminal experiments on choices in risky and riskless contexts, and to reveal the insightful social interpretations. By the simulation analysis, it was shown that the memory capacity of the individuals was inversely proportional to the speeds of the ultimate opinions formational. In addition, the opinions of all the individuals on different topics were stationary under the memory communication rule and the opinions of all the individuals on different topics were nonstationary under the memoryless communication rule.
In the future works, we will consider the number of the individuals coming from a camp by the corresponding Laplacian matrix. Moreover, the heterogeneous opinion dynamics model is another interesting topic and a system of more nodes will be discussed in our future study.

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Appendix

A1. Proof of Theorem 3.1

According to (3.5) and (3.6), (3.4) can be represented as

\[ v_i(t) = -F \sum_{j=1}^{N} l_{ij}(\chi_j(t) + (1 - \sigma) + \int_{t-h}^{t} e^{A(t-s)} \Lambda v_j(s) \, ds), \quad i \in [1, N]. \]  

(7.1)

Let \( \kappa \) and \( \mu \) be defined as

\[ \kappa = [\kappa_1^T, \kappa_2^T, \ldots, \kappa_N^T]^T \triangleq (DU)^{-1} \otimes I_n \chi, \]

and

\[ \mu = [\mu_1^T, \mu_2^T, \ldots, \mu_N^T]^T \triangleq (DU)^{-1} \otimes I_n v, \]

where \( \chi = [\chi_1^T, \chi_2^T, \ldots, \chi_N^T]^T \), \( v = [v_1^T, v_2^T, \ldots, v_N^T]^T \) and \( \otimes \) denotes the Kronecker product. The signed graph \( \mathcal{G} \) is structurally balanced and quasi-strongly connected. Hence, by Lemma 2.6, (2.3) and (7.1) can be rewritten as, respectively,

\[ \dot{\kappa}(t) = (I_N \otimes A) \kappa(t) + (\sigma I_N \otimes \Lambda) \mu(t) + ((1 - \sigma) I_N \otimes \Lambda) \mu(t - h), \]
and
\[
\mu (t) = - (J_T \otimes F) \kappa (t) - \left( (1 - \sigma) J_T \otimes F \int_{t-h}^t e^{A(t-h-s)} \Lambda \mu (s) \, ds \right).
\]

It follows from (2.1) that \( \dot{\kappa}_1 (t) = A \kappa_1 (t) \) and

\[
\begin{aligned}
\dot{\alpha} (t) &= (I_{N-1} \otimes A) \alpha (t) + (I_{N-1} \otimes \sigma \Lambda) \nu (t) \\
&\quad + (I_{N-1} \otimes (1 - \sigma) \Lambda) \nu (t-h), \\
\nu (t) &= -(J \otimes F) \varsigma (t),
\end{aligned}
\]

where \( \alpha \triangleq [\kappa_2^T, \kappa_3^T, \ldots, \kappa_N^T]^T \), \( \nu \triangleq [\mu_2^T, \mu_3^T, \ldots, \mu_N^T]^T \) and

\[
\varsigma (t) = \alpha (t) + (1 - \sigma) \int_{-h}^0 e^{A(-h-\theta)} \Lambda \nu (t+\theta) \, d\theta.
\]

Let \( \mathcal{L} (f (t)) \) denote the Laplace transformation of the time function \( f (t) \). Then,

\[
\mathcal{L} (\varsigma (t)) = X (s) + \left( (1 - \sigma) \int_{-h}^0 e^{A(-h-\theta)} \Lambda \nu (t+\theta) \, d\theta \right) V (s)
\]

\[
= X (s) + \varrho V (s),
\]

where \( \mathcal{L} (\alpha (t)) = X (s) \), \( \mathcal{L} (\nu (t)) = V (s) \) and \( \varrho = (1 - \sigma) (sI_n - A)^{-1} (e^{-Ah} - e^{-sh}) \Lambda \). Therefore, (7.2) can be written in the frequency domain as

\[
(7.3)
\begin{bmatrix}
\Omega_0 & \Omega_1 \\
J \otimes F & \Omega_2
\end{bmatrix}
\begin{bmatrix}
X (s) \\
V (s)
\end{bmatrix} = 0,
\]

where

\[
\begin{aligned}
\Omega_0 &= I_{N-1} \otimes (sI_n - A), \\
\Omega_1 &= -I_{N-1} \otimes (\sigma \Lambda + (1 - \sigma) \Lambda e^{-sh}), \\
\Omega_2 &= I_{N-1} \otimes I_n + (J \otimes F \varrho).
\end{aligned}
\]

