Communication over Finite-Chain-Ring Matrix Channels

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Abstract

Though network coding is traditionally performed over finite fields, recent work on nested-lattice-based network coding suggests that, by allowing network coding over certain finite rings, more efficient physical-layer network coding schemes can be constructed. This paper considers the problem of communication over a finite-ring matrix channel \( Y = AX + BE \), where \( X \) is the channel input, \( Y \) is the channel output, \( E \) is random error, and \( A \) and \( B \) are random transfer matrices. Tight capacity results are obtained and simple polynomial-complexity capacity-achieving coding schemes are provided under the assumption that \( A \) is uniform over all full-rank matrices and \( BE \) is uniform over all rank-\( t \) matrices, extending the work of Silva, Kschischang and Kötter (2010), who handled the case of finite fields. This extension is based on several new results, which may be of independent interest, that generalize concepts and methods from matrices over finite fields to matrices over finite chain rings.

Index Terms

Lattice network coding, finite chain rings, matrix normal form, matrix channels, channel capacity.

I. INTRODUCTION

Matrix channels provide a useful abstraction for studying error control for linear network coding schemes. Transmitted and received packets, drawn from some ambient message space \( \Omega \), can be gathered into the rows of a transmitted matrix \( X \) and a received matrix \( Y \), respectively, while error packets injected into the network can be described by the rows of an error matrix \( E \). Due to the nature of linear network coding, the linear transformation of transmitted packets \( X \) and the linear propagation of error packets \( E \) can be modelled as a multiplicative-additive matrix channel (MAMC), defined via

\[
Y = AX + BE
\]
for appropriate transfer matrices $A, B$. One typically assumes that $A, B,$ and $E$ are random matrices (drawn according to certain distributions) and independent of $X$. This type of stochastic model is appropriate in situations where random network coding is performed and the error matrix $E$ arises due to decoding errors, rather than from the malicious actions of an adversary.

When the ambient space $\Omega$ is a vector space over a finite field, tight capacity bounds and simple, asymptotically capacity-achieving, coding schemes are developed in [1], under certain distributions of $A, B,$ and $E$. Similar work along this line can be found, e.g., in [2]–[5]. Prior work on matrix channels for linear network coding has mainly focused on the finite-field case.

In this paper, we consider a more general ambient space $\Omega$ of the form

$$\Omega = T/\langle d_1 \rangle \times T/\langle d_2 \rangle \times \cdots \times T/\langle d_m \rangle,$$

(2)

where $T$ is a sub-ring of $\mathbb{C}$ forming a principal ideal domain and $d_1, d_2, \ldots, d_m \in T$ are nonzero non-unit elements. To handle such an ambient space, we need to generalize the work of [1] from finite fields to finite chain rings. The motivation for considering this generalization arises from nested-lattice physical-layer network coding [6]–[10], in which the ambient space $\Omega$ is given precisely in the form of (2). As in [1], we gather insight by first studying two variations: the noise-free multiplicative matrix channel (MMC) $Y = AX$, and the multiplication-free additive matrix channel (AMC) $Y = X + BE$.

The essential step in handling the MMC over finite fields is based on the concept of reduced row echelon form (RREF) [1]. Due to the presence of zero divisors, the extension to finite chain rings of this concept is not straightforward. Whereas over a finite field any echelon form of a matrix will have the same number of nonzero rows (equal to the matrix rank), this is not the case for matrices over finite chain rings. To address this difficulty, several possible extensions of the RREF have been proposed in the literature, including the Howell form [11], [12] and the $p$-basis [13]. In this paper, we use the row canonical form defined in the dissertation of Kiermaier [14], which is itself a variant of the matrix canonical form described in an exercise in [15], and traces back to earlier ideas of Fuller [16] and Birkhoff [17]; see Section IV for more details. This row canonical form is particularly suitable for studying matrix channels with an ambient space of the form (2). We provide a new elementary proof for the existence and uniqueness of this row canonical form. Based on these results, we introduce a notion of (combinatorially dominant) principal row canonical forms, which allows us to obtain simple, capacity-achieving, coding schemes for the MMC.

The key step in handling the AMC over finite fields is counting the number of matrices of a given rank $t$. The rank $t$ may be regarded as a measure of “noise level” of the matrix $BE$. For matrices over finite chain rings, the concept of “rank” is more subtle, and must be suitably generalized. We first show how the concept of “shape”—the appropriate chain-ring-theoretic generalization of dimension—can be used to indicate the noise level. We then derive an enumeration result that counts the number of matrices of a given shape. This enables us to obtain capacity results and simple capacity-achieving coding schemes for the MMC.

Building upon the generalizations for the two special cases, we derive tight capacity bounds and simple, polynomial-
complexity, asymptotically capacity-achieving coding schemes for the MAMC model related to (1). We also consider several possible extensions of the MAMC model.

The remainder of this paper is organized as follows. Section II motivates the study of matrix channels over finite rings. Section III reviews some basic facts about finite chain rings, modules and matrices over finite chain rings. Section IV introduces the row canonical form. Section V presents several enumeration results and construction methods for matrices over finite chain rings. These new results provide us with essential algebraic tools for extending the work of [1]. Section VI introduces a channel-decomposition technique that connects the matrix channels described in Section II to the algebraic tools developed in Sections IV and V. Three basic channel models (MMC, AMC, and MAMC) are addressed in Sections VII, VIII and IX respectively, where capacity and coding results are presented. Section X presents possible extensions. Finally, Section XI concludes the paper.

II. Motivating Examples

In this section, we explain how finite rings arise naturally in the context of nested-lattice-based physical-layer network coding (PNC). We then introduce an end-to-end matrix model for wireless relay networks based on such PNC schemes.

![Transmitted constellation](image1)

![Received constellation](image2)

(a) Transmitted constellation  
(b) Received constellation

Fig. 1: Transmitted and received constellations.

We begin with the role of finite rings. As a simple starting point, consider a PNC building block where a relay attempts to decode, at the output of a Gaussian multiple access channel with complex-valued channel gains, a function $f$ of messages $w_1 = (w_{11}, w_{12})$ and $w_2 = (w_{21}, w_{22})$ sent from two transmitters, where each transmitter uses a quaternary phase-shift-keying (QPSK) signal constellation with Gray mapping as shown in Fig. 1a. Here $w_{ij} \in \{0, 1\}$. Assume that the channel gains (at the relay) are $h_1 = 1$ and $h_2 = i$. Then Fig. 1b shows the nominal received constellation (which is perturbed by Gaussian noise), from which the relay must decode. Some points in the received constellation correspond to more than one combination of transmitted messages; for example,
$(w_1, w_2) = (01, 10)$ overlaps $(w_1, w_2) = (11, 11)$. Clearly these overlapping points must correspond to the same value $f$, since otherwise the relay cannot possibly form $f$ correctly. Interestingly, in order to achieve this, one can interpret the messages $\{w_j^1, w_j^2\}$ as elements in the finite ring $\mathbb{Z}_2[i] = \{w_{j_1} + w_{j_2}i \mid w_{j_1}, w_{j_2} \in \mathbb{Z}_2\}$. For example, 01 and 10 are interpreted as $0 + i$ and $1 + 0i$, respectively. Now, consider the function $f : \mathbb{Z}_2[i] \times \mathbb{Z}_2[i] \rightarrow \mathbb{Z}_2[i]$ given by $f(w_1, w_2) = w_1 + iw_2$. In this case we have

$$f(01, 10) = (0 + i) + i(1 + 0i) = 0 + 0i = f(11, 11),$$

i.e., the points $(01, 10)$ and $(11, 11)$ have the same function value 00. Moreover, this happens for all the overlapping points in Fig. 1b and for other channel gains as well. As such, the finite ring $\mathbb{Z}_2[i]$ seems to be a “good match” for a QPSK constellation. In fact, for every nested-lattice-based constellation, there is a matching finite ring, as we have shown in our previous work [8].

![Fig. 2: A wireless relay network with three relays.](image)

Next, we introduce an end-to-end matrix model that allows us to study wireless relay networks with PNC. Fig. 2 illustrates a wireless relay network consisting of two transmitters, three relays, and a single receiver (with three antennas). Suppose that the network employs (nested-lattice-based) PNC and the packets are over some finite ring $R$. Let $w_1, w_2$ be the packets at the transmitters, and let $w_6, w_7, w_8$ be the packets at the receiver. Using PNC, each relay node first decodes a linear combination $w_j$ ($j = 3, 4, 5$) of the packets $w_1, w_2$, and then transmits this combination simultaneously. Hence, we have $w_j = a_{1j}w_1 + a_{2j}w_2$ for some $a_{1j}, a_{2j} \in R$, where $j = 3, 4, 5$. Similarly, $w_j = a_{3j}w_3 + a_{4j}w_4 + a_{5j}w_5$, where $j = 6, 7, 8$. Clearly, the relation between the transmitted packets and the received packets is given by $Y = AX$, where

$$X = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad Y = \begin{bmatrix} w_6 \\ w_7 \\ w_8 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a_{36} & a_{46} & a_{56} \\ a_{37} & a_{47} & a_{57} \\ a_{38} & a_{48} & a_{58} \end{bmatrix} \begin{bmatrix} a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \end{bmatrix} \in R^{3 \times 2}.$$

This gives rise to a matrix channel for the receiver.
Note that relays may sometimes introduce decoding errors. Suppose that the relay at the bottom of Fig. 2 makes a decoding error, i.e., \( w_5 = a_{15}w_1 + a_{25}w_2 + e \), where \( e \) represents the error packet. In this case, the receiver observes \( Y = AX + Z \), where \( A \) is the same as before, and
\[
Z = \begin{bmatrix}
a_{56} \\
a_{57} \\
a_{58}
\end{bmatrix} e.
\]

The above example can be generalized to a large network. Suppose that we now have \( n \) transmitters, \( N \) relays, and \( N \) receivers (each with a single antenna). Suppose that these receivers are connected to a central processor (similar to the architecture of small cells or cloud-based radio access networks). Clearly, the central processor observes a matrix channel \( Y = AX + Z \), where \( A \) is of size \( N \times n \).

To sum up, the matrix model \( Y = AX + Z \) (over some finite ring) provides a general abstraction for studying wireless relay networks with nested-lattice-based PNC.

III. Preliminaries

In this section, we present some basic results for finite chain rings and modules and matrices over finite chain rings. This section establishes notation and the results that will be used later for the study of matrix channels over finite rings; nevertheless, this material is standard; see e.g., [15], [18]–[23] for more details. To make the paper more self-contained, Appendix A reviews some basic facts about rings and ideals.

A. Finite Chain Rings

All rings in this paper will be commutative with identity \( 1 \neq 0 \). A ring \( R \) is called a chain ring if the ideals of \( R \) satisfy the chain condition: for any two ideals \( I, J \) of \( R \), either \( I \subseteq J \) or \( J \subseteq I \). If \( R \) is a chain ring with finitely many elements, then \( R \) is called a finite chain ring. Clearly, a finite chain ring has a unique maximal ideal, and hence is local. It is known [15] that a finite ring is a chain ring if and only if it is a local principal ideal ring (PIR); thus, in a finite chain ring, all ideals are principal. Examples of finite chain rings include \( \mathbb{Z}_{p^n} \) (the ring of integers modulo \( p^n \) where \( p \) is a prime) and Galois rings.

Let \( R \) be a finite chain ring, and let \( \pi \in R \) be any generator of the maximal ideal of \( R \). Then \( R/\langle \pi \rangle \) is the residue field of \( R \). It can be shown (see, e.g., [15]) that every ideal \( I \) of \( R \), including the zero ideal \( \langle 0 \rangle \), is generated by a power of \( \pi \), i.e., \( I = \langle \pi^l \rangle \) for some \( l \geq 0 \). It follows that \( \pi \) is nilpotent; we denote by \( s \) the nilpotency index of \( \pi \), i.e., the smallest positive integer such that \( \pi^s = 0 \). There are, then, exactly \( s + 1 \) distinct ideals of \( R \), namely, \( R = \langle \pi^0 \rangle, \langle \pi^1 \rangle, \ldots, \langle \pi^s \rangle = \{ 0 \} \) which form a chain (with respect to set inclusion):
\[
R = \langle \pi^0 \rangle \supset \langle \pi^1 \rangle \supset \cdots \supset \langle \pi^{s-1} \rangle \supset \langle \pi^s \rangle = \{ 0 \}.
\]

Thus, \( s \) is often called the chain length of \( R \). We refer to \( R \) as a \((q,s)\) chain ring if \( R \) has a residue field of size \( q \) and a chain length of \( s \).
Example 1: The ideals of \( \mathbb{Z}_8 \) form a chain with respect to set inclusion:

\[
R = \langle 1 \rangle \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \langle 0 \rangle = \{0\}.
\]

Thus, \( \mathbb{Z}_8 \) is a finite chain ring with chain length \( s = 3 \). Since the residue field \( \mathbb{Z}_8/(2) \) is isomorphic to \( \mathbb{F}_2 \), \( \mathbb{Z}_8 \) is a \((2,3)\) chain ring.

Now let \( \mathcal{R}(R, \pi) \subseteq R \) be a complete set of residues with respect to \( \pi \) and, without loss of generality, assume that \( 0 \in \mathcal{R}(R, \pi) \). Every element \( a \in R \) then has a unique representation, called the \( \pi \)-adic decomposition of \( a \) (with respect to \( \mathcal{R}(R, \pi) \)), in the form

\[
a = a_0 + a_1 \pi + \cdots + a_{s-1} \pi^{s-1},
\]

where \( a_0, \ldots, a_{s-1} \in \mathcal{R}(R, \pi) \). It follows from the uniqueness of (3) that the size of \( R \) is \( q^s \), i.e., the number of elements in a \((q, s)\) chain ring is \( q^s \). Thus, like a finite field, a finite chain ring has a cardinality that is an integer power of a prime number.

The degree of a nonzero element \( a_0 + a_1 \pi + \cdots + a_{s-1} \pi^{s-1} \in R \), denoted by \( \deg(a) \), is defined as the least index \( j \) for which \( a_j \neq 0 \). By convention, the degree of 0 is defined as \( s \). All elements of the same degree are associates in \( R \). Further, \( a \) divides \( b \) if and only if \( \deg(a) \leq \deg(b) \). Finally, \( \deg(a + b) \geq \min\{\deg(a), \deg(b)\} \), i.e., adding two elements never results in an element of lower degree.

Example 2: Let \( \mathcal{R}(\mathbb{Z}_8, 2) = \{0, 1\} \). The 2-adic decomposition of \( 5 \in \mathbb{Z}_8 \) is \( 5 = 1 + 0 \cdot 2 + 1 \cdot 2^2 \). The elements in \( \mathbb{Z}_8 \) of degree 0 (respectively, 1, 2, and 3) are \( \{1, 3, 5, 7\} \) (respectively, \( \{2, 6\} \), \( \{4\} \), and \( \{0\} \)).

Finally, we present two methods for constructing finite chain rings.

