Is the Radial Profile of the Phase-Space Density of Dark Matter Haloes a Power-Law?

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ABSTRACT

The latest cosmological N-body simulations find two intriguing properties for dark matter haloes: (1) their radial density profile, \(\rho\), is best fit by a form that flattens to a constant at the halo center (the Einasto profile) than the widely-used NFW form; (2) the radial profile of the pseudo-phase-space density, \(\rho/\sigma_r^3\), on the other hand, continues to be well fit by a power law, as seen in earlier lower-resolution simulations. In this paper we use the Jeans equation to argue that (1) and (2) cannot both be true at all radii. We examine the implied radial dependence of \(\rho/\sigma_r^3\) over 12 orders of magnitude in radius by solving the Jeans equation for a broad range of input \(\rho\) and velocity anisotropy \(\beta\). Independent of \(\beta\), we find that \(\rho/\sigma_r^3\) is approximately a power law only over the limited range of halo radius resolvable by current simulations (down to \(\sim 0.1\%\) of the virial radius), and \(\rho/\sigma_r^3\) deviates significantly from a power-law below this scale for both the Einasto and NFW \(\rho\). The same conclusion also applies to a more general density-velocity relation \(\rho/\sigma_r^3\). Conversely, when we enforce \(\rho/\sigma_r^3 \propto r^{-\eta}\) as an input, none of the physically allowed \(\rho\) (occurring for the narrow range \(1.8 \lesssim \eta \lesssim 1.9444\)) follows the Einasto form. We expect the next-generation simulations with better spatial resolution to settle the debate: either the Einasto profile will continue to hold and \(\rho/\sigma_r^3\) will deviate from a power law, or \(\rho/\sigma_r^3\) will continue as a power law and \(\rho\) will deviate from its current parameterizations.

1 INTRODUCTION

N-body simulations of cosmological structure formation have shown that the spherically-averaged radial profiles of the mass density and velocity dispersion of dark matter haloes follow simple and nearly universal functional forms that are largely independent of halo properties such as mass, environment, and formation history. For the density profile \(\rho\), there had been considerable discussion about the value of the logarithmic slope of its central cusp, \(\gamma \equiv d \log \rho/d \log r\), whether it is \(-1\) as in the forms of, e.g., Hernquist (1990), Navarro et al. (1997), or \(-1.5\) as in Moore et al. (1999). Results from recent N-body simulations now suggest a lack of a definite inner slope – the density profile of these better-resolved dark matter haloes continues to flatten with shrinking radius (e.g., Navarro et al. 2004, Merritt et al. 2005, Graham et al. 2006, Navarro et al. 2008, Stadel et al. 2008). Functional forms such as Einasto (1969) and Prugniel & Simien (1997) motivated by the Sersic profile for the surface brightness of galaxies (Sersic 1968) appear to provide a more accurate fit to the latest simulations. In these forms, \(\gamma\) itself is a function of radius and asymptotes to zero at the halo center.

A second property that has attracted much attention lately is the radial profile of the pseudo-phase-space density, \(\rho/\sigma_r^3\), which has been reported to be well approximated by a power-law in a number of N-body simulations (e.g., Taylor & Navarro 2001, Ascasibar et al. 2004, Dehnen & McLaughlin 2005, Hoffman et al. 2007, Stadel et al. 2008, Navarro et al. 2008). Both the total velocity dispersion and the velocity dispersion in the radial direction have been used to define \(\sigma\). Unlike the controversial density profile, whose best-fit form has changed over the years with improved numerical resolution, the power-law profile of \(\rho/\sigma^3\) has withstood the scrutiny, and different studies have all reported similar findings except for a minor variation in the actual value of the slope of the power law.

A third relation was proposed when the velocity anisotropy, \(\beta(r) = 1 - \sigma_r^2/\sigma_t^2\), of simulated haloes was taken into account. Hansen & Moore (2006) advocated a linear relation between \(\beta\) and the local logarithmic slope of \(\rho\). Other studies, however, have found a large scatter in this relation, particularly in the outer parts of the haloes beyond the scale radius (Dehnen & McLaughlin 2005, Navarro et al. 2008).

