Conformal Partial Wave Expansions for
$\mathcal{N} = 4$ Chiral Four Point Functions

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The conformal partial wave analysis of four point functions of $\tfrac{1}{2}$-BPS operators belonging to the $SU(4)$ $[0, p, 0]$ representation is undertaken for $p = 2, 3, 4$. Using the results of $\mathcal{N} = 4$ superconformal Ward identities the contributions from protected short and semi-short multiplets are identified in terms of the free field theory. In the large $N$ limit contributions corresponding to long multiplets with twist up to $2p - 2$ are absent. The anomalous dimensions for twist two singlet multiplets are found to order $g^4$ and agree with other perturbative calculations. Results for twist four and six are also found.

PACS no: 11.25.Hf; 11.30.Pb

Keywords: Conformal field theory, Operator product expansion, Four point function, Anomalous Dimensions.

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1. Introduction

The spectrum of operators and their couplings in a conformal field theory can be explored by analysing the four point correlation functions for any basic set of operators $\phi_i$ such that $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$ can be calculated. Using the operator product expansion for any pair $\phi_i \phi_j$ which appear in the correlation function provides an expansion in terms of conformal partial waves, functions which depend on the spins and scale dimensions of the operators which are present in the operator product expansion. Just as many of the particles appearing in the data tables were found by a partial wave analysis of experimentally measured scattering amplitudes then the spectrum and anomalous dimensions of operators may be determined through the conformal partial wave expansion of conformal four point functions.

For the simplest example we may consider a single scalar field $\phi$ of scale dimension $\Delta_\phi$ so that

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} G(u,v) ,$$

for

$$x_{ij} = x_i - x_j , \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} ,$$

where $u, v$ are conformal invariants. The operator product expansion here takes the form

$$\phi(x_1)\phi(x_2) = \sum_{\Delta,\ell,I} C_{\phi\phi O^I} \frac{1}{(x_{12}^2)^{\frac{1}{2}(2\Delta_\phi - \Delta + \ell)}} C^{(\ell)}_{\Delta}(x_{12}, \partial x_{12}) \mu_1 ... \mu_\ell O^I_{\mu_1 ... \mu_\ell}(x_2) ,$$

where $O^I_{\mu_1 ... \mu_\ell}$ is a symmetric traceless rank $\ell$ tensor conformal primary operator with scale dimension $\Delta$, $I$ labels different operators with the same $\Delta, \ell$. $\{O^I_{\mu_1 ... \mu_\ell}\}$ are assumed to form the complete set of operators appearing in the operator product expansion of $\phi\phi$. In (1.3) $C^{(\ell)}_{\Delta}(x, \partial)$ are differential operators constructed so that (1.3) is compatible with form of the three point function $\langle \phi\phi O^I \rangle$ and the two point function $\langle O^I O^J \rangle$. The first is given by

$$\langle \phi(x_1)\phi(x_2) O^I_{\mu_1 ... \mu_\ell}(x_3) \rangle = \frac{1}{C_{O^I}} C_{\phi\phi O^I} \frac{1}{(x_{12}^2)^{\Delta_\phi}} \left( \frac{x_{12}^2}{x_{13}^2 x_{23}^2} \right)^{\frac{1}{2}(\Delta - \ell)} Z_{\mu_1 ... \mu_\ell} ,$$

where

$$Z_{\mu} = \frac{x_{13\mu}}{x_{13}^2} - \frac{x_{23\mu}}{x_{23}^2} ,$$

and $\{\ldots\}$ denotes symmetrisation and removal of traces. The two point function is also

$$\langle O^I_{\mu_1 ... \mu_\ell}(x_1) O^J_{\nu_1 ... \nu_\ell}(x_2) \rangle = C_{O^I} \delta^{IJ} \frac{1}{(x_{12}^2)^{\Delta}} I_{\mu_1 \nu_1}(x_{12}) ... I_{\mu_\ell \nu_\ell}(x_{12}) ,$$
where \( I_{\mu\nu} \) is the inversion tensor,

\[
I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}.
\] (1.7)

The differential operator \( C^{(\ell)}(x, \partial) \) is then such that \( C^{(\ell)}(x, 0)_{\mu_1 \ldots \mu_\ell} = x_{\{\mu_1 \ldots x_{\mu_\ell}\}} \).

Using the operator product expansion (1.3) in (1.1) gives rise to the conformal partial wave expansion,

\[
G(u, v) = \sum_{\Delta, \ell} a_{\Delta, \ell}^\Delta u^{\Delta/2}(\Delta-\ell) G^{(\ell)}(u, v),
\] (1.8)

where

\[
a_{\Delta, \ell}^\Delta = \sum_I \frac{1}{C_{O_I}} (C_{\phi O_I})^2,
\] (1.9)

and the partial wave amplitudes \( G^{(\ell)}(u, v) \) are explicitly known functions, at least in dimensions \( d = 2, 4, 6 \), which may be expanded in powers of \( u \) and \( 1 - v \), satisfying,

\[
G^{(\ell)}(u, v) = (-1)^\ell v^{-\frac{1}{2}(\Delta-\ell)} G^{(\ell)}(u/v, 1/v).
\] (1.10)

For functions \( G \) satisfying crossing symmetry under \( x_1 \leftrightarrow x_2 \), \( G(u, v) = G(u/v, 1/v) \), this ensures that only \( \ell \) even appears in the summation in (1.8). In general we may set \( C_{O_I} = 1 \) by a choice of normalisation. We also assume \( C_{\phi} = 1 \). For the energy momentum tensor, for which \( \Delta = d, \ell = 2 \), there is however a canonical normalisation such that

\[
C_{\phi\phi T} = -\frac{1}{S_d} \frac{\Delta_{\phi d}}{d-1}, \quad S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{1}{2}d)}.
\] (1.11)

In conformal field theories the essential function \( G(u, v) \) in (1.1) may be calculated as a perturbative series in a small parameter \( \epsilon \), either the coupling or \( 1/N \) for large \( N \), and in general it has an expansion in \( \epsilon \) of the form

\[
G(u, v) = G_0(u, v) + \sum_{r=1,2,\ldots} \epsilon^r \sum_{s=0}^r \ln^s u \ G_{r,s}(u, v),
\] (1.12)

where \( G_0 \) is the free field contribution. The conformal partial wave expansion for \( G \) may be written as in (1.3) with \( \Delta \rightarrow \Delta_I \) where \( I \) now labels different operators with the same scale dimension \( \Delta_0 \) and \( \ell \) for \( \epsilon = 0 \). With the expansions

\[
\Delta_I = \Delta_0 + \epsilon \Delta_{I,1} + \epsilon^2 \Delta_{I,2} + \ldots, \quad a_{\ell, I}^{\Delta_0} = a_{\ell, I}^{\Delta_0} + \epsilon b_{\ell, I}^{\Delta_0} + \ldots,
\] (1.13)
we may derive

\[ G_0(u, v) = \sum_{\Delta_0, \ell, I} a_{\Delta_0, \ell, I} u^{\Delta_0} G_{\Delta_0}(u, v), \]

\[ G_{1,1}(u, v) = \frac{1}{2} \sum_{\Delta_0, \ell, I} \Delta_{I,1} a_{\Delta_0, \ell, I} u^{\Delta_0} G_{\Delta_0}(u, v), \]

\[ G_{1,0}(u, v) = \sum_{\Delta_0, \ell, I} u^{\Delta_0} \left( b_{\Delta_0, \ell, I} G_{\Delta_0}(u, v) + \Delta_{I,1} a_{\Delta_0, \ell, I} G_{\Delta_0}'(u, v) \right), \]

\[ G_{2,2}(u, v) = \frac{1}{8} \sum_{\Delta_0, \ell, I} \Delta_{I,1}^2 a_{\Delta_0, \ell, I} u^{\Delta_0} G_{\Delta_0}(u, v), \]

\[ G_{2,1}(u, v) = \frac{1}{2} \sum_{\Delta_0, \ell, I} u^{\Delta_0} \left( (\Delta_{I,2} a_{\Delta_0, \ell, I} + \Delta_{I,1} b_{\Delta_0, \ell, I}) G_{\Delta_0}(u, v) + \Delta_{I,1} a_{\Delta_0, \ell, I} G_{\Delta_0}'(u, v) \right), \]

where

\[ G_{\Delta_0}'(u, v) = \frac{\partial}{\partial \Delta} G_{\Delta_0}(u, v) \bigg|_{\Delta=\Delta_0}. \]

In principle, for expansions to arbitrary orders in \( \epsilon \), we may determine \( \sum_I \Delta_{I,r}^u a_{\Delta_0, \ell, I} \). For \( r = 1 \) for instance we need to consider the expansion of the coefficient of \( (\epsilon \ln u)^n, \text{G}_{n,n}(u, v) \). However this is an inefficient procedure even for determining \( \Delta_{I,1} \) except if there is no degeneracy and it is sufficient to restrict to \( n = 1 \). Such perturbative expansions were first applied for conformal \( O(N) \) sigma models in dimensions \( 2 < d < 4 \) \([3,4]\) but were later applied to superconformal gauge theories in four dimensions \([5,6,7,8,9,10,11]\).

In this paper we apply such expansions to the four point correlation functions of \( \frac{1}{2} \)-BPS operators, whose lowest scale dimension operator belongs to the \( SU(4) \) representation with Dynkin labels \([0, p, 0]\), in \( \mathcal{N} = 4 \) superconformal theories, extending previous results and offering a more complete discussion. For such correlation functions the implications of \( \mathcal{N} = 4 \) superconformal symmetry have been analysed \([12,13,14]\) and it has been shown how this is compatible with the appearance of various possible supermultiplets in the operator product expansion \([14]\). In addition for \( p = 2, 3, 4 \) perturbative results to order \( g^4 \) in the gauge coupling have been found \([15,16,17]\) and also to order \( 1/N^2 \) using the AdS/CFT correspondence \([18,19,20]\). We use these results in our discussion. A special feature of superconformal theories is the existence of supermultiplets which satisfy shortening conditions and in consequence the operators have no anomalous scale dimensions. Only long multiplets, where the dimension of conformal primary operators is proportional to \( 2^{16} \) in \( \mathcal{N} = 4 \), can have anomalous dimensions but in order to determine these it is necessary to separate off the contributions of short, for which the lowest scale dimension or superconformal primary operator has \( \ell = 0 \), and semi-short multiplets, for which \( \ell > 0 \) is allowed. This may be accomplished using the solutions of the superconformal Ward identities.
In detail in section 2 we show how the results of superconformal symmetry, following \[14\], allow an operator product expansion for the correlation functions of $\mathcal{N} = 4$ chiral primary operators in which the contribution of entire supermultiplets is evident. It is shown how non unitary multiplets are cancelled and how there are potential ambiguities due to the decomposition of a long multiplet into semi-short multiplets at unitarity threshold. In section 3 this is applied to free field theory, which is initially expressed in terms of a general linear combination of symmetric polynomials and later specialised to the large $N$ limit. The operator product expansion is carried out in detail and contributions for various supermultiplets identified. In section 4 results from the AdS/CFT correspondence are applied in the large $N$ strong coupling limit. It is shown how low twist long multiplets which appear in the conformal wave expansion of the large $N$ results without anomalous dimensions cancel exactly corresponding contributions from free field theory. This is in accord with the expectation from string theory that these should decouple \[21\]. In section 5 some perturbative results are considered. Some technical details relevant for obtaining the operator product expansions are left to five appendices.

2. Superconformal Expansions

In a $\mathcal{N} = 4$ superconformal theory the correlation functions for chiral primary $\frac{1}{2}$-BPS operators, which belong to the $SU(4)$ $[0, p, 0]$ representation, are given by symmetric traceless tensor fields $\varphi_{r_1 \ldots r_p}(x)$ and have $\Delta = p$ and $\ell = 0$. Their correlation functions may be calculated both perturbatively and for large $N$ through the AdS/CFT correspondence. For detailed analysis for arbitrary $p$ it is very convenient to consider instead $\varphi^{(p)}(x, t) = \varphi_{r_1 \ldots r_p}(x) t_{r_1} \ldots t_{r_p}$ for $t_r$ an arbitrary six dimensional complex null vector. The four point correlation functions of interest here are then simply given by an extension of (1.1)

$$\langle \varphi^{(p)}(x_1, t_1) \varphi^{(p)}(x_2, t_2) \varphi^{(p)}(x_3, t_3) \varphi^{(p)}(x_4, t_4) \rangle = \left( \frac{t_1 \cdot t_2 \cdot t_3 \cdot t_4}{x_{12}^2 x_{34}^2} \right)^p G(u, v; \sigma, \tau),$$

for $u, v$ as in (1.2) and $\sigma, \tau$ $SU(4)$ invariants which are defined by

$$\sigma = \frac{t_1 \cdot t_3 t_2 \cdot t_4}{t_1 \cdot t_2 t_3 \cdot t_4}, \quad \tau = \frac{t_1 \cdot t_4 t_2 \cdot t_3}{t_1 \cdot t_2 t_3 \cdot t_4}. \quad (2.2)$$

Necessarily, since the correlation function is homogeneous of degree $p$ in each $t_i$, $G(u, v; \sigma, \tau)$ is a polynomial of degree $p$ in $\sigma, \tau$ (i.e it may be expanded in monomials $\sigma^r \tau^s$ with $r + s \leq p$). It may be decomposed into contributions for the differing $SU(4)$ representations in the tensor product $[0, p, 0] \otimes [0, p, 0]$ for $\varphi^{(p)}(x_1, t_1) \varphi^{(p)}(x_2, t_2)$ by writing

$$G(u, v; \sigma, \tau) = \sum_{0 \leq m \leq n \leq p} a_{nm}(u, v) Y_{nm}(\sigma, \tau), \quad (2.3)$$
where \( a_{nm} \) corresponds to the representation \([n - m, 2m, n - m]\) and \( Y_{nm} \) are two variable harmonic polynomials which, as shown in [14], are given explicitly in terms of single variable Legendre polynomials by

\[
Y_{nm}(\sigma, \tau) = \frac{P_{n+1}(y)P_{m}(\bar{y}) - P_{m}(y)P_{n+1}(\bar{y})}{y - \bar{y}}, \quad \sigma = \frac{1}{4}(1 + y)(1 + \bar{y}), \quad \tau = \frac{1}{4}(1 - y)(1 - \bar{y}).
\]  

(2.4)

In a conformal theory the operator product expansion is reflected by the expansion of \( a_{nm} \) in terms of conformal partial waves which takes the form

\[
a_{nm}(u, v) = \sum_{\Delta, \ell} a^\Delta_{nm, \ell} u^{\frac{1}{2}(\Delta - \ell)} G_\Delta^{(\ell)}(u, v),
\]  

(2.5)

where \( a^\Delta_{nm, \ell} \) corresponds to the contribution of a conformal primary operator with spin \( \ell \) and scale dimension \( \Delta \). The conformal partial wave functions \( G_\Delta^{(\ell)}(u, v) \) are explicitly known functions which have a simple expression in four dimensions. Of course as a consequence of superconformal symmetry the conformal primary operators must belong to supermultiplets in each of which there is a finite range of related possible \( \ell \) and \( \Delta \), the operator with lowest \( \Delta \) is termed the superconformal primary for the appropriate supermultiplet. Manifestly all operators belonging to a given supermultiplet must have the same anomalous dimension. In general it is non-trivial to separate the contributions in the operator product expansion of descendant operators from superconformal primary operators.

This difficulty is easily resolved by considering the solution of the superconformal Ward identities for \( G \) which require [13,14]

\[
G(x, (1 - x)(1 - \bar{x}); \alpha\bar{\alpha}, (1 - \alpha)(1 - \bar{\alpha}))|_{\bar{\alpha} = \frac{1}{x}} = k + \left(\frac{\alpha - 1}{x}\right) \hat{f}(x, 2\alpha - 1). \]  

(2.6)

The solution of (2.6) can be written as

\[
G(u, v; \sigma, \tau) = k + \hat{G}_f(u, v; \sigma, \tau) + s(u, v; \sigma, \tau)\mathcal{H}(u, v; \sigma, \tau),
\]

(2.7)

\[
s(u, v; \sigma, \tau) = v + \sigma^2 uv + \tau^2 u + \sigma v(v - 1 - u) + \tau(1 - u - v) + \sigma \tau u(u - 1 - v),
\]

where \( \hat{G}_f \) may be explicitly given in terms of \( \hat{f} \). The constant \( k \) and the function \( \hat{f}(x, y) \), a polynomial in \( y \) of degree \( p - 1 \), are determined by the free field results for \( G \) whereas dynamical effects, which lead to anomalous dimensions, are contained in the function \( \mathcal{H} \) which is a polynomial in \( \sigma, \tau \) of degree \( p - 2 \). Instead of considering the conformal partial wave expansion of \( G \) it is sufficient to expand \( \mathcal{H} \) and \( \hat{f} \) as in [14] so that

\[
\mathcal{H}(u, v; \sigma, \tau) = \sum_{0 \leq m \leq n \leq p - 2} A_{nm}(u, v) Y_{nm}(\sigma, \tau)
\]

(2.8)
As a consequence of crossing symmetry of (2.1) under \(x_1, t_1 \leftrightarrow x_2, t_2\) we have
\[
A_{nm}(u, v) = (-1)^{n+m+1} \frac{1}{v^2} A_{nm}(u/v, 1/v),
\]
and hence, from (1.10), in the conformal partial wave expansion (2.8) we have \(\ell = 0, 2, \ldots\) for \(n + m\) even and otherwise \(\ell = 1, 3, \ldots\). We may also similarly expand \(\hat{f}\) in the form
\[
\hat{f}(x, y) = -2 \sum_{0 \leq n \leq p-1} b_{n, \ell} g_{0, \ell+2}(x) P_n(y) \quad \{ \begin{array}{ll} \ell \text{ odd if } n \text{ even} & \\
\ell \text{ even if } n \text{ odd} & \end{array} \}
\]
with the definition
\[
g_{t, \ell}(x) = (-\frac{1}{2} x)^{\ell} F(t + \ell, t + \ell; 2t + 2\ell; x).
\]
In general in (2.10) \(\ell \geq 0\) except when \(n = 0\) it is necessary in general to include \(\ell = -1\) in the sum. These expansions determine the conformal partial wave expansion for \(a_{nm}(u, v)\) in a form where the contribution of each superconformal multiplet is explicit,
\[
a_{n'm'} = k \delta_{n'0} \delta_{m'0} + \sum_{0 \leq m \leq n \leq p-2} A_{nm, \ell}^\Delta a_{n'm'}(A_{nm, \ell}^\Delta)
+ \frac{1}{4} \sum_{\ell} b_{0, \ell+1} a_{n'm'}(C_{00, \ell}) + \sum_{0 \leq n \leq p-2} b_{n+1, \ell} a_{n'm'}(D_{n0, \ell}).
\]

Here \(a_{nm}(M)\) are the contributions corresponding to a supermultiplet \(M\), for each \(M\) \(a_{nm}(M)\) is given by a finite linear combination of \(u^{\frac{1}{2}(\Delta-\ell)} G_{\Delta}^{(\ell)}(u, v)\) for the appropriate \(\Delta, \ell\) corresponding to all operators belonging to the SU(4) representation labelled by \(nm\) in the multiplet \(M\). For each possible \(N = 4\) supermultiplet, as described for instance in [22], detailed results for \(a_{nm}(M)\) are given in [14]. \(A_{nm, \ell}^\Delta\) denotes a generic long multiplet whose lowest state has scale dimension \(\Delta, \ell\) and belongs to the SU(4) representation labelled by \(nm\), for unitarity \(\Delta \geq 2n + \ell + 2\). Conversely \(D_{nm, \ell}\) and \(C_{nm, \ell}\) are semi-short supermultiplets in which the scale dimension for the lowest state is determined to be \(2m + \ell\) and \(2n + \ell + 2\) respectively. These occur in the decompositions
\[
A_{nm, \ell}^{2m+\ell} \simeq D_{nm, \ell} \oplus D_{nm + 1, \ell - 1} \oplus \ldots, \quad m = 0, 1, \ldots, n - 1,
A_{nm, \ell}^{2n+\ell} \simeq D_{nm, \ell} \oplus C_{nm + 1, n + 1, \ell - 2} \oplus \ldots, \quad n = m, m + 1, \ldots,
A_{nm, \ell}^{2n+\ell} \simeq C_{nm, \ell} \oplus C_{nm + 1, m, \ell - 1} \oplus \ldots, \quad n = m, m + 1, \ldots,
\]
where in each case two multiplets are omitted which are irrelevant since they cannot contribute in the operator product expansions here. Correspondingly we have\(\footnote{For convenience the normalisation of \(a_{n'm'}(A_{nm, \ell}^\Delta)\) is changed from that used in [14] by a factor 16.}
\[
a_{n'm'}(A_{nm, \ell}^{2m+\ell}) = a_{n'm'}(D_{nm, \ell}) + \frac{m + 1}{4(2m + 1)} a_{n'm'}(D_{n m + 1, \ell - 1}),
\]
\[
(2.14a)
\]
When $\ell = 0$ in (2.14b) we may use
\[
a_{n'm'}(C_{nn,-2}) = -4a_{n'm'}(B_{nn}) + \frac{(n+1)(n+2)}{(2n+1)(2n+3)}a_{n'm'}(B_{n+1,n+1}),
\]
\[
a_{n'm'}(B_{00}) = \delta_{n'0}\delta_{m'0},
\] where, for $n \geq 1$, $B_{nn}$ denotes the $\frac{1}{2}$-BPS short multiplet corresponding to the $[0,2n,0]$ $SU(4)$ representation with $\Delta = 2n$ ($B_{00}$ is the trivial singlet multiplet for the identity operator). In (2.14c) for $\ell = 0$ we also have
\[
a_{n'm'}(C_{nm,-1}) = \frac{n+1}{2n+1}a_{n'm'}(B_{n+1,n}),
\] where, for $n > m$, $B_{nm}$ is a $\frac{3}{4}$-BPS short multiplet corresponding to the $[n-m,2m,n-m]$ representation.

