Local modulated wave model for the reconstruction of space–time energy spectra in turbulent flows

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A statistical model is developed to reconstruct space–time energy spectra in turbulent flows from a non-extensive dataset comprising a time series of velocity fluctuations at a finite number of measurement points. This model is based on a higher approximation of energetic flow structures and developed by using local modulated waves. As a result, it can correctly predict the mean wavenumbers and spectral bandwidths. In contrast, Taylor’s frozen-flow hypothesis incorrectly predicts the spectral bandwidths to be zero, and the local wavenumber model significantly under-predicts the spectral bandwidths. An analytical example is formulated to illustrate the present model, and datasets from direct numerical simulations of turbulent channel flows are used to validate this model. The present statistical model is also discussed in terms of the dominating processes of temporal decorrelation in turbulent flows.

Key words: turbulence theory, turbulence modelling, turbulent boundary layers

1. Introduction

For many decades, space–time energy spectra have been fundamental in the investigation of turbulent flows. In theoretical studies, they are used to investigate energy transfer among different spatial and temporal scales (Kraichnan 1966; Jiménez 2012) and to identify energy-containing coherent structures (Adrian & Moin 1988; Taira \textit{et al.} 2017). In engineering applications, such as turbulence-generated noise and flow–structure interactions, space–time energy spectra are used to detect noise-generating flow structures (Mancinelli \textit{et al.} 2018) and to estimate the temporal spectra of power outputs from wind farms (Bossuyt, Meneveau & Meyers 2017). Recently, space–time energy spectra have been used to develop an enriched large-eddy simulation methodology for planetary boundary layers (Ghate & Lele 2017). Therefore, it is of paramount importance to develop a statistical model for understanding space–time energy spectra and numerically reconstructing space–time energy spectra from datasets of experimental measurements or numerical simulations.

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A space–time energy spectrum can be conventionally reconstructed from a complete space–time dataset of velocity fluctuations in physical space by performing Fourier transforms. The conventional approach presents two challenges. The first is that Fourier transforms require periodic boundary conditions in physical space, which is not the case for most engineering problems. The second is that this kind of reconstruction requires a complete dataset for both the space and time domains. Space–time datasets obtained from experimental measurements and numerical simulations are often incomplete in the space and/or time domains. In experiments with hot-wire anemometers (HWA) (Hutchins et al. 2009) or laser Doppler anemometers (LDA) (Cenedese, Romano & Defelice 1991; Romano 1995), the number of measurement points is limited by the few probes that generate time records of velocity fluctuations only at a few fixed locations. In particle image velocimetry experiments (de Kat & Ganapathisubramani 2015), the field of view is limited to balance the spatial and temporal resolutions, which leads to a finite range of the spatial and temporal scales. In the case of a numerical simulation in which the large-eddy simulation or the Reynolds-averaged Navier–Stokes equations are used, the resolved scales in both space and time are limited (He, Wang & Lele 2004). Therefore, it is necessary to develop a statistical model to determine space–time energy spectra from the time series of velocity fluctuations at a limited number of measurement points. This not only represents a practical demand in the age of big data (Duraisamy, Iaccarino & Xiao 2019) but also represents a theoretical development in understanding the dynamics of turbulent passages (He, Jin & Yang 2017).

A primary requirement for the estimation of space–time energy spectra is to correctly predict the first- and second-order moments conditional on a given frequency. The first-order conditional moments represent the mean wavenumbers that provide the statistical dispersion relations of turbulent flows. The second-order conditional moments represent spectral bandwidths that characterize the well-known spectral broadening. It has been shown that the mean wavenumbers are determined by phase derivatives alone and that spectral bandwidths are determined by both phase and amplitude derivatives (Wu et al. 2017). Therefore, the goal of the present paper is to develop a statistical model to correctly predict mean wavenumbers and spectral bandwidths.

Taylor’s frozen-flow hypothesis (Taylor 1938; Moin 2009) has been a staple method for the reconstruction of space–time energy spectra. Taylor’s hypothesis approximately predicts the first-order moments of space–time energy spectra; however, it incorrectly predicts the second-order moments to be zero. In fact, Taylor’s hypothesis assumes that a flow pattern propagates at a constant mean velocity \( U \) without any distortion. Therefore, velocity fluctuations \( u(x + r, t + \tau) \) at a downstream location \( x + r \) and future time \( t + \tau \) are equal to fluctuations \( u(x + r - U\tau, t) \) at an upstream location \( x + r - U\tau \) and a previous time \( t \). Therefore, the space–time energy spectrum can be obtained from a Dirac delta function \( \delta \): \( \Phi(k_x, \omega) = \Phi_s(k_x)\delta(\omega - k_x U) \), where \( \Phi(k_x, \omega) \) is a space–time energy spectrum and \( \Phi_s(k_x) \) is a spatial energy spectrum. As a result, Taylor’s hypothesis gives a standard deviation of zero. To improve Taylor’s frozen-flow approximation, the convection velocity \( U \) is further assumed to be dependent on the spatial and temporal scales of flow patterns (Fisher & Davies 1964; Wills 1964). Del Álamo & Jiménez (2009) proposed that the convection velocity of a flow pattern can be better approximated by the phase velocity of a travelling wave. Consequently, the obtained convection velocity can account for phase contributions and is explicitly dependent on either wavenumbers (Del Álamo & Jiménez 2009) or
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Yang & Howland (2018) and Howland & Yang (2018) used the local fluid velocity as the local convective velocity to account for spatial amplitude modulation, which highlights a new approach to predict non-vanishing spectral bandwidths.

Beall, Kim & Powers (1982) proposed the local wavenumber (LW) model, in which the spatial structure of a turbulent flow can be approximately represented by a local wave, where the local wavenumber is determined by the phase difference between the velocity fluctuations at two different measurement points, and the energy spectrum in terms of local wavenumbers is thus determined by the averaged energy of the velocity fluctuations at two measurement points. Therefore, if turbulent flows can be approximately represented by a superposition of all the local waves in a flow area, the conventional space–time energy spectrum is then approximately determined by the averaged energy of all the local waves of the same wavenumber and frequency.

de Kat & Ganapathisubramani (2015) developed a frequency–wavenumber mapping approach to reconstruct transfer functions from datasets of pair-point measurements, in which the transfer functions are equivalent to space–time energy spectra. The key point in this approach is that local waves of a fixed frequency can have multiple local wavenumbers. This means that the phase velocities of local waves may be distributed over a range of local wavenumbers. Therefore, the frequency–wavenumber mapping approach can account for the phase contribution to spectral broadening. Wilczek, Stevens & Meneveau (2015) developed a model for space–time energy spectra in the logarithmic layer of wall turbulence. This model introduced an additional mean velocity into the Kraichnan–Tennekes random sweeping hypothesis (Kraichnan 1964; Tennekes 1975; Wilczek & Narita 2012) and thus included the total contribution of phases to spectral broadening.

In both Taylor’s model and the LW model, only phase velocities are used to determine space–time energy spectra. In fact, it has been shown that both the phase and amplitude derivatives contribute to the spectral bandwidth (Wu et al. 2017). Spectral bandwidths are important indications of space–time energy spectra and characterize spectral broadening. Ignoring the contribution of amplitude derivatives will lead to significant underestimation of the spectral bandwidths. In the present paper, we develop a local modulated wave (LMW) model to reconstruct space–time energy spectra from the datasets of pair-point measurements. This model assumes that a spatial structure in turbulent flow can be approximately represented by a local modulated wave. Therefore, the LMW model can account for contributions of both amplitude and phase derivatives to the spectral bandwidths and thereby correctly predict the spectral bandwidths. The LMW model is not only a model to reconstruct space–time energy spectra but also a theoretical tool to study the coupling of convection and distortion in turbulent passages (He et al. 2017; He & Zhang 2006; Zhao & He 2009).