To help elucidate the physical meanings of these empirical relations determined from simulated haloes, a typical approach is to use the Jeans equation for a spherical, self-gravitating collisionless system in equilibrium to predict the density or velocity structures of dark matter haloes under a certain set of assumptions. Most of the recent studies based on this approach have begun with the assumption of a power-law \(\rho/\sigma_r^3(r)\). These papers then explored the density profiles allowed by the Jeans equation with either isotropic velocities (e.g., Taylor & Navarro 2001, Hansen 2004), or an anisotropic velocity profile \(\beta(r)\) (e.g., Dehnen & McLaughlin 2005, Hoffman et al. 2007, Stadel et al. 2008, Navarro et al. 2008).
2 RESULT: $\rho/\sigma^3$ IS NOT A POWER-LAW

For a spherical, self-gravitating collisionless system in equilibrium, its density and velocity structures satisfy the Jeans equation

$$\frac{1}{\rho} \frac{d}{dr} \left( \rho \sigma^2 \right) + \frac{2 \sigma^2}{r} \beta = - \frac{d\Phi}{dr},$$

where $\rho$ is the radial density profile, $\beta = 1 - \sigma^2/\sigma_\gamma^2$ is the velocity anisotropy parameter, $\sigma_\gamma$ and $\sigma_r$ are the one-dimensional radial and tangential velocity dispersions, and $\Phi$ is the gravitational potential. To apply the Jeans equation, we take as input a form for $\rho$ and solve for $\sigma_\gamma$.

For the purposes of this paper, we consider two broad types of radial density profiles. The first type is cuspy, all the way to the halo center with an inner logarithmic slope $\gamma$ and an outer slope $\gamma_\infty$:

$$\rho(r) = \frac{\rho_0}{(r/r_\gamma)^\gamma [1 + r/r_\gamma]^{\gamma_\infty - \gamma}},$$

where $r_\gamma$, often referred to as the scale radius, is the radius at which $d\ln \rho/d\ln r = -2$. Examples of special cases of ($\gamma, \gamma_\infty$) that have been proposed for dark matter haloes include $(-1, -3)$ by Navarro et al. (1997), $(-1, -4)$ by Hernquist (1990) and Dubinski & Carlberg (1991), and $(-1.5, -3)$ by Moore et al. (1999). We find very similar results from our calculations for $\gamma_\infty = -3$ vs $-4$; we will thus set $\gamma_\infty = -3$ and refer to equation (2) as GNFW below.

The other type of density profile considered in this paper has a non-cuspy inner profile, given by the Einasto profile (Einasto 1969) advocated in several recent studies (e.g., Merritt et al. 2005; Graham et al. 2006; Navarro et al. 2008; Stadel et al. 2008):

$$\ln \frac{\rho(r)}{\rho_{-2}} = \frac{2}{\alpha} [1 - (r/r_{-2})^\alpha].$$

This profile has the feature that its logarithmic slope is itself a power-law in $r$: $d\ln \rho/d\ln r = -2(r/r_\gamma)^\alpha$. Unlike equation (2) that has a definite inner slope of $\gamma$, this profile continues to flatten towards the halo center. Equation (3) has the same form as the Sersic profile commonly used to fit the two-dimensional projected surface brightness of galaxies (Sersic 1968). The Sersic index $n$ is simply equal to $1/\alpha$.

Starting with either profile in equation (2) or (3), and $\beta = 0$ or some form of $\beta(r)$, we integrate the Jeans equation (1) to obtain $\sigma_\gamma(r)$. The numerical integration is performed from large radius (typically $r = 10 r_\text{vir}$) down to the radius $r$ of interest, using a standard Runge-Kutta integrator (Press et al. 1992). The velocity dispersion is assumed to be zero at the starting large radius. The $\sigma_\gamma$ from the integration is then combined with the input $\rho(r)$ to obtain $\rho/\sigma^3_\gamma(r)$.

The left two panels of Fig. 1 shows the result $\rho/\sigma^3_\gamma$ (upper panel) and its logarithmic slope (lower panel) over 12 orders of magnitude in halo radius for $\beta = 0$ and both types of input density profiles: GNFW with $\gamma = 0.5, 0.75, 1.0$, and $1.5$ (dashed curves), and Einasto with $\alpha = 0.12, 0.16$, and $0.18$ (solid curves). The range of $\alpha$ is chosen to span the best-fit values of $0.115$ to $0.179$ for the six simulated galaxy-size haloes in (Navarro et al. 2005). The right two panels of Fig. 1 show zoom-in views of the portion of the left figures that is resolvable by current simulations: $0.01 \leq r/r_{-2} \leq 10$.