The multiplets $D_{nm,\ell}$ are non unitary and their contributions must be cancelled in the conformal partial wave expansion for a unitary theory. The coefficients for non unitary long multiplets with $\Delta = 2t + \ell$, $t = 0, 1, \ldots, n$, are written $A_{nm,\ell}^{2t+\ell} \equiv A_{nm,\ell}$ and we must then have
\[
b_{n+1,\ell} + A_{n0,0\ell} = 0, \quad n = 0, 1, \ldots, p - 2,
\]
\[
\frac{m+1}{4(2m+1)}A_{nm,\ell} + A_{nm+1,m+1,\ell-1} = 0, \quad m = 0, 1, \ldots, n-1 \leq p - 3.
\] In other cases we must also require
\[
A_{nm,t\ell} = 0, \quad t = 0, \ldots, n, \quad t \neq m.
\] With the above relations we get
\[
a_{n'm'} = k\delta_{n'0}\delta_{m'0} + \sum_{0\leq m'\leq n \leq p-2} A_{nm,\ell}^{\Delta} a_{n'm'}(A_{nm,\ell}^{\Delta}) + \frac{1}{4} \sum_{\ell} b_{0,\ell+1} a_{n'm'}(C_{00,\ell})
\]
\[
+ \sum_{0 \leq n \leq p-2} \frac{(n+1)(n+2)}{16(2n+1)(2n+3)} A_{nn,n\ell} a_{n'm'}(C_{n+1,n,\ell-2}).
\] As may be seen from (2.14c) the contributions to the partial waves of the semi-short multiplets $C_{nn,\ell}$ for $n = 0, 1, \ldots, p-1$ cannot be combined to those corresponding to a
long multiplet. However using (2.14d) any particular semi-short multiplet can be taken to be part of a long multiplet at the expense of introducing the contribution for another semi-short multiplet. In an interacting theory in general all long multiplets are expected to gain anomalous dimensions but in a free theory with canonical scale dimensions there is an inherent ambiguity as a consequence of (2.14d) in writing the superconformal partial wave expansion. Necessarily, so long as the coefficients are non zero, the contributions corresponding to at least \( p \) semi-short multiplets for each appropriate \( \ell \) must be present in the expansion. In any interacting theory no contribution corresponding to \( C_{00,\ell} \) should be present as it contains higher spin conserved currents. If this is removed by using (2.14d) to be part of the associated twist 2 long multiplet \( A_{\ell+2} \), we obtain from (2.19)

\[
a_{n'n'} = C \delta_{n'0} \delta_{m'0} + \sum_{n=1}^{p} C_n a_{n'm'}(B_{nn}) + C_{20} a_{n'm'}(B_{20}) + \sum_{0 \leq m \leq n \leq p - 2} \dot{A}_{nm,\ell} a_{n'm'}(A_{nm,\ell}) - \frac{1}{24} \sum_{\ell=1,3,...} b_{0,\ell+2} a_{n'm'}(C_{10,\ell})
\]

\[
+ \sum_{0 \leq n \leq p - 2} \sum_{\ell=0,2,...} \frac{(n+1)(n+2)}{16(2n+1)(2n+3)} A_{nn,n+2} a_{n'm'}(C_{n+1,n+2})
\]

where we define for \( \Delta = \ell + 2 \),

\[
\dot{A}_{00,1\ell} = A_{00,1\ell} + \frac{1}{4} b_{0,\ell+1}, \quad \ell = 0, 2, \ldots,
\]

with otherwise \( \dot{A}_{nm,\ell} = A_{nm,\ell} \). The remaining coefficients are given by

\[
C = k - b_{0,-1}, \quad C_1 = \frac{1}{36}(b_{0,-1} - A_{00,00}),
\]

and for \( n = 2, 3, \ldots, p \)

\[
C_n = -\frac{n(n+1)}{4(2n-1)(2n+1)} A_{n-1,n-1,0} + \frac{(n-1)n^2(n+1)}{16(2n-3)(2n-1)^2(2n+1)} A_{n-2,n-2,0}.
\]

and also,

\[
C_{20} = -\frac{1}{36} b_{0,1}.
\]

For unitarity it is necessary that all coefficients are positive and that only contributions for \( \Delta \geq 2n + \ell + 2 \) arise. In general further rearrangements are necessary to achieve this as demonstrated later.
3. Free Field Results

We here consider the superconformal expansions for four point functions of BPS operators for $p = 2, 3, 4$ in the free field case. We present the results in each case first and then draw more general conclusions.

The results for $G$ in free field theory are expressible in terms of crossing symmetric polynomials $S_p(\sigma, \tau)$ which are defined by

$$S_p(\sigma, \tau) = S_p(\tau, \sigma) = \tau^p S_p(\sigma/\tau, 1/\tau).$$  \hspace{1cm} (3.1)

A basis for minimal polynomials is given for $p = 1, 2, 3, \ldots$ by first defining

$$S_{p,i}(\sigma, \tau) = \begin{cases} 
\sigma^p + \tau^p + 1, & i = 0, \\
\sigma^{p-i} \tau^i + \sigma^i \tau^{p-i} + \sigma^{p-i} + \tau^{p-i} + \sigma^i + \tau^i, & i = 1, 2, \ldots, i < \frac{1}{2}p, \\
\sigma^{\frac{1}{2}p} \tau^{\frac{1}{2}p} + \sigma^{\frac{1}{2}p} + \tau^{\frac{1}{2}p}, & i = \frac{1}{2}p, p \text{ even}.
\end{cases} \hspace{1cm} (3.2)$$

Assuming

$$S_{0,0}(\sigma, \tau) = 1,$$ \hspace{1cm} (3.3)

we then define a complete set of crossing symmetric polynomials by

$$S_{p(i,j)}(\sigma, \tau) = (\sigma \tau)^i S_{p-3j,i}(\sigma, \tau), \quad i = 0, 1, \ldots, \left[\frac{1}{3}(p - 3j)\right], \quad j = 0, 1, \ldots, \left[\frac{1}{3}p\right], \hspace{1cm} (3.4)$$

so that $2i + 3j \leq p$. For each $p$ a formula for the number of independent $(i, j)$ is given in [12]. The contributions represented by each polynomial correspond to the different possible sets of crossing symmetric free field graphs where the vertices are linked by $l, m, n$ lines, as shown in Fig. 1, where $l + m + n = p$ and $l \geq m \geq n \geq 0$ and where we identify $n = j, m = i + j$.

![Fig. 1 Free field contributions to four point function for $l = 5$, $m = 3$, $n = 2$.](image-url)
For $p = 2$ there are just two possible polynomials and the free field results are in general expressible as

$$G_0(u, v; \sigma, \tau) = S_{2(0,0)}(\sigma u, \tau u/v) + a S_{2(1,0)}(\sigma u, \tau u/v), \quad (3.5)$$

for some coefficient $a$. The first term corresponds to disconnected graphs, its coefficient is one as a consequence of normalising the two point function for the BPS operators to one. In the large $N$ limit the connected contribution is suppressed and we have

$$a = \frac{4}{N^2}. \quad (3.6)$$

From this we may determine

$$k = 3(1 + a), \quad H_0(u, v; \sigma, \tau) = 1 + \frac{1}{v^2} + a \frac{1}{v}, \quad (3.7)$$

and

$$\hat{f}(x, y) = \frac{1}{2} y \left( x^2 + x'^2 - a(x + x') \right) + \frac{1}{2} \left( x^2 - x'^2 + (2 + 3a)(x - x') \right), \quad (3.8)$$

where

$$x' = \frac{x}{x - 1}, \quad x + x' = xx' = \frac{x^2}{x - 1}. \quad (3.9)$$

The expansion of $H_0$ in (3.7) gives

$$A_{00, \ell} = 2^{\ell+1} \frac{(\ell + t + 1)!^2 (t!)^2}{(2\ell + 2t + 2)! (2t)!} \left( (\ell + 1)(\ell + 2t + 2) + a(-1)^t \right), \quad (3.10)$$

and for $\hat{f}$ in (3.8)

$$b_{1, \ell} = -2^{\ell+1} \frac{(\ell + 1)!^2}{(2\ell + 2)!} \left( (\ell + 1)(\ell + 2) + a \right), \quad \ell = 0, 2 \ldots, \quad (3.11)$$

$$b_{0, \ell} = -2^{\ell+1} \frac{(\ell + 1)!^2}{(2\ell + 2)!} \left( \ell(\ell + 3) - 3a \right), \quad \ell = -1, 1, \ldots.$$

As required by (2.17) to remove twist zero $A_{00,0,\ell} + b_{1,\ell} = 0$. The results (2.22), (2.23) and (2.24) give

$$C = 1, \quad C_1 = \frac{1}{3} a, \quad C_2 = \frac{1}{30} \left( 1 + \frac{1}{2} a \right), \quad C_{20} = \frac{2}{27} \left( 1 - \frac{3}{4} a \right). \quad (3.12)$$

The result that $C = 1$ is necessary for consistency with our normalisation since the coefficient of the identity in the operator product expansion is the same as that for the two point function. From (2.21) we get

$$\hat{A}_{00,1,\ell} = 2^{\ell+1} \frac{(\ell + 2)!^2}{(2\ell + 4)!} a. \quad (3.13)$$
The expansion for the free theory then becomes

\[
a_{n',m'} = \delta_{n'0}\delta_{m'0} + \sum_{n=1}^{2} C_n a_{n'm'}(B_{nn}) + C_{20} a_{n'm'}(B_{20}) + \sum_{\ell=0,2,\ldots,t \geq 1} \hat{A}_{00,\ell} a_{n'm'}(A_{2t+\ell}^{0}) \\
+ \frac{1}{\pi} \sum_{\ell=0,2,\ldots} \left( A_{00,0}a_{n'm'}(C_{11,\ell}) - b_{0,\ell+3} a_{n'm'}(C_{10,\ell+1}) \right).
\] (3.14)

The positivity requirements are satisfied for \(0 < a \leq \frac{4}{3}\) (in general for \(N = 4\) \(SU(N)\) superconformal theory \(a = \frac{4}{3(N^2 - 1)}\) so that \(a = \frac{4}{3}\) when \(N = 2\)).

For \(p = 3\) the free field theory result for the correlation function is in general

\[
G_0(u,v;\sigma,\tau) = S_{3(0,0)}(\sigma u,\tau u/v) + a S_{3(1,0)}(\sigma u,\tau u/v) + b S_{3(0,1)}(\sigma u,\tau u/v),
\] (3.15)

where as before the leading coefficient is taken to be 1 and for large \(N\) we have

\[
a = \frac{9}{N^2}, \quad b = \frac{18}{N^2}.
\] (3.16)

From (3.15) we may determine

\[
k = 3 + 6a + b,
\] (3.17)

and

\[
H_0(u,v;\sigma,\tau) = \frac{1}{v^3} \left( \frac{1}{2}(\sigma + \tau)u(1 + v^3) + \frac{1}{2}(\sigma - \tau)(-3u(1 - v^3) + 2(1 - v)(1 + v^3)) \right. \\
\left. + u(1 + v^3) - 1 + 2v + 2v^3 - v^4 \right) \\
+ a \frac{1}{v^2} \left( \frac{1}{2}(\sigma + \tau)u(1 + v) + \frac{1}{2}(\sigma - \tau)(1 - v)(2(1 + v) - u) + (1 + v)^2 \right) \\
+ b \frac{1}{v},
\] (3.18)

and

\[
\hat{f}(x,y) = \frac{1}{4}((y^2 + 1)(x^3 - x'^3) + 2y(x^3 + x'^3 + x^2 + x'^2) + 2(x^2 - x'^2) + 4(x - x')) \\
- \frac{1}{4}a \left((y^2 - 3)(x^2 - x'^2) - 2y(x^2 + x'^2 - 2(x + x')) - 12(x - x') \right) \\
- \frac{1}{2}b \left(y(x + x') - (x - x') \right).
\] (3.19)

Using from (2.4)

\[
Y_{00} = 1, \quad Y_{10} = 3(\sigma - \tau), \quad Y_{11} = 3(\sigma + \tau) - 1,
\] (3.20)

and letting

\[
A_{nm,\ell} = 2^{\ell-2} \frac{(\ell + t + 1)!^2 (t)!^2}{3(2\ell + 2t + 2)! (2t)!} a_{nm,\ell},
\] (3.21)
we have
\[ a_{11,\ell} = t(t+1)(\ell+1)(\ell+2t+2)(\ell+t+1)(\ell+t+2) \]
\[ + 4a((1 - (-1)^t)(\ell + t + 1)(\ell + t + 2) - (1 + (-1)^t)t(t + 1)), \]
\[ a_{10,\ell} = (t - 1)(t + 2)(\ell + 1)(\ell + 2t + 2)(\ell + t)(\ell + t + 3) \]
\[ - 4a(1 + (-1)^t)(\ell + 1)(\ell + 2t + 2), \]
\[ a_{00,\ell} = (t - 2)(t + 3)(\ell + 1)(\ell + 2t + 2)(\ell + t - 1)(\ell + t + 4) \]
\[ + 4a((1 - (-1)^t)(\ell + t + 1)(\ell + t + 2) - (1 + (-1)^t)t(t + 1)) \]
\[ + 24a(\ell + 1)(\ell + 2t + 2) + 24(2a + b)(-1)^t. \] (3.22)

In addition from (3.19) we obtain
\[ b_{2,\ell} = \frac{1}{3} 2^{\ell-1} \frac{(\ell + 1)!^2}{(2\ell + 2)!} (\ell + 1)(\ell + 2)(\ell(\ell + 3) + 4a), \]
\[ b_{1,\ell} = 2^{\ell-1} \frac{(\ell + 1)!^2}{(2\ell + 2)!} \left( (\ell + 1)(\ell + 2)((\ell - 1)(\ell + 4) - 4a) - 8a - 4b \right), \] (3.23)
\[ b_{0,\ell} = \frac{1}{3} 2^{\ell} \frac{(\ell + 1)!^2}{(2\ell + 2)!} \left( (\ell - 1)(\ell + 4)(\ell(\ell + 3) - 8a) - 12a + 6b \right). \]

As required by (2.17) the twist zero contributions are cancelled as a consequence of
\[ A_{00,0\ell} + b_{1,\ell} = 0, \quad A_{10,0\ell} + b_{2,\ell} = 0 \]
while removal of twist two contributions from the [0,2,0] partial wave follows from
\[ A_{11,1\ell} + \frac{1}{4} A_{10,\ell + 1} = 0 \]
which are satisfied by (3.22) and (3.23). Furthermore
\[ a_{11,0\ell} = a_{10,1\ell} = 0 \]
in accord with (2.18). Using (2.22) and (2.23) gives
\[ C = 1, \quad C_1 = \frac{1}{3} a, \quad C_2 = \frac{1}{60}(2a + b), \quad C_3 = \frac{1}{1050}(1 + a), \]
(3.24)
where \( C = 1 \) is necessary for consistency as before. From (2.21) we also have
\[ \hat{A}_{00,1\ell} = 2^{\ell+1} \frac{(\ell + 2)!^2}{(2\ell + 4)!} a. \] (3.25)

However \( A_{00,0\ell} < 0 \) for \( \ell \geq 2 \) and \( b_{0,\ell} > 0 \) for \( \ell > 1 \). These negative semi-short contributions may be absorbed into the corresponding long multiplets by letting \( A \rightarrow \hat{A} \) for
\[ \hat{A}_{10,2\ell} = A_{10,2\ell} - \frac{1}{24} b_{0,\ell + 2}, \quad \hat{A}_{11,2\ell} = A_{11,2\ell} + \frac{1}{24} A_{00,0\ell + 2} \] (3.26)
which gives for this case
\[ \hat{A}_{10,2\ell} = 2^\ell \frac{(\ell + 3)!^2}{3(2\ell + 6)!} ((\ell + 1)(\ell + 6)a + 2a - b), \] (3.27)
\[ \hat{A}_{11,2\ell} = 2^\ell \frac{(\ell + 3)!^2}{3(2\ell + 6)!} ((\ell + 3)(\ell + 4)a + b). \]
The remaining twist 4 contribution is given by

\[ A_{00,2\ell} = 2^\ell \frac{(\ell + 3)!^2}{3(2\ell + 6)!} ((\ell + 1)(\ell + 6)a + b). \]  

(3.28)

The expansion for the free theory then becomes

\[
a_{n'm'} = \delta_{n'0} \delta_{m'0} + \sum_{n=1}^{3} C_n\ a_{n'm'}(B_{nn}) + C_{20}\ a_{n'm'}(B_{20}) + C_{31}\ a_{n'm'}(B_{31})
\]

\[ + \sum_{0 \leq m \leq n \leq 1} \hat{A}_{nm,\ell} a_{n'm'}(A_{nm,\ell}^{2\ell+\ell}) \]

\[ + \frac{1}{160} \sum_{\ell=0,2,...} \left( 4A_{11,1\ell+2} a_{n'm'}(C_{22,\ell}) - A_{00,0\ell+4} a_{n'm'}(C_{21,\ell+1}) + b_{0,\ell+3} a_{n'm'}(C_{20,\ell}) \right), \]

(3.29)

with in general

\[ C_{31} = -\frac{3}{800} A_{00,02}. \]  

(3.30)

Hence from (2.24) and (3.30) we obtain

\[ C_{20} = \frac{1}{54} (2a - b), \quad C_{31} = \frac{3}{2000} (18 - 4a - b). \]  

(3.31)

For \( p = 4 \) the free field theory result for the correlation function is in general

\[
\mathcal{G}_0(u, v; \sigma, \tau) = S_{4(0,0)}(\sigma u, \tau u/v) + a S_{4(1,0)}(\sigma u, \tau u/v) + b S_{4(2,0)}(\sigma u, \tau u/v) + c S_{4(0,1)}(\sigma u, \tau u/v),
\]

(3.32)

where for large \( N \) we have

\[ a = b = \frac{16}{N^2}, \quad c = \frac{32}{N^2}. \]  

(3.33)

In this case from (3.32) we have

\[ k = 3(1 + 2a + b + c), \]  

(3.34)

and

\[
\mathcal{H}_0(u, v; \sigma, \tau)
\]

\[ = \frac{1}{v^4} \left( \sigma \tau u^2(1 + v^4) \right.
\]

\[ + \frac{1}{2}(\sigma - \tau)^2(2(1 - v)^2(1 + v^4) - 5u(1 + v^5) + 3uv(1 + v^3) + 4u^2(1 + v^4))
\]

\[ + \frac{1}{2}(\sigma^2 - \tau^2)(u(1 - v)(1 + v^4) - 2u^2(1 - v^4))
\]

\[ + \frac{1}{2}(\sigma + \tau)(- (1 - v)^2 + u(1 + v))(1 + v^4) \]

