Do extremists impose the structure of social networks?

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The structure and the properties of complex networks essentially depend on the way how nodes get connected to each other. We assume here that each node has a feature which attracts the others. We model the situation by assigning two numbers to each node, \( \alpha \) and \( \omega \), where \( \omega \) indicates some property of the node and \( \alpha \) the affinity towards that property. A node \( A \) is more likely to establish a connection with a node \( B \) if \( B \) has a high value of \( \omega \) and \( A \) has a high value of \( \alpha \). Simple computer simulations show that networks built according to this principle have a degree distribution with a power law tail, whose exponent is determined only by the nodes with the largest value of the affinity \( \alpha \) (the "extremists"). This means that the extremists lead the formation process of the network and manage to shape the final topology of the system. The latter phenomenon may have implications in the study of social networks and in epidemiology.

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The study of complex networks is currently one of the hottest fields of modern physics \[1, 2, 3\]. A network (or graph) is a set of items, called vertices or nodes, with connections between them, called edges. Nodes linked by an edge are neighbours and the number of neighbours of a node is called degree. Complex weblike structures describe a wide variety of systems of high technological and intellectual importance. Examples are the Internet, the World Wide Web (WWW), social networks of acquaintance or other connections between individuals, neural networks, food webs, citation networks and many others.

One of the crucial questions concerns the formation of these structures. Complex networks are in general systems in evolution, with new nodes/edges that get formed and old ones that get removed or destroyed. The currently accepted mechanism finds its roots in an old idea of Price \[4\], based on the so-called preferential attachment, which means that a newly formed node \( A \) builds an edge with a preexisting node with a probability that is proportional to the degree of the latter node. Networks constructed in this way \[5, 6, 7\] have a degree distribution with a power law tail, as observed in real networks. This simple rule, however, makes implicitly the strong assumption that each node is at any time informed about the degree of all other nodes, which is certainly not true, especially for gigantic systems which contain many millions of nodes, like the WWW. We rather believe that the key behind the building of a connection between a pair of nodes lies essentially in the mutual interaction of the two nodes, (almost) independently of the rest of the system: two persons usually become friends because they like each other.

In this letter we have social networks in mind, but nevertheless we will speak generally about networks, as we believe that our model has a more general validity. The mechanism we propose is that any node has some property (beauty, richness, power, etc.) by which the others are attracted. We indicate the property with a positive number \( \omega \), the attractiveness by another positive number \( \alpha \). We assume that high values of \( \omega \) correspond to a high degree of the property (the most beautiful people, for instance). Both \( \omega \) and \( \alpha \) are attributes of single nodes/individuals, so they take in general different values for different nodes. What we need is a knowledge of the distribution of the \( \omega \) and \( \alpha \) in the network. For the property \( \omega \), distributions which vanish for high values of \( \omega \), like exponentials or power laws, are realistic. As far as the affinity \( \alpha \) is concerned, it is less clear which distributions can be considered plausible, therefore we tested several functions. We remark that the idea that the nodes have individual appeal already exists in the literature on complex networks. Bianconi and Barabási \[8\] assigned a parameter \( \eta \) called "fitness" to each node of the network and the linking probability becomes proportional to the product of the degree with the fitness of the target node. In the same framework, Ergün and Rodgers \[9\] proposed instead a variation of the fitness theme which eliminates preferential attachment, so that the formation principle of networks lies in the attraction which nodes exert on each other by virtue of their individual quality/importance, which is actually in the spirit of our work. So, in \[9\], the linking probability is simply a function of the fitness values of the pair of nodes, and several possible choices for this function are introduced and discussed.

Our expression for the probability \( p_{AB} \) of a node \( A \) to build an edge with a node \( B \) is also a function of the individual attributes of the nodes, \( \omega \) and \( \alpha \). We adopt
the ansatz

$$p_{AB} = \frac{c_B}{\phi(\omega_B)^{\alpha A}}.$$  \hspace{1cm} (1)

