Method for High Accuracy Multiplicity Correlation Measurements

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Abstract

Multiplicity correlation measurements provide insight into the dynamics of high energy collisions. Models describing these collisions need these correlation measurements to tune the strengths of the underlying QCD processes which influence all observables. Detectors, however, often possess limited coverage or reduced efficiency that influence correlation measurements in obscure ways. In this paper, the effects of non-uniform detection acceptance and efficiency on the measurement of multiplicity correlations between two distinct detector regions (termed forward-backward correlations) are derived. This result is transformed into a correction method. Verification of the presented correction method is provided through simulations using different event generators. The result of the method allows one to correct measurements in a simulation independent manner with high accuracy and thereby shed light on the underlying processes.

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I. INTRODUCTION

The charged particles produced in high energy particle collisions are the result of the interactions in the collision along with the following hadronization processes. The dynamics of these collisions are described well by Quantum Chromodynamics (QCD). The majority of these processes occur at low momentum and are, therefore, in the non-perturbative regime of QCD. This necessitates the use of effective models to characterize the net effect of these processes. The models must be verified by (and possibly tuned to) experiments. Therefore, characterization of the properties of the distributions of the produced particles is essential for understanding the physics involved in the collisions and the underlying interactions. Such properties may be the average particle production or the multiplicity distribution. In this paper we focus on the phenomenon called the forward-backward particle multiplicity correlations (or forward-backward correlations for short).

Forward-backward correlations measure the correlation strength between the number of particles produced in regions located in opposite hemispheres separated by the plane perpendicular to the beam axis intersecting the collision point. The regions are typically equidistant (angularly) from the plane perpendicular to the beam axis. This measurement has the advantage that it is mostly influenced by the dynamics of the collision rather than the following hadronization processes [1].

While different representations exist for characterizing forward-backward correlations, in this paper we focus only on the Pearson correlation factor, which we denote as $b$. This correlation factor is defined as:

$$b \equiv \text{Cor}(N_f, N_b) = \frac{\text{Cov}(N_f, N_b)}{\sqrt{\text{Var}(N_f) \cdot \text{Var}(N_b)}}$$
$$= \frac{\langle N_f N_b \rangle - \langle N_f \rangle \langle N_b \rangle}{\sqrt{(\langle N_f^2 \rangle - \langle N_f \rangle^2) \cdot (\langle N_b^2 \rangle - \langle N_b \rangle^2)}}$$

where $N_f$ and $N_b$ are the number of particles produced in the regions in the forward and backward hemispheres, respectively.

One important property of the Pearson correlation factor is that it is a bound quantity. It can be shown that $-1 \leq b \leq 1$ [2]. This ensures that the measurement is independent of the multiplicities of the event. The property arises from the denominator of $b$, which is the square root of the product of the forward and backward multiplicity variances.
FIG. 1. The figures depict three sets of forward-backward multiplicity pairs with $b = 1, 0.6, \text{and } 0$ from left to right. The variances (used in the denominator of $b$) are the same for $n_F$ and $n_B$ in all three cases. This demonstrates that the correlation information is essentially contained in the covariance.

The correlation factor can be interpreted geometrically as how well the set of number pairs describe a line when plotted on a two dimensional figure. This is demonstrated in figure 1. The intersection and the slope of the line are irrelevant to the value of $b$. This can likewise be demonstrated by the fact that

$$\text{Cor}(\alpha X + \beta, \gamma Y + \nu) = \text{Cor}(X, Y), \text{ if } \alpha \gamma > 0$$

(2)

where $\alpha$, $\beta$, $\gamma$, and $\nu$ are constants. If $\alpha \gamma < 0$, the correlation factor switches sign. If the slope in figure 1 is negative, the corresponding correlation factor is also negative and the quantities are said to be anti-correlated.

While Equation (1) shows that only five quantities ($\langle N_f \rangle$, $\langle N_b \rangle$, $\langle N_f N_b \rangle$, $\langle N_f^2 \rangle$ and $\langle N_b^2 \rangle$) are necessary to calculate the correlation factor, the measurement is often not trivial to perform for many detector types. Any observable will be altered by the environment surrounding the collision in the experiment. Secondary particle production and partial detector acceptance and inefficiency will influence the measurement. Directly evaluating this influence in a model independent way is challenging for correlation measurements. This is especially evident when evaluating the variances in the denominator of $b$ when partial acceptance exists. The correlation between the measured and not measured regions requires more sophisticated techniques if the lack of acceptance is significant. While the effect of secondary particle production is beyond the scope of this paper (but could be the subject of a subsequent paper), the effect of detector inefficiency and partial detector acceptance...
is examined. The influence on the measured correlation strength and a means to correct the measured value for these effects is provided. The correction method is verified through studies using simulations.

II. MEASURING THE CORRELATION FACTOR

The space surrounding the collision can be divided into a forward hemisphere and a backward hemisphere separated by the plane perpendicular to the beam axis intersecting the collision point. The hemisphere where $\theta < \pi/2$ is usually termed forward, and the hemisphere where $\theta > \pi/2$ is usually termed backward, where the reference direction at $\theta = 0$ is defined by the experiment.

Forward-backward multiplicity correlations are usually measured between bins of equal width (in $\eta$) spanning the entire azimuth. Correlations between bins where only part of the azimuthal angle is taken into account (twist correlations) can also be measured. While these twist correlations are not directly computed in this paper, they require merely a subset of the information necessary to analyze the full azimuth and, therefore, the techniques presented here could be used with minor modifications to measure twist correlations. The centers of the two bins (in $\eta$) are likewise usually equidistant from $\eta = 0$. In this paper we call such a pair of geometrical regions a forward-backward bin.