13
\[ a \left( \sigma \tau u^2 (1 + v^2) + \frac{b}{v^2} \left( \sigma \tau u^2 + \frac{1}{2} (\sigma - \tau)^2 (2(1 - v)^2 - u(1 + v)) + \frac{1}{2} (\sigma^2 - \tau^2) u(1 - v) \right) + \frac{c}{v^2} \left( \frac{1}{2} (\sigma + \tau) u(1 + v) + \frac{1}{2} (\sigma - \tau)(1 - v)(2(1 + v) - u) + 3v \right) \right). \] (3.35)

In addition
\[ \hat{f}(x, y) = \frac{1}{8} \left( (y^3 + 3y)(x^4 + x'^4) + y^2(3(x^4 - x'^4) + 2(x^3 - x'^3)) \right) + \frac{1}{2} y(x^3 + x'^3 + x^2 + x'^2) \]
\[ + \frac{1}{2} \left( x^4 - x'^4 + 2(x^3 - x'^3) + 4(x^2 - x'^2) + 8(x - x') \right) \]
\[ - \frac{1}{8} a \left( (y^3 - 5y)(x^3 + x'^3 + x^2 + x'^2 + 2(x + x')) - y^2(x^3 - x'^3 - 3(x^2 - x'^2)) \right) \]
\[ - \frac{1}{8} b \left( 16y(x + x') - 3(x^3 - x'^3) - 7(x^2 - x'^2) - 24(x - x') \right), \]
\[ + \frac{1}{8} b \left( (y^3 + 3y)(x^2 + x'^2 + 2(x + x')) - y^2(x^2 - x'^2) \right) \]
\[ - \frac{1}{8} b \left( 12y(x + x') - 5(x^2 - x'^2) - 12(x - x') \right) \]
\[ - \frac{1}{8} c \left( (y^2 - 1)(x^2 - x'^2) + 6y(x + x') - 6(x - x') \right). \] (3.36)

For this case we take from (2.43) as well as (3.20)
\[ Y_{20} = 10(\sigma - \tau)^2 - 5(\sigma + \tau) + 1, \quad Y_{21} = 10(\sigma^2 - \tau^2) - 5(\sigma - \tau), \]
\[ Y_{22} = 10(\sigma^2 + \tau^2) + 40\sigma\tau - 8(\sigma + \tau) + 1, \] (3.37)

and writing
\[ A_{nm,t\ell} = 2^{\ell-5} \frac{(\ell + t + 1)!^2 (t)!^2}{45(2\ell + 2t + 2)! (2t)!} a_{nm,t\ell}, \] (3.38)

we have the expansion coefficients
\[ a_{22,t\ell} = \frac{1}{3} (t - 1) t (t + 1) (t + 2) (\ell + 1) (\ell + 2t + 2) (\ell + t)(\ell + t + 1)(\ell + t + 2)(\ell + t + 3) \]
\[ a_{21, t\ell} = (t - 2) t(t + 1)(t + 3)(\ell + 1)(\ell + 2t + 2)(\ell + t - 1)(\ell + t + 1)(\ell + t + 2)(\ell + t + 4) - 36a(1 - (-1)^t)(\ell + 1)(\ell + 2t + 2)((\ell + 3)(\ell + 2t) + 2(t + 1)(t - 2)) - 144b(1 - (-1)^t)(\ell + 1)(\ell + 2t + 2), \]
\[ a_{20, t\ell} = \frac{2}{3}(t - 2)(t - 1)(t + 2)(t + 3)(\ell + 1)(\ell + 2t + 2) \times (\ell + t - 1)(\ell + t)(\ell + t + 3)(\ell + t + 4) - 6a((1 + (-1)^t)(t - 2)(t + 3)(\ell + t)(\ell + t + 3)((\ell + 1)(\ell + 2t + 2) - 4)
+ (1 - (-1)^t)(t - 1)(t + 2)(\ell + t - 1)(\ell + t + 4)((\ell + 1)(\ell + 2t + 2) + 4)) + 48b(2(1 - (-1)^t)(\ell + 1)(\ell + 2t + 2) - (-1)^t(t - 1)(t + 2)(\ell + t - 1)(\ell + t + 4)), \]
\[ a_{11, t\ell} = 2(t - 3)t(t + 1)(t + 4)(\ell + 1)(\ell + 2t + 2) \times (\ell + t - 2)(\ell + t + 1)(\ell + t + 2)(\ell + t + 5) + 12a((t + 2 + 2)(\ell + t)(\ell + t + 1)(\ell + t + 2) + 12(-1)^t(t - 3)(t + 4))
- 6(1 - (-1)^t)(\ell + 1)(\ell + 2t + 2)((\ell + 1)(\ell + t + 2) + (t - 3)(t + 4)) + 48b(6(1 - (-1)^t)(\ell + 1)(\ell + 2t + 2) + (-1)^t(t - 3)(t + 4)(\ell + t + 1)(\ell + t + 2)) + 480c((1 + (-1)^t)(\ell + 1)(\ell + 2t + 2) - 2(-1)^t(\ell + t + 1)(\ell + t + 2)), \]
\[ a_{10, t\ell} = \frac{5}{3}(t - 3)(t - 1)(t + 2)(t + 4)(\ell + 1)(\ell + 2t + 2) \times (\ell + t - 2)(\ell + t)(\ell + t + 1)(\ell + t + 3)(\ell + t + 5) + 60a((\ell + 1)(\ell + 2t + 2)(\ell + t)(\ell + t + 3)(2(t - 1)(t + 2) + 1 + (-1)^t)
+ 60(a(t - 3)(t + 4) - 4b - 8c)(1 + (-1)^t)(\ell + 1)(\ell + 2t + 2), \]
\[ a_{00, t\ell} = (t - 3)(t - 2)(t + 3)(t + 4)(\ell + 1)(\ell + 2t + 2) \times (\ell + t - 2)(\ell + t - 1)(\ell + t + 4)(\ell + t + 5) + 12a((\ell + 1)(\ell + 2t + 2)(\ell + t - 1)(\ell + t + 4)(11(t - 2)(t + 3) - 3 + 3(-1)^t)
- (1 - (-1)^t)(\ell + 1)(\ell + 2t + 2)(t - 3)(t + 4)
+ 6(-1)^t(t - 3)(t - 2)(t + 3)(t + 4)) + 48b(63(\ell + 1)(\ell + 2t + 2)
+ (-1)^t((\ell + t + 1)(\ell + t + 2)(t^2 + t - 9) - 9(t - 1)(t + 2) + 54)) + 480c((1 + (-1)^t)(\ell + 1)(\ell + 2t + 2) - 2(-1)^t((\ell + t + 1)(\ell + t + 2) - 9)). (3.39) \]

Expanding \( \hat{f} \) from (3.36) gives
\[ b_{3, \ell} = -\frac{1}{5} 2^{\ell - 2} \left( \frac{(\ell + 1)^2}{(2\ell + 2)!} (\ell + 3) \ell (\ell + 2)(\ell + 1) \left( \frac{1}{3}(\ell + 4)(\ell - 1 + a) - 4(a - b) \right) , \right. \]
where we have

From (3.39) and (3.40) using (2.21) we get for the \( p \)

and for \( t \)

in accord with (2.18).

The results (3.39) and (3.40) satisfy the six relations required by (2.17) when \( p = 4 \) and furthermore we have \( a_{10,1t} = a_{11,0t} = a_{20,1t} = a_{20,2t} = a_{21,0t} = a_{21,2t} = a_{22,0t} = a_{22,1t} = 0 \)

in accord with (2.18).

For the \( p = 4 \) case the conformal partial wave expansion may be rewritten in the form

\[
a_{n'm'} = \delta_{n'0}\delta_{m'0} + \sum_{n=1}^{4} C_n a_{n'm'}(B_n) + \sum_{n=0}^{2} C_{n+2} a_{n'm'}(B_{n+2}) + C_{40} a_{n'm'}(B_{40})
\]

\[
+ \sum_{0 \leq m \leq n \leq 2} \hat{A}_{nm,\ell} a_{n'm'}(A_{nm,\ell}^2 + \ell)
\]

\[
+ \frac{1}{10} \sum_{\ell=0,2,\ldots} \left( 3A_{22,2\ell+2} a_{n'm'}(C_{33,\ell}) - \frac{1}{2}A_{11,1\ell+4} a_{n'm'}(C_{32,\ell+1})
\]

\[
+ \frac{1}{8} A_{00,0\ell+4} a_{n'm'}(C_{31,\ell}) - \frac{1}{8} b_{0,\ell+5} a_{n'm'}(C_{30,\ell+1}) \right),
\]

(3.41)

where we have

\[
C_{42} = -\frac{1}{490} A_{11,12}, \quad C_{40} = -\frac{1}{1960} b_{0,3},
\]

(3.42)

and for \( t = 3 \) we define

\[
\hat{A}_{20,3\ell} = A_{20,3\ell} + \frac{1}{160} b_{0,\ell+3}, \quad \hat{A}_{21,3\ell} = A_{21,3\ell} - \frac{1}{160} A_{00,0\ell+3},
\]

\[
\hat{A}_{22,3\ell} = A_{22,3\ell} + \frac{1}{40} A_{11,1\ell+2}.
\]

(3.43)

From (3.39) and (3.40) using (2.21) we get for the \( p = 4 \) case

\[
\hat{A}_{00,1\ell} = 2^{\ell+1} \frac{(\ell + 2)!^2}{(2\ell + 4)!} a,
\]

(3.44)

and from (3.26)

\[
\hat{A}_{10,2\ell} = 2^\ell \frac{(\ell + 3)!^2}{3(2\ell + 6)!} ((\ell + 1)(\ell + 6)b - 2a + 4b - c),
\]

\[
\hat{A}_{11,2\ell} = 2^\ell \frac{(\ell + 3)!^2}{3(2\ell + 6)!} ((\ell + 3)(\ell + 4)b + c).
\]

(3.45)
For the remaining twist 4 contribution we have

$$A_{00,2\ell} = 2^{\ell} \frac{(\ell + 3)!^2}{3(2\ell + 6)!} ((\ell + 1)(\ell + 6)b + c). \quad (3.46)$$

Furthermore (3.43) gives for twist 6

$$\hat{A}_{20,3\ell} = 2^{\ell-2} \frac{(\ell + 4)!^2}{15(2\ell + 8)!} ((\ell + 3)(\ell + 4)(\ell + 5)(\ell + 6)a - 2(\ell + 4)(\ell + 5)(3a + 3b + c) + 6(4a + 2b + 3c)), \quad (3.47)$$

and we also have

$$A_{10,3\ell} = 2^{\ell-3} \frac{(\ell + 4)!^2}{3(2\ell + 8)!} ((\ell + 1)(\ell + 3)(\ell + 6)(\ell + 8)a, \quad (3.48)$$

$$A_{11,3\ell} = 2^{\ell-2} \frac{(\ell + 4)!^2}{15(2\ell + 8)!} ((\ell + 4)(\ell + 5)(3(\ell + 1)(\ell + 8)a + 2c), \quad (3.49)$$

$$A_{00,3\ell} = 2^{\ell-3} \frac{(\ell + 4)!^2}{5(2\ell + 8)!} ((\ell + 1)(\ell + 2)(\ell + 7)(\ell + 8)a + 4(\ell + 1)(\ell + 8)(b + \frac{1}{3}c) + 4c).$$

Using (2.22) and (2.23) gives $C = 1$ again as required and for the coefficients for the contributions of $\frac{1}{2}$-BPS operators

$$C_1 = \frac{1}{3}a, \quad C_2 = \frac{1}{60}(2b + c), \quad C_3 = \frac{1}{350}(a + c), \quad C_4 = \frac{1}{2^{2.3.5.7.7}2}(2 + 2a + b). \quad (3.50)$$

From (2.24), (3.30) and (3.42) we obtain

$$C_{20} = \frac{1}{54}(4b - 2a - c), \quad C_{31} = \frac{9}{2000}(6a - 4b - c), \quad (3.51)$$

$$C_{42} = \frac{1}{3.5.7.7}(24 - 15a - 2b - 4c), \quad C_{40} = \frac{1}{3.5.7.7}(84 - 153a + 61b + 11c). \quad (3.52)$$

To summarise some of the above results we first note from (3.12), (3.24) and (3.49) that in each case $C_1 = \frac{1}{3}a$. The corresponding short multiplet $B_{11}$ is special in that it contains the energy momentum tensor as well as the $SU(4)$ current. The contributions of these operators in the operator product expansion are constrained by Ward identities [23]. For the energy momentum tensor the coefficient of its contribution in the operator product expansion in four dimensions to the four point function, by applying (1.9) and (1.11) in this case, is $16p^2/9C_T$, where $C_T/S_{12}^2$ is the coefficient of the energy momentum
tensor two point function. In the expansion of $a_{00}(B_{11})$ the contribution corresponding 
the energy momentum tensor is $\frac{2}{15}uG_4^{(2)}(u,v)$ so that this requires $C_1 = 40p^2/3C_T$, where 
with our normalisations $C_T = 40(N^2 - 1)$ (where $40 = 6 \times \frac{4}{3} + 4 \times 4 + 16$ reflecting the 
decomposition into the contributions of the scalars, fermions and vector in the elementary $\mathcal{N} = 4$ multiplet). Assuming that for arbitrary $p$ only $S_{p(1,0)}$ in $\mathcal{G}_0$ gives a non zero $C_1$, as we have found for $p = 2, 3, 4$ and as may be expected since this is the only contribution involving two particle reducible graphs, then its coefficient must in general be

$$a = \frac{p^2}{N^2 - 1}. \quad (3.51)$$

This result was found exactly [17] for $p = 4$ even allowing for an admixture of double trace operators in the $\frac{1}{2}$-BPS operators whose correlation function is being considered. In addition we note that the expansion coefficients for long multiplets with twist $< 2p$ are suppressed in the large $N$ limit so that there are only $1/N^2$ contributions. The results for $\hat{A}_{00,1\ell}$ given by (3.13), (3.25) and (3.44) are identical in form. This also applies to $\hat{A}_{10,2\ell}, \hat{A}_{11,2\ell}$, as shown by (3.27) and (3.45), and $\hat{A}_{00,2\ell}$, as given by (3.28) and (3.46), so long as we use the large $N$ relations $b = 2a$ from (3.16) for $p = 3$ and $b = a, c = 2a$ from (3.33) for $p = 4$. This result, which plays a crucial role in our subsequent discussions using perturbation theory and also results from large $N$ obtained via AdS/CFT, is dependent on there being only contributions for semi-short multiplets $C_{nm}$ in the superconformal partial wave expansions, as in (3.14), (3.29) or (3.41), for $n = p - 1$, with others decoupled for large $N$. In the large $N$ limit we may also note that the coefficients for $\frac{1}{2}$-BPS multiplets $B_{nn}$ are suppressed except when $n = p$ and the $\frac{1}{4}$-BPS multiplets $B_{n+2n}$, from (3.31) and (3.50), are absent for $n = 0, \ldots, p - 3$.

4. Large $N$, Strong Coupling Results

The dynamical contributions to the $\frac{1}{2}$-BPS four-point functions is contained solely in the function $\mathcal{H}$ which is constrained by crossing symmetry

$$\mathcal{H}(u,v;\sigma,\tau) = \frac{1}{v^2} \mathcal{H}(u/v, 1/v; \tau, \sigma) = \left(\frac{u}{v}\right)^p \tau^{p-2} \mathcal{H}(v, u; \sigma/\tau, 1/\tau). \quad (4.1)$$

The results obtained via the AdS/CFT correspondence, for the large $N$ limit, are expressible in terms of conformal four-point integrals which may be reduced to the two variable functions $\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$ for arbitrary $\Delta_i$. The properties of these functions have been explored in detail [24,23] and they satisfy various symmetry and other relations, some of which are listed in appendix C.
For $p = 2$ we have

$$H(u, v; \sigma, \tau) = -\frac{4}{N^2} u^2 D_{2422}(u, v), \quad (4.2)$$

and for $p = 3$ we have

$$H(u, v; \sigma, \tau) = -\frac{9}{N^2} u^3 ((1 + \sigma + \tau) D_{3533} + D_{3522} + \sigma D_{2523} + \tau D_{2532}), \quad (4.3)$$

while for $p = 4$ we have

$$H(u, v; \sigma, \tau) = -\frac{4}{N^2} u^4 \left((1 + \sigma^2 + \tau^2 + 4\sigma + 4\tau + 4\sigma\tau) D_{4644} \right.$$  
$$+ 2(D_{4633} + D_{4622}) + 2\sigma^2(D_{3634} + D_{2624}) + 2\tau^2(D_{3643} + D_{3624})$$  
$$- 4\sigma(\overline{D}_{4624} - 2D_{3623}) - 4\tau(\overline{D}_{4642} - 2D_{3632})$$  
$$- 4\sigma\tau(\overline{D}_{2644} - 2D_{2633}) \right). \quad (4.4)$$

In each case the results have been rewritten to ensure that the $D$ functions are multiplied by the maximum overall power of $u$. The crossing relations (4.1) also follow straightforwardly using symmetry relations for $D$ functions listed in appendix C.

Defining

$$s = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4), \quad (4.5)$$

then for $s = 0, 1, 2, \ldots$ $D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v)$ can be written in the form $\ln u f(u, v) + g(u, v)$ where $f(u, v), g(u, v)$ have power series expansions in powers of $u$ and $1 - v$, although for $g$ it is necessary to allow negative powers $u^{-s+m}, m = 0, 1, 2 \ldots$. The case for $s = -1, -2, \ldots$ may also be accommodated by virtue of

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v) = u^{-s} \overline{D}_{\Delta_4\Delta_3\Delta_2\Delta_1}(u, v). \quad (4.6)$$

The $\ln u$ terms of course lead to contributions to anomalous dimensions of order $1/N^2$. From (4.2), (4.3) and (4.4) the operators which gain anomalous dimensions must have a twist of at least $2p$.

However the potentially singular contributions present in $g(u, v)$ involving negative powers of $u$ play a significant role. For $s$ a positive integer the $\overline{D}$ function can be decomposed in the form

$$\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v) = \overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v)_{\text{reg.}} + \overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v)_{\text{sing.}} \quad (4.7)$$
where the first regular part has an expansion involving $u^m$ and $u^m \ln u$ for $m = 0, 1, \ldots$

\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v)_{\text{sing.}} = u^{-s} \frac{\Gamma(\Delta_1 - s)\Gamma(\Delta_2 - s)\Gamma(\Delta_3)\Gamma(\Delta_4)}{\Gamma(\Delta_3 + \Delta_4)} \times \sum_{m=0}^{s-1} (-1)^m (s - m - 1)! \frac{(\Delta_1 - s)_m (\Delta_2 - s)_m (\Delta_3)_m (\Delta_4)_m}{m! (\Delta_3 + \Delta_4)_{2m}} \times u^m F(\Delta_2 - s + m, \Delta_3 + m; \Delta_3 + \Delta_4 + 2m; 1 - v). \]  

(4.8)

In (4.2) the $D$ function has $s = 1$ while in (4.3) we have $s = 1, 2$ and in (4.4) $s = 0, 1, 2, 3$. In general with the $SU(4)$ decomposition given by (2.8) we may write

\[ A_{nm}(u, v) = B_{nm}(u, v) + O(u^p \ln u, u^p), \]  

(4.9)

where $B_{nm}$ is calculated by using (4.8). Since there are no $\ln u$ terms in $B_{mn}(u, v)$ there are no anomalous dimensions obtained in the conformal wave expansion which involve operators with twist $2 \leq \Delta - \ell < 2p$. It is the purpose here to show that these contributions cancel exactly the corresponding long supermultiplet contributions obtained by the free-field calculations for large $N$ in section 3. This indicates that the corresponding long supermultiplets $A_{nm, \ell}$, $0 < \Delta - \ell < 2p$, decouple from the spectrum in the large $N$ limit. As mentioned earlier, this depends on requiring that the contributions from semi-short multiplets $C_{nm, \ell}$ in the partial wave expansion are solely those for $n = p - 1$, with others disappearing for large $N$.