If \( R \) is itself a \((q, s)\) chain ring with maximal ideal \( \langle \pi \rangle \), then the quotient \( R/\langle \pi^l \rangle \) \( (0 < l < s) \) is a \((q, l)\) chain ring. This method constructs new finite chain rings from existing ones.

If \( T \) is a principal ideal domain (PID), and \( p \) is a prime in \( T \), then \( T/\langle p \rangle \) is a field, since \( \langle p \rangle \) is a maximal ideal of \( T \). Let \( q \) be the size of \( T/\langle p \rangle \) and suppose that \( q \) is finite. Then the quotient \( T/\langle p^l \rangle \) is a \((q, l)\) \( (l > 0) \) chain ring. This method constructs finite chain rings from PIDs.

B. Modules over Finite Chain Rings

A module is to a ring as a vector space is to a field. More formally, an \( R \)-module \( M \) is an abelian group \((M, +)\) together with an action of \( R \) on \( M \) satisfying the following conditions for all \( m, n \in M \) and for all \( a, b \in R \):

1) \( 1m = m \) and \((ab)m = a(bm)\)
2) \((a + b)m = am + bm\)
3) \(a(m + n) = am + an\)

When \( R \) is a finite chain ring, an \( R \)-module is always isomorphic to a direct product of various ideals of \( R \); this structure can be described by a “shape.” An \( s \)-shape \( \mu = (\mu_1, \mu_2, \ldots, \mu_s) \) is simply a sequence of non-decreasing non-negative integers, i.e., \( 0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_s \). We denote by \( |\mu| \) the sum of its components, i.e., \( |\mu| = \sum_{i=1}^{s} \mu_i \). For later notational convenience, we define the “zeroth component” of a shape as \( \mu_0 = 0 \).
An $s$-shape $\kappa = (\kappa_1, \ldots, \kappa_s)$ is said to be a subshape of $\mu = (\mu_1, \ldots, \mu_s)$, written $\kappa \leq \mu$, if $\kappa_i \leq \mu_i$ for all $i = 1, \ldots, s$. Thus, for example, $(1, 1, 3) \preceq (2, 4, 4)$. The number of subshapes of the $s$-shape $(m_1, \ldots, m_s)$ is given by $\binom{m+s}{s}$, which implies that the number of subshapes of $\mu = (\mu_1, \ldots, \mu_s)$ is upper-bounded by $\binom{\mu+s}{s}$.

Two $s$-shapes can be added together to form a new $s$-shape simply by adding componentwise. Thus, for example, $(1, 1, 3) + (2, 4, 4) = (3, 5, 7)$. Also, for a shape $\mu = (\mu_1, \ldots, \mu_s)$ and a positive integer $m$ we define $\mu/m = (\mu_1/m, \ldots, \mu_s/m)$ (which is an $s$-tuple, but not necessarily a shape). For convenience, we will sometimes identify the integer $t$ with the $s$-shape $(t, \ldots, t)$. Thus, for example, $\mu \preceq t$ means $\mu_i \leq t$ for all $i$, $\kappa = t$ means $\kappa_i = t$ for all $i$, and $\mu - t = (\mu_1 - t, \ldots, \mu_s - t)$, assuming $t \leq \mu$.

Let $R$ be a $(q, s)$ chain ring with maximal ideal $\langle \pi \rangle$. For any $s$-shape $\mu$, we define the $R$-module $R^\mu$ as

$$R^\mu = \bigoplus_{\mu_i} \langle 1 \rangle \times \cdots \times \langle 1 \rangle \times \langle \pi \rangle \times \cdots \times \langle \pi^{s-i} \rangle \times \cdots \times \langle \pi^{s-1} \rangle.$$ (4)

Since a positive integer $t$ is identified with the shape $(t, \ldots, t)$, it is indeed true that $R^t$ denotes the $t$-fold Cartesian product of $R$ with itself.

The module $R^\mu$ can be viewed as a collection of $\mu_s$-tuples whose components are drawn from $R$ subject to certain constraints imposed by $\mu$. Specifically, while the first $\mu_1$ components can be any element of $R$, the next $\mu_2 - \mu_1$ components must be multiples of $\pi$, and so on. Since each ideal $\langle \pi^i \rangle$ in $R$ contains $q^{s-i}$ elements ($0 \leq i < s$), it follows that the size of $R^\mu$ is $|R^\mu| = q^{\mu_1}$.

**Example 3:** Let $R = \mathbb{Z}_8$, and let $\mu = (2, 4, 4)$. Then

$$R^\mu = \langle 1 \rangle \times \langle 1 \rangle \times \langle 2 \rangle \times \langle 4-2 \rangle.$$ Note that the first two components of $R^\mu$ can each be chosen in $2^2$ ways, while the last two components can each be chosen in only $2^2$ ways. Hence, the size of $R^\mu$ is $2^{10}$.}

For every $s$-shape $\mu$, $R^\mu$ is a finite $R$-module. Conversely, the following theorem establishes that every finite $R$-module is isomorphic to $R^\mu$ for some unique $s$-shape $\mu$.

**Theorem 1:** [21, Theorem 2.2] For any finite $R$-module $M$ over a $(q, s)$ chain ring $R$, there is a unique $s$-shape $\mu$ such that $M \cong R^\mu$.

We call the unique shape $\mu$ given in Theorem 1 the shape of $M$, and write $\mu = \text{shape } M$. It is known [21] that if $M'$ is a submodule of $M$, then shape $M' \leq \text{shape } M$, i.e., the shape of a submodule is a subshape of the module. It is also known [21] that the number of submodules of $R^\mu$ whose shape is $\kappa$ is given by

$$\left[\begin{array}{c} \mu \\ \kappa \end{array}\right]_q = \prod_{i=1}^{s} q^{(\mu_i - \kappa_i)\kappa_{i-1}} \left[\begin{array}{c} \mu_i - \kappa_{i-1} \\ \kappa_i - \kappa_{i-1} \end{array}\right]_q,$$ (5)

$^1$Some authors (like Honold et al. [21]) use a different convention and define the shape of an $R$-module to be the conjugate (in the integer-partition-theoretic sense) of the shape as defined in this paper.
associates. In this paper, we shall require the diagonal entries of has a unique Smith normal form.

Finally, we will let \( A \) be a \((q,s)\) chain ring with maximal ideal \( \langle \pi \rangle \). The set of all \( n \times m \) matrices with entries from \( R \) will be denoted by \( R^{n \times m} \). If \( A \in R^{n \times m} \), we denote by \( A[i,j] \) the entry of \( A \) in the \( i \)th row and \( j \)th column, where \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). We will let \( A[i_1:i_2,j_1:j_2] \) denote the submatrix of \( A \) formed by rows \( i_1 \) to \( i_2 \) and by columns \( j_1 \) to \( j_2 \), where \( 1 \leq i_1 \leq i_2 \leq n \) and \( 1 \leq j_1 \leq j_2 \leq m \). Finally, we will let \( A[i,:) \) denote the \( i \)th row of \( A \) and \( A[:,j] \) denote the \( j \)th column of \( A \).

A square matrix \( U \in R^{n \times n} \) is invertible if \( UV = VU = I_n \) for some \( V \in R^{n \times n} \), where \( I_n \) denotes the \( n \times n \) identity matrix. The set of invertible matrices in \( R^{n \times n} \), denoted as \( GL_n(R) \), forms a group—the so-called general linear group—under matrix multiplication.

Two matrices \( A, B \in R^{n \times m} \) are said to be left-equivalent if there exists a matrix \( U \in GL_n(R) \) such that \( UA = B \). Two matrices \( A, B \in R^{n \times m} \) are said to be equivalent if there exist matrices \( U \in GL_n(R) \) and \( V \in GL_m(R) \) such that \( UAV = B \).

A matrix \( D \in R^{n \times m} \) is called a diagonal matrix if \( D[i,j] = 0 \) whenever \( i \neq j \). A diagonal matrix \( D \), which need not be square, can be written as \( D = \text{diag}(d_1, \ldots, d_r) \), where \( r = \min\{n,m\} \), and \( d_i = D[i,i] \) for \( i = 1, \ldots, r \).

Let \( A \in R^{n \times m} \). A diagonal matrix \( D = \text{diag}(d_1, \ldots, d_r) \in R^{n \times m} (r = \min\{n,m\}) \) is called a Smith normal form of \( A \), if \( D \) is equivalent to \( A \) and \( d_1 \mid d_2 \mid \cdots \mid d_r \) in \( R \). It is known \([19]\) that every matrix over a PIR (in particular, a finite chain ring) has a Smith normal form whose diagonal entries are unique up to equivalence of associates. In this paper, we shall require the diagonal entries \( d_1, \ldots, d_r \) in the Smith normal form \( D \) to be powers of \( \pi \), i.e.,

\[
(d_1, \ldots, d_r) = (\pi^{l_1}, \ldots, \pi^{l_r}),
\]

where \( 0 \leq l_1 \leq \ldots \leq l_r \leq s \) since \( d_1 \mid d_2 \mid \cdots \mid d_r \). With this constraint, once \( \pi \) is fixed, every matrix \( A \in R^{n \times m} \) has a unique Smith normal form.

**Example 4:** Consider the two matrices

\[
A = \begin{bmatrix}
4 & 6 & 2 & 1 \\
0 & 0 & 0 & 2 \\
2 & 4 & 6 & 1 \\
2 & 0 & 2 & 1
\end{bmatrix},
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
over \( \mathbb{Z}_8 \). It is easy to check that

\[
A = \begin{bmatrix}
1 & 2 & 0 & 0 \\
2 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 2 & 1 \\
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} = USV.
\]

Since \( U \) and \( V \) are invertible, \( S \) is equivalent to \( A \). Since the diagonal entries of \( S \) satisfy \( 1 \mid 2 \mid 4 \mid 0 \) in \( \mathbb{Z}_8 \), \( S \) is the Smith normal form of \( A \).

For any \( A \in \mathbb{R}^{n \times m} \), we denote by \( \text{row} \, A \) and \( \text{col} \, A \) the row span and column span of \( A \), respectively. By using the Smith normal form, it is easy to see that \( \text{row} \, A \) is isomorphic, as an \( R \)-module, to \( \text{col} \, A \). It is also easy to see that left-equivalent matrices have identical row spans and equivalent matrices have isomorphic row spans.

The \textit{shape} of a matrix \( A \) is defined as the shape of the row span of \( A \), i.e.,

\[
\text{shape} \, A = \text{shape} \, (\text{row} \, A).
\]

Clearly, \( \text{shape} \, A = \text{shape} \, (\text{col} \, A) \). Moreover, \( \text{shape} \, A = \mu \) if and only if the Smith normal form of \( A \) is given by

\[
\text{diag}(\underbrace{1, \ldots, 1}_\mu, \underbrace{\pi, \ldots, \pi}_{\mu_2 - \mu}, \ldots, \underbrace{\pi^{-1}, \ldots, \pi^{-1}}_{\mu_s - \mu_{s-1}}, 0, \ldots, 0),
\]

where \( r = \min \{n, m\} \). In particular, a matrix \( U \in \mathbb{R}^{n \times n} \) is invertible if and only if \( \text{shape} \, U = (n, \ldots, n) \).

\textbf{Example 5:} Since \( D = \text{diag}(1, 2, 4, 0) \) is the Smith normal form of \( A \) in Example 4, \( \text{shape} \, A = (1, 2, 3) \).

As one might expect, matrix shape has a number of properties similar to matrix rank.

\textbf{Proposition 1:} Let \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{m \times k} \). Then

1) \( \text{shape} \, A = \text{shape} \, A^T \), where \( A^T \) is the transpose of \( A \).

2) For any \( P \in \text{GL}_n(R), Q \in \text{GL}_m(R) \), \( \text{shape} \, A = \text{shape} \, PAQ \).

3) \( \text{shape} \, AB \preceq \text{shape} \, A, \text{shape} \, AB \preceq \text{shape} \, B \).

4) For any submatrix \( C \) of \( A \), \( \text{shape} \, C \preceq \text{shape} \, A \).

5) \( \text{shape} \, A \preceq \min \{n, m\} \).

\textbf{Proof:} 1) Since \( \text{row} \, A \cong \text{col} \, A \), we have \( \text{row} \, A \cong \text{row} \, A^T \). Hence, \( \text{shape} \, A = \text{shape} \, A^T \). 2) Since \( A \) is equivalent to \( PAQ \) for any invertible \( P \) and \( Q \), \( \text{shape} \, A = \text{shape} \, PAQ \). 3) Since \( \text{row} \, AB \) is a submodule of \( \text{row} \, B \), we have \( \text{shape} \, AB \preceq \text{shape} \, B \). Similarly, since \( \text{col} \, AB \) is a submodule of \( \text{col} \, A \), we have \( \text{shape} \, AB \preceq \text{shape} \, A \).

4) Note that any submatrix \( C \) of \( A \) is equal to \( E_1 \cdot E_2 \) for some \( E_1 \in \mathbb{R}^{k \times n} \) (selecting \( k \) rows) and \( E_2 \in \mathbb{R}^{m \times l} \) (selecting \( l \) columns). Hence, \( \text{shape} \, C = \text{shape} \, E_1 \cdot E_2 \preceq \text{shape} \, A \).

5) Since the Smith normal form of \( A \) has at most \( \min \{n, m\} \) nonzero diagonal entries, we have \( \text{shape} \, A \preceq \min \{n, m\} \).

For convenience, we say a matrix \( A \in \mathbb{R}^{n \times m} \) have \textit{rank} \( t \), if \( \text{shape} \, A = t \). Note that the rank of a matrix is not always defined. A matrix \( A \in \mathbb{R}^{n \times m} \) is called \textit{full rank} if \( \text{rank} \, A = \min \{n, m\} \). A matrix \( A \in \mathbb{R}^{n \times m} \) is called \textit{full row rank} if \( \text{rank} \, A = n \) (which requires \( n \leq m \)). The number of full-row-rank matrices in \( \mathbb{R}^{n \times m} \) is
\[ q^{nm} \prod_{i=0}^{n-1} (1 - q^i - m). \] A matrix is full column rank if its transpose is full row rank. Full-column-rank matrices have the following property.

**Lemma 1:** Let \( A \) be a full-column-rank matrix. Then \( AB \) is a zero matrix if and only if \( B \) is a zero matrix.

**Proof:** The “if” part is trivial, so we turn to the “only if” part. Let \( A \in \mathbb{R}^{n \times m} \). Suppose that \( AB = 0 \) for some matrix \( B \in \mathbb{R}^{m \times k} \). We will show that \( B \) is a zero matrix. Since \( A \) is full column rank, its Smith normal form \( S \) must have the form

\[
S = \begin{bmatrix}
I_m \\
0_{(n-m) \times m}
\end{bmatrix}
\]

and \( A = USV \) for some invertible matrices \( U \) and \( V \). Thus, we have

\[
AB = U \begin{bmatrix}
I \\
0
\end{bmatrix} VB = 0,
\]

which implies \( B = 0 \).