where $c_B$ is a normalization constant and $\phi(\omega)$ the distribution function of the property $\omega$, whereas we will indicate with $\psi(\alpha)$ the distribution function of the affinity $\alpha$. What (1) says is that the pairing probability is inversely proportional to the relative frequency of the property $\omega$ in the network. Thinking again of a social system, the idea is that there is a general tendency to be more attracted by those subjects who are characterized by high values of $\omega$. In a network of sexual relationships, for instance, the best looking people usually have the greatest chances to be chosen as sexual partners. We believe that the choices of the people are not influenced by the absolute importance of $\omega$, which is a vague and abstract concept, but rather by the perception of the importance of the property $\omega$ within the society, which is related to its distribution. This is why we associated the pairing probability to the relative frequency $\phi(\omega)$ and not directly to $\omega$, at variance with \[1\]. Accordingly, the larger $\omega$, the lower the occurrence $\phi(\omega)$ of that degree of the property in the network, and the edge-building probability gets higher. On the other hand, for a given node $B$, characterized by its property $\omega_B$, the other nodes will feel an attraction towards $B$ which varies from a subject to another. This modulation of the individual attraction is expressed by the exponent $\alpha_A$ in (1). The probability $p_{AB}$ increases with $\alpha_A$, justifying the denomination of "affinity" we have given to the parameter $\alpha$. The parameter $\alpha$ must be taken in the range $[0, 1]$ for normalization purposes \[11\].

The expression \[11\] might look ad hoc, because of the power law dependence on $\alpha$. This is not the case, due to the freedom we have in the choice of the $\alpha$’s: if we have an arbitrary probability $p_{AB}$ for the node $A$ to be linked to $B$, in most cases one is able to find a number $\alpha_A$ such that $p_{AB} = c_B/\phi(\omega_B)^{\alpha A}$, so to reproduce the wished probability.

Our simple model is a generalization of the so-called "cameo principle" which has recently been introduced by two of the authors \[12\]. There, the affinity $\alpha$ was the same for all nodes and there was consequently no correlation between pairs of nodes. In this case it was rigorously proven that the network has indeed a degree distribution with a power law tail and that the exponent $\gamma$ is a simple function of $\alpha$, more precisely

- if $\phi(\omega)$ decreases as a power law with exponent $\beta$ when $\omega \to \infty$, $\gamma = 1 + 1/\alpha - 1/\alpha\beta$;
- if $\phi(\omega)$ vanishes faster than any power law when $\omega \to \infty$, $\gamma = 1 + 1/\alpha$.

For our generalization we studied the problem numerically, by means of Monte Carlo simulations, but an analytical proof of the results we show here is in progress.

In order to build the network we pick up a node $A$ and build $m$ edges with the other nodes of the network, with probability given by \[1\]. The procedure is then repeated for all other nodes of the network. We remark that our construction process is static, i.e. all nodes of the network are there from the beginning of the process and neither nodes are added nor destroyed. However, the principle can as well be implemented in a dynamical way, with new nodes which are progressively added to the network \[12\].

We fixed the outdegree $m$ to the same value for all nodes, as it is done in the famous model of Barabási and Albert \[3\] (we set $m = 100$). The number $N$ of nodes was mostly $100\,000$.

We have always used a simple exponential for $\phi(\omega)$. Fig. \[4\] shows the cumulative degree distribution of the network constructed with a uniform affinity distribution $\psi(\alpha) = \text{const.}$, for $\alpha$ in the range $[0, 0.7]$. The cumulative distribution is the integral of the normal distribution. So, for a value $k$ of the degree we counted how many nodes have degree larger than $k$. The summation reduces considerably fluctuations and the analysis gets easier. If the degree distribution is a power law with exponent $\gamma$, its integral will be again a power law but with exponent $\gamma - 1$.

In Fig. \[4\] we see that indeed the cumulative distribution ends as a straight line in a double logarithmic plot, so it has a power law tail. We performed many trials, by varying the range of the uniform distribution $\psi(\alpha)$, and by using other kinds of distribution functions for $\alpha$, like gaussians, exponentials and power laws. We found that the result holds in all cases we considered.

Another striking feature of our findings is shown in Fig. \[4\]. Here we plot the cumulative degree distributions for two networks, where $\psi(\alpha)$ is uniform and we chose the affinity ranges such that they share the same...
upper limit $\alpha_{\text{max}}$ ([0, 0.8] and [0.3, 0.8], respectively, so $\alpha_{\text{max}} = 0.8$). We see that the tails of the two curves have the same slope, which suggests that $\gamma$ only depends on $\alpha_{\text{max}}$. We repeated this experiment several times, for different ranges and taking as well exponential and Gaussian distributions for $\alpha$. Within errors, we confirmed this remarkable result.