For $p = 2$ it is easy to see that

\[ B_{00}(u, v) = -\frac{4}{3N^2} u F(3, 2; 4; 1 - v). \]  

(4.10)

For $p = 3$ from (4.3) we get, with the aid of various $D$ identities,

\[ A_{11}(u, v) = -\frac{3}{2N^2} u^3 \left(2D_{3533}(u, v) - D_{1533}(u, v) + \frac{1}{uv^2} \right), \]

\[ A_{10}(u, v) = -\frac{3}{2N^2} u^3 \left(D_{2523}(u, v) - D_{2532}(u, v) \right), \]  

(4.11)

\[ A_{00}(u, v) = -\frac{3}{2N^2} u^3 \left(8D_{3533}(u, v) + 6D_{3522}(u, v) - D_{1533}(u, v) + \frac{1}{uv^2} \right), \]

which leads to

\[ B_{11}(u, v) = -\frac{3}{5N^2} u^2 \left(F(4, 3; 6; 1 - v) + \frac{5}{2v^2} \right), \]

\[ B_{10}(u, v) = \frac{3}{5N^2} u^2 (1 - v) F(5, 3; 6; 1 - v), \]  

(4.12)

\[ B_{00}(u, v) = -\frac{3}{5N^2} u \left(5F(3, 2; 4; 1 - v) + u F(4, 3; 6; 1 - v) + \frac{5u}{2v^2} \right), \]
using (4.8) and for $B_{10}$ a result from appendix C. It is also useful to note that $v^{-2} = F(4, 2; 4; 1 - v)$. In a similar fashion for $p = 4$ we may obtain

\[
A_{22} = -\frac{2}{5 N^2} u^4 \left( \overline{D}_{4644} + \overline{D}_{2633} - \overline{D}_{2644} - \frac{1}{3} (\overline{D}_{1634} + \overline{D}_{1643}) + \frac{2}{3} \frac{1}{uv^3} (1 + v) \right),
\]

\[
A_{20} = -\frac{4}{5 N^2} u^4 \left( -\overline{D}_{2633} - \frac{1}{3} (\overline{D}_{1634} + \overline{D}_{1643}) + \frac{2}{3} \frac{1}{uv^3} (1 + v) \right),
\]

\[
A_{11} = -\frac{4}{5 N^2} u^4 \left( 8\overline{D}_{4644} + \frac{2}{3} \overline{D}_{4633} - \frac{1}{3} \overline{D}_{2633} - \frac{14}{3} \overline{D}_{2644} - \overline{D}_{1634} + \overline{D}_{1643} \right)
+ 2 \frac{1}{uv^3} (1 + v) + \frac{10}{3} \frac{1}{u^2 v^2},
\]

\[
A_{00} = -\frac{4}{N^2} u^4 \left( \frac{5}{2} \overline{D}_{4644} + \frac{10}{3} \overline{D}_{4633} + 2\overline{D}_{4622} + \frac{1}{30} \overline{D}_{2633} - \frac{5}{6} \overline{D}_{2644} - \frac{1}{10} (\overline{D}_{1634} + \overline{D}_{1643}) \right)
+ \frac{1}{5} \frac{1}{uv^3} (1 + v) + \frac{2}{3} \frac{1}{u^2 v^2},
\]

(4.13)

and also

\[
A_{21} = -\frac{2}{5 N^2} u^4 \left( \overline{D}_{3634} - \overline{D}_{3643} - \overline{D}_{1634} + \overline{D}_{1643} - 2 \frac{1}{uv^3} (1 - v) \right),
\]

\[
A_{10} = -\frac{2}{3 N^2} u^4 \left( 4 (\overline{D}_{3623} - \overline{D}_{3632}) + 5 (\overline{D}_{3634} - \overline{D}_{3643}) \right)
- \overline{D}_{1634} + \overline{D}_{1643} - 2 \frac{1}{uv^3} (1 - v) \right).
\]

(4.14)

In (4.13) $\overline{D}_{1634} + \overline{D}_{1643}$ may be further simplified using a result in appendix C. Using (4.8) and an extension from appendix C this leads to

\[
B_{22}(u, v) = -\frac{4}{5 N^2} u^3 \left( \frac{2}{3} F(5, 3; 6; 1 - v) + \frac{6}{35} F(5, 4; 8; 1 - v) + \frac{1}{3} (1 + v) \frac{1}{v^3} \right),
\]

\[
B_{21}(u, v) = \frac{4}{5 N^2} u^3 (1 - v) \left( \frac{1}{5} F(6, 4; 8; 1 - v) + \frac{1}{v^3} \right),
\]

\[
B_{20}(u, v) = -\frac{8}{5 N^2} u^3 \left( -\frac{2}{5} F(5, 3; 6; 1 - v) + \frac{1}{3} (1 + v) \frac{1}{v^3} \right),
\]

\[
B_{11}(u, v) = -\frac{8}{5 N^2} u^2 \left( \frac{2}{3} F(4, 3; 6; 1 - v) + \frac{1}{5} \frac{1}{v^3} - \frac{2}{15} u F(5, 3; 6; 1 - v)
+ \frac{8}{35} u F(5, 4; 8; 1 - v) + (1 + v) \frac{u}{v^3} \right),
\]

\[
B_{10}(u, v) = \frac{4}{5 N^2} u^2 (1 - v) \left( \frac{4}{3} F(5, 3; 6; 1 - v) + \frac{5}{21} u F(6, 4; 8; 1 - v) + \frac{5}{3} \frac{u}{v^3} \right),
\]

\[
B_{00}(u, v) = -\frac{8}{5 N^2} u \left( \frac{10}{3} F(3, 2; 4; 1 - v) + \frac{2}{3} u F(4, 3; 6; 1 - v) + \frac{5}{3} \frac{u}{v^2}
+ \frac{1}{15} u^2 F(5, 3; 6; 1 - v) + \frac{1}{7} u^2 F(5, 4; 8; 1 - v) + \frac{1}{2} (1 + v) \frac{u^2}{v^3} \right),
\]

(4.15)

where also $v^{-3} = F(6, 3; 6; 1 - v)$. 

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The conformal partial wave expansion

\[ B_{nm}(u, v) = \sum_{t, \ell} B_{nm, t\ell} u^t G_{\ell+2t+4}^{(\ell)}(u, v), \]  

would involve contributions for operators which have no anomalous dimensions. To analyse these we need to compute \( B_{nm, t\ell}, 1 \leq t \leq p - 1 \) for \( p = 2, 3, 4 \). Since

\[ G_{\ell+2t}^{(\ell)}(0, v) = g_{t, \ell}(1 - v), \]  

where \( g_{t, \ell} \) is defined in (2.11), then for the lowest twist contributions for each \( B_{nm} \) it is sufficient to use, since starting from \( D_{n_1 n_2 n_3} \) only hypergeometric functions of this form appear in (4.10), (4.12) and (4.13),

\[ F(a, b; 2b; x) = \sum_{\ell=0, 2, \ldots} r_{a b, \ell} g_{a, \ell}(x), \]  

where, as demonstrated in appendix D,

\[ r_{a b, \ell} = \frac{1}{2^\ell (1/2)\ell} \frac{(a)_{\ell}(a - b)_{1/2 \ell}}{(a + 1/2 \ell - 1/2)_{1/2 \ell}(b + 1/2)_{1/2 \ell}}. \]

More generally for application to the required expansion of \( B_{nm} \) this is extended in appendix D to an expansion in conformal partial waves with increasing twist

\[ u^{a-2} F(a, b; 2b; 1 - v) = \sum_{\ell=0, 2, \ldots, j=0, 1, \ldots} c_{a b; j, \ell} u^{a-2+j} G_{\ell+2a+2j}^{(\ell)}(u, v), \]  

where the first few cases of \( c_{a b; j, \ell} \) are

\[ c_{a b; 0, \ell} = r_{a b, \ell}, \]
\[ c_{a b; 1, \ell} = \frac{ab(a - 2b)}{2(4b^2 - 1)} r_{a+1 b+1, \ell} - \frac{1}{4} r_{a-1 b-1, \ell+2}, \]
\[ c_{a b; 2, \ell} = a(a + 1)(b + 1) \frac{(a - 2b - 1)(a - 2b)}{16(2b + 1)^2(2b + 3)} r_{a+2 b+2, \ell} - a \frac{a(a - 3b - 1) + b}{16(2a - 1)(2b + 1)} r_{a b, \ell+2}. \]

For \( p = 2 \) we may then find for the partial wave expansion coefficients

\[ B_{00, 1\ell} = - \frac{4}{3N^2} c_{3 2; 0, \ell} = - \frac{1}{N^2} 2^{\ell+3} (\ell + 2)^2 (2\ell + 4)! , \]

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which clearly cancels \( \hat{A}_{00,1\ell} \) in (3.13) for \( a \) as in (3.6). For \( p = 3 \) we find that, using for \( B_{10} \) a recurrence relation for \((1-v)G_{\Delta}^{(\ell)}(u,v)\),

\[
B_{11,2\ell} = -\frac{3}{5N^2} \left( c_{42;0,\ell} + \frac{\ell}{2} c_{42;0,\ell} \right) = -\frac{3}{N^2} 2^\ell \left( \frac{(\ell + 3)!}{(2\ell)!} \right) \left( \ell^2 + 7\ell + 14 \right),
\]

\[
B_{10,2\ell} = -\frac{6}{5N^2} c_{53;0,\ell-1} = -\frac{3}{N^2} 2^\ell \left( \frac{(\ell + 3)!}{(2\ell)!} \right) (\ell + 1)(\ell + 6),
\]

\[
B_{00,2\ell} = -\frac{3}{N^2} \left( c_{32;1,\ell} + \frac{\ell}{5} c_{43;0,\ell} + \frac{\ell}{2} c_{42;0,\ell} \right) = -\frac{3}{N^2} 2^\ell \left( \frac{(\ell + 3)!}{(2\ell)!} \right) \left( \ell^2 + 7\ell + 8 \right),
\]

\[
B_{00,1\ell} = -\frac{3}{N^2} c_{32;0,\ell} = -\frac{9}{N^2} 2^\ell \left( \frac{(\ell + 2)!}{(2\ell + 4)!} \right),
\]

which cancel the corresponding free-field cases in (3.25), (3.26), (3.27) and (3.28) with \( a, b \) as in (3.16). For \( p = 4 \) in a similar fashion we find that

\[
B_{22,3\ell} = -\frac{1}{N^2} 2^{\ell+1} \left( \frac{(\ell + 4)!}{15(2\ell + 8)!} \right) (\ell + 4)(\ell + 5)(\ell^2 + 9\ell + 26),
\]

\[
B_{21,3\ell} = -\frac{1}{N^2} 2^{\ell+1} \left( \frac{(\ell + 4)!}{5(2\ell + 8)!} \right) (\ell + 1)(\ell + 8)(\ell^2 + 9\ell + 22),
\]

\[
B_{20,3\ell} = -\frac{1}{N^2} 2^{\ell+2} \left( \frac{(\ell + 4)!}{15(2\ell + 8)!} \right) (\ell + 1)(\ell + 2)(\ell + 7)(\ell + 8),
\]

\[
B_{11,3\ell} = -\frac{1}{N^2} 2^{\ell+2} \left( \frac{(\ell + 4)!}{15(2\ell + 8)!} \right) (\ell + 4)(\ell + 5)(3\ell^2 + 27\ell + 28),
\]

\[
B_{10,3\ell} = -\frac{1}{N^2} 2^{\ell+1} \left( \frac{(\ell + 4)!}{3(2\ell + 8)!} \right) (\ell + 1)(\ell + 3)(\ell + 6)(\ell + 8),
\]

\[
B_{00,3\ell} = -\frac{1}{N^2} 2^{\ell+1} \left( \frac{(\ell + 4)!}{15(2\ell + 8)!} \right) (3\ell^2 + 27\ell + 26),
\]

\[
B_{11,2\ell} = -\frac{1}{N^2} 2^{\ell+4} \left( \frac{(\ell + 3)!}{3(2\ell + 6)!} \right) (\ell^2 + 7\ell + 14),
\]

\[
B_{10,2\ell} = -\frac{1}{N^2} 2^{\ell+4} \left( \frac{(\ell + 3)!}{3(2\ell + 6)!} \right) (\ell + 1)(\ell + 6),
\]

\[
B_{00,2\ell} = -\frac{1}{N^2} 2^{\ell+4} \left( \frac{(\ell + 3)!}{3(2\ell + 6)!} \right) (\ell^2 + 7\ell + 8),
\]

\[
B_{00,1\ell} = -\frac{1}{N^2} 2^{\ell+5} \left( \frac{(\ell + 2)!}{(2\ell + 4)!} \right),
\]

which cancel exactly the corresponding free-field cases in (3.44), (3.45), (3.46), (3.47) and (3.48) for \( a, b, c \) as in (3.33).

These cancellations are very non trivial and provide a strong consistency check on the results (4.2), (4.3) and (4.4). The coefficients of \( \ln u \) in the large \( N \) results for \( A_{nm}(u,v) \)
may also be expanded to give results for large $N$ anomalous dimensions, for $p = 2$ and $p = 3$ these were quoted in [25] and [19] respectively, for $p = 4$ see [20].

5. Perturbation Theory Results

In order to express the results from perturbation theory it is convenient to write solutions of the crossing symmetry relations (4.1) in terms of invariant functions of $u, v$.

For $p = 2$

$$\mathcal{H}(u, v; \sigma, \tau) = \frac{u}{v} \mathcal{F}(u, v),$$

while for $p = 3$

$$\mathcal{H}(u, v; \sigma, \tau) = \frac{u}{v} \mathcal{F}(u, v) + \frac{u^2}{v^2} \left( \sigma \mathcal{F}(1/v, u/v) + \tau \mathcal{F}(v, u) \right),$$

and for $p = 4$ there are two functions $\mathcal{F}, \tilde{\mathcal{F}}$.

$$\mathcal{H}(u, v; \sigma, \tau) = \frac{u}{v} \mathcal{F}(u, v) + \frac{u^2}{v^2} \left( \sigma \tilde{\mathcal{F}}(v, u) + \tau \frac{1}{v} \tilde{\mathcal{F}}(1/v, u/v) \right) + \frac{u^3}{v^2} \left( \sigma^2 \mathcal{F}(1/v, u/v) + \tau^2 \frac{1}{v} \mathcal{F}(v, u) + \sigma \tau \tilde{\mathcal{F}}(u, v) \right).$$

In each case we must have to satisfy (4.1)

$$\mathcal{F}(u, v) = \frac{1}{v} \mathcal{F}(u/v, 1/v), \quad \tilde{\mathcal{F}}(u, v) = \frac{1}{v} \tilde{\mathcal{F}}(u/v, 1/v),$$

while for $p = 2$ we have in addition

$$\mathcal{F}(u, v) = \mathcal{F}(v, u).$$

To first order in perturbation theory there is a simple general formula, for $\lambda = g_Y^2 N / 4\pi^2$,

$$\mathcal{F}_1(u, v) = \tilde{\mathcal{F}}_1(u, v) = -\frac{p^2 \lambda}{2N^2} \Phi^{(1)}(u, v).$$

If we define

$$\mathcal{I}(u, v) = \frac{1}{4} (1 + v) \Phi^{(1)}(u, v)^2 + \Phi^{(2)}(u, v) + \frac{1}{v} \Phi^{(2)}(u/v, 1/v),$$

$$\mathcal{J}(u, v) = \frac{1}{4} u \Phi^{(1)}(u, v)^2 + \frac{1}{u} \Phi^{(2)}(1/u, v/u),$$

then the results obtained in [13] and, for $p = 3, 4$, in [17] to $O(\lambda^2)$ are

$$\mathcal{F}_2(u, v) = \frac{p^2 \lambda^2}{4N^2} \mathcal{I}(u, v), \quad p = 3, 4, \quad \tilde{\mathcal{F}}_2(u, v) = \frac{p^2 \lambda^2}{4N^2} \mathcal{J}(u, v), \quad p = 4,$$
and for \( p = 2 \)
\[
\mathcal{F}_2(u, v) = \frac{\lambda^2}{N^2} (\mathcal{I}(u, v) + \mathcal{J}(u, v)) ,
\]
which is necessary to satisfy (5.9).

In (5.6) and (5.7) \( \Phi^{(L)} \) are conformal loop integrals, \( \Phi^{(1)}(u, v) = D_{1111}(u, v) \). If
\[
u = \frac{x' \bar{x}'}{(1 - x')(1 - \bar{x}')} = x \bar{x} , \quad \nu = \frac{1}{(1 - x')(1 - \bar{x}')} = (1 - x)(1 - \bar{x}) , \quad (5.10)
\]
with \( x' \) related to \( x \) as in (3.9), then defining
\[
\hat{\Phi}^{(L)}(x', \bar{x}') = \hat{\Phi}^{(L)}(\bar{x}', x') = \Phi^{(L)}(u, v) , \quad (5.11)
\]
allows an expression in terms of single variable polylogarithms [27]. The form given by Isaev [28] is particularly simple and for \( L = 1, 2 \)
\[
v \hat{\Phi}^{(1)}(x', \bar{x}') = - \ln x' \bar{x}' \phi_1(x', \bar{x}') + 2 \phi_2(x', \bar{x}') ,
\]
\[
v \hat{\Phi}^{(2)}(x', \bar{x}') = \frac{1}{2} \ln^2 x' \bar{x}' \phi_2(x', \bar{x}') - 3 \ln x' \bar{x}' \phi_3(x', \bar{x}') + 6 \phi_4(x', \bar{x}') , \quad (5.12)
\]
where \( \phi_n \) is defined by
\[
\phi_n(x, \bar{x}) = \frac{\text{Li}_n(x) - \text{Li}_n(\bar{x})}{x - \bar{x}} , \quad \text{Li}_n(x) = \sum_{r=1}^{\infty} \frac{x^r}{r^n} , \quad \text{Li}_1(x) = - \ln(1 - x) . \quad (5.13)
\]
The \( \Phi^{(L)} \) satisfy
\[
\Phi^{(L)}(u, v) = \Phi^{(L)}(v, u) \iff \hat{\Phi}^{(L)}(x', \bar{x}') = \hat{\Phi}^{(L)}(1/x', 1/\bar{x}') , \quad (5.14)
\]
where the latter result follows from standard polylogarithm identities for \( x', \bar{x}' < 0 \). In addition we have \( \Phi^{(1)}(u, v) = \Phi^{(1)}(u/v, 1/v) \)/\( u = - \ln x \bar{x} \phi_1(x, \bar{x}) + 2 \phi_2(x, \bar{x}) \).