It yields that the characteristic equation of (7.3) is

\[
\Delta (s) = \det \begin{bmatrix} \Omega_0 & \Omega_1 \\ J \otimes F & \Omega_2 \end{bmatrix} = 0.
\]

Notice that

\[
J \otimes F (sI_n - A)^{-1} \Omega_1 + \Omega_2 = \Omega_3,
\]
where $\Omega_3 = I_{N-1} \otimes I_n + (J \otimes F_0 (\sigma + (1-\sigma)e^{-Ah}) A)$, in which $\varrho_0 = (sI_n - A)^{-1}$. It follows that

$$\Delta (s) = \det \begin{bmatrix} I_{N-1} \otimes I_n & 0 \\ -(J \otimes F_0) & I_{N-1} \otimes I_n \end{bmatrix} \begin{bmatrix} \Omega_0 & \Omega_1 \\ 0 & \Omega_3 \end{bmatrix}$$

$$= \det (\Omega_0) \det (\Omega_3)$$

$$= \det (\Omega_0 (I_{N-1} \otimes I_n + (J \otimes \varrho_0 A))$$

$$= \det (I_{N-1} \otimes (sI_n - A) + J \otimes AF)$$

$$= \prod_{i=2}^{N} |sI_n - (A - \lambda_i AF)|,$$

where we have used $\det(I_n + CE) = \det(I_n + EC)$ with $C$ and $E$ having appropriate dimensions. In view of $A + \lambda_i AF$ are Hurwitz, then we have

$$\lim_{t \to \infty} \kappa_i (t) = 0, i \in I[2, N].$$

It follows from Lemma 2.6, (2.1) and (3.5) that

$$\kappa (t) = J_\mathcal{T} \varphi (t),$$

where $\varphi \triangleq [\varphi_1^T, \varphi_2^T, \ldots, \varphi_N^T] \triangleq ((DU)^{-1} \otimes I_n) x$, in which, $x \triangleq [x_1^T, x_2^T, \ldots, x_N^T]^T$. It yields from (2.1) and (7.5) that $\kappa_1 (t) = 0$ and

$$\kappa_i (t) = \lambda_i \varphi_i (t) + \epsilon_i \varphi_{i+1} (t), i \in I[2, N],$$

where $\varphi_{N+1} (t) = 0$. By (7.4), we can further get

$$\lim_{t \to \infty} \varphi_i (t) = 0, i \in I[2, N].$$

Since the first column of $U$ is $1_N$ and $x (t) = (DU \otimes I_n) \varphi (t) = (U \otimes I_n)(D \otimes I_n) \varphi (t)$. Then,

$$\lim_{t \to \infty} \|x_i (t) - d \varphi_1 (t)\| = 0, i \in I[1, N],$$

which means that the individuals’ opinions are polarizable. We point out that the condition $A \in \mathcal{S}$ is such that $\lim_{t \to \infty} x_i (t) = c$ for nonzero initial opinions, where $|c| \neq 0$ is a bounded real number.

**A2. Proof of Theorem 3.3**
By virtue of the proof of Theorem 3.1, the opinion dynamics network consisting of dynamic model (2.3) and the memory communication rule (3.3) can be written as

$$\dot{x}(t) = (I_N \otimes A - \sigma L \otimes \Lambda F - (1 - \sigma) L \otimes e^{-Ah} \Lambda F) x(t),$$

(7.6)

by which, it is clear to see that the individuals’ opinions in the coopetitive social networks are neutralizable if and only if

$$\text{Re}\{ (I_N \otimes A - \sigma L \otimes \Lambda F - (1 - \sigma) L \otimes e^{-Ah} \Lambda F) \} = \text{Re}\{ (A - \varepsilon A \Lambda) \} < 0,$$

where $\varepsilon \in \lambda(L)$, in which, $\lambda(L)$ denotes the eigenvalue of $L$.

Sufficiency: Since $G$ does not involve an in-isolated structurally balanced subgraph, it follows that $0 \notin \lambda(L)$. In addition, notice that $(A, \Lambda)$ is controllable, then there exists a opinion adjustment matrix $F$ such that $A - \varepsilon A \Lambda, \forall \varepsilon \in \lambda(L)$ is Hurwitz. Therefore, the individuals’ opinions are neutralizable.

Necessity: If individuals’ opinions are neutralizable, then we have $\text{Re}\{ A - \varepsilon A \Lambda \} < 0, \forall \varepsilon \in \lambda(L)$. We assume that $G$ is structurally balanced or involves an in-isolated structurally balanced subgraph, which leads to $0 \in \lambda(L)$. However, $\text{Re}\{ A - \varepsilon A \Lambda \} < 0$ for $\varepsilon = 0$ implies $A$ is a Hurwitz matrix, which is contradictory to the fact that $A$ is not Hurwitz. Hence, we derive the necessity that $G$ does not involve an in-isolated structurally balanced subgraph. By adopting the same method presented in [21] to display that $(A, \Lambda)$ is controllable.