**IV. Row Canonical Form**

The main algebraic tools for studying matrix channels over finite fields include Gaussian elimination and reduced row echelon forms. The generalization of these tools to finite chain rings is, however, not straightforward. Consider the \( 3 \times 4 \) matrix

\[
A = \begin{bmatrix}
2 & 1 & 1 & 2 \\
6 & 3 & 7 & 2 \\
6 & 7 & 1 & 0
\end{bmatrix}
\]

over \( \mathbb{Z}_8 \). On the one hand, we have

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} A = \begin{bmatrix}
2 & 1 & 1 & 2 \\
0 & 4 & 0 & 4 \\
0 & 0 & 2 & 2
\end{bmatrix}.
\]

On the other hand, we have

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 1 \\
7 & 1 & 2
\end{bmatrix} A = \begin{bmatrix}
2 & 1 & 1 & 2 \\
0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

In both cases we have transformed \( A \) to echelon form using elementary row operations. Recall that, over finite fields, the rank of a matrix is precisely the number of nonzero rows in its echelon form. This property, however, does not hold for matrices over finite chain rings.

To address this difficulty, several possible generalizations of reduced row echelon forms have been proposed in the literature, including the Howell form [11], [12], the matrix canonical form [15], [16], and the \( p \)-basis [13]. In this section, we will describe a row canonical form that is particularly suitable for studying matrix channels over finite chain rings. This row canonical form is essentially the same as the reduced row echelon form defined.
in Kiermaier’s thesis \cite{14} Definition 2.2.2 (written in German), which itself is a variant of the matrix canonical form in \cite{15} p. 329, Exercise XVI.7. It appears that the key idea behind these forms was proposed by Fuller \cite{16} based on an earlier result of Birkhoff \cite{17}. We provide in this section a new elementary proof for the existence and uniqueness of the row canonical form.

Throughout this section, $R$ is a $(q, s)$ chain ring with maximal ideal $\langle \pi \rangle$. We fix a complete set of residues $R(R, \pi)$ (including 0), i.e., a representation of the residue field $R/\langle \pi \rangle$, and, for $1 < l < s$, we choose the complete set of residues for $\pi^l$ as

$$R(R, \pi^l) = \{ \sum_{i=0}^{l-1} a_i \pi^i : a_0, \ldots, a_{l-1} \in R(R, \pi) \}.$$ 

Finally, we set $R(R, \pi^0) = \{0\}$.

**A. Definitions**

We start with a few definitions.

Let $A$ be matrix with entries from $R$. The $i$th row of $A$ is said to occur *above* the $(i')$th row of $A$ (or the $(i')$th row occurs *below* the $i$th row) if $i < i'$. Similarly the $j$th column of $A$ is said to occur *earlier* than the $(j')$th column (or the $(j')$th column occurs *later* than the $j$th column) if $j < j'$. This terminology extends to the entries of $A$: $A[i, j]$ is above $A[i', j']$ if $i < i'$ and $A[i, j]$ is earlier than $A[i', j']$ if $j < j'$. If $P$ is some property obeyed by at least one of the entries in the $i$th row of $A$, then the *first* entry in row $i$ with property $P$ occurs earlier than every other entry in row $i$ having property $P$.

The *pivot* of a nonzero row of a matrix is the first entry among the entries having least degree in that row. For example, 6 and 2 are the entries of least degree in the row $[0 4 6 2]$ over $\mathbb{Z}_8$, and 6 occurs earlier. Thus, 6 is the pivot of the row $[0 4 6 2]$. Note that the pivot of a row is not necessarily the first nonzero entry of the row.

**Definition 1:** A matrix $A$ is in *row canonical form* if it satisfies the following conditions.

1) Nonzero rows of $A$ are above any zero rows.

2) If $A$ has two pivots of the same degree, the one that occurs earlier is above the one that occurs later. If $A$ has two pivots of different degree, the one with smaller degree is above the one with larger degree.

3) Every pivot is of the form $\pi^l$ for some $l \in \{0, \ldots, s-1\}$.

4) For every pivot (say $\pi^i$), all entries below and in the same column as the pivot are zero, and all entries above and in the same column as the pivot are elements of $R(R, \pi^i)$.

**Example 6:** Consider the matrix

$$A = \begin{bmatrix}
0 & 2 & 0 & 1 \\
2 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$
over \( \mathbb{Z}_8 \) with \( \pi = 2 \) and \( \mathbb{Z}_8/(2) = \{0, 1\} \), in which the pivots have been identified with an overline. Clearly, \( A \) satisfies all of the conditions to be in row canonical form.

The following facts follow immediately from the definition of row canonical form.

**Proposition 2:** Let \( A \in R^{n \times m} \) be a matrix in row canonical form, let \( p_k \) be the pivot of the \( k \)th row, let \( c_k \) be the index of the column containing \( p_k \). (If the \( k \)th row is zero, let \( p_k = 0 \) and \( c_k = 0 \).) Let \( d_k = \deg(p_k) \), and let \( w = (w_1, \ldots, w_m) \) be an arbitrary element of row \( A \).

1) Any column of \( A \) contains at most one pivot.

2) If \( A \) has more than one row, deleting a row of \( A \) results in a matrix also in row canonical form.

3) \( i \geq k \) implies \( \deg(A[i,j]) \geq d_k \).

4) \( (i \geq k \text{ and } j < c_k) \) or \( (i > k \text{ and } j \leq c_k) \) implies \( \deg(A[i,j]) > d_k \).

5) \( p_1 \) divides \( w_1, w_2, \ldots, w_m \).

6) \( j < c_1 \) implies \( \deg(w_j) > d_1 \).

The proof is provided in Appendix B. For any \( A \in R^{n \times m} \), we say a matrix \( B \in R^{n \times m} \) is a row canonical form of \( A \), if (i) \( B \) is in row canonical form, and (ii) \( B \) is left-equivalent to \( A \). We will show that any \( A \in R^{n \times m} \) has a unique row canonical form. For this reason, we denote by \( \text{RCF}(A) \) the row canonical form of \( A \).

**B. Existence and Uniqueness**

First, we demonstrate the existence of a row canonical form for any matrix \( A \) by presenting a simple algorithm that performs elementary row operations to reduce \( A \) into row canonical form. Here, the allowable elementary row operations (over \( \mathbb{R} \)) are:

- Interchange two rows.
- Add a multiple of one row to another.
- Multiply a row by a unit in \( \mathbb{R} \).

Each of these operations is invertible, and so a matrix obtained from \( A \) by any sequence of these operations will have the same row span as \( A \).

The algorithm proceeds in a series of steps. In the \( k \)th step, the algorithm selects the \( k \)th pivot, moves it to the \( k \)th row, and uses elementary row operations to reduce into row canonical form the submatrix consisting of the top \( k \) rows. The pivot selection procedure operates on any given set of rows. If the rows are all zero, the procedure should return with the result that no pivot can be found. Otherwise, among all entries of least degree in the given rows, an entry must be chosen that occurs as early as possible. This entry must certainly be the pivot of its row. The procedure should return the row and column index of the selected element.

Now we are ready to describe the algorithm in detail. In step \( k = 1 \), apply pivot selection to all of the rows of \( A \). If no pivot can be found, then \( A \) is a zero matrix, and is already in row canonical form. Otherwise, we call this pivot the first pivot and place it in the first row by an interchange of rows (if necessary). If this pivot is not of the form \( \pi^l \) \( (l = 0, \ldots, s - 1) \), we multiply the first row by a suitable unit so that the first pivot is a power of
π. Note that nonzero entries in the same column below the first pivot have degrees no less than the pivot, which means that they are all multiples of the first pivot. By a sequence of elementary row operations, these entries can be cancelled, so that we arrive at a matrix, say $A_1$, in which the first row is in row canonical form and all entries in the same column below the first pivot are zero. We can now increment $k$ and proceed to the next step.

For $k \geq 2$, we apply pivot selection to the rows of $A_{k-1}$, excluding the first $k-1$ rows. If no pivot can be found, then the remaining rows are all zero and $A_{k-1}$ is in row canonical form. Otherwise we call this pivot the $k$th pivot and place it in the $k$th row by an exchange of rows (if necessary). As in the first step, if this pivot is not an integer power of $\pi$, we multiply the $k$th row by a suitable unit so that the $k$th pivot is a power of $\pi$, say $\pi^l$. Nonzero entries in the same column below the $k$th pivot can be cancelled using elementary row operations. A nonzero entry, say $a$, in the same column above the $k$th pivot has $\pi$-adic decomposition

$$a = a_0 + \cdots + a_{s-1} \pi^{s-1}$$

$$= a_0 + \cdots + a_{l-1} \pi^{l-1} + \pi^l(a_l + \cdots + a_{s-1} \pi^{s-l-1}).$$

Thus by subtracting $(a_l + \cdots + a_{s-1} \pi^{s-l-1})$ times the $k$th row from the row containing $a$, we change $a$ to $a_0 + \cdots + a_{l-1} \pi^{l-1} \in \mathcal{R}(R, \pi^l)$, without affecting the pivot of that row. Reducing all nonzero entries in the same column as the $k$th pivot in this way, we arrive at a matrix, say $A_k$, in which the top $k$ rows are in row canonical form and all entries in the same column below the first, second, ..., $k$th pivots are zero.

The above algorithm stops when no more pivots can be found. Note that, at the end of the $k$th step, the matrix $A_k$ is left-equivalent to $A$ and the submatrix formed by the top $k$ rows of $A_k$ is in row canonical form. It follows that the final matrix must be in row canonical form.

Therefore, we have the following result.

**Proposition 3:** For any $A \in \mathbb{R}^{n \times m}$, the algorithm described above computes a row canonical form of $A$.

A simple count shows that this algorithm requires

$$\mathcal{O}(nm \min\{n, m\})$$

basic operations over $R$.

**Example 7:** Consider the matrix

$$A = \begin{bmatrix} 4 & 6 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 2 & 4 & 6 & 1 \\ 2 & 0 & 2 & 1 \end{bmatrix}$$

over $\mathbb{Z}_8$. There are three 1s in the last column of $A$, namely, $A[1,4]$, $A[3,4]$ and $A[4,4]$, which are the elements of least degree in $A$. We can choose any of them as the first pivot. Here, we choose $A[1,4]$ (indicated by an overline).
After some elementary row operations, we can make the entries below the pivot zero to obtain

\[
A_1 = \begin{bmatrix}
4 & 6 & 2 & 1 \\
0 & 4 & 4 & 0 \\
6 & 6 & 4 & 0 \\
6 & 2 & 0 & 0
\end{bmatrix}.
\]

Now consider the submatrix formed by omitting the first row of \(A_1\). There are four entries of least degree, namely, \(A_1[3,1] = 6\), \(A_1[3,2] = 6\), \(A_1[4,1] = 6\), and \(A_1[4,2] = 2\), among which \(A_1[3,1]\) and \(A_1[4,1]\) are valid choices for the second pivot. Here, we choose \(A_1[3,1]\) (indicated by an overline). We interchange the second row and third row of \(A_1\), and then multiply the new second row by 3, obtaining

\[
A'_1 = \begin{bmatrix}
4 & 6 & 2 & 1 \\
\overset{\mathbf{2}}{2} & \overset{\mathbf{2}}{2} & \overset{\mathbf{4}}{4} & 0 \\
0 & 4 & 4 & 0 \\
6 & 2 & 0 & 0
\end{bmatrix}.
\]

By some elementary row operations, we can make the entries below the second pivot zero. After that, we subtract 2 times the second row from the first row, obtaining

\[
A_2 = \begin{bmatrix}
0 & 2 & 2 & 1 \\
2 & 2 & 4 & 0 \\
0 & 4 & 4 & 0 \\
0 & 4 & 4 & 0
\end{bmatrix}.
\]

Clearly, the submatrix formed by the top two rows of \(A_2\) is in row canonical form. Next, consider the submatrix formed by omitting the top two rows of \(A_2\). We choose the entry \(A_2[3,2]\) (indicated by an overline) as the third pivot. We subtract the third row from the fourth row and obtain

\[
A_3 = \begin{bmatrix}
0 & 2 & 2 & 1 \\
2 & 2 & 4 & 0 \\
0 & 4 & 4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Clearly, the submatrix formed by the top three rows of \(A_3\) is in row canonical form (with all the pivots indicated). Since no more pivots can be found, our algorithm outputs \(A_3\), which is indeed in row canonical form.

As expected, the row canonical form is unique.

**Proposition 4:** For any \(A \in \mathbb{R}^{n \times m}\), the row canonical form of \(A\) is unique.

The proof is provided in Appendix [B].
\[ X_0 = \begin{bmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{bmatrix}, \quad X_1 = \begin{bmatrix} \ast & \ast & \ast & \ast & \ast \end{bmatrix}, \quad X_2 = \begin{bmatrix} \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \end{bmatrix} \]

\[ \mu_1 \quad \mu_2 \quad \mu_3 \]

Fig. 3: Illustration of a \( \pi \)-adic decomposition for \( s = 3 \) and \( \mu = (4, 6, 8) \).

V. MATRICES UNDER ROW CONSTRAINTS

In this section, we study a class of matrices in \( R^{n \times m} \) whose rows are constrained to be elements of \( R^\mu \). We provide several new counting results and a construction of principal row canonical forms for this class of matrices. These results are of primary importance to our study of capacities and coding schemes in later sections.

A. \( \pi \)-adic Decomposition

Let \( R^{n \times \mu} \) denote the set of matrices in \( R^{n \times m} \) whose rows are elements of \( R^\mu \). Then the size of \( R^{n \times \mu} \) is

\[ |R^{n \times \mu}| = |R^\mu|^n = q^n |\mu|, \tag{6} \]

since there are \( |R^\mu| = q^{|\mu|} \) choices for each row. Taking the logarithm on both sides of (6), we obtain

\[ \log_q |R^{n \times \mu}| = n |\mu|. \tag{7} \]

Every matrix \( X \in R^{n \times \mu} \) can be constructed based on its \( \pi \)-adic decomposition

\[ X = X_0 + \pi X_1 + \cdots + \pi^{s-1} X_{s-1}, \]

with each auxiliary matrix \( X_i (i = 0, \ldots, s - 1) \) satisfying:

1) \( X_i[1:n, 1:|\mu_{i+1}|] \) is an arbitrary matrix over \( \mathcal{R}(R, \pi) \), and
2) all other entries in \( X_i \) are zero.

The construction is illustrated in Fig. 3. Clearly, this construction provides a one-to-one mapping from sequences of \( n |\mu| \) \( q \)-ary symbols to matrices in \( R^{n \times \mu} \).