For the perturbative analysis of the operator product expansion we consider initially supermultiplets whose superconformal primaries are singlets with twist 2 in free theory. In this case it is sufficient to consider only the leading terms as \( u \to 0 \). For \( p = 2, 3, 4 \) from (5.1), (5.2), (5.3)
\[
A_{00}(u, v)_{\text{pert.}} = \frac{u}{v} \mathcal{F}(u, v) + O(u^2) = A_{00,1}(u, v) + O(u^2) , \quad (5.15)
\]
and since, as shown in appendix C there are no terms in \( \Phi^{(2)}(1/u, v/u)/u \) proportional to \( \ln u \) for small \( u \) so that \( \mathcal{J} \) in (5.9) may be neglected, we may take
\[
A_{00,1}(u, v) = \frac{p^2}{N^2} \frac{u}{v} \left( - \frac{1}{2} \lambda \Phi^{(1)}(u, v) + \frac{1}{4} \lambda^2 \mathcal{I}(u, v) + O(\lambda^3) \right) . \quad (5.16)
\]
Using (5.10) this may be further simplified giving

\[ A_{00,1}(u, v) \sim \frac{p^2}{N^2} u \sum_{r=1,2,...} \lambda^r \sum_{s=0}^r \ln^s x \bar{x} f_{rs}(x) \quad \text{as} \quad \bar{x} \to 0, \tag{5.17} \]

where, with the above results (5.12), we obtain

\[ f_{11}(x) = -\frac{1}{2} \frac{1}{x(1-x)} \ln(1-x), \quad f_{10}(x) = -\frac{1}{x(1-x)} \text{Li}_2(x), \]

\[ f_{22}(x) = \frac{1}{8} \frac{1}{x(1-x)} \left( \frac{1}{x} \ln^2(1-x) + 2 \text{Li}_2(x) \right), \]

\[ f_{21}(x) = \frac{1}{8} \frac{1}{x(1-x)} \left( -\ln^3(1-x) + \frac{4}{x} (1-x) \ln(1-x) \text{Li}_2(x) - 6 \left( \text{Li}_3(x) - \text{Li}_3(x') \right) \right). \tag{5.18} \]

Assuming only one operator for each \( \ell \) is present with zeroth order twist 2 we must have for \( \bar{x} \to 0 \) from (2.8) and (4.17)

\[ \hat{A}_{00}(u, v) + A_{00}(u, v)_{\text{pert}} \sim \frac{p^2}{N^2} u \sum_{\ell=0,2,...} a_\ell(x\bar{x})^{\hat{\eta}_\ell} g_{3,\ell}(x), \tag{5.19} \]

where \( \eta_\ell \) is the anomalous dimension for each \( \ell \) and \( \hat{A}_{00}(u, v) \) is obtained from free field theory with all contributions of protected short and semi-short multiplets subtracted. Writing

\[ \eta_\ell = \lambda \eta_{\ell,1} + \lambda^2 \eta_{\ell,2} + \ldots, \quad a_\ell = a_{\ell,0} (1 + \lambda b_{\ell,1} + \ldots), \tag{5.20} \]

then using (3.13), (3.25) and (3.44) for \( p = 2, 3 \) and 4 and the appropriate large \( N \) value of \( a \) we have

\[ a_{\ell,0} = 2^\ell \frac{(\ell + 1)! (\ell + 2)!}{(2\ell + 3)!}. \tag{5.21} \]

The determination of anomalous dimensions simplifies to matching single variable expansions in (5.17) and (5.19). Using results from appendix E we first have

\[ f_{11}(x) = \sum_{\ell=0,2,...} a_{\ell,0} h(\ell + 2) g_{3,\ell}(x), \tag{5.22} \]

where

\[ h(n) = \sum_{r=1}^n \frac{1}{r}. \tag{5.23} \]

Hence we easily find that

\[ \eta_{\ell,1} = 2h(\ell + 2), \tag{5.24} \]
in accordance with earlier results [25]. Furthermore we have

\[ f_{22}(x) = \frac{1}{2} \sum_{\ell=0,2,\ldots} a_{\ell,0} h(\ell + 2)^2 g_{3,\ell}(x), \quad (5.25) \]

which demonstrates that only a single operator for each \( \ell \) with twist two at \( \lambda = 0 \) is present. We may also write

\[ f_{10}(x) = \sum_{\ell=0,2,\ldots} a_{\ell,0} \left( (\frac{1}{2} b_{\ell,1} g_{3,\ell}(x) + h(\ell + 2) g'_{3,\ell}(x) \right) + g'_{t,\ell}(x) = \frac{\partial}{\partial t} g_{t,\ell}(x). \quad (5.26) \]

As shown in appendix E this gives

\[ b_{\ell,1} = 2h(\ell + 2)^2 - 2h(\ell + 2)h(2\ell + 4) - \sum_{r=1}^{\ell+2} \frac{1}{r^2}. \quad (5.27) \]

Similarly we have

\[ f_{21}(x) = \sum_{\ell=0,2,\ldots} a_{\ell,0} \left( (\frac{1}{2} \eta_{\ell,2} + b_{\ell,1}h(\ell + 2))g_{3,\ell}(x) + h(\ell + 2)^2 g'_{3,\ell}(x) \right), \quad (5.28) \]

which gives according to the results of appendix E

\[ \eta_{\ell,2} = -2 \sum_{r=1}^{\ell+2} \frac{1}{r^2} \sum_{s=1}^{r} (-1)^s \frac{(-1)^s}{s^2} - 2h(\ell + 2) \sum_{r=1}^{\ell+2} \frac{1}{r^2} - \sum_{r=1}^{\ell+2} \frac{1}{r^3} \left( 1 - (-1)^r \right) \]

\[ = 2 \sum_{r=1}^{\ell+2} \frac{(-1)^r}{r^2} h(r) - 2h(\ell + 2) \sum_{r=1}^{\ell+2} \frac{1}{r^2} (1 + (-1)^r) - \sum_{r=1}^{\ell+2} \frac{1}{r^3} (1 + (-1)^r), \quad (5.29) \]

where we give two equivalent expressions. We may note that \( b_{0,1} = -3, b_{2,1} = -\frac{1025}{252} \) and\( \eta_{0,2} = -3, \eta_{2,2} = -\frac{925}{216} \) which coincide with the revised results of Arutyunov et al [9]. For \( \ell = 0 \) the operator is the Konishi scalar whose second order anomalous dimension was found in [10]. The results for general \( \ell \) are the same as those obtained by very different perturbative calculations [29,30]. The corrections to the coupling \( b_{\ell,1} \) are universal in that they are independent of the specific BPS operator, or value of \( p \), in accord with results in [31].

At higher twist there are several operators for each \( \ell \) which leads to mixing effects [32,33]. In general we write the scale dimension of the superconformal primary operator belonging to the representation with Dynkin labels \([n - m, 2m, n - m]\) in a long multiplet in the form

\[ \Delta_{nm,t\ell} = 2t + \ell + \eta_{nm,t\ell} \] \( t = n + 1, n + 2, \ldots \), \quad (5.30)
where $2t$ is the twist and $\eta_{nm,t\ell}^I$ is the anomalous dimension which is perturbatively given as an expansion in $\lambda$, as in (5.20) where $\eta_{t\ell} \equiv \eta_{00,1t\ell}$ with the index $I$ redundant. In general we have

$$A_{nm}(u,v) = \frac{p^2}{N^2} \sum_{t,\ell} u^t \sum_{I} a_{nm,t\ell}^I u^2 \eta_{nm,t\ell}^I G_{2t+\ell+4+\eta_{nm,t\ell}^I}(u,v),$$

(5.31)

where for $t = n + 1, n + 2, \ldots$

$$a_{nm,t\ell}^I|_{\eta_{nm,t\ell}^I \neq 0, \lambda \rightarrow 0} = a_{nm,t\ell,0}^I \frac{p^2}{N^2} \sum_{I} a_{nm,t\ell,0}^I = \hat{A}_{nm,t\ell},$$

(5.32)

defines the zeroth order contribution of long supermultiplets in the conformal partial wave expansion. Perturbation theory generates an expansion which is expressible in the form

$$A_{nm}(u,v)_{\text{pert}} = \frac{p^2}{N^2} u^{n+1} \frac{x}{x - \bar{x}} \sum_{r=1,2,\ldots} \lambda^r \sum_{s=0}^{\infty} \ln^s x \bar{x} \sum_{k=0}^{\infty} f_{nm,rs;k}(x) \bar{x}^k.$$  

(5.33)

Comparing (5.33) with (5.31) the $\ln x \bar{x}$ terms determine the anomalous scale dimensions $\eta_{nm,t\ell}^I$ as a series in $\lambda$. For a given $k$ in the expansion in (5.33) we must have $t \geq n + 1 + k$ in (5.31). In appendix A it is shown how to define $f^{(j)}_{nm,rs} = \sum_{k=0}^{j} \beta_{j,k} f_{nm,rs;k}$, $\beta_{j,j} = 1$, so that $f^{(j)}_{nm,rs}(x)$ determines the perturbative expansion $a_{nm,t\ell,0}^I$, $\eta_{nm,t\ell}^I$ for just $t = n + 1 + j$.

Here we consider for simplicity just the first order contributions to the anomalous dimensions which are constrained by

$$f^{(j)}_{nm,11}(x) = \frac{1}{2} x^j \sum_{\ell=-j-1}^{\infty} \sum_{I} a_{nm,n+j+1,\ell,0}^I \eta_{nm,n+j+1,\ell,1}^I g_{n+j+3,\ell}(x),$$

$$f^{(j)}_{nm,22}(x) = \frac{1}{8} x^j \sum_{\ell=-j-1}^{\infty} \sum_{I} a_{nm,n+j+1,\ell,0}^I \left(\eta_{nm,n+j+1,\ell,1}^I\right)^2 g_{n+j+3,\ell}(x).$$

(5.34)

In (5.34) we note that we may take $a_{nm,n+j+1,\ell,0}^I = 0$ for $\ell = -1$ and contributions where $\ell < -1$, necessary for $j = 1, 2, \ldots$, are in general necessary but can be disregarded for the results obtained here. By using free field results for $\sum_{I} a_{nm,n+j+1,\ell,0}^I$, we are able to obtain from the expansions (5.34) results for $\langle \eta_{nm,t\ell,1} \rangle$, $\langle \eta_{nm,t\ell,1}^2 \rangle$ where $t = n + j + 1$. In general for unitarity $\langle \eta_{nm,t\ell,1} \rangle \geq \langle \eta_{nm,t\ell,1}^2 \rangle$ with equality if just one operator is present.

We here apply this discussion to twist 4 operators for $p = 3, 4$ which, as shown in section 4, are also decoupled in the large $N$ limit. At least for these $p$ we find from (5.22) and (5.3) using (5.6) and (5.9), with (5.8), we have a universal form to lowest order in an expansion in $u$,

$$A_{1m}(u,v)_{\text{pert}} = A_{1m,2}(u,v) + O(u^3), \quad m = 0, 1,$$

$$A_{00}(u,v)_{\text{pert}} = A_{00,1}(u,v) + A_{00,2}(u,v) + O(u^3),$$

(5.35)

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where

\[ A_{00,2}(u, v) = A_{11,2}(u, v) = \frac{p^2}{6N_2} \frac{u^2}{v^2} \left( -\frac{1}{2} \lambda (1 + v) \Phi^{(1)}(u, v) + \frac{1}{4} \lambda^2 K_{2,+}(u, v) + O(\lambda^3) \right), \]

\[ A_{10,2}(u, v) = \frac{p^2}{6N^2} \frac{u^2}{v^2} \left( \frac{1}{2} \lambda (1 - v) \Phi^{(1)}(u, v) - \frac{1}{4} \lambda^2 K_{2,-}(u, v) + O(\lambda^3) \right), \] (5.36)

for

\[ K_{n, \pm}(u, v) = \frac{1}{2} (1 \pm v^n) \Phi^{(1)}(u, v)^2 + \Phi^{(2)}(u, v) \pm v^{n-2} \Phi^{(2)}(u/v, 1/v). \] (5.37)

For \( n = 1, m = 0, 1 \) we may restrict the expansion in (5.33) to just \( k = 0 \). Using the results (3.27) with (3.16) or (3.45) with (3.33) for large \( N \) we have at zeroth order in \( \lambda \),

\[
\sum_{\ell} a^{I}_{10,2\ell,0} = 2^{\ell-1} \frac{\ell + 2)! (\ell + 3)!}{3(2\ell + 5)!} (\ell + 1)(\ell + 6),
\]

\[
\sum_{\ell} a^{I}_{11,2\ell,0} = 2^{\ell-1} \frac{(\ell + 2)! (\ell + 3)!}{3(2\ell + 5)!} (\ell + 3)(\ell + 4 + 2). \] (5.38)

The leading terms in the expansion in (5.33) are then given by

\[
f_{10,11,0}(x) = \frac{1}{12} \frac{1}{(1-x)^2} \ln(1-x), \quad f_{11,11,0}(x) = -\frac{1}{12} \frac{2-x}{x(1-x)^2} \ln(1-x),
\]

\[
f_{10,22,0}(x) = -\frac{1}{48} \frac{1}{x^2(1-x)^2} \left( x^2 \text{Li}_2(x) + \frac{1}{2} x (3-x) \ln^2(1-x) \right),
\]

\[
f_{11,22,0}(x) = \frac{1}{48} \frac{1}{x^2(1-x)^2} \left( x(2-x) \text{Li}_2(x) + \frac{1}{2} (2-x + x^2) \ln^2(1-x) \right).
\] (5.39)

Using the expansions of (5.39) obtained in appendix E in (5.34) we have

\[
\langle \eta_{11,2\ell,1} \rangle = 2 \frac{(\ell + 3)(\ell + 4)(h(\ell + 3) - 1)}{(\ell + 3)(\ell + 4) + 2},
\]

\[
\langle \eta_{10,2\ell,1} \rangle = 2 \frac{(\ell + 2)(\ell + 5)(h(\ell + 3) - 1) - 2}{(\ell + 1)(\ell + 6)},
\] (5.40)

and

\[
((\ell + 3)(\ell + 4) + 2) \langle \eta_{11,2\ell,1} \rangle
\]

\[
= 4(\ell + 3)(\ell + 4)(h(\ell + 3) - 1)^2 - 2(h(\ell + 3) - 1)(h(\ell + 3) + 2)
\]

\[
+ 2((\ell + 3)(\ell + 4) - 1) \sum_{r=2}^{\ell+3} \frac{(-1)^r}{r^2},
\]

\[
(\ell + 1)(\ell + 6) \langle \eta_{10,2\ell,1} \rangle
\]

\[
= 4(\ell + 3)(\ell + 4)(h(\ell + 3) - 1)^2 - 2(h(\ell + 3) - 1)(h(\ell + 3) + 2)
\]

\[
+ 2((\ell + 3)(\ell + 4) + 1) \sum_{r=2}^{\ell+3} \frac{(-1)^r}{r^2} - 6. \] (5.41)
It is evident that in general $\langle \eta_{1m,2\ell,1}^2 \rangle \neq \langle \eta_{1m,2\ell,1} \rangle^2$ so that more than one operator must contribute, unlike the twist two case. For both cases in (5.41) $\langle \eta^2 \rangle \geq \langle \eta \rangle^2$, for large $\ell$ $\langle \eta^2 \rangle - \langle \eta \rangle^2$ tends to $2 - \pi^2/6$. It is easy to check that when $\ell = 0, 1$

$$
\langle \eta_{11,20,1} \rangle = \frac{19}{7}, \quad \langle \eta_{10,21,1} \rangle = \frac{5}{2}, \quad \langle \eta_{11,20,1}^2 \rangle = \frac{16}{7}, \quad \langle \eta_{10,21,1}^2 \rangle = \frac{25}{4}.
$$

The results for $\langle \eta_{11,20,1} \rangle, \langle \eta_{11,20,1}^2 \rangle$ are in accord with [17]. Since $\langle \eta_{10,21,1}^2 \rangle = \langle \eta_{10,21,1} \rangle^2$ we expect that only one operator in this case contributes in the large $N$ limit

For twist 4 singlet operators we use formulae from appendix A to determine

$$
f^{(1)}_{00,11}(x) = xf_{11,11,0}(x) - \frac{1}{2} \frac{1}{x(1-x)},
$$

$$
f^{(1)}_{00,22}(x) = xf_{11,22,0}(x) + \frac{1}{16} \frac{1}{x^3(1-x)} \left( (2 - 2x + x^2) \ln^2(1-x) + 2x(2-x) \ln(1-x) - 4x^2 \right),
$$

and from (3.28) or (3.46)

$$
\sum_{\ell} a^\ell \langle \eta_{00,2\ell,0} \rangle = 2^{\ell-1} \left( \frac{(\ell + 2)!(\ell + 3)!}{3(2\ell + 5)!} \right) \left( (\ell + 1)(\ell + 6) + 2 \right).
$$

Using expansions from appendix E we obtain

$$
\langle \eta_{00,2\ell,1} \rangle = 2 \frac{(\ell + 3)(\ell + 4)(h(\ell + 3) - 1) - 3}{(\ell + 1)(\ell + 6) + 2},
$$

and

$$
((\ell + 1)(\ell + 6) + 2)\langle \eta_{00,2\ell,1}^2 \rangle
= 4(\ell + 3)(\ell + 4)(h(\ell + 3) - 1)^2 + 2(h(\ell + 3) - 1)(5h(\ell + 3) - 2)
+ 2((\ell + 3)(\ell + 4) + 5) \sum_{r=2}^{\ell+3} \frac{(-1)^r}{r^2} - 24 + \frac{12}{(\ell + 3)(\ell + 4)}.
$$

For $\ell = 0$ we have

$$
\langle \eta_{00,20,1} \rangle = \frac{7}{4}, \quad \langle \eta_{00,20,1}^2 \rangle = \frac{27}{8},
$$

in agreement with [17].

\textsuperscript{2} There is apparently one single trace superconformal primary operator with $\ell = 1$ and $\Delta = 5$. [34].
For completeness we also consider twist 6 operators where we may use the perturbative results for \( p = 4 \) correlation functions to obtain results for anomalous dimensions which are not suppressed for large \( N \). In this case we may extend (5.35) to

\[
A_{2m}(u, v)_{\text{pert.}} = A_{2m,3}(u, v) + O(u^4), \quad m = 0, 1, 2,
\]
\[
A_{1m}(u, v)_{\text{pert.}} = A_{1m,2}(u, v) + A_{1m,3}(u, v) + O(u^4), \quad m = 0, 1,
\]
\[
A_{00}(u, v)_{\text{pert.}} = A_{00,1}(u, v) + A_{00,2}(u, v) + A_{00,3}(u, v) + O(u^4),
\]

where we use (5.16) and (5.36) and from (5.3), (5.8) for \( p \) are not suppressed for large \( N \).

\[
A_{22,3}(u, v) = \frac{4}{15N^2} \frac{u^3}{v^3} \left( -\frac{7}{2} \lambda (1 + v + v^2) \Phi^{(1)}(u, v) + \frac{1}{2} \lambda^2 K_{3,+}(u, v) + O(\lambda^3) \right),
\]
\[
A_{21,3}(u, v) = \frac{3}{5} A_{10,3}(u, v) = \frac{4}{15N^2} \frac{u^3}{v^3} \left( \frac{1}{2} \lambda (1 - v^2) \Phi^{(1)}(u, v) - \frac{1}{4} \lambda^2 K_{3,-}(u, v) + O(\lambda^3) \right),
\]
\[
A_{20,3}(u, v) = \frac{8}{15N^2} \frac{u^3}{v^3} \left( -\frac{1}{2} \lambda (1 - \frac{1}{2} v + v^2) \Phi^{(1)}(u, v) + \frac{1}{4} \lambda^2 K_{3,+}(u, v) + O(\lambda^3) \right),
\]
\[
A_{11,3}(u, v) = \frac{8}{15N^2} \frac{u^3}{v^3} \left( -\frac{1}{2} \lambda (1 + \frac{1}{6} v + v^2) \Phi^{(1)}(u, v) + \frac{1}{4} \lambda^2 K_{3,+}(u, v) + O(\lambda^3) \right),
\]
\[
A_{00,3}(u, v) = \frac{4}{15N^2} \frac{u^3}{v^3} \left( -\frac{1}{2} \lambda (1 + \frac{1}{3} v + v^2) \Phi^{(1)}(u, v) + \frac{1}{4} \lambda^2 K_{3,+}(u, v) + O(\lambda^3) \right).
\]

To first order in \( \lambda \) we easily find from (5.49),

\[
f_{22,11,0}(x) = -\frac{1}{120} \frac{1}{x(1-x)^3} \left( 3 - 3x + x^2 \right) \ln(1-x),
\]
\[
f_{20,11,0}(x) = -\frac{1}{120} \frac{1}{x(1-x)^3} \left( 3 - 3x + 2x^2 \right) \ln(1-x),
\]
\[
f_{21,11,0}(x) = \frac{1}{40} \frac{2-x}{(1-x)^3} \ln(1-x),
\]

and taking into account the other terms in (5.48) we have

\[
f_{10,11}^{(1)}(x) = \frac{1}{24} \frac{2-x}{x(1-x)^3} (1 - x^2 + x^2) \ln(1-x) + \frac{1}{12} \frac{1}{(1-x)^2},
\]
\[
f_{11,11}^{(1)}(x) = -\frac{1}{20} \frac{1}{(1-x)^3} \left( 3 - 3x + x^2 \right) \ln(1-x) - \frac{1}{12} \frac{2-x}{x(1-x)^2},
\]
\[
f_{00,11}^{(2)}(x) = -\frac{1}{40} \frac{1}{x(1-x)^3} \left( (1 - x)^4 + 1 \right) \ln(1-x) - \frac{1}{12} \frac{2-x}{(1-x)^2}.
\]
At zeroth order from (3.47), (3.48) and (3.33)

\[ \sum_I a_{22,3\ell,0}^I = 2^{\ell-4} \frac{((\ell + 1)(\ell + 8) + 18)}{15(2\ell + 7)!} \]

\[ \sum_I a_{21,3\ell,0}^I = 2^{\ell-4} \frac{(\ell + 1)(\ell + 8)((\ell + 2)(\ell + 7) + 8)}{5(2\ell + 7)!} \]

\[ \sum_I a_{20,3\ell,0}^I = 2^{\ell-3} \frac{(\ell + 1)(\ell + 7)(\ell + 8)}{15(2\ell + 7)!} \]

\[ \sum_I a_{10,3\ell,0}^I = 2^{\ell-4} \frac{(\ell + 1)(\ell + 3)(\ell + 6)(\ell + 8)}{3(2\ell + 7)!} \]

\[ \sum_I a_{11,3\ell,0}^I = 2^{\ell-3} \frac{(\ell + 1)(\ell + 12)}{2(2\ell + 7)!} \]

\[ \sum_I a_{00,3\ell,0}^I = 2^{\ell-4} \frac{(\ell + 1)(\ell + 8) + \frac{2}{3}}{15(2\ell + 7)!} \]

Hence we obtain using expansions from appendix E

\[ \langle \eta_{22,3\ell,1} \rangle = 2 \frac{(\ell + 3)(\ell + 6)(h(\ell + 4) - \frac{3}{2})}{(\ell + 3)(\ell + 6) + 8} \]

\[ \langle \eta_{21,3\ell,1} \rangle = 2 \frac{(\ell + 4)(\ell + 5)((\ell + 2)(\ell + 7)(h(\ell + 4) - \frac{2}{2}) - 2)}{(\ell + 1)(\ell + 8)((\ell + 2)(\ell + 7) + 8)} \]

\[ \langle \eta_{20,3\ell,1} \rangle = 2 \frac{(\ell + 3)(\ell + 6)(h(\ell + 4) - \frac{3}{2})}{(\ell + 1)(\ell + 8)} - 3 \]

\[ \langle \eta_{10,3\ell,1} \rangle = 2 \frac{(\ell + 4)(\ell + 5)(h(\ell + 4) - \frac{3}{2})}{(\ell + 1)(\ell + 8)} - 4 \]

\[ \langle \eta_{11,3\ell,1} \rangle = 2 \frac{(\ell + 3)(\ell + 6)(h(\ell + 4) - \frac{3}{2}) - \frac{10}{3}}{(\ell + 1)(\ell + 8) + \frac{4}{3}} \]

\[ \langle \eta_{00,3\ell,1} \rangle = 2 \frac{((\ell + 3)(\ell + 6) + 4)((h(\ell + 4) - \frac{3}{2}) - 14)}{(\ell + 1)(\ell + 8) + \frac{4}{3}} \]

In each case the leading behaviour for large \( \ell \) is the same. The corresponding results to (5.53) for \( \langle \eta^2 \rangle \) may also be obtained but we omit these here.