The previous paper (Wu et al. 2017) focused on the exact expression of spectral bandwidths and found that Taylor’s model and the LW model significantly underestimate the spectral bandwidths. To correctly predict the spectral bandwidths, we develop a new model (the LMW model) in the present paper and present the theoretical justification and numerical validation. The present paper is organized as follows. In § 2, we will develop the LMW model. The model will be first introduced for a one-dimensional velocity field in § 2.1, and a corresponding numerical procedure will be described in § 2.3. The model will be shown to exactly predict the mean wavenumbers and spectral bandwidths in § 2.2. In § 3, the LMW model will be illustrated by using an analytical example of a propagating Gaussian function with

frequencies (Renard & Deck 2015), which leads to non-vanishing spectral bandwidths.
a non-propagating Gaussian amplitude. In § 4, datasets from the direct numerical simulation (DNS) of turbulent channel flows are used to verify the LMW model. In § 5, conclusions and discussions on the LMW model will be provided.

2. Reconstructing space–time energy spectra from the LMW model

The first subsection § 2.1 will develop an LMW model for space–time energy spectra. It is shown in the second subsection § 2.2 that the LMW model exactly predicts the mean wavenumbers and bandwidths of space–time energy spectra. The final subsection § 2.3 presents a numerical procedure for using the LMW model to reconstruct space–time energy spectra. For convenience, we will present the results for a one-dimensional random field. However, the results for three-dimensional turbulent channel flows can be readily obtained by introducing two additional spatial variables. Notably, a Dirac delta function is introduced to express the space–time energy spectra. The obtained expressions explain how phase and amplitude variations contribute to space–time energy spectra and clearly indicate the differences of the LMW model relative to Taylor’s model and the LW model.

2.1. The LMW model

We start by using temporal Fourier modes to calculate the space–time energy spectra of velocity fields. Consider a one-dimensional stationary and homogeneous random field \( u(x, t) \) with a zero mean. The space–time correlation of the velocity field is defined as

\[
R(r, \tau) = \langle u(x, t)u(x + r, t + \tau) \rangle,
\]

where \( r \) is the streamwise separation, \( \tau \) is the time delay, the angular bracket denotes an ensemble average. The space–time energy spectrum \( \Phi(k_x, \omega) \) is defined as the Fourier transform of \( R(r, \tau) \):

\[
\Phi(k_x, \omega) = \frac{1}{(2\pi)^2} \int \int R(r, \tau) e^{-ik_x r - i\omega \tau} \, dr \, d\tau.
\]

In the present paper, the upper and lower limits of the integral are positive and negative infinity, respectively, unless they are explicitly indicated. The temporal energy spectrum \( \Phi_t(\omega) \) can be obtained by integrating \( \Phi(k_x, \omega) \) with respect to the wavenumber \( k_x \):

\[
\Phi_t(\omega) = \int \Phi(k_x, \omega) \, dk_x.
\]

The space–time energy spectra can be alternatively expressed using cross-correlations of temporal Fourier modes \( \hat{u}(x, \omega) \) at two different locations \( x \) and \( x + r \):

\[
\frac{\Phi(k_x, \omega)}{\Phi_t(\omega)} = \frac{1}{2\pi} \int \frac{\langle \hat{u}^*(x, \omega)\hat{u}(x + r, \omega) \rangle}{\langle \hat{u}^*(x, \omega)\hat{u}(x, \omega) \rangle} e^{-ik_x r} \, dr.
\]

If the temporal Fourier mode \( \hat{u}(x, \omega) \) is expressed in terms of its amplitude \( a(x, \omega) \) and phase \( \theta(x, \omega) \), such as

\[
\hat{u}(x, \omega) = a(x, \omega)e^{i\theta(x, \omega)},
\]

we have

\[
\frac{\Phi(k_x, \omega)}{\Phi_t(\omega)} = \frac{1}{2\pi} \int \frac{\langle a(x, \omega)a(x + r, \omega)e^{i[\theta(x+r,\omega)−\theta(x,\omega)]} \rangle}{\langle a^2(x, \omega) \rangle} e^{-ik_x r} \, dr.
\]
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\[ \Phi(k_x, \omega) = \frac{1}{2\pi} \int \frac{\langle a(x, \omega)a(x+r, r)e^{i(\theta(x+r) - k_x r)} \rangle}{\langle a^2(x, \omega) \rangle} dr. \quad (2.6) \]

In the above derivations, the Taylor series expansion of the phase \( \theta(x + r, \omega) \) at location \( x \) is taken up to the first order, and the exponential function \( e^{-ik_x r} \) is moved into the angle brackets (ensemble average). A linear approximation of the phase difference is used in (2.6) for space–time energy spectra. The approximation is valid if the spatial separation \( r \) has the order of the characteristic length scale at which the local wavenumber varies. This can be seen by the comparison of the second-order term with the first term in the Taylor expansion of the phase difference. The characteristic length scale thus obtained is given by \( r \ll (1 + U/\sqrt{\langle u^2 \rangle})(k_{loc}^x)^{-1} \), where \( k_{loc}^x = \partial_x \theta \) is the local wavenumber, \( U \) is a local mean velocity and \( u \) is a local velocity fluctuation.

We now introduce the LMW model and postpone its mathematical derivation to the later part of this subsection (see (2.15), (2.16) and (2.17)). The space–time energy spectra in (2.4) and (2.6) can be interpreted as ensemble averages of spatial and/or temporal Fourier modes. This suggests that, if the Fourier modes are represented by certain energetic flow structures, space–time energy spectra can thus be determined from the ensemble averages of the flow structures.

The simplest representation of a temporal Fourier mode is a travelling wave of propagation speed \( U \). In this case,

\[ \begin{align*}
a(x + r, \omega) &= a(x, \omega), \\
\theta(x + r, \omega) - \theta(x, \omega) &= \frac{\omega}{U} r.
\end{align*} \quad (2.7) \]

Hence, we substitute (2.7) into (2.6) to obtain the space–time energy spectra

\[ \Phi(LW)(k_x, \omega) \Phi_t(\omega) = \delta(k_x - \frac{\omega}{U}). \quad (2.8) \]

This result is consistent with Taylor’s frozen-flow hypothesis but incorrectly predicts the spectral bandwidth to be zero.

Beall et al. (1982) introduced a local wave to represent the temporal Fourier modes, wherein the phase was approximated by a first-order Taylor series expansion, such that

\[ \begin{align*}
a(x + r, \omega) &= a(x, \omega), \\
\theta(x + r, \omega) &= \theta(x, \omega) + \partial_x \theta \cdot r.
\end{align*} \quad (2.9) \]

Substituting (2.9) into (2.6), we obtain the space–time energy spectra

\[ \frac{\Phi^{LW}(k_x, \omega)}{\Phi_t(\omega)} = \frac{\langle a^2(x, \omega)\delta(k_x - \partial_x \theta) \rangle}{\langle a^2(x, \omega) \rangle}. \quad (2.10) \]

Here, and throughout this paper, the superscript ‘LW’ denotes a result from the LW model. It will be seen in § 2.2 that this result exactly predicts the first-order moment and under-predicts the second-order moment.

To correctly predict the spectral bandwidths, we introduce local modulated waves to represent the temporal Fourier modes. In the present study, a local modulated wave is defined as a wave-like structure of a single frequency and two distinct wavenumbers. It is mathematically expressed by
\[
\hat{u}(x + r, \omega) = \hat{u}(x, \omega) \left[ \frac{e^{i(a^{-1}\partial_a r - \pi/4)}}{\sqrt{2}} + \frac{e^{-i(a^{-1}\partial_a r - \pi/4)}}{\sqrt{2}} \right] e^{i\alpha \theta - r} \\
= \hat{u}(x, \omega) \cdot \sqrt{2} \cos(a^{-1}\partial_a r - \pi/4) e^{i\alpha \theta - r}.
\] (2.11)

In this case, the amplitude and phase of the local modulated wave downstream are related to those upstream by

\[
a(x + r, \omega) = a(x, \omega) \left[ \frac{e^{i(a^{-1}\partial_a r - \pi/4)}}{\sqrt{2}} + \frac{e^{-i(a^{-1}\partial_a r - \pi/4)}}{\sqrt{2}} \right] \\
\approx a(x, \omega) + \partial_r a \cdot r,
\] (2.12a)

\[
\theta(x + r, \omega) = \theta(x, \omega) + \partial \theta \cdot r.
\] (2.12b)

It can be found from (2.12a) and (2.12b) that (2.11) is exactly the same as the first-order Taylor expansions of both amplitude and phase. Therefore, (2.11) effectively represents a spatially localized wave-like structure due to the local validity of the Taylor expansion. Beall et al. (1982) utilized the first-order Taylor expansion of phase. The new development in this paper is that we utilize the first-order Taylor expansions of both amplitude and phase. Here, \(a^{-1}\partial_a r\) represents the characteristic length scales of the amplitude variations. The phase delay \(\pi/4\) and the factor \(\sqrt{2}\) are used such that (2.11) is consistent with the first-order Taylor expansion. Meanwhile, they can also guarantee that the integral of the space–time energy spectrum with respect to the wavenumber is equal to its marginal distribution, i.e. \(\int \Phi(k_x, \omega) dk_x = \Phi(\omega)\) (temporal energy spectrum).