A3. A Useful Lemma

Lemma 7.1. Let $F = -\Lambda^T P(\gamma)$ and $P(\gamma)$ be unique positive definite solution to the parametric ARE

$$A^T P + PA - P A A^T P = -\gamma P.$$  

(7.7)

Then there exists a scalar $\gamma^* > 0$ such that the following closed-loop system

$$\left\{ \begin{array}{l} \dot{\eta}(t) = A\eta(t) + \sigma \Lambda u(t) + (1 - \sigma) \Lambda u(t - h), \\ u(t) = \mathcal{F}\eta(t), \end{array} \right.$$  

(7.8)

is asymptotically stable for all $\gamma \in (0, \gamma^*)$, where $\mathcal{F} = \lambda \varphi F$, $\varphi \geq 1/\text{Re}\{\lambda\}$ and $\text{Re}\{\lambda\} > 0.$
Proof. Let

\[ z(t) = \eta(t) + (1 - \sigma) \int_{t-h}^{t} e^{A(t-h-s)} A u(s) \, ds \]
\[ \triangleq \eta(t) + \psi(t). \]  

(7.9)

It then follows from (7.8) and (7.9) that

\[ \dot{z}(t) = Az(t) + Au(t), \]

(7.10)

where \( A = \sigma \Lambda + (1 - \sigma) e^{-Ah} \Lambda \). It yields from (7.10), (7.9) and the second equation in (7.8) that

\[ \dot{z}(t) = (A + AF) z(t) - AF \psi(t). \]

(7.11)

We now consider the Lyapunov function

\[ V_1(z(t)) = z^T(t) P z(t). \]

Therefore, for any given \( \gamma \in (0, \gamma_0) \), where \( \gamma_0 > 0 \), we get

\[ \dot{V}_1(z(t)) = z^T(t) \left( (A + AF)^T P + P(A + AF) \right) z(t) \]
\[ - 2z^T(t) P AF \psi(t) \]
\[ = z^T(t) \left( PA \Lambda^T P - \lambda_0 P \Lambda^T P - \gamma P - \lambda_0 P A \Lambda^T P \right) z(t) + 2\lambda_0 \gamma z^T(t) P A \Lambda^T P \psi(t) \]
\[ \leq z^T(t) \left( - \| \lambda \| \gamma P A \Lambda^T P - \gamma P \right) z(t) + \| \lambda \| \gamma z^T(t) P A \Lambda^T P z(t) + \| \lambda \| \gamma \psi^T(t) P A \Lambda^T P \psi(t) \]
\[ \leq - \gamma z^T(t) P z(t) + \gamma \| \lambda \| \gamma \psi^T(t) P \psi(t), \]

(7.11)

where we have used \( P A \Lambda^T P \leq n \gamma P \). By virtue of

\[ z^T(t) P z(t) = \eta^T(t) P \eta(t) + 2\eta^T(t) P \psi(t) + \psi^T(t) P \psi(t), \]

and

\[ \frac{1}{2} \gamma \eta^T(t) P \eta(t) + 2\gamma \psi^T(t) P \psi(t) \]
\[ = \left( \sqrt{\gamma} \frac{1}{2} P \frac{1}{2} \eta(t) \right)^T \left( \sqrt{\gamma} \frac{1}{2} P \frac{1}{2} \eta(t) \right) + \left( \sqrt{2} \gamma \frac{1}{2} P \frac{1}{2} \psi(t) \right)^T \left( \sqrt{2} \gamma \frac{1}{2} P \frac{1}{2} \psi(t) \right) \]
\[ \geq 2 \left( \sqrt{\gamma} \frac{1}{2} P \frac{1}{2} \eta(t) \right)^T \left( \sqrt{2} \gamma \frac{1}{2} P \frac{1}{2} \psi(t) \right) \]
\[ = 2 \gamma \eta^T(t) P \psi(t), \]

(7.12)
it follows from (7.11) and (7.12) that

\[
\dot{V}_1(z(t)) \leq -\gamma \eta^T(t) P\eta(t) - \gamma \psi^T(t) P\psi(t) + \frac{1}{2} \gamma \eta^T(t) P\eta(t) + 2 \gamma \psi^T(t) P\psi(t)
\]

\[
+ \gamma \|\lambda\| \eta \psi^T(t) P\psi(t)
\]