B. Row Canonical Forms in \( \mathcal{T}_\kappa(R^{n \times \mu}) \)

Let \( \mathcal{T}_\kappa(R^{n \times \mu}) \) denote the set of matrices in \( R^{n \times \mu} \) whose shape is \( \kappa \). Then \( |\mathcal{T}_\kappa(R^{n \times \mu})| = 0 \) unless \( \kappa \preceq n \) and \( \kappa \preceq \mu \) (written \( \kappa \preceq n, \mu \) for short). The first constraint comes from the fact that the row canonical form of a matrix in \( R^{n \times \mu} \) has at most \( n \) nonzero rows. The second constraint comes from the fact that row \( A \) is a submodule of \( R^\mu \), for any \( A \in R^{n \times \mu} \). Hence, we will assume that \( \kappa \preceq n, \mu \) in the rest of this paper. As we will see, the set \( \mathcal{T}_\kappa(R^{n \times \mu}) \), together with the row canonical forms in \( \mathcal{T}_\kappa(R^{n \times \mu}) \), plays a crucial role in our coding schemes.

We now enumerate the row canonical forms in \( \mathcal{T}_\kappa(R^{n \times \mu}) \). We need the following lemma.
**Lemma 2:** There is a one-to-one correspondence between row canonical forms in $T_\kappa(R^{n \times \mu})$ and submodules of $R^\mu$ with shape $\kappa$.

The proof is provided in Appendix C. By Lemma 2, the number of row canonical forms in $T_\kappa(R^{n \times \mu})$ is $\left[\frac{n}{\kappa}\right]_q$. It is helpful to bound this number as well as the logarithm of this number. Combining (5) and the fact that $q^{k(m-k)} \leq \left[\frac{m}{k}\right]_q \leq 4q^{k(m-k)}$ (see, e.g., [24, Lemma 4]), we have

$$q^{\sum_{i=1}^s \kappa_i(\mu_i - \kappa_i)} \leq \left[\frac{\mu}{\kappa}\right]_q \leq 4^s q^{\sum_{i=1}^s \kappa_i(\mu_i - \kappa_i)}.$$  \(\text{(8)}\)

Taking logarithms, we obtain

$$\sum_{i=1}^s \kappa_i(\mu_i - \kappa_i) \leq \log_q \left[\frac{\mu}{\kappa}\right]_q \leq \sum_{i=1}^s \kappa_i(\mu_i - \kappa_i) + s \log_q 4.$$  \(\text{(9)}\)

**Example 8:** Let $R = \mathbb{Z}_4$, and let $n = 2$, $\mu = (2, 3)$, $\kappa = (1, 2)$. Then by Lemma 2 there are 18 row canonical forms in $T_\kappa(R^{n \times \mu})$. These 18 row canonical forms can be classified into 4 categories based on the positions of their pivots:

\[
\begin{bmatrix}
1 & * & * \\
0 & 1 & 0 \\
0 & 2 & * \\
2 & 0 & *
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 1 \\
1 & * & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\quad
\begin{bmatrix}
1 & * & 0 \\
* & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 2
\end{bmatrix}.
\]

The first category contains 8 row canonical forms, namely,

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
1 & 1 & 0 \\
0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 2 \\
1 & 1 & 0 \\
0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 \\
1 & 1 & 0 \\
0 & 2 & 0 \\
0 & 2 & 0
\end{bmatrix}.
\]

The second category contains 4 row canonical forms, namely,

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 2 \\
2 & 0 & 0 \\
2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
2 & 2 & 0 \\
2 & 0 & 2
\end{bmatrix}.
\]

The third category contains 4 row canonical forms, namely,

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 2 \\
1 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 0 \\
1 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 2
\end{bmatrix}.
\]

The fourth category contains 2 row canonical forms, namely,

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 0 \\
2 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 2
\end{bmatrix}.
\]

Clearly, the first category contains a significant portion of all possible row canonical forms.

Motivated by the above example, we introduce principal row canonical forms that make up a significant portion of all possible row canonical forms in $T_\kappa(R^{n \times \mu})$. 

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A row canonical form in $\mathcal{T}_\kappa(R^{n \times \mu})$ is called principal if its diagonal entries $d_1, d_2, \ldots, d_r$ ($r = \min\{n, m\}$) have the following form:

$$d_1, \ldots, d_r = 1, \ldots, 1, \pi, \ldots, \pi, \pi^{s-1}, \ldots, \pi^{s-1}, 0, \ldots, 0.$$

Clearly, the first category in Example 8 contains all principal row canonical forms for $\mathcal{T}_\kappa(Z^{n \times \mu})$ with $n = 2$, $\mu = (2, 3)$ and $\kappa = (1, 2)$.

**Proposition 5:** Every principal row canonical form $X \in \mathcal{T}_\kappa(R^{n \times \mu})$ can be constructed based on its $\pi$-adic decomposition

$$X = X_0 + \pi X_1 + \cdots + \pi^{s-1} X_{s-1},$$

with each auxiliary matrix $X_i$ ($i = 0, \ldots, s - 1$) satisfying the following conditions:

1) $X_i[1:\kappa_i+1, 1:\kappa_i+1] = \text{diag}(0, \ldots, 0, 1, \ldots, 1)$.

2) $X_i[1:\kappa_i+1, \kappa_i+1 + 1:\mu_i+1]$ can be any matrix over $\mathcal{R}(R, \pi)$, and

3) all other entries in $X_i$ are zero.

The proof is provided in Appendix C. The construction is illustrated in Fig. 4. Clearly, this construction provides a one-to-one mapping from sequences of $\sum_{i=1}^s \kappa_i (\mu_i - \kappa_i)$ $q$-ary symbols to principal row canonical forms in $\mathcal{T}_\kappa(R^{n \times \mu})$. Note that the number of principal row canonical forms in $\mathcal{T}_\kappa(R^{n \times \mu})$ is $q \sum_{i=1}^s \kappa_i (\mu_i - \kappa_i)$, which is comparable to the number of row canonical forms in $\mathcal{T}_\kappa(R^{n \times \mu})$ in total.

**C. General Matrices in $\mathcal{T}_\kappa(R^{n \times \mu})$**

Next, we count the number of matrices in $R^{n \times \mu}$ of shape $\kappa$, which is a central result in this section. The proof is provided in Appendix C.

**Theorem 2:** The size of $\mathcal{T}_\kappa(R^{n \times \mu})$ is given by

$$|\mathcal{T}_\kappa(R^{n \times \mu})| = |R^{n \times \kappa}| \prod_{i=0}^{\kappa_2-1} (1 - q^{i-n}) \left[\begin{array}{c} \mu \\ \kappa \end{array}\right]_q.$$

(11)
In particular, when the chain length \( s = 1 \), \( R \) becomes \( \mathbb{P}_q \), and this counting result becomes \( \prod_{i=0}^{\kappa_1-1} (q^n - q^i) \left[ \mu_1 \right]/[\kappa_1]_q \), which is the number of \( n \times \mu_1 \) matrices of rank \( \kappa_1 \). We note that Theorem 2 generalizes a theorem of [25] from square matrices to general matrices and from Galois rings to finite chain rings.

Taking logarithms on both sides of (11), we have

\[
\log_q |\mathcal{T}_\kappa(R^{n \times \mu})| = \log_q \left[ \mu \right]_q + \log_q |R^{n \times \kappa}| + \log_q \prod_{i=0}^{\kappa_1-1} (1 - q^{-i n}).
\]

Combining this with (7) and (9), we obtain

\[
\sum_{i=1}^{s} \kappa_i (n + \mu_i - \kappa_i) + \log_q \prod_{i=0}^{\kappa_1-1} (1 - q^{-i n}) \\
\leq \log_q |\mathcal{T}_\kappa(R^{n \times \mu})| \leq \\
\sum_{i=1}^{s} \kappa_i (n + \mu_i - \kappa_i) + \log_q \prod_{i=0}^{\kappa_1-1} (1 - q^{-i n}) + s \log_q 4. \quad (12)
\]

D. Notational Summary

Table I summarizes the notation that will be used extensively in the study of matrix channels. Also listed are finite-field counterparts, which facilitates comparisons of this work with [1].

| notation | meaning | finite-field counterpart |
|----------|---------|-------------------------|
| \( \mu \) | shape | rank |
| \( R^\mu \) | \( R \)-module | vector space \( \mathbb{F}_q^m \) |
| \( R^{n \times \mu} \) | set of matrices with rows from \( R^\mu \) | \( \mathbb{F}_q^{n \times m} \) |
| \( \mathcal{T}_\kappa(R^{n \times \mu}) \) | set of matrices in \( R^{n \times \mu} \) with shape \( \kappa \) | set of matrices in \( \mathbb{F}_q^{n \times m} \) with rank \( t \) |
| RCF(\( A \)) | row canonical form of \( A \) | reduced row echelon form |

VI. Channel Decomposition

In this section, we introduce a channel decomposition technique that converts a matrix channel over certain finite rings into a set of independent parallel matrix channels over finite chain rings. This enables us to focus on matrix channels over finite chain rings, thereby greatly facilitating our study of capacity results and coding schemes in later sections.

As shown in our previous work [8], nested-lattice-based PNC induces a message space of the form \( \Omega = T/\langle d_1 \rangle \times \cdots \times T/\langle d_m \rangle \), where \( T \) is a PID and \( d_m | \cdots | d_1 \). Let \( R \triangleq T/\langle d_1 \rangle \). (Note that \( R \) is a PIR, but not necessarily a finite chain ring.) We can rewrite \( \Omega \) as

\[
\Omega = R \times (d_1/d_2)R \times \cdots \times (d_1/d_m)R;
\]
By the Chinese remainder theorem, we have that
\[ \Omega = \mathbb{Z}_{12} \times \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_2. \]
Then \( \Omega \) can be expressed as \( \mathbb{Z}_{12} \times 2\mathbb{Z}_{12} \times 2\mathbb{Z}_{12} \times 6\mathbb{Z}_{12} \) via the following map:
\[
(a_1 + (12), a_2 + (6), a_3 + (6), a_4 + (2)) \rightarrow (a_1, 2a_2, 2a_3, 6a_4),
\]
where \( a_1 \in \{0, \ldots, 11\}, a_2, a_3 \in \{0, \ldots, 5\}, \) and \( a_4 \in \{0, 1\} \). Clearly, this map is one-to-one.

With this expression, our matrix channel can be written as
\[
Y = AX + BE
\]
where \( X \in R^{n \times m} \) and \( Y \in R^{N \times m} \) are the input and output matrices whose rows are from \( \Omega \), \( E \in R'^{t \times m} \) is the error matrix whose rows (also from \( \Omega \)) correspond to additive (random) error packets. The transfer matrices \( A \in R^{N \times n} \) and \( B \in R^{N \times t} \) are random matrices with some joint distribution, and \( X, (A, B), E \) are statistically independent.

For simplicity of presentation, we sometimes write the channel model as \( Y = AX + Z \), where \( Z = BE \) is called the noise matrix. Clearly, the channel model is an instance of the discrete memoryless channel \((X, p_{Y|X}, Y)\) with input alphabet \( X \in R^{n \times m} \), output alphabet \( Y \in R^{N \times m} \) and channel transition probability \( p_{Y|X} \). The capacity of this channel is given by
\[
C = \max_{p_X} I(X; Y)
\]
where \( p_X \) is the input distribution.

Next, we illustrate how to decompose the matrix channel. To this end, we first decompose the message space \( \Omega \).
Since \( T \) is a PID, \( d_1 \in T \) can be factored as \( d_1 = u_1 p_{11}^{t_{1,1}} \cdots p_{1L}^{t_{1,L}} \), where \( u_1 \) is a unit in \( T \), \( p_{11}, \ldots, p_{1L} \) are primes in \( T \), and \( t_{1,1}, \ldots, t_{1,L} \) are positive integers. Since \( d_{m_1} \mid \cdots \mid d_1 \), we have \( d_j = u_j p_{11}^{t_{j,1}} \cdots p_{jL}^{t_{j,L}} \) \((j = 2, \ldots, m)\), where \( u_j \) is a unit, and \( t_{1,j}, \ldots, t_{j,j} \) are non-negative integers. Now, let
\[
\Omega_\ell \triangleq T/(p_{\ell 1}^{t_{\ell 1}}) \times \cdots \times T/(p_{\ell m}^{t_{\ell m}}), \quad \ell = 1, \ldots, L.
\]
By the Chinese remainder theorem, we have \( \Omega \cong \Omega_1 \times \cdots \times \Omega_L \). This gives rise to a decomposition of \( \Omega \).

Example 10: Let \( \Omega = \mathbb{Z}_{12} \times \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_2 \). Then
\[
\Omega \cong (\mathbb{Z}_4 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times \mathbb{Z}_2
\cong (\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3).
\]

Note that \( \Omega_\ell \) has an interesting interpretation: \( \Omega_\ell \) is a natural projection of \( \Omega \) onto some finite chain ring. Let \( R_\ell \cong T/(p_{\ell 1}^{t_{\ell 1}}) \) (which is a finite chain ring). It is easy to check that \( \Omega_\ell = R_\ell \times p_{\ell 1}^{(t_{\ell 1}-t_{\ell 2})} R_\ell \times \cdots \times p_{\ell 1}^{(t_{\ell 1}-t_{\ell m})} R_\ell \) and that
\[
\Omega_\ell = \{(r_1, \ldots, r_m) \mid (r_1, \ldots, r_m) \in \Omega\}.
\]
We are now ready to introduce the channel decomposition. For any matrix $X \in \mathbb{R}^{n \times m}$, let $X^{[\ell]} \triangleq X \mod R_{\ell}$, the projection of every entry of $X$ onto $R_{\ell}$. Applying this projection to the matrix channel, we obtain $L$ sub-channels

$$Y^{[\ell]} = A^{[\ell]} X^{[\ell]} + Z^{[\ell]},$$

for $\ell = 1, \ldots, L$, as illustrated in Fig. 5. Clearly, each row of $X^{[\ell]}$ (or, $Y^{[\ell]}$, $Z^{[\ell]}$) is from $\Omega_{\ell}$.

![Fig. 5: An illustration of the channel decomposition.](image)

These sub-channels are, in general, correlated with each other. Hence, we have $C \geq \sum_{\ell=1}^{L} C_{\ell}$, where $C_{\ell}$ is the capacity of sub-channel $\ell$. The equality is achieved for certain distributions of $A$ and $Z$. One such distribution is provided in Theorem 3. We need a few definitions. We say a matrix $A$ has rank $t$. A matrix $A \in \mathbb{R}^{n \times m}$ is full rank if $\text{rank} A = \min \{n, m\}$.

**Theorem 3:** Suppose that the transfer matrix $A \in \mathbb{R}^{N \times n}$ ($N \geq n$) is uniform over all full-rank matrices and that the noise matrix $Z \in \mathbb{R}^{N \times m}$ is uniform over all rank-$t$ matrices (whose rows are from $\Omega$). Suppose that $A$ and $Z$ are independent of each other. Then the channel decomposition induces $L$ independent sub-channels

$$Y^{[\ell]} = A^{[\ell]} X^{[\ell]} + Z^{[\ell]}, \quad \ell = 1, \ldots, L,$$

where $A^{[\ell]} \in \mathbb{R}_{\ell}^{N \times n}$ is uniform over full-rank matrices (over $R_{\ell}$), $Z^{[\ell]}$ is uniform over rank-$t$ matrices whose rows are from $\Omega_{\ell}$, and $A^{[\ell]}$ is independent of $Z^{[\ell]}$. Clearly, these sub-channels form a product discrete memoryless channel (DMC). In particular, the capacity of this product DMC is $C = \sum_{\ell=1}^{L} C_{\ell}$.