In fact, according to the first expression of (2.11), a local modulated wave can be thought of as the superposition of two waves. Meanwhile, the local modulated wave can also be thought of as the product of two waves in terms of the second expression of (2.11). The one associated with the phase derivative is the carrier wave, and the one associated with the amplitude derivative is the wave envelope. The wave envelope modulates the carrier wave, which appears as a local modulated wave. Figure 1 shows the sketches of the local wave and local modulated wave, where the former is the local wave structure of a single wavenumber and the latter is the local wave-like structure of two wavenumbers. Therefore, a local modulated wave is the wavepacket of two wavenumbers.

Upon substituting (2.11) into (2.6) and invoking the homogeneity assumption \(\langle \hat{u}^* (x, \omega) \hat{u}(x + r, \omega) \rangle = \langle \hat{u}^* (x - r, \omega) \hat{u}(x, \omega) \rangle\), we perform the Fourier transform on the resulting equation with respect to \(r\). It is easily found that the imaginary parts of the space–time energy spectra \(\Phi(k_x, \omega)\) are zero. Therefore, we obtain the LMW model

\[
\frac{\Phi_{\text{LMW}}(k_x, \omega)}{\Phi_{\text{L}}(\omega)} = \frac{\langle a^2(x, \omega) \delta(k_x - k_x^+(x, \omega)) \rangle}{2\langle a^2(x, \omega) \rangle} + \frac{\langle a^2(x, \omega) \delta(k_x - k_x^-(x, \omega)) \rangle}{2\langle a^2(x, \omega) \rangle},
\] (2.13)

where

\[
k_x^+(x, \omega) = \partial_x \theta + a^{-1}\partial_a r,\] (2.14a)

\[
k_x^-(x, \omega) = \partial_x \theta - a^{-1}\partial_a r.\] (2.14b)

Here, and throughout this paper, the superscript ‘LMW’ denotes the results from the LMW model. The form of (2.13) is similar to that of the grained probability density.
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If the velocity fluctuation $u(x, t)$ is statistically homogeneous in space, the ensemble average can be taken as the spatial average over all locations $x$. As a result, (2.13) implies that the space–time energy spectra at wavenumber $k_x$ are the spatial average of the squared amplitudes over the locations $x$ at which the wavenumber $k_x^+$ or $k_x^-$ is equal to the wavenumber $k_x$. In other words, the present model proposes that if a velocity field is approximately represented by local modulated waves of wavenumber $k_x^+$ and $k_x^-$, the energy density at each wavenumber $k_x$ is the average of the energy densities of these carrier waves and wave envelopes with the same wavenumber $k_x$, i.e., $k_x^+ = k_x$ and $k_x^- = k_x$. Therefore, (2.13) implies that space–time energy spectra are the ensemble averages of energetic wave-like structures of the wavenumber $k_x$ at given frequency $\omega$. It will be seen in § 2.2 that (2.13) can exactly predict the first- and second-order moments of space–time energy spectra.

We will show that the space–time energy spectra take the form of (2.13) under the following three conditions: (i) the amplitudes are approximated by a second-order Taylor expansion; (ii) the phase is approximated by a first-order Taylor expansion; and (iii) the amplitude and phase are weakly correlated. These conditions are less stringent than those of the LW model (Beall et al. 1982). The cross-correlation of the

Figure 1. Schematic of (a) a local wave and (b) a local modulated wave in yellow shades; (c) energy spectra of local waves and local modulated waves.
amplitudes $a(x, \omega)$ and $a(x + r, \omega)$ can be estimated using a Taylor series expansion up to the second order and the homogeneity assumption

$$\langle a(x, \omega)a(x + r, \omega) \rangle \approx \langle a^2 \rangle + \langle a\partial_x a \rangle r + \frac{1}{2} \langle a\partial_x^2 a \rangle r^2$$

$$= \langle a^2 \rangle - \frac{1}{2} \langle (\partial_x a)^2 \rangle r^2. \quad (2.15)$$

It is also noted that the following identity holds

$$\frac{1}{2}a^2[e^{ia^{-1}\partial_x a \cdot r} + e^{-ia^{-1}\partial_x a \cdot r}] = a^2\cos(a^{-1}\partial_x a \cdot r) \approx a^2 - \frac{1}{2}(\partial_x a)^2 r^2. \quad (2.16)$$

A comparison of (2.16) with (2.15) yields

$$\langle a(x, \omega)a(x + r, \omega) \rangle \approx \frac{1}{2} \langle a^2[e^{ia^{-1}\partial_x a \cdot r} + e^{-ia^{-1}\partial_x a \cdot r}] \rangle. \quad (2.17)$$

Substituting (2.17) into (2.6) with the weakly correlated assumption of the amplitude and phase leads to (2.13).

The present model implies that if a velocity field is approximately represented by local modulated waves of wavenumbers $k_+^x$ and $k_-^x$, then the energy density at wavenumber $k_x$ is the ensemble average of the energy density of these waves and their envelopes with the same wavenumber $k_x$, i.e. $k_+^x = k_x$ and $k_-^x = k_x$.

This model can be used to reconstruct space–time energy spectra from the time series of velocity components at an array of in-line spatial locations. We will show in §2.2 that the present model can exactly predict the first- and second-order moments of energy spectra. If temporal information is obtained from experimental or numerical measurements, we can calculate the temporal Fourier modes and reconstruct the space–time energy spectra.

The decomposition of the local modulated waves into two waves with the same frequency but different wavenumbers can also be performed for the same wavenumber with two distinct frequencies. The former is suitable for the time series of velocity fluctuations at two or a limited number of measurement points, which is conveniently transformed into temporal Fourier modes, and the latter is suitable for the spatial series of velocity fluctuations with a short-term record, which is readily transformed into spatial Fourier modes. In experimental measurements such as HWA and LDA, the long-term records at a few measurement points are far more likely to be obtained than the spatial series of long lengths with short-term records. Therefore, the present study focuses on the local modulated waves of one frequency and two different wavenumbers.

The present model is consistent with the dominating processes of temporal decorrelation in turbulent flows. Temporal decorrelation is dominated by the following three processes: the convection of small-scale structures by large-scale structures, the interaction between small-scale structures and the stretching of small-scale structures by large-scale structures. The first process is primarily characterized by Taylor’s frozen-flow hypothesis (Taylor 1938), and the second process is determined by the Kraichnan–Tennekes random sweeping hypothesis (Kraichnan 1964; Tennekes 1975). The present work proposes that the third process can be approximately modelled by local modulated waves. If small-scale structures are approximated as local waves, large-scale stretching modifies the amplitudes of small-scale structures and consequently generates local modulated waves. This is particularly evident for slow-evolving large-scale flow structures and rapid-distorting small-scale flow structures.
The present work proposes that space–time energy spectra can be reconstructed from statistical averages of energetic flow structures. In the LW model, a local wave of a single wavenumber is introduced to represent an energetic flow structure. The obtained result correctly predicts the first-order moments of the space–time energy spectra but under-predicts the second-order moments. In the present LMW model, a local modulated wave of two local wavenumbers is introduced to represent the energetic flow structure. The obtained result correctly predicts the first- and second-order moments of the space–time energy spectra. Consequently, a high-order approximation can be achieved if a wavepacket of more local wavenumbers is introduced to represent the energetic flow structure. This constitutes a structure-based statistical approximation.