(7.13)

\[
= -\frac{1}{2} \gamma \eta^T(t) P\eta(t) + (1 + \|\lambda\| \eta) \gamma \psi^T(t) P\psi(t).
\]

According to Lemma 2.3, we have

\[
\psi^T(t) P\psi(t) = (1 - \sigma)^2 \left( \int_{t-h}^t e^{A(t-h-s)} \Lambda u(s) \, ds \right)^T P \left( \int_{t-h}^t e^{A(t-h-s)} \Lambda u(s) \, ds \right)
\]

\[
\leq (1 - \sigma)^2 h \int_{t-h}^t u^T(s) \Lambda^T e^{AT(t-h-s)} P e^{A(t-h-s)} \Lambda u(s) \, ds
\]

(7.14)

\[
\leq (1 - \sigma)^2 h \delta_0 \int_{t-h}^t u^T(s) u(s) \, ds,
\]

where \( \delta_0 = \|\Lambda\|^2 \cdot \|e^{Ah}\|^2 \|P\| \). We note that

\[
u^T(s) u(s) = (\lambda \eta)^T(s) P \Lambda \Lambda^T P \eta(s)
\]

\[
\leq \|\lambda\|^2 \eta^T(s) P \eta(s).
\]

Hence, equation (7.14) can be written as

(7.15)

\[
\psi^T(t) P\psi(t) \leq \delta_1 \gamma \int_{t-h}^t \eta^T(s) P\eta(s) \, ds,
\]

where \( \delta_1 = (1 - \sigma)^2 \|\lambda\|^2 \rho \eta n \delta_0 \). Substituting (7.15) into (7.13) gives

\[
\dot{V}_1(z(t)) \leq -\frac{1}{2} \gamma \eta^T(t) P\eta(t) + \delta_2(\gamma) \int_{t-h}^t \eta^T(s) P\eta(s) \, ds,
\]

where \( \delta_2(\gamma) = (1 + \|\lambda\| \eta n) \delta_1 \gamma^2 \). Now, we choose another two Lyapunov functional as

\[
V_2(\eta_t) = \delta_2(\gamma) \int_0^h \left( \int_{t-s}^t \eta^T(l) P\eta(l) \, dl \right) ds,
\]

and

\[
V_3(\eta_t) = 2 \delta_1 \gamma^2 \int_{t-h}^t \eta^T(s) P\eta(s) \, ds.
\]
It follows that
\[
\dot{V}(\eta_t) = \dot{V}_1(z(t)) + \dot{V}_2(\eta_t) + \dot{V}_3(\eta_t)
\leq -\frac{1}{2} \gamma \eta^T(t) P \eta(t) + \delta_2(\gamma) \int_{t-h}^{t} \eta^T(s) P \eta(s) \, ds + \delta_2(\gamma) h \eta^T(t) P \eta(t) + 2\delta_1 \gamma^2 \eta^T(t) P \eta(t)
- \delta_2(\gamma) \int_{0}^{h} \eta^T(t-s) P \eta(t-s) \, ds - 2\delta_1 \gamma^2 \eta^T(t-h) P \eta(t-h)
\leq -\left(\frac{1}{2} - h (1 + \|\lambda\| \gamma n) \delta_1 \gamma - 2\delta_1 \gamma\right) \gamma \eta^T(t) P \eta(t).
\]

Let \(\gamma^* \in (0, \gamma_0)\) be such that
\[
\frac{1}{2} - h (1 + \|\lambda\| \gamma n) \delta_1 \gamma - 2\delta_1 \gamma \geq \frac{1}{4}, \gamma \in (0, \gamma^*).
\]
It then yields that
\[
\dot{V}(\eta_t) \leq -\frac{1}{4} \gamma \eta^T(t) P \eta(t) \leq -\frac{1}{4} \delta \gamma^{1+a} \|\eta(t)\|^2,
\]
where we have used there exists a constant \(\delta > 0\) and an integer \(a \geq 1\) such that \(P \geq \delta \gamma^a I_n\).