**Proof:** Note that $A$ is full rank over $R$, if and only if each $A^{[\ell]}$ is full rank over $R_{\ell}$. Hence, the number of full-rank matrices in $\mathbb{R}^{N \times n}$ is equal to the product of the number of full-rank matrices in $R_{\ell}^{N \times n}$ ($\ell = 1, \ldots, L$). In particular, it follows that when $A$ is uniform over full-rank matrices, each $A^{[\ell]}$ is also uniform over full-rank matrices and independent of each other. Similarly, each $Z^{[\ell]}$ is uniform over rank-$t$ matrices and independent of each other. Since $A^{[\ell]}$ and $Z^{[\ell]}$ are projections of $A$ and $Z$, respectively, $A^{[\ell]}$ and $Z^{[\ell]}$ are independent. Therefore, the sub-channels $Y^{[\ell]} = A^{[\ell]} X^{[\ell]} + Z^{[\ell]}$ are independent of each other. In particular, $C = \sum_{\ell=1}^{L} C_{\ell}$.

Theorem 3 says that when $A$ and $Z$ follow certain distributions, the channel decomposition incurs no loss of information. Hence, in this case, it suffices to study each sub-channel independently.

Next, we comment on the assumptions in Theorem 3. First, as we will soon see in later sections, these assumptions allow us to derive clean capacity results and simple coding schemes, based on which more general distributions can be studied (see Section X).
Second, we note that the full-rank assumption on $A$ and the rank-$t$ assumption on $Z$ are reasonable, when the system size is large. To see this, observe that the portion of full-rank matrices in $R^{N \times n}$ is lower-bounded by

$$1 - \sum_{\ell = 1}^{L} \frac{n}{|p_{\ell}|^2(N - n)}.$$  

Clearly, this lower bound tends to 1 as $n$ and $N$ grow. For example, if we set $n = 100$, $N = 110$, and choose $R = Z[2[i] = Z[i]/(1 + i)^2]$, then the lower bound is around 0.999976. Using the same argument, we can show that rank-$t$ matrices make up a significant portion of all possible noise matrices $Z = BE$ for large $t$, $m$, and $N$.

Third, we note that the uniformness assumptions on $A$ and $Z$ provide us with “worst-case” scenarios, which will be elaborated in Section X.

Without loss of generality, we will focus on the case $L = 1$, and so $R$ is a finite chain ring for the remainder of the paper. Suppose that $R$ be a $(q, s)$ chain ring. Let $\mu$ be the shape of $\Omega$. Then, we can write $X \in R^{N \times \mu}$ and $Y, Z \in R^{N \times \mu}$. That is, we may think of the rows of $X$, $Y$ and $Z$ as packets over the ambient space $R^{\mu}$. (To support this ambient space, the length of a packet, denoted by $m$, is equal to $\mu_s$.)

In many situations, it is useful to understand the capacity scaling as the system size and packet length grow. For that reason, we introduce a notion of asymptotic capacity

$$\tilde{C} = \lim_{m \to \infty} \frac{1}{n|\mu|} C = \lim_{m \to \infty} \frac{1}{n|\mu| m^2} C,$$

where we assume that $n = n/m$ and $\mu = (\mu_1, \ldots, \mu_s) = \mu/m$ are fixed. Here, logarithms are taken to the base $q$, so that the capacity $C$ is given in $q$-ary units per channel use and that $\tilde{C}$ is normalized such that $\tilde{C} = 1$ if the channel is noiseless (i.e., $A = I$ and $Z = 0$).

VII. THE MULTIPLICATIVE MATRIX CHANNEL

As a first special case, following [1], we consider the multiplicative matrix channel (MMC) defined by the law

$$Y = AX,$$

where $A \in R^{N \times n}$ is uniform over all full-column-rank matrices and independent from $X \in R^{n \times \mu}$. This model is a special case of the channel model [14] with $Z = 0$.

A. Capacity

The capacity of the MMC can be obtained by investigating the channel transition probabilities. Since full-column-rank matrices preserve the row span, we have row $X = \text{row } Y$. It follows that the channel transition probability $p_{Y|X}(Y|X) > 0$ if and only if row $X = \text{row } Y$. Moreover, we have the following lemma:

**Lemma 3:** The channel transition probabilities satisfy the following two properties.

1) $p_{Y|X}(Y_1|X) = p_{Y|X}(Y_2|X) > 0$, if row $X = \text{row } Y_1 = \text{row } Y_2$.
2) $p_{Y|X}(Y|X_1) = p_{Y|X}(Y|X_2) > 0$, if row $X_1 = \text{row } X_2 = \text{row } Y$.  

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Proof: Since row $Y_1 = row Y_2$, there exists some invertible matrix $P$ such that $PY_1 = Y_2$. Let $A_j = \{ A \in T_n(\mathbb{R}^{N \times n}) \mid AX = Y_j \}$ be the set of transfer matrices such that $AX = Y_j$. Then $A_1$ and $A_2$ have the same size (i.e., $|A_1| = |A_2|$), because $A \in A_1$ if and only if $PA \in A_2$. Hence, we have $p_{Y_1|X}(Y_1|X) = p_{Y_2|X}(Y_2|X)$. In particular, when row $X = row Y_1$, the set $A_1$ is non-empty, and so $p_{Y_1|X}(Y_1|X) > 0$. This proves Part 1). Similarly, we can prove Part 2).

Lemma 3 characterizes the structure of the channel transition probabilities, based on which one can show that the capacity only depends on the number of all possible submodules generated by $X$.

**Theorem 4:** The capacity of the MMC, in $q$-ary symbols per channel use, is given by

$$C_{MMC} = \log_q \sum_{\lambda \leq n, \mu} \left[ \frac{\mu}{\lambda} \right]_q.$$  

A capacity-achieving code $C \subseteq \mathbb{R}^{n \times \mu}$ consists of all possible row canonical forms in $\mathbb{R}^{n \times \mu}$.

Theorem 4 suggests that information should be encoded in the choice of submodules. That is, “transmission via submodules” is optimal here. This naturally generalizes the “transmission via subspaces” strategy in [24].

**Corollary 1:** The capacity $C_{MMC}$ is bounded by

$$\sum_{i=1}^{s} \kappa_i(\mu_i - \kappa_i) \leq C_{MMC} \leq \sum_{i=1}^{s} \kappa_i(\mu_i - \kappa_i) + \log_q 4^s \left( \frac{n + s}{s} \right)$$

where $\kappa_i = \min\{n, \lfloor \mu_i/2 \rfloor \}$ for all $i$.

Proof: First, since $\kappa = (\kappa_1, \ldots, \kappa_s) \leq n, \mu$, we have

$$C_{MMC} = \log_q \sum_{\lambda \leq n, \mu} \left[ \frac{\mu}{\lambda} \right]_q \geq \log_q \left[ \frac{\mu}{\kappa} \right]_q \geq \sum_{i=1}^{s} \kappa_i(\mu_i - \kappa_i),$$

where the second inequality follows from (9).

Second, we have

$$C_{MMC} = \log_q \sum_{\lambda \leq n, \mu} \left[ \frac{\mu}{\lambda} \right]_q \leq \log_q \sum_{\lambda \leq n, \mu} 4^s q^{\sum_i \lambda_i(\mu_i - \lambda_i)} \leq \log_q \sum_{\lambda \leq n, \mu} 4^s q^{\sum_i \kappa_i(\mu_i - \kappa_i)} \leq \log_q 4^s \left( \frac{n + s}{s} \right) q^{\sum_i \kappa_i(\mu_i - \kappa_i)} = \sum_{i=1}^{s} \kappa_i(\mu_i - \kappa_i) + \log_q 4^s \left( \frac{n + s}{s} \right).$$
where the first inequality follows from (9), the second inequality follows from the fact that \( \kappa \) maximizes the quantity \( \sum \lambda_i (\mu_i - \lambda_i) \) subject to the constraint \( \lambda \preceq n, \mu \), and the third inequality follows from the fact that the number of shapes satisfying \( \lambda \preceq n, \mu \) is upper-bounded by \( \binom{n+s}{s} \).

We next turn to the asymptotic capacity of the MMC.

**Theorem 5:** The asymptotic capacity \( \bar{C}_{\text{MMC}} \) is given by

\[
\bar{C}_{\text{MMC}} = \sum_{i=1}^{s} \bar{\kappa}_i (\bar{\mu}_i - \bar{\kappa}_i),
\]

where \( \bar{\kappa} = \kappa/m \) with \( \kappa_i = \min\{n, \lfloor \mu_i/2 \rfloor \} \) for all \( i \).

**Proof:** This follows from Corollary 1 and the fact that \( \frac{1}{m} \log_q 4^s \binom{n+s}{s} \to 0 \), as \( m \to \infty \).

Theorem 5 implies that the shape \( \kappa \) given by \( \kappa_i = \min\{n, \lfloor \mu_i/2 \rfloor \} \) \( (1 \leq i \leq s) \) is “typical” among the shapes of all possible row canonical forms in \( R^{n \times \mu} \). In other words, the row canonical forms of shape \( \kappa \) make up a significant portion of all possible row canonical forms. Hence, the transmitter may encode information in the choice of row canonical forms of shape \( \kappa \) instead of all row canonical forms.

B. A Simple Coding Scheme

In this section, we present a simple coding scheme that achieves the asymptotic capacity in Theorem 5. The key idea is to make the codebook the set of all principal row canonical forms for \( T_\kappa(R^{n \times \mu}) \). In other words, we employ two “reductions” in the code construction. First, we move from all row canonical forms in \( R^{n \times \mu} \) to all row canonical forms in \( T_\kappa(R^{n \times \mu}) \), as suggested by Theorem 5. Then, we move from all row canonical forms in \( T_\kappa(R^{n \times \mu}) \) to all principal row canonical forms in \( T_\kappa(R^{n \times \mu}) \). With these two reductions, our coding scheme not only achieves the asymptotic capacity, but also admits fast encoding and decoding.

1) **Encoding:** The input matrix \( X \) is chosen from the set of principal row canonical forms for \( T_\kappa(R^{n \times \mu}) \) by using the construction presented in Section V-B. Clearly, the encoding rate of the scheme is \( R_{\text{MMC}} = \sum_{i=1}^{s} \kappa_i (\mu_i - \kappa_i) \).

2) **Decoding:** Upon receiving \( Y = AX \), the decoder simply computes the row canonical form of \( Y \). The decoding is always correct by the uniqueness of the row canonical form. By comparing the encoding rate with the asymptotic capacity, we have the following theorem.

**Theorem 6:** The coding scheme described above achieves the asymptotic capacity (16).

VIII. THE ADDITIVE MATRIX CHANNEL

In this section, we consider the additive matrix channel (AMC) defined by the law

\[
Y = X + Z,
\]

where \( Z \) is uniform over \( T_\kappa(R^{n \times \mu}) \) and independent from \( X \). This model is a special case of the channel model (14) with \( A = I \).
A. Capacity

**Theorem 7:** The capacity of the AMC, in $q$-ary symbols per channel use, is given by

$$C_{\text{AMC}} = \log_q |R^{n\times \mu}| - \log_q |T_\tau(R^{n\times \mu})|,$$

achieved by the uniform input distribution.

**Proof:** The AMC is an example of a symmetric discrete memoryless channel, whose capacity is achieved by the uniform input distribution. Note that when $X$ is uniform over $R^{n\times \mu}$, so is $Y$. Thus, we have

$$C_{\text{AMC}} = H(Y) - H(Z) = \log_q |R^{n\times \mu}| - \log_q |T_\tau(R^{n\times \mu})|.$$

**Corollary 2:** The capacity $C_{\text{AMC}}$ is bounded by

$$\sum_{i=1}^{s} (n-\tau_i)(\mu_i - \tau_i) - \log_q 4^s \prod_{i=0}^{\tau_i-1} (1 - q^{i-n}) < C_{\text{AMC}} < \sum_{i=1}^{s} (n-\tau_i)(\mu_i - \tau_i) - \log_q 4^s \prod_{i=0}^{\tau_i-1} (1 - q^{i-n}).$$

**Proof:** It follows immediately from Theorem 7 and (7), (12).

We next turn to the asymptotic behavior of the AMC.

**Theorem 8:** The asymptotic capacity $\bar{C}_{\text{AMC}}$ is given by

$$\bar{C}_{\text{AMC}} = \frac{\sum_{i=1}^{s} (n-\bar{\tau}_i)(\bar{\mu}_i - \bar{\tau}_i)}{n|\bar{\mu}|}.$$

**Proof:** It follows from Corollary 2 and the fact that

$$\frac{1}{m^2} \log_q 4^s \prod_{i=0}^{\tau_i-1} (1 - q^{i-n}) \rightarrow 0, \text{ as } m \rightarrow \infty.$$

B. Coding Scheme

We focus on a special case when $\tau = t$, and present a coding scheme based on the idea of error-trapping in [1]. This scheme achieves the asymptotic capacity for this special case.

1) **Encoding:** Set $v \geq t$. The input matrix $X$ is constructed as

$$X = \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix},$$

where the size of $U$ is $(n-v) \times (m-v)$, and the sizes of other zero matrices are chosen to make $X$ an $n \times m$ matrix. Here, $U$ is chosen from the set $R^{(n-v)\times (\mu-v)}$ by using the construction in Section 5 (as illustrated in Fig. 6). Clearly, the encoding rate of the scheme is $R_{\text{AMC}} = \sum_{i=1}^{s} (n-v)(\mu_i - v)$. 

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2) Decoding: Following [1], we write the noise matrix $Z$ as

$$Z = BE = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \end{bmatrix},$$

where $B_1 \in R^{v \times t}$, $B_2 \in R^{(n-v) \times t}$, $E_1 \in R^{t \times v}$ and $E_2 \in R^{t \times (m-v)}$. The received matrix $Y$ is then given by

$$Y = X + Z = \begin{bmatrix} B_1 E_1 & B_1 E_2 \\ B_2 E_1 & U + B_2 E_2 \end{bmatrix}.$$

Similar to [1], we define that the error trapping is successful if shape $B_1 E_1 = t$. Assume that this is the case. Then by Proposition [3], we have shape $B_1 = \text{shape } E_1 = t$. Consider the submatrix consisting of the first $v$ columns of $Y$. Since shape $B_1 E_1 = t$, the rows of $B_2 E_1$ are completely spanned by the rows of $B_1 E_1$. That is, row $B_2 E_1 \subseteq \text{row } B_1 E_1$. Thus, there exists some matrix $\bar{T}$ such that $B_2 E_1 = \bar{T} B_1 E_1$. Since $E_1$ is full row rank, by Lemma [1] $B_2 E_1 = \bar{T} B_1 E_1$ implies $B_2 = \bar{T} B_1$. It follows that

$$T \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $T = \begin{bmatrix} I & 0 \\ -\bar{T} & I \end{bmatrix}$.

