Stability Analysis on Mangrove Forest Resource Models with Proboscis Monkey and Fish Pond Land Interaction with Time Delay

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Abstract. Mangrove is one of the most important ecosystems in Balikpapan, where the Proboscis monkeys-an endemic animal in Borneo-lives. The growth of their population is affected by the opening of pond land by human around the forest. In this research, the problem from the interaction among the growth rate of mangrove, monkey and the opened pond land is presented in a mathematics model in the form of prey-predator for those three variables. We give an attention to the model by introducing time delays for the growth rate of fish pond land. It shows that the linearized system performs a small perturbation and the value of critical delay of the characteristic equation will affects the stability of the equilibrium point and causes stability changes (Hopf bifurcation). In this work, we obtain the existence of a critical value of the time delay and the fulfillment of the transversal conditions.

Keywords: Mangrove, Proboscis Monkey, Pond Land, Prey-Predator Model, Time Delay

1. Introduction
Mangrove forest is one of biospheres in Balikpapan- a city in Kalimantan -which becomes a vital zone for live. Its existence not only brings a lot of benefits for human to do some activities but also for the proboscis monkeys for their living. For the people around the forest, the forest provides a wide land and much water to have some fish ponds. While the proboscis monkey can grow their population by living inside the forest since it acts as a food resources and home for the monkey. They eat the leaves of mangrove trees and use the trees to hide their group in order to save the population.

Nowadays, humans create so many ponds and significantly decrease the population of mangrove. It implies that the ecosystem of the monkey is disturbed. As an endemic species in Kalimantan, its population dramatically decreases every year due to the opening of pond land. The drops of population of mangrove and the monkey which are caused by opening some ponds are described as biological interactions between those three variables: human, proboscis monkey, and the mangrove forest. The interaction was studied in [1] and presented in a mathematical model. The model was then modified by the prey-predator model in [2] and other researchers in [3], [4], [5], [6], and [7] and [8].
During the interaction, there exist some changes of their rate of population. The changes need a certain time which is caused by the conversion of the consumed part of tree by the monkey, the process of opening the ponds, and also the depletion of the trees. Those times are considered as the delay times and a study of prey-predator model with no time delays was presented in [9].

Another study of stability behavior of the prey-predator model with discrete time delays was performed in [10] and [11]. It also estimated the length of delay to preserve stability. Using the Hopf Bifurcation analysis, it showed that the bifurcation could be supercritical stable (unstable), and the period of the bifurcating periodic solution increases (decreases). In this paper, we describe a prey-predator population model with a time delay in the three variables which is based on Lotka-Volterra model and then analyze its stability behavior.

2. Methodology

2.1 Model Construction

This research was started by modifying the model based on the mangrove forest resource depletion model due to opening fish ponds land owned by Nugraheni, et al [9]. The model is in the form of a nonlinear differential equation consisting of three compartments, which are modified with a time delay.

2.2 Equilibrium Point

The next step is to determine the point of equilibrium. The equilibrium point is obtained from the equilibrium solution of the equation system, when the growth rate does not change or is zero. Next, the conditions of existence of the equilibrium point are determined.

2.3 Analysis of Model Equilibrium Stability with Time Delay

The determination of stability begins with the linearization of the model using the perturbation equation from equilibrium by a small perturbation. The main purpose of discussing models with time delay is to study the behaviour of equilibrium stability by taking into account the discrete-time delay. Furthermore, investigating the equilibrium point experiences a change in the stability or bifurcation of Hopf. It is shown by determining the existence of a critical value of delay and the fulfillment of transversal conditions.

2.4 Conclusion

The final step of this study is drawing conclusion. The conclusion obtained from the results of the discussion is about the behaviour of the solutions that occur in the system or the model.

3. Result and Discussions

In this section, we present a model formulation which modifies the previous models and determine the equilibrium points and analyze the stability of the model.