Clearly, there exists a constant \(\delta_4 > 0\) such that \(V(\eta_t) \leq \delta_4 \|\eta(t)\|^2\). In the following, we will show that there exists a constant \(\delta_3 > 0\) such that \(V(\eta_t) \geq \delta_3 \|\eta(t)\|^2\). Notice that
\[
-2\eta^T(t) P \psi(t) \leq \frac{1}{2} \eta^T(t) P \eta(t) + 2\psi^T(t) P \psi(t)
\leq 2\delta_1 \gamma^2 \int_{t-h}^{t} \eta^T(s) P \eta(s) \, ds + \frac{1}{2} \eta^T(t) P \eta(t),
\]
by which, we can obtain
\[
V_1(z(t)) \geq \eta^T(t) P \eta(t) + 2\eta^T(t) P \psi(t)
\geq \frac{1}{2} \eta^T(t) P \eta(t) - 2\delta_1 \gamma^2 \int_{t-h}^{t} \eta^T(s) P \eta(s) \, ds
= \frac{1}{2} \eta^T(t) P \eta(t) - V_3(\eta_t).
\]
Then, we can further get
\[
V(\eta_t) \geq V_1(z(t)) + V_3(\eta_t)
\geq \frac{1}{2} \delta \gamma^a \|\eta(t)\|^2.
\]
Therefore, the closed-loop system (7.8) is asymptotically stable for \(\gamma \in (0, \gamma^*)\). \(\square\)
A4. Proof of Theorem 4.1

Notice that the opinion dynamic network consisting of opinion dynamics model (2.3) and the memoryless communication rule (4.4) is given by

\[
\dot{x}_i (t) = Ax_i (t) + \sigma \Delta F_t \chi_i (t) + (1 - \sigma) \Delta F_t \chi_i (t - h) \\
= Ax_i (t) + \sigma \Delta F \sum_{j \in N_i} l_{ij} x_j (t) + (1 - \sigma) \Delta F
\]

(7.16)

\[
\times \sum_{j \in N_i} h_{ij} x_j (t - h), \quad i \in I[1, N],
\]

where \( F = gF \), in which \( F = -A^T P (\gamma) \). Let \( x = [x_1^T, x_2^T, \ldots, x_N^T]^T \), then (7.16) can be expressed as the following compact form,

\[
\dot{x} (t) = (I_N \otimes A) x (t) + (\sigma L \otimes \Delta F) x (t) + ((1 - \sigma) L \otimes \Delta F) x (t - h) \\
= ((DU) \otimes I_n) (I_N \otimes A) ((DU)^{-1} \otimes I_n) x (t) + ((DU) \otimes I_n) (\sigma J_T \otimes \Delta F) ((DU)^{-1} \otimes I_n) x (t) \\
+ ((DU) \otimes I_n) ((1 - \sigma) J_T \otimes \Delta F) ((DU)^{-1} \otimes I_n) x (t - h),
\]

(7.17)

where we have used Lemma 2.6. Define a set of new variables \( \zeta = ((DU)^{-1} \otimes I_n) x = [\zeta_1^T, \zeta_2^T, \ldots, \zeta_n^T]^T \), it follows from (2.1) that (7.17) is equivalent to \( \dot{\zeta}_1 (t) = A\zeta_1 (t) \) and

\[
\dot{\zeta}_i (t) = A\zeta_i (t) + \lambda_i \sigma \Delta F \zeta_i (t) + \lambda_i (1 - \sigma) \Delta F \zeta_i (t - h) + \epsilon_i \sigma \Delta F \zeta_{i+1} (t) + \epsilon_i (1 - \sigma) \Delta F \zeta_{i+1} (t - h),
\]

where \( i \in I[2, N] \) and \( \zeta_{N+1} (t) = 0 \). Since \( x = ((DU) \otimes I_n) \zeta \), it yields from the special structure of \( U \) that the polarization is achieved if

\[
\lim_{t \to \infty} \| x_i (t) - d_i \zeta_1 (t) \| = 0, \quad i \in I[1, N],
\]

by which, we have

\[
\lim_{t \to \infty} \| x_i (t) - d_i \zeta_1 (t) \| = 0, \quad i \in I[1, N].
\]

Clearly, (7.18) is true if and only if

\[
\dot{\zeta}_i (t) = A\zeta_i (t) + \lambda_i \sigma \Delta F \zeta_i (t) + \lambda_i (1 - \sigma) \Delta F \zeta_i (t - h),
\]

(7.19)

is asymptotically stable, \( i \in I[2, N] \). In what follows, we will show that system (7.19) is indeed asymptotically stable if \( g \geq \max_{i \in I[2, N]} \{ 1/ \text{Re} \{ \lambda_i \} \} \). Notice that the stability of (7.19) is equivalent to the stability of the following system

\[
\begin{cases}
\dot{\eta} (t) = A\eta (t) + \sigma \Delta u (t) + (1 - \sigma) \Delta u (t - h), \\
u (t) = \mathcal{F} \eta (t),
\end{cases}
\]

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where $F = \lambda \varrho F$, $\varrho \geq 1/\text{Re}\{\lambda\}$ and $\text{Re}\{\lambda\} > 0$. The rest of proof is completed by Lemma 7.1.

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