Note also that $TX = X$. Thus,

$$TY = TX + TZ = \begin{bmatrix} B_1 E_1 & B_1 E_2 \\ 0 & U \end{bmatrix},$$

from which the data matrix $U$ is readily obtained.

The decoding is summarized as follows. The decoder observes $B_1 E_1$, $B_1 E_2$, and $B_2 E_1$ thanks to the error traps. The decoder then checks the condition shape $B_1 E_1 = t$. If the condition does not hold, the decoder declares a failure. Otherwise, the decoder finds a matrix $\bar{T}$ such that $B_2 E_1 = \bar{T} B_1 E_1$ (which means $B_2 = \bar{T} B_1$). Since $B_2 = \bar{T} B_1$, the decoder can recover $B_2 E_2$ by using the relation $B_2 E_2 = \bar{T} B_1 E_2$. Clearly, the error probability of the scheme is zero. The failure probability of the scheme is

$$P_f = \Pr[\text{shape } B_1 E_1 \neq t].$$

**Lemma 4:** The failure probability $P_f$ of the above scheme is upper-bounded by $P_f < \frac{2t}{q^{t+v+m}}$.

**Proof:** If $B_1$ and $E_1$ are full rank, then shape $B_1 E_1 = t$. Hence, by the union bound, the failure probability

$$P_f \leq \Pr[E_1 \text{ is not full rank}] + \Pr[B_1 \text{ is not full rank}].$$
Now consider the probability that $E_1$ is full rank. Recall that $E \in R^{t \times \mu}$ is a full-rank matrix chosen uniformly at random. An equivalent way of generating $E$ is to first generate the entries of a matrix $E' \in R^{t \times \mu}$ uniformly at random, and then discard $E'$ if it is not full rank. This suggests that

$$\Pr[E_1 \text{ is full rank}] = \Pr[E'_1 \text{ is full rank} \mid E' \text{ is full rank}] > \Pr[E'_1 \text{ is full rank}],$$

where $E'_1$ consists of the first $v$ columns of $E'$. Thus,

$$\Pr[E_1 \text{ is full rank}] > \frac{|T_t(R^{t \times v})|}{|R^{t \times v}|} = q^{tv} \prod_{i=0}^{t-1} \frac{(1 - q^{i-v})}{q^{stv}} = \prod_{i=0}^{t-1} (1 - q^{i-v}) > 1 - \frac{t}{q^{t+\nu-t}}.$$

Similarly, we can show that

$$\Pr[B_1 \text{ is full rank}] > 1 - \frac{t}{q^{t+\nu-t}}.$$

Therefore, the failure probability $P_f < \frac{2t}{q^{t+\nu-t}}$. \hfill \blacksquare

Recall that the encoding rate of the scheme is $R_{\text{AMC}} = \sum_{i=1}^s (n - v)(\mu_i - v)$. Thus, if we set $v$ such that

$$v - t \to \infty, \text{ and } \frac{v - t}{m} \to 0,$$

as $m \to \infty$, then we have $P_f \to 0$ and $\hat{R}_{\text{AMC}} = \frac{R_{\text{AMC}}}{n|\mu|} \to \bar{C}_{\text{AMC}}$. Therefore, we have the following theorem.

**Theorem 9:** The coding scheme described above can achieve the capacity expression (17) for the special case when $\tau = t$.

**Remark:** The general case can also be handled by combining the above scheme with the successive cancellation technique.

**IX. THE MULTIPLICATIVE-ADDITIONAL MATRIX CHANNEL**

In this section, we consider the multiplicative-additive matrix channel (MAMC) defined by the law

$$Y = AX + Z,$$

where $A \in T_n(R^{N \times n})$ and $Z \in T_\nu(R^{N \times \mu})$ are uniformly distributed and independent from any other variables.
A. Capacity Bounds

Since $A$ is uniform over $T_n(R^{N \times n})$, $A$ is statistically equivalent to $P \begin{bmatrix} 0 \\ I_n \end{bmatrix}$, where $P \in R^{N \times N}$ is uniform over $\text{GL}_N(R)$, $I_n \in R^{n \times n}$ is an identity matrix, and $0 \in R^{(N-n) \times n}$ is a zero matrix. Hence, we have

$$Y = P \begin{bmatrix} 0 \\ I_n \end{bmatrix} X + Z = P \begin{bmatrix} 0 \\ X \end{bmatrix} + Z = P \begin{bmatrix} 0 \\ X \end{bmatrix} + W,$$

where $W = P^{-1}Z$ is uniform over $T_{r}(R^{N \times \mu})$ and independent of $X$.

**Theorem 10:** The capacity of the MAMC, in $q$-ary symbols per channel use, is upper-bounded by

$$C_{\text{AMMC}} \leq \log_q \sum_{\lambda \leq N+n+\tau,\mu} \left[ \mu \right]_q - \log_q |T_{r}(R^{N \times \mu})| + \log_q \sum_{\tau' \geq \tau} |T_{\tau'}(R^{N \times \min(n+\tau,N)})|. \quad (18)$$

**Proof:** Let $U = \begin{bmatrix} 0 \\ X \end{bmatrix} + W$. Then $Y = PU$, and $X, U, Y$ form a Markov chain. Hence, $I(X;Y|U) = 0$. Using the chain rules, we have

$$I(X;Y) = I(U;Y) - I(U;Y|X) + I(X;Y|U) = 0$$

$$= I(U;Y) - H(U|X) + H(U|X,Y)$$

$$= I(U;Y) - H(W) + H(W|X,Y)$$

$$= I(U;Y) - \log_q |T_{r}(R^{N \times \mu})| + H(W|X,Y)$$

Next, we upper bound the terms $I(U;Y)$ and $H(W|X,Y)$. Since shape $U \leq N, n + \tau$, the row span row $U$ has at most $\sum_{\lambda \leq N+n+\tau,\mu} \left[ \mu \right]_q$ choices. Hence, $I(U;Y) \leq \log_q \sum_{\lambda \leq N+n+\tau,\mu} \left[ \mu \right]_q$.

Let $\kappa = \text{shape } Y$. Let $S$ be the Smith normal form of $Y$. Then $S$ contains $\kappa_s$ nonzero diagonal entries. Thus, $Y$ can be expressed as

$$Y = \begin{bmatrix} P_1 & P_2 \\ S_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = P_1 S_{11} Q_1,$$

where $P_1 \in R^{N \times \kappa_s}$, $Q_1 \in R^{\kappa_s \times m}$, and $S_{11} \in R^{\kappa_s \times \kappa_s}$.

Note that

$$\begin{bmatrix} 0 \\ X \end{bmatrix} + W = P^{-1}Y = P^*Q_1,$$

where $P^* = P^{-1}P_1 S_{11}$. Since $Q_1$ consists of the first $\kappa_s$ rows of an invertible matrix $Q$, $Q_1$ is a full-rank matrix. In particular, $Q_1$ contains an invertible $\kappa_s \times \kappa_s$ submatrix. By reordering columns if necessary, we can assume that the left $\kappa_s \times \kappa_s$ submatrix of $Q_1$ is invertible. Write $Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$, where $Q_{11}$, $X_1$, and $W_1$ have $\kappa_s$ columns. We have

$$\begin{bmatrix} 0 & 0 \\ X_1 & X_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} P^*Q_{11} & P^*Q_{12} \end{bmatrix}.$$
It follows that
\[ P^* = \left( \begin{bmatrix} 0 \\ X_1 \end{bmatrix} + W_1 \right) Q_{11}^{-1} \] and \( W_2 = P^* Q_{12} - \left[ \begin{bmatrix} 0 \\ X_2 \end{bmatrix} \right] \).

This suggests that \( W_2 \) can be computed from \( W_1 \) if \( X \) and \( Y \) are known. Thus,
\[ H(W|X,Y) = H(W_1|X,Y) \leq H(W_1|\text{shape } Y). \]

Since \( W_1 \) is an \( N \times \kappa_s \) matrix with shape \( W_1 \preceq \tau \), we have
\[ H(W_1|\text{shape } Y = \kappa) \leq \log_q \sum_{\tau' \preceq \tau} |T_{\tau'}(R^{N \times \kappa_s})|, \]
which is maximized when \( \kappa_s = \min\{N, n + \tau_s\} \). Hence,
\[ H(W_1|\text{shape } Y) \leq \log_q \sum_{\tau' \preceq \tau} |T_{\tau'}(R^{\min\{n+\tau_s,N\}})|. \]

So, \( H(W|X,Y) \leq \log_q \sum_{\tau' \preceq \tau} |T_{\tau'}(R^{\min\{n+\tau_s,N\}})| \), which completes the proof.

**Corollary 3:** The capacity \( C_{\text{MAMC}} \) is upper-bounded by
\[ C_{\text{MAMC}} \leq \sum_{i=1}^{s} (\mu_i - \xi_i) \xi_i + \sum_{i=1}^{s} (\min\{n + \tau_s, N\} - \mu_i) \tau_i + 2s \log_q 4 + \log_q \left( \frac{N+4}{s} \right) + \log_q \left( \frac{\tau^*_s}{s} \right) - \log_q \prod_{i=0}^{\tau_s-1} (1 - q^{-i-N}), \]
where \( \xi_i = \min\{N, n + \tau_i, \lfloor \mu_i/2 \rfloor\} \) for all \( i \). In particular, when \( \mu \geq 2N \) and \( \tau = t \), the upper bound reduces to
\[ C_{\text{MAMC}} \leq \sum_{i=1}^{s} (\min\{n + t, N\} - t)(\mu_i - \min\{n + t, N\}) + 2s \log_q 4 + \log_q \left( \frac{N+4}{s} \right) + \log_q \left( \frac{t^*_s}{s} \right) - \log_q \prod_{i=0}^{t-1} (1 - q^{-i-N}). \]

**Proof:** By \ref{lem:main2}, we have
\[ \log_q \sum_{\lambda \preceq N+n+\tau, \mu} \left[ \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \right] \leq \sum_{i=1}^{s} (\mu_i - \xi_i) \xi_i + 2s \log_q 4 + \log_q \left( \frac{N+4}{s} \right). \]

By \ref{lem:main1}, we have
\[ -\log_q |T_{\tau'}(R^{N \times s})| \leq -\sum_{i=1}^{s} (N + \mu_i - \tau_i) \tau_i - \log_q \prod_{i=0}^{\tau_s-1} (1 - q^{-i-N}). \]

Note that
\[ |T_{\tau'}(R^{N \times \min\{n+\tau_s,N\}})| \leq |R^{N \times \tau'}| \left[ \min_{\tau' \preceq \tau} \{n + \tau_s, N\} \right] \leq 4^s q^\sum_{i=1}^{\tau_s} (N+\min\{n+\tau_s,N\}-\tau'_i) \tau'_i, \]
where the first inequality comes from \ref{lem:main1}, and the second inequality comes from \ref{lem:main2} and \ref{lem:main4}. Hence,
\[ \sum_{\tau' \preceq \tau} |T_{\tau'}(R^{N \times \min\{n+\tau_s,N\}})| \leq 4^s q^\sum_{i=1}^{\tau_s} (N+\min\{n+\tau_s,N\}-\tau'_i) \tau'_i \leq \left( \frac{\tau_s + s}{s} \right) 4^s q^\sum_{i=1}^{\tau_s} (N+\min\{n+\tau_s,N\}-\tau_i) \tau_i \]
where the second inequality comes from the fact that \( \tau \) maximizes the quantity \( q^\sum_{i=1}^{\tau_s} (N+\min\{n+\tau_s,N\}-\tau'_i) \tau'_i \) and the fact that the number of shapes \( \tau' \) with \( \tau' \preceq \tau \) is upper-bounded by \( \left( \frac{\tau_s + s}{s} \right) \). Therefore, we have
\[ \log_q \sum_{\tau' \preceq \tau} |T_{\tau'}(R^{N \times \min\{n+\tau_s,N\}})| \leq \sum_{i=1}^{s} (N + \min\{n + \tau_s, N\} - \tau_i) \tau_i + s \log_q 4 + \log_q \left( \frac{\tau^*_s}{s} \right). \]
Combining all the above results, we have obtained the upper bound. In particular, when \( \mu \geq 2N \) and \( \tau = t \), we have \( \xi_i = \min\{n + t, N\} \) for all \( i \). Substituting this into the upper bound completes the proof. \qed

We next study the asymptotic behavior of \( C_{AMMC} \).

**Theorem 11:** When \( \mu \geq 2N \) and \( \tau = t \), the asymptotic capacity \( C_{MAMC} \) is upper-bounded by

\[
C_{MAMC} \leq \left\{ \begin{array}{ll}
\sum_{i=1}^{s} \frac{n(\mu_i - n - t)}{n|\mu|} & \text{if } n + t \leq N \\
\sum_{i=1}^{s} (N - t)(\mu_i - N) & \text{if } n + t > N.
\end{array} \right.
\] (19)

**Proof:** This follows directly from Corollary \([3]\) \qed

B. A Coding Scheme

We again focus on the special case when \( \mu \geq 2N \) and \( \tau = t \). We describe a coding scheme that achieves the asymptotic bound in Theorem 11.

1) **Encoding:** The encoding is a combination of the encoding strategies for the MMC and the AMC. We first consider the case when \( n + t > N \). Set \( v \geq t \). We construct the input matrix \( X \) as

\[
X = \begin{bmatrix}
0 & 0 \\
0 & \bar{X}
\end{bmatrix},
\]

where the size of \( \bar{X} \) is \( (N - v) \times (m - v) \), and the sizes of other zero matrices are readily available. Here, \( \bar{X} \) is chosen from the set of principal row canonical forms for \( \mathcal{T}_n(R^{(N-v)\times (m-v)}) \) by using the construction in Section V-B, where \( \kappa_i = \min\{N - v, \lfloor (\mu_i - v)/2 \rfloor \} \) for all \( i \). The encoding is illustrated in Fig. 7. Clearly, the encoding rate of the scheme is \( R_{MAMC} = \sum_{i=1}^{s} \kappa_i(\mu_i - v - \kappa_i) \). In particular, when \( \mu \geq 2N \), we have \( \lfloor (\mu_i - v)/2 \rfloor \geq n - v \) for all \( i \). Thus, \( \kappa_i = N - v \) for all \( i \), and the encoding rate is \( R_{MAMC} = \sum_{i=1}^{s} (N - v)(\mu_i - N) \).

![MAMC encoding scheme](image)

Fig. 7: Illustration of the MAMC encoding scheme for \( s = 3 \), \( N = 6 \), \( n = 5 \), \( v = 2 \), \( \mu = (4, 6, 8) \), so that \( \kappa = (1, 2, 3) \).