Fig. 1 illustrates that $\rho/\sigma^3_\gamma$ is not a power-law in $r$ for any of the seven input density profiles. For GNFW haloes, the slopes of $\rho/\sigma^3_\gamma$ exhibit oscillations and deviate noticeably from the critical case $\rho/\sigma^3_\gamma \propto r^{-1.9}$ (indicated by light dotted straight lines), in particular in the extreme cases of inner cusps of $\gamma = 0.5$ (red dashed) and $1.5$ (blue dashed). The Einasto haloes also deviate strongly from a power-law at small radius. The zoom-in panels show, however, that the slopes of $\rho/\sigma^3_\gamma$ happen to be quite close to $-1.9$ over the limited range of $r/r_{-2} \sim 0.01$ to $10$ that is resolvable by current simulations. This feature is particularly striking for the Einasto profiles, where all three solid curves for $\rho/\sigma^3_\gamma$ have a slope within $-2.0$ and $-1.8$ from $r/r_{-2} \sim 0.01$ to $10$, with the deviations only starting to show up at the smallest radius $r/r_{-2} \sim 0.01$ near the simulation resolution limit. It is therefore not surprising that the power-law behavior of $\rho/\sigma^3_\gamma(r)$ continues to appear to be valid even though the latest simulations find the Einasto form a better fit for $\rho(r)$ than GNFW – Fig. 1 shows the Einasto profiles in fact predict a more power-law $\rho/\sigma^3_\gamma(r)$ for $r/r_{-2} \gtrsim 0.01$.

The important point to note, however, is at the smaller radius of $r/r_{-2} \lesssim 0.01$ in Fig. 1. Here $\rho/\sigma^3_\gamma(r)$ deviates far away from a pure power law with a wide range of slopes that depend on the input $\rho$. For Einasto haloes, the shape of $\rho/\sigma^3_\gamma$ flattens continuously towards the halo center, reaching the asymptotic value of $d\ln \rho/\sigma^3_\gamma/d\ln r = 0$ at $r = 0$ regardless of the parameter $\alpha$. This is not unexpected of the Einasto profile as the density approaches an asymptotic value in the core. For GNFW, there are two possibilities. For inner slopes of $\rho$ that are steeper (shallower) than the critical $\gamma \approx 0.75$, the power-law slopes of $\rho/\sigma^3_\gamma$ are steeper (shallower) than the critical $-1.9$. At the critical $\gamma = 0.75$, the GNFW halo has $\rho/\sigma^3_\gamma \approx r^{-1.875}$ at small $r$, a result consistent with that of Taylor & Navarro (2001), which showed that starting with an exact power-law of $\rho/\sigma^3_\gamma \propto r^{-1.875}$, the resulting density profile has an inner slope of $d\ln \rho/d\ln r \approx 0.75$. It is worth noting, however, that even the $\gamma = 0.75$ GNFW halo shows
Figure 1. Radial profiles of the pseudo-phase-space density \( \rho/\sigma^3 \) (upper panels) and the corresponding logarithmic slope \( \frac{d \ln \rho}{d \ln r} \) (lower panels) obtained from the spherical Jeans equation with \( \beta = 0 \) for seven input halo density profiles: Einasto (solid) with \( \alpha = 0.18 \) (blue), 0.16 (green), and 0.12 (red), and GNFW (dashed) with \( \gamma = 1.5 \) (blue), 1 (black), 0.75 (green), and 0.5 (red). The left panels show the behavior of \( \rho/\sigma^3 \rho(r) \) over 12 orders of magnitude in \( r \), while the right panels show zoom-in views of the region \( 0.01 \lesssim \frac{r}{r_{-2}} \lesssim 10 \), which corresponds to the range resolvable by the latest N-body simulations. For ease of comparison with a power-law, the light dotted straight lines indicate the critical case \( \rho/\sigma^3 \rho(r) \propto r^{-1.9} \). All curves are scaled to have \( \rho/\sigma^3 \rho(r) = 1 \) at \( r = r_{-2} \).

wiggles in the corresponding \( \rho/\sigma^3 \rho(r) \) profile; that is, no GNFW haloes have an exact power-law \( \rho/\sigma^3 \rho(r) \).

3 FURTHER CONSIDERATIONS: \( \rho/\sigma^3 \rho(r) \) IS STILL NOT A POWER-LAW

We also find the conclusion reached in Sec. 2 to hold not just for isotropic velocity distributions, but also for anisotropic velocity distributions. To illustrate this point, we solve equation (1) using an input \( \beta(r) \) motivated by N-body simulations [Hansen & Moore 2006; Zait et al. 2008], where \( \beta \) is a function of the local logarithmic slope of the density profile:

\[
\beta(r) = -0.2 \left( \frac{d \ln \rho}{d \ln r} + 0.8 \right). \tag{4}
\]

We then compute \( \rho/\sigma^3 \rho(r) \) for a similar suite of GNFW and Einasto profiles. The results are shown in Fig. 2. What is especially notable is how insensitive the slopes of \( \rho/\sigma^3 \rho(r) \) are to the form of \( \beta(r) \) used in the calculation. This independence

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from velocity anisotropy strengthens our conclusion reached earlier, namely, that \( \rho/\sigma_0^2 \) is not a power law.