2.2. Mean wavenumbers and spectral bandwidths

A space–time energy spectrum \( \Phi(k_x, \omega) \) is a two-dimensional distribution of the energy density with respect to the wavenumber and frequency. Analogously to the case of a joint probability density function, a conditional distribution \( \Phi(k_x|\omega_0) \) can be introduced to describe the wavenumber spectrum of the energy density at a given frequency \( \omega_0 \), such that \( \Phi(k_x|\omega_0) = \frac{\Phi(k_x, \omega_0)}{\Phi_t(\omega_0)} \). Conditional distributions are cuts through space–time energy spectra at a given frequency. Furthermore, we introduce the first- and second-order conditional moments as

\[
k_{xc}(\omega_0) = \int k_x \Phi(k_x|\omega_0) \, dk_x
\]

and

\[
B(\omega_0) = \int [k_x - k_{xc}(\omega_0)]^2 \Phi(k_x|\omega_0) \, dk_x,
\]

which represent the mean wavenumbers and standard deviations of the conditional distributions, respectively.

The first-order conditional moments indicate the mean wavenumbers, which represent the mass centres of the conditional distributions of the energy densities. The mean wavenumbers are often used to estimate the frequency-dependent convection velocity, such that \( U_c(\omega) = \omega/k_{xc}(\omega) \) (Renard & Deck 2015). The second-order conditional moments indicate the bandwidths of the energy spectra. The spectral bandwidths are used to study the well-known ‘spectral broadening’ in turbulent flows (Lumley 1965; Wilczek & Narita 2012; Wilczek et al. 2015; Wu et al. 2017).

The mean wavenumber and spectral bandwidth can be exactly expressed using temporal Fourier modes, given by (Wu et al. 2017)

\[
k_{xc}(\omega) = \frac{\langle a^2 \bar{\partial}_x \theta \rangle}{\langle a^2 \rangle}
\]

and

\[
B(\omega) = \frac{\langle a^2 (\bar{\partial}_x \theta - k_{xc})^2 \rangle}{\langle a^2 \rangle} + \frac{\langle (\partial_x a)^2 \rangle}{\langle a^2 \rangle}.
\]

The mean wavenumbers are determined by phase derivatives alone, and the spectral bandwidths are determined by changes in both phase and amplitude. Our numerical simulations (Wu et al. 2017) show that the contribution of amplitude derivatives to the spectral bandwidth is not small compared to the phase contribution at a Reynolds
number of $Re_t = 550$ in turbulent channel flows. Therefore, the contribution of amplitude derivatives to spectral bandwidths cannot be ignored.

We now show that the LMW model exactly predicts the mean wavenumbers and spectral bandwidths. For this purpose, we substitute the LMW model (2.13) into (2.18) for the mean wavenumbers, which yields

$$k_{xc}^{\text{LMW}}(\omega) = \int k_x \Phi^{\text{LMW}}(k_x, \omega) \, dk_x / \Phi(t, \omega)$$

$$= \int k_x \langle a^2 \delta(k_x - \partial_x \theta - a^{-1} \partial_t a) \rangle \, dk_x / (2 \langle a^2 \rangle)$$

$$+ \int k_x \langle a^2 \delta(k_x - \partial_x \theta + a^{-1} \partial_t a) \rangle \, dk_x / (2 \langle a^2 \rangle)$$

$$= [\langle a^2 (\partial_x \theta + a^{-1} \partial_t a) \rangle + \langle a^2 (\partial_x \theta - a^{-1} \partial_t a) \rangle] / (2 \langle a^2 \rangle)$$

$$= \langle a^2 \partial_x \theta / \langle a^2 \rangle \rangle.$$

(2.22)

Therefore, upon comparing the above expression with (2.20), we have $k_{xc}^{\text{LMW}}(\omega) = k_{xc}(\omega)$. Similarly, we can show that $k_{xc}^{\text{LW}}(\omega) = \langle a^2 \partial_x \theta / \langle a^2 \rangle \rangle$. Next, we substitute the LMW model (2.13) into (2.19) for spectral bandwidths to obtain

$$B_{\text{LMW}}(\omega) = \int (k_x - k_{xc})^2 \Phi^{\text{LMW}}(k_x, \omega) \, dk_x / \Phi(t, \omega)$$

$$= \int (k_x - k_{xc})^2 \langle a^2 \delta(k_x - \partial_x \theta - a^{-1} \partial_t a) \rangle \, dk_x / (2 \langle a^2 \rangle)$$

$$+ \int (k_x - k_{xc})^2 \langle a^2 \delta(k_x - \partial_x \theta + a^{-1} \partial_t a) \rangle \, dk_x / (2 \langle a^2 \rangle)$$

$$= [\langle a^2 (\partial_x \theta + a^{-1} \partial_t a - k_{xc})^2 \rangle + \langle a^2 (\partial_x \theta - a^{-1} \partial_t a - k_{xc})^2 \rangle] / (2 \langle a^2 \rangle)$$

$$= [\langle a^2 (\partial_x \theta - k_{xc})^2 \rangle + \langle (\partial_t a)^2 \rangle] / \langle a^2 \rangle.$$

(2.23)

Accordingly, upon comparing the above expression with (2.21), we have $B_{\text{LMW}}(\omega) = B(\omega)$. Similarly, we can show that $B_{\text{LW}}(\omega) = \langle a^2 (\partial_x \theta - k_{xc})^2 \rangle / \langle a^2 \rangle$.

The exact expression (2.20) implies that the first-order conditional moments are fully determined by the energy-weighted averages of the phase derivatives. It is noted that the LW model and LMW model include the contributions of phase derivatives. As a result, these two models can correctly predict the first-order conditional moments. The exact expression (2.21) indicates that the spectral bandwidths are determined by both phase and amplitude derivatives. However, the LW model excludes the contribution of amplitude derivatives and thus under-predicts the spectral bandwidths. The LMW model can correctly predict the spectral bandwidths since it takes into account the contributions of both phase and amplitude derivatives.

2.3. Numerical procedure

This section presents a numerical procedure to reconstruct space–time energy spectra. The numerical procedure is based on the LMW model and will be illustrated and validated in §§ 3 and 4, respectively. The input for the numerical procedure is a non-periodic and incomplete dataset of velocity fluctuations $u(x_i, t_j)$ at spatial points $x_i$ and discrete times $t_j$. Although the procedure is presented for a one-dimensional random field, it can be extended to turbulent channel flows.
The numerical procedure for the LMW model is described as follows:

(i) We perform Fourier transforms to obtain the temporal Fourier modes of the velocity fluctuations \( u(x, t) \):

\[
\hat{u}(x, \omega) = \frac{1}{\sqrt{T}} \int_{0}^{T} w(t) u(x, t) e^{i \omega t} \, dt,
\]

(2.24)

wherein a Hanning window \( w(t) \) is used and the window length \( T \) is taken as the maximal time scale.

(ii) We calculate the cross-spectra \( \Psi(x, \Delta x, \omega) \) of the temporal Fourier modes at two different spatial locations \( x \) and \( x + \Delta x \):

\[
\Psi(x, \Delta x, \omega) \equiv \hat{u}^{*}(x, \omega) \hat{u}(x + \Delta x, \omega).
\]

(2.25)

Consequently, we find the cross-spectral energy \( |\Psi| \) and the phase difference \( \Delta \theta \equiv \text{arg}(\Psi) = \theta(x + \Delta x, \omega) - \theta(x, \omega) \). Here, \( \text{arg} \) is denoted as the phase of a complex variable; \( \Delta \theta \) is calculated by the unfolding technique (Buxton, de Kat & Ganapathisubramani 2013; de Kat & Ganapathisubramani 2015), such that \( \Delta \theta \) is located in the range of \( \omega \Delta x/U_c - \pi < \Delta \theta < \omega \Delta x/U_c + \pi \) with \( U_c \) being the convection velocity.

(iii) The amplitude and phase derivatives of the temporal Fourier modes are calculated using the finite-difference approximation

\[
\partial_{a} \theta = \Delta \theta / \Delta x,
\]

\[
\partial_{a} a = \frac{\Delta a}{\Delta x} \equiv \frac{a(x + \Delta x, \omega) - a(x, \omega)}{\Delta x}.
\]

(2.26)

(2.27)

(iv) Two distinct wavenumbers of a local modulated wave are determined from their definitions (2.14), such that

\[
k_{x}^{+}(x, \omega) = \frac{\Delta \theta}{\Delta x} + \frac{1}{\sqrt{|\Psi|}} \frac{\Delta a}{\Delta x},
\]

(2.28)

and

\[
k_{x}^{-}(x, \omega) = \frac{\Delta \theta}{\Delta x} - \frac{1}{\sqrt{|\Psi|}} \frac{\Delta a}{\Delta x},
\]

(2.29)

where the amplitude \( a \) is estimated by the square root of the cross-spectral energy \( \sqrt{|\Psi|} \).