We then consider the case when \( n + t \leq N \). Similarly, set \( v \geq t \). We construct the input matrix \( X \) as

\[
X = \begin{bmatrix}
0 & \bar{X}
\end{bmatrix},
\]

where the size of \( \bar{X} \) is \( n \times (m - v) \). Again, \( \bar{X} \) is chosen from the set of principal row canonical forms for \( \mathcal{T}_n(R^{n\times (m-v)}) \), where \( \kappa_i = \min\{n, \lfloor (\mu_i - v)/2 \rfloor \} \) for all \( i \). Clearly, the encoding rate is \( R_{MAMC} = \sum_{i=1}^{s} \kappa_i(\mu_i - v - \kappa_i) \). In particular, when \( \mu \geq 2N \), we have \( \kappa_i = n \) for all \( i \), and the encoding rate \( R_{MAMC} = \sum_{i=1}^{s} n(\mu_i - n - v) \).
2) Decoding: The decoder receives \( Y = P \left( \begin{bmatrix} 0 \\ X \end{bmatrix} + W \right) \) and attempts to recover \( \hat{X} \) from the row canonical form of \( Y \). We decompose the noise matrix \( W \) as 
\[
W = BE = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \end{bmatrix},
\]
as we did in Section [VIII]. Clearly, we have 
\[
\begin{bmatrix} 0 \\ X \end{bmatrix} + W = \begin{bmatrix} B_1 E_1 & B_1 E_2 \\ B_2 E_1 & X + B_2 E_2 \end{bmatrix}.
\]
Following [1], we define error trapping to be successful if \( \text{shape } B_1 E_1 = t \). Assume that this is the case. From Section [VIII] there exists some matrix \( T \in \text{GL}_N(R) \) such that 
\[
T \left( \begin{bmatrix} 0 \\ X \end{bmatrix} + W \right) = \begin{bmatrix} B_1 E_1 & B_1 E_2 \\ 0 & \hat{X} \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ 0 & \hat{X} \end{bmatrix}.
\]
Note that
\[
\text{RCF} \left( \begin{bmatrix} E_1 & E_2 \\ 0 & \hat{X} \end{bmatrix} \right) = \begin{bmatrix} \hat{Z}_1 \\ \hat{Z}_2 \end{bmatrix}
\]
for some \( \hat{Z}_1 \in R^{t \times v} \) in row canonical form and some \( \hat{Z}_2 \in R^{t \times (m-v)} \). It follows that 
\[
\text{RCF} \left( \begin{bmatrix} 0 \\ X \end{bmatrix} + W \right) = \text{RCF} \left( \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ 0 & \hat{X} \end{bmatrix} \right) = \begin{bmatrix} \hat{Z}_1 \\ \hat{Z}_2 \end{bmatrix}.
\]
Since \( P \) is invertible, \( \text{RCF}(Y) = \text{RCF} \left( \begin{bmatrix} 0 \\ X \end{bmatrix} + W \right) \), from which \( \hat{X} \) can be readily obtained. Hence, decoding amounts to computing the row canonical form, whose complexity is \( O(nm \min \{n, m\}) \) basic operations over \( R \).

The decoding can be summarized as follows. First, the decoder computes \( \text{RCF}(Y) \). Second, the decoder checks the condition \( \text{shape } B_1 E_1 = t \). If the condition does not hold, the decoder declares a failure. Otherwise, the decoder outputs \( \hat{X} \) from \( \text{RCF}(Y) \).

Let \( n' = \min \{n + v, N\} \). Let \( \hat{Y} \) denote the left-most \( n' \) columns of \( \text{RCF}(Y) \), i.e., \( \hat{Y} = \text{RCF}(Y)[1:N, 1:n'] \). We note that \( \text{shape } B_1 E_1 = t \) if and only if \( \text{shape } \hat{Y} = t + \kappa \). Hence, the error probability of the scheme is zero, and the failure probability \( P_f \) of the scheme is bounded by \( P_f < \frac{2t}{q^{2t + \kappa}} \) (as shown in Section [VIII]).

Finally, if we set \( v \) such that \( v - t \to \infty \) and \( \frac{v-t}{m} \to 0 \), as \( m \to \infty \), we have \( P_f \to 0 \), and \( \bar{R}_{\text{MAMC}} = \frac{R_{\text{MAMC}}}{n|\mu|} \) approaches the upper bound of the asymptotic capacity in Theorem [11].

**Theorem 12:** When \( \tau = t \) and \( \mu \geq 2N \), the coding scheme described above can achieve the upper bound [19].
X. Extensions

Previously, we assume that the transfer matrix \( A \in \mathbb{R}^{N \times n} \) is uniform over all full-rank matrices, and the noise matrix \( Z \in \mathbb{R}^{N \times m} \) is uniform over all rank-\( t \) matrices. In this section, we discuss possible extensions of our previous channel models.

A. Non-Uniform Transfer Matrices

We note that the uniformness assumption on \( A \) leads to a “worst-case” scenario. To see this, let us consider a model identical to the MAMC except for the fact that the transfer matrix \( A \) is chosen according to an arbitrary probability distribution on all full-rank matrices in \( \mathbb{R}^{N \times n} \). It should be clear that the capacity of this channel cannot be smaller than that of the MAMC. This is because our coding scheme does not rely on any particular distribution of \( A \) (as long as \( A \) is full-column-rank and \( Z \) is uniform over all rank-\( t \) matrices), and therefore still works for non-uniform distributions. Hence, we have the following lower bound on the asymptotic capacity \( \tilde{C} \):

\[
\tilde{C} \geq \begin{cases} 
\sum_{i=1}^{s} \frac{n(\bar{\mu}_i - \bar{\mu} - t)}{n|\bar{\mu}|} & \text{if } n + t \leq N \\
\sum_{i=1}^{s} \frac{(N-t)(\bar{\mu}_i - \bar{\mu})}{n|\bar{\mu}|} & \text{if } n + t > N.
\end{cases}
\]  

(20)

On the other hand, the capacity of the channel \( Y = AX + Z \) can be upper-bounded by assuming that the transfer matrix \( A \) is known at the receiver. One can show that the asymptotic capacity is upper-bounded by

\[
\tilde{C} \leq \begin{cases} 
\sum_{i=1}^{s} \frac{n(\bar{\mu}_i - t)}{n|\bar{\mu}|} & \text{if } n + t \leq N \\
\sum_{i=1}^{s} \frac{(N-t)(\bar{\mu}_i - \bar{\mu})}{n|\bar{\mu}|} & \text{if } n + t > N.
\end{cases}
\]  

(21)

Note that when \( \mu_1 \) is much larger than \( N \), the difference between the lower bound (20) and the upper bound (21) is small. In this case, our coding scheme is close to the capacity.

B. Noise Matrix with Variable Rank

We consider a more general case where the number of error packets is allowed to vary, while still bounded by \( t \). More precisely, we assume that \( Z \) is chosen uniform at random from rank-\( T \) matrices, where \( T \in \{0, \ldots, t\} \) is a random variable with an arbitrary probability distribution \( \Pr[T = k] = p_k \). Note that

\[
H(Z) = H(Z, T) = H(T) + H(Z|T)
\]

\[
= H(T) + \sum_k p_k H(Z|T = k)
\]

\[
= H(T) + \sum_k p_k \log q |T_k(R^{N \times \mu})| 
\]

\[
\leq H(T) + \log q |T_t(R^{N \times \mu})|.
\]

Hence, the capacity may be reduced by at most \( H(T) \leq \log q (t + 1) \) compared to the MAMC. This loss is asymptotically negligible for large \( n \) and \( N \).
The coding scheme remains the same. The only difference is that now decoding errors may occur, because the condition shape $B_1 E_1 = t$ becomes shape $B_1 E_1 = T$, which is, in general, impossible to check. Yet, the analysis of decoding is still applicable, and the error probability is bounded by $P_e < \frac{2t}{q^{v-t}}$, which goes to 0 as $v-t \to \infty$.

C. Non-uniform Noise Matrices

We note that the uniformness assumption on $Z$ again gives a “worst-case” scenario. To see this, consider a model identical to the MAMC except for the fact that the noise matrix $Z$ is chosen according to some non-uniform probability distribution on $T(R^{N \times m})$. It should be clear that the capacity can only increase, since the entropy $H(Z)$ always decreases.

To apply our coding scheme in this more general case, we need some transformation. At the transmitter side, let $X = X' Q$, where $Q \in R^{m \times m}$ is chosen uniformly at random (and independent of any other variables) from the set of matrices of the form

$$Q = \begin{bmatrix} Q'_{\mu_1 \times \mu_1} & 0 \\ 0 & I_{m-\mu_1} \end{bmatrix}.$$ 

Here, $Q'$ is an invertible matrix (of size $\mu_1 \times \mu_1$) and $I$ is an identify matrix (of size $(m-\mu_1) \times (m-\mu_1)$). Clearly, $Q$ is invertible by construction. At the receiver side, let $Y' = PY Q^{-1}$, where $P \in R^{N \times N}$ is chosen uniformly at random (and independent of any other variables) from all invertible matrices. Then

$$Y' = PY Q^{-1} = P(AX' Q + Z)Q^{-1} = (PA) X' + PZ Q^{-1}.$$ 

After this transformation, our coding scheme can be applied directly. Moreover, our error analysis still holds, and the failure probability is again bounded by $P_f < \frac{2t}{q^{v-t}}$.

XI. CONCLUSIONS

In this work, we have studied the matrix channel $Y = AX + BE$ where the packets are from the ambient space $\Omega$ of form [4]. Under the assumption that $A$ is uniform over all full-rank matrices and $BE$ is uniform over all rank-$t$ matrices, we have derived tight capacity results and provided polynomial-complexity capacity-achieving coding schemes, which naturally extend the work of [1] from finite fields to certain finite rings. Our extension is based on several new enumeration results and construction methods, for matrices over finite chain rings, which may be of independent interest.

We believe that there is still much work to be done in this area. One direction would be to further relax the assumptions on $A$ and $BE$. Following this direction, we have explored a particular case when $A$ can be any matrix and $BE = 0$ in [26]. Another direction would be to find other applications of the algebraic tools developed in this paper, especially the row canonical form.
Appendix

A. Rings and Ideals

Let $R$ be a ring. We will let $R^*$ denote the nonzero elements of $R$, i.e., $R^* = R \setminus \{0\}$. An element $a$ in $R$ is called a unit if $ab = 1$ for some $b \in R$. We will let $U(R)$ denote the units in $R$. Two elements $a, b \in R$ are said to be associates if $a = ub$ for some $u \in U(R)$. Associatedness is an equivalence relation on $R$.

Suppose $a, b \in R$. The element $a$ divides $b$, written $a \mid b$, if $ac = b$ for some $c \in R$. Let $d \in R^*$ be a nonzero element in $R$. Two elements $a, b$ are said to be congruent modulo $d$ if $d$ divides $a - b$. Congruence modulo $d$ is an equivalence relation on $R$. A set containing exactly one element from each equivalence class is called a complete set of residues with respect to $d$, and is denoted by $\mathcal{R}(R, d)$. Note that the difference $a - b$ between distinct elements $a, b \in \mathcal{R}(R, d)$, $a \neq b$, can never be a multiple of $d$.

An element $a$ of $R^*$ is a called a zero-divisor if $ab = 0$ for some $b \in R^*$. If $R$ contains no zero-divisors, then $R$ is an integral domain. If $R$ is finite and an integral domain, then $R$ is, in fact, a finite field. This latter case is not of central interest in this paper; almost all of the rings considered here will have zero divisors.

Example 11: Let $R = \mathbb{Z}_8 \triangleq \{0, \ldots, 7\}$, under integer addition and multiplication modulo 8. Then $U(\mathbb{Z}_8) = \{1, 3, 5, 7\}$. There are four equivalent classes induced by congruence modulo 4, namely, \{0, 4\}, \{1, 5\}, \{2, 6\}, and \{3, 7\}. An example of a complete set of residues with respect to the element 4 in $\mathcal{R}(\mathbb{Z}_8)$ is $\mathcal{R}(\mathbb{Z}_8, 4) = \{0, 1, 2, 3\}$. The zero-divisors of $\mathbb{Z}_8$ form the set \{2, 4, 6\}.

A nonempty subset $I$ of $R$ that is closed under subtraction, i.e., $a, b \in I$ implies $a - b \in I$, and closed under inside-outside multiplication, i.e., $a \in I$ and $r \in R$ implies $ar \in I$, is called an ideal of $R$. If $A = \{a_1, \ldots, a_m\}$ is a finite nonempty subset of $R$, we will use $\langle a_1, \ldots, a_m \rangle$ to denote the ideal generated by $A$, i.e.,

$$\langle a_1, \ldots, a_m \rangle = \{a_1c_1 + \cdots + a_mc_m : c_1, \ldots, c_m \in R\}.$$

An ideal $I$ of $R$ is said to be principal if $I$ is generated by a single element in $I$, i.e., $I = \langle a \rangle$ for some $a \in I$. A ring $R$ is called a principal ideal ring (PIR) if every ideal $I$ of $R$ is principal. If $R$ is a PIR and also an integral domain, then $R$ is called a principal ideal domain (PID).

An ideal $N$ is said to be maximal if $N \neq R$ and the only ideals containing $N$ are $N$ and $R$ (in other words, $N$ is “maximal” with respect to set inclusion among all proper ideals). If $N$ is a maximal ideal, then the quotient $R/N$ is a field, called a residue field. A ring with a unique maximal ideal is said to be local.

Example 12: The ideals of $\mathbb{Z}_8$ are $\{0\} = \langle 0 \rangle$, $\{0, 4\} = \langle 4 \rangle$, $\{0, 2, 4, 6\} = \langle 2 \rangle$, and $R = \langle 1 \rangle$. Thus, $\mathbb{Z}_8$ is a PIR, and has a unique maximal ideal $\langle 2 \rangle$. The residue field $\mathbb{Z}_8/\langle 2 \rangle$ is isomorphic to the finite field $\mathbb{F}_2$ of two elements.

B. Proofs for Section IV

1) Proof of Proposition 2 We prove the claims one by one.
1) The presence of a pivot \( p \) in a column rules out the possibility of another pivot in the same column and below \( p \), since all entries in the same column below \( p \) must be zero and hence cannot be pivots.

2) Deleting a row of \( A \) does not influence the value or the position of the pivots in the other rows; thus it easy to verify that the modified matrix satisfies the four conditions required for a matrix to be in row canonical form.

3) By definition \( p_k \) has degree smaller than or equal to that of any element in its row. If \( A \) contained an element in a row below row \( k \) of degree smaller than \( d_k \), then the pivot of that row would have degree smaller than \( d_k \), contradicting the property that pivots of smaller degree must occur above pivots of larger degree.

4) By definition \( p_k \) is the earliest element having minimum degree in row \( k \), so every element in row \( k \) occurring earlier than \( p_k \) has degree strictly larger than \( d_k \). We know from 3) that \( A \) contains no element in a row below \( k \) of degree smaller than \( d_k \). If such a row contains an element of degree equal to \( d_k \), then the pivot of that row must occur later than \( p_k \), which implies that every element occurring in that row occurring in column \( c_k \) or earlier has degree strictly larger than \( d_k \).