The analysis is carried out by determining the number of particles present in each geometrical region event-by-event. From these particle multiplicities the necessary five quantities are calculated for each event. These five values are then averaged over all events and the correlation factor is calculated. Figure 2 shows an example of how the forward-backward bins are defined.

A. The Effect of Efficiency

It is common, either by design or due to malfunction, that detectors do not register all particles impinging on them. Full hermeticity does not usually exist either. In both cases, the result is that fewer particles are detected than were actually produced in the collision. This alters the value of an observable. First order observables, like the average number of produced particles, can be corrected for in a straight-forward manner, since the value scales
FIG. 2. Left: The detector is schematically divided into two halves along the line at \( \theta = \pi/2 \). Each half then consists of solid angles which span a “small” polar angle and the full azimuthal angle. A forward-backward bin consists of two regions where their centers are equidistant from \( \theta = \pi/2 \). The regions are often termed forward when \( \theta < \pi/2 \) and backward when \( \theta > \pi/2 \). Each forward-backward bin is represented by a different color here. Right: The detector regions have been mapped to a two dimensional figure. The horizontal axis is the pseudorapidity, which is a function of the polar angle, \( \theta \). Note that the mapping is not to scale and equal size polar angle bins on the left do not correspond to equal size pseudorapidity bins on the right.

with the efficiency or acceptance. For higher order observables, the effect of efficiency or acceptance becomes more complex.

To study the effect of efficiency, a statistical approach is taken. In the case of forward-backward correlations, a joint probability distribution for the produced primary particles, \( P^P(N_f^P, N_b^P) \), contains the physics information one wants to measure. The joint probability distribution is normalized such that

\[
\sum_{N_f^P=0}^{\infty} \sum_{N_b^P=0}^{\infty} P^P(N_f^P, N_b^P) = 1
\] (3)

A moment generating function can be defined from this whose derivatives evaluated at \( t_f = 0 \) and \( t_b = 0 \) produce all of the desired moments.

\[
\text{mgf}^P(t_f, t_b) \equiv \sum_{N_f^P=0}^{\infty} \sum_{N_b^P=0}^{\infty} P^P(N_f^P, N_b^P) e^{N_f^P t_f + N_b^P t_b}
\] (4)

From the moment generating function, the cumulant generating function is defined as:

\[
\text{cgf}^P(t_f, t_b) \equiv \ln [\text{mgf}^P(t_f, t_b)]
\] (5)
where derivatives of $\text{cgf}^P$ evaluated at $t_f = 0$ and $t_b = 0$ produce the quantities desired to compute the correlation factor (and many more cumulants with further derivatives). For the purpose of this paper, the first two cumulants (the mean and the covariance) are important.

\[
\frac{\partial \text{cgf}^P}{\partial t_r}(0, 0) \equiv \text{cgf}^P_{r}(0, 0) = \langle N_r \rangle, \text{ where } r = f \text{ or } b
\] (6)

\[
\frac{\partial^2 \text{cgf}^P}{\partial t_{r_1} \partial t_{r_2}}(0, 0) \equiv \text{cgf}^P_{r_1 r_2}(0, 0) = \text{Cov}(N_{r_1}^P, N_{r_2}^P), \text{ where } r_1, r_2 = f \text{ or } b
\] (7)

In equation (6), $r$ stands for a “region” that could be forward or backward. In equation (7), $r_1$ and $r_2$ stand for “region 1” and “region 2”, respectively, and can independently be forward or backward. In the case where $r_1 = r_2$ ($= r$), the covariance becomes the variance, such that

\[
\text{Cov}(N_r^P, N_r^P) = \text{Var}(N_r^P)
\] (8)

Equation (3) can be modified to account for an efficiency in the forward and backward regions ($\varepsilon_f$ and $\varepsilon_b$, respectively) allowing for all possible outcomes.

\[
\sum_{N_f^P=0}^{\infty} \sum_{N_b^P=0}^{\infty} P^P(N_f^P, N_b^P)(\varepsilon_f + (1 - \varepsilon_f))^{N_f^P} (\varepsilon_b + (1 - \varepsilon_b))^{N_b^P} = 1
\] (9)

In equation (9), $\varepsilon_f$ and $\varepsilon_b$ define the probability that a particle is detected in the forward and backward regions, respectively, where perfect detection probability would have a value of 1. Note that equation (9) strictly assumes that the detection probability is uniform over the whole region to which it refers. Such a restriction is not realistic, but is instructive for an initial investigation where the efficiency will be taken to be the average detection efficiency in the region.

One can now arrive at the moment generating function for the detected particles (mgf$^D$). Since one particle is detected with the probability $\varepsilon_r$, one applies the term $e^{t_r}$ to the $\varepsilon_r$ terms. Likewise, one applies $e^{0-t_r} = 1$ to the $(1 - \varepsilon_r)$ terms, since no particle is detected with this probability. The resulting term, $\varepsilon_r e^{t_r} + (1 - \varepsilon_r)$, is actually the moment generating function for a specific particle to be found in the region with probability $\varepsilon_r$, which we term mgf$^E(t_r; \varepsilon_r)$. The corresponding cumulant generating function is then cgf$^E(t_r; \varepsilon_r) \equiv \ln \left[ \text{mgf}^E(t_r; \varepsilon_r) \right]$. The moment generating function for the distribution of detected particles
then becomes:

\[
\text{mgf}^D(t_f, t_b) = \sum_{N_f^P=0}^{\infty} \sum_{N_b^P=0}^{\infty} P^P(N_f^P, N_b^P) \left[ \varepsilon_f e^{t_f} + (1 - \varepsilon_f) \right]^{N_f^P} \left[ \varepsilon_b e^{t_b} + (1 - \varepsilon_b) \right]^{N_b^P}
\]

\[
= \sum_{N_f^P=0}^{\infty} \sum_{N_b^P=0}^{\infty} P^P(N_f^P, N_b^P)e^{N_f^P \text{cgf}^E(t_f; \varepsilon_f) + N_b^P \text{cgf}^E(t_b; \varepsilon_b)} \tag{10}
\]