(v) The energy spectra corresponding to wavenumbers \( k_{x}^{+} \) and \( k_{x}^{-} \) of the local modulated waves are estimated from the cross-spectra

\[
\frac{\Phi^{+}(k_{x}, \omega) \Delta k_{x}}{\Phi_{i}(\omega)} = \sum_{k_{x} \in \text{Bin}(k_{x})} \left( \frac{1}{2} |\Psi(x, \omega; \Delta x)| \right) ,
\]

(2.30)

\[
\frac{\Phi^{-}(k_{x}, \omega) \Delta k_{x}}{\Phi_{i}(\omega)} = \sum_{k_{x} \in \text{Bin}(k_{x})} \left( \frac{1}{2} |\Psi(x, \omega; \Delta x)| \right) ,
\]

(2.31)
where the interval Bin\(k_i\)  $$\equiv [k_i - \Delta k_i/2, k_i + \Delta k_i/2]$$ is centred at wavenumber \(k_i\) with a small wavenumber bandwidth \(\Delta k_i\). The summation in the numerator of (2.30) is taken for all locations \(x\) such that the local wavenumbers \(k^+_i(x, \omega)\) belong to Bin\(k_i\). Similarly, the summation in the numerator of (2.31) is taken for all locations \(x\) such that the wavenumbers \(k^-_i(x, \omega)\) belong to Bin\(k_i\). The summations in the denominators of (2.30) and (2.31) are taken for all locations \(x\). The cross-spectral energy \(|\Psi|\) in (2.28), (2.29), (2.30) and (2.31) can be replaced by the averaged energy at the two points, \([a^2(x, \omega) + a^2(x + \Delta x, \omega)]/2\), which can conserve the energy of the space–time spectra as the averaged energies at the two points. Our numerical comparisons show that the results obtained from cross-spectral energy are very close to those from the averaged energy at the two points.

(vi) We finally calculate the sum of the energy spectra \(\Phi^+\) and \(\Phi^-\) corresponding to \(k^+_i\) and \(k^-_i\), respectively, for the space–time energy spectra

\[
\Phi^{LMW}(k_i, \omega) = \Phi^+(k_i, \omega) + \Phi^-(k_i, \omega).
\] (2.32)

The numerical procedure for one-dimensional random fields can be easily extended to calculate the space–time energy spectra of a single wavenumber and frequency for three-dimensional random fields, such as the joint streamwise-wavenumber and frequency energy spectra of streamwise components of three-dimensional velocity fields in turbulent channel flows. The corresponding changes for the numerical procedure are listed as follows: (i) \(u(x, t)\) and \(\hat{u}(x, \omega)\) in (2.24) and (2.25) are replaced separately by \(u(x, t; y, z)\) and \(\hat{u}(x, \omega; y, z) \equiv a(x, \omega; y, z) \exp[i\theta(x, \omega; y, z)]\). Here, \(u\) is a velocity component in the streamwise direction; \(x, y\) and \(z\) denote the coordinates in the streamwise, wall-normal and spanwise directions, respectively. For turbulent channel flows, the space–time energy spectra are dependent on the wall-normal height \(y\). Therefore, \(k^+_i(x, \omega), \Phi^\pm(k_i, \omega), \Phi_i(\omega)\) and \(\Phi^{LMW}(k_i, \omega)\) in (2.28)–(2.32) are replaced by \(k^+_i(x, \omega; y, z), \Phi^\pm(k_i, \omega; y), \Phi_i(\omega; y)\) and \(\Phi^{LMW}(k_i, \omega; y)\), respectively. (ii) The summations in both the numerators and denominators of (2.30) and (2.31) should be carried out for all locations \(z\) and for all locations \(x\) at which certain wavenumbers are taken, since the streamwise velocity component is homogeneous in the spanwise direction \(z\).

The LMW model can be used to correctly predict the mean wavenumbers and spectral bandwidths if (i) the measurement resolution \(\Delta x\) in space is reasonably small to correctly estimate the derivatives of phase and amplitude and (ii) the length of the measurement in time is sufficiently large to provide the ensemble of space–time signals to estimate the space–time energy spectra. In other words, the length of the time record is sufficiently large to allow the wave-like structures of wavelength \(\lambda\) to pass the entire measurement domain (of length \(L_x\) in one-dimension) several times. Usually, the spatial resolution \(\Delta x\) is smaller than the domain size (\(\Delta x < L_x\)) and required to be smaller than the wavelength (\(\Delta x < \lambda\)). Therefore, if \(\lambda > L_x\), then \(\Delta x < L_x < \lambda\), which implies that the LMW model can be used for the wave-like structures of long wavelength that is comparable with the length scales of the measurement domains. However, at this moment, we do not intend to use the LMW model for reconstructing the instantaneous velocity fields, such as superstructures in the logarithmic and lower wake regions of turbulent boundary layers (Hutchins & Marusic 2007; Kevin, Monty & Hutchins 2019), which is a very interesting topic for future study.
3. An illustrative example for the LMW model

In this section, we use a simple example to illustrate and validate the LMW model. This example is a propagating Gaussian signal modulated by a non-propagating Gaussian amplitude, which takes the form of

$$u(x, t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp \left( -\frac{x^2}{2\sigma_x^2} \right) \exp \left\{ -\frac{[t - x/(U + v)]^2}{2\sigma_t^2} \right\}. \tag{3.1}$$

Here, $\sigma_x$ is the spatial standard deviation of the non-propagating Gaussian amplitude and $\sigma_t$ is the temporal standard deviation of the propagating Gaussian signal; $U$ is a constant propagation speed; $v$ is a random sweeping velocity that satisfies a Gaussian distribution of zero mean and variance $V^2$ and is independent of space and time. Figure 2 shows the temporal evolution of the function $u(x, t)$ for $\sigma_x = 3.54$, $\sigma_t = 0.24$, $U = 12$ and $v = 0$, from which we find that its amplitude and phase depend on both space and time.

To simplify the presentation, we now consider the case of $v = 0$. The results for $v \neq 0$ will be provided at the end of this section, with random phase effects included. In the case of $v = 0$, the space–time energy spectrum of $u(x, t)$ is given by

$$\Phi(k_x, \omega) = \frac{\sigma_x}{\sqrt{\pi}} \exp \left[ -\sigma_x^2 \left( k_x - \frac{\omega}{U} \right)^2 \right] \Phi_t(\omega). \tag{3.2}$$

We substitute (3.2) into (2.18) and (2.19) to find the mean wavenumbers and spectral bandwidths, respectively,

$$k_{xc}(\omega) = \int k_x \Phi(k_x | \omega) \, dk_x = \frac{\omega}{U}, \tag{3.3}$$

$$B(\omega) = \int (k_x - k_{xc})^2 \Phi(k_x | \omega) \, dk_x = \frac{1}{2\sigma_x^2}. \tag{3.4}$$

For the following discussion, we also calculate the temporal Fourier modes

$$\hat{u}(x, \omega) = a(0, \omega) \exp \left( -\frac{x^2}{2\sigma_x^2} \right) \exp \left( i\frac{\omega x}{U} \right). \tag{3.5}$$
and their amplitudes and phases

\[ a(x, \omega) = a(0, \omega) \exp \left( -\frac{x^2}{2\sigma_x^2} \right), \quad (3.6) \]

\[ \theta(x, \omega) = \frac{\omega x}{U}. \quad (3.7) \]

Taylor’s frozen-flow hypothesis cannot be applied to the present example, since \( u(x, t + \tau) \neq u(x - U\tau, t) \). The LW model also cannot be applied to the present example, since the amplitude derivative \( \partial_x a \neq 0 \) explicitly depends on the location \( x \). In fact, the local wavenumbers in this model are given by

\[ k^{LW}_x(x, \omega) = \partial_x \theta = \frac{\omega}{U}. \quad (3.8) \]

Substitution of (3.8) into the LW model, equation (2.10), yields the space–time energy spectrum

\[ \Phi^{LW}(k_x, \omega) = \Phi_t(\omega) \delta(k_x - \omega/U). \quad (3.9) \]

According to (3.9), we find the mean wavenumber \( k^{LW}_x(\omega) = \omega/U \) and the spectral bandwidth \( B^{LW}(\omega) = 0 \). The latter is different from the exact result, as given by (3.4).