Our findings are also in line with some recent work that has called into question the universality of \( \rho/\sigma_0^2 \). For instance, Schmidt et al. (2008) has advocated that individual simulated haloes are better fit by a generalized power-law relation that is not necessarily \( \rho/\sigma_0^2 \):  

\[
\frac{\rho}{\sigma_D^2} \propto r^{-\alpha},
\]

where \( \sigma_D = \sigma_r \sqrt{1 + D^2} \), and \( D \) parameterizes a generalized \( \sigma_D \); for instance, \( D = 0, -2/3 \) and \( -1 \) correspond to \( \sigma_D = \sigma, \sigma_{\text{tot}} \) (1-d), and \( \sigma_1 \), respectively. Schmidt et al. (2008) showed that the best-fit values of \( (D, \epsilon, \alpha) \) differ from halo to halo, and as a set, they roughly follow the linear relations \( \epsilon = 0.97D + 3.15 \) and \( \alpha = 0.19D + 1.94 \). For \( \sigma = \sigma_r \) (i.e. \( D = 0 \)), the optimal relation is \( \rho/\sigma_0^{2.15} \propto r^{-1.94} \), which is consistent with the reported behavior of \( \rho/\sigma_0^2 \) in N-body simulations within error bars. However, few haloes' best-fit value of \( D \) in Schmidt et al. (2008) is near \( D = 0 \).

To assess whether any of these relations is closer to a power-law than \( \rho/\sigma_0^2 \) in our calculations, we choose three sets of \( D \) and \( \epsilon \) from their relation, \( (D, \epsilon) = (0, 3.15), (-2/3, 2.50), \) and \( (-1, 2.18) \), and plot in Fig. 3 the logarithmic slopes of these three relations \( \rho/\sigma_D^2 \), using our solutions of the Jeans equation with non-zero \( \beta \) shown in Fig. 2. For clarity, only the three Einasto profiles are shown in Fig. 3, although our conclusions apply to the GNFW profiles as well. The three solid curves in Fig. 3 (for \( D = 0 \)) represent \( \rho/\sigma_0^{2.15} \), and are therefore very similar to the short-dashed curves for the Einasto \( \rho/\sigma_r^2 \) in Fig. 2. The other two sets of curves in Fig. 3 suggest that \( \rho/\sigma_0^{2.5} \) (i.e. \( D = -2/3 \)) and \( \rho/\sigma_0^{2.18} \) (i.e. \( D = -1 \)) are also far from being a power-law over the wide range of radius shown. When it is limited to the range of \( 0.01 \leq r/r_{-2} \leq 10 \) probed by simulations, \( \rho/\sigma_0^{2.5} \) appears to be slightly closer to a power-law than \( \rho/\sigma_0^2 \) for the \( \alpha = 0.12 \) (red) and 0.16 (green) Einasto profiles. Over the larger range of radius shown in Fig. 3, however, our earlier conclusion of a non-power-law \( \rho/\sigma_r^2 \) carries over to the broader range of \( \rho/\sigma_D^2 \) shown.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Effects of velocity anisotropy on the radial profile of \( \rho/\sigma_0^2 \) computed from the Jeans equation: \( \beta = 0 \) (solid for Einasto; long dashed for GNFW; same as lower left panel in Fig. 1), and \( \beta \) given by eq. (3) (short dashed for Einasto; dashed-dotted for GNFW). The seven input density profiles are the same as in Fig. 1. This figure illustrates that including velocity anisotropy in the Jeans equation does not change \( \rho/\sigma_0^2 \) significantly and still results in a non-power-law \( \rho/\sigma_0^2 \).
}\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Radial dependence of three examples of the general density-velocity relation \( \rho/\sigma_{\text{tot}}^2 \) (solid), \( \rho/\sigma_r^2 \) (short-dashed), and \( \rho/\sigma_D^2 \) (long-dashed) for the Einasto profiles shown in Figs. 1 and 2. This figure illustrates that like \( \rho/\sigma_r^2 \), \( \rho/\sigma_D^2 \) is also far from being a power-law over the wide range of radius shown.
}\end{figure}