Comparing equation (10) to equation (4) shows that the effect of detection efficiency is merely a substitution of the variables in the moment generating function of primary particles, namely \( t_r \rightarrow \text{cgf}^E(t_r; \varepsilon_r) \). The final (simple) equation relating the cumulant generating function of detected particles to the cumulant generating function of primary particles is found by:

\[
\text{mgf}^D(t_f, t_b) = \text{mgf}^P \left( \text{cgf}^E(t_f; \varepsilon_f), \text{cgf}^E(t_b; \varepsilon_b) \right) \Rightarrow
\]

\[
\text{cgf}^D(t_f, t_b) = \text{cgf}^P \left( \text{cgf}^E(t_f; \varepsilon_f), \text{cgf}^E(t_b; \varepsilon_b) \right) \tag{11}
\]

One should note that equation (11) can be generalized to allow one to evaluate the effect of acceptance or efficiency on any order correlation. The \( n \)-region equivalent of equation (11) is:

\[
\text{mgf}^D(t_1, \cdots, t_n) = \text{mgf}^P \left( \text{cgf}^E(t_1; \varepsilon_1), \cdots, \text{cgf}^E(t_n; \varepsilon_n) \right) \Rightarrow
\]

\[
\text{cgf}^D(t_1, \cdots, t_n) = \text{cgf}^P \left( \text{cgf}^E(t_1; \varepsilon_1), \cdots, \text{cgf}^E(t_n; \varepsilon_n) \right) \tag{12}
\]

Derivatives of equation (12) evaluated at \( t_1, \cdots, t_n = 0 \) reveal the effect of acceptance or efficiency on the desired moment or cumulant relative to the moments or cumulants of the primary distribution. One could use this information (as will be done here for the variance and covariance) to correct higher order correlations for these effects.

The cumulants of the distribution of detected particles can now be calculated by differentiating equation (11) and evaluating the results at \( t_f = 0 \) and \( t_b = 0 \). The first derivative gives the average number of found particles in a region.

\[
\frac{\partial \text{cgf}^D}{\partial t_r} \bigg|_{t_f, t_b=0} = \text{cgf}^P_r \left( \text{cgf}^E(0; \varepsilon_f), \text{cgf}^E(0; \varepsilon_b) \right) \left. \frac{d \left[ \text{cgf}^E(t_r; \varepsilon_r) \right]}{dt_r} \right|_{t_r=0}
\]

\[
= \text{cgf}^P_r (0, 0) \cdot \frac{\varepsilon_r e^{t_r}}{\varepsilon_r e^{t_r} + (1 - \varepsilon_r)} \bigg|_{t_r=0} \Rightarrow
\]

\[
\langle N_r^D \rangle = \langle N_r^P \rangle \cdot \varepsilon_r \tag{13}
\]
The result in equation (13) is expected, since it is intuitive that the mean value of the distribution scales with the probability that any given particle is detected. The variances or the covariance (given by the second derivative), however, yield a more complicated result.

\[
\frac{\partial^2 \text{cgf}^P}{\partial t_1 \partial t_2} \bigg|_{t_f,t_b=0} = \left. \text{cgf}^P_{r_1r_2} \left( \text{cgf}^E(0; \varepsilon_f), \text{cgf}^E(0; \varepsilon_b) \right) \cdot \frac{d \left[ \text{cgf}^E(t_{r_1}; \varepsilon_{r_1}) \right]}{dt_1} \bigg|_{t_{r_1}=0} \cdot \frac{d \left[ \text{cgf}^E(t_{r_2}; \varepsilon_{r_2}) \right]}{dt_2} \bigg|_{t_{r_2}=0} \\
+ \delta_{r_1r_2} \cdot \text{cgf}^P_{r_1} \left( \text{cgf}^E(0; \varepsilon_f), \text{cgf}^E(0; \varepsilon_b) \right) \cdot \frac{d^2 \left[ \text{cgf}^E(t_{r_1}; \varepsilon_{r_1}) \right]}{dt_1^2} \bigg|_{t_{r_1}=0} \\
+ \delta_{r_1r_2} \cdot \text{cgf}^P_{r_1} (0,0) \cdot \frac{\varepsilon_{r_1} e^{t_{r_1}}}{\varepsilon_{r_1} e^{t_{r_1}} + (1 - \varepsilon_{r_1})} \bigg|_{t_{r_1}=0} \cdot \frac{\varepsilon_{r_2} e^{t_{r_2}}}{\varepsilon_{r_2} e^{t_{r_2}} + (1 - \varepsilon_{r_2})} \bigg|_{t_{r_2}=0} \\
= \left. \text{cgf}^P_{r_1r_2} (0,0) \cdot \frac{\varepsilon_{r_1} e^{t_{r_1}}}{\varepsilon_{r_1} e^{t_{r_1}} + (1 - \varepsilon_{r_1})} \bigg|_{t_{r_1}=0} \cdot \frac{\varepsilon_{r_2} e^{t_{r_2}}}{\varepsilon_{r_2} e^{t_{r_2}} + (1 - \varepsilon_{r_2})} \bigg|_{t_{r_2}=0} \right. \\
\Rightarrow \\
\text{Cov}(N^D_{r_1}, N^D_{r_2}) = \text{Cov}(N^P_{r_1}, N^P_{r_2}) \cdot \varepsilon_{r_1} \varepsilon_{r_2} + \delta_{r_1r_2} \cdot \langle N^P_{r_2} \rangle \cdot \varepsilon_{r_1} \left( 1 - \varepsilon_{r_1} \right) \\
(14)
\]