In the LMW method, the two distinct wavenumbers are determined from (2.14)

\[ k^+_x(x, \omega) = \partial_x \theta + a^{-1} \partial_x a = \frac{\omega}{U} - \frac{x}{\sigma_x^2}, \quad (3.10) \]

\[ k^-_x(x, \omega) = \partial_x \theta - a^{-1} \partial_x a = \frac{\omega}{U} + \frac{x}{\sigma_x^2}. \quad (3.11) \]

Here, the local wavenumbers are determined by both the phase and amplitude derivatives. We use the LMW model (2.13) to calculate the space–time energy spectra

\[ \Phi^{LMW}(k_x, \omega) = \frac{\sigma_x}{\sqrt{\pi}} \exp \left[ -\sigma^2_x \left( k_x - \frac{\omega}{U} \right)^2 \right] \Phi_t(\omega). \quad (3.12) \]

In this calculation, the following equation is used, which is derived from two properties of Dirac delta functions:

\[ \delta \left[ k_x - \left( \frac{\omega}{U}, \pm \frac{x}{\sigma_x^2} \right) \right] = \sigma_x^2 \delta \left[ x \mp \sigma_x^2 \left( k_x - \frac{\omega}{U} \right) \right]. \quad (3.13) \]

It can be found by comparing (3.12) with (3.2) that the space–time energy spectra obtained from the LMW model are the same as the exact ones. In particular, they have the same mean wavenumbers and spectral bandwidths

\[ k^{LMW}_x(\omega) = k^{LW}_x(\omega) = k_x(\omega) = \omega/U, \quad (3.14) \]

\[ B^{LMW}(\omega) = B(\omega) = 1/(2\sigma_x^2) \neq B^{LW}(\omega) = 0. \quad (3.15) \]

It is noted that the rescaling technique (Wu et al. 2017) in combination with the LW model fails for the vanishing bandwidth in the LW model, since it requires that the rescale factor is non-zero. In the limit of \( \sigma_x = \infty \), we will have \( \partial_x a = 0 \); thus, \( a \) is independent of \( x \). Hence, the LMW model is reduced to the LW model, and both models yield the same spectral bandwidths.
Finally, we present the results for a non-vanishing random sweeping velocity $v \neq 0$, without a detailed derivation. In this case, the space–time energy spectra take the form of

$$
\Phi(k_x, \omega) = \Phi_t(\omega) \frac{\sigma_x}{\sqrt{2\pi V^2}} \int \exp \left[ -\left( k_x - \frac{\omega}{U + v} \right)^2 \sigma_x^2 \right] \exp \left( -\frac{v^2}{2V^2} \right) dv.
$$

It can be found from the above results that the sweeping velocity plays the role of a random phase and increases the spectral bandwidths. This is consistent with the results of Wilczek et al. (2015). However, the sweeping velocity is not related to the amplitude modulation, as shown by the first term in (3.18). The application of the LMW model to this case leads to correct predictions,

$$
\Phi_{LMW}(k_x, \omega) = \Phi(k_x, \omega),
$$

$k_{xc}^{LMW}(\omega) = k_{xc}(\omega)$ and $B^{LMW}(\omega) = B(\omega)$, which again validates the LMW model.

4. Numerical validation

We use DNS data of turbulent channel flows at $Re_t \equiv u_t h / \nu = 550$ to validate the LMW model, where $u_t$ is the friction velocity, $2h$ is the height of the channel and $\nu$ is the kinematic viscosity. The bulk Reynolds number $Re \equiv U_b h / \nu$ is approximately 10 000, where $U_b$ is the bulk velocity. The numerical set-up is sketched in figure 3(a). The computational domain is chosen to be $8\pi h$ in the streamwise direction ($x$) and $3\pi h$ in the spanwise direction ($z$). The grid numbers $1536 \times 256 \times 1152$ are used in the streamwise ($x$), wall-normal ($y$) and spanwise ($z$) directions, respectively. The streamwise and spanwise resolutions are $\Delta_+^x = 8.95$ and $\Delta_+^z = 4.47$ ('+' indicates normalization with wall units), respectively. The minimum and maximum wall-normal resolutions are $\min(\Delta_+^y) = 0.04$ near the wall and $\max(\Delta_+^y) = 6.67$ at the centre of the channel. Periodic boundary conditions are used in the streamwise and spanwise directions, and no-slip boundary conditions are employed at the bottom and top walls.
A pseudo-spectral method is adopted to numerically solve the Navier–Stokes equations. Time is advanced via a third-order, stiffly stable scheme. The time step is taken as $\Delta t = 1.25 \times 10^{-3} h/U_b$ ($\Delta t^+ \approx 0.037$). The Navier–Stokes solver used in the present study has been validated in previous studies (Deng & Xu 2012; Geng et al. 2015; Wu et al. 2017). The mean streamwise velocity profile and turbulence intensities are consistent with those of Del Álamo & Jiménez (2003) at the same Reynolds number. Figure 3(b) shows instantaneous fluctuations of the streamwise velocity component on the $x$–$z$ plane at $y^+ = 92$.

The dataset was stored at every eight time steps during a period of $20.48 h/U_b$ as the flow reached a statistically stationary state. To calculate the temporal Fourier mode of the velocity fluctuations $u(x, t; y, z)$ at a given wall-normal coordinate $y$ and spanwise coordinate $z$, we divide the dataset into overlapping intervals in the time domain with a 50% overlap and use the Hanning window to minimize spectral leakage. This leads to

$$\hat{u}(x, \omega; y, z) = \frac{1}{\sqrt{T}} \int_0^T \int_0^T w(t) u(x, t; y, z) e^{i \omega t} dt,$$

where $w(t)$ is the Hanning window, and $T$ is the interval length taken as $T = 5.12 h/U_b$ in the present calculations. The temporal energy spectrum is calculated by

$$\Phi_t(\omega; y) = \frac{\langle \hat{u}^* (x, \omega; y, z) \hat{u}(x, \omega; y, z) \rangle}{\Delta \omega},$$

where $\Delta \omega = 2\pi/T$, and the angular bracket denotes the ensemble average which is performed in time and in the streamwise and spanwise directions due to homogeneity. The space–time Fourier mode $\tilde{u}(k_x, \omega; y, z)$ is determined by

$$\tilde{u}(k_x, \omega; y, z) = \frac{1}{L_x} \int_0^{L_x} \hat{u}(x, \omega; y, z) e^{-i k_x x} dx.$$

It is noted that no window is used in the periodic streamwise direction. The space–time energy spectrum $\Phi(k_x, \omega; y)$ is calculated by

$$\Phi(k_x, \omega; y) = \frac{\langle \tilde{u}^* (k_x, \omega; y, z) \tilde{u}(k_x, \omega; y, z) \rangle}{\Delta k_x \Delta \omega},$$

where $\Delta k_x = 2\pi/L_x$, and the angular bracket denotes the ensemble average which is performed in time and in the spanwise direction. In the calculations of the LW and LMW models, $\Delta x$ is taken to be the same as the streamwise mesh size $\Delta x$. The numerical technique used in this section has been verified in our previous study (Wu et al. 2017).

Figure 4 shows the space–time energy spectra at $y^+ = 92$ in the logarithmic region of turbulent channel flows obtained from the DNS and three different models. The two-dimensional surface of the space–time energy spectrum plotted in figure 4(a) is narrow and mountain-like. It decays most slowly along the ridge line and has a bell-shaped slice (denoted by the red solid line) at a fixed frequency. The projection of the ridge line onto the $\omega - k_x$ plane is the characteristic line $\omega - k_x U_c = 0$ (denoted by the black dashed line), which relates the mean wavenumbers $k_x$ with the frequencies $\omega$ through the convection velocity $U_c$. The width of the slice (denoted by the arrow
Local modulated wave model for space–time energy spectra

Figure 4. Space–time energy spectra of the streamwise velocity fluctuations at $y^+ = 92$ in turbulent channel flows at $Re = 550$. The coloured surfaces show the energy spectra from (a) the DNS, (b) Taylor’s frozen-flow hypothesis, (c) the LW model and (d) the LMW model.