6. Conclusion

The results of this paper show that the contributions of long multiplets with twist at most \( 2p - 2 \) as \( \lambda \to 0 \) are absent from the operator product expansion of two BPS operators.
belonging to the $[0, p, 0]$ representation in the large $N$ limit. This was demonstrated by explicit calculation for $p = 2, 3, 4$ using the results obtained by the AdS/CFT correspondence. Such multiplets correspond to string states and are expected to have anomalous dimensions proportional to $\sqrt{\lambda}$ for large $\lambda$. In perturbation theory the anomalous dimensions are given by an expansion in $\lambda$ without any $1/N$ suppression. Except for the leading twist two case the anomalous dimensions cannot be determined completely from the known perturbative results for four point correlation functions. In the twist two case we were able to recover the results of perturbative calculations. For twist four and greater higher order results would be necessary depending on the number of superconformal primary operators present for each $\ell$. If only two are present with anomalous dimensions $\eta_1, \eta_2$ then we would have the relations for each $r = 0, 1, 2 \ldots$ [10],

$$\langle \eta^{r+2} \rangle - (\eta_1 + \eta_2)\langle \eta^{r+1} \rangle + \eta_1 \eta_2 \langle \eta^r \rangle = 0,$$

(6.1)

where $\langle 1 \rangle = 1$. A solution for $\eta_1, \eta_2$ is possible using the $r = 0, 1$ relations if $\langle \eta^3 \rangle$ is known in addition to $\langle \eta \rangle, \langle \eta^2 \rangle$. Using only $r = 0$ then (5.42) agrees with $\eta_{1,2} = \frac{1}{4}(5 \pm \sqrt{5})\lambda$ found in [10] for the lowest dimension scalar operators in the $[0, 2, 0]$ representation and (5.47) is in accord with $\eta_{1,2} = \frac{1}{4}(13 \pm \sqrt{41})\lambda$ obtained in [32] for singlet operators with zeroth order dimension 4. For just two operators (6.1) requires the consistency relation,

$$\langle \eta^4 \rangle - \langle \eta^2 \rangle^2 = \frac{(\langle \eta^3 \rangle - \langle \eta \rangle \langle \eta^2 \rangle)^2}{\langle \eta^2 \rangle - \langle \eta \rangle^2}.$$

(6.2)

Alternatively we may extend the operator product expansion analysis to correlation functions for BPS operators with different $p$, although such cases have not been calculated either perturbatively or in the large $N$ limit.

**Acknowledgments**

We are grateful to Gleb Arutyunov, Paul Heslop and Emeri Sokatchev for many helpful conversations.
Appendix A. Conformal Partial Wave Expansions

We consider first the general problem of expanding a general function of $u, v$ in terms of conformal partial waves

$$u^a F(u, v) = \sum_{j=0,1,\ldots} c_{j,\ell} u^{a+j} G_{\ell+2a+2j}^{(\ell)}(u, v), \quad (A.1)$$

where $F(u, v)$ is assumed to have a power series expansion in $u, 1 - v$. If $F$ satisfies

$$F(u, v) = \pm \frac{1}{v^a} F(u/v, 1/v), \quad (A.2)$$

then from (1.10) we must require $\ell$ to be respectively even, odd in (A.1).

To determine $c_{j,\ell}$ in (A.1) we use the explicit form for $G_{\Delta}^{(\ell)}$ which, with $x, \bar{x}$ defined as in (5.10) and with $g_{t,\ell}$ as in (2.11), was obtained in [23]

$$G_{\ell+2t}^{(\ell)}(u, v) = \frac{1}{x - \bar{x}} \left( x g_{t,\ell}(x) F(t-1, t-1; 2t-2; \bar{x}) - x \leftrightarrow \bar{x} \right). \quad (A.3)$$

This satisfies

$$G_{\Delta}^{(\ell)}(u, v) = -(\frac{1}{4} u)^{\ell+1} G_{\Delta}^{(-\ell-2)}(u, v), \quad (A.4)$$

so that we may require that the coefficients in (A.1) satisfy

$$-(\frac{1}{4})^{\ell+1} c_{j,\ell} = c_{j+\ell+1, -\ell-2}. \quad (A.5)$$

The analysis depends on considering an expansion of $F$ in the form

$$F(u, v) = \frac{x}{x - \bar{x}} \sum_{k=0}^{\infty} F_k(x) \bar{x}^k. \quad (A.6)$$

Using the power series expansions of $F(t-1, t-1; 2t-2; \bar{x})$ and $g_{t,\ell}(\bar{x})$ we may then match powers of $\bar{x}$ in (A.1) to find

$$F_k(x) = \sum_{j=0}^{k} \alpha_{k,j} x^j \sum_{\ell=0}^{\infty} c_{j,\ell} g_{a+j,\ell}(x) \quad (A.7)$$

where we have used the definition (2.11) of $g_{t,\ell}$ and

$$\alpha_{k,j} = \frac{1}{(k-j)!} \frac{((a+j-1)k-j)^2}{(2a+2j-2)k-j}. \quad (A.8)$$
With the aid of (A.5) this may be easily rewritten as

\[ F_k(x) = \sum_{j=0}^{k} \alpha_{k,j} x^j \sum_{\ell=-j-1}^{\infty} c_{j,\ell} g_{a+j,\ell}(x). \] (A.9)

As shown by Lang and Rühl (see the second paper in [3]) this may be inverted giving

\[ F^{(j)}(x) \equiv \sum_{k=0}^{j} \beta_{j,k} F_k(x) = x^j \sum_{\ell=-j-1}^{\infty} c_{j,\ell} g_{a+j,\ell}(x), \] (A.10)

where \( \sum_{k=0}^{j} \beta_{j,k} \alpha_{k,l} = \delta_{jl} \) which is satisfied by

\[ \beta_{j,k} = (-1)^{j-k} \frac{1}{(j-k)!} \frac{((a+k-1)j-k)^2}{(2a+k+j-3)_{j-k}}. \] (A.11)

The result (A.9) then reduces the problem of determining \( c_{j,\ell} \) to matching single variable expansions which is more straightforward. The first few \( F^{(j)} \) are given by

\[ F^{(0)}(x) = F_0(x), \quad F^{(1)}(x) = F_1(x) - \frac{1}{2} (a-1) F_0(x), \]
\[ F^{(2)}(x) = F_2(x) - \frac{1}{2} a F_1(x) + \frac{a(a-1)^2}{4(2a-1)} F_0(x). \] (A.12)

**Appendix B. Expansion Coefficients for Free Fields**

The determination of the coefficients in the conformal partial wave expansions of \( \hat{f}(x,y) \) and, for free field theory, \( \mathcal{H}_0(u,v;\sigma,\tau) \) for the cases \( p = 2, 3, 4 \) considered here may be reduced to combinations of various basic expansions which are listed in this appendix.

The expansion of \( \hat{f} \), as in (2.10), can be obtained by considering

\[ x^{n+1} = \sum_{\ell=n}^{\infty} p_{n,\ell} g_{0,\ell+1}(x), \quad x'^{n+1} = \sum_{\ell=n}^{\infty} (-1)^{\ell+1} p_{n,\ell} g_{0,\ell+1}(x). \] (B.1)

Since, with the definition (2.11), \( g_{0,\ell+1}(x) = (-\frac{1}{2})^{\ell+1} \sum_{n=\ell}^{\infty} \alpha_{n,\ell} x^{n+1} \), where \( \alpha_{n,\ell} \) is given by (A.8) with \( a = 2 \), we then have from (A.11)

\[ (-\frac{1}{2})^{\ell+1} p_{n,\ell} = \beta_{\ell,n} = (-1)^{n-\ell} \frac{(\ell)!^2}{(2\ell)! (n!)^2 (\ell-n)!}, \] (B.2)

using (A.11) for this case. For subsequent use we note that from (2.11) we have for any integer \( t \)

\[ (-\frac{1}{2} x)^t g_{t,\ell}(x) = g_{0,\ell+t}(x). \] (B.3)
For the analysis of \( \mathcal{H}_0(u, v; \sigma, \tau) \) we require expansions as in (A.1) with \( \ell = 2 \) again. In this case \( \beta_{j,k} \) is by (B.2) and (A.10) reduces to \((-\frac{1}{2}t^{j+2}x^2 F^{(j)}(x) = \sum c_j, t g_{0, j + \ell + 2}(x)\). For expansion of free field expressions it is then sufficient to use just (B.1) and (B.2).

For the leading terms in each of (3.7), (3.18) and (3.35) we require for \( n = 0, 1 \) and 2, \( u^n = \sum_{\ell=0,1, \ldots} a_{t, \ell}^{(n)} x_t^t G_{t+2t+4}^{(\ell)}(u, v) \), \( \frac{u^n}{v^{n+2}} = \sum_{\ell=0,1, \ldots} (\ell+1)! (2 \ell + 2)! (2t)! (t+n)! \times (t-1)! (n+1)! \times a_{t, \ell}^{(n)} x_t^t G_{t+2t+4}^{(\ell)}(u, v) \). (B.4)

For \( F(u, v) = u^n \) the method of obtaining the expansion coefficients described in appendix A then gives \( a_{t, \ell}^{(n)} = (-2)^\ell (\beta_{t,n} \beta_{t+\ell+1,n+1} - \beta_{t,n+1} \beta_{t+\ell+1,n}) \) or

\[
a_{t, \ell}^{(n)} = 2^\ell \frac{((\ell + t + 1)!)^t (t!)^2}{(2 \ell + 2t + 2)! (2t)!} (\ell + 1)(\ell + 2t + 2) \times \frac{(\ell + t + 1 + n)! (t + n)!}{(\ell + t + 1 - n)! (t - n)! ((n+1)! n!)^2}.
\]

More generally, the expansion coefficients \( a_{t, \ell}^{(n)} \) are sufficient to compute the expansion coefficients for the contributions from disconnected graphs in the free-field four-point function for any \( p = n + 2 \).

The sub-leading terms in the large \( N \) limit require a variety of other results. To determine

\[
\frac{1}{v} = \sum_{\ell=0,2, \ldots} b_{t, \ell} u_t G_{t+2t+4}^{(\ell)}(u, v),
\]

we use (A.6) and (A.10) with \( \sum_{k=0}^j \beta_{j,k} = (-1)^j \beta_{j,0} \) to obtain with \( F(u, v) = 1/v, x^2 F^{(j)}(x) = -\beta_{j,0}(x' + (-1)^j x) \). Hence \( b_{t, \ell} = (1 + (-1)^\ell)(-2)^{\ell+1}\beta_{t,0} \beta_{t+\ell+1,0} \), giving for \( \ell \) even

\[
b_{t, \ell} = 2^{\ell+1} \frac{((\ell + t + 1)!)^t (t!)^2}{(2 \ell + 2t + 2)! (2t)!} (\ell + 1)^{\ell}. \]

This allows the expansion coefficients for the \( a \) term in (3.7) and the \( b \) term in (3.18) to be readily obtained.

In a similar fashion for

\[
\frac{u^2}{v^2} = \sum_{\ell=0,2, \ldots} c_{t, \ell} u_t G_{t+2t+4}^{(\ell)}(u, v),
\]

we have, using now also \( \sum_{k=0}^j \beta_{j,k} k = (-1)^j \beta_{j,1} = (-1)^j (j + 1) \beta_{j,0} \), for this case from (A.10) \( x^2 F^{(j)}(x) = (1 - (-1)^j) \beta_{j,0}(x^2 - x'^2) - \beta_{j,1}((-1)^j x^2 + x'^2 + (1 + (-1)^j)(x + x')) \). Hence
\[ c_{\ell t} = (1 + (-1)^t)2^\ell \{(1 - (-1)^t)\beta_{t,0}\beta_{t+\ell+1,1} - (1 + (-1)^t)\beta_{t,1}\beta_{t+\ell+1,0} - (-1)^t\beta_{t,1}\beta_{t+\ell+1,1}\} \]

giving for \( \ell \) even

\[
c_{\ell t} = 2^\ell \frac{((\ell + t + 1)!)^2 (t!)^2}{(2\ell + 2t + 2)! (2t)!} \left( (1 + (-1)^t)t(t + 1)(\ell + t)(\ell + t + 3) - (1 - (-1)^t)(t - 1)(t + 2)(\ell + t + 1)(\ell + t + 2) \right) .
\] (B.9)

In conjunction with recurrence relations given below, the expansion coefficients \( c_{\ell t} \) are sufficient to determine those for the \( a \) term in (3.18) and the \( b \) and \( c \) terms in (3.35).

Additionally for the remaining \( a \) terms in (3.35) we require the expansions.

\[
u^2 \over v = \sum_{\ell=0,1,\ldots, t=2,3,\ldots} d_{\ell t} u^\ell G_{\ell+2t+4}^\ell(u,v), \quad \frac{u^2}{v^3} = \sum_{\ell=0,1,\ldots, t=2,3,\ldots} (-1)^t d_{\ell t} u^\ell G_{\ell+2t+4}^\ell(u,v).
\] (B.10)

For \( F(u,v) = u^2/v \) then \( x^2F^{(j)}(x) = (\beta_{j,0}(1 - (-1)^t) + \beta_{j,1})x^3 - \beta_{j,2}(x^2 + x + x') \). This gives \( d_{\ell t} = (-2)^\ell \{(1 - (-1)^t)\beta_{t,0} + \beta_{t,1}\beta_{t+\ell+1,2} - \beta_{t,2}(\beta_{t+\ell+1,1} + (1 + (-1)^{t+\ell})\beta_{t+\ell+1,0})\} \)

so that for \( \ell \) even

\[
d_{\ell t} = 2^{\ell-3} \frac{((\ell + t + 1)!)^2 (t!)^2}{(2\ell + 2t + 2)! (2t)!} \times \left( (1 + (-1)^t)t(t + 1)(\ell + t + 3)((\ell + 1)(\ell + 2t + 2) + 2) + (1 - (-1)^t)(t - 1)(t + 2)(\ell + t + 1)(\ell + t + 2)((\ell + 1)(\ell + 2t + 2) - 2) \right),
\] (B.11)

and for \( \ell \) odd

\[
d_{\ell t} = 2^{\ell-3} \frac{((\ell + t + 1)!)^2 (t!)^2}{(2\ell + 2t + 2)! (2t)!} (\ell + 1)(\ell + 2t + 2) \times \left( (1 + (-1)^t)t(t + 1)(\ell + t + 1)(\ell + t + 2) + (1 - (-1)^t)(t - 1)(t + 2)(\ell + t)(\ell + t + 3) \right).
\] (B.12)

All the above results for expansion coefficients satisfy the identity (A.3).