This result shows that a special case exists for the variance (where the differentiation is performed twice with respect to the same variable and the \( \delta \) function evaluates to 1). The final expressions for the covariance and the variances of the distribution of detected particles are:

\[
\text{Cov}(N^D_f, N^D_b) = \text{Cov}(N^P_f, N^P_b) \cdot \varepsilon_f \varepsilon_b \\
(15) \\
\text{Var}(N^D_r) = \text{Var}(N^P_r) \cdot \varepsilon_r^2 + \langle N^P_r \rangle \cdot \varepsilon_r \left( 1 - \varepsilon_r \right) \\
(16)
\]

Equation (16) shows that the detected variance has an additional dependence, beyond the variance of the primary produced particles and the efficiency, on the mean number of particles produced in the region, which the covariance does not possess.

Equations (13), (15), and (16) can be inverted to obtain the cumulants of the distribution of the primary particles from the detected quantities:

\[
\langle N^P_r \rangle = \frac{\langle N^D_r \rangle}{\varepsilon_r} \\
(17) \\
\text{Cov}(N^P_f, N^P_b) = \frac{\text{Cov}(N^P_f, N^P_b)}{\varepsilon_f \varepsilon_b} \\
(18) \\
\text{Var}(N^P_r) = \frac{\text{Var}(N^P_r) - \langle N^P_r \rangle \cdot (1 - \varepsilon_r)}{\varepsilon_r^2} \\
(19)
\]

From these expressions, the correlation factor in the case of an imperfect detector (with an
efficiency less than 1) is derived as:

\[
 b = \frac{\text{Cov}(N_f^P, N_b^P)}{\sqrt{\text{Var}(N_f^P) \cdot \text{Var}(N_b^P)}} \frac{\text{Cov}(N_f^D, N_b^D)}{\varepsilon_f \varepsilon_b} \\
= \frac{\sqrt{\text{Var}(N_f^P) - \langle N_f^P \rangle \cdot (1 - \varepsilon_f)}}{\varepsilon_f} \frac{\sqrt{\text{Var}(N_b^P) - \langle N_b^P \rangle \cdot (1 - \varepsilon_b)}}{\varepsilon_b} \\
= \frac{\text{Cov}(N_f^D, N_b^D)}{\sqrt{\text{Var}(N_f^D) - \langle N_f^D \rangle \cdot (1 - \varepsilon_f) \sqrt{\text{Var}(N_b^D) - \langle N_b^D \rangle \cdot (1 - \varepsilon_b)}}}
\]

(20)

While the overall multiplicative efficiency factors in the covariance and variance terms can cancel when calculating the correlation factor, equation (20) shows that the additive terms, proportional to the mean number of particles detected in the region, remain and must be evaluated when an inefficiency exists.

The result in equation (20) assumes that the detection efficiencies, \( \varepsilon_f \) and \( \varepsilon_b \), are the same for all particles in their respective regions. When the efficiency varies little or not at all over the region, this assumption is valid. However, variations in the efficiency of the region will affect a correlation measurement. The most extreme variation exists when a fraction of the region has no detection efficiency and the rest has perfect detection efficiency, which could be the case when the acceptance of the detector does not cover the whole region (in azimuth for instance). Additionally, a non-uniform distribution of particles (termed “event shape”) in the region will affect the measurement when the efficiency varies. In this case, when the particle multiplicity density is higher in the active region relative to the dead region, the effective efficiency is higher. The opposite is true when the particle multiplicity density is lower in the active region relative to the dead region. The net effect does not necessarily cancel out on average over many events when performing correlation measurements. The effect of efficiency variations and event shape is analyzed in section II B using the same framework developed so far and the effects they have on the correlation measurements are examined in section III.

**B. Accounting for Azimuthal Event Shape**

The effect of the event shape (in the presence of an inefficiency) can be reduced if one can select regions of the detector where the particle multiplicity density gradient is small or
the efficiency is constant over the region. This generally occurs when smaller regions of the
detector are used. We first consider the case where, event-by-event, a non-uniform azimuthal
event shape exists for the produced particles, which is, however, uniform on average over
many events. The solution is then to segment the $\eta$ regions, studied in the section II A
additionally into $\varphi$ segments. The particle multiplicity of these sub-regions will be denoted
with an extra subscript (for example, $N_{P,1}^\varphi$ for the primary multiplicity in the first $\varphi$ segment
of the forward region), where the second subscript is a value between 1 and $m_\varphi$ (the number
of $\varphi$ segments). The results in equations (13) and (14) have no assumption about the type
of segmentation and are, therefore, also true for these sub-regions. The generalization to
these sub-regions is

$$\langle N_{r,i}^D \rangle = \langle N_{r,i}^P \rangle \cdot \varepsilon_{r,i}$$  \hspace{1cm} (21)$$

$$\text{Cov}(N_{r,1}^D, N_{r,2}^D) = \text{Cov}(N_{r,1,i}^P, N_{r,2,j}^P) \cdot \varepsilon_{r,1,i} \varepsilon_{r,2,j} + \delta_{r,1,2} \cdot \delta_{i,j} \cdot \langle N_{r,1,i}^P \rangle \cdot \varepsilon_{r,1,i} (1 - \varepsilon_{r,1,i})$$  \hspace{1cm} (22)$$

where $1 \leq i, j \leq m_\varphi$.