3.1 Model Formulation

In order to describe the concentration of the proboscis monkeys, the pond lands, and the depletion of the mangrove forest, [9] introduced the mangrove forest resource depletion models due to the opening of fish pond land as follows:

\[
\frac{dM}{dt} = sM \left(1 - \frac{b}{L}\right) - \alpha MN - eM^2 P,
\]

\[
\frac{dN}{dt} = rN \left(1 - \frac{c}{K}\right) + \beta aMN - \mu N,
\]

\[
\frac{dP}{dt} = \gamma M - dP,
\]

(1)
where \( M(0) \geq 0, N(0) \geq 0, P(0) \geq 0 \). \( M(t), N(t) \) and \( P(t) \) are concentration of mangrove resources, concentration of proboscis monkey population and concentration of depletion due the opening of fish pond land. The parameter \( s \) denotes the growth rate of mangrove resources, \( \left( \frac{b}{L} \right) \) is growth rate of proboscis monkey population, concentration of depletion due the opening of fish pond land. Moreover, \( \alpha \) denotes the depletion rate of mangrove forest resources due to the proboscis monkey population, and \( e \) is the depletion rate coefficient of mangrove forest resources due to the opening of fish pond land. Proboscis monkey population grows exponentially with a rate \( r \) towards intra-class competition and carrying capacity \( \left( \frac{c}{K} \right) \). The constant \( \beta \) denotes proboscis monkey growth rate due to mangrove resources, and \( \mu \) is death rate of proboscis monkey population. The growth rate of opening fish pond land due to cutting down mangrove resources is represented by \( \gamma \), and the constant \( d \) denotes natural depletion rate. All parameters are non negatives constants.

Suppose that depletion of mangrove resources due to opening the fish pond land will not be instantaneous, but involves time delays. So, the models in [9] are modified as follows:

\[
\begin{align*}
\frac{dM}{dt} &= sM \left( 1 - \frac{b}{L} M \right) - \alpha MN - eM^2 P(t - \tau), \\
\frac{dN}{dt} &= rN \left( 1 - \frac{c}{K} N \right) - \beta \alpha MN - \mu N, \\
\frac{dP}{dt} &= \gamma M - dP(t - \tau),
\end{align*}
\]

(2)

with \( M(0) \geq 0, N(0) \geq 0, P(\vartheta) = \varphi(\vartheta) \), such that \( \varphi(\vartheta) > 0, \forall \vartheta \in [-\tau,0] \). It is assumed that all parameters are the same in system (1), except the positive constant \( r \) which represents the long period of the depletion due to opening fish pond land.

### 3.2 Equilibrium Points and Stability Analysis

Model has four equilibrium point, those are:

1. \( E_{0}(0,0,0) \) as the origin;
2. \( E_{1} \left( 0, \frac{K}{rC}(r-\mu),0 \right) \) as the survival of proboscis monkey population;
3. \( E_{2} \left( M_{1}^*, \frac{L}{d} M_{1}^*,0 \right) \), where \( M_{1}^* = -\frac{d}{2e\lambda} \left( b \frac{s}{L} \pm \sqrt{b \frac{s}{L}^2 + 4e\frac{\gamma}{d}} \right) \); and
4. \( E_{3} \left( M_{2}^*, \frac{K}{rC} (\beta \alpha M_{2}^* - \mu + r) \frac{\gamma}{d} M_{2}^* \right) \), where

\[
M_{2}^* = -\frac{d}{2e\gamma} \left( b \frac{s}{L} + \beta \alpha^2 \frac{K}{rC} \right) \pm \sqrt{b \frac{s}{L} + \beta \alpha^2 \frac{K}{rC}^2 + 4e\gamma \left( s + (\mu - r) \frac{K}{rC} \right)}.
\]

The stability behavior of model which presents time delays can be studied by linearizing the system using transformation as follows:

\[
\begin{align*}
M(t) &= M^* + \varepsilon M^*(t), \\
N(t) &= N^* + \varepsilon N^*(t),
\end{align*}
\]
\[ P(t) = P^* + \epsilon \tilde{P}(t). \]

After applying the transformation, we obtain
\[ \frac{d}{dt} \tilde{U}(t) = A \tilde{U}(t) + B \tilde{U}(t - \tau), \]
where
\[ \tilde{U}(t) = [M \quad N \quad P]^T, \quad A = \{a_{ij}\}_{3 \times 3}, \quad B = \{b_{ij}\}_{3 \times 3}, \]
\[ a_{11} = s - 2s \frac{b}{L} M^* - \alpha N^* - 2eM^* P^*, \quad a_{21} = -\beta \alpha N^*, \]
\[ a_{12} = -\alpha M^*, \quad a_{22} = r - 2r \frac{c}{K} N^* - \beta \alpha M^* - \mu, \]
\[ a_{31} = \gamma, \quad \text{all other } a_{ij} = 0, \]
\[ b_{33} = -\epsilon (M^*)^2, \quad b_{33} = -d, \quad \text{all other } b_{ij} = 0. \]