For other results we may make repeated use of the following recurrence relations,

\[
(v - 1) G_\Delta^{(\ell)}(u,v) = 2 G_\Delta^{(\ell+1)}(u,v) + \frac{1}{2} u G_\Delta^{(\ell-1)}(u,v) + \frac{1}{8} f_{\Delta+\ell} u G_\Delta^{(\ell+1)}(u,v) + \frac{1}{128} f_{\Delta-\ell-2} u^2 G_\Delta^{(\ell-1)}(u,v),
\] (B.13)

\[
(v + 1) G_\Delta^{(\ell)}(u,v) = 2 G_\Delta^{(\ell-2)}(u,v) + \frac{1}{2} f_{\Delta+\ell} G_\Delta^{(\ell+2)}(u,v) + \frac{1}{2} u G_\Delta^{(\ell)}(u,v) + \frac{1}{32} f_{\Delta-\ell-2} u^2 G_\Delta^{(\ell-2)}(u,v) + \frac{1}{128} f_{\Delta-\ell-2} f_{\Delta-\ell-2} u^2 G_\Delta^{(\ell)}(u,v),
\]

where \( f_\lambda = \lambda^2/(\lambda^2 - 1) \).
Appendix C. Results for $D$ Functions and Loop Integrals

We here list some results for the two variable functions $D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v)$ which arise from AdS/CFT integrals for large $N$ and in terms of which the perturbative results may also be expressed. Many properties are known, see [24, 23, 19], only the significant ones in the present context are listed here. With the definition (4.5) for $s$ they may in general be expressed as power series as follows

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \Gamma(-s) \frac{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3 + s) \Gamma(\Delta_4 + s)}{\Gamma(\Delta_1 + \Delta_2)}$$

$$\times G(\Delta_2, \Delta_3 + s, 1 + s, \Delta_1 + \Delta_2; u, 1 - v)$$

$$+ \Gamma(s) \frac{\Gamma(\Delta_1 - s) \Gamma(\Delta_2 - s) \Gamma(\Delta_3) \Gamma(\Delta_4)}{\Gamma(\Delta_3 + \Delta_4)}$$

$$\times u^{-s} G(\Delta_2 - s, \Delta_3, 1 - s, \Delta_3 + \Delta_4; u, 1 - v),$$

where

$$G(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\delta - \alpha)_m (\delta - \beta)_m (\alpha)_{m+n} (\beta)_{m+n}}{m! (\gamma)_m n! (\delta)_{2m+n}} x^m y^n. \quad (C.1)$$

The result (C.1) clearly satisfies (4.6). The series is convergent in the neighbourhood of $u, 1 - v \sim 0$. For other limits we may use the symmetry relations for transpositions of $\Delta_i$,

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = v^{-\Delta_2} D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u/v, 1/v)$$

$$= D_{\Delta_1 \Delta_2 \Delta_1 \Delta_4}(v, u)$$

$$= u^{-\Delta_2} D_{\Delta_4 \Delta_2 \Delta_3 \Delta_1}(1/u, v/u), \quad (C.3)$$

where other cases may be obtained by using the identities (4.3) or

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = v^{\frac{1}{2} (\Delta_1 + \Delta_4 - \Delta_2 - \Delta_3)} D_{\Delta_2 \Delta_1 \Delta_4 \Delta_3}(u, v). \quad (C.4)$$

The first relation in (C.3) is responsible for (4.11), (4.13) and (4.14) obeying (2.9). We may also note that, directly from the representation (C.1),

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = D_{\Sigma - \Delta_3 \Sigma - \Delta_4 \Sigma - \Delta_1 \Sigma - \Delta_2}(u, v), \quad \Sigma = \frac{1}{2} \sum_{i=1}^{4} \Delta_i. \quad (C.5)$$

The singularities present in the result (C.1) arising from $\Gamma(-s)$ for $s = 0, 1, 2, \ldots$ are cancelled by corresponding terms in the second expression on the right hand side of (C.1) but this leads to $\ln u$ terms. The full result is then given by (4.7) and (4.8) with

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v)_{\text{reg.}} = \frac{(-1)^s}{s!} \frac{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3 + s) \Gamma(\Delta_4 + s)}{\Gamma(\Delta_1 + \Delta_2)}$$

$$\times \left( - \ln u G(\Delta_2, \Delta_3 + s, 1 + s, \Delta_1 + \Delta_2; u, 1 - v) \right. \quad (C.6)$$

$$\left. + \sum_{m,n=0}^{\infty} \frac{(\Delta_1)_m (\Delta_4 + s)_m}{m! (s + 1)_m} \frac{(\Delta_2)_{m+n} (\Delta_3 + s)_{m+n}}{n! (\Delta_1 + \Delta_2)_{2m+n}} g_{mn} u^m (1 - v)^n \right),$$

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where

\[ g_{mn} = \psi(m + 1) + \psi(s + m + 1) + 2\psi(\Delta_1 + \Delta_2 + 2m + n) \]
\[ - \psi(\Delta_1 + m) - \psi(\Delta_4 + s + m) - \psi(\Delta_2 + m + n) - \psi(\Delta_3 + s + m + n). \]  \hspace{1cm} (C.7)

We may also note that from (4.8)

\[ D_\Delta \Delta_1 \Delta_2 \Delta+1(u, v) = -u^{-s}(1 - v) \frac{\Gamma(\Delta_1 - s)\Gamma(\Delta_2 - s + 1)\Gamma(\Delta)\Gamma(\Delta + 1)}{\Gamma(2\Delta + 2)} \]
\[ \times \sum_{m=0}^{s-1} (-1)^m (s - m - 1)! \frac{(\Delta_1 - s)_m(\Delta_2 - s + 1)_m(\Delta + 1)_m}{m!(2\Delta + 2)_m} \times u^m F(\Delta_2 - s + 1 + m, \Delta + m + 1; 2\Delta + 2m + 2; 1 - v). \]  \hspace{1cm} (C.8)

If \( \Delta_i = 0 \) the integral defining the \( D \) function reduces to that for a three point function which may be directly evaluated giving

\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v)_{\Delta_4 \rightarrow 0} = \Gamma(\Sigma - \Delta_1)\Gamma(\Sigma - \Delta_2)\Gamma(\Sigma - \Delta_3) u^{\Delta_3 - \Sigma} v^{\Delta_1 - \Sigma}, \]  \hspace{1cm} (C.9)

with results for other \( \Delta_i = 0 \) obtained from (C.3).

The derivation of (4.11) depends on the identity

\[ D_{2523}(u, v) + D_{2532}(u, v) = -D_{1533}(u, v) + u^{-1}v^{-2}, \]  \hspace{1cm} (C.10)

whereas to obtain (4.13) we make use of

\[ D_{3634}(u, v) + D_{3643}(u, v) = -D_{2644}(u, v) + D_{2633}(u, v), \]
\[ D_{2624}(u, v) + D_{2642}(u, v) = -2D_{2633}(u, v) - (D_{1634}(u, v) + D_{1643}(u, v)) \]
\[ + 2u^{-1}v^{-3}(1 + v), \]
\[ D_{3623}(u, v) + D_{3632}(u, v) = -D_{2633}(u, v) + u^{-2}v^{-2}, \]
\[ D_{4624}(u, v) + D_{4642}(u, v) = -2D_{4633}(u, v) + D_{2644}(u, v) - 2D_{2633}(u, v) + u^{-2}v^{-2}. \]  \hspace{1cm} (C.11)

To obtain (4.14) we also use

\[ D_{2624}(u, v) - D_{2642}(u, v) = -D_{1634}(u, v) + D_{1643}(u, v) - 2u^{-1}v^{-3}(1 - v), \]
\[ D_{4624}(u, v) - D_{4642}(u, v) = D_{3623}(u, v) - D_{3632}(u, v) - D_{3634}(u, v) + D_{3643}(u, v). \]  \hspace{1cm} (C.12)

The results in (4.13) may be further simplified by using

\[ D_{1634}(u, v) + D_{1643}(u, v) = -u D_{1733}(u, v)_{\text{reg.}} - 4 \frac{1}{q^3} \ln u + f(v), \]  \hspace{1cm} (C.13)
with \( f(v) \) given by a series in \( 1 - v \).

Besides appearing in the large \( N \) expansion the \( D \) functions may also be used as a generating function for the loop integrals \( \Phi^{(L)} \) which appear in perturbation theory. If \( \delta = \gamma + 1 \) in (C.2), which corresponds to \( \sum_i \Delta_i = 4 \) in (C.1), we have with the definitions (5.10) from [23]

\[
G(\alpha, \beta, \gamma, \gamma + 1; u, 1 - v) = \frac{1}{x - \bar{x}} \left( x F(\alpha, \beta; \gamma + 1; x) F(\alpha - 1, \beta - 1; \gamma - 1; \bar{x}) - x \leftrightarrow \bar{x} \right). \tag{C.14}
\]

Using this we have

\[
\overline{D}_{1+\delta 111-\delta}(u, v) = \frac{\pi}{\sin \pi \delta} \left( -G_\delta(x, \bar{x}) + u^{-\delta} G_{-\delta}(x, \bar{x}) \right), \tag{C.15}
\]

where

\[
G_\delta(x, \bar{x}) = \frac{1}{1 + \delta} \frac{1}{x - \bar{x}} \left( x F(1, 1 + \delta; 2 + \delta; x) - x \leftrightarrow \bar{x} \right). \tag{C.16}
\]

By using standard hypergeometric identities we may then obtain

\[
G_\delta(x, \bar{x}) = -\frac{1}{x \bar{x}} G_{-\delta}(1/x, 1/\bar{x}) + \frac{\pi}{\sin \pi \delta} \left( \frac{(-x)^{-\delta} - (-\bar{x})^{-\delta}}{x - \bar{x}} \right), \tag{C.17}
\]

which, used in (C.13), gives

\[
\overline{D}_{1+\delta 111-\delta}(u, v) = u^{-1-\delta} \overline{D}_{1+\delta 111-\delta}(1/u, v/u), \tag{C.18}
\]

in accord with (C.3) and (4.6).

From the identity (C.16) it is straightforward to see that

\[
G_\delta(x, \bar{x}) = \sum_{r=0}^{\infty} (-\delta)^r \phi_{r+1}(x, \bar{x}), \tag{C.19}
\]

with \( \phi_n \) defined in terms of polylogarithms as in (5.13). Using (C.19) in (C.17) is equivalent to standard identities relating \( \text{Li}_n(x) \) and \( \text{Li}_n(1/x) \). The result (C.19) leads to a corresponding expansion of \( \overline{D}_{1+\delta 111-\delta} \) by virtue of (C.13). This may be rearranged in terms of generalised loop integrals where the definitions in (5.12) for \( L = 1, 2 \) are extended [27,28], with \( u, v \) as in (5.10), to

\[
\frac{1}{v} \Phi^{(L)}(x, \bar{x}) = \frac{1}{L!} \sum_{n=L}^{2L} \frac{(-1)^n n!}{(n - L)! (2L - n)!} \ln^{2L-n} u \phi_n(x, \bar{x}), \tag{C.20}
\]
and we let $\Phi^{(L)}(u/v, 1/v) = \hat{\Phi}^{(L)}(x, \bar{x})$. The expansion obtained by using (C.19) in (C.13) may then be re-expressed in the form

$$D_{1+\delta 111-\delta}(u, v) = \frac{\pi}{\sin \pi \delta} \sum_{L=1}^{\infty} h_L(u, \delta) \frac{1}{v} \hat{\phi}^{(L)}(x, \bar{x}),$$

(C.21)

where

$$h_L(u, \delta) = \frac{(-1)^L L!}{\ln^{2L-1} u} \sum_{r=0}^{L-1} (2L - 2 - r)! (L - 1 - r)! r! \frac{\delta \ln u}{r !} \left(u^{-\delta} - (-1)^r\right).$$

(C.22)

The functions $h_L$ are linearly independent and, despite appearances, regular for $u = 1$, $h_L(1, \delta) = \frac{2(L)!^2}{(2L)!^2} \delta^{2L-1}$. It is easy to see that

$$h_L(u, \delta) = u^{-\delta} h_L(1/u, \delta),$$

(C.23)

so that (C.18) is equivalent (5.14) to for each $L$. As a consequence of (C.21) $\Phi^{(L)}$ for each $L$ may therefore be obtained in terms of an appropriate limit of the expansion of $D_{1+\delta 111-\delta}$ to $O(\delta^{2L-2})$.

For the purpose of discussing perturbative results in terms of the operator product expansion as in section 5 we need to analyse $\Phi^{(L)}(1/u, v/u)$ for small $u, 1 - v$ or equivalently $\hat{\Phi}^{(L)}(x, \bar{x})$ in the neighbourhood of $x, \bar{x} = 1$. For $L = 1$ it is sufficient to use $\Phi^{(1)}(1/u, v/u)/u = \Phi^{(1)}(u, v)$ but for higher $L$ an expression for the non analytic piece involving $\ln u$ may be obtained from

$$D_{1+\delta 111-\delta}(v, u) = D_{111+\delta 1-\delta}(u, v) \sim -\frac{\pi \delta}{\sin \pi \delta} \ln u G(1, 1 + \delta, 1, 2; u, 1 - v)
= -\frac{\pi \delta}{\sin \pi \delta} \ln u \frac{1}{x - \bar{x}} \left((1 - x)^{-\delta} - (1 - \bar{x})^{-\delta}\right),$$

(C.24)

where we have used (C.3), the $\ln u$ part of (C.6) and (C.14). Using the expansion (C.21) to $O(\delta^2)$ we may then obtain

$$\frac{1}{u} \Phi^{(2)}(1/u, v/u) \sim \frac{1}{2} \ln u \phi_1(x, \bar{x}) \ln(1 - x) \ln(1 - \bar{x}),$$

(C.25)

neglecting terms which are just a power series in $u, 1 - v$. For application in section 4 we may then note that $\ln(1 - x) \ln(1 - \bar{x}) = O(u)$. To determine a suitable expansion for the for the non $\ln u$ terms in $\Phi^{(L)}(1/u, v/u)$ we may start from (C.16) and use

$$\frac{x}{1 + \delta} F(1, 1 + \delta; 2 + \delta; x) = x^{-\delta} \left(\frac{\ln(1 - x) + f_\delta(1 - x)}{1 + \delta} \right),$$

(C.26)
where $f_\delta$ is expressible as a power series so that

$$f_\delta(x) - \psi(1 + \delta) + \psi(1) = \sum_{r=1}^{\infty} \frac{(1 + \delta)_r}{r!} x^r - \ln(1 - x)$$

$$= \sum_{r=1}^{\infty} \frac{(-\delta)_r}{r!} x^r = \int_{0}^{x} \frac{1}{u}((1 - u)^\delta - 1) \, du .$$

Using (C.16) in (C.15) we then get

$$\mathcal{D}_{1+\delta 1\bar{1}1 - \delta}(v, u) = -\frac{\pi}{\sin \pi \delta} \frac{1}{x - \bar{x}} \left( (1 - x)^{-\delta} \left( \ln x \bar{x} + f_\delta(x) + f_{-\delta}(\bar{x}) \right) - x \leftrightarrow \bar{x} \right) .$$

This may be used to obtain results for $\Phi^{(L)}$ from (C.21) if we expand $f_\delta$ in the form

$$f_\delta(x) = \sum_{r=1,2,...} (-\delta)^r \left( p_r(x) - \zeta(r + 1) \right) ,$$

where

$$p_r(x) = \frac{1}{r!} \int_{0}^{x} \frac{1}{u} (-\ln(1 - u))^r \, du = (-1)^r \sum_{n=r}^{\infty} \frac{S_{n}^{(r)}}{n! n} (-x)^n , \quad p_1(x) = \text{Li}_2(x) ,$$

where $S_{n}^{(r)}$ is a Stirling number of the first kind. It is easy to check that

$$p_r(x) = (-1)^r p_r(x') - \frac{1}{(r+1)!} (-\ln(1 - x))^r .$$

For $\Phi^{(2)}$ there is then an alternative expression of the form

$$\frac{1}{u} \Phi^{(2)}(1/u, v/u) = \left( \frac{1}{2} \ln u \phi_1(x, \bar{x}) - 3\phi_2(x, \bar{x}) \right) \ln(1 - x) \ln(1 - \bar{x}) + \frac{1}{2} \phi_2(x, \bar{x}) \ln^2 v$$

$$+ 3\phi_1(x, \bar{x}) \left( 2\zeta(3) - p_2(x) - p_2(\bar{x}) \right) + 6 \frac{p_3(x) - p_3(\bar{x})}{x - \bar{x}} ,$$

where the right hand side may be readily expanded in powers of $x, \bar{x}$.

### Appendix D. Expansion Coefficients for Large $N$ Calculations

We here demonstrate how the first few expansion coefficients in (4.20) are given by (4.21) with (4.19). We start by proving the result (4.18), namely

$$F(a, b; 2b; x) = \sum_{\ell=0,2,...} r_{ab,\ell} \left( \frac{1}{2} x \right)^\ell F(\ell + a, \ell + a; 2\ell + 2a; x) .$$

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for \(r_{ab,\ell}\) as in (4.19). The summation over \(\ell\) may be performed to rewrite the right-hand side of (D.1), using standard \(\Gamma\)-function identities, as

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \frac{(a)^2}{(2a)^n} S_n(a, b) x^n,
\]  

(D.2)

where

\[
S_n(a, b) = {}_4F_3\left( \begin{array}{cccc}
-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}, a - \frac{1}{2}, a - b \\
a + \frac{n}{2}, a + \frac{n}{2} + \frac{1}{2}, b + \frac{1}{2}
\end{array} \middle| 1 \right)
\]

\[+ \frac{4n(n-1)(a-b)}{(2a+n)(2a+n+1)(2b+1)} {}_4F_3\left( \begin{array}{cccc}
1 - \frac{n}{2}, \frac{3}{2} - \frac{n}{2}, a + \frac{1}{2}, a - b + 1 \\
a + \frac{n}{2} + 1, a + \frac{n}{2} + \frac{3}{2}, b + \frac{3}{2}
\end{array} \middle| 1 \right),
\]

(D.3)

in terms of standard \( {}_4F_3 \) hypergeometric functions. All that is required now is to prove that the latter equals

\[
T_n(a, b) = \frac{(2a)_n(b)_n}{(a)_n(2b)_n},
\]

(D.4)

when \(n \geq 0\) so that (D.1) immediately follows. This appears to be a non-standard hypergeometric identity however we may prove it (by a method similar to that used by Pfaff in 1797 to prove the Pfaff-Saalschütz \( {}_3F_2 \) identity) by first establishing the following recurrence relation, namely,

\[
S_n(a, b) = S_{n-1}(a, b) + \frac{2(n-1)(2a+1)(2a+3)(a-b)}{(2a+n-1)(2a+n)(2a+n+1)(2b+1)} S_{n-2}(a+2, b+1).
\]

(D.5)

To see this, it is perhaps helpful to note that \(S_n(a, b)\) may be rewritten as,

\[
S_n(a, b) = \sum_{k=0}^{\lfloor n/2 \rfloor} S_{n,k}(a, b), \quad S_{n,k}(a, b) = \left(1 + \frac{4k}{2a-1}\right) \frac{(-n)_{2k}(a-\frac{1}{2})_k(a-b)_k}{k!(2a+n)_{2k}(b+\frac{1}{2})_k},
\]

(D.6)

whereby we may readily verify that

\[
S_{n,k}(a, b) = S_{n-1,k}(a, b) + \frac{2(n-1)(2a+1)(2a+3)(a-b)}{(2a+n-1)(2a+n)(2a+n+1)(2b+1)} S_{n-2,k-1}(a+2, b+1),
\]

(D.7)

so that, summing the latter over \(k\), (D.5) follows. We may then note that (D.5) with \(S_0(a, b) = S_1(a, b) = 1\) uniquely defines \(S_n(a, b), n \geq 0\). Thus, as \(T_n(a, b)\) in (D.4) satisfies the same recurrence relation (D.5) and \(T_0(a, b) = T_1(a, b) = 1\) then \(S_n(a, b) = T_n(a, b)\) for \(n \geq 0\), as required.

Turning now to the proof of (4.20), (4.21) with (4.19), we make use of the general discussion in appendix A where we take

\[
F(u, v) = F(a, b; 2b; 1 - v),
\]

(D.8)
so that the definition (A.6) of $F(k)(x)$ now becomes

$$(x - \bar{x})F(a, b; 2b; x + \bar{x}(1 - x)) = x \sum_{k=0}^{\infty} F_k(x) \bar{x}^k,$$  \hspace{2cm} (D.9)

and the first few $F_k(x)$ are

$$F_0(x) = F(a, b; 2b; x),$$
$$F_1(x) = \left(\frac{1}{2}a x - 1\right) \frac{1}{x} F(a, b; 2b; x) + \frac{a(a - 2b)}{4(2b + 1)} x F(a + 1, b + 1; 2b + 2; x),$$
$$F_2(x) = \frac{1}{4} a ((a - b + 1)x - 2) \frac{1}{x} F(a, b; 2b; x)$$
$$+ \frac{a(a - 2b)}{8(2b + 1)} ((a + b + 1)x - 2(b + 1)) F(a + 1, b + 1; 2b + 2; x),$$  \hspace{2cm} (D.10)

where we have used the identity

$$(x - 1) \frac{\gamma}{\alpha \beta} \frac{d}{dx} F(\alpha, \beta; \gamma; x) = -F(\alpha, \beta; \gamma; x) + \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma + 1)} x F(\alpha + 1, \beta + 1; \gamma + 2; x),$$

in the Taylor expansion of the left-hand side of (D.9). With (D.10), we may use (A.12) to obtain

$$F^{(0)}(x) = F(a, b; 2b; x),$$
$$F^{(1)}(x) = \frac{ab(a - 2b)}{2(4b^2 - 1)} x F(a + 1, b + 1; 2b + 2; x) - \frac{1}{x} F(a - 1, b - 1; 2b - 2; x),$$
$$F^{(2)}(x) = a(a + 1)(b + 1) \frac{(a - 2b - 1)(a - 2b)}{16(2b + 1)^2(2b + 3)} x^2 F(a + 2, b + 2; 2b + 4; x)$$
$$- a \frac{a - 3b - 1 + b}{4(2a - 1)(2b + 1)} F(a, b; 2b; x),$$  \hspace{2cm} (D.12)

where we have used the identity

$$\left(1 - \frac{\gamma(\alpha + \beta - 1) - 2\alpha \beta}{\gamma(\gamma - 2)} x\right) F(\alpha, \beta; \gamma; x)$$
$$= F(\alpha - 1, \beta - 1; \gamma - 2; x) + \alpha \beta \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma^2(\gamma^2 - 1)} x^2 F(\alpha + 1, \beta + 1; \gamma + 2; x).$$  \hspace{2cm} (D.13)

Using (A.9) with (D.12) along with the expansion (D.1) we may now easily determine $c_{j,\ell}$ as in (4.21) for $j = 1, 2$. It is easy to check that $\frac{1}{4}c_{0,0} = -c_{1,-2}$ and $\frac{1}{4}c_{1,0} = -c_{2,-2}$ in accord with (A.5).
An alternative approach to finding these results is to consider an expansion of the conformal partial waves $G_\Delta^{(\ell)}$. By separating (4.3) into two separate terms we have in general
\[
G_\Delta^{(\ell)}(u, v) = f_{\Delta, \ell}(u, 1 - v) - \left(\frac{1}{4}u\right)^{\ell+1}f_{\Delta, -\ell-2}(u, 1 - v),
\]
and we may then expand $f_{\Delta, \ell}(u, 1 - v)$ in powers of $u$, extending the result (4.17). The first few terms are then
\[
f_{\Delta, \ell}(u, x) = g_\ell(x) - \frac{1}{4}(\ell - 1)ug_{\ell + 1, \ell - 2}(x) + \frac{(t + \ell)^2(2t + \ell)}{4(2t + 2l - 1)(2t + 2l + 1)}ug_{\ell + 1, \ell}(x)
\]
\[
+ \frac{1}{32}(\ell - 2)(\ell - 3)u^2g_{\ell + 2, \ell - 4}(x)
\]
\[
- \frac{1}{16(2t + 2l - 3)(2t + 2l + 1)}\left((t + \ell - 1)(t + \ell) - \frac{t - 1}{2t - 1}\right)u^2g_{\ell + 2, \ell - 2}(x)
\]
\[
+ \frac{(t + \ell)^2(2t + \ell + 1)^2(2t + \ell + 2)}{32(2t + 2l - 1)(2t + 2l + 1)^2(2t + 2l + 3)}u^2g_{\ell + 2, \ell}(x) + O(u^3). \tag{D.15}
\]
The general form is $f_{\Delta, \ell}(u, x) = \sum_{p=0,1,\ldots,q=0,1,\ldots}c_{p,q}u^pg_{t+p,\ell-2q}(x)$, in accord with a very different treatment in [9]. Each term in such an expansion is in accord with the symmetry condition (1.10) as a consequence of (4.11). Each term in such an expansion is in accord with the symmetry condition (1.10) as a consequence of (4.11). Each term in such an expansion is in accord with the symmetry condition (1.10) as a consequence of (4.11). Each term in such an expansion is in accord with the symmetry condition (1.10) as a consequence of (4.11).