The relationship of the mean and covariance of the sub-regions (for primary particles)
can be trivially derived. For the mean, this is

$$\langle N_r^P \rangle = \sum_{i_\varphi=1}^{m_\varphi} \langle N_{r,i_\varphi}^P \rangle = \sum_{i_\varphi=1}^{m_\varphi} \langle N_{r,i_\varphi}^P \rangle$$  \hspace{1cm} (23)$$

which is the expected sum of the means of the sub-regions. For the covariance, this is

$$\text{Cov}(N_{r_1}^P, N_{r_2}^P) = \langle N_{r_1}^P N_{r_2}^P \rangle - \langle N_{r_1}^P \rangle \langle N_{r_2}^P \rangle$$

$$= \sum_{i_\varphi=1}^{m_\varphi} \sum_{j_\varphi=1}^{m_\varphi} \langle N_{r_1,i_\varphi}^P N_{r_2,j_\varphi}^P \rangle - \sum_{i_\varphi=1}^{m_\varphi} \sum_{j_\varphi=1}^{m_\varphi} \langle N_{r_1,i_\varphi}^P \rangle \langle N_{r_2,j_\varphi}^P \rangle$$

$$= \sum_{i_\varphi=1}^{m_\varphi} \sum_{j_\varphi=1}^{m_\varphi} \left( \langle N_{r_1,i_\varphi}^P N_{r_2,j_\varphi}^P \rangle - \langle N_{r_1,i_\varphi}^P \rangle \langle N_{r_2,j_\varphi}^P \rangle \right)$$

$$= \sum_{i_\varphi=1}^{m_\varphi} \sum_{j_\varphi=1}^{m_\varphi} \text{Cov}(N_{r_1,i_\varphi}^P, N_{r_2,j_\varphi}^P)$$ \hspace{1cm} (24)$$

which is the sum of the covariances of each sub-region to every other sub-region. One should
note that equations (23) and (24) apply also to the detected means and covariances.
To correct the measurements properly, rotational invariance is exploited. One would expect, for example, that the mean number of primary particles produced at a certain pseudorapidity and at a certain azimuthal angle would be independent of the azimuthal angle (and only dependent on the azimuthal range of the measurement). To use this in practice, we will impose the restriction that each $\eta$ region is equally divided into $m_\phi$ azimuthal segments that span $2\pi/m_\phi$. With this restriction, many of the measurements are redundant. For the mean number of primary particles, this means that the value at each angle can be replaced by the average.

$$\langle N_{r,i_\phi}^P \rangle = \frac{\sum_{j_\phi=1}^{m_\phi} \langle N_{r,j_\phi}^P \rangle}{m_\phi}, \text{ where } 1 \leq i_\phi \leq m_\phi \tag{25}$$

Using equations (21) and (25) one can derive the (expected) correction for the mean number of primary particles.

$$\langle N_r^D \rangle = \sum_{i_\phi=1}^{m_\phi} \langle N_{r,i_\phi}^D \rangle = \sum_{i_\phi=1}^{m_\phi} \langle N_{r,i_\phi}^P \rangle \varepsilon_{r,i_\phi} = \sum_{i_\phi=1}^{m_\phi} \left( \frac{\sum_{j_\phi=1}^{m_\phi} \langle N_{r,j_\phi}^P \rangle}{m_\phi} \right) \varepsilon_{r,i_\phi}$$

$$= \left( \frac{\sum_{j_\phi=1}^{m_\phi} \langle N_{r,j_\phi}^P \rangle}{m_\phi} \right) \cdot \sum_{i_\phi=1}^{m_\phi} \varepsilon_{r,i_\phi} = \frac{\langle N_r^P \rangle}{m_\phi} \cdot \sum_{i_\phi=1}^{m_\phi} \varepsilon_{r,i_\phi}$$

$$\Rightarrow \langle N_r^P \rangle = m_\phi \cdot \frac{\sum_{i_\phi=1}^{m_\phi} \langle N_{r,i_\phi}^D \rangle}{\sum_{i_\phi=1}^{m_\phi} \varepsilon_{r,i_\phi}} \tag{26}$$

Equation (26) is simple because all quantities in the sum are equivalent (due to rotational invariance). Rotational invariance can be applied to the expression for the covariance where one expects the covariance between any two segments with equal $\phi$ displacement to be equivalent. To do this, equation (24) must be rewritten to group these quantities.

$$\text{Cov}(N_{r_1}^P, N_{r_2}^P) = \sum_{i_\phi=1}^{m_\phi} \text{Cov}(N_{r_1,i_\phi}^P, N_{r_2,i_\phi}^P)$$

$$+ \sum_{s=1}^{m_\phi-1} \left\{ \sum_{i_\phi=1}^{m_\phi-s} \text{Cov}(N_{r_1,i_\phi}^P, N_{r_2,i_\phi+s}^P) \right\} + \sum_{i_\phi=1}^{s} \text{Cov}(N_{r_1,i_\phi+m_\phi-s}^P, N_{r_2,i_\phi}^P) \tag{27}$$

The first sum in equation (27) correlates all regions with the same $\phi$. The terms within the braces in equation (27) correlate regions shifted by $s$ segments in $\phi$ (these correspond to twist correlations). Every term in the first sum must be the same (on average) by rotational
invariance as well as every term within the braces (for each value of $s$). Each of these terms can be analyzed individually to see how they relate to the detected quantities.