Fig. 3 shows how the exponent $\gamma - 1$ of the cumulative degree distribution varies with $\alpha_{\text{max}}$. The pattern of the data points follows an hyperbola $a/\alpha_{\text{max}}$, with a coefficient $a = 1.29$; this is very close to what one gets for the original Cameo principle [12], where $\gamma - 1 = 1/\alpha$. It is likely that in the limit of infinite nodes the coefficient would indeed converge to one. Since $\alpha_{\text{max}}$ can be chosen arbitrarily close to zero, from the ansatz $a/\alpha_{\text{max}}$ we deduce that, within our model, we are able to build networks with any value of $\gamma$ greater than (about) 2. This is fine, as for the great majority of complex networks $\gamma \geq 2$ as well.

So, we have discovered that the nodes with the highest affinity $\alpha$, that we call "extremists" for obvious reasons, are responsible for the exponent $\gamma$ of the power law tail of the degree distribution of the network. This is valid independently of the distribution $\psi(\alpha)$, so it works even in the case where the extremists are just a very small part of the population [13]. If we consider terrorism networks, for example, the leaders of the group (those with highest charisma/$\omega$) are the hubs of the network, i.e. the most connected individuals, but their relative importance is determined by the most fanatic followers (those with largest $\alpha$). We have then shown that there is a sort of time-dependent hierarchy among the nodes: the extremists lead the formation process, the hubs dominate the structure once the network is built.

We give here a hint to the analytical proof of this result. Let us consider the simple case of a discrete distribution $\Psi(\alpha)$ of the form

$$\Psi(\alpha) = \sum_{i=1}^{m} \lambda_i \delta(\alpha - \alpha_i),$$  \hspace{1cm} (2)

with $\lambda_i > 0$ and $\sum_{i=1}^{m} \lambda_i = 1$. So the fraction of nodes $x$ with $\alpha(x) = \alpha_i$ is $\lambda_i > 0$. The validity of the result lies in the fact that the global degree distribution is given by a superposition of the degree distributions associated to nodes with the same $\alpha_i$. Since each of those distribution has a power law tail [12], the overlap is dominated by the term having the fattest tail, i.e. the smallest exponent $\gamma_i$, which corresponds to the maximum $\alpha_{\text{max}}$ of the $\alpha_i$'s, due to the relation $\gamma = 1 + 1/\alpha_i$. In this way, any function can be considered as the limit of a sum like [2], when the number of terms goes to infinity; for more details see [12] and [13].

We know that the exponent $\gamma$ is a crucial feature of complex networks in many respects. For epidemic spreading, for example, there is no non-zero epidemic threshold [12] so long as $\gamma \leq 3$, which would have catastrophic consequences. If the network is in evolution, to control the extremists would mean to be able to exert an influence on the future topology of the network, which can be crucial in many circumstances.

From a practical point of view, it is not obvious how to model things like attractiveness (or fitness), which usually are out of the domain of quantitative investigations. However, our result on the leading role of the extremists is quite robust, as it does not depend on the specific function $\psi(\alpha)$ that one decides to adopt. The attempt to mathematically modelize apparently abstract features of social systems (here the "attractiveness") is not isolated [16]. The last few years witnessed a big effort to describe society as a physical system [17, 18, 19], with people playing the role of atoms or classical spins undergoing elementary interactions. There are meanwhile
several models to explain how hierarchies and consensus may originate in a society and in these models abstract items like opinion, confidence, etc. are associated to well-defined mathematical variables. Although one must always be careful not to demand too much from such models, the first results of this line of research are encouraging; with the consensus model of Sznajd one could reproduce the final distribution of votes among candidates in Brazilian and Indian elections.

In conclusion, we have introduced a simple criterion for the nodes of a complex network to choose each other as terminals of mutual connections: each node has a property $\omega$ which attracts the other nodes to an extent which depends on another individual parameter $\alpha$. Networks built in this way are always characterized by a degree distribution with a power law tail. The exponent of the power law is determined uniquely by those nodes which are most sensible to the property $\omega$. Acting on such nodes could be an effective way to control the structure of evolving networks.

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