For a special case a related expansion was given in the first paper in [11].

**Appendix E. Expansion Coefficients for Perturbative Calculations**

The results of section 5 depend on the expansion of various functions $f(x)$, such as listed in (5.18), in terms of $g_{t, \ell}(x)$ for appropriate $t$. We here describe some further details as to how these were obtained. By letting $x^t f(x) \rightarrow f(x)$ and using (B.3) we need only consider expansion involving $g_{t, \ell}(x)$. In consequence we need to determine coefficients $\tilde{f}_{\ell}$ such that
\[
f(x) = \sum_{\ell=1}^{\infty} 2^{\ell+1} (\ell)!^2 (2\ell)! \tilde{f}_{\ell} g_{0, \ell+1}(x). \tag{E.1}
\]
Since $g_{0, \ell}(x) = (-1)^{\ell}g_{0, \ell}(x')$ the expansion of $f(x')$ is as in (E.1) with $\tilde{f}_{\ell} \rightarrow (-1)^{\ell+1}\tilde{f}_{\ell}$. By expanding $f$ so that
\[
f(x) = \sum_{n=1}^{\infty} f_n x^{n+1}, \tag{E.2}
\]
and using (B.1), (B.2) we obtain
\[
\tilde{f}_{\ell} = \sum_{n=1}^{\ell} f_n (-1)^{n+1} \frac{2\ell+1}{(n)!^2(\ell-n)!}, \tag{E.3}
\]
The determination of $\tilde{f}_\ell$ is aided by using from (E.3)
\[
\sum_{\ell=1}^{\infty} \tilde{f}_\ell y^\ell = -\frac{1}{1-y} \sum_{n=1}^{\infty} f_n \frac{(2n)!}{(n!)^2} \left( \frac{-y}{(1-y)^2} \right)^n,
\]
where the right hand side may be determined in cases of interest. If
\[
S_k(y) = -\sum_{n=1}^{\infty} \frac{1}{n^k} \frac{(2n)!}{(n!)^2} \left( \frac{-y}{(1-y)^2} \right)^n,
\]
then
\[
\frac{1-y}{1+y} y \frac{d}{dy} S_k(y) = S_{k-1}(y),
\]
so that we may find
\[
S_0(y) = \frac{2y}{1+y}, \quad S_1(y) = -2 \ln(1-y), \quad S_2(y) = 2 \left( \text{Li}_2(y) + \ln^2(1-y) \right).
\]
Expanding $S_k(y)/(1-y)$ then gives $\tilde{f}_\ell$ in (E.3) for $f_n = n^{-k}$. In other cases we have resorted to matching expansions obtained through algebraic calculations of large numbers of terms.

Many of the results can be obtained from certain basic summations. We consider first
\[
(-x)^p \ln(1-x) = \sum_{\ell=1}^{\infty} 2^{\ell+1} \frac{\ell^2}{(2\ell)!} a_\ell^{(p)} g_{0,\ell+1}(x),
\]
and
\[
-(x)^p \text{Li}_2(x) = \sum_{\ell=1}^{\infty} 2^{\ell+1} \frac{\ell^2}{(2\ell)!} b_\ell^{(p)} g_{0,\ell+1}(x),
\]
where the coefficients $a_\ell^{(p)}$, $b_\ell^{(p)}$ are obtained by taking $f_n^{(p)} \to (-1)^{p+1}/(n-p+1)$, $(-1)^{p+1}/(n-p+1)^2$ for $n \geq p$ respectively in (E.3). We find
\[
a_\ell^{(1)} = 2h(\ell), \\
a_\ell^{(2)} = 2\ell(\ell+1)(h(\ell) - 1), \\
a_\ell^{(3)} = \frac{1}{2}(\ell - 1)\ell(\ell+1)(\ell+2)(h(\ell) - \frac{3}{2}),
\]
and
\[
b_\ell^{(1)} = 2h(\ell)^2, \\
b_\ell^{(2)} = 2\ell(\ell+1)(h(\ell) - 1)^2 - 2h(\ell) + \ell(\ell+1).
\]
With these results and (5.18) we have

\[ x^3 f_{11}(x) = \frac{1}{2} (x \ln(1 - x) - x' \ln(1 - x')) = - \sum_{\ell=2,4,...} 2^{\ell+2} \frac{(\ell)!^2}{(2\ell)!} h(\ell) g_{0,\ell+1}(x), \quad (E.12) \]

and

\[ x^3 f_{22}(x) = -\frac{1}{4} (x \text{Li}_2(x) - x' \text{Li}_2(x')) = - \sum_{\ell=2,4,...} 2^{\ell+1} \frac{(\ell)!^2}{(2\ell)!} h(\ell)^2 g_{0,\ell+1}(x). \quad (E.13) \]

(E.12) and (E.13) directly lead to the expansions for \( f_{1,1} \) and \( f_{2,2} \) given by (5.22) and (5.25).

Further relevant expansions are given by

\[ (-x')^p \text{Li}_2(x) = \sum_{\ell=1}^{\infty} 2^{\ell+1} \frac{(\ell)!^2}{(2\ell)!} b_{\ell}^{(p)} g_{0,\ell+1}(x), \quad (E.14) \]

where we find

\begin{align*}
 b_{\ell}^{(1)} &= (-1)^{\ell/2} 2 \sum_{r=1}^{\ell} (-1)^r \frac{1}{r^2}, \\
 b_{\ell}^{(2)} &= (-1)^{\ell} \left( 2\ell(\ell+1) \sum_{r=1}^{\ell} \frac{(-1)^r}{r^2} + 2h(\ell) + 1 \right) - 1. 
\end{align*} \quad (E.15)

Hence

\[ x^3 f_{10}(x) = (x + x') \text{Li}_2(x) = \sum_{\ell=1}^{\infty} 2^{\ell+1} \frac{(\ell)!^2}{(2\ell)!} (b_{\ell}^{(1)} + b_{\ell}^{(1)}) g_{0,\ell+1}(x). \quad (E.16) \]

In addition for application in (5.26) we also need to consider

\[ \sum_{\ell=2,4,...} 2^{\ell+1} \frac{(\ell)!^2}{(2\ell)!} h(\ell) g'_{0,\ell+1}(x) = - \sum_{\ell=3}^{\infty} 2^{\ell+1} \frac{(\ell)!^2}{(2\ell)!} d_{\ell} g_{0,\ell+1}(x). \quad (E.17) \]

Expanding in powers of \( x \) gives

\[ d_{\ell} = 2 \sum_{n=2}^{\ell} (-1)^{n+1} \frac{(\ell + n)!}{(\ell - n)!} \]

\[ \times \sum_{j=2,4,...} \frac{2j + 1}{(n-j)!(j+n+1)!} h(j) (h(n) - h(j) - h(j+n+1) + h(2j+1)) \]

\[ = \begin{cases} 
 h(\ell)^2 + \sum_{r=1}^{\ell} \frac{(-1)^r}{r}, & \text{if } \ell = 3,5,\ldots, \\
 3h(\ell)^2 - 2h(2\ell)h(\ell) - \sum_{r=1}^{\ell} \frac{1}{r^2}(1+(-1)^r), & \text{if } \ell = 2,4,\ldots.
\end{cases} \quad (E.18) \]
With (2.20) and (E.15) we then find
\[ \tilde{d}_\ell = d_\ell - \frac{1}{2} (b^{(1)}_\ell + b^{(1)}_\ell) = \begin{cases} 0, & \text{if } \ell = 3, 5, \ldots, \\ 2h(\ell)^2 - 2h(2\ell)h(\ell) - \sum_{r=1}^\ell \frac{1}{r^2}, & \text{if } \ell = 2, 4, \ldots, \end{cases} \quad (E.19) \]
which, with the aid of (E.16), is used in obtaining \( b_{\ell,1} = \tilde{d}_{\ell+2} \) in (5.27) from (5.26).

From the result (5.18) for \( f_{21} \) we have
\[ x^3 f_{21}(x) = \frac{3}{4}(x + x')(\operatorname{Li}_3(x) - \operatorname{Li}_3(x')) + \frac{1}{4}(x - x') \ln(1 - x) \operatorname{Li}_2(x) + \frac{1}{4}(x + x') \ln(1 - x') \operatorname{Li}_2(x'). \quad (E.20) \]
To obtain its expansion we first consider
\[ x \operatorname{Li}_3(x) = \sum_{\ell=1}^\infty 2^{\ell+1} \frac{\ell^2}{(2\ell)!} c_\ell g_{0,\ell+1}(x), \quad -x' \operatorname{Li}_3(x) = \sum_{\ell=1}^\infty 2^{\ell+1} \frac{\ell^2}{(2\ell)!} c'_\ell g_{0,\ell+1}(x), \quad (E.21) \]
which gives
\[ c_\ell = \sum_{n=1}^\ell \frac{1}{n^3} (-1)^{n+1} \frac{(\ell + n)!}{(n!)^2(\ell - n)!} = \frac{2}{3} \left( \sum_{r=1}^\ell \frac{1}{r^3} + 2h(\ell)^3 \right), \quad (E.22) \]
\[ c'_\ell = \sum_{n=1}^\ell \sum_{r=1}^n \frac{1}{r^3} (-1)^{n+1} \frac{(\ell + n)!}{(n!)^2(\ell - n)!} = 2 \sum_{r=1}^\ell (-1)^{\ell+r} \left( 2h(r) \frac{1}{r^2} - \frac{1}{r^3} \right). \]
For the rest of \( f_{21} \) it is then sufficient to expand
\[ -x \ln(1 - x) \operatorname{Li}_2(x) = \sum_{\ell=1}^\infty 2^{\ell+1} \frac{\ell^2}{(2\ell)!} e_\ell g_{0,\ell+1}(x), \quad (E.23) \]
\[ x' \ln(1 - x) \operatorname{Li}_2(x) = \sum_{\ell=1}^\infty 2^{\ell+1} \frac{\ell^2}{(2\ell)!} e'_\ell g_{0,\ell+1}(x), \]
which leads to
\[ e_\ell = 4 \left( \sum_{r=1}^\ell \frac{(-1)^r}{r^3} - 2 \sum_{r=1}^\ell \frac{1}{r} \sum_{s=1}^r \frac{(-1)^s}{s^2} - h(\ell)^3 \right), \quad (E.24) \]
\[ e'_\ell = -4 (-1)^{\ell+1} \left( \sum_{r=1}^\ell \frac{1}{r^3} (1 + 2(-1)^r) - 4 \sum_{r=1}^\ell \frac{1}{r} \sum_{s=1}^r \frac{(-1)^s}{s^2} + 3h(\ell) \sum_{r=1}^\ell \frac{(-1)^r}{r^2} \right). \]
Hence (E.20) gives
\[ x^3 f_{21}(x) = \sum_{\ell=1}^\infty 2^{\ell-1} \frac{\ell^2}{(2\ell)!} \left( (3(c_\ell - c'_\ell) - e_\ell') (1 + (-1)^{\ell}) - e_\ell (1 - (-1)^{\ell}) \right) g_{0,\ell+1}(x). \quad (E.25) \]
It remains to calculate
\[ \sum_{\ell=2,4,\ldots} 2^{\ell+1} \left( \frac{\ell!}{(2\ell)!} \right)^2 h(\ell)^2 g_{\ell_0,\ell+1}(x) = - \sum_{\ell=3}^{\infty} 2^{\ell+1} \left( \frac{\ell!}{(2\ell)!} \right)^2 f_{\ell} g_{\ell_0,\ell+1}(x). \] (E.26)

By using an expression similar to (E.18) we may find
\[
\begin{align*}
\ell = 3, 5, \ldots, \\
3h(\ell)^3 - 2h(2\ell)h(\ell)^2 - 2h(\ell) \sum_{r=1}^{\ell} \frac{1}{r^2} - 2 \sum_{r=1}^{\ell} \frac{1}{r} (1 + (-1)^r), \\
\end{align*}
\]
(E.27)

Using these results in (5.28) we have for odd \(\ell\)
\[ f_{\ell} + \frac{1}{4} \epsilon \ell = 0, \] (E.28)

whereas for even \(\ell\),
\[
\begin{align*}
\tilde{f}_{\ell} = f_{\ell} - \frac{3}{4} (c_{\ell} - c'_{\ell}) + \frac{1}{4} \epsilon \ell = 2h(\ell)^3 - 2h(2\ell)h(\ell)^2 - 2h(\ell) \sum_{r=1}^{\ell} \frac{1}{r^2} - \sum_{r=1}^{\ell} \frac{1}{r} \sum_{s=1}^{r} \frac{(-1)^s}{s^2} - \frac{3}{2} \sum_{r=1}^{\ell} \frac{1}{r^3} (1 + (-1)^r),
\end{align*}
\] (E.29)

so that from (5.28) we have \(\eta_{\ell,2} = 2(\tilde{f}_{\ell+2} - \tilde{d}_{\ell+2} h(\ell + 2))\), in accord with (5.29).

For the analysis of the twist 4 case we have from (5.39)
\[
\begin{align*}
x^4 f_{11,11;0}(x) &= \frac{1}{12} \left( x^2 \ln(1 - x) + x'^2 \ln(1 - x') \right), \\
x^4 f_{10,11;0}(x) &= \frac{1}{12} \left( (x^2 + 2x) \ln(1 - x) - (x'^2 + 2x') \ln(1 - x') \right).
\end{align*}
\] (E.30)

Hence from (E.10) we have
\[
\begin{align*}
x^4 f_{11,11;0}(x) &= \sum_{\ell=1,3,\ldots} 2^{\ell} \left( \frac{\ell!}{3(2\ell)!} \right)^2 b^{(2)}_{\ell} g_{\ell_0,\ell+1}(x), \\
x^4 f_{10,11;0}(x) &= \sum_{\ell=2,4,\ldots} 2^{\ell} \left( \frac{\ell!}{3(2\ell)!} \right)^2 \left( b^{(2)}_{\ell} - 4h(\ell) \right) g_{\ell_0,\ell+1}(x).
\end{align*}
\] (E.31)

At the next order (5.39) gives
\[
\begin{align*}
x^4 f_{11,22;0}(x) &= -\frac{1}{48} \left( (2x^2 + x'^2 + x + x') \text{Li}_2(x) + (2x'^2 + x^2 + x + x') \text{Li}_2(x') \right), \\
x^4 f_{10,22;0}(x) &= -\frac{1}{48} \left( (2x^2 - x'^2 + x + x') \text{Li}_2(x) - (2x'^2 - x^2 + x + x') \text{Li}_2(x') \right).
\end{align*}
\] (E.32)
The results (E.10), (E.11) and (E.14) then give
\[ x^4 f_{11,22;0}(x) = \sum_{\ell=3,4,\ldots} 2^{\ell-2} \frac{(\ell)!^2}{3(2\ell)!} \left( 2b_\ell^{(2)} - b_\ell^{(4)} - b_\ell^{(1)} - b_\ell^{(1)} \right) g_{0,\ell+1}(x), \]
\[ x^4 f_{10,22;0}(x) = \sum_{\ell=2,4,\ldots} 2^{\ell-2} \frac{(\ell)!^2}{3(2\ell)!} \left( 2b_\ell^{(2)} + b_\ell^{(4)} - b_\ell^{(1)} - b_\ell^{(1)} \right) g_{0,\ell+1}(x). \]

(E.33)

For singlet operators we may use from (5.43)
\[ x^3 f_{00,11}^{(1)}(x) = x^4 f_{11,11;0}(x) + \frac{1}{2}(x+x') = \sum_{\ell=1,3,\ldots} 2^{\ell+1} \frac{(\ell)!^2}{2^{\ell+1}[(\ell+1)!]} \left( \frac{1}{6} b_\ell^{(2)} - 1 \right) g_{0,\ell+1}(x), \]
and
\[ x^3 f_{00,22}^{(1)}(x) = x^4 f_{11,22;0}(x) = \frac{1}{8}(x+x'-2)\left( \text{Li}_2(x) + \text{Li}_2(x') \right) + \frac{1}{8}(x-x')\ln(1-x) + \frac{1}{4}(x+x') \]
\[ = \sum_{\ell=1,3,\ldots} 2^{\ell-1} \frac{(\ell)!^2}{(2\ell)!} \left( b_\ell^{(1)} + b_\ell^{(1)} - 2h(\ell) - 2 + \frac{2}{\ell(\ell+1)} \right) g_{0,\ell+1}(x). \]

(E.35)

To obtain (E.35) we use
\[ -\text{Li}_2(x) - \text{Li}_2(x') = \frac{1}{2} \ln^2(1-x) = \sum_{\ell=1,3,\ldots} 2^{\ell+1} \frac{(\ell)!^2}{2^{\ell+1}[(\ell+1)!]} \frac{2}{\ell(\ell+1)} g_{0,\ell+1}(x). \]

(E.36)

From (5.50) we have
\[ x^5 f_{22,11;0}(x) = \frac{1}{120} \left( x^3 \ln(1-x) - x^3 \ln(1-x') \right), \]
\[ x^5 f_{20,11;0}(x) = \frac{1}{120} \left( 2x^3 + 3x^2 + 6x \right) \ln(1-x) - \left( 2x^3 + 3x^2 + 6x' \right) \ln(1-x') \],
\[ x^5 f_{21,11;0}(x) = \frac{1}{40} \left( (x^3 + x^2) \ln(1-x) + (x^3 + x^2) \ln(1-x') \right), \]

(E.37)

and from (5.51)
\[ x^4 f_{10,11}^{(1)}(x) = \frac{1}{24} \left( x^3 \ln(1-x) + x^3 \ln(1-x') \right) + \frac{1}{12} \left( x^2 + 2x + x'^2 + 2x' \right), \]
\[ x^4 f_{11,11}^{(1)}(x) = \frac{1}{20} \left( x^3 \ln(1-x) - x^3 \ln(1-x') \right) + \frac{1}{12} (x^2 - x^2), \]
\[ x^3 f_{00,11}^{(2)}(x) = \frac{1}{40} \left( (x^3 - x^2) \ln(1-x) - (x^3 - x^2) \ln(1-x') \right) + \frac{1}{12} (x^2 - x^2). \]

(E.38)

Using (E.8) and (E.10) the required expansions may be easily read off.
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