\section{Forcing \( \rho/\sigma_r^2 \) to be a Power Law}

Thus far we have solved the Jeans equation assuming an input \( \rho(r) \). As a point of comparison, we have also solved the Jeans equation following the works of previous authors with a starting assumption of a power-law \( \rho/\sigma_r^2 \). Most notably, Taylor & Navarro (2001) presented results for the special case of \( \eta = 15/8 = 1.875 \), whereas Dehnen & McLaughlin (2005) (see also Austin et al. (2005) argued that only a single realistic solution exists for \( \rho \), given when \( \eta \) takes the particular value of \( 35/18 = 1.9444 \) for isotropic velocities, and \( \eta = 35/18 = 2\beta_0/9 \) for anisotropic velocities, where \( \beta_0 \equiv \beta(r = 0) \).

Here we examine a range of \( \eta \) and show in Fig. 4 our numerical solutions (for \( \beta = 0 \)) from the Jeans equation for \( \rho(r) \) for four values of input power-law \( \rho/\sigma_r^2 \); \( \eta = 1.8, 1.875, 1.9444 \) and 2.0. As \( \eta \) moves away from the critical value 1.9444, \( \rho \) drops to zero rapidly at some small radius if \( \eta > 1.9444 \) (short-dashed curve), while \( \rho \) is cut-
off sharply at some large $r$ if $\eta < 1.9444$ (long-dashed and dotted curves). The former solution with a central hole is unphysical for dark matter haloes. The latter, however, is not automatically ruled out. Only for $\eta \lesssim 1.8$ do we find the outer $\rho$ to drop off too steeply to represent realistic $\Lambda$CDM haloes. It therefore appears that the narrow range of $1.8 \lesssim \eta \lesssim 1.9444$ may admit physical solutions for $\rho$. It is important to keep in mind, however, that $\rho$ in these cases are not well described at small radius by either the Einasto profile advocated by recent simulations, or the original $\gamma = 1$ GNFW profile. Lowering the inner slope of the GNFW profile to $\gamma \approx 0.75$ provides a better fit. Dark matter haloes therefore cannot be well fit by the Einasto $\rho$ and a power-law $\rho/\sigma^3$ simultaneously at all radii.

5 CONCLUSIONS

Motivated by the apparent power-law radial profile of the pseudo-phase space density, $\rho/\sigma^3$, reported in various $N$-body simulations (e.g., Taylor & Navarro 2001; Ascasibar et al. 2004; Dehnen & McLaughlin 2005; Hoffman et al. 2007; Stadel et al. 2008; Navarro et al. 2008), we solve the Jeans equation to study the implied pseudo-phase-space density for given parameterizations of $\rho$ suggested by $N$-body simulations, i.e., Einasto and GNFW. We find that independent of the velocity anisotropy, $\rho/\sigma^3$ is not a pure power law in radius for either the Einasto or GNFW profiles (left panels of Fig. 1). In the radial ranges that are probed by current $N$-body simulations (down to $\sim 10^{-3}r_{cr}$, $\sim 10^{-2}r_{-2}$), we find that $\rho/\sigma^3$ happens to be approximately a power law, in particular for the Einasto profile (right panels of Fig. 1). For radial scales right below the resolution of current simulations, however, we see significant deviations from a power law profile for $\rho/\sigma^3$ if either Einasto or GNFW continues to hold as a suitable parameterization of $\rho$. This result is unchanged when velocity anisotropy is included in the calculation (Fig. 2), and when a more general density-velocity relation of $\rho/\sigma^3$ is considered (Fig. 3). The hope to gain deep insight into the process of dark matter halo formation using a simple and universal power law scaling of $\rho/\sigma^3$ may therefore be misleading. Conversely, when $\rho/\sigma^3 \propto r^{-\eta}$ is assumed as an input in the Jeans equation, none of the realistic solutions for $\rho$, which occur only for the narrow range $1.8 \lesssim \eta \lesssim 1.9444$, take the Einasto form (Fig. 4).

We therefore conclude that the two halo properties seen in current simulations – the Einasto $\rho$ and power-law $\rho/\sigma^3$ – cannot hold simultaneously at all radii. We expect that the upcoming simulations with better spatial resolution will settle this debate: either the Einasto profile will continue to hold and $\rho/\sigma^3$ will show a break from a power law, or $\rho/\sigma^3$ will continue as a power law inward and $\rho$ will deviate from its current parameterizations.

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