The line) indicates the spectral bandwidths of the conditional distribution of the energy density at a given frequency. It can be clearly observed from figure 4(b) that Taylor’s frozen-flow model displays the same characteristic line and vanishing bandwidth. The two-dimensional surfaces in figures 4(c) and 4(d) look similar to the one in figure 4(a). All three surfaces present the same ridge lines. However, the bandwidth of the slice in figure 4(c) is narrower than that of the slice in figure 4(a). The bandwidth of the slice in figure 4(d) is in agreement with that of the slice in figure 4(a). These observations are consistent with the results from the LW model and LMW model. We will quantify the differences in spectral bandwidths in figures 8 and 9.

Figures 5(a) and 5(c) compare the contours of space–time energy spectra at $y^+ = 92$ from the DNS, the LW model and the LMW model. Wavenumbers and frequencies are also normalized by wall units, wherein $\delta_v \equiv v/u_\tau$ is the viscous length scale. It is observed that their shapes are very similar. However, the line contours from the LW model are located within those from the DNS. The line contours from the LMW model are in good agreement with those from the DNS. In figures 5(b) and 5(d), we further compare slices of the space–time energy spectra at three different frequencies.
from the DNS, the LW model and the LMW model. The shapes of the curves are very similar. However, the slices from the LMW model are closer to those from the DNS than those from the LW model. In other words, the LMW model presents more spreading in the slices than does the LW model, which is in agreement with the discussion in § 2. Wu et al. (2017) use the rescaling technique in combination with the LW model to obtain the space–time energy spectra. However, the rescaling technique needs the exact value of the spectral bandwidth \( a \text{ priori} \). The LMW model does not need this value for spectral bandwidths.

Figures 6(a) and 6(c) compare the contours of the premultiplied spectra of energy density at \( y^+ = 92 \) from the DNS, the LW model and the LMW model. The premultiplied spectrum for the space–time energy spectrum is defined as

\[
k_s \omega \Phi'(k_s, \omega) = k_s \omega [\Phi(k_s, \omega) + \Phi(-k_s, \omega)].
\]
Local modulated wave model for space–time energy spectra

$\frac{\omega h}{U_b} = 6.14$

$\frac{\omega h}{U_b} = 20.9$

$\frac{\omega h}{U_b} = 35.6$

It is observed that the line contours of the premultiplied spectra from the LMW model are closer to the DNS results than those from the LW model. In figures 6(b) and 6(d),
we further compare the slices of the premultiplied spectra at three different frequencies from the DNS, the LW model and the LMW model. The slices from the LMW model are in good agreement with those from the DNS. However, the peaks from the LW model overshoot those from the DNS. Therefore, the LMW model better represents the energetic flow structures than does the LW model.

To quantify the differences between the models and the DNS results, we compare the Hellinger distance between the conditional spectral distributions from the models and the DNS. The Hellinger distance (Gibbs & Su 2002; Liese & Vajda 2006) is defined as the total of the point-to-point differences between the models and the DNS results, which is given by

\[ d_H(\Phi_{\text{model}}, \Phi_{\text{DNS}}) = \sqrt{\int \left( \frac{\Phi_{\text{model}}(k_x|\omega) - \Phi_{\text{DNS}}(k_x|\omega)}{\Phi_{\text{DNS}}(k_x|\omega)} \right)^2 \, dk_x} \]  

(4.7)

The Hellinger distance \( d_H = 0 \) if and only if \( \Phi_{\text{model}} = \Phi_{\text{DNS}} \). As shown in figure 7, the Hellinger distances between the LMW model and the DNS results are smaller than those between the LW model and the DNS results, which implies that the LMW model is better than the LW model.

Figure 8 shows the bandwidths of the space–time energy spectra obtained from the DNS, the LW model and the LMW model at five different heights in the channel. These heights cover the typical wall layers from the viscous sublayer to the outer layer. The frequencies in the present figures are normalized using the peak frequency \( \omega_p(y^+) \) of the dissipation spectra \( \omega^2 \Phi_t(\omega; y^+) \), since the characteristic frequencies in different wall layers are not the same. For all heights in the channel, the bandwidths increase with increasing frequencies. We again find that the results from the LMW model are in good agreement with those from the DNS. However, the results from the LW model are significantly smaller than those from the DNS.
Local modulated wave model for space–time energy spectra

Figure 8. Bandwidths of the conditional distributions of the space–time energy spectra at the given frequencies for streamwise velocity fluctuations in turbulent channel flows at $Re_{\tau} = 550$. The results from the DNS are denoted by red lines with squares, the results from the LW model are denoted by blue lines with triangles and the results from the LMW model are denoted by green lines with circles. (a) $y^+ = 5$. (b) $y^+ = 12$. (c) $y^+ = 44$. (d) $y^+ = 92$. (e) $y^+ = 270$. Note that the frequencies are normalized by the peak frequency $\omega_p(y^+)$ of the dissipation spectrum $\omega^2 \Phi_t(\omega; y^+)$. Figure 9 shows the errors in the bandwidths, $\varepsilon_B = |B_{\text{model}} - B_{\text{DNS}}|/B_{\text{DNS}}$, obtained from the LW model and the LMW model relative to the DNS results at the five
different heights in the channel. The relative errors of the LW model remain between 40% and 50%. The relative errors in the bandwidths obtained from the LMW model are almost zero, which again demonstrates the effectiveness of the LMW model. The nominator of the relative error that is the absolute error must be bounded by a constant since the LMW model exactly predicts the spectral bandwidths. Meanwhile, the denominator of the relative error that is the spectral bandwidth decreases at lower frequencies. Therefore, their ratio becomes larger at lower frequencies.

In figures 10 and 11, the effects of the separation distance $\Delta x$ between two measurement locations on the accuracy of the LMW model are investigated using the DNS data at $y^+=92$. Figures 10(a) and 10(b) compare the contours of space–time energy spectra from the LMW model at different separations $\Delta x$ with the result at $\Delta x = \Delta_x$. The result with $\Delta x = 10\Delta_x$ ($\Delta x^+ = 89.5$) is better than that with $\Delta x = 20\Delta_x$ ($\Delta x^+ = 179$). As expected, the accuracy of the LMW model becomes better with decreasing separation distance. Figures 10(c) and 10(d) compare the contours of the premultiplied spectra of energy density at different separations $\Delta x$ with the result at $\Delta x = \Delta_x$. The results obtained are in consistent with figures 10(a) and 10(b). It can be observed from figure 11(a) that the spectral bandwidths at $\Delta x = 10\Delta_x$ are close to the DNS results, while the spectral bandwidths at $\Delta x = 20\Delta_x$ deviate from the DNS results. The relative errors of the LMW model shown in figure 11(b) are consistent with figures 10 and 11(a). It is noted that the relative error is less than 10% for low and moderate frequencies at the separation distance $\Delta x = 10\Delta_x$ ($\Delta x^+ = 89.5$).
5. Conclusions and future work

We develop a statistical model for reconstructing space–time energy spectra in turbulent flows. It is shown that this model can exactly predict the mean wavenumbers and spectral bandwidths of the space–time energy spectra. Both Taylor’s frozen-flow model and the LW model can predict the mean wavenumbers. However, the former incorrectly predicts the spectral bandwidths to be zero, and the latter excludes the contributions of amplitude derivatives, thus significantly under-predicting the spectral bandwidths. It is noted that the LW model uses the amplitudes as weights but does not explicitly use the amplitude derivatives. Instead, the present model includes both
Figure 11. (a) Bandwidths and (b) the relative errors of the bandwidths of the space–time energy spectra reconstructed from the LMW model at $\Delta x = 10\Delta_x$ ($\Delta x^+ = 89.5$) and $\Delta x = 20\Delta_x$ ($\Delta x^+ = 179$).

Phase and amplitude derivatives so that it can correctly predict the mean wavenumbers and spectral bandwidths.