We first analyze the terms inside the braces of equation (27), but investigate the result as if it was detected quantities instead. This yields the following result, if rotational invariance is applied for each twisted quantity.

\[
\sum_{i_{i_{φ}}=1}^{m_{φ}-s} \text{Cov}(N_{r1,i_{φ}}^{P}, N_{r2,i_{φ}}^{P}+s) + \sum_{i_{i_{φ}}=1}^{s} \text{Cov}(N_{r1,m_{φ}+i_{φ}-s}^{P}, N_{r2,i_{φ}}^{P}) =
\sum_{i_{i_{φ}}=1}^{m_{φ}-s} \text{Cov}(N_{r1,i_{φ}}^{P}, N_{r2,i_{φ}}^{P}+s) \cdot \varepsilon_{r1,i_{φ}} \varepsilon_{r2,i_{φ}} + s \sum_{i_{i_{φ}}=1}^{s} \text{Cov}(N_{r1,m_{φ}+i_{φ}-s}^{P}, N_{r2,i_{φ}}^{P}) \cdot \varepsilon_{r1,m_{φ}+i_{φ}-s} \varepsilon_{r2,i_{φ}}
\]

Equation (28) uses the result in equation (22) to relate the detected quantities to the primary quantities. In the case here (where $s \geq 1$), the second piece of equation (22) is always 0, because the terms never have the same $φ$. Equation (28) can be inverted to allow one to compute the sum of invariant twisted covariances for primary particles from detected values.

\[
\sum_{i_{i_{φ}}=1}^{m_{φ}-s} \text{Cov}(N_{r1,i_{φ}}^{P}, N_{r2,i_{φ}}^{P}+s) + \sum_{i_{i_{φ}}=1}^{s} \text{Cov}(N_{r1,m_{φ}+i_{φ}-s}^{P}, N_{r2,i_{φ}}^{P}) =
\sum_{i_{i_{φ}}=1}^{m_{φ}-s} \text{Cov}(N_{r1,i_{φ}}^{P}, N_{r2,i_{φ}}^{P}+s) \cdot \varepsilon_{r1,i_{φ}} \varepsilon_{r2,i_{φ}} + s \sum_{i_{i_{φ}}=1}^{s} \text{Cov}(N_{r1,m_{φ}+i_{φ}-s}^{P}, N_{r2,i_{φ}}^{P}) \cdot \varepsilon_{r1,m_{φ}+i_{φ}-s} \varepsilon_{r2,i_{φ}}
\]

The same analysis can be performed on the first term in equation (27), but now, when invoking equation (22) the second piece must be kept as it may not vanish (when calculating
a variance for example).

\[ \sum_{i\varphi=1}^{m\varphi} \text{Cov}(N^D_{r1,i\varphi}, N^D_{r2,i\varphi}) = \sum_{i\varphi=1}^{m\varphi} \left( \text{Cov}(N^P_{r1,i\varphi}, N^P_{r2,i\varphi}) \cdot \varepsilon_{r1,i\varphi} \varepsilon_{r2,i\varphi} 
+ \delta_{r1r2} \cdot \langle N^P_{r1,i\varphi} \rangle \cdot \varepsilon_{r1,i\varphi} (1 - \varepsilon_{r1,i\varphi}) \right) \]
\[ = \sum_{i\varphi=1}^{m\varphi} \text{Cov}(N^P_{r1,i\varphi}, N^P_{r2,i\varphi}) \cdot \sum_{i\varphi=1}^{m\varphi} \varepsilon_{r1,i\varphi} \varepsilon_{r2,i\varphi} 
+ \delta_{r1r2} \cdot \sum_{i\varphi=1}^{m\varphi} \langle N^P_{r1,i\varphi} \rangle \cdot \sum_{i\varphi=1}^{m\varphi} \varepsilon_{r1,i\varphi} (1 - \varepsilon_{r1,i\varphi}) \]  

(30)

Equation (30) can similarly be inverted to compute the sum of the non-twisted portion of equation (27) for primary particles:

\[ \sum_{i\varphi=1}^{m\varphi} \text{Cov}(N^P_{r1,i\varphi}, N^P_{r2,i\varphi}) = m\varphi \cdot \sum_{i\varphi=1}^{m\varphi} \text{Cov}(N^D_{r1,i\varphi}, N^D_{r2,i\varphi}) \]
\[ \cdot \sum_{i\varphi=1}^{m\varphi} \varepsilon_{r1,i\varphi} \varepsilon_{r2,i\varphi} 
- \delta_{r1r2} \cdot m\varphi \cdot \sum_{i\varphi=1}^{m\varphi} \varepsilon_{r1,i\varphi} (1 - \varepsilon_{r1,i\varphi}) \cdot \sum_{i\varphi=1}^{m\varphi} \langle N^D_{r1,i\varphi} \rangle \]  

(31)

where equation (26) was used to relate the mean number of primary particles to the mean number of detected particles.

The final expression for the corrected covariance is obtained by inserting equations (29) and (31) into equation (27):

\[ \text{Cov}(N^P_{r1}, N^P_{r2}) = 
\]
\[ m\varphi \cdot \sum_{i\varphi=1}^{m\varphi} \text{Cov}(N^D_{r1,i\varphi}, N^D_{r2,i\varphi}) \]
\[ \cdot \sum_{i\varphi=1}^{m\varphi} \varepsilon_{r1,i\varphi} \varepsilon_{r2,i\varphi} 
+ m\varphi \cdot \sum_{s=1}^{m\varphi-1} \left\{ \sum_{i\varphi=1}^{m\varphi-s} \text{Cov}(N^D_{r1,i\varphi}, N^D_{r2,i\varphi+s}) + \sum_{i\varphi=1}^{s} \text{Cov}(N^D_{r1,m\varphi+i\varphi-s}, N^D_{r2,i\varphi}) \right\} \]
\[ - \delta_{r1r2} \cdot m\varphi \cdot \sum_{i\varphi=1}^{m\varphi} \varepsilon_{r1,i\varphi} (1 - \varepsilon_{r1,i\varphi}) \cdot \sum_{i\varphi=1}^{m\varphi} \langle N^D_{r1,i\varphi} \rangle \]  