5) Consider \( w_j \). From 3) we know that \( p_1 \) divides every element of \( A \); in particular, \( p_1 \) divides every element of column \( j \) of \( A \). Since \( w_j \) is a linear combination of these elements, it must be that \( p_1 \) divides \( w_j \).

6) If \( j < c_1 \), we know from 4) that every element in column \( j \) of \( A \) has degree strictly greater than \( d_1 \) and so does every linear combination of these elements, in particular \( w_j \).

2) Proof of Proposition 2 If \( A \) is the zero matrix, then its row canonical form must also be the zero matrix, which is therefore unique. Thus let us assume that \( A \) is nonzero.

We will proceed by induction on \( n \). For \( n = 1 \), the proof is obvious. Thus suppose that \( n > 1 \), and let \( B \) and \( C \) be two row canonical forms of \( A \). Clearly, \( \text{row } B = \text{row } C \), and each row of \( B \) and \( C \) are elements of \( \text{row } A \). Let \( B[1, j_1] \) and \( C[1, j_2] \) be the pivots in the first row of \( B \) and \( C \), respectively. From Proposition 2-5 we have that \( B[1, j_1] \mid C[1, j_2] \) and \( C[1, j_2] \mid B[1, j_1] \); thus \( B[1, j_1] \) and \( C[1, j_2] \) are associates. However, since pivot elements must take the form \( x^i \) for some \( i \), we conclude that \( B[1, j_1] = C[1, j_2] \). Suppose \( j_1 < j_2 \). By Proposition 2-6 we have \( \deg(B[1, j_1]) > \deg(C[1, j_2]) \), contradicting the fact that \( B[1, j_1] = C[1, j_2] \). A similar contradiction arises if \( j_1 > j_2 \). We conclude that \( j_1 = j_2 \), i.e., both \( B \) and \( C \) must have exactly the same pivot element in exactly the same position in their first row.

Now let \( j_1 = j_2 = j \). Consider the submodule of \( \text{row } A \) in which every element has zero in its \( j \)th component. Every element \( a \) of this submodule is a linear combination

\[
a = \sum_{i=1}^{n} b_i B[i, j];
\]

for some choice of coefficients \( b_1, \ldots, b_n \). However, since \( a_j = 0 \), and \( B[i, j] = 0 \) for \( i > 2 \), we must have \( b_1 B[1, j] = 0 \). Since \( B[1, j] \) is the pivot element of the first row of \( B \), it divides every element of that row; thus if \( b_1 B[1, j] = 0 \), then \( b_1 B[1, j] = 0 \), i.e., the first row can only contribute 0 to \( a \). This means that the given submodule is equal to \( \text{row } B[2:n, 1:m] \). Similarly, the given submodule is also equal to \( \text{row } C[2:n, 1:m] \). By Proposition 2-2, both \( B[2:n, 1:m] \) and \( C[2:n, 1:m] \) are in row canonical form. Thus by induction, we have \( B[2:n, 1:m] = C[2:n, 1:m] \).
This implies that $B$ and $C$ can differ in their first row only.

Let us assume that $B[1, :] \neq C[1, :],$ i.e., that the first rows of $B$ and $C$ are not equal, so that $\Delta = (\delta_1, \ldots, \delta_m) = B[1, :] - C[1, :]$ is nonzero. Since $\Delta$ is an element of row $A$ with zero in its $j$th component, we have $\Delta \in \text{row } B[2:n, 1:m]$ from which it follows that

$$\Delta = \sum_{i=2}^{n} c_i B[i, :],$$

for some $c_2, \ldots, c_n \in R$. If $B[2:n, 1:m]$ is the zero matrix, then $\Delta = 0$, which is a contradiction. Otherwise, let $B[2, j_3]$ be the pivot of $B[2, :]$. Note, on the one hand, that $B[i, j_3] = 0$ for all $i > 2$; thus $\delta_{j_3} = c_2 B[2, j_3]$, i.e., $\delta_{j_3}$ must be a multiple of $B[2, j_3]$. On the other hand, because $B[2, j_3]$ and $C[2, j_3]$ are (identical) pivots, $B[1, j_3]$, $C[1, j_3] \in \mathcal{R}(R, B[2, j_3])$. If $B[1, j_3]$ and $C[1, j_3]$ are distinct, their difference, $\delta_{j_3}$, cannot be a multiple of $B[2, j_3]$.

We conclude that $\delta_{j_3} = 0$, i.e., $B[1, j_3]$ and $C[1, j_3]$ are not distinct. Since $B[2, j_3]$ is the pivot of $B[2, :]$ it divides every element of $B[2, :]$; thus if $c_2 B[2, j_3] = 0$, then $c_2 B[2, :] = 0$. Continuing this argument, we have $c_i B[i, :] = 0$ for all $i \geq 2$. Therefore, we have $\Delta = 0$, which is a contradiction. This establishes uniqueness.

C. Proofs for Section $\nabla$

1) Proof of Lemma $\mathcal{G}$ Let $\mathcal{S}$ denote the set of row canonical forms in $\mathcal{T}_c(R^{n \times \mu})$, and let $\mathcal{G}$ denote the set of submodules of $R^\mu$ with shape $\kappa$. Let $\phi : \mathcal{S} \to \mathcal{G}$ be the map that takes a matrix $B \in \mathcal{S}$ to its row module row $B$.

We will show that $\phi$ is a one-to-one correspondence.

If $\phi(B_1) = \phi(B_2)$ then $B_1$ and $B_2$ are left-equivalent, and so $B_1$ is a row canonical form of $B_2$ and vice-versa.

By the uniqueness of the row canonical form, we have $B_1 = B_2$; thus $\phi$ is injective.

Now let $M$ be a submodule of $R^\mu$ with shape $M = \kappa$, and construct a matrix $A$ such that every element in $M$ is a row of $A$. Clearly, row $A = M$ and shape $A = \kappa$. Since $\kappa \leq n$, RCF($A$) has at most $n$ nonzero rows.

Let $B$ be the submatrix of RCF($A$) consisting of the top $n$ rows. Then we have shape $B = \text{shape } A = \kappa$. Hence, $B \in \mathcal{T}_c(R^{n \times \mu})$, and the map $\phi$ is surjective.

2) Proof of Proposition $\mathcal{H}$ We will show that (i) every $X$ constructed as above is a principal row canonical form, and (ii) every principal row canonical form has a $\pi$-adic decomposition following the above conditions.

We begin with Claim (i). First, we track the diagonal entries in $X$. Clearly, by construction, the first $\kappa_1$ diagonal entries in $X$ are 1; they are contributed by $X_0$. The next $\kappa_2 - \kappa_1$ diagonal entries in $X$ are $\pi$; they are contributed by $X_1$. Continuing this argument, we conclude that the diagonal entries in $X$ are indeed of the form $\mathcal{I}$.

Second, we show that $X$ satisfies all the four conditions for row canonical forms.

1) By construction, the first $\kappa_\pi$ rows of $X$ are the only nonzero rows. Hence, $X$ satisfies Condition 1.

2) It suffices to show that the nonzero diagonal entries are precisely the pivots in $X$. Suppose that the $i$th diagonal entry $X_{i, i} = \pi^l$. Then by construction, $\pi^l$ is contributed by $X_1$ and $\kappa_l \leq i \leq \kappa_{l+1}$. Note that for each auxiliary matrix $X_{\pi^l}$, only the first $\kappa_{\pi^l+1}$ rows are nonzero. Thus, the $i$th row in $X_{\pi^l}$ is zero for all $l = 0, \ldots, l - 1$. In particular, $X_{\pi^l}[i, j] = 0$, for all $l = 0, \ldots, l - 1$ and for all $j > i$. Therefore, we have,
for all $j > i$,

$$X[i, j] = \sum_{l'=0}^{s-1} \pi'^l X[i, j] = \sum_{l'=l}^{s-1} \pi'^l X[i, j] = \pi^l \sum_{l'=l}^{s-1} \pi'^{l-l} X[i, j].$$

That is, every $X[i, j]$ is a multiple of $\pi^l$ whenever $j > i$. On the other hand, by construction, $X[i, j] = 0$ whenever $j < i$. It follows that $X[i, i]$ is indeed the pivot of row $i$. Hence, $X$ satisfies Condition 2.

3) Since the nonzero diagonal entries are the pivots, $X$ satisfies Condition 3.

4) Suppose that the $i$th pivot $X[i, i] = \pi^l$. Then, we have $\kappa_l < i \leq \kappa_{l+1}$. Note that for each auxiliary matrix $X_i$, all other entries in column $i$ are zero as long as $l' \geq l$. Thus, we have, for all $j \neq i$,

$$X[j, i] = \sum_{l'=0}^{s-1} \pi'^l X[i, j] = \sum_{l'=0}^{l-1} \pi'^l X[i, j].$$

It follows that $X[j, i] \in R(R, \pi^l)$ for all $j \neq i$. Hence, $X$ satisfies Condition 4.

We turn now to Claim (ii). Let $X$ be a principal row canonical form in $T_n(R^{\times n}).$ Then the diagonal entries in each $X_i$ must satisfy

$$X_i[1, 1], \ldots, X_i[\kappa_{i+1}, \kappa_{i+1} + 1] = 0, \ldots, 0, 1, \ldots, 1 \text{.}$$

Moreover, since $X$ satisfies Condition 4, it follows that each $X_i$ satisfies the first condition described above. Since $X$ satisfies Condition 2, it follows that $X_i[\kappa_{i+1} + 1:n, 1:m]$ is a zero matrix. Finally, due to the constraints imposed by $\mu$, $X_i[1:n, \mu_{i+1} + 1:m]$ is a zero matrix for all $i$. Therefore, each $X_i$ satisfies the second and third conditions. This completes the proof.

3) **Proof of Theorem** We need two technical lemmas. The first lemma is a natural extension of the well-known rank decomposition.

**Lemma 5:** Let $B$ be the row canonical form of $A \in R^{n \times m}$. Let $\tilde{B}$ be the submatrix of $B$ consisting of only nonzero rows. Then $A$ can be decomposed as a product $P_1 \tilde{B}$ of some full-column-rank matrix $P_1$ and the matrix $\tilde{B}$. Moreover, the number of $P_1$ producing such a decomposition is $q^n \Sigma_{i=1}^{\kappa} i(\kappa_{i+1} - \kappa_i)$, where $\kappa =$ shape $A$.

**Proof:** Since $B$ is the row canonical form of $A$, $A = PB$ for some invertible matrix $P \in GL_n(R)$. Since $\kappa =$ shape $A = \text{shape } B$, $B$ has $\kappa_s$ nonzero rows, and $\tilde{B} \in R^{\kappa_s \times m}$. Let $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$, where $P_1 \in R^{n \times \kappa_s}$ and $P_2 \in R^{n \times (n-\kappa_s)}$. Then we have

$$A = PB = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix} = P_1 \tilde{B}. $$
Since $P$ is invertible, $P_1$ is full column rank.

Next, we count the number of such decompositions. Consider the matrix equation $X \tilde{B} = P_1 \tilde{B}$, in unknown $X$. Clearly, the number of decompositions of $A$ is equal to the number of solutions to this matrix equation. Let $\tilde{B}[i, j_i]$ be the pivot of the $i$th row of $\tilde{B}$, for all $i = 1, \ldots, s$. Then $\tilde{B}[i, j_i]$ divides the $i$th row of $\tilde{B}$. It follows that $\tilde{B} = DB'$, where $D = \text{diag} \left( \tilde{B}[1, j_1], \ldots, \tilde{B}[\kappa_s, j_{\kappa_s}] \right)$, and the $i$th row of $B'$ is equal to the $i$th row of $\tilde{B}$ divided by $\tilde{B}[i, j_i]$. Clearly, $B'[i, j_i] = 1$ for all $i = 1, \ldots, \kappa_s$. Since $j_1, \ldots, j_{\kappa_s}$ are all distinct, shape $B' = (\kappa_s, \ldots, \kappa_s)$, which implies that $B'$ is full row rank. By Lemma 2, the number of such classes is $|R^{n \times \mu}|$. Thus, it suffices to count the number of solutions to $XD = P_1 D$. Note that $XD = P_1 D$ is equivalent to the following system of equations
\[ X[i, k] \tilde{B}[k, j_k] = P_1[i, k] \tilde{B}[k, j_k], i = 1, \ldots, n, \ k = 1, \ldots, \kappa_s. \] (22)

Suppose that $\tilde{B}[k, j_k] = \pi^{l_k}$ for some $0 \leq k < s$. Then it is easy to check that the equation $X[i, k] \pi^{l_k} = P_1[i, k] \pi^{l_k}$ has exactly $q^{l_k}$ solutions for $X[i, k]$. It follows that (22) has exactly $q^{n(l_1 + \cdots + l_{\kappa_s})}$ solutions. Finally, by using the fact that $\sum_{k=1}^{s} l_k = \sum_{i=1}^{s} i(\kappa_{i+1} - \kappa_i)$, we complete the proof.

**Lemma 6:** The number of matrices in $R^{n \times \mu}$ having a given row canonical form in $T_\kappa(R^{n \times \mu})$ is equal to
\[ |R^{\kappa_s \times \mu}| \prod_{i=0}^{\kappa_s-1} (1 - q^{i-n}). \]

**Proof:** Let $B$ be a row canonical form in $T_\kappa(R^{n \times \mu})$. Let $\tilde{B}$ be the submatrix of $B$ consisting of only nonzero rows. Clearly, $\tilde{B} \in R^{\kappa_s \times \mu}$. We would like to count the number of matrices in $R^{n \times \mu}$ having the row canonical form $B$.

By Lemma 5, every matrix $A$ with RCF($A$) = $B$ has $q^n \sum_{i=1}^{s-1} i(\kappa_{i+1} - \kappa_i)$ decompositions of the form $A = CB$ for some full-column-rank $C \in R^{n \times \kappa_s}$. Hence, the number of matrices in $R^{n \times \mu}$ having the row canonical form $B$ is equal to the number of full-column-rank matrices of size $n \times \kappa_s$ divided by $q^n \sum_{i=1}^{s-1} i(\kappa_{i+1} - \kappa_i)$, which can be simplified to $|R^{\kappa_s \times \mu}| \prod_{i=0}^{\kappa_s-1} (1 - q^{i-n})$.

We can partition all the matrices in $T_\kappa(R^{n \times \mu})$ based on their row canonical forms: two matrices belong to the same class if and only if they have the same row canonical form. By Lemma 2, the number of such classes is $\left[ \begin{array}{c} \kappa_s \\ n \end{array} \right]_q$. By Lemma 6, the number of matrices in each class is $|R^{\kappa_s \times \mu}| \prod_{i=0}^{\kappa_s-1} (1 - q^{i-n})$. Combining these two results gives us Theorem 2.

**ACKNOWLEDGMENT**

The authors would like to thank Michael Kiermaier for useful discussions on the topic of row canonical forms for matrices over finite chain rings.

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