The LMW model can be used to numerically reconstruct space–time energy spectra from an incomplete dataset in physical space. The dataset required for the LMW model is the time series of velocity fluctuations at a finite number of measurement points, wherein extensive spatial sampling is not needed. As an analytical example, a propagating Gaussian signal modulated by a non-propagating Gaussian function is designed to illustrate the LMW model. Furthermore, the DNS of turbulent channel flows is performed to verify the LMW model. All the results validate the LMW model.

Taylor’s frozen-flow hypothesis proposed a physical picture for turbulent passages, in which the spatial patterns of turbulent motions are carried past a fixed point at the convection velocity without significantly changing. The hypothesis highlights an attack line to reconstruct space–time energy spectra of turbulent flows. As a first approximation, flow patterns are assumed to be local waves, which leads to the LW model. The present paper suggests a second approximation, in which the energetic flow patterns are represented by local modulated waves, such that the proposed model can exactly predict the spectral bandwidths. Wavepackets that include more wavenumbers can provide a higher approximation of the space–time energy spectra.

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Declaration of interests

The authors report no conflict of interest.
REFERENCES

ADRIAN, R. J. & MOIN, P. 1988 Stochastic estimation of organized turbulent structure: homogeneous shear flow. J. Fluid Mech. 190, 531–559.

BEALL, J. M., KIM, Y. C. & POWERS, E. J. 1982 Estimation of wavenumber and frequency spectra using fixed probe pairs. J. Appl. Phys. 53 (6), 3933–3940.

BOSSUYT, J., MENEVEAU, C. & MEYERS, J. 2017 Wind farm power fluctuations and spatial sampling of turbulent boundary layers. J. Fluid Mech. 823, 329–344.

BUXTON, O. R. H., DE KAT, R. & GANAPATHISUBRAMANI, B. 2013 The convection of large and intermediate scale fluctuations in a turbulent mixing layer. Phys. Fluids 25, 125105.

CENEDESE, A., ROMANO, G. P. & DE Felice, F. 1991 Experimental testing of Taylor’s hypothesis by LDA in highly turbulent flow. Exp. Fluids 11, 351–358.

DEL ÁLAMO, J. C. & JIMÉNEZ, J. 2003 Spectra of the very large anisotropic scales in turbulent channels. Phys. Fluids 15, L41–L44.

DEL ÁLAMO, J. C. & JIMÉNEZ, J. 2009 Estimation of turbulent convection velocities and corrections to Taylor’s approximation. J. Fluid Mech. 640, 5–26.

DENG, B. Q. & XU, C. X. 2012 Influence of active control on STG-based generation of streamwise vortices in near-wall turbulence. J. Fluid Mech. 710, 234–259.

DURAIANDAYAM, K., IACCARINO, G. & XIAO, H. 2019 Turbulence modeling in the age of data. Annu. Rev. Fluid Mech. 51, 357–377.

FISHER, M. J. & DAVIES, P. O. A. 1964 Correlation measurements in a non-frozen pattern of turbulence. J. Fluid Mech. 18, 97–116.

GENG, C. H., HE, G. W., WANG, Y. S., XU, C. X., LOZANO-DURÁN, A. & WALLACE, J. M. 2015 Taylor’s hypothesis in turbulent channel flow considered using a transport equation analysis. Phys. Fluids 27, 025111.

GHATE, A. S. & LELE, S. K. 2017 Subfilter-scale enrichment of planetary boundary layer large eddy simulation using discrete Fourier-Gabor modes. J. Fluid Mech. 819, 494–539.

GIBBS, A. L. & SU, F. E. 2002 On choosing and bounding probability metrics. Intl Stat. Rev. 70 (3), 419–435.

HE, G. W., JIN, G. D. & YANG, Y. 2017 Space–time correlations and dynamic coupling in turbulent flows. Annu. Rev. Fluid Mech. 49, 51–70.

HE, G. W., WANG, M. & LELE, S. K. 2004 On the computation of space–time correlations by large-eddy simulation. Phys. Fluids 16 (11), 3859–3867.

HE, G. W. & ZHANG, J. B. 2006 Elliptic model for space–time correlations in turbulent shear flows. Phys. Rev. E 73, 055303.

HOWLAND, M. F. & YANG, X. I. A. 2018 Dependence of small-scale energetics on large scales in turbulent flows. J. Fluid Mech. 852, 641–662.

HUTCHINS, N. & MARUSIC, I. 2007 Evidence of very long meandering features in the logarithmic region of turbulent boundary layers. J. Fluid Mech. 579, 1–28.

HUTCHINS, N., NICKELS, T. B., MARUSIC, I. & CHONG, M. S. 2009 Hot-wire spatial resolution issues in wall-bounded turbulence. J. Fluid Mech. 635, 103–136.

JIMÉNEZ, J. 2012 Cascades in wall-bounded turbulence. Annu. Rev. Fluid Mech. 44, 27–45.

DE KAT, R. & GANAPATHISUBRAMANI, B. 2015 Frequency-wavenumber mapping in turbulent shear flows. J. Fluid Mech. 783, 166–190.

KEVIN, K., MONTY, J. & HUTCHINS, N. 2019 The meandering behaviour of large-scale structures in turbulent boundary layers. J. Fluid Mech. 865, R1.

KRAICHNAN, R. H. 1964 Kolmogorov’s hypotheses and Eulerian turbulence theory. Phys. Fluids 7, 1723–1734.

KRAICHNAN, R. H. 1966 Isotropic turbulence and inertial-range structure. Phys. Fluids 9 (9), 1728–1752.

LIESE, F. & VAJDA, I. 2006 On divergences and informations in statistics and information theory. IEEE Trans. Inf. Theory 52 (10), 4394–4412.

LUMLEY, J. L. 1965 Interpretation of time spectra measured in high-intensity shear flows. Phys. Fluids 8, 1056–1062.
Mancinelli, M., Pagliaroli, T., Camussi, R. & Castelain, T. 2018 On the hydrodynamic and acoustic nature of pressure proper orthogonal decomposition modes in the near field of a compressible jet. J. Fluid Mech. 836, 998–1008.

Moin, P. 2009 Revisiting Taylor’s hypothesis. J. Fluid Mech. 640, 1–4.

Pope, S. B. 2000 Turbulent Flows. Cambridge University Press.

Renard, N. & Deck, S. 2015 On the scale-dependent turbulent convection velocity in a spatially developing flat plate turbulent boundary layer at Reynolds number $Re_\theta = 13000$. J. Fluid Mech. 775, 105–148.

Romano, G. P. 1995 Analysis of two-point velocity measurements in near-wall flows. Exp. Fluids 20, 68–83.

Taira, K., Brunton, S. L., Dawson, S. T. M., Rowley, C. W., Colonius, T., McKeon, B. J., Schmidt, O. T., Gordeyev, S., Theofilis, V. & Ukeiley, L. S. 2017 Modal analysis of fluid flows: an overview. AIAA J. 4013–4041.

Taylor, G. I. 1938 The spectrum of turbulence. Proc. R. Soc. Lond. A 164, 476–490.

Tennekes, H. 1975 Eulerian and Lagrangian time microscales in isotropic turbulence. J. Fluid Mech. 67, 561–567.

Wilczek, M. & Narita, Y. 2012 Wave-number-frequency spectrum for turbulence from a random sweeping hypothesis with mean flow. Phys. Rev. E 86, 066308.

Wilczek, M., Stevens, R. J. A. M. & Meneveau, C. 2015 Spatio-temporal spectra in the logarithmic layer of wall turbulence: large-eddy simulations and simple models. J. Fluid Mech. 769, R1.

Wills, J. A. B. 1964 On convection velocities in turbulent shear flows. J. Fluid Mech. 20, 417–432.

Wu, T., Geng, C. H., Yao, Y. C., Xu, C. X. & He, G. W. 2017 Characteristics of space–time energy spectra in turbulent channel flows. Phys. Rev. Fluids 2 (8), 084609.

Yang, X. I. A. & Howland, M. F. 2018 Implication of Taylor’s hypothesis on measuring flow modulation. J. Fluid Mech. 836, 222–237.

Zhao, X. & He, G. W. 2009 Space–time correlations of fluctuating velocities in turbulent shear flows. Phys. Rev. E 79, 046316.