(32)

While the result using equation (32) must deviate from the result obtained from the distribution of primary particles (due to the imperfect detector response resulting in partial information loss), tests show a vast improvement over using equations (18) and (19). Results using equation (32) often agree within statistical error with the results obtained from the primary distribution as will be shown in section III. One should note that there is a limitation to this method that greater than 50% of the acceptance must be present in each region. If this requirement is not satisfied, one or more of the denominators summing over multiplications of efficiency factors in equations (29) and (31) will be 0.
III. VERIFICATION

The validity of the correction formula is verified through studies using simulations of proton-proton collisions. The event generator used here is Pythia 6.4 \cite{4}. It has been chosen, because many pre-configured tunes exist which predict substantially different quantities for different observables and specifically, in this case, for forward-backward correlations. To test the validity of the method, this difference is essential to establish if any residual dependence on the generator exists. The properties of the tunes can be found in \cite{5}. The primary Pythia tune used in the following examples is Perugia3. The other tunes used are Perugia0 and DW. The DW tune results in quite different correlation factors when compared to the other two tunes (see figure \ref{fig:3}). One should note that the bin width in $\eta$ affects the value of $b$ with $b \rightarrow 0$ as $\Delta_{\text{bin}} \rightarrow 0$. For all plots shown in the rest of this paper $\Delta_{\text{bin}} = 0.5$ is chosen as the bin width unless otherwise specified.

![Graph showing correlation factors for different tunes and bin widths.](image)

**FIG. 3.** Correlation factors ($b$) obtained using the particle multiplicities computed at $\eta = \frac{\Delta\eta}{2}$ and $\eta = -\frac{\Delta\eta}{2}$ from the Pythia6 event-generator. Three different tunes were selected: Perugia3, Perugia0, and DW. Three different bin widths, $\Delta_{\text{bin}}$, in $\eta$ were used. Left: $\Delta_{\text{bin}} = 1$. Middle: $\Delta_{\text{bin}} = 0.5$. Right: $\Delta_{\text{bin}} = 0.25$.

A. Reduced Acceptance

The initial study involves the reduction of the acceptance of each $\eta$ bin. Two examples are studied: a simple case, where inactive regions have identical $\varphi$ locations in all $\eta$ bins, and a realistic case, where inactive regions have been placed randomly into each $\eta$ bin. In
both cases, geometrical areas are chosen to be inactive with respect to particle detection, meaning that any particle with a momentum vector pointing toward an inactive region is excluded from the detected quantities.

1. **The Simple Case**

Four simple examples are investigated in this section. The inactive areas are chosen such that they begin at $\varphi = 0$ and extend to $n \cdot \frac{2\pi}{10}$ where $n = 1, 2, 3,$ and $4$. This results in geometric acceptances for each $\eta$ bin of 90%, 80%, 70%, and 60%. The acceptance maps are shown in figure 4.

![Acceptance Maps](image)

**FIG. 4.** The four simple examples of the acceptance maps of the $\eta$ bins. The four panes show the inactive regions with acceptances of 90%, 80%, 70%, and 60%.

Regardless of the cause, undetected particles will result in a loss of information and...
will affect the measured correlation factor. We would intuitively expect that the correlation factors are attenuated when the efficiency of a bin is less than 1. Equation (20) demonstrates this. This effect is illustrated in left pane of figure 5. The correlation factor at the event-generator level is black while other colors are used for each case of reduced acceptance. The graph shows that more attenuation exists when the size of the inactive areas is increased.

FIG. 5. Left: Attenuation of measured correlation factors as a result of decreased $\varphi$ acceptance in the $\eta$ bins (see figure 4). The ratio on the bottom is the measured correlation factor divided by the corresponding primary correlation factor. Right: The corrected correlation factors using equation (20), which uses no $\varphi$ segmentation. The simple assumption of uniform detection probability reduces the discrepancy from the primary correlation factor by more than a factor of 3, but still leaves discrepancies between the corrected correlation factors and those from the generator output.

To illustrate the necessity of segmentation, the results are first corrected without any segmentation (using equation (20)). The results from this correction are shown in the right pane of figure 5. Although the correlation factors are now less than 10% from the primary correlation factors, the discrepancy is still sizable. To further reduce this discrepancy, the detected values can be corrected back to the primary values by dividing the $\eta$ bins into segments of equal size in $\varphi$. Then equation (32) can be used to calculate the corrected correlation factors. In the left pane of figure 6, this correction has been performed using 10 $\varphi$ segments. While up to 40% of the bin is inactive, detecting down to 60% of the particles,
the correlation factor can be corrected back to within a few per mill of the primary values.

FIG. 6. Left: Corrected correlation factors computed using 10 φ segments to account for the azimuthal event shape. Note that the scale in the ratio has changed (with a much smaller range) from the previous plots with no φ segmentation. Right: Results of correcting the 70% acceptance case with a varied number of φ segments in each η bin. Note that there is no improvement (nor should there be in this example) when increasing from 10 to 20 φ segments.

The chosen number of segments in the analysis influences the accuracy of the correction. This is shown in the right pane of figure 6 where the correction has been performed using 1, 5, 10, and 20 φ segments. Using one φ segment produces the same result as in the right pane of figure 5 while choosing more segments improves the result up to having 10 segments. The results when using 10 and 20 segments are identical. This is true, because every adjacent pair, in φ, of acceptance values are the same and, therefore, the 20 segment version of equation (32) simplifies identically into the 10 segment version of that equation. If one had, for instance, the same acceptance value for every φ segment in an η bin, equation (32) would identically simplify to equation (18) or (19) depending on whether it corresponded to a covariance or a variance. In the example in the right pane of figure 6, 10 segments is enough to ensure segments of equal size, while also ensuring that all segments have the same detection efficiency of either 1 or 0. This is not the case when the correction is performed using 5 segments. In that case, one (or more) segments have an average efficiency of 0.5.
This makes the 5 segment case more inaccurate because the assumption of uniform efficiency in the bin is violated. This study shows that, while finer segmentation can produce more accurate results, there may exist a limit beyond which no further accuracy is attained. In fact, if possible, the segmentation used in the analysis should only be fine enough to ensure that all segments have an efficiency of either 1 or 0, if acceptance is the only effect being corrected for, since this will reduce the required storage of information to perform the measurement.