Let denote \( \tilde{U}(t) = \tilde{C} e^{\omega t} \), then the system (3) solution defined as follows,
\[ \lambda^3 + a_1 \lambda^2 + a_2 \lambda + (a_3 \lambda^2 + a_4 \lambda + a_5)e^{-\tau \lambda} = 0, \]
where,
\[ a_1 = -(a_{11} + a_{22}), \]
\[ a_2 = -(a_{12}a_{12} - a_{11}a_{22}), \]
\[ a_3 = -b_{33}, \]
\[ a_4 = a_{23}b_{33} + a_{13}b_{33}, \]
\[ a_5 = -a_1a_{22}b_{33} + a_{22}a_3b_{33} + a_2a_1b_{33}. \]

The actual condition for stability behavior of equilibrium point are fulfilled when characteristic polynomial have only imaginary solutions. Assuming that \( \lambda = i \omega (\omega \neq 0) \), and substituting \( \lambda = i \omega \), to (4), and separating real and imaginary parts, we obtain
\[ a_1 \omega^2 = (a_5 - a_4 \omega^2) \cos \omega \tau + a_4 \omega \sin \omega \tau, \]
\[ -\omega^3 + a_2 \omega = (a_3 - a_4 \omega^2) \sin \omega \tau - a_4 \omega \cos \omega \tau. \]
By squaring and adding two equations in (5), it shows that
\[ \omega^6 + c_1 \omega^4 + c_2 \omega^2 + c_3 = 0, \]
where
\[ d_1 = (a_1^2 - a_3^2 - 2a_2); \]
\[ d_2 = (a_2^2 - a_4^2 + 2a_3a_4); \]
\[ d_4 = -a_5^2. \]
Substituting \( v = \omega^2 \), into the equation (6) obtains
\[ v^3 + b_1 v^2 + b_2 v + b_3 = 0. \]
According to a lemma in [12], there exists positive roots which follows,
\[ \omega_1 = \sqrt{v_1}, \omega_2 = \sqrt{v_2}, \omega_3 = \sqrt{v_3}. \]
Furthermore, following the method in [13], the equation (5) derives a solution \( \tau^* \) as
From the equation (7), it is known that there exists infinite values of \( \tau_k^* \) that satisfy the equation (5).

The smallest value \( \tau_0^* \) of the delay time is less than the critical value of the delay and unstable if the value around \( \tau_0^* \). The Transversal conditions are stated as follows

\[
\frac{d}{d\tau} \text{Re}(\lambda(\tau)) \bigg|_{\lambda=0, \tau=\tau_0^*} \neq 0.
\]

According to [13], the transverse conditions are met if and only if the conditions below are met

\[
R_1(\omega_0)R_1'(\omega_0) + Q_1(\omega_0)Q_1'(\omega_0) \neq R_2(\omega_0)R_2'(\omega_0) + Q_2(\omega_0)Q_2'(\omega_0),
\]

Let \( \lambda(\tau) = \alpha(\tau) \pm i\omega(\tau) \) as the roots of the equation (4) that satisfy \( \alpha(\tau_0) = 0 \) and \( \omega(\tau_0) = \omega_0 \). From the equation (5), it gives

\[
2a_1^2\omega^3 + 3\omega^5 - 4a_2\omega^3 + a_4\omega \neq -2a_1a_3\omega + 2a_3^2\omega^3 + a_4^2\omega.
\]

If the equation (9) occurs then the characteristic equation (4) satisfies the transversal condition. This fulfillment results in changes of instability, named asymptotic stable when \( \tau < \tau_0 \) and unstable when \( \tau_0 > \tau \) or vice versa with \( \text{Re}(\lambda) \neq 0 \).

4. Conclusion

This paper studies a stability analysis mangrove forest resource models with proboscis monkey and fish pond land interaction with time delay. It is clearly found that the model has four equilibrium points. The stability analysis of the presence of time delays can be studied by linearizing the system using transformation. Stability behavior of equilibrium point is fulfilled when the characteristic polynomial has only imaginary solutions. The delay time in the model affects the stability of the equilibrium point and causes stability changes (Hopf bifurcation). The equilibrium point is asymptotically stable if the delay time is less than the critical value of the delay and unstable if the delay time is greater than the critical value of the delay.

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