2. A Realistic Case

The simple test shown in section III A 1 demonstrates the general effect of reduced acceptance. Realistic detector acceptances lack that simplicity though. To test the method more generally, 20 inactive regions were placed randomly over the analysis region. The only restriction placed on the randomness was that there must be greater than 50% acceptance in every $\eta$ bin to ensure that the correction can be calculated using this method. The resulting acceptance map is shown in the left pane of figure 7.

![Acceptance map with 20 randomly placed inactive regions. The individual regions span 1 unit in $\eta$ and $\frac{2\pi}{10}$ in azimuth. Right: The correlation factors found after correcting for randomly placed inactive regions using different $\varphi$ segmentations.](image)

FIG. 7. Left: Acceptance map with 20 randomly placed inactive regions. The individual regions span 1 unit in $\eta$ and $\frac{2\pi}{10}$ in azimuth. Right: The correlation factors found after correcting for randomly placed inactive regions using different $\varphi$ segmentations.
The right pane of figure 7 shows the result of the analysis with different numbers of $\varphi$ segments. When the acceptance varies in each $\eta$ bin, structure can be seen in the corrected correlation factors with no $\varphi$ segmentation that is not present in the simple case presented in section III A 1. Including $\varphi$ segmentation minimizes this effect. Increasing the number of $\varphi$ segments to 10 gives the same accuracy as seen in the simple case. Also as for the simple case, increasing the segmentation beyond 10 $\varphi$ segments in these examples does not produce a more accurate measurement.

B. Efficiency

In this section we address the case where the detection efficiency can have any value between 0 and 1. This is in contrast to the previous cases where the detection efficiency was 1 for active regions and 0 for inactive regions. This case is quite realistic for most detectors since perfect detection efficiency is never achieved.

A continuous efficiency gradient is applied to the primary particles from the generator. The function used here to impose the gradient is a sine function of the form $\varepsilon(\varphi) = A\sin(B\varphi + C) + D$ where $A$, $B$, $C$, and $D$ have been chosen such that the range of efficiency values is $0.2 < \varepsilon < 0.8$. The resulting efficiency map is shown in the left pane of figure 8. Note that, due to binning, the values portrayed in the figure show the average efficiency of the detection regions and not the continuous distribution which is actually imposed on the particles.

The results from applying a continuous efficiency gradient in $\varphi$ are shown in the right pane of figure 8. In principle, the accuracy can always be improved by increasing the number of segments, because the gradient never completely goes away (depending on the available statistics). In this case, one must choose the number of segments corresponding to the desired accuracy and available data. In this analysis an accuracy of better than 1% is already achieved by using 5 $\varphi$ segments.

C. Comparison between Different Tunes

The need for such accuracy in this correction method can be shown by looking at the results from the different generators. Figure 9 shows the result using the simple acceptance
configuration with 60% acceptance in each $\eta$ bin for the different tunes of Pythia. The bins have been divided into 10 azimuthal segments. The correction method has been applied with no simulation input and only the knowledge of the acceptance for all results. The results show no particularly different behavior for the discrepancies from the true values for any specific generator (tune). For the vast majority of points, for all bin widths, the accuracy of the corrected values is within 1%.

Figure 9 should (and does) reproduce figure 3. For small $\Delta \eta$ (which many detectors possess), one must achieve high accuracy and precision to distinguish between different tunes and, consequently, the relative strengths of the underlying physical processes. The methods presented here allow one to make correlation measurements with high enough accuracy to achieve this goal.
FIG. 9. Results of the corrected correlation factors using different tunes of Pythia and three different \( \eta \) bin widths (\( \Delta_{\text{bin}} = 1, 0.5, \) and 0.25). Data from each tune are subjected to the same geometrical acceptance. The figure shows no particularly significant dependence of the deviations on the tune.

IV. CONCLUSIONS

The effect of reduced acceptance and imperfect detection efficiency on forward-backward correlations is derived using a statistical approach. No assumptions about the distribution of primary particles were made and, therefore, the derived results are valid for physical data as well as the simulated data studied here. Furthermore a framework to evaluate the effect of detector acceptance and efficiency on any order multiplicity correlation has been established. If the acceptance and the efficiency are well determined, the method can correct forward-backward correlations very accurately depending on the capabilities of the detector (segmentation) as long as the inactive regions are smaller than 50% in all \( \eta \) regions. Considerations must be made concerning the desired segmentation used in the analysis. The number of segments should be large enough to ensure nearly constant efficiency within the segments while balancing against the storage required for recording the necessary information for the analysis.

The presented method allows one to achieve high accuracy for computing multiplicity correlations necessary to distinguish between the underlying processes governing particle production in the collision. The framework could be further used to investigate higher order multiplicity correlations that could put additional constraints on models. To further gain the power to distinguish between the underlying processes, one must allow for these
correlation measurements to be perform accurately with large $\Delta \eta$. This often requires using detectors which have little ability to reject secondary particles (which this paper has not investigated). Extending this framework to deal with this effect would provide a powerful tool in the analysis of correlations over wide $\eta$ ranges.

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